EFFECTIVE GEOMETRY IN ASTROPHYSICS

S. E. PEREZ BERGLIAFFA

Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150,
Urca 22290-180 Rio de Janeiro, RJ – Brazil
sepb@cbpf.br

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The effective metric is introduced by means of two examples (non-linear electromagnetism and hydrodynamics), along with applications in Astrophysics. A sketch of the generality of the effect is also given.

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1. Introduction

One of the fundamental building blocks of Einstein’s General Relativity is the fact that every form of matter and energy couples to gravity. As M. Novello put in a picturesque way, “I fall, then I exist”\(^1\). If this universal coupling were broken, gravitation could not have been identified with the geometry of spacetime, experienced by every form of matter. This feature distinguishes gravity from the other forces that we know: no other interaction has this universal character. In spite of this, in certain circumstances it would be useful to have partial geometrization schemes, in which particles move following geodesics of a metric determined by the interaction with a background field. Quite unexpectedly, this geometrization program has been carried out in systems that are very different in nature: ordinary non-viscous fluids, superfluids, flowing and non-flowing dielectrics, non-linear electromagnetism in vacuum, and Bose-Einstein condensates, to name some of them\(^2\). As we shall see below, the underlying feature shared by these systems is that the behaviour of the fluctuations around a background solution of the equations of motion (EOM) is governed by an “effective metric”. More precisely, the particles associated to the perturbations do not follow geodesics of the background spacetime but of a Lorentzian geometry
described by the effective metric, which depends on the background solution and on the features of the interaction.

The effective geometry has permitted the simulation of several configurations of the gravitational field, such as wormholes and closed space-like curves for photons, and warped spacetimes for phonons. Particular attention has been paid to analog black holes, because these would emit Hawking radiation exactly as gravitational black holes do, and are obviously much easier to generate in the laboratory. The fact that analog black holes emit thermal radiation was shown first by Unruh in the case of dumb black holes, and it is the prospect of observing this radiation (thus testing the hypothesis that the thermal emission is independent of the physics at arbitrarily short wavelengths) that motivates the quest for a realization of analog black holes in the laboratory.

A less explored consequence of the effective metric approach is that the geometrical tools of General Relativity can be used to study non-gravitational systems. In particular, we shall see below that in a few instances the idea of effective geometry has been applied in Astrophysics. I shall begin by presenting in Sec. 2 the basics of the idea of effective geometry in the example of non-linear electromagnetism, along with its application to magnetars. In Sec. 3 it will be shown that the notion of effective metric also arises naturally in hydrodynamics, and we shall see that it can be fruitfully applied to the problem of accretion onto a black hole. In an appendix, it will be shown that the effective geometry is a generic consequence of perturbing the equations of motion of a given field theory around a fixed background. We shall close with a discussion.

2. The effective metric in non-linear electromagnetism

Historically, the first example of the idea of effective metric was presented by W. Gordon in 1923, who showed that light with wave-vector \( k_\mu \) in a moving non-dispersive medium with slowly varying refractive index \( n \) and 4-velocity \( u^\mu \) moves according to \( g^{\mu\nu}k_\mu k_\nu = 0 \) (see Ref. 11 for details), where

\[
g^{\mu\nu} = \eta^{\mu\nu} + (n^2 - 1)u^\mu u^\nu \tag{1}
\]

is the effective metric for this problem. It must be noted that only photons in the geometric optics approximation move on geodesics of \( g^{\mu\nu} \): the particles that compose the fluid couple instead to the background flat metric. After this early contribution, the effective metric occasionally resurfaced in the literature. A good deal of work has been devoted lately to non-linear electromagnetism, the case we shall examine next. We shall concentrate in non-linear electromagnetic theories in vacuum. Let us

\( ^a \)Only some kinematical aspects of General Relativity can be imitated by this method, but not its dynamical features (see however Refs. 9 and 13).
start with the action
\[ S = \int \sqrt{-\gamma} \mathcal{L}(F) \, d^4x, \]  
(2)
where \( F \equiv F^{\mu\nu}F_{\mu\nu} \), and \( \mathcal{L} \) is an arbitrary function of \( F \). Notice that \( \gamma \) is the determinant of the background metric, which we take in the following to be that of flat spacetime\(^b\). Varying this action w.r.t. the potential \( A_\mu \), related to the field by the expression \( F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu} \), we obtain the Euler-Lagrange EOM
\[ (\sqrt{-\gamma} \mathcal{L}_F F^{\mu\nu})_{,\nu} = 0, \]  
(3)
where \( \mathcal{L}_F \equiv \frac{\delta \mathcal{L}}{\delta F} \). In the particular case of a linear dependence of the Lagrangian with the invariant \( F \) we recover Maxwell’s equations of motion.

As mentioned in the Introduction, we want to study the behaviour of perturbations of these EOM around a fixed background solution. Instead of using the traditional perturbation method\(^{15}\), we shall use a more elegant method set out by Hadamard\(^{16}\), in which the propagation of high-energy photons is studied by following the evolution of the wave front. It is assumed that the field is is continuous through the wave front, but its first derivative is not. To be specific, let \( \Sigma \) be the surface of discontinuity defined by the equation \( \Sigma(x^\mu) = \text{constant} \). The discontinuity of a function \( J \) through the surface \( \Sigma \) will be represented by \( [J]_\Sigma \), and its definition is
\[ [J]_\Sigma \equiv \lim_{\delta \to 0^+} (J|_{\Sigma+\delta} - J|_{\Sigma-\delta}). \]
The discontinuities of the field and its first derivative are given by
\[ [F_{\mu\nu}]_\Sigma = 0, \quad [F_{\mu\nu,\lambda}]_\Sigma = f_{\mu\nu}k_\lambda, \]  
(4)
where the vector \( k_\lambda \) is nothing but the normal to the surface \( \Sigma \), that is, \( k_\lambda = \Sigma_{,\lambda} \). Let us apply this technique to the case of a nonlinear Lagrangian for the electromagnetic field, given by \( \mathcal{L}(F) \). Taking the discontinuity of the EOM (3), we get
\[ \mathcal{L}_F f^{\mu\nu}k_\nu + 2\eta \mathcal{L}_F F^{\mu\nu}k_\nu = 0, \]  
(5)
where \( \eta \equiv F^{\alpha\beta}f_{\alpha\beta} \). Defining \( F^*_{\mu\nu} \equiv \frac{1}{2} \eta_{\mu\nu\alpha\beta}F^{\alpha\beta} \), where \( \eta_{\mu\nu\alpha\beta} \) is the completely antisymmetric Levi-Civita tensor, the second Maxwell equation is given by \( F^{*\mu\nu} ;\nu = 0 \) or equivalently,
\[ F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0. \]  
(6)
The discontinuity of this equation yields
\[ f_{\mu\nu}k_\lambda + f_{\nu\lambda}k_\mu + f_{\lambda\mu}k_\nu = 0. \]  
(7)
Multiplying this equation by \( k^\lambda F^{\mu\nu} \) gives
\[ \eta k^2 + F^{\mu\nu} f_{\nu\lambda}k^\lambda k_\mu + F^{\mu\nu} f_{\lambda\mu}k^\lambda k_\nu = 0, \]  
(8)
\(^b\)The same techniques can be applied when the background is curved (see for instance Ref. 14).
where \( k^2 \equiv k_{\mu}k_{\nu}\gamma^{\mu\nu} \). Substituting in this equation the term \( f^{\mu\nu}k_{\nu} \) from Eq.(5) yields

\[
\eta k^2 - 2\frac{\mathcal{L}_{FF}}{\mathcal{L}_F}\eta(F^{\mu\lambda}k_\mu k_\lambda - F^{\lambda\mu}k_\mu k_\lambda) = 0,
\]

which can be written as \( g^{\mu\nu}k_{\mu}k_{\nu} = 0 \), where

\[
g^{\mu\nu} = \gamma^{\mu\nu} - 4\frac{\mathcal{L}_{FF}}{\mathcal{L}_F}F^{\mu\nu}.
\]

We then conclude that the high-energy photons\(^6\) of a nonlinear theory of electrodynamics with \( \mathcal{L} = \mathcal{L}(F) \) do not propagate on the null cones of the background metric but on the null cones of an effective metric, generated by the self-interaction of the electromagnetic field. It is important to realize that all the quantities in these expressions are evaluated at the background configuration. In other words, the nonlinear EOM must be solved for a given source in order to calculate the effective metric for that background. Note also that there are two metrics in the problem: the background metric (which is seen by all every type of matter but photons), and the effective metric (experienced only by high-energy photons).

Let us briefly sketch the case in which the Lagrangian depends also on the invariant \( G = F^{\mu\nu}F^{\ast\mu\nu} \). The EOM for a nonlinear \( \mathcal{L} = \mathcal{L}(F,G) \) are

\[
(\mathcal{L}_F F^{\mu\nu} + \mathcal{L}_G F^{*\mu\nu})_{;\nu} = 0, \quad F^{*\mu\nu}_{;\mu} = 0.
\]

Following the procedure employed in the case \( \mathcal{L} = \mathcal{L}(F) \), it can be shown that light rays do not follow geodesics of the background spacetime but of the effective metric\(^7\)

\[
g^{\mu\nu} = \mathcal{L}_F\eta^{\mu\nu} - 4\left[ (\mathcal{L}_{FF} + \Omega_{i} \mathcal{L}_{FG}) F^{\nu}_{\lambda} F^{\lambda\nu} + (\mathcal{L}_{FG} + \Omega_{i} \mathcal{L}_{GG}) F^{\nu\lambda}_{\chi} F^{*\nu\lambda} \right]
\]

where

\[
\Omega_{\pm} = -\frac{\Omega_2 \pm \sqrt{\Delta}}{2\Omega_1}, \quad \Delta = (\Omega_2)^2 - 4\Omega_1\Omega_3,
\]

and \( \Omega_i (i = 1, 2, 3) \) depend on \( \mathcal{L}_i \) and \( \mathcal{L}_{ij} \), with \( i, j = F, G \). We see that in general nonlinear electrodynamics displays bi-refringence. That is, the two polarization states of the photon propagate in a different way. In some special cases, there is also bi-metricity (one effective metric for each state)\(^8\). Bi-metricity is absent in every theory in which \( \Delta = 0 \), as for instance in Born-Infeld\(^7\).

It is known that non-linear EOM for the electromagnetic field arise in the strong-field regime, when quantum corrections must be taken into account. Objects like as magnetars\(^9\), charged black holes, and super-conducting cosmic strings may have such very intense fields. As an example of the usefulness of the effective metric in this realm, let us discuss the corrections to gravitational redshift in magnetars\(^10\).

\(^6\)For considerations on the regime in which Hadamard’s approach is valid, see Ref. 13.
Gravity effects cause the observed energies of the spectral lines of excited atoms in a compact object to be shifted to lower values by a factor
\[
\frac{1}{(1+z)} \equiv \left(1 - \frac{2G}{c^2} \frac{M}{R}\right)^{1/2}.
\] (14)

If, as in the case of magnetars, the object is endowed with magnetic field larger than the critical value \(B_{\text{crit}} \approx 10^{14} \text{G}\), corrections to this expression coming from non-linearities originating in vacuum polarization are to be expected. These corrections can be calculated using Eq. (10). As shown in Ref. 19, Eq. (14) must be replaced by
\[
z + 1 \approx \left(1 - \frac{2GM}{c^2 R}\right)^{-\frac{1}{2}} = \left(1 - 0.3 \frac{M}{R}\right)^{-\frac{1}{2}}
\] (15)
where \(M\) is the mass of the star in units of \(M_\odot\), \(R\) its radius in units of 10 km, and \(B_{15}\) is the B-field in units of \(10^{15}\) G. We see than that in certain cases the assumed gravitational effects may be in fact a mixture of gravitation with non-linear electromagnetism. This correction may be pointing out to inconsistencies in the determination of the quotient mass/ratio, inferred from the redshift\(^{19}\).

3. Acoustics in flowing fluids

Another example of the idea of effective metric comes from fluid mechanics\(^{8,20}\). The equations of motion for inviscid fluids in a Newtonian background are the continuity equation,
\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0,
\] (16)
and the Euler equation,
\[
\rho (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) = -\mathbf{p} - \rho \mathbf{v} \phi - \rho \mathbf{v} \Phi,
\] (17)
where \(\phi\) is the gravitational potential and \(\Phi\) is the potential associated to other external forces. Assuming that there is no vorticity\(^d\) means that \(\mathbf{v} = -\nabla \psi\). If the fluid is barotropic we can write \(\nabla h = \frac{1}{\rho} \nabla p\). Under these assumptions, the Euler equation reduces to
\[
-\partial_t \psi + h + \frac{1}{2} (\nabla \psi)^2 + \phi + \Phi = 0
\] (18)
Next we shall linearize the EOM around a given background using \(\rho = \rho_0 + \epsilon \rho_1 + O(\epsilon^2)\), and similar developments for \(p\) and \(\psi\), where the background quantities have a 0 subindex. Keeping terms up to first order in \(\epsilon\), we get from the linearized EOM:
\[
-\partial_t \left( \frac{\partial \rho}{\partial p} \rho_0 (\partial_t \psi_1 + \psi_0 \nabla \psi_1) \right) + \nabla \cdot \left( \rho_0 \nabla \psi_1 - \frac{\partial \rho}{\partial p} \rho_0 \psi_0 (\partial_t \psi_1 + \psi_0 \nabla \psi_1) \right) = 0.
\] (19)
\(^d\)The restriction of nonzero vorticity is lifted in Ref. 21.
This messy equation can be rearranged by introducing \( c_s^{-2} = \frac{\partial p}{\partial \rho} \), and the metric

\[
g_{\mu\nu} = \rho_0 \frac{c_s}{c_s} \begin{pmatrix}
-(c_s^2 - v_0^2) & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
-v_0^2 & \cdots & \delta_{ij}
\end{pmatrix}.
\]

With these definitions, Eq. (19) can be written as a wave equation for the scalar field \( \psi_1 \), namely

\[
\Delta \psi_1 = 0,
\]

where \( \Delta \) is the d'Alembertian in curved spacetime, defined in terms of the effective metric \( g_{\mu\nu} \) by

\[
\Delta \psi_1 = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \psi_1).
\]

(20)

As in the case of the non-linear electromagnetic field, the dynamics of the fluctuations of the scalar field are governed by an effective metric, but note that fluid particles couple only to the Minkowski background metric. Sound waves instead couple only to the effective metric, which has only two degrees of freedom (to be chosen from \( \psi_0(x), \rho_0(x), \) and \( c_s(x) \)) instead of the 6 degrees of freedom that a solution of Einstein's eqns. may have. Note also that in Einstein gravity, the metric is determined by the distribution of matter through Einstein equations. The effective metric instead is related algebraically to the distribution of matter.

With a metric to our disposal, many of the notions of GR (like horizon and ergosphere) can be translated to this context. For example, take a closed 2-surface \( S \). If \( \vec{v}_0 \) is everywhere inward pointing on \( S \), and \( \vec{v}_0 \perp \) is greater than the local \( c_s \), then the sound will be swept inwards by the flow, and trapped in \( S \), which is then an outer-trapped surface. The trapped region is the region containing outer-trapped surfaces, and the acoustic (future) event horizon is the boundary of the trapped region.

These ideas are relevant in the problem of the stability of accretion onto a black hole. Moncrief\(^{22}\) showed that the perturbations \( \psi_1 \) of a non-self-gravitating perfect fluid in stationary potential flow accreting onto a Schwarzschild black hole are governed by the equation \( \Delta \psi_1 = 0 \), where \( \Delta \) is built with the effective metric \( \gamma_{\mu\nu} = \frac{n}{h} \left( \frac{c}{v_s} \right) \left[ g_{\mu\nu} + \left( 1 - \frac{v_s^2}{c^2} \right) u_\mu u_\nu \right] \).

In this equation, \( n \) is the particle number density, \( h \) is the enthalpy, \( g_{\mu\nu} \) is the Schwarzschild metric, and the velocity of sound in the background flow is given by

\[
\left( \frac{v_s}{c} \right)^2 = \left( \frac{\partial p}{\partial \rho} \right)_s.
\]

The causal properties of sound propagation are determined by \( \gamma_{\mu\nu} \). In particular, the sound cones of \( \gamma_{\mu\nu} \) lie inside of the light cones defined by \( g_{\mu\nu} \). It is easy to see that there is a sonic horizon, which in the spherically symmetric case, is located at \( \gamma_{tt}(r_s) = 0 \), or

\[
\left( -g_{00} + \frac{v_s^2}{c^2} \right)_{r=r_s} = 1.
\]
Using arguments based on energy-momentum conservation, Moncrief was able to show that the norm of $\nabla_\mu \psi_1$ is bounded by the initial value of the energy, $E_0$. In particular, no unstable perturbation can exist outside the sonic horizon.

4. Conclusion

We have seen two examples in Astrophysics in which the concept of effective metric is useful: magnetars and accretion disks. Other applications of this idea in the same field are the effect of lensing in charged black holes, the influence of non-linearities in the velocity of gravitational waves, and the stability of stellar coronas and winds. The effective metric has been applied even in Cosmology. In fact, as shown in the Appendix, in every problem in which the dynamics is non-linear and we are interested in the perturbations, the effective metric is bound to play a role. It is precisely this generality, joined with the power of the techniques of General Relativity, that makes the effective geometry a useful tool in several branches of Physics and Astrophysics.

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Appendix

As shown by Barceló et al., the emergence of a curved Lorentzian geometry is a generic result of the linearization of a classical field theory around a background. A sketch of the proof of this statement follows. Let us take a scalar field Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \nabla \phi)$. Using the expansion

$$\phi(t, \vec{x}) = \phi_0(t, \vec{x}) + \epsilon \phi_1(t, \vec{x}) + \frac{\epsilon^2}{2} \phi_2(t, \vec{x}) + O(\epsilon^3),$$

we can linearize the action for $\mathcal{L}$. With the help of the EOM for the background field, the EOM for the fluctuations are

$$\partial_\mu \left( \frac{\partial^2 \mathcal{L}}{\partial (\partial_\mu \phi) \partial (\partial_\nu \phi)} \right)_{\phi_0} \partial_\nu \phi_1 - \left( \frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi} - \partial_\mu \left( \frac{\partial^2 \mathcal{L}}{\partial (\partial_\mu \phi) \partial \phi} \right) \right)_{\phi_0} \phi_1 = 0.$$

The subindex 0 means that the quantities are evaluated at the background. Defining

$$\sqrt{g} g^{\mu \nu} = \left\{ \frac{\nabla^2 \mathcal{L}}{\nabla(\nabla_\mu \phi) \nabla(\nabla_\nu \phi)} \right\}_{\phi_0},$$

*See Ref. 9 for details.
the EOM for the fluctuations takes the form
\[ \{ \Delta (g(\phi_0)) - V(\phi_0) \} \phi_1 = 0, \]
where \( \Delta \) is the DAlembertian built with \( g_{\mu\nu}(\phi_0) \) and the background-dependent potential is given by
\[ V(\phi_0) = \frac{1}{\sqrt{g}} \left\{ \frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial (\partial_\mu \phi) \partial \phi} \right\} \right\}. \]
We obtain the result that the kinematics of the fluctuations of the scalar field Lagrangian \( \mathcal{L} = \mathcal{L}(\phi, \nabla \phi) \) is governed by an effective (curved) geometry\(^4\).

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