VARIATIONAL ITERATION HOMOTOPY PERTURBATION METHOD FOR THE SOLUTION OF SEVENTH ORDER BOUNDARY VALUE PROBLEMS

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Abstract. The induction motor behaviour is represented by a fifth order differential equation model. Addition of a torque correction factor to the model accurately reproduces the transient torques and instantaneous real and reactive power flows of the full seventh order differential equation model. The variational iteration homotopy perturbation method (VIHPM) is employed to solve the seventh order boundary value problems. The approximate solutions of the problems are obtained in terms of a rapidly convergent series. Several numerical examples are given to illustrate the implementation and efficiency of the method.

Keywords: Variational iteration homotopy perturbation method; Boundary value problems; Linear and nonlinear problems; Approximate solution.

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INTRODUCTION

The theory of seventh order boundary value problems is not much available in the numerical analysis literature. These problems are generally arise in modelling induction motors with two rotor circuits.

The induction motor behavior is represented by a fifth order differential equation model. This

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model contains two stator state variables, two rotor state variables and one shaft speed. Normally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters.

To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth order and rotor impedance is algebraically altered as function of rotor speed under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when rotor frequency is not a single value. So, the behavior of such models show up in the seventh order (Richards and Sarma, 1994)

Siddiqi and Ghazala (Siddiqi and Ghazala, 2006a and 2006b) presented the solutions of fifth and sixth order boundary value problems using non-polynomial spline. Noor and Mohyud-Din (Noor and Mohyud-Din, 2009) applied modified variational iteration method for solving the boundary layer problem in unbounded domain. Matinfar *et al* (Matinfar *et al*, 2010) implemented the variational homotopy perturbation method to obtain the solution of Fisher’s equation. (Siddiqi and Iftikhar, 2012) used the variation of parameter method to solve the seventh order boundary value problems.

Odibat discussed the convergence of variational iteration method in (Odibat, 2010). Tatari and Dehghan presented the sufficient conditions to guarantee the convergence of the variation iteration method (Tatari and Dehghan, 2007).

The aim of this study is to solve the seventh order boundary value problems and the variational iteration homotopy perturbation method is used for this purpose.

**Variational Iteration Homotopy Perturbation Method (Noor and Mohyud-Din, 2009, Matinfar *et al*, 2010, Mohyud-Din *et al*, 2010)**

Variational iteration homotopy perturbation method is formulated by the coupling of variational iteration method and homotopy perturbation method. The boundary value problem is
considered as under
\[ L[u(x)] + N[u(x)] = g(x), \]  
(1)

where \( L \) and \( N \) are linear and nonlinear operators respectively and \( g(x) \) is a forcing term. Following the variational iteration method used by He (He, 1998, 1999, 2000a, 2001). The correct functional for the problem \(^{11}\) can be written as
\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t) \{ Lu_n(t) + N\tilde{u}_n(t) - g(t) \} dt, \]  
(2)

where \( \lambda \) is a Lagrange multiplier, that can be identified optimally via variational iteration method. Here, \( \tilde{u}_n \) is considered to be a restricted variation which shows that \( \delta \tilde{u}_n = 0 \). Making the correct functional (2) stationary, yields
\[ \delta v_{n+1}(x) = \delta v_n(x) + \delta \int_0^x \lambda(t) \{ Lv_n(t) + N\tilde{v}_n(t) - g(t) \} dt \]  
\[ = \delta v_n(x) + \int_0^x \delta \{ \lambda(t)Lv_n(t) \} dt. \]  
(3)

Its stationary conditions can be obtained using integration by parts in Eq. (3). Therefore, the Lagrange multiplier can be written as
\[ \lambda = \frac{(-1)^m(t-x)^{m-1}}{(m-1)!}. \]  
(4)

Applying the homotopy perturbation method the following relation is obtained as
\[ \sum_{i=0}^{\infty} p^i u_i(x) = u_0(x) + \int_0^x \lambda(t) \left\{ L \left( \sum_{i=0}^{\infty} p^i u_i \right) + N \left( \sum_{i=0}^{\infty} p^i \tilde{u}_i \right) \right\} dt - \int_0^x \lambda(t)g(t)dt, \]  
(5)

Equating the like powers of \( p \) gives \( u_0, u_1, \cdots \). The embedding parameter \( p \in [0, 1] \) can be used as an expanding parameter. The approximate solution of the problem \(^{11}\), therefore, can be expressed as
\[ u = \lim_{p \to 1} \sum_{i=0}^{\infty} p^i u_i = u_0 + u_1 + u_2 + \cdots \]  
(6)
The series (6) is convergent for most of the cases. It is assumed that (6) has a unique solution.

In fact, the solution of the problem (1) is considered as the fixed point of the following functional under the suitable choice of the initial term $v_0(x)$.

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(t)\{Lv_n(t) + Nv_n(t) - g(t)\} dt. \quad (7)$$

**Convergence**

In this section, we will present Banach’s theorem about the convergence of the VIHPM. The VIHPM changes the given differential equation into a recurrence sequence of functions. The limit of this sequence is considered as the solution of the given differential equation.

**Theorem 1** (Banach’s fixed point theorem) (Tatari and Dehghan, 2007) Suppose that $X$ is a Banach space and $B : X \to X$ is a nonlinear mapping, and assume that

$$\|B[u] - B[\bar{u}]\| \leq \gamma \|u - \bar{u}\|, \quad \forall \ u, \bar{u} \in X. \quad (8)$$

for some constant $\gamma < 1$. Then $B$ has a unique fixed point. Moreover, the sequence

$$u_{n+1} = B[u_n] \quad (9)$$

with an arbitrary choice of $u_0 \in X$ converges to the fixed point of $B$ and

$$\|u_k - u_1\| \leq \sum_{j=1}^{k-2} \gamma^j \|u_1 - u_0\|, \quad (10)$$

According to Theorem 1, for the nonlinear mapping

$$B[u] = u(x) + \int_0^x \lambda(t)\{Lv_n(t) + Nv_n(t) - g(t)\} dt, \quad (11)$$

is a sufficient condition for convergence of the variational iteration homotopy perturbation method is strictly contraction of $B$. Furthermore, the sequence (9) converges to the fixed point of $B$ which also is the solution of the problem (1).

To implement the method, some numerical examples are considered in the following section.
Example 1 The following seventh order linear boundary value problem is considered

\[
\begin{align*}
    u^{(7)}(x) &= -u(x) - e^x(35 + 12x + 2x^2), 0 \leq x \leq 1, \\
    u(0) &= 0, \quad u(1) = 0, \\
    u(1)(0) &= 1, \quad u(1)(1) = -e, \\
    u^{(2)}(0) &= 0, \quad u^{(2)}(1) = -4e, \\
    u^{(3)}(0) &= -3.
\end{align*}
\]

The exact solution of the Example 1 is \( u(x) = x(1 - x)e^x \), (Siddiqi and Iftikhar, 2013).

The correct functional for the problem (12) can be written as

\[
v_{n+1}(x) = v_n(x) + \int_0^x \lambda(t)\{v^{(7)}_n(t) - v_n(t) + e^t(35 + 12t + 2t^2)\}dt,
\]

Making the correct functional (13) stationary, yields

\[
\delta v_{n+1}(x) = \delta v_n(x) + \delta \int_0^x \lambda(t)\{v^{(7)}_n(t) - v_n(t) + e^t(35 + 12t + 2t^2)\}dt
\]

Hence, the following stationary conditions can be determined

\[
\begin{align*}
    \lambda^{(7)}(t) &= 0, \\
    \lambda(t)|_{t=x} &= 0, \\
    \lambda'(t)|_{t=x} &= 0, \\
    : &
\end{align*}
\]

which yields

\[
\lambda = (-1)^7(t - x)^6 \frac{6!}{(6)!}.
\]
The Lagrange multiplier can be identified as follows

\[ \lambda = \frac{(-1)^m(t-x)^{m-1}}{(m-1)!}. \]  

(16)

According to (5), the following iteration formulation is obtained

\[ \sum_{i=0}^{\infty} p_i u_i(x) = u_0(x) + \int_0^x \frac{(-1)^7(t-x)^6}{6!} \left\{ \left( \sum_{i=0}^{\infty} p_i u_i \right)^{(7)} + \sum_{i=0}^{\infty} p_i u_i + e^t(35 + 12t + 2t^2) \right\} dt. \]  

(17)

Now, assume that an initial approximation has the form

\[ u_0(x) = x - \frac{x^3}{2} + Ax^4 + Bx^5 + Cx^6. \]  

(18)

Comparing the coefficient of like powers of \( p \)

\[ p^0 : \quad u_0(x) = x - \frac{x^3}{2} + Ax^4 + Bx^5 + Cx^6, \]

\[ p^1 : \quad u_1(x) = -\frac{x^7}{144} - \frac{x^8}{840} - \frac{x^9}{5760} - \frac{x^{10}}{45360} + \left( -\frac{107}{39916800} - \frac{A}{1663200} \right) x^{11} \]

\[ + \left( -\frac{1}{3548160} - \frac{B}{3991680} \right) x^{12} + O(x)^{13}, \]

where \( A, B \) and \( C \) are unknown constants to be determined later.

Using the first two approximations the series solution can be written as

\[ u(x) = x - \frac{x^3}{2} + Ax^4 + Bx^5 + Cx^6 - \frac{x^7}{144} - \frac{x^8}{840} - \frac{x^9}{5760} - \frac{x^{10}}{45360} + \left( -\frac{107}{39916800} - \frac{A}{1663200} \right) x^{11} \]

\[ + \left( -\frac{1}{3548160} - \frac{B}{3991680} \right) x^{12} + O(x)^{13}. \]

Using the boundary conditions (12), the values of the unknown constants can be determined as follows

\[ A = -0.3333333170467781, \quad B = -0.12500003614813987, \quad C = -0.03333331303032349. \]
Finally, the series solution is

\[ u(x) = x - (0.5)x^3 - 0.333333x^4 - 0.125x^5 - 0.0333333x^6 - 0.00694444x^7 - 0.00119048x^8 - 0.000173611x^9 - 0.0000220459x^{10} - \left(2.48016 \times 10^{-6}\right)x^{11} - \left(2.50521 \times 10^{-7}\right)x^{12} + O(x^{13}). \]

The comparison of the values of maximum absolute errors of the present method with the variation of parameter method (Siddiqi and Iftikhar, 2013) for the Example 1 is given in Table 1, which shows that the present method is more accurate. In Figure 1 the comparison of exact and approximate solutions is given and absolute errors are plotted in Figure 2 for Example 1.

**Example 2**

The following seventh order nonlinear boundary value problem is considered

\[
\begin{aligned}
&u^{(7)}(x) = e^{-x}u^2(x), \quad 0 < x < 1, \\
u(0) = u^{(1)}(0) = u^{(2)}(0) = u^{(3)}(0) = 1, \\
u(1) = u^{(1)}(1) = u^{(2)}(1) = e.
\end{aligned}
\]  

(19)

The exact solution of the Example 2 is \( u(x) = e^x \), (Siddiqi and Iftikhar, 2013)

The Lagrange multiplier can be identified as follows

\[
\lambda = \left(-1\right)^m \frac{(t-x)^{m-1}}{(m-1)!}.
\]  

(20)

According to (15), the following iteration formulation is obtained

\[
\sum_{i=0}^{\infty} p^i u_i(x) = u_0(x) + \int_0^x \left(-1\right)^7 \frac{(t-x)^6}{6!} \left\{ \left(\sum_{i=0}^{\infty} p^i u_i\right)^{(7)} - \left(\sum_{i=0}^{\infty} p^i u_i\right)^2 e^{-t} \right\} dt.
\]  

(21)

Now, assume that an initial approximation has the form

\[
u_0(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + Ax^4 + Bx^5 + Cx^6.
\]  

(22)
Comparing the coefficient of like powers of $p$

\[ p^0: \ u_0(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + Ax^4 + Bx^5 + Cx^6, \]

\[ p^1: \ u_1(x) = \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \left( -\frac{1}{39916800} + \frac{A}{831600} \right) x^{11} \]

\[ + \left( -\frac{1}{479001600} + \frac{B}{1995840} \right) x^{12} + O(x)^{13}, \]

where $A$, $B$ and $C$ are unknown constants to be determined later.

Using the first two approximations the series solution can be written as

\[ u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + Ax^4 + Bx^5 + Cx^6 + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} \]

\[ + \left( -\frac{1}{39916800} + \frac{A}{831600} \right) x^{11} + \left( -\frac{1}{479001600} + \frac{B}{1995840} \right) x^{12} + O(x)^{13}. \]

Using the boundary conditions (19), the values of the unknown constants can be determined as follows

\[ A = 0.041666667529862395, \quad B = 0.008333331197193119, \quad C = 0.001388890268167299. \]

Finally, the series solution is

\[ u(x) = 1 + x + 0.5x^2 + 0.166667x^3 + 0.0416667x^4 + 0.00833333x^5 + 0.00138889x^6 \]

\[ + 0.000198413x^7 + 0.0000248016x^8 + (2.75573 \times 10^{-6})x^9 + (2.75573 \times 10^{-7})x^{10} \]

\[ + (2.50521 \times 10^{-8})x^{11} + (2.08768 \times 10^{-9})x^{12} + O(x)^{13}. \]

In Table 2, errors obtained by the present method are compared with errors obtained using the variation of parameters method (Siddiqi and Iftikhar, 2013) for the Example 2. It is observed that the maximum absolute error value for the present method is $4.5614 \times 10^{-9}$ which is better than the maximum absolute error value $7.7176 \times 10^{-7}$, of the variation of parameters method (Siddiqi and Iftikhar, 2013). Figure 2 shows the comparison of exact and approximate solutions and absolute errors are plotted in Figure 4 for Example 2. The results reveals that the present
method is more accurate.

**Example 3** The following seventh order nonlinear boundary value problem is considered

\[
\begin{align*}
\begin{aligned}
 u^{(7)}(x) &= -u(x)u'(x) + g(x), \quad 0 \leq x \leq 1, \\
u(0) &= 0, \quad u(1) = 0, \\
u^{(1)}(0) &= 1, \quad u^{(1)}(1) = -e, \\
u^{(2)}(0) &= 0, \quad u^{(2)}(1) = -4e, \\
u^{(3)}(0) &= -3.
\end{aligned}
\end{align*}
\]

where \( g(x) = e^x(-35 + (-13 + e^x)x - (1 + 2e^x)x^2 + e^xx^4) \).

The exact solution of the Example 3.3 is

\[ u(x) = x(1 - x)e^x. \]

Following the procedure of the previous examples the series solution, using the first two approximations, can be written as

\[
\begin{align*}
 u(x) &= x - (0.5)x^3 - 0.333333x^4 - 0.125x^5 - 0.0333332x^6 - 0.00694444x^7 \\
 &= -0.00119048x^8 - 0.000173611x^9 - 0.0000220459x^{10} - (2.48016 \times 10^{-6})x^{11} \\
 &= -(2.50521 \times 10^{-7})x^{12} + O(x^{13}).
\end{align*}
\]

The comparison of the exact solution with the series solution of the Example 3 is given in Table 3. In Figure 5 absolute errors are plotted.

**Example 4** The following seventh order nonlinear three point boundary value problem is considered

\[
\begin{align*}
\begin{aligned}
 u^{(7)}(x) &= u(x)u'(x) - e^x(6 + x - e^x x + e^x x^2), \quad 0 \leq x \leq 1, \\
u(0) &= 1, \quad u(\frac{1}{2}) = \frac{e^2}{2}, \\
u^{(1)}(0) &= 0, \quad u^{(1)}(\frac{1}{2}) = -\frac{e^2}{2}, \\
u^{(2)}(0) &= -1, \quad u^{(2)}(1) = -2e, \\
u(1) &= 0.
\end{aligned}
\end{align*}
\]
The exact solution of the Example 4 is $u(x) = (1 - x)e^x$.

Following previous examples the series solution, using the first two approximations, can be written as

$$u(x) = 1 - (0.5)x^2 - 0.333333x^3 - 0.125001x^4 - 0.033319x^5 - 0.0069445x^6$$

$$- 0.00119048x^7 - 0.000173611x^8 - 0.0000220459x^9 - (2.48016 \times 10^{-6})x^{10}$$

$$- (2.50517 \times 10^{-7})x^{11} - (2.29651 \times 10^{-8})x^{12} + O(x)^{13}.$$

The comparison of the exact solution with the series solution of the Example 4 is given in Table 4. Absolute errors are plotted in Figure 6.

![Figure 1](image1.png)

Figure 1. Comparison between the exact solution and the approximate solution for Example (1). Dotted line: approximate solution, solid line: the exact solution.

![Figure 2](image2.png)

Figure 2. Absolute errors for Example (1).
Figure 3. Comparison between the exact solution and the approximate solution for Example (2). Dotted line: approximate solution, solid line: the exact solution.

Figure 4. Absolute errors for Example (2).

Figure 5. Absolute errors for Example (3).
1. Conclusion

In this paper, variational iteration homotopy perturbation method has been applied to obtain the numerical solutions of linear and nonlinear seventh order boundary value problems. The variational iteration homotopy perturbation method solves nonlinear problems without using He’s or Adomian’s polynomials. The method gives rapidly converging series solutions in both linear and nonlinear cases. The numerical results revealed that the present method is a powerful mathematical tool for the solution of seventh order boundary value problems. Numerical examples also show the accuracy of the method.

Table 1. Comparison of maximum absolute errors for Example 1

| present method | Variation of Paramters method |
|----------------|------------------------------|
| 2.46782 × 10^{-10} | 2.1729 × 10^{-09} |

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Table 2. Comparison of numerical results for Example 2

| $x$  | Exact solution | Approximate solution | Absolute Error present method | Absolute Error (Siddiqi and Iftikhar, 2013) |
|------|----------------|----------------------|-------------------------------|------------------------------------------|
| 0.0  | 1.0000         | 1.0000               | 1.32455E-09                  | 0.0000                                   |
| 0.1  | 1.1051         | 1.1051               | 5.26137E-10                  | 2.26257E-07                             |
| 0.2  | 1.2214         | 1.2214               | 1.64015E-09                  | 4.38942E-07                             |
| 0.3  | 1.3498         | 1.3498               | 4.56139E-09                  | 6.1274E-07                              |
| 0.4  | 1.4918         | 1.4918               | 2.9619E-09                   | 7.71759E-07                             |
| 0.5  | 1.6487         | 1.6487               | 7.54889E-10                  | 7.71759E-07                             |
| 0.6  | 1.8221         | 1.8221               | 2.67612E-09                  | 7.37682E-07                             |
| 0.7  | 2.0137         | 2.0137               | 8.42306E-10                  | 6.25932E-07                             |
| 0.8  | 2.2255         | 2.2255               | 1.16866E-09                  | 4.68244E-07                             |
| 0.9  | 2.4596         | 2.4596               | 4.4716E-09                   | 2.95852E-07                             |
| 1.0  | 2.7183         | 2.7183               | 1.02746E-09                  | 1.25922E-07                             |

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Table 3. Comparison of numerical results for Example 3

| $x$  | Exact solution | Approximate solution | Absolute Error |
|------|----------------|---------------------|---------------|
| 0.0  | 0.0000         | 0.0000              | 0.0000        |
| 0.1  | 0.0994654      | 0.0994654           | 5.28944E-12   |
| 0.2  | 0.195424       | 0.195424            | 6.44606E-11   |
| 0.3  | 0.28347        | 0.28347             | 2.38427E-10   |
| 0.4  | 0.358038       | 0.358038            | 5.20559E-10   |
| 0.5  | 0.41218        | 0.41218             | 8.11431E-10   |
| 0.6  | 0.437309       | 0.437309            | 9.55209E-10   |
| 0.7  | 0.422888       | 0.422888            | 8.30543E-10   |
| 0.8  | 0.356087       | 0.356087            | 4.67351E-10   |
| 0.9  | 0.221364       | 0.221364            | 1.04882E-10   |
| 1.0  | 0.0000         | 3.90259E-12         | 3.90259E-12   |

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Table 4. Comparison of numerical results for Example 4

| $x$ | Exact solution | Approximate series solution | Absolute Error |
|-----|----------------|-----------------------------|----------------|
| 0.0 | 1.0000         | 1.0000                      | 0.0000         |
| 0.1 | 0.9946         | 0.9946                      | 9.48615E-11    |
| 0.2 | 0.9771         | 0.9771                      | 3.7371E-10     |
| 0.3 | 0.9449         | 0.9449                      | 4.8626E-10     |
| 0.4 | 0.8950         | 0.8950                      | 2.46565E-10    |
| 0.5 | 0.8243         | 0.8243                      | 8.16711E-11    |
| 0.6 | 0.7288         | 0.7288                      | 8.67514E-11    |
| 0.7 | 0.6041         | 0.6041                      | 9.51461E-11    |
| 0.8 | 0.4451         | 0.4451                      | 2.63398E-09    |
| 0.9 | 0.2459         | 0.2459                      | 1.44494E-08    |
| 1.0 | 0.0000         | -4.90417E-08                | 4.90417E-08    |