Some mass relations for mesons and baryons in Regge phenomenology

Xin-Heng Guo* and Ke-Wei Wei†

College of Nuclear Science and Technology, Beijing Normal University, Beijing 100875, China

Xing-Hua Wu‡

College of Nuclear Science and Technology, Beijing Normal University, Beijing 100875, China; College of Physics and Information Engineering, Henan Normal University, Xinxiang 453007, China

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Abstract

In the quasilinear Regge trajectory ansatz, some useful linear mass inequalities, quadratic mass inequalities and quadratic mass equalities are derived for mesons and baryons. Based on these relations, mass ranges of some mesons and baryons are given. The masses of $\bar{b}c$ and $s\bar{s}$ belonging to the pseudoscalar, vector and tensor meson multiplets are also extracted. The $J^P$ of the baryon $\Xi_{cc}^+(3520)$ is assigned to be $\frac{1}{2}^+$. The numerical values for Regge slopes and intercepts of the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ $SU(4)$ baryon trajectories are extracted and the masses of the orbital excited baryons lying on the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories are estimated. The $J^P$ assignments of baryons $\Xi_c(2980)$, $\Xi_c(3055)$, $\Xi_c(3077)$ and $\Xi_c(3123)$ are discussed. The predictions are in reasonable agreement with the existing experimental data and those suggested in many other different approaches. The mass relations and the predictions may be useful for the discovery of the unobserved meson and baryon states and the $J^P$ assignment of these states.

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* e-mail: xhguo@bnu.edu.cn
† Corresponding author, e-mail: weikw@brc.bnu.edu.cn
‡ e-mail: singhwa.wu@gmail.com
I. INTRODUCTION

The study of hadronic physics has been a subject of intense interest. There are many hadronic states reported in recent years: $B^{*}_2\ [1]$, $B^{*}_{s2}\ [2]$, $\Xi^+_2(3520)\ [3]$, $\Lambda_2^+(2880)\ [4, 5, 6]$, $\Lambda_c^+(2940)\ [5, 6]$, $\Xi_c^{0,+}(2980, 3077)\ [7, 8]$, $\Xi_c^+(3055, 3123)\ [9]$, $\Sigma_b^{(*)\pm}\ [10]$ and $\Xi_b^{-}\ [11]$. More and more states will be discovered in the near future. However, the properties of some states such as $\Xi_c^+(3520)$ are still not very clear. $\Xi_c^+(3520)$ was reported as the doubly charmed baryon state by SELEX in two different decay modes \[3\], but the $J^P$ number has not been determined. Moreover, it has not been confirmed by other experiments (notably by BABAR \[12\], BELLE \[13\] and FOCUS \[14\]). According to the Particle Data Group's “Review of Particle Physics” in 2006 \[15\], many hadrons, especially heavy hadrons, are still absent from the summary tables. Obviously, there is still a lot of work to be done both theoretically and experimentally.

The eightfold way and the standard SU(3) Gell-Mann–Okubo (GMO) formula \[16\] have played an important role in the historical progress in particle physics. However, the direct generalization of the GMO formula to the charmed and bottom hadrons cannot agree well with experimental data due to higher-order breaking effects. Consequently, there are many works focused on the mass relations, including inequalities \[17, 18, 19, 20\] and equalities \[21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\].

Quantum chromodynamics (QCD) has been verified as an appropriate theory to describe strong interaction at short distances. However, the application of QCD to the processes of hadronic interactions at large distances is still limited by the unsolved confinement problem. Nowadays calculations of hadronic properties, which are related to the nonperturbative effects, are frequently carried out with the help of phenomenological models. Regge phenomenology (which was derived from the analysis of the properties of the scattering amplitude in the complex angular momentum plane \[37\]) is one of the simplest ones among these phenomenological models. Regge theory is concerned with almost all aspects of strong interactions, including the particle spectra, the forces between particles, and the high energy behavior of scattering amplitudes \[38\]. The quasilinear Regge trajectory ansatz, which is one of the most effective and popular approaches for studying hadron spectra, can (at least at present) give a reasonable description for the hadron spectroscopy \[21, 22, 23, 39, 40\], although some suggestions that the realistic Regge trajectories could be nonlinear exist \[41\].

As pointed out in Refs. \[24, 42\], Regge intercepts and slopes are useful for many spectral and nonspectral purposes, for example, in the recombination \[43\] and fragmentation \[44\] models. There-
fore, as pointed out in Ref. [45], the slopes and intercepts of the Regge trajectories are fundamental constants of hadron dynamics, perhaps in general more important than the masses of particular states. Thus, the determination of Regge slopes and intercepts of hadrons is of great importance since this provides opportunities for a better understanding of the dynamics of strong interactions [42].

In the quasilinear Regge trajectory ansatz, the numerical values of the parameters of the Regge trajectories were extracted for mesons of different flavors [21, 22, 39, 40, 46]. Under the approximation that mesons or baryons in the light quark sector have the common Regge slopes, Burakovsky et al. derived two 6th power and one 14th power meson mass relations in Ref. [22], and derived some new quadratic Gell-Mann–Okubo–type baryon mass equalities in Ref. [23]. Using those new quadratic baryon mass relations they predicted the masses of $\frac{1}{2}^+$ and $\frac{3}{2}^+$ charmed baryon states absent from the baryon summary table then. (Here and below, $\frac{1}{2}^+$ and $\frac{3}{2}^+$ multiplets refer to the ground multiplets in which the total orbital angular momenta $L=0$.) However, the numerical values for the parameters of the charmed baryon Regge trajectories were not given in Ref. [23].

In the present work, under the assumption that the quasilinear Regge trajectory ansatz is suitable to describe meson spectra and baryon spectra with the requirements of the additivity of intercepts and inverse slopes, the relations between slope ratios and masses of hadrons with different flavors and the mass relations among hadrons will be studied. We will show that the linear mass GMO formula is virtually an inequality and the quadratic mass GMO formula is also an inequality with the sign opposite to the linear case. We will get a high-power mass equation which is very useful to predict the masses of $\bar{b}c$ states and the masses of pure $s\bar{s}$ states. We will also get some useful quadratic mass equations for baryons. The $J^P$ assignment of $\Xi_{cc}^+(3520)$, $\Xi_{c}(2980)$, $\Xi_{c}(3055)$, $\Xi_{c}(3077)$ and $\Xi_{c}(3123)$ baryons will be discussed. The numerical values for the parameters of the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories will be extracted and the masses of the baryon states lying on the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories will be estimated.

The remainder of this paper is organized as follows. In Sec. II we briefly introduce the quasilinear Regge trajectory ansatz. Then, we extract the mass inequalities and mass equalities for mesons and baryons. In Sec. III we present some applications of the relations derived in Sec. II and discuss the $J^P$ assignment of $\Xi_{cc}^+(3520)$, $\Xi_{c}(2980)$, $\Xi_{c}(3055)$, $\Xi_{c}(3077)$ and $\Xi_{c}(3123)$ baryons. The parameters of the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories are extracted and the masses of the baryon states lying on the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories are estimated. Finally, we give a discussion and conclusion in Sec. IV.
II. FRAMEWORK

It is known from Regge theory that all mesons and baryons are associated with Regge poles which move in the complex angular momentum plane as a function of energy. The trajectory of a particular pole (Regge trajectory) is characterized by a set of internal quantum numbers (baryon number $B$, intrinsic parity $P$, strangeness $S$, charmness $C$, bottomness $B$, etc.) and by the evenness or oddness of the total spin $J$ for mesons ($J - \frac{1}{2}$ for baryons) \[47\]. The plots of Regge trajectories of hadrons in the $(J, M^2)$ plane are usually called Chew-Frautschi plots (where $J$ and $M$ are respectively the total spins and the masses of the hadrons). In Fig. 1, we draw the Chew-Frautschi plots for some meson and baryon Regge trajectories.

Assuming the existence of the quasilinear Regge trajectories for both light and heavy hadrons, one can have

\[ J = \alpha(M) = a(0) + \alpha' M^2, \]
where $a(0)$ and $\alpha'$ are respectively the intercept and slope of the trajectory on which the particles lie. Hadrons lying on the same Regge trajectory which have the same internal quantum numbers are classified into the same family. The difference between the total spins of these hadrons is $2n$ ($n=1,2,3,\cdots$), e.g., mesons with the quantum numbers $N^{2S+1}L_J, N^{2S+1}(L+2)_{J+2}, N^{2S+1}(L+4)_{J+4}, \cdots$ (where $N$, $L$ and $S$ denote the radial excited quantum number, the orbital quantum number and the intrinsic spin, respectively) lying on the same Regge trajectory. These features can be seen from the well-known Chew-Frautschi plots (Fig. 1).

For a meson multiplet with spin-parity $J^P$ (more exactly speaking, with quantum numbers $N^{2S+1}L_J$), the parameters for different quark constituents can be related by the following relations:

the additivity of intercepts \[ a_{ii}(0) + a_{jj}(0) = 2a_{ij}(0), \] (2)

the additivity of inverse slopes \[ \frac{1}{\alpha'_{ii}} + \frac{1}{\alpha'_{jj}} = \frac{2}{\alpha'_{ij}}, \] (3)

where $i$ and $j$ represent quark flavors. Equations (2) and (3) were derived in a model based on the topological expansion and the $q\bar{q}$-string picture of hadrons \[ 46]. This model provides a microscopic approach to describe Regge phenomenology in terms of quark degrees of freedom \[ 53]. In fact, Eq. (2) was first derived for light quarks in the dual-resonance model \[ 48], and was found to be satisfied in two-dimensional QCD \[ 49], the dual-analytic model \[ 50], and the quark bremsstrahlung model \[ 51]. Also, it saturates the inequality for Regge intercepts \[ 52] which follows from the Schwarz inequality and the unitarity relation. The above two relations are usually generalized to the baryon case \[ 23, 42, 51\], in which one has

\[ a_{iiq}(0) + a_{jjq}(0) = 2a_{ijq}(0), \] (4)

\[ \frac{1}{\alpha'_{iiq}} + \frac{1}{\alpha'_{jjq}} = \frac{2}{\alpha'_{ijq}}, \] (5)

where $q$ represents a quark.

There are also relations about the factorization of slopes for mesons \[ 54, 55\] and baryons \[ 55\]:

\[ \alpha'_{ii} \cdot \alpha'_{jj} = \alpha'_{ij}^2, \] (6)

\[ \alpha'_{iiq} \cdot \alpha'_{jjq} = \alpha'_{ijq}^2, \] (7)
which follow from the factorization of residues of the $t$-channel poles. The paper by Burakovsky and Goldman \[42\] showed that only the additivity of inverse Regge slopes is consistent with the formal chiral and heavy quark limits for both mesons and baryons, and that the factorization of Regge slopes, although consistent with the formal chiral limit, fails in the heavy quark limit. Besides, in Sec. III B, we will show that the high-power equation (63) derived from the relations (11), (2) and (6) is not as good as the high-power equation (16) derived from the relations (1), (2) and (3) compared with the well-established meson multiplets. Therefore, we will use the relations (3) and (5) (the additivity of inverse slopes) rather than the relations (6) and (7) (the factorization of slopes) in this study. There are also studies about the relations between the ground state and its radial excited states \[39, 56, 57\] and there are suggestions that the radial excited states lie on daughter trajectories of the ground state \[38\]. However, we do not discuss these relations in the present work.

A. Relations between slope ratios and hadron masses

For mesons, using Eqs. (1) and (2), one obtains

$$\alpha'_{\tilde{i}i}M_{\tilde{i}i}^2 + \alpha'_{\tilde{j}j}M_{\tilde{j}j}^2 = 2\alpha'_{ij}M_{ij}^2,$$  

(8)

where the meson states $\tilde{i}i$, $\tilde{j}j$ and $ij$ belong to the same $N^{2S+1}L_J$ multiplet. This relation can be reduced to the quadratic Gell-Mann–Okubo-type formula by assuming that all the slopes are independent of flavors ($\alpha'_{\tilde{i}i} = \alpha'_{\tilde{j}j} = \alpha'_{ij}$). Combining the relations (3) and (8), one can get two pairs of solutions. The first pair of solutions are

$$\begin{align*}
\frac{\alpha'_{\tilde{j}j}}{\alpha'_{\tilde{i}i}} &= \frac{1}{2M_{\tilde{j}j}^2} \times [(4M_{\tilde{j}j}^2 - M_{\tilde{i}i}^2 - M_{\tilde{j}j}^2) + \sqrt{(4M_{\tilde{j}j}^2 - M_{\tilde{i}i}^2 - M_{\tilde{j}j}^2)^2 - 4M_{\tilde{i}i}^2M_{\tilde{j}j}^2}], \\
\frac{\alpha'_{\tilde{j}j}}{\alpha'_{\tilde{i}i}} &= \frac{1}{4M_{\tilde{j}j}^2} \times [(4M_{\tilde{j}j}^2 + M_{\tilde{i}i}^2 - M_{\tilde{j}j}^2) + \sqrt{(4M_{\tilde{j}j}^2 + M_{\tilde{i}i}^2 - M_{\tilde{j}j}^2)^2 - 4M_{\tilde{i}i}^2M_{\tilde{j}j}^2}], 
\end{align*}$$  

(9)

while the second pair of solutions are

$$\begin{align*}
\frac{\alpha'_{\tilde{j}j}}{\alpha'_{\tilde{i}i}} &= \frac{1}{2M_{\tilde{j}j}^2} \times [(4M_{\tilde{j}j}^2 - M_{\tilde{i}i}^2 - M_{\tilde{j}j}^2) - \sqrt{(4M_{\tilde{j}j}^2 - M_{\tilde{i}i}^2 - M_{\tilde{j}j}^2)^2 - 4M_{\tilde{i}i}^2M_{\tilde{j}j}^2}], \\
\frac{\alpha'_{\tilde{j}j}}{\alpha'_{\tilde{i}i}} &= \frac{1}{4M_{\tilde{j}j}^2} \times [(4M_{\tilde{j}j}^2 + M_{\tilde{i}i}^2 - M_{\tilde{j}j}^2) - \sqrt{(4M_{\tilde{j}j}^2 + M_{\tilde{i}i}^2 - M_{\tilde{j}j}^2)^2 - 4M_{\tilde{i}i}^2M_{\tilde{j}j}^2}], 
\end{align*}$$  

(10)

From Eq. (11), one has

$$\alpha' = \frac{(J + 2) - J}{M_{j+2}^2 - M_j^2}.$$  

(11)
Table 1. The values of $\frac{\alpha'_{c\bar{c}}}{\alpha_{n\bar{n}}}$ and $\frac{\alpha'_{b\bar{b}}}{\alpha_{n\bar{n}}}$ $(n$ denotes $u$ or $d$ quark) obtained from Eqs. (9) and (10).

| $N^{2S+1}L_J$ | (9)      | (10)      |
|----------------|----------|-----------|
| $1^1S_0$       | 0.5636   | 0.0038    |
| $1^1P_1$       | 0.5433   | 0.2238    |
| $1^3S_1$       | 0.4921   | 0.1274    |
| $1^3P_2$       | 0.5041   | 0.2726    |
| $1^1S_0$       | 0.2880   | 0.0008    |
| $1^3S_1$       | 0.2361   | 0.0290    |
| $1^3P_2$       | 0.2562   | 0.0690    |

It is obvious that the Regge slope $\alpha'$ should be a single positive real number. Thus, $\frac{\alpha'_{c\bar{c}}}{\alpha_{n\bar{n}}}$ should take only one value for a multiplet with certain $i$ and $j$. Since the relations (3) and (8) are symmetric under the exchange of the quark flavors $i$ and $j$, we only consider the case in which quark masses satisfy $m_i < m_j$ for mesons here and after.

From Eqs. (9) and (10), we have the values of $\frac{\alpha'_{c\bar{c}}}{\alpha_{n\bar{n}}}$ and $\frac{\alpha'_{b\bar{b}}}{\alpha_{n\bar{n}}}$ (n denotes $u$ or $d$ quark) for the well-established multiplets. In the calculation, we do not consider the small mass splittings caused by isospin breaking effects due to electromagnetic interaction. Here and below, all the masses of hadrons used in calculation are taken from PDG2006 [15] except for the newly observed hadrons. The results are shown in Table 1.

The values of $\alpha'_{c\bar{c}}$ for light nonstrange meson trajectories of different multiplets are in the range $0.7$–$0.9$ GeV$^{-2}$ [21, 22, 39, 46, 58]. The values of $\alpha'_{c\bar{c}}$ and $\alpha'_{b\bar{b}}$ for charmonium and bottomonium trajectories of different multiplets are in the ranges $0.3$–$0.5$ GeV$^{-2}$ and $0.18$–$0.25$ GeV$^{-2}$, respectively [21, 22, 46, 57]. Then, we have $\frac{\alpha'_{c\bar{c}}}{\alpha_{n\bar{n}}} \sim 0.5$ and $\frac{\alpha'_{b\bar{b}}}{\alpha_{n\bar{n}}} \sim 0.27$. From Table 1, one can see that the values of $\frac{\alpha'_{c\bar{c}}}{\alpha_{n\bar{n}}}$ (or $\frac{\alpha'_{b\bar{b}}}{\alpha_{n\bar{n}}}$) given by Eq. (9) are approximately the same for different multiplets as they should to be. However, the values of $\frac{\alpha'_{c\bar{c}}}{\alpha_{n\bar{n}}}$ (or $\frac{\alpha'_{b\bar{b}}}{\alpha_{n\bar{n}}}$) given by Eq. (10) are quite different for different multiplets. Furthermore, the values of $\frac{\alpha'_{c\bar{c}}}{\alpha_{n\bar{n}}}$ and $\frac{\alpha'_{b\bar{b}}}{\alpha_{n\bar{n}}}$ given by Eq. (10) are too small to be accepted. Therefore, we take the first pair of solutions (Eq. (9)) and discard the second pair of solutions (Eq. (10)).

For baryons, using Eqs. (1) and (4), one obtains

$$\alpha'_{iiq}M_{iiq}^2 + \alpha'_{jjq}M_{jjq}^2 = 2\alpha'_{ijq}M_{ijq}^2,$$  \hspace{1cm} (12)
where \( q \) denotes an arbitrary light or heavy quark. Combining the relations (5) and (12), one can get two pairs of solutions,

\[
\begin{aligned}
\frac{\alpha'_{jjq}}{\alpha'_{iiq}} &= \frac{1}{2M_{jjq}^2} \times [(4M_{ijq}^2 - M_{ijq}^2 - M_{jjq}^2) + \sqrt{(4M_{ijq}^2 - M_{ijq}^2 - M_{jjq}^2)^2 - 4M_{ijq}^2M_{jjq}^2}], \\
\frac{\alpha'_{ijq}}{\alpha'_{iiq}} &= \frac{1}{4M_{ijq}^2} \times [(4M_{ijq}^2 + M_{ijq}^2 - M_{jjq}^2) + \sqrt{(4M_{ijq}^2 - M_{ijq}^2 - M_{jjq}^2)^2 - 4M_{ijq}^2M_{jjq}^2}],
\end{aligned}
\tag{13}
\]

and

\[
\begin{aligned}
\frac{\alpha'_{jjq}}{\alpha'_{iiq}} &= \frac{1}{2M_{jjq}^2} \times [(4M_{ijq}^2 - M_{ijq}^2 - M_{jjq}^2) - \sqrt{(4M_{ijq}^2 - M_{ijq}^2 - M_{jjq}^2)^2 - 4M_{ijq}^2M_{jjq}^2}], \\
\frac{\alpha'_{ijq}}{\alpha'_{iiq}} &= \frac{1}{4M_{ijq}^2} \times [(4M_{ijq}^2 + M_{ijq}^2 - M_{jjq}^2) - \sqrt{(4M_{ijq}^2 - M_{ijq}^2 - M_{jjq}^2)^2 - 4M_{ijq}^2M_{jjq}^2}],
\end{aligned}
\tag{14}
\]

From the Chew-Frautschi plots (Fig. 1), it is obvious that the Regge slope \( \alpha' \) should be a single positive real number. Thus, \( \frac{\alpha'_{jjq}}{\alpha'_{iiq}} \) should take only one value for a multiplet with certain \( i, j \) and \( q \).

Since the relations (5) and (12) are symmetric under the exchange of the quark flavors \( i \) and \( j \), we only consider the case in which quark masses satisfy \( m_i < m_j \) for baryons here and after.

For the \( \frac{1}{2}^+ \) multiplet, when \( i = n, j = s, \) and \( q = n \), we have \( M_{nnn} = M_{N(939)}, M_{nss} = M_\Xi, \) and \( M_{nns} = \frac{1}{4}(3M_\Lambda^2 + M_\Sigma^2) \) \[23\]. Then, we have \( \frac{\alpha'_{ss}}{\alpha'_{ NN}} = 0.89 \) from Eq. (13) and \( \frac{\alpha'_{s\bar{s}}}{\alpha'_{ N\bar{N}}} = 0.57 \) from Eq. (14). For the \( \frac{3}{2}^+ \) multiplet, when \( i = n, j = s, \) and \( q = n \), we have \( M_{nnn} = M_\Delta, M_{nss} = M_\Xi^*, \) and \( M_{nns} = M_\Sigma^* \). Then, we have \( \frac{\alpha'_{s\bar{s}}}{\alpha'_{ \Delta \bar{\Delta}}} = 0.89 \) from Eq. (13) and \( \frac{\alpha'_{s\bar{s}}}{\alpha'_{ \Delta \bar{\Delta}}} = 0.72 \) from Eq. (14). Since the Regge trajectories of light baryons are approximately parallel, the values of \( \frac{\alpha'_{ss}}{\alpha'_{ NN}} \) and \( \frac{\alpha'_{s\bar{s}}}{\alpha'_{ N\bar{N}}} \) should be close to 1. Therefore, Eqs. (14) should be discarded in the case of quark masses \( m_i < m_j \). Furthermore, Eqs. (13) and (14) can be considered as the generalization of Eqs. (9) and (10) respectively from the meson case to the baryon case. Therefore, we take Eq. (13) and discard Eq. (14).

## B. High-power mass equalities

From Eqs. (9) and (13), high-power mass equalities can be derived for mesons and baryons, respectively. For mesons, using

\[
\frac{\alpha'_{jjq}}{\alpha'_{iiq}} = \frac{\alpha'_{kkq}}{\alpha'_{iiq}} \times \frac{\alpha'_{jjq}}{\alpha'_{kkq}},
\tag{15}
\]

and Eq. (9), when \( m_i < m_j < m_k \), we have
This relation can be simplified to

\[
\frac{(4M_{ij}^2 - M_{ii}^2 - M_{jj}^2) + \sqrt{(4M_{ij}^2 - M_{ii}^2 - M_{jj}^2)^2 - 4M_{ii}^2M_{jj}^2}}{2M_{jj}^2}
\]

\[
= \frac{((4M_{ik}^2 - M_{ii}^2 - M_{kk}^2) + \sqrt{(4M_{ik}^2 - M_{ii}^2 - M_{kk}^2)^2 - 4M_{ii}^2M_{kk}^2})/2M_{kk}^2}{((4M_{jk}^2 - M_{jj}^2 - M_{kk}^2) + \sqrt{(4M_{jk}^2 - M_{jj}^2 - M_{kk}^2)^2 - 4M_{jj}^2M_{kk}^2})/2M_{kk}^2}.
\]

(16)

For baryons, using

\[
\frac{\alpha'_{jjq}}{\alpha'_{iiq}} = \frac{\alpha'_{kkq}}{\alpha'_{iiq}} \times \frac{\alpha'_{jjq}}{\alpha'_{kkq}},
\]

(17)

and Eq. (13), when \(m_i < m_j < m_k\), we have

\[
\frac{(4M_{ij}^2 - M_{ii}^2 - M_{jj}^2) + \sqrt{(4M_{ij}^2 - M_{ii}^2 - M_{jj}^2)^2 - 4M_{ii}^2M_{jj}^2}}{2M_{jj}^2}
\]

\[
= \frac{((4M_{ik}^2 - M_{ii}^2 - M_{kk}^2) + \sqrt{(4M_{ik}^2 - M_{ii}^2 - M_{kk}^2)^2 - 4M_{ii}^2M_{kk}^2})/2M_{kk}^2}{((4M_{jk}^2 - M_{jj}^2 - M_{kk}^2) + \sqrt{(4M_{jk}^2 - M_{jj}^2 - M_{kk}^2)^2 - 4M_{jj}^2M_{kk}^2})/2M_{kk}^2}.
\]

(18)

where \(q\) denotes an arbitrary light or heavy quark.

Relations (16) and (18) are the high-power mass equalities among one \(J^P\) multiplet. They can be used to predict the masses of unobserved states. In Sec. III, we will apply Eq. (16) to predict the masses of \(\bar{b}c\) meson states and the masses of the pure \(s\bar{s}\) meson states.

C. Linear mass inequalities and quadratic mass inequalities

From Eqs. (9) and (13), two kinds of interesting inequalities can be derived for mesons and baryons, respectively. For mesons, as mentioned in the above discussion, \(\alpha'_{jj}\) and \(\alpha'_{ii}\) ought to be positive real numbers. Thus \(\frac{\alpha'_{jj}}{\alpha'_{ii}}\) should also be a real number. Then from Eq. (9), we have

\[
|4M_{ij}^2 - M_{ii}^2 - M_{jj}^2| \geq 2M_{ii}M_{jj}.
\]

(19)

When \(i = j\), \(4M_{ij}^2 - M_{ii}^2 - M_{jj}^2 \leq 0\) cannot be held; when \(i \neq j\), \(4M_{ij}^2 - M_{ii}^2 - M_{jj}^2 \leq 0\) can be easily ruled out by the data of the well-established meson multiplets. Therefore, \(4M_{ij}^2 - M_{ii}^2 - M_{jj}^2 \geq 0\). Thus, Eq. (19) can be written as the following:

\[
4M_{ij}^2 - M_{ii}^2 - M_{jj}^2 \geq 2M_{ii}M_{jj}.
\]

(20)

This relation can be simplified to

\[
2M_{ij} \geq M_{ii} + M_{jj}.
\]

(21)
If \( i = j \), \( M_{i\bar{i}} = M_{ij} = M_{jj} \), then we have \( 2M_{ij} = M_{i\bar{i}} + M_{jj} \). On the other hand, if \( 2M_{ij} = M_{i\bar{i}} + M_{jj} \), using Eq. (9), we have

\[
\frac{\alpha'_{ij}}{\alpha'_{ii}} = \frac{M_{i\bar{i}}}{M_{jj}}.
\]

From the derivation of Eq. (22), we can see that this equation is valid for hadrons belonging to the same multiplet. Since hadrons lying on the same Regge trajectory (which have the total angular momenta \( J, J + 2, J + 4, \ldots \)) have the same slope, we have

\[
\frac{\alpha'_{ij}}{\alpha'_{ii}} = \frac{M_{i\bar{i},J}}{M_{jj,J}} = \frac{M_{i\bar{i},J+2}}{M_{jj,J+2}}.
\]

From Eq. (11), we have

\[
\alpha'_{ii} = \frac{2}{M_{i\bar{i},J+2}^2 - M_{i\bar{i},J}^2}, \quad \alpha'_{jj} = \frac{2}{M_{jj,J+2}^2 - M_{jj,J}^2}.
\]

Combining Eqs. (23) and (24), we have

\[
\frac{\alpha_{ij}}{\alpha_{ii}} = \frac{M_{i\bar{i},J+2} + M_{i\bar{i},J}}{M_{jj,J+2} + M_{jj,J}} \times \frac{M_{i\bar{i},J+2} - M_{i\bar{i},J}}{M_{jj,J+2} - M_{jj,J}} = \left( \frac{\alpha'_{ij}}{\alpha'_{ii}} \right)^2.
\]

As mentioned before, the Regge slope \( \alpha' \) is a positive real number. Therefore, \( \frac{\alpha'_{ij}}{\alpha'_{ii}} = 1 \) when \( 2M_{ij} = M_{i\bar{i}} + M_{jj} \). Consequently we have \( M_{i\bar{i},J} = M_{jj,J} \) and \( M_{i\bar{i},J+2} = M_{jj,J+2} \) from Eq. (23). This leads to \( i = j \) since the \( i\bar{i} \) and \( jj \) states have the same \( J^P \).

From the above analysis, we can conclude that if and only if \( i = j \), \( 2M_{ij} = M_{i\bar{i}} + M_{jj} \). Therefore, when \( i \neq j \), we have

\[
2M_{ij} > M_{i\bar{i}} + M_{jj}.
\]

Many authors argued recently that the slopes of Regge trajectories decrease with quark mass increase [21, 22, 40, 41, 45, 46, 55, 59, 60]. Therefore, \( \frac{\alpha'_{JJ}}{\alpha'_{ii}} < 1 \) when \( j \)-quark is heavier than \( i \)-quark. Then, from Eq. (9) one can have

\[
\frac{1}{2M_{jj}^2} \times [(4M_{ij}^2 - M_{i\bar{i}}^2 - M_{jj}^2) + \sqrt{(4M_{ij}^2 - M_{i\bar{i}}^2 - M_{jj}^2)^2 - 4M_{i\bar{i}}^2M_{jj}^2}] < 1.
\]

From this relation, we obtain

\[
\begin{cases}
2M_{jj}^2 - (4M_{ij}^2 - M_{i\bar{i}}^2 - M_{jj}^2) > 0, \\
(4M_{ij}^2 - M_{i\bar{i}}^2 - M_{jj}^2)^2 - 4M_{i\bar{i}}^2M_{jj}^2 < [2M_{jj}^2 - (4M_{ij}^2 - M_{i\bar{i}}^2 - M_{jj}^2)]^2.
\end{cases}
\]

These two inequalities can be simplified to

\[
2M_{ij}^2 < M_{i\bar{i}}^2 + M_{jj}^2.
\]
The relation (29) can also be derived in the same way if we use the second equation in Eq. (9) considering $\alpha' \bar{i} \bar{j} < 1$.

The baryon mass inequalities can be extracted in the same way as that in the meson case. Then, we have

\begin{align*}
2M_{ijq} > M_{iiq} + M_{jjq}, & \quad (30) \\
2M_{ijq}^2 < M_{iiq}^2 + M_{jjq}^2. & \quad (31)
\end{align*}

It is very interesting that the inequalities (26), (29), (30) and (31) are the concave and convex relations. These mass inequalities can be used to give constraints (lower limits and upper limits) for masses of hadrons which have not been discovered. For example, we have from the inequalities (26) and (29) that

\begin{equation}
\frac{M_{ii} + M_{jj}}{2} < M_{ij} < \sqrt{\frac{M_{ii}^2 + M_{jj}^2}{2}}, \tag{32}
\end{equation}

in which one inequality gives an upper limit while the other gives a lower limit for $M_{ij}$. For baryons, we have from the inequalities (30) and (31) that

\begin{equation}
\frac{M_{iiq} + M_{jjq}}{2} < M_{ijq} < \sqrt{\frac{M_{iiq}^2 + M_{jjq}^2}{2}}. \tag{33}
\end{equation}

We will use Eqs. (32) and (33) to give mass ranges for mesons and baryons in Sec. III.

**D. Quadratic mass equalities**

To evaluate the deviations of relations (29) and (31) from the equalities that would be obtained by changing the signs of inequalities to equal signs, we introduce a parameter $\delta$, which is denoted by $\delta^m_{ij}$ for mesons,

\begin{equation}
\delta^m_{ij} = M_{ii}^2 + M_{jj}^2 - 2M_{ij}^2, \tag{34}
\end{equation}

and by $\delta^b_{ij}$ for baryons,

\begin{equation}
\delta^b_{ij} = M_{iiq}^2 + M_{jjq}^2 - 2M_{ijq}^2, \tag{35}
\end{equation}

where $i$, $j$ and $q$ are arbitrary light or heavy quarks. From relations (29) and (31), we know $\delta^m(b) > 0$. It will be shown later that $\delta^b_{ij}$ is independent of $q$.

For mesons, from Eqs. (2) and (3), we have

\begin{equation}
a_{\bar{i}i}(0) - a_{\bar{j}j}(0) = a_{\bar{i}j}(0) - a_{\bar{j}j}(0), \tag{36}
\end{equation}
\[
\frac{1}{\alpha_{ii}' - \alpha_{ij}'} = \frac{1}{\alpha_{ij}'} - \frac{1}{\alpha_{jj}'}. \tag{37}
\]

Let
\[
\lambda_i \equiv a_{nn}(0) - a_{ni}(0), \quad \gamma_i \equiv \frac{1}{\alpha_{ni}'} - \frac{1}{\alpha_{nn}'}, \tag{38}
\]
where \(n\) denotes light nonstrange quark \(u\) or \(d\). Using Eqs. (36), (37) and (38) we have
\[
\lambda_i = a_{nn}(0) - a_{ni}(0) = a_{ni}(0) - a_{ii}(0), \tag{39}
\]
\[
\gamma_i = \frac{1}{\alpha_{ni}'} - \frac{1}{\alpha_{nn}'} = \frac{1}{\alpha_{ii}'} - \frac{1}{\alpha_{ni}'} \tag{40}
\]
Hence,
\[
a_{ii}(0) = a_{nn}(0) - 2\lambda_i, \tag{41}
\]
\[
\frac{1}{\alpha_{ii}'} = \frac{1}{\alpha_{nn}'} + 2\gamma_i. \tag{42}
\]

With the help of Eqs. (41) and (42), we have from Eqs. (2) and (3)
\[
a_{ij}(0) = \frac{1}{2} \left[ a_{ii}(0) + a_{jj}(0) \right] = a_{nn}(0) - \lambda_i - \lambda_j, \tag{43}
\]
\[
\frac{1}{\alpha_{ij}'} = \frac{1}{\alpha_{nn}'} + \frac{1}{\alpha_{ii}'} + \frac{1}{\alpha_{jj}'} = \frac{1}{\alpha_{nn}'} + \gamma_i + \gamma_j. \tag{44}
\]
Similarly for baryons, from Eqs. (4) and (5), we have
\[
a_{ijq}(0) = a_{nnn}(0) - \lambda_i - \lambda_j - \lambda_q, \tag{45}
\]
\[
\frac{1}{\alpha_{ijq}'} = \frac{1}{\alpha_{nnn}'} + \gamma_i + \gamma_j + \gamma_q, \tag{46}
\]
where \(\lambda_x \equiv a_{nnn}(0) - a_{nxx}(0), \gamma_x \equiv \frac{1}{\alpha_{nxx}'} - \frac{1}{\alpha_{nnn}'} (x\) denotes \(i, j\) or \(q\)). It should be pointed out that the values of \(\lambda_x\) and \(\gamma_x\) can be different for different multiplets.

For \(n\bar{n}\) and \(i\bar{j}\) states in a meson multiplet, from Eq. (11), we have
\[
J = a_{n\bar{n}}(0) + \alpha_{n\bar{n}}' M_{n\bar{n}}^2, \tag{47}
\]
\[
J = a_{ij}(0) + \alpha_{ij}' M_{ij}^2. \tag{48}
\]

With the help of Eqs. (43), (44) and (47), we have from Eq. (48)
\[
M_{ij}^2 = (\alpha_{n\bar{n}}' M_{n\bar{n}}^2 + \lambda_i + \lambda_j)(\frac{1}{\alpha_{n\bar{n}}'} + \gamma_i + \gamma_j). \tag{49}
\]
Therefore, from Eqs. (34) and (49), we have

\[ \delta_{ij}^m = (\alpha'_{nn} M_{nn}^2 + 2 \lambda_i - 2 \gamma_i) + (\alpha'_{nn} M_{nn}^2 + 2 \lambda_j) \frac{1}{\alpha'_{nn}} + 2 \gamma_j \]

\[ - 2 (\alpha'_{nn} M_{nn}^2 + \lambda_i + \lambda_j) \frac{1}{\alpha'_{nn}} + \gamma_i + \gamma_j \]

\[ = 2 (\lambda_i - \lambda_j) (\gamma_i - \gamma_j). \tag{50} \]

For baryons, in the same way, we have

\[ \delta_{ij}^b = M_{ijq} + M_{jjq}^2 - 2 M_{ijq} \]

\[ = (\alpha'_{nnn} M_{nnn}^2 + 2 \lambda_i + \lambda_q) \frac{1}{\alpha'_{nnn}} + 2 \gamma_i + \gamma_q + (\alpha'_{nnn} M_{nnn}^2 + 2 \lambda_j + \lambda_q) \frac{1}{\alpha'_{nnn}} + 2 \gamma_j + \gamma_q \]

\[ - 2 (\alpha'_{nnn} M_{nnn}^2 + \lambda_i + \lambda_j + \lambda_q) \frac{1}{\alpha'_{nnn}} + \gamma_i + \gamma_j + \gamma_q \]

\[ = 2 (\lambda_i - \lambda_j) (\gamma_i - \gamma_j). \tag{51} \]

It can be seen from Eq. (51) that \( \delta_{ij}^b \) is independent of \( q \).

From Eq. (38), we know that \( \lambda_n = \gamma_n = 0 \). Since we choose \( m_i < m_j, \alpha'_{ii} > \alpha'_{jj} \). Hence from the definition of \( \gamma_i \) (Eq. (38)), we have \( \gamma_i < \gamma_j \). Therefore, \( 0 = \gamma_n < \gamma_s < \gamma_c < \gamma_b \). From Eqs. (9) and (26), we know that \( \frac{\alpha'_{jj}}{\alpha'_{ii}} > \frac{M_{jj}}{M_{ii}} \). Hence \( \alpha'_{jj} M_{jj} > \alpha'_{ii} M_{ii} \cdot M_{ii} \). With the help of Eqs. (11) and (41), we have \( \lambda_i < \lambda_j \). Therefore, \( 0 = \lambda_n < \lambda_s < \lambda_c < \lambda_b \). Consequently, we have \( 0 < \delta_{ns}^m < \delta_{nc}^m < \delta_{nb}^m \), \( 0 < \delta_{cb}^m < \delta_{sb}^m < \delta_{nb}^m \), \( 0 < \delta_{sc}^m < \delta_{nc}^m \), and \( 0 < \delta_{sc}^m < \delta_{sb}^m \). If we assume that \( \gamma_s < 1 \gamma_c < 1 \gamma_b \) and \( \gamma_s < 1 \gamma_c < 1 \gamma_b \), with the above analysis, we can have \( \delta_{ns}^m < \delta_{sc}^m < \delta_{nc}^m < \delta_{cb}^m < \delta_{sb}^m < \delta_{nb}^m \). We will show later that these relations hold indeed. For baryons, we can have \( \delta_{ns}^b < \delta_{sb}^b < \delta_{nc}^b < \delta_{cb}^b < \delta_{sb}^b < \delta_{nb}^b \) in the same way.

Inserting the corresponding masses into relation (34), we have the values of \( \delta_{ij}^m \) for some meson multiplets which are shown in Table 2. From Table 2, we can see that the relation \( \delta_{ns}^m < \delta_{sc}^m < \delta_{nc}^m < \delta_{cb}^m < \delta_{sb}^m < \delta_{nb}^m \) is indeed satisfied for different meson multiplets. These inequalities imply that the higher-order breaking effects become more pronounced with the quark mass increase.

Table 2. The values of \( \delta_{ij}^m \) for some multiplets (in units of GeV²).

| Multiplet | \( \delta_{ns}^m \) | \( \delta_{sc}^m \) | \( \delta_{nc}^m \) | \( \delta_{cb}^m \) | \( \delta_{sb}^m \) | \( \delta_{nb}^m \) |
|-----------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( ^1 S_0 \) | 0.016 | 1.623 | 1.931 | 16.898 | 29.179 | 30.769 |
| \( ^3 S_1 \) | 0.015 | 1.682 | 2.125 | 18.294 | 31.930 | 33.387 |
| \( ^3 P_2 \) | 0.018 | 1.785 | 2.281 | 18.042 | 32.434 | 34.018 |
| \( ^1 P_1 \) | | | | | 2.198 | |
1. Mass relations for the $\frac{3}{2}^+$ multiplet

For the $\frac{3}{2}^+$ multiplet, noticing that $\delta_{ij}^{\frac{3}{2}^+}$ in the above relation (51) is independent of $q$, we have some equalities which are given in the following.

1) When $i = n, j = s, q = n, s, c, b$,

$$\delta_{ns}^{\frac{3}{2}^+} = M_{S}^2 + M_{\Xi}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Omega} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi}. \quad (52a)$$

2) When $i = n, j = c, q = n, s, c, b$,

$$\delta_{nc}^{\frac{3}{2}^+} = M_{S}^2 + M_{\Xi}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Omega} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi}. \quad (52b)$$

3) When $i = s, j = c, q = n, s, c, b$,

$$\delta_{sc}^{\frac{3}{2}^+} = M_{S}^2 + M_{\Xi}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Omega} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi}. \quad (52c)$$

4) When $i = n, j = b, q = n, s, c, b$,

$$\delta_{nb}^{\frac{3}{2}^+} = M_{S}^2 + M_{\Xi}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Omega} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi}. \quad (52d)$$

5) When $i = s, j = b, q = n, s, c, b$,

$$\delta_{sb}^{\frac{3}{2}^+} = M_{S}^2 + M_{\Xi}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Omega} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi}. \quad (52e)$$

6) When $i = c, j = b, q = n, s, c, b$,

$$\delta_{cb}^{\frac{3}{2}^+} = M_{S}^2 + M_{\Xi}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Omega} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi} = M_{S}^2 + M_{\Omega}^2 - 2M_{S}^2_{\Xi}. \quad (52f)$$

From Eqs. (52a)-(52c), one can get the quadratic mass Eqs. (25)-(29) in Ref. [23], derived by Burakovsky et al. The linear forms of Eqs. (52a)-(52c) were obtained by Hendry and Lichtenberg in the quark model [26], by Verma and Khanna considering the second-order effects arising from the $84$ representation of SU(4) [27], and in the framework of $SU(8)$ symmetry [28], and by Singh et al. studying SU(4) second-order mass-breaking effects with a dynamical consideration [29] (bottom baryons were not included in Refs. 23, 26, 27, 28, 29). The linear forms of Eqs. (52a)-(52f) were derived by Singh and Khanna in the nonrelativistic additive quark model [30] and by Singh using broken SU(6) internal symmetry including second-order mass contributions [31]. We will show some arguments in Sec. IV which support the quadratic form mass formulas for mesons and baryons rather than the linear form.
2. Mass relations for the $\frac{1}{2}^+$ multiplet

For the $\frac{1}{2}^+$ multiplet, it is very different from the $\frac{3}{2}^+$ multiplet because there are different ways for the spins of the constituent quarks to form the total spin $S = \frac{1}{2}$. Three constituent quarks in a $\frac{1}{2}^+$ baryon can be regarded as a quark and a scalar diquark or regarded as a quark and an axial-vector diquark. Regge slopes of $\Lambda$, $\Lambda_c$, $\Lambda_b$, $\Xi_c$ and $\Xi_b$ are slightly bigger than those of $\Sigma$, $\Sigma_c$, $\Sigma_b$, $\Xi'_c$ and $\Xi'_b$, respectively, although sometimes they can be considered to be approximately equal [23, 61]. Regge intercepts of $\Lambda$, $\Lambda_c$, $\Lambda_b$, $\Xi_c$ and $\Xi_b$ are much bigger than those of $\Sigma$, $\Sigma_c$, $\Sigma_b$, $\Xi'_c$ and $\Xi'_b$, respectively. However, these cannot be reflected from Eqs. (45) and (46). Therefore, some of the $\frac{1}{2}^+$ baryons may not be related as the $\frac{3}{2}^+$ baryons.

The ‘$Qq'q$’ and ‘$QQ'q'$ (where $q$ and $q'$ denote the light quarks while $Q$ and $Q'$ denote the heavy quarks $c$ or $b$) baryon states are believed to be described by the quark-diquark picture: Two light quarks $qq'$ are bound into a color antitriplet system with the size comparable to the QCD scale in the ‘$Qq'q$’ baryon state [62, 63]; Two heavy quarks $QQ'$ are bound into a small (compared with the QCD scale) color antitriplet system in the ‘$QQ'q'$ baryon state [62, 64]. The heavy baryons which are composed of a heavy quark and a light axial-vector diquark ($\Sigma Q$, $\Xi Q$ and $\Omega Q$) belong to a 6 representation of flavor $SU(3)$ [15]. Therefore, $\delta_{ns}^{\frac{1}{2}^+}$ can be expressed as

$$\delta_{ns}^{\frac{1}{2}^+} = M_{\Sigma c}^2 + M_{\Omega c}^2 - 2M_{\Xi'_c}^2 = M_{\Sigma b}^2 + M_{\Omega b}^2 - 2M_{\Xi'_b}^2. \quad (53)$$

For the doubly heavy baryons which are composed of a light quark and a heavy axial-vector diquark, $\delta_{bc}^{\frac{1}{2}^+}$ can be expressed as

$$\delta_{bc}^{\frac{1}{2}^+} = M_{\Xi_{cc}}^2 + M_{\Xi_{bb}}^2 - 2M_{\Xi'_{bc}}^2 = M_{\Omega_{cc}}^2 + M_{\Omega_{bb}}^2 - 2M_{\Omega'_{bc}}^2. \quad (54)$$

Since $\delta_{qq'}^b$ is determined by the dynamics of the light diquark system ‘$qq'$’ inside a heavy baryon ‘$Qq'q$’ and since this dynamics is independent of flavor and spin of the heavy quark due to the $SU(2)_f \otimes SU(2)_s$ symmetry in the heavy quark limit [65], we assume that $\delta_{ns}^{\frac{1}{2}^+}$ for the $\frac{1}{2}^+$ charmed (bottom) sextet equals $\delta_{ns}^{\frac{3}{2}^+}$ for the $\frac{3}{2}^+$ charmed (bottom) sextet, $\delta_{ns}^{\frac{1}{2}^+} = \delta_{ns}^{\frac{3}{2}^+}$. This relation holds exactly when the masses of charmed and bottom quarks are taken to be infinitely large. Deviations from this relation are due to $\frac{1}{m_c}$ and $\frac{1}{m_b}$ corrections. Then, one can have

$$M_{\Sigma c}^2 + M_{\Omega c}^2 - 2M_{\Xi'_{c}}^2 = M_{\Sigma b}^2 + M_{\Omega b}^2 - 2M_{\Xi'_{b}}^2. \quad (55)$$

$$M_{\Sigma b}^2 + M_{\Omega b}^2 - 2M_{\Xi'_{b}}^2 = M_{\Sigma'_{c}}^2 + M_{\Omega'_{c}}^2 - 2M_{\Xi'_{b}}^2. \quad (56)$$
There are two linear mass equations similar to the above quadratic mass equations,

\[ M_{\Sigma_c} + M_{\Omega_c} - 2M_{\Xi_c} = M_{\Sigma_c^*} + M_{\Omega_c^*} - 2M_{\Xi_c^*}, \]  
\[ (57) \]

\[ M_{\Sigma_b} + M_{\Omega_b} - 2M_{\Xi_b} = M_{\Sigma_b^*} + M_{\Omega_b^*} - 2M_{\Xi_b^*}, \]  
\[ (58) \]

which were extracted by Jenkins in the \(1/m_Q\) and \(1/N_c\) expansions \[33\]. Similarly, assuming that \(\delta_{bc}^{3+} = \delta_{bc}^{3+}\), one can have

\[ M_{\Xi_{cc}}^2 + M_{\Xi_{bb}}^2 - 2M_{\Xi_{cb}}^2 = M_{\Omega_{cc}}^2 + M_{\Omega_{bb}}^2 - 2M_{\Omega_{cb}}^2 = M_{\Xi_{cc}}^2 + M_{\Xi_{bb}}^2 - 2M_{\Xi_{cb}}^2 = M_{\Omega_{cc}}^2 + M_{\Omega_{bb}}^2 - 2M_{\Omega_{cb}}^2. \]  
\[ (59) \]

From Eq. (52), we can have a relation for the \(3^+\) baryons,

\[ (M_{\Omega_{cc}}^2 - M_{\Omega_{cc}}^2) + (M_{\Xi_{cc}}^2 - M_{\Xi_{cc}}^2) = (M_{\Omega_{cc}}^2 - M_{\Omega_{cc}}^2). \]  
\[ (60) \]

Its corresponding relation for the \(\frac{1}{2}^+\) baryons is

\[ (M_{\Omega_{cc}}^2 - M_{\Omega_{cc}}^2) + (M_{\Xi_{cc}}^2 - M_{\Xi_{cc}}^2) = (M_{\Omega_{cc}}^2 - M_{\Omega_{cc}}^2). \]  
\[ (61) \]

The linear form of Eq. (61) can satisfy the instanton model \[25\] and has been given by Verma and Khanna considering the second-order effects arising from the \(84\) representation of \(SU(4)\) \[27\]. A different relation,

\[ (M_{\Omega_{cc}}^2 - M_{\Xi_{cc}}^2) + \left(3M_{\Lambda_c}^2 + M_{\Sigma_c}^2 - M_{\Xi_c}^2\right) = 2 \left(M_{\Xi_c}^2 + M_{\Xi_c}^2 - 3M_{\Lambda_c}^2 + M_{\Sigma_c}^2\right), \]  
\[ (62) \]

has been proposed in Ref. \[23\]. However, the linear form of Eq. (62) cannot satisfy the instanton model \[25\]. Furthermore, the value of \((M_{\Omega_{cc}}^2 - M_{\Xi_{cc}}^2)\) given by Eq. (61) (\(\sim 0.94 \text{ GeV}^2\)) is close to the value of \((M_{\Omega_{cc}}^2 - M_{\Xi_{cc}}^2)\) given by Eq. (60) (\(\sim 0.89 \text{ GeV}^2\)) while the value of \((M_{\Omega_{cc}}^2 - M_{\Xi_{cc}}^2)\) given by Eq. (62) (\(\sim 1.39 \text{ GeV}^2\)) is much larger. We will use Eq. (61) rather than Eq. (62) to extract the mass of \(\Omega_{cc}\) in Sec. III.

### III. SOME APPLICATIONS

In this section, we will apply the relations we have obtained in Sec. II to discuss the mass ranges of mesons and baryons, the masses of the \(\bar{b}c\) and \(s\bar{s}\) meson states, the properties of \(\Xi_{cc}^+(3520)\), the parameters of the Regge trajectories for the \(\frac{1}{2}^+\) and \(\frac{3}{2}^+\) multiplet, and the properties of the charm-strange baryons (some of which have just been observed).
A. Mass ranges of mesons and baryons

Using Eqs. (32) and (33), we calculate the upper and lower mass limits for some meson states \((s\bar{s}, c\bar{n}, \bar{b}c, c\bar{s} and \bar{b}s)\) of different multiplets and some baryon states of \(\frac{1}{2}^{+}\) and \(\frac{3}{2}^{+}\) multiplets. The results for mesons are shown in Table 3-1 and Table 3-2 in comparison with the measured meson masses \([15]\). The results for baryons are shown in Table 4 in comparison with the measured baryon masses \([15]\).

The masses of the pure \(s\bar{s}\) states cannot be directly measured experimentally because of the usual mixing of the pure isoscalar \(n\bar{n}\) and \(s\bar{s}\) states. The way to extract masses of the pure \(s\bar{s}\) states will be displayed in the next section. In calculating the mass limits about the \(c\bar{s}\) and \(\bar{b}s\) states in Table 3-2, we approximately use the values of \(\sqrt{2M_{K}^{2} - M_{\pi}^{2}}\) (given by the quadratic GMO formula \(M_{\pi}^{2} + M_{s\bar{s}(1S_{0})}^{2} = 2M_{K}^{2}\)), \(M_{\phi}\) and \(M_{f^{'}_{2}(1525)}\) to replace \(M_{s\bar{s}(1S_{0})}\), \(M_{s\bar{s}(3S_{1})}\) and \(M_{s\bar{s}(3P_{2})}\), respectively. \(f^{'}_{2}(1525)\) was proved to be a nearly pure tensor \(s\bar{s}\) state (\(\sim 98.2\%\)) \([66]\). These approximations shift the mass limits of the \(c\bar{s}\) and \(\bar{b}s\) states only a few MeV.

It can be seen from Tables 3-1, 3-2 and 4 that the inequalities (32) and (33) (which were given from the inequalities (26), (29), (30) and (31)) agree well with the existing experimental data \([15]\). The inequalities (32) and (33) also give predictions for the mass ranges of some hadrons which have not been observed. More detailed discussions about the inequalities derived in this work and those in Refs. \([17, 18, 19, 20]\) will be given in Sec. IV.

B. Masses of the \(\bar{b}c\) and \(s\bar{s}\) meson states

1. Masses of the \(\bar{b}c\) meson states

The \(\bar{b}c\) (or \(b\bar{c}\)) meson states are special systems with two heavy quarks of different flavors. The presence of both such quarks impacts on the production, decay and mass properties of the \(\bar{b}c\) mesons. Until recently, only the pseudoscalar mesons \(B_{c}^{\pm}\) have been observed experimentally \([15, 67, 68]\). The copious productions of \(B_{c}\) mesons and their radial and orbital excitations are expected at the experimental facilities such as the Large Hadron Collider (LHC) at CERN. The masses of \(\bar{b}c\) mesons have been predicted in many different approaches \([21, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80]\).

In the following, we will use Eq. (16) to calculate the masses of \(B_{c}, B_{c}^{*}\) and \(B_{c}^{*2}\) meson states and compare the results with those given in Refs. \([21, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80]\).

For the \(1^{1}S_{0}\) multiplet, when \(i = n, j = c, and k = b\), inserting the masses of \(\pi, \eta_{c}(1S), \eta_{b}(1S)\),

\[17\]
Table 3-1. The numerical results for upper and lower limits for the masses of mesons (s\(\bar{s}\), c\(\bar{c}\) and b\(\bar{b}\)) obtained from Eqs. (26) and (29) in comparison with the experimental data (in units of GeV).

| \(N^{2S+1}L_J\) | Inequalities | Lower and upper limits |
|-----------------|--------------|------------------------|
| s\(\bar{s}\) sector | \(\sqrt{2M^2_{ns} - M^2_{n0}} < M_{s\bar{s}} < 2M_{ns} - M_{n\bar{n}}\) | |
| 1 \(^1S_0\) | \(\sqrt{2M^2_K - M^2_\pi} < M_{s\bar{s}} < 2M_K - M_\pi\) | 0.687 < M_{s\bar{s}} < 0.854 |
| 1 \(^3S_1\) | \(\sqrt{2M^2_{K^*} - M^2_\rho} < M_{s\bar{s}} < 2M_{K^*} - M_\rho\) | 0.998 < M_{s\bar{s}} < 1.012 |
| 1 \(^3P_2\) | \(\sqrt{2M^2_{K^*_2} - M^2_{a_2(1320)}} < M_{s\bar{s}} < 2M_{K^*_2} - M_{a_2(1320)}\) | 1.538 < M_{s\bar{s}} < 1.547 |
| 1 \(^1D_2\) | \(\sqrt{2M^2_{K^*_2(1770)} - M^2_{\pi_2(1670)}} < M_{s\bar{s}} < 2M_{K^*_2(1770)} - M_{\pi_2(1670)}\) | 1.868 < M_{s\bar{s}} < 1.874 |
| 1 \(^3D_3\) | \(\sqrt{2M^2_{K^*_3} - M^2_{\rho_3}} < M_{s\bar{s}} < 2M_{K^*_3} - M_{\rho_3}\) | 1.859 < M_{s\bar{s}} < 1.863 |
| c\(\bar{c}\) sector | \((M_{n\bar{n}} + M_{c\bar{c}})/2 < M_{c\bar{n}} < \sqrt{(M^2_{n\bar{n}} + M^2_{c\bar{c}})/2}\) | |
| 1 \(^1S_0\) | \((M_\pi + M_{\eta_c(1S)})/2 < M_D < \sqrt{(M^2_\pi + M^2_{\eta_c(1S)})/2}\) | 1.559 < 1.867 (exp.) < 2.110 |
| 1 \(^3S_1\) | \((M_\rho + M_{J/\psi(1S)})/2 < M_{D^*} < \sqrt{(M^2_\rho + M^2_{J/\psi(1S)})/2}\) | 1.936 < 2.008 (exp.) < 2.257 |
| 1 \(^3P_2\) | \((M_{a_2(1320)} + M_{\chi_{c2}(1P)})/2 < M_{D^*_2} < \sqrt{(M^2_{a_2(1320)} + M^2_{\chi_{c2}(1P)})/2}\) | 2.437 < 2.460 (exp.) < 2.682 |
| 1 \(^1P_1\) | \((M_{b_1(1235)} + M_{h_c(1P)})/2 < M_{D_1(2420)} < \sqrt{(M^2_{b_1(1235)} + M^2_{h_c(1P)})/2}\) | 2.378 < 2.423 (exp.) < 2.640 |
| 1 \(^3P_1\) | \((M_{a_1(1260)} + M_{\chi_{c1}(1P)})/2 < M_{D_1(13P_1)} < \sqrt{(M^2_{a_1(1260)} + M^2_{\chi_{c1}(1P)})/2}\) | 2.370 < 2.630 (exp.) < 2.931 |
| 1 \(^3D_1\) | \((M_\rho(1700) + M_{\psi(3770)})/2 < M_{D^*_1(13D_1)} < \sqrt{(M^2_\rho(1700) + M^2_{\psi(3770)})/2}\) | 2.746 < 2.931 (exp.) < 2.931 |
| 2 \(^1S_0\) | \((M_\pi(1300) + M_{h_c(2S)})/2 < M_{D(21S_0)} < \sqrt{(M^2_\pi(1300) + M^2_{h_c(2S)})/2}\) | 2.419 < 2.756 (exp.) < 2.756 |
| 2 \(^3S_1\) | \((M_\rho(1450) + M_{\psi(2S)})/2 < M_{D^*_1(23S_1)} < \sqrt{(M^2_\rho(1450) + M^2_{\psi(2S)})/2}\) | 2.573 < 2.803 (exp.) < 2.803 |
| b\(\bar{b}\) sector | \((M_{n\bar{n}} + M_{b\bar{b}})/2 < M_{b\bar{n}} < \sqrt{(M^2_{n\bar{n}} + M^2_{b\bar{b}})/2}\) | |
| 1 \(^1S_0\) | \((M_\pi + M_{b(1S)})/2 < M_B < \sqrt{(M^2_\pi + M^2_{b(1S)})/2}\) | 4.719 < 5.279 (exp.) < 6.577 |
| 1 \(^3S_1\) | \((M_\rho + M_{T(1S)})/2 < M_{B^*} < \sqrt{(M^2_\rho + M^2_{T(1S)})/2}\) | 5.118 < 5.325 (exp.) < 6.712 |
| 1 \(^3P_2\) | \((M_{a_2(1320)} + M_{\chi_{b2}(1P)})/2 < M_{B^*_2} < \sqrt{(M^2_{a_2(1320)} + M^2_{\chi_{b2}(1P)})/2}\) | 5.615 < 5.743 (exp.) < 7.071 |
| 1 \(^3P_1\) | \((M_{a_1(1260)} + M_{\chi_{b1}(1P)})/2 < M_{B_1(13P_1)} < \sqrt{(M^2_{a_1(1260)} + M^2_{\chi_{b1}(1P)})/2}\) | 5.561 < 5.615 (exp.) < 7.049 |
| 2 \(^3S_1\) | \((M_{\rho(1450)} + M_{T(2S)})/2 < M_{B^*_1(23S_1)} < \sqrt{(M^2_{\rho(1450)} + M^2_{T(2S)})/2}\) | 5.741 < 7.162 (exp.) < 7.162 |
| 2 \(^3P_2\) | \((M_{a_2(1700)} + M_{\chi_{b2}(2P)})/2 < M_{B^*_2(23P_2)} < \sqrt{(M^2_{a_2(1700)} + M^2_{\chi_{b2}(2P)})/2}\) | 6.000 < 7.363 (exp.) < 7.363 |
Table 3-2. The numerical results for upper and lower limits for the masses of mesons (\(\bar{b}c\), \(c\bar{s}\) and \(\bar{b}s\)) obtained from Eqs. [26] and [29] in comparison with the experimental data (in units of GeV).

| \(N^{2S+1}L_J\) | Inequalities | Lower and upper limits |
|-----------------|--------------|-----------------------|
| \(\bar{b}c\) sector | \((M_{\bar{b}c} + M_{bb})/2 < M_{bc} < \sqrt{(M_{\bar{b}c}^2 + M_{bb}^2)/2}\) | 6.140 < 6.286(\(\text{exp.}\)) < 6.906 |
| 1 \(S_0\) | \((M_{\bar{b}c}(1S) + M_{\bar{b}c}(1S))/2 < M_{B_c} < \sqrt{(M_{\bar{b}c}(1S)^2 + M_{\bar{b}c}(1S)^2)/2}\) | 6.279 < 7.039 |
| 1 \(S_1\) | \((M_{\bar{b}c}(1S) + M_{\bar{b}c}(1S))/2 < M_{B_c} < \sqrt{(M_{\bar{b}c}(1S)^2 + M_{\bar{b}c}(1S)^2)/2}\) | 6.734 < 7.446 |
| 1 \(P_0\) | \((M_{\bar{b}c}(1P) + M_{\bar{b}c}(1P))/2 < M_{B_c} < \sqrt{(M_{\bar{b}c}(1P)^2 + M_{\bar{b}c}(1P)^2)/2}\) | 6.637 < 7.378 |
| 1 \(P_1\) | \((M_{\bar{b}c}(1P) + M_{\bar{b}c}(1P))/2 < M_{B_c} < \sqrt{(M_{\bar{b}c}(1P)^2 + M_{\bar{b}c}(1P)^2)/2}\) | 6.702 < 7.423 |
| 2 \(S_1\) | \((M_{\bar{b}c}(2S) + M_{\bar{b}c}(2S))/2 < M_{B_c}(2S) < \sqrt{(M_{\bar{b}c}(2S)^2 + M_{\bar{b}c}(2S)^2)/2}\) | 6.855 < 7.552 |
| \(c\bar{s}\) sector | \((M_{c\bar{s}} + M_{s\bar{s}})/2 < M_{c\bar{s}} < \sqrt{(M_{c\bar{s}}^2 + M_{s\bar{s}}^2)/2}\) | 1.834 < 1.968(\(\text{exp.}\)) < 2.163 |
| 1 \(S_0\) | \((M_{c\bar{s}}(1S) + M_{s\bar{s}}(1S))/2 < M_{D_s} < \sqrt{(M_{c\bar{s}}(1S)^2 + M_{s\bar{s}}(1S)^2)/2}\) | 2.058 < 2.112(\(\text{exp.}\)) < 2.305 |
| 1 \(S_1\) | \((M_{c\bar{s}}(1S) + M_{s\bar{s}}(1S))/2 < M_{D_s} < \sqrt{(M_{c\bar{s}}(1S)^2 + M_{s\bar{s}}(1S)^2)/2}\) | 2.541 < 2.574(\(\text{exp.}\)) < 2.736 |
| \(\bar{b}s\) sector | \((M_{bb} + M_{s\bar{s}})/2 < M_{bs} < \sqrt{(M_{bb}^2 + M_{s\bar{s}}^2)/2}\) | 4.994 < 5.368(\(\text{exp.}\)) < 6.594 |
| 1 \(S_0\) | \((M_{\bar{b}s}(1S) + M_{s\bar{s}}(1S))/2 < M_{B_s} < \sqrt{(M_{\bar{b}s}(1S)^2 + M_{s\bar{s}}(1S)^2)/2}\) | 5.240 < 5.413(\(\text{exp.}\)) < 6.728 |
| 1 \(S_1\) | \((M_{\bar{b}s}(1S) + M_{s\bar{s}}(1S))/2 < M_{B_s} < \sqrt{(M_{\bar{b}s}(1S)^2 + M_{s\bar{s}}(1S)^2)/2}\) | 5.719 < 5.840(\(\text{exp.}\)) < 7.091 |

\(D\) and \(B\) into Eq. [16], the mass of \(B_c\) can be extracted. For the \(1^3S_1\) multiplet, when \(i = n, j = c,\) and \(k = b,\) inserting the masses of \(\rho, J/\psi(1S), T(1S), D^*\) and \(B^*\) into Eq. [16], the mass of \(B_c^*\) can be extracted. For the \(1^3P_2\) multiplet, when \(i = n, j = c,\) and \(k = b,\) inserting the masses of \(a_2(1320),\) \(\chi_{c2}(1P), \chi_{b2}(1P), D_2^*(2460)\) and \(B^*_c(5740)\) which was observed recently [4] into Eq. [16], the mass of \(B^*_c\) can be extracted. Comparison of the masses of \(B_c, B_c^*\) and \(B^*_c\) extracted in the present work and those given by other references is shown in Table 5. The application of Eq. [18] (baryon case) will be performed in Subsection D of this section.

If Eq. (2) (the additivity of inverse slopes) were replaced by Eq. (6) (the factorization of slopes)
in the derivation of Eq. (16), we would have the following equation instead of Eq. (16),

\[
\frac{(2M_{ij}^4 - M_{ii}^2 M_{jj}^2) + 2M_{ij}^2 \sqrt{(2M_{ij}^4 - M_{ii}^2 M_{jj}^2)}}{M_{jj}^4} = \frac{[(2M_{ik}^4 - M_{ii}^2 M_{kk}^2) + 2M_{ik}^2 \sqrt{(2M_{ik}^4 - M_{ii}^2 M_{kk}^2)]/M_{kk}^4}}{[(2M_{jk}^4 - M_{jj}^2 M_{kk}^2) + 2M_{jk}^2 \sqrt{(2M_{jk}^4 - M_{jj}^2 M_{kk}^2)]/M_{kk}^4}}
\]

(63)

Applying this equation to the $1^1S_0$, $1^3S_1$ and $1^3P_2$ multiplets we would extract the masses of $B_c$, $B'_c$ and $B''_c$ which are also shown in Table 5.

In Ref. [22], under the approximation that mesons in the light quark sector have the common
Regge slopes, a 14th power meson mass relation,

\[ (M_{ss}^2 - M_{nn}^2)(M_{cc}^2 M_{nb}^2 (M_{cs}^2 - M_{ch}^2) + M_{bb}^2 M_{cn}^2 (M_{sb}^2 - M_{nb}^2)) - M_{nn}^2 (M_{cc}^2 + M_{bb}^2) (M_{cs}^2 - M_{ch}^2) (M_{sb}^2 - M_{nb}^2) \]

\[ \times [(M_{ss}^2 - M_{nn}^2)(M_{nb}^2 (M_{cs}^2 - M_{ch}^2) + M_{cn}^2 (M_{sb}^2 - M_{nb}^2)) - 2M_{nn}^2 (M_{cs}^2 - M_{ch}^2) (M_{sb}^2 - M_{nb}^2)] \]

\[ = 4M_{bc}^2 (M_{cs}^2 - M_{cn}^2)(M_{sb}^2 - M_{nb}^2)(M_{cn}^2M_{ss}^2 - M_{cs}^2M_{nn}^2)(M_{nb}^2M_{ss}^2 - M_{sb}^2M_{nn}^2), \]

(64)

was derived to predict the mass of \( B_c^* \) with the value \( M_{B_c^*} = 6.285 \text{ GeV} \). The results of applying Eq. (63) with the existing experimental data \[15\] for the \( 1^1S_0, 1^3S_1 \) and \( 1^3P_2 \) multiplets to extract the masses of \( B_c, B_c^* \) and \( B_{c2}^* \) are also shown in Table 5.

\[ \text{Table 5. The masses of } B_c, B_c^*, \text{ and } B_{c2}^* \text{ (in units of GeV).} \]

| States \((N^{2S+1}L_J)\) | Present work | Eq. [63] | Eq. [64] | Exp. | [21] | [69] | [70] | [71] | [72] |
|-------------------------|-------------|--------|--------|-----|-----|-----|-----|-----|-----|
| \( B_c (1^1S_0) \)     | 6.264       | 6.404  | 6.142  | 6.276\(^a\)  | 6.263 | 6.270 | 6.253 | 6.264 | 6.247 |
| \( B_c^* (1^3S_1) \)   | 6.356       | 6.502  | 6.292  | 6.354 | 6.332 | 6.317 | 6.337 | 6.308 |
| \( B_{c2}^* (1^3P_2) \)| 6.814       | 6.940  | 6.767  | 6.781 | 6.762 | 6.743 | 6.747 | 6.773 |

\(^a\)The CDF Collaboration confirms their earlier report \[67\] with higher statistical samples with a significance greater than 8\( \sigma \) \[68\].

2. Masses of the pure \( s\bar{s} \) states

The masses of the pure \( s\bar{s} \) states cannot be directly measured experimentally because of the usual mixing of the pure isoscalar \( n\bar{n} \) and \( s\bar{s} \) states. However, the comparison of the mass of the pure \( s\bar{s} \) state with that of the physical state can help us to understand the mixing of the two isoscalar states of a meson nonet.

The masses of the pure \( s\bar{s} \) states can be calculated from Eq. (16). When \( i = n, j = s, k = b \) or \( c \), inserting the corresponding masses into Eq. (16), the masses of \( s\bar{s} \) for the \( 1^1S_0, 1^3S_1 \) and \( 1^3P_2 \) multiplets are extracted and shown in Table 6.
Table 6. The masses of the pure $s\bar{s}$ states in pseudoscalar, vector and tensor meson multiplets given by Eqs. (16) and (65) (in units of GeV).

| $N^{2S+1L_J}$ | Eq. (16) i,j,k=n,s,c | Eq. (16) i,j,k=n,s,b | Eq. (65) Q=c | Eq. (65) Q=b |
|---------------|----------------------|----------------------|--------------|--------------|
| $1^{1}S_{0}$  | 0.697                | 0.698                | 0.761 or 0.157 | 0.927 or 0.147 |
| $1^{3}S_{1}$  | 1.009                | 1.006                | 0.891 or 1.079 | 0.841 or 1.145 |
| $1^{3}P_{2}$  | 1.546                | 1.544                | 1.492 or 1.582 | 1.423 or 1.627 |

In Ref. [22], under the approximation that mesons in the light quark sector have the common Regge slopes, two 6th power meson mass relations were derived to predict the masses of $c\bar{c}$ and $b\bar{b}$ meson states, respectively. Those two 6th power meson mass relations can be written as follows,

\[
(M_{s\bar{s}}M_{nQ}^2 - M_{hn}^2M_{nQ}^2)(M_{\bar{s}s}^2 - M_{n\bar{n}}^2) + M_{QQ}^2(M_{sQ}^2 - M_{nQ}^2)(M_{\bar{s}s}^2 - M_{n\bar{n}}^2)
\]

\[
= 4(M_{s\bar{s}}^2M_{nQ}^2 - M_{hn}^2M_{nQ}^2)(M_{sQ}^2 - M_{nQ}^2),
\]

(65)

where $Q$ denotes $c$ or $b$. The results of applying Eq. (65) for the $1^{1}S_{0}$, $1^{3}S_{1}$ and $1^{3}P_{2}$ multiplets to extract the masses of the $s\bar{s}$ states are also shown in Table 6.

From Table 5, one can see that the masses of $B_c$, $B_c^*$ and $B_c^{*2}$ given by Eq. (63) are bigger than those given in Refs. [21, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80]. The mass of the $B_c$ meson given by Eq. (16) (present work) is better than those given by Eqs. (63) and (64) comparing with experimental data. The masses of $B_c$, $B_c^*$ and $B_c^{*2}$ given by Eq. (16) (present work) are in reasonable agreement with those given in Refs. [21, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80]. From Table 6, one can see that the masses of the pure $s\bar{s}$ state in the same multiplet given by Eq. (16) are approximately the same when we choose $k = c$ and $k = b$ and they all satisfy the mass ranges shown in Table 3-1 which are given by the linear mass inequality (26) and quadratic mass inequality (29). However, the masses of the pure $s\bar{s}$ states given by Eq. (65) do not satisfy these constrains.

As mentioned above, Eq. (65) was derived under the approximation that mesons in the light quark sector have the common Regge slopes and was applied for predicting the masses of charmonium and bottomonium [22]. Obviously, Eq. (65) may be limited by this approximation while predicting the masses of light hadrons. Equation (64) was extracted under the same arguments on which Eq. (65) is based [22]. When $i = n$, $j = s$, and $k = Q$, Eq. (16) can be reduced to Eq. (65) if we choose $\alpha_{s\bar{s}} = 1$. Furthermore, with Eq. (16) one needs less meson states than those in the case of Eq. (64) to predict the masses of $\bar{b}c$ states. Therefore, Eq. (16) can properly describe the present meson spectroscopy [15].
C. Doubly charmed baryon \( \Xi_{cc}^+(3520) \)

The doubly charmed baryon \( \Xi_{cc}^+(3520) \) (ccd) was first reported in the charged decay mode \( \Xi_{cc}^+ \rightarrow \Lambda_c^+ K^- \pi^+ \) (SELEX 2002) and confirmed in the decay mode \( \Xi_{cc}^+ \rightarrow pD^+ K^- \) (SELEX 2005) \[3\]. These reports were adopted by the Particle Data Group \[15\] with the average mass \( 3518.9 \pm 0.9 \) MeV. However, the \( J^P \) number has not been determined experimentally. Moreover, it has not been confirmed by other experiments (notably by BABAR \[12\], BELLE \[13\] and FOCUS \[14\]), even though they have \( O(10) \) (FOCUS) and \( O(100) \) (BABAR, BELLE) more reconstructed charm baryons than SELEX. This experimental puzzle raised many theoretical discussions \[81, 82, 85\]. It was suggested that \( \Xi_{cc}^+ \) should be the ground state \((L = 0)\) with \( J^P = \frac{1}{2}^+ \) or \( \frac{3}{2}^+ \) due to its mass \[81, 82, 85\].

Now we will see whether the state \( \Xi_{cc}^+ \) could be assigned as a \( \frac{3}{2}^+ \) doubly charmed baryon. Let us first assume that \( \Xi_{cc}^+ \) belongs to the \( \frac{3}{2}^+ \) multiplet. When \( j = c, i = n, \) and \( q = n \), from Eq. \[13\], we have

\[
\frac{\alpha'_{\Sigma^*_c}}{\alpha'_{\Delta}} = \frac{1}{2M_{\Xi_{cc}^+(3520)}^2} \times \left[ (4M_{\Sigma^*_c}^2 - M_\Delta^2 - M_{\Xi_{cc}^+(3520)}^2) + \sqrt{(4M_{\Sigma^*_c}^2 - M_\Delta^2 - M_{\Xi_{cc}^+(3520)}^2)^2 - 4M_\Delta^2 M_{\Xi_{cc}^+(3520)}^2} \right],
\]

\[
\frac{\alpha'_{\Sigma^*_c}}{\alpha'_{\Delta}} = \frac{1}{4M_{\Sigma^*_c}^2} \times \left[ (4M_{\Sigma^*_c}^2 + M_\Delta^2 - M_{\Xi_{cc}^+(3520)}^2) + \sqrt{(4M_{\Sigma^*_c}^2 + M_\Delta^2 - M_{\Xi_{cc}^+(3520)}^2)^2 - 4M_\Delta^2 M_{\Xi_{cc}^+(3520)}^2} \right].
\]

When \( j = c, i = s, \) and \( q = n \), from Eq. \[13\], we have

\[
\frac{\alpha'_{\Xi^*}}{\alpha'_{\Delta}} = \frac{1}{2M_{\Xi_{cc}^+(3520)}^2} \times \left[ (4M_{\Xi^*}^2 - M_\Sigma^* - M_{\Xi_{cc}^+(3520)}^2) + \sqrt{(4M_{\Xi^*}^2 - M_\Sigma^* - M_{\Xi_{cc}^+(3520)}^2)^2 - 4M_\Sigma^* M_{\Xi_{cc}^+(3520)}^2} \right],
\]

From Eq. \[46\], we have

\[
\frac{1}{\alpha'_{\Delta}} + \frac{2}{\alpha'_{\Omega}} = \frac{3}{\alpha'_{\Sigma^*_c}}.
\]

Inserting the masses of \( \Delta, \Sigma^*_c \) and \( \Xi_{cc}^+(3520) \) into Eq. \[67\], we have

\[
\alpha'_{\Sigma^*_c} = 0.867 \alpha'_{\Delta}.
\]

Inserting the masses of \( \Delta, \Sigma^*_c, \Xi^*, \Xi_c^* \) and \( \Xi_{cc}^+(3520) \) into Eqs. \[66\] and \[68\], with the aid of Eq. \[69\], we have

\[
\alpha'_{\Omega} = 0.860 \alpha'_{\Delta}.
\]

Therefore, \( \alpha'_{\Omega} \lesssim \alpha'_{\Sigma^*_c} \). This does not agree with the usual belief that the slopes of charmed baryons should be much smaller than the slopes of light noncharmed baryons. We have calculated
the numerical results of \( \frac{\alpha'}{\alpha_{cc}^*} \) and find that it increases with the mass increase of \( \Xi_{cc}^* \). Therefore, the mass of \( \Xi_{cc}^* \) should be much bigger than the mass of \( \Xi_{cc}^+ (3520) \). In other words, the mass of \( \Xi_{cc}^+ (3520) \) is too small to be assigned as the \( \frac{3}{2}^+ \) doubly charmed baryons.

According to the quark model, the lowest lying baryon states should be the ground states \( (L = 0) \) including the \( J = \frac{1}{2}^+ \) and \( J = \frac{3}{2}^+ \) doublets. In the above discussion, we have manifested that the mass of \( \Xi_{cc}^+ (3520) \) is too small to be assigned as the \( \frac{3}{2}^+ \) doubly charmed baryons in Regge phenomenology. Therefore, we can conclude that \( \Xi_{cc}^+ (3520) \) should be the ground state with its \( J^P \) as \( \frac{1}{2}^+ \). This assignment coincides with the fact that \( \Xi_{cc}^+ (3520) \) is observed to decay only weakly \( [3] \) (if the \( J^P \) of \( \Xi_{cc}^+ (3520) \) were \( \frac{3}{2}^+ \), it should decay electromagnetically \( [81] \)).

Inserting the masses of \( \Sigma, \Xi, \Sigma_c, \Omega_c \) and \( \Xi_{cc}^+ (3520) \) into Eq. (60), we can get the mass of \( \Omega_{cc} \),

\[
M_{\Omega_{cc}} = 3650.4 \pm 6.3 \text{GeV},
\]

where the uncertainty comes from the errors of the input data. Comparison of the masses of \( \Xi_{cc} \) and \( \Omega_{cc} \) extracted in the present work and those given in other references is shown in Table 7.

D. Parameters of Regge trajectories for the \( \frac{3}{2}^+ \) \( SU(4) \) multiplet

In Ref. \([21]\), the parameters of Regge trajectories for different meson multiplets and the masses of the meson states lying on those Regge trajectories were estimated. In this section, we will first extract the masses of the \( \frac{3}{2}^+ \) \( SU(4) \) baryons absent from the baryon summary table so far. And then, with all the \( \frac{3}{2}^+ \) \( SU(4) \) baryon masses and the value of \( \alpha_{\Delta} \), we will calculate all the parameters (Regge slopes and intercepts) for the \( \frac{3}{2}^+ \) baryon trajectories. After that, we will estimate the masses of the orbital excited baryons lying on these Regge trajectories.

All the masses of \( \frac{3}{2}^+ \) light baryons and charmed baryons are known experimentally. We need to know one of the masses of the baryons \( \Xi_{cc}^*, \Omega_{cc}^* \) and \( \Omega_{ccc} \) to calculate the masses of the other two states using the quadratic mass equalities \([52]\). First, we apply Eq. (18) to extract the mass of \( \Xi_{cc}^*, \Omega_{cc}^* \) or \( \Omega_{ccc}^* \). When \( i = n, j = c, \) and \( q = s \), we could insert the masses of \( \Delta, \Sigma^*, \Xi^*, \Sigma_c^* \) and \( \Xi_{cc}^* \) into the relation (18) to calculate \( M_{\Xi_{cc}} \). When \( i = s, j = c, \) and \( q = s \), we could insert the masses of \( \Sigma^*, \Xi^*, \Omega^*, \Xi_{ccc}^* \) and \( \Omega_{ccc}^* \) into the relation (18) to calculate \( M_{\Omega_{ccc}} \). However, we find that the numerical results of \( M_{\Xi_{cc}} \) and \( M_{\Omega_{ccc}} \) are very sensitive to the errors of the light baryon masses. Therefore, another way is needed to calculate the mass of \( \Xi_{cc}^* \) or \( \Omega_{cc}^* \). In Sec. III C, \( \Xi_{cc}^+ (3520) \) was assigned as the ground \( \frac{1}{2}^+ \) doubly charmed baryon. This may open a window to extract the masses of \( \frac{3}{2}^+ \) doubly charmed baryons.
Table 7. The masses of doubly and triply charmed baryons (in units of MeV). The numbers in boldface are the experimental values taken as the input.

|       | Ξ_{cc}  | Ω_{cc}  | Ξ^*_{cc} | Ω^*_{cc} | Ω_{ccc}  |
|-------|---------|---------|----------|----------|----------|
| Pre.  | 3518.9\pm0.9 | 3650.4\pm6.3 | 3684.4\pm4.4 | 3808.4\pm4.3 | 4818.9\pm6.8 |
| [23]  | 3610 \pm 3  | 3804 \pm 8  | 3735 \pm 17 | 3850 \pm 25 | 4930 \pm 45 |
| [83]  | 3511      | 3664      | 3630      | 3764      | 4747      |
| [84]  | 3524      | 3524      | 3548      | 3548      | 4632      |
| [85]  | 3510      | 3719      | 3548      | 3746      | 4803      |
| [86]  | 3642      | 3732      | 3723      | 3765      | 4473      |
| [87]  | 3676      | 3815      | 3753      | 3876      | 4965      |
| [88]  | 3635      | 3800      | 3695 \pm 60 | 3840 \pm 60 | 4925 \pm 90 |
| [89]  | 3549\pm13\pm19\pm92 | 3663\pm11\pm17\pm95 | 3641\pm18\pm8\pm95 | 3734\pm14\pm8\pm97 |
| [90]  | 3660\pm70  | 3740\pm80  | 3740\pm70  | 3820\pm80  |
| [91]  | 3620      | 3778      | 3727      | 3872      |
| [92]  | 3520      | 3619      | 3630      | 3721      |
| [93]  | 3478      | 3594      | 3610      | 3730      |
| [94]  |          | 3737      | 3797      | 4787      |
| [95]  | 3550\pm80  | 3650\pm80  |          | 4760\pm60  |
| [96]  |          |          |          |          | 4790      |
| [97]  |          |          |          |          |          |

The first-order GMO formula for the baryon octet,

\[ 2(M_N + M_\Xi) = (3M_\Lambda + M_\Sigma), \]  

is usually generalized to charmed cases by replacing s-quark with c-quark,

\[ 2(M_N + M_{\Xi_{cc}}) = 3M_{\Lambda_c} + M_{\Sigma_c}. \]  

The quadratic form of Eq. (71) is

\[ 2(M_N^2 + M_{\Xi_{cc}}^2) = 3M_{\Lambda_c}^2 + M_{\Sigma_c}^2. \]  

However, the existence of high-order breaking effects in Eqs. (71) and (72) is obvious \cite{23}. We use
of the input masses of $\Delta(1950)$ and $\Delta$, we have all the Regge slopes of (13). Then, with these masses and the obtained Regge slopes, we have all the Regge intercepts of $\alpha$ tripled charmed baryons predicted by us agree well with those given in most other references. The trajectories can be calculated. The Regge intercepts and the Regge slopes of the $3^+$ baryons lying on the trajectories are shown in Table 8. The masses of light baryons, charmed baryons, and doubly and triply charmed baryons are known. With these masses and the value of $\alpha' = 2/(M_{\Delta(1950)}^2 - M_{\Delta}^2) = 0.9022 \pm 0.0285$ GeV$^{-2}$ (where the uncertainty comes from the errors of the input masses of $\Delta(1950)$ and $\Delta$), we have all the Regge slopes of $3^+$ trajectories from Eq. (13). Then, with these masses and the obtained Regge slopes, we have all the Regge intercepts of $3^+$ trajectories from Eq. (1).

From Eq. (1), one has

$$M_{J+2} = \sqrt{M_J^2 + \frac{2}{\alpha'}}.$$  \hspace{1cm} (77)

Then, using this equation, the masses of the orbital excited baryons ($J^P = \frac{3}{2}^+, \frac{5}{2}^+$) lying on the $3^+$ trajectories can be calculated. The Regge intercepts and the Regge slopes of the $3^+$ trajectories are shown in Table 8. The masses of light baryons, charmed baryons, and doubly and triply charmed baryons lying on the $3^+$ trajectories are shown in Tables 9-1, 9-2 and 9-3, respectively.

The masses of $\Xi_{cc}^*$, $\Omega_{cc}$ and $\Omega_{ccc}$ extracted in the present work and those given in other references are also shown in Table 7. From Table 7, we can see that the masses of $\frac{1}{2}^+$ and $\frac{3}{2}^+$ doubly and triply charmed baryons predicted by us agree well with those given in most other references. The...
Table 8. The Regge slopes (in units of $GeV^{-2}$) and the Regge intercepts of the $^{3/2}_2$ trajectories.

|   | $\Delta$ | $\Sigma^*$ | $\Xi^*$ | $\Omega$ | $\Sigma^*_c$ | $\Xi^*_c$ | $\Omega^*_c$ | $\Xi^*_{cc}$ | $\Omega^*_{cc}$ | $\Omega^*_{ccc}$ |
|---|-----------|------------|---------|---------|-------------|-----------|-------------|-------------|-------------|---------------|
| $\alpha'$ | 0.902     | 0.862      | 0.825   | 0.791   | 0.644       | 0.623     | 0.604       | 0.501       | 0.488       | 0.410         |
|       | $\pm 0.029$ | $\pm 0.036$ | $\pm 0.042$ | $\pm 0.047$ | $\pm 0.023$ | $\pm 0.026$ | $\pm 0.029$ | $\pm 0.019$ | $\pm 0.021$ | $\pm 0.016$   |
| $a(0)$ | 0.131     | -0.151     | -0.432  | -0.713  | -2.583      | -2.864    | -3.145      | -5.296      | -5.577      | -8.009        |
|       | $\pm 0.046$ | $\pm 0.074$ | $\pm 0.102$ | $\pm 0.133$ | $\pm 0.147$ | $\pm 0.174$ | $\pm 0.203$ | $\pm 0.249$ | $\pm 0.276$ | $\pm 0.351$   |

Predictions in Ref. [23] are bigger than ours because of the approximation adopted there that baryons in the light quark sector have common Regge slopes. The mass splitting obtained in the framework of nonrelativistic effective field theories of QCD, $M_{\Xi_{cc}} - M_{\Xi_{cc}} = 120 \pm 40 MeV$ (see Ref. [98] and references therein), agrees with our present results shown Table 7.

E. Parameters of Regge trajectories for the $^{1/2}_2 SU(4)$ multiplet

Up to now, all the masses of ground $^{1/2}_2 SU(4)$ baryons are known. We will determine the Regge slopes and intercepts of the $^{1/2}_2 SU(4)$ multiplet and give predictions for masses of the $^{5/2}_2$ and $^{9/2}_2$ baryon states lying on these Regge trajectories.

Recently, the spin-parity of the $\Lambda^+_c(2880)$ baryon was determined by experiment. $\Lambda^+_c(2880)$ was observed by CLEO in the $\Lambda_c^+\pi^-\pi^+$ mode [4] and then confirmed by BABAR in the $D^0\rho$ mode recently [6]. From the analysis of the angular distribution in its $\Sigma_c(2455)\pi$ decays and the small ratio, $\Gamma_{\Sigma(2520)\pi}/\Gamma_{\Sigma(2455)\pi} \approx 0.23$, measured by BELLE it is concluded that the $J^P$ of $\Lambda^+_c(2880)$ is $^{5/2}_2$ [5]. This spin-parity assignment is in agreement with the theoretical investigation that $\Lambda^+_c(2880)$ is the orbital ($L = 2$) excitation of $\Lambda^+_c$ [91, 105]. Therefore, $\Lambda^+_c(2880)$ and $\Lambda^+_c$ lie on the common Regge Trajectory. We can have the Regge slope of $\Lambda^+_c$ from Eq. (11),

$$\alpha'_{\Lambda_c} = \frac{\frac{5}{2} - \frac{1}{2}}{M^2_{\Lambda_c(2880)} - M^2_{\Lambda^+_c}} = 0.650 \pm 0.005 \text{ GeV}^{-2}. \quad (78)$$

From Eq. (11), we also have

$$\alpha'_{\Lambda} = \frac{2}{M^2_{\Lambda(1820)} - M^2_{\Lambda}} = 0.967 \pm 0.009 \text{ GeV}^{-2}, \quad (79)$$

$$\alpha'_{N} = \frac{2}{M^2_{N(1680)} - M^2_{N}} = 1.022 \pm 0.009 \text{ GeV}^{-2},$$
Table 9-1. The masses of the light baryons lying on the $^3_2$ trajectories (in units of MeV). The numbers in boldface are the experimental values taken as the input.

|       | $M_\Delta$ |       | $M_\Sigma^*$ |       |
|-------|------------|-------|--------------|-------|
|       | J=3/2     | J=7/2 | J=11/2       | J=3/2 | J=7/2 | J=11/2 |
| Pre.  | 1232±1    | 1932.5±17.5 | 2440±28 | 1383.9±2.3 | 2058±22 | 2560±36 |
| Exp.  | 1232±1    | 1915~1950 | 2300~2500 | 1384.6±2.6 | 2015~2040 |
| [86]  | 1261      | 1951   | 2442        | 1411  | 2027  |
| [99]  | 1232      | 1921   | 2175        |       |       |
| [100] | 1232      | 1950   | 2467        | 1394  | 2056  |
| [101] | 1290      | 1954   |             | 1377  | 2029  |
| [102] | 1232.9±1.2| 1923.3±0.5|          |       |       |
| [103] | 1230      | 1940   | 2450        | 1370  | 2060  |
| [104] | 1240      | 1915   |             | 1390  | 2015  |

|       | $M_\Xi^*$ |       | $M_\Omega$ |       |
|-------|-----------|-------|------------|-------|
|       | J=3/2     | J=7/2 | J=11/2     | J=3/2 | J=7/2 | J=11/2 |
| Pre.  | 1530.2±1.9| 2183±27 | 2681±45 | 1672.45±0.29 | 2308±32 | 2802±54 |
| Exp.  | 1533.4±2.1|       |           | 1672.45±0.29 |       |       |
| [86]  | 1539      | 2169  |           | 1636  | 2292  |
| [99]  |           |       |           |       |       |
| [100] | 1540      | 2157  |           | 1672  |       |
| [101] | 1502      | 2142  |           | 1665  | 2293  |
| [102] |           |       |           |       |       |
| [103] | 1505      | 2180  |           | 1635  | 2295  |
| [104] | 1530      |       |           | 1675  |       |

We assume that $\alpha'_\Sigma = \alpha'_\Sigma^*$, $\alpha'_\Xi = \alpha'_\Xi^*$, $\alpha'_\Sigma_c = \alpha'_\Sigma^*_c$, $\alpha'_\Xi_c = \alpha'_\Xi^*_c$, $\alpha'_\Omega_c = \alpha'_\Omega^*_c$, $\alpha'_\Xi_{cc} = \alpha'_\Xi^*_{cc}$, and $\alpha'_\Omega_{cc} = \alpha'_\Omega^*_{cc}$. Although the slopes of a heavy baryon containing a scalar diquark and that containing an axial-vector diquark are different, we assume that $\gamma_s$ for the heavy baryons containing scalar diquarks is approximately the same as $\gamma_s$ for heavy baryons containing axial-vector diquarks, i.e., $\frac{1}{\alpha'_\Xi} - \frac{1}{\alpha_{\Lambda_c}} = \frac{1}{\alpha'_\Xi^*} - \frac{1}{\alpha_{\Sigma_c}}$. Then, all the Regge slopes of $^1_2 SU(4)$ baryons are known and shown in Table 10.
Table 9-2. The masses of the charmed baryons lying on the $\frac{3}{2}^+$ trajectories (in units of MeV). The numbers in boldface are the experimental values taken as the input.

|       | $M_{\Xi^*_c}$ | $M_{\Xi^*_c}$ | $M_{\Omega^*_c}$ |
|-------|----------------|----------------|------------------|
| $J=3/2$ | 2518.0±1.9 | 3073±18 | 3543±30 |
| $J=7/2$ | 2646.4±1.6 | 3196±22 | 3664±37 | 2774.1±5.5 | 3318±28 | 3784±46 |

Table 9-3. The masses of the doubly and triply charmed baryons lying on the $\frac{3}{2}^+$ trajectories (in units of MeV).

|       | $M_{\Xi^{*}_{cc}}$ | $M_{\Omega^{*}_{cc}}$ | $M_{\Omega^{*}_{ccc}}$ |
|-------|--------------------|------------------------|------------------------|
| $J=3/2$ | 3684.4±4.4 | 4313±23 | 4765±39 |
| $J=7/2$ | 4192±19 | 4644±32 | 5302±21 |
| $J=11/2$ | 4965 | 5744±34 |

Table 10. The Regge intercepts and Regge slopes of the $1^+$ trajectories.

|       | $N$ | $\Lambda$ | $\Sigma$ | $\Xi$ | $\Lambda_c$ | $\Sigma_c$ | $\Xi_c$ | $\Xi'_c$ | $\Omega_c$ | $\Xi_{cc}$ | $\Omega_{cc}$ |
|-------|-----|-----------|----------|------|------------|----------|--------|---------|----------|----------|--------------|
| $a(0)$ | -0.401 | -0.704 | -0.727 | -0.933 | -2.900 | -3.377 | -3.337 | -3.638 | -3.892 | -5.699 | -6.002 |
|       | ±0.010 | ±0.011 | ±0.059 | ±0.082 | ±0.003 | ±0.137 | ±0.043 | ±0.184 | ±0.217 | ±0.228 | ±0.291 |
| $\alpha'$ | 1.022 | 0.967 | 0.862 | 0.825 | 0.650 | 0.644 | 0.629 | 0.623 | 0.604 | 0.501 | 0.488 |
|       | ±0.009 | ±0.009 | ±0.036 | ±0.042 | ±0.005 | ±0.022 | ±0.006 | ±0.026 | ±0.029 | ±0.018 | ±0.020 |
Table 11-1. The masses of the light baryons lying on the $\frac{1}{2}^+$ trajectories (in units of MeV). The numbers in boldface are the experimental values taken as the input.

|       | $M_N$ |       | $M_A$ |       | $M_{\Lambda}$ |       | $M_{\Sigma}$ |       | $M_{\Xi}$ |       | $M_{\bar{\Xi}}$ |       | $M_{\Omega}$ |       |
|-------|-------|-------|-------|-------|----------------|-------|-------------|-------|----------|-------|----------------|-------|-------------|-------|
|       | $J=1/2$ | $J=5/2$ | $J=9/2$ | $J=1/2$ | $J=5/2$ | $J=9/2$ | $J=1/2$ | $J=5/2$ | $J=9/2$ | $J=1/2$ | $J=5/2$ | $J=9/2$ | $J=1/2$ | $J=5/2$ | $J=9/2$ |
| Pre.  | 938.92 | 1685 | 2190 | 1115.883 | 1820 | 2319 | 1193.17 | 1945 | 2463 | 1318.07 | 2048 | 2566 |
| ±0.65 | ±5 | ±8.0 | ±0.006 | ±5 | ±7.8 | ±4.11 | ±27 | ±41 | ±4.31 | ±33 | ±50 |
| Exp.  | 938.92 | 1680 | 2200 | 1115.683 | 1815 | 2340 | 1190.17 | 1900 | 2460 | 1318.07 | 2025 |          |
| ±0.65 | ~±1690 | ~±2300 | ±0.006 | ~±1825 | ~±2370 | ±4.11 | ~±1935 | ±4.31 | ~±5 |        |

Table 11-2. The masses of the charmed baryons lying on the $\frac{1}{2}^+$ trajectories (in units of MeV). The numbers in boldface are the experimental values taken as the input.

|       | $M_N$ |       | $M_A$ |       | $M_{\Lambda}$ |       | $M_{\Sigma}$ |       | $M_{\Xi}$ |       | $M_{\bar{\Xi}}$ |       | $M_{\Omega}$ |       |
|-------|-------|-------|-------|-------|----------------|-------|-------------|-------|----------|-------|----------------|-------|-------------|-------|
|       | $J=1/2$ | $J=5/2$ | $J=9/2$ | $J=1/2$ | $J=5/2$ | $J=9/2$ | $J=1/2$ | $J=5/2$ | $J=9/2$ | $J=1/2$ | $J=5/2$ | $J=9/2$ | $J=1/2$ | $J=5/2$ | $J=9/2$ |
| Pre.  | 2286.46 | 2881.5 | 3737 | 2453.56 | 3046 | 3529 | 2576.9 | 3138 | 3614 | 2697.5 | 3254 | 3729 |
| ±0.14 | ±0.3 | ±0.61 | ±0.85 | ±18 | ±31 | ±20 | ±7 | ±10 | ±4.2 | ±24 | ±40 | ±2.6 | ±26 | ±44 |
| Exp.  | 2286.46 | 2881.5 | 3737 | 2453.56 | 3046 | 3529 | 2576.9 | 3138 | 3614 | 2697.5 |          |
| ±0.14 | 0.3 | ±0.85 | ±1.2 | ±4.2 | ±2.6 |        |

With the masses and the obtained Regge slopes for the $\frac{1}{2}^+$ baryons, we have all the Regge intercepts of $\frac{1}{2}^+$ trajectories from Eq. (11). Then, using Eq. (77), the masses of orbital excited baryons ($JP = \frac{5}{2}^-, \frac{9}{2}^+$) lying on the $\frac{1}{2}^+$ trajectories can be calculated. The Regge intercepts of the $\frac{1}{2}^+$ trajectories are also shown in Table 10. The masses of light baryons, charmed baryons and doubly charmed baryons lying on the $\frac{1}{2}^+$ trajectories are shown in Tables 11-1, 11-2 and 11-3, respectively.

F. Charm-strange baryons

There are five charm-strange baryons presented in PDG 2006 [15]: $\Xi_c$, $\Xi'_c$, $\Xi^+_c$, $\Xi_c(2790)$ and $\Xi_c(2815)$. $\Xi_c(2790)$ and $\Xi_c(2815)$ were assigned as the first orbital (1P) excitations of $\Xi_c$ with $JP = \frac{1}{2}^-$ and $JP = \frac{3}{2}^-$, respectively.

Recently, $\Xi_c(2980)$ and $\Xi_c(3077)$ were first reported by BELLE [7] and then confirmed by BABAR [8]. BABAR also reported the observation of $\Xi^+_c(3055)$ and $\Xi^+_c(3123)$ [9]. The $JP$ of $\Xi_c(2980)$,
Table 11-3. The masses of the doubly charmed baryons lying on the $\frac{1}{2}^+$ trajectories (in units of MeV).

The numbers in boldface are the experimental values taken as the input.

|          | $M_{\Xi_{cc}}$ |                      | $M_{\Omega_{cc}}$ |
|----------|----------------|----------------------|-------------------|
|          | $J=1/2$        | $J=5/2$              | $J=9/2$           | $J=1/2$        | $J=5/2$              | $J=9/2$           |
| Pre.     | 3518.9±0.9     | 4047±19              | 4514±33           | 3650.4±6.3     | 4174±26              | 4639±41           |
| Exp.     | 3518.9±0.9     |                      |                   |                |                      |                   |
| [87]     | 3676           | 4047                 |                   | 3815           | 4202                 |                   |
| [93]     | 3478           | 4050                 |                   | 3594           |                      |                   |

$\Xi_c(3055)$, $\Xi_c(3077)$ and $\Xi_c(3123)$ have not been measured. The masses of these states imply that they could be the states with the total quark orbital angular momentum $L = 2$. Here we attempt to study which Regge trajectory these states may lie on.

From Table 11-2, it can be seen that the mass of $\Xi_c(3123)$ coincides with the mass of $\Xi'_c(\frac{5}{2}^+)$. Therefore, $\Xi_c(3123)$ probably lies on the Regge trajectory of $\Xi'_c$. In other words, $\Xi_c(3123)$ may be the orbital excited ($J^P = \frac{5}{2}^+$) state of $\Xi'_c$ containing an axial-vector diquark. This assignment is in agreement with Ebert’s assignment in the relativistic quark model [91].

We can also see that both the masses of $\Xi_c(3055)$ and $\Xi_c(3077)$ are near the mass of $\Xi_c(\frac{5}{2}^+)$. The mass of $\Xi_c(2980)$ is lower compared with that of $\Xi_c(\frac{5}{2}^+)$ or $\Xi'_c(\frac{5}{2}^+)$. The above comments can be seen more clearly when combining with the slopes of these baryons.

As mentioned above, the slopes of Regge trajectories decrease with quark mass increase. Therefore, the slope of $\Xi_c$ ($\Xi'_c$, $\Xi^*_c$) is less than the slope of $\Lambda_c$,

$$\alpha^{\prime}_{\Xi_c(\Xi'_c, \Xi^*_c)} < 0.650 \text{ GeV}^{-2}.$$  \hspace{1cm} (80)

Assuming that $\Xi_c(2980)$, $\Xi_c(3055)$, $\Xi_c(3077)$ or $\Xi_c(3123)$ lies on the same Regge trajectory with $\Xi_c(\frac{5}{2}^+)$, respectively, so that the difference between the angular momenta of these baryons with those of $\Xi_c(\frac{5}{2}^+)$ is $\Delta L = 2$, we obtain the values of the Regge slopes for $\Xi_c(\frac{5}{2}^+)$ shown in Table 12.

From the relation (80), Table 10, Table 11-2 and Table 12, we can conclude that: $\Xi_c(2980)$ cannot lie on the Regge trajectory of $\Xi_c$, $\Xi'_c$ and $\Xi^*_c$. ($\Xi_c(2980)$ can be interpreted in the relativistic quark model as the first radial (2S) excitation of the $\Xi_c$ with $J^P = \frac{1}{2}^+$ containing the light axial-vector diquark [91].) Both $\Xi_c(3055)$ and $\Xi'_c(3077)$ can be assigned as the $J^P = \frac{5}{2}^+$ state. $\Xi_c(3123)$ probably lies on the Regge trajectory of $\Xi'_c$. In other words, $\Xi_c(3123)$ may be the orbital excited ($\Delta L=2$) state of $\Xi'_c$ with $J^P = \frac{5}{2}^+$ containing an axial-vector diquark. Further study is needed to determine
Table 12. The values (in units of GeV$^{-2}$) of the Regge slope for $\Xi_c^{(t,s)}$ given from Eq. (11) under the assumption that $\Xi_c(2790)$, $\Xi_c(2815)$, $\Xi_c^+(3055)$ or $\Xi_c(3123)$ lies on the same Regge trajectory with $\Xi_c^{(t,s)}$, respectively.

|        | $\Xi_c(2980)$ | $\Xi_c(3055)$ | $\Xi_c(3077)$ | $\Xi_c(3123)$ |
|--------|---------------|---------------|---------------|---------------|
| $\alpha_{\Xi_c}^{t}$ | 0.728         | 0.619         | 0.591         | 0.547         |
| $\alpha_{\Xi_c}^{s}$ | 0.907         | 0.944         | 0.703         | 0.643         |
| $\alpha_{\Xi_c}^{c}$ | 1.086         | 0.860         | 0.806         | 0.727         |

the $J^P$ of these states more accurately.

IV. DISCUSSION AND CONCLUSION

In this work, under the main assumption that the quasilinear Regge trajectory ansatz is suitable to describe meson spectra and baryon spectra, with the requirements of the additivity of intercepts and inverse slopes, some useful linear mass inequalities, quadratic mass inequalities and quadratic mass equalities are derived for mesons and baryons.

Based on these relations, we have given upper limits and lower limits for some mesons and baryons. The masses of $\bar{b}c$ and $s\bar{s}$ belonging to the pseudoscalar ($1^1S_0$), vector ($1^3S_1$) and tensor ($1^3P_2$) meson multiplets are also extracted. We suggest that the $J^P$ of $\Xi_c^+(3520)$ should be $\frac{1}{2}^+$. The numerical values for the parameters of the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ $SU(4)$ baryon trajectories are extracted and the masses of the orbital excited baryons lying on the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories are estimated. We propose that $\Xi_c(3123)$ may be a candidate for the orbital excited ($\Delta L=2$) state of $\Xi_c'$ with $J^P = \frac{5}{2}^+$ containing an axial-vector diquark. The predictions are in reasonable agreement with the existing experimental data and those suggested in many other different approaches.

In Sec. II C, we showed that the linear mass GMO formula is an inequality in fact and the quadratic mass GMO formula is also an inequality with the sign opposite to the linear case. Encouragingly, the linear meson mass inequalities (26) and the linear baryon mass inequalities (30) are similar to those derived from a general illation in QCD for the ground hadron states [17, 18, 19] (The authors of Ref. [19] also point out that the linear mass inequalities (26) and (30) hold for many potentials, although the linear baryon mass inequality (30) does not hold for some special potentials). In Ref. [18], Nussinov and Lampert showed that the linear meson mass inequality (26)
satisfies the experimental data of the well-established meson multiplets (vector $1^3S_1$, tensor $1^3P_2$, axial-vector $1^3P_1$ and scalar $1^3P_0$) with different flavor combinations of $i$ and $j$, and the linear baryon mass inequality (30) satisfies the experimental data of the baryon octet and the baryon decuplet. They gave the lower limits for the masses of some unobserved mesons and baryons with the linear mass inequalities. In our work, in addition to the lower limits, we also give the upper limits for the masses of hadrons. We can see from Table 3-1, 3-2 and 4 that these limits agree with the existing data. The mass ranges in Table 3 and 4 are narrow (smaller than 0.5 GeV) for hadrons which do not contain $b$-quark. These mass ranges will be useful for the discovery of the unobserved hadron states. When $b$-quark is involved, the mass ranges in Tables 3 and 4 become large (could be as large as 1 to 2 GeV) and consequently, the constraints become weaker. However, since many hadrons containing $b$-quark have not been observed in experiments, these mass ranges may also provide helpful guidance for the discovery of these hadrons.

As far as we know, there is only one work to study the quadratic meson mass inequalities. In Ref. [20], with the current-algebra technique, corrections to the GMO quadratic mass formula due to second-order SU(4) breaking was discussed by Simard and Suzuki. They gave a quadratic mass inequality for pseudoscalar mesons,

$$\frac{1}{2} \left[ M_\pi^2 + \left( \frac{2}{3} M_\eta^2 + \frac{1}{3} M_\eta' \right) \right] + M_{\eta,(1S)}^2 - 2M_D^2 > 0,$$

and two quadratic mass inequalities for vector mesons,

$$\frac{1}{2} (M_\rho^2 + M_\omega^2) + M_{J/\psi,(1S)}^2 - 2M_D^2 < 0,$$

$$M_\phi^2 + M_{J/\psi,(1S)}^2 - 2M_{D}\bar{s} < 0.$$

The sign of the quadratic mass inequality (81) is the same as that of our quadratic mass inequality (29), but the signs of the quadratic mass inequalities (82) and (83) are opposite to that of our quadratic mass inequality (29). The calculations (shown in Table 3-1 and Table 3-2) manifest that the quadratic mass inequalities (29) and (81) do satisfy the present experimental data [15] while the quadratic mass inequalities (82) and (83) do not.

We stress that quadratic baryon mass inequality (31) has not been given before. From Tables 3-1, 3-2 and 4, we can see that the inequalities (26), (29), (30) and (31) agree well with the existing experimental data [15]. These inequalities (26), (29), (30) and (31) indicate the existence of higher-order breaking effects.
For the Regge slopes of $\frac{3}{2}^+$ SU(4) baryons, from Table 8, we can see that $\alpha_\Delta' > \alpha_{\Xi^*}' > \alpha_{\Omega}' > \alpha_{\Omega^*}' > \alpha_{\Xi_t}' > \alpha_{\Xi^*_t}' > \alpha_{\Omega_{ccc}^*}'$ and $a_\Delta(0) > a_{\Sigma^*}(0) > a_{\Xi^*}(0) > a_{\Omega}(0) > a_{\Sigma^*_t}(0) > a_{\Omega_t}(0) > a_{\Xi^*_t}(0) > a_{\Xi_{ccc}^*}(0) > a_{\Omega_{ccc}}(0)$. These inequalities coincide with the expectation that the slopes of Regge trajectories decrease with quark mass increase (flavor dependent).

From Table 2, we can see that the values of $\delta_{ij}^m$ are very sensitive to quark flavors $i$ and $j$. For the same $i$ and $j$, $\delta_{ij}^m$ are approximately a constant (only a little different among different multiplets). This character may be used to predict meson masses approximately in some cases. The calculations (Table 2) show that $\delta_{ns} < \delta_{sc} < \delta_{nc} < \delta_{cb} < \delta_{sb} < \delta_{nb}$. For the light mesons and baryons, $\delta_{ns}$ is close to zero. Letting $\delta \to 0$, one can get the usual Gell-Mann–Okubo quadratic relations, namely the first order of Gell-Mann–Okubo relations. For the heavy mesons or baryons, $\delta_{Qq}$ are large. In this case, the quadratic mass inequalities are far from equalities. These features imply that the higher-order breaking effects arise with the quark mass increase.

To the second order, for baryons, as shown by Okubo long ago [34], both the well known mass relation for the baryon octet (Eq. (70)) and the equal spacing rule for the baryon decuplet ($M_\Omega - M_{\Xi^*} = M_{\Xi^*} - M_{\Sigma^*} = M_{\Sigma^*} - M_\Delta$) do not hold. Only one relation remains,

$$M_\Omega - M_\Delta = 3(M_{\Xi^*} - M_{\Sigma^*}).$$  (84)

This second-order linear mass equation was given by Morpurgo in the relativistic field theory [35] and by Lebed in the chiral perturbation theory [36] and was also given in Refs. [26, 27, 28, 29, 30, 31] mentioned above.

A special equation among the masses of baryons involving only two flavors can be derived by taking $\delta_{ij}^b|_{q=i} = \delta_{ij}^b|_{q=j}$ in Eq. (51),

$$\delta_{ij}^b|_{q=i} = M_{iii}^2 + M_{jjj}^2 - 2M_{ijj}^2 = \delta_{ij}^b|_{q=j} = M_{ijj}^2 + M_{jjj}^2 - 2M_{ijj}^2,$$  (85)

namely,

$$M_{jjj}^2 - M_{iii}^2 = 3(M_{ijj}^2 - M_{ijj}^2).$$  (86)

In the light quark sector, when $i = n, j = s$, for the $\frac{3}{2}^+$ multiplet, we have

$$M_\Omega^2 - M_\Delta^2 = 3(M_{\Xi^*}^2 - M_{\Xi^*_t}^2).$$  (87)

The quadratic equation (87) was also given by Tait in the study of the unification $SO(6,1)$ as a spectrum generating algebra [32].
In the light sector, both the linear mass equation, Eq. (84), and the quadratic mass equation, Eq. (87), can be satisfied by the experimental data. The deviations from both of them are not more than 2%.

However, generally speaking, the linear mass relation and the quadratic mass relation may not be held at the same time. On the other hand, the quadratic mass equation (86) and the linear form of Eq. (86) should give very different mass values for heavy baryons. The masses of the charmed and bottom particles discovered in the near future will numerically test which of them is realized in nature.

Theoretically, we also have some reasons besides the Regge theory to believe that mass formulas for mesons and baryons should take the quadratic form rather than the linear form: 1) The square of the mass operator ($M^2$) is the Casimir invariant of the Poincare group independent of any certain frame [106]; 2) Formulas given by asymptotic chiral symmetry are indeed in quadratic form [107]; 3) In the infinite-momentum frame, formulas between energy eigenvalues of hadrons spontaneously lead to quadratic mass formulas [108]; 4) Analysis on the algebraic approach indeed leads to quadratic mass formulas [32, 109]. It was pointed out that the quadratic mass formula can be approximately written as the relevant linear mass formula when the mass splittings between the hadrons of the formula are small compared with the hadron masses [106, 108].

To sum up, we conclude that quasilinear Regge trajectory and the additivity of intercepts and inverse slopes are indeed suitable to describe meson spectra and baryon spectra at present. The mass relations and the predictions may be useful for the discovery of the unobserved meson and baryon states and the $J^P$ assignment of the meson and baryon states which will be observed in the future.

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