Dynamics of dust ion acoustic waves in the Low Earth Orbital (LEO) plasma region

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We consider the system consisting of the plasma environment in the Low Earth Orbital (LEO) region in presence of charged space debris objects. This system is modelled for the first time as a weakly coupled dusty plasma; where the charged space debris objects are treated as weakly coupled dust particles with two dimensional space and time dependences. The dynamics of dust ion acoustic waves in the system is found to be governed by a forced Kadomtsev-Petviashvili (KP) type model equation, where the forcing term depends on the distribution of debris objects. Accelerated planar solitary wave solutions are obtained from the forced KP equation upon transferring the frame of reference, and applying a specific non holonomic constraint condition. For a different constraint condition, the forced KP equation also admits lump wave solutions. The dynamics of accelerated lump wave solutions, which are happened to be pinned, is also explored. Approximate dust ion acoustic wave solutions for different types of localized space debris or forcing functions are analysed using different perturbation methods. Our work provides a much clearer insight of the debris dynamics in the plasma medium in the LEO region, revealing some novel results that are immensely helpful for various space missions by space agencies in the world. Different perspectives for practical applications of our theoretical results are discussed in detail.

I. INTRODUCTION

Upsurge in research endeavours encompassing dynamics of space debris objects in near-earth atmospheres has been gaining significant attention by numerous scientists across countries from recent years. Space debris objects [1] include dead satellites, meteoroids, destroyed spacecrafts, other inactive materials resulting from many natural phenomena etc. which are being levitated in extraterrestrial regions especially in near-earth space. These are also combingingly referred to as space junk. The space debris objects are substantially found in the Low Earth Orbital (LEO) [2] and Geosynchronous Earth Orbital (GEO) regions. Also, their number is continuously being increased nowadays due to various artificial space missions which result in dead satellites, destroyed spacecrafts etc. and many natural hazards occurring in space. These debris objects are of varying sizes and shapes, and move with different velocities [3]; thus, cause significant harm to running spacecrafts. Therefore, to avoid these deteriorating effects, active debris removal (ADR) has become a challenging problem in twenty-first century. Some indirect detection techniques for space debris have also been developed by different authors [4–6]. This paper interprets a much more realistic situation; which is not considered by these authors and may provide more justifiable indirect evidence of presence of debris objects.

We model, for the first time, the system consisting of the plasma environment in presence of space debris objects in the Low Earth Orbital (LEO) region as a weakly coupled dusty plasma system. Space debris objects become charged in a plasma medium because of different mechanisms such as photo-emission, electron and ion collection, secondary electron emission [7] etc. These charged debris objects of varying sizes ranging from as small as microns to as big as centimetres, and, in certain conditions, even more than centimetres [3, 8] move with different velocities. Therefore, there can be finite chances that the individual dynamics of charged debris objects are mutually correlated; which results in weak coupling among them. Numerous recent works on dynamics of space debris are performed without taking into account this paramount effect as far as our knowledge goes. In this work, the weak coupling effect among charged debris objects is accomplished with the introduction of a two dimensional space and time dependent forcing function arising out of debris objects. Consequently the forcing function depends physically on the distribution of space debris objects in the LEO plasma region, and their possible relative motions. This new generalized forcing function represents a two dimensional extension of recent works done by Sen et al. [5], and Mukherjee et al. [6].

This paper is organized in the following manner. The detailed derivation of nonlinear evolution equation in the form of forced Kadomtsev-Petviashvili (KP) equation is shown in Section II. The dynamics of accelerated planar solitary wave solutions is discussed in section III. Similarly, lump wave solutions arising from the nonlinear evolution equation in the form of forced Kadomtsev-Petviashvili equation is explored in section IV. In section V, we investigate approximate solutions of the nonlinear evolution equation taking into account different types of localized forcing.
functions representing the effect of debris objects. The thorough discussions of our obtained results along with possible applications are provided in section VI. Conclusive remarks are given in section VII; followed by acknowledgements and bibliography.

II. DERIVATION OF NONLINEAR EVOLUTION EQUATION

We consider the propagation of finite amplitude nonlinear dust ion acoustic waves (DIAWs) in the Low Earth Orbital (LEO) region due to the motion of orbital charged debris objects. The LEO region consists of a low temperature low density plasma along with the abundance of debris objects. We assume that the ion species is treated as a cold species, i.e. the ion pressure is neglected and the electrons obey the Boltzmann distribution. Following Sen et al. [5], and Amin et al. [9], the basic normalized system of equations in our system in (2+1) dimensions is given by

\[
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) + \frac{\partial}{\partial y}(nv) = 0, \tag{1}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial x} = 0, \tag{2}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial y} = 0, \tag{3}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + n - e^\phi - Z_d n_d = Z_d S(x, y, t), \tag{4}
\]

where the following normalizations have been used:

\[
x \rightarrow x/\lambda_d; \quad y \rightarrow y/\lambda_d; \quad t \rightarrow C_s \lambda_d t; \quad n \rightarrow n/n_0; \quad u \rightarrow u/C_s; \quad v \rightarrow v/C_s; \quad \phi \rightarrow e^\phi/k_B T_e, \tag{7}
\]

where \( \lambda_d \) is electron Debye length, \( C_s \) is ion acoustic speed, \( k_B \) is Boltzmann constant, \( T_e \) is electron temperature, and \( n_0 \) is equilibrium ion density. Equations 1, 2, 3, and 4 represent ion continuity equation, ion momentum conservation equations in \( x \) and \( y \) directions, and Poisson’s equation respectively; where \( n, u, v, \) and \( \phi \) denote the density, \( x \) and \( y \) component of velocity, and electrostatic potential of the ion species respectively. Similarly, \( n_e, n_d, z_d, \) and \( v_d \) denote electron density, dust density, dust charge in electron units, and dust velocity respectively. The term \( S(x, y, t) \) in the RHS of equation 4 represents a charge density source arising due to the weakly coupled charged debris objects having two dimensional space and time dependences. Therefore, \( S(x, y, t) \) also depends on the distribution of debris objects in the LEO region. In our model, the effects of space debris objects modelled as weakly coupled dust particles enter the system through quasi-neutrality condition and forcing term in the Poisson equation 4. In the pioneering work done by Sen et al. [5], they have considered the source term to have one dimensional space and time dependences. Also, Sen et al. have taken both ion acoustic solution and forcing term to have the form of line solitons with constant amplitudes and constant velocities; which is subsequently generalized by Mukherjee et al. [6] by considering more realistic time dependent amplitudes and velocities for both ion acoustic solution and forcing term. We generalize both the work done by Sen et al. and Mukherjee et al. to consider a forcing term in two spatial and one temporal dimensions in order to accomplish the weakly coupled nature of orbital debris objects. We do not follow any restrictions as taken by Sen et al. as well.

We derive the evolution equation corresponding to the nonlinear DIAWs in our system following the well-known reductive perturbation technique (RPT) [10]; where we expand the dependent variables of the system as:

\[
n = 1 + e^2 n_1 + e^4 n_2 + ..., \tag{8}
\]

\[
u = e^2 u_1 + e^4 u_2 + ..., \tag{9}
\]

\[
v = e^2 v_1 + e^4 v_2 + ..., \tag{10}
\]

\[
\phi = e^2 \phi_1 + e^4 \phi_2 + ..., \tag{11}
\]

\[
Z_d = 1 + e^{5/2} Z_{d1} + e^{7/2} Z_{d2} + ..., \tag{12}
\]

\[
n_d = e^2 n_{d1} + e^4 n_{d2} + ..., \tag{13}
\]
where $\epsilon$ is a small dimensionless expansion parameter characterizing the strength of nonlinearity in the system. We consider a weak space-time dependent localized debris function which vanishes at space infinities. After scaling we have

$$S(x, y, t) = \epsilon^4 f(x, y, t),$$

where $f(x, y, t)$ can have any spatially localized form that is consistent with the weakly coupled charged debris dynamics as per our approach. Similarly, the independent variables are also rescaled as:

$$\xi = \epsilon(x - v_p t); \tau = \epsilon^3 t; \eta = \epsilon^2 y,$$

where $v_p$ is the phase velocity of the wave in $x$ direction. Accordingly, the differential operators are expressed in terms of the rescaled variables as:

$$\frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial \xi}; \frac{\partial}{\partial t} = -\epsilon v_p \frac{\partial}{\partial \xi} + \epsilon^3 \frac{\partial}{\partial \tau} = \epsilon^2 \frac{\partial}{\partial \eta}.$$

Putting these expanded and rescaled variables in equation 1 and collecting different powers of $\epsilon$, we get

$$O(\epsilon^3): -v_p \frac{\partial n_1}{\partial \xi} + \frac{\partial u_1}{\partial \xi} = 0,$$  

$$O(\epsilon^5): -v_p \frac{\partial n_2}{\partial \xi} + \frac{\partial n_1}{\partial \tau} + \frac{\partial}{\partial \xi}\left(u_2 + n_1 u_1\right) + \frac{\partial v_1}{\partial \eta} = 0.$$

Similarly, using equation 2, we get

$$O(\epsilon^3): -v_p \frac{\partial u_1}{\partial \xi} + \frac{\partial \phi_1}{\partial \xi} = 0,$$  

$$O(\epsilon^5): -v_p \frac{\partial u_2}{\partial \xi} + \frac{\partial u_1}{\partial \tau} + u_1 \frac{\partial u_1}{\partial \xi} + \frac{\partial \phi_2}{\partial \xi} = 0.$$

Again, using equation 3, we get

$$O(\epsilon^4): -\frac{\partial v_1}{\partial \xi} + \frac{\partial \phi_1}{\partial \eta} = 0,$$

Finally, equation 4 yields:

$$O(\epsilon^2): -\phi_1 + n_1 = 0,$$  

$$O(\epsilon^4): \frac{\partial^2 \phi_1}{\partial \xi^2} - \frac{\phi_1^2}{2} + n_2 = f.$$  

Now equations 17, 19, 21, and 22 give

$$n_1 = \phi_1 = \pm u_1; \quad v_p^2 = 1; \quad \frac{\partial v_1}{\partial \xi} = \pm \frac{\partial \phi_1}{\partial \eta}.$$  

From equations 18 and 20, we get

$$\frac{\partial n_2}{\partial \xi} = \frac{1}{v_p}\left[\frac{\partial n_1}{\partial \tau} + \frac{\partial}{\partial \xi}\left(u_2 + n_1 u_1\right) + \frac{\partial v_1}{\partial \eta}\right],$$  

$$\frac{\partial u_2}{\partial \xi} = \frac{1}{v_p}\left[\frac{\partial u_1}{\partial \tau} + u_1 \frac{\partial u_1}{\partial \xi} + \frac{\partial \phi_2}{\partial \xi}\right].$$

Then, we differentiate partially equation 23 wrt $\xi$ once and use the relations given by equations 24, 25, and 26. After some simplifications, we get the nonlinear evolution equation as:

$$\left(\pm n_1 + n_1 n_1 + \frac{1}{2} n_1 n_1 \xi\right) \xi + \frac{1}{2} n_1 n_1 \eta = \frac{1}{2} f_{\xi \xi}.$$
where the subscripted variables denote partial derivatives. This is the forced Kadmotsev-Petviashvili (KP) equation, i.e. the generalization of forced KdV equation to two dimensional space. In particular, this is categorized as KPII equation. In order to get a convenient form, we apply the frame transformation:

\[ n_1 = 3U; \quad \tau = 2T; \quad \xi = \pm X; \quad \eta = \frac{1}{\sqrt{3}} Y; \quad f = 3F. \]  

(28)

Then, equation 27 becomes

\[ (U_T + 6UU_X + U_{XXX})_X + 3U_{YY} = F_{XX}. \]  

(29)

This is the final nonlinear evolution equation describing the dynamical evolution of DIAWs in presence of space debris in the LEO plasma region; which contains a low density and low temperature plasma. We know that KP, modified KP, coupled KP [11], generalized KP [12], and nonplanar KP [13] equations are very well-known and different methods of solutions are already known [14, 15]. Also, KP Burgers equation, different solutions [16] and their dynamical evolution have been analysed for plasmas. These analyses have been performed without taking into account the forcing term. We analyse different types of solutions of forced KP equation 29 in the following sections.

III. DYNAMICS OF PLANAR SOLITARY WAVE SOLUTIONS OF NONLINEAR EVOLUTION EQUATION

The equation 29, upon applying a transformation \( X \rightarrow -X \) and \( F \rightarrow -F \), admits exact dust ion acoustic accelerated planar soliton solution when the forcing function \( F \) satisfies a specific non holonomic constraint:

\[ (F_{XXX} + 4UF_X + 2U_XF)_X + F_{YY} = -2a(T)U_{XX}, \]  

(30)

where \( a(T) \) denotes time dependent amplitude of the forcing function \( F \). Here, we generalize the constraint condition, taken by Mukherjee et al. [6] to solve forced Korteweg-de Vries (KdV) equation, to two dimensions in order to apply the same for forced KP equation. Then, after some simplifications, the exact dust ion acoustic accelerated planar soliton solution is given by

\[ U(X,Y,T) = 2(k_x^2 + k_y^2)sech^2[k_xX + k_yY + (k_x + k_y)V(T)T + \theta]; \quad V(T) = 4(k_x^2 + k_y^2)^2 \left( \int a(T) dT \right) \frac{2(k_x^2 + k_y^2)^2}{T}, \]  

(31)

where \( k_x \) and \( k_y \) represent the \( x \) and \( y \) components of wave number, and \( \theta \) denotes the phase of the solitary wave solution respectively. The acceleration of the solitary wave solution comes due to presence of \( V(T) \). This planar solitary wave solution typically looks like as shown in figure 1. In order to analyse the characteristic time evolution of this planar solitary wave solution, we plot it at origin, i.e. at \( X = 0 \), and \( Y = 0 \). This is as shown in figure 2; typically looking like a one dimensional soliton. The complete dynamical evolution of the planar soliton can be picturized by superposing evolutions given by figures 1, and 2. Similarly, the forcing function is given by

\[ F(X,Y,T) = -a(T)sech^2[k_xX + k_yY + (k_x + k_y)V(T)T + \theta]. \]  

(32)

Therefore, it is concluded that both amplitude and velocity of the forcing function given by equation 32 change whereas only velocity of the dust ion acoustic soliton solution given by equation 31 changes wrt time. The typical plot for the forcing function \( F \) is shown in figure 3, and its temporal evolution at origin is shown in figure 4.

IV. DYNAMICS OF LUMP WAVE SOLUTIONS OF NONLINEAR EVOLUTION EQUATION

We know that unforced KP equation admits lump wave solutions. It is also very well-known that KP equation is further categorized as KPI and KPII equations depending upon the signs of coefficients. In our work, we obtain forced KPII equation, as mentioned in section II. Numerous works have been done for KPI equation and its solutions by many researchers whereas that for KPII equation is very less. Therefore, in order to solve the forced KPII equation 29, we proceed by applying some novel methods as given in the following subsections.
FIG. 1: Planar soliton solution for \( k_x = 1, k_y = 1, a(T) = \cos T \) and \( \theta = 0 \) at time \( T = 0.5\pi \).

FIG. 2: Variation of planar soliton solution at origin against time \( T \) for \( k_x = 1, k_y = 1, a(T) = \cos T \) and \( \theta = 0 \).

A. Constrained solution

Recently, Yong et al. [17] reported a self consistent model for forced KPI equation, namely KPIESCS, i.e. KPI equation with a self-consistent source; where the source or forcing term obeys a specific constraint condition. For solving forced KPII equation 29, we closely follow their procedure. Firstly, we approximate the forcing function as:

\[
F = -8\Omega \Omega^*,
\]

(33)

where \( \Omega \) denotes a complex variable. This is identical to that taken by Yong et al. Then, equation 29 becomes

\[
[U_T + 6UU_X + U_{XXX} + 8(\Omega \Omega^*)_X]_X + 3U_{YY} = 0.
\]

(34)

Secondly, in contrast to Yong et al., we have taken a slightly different constraint condition to be satisfied by forcing function \( F \), which is given by

\[
\pm \Omega_Y = \Omega_{XX} + U\Omega.
\]

(35)
Likewise, the system consisting of equations 34 and 35 is modelled in our work as KPIIESCS, i.e. KPI equation with a self-consistent source. This is analogous to KPIIESCS which is modelled by Yong et al. for KPI equation. Following their work to implement Hirota bilinear method [18], we proceed further by applying a transformation $U = 2(\ln H)_{XX}$ and $\Omega = \frac{G}{H}$ to equations 34, and 35 to obtain the bilinear equation

$$(D_X D_T + D_X^4 + 3D_Y^2)H.H + 8GG^* - cH^2 = 0, \quad (36)$$

$$(\pm D_Y - D_X^2)G.H = 0, \quad (37)$$

where $D$ is the famous Hirota bilinear operator, and $c$ is the constant of integration arose during the simplification process. In order to investigate lump solutions for the above bilinear equations 36, and 37, we assume

$$H = 1 + \xi_1^2 + \xi_2^2, \quad (38)$$

$$G = G_R + iG_I, \quad (39)$$

where $\xi_1$ and $\xi_2$ are two new variables, and $G_R$ and $G_I$ are the real and imaginary parts of the complex variable $G$ respectively. Here $i$ denotes the imaginary number. Our assumption for $H$ guarantees analyticity and rational
localization of solutions. These new variables are defined in terms of the old variables as
\[ \begin{align*}
\xi_1 &= a_1 X + a_2 Y + a_4 T + a_4; \\
\xi_2 &= a_5 X + a_6 Y + a_7 T + a_8, \\
G_R &= b_0 + b_1 \xi_1 + b_2 \xi_2 + b_3 \xi_1^2 + b_4 \xi_2^2; \\
G_I &= c_0 + c_1 \xi_1 + c_2 \xi_2 + c_3 \xi_1^2 + c_4 \xi_2^2,
\end{align*} \tag{40} \]
where \( a_i (1 \leq i \leq 8) \), \( b_j (0 \leq j \leq 4) \), and \( c_l (0 \leq l \leq 4) \) are real constants to be determined subsequently. After rigorous calculations taking the constant of integration \( c = 8 \) for simplicity, we observe that these real constants are related by the following constraints.
\[ \begin{align*}
a_3 &= \frac{(a_2^2 - a_1^4)[3(a_1^4 + a_2^2)^2 + 16a_1^4]}{a_1(a_1^4 + a_2^2)^2}, \\
a_5 &= 0, \\
a_6 &= a_1^2, \\
a_7 &= \frac{2a_1 a_2[3(a_1^4 + a_2^2)^2 - 16a_1^4]}{(a_1^4 + a_2^2)^2}, \\
b_0 &= \frac{b_3(a_2^2 - 3a_1^4)}{a_1^4 + a_2^2}, \\
b_1 &= kc_1, \\
b_2 &= kc_2, \\
b_4 &= b_3,
\end{align*} \tag{41} \]
where \( k \) is a constant which requires to satisfy \( b_3^2(1 + k^2) = 1 \). The localization of the associated solutions is also guaranteed here because the non-zero determinant condition as illustrated in \([19]\) is satisfied, i.e. \( a_1 a_6 - a_2 a_5 \neq 0 \). Incidentally these constraints given in above equation 42 are identical to those obtained by Yong et al. One admissible lump solution \( U \) typically looks like
\[ U(X, Y, T) = \frac{4[(X - 2Y + 10.9T - 2)^2 + (Y - 9.4T)^2]}{1 + (X - 2Y + 10.9T - 1)^2 + (Y - 9.4T)^2}. \tag{43} \]
We plot the above solution 43 in figure 5 at time \( T = 0 \), and, then, plot it at origin, i.e. at \( X = 0 \) and \( Y = 0 \) against time \( T \); which is as shown in figure 6. The complete dynamical evolution of the lump solution for our KPII type model equation 29 can be pictured by the superposition of the dynamical behaviours as represented in figures 5, and 6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Lump solution at \( T = 0 \).}
\end{figure}

### B. Pinned solution

In this subsection, we discuss the solution of forced KPII equation 29 employing the following approximation method. The equation 29 can be rewritten as:
\[ (U_T + 6UU_X + U_{XXX} + 3 \int U_{YY}dX)_X = F_{XX}. \tag{44} \]
FIG. 6: Variation of $U$ against $T$ at origin, i.e. $X = 0$, and $Y = 0$.

We approximate the forcing function as

$$ F = a(t)U. \quad (45) $$

Therefore, equation (46) becomes

$$ (U_T + 6UU_X + U_{XXX} + 3 \int U_Y dX)_X = a(t)U_{XX}. \quad (46) $$

Now we apply a frame transformation: $X' = X + b(T); Y' = Y; T' = T$. This implies

$$ \frac{\partial}{\partial X} = \frac{\partial}{\partial X'}, \frac{\partial}{\partial Y} = \frac{\partial}{\partial Y'}, \frac{\partial}{\partial T} = b_t \frac{\partial}{\partial X'} + \frac{\partial}{\partial T'}. \quad (47) $$

Substituting equation (47) in equation (46), we obtain an unforced KP equation for the condition $b(T) = \int_0^T a(T) dT$ after some simplifications, which is as given below:

$$ (U_{T'} + 6UU_{X'} + U_{X'X'} X')_X + 3U_{Y'Y'} = 0. \quad (48) $$

This represents unforced KPII type model equation which admits lump solution of the form

$$ U = 4 - \frac{[X' + miY' + 3(m^2 - n^2)T']^2 + n^2(iY' + 6mT')^2 + 1/n^2}{[[X' + miY' + 3(m^2 - n^2)T']^2 + n^2(iY' + 6mT')^2 + 1/n^2]^2}, \quad (49) $$

where $m$ and $n$ are free parameters. Substituting the values of $X'$, $Y'$ and $T'$, we get

$$ U = 4 - \frac{[X + \int_0^T a(T) dT + miY + 3(m^2 - n^2)T]^2 + n^2(iY + 6mT)^2 + 1/n^2}{[[X + \int_0^T a(T) dT + miY + 3(m^2 - n^2)T]^2 + n^2(iY + 6mT)^2 + 1/n^2]^2}. \quad (50) $$

For simplicity, we plot the solution (50) at $Y = 0$. This is as shown in figure 7. In order to analyse the temporal evolution of the above solution, particularly, due to the term $\int_0^T a(T) dT$, we also plot the solution at origin, i.e. at $X = Y = 0$. At origin, the solution (50) looks like

$$ U = 4 - \frac{[\int_0^T a(T) dT + 3(m^2 - n^2)T]^2 + 36m^2n^2T^2 + 1/n^2}{[[\int_0^T a(T) dT + 3(m^2 - n^2)T]^2 + 36m^2n^2T^2 + 1/n^2]^2}. \quad (51) $$
and the plot wrt time $T$ is as shown in figure 8. Similarly, the expression for the forcing function $F$ is obtained using equation 45 as:

$$F = 4a(t) \frac{-[X + \int_0^T a(T)dT + miY + 3(m^2 - n^2)T]^2 + n^2(iY + 6mT)^2 + 1/n^2}{\{[X + \int_0^T a(T)dT + miY + 3(m^2 - n^2)T]^2 + n^2(iY + 6mT)^2 + 1/n^2\}^2}. \quad (52)$$

This forcing function $F$, for $Y = 0$, is typically as shown in figure 9. At origin, the solution becomes

$$F = 4a(t) \frac{-[\int_0^T a(T)dT + 3(m^2 - n^2)T]^2 + 36m^2n^2T^2 + 1/n^2}{\{[\int_0^T a(T)dT + 3(m^2 - n^2)T]^2 + 36m^2n^2T^2 + 1/n^2\}^2}. \quad (53)$$

and its plot is given in figure 10. From equations 50 and 52, it is clear that the solution $U$ and forcing function $F$ move with the same time dependent velocity. That’s why these are referred to as ”pinned accelerated lump solutions”. The complete dynamical evolution of these pinned lump solutions is also given by equations 50, and 52.
V. MORE REALISTIC SOLUTIONS FOR NONLINEAR EVOLUTION EQUATION

In order to get a more realistic solution of the nonlinear evolution equation in form of forced KP equation 29, we proceed by taking different kinds of possible localized debris functions or forcing functions $F$. For this, we proceed in a gradual manner; starting from spatially independent, and only temporally dependent simple forcing function, i.e. $F = F(T)$ to different types of both spatially and temporally dependent complicated forcing functions, i.e. $F = F(X, Y, T)$. These forcing functions are necessarily localized; which is discussed in detail in this section, and is as given below.

A. $f = f(T)$

For dealing with this kind of special spatially independent, and only temporally dependent forcing function, we follow closely the recent work by Mukherjee [20]. It is found that the forced KPII equation becomes exactly
We approximate the solution as the following perturbation series:

\[
(\ddot{U}_T + 6\dot{U}_X + \ddot{U}_{XX})_X + 3\ddot{U}_Y = 0,
\]

(54)

where we \( A(T) \) is an arbitrary function of time, and we assume that it is correlated to the forcing function as \( \frac{dA}{dT} = 6 \int F_X(T)dT \). We know that this is unforced KPII equation whose solution is given by

\[
\ddot{U} = 4\frac{[X + miY + 3(m^2 - n^2)T]^2 + n^2(iY + 6mT)^2 + 1/n^2}{([X + miY + 3(m^2 - n^2)T]^2 + n^2(iY + 6mT)^2 + 1/n^2)^2}. 
\]

(55)

Back-boosting to the original variables, finally we get the solution as

\[
U = 4\frac{[X - A(T) + miY + 3(m^2 - n^2)T]^2 + n^2(iY + 6mT)^2 + 1/n^2}{([X - A(T) + miY + 3(m^2 - n^2)T]^2 + n^2(iY + 6mT)^2 + 1/n^2)^2} + \int F_X(T)dT.
\]

(56)

This equation 56 represents the final solution for the forced KP equation 29. The dynamical evolution of the DIAWs in the LEO plasma region can be given by this equation for different kinds of time dependent debris functions or forcing functions. These functions can be of different types such as Gaussian, hyperbolic, functions etc. These functions have to be localized in order to represent debris dynamics.

**B. \( f=f(X,Y,T) \)**

For this kind of general space time dependent forcing function, we proceed to find approximate solution using perturbation theory following the work done by Moroz [21]. For implementing this, we assume that the forcing function \( F(X,Y,T) \) is a rapidly varying function. This implies the introduction of two time scales in the calculation. Therefore, we introduce a new time scale: \( \tau = \frac{T}{\epsilon_1} \) with \( 0 < \epsilon_1 << 1 \). This results in the transformation of the time derivative as

\[
\frac{\partial}{\partial T} \rightarrow \frac{1}{\epsilon_1} \frac{\partial}{\partial \tau} + \frac{\partial}{\partial T}.
\]

(57)

In accordance with this, we also take for simplicity: \( F'' = F_{XX} = F''(X,Y,\tau) \). After this, equation 29 becomes

\[
(U_T + 6U_X + U_{XX})_X + 3U_{YY} + \frac{1}{\epsilon_1}U_{X\tau} = F''(X,Y,\tau).
\]

(58)

We approximate the solution as the following perturbation series:

\[
U = \sum_{n=0}^{\infty} U_n(X,Y,T,\tau)\epsilon_1^n.
\]

(59)

We put this perturbative form of \( U \) in equation 58, and collect different powers of \( \epsilon_1 \) as given below.

\[
O(1/\epsilon_1) : U_{0X\tau} = 0,
\]

(60)

\[
O(1) : U_{0XT} + 6(U_0U_{0X})_X + U_{0XX} + 3U_{0YY} + U_{1X\tau} = F'',
\]

(61)

\[
O(\epsilon_1) : U_{1XT} + 6(U_0U_{1X})_X + 6(U_1U_{0X})_X + U_{1XX} + 3U_{1YY} + U_{2X\tau} = 0,
\]

(62)

and so on. We need to solve these perturbative equations at each order to evaluate the full perturbative solution. Integrating equation 60 twice, we obtain

\[
U_0(X,Y,T,\tau) = V_0(X,Y,T) + W(Y,T,\tau),
\]

(63)

where \( V_0 \) and \( W \) are new functions to be determined later. We know that lump solutions are spatially localized solutions implying that the lump wave vanishes as \( X \rightarrow \pm \infty \) with fixed \( Y \). This implies that \( W = 0 \). Therefore, we get

\[
U_0(X,Y,T,\tau) = V_0(X,Y,T).
\]

(64)
This is evident from the above equation 64 that zeroth order solution $U_0$ does not depend upon the fast time scale $\tau$. This is because when $\epsilon_1$ is zero, there is no forcing function, and we get the solution of unforced KPII equation. Now we separate the $\tau$ independent and dependent parts of equation 61, and apply equation 64 to get

$$V_{0XT} + 6(V_0V_{0X})_X + V_{0XXXX} + 3V_{0YY} = 0, \quad (65)$$

$$U_{1X\tau} = F''(X,Y,\tau). \quad (66)$$

It can easily be seen from equation 65 that $V_0$ is the solution of unforced KPII equation. On solving equation 66, we get

$$U_1(X,Y,T,\tau) = V_1(X,Y,T) + G_1(X,Y,T,\tau) = \int \int F''(X,Y,\tau) dX d\tau, \quad (67)$$

where $V_1$ is to be determined in the next order. From equation 62, we again separate the $\tau$ independent and dependent parts to get

$$V_{1XT} + 6(V_0V_{1X})_X + 6(V_1V_{0X})_X + V_{1XXXX} + 3V_{1YY} = 0, \quad (68)$$

$$G_{1XT} + 6(V_0G_{1X})_X + 6(G_1V_{0X})_X + G_{1XXXX} + 3G_{1YY} + U_{2X\tau} = 0, \quad (69)$$

where we have used equation 67. Differentiating equation 65 wrt $X$, one can see that equation 68 is identical provided $V_1 = V_0X$. Integrating equation 69, we obtain

$$U_2(X,Y,T,\tau) = V_2(X,Y,T) + G_2(X,Y,T,\tau), \quad (70)$$

where

$$G_2(X,Y,T,\tau) = -\int (G_{1T} + 6V_0G_{1X} + 6G_1V_{0X} + G_{1XXX} + 3\int G_{1YY} dX) d\tau. \quad (71)$$

Following the same procedure, we can get $V_2(X,Y,T)$ in the next order as $V_2 = \frac{1}{2}V_0XX$. By interpreting all these results, we can summarize the solutions at different orders as given below.

$$U_0(X,Y,T,\tau) = V_0(X,Y,T), \quad (72)$$

$$U_1(X,Y,T,\tau) = V_0X + \int \int F''(X,Y,\tau) dX d\tau, \quad (73)$$

$$U_2(X,Y,T,\tau) = \frac{1}{2}V_0XX - \int (G_{1T} + 6V_0G_{1X} + 6G_1V_{0X} + G_{1XXX} + 3\int G_{1YY} dX) d\tau, \quad (74)$$

and so on. Therefore, it is concluded that the zeroth order solution is given by the solution of unforced KPII equation whereas higher order solutions depend upon the zeroth order solution, and nature of forcing function. The complete solution of forced KPII equation 29 can be obtained by substituting these perturbative solutions at different orders in equation 59. This kind of perturbative solution is also discussed by Mukherjee [20] recently in the context of forced KPI equation. Substituting different types of space time dependent localized forcing functions like Gaussian, hyperbolic etc. in these perturbative solutions at different orders upto desired accuracy, we can get the final solution describing evolution of DIAWs in the LEO region. Although this solution is approximate, this is very much useful as it represents a more realistic solution through perturbation methods.

**VI. DISCUSSIONS AND APPLICATIONS**

In this work, we obtain exact accelerated lump wave solutions, as explored in section IV, due to the presence of forcing function $F$ in (2+1) dimensions. Again, these accelerated lump solutions are happened to be 'pinned', i.e. both source or forcing function and analytical solution move with the same velocity. We also obtain accelerated planar solitary wave solution when the forcing function $F$ satisfies a certain constraint condition as represented by equation.
30, and lump wave solutions when the constraint condition changes to that represented by equation 35. But the condition that forcing function obey a certain constraint is not more realistic. Therefore, from a practical point of view, there are more chances that pinned accelerated lump wave solutions are to be resulted as consequences of the presence of weakly coupled debris objects, which are self-consistently related to the dust ion acoustic solution. We obtain this inference after we generalize the debris problem taken by Sen et al. [5] to two dimensions to model as a weakly coupled dusty plasma system in order to make the solutions more realistic and practicable, after considering time dependent amplitudes of the forcing functions.

We know that lump wave solutions are special kinds of rational function solutions that are localized in all directions in space whereas solitary wave solutions are exponentially localized solutions in certain directions. Therefore, lump waves can be more stable as compared to solitary waves resulting from KP equation and are detectable by external means. Recently, Sen et al. [5] devise an indirect method of detection of centimetre-sized debris objects by observation of precursor line solitons. But they neglected many crucial effects including the time dependence of amplitudes of forcing function which represents debris objects. This may imply the observation of a debris object even when no line soliton is to be detected in its vicinity. Therefore, our observation of pinned accelerated lump wave solutions due to presence of debris objects may pave a novel way of detection of debris objects through observation of lump waves irrespective of the sizes and shapes of debris objects by advanced sensors or technologies equipped in space-crafts. This also happens to be the generalization of the work done by Kulikov et al. [4] where they conclude the detection of debris objects through growth of amplitudes. From an outsider point of view, our work provides a much clearer insight of space debris dynamics in the LEO plasma region through mathematical modelling. The novel results of our work can be verified by ground-based or in situ observations by different advanced techniques.

VII. CONCLUSIONS

We obtain a forced KP type model equation in our work describing the dynamical evolution of dust ion acoustic waves in presence of space debris in the LEO plasma region; which is a low density and low temperature plasma containing an enormous number of space debris objects. Thereafter, we analyse different types of possible solutions of this nonlinear evolution equation through various analytical techniques. Planar solitary wave solutions, and lump wave solutions are discovered in our work as a result of charged space debris motion. Particularly pinned accelerated lump wave solution is resulted showing a special feature that both the analytical solution, and debris or forcing function move with the same velocity self-consistently. We propose that this lump wave solution, being localized in nature, can represent an indirect evidence of presence of debris objects. Solutions for only time dependent, and both space and time dependent forcing functions are explored as well. Our work provides a detailed dynamical evolution of nonlinear dust ion acoustic waves generated in the LEO plasma region by charged space debris motion; taking into account various realistic circumstances that are not discussed by researchers working in this field till now as far as our knowledge goes.

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