Information paradox and island in quasi-de Sitter space

Min-Seok Seo

Department of Physics Education, Korea National University of Education, Cheongju 28173, Republic of Korea

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Abstract Whereas a static observer in de Sitter (dS) space detects thermal radiation emitted by the horizon, the dS isometries impose that the radiation is in equilibrium with the background. This implies that for the static observer to find the information paradox, the background must be deformed to quasi-dS space in which the dS isometries are spontaneously broken. We study the condition that the information paradox arises in quasi-dS space with the monotonically increasing horizon size which is used to describe the inflationary cosmology. For this purpose, the dimensional reduction of three-dimensional dS space with thermal radiation modelled by the JT gravity coupled to CFT is considered. We argue that when the central charge monotonically increases in time, the information paradox arises but the conditions for the existence of the island become more restrictive. As the central charge can be interpreted as the number of degrees of freedom, the absence of the island in quasi-dS space supports the entropy argument for the dS swampland conjecture.

1 Introduction

The black hole information paradox [1] has been one of the most puzzling issues in quantum gravity over several decades. It arises as the semiclassical entropy of the Hawking radiation monotonically increases in time, eventually exceeding the black hole entropy given by $S_{\text{BH}} = (\text{Horizon Area})/(4G)$. This is incompatible with so-called central dogma, which claims that black hole as seen from the outside is described by the unitarily evolving quantum system with $\exp[S_{\text{BH}}]$ degrees of freedom. Recently, a resolution to the paradox at the semiclassical level has been suggested [2–6] (see also [7, 8] for reviews). It is based on the finding that the dominant saddle of the Euclidean path integral on the replica manifolds, which is used to calculate the radiation entropy, includes the black hole interior called an “island” at late time. Then the full entropy of radiation after some time called Page time is decreased by the entanglement between radiation and island, realizing the Page curve [9, 10], the time evolution of the radiation entropy consistent with the central dogma.

This remarkable progress in black hole physics motivates the application of the ‘island rule’ to other quantum gravity systems. Among possible backgrounds composing the quantum gravity system, de Sitter (dS) space has drawn considerable interest [11–21] since an observer in dS space can access the finite portion of spacetime surrounded by the event horizon only. The observer cannot receive signals from the region beyond the horizon due to the accelerating expansion, which leads to the production of thermal radiation near the horizon [22]. Since it is quite similar to the Hawking radiation process in black hole, one may naively expect that the information paradox arises in dS space as well. However, in spite of several similarities, the thermodynamics of dS space is quite different from that of black hole (see, e.g., [23] for a review). The difference mainly comes from the fact that the dS horizon is not a localized object in space, but an observer dependent one. The radiation emitted by the dS horizon will pass by the observer and recede beyond the horizon. The dS isometries guarantee that the radiation is emitted and absorbed by the horizon in all directions at the same rate, resulting in the thermal equilibrium of the radiation with the surroundings. Thus unlike the evaporating black hole, dS space maintains the constant horizon size and we can find a time direction along which the radiation entropy is static such that the information paradox does not arise in the static patch, the region inside the horizon.\(^1\)

\(^1\) Whereas black hole formed by the collapse of star evaporates, the eternal black hole maintains the constant size through the balance between the energy flux emitted by the past horizon and that absorbed by the future horizon. The quantum state for the collapsing black hole is called...
The situation can be changed in quasi-dS space, in which some of dS isometries are slightly broken by the background. In this case, the energy flux absorbed by the horizon is not balanced with that emitted by the horizon any longer. Then the radiation entropy evolves in time, and furthermore, the background geometry may be deformed by the back-reaction of the nonzero net energy flux \( [28, 29] \) (see also [16]). When the horizon radius of quasi-dS space \( H^{-1} \) varies in time, we can compare the time evolution of the radiation entropy with that of the geometric entropy given by \( S_{\text{dS}} = (\text{Horizon Area})/(4G) \). An observer in the static patch can find the information paradox if the radiation entropy exceeds the geometric entropy (see also [17] for a relevant discussion).

Meanwhile, the universe is believed to have experienced inflation at the early stage, which is well described by the quasi-dS background. Since the current universe has only a small amount of dark energy giving \( H \sim 10^{-60} G^{-1/2} \), we expect that \( H \) has decreased, or equivalently, the geometric entropy has increased in time during the inflation. Therefore, for the inflationary quasi-dS space to have the information paradox, the radiation entropy needs to increase much faster than the geometric entropy. In this article, we investigate when this condition is satisfied and whether we can find the island in the inflationary quasi-dS space. For this purpose, we consider the two-dimensional Jackiw–Teitelboim (JT) gravity \([30, 31]\) coupled to conformal field theory (CFT). As reviewed in Sect. 2, the JT gravity can be obtained by the dimensional reduction of three-dimensional dS space, so we in fact address the information paradox in the three-dimensional quasi-dS background. Moreover, our discussion is made in terms of the static coordinates, which are appropriate to describe the dynamics of the horizon as seen by an observer in the static patch.

In Sect. 3, we argue that the inflationary quasi-dS space can have the information paradox when the central charge increases in time. Intriguingly, since the central charge can be interpreted as the number of degrees of freedom, the appearance of the information paradox in this case can be connected to the entropy argument in [35] supporting the dS swampland conjecture [32–35]. That is, as the modulus responsible for the vacuum energy traverses along the trans-Planckian geodesic distance, towers of states descend from UV [36], resulting in the rapid increase in the number of low energy degrees of freedom. We note here that the distance conjecture is not so well established in two dimensions [37]. Since two dimensional gravity we consider is obtained by a dimensional reduction of three dimensional gravity, we assume the increase in the number of degrees of freedom to be an effect of three dimensional gravity. In two dimensional CFT perspective, this may not contradict to the Zamolodchikov’s \( c \)-theorem which states \( c_{\text{UV}} > c_{\text{IR}} \) \([38]\) as the increase in the number of degrees of freedom does not mean that IR degrees of freedom are newly generated but UV degrees of freedom descend. Since the distance conjecture assumes a tower of states in UV, we expect many more degrees of freedom still present in UV.

If quasi-dS space does not have the island, the radiation entropy produced in this way keeps monotonically increasing in time, contradict to the central dogma [19]. Then we expect the quasi-dS background cannot persist for arbitrarily long times [35] (see also [39–41] for more discussions). In Sect. 4, we check if the island can exist in the inflationary quasi-dS space when the central charge increases in time. For this purpose, we use three conditions in [12] which restrict the spacetime region allowing the island to exist. From this we conclude that the island does not exist in the inflationary quasi-dS space thus for the consistency with the central dogma, the background must be strongly deformed, as the entropy argument for the dS swampland conjecture claims.

### 2 JT gravity on dS2 coupled to CFT

This section is devoted to the review on the essential ingredients needed for our discussion. We first address the features of the Jackiw–Teitelboim (JT) gravity, focusing on the dS2 background. When the JT gravity is coupled to CFT2, the entanglement entropy of radiation can be produced, the expression of which will be presented as well.

#### 2.1 JT gravity on dS2

The JT gravity is a theory of two-dimensional gravity coupled to dilaton, which can be obtained by the dimensional reduction of three-dimensional gravity. For the dS3 background, the action is written as

\[
S = \frac{1}{16\pi G(3)} \int d^3x \sqrt{-g^{(3)}} \left( R^{(3)} - 2H^2 \right) - \frac{1}{8\pi G(3)} \int d^2x \sqrt{-h^{(2)}} K^{(2)},
\]

where \( R^{(3)} = R^{(2)} - \frac{2\Phi}{\Phi} \Box^{(2)}\Phi \).
Fig. 1 Penrose diagram for dS$_2$. The conformal coordinates ($\sigma, \theta$) cover the whole region, regions I, II, III, and IV

\[ K^{(3)} = K^{(2)} + h^{(3)\phi \phi} K_{\phi \phi} = K^{(2)} + \frac{1}{\Phi} n^\mu \nabla_\mu \Phi, \]  \hspace{1cm} \text{(3)}

where $n^\mu$ is the unit vector normal to the boundary surface, from which the action is reduced to that of the JT gravity,

\[ S = \int d^2 x \sqrt{-g^{(2)}} \frac{\Phi}{16\pi G} \left( R^{(2)} - 2H^2 \right) \]
\[ - \int d^2 x \sqrt{-h^{(2)}} \frac{\Phi h}{8\pi G} K^{(2)}. \]  \hspace{1cm} \text{(4)}

Here the two-dimensional Newton’s constant is given by $G = G^{(3)} H$. Since the JT gravity and dS$_3$ are connected through the dimensional reduction, the solutions ($g^{(2)}_{ij}, \Phi$) to the Einstein equation can be immediately read off from the components of dS$_3$ metric. Here we list the solutions in various coordinates for dS space.

- Conformal coordinates ($\sigma, \theta$): The metric and dilaton are given by

\[ ds^2_2 = \frac{1}{H^2 \cos^2 \sigma} (-d\sigma^2 + d\theta^2), \quad \Phi = 2\pi \frac{\sin \theta}{\cos \sigma}. \]  \hspace{1cm} \text{(5)}

Here $\sigma \in (-\pi/2, \pi/2)$ and $\theta \in (0, \pi)$, which cover the whole dS$_2$ manifold (regions I, II, III, and IV) as depicted in Fig. 1. When we describe dS$_2$ as the hyperboloid embedded in 3-dimensional Minkowski space satisfying $-(X^0)^2 + (X^1)^2 + (X^2)^2 = H^{-2}$, the conformal coordinates correspond to the parametrization

\[ X^0 = H^{-1} \tan \sigma, \quad X^1 = H^{-1} \frac{\sin \theta}{\cos \sigma}, \quad X^2 = H^{-1} \frac{\cos \theta}{\cos \sigma}. \]  \hspace{1cm} \text{(6)}

- Static coordinates ($t, r$): The metric and dilaton are given by

\[ ds^2_2 = -(1 - H^2 r^2) dt^2 + \frac{dr^2}{1 - H^2 r^2}, \quad \Phi = 2\pi H r. \]  \hspace{1cm} \text{(7)}

For the parametrization giving above solutions,

\[ X^0 = H^{-1} \sqrt{1 - H^2 r^2} \sinh(\Phi t), \quad X^1 = r, \]
\[ X^2 = H^{-1} \sqrt{1 - H^2 r^2} \cosh(\Phi t), \]  \hspace{1cm} \text{(8)}

the coordinates $t \in (-\infty, +\infty)$ and $r \in (0, H^{-1})$ cover the static patch (region I in Fig. 1). Meanwhile, region II can be covered by another parametrization

\[ X^0 = H^{-1} \sqrt{H^2 r^2 - 1} \cosh(\Phi t), \quad X^1 = r, \]
\[ X^2 = H^{-1} \sqrt{H^2 r^2 - 1} \sinh(\Phi t), \]  \hspace{1cm} \text{(9)}

as $t \in (-\infty, +\infty)$ and $r \in (H^{-1}, \infty)$. The covering of regions III and IV can be obtained by the replacement of $(X^0, X^1, X^2)$ in (8) and (9) by $(-X^0, X^1, -X^2)$, respectively, which corresponds to $\sigma \rightarrow -\sigma$ and $\theta \rightarrow \pi - \theta$ in terms of (6). In any case, the static coordinates are regular only in the region they are defined and different regions are separated by the coordinate singularity at the horizon, $r = H^{-1}$. The covering of region I and II by the static coordinates is depicted in the left panel of Fig. 2.

- Flat coordinates ($\tau, \rho$): The metric and dilaton are given by

\[ ds^2_2 = -d\tau^2 + e^{2\Phi \tau} d\rho^2, \quad \Phi = 2\pi H \rho e^{H \tau}, \]  \hspace{1cm} \text{(10)}

where $\tau \in (-\infty, +\infty)$ and $\rho \in (0, \infty)$. This corresponds to the parametrization

\[ X^0 = H^{-1} \left( \sinh(\Phi \tau) + \frac{1}{2} (H \rho^2) e^{H \tau} \right), \quad X^1 = \rho e^{H \tau}, \]
\[ X^2 = H^{-1} \left( \cosh(\Phi \tau) - \frac{1}{2} (H \rho^2) e^{H \tau} \right). \]  \hspace{1cm} \text{(11)}

The static and flat coordinates are related by

\[ t = \tau - \frac{1}{2H} \log \left( 1 - H^2 \rho^2 e^{2H \tau} \right), \quad r = \rho e^{H \tau} \]  \hspace{1cm} \text{(12)}

in region I and

\[ t = \tau - \frac{1}{2H} \log \left( H^2 \rho^2 e^{2H \tau} - 1 \right), \quad r = \rho e^{H \tau} \]  \hspace{1cm} \text{(13)}

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in region II, respectively. The flat coordinates cover the inflating patch, the union of regions I and II, without the coordinate singularity, as can be seen in the right panel of Fig. 2.

In the following discussions, we consider $H$ varying with respect to some timelike direction which we will call $\tau$. In this case, the hypersurface orthogonal to the $\tau$ direction has a rotational invariance as an isometry. Then we expect the corrected metric to be in the form of FRW metric, in which the $dS_2$ coordinates (time and radial coordinates) are corrected metric to be in the form of FRW metric, in which $H$ is close to the constant $\Omega$. Indeed, the dimensional reduction shows that, up to surface term which does not affect the equations of motion, the action for a $\tau$ dependent $H$ can be obtained by replacing the constant $H$ by $H(\tau)$. We sketch the derivation of this in Appendix A.

We remark that the value of dilaton $\Phi$ is positive in any coordinates and interpreted as an area of surface in $dS_3$ on their own static coordinates. Here $t$ (grey) lines is constant on grey dashed (grey) lines. (Right): Covering of the inflating patch, the union of regions I and II by the flat coordinates. Here $t$ (grey) lines is constant on grey dashed (grey) lines.

2.2 CFT entropy in $dS_2$ background

We now consider CFT$_2$ coupled to the JT gravity. In terms of the lightcone coordinates

$$x^+ = e^{i(\theta - \sigma)}, \quad x^- = e^{-i(\theta + \sigma)},$$

the $dS_2$ metric is written in the form of the Weyl rescaling of the flat metric,

$$ds^2 = \frac{dx^+dx^-}{\Omega(x^+, x^-)^2}, \quad \Omega(x^+, x^-) = \frac{H}{2}(1 + x^+x^-). \quad (16)$$

Then the entanglement entropy, or the von Neumann entropy of the CFT matter on a segment between $(x_1^+, x_1^-)$ and $(x_2^+, x_2^-)$ in $dS_2$ can be written as

$$S_{\text{mat}} = \frac{c}{6} \log \left[ \frac{1}{\epsilon_{\text{UV}}} \frac{\Omega(x_1)\Omega(x_2)}{(x_1^+ - x_1^-)(x_2^- - x_2^+)} \right]$$

$$\quad \quad = \frac{c}{6} \log \left[ \frac{2}{\epsilon_{\text{UV}}^2H^2} \frac{\cos(\sigma_1 - \sigma_2) - \cos(\theta_1 - \theta_2)}{\cos(\sigma_1 \cos \sigma_2)} \right]. \quad (17)$$

where $c$ is the central charge of CFT$_2$ and $\epsilon_{\text{UV}}$ is the UV cutoff length [43–45]. While the expression above is given in terms of the conformal coordinates, it is straightforward to rewrite this in terms of other coordinates by noting that the argument of the logarithm is nothing more than the spacetime interval in three-dimensional Minkowski space $(X_1 - X_2)^2$, i.e.,

$$\frac{\cos(\sigma_1 - \sigma_2) - \cos(\theta_1 - \theta_2)}{\cos(\sigma_1 \cos \sigma_2)} = 1 + \tan \sigma_1 \tan \sigma_2$$

$$\frac{\sin \theta_1 \sin \theta_2}{\cos \sigma_1 \cos \sigma_2} = \frac{\cos(\sigma_1 \cos \sigma_2)}{\cos(\sigma_1 \cos \sigma_2)}$$

$$1 + H^2(X_1^0X_2^0 - X_1^1X_2^1 - X_1^2X_2^2) = 1$$

$$-H^2X_1 \cdot X_2 = \frac{H^2}{2}(X_1 - X_2)^2, \quad (18)$$

such that

$$S_{\text{mat}} = \frac{c}{6} \log \left[ \frac{(X_1 - X_2)^2}{\epsilon_{\text{UV}}^2} \right]. \quad (19)$$

In the flat coordinates, it can be written as

$$S_{\text{mat}} = \frac{c}{6} \log \left[ \frac{2}{\epsilon_{\text{UV}}^2H^2} \left[ 1 - \cosh[H(\tau_1 - \tau_2)] \right] \right.$$}

$$\left. + \frac{1}{2} e^{H(\tau_1 + \tau_2)}H^2(\rho_1 - \rho_2)^2 \right]. \quad (20)$$
In the static coordinates, it is written as
\[ S_{\text{mat}} = \frac{c}{6} \log \left[ \frac{2}{\epsilon_{\text{UV}} H^2} \left( 1 - H^2 r_1 r_2 \right) \right. \\
\left. - \sqrt{1 - H^2 r_1^2} \sqrt{1 - H^2 r_2^2} \cosh[H(t_1 - t_2)] \right] \] (21)
for region I and
\[ S_{\text{mat}} = \frac{c}{6} \log \left[ \frac{2}{\epsilon_{\text{UV}} H^2} \left( 1 - H^2 r_1 r_2 \right) \\
+ \sqrt{H^2 r_1^2 - 1} \sqrt{H^2 r_2^2 - 1} \cosh[H(t_1 - t_2)] \right] \] (22)
for region II, respectively.

3 Information paradox in quasi-dS static patch

3.1 Choice of spacelike surface collecting radiation

Since a static observer in (quasi-)dS space detects the thermal radiation emitted by the horizon, the observer’s semiclassical description of (quasi-)dS space is expected to be quite similar to that of black hole as seen from far outside the horizon. This may motivate one to postulate the central dogma for (quasi-)dS space as follows:

(quasi-)dS space as seen by an static observer surrounded by the horizon is described by the unitarily evolving quantum system with \( \exp[(\text{Horizon Area})/4G] \) degrees of freedom.

This slight modification of the central dogma for black hole, however, conceals a number of subtleties. For black hole, a compact object in space, it is natural to introduce the cutoff surface outside which gravitational effects are small enough to be neglected. Then the region inside the cutoff surface is treated as the quantum black hole, the quantum gravity system in the central dogma. The radiation entropy is measured by collecting the radiation passing through the spacelike surface extending over the region outside the cutoff surface. In contrast, for (quasi-)dS space, the curvature does not vanish at any point: the Ricci scalar of perfect dS space is given by a constant, \( R^{(d)} = d(d - 1)H^2 \). Thus, gravity cannot be neglected at any point in (quasi-)dS space and the separation of the observer collecting the radiation from the quantum gravity system is challenging. Nevertheless, when \( H(G^{(d)})^{1/(d-2)} \ll 1 \), the most part of the static patch can be treated semiclassically, in which the curved spacetime is just a non-dynamical background.\(^2\) Then we can consider the spacelike surface \( \Sigma \) in this ‘radiation region’ (or thermal bath) to collect the radiation. At the same time, the union of the rest part of the static patch and the region beyond the horizon is regarded as the quantum gravity system in the central dogma.

Before considering quasi-dS space, we address the radiation entropy in perfect dS space in which \( H \) is constant. If we choose the spacelike surface \( \Sigma \) to be the equal-time surface in the conformal or flat coordinates, \( \sigma = \) (constant) or \( \tau = \) (constant), the entanglement entropy of the CFT radiation given by
\[ S_{\text{mat}} = \frac{c}{6} \log \left[ \frac{2}{\epsilon_{\text{UV}} H^2} \left( \frac{1 - \cos(\theta_1 - \theta_2)}{\cos^2 \sigma} \right) \right] \]
\[ \sigma = \) (constant), or
\[ S_{\text{mat}} = c \log \left[ \frac{1}{\epsilon_{\text{UV}}} e^{2H \tau} (\rho_1 - \rho_2)^2 \right] \]
\[ \tau = \) (constant),
(23)
where \( \theta_{1,2} \) and \( \rho_{1,2} \) being the coordinates of two endpoints of \( \Sigma \) in each case, explicitly depends on the time coordinate \( \sigma \) or \( \tau \). Let us focus on the flat coordinates which are appropriate to describe the homogeneous and isotropic universe. They are also useful if we are interested in the spacelike future infinity, which is taken as the boundary of dS space in the dS/CFT correspondence [46]. We infer from (23) that in the flat coordinates, \( S_{\text{mat}} \) increases in \( \tau \) for fixed \( \rho_1 - \rho_2 \). Then one is tempted to argue that even in perfect dS space, we can find the direction of irreversibility parametrized by \( \tau \) (for previous works along the similar line, see, e.g., [47–53]). However, as often summarized as “perfect dS space does not have a clock”, the geometries at \( \tau \) and \( \tau + a \) are not distinct. This is because the shift in time \( \tau \to \tau + a \) can be compensated by the rescaling \( \rho \to e^{-aH} \rho \) to leave the geometry unchanged, which indeed is one of dS isometries. Since the static radial coordinate given by \( r = \rho e^{H \tau} \) is invariant under this transformation, the absence of the clock is obvious if we choose \( \Sigma \) to be a surface \( t = \) (constant) where \( t \) is the static time coordinate, with the \( r \) coordinates of the endpoints of \( \Sigma \) fixed. Then we expect that \( S_{\text{mat}} \) on this surface does not evolve in time. This is consistent with the observation that in the Bunch–Davies vacuum in which dS isometries are preserved, the perfect dS space is in thermal equilibrium so that the thermodynamic quantities like the horizon size are static. The clock can be introduced in quasi-dS space in which \( H \) varies in time, as the dS isometry associated with the shift in \( \tau \) and the rescaling of \( \rho \) is spontaneously broken. In this case, \( S_{\text{mat}} \) on the surface \( t = \) (constant) will evolve in time, measuring the irreversibility generated by the out-of-equilibrium process analogous to the black hole evaporation. Moreover, in the static coordinates, unlike the flat coordinates, regions I (inside the horizon) and II (beyond the horizon) are separated by the coordinate singularity hence the role of the horizon is

\(^2\) Even though this condition does not apply to the case of \( d = 2 \), the two dimensional gravity we are discussing is a dimensional reduction of three dimensional gravity, thus the condition \( H G^{(3)} \ll 1 \) is assumed.
emphasized. This indicates that the static coordinates enable one to investigate thermodynamics relevant to the horizon more manifestly from the direct comparison with the black hole thermodynamics.

From the discussion above, we take $\Sigma$ to lie on the spacelike surface $t = \text{constant}$ within the static patch such that the static coordinates of two endpoints are given by $(t_1, r_1) = (t, 0)$ and $(t_2, r_2) = (t, r)$ ($r < H^{-1}$), respectively, as shown in Fig. 3. Then for perfect $dS$ space, the radiation entropy on $\Sigma$ is written as

$$S_{\text{mat}}(\Sigma) = \frac{c}{6} \log \left[ \frac{2}{\epsilon_{\text{UV}} H^2} \left( 1 - \sqrt{1 - H^2 r^2} \right) \right].$$

(24)

This explicitly shows that $S_{\text{mat}}(\Sigma)$ does not evolve in time $t$, reflecting the thermal equilibrium under the $dS$ isometries. While the central charge is typically taken to be $c \gg 1$ in order to suppress the subleading corrections which are not taken into account in our discussion, it cannot be arbitrarily large since $S_{\text{mat}}(\Sigma)$ is restricted to be smaller than the $dS$ entropy. Taking the shift of the dilaton $\Phi \rightarrow \Phi + (2/3)cG$ coming from (14) into account, the $dS$ entropy is given by

$$S_{dS} = \frac{\text{Horizon Area}}{4G} = \frac{\pi}{2G} + \frac{c}{6} = \frac{\pi}{2G (3) H} + \frac{c}{6},$$

(25)

where the horizon area is given by $\Phi(r) = 2\pi H r$ with $r = H^{-1}$.

We note that in the JT gravity, the dilaton multiplied to the Ricci tensor in the action (4) seems to enhance the strength of the gravitational interaction at $r = 0$, the endpoint of $\Sigma$, as the Newton’s constant reads $G/\Phi = G^{(3)}/(2\pi r)$ effectively. This may invalidate our setup that gravity is negligible over the region of $\Sigma$. When we regard the JT gravity as a dimensional reduction of three dimensional gravity, such an enhancement is an artifact caused by promoting the factor $r$ in $(-g^{(3)})^{1/2}$ which was originally the component of the three dimensional metric in the angular direction to the dilaton field $\Phi$. Nevertheless, one may circumvent the issue of strong gravity by taking one of endpoints of $\Sigma$ to be $(t_1, r_1) = (t, \epsilon_{\text{UV}})$ and imposing $G^{(3)} \ll \epsilon_{\text{UV}}$. So far as $\epsilon_{\text{UV}}$ is much smaller than $H^{-1}$, we can neglect the $\epsilon_{\text{UV}}$ dependence in $S_{\text{mat}}(\Sigma)$.

3.2 Radiation entropy in quasi-$dS$ space

In quasi-$dS$ space, $H$ is no longer constant in time as the $dS$ isometries are spontaneously broken. The radiation in this case is not in equilibrium with the quantum gravity system, which leads to the evolution of the radiation entropy in time. In order to see the implications to the inflationary cosmology, we consider $H = H(t)$ as a monotonically decreasing function of the flat time coordinate $t$. Then the deviation of the background from perfect $dS$ space can be treated in a controllable way through the expansion in terms of slow-roll parameters,

$$\epsilon_H = -\frac{1}{H^2} \frac{dH}{dt}, \quad \eta_H = -\frac{1}{2H} \frac{d^2H/d\tau^2}{dH/d\tau},$$

(26)

where $\epsilon_H > 0$.

Meanwhile, as discussed in Sect. 3.1, the surface $\Sigma$ is taken to be the equal-static-time ($t$) surface. It is reasonable to assume here that the leading term of $S_{\text{mat}}(\Sigma)$ for the quasi-$dS$ background is given by $(c/6) \log[(X_1 - X_2)^2/\epsilon_{\text{UV}}^2]$, the same form as $S_{\text{mat}}(\Sigma)$ for the perfect $dS$ background. Then different $H$ values at two endpoints of $\Sigma$ make $S_{\text{mat}}(\Sigma)$ time dependent. More explicitly, from the relation between $t$ and $\tau$ in (12), the flat time coordinates of $X_1$ and $X_2$ are given by

$$t_1 = t, \quad t_2 = t + \frac{1}{2H_2} \log(1 - H_2^2 r^2),$$

(27)

respectively, where $H_{1,2} = H(t_{1,2})$. Since $1 - H_2^2 r^2 < 1$ one finds that $t_2 > t$ hence $H_1 < H_2$. For $\epsilon_H \ll 1$, the values of $H_1$ and $H_2$ are similar in magnitude. Then we can expand $H_2$ as

$$H_2 = H_1 \left[ 1 - \epsilon_H(t) \left( \frac{H_1}{2H_2} \log(1 - H_2^2 r^2) + \epsilon_H \eta_H \left( \frac{H_1}{2H_2} \log(1 - H_2^2 r^2) \right)^2 + \cdots \right) \right].$$

(28)

Given $\eta_H \ll \epsilon_H \ll 1$, we focus on the value of $r$ satisfying

$$\epsilon_H(t) \left[ \frac{H_1}{2H_2} \log \left( \frac{1}{1 - H_2^2 r^2} \right) \right] < 1,$$

(29)

for the controllable expansion in terms of slow-roll parameters. While this obviously requires $\epsilon_H |\log(1 - H_1^2 r^2)| < 1$, the expansion of LHS,

$$\frac{\epsilon_H}{2} \log(1 - H_1^2 r^2) + \frac{\epsilon_H}{2} \log(1 - H_1^2 r^2) \left[ \frac{1}{2} \log(1 - H_1^2 r^2) + \frac{H_1^2 r^2}{1 - H_1^2 r^2} + \mathcal{O}(\epsilon_H^3) \right],$$

(30)
Since this term is required to be positive, we find a bound
\[ \frac{\epsilon_H H_1^2 r^2}{1 - H_1^2 r^2} < 1, \]  
preventing \( r \) from being too close to the horizon \( H^{-1} \).

When the radiation region occupies the most part of the static patch, \( r \) is taken to be close to the horizon so far as (31) is satisfied. In this case, it is convenient to express \( S_{\text{mat}}(\Sigma) \) in terms of \( \ell \) defined as
\[ H_2^2 \ell^2 = 1 - H_2^2 r^2, \]  
which is much less than 1, measuring the deviation of the endpoint \( X_2 \) from the horizon. Then the condition (31) reads \( \epsilon_H < (H_1 \ell)^2 \) (since \( H_1 \simeq H_2 \) and \( H_2 \ell \ll 1 \)) and (28) is rewritten as
\[ H_1 - H_2 = \epsilon_H \frac{H_2^2}{2H_2} \log(H_2^2 \ell^2) + \mathcal{O}(\epsilon_H^2) \]
\[ = -\epsilon_H H_1 \log \left( \frac{1}{H_2 \ell} \right) + \mathcal{O}(\epsilon_H) \].  
(33)

Therefore, given the radiation entropy,
\[ S_{\text{mat}}(\Sigma) = \frac{c}{6} \log \left( \frac{1}{\epsilon_{UV}^2} \right) + \mathcal{O}(\epsilon_H^2) \]
\[ = \frac{c}{6} \log \left[ \frac{1}{\epsilon_{UV}^2} H_1 H_2 \left( \frac{H_1}{H_2} + \frac{H_2}{H_1} \right) \right. \]
\[ - 2 \sqrt{1 - H_2^2 r^2} \cosh[(H_1 - H_2) t] \right] \}. \]  
(34)

the leading term in the expansion with respect to \( \epsilon_H \) is written as
\[ S_{\text{mat}}(\Sigma) = \frac{c}{6} \log \left[ \frac{2}{\epsilon_{UV}^2 H_1 H_2} \left( 1 - H_2 \ell \cosh \left( \frac{\epsilon_H H_1 t \log \left( \frac{1}{H_2 \ell} \right)}{} \right) \right) \right. \]
\[ \times \left. \left( \frac{1}{\epsilon_{UV}^2 H_1 H_2} \right) \right] \}. \]  
(35)

We will observe the time evolution of (35) with \( \ell \) fixed. We note that the expression (35) is not valid for arbitrary large \( t \). For \( H_1 t \gg 1 \), the argument of hyperbolic cosine becomes larger than 1 such that the term in the parentheses is approximated as
\[ 1 - H_2 \ell \cosh \left( \frac{\epsilon_H H_1 t \log \left( \frac{1}{H_2 \ell} \right)}{} \right) \]
\[ \simeq 1 - \frac{1}{2} \left( H_2 \ell \right)^{-1 - \epsilon_H H_1 t}. \]  
(36)

Since this term is required to be positive,\(^3\) we find a bound
\[ t < \left( \epsilon_H H_1 \right)^{-1} \frac{1}{\log \left[ H_1 \ell / 2 \right] / \log \left( H_1 \ell \right)}, \]  
or roughly \( t < \left( \epsilon_H H_1 \right)^{-1} \) up to \( \mathcal{O}(1) \) coefficient. In fact, the bound on \( t \) we find is nothing more than the time scale after which \( H_2 \) is no longer close to \( H_1 \). One way to see this is to observe the explicit form of \( H(t) \) for the special case of a constant \( \epsilon_H \), \( H(t) = H_0 \left( 1 + \epsilon_H H_0 t \right) \), which significantly deviates from \( H_0 \) after \( t > \left( \epsilon_H H_0 \right)^{-1} \).

Finally, the central charge \( c \) which is proportional to the number of degrees of freedom may vary in time. Here we will focus on the case that \( c \) is monotonically increasing in time. This is motivated by the entropy argument for the dS swampland conjecture [35], which is based on the claim that the number of low energy degrees of freedom increases exponentially along the trans-Planckian geodesic trajectory of the modulus responsible for the vacuum energy [36]. In our analysis, for simplicity, we assume that in region 1, \( c \) depends only on the static time coordinate \( t \) such that it is constant on the slice \( \Sigma \). In fact, for the application to the inflationary cosmology in which homogeneity and isotropy are emphasized, it is reasonable to take \( c \) to be a function of the flat time coordinate \( \tau \), just like \( H \). To compare the difference between the values of \( c \) at two endpoints of \( \Sigma \) in this case, we employ the ansatz motivated by the distance conjecture [36], \( c = c_0 \exp[\lambda \Delta \phi] \), where \( \Delta \phi \) is the geodesic distance traversed by the modulus in Planck unit and \( \lambda \) is an \( \mathcal{O}(1) \) constant. In the four-dimensional case, the equations of motion are solved to give \( d\phi/d\tau = \sqrt{2\epsilon_H H} \), implying \( dc/d\tau = \lambda \sqrt{2\epsilon_H H} c \). This suggests to consider \( c(\tau) \) satisfying \( [c(\tau_1) - c(\tau_2)]/\epsilon_H = \mathcal{O}(1) \), in which case \( S_{\text{mat}}(\Sigma) \) containing \( c(t) \) and \( c(\tau) \) differ by the \( \mathcal{O}(\epsilon_H^2) \) correction. While this dominates over the \( \mathcal{O}(\epsilon_H) \) correction coming from difference between \( H_1 \) and \( H_2 \) discussed above, it does not change the qualitative feature of the conclusion we will draw so we will restrict our attention to \( c(t) \) rather than \( c(\tau) \).

3.3 Condition for information paradox

Now we consider the condition for the information paradox in three-dimensional quasi-dS space, regarding the JT gravity as a dimensional reduction of dS\(^3\) space. For this purpose, we set \( G^{(3)} \) rather than \( G = G^{(3)} H \) to be a constant. For the information paradox to arise in quasi-dS space, the increasing rate of \( S_{\text{mat}}(\Sigma) \) in time must be larger than that of \( S_{\text{DS}} \) such that \( S_{\text{mat}}(\Sigma) \) eventually exceeds \( S_{\text{DS}} \). Since \( H \) depends only on \( \tau \), not on \( \rho \), we have \( dH/d\tau = dH/d\tau = -\epsilon_H H^2 \),\(^4\) from

\[ \frac{d}{d\tau} = \frac{\partial \tau}{\partial \sigma} \frac{\partial \sigma}{\partial \tau} + \frac{\partial \rho}{\partial \tau} \frac{\partial \tau}{\partial \rho} = \frac{\partial \rho}{\partial \tau} - \frac{\partial \rho}{\partial \rho}. \]  
(37)

\(^3\) More precisely, the term \( 1 - H_2 \ell \cosh[\epsilon_H H_1 t \log \left( 1/(H_2 \ell) \right)] \) in the logarithm of (35) is bounded by \( \epsilon_{\text{UV}}^2 H_1 H_2 \) as \( (X_1 - X_2)^2 \geq \epsilon_{\text{UV}}^2 \). But since \( \epsilon_{\text{UV}}^2 H_1 H_2 \ll 1 \), our estimation is not much affected.

\(^4\) Derivative with respect to the static time coordinate \( t \) is written in terms of derivatives with respect to the flat coordinates \( (\tau, \rho) \) as
which one finds that the derivative of $S_{\text{DS}}$ given by (25) with respect to $t$ becomes

$$\frac{dS_{\text{DS}}}{dt} = -\epsilon H H_1^2 \frac{dS_{\text{DS}}}{dH_1} = \frac{\pi \epsilon_H}{2G^{(3)}} + \frac{1}{6} \frac{dc}{dt} > 0.$$  \hspace{1cm} (38)

This shows that $S_{\text{DS}}$ increases in time provided $dc/dt$ is positive or suppressed to $O(\epsilon_H)$. The same feature is found in the four-dimensional case, in which $S_{\text{DS}} = \pi/(GH^2)$, giving $dS_{\text{DS}}/dt = 2\pi \epsilon_H/(GH) > 0$.

On the other hand, the radiation entropy $S_{\text{mat}}(\Sigma)$ given by (35) decreases in time as $t$ approaches the bound $(\epsilon_H H_1)^{-1}$, since the argument of the logarithm vanishes for $t \sim (\epsilon_H H_1)^{-1}$. However, $S_{\text{mat}}(\Sigma)$ also has an implicit $t$ dependence through $H_1$ and $c$, which provides a chance for $S_{\text{mat}}(\Sigma)$ to increase in $t$ and exceed $S_{\text{DS}}$ during some period. Then the information paradox arises in quasi-dS space. To see this more explicitly, we take the derivative of $S_{\text{mat}}(\Sigma)$ with respect to $t$ using

$$\frac{d}{dt} = \frac{\partial}{\partial t} - \epsilon H H_1^2 \frac{\partial}{\partial H_1} + \frac{dc}{dt} \frac{\partial}{\partial c}.$$  \hspace{1cm} (39)

To leading order in $\epsilon_H$, $dS_{\text{mat}}(\Sigma)/dt$ can be written as a sum of four terms, $dS_{\text{mat}}(\Sigma)/dt = \dot{\Sigma}_c + \dot{\Sigma}_{H1} + \dot{\Sigma}_t + \dot{\Sigma}_{H2}$:

- $\dot{\Sigma}_c$: This term comes from the time evolution of $c$.

$$\dot{\Sigma}_c = \frac{dc}{dt} S_{\text{mat}}(\Sigma) = \frac{dc}{dt} \log \left[ \frac{2}{\epsilon_{UV} H_1 H_2} \right] \times \left( 1 - H_2 \epsilon \cosh \left[ \epsilon_H H_1 t \log \left( \frac{1}{H_2 \epsilon} \right) \right] \right).$$  \hspace{1cm} (40)

- $\dot{\Sigma}_{H1}$: This term comes from the time evolution of $H_1$ in $(c/6) \log(2/(\epsilon_{UV} H_1 H_2))$ part of (35).

$$\dot{\Sigma}_{H1} = \frac{c}{3} \epsilon_H H_1.$$  \hspace{1cm} (41)

- $\dot{\Sigma}_t$: This term comes from the explicit $t$ dependence in $(c/6) \log[X(t)]$ part of (35), where

$$X(t) = 1 - H_2 \epsilon \cosh \left[ \epsilon_H H_1 t \log \left( \frac{1}{H_2 \epsilon} \right) \right].$$  \hspace{1cm} (42)

$$\dot{\Sigma}_t = -\frac{c}{6} \epsilon_H H_1 \left( H_1 t \log \left( \frac{1}{H_1 \epsilon} \right) \right) \times \sinh \left[ \epsilon_H H_1 t \log \left( \frac{1}{H_1 \epsilon} \right) \right] \times \frac{1}{1 - H_2 \epsilon \cosh \left[ \epsilon_H H_1 t \log \left( \frac{1}{H_2 \epsilon} \right) \right]}. \hspace{1cm} (43)$$

- $\dot{\Sigma}_{H2}$: This term comes from the time evolution of $H_1$ in $(c/6) \log[X(t)]$.

$$\dot{\Sigma}_{H2} = \frac{c}{6} \epsilon_H H_1 (H_1 \epsilon) \cosh \left[ \epsilon_H H_1 t \log \left( \frac{1}{H_2 \epsilon} \right) \right] \cosh \left[ \epsilon_H H_1 t \log \left( \frac{1}{H_2 \epsilon} \right) \right].$$  \hspace{1cm} (44)

We note that whereas the negative $\dot{\Sigma}_t$ leads to the decrease of $S_{\text{mat}}(\Sigma)$ in $t$ through the explicit $t$ dependence as expected from the behavior of $S_{\text{mat}}(\Sigma)$ at $t \sim (\epsilon_H H_1)^{-1}$, the decrease of $H_1$ in $t$ giving the positive $\dot{\Sigma}_{H1}$ and $\dot{\Sigma}_{H2}$ leads to the increase of $S_{\text{mat}}(\Sigma)$ in $t$. For $t$ close to $(\epsilon_H H_1)^{-1}$, $\dot{\Sigma}_t$ can easily dominate over $\dot{\Sigma}_{H1}$ as $X(t)$ in denominator of $\dot{\Sigma}_t$ is close to zero, and over $\dot{\Sigma}_{H2}$ as sinh and cosh are comparable in size but $\dot{\Sigma}_t$ contains the logarithmic enhancement factor $\log[1/(H_1 \epsilon)]$. The enhancement of $\dot{\Sigma}_t$ through $1/X(t)$ does not take place when $t \ll (\epsilon_H H_1)^{-1}$ as $X(t) \lesssim 1$.\footnote{This is also helpful for controlling the expansion of $S_{\text{mat}}(\Sigma)$ with respect to $\epsilon_H$ since the subleading terms in the expansion contain positive powers of $1/X(t)$ which is enhanced for $X(t)$ close to 0 (more precisely, $\epsilon_{UV} H_1^2$).} In this case, since $H_2 \epsilon \ll 1$, $S_{\text{mat}}(\Sigma)$ is approximated as

$$S_{\text{mat}}(\Sigma) \sim \frac{c}{6} \log \left[ \frac{2}{\epsilon_{UV} H_1 H_2} \right] \sim \frac{c}{6} \log \left[ \frac{2}{\epsilon_{UV} H_1^2} \right] - \frac{c}{6} \epsilon_H \log \left[ \frac{1}{H_1 \epsilon} \right].$$  \hspace{1cm} (45)

and $\dot{\Sigma}_{H1}$ can be larger than $\dot{\Sigma}_t$ and $\dot{\Sigma}_{H2}$. If $\dot{\Sigma}_c$ is negligibly small, $dS_{\text{mat}}(\Sigma)/dt \approx \dot{\Sigma}_{H1}$ is larger than $dS_{\text{DS}}/dt$ provided $c > \frac{3\pi}{2} \frac{1}{G^{(5)} H_1}$.

or equivalently, $S_{\text{mat}}(\Sigma) \gtrsim (S_{\text{DS}}/2) \log[2/(\epsilon_{UV} H_1^2)]$. Then for the weak gravity case $\epsilon_{UV} H_1 \sim G^{(5)} H_1 \ll \sqrt{2}/\epsilon \sim 0.5$, $S_{\text{mat}}(\Sigma)$ is already larger than $S_{\text{DS}}$ from beginning, contradict to our setup. Therefore, the only way to realize the information paradox in the acceptable parameter region is allowing $c$ to depend on $t$ such that $\dot{\Sigma}_c$ driven by $dc/dt$ is positive and dominant over $O(\epsilon_H)$ contributions $\dot{\Sigma}_{H1}$, $\dot{\Sigma}_t$, and $\dot{\Sigma}_{H2}$. For instance, for the ansatz $dc/dt = \lambda \sqrt{2 \epsilon_H} H_1 c$ motivated by the entropy argument for the dS swampland conjecture, $\dot{\Sigma}_c \simeq \lambda \sqrt{2 \epsilon_H} H_1 S_{\text{mat}}(\Sigma)$ becomes a dominant contribution to $dS_{\text{mat}}(\Sigma)/dt$ as it is $O(\epsilon_H^{1/2})$. While $dS_{\text{DS}}/dt$ in this case is dominated by $(1/6)dc/dt$ term which originates from the trace anomaly, $S_{\text{mat}}$ is larger than $c/6$ by the logarithmic enhancement by $\log[1/(\epsilon_{UV} H_1^2)]$, so we expect that $\dot{\Sigma}_c$ is larger than $dS_{\text{DS}}/dt$.

Since time scale we consider must be much smaller than $\epsilon_H H_1)^{-1}$, it is a good approximation to take $H_1$ to be almost constant until $S_{\text{mat}}(\Sigma)$ becomes close to $S_{\text{DS}}$. Then we can set $c \simeq c_0 \exp[\gamma \sqrt{2 \epsilon_H} H_1 \Delta t]$, from which one finds that given...
the initial condition $S_{\text{mat}}(\Sigma) \ll S_{\text{dS}}, S_{\text{mat}}(\Sigma)$ given by (45) is comparable to $S_{\text{dS}}$ when

$$\Delta t = \frac{1}{\gamma} \frac{1}{\sqrt{2\kappa H}H_1} \left[ \log(S_{\text{dS}}) - \log \left( \frac{6}{H_1} \log \left( \frac{2}{\epsilon H_1} \right) \right) \right]. \tag{47}$$

We note here that whereas $dS_{\text{dS}}/dt$ is dominated by $(1/6)c\dot{c}/dt$, $c/6$ in $S_{\text{dS}}$ is required to be much smaller than the area term as $c$ is restricted to be (46). For $\epsilon \kappa \simeq G(3)$, the second term is comparable to $\log^2 S_{\text{dS}}$ in size, which is much smaller than $S_{\text{dS}}$ for $S_{\text{dS}} \gg 1 \text{. Then we arrive at } \Delta t \simeq (\gamma \sqrt{2\kappa H}H_1)^{-1} \log(S_{\text{dS}})$, which is acceptable for $\Delta t \ll (\epsilon H H_1)^{-1}$, or $\epsilon H \leq 1 \log(S_{\text{dS}})$.

We have seen that for the information paradox to arise in quasi-dS space, the radiation entropy which was initially much smaller than the dS entropy has to increase in time as the central charge increases, eventually exceeding the dS entropy after $\Delta t$ given by (47). It is remarkable that the situation is equivalent to that considered in the entropy argument for the dS swampland conjecture [35] when we identify the central charge with the number of degrees of freedom. That is, the entropy argument claims that when the number of low energy degrees of freedom increases exponentially as expected from the distance conjecture, the entropy of matter within the horizon will exceed the bound given by the dS entropy. Then Bousso’s covariant entropy bound is violated and the backreaction of matter deforms the horizon, from which we can say that dS space is unstable. In this regard, it is not strange that the time scale (47) for the information paradox is consistent with the time scale $(\sqrt{\kappa H} H)^{-1} \log(S_{\text{dS}})$ at which the entropy bound is saturated by the matter entropy [40,54]. If quasi-dS space has the island, the true radiation entropy $S_{\text{gen}}(I \cup \Sigma)$ (which will be defined in (49)) cannot exceed $S_{\text{dS}}$ by the entanglement between the radiation and the island hence the information paradox does not arise. As pointed out in [19], this in turn means that in the presence of the island in quasi-dS space, the situation considered in the entropy argument for the dS swampland conjecture is not realized, which tells us that quasi-dS space is quite stable. This motivates the search for the island in the inflationary quasi-dS space, which will be addressed in Sect. 4.

4 Island in inflationary (quasi-)dS space

In the previous section, we argue that for the central dogma to hold without strong deformation of the quasi-dS background, we must be able to find the island within quasi-dS space. Meanwhile, as pointed out in [12], the island must satisfy three conditions, which forbid the island in pure dS space. Small corrections by slow-roll parameters do not change the conclusion significantly, and as we will see, the leading correction coming from the increase of the central charge in time makes the existence of the island more difficult.

4.1 Conditions on island

To begin with, we briefly review the conditions for the existence of the island considered in [12]. Let us denote the region collecting radiation by $\Sigma$ and the island by $I$, respectively. We can always find the region $\Sigma'$ which contains $\Sigma$ but does not intersect with $I$ $(\Sigma' \supset \Sigma$ and $\Sigma' \cap I = \emptyset)$. Then basic inequalities on entropy like the strong subadditivity,

$$S(A) + S(A') \geq S(A \cup A') + S(A \cap A'), \tag{48}$$

restrict the region in which the island can exist. To see this, we define following quantities:

- Generalized entropy of $I \cup \Sigma$

$$S_{\text{gen}}(I \cup \Sigma) = \frac{\text{Area}(\partial I)}{4G} + S_{\text{mat}}(I \cup \Sigma) \tag{49}$$

- Generalized entropy of $I$

$$S_{\text{gen}}(I) = \frac{\text{Area}(\partial I)}{4G} + S_{\text{mat}}(I) \tag{50}$$

- Mutual information between $I$ and $\Sigma$$^6$

$$I(I, \Sigma) = S_{\text{gen}}(I) + S_{\text{gen}}(\Sigma) - S_{\text{gen}}(I \cup \Sigma) = S_{\text{mat}}(I) + S_{\text{mat}}(\Sigma) - S_{\text{mat}}(I \cup \Sigma). \tag{51}$$

We also note that for the island to resolve the information paradox,

$$S_{\text{gen}}(I \cup \Sigma) < S_{\text{mat}}(\Sigma) \tag{52}$$

must be satisfied. Among possible $I$ satisfying (51), the island is chosen to extremize $S_{\text{gen}}(I \cup \Sigma)$.

4.1.1 Condition 1: violation of the area bound on $I$

Condition 1 is a result of two facts. First, application of the island condition (52) to the first equality of (51) gives

$$I(I, \Sigma) > S_{\text{gen}}(I). \tag{53}$$

In addition, the strong subadditivity (48) for $A = I \cup \Sigma$ and $A' = \Sigma'$ (hence $A \cup A' = I \cup \Sigma'$ and $A \cap A' = \Sigma$) gives

$$I(I, \Sigma') \geq I(I, \Sigma). \tag{54}$$

$^6$ For the role of the mutual information in the island rule, see., e.g., [55].
Choosing $\Sigma^\prime$ to satisfy $(I \cup \Sigma^\prime)^c = \emptyset$ (hence $S(I \cup \Sigma^\prime) = S((I \cup \Sigma^\prime)^c) = 0$), we obtain the relation $S_{\text{mat}}(\Sigma^\prime) = S_{\text{mat}}(\Sigma^\prime) = S_{\text{mat}}(I)$, resulting in $I(I, \Sigma') = 2S_{\text{mat}}(I)$. Then above two inequalities give

$$I(I, \Sigma') = 2S_{\text{mat}}(I) \geq I(I, \Sigma') > S_{\text{gen}}(I) = S_{\text{mat}}(I) + \frac{\text{Area}(\partial I)}{4G},$$

from which condition 1

$$S_{\text{mat}}(I) > \frac{\text{Area}(\partial I)}{4G}$$

is obtained.

### 4.1.2 Condition 2: quantum normality of $I$

Consider the deformation of $\partial I$ in the lightlike direction, $X^\mu \rightarrow X^\mu + \lambda^\mu \mu$ with $k^2 = 0$. The extremality condition $dS_{\text{gen}}(I \cup \Sigma)/d\lambda = 0$ in this case reads

$$\frac{d}{d\lambda} S_{\text{gen}}(I) = -d(I(I, \Sigma)), \quad \text{(57)}$$

where $dS_{\text{mat}}(\Sigma)/d\lambda = 0$ (since $\Sigma$ is not affected by the deformation of $\partial I$) is used. Now suppose $\lambda \equiv \lambda_+$ is chosen such that the deformed $I$ contains $I$ before deformation: $I(\lambda_+) \subset I(\lambda_+ + d\lambda)$. From the strong subadditivity (48) with $A = I(\lambda) \cup \Sigma$ and $A' = I(\lambda + d\lambda)$ (hence $A \cup A' = I(\lambda + d\lambda) \cup \Sigma$ and $A \cap A' = I(\lambda)$) we obtain $dI(I, \Sigma)/d\lambda_+ \geq 0$, which is equivalent to $dS_{\text{gen}}(I)/d\lambda_+ \geq 0$ by (57). We can do the same calculation by choosing $\lambda \equiv \lambda_-$ such that the deformed $I$ is contained in $I$ before deformation, $I(\lambda_+ + d\lambda) \subset I(\lambda_0)$, which results in $dS_{\text{gen}}(I)/d\lambda_- \leq 0$. In summary, the quantum normality condition of $I$ reads

$$\pm \frac{d}{d\lambda_{\pm}} S_{\text{gen}}(I) \geq 0. \quad \text{(58)}$$

### 4.1.3 Condition 3: quantum normality of $G = (I \cup \Sigma')^c$

From the strong subadditivity with $A = I(\lambda_+ + d\lambda) \cup \Sigma$ and $A' = I(\lambda_+) \cup \Sigma'$ (hence $A \cup A' = I(\lambda_+ + d\lambda) \cup \Sigma'$ and $A \cap A' = I(\lambda_+) \cup \Sigma$) we obtain

$$\frac{d}{d\lambda_+} S_{\text{mat}}(I \cup \Sigma') \leq \frac{d}{d\lambda_+} S_{\text{mat}}(I \cup \Sigma). \quad \text{(59)}$$

Using (57), one finds

$$\frac{d}{d\lambda_+} S_{\text{gen}}(I) = \frac{d}{d\lambda_+} I(I, \Sigma) = \frac{d}{d\lambda_+} (S_{\text{mat}}(I) - S_{\text{mat}}(I \cup \Sigma')) \leq \frac{d}{d\lambda_+} (S_{\text{mat}}(I) - S_{\text{mat}}(I \cup \Sigma')) \leq \frac{d}{d\lambda_+} (S_{\text{mat}}(I) - S_{\text{mat}}(G)). \quad \text{(60)}$$

where for the last equality we use the fact that $I$ and $G$ share the boundary. Together with the result for $\lambda_-$ under the same step, the quantum normality condition of $G$ is written as

$$\pm \frac{d}{d\lambda_{\pm}} S_{\text{gen}}(G) \leq 0. \quad \text{(62)}$$

We note that condition 3 must be satisfied by any possible $G$. Thus, even if we find $G$ satisfying condition 3, it does not tell us the island is allowed if we can also find another $G$ violating condition 3.

### 4.2 Island in inflationary quasi-dS space

We now show that while the increase of $c$ in time gives rise to the information paradox in the inflationary quasi-dS space, it makes more difficult for the background to have the island satisfying three conditions listed in Sect. 4.1. For this purpose, we keep terms up to $O(c^{1/2})$ only, by treating $H$ as a constant but taking $(dc/d\tau)/c = \sqrt{2\epsilon_H}H$ into account in the analysis when we consider region I, the static patch. In region II, since $r$ instead of $\tau$ is timelike and we will consider the island on which $r$ is constant, it is convenient to consider $dc/dr$. Comparing with $c(\tau)$ as a function of the flat time coordinate $\tau$, we have

$$dc/dr = [(Hr)/(H^2r^2 - 1)](dc/d\tau)^7$$

so when we consider region II, we replace $dc/d\tau$ by $dc/dr = [(Hr)/(H^2r^2 - 1)](\sqrt{2\epsilon_H}Hc)$. Without the loss of generality, we consider two possible types of island:

- **Type 1**: The island entirely belongs to the region beyond the horizon.
- **Type 2**: While the most part of the island belongs to the region beyond the horizon, the island extends to region I as well. That is, one endpoint of the island is located in region I.

The type 2 island appears because the part of region I just inside the horizon can belong to the quantum system in the central dogma. Whereas the island in this case crosses the horizon, the coordinate singularity in the static coordinates, it does not matter as the radiation entropy depends only on

$$\frac{d}{dr} = \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} = \frac{Hr}{H^2r^2 - 1} \frac{\partial}{\partial r} - \frac{e^{-Hr}}{H^2r^2 - 1} \frac{\partial}{\partial \rho}. \quad \text{(63)}$$
the location of endpoints and the coordinate singularity is not a real singularity.

Before moving onto details, we note that the null directions parametrized by \( \lambda_{\pm} \) are well described by the Kruskal–Szekeres null coordinates. From the tortoise coordinate defined by \( dr_* = dr/(1 - H^2 r^2) \), or

\[
\begin{cases}
    \frac{1}{2\pi} \log \left[ \frac{1 + r_H}{1 - r_H} \right] & \text{Region I} \\
    \frac{1}{2\pi} \log \left[ \frac{r_H + 1}{r_H - 1} \right] & \text{Region II}
\end{cases}
\]

the Kruskal–Szekeres null coordinates are defined as

\[
U = \left. \frac{1}{H} e^{H(t - r_*)} \right|_{t=1} = \left. \frac{1}{H} e^{H t} \right|_{t=1} \sqrt{1 - H r}, \\
V = \left. -\frac{1}{H} e^{-H(t + r_*)} \right|_{t=1} = \left. -\frac{1}{H} e^{-H t} \right|_{t=1} \sqrt{1 + H r},
\]

in region I, and

\[
U = \left. \frac{1}{H} e^{H(t - r_*)} \right|_{t=1} = \left. \frac{1}{H} e^{H t} \right|_{t=1} \sqrt{\frac{H r - 1}{H r + 1}}, \\
V = \left. -\frac{1}{H} e^{-H(t + r_*)} \right|_{t=1} = \left. -\frac{1}{H} e^{-H t} \right|_{t=1} \sqrt{\frac{H r - 1}{H r + 1}},
\]

in region II, respectively. They give

\[
\begin{align*}
\partial_U &= \frac{e^{-H t}}{2} \sqrt{\frac{1 + H r}{1 - H r}} \left( \partial_t - (1 - H^2 r^2) \partial_r \right), \\
\partial_V &= -\frac{e^{-H t}}{2} \sqrt{\frac{1 + H r}{1 - H r}} \left( \partial_t + (1 - H^2 r^2) \partial_r \right),
\end{align*}
\]

in region I and

\[
\begin{align*}
\partial_U &= \frac{e^{H t}}{2} \sqrt{\frac{H r + 1}{H r - 1}} \left( \partial_t + (H^2 r^2 - 1) \partial_r \right), \\
\partial_V &= -\frac{e^{H t}}{2} \sqrt{\frac{H r + 1}{H r - 1}} \left( \partial_t - (H^2 r^2 - 1) \partial_r \right),
\end{align*}
\]

in region II.

4.2.1 Type 1 island

We consider the island on the surface \( r_i = \text{(constant)} \) and \( t \in [-t_i, t_i] \) in region II as depicted in Fig. 4. The radiation entropy and the geometric entropy of the island in this case are given by

\[
S_{\text{mat}}(I) = \frac{c}{3} \log \left[ \frac{2}{e^{\frac{r_i}{G(3)}} H^2} \times \left( 1 - H^2 r_i^2 + (H^2 r_i^2 - 1) \cosh[2H t_i] \right) \right],
\]

\[
S_{\text{geo}}(I) = 2 \times \frac{\pi r_i}{2G(3)} + \frac{c}{3},
\]

respectively, the sum of which form the generalized entropy,

\[
S_{\text{gen}}(I) = S_{\text{mat}}(I) + S_{\text{geo}}(I).
\]

Then three conditions can be written as following inequalities:

- **Condition 1:** This is simply given by

\[
S_{\text{mat}}(I) > S_{\text{geo}}(I).
\]

- **Condition 2:** For positive \( t_i \), type 1 island \( I \) expands (shrinks) as \( \partial I \) moves in the direction of \( U(V) \), indicating \( d/d\lambda_+ = \partial/\partial U \) and \( d/d\lambda_- = \partial/\partial V \). Using (68), the quantum normality condition of \( I \) is written as

\[
\begin{align*}
\partial_U S_{\text{gen}}(I) &\geq 0 \Rightarrow \left( \partial_t + (H^2 r_i^2 - 1) \partial_r \right) S_{\text{gen}}(I) \geq 0 \\
&\Rightarrow (H^2 r_i^2 - 1) \left[ \frac{3\pi}{e^{G(3)} H} + \frac{1}{cH} \frac{dc}{dr_i} \right] + \cosh[H t_i] + H r_i \geq 0, \\
\partial_V S_{\text{gen}}(I) &\leq 0 \Rightarrow -\left( \partial_t - (H^2 r_i^2 - 1) \partial_r \right) S_{\text{gen}}(I) \leq 0 \\
&\Rightarrow -(H^2 r_i^2 - 1) \left[ \frac{3\pi}{cG(3) H} + \frac{1}{cH} \frac{dc}{dr_i} \right] + \cosh[H t_i] - H r_i \geq 0.
\end{align*}
\]

- **Condition 3:** When we take \( \Sigma' \) to be region I, the region \( G \) can be chosen to be the subregion of the \( r_i = \text{(constant)} \) surface in region II complement to \( I \) (the thick red dashed line in Fig. 4). Then \( G \) consists of two disconnected regions, each of which shares the boundary with \( I \). The generalized entropy of \( G \) is given by

\[
S_{\text{gen}}(G) = \frac{\pi r_i}{G(3)} + \frac{c}{3} + S_{\text{mat}}(G)
\]

\[
= \frac{\pi r_i}{G(3)} + \frac{c}{3} + \frac{c}{3} \log \left[ \frac{2}{e^{\frac{r_i}{G(3)}} H^2} \left( 1 - H^2 r_i^2 \right) \right] + \log \left[ \frac{2}{e^{\frac{r_i}{G(3)}} H^2} \right]
\]

Fig. 4 An example of type 1 island \( I \) (thick blue line) and region \( G \) (think red dashed line). The directions of the Kruskal–Szekeres coordinates \( U \) and \( V \) are also shown.
\[+(H^2 r_i^2 - 1) \cosh[H(T_0 - t_i)]\],  \quad (74)\]

where \(T_0\) is taken to be infinity, from which the quantum normality condition of \(G\) is written as

\[
\partial_U S_{\text{gen}}(G) \leq 0 \\
\implies (H^2 r_i^2 - 1) \left[ \frac{3\pi}{cG^3 H} + \frac{1}{cHdri} \right] + \frac{3dc/dr_i}{c^2 H} S_{\text{mat}}(G) \\
- \coth \left[ \frac{H}{2}(T_0 - t_i) \right] + 2Hr_i \leq 0,  \quad (75)\]

\[
\partial_V S_{\text{gen}}(G) \geq 0 \\
\implies -(H^2 r_i^2 - 1) \left[ \frac{3\pi}{cG^3 H} + \frac{1}{cHdri} \right] + \frac{3dc/dr_i}{c^2 H} S_{\text{mat}}(G) \\
- \coth \left[ \frac{H}{2}(T_0 - t_i) \right] - 2Hr_i \leq 0.  \quad (76)\]

We first consider the \(O(e_H^{1/2})\) effects only by setting the \(O(e_H^{1/2})\) correction \((dc/dr_i)/(cH)\) zero. Condition 1 is easily satisfied for \(Hr_i \gg 1\). The region \(Hr_i \approx O(1)\) is allowed by condition 1 when \(cG^3 H/(6\pi) > 1\) but this is not an appropriate parameter value for \(dc/dr_i = 0\) as \(S_{\text{mat}}(\Sigma)\) in this case can be larger than \(S_{\text{AS}}\), i.e., \((c/6) \log[2/(c^2 U V H^2)] > \pi/(2G^3 H)\). We allow the case of \(cG^3 H/(6\pi) > 1\) as a consequence of the increase of \(c\) by nonzero \(dc/dr_i\) only, in which the information paradox arises. For condition 2, \(\partial_U S_{\text{gen}}(I) \geq 0\), or equivalently,

\[
Hr_i \geq -\left( \frac{cG^3 H}{6\pi} \right) \\
+ \sqrt{1 - 2\left( \frac{cG^3 H}{6\pi} \right) \coth(Ht_i) + \left( \frac{cG^3 H}{6\pi} \right)^2},  \quad (77)\]

is approximated as

\[
Hr_i \gtrsim -\left( \frac{cG^3 H}{6\pi} \right) + 1 - \left( \frac{cG^3 H}{6\pi} \right)  \quad (78)\]

for \(Ht_i \gg 1\) (\(\coth(Ht_i) \simeq 1\)), which is trivially satisfied in the region \(Hr_i > 1\) regardless of the size of the ‘strength of gravitation’ \(cG^3 H/(6\pi)\). On the other hand, \(\partial_V S_{\text{gen}}(I) \leq 0\) is written as

\[
Hr_i \leq \left[ -\left( \frac{cG^3 H}{6\pi} \right) + \left( \frac{3\pi}{cG^3 H} \right) \right] 1 + 2\left( \frac{cG^3 H}{6\pi} \right) \coth(Ht_i) + \left( \frac{cG^3 H}{6\pi} \right)^2,  \quad (79)\]

which is approximated as

\[
Hr \lesssim -\left( \frac{cG^3 H}{6\pi} \right) + 1 + \left( \frac{cG^3 H}{6\pi} \right) = 1  \quad (80)\]

for \(Ht_i \gg 1\), excluding the region \(Hr_i > 1\) we are considering. Finally, from \(\coth(H/2)(T_0 - t_i) \rightarrow 1\) as \(T_0 \rightarrow \infty\), condition 3 reads

\[
Hr_i \leq -\left( \frac{cG^3 H}{3\pi} \right) + \sqrt{1 + \frac{cG^3 H}{3\pi} + \left( \frac{cG^3 H}{3\pi} \right)^2},  \quad (81)\]

Since RHS of both inequalities are smaller than 1, the first one excludes while the second one allows the whole region \(Hr_i > 1\). Our analysis so far shows that the region satisfying three conditions simultaneously does not exist when \(c\) is constant thus type 1 island is not allowed in perfect dS space. In the left panel of Fig. 5 we show our conclusion explicitly for specific choice of parameters.

Taking nonzero \((dc/dr_i)/(cH) \sim O(e_H^{1/2})\) into account makes the situation even worse, as can be found in the right panel of Fig. 5.\(^8\) For \(Hr_i > 1\), both \(S_{\text{mat}}(I)\) and \(S_{\text{mat}}(G)\) are positive unless \(Hr_i\) is very close to 1. In this case, the addition of positive term \((H^2 r_i^2 - 1)((3cH)^{-1}dc/dr_i + (dc/dr_i)c^{-1}S_{\text{mat}}(I))\) to \(\partial_U S_{\text{gen}}(I)\) in condition 2 and \((H^2 r_i^2 - 1)((3cH)^{-1}dc/dr_i + (dc/dr_i)c^{-1}S_{\text{mat}}(G))\) to \(\partial_V S_{\text{gen}}(I)\) in condition 3 make the inequalities \(\partial_U S_{\text{gen}}(I) \geq 0\) and \(\partial_V S_{\text{gen}}(I) \geq 0\) easier but the inequalities \(\partial_U S_{\text{gen}}(I) \leq 0\) and \(\partial_V S_{\text{gen}}(I) \leq 0\) more difficult to be satisfied. This also can be seen by noticing that RHS of (79) coming from \(\partial_V S_{\text{gen}}(I) \leq 0\) in condition 2 is larger than 1 (since \(\coth(Ht_i) > 1\), and decreases as \(3\pi/c(G^3 H)\) becomes larger. In the presence of positive \(dc/dr_i,(3\pi)/(cG^3 H)\) in condition 2 is modified as

\[
\frac{3\pi}{cG^3 H} + \frac{1}{cHdri} \frac{3dc/dr_i}{c^2 H} S_{\text{mat}}(I),  \quad (82)\]

so the addition of positive term \((cH)^{-1}dc/dr_i + [3dc/dr_i]/(c^2 H)S_{\text{mat}}(I)\) may be regarded as the increase of \(3\pi/c(G^3 H)\) effectively. This makes RHS of (79) smaller, which means that the inequality becomes more restrictive. On the

\(^8\) We note that while \(S_{\text{gen}}\) contains a term \(c/6\) reflecting the trace anomaly, it does not play the crucial role as it is suppressed compared to \(S_{\text{mat}}\) as the latter has a logarithmic enhancement.
other hand, all the RHS of other inequalities (77) and (81) get larger as $(3\pi)/(cG^{(3)}H)$ increases. However, observing changes of these bounds in the presence of the positive $dc/dr_i$ is not meaningful since all these values are smaller than 1, while we are considering $Hr_i$ larger than 1.

The increase of $c$ driven by nonzero $dc/dr_i$ leads to the information paradox as $S_{\text{mat}}(\Sigma)$ becomes larger than $S_{\text{DS}}$. In this case, condition 1 is satisfied even in the region $Hr_i \lesssim 1$, which may allow the overlap with the region satisfying $\partial_U S_{\text{gen}}(I) \leq 0$. Indeed, since $(cG^{(3)}H)/(6\pi r)$ is no longer much less than 1, $\partial_U S_{\text{gen}}(I) \leq 0$ in condition 2 reads

$$Hr_i \lesssim \coth[Ht_i] - (H^2r_i^2 - 1) \left( \frac{1}{cH \, dr_i} + \frac{3dc/dr_i}{c^2H} S_{\text{mat}}(I) \right).$$

While this can be satisfied in some part of the region $Hr_i > 1$ as $\coth[Ht_i]$ is larger than 1, the negative term containing $dc/dr_i$ still makes the inequality restrictive even if $(3\pi)/(cG^{(3)}H)$ is suppressed. Moreover, $\partial_U S_{\text{gen}}(G) \leq 0$ in condition 3 becomes

$$Hr_i \lesssim \frac{1}{2} - (H^2r_i^2 - 1) \left( \frac{1}{2cH \, dr_i} + \frac{3dc/dr_i}{2c^2H} S_{\text{mat}}(G) \right),$$

in which RHS is evidently less than 1, excluding the whole region $Hr_i > 1$. This shows that type 1 island does not exist in the presence of the information paradox.

---

**Fig. 5** Regions satisfying three conditions. We set $cG^{(3)}H/(6\pi) = 0.09$ and $G^{(3)}H = \epsilon U/V = 0.1$. (Left): Constant central charge $(dc/dr_i = 0)$ case. Regions allowed by condition 1 and $\partial_V S_{\text{gen}}(I) \leq 0$ in condition 2 are colored in blue and orange, respectively. They do not overlap. While $\partial_V S_{\text{gen}}(G) \leq 0$ in condition 3 is not satisfied in any region, other conditions are satisfied in the whole region $Hr_i > 1$. (Right): Evolving central charge with $(dc/dr_i)/(cH) = 0.5[(Hr_i)/(H^2r_i^2 - 1)]$. While condition 1 is unchanged and $\partial_V S_{\text{gen}}(I) \geq 0$ in condition 2 still holds in whole region $Hr_i > 1$, the region satisfying $\partial_V S_{\text{gen}}(I) \leq 0$ in condition 2 shrinks. Moreover, $\partial_U S_{\text{gen}}(G) \leq 0$ in condition 3 is not satisfied in any region.

---

**Fig. 6** The choice of Cauchy slice, the union of $\Sigma$, $I$, and the red dashed line

We can also understand the nonexistence of type 1 island in a more direct way. Assuming the quantum state on the Cauchy slice to be pure, $S_{\text{mat}}(I \cup \Sigma)$ is identified with $S_{\text{mat}}((I \cup \Sigma)^c)$, the matter entropy on the red dashed line in Fig. 6. Adding the boundary area to this, we obtain

$$S_{\text{gen}}(I \cup \Sigma) = \frac{\pi r}{G^{(3)}} + 2\frac{\pi r_i}{G^{(3)}} + c + \frac{c}{6} \log \left[ \frac{2}{\epsilon V H^2} \left( 1 - H^2 r_i \right) \sqrt{(1 - H^2 r_i^2)(H^2r_i^2 - 1) \sinh[H(t - t_1)]} \right]$$
4.2.2 Type 2 island

In order to collect the radiation in (quasi-)dS space, we choose the hypersurface \( \Sigma \) in the part of region I in which gravity is treated as a nondynamical background. The quantum gravity region extends over the rest part of region I as well as the region beyond the horizon. Since the island can exist in any part of quantum gravity region, we can consider type 2 island as depicted in Fig. 7: whereas the most part of the island belongs to region II, the island also extends to region I and III. Let the static coordinates of two endpoints are given by \((t_i, r_i)\) (in region I) and \((-t_i, r_i)\) (in region III), respectively. Here the right end of the island in region III is the reflection of the left end of the island in region I with respect to the line \( U = V \) in Fig. 7, \( X^0,1 \to X^0,1 \) and \( X^2 \to -X^2 \), or equivalently, \( \sigma \to \sigma \) and \( \theta \to \pi - \theta \) in the conformal coordinates. Thus, the embeddings in three-dimensional Minkowski space are given by

\[
\begin{align*}
X^0_R &= H^{-1} \sqrt{1 - H^2 r_i^2 \sinh(H t_i)}, \quad X^1_R = r_i, \\
X^0_L &= H^{-1} \sqrt{1 - H^2 r_i^2 \cosh(H t_i)},
\end{align*}
\tag{87}
\]

for the right end of the island, and

\[
\begin{align*}
X^0_L &= H^{-1} \sqrt{1 - H^2 r_i^2 \sinh(H t_i)}, \quad X^1_L = r_i, \\
X^0_R &= H^{-1} \sqrt{1 - H^2 r_i^2 \cosh(H t_i)},
\end{align*}
\tag{88}
\]

for the left end of the island, respectively. From this, the radiation entropy of the island is written as

\[
S_{\text{mat}}(I) = \frac{c}{6} \log \left[ \frac{2}{\epsilon_U H^2} (1 - H^2 X_L \cdot X_R) \right]
= \frac{c}{6} \log \left[ \frac{2}{\epsilon_U H^2} \right. \\
\times \left. \left(1 - H^2 r_i^2 + (1 - H^2 r_i^2 \cosh[2 H t_i]) \right) \right].
\tag{89}
\]

Adding this to the geometric entropy,

\[
S_{\text{geo}}(I) = 2 \times \frac{\pi r_i}{2G(3)} + \frac{c}{3}
\tag{90}
\]

we obtain the generalized entropy of the island,

\[
S_{\text{gen}}(I) = S_{\text{mat}}(I) + S_{\text{geo}}(I).
\tag{91}
\]

- **Condition 1:** This is simply given by

\[
S_{\text{mat}}(I) > S_{\text{geo}}(I).
\tag{92}
\]

- **Condition 2:** Type 2 island \( I \) expands (shrinks) as \( \partial I \) moves in the direction of \( U(V) \), indicating \( d/d\lambda_+ = \partial/\partial U \) and \( d/d\lambda_- = \partial/\partial V \). From (67), we obtain the quantum normality condition on \( I \),

\[
\begin{align*}
\partial_U S_{\text{gen}}(I) &\geq 0 \implies (\partial_{t_i} - (1 - H^2 r_i^2) \partial_{r_i}) S_{\text{gen}}(I) \geq 0, \\
\implies &\frac{1}{c H} \frac{dc}{dt_i} + \frac{3dc/dt_i}{c^2 H} S_{\text{mat}}(I) \\
&- (1 - H^2 r_i^2) \frac{3\pi}{c G(3) H} + \tan[H t_i] + H r_i \geq 0 \tag{93}
\end{align*}
\]

\[
\begin{align*}
\partial_V S_{\text{gen}}(I) &\leq 0 \implies (\partial_{t_i} + (1 - H^2 r_i^2) \partial_{r_i}) S_{\text{gen}}(I) \leq 0, \\
\implies &\frac{1}{c H} \frac{dc}{dt_i} + \frac{3dc/dt_i}{c^2 H} S_{\text{mat}}(I) \\
&+ (1 - H^2 r_i^2) \frac{3\pi}{c G(3) H} + \tan[H t_i] - H r_i \leq 0. \tag{94}
\end{align*}
\]

- **Condition 3:** We consider the following choice of \( G \) as shown in Fig. 7. First, in order to avoid the overlap with \( \Sigma \) extending over the most part of region I, the part of \( G \) in region I occupies only a narrow interval of length \( \epsilon_{UV} \) thus the radiation entropy in this part is negligible.
While one of boundaries of this part is not shared by \( I \), it is not relevant to our discussion since we are interested in the variation of \( G \) under the deformation of boundary shared by \( J \). Meanwhile, another part of \( G \) lies on \(-t_i = \text{(constant)}\) surface in region III such that the endpoints are given by \((-t_i, 0)\) and \((-t_i, r_i)\). Then up to the addition of irrelevant term, the generalized entropy of \( G \) is given by

\[
S_{\text{gen}}(G) = \frac{c}{6} \log \left[ \frac{2}{c^{\frac{7}{2}} V H^2} \left( 1 - \sqrt{1 - H^2 r_i^2} \right) \right] + S_{\text{geo}}(I). \tag{95}
\]

Then the quantum normality condition on \( G \) is written as

\[
\partial_U S_{\text{gen}}(G) \leq 0 \quad \Rightarrow 2 \frac{dc}{cH \, dt} + \frac{6dc/dt}{\ell^2 H} S_{\text{mat}}(G) - (1 - H^2 r_i^2) \frac{6\pi}{c G^{(3)} H} - \frac{\sqrt{1 - H^2 r_i^2}}{1 - \sqrt{1 - H^2 r_i^2}} H r_i \leq 0,
\]

\[
\partial_V S_{\text{gen}}(I) \geq 0 \quad \Rightarrow 2 \frac{dc}{cH \, dt} + \frac{6dc/dt}{\ell^2 H} S_{\text{mat}}(G) + (1 - H^2 r_i^2) \frac{6\pi}{c G^{(3)} H} + \frac{\sqrt{1 - H^2 r_i^2}}{1 - \sqrt{1 - H^2 r_i^2}} H r_i \geq 0. \tag{96}
\]

We note that whereas two conditions are equivalent for \( dc/dt_i = 0 \), when \( dc/dt_i > 0 \) the first one gives more stringent bound.

We first consider the \( \mathcal{O}(c^0_H) \) effects, in which \( dc/dt_i \) is neglected. In addition, for the time being, we take \( t_i \) to be positive. In fact, whereas \( t_i \) appears in condition 1 and 2, \( S_{\text{mat}}(I) \) is even in \( t_i \) so condition 1 is just a condition on \( |t_i| \). For condition 2, when \( dc/dt_i \) term is neglected, the only difference between \( \partial_U S_{\text{gen}}(I) \geq 0 \) and \( \partial_V S_{\text{gen}}(I) \leq 0 \) is the sign of \( \text{tanh}(H t_i) \) term, hence \( \partial_U S_{\text{gen}}(I) \geq 0 \) for positive \( t_i \) is equivalent to \( \partial_V S_{\text{gen}}(I) \leq 0 \) for negative \( t_i \) and vice versa. Just like the case of type 1 island, condition 1 is easily satisfied when \( H t_i \gg 1 \). The inequality \( \partial_U S_{\text{gen}}(I) \geq 0 \) in condition 2 reads

\[
H r_i \geq - \frac{c G^{(3)} H}{6\pi} + \sqrt{1 - 2 \left( \frac{c G^{(3)} H}{6\pi} \right) \text{tanh}(H t_i) + \left( \frac{c G^{(3)} H}{6\pi} \right)^2}. \tag{97}
\]

Here \( \text{RHS} \) is smaller than 1 since \( \text{tanh}(H t_i) < 1 \), and for \( H t_i \gg 1 \), the inequality becomes

\[
H r_i \geq - \frac{c G^{(3)} H}{6\pi} + \left| 1 - \frac{c G^{(3)} H}{6\pi} \right|. \tag{98}
\]

For the weak gravity case, \( (c G^{(3)} H)/(6\pi) \ll 1 \), this condition becomes \( H r_i \geq 1 - (c G^{(3)} H)/(3\pi) \) which allows the region close to the horizon. For the strong gravity case, \( (c G^{(3)} H)/(6\pi) \gg 1 \), the condition becomes \( H r_i \gg -1 \), which is trivially satisfied in all region inside the horizon. In addition, the region satisfying condition 1 is extended as well. This will be considered later with nonzero \( dc/dt_i \) effect to discuss the information paradox as \( S_{\text{mat}}(\Sigma) \) in this case is no longer smaller than \( S_{\text{DS}} \). Next, the condition \( \partial_V S_{\text{gen}}(I) \leq 0 \) is written as

\[
H r_i \geq - \frac{c G^{(3)} H}{6\pi} + \sqrt{1 + 2 \left( \frac{c G^{(3)} H}{6\pi} \right) \text{tanh}(H t_i) + \left( \frac{c G^{(3)} H}{6\pi} \right)^2}. \tag{99}
\]

For \( H t_i \gg 1 \), this condition reads

\[
H r_i \geq - \frac{c G^{(3)} H}{6\pi} + \left| 1 + \left( \frac{c G^{(3)} H}{6\pi} \right) \right| \simeq 1, \tag{100}
\]

which allows the region around the horizon only. Since two inequalities \( \partial_U S_{\text{gen}}(I) \geq 0 \) and \( \partial_V S_{\text{gen}}(I) \leq 0 \) can be interchanged by \( t_i \to -t_i \), the region allowed by them can overlap for \( t_i \simeq -t_i \), i.e., \( H t_i \simeq 0 \). For the weak coupling case \( (c G^{(3)} H)/(6\pi) \ll 1 \), the value of \( H r_i \) satisfying two inequalities simultaneously for \( H t_i \simeq 0 \) is close to 1, which is easily excluded by condition 1. More concretely, as \( \cos(2 H t_i) \simeq 1 \) and \(- H^2 r_i^2 \ll 1 \), \( S_{\text{mat}}(I) \) becomes much smaller than \( S_{\text{DS}}(\Sigma) \simeq (c/6) \log[2/(c G^{(3)} H^2)] \) which is required to be bounded by \( S_{\text{DS}} = \pi/(2 G^{(3)} H) \). But at the same time, since \( r_i \) is close to \( H^{-1} \), \( S_{\text{geo}}(I) \) becomes close to \( S_{\text{DS}} \), leading to \( S_{\text{mat}}(I) < S_{\text{geo}}(I) \), contradict to condition 1. Indeed, when \( H t_i \simeq 0 \), the value of \( H r_i \) satisfying two inequalities in condition 2 is given by

\[
H r_i \simeq - \frac{c G^{(3)} H}{6\pi} + \sqrt{1 + \left( \frac{c G^{(3)} H}{6\pi} \right)^2}. \tag{101}
\]
Then for the weak gravity case \( (c G^{(3)} H)/(6\pi) \ll 1 \), we have

\[
S_{\text{mat}}(I) \simeq \frac{c}{6} \log \left[ \frac{8}{\epsilon_{UV}^2 H^2} \frac{c G^{(3)} H}{6\pi} \right] \\
\simeq \frac{c}{6} \log \left[ \frac{8c}{(6\pi)^2} \frac{6\pi}{G^{(3)} H} \right],
\]

where we use \( \epsilon_{UV} \approx G^{(3)} \). Comparing this with \( S_{\text{geo}}(I) \simeq \pi (G^{(3)} H)^{-1} \), we find that \( S_{\text{geo}}(I) \) is larger than \( S_{\text{mat}}(I) \) since they are linear and logarithmic in large value \((6\pi)/(c G^{(3)} H)\), respectively. This is not consistent with condition 1. Finally, one finds that for \( dc/dt_i = 0 \), two inequalities in condition 3 give the same condition and trivially satisfied. Features of three conditions discussed so far are summarized in the left panel of Fig. 8, showing that the region satisfying all three conditions does not exist.

When we take the nonzero, positive \( dc/dt_i \) into account, as shown in the right panel of Fig. 8, conditions \( \partial_U S_{\text{gen}}(G) \leq 0 \) and \( \partial_V S_{\text{gen}}(I) \leq 0 \) become more difficult to be satisfied, just like the case of island 1. In particular, the region allowed by \( \partial_U S_{\text{gen}}(G) \leq 0 \) is more restricted to \( H_{t_i} \approx 0 \) and \( H_r \approx 1 \), in which condition 1 is violated as explained above.

As \( c \) increases in \( t \), \( (c G^{(3)} H)/(6\pi) \) gets close to \( O(1) \) such that \( S_{\text{mat}}(\Sigma) \gtrsim S_{\text{IS}} \) is satisfied and the information paradox arises. Then the region \( H_{t_i} \lesssim 1 \) begins to be allowed by condition 1. Moreover, the only negative term \(-1 + H^2 r_i^2 \) in LHS of the first inequality in (94) is no longer enhanced so \( \partial_V S_{\text{mat}}(I) \gtrsim 0 \) is easier to be satisfied. That is, when \( (3\pi)/(c G^{(3)} H) \) is suppressed by the exponentially large \( c \), the inequality reads

\[
H_{r_i} \gtrsim -\tanh[H_{t_i}] - \frac{1}{c H} \frac{dc}{dt_i} - 3\frac{dc}{dt_i} S_{\text{mat}}(I),
\]

which is trivially satisfied as RHS is negative. At the same time, \( \partial_V S_{\text{mat}}(I) \leq 0 \) in condition 2 becomes

\[
H_{r_i} \gtrsim \tanh[H_{t_i}] + \frac{1}{c H} \frac{dc}{dt_i} + 3\frac{dc}{dt_i} S_{\text{mat}}(I),
\]

which prefers the region \( H_{t_i} < 1 \). If \( dc/dt_i \) were negligibly small, since the region \( H_{t_i} \lesssim 1 \) is now allowed by condition 1, we can find the region satisfying condition 1 and two inequalities in condition 2 simultaneously. Moreover, two inequalities in condition 3 are equivalent and trivially satisfied even if \( (c G^{(3)} H)/(6\pi) \gtrsim 1 \), so the island seems to be allowed as shown in the left panel of Fig. 9. However, we need to note here that \( t_i \) for nonzero \( dc/dt_i \) is interpreted as a time interval taken by the central charge to have the value \( c(t_i) \) through the time evolution. So the statement that \( c \) is so large to satisfy \( S_{\text{mat}}(\Sigma) \gtrsim S_{\text{IS}} \) at \( H_{t_i} \approx 0 \) means that \( c \) is given by this large value from beginning, contradict to our setup. As can be inferred from (47), when \( dc/dt_i \) is close to
thus one finds

\[ \begin{align*}
S_{\text{gen}}(\Sigma \cup I) &= \frac{\pi r}{G^{(3)}} + 2 \frac{\pi r_i}{G^{(3)}} + c \\
&\quad + \frac{c}{6} \log \left[ \frac{2}{e_{\text{UV}} H^2 (1 - \sqrt{1 - H^2 r_i^2})} \right].
\end{align*} \tag{105} \]

Since it depends on \( t_i \) through \( c \) satisfying \( dc/dt_i > 0 \), \( dS_{\text{gen}}(\Sigma \cup I)/dt_i > 0 \) thus \( S_{\text{gen}}(\Sigma \cup I) \) is not extremized in \( t_i \) direction. Moreover, from

\[ \begin{align*}
\frac{S_{\text{gen}}(\Sigma \cup I)}{dt_i} &= \frac{2\pi}{G^{(3)}} \\
&\quad + \frac{c}{6} \left[ \frac{H^2 r_i}{\sqrt{1 - H^2 r_i^2} (1 - \sqrt{1 - H^2 r_i^2})} \right] > 0, \tag{106} \end{align*} \]

\( S_{\text{gen}}(\Sigma \cup I) \) is not extremized in \( r_i \) direction. This shows that the island does not exist even if \( c \) is constant in \( t_i \).

From analysis of two types of island so far, we find that when the positive increasing rate of the central charge \( c \) in time is the main reason to the increase of the radiation entropy in time, the information paradox arises but at the same time, \( \partial_t S_{\text{gen}}(I) \leq 0 \) in condition 2 and \( \partial_U S_{\text{gen}}(G) \leq 0 \) in condition 3 become more restrictive. We also check that the islands we considered cannot extremize \( S_{\text{gen}}(\Sigma \cup I) \) in a direct way. Even if we may find the parameter region allowing the island, we expect that this is a result of fine-tuning.

### 5 Conclusion

In this article, we investigate when quasi-dS space realizing cosmological inflation can have the information paradox and whether the island can exist without deforming the background far away from dS space, emphasizing the connection...
between the thermal equilibrium and the dS isometries. For this purpose, we restrict the region collecting the radiation to the static patch. Moreover, the radiation entropy is written in terms of the static coordinates such that it is independent of the static time coordinate in the perfect dS background, reflecting the equilibrium between the radiation and the background.

Our analysis shows that the information paradox in static patch can arise when the time evolution of the radiation entropy \( S_{\text{mat}}(\Sigma) \) is dominated by the increase of the central charge. Since the central charge can be interpreted as the number of degrees of freedom, this is a reminiscent of the entropy argument supporting the dS swampland conjecture which states that the rapid descent of UV degrees of freedom violates the covariant entropy bound. If the island can exist within the (quasi-)dS background, the radiation entropy would not increase any longer, so the entropy argument does not work.

However, the dominance of \( dS_{\text{mat}}(\Sigma)/dt \) by the increase of the central charge in time, nothing more than the origin of the information paradox, also prevents \( S_{\text{gen}}(I) \) and \( S_{\text{gen}}(G) \) from decreasing in one of the lightlike future direction. Then the island is difficult to exist in the quasi-dS background. Therefore, we can conclude that the information paradox is not resolved within the (quasi-)dS space unless the parameters are fine-tuned, supporting the instability of dS space as claimed by the dS swampland conjecture: for the central dogma to hold, spacetime needs to be strongly deformed from dS space.

As a possible way of strong deformation of the background, we can consider the backreaction of information collected by an observer, which breaks the dS isometries. If the huge amount of collected information is concentrated on the small region, black hole may be formed. Then as considered in literatures discussing the island in dS space \([11–13,17–19]\), the island is allowed in the region inside the black hole horizon. On the other hand, since the modulus responsible for the positive vacuum energy has a slow-roll potential, spacetime eventually becomes Minkowski space, which the observer does not find the information paradox. We note here that if \( \epsilon_H \) is very tiny (less than \( H^2G \) in four-dimensional case), quantum fluctuation can prevent the modulus from slow-roll so some region of spacetime may keep inflating, realizing the eternal inflation \([56–60]\) (For a review, see, e.g., \([61]\)). Intriguingly, the condition allowing the eternal inflation is equivalent to the condition that black hole is formed from the quantum fluctuation \([62]\) (See also \([63,64]\)). This is not strange since for black hole to be formed in dS space, the quantum fluctuation must be large enough to produce the large mass concentration overwhelming the accelerating expansion of spacetime. While both the black hole formation and the slow roll with a sizeable \( \epsilon_H \) (but still much smaller than 1) can be ways to resolve the information paradox through the deformation of the background far away from dS space, these two have different nature since in four-dimensional spacetime, \( \epsilon_H \) is smaller than \( H^2G \) for the former and larger than \( H^2G \) for the latter, respectively. That is, the unstable quasi-dS space can evolve into two different phases having different thermodynamic properties, depending on the size of the slow-roll parameter.

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is a theoretical study which is not relevant to the experimental data.]

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**Appendix A: Modification of the action for time dependent \( H \)**

Here we find the two-dimensional action for a \( \tau \) dependent \( H \) from the dimensional reduction. Since \( H \) is no longer a constant in this case, \( \Phi \) in \((3)\) is replaced by \( \Phi/H(\tau) \) as the three-dimensional metric \((2)\) depends on \( \Phi \) through the combination \( \Phi/(2\pi H) \). If we restrict our attention to the case of \( g_{\tau\tau} = 0 = g^{\tau\tau} \) we obtain

\[
R^{(3)} = R^{(2)} - \frac{2H}{\Phi} \square^{(2)} \Phi - \frac{2H}{\Phi} g^{\tau\tau} \frac{d}{d\tau} (\epsilon_H \Phi) - \epsilon_H \frac{2H}{\Phi} \frac{1}{\sqrt{-g^{(2)}}} \partial_{\tau} \left( \sqrt{-g^{(2)}} g^{\tau\tau} \Phi \right),
\]

where \( \epsilon_H = -H^{-2}dH/d\tau = dH^{-1}/d\tau \) as usually defined. Combining this with \( \sqrt{-g^{(3)}} = \sqrt{-g^{(2)}} [\Phi/(2\pi H)] \) and integrating over \( \varphi \), we obtain the dimensionless reduction of the Einstein–Hilbert term,

\[
16\pi G^{(3)} S_{EH} = \int d^2x \sqrt{-g^{(2)}} \left( \frac{\Phi}{H} R^{(2)} - \frac{2}{H} \square^{(2)} \Phi - 2g^{\tau\tau} \frac{d}{d\tau} (\epsilon_H \Phi) - \frac{2 \epsilon_H H}{\sqrt{-g^{(2)}}} \partial_{\tau} \left( \sqrt{-g^{(2)}} g^{\tau\tau} \Phi \right) \right).
\]
The first term is what we can find in (4). The second term and the last term can be rewritten as
\[
-2 \frac{2}{H} \sqrt{-g^{(2)} d^2(2)} \Phi = 2 \varepsilon H \sqrt{-g^{(2)} g^{rr} \varepsilon d \Phi}
- \partial_t \left( 2 \frac{H}{\sqrt{-g^{(2)}}} g^{(2)} g^{rr} \Phi \right),
\]
and
\[
-2 \varepsilon H \partial_t (\sqrt{-g^{(2)} g^{rr} \Phi}) = 2 \frac{deH}{dt} \sqrt{-g^{(2)} g^{rr} \Phi}
- \partial_t (2 \varepsilon H \sqrt{-g^{(2)} g^{rr} \Phi}),
\]
respectively. Ignoring the surface term, the addition of these two terms gives
\[
2 \sqrt{-g^{(2)} g^{rr} \partial_t (\varepsilon H \Phi)},
\]
which exactly cancels the third term in \( S_{EH} \). Therefore, the dimensional reduction of the \( S_{EH} \) is simply given by
\[
S_{EH} = \int d^2 x \sqrt{-g^{(2)}} \frac{\Phi}{16 \pi G^{(3)} H} R^{(2)}.
\]
Since \( H \) now varies with respect to \( \tau \), the Newton’s constant \( G^{(3)} H \) is no longer a constant, even though it does not vary much over the time scale we consider. Indeed, since we are interested in \( O(\varepsilon^{1/2}) \) corrections, the change of the Newton’s constant of \( O(\varepsilon H) \) does not affect our analysis. Meanwhile, in the flat coordinates, the dilaton can be approximated as \( \Phi \approx 2 \pi H \rho e^{H \tau} \) if \( \varepsilon H \ll 1 \). Then it is suggestive to introduce some fiducial constant \( H_0 \) such that we take a constant \( G^{(3)} H_0 \) to be a Newton’s constant \( G \) in which case the combination \( \Phi/(G^{(3)} H) \) becomes \( (H_0/H) \Phi/G \). Then \( (H_0/H) \Phi \) with \( \Phi = 2 \pi H \rho a(\tau) \) depends on \( \tau \) only through the scale factor \( a(\tau) \), which is \( e^{H \tau} \) for a constant \( H \). Since the cosmological constant term is not affected by the \( \tau \) dependence of \( H \), the bulk action is the same form as that given by (4), which is just a replacement of a constant \( H \) by a \( \tau \) dependent \( H(\tau) \). Since the hypersurface orthogonal to direction has a rotation as an isometry, the FRW metric with the scale factor \( a(\tau) \) satisfying \( H(\tau) = a^{-1} da/d\tau \) is expected to solve the equations of motion.

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