FACTORIZATION OF DIFFERENTIAL OPERATORS, QUASIDETERMINANTS, AND NONABELIAN TODA FIELD EQUATIONS

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Abstract

We integrate nonabelian Toda field equations [Kr] for root systems of types $A$, $B$, $C$, for functions with values in any associative algebra. The solution is expressed via quasideterminants introduced in [GR1],[GR2], [GR4]. In the appendix we review some results concerning noncommutative versions of other classical integrable equations.

Introduction

Nonabelian Toda equations for the root system $A_{n-1}$ were introduced by Polyakov (see [Kr]). They are equations with respect to $n$ unknowns $\phi = (\phi_1, ..., \phi_n) \in A[[u,v]]$, where $A$ is some associative (not necessarily commutative) algebra with unit:

$$\frac{\partial}{\partial u} \left( \frac{\partial \phi_j}{\partial v} \phi_j^{-1} \right) = \begin{cases} \phi_2 \phi_1^{-1}, & j = 1 \\ \phi_{j+1} \phi_j^{-1} - \phi_j \phi_{j-1}^{-1}, & 2 \leq j \leq n-1 \\ -\phi_n \phi_{n-1}^{-1}, & j = n \end{cases}$$

Suppose that $A$ is a $*$-algebra, i.e. it is equipped with an involutive antiautomorphism $*: A \to A$. Then, setting in (1) $\phi_{n+1-i} = (\phi_i^*)^{-1}$, we obtain a new system of equations. If $n = 2k$, we get the nonabelian Toda system for root system $C_k$:

$$\frac{\partial}{\partial u} \left( \frac{\partial \phi_j}{\partial v} \phi_j^{-1} \right) = \begin{cases} \phi_2 \phi_1^{-1}, & j = 1 \\ \phi_{j+1} \phi_j^{-1} - \phi_j \phi_{j-1}^{-1}, & 2 \leq j \leq k-1 \\ (\phi_k^*)^{-1} \phi_k^{-1} - \phi_k \phi_{k-1}^{-1}, & j = k \end{cases}$$

If $n = 2k + 1$, we get the nonabelian Toda system for root system $B_k$:

$$\frac{\partial}{\partial u} \left( \frac{\partial \phi_j}{\partial v} \phi_j^{-1} \right) = \begin{cases} \phi_2 \phi_1^{-1}, & j = 1 \\ \phi_{j+1} \phi_j^{-1} - \phi_j \phi_{j-1}^{-1}, & 2 \leq j \leq k \\ (\phi_k^*)^{-1} \phi_{k+1}^{-1} - \phi_{k+1} \phi_{k}^{-1}, & j = k+1, \end{cases}$$

where $\phi_{k+1} = \phi_{k+1}^{-1}$.

Quasideterminants were introduced in [GR1], as follows. Let $X$ be an $m \times m$-matrix over $A$. For any $1 \leq i, j \leq m$, let $r_i(X), c_j(X)$ be the i-th row and the j-th column of $X$. Let $X^{ij}$ be the submatrix of $X$ obtained by removing the i-th
row and the j-th column from X. For a row vector r let \( r^{(j)} \) be r without the j-th entry. For a column vector c let \( c^{(i)} \) be c without the i-th entry. Assume that \( X^{ij} \) is invertible. Then the quasideterminant \( |X|_{ij} \in A \) is defined by the formula

\[
|X|_{ij} = x_{ij} - r_i(X)^{(j)}(X^{ij})^{-1}c_j(X)^{(i)},
\]

where \( x_{ij} \) is the ij-th entry of X.

In this paper we will use quasideterminants to integrate nonabelian Toda equations for root systems of types \( A, B, C \) (see [LS]; see also [FF] and references therein; for the one variable case see [Ko]).

Our method of integration, which is based on interpreting the Toda flow as a flow on the space of factorizations of a fixed ordinary differential operator. This method and explicit solutions of Toda equations are well known in the commutative case (see [LS]; see also [FF] and references therein; for the one variable case see [Ko]).

1. Factorization of differential operators

Let \( R \) be an associative algebra over a field \( k \) of characteristic zero, and \( D : R \to R \) be a k-linear derivation.

Let \( f_1, \ldots, f_m \) be elements of \( R \). By definition, the Wronski matrix \( W(f_1, \ldots, f_m) \) is

\[
W(f_1, \ldots, f_m) = \begin{pmatrix}
 f_1 & \cdots & f_m \\
 Df_1 & \cdots & Df_m \\
 \vdots & \cdots & \vdots \\
 D^{m-1}f_1 & \cdots & D^{m-1}f_m
\end{pmatrix}
\]

We will call a set of elements \( f_1, \ldots, f_m \in R \) nondegenerate if \( W(f_1, \ldots, f_m) \) is invertible.

Denote by \( R[D] \) the space of polynomials of the form \( a_0D^n + a_1D^{n-1} + \ldots + a_n, a_i \in R \). It is clear that any element of \( R[D] \) defines a linear operator on \( R \).

Example: \( R = C^\infty(\mathbb{R}) \), \( D = \frac{d}{dt} \). In this case, \( R[D] \) is the algebra of differential operators on the line.

By analogy with this example, we will call elements of \( R[D] \) differential operators.

We will consider operators of the form \( L = D^n + a_1D^{n-1} + \ldots + a_n \). We will call such \( L \) an operator of order \( n \) with highest coefficient 1. Denote the space of all such operators by \( R_n(D) \).

**Theorem 1.1.** (i) Let \( f_1, \ldots, f_n \in R \) be a nondegenerate set of elements. Then there exists a unique differential operator \( L \in R[D] \) of order \( n \) with highest coefficient 1, such that \( Lf_i = 0 \) for \( i = 1, \ldots, n \). It is given by the formula

\[
Lf = |W(f_1, \ldots, f_n, f)|_{n+1,n+1}.
\]

(ii) Let \( L \) be of order \( n \) with highest coefficient 1, and \( f_1, \ldots, f_n \) be a set of solutions of the equation \( Lf = 0 \), such that for any \( m \leq n \) the set of elements \( f_1, \ldots, f_m \) is nondegenerate. Then \( L \) admits a factorization \( L = (D-b_n)\ldots(D-b_1) \), where

\[
b_i = (DW_i)W_i^{-1}, \ W_i = |W(f_1, \ldots, f_i)|_{ii}.
\]

**Proof.** (i) We look for \( L \) in the form \( L = D^n + a_1D^{n-1} + \ldots + a_n \). From the equations \( Lf_i = 0 \) it follows that

\[
(a_n, \ldots, a_1) = -(D^n f_1, \ldots, D^n f_n)W(f_1, \ldots, f_n)^{-1}.
\]
By definition,

\[ |W(f_1, \ldots, f_n, f)|_{n+1, n+1} = D^n f - (D^n f_1, \ldots, D^n f_n)W(f_1, \ldots, f_n)^{-1}(f, D f, \ldots, D^{n-1} f)^T = D^n f + (a_n, ..., a_1)(f, D f, \ldots, D^{n-1} f)^T = L f. \]

(ii) We will prove the statement by induction in \( n \). For \( n = 1 \), the statement is obvious. Suppose it is valid for the differential operator \( L_{n-1} \) of order \( n - 1 \) with highest coefficient 1, which annihilates \( f_1, \ldots, f_{n-1} \) (by (i), it exists and is unique). Set \( b_n = (DW_n)W_n^{-1} \), and consider the operator \( \tilde{L} = (D - b_n)L_{n-1} \). It is obvious that \( \tilde{L}f_i = 0 \) for \( i = 1, \ldots, n - 1 \). Also, by (i)

\[ \tilde{L}f_n = (D - b_n)L_{n-1}f_n = (D - b_n)W_n = 0. \]

Therefore, by (i), \( \tilde{L} = L \). \( \square \)

Now consider the special case: \( R = A[[t]] \), where \( A \) is an associative algebra over \( k \), and \( D = \frac{d}{dt} \) (here \( t \) commutes with everything). In this case, it is easy to show that nondegenerate sets of solutions of \( L f = 0 \) exist, and are in 1-1 correspondence with elements of the group \( GL(A) \), via \( f = (f_1, \ldots, f_n) \rightarrow W(f)(0) \).

It is clear that two different sets of solutions of the equation \( L f = 0 \) can define the same factorization of \( L \). However, to each factorization \( \gamma \) of \( L \) we can assign a set \( f_\gamma = (f_1, \ldots, f_n) \) of solutions of \( L f = 0 \), which gives back the factorization \( \gamma \) under the correspondence of Theorem 1.1(ii). This set is uniquely defined by the condition that the matrix \( W(f_\gamma)(0) \) is lower triangular with 1-s on the diagonal.

Here is a formula for computing \( f_\gamma \), which is well known in the commutative case.

**Proposition 1.2.** If \( \gamma \) has the form

\[ L = (D - (Dg_n)g_n^{-1}) \ldots (D - (Dg_1)g_1^{-1}), \]

where \( g_i(0) = 1 \), then \( f_\gamma = (f_1, \ldots, f_n) \), where

\[ (1.3) \quad f_j(t) = \int_0^t \int_0^{t_1} \ldots \int_0^{t_{j-2}} g_1(t)g_1(t_1)^{-1}g_2(t_1)g_2(t_2)^{-1}\ldots g_j(t_{j-1})dt_{j-1} \ldots dt_2 dt_1, \]

where \( \int_0^u (\sum a_i t^i) dt := \sum a_i \frac{u^{i+1}}{i+1} \).

**Proof.** It is easy to see that if \( f = (f_1, \ldots, f_n) \), with \( f_j \) given by (1.3), then \( W(f)(0) \) is strictly lower triangular. So it remains to show that \( f_j \) is a solution of the equation \( L_j f = 0 \), where \( L_j = (D - (Dg_j)g_j^{-1}) \ldots (D - (Dg_1)g_1^{-1}) \).

We prove this by induction in \( j \). The base of induction is clear, since from (1.3) we get \( f_1 = g_1 \). Let us perform the induction step. By the induction assumption, from (1.3), we have

\[ f_j(t) = g_1(t) \int_0^t g_1(s)^{-1} h(s) ds, \]

where \( h \) obeys the equation \( (D - (Dg_j)g_j^{-1}) \ldots (D - (Dg_2)g_2^{-1}) h = 0 \). Thus, we get

\[ (D - (Dg_1)g_1^{-1})f_j = h. \]
This proves that \( L_j f_j = 0 \). □

Now consider an application of these results to the noncommutative Vieta theorem [GR3]. Let \( A \) be an associative algebra. Call a set of elements \( x_1, ..., x_n \in A \) generic if their Vandermonde matrix \( V(x_1, ..., x_n) \) (\( V_{ij} := x_j^{i-1} \)) is invertible.

Consider an algebraic equation

\[
x^n + a_1 x^{n-1} + ... + a_n = 0.
\]

with \( a_i \in A \). Let \( x_1, ..., x_n \in A \) be solutions of (1.4) such that \( x_1, ..., x_i \) form a generic set for each \( i \). Let \( V(i) = V(x_1, ..., x_i) \), and \( y_i = |V(i)|_{ii} x_i |V(i)|_{ii}^{-1} \).

**Theorem 1.3.** [GR3] (Noncommutative Vieta theorem)

\[
a_r = (-1)^r \sum_{i_1 < \ldots < i_r} y_{i_r} \ldots y_{i_1}.
\]

**Proof.** (Using differential equations.) Consider the differential operator with constant coefficients in \( R = A[[t]] \):

\[
L = D^n + a_1 D^{n-1} + ... + a_n.
\]

We have solutions \( f_i = e^{tx_i} \) of the equation \( L f = 0 \), and for any \( i \) the set \( f_1, ..., f_i \) is nondegenerate, since its Wronski matrix \( W(i) \) is of the form

\[
W(i) = V(i) \text{diag}(e^{tx_1}, ..., e^{tx_i}).
\]

Thus, by Theorem 1(ii), the operator \( L \) admits a factorization

\[
L = (D - b_n) ... (D - b_1),
\]

where \( b_i = (D|W(i)|_{ii})|W(i)|_{ii}^{-1} \). Substituting (1.6) into this equation, we get

\[
b_i = |V(i)|_{ii} x_i |V(i)|_{ii}^{-1} = y_i.
\]

Since \([D, b_i] = 0\), we obtain the theorem.

### 2. Toda equations

In this section we will solve equations (1),(2),(3).

Let \( M = \{(g, h) \in A[[t]] \oplus A[[t]] : g(0) = h(0) \in A^*\} \), where \( A^* \) is the set of invertible elements of \( A \). We have the following obvious proposition, which says that solutions of Toda equations are uniquely determined by initial conditions.

**Proposition 2.1.** The assignment \( \phi(u, v) \rightarrow (\phi(u, 0), \phi(0, v)) \) is a bijection between the set of solutions of the Toda equations (1) and the set \( M \).

Let \( B \) be an algebra over \( k \), \( R = B[[v]] \), \( D = \frac{d}{dv} \). For \( \phi = (\phi_1, ..., \phi_n) \), where \( \phi_i \in B[[v]] \) are invertible, define \( L_i^\phi \in R[D] \) by

\[
L_i^\phi = (D - (D \phi_i) \phi_i^{-1}) ... (D - (D \phi_1) \phi_1^{-1}).
\]

Now set \( B = A[[u]] \). Then \( B[[v]] = A[[u, v]] \), so for any \( \phi = (\phi_1, ..., \phi_n) \), with all \( \phi_i \in A[[u, v]] \) invertible, we can define \( L_i^\phi \).
Proposition 2.2. A vector-function $\phi(u, v)$ is a solution of Toda equations (1) if and only if

$$\frac{\partial L^\phi_i}{\partial u} = -\phi_{i+1} \phi_i^{-1} L^\phi_{i-1}, \quad i \leq n - 1; \quad \frac{\partial L^\phi_i}{\partial v} = 0. \tag{2.2}$$

Proof. Let $\phi$ be a solution of the Toda equations. Set $b_i = (D \phi_i) \phi_i^{-1}$. We have $L_i^\phi = L_i = (D - b_i) \ldots (D - b_1)$. Therefore, for $i \leq n - 1$ we have

$$\frac{\partial L_i}{\partial u} = - \sum_{j=1}^{i} (D - b_i) \ldots (D - b_{j+1}) \phi_{j+1} \phi_j^{-1} (D - b_{j-1}) \ldots (D - b_1)$$

$$+ \sum_{j=1}^{i-1} (D - b_i) \ldots (D - b_{j+2}) \phi_{j+1} \phi_j^{-1} (D - b_j) \ldots (D - b_1). \tag{2.3}$$

But

$$(D - b_{j+1}) \circ \phi_{j+1} \phi_j^{-1} = \phi_{j+1} \phi_j^{-1} (D - b_j).$$

Therefore, the right hand side of (2.3) equals $-\phi_{i+1} \phi_i^{-1} L_{i-1}$. The same argument shows that for $i = n$ the derivative $\frac{\partial L_i}{\partial u}$ vanishes.

Conversely, it is easy to see that equations (2.2) imply (1). □

Now we will compute the solutions of Toda equations explicitly. Let $\eta_1, \ldots, \eta_n, \psi_1, \ldots, \psi_n \in A[[t]]$ be such that $\eta_i(0) = \psi_i(0) \in A^*$. We will find the solution of the following initial value problem for Toda equations:

$$\phi_i(u, 0) = \psi_i(u), \quad \phi_i(0, v) = \eta_i(v). \tag{2.4}$$

Proposition 2.1 states that this solution exists and is unique.

Let $g_i(v) = \eta_i(v) \eta_i(0)^{-1}$. Let $f = (f_1, \ldots, f_n)$ be given by formula (1.3) in terms of $g_i$. Define the lower triangular matrix $\Delta(u)$ whose entries are given by the formula

$$\Delta_{ij}(u) = \int_0^u \int_0^{t_1} \ldots \int_0^{t_{i-j-1}} \psi_i(t_{i-j}) \psi_{i-1}^{-1}(t_{i-j}) \psi_{i-2}^{-1}(t_{i-j-1}) \ldots \psi_j^{-1}(t_1) \psi_i(u) dt_{i-j} \ldots dt_1$$

Let $f^u = (f_1^u, \ldots, f_n^u)$ be defined by the formula

$$f^u = f \Delta(u).$$

Then we have

**Theorem 2.3.** The solution of the problem (2.3) is given by the formula

$$\phi_i(u, v) = [W(f_1^u, \ldots, f_i^u)]_{ii} \tag{2.5}$$

Proof. Let $L_i^u = L_i^{\phi(u,v)}$, where $\phi(u, v)$ is defined by (2.5). By Theorem 1.1(ii), we have $L_i^u f_j^u = 0$ for $j \leq i$. Differentiating this equation with respect to $u$, we get

$$L_i^u \frac{\partial f_j^u}{\partial u} + \frac{\partial L_i^u}{\partial u} f_j^u = 0. \tag{2.6}$$
On the other hand, it is easy to see that
\[ \Delta'(u) = \Delta(u) \Theta(u), \]
where
\[
\Theta_{ij}(u) = \begin{cases} 
\psi_i^{-1}(u) \psi'_i(u) & i = j \\
1 & i = j + 1 \\
0 & \text{otherwise}
\end{cases}
\]
This implies that
\[
\frac{\partial f^u_i}{\partial u} = f^u_{i+1} + f^u_i \psi_i^{-1} \psi'_i.
\]
Therefore, \( L^u_i \frac{\partial f^u_i}{\partial u} = 0 \) for \( i \geq j + 1 \), and
\[
L^u_i \frac{\partial f^u_i}{\partial u} = L^u_i f^u_{i+1} = |W(f^u_1, \ldots, f^u_{i+1})|_{i+1} = \phi_{i+1}.
\]
Thus, from (2.6) we get
\[
\frac{\partial L^u_i}{\partial u} f^u_j = -\phi_{i+1} \phi_i^{-1} L_{i-1} f^u_j, \ i \geq j.
\]
By Theorem 1.1(i), this implies equations (2.2), which are equivalent to Toda equations.

It is obvious that \( \phi(u, v) \) satisfies the required initial conditions. The theorem is proved. □

If initial conditions (2.4) satisfy the symmetry property \( \phi_{n+1-i}(u, v) = (\phi_i^*)^{-1} \), then we obtain a solution of the initial value problem for equations (2) (for even \( n \)), and (3) (for odd \( n \)). This gives a complete description of solutions of systems of equations (1),(2),(3).

In the case \( \phi(u, 0) = 1 \), the Toda flow can be interpreted as a flow on the space of factorizations of a differential operator. Namely, let \( L \) be a differential operator of order \( n \) with highest coefficient 1. Let \( F(L) \) be the space of factorizations of \( L \). Let \( N^\sim_\pi \) be the group of strictly lower triangular matrices over \( A \). It is easy to see that the map \( \pi : F(L) \to N^\sim_\pi \), given by \( \pi(\gamma) = W(\mathbf{f}_\gamma)(0) \), is a bijection. We will identify \( F(L) \) with \( N^\sim_\pi \) using \( \pi \).

Let \( \gamma(u) = \gamma(0)e^{u J_n} \) be a curve on \( N^\sim_\pi \) generated by the 1-parameter subgroup \( e^{u J_n} \), where \( J_n \) is the lower triangular nilpotent Jordan matrix \( (J_n)_{ij} = \delta_{i,j+1} \). Let \( L = (D - b_n^u)(D - b_i^u) \) be the factorization of \( L \) corresponding to the point \( \gamma(u) \). Let \( \phi_i(u, v) \) be such that \( (D \phi_i) \phi_i^{-1} = b_i^v \), and \( \phi_i(u, 0) = 1 \). (here \( D = \frac{\partial}{\partial v} \)). Then we have the following Corollary from Theorem 2.3.

**Corollary 2.4.** \( \phi = (\phi_1, \ldots, \phi_n) \) is a solution of Toda equations (1), and all solutions with \( \phi(u, 0) = 1 \) are obtained in this way.

**Proof.** For the proof it is enough to observe that if \( \psi_i(u) = 1 \) then \( \Theta(u) = J_n \), and \( \Delta(u) = e^{u J_n} \). □

**Remark.** An analogous statement can be made for equations (2) and (3). In this case, instead of an arbitrary differential operator \( L \in R_n(D) \) one should consider a selfadjoint (respectively, skew-adjoint) operators, instead of the group \( N^\sim_\pi \) – a
maximal nilpotent subgroup in $Sp_n(A)$ (respectively, $O_n(A)$), and instead of $J_n$ the sum of simple root elements in the Lie algebra of this group. In the commutative case, such a picture of the Toda flow is known for all simple Lie groups [FF].

Consider now the statement of Theorem 2.3 for $i = 1$. In this case we have

$$\phi_1(u, v) = f_1^u(v) = \sum_{i=1}^{n} f_i^0(v) \Delta_i^(u)$$

Thus, we get

Corollary 2.4. [RS] If $\phi$ is a solution of the Toda equations then $\phi_1(u, v) = \sum_{i=1}^{n} p_i(u)q_i(v)$, where $p_i, q_i$ are some formal series.

Now let us discuss infinite Toda equations. These are equations (1) with $n = \infty$. These equations allow to express $\phi_i$ recursively in terms of $f = \phi_1$, which can be done using quasideterminants:

$$\phi_i = |Y_i(f)|_{ii}, \ Y_i(f) := W\left(f, \frac{\partial f}{\partial u}, \ldots, \frac{\partial^{i-1} f}{\partial u^{i-1}}\right),$$

where $\frac{\partial^i f}{\partial u^i}$ are regarded as functions of $v$ for a fixed $u$. This formula is easily proved by induction. It appears in [GR2], section 4.5 in the case $u = v$, and in [RS] in the 2-variable case.

Formula (2.8) can be used to give another expression for the general solution of finite Toda equations, which appears in [RS]. Indeed, we have the following easy proposition.

Proposition 2.5. Let $f \in A[[u, v]]$ be such that $Y_1(f), ..., Y_n(f)$ are invertible matrices. Then $|Y_{n+1}(f)| = 0$ if and only if $f$ is “kernel of rank $n$”, i.e.

$$f = \sum_{i=1}^{n} p_i(u)q_i(v).$$

Thus, taking $f = \phi_1$ of the form (2.9), with $Y_1, ..., Y_n$ invertible, and using formula (2.6) we will get a solution $(\phi_1, ..., \phi_n)$ of the finite Toda system of length $n$. It is not difficult to show that in this way one gets all possible solutions.

Example. Consider the nonabelian Liouville equation

$$\frac{\partial}{\partial u}\left(\frac{\partial \phi}{\partial v} \phi^{-1}\right) = (\phi^*)^{-1}\phi^{-1},$$

which is a special case of (2) for $k = 1$. Consider the initial value problem

$$\phi(0, v) = \eta(v), \phi(u, 0) = \psi(u), \eta(0) = \psi(0) = a.$$

By Theorem 2.3, we get the following formula for the solution:

$$\phi(u, v) = \eta(v)\left(a^{-1} + \int_0^u \int_0^u \eta^{-1}(t)(\eta^*)^{-1}(t)a^*(\psi^*)^{-1}(s)\psi^{-1}(s)dsdt\right)\psi(u).$$
For example, if $\psi(u) = 1$, we get

$$\phi(u, v) = \eta(v) \left( 1 + u \int_0^v \eta^{-1}(t)(\eta^*)^{-1}(t)dt \right).$$

In the commutative case, these formulas coincide with the standard formulas for solutions of the Liouville equation.

**Appendix: Noncommutative soliton equations**

In this appendix we will review some results about noncommutative versions of classical soliton hierarchies (KdV, KP). These results are mostly known or can be obtained by a trivial generalization of the corresponding commutative results, but they have never been exposed systematically.

We will follow Dickey’s book [D].

**A1. Nonabelian KP hierarchy.**

Nonabelian KP hierarchy is defined in the same way as the usual KP hierarchy [D]. Let

$$L = \partial + w_0\partial^{-1} + w_1\partial^{-2} + ....$$

be a formal pseudodifferential operator. Here $\partial = \frac{d}{dx}$, $w_i \in A[[x]]$, where $A$ is an associative (not necessarily commutative) algebra with 1. Consider the following infinite system of differential equations:

$$\frac{\partial L}{\partial t_m} = [B_m, L], B_m = (L^m)_+,$$

where for a pseudodifferential operator $M$, we denote by $M_+, M_-$ the differential and the integral parts of $M$ (i.e. all terms with nonnegative, respectively negative, powers of $\partial$). For brevity we will write $L^m_\pm$ instead of $(L^m)_\pm$.

Each of the differential equations (A2) defines a formal flow on the space $P$ of pseudodifferential operators of the form (A1). Indeed, $[L^m_+, L] = -(L^m_-)_+$, and the order of $[L^m, L]$ is at most zero.

**Proposition A1.** The flows defined by equations (A2) commute with each other.

**Proof.** The proof is the same as the proof of Proposition (5.2.3) in [D]. Namely, one needs to check the zero curvature condition

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} - [B_m, B_n] = 0,$$

which is done by a direct calculation given in [D].

The hierarchy of flows defined by (A2) for $m = 1, 2, 3, ...$ is called the noncommutative KP hierarchy.

**A2. Noncommutative KdV and nKdV hierarchies.**

As in the commutative case, we can restrict the KP hierarchy to the subspace $P_n \subset P$ of all operators $L$ such that $L^n$ is a differential operator, i.e. $L^n = 0$ [D].

This subspace is invariant under the KP flows, as $\frac{\partial L^n}{\partial t_m} = [L^m_+, L^n]_-$. The space $P_n$ can be identified with the space $D_n$ of differential operators of the form

$$M = \partial^n + u_2\partial^{n-2} + ... + u_n,$$
by the map \( L \to M = L^n \). The KP hierarchy induces a hierarchy of flows on \( D_n \) called the nKdV hierarchy:

\[
\frac{\partial M}{\partial t_m} = [M_{+}^{m/n}, M].
\]

Among these flows, the flows corresponding to \( m = nl, l \in \mathbb{Z}_{+} \), are trivial, but the other flows are nontrivial.

For \( n = 2 \) the nKdV hierarchy is the usual KdV hierarchy. The first two nontrivial equations of this hierarchy are

\[
\begin{align*}
& u_{t_1} = u_x, \\
& u_{t_3} = \frac{1}{4}(u_{xxx} + 3u_x u + 3uu_x).
\end{align*}
\]

The second equation is the noncommutative KdV equation.

**A3. Finite-zone solutions of the nKdV hierarchy.**

Denote the vector fields of the noncommutative nKdV hierarchy by \( V_m, m \neq nl \). Let \( V = \sum_{i=1}^{p} a_i V_i, a_p \neq 0 \), be a finite linear combination of \( V_i \) with coefficients \( a_i \in l \). Let \( S(a), a = (a_1, ..., a_p) \) be the set of stationary points of \( V \) in \( D_n = A[[x]]^{n-1} \).

The set \( S(a) \) can be identified with \( A^{(n-1)p} \). Indeed, \( S(a) \) is the subset of \( D_n \) defined by the differential equations \( V(M) = 0 \), which have the form

\[
\frac{d^p u_i}{dx^p} = F_i(u_j, u'_j, ..., u_j^{(p-1)}).
\]

So \( u_i \) are uniquely determined by their first \( p \) derivatives at 0.

Since the vector field \( V \) commutes with the nKdV hierarchy, \( S(a) \) is invariant under the flows of this hierarchy. Using the identification \( S(a) \to A^{(n-1)p} \), we can rewrite each of the nKdV flows as a system of ordinary differential equations on \( A^{(n-1)p} \). Thus, solutions of nKdV equations belonging to \( S(a) \) can be computed by solving ordinary differential equations. Such solutions are called finite-zone solutions.

**Remark.** If \( M \in S(a) \) then \( [Q(M^{1/n})_+, M] = 0 \), where \( Q(x) = \sum a_i x^i \). Thus, we have two commuting differential operators \( M \) and \( N = Q(M^{1/n})_+ \) in 1 variable. If the algebra \( A \) is finite dimensional over its center then there exists a nonzero polynomial \( R(x, y) \) with coefficients in the center of \( A \), such that \( R(M, N) = 0 \). This polynomial defines an algebraic curve, called the spectral curve. The operator \( M \) can then be computed explicitly using the method of Krichever [Kr2]. Similarly, all nKdV equations restricted to the space of such operators can be solved explicitly in quadratures. However, if \( A \) is infinite-dimensional over its center, the polynomial \( R \) does not necessarily exist, and we do not know any way of computing \( M \) explicitly.

**A4. Multisoliton solutions of the noncommutative KP hierarchy.**

Here we will construct N-soliton solutions of the KP and KdV hierarchies in the noncommutative case. We will use the dressing method. In the exposition we will closely follow Dickey’s book [D].

We will now construct a solution of the KP hierarchy. Let \( t = (t_1, t_2, ...) \) Consider the formal series

\[
\xi(x, t, \alpha) = (x + t_1)\alpha + t_2\alpha^2 + ... + t_r\alpha^r + ..., \alpha \in A.
\]
Fix $\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N, a_1, ..., a_N \in A$, and set

\[ y_s(x, t) = e^{\xi(x,t,\alpha_s)} + a_s e^{\xi(x,t,\beta_s)}. \]

Define the differential operator of order $N$, with highest coefficient 1, by

\[ \Phi f(x) = |W(y_1, ..., y_N, f)|_{N+1,N+1}(x), \]

where the derivatives in the Wronski matrix are taken with respect to $x$, and we assume that the functions $y_1, ..., y_N$ are a generic set (in the sense of Chapter 1). Set

\[ L = \Phi \partial \Phi^{-1}. \]

**Proposition A2.** The operator-valued series $L(x, t)$ is a solution of the KP hierarchy.

**Proof.** The proof is the same as the proof of Proposition 5.3.6 in [D].

Such solutions are called N-soliton solutions.

Proposition A2 can be used to construct N-soliton solutions of the nKdV hierarchy. Namely, we should restrict the above construction to the case when $\beta_k = \varepsilon_k \alpha_k$, where $\varepsilon_k$ is an $n$-th root of unity. In this case it can be shown as in [D] that the operator $M = L^n = \Phi \partial^n \Phi^{-1}$ is a differential operator of order $n$, and it is a solution of the nKdV hierarchy.

As an example, let us consider the N-soliton solutions of the KdV hierarchy. In this case, we may set $t_{2k} = 0$, and we have

\[ y_s = e^{\xi(x,t,\alpha_s)} + a_s e^{-\xi(x,t,\alpha_s)}. \]

**Proposition A3.** Let

\[ b_i = (\partial W_i) W_i^{-1}, \quad W_i := |W(y_1, ..., y_i)|_{ii}. \]

Then the function

\[ u(x, t) = 2 \partial (\sum_{i=1}^N b_i). \]

is a solution of the noncommutative KdV hierarchy.

**Proof.** Let $\Phi = \partial^N + v_1 \partial^{N-1} + ...$. Then from the equation $(\partial^2 + u) \Phi = \Phi \partial^2$ we obtain $u = -2 \partial v_1$, and from Theorem 1.1(ii) $v_1 = -\sum_i b_i$, where $b_i$ are given by (A13). This implies (A14a).

Another formula for $u$, which is equivalent to (A14a), is the following. Let $Y(y_1, ..., y_N)$ be the matrix which coincides with the Wronski matrix $W(y_1, ..., y_N)$ except at the last row, where it has $y_i^{(N)}$ instead of $y_i^{(N-1)}$. Let $Y_N = |Y(y_1, ..., y_N)|_{NN}$. Then the function $u$ given by (A14a) can be written as

\[ u = 2 \partial (Y_N W_N^{-1}), \]
In the commutative case formulas (A14a),(A14b) reduce to the classical formula

\begin{equation}
A15\quad u = 2 \partial^2 \ln \det W(y_1, \ldots, y_N). 
\end{equation}

In particular, we can obtain N-soliton solutions of the noncommutative KdV equation \( u_t = \frac{1}{4} (u_{xxx} + 3u_x u + 3uu_x) \). For this purpose set \( t_i = 0, i \neq 3 \), and \( t_3 = t \). Then we get

\begin{equation}
A16\quad y_s = e^{\beta_s x + \beta_3 t} + a_s e^{-\beta_s x - \beta_3 t},
\end{equation}

and the solution \( u(x, t) \) is given by (A14a),(A14b).

For example, consider the 1-soliton solution. According to (A14a), it has the form

\begin{equation}
A17\quad u = 2 \frac{\partial}{\partial x} \left[ a x + a^3 t - a e^{-\alpha x - \alpha^3 t} \alpha (e^{\alpha x + a^3 t} + ae^{-\alpha x - \alpha^3 t})^{-1} \right].
\end{equation}

In the commutative case, it reduces to the well known solution

\begin{equation}
A18\quad u = \frac{2\alpha^2}{\cosh^2(\alpha x + \alpha^3 t - c)}, \quad c = \frac{1}{2} \ln a,
\end{equation}

—the solution corresponding to the solitary wave which was observed by J.S.Russell in August of 1834.

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