UNO and TVD schemes for calculation of waves in an elastic-plastic body

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Abstract. TVD and UNO schemes for efficiently calculating one-dimensional waves in an elastic-plastic body are presented. Both schemes are of the second order of accuracy, with an exception for the TVD scheme in the local extrema of the solution where its accuracy is reduced to the first order. The waves in the body are governed by differential equations in the hydrostatic pressure and the components of the velocity and the deviator of the stress tensor. The plasticity of the body is taken into account using the Mises condition. The classical Godunov method is used for estimating the effectiveness of the presented schemes. It is shown that on the same computational grids, the accuracy of the numerical results by the presented schemes significantly exceeds the accuracy of the results by the Godunov method. At the same time, the UNO scheme is more preferable, since the TVD scheme cuts the solution extrema because it strictly satisfies the TVD condition.

1. Introduction
The classical Godunov method [1], thanks to its stability and monotonicity, is widely used for studying the dynamics of gas, liquid, and elastic-plastic bodies. However, it is of the first order of accuracy, and therefore its application to complex problems may require rather large amount of computer resources. In the case of the body dynamics, similar situations may occur in simulating long-term propagation and interaction of various waves, the residual stresses and deformations inside a solid or on its surface, etc. The present paper presents TVD (Total Variation Diminishing) and UNO (Uniformly Non Oscillatory) schemes, based on the local characteristic approach, for computing wave processes in an elastic-plastic body. The first scheme strictly satisfies the TVD condition, whereas the second one meets it approximately, on the level of approximation errors. An example of local characteristic methods for solving gas dynamics problems along with some results of its application can be found in [2]. In estimating the effectiveness of the schemes considered, their results are compared with those by the Godunov method.

2. Equations of body dynamics and their characteristic form
The body dynamics is governed by a system of linear differential equations in the components of the velocity vector \( \mathbf{u} \), the deviator \( S \) of the stress tensor, and the all-round (hydrostatic) pressure \( P \) [3]. In the one-dimensional case in the vector-matrix form, it can be presented as follows
\[ \frac{\partial \mathbf{q}}{\partial t} + A \frac{\partial \mathbf{q}}{\partial x} = 0, \]  

(1)

where

\[ \mathbf{q} = \begin{pmatrix} u \\ S_{xx} \\ P \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\frac{1}{\rho} & \frac{1}{\rho} \\ -\frac{4}{3} \mu & 0 & 0 \\ K & 0 & 0 \end{pmatrix}. \]

Here \( t \) is the time, \( x \) is the Cartesian coordinate, \( u, S_{xx} \) are respectively the components of the velocity and the deviator, \( \rho \) is unperturbed density of the body material, \( K = \lambda + (2/3)\mu, \lambda = E\nu/(1+\nu)/(1-2\nu) \) and \( \mu = 0.5E/\nu \) are the Lamé parameters, \( E \) is the Young modulus, \( \nu \) is the Poisson ratio.

In the plastic zones, the von Mises yield condition is satisfied [4]

\[ \sigma_i = Y_0, \]  

(2)

where \( Y_0 \) is the yield strength of the body material, \( \sigma_i \) is the von Mises stress, which in the one-dimensional case (1) is determined by the expression \( \sigma_i = 1.5 |S_{xx}| \). Deformations in the body are assumed small.

TVD and UNO schemes for calculating one-dimensional problems governed by (1), (2) within the framework of local characteristic approach are considered. The local characteristic approach is based on the characteristic form of equations (1). It is derived using the eigenvalues of the matrix \( A \) and the matrix \( L_A \) of the left eigenvectors of \( A \).

The eigenvalues \( \alpha' \) of \( A \) are determined from the equation \( |A - \alpha E| = 0 \). Its solution is \( \alpha^1 = 0, \alpha^2 = c_1, \alpha^3 = -c_1 \), where \( c_1 = \sqrt{(\lambda + 2\mu)/\rho} \) is the velocity of propagation of the elastic perturbations along the \( x \)-axis. The matrix \( L_A \) of the left eigenvectors of the matrix \( A \) is determined from the equation

\[ L_A A = \text{diag}\{\alpha^1, \alpha^2, \alpha^3\} L_A, \]

where \( \text{diag}\{\alpha^1, \alpha^2, \alpha^3\} \) is a diagonal matrix with the eigenvalues of the matrix \( A \) on the diagonal.

The characteristic form of system (1) is obtained by multiplying the matrix \( A \) by the matrix \( L_A \) from the left, which gives

\[ \frac{\partial \mathbf{w}}{\partial t} + \text{diag}\{\alpha^1, \alpha^2, \alpha^3\} \frac{\partial \mathbf{w}}{\partial x} = 0, \]

where \( \mathbf{w} = L_A \mathbf{q} = (w^1, w^2, w^3)^T \) is the vector of invariants.

If necessary (in case of computation of the von Mises stress or analysis of the results obtained, etc.), the vector \( \mathbf{q} \) is determined by the formula \( \mathbf{q} = R_A \mathbf{w} \), where \( R_A \) is the matrix of the right eigenvectors of the matrix \( A \), and \( R_A = L_A^{-1} \).

### 3. The main points of the TVD and UNO schemes

Let \( i \) be the cell number of a spatially uniform grid with a step \( \Delta x \), \( n \) be the number of the time layer of a uniform time grid with a step \( \Delta t \).

The TVD and UNO schemes under consideration can be written as follows

\[ (w^k)^{i+1} = (w^k)^i - \beta^k \left( (w^k)^{i+1/2} - (w^k)^{i-1/2} \right), \quad \beta^k = \alpha^k \Delta t / \Delta x, \]

where (omitting the time index \( n \))

\[ w_{i+1/2}^k = w_i^k + \frac{1-\beta^k}{2} \Delta w_i^k \quad \text{at} \quad \beta^k \geq 0, \]

\[ w_{i-1/2}^k = w_{i+1}^k - \frac{1-\beta^k}{2} \Delta w_{i+1}^k \quad \text{at} \quad \beta^k < 0, \]
\[ \Delta w_i^k = \min \left( \Delta w_i^k + \frac{\eta}{2} \Delta^2 w_i^k, \Delta w_i^k - \frac{\eta}{2} \Delta^2 w_i^k \right), \]

\[ \Delta w_i^{s+1/2} = w_i^{s+1/2} - w_i^s, \]

\[ \Delta^2 w_i^{s+1/2} = \min \left( \Delta^2 w_i^s, \Delta^2 w_i^{s+1} \right), \]

\[ \Delta^2 w_i^s = w_i^{s+1} - 2w_i^s + w_i^{s-1}, \]

\[ \min \text{mod}(x, y) = \frac{1}{2} (\text{sgn}(x) + \text{sgn}(y)) \min(|x|, |y|). \]

Here \( \eta = 0 \) corresponds to the TVD scheme and \( \eta = 1 \) corresponds to the UNO scheme.

In both schemes, the computations at each time step are completed by taking into account the plasticity effect. To this end, the von Mises stress \( \sigma_i \) is calculated in each grid cell. If in a cell the value of \( \sigma_i \) exceeds the yield strength of the material, then the deviator of the stress tensor in that cell is corrected [4] by the formula

\[ S_{xx}^{corr} = \left( Y_0 / \sigma_i \right) S_{xx}, \]

After such a correction, the von Mises stress \( \sigma_i \) in that cell becomes exactly equal to the yield strength, which means that the von Mises condition (2) is satisfied in that cell.

The time step is determined from the Courant condition [5]

\[ \Delta t = \delta \Delta x / c, \]

where \( c \) is the maximum velocity of the calculated waves, \( \delta < 1 \) is a parameter, the value of which is usually taken close to 1.

4. Numerical results

A problem of breakdown of two discontinuities in the all-round pressure inside a body is considered to illustrate the effectiveness of the presented schemes. The body is an aluminum alloy with the Young modulus \( E = 71 \) GPa, the Poisson ratio \( \nu = 0.34 \), the density \( \rho = 2700 \) kg / m\(^3\), the yield strength \( Y_0 = 325 \) MPa. At the initial time \( t = 0 \),

\[ u = 0, S_{xx} = 0 \text{ at } -\infty < x < \infty, \]

\[ P = \begin{cases} P_0 \text{ at } R/3 \leq x \leq 2R/3 \\ 0 \text{ at } x < R/3 \text{ and } x > 2R/3 \end{cases}, \]

where \( P_0 \) and \( R \) are some constants.

Figure 1 demonstrates the solution of this problem obtained in the computational domain \( 0 \leq x \leq R \) covered with a uniform grid of 200 cells. Three successive instants of time, \( t_{1-3} \), are shown. The value of \( P_0 \) is taken at which the yield point is not reached at \( t_1 \) and \( t_2 \) so that at these moments, the deformations in the body keep being elastic. Plasticity of the body manifests itself by moment \( t_3 \). To estimate the effectiveness of the schemes considered, the results of applying the classical Godunov scheme of the first order of accuracy are used.

It can be seen that at all the three presented instants of time, the numerical solutions of the UNO and TVD schemes are very close to one another in the vicinities of the discontinuities and in the smooth regions of the solution. Their resolution of the discontinuities is better than it is in the numerical solution by the Godunov scheme, which noticeably “smears” them. At moments \( t_1 \) and \( t_2 \), when the yield strength is still not reached, rather rapid changes in the solution take place in some regions and its extrema are attained in relatively narrow areas. The extrema are better resolved by the UNO scheme, whereas they are slightly “cut” by the TVD scheme and strongly smoothed by the Godunov scheme. At moment \( t_1 \) the errors of the results of the TVD and Godunov schemes relative to those of the UNO scheme are 3\% and 24\%, respectively. At moment \( t_2 \), they amount to 20\% and 64\%, respectively.
Figure 1. The von Mises stress $\sigma_i$ distribution at three successive time moments. The solid, dotted, and dashed lines represent the numerical solutions obtained by the UNO, TVD, and Godunov schemes, respectively.

5. Conclusions
Local characteristic TVD and UNO schemes for calculation of perturbations propagating in elastic-plastic bodies are presented. Both schemes are of the second order of accuracy, except for local extrema of the solution, where the accuracy of the TVD scheme is reduced to the first order. The effectiveness of the presented schemes is estimated by their comparison with the classical Godunov method of the first order of accuracy. It is shown that on the same computational grids both the high-accuracy schemes resolve the elastic-plastic waves in the body much better than the classical Godunov method. Along with that, the UNO scheme is more preferable, since the TVD scheme cuts the solution extrema due to strict satisfying the TVD condition.

References
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