Statefinder diagnostic for Yang-Mills dark energy model

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Abstract

We study the statefinder parameters in the Yang-Mills condensate dark energy models, and find that the evolving trajectories of these models are different from those of other dark energy models. We also define two eigenfunctions of the Yang-Mills condensate dark energy models. The values of these eigenfunctions are quite close to zero if the equation-of-state of the Yang-Mills condensate is not far from $-1$, which can be used to simply differentiate between the Yang-Mills condensate models and other dark energy models.

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I. INTRODUCTION

Physicists and astronomers begin to consider the dark energy cosmology seriously and to explore the nature of dark energy actively since the expansion of our universe is proven to be accelerating at present time by the Type Ia supernovae observations\cite[1]{1}. The analysis of cosmological observations, in particular of the WMAP (Wilkinson Microwave Anisotropy Probe) experiment\cite[2]{2}, indicates that dark energy occupies about 70\% of the total energy of our universe, and dark matter about 26\%. The accelerated expansion of the present universe is attributed to that dark energy is an exotic component with negative pressure, and the simplest candidate for dark energy is the cosmological constant. However, two difficulties arise from this scenarios, namely the fine-tuning problem and the cosmic coincidence problem\cite[3]{3}. So the dynamical models are considered by a number of authors, such as the quintessence, phantom, k-essence, and quintom models\cite[4]{4}. The effective Yang-Mills condensate (YMC) as a kind of candidate for dark energy has been detailed discussed in the Refs.\cite[5, 6, 7]{5, 6, 7}. The effective lagrangian up to 1-loop order is \cite[8, 9]{8, 9}

\[
L_{\text{eff}} = \frac{b}{2} F \ln \left| \frac{F}{\kappa} \right| ,
\]

where \( b = \frac{11}{24\pi^2} N \) for the generic gauge group \( SU(N) \) is the Callan-Symanzik coefficient\cite[10]{10}. \( F = -(1/2) F_{\mu \nu} F^{\mu \nu} \) plays the role of the order parameter of the YMC, \( \kappa \) is the renormalization scale, the only model parameter. The attractive features of this effective lagrangian include the gauge invariance, the Lorentz invariance, the correct trace anomaly, and the asymptotic freedom\cite[8]{8}. The effective YMC was firstly put into the expanding Friedmann-Robertson-Walker (FRW) spacetime to study inflationary expansion \cite[5]{5} and the dark energy \cite[6]{6}. We work in a spatially flat FRW spacetime with a metric

\[
ds^2 = a^2(\tau)(d\tau^2 - \delta_{ij} dx^i dx^j),
\]

where \( \tau = \int (a_0/a) dt \) is the conformal time. For simplicity we study the \( SU(2) \) group. Compared with the scalar field, the YM field is the indispensable cornerstone to particle physics and the gauge bosons have been observed. There is no room for adjusting the form of effective YM lagrangian as it is predicted by quantum corrections according to field theory. In the previous works, we have deeply investigated the 1-order YMC models and found attractive features: a. this dark energy can naturally get the equation-of-state (EOS)
of $w > -1$ and $w < -1$, which is different from the scalar quintessence models; $b$. in the free field models, with the expansion of the universe, the EOS of the YMC naturally turns to the critical state of $w = -1$, consistent to the observations; $c$. the cosmic coincidence problem is naturally avoided in the models; $d$. the EOS of the dark energy can cross $-1$ in the double-field models or coupled models; $e$. the big rip is naturally avoided in the models; $f$. the magnetic component of the YMC naturally decreases to zero with the expansion of the universe.

In the recent paper, the authors have detailed discussed the 2-loop YMC dark energy. In this model, the effective lagrangian is

$$\mathcal{L}_{\text{eff}} = \frac{b}{2} F \left[ \ln \left| \frac{F}{\kappa^2} \right| + \eta \ln \left| \ln \left| \frac{F}{\kappa^2} \right| + \delta \right| \right],$$  

(3)

where $\eta \simeq 0.84$ and the dimensionless constant $\delta$ is a parameter representing higher order corrections. In this 2-loop model, the cosmic coincidence problem is also naturally avoided. This feature is same with the 1-loop models. From the Einstein equation, we can easily find that, in the free field models, the late time attractor exists, which satisfies the relation

$$\beta_c = -\eta \left( \log |\beta_c - 1 + \delta| + \frac{1}{\beta_c - 1 + \delta} \right),$$  

(4)

where $\beta = \ln |F/\kappa^2|$. It is easily found that $w = -1$, when $\beta = \beta_c$. So in these models, the EOS of YMC also naturally turns to the critical state $w = -1$. However, the cosmological constant crossing is also naturally realized, if considering the interaction between the YMC and matter. The discussion in shows that, although the 2-loop models are much more general and complicated than the 1-loop model, the most physics features are not unchanged.

In this paper, we only consider the 1-loop models. As in previous works, we only consider the electric case with $B^2 \equiv 0$. The energy density and pressure of YMC are given by

$$\rho_y = \frac{E^2}{2} (\epsilon + b), \quad p_y = \frac{E^2}{2} \left( \frac{\epsilon}{3} - b \right),$$  

(5)

where the dielectric constant is

$$\epsilon = b \ln \left| \frac{F}{\kappa^2} \right|. \quad \quad (6)$$

and the EOS is obtained

$$w = \frac{p_y}{\rho_y} = \frac{\beta - 3}{3\beta + 3}. \quad \quad (7)$$
where $\beta \equiv \epsilon/b$. At the critical point with the condensate order parameter $F = \kappa^2$, one has $\beta = 0$ and $w = -1$. Around this critical point, $F < \kappa^2$ gives $\beta < 0$ and $w < -1$, and $F > \kappa^2$ gives $\beta > 0$ and $w > -1$. So in the YMC model, EOS of $w > -1$ and $w < -1$ can be naturally realized. When $\beta \gg 1$, the YM field has a state of $w = 1/3$, becoming a radiation component.

In the free field models, the effective YM equations are

$$\partial_{\mu}(a^4\epsilon F^{a\mu\nu}) + f^{abc}A^b_{\mu}(a^4\epsilon F^{c\mu\nu}) = 0,$$

(8)

which reduces to

$$\partial_{\tau}(a^2\epsilon E) = 0.$$

(9)

At the critical point ($\epsilon = 0$), this equation is an identity. When $\epsilon \neq 0$, this equation has an exact solution:

$$\beta e^{-\beta/2} \propto a^{-2},$$

(10)

where the coefficient of proportionality depends on the initial condition. For a fixed initial condition, we can obtain the evolution of the EOS of the YMC by using the YM equation (10). In the previous works, we found the free YMC can be separated into two kinds, the quintessence-like or phantom-like, which only depends on the choice of the initial condition. In order to differentiate between the YMC dark energy models and other models, a sensitive and robust diagnostic for dark energy models is needed. For this purpose a diagnostic proposal that makes use of parameter pair $\{r, s\}$, the so-called “statefinder”, was introduced by Sahni et al. [12]. The statefinder probes the expansion dynamics of the universe through higher derivatives of the expansion factor $\ddot{a}$ and is a natural companion to the deceleration parameter $q$ which depends upon $\ddot{a}$. The statefinder pair $\{r, s\}$ is defined:

$$r \equiv \frac{\ddot{a}}{aH^3}, \quad s \equiv \frac{r - 1}{3(q - 1/2)}.$$

(11)

The statefinder is a “geometrical” diagnostic in the sense that it depends upon the expansion factor and hence upon the metric describing space-time.

Trajectories in the $s-r$ plane corresponding to different cosmological models exhibit qualitatively different behaviors. The spatially flat LCDM (cosmological constant $\Lambda$ with cold dark matter) scenario corresponds to a fixed point in the diagram $\{s, r\} = \{0, 1\}$. Departure of a given dark energy model from this fixed point provides a good way of establishing the
“distance” of this model from LCDM [12, 13]. As demonstrated in Refs. [12, 13, 14] the
statefinder can successfully differentiate between a wide variety of dark energy models in-
cluding the cosmological constant, quintessence, the Chaplygin gas, braneworld models and
interacting dark energy models. We can clearly identify the “distance” from a given dark
energy model to the LCDM scenario by using the $r(s)$ evolution diagram.

The current location of the parameters $s$ and $r$ in these diagrams can be calculated in
models, and on the other hand it can also be extracted from data coming from SNAP (Su-
perNovae Acceleration Probe) type experiments [15]. Therefore, the statefinder diagnostic
combined with future SNAP observations may possibly be used to discriminate between
different dark energy models. In this letter we apply the statefinder diagnostic to YMC
dark energy models.

II. STATEFINDER FOR YMC DARK ENERGY

The statefinder parameters $r$ and $s$ in (11) can be rewritten as

$$r = 1 + \frac{9}{2} w (1 + w) \Omega_y - \frac{3}{2} w' \Omega_y,$$

$$s = 1 + w - \frac{1}{3} \frac{w'}{w},$$

where $w$ is the EOS of the YMC, and $\Omega_y$ is the fractional energy density of YMC. A prime
denotes derivation with respect to the e-folding time $N \equiv \ln a$. From the previous discussion,
we know

$$w = \frac{\beta - 3}{3 \beta + 3}, \text{ and } w' = \beta \frac{dw}{d\beta}. (14)$$

So the statefinder for YMC only depends on the evolution of parameter $\beta$, which can be
exactly determined by the YM equation (10). From the YM equation, we obtain that

$$\beta' = -\frac{4 \beta}{2 + \beta}, \text{ and } w' = \frac{-16 \beta}{3(1 + \beta)^2(2 + \beta)},$$

(15)

and the statefinder parameters become

$$r = \frac{2 + (3 - 4 \Omega_y) \beta + (1 + 2 \Omega_y) \beta^2}{2 + 3 \beta + \beta^2},$$

$$s = \frac{4 \beta (\beta - 2)}{3(\beta^2 - \beta - 6)}.$$

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The deceleration parameter is also obtained

\[ q = \frac{1 + \beta - 3\Omega_y + \beta\Omega_y}{2 + 2\beta}. \tag{18} \]

We first consider the case with \( w > -1 \), quintessence-like, where \( \beta > 0 \) is kept for all time. In the very early universe with \( \beta \gg 1 \) and \( \Omega_y \rightarrow 0 \), we obtain

\[ r \rightarrow 1, \quad s \rightarrow \frac{4}{3}, \quad \text{and} \quad q \rightarrow \frac{1}{2}, \tag{19} \]

which is independently of the initial condition, and obviously different from the SCDM (standard cold dark matter) model with \( (r, s, q) = (1, 1, 1/2) \). In the later stage of the universe with \( \beta \rightarrow 0 \) and \( \Omega_y \rightarrow 1 \), we have

\[ r \rightarrow 1, \quad s \rightarrow 0, \quad \text{and} \quad q \rightarrow -1. \tag{20} \]

The universe approaches an exact de Sitter expansion, which is same with the late stage of the LCDM model. We also notice that the value of \( s \) is infinite when \( \beta = 3 \), where the EOS of the YMC is \( w = 0 \). This is also a character of the YMC models.

If the YMC is phantom-like with \( w < -1 \) and \( \beta < 0 \). In the later stage of the universe, we have \( \beta \rightarrow 0 \) and \( \Omega_y \rightarrow 1 \), which follows that

\[ r \rightarrow 1, \quad s \rightarrow 0, \quad \text{and} \quad q \rightarrow -1. \tag{21} \]

which is same with the quintessence-like case. Here we should point out that, in the very early universe with \( a \rightarrow 0 \), the YM kinetic equation \( (10) \) has no solution for the case with \( w_0 < -1 \), where \( w_0 \) is the present EOS of the YMC. So the free YMC is not applied in the very early universe, where the interaction between the YMC and matter \([7, 11]\), or the phase transition of the YMC must be considered.

The evolution of the parameter \( \beta \) and \( \Omega_y \) can be obtained by Eq.\( (8) \) for a fixed initial condition. In Fig.1, we show the evolution of the statefinder pair \( s, r \), where the initial conditions of YMC are \( w_0 = -1.1, -0.9, -0.8 \), respectively, and the present fraction energy density of YMC is \( \Omega_y = 0.7 \). It can be found that the trajectories of these models never cross the LCDM fixed point. However, with the expansion of the universe, they will approach this fixed point, which is independent of the choice of the initial condition. The only difference for the quintessence-like and phantom-like cases is the direction of the trajectories, when the models approach the fixed point. The coordinate of today’s points
are \((-0.033, 1.113), (0.034, 0.903), (0.070, 0.824)\), respectively, thus the distance from these models to the LCDM can be easily identified in this diagram.

We also plot the evolution trajectories of statefinder pair \(r, q\) in Fig.2.

### III. STATEFINDER OF FIRST ORDER

Cosmological observations show that the EOS of the dark energy is closer to \(-1\). In the YMC dark energy models, from the expression of EOS of YMC in (17), we find that \(w \to -1\) follows that \(|\beta| \ll 1\). So we can Taylor expand the EOS and the statefinder of the YMC with parameter \(\beta\) at the critical state of \(w = -1\). Keeping the first order of the smaller quantity \(\beta\), we can rewrite the EOS of the YMC as

\[
w = -1 + \frac{4}{3} \beta + O(\beta^2), \quad \text{and} \quad w' = -\frac{8}{3} \beta + O(\beta^2). \tag{22}
\]

From the expressions in (16), (17) and (18), we obtain

\[
r = 1 - 2 \Omega_y \beta + O(\beta^2), \quad s = \frac{4}{9} \beta + O(\beta^2) \tag{23}
\]
and the deceleration parameter

\[
q = \left(\frac{1}{2} - \frac{3 \Omega_y}{2}\right) + 2 \Omega_y \beta + O(\beta^2). \tag{24}
\]

These functions only depend on the quantities \(\beta\) and \(\Omega_y\), which are all determined by the initial condition, and the initial condition of the YMC directly relates to the present EOS of the YMC.

In order to differentiate between the YMC dark energy models and other models, such as the quintessence, phantom, k-essence, or chaplygin gas, we can define an eigenfunction of first order for the YMC models

\[
\xi_1 = \frac{2r - 1}{9 \Omega_y} + s. \tag{25}
\]

From the expressions of (23), we find that the value of this eigenfunction is \(\xi_1 = 0 + O(\beta^2)\), which is independent of the initial condition of the YM dark energy models. It is easy to find that this feature is not right for other dark energy models. So we can differentiate the YMC dark energy models from other models by the observable quantity \(\xi_1\).
IV. STATEFINDER OF SECOND ORDER

We also can expand the EOS and statefinder of YMC to the second order of $\beta$. From the expressions in (14), we obtain

$$w = -1 + \frac{4}{3} \beta - \frac{4}{3} \beta^2 + O(\beta^3), \text{ and } w' = -\frac{8}{3} \beta + \frac{20}{3} \beta^2 + O(\beta^3).$$  \hspace{1cm} (26)

The statefinder parameters are

$$r = 1 - 2\Omega_y \beta + 4\Omega_y \beta^2 + O(\beta^3),$$  \hspace{1cm} (27)
$$s = \frac{4}{9} \beta - \frac{8}{27} \beta^2 + O(\beta^3),$$  \hspace{1cm} (28)

and the deceleration parameter is

$$q = \left(\frac{1}{2} - \frac{3\Omega_y}{2}\right) + 2\Omega_y \beta - 2\Omega_y \beta^2 + O(\beta^3).$$  \hspace{1cm} (29)

From these expressions, we can also define an eigenfunction of second order

$$\xi_2 = -\frac{8}{27} \frac{r - 1}{\Omega_y} + \frac{512}{9} (w + 1) - 44s.$$  \hspace{1cm} (30)

It is easy to find that the value of this eigenfunction $\xi_2 = 0 + O(\beta^3)$. In Fig.3, we plot the evolution of the eigenfunctions $\xi_1$ and $\xi_2$ in the different YMC dark energy models. We find that, in all these models, the values of $\xi_1$ and $\xi_2$ are all very closer to 0 if the EOS of the YMC is not very far from $-1$. From this figure, we also find that the value of $\xi_2$ is much more closer to 0 than which of $\xi_1$. The former is a more effective function for affirming the YMC dark energy models. Of course, in the LCDM models, the values of $\xi_1$ and $\xi_2$ are all exact 0, so it is difficult to differentiate between the YMC dark energy model and LCDM model.

V. SUMMARY

In summary, we have investigated the statefinder of the YMC dark energy models in this letter. We analyze two cases of the models, the quintessence-like case and the phantom-like case, and perform a statefinder diagnostic to both cases. It is shown that the evolving trajectory of this scenario in the $s - r$ plane is quite different from those of other models. We also define two eigenfunctions of YMC dark energy model. If the EOS of the YMC is not far from $-1$, the values of the eigenfunctions are very closer to 0, which can be used to simply differentiate between the YMC and other dark energy models.
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FIG. 1: The $s - r$ diagram of the YMC dark energy models. Dots locate the current values of the statefinder pair $\{s, r\}$, and the arrows denote the evolution direction of the statefinders with expansion of the universe.
FIG. 2: The $q - r$ diagram of the YMC dark energy models. Dots locate the current values of the statefinder pair \( \{q, r\} \), and the arrows denote the evolution direction of the statefinders with expansion of the universe. The point of \((-1, 1)\) corresponds to the steady state models (SS) - the de Sitter expansion.
FIG. 3: The evolution of the eigenfunctions $\xi_1$ and $\xi_2$ with the scale factor $a$. 