THE CAHN-HILLIARD EQUATION AND THE BIHARMONIC HEAT KERNEL ON EDGE MANIFOLDS

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Abstract. We construct the biharmonic heat kernel of a suitable closed self-adjoint extension of the bi-Laplacian on a manifold with incomplete edge singularities. We establish mapping properties of the biharmonic heat operator and derive short time existence for certain semi-linear equations of fourth order. We apply the analysis to a class of semi-linear partial differential equations in particular the Cahn-Hilliard equation and obtain asymptotics of the solutions near the edge singularity.

1. Introduction

The Cahn-Hilliard equation was proposed by Cahn and Hilliard in [Cah61], [CaHi58] as a simple model of the phase separation process, where at a fixed temperature the two components of a binary fluid spontaneously separate and form domains that are pure in each component. Let \( \Delta \) denote a self-adjoint extension of a Laplace operator. Then the Cahn-Hilliard equation may be stated in the following form

\[
\partial_t u(t) + \Delta^2 u(t) + \Delta(u(t) - u(t)^3) = 0, \quad u(0) = u_0, \quad t \in (0,T).
\]

Global existence for solutions to the Cahn-Hilliard equation has been established by Elliott and Songmu [ElSo86], and Caffarelli and Muler [CaMu95]. In the setup of singular manifolds however, there is still a question of asymptotics of solutions at the singular strata. This aspect has been studied by the recent work of Roidos and Schrohe [RoSc12] in the context of manifolds with isolated conical singularities, which has motivated the present discussion here. Using the notion of maximal regularity, they establish short time existence of solutions to the Cahn-Hilliard equation in certain weighted Mellin-Sobolev spaces which then yields regularity and asymptotics of solutions near the conical point.

In this paper we study the Cahn-Hilliard equation in the geometric setup of spaces with incomplete edges, which generalizes the notion of isolated conical singularities. Our method is different from [RoSc12] and uses the microlocal construction of the heat kernel for the bi-Laplacian. Precise analysis of the asymptotic behaviour of the heat kernel allows for a derivation of the mapping properties of the heat operator and a subsequent argument on the short time existence for certain semi-linear equations of fourth order, in analogy to [JL03] and [BDV11]. Existence and asymptotics for solutions to the Cahn-Hilliard equation comes as an application of the general results.

The microlocal analysis of the biharmonic heat kernel on edge spaces also allows for derivation of Schauder estimates and ultimately leads to short time existence results for fourth order PDE’s, including the Bock flow [BaDy11] on singular spaces.

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This will be the subject of forthcoming analysis. We also point out that our approach is not limited to squares of Laplacians on functions, but yields similar results for general powers of Hodge Laplacians on differential forms along the same lines.

Our main result may be summarized briefly as follows.

**Theorem 1.1.** Let \((M, g)\) be a Riemannian manifold with an admissible Riemannian incomplete edge metric \(g\). Then the biharmonic heat kernel \(H\) for the Friedrichs self-adjoint extension of the Laplace-Beltrami operator \(\Delta_g\) lifts to a polyhomogeneous distribution on a blowup of \(\mathbb{R}^+ \times M^2\) and defines a biharmonic heat operator with various mapping properties, such that certain semi-linear fourth order equations, among then the Cahn-Hilliard equation, admit a short time solution of a certain regularity.

This paper is organized as follows. We first introduce the basic geometry of incomplete edge spaces in §2. We then provide a microlocal construction of the heat kernel for a certain self-adjoint extension of the bi-Laplacian in §3. We proceed with with establishing mapping properties of the biharmonic heat operator in §4. We conclude with a short time existence result for certain semi-linear equations of fourth order and apply the analysis to the particular example of the Cahn-Hilliard equation in §5.

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2. Spaces with incomplete edge singularities

We introduce the fundamental geometric aspects of spaces with incomplete edge singularities, described in detail in [Maz91], compare also [MAVe12]. Let \(\tilde{M}\) be a compact stratified space with a single top-dimensional stratum \(\tilde{M}\) and a single lower dimensional stratum \(B\). By the definition of stratified spaces, \(B\) is a smooth closed manifold. Moreover there is an open neighbourhood \(U \subset \tilde{M}\) of \(B\) and a radial function \(x : U \cap M \to \mathbb{R}\), such that \(U \cap M\) is a smooth fibre bundle over \(B\) with fibre \(E(F) = (0, 1) \times F\), a finite open cone over a compact smooth manifold \(F\). The restriction of \(x\) to each fibre is the radial function of that cone.
The singular stratum $B$ in $\overline{M}$ may be resolved and defines a compact manifold $\tilde{M}$ with boundary $\partial M$, where $\partial M$ is the total space of a fibration $\phi: \partial M \to B$ with the fibre $F$. The resolution process is described in detail for instance in [Maz91]. The neighborhood $U$ lifts to a collar neighborhood $\tilde{U}$ of the boundary, which is a smooth fibration of cylinders $[0, 1) \times F$ over $B$ with the radial function $x$.

**Definition 2.1.** A Riemannian manifold $(\tilde{M}\setminus \partial M, g)$ has an incomplete edge singularity at $B$ if over $\tilde{U}$ the metric $g$ equals $g = g_0 + h$, where $|h|_{g_0} = O(x)$ as $x \to 0$ and

$$g_0 | \tilde{U} \setminus \partial M = dx^2 + s^2 g^F + \phi^* g^B,$$

with $g^B$ being a Riemannian metric on the closed manifold $B$, and $g^F$ a symmetric 2-tensor on the fibration $\partial M$ which restricts to a Riemannian metric on each fibre $F$.

We set $m = \dim M, b = \dim B$ and $f = \dim F$. Clearly, $m = 1 + b + f$. We assume henceforth $f = \dim F \geq 1$. Otherwise the incomplete edge manifold reduces to a compact manifold with boundary, where our discussion below is no longer applicable.

Similarly to other discussions in the singular edge setup, see [Alb07], [BDV11],[BaVe11] and [MaVe12], we additionally require $\phi: (\partial M, g^F + \phi^* g^B) \to (B, g^B)$ to be a Riemannian submersion. If $p \in \partial M$, then the tangent bundle $T_p \partial M$ splits into vertical and horizontal subspaces as $T^V_p \partial M \oplus T^H_p \partial M$, where $T^V_p \partial M$ is the tangent space to the fibre of $\phi$ through $p$ and $T^H_p \partial M$ is the orthogonal complement of this subspace. $\phi$ is said to be a Riemannian submersion if the restriction of the tensor $g^F$ to $T^H_p \partial M$ vanishes. We say that an incomplete edge metric $g$ is feasible if $\phi$ is a Riemannian submersion and in addition the Laplacians associated to $g^F$ at each $y \in B$ are isospectral.

The reasons behind the feasibility assumptions are as follows. Let $y = (y_1, ..., y_b)$ be the local coordinates on $B$ lifted to $\partial M$ and then extended inwards. Let $z = (z_1, ..., z_f)$ restrict to local coordinates on $F$ along each fibre. Then $(x, y, z)$ are the local coordinates on $M$ near the boundary. Consider the Laplace Beltrami operator $\Delta$ associated to $(M, g)$ and its normal operator $N(x^2 \Delta)_{g_0}$, defined as the limiting operator with respect to the local family of dilatations $(x, y, z) \to (\lambda x, \lambda (y - y_0), z)$ and acting on functions on the model edge $\mathbb{R}^+_x \times F \times \mathbb{R}^+_y$ with incomplete edge metric $g_{ie} = ds^2 + s^2 g^F + |du|^2$. Under the first admissibility assumption, $N(x^2 \Delta)_{g_0}$ is naturally identified with $s^2$ times the Laplacian for that model metric.

The second condition on isospectrality is imposed to ensure polyhomogeneity of the associated heat kernels when lifted to the corresponding parabolic blowup space. More precisely we only need that the eigenvalues of the Laplacians on fibres are constant in a fixed range $[0, 1]$, though we still make the stronger assumption for a clear and convenient representation.

**Definition 2.2.** Let $(M, g)$ be a Riemannian manifold with a feasible edge metric. This metric $g = g_0 + h$ is said to be admissible if in addition to feasibility

(i) the lowest non-zero eigenvalue $\lambda_0 > 0$ of the Laplace-Beltrami operator $\Delta_{F, y}$ associated to $(F, g^F |_{\phi^{-1}(y)})$ for any $y \in B$, satisfies $\lambda_0 > \dim F$.

(ii) $h$ vanishes to second order at $x = 0$, i.e. $|h|_{g_0} = O(x^2)$ as $x \to 0$. 

The reasons behind the admissibility assumptions are of technical rather than geometric nature and somewhat less straightforward to explain. However we point out that condition (i) is satisfied by a rescaling of $g^F$ which geometrically corresponds to decreasing the cone angle uniformly along the edge. Condition (ii) in particular holds for even metrics which depend on $x^2$ instead of $x$. The admissibility assumptions yield particular information on the heat kernel expansion, which is then used in Proposition 3.5.

An important ingredient in the analysis of singular edge spaces is the vector space $\mathcal{V}_e$ of edge vector fields smooth in the interior of $\tilde{M}$ and tangent at the boundary $\partial M$ to the fibres of the fibration. This space $\mathcal{V}_e$ is closed under the ordinary Lie bracket of vector fields, hence defines a Lie algebra. Its description in local coordinates is as follows. Consider the local coordinates $(x,y,z)$ on $\tilde{M}$ near the boundary. Then the edge vector fields $\mathcal{V}_e$ are locally generated by $\{ x\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_b}, \partial_{z_1}, \ldots, \partial_{z_f} \}$.

We may now define the Banach space of continuous sections $C^\infty_0(M)$, continuous on $\tilde{M}$ up to the boundary and fibrewise constant at $x = 0$. This is precisely the space of continuous sections with respect to the topology on $M$ induced by the Riemannian metric $g$. Banach spaces of higher order are defined as follows, compare [BDV11].

**Definition 2.3.** Let $(M,g)$ be a Riemannian manifold with an incomplete edge metric. Let $\mathcal{D}$ denote a collection of $\Delta$ and a choice of derivatives in $\{x^{-1}\mathcal{V}_e^2, x^{-1}\mathcal{V}_e, \mathcal{V}_e\}$, which will be specified later. Then for each $k \in \mathbb{N}$ we define a Banach space $C^{2k}_e(M, \mathcal{D}) := \{ u \in C^{2k}(M) \cap C^0_e(M) \mid X \circ \Delta^j u \in C^0_e(M), X \in \mathcal{D}, j = 0, \ldots, k - 1 \}$, with the norm $\| u \|_{2k} := \| u \|_\infty + \sum_{j=0}^k \sum_{X \in \mathcal{D}} \| X \circ \Delta^j u \|_\infty$.

3. Microlocal heat kernel construction

### 3.1. Self-adjoint extension of the bi-Laplacian

Let $\Delta$ denote the Laplace-Beltrami operator acting on functions on an incomplete edge space $(M,g)$ with a feasible incomplete edge metric $g$. As a first step we fix a self-adjoint extension of the bi-Laplacian $\Delta^2$. Consider the space of square-integrable forms $L^2_2(M,g)$ with respect to $g$. The maximal and minimal closed extensions of $\Delta$ are defined by the domains

$$
\mathcal{D}_{\text{max}}(\Delta) := \{ u \in L^2(M,g) \mid \Delta u \in L^2(M,g) \},
\mathcal{D}_{\text{min}}(\Delta) := \{ u \in \mathcal{D}_{\text{max}}(\Delta) \mid \exists u_j \in C^\infty_0(M) \text{ such that } u_j \to u \text{ and } \Delta u_j \to \Delta u \text{ both in } L^2(M,g) \}.
$$

where $\Delta u \in L^2$ is a priori understood in the distributional sense. Under the unitary rescaling transformation in the singular edge neighborhood

$$
\Phi : L^2(\mathcal{H}, d\text{vol}(g)) \to L^2(\mathcal{H}, x^{-f} d\text{vol}(g)), \ u \mapsto x^{f/2} u,
$$

(3.2)
the Laplacian $\Delta$ takes the form
\[
\Delta^{\Phi} := \Phi \circ \Delta \circ \Phi^{-1} = -\frac{\partial^2}{\partial x^2} + 1 \frac{1}{x^2} \left( \Delta_{F,y} + \left( \frac{f-1}{2} \right)^2 \frac{1}{4} \right) .
\]

**Lemma 3.1** ([MaVe12]). Let $(M, g)$ be an incomplete edge space with a feasible edge metric. Consider the increasing sequence of eigenvalues $(\sigma_j)_{j=1}^p$ of $\Delta_{F,y}$, counted with their multiplicities, such that $\nu_j^2 := \sigma_j + (f-1)^2/4 \in [0,1)$. The associated indicial roots are given by $\gamma_j^\pm = \pm \nu_j + 1/2$. Then any $u \in D_{\text{max}}(\Delta)$ expands under the rescaling (3.2) as $x \to 0$

\[
\Phi u \sim \sum_{j=1}^p (c_j^+ [u] \psi_j^+(x,z;y) + c_j^- [u] \psi_j^-(x,z;y)) + \Phi \ddot{u}, \ddot{u} \in D_{\text{min}}(\Delta),
\]

where the leading order term of each $\psi_j^\pm$ is the corresponding solution of the indicial operator. More precisely, let $\phi_j$ denote the normalized $\sigma_j$-eigenfunctions of $\Delta_{F,y}$. Then

\[
\psi_j^+(x,z;y) = x^{\gamma_j^+} \phi_j(z;y), \quad \psi_j^-(x,z;y) = \begin{cases} \sqrt{x} (\log x) \phi_j(z;y), & \nu_j = 0, \\ x^{\gamma_j^-} (1 + a_j x) \phi_j(z;y), & \nu_j > 0, \end{cases}
\]

with $a_j \in \mathbb{R}$ uniquely determined by $\Delta$. The coefficients $c_j^\pm [u]$ are of negative regularity in $y$ and the asymptotic expansion holds only in a weak sense, i.e. there is an expansion of the pairing $\int_B u(x,y,z) \chi(y) \, dy$ for any test function $\chi \in C^\infty(B)$.

The Friedrichs self-adjoint extension of $\Delta$ has been identified in [MaVe12] as
\[
D(\Delta_{\mathcal{F}}) = \{ u \in D_{\text{max}}(\Delta) \mid \forall j=1,\ldots,p: c_j^- [u] = 0 \}. \tag{3.3}
\]

Note that the sequence of eigenvalues $(\sigma_j)_{j=1}^p$ of $\Delta_{F,y}$ starts with $\sigma_j = 0$ for $j = 1,\ldots,\dim H^0(F)$. The corresponding indicial roots compute to $\gamma_j = 1/2 \pm (f-1)/2$ and the coefficient are constant in $z \in F$. Consequently, in view of the explicit form of the rescaling transformation in (3.2), the Friedrichs domain contains precisely those elements in the maximal domain whose leading term in the weak expansion as $x \to 0$ is given by $x^0$ with fibrewise constant coefficients. In particular
\[
D_{\text{max}}(\Delta) \cap \mathcal{E}_{\text{le}}^0(M) \subset D(\Delta_{\mathcal{F}}). \tag{3.4}
\]

We fix a self-adjoint extension of the bi-Laplacian as the square of $\Delta_{\mathcal{F}}$
\[
D(\Delta_{\mathcal{F}}^2) = \{ u \in D_{\mathcal{F}} \mid \Delta u \in D_{\mathcal{F}} \}. \tag{3.5}
\]

**3.2. Biharmonic heat kernel on a model edge.** In the second step we construct the heat kernel for $\Delta_{\mathcal{F}}^2$. We begin with studying the homogeneity properties of the heat kernel for the bi-Laplacian in the model case of an exact edge $(\mathcal{E} = \mathbb{R}^b \times \mathcal{C}(F), dy^2 + g)$ where $(\mathcal{E}(F) = (0, \infty) \times F, g = ds^2 + s^2 g^F)$ is an exact unbounded cone over a closed Riemannian manifold $(F, g^F)$. The Laplacian $\Delta_{\mathcal{E}}$ on the exact edge is then a sum of the Laplacian on $(\mathcal{E}(F), g)$ and the Euclidean Laplacian on $\mathbb{R}^b$. Consider the scaling operation ($\lambda > 0$)
\[
\Psi_\lambda : C^\infty(\mathbb{R}^+ \times \mathcal{E}^2) \to C^\infty(\mathbb{R}^+ \times \mathcal{E}^2),
\]

\[
(\Psi_\lambda u)(t, (s, y, z), (\tilde{s}, \tilde{y}, \tilde{z})) = u(\lambda^4 t, (\lambda s, \lambda (y - \tilde{y}), z), (\lambda \tilde{s}, \lambda \tilde{y}, \tilde{z})).
\]
Under the scaling operation we find
\[(\partial_t + \Delta^2_E)\Psi_\lambda u = \lambda^4 \Psi_\lambda (\partial_t + \Delta^2_E) u.\]  
(3.6)

Consequently, given the heat kernel $H_\xi$ for the Friedrichs extension of $\Delta^2_E$ (or at that stage any other self-adjoint extension), any multiple of $\Psi_\lambda H_\xi$ still solves the heat equation and also maps into the domain of $\Delta^2_E$. For the initial condition we obtain substituting $\widetilde{\nu} = \lambda \widetilde{s}, \widetilde{S} = \lambda \widetilde{s}$

\[
\lim_{t \to 0} \int_{\xi} (\Psi_\lambda H_\xi)(t, s, y, z, \widetilde{s}, \widetilde{y}, \widetilde{z}) u(\widetilde{s}, \widetilde{y}, \widetilde{z}) \widetilde{s}^f d\widetilde{s} d\widetilde{y} d\widetilde{z}
\]

\[
= \lim_{t \to 0} \lambda^{-1-b-f} \int_{\xi} H_\xi(\lambda^4 t, s, \lambda y - \widetilde{\nu}, z, \widetilde{s}, \widetilde{y}, \widetilde{z}) u(\widetilde{S}/\lambda, \widetilde{Y}/\lambda, \widetilde{z}) \widetilde{S}^f d\widetilde{S} d\widetilde{Y} d\widetilde{z}
\]

\[
= \lambda^{-1-b-f} u(\lambda s/\lambda, \lambda y/\lambda, z) = \lambda^{-1-b-f} u(s, y, z).
\]

By uniqueness of the heat kernel we obtain
\[
\Psi_\lambda H_\xi = \lambda^{-1-b-f} H_\xi.
\]

(3.7)

In addition to the homogeneity properties of $H_\xi$, we also require a full asymptotic expansion of the biharmonic heat kernel as $(s, \widetilde{s}) \to 0$. We accomplish this by establishing an explicit integral representation of $H_\xi$. Under the unitary rescaling (3.2) and a spectral decomposition of $L^2(F, g^F)$ into $\sigma^2$-eigenspaces of $\Delta_F$, we may write for the rescaled model edge Laplacian

\[
\Delta^\phi_\xi = -\partial^2_s + s^{-2} \left( \Delta_F + \left( \frac{f-1}{2} \right)^2 - \frac{1}{4} \right) + \Delta_{\mathbb{R}^b}
\]

\[
= \bigoplus_{\sigma} -\partial^2_s + s^{-2} \left( \sigma^2 + \left( \frac{f-1}{2} \right)^2 - \frac{1}{4} \right) + \Delta_{\mathbb{R}^b} =: \bigoplus_{\sigma} l_{\nu(\sigma)} + \Delta_{\mathbb{R}^b},
\]

where $\nu(\sigma) := \sqrt{\sigma^2 + (f-1)^2/4}$ and $l_{\nu(\sigma)}$ is defined on $C_0^\infty(0, \infty)$. The Friedrichs extension of $\Delta_\xi$ is compatible with the decomposition, compare a similar discussion in ([Ver08], Proposition 4.9). As a special case of Lemma 3.1, $l_{\nu}$ has unique self-adjoint extension $L_{\nu}$ in $L^2(\mathbb{R}^+) \,$ for $\nu \geq 1$, and in case of $\nu \in [0, 1)$, solutions $u \in \mathcal{D}(l_{\nu, \text{max}})$ admit a partial asymptotic expansion as $s \to 0$

\[
u(\nu) = \widetilde{u} + c^+[u] s^{\nu+1/2} + c^-[u] \left\{ \begin{array}{ll}
\sqrt{s} \log(s), & \nu = 0,
\nu \in (0, 1),
\end{array} \right.
\]

Then the Friedrichs extension $L_{\nu}$ of $l_{\nu}$ is defined, similar to (3.3), by requiring $c^- [u] = 0$, and moreover, identifying $\Delta_{\mathbb{R}^b}$ with its unique self-adjoint extension in $L^2(\mathbb{R}^b)$, we may write

\[
\Delta^\phi_{\xi, \mathcal{F}} = \bigoplus_{\sigma} L_{\nu(\sigma)} + \Delta_{\mathbb{R}^b}.
\]

(3.8)

Consequently, it suffices to construct the biharmonic heat kernel for $L_{\nu} + \Delta_{\mathbb{R}^b}$ in $L^2(\mathbb{R}^+ \times \mathbb{R}^b)$. Denote by $J_{\nu}$ the $\nu$-th Bessel function of first kind and consider the Hankel transform of order $\nu \geq 0$

\[
(\mathcal{H}_{\nu} u)(s) := \int_0^\infty \sqrt{ss'} J_{\nu}(ss') u(s') ds', \ u \in C_0^\infty(0, \infty).
\]

(3.9)
By ([Co59], Chapter III) and also by ([Les97], Proposition 2.3.4), the Hankel transform extends to a self-adjoint isometry on $L^2(\mathbb{R}^+ \times \mathbb{R}^b)$. We denote by

$$(\mathcal{F} u)(\xi) := (2\pi)^{-b/2} \int_{\mathbb{R}^b} u(y) e^{-iy\cdot \xi} dy, \quad u \in C_0^\infty(\mathbb{R}^b),$$

the Fourier transform on $\mathbb{R}^b$, which extends to an isometric automorphism of $L^2(\mathbb{R}^b)$. Consequently, $\mathcal{G}_\nu := \mathcal{H}_\nu \circ \mathcal{F}$ defines an isometric automorphism of $L^2(\mathbb{R}^+ \times \mathbb{R}^b)$ such that $\mathcal{G}_\nu^{-1} = \mathcal{H}_\nu \circ \mathcal{F}^{-1}$. Applying ([Les97], Proposition 2.3.5), we arrive at the following

**Proposition 3.2.** The isometric automorphism $\mathcal{G}_\nu$ diagonalizes $L_\nu + \Delta_{\mathbb{R}^b}$. More precisely,

$$\mathcal{D}(L_\nu + \Delta_{\mathbb{R}^b}) = \{ u \in L^2(\mathbb{R}^+ \times \mathbb{R}^b) \mid (S^2 + |\Xi|^2) \, \mathcal{G}_\nu u \in L^2(\mathbb{R}^+ \times \mathbb{R}^b) \},$$

$$\mathcal{G}_\nu (L_\nu + \Delta_{\mathbb{R}^b}) \mathcal{G}_\nu^{-1} = S^2 + |\Xi|^2,$$

where $X, \Xi$ denote multiplication operators by $x \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^b$, respectively.

Similarly, the isometry $\mathcal{G}_\nu$ diagonalizes the squared operator $(L_\nu + \Delta_{\mathbb{R}^b})^2$, identifying its action with $(S^2 + |\Xi|^2)^2$. Consequently we may express the biharmonic heat kernel of $(L_\nu + \Delta_{\mathbb{R}^b})^2$ as an integral in terms of Bessel functions. For $u \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^b)$ we find

$$\left( e^{-t(L_\nu + \Delta_{\mathbb{R}^b})^2} u \right)(s, y) = \left( \mathcal{G}_\nu e^{-t(S^2 + |\Xi|^2)^2} \mathcal{G}_\nu^{-1} u \right)(s, y)$$

$$= (2\pi)^{-b/2} \int_{\mathbb{R}^b} \int_0^\infty \left( \int_{\mathbb{R}^b} \int_0^\infty e^{i(y-\tilde{y})\xi} \sqrt{s\tilde{s}} J_\nu(s\rho) J_\nu(\tilde{s}\rho) \rho e^{-t(\rho^2 + |\xi|^2)} d\rho d\xi \right) u(s, \tilde{s}) d\tilde{s} d\tilde{y}.$$ 

Denote by $\phi_\sigma$ the normalized $\sigma^2$-eigenfunction of $\Delta_F$, where we count the eigenvalues $\sigma^2 \in \text{Spec}(\Delta_F)$ with their multiplicities. Then, as a consequence of (3.8), we finally obtain for the $\Phi$-rescaled biharmonic heat kernel on a model edge

$$H_{\xi}^\Phi = (2\pi)^{-b/2} \bigoplus_\sigma \int_{\mathbb{R}^b} \int_0^\infty e^{i(y-\tilde{y})\xi} \sqrt{s\tilde{s}} J_{\nu(\sigma)}(s\rho) J_{\nu(\sigma)}(\tilde{s}\rho) \rho e^{-t(\rho^2 + |\xi|^2)} d\rho d\xi \cdot \phi_\sigma(z) \otimes \phi_\sigma(\tilde{z}).$$

The $\nu$-th Bessel function of first kind admits an asymptotic expansion for small arguments $J_\nu(\zeta) \sim \sum_{j=0}^\infty a_j \zeta^{\nu+2j}$, as $\zeta \to 0$. This yields an asymptotic expansion of $H_{\xi}^\Phi$ as $(s, \tilde{s}) \to 0$ and consequently, rescaling back, we obtain as $s \to 0$

$$H_{\xi}(t, s, y, \tilde{s}, \tilde{y}, z, \tilde{z}) \sim \sum_{\gamma} a_{\gamma, j}(t, \tilde{s}, y, \tilde{y}, z, \tilde{z}) s^{\gamma + 2j},$$

(3.10)

where the summation is over all $\gamma = -(f - 1)/2 + \sqrt{\sigma^2 + (f - 1)^2}/4$ with $\sigma^2 \in \text{Spec}\Delta_F$, counted with multiplicity, and each coefficient $a_{\nu, j}$ lies in the corresponding $\sigma^2$-eigenspace. We summarize the properties of $H_{\xi}$, established above, in a single proposition for later reference.

**Proposition 3.3.** Consider the model edge $(E = \mathbb{R}^b \times C(F), dy^2 + g)$, where $(C(F) = (0, \infty) \times F, g = ds^2 + s^2 g^F)$ is an exact unbounded cone over a closed Riemannian manifold $(F^1, g^F)$. Fix the Friedrichs self-adjoint extension of the associated Laplace
Beltrami operator \( \Delta_{\mathcal{E}} \). Then the biharmonic heat kernel \( H_{\mathcal{E}} \) of \( \Delta_{\mathcal{E}}^2 \) is homogeneous of order \((-1 - b - f)\) under the scaling operation \((\lambda > 0)\)

\[
\Psi_{\lambda} : C^\infty(\mathbb{R}^+ \times \mathcal{E}^2) \to C^\infty(\mathbb{R}^+ \times \mathcal{E}^2),
\]

\[
(\Psi_{\lambda} u)(t, (s, y, z), (\tilde{s}, \tilde{y}, \tilde{z})) = u(\lambda^4 t, (\lambda s, \lambda(y - \tilde{y}), z), (\lambda\tilde{s}, \lambda\tilde{y}, \tilde{z})).
\]

Moreover, \( H_{\mathcal{E}} \) admits an asymptotic expansion as \((s, \tilde{s}) \to 0\) with the index set given by \( E + 2N_0 \), where

\[
E = \{ \gamma \geq 0 \mid \gamma = -\frac{(f - 1)}{2} + \sqrt{\frac{(f - 1)^2}{4} + \sigma^2}, \sigma^2 \in \text{Spec } \Delta_{\mathcal{E}} \},
\]

uniformly in other variables and with coefficients taking value in the corresponding \( \sigma^2 \)-eigenspace.

### 3.3. Construction of the biharmonic heat kernel

We can now proceed from the analysis of the heat kernel on the model edge to the construction of the heat kernel \( H \) for the bi-Laplacian on a space \((M, g)\) with an incomplete feasible edge metric. The heat kernel construction here follows ad verbatim the discussion in [MaVe12] for the edge Laplacian, with the only difference that for the bi-Laplacian now rather \( t^{1/4} \) instead of \( \sqrt{t} \) is treated as a smooth variable.

The heat kernel is a function on \( M^2_b = \mathbb{R}^+ \times \tilde{M}^2 \). Let \((x, y, z)\) and \((\tilde{x}, \tilde{y}, \tilde{z})\) be the coordinates on the two copies of \( M \) near the edge. Then the local coordinates near the corner in \( M^2_b \) are given by \((t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))\). The kernel \( H(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) \) has a non-uniform behaviour at the submanifolds

\[
A = \{(t = 0, (y, z), (\tilde{y}, \tilde{z})) \in \mathbb{R}^+ \times \partial M^2 \mid y = \tilde{y}\},
\]

\[
D = \{(t = 0, (x, z), (\tilde{x}, \tilde{z})) \in \mathbb{R}^+ \times \tilde{M}^2 \mid x = \tilde{x}, y = \tilde{y}, z = \tilde{z}\}.
\]

Exactly as in the case of the Hodge Laplacian on edges, see [MaVe12], we introduce an appropriate blowup of the heat space \( M^2_b \), such that the corresponding heat kernel lifts to a polyhomogeneous distribution in the sense of the definition below. This procedure has been introduced by Melrose in [Mel93]. For self-containment of the paper we repeat the definition of polyhomogeneity as well as the blowup process here.

**Definition 3.4.** Let \( \mathcal{W} \) be a manifold with corners, with all boundary faces embedded, and \( \{(H_i, \rho_i)\}_{i=1}^{N} \) an enumeration of its boundaries and the corresponding defining functions. For any multi-index \( b = (b_1, \ldots, b_N) \in \mathbb{C}^N \) we write \( \rho^b = \rho_1^{b_1} \ldots \rho_N^{b_N} \).

Denote by \( \mathcal{V}_b(\mathcal{W}) \) the space of smooth vector fields on \( \mathcal{W} \) which lie tangent to all boundary faces. A distribution \( \omega \) on \( \mathcal{W} \) is said to be conormal, if \( \omega \in \rho^b L^\infty(\mathcal{W}) \) for some \( b \in \mathbb{C}^N \) and \( V_1 \ldots V_l \omega \in \rho^b L^\infty(\mathcal{W}) \) for all \( V_j \in \mathcal{V}_b(\mathcal{W}) \) and for every \( \ell \geq 0 \). An index set \( E_i = \{(\gamma, p) \} \subset \mathbb{C} \times \mathbb{N} \) satisfies the following hypotheses:

(i) \( \text{Re}(\gamma) \) accumulates only at plus infinity,
(ii) For each \( \gamma \) there is \( P_\gamma \in \mathbb{N}_0 \), such that \( (\gamma, p) \in E_i \) if and only if \( p \leq P_\gamma \),
(iii) If \( (\gamma, p) \in E_i \), then \( (\gamma + j, p') \in E_i \) for all \( j \in \mathbb{N} \) and \( 0 \leq p' \leq p \).

An index family \( E = (E_1, \ldots, E_N) \) is an \( N \)-tuple of index sets. Finally, we say that a conormal distribution \( \omega \) is polyhomogeneous on \( \mathcal{W} \) with index family \( E \), we write
\( \omega \in \mathcal{A}_{\text{phg}}^E(\mathcal{M}) \), if \( \omega \) is conormal and if in addition, near each \( H_i \),

\[
\omega \sim \sum_{(\gamma,p) \in E_i} a_{\gamma,p} \rho_i^p (\log \rho_i)^p, \quad \text{as} \quad \rho_i \to 0,
\]

with coefficients \( a_{\gamma,p} \) conormal on \( H_i \), polyhomogeneous with index \( E_j \) at any \( H_i \cap H_j \).

The homogeneity property (3.7) contains the information how precisely the submanifolds \( A,D \subset M_h^2 \) need to be blown up such that the heat kernel becomes polyhomogeneous. To get the correct blowup of \( M_h^2 \) we first bi-parabolically \((t^{1/4} \text{ is viewed as a coordinate function})\) blow up the submanifold

\[
A = \{(t, (0, y, z), (0, \tilde{y}, \tilde{z})) \in \mathbb{R}^+ \times \partial M^2 : t = 0, y = \tilde{y} \} \subset M_h^2.
\]

The resulting heat-space \([M_h^2, A]\) is defined as the union of \( M_h^2 \setminus A \) with the interior spherical normal bundle of \( A \) in \( M_h^2 \). The blowup \([M_h^2, A]\) is endowed with the unique minimal differential structure with respect to which smooth functions in the interior of \( M_h^2 \) and polar coordinates on \( M_h^2 \) around \( A \) are smooth. As in [MAVe12], this blowup introduces four new boundary hypersurfaces; we denote these by ff (the front face), rf (the right face), lf (the left face) and tf (the temporal face).

The actual heat-space blowup \( \mathcal{M}_h^2 \) is obtained by a bi-parabolic blowup of \([M_h^2, A]\) along the diagonal \( D \), lifted to a submanifold of \([M_h^2, A]\). The resulting blowup \( \mathcal{M}_h^2 \) is defined as before by cutting out the submanifold and replacing it with its spherical normal bundle. It is a manifold with boundaries and corners, visualized in Figure below.

![Figure 1. Heat-space Blowup \( \mathcal{M}_h^2 \).](image)

The projective coordinates on \( \mathcal{M}_h^2 \) are then given as follows. Near the top corner of the front face ff, the projective coordinates are given by

\[
\rho = t^{1/4}, \quad \xi = \frac{x}{\rho}, \quad \tilde{\xi} = \frac{\tilde{x}}{\rho}, \quad u = \frac{y - \tilde{y}}{\rho}, \quad z, \tilde{y}, \tilde{z},
\]

(3.11)

where in these coordinates \( \rho, \xi, \tilde{\xi} \) are the defining functions of the boundary faces ff, rf and lf respectively. For the bottom corner of the front face near the right hand side projective coordinates are given by

\[
\tau = \frac{t}{\tilde{x}^4}, \quad s = \frac{x}{\tilde{x}}, \quad u = \frac{y - \tilde{y}}{\tilde{x}}, \quad z, \tilde{x}, \tilde{y}, \tilde{z},
\]

(3.12)

where in these coordinates \( \tau, s, \tilde{x} \) are the defining functions of tf, rf and ff respectively. For the bottom corner of the front face near the left hand side projective coordinates
are obtained by interchanging the roles of \( x \) and \( \tilde{x} \). Projective coordinates on \( \mathcal{M}_h^2 \) near temporal diagonal are given by

\[
\eta = \frac{t^{1/4}}{x}, \quad S = \frac{(x - \tilde{x})}{t^{1/4}}, \quad U = \frac{y - \tilde{y}}{t^{1/4}}, \quad Z = \frac{\tilde{x}(z - \tilde{z})}{t^{1/4}}, \quad \tilde{x}, \tilde{y}, \tilde{z}.
\]

(3.13)

In these coordinates \( \text{tf} \) is the face in the limit \(|(S, U, Z)| \to \infty\), \( \text{ff} \) and \( \text{td} \) are defined by \( \tilde{x}, \eta \), respectively. The blowdown map \( \beta : \mathcal{M}_h^2 \to \mathcal{M}_h^2 \) is in local coordinates simply the coordinate change back to \((t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))\).

In case the edge manifold is an exact edge \((\mathcal{E} = \mathbb{R}^b \times \mathcal{C}(F), dy^2 + g)\) where \((\mathcal{C}(F) = (0, \infty) \times F^f, g = ds^2 + s^2 g^\mathcal{F})\), Proposition 3.3 implies that \( H_\mathcal{E} \) lifts to a polyhomogeneous conormal distribution on the heat space blow up, of order \((-m), m = 1 + b + f\), at the front and the temporal diagonal faces, vanishing to infinite order at \( \text{tf} \), and with the index set at \( \text{rf} \) and \( \text{lf} \) given by \( E + 2\mathbb{N}_0 \), where

\[
E = \{ \gamma \geq 0 \mid \gamma = -\frac{(f - 1)}{2} + \sqrt{\frac{(f - 1)^2}{4} + \sigma^2}, \sigma^2 \in \text{Spec } \Delta_F \}.
\]

In the general case of a feasible edge space \((M, g), H_\mathcal{E}\) is only an initial parametrix and solves the heat equation only to first order. Repeating almost ad verbatim the heat kernel construction in case of the edge Laplacian in [MaVE12], we arrive at the following

**Proposition 3.5.** Let \((M^m, g)\) be an incomplete edge space with a feasible edge metric \( g \). Then the lift \( \beta^* H \) is polyhomogeneous on \( \mathcal{M}_h^2 \) of order \((- \dim M)\) at \( \text{ff} \) and \( \text{td} \), vanishing at infinite order at \( \text{tf} \), and with the index set at \( \text{rf} \) and \( \text{lf} \) given by \( E + \mathbb{N}_0 \) where

\[
E = \{ \gamma \geq 0 \mid \gamma = -\frac{(f - 1)}{2} + \sqrt{\frac{(f - 1)^2}{4} + \sigma^2}, \lambda \in \text{Spec } \Delta_{F,g} \}.
\]

More precisely, if \( s \) denotes the boundary defining function of \( \text{rf} \), we obtain

\[
\beta^* H \sim \sum_{\gamma \in E} \left( \sum_{j=0}^{\infty} s^{\gamma+2j} a_{\gamma,j}(\beta^* H) + \sum_{j=0}^{\infty} s^{\gamma+2j+1} a'_{\gamma,j}(\beta^* H) \right) \text{ as } s \to 0,
\]

where the coefficients \( a_{\gamma,j}(H) \) are of order \((-m)\) at the front face and lie in their corresponding \( \Delta_{F,g} \) eigenspaces. The higher coefficients \( a'_{\gamma,j}(\beta^* H) \) are of order \((-m+1)\) at \( \text{ff} \).

**Proof.** Recall the heat kernel construction in [MaVE12], which we basically follow here. Denote by \( \Delta \) the Laplace Beltrami operator on \((M, g)\). We write \( \mathcal{L} := \partial_t + \Delta^2 \) for the heat operator. The restriction of the lift \( \beta^*(t\mathcal{L}) \) to \( \text{ff} \) is called the normal operator \( N_\text{ff}(t\mathcal{L})_{y_0} \) at the front face (at the fibre over \( y_0 \in B \)) and is given in projective coordinates (3.12) explicitly as follows

\[
N_\text{ff}(t\mathcal{L})_{y_0} = \tau \left( \partial_x + \left( -\partial_s^2 - fs^{-1}\partial_s + s^{-2}\Delta_{F,y_0} + \Delta_u^b \right)^2 \right)
\]

\[
= : \tau \left( \partial_x + \left( \Delta_{F,y_0}^e + \Delta_u^b \right)^2 \right).
\]

\( N_\text{ff}(t\mathcal{L}) \) does not involve derivatives with respect to \((y_0, \tilde{x}, \tilde{y}, \tilde{z})\) and hence acts tangentially to the fibres of the front face. Consequently in our choice of an initial
The next step in the construction of the heat kernel involves adding a kernel with each coefficient $a_{\gamma,j}$ of the initial parametrix $H_0$ is the heat equation for the bi-Laplace operator on the model edge $C^\infty(F) \times \mathbb{R}^b$. Consequently, the initial parametrix $H_0$ is defined by choosing $N_{\mathcal{H}}(H_0)$ to equal the fundamental solution for the heat operator $N_{\mathcal{H}}(t\mathcal{L})$, and extending $N_{\mathcal{H}}(H_0)$ trivially to a neighbourhood of the front face. More precisely, consider the biharmonic heat kernel $H_{E,y}$ on the model edge $(C^\infty(F) \times \mathbb{R}^b, ds^2 + s^2g_{y_0} + du^2)$ with the parameter $y_0 \in B$. Then in projective coordinates $(\tau,s,y_0,z,\tilde{x},u,\tilde{z})$ near the right corner of $\text{ff}$, where $\tilde{x}$ is the defining function of the front face, we set

$$H_0(\tau,s,u,y_0,z,\tilde{x},u,\tilde{z}) := \tilde{x}^{-m}\phi(\tilde{x})H_{E,y_0}(\tau,s,u,z,\tilde{s} = 1,\tilde{u} = 0,\tilde{z}),$$

(3.14)

where $\phi$ is a smooth cutoff function, $\phi \equiv 1$ in an open neighborhood of $\tilde{x} = 0$, and with compact support in $[0,1)$. By Proposition 3.3, our initial parametrix $H_0$ is polyhomogeneous on $M^2_{\mathcal{H}}$ and solves the heat equation to first order at the front face $\text{ff}$ of $\mathcal{H}^2$. Moreover Proposition 3.3 asserts

$$H_0 \sim \sum_{\gamma \in E} \sum_{j=0}^{\infty} s^{\gamma+2j}a_{\gamma,j}(H_0), \ s \to 0,$$

(3.15)

with each coefficient $a_{\gamma,j}(H_0)$ lying in the corresponding $\Delta_{F,y_0}$ eigenspace. The error of the initial parametrix $H_0$ is given by

$$\beta^*(t\mathcal{L})H_0 = (\beta^*(t\Delta_g^2) - \tau(\Delta_{F,y_0}^e + \Delta_{u}^e)^2)H_0 =: P_0.$$

The leading order term in the expansion of $\beta^*(t\mathcal{L})$ at $t d$ does not depend on the edge geometry and corresponds to the bi-Laplacian on a closed manifold. Consequently, classical arguments allow to refine the initial parametrix such that the error term $P_0$ is vanishing to infinite order at $t d$, compare the corresponding discussion in ([MAVe12], Section 3.2). We need to understand the explicit structure of the asymptotic expansion of $P_0$ at $\text{ff}$ and $\text{rf}$. By Definition 2.2 (ii) we find

$$\beta^*\Delta_g = \tilde{x}^{-2}(\Delta_{s,y_0}^e + \Delta_{u}^e) + \tilde{x}^{-1}L_1 + L_2,$$

(3.16)

where $L_1$ is comprised of the derivatives $\partial_\tau\partial_\sigma$ and $L_2$ consists of edge derivatives $\nabla_{\tilde{e}}^e$. Consequently, we obtain after taking squares

$$\beta^*(t\Delta_g^2) - \tau(\Delta_{s,y_0}^e + \Delta_{u}^e)^2 = \tau x^{-2}\left((\Delta_{s,y_0}^e + \Delta_{u}^e)L_1 + L_1(\Delta_{s,y_0}^e + \Delta_{u}^e)\right) + \tau x^{-2}\left((\Delta_{s,y_0}^e + \Delta_{u}^e)L_2 + L_2(\Delta_{s,y_0}^e + \Delta_{u}^e) + L_2^2 + \tau x^{-4}L_2^2.$$

We now apply each of the summands above to the asymptotic expansion (3.15) of $H_0$. Note that $\Delta_{s,y_0}^e$ annihilates each $s^{\gamma}a_{\gamma,j}(H_0), \gamma \in E$, and lowers the $s$-order of $s^{\gamma+2j}a_{\gamma,j}(H_0)$ by 2, if $j \geq 1$. Consequently, we obtain as $s \to 0$

$$P_0 = \beta^*(t\mathcal{L})H_0 \sim \tilde{x}^{-m+1}\sum_{\gamma \in E} \sum_{j=0}^{\infty} s^{\gamma+j-2}c_{\gamma,j}.$$

The next step in the construction of the heat kernel involves adding a kernel $H'_0$ to $H_0$, such that the new error term is vanishing to infinite order at $\text{rf}$. In order
to eliminate the term $s^k a$ in the asymptotic expansion of $P_0$ at rf, we only need to solve

$$(-\partial_\tau^2 - fs^{-1}\partial_s + s^{-2}\Delta_{F,y_0})^2 u = s^k(\tau^{-1} a).$$

(3.17)

This is because all other terms in the expansion of $tL$ at rf lower the exponent in $s$ by at most one, while the indicial part lowers the exponent by two. The variables $(\tau, u, \bar{x}, y_0, \bar{y}, z)$ enter the equation only as parameters. The equation is solved by Mellin transform as well as spectral decomposition over $F$. The solution is polyhomogeneous in all variables, including parameters and is of leading order $(k+4)$. Consequently, the correcting kernel $H'_0$ must be of leading order 2 at rf and of leading order $(-m+1)$ at ff, since $P_0$ is of order $(-2)$ at rf and $(-m+1)$ at ff and the defining function $\bar{x}$ of the front face enters (3.17) only as a parameter. Hence

$$H_1 := H_0 + H'_0 \sim \sum_{\gamma \in E} \left( \sum_{j=0}^{\infty} s^{\gamma+2j} a_{\gamma,j}(H_1) + \sum_{j=0}^{\infty} s^{\gamma+2+j} a'_{\gamma,j}(H_1) \right) \text{ as } s \to 0,$$

where the coefficients $a_{\gamma,j}(H_1)$ each lie in the corresponding $\Delta_{F,y_0}$ eigenspace. In the following correction steps the exact heat kernel is obtained from $H_1$ by an iterative correction procedure, adding terms of the form $H_1 \circ (P_1)^k$, where $P_1 := tL H_1$ is vanishing to infinite order at rf and td. This leads to an expansion

$$\beta^* H \sim \sum_{\gamma \in E} \left( \sum_{j=0}^{\infty} s^{\gamma+2j} a_{\gamma,j}(\beta^* H) + \sum_{j=0}^{\infty} s^{\gamma+2+j} a'_{\gamma,j}(\beta^* H) \right) \text{ as } s \to 0,$$

(3.18)

where the coefficients $a_{\gamma,j}(H)$ still lie in their corresponding $\Delta_{F,y_0}$ eigenspaces, and are of order $(-m)$ at the front face. The higher coefficients $a'_{\gamma,j}(\beta^* H)$ are of order $(-m+1)$ at ff.

Note that in the construction above, we have only used feasibility of the edge metric and Definition 2.2 (ii). The assumption of Definition 2.2 (i) for admissible edge metrics is not required there, but plays an essential role in the argument that $H$ indeed takes values in $\mathcal{D}(\Delta^2_F)$. First note that $H$ indeed takes values in $\mathcal{D}(\Delta_F)$, since the expansion (3.18) satisfies the characterization of maximal solutions in Lemma 3.1 and also the condition in (3.3) under the rescaling $\Phi$.

The conclusion that $\Delta H$ also takes values in $\mathcal{D}(\Delta_F)$ is more intricate. Recall (3.16). It is easily checked from (3.18) that $\bar{x}^{-2} \left( \Delta^x(F) + \Delta^y_{y_0} \right) H$ indeed takes values in $\mathcal{D}(\Delta_F)$ without any further assumptions. Application of $(\bar{x}^{-1} L_1 + L_2)$ to $H$ preserves the expansion (3.18), however the coefficients in the expansion need not lie in the correct $\Delta_{F,y_0}$ eigenspaces, and hence we cannot deduce that $(\bar{x}^{-1} L_1 + L_2) H$ maps into $\mathcal{D}(\Delta_F)$ in general.

By condition (i) of Definition 2.2, any $\gamma \neq 0$ is automatically $\gamma > 1$, and hence it then suffices to check whether the $s^0$ coefficient in the expansion of $(\bar{x}^{-1} L_1 + L_2) H$ lies in the zero-eigenspace of $\Delta_F$, in other words is harmonic on fibres and hence constant in $z$. The leading term $s^0 a_{0,0}(\beta^* H)$ in the expansion of $\beta^* H$ is annihilated by $(s \partial_x, \partial_z$ and is increased by $\beta^* x \partial_x = s \partial_x + \bar{x} s \partial_y$. Consequently, $(\bar{x}^{-1} L_1 + L_2) H$ admits no $s^0$ term and hence trivially maps into $\mathcal{D}(\Delta_F)$. 


The kernels \( H \) and \( \Delta H \) thus both map into \( \mathcal{D}(\Delta_x) \) and hence by definition, \( H \) indeed maps into \( \mathcal{D}(\Delta_x^2) \) and thus is the biharmonic heat kernel associated to \( \Delta_x^2 \).

\[ \square \]

4. Mapping properties of the biharmonic heat operator

In this section we prove boundedness and strong continuity of the biharmonic heat operator with respect to Banach spaces introduced in Definition 2.3.

**Theorem 4.1.** Let \( (M^m, g) \) be an incomplete edge space with a feasible edge metric \( g \). Fix the Friedrichs extension \( \Delta_x \) of the corresponding Laplace Beltrami operator. Put \( \mathcal{D}_0 = \{ \Delta \} \) and \( \mathcal{D} = \{ \Delta, x^{-1} \mathcal{V}_e' \}, \) where \( \mathcal{V}_e' \subset \mathcal{V}_e \) consists locally of all edge vector fields where \( x \partial_y \) is weighted with functions that are fibrewise constant.

Then the associated biharmonic heat operator \( e^{-t\Delta_x^2} \) is a bounded map between the (weighted) Banach spaces

\[ e^{-t\Delta_x^2} : \mathcal{C}^2_{iw}(M, D_0) \to e^{-1/4t}\mathcal{C}^{2(k+1)}(M, D). \]

**Proof.** First we prove the statement for \( k = 0 \). The explicit structure of the heat kernel expansion in Proposition 3.5 implies that for any \( X \in \mathcal{D} \) applied to the biharmonic heat kernel \( H \), the lift \( \beta^*(XH) \) admits the following behaviour near the front face of the heat space \( \mathcal{M}_h^2 \)

\[ \beta^*(XH) = O \left( (\rho_{if}\rho_{if})^{m-2}\rho_{it}^{\infty} \right), \quad (4.1) \]

where \( \rho_* \) denotes a defining function of a boundary face \( *, * \in \{ rf, lf, ff, td, tf \} \).

Consider the lift of the volume form in the various projective coordinates near \( ff \). We exemplify the transformation rules for the volume form near the lower left, lower right and the top corner of the front face

- near left corner: \( \tau = \frac{t}{x^4} = \rho_{if}, s = \frac{x}{x}, u = \frac{y - y}{x}, x = \rho_{if}, y, z, \tilde{z}, \)

\[ \beta^*(\tilde{x} \tilde{f} \tilde{x} \tilde{d} \tilde{x} \tilde{d} \tilde{v}_\Theta_1(\tilde{x})) = h \cdot x^m s^l ds du \tilde{d} \tilde{z}, \]

- near right corner: \( \tau = \frac{t}{x^4} = \rho_{if}, s = \frac{x}{x}, u = \frac{y - \tilde{y}}{x}, z, \tilde{x} = \rho_{if}, \tilde{y}, \tilde{z}, \)

\[ \beta^*(\tilde{x} \tilde{f} \tilde{x} \tilde{d} \tilde{x} \tilde{d} \tilde{v}_\Theta_1(\tilde{x})) = h \cdot \tilde{x}^{m-1} \tilde{d} \tilde{t} du \tilde{d} \tilde{z}, \]

- near top corner: \( \rho = t^{1/4} = \rho_{if}, \xi = \frac{x}{\rho} = \rho_{if}, \tilde{\xi} = \frac{x}{\rho} = \rho_{it}, u = \frac{y - \tilde{y}}{\rho}, y, z, \tilde{z}, \)

\[ \beta^*(\tilde{x} \tilde{f} \tilde{x} \tilde{d} \tilde{x} \tilde{d} \tilde{v}_\Theta_1(\tilde{x})) = h \cdot \rho^m \tilde{\xi}^l du \tilde{d} \tilde{z}. \]

(4.2)

The projective coordinates and the transformation rule for the volume form where the front and the temporal diagonal faces meet, is as follows

\[ \eta = \frac{t^{1/4}}{x} = \rho_{ad}, \quad S = \frac{x - \tilde{x}}{t^{1/4}}, \quad U = \frac{y - \tilde{y}}{t^{1/4}}, \quad Z = \frac{x(z - \tilde{z})}{t^{1/4}}, x = \rho_{if}, y, z, \quad (4.3) \]

\[ \beta^*(\tilde{x} \tilde{f} \tilde{x} \tilde{d} \tilde{x} \tilde{d} \tilde{v}_\Theta_1(\tilde{x})) = h \cdot x^m \eta^m dS du dZ. \]

When we combine the asymptotics of \( \beta^*(XH) \) in (4.1) with the behaviour of the volume form in the various projective coordinates (4.2) and (4.3), we find that...
\[ \beta^*(XH\bar{\tau}^l d\bar{x} \text{dvol}_M(\bar{x})) \] has a singular behaviour of \((\rho_t \rho_{ad})^{-2} \leq ct^{-1/4}\). Consequently, we may estimate for \(X \in \mathcal{D}\) and any continuous function, in particular for any \(u \in \mathcal{C}^0_{\text{ie}}(M)\)

\[
\|X e^{-t\Delta^2} u\|_\infty \leq C t^{-1/4}\|u\|_\infty,
\]

for some constant \(C > 0\) independent of \(u\). Furthermore, by Proposition 3.5, \(\beta^* X H \sim \rho_0 \rho_{ad}^{-1}\), as \(\rho_0 \rightarrow 0\) for \(X \in \mathcal{D}\), where \(\rho_0\) is fibrewise constant, i.e. independent of \(z \in F\). Here the fact that for \(X \in x^{-1}\mathcal{V}_e\) its \(\partial_y\) component is weighted with a fibrewise constant function, is essential. Hence, indeed \(X e^{-t\Delta^2} u \in t^{-1/4}\mathcal{C}^0_{\text{ie}}(M)\). This proves the statement for \(k = 0\). The general statement follows from the fact that due to (3.4), \(\mathcal{C}^{2k}_{\text{ie}}(M, \mathcal{D}) \subset \mathcal{D}(\Delta^k_{\text{ie}})\) and for any \(u \in \mathcal{D}(\Delta^k_{\text{ie}})\), \(\Delta^k e^{-t\Delta^2} u = e^{-t\Delta^2} \Delta^k u\), by uniqueness of solutions to the biharmonic heat equation with fixed initial condition.

Finally note that while \(t^{1/4} X e^{-t\Delta^2} u\) is continuous even for \(X \in x^{-1}\mathcal{V}_e\), it need not remain fibrewise constant at \(x = 0\) since in general \(X\) may include vector fields weighted with \(z\)-dependent functions. Hence \(x^{-1}\mathcal{V}_e\) is replaced by \(x^1\mathcal{V}_e\) in \(\mathcal{D}\), where \(\mathcal{V}_e \subset \mathcal{V}_e\) consists locally of linear combinations of \(\{x\partial_x, x\partial_y, \partial_z\}\), weighted with functions that are fibrewise constant.

**Remark 4.2.** A particular property of \(u \in \mathcal{C}^{2}_{\text{ie}}(M, \mathcal{D})\) with \(\{\partial_x, \partial_y, x^{-1}\partial_z\} \subset \mathcal{D}\) is worth noticing. In the singular neighborhood of the edge, the distance defined by the Riemannian edge metric \(g\) is equivalent to

\[
\text{d}((x, y, z), (\bar{x}, \bar{y}, \bar{z})) = \left( |x - \bar{x}|^2 + |y - \bar{y}|^2 + (x + \bar{x})^2 |z - \bar{z}|^2 \right)^{1/2}.
\]

In local coordinates near the edge we find

\[
u(x, y, z) = u(\bar{x}, \bar{y}, \bar{z})
\]

\[
u(x, y, z) = u(x, y, z) - u(\bar{x}, y, z) + u(\bar{x}, y, z) - u(\bar{x}, \bar{y}, z) + u(\bar{x}, \bar{y}, z) - u(\bar{x}, \bar{y}, \bar{z})
\]

\[
u(x, y, z) = \partial_x u(\xi, y, z)(x - \bar{x}) + \partial_y u(\bar{x}, \gamma, z)(y - \bar{y}) + \bar{x}^{-1}\partial_z u(\bar{x}, \bar{y}, \zeta)(z - \bar{z}).
\]

Consequently we obtain

\[
|u(x, y, z) - u(\bar{x}, \bar{y}, \bar{z})| \leq \|u\|_2 \left( |x - \bar{x}| + |y - \bar{y}| + (x + \bar{x}) |z - \bar{z}| \right)
\]

\[
\leq \sqrt{2} \|u\|_2 \text{d}((x, y, z), (\bar{x}, \bar{y}, \bar{z})).
\]

In other words, \(u \in \mathcal{C}^{2}_{\text{ie}}(M, \mathcal{D})\) is automatically Lipschitz with respect to \(\text{d}\).

Our argument on existence of solutions to certain semi-linear fourth order equations in the section below crucially depends on the strong continuity property of the biharmonic heat operator with respect to the Banach space \(\mathcal{C}^{2k}_{\text{ie}}(M, \mathcal{D})\). This is the content of the next theorem. Note that for strong continuity we will choose a different space \(\mathcal{D}'\) of allowable operators, smaller than in Theorem 4.1. Beforehand we note the following well-known function analytic result.

**Lemma 4.3.** Let \(D\) be a self-adjoint non-negative unbounded operator in a Hilbert space \(H\). Then the following is true.

(i) The continuously differentiable solution to \((\partial_t + D^2) u = 0\), with \(u(t) \in \mathcal{D}(D^2)\) for \(t > 0\) and \(\lim_{t \to 0} u(t) = u_0 \in H\) is unique, for any given \(u_0 \in H\).

(ii) For any \(u_0 \in \mathcal{D}(D)\), we have \(D e^{-tD^2} u_0 = e^{-tD^2} D u_0\).
Proof. (i) For $s \in (0, t]$ we compute
\[
\partial_s e^{-(t-s)D^2}u(s) = -\partial_t e^{-(t-s)D^2}u(s) + e^{-(t-s)D^2}\partial_s u(s)
\]
\[
= e^{-(t-s)D^2}D^2u(s) - e^{-(t-s)D^2}D^2u(s) = 0.
\]
Consequently, $e^{-(t-s)D^2}u(s)$ is constant for $s \in (0, t]$. Since $e^{-tD^2}$ converges to $\text{Id}$ in the Hilbert space norm as $t \to 0$ and, by assumption, $u(t)$ is continuous at $t = 0$, we find that $e^{-(t-s)D^2}u(s)$ is constant for $s \in [0, t]$. Considering the limit of $e^{-(t-s)D^2}u(s)$ as $s \to t$ and as $s \to 0$ proves for any $u_0 \in \mathcal{H}$
\[
u(t) = e^{-tD^2}u_0.
\]
(ii) Consider $\lambda \in \text{Res}(D^2)$ in the resolvent set of $D^2$. Then for any $u_0 \in \mathcal{D}(D)$, the resolvent $(D^2 - \lambda)^{-1}u_0 \in \mathcal{D}(D^2)$ and we compute
\[
(D^2 - \lambda)D(D^2 - \lambda)^{-1}u_0 = D(D^2 - \lambda)(D^2 - \lambda)^{-1}u_0 = Du_0,
\]
\[
\Rightarrow D(D^2 - \lambda)^{-1}u_0 = (D^2 - \lambda)^{-1}Du_0.
\]
The statement now follows by closedness of $D$ and definition of the heat operator as the strong limit $e^{-tD^2} := \lim_{n \to \infty} (I + tD^2/n)^{-n}$. □

**Theorem 4.4.** Let $(M^n, g)$ be an incomplete edge space with a feasible edge metric $g$. Fix the Friedrichs extension $\Delta_{\mathcal{F}}$ of the corresponding Laplace-Beltrami operator. Put $\mathcal{D}' = \{\Delta, \mathcal{V}_o^2, \mathcal{V}_e\}$. Then the associated biharmonic heat operator $e^{-t\Delta_{\mathcal{F}}^2}$ is a strongly continuous bounded map between Banach spaces
\[
e^{-t\Delta_{\mathcal{F}}^2} : \mathcal{C}^{2k}_\text{ie}(M, \mathcal{D}') \to \mathcal{C}^{2k}_\text{ie}(M, \mathcal{D'}).
\]
Proof. By (3.4), we have $\mathcal{C}^{2k}_\text{ie}(M, \mathcal{D}') \subset \mathcal{D}(\Delta_{\mathcal{F}}^k)$ and hence for any $u \in \mathcal{C}^{2k}_\text{ie}(M, \mathcal{D}')$ we infer by the previous Lemma 4.3, $\Delta_{\mathcal{F}}^k e^{-t\Delta_{\mathcal{F}}^2}u = e^{-t\Delta_{\mathcal{F}}^2}\Delta_{\mathcal{F}}^ku$. This reduces the statement to $k = 1$ and $k = 0$. Proof of both cases requires stochastic completeness of the biharmonic heat kernel, which we explain below. Solutions to the initial value problem
\[
\partial_t u + \Delta^2 u = 0, \quad u(0) = u_0, \quad u(t) \in \mathcal{D}(\Delta_{\mathcal{F}}^2), \quad t > 0,
\]
are unique and in fact given by $u(t) = e^{-t\Delta_{\mathcal{F}}^2}u_0 \in \mathcal{D}(\Delta_{\mathcal{F}}^2)$. We have observed in subsection 3.1 that, reversing eventual rescaling, the Friedrichs domain contains precisely those elements in the maximal domain whose leading term in the weak expansion as $x \to 0$ is given by $x^\alpha$, with a fibrewise constant coefficient, cf. (3.4). Consequently, the constant function $1 \in \mathcal{D}(\Delta_{\mathcal{F}})$. Moreover, $\Delta_{\mathcal{F}} 1 = 0 \in \mathcal{D}(\Delta_{\mathcal{F}})$ and consequently $1 \in \mathcal{D}(\Delta_{\mathcal{F}}^2)$. The constant function $1$ solves the heat equation and hence we deduce by uniqueness of solutions the stochastic completeness
\[
e^{-t\Delta_{\mathcal{F}}^2}1 \equiv \int_M H(t, x, \tilde{p}) \mathcal{d}\mathcal{v}_g(\tilde{p}) = 1, \quad \text{for all } p \in M, t > 0.
\]
This reduces the case to $k = 0, 1$. We can now prove the statement for $k = 0$, basically repeating the arguments in ([BDV11]) where the classical proof of strong continuity of the heat operator on closed (non-singular) manifolds is adapted to the present setup. Using stochastic completeness we find
\[
(e^{-t\Delta_{\mathcal{F}}^2}u)(p, t) - u(p) = \int_M H(t, x, \tilde{p}) (u(\tilde{p}) - u(p)) \mathcal{d}\mathcal{v}_g(\tilde{p}).
\]
Consider the distance function $d(p, \bar{p})$ induced by the incomplete edge metric $g$. In the singular neighborhood of the edge, the distance is equivalent to

$$d((x, y, z), (\bar{x}, \bar{y}, \bar{z})) = \left( |x - \bar{x}|^2 + |y - \bar{y}|^2 + (x + \bar{x})^2 |z - \bar{z}|^2 \right)^{1/2}.$$ 

Note that $u \in \mathcal{C}^0(M)$ is continuous with respect to the topology induced by the Riemannian metric $g$ and hence by the distance function $d$. Hence for any $\epsilon > 0$ there exists some $\delta(\epsilon) > 0$, such that for $d(p, \bar{p}) \leq \delta(\epsilon)$ one has $|u(p) - u(\bar{p})| \leq \epsilon$. For any given $\epsilon > 0$ we separate the integration region into

$$M^+_\epsilon := \{ \bar{p} \mid d(p, \bar{p}) \geq \delta(\epsilon) \},$$

$$M^-_\epsilon := \{ \bar{p} \mid d(p, \bar{p}) \leq \delta(\epsilon) \}.$$  

Employing continuity of $u$ we find

$$|e^{-t\Delta g} u - u| = \left| \int_M H(t, p, \bar{p}) (u(\bar{p}) - u(p)) \, d\text{vol}_g(\bar{p}) \right| \leq \int_{M^+} |H(t, p, \bar{p})| \cdot |u(\bar{p}) - u(p)| \, d\text{vol}_g(\bar{p}) + \int_{M^-} |H(t, p, \bar{p})| \cdot |u(\bar{p}) - u(p)| \, d\text{vol}_g(\bar{p}) \leq 2 \frac{t^{1/4}}{\delta(\epsilon)} \| u \|_0 \int_{M^+} |H(t, p, \bar{p})| \frac{d(p, \bar{p})}{t^{1/4}} \, d\text{vol}_g(\bar{p}) + \epsilon \int_{M^-} |H(t, p, \bar{p})| \, d\text{vol}_g(\bar{p}).$$

It may be checked in the various projective coordinates around the front face in the heat space $\mathcal{M}_h^2$, that $\beta^*(\|H\| \, d\text{vol}_g)$ and $\beta^*(d(p, \bar{p}) t^{-1/4}) \rho_{td}$ is bounded. Since $\beta^*|H|$ is vanishing to infinite order at $t_f$, we find that both integrals above are bounded uniformly in $(t, p, \epsilon)$. Therefore we obtain

$$\|e^{-t\Delta g} u - u\|_0 \leq C \left( \frac{t^{1/4}}{\delta(\epsilon)} \| u \|_0 + \epsilon \right).$$

Thus, for any given $\epsilon > 0$ we can estimate $\|e^{-t\Delta g} u - u\|_0 \leq 2\epsilon C$ for $t^{1/4} < \epsilon \delta(\epsilon)/\| u \|_0$. This proves strong continuity of the biharmonic heat operator on $\mathcal{C}^0(M)$. It remains to prove the case $k = 1$. Strong continuity of the biharmonic heat operator with respect to $\mathcal{C}^0(M)$ means $\|X^2 (e^{-t\Delta g} u - u)\|_0 \to 0$ as $t \to 0$, for $u \in \mathcal{C}^0(M)$ and $X \in \mathcal{D}$. If $X = \Delta$, this follows from the case $k = 0$, since $\Delta e^{-t\Delta g} u = e^{-t\Delta g} \Delta u$ for $u \in \mathcal{C}^0(M) \subset \mathcal{C}^0(\mathcal{D})$. For $X \in \{ \mathcal{V}^2, \mathcal{V}_0 \}$ the leading order of $\beta^*H$ at the front face is preserved under $X$, so that away from $t_d$, the estimates reduce to the case $k = 0$. Near $t_d$, a priori $X$ leads to $\rho_{td}^{-2}$ singular behaviour. However, integration by parts, exactly as worked out in detail in [BDV11] allows to pass $X$ to $u$, so that the estimates again reduce to the case $k = 0$. We write down the argument for completeness. The edge vector fields obey the following transformation rules in projective coordinates (3.13) near the temporal diagonal

$$\beta^*(x \partial_x) = -\eta \partial_\eta + \frac{1}{\eta} \partial_s + Z \partial_z + x \partial_x, \quad \beta^*(x \partial_y) = \frac{1}{\eta} \partial_u + x \partial_y, \quad \beta^*(\partial_z) = \frac{1}{\eta} \partial_z + \partial_\zeta.$$ 

By Proposition 3.5

$$\beta^*H(\eta, S, U, Z, x, y, z) = x^{-m} \eta^{-m} G(\eta, s, U, Z, x, y, z),$$

$$\beta^*(\bar{x}^i d\bar{x} \, d\text{vol}_{\text{out}}(\bar{x})) = h(x \eta)^m (1 - \eta S)^f \, dS \, dU \, dZ,$$
where $G$ is bounded in its entries, and in fact infinitely vanishing as $|(S, U, Z)| \to \infty$, and $h = h(\eta, x(1 - \eta S), y - x\eta U, z - \eta Z, x, y, z)$ is a bounded distribution on $\mathcal{M}_h^2$. We consider $||x\partial_x(e^{-t\Delta}u - u)||_0$. Using stochastic completeness of the heat kernel, we find

\[
F := x\partial_x(e^{-t\Delta}u - u) = \int (x\partial_x H)u(\tilde{x}, \tilde{y}, \tilde{z})d\tilde{x}d\tilde{y}d\tilde{z}

- \int (x\partial_x)\left[Hu(x, y, z)d\tilde{x}d\tilde{y}d\tilde{z}\right] =: F_1 - F_2.

Next we transform to projective coordinates and integrate by parts in $S$, where the boundary terms lie away from the diagonal and hence are vanishing to infinite order for $t \to 0$ by the asymptotic behaviour of the heat kernel. Omitting these irrelevant terms, we obtain

\[
F_1 = \int \left( -\eta\partial_\eta + \frac{1}{\eta}\partial_S + Z\partial_Z + x\partial_x \right) [(x\eta)^{-m}G(\eta, S, U, Z, x, y, z)]x\partial_x \left[(x\eta)^{-m}G(\eta, S, U, Z, x, y, z)\right]
\]

\[
\times u(x(1 - \eta S), y - x\eta U, z - \eta Z) h(x\eta)^m(1 - \eta S)^f dS dU dZ
\]

\[
= \int \left[ (-\eta\partial_\eta + Z\partial_Z + x\partial_x)(x\eta)^{-m}G \right] u h(x\eta)^m(1 - \eta S)^f dS dU dZ
\]

\[
- \int G \left[ \left( \frac{1}{\eta}\partial_S \right) \right] h(1 - \eta S)^f dS dU dZ
\]

\[
- \int (x\eta)^{-m}G \cdot u \left( \left( \frac{1}{\eta}\partial_S \right) h(x\eta)^m(1 - \eta S)^f \right) dS dU dZ.
\]

We perform similar computations for $F_2$:

\[
F_2 = \int \left[ (x\partial_x H)u(x, y, z) + Hu(x, y, z) \right] \tilde{x}d\tilde{x}d\tilde{y}d\tilde{z}
\]

\[
= \int \left[ (-\eta\partial_\eta + \frac{1}{\eta}\partial_S + Z\partial_Z + x\partial_x) \right] (x\eta)^{-m}G \cdot u h(x\eta)^m(1 - \eta S)^f dS dU dZ
\]

\[
+ \int G(\eta, S, U, Z, x, y, z)(x\partial_x u(x, y, z))h(1 - \eta S)^f dS dU dZ
\]

\[
= \int \left[ (-\eta\partial_\eta + Z\partial_Z + x\partial_x)(x\eta)^{-m}G \right] \cdot u h(x\eta)^m(1 - \eta S)^f dS dU dZ
\]

\[
- \int (x\eta)^{-m}G \cdot u \left( \left( \frac{1}{\eta}\partial_S \right) h(x\eta)^m(1 - \eta S)^f \right) dS dU dZ
\]

\[
+ \int G(\eta, S, U, Z, x, y, z)(x\partial_x u(x, y, z))h(1 - \eta S)^f dS dU dZ.
\]
Thus $F = F_1 - F_2$ becomes

$$
F = \int \left[ (-\eta \partial_{\eta} + Z \partial_Z + x \partial_x)(x\eta)^{-m}G(\eta, S, U, Z, x, y, z) \right] h(x\eta)^m (1 - \eta S)^f \\
\times (u(x(1 - \eta S), y - x\eta U, z - \eta Z) - u(x, y, z)) dS dU dZ \\
- \int G(\eta, S, U, Z, x, y, z) \left[ \left( \frac{1}{\eta} \partial_{\eta} \right) h \cdot (1 - \eta S)^f \right] \\
\times (u(x(1 - \eta S), y - x\eta U, z - \eta Z) - u(x, y, z)) dS dU dZ \\
- \int G \left[ \frac{1}{\eta} \partial_{\eta} u(x(1 - \eta S), y - x\eta U, z - \eta Z) + x \partial_x u(x, y, z) \right] h(1 - \eta S)^f dS dU dZ.
$$

Now, each of the three integrals is estimated as above for $k = 0$ by separating the integration region into $M^\varepsilon_+$ and $M^{-\varepsilon}_-$ for any given $\varepsilon > 0$. Note that in the final integral we use the fact that $u \in C^2(M, D')$ so that $\eta^{-1} \partial_{\eta} u$ and $x \partial_x u$ are bounded. Higher order and other edge derivatives may be estimated in a similar way. This proves strong continuity in general and as a trivial consequence boundedness of the biharmonic heat operator.

\begin{remark}
We point out that Theorem 4.1 holds also when the basic space $C^0_{ie}(M)$ is replaced by the Banach space of sections continuous up to $x = 0$, without the requirement of being fibrewise constant at $\partial M$. Also, we may set $D = \{\Delta, x^{-1} V^2, x^{-1} V_e, V_e\}$. The use of the refined space $C^1_{ie}(M)$ and the restriction of $x^{-1} V_e$ to $x^{-1} V'_e$ in $D$ becomes however crucial in Theorem 4.4.
\end{remark}

5. \textbf{Short time existence of semi-linear equations of fourth order}

In this section we explain how the mapping properties of the biharmonic heat operator and its strong continuity yields short-time existence of solutions to certain semilinear equations of fourth order. The underlying idea is based on the following theorem.

\textbf{Theorem 5.1.} \textbf{[TAY96, Proposition 15.1.1]} Let $P$ be some, possibly unbounded, linear operator in a Hilbert space $H$, bounded from below. Suppose that $V, W \subset H$ are Banach spaces, such that $P : V \to W$ is bounded and moreover

(i) $e^{-tP} : V \to V$ is a strongly continuous semigroup, for $t \geq 0$.
(ii) $Q : V \to W$ is locally Lipschitz.
(iii) $e^{-tP} : W \to t^{-\gamma} V$ bounded for some $\gamma < 1$.

Then for any $u_0 \in W$, the initial value problem

$$
\partial_t u - Pu = Q(u), \quad u(0) = u_0 \in W
$$

has a unique solution $u \in C([0, T], C^{2(k+1)}_{ie}(M, D))$, for some $T > 0$, where $T$ may be estimated from below in terms of $\|u_0\|_W$. The solution $u$ is the fixed point of the operator $F : V \to V$ with

$$
F(u) = e^{-tP} u_0 + \int_0^t e^{-(t-s)P} Q(u) ds.
$$

\textbf{Corollary 5.2.} Let $(M, g)$ be an incomplete edge space with a feasible edge metric $g$. Put $D' = \{\Delta, V^2_e, V_e\}$ and $D = \{\Delta, x^{-1} V^2, x^{-1} V_e, V_e\}$, where $V'_e \subset V_e$ consists locally of linear combinations of $\{x \partial_x, x \partial_y, \partial_z\}$, where $x \partial_y$ is weighted with functions

$$
\text{...}
$$
that are fibrewise constant. Assume $Q : \mathcal{C}^{2(k+1)}(\mathcal{M}, \mathcal{D}') \to \mathcal{C}^{2k}(\mathcal{M}, \mathcal{D}')$ is locally Lipschitz. Then the semilinear equation

$$\partial u + \Delta^2 u = Q(u), \quad u(0) = u_0 \in \mathcal{C}^{2k}(\mathcal{M}, \mathcal{D}')$$

has a unique solution $u \in C([0, T], \mathcal{C}^{2(k+1)}(\mathcal{M}, \mathcal{D}))$, for some $T > 0$, where $T$ may be estimated from below in terms of $\|u_0\|_{2k}$.

**Proof.** Consider first a slightly smaller set of operators $\mathcal{D}' = \{\Delta, \nabla^2, \nabla_e\}$ and set $W = \mathcal{C}^{2k}(\mathcal{M}, \mathcal{D}'), V = \mathcal{C}^{2(k+1)}(\mathcal{M}, \mathcal{D}')$. In view of Theorem 4.1 and Theorem 4.4, the heat operator associated to $\Delta^2$ satisfies the conditions of Theorem 5.1 with $\gamma = 1/4$.

Consequently, by Theorem 5.1 the unique solution $u$ exists and lies in $\mathcal{C}^{2(k+1)}(\mathcal{M}, \mathcal{D}')$. This solution is the fixed point of the map $F : \mathcal{C}^{2(k+1)}(\mathcal{M}, \mathcal{D}') \to \mathcal{C}^{2(k+1)}(\mathcal{M}, \mathcal{D}')$ with

$$F(u) = e^{-t\Delta^2}u_0 + \int_0^t e^{-(t-s)\Delta^2}Q(u)ds.$$ 

However, by Theorem 4.1, $F$ actually maps $\mathcal{C}^{2(k+1)}(\mathcal{M}, \mathcal{D}') \subset \mathcal{C}^{2(k+1)}(\mathcal{M}, D_0)$ to $\mathcal{C}^{2(k+1)}(\mathcal{M}, \mathcal{D})$. Consequently, $u \in \mathcal{C}^{2(k+1)}(\mathcal{M}, \mathcal{D})$, with a slightly better regularity, as claimed. \hfill \Box

We now apply this general existence result to the example of the Cahn-Hilliard equation on an incomplete edge manifold. We define

$$Q : \mathcal{C}^{2k+2}(\mathcal{M}, \mathcal{D}) \to \mathcal{C}^{2k}(\mathcal{M}, \mathcal{D}), \quad Q(u) := \Delta(u - u^3).$$

The mapping $Q$ is indeed locally Lipschitz, since for any $u, v \in \mathcal{C}^{2k+2}(\mathcal{M}, \mathcal{D})$

$$\|Q(u - v)\|_{2k} \leq \|\Delta(u - v)\|_{2k} + \|\Delta(u - v)^3\|_{2k}$$

$$\leq \|u - v\|_{2(k+1)} \left(1 + \|u - v\|_{2(k+1)}^2\right).$$

We hence arrive at our final result.

**Corollary 5.3.** Let $(\mathcal{M}, g)$ be an incomplete edge space with a feasible edge metric $g$. Put $\mathcal{D}' = \{\Delta, \nabla^2, \nabla_e\}$ and $\mathcal{D} = \{\Delta, x^{-1}\nabla^2, x^{-1}\nabla_e, \nabla_e\}$, where $\nabla_e \subset \nabla_c$ consists locally of linear combinations of $\{x\partial_x, x\partial_y, \partial_z\}$, where $x\partial_y$ is weighted with functions that are fibrewise constant. Then the Cahn-Hilliard equation

$$\partial_t u + \Delta^2 u + \Delta(u - u^3) = 0, \quad u(0) = u_0 \in \mathcal{C}^{2k}(\mathcal{M}, \mathcal{D}')$$

has a unique solution $u \in C([0, T], \mathcal{C}^{2k+2}(\mathcal{M}, \mathcal{D}))$, for some $T > 0$.

It should be noted that in correspondence with [RoSc12] our approach leads to an explicit identification of the asymptotics of the Cahn-Hilliard solution at $x = 0$. Indeed, $u \in \mathcal{C}^{2k+2}(\mathcal{M}, \mathcal{D}) \subset \mathcal{D}(\Delta^{k+1})$, which yields a partial asymptotics of $u$ to higher and higher order, depending on $k \in \mathbb{N}$, by an iterative application of Lemma 3.1 for $k$ steps.

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