Killing spinors for the bosonic string

H. Lü\textsuperscript{1,2} and Zhao-Long Wang\textsuperscript{3}

\textsuperscript{1} China Economics and Management Academy, Central University of Finance and Economics
Beijing 100081, China
\textsuperscript{2} Institute for Advanced Study, Shenzhen University - Nanhai Ave 3688, Shenzhen 518060, China
\textsuperscript{3} School of Physics, Korea Institute for Advanced Study - Seoul 130-722, Korea

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Abstract – We obtain the effective action for the bosonic string with arbitrary Yang-Mills fields, up to the $\alpha'$ order, in general dimensions. The form of the action is determined by the requirement that the action admit well-defined Killing spinor equations, whose projected integrability conditions give rise to the full set of equations of motion. The success of the construction suggests that the hidden "pseudo-supersymmetry" associated with the Killing spinor equations may be a property of the bosonic string itself.

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Introduction. – One of the most important advances in quantum gravity was the invention of string theory. Quantum consistency in general requires that string theories be supersymmetric in the critical dimension ten\textsuperscript{1}. With the discovery of the AdS/CFT correspondence [2], the application of string theory goes beyond quantum gravity, but also provides new understandings and techniques in non-perturbative quantum field theories including nuclear and condensed matter physics. However, these low-energy arenas are necessarily non-supersymmetric, and hence the hitherto abandoned bosonic string theory may provide an alternative and useful approach.

One advantage of supersymmetry is that it provides a powerful organizing tool for the non-perturbative region. In particular the characteristic properties of the BPS solutions of supergravities, such as the mass-charge relation, are expected to survive the higher-order quantum corrections. The defining property for the BPS solutions is that they admit Killing spinors. Killing spinors, however, were introduced in ordinary Riemannian geometry which predated supersymmetry. Killing spinors in supergravities are generalizations of those in the Riemannian geometry. They all have an important defining property that certain projected integrability conditions for the Killing spinor equations give rise to the full set of equations of motion of the bosonic fields. This property is a necessary condition for constructing supergravities, and conversely, all supergravities have consistent Killing spinor equations. It is natural to ask whether there exist non-supersymmetric theories other than pure gravities that also admit well-defined Killing spinors.

Recently, an intriguing connection between the consistency of the Kluza-Klein sphere reduction and Killing spinors was observed in [3]. Inspired by this and also the fact that it is consistent to perform $S^3$ and $S^{D-3}$ reductions on the effective action of the bosonic string [4], Killing spinor equations were proposed for this action [3]. It was shown that the projected integrability condition gives rise precisely to the full set of equations of motion. For non-supersymmetric theories, such examples are uncommon [5]. The only known non-trivial examples are pure gravity, certain scalar-gravity theories [6] and the bosonic string. This implies that the bosonic string has a non-trivial generalized geometric structure arising from the "pseudo-supersymmetry", a terminology we introduce here to refer to the hidden symmetry associated with the existence of well-defined Killing spinor equations in non-supersymmetric theories. From the AdS/CFT point of view, where the bulk gravity is classical, the pseudo-supersymmetric effective action of the bosonic string can be put on an equal footing as in supergravity.

Effective action with Yang-Mills fields. – The pseudo-supersymmetry obtained in [3] is for the tree-level action of the bosonic string. If the pseudo-supersymmetry is indeed a generic stringy property, we would expect that the Killing spinor equations should remain well defined when higher-order corrections are included. In this paper,
we extend the discussion of [3] by considering the \(\alpha'\) correction to the low-energy effective action. We start by first introducing arbitrary Yang-Mills fields, which originate in diverse ways in string theory. For example, wrapping the string on singular cycles of the internal manifold can lead to gauge symmetry enhancement. One can also consider hybrid mixing between the left- and right-moving sectors, with the mismatching dimensions compactified on some suitable self-dual lattices, à la heterotic string [7]. The effective action with arbitrary Yang-Mills fields in \(D\) dimensions is given by

\[
\mathcal{L}_D = \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{12} e^{\alpha \phi} G^2 - \frac{1}{4} e^{\frac{3}{2} \alpha \phi} \tau' (F^2) \right),
\]

where \(a^2 = 8/(D - 2)\) and the various form fields are given by

\[
\begin{align*}
G(3) &= \text{dB}(2) - \frac{1}{2} \omega(3), \\
\omega(3) &= \text{tr}' (F_2 \wedge A_1 - \frac{1}{3} A_1 \wedge A_1 \wedge A_1), \\
F(2) &= \text{d}A_1 + A_1 \wedge A_1, \\
g(3) &= -\frac{1}{2} \text{tr}' (F_2 \wedge F_2).
\end{align*}
\]

The Yang-Mills 1-forms are defined by \(A_1 = A^T T_I\), where the generators \(T_I\) are anti-Hermitian, obeying the Lie algebra \([T_I, T_J] = f^I_{JK} T_K\), and normalized in the fundamental representation as \(\text{tr}(T_I T_J) = \delta_{IJ}\). Then the trace \(\text{tr}'\) is defined by \(\text{tr}' = \frac{1}{2} \text{tr}\). Note that for simplicity, we have set the Yang-Mills coupling to unity.

We propose that the defining equations for the Killing spinors for (1) are given by

\[
\begin{align*}
D_M \epsilon + \frac{1}{96} e^{\frac{1}{2} \alpha \phi} \left( a^2 \text{M} \Gamma^{NPQ} - 12 \delta_M^{NPQ} \right) G_{NPQ} \eta &= 0, \\
\Gamma^M \partial_M \phi \eta + \frac{1}{12} a e^{\frac{1}{2} \alpha \phi} \Gamma^{MNP} G_{MNP} \eta &= 0, \\
\Gamma_{M}^{M} \partial_{M} F_{M} \eta &= 0.
\end{align*}
\]

When \(D = 10\), these are precisely the Killing spinor equations for \(D = 10\), \(N = 1\) supergravity [8] with Yang-Mills matter multiplets. Let us now examine the consistency of these equations. For Riemannian Killing spinors, satisfying \(D_M \epsilon = 0\), the integrability condition is given by \(D_M, D_N)\epsilon = \frac{1}{2} R_{MN} \Gamma^{PQ} \epsilon = 0\). The projected integrability condition, namely \(\Gamma^M D_M, D_N)\epsilon = \frac{1}{2} R_{MN} \Gamma^M \epsilon = 0\), is satisfied by virtue of the Einstein equation. The projected integrability conditions for the equations (3)-(5) are much more involved. We find that the condition associated with \(\Gamma^M D_M, D_N)\epsilon\) is given by

\[
\left[ R_{MN} - \frac{1}{2} \partial_M \phi \partial_N \phi - \frac{1}{4} e^{\alpha \phi} \left( G_{MN}^2 - \frac{2}{3(D - 2)} G^2 g_{MN} \right) \right] \Gamma^N \eta
\]

\[
- \frac{1}{2} e^{\frac{3}{2} \alpha \phi} (F^2 - \frac{1}{2(D - 2)} F^2 g_{MN}) \right] \Gamma^N \eta
\]

\[
\frac{1}{2} e^{\frac{3}{2} \alpha \phi} (F^2 - \frac{1}{2(D - 2)} F^2 g_{MN}) \right] \Gamma^N \eta
\]

\[
L = \sqrt{-g} e^{-2 \Phi} \left( R + 4(\partial \phi)^2 - \frac{1}{12} G_{(3)}^2 - \frac{1}{4} \text{tr}' (F^2) \right).
\]

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The defining equations for the Killing spinors, which are scaled by a factor $e^{-\Phi}$ in the string frame, are now given by

$$
D_M(\omega_-)\eta = 0, \quad \Gamma^M \partial_M \Phi \eta - \frac{1}{12} \Gamma^{MNP} G_{MNP} \eta = 0,
$$

$$
\Gamma^{M,M_2} F_{M,M_2} \eta = 0,
$$

where $\omega_-$ is the torsional spin connection, defined as

$$
\omega_{\mu}^{ab} = \omega_{ab}^{\mu} = \frac{1}{2} G_{\mu}^{\ ab}.
$$

A major advantage of the string frame is that there is no manifest dimensional dependence in either the action or the Killing spinor equations. At the $\alpha'$ order, anomaly cancelation requires that the quadratic curvature terms enter the Bianchi identity of the 3-form in the form of $d G_{(3)} \sim \frac{1}{2} \alpha (tr(R_{(2)} \wedge R_{(2)}) - tr'(F_{(2)} \wedge F_{(2)}))$, where $R_{(2)} = d \omega + \omega \wedge \omega$ is the curvature 2-form and $tr(R_{(2)} \wedge R_{(2)}) \equiv (R_{(2)})^a_{\ b}(R_{(2)})^b_{\ a}$. (Note that for the ten-dimensional superstring, $\alpha = \frac{1}{2} \alpha'$, and we expect this to hold in general. Note also that we have scaled the Yang-Mills fields appropriately so that they contribute the same $\alpha'$-order anomaly as that of the curvature-squared term.) This suggests that we do not need to modify the Killing spinors equations (12), but instead add an additional projection associated with the curvature 2-form à la Yang-Mills. This projection in fact already exists at the $\alpha'$ order, since [10]

$$
0 = [D_M(\omega_-), D_N(\omega_-)]\eta = \frac{1}{4} R_{MN}^{\ \ ab}(\omega_-) \Gamma_{ab} \eta = \frac{1}{4} R_{MN}^{\ \ ab}(\omega_+) \Gamma_{ab} \eta + O(\alpha).
$$

This projection is analogous to that for the Yang-Mills fields (5), and hence the proper Bianchi identity for the 3-form is given by

$$
d G_{(3)} = \frac{1}{2} \alpha \left( tr(R_{(2)}(\omega_+) \wedge R_{(2)}(\omega_-)) - tr'(F_{(2)} \wedge F_{(2)}) \right).
$$

Note that we adopt the supergravity convention that the torsionful Riemann tensors are defined as $(R_{(2)}(\omega_+))^a_{\ b} = \frac{1}{2} R_{MN}^{\ \ ab}(\omega_+) dx^M \wedge dx^N$. Following the same strategy as in the earlier calculation, we find that the full set of equations of motion can now be obtained by the projected integrability conditions, together with the Bianchi identity (15). They are given by

$$
R - 4 (\partial \Phi)^2 + 4 \Box \Phi - \frac{1}{12} G_{(3)}^2 - \frac{1}{4} \alpha \left( tr' F_{(2)}^2 - R_{MNAB}(\omega_+)^a_{\ b} R^{MNAB}(\omega_+) \right) = 0,
$$

$$
R_{MN} + 2 \nabla_M \nabla_N \Phi - \frac{1}{4} G_{MN}^2 - \frac{1}{2} \alpha \left( tr' F_{MN}^2 - R_{MPAB}(\omega_+) R^{PAB}(\omega_+) \right) = 0,
$$

$$
d (e^{-2\Phi} G_{(3)}) = 0,
$$

$$
D (e^{-2\Phi} F_{(2)}) + (-1)^D e^{-2\Phi} F_{(2)} \wedge * G_{(3)} = 0.
$$

Up to the $\alpha'$ order, these equations can be derived from the Lagrangian

$$
L_D = \sqrt{-g} e^{-2\Phi} \left[ R + 4 (\partial \Phi)^2 - \frac{1}{12} G_{(3)}^2 - \frac{1}{4} \alpha \left( tr' F_{(2)}^2 - R_{MNAB}(\omega_+) R^{MNAB}(\omega_+) \right) \right].
$$

This Lagrangian in general dimensions takes the exact form of the bosonic effective action of the heterotic string with curvature-squared terms [10]. The equations of motion (16) also take the same form as those in the corresponding supergravity [11]. Note that the terms arising from the variation of $\omega_+$ in the torsional Riemann tensor do not appear in the equations of motion (16). These terms could at least contribute to the $\alpha'$ order. As in the $R^2$ supergravity in $D = 10$ [10], at the $\alpha'$ order, we find that they vanish by virtue of the leading-order equations of motion. Thus the effective action of the bosonic string are now fixed, up to the $\alpha'$ order, by the assumption that the Killing spinors can be well defined. Conversely the above derivation also indicates strongly that the pseudo-supersymmetry may be a property of the full bosonic string.

**Generalized holonomy.** – We now turn to the solutions of the bosonic string. Let us first set $\alpha' = 0$. The theory admits the electric string solution

$$
\text{ds}^2_{\text{str}} = H^{-1} (-dt^2 + dx^2) + dr^2 + r^2 d\Omega^2_{D-3},
$$

$$
e^{-2\Phi} = H = 1 + \frac{Q}{r^{D-4}},
$$

and the magnetic $(D - 5)$-brane

$$
\text{ds}^2_{\text{str}} = -\eta_{ab} dx^a dx^b + H (dr^2 + r^2 d\Omega^2_3),
$$

$$
e^{-2\Phi} = H = 1 + \frac{P}{r^2},
$$

and their intersection

$$
\text{ds}_{\text{str}} = H_1^{-1} (-dt^2 + dx^2) + H_2 (dr^2 + r^2 d\Omega^2_3) + dx^i dx^i,'
$$

$$
e^{-2\Phi} = \frac{H_2}{H_1},
$$

$$
F_{(3)} = dt \wedge dx \wedge dH^{-1} + e^{-2\Phi} * dt \wedge dx \wedge d(\partial^{(D-4)}) x \wedge dH_2^{-1},
$$

$$
H_1 = 1 + \frac{Q}{r^{D-4}}, \quad H_2 = 1 + \frac{P}{r^2}.
$$

In the decoupling limit when the “1” in the harmonic functions $H_1$ can be dropped, the metric (20) becomes $\text{AdS}_3 \times S^3 \times \mathbb{R}^{D-6}$. As in supergravities, we shall call these solutions “BPS” since they preserve fractions of the maximally allowed Killing spinors in the vacuum.

The Bianchi identity for $G_{(3)}$ implies that the $(D - 5)$-brane can be supported by Yang-Mills instantons. For a
single SU(2) instanton, the Yang-Mills fields are given by \( A_{(1)} = \frac{a^2}{r^2 + a^2} \sigma_i \), where \( \sigma_i \) are the SU(2) left-invariant 1-forms. The \( S^4 \) metric in (19) can be expressed as \( d\Omega^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \). The \((D - 5)\)-brane takes the same form as (19) but with \( H \) now given by

\[
H = 1 + \frac{r^2 + 2a^2}{(r^2 + a^2)^2}.
\]

(21)

This solution in the heterotic string was obtained in [12]. For multi-instanton-supported solutions, see [13].

We now examine the \( \alpha' \) corrected solutions. It turns out that, up to the \( \alpha' \) order, (19) is unmodified as a solution for the Lagrangian (17). This is because the torsion for this solution is parallelizing and hence the curvature 2-form \( R_{(2)}(\omega_+) \) vanishes. If the existence of the Killing spinors would restrict the higher-order corrections to be scalar polynomials of \( R_{(2)}(\omega_+) \), the solution (19) would survive all the quantum corrections. This was first observed for heterotic string in [12], and we expect the same may be true for the bosonic string with Killing spinors. On the other hand, both solutions (18) and (20) are modified by the \( \alpha' \) correction; however, the near-horizon geometry \( \text{AdS}_3 \times S^3 \times \mathbb{R}^{D-6} \) of (20) remains unchanged under the \( \alpha' \) correction.

In supergravities, the supersymmetry implies the existence of the superspace geometry. The solutions can be classified by a generalized holonomy. The generalized holonomy for the M-theory was studied in [14,15]. The existence of the well-defined Killing spinors for the bosonic string suggests that the theory may also have a generalized geometrical structure which can be called pseudo-superspace. The Killing spinor equations (12) imply that the generalized holonomy remains in the same \( SO(1, D - 1) \) as Einstein gravity since it merely adds a totally antisymmetric torsion to the usual spin connection. We now examine the reduced holonomy for the 4-BPS solutions. We first look at the electric string solution. We can replace the metric of the \((D - 2)\)-dimensional transverse space by the one in the Cartesian coordinates \( dy^m dy^m \), in which \( H \) is an arbitrary harmonic function. For this background, the “super”-covariant derivative is given by

\[
\mathcal{D}_\mu = \partial_\mu - \frac{1}{2} H^{-\frac{1}{2}} \partial_m H P^+ \Gamma^{\hat{n}}_{\mu},
\]

\[
\mathcal{D}_m = \partial_m - \frac{1}{4} H \partial_m H^{-1} \Gamma^{\hat{n}}_{\hat{n}},
\]

(22)

where the projection operator is given by \( P^\pm = \frac{1}{2}(1 \pm \Gamma^{\hat{n}}_{\hat{n}}) \).

Letting \( \mathcal{M}_{MN} = [\mathcal{D}_M, \mathcal{D}_N] \), we find

\[
\mathcal{M}_{\mu\nu} = 0 = \mathcal{M}_{mn},
\]

\[
\mathcal{M}_{\mu m} = \frac{1}{2} H^{-\frac{1}{2}} \partial_n \partial_m \ln H P^+ \Gamma^{\hat{n}}_{\mu}.
\]

(23)

Since only the commuting generators \( K_{\mu}^{\hat{n}} = P^+ \Gamma^{\hat{n}}_{\mu} \) are present, the reduced holonomy for the electric string solution is given by

\[
\mathcal{H}_{\text{string}} = \mathbb{R}^{D-2}.
\]

(24)

The analysis for the magnetic \((D - 5)\)-brane is similar. After replacing the transverse space in (19) with the Cartesian system, the “super”-covariant derivative is given by

\[
\mathcal{D}_\mu = \partial_\mu, \quad \mathcal{D}_m = \partial_m + \frac{1}{2} H^{-1} \partial_n H P^+ \Gamma^{\hat{n}}_{\mu},
\]

(25)

where the projection operator is given by \( P^\pm = \frac{1}{2}(1 \pm \Gamma^{\hat{n}}_{\hat{n}}) \). We now have

\[
\mathcal{M}_{\mu\nu} = 0, \quad \mathcal{M}_{mn} = 0,
\]

\[
\mathcal{M}_{mn} = \frac{1}{2} \left( \partial_m \partial_n f - (\partial_m f)(\partial_n f) \right) P^+ \Gamma^{\hat{n}}_{\hat{n}},
\]

\[
- \frac{1}{2} \left( \partial_n \partial_m f - (\partial_n f)(\partial_m f) \right) P^+ \Gamma^{\hat{n}}_{\hat{n}},
\]

\[
- \frac{1}{2} \left( \partial_n f \right)(\partial_m f) P^+ \Gamma^{\hat{n}}_{\hat{n}},
\]

(26)

where \( f = \ln H \). The non-vanishing generators are \( T^a_{\hat{n}\hat{n}} = P^+ \Gamma^{\hat{n}}_{\hat{n}}. \) Since

\[
[P^+ \Gamma^{\hat{n}_1\hat{n}_2}, P^+ \Gamma^{\hat{n}_1\hat{n}_2}] = P^+ [\Gamma^{\hat{n}_1\hat{n}_2}, \Gamma^{\hat{n}_1\hat{n}_2}],
\]

(27)

it follows that \( T^a_{\hat{n}\hat{n}} \) generates the \( so(4) \) algebra. Thus, the reduced holonomy for the \((D - 5)\)-brane is given by

\[
\mathcal{H}_{(D - 5)\text{-brane}} = SO(4)_+.
\]

(28)

where \( + \) refers to the sign of the \( P_+ \) projection. For the \( \frac{1}{2}\)-BPS intersecting solution, we find that the reduced holonomy is \( \mathcal{H} = SO(4)_+ \times 2\mathbb{R}^{(2)} \), where \( \mathbb{R}^{(2)} \) is the two-dimensional spinor representation of the \( SO(4)_+ \).

**Conclusions.** – To conclude, we obtain the effective action for the bosonic string up to the \( \alpha' \) order. The guiding principle in the construction is that the theory has well-defined Killing spinor equations, whose projected integrability condition yields the full set of equations of motion. The success of our construction suggests that the hidden pseudo-supersymmetry associated with the Killing spinor equations may be a property of the full bosonic string. The pseudo-supersymmetry enables us to classify solutions with respect to the fractions of the surviving Killing spinors. These solutions are characterized by the different reduced holonomy groups which are subgroups of \( SO(1, D - 1) \). It is tempting to expect, as in the case of the superstring and M-theory, that the properties of these classical solutions with Killing spinors, such as the mass/charge relation, may also be protected from quantum corrections and hence they provide tools for
studying the non-perturbative aspects of the bosonic string.

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