Abstract

Initiated by Mizutani and Ito’s work in 1987, Kawamura and Allen recently showed that certain self-similar sets generalized by two similar contractions have a natural complex power series representation, which is parametrized by past-dependent revolving sequences.

In this paper, we generalize the work of Kawamura and Allen to include a wider collection of self-similar sets. We show that certain self-similar sets consisting of more than two similar contractions also have a natural complex power series representation, which is parametrized by $\Delta$-revolving sequences. This result applies to several other famous self-similar sets such as the Heighway dragon, Twindragon, and Fudgeflake.

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1 Introduction

The history of systematic mathematical research on self-similar fractals dates back to 1981, when Hutchinson proved the following celebrated theorem [5]:

\[ \text{ } \]
For any finite family of similar contractions \( \{\psi_0, \psi_1, \ldots, \psi_{m-1}\} \) on \( \mathbb{C} \), which is called an iterated function system or IFS, there exists a unique non-empty compact solution \( E \) of the set equation:

\[
E = \bigcup_{i=0}^{m-1} \psi_i(E).
\]

We call \( E \) an attractor or a self-similar set for the IFS.

It is well known that any point of \( E \) can be represented by at least one coding sequence \((x_i)_{i=1}^{\infty}\) such that

\[
E = \left\{ \lim_{n \to \infty} \psi_{x_1} \circ \psi_{x_2} \circ \cdots \psi_{x_n}(0) : (x_i)_{i=1}^{\infty} \in \{0, 1, 2, \ldots, m-1\}^\mathbb{N} \right\}. \quad (1.1)
\]

For more details, see Falconer’s book [3].

Eleven years prior to Hutchinson’s paper [5], C. Davis and D. E. Knuth [1] in 1970 introduced the idea that some tiling fractals are associated with the complex number system. This idea was further discussed by Gilbert [4]. In [1], Davis and Knuth introduced the concept of revolving sequences to represent Gaussian integers: for any \( z = x + iy \) with \( x, y \in \mathbb{Z} \), there exists a finite revolving sequence with length \( N \): \((\delta_0, \delta_1, \ldots, \delta_N)\) such that

\[
z = \sum_{n=0}^{N} \delta_{N-n}(1 + i)^n,
\]

where \( \delta_j \in \{0, 1, -1, i, -i\} \) with the restriction that the non-zero values must follow the cyclic pattern from left to right:

\[
\cdots \rightarrow 1 \rightarrow (-i) \rightarrow (-1) \rightarrow i \rightarrow 1 \rightarrow \cdots.
\]

Notice that a revolving sequence is created by a cycle of 90 degree clockwise rotation on the unit circle.

Let \( W \) be the set of all revolving sequences. M. Mizutani and S. Ito in 1987 [10] considered the following set of points in the complex plane:

\[
X := \left\{ \sum_{n=1}^{\infty} \delta_n(1 + i)^{-n} : (\delta_1, \delta_2, \ldots) \in W \right\}. \quad (1.2)
\]

Using techniques from symbolic dynamics, they proved that \( X \) (which they called Tetradragon) is tiled by four rotated paper-folding dragons. Paper-folding dragon is generated by the IFS:

\[
\begin{align*}
\psi_0(z) &= (\frac{1-i}{2})z, \\
\psi_1(z) &= (\frac{-1-i}{2})z + \frac{1-i}{2}.
\end{align*} \quad (1.3)
\]
Observe that revolving sequences \((\delta_n)_{n=1}^{\infty}\) are past-dependent sequences.

In the same paper, they mentioned an interesting conjecture: suppose \(\delta_n\) moves on the unit circle counterclockwise instead of clockwise, then

\[
X^R := \left\{ \sum_{n=1}^{\infty} \delta_n (1 + i)^{-n} : (\delta_1, \delta_2, \ldots) \in W \right\}
\]

is a union of four Lévy’s dragon curves. Lévy’s dragon is a well-known continuous curve with positive area. Mizutani and Ito’s computer simulation of \(X^R\) empirically confirmed the conjecture; however, they could not give a mathematical proof. Kawamura in 2002 [6] proved that their conjecture is correct using a different approach from the viewpoint of functional equations.

These two concrete examples indicate that not only tiling fractals but more generally some self-similar attractors can be parametrized by past-dependent sequences.

Recently, Kawamura and Allen [7] defined Generalized Revolving Sequences (GRS), where the 90 degree angle of rotation is replaced with a more general angle \(|\theta| = \frac{2\pi q}{p} \leq \pi\). In other words, \(\delta_k \in \{0, 1, e^{i\theta}, e^{2i\theta}, \ldots, e^{(p-1)i\theta}\}\) with the restriction that the non-zero values must follow the following cyclic pattern from left to right:

\[
\cdots \rightarrow 1 \rightarrow e^{i\theta} \rightarrow e^{2i\theta} \rightarrow e^{3i\theta} \rightarrow \cdots
\]

Define \(W_\theta\) to be the set of all generalized infinite revolving sequences with parameter \(\theta\). Kawamura and Allen considered the following set.

\[
X_{\alpha,\theta} := \left\{ \sum_{n=1}^{\infty} \delta_n \alpha^n : (\delta_1, \delta_2, \ldots) \in W_\theta \right\}.
\]  

(1.4)

where \(\alpha \in \mathbb{C}\) such that \(|\alpha| < 1\) and \(|\theta| = \frac{2\pi q}{p} \leq \pi\). Note that generalized revolving sequences are past-dependent.

Kawamura and Allen proved that \(X_{\alpha,\theta}\) is a union of \(p\) rotated copies of the self-similar attractor \(M_{\alpha,\theta}\) of an IFS consisting of two similarities with the same scale factor, one of which involves a rotation through an angle \(\theta\):

\[
\begin{align*}
\psi_0(z) &= \alpha z, \\
\psi_1(z) &= (\alpha e^{i\theta})z + \alpha.
\end{align*}
\]  

(1.5)

Figure 1 shows two examples of \(X_{\alpha,\theta}\). Observe that this is a generalization of Mizutani-Ito’s and Kawamura’s results: if \(\alpha = (1 - i)/2\) and \(\theta = -\pi/2\)
then $X_{\alpha,\theta}$ is a union of four paper-folding dragons \cite{10}. If $\alpha = (1 - i)/2$ and $\theta = \pi/2$ then $X_{\alpha,\theta}$ is a union of four Lévy dragon curves.

In this paper, we continue building on the work initiated by Mizutani and Ito to include a wider collection of self-similar sets. In particular, the following questions are discussed.

1. Does replacing the constant term $\alpha$ in (1.5) by an arbitrary constant give any major influence to the property of generalized revolving sequences in (1.4)?

2. Consider self-similar sets generated by the IFS consisting of more than two similar contractions. Can we find a complex power series representation, which is parametrized by past-dependent sequences? How does the number of contractions influence the properties of generalized revolving sequences?

In section 2, we introduce $\Delta$-revolving sequences allowing revolutions on the unit circle by more than one angle, while keeping the current-dependency. We illustrate the difference between generalized revolving sequences and $\Delta$-revolving sequences with examples.

In section 3 we show that certain self-similar attractors of IFSs consisting of $m \geq 2$ similar contractions have a natural complex power series representation, which is parametrized by $\Delta$-revolving sequences. This main theorem applies to several other famous self-similar sets such as the Heighway dragon, Twindragon, and Fudgeflake.

Lastly in section 4 we introduce a slight modification of $\Delta$-revolving sequences called $\Delta_0$-revolving sequences. $\Delta_0$-revolving sequences are a more
natural extension of the notion of generalized revolving sequences than \(\Delta\)-revolving sequences. As a corollary of the main theorem, we give a condition when certain self-similar attractors can also be parametrized by \(\Delta_0\)-revolving sequences.

\section{\(\Delta\)-Revolving Sequences}

First, we review some notations and a result from [7].

\textbf{Definition 2.1 (Revolving Angle).} We say \(\theta \in (-\pi, \pi]\) is a \textit{revolving angle} if \(\theta\) is a nonzero rational multiple of \(2\pi\), that is, \(|\theta| = (2\pi q)/p\) where \(p \in \mathbb{N}, q \in \mathbb{N}_0\).

\textbf{Definition 2.2 (Generalized Revolving Sequence).} Define

\[
\Delta_\theta := \{0, 1, e^{i\theta}, e^{2i\theta}, \ldots, e^{(p-1)i\theta}\},
\]

where \(\theta\) is a revolving angle. We say that a sequence \((\delta_n) \in \Delta^\mathbb{N}_\theta\) satisfies the Generalized Revolving Condition (GRC), if

1. \(\delta_1\) is free to choose.
2. If \(\delta_1 = \delta_2 = \cdots = \delta_n = 0\), then \(\delta_{n+1}\) is free to choose.
3. Otherwise, \(\delta_{n+1} = 0 \text{ or } \delta_{n+1} = \delta_{j_0(n)}e^{i\theta}\), where \(j_0(n) = \max\{j \leq n : \delta_j \neq 0\}\).

\textbf{Example 2.3.} If \(\theta = \pi/3\), \(\Delta_\theta := \{0, 1, e^{i\pi/3}, e^{2i\pi/3}, e^{i\pi}, e^{4i\pi/3}, e^{5i\pi/3}\}\).

\[
(\delta_n) = (0, 1, e^{i(\pi/3)}, 0, e^{i(2\pi/3)}, 0, 0, e^{i\pi}, e^{i(4\pi/3)}, 0, \ldots)
\] (2.1)

is a generalized revolving sequence.

Define \(W_\theta\) as the set of all generalized revolving sequences with parameter \(\theta\):

\[
W_\theta := \{(\delta_1, \delta_2, \cdots) \in \Delta^\mathbb{N}_\theta : (\delta_1, \delta_2, \cdots) \text{ satisfies the GRC}\}.
\]

Observe that \(\delta_n\) moves on the unit circle counterclockwise if \(\theta > 0\), and clockwise if \(\theta < 0\). Roughly speaking, \(j_0(n)\) is the last time before \(n\) that \(\delta_j\) is on the unit circle. Therefore, generalized revolving sequences are past-dependent.
Theorem 2.4 (Kawamura-Allen, 2020). For a given $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$ and $\theta$ is a revolving angle, define

$$X_{\alpha,\theta} := \left\{ \sum_{n=1}^{\infty} \delta_n \alpha^n : (\delta_n) \in W_\theta \right\}.$$ 

Then $X_{\alpha,\theta}$ is a union of $p$ rotated copies of $M_{\alpha,\theta}$:

$$X_{\alpha,\theta} = \bigcup_{l=0}^{p-1} (e^{i\theta})^l M_{\alpha,\theta},$$

where $M_{\alpha,\theta} = \psi_0(M_{\alpha,\theta}) \cup \psi_1(M_{\alpha,\theta})$ is the self-similar set generated by the IFS

$$\begin{cases} 
\psi_0(z) = \alpha z, \\
\psi_1(z) = (\alpha e^{i\theta}) z + \alpha.
\end{cases}$$

Next, we loosen the restriction of a generalized revolving sequence, allowing revolutions on the unit circle by more than one angle.

Definition 2.5 (Generator Set). Let $S := \{\theta_0, \theta_1, \ldots, \theta_{m-1}\}$ where $\theta_0, \theta_1, \ldots, \theta_{m-1}$ are distinct angles and $\theta_0 = 0$. Its elements can be written as $\theta_j = \frac{2\pi q_j}{p_j}$. We refer to the set $S$ as a generator set.

Definition 2.6 (Revolving Group Generated by $S$). Let $S = \{\theta_0, \theta_1, \ldots, \theta_{m-1}\}$ be a generator set. Define

$$\Delta := \left\{ e^{i\sum_{n=1}^{m-1} k_n \theta_n} : k_n \in \{0, 1, \ldots, p_n - 1\} \right\}.$$ 

Example 2.7. If $S = \{0, \pi, \frac{2\pi}{3}\}$, then

$$\Delta = \{1, e^{i\pi}, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}, e^{i(\pi+\frac{2\pi}{3})}, e^{i(\pi+\frac{4\pi}{3})}\}.$$ 

From this example, it is clear that $\Delta$ is a group, which we call the revolving group generated by $S$. Notice that $\Delta$ does not contain 0, and $|\Delta| = \text{lcm}(|p_1|, \ldots, |p_{m-1}|)$.

Definition 2.8 (\(\Delta\)-Revolving Sequence). We say that a sequence $(\gamma_n) \in \Delta^\mathbb{N}$ satisfies the \(\Delta\)-Revolving Condition (DRC), if

1. $\gamma_1$ is free to choose in $\Delta$,

2. For $k \geq 1$, $\gamma_{k+1} = \gamma_k e^{i\theta_j}$ for some $\theta_j \in S$. 


Figure 2: Cayley Digraph for generator set $S = \{0, \frac{2\pi}{3}, \pi\}$

If $(\gamma_n)$ satisfies the DRC, we call $(\gamma_n)$ a $\Delta$-Revolving Sequence.

**Example 2.9.** If $S = \{0, \pi, \frac{2\pi}{3}\}$, $\Delta = \{1, e^{i(\pi)}, e^{i(\frac{2\pi}{3})}, e^{i(\pi + \frac{2\pi}{3})}, e^{i(\pi + \frac{4\pi}{3})}\}$. And, for examples,

$$(\gamma_n) = (1, e^{i\pi}, e^{i(\pi + \frac{2\pi}{3})}, e^{i(\pi + \frac{4\pi}{3})}, e^{i(\frac{2\pi}{3})}, \ldots) \quad (2.2)$$

is a $\Delta$-revolving sequence.

Define $W^\Delta$ as the set of all $\Delta$-revolving sequences with generator set $S$ and $W_1^\Delta$ be the subset of $W^\Delta$ with the restriction: $\gamma_1 = 1$.

Compare examples: $(2.1)$ and $(2.2)$. It is clear that the generalized revolving condition is associated with a single angle of rotation on the unit circle, while the $\Delta$-revolving condition is associated with multiple angles of rotation on the unit circle. Observe also that generalized revolving sequences allow that $\delta_n$ goes back to the origin and stays there as long as it wants, while $\Delta$-revolving sequences do not allow that $\gamma_n$ goes back to the origin; however, they allow "staying in place". Informally, each digit $\gamma_n$ can choose freely either to stay ($\theta_0 = 0$) or move ($\theta_j \in S \setminus \{\theta_0\}$). However, the next digit $\gamma_{k+1}$ is determined based on the current position $\gamma_k$. Therefore, $\Delta$-revolving sequences are current-dependent, while the generalized revolving sequences are past-dependent.

Alternatively, due to the group structure of $\Delta$, a $\Delta$-Revolving Sequence can be thought of as an infinite walk on the Cayley Digraph of the group $\Delta$ generated by the set with elements corresponding to $S$. 
3 Main results

Lemma 3.1. Consider a standard IFS \(\{\psi_0, \psi_1, \ldots, \psi_{m-1}\}\) given by

\[
\psi_k(z) = \alpha_k z + c_k, \tag{3.1}
\]

where \(\alpha_k \in \mathbb{C}\) such that \(|\alpha_k| < 1\) and \(c_k \in \mathbb{C}\) for \(k = 0, 1, \ldots, m - 1\).

Then the self-similar set \(E = \bigcup_{i=0}^{m-1} \psi_i(E)\) generated by (3.1) has the following natural series representation:

\[
E = \left\{ \sum_{n=1}^{\infty} c_{x_n} \prod_{k=0}^{m-1} \alpha_k^{I_k(x,n-1)} : x = (x_n) \in \{0, 1, \ldots, m - 1\}^\mathbb{N} \right\}, \tag{3.2}
\]

where \(I_k(x,n) := \#\{j \leq n : x_j = k\}\) for \(k = 0, 1, \ldots, m - 1\) and \(n \in \mathbb{N}\).

Notice that \(I_k(x,n)\) is the number of \(k\)'s occurring in the first \(n\) digits of the sequence \((x_n)\).

Proof. From (1.1), it follows that every point in \(E\) can be represented by at least one coding sequence \((x_n)\) given in (3.2). Vice versa, the set on the right of (3.2) satisfies the set equation \(E = \bigcup_{i=0}^{m-1} \psi_i(E)\). Let \(M := \{0, 1, \ldots, m - 1\}^\mathbb{N}\). It is well known that \(M\) is compact in the product topology. Define the function \(f : M \to \mathbb{C}\) by

\[
f(x) = \sum_{n=1}^{\infty} c_{x_n} \prod_{k=0}^{m-1} \alpha_k^{I_k(x,n-1)}.\]

Since it is easily seen that \(f\) is continuous and \(f(M) = E\), it follows that \(E\) is compact so it is closed. \(\square\)

Before presenting the main theorem, let us define one more notation.

Definition 3.2 (Constant Sequence of \((\gamma_n)\)). Let \(S = \{\theta_0, \theta_1, \ldots, \theta_{m-1}\}\) be a generator set and \(\Delta\) be the revolving group generated by \(S\).

For \(\Sigma := (\gamma_n) \in W^\Delta\) and \(c_k \in \mathbb{C}\) for \(k = 0, 1, \ldots, m - 1\), define the function \(s_{\Sigma} : \mathbb{N} \to \{c_0, c_1, \ldots, c_{m-1}\}\) by

\[
s_{\Sigma}(n) = c_k, \text{ if } \gamma_{n+1} = \gamma_n e^{i\theta_k}.\]

We say that the sequence \((s_n)\) given by \(s_n = s_{\Sigma}(n)\) is the constant sequence of \((\gamma_n)\). Notice that we allow \(c_l = c_k\) for \(l \neq k\).
Theorem 3.3. Let $S = \{\theta_0, \theta_1, \ldots, \theta_{m-1}\}$ be a generator set and $\Delta$ be the revolving group generated by $S$. For $\alpha \in \mathbb{C}$ with $|\alpha| < 1$ and $c_0, c_1, \ldots, c_{m-1} \in \mathbb{C}$, define $X_{\alpha, S}$ as follows.

$$X_{\alpha, S} = \left\{ \sum_{n=1}^{\infty} \alpha^{n-1} s_n \gamma_n : \Sigma = (\gamma_n) \in W^\Delta, s_n = s_{\Sigma(n)} \right\}. \quad (3.3)$$

Then $X_{\alpha, S}$ is the union of $|\Delta|$ many rotated copies of the self-similar attractor $T_{\alpha, S}$:

$$X_{\alpha, S} = \bigcup_{\gamma \in \Delta} \gamma \cdot T_{\alpha, S},$$

where $T_{\alpha, S}$ is generated by the IFS:

$$\begin{align*}
\psi_0(z) &= \alpha z + c_0, \\
\psi_k(z) &= (\alpha e^{i\theta_k}) z + c_k, \quad k = 1, 2, \ldots, m - 1.
\end{align*}$$

Proof.

For a given generator set $S = \{\theta_0, \theta_1, \ldots, \theta_{m-1}\}$, set $\alpha_k = \alpha e^{i\theta_k}$ for $k = 0, 1, \ldots, m - 1$ in Lemma 3.1. Note that $\alpha_0 = \alpha$ since $\theta_0 = 0$. Then it follows from (3.2) that the self-similar set $T_{\alpha, S}$ has the following representation.

$$T_{\alpha, S} = \left\{ \sum_{n=1}^{\infty} c_{x_n} \prod_{k=0}^{m-1} (\alpha e^{i\theta_k}) I_k(x, n-1) : x = (x_n) \in \{0, 1, \ldots, m-1\}^\mathbb{N} \right\}. \quad (3.4)$$

Observe that $\sum_{k=0}^{m-1} I_k(x, n-1) = n - 1$ and

$$\prod_{k=0}^{m-1} e^{i\theta_k \cdot I_k(x, n-1)} = e^{i \sum_{k=0}^{m-1} I_k(x, n-1) \cdot \theta_k} = e^{i \sum_{j=1}^{n-1} \theta x_j},$$

Thus,

$$T_{\alpha, S} = \left\{ \sum_{n=1}^{\infty} \alpha^{n-1} c_{x_n} e^{i \sum_{j=1}^{n-1} \theta x_j} : (x_n) \in \{0, 1, \ldots, m-1\}^\mathbb{N} \right\}. \quad (3.4)$$

Define $X_{1, \alpha, S}$ given by:

$$X_{1, \alpha, S} = \left\{ \sum_{n=1}^{\infty} \alpha^{n-1} s_n \gamma_n : \Sigma = (\gamma_n) \in W^\Delta, \gamma_1 = 1, s_n = s_{\Sigma(n)} \right\}.$$
We prove that $X_{1,\alpha,S} = T_{\alpha,S}$. Let
\[
\gamma_{n} = e^{i \sum_{j=1}^{n-1} \theta_{j}}.
\]
Then it is clear that there exists a coding:
\[
\gamma_{1} = 1, \quad \gamma_{n+1} = \gamma_{n} \cdot e^{i \theta_{n}}
\]
for $n = 1, 2, \ldots$. Therefore, $(\gamma_{n})$ satisfies the DRC with the restriction that the first term is 1.

Let $s_{n} = c_{x_{n}}$. It is clear that $(s_{n})$ is the constant sequence of $(\gamma_{n})$.

Conversely, take a point $z \in X_{1,\alpha,S}$, which is generated by a sequence $(\gamma_{n})_{n \in \mathbb{N}}$. Construct a sequence $(x_{n}) \in \{0, 1, \ldots, m-1\}^{\mathbb{N}}$ by
\[
x_{n} = k, \quad \text{if } \gamma_{n+1} = \gamma_{n} e^{i \theta_{k}},
\]
for $k = 0, 1, 2, \ldots (m-1)$. Notice that $\theta_{k}$ is a unique revolving angle for each $k$ so that $(x_{n})$ is determined uniquely by $(\gamma_{n})$.

Recall that for a given generator set $S$, the revolving group generated by $S$ has lcm$(|p_{1}|, \ldots, |p_{m-1}|)$ many elements where $p_{i}$ is the denominator of $\theta_{i}$ in reduced form. Thus,
\[
X_{\alpha,S} = \bigcup_{\gamma \in \Delta} \gamma \cdot X_{1,\alpha,S} = \bigcup_{\gamma \in \Delta} \gamma \cdot T_{\alpha,S}.
\]

Example 3.4. Mandelbrot introduced a tiling fractal, known as the fudge-flake in his classic book (see page 72 in [8]). The fudgeflake is a self-similar attractor generated by three similar contractions:
\[
\begin{cases}
\psi_{1}(z) = \alpha z, \\
\psi_{2}(z) = (\alpha e^{i \theta_{1}})z + \alpha, \\
\psi_{3}(z) = (\alpha e^{i \theta_{2}})z + \bar{\alpha},
\end{cases}
\tag{3.5}
\]
where $\alpha = \frac{1}{2} - \frac{\sqrt{3}}{6} i$, $\theta_{1} = \pi/3$ and $\theta_{2} = -2\pi/3$. For more details, see page 22-23 in Edger’s book [2].

It is clear that Theorem 3.3 includes this particular case. Figure 3 shows several examples of $X_{\alpha,S}$; in particular, the self-similar attractors of (3.5).

Terdragon is another famous tiling self-similar attractor generated by (3.5) if $\alpha = \frac{1}{2} - \frac{\sqrt{3}}{6} i$, $\theta_{1} = 2\pi/3$ and $\theta_{2} = 0$. (See page 163 in [2]). However, notice that Theorem 3.3 excludes this case since $\theta_{2} = 0$. 
Figure 3: Examples of $X_{\alpha,S}$: $(m, c_0, c_1, c_2, \alpha) = (3, 0, \alpha, \bar{\alpha}, \frac{1}{2} - \frac{\sqrt{3}i}{3})$. $(\theta_1, \theta_2) = (\pi/3, -2\pi/3)$: a union of 3 fudgeflake (top), $(\theta_1, \theta_2) = (\pi/2, -\pi/2)$: (bottom left), $(\theta_1, \theta_2) = (\pi/3, -\pi/3)$: (bottom right).
Example 3.5. The Heighway dragon and the Twindragon are famous self-similar attractors, not generated by (1.5) but by a slightly different pair of two similar contractions:

$$\begin{align*}
\phi_0(z) &= \alpha z, \\
\phi_1(z) &= (\alpha e^{i\theta})z + 1.
\end{align*}$$

(3.6)

In particular, if $\alpha = (1 + i)/2$ and $\theta = \pi/2$, then the IFS generates the Heighway dragon. If $\alpha = (1 + i)/2$ and $\theta = \pi$, then the Twindragon is generated [2].

Let $S = \{0, \theta\}$, $c_0 = 0$ and $c_1 = 1$ in Theorem 3.3. Since

$$s_n = s_\Sigma(n) = \begin{cases} 
0, & \text{if } \gamma_{n+1} = \gamma_n, \\
1, & \text{if } \gamma_{n+1} = \gamma_ne^{i\theta},
\end{cases}$$

$X_{\alpha,S}$ can also be parametrized by generalized revolving sequences $(\delta_n)$.

$$X_{\alpha,S} = \left\{ \sum_{n=0}^{\infty} \delta_n \alpha^n : (\delta_n) \in W_\theta \right\}. \quad (3.7)$$

Note that the only difference between (3.7) and (1.4) is whether the sum begins from $n = 0$ or $n = 1$. It is not surprising. Let $H_{\alpha,\theta} = \phi_0(H_{\alpha,\theta}) \cup \phi_1(H_{\alpha,\theta})$ be the self-similar set generated by the IFS (3.6). Notice that $\psi_0(z) = \alpha \phi_0(z/\alpha)$ and $\psi_1(z) = \alpha \phi_1(z/\alpha)$ in (1.5) and (3.6). Thus, it is easy to see that $H_{\alpha,\theta} = M_{\alpha,\theta}/\alpha$.

Remark 3.6. Theorem 2.4 is a special case of Theorem 3.3. Let $S = \{0, \theta_1\}$, $c_0 = 0$ and $c_1 = \alpha$ so that

$$s_n = \begin{cases} 
0, & \text{if } \gamma_{n+1} = \gamma_n, \\
\alpha, & \text{if } \gamma_{n+1} = \gamma_ne^{i\theta_1}.
\end{cases}$$

Observe that $\delta_n = \alpha^{-1} \gamma_n s_n$ since (i) $\delta_n = 0$ if $\gamma_{n+1} = \gamma_n$ and (ii) $\delta_n = \gamma_n$ if $\gamma_{n+1} = \gamma_ne^{i\theta_1}$. Therefore,

$$\left\{ \sum_{n=1}^{\infty} \alpha^{-1} \gamma_n s_n : \Sigma = (\gamma_n) \in W^\Delta, \ s_n = s_\Sigma(n) \right\} = \left\{ \sum_{n=1}^{\infty} \delta_n \alpha^n : (\delta_n) \in W_\theta \right\},$$

where $W^\Delta$ is the set of all $\Delta$-revolving sequences, and $W_\theta$ is the set of all generalized revolving sequences.
4 Remark

First, we define a slight modification of $\Delta$-revolving sequences as follows.

**Definition 4.1 ($\Delta_0$-Revolving Sequence).** Let $\Delta_0 := \Delta \cup \{0\}$. We say that a sequence $(\delta_n) \in \Delta_0^N$ satisfies the $\Delta_0$-Revolving Condition (DZRC) if

1. $\delta_1$ is free to choose in $\Delta_0$
2. If $\delta_1 = \delta_2 = \cdots = \delta_n = 0$, then $\delta_{n+1}$ is free to choose in $\Delta_0$.
3. Otherwise, $\delta_{n+1} = 0$ or $\delta_{n+1} = \delta_{j_0(n)} e^{i\theta_k}$ for some $\theta_k \in S \backslash \{0\}$,

where $j_0(n) = \max\{j \leq n : \delta_j \neq 0\}$.

**Example 4.2.** If $S = \{0, \pi, \frac{2\pi}{3}\}$, $\Delta_0 = \{0, 1, e^{i\pi}, e^{i(\frac{2\pi}{3})}, e^{i(\frac{4\pi}{3})}, e^{i(\pi + \frac{2\pi}{3})}, e^{i(\pi + \frac{4\pi}{3})}\}$, and, for examples,

$$ (\delta_n) = (1, e^{i\pi}, 0, e^{i(\pi + \frac{2\pi}{3})}, 0, 0, e^{i(\frac{2\pi}{3})}, \cdots) \quad (4.1) $$

is a $\Delta_0$-revolving sequence.

We denote the set of all $\Delta_0$-revolving sequences with generator set $S$ by $W_{\Delta_0}$.

Compare examples: (2.2) and (4.1). Observe that $\Delta_0$-revolving sequences are a more natural extension of the notation of generalized revolving sequences than $\Delta$-revolving sequences. In fact, $\Delta_0$-revolving sequences are past-dependent.
Now, a question arises naturally: for a given $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$ and a generator set $S$, define

$$X^*_{\alpha,S} := \left\{ \sum_{n=1}^{\infty} \delta_n \alpha^n : (\delta_n) \in W^{\Delta_0} \right\}.$$ 

It is not hard to imagine that $X^*_{\alpha,S}$ is a union of self-similar sets; however, it is not immediately clear which IFS generates these self-similar sets.

**Corollary 4.3.** $X^*_{\alpha,S}$ is the union of $|\Delta|$ many rotated copies of the self-similar attractor $T^*_{\alpha,S}$:

$$X^*_{\alpha,S} = \bigcup_{\gamma \in \Delta} \gamma \cdot T^*_{\alpha,S},$$

where $T^*_{\alpha,S}$ is generated by the IFS:

$$\begin{align*}
\psi_0(z) &= \alpha z, \\
\psi_k(z) &= (\alpha e^{i\theta_k}) z + \alpha, \quad k = 1, 2, \ldots, m - 1.
\end{align*}$$

**Remark 4.4.** In general, self-similar attractors generated by

$$\begin{align*}
\psi_0(z) &= \alpha z, \\
\psi_k(z) &= (\alpha e^{i\theta_k}) z + c_k, \quad k = 1, 2, \ldots, m - 1.
\end{align*}$$

cannot be parametrized by $\Delta_0$-revolving sequences. Consider (3.4) in the proof of Theorem 3.3. Since $c_0 = 0$, let $\delta_n = I(x_n) e^{i \sum_{j=1}^{n-1} \theta_{x_j}}$, where

$$I(x_n) = \begin{cases} 
0, & \text{if } x_n = 0, \\
1, & \text{if } x_n \neq 0.
\end{cases}$$

Then it is clear that there exists a $\Delta_0$-revolving sequence $(\delta_n)$ given by

$$\delta_n = \begin{cases} 
0, & \text{if } x_n = 0, \\
\delta_{j_0(n-1)} e^{i \theta_{x_{j_0(n-1)}}}, & \text{if } j_0(n-1) \text{ exists}, \\
1, & \text{if } j_0(n-1) \text{ does not exist},
\end{cases}$$

where $j_0(n) = \max\{j \leq n : x_j \neq 0\}$. However, notice that there is an issue to define $(x_n)$ from $(\delta_n)$, if $(\delta_n)$ is a sequence whose terms are eventually all 0.
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