On the maximum rank of a real binary form

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Abstract

We show that a real binary form \( f \) of degree \( n \geq 3 \) has \( n \) distinct real roots if and only if for any \( (\alpha, \beta) \in \mathbb{R}^2 \setminus \{0\} \) all the forms \( \alpha f_x + \beta f_y \) have \( n - 1 \) distinct real roots. This answers to a question of P. Comon and G. Ottaviani in [1], and allows to complete their argument to show that \( f \) has symmetric rank \( n \) if and only if it has \( n \) distinct real roots.

1 Introduction

This paper deals with the following problem: Given a degree \( n \) polynomial \( f \in K[x_1, \ldots, x_m] \) find the rank (or Waring rank) of \( f \), i.e. the minimum number of summands which achieve the following decomposition:

\[
f = \lambda_1 l_1^n + \cdots + \lambda_r l_r^n \quad \text{with} \quad \lambda_i \in K \text{ and } l_i \text{ linear forms}.
\]

For \( K = \mathbb{C} \) and \( f \) generic the answer has been given (see [2, 3]), nevertheless some questions remain unsolved, e.g. it’s not yet known which is the stratification of the set of complex polynomials by the rank. However one can see [4] for an answer in the binary case.

In the real case, i.e. \( K = \mathbb{R} \), the situation becomes more complicated. In contrast to the complex case which has a generic rank, in the real case the generic rank is substituted by the concept of typical rank. A rank \( k \) is said typical for a given degree \( n \) if there exists a euclidean open set in the space of real degree \( n \) polynomials such that any \( f \) in such open set has rank \( k \). We will prove the following theorem, posed as a question in [1].

**Theorem 1.** Let \( f(x, y) \) be a real homogeneous polynomial of degree \( n \geq 3 \) without multiple roots in \( \mathbb{C} \). Then \( f \) has all real roots if and only if for any \( (0, 0) \neq (\alpha, \beta) \in \mathbb{R}^2 \) the polynomial \( \alpha f_x + \beta f_y \) has \( n - 1 \) distinct real roots.

Notice that the “only if” part of the theorem is easy. Indeed given any \( (\alpha, \beta) \neq (0, 0) \) one may consider a new coordinate system \( l, m \) on the projective line, such that \( x = \alpha l + \alpha' m \), and \( y = \beta l + \beta' m \), so that \( \partial_l = \alpha \partial_x + \beta \partial_y \). Writing \( f \) as a function of \( l, m \) and de-homogenizing by setting \( m = 1 \) one sees that \( f_l \) has \( n - 1 \) distinct roots by the theorem of Rolle.

In [1] the result above has been considered in connection with the problem of determining the rank of a real binary form, that is the minimum number \( r \) such that \( f(x, y) = \lambda_1 l_1^n + \cdots + \lambda_r l_r^n \), with \( \lambda_i \in \mathbb{R} \) and \( l_i = \alpha_i x + \beta_i y \in \mathbb{R}[x, y] \).
for \( i = 1, \ldots, r \). Using the arguments already given in [1] and applying Theorem [1] one gets the following result.

**Corollary 1.** A real binary form \( f(x, y) \) of degree \( n \geq 3 \) without multiple roots in \( \mathbb{C} \) has rank \( n \) if and only if it has \( n \) distinct real roots.

We leave the following question open for further investigations. Partial evidence for it has been given from the results in [1], where it has given a positive answer for \( n \leq 5 \), and where the reader can find references for the existing literature on rank problems for real tensors.

**Question 1.** Are all the ranks \( \lfloor n/2 \rfloor + 1 \leq k \leq n \) typical for real binary forms of degree \( n \)?

## 2 Main Theorem

Let \( f(x, y) \) be a real homogeneous polynomial of degree \( n \geq 3 \) without multiple roots in \( \mathbb{C} \). Then \( \nabla f(x, y) \neq (0, 0) \) for any \( (x, y) \neq (0, 0) \) and one can define the maps \( \tilde{\phi} : S^1 \to S^1 \) and \( \tilde{\psi} : S^1 \to S^1 \) setting, for any \( (x, y) \) with \( x^2 + y^2 = 1 \), \( \tilde{\phi}(x, y) = |\nabla f|^{-1}(f_x, f_y) \) and \( \tilde{\psi}(x, y) = |\nabla f|^{-1}(xf_x + yf_y, -yf_x + xf_y) \), with \( |\nabla f| = (f_x^2 + f_y^2)^{1/2} \). Setting \((x, y) = (\cos \theta, \sin \theta)\), one can also write \( \phi \) and \( \psi \) as functions of \( \theta \).

**Notation.** We denote \( \partial_0 = -y\partial_x + x\partial_y \) the basis tangent vector to \( S^1 \) at the point \((x, y)\). Given any differentiable map \( \phi : S^1 \to M \) to a differentiable manifold \( M \), we denote \( \phi_* : T_0S^1 \to T_{\phi(\theta)}M \) the associated tangent map. If \( M = S^1 \), and the map \( \phi \) is defined in terms of angular coordinates by the function \( \theta_1(\theta) \), we recall that the degree, or winding number, of \( \phi \) is the number

\[
\deg \phi = \frac{1}{2\pi} \int_0^{2\pi} \theta'_1(\theta)d\theta.
\]

This is always an integer number, and for any \( z \in S^1 \) one has \( \#\phi^{-1}(z) \geq |\deg \phi| \).

The following lemmas are straightforward calculations and their proofs are omitted.

**Lemma 1.** Assume that \( \theta'_1(\theta) \) never vanishes. Then \( \#\phi^{-1}(z) = |\deg \phi| \) for any \( z \in S^1 \).

We assume that for any \((\alpha, \beta) \in \mathbb{R}^1(\mathbb{R})\) the polynomial \( \alpha f_x + \beta f_y \) has \( n - 1 \) distinct roots in \( \mathbb{R} \). Under this assumption, we want to show that the absolute value of the degree of \( \psi \) is \( n \). Since \( f(x, y) = 0 \) if and only if \( \psi(x, y) = (0, \pm 1) \) this implies that \( f \) has all its roots in \( \mathbb{R} \). Indeed \( \psi(-x, -y) = (-1)^n \psi(x, y) \), henceforth when \( n \) is even \( n/2 \) real roots of \( f(x, y) = 0 \) are in \( \psi^{-1}(0, 1) \) and the other \( n/2 \) roots are in \( \psi^{-1}(0, -1) \); otherwise when \( n \) is odd one gets \( \psi^{-1}(0, 1) = \psi^{-1}(0, -1) \), hence \( \psi^{-1}(0, 1) \) is the set of the \( n \) real roots of \( f(x, y) = 0 \).
Lemma 2. Let $F : S^1 \to \mathbb{R}^2$ be a differentiable function defined by $F(x,y) = (F_1(x,y), F_2(x,y)) = (a(\theta), b(\theta))$. Then $F_*(\partial_\theta) = A\partial_x + B\partial_y$ with

$$A = -yF_{1x} + xF_{1y} = a'(\theta)$$
$$B = -yF_{2x} + xF_{2y} = b'(\theta).$$

Notation. Given a map $f : S^1 \to \mathbb{R}^2$, which one can write $f(\theta) = (a(\theta), b(\theta))$, we denote with $(f,f_\theta)$ the matrix

$$\begin{pmatrix}
a(\theta) & b(\theta) \\
a'(\theta) & b'(\theta)
\end{pmatrix}.$$  

Notice that the sign of the determinant of this matrix expresses if $f_*$ is orientation-preserving at the point $f(\theta)$.

Lemma 3. Let $g : S^1 \to \mathbb{R}^2$ and $\rho : S^1 \to \mathbb{R}_+$ be differentiable functions. Then $\det(g,g_\theta) = \rho^{-2}\det(\rho g, (\rho g)_\theta)$.

Notice that if $\bar{g} : S^1 \to S^1$ is the map $\bar{g}(x,y) = |\nabla f|^{-1}(f_x, f_y)$ then one may calculate the sign of $\det(\bar{g}, \bar{g}_\theta)$ by reducing to the simpler map $\phi = (f_x, f_y) : S^1 \to \mathbb{R}^2$.

Notation. We denote by $H(f) = \det(f_{xx} f_{xy} f_{yx} f_{yy})$, the hessian of $f$.

Proposition 1. Let $\phi : S^1 \to \mathbb{R}^2$ be the map defined above. Then $\det(\phi, \phi_\theta) = (n-1)^{-1}H(f)$.

Proof. We have $\phi_*(\partial_\theta) = A\partial_x + B\partial_y$ with $A$ and $B$ determined as in Lemma 2, hence

$$\det(\phi, \phi_\theta) = \det\begin{pmatrix} f_x & f_y \\ -yf_{xx} + xf_{xy} & -yf_{yx} + xf_{yy} \end{pmatrix} = \frac{1}{n-1} \det\begin{pmatrix} xf_{xx} + yf_{xy} & xf_{yx} + yf_{yy} \\ -yf_{xx} + xf_{xy} & -yf_{yx} + xf_{yy} \end{pmatrix} = \frac{1}{n-1} \det\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \frac{1}{n-1}H(f).$$

Proposition 2. Let $\phi : S^1 \to \mathbb{R}^2 \cong \mathbb{C}$ defined by $\phi(\theta) = a(\theta) + ib(\theta)$ and $\psi : S^1 \to \mathbb{C}$ defined by $\psi(\theta) = e^{-i\theta}\phi(\theta)$. Then $\det(\psi, \psi_\theta) = \det(\phi, \phi_\theta) - a^2 - b^2$.

Proof. We calculate $\psi'(\theta) = (a' + b + i(b' - a))e^{-i\theta}$. It follows that

$$\det(\psi, \psi_\theta) = \det\begin{pmatrix} a & b \\ a' + b & b' - a \end{pmatrix} = \det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} - a^2 - b^2.$$
Applying this result to \( g \), we find that

\[
\text{Corollary 2. In the notations above, the following statements hold.}
\]

1. \( \det(\tilde{\phi}, \tilde{\phi}_\theta) = \theta'_1(\theta) = (n - 1)^{-1}\nabla f^{-2}H(f) \).
2. \( \deg \bar{\psi} = \deg \tilde{\phi} - 1 \).

**Proof.** The first statement follows form Proposition 1 and Lemma 3. The second one follows from Proposition 2 and Lemma 3 since

\[
\deg \bar{\psi} = \frac{1}{2\pi} \int \det(\bar{\psi}, \bar{\psi}_\theta) = \frac{1}{2\pi} \int \det(\tilde{\phi}, \tilde{\phi}_\theta) - 1 = \deg \tilde{\phi} - 1.
\]

Now we are ready to complete the proof of Theorem 1.

**Proof of Theorem 1.** Since for any \((\alpha, \beta) \in \mathbb{R}^2 \setminus \{0\}\) the polynomial \( \alpha f_x + \beta f_y \) has \( n - 1 \) distinct real roots, then the map \((f_x, f_y): \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}\) has no ramification at any real point of \( \mathbb{P}^1_{\mathbb{R}} \). Equivalently, the jacobian of \( \phi \) which is equal to the hessian \( H(f) \) is always non-zero at the real points of \( \mathbb{P}^1_{\mathbb{R}} \). We call the map \( \phi: S^1 \to S^1 \) defined by \( \phi = |\nabla f|^{-1}(f_x, f_y) \) and we also express it as \( \theta_1 = \theta_1(\theta) \) in angular coordinates. By the observation above and Corollary 2, it follows that the derivative \( \theta'_1(\theta) \) is non vanishing at any \( \theta \in S^1 \). Hence \( \theta'_1(\theta) \) is either always positive or always negative.

**Claim:** \( \theta'_1(\theta) < 0 \) for any \( \theta \).

The sign of \( \theta'_1(\theta) \) is the same as the sign of \( H(f) \). Since we already know that it is constant it will be sufficient to evaluate it at a single point \((x, y) \in S^1 \).

We choose to examine the point \((1, 0)\). One observes that for any binary form

\[
g(x, y) = \binom{m}{0} a_0 x^m + \binom{m}{1} a_1 x^{m-1} y + \cdots + \binom{m}{m} a_m y^m
\]

of degree \( m \geq 3 \), the Hessian \( H(g) \) calculated at \((1, 0)\) is equal to

\[
m(m - 1) \det \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}.
\]

Similarly the Hessian of its derivative \( g_x \) at \((1, 0)\) is given by

\[
m(m - 1)(m - 2) \det \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}.
\]

Therefore we find that

\[
H(g)(1, 0) = (m - 2)^{-1}H(g_x)(1, 0).
\]

Applying this result to \( g = f \), we are reduced to compute the sign of \( H(f_x) \). We know that \( f_x \) has \( n - 1 \) distinct real roots, so all of its derivatives \( \partial^j_x(f_x) \) have all
their roots real and distinct, up to \( i = n - 3 \). The last of these derivatives is \( h = \partial_{x}^{n-2}f \), and its Hessian is a constant equal to \(-\Delta(h)\), hence \( H(h) < 0 \). Applying recursively the reduction step, we find that \( H(f)(1,0) = (n-2)^{-1}H(f_{x})(1,0) = ((n-2)!)^{-1}H(h) < 0 \), proving the claim.

By Corollary 2(1), Lemma 1 and applying the claim above, we get that \( \deg \overline{\phi} < 0 \) and \( \#\overline{\phi} - 1(z) = |\deg \phi| \) for any \( z \in S^{1} \), hence \( \deg \overline{\phi} = -n + 1 \). Moreover, by Corollary 2(2), we have \( \deg \overline{\psi} = \deg \overline{\phi} - 1 = -n \), hence \( \#\text{real roots}(f) \geq |\deg \overline{\psi}| = n \). This completes the proof of the Theorem.

We conclude giving a self-contained proof of the result on the rank of a real binary form mentioned in the introduction. The arguments given are all already in [1].

Proof of Corollary 7. The statement holds for \( n = 3 \), as shown in [1], Proposition 2.2. Assuming \( n > 3 \), suppose the statement holds in degree \( n - 1 \). Assume \( \text{rank}(f) = r \), so one can write \( f = \lambda_{1}l_{1}^{n} + \cdots + \lambda_{r}l_{r}^{n} \), with \( r \) minimal. Then one can consider \( l = l_{1} \) and \( m = l_{r} \), and \( g(t) = m^{-n}f \), with \( t = l/m \). One sees that \( m^{-n+1}f_{t} = g'(t) \) can be expressed as a sum of at most \( r - 1 \) \( n \)-th powers of linear forms in \( t \). If \( f \) has \( n \) distinct real roots then, by induction hypothesis \( f_{t} \) has \( n - 1 \) distinct real roots and we find \( r - 1 \geq n - 1 \), i.e. \( r \geq n \). Since the inequality \( r \leq n \) always holds, as shown in [1] Proposition 2.1, we have \( r = n \). Conversely, if the rank of \( f \) is \( n \) then take \( r = n \) and consider any derivative \( \alpha f_{x} + \beta f_{y} = f_{t} \), after defining a suitable coordinate system \( l, m \), as explained in the introduction. We can consider the polynomial \( g'(t) = m^{-n+1}f_{t} \). If it has rank \( < n - 1 \), then by indefinite integration over \( t \) one sees easily that \( f \) has rank \( < n \), contrary to the assumption. So \( \text{rank}(f_{t}) = n - 1 \) and, by induction hypothesis, it also holds that \( f_{t} \) has \( n - 1 \) distinct real roots. By the arbitrariness of \( l \) and by Theorem 1 we conclude that \( f \) has \( n \) distinct roots. 

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