Cosmological production of charged black hole pairs

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Abstract

We investigate the pair creation of charged black holes in a background with a positive cosmological constant. We consider $C$ metrics with a cosmological constant, and show that the conical singularities in the metric only disappear when it reduces to the Reissner-Nordström de Sitter metric. We construct an instanton describing the pair production of extreme black holes and an instanton describing the pair production of non-extreme black holes from the Reissner-Nordström de Sitter metric, and calculate their actions. There are a number of striking similarities between these instantons and the Ernst instantons, which describe pair production in a background electromagnetic field. We also observe that the type I instanton in the ordinary $C$ metric with zero cosmological constant is actually the Reissner-Nordström solution.
1 Introduction

There has been considerable interest recently in studying black hole pair creation by instanton methods, and a number of interesting results have been obtained [1, 2, 3]. The study of black hole pair creation has so far been mostly restricted to the pair creation of oppositely-charged black holes by an electromagnetic field, where pair creation is possible because the negative potential energy of the created pair of black holes in the background electromagnetic field balances their rest mass energy. Black holes can, however, also be pair created by a cosmological background, as a positive cosmological constant supplies the necessary negative potential energy. We will examine the pair creation of electrically or magnetically charged black holes in a cosmological background. There are two instantons, which describe pair production of non-extreme and extreme black holes respectively. There is another instanton, the charged generalisation of the instanton found in [4], but careful analysis suggests this latter instanton does not formally represent black hole pair creation.

We find that some of the most interesting results of the electromagnetic case can be reproduced in the cosmological case; in particular, the pair creation rate is still determined by the entropy of the solutions. Indeed, there is an interesting similarity between the instantons we find and the instantons found in [1, 2] for the pair creation of black holes in an electromagnetic field. We feel that these similarities encourage the idea that the results of these instanton calculations represent real quantum gravity effects, and will not be qualitatively modified by the inclusion of quantum corrections. In other words, since the main features are similar in these two disparate models, we think that they represent features that should also be present in the full quantum theory.

We will begin by considering the charged $C$ metric, which can be interpreted as representing a pair of oppositely-charged black holes accelerating away from each other in a flat background spacetime. As there is no force to accelerate the black holes, this metric is not in general regular. The approach taken in [5, 6] was to add a background electromagnetic field to this metric by a Harrison transformation, giving the Ernst metric, which could be made regular by a suitable choice of field strength. The background field provides the necessary force to accelerate the black holes. We will consider adding a cosmological constant to the $C$ metric instead, and see if we can obtain a non-singular solution by using the cosmological acceleration to accelerate the black holes. We find that we can, but that the $C$ metric reduces
to the Reissner-Nordström de Sitter solution whenever the acceleration of
the black holes is matched to the cosmological acceleration in this way.

There are in fact two ways to obtain a non-singular solution in the
charged $C$ metric itself. One is to set the acceleration to zero, in which
case the $C$ metric reduces to the Reissner-Nordström metric. The other
way to get rid of the conical singularities is to allow the black hole and
acceleration horizons to coincide, which has been referred to as the Type I
instanton [1]. We have found that the $C$ metric in this special case has the
same functional form as the Reissner-Nordström metric, but it is now what
is usually regarded as the azimuthal coordinate which is playing the role of
time. That is, one can obtain this metric by analytically continuing $\phi \to i\Phi$
in the Euclidean Reissner-Nordström metric.

As the only non-singular versions of the charged $C$ metric with a cos-
mological constant reduce to the Reissner-Nordström de Sitter metric, we
would like to know what instantons can be made from this metric. We find
that there are four types of instantons, referred to as the lukewarm, cold,
ultracold [3], and charged Nariai solutions. The Euclidean section of the
lukewarm solution has topology $S^2 \times S^2$, and it can be thought of as repre-
senting pair creation of non-extreme black holes. The Euclidean section of the
cold solution has topology $S^2 \times R^2$, and it can be thought of as represent-
ing pair creation of extreme black holes (by an extreme black hole, we mean
one in which the inner and outer horizons coincide). The charged Nariai
solution has topology $S^2 \times S^2$; its interpretation is not clear. The ultracold
solution is just a special case of the cold solution where all three horizons
coincide, but the metric is quite different in this special case. There is an
analogy between this set of instantons and the instantons constructed from
the Ernst solution in [2]: there one had an extreme black hole instanton, a
non-extreme black hole instanton, and the Type I instanton; here we have
the cold, lukewarm and charged Nariai instantons.

In the instanton approach, the wavefunction is approximated by $\Psi \approx e^{-I}$, where $I$ is the action of the instanton, so the partition function, which
gives the pair creation rate, is approximated by $Z \approx e^{-2I}$. The action for
Einstein-Maxwell with a cosmological constant is

$$I = -\frac{1}{16\pi} \int d^4x \sqrt{g} (R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) + \frac{1}{8\pi} \int_{\Sigma} d^3x \sqrt{h}K,$$

(1)

where $R$ is the Ricci scalar of the metric $g_{\mu\nu}$, $\Lambda$ is the cosmological constant
(we assume $\Lambda > 0$), and $F_{\mu\nu}$ is the electromagnetic field tensor. The bound-
ary of the manifold is $\Sigma$, which has metric $h_{ij}$, and extrinsic curvature $K_{ij}$. 

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Since there is no asymptotic region for these solutions, this action is the microcanonical action \( I \), and thus the entropy is \( S \)

\[
S = \log Z = -2I. 
\] (2)

In section 5, we calculate the action for the instantons described in section 4. We find that the pair creation rate given by these instantons is smaller than the rate which de Sitter space gives to propagate from nothing to an \( S^3 \). Thus the contribution from de Sitter space will dominate that of these instantons in the path integral, in agreement with experience. We also find that the entropy of these solutions is always equal to one quarter of the area of the horizons which appear on the Euclidean section, as expected. These instantons are thus very similar to the Ernst instantons. We summarise our results in section 6.

## 2 Cosmological \( C \) metrics

The well-known \( C \) metric solution of the Einstein-Maxwell equations, which describes a pair of charged black holes undergoing uniform acceleration, was found in [9]. There is a less well-known generalisation of this solution to include a cosmological constant [10]. In the usual \( C \) metric, it is not possible to eliminate the possible conical singularities at both poles in general. The \( C \) metric is interpreted as describing two oppositely-charged black holes undergoing uniform acceleration, and these singularities are interpreted as representing “rods” or “strings” which provide the force necessary to accelerate the black holes.

If we consider the \( C \) metric with a positive cosmological constant, one might think that the cosmological constant can provide the necessary force, and so we should be able to eliminate the conical singularities, and thus obtain a completely regular metric. What we find is that the elimination of the conical singularities, which we regard as fixing the acceleration parameter \( A \), causes the metric to reduce to the Reissner-Nordström de Sitter metric.

The cosmological \( C \) metric can be written as

\[
ds^2 = \frac{1}{A^2(x-y)^2} \left[ H(y)dt^2 - H^{-1}(y)dy^2 + G^{-1}(x)dx^2 + G(x)d\varphi^2 \right], \] (3)

where

\[
G(x) = 1 - x^2(1 + r_+ Ax)(1 + r_- Ax) - \frac{\Lambda}{3A^2}, \] (4)
and

\[ H(y) = 1 - y^2(1 + r_Ay)(1 + r_Ay), \]  

(5)

The cosmological constant is \( \Lambda \), which we will assume is positive. The gauge field in the magnetic case is

\[ F = -qd x \wedge d\varphi, \]  

(6)

and the gauge field in the electric case is

\[ F = -q dt \wedge dy, \]  

(7)

where \( q = \sqrt{r_+r_-} \). We will denote by \( x_1, x_2, x_3, x_4 \) the roots of \( G(x) \) in ascending order, and by \( y_1, y_2, y_3, y_4 \) the roots of \( H(y) \) in ascending order. We will restrict the parameters \( \Lambda, r_+, r_- \) and \( A \) so that all these roots are real. As with the usual \( C \) metric, \( y = y_1 \) is interpreted as the inner black hole horizon, \( y = y_2 \) is interpreted as the outer black hole horizon, and \( y = y_3 \) is interpreted as the acceleration or cosmological horizon.

We will first restrict \( x \) to \( x_3 \leq x \leq x_4 \) in order to obtain a metric of the appropriate signature. We will restrict \( y \) to \( -\infty < y \leq x \), as when \( y = x \) the conformal factor in the metric diverges, and so this corresponds to infinity. In order to have a regular solution, we must avoid having conical singularities at \( x = x_3 \) or \( x = x_4 \), so we must demand that

\[ G'(x_3) = -G'(x_4), \]  

(8)

and identify \( \varphi \) periodically with period \( \Delta \varphi = 4\pi/|G'(x_3)| \). This assumes that \( x_3 \) is at a finite distance from \( x_4 \), but if one took \( x_3 = x_2 \), then \( x = x_3 \) would lie at infinite proper distance from any other point, so that there could be no conical singularity there. That is, the \((x, \varphi)\) sections would no longer be compact. This is analogous to what happens with the Euclidean section of an extreme black hole [3]. However, for positive cosmological constant, \( H(y) \) will have just two real roots if \( x_2 = x_3 \), so this contradicts our restriction of the parameters. Therefore we must satisfy (8), which corresponds to

\[(x_3 - x_4)(x_3 - x_2)(x_3 - x_1) = (x_3 - x_4)(x_4 - x_2)(x_4 - x_1), \]  

(9)

and can only be satisfied by taking \( x_3 = x_4 \).

This seems to imply that the \((x, \varphi)\) section shrinks to a point, but this is just due to a poor choice of coordinate system. In the limit that \( x_3 = x_4 \), the proper distance between \( x_3 \) and \( x_4 \) remains finite, as can be seen by the
following coordinate transformation. The limit $x_3 \to x_4$ corresponds to the limit $A \to \sqrt{\Lambda}/3$ from above. Let us write $1 - \Lambda/(3A^2) = \epsilon^2$, so that the appropriate limit is $\epsilon \to 0$. With this parametrisation,

$$G(x) = \epsilon^2 - x^2(1 + r_+Ax)(1 + r_-Ax), \tag{10}$$

so $x_3 \approx -\epsilon$ and $x_4 \approx \epsilon$. Then set

$$x = \epsilon \cos \theta, \varphi = \frac{\phi}{\epsilon}. \tag{11}$$

In the limit $\epsilon \to 0$, the metric becomes

$$ds^2 = \frac{3}{\Lambda y^2} \left[ H(y)dt^2 - H^{-1}(y)dy^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right], \tag{12}$$

and if we make the further coordinate transformation

$$r = -\sqrt{3/\Lambda}y, T = \sqrt{3/\Lambda}t, \tag{13}$$

it becomes

$$ds^2 = -V(r)dT^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{14}$$

where

$$V(r) = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right) - \frac{\Lambda r^2}{3}. \tag{15}$$

Note also that as $\varphi$ has period $4\pi/|G'(x_3)|$, $\phi$ has period $2\pi$. The range of $r$ is from 0 to $\infty$, and $\theta$ runs from 0 to $\pi$. The gauge field is now

$$F = q \sin \theta d\theta \wedge d\phi \tag{16}$$

in the magnetic case, and

$$F = -\frac{q}{r^2}dT \wedge dr \tag{17}$$

in the electric case. Thus, this can be identified as a Reissner-Nordström de Sitter solution with charge $q = \sqrt{r_+r_-}$ and ‘mass’ $M = (r_+ + r_-)/2$. Note that if we let $\Lambda = 0$, the limit $x_3 = x_4$ is just the limit $A \to 0$, and what

\footnote{This coordinate transformation is inspired by that in $[4]$.}
we have just done specialises to the familiar statement that the $C$ metric reduces to the Reissner-Nordström metric in this limit [1].

We have assumed earlier that $-\infty < y < x$ to avoid divergence in the conformal factor, but we could equally well have taken $x < y < \infty$. Then $y = y_2$ is interpreted as the cosmological horizon, $y = y_3$ is the outer black hole horizon, and $y = y_4$ is the inner black hole horizon. We now have to take $x_1 \leq x \leq x_2$ to get a metric of the right signature.

To make this metric regular, we must take $\varphi$ to be periodic with period $\Delta \varphi = 4\pi/|G'(x_1)|$, and require

$$G'(x_1) = -G'(x_2).$$

This implies

$$(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) = (x_1 - x_2)(x_2 - x_3)(x_2 - x_4),$$

and thus $x_1 = x_2$. We have found that this occurs when $A = A_c$, where

$$A_c^2 = \frac{\Lambda}{3} - \frac{[3(r_+ + r_-) + \gamma]^2[(r_+ + r_- - \gamma)^2 - 16(r_+ - r_-)^2]}{4096r_+^3r_-^3},$$

and

$$\gamma = \sqrt{9(r_+^2 + r_-^2) - 14r_+r_-}.$$ 

If we let $A_c^2/A_2^2 = 1 - \epsilon^2$, and $X = x - x_1$, then

$$G(x) = \epsilon^2 - \frac{X^2}{16r_+r_-}(a + bAX + cA^2X^2),$$

where

$$a = (3(r_+ + r_-) + \gamma)\gamma,$$

$$b = -8r_+(r_- + r_+ + \gamma),$$

and

$$c = 16r_+^2r_-^2.$$ 

Therefore, if we make coordinate transformations

$$X = \sqrt{\frac{16r_+r_-}{a}}\epsilon \cos \theta,$$

and

$$\varphi = \sqrt{\frac{16r_+r_-}{a}}\phi/\epsilon,$$
the metric becomes, in the limit $\epsilon \to 0$,

$$ds^2 = \frac{1}{A_c^2(y - x_1)^2} \left[ H(y) dt^2 - H^{-1}(y) dy^2 + \frac{16 r_+ r_-}{a}(d\theta^2 + \sin^2 \theta d\phi^2) \right].$$

(28)

If we make the further coordinate transformations

$$r = \sqrt{\frac{16 r_+ r_-}{a}} A_c(y - x_1), \quad T = \sqrt{\frac{a}{16 r_+ r_-}} t,$$

(29)

the metric will become

$$ds^2 = -V(r) dT^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(30)

where

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2,$$

(31)

and

$$M = -\sqrt{\frac{16 r_+ r_-}{a}} \frac{b}{2a}, \quad Q^2 = \frac{16 r_+ r_- c}{a^2}.$$  

(32)

The gauge field becomes

$$F = \frac{16 r_+ r_-}{a} q \sin \theta d\theta \wedge d\phi = Q \sin \theta d\theta \wedge d\phi$$

(33)

in the magnetic case, and

$$F = \frac{Q}{r^2} dT \wedge dr$$

(34)

in the electric case. Therefore, this can be identified as a Reissner-Nordström de Sitter metric as well. Note that although the equations are more complicated, $M$ and $Q$ are still just functions of $r_+$ and $r_-$, and $M$ is positive if $r_+$ and $r_-$ are positive.

If the cosmological constant is set to zero, we see that the $C$ metric is again only non-singular when it reduces to the Reissner-Nordström metric, that is, when the acceleration of the black holes vanishes. However, in this case, this happens for non-zero $A$, as we can easily see from (20). This is just an indication that we shouldn’t think of $A$ as simply parametrising the acceleration in this case.
3 The Type I instanton

When the cosmological constant is set to zero, we have seen that the ordinary C metric has no conical singularities when the acceleration vanishes, as we should have expected. However, when \( \Lambda = 0 \), we can eliminate the conical singularities in another way, as was first observed by Dowker et al [1]. It is now possible to set \( x_3 = x_2 \), as \( x_3 = y_3 = \xi_3 \) and \( x_2 = y_2 = \xi_2 \). Again, the apparent degeneracy of the two roots is merely an artifact of our coordinate system. As explained in [1], we may make a transformation so that the metric remains regular when \( \xi_3 = \xi_2 \).

The C metric is given by (3) with \( \Lambda = 0 \). However, for consistency with [1], we will adopt a slightly different coordinate system in this section, and write the C metric as

\[
\frac{1}{A^2(x-y)^2} \left[ G(y)dt^2 - G^{-1}(y)dy^2 + G^{-1}(x)dx^2 + G(x)d\varphi^2 \right]
\]

with

\[
G(\xi) = [1 - \xi^2(1 + r+A\xi)](1 + r-A\xi).
\]

The transformation between the two forms of the C metric is discussed in [3]. The gauge field is still (6) or (7). We will refer to the roots of \( G(\xi) \) as \( \xi_1, \xi_2, \xi_3, \xi_4 \) in ascending order. As before, \( x \) is restricted to \( \xi_3 \leq x \leq \xi_4 \) to obtain the appropriate signature. In fact, if \( \xi_2 = \xi_3 \), the appropriate range is actually \( \xi_3 < x \leq \xi_4 \), and the \((x, \varphi)\) section becomes topologically \( R^2 \). We can then obtain a regular solution by identifying \( \varphi \) with period \( 4\pi/G'(\xi_4) \).

The two roots \( \xi_3 \) and \( \xi_4 \) will coincide when \( A = A_c = 2/(3\sqrt{3}r_+) \). Following [1], we let \( r_+ A = 2/(3\sqrt{3}) - \epsilon^2/\sqrt{3} \), so that the limit of coincident roots is \( \epsilon \to 0 \). If we make the coordinate transformation

\[
y = \sqrt{3}(-1 + \epsilon \cos \chi), \quad \psi = \sqrt{3}\epsilon t
\]

the metric is, in the limit \( \epsilon \to 0 \),

\[
\frac{1}{A_c^2(x + \sqrt{3})^2} \left[ -\alpha \sin^2 \chi d\psi^2 + \alpha^{-1}d\chi^2 + G^{-1}(x)dx^2 + G(x)d\varphi^2 \right],
\]

where now

\[
G(x) = -\frac{2}{3\sqrt{3}}(x + \sqrt{3})^2(x - \sqrt{3}/2) \left( 1 + r_-x \right)
\]

(39)
and
\[ \alpha = 1 - \frac{2r_-}{3r_+}. \]  \hfill (40)
If we analytically continue \( \psi \to i\Psi \), and identify \( \Psi \) with period \( 2\pi \alpha^{-1} \), we obtain an instanton with topology \( S^2 \times R^2 \), referred to as the type I instanton. It was not initially clear how to interpret this instanton. However, our experience with the cosmological \( C \) metric above suggests that it is related to the Reissner-Nordström instanton. We are encouraged in this guess by the fact that the \( (\Psi, \chi) \) two-sphere sections are round.

The range of \( x \) in this solution is \(-\sqrt{3} < x \leq \sqrt{3}/2 \). Let \( \tilde{x} = x + \sqrt{3} \), and \( \phi = \alpha \Psi \), so that \( \phi \) has period \( 2\pi \), and the Euclideanised metric becomes
\[ ds^2 = \frac{\alpha^{-1}}{A_c^2 \tilde{x}^2} \left[ d\chi^2 + \sin^2 \chi d\phi^2 + \alpha(G^{-1}(\tilde{x})d\tilde{x}^2 + G(\tilde{x})d\varphi^2) \right]. \]  \hfill (41)
If we make a further coordinate transformation,
\[ r = \frac{\alpha^{-1/2}}{A_c \tilde{x}}, \tau = \frac{\alpha^{1/2} \varphi}{A_c}, \]  \hfill (42)
the metric becomes
\[ ds^2 = r^2(d\chi^2 + \sin^2 \chi d\phi^2) + V(r)d\tau^2 + \frac{dr^2}{V(r)}, \]  \hfill (43)
where
\[ V(r) = \left(1 - \frac{\tilde{x}_+}{r}\right)\left(1 - \frac{\tilde{x}_-}{r}\right). \]  \hfill (44)
The new parameters are \( \tilde{x}_+ = r_+ \alpha^{-1/2} \) and \( \tilde{x}_- = -r_- \alpha^{-3/2} \). We also note that \( r \) runs from \( \tilde{x}_+ \) to \( \infty \) on the Euclidean section. If the gauge field is (6), it becomes
\[ F = \frac{q\alpha^{-1}}{r^2}dr \wedge d\tau = -\frac{iQ}{r^2}d\tau \wedge dr, \]  \hfill (45)
and if it is (7), it becomes
\[ F = -iq\alpha^{-1}\sin \chi d\chi \wedge d\Psi = Q \sin \chi d\chi \wedge d\phi, \]  \hfill (46)
where \( Q^2 = \tilde{x}_+ \tilde{x}_- \). Therefore, this instanton is seen to be the Euclidean Reissner-Nordström solution with charge \( Q \) and mass \( M = \frac{1}{2}(\tilde{x}_+ + \tilde{x}_-) \), with the magnetic instanton being identified with the electric Reissner-Nordström solution and vice-versa. Thus we see again that eliminating the conical singularities is only possible when the \( C \) metric reduces to Reissner-Nordström.
However, the coordinate we analytically continued to obtain a Euclidean section has been identified with the azimuthal coordinate in the Euclidean Reissner-Nordström solution. This means that the problem of understanding the physical significance, if any, of the $C$ metric in this special case is equivalent to giving a physical interpretation of the metric (43) with $\phi = i\Phi$. Because $\tau$ is not analytically continued, this metric has one compact direction. This leads us to suspect that the interpretation is similar to that given for the five-dimensional black holes in [11]; that is, that this instanton should be interpreted as representing the decay of a Kaluza-Klein vacuum, $\text{Mink}^{(2,1)} \times S^1$.

The analytically continued metric is

$$\begin{align*}
ds^2 &= -r^2 \sin^2 \chi d\Phi^2 + r^2 d\chi^2 + V(r) d\tau^2 + \frac{dr^2}{V(r)},
\end{align*}$$

where $V(r)$ is given in (44). So long as the period $\Delta \tau$ around the compact direction is chosen appropriately, the coordinate singularity at $r = \tilde{r}_+$ is harmless. Let us ignore for the moment the factors of $V(r)$, and consider just the three-dimensional space $(r, \chi, \Phi)$, that is, consider

$$\begin{align*}
ds^2 &= -r^2 \sin^2 \chi d\Phi^2 + dr^2 + r^2 d\chi^2.
\end{align*}$$

This metric describes a portion of three-dimensional Minkowski space. In fact, if we make the coordinate transformations

$$\begin{align*}
z &= r \sin \chi \cosh \Phi, \\
t &= r \sin \chi \sinh \Phi, \\
y &= r \cos \chi
\end{align*}$$

the metric (48) becomes

$$\begin{align*}
ds^2 &= -dt^2 + dz^2 + dy^2,
\end{align*}$$

the usual metric on Minkowski space. Because $\chi$ is restricted to $0 \leq \chi \leq \pi$, the original coordinates only cover the part $z \geq 0$ of the Minkowski space; however there is no obstruction to extending the solution to the whole of Minkowski space. We could also set

$$\begin{align*}
z &= R \sin \psi, \\
y &= R \cos \psi,
\end{align*}$$

and rewrite this metric as

$$\begin{align*}
ds^2 &= -dt^2 + dR^2 + R^2 d\psi^2.
\end{align*}$$
Now let us consider the effect of the factors of $V(r)$ in (47). The metric (47) approaches the product manifold $\text{Mink}^{(2,1)} \times S^1$ asymptotically, but the coordinate $r$ is now restricted to $\tilde{r}_+ \leq r \leq \infty$. We can still make the same coordinate transformations, (49, 51). The resulting metric approaches $\text{Mink}^{(2,1)} \times S^1$ asymptotically, but the restriction $r \geq \tilde{r}_+$ implies $R^2 - t^2 \geq \tilde{r}_+^2$. Note that it is not possible to continue the metric beyond $r = \tilde{r}_+$, as the radius of the circle direction vanishes there.

The physical interpretation is exactly the same as in [11]: if we assume the radius of the circle direction is relatively small, putative observers living in this space who didn’t go too close to $r = \tilde{r}_+$ would think (47) described three-dimensional Minkowski space with the interior of the hyperboloid $R^2 - t^2 = \tilde{r}_+^2$ omitted. The type I instanton should be interpreted as representing tunnelling from the vacuum (52) cross a circle to (47); that is, it describes the decay of a three-dimensional Kaluza-Klein vacuum.

### 4 Reissner-Nordström de Sitter instantons

Since the only special cases of the cosmological $C$ metrics for which the metric is regular reduce to the Reissner-Nordström de Sitter metrics, consideration of the pair creation of charged black holes in a background with a positive cosmological constant reduces to a consideration of the non-singular instantons that can be constructed from the Reissner-Nordström de Sitter metric. This question has of course been studied before, notably in [6, 12], and in the uncharged case in [4], but we hope to present a unified picture which makes the relations between the instantons clear.

The Reissner-Nordström de Sitter metric is

$$ds^2 = -V(r) dt^2 + \frac{dr^2}{V(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

where

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3} \Lambda r^2.$$  

(54)

The gauge field is

$$F = -\frac{Q}{r^2} dt \wedge dr$$

(55)

for an electrically-charged solution, and

$$F = Q \sin \theta \, d\theta \wedge d\phi$$

(56)
for a magnetically-charged solution. For the sake of simplicity, we will not consider dyonic solutions. This solution has three independent parameters, the ‘mass’ $M$, charge $Q$ and cosmological constant $\Lambda$, which we will assume are all positive. There are then four roots of $V(r)$, which we designate by $r_1, r_2, r_3, r_4$ in ascending order. The first root is negative, and therefore has no physical significance. The remaining roots are interpreted as various horizons: $r = r_2$ is the inner (Cauchy) black hole horizon, $r = r_3$ is the outer (Killing) black hole horizon, and $r = r_4$ is the cosmological (acceleration) horizon. In the Lorentzian section, $0 \leq r < \infty$.

We want to construct instantons from this metric by analytically continuing $t \to i\tau$. To obtain a positive-definite metric, we must restrict $r$ to $r_3 \leq r \leq r_4$. There is then potentially a conical singularity at $r = r_3$ and at $r = r_4$. However, if $r_3 = r_2$, the range of $r$ in the Euclidean section will be $r_3 < r \leq r_4$, as the double root in $V(r)$ implies that the proper distance from any other point to $r = r_3$ along spacelike directions is infinite. In this case, we may obtain a regular instanton by identifying $\tau$ periodically with period $2\pi/\kappa_4$, where $\kappa_4$ is the surface gravity of the horizon $r = r_4$. This instanton will be referred to as the cold instanton, following [6].

If we do not have $r_2 = r_3$, then we must have

$$\kappa_3 = \kappa_4, \quad (57)$$

and identify $\tau$ with the same period. There are two ways to satisfy this condition; one is $r_3 = r_4$, which gives an instanton analogous to that constructed out of the Nariai metric in [4], which we shall refer to as the charged Nariai instanton. This is very similar to the way in which the Type I instanton is obtained. The other is to set $Q = M$, which implies (57) [12]; we shall refer to this as the lukewarm instanton, following [6]. There is also a special case, when $r_2 = r_3 = r_4$, which we refer to as the ultracold instanton, again following [6].

The form of all these instantons in terms of the metric (53) has been given in detail in [6]; we will briefly summarise that discussion here. If there is a double root $\rho$ of $V(r)$, we can write

$$V_d(r) = \left(1 - \frac{\rho}{r}\right)^2 \left(1 - \frac{1}{3} \Lambda \rho^2 + 2\rho r + 3\rho^2 \right), \quad (58)$$

and the mass and charge are thus given by

$$M = \rho \left(1 - \frac{2}{3} \Lambda \rho^2 \right), \quad (59)$$
\[ Q^2 = \rho^2(1 - \Lambda \rho^2). \]  

(60)

For positive \( \Lambda \), the double root \( \rho \) must lie in \( 0 < \rho < \Lambda^{-1/2} \). The other positive root of \( V_d(r) \) is

\[ b = \sqrt{3\Lambda^{-1} - 2\rho^2 - \rho}. \]  

(61)

For \( 0 < \rho^2 < \Lambda^{-1/2} \), \( b > \rho \), so \( r_2 = r_3 = \rho \) : this solution gives the cold instanton. For \( \Lambda^{-1/2} < \rho^2 < \Lambda^{-1} \), \( b < \rho \), so \( r_3 = r_4 = \rho \) : this solution gives the charged Nariai instanton. If \( \rho^2 = \Lambda^{-1/2} \), then \( b = \rho = r_2 = r_3 = r_4 \) : this solution gives the ultracold instanton. The function \( V_d(r) \) can also be rewritten as

\[ V_d(r) = \frac{-r^2}{(b^2 + 2\rho b + 3\rho^2)} \left( 1 - \frac{\rho}{r} \right)^2 \left( 1 - \frac{b}{r} \right) \left( 1 + \frac{2\rho + b}{r} \right), \]  

(62)

and we could also write \( M, Q \) and \( \Lambda \) as functions of \( \rho \) and \( b \) (see [6] for details).

If \( V(r) \) does not have a double root, then it must have two roots \( r_3 \) and \( r_4 \) such that

\[ \kappa_3 = \frac{1}{2} |V'(r_3)| = \kappa_4 = \frac{1}{2} |V'(r_4)|. \]  

(63)

This fixes \( V(r) \) to have the form

\[ V_l(r) = \left( 1 - \frac{r_3r_4}{(r_3 + r_4)r} \right)^2 - \frac{r^2}{(r_3 + r_4)^2}, \]  

(64)

from which one sees immediately that \( Q = M \). We will now comment briefly on the nature of each of these instantons in turn.

For the lukewarm instanton, the topology of the Euclidean section is \( S^2 \times S^2 \). The common temperature of the two horizons is

\[ T = \frac{1}{2\pi} \sqrt{\frac{\Lambda}{3} \left( 1 - 4M \sqrt{\frac{\Lambda}{3}} \right)}. \]  

(65)

The Lorentzian section describes two black holes in de Sitter space, so this instanton represents pair creation of non-extreme black holes in thermal equilibrium with the cosmological acceleration radiation. For the cold instantons, the horizon at \( r = r_3 \) is at infinite distance, so the Euclidean section has topology \( S^2 \times R^2 \). There is a boundary \( B^\infty \) at the internal infinity \( r = r_3 = \rho \). In the calculation of the action, we will take the boundary...
to lie at \( r = \rho + \epsilon \), and then take the limit as \( \epsilon \to 0 \), that is, as the boundary approaches \( B^\infty \). The relation between the charge and mass is given parametrically by (59,60), and is displayed on Figure 1. The temperature of the horizon at \( r = r_4 \) is

\[
T = \frac{b}{2\pi (b^2 + 2\rho b + 3\rho^2)} \left( 1 - \frac{\rho}{b} \right)^2 \left( 1 + \frac{\rho}{b} \right). \tag{66}
\]

The Lorentzian section describes two extreme black holes in de Sitter space, so this instanton represents pair creation of extreme black holes in thermal equilibrium with the acceleration radiation. For the charged Nariai instantons, where \( r_3 = r_4 \), it is necessary to make a coordinate transformation and rewrite the metric as

\[
ds^2 = \frac{1}{A} (d\chi^2 + \sin^2 \chi d\psi^2) + \frac{1}{B} (d\theta^2 + \sin^2 \theta d\phi^2), \tag{67}
\]

where \( A \) and \( B \) are constants, with \( B > A \), \( \chi \) and \( \theta \) run from 0 to \( \pi \), and \( \psi \) and \( \phi \) are periodic coordinates with period \( 2\pi \). The gauge field becomes

\[
F = Q \sin \theta d\theta \wedge d\phi \tag{68}
\]

in the magnetic case, and

\[
F = -iQ \frac{B}{A} \sin \chi d\chi \wedge d\psi \tag{69}
\]

in the electric case. The cosmological constant is \( \Lambda = \frac{1}{4}(A + B) \), while \( \rho^2 = 1/B \), and \( M \) and \( Q \) are given by (59,60). This instanton has topology \( S^2 \times S^2 \); indeed, it is just the direct product of two round two-spheres with different radii. The relation between the mass and charge is still given parametrically by (59,60), and is also displayed on Figure 1. However, although this solution is just a special case of Reissner-Nordström de Sitter, we find that the singularity retreats to infinite proper distance when \( r_3 = r_4 \), and so there is no longer a global event horizon, and the Lorentzian section is just the direct product of two-dimensional de Sitter space and a two-sphere of fixed radius, \( dS^2 \times S^2 \). Thus the instanton doesn’t represent pair creation of black holes. However, as in \( [3] \), higher-order quantum corrections will break the degeneracy of the two roots, so the charged Nariai solution will revert to an ordinary Reissner-Nordström de Sitter spacetime once these effects are included.
Figure 1: The values of $Q$ and $M$ for which instantons can be obtained in the cosmological case. The plot is of the dimensionless quantities $Q\sqrt{\Lambda}$ vs. $M\sqrt{\Lambda}$. The curve DU represents the cold solutions, DC represents the lukewarm solutions, and NU represents the charged Nariai solutions. The point at D is de Sitter space, while U is the ultracold case, and N is the Nariai solution.
The ultracold case deserves a more detailed description. In this case \( r_2 = r_3 = r_4 \), which may be regarded as the limit \( b = \rho \) in the metric (53) with \( V(r) \) having the double root form (62). In this case, the mass and charge are given by (59,60) with \( \rho = 1/\sqrt{2\Lambda} \), that is,

\[
M = \frac{2}{3\sqrt{2\Lambda}}, \quad Q^2 = \frac{1}{4\Lambda}.
\] (70)

Suppose \( \rho = 1/\sqrt{2\Lambda} - \epsilon \), and \( b = 1/\sqrt{2\Lambda} + \epsilon \), and consider the limit \( \epsilon \to 0 \). We can construct two different metrics in this limit. First, define a new coordinate \( R \) by

\[
r = 1/\sqrt{2\Lambda} + \epsilon \cos \left( \sqrt{\frac{4\epsilon(2\Lambda)^{3/2}}{3}} R \right),
\] (71)

and take

\[
\psi = \frac{4(2\Lambda)^{3/2}}{3} \epsilon^2 \tau
\] (72)
in (53), with \( t \to i\tau \). Then

\[
V_d(r) = \frac{2(2\Lambda)^{3/2}}{3} \epsilon^3 \sin^2 \left( \sqrt{\frac{4\epsilon(2\Lambda)^{3/2}}{3}} R \right) \cos \left( \sqrt{\frac{4\epsilon(2\Lambda)^{3/2}}{3}} R \right) + 1],
\] (73)

and the metric in the limit \( \epsilon \to 0 \) is

\[
ds^2 = R^2 d\psi^2 + dR^2 + \frac{1}{2\Lambda} (d\theta^2 + \sin^2 \theta d\phi^2).
\] (74)

Thus this instanton has topology \( S^2 \times R^2 \). In fact, the metric is the direct product of a flat \( R^2 \) and a round two-sphere of radius \( 1/\sqrt{2\Lambda} \). The internal infinity \( B^\infty \) is now at \( R = \infty \). We will take \( R = R_0 \) in the calculations, and then take the limit \( R_0 \to \infty \). The angle \( \psi \) is interpreted as the imaginary time. We could construct the same instanton from (67) in the limit \( A \to 0 \), by taking \( \chi = \sqrt{A} R \). The gauge field is

\[
F = \frac{1}{2\sqrt{\Lambda}} \sin \theta d\theta \wedge d\phi
\] (75)
in the magnetic case, and becomes

\[
F = -i\sqrt{\Lambda} R dR \wedge d\psi
\] (76)
in the electric case. The Lorentzian metric obtained by taking \( \psi \to i\Psi \) represents \( \text{Mink}^{(1,1)} \times S^2 \) in Rindler coordinates; the horizon at \( R = 0 \) is the Rindler horizon.

Alternatively, we could define \( x \) by

\[
r = \frac{1}{\sqrt{2\Lambda}} + \sqrt{\frac{2(2\Lambda)^{3/2}}{3}} \epsilon^{3/2} \tau,
\]

and take

\[
\gamma = \sqrt{\frac{2(2\Lambda)^{3/2}}{3}} \epsilon^{3/2} \tau.
\]

Then

\[
V_d(r) = \frac{2\epsilon^3(2\Lambda)^{3/2}}{3} \left( 1 + \sqrt{\frac{2(2\Lambda)^{3/2}}{3}} \epsilon^{1/2} x \right)^2 \left( 1 - \sqrt{\frac{2(2\Lambda)^{3/2}}{3}} \epsilon^{1/2} x \right),
\]

and the metric in the limit \( \epsilon \to 0 \) is

\[
ds^2 = d\gamma^2 + dx^2 + \frac{1}{2\Lambda} (d\theta^2 + \sin^2 \theta d\phi^2).
\]

This instanton also has topology \( S^2 \times R^2 \), but the internal infinity \( B^\infty \) now has two components, \( x = \pm \infty \). We will evaluate the action for a region bounded by \( x = \pm x_0 \), and then take \( x_0 \to \infty \). It is \( \gamma \) that is interpreted as the analogue of imaginary time, and \( \gamma \) runs from \( -\infty < \gamma < \infty \), so this looks just like flat space. The gauge field in this case is

\[
F = \frac{1}{2\sqrt{\Lambda}} \sin \theta d\theta \wedge d\phi
\]

in the magnetic case, and

\[
F = i\sqrt{\Lambda} d\gamma \wedge dx
\]

in the electric case. This solution describes the neighbourhood of a point, when both horizons have receded to infinity. That is, the Lorentzian section is just \( \text{Mink}^{(1,1)} \times S^2 \) in the usual coordinates.

In summary: for sufficiently small mass, there are two solutions, the lukewarm and cold solutions, which correspond to pair creation of non-extreme and extreme black holes respectively. At given mass, the cold solution has higher charge than the lukewarm solution. Once the mass
reaches $M = 1/(3\sqrt{\Lambda})$, there is a third solution, the charged Nariai solution, which has lower charge than the other two. When the mass reaches $M = 3/(4\sqrt{3\Lambda})$, the lukewarm and charged Nariai solutions coincide, and there is no lukewarm solution with higher mass. The cold and charged Nariai solutions coincide, in the ultracold solution, when the mass reaches $M = 2/(3\sqrt{2\Lambda})$, and there are no regular solutions where the mass is larger than this. The ratio of charge to mass is also at its largest at this point, where $Q/M = 3/(2\sqrt{2})$. There is an interesting analogy between the situation here and the Ernst instantons; with the Ernst solution, there are also three ways in which a non-singular instanton can be achieved. These are the non-extreme (or type II) instanton, the extreme instanton, and the type I instanton [1, 2]. The lukewarm solution may be thought of as the analogue of the non-extreme instanton, the cold solution as the analogue of the extreme instanton, and the charged Nariai solution as the analogue of the Type I instanton. The plots of mass versus charge for the Reissner-Nordström de Sitter and Ernst instantons are given in Figure 1 and Figure 2 respectively. While the numerical values are not the same, the qualitative features of these two plots are strikingly similar.
5 Pair creation rate and entropy

In the previous section, we presented the Euclidean solutions which provide the instantons for pair creation of charged black holes in a cosmological background. We will now review their interpretation as instantons, and use them to obtain approximate rates for these processes.

The pair creation of non-extreme charged black holes in a cosmological background is described by propagation from nothing to a surface $\Sigma$ with topology $S^2 \times S^1$, since such a surface may be thought of as a Wheeler wormhole attached to an $S^3$, and $S^3$ is the topology of the spatial sections of de Sitter space. The pair creation of extreme charged black holes is similarly described by propagation to a surface with topology $S^2 \times R^1$. In this case there is also a boundary component $B^\infty$, which represents an “internal infinity”. We will use $\Sigma$ to denote the whole boundary in this case, and $\Sigma_s$ to denote the part with topology $S^2 \times R^1$. The amplitude for these processes will, at least formally, be given by a path integral;

$$\Psi = \int d[g] d[A] e^{-I}. \quad (83)$$

The integral is over all metrics and gauge fields which agree with the given boundary data on $\Sigma$, and $\Psi$ may thus be thought of as a functional of the boundary data. We assume that, if there is a Euclidean classical solution which interpolates within the given boundary, then the integral is dominated by the contribution from it. That is, if there is an appropriate instanton, $\Psi$ will be approximately

$$\Psi \approx e^{-I}, \quad (84)$$

where $I$ is the action of this instanton. The partition function, and thus the pair creation rate, is given by the square of this amplitude.

A spatial section of the lukewarm solution has topology $S^2 \times S^1$, so half of the Euclidean section provides an instanton for the pair creation of non-extreme charged black holes. We have to use half of the solution, as we want the extrinsic curvature of $\Sigma$ to vanish, so that $\Sigma$ can be interpreted as the zero-momentum initial data for the Lorentzian extension. Similarly, half of the cold solution can be used as an instanton to describe the pair creation of extreme charged black holes. Note that these instantons only exist when the data on $\Sigma$ specified in the path integral agrees with the data induced by the solutions.

The whole of the Euclidean section in each case provides a “bounce” solution. In asymptotically-flat situations, one can deal with the bounces
rather than the instantons, but in a cosmological situation this is no longer possible, as the boundary data on the surface Σ provide crucial information. We will therefore be interested in the calculation of the action for the instantons.

We also need to consider what action we should use in the calculation of (84). We want to use the action for which it is natural to fix the boundary data on Σ specified in the path integral (83). That is, we want to use an action whose variation gives the Euclidean equations of motion when the variation fixes these boundary data on Σ. If we consider the action (1), we can see that its variation will be

$$\delta I = \text{(terms giving the equations of motion)} + \text{(gravitational boundary terms)}$$

$$+ \frac{1}{4\pi} \int_{\Sigma} d^3 x \sqrt{h} F^{\mu\nu} n_\mu \delta A_\nu,$$

where $n_\mu$ is the normal to Σ and $h_{ij}$ is the induced metric on Σ (see [8] for a more detailed discussion of the gravitational boundary terms). Thus, the variation of (1) will only give the equations of motion if the variation is at fixed gauge potential on the boundary, $A_i$. Note that it is not necessary to fix the component $A_\mu n^\mu$ normal to the boundary.

For the magnetic Reissner-Nordström solutions, fixing the gauge potential fixes the charge on each of the black holes, as the magnetic charge is just given by the integral of $F_{ij}$ over a two-sphere lying in the boundary. However, in the electric case, fixing the gauge potential $A_i$ can be regarded as fixing a chemical potential $\omega$ which is conjugate to the charge [7]. Holding the charge fixed in the electric case is equivalent to fixing $n_\mu F^{\mu i}$ on the boundary, as the electric charge is given by the integral of the dual of $F$ over a two-sphere lying in the boundary. Therefore, the appropriate action is

$$I_{el} = I - \frac{1}{4\pi} \int_{\Sigma} d^3 x \sqrt{h} F^{\mu\nu} n_\mu A_\nu,$$

as its variation is

$$\delta I_{el} = \text{(terms giving the equations of motion)} + \text{(gravitational boundary terms)}$$

$$- \frac{1}{4\pi} \int_{\Sigma} d^3 x \delta(\sqrt{h} F^{\mu\nu} n_\mu) A_\nu,$$

and so it gives the equations of motion when $\sqrt{h} n_\mu F^{\mu i}$, and thus the electric charge, is held fixed.
Let us consider the pair creation where the electric and magnetic charge are held fixed on $\Sigma$. Then the appropriate action will be (1) in the magnetic case, and (86) in the electric case. It is also worth pointing out that, since we identify $\Sigma$ ($\Sigma_s$ in the cold case) with a surface of zero extrinsic curvature in the Euclidean section, the gravitational boundary term in the action (1) will make no contribution to the action. Thus the action (1) for the instanton will just be half that of the whole Euclidean section.

We will now consider each of the instantons derived in section 4, and calculate the actions (1) in the magnetic case and (86) in the electric case. For the lukewarm solution we have, in the magnetic case,

$$F^2 = \frac{2Q^2}{r^4},$$

thus the action (1) is

$$I_L = -\frac{\Lambda V^{(4)}}{8\pi} + \frac{1}{16\pi} \int d^4x \sqrt{-g} F^2$$

$$= -\beta \Lambda \frac{r_3^3 - r_3^3}{12} + \frac{Q^2}{4} \beta \left( \frac{1}{r_3} - \frac{1}{r_4} \right) = -\frac{3\pi}{2\Lambda} + \pi M \sqrt{\frac{3}{\Lambda}},$$

(88)

where $V^{(4)}$ is the four-volume of the instanton, and $\beta$ is the period of $\tau$. In the electric case, $F^2 = -2Q^2/r^4$, and we find that (1) gives

$$I_L^E = -\beta \Lambda \frac{r_4^3 - r_3^3}{12} + \frac{Q^2}{4} \beta \left( \frac{1}{r_3} - \frac{1}{r_4} \right) = -\frac{3\pi}{2\Lambda}.$$

(89)

To calculate the additional boundary term in (86), we have to pick a gauge for the Maxwell field. To obtain a unique result, we have to constrain the gauge choice to be regular at both horizons. A suitable gauge choice for the lukewarm solution is

$$A = -i\frac{Q}{r^2} \tau dr.$$  

(90)

It might seem that this gauge choice involves a discontinuity at the horizons, but in fact it does not. To consider whether there is a discontinuity at the horizon, we should look at the gauge potential in orthonormal coordinates. An orthonormal frame for the metric (53) is

$$e_0 = V(r)^{1/2} dt, e_1 = V(r)^{-1/2} dr, e_2 = r d\theta, e_3 = r \sin \theta d\phi,$$

(91)

and the gauge potential (90) is

$$A = -i V(r)^{1/2} \frac{Q}{r^2} \tau e_1,$$

(92)

which vanishes at $r = r_3$ and $r = r_4$. 

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To evaluate the additional boundary term in (86), we take a coordinate system such that the boundary is the surface \( \tau = 0, \beta/2 \) in the Euclidean section, and we take the integral in the \( r \) direction on the boundary to run from the black hole horizon to the acceleration horizon along \( \tau = 0 \), and back along \( \tau = \beta/2 \). The additional term is

\[
\frac{1}{4\pi} \int_\Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu = -\frac{Q^2}{2} \beta \left( \frac{1}{r_3} - \frac{1}{r_4} \right) = -\pi M \sqrt{\frac{3}{\Lambda}} \tag{93}
\]

and thus the action (86) in the electric case is

\[
I^E_{el} = I^E_0 - \frac{1}{4\pi} \int_\Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu = -\frac{3\pi}{2\Lambda} + \pi M \sqrt{\frac{3}{\Lambda}}. \tag{94}
\]

The relevant action is thus the same for the electric and magnetic lukewarm instantons. It lies in the range \(-3\pi/2\Lambda \leq I^E \leq -3\pi/4\Lambda\), as \( M < \sqrt{3/\Lambda}/4 \).

For the cold solution, \( F^2 = 2Q^2/r^4 \) in the magnetic solution, and the action (1) is

\[
I^C = \beta \Lambda \frac{b^3 - \rho^3}{12} + \frac{Q^2}{4} \beta \left( \frac{1}{\rho} - \frac{1}{b} \right) = -\frac{\pi}{2} b^2. \tag{95}
\]

There is also an extrinsic curvature boundary term at \( B^\infty \), but this vanishes. In the electric case, \( F^2 = -2Q^2/r^4 \), so the action (1) is

\[
I^E_C = \beta \Lambda \frac{b^3 - \rho^3}{12} - \frac{Q^2}{4} \beta \left( \frac{1}{\rho} - \frac{1}{b} \right) = -\frac{\pi}{2} b^2 - \frac{Q^2}{2} \beta \left( \frac{1}{\rho} - \frac{1}{b} \right). \tag{96}
\]

A suitable gauge potential, which is regular everywhere on the instanton, is

\[
A = -iQ \left( \frac{1}{r} - \frac{1}{b} \right) d\tau. \tag{97}
\]

The integral over \( \Sigma \) now consists of two parts; there is an integral over the \( S^2 \times R^1 \) factor, which we take to be from \( r = \rho + \epsilon \) to the acceleration horizon at \( r = b \) along \( \tau = 0 \), and back along \( \tau = \beta/2 \), and an integral over the internal infinity \( r = \rho + \epsilon \), which is in the direction of decreasing \( \tau \). The additional boundary term in (86) is

\[
\frac{1}{4\pi} \int_\Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu = -\frac{Q^2}{2} \beta \left( \frac{1}{\rho} - \frac{1}{b} \right), \tag{98}
\]

and thus (86) is

\[
I^C_{el} = -\frac{\pi}{2} b^2. \tag{99}
\]
Again, the action is the same in the electric and magnetic cases. It lies in the range $-3\pi/2\Lambda \leq I^C \leq -\pi/4\Lambda$.

The actions in the charged Nariai case have already been computed in \([7]\). In the magnetic case, $F^2 = 2Q^2/B^2$, and the action (100) is

$$I^{CN} = -\frac{\pi}{B}. \quad (100)$$

In the electric case, $F^2 = -2Q^2/B^2$, so the action (101) is

$$I^{CN}_E = -\frac{\pi}{A}. \quad (101)$$

A suitable gauge potential is

$$A = i\frac{QB}{A} \sin(\chi) \psi d\chi. \quad (102)$$

We take the boundary to be the surface $\psi = 0, \psi = \pi$, and integrate from the black hole horizon ($\chi = \pi$) to the acceleration horizon ($\chi = 0$) along $\psi = 0$, and back along $\psi = \pi$, so the additional boundary term in (103) is

$$\frac{1}{4\pi} \int \Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu n_\nu = -\frac{1}{4\pi} \frac{Q^2B}{A} \int \psi \sin \chi \sin \theta d\chi d\theta d\phi = -2\pi Q^2 \frac{B}{A},$$

and thus (103) is

$$I^{CN}_E = -\frac{\pi}{A} + 2\pi Q^2 \frac{B}{A} = -\frac{\pi}{B}. \quad (104)$$

The relevant action is the same in the electric and magnetic cases, and it lies in the range $-\pi/\Lambda \leq I^{CN} \leq -\pi/2\Lambda$.

For both metrics which can be constructed in the ultracold case, $F^2 = 2\Lambda$ in the magnetic solution, so the volume contribution to the action (100) vanishes. Let us consider first the metric (74). Then

$$I^{UC1} = -\frac{1}{8\pi} \int_{B^\infty} \sqrt{h} K = -\frac{\pi}{4\Lambda}. \quad (105)$$

In the electric solution, $F^2 = -2\Lambda$, so (101) gives

$$I^{UC1}_E = -\Lambda V^{(4)} \frac{1}{4\pi} + \frac{1}{8\pi} \int_{B^\infty} \sqrt{h} K = -\pi R_0^2/4 - \frac{\pi}{4\Lambda} \quad (106)$$

(the boundary $B^\infty$ in this case is the surface $R = R_0$). One could take the electric gauge potential to be

$$A = -\frac{i}{2} \sqrt{\Lambda} R^2 d\psi. \quad (107)$$
We define $\Sigma$ to be the surfaces $\psi = 0, \psi = \pi$, together with the semi-circle at $R = R_0$ lying between them, and take the integral around the boundary to be from $R = R_0$ to $R = 0$ along $\psi = 0$, back along $\psi = \pi$, and around $R = R_0$ in the direction of decreasing $\psi$. The additional boundary term in (86) is

$$\frac{1}{4\pi} \int_{\Sigma} d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu = -\pi R_0^2/4,$$

so (86) is

$$\int_{el}^{UC1} \approx -\frac{\pi}{4\Lambda}. \tag{109}$$

This action agrees with the limit of the action of the cold solution as it approaches ultracold.

If we consider instead the metric (80), the action vanishes in the magnetic case, $I^{UC2} = 0$, as the extrinsic curvature surface term at $x = \pm x_0$ vanishes as well. In the electric case, $F^2 = -2\Lambda$, so if we consider the action for the region between two surfaces $\gamma = \pm \gamma_0$, (1) gives

$$I_{el}^{UC2} = -\frac{\Lambda V^{(4)}}{4\pi} = -2x_0\gamma_0. \tag{110}$$

One could take the electric gauge potential to be

$$A = -i\sqrt{\Lambda} x d\gamma. \tag{111}$$

The additional boundary term in (86) is then

$$\frac{1}{4\pi} \int_{\Sigma} d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu = -2x_0\gamma_0,$$

so (86) vanishes as well, $I_{el}^{UC2} = 0$.

One can use the action of an instanton to approximate the wavefunction for the propagation to some final surface, so the action gives the approximate amplitude at which this process occurs. One can square this to get the rate. The actions we have just calculated for the cold and lukewarm instantons give the rate for pair creation of black holes in a cosmological background by these instantons. The pair creation rate is approximately

$$\Gamma = \Psi^2 \approx e^{-2I}, \tag{113}$$

in each case, where $I$ is the relevant action. Since de Sitter space has action $I_{de Sitter} = -3\pi/2\Lambda$, which is the lower bound of the action for the cold and
Figure 3: The action for the various instantons in the cosmological case. The action as a fraction of the action for de Sitter space, $I/I_{\text{de Sitter}}$, is plotted against the dimensionless mass $M\sqrt{\lambda}$. The curve DU1 represents the cold solutions, DC represents the lukewarm solutions, and NU represents the charged Nariai solutions. The point at D is de Sitter space, N is the Nariai solution, and U1 and U2 represent the actions of the first and second type of ultracold solutions. Note that U does not correspond to one of the ultracold solutions.
lukewarm instantons, the rate at which black hole pairs are created relative to the rate at which de Sitter space is itself created is less than one. That is, the pair creation of black holes is suppressed. The situation is illustrated in Figure 3.

Since the instantons are all cosmological solutions, there is no asymptotic region in the Euclidean section (that is, there is no ‘point at infinity’). This can be interpreted as meaning that these solutions are closed systems, and thus necessarily have fixed energy. They should therefore be interpreted as a contribution to the microcanonical ensemble, as this is the thermodynamical ensemble at fixed energy. The partition function $Z = \Psi^2$ should therefore be interpreted as the density of states, and thus the entropy will be just the ln of this partition function, $S = \ln Z$. The contribution to the entropy from the instantons is thus just

$$S = -2I. \tag{114}$$

As we might expect, the entropy turns out to be just a quarter of the total area of the horizons which appear in the instanton (which we denote by $A$). In the lukewarm case, there are two horizons, at $r = r_3$ and $r = r_4$, so $A/4 = \pi r_3^2 + \pi r_4^2$, but

$$r_{3,4} = \frac{1}{2} \left[ \frac{3}{\Lambda} \pm \sqrt{\frac{3}{\Lambda} - 4M \frac{3}{\Lambda}} \right], \tag{115}$$

so

$$A/4 = 3\pi \frac{1}{\Lambda} - 2\pi M \sqrt{\frac{3}{\Lambda}}, \tag{116}$$

and thus

$$S^L = -2I^L = A/4. \tag{117}$$

In the cold case, only the acceleration horizon at $r = b$ is part of the instanton, and it has area $A = 4\pi b^2$, so

$$S^C = -2I^C = \pi b^2 = A/4. \tag{118}$$

In the charged Nariai case, there are again two horizons, which both have area $4\pi/B$. Thus $A/4 = 2\pi/B$, and

$$S^{CN} = -2I^{CN} = 2\pi/B = A/4. \tag{119}$$
For the first type of ultracold solution, the surface $R = 0$ is interpreted as a Rindler horizon, which has area $A = 2\pi/\Lambda$. Thus,

$$S^{UC1} = -2I^{UC1} = \pi/2\Lambda = A/4.$$ (120)

For the other type of ultracold solution, there are no horizons, and the entropy vanishes, $S^{UC2} = -2I^{UC2} = 0$, as we expect. Thus we see that horizons contribute to the gravitational entropy only if they are in the instanton; in particular, extreme black hole horizons make no contribution to the entropy, even if they have non-zero area, as discovered in [3].

Thus, the usual relation between the entropy and the area of the horizons extends to all these solutions. Just as for the Ernst instantons, the pair creation of extreme black holes is suppressed relative to the pair creation of non-extreme black holes by a factor of $e^{S_{bh}}$, where $S_{bh}$ is the entropy associated with the black hole horizon.

6 Conclusions

The pair creation of charged black holes by a strong electromagnetic field has been a subject of considerable recent interest. We have seen that charged black holes can also be pair created in a background with a positive cosmological constant. The nature of the instantons describing this pair creation is very similar to that of the instantons describing pair creation in an electromagnetic field. There is a non-extreme and an extreme instanton. As in the electromagnetic case, the pair creation of extreme black holes is suppressed relative to that of non-extreme black holes by a factor of $e^{S_{bh}}$, where $S_{bh}$ is the entropy associated with the black hole horizon. This is further evidence that $e^{S_{bh}}$ should be regarded as the number of internal states of the black holes.

The pair creation rate obtained from the instantons is in fact just $e^{S}$, where $S$ is the total gravitational entropy of the instanton, that is, a quarter the area of the black hole and the cosmological horizons. Seen from this point of view, black hole pair creation in de Sitter space is suppressed simply because de Sitter space has a higher entropy; that is, the single horizon of de Sitter space has an area larger than the combined area of the horizons in the instantons. This is also similar to the electromagnetic case, where the pair creation rate was $e^{\Delta A/4 + A_{bh}/4}$ in the non-extreme case, with the suppression being due to the fact that the difference in acceleration horizon area $\Delta A$ was negative.
Because the conical singularities in the charged $C$ metric with a cosmological constant can only be eliminated when it reduces to the Reissner-Nordström de Sitter metric, we are fairly confident that the instantons we have described in this paper are the only ones which can be interpreted as representing pair creation of charged black holes in a cosmological background. It might be interesting to see if these solutions could be extended to dilaton gravity with some kind of effective cosmological constant, as the presence of the dilaton field might allow some more possibilities.

We have also observed that the Type I instanton discovered in [1], where the conical singularities in the ordinary charged $C$ metric are eliminated by allowing $\xi_2$ and $\xi_3$ to coincide, is in fact the Reissner-Nordström metric.

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