Surfaces and the Sklyanin bracket

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Abstract: We discuss the Lie Poisson groups structures associated to splittings of the loop group $LGL(N, \mathbb{C})$, due to Sklyanin. Concentrating on the finite dimensional leaves of the associated Poisson structure, we show that the geometry of the leaves is intimately related to a complex algebraic ruled surface with a $\mathbb{C}^\times$-invariant Poisson structure. In particular, Sklyanin's Lie Poisson structure admits a suitable abelianisation, once one passes to an appropriate spectral curve. The Sklyanin structure is then equivalent to one considered by Mukai, Tyurin and Bottacin on a moduli space of sheaves on the Poisson surface. The abelianization procedure gives rise to natural Darboux coordinates for these leaves, as well as separation of variables for the integrable Hamiltonian systems associated to invariant functions on the group.

1. Introduction

The aim of this note is to close up a circle of ideas linking algebraically integrable systems associated to loop groups and loop algebras on $Gl(N, \mathbb{C})$ or $Sl(N, \mathbb{C})$, to symmetric products of certain symplectic surfaces (more properly, Hilbert schemes of zero dimensional subschemes on the surfaces). The general idea is to show that the phase spaces of these systems are birationally symplectomorphic to the Hilbert schemes, in such a way that the leaves of the Lagrangian foliation are given by the space of divisors on spectral curves; these spectral curves lie in the surface, and the inclusion of Lagrangian leaves (symmetric products of the curves) into the symplectic leaves (symmetric product of the surface) is induced by the inclusion of the curves into the surface. This isomorphism, when made explicit, gives simple separations of variables for the systems.

Integrable systems on loop algebras and loop groups include as special cases most of the frequently studied integrable systems. Indeed, appropriate choices of rank, location of poles, residues at the poles, etc, give us most of the classical systems: the Neumann oscillator, the various tops, the finite gap solutions to the KdV, the NLS, the CNLS and the Boussinesq equations, the various Gaudin models, the Landau-Lifshitz equation are just some of the examples. References include the book [FT], the survey [RS2], and the references therein, or the articles [Mo, AvM, RS1, AHP, HaH]. All of these systems are associated to splittings of the loop algebra $Lg = Lg_+ \oplus Lg_-$ or the corresponding local decompositions of the loop groups $LG = LG_+ LG_-;$ this splitting gets encoded in terms of $r$-matrices. One has three main types of splitting, given by the rational, trigonometric and elliptic $r$-matrices. The splittings allow us to define a bracket on the loop algebra (linear, or Lie Poisson bracket), and on the group (quadratic, or Sklyanin bracket [Sk1, S]). For all of these brackets, there are integrable systems whose Hamiltonians are the coefficients of the equations defining the spectral curve of the loop.

The linear brackets admit an important generalisation: the generalised Hitchin systems [Hi1, Hi2, Bo1, M]. These systems are defined on moduli spaces of pairs $(E, \phi)$, where $E$ is a principal $G$ bundle over a Riemann surface $\Sigma$, and $\phi$ is a meromorphic 1-form valued

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section of the associated adjoint bundle. The $r$-matrix systems correspond to the cases when the bundles are rigid: choosing trivialisations, the sections $\phi$ take their values in suitable subspaces of the loop algebra. For $\text{Sl}(N, \mathbb{C})$, these cases occur when the curve $\Sigma$ is rational (rational $r$-matrix), elliptic (elliptic $r$-matrix) or rational nodal (trigonometric $r$-matrix). In short, there is a table: (L.P.: Lie Poisson, Sk.: Sklyanin)

|                | Rational curve | Rat'l nodal curve | Elliptic curve | General curve |
|----------------|----------------|------------------|----------------|--------------|
| Algebra        | Rational L.P. | Trigonometric L.P. | Elliptic L.P. | Generalised Hitchin |
| Group          | Rational Sk.  | Trigonometric Sk. | Elliptic Sk.  | -             |

The first (general) case of the isomorphism of these systems with Hilbert schemes was given in [AHH] (special cases were considered in [NV]). The case considered in [AHH] is the linear Lie-Poisson (rational $r$-matrix) bracket on the dual $Lg^*$ of the loop algebra $Lg_+$ of polynomial loops in the Lie algebra $gl(N, \mathbb{C})$. The (reduced) symplectic leaves are reductions of coadjoint orbits in $Lg^*$ by the action of $gl(N, \mathbb{C})$. It was shown that one has natural Darboux coordinates on the symplectic leaves, which establish a birational symplectomorphism between the symplectic leaves and the Hilbert scheme of a rational symplectic surface. The surface is a blow-up of the total space of the line bundle $K_{\mathbb{P}^1(D)}$ over $\mathbb{P}^1(\mathbb{C})$, where $D$ is the divisor of poles of the meromorphic section. The blow up is taken at the intersection of the spectral curve with the inverse image of the divisor $D$. This picture was generalised in [HK], following [H], to cover the case of the generalised Hitchin systems, which then specialises to the case of linear elliptic and linear trigonometric $r$-matrices. The surfaces one obtains are blow-ups of line bundles over the base curve $\Sigma$.

For the quadratic brackets, the rational case was treated by Scott in [Sc]; one again obtains a blow up of a line bundle over $\mathbb{P}^1(\mathbb{C})$, but now blowing up at the intersection of the spectral curve with the zero section. The main purpose of this note is to prove the corresponding result in the elliptic and trigonometric cases, which completes the list of six cases enumerated above; at the same time, we give a general, simplified exposition applicable to all of the cases, as well as a geometric indication of why the quadratic brackets do not generalise to higher genus curves. The finite dimensional leaves of the quadratic systems are shown to be symplectic leaves of moduli spaces of Higgs bundles with a rigid $\text{Sl}(N, \mathbb{C})$-bundle and a generically invertible Higgs field. The abelianization of the moduli space of Higgs bundles amounts to its realization as a moduli space of sheaves on a ruled surface. The Poisson structure is then equivalent to one considered by Mukai, Tyurin and Bottacin. In turn, this structure is induced by a structure on the ruled surface. A similar situation holds for the generalized Hitchin system. The structure there is invariant under translation on the fibers of the ruling; this translates into a linear Poisson structure on the space of matrices. In contrast, the Sklyanin brackets, which are quadratic on the space of matrices, correspond to a $\mathbb{C}^\times$-invariant Poisson structure on the surface (see Theorem 3.24 and the following remark). The ruled surface admits such a $\mathbb{C}^\times$-invariant Poisson structure when the base curve is rational or elliptic. Indeed, one of the virtues of this approach is that it classifies to a certain degree the possible Poisson brackets on the loop algebras or loop groups: the linear and quadratic brackets, between them, exhaust the possible Poisson structures on the ruled surface, when the base curve is elliptic or rational.
When the genus of the base curve is higher, one only has Poisson structures on the surface which correspond to the linear brackets, so that the table above arises in some sense as a consequence of the classification of Poisson structures on a ruled surface. Indeed, we will see that, in trying to generalise the Sklyanin structure to the higher genus base curve case, we do not obtain integrable systems on symplectic varieties, but rather on varieties equipped with a degenerate closed form.

We will generalize the construction to arbitrary reductive groups in a separate paper [HM]. Rigid bundles on an elliptic curve do not exist for simple groups of type other than $A_n$ ([FM] Theorem 5.13). This translates into the fact that integrable systems with a quadratic Poisson structure can not be constructed on loop groups of type other than $A_n$ [BD]. Nevertheless, allowing the principal bundle to deform, we do get integrable systems with a quadratic bracket on moduli spaces of pairs $(P, g)$ consisting of a principal $G$-bundle $P$ and a meromorphic section of its adjoint group bundle. These are equivalent to those obtained from the dynamic r-matrix formalism [F, EV].

The paper is organized as follows: Section 2 recalls some basic facts about the Sklyanin bracket, specialised to the context of loop algebras. In section 3 we exhibit the birational isomorphism between the Sklyanin systems and Hilbert schemes of symplectic surfaces, in the elliptic case. We show that the Sklyanin systems are supported on the moduli space of Higgs pairs and the Poisson structure is a natural translation of the Mukai-Tyurin Poisson structure, via the one-to-one correspondence between a Higgs pair and its spectral data. Sections 4 and 5 discuss the trigonometric and rational cases, respectively. Section 6 discusses the case of general base curves.

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2. Loop groups and the Sklyanin bracket

In this section we recall certain basic facts about the Sklyanin bracket in the case which concerns us here, that of a loop group with a (local) decomposition into a product of two subgroups. Our main aim is to identify the finite dimensional symplectic leaves. After normalisation, these correspond to meromorphic maps on the curve. A good general reference is [RS2], section II.12

Let $G$ be a reductive complex group, thought of as a subgroup of some $GL(N, \mathbb{C})$, and consider the loop group $LG$ of analytic mappings of the circle into $G$. Let us now assume that this circle is embedded into a Riemann surface $\Sigma$, and bounds a disk $U_-$ whose centre $p$ will correspond to $z = \infty$; set $U_+ = \Sigma \setminus p$. We furthermore choose a transition matrix $T(z)$ defined over the punctured disk, possibly taking values in a larger group $\hat{G}$, but such that $G$ is a normal subgroup of $\hat{G}$. We assume that the Lie algebra $Lg$ of $LG$ splits into a sum $Lg = Lg_- \oplus Lg_+$, where

- $Lg_-$ consists of holomorphic functions defined over $U_-$ into $g$,
- $Lg_+$ consists of elements of the form $Ad_{T(z)}a(z)$ where $a(z)$ is a holomorphic function from $U_+$ into $g$. 

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If $P$ is the principal $\hat{G}$-bundle defined by $T$, the existence of a splitting is equivalent to the vanishing of the first cohomology of the adjoint bundle $H^1(\Sigma, P(\mathfrak{g})) = 0$. When $\hat{G} = G$, standard deformation theory shows that the vanishing is tantamount to $P$, defined by the transition matrix $T$, being rigid. Rigidity of $P$ means that any small deformation of $P$ is isomorphic to $P$. If $\hat{g}$ is reductive with semi-simple part $\mathfrak{g}$, then the vanishing of $H^1(\Sigma, P(\mathfrak{g}))$ means that the associated principal $G_{\text{adj}}$-bundle is rigid, where $G_{\text{adj}}$ is the adjoint group. In other words, the deformations of $P$ all arise from the center of $\hat{G}$. Such rigidity happens only for curves of genus zero or one: if the curve is rational, $G$ can be any reductive group; if the curve is elliptic, $G$ must be of type $A_n$ ([FM] Theorem 5.13). In the elliptic case we choose $G = SL(n, \mathbb{C})$ and $\hat{G} = GL(n, \mathbb{C})$. Uniqueness in the decomposition is linked to the absence of sections of the adjoint bundle: $H^0(\Sigma, P(\mathfrak{g})) = 0$.

We write the corresponding (local) decomposition of groups as

$$LG = LG_+ \cdot LG_-.$$  \hspace{1cm} (2.1)

and decompose elements of $LG$ as follows:

$$g = g_+ \cdot g_-^{-1}.$$  \hspace{1cm} (2.2)

Let $\omega$ be a non-vanishing holomorphic one-form on $U_-$. If $(a, b) \mapsto \text{tr}(ab)$ denotes the Killing form on $\mathfrak{g}$, we can define a pairing on $Lg$ by

$$< f, g > = \int tr(fg) \omega.$$  \hspace{1cm} (2.3)

With respect to this pairing, both $Lg_+$ and $Lg_-$ are isotropic.

Now let $P_+, P_-$ denote the projections of $Lg$ onto $Lg_+$ and $Lg_-$ respectively. We set

$$R = P_+ - P_-.$$  \hspace{1cm} (2.4)

If $\psi$ is a function on $LG$, the left derivative $D\psi$ and the right derivative $D'\psi$ in $Lg$ are defined at $g$ by

$$< D\psi(g), h > = \frac{d}{dt} \psi(\exp(th) \cdot g)|_{t=0}, \quad \forall h \in Lg,$$

$$< D'\psi(g), h > = \frac{d}{dt} \psi(g \cdot \exp(th))|_{t=0}, \quad \forall h \in Lg.$$  \hspace{1cm} (2.5)

In terms of the Maurer-Cartan forms $\theta = dg \cdot g^{-1}, \theta' = g^{-1} dg$, thought of as maps $TLG \to Lg$, we have

$$D\psi = (\theta^{-1})^*(d\psi),$$

$$D'\psi = (\theta'^{-1})^*(d\psi).$$  \hspace{1cm} (2.6)

At an element $g$ of $LG$,

$$D\psi = Ad_g(D'\psi).$$  \hspace{1cm} (2.7)
The Sklyanin bracket of \( \psi \) and \( \phi \) is then defined by
\[
\{ \psi, \phi \} = \frac{1}{2} < R(D\psi), D\phi > - \frac{1}{2} < R(D'\psi), D'\phi > .
\] (2.8)

Alternately, one can write
\[
\{ \psi, \phi \} = \frac{1}{2} < D\psi_+ - D\psi_- - Ad_g(D'\psi_+ - D'\psi_-), D\phi >
= < D\psi_+ - Ad_g(D'\psi_+), D\phi >
= < -D\psi_- + Ad_g(D'\psi_-), D\phi >,
\] (2.9)

The dressing action. One has a (right) action of \( LG_- \times LG_+ \) on \( LG \), defined by
\[
g(h_+, h_-) = ((gh_+h_-^{-1}g^{-1})_+^{-1}gh_+ = ((gh_+h_-^{-1}g^{-1})_-^{-1}gh_-
\] (2.10)

For \( g \) lying in \( LG_- \), for example, the action of \( h_- \) is trivial, and the action of \( h_+ \) is
\[g \mapsto ((gh_+)_-^{-1} = ((gh_+)_+^{-1}gh_+.

Referring to (2.2), this is in essence the right action of \( h_+ \) on \( g \), followed by projection to \( G_- \). A more conceptual definition can be given of the dressing action, involving projection of simple flows on a larger space; see, e.g., proposition 12.20 of [RS2]. Set, for an element \( \zeta \) of \( Lg \):
\[\zeta_+ = P_+(\zeta), \quad \zeta_- = P_-(\zeta).
\]

We note that this differs from the infinitesimal version of (2.2) by a sign. If \( \xi_+ \in Lg_+, \xi_- \in Lg_- \), the actions of the one parameter subgroups \( exp(\epsilon \xi_\pm) \) are given by
\[
g \mapsto (1 + \epsilon(Ad_g\xi_-) + O(\epsilon^2))g, \quad g \mapsto (1 + \epsilon(Ad_g\xi_-) + O(\epsilon^2))g.
\] (2.11)

In other words, for a function \( f \), if \( v_{\xi_\pm} \) denotes the vector field corresponding to \( \xi_\pm \),
\[v_{\xi_+}(f) = < (Ad_g(\xi_+))_-, Df >, \quad v_{\xi_-}(f) = < (Ad_g(\xi_-))_+, Df > .
\] (2.12)

One has (see [RS2], [S]):

**Theorem (2.13)** The symplectic leaves of the Sklyanin bracket are given by the orbits of the dressing action.

From now on, we take \( G = Gl(n, \mathbb{C}) \) or \( Sl(n, \mathbb{C}) \). We can analyse the finite dimensional symplectic leaves as follows:

**Theorem (2.14)**

a) The finite dimensional leaves in \( LGL(n, \mathbb{C}) \) are orbits of elements of the form \( f(z)g(z) \) where \( f(z) \) is a scalar function, \( g(z) \) has a pole of finite order at \( p \) and \( T(z)g(z)T(z)^{-1} \) is a meromorphic matrix-valued function on \( \Sigma - \{ p \} \), with a finite number of poles.
b) The location and the order of the poles is constant along the orbit.

c) The points over which $\det(g)$ vanishes are constant along the orbit.

Proof: Normalise one of the matrix coefficients of $g^{-1}$, say $g_{11}^{-1}$, to 1; this accounts for the function $f$. Now look at the action of $Lg_-$, to analyse the pole at $\infty$. Filter $Lg_-$ as ...$(Lg_-)^{-n-1} \subset (Lg_-)^{-n} \subset ...$ by order of vanishing at $\infty$. Finite codimension of the stabiliser $W_-$ implies that $(Lg_-)^{-n} \subset W_-$ for some $n$: there can only be a finite set of $n$’s such that the map $(Lg_-)^{-n} \cap W_- \to g = (Lg_-)^{-n}/(Lg_-)^{-n-1}$ is not surjective. Next consider elements of $(Lg_-)^n$ of the form $z^n \epsilon_{j_1}$, where $\epsilon_{j_1}$ is the $(j, 1)$-th elementary matrix and using the fact that $g_{ij}z^n = g_{ij}z^n \epsilon_{j_1}g_{11}^{-1}$ is holomorphic at $\infty$ tells us that $g$ has a pole of order at most $n$.

Next we consider the action of $Lg_+$. The stabiliser $W_+$ is an $\mathcal{O}(\Sigma - \infty)$-module and so we can localise over the points $q$ of $\Sigma - \{\infty\}$, and consider the quotients $(Lg_+)_q/(W_+)_q$. Again, the finiteness of the codimension of $W_+$ tells us that only a finite number of these quotients are non-zero, and that at each point the quotient module is supported on some finite formal neighbourhood of the point. This tells us that there is a function $f$ on $\Sigma - \{\infty\}$ with finite order poles at a finite number of points such that $f \cdot Lg_+$ maps to zero in $(Lg_+)_q/(W_+)_q$ at each point $q$ of $\Sigma - \{\infty\}$, and so $f \cdot Lg_+ \subset W_+$. Again this tells us that the components $f g_{ij}$ lie in $\mathcal{O}(\Sigma - \{\infty\})$.

For b), note that we can write the dressing action of $\xi_+$, using (2.11), as:

$$\dot{g} = (g \xi_+ g^{-1})_+ g = g \xi_+ - (g \xi_+ g^{-1})_+ g,$$

and similarly, for $\xi_-$

$$\dot{g} = (g \xi_- g^{-1})_+ g = g \xi_- - (g \xi_- g^{-1})_+ g,$$

from which it follows that the poles of $\dot{g}$ are included amongst those of $g$: the dressing action preserves the singularities of $g$. Part c) follows by considering the explicit form (2.10) of the dressing action.

Remark: In keeping with our interpretation of the decomposition (2.1) in terms of a holomorphic bundle, we note that there are two operations we can perform on the brackets:

(2.15) Changing trivialisations. As we are thinking of our splitting $Lg = Lg_+ \oplus Lg_-$ in terms of sections of a rigid bundle $E$, we should take advantage of this and allow ourselves to change trivialisations. Let us then set $\tilde{T} = T_- T_+^{-1}$, where $T_\pm : U_\pm \to G$ are holomorphic maps. Define $\widetilde{Lg}_+$ to be $T_- (Lg_+) T_-$, and define the map

$$\rho : Lg_+ \times Lg_- \to \widetilde{Lg}_+ \times Lg_-$$

$$(a, b) \mapsto (T_- a T_-^{-1}, T_- b T_-^{-1}).$$

If we define modified projections by

$$\dot{P}_\pm(a) = T_- P_\pm(T_-^{-1} a T_-) T_-^{-1},$$

Then $\rho(P_+ P_-) = \dot{P}_+ \dot{P}_-$. It is also clear that $\rho$ is $\mathcal{O}(\Sigma)$-equivariant, and so is well-defined on the quotient $\mathcal{O}^\Sigma$.
and set \( \hat{R} = \hat{P}_+ - \hat{P}_- \), we have that the map \( \rho \) intertwines the two Poisson brackets defined by \( R, \hat{R} \). In other words, we may work with the trivialisation we wish.

(2.16) Adding points. A given bundle of course admits not only several trivialisations with respect to a fixed covering by open sets, but also trivialisations with respect to different coverings. In particular, let us suppose that we have not only a covering by \( U_+ = \Sigma - \{p\}, U_- = \text{disk containing } p \), but also a cover by \( \bar{U}_+ = \Sigma - \{p, q\}, \bar{U}_- = \text{disk containing } q \), where the two disks \( \bar{U}_- , p, \bar{U}_- , q \) do not overlap. For this second cover, the functions on the overlaps \( \bar{U}_+ \cap (\bar{U}_- , p \cup \bar{U}_- , q) \) correspond to a sum of two copies of the loop algebra \( L_\mathfrak{g} \oplus L_\mathfrak{g} \). We keep the transition function \( T \) on \( \bar{U}_+ \cap \bar{U}_- , p \), and take the identity as transition function on \( \bar{U}_+ \cap \bar{U}_- , q \). We can decompose \( L_\mathfrak{g} \oplus L_\mathfrak{g} \) into a sum \( \bar{L}_\mathfrak{g}_+ \oplus \bar{L}_\mathfrak{g}_- \), where \( \bar{L}_\mathfrak{g}_+ \) corresponds to sections of \( \text{ad}(E) \) over \( \bar{U}_+ \), and \( \bar{L}_\mathfrak{g}_- = L_\mathfrak{g}_- , p \oplus L_\mathfrak{g}_- , q \) consists of sections over the two open disks. There are corresponding projections \( \bar{P}_+ , \bar{P}_- \), and a corresponding Sklyanin bracket. One can show that the projection \( \pi : L_\mathfrak{g} \oplus L_\mathfrak{g} \to L_\mathfrak{g} \) onto the first factor is a Poisson map. More generally, we can add and subtract points, which shows that the intrinsic object we are considering is really the space of sections of a rigid bundle.

3. Spectral curves and Abelianisation: the elliptic case

As we have seen, we are in essence considering sections over a punctured disk of the automorphisms of rigid bundle over a Riemann surface; the finite dimensional symplectic leaves are those of meromorphic sections of the automorphisms over the whole surface. There are three cases that one can consider, those of an elliptic curve, a rational nodal curve, and a rational curve. This section is devoted to the elliptic case.

3.a A rigid bundle on an elliptic curve. Let \( \Sigma \) be an elliptic curve and \( D \) a positive divisor on \( \Sigma \). We will take as vector bundle \( E \) a stable vector bundle of rank \( N \), degree 1, and we fix the top exterior power of \( E \). This makes the bundle rigid, and in fact determines the bundle. The bundle \( E \) can be defined as follows. Let \( q = \exp(2\pi i/N) \), and set

\[
I_1 = \text{diag}(1, q, q^2, \ldots, q^{N-1}), \quad I_2 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Now let us represent the elliptic curve \( \Sigma \) as \( \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \). Puncture the curve at a point \( p \). One can lift \( E \) to \( \mathbb{C} \); sections of \( E \) over \( \Sigma \) will be given by \( N \)-tuples \( F \) of \( N \)-valued functions defined over the inverse image in \( \mathbb{C} \) of \( \Sigma - p \), satisfying:

- \( F(z + \omega_i) = I_i F(z) \),
- \( F \) is of the form \( z^{-1/N} \)-holomorphic, near the inverse images in \( \mathbb{C} \) of the puncture \( p \), where \( z = 0 \) corresponds to \( p \).
In a similar vein, sections of $\text{End}(E)$ are given by holomorphic matrix-valued functions $M$ (this time single-valued) on $\mathbb{C}$, satisfying $M(z + \omega_i) = I_i M(z) I_i^{-1}$. We will consider the subspace $H^0(\Sigma, \text{End}(E)(D))$ of endomorphisms of $E$, with the order of the poles bounded by the divisor $D$.

We note that, because the degree is 1, we have a bundle with structure group $Gl(N, \mathbb{C})$. The group that we consider for our splitting, however, is $Sl(N, \mathbb{C})$, with Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, or, alternately, the group $PGL(N, \mathbb{C})$. This will correspond to the trace-less endomorphisms $\text{End}^0(E)$. We have

$$H^0(\Sigma, \text{End}^0(E)) = H^1(\Sigma, \text{End}^0(E)) = 0,$$

so that we are indeed in the case of a unique decomposition, as in (2.1). The decompositions we consider will thus be of the group of sections of $Sl(E)$ over the punctured disk, or more generally, sections of $Gl(E)$ with fixed determinant. While the bundle has been defined using automorphy factors, rather than transition matrices, a change of trivialisations, as in remark (2.15), allows one to go from one formalism to the other.

3.b The Mukai structure. We have already seen in the previous sections, that the space of sections of $H^0(\Sigma, \text{End}(E)(D))$ with a fixed determinant is a union of symplectic leaves for the Sklyanin structure. We will now construct on $H^0(\Sigma, \text{End}(E)(D))$ another Poisson structure, whose symplectic leaves will again be subvarieties of $H^0(\Sigma, \text{End}(E)(D))$ with fixed determinant. We will proceed by reduction by a $\mathbb{C}^*$-action of a larger space $\mathcal{M}$ of pairs $(E', g)$ where $E' = E \otimes L$ for a line bundle $L$, and $g \in H^0(\Sigma, \text{End}(E)(D))$ is generically invertible. Symplectic leaves of $\mathcal{M}$ are determined by the zero divisor of the determinant of $g$, so that $\det(g)$ is fixed only up to a scalar factor. The $\mathbb{C}^*$-action is defined by

$$c(E, g) = (E, c \cdot g), \quad (3.2)$$

and so, up to a finite cover corresponding to action by roots of unity, taking the quotient by $\mathbb{C}^*$ corresponds to fixing the determinant.

We have not mentioned yet the Hamiltonians that will define our integrable systems on the finite dimensional symplectic leaves; this system is closely tied to the Mukai structure. The Hamiltonians are given by the coefficients of the defining equation $F$ of the spectral curve $S$ of $g \in H^0(\Sigma, \text{End}(E)(D))$:

$$F(z, \lambda) = \det(g(z) - \lambda \mathbb{I}) = 0. \quad (3.3)$$

In short, the Lagrangian leaves of the integrable system are given by fixing the spectral curve. If $D$ is the divisor of poles of $g$, the equation (3.3) defines a compact curve $S$ embedded in the total space $\mathcal{T}$ of the line bundle $\mathcal{O}(D)$ over $\Sigma$; there is an $n$-sheeted projection $\pi : S \to \Sigma$. One can also define a sheaf $L$ supported over the spectral curve as a cokernel of $g - \lambda \mathbb{I}$; generically it is a line bundle over $S$,

$$0 \longrightarrow \pi^* E \otimes \mathcal{O}(-D) \overset{\partial}{\longrightarrow} \pi^* E \rightarrow L \rightarrow 0. \quad (3.4)$$

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We have:

**Proposition (3.5)** [H]

a) The push-down $\pi_* L$ is isomorphic to $E$.

b) The map $g$, up to conjugation by the global automorphisms of $E$, is the push-down of the action on $L$ given by multiplication by the fiber coordinate $\lambda$.

The automorphisms of our family of $E$’s are multiples of the identity, so that one recovers the pair $(E, g)$ from $(S, L)$.

**Proposition (3.6)** Let $S$ be the family of smooth curves $S'$ in the linear system of $S$ on the surface $T$. Then $\mathcal{M}$ contains the Jacobian fibration of $S$ (of degree $g$ line-bundles) as a Zariski open subset.

Let us consider the deformation theory of $L$, first as a line bundle supported over a smooth curve. The normal bundle of the spectral curve is given by the twist $K_S(D)$ of the canonical bundle of $S$ by $\pi^*(\mathcal{O}(D))$ and so the space of infinitesimal deformations of the curve is then $H^0(S, K_S(D))$. If one constrains the sections of the normal bundle to vanish on the zero-section in $T$, one gets a space of sections isomorphic to $H^0(S, K_S)$. Deformations of the line bundle, fixing the curve, are given by $H^1(S, \mathcal{O})$. On the other hand (and more generally), one can think of $L$ as a sheaf on $T$: deformations of $L$ as a sheaf on $T$ include both deformations of its support, and deformations of the line bundle. These deformations are classified by the extension group $Ext^1_T(L, L)$. On $\mathcal{M}$, we have an exact sequence for the tangent bundle, linked to the fact that it is the Jacobian fibration:

$$0 \to H^1(S, \mathcal{O}) \to TM = Ext^1(L, L) \to H^0(S, K_S(D)) \to 0 \quad (3.7)$$

We will show that the Sklyanin structure is equivalent to one defined by Tyurin and Bottacin [Bo2,T] for sheaves on a Poisson surface (generalizing the work of Mukai [Mu]). The surface that we are considering is $T$; the top exterior power of the tangent bundle of $T$ is simply $\pi^*\mathcal{O}(D)$. This has a $\text{deg}(D)$-dimensional family of sections lifted from $\Sigma$; it also has a tautological section $\lambda$, which vanishes along the zero section in $\mathcal{O}(D)$. Each of these sections defines a Poisson structure on $T$; the one we will use is $\lambda$.

In turn, each Poisson structure on the surface $T$ induces a Poisson structure on moduli spaces of sheaves on $T$ [Bo2,T]. The moduli space we consider is that of the sheaves $L$ defined above, which are supported along the spectral curves. The tangent space to the moduli space at $L$ is $Ext^1(L, L)$; dually, the cotangent space is $Ext^1(L, L \otimes K_T)$. The Poisson structure can be thought of as a skew map from the cotangent space to the tangent space; it is given here by multiplication by $\lambda$.

$\hat{\lambda} : Ext^1(L, L \otimes K_T) \to Ext^1(L, L) \quad (3.8)$

To compute the $Ext$-groups, one can first take a locally free resolution $R$ of $L$, take the induced complex $Hom(R, L \otimes K_T)$, and then compute the first hypercohomology group of this complex. We have already found a resolution; it is given by the sequence (3.4).
Applying $\text{Hom}$, and recalling that $K_T = \pi^*\mathcal{O}(-D)$, the cotangent space will be the first hypercohomology of the complex

\[(\pi^*E)^* \otimes L \otimes \pi^*\mathcal{O}(-D) \xrightarrow{(g-\lambda I)^*} (\pi^*E)^* \otimes L, \tag{3.9}\]

and the tangent space will be the first hypercohomology of

\[(\pi^*E)^* \otimes L \xrightarrow{(g-\lambda I)^*} (\pi^*E)^* \otimes L \otimes \pi^*\mathcal{O}(D). \tag{3.10}\]

The map between the two complexes is multiplication by the tautological section $\lambda$. Pushing this down to $\Sigma$ we obtain for the cotangent and tangent spaces the first hypercohomology groups of

\[\text{End}(E)(-D) \xrightarrow{-ad_g} \text{End}(E), \quad \text{End}(E) \xrightarrow{ad_g} \text{End}(E)(D), \tag{3.11}\]

respectively. The map between the complexes in (3.11) is left multiplication by $g$. The tangent and cotangent spaces then fit into exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & H^0(\Sigma, \text{End}(E)) & \rightarrow & T^*_{(E,g)}\mathcal{M} & \rightarrow & H^1(\Sigma, \text{End}(E)(-D)) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^0(\Sigma, \text{End}(E)(D)) & \rightarrow & T_{(E,g)}\mathcal{M} & \rightarrow & H^1(\Sigma, \text{End}(E)) & \rightarrow & 0.
\end{array} \tag{3.12}\]

The vertical arrows are left multiplication by $g$. Explicitly, in Čech terms with respect to a covering $U_\alpha$, the cocycles for the first hypercohomology group for a complex $A \xrightarrow{\sigma} B$ are given by pairs $(a_{\alpha,\beta}, b_\alpha)$, where $a_{\alpha,\beta}$ is a 1-cocycle for $A$, and $b_\alpha$ a 0-cochain for $B$ satisfying

\[\sigma(a_{\alpha,\beta}) - b_\alpha + b_\beta = 0 \tag{3.13}\]

on overlaps. The coboundaries in turn, are given by taking a cochain $a_\alpha$ for $A$ and mapping it to $(a_\alpha - a_\beta, \sigma(a_\alpha))$.

Reduction by the $\mathbb{C}^*$-action (3.2) corresponds to fixing the top exterior power of $E$ and taking the quotient by the $\mathbb{C}^*$-action. A Zariski open subset of the moduli $\mathcal{M}$ consists of Higgs pairs with a stable vector bundle of degree 1. Since such a stable vector bundle $E$ is unique up to tensoring by a line bundle, a component of the reduced moduli space is the projective space $\mathbb{P}H^0(\Sigma, \text{End}(E)(D))$, endowed with a Poisson structure. We have the Casimir determinant morphism, from the generically invertible locus in $\mathbb{P}H^0(\Sigma, \text{End}(E)(D))$, to the linear system $\mathbb{P}H^0(\Sigma, \mathcal{O}(N \cdot D))$. The generic fiber contains a maximal dimensional symplectic leaf. Fix a non-zero section $\delta$ of $H^0(\Sigma, \mathcal{O}(N \cdot D))$. The locus $H^0(\Sigma, \text{End}(E)(D))_{\delta}$, of sections with determinant $\delta$, is a cyclic $N$-sheeted étale covering of the symplectic leaf in $\mathbb{P}H^0(\Sigma, \text{End}(E)(D))$ determined by the zero divisor of $\delta$. We abuse notation and denote this symplectic cyclic cover by $\mathcal{M}_{\text{red}}(E, \delta)$, or $\mathcal{M}_{\text{red}}$ for short.

Next we identify the tangent and cotangent spaces of $\mathcal{M}_{\text{red}}$. Denote by $\text{End}^0(E)$ the subbundle of traceless endomorphisms. Let $\text{End}^g(E)$ be the subbundle of $\text{End}(E)$, which, away from the singularities of $g$, is the image of $\text{End}^0(E)$ under right multiplication by $g$
(left multiplication results with the same subbundle). \( \text{End}^g(E) \) is the subsheaf of \( \text{End}(E) \) of sections satisfying
\[
\{ f \in \text{End}(E) : \text{tr}(g^{-1} f) = 0 \}.
\]
If \( g^{-1} \) is a nowhere vanishing holomorphic section of \( \text{End}(E)(D') \), then it defines a line subbundle \( L \) of \( \text{End}(E) \) isomorphic to \( \mathcal{O}_\Sigma(-D') \). \( \text{End}^g(E) \) is the subbundle \( L^\perp \) orthogonal to \( L \) with respect to the trace pairing. It is isomorphic to the dual of the quotient \( \text{End}(E)/L \). Thus, \( \text{deg}(\text{End}^g(E)) = \text{deg}(L) = -\text{deg}(D') \). The tangent space of \( \mathcal{M}_{\text{red}} \) at \( g \) is given by the first hypercohomology of the complex (in degrees 0 and 1)
\[
\text{End}^0(E) \xrightarrow{ad_g} \text{End}^g(E)(D).
\]
The cotangent space is given by the first hypercohomology of the dual complex (in degrees 0 and 1)
\[
\text{End}^g(E)^*(-D) \xrightarrow{-ad_g^*} \text{End}^0(E)^*.
\]
The Poisson structure is induced by a homomorphism \( \Lambda \) from the cotangent complex to the tangent complex. In degree 1, \( \Lambda_1 \) is the composition of the isomorphism \( \text{End}^0(E)^* \cong \text{End}^0(E) \) with left multiplication by \( g \) (which takes \( \text{End}^0(E) \) into \( \text{End}^g(E)(D) \)). It is simpler to describe the dual of the homomorphism \( \Lambda_0 \) in degree 0. \( \Lambda_0^* \) is the composition of the isomorphism \( \text{End}^0(E)^* \cong \text{End}^0(E) \) with right multiplication by \( g \). The commutativity \( ad_g \circ \Lambda_0 = -\Lambda_1 \circ ad_g^* \) follows from that in the \( \text{Gl}(N) \) case. (Note that the transpose of right multiplication by \( g \) is given by left multiplication by \( g \) and the transpose of \( ad_g \) is \( -ad_g \)).

Taking the first hypercohomologies and recalling that both \( H^0(\Sigma, \text{End}^0(E)) \) and \( H^1(\Sigma, \text{End}^0(E)) \) vanish, we find:
\[
T_{(E,g)}^* \mathcal{M}_{\text{red}} \cong H^1(\Sigma, \text{End}^g(E)^*(-D)) \\
H^0(\Sigma, \text{End}^g(E)(D)) \cong T_{(E,g)} \mathcal{M}_{\text{red}}
\]
This procedure endows the Zariski open subset of \( H^0(\Sigma, \text{End}(E)(D)) \), of generically invertible sections, with a Poisson structure. As a homomorphism from \( T_{(E,g)}^* H^0(\Sigma, \text{End}(E)(D)) \)
\[
= H^1(\text{End}(E)(-D)) \to H^0(\Sigma, \text{End}(E)(D)) = T_{(E,g)} H^0(\Sigma, \text{End}(E)(D)) ,
\]
it factors through the homomorphisms (3.15).

**Lemma (3.16)** If \( D > 0 \) and \( N > 1 \), the Poisson structure extends to the whole of \( H^0(\Sigma, \text{End}(E)(D)) \).

**Proof:** One shows that the locus of non-invertible sections of \( H^0(\Sigma, \text{End}(E)(D)) \) has codimension \( \geq 2 \). Since \( N > 1 \), it suffices to estimate the codimension in the subspace of traceless sections. Let \( \text{End}^0(E_D) \) be its restriction to \( D \). The evaluation homomorphism \( \text{End}^0(E)(D) \to \text{End}^0(E_D) \) is an isomorphism because \( H^1(\text{End}^0(E)) = 0 \). The determinant divisor in \( \text{sl}_N \) is irreducible. If \( D > 0 \) and \( x \) is a point in \( D \), we get an irreducible divisor in \( H^0(\text{End}^0(E_D)) \) of sections which are not invertible at \( x \). It suffices that one
of those sections $\varphi$ is generically invertible. Indeed, a line bundle $L$ on a reduced and irreducible sectral curve passing through the zero point in the fiber over $x$ will give rise to such a section $\varphi$.

**Remarks.** 1) The left multiplication, appearing in the construction of the Poisson structure, corresponds to an embedding of the Lie group $GL(N)$ in its Lie algebra. This embedding has been implicitly used when we described meromorphic elements of the loop group as Higgs fields, i.e., as meromorphic sections of a Lie algebra bundle.

2) We could use, instead, right multiplication. The resulting Poisson structures will be equal to the one coming from left multiplication. Indeed, their difference $ad_g$ is a homomorphism between the complexes in (3.11), which is homotopic to zero. The homotopy $h$, as a homomorphism of degree $-1$ between the complexes, is given by the identity from $\text{End}(E)$ to $\text{End}(E)$.

**3.c. Comparing the Sklyanin and the Mukai brackets.** We start with an element

$$c \in H^1(\Sigma, \text{End}^g(E)(-D)) = H^0(\Sigma, \text{End}^g(E)^*(D))^*.$$

We choose an open cover $U_+, U_-$ compatible with $D$ a divisor disjoint from the open disk $U_-$. We can represent $c$ as a cocycle $c_\pm$ with respect to our cover. Lifting to $\text{Ext}^1$, we have a class represented by $(c_\pm, \rho_+, \rho_-)$, with $gc_\pm - c_\pm g - \rho_+ + \rho_- = 0$ on $U_+ \cap U_-$. Now note that $gc, cg \in H^1(\Sigma, \text{End}(E))$ can be split as

$$gc_\pm = \mu_+ - \mu_- + \frac{1}{N} \text{tr}(gc_\pm) I, \quad cg_\pm = \nu_+ - \nu_+ + \frac{1}{N} \text{tr}(cg_\pm) I,$$

(3.17)

since $H^1(\Sigma, \text{End}^0(E)) = 0$. The hypercohomology cocycle condition implies that one can choose $\mu_+, \mu_-, \nu_+, \nu_-$ to satisfy $\rho_+ = \mu_+ - \nu_+$ and $\rho_- = \mu_- - \nu_-$. With this, we can compute the explicit form of the Poisson structure $\Lambda$

$$\Lambda : T^*\mathcal{M}_{\text{red}} \to T\mathcal{M}_{\text{red}}$$

$$(c_\pm, \mu_+ - \nu_+, \mu_- - \nu_-) \mapsto (gc_\pm, g\mu_+ - g\nu_+, g\mu_- - g\nu_-).$$

(3.18)

Now we modify the class on the right by the coboundary $-(\mu_+ - \mu_-, \mu_+ + \mu_+ - \mu_+ g, \mu_- - \mu_- g)$, which rewrites the map (3.18) as:

$$(c_\pm, \mu_+ - \nu_+, \mu_- - \nu_-) \mapsto (0, \mu_+ g - g\nu_+, \mu_- g - g\nu_-).$$

(3.19)

The cocycle condition on the right hand side of (3.19) tells us that $\mu_+ g - g\nu_+ = \mu_- g - g\nu_-$, and so defines a global section of $H^0(\Sigma, \text{End}^g(E)(D))$.

Given two classes $c$ and $d$ in the cotangent space $H^1(\Sigma, \text{End}^g(E)^*(D))$ at $(E, g)$, the Poisson structure is given by

$$\langle \Lambda(c), d \rangle = \langle \mu_+ g - g\nu_+, d_\pm \rangle = \langle \mu_+ - Ad_g(\nu_+), gd_\pm \rangle.$$

(3.20)

Let us compute the Poisson bracket corresponding to this, on a pair of functions $f, h$ on $H^0(\Sigma, \text{End}^g(E)(D))$. The differentials $df, dh$ of these functions at $g$ are naturally

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identified with classes in $H^1(\Sigma, \text{End}^g(E)^*(-D))$ via Serre’s Duality and the trace pairing. We will need the following elementary Lemma.

**Lemma (3.21)** Trivialize the tangent bundle of $GL(N)$ via the inclusion $GL(N) \subset gl(N)$. Let $\rho_g : GL(N) \to GL(N)$ denote the right multiplication by $g$. Identify a one form $df$ on $GL(N)$ with a vector field $\phi$ via the above trivialization and the trace multiplication pairing:

$$<\xi, df> = \text{tr}(\xi \cdot \phi), \quad \forall \xi \in gl(N).$$

Then the pull back of a 1-forms $df$ by $\rho_g$ corresponds to left multiplication of $\phi$ by $g$.

**Proof:**

$$<\xi, d\rho_g^*(df)> = <d\rho_g(\xi), df> = \text{tr}(\xi \cdot g \cdot \phi).$$

For an infinitesimal variation $\dot{g}$ through $g$,

$$<Df, \dot{g}^{-1}g^{-1}> = <df, \dot{g}> = <D'f, g^{-1}\dot{g}>.$$ 

Thus, $(Df)(g)$ is identified with $d\rho_g^*(df)$. Using the above Lemma, we can identify $Df$ with $g \cdot df$, and $D'f$ with $df \cdot g$, and similarly for $dh$. In particular, if we represent $df$ by a 1-cocycle $c_\pm$ as above, then $P_+(Df) = P_+(gc_\pm) = \mu_+$. Similarly, we have:

$$P_\pm(Df) = \mu_+, \quad P_\pm(D'f) = \nu_+. \quad (3.22)$$

Substituting this into the expression (3.20) for the Poisson bracket gives

$$\{f, h\} = <\Lambda(df), dh> = <P_+(Df) - \text{Ad}_g(P_+(D'f)), Dh>.$$ 

(3.23)

As this is the expression given above in (2.13) for the Sklyanin bracket, we have:

**Theorem (3.24)** The Mukai bracket and the Sklyanin bracket coincide on the reduced symplectic leaf of $H^0(\Sigma, \text{End}(E)(D))$ consisting of endomorphisms with a fixed determinant $\delta$.

**Remark:** 1) There is a natural $\mathbb{C}^\times$-action on the surface $\mathcal{T}$, and consequently on the moduli spaces $\mathcal{M}$ and $\mathcal{M}_{\text{red}}$ of sheaves on $\mathcal{T}$. The Poisson structure we constructed is $\mathbb{C}^\times$-invariant with respect to the natural $\mathbb{C}^\times$-action on $H^0(\mathcal{M}, \wedge^2 TM)$. So is the Poisson structure we started with on the surface $\mathcal{T}$. The $\mathbb{C}^\times$-invariance is related to the quadratic nature of the Poisson structure. Indeed, Lemma (3.16) produced a Poisson structure on the vector space $V = H^0(\Sigma, \text{End}(E)(D))$, which must come from an element of $\text{Sym}^2(V^*) \otimes \wedge^2 V$.

2) Polishchuk constructed a related quadratic Poisson structure on the moduli space $\mathcal{N}$ of stable triples $(E_1, E_2, \phi : E_2 \to E_1)$ (see [Po]). There is a natural morphism from our moduli space $\mathcal{M}$ of stable Higgs pairs to $\mathcal{N}$ (it involves taking the quotient my the $\mathbb{C}^\times$-action). The morphism is Poisson.

**3.d. Birational symplectic isomorphisms with Hilbert schemes.** We can compute simple Darboux coordinates for the Mukai symplectic form in the $GL(N, \mathbb{C})$-case. This will, incidentally, also show explicitly that we do have an integrable system, as well as characterise
the symplectic leaves. To do this, we construct different resolutions for $L$, to compute the Ext-groups of (3.8). Let us extend $L$ to a sheaf $L_U$ defined on an analytic neighbourhood $U$ of a smooth spectral curve. We then have the resolution, on $U$:

$$0 \to L_U(-nD) \xrightarrow{\det(g-\lambda I)} L_U \to L \to 0 \quad (3.25)$$

taking duals, and tensoring with $L(-D)$, the cotangent space of our moduli will be the hypercohomology, over $S$, of the sequence

$$\mathcal{O}(-D) \to \mathcal{O}((n-1)D) \quad (3.26)$$

and the tangent space, that of the sequence

$$\mathcal{O} \to \mathcal{O}(nD) \quad (3.27)$$

The maps in (3.26), (3.27) induced by (3.25) are simply the zero map, and so, since $K_S = \mathcal{O}((n-1)D)$, the cotangent space splits as

$$T^*\mathcal{M} \simeq H^1(S,\mathcal{O}(-D)) \oplus H^0(S,K_S) \quad (3.28)$$

and the tangent space as

$$T\mathcal{M} \simeq H^1(S,\mathcal{O}) \oplus H^0(S,K_S(D)) \quad (3.29)$$

The Poisson structure $\Lambda : T^*\mathcal{M} \to T\mathcal{M}$ is given, as above, by multiplication by the tautological section $\lambda$ of $\mathcal{O}(D)$ on both summands.

Let us define a subspace $\mathcal{M}^Z$ of $\mathcal{M}$ of pairs $(E, g)$ whose spectral curve intersects the zero section in $\mathcal{T}$ in a fixed divisor $Z$. $\mathcal{M}^Z$, by (2.14), is a union of symplectic leaves of the Poisson structure. Now identify $H^0(S,K_S)$ as the subspace of $H^0(S,K_S(D))$ of sections vanishing along the zero-section. We have

$$T\mathcal{M}^Z \simeq H^1(S,\mathcal{O}) \oplus H^0(S,K_S) \quad (3.30)$$

and dually

$$T^*\mathcal{M}^Z \simeq H^1(S,\mathcal{O}) \oplus H^0(S,K_S) \quad (3.31)$$

Under our identifications, the Poisson structure $T^*\mathcal{M}^Z \to T\mathcal{M}^Z$ is simply the identity map. In other words, the Poisson tensor is the canonical one on the sum (3.30), under the identifications we have made. As the Serre pairing is non-degenerate, this shows, fairly immediately, several important things:

**Theorem (3.32):** The space $\mathcal{M}^Z$ has an open set which is a symplectic leaf. The foliation on this leaf obtained by fixing the spectral curve is Lagrangian. The dimension of $\mathcal{M}^Z$ is twice the genus $g$ of the spectral curve, which is given by

$$g = \frac{N(N-1)d}{2} + 1$$
The constant $d$ is the degree of the divisor $D$. The computation of the genus is a simple application of the adjunction formula.

The canonical forms have simple Darboux coordinates. Indeed, the bundle $E$ has a one dimensional space of sections $\mathcal{A}_t$, and so, by (3.6), does $L$. In other words, the bundle $L$ is represented in a unique way as a divisor $\sum \mu \ p_\mu$, $\ p_\mu \in S$. Now we note that the space $\mathcal{T}$ admits a symplectic form, unique up to scale, with a pole along the zero-divisor. If $\lambda$ is a linear fibre coordinate on $\mathcal{T}$, and $\omega = dz$ is a one-form on the base elliptic curve $\Sigma$ (where $z$ is a standard linear coordinate on the curve), the symplectic form on $\Sigma$ is given by $\Omega_\mathcal{T} = \frac{d\lambda}{\lambda} \wedge \pi^* \omega$. The points $p_\mu$, which we will suppose distinct (as they are, generically), are given by a pair of coordinates $(z_\mu, \lambda_\mu)$. These pairs not only determine the line bundle but, generically, also the curve $S$, as it must pass through the points $p_\mu$.

**Proposition (3.33):** The Mukai form on the symplectic leaves can be written as $\Omega = \sum \mu \frac{d\lambda_\mu}{\lambda_\mu} \wedge d\lambda_\mu$.

**Proof:** The proof follows verbatim that given, e.g., in [HK]. It is mostly a question of writing down the explicit form of the duality pairings.

More invariantly, the proposition establishes a birational symplectomorphism between the open symplectic leaves of $\mathcal{M}^2$ and the $g$-th symmetric product of a blow up $\tilde{T}$ of $\mathcal{T}$. Indeed, along $\mathcal{M}^2$, the intersection $S \cap Z$ of the spectral curves with the zero-section is fixed. Let us blow up the points of $S \cap Z$, and call the resulting surface $\mathcal{T}$. The form $\Omega_\mathcal{T}$ lifts to a form $\tilde{\Omega}_\mathcal{T}$ on $\tilde{T}$, which is holomorphic away from the proper transform of the zero section. Let $\text{Hilb}_g(\tilde{T})$ denote the Hilbert scheme of 0-dimensional length $g$ subschemes in $\tilde{T}$; this is a desingularisation of the symmetric product, and it is symplectic. Proposition (3.33) then becomes:

**Proposition (3.34):** On the generic symplectic leaves of the Mukai bracket, the map which associates to a pair $(S, L)$ its divisor $\sum \mu \ p_\mu$ is a birational symplectic map between $\mathcal{L}$ and $\text{Hilb}_g(\tilde{T})$.

To deal with the Sklyanin bracket, we must reduce, both on the space of sections of $\text{End}(E)(D)$ and on the Hilbert scheme. For the first, as we indicated, the reduction amounts to fixing the top exterior power of $E$, and then quotienting by the action of $\mathbb{C}^*$ on the section $g$; equivalently, up to a finite cover, we fix the scale of the determinant; its zeroes are fixed on the symplectic leaf. For the Hilbert scheme, the surface $\tilde{T}$ admits a $\mathbb{C}^*$-action along the fibers of the projection $\tilde{T} \rightarrow \Sigma$. This action is symplectic, and its moment map (with values in $\Sigma$) is given by projection. The action extends to $\text{Hilb}_g(\tilde{T})$; the moment map is then the sum in $\Sigma$ of the points $\pi(p_\mu)$. To reduce under this action, we must fix the sum of the points, and then quotient by the $\mathbb{C}^*$ action. Note that, as in [HK], the sum of the points in $\Sigma$ is essentially the divisor corresponding to the top exterior power of the push-down $E$ of the line bundle $L$. Fixing the determinant of $g$, once one has its zeroes, results in a cyclic cover of the quotient by the $\mathbb{C}^*$ action. In short, the reductions by the $\mathbb{C}^*$ actions are compatible. We have:

**Proposition (3.35):** On the symplectic leaves $\mathcal{L}$ of the Sklyanin bracket, the map
which associates to a pair \((S,L)\) its divisor \(\sum \mu p_\mu\) is a symplectic map between \(L\) and \(\text{Hilb}_g(\tilde{T})/\mathbb{C}^*\).

It is perhaps worth emphasizing that the above description is quite amenable to explicit calculation. Indeed, as we saw, using the projection \(\pi: \mathbb{C} \to \Sigma\), elements of the symplectic leaf \(L\) can be described as matrix valued functions \(M\) on \(\mathbb{C}\) with poles at \(\pi^{-1}(D)\) satisfying \(M_i(z + \omega_i) = I_i M(z) I_i^{-1}\); these can be represented using theta-functions. The points \((z_\mu, \lambda_\mu)\) can be computed as zeroes of the equation

\[
(M(z) - \lambda I)\text{adj} S = 0,
\]

where \(\text{adj}\) denotes the matrix of cofactors, and \(S\) is a column vector of functions representing the section of \(E\). It can be computed explicitly using theta-functions, and the explicit formula is given in [HK], section 4.

The coordinates \((z_\mu, \lambda_\mu)\) allow a simple linearisation of the flows. Indeed, we note that fixing the Hamiltonians \(H_1, \ldots, H_k\) fixes the spectral curve, and so determines \(\lambda\) as a function of \(z\): \(\lambda = \lambda(z, H_1, \ldots, H_k)\). Choosing a base point \(z_0\) on \(\Sigma\), we set

\[
F(z_\mu, H_i) = \sum_\mu \int_{z_0}^{z_\mu} \ln(\lambda(z, H_i)) dz.
\]

One can show that these are sums of Abelian integrals.

4. Rational nodal, or trigonometric case.

One can allow the elliptic curve \(\Sigma\) to degenerate, and obtain a rational nodal curve \(\Sigma_0\) which is equivalent to the Riemann sphere \(\mathbb{P}^1\) with two points \(z = 0, \infty\) identified. We take the bundle \(O \oplus O \oplus \ldots \oplus O(1)\) of degree one on \(\mathbb{P}^1\), and identify the fibers over \(0, \infty\) to obtain a bundle \(E\) on the rational nodal curve. If one takes the transition matrix from \(z \neq \infty\) to \(z \neq 0\)

\[
T(z) = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
z^{-1} & 0 & 0 & \ldots & 0
\end{pmatrix},
\]

the identification between the fibers can be taken to be the identity matrix. Alternately, we can pass to the universal cover \(\mathbb{C}\) of \(\mathbb{C}^* = \mathbb{P}^1 - \{0, \infty\}\) and use an automorphy factor representation, so that sections of \(E\) are represented by vector functions satisfying \(F(x + \ldots)

\]

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1) = \text{I}_1 F(x)$, and suitable boundary behaviour as $ix \to \pm \infty$. Endomorphisms again become matrix valued functions with $M(x + 1) = I_1 M(x) I_1^{-1}$. We refer to [HK], section 5.

Again these bundles are rigid, up to the top exterior power. There is again a spectral curve $S$, covering the curve $\Sigma_0$, and a line bundle $L$ on $S$, which can as above be represented by a divisor $\sum (z_\mu, \lambda_\mu)$. We can go through the proof of the identity of the reduced Mukai bracket with the Sklyanin bracket, essentially verbatim. There is a splitting of the loop group into the sum of two subgroups, one corresponding to sections on a neighbourhood of $x = 0$ (that is, $z = 1$), and the other to sections of $\text{End}(E)$ on the complement of $z = 1$. Again, the Mukai symplectic form on the leaves has the form $\sum_\mu \frac{d\lambda_\mu}{\lambda_\mu} \wedge dz_\mu$; the reduction to the Sklyanin form amounts to fixing the determinant of the curve, and fixing the product of the $z_\mu$. The formula for computing the $(z_\mu, \lambda_\mu)$ are similar.

5. Rational case.

While this case has already been computed explicitly in [Sc], the proof given above adapts in a straightforward way to cover this case, too. Our bundle $E$, now, is simply the trivial rank $N$ bundle over $\mathbb{P}_1$. The bundle is, indeed, rigid; however, $H^0(\mathbb{P}_1, \text{End}(E)) \neq 0$, and so there is no unique splitting of the sections of $\text{End}(E)$ over the punctured disk. The groups $H^0(\mathbb{P}_1, \text{End}(E)(-1))$ and $H^1(\mathbb{P}_1, \text{End}(E)(-1))$ are zero, however, and this gives a decomposition of sections of $\text{End}(E)$ over the punctured disk into a direct sum of

- the subalgebra of sections of $\text{End}(E)$ over the disk which vanish at the origin, and

- the subalgebra of sections of $\text{End}(E)$ which are defined on the complement of the origin.

The spectral curves of elements $g$ of $H^0(\Sigma, \text{End}(E)(D))$ lie in the total space $\mathcal{T}$ of the line bundle $\mathcal{O}(D)$ over $\mathbb{P}_1$. Let $z$ be the standard coordinate on $\mathbb{P}_1$, and let $\lambda$ be a linear coordinate along the fibers of $\mathcal{T}$. The symplectic leaves lying in $H^0(\Sigma, \text{End}(E)(D))$ correspond to sections with a fixed determinant, as well as spectrum, which is fixed to order two over the point at infinity in $\mathbb{P}_1$. In other words, the spectral curves have fixed intersection with the zero-section $\lambda = 0$, as well as with $(z^{-2}) = 0$. This foliation by symplectic leaves corresponds to the choice of a Poisson structure on $\mathcal{T}$, whose divisor is precisely the zero-section $\lambda = 0$ and twice the fiber over $z = \infty$.

The space $H^0(\Sigma, \text{End}(E)(D))$ is acted on by $\text{PGL}(N, \mathbb{C})$ via the adjoint action of the group of automorphisms of the trivial bundle. We can take the Poisson quotient, to obtain a reduced space $H^0(\Sigma, \text{End}(E)(D))/\text{PGL}(N, \mathbb{C})$. We have, in a fashion analogous to what is given above:

**Proposition (5.1)** The Mukai Poisson structure and the reduced Sklyanin structure coincide. If elements $g$ correspond to a line bundle $L$ over the spectral curve, represented by a divisor $\sum_\mu (z_\mu, \lambda_\mu)$, the symplectic form on the leaves is $\sum_\mu \frac{d\lambda_\mu}{\lambda_\mu} \wedge dz_\mu$.

6. Higher genus.
One can ask how the above extends to higher genus base curves. One still, of course, has a space of pairs \((E, g)\), consisting of rank \(N\) bundles \(E\) and sections \(g\) of \(H^0(\Sigma, \text{End}(E)(D))\). If \(D\) is the sum of a canonical divisor and an effective divisor, then there is a Poisson structure on this space. The Poisson structure corresponds to the generalised Hitchin systems. Following the procedure of Mukai, it corresponds to a Poisson structure on \(\mathcal{O}(D)\) which is constant along the fibers of the projection \(\pi\) to the base curve \(\Sigma\). The Sklyanin systems, on the other hand, correspond to Poisson structures which are linear along the fibers. These only exist if the genus is at most one; if the genus is greater, one only has meromorphic Poisson structures, of the form \(\lambda \frac{\partial}{\partial \lambda} \wedge \pi^*\omega^{-1}\), where \(\omega\) is a holomorphic form on \(\Sigma\), and \(\lambda\) is a coordinate along the fiber. These forms correspond to degenerate symplectic forms on the Jacobian fibration \((S, L) \to S\). The form is null on certain directions in the fibers of the Jacobian fibration: if \(Z\) denotes the zero locus of \(\omega\), the null direction in the Jacobian corresponds to the coboundary \(\delta(H^0(S \cap \pi^{-1}(Z), \mathcal{O}(\pi^{-1}(Z))))\) in the exact sequence

\[ \ldots \to H^0(S \cap \pi^{-1}(Z), \mathcal{O}(\pi^{-1}(Z))) \to H^1(S, \mathcal{O}) \to H^1(S, \mathcal{O}(\pi^{-1}(Z))). \]

In any case, we can see that the Poisson geometry of rational surfaces suggests quite strongly that there is no nice Poisson extension of the Sklyanin bracket to arbitrary base curves.

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