A Note on the Minimum Number of Edges in Hypergraphs with Property O

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Abstract

An oriented k-graph is said to have Property O if for every linear order of the vertex set, there is some edge oriented consistently with the linear order. Recently Duffus, Kay and Rödl investigated the minimum number $f(k)$ of edges in a k-uniform hypergraph with Property O. They proved that $k! \leq f(k) \leq (k^2 \ln k)k!$, where the upper bound holds for sufficiently large $k$. In this short note we improve their upper bound by a factor of $k \ln k$ showing that $f(k) \leq (\lceil \frac{k}{2} \rceil + 1)k! - \lfloor \frac{k}{2} \rfloor (k - 1)!$ for every $k \geq 3$. We also show that their lower bound is not tight. Furthermore, Duffus, Kay and Rödl also studied the minimum possible number $n(k)$ of vertices in an oriented k-graph with Property O. For $k = 3$ they showed that $n(3) \in \{6, 7, 8, 9\}$, and asked for the precise value of $n(3)$. Here we show that $n(3) = 6$.

1 Introduction

Extremal combinatorics is one of the central branches of modern combinatorial theory, which has developed spectacularly over the last few decades. A typical problem in extremal combinatorics has the following form: What is the maximum or minimum size of a finite structure which satisfies certain properties? One of the most famous examples is Property B, first introduced by Bernstein in 1908 [2] (see [1] for a good overview), asking for the minimum number of edges in a k-uniform hypergraph such that every two colouring of its vertex set has a monochromatic edge.

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Recently Duffus, Kay and Rödl [3] introduced an analogue to Property B, called *Property O*, with colouring replaced by order. An *oriented* $k$-*graph* is a pair $\mathcal{H} = (V, \mathcal{E})$, where $V$ is a finite set, and $\mathcal{E}$ is a family of $k$-tuples of $V$ such that no $k$-tuple has a repeated element, and no two $k$-tuples induce the same set. We say $V$ is the vertex set of $\mathcal{H}$, and $\mathcal{E}$ is the edge set of $\mathcal{H}$. In the case that $\mathcal{E}$ contains a $k$-tuple for each $k$-subset of $V$, call $\mathcal{H}$ a $k$-*tournament*. A tuple $(x_1, x_2, \ldots, x_\ell)$ of distinct elements of $V$ is said to be consistent with a linear order $<$ on $V$ if $x_1 < x_2 < \ldots < x_\ell$.

**Definition 1.1 (Property O).** Given an oriented $k$-graph $\mathcal{H} = (V, \mathcal{E})$, we say that $\mathcal{H}$ has *Property O* if for every linear order $<$ of $V$, there exists an edge $\vec{e} \in \mathcal{E}$ that is consistent with $<$. Furthermore, let $f(k)$ be the minimum number of edges in an oriented $k$-graph with Property O. That is,

$$f(k) := \min\{|\mathcal{E}| : \text{there exists an oriented } k\text{-graph } \mathcal{H} = (V, \mathcal{E}) \text{ with Property O} \}.$$ 

It is easy to check that $f(2) = 3$ and an example for the upper bound is a cyclically ordered triangle. In [3], Duffus, Kay and Rödl initiated the study of $f(k)$, and established the following general bounds.

**Theorem 1.2 (Duffus–Kay–Rödl).** The function $f(k)$ satisfies

$$k! \leq f(k) \leq (k^2 \ln k)k!,$$

where the lower bound holds for all $k$ and the upper bound holds for $k$ sufficiently large.

Note that the lower bound follows from a simple counting argument, which we include here for convenience of the reader. Suppose that $\mathcal{H} = ([n], \mathcal{E})$ is an oriented $k$-graph that has Property O. Then every edge $\vec{e} \in \mathcal{E}$ is consistent with exactly $\binom{n}{k} (n-k)! = \frac{n!}{k!}$ linear orders on $[n]$. Since $\mathcal{H}$ has Property O, we must have $|\mathcal{E}| \cdot \frac{n!}{k!} \geq n!$, implying $f(k) \geq k!$.

For the upper bound, Duffus et al. showed that a random $k$-tournament on $n = \frac{k^2}{\pi} \exp\left(\frac{e^2}{2}\right) \cdot k^3 \ln k \cdot k^1/k$ vertices with $\binom{n}{k} < k^2 \ln(k)k!$ edges has Property O with positive probability. Furthermore they proved that almost all $k$-tournaments with $(1 - o(1))\sqrt{k} \cdot k!$ edges do not have Property O.

In this note we show that the lower bound in Theorem 1.2 is not tight, and the upper bound in Theorem 1.2 can be improved by a factor of $k \ln k$.

**Theorem 1.3.** For every integer $k \geq 3$, we have

$$k! + 1 \leq f(k) \leq \left(\left\lceil \frac{k}{2} \right\rceil + 1 \right) k! - \left\lceil \frac{k}{2} \right\rceil (k-1)!.$$

In Subsection 2.1 we provide an explicit construction for showing the upper bound, and in Subsection 2.2 we give a proof for the lower bound.

Another natural problem posed by Duffus, Kay and Rödl [3] is to determine the minimum number of vertices in an oriented $k$-graph with Property O.

**Definition 1.4.** For $k \geq 2$ we define

$$n(k) := \min\{|V| : \text{there exists an oriented } k\text{-graph } \mathcal{H} = (V, \mathcal{E}) \text{ having Property O} \}.$$ 

Duffus et al. proved $6 \leq n(3) \leq 9$; for the upper bound they gave a construction and the lower bound was obtained via an exhaustive computer search. In Section 3 we show $n(3) = 6$ by providing two different constructions.
2 Proof of Theorem 1.3

2.1 Upper bound

In this subsection we will construct an oriented $k$-graph with $(\lceil \frac{k}{2} \rceil + 1) k! - \lceil \frac{k}{2} \rceil (k - 1)!$ edges possessing Property O. To aid the reader, we first describe the idea behind our construction for the case $k = 3$.

We start by defining two edges $(x, y, a)$ and $(y, x, b)$. Any ordering is consistent with the relative order of exactly one of these edges with respect to the positions of $x$ and $y$. If it happens to be $x < y$, but the edge $(x, y, a)$ is not consistent with the ordering, then there are two possibilities for the position of $a$ with respect to both $x$ and $y$. For each possibility we introduce one new vertex and two edges, such that at least one of them is consistent with the ordering.

**Warm-up.** There is an oriented 3-graph with 10 edges having Property O.

**Proof.** Let $\mathcal{H}$ be an oriented 3-graph with vertex set $V = \{x, y, a, b, c, d, e, f\}$, and edge set

$$
\mathcal{E} = \{(x, y, a), (a, x, c), (c, x, y), (x, a, d), (d, a, y), (y, x, b), (b, y, e), (e, y, x), (y, b, f), (f, b, x)\}.
$$

Clearly $\mathcal{H}$ has 10 edges. It remains to show that $\mathcal{H}$ possesses Property O. Let $<$ be an arbitrary ordering of $V$. Since either $x < y$ or $y < x$, let us first suppose $x < y$. If the edge $(x, y, a)$ is not consistent with $<$, then we either have $a < x < y$ or $x < a < y$. If $a < x < y$, then either $(a, x, c)$ or $(c, x, y)$ is consistent with $<$. On the other hand, if $x < a < y$, then one of the edges $(x, a, d)$ and $(d, a, y)$ is consistent with $<$. Now, if $y < x$ but $(y, x, b)$ is not consistent with $<$, then either $b < y < x$ or $y < b < x$. In the first case one of the edges $(b, y, e)$ and $(e, y, x)$ is consistent with $<$, and in the latter one of the edges $(y, b, f)$ and $(f, b, x)$ is consistent with $<$. Hence $\mathcal{H}$ has Property O, as desired. $\square$

To prove the upper bound in Theorem 1.3, we will generalise the above construction. We begin with $(k - 1) + (k - 1)!$ vertices $x_1, \ldots, x_{k-1}, a_1, \ldots, a_{(k-1)!}$, and $(k - 1)!$ edges $(\pi_1, a_1), \ldots, (\pi_{(k-1)!}, a_{(k-1)!})$, where $\pi_1, \ldots, \pi_{(k-1)!}$ are permutations (viewed as $(k-1)$-tuples) of $\{x_1, \ldots, x_{k-1}\}$. Let $<$ be any linear order of the vertices. We can find $j \in [(k - 1)!]$ such that $\pi_j$ is consistent with $<$. If the edge $(\pi_j, a_j)$ is not consistent with $<$, then there are $(k - 1)$ possible locations for $a_j$ (in the restriction of $<$ to $x_1, \ldots, x_{k-1}, a_i$). For each possibility we introduce one new vertex and $\lceil \frac{k}{2} \rceil + 1$ edges such that at least one of the edges will be consistent with $<$. More details can be found below.

**Proof of Theorem 1.3 (upper bound).** We shall construct an oriented $k$-graph $\mathcal{H} = (V, \mathcal{E})$ with the desired property. Let

$$
V = \left\{x_1, x_2, \ldots, x_{k-1}, a_1, \ldots, a_{(k-1)!}, a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(k-1)}, a_2^{(1)}, \ldots, a_2^{(k-1)}, \ldots, a_{(k-1)!}^{(1)}, a_{(k-1)!}^{(2)}, \ldots, a_{(k-1)!}^{(k-1)}\right\},
$$

and let $\mathcal{E}$ be the set of $k$-tuples of $V$ defined as follows. Let $\pi_1, \ldots, \pi_{(k-1)!}$ be all possible permutations of $\{x_1, \ldots, x_{k-1}\}$ viewed as $(k - 1)$-tuples.

1. We put all $k$-tuples of the form $(\pi_j, a_j)$ into $\mathcal{E}$. There are $(k - 1)!$ such $k$-tuples.

2. For each $j \in [(k - 1)!]$, we will add $(k - 1) \left(\lceil \frac{k}{2} \rceil + 1\right)$ edges to $\mathcal{E}$ as follows. For every $i \in [k - 1]$ and $\ell \in \{1, 3, \ldots, 2 \lceil \frac{k}{2} \rceil - 1, k\}$, let $\pi_j^{(i)}$ be the $k$-tuple obtained by inserting
We associate to each permutation $\sigma$ one edge. Suppose for the contrary that there exists an oriented $\sigma$-graph $H$.

In this subsection we will prove the lower bound $f(k) \geq k! + 1$.

Proof of Theorem 1.3 (lower bound). Suppose for the contrary that there exists an oriented $k$-graph $H = ([n], E)$ with Property O such that $|E| \leq k!$. Observe that each edge $\vec{e} \in E$ is consistent with exactly $\binom{n}{k}(n-k)! = \frac{n^k}{k!}$ linear orders of $[n]$. Since $H$ has Property O, we must have $|E| \cdot \frac{n!}{k!} \geq n!$. Hence $|E| = k!$, and each linear order of $[n]$ is consistent with exactly one edge.

Write $E = \{\vec{e}^1, \ldots, \vec{e}^{k!}\}$. Without loss of generality we can suppose that $\vec{e}^1 = (1, 2, \ldots, k)$. We associate to each permutation $\sigma: [k] \to [k]$ a linear order $\bar{\sigma}$ on $[n]$, defined by

$$\sigma(1) < \bar{\sigma}(2) < \ldots < \sigma(k) \bar{\sigma} < k + 1 \bar{\sigma} < k + 2 \bar{\sigma} < \ldots < n.$$ 

For $1 \leq i \leq k!$, let $S_i = \{\sigma: \vec{e}^i_1 \text{ is consistent with } \bar{\sigma}\}$. From the discussion above, we learn that each permutation $\sigma: [k] \to [k]$ is contained in exactly one $S_i$, and so

$$\sum_{1 \leq i \leq k!} |S_i| = k!.$$ 

Note that

$$|S_i| = \frac{k!}{(|\vec{e}^i_1 \cap [k]|)!}$$

for every $i \in [k!]$ such that $S_i \neq \emptyset$, since whether $\sigma \in S_i$ depends only on the relative order of $\vec{e}^i_1 \cap [k]$. This together with the assumption that $\vec{e}^i_1$ and $\vec{e}^j_1$ induce different $k$-sets implies that $|S_1| = 1$ and $|S_i|$ is divisible by $k$ for $i > 1$. Therefore, we get a contradiction

$$0 \equiv k! = \sum_{1 \leq i \leq k!} |S_i| \equiv 1 \pmod{k},$$

finishing our proof. \qed
3 Oriented 3-graphs with Property O

In this section we will construct two oriented 3-graphs on 6 vertices having Property O. Combined with the lower bound \( n(3) \geq 6 \) given in [3], this shows \( n(3) = 6 \).

**First construction** For each \( k \geq 2 \) we construct an oriented \( k \)-graph \( \mathcal{H}_k = (V_k, E_k) \) with Property O, where
\[
|V_k| = 3 \cdot 2^{k-2} \quad \text{and} \quad |E_k| = 3^{k-1} \cdot 2^{\binom{k-1}{2}},
\]
(1)
Let \( \mathcal{H}_2 = (V_2, E_2) \) be an oriented 3-cycle. It is easy to see that \( \mathcal{H}_2 \) has Property O, and the sizes of its vertex set and edge set are given by (1).

Suppose that we have constructed an oriented \( k \)-graph \( \mathcal{H}_k = (V_k, E_k) \) with the desired properties. Now make two disjoint copies \( \mathcal{H}_X = (X, E_X) \) and \( \mathcal{H}_Y = (Y, E_Y) \) of \( \mathcal{H}_k = (V_k, E_k) \). Let \( \mathcal{H}_{k+1} = (V_{k+1}, E_{k+1}) \) be an oriented \( (k+1) \)-graph with vertex set \( V_{k+1} = X \cup Y \), and edge set
\[
E_{k+1} = \{(x_1, \ldots, x_k, y) : y \in Y, (x_1, \ldots, x_k) \in E_X\} \cup \{(y_1, \ldots, y_k, x) : x \in X, (y_1, \ldots, y_k) \in E_Y\}
\]
(see Fig. 1).

![Figure 1: The oriented 3-graph \( \mathcal{H}_3 \) with the edge \((a_0, a_1, b_1)\) depicted.](image)

Let us show that \( \mathcal{H}_{k+1} = (V_{k+1}, E_{k+1}) \) satisfies (1). Indeed, from the definition of \( V_{k+1} \) and \( E_{k+1} \) we find \( |V_{k+1}| = 2|V_k| = 2 \cdot 3 \cdot 2^{k-2} = 3 \cdot 2^{k-1} \), and
\[
|E_{k+1}| = |V_{k+1}| \cdot |E_k| = 3 \cdot 2^{k-1} \cdot 3^{k-1} \cdot 2^{\binom{k-1}{2}} = 3^k \cdot 2^{\binom{k}{2}}.
\]

To see that \( \mathcal{H}_{k+1} \) has Property O, let \( < \) be any linear order on \( V_{k+1} \). We can find an edge of \( E_{k+1} \) which is consistent with \( < \) as follows.

1. Suppose first that there is some vertex \( y \in Y \) such that \( y > x \) for all \( x \in X \). Because \( \mathcal{H}_X \) is isomorphic to \( \mathcal{H}_k \), some tuple \( (x_1, \ldots, x_k) \in E_X \) must be consistent with \( < \). Since \( y > x_k \), the edge \( (x_1, \ldots, x_k, y) \) is consistent with \( < \).

2. Suppose now that there exists \( x \in X \) such that \( x > y \) for every \( y \in Y \). By the same argument as above, we can show that some edge of the form \( (y_1, \ldots, y_k, x) \) is consistent with \( < \).
Second construction  The second example is a simple modification of the construction given in Section 2. Instead of using the vertices $e$ and $f$, we can use $c$ and $d$ again: Simply replace $e$ by $d$ and $f$ by $c$. So we get an oriented 3-graph $H$ with vertex set $V = \{x, y, a, b, c, d\}$ and edge set

\[ E = \{(x, y, a), (a, x, c), (c, x, y), (x, a, d), (d, a, y), (y, x, b), (b, y, d), (d, y, x), (y, b, c), (c, b, x)\}. \]

One can use the same proof to show that $H$ possesses Property O.

4 Concluding Remarks

In this note we have shown that $k! + 1 \leq f(k) \leq \left(\left\lceil \frac{k}{2} \right\rceil + 1\right) k! - \left\lceil \frac{k}{2} \right\rceil (k - 1)!$ for every $k \geq 3$. The main open problem regarding the asymptotic behaviour of $f(k)$ is the following.

Problem 4.1 (Duffus–Kay–Rödl [3]). Is it true that $\frac{f(k)}{k!} \to \infty$ as $k \to \infty$?

We believe that the answer should be yes. However, we have not even been able to show that there is some absolute constant $c > 1$ such that $f(k) > ck!$ for $k$ sufficiently large. Of course, any improvement of the upper bound would be interesting as well.

Recall that $n(k)$ is the minimum possible number of vertices in an oriented $k$-graph with Property O. It is easy to see that $n(2) = 3$, while showing $n(3) = 6$ requires some effort (see Section 3). This is all we know about the precise values of $\{n(k)\}_{k \geq 2}$. However, for large $k$, one can determine $n(k)$ asymptotically. Indeed, a trivial lower bound on $n(k)$ is $\left(\frac{k}{e}\right)^2$, since the number of edges is at least $k!$ and at most $\left(\binom{n(k)}{k}\right)$. On the other hand, Duffus et al. [3, pp. 3–4] showed that a random $k$-tournament on $n = \left(\frac{k}{e}\right)^2 \left(\pi \cdot \exp(e^2/2) \cdot k^3 \ln k\right)^{1/k}$ vertices has Property O with positive probability. Hence

\[ n(k) \leq \left(\frac{k}{e}\right)^2 \left(\pi \cdot \exp(e^2/2) \cdot k^3 \ln k\right)^{1/k} = (1 + o(1)) \left(\frac{k}{e}\right)^2, \]

and so $n(k) = (1 + o(1)) \left(\frac{k}{e}\right)^2$.

The second construction in Section 3 has fairly few edges, namely 10, and $n(3) = 6$ vertices. This naturally leads us to the following question.

Problem 4.2. Let $k \geq 3$ be an integer. Is there an oriented $k$-graph with $n(k)$ vertices and $f(k)$ edges having Property O?

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