HOMOGENEOUS BETA-TYPE FUNCTIONS

MARTIN HIMMEL AND JANUSZ MATKOWSKI

Abstract. All beta-type functions, i.e. the functions $B_f : (0, \infty)^2 \to (0, \infty)$ of the form

$$B_f (x, y) = \frac{f (x) f (y)}{f (x + y)}, \quad x, y > 0,$$

for some $f : (0, \infty) \to (0, \infty)$, which are $p$-homogeneous, are determined. Applying this result, we show that a beta-type function is a homogeneous mean iff it is the harmonic one. A reformulation of a result due to Heuvers in terms of a Cauchy difference and the harmonic mean is given.

1. Introduction

For a given $f : (0, \infty) \to (0, \infty)$, the function $B_f : (0, \infty)^2 \to (0, \infty)$ defined by

$$B_f (x, y) = \frac{f (x) f (y)}{f (x + y)}, \quad x, y > 0,$$

is called the beta-type function, and $f$ is called its generator (2). The notion the beta-type function arises from the well-known relation between the Euler Beta function $B : (0, \infty)^2 \to (0, \infty)$ and the Euler Gamma function $\Gamma : (0, \infty) \to (0, \infty)$

$$B (x, y) = \frac{\Gamma (x) \Gamma (y)}{\Gamma (x + y)}, \quad x, y > 0.$$

Given $p \in \mathbb{R}$, we examine when the beta-type function $B_f$ is $p$-homogeneous, i.e. when

$$B_f (tx, ty) = t^p B_f (x, y), \quad x, y > 0.$$

Theorem 1, the main result, says that, under some regularity assumptions of the generator $f$, the beta-type function is $p$-homogeneous if, and only if, there exist $a, b > 0$ such that $f (x) = b x a^x$ for all $x > 0$. As a corollary we obtain that a beta-type function is a homogeneous pre-mean if, and only if, there exists $a > 0$ such that $f (x) = 2 x a^x$ for all $x > 0$, or, equivalently, that $B_f$ is the harmonic mean, that is $B_f = H$, where

$$H (x, y) = \frac{2xy}{x+y}, \quad x, y > 0.$$

A related companion of the beta-type function is the Cauchy difference $C_g : (0, \infty)^2 \to \mathbb{R}$ defined by

$$C_g (x, y) = g (x + y) - g (x) - g (y).$$

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for a function $g : (0, \infty) \to \mathbb{R}$. The relationship

$$B_f = \exp \circ (-C_{\log f})$$

allows to reformulate Theorem 1 in terms of logarithmical homogeneity of the Cauchy difference (Corollary 3).

At the end we remark that Heuvers result [4] on a characterization of logarithmic functions can be reformulated in terms of the Cauchy difference and the harmonic mean.

2. Main result

**Theorem 1.** Let a function $f : (0, \infty) \to (0, \infty)$ be continuous or Lebesgue measurable. Then the following conditions are equivalent:

(i) the beta-type function $B_f$ is $p$-homogeneous, i.e.

$$B_f (tx, ty) = t^p B_f (x, y), \quad x, y, t > 0;$$

(ii) there exist $a, b \in (0, \infty)$ such that

$$f(x) = bxa^x, \quad x > 0$$

and

$$B_f (x, y) = b \left( \frac{xy}{x+y} \right)^p, \quad x, y > 0.$$

**Proof.** Assume (i) holds. Hence, by the definition of $B_f$, we have

$$\frac{f(tx) f(ty)}{f(t(x+y))} = t^p f(x) f(y), \quad x, y, t > 0,$$

which can be written in the form

$$\frac{f(t(x+y))}{t^p f(x+y)} = \frac{f(tx)}{t^p f(x)} \frac{f(ty)}{t^p f(y)}, \quad x, y, t > 0.$$

For every fixed $t > 0$ define $\varphi_t : (0, \infty) \to (0, \infty)$ by

$$\varphi_t (x) := \frac{f(tx)}{t^p f(x)}, \quad x > 0.$$

Thus, from (2.2), for arbitrary fixed $t > 0$, it holds

$$\varphi_t (x+y) = \varphi_t (x) \varphi_t (y), \quad x, y > 0,$$

stating that $\varphi_t$ is an exponential function. Hence (see, for instance, [1] p. 39), for every $t > 0$, there exists a unique additive function $\alpha_t : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi_t (x) = e^{\alpha_t(x)}, \quad x > 0.$$

From the definition of $\varphi_t$, we have

$$e^{\alpha_t(x)} t^p f(x) = f(tx), \quad x > 0.$$

Since the right hand side is symmetric in $x$ and $t$, so is the left hand side; thus

$$e^{\alpha_t(x)} x^p f(t) = f(xt) = f(tx) = e^{\alpha_t(x)} t^p f(x), \quad x, t > 0.$$

Setting here $t = 1$ gives

$$e^{\alpha_1(x)} f(x) = f(x) = e^{\alpha_1(1)} x^p f(1), \quad x > 0,$$

and as, by assumption, $f$ is positive, it follows that

$$\alpha_1 (x) = 0, \quad x > 0.$$
and, consequently,

\[ f(x) = f(1) x^p e^{\alpha(1)}, \quad x > 0. \]

Putting, for convenience, \( \lambda : (0, \infty) \to \mathbb{R}, \)

\[ \lambda(x) := \alpha_x(1), \quad x > 0, \]

we have

(2.3) \[ f(x) = f(1) x^p e^{\lambda(x)}, \quad x > 0. \]

Inserting this into (2.1), we obtain,

\[
\begin{align*}
\frac{f(1)(tx)^p e^{\lambda(tx)} f(1)(ty)^p e^{\lambda(ty)}}{f(1)(t(x+y))^p e^{\lambda(t(x+y))}} &= t^p \frac{f(1)x^p e^{\lambda(x)} f(1)y^p e^{\lambda(y)}}{f(1)(x+y)^p e^{\lambda(x+y)}}, \quad x, y, t > 0,
\end{align*}
\]

that reduces to

\[ e^{\lambda(tx) + \lambda(ty) - \lambda(t(x+y))} = e^{\lambda(x) + \lambda(y) - \lambda(x+y)}, \quad x, y, t > 0, \]

whence

\[ \lambda(tx) + \lambda(ty) - \lambda(t(x+y)) = \lambda(x) + \lambda(y) - \lambda(x+y), \quad x, y, t > 0. \]

Writing this in the form

\[ \lambda(t(x+y)) - \lambda(x+y) = [\lambda(tx) - \lambda(x)] + [\lambda(ty) - \lambda(y)], \quad x, y, t > 0, \]

we conclude that, for any \( t > 0, \) the function \( \omega = \omega_t : (0, \infty) \to \mathbb{R}, \) defined by

(2.4) \[ \omega(x) := \lambda(tx) - \lambda(x), \quad x > 0, \]

is additive. From (2.3) and the assumed regularity of \( f \) we get that \( \omega \) is continuous or Lebesgue measurable. Thus, \( \omega, \) being additive and continuous or measurable, is of the form \([6], \) p. 129, see also [4]

\[ \omega(x) = \omega(1)x, \quad x > 0, \]

and hence, by (2.4),

\[ \lambda(tx) - \lambda(x) = (\lambda(t) - \lambda(1))x, \quad x, t > 0, \]

whence

\[ \lambda(tx) = \lambda(x) + (\lambda(t) - \lambda(1))x, \quad x, t > 0. \]

The symmetry in \( t \) and \( x \) of the left hand side implies that

\[ \lambda(x) + (\lambda(t) - \lambda(1))x = \lambda(t) + (\lambda(x) - \lambda(1))t, \quad x, t > 0, \]

whence

\[ \lambda(x)(1-t) + \lambda(1)t = \lambda(t)(1-x) + \lambda(1)x, \quad x, t > 0. \]

Subtracting \( \lambda(1) \) from both sides yields

\[ \lambda(x)(1-t) + \lambda(1)t - \lambda(1) = \lambda(t)(1-x) + \lambda(1)x - \lambda(1), \quad x, t > 0, \]

whence

\[ \lambda(x)(1-t) - \lambda(1)(1-t) = \lambda(t)(1-x) - \lambda(1)(1-x), \quad x, t > 0, \]

and, consequently,

\[
\frac{\lambda(x) - \lambda(1)}{1-x} = \frac{\lambda(t) - \lambda(1)}{1-t}, \quad x, t > 0, \quad x \neq 1 \neq y.
\]
It follows that there exists $c \in \mathbb{R}$ such that
\[
\frac{\lambda(x) - \lambda(1)}{1 - x} = -c, \quad x > 0, x \neq 1,
\]
whence,
\[
\lambda(x) = c(x - 1) + \lambda(1), \quad x > 0,
\]
and we obtain
\[
\lambda(x) = cx + d, \quad x > 0,
\]
where $d := \lambda(1) - c$. Inserting this function $\lambda$ into (2.3), we obtain
\[
f(x) = f(1) e^{dx} (e^{c})^x, \quad x > 0,
\]
whence, setting
\[
a := e^c, \quad b := f(1) e^d,
\]
we get
\[
f(x) = bx^a, \quad x > 0,
\]
and
\[
B_f(x, y) = b \left( \frac{xy}{x + y} \right)^p, \quad x, y > 0,
\]
which proves (ii). The implication (ii) $\implies$ (i) is obvious.

3. Applications to pre-means

Definition 1. Let $I \subseteq \mathbb{R}$ be an interval and $M : I^2 \to \mathbb{R}$. The $M$ is reflexive, if
\[
M(x, x) = x, \quad x \in I;
\]
$M$ is called a pre-mean in $I$ ($\mathbb{R}$), if it is reflexive and $M(I^2) \subseteq I$;
$M$ is called a mean in $I$, if
\[
\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.
\]

Remark 1. If $M : I^2 \to \mathbb{R}$ is reflexive, then $I \subseteq M(I^2)$; so a reflexive function is a pre-mean if, and only if, $M(I^2) = I$.

Remark 2. Obviously, every mean is a pre-mean, but, in general, not vice versa. Indeed, the function $M : (0, \infty)^2 \to (0, \infty)$ defined by
\[
M(x, y) = \frac{2x^2 + y^2}{x + 2y}
\]
is a pre-mean. Since $M(2, 1) = 3 \notin [2, 1]$ the function is not a mean. So $M$ is not increasing in both variables because, otherwise, it would be a mean.

Remark 3. If $M : (0, \infty)^2 \to \mathbb{R}$ is reflexive and, for some $p \in \mathbb{R}$, $p$-homogenous, then $p = 1$.

Corollary 1. Let $f : (0, \infty) \to (0, \infty)$ be a continuous function. Then the following conditions are equivalent:
(i) the beta-type function $B_f$ is a homogeneous pre-mean;
(ii) there exists $a \in (0, \infty)$ such that
\[
(3.1) \quad f(x) = 2xa^x, \quad x > 0;
\]
(iii) the beta-type function coincides with the harmonic mean, i.e.
\[
B_f(x, y) = \frac{2xy}{x + y}, \quad x, y > 0.
\]
Proof. Assume (i). By Theorem 1 and remark 3 its generator $f$ is of the form

$$f(x) = bxa^x, \quad x > 0,$$

for some $a, b \in (0, \infty)$. Since $B_f(x, x) = x$ for all $x \in (0, \infty)$.

Substituting here $x = 2$ and using Theorem 1 (ii), yields

$$2 = B_f(2, 2) = \frac{f(2)f(2)}{f(2+2)} = \frac{b \cdot 2}{2+2} = b,$$

whence we get (3.1), which proves (ii).

Assume (ii). From (3.1) and the definition of $B_f$ we get (iii).

The implication (iii) $\implies$ (i) is obvious.

Because every homogeneous quasi-arithmetic mean is a power mean (2, p. 249), our result implies the following

**Corollary 2.** A homogeneous beta-type function is a quasi-arithmetic mean if, and only if, it is the harmonic mean.

For another result connecting harmonic mean and the Euler Gamma function see [3].

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### 4. Cauchy Differences and a Corollary

Applying our main result, we obtain the following

**Corollary 3.** Let $g : (0, \infty) \to \mathbb{R}$ be an arbitrary continuous function and let $p \in \mathbb{R}$. The following conditions are equivalent:

(i) the Cauchy difference is $p \log t$-homogeneous, that is

$$C_g(tx, ty) = C_g(x, y) + p \log t, \quad x, y, t > 0;$$

(ii) there exist $c, d \in \mathbb{R}$ such that

$$g(x) = cx + d - p \log t, \quad x > 0$$

and

$$C_g(x, y) = \log \left( \frac{xy}{x+y} \right)^p - d, \quad x, y > 0.$$  

**Proof.** Setting $f := \exp \circ g$, we observe that condition (i) is equivalent to

$$B_f(tx, ty) = t^{-p}B_f(x, y), \quad x, y, t > 0,$$

since, using the definition of beta-type function, we have, for all $x, y > 0$,

$$e^{g(tx)+g(ty)-g(t(x+y))} = t^{-p}e^{g(x)+g(y)-g(x+y)}.$$  

Taking the logarithm of both sides, we indeed obtain

$$-C_g(tx, ty) = \log t^{-p} - C_g(x, y), \quad x, y > 0,$$

and thus $g$ satisfies (4.1).

By Theorem 1, there exist $a, b > 0$

$$f(x) = bx^{-p}a^x, \quad x > 0.$$  

Thus, by the definition of $f$, we get, for all $x > 0$,

$$g(x) = \log b + p \log x + x \log a;$$

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$$f(x) = bxa^x, \quad x > 0,$$

for some $a, b \in (0, \infty)$. Since $B_f(x, x) = x$ for all $x \in (0, \infty)$.

Substituting here $x = 2$ and using Theorem 1 (ii), yields

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Thus, by the definition of $f$, we get, for all $x > 0$,

$$g(x) = \log b + p \log x + x \log a;$$
whence, putting $c := \log a$ and $d := \log b$, we obtain,

$$g(x) = cx + d + p \log x, \quad x > 0,$$

and consequently, for all $x, y > 0$,

$$C_g(x, y) = g(x + y) - g(x) - g(y)$$

$$= \log \left(\frac{xy}{x + y}\right)^p - d,$$

which proves the implication $(i) \implies (ii)$.

The second implication is easy to verify. □

In connection with Cauchy differences and harmonic mean, let us note that Heuvers result [4] (see also Kannappan [5], p. 31) can be reformulated as

**Remark 4.** The Cauchy difference of a function $f : (0, \infty) \to \mathbb{R}$ satisfies the functional equation

$$C_f(x, y) = f \left(\frac{2}{H(x, y)}\right), \quad x, y > 0$$

if, and only if, $f$ is a logarithmic function, i.e.

$$f(xy) = f(x) + f(y), \quad x, y > 0.$$

**References**

[1] J. Aczél, Lectures on Functional Equations and Their Applications, Academic Press, New York and London, 1966.

[2] J. Aczél, J. G. Dhombres, Functional Equations in Several Variables, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1989.

[3] H. Alzer, A harmonic mean inequality for the Gamma function, J. Comput. Appl. Math. 87 (1997), 195-198.

[4] K. J. Heuvers, Another logarithmic functional equation, Aeq. Math., 58 (1999), 260-264.

[5] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer Monographs in Mathematics, Springer, New York, 2009.

[6] M. Kuczma, A. Gilányi, An Introduction to the Theory of Functional Equations and Inequalities, 2009, Birkhäuser Verlag AG, Basel – Boston – Berlin.

[7] M. Himmel, J. Matkowski, Directional convexity and characterizations of Beta and Gamma functions (submitted).

[8] J. Matkowski, Convergence of iterates of pre-mean type mappings, ESAIM: Proceedings and Surveys, ECTT 2012. Witold Jarczyk, Daniele Fournier-Prunaret, João Manuel Gonçalves Cabral, November 2014, Vol. 46, 196-228.

Current address: Faculty of Mathematics, Computer Science and Econometrics University of Zielona Góra, Szafrana 4A, PL 65-516 Zielona Góra, Poland
E-mail address: himmel@mathematik.uni-mainz.de

Current address: Faculty of Mathematics, Computer Science and Econometrics University of Zielona Góra, Szafrana 4A, PL 65-516 Zielona Góra, Poland
E-mail address: J.Matkowski@wmie.uz.zgora.pl