Physical fields and Clifford algebras II.
Neutrino field

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Abstract
The neutrino field is considered in the framework of a complex Clifford algebra $C_3 \cong C_2 \oplus \hat{C}_2$. The factor-algebras $C_2$ and $\hat{C}_2$, which are obtained by means of homomorphic mappings $C_3 \rightarrow C_2$ and $C_3 \rightarrow \hat{C}_2$, are identified with the neutrino and antineutrino fields, respectively. In this framework we have natural explanation for absence of right-handed neutrino and left-handed antineutrino.

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1 Fundamental automorphisms of Clifford algebra

In Clifford algebra \( ^lK_n \), where \( K \) is a field of characteristic 0 (\( K = R, \ K = \Omega, \ K = C \)), there exists four fundamental automorphisms [1, 2]:

1) An automorphism \( A \rightarrow A \).

This automorphism, obviously, be an identical automorphism of algebra \( ^lK_n \), \( A \) is an arbitrary element of \( ^lK_n \).

2) An automorphism \( A \rightarrow A^\star \).

In more details, for arbitrary element \( A \in ^lK_n \) there exists decomposition

\[
A = A' + A'',
\]

where \( A' \) is an element consisting of homogeneous odd elements, and \( A'' \) is an element consisting of homogeneous even elements, respectively. Then the automorphism \( A \rightarrow A^\star \) is that element \( A'' \) is not changed, and element \( A' \) is changed the sign:

\[
A^\star = -A' + A''.
\]

If \( A \) is a homogeneous element, then

\[
A^\star = (-1)^k A,
\]

(1)

where \( k \) is a degree of element.

3) An anti-automorphism \( A \rightarrow \tilde{A} \).

The anti-automorphism \( A \rightarrow \tilde{A} \) be a reversion of the element \( A \), that is the substitution of each basis element \( e_{i_1i_2...i_k} \in A \) by element \( e_{i_ki_{k-1}...i_1} \):

\[
e_{i_ki_{k-1}...i_1} = (-1)^{\frac{k(k-1)}{2}} e_{i_1i_2...i_k}.
\]

Therefore, for any \( A \in ^lK_n \) we have

\[
\tilde{A} = (-1)^{\frac{k(k+1)}{2}} A.
\]

(2)

4) An anti-automorphism \( A \rightarrow \tilde{A}^\star \).

This anti-automorphism be a composition of the anti-automorphism \( A \rightarrow \tilde{A} \) with the automorphism \( A \rightarrow A^\star \). In the case of homogeneous element from formulae (2) and (3) follows

\[
\tilde{A}^\star = (-1)^{\frac{k(k+1)}{2}} A.
\]

(3)

It is obvious that \( \tilde{\tilde{A}} = A \), \( (A^\star)^\star = A \), and \( (\tilde{A}^\star)^\star = A \).

For example consider the anti-automorphism \( A \rightarrow \tilde{A} \) for electromagnetic field. It is known [3, 4] that the components of electric and magnetic vectors may be represented by the following element of algebra \( R_3 \):

\[
A = E^1 e_1 + E^2 e_2 + E^3 e_3 + H^1 e_{23} + H^2 e_{31} + H^3 e_{12}.
\]

(4)
Since the maximal basis element \( \omega = e_{123} \) (\( \omega^2 = -1 \)) is belong to a center of algebra \( R_3 \) the element \( \frac{1}{2} \) (by force of identity \( R_3 = C_2(\frac{1}{2}) \)) may be rewritten as

\[
A = (E^1 + \omega H^1) e_1 + (E^2 + \omega H^2) e_2 + (E^3 + \omega H^3) e_3.
\]  

Further, under action of anti-automorphism \( A \rightarrow \bar{A} \) the elements \( \frac{1}{2} \) and \( \bar{\frac{1}{2}} \) are adopt the following form (by force of (2))

\[
\bar{A} = E^1 e_1 + E^2 e_2 + E^3 e_3 - H^1 e_{23} - H^2 e_{31} - H^3 e_{12},
\]

\[
\bar{\frac{1}{2}} = (E^1 - \omega H^1) e_1 + (E^2 - \omega H^2) e_2 + (E^3 - \omega H^3) e_3.
\]

It is easy to see that elements \( \frac{1}{2} \) and \( \bar{\frac{1}{2}} \) are isomorphic to the general elements of algebras \( C_2 \) and \( C_2^* \): \( A = F^0 e_0 + F e_1 + F^2 e_2 + F^3 e_{12} \) and \( \bar{A} = \bar{F}^0 \bar{e}_0 + \bar{F} e_1 + \bar{F}^2 e_2 + \bar{F}^3 e_{12} \), where \( F^0 = \bar{F}^0 = 0 \). The algebras \( C_2 \) and \( C_2^* \) are described the photon fields with left-handed and right-handed polarization, respectively. Therefore, under action of anti-automorphism \( A \rightarrow \bar{A} \) the photon field with left-handed polarization turn into the photon field with right-handed polarization and back.

The other example, also impotent in physical applications, we have for the anti-automorphism \( A \rightarrow \bar{A}^* \) which may be connected with a charge conjugation \( \bar{\frac{1}{2}} \).

## 2 Clifford algebras with odd dimensionality and spinor representations

Later on we shall restricted by the case of a field \( K = C \), because this case is a most interesting for physics. Consider an algebra \( C_3 \), the general element of which has a following form:

\[
A = a^0 e_0 + \sum_{i=1}^{3} a^i e_i + \sum_{i=1}^{3} \sum_{j=1}^{3} a^{ij} e_{ij} + a^{123} e_{123}.
\]

Here the volume element \( \omega = e_{123} \) (\( \omega^2 = -1 \)) is belong to a center of algebra \( C_3 \) (that is commutes with all elements of \( C_3 \)). Let

\[
\lambda_+ = \frac{1 + i \omega}{2}, \quad \lambda_- = \frac{1 - i \omega}{2}
\]

the mutually orthogonal idempotents of algebra \( C_3 \) are satisfying to conditions

\[
\lambda_+ + \lambda_- = 1, \quad \lambda_+ \lambda_- = 0, \\
\lambda_+^2 = \lambda_+, \quad \lambda_-^2 = \lambda_-.
\]
At this the idempotents $\lambda_+$ and $\lambda_-$ together with volume element $\omega$ are commutes with all elements of $C_3$ whose general element now may be represented as

$$A = (\lambda_+ + \lambda_-)A = \lambda_+ A + \lambda_- A.$$  

Consider the elements of the form $\lambda_. A$. Choosing by turns $A = e_0, e_1, e_2, e_3$ we obtain

$$\frac{1}{2}(1 - i e_{123}), \quad \frac{1}{2}(e_1 - i e_{23}), \quad \frac{1}{2}(e_2 - i e_{31}), \quad \frac{1}{2}(e_3 - i e_{12}).$$  

(9)

Analogously, for the elements of the form $\lambda_+. A$

$$\frac{1}{2}(1 + i e_{123}), \quad \frac{1}{2}(e_1 + i e_{23}), \quad \frac{1}{2}(e_2 + i e_{31}), \quad \frac{1}{2}(e_3 + i e_{12}).$$  

(10)

It is obvious that the same result we obtain if $A = e_{23}, e_{31}, e_{12}, e_{123}$. It is easy to see that elements $\lambda_+. A$ are make a subalgebra of $C_3$ since

$$\lambda_+ A + \lambda_- A' = \lambda_+ A + A', \quad \lambda_+ AA' = \lambda_+ A' A,$$

$$a\lambda_+ A = \lambda_+ aA.$$  

The elements of the form $\lambda_. A$ also are make a subalgebra of $C_3$. The subalgebras $\{\lambda_. A\}$ and $\{\lambda_+. A\}$ are isomorphic (by force of identity $i \equiv \omega$) to algebras $C_2$ and $C_2^\ast$, respectively. Thus, the algebra $C_3$ is decomposed into a direct sum of two subalgebras $\{\lambda_. A\} \cong C_2$ and $\{\lambda_+. A\} \cong C_2^\ast$ (the product of elements from different subalgebras is equal to zero since $\lambda_+ \lambda_- = 0$). Analogously, for the algebra $C_5$ we have the same decomposition. By means of idempotents $\lambda_+ = \frac{1 + \omega}{2}$ and $\lambda_- = \frac{1 - \omega}{2}$ ($\omega = e_{12345}, \omega^2 = 1$) the algebra $C_5$ is decomposed into a direct sum of two subalgebras $\{\lambda_+. A\} \cong C_4$ and $\{\lambda_. A\} \cong C_4^\ast$. In general

$$C_{4m'+1} \cong C_{4m'} \oplus C_{4m'},$$  

$$C_{4m'-1} \cong C_{4m'-2} \oplus C_{4m'-2},$$  

(11)

$m' = 1, 2, \ldots$.

At this the mutually orthogonal idempotents have the following form

$$\lambda_+ = \frac{1 + i \omega}{2}, \quad \lambda_- = \frac{1 - i \omega}{2}, \quad \text{if } \omega^2 = -1,$$

$$\lambda_+ = \frac{1 + \omega}{2}, \quad \lambda_- = \frac{1 - \omega}{2}, \quad \text{if } \omega^2 = 1.$$  

It is well-known that Clifford algebras $C_n$ with even dimensionality ($n = 2\nu$) are isomorphic to matrix algebras of order $2^\nu (M_{2^\nu}(C))$. Hence it immediately follows that
Consider this homomorphism in details. First of all, the volume elements \( \omega \) between Clifford algebras \( \mathbb{C} \) and \( \mathbb{M} \) with \( n \) \( \omega \) elements are not contained in \( \mathbb{C} \). This follows that the linear representations of algebras \( n \) \( \mathbb{C} \) are written in the following form

\[
\begin{align*}
C_{4m'} & \cong M_{2m'}(\mathbb{C}), \\
C_{4m+2} & \cong M_{2m+1}(\mathbb{C}), \\
C_{4m'+1} & \cong M_{2m'}(\mathbb{C}) \oplus M_{2m'}(\mathbb{C}), \\
C_{4m'-1} & \cong M_{2m'-1}(\mathbb{C}) \oplus M_{2m'-1}(\mathbb{C}),
\end{align*}
\]

\( m = 0, 1, \ldots, m' = 1, 2, \ldots \).

### 3 A homomorphism \( \mathbb{C}_{n+1} \longrightarrow \mathbb{C}_n \)

From the latest isomorphisms follows that the linear representations of algebras \( \mathbb{C}_n \) with \( n \) is odd are isomorphic to the direct sums of matrix algebras \( M_{2m'}(\mathbb{C}) \oplus M_{2m'}(\mathbb{C}) \) and \( M_{2m'-1}(\mathbb{C}) \oplus M_{2m'-1}(\mathbb{C}) \), respectively. However, there exists a homomorphism between Clifford algebras \( C_{4m'+1}, C_{4m'-1} \) and matrix algebras \( M_{2m'}(\mathbb{C}), M_{2m'-1}(\mathbb{C}) \). Consider this homomorphism in details. First of all, the volume elements \( \omega = e_{12...4m'+1} \) and \( \omega = e_{12...4m'-1} \) are belong to the centers of algebras \( C_{4m'+1} \) and \( C_{4m'-1} \), and therefore are commutes with all basis elements of these algebras. Further, since \( \omega^2 = e_{12...n}^2 = (-1)^{\frac{n(n-1)}{2}} \), where \( n = 4m' + 1 \) or \( n = 4m' - 1 \), then denoting \( \varepsilon = \pm i^{\frac{n(n-1)}{2}} \) we shall have always \( (\varepsilon \omega)^2 = 1 \). Recalling that the each Clifford algebra \( \mathbb{C}_n \) is associate with a complex vector space \( \mathbb{C}_n \) we see that the basis vectors \( \{e_1, e_2, \ldots, e_n\} \) are spanned a subspace \( \mathbb{C}_n \subset \mathbb{C}_{n+1} \). Thus, the algebra \( \mathbb{C}_n \) in \( \mathbb{C}_n \) be a subalgebra of \( \mathbb{C}_{n+1} \) and consist of the elements which are not contain the symbol \( e_{n+1} \) (in our case \( e_{4m'+1} \) and \( e_{4m'-1} \)). A decomposition of the each element \( A \in \mathbb{C}_{n+1} \) may be written in the following form

\[
A = A^1 + A^0,
\]

where \( A^0 \) be a set whose elements are contain the unit \( e_{n+1} \), and \( A^1 \) be a set whose elements are not contain \( e_{n+1} \), therefore \( A^1 \in \mathbb{C}_n \). If multiply \( A^0 \) by \( \varepsilon \omega \), then the units \( e_{n+1} \) are mutually annihilate, therefore \( \varepsilon \omega A^0 \in \mathbb{C}_n \). Denoting \( A^2 = \varepsilon \omega A^0 \) we obtain

\[
A = A^1 + \varepsilon \omega A^2,
\]

where \( A^1, A^2 \in \mathbb{C}_n \). Define now a homomorphism \( \epsilon : \mathbb{C}_{n+1} \longrightarrow \mathbb{C}_n \), the action of which satisfy to the following rule

\[
\epsilon : A^1 + \varepsilon \omega A^2 \longrightarrow A^1 + A^2.
\]

(12)

Obviously, at this the all operations (addition, multiplication, and multiplication by number) are remained. Indeed, let

\[
A = A^1 + \varepsilon \omega A^2, \quad B = B^1 + \varepsilon \omega B^2,
\]

and...
then for multiplication by force of \((\varepsilon \omega)^2 = 1\) and commutativity \(\omega\) with all elements we have

\[
AB = (A^1B^1 + A^2B^2) + \varepsilon \omega (A^1B^2 + A^2B^1) \xrightarrow{\varepsilon \omega} (A^1B^1 + A^2B^2) + (A^1B^2 + A^2B^1) = (A^1 + A^2)(B^1 + B^2).
\]

Thus, the image of product is equal to the product of multiplier images in the same order.

In the particular case when \(A = \varepsilon \omega\) we have \(A^1 = 0\) and \(A^2 = 1\), therefore

\[
\varepsilon \omega \rightarrow 1.
\]

This way, a kernel of homomorphism \(\varepsilon\) consist of the all elements of form \(A^1 - \varepsilon \omega A^1\).

Recalling that algebra \(C_n\), where \(n\) is even, is isomorphic to a matrix algebra \(M_{2^n/2}(\mathbb{C})\) we obtain by force of \(\varepsilon: C_{n+1} \rightarrow C_n \subset C_{n+1}\) that algebra \(C_{n+1}\) is isomorphic to a matrix algebra \(M_{2^{n+1}/2}(\mathbb{C})\). For more details

\[
C_{4m'} = M_{2^{m'}(\mathbb{C})},
\]

\[
C_{4m'-1} = M_{2^{m'-1}(\mathbb{C})}.
\]

Consider now the fundamental automorphisms \(A \rightarrow \tilde{A}, A \rightarrow A^*, A \rightarrow \tilde{A}^*\) are defined in \(C_{4m'+1}\) or \(C_{4m'-1}\). It is interest to us what form these automorphisms are adopted after homomorphic mapping \(\varepsilon: C_{n+1} \rightarrow C_n \subset C_{n+1}\). First of all, in the case of anti-automorphism \(A \rightarrow \tilde{A}\) the elements \(A, B, \ldots \in C_{n+1}\), which are mapped into one element \(D \in C_n\) (the kernel of homomorphism \(\varepsilon\) if \(D = 0\)) after transformation \(A \rightarrow \tilde{A}\) are must transit to the elements \(\tilde{A}, \tilde{B}, \ldots \in C_{n+1}\) which are also mapped into one element \(\tilde{D} \in C_n\). Otherwise the transformation \(A \rightarrow \tilde{A}\) not be one-to-one.

In particular, it is necessary that \(\tilde{\varepsilon} \omega = \varepsilon \omega\), since 1 and element \(\varepsilon \omega\) under action of homomorphism \(\varepsilon\) are equally mapped into unit, then \(\tilde{1} \rightarrow 1\), and \(\tilde{\varepsilon} \omega \rightarrow \pm 1\) (by force of (2)), therefore we must assume

\[
\tilde{\varepsilon} \omega = \varepsilon \omega.
\]

The condition (13) be sufficient for transition of anti-automorphism \(A \rightarrow \tilde{A}\) from \(C_{n+1}\) to \(C_n\). Indeed, in this case we have

\[
A^1 - A^1\varepsilon \omega \rightarrow \tilde{A}^1 - \tilde{\varepsilon} \omega \tilde{A}^1 = \tilde{A}^1 - \varepsilon \omega \tilde{A}^1.
\]
Therefore, the elements of the form $A^1 - A^1 \varepsilon \omega$, which are belong to a kernel of homomorphism $\epsilon$, at the transformation $A \rightarrow \tilde{A}$ are transit to the elements of the same form.

The analogous conditions we have for other fundamental automorphisms. However, for automorphism $A \rightarrow A^*$ a condition $(\varepsilon \omega)^* = \varepsilon \omega$ is not execute, since $\omega$ is odd, and in accordance with (13) we have

$$\omega^* = -\omega.$$ 

Therefore, the automorphism $A \rightarrow A^*$ never transfer from $C_{n+1}$ to $C_n$.

Further, in general case we have for a field $K = \mathbb{C}$ two homomorphisms:

$$C_{4m'+1} \rightarrow C_{4m'},$$
$$C_{4m'-1} \rightarrow C_{4m'-2}.$$ 

The multiplier $\varepsilon = \pm i^{\frac{n(n-1)}{2}}$ (here $n$ is equal to $4m' + 1$ or $4m' - 1$) in this case has the following values

$$\varepsilon = \begin{cases} 
\pm 1, & \text{if } n = 4m' + 1, \\
\pm i, & \text{if } n = 4m' - 1;
\end{cases}$$

Thus, for anti-automorphism $A \rightarrow \tilde{A}$ in accordance with (2) we have

$$\tilde{\omega} = \tilde{e}_{12...4m'+1} = (-1)^{(4m'+1)4m'} \omega = \omega, \quad (14)$$
$$\tilde{i\omega} = i\tilde{e}_{12...4m'-1} = (-1)^{(4m'-1)(2m'-1)}i\omega = -i\omega. \quad (15)$$

Therefore, by force of condition (13) the anti-automorphism $A \rightarrow \tilde{A}$ is transfer only in the case

$$C_{4m'+1} \rightarrow C_{4m'}.$$ 

Consider now the anti-automorphism $A \rightarrow \tilde{A}^*$. Obviously, for transfer of $A \rightarrow \tilde{A}^*$ from $C_{n+1}$ to $C_n$ it is necessary that the following condition be executed:

$$(\varepsilon \omega)^* = \varepsilon \omega. \quad (16)$$

It is easy to see that by force of (15) this condition be executed only in the case

$$C_{4m'-1} \rightarrow C_{4m'-2},$$

since

$$(\varepsilon \omega)^* = \varepsilon e_{12...4m'-1}^* = (-1)^{(4m'-1)(2m'-1)}\varepsilon \omega^* = -\varepsilon \omega^* = \varepsilon \omega.$$ 

Finally, from $C_{n+1}$ to $C_n$ are transfer the following fundamental automorphisms:

$$A \rightarrow \tilde{A} \quad \text{at} \quad C_{4m'+1} \rightarrow C_{4m'}, \quad (17)$$
$$A \rightarrow \tilde{A}^* \quad \text{at} \quad C_{4m'-1} \rightarrow C_{4m'-2}. \quad (18)$$
4 Neutrino field

The main goal of present paper be a consideration of neutrino field in the framework of algebra $\mathbb{C}_3$, which be a simplest algebra with odd dimensionality. At this the central role in this consideration played a homomorphism $\epsilon : \mathbb{C}_{n+1} \rightarrow \mathbb{C}_n$. First of all, in accordance with (11) the algebra $\mathbb{C}_3$ is decomposed into a direct sum of two subalgebras $\mathbb{C}_2$ and $\mathbb{C}_2^*$. This decomposition may be represented by a following diagram:

\[ \begin{array}{c}
\mathbb{C}_3 \\
\downarrow \lambda_- \quad \downarrow \lambda_+ \\
\mathbb{C}_2 \oplus \mathbb{C}_2^* \\
\end{array} \]

Here we identify the idempotents $\lambda_-, \lambda_+$ with the helicity projection operators. Further, after homomorphic mappings $\mathbb{C}_3 \rightarrow \mathbb{C}_2$ and $\mathbb{C}_3 \rightarrow \mathbb{C}_2^*$ we obtain in result the factor-algebras $\mathbb{C}_2^*$ and $\mathbb{C}_2^*$ (we introduce here the sign of homomorphism $\epsilon$ for difference of neutrino fields from the photon fields with left-handed and right-handed polarizations, which are described by the algebras $\mathbb{C}_2$ and $\mathbb{C}_2^*$). Later on we shall connected the factor-algebras $\mathbb{C}_2^*$ and $\mathbb{C}_2^*$ with the neutrino and antineutrino fields, respectively. Indeed, the factor-algebra $\mathbb{C}_2^*$ (like to $\mathbb{C}_2$) is isomorphic to a matrix algebra $\mathbb{M}_2(\mathbb{C})$ which is represent the algebra of linear operators in a 2-dimensional complex space $\mathbb{S}_2$ (so-called a spinor space). Analogously, by force of isomorphism $\mathbb{C}_2^* \cong \mathbb{M}_2(\mathbb{C})$ we have a space $\mathbb{S}_2^*$ (a co-spinor space). The linear transformations of the vectors (spinors) of these spaces are defined by the following expressions:

\[
\begin{align*}
\xi^1' &= \alpha \xi^1 + \beta \xi^2, \\
\xi^2' &= \gamma \xi^1 + \delta \xi^2,
\end{align*}
\]

\[
\begin{align*}
\xi_1' &= \alpha^* \xi_1^* + \beta^* \xi_2^*, \\
\xi_2' &= \gamma^* \xi_1^* + \delta^* \xi_2^*,
\end{align*}
\]

\[
\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \sigma^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}.
\]

It is well-known that a group of all complex matrices of second order is isomorphic to a group $\text{SL}(2; \mathbb{C})$, which be a double-meaning representation of the Lorentz group. We come to van der Waerden 2-spinor formalism [9, 10, 11] if suppose $\xi^1 \sim \xi_1^*$, $\xi^2 \sim \xi_2^*$. At

\[ ^1\text{It is obvious that this space is homeomorphic to extended complex plane. Moreover, the space } S_2 \text{ may be identified with an absolute (a totality of infinitely distant points) of Lobatchevskii space } ^1S_3 \text{, the motion group of which is a Lorentz group (at this the space } ^1S_3 \text{ be an absolute of Minkowski space-time } ^1R_4 \text{).} \]
this the spaces of un-dotted and dotted spinors $S_2$ and $\hat{S}_2$ be respectively the spaces of un-dotted and dotted representations of the own Lorentz group. In this case an any spin-tensor may be represented as $a_{\mu_1\cdots\nu_s} = \xi^n_1 \cdots \xi^n_r \xi^1_{m_1} \cdots \xi^r_{m_s}$ \((n_i, m_j = 1, 2)\). Let

$$\xi^\lambda = \left( \begin{array}{c} \xi^1 \\ \xi^2 \end{array} \right), \quad \eta_{\dot{\mu}} = \left( \begin{array}{c} \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{array} \right)$$

are two-component spinors, obviously, be the vectors of the spaces $S_2$ and $\hat{S}_2$, and are satisfying to conditions

$$\xi_1 = \xi^2, \quad \xi_2 = -\xi^1, \quad \eta_{\dot{1}} = -\eta_{\dot{2}}, \quad \eta_{\dot{2}} = \eta_{\dot{1}}. \quad (19)$$

Further, let

$$\left( \begin{array}{cc} \partial_{1\dot{1}} & \partial_{1\dot{2}} \\ \partial_{2\dot{1}} & \partial_{2\dot{2}} \end{array} \right) = \left( \begin{array}{cc} \partial_0 - \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 + \partial_3 \end{array} \right),$$

$$\left( \begin{array}{cc} \partial^{1\dot{1}} & \partial^{2\dot{1}} \\ \partial^{1\dot{2}} & \partial^{2\dot{2}} \end{array} \right) = \left( \begin{array}{cc} \partial_0 + \partial_3 & \partial_1 + i\partial_2 \\ \partial_1 - i\partial_2 & \partial_0 - \partial_3 \end{array} \right).$$

are matrices of the symmetric spin-tensors $\partial_{\lambda\dot{\mu}} = \partial_{\dot{\mu}\lambda}$, $\partial^{\lambda\dot{\mu}} = \partial^{\dot{\mu}\lambda}$. The matrix of spin-tensor $\partial^{\lambda\dot{\mu}}$ is obtained from the matrix of $\partial_{\lambda\dot{\mu}}$ by means of conjugation and formulae \(19\).

Compose now the equations $\partial_{\lambda\dot{\mu}} \xi^\lambda = 0$, $\partial^{\lambda\dot{\mu}} \eta_{\dot{\mu}} = 0$:

$$\partial_{1\dot{1}} \xi_1^1 + \partial_{1\dot{2}} \xi_1^2 = 0,$$

$$\partial_{2\dot{1}} \xi_2^1 + \partial_{2\dot{2}} \xi_2^2 = 0,$$

$$\partial^{1\dot{1}} \eta_{\dot{1}} + \partial^{2\dot{1}} \eta_{\dot{2}} = 0,$$

$$\partial^{1\dot{2}} \eta_{\dot{1}} + \partial^{2\dot{2}} \eta_{\dot{2}} = 0. \quad (20)$$

It is well-known that equations $\partial_{\lambda\dot{\mu}} \xi^\lambda = 0$, $\partial^{\lambda\dot{\mu}} \eta_{\dot{\mu}} = 0$ be the Weyl equations for neutrino and antineutrino fields, respectively. In connection with this consider one interesting correlation between Weyl equations and Maxwell equations in vacuum. Maxwell equations in the spinor form may be written as

$$\partial^\mu_{\lambda} f^\lambda = 0,$$

$$\partial^\mu_{\dot{\lambda}} f^\lambda = s^\mu_{\dot{\rho}}.$$  

Or, in the case of $s^\mu_{\dot{\rho}} = 0$:

$$\partial^1_{\dot{1}} f^1 \partial^2_{\dot{1}} f^2 = 0,$$

$$\partial^1_{\dot{1}} f^2 \partial^2_{\dot{2}} f^2 = 0,$$

$$\partial^1_{\dot{1}} f^1 \partial^1_{\dot{1}} f^2 = 0,$$

$$\partial^2_{\dot{2}} f^1 \partial^2_{\dot{2}} f^2 = 0.$$
Using \((19)\) the latest system may be rewritten in the form

\[
\begin{align*}
\partial_{21} f_{12} + \partial_{22} f_{22} &= 0, \\
\partial_{11} f_{12} + \partial_{12} f_{12} &= 0, \\
\partial_{11} f_{11} + \partial_{12} f_{12} &= 0, \\
\partial_{21} f_{11} + \partial_{22} f_{12} &= 0,
\end{align*}
\]

Raising the indexes in the first two equations of the latest system and conjugate these equations (using the property \(\ddot{\xi} = \xi\)) we obtain

\[
\begin{align*}
\partial^{1\dot{1}} f^{1\dot{1}} + \partial^{2\dot{1}} f^{1\dot{2}} &= 0, \\
\partial^{1\dot{2}} f^{1\dot{1}} + \partial^{2\dot{2}} f^{1\dot{2}} &= 0, \\
\partial^{1\dot{1}} f_{11} + \partial^{1\dot{2}} f_{12} &= 0, \\
\partial^{2\dot{1}} f_{11} + \partial^{2\dot{2}} f_{12} &= 0,
\end{align*}
\]

It is easy to see that system \((20)\) is coincide with the system \((21)\) if suppose

\[
\xi^{\dot{\lambda}} \sim \begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix}, \quad \eta^{\dot{\mu}} \sim \begin{pmatrix} \dot{f}^{1\dot{1}} \\ \dot{f}^{1\dot{2}} \end{pmatrix} = \begin{pmatrix} F^3 \\ F^1 + i F^2 \end{pmatrix}.
\]

Thus, we assume that neutrino and antineutrino fields are described by the factor-algebras \(\mathfrak{C}_2\) and \(\mathfrak{C}_2\). It is well-known that in Nature there exists only left-handed neutrino and right-handed antineutrino, and no there exists right-handed neutrino and left-handed antineutrino. In other words, the transformations of neutrino (antineutrino) field are not contain spatial reflections, unlike to the photon fields, for which we have both left-handed and right-handed photons are described by the algebras \(\mathfrak{C}_2\) and \(\mathfrak{C}_2\), and are transformed into each other under action of anti-automorphism \(A \rightarrow \tilde{A}\).

The absence of spatial reflections in the case of neutrino field has a natural explanation in the framework of algebra \(\mathfrak{C}_3\). Indeed, in accordance with \((18)\) at the homomorphic mappings \(\mathfrak{C}_3 \rightarrow \mathfrak{C}_2\) and \(\mathfrak{C}_3 \rightarrow \mathfrak{C}_2\) the anti-automorphism \(A \rightarrow \tilde{A}\) is not transfer. Thus, for the factor-algebras \(\mathfrak{C}_2\) and \(\mathfrak{C}_2\), which are obtained in the result of these mappings, the anti-automorphism \(A \rightarrow \tilde{A}\) is not defined (that is the factor-algebras \(\mathfrak{C}_2\) and \(\mathfrak{C}_2\) are not transformed into each other under action of \(A \rightarrow \tilde{A}\)). Therefore, the neutrino and antineutrino fields, which are described by \(\mathfrak{C}_2\) and \(\mathfrak{C}_2\), are possess the fixed helicities (that is the transformations of these fields are not contain the spatial reflections). However, in accordance with \((18)\) the factor-algebras \(\mathfrak{C}_2\) and \(\mathfrak{C}_2\) are transformed into each other under action of anti-automorphism \(A \rightarrow \tilde{A}\) (this property of neutrino field known as \(CP\)-invariance also has place in the framework of algebra \(\mathfrak{C}_3\)).
5 Summary

In conclusion we shall summarize the results obtained above, and also in previous paper [3], in the form of following schedule:

- **C\textsubscript{2}**
  The simplest Clifford algebra with even dimensionality $C\textsubscript{2}$ and obtained from it under action of anti-automorphism $A \rightarrow \tilde{A}$ the algebra $\hat{C}\textsubscript{2}$, are described the photon fields with left-handed and right-handed polarization, respectively. The algebras $C\textsubscript{2}$ and $\hat{C}\textsubscript{2}$ are transformed into each other under action of anti-automorphism $A \rightarrow \tilde{A}$.

- **C\textsubscript{3}**
  The simplest Clifford algebra with odd dimensionality $C\textsubscript{3} \cong C\textsubscript{2} \oplus \hat{C}\textsubscript{2}$. The factor-algebras $C\textsubscript{2}$ and $\hat{C}\textsubscript{2}$, which are obtained in the result of homomorphic mappings $C\textsubscript{3} \rightarrow C\textsubscript{2}$ and $C\textsubscript{3} \rightarrow \hat{C}\textsubscript{2}$, are described the left-handed neutrino and right-handed antineutrino fields, respectively. The factor-algebras $C\textsubscript{2}$ and $\hat{C}\textsubscript{2}$ are transformed into each other under action of anti-automorphism $A \rightarrow \tilde{A}$.

- **C\textsubscript{4}**
  The algebra $C\textsubscript{4}$ is described the electron field, which by force of isomorphism $C\textsubscript{4} \cong C\textsubscript{2} \otimes \hat{C}\textsubscript{4}$ may be represented as a tensor product of two photon fields. Analogously, the algebra $\hat{C}\textsubscript{4}$ is described the positron field, which obtained from $C\textsubscript{4}$ by means of anti-automorphism $A \rightarrow A^\ast$.

It is obvious that this schedule may be continued. Moreover, we assume that in like manner (that is in the terms of Clifford algebras) may be constructed the full particle systematics.

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