Passing the Einstein–Rosen bridge

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Abstract

We relax the requirement of geodesic completeness of a space-time. Instead, we require test particles trajectories to be smooth only in the physical sector. Test particles trajectories for Einstein–Rosen bridge are proved to be smooth in the physical sector, and particles can freely penetrate the bridge in both directions.

The metric of Einstein–Rosen bridge is [1]

\[ ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta = \frac{u^2}{u^2 + 4M} dt^2 - (u^2 + 4M) du^2 - \frac{1}{4}(u^2 + 4M)(d\theta^2 + \sin^2 \theta d\varphi^2), \tag{1} \]

where \( \alpha = 0, 1, 2, 3; x^0 = t, x^1 = u \) are time and space coordinates, \( x^2 = \theta, x^3 = \varphi \) are angles of the spherical coordinate system, and \( M \) is the Schwarzschild mass. This metric is smooth on the \( t, u \) plane, but degenerate on the line \( u = 0 \). Metric (1) glues smoothly two copies of the external Schwarzschild solution along the line \( u = 0 \) which corresponds to the horizon. One external Schwarzschild solution is isometric to the half plane \( u > 0 \) multiplied by a sphere, and the other to the half plane \( u < 0 \).

The well known problem for the Einstein–Rosen bridge is its geodesic incompleteness at \( u = 0 \). At present, the common point of view is that metric (1) satisfies Einstein equations with nontrivial exotic matter energy–momentum tensor concentrated at \( u = 0 \). This conclusion depends on how we interpret the singularity at \( u = 0 \).

In the present paper, we propose another point of view. A test particle of mass \( m \) moving in the space-time with metric (1) is described by the action

\[ S = \int_\gamma d\tau L(q, \dot{q}) := -m \int_\gamma d\tau \sqrt{g_{\alpha \beta} \dot{q}^\alpha \dot{q}^\beta}, \tag{2} \]

where \( q^\alpha(\tau), \tau \in \mathbb{R}, \) is the world line of a test particle. The action (2) is reparameterization invariant and therefore is a gauge model. A test particle has three physical propagating degrees of freedom corresponding to space coordinates \( q^\mu, \mu = 1, 2, 3 \) (Greek indices from the middle of the alphabet are assumed to take only space values). The time coordinate \( q^0 \) is a gauge degree of freedom, and is not important from the physical standpoint.

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The natural and important requirement to any solution of Einstein equations is its geodesic completeness (maximal extension along geodesics). In this way, we obtain the Kruskal–Szekeres extension [2, 3] of the Schwarzschild solution which describes black and white holes. Here we relax the requirement of geodesic completeness. Instead, we require the trajectories of test particles to be smooth only in the physical sector. We prove that trajectories of test particles for metric (1) are smooth and complete in the physical sector. All singularities are moved to unphysical sector and can be considered as artifacts of the gauge condition. Thus test particles can freely penetrate the Einstein–Rosen bridge in both directions.

Action (2) yields equations of motion (geodesic equations)

\[ \ddot{q}^\alpha = -\Gamma^\alpha_{\beta\gamma} \dot{q}^\beta \dot{q}^\gamma, \]  

where \( \Gamma^\alpha_{\beta\gamma} \) are Christoffel’s symbols, if we choose the canonical parameter \( \tau \) along the worldline.

We consider only radial geodesics for simplicity. Then geodesic equations (3) for the Einstein–Rosen bridge reduce to two equations

\[ \ddot{t} = -\frac{8M}{u(u^2 + 4M)} \dot{u}, \]
\[ \ddot{u} = -\frac{4Mu}{(u^2 + 4M)^2} \dot{t}^2 - \frac{u}{u^2 + 4M} \dot{u}^2. \]

They have two integrals of motion

\[ C_0 = g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = \frac{u^2}{u^2 + 4M} \dot{t}^2 - (u^2 + 4M) \dot{u}^2, \]
\[ C_1 = K_\alpha \dot{x}^\alpha = \frac{u^2}{u^2 + 4M} \dot{t}. \]

The last integral is due to the existence of the Killing vector field \( K = \partial_0 \). Now we can easily obtain the equation defining the form of radial timelike geodesics

\[ \frac{dt}{du} = \pm \frac{u^2 + 4M}{u \sqrt{1 - C \frac{u^2}{u^2 + 4M}}}, \]  

where \( C := C_0/C_1^2 \). The qualitative behavior of timelike geodesics is shown in Fig.1. They are incomplete at \( u = 0 \).

Strictly speaking, the space-time with metric (1) is geodesically incomplete at \( u = 0 \). The requirement that wordlines of test particles be smooth and defined for all \( \tau \in \mathbb{R} \) in the physical sector yields a natural way to identify geodesics on different sides of the line \( u = 0 \), thus making the space-time of Einstein–Rosen bridge geodesically complete. To separate unphysical degree of freedom we use the well known Hamiltonian formulation (see, i.e. [4]). We denote canonically conjugate variable as \( q^\alpha, p_\alpha \). The Hamiltonian action for a test particle can be written in the form

\[ S = \int dt (p_\alpha \dot{q}^\alpha - H) = \int dt \left( p_\mu \dot{q}^\mu - |\dot{q}^0 N| \sqrt{\hat{p}^2 + m^2} + \dot{q}^0 N^\mu p_\mu \right), \]

where \( \hat{p}^2 := \hat{p}_\mu \hat{p}^\mu \).
Figure 1: Qualitative behavior of timelike geodesics, $C > 0$. Null geodesics correspond to $C = 0$. Every geodesic can be moved along $t$ axis.

where $N$ and $N^\mu$ are the lapse and shift functions in the ADM parameterization of the metric $ds^2 = N^2 dt^2 + g_{\mu\nu}(dx^\mu + N^\mu t)(dx^\nu + N^\nu t)$. We also introduced notation $\hat{p}^2 := -\hat{g}^{\mu\nu} p_\mu p_\nu$ for positive definite square of space momenta, where $\hat{g}^{\mu\nu}$ is the three dimensional matrix which is inverse to $g_{\mu\nu}$. We keep in mind two possible signs of the lapse function $N > 0$ and $N < 0$. The zeroes component of momentum is defined by the first class constraint

$$G := \sqrt{\hat{p}^2 + m^2} - \left| p_0 - N^\mu p_\mu / N \right| = 0. \quad (6)$$

The component $q^0$ is a free function to be fixed by the gauge condition.

Consider the case $\dot{q}^0 N > 0$. Then Hamiltonian equations of motion for physical degrees of freedom are

$$\dot{q}^\mu = \dot{q}_0 \frac{\partial H_{\text{eff}}}{\partial p_\mu},$$

$$\dot{p}_\mu = -\dot{q}_0 \frac{\partial H_{\text{eff}}}{\partial q^\mu}. \quad (7)$$

where we introduced the effective Hamiltonian

$$H_{\text{eff}} := N \sqrt{\hat{p}^2 + m^2} - N^\mu p_\mu$$

for physical degrees of freedom. One can easily check that the energy $E := H_{\text{eff}}$ is conserved for arbitrary function $q^0(\tau)$. This means that the function $q^0$ describes the freedom in parameterization of the world line and is to be fixed by a gauge condition.

The lapse function is defined by the metric up to a sign. We choose it to be

$$N = \frac{u}{\sqrt{u^2 + 4M}}. \quad (8)$$

It is positive on the right half plane $u > 0$ and negative on the left $u < 0$. This is important, because if we choose positive lapse function $|N|$ everywhere then it will produce the nontrivial energy-momentum tensor on the right hand side of Einstein’s equations.
proportional to $\delta(u)$ due to discontinuity in the first derivative [5], and the Einstein–Rosen solution (11) is no longer a solution to the vacuum Einstein’s equations. On the other hand, we proved that the Schwarzschild solution in isotropic coordinates describes the gravitational field around point particle [6]. This solution obtained within the Hamiltonian formulation is globally isometric to Einstein–Rosen bridge, and the lapse function does change its sign when crossing the critical sphere corresponding to $u = 0$.

Now we fix the gauge and analyze equations of motion (7) for metric (1). To simplify notation, we put $q_1 := q$ and $p_1 := p$. Then equations (7) take the form

$$\dot{q} = \frac{\dot{q}^0 N p}{(q^2 + 4M)\sqrt{\hat{p}^2 + m^2}}, \tag{9}$$

$$\dot{p} = -\dot{q}^0 \sqrt{\hat{p}^2 + m^2} \partial_q N + \frac{\dot{q}^0 N q p^2}{(q^2 + 4M)^2 \sqrt{\hat{p}^2 + m^2}}, \tag{10}$$

where the lapse function is given by Eq.(8) with $u \to q$, and

$$\hat{p}^2 = \frac{p^2}{q^2 + 4M}. \tag{12}$$

The obtained equations can be solved because of the energy conservation:

$$E = N \sqrt{\hat{p}^2 + m^2} = \text{const.}$$

From here we can find the momentum

$$p = \pm \frac{1}{N} \sqrt{(q^2 + 4M)(E^2 - m^2 N^2)}. \tag{12}$$

Substituting this solution into Eq.(9) yields

$$\dot{q} = \pm \frac{\dot{q}^0 N}{E(q^2 + 4M)} \sqrt{(q^2 + 4M)(E^2 - m^2 N^2)}. \tag{11}$$

It defines the form of geodesics $dt/du = \dot{q}^0/\dot{q}$ as given by Eq.(4) where $C = m^2/E^2$. We see that Lagrangian and Hamiltonian description result in the same geodesic lines as it should be.

Now we fix the gauge $q^0 N = 1$. Then Eq.(11) reduces to

$$\dot{q} = \pm \frac{1}{\sqrt{q^2 + 4M}} \sqrt{1 - C \frac{q^2}{q^2 + 4M}}. \tag{12}$$

This is the final equation. Its solutions are smooth and defined for all $\tau \in \mathbb{R}$. Qualitative behavior of physical trajectories $q(\tau)$ are shown in Fig.2. Trajectories are oscillating for $C > 1$. At the most interesting point $u = q = 0$ Eq.(12) takes the form $\dot{q} = \pm 1/\sqrt{4M}$ and is nonsingular. We see that a test particle freely penetrate the line $u = 0$ if we exclude unphysical degree of freedom. From physical point of view this is all that matters, because we may not bother about the gauge degree of freedom.

In Fig.1 arrows show the direction of increasing $\tau$ which coincides with increasing of proper time. On the left hand side $\dot{q}^0 < 0$, and the observer sees antiparticle (a particle of the same mass but opposite charge [4]).
The singularity in the Einstein–Rosen metric is avoided because it is moved to the unphysical sector due to the gauge condition which can be written in the form

$$q^0 = \int q(\tau')^2 + 4M q(\tau') d\tau'$$.

It is divergent at $q \to 0$ for solutions of Eq. (12).

Physical interpretation of the obtained solution is strange. External observer at a given moment of time $q^0$ sees simultaneously two particles: a particle on the right and an antiparticle on the left. For $C > 1$, they move simultaneously either from $u = 0$ or towards $u = 0$. For $C > 1$, the observer sees the birth of particle and antiparticle at infinite past $q^0 = -\infty$ and annihilation of the pair at infinite future $q^0 = \infty$.

Recently, we proved that gravitational field of a point particle is described by the Schwarzschild solution in isotropic coordinates [6]. This solution is globally isometric to Einstein–Rosen bridge. The results of the present paper are also applicable to this case: test particles can freely penetrate the critical sphere corresponding to $u = 0$. In this sense, the solution is complete and does not require further continuation.

**References**

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