Non-Perturbative Corrections to Heavy Quark Fragmentation in $e^+e^-$ Annihilation

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Abstract

We estimate the non-perturbative power-suppressed corrections to heavy flavour fragmentation and correlation functions in $e^+e^-$ annihilation, using a model based on the analysis of one-loop Feynman graphs containing a massive gluon. This approach corresponds to the study of infrared renormalons in the large-$n_f$ limit of QCD, or to the assumption of an infrared-finite effective coupling at low scales. We find that the leading corrections to the heavy quark fragmentation function are of order $\lambda/M$, where $\lambda$ is a typical hadronic scale ($\lambda \sim 0.4$ GeV) and $M$ is the heavy quark mass. The inclusion of higher corrections corresponds to convolution with a universal function of $M(1 - x)$ concentrated at values of its argument of order $\lambda$, in agreement with intuitive expectations. On the other hand, corrections to heavy quark correlations are very small, of the order of $(\lambda/Q)^p$, where $Q$ is the centre-of-mass energy and $p \geq 2$.
1 Introduction

The production of hadrons containing heavy quarks in $e^+e^-$ annihilation has proved to be a process of great importance for testing the Standard Model and searching for new physics. Heavy flavour processes are also valuable as a testing-ground for new techniques in QCD, because the heavy quark mass $M$ provides a second large momentum scale, in addition to the overall hard scale $Q$, set by the centre-of-mass energy in $e^+e^-$ annihilation. This has led to a rather good understanding of heavy quark production and fragmentation within the context of perturbation theory [1–4]. On the other hand, it has become clear as a result of this understanding that the $c$ and $b$ quark masses are not large enough for a purely perturbative approach to provide a good description of the data for these flavours [2,3]. This is because non-perturbative effects can give rise to contributions of order $\lambda/M$, where $\lambda$ is a typical soft physics scale. One may also worry about the possibility of non-perturbative effects of order $\lambda/Q$, which could be significant at present energies. Thus it becomes important to study power-suppressed corrections in QCD from as many viewpoints as possible, and to apply the resulting insight to heavy flavour processes in particular.

One approach to the study of power-suppressed corrections, which has proved popular recently, arose from the study of infrared renormalons [5]. Here a divergent series of perturbative contributions gives rise to a power-suppressed renormalon ambiguity in the prediction of perturbation theory. One can then argue that a non-perturbative contribution with the same power behaviour should be present, with an ambiguity in its coefficient which cancels that associated with the renormalon. Conversely, when the renormalon ambiguity involves a high inverse power of the hard scale, one expects non-perturbative corrections to be especially small. This approach can be reformulated without reference to renormalons by postulating the existence of an infrared-regular effective coupling at low scales (the ‘dispersive approach’ of ref. [6]). The expected power-suppressed corrections are consistent with those predicted by more rigorous approaches such as the operator product expansion where applicable, and have been found to agree fairly well with those suggested by experimental data [7–12].

The aim of the present paper is to study power suppressed effects in heavy flavour production in $e^+e^-$ annihilation, using either the ‘renormalon’ or ‘dispersive’ approach, which are equivalent for the purpose of the present work. Our aim will be to test some standard assumption about the form of non-perturbative corrections that enter the fragmentation function. Furthermore, we will also examine the correlation between the quark and antiquark momenta. This topic is of practical interest, since this correlation affects the determination of $\Gamma_b$, the partial width for the decay of the $Z^0$ boson into $b$-flavoured hadrons.
1.1 Renormalons and power-suppressed corrections

In the renormalon approach, described in detail for $e^+e^-$ shape variable calculations in ref. [13], one starts from the first-order perturbative contributions to the process $e^+e^- \rightarrow Q\bar{Q}X$, involving emission of a single real or virtual gluon. One then identifies a factorially divergent series of perturbative contributions associated with light quark loop insertions on the gluon line. These contributions will be dominant at high orders for sufficiently large values of $n_f$, the number of light flavours. One now argues [14–16] that the true high-order behaviour of perturbative QCD can be approximated by making the replacement $n_f \rightarrow n_f - 33/2$ in the large-$n_f$ behaviour. There is support for this ‘naive non-Abelianization’ assumption in the high-order behaviour of the average plaquette in quenched ($n_f = 0$) lattice QCD [17].

The resulting factorial divergences of the perturbation series can be of two types. Those associated with high-momentum regions of integration (ultraviolet renormalons) correspond to divergent series with alternating signs, which can be summed unambiguously using standard techniques such as Borel summation. We shall not be concerned with them in the present paper. Those due to low momenta flowing in loop integrals (infrared renormalons) produce same-sign asymptotic series, which are intrinsically ambiguous. The ambiguity is of the order of the smallest term in the series, which turns out to be a power-suppressed quantity, of order $(\lambda/Q)^p$ where $p$ is a number that depends on the observable being computed.

In the full theory of QCD, the infrared renormalon ambiguities of perturbation theory must be cancelled by non-perturbative contributions. We shall assume that the presence of an infrared renormalon with a particular value of the power $p$ indicates that a comparable power-suppressed non-perturbative contribution is actually present in the full theory.

The dispersive approach of ref. [6] involves similar calculations, with a slightly different interpretation. A (formal) dispersion relation for the QCD running coupling of the form

$$\alpha_s(k^2) = -\int_0^\infty \frac{d\mu^2}{\mu^2 + k^2} \rho_s(\mu^2), \quad \rho_s(\mu^2) = -\frac{1}{2\pi i} \text{Disc} \{\alpha_s(-\mu^2)\},$$

is assumed. In QCD, unlike QED, the running of the coupling cannot be associated with vacuum polarization effects alone. However, in the same spirit as the ‘naive non-Abelianization’ assumption, it is further assumed that the dominant effect on some QCD observable $F$ of the running of $\alpha_s$ in one-loop graphs may be represented in terms of the spectral function $\rho_s(\mu^2)$ and a characteristic function $F(\mu^2)$, as follows:

$$F = \alpha_s(0) F(0) + \int_0^\infty \frac{d\mu^2}{\mu^2} \rho_s(\mu^2) \cdot F(\mu^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \rho_s(\mu^2) \cdot [F(\mu^2) - F(0)],$$
where the relation (1.1) has been used to eliminate \( \alpha_s(0) \). The characteristic function \( F(\mu^2) \) is obtained by computing the relevant one-loop graphs with a non-zero gluon mass \( \mu \) \(^{[17]}\) and dividing by \( \alpha_s \). Integrating Eq. (1.2) by parts, we can write

\[
F = \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}(\mu^2) \cdot \dot{\mathcal{F}}(\mu^2),
\]

where

\[
\dot{\mathcal{F}} \equiv -\frac{\partial \mathcal{F}}{\partial \ln \mu^2}
\]

and we have introduced the effective coupling \( \alpha_{\text{eff}}(\mu^2) \), defined in terms of \( \rho_s(\mu^2) \) by

\[
\rho_s(\mu^2) = \frac{d}{d \ln \mu^2} \alpha_{\text{eff}}(\mu^2).
\]

It follows from this definition and Eq. (1.1) that

\[
\alpha_{\text{eff}}(\mu^2) = \alpha_s(\mu^2) - \frac{\pi^2}{3!} \frac{d^2 \alpha_s}{d \ln^2 \mu^2} + \frac{\pi^4}{5!} \frac{d^4 \alpha_s}{d \ln^4 \mu^2} - \ldots = \alpha_s + \mathcal{O}(\alpha_s^3),
\]

and therefore in the perturbative domain \( \alpha_s \ll 1 \) the standard and effective couplings are approximately the same. In the large-\( n_f \) renormalon approach\(^\dagger\) we have the explicit expression, obtained by substituting the one-loop running coupling into Eq. (1.6):

\[
\alpha_{\text{eff}}(\mu^2) = \frac{1}{\pi b_0} \arctan \left( \pi b_0 \alpha_s(\mu^2) \right)
\]

where \( b_0 = (33 - 2n_f)/12\pi \).

The characteristic function \( F(\mu^2) \) is more precisely a function of the dimensionless ratio \( \epsilon = \mu^2/Q^2 \), where \( Q^2 \) is the characteristic scale of the hard process. If the effective coupling in Eq. (1.3) has a non-perturbative component \( \delta \alpha_{\text{eff}}(\mu^2) \), with support limited to low values of \( \mu^2 \), the corresponding correction to \( F \),

\[
\delta F = \int \frac{d\mu^2}{\mu^2} \delta \alpha_{\text{eff}}(\mu^2) \dot{\mathcal{F}}(\mu^2),
\]

will therefore have a \( Q^2 \) dependence determined by the low-\( \mu^2 \) behaviour of \( F \).

A crucial point is that only those terms in \( F \) that are non-analytic at \( \mu^2 = 0 \) (\( \epsilon = 0 \)) can produce power-suppressed contributions to \( \delta F \). This is because the integer \( \mu^2 \)-moments of \( \delta \alpha_{\text{eff}}(\mu^2) \) are required to vanish, for consistency with the operator product expansion. The same result may be seen in the renormalon analysis of ref. \([13]\): for any behaviour of \( \mathcal{F}(\epsilon) \) of the form \( \epsilon^p \) as \( \epsilon \to 0 \), the renormalon contribution is proportional

\(^\dagger\)We stress that this model is not physically fully consistent, because of the presence of the Landau pole, which implies that the support of the spectral function of \( \alpha_s \) must be extended to negative values of the argument for Eq. (1.1) to be valid.
to Im(e^{i\nu\pi}) and therefore vanishes for integer \( p \). On the other hand, a \( \sqrt{\epsilon} \) behaviour implies a non-vanishing correction proportional to \( 1/Q \), while \( \epsilon \ln \epsilon \) gives \( 1/Q^2 \), etc. Thus our objective is to identify the leading non-analytic terms in the behaviour of the characteristic function at small values of the gluon mass-squared, which will tell us the \( Q^2 \)-dependence of the leading power-suppressed corrections.

If one makes the additional assumption that the effective coupling modification \( \delta\alpha_{\text{eff}}(\mu^2) \) in Eq. (1.8) is universal, one obtains a factorization property for power-suppressed corrections, which leads to relationships between the coefficients of the corrections to different observables. For variables like event shapes, this type of factorization is only approximate, due to the fact that a cut dressed gluon line is weighted differently, depending upon the value of the shape variable for the particular final-state structure of the cut gluon \[13\]. In the present case, however, the dressed gluon is cut fully inclusively, without any weight, and therefore factorization may be more reliable.

In the case of heavy flavour processes, we also want to study corrections that are suppressed by powers of the heavy quark mass, \( M \). As long as we treat both \( M \) and \( Q \) as large parameters, and keep track of the dependence on their ratio, this will be done automatically when we extract the non-analytic terms in \( \epsilon \). Defining \( \rho = 4M^2/Q^2 \), a \( \sqrt{\epsilon/\rho} \) term will indicate a correction proportional to \( 1/M \), \( \epsilon \ln \epsilon/\rho \) implies \( 1/M^2 \), and so on.

In our terminology, mass corrections of the form \( (M/Q)^p \) will not be called power-suppressed, since we are always assuming that \( M \) is not small.

\section{2 Calculations}

\subsection{2.1 Massive gluon cross sections}

Considering first the vector current contribution, the distribution of the heavy quark and antiquark energy fractions \( x \) and \( \bar{x} \) with emission of a gluon of mass \( \mu \) in the process \( e^+e^- \rightarrow QQg \) is given by

\[
\frac{1}{\sigma_V} \frac{d^2\sigma_V}{dx d\bar{x}} = \frac{\alpha_s C_F}{2\pi} \frac{(x + \eta)^2 + (\bar{x} + \eta)^2 + \zeta_V}{(1 + \frac{1}{2}\rho)(1 - x)(1 - \bar{x})} - \frac{\eta}{(1 - x)^2} - \frac{\eta}{(1 - \bar{x})^2} .
\]

(2.1)

Here \( \eta = (\mu^2 + 2M^2)/Q^2 = \epsilon + \frac{1}{2}\rho \) where \( \epsilon = \mu^2/Q^2 \), \( M \) is the quark mass, \( \rho = 4M^2/Q^2 \),

\[
\zeta_V = -2\rho(1 + \eta) ,
\]

(2.2)

\( \beta = \sqrt{1 - \rho} \) is the heavy quark velocity, and

\[
\sigma_V = \sigma_0 \left(1 + \frac{1}{2}\rho\right) \beta
\]

(2.3)

is the Born cross section for heavy quark production by a vector current, \( \sigma_0 \) being the massless quark Born cross section.
The phase space is determined by the triangle relation

\[ \Delta(x^2 - \rho, \bar{x}^2 - \rho, x_g^2 - 4\epsilon) \leq 0 \]  

(2.4)

where \( \Delta(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca \) and \( x_g = 2 - x - \bar{x} \). This gives \( x_- \leq \bar{x} \leq x_+ \) where

\[
x_\pm = \frac{(2-x)(1-x-\epsilon + \frac{1}{2}\rho) \pm \Xi(x, \rho, \epsilon)}{2(1-x)+\frac{1}{2}\rho} \tag{2.5}
\]

and

\[
\sqrt{\rho} \leq x \leq 1 - \epsilon - \sqrt{\epsilon\rho} \tag{2.6}
\]

In the case of the axial current contribution, instead of Eq. (2.1) we have

\[
\frac{1}{\sigma_A} \frac{d^2\sigma_A}{dx d\bar{x}} = \frac{\alpha_s C_F}{2\pi \beta} \left[ \frac{(x + \eta)^2 + (\bar{x} + \eta)^2 + \zeta_A}{(1-\rho)(1-x)(1-\bar{x})} - \frac{\eta}{(1-x)^2} - \frac{\eta}{(1-\bar{x})^2} \right] , \tag{2.7}
\]

where

\[
\zeta_A = \frac{1}{2}\rho[(3 + x_g)^2 - 19 + \rho - 8\epsilon] , \tag{2.8}
\]

\( \sigma_A \) being the Born cross section for heavy quark production by the axial current:

\[
\sigma_A = \sigma_0 \beta^3 = \sigma_0 (1-\rho)\beta . \tag{2.9}
\]

### 2.2 Leading power corrections

Clearly the expressions (2.4) and (2.7) are analytic functions of \( \epsilon \) at \( \epsilon = 0 \) except possibly for \( x \) and \( \bar{x} \) near 1. The phase space is also analytic in \( \epsilon \) whenever the gluon momentum is large, since in this region one can always expand kinematic variables in powers of \( \epsilon \). As discussed above, this implies that there are no non-perturbative corrections of the type we are considering for \( x, \bar{x} < 1 \). Non-analytic behaviour may only arise in the region where \( x_g \approx \sqrt{\epsilon} \) \( (x, \bar{x} \approx 1) \). In order to investigate the corrections associated with this region we take moments of the form

\[
\mathcal{M}(N, \bar{N}, \epsilon) = \int dx d\bar{x} x^N \bar{x}^{\bar{N}} \frac{1}{\sigma} \frac{d^2\sigma}{dx d\bar{x}} \tag{2.10}
\]

and expand

\[
x^N \bar{x}^{\bar{N}} = 1 - Ny - \bar{N}\bar{y} + \cdots , \tag{2.11}
\]

where \( y = 1 - x \) and \( \bar{y} = 1 - \bar{x} \). The first term corresponds to the total heavy flavour cross section, whose dominant power correction is of order \((\lambda/Q)^4\) or smaller. However, the next two terms give contributions proportional to \( \sqrt{\epsilon} \) at small \( \epsilon \), which could give rise to \( \lambda/Q \) and/or \( \lambda/M \) corrections. To evaluate them we note that their
contribution to the difference \( \delta \mathcal{M} = \mathcal{M}(N, \bar{N}, 0) - \mathcal{M}(N, \bar{N}, \epsilon) \) for small \( \epsilon \) can be written, for both the vector and axial current contributions, as \( C_F \alpha_s \mathcal{F} / 2\pi \) where

\[
\mathcal{F} = \frac{1}{\beta} \int_R dy d\bar{y} (Ny + \bar{N}\bar{y}) \left( \frac{2 - \rho}{y\bar{y}} - \frac{\rho}{2y^2} - \frac{\rho}{2\bar{y}^2} \right) 
\]

\[
= \frac{1}{2\beta} (N + \bar{N}) \int_R dy d\bar{y} \left( \frac{4 - 3\rho}{y} - \frac{\rho\bar{y}}{y^2} \right) ,
\]

\( \mathcal{F} \) being the region between the phase space boundaries for \( \epsilon = 0 \) and \( \epsilon > 0 \). Eq. (2.4) may be expanded in the region \( x_g = y + \bar{y} \approx \sqrt{\epsilon} \), so that the boundary of phase space in this region is given by

\[
\frac{1}{4} \Delta(x^2 - \rho, \bar{x}^2 - \rho, x^2_g - 4\epsilon) \approx \rho(y^2 + \bar{y}^2) - 2(2 - \rho)y\bar{y} + 4\epsilon(1 - \rho) \leq 0 ,
\]

which is the equation of a hyperbola. Changing variables to \( r, \phi \) where \( y = r \cos \phi \) and \( \bar{y} = r \sin \phi \), the region \( R \) may be written as

\[
R : \quad \delta < \phi < \frac{1}{2}\pi - \delta \, , \quad 0 < r < \sqrt{\frac{2\epsilon (1 - \sin 2\delta)}{\sin 2\phi - \sin 2\delta}}
\]

(2.14)

where \( \sin 2\delta = \rho/(2 - \rho) \). Performing the integration one gets

\[
\mathcal{F} = \frac{\pi}{2\beta^2} (N + \bar{N}) \sqrt{\frac{2\epsilon (1 - \sin 2\delta)}{\sin 2\delta}} \left( 4 - 3\rho - \frac{\rho}{\sin 2\delta} \right)
\]

\[
= 2\pi \beta^2 (N + \bar{N}) \sqrt{\frac{\epsilon}{\rho}} = \pi (N + \bar{N}) \left( 1 - \frac{M^2}{Q^2} \right) \frac{\mu}{M} .
\]

(2.15)

Therefore the leading power correction is of order \( \mu/M \) rather than \( \mu/Q \). The leading correction with an explicit dependence on \( Q \) is of order \( \mu M/Q^2 \). This is consistent with the finding for light quark fragmentation functions \([6,10,11,18]\): the leading power correction in \( Q \) is of order \( 1/Q^2 \).

The linear dependence on the moment index \( N \) in the result (2.15) implies a behaviour in \( x \)-space of the form

\[
\mathcal{F}(x, \bar{x}, \mu^2) = \pi \beta^2 \frac{\mu}{M} [\delta(1 - x)\delta'(1 - \bar{x}) + \delta'(1 - x)\delta(1 - \bar{x})] .
\]

(2.16)

This means that, as far as the leading power correction is concerned, the two-particle heavy quark distribution factorizes. In the dispersive approach of ref. \([3]\), the non-perturbative correction is given in terms of the low-energy modification to the effective coupling, \( \delta \alpha_{\text{eff}}(\mu^2) \), by Eq. (1.8). Defining the non-perturbative parameter \( A_1 \)

\[
A_1 = \frac{C_F}{2\pi} \int \frac{d\mu^2}{\mu^2} \mu \delta \alpha_{\text{eff}}(\mu^2) ,
\]

(2.17)
Eqs. (1.8) and (2.16) then imply that
\[
\frac{1}{\sigma} \frac{d^2 \sigma}{d x d \bar{x}} \approx \left[ \delta(1 - x) - \frac{\pi A_1}{2M} \beta^2 \delta'(1 - x) \right] \cdot \left[ \delta(1 - \bar{x}) - \frac{\pi A_1}{2M} \beta^2 \delta'(1 - \bar{x}) \right]
\]
\[
\approx \delta \left( 1 - x - \beta^2 \frac{\lambda}{M} \right) \cdot \delta \left( 1 - \bar{x} - \beta^2 \frac{\lambda}{M} \right)
\]
(2.18)

where \( \lambda = \pi A_1 / 2 \). Thus we see that the main non-perturbative effect is a shift in the heavy-quark momentum fractions by an amount \( \delta x \sim \lambda / M \). Assuming approximate universality of \( \delta \alpha_{\text{eff}} \), one may estimate from light-quark event shape data that \( A_1 \approx 0.25 \text{ GeV} \) [6], which gives \( \lambda \sim 0.4 \text{ GeV} \). This agrees with the order of magnitude of the non-perturbative shift estimated from \( \langle x \rangle \) in heavy flavour fragmentation [2].

2.3 Higher power corrections

Power-suppressed effects in the heavy flavour fragmentation functions should be equivalent to a convolution with a non-perturbative initial condition of the form \( Mf(M(1 - x)) \), and therefore should approach a delta function as \( M \to \infty \). This can be inferred by intuitive reasoning, but can also be derived more rigorously in the context of the heavy quark mass expansion [19]. In this section we will show that this expectation is also fulfilled in our model.

First of all we note that Eq. (2.18) can be written as
\[
\frac{1}{\sigma} \frac{d^2 \sigma}{d x d \bar{x}} \approx M \delta \left( M(1 - x) - \beta^2 \lambda \right) \cdot M \delta \left( M(1 - \bar{x}) - \beta^2 \lambda \right)
\]
and therefore the expected form is indeed obtained when one includes only the leading power correction.

To go beyond the leading correction, we have to consider higher moments with respect to \( y \) and \( \bar{y} \) in Eq. (2.12). We examine first the moments of the single-particle distribution \( \bar{N} = 0 \). For moments weighted by \( y^p \) with \( p > 1 \) we have to define the integration region \( R \) more carefully. Consider for simplicity the case that \( \rho \) is small (i.e. \( M^2 \ll Q^2 \)), so that \( \sin 2\delta \approx 2\delta \approx \frac{1}{2} \rho \). The upper limit of the \( r \)-integration becomes a constant of order unity when \( \delta < \phi < \delta + \epsilon \). Hence this region gives a term that is analytic in \( \epsilon \), which will not contribute to power corrections. The important region is \( \delta + \epsilon < \phi < \frac{1}{2} \pi - \delta - \epsilon \). The leading non-analytic term coming from this region is proportional to \( (\epsilon / \delta)^{p/2} \) when \( p \) is odd, and proportional to \( (\epsilon / \delta)^{p/2} \ln \epsilon \) when \( p \) is even. Hence for every value of \( p \) there is a power correction of order \( (\epsilon / \delta)^{p/2} = (\mu / M)^p \).

In detail, for \( \rho \) small and \( \bar{N} = 0 \) Eq. (2.12) becomes
\[
\mathcal{F} = \sum_{p=1}^{\infty} \left( \frac{N}{p} \right) \int_R dy d\bar{y} y^p \left( \frac{2}{y\bar{y}} - \frac{\rho}{2y^2} - \frac{\rho}{2\bar{y}^2} \right)
\]
\[
\approx \sum_{p=1}^{\infty} \left( \frac{N}{p} \right) \frac{2}{p} \epsilon^{p/2} \int_{\epsilon+\delta}^{1} \frac{d\phi}{\phi^{2}} (\phi - \delta)^{(2-p)/2}.
\]
(2.20)
Now
\[
\int_{\epsilon+\delta}^{1} \frac{d\phi}{\phi^2} (\phi - \delta)^{(2-p)/2} \approx (-1)^{(p-1)/2} \left(1 - \frac{1}{2}p\right) \pi \delta^{-p/2} + \cdots \quad \text{for } p \text{ odd},
\]
\[
\approx (-1)^{p/2} \left(1 - \frac{1}{2}p\right) \delta^{-p/2} \ln \epsilon + \cdots \quad \text{for } p \text{ even},
\]
where the dots correspond to terms giving contributions that are analytic and/or higher-order in \(\epsilon\). Thus, keeping only the leading non-analytic parts, we find
\[
\mathcal{F} = \text{Re} \left\{ (\ln \epsilon - i\pi) \sum_{p=1}^{\infty} \left( \frac{N}{p} \right) \left( \frac{2}{p} - 1 \right) \left( \frac{i\mu}{M} \right)^p \right\},
\]
which corresponds in \(x\)-space to
\[
\mathcal{F}(x, \mu^2) = Mf \left(M(1-x)\right)
\]
\[
f(z) = \text{Re} \left\{ (\ln \epsilon - i\pi) \sum_{p=1}^{\infty} \frac{1}{p!} \left( \frac{2}{p} - 1 \right) (i\mu)^p \delta^{(p)}(z) \right\}.
\]
We can therefore see the expected scaling of the fragmentation function. We also see that while the leading power correction, eq. (2.18), corresponds to a simple shift in the value of \(x\), this behaviour is not preserved by the higher power corrections.

3 Correlations

We consider now the higher power corrections to the two-particle distribution, i.e. the inclusion of higher powers of \(\mu/M\) in Eq. (2.19). We have to examine the double moments corresponding to Eq. (2.12) with general weights \(y^p\bar{y}^q\). Again we treat only the case of small \(\rho\). Then we find that for any \(q\) such that \(0 < q \leq p\), the leading non-analytic term is suppressed by a factor of \(\rho^q\) relative to that for weight \(y^p\bar{y}^q\). Therefore the leading terms in each order of \(\mu/M\) remain of the form (2.16):
\[
\mathcal{F}(x, \bar{x}, \mu^2) = \delta(1 - x)\mathcal{F}(\bar{x}, \mu^2) + \mathcal{F}(x, \mu^2)\delta(1 - \bar{x})
\]
where \(\mathcal{F}(x, \mu^2)\) is as given in Eq. (2.23). Thus the two-particle distribution including these terms still factorizes and can be expressed as a function of \(M(1-x)\) and \(M(1-\bar{x})\), although beyond leading order in \(1/M\) it differs from the simple product of delta functions given in Eq. (2.19).

Because of its possible impact on the determination of \(\Gamma_b\) in \(Z^0\) decays, it is interesting to determine what is the leading power correction to the momentum correlation \(\langle y\bar{y} \rangle - \langle y \rangle^2\). As stated before, the correction to \(\langle y\bar{y} \rangle\) behaves as \(\rho(\sqrt{\epsilon}/\rho)^2 = \epsilon\) at small \(\epsilon\). The term \(\langle y \rangle^2\) gives zero at the order we are considering. In fact in the large-\(n_f\) limit it is subleading, and in the dispersive approach it is of second order in the effective coupling. Therefore the momentum correlation is of order \(\epsilon\).
In order to confirm this conclusion, we also calculated the difference $\langle y\bar{y} \rangle - \langle y\bar{y} \rangle_{\epsilon=0}$ using the exact phase space and matrix elements. We found that the leading term at small $\epsilon$ is proportional to $\epsilon \ln \rho$. Thus, corrections to the momentum correlation in our model are suppressed by at least two powers of $Q$, and should therefore be completely negligible at LEP energies. Whether this result survives higher-order corrections is an open question, and in fact a very difficult one. We simply point out here that, while Monte Carlo models seem to indicate the presence of $\lambda/Q$ corrections to correlations (see ref. [4] and references therein), the simple model that we have adopted in this work does not provide support for the presence of such corrections. This is also consistent with the findings of ref. [10], where power corrections to fragmentation functions were computed in the strictly massless limit.

4 Discussion

We have examined the heavy flavour fragmentation function and correlations in a simple model, and found the following results.

At leading order, non-perturbative effects in the fragmentation function can be represented as a convolution with a function of the form

$$M f (M (1-x)),$$

which approaches a $\delta$ function as $M \to \infty$. The leading power correction has the form $\lambda/M$, where $\lambda$ is a typical soft hadronic scale. An estimate of this correction based on the approach proposed in ref. [6] gives the correct sign and order of magnitude.

In the two-particle distribution, corrections of the order of $(\lambda/M)^p$ factorize and therefore no large correlations of this order arise. Since correlations are important for their possible impact on the determination of $\Gamma_b$ in $Z^0$ decays, and since the perturbative value of the correlation is of the order of 1% [4], it is also important to understand whether corrections of the order of $\lambda/Q$ are present. In our analysis, consistently with ref. [10], terms of this order do not arise. In fact, we also verified numerically that the $\epsilon$ dependence of the correlation $\langle y\bar{y} \rangle - \langle y \rangle^2$ is of order $\epsilon \log \rho$, and therefore the leading power correction is less than order $(\lambda/Q)^2 \log \rho$.

We end with a comment on the relationship between our results and those of ref. [3]. In that paper a resummed expression for the heavy quark spectrum was derived and numerical results were presented using various models for the behaviour of the QCD running coupling at low scales. The non-perturbative component of the coupling generates $1/M$ corrections which should correspond to those considered here, after convolution with the perturbative fragmentation function.

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Appendix

We give here for reference the single-particle inclusive distribution which results from integrating the vector two-particle distribution (2.1) for $x < 1$:

\[
\frac{1}{\sigma_V} \frac{d\sigma_V}{dx} = \int_{x_-}^{x_+} \frac{d\bar{x}}{x} \frac{d^2\sigma_V}{dx d\bar{x}} = \frac{\alpha_s}{2\pi} \frac{C_F}{\beta(1 + \frac{1}{2}\rho)} \left\{ \left( \frac{2(1 + \epsilon)^2 - \frac{1}{2}\rho^2}{1 - x} - 1 - 2\epsilon - \rho \right) \times \ln \left[ \frac{(1 - x)(x - \frac{1}{2}\rho) + \epsilon(2 - x) + \Xi(x, \rho, \epsilon)^2}{(1 - x + \frac{1}{2}\rho)[\rho(1 - x)^2 + 4\epsilon(x + \epsilon - \rho)]} \right] \right. 
\]

\[
- \left. \frac{\Xi(x, \rho, \epsilon) \Phi(x, \rho, \epsilon)}{8(1 - x)^2(1 - x + \frac{1}{4}\rho)^2[\rho(1 - x)^2 + 4\epsilon(x + \epsilon - \rho)]} \right\} 
\]

(A.1)

where $x_{\pm}$ and $\Xi$ are given in Eq. (2.5) and

\[
\Phi(x, \rho, \epsilon) = \rho(1 - x)^2[4(1 - x)^2(8 - x) + 2\rho(1 - x)(17 - 9x) + 2\rho^2(5 - 4x) + \rho^3] 
+ 2\epsilon[8(1 - x)^2(2 + x^2) - 2\rho(1 - x)(4 - 7x - 12x^2 + 7x^3) 
- 2\rho^2(13 - 19x + x^2 + 4x^3) - \rho^3(9 - 7x - x^2) - \rho^4] 
+ 2\epsilon^2[8(1 - x)(4 - x - 2x^2) + 4\rho(1 + 5x - 5x^2) - 10\rho^2(1 - x) - \rho^3] 
+ 4\epsilon^3[4(1 - x)(4 - 3x) + 2\rho(5 - 4x) + \rho^2]. 
\]

(A.2)

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