Planckian Vertices
on High Genus Riemann Surfaces

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Abstract

We suggest a method to compute leading contribution at Planckian energies for superstring scattering amplitudes of any genus. In particular we test the method at one-loop level by comparison with previous result for the Regge trajectory renormalization. Modular invariance of these asymptotic terms are also discussed.
1. Introduction

Gravitational physics at Planckian energies has been studied in the past through superstring scattering amplitude at fixed angle [1] or in the Regge regime [2,3] and it is still under discussion [4,5] in order to understand the role of short distances in string theory and quantum gravity.

Most of the interest in superstring originates in its unique role as a finite [6] quantum theory containing gravity. Since gravitational interactions couple to energy, the Regge regime of very large center-of-mass energies $\sqrt{s}$ and fixed transferred momentum $\sqrt{-t}$ turns out to be very promising.

It has been clarified [7] that in this asymptotic limit only pinched Riemann surfaces give the main contribution to the Polyakov correlation function; in a different language only particular regions of modular space contribute. It was also shown that the Koba-Nilsen variables of external fast particles get closer and closer as the energy is increased, their distance being of the order $1/s$.

In a previous paper [8] the relation between high energy and O.P.E. on the string world sheet was exploited to introduce a reggeon emission vertex $V_{\alpha,k}(z)$ suitable to study the Regge behavior of superstring amplitudes. The main achievement was the construction of a slightly modified vertex operator for inserting on the Riemann surface all the states lying on the leading Regge trajectory. Although the method could be used to compute Regge amplitudes of arbitrary genus only tree amplitudes were considered explicitly [9].

Recently the one-loop correction to the graviton trajectory has been reconsidered [10] due to its contribution to the superstring eikonal. Starting from the one-loop superstring amplitude, after a proper subtraction of the leading Regge cut term, the first subleading high-energy contribution at large impact parameter has been obtained.

In this letter we present an alternative derivation of the same integral representation which makes use of the Regge emission vertices. Since these vertices describe external Regge poles in the angular momentum plane, and the trajectory renormalization at first order arises as a double pole correction, it is natural to compute this one by the insertion of two Regge vertices on the torus world-sheet. It turns out that the latter is a simple
method to study the Regge behavior of string amplitudes of any genus.

2. Off-shell vertex technique

In a previous paper [10] we have investigated the pole and cut regions of a four point amplitude directly on the torus, and we have found that in the pole region the external punctures are close in pairs, because

\[ |\nu_a - \nu_d| \sim |\nu_b - \nu_c| \sim \frac{1}{\sqrt{s}}. \]

This feature was used in ref. [9] as the starting point for the definition of Regge vertices attached to the punctures.

We expect to obtain in a natural way the entire trajectory renormalization at one loop by inserting two gravireggeon vertices on the torus world-sheet as shown in fig.1a. This will turn out to be in agreement with the stationary phase method applied to the four point amplitude at high energies used by Sundborg in his analysis [11].

The gravireggeon emission vertex takes the form

\[ V_{a,\tilde{k}}(z, \bar{z}) = \int \frac{d^2 \delta}{|\delta|^2} |\delta|^2 q^2 V_R(z, \delta)V_L(\bar{z}, \bar{\delta}) \]  \hspace{1cm} (2.1)

where \( V_R \) is given in term of the right moving operators by

\[ V_R(z, \delta) =: \left[ -i\tilde{k} \cdot \partial_z X_R(z) + \tilde{k} \cdot \psi_R(z) \hat{q} \cdot \psi_R(z) \right] e^{i\hat{q} \cdot X_R(z)} : \]

\[ \hat{q} \equiv q + \tilde{k} \delta \frac{\partial}{\partial z} \]  \hspace{1cm} (2.2)

and similarly for \( V_L \) with the obvious substitution. We remind that \( q \) is the off-shell momentum transfer coupled with to the surface (fig 1a), while \( \tilde{k} \) is the external momentum flowing in the fast legs, which in Eq. (2.2) plays the role of polarization vector, with \( \tilde{k} \cdot q = 0 \).

We now describe how to compute the correlation functions of off-shell vertices (2.1) over a genus one Riemann surface in superstring type II formalism. First of all we remind that on the torus there are 4 spin structures, one odd corresponding to periodic-periodic
boundary condition for all world-sheet spinors $\psi$ and three even ones, containing at least one anti-periodic boundary condition. These condition must be realized independently for left and right components.

The amplitude for two external punctures turns out to be an integral over modular space of the following density

$$\tilde{A} = \sum_{(a,\bar{a})} M_a \tilde{M}_a C_{a,\bar{a}} \ll V_{\alpha,k_1}(z_1, \bar{z}_1) V_{\alpha,k_2}(z_2, \bar{z}_2) \gg_{a,\bar{a}}$$

where the factor $M_a \tilde{M}_a C_{a,\bar{a}}$ is the contribution of spin structure $(a,\bar{a})$ to the one-loop partition function, that for even spin structures [12] is:

$$M_a = \vartheta[a] (0|\tau)^4$$

$$\vartheta_{00}(\nu|\tau) = \vartheta_3(\nu|\tau) \quad \vartheta_{01}(\nu|\tau) = \vartheta_4(\nu|\tau) \quad \vartheta_{10}(\nu|\tau) = \vartheta_2(\nu|\tau)$$

and the factor $C_{a,\bar{a}}$, discussed by Alvarez-Gaume and Vafa [13], is for type II superstring only a phase $(-)^{a_1+a_2+\bar{a}_1+\bar{a}_2}$. We can limit ourselves to even spin structures because to get a non zero result from odd spin structures we need at least six external particles.

We recall [6] that partition function, tadpole insertion, self-energy correction and 3-point amplitude vanishes at one-loop level due to the Jacobi relation for theta functions [14]

$$\vartheta_4^4(0|\tau) = \vartheta_2^4(0|\tau) + \vartheta_4^4(0|\tau)$$

and the constraint of on-shell vertices. We shall see that this result cannot be extended to off-shell vertices.

We proceed in the computation of Eq.(2.3) by introducing the sum over loop momenta to exhibit factorization of left and right modes which is allowed due to the absence of non-holomorphic odd spin structures:

$$\ll V_{\alpha_1,k_1}(z_1, \bar{z}_1) V_{\alpha_2,k_2}(z_2, \bar{z}_2) \gg_{a,\bar{a}} = \int d^{10}p \frac{d^2\delta_1 d^2\delta_2}{|\delta_1 \delta_2|^2} |\delta_1 \delta_2|^2 a^q \Lambda_{q_1,q_2}(z_i, \delta_i) \tilde{\Lambda}_{q_1,q_2}(\bar{z}_i, \bar{\delta}_i)$$

where we have introduced the shorthand notation $\Lambda_{q_1,q_2}(z_i, \delta_i)$ to mean the following chiral component

$$\Lambda_{q_1,q_2}^a = \langle \left( -\frac{\partial}{\partial \delta_1} + : \tilde{k}_1 \cdot \psi(z_1) \tilde{q}_1 \cdot \psi(z_1) : \right) \left( -\frac{\partial}{\partial \delta_2} + : \tilde{k}_2 \cdot \psi(z_2) \tilde{q}_2 \cdot \psi(z_2) : \right) \rangle_a$$

$$\exp \left[ i\pi p^2 \tau + 2i\pi p \cdot (\tilde{q}_1 z_1 + \tilde{q}_2 z_2) - \tilde{q}_1 \cdot \tilde{q}_2 G(z_1, z_2) \right]$$

(2.4)
and the chiral $X$-propagator $G(z_1, z_2)$ [12] or correlator of chiral bosonic fields

$$G(z_1, z_2) = \langle X(z_1)X(z_2) \rangle = -\ln E(z_1, z_2)$$

over non zero modes. It may be useful to recall that the prime form $E$ takes a simple form for the torus

$$E(z_1, z_2) = \frac{\vartheta(z_1 - z_2 | \tau)}{\vartheta_1(0 | \tau)}$$

and that the Dirac propagator for even spin structures

$$\langle \psi^\mu(z_1)\psi^\nu(z_2) \rangle_a = g^{\mu\nu}S_a(z_1, z_2)$$

is given by the Szegö kernel

$$S_a(z_1, z_2) = \frac{\vartheta[a](z_1 - z_2 | \tau)\vartheta_1'(0 | \tau)}{\vartheta[a](0 | \tau)\vartheta_1(z_1 - z_2 | \tau)}, \quad (2.5)$$

Keeping into account the normal ordering condition and the transversality of $\tilde{k}_i$ with respect to $q_i$, the fermion correlator of equation (2.4) gives for a specific spin structure the following result

$$\Lambda^a_{q_1, q_2} = \left[ \frac{\partial^2}{\partial \delta_1 \partial \delta_2} + \tilde{k}_1 \cdot \tilde{k}_2 q^2 S_a(z_1, z_2) + (\tilde{k}_1 \cdot \tilde{k}_2)^2 \delta_1 \delta_2 \frac{\partial^2 S_a(z_1, z_2)}{\partial z_1^2} \frac{\partial S_a(z_1, z_2)}{\partial z_2} \right. +
$$

$$\left. - (\tilde{k}_1 \cdot \tilde{k}_2)^2 \delta_1 \delta_2 S_a(z_1, z_2) \frac{\partial^2 S_a(z_1, z_2)}{\partial z_1 \partial z_2} \right] \exp \left[ i\pi p^2 \tau + 2i\pi p \cdot q(z_1 - z_2) +
$$

$$+ 2i\pi p \cdot (\tilde{k}_1 \delta_1 + \tilde{k}_2 \delta_2) + q_1 \cdot q_2 \ln E(z_1, z_2) + \tilde{k}_1 \cdot \tilde{k}_2 \delta_1 \delta_2 \frac{\partial^2}{\partial z_1 \partial z_2} \ln E(z_1, z_2) \right] \quad (2.6)$$

Summing now over the spin structures of the right and left sector we have for the density $\tilde{A}$

$$\tilde{A} = \int d^D p \int |\delta_1 \delta_2|^q \left( \sum_{(a)} (-)^{\alpha_1 + \alpha_2} M_a \Lambda^a \right) (c.c.) \frac{d^2 \delta_1}{|\delta_1|^2} \frac{d^2 \delta_2}{|\delta_2|^2} \quad (2.7)$$

and using the general Riemann identities we obtain a non-zero result only from the last two terms in the first factor of Eq. (2.6), proportional to the off-shell deformations $\delta_1, \delta_2$. This was to be expected because going on-shell we should recover a null result for the two point amplitude. We now discuss in more detail this last contribution

$$-(\tilde{k}_1 \cdot \tilde{k}_2)^2 \delta_1 \delta_2 \sum_{(a)} (-)^{\alpha_1 + \alpha_2} S_a^2(z_1 - z_2) \frac{\partial^2}{\partial z_1^2} \ln S_a(z_1 - z_2)$$
which in terms of theta function reads

\[-(\tilde{k}_1 \cdot \tilde{k}_2)^2 \delta_1 \delta_2 \sum_{(a)} (-)^{a_1+a_2} \vartheta[a](0|\tau)^2 \left( \vartheta'[a](z_1-z_2|\tau)^2 - \vartheta[a](z_1-z_2|\tau) \vartheta''[a](z_1-z_2|\tau) \right)\]

(2.8)

Using the identity

\[\sum_{(a)} (-)^{a_1+a_2} \vartheta[a](0|\tau)^2 \vartheta[a](u|\tau) \vartheta[a](v|\tau) = 2 \vartheta_1^2(\frac{u+v}{2}|\tau) \vartheta_2(\frac{u-v}{2}|\tau)\]

and its derivatives computed for \(u = v\), the above expression simplifies to \(-2(\tilde{k}_1 \cdot \tilde{k}_2)^2 \delta_1 \delta_2\) and similarly for the left sector. Finally integrating Eq. (2.7) over the ten dimensional loop momenta and introducing the toroidal compactification factor \(F_c\) we obtain the following amplitude

\[A = 4g^2(\tilde{k}_1 \cdot \tilde{k}_2)^4 \int \frac{d^2 \tau}{\tau_2^2} F_c(\tau, R_c) d^2 z d^2 \delta_1 d^2 \delta_2 |\delta_1 \delta_2| q^2 e^{-2\tilde{k}_1 \cdot \tilde{k}_2 \frac{2\pi}{\tau_2} \text{Im} \delta_1 \text{Im} \delta_2} \left\{ \frac{\vartheta_1(z|\tau)}{\vartheta_1(0|\tau)} \frac{\pi}{\tau_2} \text{Im}^2 z \right\}^{-2q^2} e^{-2\tilde{k}_1 \cdot \tilde{k}_2 \text{Re} \left[ \delta_1 \delta_2 \frac{\partial^2 \ln \vartheta_1(z|\tau)}{\partial z^2} \right]}\]

(2.9)

We note the agreement of the above expression with Eq. (5.6) of ref. [11], where \(\delta_1, \delta_2\) are the variables of the saddle-point approximation used by Sundborg to compute the double-pole correction (in the angular momentum plane) starting from the four dilatons scattering amplitude on the torus. It is worthwhile to discuss more the last expression.

Generally a superstring amplitude is Regge behaved and admits the following expansion

\[A = \sum_{i,k} s^{\alpha_{i,k}(t)} (\ln s)^k \beta_{i,k}(t)\]

where, in formal analogy with O. P. E. in quantum field theory, the physical quantities are the Regge trajectories \(\alpha_{i,k}(t)\) (anomalous dimension) and their residues \(\beta_{i,k}(t)\) (non trivial vacuum expectation values) and the large energy replace the short distance expansion.

Indeed for the amplitude (2.9) the polarization-like vectors \(\tilde{k}\) give rise to the large energy factor \(s = 2\tilde{k}_1 \cdot \tilde{k}_2\) in the exponent and require the smallness of \(\delta_1 \delta_2\). The latter are interpreted like distances between the Koba-Nilsen variables of the external fast legs.
(see fig. 1a) $\delta_1 = \nu_a - \nu_d$ and $\delta_2 = \nu_b - \nu_c$, where we use the same notation of ref. [10] for comparison of the result.

The aforementioned Regge behavior of $A$ is naively factorized-out integrating $\delta_1$ and $\delta_2$ over two small disks of radius $\sim 1/\sqrt{s}$ with the result

$$A = g^2 \beta_1(t) s^{2+t} \ln s + \text{subleading terms}$$

where the logarithmic factor is typical of the rapidity integral in the Regge-Gribov calculus.

Adding this term to the asymptotic tree amplitude $s^{2+t} \beta_0(t)$ and assuming exponentiation at higher genus for the full amplitude

$$A_f = s^{2+t} + g^2 \frac{\beta_1(t)}{\beta_0(t)} + \beta_0(t) + \text{higher order terms},$$

we identify the first order renormalization to the graviton trajectory.

Finally we show that the integral representation (2.9) is modular invariant. We refer to [6,12] for the definition of a modular transformation, we only quote here that the quantity in braces in (2.9) is a modular form $\chi(z|\tau)$ of weight one [14], with the property

$$\chi(z'|\tau') = |c\tau + d|^{-1} \chi(z|\tau)$$

while the theta function transform as

$$\vartheta_1(z'|\tau') = \eta(c\tau + d)^{\frac{1}{2}} \exp \left( \frac{i\pi cz^2}{c\tau + d} \right) \vartheta(z|\tau),$$

where $\eta$ is a constant phase.

The exponent in (2.9), proportional to the large kinematical variable $s$, is transformed into

$$- \frac{2\pi |c\tau + d|^2}{\tau_2} \text{Im} \frac{\delta_1}{c\tau + d} \text{Im} \frac{\delta_2}{c\tau + d} + \text{Re} \left[ \delta_1 \delta_2 \frac{2i\pi}{c\tau + d} + \delta_1 \delta_2 \frac{\partial^2 \ln \vartheta(z|\tau)}{\partial z^2} \right]$$

and it is matter of simple algebra to show the invariance of the integrand. The only change is in the radius of the small disks’s integration regions. We get finally the modular

* the actual asymptotic behavior is the one typical of Regge cut $s^{3+\frac{1}{2}t}$ due to an overlap of the relevant integration regions, and make expression (2.9) ill-defined without a proper subtraction as discussed in ref. [10].
invariance of $\beta(t)$ noticing that the actual extension of these circular regions is irrelevant [10] for the leading asymptotic behavior.

We conclude stating that the off-shell vertices used before are a useful concept in high energy superstring computations, when only pinched Riemann surfaces are relevant for the asymptotic behavior.

We finally point out that the contribution (2.9), which corresponds to the pinched surface (fig. 1b), is the same as the one obtained by the Regge-Gribov diagram with two external Regge poles shown in fig. 1a. This is in general very different from the Feynman diagram calculation. In particular the $\ln s$ factor, typical of the above double-pole correction, is produced (in a covariant gauge) only as a subleading term of a box diagram [10] and not by the insertion of a renormalized graviton propagator as fig. 1b could erroneously suggest.

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References

[1] D.J.Gross and P.F.Mende, Phys. Lett B 197 (1987) 129; Nucl. Phys. B303 (1988) 407.

[2] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B197 (1987) 81; Int. J. Mod. Phys. 3A (1988) 1615.

[3] D. Amati, M. Ciafaloni and G. Veneziano, Nucl. Phys. B347 (1990) 550.

[4] D. Amati, M. Ciafaloni and G. Veneziano, Planckian scattering beyond the semiclassical approximation, CERN preprint TH.6395/92.

[5] E. Verlinde and H. Verlinde, Scattering at planckian energies, Princeton preprint PUTP-1279 (1991); see also R.Kallosh, Geometry of scattering at Planckian energies, Stanford preprint SU-ITP 903 (1991).

[6] M. B. Green, J. H. Schwarz and E. Witten, Superstring theory (Cambridge University Press, Cambridge, 1987).

[7] E.Gava, R.Iengo and C.J.Zhu, Nucl. Phys. B323 (1989) 585; A. Bellini, G. Cristofano, M. Fabbrichesi and K. Roland, Nucl. Phys. B356 (1991) 69.

[8] M. Ademollo, A. Bellini and M. Ciafaloni, Phys. Lett. B223 (1989) 318.

[9] M. Ademollo, A. Bellini and M. Ciafaloni, Nucl. Phys. B338 (1990) 114.

[10] M. Ademollo, A. Bellini and M. Ciafaloni, Superstring One-Loop and Gravitino Contribution to Planckian Scattering, Nordita preprint 92/45 (July 1992).

[11] B. Sundborg, Nucl. Phys. B306 (1988) 545.

[12] E. D’Hoker and D. H. Phong, Rev. Mod. Phys. 60 (1988) 917.

[13] L. Alvarez-Gaume, G. Moore and C. Vafa, Commun. Math. Phys. 106 (1986) 1.

[14] D. Mumford Tata Lectures on Theta Functions, (2 vol.) Birkhauser (1983).