CLASSIFICATION OF FOLIATIONS BY CURVES OF LOW DEGREE ON THE THREE-DIMENSIONAL PROJECTIVE SPACE

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Abstract. We study foliations by curves on the three-dimensional projective space with no isolated singularities, which is equivalent to assuming that the conormal sheaf is locally free. We provide a classification of such foliations by curves up to degree 3, also describing the possible singular schemes. In particular, we prove that foliations by curves of degree 1 or 2 are either contained on a pencil of planes or legendrian, and are given by the complete intersection of two codimension one distributions. We prove that the conormal sheaf of a foliation by curves of degree 3 with reduced singular scheme either splits as a sum of line bundles or is an instanton bundle. For degree larger than 3, we focus on two classes of foliations by curves, namely legendrian foliations and those whose conormal sheaf is a twisted null correlation bundle. We give characterizations of such foliations, describe their singular schemes and their moduli spaces.

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1. Introduction

The qualitative study of polynomial differential equations was initiated in classical works by Poincaré, Darboux, and Painlevé, see for instance [11, 22, 23]. In the modern terminology, these works provided us with study of holomorphic foliations by curves on \( \mathbb{P}^2 \) by analyzing their possible algebraic leaves.
Since then, classifications of codimension one foliations on higher dimensional projective spaces have also been obtained. To be more precise, Jouanolou classified codimension one foliations of degrees 0 and 1 in [18]; Cerveau and Lins Neto showed in [5] that there exist six irreducible components of foliations of degree 2 on projective spaces and in [6] they proved that foliations of degree three are either transversely affine foliations, or are rational pullbacks of foliations on $\mathbb{P}^2$.

Recently, the authors of [4] initiated a systematic study of codimension one holomorphic distributions on $\mathbb{P}^3$, analyzing the properties of their singular schemes and tangent sheaves. In particular, a classification of codimension one distributions of degree at most 2 with locally free tangent sheaves was provided, together with a description of the geometry of certain the moduli space of distributions.

By contrast, classifications for foliations by curves of are not widely known. Note that generic foliations by curves, understood as a twisted holomorphic vector fields, have only isolated singularities, and the challenge is to understand foliations by curves with non-isolated singularities. In this work we are interested on the class of non-generic foliations whose singular schemes are of pure dimension one. We say that such foliations are of local complete intersection type, since they are given by twisted 2-forms which are locally decomposable even along their singular scheme, i.e., the foliation is given locally by the intersection of two codimension one distributions. This also means that the conormal sheaf is locally free, see Lemma 2.1. Therefore, in order to classify locally complete intersection foliations it is sufficient to describe the geometry of their conormal bundles and their singular schemes in the spirit of [4].

Our first goal is to provide a complete classification of locally complete intersection foliations by curves of degree at most 3 on $\mathbb{P}^3$ in terms of the geometric and algebraic invariants of the conormal sheaf (Chern classes, cohomology module) and of the singular curve (degree, genus and the Rao module). Since there are no locally complete intersection foliations by curves of degree 0, our classification starts in degree 1 and 2.

**Main Theorem 1.** Let $\mathcal{F}$ be a foliation by curves on $\mathbb{P}^3$ of degree $d \in \{1, 2\}$. If $\text{Sing}(\mathcal{F})$ is a curve, then its conormal sheaf $N^*_\mathcal{F}$ splits as a sum of line bundles. More precisely, we have that

1. if $d = 1$, then $N^*_\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3)$ and $\text{Sing}(\mathcal{F})$ consists of two skew lines;
2. if $d = 2$, then $N^*_\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3)$ and $\text{Sing}(\mathcal{F})$ is a connected curve of degree 5 and arithmetic genus 1.

In particular, $\mathcal{F}$ is contained on a pencil of planes or legendrian, and is given by the complete intersection of two codimension one distributions.

Our next step is to classify foliations by curves of degree 3, where we find examples of foliations with conormal sheaves that do not split as sum of line bundles. Recall that an instanton bundle on $\mathbb{P}^3$ is a stable rank 2 locally free sheaf $E$ satisfying $h^1(E(-2)) = 0$; $c_2(E)$ is called the charge of $E$. Moreover, $E$ is said to be a ’t Hooft instanton bundle if, $h^0(E(1)) = 1$, and a special ’t Hooft instanton bundle if, in addition, $h^0(E(1)) = 2$, see [16].

**Main Theorem 2.** Let $\mathcal{F}$ be a foliation by curves on $\mathbb{P}^3$ of degree 3. If $\mathcal{F}$ is of local complete intersection type, then one of the following possibilities hold:

1. $N^*_\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-4)$, and $\text{Sing}(\mathcal{F})$ is a connected curve of degree 10 and arithmetic genus 5;
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(2) $N^*_F = \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2}$, and $\text{Sing}(F)$ is a connected curve of degree 9 and arithmetic genus 3;

(3) $N^*_F = E(-3)$, where $E$ is a stable rank 2 locally free sheaf with $c_1(E) = 0$ and $1 \leq c_2(E) \leq 5$; the singular scheme $\text{Sing}(F)$ is a curve of degree $9 - c_2(E)$ and arithmetic genus $p_a(C) = 8 - 3c_2(E)$.

If, in addition, $\text{Sing}(F)$ is reduced, then $1 \leq c_2(E) \leq 4$, $E$ is an instanton bundle (though not a special 't Hooft instanton bundle of charge 3 or a 't Hooft instanton bundle of charge 4), and $\text{Sing}(F)$ is connected if and only if $c_2(E) = 1, 2$.

Conversely, for each $n \in \{1, 2, 3, 4\}$, there is foliation by curves $F$ of degree 3 on $\mathbb{P}^3$ such that $N^*_F(3)$ is an instanton bundle of charge $n$.

One consequence of the our classification is that degree and genus of the singular scheme are not enough to distinguish the possible foliations of local complete intersection type which are not of global complete intersections. The new invariant that comes into play is the cohomology module of the conormal sheaf

$$M_F := H^1(N^*_F) := \bigoplus_{p \in \mathbb{Z}} H^1(N^*_F(p)),$$

regarded as a graded $\mathbb{C}[x_0, x_1, x_2, x_3]$-module.

| $(\text{deg}(C), p_a(C))$ | $c_2(N^*_F(3))$ | $\dim M_F$ | $h^0(\mathcal{O}_C)$ |
|---------------------------|------------------|-------------|---------------------|
| $(8, 5)$                  | 1                | 1           | 1                   |
| $(7, 2)$                  | 2                | 4           | 1                   |
| $(6, -1)$                 | 3                | 8           | 2                   |
| $(5, -4)$                 | 4                | 14          | 5                   |

Table 1. Classification of foliations of degree 3 which are not global complete intersection, with reduced singular scheme.

Regarding the existence part of Main Theorem 2, we prove a somewhat stronger statement: every instanton bundle of charge up to 3 (except for the special 't Hooft instanton bundles of charge 3) arises as the conormal sheaf, up to twist, of a foliation by curves of degree 3; see Propositions 6.12, 6.13 and 6.14 below.

Beyond degree 3, we focus on two particular classes of foliations by curves. First, we consider the so called legendrian foliations; a foliation by curves is called legendrian if it is a sub-distribution of a contact distribution on $\mathbb{P}^3$, see details in Section 7. We prove that such foliations are globally complete intersections, and establish the following characterization.

**Main Theorem 3.** Every legendrian foliation $F$ of degree $d$ is of the form $\omega_0 \wedge \omega$, where $\omega_0$ is a contact form and $\omega \in H^0(\Omega^1_{\mathbb{P}^3}(d + 1))$. In addition, the moduli space of the legendrian foliations of degree $d$ is an irreducible quasi-projective variety of dimension

$$d \cdot \left(\binom{d + 3}{2} - \binom{d + 2}{3}ight) + 4 \quad \text{if} \quad d \geq 2$$

and of dimension 8 if $d = 1$. 
Finally, we consider those locally complete intersection foliations by curves whose conormal sheaf is a twisted null-correlation bundle, which is the simplest rank 2 locally free sheaf on \( \mathbb{P}^3 \) which does not split as a sum of line bundles.

These foliations can also be regarded as the simplest locally complete intersections foliations which are not globally complete intersections from an algebraic point of view. Indeed, one important algebraic invariant of a foliation is the \( \text{Rao module} \) of its singular scheme \( Z \), namely the graded \( \mathbb{C}[x_0, x_1, x_2, x_3] \)-module which is closely related with the graded module \( H^1_*(N^*_F) \) mentioned above. One can prove the conormal sheaf of a foliation by curves splits as a sum of line bundles if and only if the Rao module of singular scheme is 1-dimensional over \( \mathbb{C} \), see [8, Theorem 2]. Our last main result shows that the foliations by curves whose singular scheme has a 2-dimensional Rao module are precisely the ones for which the conormal sheaf is a null correlation bundle, up to twist.

**Main Theorem 4.** Let \( \mathcal{F} \) be a foliation by curves. The conormal sheaf is a twisted null correlation bundle if and only if the singular scheme is a curve with 2-dimensional Rao module. Furthermore, the moduli space of foliations by curves of degree \( 2k + 1 \) \((k \geq 1)\) whose conormal sheaf is a twisted null correlation bundle is an irreducible quasi-projective variety of dimension
\[
8\left( \frac{k + 4}{3} \right) - 2\left( \frac{k + 5}{3} \right) - 3k - 3.
\]

The singular scheme of such foliations is always connected, and also smooth for a generic one.

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## 2. Foliations by curves

Let \( X \) be a nonsingular projective variety of dimension \( n \). Recall that a \textit{codimension} \( r \) \textit{distribution} \( \mathcal{F} \) on \( X \) is given by an exact sequence
\[
\mathcal{F} : 0 \rightarrow T_\mathcal{F} \xrightarrow{\phi} TX \xrightarrow{\pi} N_\mathcal{F} \rightarrow 0,
\]
where \( T_\mathcal{F} \) is a coherent sheaf of rank \( s := n - r \), and \( N_\mathcal{F} \) is a torsion free sheaf. The sheaves \( T_\mathcal{F} \) and \( N_\mathcal{F} \) are called the \textit{tangent} and the \textit{normal} sheaves of \( \mathcal{F} \), respectively. Note that \( T_\mathcal{F} \) must be reflexive [13, Proposition 1.1].

The \textit{singular scheme} of \( \mathcal{F} \) is defined as follows. Taking the maximal exterior power of the dual morphism \( \phi^\vee : \Omega^1_X \rightarrow T^*_\mathcal{F} \) we obtain a morphism \( \Omega^1_X \rightarrow \det(T_\mathcal{F})^* \); the image of such morphism is an ideal sheaf \( I_Z \) of a subscheme \( Z \subset X \), which is called the singular scheme of \( \mathcal{F} \), twisted by \( \det(T_\mathcal{F})^* \).

Finally, we introduce the notion of integrability. A \textit{foliation} is an integrable distribution, which means a distribution whose tangent sheaf is closed under the Lie bracket of vector fields, that is \([\phi(T_\mathcal{F}), \phi(T_\mathcal{F})] \subset \phi(T_\mathcal{F})\).
In this paper we focus on the case \( r = n - 1 \). Clearly, every distribution of codimension \( n - 1 \) is integrable, and it is called a foliation by curves. In addition, \( T_\mathcal{F} \) must be a line bundle on \( X \), which we denote by \( \mathcal{L} \) from now on, while the normal sheaf \( N_\mathcal{F} \) is a torsion-free sheaf of rank \( n - 1 \). Therefore, a foliation by curves is simply given by a nontrivial section \( \phi \in H^0(TX \otimes \mathcal{L}^*) \), whose cokernel is a torsion free sheaf.

Dualizing the sequence (1), we obtain

\[
0 \to N^*_\mathcal{F} \to \Omega^1_X \xrightarrow{\phi^*} \mathcal{L}^* \to \text{Ext}^1(N_\mathcal{F}, \mathcal{O}_X) \to 0,
\]

thus \( \text{Ext}^1(N_\mathcal{F}, \mathcal{O}_X) \simeq \mathcal{O}_Z \otimes \mathcal{L}^* \), where \( Z \) is the singular scheme of \( \mathcal{F} \). In other words, the singular set of \( N_\mathcal{F} \) coincides with the singular locus of \( \mathcal{F} \) as a set, which can also be regarded as the vanishing locus of \( \phi \) as a section in \( H^0(TX \otimes \mathcal{L}^*) \). We also conclude that \( \text{Ext}^p(N_\mathcal{F}, \mathcal{O}_X) = 0 \) for \( p \geq 2 \).

Cutting sequence (2), we obtain the following short exact sequence

\[
0 \to N^*_\mathcal{F} \to \Omega^1_X \xrightarrow{\phi^*} \mathcal{L}^* \to 0,
\]

which will play an important role in this paper; the sheaf \( N^*_\mathcal{F} \) is called the conormal sheaf of the foliation \( \mathcal{F} \).

Conversely, we dualize the sequence in display (3) to obtain

\[
0 \to \mathcal{L} \xrightarrow{\phi} TX \to N^*_\mathcal{F} \to \text{Ext}^1(I_Z, \mathcal{O}_X) \otimes \mathcal{L} \to 0.
\]

Since \( N_\mathcal{F} = \text{coker} \phi \) by definition, we recover the original foliation by curves

\[
0 \to \mathcal{L} \xrightarrow{\phi} TX \to N_\mathcal{F} \to 0,
\]

and conclude that

\[
0 \to N_\mathcal{F} \to N^*_\mathcal{F} \to \text{Ext}^2(O_Z, \mathcal{O}_X) \otimes \mathcal{L} \to 0.
\]

Note that \( Z \) might not be pure dimensional; let \( R \) be the maximal subsheaf of \( \mathcal{O}_Z \) of codimension greater than 2; the quotient \( \mathcal{O}_Z/R \) is the structure sheaf of a (possibly empty) scheme of pure codimension 2, which we denote by \( C \). These facts are described in the short exact sequence

\[
0 \to R \to \mathcal{O}_Z \to \mathcal{O}_C \to 0.
\]

It follows that \( \text{Ext}^2(O_Z, \mathcal{O}_X) \simeq \text{Ext}^2(O_C, \mathcal{O}_X) = \omega_C \otimes \omega_X \), where \( \omega_C \) and \( \omega_X \) are the dualizing sheaves, and \( \text{Ext}^p(O_Z, \mathcal{O}_X) \simeq \text{Ext}^p(R, \mathcal{O}_X) \) for \( p \geq 3 \). This observation has three interesting consequences; first, the sequence in display (5) can be rewritten in the following manner

\[
0 \to N_\mathcal{F} \to N^*_\mathcal{F} \to \omega_C \otimes \omega_X \otimes \mathcal{L} \to 0.
\]

The other two consequences are stated in the following lemma.

**Lemma 2.1.** Let \( \mathcal{F} \) be a foliation by curves on a projective variety \( X \).

1. \( N^*_\mathcal{F} \) is locally free if and only if its singular locus has pure codimension 2.
2. \( N^*_\mathcal{F} \) is reflexive but not locally free if and only if its singular locus has codimension 3.

**Proof.** The dualization of the sequence in display (3) also leads to the isomorphisms

\[
\text{Ext}^p(N^*_\mathcal{F}, \mathcal{O}_X) \simeq \text{Ext}^{p+1}(I_Z, \mathcal{O}_X) \otimes \mathcal{L} \simeq \text{Ext}^{p+2}(R, \mathcal{O}_X) \otimes \mathcal{L}.
\]
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The rightmost sheaf vanishes for all \( p \geq 1 \) if and only if \( R = 0 \), which is equivalent to saying that \( Z \) has pure codimension 2.

\[ \square \]

### 3. Moduli spaces of foliations by curves

In [12] Section 2.3] the authors developed a general construction of the moduli spaces of distributions on projective varieties. The present section is dedicated to a slightly different dual construction more suitable to understand foliations by curves.

The set of all foliations by curves with a fixed line bundle \( \mathcal{L} \) as tangent sheaf is simply the open subset of saturated sections \( \phi \in H^0(TX \otimes \mathcal{L}^*) \), that is

\[ H^0(TX \otimes \mathcal{L}^*)^{\text{sat}} := \{ \phi \in H^0(TX \otimes \mathcal{L}^*) \mid \text{coker} \phi \text{ is torsion free} \}. \]

This set can be stratified according with the Hilbert polynomial of the vanishing locus of \( \phi \). With this in mind, let \( P \) be a fixed polynomial of degree at most \( \dim X - 2 \); we define the set

\[ \mathcal{D}^P_{\mathcal{L}} := \{ \phi \in H^0(TX \otimes \mathcal{L}^*)^{\text{sat}} \mid P_{\mathcal{O}_{Z_\phi} \otimes \mathcal{L}^\vee}(t) = P \}, \]

where \( Z_\phi \) is the vanishing locus of \( \phi \), that is \( \mathcal{O}_{Z_\phi} = \text{coker} \phi \otimes \mathcal{L} \). Note that \( \mathcal{D}^P_{\mathcal{L}} \) can be regarded as a locally closed subscheme of \( H^0(TX \otimes \mathcal{L}^*) \). Here, \( P_\mathcal{F}(t) \) denotes the Hilbert polynomial of the sheaf \( \mathcal{F} \) on \( X \), defined by

\[ P_\mathcal{F}(t) := \sum_{i=0}^n (-1)^i \dim H^i(F \otimes \mathcal{O}_X(t)), \]

where \( \mathcal{O}_X(1) \) is a fixed ample line bundle on \( X \).

The set \( \mathcal{D}^P_{\mathcal{L}} \) can be given an alternative description in terms of the Grothendieck quot-scheme for the cotangent bundle \( \Omega^1_X \). Let us briefly recall its definition, using [17, Section 2.2] as main reference.

Let \( \mathfrak{Sch}_{/\mathbb{C}} \) denote the category of schemes of finite type over \( \mathbb{C} \), and \( \mathfrak{Set} \) be the category of sets. Fix a polynomial \( P \in \mathbb{Q}[t] \), and consider the functor

\[ \text{Quot}^P : \mathfrak{Sch}_{/\mathbb{C}} \to \mathfrak{Set}, \quad \text{Quot}^P(S) := \{(N, \eta)\}/\sim \]

where

(i) \( N \) is a coherent sheaf of \( \mathcal{O}_{X \times S} \)-modules, flat over \( S \), such that the Hilbert polynomial of \( N_s := N|_{X \times \{s\}} \) is equal to \( P \) for every \( s \in S \);

(ii) \( \eta : \pi_X^* \Omega^1_X \to N \) is an epimorphism, where \( \pi_X : X \times S \to X \) is the standard projection onto the first factor.

In addition, we say that \( (N, \eta) \sim (N', \eta') \) if there exists an isomorphism \( \gamma : N \to N' \) such that \( \gamma \circ \eta = \eta' \).

Finally, if \( f : R \to S \) is a morphism in \( \mathfrak{Sch}_{/\mathbb{C}} \), we define \( \text{Quot}^P(f) : \text{Quot}^P(S) \to \text{Quot}^P(R) \) by \((N, \eta) \mapsto (f^*N, f^*\eta)\). Elements of the set \( \text{Quot}^P(S) \) will be denoted by \([N, \eta]\).

Let us recall the following result, which is just an adaptation of [17, Proposition 2.2.8] suitable for our purposes.
**Theorem 3.1.** The functor $\text{Quot}^P$ is represented by a projective scheme $Q^P$ of finite type over $\mathbb{C}$, that is, there exists an isomorphism of functors $\text{Quot}^P \cong \text{Hom}(\cdot, Q^P)$. In addition, if $\text{Ext}^1(\ker \eta, N) = 0$, then $Q^P$ is smooth at a point $[N, \eta]$, and $\dim T_{[N, \eta]}Q^P = \dim \text{Hom}(\ker \eta, N)$.

We assume from now on that $\text{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$, so that the isomorphism class of a line bundle on $X$ is uniquely determined by its degree; for each $d \in \mathbb{Z}$, let $P_d := \mathcal{P}_{\mathcal{O}_X(d)}$ and set $D_d := D^P_{\mathcal{O}_X(d)}$. We argue that there is a set theoretical bijection between $D'_d$ and the open subset of $Q^P$, where $P' := P_d - P$, consisting of those pairs $[L, \eta] \in Q^P$ such that $L$ is a (rank 1) torsion free sheaf.

Indeed, take $\phi \in D'_d$; the sheaf $\text{Im}(\phi^\vee) = \mathcal{I}_Z \otimes \mathcal{O}_X(d)$ (compare with the exact sequence in display (3)) is a quotient of $\Omega^1_X$ whose Hilbert polynomial is precisely $P'$ as above. Conversely, given a quotient $\eta : \Omega^1_X \to L$ with $L$ being a torsion free sheaf with $P_L(t) = P'$; it follows that $\text{rk}(L) = 1$, thus $L^*$ is a line bundle and $\eta^* \in H^0(TX \otimes L^*)$.

From now on, we will regard the set $D'_d$ as a scheme with the schematic structure inherited from the quot-scheme $Q^P$.

Let $D''_d$ denote the open subset of $D'_d$ consisting of foliations by curves on $X$ whose conormal sheaf is $\mu$-stable. Let also $M^{P'}$ denote the moduli space of reflexive sheaves on $X$ with Hilbert polynomial equal to $P'$. Following the same ideas and proof as in [1] Section 2.3], especially Lemmas 2.5 and 2.6 there, we obtain the following statement.

**Lemma 3.2.** Let $X$ be a non singular projective variety of dimension $n$ with rank 1 Picard group, and let $P$ be polynomial of degree at most $n - 2$. There exists a forgetful morphism

$$\varpi : D''_{-d} \to M^{P_{-d}}, [L, \eta] \mapsto \ker \eta,$$

sending a foliation by curves to its conormal sheaf. In addition, if $N^*_{\mathcal{F}} = \ker \eta$ satisfies $\text{Ext}^1(N^*_{\mathcal{F}}, \Omega^1_X) = \text{Ext}^2(N^*_{\mathcal{F}}, N^*_{\mathcal{F}}) = 0$, then $[L, \eta]$ is a non-singular point of $D''_{-d}$, $\varpi$ is a submersion, and

$$\dim_{[L, \eta]} D''_{-d} = \dim \text{Ext}^1(N^*_{\mathcal{F}}, N^*_{\mathcal{F}}) + \dim \text{Hom}(N^*_{\mathcal{F}}, \Omega^1_X) - 1.$$

Note that $\dim \text{Ext}^1(N^*_{\mathcal{F}}, N^*_{\mathcal{F}})$ is precisely the dimension of $M^{P'}$ at the isomorphism class of $N^*_{\mathcal{F}}$, while $\dim \text{Hom}(N^*_{\mathcal{F}}, \Omega^1_X) - 1$ gives the dimension of the set of monomorphisms $N^*_{\mathcal{F}} \to \Omega^1_X$ with torsion free cokernel, up to scalar multiplication. Therefore, a family of foliations by curves of the form

$$\mathcal{F} : 0 \to N^*_{\mathcal{F}} \to \Omega^1_X \to \mathcal{I}_Z \otimes \mathcal{O}_X(d) \to 0$$

where $Z := \text{Sing}(\mathcal{F})$ satisfying the two vanishing conditions on the previous lemma forms an irreducible component of $D''_{-d}$, understood as the moduli space of foliations by curves with stable conormal sheaf.

4. Foliations by curves on $\mathbb{P}^3$

From now on we will only consider foliations by curves in $X = \mathbb{P}^3$. The sequence in display (3) then simplifies to

$$(8) \quad \mathcal{F} : 0 \to N^*_{\mathcal{F}} \to \Omega^1_{\mathbb{P}^3} \xrightarrow{\phi^\vee} \mathcal{I}_Z(d - 1) \to 0,$$
with $\phi \in H^0(T\mathbb{P}^3(d - 1))$, where $d \geq 0$ is called the degree of $\mathcal{F}$. The sheaf $R$ defined by the sequence in display (4) is a sheaf of dimension 0, while the scheme $C$ is a curve; we will often denote it by $\text{Sing}_1(\mathcal{F})$, the 1-dimensional component of the singular locus of the foliation $\mathcal{F}$.

Our first step step towards a deeper understanding of foliations by curves in $\mathbb{P}^3$ is to determine a relation between the Chern classes of the conormal sheaf and the numerical invariants of the singular scheme.

**Theorem 4.1.** Let $\mathcal{F}$ be a foliation by curves of degree $d$ with $C$ and $R$ as defined by the sequence in display (4). One has

(i) $c_1(N^*_{\mathcal{F}}) = -3 - d$;
(ii) $c_2(N^*_{\mathcal{F}}) = d^2 + 2d + 3 - \deg(C)$;
(iii) $c_3(N^*_{\mathcal{F}}) = h^0(R) = d^3 + d^2 + d + 1 - 3\deg(C)(d - 1) - 2\chi(O_C)$.

We observe that in [7, 20, 27] the authors determine the number of isolated singularities under the hypothesis that $R$ is the structure sheaf of a 0-dimensional scheme disjoint from $C$.

**Proof.** Consider the exact sequence (5), and use $c(\Omega^1_{\mathbb{P}^3}) = c(N^*_{\mathcal{F}}) \cdot c(I_Z(d - 1))$ to obtain

\begin{equation}
-4 = c_1(N^*_{\mathcal{F}}) + c_1(I_Z(d - 1)) \tag{9}
\end{equation}

\begin{equation}
6 = c_1(N^*_{\mathcal{F}}) \cdot c_1(I_Z(d - 1)) + c_2(N^*_{\mathcal{F}}) + c_2(I_Z(d - 1)) \tag{10}
\end{equation}

\begin{equation}
-4 = c_3(N^*_{\mathcal{F}}) + c_3(I_Z(d - 1)) + c_1(N^*_{\mathcal{F}}) \cdot c_2(I_Z(d - 1)) + c_2(N^*_{\mathcal{F}}) \cdot c_1(I_Z(d - 1)) \tag{11}
\end{equation}

The first equality gives $c_1(N^*_{\mathcal{F}}) = -3 - d$, since $c_1(I_Z(d - 1)) = d - 1$.

Since $c_2(I_Z(d - 1)) = \deg(C)$, the substitution of the values of the first Chern classes in the second equation, implies

\[c_2(N^*_{\mathcal{F}}) = d^2 + 2d + 3 - \deg(C).\]

Moreover, the substitution from the values of the first and second Chern classes in the third equation, we obtain

\[d^3 + d^2 + d + 1 + c_3(I_Z(d - 1)) + c_3(N^*_{\mathcal{F}}) - 3d\deg(C) = 0.\]

On the other hand, we have that

\[c_3(I_Z(d - 1)) = c_3(I_Z) - (d - 1)\deg(C),\]

and

\[c_3(I_Z) = c_3(I_C) - c_3(O_R) = 4\deg(C) - 2\chi(O_C) - 2h^0(R).\]

The substitution of the expression in display (11) and (12) in equation (10), together with the fact that $c_3(N^*_{\mathcal{F}}) = h^0(R)$ leads to

\[h^0(R) = d^3 + d^2 + d + 1 - 3\deg(C)(d - 1) - 2\chi(O_C),\]

as claimed. $\square$

Let us analyse two extreme situations. First, if the foliation $\mathcal{F}$ has only isolated singularities, that is $O_Z = R$, then $N_{\mathcal{F}}$ is reflexive by Lemma 2.1 with Chern classes

\[c_3(N_{\mathcal{F}}) = d^2 + 2d + 3 \quad \text{and} \quad c_3(N_{\mathcal{F}}) = d^3 + d^2 + d + 1.\]
On the other hand, if \( Z \) has pure dimension 1, that is \( R = 0 \), then \( N_{\mathcal{F}}^* \) is locally free by Lemma 2.1 and one obtains the following expressions for the degree and arithmetic genus of \( C \) in terms of the degree of the distribution and the second Chern class of the conormal sheaf:

\[
\deg(C) = d^2 + 2d + 3 - c_2(N_{\mathcal{F}}^*) \\
p_a(C) = d^3 + d^2 + d - 3(d - 1)c_2(N_{\mathcal{F}}^*)/2 - 4.
\]

**Lemma 4.2.** If \( \mathcal{F} \) be a foliation by curves of degree \( d \) on \( \mathbb{P}^3 \), then

\[
d + 2 \leq c_2(N_{\mathcal{F}}^*) \leq d^2 + 2d + 3.
\]

If, in addition, \( N_{\mathcal{F}}^* \) is locally free, then \( c_2(N_{\mathcal{F}}^*) \leq d^2 + 2d + 1 \).

**Proof.** The upper bound follows from the second equality in Theorem 4.1 by noticing that \( \deg(C) \geq 0 \). The equality is attained when \( \mathcal{F} \) is a generic foliation, so that \( \deg(C) = 0 \). Assume that \( \mathcal{F} \) is not generic, i.e., \( C \neq \emptyset \). It follows from [26, Theorem 1.1] that

\[
\deg(C) \leq d^2 + d + 1;
\]

substituting this in the second equality in Theorem 4.1 gives the lower bound in the statement of the lemma.

If \( N_{\mathcal{F}}^* \) is locally free, then \( \deg(C) \geq 1 \); if the equality holds, it follows that \( C \) must be a line, so \( p_a(C) = 0 \). The equality in display (15) would imply that the polynomial equation

\[
d^3 + d^2 - 2d + 2 = 0
\]

has an integer solution, which it does not. Thus \( \deg(C) \geq 2 \), and we obtain the improved upper bound in the second part of the statement. \( \Box \)

Next, we give a cohomological criterion for connectedness of the singular scheme of foliations by curves analogous to the criterion given for codimension one distributions in [11, Theorem 3.8].

**Proposition 4.3.** Let \( \mathcal{F} \) be a foliation by curves on \( \mathbb{P}^3 \) of degree \( d \geq 2 \) with locally free conormal sheaf. If \( h^2(N_{\mathcal{F}}^*(1 - d)) = 0 \), then \( Z := \text{Sing}_1(\mathcal{F}) \) is connected. Otherwise, \( Z \) has \( h^2(N_{\mathcal{F}}^*(1 - d)) + 1 \) connected components, when it is reduced.

In particular, the singular scheme of a foliation by curves of global complete intersection type is always connected.

**Proof.** Taking cohomology on the following exact sequence

\[
0 \rightarrow N_{\mathcal{F}}^*(1 - d) \xrightarrow{\phi} \Omega^1_{\mathbb{P}^3}(1 - d) \rightarrow \mathcal{I}_Z \rightarrow 0,
\]

we obtain the equality \( h^1(\mathcal{I}_Z) = h^2(N_{\mathcal{F}}^*(1 - d)) \), since \( d \geq 2 \). It follows that

\[
h^0(\mathcal{O}_Z) = h^1(\mathcal{I}_Z) + 1 = h^2(N_{\mathcal{F}}^*(1 - d)) + 1.
\]

If \( h^2(N_{\mathcal{F}}^*(1 - d)) = 0 \), then \( Z \) must be connected. Otherwise, if \( Z \) is reduced, then the number of connected components of \( Z \) is precisely \( h^0(\mathcal{O}_Z) \). \( \Box \)

**Remark 4.4.** The hypothesis \( d \geq 2 \) is necessary. In fact, the conormal sheaf of a foliation by curves of degree 0 is reflexive but not locally free, and its singular set consists of a single point,
see the first paragraph of Section 5 below. Furthermore, there exist foliations of degree 1 with $N^*_\mathscr{F} = \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2}$ whose singular set consists of two skew lines, see Example 5.4 below.

We complete this section by proving a useful technical result which partially describes the cohomology of the normal sheaves of foliations by curves.

**Lemma 4.5.** If $\mathcal{F}$ is a foliation by curves on $\mathbb{P}^3$ of degree $d \geq 1$, then

(i) $h^0(N^*_\mathscr{F}(p)) = 0$ for $p \leq 1$;
(ii) $h^1(N^*_\mathscr{F}(p)) = 0$ for $p \leq 1 - d$;
(iii) $h^3(N^*_\mathscr{F}(p)) = 0$ for $p \geq d - 2$.

If, in addition, $\text{Sing}_1(\mathcal{F})$ is reduced, then $h^2(N^*_\mathscr{F}(p)) = c_3(N^*_\mathscr{F})$ for $p \leq -d$.

**Proof.** The first item follows easily from the exact sequence in display (3), since $h^0(\Omega^1_{\mathbb{P}^3}(p)) = 0$ for $p \leq 1$. Item (iii) is then obtained via Serre duality (see [13], Thm. 2.5), noticing that

$$(N^*_\mathscr{F})^* = N^*_\mathscr{F} \otimes \text{det}(N^*_\mathscr{F}) = N^*_\mathscr{F}(d + 3)$$

since $N^*_\mathscr{F}$ is a rank two reflexive sheaf, see [13], Prop 1.10.

For item (ii), we have the exact sequence in cohomology

$$H^0(\mathcal{I}_Z(d + p - 1)) \to H^1(N^*_\mathscr{F}(p)) \to H^1(\Omega^1_{\mathbb{P}^3}(p)).$$

The term on the left vanishes for $p + d - 1 \leq 0$, while the term of the right vanishes for all $p \neq 0$.

When $C = \text{Sing}_1(\mathcal{F})$ is reduced, we have that $h^0(\mathcal{O}_C(k)) = 0$ for $k \leq -1$, thus

$$h^1(\mathcal{I}_Z(k)) = h^0(\mathcal{O}_Z(k)) = h^0(R) = c_3(N^*_\mathscr{F})$$

in the same range. The first part of item (v) then follows from the cohomology sequence

$$H^1(\Omega^1_{\mathbb{P}^3}(p)) \to H^1(\mathcal{I}_Z(p + d - 1)) \to H^2(N^*_\mathscr{F}(p)) \to H^2(\Omega^1_{\mathbb{P}^3}(p)),$$

since the leftmost and rightmost terms vanish for $p \leq -d < 0$.  

If $N^*_\mathscr{F}$ is locally free, then Serre duality implies that $h^1(N^*_\mathscr{F}(k)) = h^2(N^*_\mathscr{F}(d - k - 1))$, and the following claim follows easily from the previous lemma.

**Corollary 4.6.** If $\mathcal{F}$ is a foliation by curves on $\mathbb{P}^3$ of degree $d \geq 1$ such that $N^*_\mathscr{F}$ is locally free and $\text{Sing}_1(\mathcal{F})$ is reduced, then $h^1(N^*_\mathscr{F}(p)) = h^2(N^*_\mathscr{F}(p)) = 0$ for $p \geq 2d + 1$.

5. **Foliations with locally free conormal sheaf of degree 1 and 2**

Foliations by curves of degree 0 on $\mathbb{P}^3$ are quite simple to describe. These are given by the choice of a nontrivial section $\sigma \in H^0(\mathbb{P}^3(-1)), \text{leading to the exact sequence}$

$$\mathcal{F} : 0 \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathbb{T}\mathbb{P}^3 \to S_p \to 0,$$

where $S_p$ is the rank 2 reflexive sheaf defined by the resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3} \to S_p \to 0,$$

with $p$ being the point where the three linear sections of the first morphism vanish simultaneously. Note that $\text{Sing}(\mathcal{F}) = \{p\}$. Such sheaves are called 1-tail and have been studied, more in general, in [10].
In particular, the conormal sheaf of a foliation by curves of degree 0 never is locally free. The dual description of such foliations is given by the exact sequence
\[ \mathcal{F} : 0 \to S_p(-3) \to \Omega_{\mathbb{P}^3}^1 \to \mathcal{I}_p(-1) \to 0. \]

Let us now consider foliations by curves of degree 1 on \( \mathbb{P}^3 \) with locally free conormal sheaf.

**Theorem 5.1.** If \( \mathcal{F} \) be a foliation by curves on \( \mathbb{P}^3 \) of degree 1 with locally free conormal sheaf, then \( N^*_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \) and \( \text{Sing}(\mathcal{F}) \) consists of two skew lines.

**Proof.** Since \( c_1(N^*_{\mathcal{F}}) = -4 \), \([25, \text{Corollary 2.2}] \) tells us that if \( N^*_{\mathcal{F}} \) does not split as a sum of line bundles, then \( h^1(N^*_{\mathcal{F}}(1)) \neq 0 \).

Note from the sequence
\[ 0 \to N^*_{\mathcal{F}} \to \Omega_{\mathbb{P}^3}^1 \to \mathcal{I}_C \to 0, \]
where \( C := \text{Sing}(\mathcal{F}) \) is a curve of degree \( 6 - c_2(N^*_{\mathcal{F}}) \), that \( h^1(N^*_{\mathcal{F}}(1)) = h^0(\mathcal{I}_C(1)) \), so if \( N^*_{\mathcal{F}} \) does not split as a sum of line bundles, then \( C \) is a plane curve. However, the expressions in display \([15] \) imply that \( p_a(C) = -1 \), and no plane curve can have negative genus.

We conclude that \( N^*_{\mathcal{F}} \) must split as a sum of line bundles, and the only possibility in degree 1 is \( \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \). In addition, since \( \deg(C) = 2 \) and \( p_a(C) = -1 \), \( C \) must consist of the union of two skew lines. \( \square \)

Our next goal is the classification of degree 2 foliations.

**Theorem 5.2.** Let \( \mathcal{F} \) be a foliation by curves on \( \mathbb{P}^3 \) of degree 2 with locally free conormal sheaf. Then \( N^*_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3) \) and \( \text{Sing}(\mathcal{F}) \) is a connected curve of degree 5 and arithmetic genus 1.

**Proof.** Lemma \([4.2] \) tells us that \( 4 \leq c_2(N^*_{\mathcal{F}}) \leq 9 \), while the expressions in displays \([14] \) and \([15] \) yield
\[
\deg(C) = 11 - c_2(N^*_{\mathcal{F}}) \quad \text{and} \quad p_a(C) = 10 - 3c_2(N^*_{\mathcal{F}})/2,
\]
where \( C := \text{Sing}(\mathcal{F}) \). Since \( c_2(N^*_{\mathcal{F}}) \) must be even, there are only 3 possible values: 4, 6 and 8.

Set \( E := N^*_{\mathcal{F}}(2) \), and note that \( c_1(E) = -1 \) and \( c_2(E) = c_2(N^*_{\mathcal{F}}) - 6 \). Lemma \([4.5] \) implies that \( h^0(E(k)) = 0 \) for \( k \leq -1 \).

If \( c_2(N^*_{\mathcal{F}}) = 4 \), then \( c_1(N^*_{\mathcal{F}})^2 - 4c_2(N^*_{\mathcal{F}}) = 9 > 0 \), so \( N^*_{\mathcal{F}} \) cannot be \( \mu \)-stable. It follows that \( h^0(E) \neq 0 \), so let \( \sigma \in H^0(E) \) be a nontrivial section. If \( \sigma \) does not vanish, then \( E = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \), contradiction \( c_2(E) = -2 \); so let \( Y := (\sigma)_0 \). It would follow that \( \deg(Y) = c_2(E) = -2 \), again a contradiction.

Now let \( c_2(N^*_{\mathcal{F}}) = 6 \). Again \( c_1(N^*_{\mathcal{F}})^2 - 4c_2(N^*_{\mathcal{F}}) = 1 > 0 \), so \( N^*_{\mathcal{F}} \) cannot be \( \mu \)-stable. Again, it follows that \( h^0(E) \neq 0 \), but since \( c_2(E) = 0 \) any section \( \sigma \in H^0(E) \) must be nowhere vanishing, thus \( E = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \), implying that \( N^*_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3) \). The connectedness of \( \text{Sing}(\mathcal{F}) \) in this case is an immediate consequence of Proposition \([4.3] \).
Finally, assume that \( c_2(N_\mathbb{P}^3) = 8 \), so \( c_2(E) = 2 \). If \( h^0(E) \neq 0 \), then we obtain the diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\mathcal{O}_{\mathbb{P}^3} & \rightarrow & \mathcal{O}_{\mathbb{P}^3} \\
\downarrow & & \downarrow & & & & \\
0 & \rightarrow & E & \rightarrow & \Omega^1_{\mathbb{P}^3}(2) & \rightarrow & \mathcal{I}_C(5) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & \mathcal{I}_Y(-1) & \rightarrow & N(1) & \rightarrow & \mathcal{I}_C(5) & \rightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

where \( N \) is a non locally free, null correlation sheaf, and \( Y := (\sigma)_0 \) is a curves of degree \( c_2(E) = 2 \). However, there can be no injective morphism \( \mathcal{I}_Y(-1) \hookrightarrow N(1) \) when \( \deg(Y) > 1 \), so we obtain a contradiction.

It follows that \( E \) must be a \( \mu \)-stable bundle with \( (c_1(E), c_2(E)) = (-1, 2) \). Every such bundle is the cohomology of a monad of the form

\[
\mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1).
\]

We will now prove that this situation cannot occur. Indeed, suppose that \( E \) defines a foliation described by the exact sequence

\[
0 \rightarrow E \rightarrow \Omega^1_{\mathbb{P}^3}(2) \rightarrow \mathcal{I}_C(3) \rightarrow 0.
\]

from which we can obtain the following commutative diagram

\[
\begin{array}{ccccccc}
0 & & & & & & \\
| & & & & & & \\
\mathcal{O}_{\mathbb{P}^3}(-2) & & & & & & \\
\downarrow & & & & & & \\
\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4} & \rightarrow & E^* & \rightarrow & G & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
TP^3(-2) & \rightarrow & E^* & \rightarrow & G & \rightarrow & 0 \\
\downarrow & & & & & & \\
0 & & & & & & 
\end{array}
\]

where \( G \) is a rank 0 sheaf. Recall that \( E^* \simeq E(1) \).
Example 5.4. Let \( \mathcal{F} \) be a one-codimensional distribution on \( \mathbb{P}^3 \) induced, respectively, by the \( 1 \)-forms \( \omega_1 = z_0z_2d_1 - z_1z_2d_0 \) and \( \omega_2 = z_0d_1 - z_1d_2 + z_2d_3 - z_3d_2 \). We have that the complete intersection \( \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \), of degree one, is given by
\[
\omega = \omega_1 \land \omega_2 = z_0z_2d_1 \land d_2 - z_0z_3d_1 \land d_2 - z_1d_2d_0 \land d_3 + z_1z_3d_0 \land d_2
\]
with \( \text{Sing}(\mathcal{F}) = \{ z_0 = z_1 = 0 \} \cup \{ z_2 = z_3 = 0 \} \). Since \( \omega_2 \) induces a contact distribution and \( \omega_1 \) induces a pencil of planes on \( \mathbb{P}^3 \), we have that \( \mathcal{F} \) is a legendrian foliation whose leaves are contained on a pencil of planes.

6. Foliations with Locally Free Conormal Sheaf of Degree 3

In this section we will prove the classification of foliations by curves of degree three with locally free conormal sheaf stated in Main Theorem 2. Note that a foliation \( \mathcal{F} \) of degree 3 is given by the
short exact sequence

\[ 0 \to N^*_{\mathcal{F}} \to \Omega^1_{\mathbb{P}^3} \to \mathcal{I}_L(2) \to 0. \]

We begin by considering complete intersection foliations, that is, the case when the conormal sheaf \( N_{\mathcal{F}}^* \) splits as a sum of line bundles. These correspond to cases (1) and (2) of Main Theorem 2.

Since \( c_1(N_{\mathcal{F}}^*) = -6 \), it is easy to see that \( \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-4) \) and \( \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \) are the only possibilities in degree 3. The connectedness of the singular scheme is a straightforward consequence of Proposition 4.3 while the calculation of its degree and genus uses the formulas in display (14).

To start addressing the third item of Main Theorem 2, we establish the following result.

**Proposition 6.1.** Let \( \mathcal{F} \) be a foliation by curves of degree 3 on \( \mathbb{P}^3 \) of local complete intersection type. If \( N_{\mathcal{F}}^* \) does not split as a sum of line bundles, then it is stable and \( 10 \leq c_2(N_{\mathcal{F}}^*) \leq 16 \).

**Proof.** We start by showing that \( h^0(N_{\mathcal{F}}^*(2)) = 0 \). Suppose that \( h^0(N_{\mathcal{F}}^*(2)) \neq 0 \); a nontrivial section \( \sigma \in H^0(N_{\mathcal{F}}^*(2)) \) induces a codimension one distribution of degree 0 on \( \mathbb{P}^3 \)

\[ \mathcal{G} : 0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \Omega^1_{\mathbb{P}^3} \to \Omega_{\mathcal{G}} \to 0 \]

which contains \( \mathcal{F} \). Then, by the classification of such distributions (see [4, Proposition 7.1]), either \( \mathcal{G} \) is the non-singular contact distribution or \( \mathcal{G} \) is pencil of planes. In the first case, the section \( \sigma : \mathcal{O}_{\mathbb{P}^3}(-2) \to N_{\mathcal{F}}^* \) cannot vanish, which implies that \( N_{\mathcal{F}}^* \) must split as a sum of line bundles, contradicting our hypothesis. In the second case, the zero locus of \( \sigma \) is a line since this is the singular set of the pencil of planes \( \mathcal{G} \). We therefore have an exact sequence of the form

\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to N_{\mathcal{F}}^* \to \mathcal{I}_L(2) \to 0 \]

where \( L \) is a line in \( \mathbb{P}^3 \). However, \( L \) is a complete intersection curve, so [12, Corollary 1.2] implies that \( N_{\mathcal{F}}^* \) would again split as a sum of line bundles.

Notice that the stability of \( N_{\mathcal{F}}^* \) is equivalent to \( h^0(N_{\mathcal{F}}^*(3)) = 0 \), since \( N_{\mathcal{F}}^*(3) \) is the normalization of \( N_{\mathcal{F}}^* \).

Let us suppose that, on the contrary, \( N_{\mathcal{F}}^*(3) \) admits a non trivial global section, which we also denote by \( \sigma \). Since \( h^0(N_{\mathcal{F}}^*(2)) = 0 \), its cokernel must be a torsion free sheaf of rank 1, so it must be the twisted ideal sheaf of a curve \( S \) in \( \mathbb{P}^3 \), that is \( \text{coker} \, \sigma \simeq \mathcal{I}_S(3) \).

In addition, \( \sigma \) induces the commutative diagram in display (17) below, with the middle column being a codimension one distribution \( \mathcal{G} \) of degree 1.

It follows from the bottom row of diagram (17) that \( S \subset \text{Sing}(\mathcal{F}) \); let \( S' \) denote the maximal 1-dimensional subscheme of \( \text{Sing}(\mathcal{F}) \). According to the classification of distribution of degree one [4, Section 8], \( S' \) is a curve of degree at most 3 with \( p_a(S') = 0 \). It follows that \( 1 \leq \deg(S) \leq 3 \), and \( p_a(S) = 1 - 2\deg(S) \) [12, Proposition 2.1]. Let us now look at each of the three possible cases.

- If \( \deg(S) = 1 \), then \( S \) is a line, contradicting \( p_a(S) = -1 \).
- Assume that \( \deg(S) = 2 \), so either \( \deg(S') = 2 \) or \( \deg(S') = 3 \). In the first case, we would have \( S = S' \), but \( p_a(S) = -3 \) while \( p_a(S') = 0 \). In the second case, we have that \( T_\mathcal{G} = \Omega_{\mathcal{G}}^* \) splits as a sum of line bundles, namely \( T_\mathcal{G} \simeq \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3} \), see [4, Section 8]; we argue that \( \text{Sing}(\mathcal{F}) \subset S' = \text{Sing}(\mathcal{F}) \).
Indeed, take a point \( p \in \text{Sing}(\mathcal{F}) \). Since \( T\mathcal{F} \subset T\mathcal{G} \) and \( T\mathcal{G} \) is locally free we can consider a neighborhood \( U \) of \( p \in \text{Sing}(\mathcal{F}) \), with an analytic local coordinates system \((x_1, x_2, x_3)\), such that \( \mathcal{G} \) is induced by a 1-form \( \omega = i_v i_u (dx_1 \wedge dx_2 \wedge dx_3) \), where \( v \) and \( u \) are vector fields with \( v \) tangent to \( \mathcal{F} \). This shows that \( \text{Sing}(\mathcal{F})|_U = \{ v = 0 \} \subset \{ \omega = 0 \} = \mathcal{S}'|_U \).

Now, from Theorem 4.1 we have that \( \deg(\text{Sing}(\mathcal{F})) = 18 - c_2(\mathcal{N}^*_F) = 18 - (c_2(\mathcal{N}^*_F(3)) + 9) = 9 - \deg(\mathcal{S}) = 7 \), since \( c_2(\mathcal{N}^*_F(3)) = \deg(\mathcal{S}) = 2 \). However, a curve of degree 7 cannot be contained in a curve of degree 3, and we obtain the desired contradiction.

- If \( \deg(\mathcal{S}) = 3 \), then \( \mathcal{S} = \mathcal{S}' \) but \( p_a(\mathcal{S}) = -5 \) while \( p_a(\mathcal{S}') = 0 \).

(17)

\[
\begin{array}{cccc}
0 & 0 & \mathcal{O}_{\mathbb{P}^3}(-3) & \mathcal{O}_{\mathbb{P}^3}(-3) \\
\sigma & \mathcal{N}_{\mathcal{F}} & \Omega^1_{\mathbb{P}^3} & \mathcal{I}_{\mathcal{C}}(2) & 0 \\
\mathcal{I}_{\mathcal{S}}(3) & \Omega_{\mathcal{G}} & \mathcal{I}_{\mathcal{C}}(2) & 0 \\
0 & 0 & 0 & 0
\end{array}
\]

The lower bound on \( c_2(\mathcal{N}^*_F) \) is a direct consequence of the fact that \( \mathcal{N}^*_F \) is stable via the Bogomolov inequality, see [12, Lemma 3.2]. The upper bound is the one given in Lemma 4.2.

The next step is to rule out the possibilities \( c_2(\mathcal{N}^*_F) = 16 \) and \( c_2(\mathcal{N}^*_F) = 15 \), which requires a detailed analysis of the possible conormal sheaves and singular schemes. For this purpose, recall that every rank 2 locally free sheaf \( \mathcal{F} \) on \( \mathbb{P}^3 \) with \( c_1(\mathcal{F}) = 0 \) is isomorphic the the cohomology of a monad of the form

\[
\bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^3}(-c_i) \to \bigoplus_{j=1}^{s+1} \mathcal{O}_{\mathbb{P}^3}(-b_j) \oplus \mathcal{O}_{\mathbb{P}^3}(b_j) \to \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^3}(c_i),
\]

with \( 1 \leq c_1 \leq \cdots \leq c_s \) and \( 0 \leq b_1 \leq \cdots \leq b_{s+1} \). Furthermore, recall also that a sheaf \( \mathcal{F} \) on \( \mathbb{P}^n \) is \( k \)-regular (in the sense of Castelnuovo–Mumford) if \( h^p(\mathcal{F}(k - p)) = 0 \) for every \( p > 0 \); moreover, every \( k \)-regular sheaf is also \( k' \)-regular for every \( k' \geq k \).

With these facts in mind, we now state an useful technical result.

**Lemma 6.2.** Let \( \mathcal{F} \) be a stable rank 2 locally free sheaf on \( \mathbb{P}^3 \) with \( c_1(\mathcal{F}) = 0 \).

1. If \( c_2(\mathcal{F}) = 7 \), then \( \mathcal{F} \) is 13-regular.
(2) If \( c_2(F) = 6 \), then \( F \) is 10-regular. Furthermore, if \( F \) is not isomorphic to the cohomology of one of the following two monads

\[
\begin{align*}
(19) & \quad 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 4 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}(2) \\
(20) & \quad \mathcal{O}_{\mathbb{P}^3}(-3) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 4 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(3)
\end{align*}
\]

then \( F \) is 8-regular.

**Proof.** According to [9, Theorem 3.2], the cohomology of a monad of the form (18) is at least \( k \)-regular with

\[
k = 2c_s + b_3 + \cdots + b_{s+1} + c_1 + \cdots + c_s - 2.
\]

On the other hand, Hartshorne and Rao classify in [15, Section 5.3] all possible monads for stable rank 2 bundles \( F \) on \( \mathbb{P}^3 \) with \( c_1(F) = 0 \) and \( c_2(F) \leq 8 \). The claims in the statement of the lemma are obtained simply by applying the formula in display (21) to all monads listed in [15, Section 5.3]. \( \square \)

In order to use the previous result, we introduce the notation \( E_{\mathcal{F}} := N_{\mathcal{F}}^* (3) \) for any foliation \( \mathcal{F} \) by curves of degree 3 of local complete intersection type. Note that \( c_1(E_{\mathcal{F}}) = 0 \), so \( E_{\mathcal{F}} \) is the normalization of the conormal sheaf \( N_{\mathcal{F}}^* \); in addition, we have \( c_2(E_{\mathcal{F}}) = c_2(N_{\mathcal{F}}^*) - 9 \) and

\[
h^1(E_{\mathcal{F}}(k)) = h^2(E_{\mathcal{F}}(-4 - k)) = h^2(N_{\mathcal{F}}^*(-k - 1)) = h^1(\mathcal{I}_C(1 - k))
\]

by Serre duality, where \( C = \text{Sing}(\mathcal{F}) \).

**Proposition 6.3.** There are no foliations by curves of local complete intersection type and degree 3 on \( \mathbb{P}^3 \) with \( c_2(N_{\mathcal{F}}^*) = 16 \).

**Proof.** Let \( \mathcal{F} \) be a foliation as in the statement of the lemma. According to the formulas in displays (13) and (15), the singular locus of \( \mathcal{F} \) is a curve \( C \) of degree 2 and genus \(-13\); let \( L := C_{\text{red}} \), so that \( C \) is a multiplicity 2 structure on the line \( L \). Following [19, Proposition 1.4 and Remark 1.5], we must have the exact sequence

\[
0 \to \mathcal{O}_L(12) \to \mathcal{O}_C \to \mathcal{O}_L \to 0,
\]

thus, by the equalities in display (22), \( h^1(E_{\mathcal{F}}(13)) = h^1(\mathcal{I}_C(-12)) = 1 \), meaning that \( E_{\mathcal{F}} \) is not 12-regular, thus contradicting the first part of Lemma 6.2 since \( c_2(E_{\mathcal{F}}) = 7 \). \( \square \)

Let us know shift our attention to the case \( c_2(N_{\mathcal{F}}^*) = 15 \).

**Lemma 6.4.** Let \( \mathcal{F} \) be a foliation by curves of local complete intersection type and degree 3 on \( \mathbb{P}^3 \).

If \( c_2(N_{\mathcal{F}}^*) = 15 \), then \( C \) is a multiplicity 3 structure on a line \( L \) satisfying the exact sequence

\[
0 \to \mathcal{O}_L(a) \oplus \mathcal{O}_L(c) \to \mathcal{O}_C \to \mathcal{O}_L \to 0,
\]

with \((a,c)\) equal either to \((1,7)\) or to \((2,6)\).

**Proof.** Let \( \mathcal{F} \) be a foliation by curves of local complete intersection type and degree 3 on \( \mathbb{P}^3 \) with \( c_2(N_{\mathcal{F}}^*) = 15 \), so that \( c_2(E_{\mathcal{F}}) = 6 \). It follows from the formulas in displays (13) and (15) that the singular locus of \( \mathcal{F} \) is a curve \( C \) of degree 3 and genus \(-10\). Letting \( L := C_{\text{red}} \), three possibilities may occur:
Proof. Let \( h \) be a foliation as in the statement of the lemma, and set \( E_\mathcal{F} = N^*_\mathcal{F}(3) \). We have that 
\[
h^3(E_\mathcal{F}(1)) = h^0(E_\mathcal{F}(-3)) = 0,
\]
and 
\[
h^1(E_\mathcal{F}(1)) = h^0(E_\mathcal{F}(1)) + h^2(E_\mathcal{F}(1)) - \chi(E_\mathcal{F}(1)) = 10 + h^0(E_\mathcal{F}(1)) + h^2(E_\mathcal{F}(1)).
\]
As observed above, \( E_\mathcal{F} \) must be the cohomology of monad either as in display (19) or as in display (20). In both case, \( h^0(E_\mathcal{F}(1)) \neq 0 \), see [15, Section 5.3]. It follows, using the equality in display (22), that 
\[
h^1(I_C) = h^1(E_\mathcal{F}(1)) > 10.
\]
On the other hand, let us examine the exact sequence 
\[
0 \to I_C \to I_L \to O_L(a) \oplus O_L(c) \to 0,
\]
which is equivalent to the sequence in the statement of Lemma (6.4) with \( (a, c) = (1, 7), (2, 6) \). Since \( h^1(I_L) = 0 \), we have that 
\[
h^1(I_C) = h^0(O_L(a)) + h^0(O_L(c)) = a + c + 2 = 10.
\]

In particular, note that \( h^1(E_\mathcal{F}(7)) \neq 0 \), so if \( \mathcal{F} \) is a foliation by curves of local complete intersection type and degree 3 with \( c_2(N_\mathcal{F}^*) = 15 \), then 
\[
\begin{align*}
h^1(E_\mathcal{F}(7)) &= h^1(I_C(-8)) = h^0(O_L(2a + b - 8)) = 0, \quad \text{so } E_\mathcal{F} \text{ is not } 10\text{-regular, again contradicting the second part of Lemma } 6.2 \quad \text{setting } c := 2a + b,
\end{align*}
\]
we obtain the desired result.

\[\square\]

Proposition 6.5. There are no foliations by curves of local complete intersection type and degree 3 on \( \mathbb{P}^3 \) with \( c_2(N_\mathcal{F}^*) = 15 \).

Proof. Let \( \mathcal{F} \) be a foliation as in the statement of the lemma, and set \( E_\mathcal{F} = N^*_\mathcal{F}(3) \). We have that 
\[
h^3(E_\mathcal{F}(1)) = h^0(E_\mathcal{F}(-3)) = 0,
\]
and 
\[
h^1(E_\mathcal{F}(1)) = h^0(E_\mathcal{F}(1)) + h^2(E_\mathcal{F}(1)) - \chi(E_\mathcal{F}(1)) = 10 + h^0(E_\mathcal{F}(1)) + h^2(E_\mathcal{F}(1)).
\]
As observed above, \( E_\mathcal{F} \) must be the cohomology of monad either as in display (19) or as in display (20). In both case, \( h^0(E_\mathcal{F}(1)) \neq 0 \), see [15, Section 5.3]. It follows, using the equality in display (22), that 
\[
h^1(I_C) = h^1(E_\mathcal{F}(1)) > 10.
\]

On the other hand, let us examine the exact sequence 
\[
0 \to I_C \to I_L \to O_L(a) \oplus O_L(c) \to 0,
\]
which is equivalent to the sequence in the statement of Lemma (6.4) with \( (a, c) = (1, 7), (2, 6) \). Since \( h^1(I_L) = 0 \), we have that 
\[
h^1(I_C) = h^0(O_L(a)) + h^0(O_L(c)) = a + c + 2 = 10,
\]

providing the desired contradiction.

We have so far proved the first part of Main Theorem concerning items (1), (2) and (3).

6.1. Foliations with reduced singular scheme. We now move to the second part of Main Theorem making the further assumption that $\text{Sing}(\mathcal{F})$ is reduced.

Recall that a locally free sheaf is said to have natural cohomology if for each $i = 0, 1, 2, 3$ there can be only one $p \in \mathbb{Z}$ such that $h^i(E(p)) \neq 0$ [13]; note that if $E$ is stable and has rank 2, then $\chi(E(-2)) = h^2(E(-2)) - h^1(E(-2)) = 0$, so stable rank 2 locally free sheaves with natural cohomology are necessarily instanton bundles. However, not every instanton bundle has natural cohomology (‘t Hooft instanton bundle of charge at least 3 are the most well known exceptions). In any case, instanton bundles with natural cohomology form an open subset in the moduli space of instanton bundles.

To be precise, we establish the following result.

**Proposition 6.6.** There are no foliations by curves $\mathcal{F}$ of degree 3 such that $\text{Sing}(\mathcal{F})$ is reduced and either of the following conditions hold:

- $c_2(N^*_{\mathcal{F}}) = 14$;
- $c_2(N^*_{\mathcal{F}}) = 13$ and $E_{\mathcal{F}} = N^*_{\mathcal{F}}(3)$ is not an instanton bundle of charge 4 with natural cohomology;
- $c_2(N^*_{\mathcal{F}}) = 12$ and $E_{\mathcal{F}} = N^*_{\mathcal{F}}(3)$ is not an instanton bundle of charge 3 with $h^0(E(1)) \leq 1$.

Letting $E_{\mathcal{F}} := N^*_{\mathcal{F}}(3)$, a foliation by curves of local complete intersection type and degree 3 can be described as the following a short exact sequence

$$0 \rightarrow E_{\mathcal{F}}(-3) \rightarrow \Omega^1_{\mathbb{P}^3} \rightarrow \mathcal{I}_C(2) \rightarrow 0,$$

with $C$ being a curve. Observe that $h^3(E_{\mathcal{F}}(1)) = h^0(E_{\mathcal{F}}(-5)) = 0$ since $E_{\mathcal{F}}$ is $\mu$-stable and $c_1(E_{\mathcal{F}}) = 0$, while $h^2(E_{\mathcal{F}}(1)) = h^1(E_{\mathcal{F}}(-5)) = 0$ since $c_2(E_{\mathcal{F}}) \leq 5$. It follows that

$$\chi(E_{\mathcal{F}}(1)) = h^0(E_{\mathcal{F}}(1)) - h^1(E_{\mathcal{F}}(1)) = 8 - 3c_2(E_{\mathcal{F}}).$$

Since $h^2(N^*_{\mathcal{F}}(-2)) = h^1(E_{\mathcal{F}}(1))$, the argument in the proof of Proposition yields

$$h^0(O_Z) = 1 + h^1(E_{\mathcal{F}}(1)) = 3c_2(E_{\mathcal{F}}) - 7 + h^0(E_{\mathcal{F}}(1)).$$

This is the key fact to be explored in the proof of Proposition. We will also require the following additional fact.

**Lemma 6.7.** Let $C$ be the disjoint union of five lines in $\mathbb{P}^3$. Then, there exists an epimorphism

$$\Omega^1_{\mathbb{P}^3} \xrightarrow{\varpi} \mathcal{I}_C(2)$$

if and only if $Z$ has no 5-secant line. Furthermore, $\ker \varpi(3)$ an instanton bundle of charge 4 with natural cohomology.

**Proof.** If $C$ has no 5-secant line, then the result is proven in [2] Lemma 2.

Let us suppose the existence of a 5-secant line $L$. Directly from the canonical exact sequence of the canonical sheaf of $C$, we have, restricting on $L$, a surjective map

$$(\mathcal{I}_C)|_L \rightarrow \mathcal{O}_L(-5).$$
If a surjective map as in (24) exists, then, restricting to the line $L$, we would obtain the following composition, which is again surjective

$$
(\Omega^1_{\mathbb{P}^3})|_L \longrightarrow (\mathcal{I}_C(2))|_L \longrightarrow \mathcal{O}_L(-3).
$$

Being $(\Omega^1_{\mathbb{P}^3})|_L \simeq \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^2$, we get a contradiction.

Now let $F := \ker \varpi(3)$, and note that $c_1(F) = 0$, $c_2(F) = 4$ and $c_3(F) = 0$; in other words,

$$
0 \rightarrow F(-3) \rightarrow \Omega^1_{\mathbb{P}^3} \xrightarrow{\varpi} \mathcal{I}_C(2) \rightarrow 0,
$$

is a foliation by curves of local complete intersection type and degree 3. Since $Z$ is not an ACM curve, $F$ does not split as a sum of line bundles, and therefore, by Proposition 6.1, $F$ must be stable. As it was observed in [2, Lemma 5], the fact that $C$ does not have a 5-secant line implies that $h^i(\mathcal{I}_C(3)) = 0$ for $i = 0, 1$. It follows that $h^1(F(-2)) = 0$, forcing $F$ to be an instanton bundle; in addition, we have that

$$
h^1(F(2)) = h^2(F(-6)) = h^3(\mathcal{I}_Z(-3)) = 0,
$$

thus $F$ has natural cohomology, since $h^3(F(t)) = 0$ for $t \geq 3$ as every instanton bundle of charge 4 is 4-regular.

We are finally in position to complete the proof of Proposition 6.6. We go over each item separately.

First, if $c_2(N^*_\mathcal{F}) = 14$, we have from the second item in Theorem 1.11 that $\deg(C) = 4$ while $h^0(\mathcal{O}_C) \geq 8$ by the formula in display (23); but this is impossible for a reduced curve.

Similarly, if $c_2(N^*_\mathcal{F}) = 13$, then we have that $\deg(C) = 5$ and $p_a(C) = -4$, while $h^0(\mathcal{O}_C) = 5 + h^0(E,\mathcal{F}(1)) \geq 5$. If $C$ is reduced, we must have that $Z$ consists of 5 skew lines; Lemma 6.7 then implies that $E,\mathcal{F}$ must be an instanton bundle of charge 4 with natural cohomology.

Finally, if $c_2(N^*_\mathcal{F}) = 12$, then we have that $\deg(C) = 6$ and $p_a(C) = -1$, while $h^0(\mathcal{O}_C) = 2 + h^0(E,\mathcal{F}(1))$. Note that a curve of degree 6 with 4 connected components, must either be the disjoint union of two conics and two lines, or the disjoint union of a cubic and three lines. Since neither has arithmetic genus equal to $-1$, we conclude that $h^0(E,\mathcal{F}(1)) \leq 1$. This restriction not only rules out $E,\mathcal{F}$ being a special ’t Hooft instantons of charge 3, but also the generalized null correlation bundle given as the cohomology of a monad of the form

$$
\mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(2).
$$

Since, according the classification by Hartshorne and Rao [15, Table 5.3], a stable rank 2 bundle $E$ with $c_1(E) = 0$ and $c_1(E) = 3$ is either an instanton bundle or a generalized null correlation bundle as above, this completes the proof of Proposition 6.6.

We complete this section by observing that it is also possible to discard some of the possible foliations by curves of local complete intersection type and degree 3 with $c_2(N^*_\mathcal{F}) = 13$ or 14 without the hypothesis of $\text{Sing}(\mathcal{F})$ being reduced. Here are two cases.

**Lemma 6.8.** Let $E$ be an instanton bundle of charge 5 with natural cohomology. Then there are no foliations by curves $\mathcal{F}$ such that $N^*_\mathcal{F} = E(-3)$. 

Proof. If $E$ is a stable rank 2 locally free sheaf with $c_2(E) = 5$, then
\[ \chi(E(2)) = h^0(E(2)) - h^1(E(2)) = 0. \]
If $E$ has natural cohomology, then $h^0(E(2)) = h^1(E(2)) = 0$. However, $E \otimes \Omega^1_{\proj^3}(3)$ is a subsheaf of $E(2)^{\oplus 4}$, hence $\text{Hom}(E(-3), \Omega^1_{\proj^3}) = 0$, so there can be no foliation by curves with $N^*_\mathcal{F} = E(-3)$. □

**Lemma 6.9.** Let $E$ be a 't Hooft instanton bundle of charge 4 or 5. Then there are no foliations by curves $\mathcal{F}$ such that $N^*_\mathcal{F} = E(-3)$.

Proof. If $E$ be a 't Hooft instanton bundle of charge, then $E(1)$ has a global section that vanishes along $n + 1$ skew lines. We will discuss the case $n = 4$ in detail; the case $n = 5$ can be dealt with similarly (the argument is even simpler).

Assume that
\[ 0 \to E(-3) \xrightarrow{\varphi} \Omega^1_{\proj^3} \to \mathcal{I}_C(2) \to 0 \]
be a foliation by curves, and $\sigma \in H^0(E(1))$ be a nontrivial global section. The composition of monomorphisms
\[ \mathcal{O}_{\proj^3}(-4) \to E(-3) \xrightarrow{\varphi} \Omega^1_{\proj^3} \]
duces a codimension 1 distribution of degree 2
\[ 0 \to G^* \to T\proj^3 \to \mathcal{I}_W(4) \to 0, \]
where $G := \text{coker}(\varphi \circ \sigma)$, and $W$ is its singular scheme. Setting $C_0 := (\sigma)_0$ (in the case at hand, $C_0$ consists of 5 skew lines), observe that $C_0 \subseteq W$. According to the classification of codimension 1 distributions of degree 2 studied in [9] Section 9], we know that $\deg(W) \leq 7$, so there are 3 possibilities to be considered.

First, if $\deg(W) = 5$, then actually $W = C_0$; however, either $p_a(W) = 1$ or $p_a(W) = 2$, contradicting that $p_a(C_0) = -4$.

If $\deg(W) = 6$, then $p_a(W) = 3$ (cf. [9] Theorem 9.5]), and one must consider two possibilities. First, assume that $W$ is reduced, so that $W = C_0 \cup L$, where $L$ is a line, implying that $p_a(W) = k - 5 \leq 0$, where $k$ is the number of points in $C_0 \cap L$ (note that $0 \geq k \geq 5$). If $W$ is not reduced, then $W = C' \cup \hat{L}$, where $C' \subset C_0$ consists of 4 skew lines, and $\hat{L}$ is a double structure on the remaining line $C_0 \setminus C'$, which leads to $p_a(W) = p_a(\hat{L}) - 4 \leq -4$. We end up with contradictions in both cases.

If $\deg(W) = 7$, then $p_a(W) = 5$ (cf. [9] Theorem 9.5]), and one must again consider two possibilities. Either $W = C_0 \cup Q$ where $Q$ is a degree 2 scheme (possibly non-reduced, so $p_a(Q) \leq 0$), or $W = C' \cup \hat{L}$, where $C' \subset C_0$ consists of 4 skew lines, and $\hat{L}$ is a triple structure on the remaining line $C_0 \setminus C'$ (so $p_a(\hat{L}) \leq 1$). In both situations, $p_a(W) \leq 1$, providing a contradiction as in the previous paragraph. □

### 6.2. Existence of foliations

The goal of this section is to provide sufficient and necessary conditions on a stable rank 2 locally free sheaf $E$ with $c_1(E) = 0$ that guarantee the existence of a monomorphism $E(-3) \to \Omega^1_{\proj^3}$ with torsion free cokernel. This will allows us to complete the proof of Main Theorem [2] and fill out all the information in Table [1]. We remark that the results in this section do not assume that the singular scheme is reduced.
We start by presenting two criteria which ensure the existence of a monomorphism of a locally free sheaf into the cotangent bundle (first criterion) whose cokernel is torsion free (second criterion).

**Lemma 6.10.** Let $E$ be a stable rank 2 vector bundle on $\mathbb{P}^3$ with $c_1(E) = 0$.

- If $h^0(E(1)) = 0$, then every non trivial morphism $\varphi : E(-3) \to \Omega^1_{\mathbb{P}^3}$ is a monomorphism.
- If $h^0(E(1)) \geq 1$, then there is a monomorphism $\varphi : E(-3) \to \Omega^1_{\mathbb{P}^3}$ if and only if $h^0(E \otimes \Omega^1_{\mathbb{P}^3}(3)) \geq 7$.

**Proof.** Let us consider a non trivial morphism $\varphi$ as above, and let us suppose it not to be injective. This means that $\text{rk}(\ker \varphi) = 1$ and it is necessarily a reflexive sheaf, henceforth a line bundle. We obtain the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(k) & \xrightarrow{\varphi} & \Omega^1_{\mathbb{P}^3} \\
\downarrow & & \downarrow \\
\mathcal{I}_X(-k-6) & \xrightarrow{\tau} & \Omega^1_{\mathbb{P}^3}
\end{array}
$$

where $X \subset \mathbb{P}^3$ is a curve. Since $\tau$ induces a nontrivial section in $H^0(\Omega^1_{\mathbb{P}^3}(k+6))$, it follows that $k \geq -4$. On the other hand, being $E$ stable and therefore not having global section, forces $k \leq -4$. Hence we must have $k = -4$. This implies straightforwardly that if $h^0(E(1)) = 0$ then $\ker \varphi = 0$, thus $\varphi$ is a monomorphism.

Finally, if $h^0(E(1)) \geq 1$ and $\varphi$ is not injective, we have proved that $\varphi$ factors through $\tau$. Since $\text{Hom}(\mathcal{I}_X(-2), \Omega^1_{\mathbb{P}^3}) \cong \text{Hom}(\mathcal{O}_{\mathbb{P}^3}(-2), \Omega^1_{\mathbb{P}^3}) \cong H^0(\Omega^1_{\mathbb{P}^3}(2))$, which is a 6 dimensional vector space, we conclude that if $\text{hom}(E, \Omega^1_{\mathbb{P}^3}(3)) \geq 7$ we get a morphism $\varphi$ which does not factor through $\tau$ and must therefore be injective.

If $h^0(E \otimes \Omega^1_{\mathbb{P}^3}(3)) = 6$, we can apply the functor $\text{Hom}(\cdot, \Omega^1_{\mathbb{P}^3}(3))$ to the exact sequence

$$
0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to E \to \mathcal{I}_X(1) \to 0
$$

induced by a non trivial section in $H^0(E(1))$, with $X$ being its zero locus, by $\Omega^1_{\mathbb{P}^3}(-2)$ and conclude that

$$
\text{Hom}(E(-3), \Omega^1_{\mathbb{P}^3}) \cong \text{Hom}(\mathcal{I}_X(-2), \Omega^1_{\mathbb{P}^3}).
$$

This means that every $\phi : E(-3) \to \Omega^1_{\mathbb{P}^3}$ must factor through $\mathcal{I}_X(-2)$, and so it cannot be injective. \hfill $\square$

Let us now turn to our second criterion.

**Lemma 6.11.** Let $E$ be a stable rank 2 vector bundle on $\mathbb{P}^3$ with $c_1(E) = 0$.

- If $h^0(E(1)) \leq 1$, then every monomorphism $\varphi : E(-3) \to \Omega^1_{\mathbb{P}^3}$ has a torsion free cokernel.
- If $h^0(E(1)) = 2$, then there exists a monomorphism $\varphi : E(-3) \to \Omega^1_{\mathbb{P}^3}$ with torsion free cokernel if and only if $h^0(E \otimes \Omega^1_{\mathbb{P}^3}(3)) \geq 13$.

**Proof.** Let $\varphi : E(-3) \to \Omega^1_{\mathbb{P}^3}$ be a monomorphism, and suppose that $K := \text{coker} \varphi$ is not torsion free; let $P$ be the maximal torsion subsheaf of $K$, so that $K/P$ is torsion free.
Let us first prove $P$ has codimension one; indeed, let $G$ be the kernel of the composed epimorphism $\Omega^1_{\mathbb{P}^3} \to K \to K/P$. Using the Snake Lemma, we get a short exact sequence
\begin{equation}
0 \to E(-3) \xrightarrow{\tau} G \to P \to 0.
\end{equation}

If $\text{codim} \, P \geq 2$, then $E(-3)^* \simeq G^*$, which implies that
\[ E(-3) \simeq E(-3)^{**} \simeq G^{**} \simeq G, \]
since $G$ is reflexive (it is the kernel of an epimorphism from a locally free to a torsion free sheaf).

Hence we would have that $K$ is torsion free, contradicting our hypothesis.

It follows that $c_1(P) > 0$, thus $c_1(G) > c_1(E(-3)) = -6$. In addition,
\[ 0 \to G \xrightarrow{\phi} \Omega^1_{\mathbb{P}^3} \to K/P \to 0 \]
is a foliation by curves $\mathcal{G}$ with conormal sheaf $G$ of degree $d < 3$. We argue that $G$ must be locally free. Indeed, let $\omega := \wedge^2 \varphi \in H^0(\Omega^2_{\mathbb{P}^3}(6))$ and $\tilde{\omega} := \wedge^2 \phi \in H^0(\Omega^2_{\mathbb{P}^3}(d+3))$; note that $\omega = f \cdot \tilde{\omega}$, where $f \in H^0(\mathcal{O}_{\mathbb{P}^3}(3-d))$ (the hypersurface $\{f = 0\}$ is precisely the support of the torsion sheaf $P$). By definition, $\text{Sing}(\mathcal{G}) = \{\tilde{\omega} = 0\}$, so if $\text{Sing}_0(\mathcal{G})$ is non-empty, then $\{\omega = 0\}$, which is the degeneracy scheme of $\varphi$, also has an isolated or embedded point, contradicting the fact the $E$ is locally free (see [1], Chapter II, Section 4). Therefore $\text{Sing}_0(\mathcal{G})$ must be empty, and $G$ is locally free.

As observed in the first paragraph of Section [5], the conormal sheaf of a foliation by curves of degree 0 is never locally free. Therefore, either $\deg(\mathcal{G}) = 1$ or $\deg(\mathcal{G}) = 2$, and according to Main Theorem [1] we must have that either $G = 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2)$ or $G = \mathcal{O}_{\mathbb{P}^3}(-2) + \mathcal{O}_{\mathbb{P}^3}(-3)$, respectively.

It is easy to see that the second possibility cannot occur: a morphism
\[ \tau \in \text{Hom}(E(-3), \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3)) = H^0(E(1)) \oplus H^0(E) \]
cannot be injective since $h^0(E) = 0$ by the stability of $E$, thus contradicting the existence of the exact sequence in display (25).

Regarding the first possibility, we have that
\[ \tau \in \text{Hom}(E(-3), 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2)) = H^0(E(1)) \oplus H^0(E(1)). \]
If $h^0(E(1)) \leq 1$, then $\tau$ cannot be injective (otherwise we would again fall in contradiction with the sequence in display (25)), so every monomorphism $\varphi \in \text{Hom}(E(-3), \Omega^1_{\mathbb{P}^3})$ must have torsion free cokernel.

In order to establish the second item of the lemma, suppose now that $h^0(E(1)) = 2$, and choose a monomorphism $\tau : E(-3) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2)$. Since coker $\tau$ is a torsion sheaf, we obtain the injective map
\[ 0 \to \text{Hom}(2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2), \Omega^1_{\mathbb{P}^3}) \to \text{Hom}(E(-3), \Omega^1_{\mathbb{P}^3}) \to \cdots \]
given by composition with $\tau$. Clearly, we must have $h^0(E \otimes \Omega^1_{\mathbb{P}^3}(3)) \geq 2 \cdot h^0(\Omega^1_{\mathbb{P}^3}(2)) = 12$. If $h^0(E \otimes \Omega^1_{\mathbb{P}^3}) \geq 13$, then there is a morphism $\varphi : E(-3) \to \Omega^1_{\mathbb{P}^3}$ which does not factor through $2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2)$. By the second item in Lemma [6.10] we can choose this morphism to be injective; if coker $\varphi$ is not torsion free, we would again get in contradiction with the existence of the sequence in display (25). Conversely, if $h^0(E \otimes \Omega^1_{\mathbb{P}^3}) = 12$, then every monomorphism $\varphi : E(-3) \to \Omega^1_{\mathbb{P}^3}$ factors through $\tau$ as above, and hence $\text{coker} \, \varphi$ cannot be torsion free since it contains $\text{coker} \, \tau$ as a subsheaf. \qed
We will now use the Lemma 6.10 and Lemma 6.11 to address the existence part of Main Theorem 2.

We will prove stronger statements regarding instanton bundles $E$ of charge up to 3. First, note that twisting the Euler sequence for the cotangent bundle by $E(3)$ we obtain

$$0 \to E \otimes \Omega^1_{\mathbb{P}^3}(3) \to 4 \cdot E(2) \to E(3) \to 0$$

from which we conclude that

(26) \quad \text{hom}(E(-3), \Omega^1_{\mathbb{P}^3}) \geq 4 \cdot h^0(E(2)) - h^0(E(3)) = 40 - 11c_2(E),

with the last equality following from the fact that $h^0(E(p)) = \chi(E(p))$ for $p \geq 2$ since instanton bundles of charge $n$ are $n$-regular. This last fact also allows us to easily compute $h^1(E(p))$ for every $p \in \mathbb{Z}$; we have (we only write the dimensions of the nonzero cohomologies):

- for $n = 1$, $h^1(E(-1)) = 1$;
- for $n = 2$, $h^1(E(-1)) = h^1(E) = 2$;
- for $n = 3$, $h^1(E(-1)) = 3$, $h^1(E) = 4$ and $h^1(E(1)) = 1 + h^0(E(1))$;

In addition, if $E$ is an instanton bundle of charge 4 with natural cohomology, then $h^1(E(-1)) = 4$, $h^1(E) = 6$ and $h^1(E(1)) = 4$.

**Proposition 6.12.** For each instanton bundle $E$ of charge 1 there is a foliation by curves $\mathcal{F}$ of degree 3 such that $N^*_{\mathcal{F}}(3) = E$. Furthermore, $\text{Sing}(\mathcal{F})$ is a curve of degree 8 and arithmetic genus 5 that is connected whenever it is reduced, and $\dim_{\mathbb{C}} M_\mathcal{F} = 1$.

The second statement in the previous proposition follows from the formulas of Theorem 4.1 and in display (23), as well as the considerations in the paragraph just above it. We proceed in the same way in the next three proposition that close the proof of Main Theorem 2.

**Proof.** Since $h^0(E(1)) = 5$ we cannot apply the previous results in this case. However, instanton bundles of charge 1 are 1-regular, so $E(1)$ is globally generated. It follows that $E \otimes \Omega^1_{\mathbb{P}^3}(3)$ is globally generated as well; Ottaviani’s Bertini type Theorem [21, Theorema 2.8] implies that there is a monomorphism $E(-3) \to \Omega^1_{\mathbb{P}^3}$ whose cokernel is a torsion free sheaf (see also the Appendix in [4]). \hfill \Box

**Proposition 6.13.** For each instanton bundle $E$ of charge 2 there is a foliation by curves $\mathcal{F}$ of degree 3 such that $N^*_{\mathcal{F}}(3) = E$. Furthermore, $\text{Sing}(\mathcal{F})$ is a curve of degree 7 and arithmetic genus 2 that is connected whenever it is reduced, and $\dim_{\mathbb{C}} M_\mathcal{F} = 4$.

**Proof.** Every instanton bundle of charge 2 satisfies $h^0(E(1)) = 2$ and $\text{hom}(E(-3), \Omega^1_{\mathbb{P}^3}) \geq 18$, see the formula in display (26) above. The second part of Lemma 6.11 guarantees the existence of a monomorphism $E(-3) \to \Omega^1_{\mathbb{P}^3}$ with torsion free cokernel. \hfill \Box

**Proposition 6.14.** For each instanton bundle $E$ of charge 3 with $h^0(E(1)) \leq 1$ there is a foliation by curves $\mathcal{F}$ of degree 3 such that $N^*_{\mathcal{F}}(3) = E$. Furthermore, $C := \text{Sing}(\mathcal{F})$ is a curve of degree 6 and arithmetic genus 1 such that $h^0(O_C) = 2 + h^0(E(1))$, and $\dim_{\mathbb{C}} M_\mathcal{F} = 8 + h^0(E(1))$.

**Proof.** If $E$ is an instanton bundle of charge 3, then the formula in display (26) yields $\text{hom}(E(-3), \Omega^1_{\mathbb{P}^3}) \geq 7$. Assuming that $h^0(E(1)) \leq 1$, the second part of Lemma 6.10 and the
first item in Lemma 6.11 guarantee the existence of a monomorphism $E(-3) \to \Omega^1_{\mathbb{P}^3}$ with torsion free cokernel.

**Proposition 6.15.** There exists a foliation by curves $\mathcal{F}$ of degree 3 such that $N^*_{\mathcal{F}}(3)$ is an instanton bundle of charge 4 with natural cohomology. Furthermore, $\text{Sing}(\mathcal{F})$ consists of five disjoint lines, and $\dim \mathbb{C} \cdot M_{\mathcal{F}} = 14$.

**Proof.** Lemma 6.7 already guarantees the existence of a foliation by curves of degree 3 satisfying the conditions claimed.

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### 7. Legendrian foliations

A curve $C \subset \mathbb{P}^3$ is called legendrian if it is tangent to a contact structure given by

$$\mathcal{D} : 0 \to T\mathcal{D} \to TP^3 \to \mathcal{O}_{\mathbb{P}^3}(2) \to 0$$

We recall that $T\mathcal{D} = N(1)$, where $N$ is the null correlation bundle.

**Definition 7.1.** A foliation by curves $\mathcal{F}$ on $\mathbb{P}^3$ is called by Legendrian if $T\mathcal{F} \subset T\mathcal{D}$. That is, the leaves of $\mathcal{F}$ are legendrian curves outside the singular set of $\mathcal{F}$.

See [3] for details about legendrian curves on $\mathbb{P}^3$.

**Theorem 7.2.** Every Legendrian foliation $\mathcal{F}$ by curves of degree $d$ is of the form $\omega_0 \wedge \omega$, where $\omega_0$ is a contact form and $\omega \in H^0(\Omega^1_{\mathbb{P}^3}(d+1))$. In addition, the moduli space of the legendrian foliations of degree $d$ is an irreducible quasi-projective variety of dimension

$$d \cdot \left( \frac{d+3}{2} \right) - \left( \frac{d+2}{3} \right) + 4 \quad \text{if} \quad d \geq 2$$

and of dimension 8 if $d = 1$.

In particular, it follows from [8, Theorem 2] that the singular scheme of a Legendrian foliation is a Buchsbaum curve which, by Proposition 4.3, is connected for $d \geq 2$.

**Proof.** In fact, we have that an induced section $\sigma : T\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(1-d) \to T\mathcal{D} = N(1)$ where $N$ is the null correlation bundle. That is, the legendrian foliation $\mathcal{F}$ induces a global section of
\( \sigma \in H^0(\mathbb{P}^3, N(d)) \), which fits into the following commutative diagram

\[
\begin{array}{cccccccc}
0 & 0 \\
\downarrow & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(1-d) & \mathcal{O}_{\mathbb{P}^3}(1-d) \\
\downarrow & \downarrow \\
0 & N(1) & \rightarrow & T\mathbb{P}^3 & \mathcal{O}_{\mathbb{P}^3}(2) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{I}_C(d+1) & \rightarrow & G & \mathcal{O}_{\mathbb{P}^3}(2) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

where the central column is the one defining the Legendrian foliation. Considering the dual exact sequence of the bottom row, we have

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow G^* \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1-d) \rightarrow 0
\]

which directly implies that \( G^* \simeq \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1-d) \). Therefore we can conclude that the Legendrian foliation is of the required form \( \omega_0 \wedge \omega \), where \( \omega_0 \) is the contact form such that \( \ker(\omega_0) = T\sigma = N(1) \) and \( \omega \in H^0(\Omega^1_{\mathbb{P}^3}(d+1)) \).

In order to prove the last statements of the result, we want to apply Theorem 3.1. First of all, we need the vanishings required in the mentioned result, which means we must compute

\[
\dim(\text{Ext}^1(G^*, \mathcal{I}_Z(d-1))) = h^1(\mathcal{I}_Z(d+1)) + h^1(\mathcal{I}_Z(2d)) = 0,
\]

where \( Z \) denotes the singular locus if the Legendrian foliation. Hence, we can apply the theorem and prove the statement, recalling that the dimension of the moduli space is given by

\[
(27) \quad \dim(\text{Hom}(G^*, \mathcal{I}_Z(d-1))) = h^0(\mathcal{I}_Z(d+1)) + h^0(\mathcal{I}_Z(2d)),
\]

which is equal to

\[
d \cdot \left( \frac{d+3}{2} \right) - \left( \frac{d+2}{3} \right) + 4 \quad \text{if} \quad d \geq 2
\]

and to 8 if \( d = 1 \), as required. \( \square \)

Let us give a closer look into the formula expressed in display (27). As proven in the previous result, every Legendrian foliation can be expressed, up to a change of coordinates, as the wedge product of the canonical contact form with a 1-form \( \omega \in H^0(\Omega^1_{\mathbb{P}^3}(d+1)) \). This means that, set theoretically, once we have changed a given contact form in the canonical one, every Legendrian foliation is given by the choice of an element in \( \mathbb{P}^0H^0(\Omega^1_{\mathbb{P}^3}(d+1)) \). In other words, we expect to have a fibration

\[
\begin{array}{c}
\{ \text{Legendrian foliations} \} \\
\downarrow \\
\{ \text{Contact forms} \}
\end{array}
\]
whose the fiber over a contact structure $w_0$ is given by $H^0(\Omega^1_{\mathbb{P}^3}(d+1)) - \{h\omega_0; \ f \in \mathcal{O}_{\mathbb{P}^3}(d-1)\}$.

Observe that
\[ H^0(\Omega^1_{\mathbb{P}^3}(d+1)) - \{h\omega_0; \ f \in \mathcal{O}_{\mathbb{P}^3}(d-1)\} = 4(d+3) - \left(\frac{d+4}{3}\right) - \left(\frac{d+2}{3}\right) = d \cdot \left(\frac{d+3}{2}\right) - \left(\frac{d+2}{3}\right), \]

since
\[ 4(d+3) - \left(\frac{d+4}{3}\right) = d \cdot \left(\frac{d+3}{2}\right). \]

Such description fits perfectly in the computation of the dimension of the moduli space.

**Remark 7.3.** Another consequence of the previous result is that the conormal sheaf of a Legendrian foliation of degree $d$ always splits as a sum of line bundles $\mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-d)$; in other words, Legendrian foliation are of global complete intersection type.

More generally, one can also consider foliations by curves of global complete intersection type given by exact sequences of the form
\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2 - d_1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2 - d_2) \rightarrow \Omega^1_{\mathbb{P}^3} \rightarrow \mathcal{I}_C(d_1 + d_2) \rightarrow 0, \]

with $d_1, d_2 \geq 0$. Such foliations have degree $d_1 + d_2 + 1$ and their singular schemes are, by Proposition 4.3 connected curves satisfying
\[ \deg(C) = (d_1 + d_2)^2 - d_1 d_2 + 2(d_1 + d_2 + 1) \quad \text{and} \]
\[ p_a(C) = (d_1 + d_2 + 1)^3 - 2(d_1 + d_2 + 1)^2 - (d_1 + d_2)(3d_1 d_2 - 2)/2, \]

according to the formulas in displays (14) and (15). In addition, since $\Omega^1(d_1 + 2) \oplus \Omega^1(d_2 + 2)$ is globally generated, Ottaviani’s Bertini type Theorem [21, Teorema 2.8] implies that the singular scheme of a generic foliation of the form described in display (28) is smooth. The moduli spaces of foliations by curves of the form (28) can be described similarly to Legendrian foliations, following the arguments in the proof of Theorem 7.2 above.

### 8. The Rao module of singular schemes of foliations by curves

As it was mentioned in the Introduction, the first cohomology module of the conormal sheaf (or equivalently, the Rao module of the singular scheme) is an important piece of algebraic information attached to a foliation by curves. We will now focus on explaining the relation between them, and discuss how they can be useful in the classification of particular classes of foliations by curves.

So let $\mathcal{F}$ be a foliation by curves on $\mathbb{P}^3$, and consider the following two graded modules
\[ R_{\mathcal{F}} := H^1_*(\mathcal{I}_Z) \quad \text{and} \quad M_{\mathcal{F}} := H^1_*(N_{\mathcal{F}}^\ast); \]

they will be called the **Rao module** and the **first cohomology module** of the foliation $\mathcal{F}$. Note that $R_{\mathcal{F}}$ is finite dimensional (as a $\mathbb{C}$-vector space) if and only if $Z := C$ has pure dimension 1, or equivalently, if and only if $\mathcal{F}$ is of local complete intersection type. On the other hand, $M_{\mathcal{F}}$ is always finite dimensional.

**Lemma 8.1.** If $\mathcal{F}$ be a foliation by curves on $\mathbb{P}^3$ of local complete intersection type, then
\[ \dim_{\mathbb{C}} M_{\mathcal{F}} \leq \dim_{\mathbb{C}} R_{\mathcal{F}} \leq \dim_{\mathbb{C}} M_{\mathcal{F}} + 1. \]
Moreover, if $h^1(N^*_F) = 0$, then the second equality holds.

Proof. Starting with the exact sequence
\[ 0 \to N^*_F(k - d + 1) \to \Omega^1(k - d + 1) \to H(k) \to 0 \]
where $d$ is the degree of $\mathcal{F}$, it is easy to see that
\[ h_1(H(k)) = h^2(N^*_F(k - d + 1)) = h^1(N^*_F(2d + 2 - k)). \]
whenever $k \neq d - 1$. When $k = d - 1$, we obtain the following exact sequence in cohomology:
\[ 0 \to H^0(H(d - 1)) \to H^1(N^*_F) \to H^1(\Omega^1_{\mathbb{P}^3}) \to H^1(H(d - 1)) \to H^2(N^*_F) \to 0, \]
thus $h^1(H(d - 1)) - h^1(N^*_F(d + 3))$ is either 0 or 1, proving the two inequalities in the claim. If $h^1(N^*_F) = 0$, then $h^1(H(d - 1)) - h^1(N^*_F(d + 3)) = 1$ and we get that the second equality must hold. \[ \Box \]

It is easy to see that the Rao module $R_{\mathcal{F}}$ of a foliation of local complete intersection type is always nontrivial. Indeed, if $\dim H^0(\mathcal{F}) = 0$, then $\dim H^0_{\mathcal{F}} = 0$ as well, implying, by the Horrocks splitting criterion, that $N_{\mathcal{F}}$ splits as a sum of line bundles; but then $h^1(N_{\mathcal{F}}) = 0$, hence the second equality in display (29) must be satisfied, which yields a contradiction.

Note also that if $N_{\mathcal{F}}$ splits as a sum of line bundles, then $\dim H^0_{\mathcal{F}} = 0$ and $h^1(N_{\mathcal{F}}) = 0$, so it follows from Lemma 8.1 that $\dim H^0_{\mathcal{F}} = 0$. This claim and its converse were already established as a particular case of [8] Theorem 2]; for the sake of completeness, we reproduce the result here.

**Theorem 8.2.** Let $\mathcal{F}$ be a foliation by curves. The conormal sheaf $N^*_F$ splits as a sum of line bundles if and only if $\dim H^0(\mathcal{F}) = 1$.

The goal of this section is to consider the case $\dim H^0(\mathcal{F}) = 2$, and establish the proof of Main Theorem 4.

First, recall that a curve $C \subset \mathbb{P}^3$ is said to be arithmetically Buchsbaum if its Rao module $H^1_1(\mathcal{I}_C)$ is trivial as graded $\mathbb{C}[x_0, x_1, x_2, x_3]$-module, that is, the multiplication map
\[ H^1_1(\mathcal{I}_C(k)) \to H^1_1(\mathcal{I}_C(k + 1)) \]
is zero for every $f \in H^0_1(\mathcal{O}_{\mathbb{P}^3}(1))$. A particular class of arithmetically Buchsbaum curves are those with Rao module is concentrated in a single degree, that is, there is $\delta \in \mathbb{Z}$ such that $h^1(\mathcal{I}_C(k)) = 0$ for every $k \neq \delta$. In particular, of $\dim H^1_1(\mathcal{I}_C) = 1$, then $C$ must be arithmetically Buchsbaum, so the singular scheme of a foliation by curves of global complete intersection type is always arithmetically Buchsbaum, as it was observed in [8].

Another class of foliations by curves with arithmetically Buchsbaum singular schemes arises as follows. Recall that the null correlation bundle $N$ on $\mathbb{P}^3$ is defined by the exact sequence
\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega^1_{\mathbb{P}^3} \to N \to 0. \]
Note that $N(k)$ is globally generated for every $k \geq 1$, so that $N \otimes \Omega^1_{\mathbb{P}^3}(k + 2)$ is also globally generated for every $k \geq 1$. It then follows from Ottaviani's Bertini type Theorem [21] Teorema 2.8, that there is a monomorphism $N(2 - k) \to \Omega^1_{\mathbb{P}^3}$ for each $k \geq 1$ whose cokernel is a torsion free sheaf.
This observation guarantees the existence of foliations by curves
\begin{equation}
0 \to N(-k - 2) \to \Omega^1_F \to \mathcal{I}_C(2k) \to 0
\end{equation}
of degree $2k + 1$, $k \geq 1$, whose conormal sheaves are a twisted null correlation bundle. The equalities in Theorem 4.1 yield
\begin{align*}
\deg(C) &= (3k + 1)(k + 1) \\
\rho_a(C) &= 5k^3 + 4k^2 - 3k - 1.
\end{align*}
Furthermore, since null correlation bundles coincide with instantoons bundles of charge 1, we remark that the case $k = 1$ coincides with foliation obtained in item (3) of Main Theorem 2 for $c_2(E) = 1$; see also Proposition 6.12.

On the other hand, note that there are no morphisms $N(-2 - k) \to \Omega^1_F$ when $k \leq -1$; just twist the Euler sequence
\begin{equation*}
0 \to \Omega^1 \to \mathcal{O}_F \to \mathcal{O}_C \to 0
\end{equation*}
by $N(k + 2)$ and recall that $H^0(N(k + 1)) = 0$ when $k \leq -1$.

For $k = 0$, even though $\dim \text{Hom}(N(-2), \Omega^1_F) \neq 0$, the curve $C$ in display (31) would have degree 1 and genus $-1$, which is impossible; so there can be no monomorphisms $N(-2) \to \Omega^1_F$ with torsion free cokernel.

**Proposition 8.3.** Let $\mathcal{F}$ be a foliation by curves on $\mathbb{P}^3$. If $N_{\mathcal{F}}$ is a twisted null correlation bundle, then $\text{Sing}(\mathcal{F})$ is a connected, arithmetically Buchsbaum curve and $\dim_C R_{\mathcal{F}} = 2$. In addition, for a generic such foliation, $\text{Sing}(\mathcal{F})$ is smooth.

**Proof.** Consider a foliation by curves like the one in display (31), so that $N_{\mathcal{F}} = N(-k - 2)$ for some null correlation bundle $N$ and some $k \geq 1$. It follows that $\dim_C M_{\mathcal{F}} = 1$, and Lemma 8.1 implies that $\dim_C R_{\mathcal{F}} = 2$ because $h^0(N_{\mathcal{F}}) = h^0(N(-k - 2)) = 0$ for every $k \geq 1$. The connectedness of $\text{Sing}(\mathcal{F})$ is a simple consequence of Proposition 4.3 since $h^2(N_{\mathcal{F}}(1 - d)) = h^2(N(-3k - 2)) = 0$ for every $k \geq 1$. The smoothness of $\text{Sing}(\mathcal{F})$ for a generic foliation is an immediate consequence of Ottaviani’s Bertini type Theorem 21, Teorema 2.8.

In addition, we show that $C := \text{Sing}(\mathcal{F})$ is arithmetically Buchsbaum. First, we check that $H^1(\mathcal{I}_C(p)) = 0$ for $p \neq 2k, 3k - 1$. Indeed, the cohomology sequence associated to the exact sequence in display (31) yields
\begin{equation*}
H^1(\Omega^1_F(p - 2k)) \to H^1(\mathcal{I}_C(p)) \to H^2(N(p - 3k - 2)).
\end{equation*}
The left term vanishes when $p \neq 2k$, while the right one vanishes when $p \neq 3k - 1$.

If $k = 1$ (so that $2k = 3k - 1 = 2$), then we have
\begin{equation*}
0 \to H^1(\Omega^1_F) \to H^1(\mathcal{I}_C(2)) \to H^1(N(-3)) \to 0,
\end{equation*}
thus $h^1(\mathcal{I}_C(p)) = 0$ for $p \neq 2$, and $h^1(\mathcal{I}_C(2)) = 2$; in particular, $C$ must be arithmetically Buchsbaum.

If $k \geq 2$, then $h^1(\mathcal{I}_C(p)) = 0$ for $p \neq 2k, 3k - 1$, and $h^1(\mathcal{I}_C(2k)) = h^1(\mathcal{I}_C(3k - 1)) = 1$. Note that $2k$ and $3k - 1$ are not consecutive when $k \geq 3$, so it easily follows that $C$ must also be arithmetically Buchsbaum.
In the case $k = 2$, it is enough to show that the multiplication map $f : H^1(I_C(4)) \to H^1(I_C(5))$ is zero for every $f \in H^0(O_{\mathbb{P}^3}(1))$. To see this, consider the following commutative diagram

\[
\begin{array}{ccc}
H^1(\Omega^1_{\mathbb{P}^3}) & \xrightarrow{f} & H^1(\Omega^1_{\mathbb{P}^3}(1)) = 0 \\
\simeq & \downarrow & \downarrow \\
H^1(I_C(4)) & \xrightarrow{f} & H^1(I_C(5))
\end{array}
\]

with the left vertical arrow being an isomorphism. It clearly follows that the lower horizontal map must vanish. \hfill \Box

The converse of Proposition 8.3 requires a technical lemma about rank 2 bundles on $\mathbb{P}^3$.

**Lemma 8.4.** There are no rank 2 locally free sheaves $E$ on $\mathbb{P}^3$ such that $\dim \mathcal{H}_1^*(E) = 2, 3$.

**Proof.** The claim is an immediate consequence of the various classification results in [24]. If $H^1_1(E)$ is concentrated in a single degree, then [24] Example 1 implies that $\dim \mathcal{H}_1^*(E) = 1$. If $H^1_1(E)$ is concentrated in two different degrees, then [24] Proposition 4 implies that $\dim \mathcal{H}_1^*(E) = 4$. Finally, $H^1_1(E)$ is concentrated in three different degrees, then [24] Theorem 2 implies that $\dim \mathcal{H}_1^*(E) \geq 4$. \hfill \Box

Our next result completes the proof of the first part of Main Theorem 4.

**Proposition 8.5.** Let $\mathcal{F}$ be a foliation by curves on $\mathbb{P}^3$. If $\dim \mathcal{H}_R^*(\mathcal{F}) = 2$, then $N^*_\mathcal{F}$ is a twisted null correlation bundle.

**Proof.** By Lemma 8.1, either $\dim \mathcal{H}_M^*(\mathcal{F}) = 2$ or $\dim \mathcal{H}_M^*(\mathcal{F}) = 1$. The first possibility is ruled out by Lemma 8.1 while second possibility implies, by [24] Example 1], that $N^*_\mathcal{F}$ must be a twisted null correlation bundle. \hfill \Box

**Remark 8.6.** Lemma 8.1 and Lemma 8.4 have another interesting consequence: there are no foliations by curves $\mathcal{F}$ such that $\dim \mathcal{H}_R^*(\mathcal{F}) = 3$.

If $E$ is a stable rank 2 locally free sheaf with $c_1(E) = 0$ and $c_2(E) = 2$, then $E(-3)$ is the conormal sheaf of a foliation by curves $\mathcal{F}$ of degree 3 (see item (3) of Main Theorem 2) with $\dim \mathcal{H}_R^*(\mathcal{F}) = 5$, since $\dim \mathcal{H}_M^*(\mathcal{F}) = 4$ and $h^1(N^*_\mathcal{F}) = h^1(E(-3)) = 0$.

However, we have not been able to find an example of a foliation by curves $\mathcal{F}$ satisfying $\dim \mathcal{H}_R^*(\mathcal{F}) = 4$. \hfill \Box

Regarding the proof of Main Theorem 4, we are left with the task of describing the moduli space of the foliations of degree $2k + 1$ whose conormal sheaf is a twisted null correlation bundle, as defined in display (31). The strategy is to check the vanishing conditions required in the hypotheses of Lemma 8.2.

Being $N$ the null correlation bundle, we already know that

$$\text{Ext}^2(N^*_\mathcal{F}, N^*_\mathcal{F}) = \text{Ext}^2(N, N) = 0.$$ 

Let us now consider

$$\text{Ext}^1(N^*_\mathcal{F}, \Omega^1_{\mathbb{P}^3}) \simeq H^1(\Omega^1_{\mathbb{P}^3} \otimes N(k + 2));$$
Tensoring the sequence in display (30) by $\Omega^1_{P^3}(k + 2)$ and taking cohomology we have that

$$H^1(\Omega^1_{P^3} \otimes N(k + 2)) \simeq H^1(\Omega^1_{P^3} \otimes \Omega^1_{P^3}(k + 3)),$$

since $h^p(\Omega^1_{P^3}(k + 1)) = 0$ for $p = 1, 2$ and every $k \geq 1$. Taking now the following resolution of the cotangent bundle

$$0 \to \mathcal{O}_{P^3}(-3) \to \mathcal{O}_{P^3}(-2)^4 \to \mathcal{O}_{P^3}(-1)^6 \to \Omega^1_{P^3}(1) \to 0,$$

and tensoring it by $\Omega^1_{P^3}(k + 2)$, it is straightforward to see that

$$h^1(\Omega^1_{P^3} \otimes \Omega^1_{P^3}(k + 3)) = 0,$$

as required.

The next step is to compute the two terms in the formula for the dimension. Recall that

$$\dim \text{Ext}^1(N^*_F, N^*_F) = \dim \text{Ext}^1(N, N) = 5,$$

and note that

$$\dim \text{Hom}(N^*_F, \Omega^1_{P^3}) = h^0(\Omega^1_{P^3} \otimes N(k + 2)).$$

To compute the latter, consider the Euler short exact sequence tensored by $N(k + 1)$, that is

$$0 \to \Omega^1_{P^3} \otimes N(k + 2) \to N(k + 1)^{\oplus 4} \to N(k + 2) \to 0.$$

Recalling that

$$h^0(N(t)) = 2\binom{t + 3}{3} - (t + 2),$$

we have

$$h^0(\Omega^1_{P^3} \otimes N(k + 2)) = 4h^0(N(k + 1)) - h^0(N(k + 2)) = 8\binom{k + 4}{3} - 2\binom{k + 5}{3} - 3k - 8,$$

and hence the required dimension of the moduli space. Together with the considerations made in the paragraph below the proof of Lemma 3.2, the proof of Main Theorem 4 is finally complete.

At last, as a by product of the results in this section, we close this paper by providing a full characterization of those foliations by curves $\mathcal{F}$ such that $R_\mathcal{F}$ is concentrated in a single degree.

**Theorem 8.7.** Let $\mathcal{F}$ be a foliation by curves such that $R_\mathcal{F}$ is concentrated in degree $\delta$. Then

(i) either $\mathcal{F}$ has degree $\delta + 1$, $N^*_\mathcal{F}$ splits as a sum of line bundles, and $\dim \mathcal{C} = 1$;

(ii) or $\mathcal{F}$ has degree 3, $\delta = 4$, $\dim \mathcal{C} = 2$, and $N^*_\mathcal{F} \simeq N(-3)$ for some null correlation bundle $N$.

**Proof.** The hypothesis implies that $C := \text{Sing} (\mathcal{F})$ is a curve; assume that $h^1(\mathcal{I}_C(p)) = 0$ for every $p \neq \delta$, and $\dim \mathcal{C} R_\mathcal{F} = h^1(\mathcal{I}_C(\delta)) = t \neq 0$.

Consider the exact sequence

$$0 \to \mathcal{I} \to \mathcal{I}_C(d - 1) \to 0;$$

by hypothesis, $N^*_\mathcal{F}$ is locally free. Since the map $H^1(\mathcal{I}_C(p)) \to H^2(N^*_\mathcal{F}(p - d + 1))$ is surjective, we conclude that $H^2(N^*_\mathcal{F}(k)) = 0$ for every $k \neq \delta - d + 1$, and $h^2(N^*_\mathcal{F}(\delta - d + 1)) \leq t$. 


If $h^2(N^*_P(\delta - d + 1)) = 0$, then $N^*_P$ must split as a sum of line bundles by the Horrocks splitting criterion. It then follows that $h^1(N^*_P(k)) = 0$ for every $k$, so $H^1(\Omega^1(\delta - d + 1) \cong H^1(\mathcal{I}_C(\delta))$, thus $d = \delta + 1$, and $h^1(\mathcal{I}_C(\delta)) = 1$.

Recall that the every rank 2 locally free sheaf $E$ such that $h^2(E(k)) = 0$ for every $k \neq l$ and $h^2(E(l)) \neq 0$ must be of the form $N(-l - 3)$ for some null correlation bundle $N$. In this case, in which case $h^2(E(l)) = 1$.

Therefore, if $h^2(N^*_P(\delta - d + 1)) \neq 0$, then $N^*_P = N(d - \delta - 4)$ for some null correlation bundle $N$; from the exact sequence

$$H^1(N(p - \delta - 3)) \rightarrow H^1(\Omega^1_{\mathbb{P}^3}(p - d + 1)) \rightarrow H^1(\mathcal{I}_C(p)) \rightarrow H^2(N(p - \delta - 3)) \rightarrow 0$$

we conclude that $d = \delta + 1$, so $N^*_P = N(-3)$, from which it follows that $d = 3$ and $t = 2$.

\[ \square \]

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