ON LOCALLY $\phi$-SEMISYMMETRIC KENMOTSU MANIFOLDS

ABSOS ALI SHAIKH AND ALI AKBAR

MSC 2010 Classifications:53C25, 53D15.

Keywords and phrases: Kenmotsu manifold, locally $\phi$- symmetric, $\phi$-semisymmetric, manifold of constant curvature.

Abstract The object of the present paper is to study the locally $\phi$- semisymmetric Kenmotsu manifolds along with the characterization of such notion.

1. Introduction

Let $M$ be an $n$-dimensional, $n \geq 3$, connected smooth Riemannian manifold endowed with the Riemannian metric $g$. Let $\nabla$, $R$, $S$ and $r$ be the Levi-Civita connection, curvature tensor, Ricci tensor and the scalar curvature of $M$ respectively. The manifold $M$ is called locally symmetric due to Cartan (\cite{2}, \cite{3}) if the local geodesic symmetry at $p \in M$ is an isometry, which is equivalent to the fact that $\nabla R = 0$. Generalizing the concept of local symmetry, the notion of semisymmetric manifold was introduced by Cartan \cite{4} and fully classified by Szabo (\cite{11}, \cite{12}, \cite{13}). The manifold $M$ is said to be semisymmetric if $\left(R(U,V) R(X,Y) Z\right) = 0$, for all vector fields $X$, $Y$, $Z$, $U$, $V$ on $M$, where $R(U,V)$ is considered as the derivation of the tensor algebra at each point of $M$.

In 1977 Takahashi \cite{14} introduced the notion of local $\phi$- symmetry on a Sasakian manifold. A Sasakian manifold is said to be locally $\phi$-symmetric if

$$\phi^{2}((\nabla_{W}R)(X,Y)Z) = 0,$$

for all horizontal vector fields $X$, $Y$, $Z$, $W$ on $M$ that is all vector fields orthogonal to $\xi$, where $\phi$ is the structure tensor of the manifold $M$. The concept of local $\phi$- symmetry on various structures and their generalizations or extension are studied in ( \cite{6}, \cite{7}, \cite{8}, \cite{9}). By extending the notion of semisymmetry and generalizing the concept of local $\phi$-symmetry of Takahashi \cite{14}, the first author and his coauthor introduced \cite{10} the notion of local $\phi$-semisymmetry on a Sasakian manifold. A Sasakian manifold $M$, $n \geq 3$, is said...
to be locally $\phi$-semisymmetric if

$$
\phi^2((R(U, V).R)(X, Y)Z) = 0,
$$

for all horizontal vector fields $X, Y, Z, U, V$ on $M$. In the present paper we study locally $\phi$-semisymmetric Kenmotsu manifolds. The paper is organized as follows:

In section 2 some rudimentary facts and curvature related properties of Kenmotsu manifolds are discussed. In section 3 we study locally $\phi$-semisymmetric Kenmotsu manifolds and obtained the characterization of such notion.

2. Preliminaries

Let $M$ be a $(2n + 1)$-dimensional connected smooth manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a tensor field of type $(1, 1)$, $\xi$ is a vector field, $\eta$ is an 1-form and $g$ is a Riemannian metric on $M$ such that [1]

$$
\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1.
$$

(2.2) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$

for all vector fields $X, Y$ on $M$.

Then we have [1]

(2.3) $\phi \xi = 0$, \quad $\eta(\phi X) = 0$, \quad $\eta(X) = g(X, \xi)$.

(2.4) $g(\phi X, X) = 0$.

(2.5) $g(\phi X, Y) = -g(X, \phi Y)$

for all vector fields $X, Y$ on $M$.

If

(2.6) $(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X$,

(2.7) $\nabla_X \xi = X - \eta(X)\xi$,
holds on $M$, then it is called a Kenmotsu manifold \[5\].

In a Kenmotsu manifold the following relations hold \[5\]

\[
\begin{align*}
(\nabla_X \eta)Y &= g(X, Y) - \eta(X)\eta(Y), \\
\eta(R(X, Y)Z) &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\
R(X, \xi)Z &= g(X, Z)\xi - \eta(Z)X, \\
R(X, \xi)\xi &= \eta(X)\xi - X, \\
S(X, \xi) &= -2n\eta(X), \\
(\nabla_W R)(X, Y)\xi &= g(X, W)Y - g(Y, W)X - R(X, Y)W, \\
(\nabla_W R)(X, \xi)Z &= g(X, Z)W - g(W, Z)X - R(X, W)Z,
\end{align*}
\]

for all vector fields $X, Y, Z$ and $W$ on $M$.

In a Kenmotsu manifold we also have \[5\]

\[
\begin{align*}
R(X, Y)\phi W &= g(Y, W)\phi X - g(X, W)\phi Y + g(X, \phi W)Y - g(Y, \phi W)X + \phi R(X, Y)W.
\end{align*}
\]

Applying $\phi$ and using (2.1) we get from (2.16)

\[
\begin{align*}
\phi R(X, Y)\phi W &= -g(Y, W)X + g(X, W)Y + g(X, \phi W)\phi Y - g(Y, \phi W)\phi X - R(X, Y)W.
\end{align*}
\]

In view of (2.17) we obtain from (2.14)

\[
\begin{align*}
(\nabla_W R)(X, Y)\xi &= g(Y, \phi W)\phi X - g(X, \phi W)\phi Y + \phi R(X, Y)\phi W.
\end{align*}
\]
3. Locally $\phi$-semisymmetric Kenmotsu Manifolds

**Definition 3.1.** A Kenmotsu manifold $M$ is said to be locally $\phi$-semisymmetric if

$$
\phi^2((R(U, V)\cdot R)(X, Y)Z) = 0,
$$

for all horizontal vector fields $X, Y, Z, U, V$ on $M$.

First we suppose that $M$ is a Kenmotsu manifold such that

$$
\phi^2((R(U, V)\cdot R)(X, Y)\xi) = 0,
$$

for all horizontal vector fields $X, Y, U$ and $V$ on $M$.

Differentiating (2.18) covariantly with respect to a horizontal vector field $U$, we get

$$
(\nabla_U \nabla_V R)(X, Y)\xi = [g(Y, V)g(U, X) - g(X, V)g(U, Y) + g(R(X, Y)V, U)]\xi + \phi(\nabla_U R)(X, Y)\phi V.
$$

Using (2.16) we obtain from (3.3)

$$
(\nabla_U \nabla_V R)(X, Y)\xi = [g(Y, V)g(U, X) - g(X, V)g(U, Y) + g(R(X, Y)V, U)]\xi + \phi(\nabla_U R)(X, Y)\phi V.
$$

Interchanging $U$ and $V$ on (3.4) we get

$$
(\nabla_V \nabla_U R)(X, Y)\xi = [g(Y, V)g(U, X) - g(X, V)g(U, Y) + g(R(X, Y)U, V)]\xi + \phi(\nabla_V R)(X, Y)\phi U.
$$

From (3.4) and (3.5) it follows that

$$
(R(U, V)\cdot R)(X, Y)\xi = 2[g(Y, V)g(U, X) - g(X, V)g(U, Y) - R(X, Y, U, V)]\xi + \phi\{(\nabla_U R)(X, Y)\phi V - (\nabla_V R)(X, Y)\phi U\}.
$$

Again from (3.2) we have

$$
(R(U, V)\cdot R)(X, Y)\xi = 0,
$$
From (3.6) and (3.7) we have

\[
2[g(Y, V)g(U, X) - g(X, V)g(U, Y) - R(X, Y, U, V)]\xi \\
+ \phi\{(\nabla_U R)(X, Y)\phi V - (\nabla_V R)(X, Y)\phi U\} = 0.
\]

Applying \(\phi\) on (3.8) and using (2.16), (2.18) and (2.3) we get

\[
(\nabla_U R)(X, Y)\phi V - (\nabla_V R)(X, Y)\phi U = 0.
\]

In view of (3.8) and (3.9) we get

\[
R(X, Y, U, V) = g(Y, V)g(U, Y) - g(X, V)g(U, Y),
\]

\[
R(X, Y, U, V) = -\{g(X, V)g(U, Y) - g(Y, V)g(U, X)\},
\]

for all horizontal vector fields \(X, Y, U\) and \(V\) on \(M\). Hence \(M\) is of constant \(\phi\)-holomorphic sectional curvature -1 and hence of constant curvature -1. This leads to the following:

**Theorem 3.1.** If a Kenmotsu manifold \(M\) satisfies the condition \(\phi^2((R(U, V).R)(X, Y)\xi) = 0\), for all horizontal vector fields \(X, Y, Z, U\) and \(V\) on \(M\), then \(M\) is a manifold of constant curvature -1.

We consider a Kenmotsu manifold which is locally \(\phi\)-semisymmetric. Then from (3.1) we have

\[
(R(U, V).R)(X, Y)Z = g((R(U, V).R)(X, Y)Z, \xi)\xi,
\]

from which we get

\[
(R(U, V).R)(X, Y)Z = -g((R(U, V).R)(X, Y)\xi, Z)\xi
\]

for all horizontal vector fields \(X, Y, Z, U, V\) on \(M\).

Now taking inner product on both side of (3.6) with a horizontal vector field \(Z\), we obtain

\[
g((R(U, V).R)(X, Y)\xi, Z) = g(\phi(\nabla_U R)(X, Y)\phi V, Z) - g(\phi(\nabla_V R)(X, Y)\phi U, Z).
\]

Using (2.5) and (3.13) we get from (3.14)

\[
(R(U, V).R)(X, Y)Z = [g((\nabla_U R)(X, Y)\phi V, \phi Z) - g((\nabla_V R)(X, Y)\phi U, \phi Z)]\xi
\]
Differentiating (2.16) covariantly with respect to a horizontal vector field $V$, we get

$$(\nabla_V R)(X, Y)\phi Z$$

(3.16)

$$= [-g(Y, Z)g(V, \phi X) + g(X, Z)g(V, \phi Y) - g(V, R(X, Y)Z)]\xi + \phi(\nabla_V R)(X, Y)Z.$$  

Taking inner product on both sides of (3.16) with a horizontal vector field $U$, we obtain

$$g\{(\nabla_V R)(X, Y)\phi Z, U\} = g\{\phi(\nabla_V R)(X, Y)Z, U\}.$$  

(3.17)

Using (2.5) we get from above

$$g\{((\nabla_V R)(X, Y)\phi Z, U\} = -g\{((\nabla_V R)(X, Y)Z, \phi U\}.$$  

(3.18)

In view of (3.18) we obtain from (3.15)

$$R(U, V).R)(X, Y)Z = [-g((\nabla_V R)(X, Y)V, \phi^2 Z) + g((\nabla_V R)(X, Y)U, \phi^2 Z)]\xi,$$

which implies that

$$R(U, V).R)(X, Y)Z = [g((\nabla_V R)(X, Y)V, Z) - g((\nabla_V R)(X, Y)U, Z)]\xi,$$

(3.19)

i.e.

$$R(U, V).R)(X, Y)Z = -((\nabla_V R)(X, Y, Z, V) + (\nabla_V R)(X, Y, Z, U)]\xi,$$

(3.20)

for any horizontal vector field $X, Y, Z, U, V$ on $M$. Hence we can state the following:

**Theorem 3.2.** A Kenmotsu manifold $M$, $n \geq 3$, is locally $\phi$-semisymmetric if and only if the relation (3.21) holds for all horizontal vector fields $X, Y, Z, U, V$ on $M$.

4. **Characterization of Locally $\phi$-semisymmetric Kenmotsu Manifolds**

In this section we investigate the condition of local $\phi$-semisymmetry of a Kenmotsu manifold for arbitrary vector fields on $M$. To find this we need the following results.

**Lemma 4.1.** For any horizontal vector field $X, Y$ and $Z$ on a Kenmotsu manifold $M$, we have

$$\ell\xi Z = (\ell\xi Z) + 2R(X, Y)Z.$$  

(4.1)
Proof. Let $X^*, Y^*$ and $Z^*$ be $\xi$-invariant horizontal vector field extensions on $X$, $Y$ and $Z$ respectively. Since $X^*$ is $\xi$-invariant of $X$, we get by using \((2.7)\)

\[
(4.2) \quad \nabla_\xi X^* = \nabla_{X^*} \xi = X^*
\]

Now making use of invariance of $X^*$, $Y^*$ and $Z^*$ by $\xi$ and using \((4.2)\) we get

\[
(\ell_\xi R)(X^*, Y^*)Z^* = [\xi, R(X^*, Y^*)Z^*] = \nabla_\xi (R(X^*, Y^*)Z^*) - \nabla_{R(X^*, Y^*)Z^*} \xi
\]

\[
= (\nabla_\xi R)(X^*, Y^*)Z^* + R(\nabla_\xi X^*, Y^*)Z^* + R(X^*, \nabla_\xi Y^*)Z^*
+ R(X^*, Y^*)Z^* - R(X^*, Y^*)Z^*
= (\nabla_\xi R)(X^*, Y^*)Z^* + 2R(X^*, Y^*)Z^*
\]

Hence we get the conclusion. \qed

Lemma 4.2. For any vector field $X$, $Y$ and $Z$ on a Kenmotsu manifold $M$ we have

\[
(4.4) \quad R(\phi^2 X, \phi^2 Y)\phi^2 Z = -R(X, Y)Z + \eta(Z)\{\eta(X)Y - \eta(Y)X\}
\]

\[
+ \{\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\}\xi
\]

Now lemma \((4.1)\) and lemma \((4.2)\) together imply the following:

Lemma 4.3. For any vector field $X$, $Y$, $Z$ and $U$ on a Kenmotsu manifold $M$, we have

\[
(\nabla_{\phi^2 U} R)(\phi^2 X, \phi^2 Y)\phi^2 Z
\]

\[
= (\nabla_U R)(X, Y)Z - \eta(X)H_1(Y, U)Z + \eta(Y)H_1(X, U)Z + \eta(Z)H_1(X, Y)U
\]

\[
+ \eta(U)\{\eta(Z)\{\eta(X)\ell_\xi Y - \eta(Y)\ell_\xi X\} - (\ell_\xi R)(X, Y)Z\}
\]

\[
+ 2\eta(U)[R(X, Y)Z - \eta(Z)\{\eta(X)Y - \eta(Y)X\} - \{\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\}\xi].
\]

where the tensor field $H_1$ of type $(1, 3)$ is given by

\[
(4.6) \quad H_1(X, Y)Z = R(X, Y)Z - g(X, Z)Y + g(Y, Z)X,
\]

for all vector fields $X$, $Y$, $Z$ on $M$.

Now let $X$, $Y$, $Z$, $U$, $V$ be arbitrary vector fields on $M$.

Now we compute $(R(\phi^2 U, \phi^2 V) R)(\phi^2 X, \phi^2 Y)\phi^2 Z$ in two different ways. Firstly from
\[ (R(\phi^2 U, \phi^2 V)R)(\phi^2 X, \phi^2 Y)\phi^2 Z = \{ (\nabla_U R)(X, Y, Z, V) - (\nabla_V R)(X, Y, Z, U) \}\xi \]
\[ + \{ \eta(U)\eta[ (\nabla_V R)(X, Y)Z] - \eta(V)\eta[(\nabla_U R)(X, Y)Z] \}\xi \]
\[ - \eta(X)\{ H(Y, U, Z, V) - H(Y, V, Z, U) \}\xi \]
\[ + \eta(Y)\{ H(X, U, Z, V) - H(X, V, Z, U) \}\xi \]
\[ + \eta(Z)\{ H(X, Y, U, V) - H(X, Y, V, U) \}\xi \]
\[ + \eta(X)\eta(Z)\{ \eta(U)g(\ell_X Y, V) - \eta(V)g(\ell_X Y, U) \}\xi \]
\[ - \eta(Y)\eta(Z)\{ \eta(U)g(\ell_X Y, V) - \eta(V)g(\ell_X Y, U) \}\xi \]
\[ + 2\{ \eta(U)R(X, Y, Z, V) - \eta(V)R(X, Y, Z, U) \}\xi \]
\[ + 2\eta(Z)\eta(V)\{ \eta(X)g(Y, U) - \eta(Y)g(X, U) \}\xi \]
\[ - 2\eta(Z)\eta(U)\{ \eta(X)g(Y, V) - \eta(Y)g(X, V) \}\xi, \tag{4.7} \]

where \( H(X, Y, Z, U) = g(H_1(X, Y)Z, U) \) and the tensor field \( H_1 \) of type \((1, 3)\) is given by \((4.6)\).

Secondly we have
\[ (R(\phi^2 U, \phi^2 V)R)(\phi^2 X, \phi^2 Y)\phi^2 Z = R(\phi^2 U, \phi^2 V)R(\phi^2 X, \phi^2 Y)\phi^2 Z \]
\[ - R(R(\phi^2 U, \phi^2 V)\phi^2 X, \phi^2 Y)\phi^2 Z - R(\phi^2 X, R(\phi^2 U, \phi^2 V)\phi^2 Y)\phi^2 Z \]
\[ - R(\phi^2 X, \phi^2 Y)R(\phi^2 U, \phi^2 V)\phi^2 Z. \tag{4.8} \]

By straightforward calculation from \((4.8)\) we get
\[ (R(\phi^2 U, \phi^2 V)R)(\phi^2 X, \phi^2 Y)\phi^2 Z \]
\[ = -(R(U, V)R)(X, Y)Z \]
\[ + \eta(X)\{ \eta(V)H_1(U, Y)Z - \eta(U)H_1(V, Y)Z \} \]
\[ + \eta(Y)\{ \eta(V)H_1(X, U)Z - \eta(U)H_1(X, V)Z \} \]
\[ + \eta(Z)\{ \eta(V)H_1(X, Y)U - \eta(U)H_1(X, Y)V \} \]
\[ + \{ \eta(V)g(H(X, Y, Z, U) - \eta(U)g(H(X, Y, Z, V) \} \xi, \tag{4.9} \]

where \( H(X, Y, Z, U) = g(H_1(X, Y)Z, U) \) and the tensor field \( H_1 \) of type \((1, 3)\) is given by \((4.6)\).
From (4.7) and (4.9) we obtain

\[(4.10)\]

\[
(R(U, V), R)(X, Y)Z = \frac{\eta(V)\eta(\{\eta(Y)g(\ell_X Y, U) - \eta(Y)g(\ell_X U, Y)\} \xi + 2\eta(Z)\eta(U)\eta(V)g(X, Y, U) - \eta(Y)g(X, U)\xi)}{2}.
\]

Thus in a locally $\phi$-semisymmetric Kenmotsu manifold the relation (4.10) holds for all arbitrary vector fields $X, Y, Z, U, V$ on $M$. Next if the relation (4.10) holds in a Kenmotsu manifold, then for any horizontal vector field $X, Y, Z, U, V$ on $M$, we get the relation (3.21) and hence the manifold is locally $\phi$-semisymmetric.

Thus we can state the following:

**Theorem 4.1.** A Kenmotsu manifold $M$ is locally $\phi$-semisymmetric if and only if the relation (4.10) holds for any arbitrary vector field $X, Y, Z, U, V$ on $M$.

**References**

[1] Blair, D. E., Contact manifolds in Riemannian geometry. Lecture Notes in Math. No. 509. Springer 1976.

[2] Cartan, E., Sur une class remarquable despace de Riemann, I, Bull. de la Soc. Math. de France , 54(1926), 214-216.

[3] Cartan, E., Sur une class remarquable despace de Riemann, II, Bull. de la Soc. Math. de France , 55(1927), 114-134.

[4] Cartan, E., Lecons sur la geometric des espaces de Riemann, 2nd ed., Paris 1946.

[5] Kenmotsu, K., A class of almost contact Riemannian manifolds, Tohoku Math. J., 24(1972), 93-102.
[6] Shaikh, A. A., Baishya, K. K., On \(\phi\)-Symmetric LP- Sasakian manifolds, Yokohama Math. J., 52(2005), 97-112.

[7] Shaikh, A. A., Baishya, K. K. and Eyasmin, S., On \(\phi\)-recurrent generalized \((k, \mu)\)-contact metric manifolds, Lobachevskii J. Math., 27(2007), 3-13.

[8] Shaikh, A. A., Basu, T. and Eyasmin, S., On locally \(\phi\)-symmetric \((LCS)_n\)-manifolds, Int. J. of Pure and Appl. Math., 41(8)(2007), 1161-1170.

[9] Shaikh, A. A., Basu, T. and Eyasmin, S., On the existence of \(\phi\)-recurrent \((LCS)_n\)-manifolds, Extracta Mathematica, 23(1)(2008), 71-83.

[10] Shaikh, A. A., Mondal, C.K. and Ahmad, H., On locally \(\phi\)-semisymmetric Sasakian manifolds, arXive: 1302. 2139v3 [math.DG] 11 Feb 2017.

[11] Szabó, Z. I., Structure theorems on Riemannian spaces satisfying \(R(X,Y).R = 0\), I, The local version, J. Diff. Geom. 17(1982), 531-582.

[12] Szabó, Z. I., Structure theorems on Riemannian spaces satisfying \(R(X,Y).R = 0\), II, Global version, Geom. Dedicata, 19(1983), 65-108.

[13] Szabó, Z. I., Classification and construction of complete hypersurfaces satisfying \(R(X,Y).R = 0\), Acta. Sci. Math., 47(1984), 321-348.

[14] Takahashi. T., Sasakian \(\phi\)-symmetric spaces, Tohoku Math. J., 29(1977), 91-113.

Department of Mathematics, Aligarh Muslim University, Aligarh, Uttar Pradesh, INDIA

E-mail address: aask2003@yahoo.co.in, aashaikh@math.buruniv.ac.in

Department of Mathematics, Rampurhat College, Birbhum, West Bengal 731224, INDIA

E-mail address: aliakbar.akbar@rediffmail.com