GENUS EXPANSIONS OF HERMITIAN ONE-MATRIX MODELS:
FAT GRAPHS VS. THIN GRAPHS

JIAN ZHOU

ABSTRACT. We consider two different genus expansions of the free energy functions of Hermitian one-matrix models, one using fat graphs, one using ordinary graphs (thin graphs). Some structural results are first proved for the thin version of genus expansion using renormalized coupling constants, and then applied to the fat version.

1. INTRODUCTION

This is a sequel to [12] where we presented a formula for the $n$-point correlators in Hermitian one-matrix models. This result was inspired by a work of Dubrovin-Yang [4] where a connection between Hermitian one-matrix models and Toda lattice hierarchy was used. In [12], a connection to the KP hierarchy was used instead, and this enabled us to apply the formula for $n$-point correlations associated to a $\tau$-function of the KP hierarchy developed in an earlier work [11]. One of the goals of this paper is to apply the results on the $n$-point correlations obtained in [4] and [12] to provide some new way to understand about the structure of the free energy functions of the Hermitian one-matrix models.

To achieve this goal, we will first distinguish two kinds of genus expansions for the free energy. Ever since [9], most authors have focused on the genus expansion induced by the genus of fat graphs. In this paper, we will introduce another genus expansion induced by considering the thin graphs obtained by the skeletons of the fat graphs. These two kinds of genus expansions will be referred to as the fat and thin genus expansions respectively.

The motivation for introducing the thin genus expansion comes from another earlier work of the author [10]. Again influenced by [9], a lot of work on matrix models have focused on the large $N$ limits and in particular double scaling limit of the matrix models to make connections to topological 2D gravity. See e.g. the survey [3]. Going against this direction, the author considered the case of $N = 1$ and referred to the resulting theory as topological 1D gravity in [10]. In that setting, ordinary graphs instead of fat graphs were used. More importantly, a version of renormalization was developed in that theory and some structural results were proved for the free energy function. This inspires us to introduce the thin genus expansions for Hermitian matrix models with finite size $N$ in this work.

The advantage of thin genus expansion against the fat one will only become clear after we apply the renormalized coupling constants introduced in [6] and developed in [10]. Their definitions will be recalled in §5. In the same spirit of [10], we will prove in two different ways the following main results of this paper: The thin free...
energy of the Hermitian $N \times N$-matrix model has the following structure:

\begin{equation}
F_{0,N} = N \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1},
\end{equation}

\begin{equation}
F_{1,N} = \frac{N^2}{2} \ln \frac{1}{1 - I_1},
\end{equation}

and for $g \geq 1$,

\begin{equation}
F_{g,N} = \sum_{\sum_{j=3}^{2g} (j-2) I_j = 2g-2} \langle p_{12}^{I_2} \cdots p_{2g}^{I_{2g}} \rangle_{g,N} \prod_{j=3}^{2g} \frac{1}{I_j^{I_j^2}} \left( \frac{I_j}{(1 - I_1)^{I_j}} \right)^{I_j}.
\end{equation}

In particular, for each $g$, the computation of $F_{g,N}$ is reduced to finitely many correlators, hence we get a very effective way to compute them.

Our first proof is based on combining the Virasoro constraints well-known in the matrix model literature with the renormalized coupling constants. By applying the loop equations for Hermitian one-matrix models to the fat and thin genus expansions, we get the fat and thin Virasoro constraints respectively. These constraints provide alternative ways to compute the fat and thin correlators, other than the formulas in [4, 12]. The first two Virasoro constraints are often called the puncture equation and the dilaton equation in the literature. As a common practice they are used to remove the two lowest degree operators in the correlators. However, when combined with the renormalized coupling constants, these two constraints become much more powerful. As noticed in [10] in the case of topological 1D gravity, when these constraints are expressed in the normalized coupling constants $\{I_k\}_{k \geq 0}$, these constraints reduce each thin free energy in genus $g$ to finitely many correlators. It turns out that the same holds for Hermitian one-matrix models, and this is one of the ways that we use to prove the above results.

Our second proof is also based on combining a technique well-known in the matrix model literature with the use of renormalized coupling constants. To evaluate the Gaussian integral on the space of Hermitian $N \times N$-matrices, one can reduce it to an integral on $\mathbb{R}^N$. One can then apply the change of coupling constants trick developed in [10] for formal Gaussian integrals on $\mathbb{R}^1$.

Both of the above proofs cease to work for the fat genus expansion, nevertheless, similar results still hold but now infinitely many fat correlators are involved at each genus. The reason is that the fat and thin correlators satisfy different selection rules, and the fat selection rule no longer exclude enough fat correlators. An alternative way to get results for the fat genus expansion is to apply the results for the thin genus expansion. This will be explained in §7.

The rest of the paper is arranged as follows. In §2 we define two kinds of genus expansions of the free energies of Hermitian one-matrix models. We then recall the derivation of Virasoro constraints and specialize them to the two genus expansion. After we apply the thin Virasoro constraints to obtain some general results on the thin genus $g$ free energy $F_{g,N}$ in §3, we prove our main results on $F_{g,N}$ in §5 and §6 in two different ways. In the final §7 we derive similar results for the fat free energy.
2. Two Kinds of Genus Expansions in Hermitian Matrix Models

In this Section we introduce two kinds of genus expansions for the free energy functions of Hermitian one-matrix models. We also present the fat and thin selection rules. For general references on matrix models, see [3, 11, 2].

2.1. Hermitian one-matrix models. For each $N$, the partition function of the Hermitian $N \times N$-matrix model is defined by the formal Gaussian integral:

\begin{equation}
Z_N = \int_{H_N} dM \exp \left( \sum_{n=1}^{\infty} \frac{g_n - \delta_n}{n g_s} M^n \right),
\end{equation}

where $H_N$ is the space of Hermitian $N \times N$-matrices. Its free energy $F_N$ is defined by:

\begin{equation}
F_N := \log Z_N.
\end{equation}

The first few terms of $F_N$ are given by:

\begin{equation}
F_N = \frac{1}{2} N^2 g_2 + \frac{1}{2} N g_s^{-1} g_1^2 + \left( \frac{1}{4} N^3 + \frac{1}{4} N \right) g_s g_4 + N^2 g_3 g_1 + \frac{N^2}{4} g_2^2 + \frac{N}{2} g_s^{-1} g_2 g_1^2 + \cdots
\end{equation}

For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, the correlator $\langle \frac{1}{z_\lambda} p_\lambda \rangle_N$ is defined by

\begin{equation}
\langle \frac{1}{z_\lambda} p_\lambda \rangle_N := \frac{\partial^l F_N}{\partial g_{\lambda_1} \cdots \partial g_{\lambda_l}} \bigg|_{g_i=0, i=1, \ldots},
\end{equation}

where $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}$. Here we are following the notations of [7]. By using the fat graphs introduced in [9], one can get (see e.g. [12, (21)]):

\begin{equation}
\langle \frac{1}{z_\lambda} p_\lambda \rangle_N = \sum_{\Gamma \in \Gamma^\lambda_c} \frac{1}{| \text{Aut}(\Gamma) |} g_s^{\sharp(\lambda) - \ell(\lambda)} N^{F(\Gamma)},
\end{equation}

where $\Gamma^\lambda_c$ is the set of connected fat graphs of type $\lambda$. The free energy function $F_N$ is given by these correlators as follows:

\begin{equation}
F_N = \sum_\lambda \langle \frac{1}{z_\lambda} p_\lambda \rangle_N g_\lambda.
\end{equation}

2.2. Genus expansion by thin graphs. The first kind of genus expansion of $F_N$ we consider is of the following form:

\begin{equation}
F_N = \sum_{g \geq 0} g^{g-1} F_{g,N}.
\end{equation}
We will refer to it as the thin genus expansion of $F_N$. For example,

\[
F_{0,N} = \frac{1}{2} N g_2^2 + \frac{N}{2} g_2 g_1^2 + \frac{N}{3} g_4 g_1^3 + \frac{N}{2} g_2^2 g_1^2 + \frac{N}{2} g_3^2 g_1^2 + N g_2 g_3 g_1^3 + \frac{N}{4} g_4 g_1^4 + \cdots
\]

\[
F_{1,N} = \frac{1}{2} N^2 g_2 + N^2 g_3 g_1 + \frac{N^2}{4} g_2^2 + \frac{3N^2}{2} g_4 g_1^2 + 2N^2 g_3 g_2 g_1 + \frac{N^2}{6} g_2^3 + \cdots,
\]

\[
F_{2,N} = (\frac{1}{2} N^3 + \frac{1}{4} N) g_4 + (N + 2N^3) g_5 g_1 + (\frac{N}{2} + N^3) g_4 g_2
\]

\[
+ \left( \frac{N}{6} + \frac{2N^3}{3} \right) g_3^2 + \cdots,
\]

\[
F_{3,N} = \left( \frac{5N^2}{3} + \frac{5N^4}{6} \right) g_6 + \cdots.
\]

2.3. Thin correlators and thin selection rule. Suppose that the correlator $\langle p_{a_1} \cdots p_{a_n} \rangle_c^c$ contributes to $F_{g,N}$. It is a summation over thin graphs $\hat{\Gamma}$, such that the number of vertices $V(\hat{\Gamma}) = n$, the number of edges $E(\hat{\Gamma}) = \frac{1}{2} \sum_{j=1}^{n} a_j$. The number of loops of the thin graph $\hat{\Gamma}$ is

\[
g(\hat{\Gamma}) = V(\hat{\Gamma}) - E(\hat{\Gamma}) + 1 = n - \frac{1}{2} \sum_{j=1}^{n} a_j + 1.
\]

Therefore, $\langle p_{a_1} \cdots p_{a_n} \rangle_c^c$ contributes to $F_{g,N}$ iff

\[
\sum_{j=1}^{n} a_j = 2g - 2 + 2n.
\]

This will be referred to as the thin selection rule. We will write $\langle p_{a_1} \cdots p_{a_n} \rangle_{g,N}^c$ instead of $\langle p_{a_1} \cdots p_{a_n} \rangle_{g,N}$ when this selection rule is satisfied for genus $g$. It will be referred to as a genus $g$ thin correlator.

For example, in degree two we have

\[
\langle \frac{p_2}{2} \rangle_{1,N}^c = \frac{1}{2} N^2, \quad \langle p_2 \rangle_{0,N}^c = N,
\]

in degree four,

\[
\langle \frac{p_4}{4} \rangle_{2,N}^c = \frac{1}{2} N^3 + \frac{1}{4} N, \quad \langle \frac{p_3}{3} p_1 \rangle_{1,N}^c = N^2,
\]

\[
\langle \frac{p_2}{2} \rangle_{1,N}^c = \frac{1}{2} N^2, \quad \langle \frac{p_2}{2} \rangle_{0,N}^c = N,
\]

\[
\langle p_4 \rangle_{g,N}^c = 0, \quad g \geq 0,
\]
2.4. The case of $N = 1$. As mentioned in the Introduction, the motivation of considering the thin-graph expansion comes from the $N = 1$ case of the $N \times N$-matrix model studied in [10]. Its partition function is defined by the formal Gaussian integral:

$$Z^{1D} = \frac{1}{\sqrt{2\pi}} \int dx \exp \left( -\frac{1}{2} x^2 + \sum_{n \geq 1} t_n - 1 \frac{x^n}{n!} \right).$$

This is just $Z_{N=1}$ with

$$g_s = \lambda^2,$$

$$g_n = (n-1)!t_{n-1}, \quad n \geq 1.$$

The first few terms of the free energy $F^{1D} = \log Z^{1D}$ are given by:

$$F^{1D} = \frac{1}{2}\lambda^{-2}t_0^2 + \frac{1}{2}t_1 + \left( \frac{1}{2}\lambda^2t_1\lambda^{-2} + \frac{1}{2}t_0t_2 + \frac{1}{4}t_3 + \frac{1}{8}t_5\lambda^2 \right) + \frac{1}{24}\left( \frac{5}{4}\lambda^2 + \frac{1}{8}\lambda^2t_4 + \frac{1}{4}\lambda^2t_3 + \frac{1}{4}\lambda^2t_3t_0 \right) + \frac{1}{8}\lambda^2t_0^2 + \frac{1}{2}\lambda^2t_1t_2 + \frac{1}{8}\left( \frac{3}{4}\lambda^2t_3 + \frac{1}{4}\lambda^2t_3t_0 \right) + \frac{1}{4}\lambda^2t_5 + \frac{1}{8}\lambda^2t_4 + \frac{1}{4}\lambda^2t_4t_0 + \frac{1}{4}\lambda^2t_2^2 + \frac{1}{8}\lambda^2t_2^2t_0 + \frac{1}{4}\lambda^2t_2^2t_1 + \frac{1}{3}\lambda^2t_2^2t_2 + \frac{1}{8}\lambda^2t_2^2t_3 + \frac{1}{16}\lambda^2t_2^2t_4 + \frac{1}{12}\lambda^2t_2^2t_5 + \frac{1}{16}\lambda^2t_2^2t_6 + \frac{1}{32}\lambda^2t_2^2t_7 + \cdots,$$

and one has

$$F_0^{1D} = \frac{1}{2}t_2^3 + \frac{1}{2}t_2^2t_1 + \frac{1}{2}t_2^2t_0 + \frac{1}{6}t_2t_0^2 + \frac{1}{2}t_2t_0t_2 + \frac{1}{2}t_2t_2t_0 + \frac{1}{2}t_2t_2t_1 + \frac{1}{2}t_2t_2t_2 + \frac{1}{2}t_2t_2t_3 + \frac{1}{2}t_2t_2t_4 + \cdots,$$

$$F_1^{1D} = \frac{1}{2}t_1 + \frac{1}{2}t_0t_2 + \frac{1}{2}t_1t_2 + \frac{1}{2}t_1t_3 + \frac{1}{2}t_1t_4 + \frac{1}{2}t_1t_5 + \frac{1}{2}t_1t_6 + \frac{1}{2}t_1t_7 + \cdots.$$
\[ F_2^{1D} = \frac{1}{8} t_3 + \frac{5}{24} t_2^2 + \frac{1}{8} t_0 t_4 + \frac{1}{4} t_1 t_3 + \frac{5}{8} t_2 t_1 + \frac{3}{8} t_4 t_0 t_1 + \frac{3}{8} t_3 t_1^2 + \frac{1}{16} t_5 t_2^2 + \frac{2}{3} t_3 t_0 t_2 + \cdots, \]
\[ F_3^{1D} = \frac{1}{48} t_5 + \frac{1}{16} t_5 t_1 + \frac{1}{12} t_3^2 + \frac{1}{48} t_0 t_6 + \frac{7}{48} t_4 t_2 + \cdots, \]
\[ F_4^{1D} = \frac{1}{384} t_7 + \cdots. \]

One can check that they match with the first few terms of \( F_{g,N=1}. \)

2.5. Genus expansion by fat graphs. Another way to define a genus expansion of \( F_N \) is to introduce the 't Hooft coupling constant

\[ t = N g_s. \]

With this one can substitute \( N \) by \( t g_s^{-1} \) in \( F_N \) to get:

\[ F_N = \frac{1}{2} t^2 g_s^{-2} g_2 + \frac{1}{2} t g_s^{-2} g_1^2 + \left( \frac{1}{2} t^3 g_s^{-2} + \frac{1}{4} t \right) g_4 + t^2 g_s^{-2} g_3 g_1 + \frac{t^2 g_s^{-2}}{4} g_2 + \frac{t^2 g_s^{-2}}{2} g_2 g_1^2 + \left( \frac{5 t^2}{3} + \frac{5 t^4}{6} g_s^{-2} \right) g_6 + (t + 2 t^3 g_s^{-2}) g_3 g_1 + \frac{t}{3} g_s^{-2} g_3 g_1^3 + \frac{t^2}{6} g_s^{-2} g_2^3 + \frac{t}{2} g_s^{-2} g_2 g_1^2 + \cdots. \]

We will write

\[ F_N = \sum_{g \geq 0} g_s^{2g-2} F_g(t), \]

and refer to it as the fat genus expansion. For example,

\[ F_0(t) = \frac{1}{2} t^2 g_2 + \frac{1}{2} t g_1^2 + \frac{1}{2} t^3 g_4 + t^2 g_3 g_1 + \frac{t^2}{4} g_2^2 + \frac{t}{2} g_2 g_1 + \frac{5 t^4}{6} g_6 + 2 t^3 g_3 g_1 + t^3 g_4 g_2 + \frac{3 t^2}{2} g_2 g_1^2 + \frac{2 t^3}{3} g_3 g_2 g_1 + \frac{t}{3} g_3 g_1 g_3 + \frac{t^2}{6} g_3 g_1^2 + \frac{t^2}{2} g_2 g_1^2 + \cdots = t \left( \frac{1}{2} g_1 + \frac{1}{2} g_2 g_1 + \frac{1}{3} g_3 g_1 + \frac{1}{2} g_2 g_1^2 + \cdots \right) + t^2 \left( \frac{1}{2} g_2 + g_3 g_1 + \frac{3}{2} g_4 g_2 + 2 g_3 g_2 g_1 + \frac{1}{2} g_1^2 + \cdots \right) + t^3 \left( \frac{1}{2} g_4 + 2 g_3 g_1 + g_4 g_2 + \frac{3}{2} g_5 + \cdots \right) + t^4 \left( \frac{5}{6} g_6 + \cdots \right) + \cdots, \]
\[ F_1(t) = \frac{1}{4} t g_4 + \frac{5 t^2}{3} g_6 + t g_5 g_1 + t^2 g_4 g_2 + \frac{t}{6} g_6^2 + \cdots. \]
It is clear that $F_g(t)$ is a formal power series in $t$:

$$F_g(t) = \sum_{m \geq 1} F_{g,m} t^m. \quad (16)$$

### 2.6. Fat correlators and the fat selection rule.

The fat correlators are defined by:

$$\langle \frac{1}{z^\lambda} p_\lambda \rangle_g^c(t) := \left. \frac{\partial^l F_g(t)}{\partial g_{\lambda_1} \cdots \partial g_{\lambda_l}} \right|_{g_i=0, i=1, \ldots, l}, \quad (17)$$

By (17) we have

$$\langle \frac{1}{z^\lambda} p_\lambda \rangle_N^c(t) = \sum_{\Gamma \in \Gamma_{\lambda g}^c} \frac{1}{|\text{Aut}(\Gamma)|} g^{\frac{1}{2}(|\lambda| - l(\lambda) - |F(\Gamma)|)} t^{|F(\Gamma)|}, \quad (18)$$

where $\Gamma_{\lambda g}^c$ is the set of connected fat graphs of type $\lambda$ and of genus $g$. It follows that

$$\langle \frac{1}{z^\lambda} p_\lambda \rangle_g^c(t) = \sum_{\Gamma \in \Gamma_{\lambda g}^c} \frac{1}{|\text{Aut}(\Gamma)|} g^{\frac{1}{2}(|\lambda| - l(\lambda) - |F(\Gamma)|)}, \quad (19)$$

and so $\langle \frac{1}{z^\lambda} p_\lambda \rangle_g^c(t) \neq 0$ only if

$$2g - 2 = \frac{1}{2} |\lambda| - l(\lambda) - m \quad (20)$$

for some $m \geq 1$. In other words, a fat correlator $\langle p_{a_1} \cdots p_{a_n} \rangle_g^c(t)$ is nonzero only when

$$\sum_{i=1}^n a_i = 4\tilde{g} - 4 + 2n + 2m \quad (21)$$

for some $m \geq 1$. We will refer to this as the fat selection rule. By comparing with the thin selection rule, we see that when the thin correlator $\langle p_{a_1} \cdots p_{a_n} \rangle_{g,N}^c(t)$ and the fat correlator $\langle p_{a_1} \cdots p_{a_n} \rangle_g^c(t)$ are both nonzero,

$$g = 2\tilde{g} + m - 1 \quad (22)$$

for some $m \geq 1$.

The following are some examples of fat correlators. In degree two we have

$$\langle \frac{1}{2} p_2 \rangle_0^c(t) = \frac{1}{2} t^2, \quad \langle p_1 \rangle_0^c(t) = t,$$

in degree four,

$$\langle \frac{1}{4} p_4 \rangle_0^c(t) = \frac{1}{2} t^2, \quad \langle \frac{1}{4} p_1^3 \rangle_0^c(t) = \frac{1}{4} t, \quad \langle \frac{1}{3} p_1 p_3 \rangle_0^c(t) = t^2, \quad \langle \frac{1}{2} p_2^2 \rangle_0^c(t) = \frac{1}{2} t^2, \quad \langle p_2 p_1 \rangle_0^c(t) = t, \quad \langle p_1^4 \rangle_0^c(t) = 0, \quad g \geq 0,$$
3.1. Loop operator and loop equations. We now recall the derivation of loop equations in Hermitian matrix models. See e.g. Kazakov’s contribution to [2]. For simplicity of notations, rewrite \( Z \) as follows:

\[
\begin{align*}
\langle p_6 \rangle_0^c(t) &= \frac{5t^4}{6}, & \langle p_6 \rangle_1^c(t) &= \frac{5t^2}{3}, & \langle p_5 \rangle_0^c(0) &= 2t^3, \\
\langle p_5 \rangle_1^c(t) &= t, & \langle p_4 p_2 \rangle_0^c(0) &= t^3, & \langle p_4 p_2 \rangle_1^c(t) &= \frac{t}{2}, \\
\langle p_4 p_2 \rangle_0^c(0) &= 3t^2, & \langle (p_3)^2 \rangle_0^c(0) &= \frac{4t^3}{3}, & \langle (p_3)^2 \rangle_1^c(t) &= \frac{t}{3}, \\
\langle p_3 p_1 \rangle_0^c(0) &= 2t^2, & \langle p_3 p_1 \rangle_1^c(0) &= 2t, & \langle (p_2)^2 \rangle_0^c(0) &= t^2, \\
\langle (p_2)^2 \rangle_1^c(0) &= 2t, & \langle p_2 p_1 \rangle_0^c_{g,N} &= 0, & g \geq 0, & \langle p_6 \rangle_g^c = 0, & g \geq 0. 
\end{align*}
\]

3. Virasoro Constraints

We recall the derivation of Virasoro constraints for matrix models for finite \( N \) in the literature. We specialize them to the two genus expansions discussed above.

Consider the collective loop operator:

\[
W_N(z) = \frac{1}{z - \lambda_k} = \text{tr} \left( \frac{1}{z - M} \right).
\]

Start with the identity:

\[
\int_{\mathbb{R}^N} \prod_{i=1}^N d\lambda_i \cdot \sum_{k=1}^N \frac{\partial}{\partial \lambda_k} \left( \frac{1}{z - \lambda_k} \right) \prod_{i=1}^N \exp \left( \sum_{n=0}^{\infty} \frac{T_n}{n!} \lambda_i^n \right) \cdot \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 = 0.
\]

Rewrite the left-hand side as follows:

\[
\left\langle \sum_{k=1}^N \left( \frac{1}{(z - \lambda_k)^2} + \frac{1}{z - \lambda_k} \cdot \left( \sum_{n=0}^{\infty} n T_n \lambda_k^{n+1} + 2 \sum_{j \neq k} \frac{1}{\lambda_k - \lambda_j} \right) \right) \right\rangle_{N,T} = 0.
\]

Using the identity

\[
\sum_{k=1}^N \frac{1}{(z - \lambda_k)^2} + 2 \sum_{1 \leq j \neq k \leq N} \frac{1}{z - \lambda_k} \frac{1}{\lambda_k - \lambda_j} = \left( \sum_{k=1}^N \frac{1}{z - \lambda_k} \right)^2,
\]

and in degree 6:

\[
\langle p_6 \rangle_0^c(t) = \frac{5t^4}{6}, & \langle p_6 \rangle_1^c(t) = \frac{5t^2}{3}, & \langle p_5 \rangle_0^c(0) = 2t^3, \\
\langle p_5 \rangle_1^c(t) = t, & \langle p_4 p_2 \rangle_0^c(0) = t^3, & \langle p_4 p_2 \rangle_1^c(t) = \frac{t}{2}, \\
\langle p_4 p_2 \rangle_0^c(0) = 3t^2, & \langle (p_3)^2 \rangle_0^c(0) = \frac{4t^3}{3}, & \langle (p_3)^2 \rangle_1^c(t) = \frac{t}{3}, \\
\langle p_3 p_1 \rangle_0^c(0) = 2t^2, & \langle p_3 p_1 \rangle_1^c(0) = 2t, & \langle (p_2)^2 \rangle_0^c(0) = t^2, \\
\langle (p_2)^2 \rangle_1^c(0) = 2t, & \langle p_2 p_1 \rangle_0^c_{g,N} = 0, & g \geq 0, & \langle p_6 \rangle_g^c = 0, & g \geq 0.
\]
one finds:

\[ \left( W_N^2(z) + \sum_{k=1}^{N} \frac{1}{z - \lambda_k} \sum_{n \geq 0} n \hat{T}_n \lambda_k^{n-1} \right)_{N,T} = 0. \]

The summation over \( k \) can be reexpressed as a residue:

\[ \left( W_N(z)^2 + \oint_C \frac{dz}{2\pi i} i_{z,\hat{z}} \frac{1}{z - \hat{z}} \cdot W_N(z) \sum_{n \geq 1} n \hat{T}_n z^{n-1} \right)_{N,T} = 0, \]

where \( C \) is a large enough circle, and

\[ i_{z,\hat{z}} \frac{1}{z - \hat{z}} = \sum_{n \geq 0} \hat{z}^n \frac{1}{z^{n+1}}. \]

This is called the loop equation.

### 3.2. Reformulation in terms of a bosonic field.

The loop equation can be further reformulated by introducing the collective field

\[ \Phi_N(z) = \frac{1}{\sqrt{2}} \sum_{n \geq 1} \hat{T}_n z^n - \sqrt{2} \text{tr} \log \left( \frac{1}{z - M} \right) \]

\[ = \frac{1}{\sqrt{2}} \sum_{n \geq 1} \hat{T}_n z^n + \sqrt{2} N \log z - \sqrt{2} \sum_{n \geq 1} \hat{z}^n \frac{\partial}{\partial T_n}. \]

The second line follows from the fact that the insertion of the operator \( \text{tr}(M^n) = \sum_{i=1}^{N} \lambda_i^n \) can be realized by taking a partial derivative with respect to \( T_n \). Similarly,

\[ W_N(z) = \frac{N}{z} + \sum_{n \geq 1} \frac{1}{z^{n+1}} \frac{\partial}{\partial T_n}. \]

Note

\[ \partial_z \Phi_N(z) = \frac{1}{\sqrt{2}} \sum_{n \geq 1} n \hat{T}_n z^{n-1} + \sqrt{2} \left( \frac{N}{z} + \sum_{n \geq 1} \hat{z}^{n-1} \frac{\partial}{\partial T_n} \right) \]

\[ = \frac{1}{\sqrt{2}} \sum_{n \geq 1} n \hat{T}_n z^{n-1} + \sqrt{2} W_N(z). \]

The loop equation \( \text{(28)} \) can now be rewritten as

\[ \oint_C \frac{dz}{2\pi i} i_{z,\hat{z}} \frac{1}{z - \hat{z}} \left( (\partial_z \Phi_N(z))^2 \right)_{N,T} = 0, \]

or

\[ \oint_C \frac{dz}{2\pi i} i_{z,\hat{z}} \frac{1}{z - \hat{z}} \cdot T_N(\hat{z}) Z_N[T] = 0, \]

where \( T_N(z) \) is the energy-momentum defined by:

\[ T_N(z) = \frac{1}{2} : (\partial_z \Phi_N(z))^2 : \]
3.3. **Virasoro constraints.** Expand $T_N(z)$ in the following form:

$$T_N(z) := \sum_{n \in \mathbb{Z}} L_{n,N} z^{-n-2},$$

where

$$L_{-1,N} = \sum_{n \geq 1} (n+1) \tilde{T}_{n+1} \frac{\partial}{\partial \tilde{T}_n} + NT_1,$$

$$L_{0,N} = \sum_{n \geq 1} n \tilde{T}_n \frac{\partial}{\partial \tilde{T}_n} + N^2,$$

$$L_{1,N} = \sum_{n \geq 1} n \tilde{T}_n \frac{\partial}{\partial \tilde{T}_n} + 2N \frac{\partial}{\partial \tilde{T}_1},$$

$$L_{n,N} = \sum_{k \geq 1} k \tilde{T}_k \frac{\partial}{\partial \tilde{T}_{k+n}} + \sum_{k=1}^{n} \frac{\partial}{\partial \tilde{T}_k} \frac{\partial}{\partial \tilde{T}_{n-k}} + 2N \frac{\partial}{\partial \tilde{T}_n}, \quad n \geq 2.$$

The loop equation can be rewritten as a set of linear differential equations

$$L_{n,N} Z_N[t] = 0 \quad (n \geq -1).$$

These are called **Virasoro constraints** because

$$[L_{m,N}, L_{n,N}] = (m-n)L_{m+N,N}.$$

3.4. **Virasoro constraints for thin genus expansion.** Now if we take $\tilde{T}_n = \frac{2^{-b_{10}}}{ng_s}$, then the operators $L_{n,N}$ become:

$$L_{-1,N} = -\frac{\partial}{\partial g_1} + \sum_{n \geq 1} n g_{n+1} \frac{\partial}{\partial g_n} + N g_1 g_s^{-1},$$

$$L_{0,N} = -2 \frac{\partial}{\partial g_2} + \sum_{n \geq 1} n g_n \frac{\partial}{\partial g_n} + N^2,$$

$$L_{1,N} = -3 \frac{\partial}{\partial g_3} + \sum_{n \geq 1} (n+1) g_n \frac{\partial}{\partial g_{n+1}} + 2N g_s \frac{\partial}{\partial g_1},$$

$$L_{m,N} = \sum_{k \geq 1} (k+m)(g_k - \delta_{k,2}) \frac{\partial}{\partial g_{k+m}} + g_s^2 \sum_{k=1}^{m-1} k(m-k) \frac{\partial}{\partial g_k} \frac{\partial}{\partial g_{m-k}}$$

$$+ 2Nm g_s \frac{\partial}{\partial g_m}, \quad m \geq 2.$$

We will refer to the constraints

$$L_{m,N} Z_N = 0$$

as the **thin Virasoro constraints**.

3.5. **Thin Virasoro constraints in terms of thin correlators.** It is useful for practical computations to rewrite the thin Virasoro constraints in terms of thin correlators. The thin puncture equation can be written as

$$\langle p_1 \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_n}}{a_n} \rangle_{g,N} = \sum_{j=1}^{n} (a_j - 1) \cdot \langle p_{a_1} \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_{j-1}}}{a_{j-1}} \cdots \frac{p_{a_n}}{a_n} \rangle_{g,N},$$

where
together with initial value:

\begin{equation}
\langle p_1^2 \rangle_{0,N}^c = N
\end{equation}

The thin dilaton equation can be written as

\begin{equation}
\langle p_2 \frac{p_{a_1}}{a_1} \ldots \frac{p_{a_n}}{a_n} \rangle_{g,N}^c = \sum_{j=1}^{n} a_j \cdot \langle \frac{p_{a_1}}{a_1} \ldots \frac{p_{a_n}}{a_n} \rangle_{g,N}^c,
\end{equation}

together with initial value:

\begin{equation}
\langle p_2 \rangle_{1,N}^c = N^2.
\end{equation}

The third equation in the sequence can be written as

\begin{equation}
\langle p_3 \frac{p_{a_1}}{a_1} \ldots \frac{p_{a_n}}{a_n} \rangle_{g,N}^c = \sum_{j=1}^{n} (a_j + 1) \cdot \langle \frac{p_{a_1}}{a_1} \ldots \frac{p_{a_{j+1}}}{a_{j+1}} \ldots \frac{p_{a_n}}{a_n} \rangle_{g,N}^c
+ 2N \langle p_1 \frac{p_{a_1}}{a_1} \ldots \frac{p_{a_n}}{a_n} \rangle_{g-1,N}^c,
\end{equation}

and for \( m \geq 2 \)

\begin{equation}
\langle p_{m+2} \frac{p_{a_1}}{a_1} \ldots \frac{p_{a_n}}{a_n} \rangle_{g,N}^c = \sum_{j=1}^{n} (a_j + m) \cdot \langle \frac{p_{a_1}}{a_1} \ldots \frac{p_{a_{j+m}}}{a_{j+m}} \ldots \frac{p_{a_n}}{a_n} \rangle_{g,N}^c
+ 2N \langle p_m \frac{p_{a_1}}{a_1} \ldots \frac{p_{a_n}}{a_n} \rangle_{g-1,N}^c
+ \sum_{k=1}^{m} \langle pk \frac{p_{a_1}}{a_1} \ldots \frac{p_{a_n}}{a_n} \rangle_{g-2,N}^c
+ \sum_{k=1}^{m} \sum_{g_1+g_2=g-1} \langle pk \prod_{i \in I_1} \frac{p_{a_i}}{a_i} \rangle_{g_1,N}^c \cdot \langle p_{m-k} \prod_{i \in I_2} \frac{p_{a_i}}{a_i} \rangle_{g_2,N}^c,
\end{equation}

where \([n] = \{1, \ldots, n\}\).

Now we present some examples of computations of thin correlators by the thin Virasoro constraints and compare with the results in [4]. In degree 8 we have:

\begin{align*}
\langle \frac{p_3}{3} \frac{p_5}{5} \rangle_{3,N}^c &= \frac{1}{3} \cdot \left( 6 \langle \frac{p_6}{6} \rangle_{3,N}^c + 2N \langle \frac{p_5}{3} \rangle_{2,N}^c \right) \\
&= \frac{1}{3} \cdot \left( 6 \cdot \left( \frac{5N^2}{3} + \frac{5N^4}{6} \right) + 2N \cdot (N + 2N^3) \right) \\
&= 4N^2 + 3N^4.
\end{align*}

Here is another example in degree 8:

\begin{align*}
\langle \frac{p_4}{4} \rangle_{5,N}^c &= \frac{1}{4} \cdot \left( 6 \langle \frac{p_4}{6} \rangle_{5,N}^c + 2N \langle \frac{p_4}{4} \rangle_{3,N}^c + \langle p_1 \frac{p_4}{4} \rangle_{5,N}^c \right) \\
&= \frac{1}{4} \cdot \left( 6 \cdot \left( \frac{5N^2}{3} + \frac{5N^4}{6} \right) + 2N \cdot 2 \left( \frac{N}{2} + N^3 \right) + 3N^2 \right) \\
&= \frac{1}{4} (15N^2 + 9N^4).
\end{align*}
This matches with \cite{4} Example 3.2.5. Here is an example in degree 10:
\[
\langle \left( \frac{p_3}{3} \right)^2 \left( \frac{p_4}{4} \right) \rangle_{3,N}^c = \frac{1}{3} \left( 4 \cdot \langle \left( \frac{p_4}{4} \right) \left( \frac{p_3}{3} \right) \rangle_{2,N}^c + 5 \cdot \langle \left( \frac{p_4}{4} \right) \left( \frac{p_4}{4} \right) \rangle_{2,N}^c + 2N \langle \frac{p_3}{3} \rangle_{2,N}^c \right)
\]
\[
= \frac{1}{3} \left( 4 \cdot \frac{1}{4} (15N^2 + 9N^4) + 5(4N^2 + 3N^4) + 2N(2N + 6N^3) \right)
\]
\[
= 13N^2 + 12N^4.
\]
This matches with \cite{4} Example 3.2.6]. In the above we have used the following computation:
\[
\langle \left( \frac{p_1}{p_3} \right)^3 \rangle_{2,N}^c = 2 \cdot \langle \left( \frac{p_2}{2} \right)^2 \rangle_{2,N}^c + 3 \cdot \langle \frac{p_3}{3} \rangle_{2,N}^c
\]
\[
= 2 \cdot \left( \frac{N}{2} + N^3 \right) + 3 \cdot \left( \frac{N}{3} + \frac{4N^3}{3} \right)
\]
\[
= 2N + 6N^3.
\]
Finally, we present an example in degree 12:
\[
\langle \left( \frac{p_3}{3} \right)^4 \rangle_{3,N}^c = \frac{1}{3} \left( 3 \cdot 4 \cdot \langle \left( \frac{p_4}{4} \right)^2 \rangle_{2,N}^c + 2N \langle \left( \frac{p_3}{3} \right)^3 \rangle_{2,N}^c \right)
\]
\[
= \frac{1}{3} \left( 12(13N^2 + 12N^4) + 2N(6N + 24N^3) \right)
\]
\[
= 56N^2 + 64N^4,
\]
where we have used:
\[
\langle \left( \frac{p_3}{3} \right)^3 \rangle_{2,N}^c = 3 \cdot 2 \cdot \langle \frac{p_2}{2} \rangle_{2,N}^c
\]
\[
= 3 \cdot (3 + 3) \cdot \langle \left( \frac{p_3}{3} \right)^3 \rangle_{2,N}^c = 18 \cdot \left( \frac{N}{3} + \frac{4N^3}{3} \right)
\]
\[
= 6N + 24N^3,
\]
This matches with \cite{4} Example 3.2.7]. The results of these concrete computations can also be double checked by the results in \cite{12} Appendix].

3.6. **Virasoro constraints for fat genus expansion.** If one introduces the ’t Hooft coupling constant \( t = Ng_s \) and take \( \tilde{T}_n = \frac{N(g_n - \delta_{n,2})}{nt} = \frac{g_n - \delta_{n,2}}{ng_s} \), then the Virasoro operators become:

\[
L_{-1,t} = -\frac{\partial}{\partial g_1} + \sum_{n \geq 1} \frac{ng_n}{\partial g_n} t g_1 g_s^{-2},
\]
\[
L_{0,t} = -2 \frac{\partial}{\partial g_2} + \sum_{n \geq 1} \frac{ng_n}{\partial g_n} t^2 g_s^{-2},
\]
\[
L_{1,t} = -3 \frac{\partial}{\partial g_3} + \sum_{n \geq 1} \frac{(n + 1)g_n}{\partial g_n} t \frac{\partial}{\partial g_1} + 2t \frac{\partial}{\partial g_1},
\]
\[
L_{m,t} = \sum_{k \geq 1} \frac{(k + m)(g_k - \delta_{k,2})}{\partial g_{k+m}} + g_s^2 \sum_{k=1}^{m-1} \frac{k(m - k)}{\partial g_k} \frac{\partial}{\partial g_{m-k}} + 2tm \frac{\partial}{\partial g_m},
\]

where \( m \geq 2 \). We will refer to the corresponding constraints on the fat free energy function as the *fat Virasoro constraints*. 
3.7. Fat Virasoro constraints in terms of fat correlators. As in the thin case, we rewrite the fat Virasoro constraints in terms of fat correlators. The fat puncture equation can be written as

\[
\langle p_1 \cdot \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_n}}{a_n} \rangle_g^c(t) = \sum_{j=1}^{n} (a_j - 1) \cdot \langle \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_j}}{a_j} \cdots \frac{p_{a_n}}{a_n} \rangle_g^c(t),
\]

together with initial value:

\[
\langle p_1^2 \rangle_0^c(t) = t
\]

The fat dilaton equation can be written as

\[
\langle p_2 \cdot \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_n}}{a_n} \rangle_g^c(t) = \sum_{j=1}^{n} a_j \cdot \langle \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_j}}{a_j} \cdots \frac{p_{a_n}}{a_n} \rangle_g^c(t),
\]

together with initial value:

\[
\langle p_2 \rangle_0^c(t) = t^2.
\]

Note in the thin case the correlator in the initial value is in genus 1. The third equation in the sequence can be written as

\[
\langle p_3 \cdot \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_n}}{a_n} \rangle_g^c(t) = \sum_{j=1}^{n} (a_j + 1) \cdot \langle \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_j}}{a_j+1} \cdots \frac{p_{a_n}}{a_n} \rangle_g^c(t)
\]

\[
+ 2t \langle p_1 \cdot \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_n}}{a_n} \rangle^c_g(t).
\]

Note in the second line, the correlator is in genus \( g \). This is different from the thin case \([19]\) where the corresponding correlator is in genus \( g - 1 \). And for \( m \geq 2 \),

\[
\langle p_{m+2} \cdot \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_n}}{a_n} \rangle^c_g(t) = \sum_{j=1}^{n} (a_j + m) \cdot \langle \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_j}}{a_j} \cdots \frac{p_{a_n}}{a_n} \rangle^c_g(t)
\]

\[
+ 2t \langle p_{m+1} \cdot \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_n}}{a_n} \rangle^c_{g-1}(t)
\]

\[
+ \sum_{k=1}^{m} \langle p_k p_{m-k} \cdot \frac{p_{a_1}}{a_1} \cdots \frac{p_{a_n}}{a_n} \rangle^c_{g-1}(t)
\]

\[
+ \sum_{k=1}^{m} \sum_{g_1+g_2=g} \langle p_k \cdot \prod_{i \in I_1} \frac{p_{a_i}}{a_i} \rangle^c_{g_1,N} \cdot \langle \prod_{i \in I_2} \frac{p_{a_i}}{a_i} \rangle^c_{g_2,N},
\]

where \([n] = \{1, \ldots, n\}\).

Now we present some examples of computations of fat correlators by the fat Virasoro constraints. In degree 8 we have:

\[
\langle \frac{p_3}{3} \frac{p_5}{5} \rangle_0^c(t) = \frac{1}{3} \cdot \left( 6 \langle \frac{p_6}{6} \rangle_0^c(t) + 2t \langle \frac{p_5}{5} \rangle_0^c(t) \right)
\]

\[
= \frac{1}{3} \cdot \left( 6 \cdot \frac{5t^4}{6} + 2t \cdot 2t^3 \right) = 3t^4,
\]
and in genus one,

\[
\langle \frac{p_3}{3} \frac{p_5}{5} \rangle \xi_1(t) = \frac{1}{3} \left( 6 \langle \frac{p_6}{6} \rangle \xi_1(t) + 2N \langle \frac{p_5}{5} \rangle \xi_1(t) \right) = \frac{1}{3} \left( 6 \cdot \frac{5t^2}{3} + 2t \cdot t \right) = 4t^2.
\]

By the fat selection rule (21), \( \langle \frac{p_3}{3} \frac{p_5}{5} \rangle \xi_g(t) \) is nonvanishing only for \( g = 0 \) and 1.

4. Some More Applications of the Thin Virasoro Constraints

In this Section we present some applications of thin Virasoro constraints to compute \( F_{g,N} \).

4.1. Computation of \( F_{0,N} \) by thin Virasoro constraints. The thin Virasoro constraints in genus zero are:

\[
\frac{\partial F_{0,N}}{\partial g_1} = \sum_{n \geq 1} n g_{n+1} \frac{\partial F_{0,N}}{\partial g_n} + Ng_1,
\]

\[
2 \frac{\partial F_{0,N}}{\partial g_2} = \sum_{n \geq 1} n g_n \frac{\partial F_{0,N}}{\partial g_n},
\]

\[
(n + 2) \frac{\partial F_{0,N}}{\partial g_{n+2}} = \sum_{k \geq 1} (k + n) g_k \frac{\partial F_{0,N}}{\partial g_{k+n}}, \quad n \geq 1.
\]

Together with the initial value \( F_{0,N}(0,0,\ldots) = 0 \), these determine \( F_{0,N} \) uniquely. Note \( f_0 = \frac{1}{N} F_{0,N} \) satisfies the following recursion relations:

\[
\frac{\partial f_0}{\partial g_1} = \sum_{n \geq 1} n g_{n+1} \frac{\partial f_0}{\partial g_n} + g_1,
\]

\[
2 \frac{\partial f_0}{\partial g_2} = \sum_{n \geq 1} n g_n \frac{\partial f_0}{\partial g_n},
\]

\[
(n + 2) \frac{\partial f_0}{\partial g_{n+2}} = \sum_{k \geq 1} (k + n) g_k \frac{\partial f_0}{\partial g_{k+n}}, \quad n \geq 1.
\]

These are just the \( N = 1 \) case of the thin Virasoro constraints in genus zero. On the other hand, these are exactly the Virasoro constraints for \( F_{1,0}^{1D} \) of genus zero free energy of topological 1D gravity studied in an earlier work [10].

**Theorem 4.1.** [10, Theorem 7.8] The partition function \( Z \) of topological 1D gravity satisfies the following equations for \( m \geq -1 \):

\[
\tilde{L}_m Z = 0,
\]
where

\begin{align}
\tilde{L}_{-1} &= \frac{t_0}{\lambda^2} + \sum_{m \geq 1} (t_m - \delta_{m,1}) \frac{\partial}{\partial t_{m-1}}, \\
\tilde{L}_{0} &= 1 + \sum_{m \geq 0} (t_m - \delta_{m,1})(m+1) \frac{\partial}{\partial t_m}, \\
\tilde{L}_{1} &= 2\lambda^2 \frac{\partial}{\partial t_0} + \sum_{n \geq 0} (t_n - \delta_{n,1})(n+2)! \frac{\partial}{\partial t_{n+1}}, \\
\tilde{L}_{m} &= 2\lambda^2 m! \frac{\partial}{\partial t_{m-1}} + \lambda^4 \sum_{\substack{m_1 + m_2 = m \\ m_1, m_2 \geq 1}} m_1! \frac{\partial}{\partial t_{m_1-1}} m_2! \frac{\partial}{\partial t_{m_2-1}} \\
&\quad + \sum_{n \geq 0} (t_n - \delta_{n,1})(m+n+1)! \frac{\partial}{\partial t_{m+n}},
\end{align}

for \( m \geq 2 \). Furthermore, \( \{\tilde{L}_m\}_{m \geq 1} \) satisfies the following commutation relations:

\begin{equation}
[\tilde{L}_m, \tilde{L}_n] = 0,
\end{equation}

for \( m, n \geq -1 \).

For \( F^{1D}_0 \), these Virasoro constraints give:

\begin{align}
\frac{\partial F^{1D}_0}{\partial t_0} &= \sum_{m \geq 1} t_m \frac{\partial F^{1D}_0}{\partial t_{m-1}} + t_0, \\
2 \frac{\partial F^{1D}_0}{\partial t_1} &= \sum_{m \geq 0} (m+1)t_m \frac{\partial F^{1D}_0}{\partial t_m}, \\
(m+2) \frac{\partial F^{1D}_0}{\partial t_{m+1}} &= \sum_{n \geq 1} t_{n-1} (m+n)! \frac{\partial F^{1D}_0}{(n-1)! \partial t_{m+n-1}},
\end{align}

for \( m \geq 1 \). After changing \( t_n \) to \( n!g_{n+1} \), these match with (53)–(55). So we have proved the following:

**Theorem 4.2.** The thin genus zero part \( F_{0,N} \) of \( F_N \) is related to \( F_{0,N=1} \) in the following way:

\begin{equation}
F_{0,N} = N \cdot F^{1D}_0.
\end{equation}

Recall the following result in [10]:

**Theorem 4.3.** [10 Theorem 5.6] The following formulas hold:

\begin{align}
F^{1D}_0 &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{p_1+\cdots+p_{k+1}=k-1} \frac{t_{p_1}}{p_1!} \cdots \frac{t_{p_{k+1}}}{p_{k+1}!}, \\
&= \sum_{k=1}^{\infty} \frac{1}{k(k+1)(1-t_1)^k} \sum_{\substack{p_1+\cdots+p_{k+1}=k-1 \\ p_1,\ldots,p_{k+1} \neq 1}} \frac{t_{p_1}}{p_1!} \cdots \frac{t_{p_{k+1}}}{p_{k+1}!}.
\end{align}

And so as a consequence, we get:
Theorem 4.4. The following formulas hold:

\[ F_{0,N} = N \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{j_1+\cdots+j_{k+1}=2k, j_1,\ldots,j_{k+1} \geq 1} g_{j_1} \cdots g_{j_{k+1}} \]

(66)

\[ = N \sum_{k=1}^{\infty} \frac{1}{k(k+1)(1-g_2)^k} \sum_{j_1+\cdots+j_{k+1}=2k} g_{j_1} \cdots g_{j_{k+1}} \]

(67)

For example,

\[ F_{0,N} = N \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{j_1+\cdots+j_{k+1}=2k} g_{j_1} \cdots g_{j_{k+1}} \geq 1 \]

(68)

4.2. Computations for \( F_{1,N} \) by thin Virasoro constraints. In the above we have computed \( F_{0,N} \) by Virasoro constraints and identified it with \( NF_{1D} \). Now apply the same idea to compute \( F_{1,N} \). The Virasoro constraints for \( F_{1,N} \) are:

\[ \frac{\partial F_{1,N}}{\partial g_1} = \sum_{n \geq 1} n g_{n+1} \frac{\partial F_{1,N}}{\partial g_n}, \]

\[ 2 \frac{\partial F_{1,N}}{\partial g_2} = \sum_{n \geq 1} n g_n \frac{\partial F_{1,N}}{\partial g_n} + N^2, \]

\[ 3 \frac{\partial F_{1,N}}{\partial g_3} = \sum_{n \geq 1} n g_n \frac{\partial F_{1,N}}{\partial g_{n+1}} + 4N \frac{\partial F_{0,N}}{\partial g_1}, \]

\[ (n+2) \frac{\partial F_{1,N}}{\partial g_{n+2}} = \sum_{k \geq 1} (k+n) g_k \frac{\partial F_{1,N}}{\partial g_{k+n}} + \sum_{k \geq 1} k(n-k) \frac{\partial F_{0,N}}{\partial g_k} \frac{\partial F_{0,N}}{\partial g_{n-k}} + 2N \frac{\partial F_{0,N}}{\partial g_n}, \]

where \( n \geq 2 \). By taking \( N = 1 \), we get the Virasoro constraints of \( F_{1D} \), because we have \( F_{g,N=1} = F_{1D} \). It follows that if one sets \( f_1 = N^{-2} F_{1,N} \) and \( f_0 = N^{-1} F_{0,N} \), then dividing by \( N^2 \) on both sides of the above equalities, one sees that \( f_0 \) and \( f_1 \) satisfy the Virasoro constraints and initial value conditions for \( F_{1D} \) and \( F_{0D} \) respectively. Therefore, we have

\[ F_{1,N} = N^2 \cdot F_{1D}. \]

4.3. Higher genera case. However, we do not have

\[ F_{g,N} = N^{g+1} \cdot F_{gD} \]
for \( g \geq 2 \). For example, the Virasoro constraints for \( F_{2,N} \) for \( n = -1, 0, 1 \) are

\[
\begin{align*}
\frac{\partial F_{2,N}}{\partial g_1} &= \sum_{n \geq 1} n g_{n+1} \frac{\partial F_{2,N}}{\partial g_n}, \\
2 \frac{\partial F_{2,N}}{\partial g_2} &= \sum_{n \geq 1} n g_n \frac{\partial F_{2,N}}{\partial g_n}, \\
3 \frac{\partial F_{2,N}}{\partial g_3} &= \sum_{n \geq 1} n g_n \frac{\partial F_{2,N}}{\partial g_{n+1}} + 4N \frac{\partial F_{1,N}}{\partial g_1}.
\end{align*}
\]

These are similar to the case of \( F_{1,N} \). But for \( n \geq 2 \),

\[
(n + 2) \frac{\partial F_{2,N}}{\partial g_{n+2}} = \sum_{k \geq 1} (k + n) g_k \frac{\partial F_{2,N}}{\partial g_{k+n}} + 2 \sum_{k=1}^{n} k(n - k) \frac{\partial F_{0,N}}{\partial g_k} \frac{\partial F_{1,N}}{\partial g_{n-k}} + 2N \frac{\partial F_{1,N}}{\partial g_n} + \sum_{k=1}^{n} k(n - k) \frac{\partial^2 F_{0,N}}{\partial g_k \partial g_{n-k}},
\]

the extra term on the second line spoils the homogeneity in \( N \). Write

\[
F_{2,N} = f_{2,1} N + f_{2,3} N^3,
\]

then one gets for \( f_{2,3} \) the following recursion relations:

\[
\begin{align*}
\frac{\partial f_{2,3}}{\partial g_1} &= \sum_{n \geq 1} n g_{n+1} \frac{\partial f_{2,3}}{\partial g_n}, \\
2 \frac{\partial f_{2,3}}{\partial g_2} &= \sum_{n \geq 1} n g_n \frac{\partial f_{2,3}}{\partial g_n}, \\
3 \frac{\partial f_{2,3}}{\partial g_3} &= \sum_{n \geq 1} n g_n \frac{\partial f_{2,3}}{\partial g_{n+1}} + 4 \frac{\partial f_1}{\partial g_1},
\end{align*}
\]

and for \( n \geq 2 \),

\[
(n + 2) \frac{\partial f_{2,3}}{\partial g_{n+2}} = \sum_{k \geq 1} (k + n) g_k \frac{\partial f_{2,3}}{\partial g_{k+n}} + 2 \sum_{k=1}^{n} k(n - k) \frac{\partial f_0}{\partial g_k} \frac{\partial f_1}{\partial g_{n-k}} + 2n \frac{\partial f_1}{\partial g_n}.
\]

and for \( f_{2,1} \) one gets the following recursion relations:

\[
\begin{align*}
\frac{\partial f_{2,1}}{\partial g_1} &= \sum_{n \geq 1} n g_{n+1} \frac{\partial f_{2,1}}{\partial g_n}, \\
2 \frac{\partial f_{2,1}}{\partial g_2} &= \sum_{n \geq 1} n g_n \frac{\partial f_{2,1}}{\partial g_n}, \\
3 \frac{\partial f_{2,1}}{\partial g_3} &= \sum_{n \geq 1} n g_n \frac{\partial f_{2,1}}{\partial g_{n+1}}.
\end{align*}
\]

and for \( n \geq 2 \),

\[
(n + 2) \frac{\partial f_{2,1}}{\partial g_{n+2}} = \sum_{k \geq 1} (k + n) g_k \frac{\partial f_{2,1}}{\partial g_{k+n}} + \sum_{k=1}^{n} k(n - k) \frac{\partial^2 f_0}{\partial g_k \partial g_{n-k}}.
\]

One can use these recursion relations to compute \( f_{2,3} \) and \( f_{2,1} \) from \( f_0 = F_0^{1D} \) and \( f_1 = F_1^{1D} \).
4.4. An application of the dilaton equation. Now we generalize the result of \[10\] §6.2 to matrix models. The dilaton equation

\( L_{0,N} Z_N = 0 \)

can be rewritten as

\[ \frac{\partial F_N}{\partial g_2} = \sum_{m \geq 1} \frac{m}{2} g_m \frac{\partial F}{\partial g_m} + \frac{N^2}{2}. \]

In terms of correlators,

\[ \langle p_{2}^{2} \rangle_{c_{1},N} = N^2/2, \]

\[ \langle \prod_{j=1}^{n} \frac{p_{a_{j}}}{a_{j}} \rangle_{g,N} = \sum_{j=1}^{n} \frac{a_{j}}{2} \prod_{j=1}^{n} \frac{p_{a_{j}}}{a_{j}}. \]

Therefore,

\[ \langle (\prod_{j=1}^{m} \frac{p_{a_{j}}}{a_{j}})^{m} \rangle_{c_{1},N} = \frac{N^2}{2} (m-1)!, \]

and for \( a_2, \ldots, a_n \neq 1 \) which satisfies the selection rule \[10\],

\[ a_1 + \cdots + a_n = 2g - 2 + 2n, \]

we have

\[ \langle (\prod_{j=1}^{m} \frac{p_{a_{j}}}{a_{j}})^{m} \rangle_{g,N} = \prod_{k=0}^{m-1} (g - 1 + n + k) \prod_{j=1}^{n} \frac{p_{a_{j}}}{a_{j}}. \]

It follows that we have

**Theorem 4.5.** The free energy \( F_N \) can be rewritten in the following form:

\[ F_N = \frac{1}{2} \log(1 - g_2) + \sum_{g \geq 0, n > 0} \sum_{a_1, \ldots, a_n \neq 2} \frac{\langle \prod_{j=1}^{n} \frac{p_{a_{j}}}{a_{j}} \rangle_{0,N}}{(1 - t_2)^n - 1} \frac{1}{n!} \prod_{j=1}^{n} g_{a_{j}}, \]

For example,

\[ F_{0,N} = \sum_{n>0} \sum_{a_1, \ldots, a_n \neq 2} \frac{\langle \prod_{j=1}^{n} \frac{p_{a_{j}}}{a_{j}} \rangle_{0,N}}{(1 - t_2)^{n-1}} \frac{1}{n!} \prod_{j=1}^{n} g_{a_{j}}, \]

\[ F_{1,N} = \log(1 - g_2) + \sum_{n>0} \sum_{a_1, \ldots, a_n \neq 2} \frac{\langle \prod_{j=1}^{n} \frac{p_{a_{j}}}{a_{j}} \rangle_{1,N}}{(1 - t_2)^n} \frac{1}{n!} \prod_{j=1}^{n} g_{a_{j}}, \]

\[ F_{g,N} = \sum_{n>0} \sum_{a_1, \ldots, a_n \neq 2} \frac{\langle \prod_{j=1}^{n} \frac{p_{a_{j}}}{a_{j}} \rangle_{g,N}}{(1 - t_2)^{g-1+n}} \frac{1}{n!} \prod_{j=1}^{n} g_{a_{j}}, \quad g > 1. \]

In particular, by the dilaton equation \[70\], one can reduce the calculations of \( F_{g,N} \) to the calculations of correlators \( \langle p_{a_1} \cdots p_{a_n} \rangle_N \) with \( a_i \neq 2 \), \( i = 1, \ldots, n \). Similarly, one can use the string equation

\[ L_{-1,N} Z_N = 0 \]
to further reduce to the correlators \( \langle p_{a_1} \cdots p_{a_n} \rangle_N \) with \( a_i > 2, i = 1, \ldots n \). An even better way to use such ideas is to introduce suitable changes of coordinates on the space of coupling constants to be discussed in next Section.

5. Thin Virasoro Constraints in Renormalized Coupling Constants

We now combine the Virasoro constraints with the renormalized coupling constants to prove some structure results for \( F_{g,N} \).

5.1. The renormalized coupling constants. Let us recall the following coordinate change on the big phase space of coupling constants:

\[
I_0 = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_1+\cdots+p_k=k-1} t_{p_1} \cdots t_{p_k} \frac{p_1! \cdots p_k!}{k!}
\]

(78)

\[
I_k = \sum_{n=0}^{\infty} t_{n+k} \frac{I_n^n}{n!}, \quad k \geq 1.
\]

(79)

These will be referred to as the renormalized coupling constants. These series were introduced in [6] to express the free energy of topological 2D gravity. In [10] they were understood as new coordinates on the big phase space. By [10, Proposition 2.4],

\[
t_k = \sum_{n=0}^{\infty} \frac{(-1)^n I_n^n}{n!} I_{n+k}.
\]

(80)

The renormalized coupling constants were used to gain better understanding of the global nature of the behavior of the theory on the big phase space. For example, two different methods were used to show that for topological 1D gravity,

\[
F_{0}^{1D} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1},
\]

(81)

\[
F_{1}^{1D} = \frac{1}{2} \ln \frac{1}{1-1},
\]

(82)

\[
F_{g}^{1D} = \sum_{g=2}^{2g-1} \sum_{\tau_2^2 \cdots \tau_{2g-1}^2} \prod_{j=2}^{2g-1} \frac{1}{l_j} \left( \frac{I_j}{(1-I_1)(j+1/2)} \right)^{l_j}, \quad g \geq 2.
\]

(83)

5.2. Puncture operator \( L_{-1,N} \) in I-coordinates. As an application of [62] and [81], we have the following result:

\[
F_{0,N} = N \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}.
\]

(84)

We now give a direct proof of it in the spirit of [10]. By [10] (34) and [80], we have

\[
L_{-1,N} = -\frac{\partial}{\partial I_0} + \frac{N}{g_s} \sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_n.
\]

(85)
It follows from (85) that
\[
\frac{\partial F_{0,N}}{\partial I_0} = N \sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_n,
\]
(86)
\[
\frac{\partial F_{g,N}}{\partial I_0} = 0, \quad g \geq 1.
\]
(87)

Therefore, we get the following results generalizing [10, Theorem 6.4]:

**Theorem 5.1.** The thin genus zero part \( F_{0,N} \) of the free energy \( F_N \) is given in \( I \)-coordinates by:
\[
F_{0,N} = N \left( \frac{1}{2} I_0^2 + \sum_{n=0}^{\infty} \frac{(-1)^n I_0^{n+1}}{(n+1)!} I_n \right).
\]
(88)

Furthermore, when \( g \geq 1 \), \( F_{g,N} \) is independent of \( I_0 \).

5.3. **Dilaton operator \( L_{0,N} \) in \( I \)-coordinates.** Similar to [10, Lemma 6.5], the dilaton operator \( L_{0,N} \) is given in \( I \)-coordinates by:
\[
L_{0,N} = -I_0 \frac{\partial}{\partial I_0} - 2 \frac{\partial}{\partial I_1} + \sum_{l \geq 1} \left( l + 1 \right) I_l \frac{\partial}{\partial I_l} + N^2.
\]
(89)

From the dilaton equation, one gets
\[
\frac{\partial F_{0,N}}{\partial I_1} = \sum_{l \geq 1} \frac{l + 1}{2} I_l \frac{\partial F_{0,N}}{\partial I_l} - \frac{1}{2} \frac{\partial F_{0,N}}{\partial I_0},
\]
(90)
\[
\frac{\partial F_{1,N}}{\partial I_1} = \sum_{l \geq 1} \frac{l + 1}{2} I_l \frac{\partial F_{1,N}}{\partial I_l} + N^2, \quad g \geq 2.
\]
(91)
\[
\frac{\partial F_{g,N}}{\partial I_1} = \sum_{l \geq 1} \frac{l + 1}{2} I_l \frac{\partial F_{g,N}}{\partial I_l}, \quad g \geq 2.
\]
(92)

By slightly modifying the proof of [10, Theorem 6.6], one can prove the following:

**Theorem 5.2.** In \( I \)-coordinates we have
\[
F_{1,N} = \frac{N^2}{2} \ln \frac{1}{1 - I_1}
\]
(93)

and for \( g \geq 1 \),
\[
F_{g,N} = \sum_{\sum_{j=3}^{2g} (j-2)I_j = 2g-2} \langle p_3^{I_3} \cdots p_{2g}^{(2g)_{I_{2g}}} \rangle_{i_{g,N}} \prod_{j=3}^{2g} \frac{1}{I_j!(j-1)!^2} \left( \frac{I_j-1}{1-(I_1)^{j/2}} \right)^{I_j}.
\]
(94)

By looking at the summation in (94), one sees surprisingly that it is a summation over partition over \( 2g-2 \). Introducing new variables \( q_n \) that plays the role of Newton power function \( p_n \):
\[
q_n := \frac{I_n+1}{(1-I_1)^{(n+2)/2}},
\]
(95)
\[
\begin{align*}
F_{g,N} &= \sum_{\sum_{k=1}^{2g-2} k m_k = 2g-2} \langle p_3^{m_3} \cdots p_{2g}^{(2g)_{m_{2g}}} \rangle_{i_{g,N}} \prod_{k=1}^{2g-2} \frac{1}{m_k!(k+1)!^{m_k}} q_k^{m_k}.
\end{align*}
\]
(96)
For example, for $g = 2$,

$$F_{2,N} = \frac{1}{2!} \left( \frac{p_3}{3} \right)^2 \frac{I_2^2}{(2l)^2(1-I_1)^3} + \frac{1}{4!} \left( \frac{p_4}{4} \right)^2 \frac{I_3}{3!(1-I_1)^2}$$

$$= \left( \frac{2N^3}{3} + \frac{N}{6} \right) \frac{I_2^2}{2(1-I_1)^3} + \left( \frac{1}{2} N^3 + \frac{1}{4} N \right) \frac{I_3}{3!(1-I_1)^2}.$$

In terms of the new variables $q_n$:

$$F_{2,N} = \left( \frac{N^3}{6} + \frac{N}{24} \right) q_1^2 + \left( \frac{N^3}{12} + \frac{N}{24} \right) q_2$$

Similarly, for $g = 3$,

$$F_{3,N} = \frac{1}{4!} \left( \frac{p_3}{3} \right)^4 \frac{I_3}{(2l)^4(1-I_1)^6} + \frac{1}{2!} \left( \frac{p_4}{4} \right)^2 \frac{I_3}{(2l)^2 3!(1-I_1)^5} + \frac{1}{2!^2} \frac{I_3}{4!(1-I_1)^4}$$

$$+ \frac{1}{2!} \frac{1}{4} \left( 15N^2 + 9N^2 \right) \frac{I_3}{(3l)^2(1-I_1)^4} + \frac{1}{2!^2} \frac{I_3}{4!(1-I_1)^4}$$

$$+ \left( \frac{5N^2}{3} + \frac{5N^4}{6} \right) \frac{I_5}{5!(1-I_1)^3}.$$

By computing the correlators, one gets:

$$F_{3,N} = \frac{1}{6} N^4 + \frac{7}{48} N^2 q_1^2 + \left( \frac{1}{4} N^4 + \frac{13}{48} N^2 \right) q_2 q_2 + \left( \frac{1}{32} N^4 + \frac{5}{96} N^2 \right) q_2^2$$

$$+ \left( \frac{1}{16} N^4 + \frac{1}{12} N^2 \right) q_1 q_3 + \left( \frac{1}{144} N^4 + \frac{1}{72} N^2 \right) q_4$$

For $g = 4$,

$$F_{4,N} = \langle \left( \frac{p_3}{3} \right)^6 \rangle_4 \frac{I_3^6}{6! (2l)^6(1-I_1)^9} + \langle \left( \frac{p_4}{4} \right)^4 \rangle_4 \frac{I_4^4}{4! (2l)^4 3!(1-I_1)^8}$$

$$+ \langle \left( \frac{p_3}{3} \right)^2 (\frac{p_4}{4})^2 \rangle_4 \frac{I_2^2 I_4}{2! (2l)^3 (3l)^2(1-I_1)^7} + \langle \left( \frac{p_3}{3} \right)^3 \rangle_4 \frac{I_3^3}{3! (3l)^2(1-I_1)^6}$$

$$+ \langle \left( \frac{p_3}{3} \right)^3 \frac{p_4}{4} \rangle_4 \frac{I_2^3 I_4}{3! (2l)^3 (3l)^4 4!(1-I_1)^7} + \langle \left( \frac{p_3}{3} \right)^3 \frac{p_5}{5} \rangle_4 \frac{I_2^3 I_5}{2! (2l)^3 5!(1-I_1)^6}$$

$$+ \langle \left( \frac{p_3}{3} \right)^3 \frac{p_4}{4} \rangle_4 \frac{I_2^3 I_4}{2! (2l)^3 4!(1-I_1)^7} + \langle \left( \frac{p_3}{3} \right)^3 \frac{p_5}{5} \rangle_4 \frac{I_2^3 I_5}{2! (2l)^3 5!(1-I_1)^6}$$

$$+ \langle \left( \frac{p_3}{3} \right)^4 \frac{p_4}{4} \rangle_4 \frac{I_2^4 I_5}{2! (2l)^4 6!(1-I_1)^8} + \langle \left( \frac{p_3}{3} \right)^4 \frac{p_5}{5} \rangle_4 \frac{I_2^4 I_5}{2! (2l)^4 7!(1-I_1)^8}.$$
By computing the correlators, one gets:

\[
F_{4,N} = (4736N^5 + 7104N^3 + 840N) \cdot \frac{I_2^6}{6!(2!)^6(1 - I_1)^9} + (1632N^5 + 3648N^3 + 630N) \cdot \frac{I_2^3I_3}{4!(2!)^43!(1 - I_1)^8} + (156N^5 + 384N^3 + 70N) \cdot \frac{I_3^2}{2!(2!)^2(3!)^2(1 - I_1)^6} + (27N^5 + 99N^3 + \frac{45}{2}N) \cdot \frac{I_4^3}{3!(3!)^3(1 - I_1)^9} + (240N^5 + 612N^3 + 114N) \cdot \frac{I_3^2I_4}{3!(2!)^34!(1 - I_1)^7} + (18N^5 + 54N^3 + 12N) \cdot \frac{I_2I_3I_4}{2!(2!)^23!(1 - I_1)^6} + (\frac{36}{5}N^5 + 24N^3 + \frac{33}{5}N) \cdot \frac{I_4^3}{2!(4!)^2(1 - I_1)^5} + (25N + \frac{370}{3}N^3 + 40N^5) \cdot \frac{I_2I_5}{2!(2!)^25!(1 - I_1)^6} + (7N + 30N^3 + 8N^5) \cdot \frac{I_2I_6}{2!(2!)^2(3!)^2(1 - I_1)^5} + \frac{1}{8} (21N + 70N^3 + 14N^5) \cdot \frac{I_7}{7!(1 - I_1)^4}.
\]

In the \(q_0\)-coordinates,

\[
F_{4,N} = (\frac{37}{360}N^5 + \frac{37}{240}N^3 + \frac{7}{384}N)q_1^6 + (\frac{17}{24}N^5 + \frac{19}{12}N^3 + \frac{35}{128}N)q_1q_2 + (\frac{13}{48}N^5 + \frac{2}{3}N^3 + \frac{35}{288}N)q_1^2q_2^2 + (\frac{1}{48}N^5 + \frac{1}{144}N^3 + \frac{5}{288}N)q_2^3 + (\frac{5}{24}N^5 + \frac{17}{32}N^3 + \frac{19}{192}N)q_1^3q_3 + (\frac{1}{16}N^5 + \frac{3}{16}N^3 + \frac{1}{24}N)q_1^2q_2q_3 + (\frac{1}{160}N^5 + \frac{1}{48}N^3 + \frac{11}{1920}N)q_1q_3^2 + (\frac{5}{192}N^5 + \frac{37}{288}N^3 + \frac{1}{24}N)q_1^2q_4 + (\frac{7}{1440}N^5 + \frac{1}{48}N^3 + \frac{1}{180}N)q_1q_5 + (\frac{1}{1920}N^5 + \frac{1}{576}N^3 + \frac{1}{2880}N)q_6.
\]

6. Renormalizations of Hermitian One-Matrix Models

We first recall the renormalization of the universal action function studied in [10], then apply it to Hermitian one-matrix models to give another proof of the main results in last Section.

6.1. Renormalization of the universal action function. By the universal action function we mean the following formal power series in \(x\) depending on infinitely many parameters \(t_0, \ldots, t_n, \ldots\):

\[
S = -\frac{1}{2}x^2 + \sum_{n \geq 1} t_n - \frac{x^n}{n!}.
\]
The coefficients $t_n$’s will be called the coupling constants. Let $x_\infty$ be the solution of
\[
\frac{\partial S}{\partial x} = 0,
\]
i.e., $x_\infty$ satisfies the equation:
\[
x_\infty = \sum_{n \geq 0} t_n \frac{x^n}{n!}.
\]
By [10] Proposition 2.2, the following formula for $x_\infty$ holds:
\[
x_\infty = \sum_{n=1}^{\infty} \frac{1}{k} \sum_{p_1 + \ldots + p_k = k-1} \frac{t_{p_1}}{p_{p_1}} \ldots \frac{t_{p_k}}{p_{p_k}}.
\]
By [10] Theorem 2.3, (56)],
\[
S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1} + \sum_{n=2}^{\infty} \frac{I_{n-1} - \delta_{n,2}}{n!} (x - I_0)^n,
\]
where $I_0 = x_\infty$ and $I_k$ ($k \geq 1$) have already been recalled earlier in (78) and (79).

6.2. Renormalization of Hermitian one-matrix models. Now in [24] we take
\[
\tilde{T}_n = \frac{\bar{t}_{n-1} - \delta_{n,2}}{n! g_s} \cdot \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2
\]

\[
Z_N = c_N \int_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i \cdot \exp\left(\sum_{i=1}^{N} \sum_{n=2}^{\infty} \frac{I_{n-1} - \delta_{n,2}}{n!} \lambda_i^n\right) \cdot \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2
\]

\[
= c_N \int_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i \cdot \exp\left(\sum_{i=1}^{N} \frac{1}{g_s} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}
\right.ight.
\]

\[
+ \sum_{n=2}^{\infty} \frac{I_{n-1} - \delta_{n,2}}{n!} (\lambda_i - I_0)^n) \right) \cdot \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2
\]

\[
= \exp\left(\sum_{i=1}^{N} \frac{1}{g_s} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}
\right.ight.
\]

\[
\cdot c_N \int_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i \cdot \exp\left(\sum_{i=1}^{N} \frac{1}{g_s} \left(\sum_{n=2}^{\infty} \frac{I_{n-1} - \delta_{n,2}}{n!} \lambda_i^n\right) \cdot \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2
\]

\[
= \exp\left(\sum_{i=1}^{N} \frac{1}{g_s} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}
\right.ight.
\]

\[
\cdot c_N \int_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i \cdot \exp\left(\sum_{i=1}^{N} \frac{1}{g_s} \left(-\frac{1}{2} \lambda_i^2 + \sum_{n=3}^{\infty} \frac{I_{n-1}}{n!(1-I_1)^{n/2}} \lambda_i^n\right) \cdot \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2
\]

\[
= \exp\left(\sum_{i=1}^{N} \frac{1}{g_s} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}
\right.ight.
\]

\[
\cdot \int_{\mathcal{H}_N} dM \cdot \exp\left(-\frac{1}{g_s} \frac{1}{2} \text{tr}(M^2) + \sum_{n=3}^{\infty} \frac{I_{n-1}}{n!(1-I_1)^{n/2}} \text{tr}(M^n)\right)
\]

\[
= \int_{\mathcal{H}_N} dM \exp\left(-\frac{1}{g_s} \frac{1}{2} \text{tr}(M^2)\right).
\]
So we have proved the following generalization of [10, (124)]:

**Theorem 6.1.** The partition function $Z_N$ of Hermitian $N \times N$-matrix models remains the same under the following renormalizations of coupling constants:

$$Z_N = \exp \left( \frac{N}{g_s} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}(I_k + \delta_{k,1})I_0^{k+1} + \frac{N^2}{2} \log \frac{1}{1-I_1} \right)$$

$$Z_N \big|_{g_1=g_2=0, g_n=\frac{t_{n-1}}{(n-1)! (1-I_1)^{n/2}}, n \geq 3}.$$

From this Theorem, one can rederive Theorem 5.1 and Theorem 5.2.

7. FROM THIN GENUS EXPANSION TO FAT GENUS EXPANSION

In this Section we show how to get the fat genus expansion from the thin genus expansion.

Recall the thin genus expansion is of the form:

$$F_N = g_s^{-1} F_{0,N} + F_{1,N} + g_s F_{2,N} + g_s^2 F_{3,N} + g_s^3 F_{4,N} + \cdots,$$

where each $F_{g,N}$ is a polynomial in $N$. Now we change $N$ to $t g_s^{-1}$, where $t = N g_s$ is the ’t Hooft coupling constant:

$$\sum_{g \geq 0} g_s^{2g-2} F_g(t)$$

$$= g_s^{-2} t \frac{1}{2} I_0^2 + \sum_{n=0}^{\infty} \frac{(-1)^n n^{n+1}}{(n+1)!} I_n + t^2 \log \frac{1}{1-I_1}$$

$$+ g_s \left( \frac{N^3}{6} + \frac{N}{24} q_1^2 + \frac{N^3}{12} + \frac{N}{24} q_2 \right)$$

$$+ g_s^2 \left( \frac{1}{3!} (56t^2 g_s^{-2} + 64t^4 g_s^{-4}) \cdot \frac{I_2}{(2!)^2 (1-I_1)^4} + \frac{1}{2!} (13t^2 g_s^{-2} + 12t^4 g_s^{-4}) \cdot \frac{I_3}{(2!)^2 (1-I_1)^5} \right.$$

$$+ \frac{1}{3!} \frac{1}{4} (15t^2 g_s^{-2} + 9t^4 g_s^{-4}) \cdot \frac{I_5}{(3!)^2 (1-I_1)^4} + (4t^2 g_s^{-2} + 3t^4 g_s^{-4}) \cdot \frac{I_2 I_4}{2! 4! (1-I_1)^4}$$

$$+ \left. \frac{5t^2 g_s^{-2}}{3} + \frac{5t^4 g_s^{-4}}{6} \right) \frac{I_5}{5! (1-I_1)^3} \right).$$
Therefore we get:

\[
F_0(t) = t \left( \frac{1}{2} I_0^2 + \sum_{n=0}^{\infty} \frac{(-1)^n I_0^{n+1}}{(n+1)!} I_n \right) + t^3 \left( \frac{2 I_2^2}{3 2^2 (1-I_1)^2} + \frac{1}{2} \frac{I_3}{3! (1-I_1)^2} \right) + t^4 \left( \frac{4 I_4}{4!} \cdot 64 \cdot \frac{I_2^2}{(2!)^4 (1-I_1)^3} + \frac{1}{2!} \cdot 12 \cdot \frac{I_2^3 I_4}{(2!)^2 3! (1-I_1)^5} + \frac{1}{2!} \cdot 1 \cdot 9 \cdot \frac{I_3^2}{3! (1-I_1)^4} + 3 \cdot \frac{I_2 I_4}{2!4! (1-I_1)^3} + \frac{5}{6} \cdot \frac{I_5}{5! (1-I_1)^3} \right)
\]

plus

\[
\begin{align*}
&+ t^5 \left[ 4736 \cdot \frac{1}{6!} \frac{I_2^6}{(2!)^6 (1-I_1)^9} + 1632 \cdot \frac{1}{4!} \frac{I_2 I_3}{(2!)^4 3! (1-I_1)^8} + 156 \cdot \frac{1}{2!2!} \frac{I_2^2 I_3}{(2!)^2 (3!)^2 (1-I_1)^6} + 27 \cdot \frac{1}{3!} \frac{I_3^3}{(3!)^3 (1-I_1)^6} \right] \\
&+ 240 \cdot \frac{1}{3!} \frac{I_2^4 I_4}{(2!)^4 4! (1-I_1)^7} + 18 \cdot \frac{1}{2!3!4!} \frac{I_2 I_3 I_4}{(2!)^2 5! (1-I_1)^6} \\
&+ 36 \cdot \frac{1}{5} \frac{I_2 I_5}{2!4! (1-I_1)^5} + 40 \cdot \frac{1}{2!} \frac{I_2^3 I_5}{(2!)^2 5! (1-I_1)^5} \\
&+ 8 \cdot \frac{I_2 I_6}{2!6! (1-I_1)^5} + \frac{1}{8} \cdot 14 \cdot \frac{I_7}{7! (1-I_1)^4} \right] + \ldots.
\end{align*}
\]
\[ F_1(t) = t^2 \log \frac{1}{1-I_1} + t \left( \frac{1}{6} \cdot \frac{I_2^2}{2!^2 (1-I_1)^3} + \frac{1}{4} \cdot \frac{I_3^2}{3! (1-I_1)^2} \right) + t^2 \left( \frac{1}{4!} \cdot 56 \cdot \frac{I_2^3 I_3}{(2!)^4 (1-I_1)^6} + \frac{1}{2!} \cdot 13 \cdot \frac{I_2^3 I_3}{(2!)^2 3! (1-I_1)^5} + \frac{1}{4!} \cdot 15 \cdot \frac{I_3^3}{(3!)^2 (1-I_1)^4} + 4 \cdot \frac{I_2 I_4}{2!^4 (1-I_1)^4} + \frac{5}{3} \cdot \frac{I_5}{5! (1-I_1)^5} \right) \]

plus

\[ + t^3 \left[ 7104 \cdot \frac{1}{6!} \frac{I_2^6}{(2!)^6 (1-I_1)^9} + 3648 \cdot \frac{1}{4!} \frac{I_2^4 I_3}{(2!)^4 3! (1-I_1)^8} + 384 \cdot \frac{1}{2!^2} \frac{I_2^6 I_3}{(2!)^2 (3!)^2 (1-I_1)^7} + 99 \cdot \frac{1}{3!} \frac{I_3^3}{(3!)^3 (1-I_1)^6} + 612 \cdot \frac{1}{3!} \frac{I_2^3 I_4}{(2!)^3 4! (1-I_1)^6} + 54 \cdot \frac{I_2 I_3 I_4}{2!^3 4! (1-I_1)^6} + 24 \cdot \frac{I_4}{2! (4!)^2 (1-I_1)^5} + 370 \cdot \frac{I_2^2 I_5}{3} + \frac{1}{8} \cdot \frac{I_2 I_6}{7! (1-I_1)^5} \right] + \ldots, \]

\[ F_2(t) = t \left[ 840 \cdot \frac{1}{6!} \frac{I_2^6}{(2!)^6 (1-I_1)^9} + 630 \cdot \frac{1}{4!} \frac{I_2^4 I_3}{(2!)^4 3! (1-I_1)^8} + 70 \cdot \frac{1}{2!^2} \frac{I_2^6 I_3}{(2!)^2 (3!)^2 (1-I_1)^7} + 45 \cdot \frac{1}{2!} \frac{I_3^3}{(3!)^3 (1-I_1)^6} + 114t \cdot \frac{1}{3!} \frac{I_2^3 I_4}{(2!)^3 4! (1-I_1)^6} + 12t \cdot \frac{I_2 I_3 I_4}{2!^3 4! (1-I_1)^6} + \frac{33}{5} \cdot \frac{I_4}{2! (4!)^2 (1-I_1)^5} + 25 \cdot \frac{1}{2!} \frac{I_2^2 I_5}{(4!)^2 5! (1-I_1)^5} + 7t \cdot \frac{I_2 I_6}{2! 6! (1-I_1)^5} + \frac{1}{8} \cdot \frac{I_2 I_7}{7! (1-I_1)^5} \right] + \ldots. \]

One can also use the coordinates \( \{q_n\} \) defined in Eq. 12 to simplify the expressions and to see a connection to partitions of even integers. Such expression should be useful to the study of singular behavior of the free energy and the (multi)critical phenomenon in Hermitian matrix models. We hope to address such applications in the future investigations.

**Acknowledgements.** The author is partly supported by NSFC grant 11661131005.

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Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China
E-mail address: jzhou@math.tsinghua.edu.cn