Random Operator Approach for Word Enumeration in Braid Groups

ALAIN COMTET†

SERGEI NECHAEV‡†

† Institut de Physique Nucléaire, Division de Physique Théorique*,
91406 Orsay Cedex, France

‡ L.D. Landau Institute for Theoretical Physics,
117940, Moscow, Russia

Abstract

We investigate analytically the problem of enumeration of nonequivalent primitive words in the braid groups $B_n$ for $n \gg 1$ by analysing the random word statistics and target space on the basis of the locally free group approximation. We develop a ”symbolic dynamics” method for exact word enumeration in locally free groups and bring arguments in support of the conjecture that the number of very long primitive words in the braid group is not sensitive to the precise local commutation relations. We consider the connection of these problems with the conventional random operator theory, localization phenomena, and statistics of systems with quenched disorder. Also we discuss the relation of the particular problems of random operator theory to the theory of modular functions.

Key words: braid group, graph of the group, primitive word, symbolic dynamics, random operator.

Submitted to Nuclear Physics B [Physical Mathematics]

PACS: 02.20.Nq; 02.30.Tb; 05.40.+j

* Unité de Recherche des Universités Paris XI et Paris VI associée au C.N.R.S.
1 Introduction: Problems and Motivations

Recent years have been marked by the emergence of more and more problems related to the consideration of physical processes on noncommutative groups. In trying to classify such problems, we distinguish between the following categories in which the noncommutative origin of phenomena appear with perfect clarity:

1. Problems connected with the spectral properties of the Harper–Hofstadter equation [1] dealing with the electron dynamics on the lattice in a constant magnetic field. We mean primarily the consideration of groups of magnetic translations [2, 3] and properties of quantum planes [4].
2. Problems of classical and quantum chaos on hyperbolic manifolds: spectral properties of dynamical systems and derivation of trace formulae \[5, 6, 7\] as well as construction of probability measures for random walks on modular groups \[8\].

3. Problems giving rise to application of quantum group theory in physics: deformations of classical abelian objects such as harmonic oscillators \[9\] and standard random walks \[10\].

4. Problems of knot theory and statistical topology: construction of nonabelian topological invariants \[11, 12\], consideration of probabilistic behavior of the words on the simplest noncommutative groups related to topology (such as braid groups) \[13\], statistical properties of "anyonic" systems \[14\].

5. Classical problems of random matrix and random operator\[1\] theory and localization phenomena: determination of Lyapunov exponents for products of random noncommutative matrices \[13, 15, 17\], study of the spectral properties and calculation of the density of states of large random matrices \[18, 19\].

Certainly, such a division of problems into these categories is very speculative and reflects to a marked degree the authors’ personal point of view. However, we believe that the enumerated items reflect, at least partially, the currently growing interest in theoretical physics of the ideas of noncommutative analysis.

Let us stress that we do not touch upon the pure mathematical aspects of noncommutative analysis in this paper and the problems discussed in the present work mainly concern the points 4 and 5 of the list above.

It is widely considered that a new fresh stream in topology was brought about by the recognition of the fact that there exists a deep relation between the Temperley–Lieb algebra and the Hecke algebra representation of the braid group. This fact resulted in the remarkable geometrical analogy between the Yang–Baxter equations, appearing as a necessary condition of the transfer matrix commutativity in the theory of integrable systems on the one hand, and the Reidemeister moves, used in the knot invariant construction on the other hand. We can mention several nice reviews \[11, 20, 21\] and books

\[n \times n\] tables having of order \(n\) random entries; if the number of random entries grows faster than \(n\) when \(n \to \infty\), we call such a table as a random matrix.

\[1\] Following L.A. Pastur, we will distinguish random matrices and matrix representations of random operators. To the random operators we attribute the \(n \times n\) tables having of order \(n\) random entries;
regarding the construction of new knot and link invariants in terms of integrable 2D statistical models, as well as their relation with different matrix and tensor representations of some noncommutative groups.

Besides the traditional topological issues such as construction of topological invariants, investigation of homotopic classes and fibre bundles, we consider a set of adjacent but much less studied problems lying at the border between statistical physics, probability theory and topology. First of all, we should mention problems related to the so-called “knot entropy” calculation. Most generally this set of problems can be formulated as follows. Take the lattice \( \mathbb{Z}^3 \) embedded in the space \( \mathbb{R}^3 \). Let \( \Omega_N \) be the ensemble of all possible closed non-self-intersecting \( N \)-step loops with one common fixed point on \( \mathbb{Z}^3 \); by \( \omega \) we denote the particular trajectory configuration. The question is: what is the probability \( P_N \) of the trajectory \( \omega \in \Omega_N \) belonging to some specific homotopy class. In [13] it has been shown that many non-trivial properties of the limit behavior of knot statistics can be explained in the context of the limit behavior of random walks over the elements of some nonabelian (hyperbolic) group related to the braid representation of knots.

In the context of “topologically–probabilistic” consideration, the problems dealing with the limit distributions of noncommutative random walks were not discussed practically except very few specific cases [16, 17, 23, 24]. In particular, in these works it has been shown that statistics of random walks with a fixed topological state with respect to the regular arrays of obstacles on the plane can be mapped to random walks on the free group \( \Gamma_2 \) which has the topology of a simply connected tree. The analytic construction of nonabelian topological invariants for trajectories on a double punctured plane as well as the statistics of the simplest nontrivial random braid \( B_3 \), were discussed briefly in [25, 26].

A preliminary analytical and numerical study of the statistics of random walks (Markov chains) on braid and so-called ”locally free” groups \( \mathcal{L} \) (see definition below) was recently undertaken in works [27, 28]. In the case of the braid group, the rather complicated group structure prevents us from applying the simple geometrical pictures of the free group \( \Gamma_2 \) (see [29]). Nevertheless the problem of the limit distribution for random walks on \( B_n \) can be reduced to the problem of a random walk on some graph \( \mathcal{L} \). In case of the group \( B_3 \) we were able to construct this graph explicitly, whereas

\[ ^{2}\text{This notation has been introduced by A.M. Vershik in [27].} \]
for the group $B_n$ ($n \geq 4$) we gave only an upper estimate for the limit distribution of random walks analysing statistics of Markov chains on "locally free groups".  

The consideration of problems dealing with the limit distributions of Markov chains on braid groups $B_n$ requires examination of the "target space" of this group, i.e., the space where the random walk takes place. The structure of the target space is uniquely determined by the group relations and in the general case of the group $B_n$ ($n \geq 4$), is still unknown.

In the present work we study the target space of the braid group $B_n$ when $n \gg 1$, trying to develop a new "statistical approach" for words enumeration in this group.

We should stress that our presentation offers a mathematical analysis which is far from rigorous, and ideas expressed here are mainly supported by numerical simulations. Moreover, we skip here some important but hard questions, like the problem of "words identity" in the braid group (deep advances concerning this subject can be found in recent work [30]). Our aim is to describe a constructive algorithm which, of course, has to be be justified and verified later.

The structure of the paper is as follows: in the next section we give some necessary definitions concerning braid and "locally free" groups and describe the model under consideration; Section 3 is devoted to developing a "symbolic dynamics" method for words enumeration in the locally free group $\mathcal{LF}_n$ (for $n \gg 1$); the target space of the braid group is studied in Section 4 by means of a statistical approach based on the concept of "locally free group with errors". In this section we discuss also some additional links between this problems and the conventional random matrix theory, localization phenomena and statistics of systems with "quenched" disorder; we present here some speculations dealing with the possible relation of particular problems of random matrix theory to the theory of Dedekind function and modular groups.

2 Basic Definitions and Statistical Model

We first recall some points concerning the definitions of braid and "locally free" groups.

**Braid Group.** The braid group $B_n$ of "$n$–strings" has $n−1$ generators $\{\sigma_1, \sigma_2, \ldots, \sigma_{n−1}\}$
with the following commutation relations:

\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (1 \leq i < n - 1) \]

\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i - j| \geq 2) \]  \hspace{1cm} (1)

\[ \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = e \]

Let us mention that:

– A word written in terms of "letters" — generators from the set \{\sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}\} gives a particular braid. Schematically the generators \(\sigma_i\) and \(\sigma_i^{-1}\) may be represented as follows:

```
|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
|   |   |   |   |   |   |   |   |   |
|   |   |   |   |   |   |   |   |   |

= \sigma_i
```

```
|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
|   |   |   |   |   |   |   |   |   |
|   |   |   |   |   |   |   |   |   |

= \sigma_i^{-1}
```

– The length of the braid is the total number of letters used, while the minimal irreducible length hereafter referred to as "primitive word" is the shortest noncontractible length of a particular braid remaining after all possible group relations Eq. (1) are applied. Diagramatically, the braid can be represented as a set of crossed strings going from the top to the bottom after gluing the braid generators.

– The closed braid appears after gluing the "upper" and the "lower" free ends of the braid on the cylinder.

– Any braid corresponds to some knot or link. So, there is a possibility to use the braid group representation for the construction of topological invariants of knots and links, but the correspondence between braids and knots is not mutually single valued and each knot or link can be represented by an infinite series of different braids.

**Locally Free Group.** The group \(\mathcal{LF}_n(d)\) is called locally free if the generators, \{\(f_1, \ldots, f_{n-1}\}\) obey the following commutation relations:
(a) Each pair $(f_j, f_k)$ generates the free subgroup of the group $LF_n(d)$ if $|j - k| < d$; 
(b) $f_j f_k = f_k f_j$ for $|j - k| \geq d$

We will be concerned mostly with the case $d = 2$ for which we define $LF_n(2) \equiv LF_n$.

- The length of the word written in terms of letters $\{f_1, \ldots, f_{n-1}, f_1^{-1}, \ldots, f_{n-1}^{-1}\}$ is the total number of generators used, and the ”primitive word” is the shortest noncontractible length of a particular word after applying all relations of the group $LF_n(2)$ (compare to the case of the braid group). The graphical representation of generators $g_i$ and $g_i^{-1}$ is also rather similar to that of braid group:

\[
\begin{array}{c}
| & | & \cdots & | & | & | \\
1 & 2 & \cdots & i & i+1 & \cdots & n-1 & n
\end{array}
= f_i
\]

\[
\begin{array}{c}
| & | & \cdots & | & | & | \\
1 & 2 & \cdots & i & i+1 & \cdots & n-1 & n
\end{array}
= f_i^{-1}
\]

It is easy to understand that the following geometrical identity is valid:

\[
\begin{array}{c}
\begin{array}{c}
| & \cdots & | & | & | & | \\
& \cdots & | & | & | & |
\end{array}
= \\
\begin{array}{c}
\begin{array}{c}
| & \cdots & | & | & | & | \\
i & i+1 & | & | & | & |
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
| & \cdots & | & | & | & | \\
i & i+1 & | & | & | & |
\end{array}
\end{array}
\equiv \\
\begin{array}{c}
\begin{array}{c}
| & \cdots & | & | & | & | \\
i & i+1 & | & | & | & |
\end{array}
\end{array}
\]

hence, it is unnecessary to distinguish between ”left” and ”right” operators $f_i$.

It can be seen that the only difference between the braid and locally free groups consists in the elimination of the Yang-Baxter relations (first line in Eq. (1)).

**Statistical Model.** Our aim is to calculate a specific ”partition function”, $V_n(\mu, d)$, giving the number of all nonequivalent primitive words of length $\mu$ in the groups $LF_{n+1}(d)$ and $B_{n+1}$ for $n \gg 1$. 

6
Remark. To have a geometrical picture of the group $\mathcal{LF}_{n+1}$ let us describe the recursion procedure of raising the graph (the "target space") associated with this group.

Take the free group $\Gamma_n$ with generators $\{\tilde{f}_1, \ldots, \tilde{f}_n\}$ where all $\tilde{f}_i$ $(1 \leq i \leq n)$ do not commute. It is well known that the group $\Gamma_n$ has the structure of a $2n$-branching Cayley tree, $C(\Gamma_n)$—see Fig.1a—where the number of distinct primitive words of length $\mu$ is equal to the number $\tilde{V}_n(\mu)$ of vertices of the tree $C(\Gamma_n)$ lying at a distance of $\mu$ steps from the origin:

$$\tilde{V}_n(\mu) = 2n(2n-1)^{\mu-1} \tag{2}$$

The graph $C(\mathcal{LF}_{n+1})$ corresponding to the group $\mathcal{LF}_{n+1}$ can be constructed from the graph $C(\Gamma_n)$ in accordance with the following recursion procedure:

(a) Take the root vertex of the graph $C(\Gamma_n)$ and consider all vertices on the distance $\mu = 2$ from it. Identify those vertices which correspond to the equivalent words in the group $\mathcal{LF}_{n+1}$. (See example in Fig.1b).

(b) Repeat this procedure taking all vertices at the distance $\mu = (1, 2, \ldots)$ and "gluing" them at the distance $\mu + 2$ according to the definition of the locally free group.

By means of the procedure described, we raise a graph ("target space") corresponding to the locally free group $\mathcal{LF}_{n+1}$. Now our main problem can be reformulated as follows: how many distinct vertices has the graph $C(\mathcal{LF}_{n+1})$ at a distance of $\mu$ steps from the origin (for $n \gg 1$).

In the next Section we give an exact answer to that question, which we use, in turn, as the basis for consideration of the much trickier case of the braid group.

It is worthwhile to mention that the graph $C(\mathcal{E}_{n+1})$ of the complete commutative group $\mathcal{E}_{n+1}$ (all generators of $\mathcal{E}_{n+1}$ commute with each others) has the topology of the lattice embedded in $\mathbb{R}^{2n}$ (see Fig.1c). Hence, the number of nonequivalent words $V_{n}^{\text{comm}}(\mu)$ of length $\mu$ can be roughly estimated as the number of lattice points lying on the surface of the $2n$–dimensional sphere, i.e.

$$V_{n}^{\text{comm}}(\mu) \simeq \text{const} \mu^{2n} \tag{3}$$
Comparing (2) and (3) we get

\[
\begin{align*}
\lim_{\mu \to \infty} \frac{1}{\mu} \ln \tilde{V}_n(\mu) & \approx \ln(2n - 1) > 0 \\
\lim_{\mu \to \infty} \frac{1}{\mu} \ln V_n^{\text{comm}}(\mu) & = 0
\end{align*}
\]

(4)

Naively we could expect that the behavior

\[
\lim_{\mu \to \infty} \frac{1}{\mu} \ln V_n(\mu) = 0
\]

for "locally free" and braid groups remains unchanged (i.e. is the same as in the completely commutative case) because for \( n \gg 1 \) we have of order \( \sim n^2 \) commutative relations and only of order \( \sim n \) noncommutative ones. However, it is proven for \( \mathcal{LF}_n \) and conjectured for \( B_n \) that

\[
\lim_{\mu \to \infty} \frac{1}{\mu} \ln V_n(\mu) = \text{const} > 0
\]

which clearly reflects the hyperbolic character of these groups.

### 3 Exact Words Enumeration in Locally Free Groups

We derive here an explicit expression for the number \( V_n(\mu, d) \) of all nonequivalent primitive words of length \( \mu \) in the group \( \mathcal{LF}_{n+1}(d) \) (when \( d = 2 \) and \( n \gg 1 \)) on the basis of the so-called "normal order" representation of words proposed by A.M. Vershik in [31] and developed in [27, 28] which is reminiscent of the enumeration of "partially commutative monoids" known in combinatorics [32].

#### 3.1 "Normal Order" Representation of Words

Let us represent each primitive word \( W_p \) of length \( \mu \) in the group \( \mathcal{LF}_{n+1}(d) \) in the normal order similar to the so-called "symbolic dynamics" appearing in the context of chaotic systems (see, for instance, [7])

\[
W_p = (f_{\alpha_1})^{m_1} (f_{\alpha_2})^{m_2} \ldots (f_{\alpha_s})^{m_s}
\]

(5)

where \( \sum_{i=1}^{s} |m_i| = \mu \) (\( m_i \neq 0 \quad \forall \, i; \quad 1 \leq s \leq \mu \)) and the sequence of generators \( f_{\alpha_i} \) in Eq.(6) for all distinct \( f_{\alpha_i} \) satisfies the following local rules [27]:

(i) If $f_{\alpha_i} = f_1$, then $f_{\alpha_{i+1}} \in \{f_2, f_3, \ldots, f_n\}$;
(ii) If $f_{\alpha_i} = f_k$ $(1 < k \leq n - 1)$, then $f_{\alpha_{i+1}} \in \{f_{k-d+1}, \ldots, f_{k-1}, f_{k+1}, \ldots, f_n\}$;
(iii) If $f_{\alpha_i} = f_n$, then $f_{\alpha_{i+1}} \in \{f_{k-d+1}, \ldots, f_{n-1}\}$.

These local rules could be represented diagramatically as follows:

```
  f_1
 / \  /
/ \  /
/ f_2 / f_3 / \ ...
/ f_1 / f_3 / f_n / f_2 / f_4 / f_n / f_2 / f_4 / ...
/   / f_n / f_n / f_{n-2} / f_n / f_n / f_{n-1}
```

The rules (i)–(iii) give the prescription how to encode and enumerate all distinct primitive words in the group $\mathcal{LF}_{n+1}(d)$. If the sequence of generators in the primitive word $W_p$ does not satisfy the rules (i)-(iii), we commute the generators in the word $W_p$ until the normal order is restored. Hence, the normal order representation enables one to give the unique coding of all nonequivalent primitive words in the group $\mathcal{LF}_{n+1}(d)$.

**Example 1.** Take an arbitrary primitive word of length $\mu = 10$ in the group $\mathcal{LF}_{8+1}(2)$:

$$W_p = f_5^{-1} f_3 f_8 f_1^{-1} f_2 f_4 f_8 f_4 f_7$$

$$\equiv (f_5)^{-1} (f_3) (f_8) (f_1)^{-1} (f_2) (f_4) (f_8)^2 (f_4) (f_7)$$

To represent the word $W_p$ in the "normal order" we have to push all generators with smaller indices to the left when it is allowed by the commutation relations of the locally free group $\mathcal{LF}_9(2)$. We get:

$$W_p = (f_1)^{-1} (f_3)^1 (f_2)^1 (f_5)^{-1} (f_4)^2 (f_8)^3 (f_7)^1$$

(the "normal order" for this word is the sequence of used generators: $\{1, 3, 2, 5, 4, 8, 7\}$).

To compute the number of different primitive words of length $\mu = 10$ with the same normal order as in Eq.(7), we have to sum up all the words like

$$W_p = (f_1)^{m_1} (f_3)^{m_2} (f_2)^{m_3} (f_5)^{m_4} (f_4)^{m_5} (f_8)^{m_6} (f_7)^{m_7}$$
under the condition $\sum_{i=1}^{7} |m_i| = 10; m_i \neq 0 \forall m_i \in [1, 7]$.

The calculation of the number of distinct primitive words, $V_n(\mu)$, of the given length $\mu$ is now rather straightforward:

$$V_n(\mu, d) = \sum_{s=1}^{\mu} \sum_{\{m_1, \ldots, m_s\}} R_n(s, d) \sum' \{m_i\} \Delta \left[ \sum_{i=1}^{s} |m_i| - \mu \right]$$

(9)

where:

• $R_n(s, d)$ is the number of all distinct sequences of $s$ generators taken from the set \{f_1, \ldots, f_n\} and satisfying the local rules (i)-(iii)

• the second sum gives the number of all possible representations of the primitive path of length $\mu$ for the fixed sequence of generators—(see the example above); “prime” denotes that the sum runs over all $m_i \neq 0$ for $1 \leq i \leq s$; $\Delta$ is the Kronecker function: $\Delta = 1$ for $x = 0$ and $\Delta = 0$ for $x \neq 1$

• special attention should be paid to the sequences built on the basis of one generator only, i.e. for primitive words of type $W_p = (f_k)^\mu \ \forall k \in [1, n]$ (see definition of $R_n(s, d)$ below).

To get the partition function $R_n(s, d)$ let us mention that the local rules (i)-(iii) define a generalized Markov chain with the states given by the $n \times n$ “incidence” matrix $\tilde{M}_n(d)$, the rows and columns of which correspond to the generators $f_1, \ldots, f_n$ as it is shown below:

$$\begin{array}{ccccccc}
\text{ } & f_1 & f_2 & f_3 & f_4 & \ldots & f_{n-1} & f_n \\
\hline
f_1 & 0 & 1 & 1 & 1 & \ldots & 1 & 1 \\
f_2 & 1 & 0 & 1 & 1 & \ldots & 1 & 1 \\
f_3 & 0 & 1 & 0 & 1 & \ldots & 1 & 1 \\
f_4 & 0 & 0 & 1 & 0 & \ldots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
f_{n-1} & 0 & 0 & 0 & \ldots & 0 & 1 \\
f_n & 0 & 0 & 0 & \ldots & 1 & 0 \\
\end{array}$$

(10)

The matrix $\tilde{M}_n(d)$ has a rather simple structure: above the diagonal we put everywhere ”1” and below diagonal we have $d - 1$ subdiagonals completely filled by ”1”; in all other places we have ”0” (the case with $d = 2$ is shown in Eq.(10)).
The number of all distinct normally ordered sequences of words of length \( s \) with allowed commutation relations is given by the following partition function

\[
R_n(s, d) = \tilde{v}_{in} \left[ \tilde{M}_n(d) \right]^{s-1} v_{out} \tag{11}
\]

where

\[
\tilde{v}_{in} = \begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix} \quad \text{and} \quad v_{out} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{12}
\]

For \( s = 1 \) we have \( R_n(1, d) = \tilde{v}_{in} v_{out} = 2n \) as it should be.

The remaining sum in Eq. (11) is independent on \( R_n(s, d) \), so its calculation is very simple (see also Appendix A for some generalizations):

\[
\sum_{\{m_1, \ldots, m_s\}}' \Delta \left[ \sum_{i=1}^{s} |m_i| - \mu \right] = 2^s C_{\mu-1}^{s-1} \tag{13}
\]

Substituting Eq. (13) and Eq. (11) into Eq. (9) we get

\[
V_n(\mu) \equiv V_n(\mu, d) = \sum_{s=1}^{\mu} 2^s C_{\mu-1}^{s-1} R_n(s, d) = 2 \tilde{v}_{in}(2\tilde{M}_n(d) + \tilde{I})^{\mu-1} v_{out} \equiv 2 \text{ Trace } (2\tilde{M}_n(d) + \tilde{I})^{\mu-1} \tag{14}
\]

where \( \tilde{I} \) is the identity matrix.

Such a quantity is rather difficult to evaluate exactly. A reasonable approximation is to replace (14) by

\[
V_n^*(\mu) = 2 \sum_{i=1}^{n} (2\lambda_i + 1)^{\mu-1} \tag{15}
\]

where \( \lambda_i \) are the eigenvalues of the matrix \( \tilde{T}_n \) which can be shown all to be real (see the later discussion). In order to check the validity of approximation (14) we have considered the case \( \{n = 3, d = 2\} \) where the exact value reads

\[
V_3(\mu) = \left( \frac{15 + 7\sqrt{5}}{5} \right) (2 - \sqrt{5})^{\mu-1} + \left( \frac{15 - 7\sqrt{5}}{5} \right) (2 + \sqrt{5})^{\mu-1}
\]

whereas the approximation (14) gives

\[
V_3^*(\mu) = 2(2 - \sqrt{5})^{\mu-1} + 2(2 + \sqrt{5})^{\mu-1} + 2(-1)^{\mu-1}
\]

It can be seen that this approximation works reasonably well even for small values of \( \mu \).
The value \( V_n(\mu, d) \) is growing exponentially fast with \( \mu \) and the "speed" of this growth is clearly represented by the fraction

\[
q(d) = \left. \frac{V_n(\mu + 1, d)}{V_n(\mu, d)} \right|_{\mu \gg 1}
\]

which has the meaning of the effective coordinational number of the graph \( C(\mathcal{L}F_n) \).

In the next section we present calculations of the asymptotic expression of (13) when \( n \gg 1 \).

### 3.2 Calculation of Eigenvalues of Matrix \( \hat{M}_n(2) \)

Consider the determinant

\[
a_n(\lambda) = \det (\hat{M}_n - \lambda I) = \begin{pmatrix} -\lambda & 1 & 1 & \ldots \\ 1 & -\lambda & 1 & \ldots \\ 0 & 1 & -\lambda & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\]

(17)

It satisfies the recursion relation

\[
a_n(\lambda) = -(\lambda + 1)a_{n-1}(\lambda) - (\lambda + 1)a_{n-2}(\lambda)
\]

(18)

with the boundary conditions

\[
\begin{cases} 
  a_0(\lambda) = 1 \\
  a_1(\lambda) = -\lambda 
\end{cases}
\]

(19)

For \( \lambda > -1 \) one may set

\[
a_n(\lambda) = (\lambda + 1)^{n-1}(-1)^n \varphi_n(\lambda)
\]

(20)

which gives

\[
\varphi_n(\lambda) = \sqrt{\lambda + 1} \varphi_{n-1}(\lambda) - \varphi_{n-2}(\lambda)
\]

(21)

The general solution of (21) satisfying the previously defined boundary conditions (19) is given in terms of Chebyshev’s polynomials of second kind

\[
\varphi_n(\lambda) = U_{n+1}(\cos \vartheta)
\]

(22)
where
\[
\cos \vartheta = \frac{\sqrt{\lambda + 1}}{2} \quad \left(0 < \vartheta < \frac{\pi}{2}\right)
\]

(23)

Therefore
\[
a_n(\lambda) = (-1)^n(\lambda + 1)^{n-1} u_{n+1}(\cos \vartheta)
\]
\[
= (-1)^n(\lambda + 1)^{n-1} \frac{\sin(n+2)\vartheta}{\sin \vartheta}
\]

(24)

The last expression enables us to obtain all the eigenvalues of the matrix \(\hat{M}_n\). In fact, it is convenient to distinguish them according to the parity in \(n\):

1. \(n = 2m + 1\)
   \[
   \lambda_0 = (-1) \quad (m \text{ such values}); \quad \lambda_k = 4 \cos^2 \frac{k\pi}{2m + 3} - 1 \quad (k = [1, m + 1])
   \]

(25)

2. \(n = 2m\)
   \[
   \lambda_0 = (-1) \quad (m \text{ such values}); \quad \lambda_k = 4 \cos^2 \frac{k\pi}{2m + 2} - 1 \quad (k = [1, m])
   \]

(26)

Since in each case we have exactly \(n\) states, this exhaust the complete set of eigenvalues, showing they all are real in the interval \([-1, 3]\). One also recovers the result obtained earlier in \([27, 28]\) for the asymptotics of the highest eigenvalue of matrix \(\hat{M}_n\) (in the limit \(n \gg 1\)):
\[
\lambda_{\text{max}} = 4 \cos^2 \frac{\pi}{n+2} - 1 \quad \bigg|_{n \gg 1} \approx 3 - \frac{4\pi^2}{n^2}; \quad (k = 1)
\]

(27)

Now we are in position to compute the number of nonequivalent words \(V_n^*(\mu)\) of the primitive length \(\mu\) in the locally free group \(L^F_n(2)\) for \(n \gg 1\), \(n = \text{const}\) (see Eq.(13)). Using the definition (13) and Eqs.(25)–(26) we get for \(n = 2m + 1\):
\[
V_n^*(\mu) = (n - 1)(-1)^{\mu-1} + 2 \sum_{k=1}^{\frac{\mu+1}{2}} \left(8 \cos^2 \frac{k\pi}{n + 2} - 1\right)^{\mu-1}
\]

(28)

Define \(\varphi_n(\mu)\) as follows:
\[
V_n^*(\mu) = (n - 1)(-1)^{\mu-1} + 2\varphi_n(\mu)
\]
By means of Euler–Mac Laurin formula we may compute the asymptotic expression of $\varphi_n(\mu)$:

$$
\varphi_n(\mu) = \frac{1}{2} \sum_{k=0}^{n+1} \left( 8 \cos^2 \frac{k\pi}{n+2} - 1 \right)^{\mu-1} = \frac{n + 2}{2\pi} \int_0^\pi (8 \cos^2 x - 1)^{\mu-1} dx - \frac{1}{2} 7^{\mu-1} \tag{29}
$$

For $\mu \gg 1$, $n=\text{const} \gg 1$ the last integral is evaluated by a saddle point approximation which yields

$$
\varphi_n(\mu) = \frac{n + 2}{2\pi} \frac{1}{2} \sqrt{\frac{\pi}{2\mu}} 7^{\mu-1} - \frac{1}{2} 7^{\mu-1} \tag{30}
$$

Thus, for the number of nonequivalent words in the locally free group $\mathcal{L}F_n(2)$ we have the following limiting behavior:

$$
\lim_{\mu \to \infty, n = \text{const} \gg 1} \frac{1}{\mu} \ln V_n^*(\mu) = \ln 7 \tag{31}
$$

and Eq.(16) gives

$$
q(d = 2) = 7
$$

Hence, the graph corresponding to the locally free group can be viewed as an effective tree with the branching number $q = 7$.

4 Approximate Statistical Approach for Words Enumeration in Braid Group $B_n$

The construction of an effective algorithm for enumeration of the words in the braid group $B_n$ for $n > 2$ is of the most intriguing problems in group theory.

In the present section we propose an approximate statistical approach for enumeration of all distinct primitive words in the group $B_n$ for $n \gg 1$ which exploits some properties of locally free groups $\mathcal{L}F_n$ considered above.

The main idea is as follows. Let us deal with the sequences of words in the braid group $B_n$ from the point of view of the locally free group $\mathcal{L}F_n^{\text{err}}$ "with errors". To be more specific let us start with the following example:
Example 2. Write a random word $W$ in the group $B_7$ consisting of 8 letters. Let this word be for instance:

$$W = (\sigma_1)^{-1} \sigma_4 (\sigma_5)^{-1} (\sigma_6)^{-1} \sigma_5 \sigma_1 \sigma_6 (\sigma_2)^{-1}$$

We reduce this word to the primitive one in two steps.

1. On the first step we act in the same way as in the case of locally free group $\mathcal{LF}_7$ and push all generators with smaller indices to the left assuming that nearest neighbors do not commute at all. We get:

$$W_{\text{reduced}} = (\sigma_2)^{-1} \sigma_4 (\sigma_5)^{-1} \underbrace{(\sigma_6)^{-1} \sigma_5 \sigma_6}_{\sigma_5 \sigma_6 (\sigma_5)^{-1}}$$

2. Now we can apply the Yang-Baxter relations to the triple $(\sigma_6)^{-1} \sigma_5 \sigma_6$ and obtain after the cancellation of $(\sigma_5)^{-1}$ and $\sigma_5$ the primitive word

$$W_p = (\sigma_2)^{-1} \sigma_4 \sigma_6 (\sigma_5)^{-1}$$

The first step of the braid contracting procedure completely coincides with what we did for the locally free group, while the second step we could regard (approximately, of course) as follows:

Consider some pair, for instance, $(\sigma_6)^{-1} \sigma_5$. We commute it with the probability $p$. Such commutation we denote as an error. The probability to meet the generator $\sigma_i$ in the Markov chain with uniform distribution over the generators in the braid group $B_n$ is of order of $p = \frac{1}{2n}$. Later on we consider more general case taking $p$ as the variational parameter.

So, let $V_n^{\text{braid}}(\mu)$ be the number of all primitive words of length $\mu$ in the braid group $B_n$. Our main idea is as follows: we would like to relate the quantity $V_n^{\text{braid}}(\mu)$ to the number of primitive words in the "group" $\mathcal{LF}_n^{\text{err}}(2)$ averaged over the uniform distribution of "errors" in commutation relations.

It should be pointed out that the object $\mathcal{LF}_n^{\text{err}}(2)$ cannot be considered as a real group anymore. So everywhere below we understand under $\mathcal{LF}_n^{\text{err}}(2)$ just the ensemble of random words written in terms of the alphabet with specific commutation relations among the letters.
4.1 Statistics of Words with ”Errors” in Locally Free Groups

The methods of theoretical description of the systems with disorder are rather well developed, especially in regard to the investigation of spin glass models [33].

Central for these methods is the concept of self-averaging which can be explained as follows. Take some additive function $F$ (the free energy, for instance) of some disordered system. The function $F$ is the self-averaging quantity if the observed value, $F_{\text{obs}}$, of any macroscopic sample of the system coincides with the value $F_{\text{av}}$ averaged over the ensemble of disorder realizations:

$$F_{\text{obs}} = \langle F \rangle_{\text{av}}$$

The phenomenon of self-averaging takes place in the systems with sufficiently weak long-range correlations: in this case only $F$ can be considered as a sum of contributions from different volume domains, containing statistically independent realizations of disorder (for more details see [18]).

The central technical problem of systems with quenched disorder deals with the calculation of the free energy $F(\mu)$ averaged over the randomly distributed quenched pattern. In our case we could associate the number of topologically different words with the partition function, hence the free energy would be $F(\mu) = -\langle \ln V_n^{\text{err}}(\mu) \rangle$ and the ”quenched pattern” is just the set of ”errors” in commutation relations.

A problem closely related to that mentioned above arises when averaging the correlation functions of some statistical system over the disorder. In this case the computations are based on finding the averaged density of states of some random matrix over the prescribed distribution of random entries. Below we show that the calculation of the mean value $\langle V_n^{\text{err}}(\mu) \rangle$ belongs precisely to this class of problems.

**Conjecture 1** The number of nonequivalent primitive words, $V_n^{\text{braid}}(\mu)$, of length $\mu$ in the braid group $B_n$ can be estimated in the limits $n = \text{const} \gg 1$, $\mu \gg 1$ as follows:

$$V_n^{\text{braid}}(\mu) \approx \langle V_n(\mu, d = 2, p) \rangle$$  \hfill (32)

where $V_n(\mu, d = 2, p)$ is the number of all distinct primitive words of length $\mu$ with the ”errors” in the commutation relations in the locally free group $\mathcal{LF}_n(2)$. We allow to commute the neighboring generators with the probability $p$ and the averaging is performed
over the uniform probability distribution of "errors". The question concerning the choice of \( p \) is considered below.

In support of our conjecture we bring the numerical computations performed in the work [28], where we have constructed the (right-hand) random walk (the random word) on the group \( \mathcal{G}_n = \{ \mathcal{L}F_{n, \text{err}}, B_n \} \) with a uniform distribution over generators \( \{g_1, \ldots, g_{n-1}, g_1^{-1}, \ldots, g_{n-1}^{-1}\} \in \mathcal{G}_n \). It means that with the probability \( \frac{1}{2^{n-2}} \) we have added the element \( g_{\alpha_N} \) or \( g_{\alpha_N}^{-1} \) to the given word of \( N - 1 \) generators (letters) from the right-hand side. In [28] the following question has been raised: what is the averaged length of the primitive path \( \langle \mu \rangle \) for the \( N \)-step random walk on the group \( \mathcal{G}_n \)?

In the Table 1 we show the results of numerical simulations carried on in [28] for the expectation value \( \langle \mu \rangle / N \) of the \( N \)-step random walk on the "locally free structure with errors", \( \mathcal{L}F_{n, \text{err}}^\text{(2)} \), and compare them to the same value for the \( N \)-step random walk on the braid group \( B_n \).

| \( n \) | \( \mathcal{L}F_{n, \text{err}}^\text{(2)} \) with \( p = 1/5 \) | \( B_n \) |
|------|-----------------|-----|
| 5    | 0.55            | 0.49|
| 10   | 0.58            | 0.56|
| 20   | 0.59            | 0.59|
| 50   | 0.60            | 0.61|
| 100  | 0.60            | 0.61|
| 200  | 0.61            | 0.61|

We have found asymptotically very good correspondence of the mean values \( \langle \mu \rangle / N \) for the braid group and the "locally free structure with the errors" for \( p = \frac{1}{5} \).

Of course, our conjecture is not exact and has a "mean–field" nature because we admit two subsequent generators \( (\sigma_{\alpha_k}, \sigma_{\alpha_{k+1}}) \) with nearest neighbor indices \( \alpha_{k+1} = \alpha_k \pm 1 \) to commute with the probability \( p \) regardless the value of the generator \( \sigma_{\alpha_{k+2}} \) in the sequence of letters in the word. Hence the problem of taking right value of \( p \) appears.

Two limiting cases are trivial:
(i) For $p = 1$ we have a complete commutative group and the obvious inequality is fulfilled

$$V_n^{\text{braid}}(\mu) \geq \langle V_n(\mu, d = 2, p = 1) \rangle \quad (33)$$

(ii) For $p = 0$ we return to locally free group $L\mathcal{F}_2$, for which we should have

$$V_n^{\text{braid}}(\mu) \leq V_n(\mu, d = 2, p = 0) \quad (34)$$

In the framework of the mean-field approximation and taking into account Eq.(32) we claim:

$$\langle V_n(\mu, d = 2, p) \rangle \bigg|_{p \to 1^-} \leq V_n^{\text{braid}}(\mu) \leq V_n(\mu, d = 2, p = 0) \quad (35)$$

We show below that in the limit $n \to \infty, \mu \to \infty$ the following equality takes place

$$\langle V_n(\mu, d = 2, p) \rangle \bigg|_{p \to 1^-} = V_n(\mu, d = 2, p = 0) \quad (36)$$

**The Main Statement.** For the value $V_n^{\text{braid}}(\mu)$ we have the following asymptotic behavior

$$\lim_{\mu \to \infty} \frac{1}{n} \ln V_n^{\text{braid}}(\mu) = \left[ \frac{1}{\mu} \ln \langle V_n(\mu, d = 2, p) \rangle \right]_{\text{independent \ on \ } p} = \ln 7 \quad (37)$$

It is easy to understand that the number of nonequivalent primitive words $V_n(\mu, d = 2, p)$ in the "locally free group with errors" can be calculated by means of averaging of Eq.(14) if we slightly change the matrix $\tilde{M}_n$ replacing it by the random incidence matrix $\hat{M}_n$:

\[
\hat{M}_n^{\text{err}}(d = 2) =
\begin{array}{cccccc}
  f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-1} & f_n \\
  f_1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\
  f_2 & x_{n-1} & 0 & 1 & 1 & \cdots & 1 & 1 \\
  f_3 & 0 & x_{n-2} & 0 & 1 & \cdots & 1 & 1 \\
  f_4 & 0 & 0 & x_{n-3} & 0 & \cdots & 1 & 1 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  f_{n-1} & 0 & 0 & 0 & \cdots & 0 & 1 \\
  f_n & 0 & 0 & 0 & 0 & \cdots & x_1 & 0 \\
\end{array}
\]
It is now parametrized by the random sequence of "0" or "1", i.e.
\[ \{ x^{(n)} \} = \{ x_{n-1}, x_{n-2}, \ldots, x_2, x_1 \} \] (39)

where
\[
\begin{align*}
\text{Prob}(x_j = 1) &= 1 - p \\
\text{Prob}(x_j = 0) &= p
\end{align*}
\] (40)

4.2 Density of States of Random Operator and Averaged Number of Nonequivalent Words

The determinant of the matrix \( \tilde{M}^{\text{err}}(d = 2) - \lambda \tilde{I} \) (see Eq.(38)) satisfies the random recursion relation
\[
a_{k+1} + a_k(\lambda + x_k) + a_{k-1}(1 + \lambda)x_k = 0 \tag{41}
\]

Introducing the Ricatti-like variable
\[
\rho_k = \frac{a_{k+1}}{a_k}
\]
we arrive at the recursion relation
\[
\rho_k = -(\lambda + x_k) - \frac{(1 + \lambda)x_k}{\rho_{k-1}} \quad k \in [0, n] \tag{42}
\]
with the boundary condition \( \rho_0 = -\lambda \).

For the continuous sequence of \( \{1\} \), i.e. \( \{ x^{(n)} \} = \{1 \ 1 \ 1 \ 1 \ldots \ 1 \} \) we have the following (nonrandom) transformation
\[
\rho_{k+1} = -(\lambda + 1) \left( 1 + \frac{1}{\rho_k} \right) \tag{43}
\]

As soon as a zero appears in the random sequence, \( \rho_k \) in (42) is set to \(-\lambda\) which coincides precisely with the initial value \( \rho_0 \). Since one returns back to the initial value, the process can be easily iterated for arbitrary random sequences \( \{ x^{(n)} \} \) when \( n \gg 1 \). Such a property of the map (42) is equivalent to the factorization of the determinant \( a_n(\lambda) \) of the random matrix \( \tilde{M}^{\text{err}}(d = 2) - \lambda \tilde{I} \):

Example 2. Consider the sequence
\[
\{ x^{(n)} \} = \begin{pmatrix}
1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 1 & \ldots & 1 \\
\hline
i_1 & & & & m_1 & & & & m_1 & &
\end{pmatrix}
\] (44)
The corresponding determinant $a_n(\lambda)$ factorizes:

$$a_n(\lambda) = \prod_{\{l_j\}} a_{l_j}(\lambda) \prod_{\{m_j\}} (-\lambda)^{m_j-1}$$  \hspace{1cm} (45)$$

It is worth pointing out that a recursion relation similar to (43) appears also in the study of the binary product of random $2 \times 2$ matrices where one of the matrix is singular [37]. Such a structure also occurs in the case of Ising model in a random magnetic field—see [38]. Following Derrida and Hilhorst [38], one can write down the invariant measure associated to (42):

$$P(\rho) = \sum_{k=1}^{\infty} p(1-p)^k \delta(\rho - \rho_k) + (1-p) \sum_{k=1}^{\infty} p^k \delta(\rho - \rho_0)$$  \hspace{1cm} (46)$$

The first (the second) term comes from the complete sequences of $\{1\}$ ($\{0\}$) of arbitrary length $k = 1, 2, \ldots$. From the invariant measure one can compute $\overline{\ln a(\lambda)}$ which can be interpreted either as a Lyapunov exponent or as the free energy (depending on the physical context). So, we get:

$$\overline{\ln a(\lambda)} = \lim_{n \to \infty} \frac{1}{n} \ln a_n(\lambda) = \int \mathcal{P}(\rho) \ln \rho d\rho$$  
$$= \sum_{k=1}^{\infty} p^2(1-p)^{k-1} \ln a_k(\lambda)$$  \hspace{1cm} (47)$$

Returning to our problem, we may use this expression to write down the averaged density of states

$$\rho(\lambda) = \frac{1}{\pi} \frac{\partial}{\partial \lambda} \text{Im} \ln a(\lambda) \equiv \sum_{k=1}^{\infty} p^2(1-p)^{k-1} \rho_k(\lambda)$$  \hspace{1cm} (48)$$

where $\rho_n(\lambda) \equiv \frac{1}{\pi} \text{Im} \ln a_n(\lambda)$ is the density of states of a pure system (i.e. without randomness) of length $n$. From the density of states we can find the average number of words in the limit $n \to \infty$

Define $\langle V^*(\mu) \rangle$ as follows:

$$\langle V^*(\mu) \rangle = \lim_{n \to \infty} \frac{\langle V^*_{n}(\mu) \rangle}{n} = 2 \int (2\lambda + 1)^{\mu-1} \frac{1}{\rho(\lambda)} d\lambda$$  \hspace{1cm} (49)$$

where the integration over $\lambda$ runs over the whole spectrum. Let us repeat once more that the function $\frac{1}{\pi} \text{Im} \ln a_n(\lambda)$ is the density of states of the random matrix $\hat{M}_n(d = 2) - \lambda \hat{I}$ $(n \to \infty$ averaged over the disordered pattern $\{x^{(n)}\}$.)
The content of Eq.(48) is as follows. The density of states $\rho(\lambda)$ can be obtained by averaging the spectrum of the regular case "weighted" with associated sequences of \{1\}:

$$\text{Prob}\left\{ x = (0 \underbrace{1 1 1 1 \ldots 1 1 1}_{\text{complete set of } \{1\}} 0) \right\} = p^2(1-p)^n \quad (50)$$

One should also add the contribution coming from zero’s energy state corresponding to sequences of \{0\}. The resulting expression reads:

$$\overline{\rho(\lambda)} = \sum_{m=1}^{\infty} p^2(1-p)^{2m-1} \left\{ m\delta(\lambda + 1) + \sum_{k=1}^{m} \delta \left( \lambda + 1 - 4\cos^2 \frac{k\pi}{2m+2} \right) \right\}$$

$$+ \sum_{m=0}^{\infty} p^2(1-p)^{2m} \left\{ m\delta(\lambda + 1) + \sum_{k=1}^{m+1} \delta \left( \lambda + 1 - 4\cos^2 \frac{k\pi}{2m+3} \right) \right\} \quad (51)$$

which may be rewritten as

$$\overline{\rho(\lambda)} = \frac{1}{2} - \frac{p}{p} \delta(\lambda + 1) + \sum_{n=0}^{\infty} p^2(1-p)^n \left[ \sum_{k=1}^{[\frac{n+2}{2}]} \delta \left( \lambda + 1 - 4\cos^2 \frac{k\pi}{n+3} \right) \right] \quad (52)$$

where $[x]$ denotes the integer part of $x$.

Using Eq.(52) we may check that the function $\overline{\rho(\lambda)}$ is properly normalized:

$$\int_{-\infty}^{+\infty} \overline{\rho(\lambda)} = \int_{-1}^{3} \overline{\rho(\lambda)} = 1$$

Returning to (49) we get

$$\langle V^*(\mu, p) \rangle = 2 \left( \frac{1-p}{2-p} \right) (-1)^{\mu-1} + 2 \sum_{n=0}^{\infty} p^2(1-p)^n S_n(\mu) \quad (53)$$

where

$$S_n(\mu) = \sum_{k=1}^{[\frac{n}{2}]} \left( 8\cos^2 \frac{k\pi}{n+1} - 1 \right)^{\mu-1} \quad (54)$$

(compare to (28)).

In order to check the algebra we have computed $\langle V^*(\mu) \rangle$ for small values of $\mu$. One gets:

$$\langle V^*(1) \rangle = \langle V^*(2) \rangle = 2$$

which can be readily obtained through a direct calculation of $\lim_{n \to \infty} 2\text{Trace} \left( 2\hat{M}_n + \hat{I} \right)^{\mu-1}$. We are however mainly interested in the limit $\mu \to \infty$. 21
Using (29)–(31) we may resum the series (53) by isolating the contribution \( n < \mu \) and \( n > \mu \). After some algebra one obtains for \( \mu < n + 1 \):

\[
S_n(\mu) = (n + 1) \sum_{p=1}^{\mu-1} C_{\mu-1}^p (-1)^{\mu-p} C_{2p-1}^{\mu-1} - \frac{1}{2} \left\{ 7^{\mu-1} + (-1)^\mu \right\} - \left[ \frac{n}{2} \right] (-1)^\mu
\]

(55)

which gives

\[
S_n(\mu) \Bigr|_{\mu \gg 1} \simeq \begin{cases} 
7^\mu & n_0 < n < \mu \\
(n + 1) 7^\mu & \mu < n 
\end{cases}
\]

(56)

where \( n_0 \) is some constant of order of unity. The Eq.(55) is plotted in Fig.2a,b.

Thus, we can rewrite Eqs.(53)–(54) as follows

\[
\langle V^*(\mu, p) \rangle = 2 \left( \frac{1-p}{2-p} \right) (-1)^{\mu-1} + \sum_0^\mu p^2(1-p)^n S_{n+2}(\mu) + \sum_\mu^\infty p^2(1-p)^n S_{n+2}(\mu)
\]

(57)

The corresponding behavior of the function \( Q(p|\mu) \) where

\[
Q(p|\mu) = \frac{\ln \langle V^*(\mu, p) \rangle}{\mu} \quad (0 < p < 1)
\]

(58)

is shown in Fig.3 for few fixed values \( \mu = \{10, 30, 150\} \).

The plot in Fig.3 enables us to come to the following conclusion. If the number of "errors" is small \( (p \to 0^+) \), the volume of the group grows exponentially with the Lyapunov exponent \( \ln 7 \) (for \( \mu \to \infty \)). For the arbitrary number of "errors", \( p \), the corresponding Lyapunov exponent approaches the same value \( \ln 7 \) for all \( p < 1 \) in the limit \( \mu \to \infty \) and exhibits a singular behavior just at the point \( p = 1 \) (which corresponds to the completely commutative group).

The asymptotic expression (57) allows us to conclude that the limit behavior of the function \( V^*(\mu) \) is independent on \( p \), \( \forall p \in [0, 1] \) and is the same as for the locally free group \( \mathcal{LF}_n \) without any errors. This fact supports our conjecture (37).

It should be emphasized that these results are expected to hold only in the thermodynamic limit \( n \to \infty \). It would be more desirable to consider the limit in which the number of generators \( n \) is kept fixed and the length of the word, \( \mu \), is much larger than \( n \).
4.3 Functional Equation, Continued Fractions and Invariant Measure

The behavior of the spectral density of our model is very similar to the one encountered in the study of harmonic chains with binary random distribution of masses. This problem, which goes back to Dyson has been investigated by Domb et al [34] and then thoroughly discussed by Nieuwenhuizen and Luck [35]. One considers a chain of oscillators where the masses can take two values:

\[
\begin{cases}
  m & \text{with the probability } 1 - p \\
  M > m & \text{with the probability } p
\end{cases}
\]

In the limit \( M \to \infty \) the system breaks into islands, each of which consisting of \( n \) light masses surrounded by two infinite heavy masses. The probability of occurrence of such an island is \( p^2 (1 - p)^n \). There is clearly a mapping to our model if one replaces the sequences of heavy and light masses by the sequences of “0” and “1”. Many results may therefore be borrowed from the works [34, 35]. In particular, by adapting the calculations of Nieuwenhuizen and Luck to our case one may rewrite the integrated density of states

\[
\overline{N}(\lambda) = \int_{-\infty}^{\lambda} \rho(\lambda') d\lambda'
\]

in the form

\[
\overline{N}(\lambda) = 1 - \frac{p}{(1 - p)^2} \sum_{n=1}^{\infty} (1 - p)^{\text{Int}(\frac{\lambda}{\pi})}
\]

where the relation between \( \lambda \) and \( \vartheta \) is given in Eq. (23).

For \( \lambda \to -1 \) one gets \( \overline{N}(\lambda) \to \frac{1 - p}{2 - p} \) which corresponds to the contribution of the states \( \lambda = -1 \) at the bottom of the spectrum.

At the upper edge of the spectrum (namely for \( \lambda \to 3^- \)) one gets \( \overline{N}(\lambda) \to 1 \) which means that all the states are counted. Equation (59) shows that the behavior around \( \lambda = 3 \) (corresponding to \( \vartheta = 0 \)) is in fact dominated by the first term \((n = 1)\) of the series. One has:

\[
\overline{N}(\lambda) \simeq 1 - \frac{p}{(1 - p)^2} (1 - p)^{2\pi \sqrt{3 - \lambda}} \\
\equiv 1 - \frac{p}{(1 - p)^2} \exp \left[ \frac{2\pi}{\sqrt{3 - \lambda}} \ln(1 - p) \right]
\]

(60)
The behavior (60) signals the appereance of Lifshits’ singularity in the density of states. A more precise analysis shows that this result is in fact modulated by a periodic function \[35\].

Equation (59) displays many interesting features. In particular, the function \(\mathcal{N}(\lambda)\) occurs in the mathematical literature as a generating function of the continued fraction expansion of \(\frac{\pi}{\vartheta}\).

Let us briefly sketch this connection. Consider the continued fraction expansion

\[
\frac{\pi}{\vartheta} = \frac{1}{c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \ldots}}}
\]

where all \(c_n\) are natural integers. Truncating this expansion at level \(n\) we get a rational number \(\frac{p_n}{q_n}\) which converges to \(\frac{\pi}{\vartheta}\) when \(n \to \infty\).

A theorem of Böhmer [36] states that the generating function of the integer part of \(\frac{\pi}{\vartheta}\)

\[
G(z) = \sum_{n=1}^{\infty} z^\text{Int}(\frac{n\pi}{\vartheta})
\]

is given by the continued fraction expansion

\[
G(z) = \frac{z}{1-z} A_0 + \frac{1}{1 - \frac{1}{A_1 + \frac{1}{A_2 + \ldots}}}
\]

where

\[
A_n(z) = \left(\frac{1}{z}\right)^{q_n} - \left(\frac{1}{z}\right)^{q_n-2} - 1
\]

and \(q_n\) is the denominator of the fraction \(\frac{p_n}{q_n}\) approximating the value \(\frac{\pi}{\vartheta}\).

In order to make connection with our problem it is sufficient to set \(z = 1 - p\) and express \(\mathcal{N}(\lambda)\) in terms of \(G(z)\).

Equation (59) shows that the invariant measure is a very singular object. However it satisfies a simple functional equation reminiscent of that which arises in the theory of automorphic forms. The equation can be derived either by a Dyson-Schmidt approach (see, for instance [18]) or just by looking at the explicit expression of \(\mathcal{P}(\rho)\).
It is in fact simpler to work with the rescaled variable

\[ z_n = -\frac{1}{\sqrt{\lambda + 1}} \rho_n \]

which satisfies the recursion relation equivalent to Eq. (13)

\[ z_n = \mu - \frac{1}{z_{n-1}} \]

(62)

where \( \mu = \sqrt{\lambda + 1} \). The transformation (62) may be obtained from the two matrices belonging to the group \( SL(2, R) \)

\[ \hat{T} = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} ; \quad \hat{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \]

The \( SL(2, R) \)–transformation \( \hat{T}\hat{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) acts on \( z \) by the fractional linear transformation

\[ z_n = \frac{az_{n-1} + b}{cz_{n-1} + d} = \mu - \frac{1}{z_{n-1}} . \]

(63)

The invariant measure of (62), which may be rewritten as

\[ \mathcal{P}(z) = \sum_{n=0}^{\infty} p(1-p)^n \delta \left( z - [\hat{S}\hat{T}]^nz_0 \right) \]

(64)

can easily be shown to satisfy the fundamental equation

\[ \mathcal{P}(z) = \frac{1}{1-p} (cz+d)^2 \mathcal{P}(\hat{S}\hat{T}z) \]

(65)

up to some singular terms.

By suitable rescaling it is in fact possible to absorb the prefactor \( 1/(1-p) \) and rewrite Eq. (65) as

\[ \mathcal{P}(\hat{g}z) = (cz+d)^{-2} \mathcal{P}(z) \]

(66)

where \( \hat{g} \in SL(2, R) \).

An analytical continuation of this expression into the complex \( z \)–plane would eventually permit one to interpret \( \mathcal{P} \) as an automorphic form. From the theory of automorphic functions it is well known that Eq. (66) is satisfied by the Dedekind modular function \( \eta(z) \):

\[ \eta(z) = \frac{1}{(cz+d)^2} \eta \left( \frac{az+b}{cz+d} \right) \quad (ad-bc = 1) \]
Such objects although perfectly smooth in the upper half–plane \( \text{Im} z > 0 \) display highly non–trivial fractal behavior on the boundary \( \text{Im} z = 0 \) (see, for instance [7, 39]).

Another interesting connection which would be worth investigating is the fact that \( \hat{S} \) and \( \hat{T} \) generate the so-called Hecke group \( \Gamma(h) \) for \( h = 2 \cos \frac{\pi}{q} \) \((q \geq 3 \text{ is integer})\). Surprisingly, these values of \( h \) coincide with a subset of the spectrum of the matrix \( \hat{M}_n \) (see Eq.(25)).

5 Final Remarks

5.1 The Geometrical View on the Word Enumeration Problem

The number of primitive words in the locally free or braid groups allows a rather straightforward geometrical description. Namely, the matrix \( \hat{M}_n(d = 2) \) can be regarded as the transfer matrix for the model of a ”biased Levy–flight”–like ("BLF"–like) one–dimensional random walk on the finite support. Actually, let us compute the statistical sum of the process described below. Take \( n \) integers on the line: 1, 2, \ldots, \( n \) and consider the random walk when the walker can jump with equal probabilities from the vertex with the coordinate \( m_1 \) \((1 \leq m_1 < n)\) to:

(i) each vertex with the coordinate \( m_2 \) \((m_2 \in [m_1 + 1, n])\);

(ii) the vertex with the coordinate \( m_2 = m_1 - 1 \) (i.e. one step back).

The corresponding process is represented schematically in Fig.4.

Analogously, we can associate the random operator \( \hat{M}_n^{err} \) with the transfer matrix of the generalized BLF–like random process which is described via the same rules (i) and (ii) but with additional requirement that the jump (ii) is blocked with probability \( p \) and allowed with probability \( 1 - p \), independently of the position of the vertex.

5.2 Conclusion

We have proposed a statistical method for enumerating the primitive words in the braid group \( B_n \) based on the consideration of locally free groups with errors in commutation relations. We brought arguments in support of the conjecture that the number of
long primitive words in the braid group is not sensitive to the precise local commutation relations. We discussed the connection of the abovementioned problems with the conventional random operator theory, localization phenomena and statistics of systems with quenched disorder, and showed the connection between some particular problems of random matrix theory and the theory of automorphic functions.

We believe that the problem of discovering the integrable models associated with the proposed locally free groups and developing the corresponding conformal field theory could help establish a bridge between the statistics of random walks on the noncommutative groups, spectral theory on multiconnected Riemann surfaces, and topological field theory.

Acknowledgments

We are very grateful to J. Desbois for elucidating for us many questions concerning the limiting behavior of random walks on locally free and braid groups; we would like to thank as well M. Tsypin for useful remarks and for help in the numerical confirmation of some of our conjectures. S.N. acknowledges the fruitful discussions with S. Fomin, L. Pastur, Ya. Sinai and A. Vershik on many aspects of the work. We highly appreciate the assistance of O.Martin in the final preparation of the paper and critical reading of the manuscript.
Appendix A

The calculation of the number of distinct primitive words \( V_n^p(\mu) \) can be easily extended to the case of the group \( LF_n^{\text{perm}} \) which is defined by the old relations:

(a) Each pair \((g_j, g_k)\) generates the free subgroup of the group \( LF_n^{\text{perm}} \) if \(|j - k| < 2\);

(b) \( g_j g_k = g_k g_j \) for \(|j - k| \geq 2\)

completed with the additional requirement\(^3\) \( (g_i)^p = 1 \) which may be rewritten also as:

\[
g_j = 1 \mod p \quad \forall j \in [1, n]
\]  

(A.1)

The calculation of the number of distinct primitive words, \( V_n^p(\mu) \), can be carried out using Eq.(9):

\[
V_n^{\text{perm}}(\mu) = \sum_{s=1}^{\mu} R_n(s, d) \sum_{\{m_1, \ldots, m_s\}}^{''} \Delta \left[ \sum_{i=1}^{s} |m_i| - \mu \right]
\]  

(A.2)

where "double prime" means that the sum runs over all \( m_i \neq 0 \) consistent with Eq.(A.1).

Taking into account that the summation over all \( \{m_1, \ldots, m_s\} \) obeying the condition (A.1) is independent on computation of \( R_n(s) \), we have to replace the Eq.(13) by the following one

\[
\mathcal{N} \equiv \sum_{\{m_1, \ldots, m_s\}}^{''} \Delta \left[ \sum_{i=1}^{s} |m_i| - \mu \right] = \frac{2^s}{2\pi i} \oint_{(C)} dz \frac{\left( \sum_{m=1}^{\mu-1} \frac{1}{z^m} \right)^s}{z^{1-\mu+s}}
\]  

(A.3)

where the contour \( C \) encloses the origin of the complex plane \( z \).

We used the integral representation of Kronecker \( \Delta \)-function

\[
\Delta(x) = \frac{1}{2\pi i} \oint \frac{dz}{z^{1+x}} = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}
\]  

(A.4)

Let us consider three special cases:

\(^3\)Compare to the definition of the group of permutations.

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1. $p \to \infty$. From Eq.(A.3) we get

$$N = \frac{2^s}{2\pi i} \oint dz \, z^{-1+\mu-s}(1-z)^{-s} = 2^s \frac{(\mu - 1)!}{(s - 1)!(\mu - s)!} \tag{A.5}$$

which coincides with Eq.(13).

2. $p = 2$. From Eq.(A.3) we get

$$N = \frac{2^s}{2\pi i} \oint dz \, z^{-1+\mu-s} = 2^s \delta_{\mu,s} \tag{A.6}$$

3. $p = 3$. From Eq.(A.3) we get

$$N = \frac{2^s}{2\pi i} \oint dz \, z^{-1+\mu-s}(1+z)^s = 2^s \frac{s!}{(2s - \mu)!(\mu - s)!} \tag{A.7}$$

Substituting Eqs.(A.5)–(A.6) and Eq.(13) into Eq.(9) we may easily compute the corresponding values $V_n^{\text{perm}}(\mu)$. 

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Appendix B

The computation of the averaged density of states $\bar{\rho}(\lambda)$ of the random matrix $\hat{M}_n^{err} - \lambda \hat{I}$ belongs to a class of problems with the "quenched disorder" (see the discussion in the section 4.1). Another class of problems closely related to the mentioned one arises when averaging the partition function (but not the free energy) over the disorder. Problems corresponding to the case of annealed disorder usually seem to be simpler from computational point of view than that of quenched disorder but usually the thermodynamic behavior of systems with annealed disorder is less rich and less interesting than that of systems of quenched disorder. However in our case it would be very desirable to compute the value $a_n(\lambda)$ where $a_n(\lambda)$ is determined by Eq.(45) in order to compare the values $\ln a_n(\lambda)$ and $\ln a_n'(\lambda)$.

Averaging $a_n(\lambda)$ over the random distributions of $\{1\}$ and $\{0\}$ in the sequence $\{x^{(n)}\}$ (see (44)) and taking into account the factorization of corresponding determinant (45), we get

$$a_n(\lambda) = \sum_{s=1}^{\infty} \sum' \sum' \prod_{j=1}^{s} \left[ a_{l_j+1}(\lambda) (-\lambda)^{m_j-1} \right]$$

(B.1)

where $\sum'$ means that summation runs over the sequences $\{l_1 \ldots l_s\}$ and $\{m_1 \ldots m_s\}$ obeying the conditions:

$$\sum_{j=1}^{s} l_j = n (1 - p)$$

$$\sum_{j=1}^{s} m_j = n p$$

(B.2)

Recall that $p$ is the fraction of zeros in the random matrix $\hat{M}_n^{err}$ and according to Eq.(B.2) the quantity $(n p)$ is always integer.

Using the integral representation of the Kronecker $\Delta$–function (see Eq.(A.3)), we may rewrite Eq.(B.2) in the form of a grand canonical distribution:

$$a_n(\lambda) = \sum_{s=1}^{\infty} (-\lambda)^{np-s} \prod_{l_1 \ldots l_s} \prod_{j=1}^{s} a_{l_j+1}(\lambda) \Delta \left[ \sum_{j=1}^{s} l_j - n(1 - p) \right]$$

$$= (-\lambda)^{np} \frac{1}{2\pi i} \oint dzz^{-1+n(1-p)} \sum_{s=1}^{\infty} (-\lambda)^{-1} \sum_{l=0}^{\infty} a_l(\lambda) z^{-l} \right]^s$$

(B.3)
where the function $b(\lambda, z)$ plays a role of generating function for $a_l(\lambda)$:

$$b(\lambda, z) = \sum_{l=1}^{\infty} a_l(\lambda) z^{-l}$$

Recall that

$$a_l(\lambda) = \frac{2(-1)^n}{\sqrt{(3-\lambda)(1+\lambda)}} (1 + \lambda)^{l/2} \sin[(l + 2)\theta]$$

$$\sin \theta = \frac{1}{2} \sqrt{3 - \lambda}$$

$$\cos \theta = \frac{1}{2} \sqrt{1 + \lambda} \tag{B.4}$$

Collecting all equations together, we arrive at the following expression

$$\overline{a_n(\lambda)} = (-\lambda)^{np} \frac{1}{2\pi i} \oint dz z^{n(1-p)} \frac{\lambda(1+z^{-1}) + z^{-1}}{\lambda + \lambda^2 z^{-1} + (\lambda^2 - 1) z^{-2}} \tag{B.5}$$

Now we can use the definition of the Chebyshev’s polynomials generating function:

$$\frac{1}{1 - 2tx + t^2} = \sum_{k=0}^{\infty} U_k(x) t^k$$

where

$$U_n(x) = \frac{\sin \left[ (n + 1) \arccos x \right]}{\sin \left[ \arccos x \right]}$$

After rather simple algebra we get the desired equation for the averaged determinant:

$$\overline{a_n(\lambda)} = (-1)^n \lambda^{np+1} \left( \frac{\lambda^2 - 1}{\lambda} \right)^{\frac{1}{2}(1-p)} \left[ U_{n(1-p)}(y) - \left( \frac{\lambda + 1}{\lambda(\lambda + 1)} \right)^{1/2} U_{n(1-p) - 1}(y) \right] \tag{B.6}$$

where

$$y = \frac{\lambda^{3/2}}{2\sqrt{\lambda^2 - 1}} \tag{B.6a}$$

As we can see, this result differs dramatically from the ”quenched case” (see Eq.(47)).
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**Figure Captions**

**Fig.1.** Graphs, corresponding to: (a) free group $\Gamma_n$; (b) locally free group $\mathcal{LF}_{n+1}$; (c) complete commutative group. In case of locally free group the vertices $A$ and $B$ should be glued because they represent one and the same word in group $\mathcal{LF}_{n+1}$.

**Fig.2.** Plot of the function $\ln S_n(\mu)$ in two regimes (Eq.(59)).

**Fig.3.** Plot of the function $Q(p|\mu)$ for three fixed values of $\mu = \{10, 30, 150\}$—see Eq.(58).

**Fig.4.** Schematic representation of the process associated with the ”biased Levy–flight” (BLF)–like random walk.