Singular values of products of Ginibre random matrices, multiple orthogonal polynomials and hard edge scaling limits

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Abstract

Akemann, Ipsen and Kieburg recently showed that the squared singular values of products of $M$ rectangular random matrices with independent complex Gaussian entries are distributed according to a determinantal point process with a correlation kernel that can be expressed in terms of Meijer G-functions. We show that this point process can be interpreted as a multiple orthogonal polynomial ensemble. We give integral representations for the relevant multiple orthogonal polynomials and a new double contour integral for the correlation kernel, which allows us to find its scaling limits at the origin (hard edge). The limiting kernels generalize the classical Bessel kernels. For $M = 2$ they coincide with the scaling limits found by Bertola, Gekhtman, and Szmigielski in the Cauchy-Laguerre two-matrix model, which indicates that these kernels represent a new universality class in random matrix theory.

1 Introduction

1.1 Products of Ginibre random matrices

Random matrix theory is a broad field with many applications in mathematics, physics, and beyond, as is witnessed by the survey volume [1] and the
recent monographs [7, 20, 22, 34]. Of particular importance for the development of the theory is the connection with determinantal point processes. Whenever the eigenvalues of a random matrix ensemble are a determinantal point process, one has explicit expressions for the eigenvalue distributions in terms of the correlation kernel. Tools from integrable systems may then be used to further analyze the correlation kernel in the large $n$ limit, in order to establish, for example, universality of local eigenvalue correlations. It is a recent discovery that products of random matrices can fall in the framework of determinantal point processes.

The topic of products of random matrices can be traced back to the work of Furstenberg and Kesten [23], where the interest lies in the asymptotic behavior as the number of factors in the product tends to infinity. This work has been highly influential with important applications in Schrödinger operator theory [14] and in statistical physics relating to disordered and chaotic dynamical systems [18].

A more recent development is the study of eigenvalue and singular value distributions for the products of random matrices with a fixed number of factors, but allowing the size of the matrices to tend to infinity. With tools from free probability and diagrammatic expansions, one may find the limiting global eigenvalue distributions as in [8, 15, 16, 33]. It turns out that, as in the theory of a single random matrix, the various limits exhibit a rich and interesting mathematical structure, which also show a large degree of universality, see e.g. [21, 32]. Apart from physical applications, the study is also motivated by other fields like MIMO (multiple-input and multiple-output) networks in telecommunication [36].

Akemann and Burda [2] proved that the eigenvalues of products of complex Ginibre matrices are determinantal in the complex plane, see [25] for an extension to quaternionic Ginibre matrices. A similar determinantal structure holds for the eigenvalues of products of truncated unitary matrices [3]. The determinantal structure opens up the way to a more detailed analysis at the finite $n$ level [3, 6]. Very recently, Akemann, Kieburg, and Wei [5] found that the squared singular values of products of complex Ginibre matrices are a determinantal point process on the positive real line. This was further extended to the case of products of rectangular Ginibre matrices by Akemann, Ipsen and Kieburg [4]. The correlation kernels in [2, 3, 4, 5, 25] are all expressed in terms of Meijer G-functions.

In this paper we follow [4]. We take $M \geq 1$ and let $X_1, X_2, \ldots, X_M$ be complex random matrices whose entries are independent with a complex Gaussian distribution, also known as Ginibre random matrices. We assume
\(X_j\) has size \(N_j \times N_{j-1}\) and form the product

\[
Y_M = X_M X_{M-1} \cdots X_1. \tag{1.1}
\]

Our interest lies in the squared singular values of \(Y_M\), that is the eigenvalues of \(Y_M^* Y_M\), where the superscript * stands for conjugate transpose. We assume \(N_0 = \min\{N_0, \ldots, N_M\}\), and write

\[
\nu_j = N_j - N_0, \quad j = 0, \ldots, M, \quad n = N_0. \tag{1.2}
\]

Thus \(\nu_0 = 0\) and \(Y_M^* Y_M\) is a square matrix of size \(n\).

The case for the products of square matrices (i.e., \(\nu_j = 0\) for every \(j\)) was considered by Akemann, Kieburg and Wei [5], who showed that the squared singular values are distributed according to a determinantal point process with a correlation kernel that can be expressed in terms of Meijer G-functions. This was extended by Akemann, Ipsen and Kieburg [4] to the general rectangular case. The determinantal point process is a biorthogonal ensemble [13] with joint probability density function (see [4, formula (18)])

\[
P(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{j<k} (x_k - x_j) \det [w_{k-1}(x_j)]_{j,k=1,\ldots,n}, \tag{1.3}
\]

where \(x_j > 0, \ j = 1, \ldots, n\), are the squared singular values of \(Y_M\),

\[
w_k(x) = G^{M,0}_{0,M}(x; \nu_M, \nu_{M-1}, \ldots, \nu_2, \nu_1 + k | x), \tag{1.4}
\]

and normalization constant (see [4, formula (21)])

\[
Z_n = n! \prod_{i=1}^{n} \prod_{j=0}^{M} \Gamma(i + \nu_j).
\]

The function \(w_k\) is a Meijer G-function (see e.g. [9, 30] and the Appendix for a brief introduction) which can be written as a Mellin-Barnes integral

\[
w_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s + \nu_1 + k) \prod_{j=2}^{M} \Gamma(s + \nu_j) x^{-s}}{\prod_{j=2}^{M} \Gamma(s + \nu_j)} ds, \quad k = 0, 1, \ldots, \tag{1.5}
\]

with \(c > 0\). By the inversion formula for the Mellin transform we have

\[
\int_0^{\infty} w_k(x) x^{s-1} dx = \Gamma(s + \nu_1 + k) \prod_{j=2}^{M} \Gamma(s + \nu_j), \quad s > 0, \tag{1.6}
\]
which in particular shows that the moments of $w_k$ are given as products of Gamma functions.

By (1.5) and the functional equation of the Gamma function $\Gamma(z+1) = z\Gamma(z)$, we have

$$w_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s + \nu_1)_k \prod_{j=1}^{M} \Gamma(s + \nu_j)x^{-s} ds,$$

where the Pochhammer symbol

$$(s + \nu_1)_k = \frac{\Gamma(s + \nu_1 + k)}{\Gamma(s + \nu_1)} = (s + \nu_1)(s + \nu_1 + 1) \cdots (s + \nu_1 + k - 1)$$

is a polynomial of degree $k$ in the variable $s$. Then by taking linear combinations of the weights we could alternatively take

$$\tilde{w}_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^k \prod_{j=1}^{M} \Gamma(s + \nu_j)x^{-s} ds,$$  \hspace{1cm} (1.7)

in the definition of (1.3). This representation shows that (1.3) is fully symmetric in all parameters $\nu_1, \ldots, \nu_M$. Note that

$$\tilde{w}_k(x) = \left(-x\frac{d}{dx}\right)^k w_0(x),$$

which can be easily obtained from (1.5).

### 1.2 Biorthogonal functions and the correlation kernel

From general properties of biorthogonal ensembles [13], it is known that (1.3) is a determinantal point process with correlation kernel

$$K_n(x, y) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x^j (M_n^{-1})_{k,j} w_k(y),$$  \hspace{1cm} (1.8)

where $M_n$ is the matrix of moments of size $n \times n$,

$$M_n = \left(\int_{0}^{\infty} x^j w_k(x) dx\right)_{j,k=0,\ldots,n-1}.$$  \hspace{1cm} (1.9)

In addition we have

$$K_n(x, y) = \sum_{k=0}^{n-1} P_k(x)Q_k(y),$$  \hspace{1cm} (1.10)
where for each \( k = 0, 1, \ldots \), \( P_k \) is a monic polynomial of degree \( k \) and \( Q_k \) belongs to the linear span of \( w_0, \ldots, w_k \) in such a way that
\[
\int_0^\infty P_j(x)Q_k(x) \, dx = \delta_{j,k}. \tag{1.11}
\]
Thus the \( P_k \) and \( Q_k \) are biorthogonal functions that we consider for every non-negative integer \( k \), not just for \( k \leq n - 1 \).

Akemann et al. [4, 5] studied an extension of (1.3) to a two-matrix model and obtained in this framework that for certain polynomials \( \tilde{Q}_k \),
\[
\int_0^\infty \int_0^\infty P_j(x)\tilde{Q}_k(y)w_\nu^M(x, y) \, dx \, dy = h_j^M \delta_{j,k}, \tag{1.12}
\]
with
\[
w_\nu^M(x, y) = y^{\nu_1-1}e^{-y}G_{0,M-1}^{M-1,0}\left( -\frac{x}{y} \right)
\]
and
\[h_j^M = \prod_{m=0}^M (j + \nu_m)!.\]
see [4] formulas (25), (27) and (37)]. We emphasize that \( \tilde{Q}_k \neq Q_k \), since indeed \( Q_k \) is not a polynomial and \( \tilde{Q}_k \) is a multiple of the Laguerre polynomial \( L_k^{(\nu)} \); see [4] formula (42)]. The biorthogonality (1.12) is related to (1.11), since
\[Q_k(x) = \frac{1}{h_k^M} \int_0^\infty \tilde{Q}_k(y)w_\nu^M(x, y) \, dy,
\]
but we will not use this fact.

The starting point of this paper is the biorthogonality (1.11) and we first show that the polynomials \( P_k \) can be characterized as multiple orthogonal polynomials with respect to the first \( M \) weight functions \( w_0, \ldots, w_{M-1} \). Hence, the point process (1.3) is a multiple orthogonal polynomial (MOP) ensemble in the sense of [28, 29]. This further implies a representation of the correlation kernel \( K_n \tag{1.10} \) in terms of the associated Riemann-Hilbert problem, which is helpful for future asymptotic analysis.

In [4] it is shown that the biorthogonal functions \( P_k \) and \( Q_k \) have integral representations as Meijer G-functions. We rederive these results in Section 3 using only the biorthogonality (1.11). The recurrence relations of the biorthogonal functions are explicitly given in Section 4. We turn to the study of the function \( K_n \) in Section 5. We derive a double contour integral representation of \( K_n \), which allows us to find its scaling limit at the origin.
The limiting kernels generalize the classical Bessel kernel, and if \( M = 2 \), it coincides with the limiting kernels in the Cauchy-Laguerre two-matrix model recently studied by Bertola, Gekhtman and Szmigielski in [12]. Universality suggests that the new limiting kernels should apply to more general situations for the products of independent complex random matrices, thus, representing a new universality class. Finally, we present the integrable form of the limiting kernels in the sense of Its-Izergin-Korepin-Slavnov [27]. For convenience of the reader, we include a short introduction to the Meijer G-function in the Appendix.

Remark 1.1. It is possible to consider the probability density function \( (1.3) \) for general parameters \( \nu_1, \ldots, \nu_M > -1 \). The condition \( \nu_j > -1 \) is needed in order to guarantee the existence of the moments in \( (1.9) \). All the constructions in this paper go through in that more general case.

However, we do not have a proof that \( (1.3) \) is a probability density function in the case of non-integer parameters, in particular we do not know that \( (1.3) \) is non-negative for all \( x_1, \ldots, x_n \), although we strongly suspect that it will be the case.

2 Multiple orthogonal polynomial ensemble

2.1 Multiple orthogonality

Our first result is that the point process \( (1.3) \) is a MOP ensemble [28, 29] with \( M \) weight functions \( w_0, \ldots, w_{M-1} \), where the \( w_k \) are defined in \( (1.4) \). This follows from the following lemma.

Lemma 2.1. The linear span of the functions \( w_0, w_1, \ldots, w_{n-1} \) is equal to the linear span of the functions

\[
x \mapsto x^j w_k(x), \quad k = 0, \ldots, M - 1, \quad k + jM < n.
\]

(2.1)

Proof. The linear span of \( w_0, w_1, \ldots, w_{n-1} \) consists of all functions that can be written as

\[
x \mapsto \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q(s) \prod_{j=1}^{M} \Gamma(s + \nu_j) x^{-s} ds, \quad \deg q(s) \leq n - 1.
\]

(2.2)
We have by (1.5) and a change of variables $s \mapsto s + j$,

$$x^j w_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s + \nu_1)_k \prod_{l=1}^{M} \Gamma(s + \nu_l) x^{s-j} \, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s + \nu_1 + j)_k \prod_{l=1}^{M} (s + \nu_l)_j \prod_{l=1}^{M} \Gamma(s + \nu_l) x^{-s} \, ds.$$ 

This is of the form (2.2) with polynomial

$$q(s) = (s + \nu_1 + j)_k \prod_{l=1}^{M} (s + \nu_l)_j$$

of degree $k + jM$. Thus the functions (2.1) belong to the linear span of $w_0, \ldots, w_{n-1}$. It is readily seen that these are independent since they correspond to polynomials $q(s)$ that have different degrees.

The polynomials $P_k$ are therefore MOPs of type II with respect to the weights $w_0, \ldots, w_{M-1}$ and diagonal multiple indices, i.e.,

$$\int_0^{\infty} P_n(x) x^j w_k(x) \, dx = 0, \quad j = 0, \ldots, \left\lfloor \frac{n-k}{M} \right\rfloor - 1, \quad k = 0, \ldots, M-1,$$

where $\left\lfloor x \right\rfloor$ denotes the smallest integer $\geq x$; see [26, 37].

### 2.2 Riemann-Hilbert problem

As a consequence of Lemma 2.1, the polynomial $P_n$ is characterized by the following Riemann-Hilbert problem. We look for a $(M+1) \times (M+1)$ matrix-valued function $Y : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}^{(M+1) \times (M+1)}$ that is analytic with jump condition

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_0(x) & \cdots & w_{M-1}(x) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad x \in (0, \infty), \quad (2.3)$$

where $Y_+ (Y_-)$ denotes the limiting value from the upper (lower) half-plane. As $z \to \infty$, we require

$$Y(z) = (I + O(1/z)) \text{diag} \left( z^n \ z^{-n_0} \ \cdots \ z^{-n_{M-1}} \right), \quad (2.4)$$
where \( n_k = \lceil \frac{n-k}{M} \rceil \). Combined with appropriate local conditions near the origin that depend on the parameters \( \nu_1, \nu_2, \ldots, \nu_M \), the Riemann-Hilbert problem (2.3)–(2.4) has a unique solution and the (1,1) entry of \( Y \) is \( P_n \); see [37]. Also, one has

\[
K_n(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & w_0(y) & \cdots & w_{M-1}(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (2.5)
\]

which is a manifestation of the Christoffel-Darboux formula for multiple orthogonal polynomials; see [19]. The representation (2.5) is potentially useful for asymptotic analysis although we will not pursue this here.

### 2.3 Special case \( M = 2 \)

We now take a look at the case \( M = 2 \). If \( M = 2 \), then

\[
w_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s + \nu_1) \Gamma(s + \nu_2) x^{-s} ds.
\]

This can be expressed in terms of the modified Bessel function of second kind (a.k.a. the Macdonald function). The formula 10.32.13 of [31] says that

\[
2K_\nu(2\sqrt{x}) = \frac{x^{\nu/2}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \Gamma(s - \nu) x^{-s} ds, \quad c > \max(\nu, 0),
\]

which after a change of variables \( s \mapsto s + \nu + \alpha \) leads to

\[
2K_\nu(2\sqrt{x}) = \frac{x^{-\nu/2-\alpha}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s+\nu+\alpha) \Gamma(s+\alpha) x^{-s} ds, \quad c > \max(-\nu-\alpha, -\alpha).
\]

We take \( \alpha = \nu_2, \nu = \nu_1 - \nu_2 \), to find that

\[
w_0(x) = 2x^{(\nu_1+\nu_2)/2} K_{\nu_1-\nu_2}(2\sqrt{x}). \quad (2.6)
\]

It will be convenient to assume that \( \nu_1 \geq \nu_2 \), which we can do without loss of generality.
Similarly, 

\[ w_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s + \nu_1 + 1)\Gamma(s + \nu_2) x^{-s} \, ds \]
\[ = 2x^{(\nu_1 + \nu_2 + 1)/2} K_{\nu_1 + 1 - \nu_2} (2\sqrt{x}). \]  

(2.7)

Thus if \( \rho_\nu(x) = 2x^{\nu/2} K_\nu (2\sqrt{x}) \), we have

\[ w_0(x) = x^\alpha \rho_\nu(x), \quad w_1(x) = x^\alpha \rho_{\nu+1}(x). \]

Multiple orthogonal polynomials associated with the two weights (2.6)–(2.7) were considered by Van Assche and Yakubovich [38] for which they obtained four term recurrence relations; see also [17] and [40] for asymptotic results for these polynomials. In the random matrix context (i.e., the case where \( \nu_j = N_j - N_0 \) are integers), we have

\[ \nu = N_1 - N_2, \quad \alpha = N_2 - N_0. \]

For the special case \( \nu = \alpha = 0 \) (i.e., the products of two square matrices), this relation was first observed in [39].

For general \( M \), there is an \( M + 2 \) term recurrence relation (this follows from general theory of MOP, cf. [26, Section 23.1.4]) and we will determine the recurrence coefficients explicitly in Section 4.

3 Integral representations

Integral representations for the biorthogonal polynomials \( P_k \) and their dual functions \( Q_k \) are given in [4] where they were derived from a two matrix model. We rederive these results directly from the biorthogonality (1.11).

3.1 Integral representation for \( Q_k \)

Recall the biorthogonality (1.11). The biorthogonal function \( Q_k \) has the form

\[ Q_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q_k(s) \prod_{j=1}^{M} \Gamma(s + \nu_j) x^{-s} \, ds, \]

where \( q_k \) is a polynomial of degree \( k \). The biorthogonality (1.11) then says that

\[ \frac{1}{2\pi i} \int_0^\infty \int_{c-i\infty}^{c+i\infty} P_l(x)q_k(s) \prod_{j=1}^{M} \Gamma(s + \nu_j) x^{-s} \, ds \, dx = \delta_{l,k}. \]
It turns out that we can write down $q_k$ explicitly as stated in the following proposition.

**Proposition 3.1.** We have

$$Q_k(x) = \frac{(-1)^k}{\prod_{j=0}^M \Gamma(k + 1 + \nu_j)} \left( \frac{d}{dx} \right)^k \left( x^k w_0(x) \right), \quad (3.1)$$

and

$$q_k(s) = \frac{(s - k)_k}{\prod_{j=0}^M \Gamma(k + 1 + \nu_j)}. \quad (3.2)$$

**Proof.** It is easy to see after applying an integration by parts $k$ times that

$$\int_0^\infty x^l \left( \frac{d}{dx} \right)^k (x^k w_0(x)) \, dx = 0, \quad \text{for } l < k.$$ 

Note that integrated terms do not contribute, since

$$w_0(x) = O(x^\alpha \log^r x), \quad \text{as } x \to 0^+,$$ 

with $\alpha = \min(\nu_1, \ldots, \nu_M) > -1$ and $r = \# \{ j \mid \nu_j = \alpha \}$, which can be deduced from properties of the Mellin transform [1.6]; see e.g. [21, Theorem 4], and since for $x \to +\infty$, we have

$$w_0(x) = O(x^\theta e^{-Mx^{1/M}}), \quad \theta = \frac{1}{M} \left( \frac{1}{2}(1 - M) + \sum_{j=1}^M \nu_j \right);$$

see [30, Theorem 5.7.5].

Similarly,

$$\int_0^\infty x^k \left( \frac{d}{dx} \right)^k (x^k w_0(x)) \, dx = (-1)^k k! \int_0^\infty x^k w_0(x) \, dx$$

$$= (-1)^k \prod_{j=0}^M \Gamma(k + 1 + \nu_j),$$

where we recall [1.6] and the fact that $\nu_0 = 0$. Thus if $Q_k$ is defined by (3.1), then we have

$$\int_0^\infty x^l Q_k(x) \, dx = \delta_{l,k}, \quad \text{for } l = 0, 1, \ldots, k. \quad (3.4)$$
Since
\[ x^k w_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \prod_{j=0}^{M} \Gamma(s + \nu_j) x^{k-s} \, ds, \]
we find by taking k derivatives that
\[ \left( \frac{d}{dx} \right)^k x^k w_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (-1)^k (s - k) \prod_{j=0}^{M} \Gamma(s + \nu_j) x^{-s} \, ds. \]
Thus
\[ Q_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q_k(s) \prod_{j=0}^{M} \Gamma(s + \nu_j) x^{-s} \, ds, \quad (3.5) \]
with \( q_k \) as in (3.2). This proves that \( Q_k \) belongs to the linear span of \( w_0, \ldots, w_{k-1} \) and (3.4) shows that it is indeed the biorthogonal function. \( \square \)

Note that (3.1) is a Rodrigues-type formula for \( Q_k \). Note also that (3.5) is an integral representation, which because of (3.2) we may also write as
\[ Q_k(x) = \frac{1}{2\pi i \prod_{j=0}^{M} \Gamma(k + \nu_j + 1)} \int_{c-i\infty}^{c+i\infty} \prod_{j=0}^{M} \Gamma(s + \nu_j) \Gamma(s - k) x^{-s} \, ds. \quad (3.6) \]
By (A.1), we can identify (3.6) as a Meijer G-function:
\[ Q_k(x) = \frac{1}{\prod_{j=0}^{M} \Gamma(k + \nu_j + 1)} G_{M+1}^{1,0} \left( \begin{array}{c} -k \\ \nu_0, \nu_1, \ldots, \nu_M \end{array} \middle| x \right). \quad (3.7) \]
Up to a multiplicative constant and an easy transformation of the Meijer G-function, (3.7) is the same as [4, formula (49)].

3.2 Integral representation for \( P_n \)
There is a similar integral representation for \( P_n \).

Proposition 3.2. We have for \( x > 0 \),
\[ P_n(x) = \frac{\prod_{j=0}^{M} \Gamma(n + \nu_j + 1)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(t-n)}{\prod_{j=0}^{M} \Gamma(t + \nu_j + 1)} x^t \, dt, \quad (3.8) \]
where \( \Sigma \) is a closed contour that encircles 0,1,\ldots,n once in the positive direction.
Proof. In the proof we assume that $P_n$ is given by (3.8) and we show that $P_n$ is a monic polynomial of degree $n$ satisfying

$$\int_0^{\infty} P_n(x) \tilde{w}_k(x) \, dx = 0, \quad k = 0, \ldots, n - 1,$$

where $\tilde{w}_k$ is defined in (1.7).

The integrand in the right-hand side of (3.8) is meromorphic on $\mathbb{C}$ with simple poles at $0, 1, \ldots, n$ (the poles of the numerator at the negative integers are cancelled by the poles of the factor $\Gamma(t + 1)$ in the denominator). Thus by the residue theorem

$$P_n(x) = \prod_{j=0}^{M} \Gamma(n + \nu_j + 1) \sum_{l=0}^{n} \text{Res}_{t=\lambda} \left( \frac{\Gamma(t - n)}{\prod_{j=0}^{M} \Gamma(t + \nu_j + 1)} \right) x^l.$$

We can evaluate the residues to obtain

$$P_n(x) = \sum_{l=0}^{n} \frac{(-1)^{n-l} \prod_{j=0}^{M} \Gamma(n + \nu_j + 1)}{(n-l)! \prod_{j=0}^{M} \Gamma(l + \nu_j + 1)} x^l,$$

which shows that $P_n$ is a monic polynomial of degree $n$.

To verify (3.9) we use

$$\int_0^{\infty} x^t \tilde{w}_k(x) \, dx = (t+1)^k \prod_{j=1}^{M} \Gamma(t + \nu_j + 1),$$

which follows from (1.7) and the inversion formula for Mellin transforms. Then we can compute by (3.8) and an interchange of integrals,

$$\int_0^{\infty} P_n(x) \tilde{w}_k(x) \, dx$$

$$= \frac{\prod_{j=0}^{M} \Gamma(n + \nu_j + 1)}{2\pi i} \oint_{\Sigma} \frac{\Gamma(t-n)}{\prod_{j=0}^{M} \Gamma(t + \nu_j + 1)} (t+1)^k \prod_{j=1}^{M} \Gamma(t + \nu_j + 1) \, dt$$

$$= \frac{\prod_{j=0}^{M} \Gamma(n + \nu_j + 1)}{2\pi i} \oint_{\Sigma} \frac{\Gamma(t-n)(t+1)^k}{\Gamma(t+1)} \, dt$$

$$= \frac{\prod_{j=0}^{M} \Gamma(n + \nu_j + 1)}{2\pi i} \oint_{\Sigma} \frac{(t+1)^k}{t(t-1) \cdots (t-n)} \, dt.$$

The remaining integrand is a rational function that behaves like $O(t^{k-n-1})$ as $t \to \infty$. The contour $\Sigma$ encircles all the poles once in the positive direction. Thus by moving the contour to infinity, we find that the integral vanishes for $k \leq n - 1$, which is the required biorthogonality (3.9).
The formula (3.10) shows that $P_n$ is a hypergeometric polynomial

$$P_n(x) = (-1)^n \prod_{j=1}^{M} \frac{\Gamma(n + \nu_j + 1)}{\Gamma(\nu_j + 1)} {\binom{-n}{1 + \nu_1, \ldots, 1 + \nu_M}}_1 F_M \left(1 + \nu_1, \ldots, 1 + \nu_M \left| x \right. \right),$$

as in [4, formula (44)]. We can also identify $P_n$ in (3.8) as a Meijer G-function:

$$P_n(x) = - \prod_{j=0}^{M} \Gamma(n + \nu_j + 1) G_{0,1}^{1,1} \left(\begin{array}{c}n + 1 \\ -\nu_0, -\nu_1, \ldots, -\nu_{M-1}, -\nu_M \end{array} \left| x \right. \right),$$

(3.11)

which is equivalent to [4, formula (45)].

### 4 Recurrence relations

By Lemma 2.1 and general theory of MOPs (cf. [26, Chapter 23]), it follows that the polynomials $P_n$ satisfy an $M + 2$ term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \sum_{k=0}^{M} a_{k,n} P_{n-k}(x).$$

(4.1)

There is a dual recurrence relation

$$xQ_n(x) = Q_{n-1}(x) + \sum_{k=0}^{M} b_{k,n} Q_{n+k}(x),$$

(4.2)

where because of the biorthogonality (1.11),

$$a_{k,n} = \int_0^\infty xP_n(x) Q_{n-k}(x) \, dx, \quad b_{k,n} = \int_0^\infty P_{n+k}(x) xQ_n(x) \, dx.$$

Therefore

$$a_{k,n} = b_{k,n-k}.$$  \hspace{2cm} (4.3)

It is the aim of this section to calculate these recurrence coefficients explicitly.
4.1 Coefficients $b_{k,n}$

**Proposition 4.1.** We have for $k = 0, \ldots, M$,

$$b_{k,n} = \left( \prod_{j=0}^{M} (n + \nu_j + 1) \right)^{k+1} \sum_{j=0}^{\lfloor (k+1) - j \rfloor} \frac{(-1)^{k+1-j}}{j!} \prod_{i=0}^{M} (n + j + \nu_i). \quad (4.4)$$

**Proof.** We have from (3.5), after a change of variable $s \mapsto s+1$,

$$xQ_n(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} q_n(s) \left( \prod_{j=1}^{M} \Gamma(s + \nu_j) \right) x^{-s+1} ds$$

$$= \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} q_n(s+1) \left( \prod_{j=1}^{M} \Gamma(s + \nu_j + 1) \right) x^{-s} ds$$

$$= \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} q_n(s+1) \left( \prod_{j=1}^{M} (s + \nu_j) \prod_{j=1}^{M} \Gamma(s + \nu_j) \right) x^{-s} ds.$$  

Then $q_n(s+1) \prod_{j=1}^{M} (s + \nu_j)$ is a polynomial in $s$ of degree $n + M$ and it is our task to show that

$$q_n(s+1) \prod_{j=1}^{M} (s + \nu_j) = q_{n-1}(s) + \sum_{k=0}^{M} b_{k,n} q_{n+k}(s) \quad (4.5)$$

with $b_{k,n}$ given by (4.4).

By (3.2) we have that all terms in (4.5) are zero for $s = 1, \ldots, n-1$, i.e., all terms are divisible by $q_{n-1}(s)$. If we do this division and use (3.2) then we find that we have to prove

$$\prod_{j=0}^{M} \frac{s + \nu_j}{n + \nu_j} = 1 + \sum_{k=0}^{M} \frac{b_{k,n}}{\prod_{j=0}^{M} (n + \nu_j) + 1} (s - n - k)_{k+1}.$$  

Write $s = t + n$. Then we have to prove

$$f(t) = \prod_{j=0}^{M} (n + \nu_j) + \sum_{k=0}^{M} \frac{b_{k,n}}{\prod_{j=0}^{M} (n + \nu_j + 1)_{k+1}} (t - k)_{k+1} \quad (4.6)$$

as an identity for polynomials in $t$, where

$$f(t) = \prod_{j=0}^{M} (t + n + \nu_j). \quad (4.7)$$

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Both sides of (4.6) have degree $M + 1$ and for $t = 0$ the identity (4.6) is valid. The polynomials $t \mapsto (t - k)_{k+1}$ for $k = 0, \ldots, m$ are a basis for the vector space of polynomials of degree $\leq M + 1$ that vanish at $t = 0$. Then it is clear that there exists coefficients $b_{k,n}$ such that (4.6) holds.

By contour integration we obtain from (4.6)

$$
\frac{b_{k,n}}{\prod_{j=0}^{M} (n + \nu_j + 1)_k} = \frac{1}{2\pi i} \oint_{\Sigma} \frac{f(t)}{(t - k - 1)_{k+2}} dt, \quad k = 0, \ldots, M,
$$

(4.8)

where $\Sigma$ is a closed contour that encircles the points $0, \ldots, k$ once in the positive direction. This leads by the residue theorem to

$$
b_{k,n} = \left( \prod_{j=0}^{M} (n + \nu_j + 1)_k \right) \sum_{j=0}^{k+1} (-1)^{k+1-j} \frac{f(j)}{j!(k+1-j)!},
$$

which gives (4.4) in view of the definition (4.7) of $f(t)$.

4.2 Coefficients $a_{k,n}$

Because of (4.3) we immediately find an expression for the recurrence coefficients $a_{k,n}$.

**Corollary 4.2.** We have for $k = 0, \ldots, M$,

$$
a_{k,n} = \left( \prod_{j=0}^{M} (n - k + \nu_j + 1)_k \right) \sum_{j=0}^{k+1} (-1)^{k+1-j} \frac{\prod_{i=0}^{M} (n - k + j + \nu_i)}{j!(k+1-j)!}. \quad (4.9)
$$

Reversing the order of summation we also have

$$
a_{k,n} = \left( \prod_{j=0}^{M} (n - k + \nu_j + 1)_k \right) \sum_{j=0}^{k+1} (-1)^{j} \frac{\prod_{i=0}^{M} (n + 1 - j + \nu_i)}{j!(k+1-j)!}.
$$

**Proof.** Use (4.3), (4.4) and reverse the order of summation.

From (4.9) we see that $a_{k,n}$ is a polynomial expression in $n$, which seems to be of degree $k(M + 1) + M + 1 = (k + 1)(M + 1)$. However there is a cancellation in the leading order terms and $a_{k,n}$ is actually a polynomial in $n$ of degree $(k + 1)M$.

**Lemma 4.3.** For every $k$ we have

$$
a_{k,n} = \binom{M + 1}{k + 1} n^{(k+1)M} + O\left(n^{(k+1)M - 1}\right).
$$
Proof. From (4.3) and the contour integral representation (4.8) for \( b_{k,n} \) we find

\[
a_{k,n} = \left( \prod_{j=0}^{M} (n - k + \nu_j + 1) \right) \frac{1}{2\pi i} \oint_{\Sigma} \frac{g_n(t)}{(t - k - 1)_{k+2}} \, dt,
\]

where

\[
g_n(t) = \prod_{j=0}^{M} (t + n - k + \nu_j) = \sum_{l=0}^{M+1} p_l(t) n^l
\]

is a polynomial of degree \( M + 1 \) in \( n \). The coefficient \( p_l(t) \) is a polynomial in \( t \) of degree \( \text{deg} p_l(t) = M + 1 - l \). Thus

\[
\frac{1}{2\pi i} \oint_{\Sigma} \frac{g_n(t)}{(t - k - 1)_{k+2}} \, dt = \sum_{l=0}^{M+1} \left( \frac{1}{2\pi i} \oint_{\Sigma} \frac{p_l(t)}{(t - k - 1)_{k+2}} \, dt \right) n^l.
\]

The integral vanishes if \( p_l \) is a polynomial of degree \( \leq k \) since in that case the integrand is \( O(t^{-2}) \), and we can move the contour to infinity. This happens for \( l \geq M - k + 1 \). For \( l = M - k \), we have

\[
p_{M-k}(t) = \binom{M+1}{k+1} t^{k+1} + O(t^k), \quad \text{as } t \to \infty,
\]

and by a residue calculation at infinity we obtain

\[
\frac{1}{2\pi i} \oint_{\Sigma} \frac{p_{M-k}(t)}{(t - k - 1)_{k+2}} \, dt = \binom{M+1}{k+1}.
\]

Thus the second factor in the right-hand side of (4.10) is a polynomial of degree \( M - k \) in \( n \) with leading coefficient \( \binom{M+1}{k+1} \).

The other factor is a monic polynomial in \( n \) of degree \( k(M + 1) \). Thus \( a_{k,n} \) has degree \( k(M + 1) + M - k = (k + 1)M \) with leading coefficient \( \binom{M+1}{k+1} \), as claimed in the lemma.

Let’s finally write down (4.10) for small values of \( M \).

Case \( M = 1 \)  For \( M = 1 \) we have a three term recurrence

\[
xP_n(x) = P_{n+1}(x) + a_{0,n} P_n(x) + a_{1,n} P_{n-1}(x)
\]

with

\[
a_{0,n} = 2n + \nu_1 + 1, \quad a_{1,n} = n(n + \nu_1).
\]

This is the recurrence relation for monic Laguerre polynomials with parameter \( \nu_1 \).
Case $M = 2$  For $M = 2$ we have a four term recurrence

$$xP_n(x) = P_{n+1}(x) + a_{0,n}P_n(x) + a_{1,n}P_{n-1}(x) + a_{2,n}P_{n-2}(x)$$

with

$$a_{0,n} = 3n^2 + (3 + 2\nu_1 + 2\nu_2)n + (1 + \nu_1 + \nu_2 + \nu_1\nu_2),$$
$$a_{1,n} = n(n + \nu_1)(n + \nu_2)(3n + \nu_1 + \nu_2),$$
$$a_{2,n} = n(n-1)(n + \nu_1)(n + \nu_1 - 1)(n + \nu_2)(n + \nu_2 - 1).$$

This agrees with the recurrence coefficients given in [38, Theorem 4] if we use $\alpha = \nu_2, \nu = \nu_1 - \nu_2$.

5  Double integral representation and large $n$ limit of $K_n$

In this section, we are concerned with the correlation kernel $K_n(x, y)$ defined in (1.10).

5.1 Double integral formula for $K_n$

The correlation kernel admits a double contour integral representation.

**Proposition 5.1.** We have

$$K_n(x, y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} ds \oint_{\Sigma} dt \prod_{j=0}^{M} \frac{\Gamma(s + \nu_j + 1) \Gamma(t - n + 1)}{\Gamma(t + \nu_j + 1) \Gamma(s - n + 1)} x^t y^{-s-1},$$

where $\Sigma$ is a closed contour going around $0, 1, \ldots, n$ in the positive direction and $\text{Re } t > -1/2$ for $t \in \Sigma$.

**Proof.** The correlation kernel (1.10) can be written as a double integral

$$K_n(x, y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} ds \oint_{\Sigma} dt \prod_{j=0}^{M} \frac{\Gamma(s + \nu_j)}{\Gamma(t + \nu_j + 1)} \sum_{k=0}^{n-1} \frac{\Gamma(t - k)}{\Gamma(s - k)} x^t y^{-s},$$

where we used the integral representation (3.3) for $P_k$ and (3.6) for $Q_k$. From the functional equation $\Gamma(z + 1) = z\Gamma(z)$, one can easily check that

$$\frac{(s - t - 1) \Gamma(t - k)}{\Gamma(s - k)} = \frac{\Gamma(t - k)}{\Gamma(s - k - 1)} - \frac{\Gamma(t - k + 1)}{\Gamma(s - k)}.$$
which means that there is a telescoping sum

\[(s - t - 1) \sum_{k=0}^{n-1} \frac{\Gamma(t - k)}{\Gamma(s - k)} = \frac{\Gamma(t - n + 1)}{\Gamma(s - n)} - \frac{\Gamma(t + 1)}{\Gamma(s)}. \tag{5.3}\]

We are going to make sure that \(s - t - 1 \neq 0\) when \(s \in c + i\mathbb{R}\) and \(t \in \Sigma\). We do this by taking \(c = 1/2\) and let \(\Sigma\) go around \(0, 1, \ldots, n\) but with \(\text{Re} \, t > -1/2\) for \(t \in \Sigma\). Then we insert (5.3) into (5.2) and get

\[
K_n(x, y) = \frac{1}{(2\pi i)^2} \int_{1/2 - i\infty}^{1/2 + i\infty} ds \int_{\Sigma} dt \prod_{j=0}^{M} \frac{\Gamma(s + \nu_j)}{\Gamma(t + \nu_j + 1)} \frac{\Gamma(t - n + 1)}{\Gamma(s - n)} \frac{x^t y^{-s}}{s - t - 1} \prod_{j=0}^{M} \frac{\Gamma(s + \nu_j)}{\Gamma(t + \nu_j + 1)} \frac{x^t y^{-s}}{s - t - 1}.
\]

The \(t\)-integral in the second double integral vanishes by Cauchy’s theorem, since the integrand does not have any singularities inside \(\Sigma\). We change \(s \mapsto s + 1\) in the first double integral and we obtain (5.1).

We can rewrite the kernel in terms of Meijer G-functions

Corollary 5.2. We have

\[
K_n(x, y) = \int_0^1 G_{1, M+1}^{0,1} \left( \begin{array}{c} n \\ -\nu_0, \ldots, -\nu_M \end{array} \right| ux \right) C_{M+1,0}^{M,1} \left( \begin{array}{c} -n \\ \nu_0, \ldots, \nu_M \end{array} \right| uy \right) du
\]

\[
= -\prod_{j=1}^{M} (n + \nu_j) \int_0^1 P_{n-1}(ux)Q_n(uy) du. \tag{5.4}
\]

Proof. Note that

\[
\frac{x^t y^{-s-1}}{s - t} = -\int_0^1 (ux)^t (uy)^{-s-1} du. \tag{5.5}
\]

The kernel (5.1) then is

\[
K_n(x, y) = -\int_0^1 \left( \frac{1}{2\pi i} \int_{\Sigma} \prod_{j=0}^{M} \frac{\Gamma(t - n + 1)}{\Gamma(t + \nu_j + 1)} (ux)^t \, dt \right) \times \left( \frac{1}{2\pi i} \int_{-1/2 - i\infty}^{-1/2 + i\infty} \prod_{j=0}^{M} \frac{\Gamma(s + \nu_j + 1)}{\Gamma(s - n + 1)} (uy)^{-s-1} \, ds \right) du. \tag{5.6}
\]

By the definition (A.1) and change of variables \(t \mapsto -t, s \mapsto s + 1\), both factors in the \(u\) integral can be identified as Meijer G-functions and the first identity in (5.4) follows.

The second identity in (5.4) follows from (3.7) and (3.11).
5.2 Microscopic limit of $K_n$ at the hard edge

With the help of the contour integral representation for $K_n$, we derive its scaling limit near the origin (hard edge). The limiting kernels are denoted by $K^{M}_\nu$, where $\nu$ stands for the collection of parameters $\nu_1, \ldots, \nu_M$.

**Theorem 5.3.** With $\nu_1, \ldots, \nu_M$ being fixed, we have

$$
\lim_{n \to \infty} \frac{1}{n} K_n \left( \frac{x}{n}, \frac{y}{n} \right) = K^{M}_\nu (x, y),
$$

uniformly for $x, y$ in compact subsets of the positive real axis, where

$$
K^{M}_\nu (x, y) = \frac{1}{(2\pi i)^2} \int_{-1/2-i\infty}^{-1/2+i\infty} ds \int_{\Sigma} dt \prod_{j=0}^{M} \frac{\Gamma(s+\nu_j+1) \sin \pi s x^j y^{-s-1}}{\Gamma(t+\nu_j+1) \sin \pi t s-t}.
$$

and where $\Sigma$ is a contour starting from $+\infty$ in the upper half plane and returning to $+\infty$ in the lower half plane which encircles the positive real axis and $\text{Re} \ t > -1/2$ for $t \in \Sigma$; see Figure 1 for an illustration.

**Proof.** The reflection formula of the Gamma function says that

$$
\Gamma(t) \Gamma(1-t) = \frac{\pi}{\sin \pi t},
$$

Figure 1: The two contours of the double integral in (5.7).
which means that
\[
\frac{\Gamma(t - n + 1)}{\Gamma(s - n + 1)} = \frac{\Gamma(n - s)}{\Gamma(n - t)} \frac{\sin \pi s}{\sin \pi t}.
\] (5.9)

As \( n \to \infty \), we have the following ratio asymptotics of Gamma functions (cf. [31, formula 5.11.13])
\[
\frac{\Gamma(n - s)}{\Gamma(n - t)} = n^{t-s} \left(1 + O(n^{-1})\right),
\] (5.10)
which can be easily verified using Stirling’s formula. By modifying the contour \( \Sigma \) in (5.1) from a closed contour around \( 0, 1, \ldots, n \) to a two sided unbounded contour as in Figure 1 and applying (5.9) and (5.10), we readily obtain the first identity in (5.7), provided that we can take the limit inside of the integral.

The \( t \)-integral in (5.7) converges since \( \Gamma(t + \nu_j + 1) \) increases if we go to infinity along \( \Sigma \) and
\[
|\sin \pi t| \geq |\sinh \pi \text{Im} t|.
\]
Also the \( s \) integral converges since
\[
|\Gamma(x + iy)| \sim \sqrt{2\pi} |y|^{x-(1/2)} e^{-\pi |y|/2},
\]
as \( y \to \pm\infty \) for bounded real value of \( x \); see [31, formula 5.11.9]. Therefore, \( \Gamma(s + \nu_j + 1) \) tends to 0 at an exponential rate if \( |s| \to \infty \) with \( \text{Re} \, s = -1/2 \).

We can then indeed justify the interchange of limit and integrals for every \( M \) by the dominated convergence theorem.

By (5.8), we see
\[
\frac{\sin \pi s}{\sin \pi t} = \frac{\Gamma(1 + t)\Gamma(-t)}{\Gamma(1 + s)\Gamma(-s)},
\]
and using the trick (5.5) as in the proof of Proposition 5.1, we obtain
\[
K^M_\nu(x, y) = -\int_0^1 \left( \frac{1}{2\pi i} \int_\Sigma \prod_{j=1}^{M} \frac{\Gamma(-t)}{\Gamma(t + \nu_j + 1)} (ux)^t \, dt \right)
\times \left( \frac{1}{2\pi i} \int_{-1/2 - i\infty}^{-1/2 + i\infty} \prod_{j=1}^{M} \frac{\Gamma(s + \nu_j + 1)}{\Gamma(-s)} (uy)^{-s-1} \, ds \right) \, du.
\]
The change of variables \( t \mapsto -t \) and \( s \mapsto s + 1 \) takes both integrals into the form (A.1) of a Meijer G-function, and the second identity in (5.7) follows. \( \square \)
It is known that the limiting mean distribution of the squared singular values for the products of $M$ Ginibre matrices blows up with a rate $x^{-M/(M+1)}$ near the origin (see [16,33]). Extending the notion of universality at the hard edge, we are led to the expectation that the kernels described in Theorem 5.3 should appear in more general situations of the products of independent complex random matrices, and possibly in other models of random matrix theory.

5.3 Special case $M = 1$

Let’s now take a closer look at the limiting kernels $K^{M}_{\nu}(x,y)$ for special values of $M$. If $M = 1$ and $\nu_1 = \nu$, one has since $\nu_0 = 0$ (we drop the superscript $M = 1$)

$$K_{\nu}(x,y) = \int_{0}^{1} G_{0,2}^{1,0} \left( \begin{array}{c} -u_x \\ 0, -\nu \end{array} \right) G_{0,2}^{1,0} \left( \begin{array}{c} -u_y \\ \nu, 0 \end{array} \right) du.$$

Since

$$G_{0,2}^{1,0} \left( \begin{array}{c} -u_x \\ 0, -\nu \end{array} \right) = (u_x)^{-\nu/2} J_\nu(2\sqrt{u_x}),$$

$$G_{0,2}^{1,0} \left( \begin{array}{c} -u_y \\ \nu, 0 \end{array} \right) = (u_y)^{\nu/2} J_\nu(2\sqrt{u_y}),$$

where $J_\nu$ denotes the Bessel function of the first kind of order $\nu$ (see [31 formula 10.9.23]), it then follows that

$$K_{\nu}(x,y) = \left( \frac{y}{x} \right)^{\nu/2} \int_{0}^{1} J_\nu(2\sqrt{ux})J_\nu(2\sqrt{uy}) du$$

$$= 4 \left( \frac{y}{x} \right)^{\nu/2} K^{\text{Bes},\nu}(4x,4y),$$

where

$$K^{\text{Bes},\nu}(x,y) = \frac{J_\nu(\sqrt{x})\sqrt{y} J'_\nu(\sqrt{y}) - \sqrt{x} J'_\nu(\sqrt{x}) J_\nu(\sqrt{y})}{2(x-y)}, \quad \nu > -1,$$

is the Bessel kernel of order $\nu$ that appears as the scaling limit of the Laguerre or Jacobi unitary ensembles at the hard edge [35], as expected.
5.4 Special case $M = 2$

If $M = 2$, one has from (5.7) that (we drop the superscript $M = 2$)

$$K_{\nu_1, \nu_2}(x, y) = \int_0^1 G_{0, 3}^{1, 0} \left( 0, -\nu_1, -\nu_2 \mid ux \right) G_{0, 3}^{2, 0} \left( \nu_1, \nu_2, 0 \mid uy \right) du.$$  \hfill (5.11)

It is interesting that these kernels appeared earlier in another random matrix model, namely in the Cauchy two-matrix model with linear potentials, see [10, 12].

The Cauchy two matrix model is defined by the probability measure

$$\frac{1}{Z_n} \det(M_1)^a \det(M_2)^b e^{-\text{Tr}(V_1(M_1) + V_2(M_2))} \frac{dM_1 dM_2}{\det(M_1 + M_2)^n}, \quad a, b > -1, a + b > -1,$$

defined on the space of two $n \times n$ positive semidefinite Hermitian matrices $M_1$ and $M_2$, with two scalar potentials $V_1, V_2$ defined on the positive real axis that grow sufficiently fast as $x \to +\infty$.

The eigenvalues of $M_1$ and $M_2$ form a determinantal point process with a correlation kernel which is defined in terms of the Cauchy biorthogonal polynomials [11] $p_l(x)$ and $q_m(y)$ satisfying

$$\int_{-\infty}^\infty \int_{-\infty}^\infty x^a y^b e^{-V_1(x) - V_2(y)} \frac{x+y}{p_l(x) q_m(y)} dx dy = \delta_{l,m}.$$  \hfill (A.2)

For the linear case $V_1(x) = x$ and $V_2(y) = y$, it was established in [12, Theorem 2.2] that the correlation kernel for the eigenvalues of $M_1$ has a scaling limit at the origin given by

$$\int_0^1 G_{0, 3}^{1, 0} \left( a, 0, -b \mid ux \right) G_{0, 3}^{2, 0} \left( b, 0, -a \mid uy \right) du. \quad \hfill (5.12)$$

This is slightly different from (5.11), since we cannot freely permute the parameters $\nu_1, \nu_2, 0$ in (5.11).

However, from (A.2) we see that

$$G_{0, 3}^{1, 0} \left( a, 0, -b \mid ux \right) = (ux)^a G_{0, 3}^{1, 0} \left( 0, -a, -b - a \mid ux \right),$$

$$G_{0, 3}^{2, 0} \left( b, 0, -a \mid uy \right) = (uy)^{-a} G_{0, 3}^{2, 0} \left( b + a, a, 0 \mid uy \right).$$

Hence,

$$\int_0^1 G_{0, 3}^{1, 0} \left( a, 0, -b \mid ux \right) G_{0, 3}^{2, 0} \left( b, 0, -a \mid uy \right) du = \left( \frac{x}{y} \right)^a K_{a+b,a}(x, y).$$
The prefactor \( \left( \frac{x}{y} \right)^a \) is irrelevant in a kernel for a determinantal point process as it does not change the determinants that give the point correlations. Therefore we see that the limiting kernels \((5.12)\) in the Cauchy two matrix models are the same kernels as the limiting kernels for squared singular values of products of two complex Ginibre matrices. This supports our conjecture that the kernels \((5.7)\) have a universal character and appear in a wider context.

### 5.5 Integrable form of the limiting kernels

An integral operator with kernel \( K(x, y) \) is called integrable if

\[
K(x, y) = \frac{\sum_{i=1}^{n} f_i(x)g_i(y)}{x - y}, \quad \text{with}\quad \sum_{i=1}^{n} f_i(x)g_i(x) = 0,
\]

for some \( n \in \{2, 3, \ldots\} \), and certain functions \( f_i \) and \( g_i \). Integral operators of this form benefit from the fact that there is a Riemann-Hilbert setting for the study of the associated resolvent kernels, determinants, etc.; see \cite{27}. The kernels of standard universality classes (sine, Airy, Bessel) encountered in random matrix theory all belong to the class of integrable operators. The representation \((2.5)\) of \( K_n \) in terms of the solution of a Riemann-Hilbert problem is also of the integrable form.

We conclude this paper by giving the integrable form of the limiting kernels derived in Theorem 5.3. Our argument follows \cite[Section 5]{12}, where this was shown for the case \( M = 2 \).

**Proposition 5.4.** With \( K^M_\nu (x, y) \) defined in \((5.7)\), we have

\[
K^M_\nu (x, y) = \mathcal{B} \left( G^{1,0}_{0,M+1} \left( -\nu_0, -\nu_1, \ldots, -\nu_M \mid x \right), G^{M,0}_{0,M+1} \left( \nu_1, \ldots, \nu_M, \nu_0 \mid y \right) \right),
\]

where \( \mathcal{B}(\cdot, \cdot) \) is a bilinear operator defined by

\[
\mathcal{B} \left( f(x), g(y) \right) = (-1)^M \sum_{j=0}^{M} (-1)^j (\Delta_x)^j f(x) \left( \sum_{i=0}^{M-j} a_{i+j} (\Delta_y)^i g(y) \right),
\]

with \( \Delta_x = x \frac{d}{dx} \) and \( \Delta_y = y \frac{d}{dy} \). The constants \( a_i \) in \((5.14)\) are determined by

\[
\prod_{i=1}^{M} (x - \nu_i) = \sum_{i=0}^{M} a_i x^i,
\]

\[\text{(5.15)}\]
that is,

\[ a_i = (-1)^i e_{M-i}(\nu_1, \ldots, \nu_M) \]  

(5.16)

with \( e_i(\nu_1, \ldots, \nu_M) \) being the elementary symmetric polynomial.

The bilinear operator \( B \) is called a point-split bilinear concomitant in \[.\[12\]

Proof. We set

\[
    f(x) = G_{0,M+1}^{1,0}\left( -\nu_0, -\nu_1, \ldots, -\nu_M \left| x \right. \right),
\]

(5.17)

\[
    g(y) = G_{0,M+1}^{M,0}\left( \nu_1, \ldots, \nu_M, \nu_0 \left| y \right. \right).
\]

(5.18)

By (5.7), our aim is then to evaluate the integral

\[
    K^M_\nu(x, y) = \int_0^1 f(tx)g(ty) \, dt.
\]

(5.19)

Note that the Meijer-G function satisfies the differential equation (A.3).

For \( f \) and \( g \) given by (5.17) and (5.18), this implies that for every \( t \),

\[
    g(ty) \prod_{j=0}^M (\Delta_x + \nu_j)f(tx) = -txf(tx)g(ty),
\]

(5.20)

\[
    f(tx) \prod_{j=0}^M (\Delta_y - \nu_j)g(ty) = (-1)^M tyf(tx)g(ty).
\]

(5.21)

If \( M \) is odd we subtract these two identities, while if \( M \) is even we add them together. Since the arguments in both cases are similar, we restrict to the case where \( M \) is odd.

Subtracting (5.20) from (5.21) we obtain

\[
    (x - y)f(tx)g(ty)
\]

\[ = \frac{1}{t} \left( f(tx) \prod_{j=0}^M (\Delta_y - \nu_j)g(ty) - g(ty) \prod_{j=0}^M (\Delta_x + \nu_j)f(tx) \right) \]

\[ = \frac{1}{t} \sum_{i=0}^M a_i \left( f(tx)(\Delta_y)^{i+1}g(ty) + (-1)^i g(ty)(\Delta_x)^{i+1}f(tx) \right), \]

(5.22)

24
where the constants $a_i$ are defined in (5.15) and (5.16). We next observe that

$$\frac{\partial}{\partial t} \left( \sum_{j=0}^{i} (-1)^j (\Delta x)^j f(tx)(\Delta y)^{i-j} g(ty) \right)$$

$$= \frac{1}{t} \left( f(tx)(\Delta y)^{i+1} g(ty) + (-1)^i g(ty)(\Delta x)^{i+1} f(tx) \right),$$

which by (5.14) and (5.22) implies that

$$(x - y)f(tx)g(ty) = \frac{\partial}{\partial t} B(f(tx), g(ty)). \quad (5.23)$$

Using (5.23) in (5.19) we find

$$(x - y)K_M^L(x, y) = B(f(x), g(y)) - \lim_{t \to 0^+} B(f(tx), g(ty)).$$

It thus remains to show that

$$\lim_{t \to 0^+} B(f(tx), g(ty)) = 0, \quad (5.24)$$

and to do this we need to understand the behavior of $f$ and $g$ at the origin.

First of all, we have by [31, formula 16.18.1] and (5.17) that $f$ is a hypergeometric function

$$f(x) = \frac{1}{\prod_{j=1}^{M} \Gamma(1 - \nu_j)} {}_0F_M\left( 1 - \nu_1, \ldots, 1 - \nu_M \bigg| -x \right), \quad (5.25)$$

so that $f$ is analytic at the origin. Next by (5.18), the definition of (A.1), and the properties of the Mellin transform (see e.g. [21]), we find

$$\int_0^{\infty} (\Delta y)^i g(y)y^{s-1} \, dy = (-s)^i \prod_{j=1}^{M} \frac{\Gamma(s + \nu_j)}{\Gamma(1 - s)}.$$  

Then it follows in the same way as we obtained (3.3) that

$$(\Delta y)^i g(y) = O(y^\alpha \log y)^{r-1}) \quad \text{as } y \to 0^+,$$  

with $\alpha = \min(\nu_1, \ldots, \nu_M) > -1$ and $r = \#\{j \mid \nu_j = \alpha\}$.

Now we look at the $j = 0$ term in (5.14) which is

$$f(x) \sum_{i=0}^{M} a_i(\Delta y)^i g(y) = f(x) \prod_{i=0}^{M} (\Delta_y - \nu_i) g(y) = -f(x)yg(y),$$

25
where in the last step we used (5.21) with $t = 1$. Replacing $x \mapsto tx$, $y \mapsto ty$, we find by (5.25) and (5.26) that the limit is 0 as $t \to 0^+$. For $j \geq 1$ we have 
\[(\Delta_x)^j f(x) = O(x) \quad \text{as } x \to 0,\]
and then it follows from (5.26) that the terms in (5.14) with $j \geq 1$ are all $O(x)$ as $x \to 0^+$, Replacing $x \mapsto tx$, $y \mapsto ty$, we then find that these terms tend to 0 as well as $t \to 0^+$. This proves (5.24) and it completes the proof of Proposition 5.4.

A The Meijer G-function

We give a brief introduction to the Meijer G-function in this appendix. By definition, the Meijer G-function is given by the following contour integral in the complex plane:

\[
G_{m,n}^{p,q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \; \bigg| \; z \right) = G_{m,n}^{p,q} \left( \begin{array}{c} a_p \\ b_q \end{array} \; \bigg| \; z \right) = \frac{1}{2\pi i} \int_{\gamma} \left( \prod_{j=1}^{m} \Gamma(b_j + u) \prod_{j=1}^{n} \Gamma(1-a_j-u) \right) \left( \prod_{j=m+1}^{n} \Gamma(1-b_j-u) \prod_{j=n+1}^{p} \Gamma(a_j+u) \right) z^{-u} du, \tag{A.1}
\]

where $\Gamma$ denotes the usual gamma function and the branch cut of $z^{-u}$ is taken along the negative real axis. It is also assumed that

- $0 \leq m \leq q$ and $0 \leq n \leq p$, where $m, n, p$ and $q$ are integer numbers;

- The real or complex parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ satisfy the conditions

\[ a_k - b_j \neq 1, 2, 3, \ldots, \quad \text{for } k = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m, \]

i.e., none of the poles of $\Gamma(b_j + u)$, $j = 1, 2, \ldots, m$ coincides with any poles of $\Gamma(1-a_k-u)$, $k = 1, 2, \ldots, n$.

The contour $\gamma$ is chosen in such a way that all the poles of $\Gamma(b_j + u)$, $j = 1, \ldots, m$ are on the left of the path, while all the poles of $\Gamma(1-a_k-u)$, $k = 1, \ldots, n$ are on the right, which is usually taken to go from $-i\infty$ to $i\infty$. In particular, it can be a loop starting and ending at $+\infty$ if $p > q$, or a loop beginning and ending at $-\infty$ if $p < q$. Most of the known special functions can be viewed as special cases of the Meijer G-functions, we refer to [30, 31] for more details. We end this appendix with several formulas used in this paper.

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From the definition (A.1), it is easily seen that
\[ z^\rho G^{m,n}_{p,q}(a_p | b_q | z) = G^{m,n}_{p,q}(a_p + \rho | b_q + \rho | z). \]  
(A.2)

The Meijer G-function \( G^{m,n}_{p,q}(a_p | b_q | z) \) satisfies the following linear differential equation of order \( \max(p, q) \):
\[
\left[ (-1)^{p-m-n} z \prod_{j=1}^{p} \left( z \frac{d}{dz} - a_j + 1 \right) \right. \\
\left. - \prod_{j=1}^{q} \left( z \frac{d}{dz} - b_j \right) \right] G^{m,n}_{p,q}(a_p | b_q | z) = 0; \]  
(A.3)

see [31] formula 16.21.1).

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