The homotopy groups of the automorphism groups of Cuntz-Toeplitz algebras

Taro Sogabe
Graduate School of Science Kyoto University Sakyo-ku, Kyoto 606-8502, Japan
staro@math.kyoto-u.ac.jp

September 17, 2019

Abstract

The Cuntz-Toeplitz algebra $E_{n+1}$ for $n \geq 1$ is the universal C*-algebra generated by $n+1$ isometries with mutually orthogonal ranges. In this paper, we determine the homotopy groups of the automorphism group of $E_{n+1}$.

1 Introduction

The Cuntz-Toeplitz algebra $E_{n+1}$ for $n \geq 1$ is the universal C*-algebra generated by $n+1$ isometries with mutually orthogonal ranges. In this paper, we investigate the automorphism groups of the Cuntz-Toeplitz algebras and determine their homotopy groups.

The homotopy groups of the automorphism groups are necessary to classify the locally trivial continuous fields of C*-algebras. However, there are only few classes of C*-algebras whose homotopy groups of the automorphism groups are determined. To the best knowledge of the author, the homotopy groups are known only for Kirchberg algebras [6, 12, 4], strongly self-absorbing C*-algebras [9], and simple AF-algebras [22, 30]. The rough strategy of computation of the homotopy groups in the previous work is as follows. First, we show the weak homotopy equivalence between the automorphism group and the endomorphism semi-group. Then we compute the homotopy groups of the endomorphism semi-group from the K-theoretic or KK-theoretic data of the C*-algebra.

We illustrate the strategy in the case of Kirchberg algebras where we have a powerful tool, Kirchberg-Phillips’ classification theorem. Regarding a continuous map $f: X \to \text{End}_E$, we can associate a KK-class $[f]$ to $f$. Therefore the homotopical data of $\text{End}_E$ are recovered from the KK-theoretic data, and we can directly compute the general homotopy sets by the map $[X, \text{Aut}_E] \to [K(X) \otimes A]$ (see [6, Proposition 5.8, Theorem 4.6]).

In general, there is no such powerful tool and the homotopy groups are computed for only exceptional classes of C*-algebras. The strongly self-absorbing C*-algebras are such examples. Dadarlat and Pennig show in [9, Theorem 2.3] that the automorphism group $\text{Aut}_D$ is contractible for every unital strongly self-absorbing C*-algebra $D$. Using a certain fibration, they determine the homotopy groups of $\text{Aut}(D \otimes \mathbb{K})$. In this paper, we use similar fibrations in the case mentioned above.

Let $\{T_i\}_{i=1}^{n+1}$ be the canonical generators of $E_{n+1}$ and let $\text{End}_0 E_{n+1}$ be the path component of $\text{id}_{E_{n+1}}$ of the semi-group of unital endomorphisms of $E_{n+1}$. We denote by $e$ the minimal projection $1 - \sum_{i=1}^{n+1} T_i T_i^*$. Then the map $\text{End}_0 E_{n+1} \ni \rho \mapsto \sum_{i=1}^{n+1} \rho(T_i) T_i^* \in U_{E_{n+1}}(1-e)$ is a homeomorphism where $U_{E_{n+1}}$ is the unitary group of $E_{n+1}$. Our proof of the main result is based on the fact that the map $U_{E_{n+1}} \to U_{E_{n+1}}(1-e)$ defined by the right multiplication by $1-e$ gives a fibration with a fibre $S^1$.

Theorem 1.1. The homotopy groups of $\text{Aut} E_{n+1}$ are as follows:

$$
\pi_1(\text{Aut} E_{n+1}) = \mathbb{Z}_n, \quad \pi_{2k+1}(\text{Aut} E_{n+1}) = \mathbb{Z}, \quad \pi_{2k}(\text{Aut} E_{n+1}) = 0, \quad k \geq 1.
$$

To prove Theorem 1.1, we show that the inclusion map $\text{Aut} E_{n+1} \to \text{End}_0 E_{n+1}$ is a weak homotopy equivalence (Theorem 3.14).

Corollary 1.2. Let $X$ be a CW complex. The following sequence is an exact sequence of pointed sets and the first four terms give an exact sequence of groups:

$$
H^1(X) \to K^1(X) \to [X, \text{Aut} E_{n+1}] \to H^2(X) \to [X, \text{BAut}_e E_{n+1}] \to [X, \text{BAut} E_{n+1}] \to H^3(X).
$$

The group $\text{Aut}_e E_{n+1}$ is the subgroup of all automorphisms that fix the minimal projection $e \in E_{n+1}$.
The original motivation of this work is to investigate the structure of continuous fields of the Cuntz algebras beyond Dadarlat’s work [10] using the Cuntz-Toeplitz extensions, and we will hopefully come back to this subject in the near future. We discuss the group structure of the homotopy sets \([X, \text{Aut } E_{n+1}]\) and \([X, \text{Aut } O_{n+1}]\) in [16] We organize this paper as follows. In section 2, we give some preliminaries to compute the homotopy groups. We introduce several fibrations sequences with the help of [9, Lemma 2.8, 2.16 and Corollary 2.9]. As a consequence, the homotopy groups of the connected component of the endomorphism semi-group, denoted by \(\text{Endo}, E_{n+1}\), are obtained.

In section 3, we show the weak homotopy equivalence of \(\text{Endo}, E_{n+1}\) and \(\text{Aut } E_{n+1}\). The main ingredient of the proof is Pimsner–Popa–Voiculescu’s non-commutative Weyl–von Neumann type theorem.

2 Preliminaries

2.1 Notation and the basic facts of the theory of C*-algebras

Let \(A\) be a unital C*-algebra and let \(U_A\) be the group of unitary elements in \(A\). We denote by \(U_{0A}\) the path component of \(1_A\) of \(U_A\). For a non-unital C*-algebra \(B\), we denote its unitization by \(B^*\). The K-groups of \(A\) are denoted by \(K_i(A), i = 0, 1\). We denote by \([p]\) the class of the projection \(p\) in \(K_0(A)\), and denote by \([u]\) the class of the unitary \(u\) in \(K_1(A)\). Let \(SA\) be the suspension of \(A\), the set of \(A\)-valued functions on \([0, 1]\) that vanish at 0 and 1. For the K-theory, we refer to [1, 17]. We denote by \(\mathbb{K}\) the algebra of compact operators of the infinite dimensional separable Hilbert space \(H\).

For a topological space \(Y\) and two elements \(y_0\) and \(y_1\), we denote \(y_0 \sim_h y_1\) in \(Y\) if there is a continuous path from \(y_0\) to \(y_1\). Two unitaries \(u, v \in U_A\) are homotopy equivalent if \(u \sim_h v \in U_A\). There is a natural map \(U_A/\sim_h \rightarrow K_1(A)\) from the set of homotopy classes of unitaries to the \(K_1\)-group. We say \(A\) is \(K_1\)-injective if the map is injective. For the non-unital C*-algebra \(B\), it is \(K_1\)-injective if the natural map \(U_B/\sim_h \rightarrow K_1(B)\) is injective. For example, the algebra \(A \otimes \mathbb{K}\) is \(K_1\)-injective by the definition of the \(K_1\)-group.

For \(A \otimes \mathbb{K}\), we denote by \(M(A \otimes \mathbb{K})\) the multiplier algebra of \(A \otimes \mathbb{K}\), and denote by \(\mathcal{Q}(A \otimes \mathbb{K})\) its quotient by \(A \otimes \mathbb{K}\). We denote the quotient map by \(\pi\). We remark that \(\mathcal{Q}(A \otimes \mathbb{K}) = K_1\)-injective, (see [21, Section 1.13]). We identify \(\mathcal{M}(\mathbb{K})\) with \(\mathcal{B}(H)\) where \(\mathcal{B}(H)\) is the algebra of the bounded operators on \(H\). For \(A = C(X)\), we denote by \(C_b^*(X, \mathcal{B}(H))\) the set of \(\mathcal{B}(H)\)-valued bounded continuous functions on \(X\) with respect to the strong* operator topology (abbreviated to SOT*). This is a realization of the multiplier algebra \(\mathcal{M}(C(X) \otimes \mathbb{K})\) (see [29, Proposition 2.57]).

We refer to [13, Theorem 1] for the K-theory of the multiplier algebra, and a generalization of Kuiper’s theorem.

**Theorem 2.1.** Let \(A\) be a unital C*-algebra. Then \(U_{\mathcal{M}(A \otimes \mathbb{K})}\) is contractible with respect to the norm topology, and we have \(K_i(\mathcal{M}(A \otimes \mathbb{K})) = 0\), \(i = 1, 2\).

Let \(A, B\) and \(C\) be C*-algebras. An extension \(C\) of \(A\) by \(B \otimes \mathbb{K}\) is an exact sequence

\[
0 \rightarrow B \otimes \mathbb{K} \rightarrow C \rightarrow A \rightarrow 0,
\]

and the Busby invariant of the extension is the induced map \(\tau : A \rightarrow \mathcal{Q}(B \otimes \mathbb{K})\). We refer to [1] for the definition of the Busby invariant. The extension is called trivial if the above exact sequence splits. The extension is called essential if \(\tau\) is injective, and called unital if \(\tau\) is unital. We refer to [1] for the basic facts of the theory of extensions of C*-algebras. There are two equivalence relations of unital extensions, the strong unitary equivalence and the weak unitary equivalence.

**Definition 2.2.** Let \(A\) and \(B\) be C*-algebras. Two Busby invariants \(\tau_i : A \rightarrow \mathcal{Q}(B \otimes \mathbb{K})\), \(i = 1, 2\) are said to be strongly unitarily equivalent if there exists a unitary \(U \in U_{\mathcal{M}(B \otimes \mathbb{K})}\) satisfying \(\tau_1 = Ad_U(U) \circ \tau_2\). They are said to be weakly unitarily equivalent if there exists a unitary \(u \in U_{\mathcal{M}(B \otimes \mathbb{K})}\) with \(\tau_1 = Ad_u \circ \tau_2\). We denote the strong unitary equivalence by \(\sim_{s,u,e}\) and denote the weak unitary equivalence by \(\sim_{w,u,e}\). We denote \(\tau_1 \sim_{s,u,e} \tau_2\) if there exists two trivial extensions \(\rho_1\) and \(\rho_2\) satisfying \(\tau_1 \oplus \rho_1 \sim_{s,u,e} \tau_2 \oplus \rho_2\). We denote by \(\text{Ext}(A, B \otimes \mathbb{K})\) the set of the equivalence classes of the Busby invariants with respect to the equivalence relation \(\sim_{s,u,e}\).

We note that the weak unitary equivalence, \(\sim_{w,u,e}\) induces the equivalence \(\sim_{s}(\text{see [1, Proposition 5.6.4]})\). In this paper, we deal with the extensions of the Cuntz algebras by \(C(X) \otimes \mathbb{K}\). One has a universal coefficient theorem of Ext-groups.

**Theorem 2.3** ([1, Theorem 23.1.1].) Let \(A\) and \(B\) be separable C*-algebras, with \(A\) in the bootstrap class. Then there is an unnaturally splitting short exact sequence

\[
0 \rightarrow \bigoplus_{i=0,1} \text{Hom}(K_i(A), K_i(B)) \rightarrow \text{Ext}(A, B \otimes \mathbb{K}) \rightarrow \bigoplus_{i=0,1} \text{Hom}(K_i(A), K_{i+1}(B)) \rightarrow 0.
\]

If \(\bigoplus_{i=0,1} \text{Hom}(K_i(A), K_{i+1}(B)) = 0\), we have an isomorphism \(\text{Ext}(A, B \otimes \mathbb{K}) \rightarrow \bigoplus_{i=0,1} \text{Ext}_1^*(K_i(A), K_i(B))\) that sends a class of Busby invariant \(\tau\) of an extension \(0 \rightarrow B \otimes \mathbb{K} \rightarrow C_r \rightarrow A \rightarrow 0\) to the class of group extension of the commutative groups \([K_i(B) \rightarrow K_i(C_r) \rightarrow K_i(A)] \in \text{Ext}_1^*(K_i(A), K_i(B))\) for \(i = 0, 1\).
Let $E_{n+1}$ be the universal $C^*$-algebra generated by $n+1$ isometries with mutually orthogonal ranges and let $(T_i)_{i=1}^{n+1}$ be the canonical generators of $E_{n+1}$. It is called the Cuntz-Toeplitz algebra. The closed two-sided ideal generated by the minimal projection $e := 1 - \sum_{i=1}^{n+1} T_i T_i^*$ is isomorphic to the compact operators $K$, which is known to be the only closed non-trivial two-sided ideal. Consider the full Fock space $\mathcal{F}(\mathbb{C}^{n+1})$ and the left creation operators $(T_i)_{i=1}^{n+1}$ (see [26, Section 1]). Then one has $K \subset C^*(\{T_i\}_{i=1}^{n+1}) = E_{n+1} \subset \mathcal{B}(\mathcal{F}(\mathbb{C}^{n+1}))$. In this paper, we frequently identify $K$ with $K + C_1 E_{n+1} \subset \mathcal{B}(\mathcal{F}(\mathbb{C}^{n+1}))$. Let $\pi : E_{n+1} \to \mathcal{O}_{n+1}$ be the quotient map by the ideal $K$, and let $S_i := \pi(T_i)$. The quotient algebra $\mathcal{O}_{n+1}$ is the universal simple $C^*$-algebra generated by $n+1$ isometries with the relation : $S_i^* S_i = \delta_{ij}, \quad 1 = \sum_{i=1}^{n+1} S_i S_i^*$. We denote by $\mathcal{O}_\infty$ the universal $C^*$-algebra generated by the countably infinite isometries with mutually orthogonal ranges. The algebras $\mathcal{O}_{n+1}$ and $\mathcal{O}_\infty$ are called the Cuntz algebras, whose $K$-groups are the following :

$$K_0(\mathcal{O}_{n+1}) = \mathbb{Z}_n, \quad K_1(\mathcal{O}_{n+1}) = 0, \quad K_0(\mathcal{O}_\infty) = \mathbb{Z}, \quad K_1(\mathcal{O}_\infty) = 0.$$ See [4, Theorem 3.7, 3.8, Corollary 3.11].

The Cuntz algebras are the Kirchberg algebras, and they tensorially absorb $\mathcal{O}_\infty$, $\mathcal{O}_{n+1} \otimes \mathcal{O}_\infty \cong \mathcal{O}_{n+1}$. The algebra that tensorially absorbs $\mathcal{O}_\infty$ has $K_1$-injectivity by the lemma below.

**Lemma 2.4** ([25, Lemma 2.1.7]). Let $A$ be a unital $C^*$-algebra. Then the natural map $U_A \otimes \mathcal{O}_\infty / \sim_h \to K_1(A \otimes \mathcal{O}_\infty)$ is bijective. In particular, every unital $C^*$-algebra that tensorially absorbs $\mathcal{O}_\infty$ is $K_1$-injective.

**Definition 2.5.** We denote by $\tau_0$ the Busby invariant of the extension

$$0 \to K \to E_{n+1} \to \mathcal{O}_{n+1} \to 0.$$

The inclusion map $C(X) \otimes K \hookrightarrow C(X) \otimes E_{n+1}$ induces the Busby invariant $\tau = \text{id}_{C(X)} \otimes \tau_0 : C(X) \otimes \mathcal{O}_{n+1} \hookrightarrow Q(C(X) \otimes K)$ of the unital essential extension

$$0 \to C(X) \otimes K \to C(X) \otimes E_{n+1} \xrightarrow{\text{id}_{C(X)} \otimes \pi} C(X) \otimes \mathcal{O}_{n+1} \to 0.$$

Since $K$ and $E_{n+1}$ are KK-equivalent to $\mathbb{C}$ (see [26, Theorem 4.4]) the above exact sequence induces the following 6-term exact sequence :

$$K_0(C(X)) \xrightarrow{-n} K_0(C(X)) \xrightarrow{\rho} K_0(C(X) \otimes \mathcal{O}_{n+1}) \xrightarrow{\text{id}} K_1(C(X)) \xrightarrow{-m} K_1(C(X)).$$

For a pointed topological space $(X, x_0)$, we denote by $\Sigma X$ its reduced suspension with the base point $x_0$. We denote by $C_0(X, x_0)$ the set of continuous functions vanishing at $x_0$. For pointed topological spaces $(X, x_0), (Y, y_0)$, we denote the set of continuous maps from $X$ to $Y$ by $\text{Map}(X, Y)$ and denote the set of base point preserving continuous maps by $\text{Map}_0(X, Y)$. We denote the homotopy set $\text{Map}(X, Y) / \sim_h$ by $[X, Y]$ and denote $\text{Map}_0(X, Y) / \sim_h$ by $[X, Y]_0$. We remark that if $Y$ is an H-space, the natural map $[X, Y]_0 \to [X, Y]$ is bijective.

**Lemma 2.6.** Let $(X, x_0)$ be a based compact Hausdorff space. Then the natural map

$$U(C_0(X, x_0) \otimes \mathcal{O}_{n+1}) / \sim_h \to K_1(C_0(X, x_0) \otimes \mathcal{O}_{n+1})$$

is a surjective isomorphism.

**Proof.** Since $K_1(\mathcal{O}_{n+1}) = 0$, we have $K_1(C_0(X, x_0) \otimes \mathcal{O}_{n+1}) = K_1(C(X) \otimes \mathcal{O}_{n+1})$. By Lemma 2.4, the natural map

$$[X, U_{\mathcal{O}_{n+1}}] = U_{C(C(X) \otimes \mathcal{O}_{n+1})} / \sim_h \to K_1(C(X) \otimes \mathcal{O}_{n+1})$$

is an isomorphism. Since $U_{\mathcal{O}_{n+1}}$ is an H-space, we have

$$U(C_0(X, x_0) \otimes \mathcal{O}_{n+1}) / \sim_h = [X, U_{\mathcal{O}_{n+1}}]_0 = [X, U_{\mathcal{O}_{n+1}}].$$

Therefore we have the conclusion. \qed

**Lemma 2.7** ([2, Proposition 6.6]). Let $A$ be a $C^*$-algebra and let $I$ be a two-sided closed ideal of $A$. If $A/I$ and $I$ are $K_1$-injective and the natural map $U_{S(A/I)} / \sim \to K_1(S(A/I))$ is surjective, then $A$ is $K_1$-injective.

We refer to [28] for the proof of the surjectivity. We also refer to [28] for the definition of the properly infinite full projections and the properly infinite $C^*$-algebras.

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Lemma 2.8 ([28, Exercise 8.9]). Let $A$ be a unital properly infinite C*-algebra. Then the natural map $U_A/\sim_h \to K_1(A)$ is surjective.

Lemma 2.9 ([28, Exercise 4.9]). Let $A$ be a unital C*-algebra, and let $p$ and $q$ be properly infinite full projections. Then there exists a partial isometry $v$ with $p = vv^*$, $q = v^*v$, if and only if $[p] = [q]_0$ in $K_0(A)$.

We show that the algebra $C(X) \otimes E_{n+1}$ is $K_1$-injective.

Proposition 2.10. Let $X$ be a compact Hausdorff space. Then, the map

$$U_{C(X) \otimes E_{n+1}}/\sim_h \to K_1(C(X) \otimes E_{n+1})$$

is an isomorphism.

Proof. Surjectivity follows from the fact that $C(X) \otimes E_{n+1}$ is properly infinite and Lemma 2.8. We identify $SC_0(X, x_0) \otimes \mathcal{O}_{n+1}$ with $C_0([0, X, x_0]) \otimes \mathcal{O}_{n+1}$. Since $C(X) \otimes \mathcal{O}_{n+1}$ is $K_1$-injective by Lemma 2.4 and $C(X) \otimes \mathbb{K}$ is $K_1$-injective, it is sufficient to prove the surjectivity of the natural map $U_{(SC(X) \otimes \mathcal{O}_{n+1})} \sim K_1(SC(X) \otimes \mathcal{O}_{n+1})$. For the space $Y := [0, 1] \times X/([0] \times X \cup \{1\} \times X)$, we have $SC(X) = C_0(Y, y_0)$. So we have the conclusion by Lemma 2.7.

Let $End E_{n+1}$ be the semi-group of unital $*$-endomorphisms of $E_{n+1}$. We topologize $End E_{n+1}$ by the point-wise norm topology, and let $End_0 E_{n+1}$ be the path component of $id_{E_{n+1}}$ in $End E_{n+1}$. We denote by $End E_{n+1}$ (resp. $Aut E_{n+1}$) the subset of $End E_{n+1}$ (resp. $Aut E_{n+1}$) consisting of all elements fixing the minimal projection $e$. Every automorphism of $E_{n+1}$ preserves the ideal of compact operators and induces an automorphism of $\mathcal{O}_{n+1}$. For $\alpha$ in $Aut E_{n+1}$, we denote by $\alpha$ the induced automorphism of $\mathcal{O}_{n+1}$. This gives a group homomorphism $Aut E_{n+1} \to Aut \mathcal{O}_{n+1}$.

Lemma 2.11. The set $End_0 E_{n+1}$ is equal to a subset $\{ \rho \in End E_{n+1} \mid \rho(e) \text{ is a minimal projection of } \mathbb{K} \}$ of $End E_{n+1}$, and the map

$$End_0 E_{n+1} \ni \rho \mapsto u_\rho := \sum_{i=1}^{n+1} \rho(T_i)T_i^* \in U_{E_{n+1}}(1-e) := \{ u(1-e) \in E_{n+1} \mid u \in U_{E_{n+1}} \}$$

is a homeomorphism.

Proof. First, we show $End_0 E_{n+1} \ni \rho \mapsto u_\rho \in U_{E_{n+1}}(1-e)$ is well-defined. If $\rho(e)$ is a minimal projection, there exists a partial isometry $v$ with $v^*v = \rho(e)$, $v^*v = e$. Then the unitary $u + u_\rho$ is in the path component of $1_{E_{n+1}}$ by the $K_1$-injectivity of $E_{n+1}$. We take a norm continuous path of unitaries $\{ u_t \}_{t \in [0, 1]}$ in $U_{E_{n+1}}$ from $u + u_\rho$ to $1_{E_{n+1}}$, and we have the continuous path $\rho_t : T_i \mapsto u_T$, from $\rho$ to $id_{E_{n+1}}$.

Second, we show the map $End_0 E_{n+1} \ni \rho \mapsto u_\rho \in U_{E_{n+1}}(1-e)$ is a homeomorphism. For every $w \in U_{E_{n+1}}(1-e)$, we have the map $\rho_w : T_i \mapsto wT_i$ by the universality of $E_{n+1}$. The map $U_{E_{n+1}}(1-e) \ni w \mapsto \rho_w \in End E_{n+1}$ is continuous because $\{ T_i \}_{i=1}^{n+1}$ is the generator of $E_{n+1}$. This gives the inverse of the map $End_0 E_{n+1} \ni \rho \mapsto u_\rho \in U_{E_{n+1}}(1-e)$.

2.2 Section algebras and the theory of extensions of C*-algebras

We use the following elementary fact.

Lemma 2.12. Let $A$ be a unital C*-algebra, and let $X$ be a compact metrizable space. Let $P_1$ and $P_2$ be principal $Aut A$ bundles over $X$. Let $A_1$ and $A_2$ be the section algebras of the associated bundles of $P_1$ and $P_2$ with fibre $A$ respectively. Then $P_1$ and $P_2$ are isomorphic if and only if there exists a $C(X)$-linear isomorphism $\varphi : A_1 \to A_2$.

Let $\pi : \mathcal{M}(C(X) \otimes \mathbb{K}) \to \mathcal{Q}(C(X) \otimes \mathbb{K})$ be the quotient map by the ideal $C(X) \otimes \mathbb{K}$. We need the following technical theorem of the theory of extensions of C*-algebras.

Theorem 2.13 ([27, Theorem 2.10]). Let $X$ be a finite CW complex. Let $A$ be a separable simple unital C*-algebra, and let $\mu : A \to \mathcal{M}(C(X) \otimes \mathbb{K})$ and $\sigma : A \to \mathcal{Q}(C(X) \otimes \mathbb{K})$ be unital $*$-homomorphisms. Then $\sigma \circ \pi \circ \mu$ and $\sigma$ are strongly unitarily equivalent.

The theorem above is a special case of [27, Theorem 2.10]. Since $A$ is simple, the assumptions for it are satisfied.

Lemma 2.14. Let $X$ be a finite CW complex. Let $A$ be a separable simple unital C*-algebra. Suppose that $A$ has a unital essential trivial extension $\pi \circ \mu$ where $\mu : A \to \mathcal{M}(C(X) \otimes \mathbb{K})$ is a unital embedding. Then two unital essential extensions $\tau_1$ and $\tau_2$ are weakly unitarily equivalent if and only if $[\tau_1] = [\tau_2]$ in $\text{Ext}(A, C(X) \otimes \mathbb{K})$.

Proof. We show that $[\tau_1] = [\tau_2]$ implies $\tau_1 \sim_{w,u,e} \tau_2$ as the other implication is always true. By definition, there exists a trivial extension $\pi \circ \rho_1$ such that $\tau_1 \sim \pi \circ \rho_1 \sim_{w,u,e} \tau_2 \sim \pi \circ \rho_2$. Adding $\pi \circ \mu$ to the both side, we may assume that $\rho_1(1_A)$ is a properly infinite full projection in $\mathcal{M}(C(X) \otimes \mathbb{K})$. Since $K_0(\mathcal{M}(C(X) \otimes \mathbb{K})) = 0$ and $\rho_1(1_A)$ is properly infinite full, there exists an isometry $V_i$ with $V_iV_i^* = \rho_1(1_A)$. Now we have $\tau_1 \sim \pi \circ (AdV_1^* \circ \rho_1) \sim \tau_2 \sim \pi \circ (AdV_2^* \circ \rho_2)$. It follows from Theorem 2.13 that $\tau_1 \sim_{w,u,e} \tau_2 \sim \pi \circ (AdV_1^* \circ \rho_1)$, and we have the conclusion. \qed
We have the following theorem of Paschke and Valette.

**Theorem 2.15** ([32, Proposition 3], [23, Theorem 6]). Let $A$ and $B$ be unital separable C*-algebras, and assume that $A$ is nuclear. Let $\mu: A \to \mathcal{M}(\mathbb{K})$ be a unital embedding with $\mu(A) \cap \mathbb{K} = \{0\}$. For the unital *-homomorphism $\tau := \pi \circ (1_B \otimes \mu)$, we have an isomorphism

$$\alpha_\tau: K_1(\tau(A)' \cap Q(B \otimes \mathbb{K})) \to \text{Ext}(SA, B \otimes \mathbb{K})$$

which sends the class of a unitary $u \in \tau(A)' \cap Q(B \otimes \mathbb{K})$ to the class of extension

$$\tau_u: SA \ni (e^{2\pi i} - 1) a \mapsto (u - 1) \tau(a) \in Q(B \otimes \mathbb{K}).$$

The following theorem holds from the argument of [24, Section 1, Theorem 1.5].

**Theorem 2.16** ([24, Section 1]). Let $\tau_1$ and $\tau_2$ be unital extensions of $\mathcal{O}_{n+1}$ by $\mathbb{K}$. Then $\tau_1 \sim_{\kappa, u, e} \tau_2$ if and only if $\tau_1 \sim \tau_2$.

**Proposition 2.17.** Let $X$ be a compact Hausdorff space with $\text{Tor}(K_0(C(X)), \mathbb{Z}_n) = 0$, and let $\sigma: \mathcal{O}_{n+1} \to Q(C(X) \otimes \mathbb{K})$ be an arbitrary unital extension. Then every element of $K_1(\sigma(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K}))$ is an $n$-torsion element, and the set $U_{\sigma(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K})}$ is contained in the path component of $1$ of $U_{Q(C(X) \otimes \mathbb{K})}$.

**Proof.** By Theorem 2.3, all elements of $\text{Ext}(SO_{n+1}, C(X) \otimes \mathbb{K})$ are $n$-torsion elements. We define $\tau := \pi \circ (1_{C(X)} \otimes \mu)$ where $\mu: \mathcal{O}_{n+1} \to \mathcal{M}(\mathbb{K})$ is a unital embedding. Since $\mathcal{O}_{n+1}$ is simple, we have $\mu(\mathcal{O}_{n+1}) \cap \mathbb{K} = \{0\}$. By Theorem 2.15, we have

$$K_1(\tau(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K})) \cong \text{Ext}(SO_{n+1}, C(X) \otimes \mathbb{K}).$$

So we have $[\sigma^{n \mu}] = n[\tau] = [\tau] = 0$, and Lemma 2.14 gives a unitary $w \in U_{Q(C(X) \otimes \mathbb{K})}$ with $\sigma^{\mu w} = Adw \circ \tau$. We have an isomorphism

$$Adw: (\sigma^{\mu w}(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K})) \cong (\tau(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K})).$$

We also have $(\sigma(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K})) \otimes \mathbb{M}_n \cong (\sigma^{\mu w}(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K}))$. So we have

$$K_1(\sigma(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K})) \cong K_1(\tau(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K})).$$

Since $K_0(C(X)) = K_1(Q(C(X) \otimes \mathbb{K}))$ has no $n$-torsion, we have $[w]_1 = 0 \in K_1(Q(C(X) \otimes \mathbb{K}))$ for every $w \in U_{\sigma(\mathcal{O}_{n+1})' \cap Q(C(X) \otimes \mathbb{K})}$. So we have the conclusion by $K_1$-injectivity of $Q(C(X) \otimes \mathbb{K})$ (see [21, Section 1.13]).

As an application of Lemma 2.14, we show in Proposition 2.22 that the group $\text{Aut} E_{n+1}$ is path connected. A straightforward computation yields the lemma below.

**Lemma 2.18.** We have the following isomorphisms of $K$-groups and $\text{Ext}$-groups:

$$ev_{pd}: \text{Ext}(\mathcal{O}_{n+1}, C(S^{2m-1}) \otimes \mathbb{K}) \to \text{Ext}(\mathcal{O}_{n+1}, \mathbb{K}),$$

$$K_1(ev_{pd}): K_1(Q(C(S^{2m-1}) \otimes \mathbb{K})) \to K_1(Q(\mathbb{K})),$$

$$K_1(ev_{pd}): K_1(Q(C([0,1]) \otimes \mathbb{K})) \to K_1(Q(\mathbb{K})),$$

for $m \geq 1$.

We need the following lemma.

**Lemma 2.19** ([19, Lemma 2.3]). Let $\gamma_0: \mathcal{O}_{n+1} \to Q(\mathbb{K})$ be the Busby invariant in Definition 2.5. If a unitary $u$ in $U_{\mathcal{M}(\mathbb{K})}$ commutes with $E_{n+1}$ up to compact operators (i.e. $[u, d] \in \mathbb{K}$ for every $d \in E_{n+1}$), there exists a self adjoint element $h$ in $Q(\mathbb{K})$ such that $e^{2\pi i h} = \pi(u)$ and $[h, a] = 0$ for every $a$ in $\mathcal{O}_{n+1}$.

**Corollary 2.20.** The group $\mathcal{N} := \{ u \in U_{\mathcal{M}(\mathbb{K})} \mid [u, E_{n+1}] \subset \mathbb{K} \}$ is path connected.

Let $\{e_{ij}\}_{ij}$ be a system of matrix units of $\mathbb{K}$.

**Lemma 2.21.** Let $\alpha$ be an automorphism of $E_{n+1}$, and $U_{n} := \sum_i \alpha(e_{ii})ve_{1i}$ be an implementing unitary of $\alpha$ on $\mathbb{K}$. Then $\alpha = AdU_{n} |_{E_{n+1}}$.

**Proof.** We show $AdU_{n} |_{E_{n+1}} = \alpha$. Let $F \subset \mathbb{K}$ be the set of all finite rank projections. Since $\alpha$ is an automorphism, the image $\alpha(\mathbb{K}) = \mathbb{K}$ contains a net $\{\alpha(p)\}_{p \in F}$ that weakly converges to 1. For every $d \in E_{n+1}$, we have $\alpha(p)\alpha(d) = \alpha(pd) = AdU_{n}(pd) = \alpha(p)AdU_{n}(d)$, and $\alpha = AdU_{n} \mid_{E_{n+1}}$ holds.

**Proposition 2.22.** The group $\text{Aut} E_{n+1}$ is path connected.

5
Proof. Let $\alpha$ be an automorphism of $E_{n+1}$ and let $\tilde{\alpha}$ be an induced automorphism of $O_{n+1}$. Since $\text{Aut } O_{n+1}$ is path connected, we take a path $h_t$ with $h_0 = \tilde{\alpha}$, $h_1 = \text{id}_{O_{n+1}}$. We take two unital essential extensions

$$
\tau_1: = \text{id}_{C[0,1]} \otimes \tau_0: C[0,1] \otimes O_{n+1} \ni f(t) \mapsto f(t) \in Q(C[0,1] \otimes K)
$$

$$
\tau_2: = \alpha \circ h: C[0,1] \otimes O_{n+1} \ni f(t) \mapsto h_t(f(t)) \in Q(C[0,1] \otimes K),
$$

where $\tau_0$ is the Busby invariant in Definition 2.5, and we regard $h: C[0,1] \otimes O_{n+1} \rightarrow C[0,1] \otimes O_{n+1}$ as a $C[0,1]$-linear isomorphism. Since $\tau_1 \sim \tau_2$, we have $[\tau_1] = [\tau_2]$ in $\text{Ext}(C[0,1] \otimes O_{n+1}, C[0,1] \otimes K)$. We have $[\tau_1 \circ (1_{C[0,1]} \otimes \text{id}_{O_{n+1}})] = [\tau_2 \circ (1_{C[0,1]} \otimes \text{id}_{O_{n+1}})]$ in $\text{Ext}(O_{n+1}, C[0,1] \otimes K)$. By Lemma 2.14, there exists a unitary $v \in U_{Q(C[0,1] \otimes K)}$ with $\tau_2 \circ (1_{C[0,1]} \otimes \text{id}_{O_{n+1}}) = \text{Ad} v \circ \tau_1 \circ (1_{C[0,1]} \otimes \text{id}_{O_{n+1}})$. Since $C[0,1]$ is in the center of $Q(C[0,1] \otimes K)$, we have $\tau_2 = \text{Ad} v \circ \tau_1$.

We show $[v]_1 = 0$ in $K_1(Q(C[0,1] \otimes K))$. By the construction of $\tau_2$ and $v$, the unitary $v_1$ is in $\tau_0(O_{n+1})' \cap Q(K)$. By Proposition 2.17, we have $[v]_1 = 0$ in $K_1(Q(K))$. Since the map $v_1: K_1(Q(C[0,1] \otimes K)) \rightarrow K_1(Q(K))$ is an isomorphism from Lemma 2.18, we have $[v]_1 = 0$.

We take a unitary lift $V \in U_{M(C[0,1] \otimes K)}$ of $v$. It follows that $\text{Ad} V$ is a $C[0,1]$-linear isomorphism of $C[0,1] \otimes E_{n+1}$. Therefore the map $[0,1] \ni t \mapsto \text{Ad} V_t \in \text{Aut } E_{n+1}$ is continuous. Let $U_n := \sum_i \alpha(e_i)v_{e_i}$ be an implementing unitary of $\alpha$ restricted to $K$ where $w$ is a partial isometry satisfying $ww^* = \alpha(e_{i_1})$, $w^* w = \alpha(e_1)$, and $\{e_i\}$ is a system of matrix units. By Lemma 2.21, we have $\text{Ad} U_n \mid E_{n+1} = \alpha$. We have $\text{Ad} (V_0) = h_0 = \tilde{\alpha} = \text{Ad} (U_n)$, and it follows that $V_n U_n$ commutes with $E_{n+1}$ up to compact operators. By Corollary 2.20, the automorphism $\text{Ad} V_n U_n$ is in the path component of $\text{id}_{E_{n+1}}$ in $\text{Aut } E_{n+1}$. Similary there is a continuous path from $\text{id}_{E_{n+1}}$ to $\text{Ad} V_1$ in $\text{Aut } E_{n+1}$. Therefore we have $\alpha \sim h \text{Ad} V_1 \circ \text{Ad} V_0 U_n \sim h \text{Ad} V_1 \sim h \text{id}_{E_{n+1}}$.

\[\square\]

2.3 Implementing unitaries of $\text{Aut}_{C(X)}(C(X) \otimes E_{n+1})$

Let $\text{Aut}_{C(X)}(C(X) \otimes E_{n+1})$ be the group of $C(X)$-linear automorphisms of $C(X) \otimes E_{n+1}$. We remark that the homotopy set $[X, \text{Aut } E_{n+1}]$ is identified with the set of homotopy equivalence classes of the elements of $\text{Aut}_{C(X)}(C(X) \otimes E_{n+1})$.

Let $G$ be a compact topological group. We denote by $B G$ its classifying space, and denote by $E G$ the universal principal $G$-bundle over $B G$. Realization of those spaces is as follows. For a contractible space $X$ equipped with a free $G$ action, the quotient map $X \rightarrow X/G$ gives the universal bundle. We refer to [14] for the basic facts about the classifying spaces.

Let $H_1$ be the set vectors of norm 1 in a separable Hilbert space with the norm topology. We denote $H_1$ with the set $\{f \in L^2([0,1]) \mid \|f\| = 1\}$. There is a map $h_t: H_1 \times [0,1] \rightarrow H_1$ that sends $(f,t)$ to $(1_{[0,0]} f + 1_{[t,1]} f)/\|1_{[0,0]} f + 1_{[t,1]} f\|_2$ where $1_{[a,b]}$ is the characteristic function of $[a,b]$. This gives the deformation retraction to the set $\{1_{[0,1]}\}$. This is a commutative diagram below:

\[\begin{array}{ccc}
X & \xrightarrow{\pi} & X/G \\
\downarrow \quad & & \downarrow \\
\text{Map}(X, B G) & \xrightarrow{\text{ev}} & \text{Map}(X, X/G) \\
\end{array}\]

Proposition 2.23. Let $X$ be a compact Hausdorff space, and let $\alpha: X \rightarrow Aut E_{n+1}$ be a continuous map. Then $\eta$ is the map $Aut E_{n+1} \ni \alpha \mapsto \alpha(e) \in B S^1$. If the image of $[\alpha]$ by the map $[X, Aut E_{n+1}] \xrightarrow{\text{ev}} [X, B S^1]$ is zero, then there exists a unitary $U$ in $U_{M(C(X) \otimes K)}$ such that $\text{Ad} U = \alpha$. 

Proof. Let $\xi_0$ be a norm 1 eigenvector corresponding to the minimal projection $e$. By assumption, there exists a norm continuous section $\xi: X \rightarrow H_1$ with $\xi_x \otimes \xi^* = \alpha(e)$. Using a system of matrix units $\{1_{C(X)} \otimes e_i\}$ with $e = e_{i_1} \cdots e_{i_n}$, we have a unitary $U_x := \sum_i \alpha(e_i)\xi_x \otimes \xi_{e_i}$. Since $\xi_x$ is norm continuous, $U_x: X \ni x \mapsto U_x \in M(K)$ in $M(K)$ is $\text{SOT}^*$-continuous. In particular, $U: X \rightarrow U_{M(K)}$ is $\text{SOT}^*$-continuous and $U \in U_{M(C(X) \otimes K)}$. Lemma 2.21 shows $\text{Ad} U \mid E_{n+1} = \alpha$ for every $x \in X$. 

\[\square\]

Lemma 2.24. Let $X$ be a compact Hausdorff space and let $\alpha$ be an element of $\text{Map}(X, \text{Aut } O_{n+1})$. Then the map $\alpha: C(X) \otimes E_{n+1} \rightarrow C(X) \otimes E_{n+1}$ induces the identity map of the $K$-groups, $K_i(\alpha) = \text{id}_{K_i(C(X) \otimes E_{n+1})}$, $i = 1,2$. 

Proof. Since $\alpha$ is $C(X)$-linear, we have the commutative diagram below:

\[\begin{array}{ccc}
C(X) & \xrightarrow{\alpha} & C(X) \\
\downarrow & & \downarrow \\
C(X) \otimes E_{n+1} & \xrightarrow{\alpha} & C(X) \otimes E_{n+1}.
\end{array}\]

We have the conclusion from the KK-equivalence of $C$ and $E_{n+1}$. 

\[\square\]
Let \( r : \operatorname{Aut} E_{n+1} \to \operatorname{Aut} \mathbb{K} \) be the restriction map. Then we have a commutative diagram below

\[
\begin{array}{ccc}
[X, \operatorname{Aut} E_{n+1}] & \xrightarrow{r_*} & [X, \operatorname{Aut} \mathbb{K}] \\
\downarrow{\eta_*} & & \downarrow{\eta_*} \\
[X, \mathbb{B} S^1] & & [X, \mathbb{B} S^1].
\end{array}
\]

We remark that the map \( \eta : \operatorname{Aut} \mathbb{K} \to \mathbb{B} S^1 \) gives the homotopy equivalence (see [9, Lemma 2.8]).

**Lemma 2.25.** The map \( \eta_* : [S^k, \operatorname{Aut} E_{n+1}] \to [S^k, \mathbb{B} S^1] \) is the zero map for \( k \geq 1 \). Hence the map \( r_* : [S^k, \operatorname{Aut} E_{n+1}] \to [S^k, \operatorname{Aut} \mathbb{K}] \) is also zero.

**Proof.** If \( k \neq 2 \), we have \([S^k, \mathbb{B} S^1] = H^2(S^k) = 0\). We show the statement in the case of \( k = 2 \). For every \( \alpha \) in \( \operatorname{Map}(S^2, \operatorname{Aut} E_{n+1}) \), the map \( K_0(\alpha) : K_0(C(S^2) \otimes E_{n+1}) \to K_0(C(S^2) \otimes E_{n+1}) \) is the identity by Lemma 2.24. Since the map \( K_0(C(S^2) \otimes \mathbb{K}) \xrightarrow{\alpha} K_0(C(S^2) \otimes E_{n+1}) \) is injective, \([e] = [\alpha(e)]_0 \) in \( K_0(C(S^2) \otimes \mathbb{K}) \). Therefore we have \( \eta_*(\alpha) = 0 \) in \( H^2(S^2) \).

### 2.4 Some fibration sequences

In this section, we introduce several fibrations to compute the homotopy groups of \( \operatorname{Aut} E_{n+1} \). We refer to [5, Chap. 6] for the definition and the basic facts about fibrations.

**Definition 2.26.** Let \( X, Y \) and \( Z \) be topological spaces, and let \( \pi : X \to Y \) be a continuous map. The map \( \pi \) has the homotopy lifting property (abbreviated to HLP) for \( Z \), if for every commuting diagram

\[
\begin{array}{ccc}
[0] \times Z & \xrightarrow{g} & X \\
\downarrow{f} & & \downarrow{\pi} \\
[0, 1] \times Z & \xrightarrow{f} & Y,
\end{array}
\]

there exists a continuous map \( \tilde{g} : [0, 1] \times Z \to X \) such that \( \tilde{g}(0, z) = g(z) \) for every \( z \) in \( Z \) and \( \pi \circ \tilde{g} = f \).

The map \( \pi : X \to Y \) is a Serre fibration, if \( \pi \) has HLP for every \( n \)-disc, \( \mathbb{D}^n \).

We remark that a Serre fibration has HLP for every CW complex. A fibration gives a long exact sequence of homotopy sets. We denote by \( \Omega X \) or \( \Omega_x X \) the loop space of the pointed set \( (X, x_0) \).

**Theorem 2.27.** Let \( (Z, z_0) \) be a pointed CW complex. Let \( \pi : (X, x_0) \to (Y, y_0) \) be a Serre fibration with the fibre \( F := \pi^{-1}(y_0) \). Then, there is a long exact sequence of groups \( (i \geq 1) \), and exact sequence of pointed sets \( (i \geq 0) \)

\[
\cdots \to [Z, \Omega Y]_0 \to [Z, \Omega X]_0 \to [Z, \Omega^2 Y]_0 \to \cdots \to [Z, F]_0 \to [Z, X]_0 \to [Z, Y]_0.
\]

In particular, we have a long exact sequence of the homotopy groups in the case of \( Z = \{z_0\} \).

We have the following fact.

**Proposition 2.28** ([5, Theorem 6.42]). Let \( (X, x_0), (Y, y_0) \) be pointed topological spaces. Then the natural map \( [\Sigma X, Y]_0 \to [X, \Omega Y]_0 \) is a bijection.

By the theorem of Hurewicz, every principal \( G \)-bundle is a fibration. Therefore we use the long exact sequence to compute the homotopy groups of the topological group \( G \). We refer to the argument in [9, Lemma 2.8, 2.16, Corollary 2.9] for the proof of the following lemmas.

**Lemma 2.29.** Let \( p : U_{E_{n+1}} \to U_{E_{n+1}(1-e)} \) be the multiplication by \( 1 - e \). Then, the map \( p \) is a principal \( S^1 \)-bundle that has the \( S^1 \) action by the right multiplication of \( (1-e) + ze \), \( z \in S^1 \).

**Lemma 2.30.** Let \( H_1 \) be the set of vectors of norm 1 of the Hilbert space \( H \), and \( \xi_0 \in H_1 \) be a vector corresponding the minimal projection \( e \). The map \( q : U_{E_{n+1}} \to H_1 \) that sends a unitary \( u \) to \( u\xi_0 \) is a fibration with the fiber \( U_{(1-e)E_{n+1}(1-e)} \).

**Remark 2.31.** Since \( H_1 \) is contractible, it follows from the long exact sequence of the homotopy groups induced by the fibration of Lemma 2.30 that the map

\[
U_{(1-e)E_{n+1}(1-e)} \ni w \mapsto w + e \in U_{E_{n+1}}
\]

is a weak homotopy equivalence. Hence the map \( \operatorname{End}_e E_{n+1} \ni \rho \mapsto e + \sum_i n(T_i)T_i^* \in U_{E_{n+1}} \) is a weak homotopy equivalence.
Lemma 2.32. Let \( \eta : \text{End}_k E_{n+1} \to B S^1 \) be the map that sends \( \alpha \) to \( \alpha(e) \) and let \( \text{Aut}_e E_{n+1} \) be the stabilizer subgroup of the minimal projection \( e \). Then there is a principal \( \text{Aut}_e E_{n+1} \)-bundle

\[ \text{Aut}_e E_{n+1} \to \text{Aut} E_{n+1} \xrightarrow{\eta} B S^1. \]

Remark 2.33. Since the map \( \eta_* : [S^k, \text{Aut} E_{n+1}] \to [S^k, B S^1] \) is the zero map for \( k \geq 1 \) by Lemma 2.25, it follows from Lemma 2.32 that for every \( a \) in \( \text{Map}(S^k, \text{Aut} E_{n+1}) \), there exists \( \alpha' \) in \( \text{Map}(S^k, \text{Aut} E_{n+1}) \) that is homotopic to \( \alpha \) in \( \text{Map}(S^k, \text{Aut} E_{n+1}) \).

Lemma 2.34. The following sequence gives a fibration:

\[ \text{End}_e E_{n+1} \to \text{End}_0 E_{n+1} \xrightarrow{\eta} B S^1. \]

Remark 2.35. In section 3, we show that the map \( \text{Aut} E_{n+1} \to \text{End}_0 E_{n+1} \) is a weak homotopy equivalence. Hence the groups \( \text{Aut}_e E_{n+1} \) and \( \text{End}_e E_{n+1} \) are weakly homotopy equivalent from the long exact sequences and 5-lemma. Then the map \( \text{Aut}_e E_{n+1} \ni \alpha \to e + \sum \alpha(T_i) T_i^* \in U_{E_{n+1}} \) is a weak homotopy equivalence by Remark 2.31.

By the fiberation in Lemma 2.29, we know the homotopy groups of \( \text{End}_0 E_{n+1} \).

Theorem 2.36. The homotopy groups of \( \text{End}_0 E_{n+1} \) are as follows:

\[
\pi_1(\text{End}_0 E_{n+1}) = \mathbb{Z}_m, \quad \pi_{2k+1}(\text{End}_0 E_{n+1}) = \mathbb{Z}, \quad \pi_{2k}(\text{End}_0 E_{n+1}) = 0, \quad \text{where} \ k \geq 1.
\]

Proof. By Lemma 2.11, it is sufficient to compute the homotopy groups of \( U_{E_{n+1}}(1 - e) \). By the fiberation sequence

\[ S^1 \to U_{E_{n+1}} \xrightarrow{\pi_e} U_{E_{n+1}}(1 - e), \]

we have the long exact sequence of the homotopy groups

\[ \cdots \to 0 \to \pi_k(U_{E_{n+1}}) \to \pi_k(U_{E_{n+1}}(1 - e)) \to \cdots \to \pi_1(S^1) \to \pi_1(U_{E_{n+1}}) \to \pi_1(U_{E_{n+1}}(1 - e)) \to 0. \]

The map \( S^1 \to U_{E_{n+1}} \) sends a complex number \( z \) to a unitary \( 1 - e + 2\pi \). We have \( [S^k, U_{E_{n+1}}] = 0 \). Hence \( U_{E_{n+1}} = K^1(S^k) \) by \( K_1 \)-injectivity of \( C(S^1) \cong \mathbb{Z} \). The map

\[
Z = [S^1, S^1] \ni [z] \mapsto [1 - e + 2\pi] \in [S^1, U_{E_{n+1}}] = \mathbb{Z}
\]

is the multiplication by \(-n\), and so we have the conclusion.

We remark that a generator of \( \pi_1(\text{End}_0 E_{n+1}) = \mathbb{Z}_n \) is the canonical gauge action of \( S^1 \) that is \( \lambda_z : T_i \mapsto zT_i \) for every \( z \in S^1 \).

3 The main result

3.1 The homotopy groups of \( \text{Aut} E_{n+1} \)

In this section, by using the theory of extensions, we show that the inclusion map \( \text{Aut} E_{n+1} \to \text{End}_0 E_{n+1} \) is a weak homotopy equivalence. First, we show that \( \text{End}_0 E_{n+1} \) is trivial for \( m \geq 1 \). Second, we show the surjectivity of the map \( [S^{2m+1}, \text{Aut} E_{n+1}] \to [S^{2m+1}, \text{End}_0 E_{n+1}] \) for \( m \geq 1 \). Finally, we show the injectivity of the map.

Let \( X \) be a compact Hausdorff space. It is well-known in homotopy theory that every principal \( \text{Aut} E_{n+1} \)-bundle \( \mathcal{P} \) over \( X \) comes from the classifying map \( X \to B \text{Aut} E_{n+1} \) [14, Section 4, Proposition 10.6]. So we identify the isomorphism class of a principal bundle \( \mathcal{P} \) with the homotopy equivalence class of its classifying map and denote \( [\mathcal{P}] \in [X, B \text{Aut} E_{n+1}] \). For a bundle \( \mathcal{P} \), the section algebra of the associated bundle \( \mathcal{P} \times_{\text{Aut} E_{n+1}} E_{n+1} \) is a locally trivial continuous \( C(X) \)-algebra \( \Gamma(X, \mathcal{P} \times_{\text{Aut} E_{n+1}} E_{n+1}) \).

Let \( k \) be a natural number. For every \( \alpha \) in \( \text{Map}(S^k, \text{Aut} E_{n+1}) \), there is a principal \( \text{Aut} E_{n+1} \)-bundle \( P_\alpha \) representing the class \( [\alpha] \) in \( [S^k, \text{Aut} E_{n+1}] \cong [S^{k+1}, B \text{Aut} E_{n+1}] \). We construct a continuous field of \( E_{n+1} \) over \( S^{k+1} \) corresponding to \( P_\alpha \), as follows. We denote the interior of the closed \( k+1 \)-disc by \( (D^{k+1})^c \). We view \( S^{k+1} \) as a non-reduced suspension of \( S^k \), that is, \( (D^{k+1})^c \cup S^k \cup (D^{k+1})^c \), and view \( \alpha \) a clutching function on \( S^k \) of two trivial bundles over \( (D^{k+1})^c \cup S^k \) and \( S^k \cup (D^{k+1})^c \). By the following lemma, we have \( [S^k, \text{Aut} E_{n+1}] = [S^{k+1}, B \text{Aut} E_{n+1}] \).

Lemma 3.1 ([14, Corollary 8.3]). Let \( G \) be a path connected group. Let \( X \) be a topological space, and let \( SX \) be its non-reduced suspension. Then the map

\[ [X, G] \ni [\alpha] \mapsto [P_\alpha] \in [SX, BG] \]

is bijective.
Definition 3.2. We identify the section algebra of $P_{n+1} \times \text{Aut} E_{n+1}$ with the following algebra:

$$B_{\alpha} := \{(F_1, F_2) \in (C([0, 1] \times S^k) \otimes E_{n+1}) \otimes \mathbb{K} | F_i(0) \in 1_{C(S^k)} \otimes E_{n+1}, F_i(1) = \alpha(F_2(1)) \in C(S^k) \otimes E_{n+1}\},$$

and denote by $C_{\alpha}$ the essential ideal

$$C_{\alpha} := \{(F_1, F_2) \in (C([0, 1] \times S^k) \otimes \mathbb{K}) \otimes \mathbb{K} | F_i(0) \in 1_{C(S^k)} \otimes \mathbb{K}, F_i(1) = \alpha(F_2(1)) \in C(S^k) \otimes \mathbb{K}\}.$$

Let $A_{\alpha}$ be the quotient algebra of $B_{\alpha}$ by $C_{\alpha}$:

$$A_{\alpha} := \{(a_1, a_2) \in (C([0, 1] \times S^k) \otimes \mathbb{K}) \otimes \mathbb{K} | a_i \in 1_{C(S^k)} \otimes \mathbb{K}_{n+1}, a_1(1) = \alpha(a_2(1)) \in C(S^k) \otimes \mathbb{K}_{n+1}\}.$$

The algebra $A_{\alpha}$ is isomorphic to the section algebra $\Gamma(S^{k+1}, P_{\alpha} \times \text{Aut} \mathbb{C} \mathbb{C}_{n+1} \times \mathbb{K})$, where $\alpha$ is the induced map in $\text{Map}(S^k, \mathbb{K}_{n+1})$. We remark that $C(S^{k+1})$ is identified with the algebra

$$\{(f_1, f_2) \in (C([0, 1] \times S^k)) \otimes \mathbb{K} | f_i(0) \in \mathbb{C}, f_i(1) = f_2(1) \in C(S^k)\},$$

which is the center of $B_{\alpha}$. Since the map $[S^k, \text{Aut} E_{n+1}] \to [S^k, \text{Aut} \mathbb{K}]$ is zero map by Lemma 2.25, the associated bundle $P_{\alpha} \times \text{Aut} E_{n+1} \mathbb{K}$ is trivial. We fix a trivialization and obtain $\theta_{\alpha}: C_{\alpha} \to C(S^{k+1}) \otimes \mathbb{K}$. Thus we get a unitarized essential extension $\tau_{\theta_{\alpha}}$.

$$\begin{array}{cccccc}
C_{\alpha} & \xrightarrow{\theta_{\alpha}} & B_{\alpha} & \xrightarrow{\pi_{\alpha}} & A_{\alpha} & \xrightarrow{\tau_{\theta_{\alpha}}} \\
\downarrow & & \downarrow & & \downarrow & \\
C(S^{k+1}) \otimes \mathbb{K} & \xrightarrow{\mathcal{M}(C(S^{k+1}) \otimes \mathbb{K})} & Q(C(S^{k+1}) \otimes \mathbb{K})
\end{array}$$

where the isomorphism $\theta_{\alpha}: C_{\alpha} \to C(S^{k+1}) \otimes \mathbb{K}$ depends on the trivialization of the bundle $P_{\alpha} \times \text{Aut} E_{n+1} \mathbb{K}$.

Lemma 3.3. Let $m \geq 1$ be a natural number. Then we have $[S^{2m}, \text{Aut} E_{n+1}] = 0$.

Proof. Since $[S^{2m}, \text{Aut} O_{n+1}] = 0$ by [12, Theorem 7.4], there is a trivialization $\varphi_{\alpha}: C(S^{2m+1}) \otimes O_{n+1} \to A_{\alpha}$ that is $C(S^{2m+1})$-linear isomorphism for every $\alpha \in \text{Map}(S^{2m}, \text{Aut} O_{n+1})$. Consider two extensions of $O_{n+1}$ by $C(S^{2m+1}) \otimes \mathbb{K}$:

$$\sigma_{\alpha} := \tau_{\theta_{\alpha}} \circ \varphi_{\alpha} \circ (1_{C(S^{2m+1})} \otimes \text{id}_{O_{n+1}}),$$

$$\sigma := 1_{C(S^{2m+1})} \otimes \tau_0,$$

where the map $\tau_0$ is the Busby invariant in Definition 2.5. It follows from the construction that $[\text{ev}_{pd} \circ \sigma_{\alpha}] = [\text{ev}_{pd} \circ \sigma]$ in $\text{Ext}(O_{n+1}, \mathbb{K})$. By Lemma 2.18, we have $[\sigma_{\alpha}] = [\sigma]$ in $\text{Ext}(O_{n+1}, C(S^{2m+1}) \otimes \mathbb{K})$, and Lemma 2.14 yields that there exists a unitary $w$ in $U_{Q(C(S^{2m+1}) \otimes \mathbb{K})}$ satisfying $\text{Ad}_w \circ \sigma = \sigma_{\alpha}$. There is another unitary $U$ in $U_{Q(C(S^{2m+1}) \otimes \mathbb{K})}$ with $\text{Ad}_U \circ \text{ev}_{pd} \circ \sigma = \text{ev}_{pd} \circ \sigma$, by Theorem 2.16. By Proposition 2.17, we have $[\text{ev}_{pd}(w)] = [\text{ev}_{pd}(w) \text{Ad}(U')]= 0$ in $K_1(Q(\mathbb{K}))$, and Lemma 2.18 yields that $[w] = 0$. Therefore we have a unitary $W$ that is a lift of $w$, and the map $\text{Ad}_W: C(S^{2m+1}) \otimes E_{n+1} \to \theta_{\alpha}(B_{\alpha})$ is a $C(S^{2m+1})$-linear isomorphism. From Lemma 2.12, the bundle $P_{\alpha}$ is isomorphic to the trivial bundle, and we have $[\alpha] = 0$ in $[S^{2m}, \text{Aut} E_{n+1}]$ by Lemma 3.1 and Proposition 2.22. 

Lemma 3.4. Let $m \geq 1$ be a natural number. Then the map

$$[S^{2m-1}, \text{Aut} E_{n+1}] \ni [\alpha] \mapsto [\alpha] \in [S^{2m-1}, \text{Aut} O_{n+1}]$$

is surjective.

Proof. Let $\sigma$ be the map $\text{id}_{C(S^{2m-1})} \otimes \tau_0$ where $\tau_0$ is the Busby invariant in Definition 2.5. We show that for every $\gamma \in \text{Map}(S^{2m-1}, \text{Aut} O_{n+1})$ there exists a lift $\Gamma \in \text{Map}(S^{2m-1}, \text{Aut} E_{n+1})$ with $\Gamma_x = \gamma_x$ for every $x \in S^{2m-1}$. We recall the notation that $\Gamma_x$ is an induced automorphism of $O_{n+1}$ from $\Gamma_x$. For every $\gamma$ in $\text{Map}(S^{2m-1}, \text{Aut} O_{n+1})$, we regard $\gamma$ as an element of $\text{Aut}_{C(S^{2m-1})}(C(S^{2m-1}) \otimes O_{n+1})$, and there are two extensions of $O_{n+1}$

$$\sigma_{\gamma} := \tau \circ \gamma \circ (1_{C(S^{2m-1})} \otimes \text{id}_{O_{n+1}}),$$

$$\sigma := \tau \circ (1_{C(S^{2m-1})} \otimes \text{id}_{O_{n+1}}).$$

For every $x \in S^{2m-1}$, the map $\gamma_x$ is homotopic to $\text{id}_{O_{n+1}}$ in $\text{Aut} O_{n+1}$ because $\text{Aut} O_{n+1}$ is path connected by [12, Theorem 1.1]. Hence we have $\text{ev}_x \circ \sigma_{\gamma} \simeq_{\text{ev}_x, \text{nd}} \text{ev}_x \circ \sigma$ by Theorem 2.16 because $\text{ev}_x \circ \gamma \simeq_{\text{nd}} \text{ev}_x \circ \sigma$. From Lemma 2.18, we have $[\sigma_{\gamma}] = [\sigma]$ in $\text{Ext}(O_{n+1}, C(S^{2m-1}) \otimes \mathbb{K})$, and $\sigma_{\gamma} \simeq_{\text{ev}_x, \text{nd}} \sigma$ by Lemma 2.14. We have two unitaries $v \in Q(C(S^{2m-1}) \otimes \mathbb{K})$ and $V \in \mathcal{M}(\mathbb{K})$ satisfying $\sigma_{\gamma} = \text{Ad}_v \circ \sigma$ and $\text{ev}_{pd} \circ \sigma_{\gamma} = \text{Ad}(V) \circ \text{ev}_{pd} \circ \sigma$. So we have
\[ [\varepsilon pt(v)]_1 = [\pi(V)^*\varepsilon pt(v)]_1 = 0 \] by Proposition 2.17, and Lemma 2.18 yields \([v]_1 = 0 \in K_1(Q(C(S^{2m-1}) \oplus K)).\] Therefore we have \(\sigma_{\gamma} \sim_{u.e.} \sigma\), and there is a unitary \(U_{\gamma} \in U_{M(C(S^{2m-1}) \oplus K)}\) with
\[
\text{Ad}(U_{\gamma})(\tau(1 \otimes a)) = \tau(\gamma(1 \otimes a)), \quad a \in O_{n+1}.
\]
We have the following commutative diagram
\[
\begin{array}{ccc}
C(S^{2m-1}) \otimes E_{n+1} & \xrightarrow{\text{Ad}U_{\gamma}} & C(S^{2m-1}) \otimes E_{n+1} \\
\gamma & \nearrow & \\
C(S^{2m-1}) \otimes O_{n+1} & \xrightarrow{\tau} & C(S^{2m-1}) \otimes O_{n+1}.
\end{array}
\]
The map \(\Gamma : S^{2m-1} \ni x \mapsto \text{Ad}(U_{\gamma})x \in \text{Aut } E_{n+1}\) is continuous and it is a lift of the map \(\gamma\).

For \(\alpha'\) in \(\text{Map}(S^{2m-1}, \text{Aut } E_{n+1})\) with \(m \geq 1\), we take the map \(\theta_{\alpha'}\) as follows. Let \(U_{\alpha'}\) be a unitary in \(U_{M(C(S^{2m-1}) \oplus \ell_2(\ell_{2m}))}\) of the form \(U_{\alpha'} = \sum_{i \leq m} \alpha'(1_{C(S^{2m-1})} \otimes e_{ii})\), where \(\{e_{ii}\}\) is a system of matrix units with \(e_{11} = e\). By Theorem 2.1, there is a norm continuous path from \(1-e\) to \(U_{\alpha'}\) in \(U_{M(C(S^{2m-1}) \oplus \ell_{2m})}\). Adding the projection \(1_{C(S^{2m-1})} \otimes e\) to the path, we have a norm continuous path \(U \in C[0, 1] \otimes M(C(S^{2m-1}) \oplus K)\) satisfying
\[
U_t \in U_{M(C(S^{2m-1}) \oplus K)}, \quad \text{Ad}U_t[1_{C^{T}} = \alpha', \quad U_0 = 1, \quad eU_t = U_t e = e, \quad t \in [0, 1],
\]
where we write \(1_{C(S^{2m-1})} \otimes e\) simply by \(e\). We define two \(C(S^{2m})\)-algebras \(M\) and \(M_{\alpha'}:\)
\[
M := \{(F_1, F_2) \in M(C([0, 1] \times S^{2m-1}) \otimes K)_{S^2} | F_1(0) \in 1 \otimes M(K), \quad F_1(1) = F_2(1) \in M(C(S^{2m-1}) \otimes K)\},
\]
\[
M_{\alpha'} := \{(F_1, F_2) \in M(C([0, 1] \times S^{2m-1}) \otimes K)_{S^2} | F_1(0) \in 1 \otimes M(K), \quad F_1(1) = \text{Ad}U_t(F_2(1)) \in M(C(S^{2m-1}) \otimes K)\}.
\]
The algebras \(M\) and \(M_{\alpha'}\) are \(C(S^{2m})\)-linearly isomorphic to \(M(C(S^{2m}) \otimes K)\) and we identify \(M\) with \(M(C(S^{2m}) \otimes K)\).

**Definition 3.5.** We define a map \(\theta_{\alpha'} : M_{\alpha'} \to M\) by the \(C(S^{2m})\)-linear isomorphism
\[
\theta_{\alpha'}(F_1, F_2) := (F_1, \text{Ad}U(F_2)), \quad F_1 \in M(C(S^{2m-1}) \otimes K).
\]
The algebras \(B_{\alpha'}\) and \(C_{\alpha'}\) defined in Definition 3.2 are subalgebras of \(M_{\alpha'}\). We denote by \(\tilde{l}\) the constant map \(S^{2m-1} \to \{1 \oplus O_{n+1}\}\) and denote by \(\tilde{l}\) the induced map \(S^{2m-1} \to \{1 \oplus O_{n+1}\}\). If \(\alpha'\) is homotopic to \(l\) in \(\text{Map}(S^{2m-1}, \text{End } E_{n+1})\), then \([\alpha] = 0\) in \([S^{2m-1}, \text{End } O_{n+1}]\) because of \([S^{2m-1}, \text{End } O_{n+1}] = [S^{2m-1}, \text{Aut } O_{n+1}]\) by [12, Proposition 6.1], and there is a trivialization \(\varphi_{\alpha'} : C(S^{2m}) \otimes O_{n+1} \to A_{\alpha'}\). We can explicitly construct \(\varphi_{\alpha'}\) from the homotopy between \(\alpha'\) and \(l\).

**Definition 3.6.** Let \(\alpha'\) be an element of \(\text{Map}(S^{2m-1}, \text{Aut } E_{n+1})\) homotopic to \(l\) in \(\text{Map}(S^{2m-1}, \text{End } E_{n+1})\), and let \(h_t : [0, 1] \times S^{2m-1} \to \text{Aut } O_{n+1}\) be a path from \(l = h_0\) to \(\alpha' = h_1\). Then we define the map \(\varphi_{\alpha'}\) as a \(C(S^{2m})\)-linear isomorphism of the form
\[
\varphi_{\alpha'} : C(S^{2m}) \otimes O_{n+1} \ni (a_1(s), a_2(t)) \mapsto (a_1(s), h_t(a_2(t))) \in A_{\alpha'}, \quad s, t \in [0, 1],
\]
where \(C(S^{2m}) \otimes O_{n+1}\) is identified with the algebra
\[
A_t = \{(a_1, a_2) \in C([0, 1] \times S^{2m-1}) \otimes O_{n+1} | a_1(0) \in 1_{C(S^{2m-1})} \otimes O_{n+1}, \quad a_1(1) = a_2(1) \in C(S^{2m-1}) \otimes O_{n+1}\}.
\]

The map \(\tau_{\alpha'} \circ \varphi_{\alpha'}\) is the Busby invariant of a unital essential extension of \(C(S^{2m}) \otimes O_{n+1}\). The following lemma says that \(\tau_{\alpha'} \circ \varphi_{\alpha'} \sim_{u.e.} \tau = id_{C(S^{2m})} \circ \tau_0\) where \(\tau_0\) is the Busby invariant in Definition 2.5.

**Lemma 3.7.** Let \(m \geq 1\) be a natural number and let \(\alpha'\) be an element of \(\text{Map}(S^{2m-1}, \text{Aut } E_{n+1})\) which is homotopic to \(l\) in \(\text{Map}(S^{2m-1}, \text{End } E_{n+1})\). Let \(\theta_{\alpha'} \circ \varphi_{\alpha'}\) and \(\tau\) be as above. Let \(i : C(S^{2m}) \oplus \ell_2 \to \{f_1, f_2\} \in B_{\alpha'}\) be the canonical unital embedding and let \(j : C(S^{2m}) \otimes K \subset \theta_{\alpha'}(B_{\alpha'})\) be the inclusion map. Then the following hold.

1. \((\theta_{\alpha'} \circ i)_* : K_0(C(S^{2m})) \cong K_0(\theta_{\alpha'}(B_{\alpha'})))\).

2. We denote \(g_1 := (\theta_{\alpha'} \circ i)_*, \quad g_2 := (\theta_{\alpha'} \circ i)_* (b_1)\), where \(b_1\) is a generator of \(K_0(C(S^{2m})))

3. We have \(j_* (1_{C(S^{2m})} \otimes e_0) = -g_1\), and there exists a generator \(b_2 \in K_0(C(S^{2m}) \otimes K)\) with \(j_* (b_2) = -g_2\).

In particular, it follows that \([\tau_{\alpha'} \circ \varphi_{\alpha'}] = [\tau] \in \text{Ext}(C(S^{2m}) \otimes O_{n+1}, C(S^{2m}) \otimes K)\). We note that both \(b_1 \in K_0(C(S^{2m}))\) and \(b_2 \in K_0(C(S^{2m}) \otimes K)\) correspond to the generator of \(K_0(C_0(S^{2m}, pt)) = \mathbb{Z}\).
Proof. We identify the sphere $S^{2m}$ with the space $D \cup S^{2m-1} \cup D$ where $D$ is the interior of the 2m-disc, and we identify $C_0(D)$ with the algebra $\{ F \in C_0([0,1]) \mid C(S^{2m-1}) \mid F(0) \in C_1(C(S^{2m-1})) \}$. Let $x_0 \in S^{2m-1}$ be the base point of $S^{2m}$ and $S^{2m-1}$. The map $\alpha : C_0(D) \ni F \to (F, 0) \in C_0(S^{2m}, x_0)$ induces an isomorphism of K-groups. An element $b_1$ is the generator of $K_0(C_0(S^{2m}, x_0))$. Let $i : (C_0(D) \otimes E_{n+1}) \otimes \mathbb{Z} \to (F_1, F_2) \mapsto (F_1, F_2) \in B_{\alpha'}$ be an embedding, and let $r : B_{\alpha'} \ni (F_1, F_2) \mapsto F_1(1) \in C(S^{2m-1}) \otimes E_{n+1}$ be the restriction map.

First, we show (1). We have the following commutative diagram

\[
\begin{array}{ccc}
(C_0(D) \otimes E_{n+1}) \otimes \mathbb{Z} & \xrightarrow{r} & C(S^{2m-1}) \otimes E_{n+1} \\
\downarrow \alpha & & \downarrow \alpha \\
(C_0(D) \otimes E_{n+1}) \otimes \mathbb{Z} & \xrightarrow{r} & C(S^{2m-1}) \otimes E_{n+1}
\end{array}
\]

From the KK-equivalence of $E_{n+1}$ and $C$, the vertical maps $(id_{C_0(D)} \otimes 1_{E_{n+1}}) \otimes \mathbb{Z}$ and $id_{C(S^{2m-1})} \otimes 1_{E_{n+1}}$ induce isomorphisms of K-groups. Therefore the map $K_0(i) : K_1(C(S^{2m})) \to K_1(B_{\alpha'})$ is an isomorphism by 6-term exact sequences and the 5-lemma.

Second, we find $b_2$. We denote by $t_1$ the inclusion $C_0(D) \otimes E_{n+1} \ni F_1 \to (F_1, 0) \in (C_0(D) \otimes E_{n+1}) \otimes \mathbb{Z}$ and denote by $t_1$ the restriction to $C_0(D) \otimes K$. We consider the following commutative diagram

\[
\begin{array}{ccc}
C_0(D) \otimes K & \xrightarrow{t_1} & (C_0(D) \otimes K) \otimes \mathbb{Z} \\
\downarrow \alpha \otimes \alpha & & \downarrow \alpha \otimes \alpha \\
(C_0(D) \otimes E_{n+1}) & \xrightarrow{t_1} & (C_0(D) \otimes E_{n+1}) \otimes \mathbb{Z}
\end{array}
\]

Since $\theta_{\alpha'} \circ t \circ t_1 = t_0 \otimes id_K$ and $K_0(t_0)$ is an isomorphism of K-groups, from diagram chasing we can find a generator $b_2 \in K_0(C_0(D) \otimes K)$ that is sent to $-n \theta_1 \in K_0(C_0(S^{2m}, x_0))$ by the map $K_0(\theta_{\alpha'})^{-1} \circ K_0(f) \circ K_0(\theta_{\alpha'} \circ t \circ t_1)$. Hence we have $b_2 := K_0(\theta_{\alpha'} \circ t \circ t_1)(b_2)$.

Third, we show $j_*([1 \otimes e]) = -n g_1$. From the assumptions, there exists the map $h' : [0, 1] \times S^{2m-1} \to \text{End}_* E_{n+1}$ with $h'_1 = \alpha'$, $h'_0 = l$. We have the unital $*$-homomorphism

\[
\eta : B_{\alpha'} \ni (F_1(s), F_2(t)) \mapsto (F_1(s), h'_1(F_2(t)) \in B_1 = C(S^{2m}) \otimes E_{n+1}
\]

which sends $(e, e) \in B_{\alpha'}$ to $(e, e) \in B_1 = C(S^{2m}) \otimes E_{n+1}$. We have $\theta_{\alpha'}^{-1}(j(1_{C(S^{2m})} \otimes e)) = (e, e) \in B_{\alpha'}$ and $(e, e) = 1_{C(S^{2m})} \otimes e \in B_1 = C(S^{2m}) \otimes E_{n+1}$, and the following commutative diagram holds

\[
\begin{array}{ccc}
K_0(B_{\alpha'}) & \xrightarrow{K_0(\eta)} & K_0(B_1) \\
\downarrow K_0(\theta_{\alpha'}) & & \downarrow K_0(\theta_{\alpha'}(B_1)) \\
K_0(C(S^{2m})) & \xrightarrow{K_0(\theta_{\alpha'}(B_1))} & K_0(C(S^{2m}))
\end{array}
\]

We have

\[
K_0(\eta)([\theta_{\alpha'}^{-1}(j(1_{C(S^{2m})} \otimes e))]) = [(e, e)] = -n[1_{\theta_{\alpha'}}] = K_0(\eta)((-n)[1_{\theta_{\alpha'}}]) = 0.
\]

Since $i_*$ is an isomorphism, the map $\eta_*$ is also an isomorphism, and we have $j(1_{C(S^{2m})} \otimes e)] = -n[1_{\theta_{\alpha'}}] = 0 = -ng_1$.

Finally, we show that $[\tau_{\alpha'} \circ \varphi_{\alpha'}] = [\tau]$. By Theorem 2.3, we identify $Ext(C(S^{2m}) \otimes \mathbb{Z}, C(S^{2m}) \otimes \mathbb{Z})$ with $Ext^2_\mathbb{Z}(K_0(C(S^{2m}) \otimes \mathbb{Z}), K_0(C(S^{2m}) \otimes \mathbb{Z}))$. The element $[\tau_{\alpha'} \circ \varphi_{\alpha'}]$ is identified with the class of extension

\[
[K_0(C(S^{2m}) \otimes \mathbb{Z}) \to K_0(\theta_{\alpha'}(B_{\alpha'})) \to K_0(C(S^{2m}) \otimes \mathbb{O}_{n+1})].
\]

By the computation above, it is equal to the class $[\mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z} \otimes \mathbb{Z}] = [\tau]$.

We have the Busby invariants of two extensions of $\mathbb{O}_{n+1}$ by $C(S^{2m}) \otimes \mathbb{Z}$:

\[
\sigma_{\alpha'} := \tau_{\alpha'} \circ \varphi_{\alpha'} \circ (1_{C(S^{2m})} \otimes id_{\mathbb{O}_{n+1}}),
\sigma := 1_{C(S^{2m})} \otimes \tau_0.
\]
From the lemma above, we have
\[ [\sigma_{\alpha'}] = (1 \otimes \text{id}_{\mathcal{O}_{n+1}})^*([\tau_{\alpha'} \circ \varphi_{\alpha'}]) = (1 \otimes \text{id}_{\mathcal{O}_{n+1}})^*([\tau]) = [\sigma] \]
in $\text{Ext}(\mathcal{O}_{n+1}, C(S^{2m}) \otimes \mathbb{K})$. Hence there exists a unitary $u_{\alpha'}$ in $U_{Q(C(S^{2m}) \otimes \mathbb{K})}$ satisfying $\sigma_{\alpha'} = \text{Ad}u_{\alpha'} \circ \sigma$ by Lemma 2.14.

Let $\varphi_0: H \to H^{2n+1}$ be a unitary operator and let $w$ be an element of the form $1_{C(S^{2m})} \otimes w_0$. The element $w \in M_{n+1}(\mathcal{M}(C(S^{2m}) \otimes \mathbb{K}))$ is a partial isometry with $ww^* = 1_{n+1}$ and $w^*w = 1$ in $M_{n+1}(\mathcal{M}(C(S^{2m}) \otimes \mathbb{K}))$. For a unital essential extension $\nu: \mathcal{O}_{n+1} \to \mathcal{O}(C(S^{2m}) \otimes \mathbb{K})$, we take a unitary $V_\nu$ introduced in [24, Section 1] :
\[
V_\nu = \begin{pmatrix} 0 & \nu(S) \\ \pi(w) & 0_{n+1} \end{pmatrix} \in M_{n+2}(\mathcal{O}(C(S^{2m}) \otimes \mathbb{K})).
\]
We denote $(S_1, \ldots, S_{n+1})$ by $S$. We claim that, for the above $\sigma$, we have $\text{ind}(V_{\nu}) = -|1_{C(S^{2m})} \otimes e|_0 \in K_0(C(S^{2m}) \otimes \mathbb{K})$. Indeed, there is a unitary lift
\[
\begin{pmatrix}
1_{C(S^{2m})} \otimes \mathbb{T} & 1_{C(S^{2m})} \otimes \varepsilon \\
0 & 0 \\
0_{n+1} & 0_{n+1} & w^* \\
1 & 0 & 0 & 0
\end{pmatrix}
\]
of $V_\nu \oplus V_{\nu}'$, where $\mathbb{T} = (T_1, \ldots, T_{n+1})$.

From direct computation of the index map, we have $\text{ind}(V_{\nu}) = -|1_{C(S^{2m})} \otimes e|_0 \in K_0(C(S^{2m}) \otimes \mathbb{K})$. Direct computation yields
\[
V_{\sigma_{\alpha'}} V_{\varepsilon}^* = \sum_{i=1}^{n+1} w_{\alpha'}(S_i) w_{\sigma_{\alpha'}}(S_i^*) \otimes 1_{n+1} = (w_{\alpha'} \otimes 1_{n+1}) \left( \begin{array}{ccc} S & 0 & 0 \\
0_{n+1} & S^* & 0 \\
0 & 0 & 1_1 \end{array} \right) \left( \begin{array}{ccc} S^* & 0 & 0 \\
0_{n+1} & S & 0 \\
0 & 0 & 1_1 \end{array} \right).
\]
Hence we have $|V_{\sigma_{\alpha'}}|_1 - |V_{\varepsilon}|_1 = -n|w_{\alpha'}|_1$ in $K_1(Q(C(S^{2m}) \otimes \mathbb{K}))$.

We show $|w_{\alpha'}|_1 = 0 = |w_{\varepsilon}|_1$ in $K_1(Q(C(S^{2m}) \otimes \mathbb{K}))$ in Theorem 3.11, and we need the following three lemmas for that. Recall the path $h: [0, 1] \times S^{2m-1} \to \text{Aut} \mathcal{O}_{n+1}$ from $h_0 = h_1$ to $\alpha' = h_1$ in Definition 3.6. Here and subsequently, we write the unitary $\sum_{i=1}^{n+1} h(S_i) 1 \otimes S_i^* \in U_{C([0, 1] \times S^{2m-1}) \otimes \mathcal{O}_{n+1}}$ by $v$ where we denote $1_{C([0, 1] \times S^{2m-1})} \otimes S_i$ by $1 \otimes S_i$ for simplicity. We denote
\[
W = \begin{pmatrix} 0 & 0 & 1_{n+2} \\
w & 0 & 0 \\
0 & 0 & 1_{n+1} \end{pmatrix} \in M_{2n+4}(M).
\]
By the definition of $\tau_{\sigma_{\alpha'}}$ and $\varphi_{\alpha'}$, the following lemma holds.

**Lemma 3.8.** Let $y_{\alpha'}$ be an element of the form
\[
y_{\alpha'} := \left( \begin{array}{ccc} 0_1 & 0 & 0 \\
0 & S & 0 \\
0 & 0_{n+1} & 0_1 \\
1_{n+2} & 0 & 0 \\
0 & 0 & 0_{n+1} \end{array} \right) \in M_{2n+4}(A_{\alpha'})
\]
where we write $1_{C([0, 1] \times S^{2m-1})} \otimes S$ simply by $S$. Then we have
\[
V_{\sigma_{\alpha'}} \oplus V_{\varepsilon}^* = \pi(W) + \tau_{\sigma_{\alpha'}} \otimes \text{id}_{\mathcal{O}_{n+1}}(y_{\alpha'}).
\]

In the lemma below, we regard an element $x \in C([0, 1] \times S^{2m-1}) \otimes E_{n+1}$ as a $C(S^{2m-1}) \otimes E_{n+1}$ valued continuous function on $[0, 1]$ and denote this by $x_t$, $t \in [0, 1]$, and frequently write $1_{C(S^{2m-1}) \otimes E_{n+1}}$ for simplicity.

**Lemma 3.9.** Let $V \in U_{C([0, 1] \times C(S^{2m-1}) \otimes \mathcal{O}_{n+1})}$ be a unitary with $V_0 = 1_{C(S^{2m-1}) \otimes \mathcal{O}_{n+1}}$. Then we can choose a unitary $V \in U_{C([0, 1] \times C(S^{2m-1}) \otimes \mathcal{O}_{n+1})}$ satisfying the following
\[
V_0 \equiv 1_{C(S^{2m-1}) \otimes E_{n+1}},
\]
\[
1_{C(S^{2m-1}) \otimes E_{n+1}} - V_t \in C(S^{2m-1}) \otimes \mathbb{K},
\]
\[
V_t V_1(1_{C(S^{2m-1}) \otimes e}) = (1_{C(S^{2m-1}) \otimes e}) V_t V_1 = (1_{C(S^{2m-1}) \otimes e}), \ t \in [0, 1].
\]
Proof. There is a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ satisfying,

$$||V_t(1_{C(S^{2m-1})} \otimes e)V_t^* - V_{t_k}(1_{C(S^{2m-1})} \otimes e)V_{t_k}^*|| < 1, \ t \in [t_k,t_{k+1}].$$

We construct the unitary $\mathcal{V}$ by induction. For $t \in [t_0,t_1]$, we have a polar decomposition

$$V_t(1_{C(S^{2m-1})} \otimes (1-e))V_t^* = w_t^0[V_t(1_{C(S^{2m-1})} \otimes (1-e))V_t^*(1_{C(S^{2m-1})} \otimes (1-e))] \quad (1)$$

for $t \in [t_0,t_1]$, and there exists a unitary

$$\mathcal{V}^0_t := \begin{cases} w_t^0 + V_t(1_{C(S^{2m-1})} \otimes e), & t \in [t_0,t_1] \\ w_t^0 + V_t(1_{C(S^{2m-1})} \otimes e), & t \in [t_1,t_m]. \end{cases}$$

with $\mathcal{V}^0_t = 1_{C(S^{2m-1})}$. Since $\pi(V_t(1_{C(S^{2m-1})} \otimes (1-e))V_t^*(1_{C(S^{2m-1})} \otimes (1-e))) = 1_{C(S^{2m-1})} \otimes C_{n+1}$ and (1), we have

$$1_{C(S^{2m-1})} - \mathcal{V}^0_t \in C(S^{2m-1}) \otimes K.$$ The unitary $\mathcal{V}^0_t \in U(\mathcal{A}_{n+1},\mathcal{R}(S^{2m-1}) \otimes E_{n+1})$ satisfies the following

$$V_t^0*V_t(1_{C(S^{2m-1})} \otimes e)V_t^0 - V_{t_k}^0*V_{t_k}(1_{C(S^{2m-1})} \otimes e)V_{t_k}^0 \mathcal{V}^0_t < 1, \ t \in [t_k,t_{k+1}], \ m - 1 \geq k \geq 0. \quad (2)$$

The condition (2) is satisfied by the computation below

$$\mathcal{V}^0_t = V_t(1_{C(S^{2m-1})} \otimes e)V_t^0 - V_{t_k}^0*V_{t_k}(1_{C(S^{2m-1})} \otimes e)V_{t_k}^0 \mathcal{V}^0_t = \begin{cases} 0, & t \in [t_0,t_1] \cap [t_k,t_{k+1}] \\ \text{AdV}_{t_k}^0(V_t(1_{C(S^{2m-1})} \otimes e)V_t^0 - V_{t_k}(1_{C(S^{2m-1})} \otimes e)V_{t_k}^0), & t \in [t_1,t_m] \cap [t_k,t_{k+1}]. \end{cases}$$

Let $l$ be a number with $m - 1 \geq l \geq 0$. Assume that there exist unitaries $\mathcal{V}^0, \cdots, \mathcal{V}^l$ satisfying

$$1_{C(S^{2m-1})} \otimes \mathcal{E}_{n+1} - \mathcal{V}^l \in C(S^{2m-1}) \otimes K, \quad \mathcal{V}^0 = 1_{C(S^{2m-1})} \otimes \mathcal{E}_{n+1}, \ l \geq i \geq 0, \quad U_i^*(1_{C(S^{2m-1})} \otimes e) = (1_{C(S^{2m-1})} \otimes e)U_i^* = (1_{C(S^{2m-1})} \otimes e), \ t \in [t_0,t_{l+1}]. \quad (3)$$

where we denote $U_i^l = V_i^l \mathcal{V}^0 \cdots \mathcal{V}^l$. Now we construct a unitary $\mathcal{V}^{l+1}$ satisfying

$$1_{C(S^{2m-1})} \otimes \mathcal{E}_{n+1} - \mathcal{V}^{l+1} \in C(S^{2m-1}) \otimes K, \quad \mathcal{V}^{l+1} = 1_{C(S^{2m-1})} \otimes \mathcal{E}_{n+1}, \quad (4)$$

$$||U_i^{l+1}U_i^*(1 \otimes e)U_i^{l+1} - U_i^{l+1}U_i^*(1 \otimes e)U_i^{l+1}|| < 1, \ t \in [t_k,t_{k+1}], \ m - 1 \geq k \geq 0, \quad (5)$$

where we write $1_{C(S^{2m-1})} \otimes e$ simply by $1 \otimes e$. By the assumption (3), we have a partial isometry $w_i^{l+1}$ from a polar decomposition

$$(1_{C(S^{2m-1})} - U_i^l(1 \otimes e)U_i^l)(1_{C(S^{2m-1})} - U_i^l(1 \otimes e)U_i^l)^* = w_i^{l+1}(1_{C(S^{2m-1})} - U_i^l(1 \otimes e)U_i^l)(1_{C(S^{2m-1})} - U_i^l(1 \otimes e)U_i^l)^*, \ t \in [t_{k+1},t_{k+2}]. \quad (7)$$

Let $\mathcal{V}^{l+1}$ be a unitary of the form

$$\mathcal{V}^{l+1} := \begin{cases} 1_{C(S^{2m-1})}, & t \in [0,t_{l+1}] \\ w_i^{l+1} + U_i^{l+1}V_i^{l+1}(1 \otimes e)U_i^{l+1}, & t \in [t_{l+1},t_{l+2}] \\ w_i^{l+1} + U_i^{l+1}V_i^{l+1}(1 \otimes e)U_i^{l+1}, & t \in [t_{l+2},t_m]. \end{cases}$$

Since $\pi(1_{C(S^{2m-1})} - U_i^{l+1}(1 \otimes e)U_i^{l+1})(1_{C(S^{2m-1})} - U_i^{l+1}(1 \otimes e)U_i^{l+1}) = 1_{C(S^{2m-1})} \otimes C_{n+1}$ and (7), we have (4). By the construction of $\mathcal{V}^{l+1}$, we have

$$\mathcal{V}^{l+1}U_i^*(1 \otimes e) = \begin{cases} (1 \otimes e)\mathcal{V}^{l+1}U_i^*(1 \otimes e), & t \in [0,t_{l+1}] \\ U_i^{l+1}(1 \otimes e)U_i^{l+1}U_i^*(1 \otimes e) = (1 \otimes e)U_i^{l+1}U_i^*(1 \otimes e)U_i^{l+1}U_i^*(1 \otimes e)\mathcal{V}^{l+1}U_i^*(1 \otimes e), & t \in [t_{l+1},t_{l+2}] \end{cases}$$
and $y^{i+1}$ satisfies (5). For every $k, m - 1 \geq k \geq 0$, direct computation yields

$$y^{i+1} = y^{i+1}((1 \times e) U^i V^{i+1} - y^{i+1} t_k + 1 (1 \times e) U^i k_{i+1} y^{i+1} t_k + 1 \times e) U^i k_{i+1} y^{i+1} t_k + 1 \times e) U^i k_{i+1} \times e) U^i k_{i+1} y^{i+1} t_k + 1 \times e) U^i k_{i+1} y^{i+1} t_k + 1 \times e) U^i k_{i+1}$$

$$= 0, t \in [t_0, t_k + 1] \cap [t_k, k_{i+1}]$$

$$\text{Adj} y^{i+1} U^i y^{i+1} (1 \times e) U^i y^{i+1} (1 \times e) U^i y^{i+1} (1 \times e) U^i y^{i+1} (1 \times e) U^i y^{i+1} (1 \times e) U^i y^{i+1}$$

and the condition (6) is satisfied by (3). Now we have a sequence of unitaries $Y^0, \ldots, Y^{m-1}$ by induction, and a unitary $V_i := Y^0 \ldots Y^{m-1}$ satisfies the assertion of the lemma.

In the sequel, we denote by $u_{\alpha'}$ the element $\sum_{i=1}^{n+1} (1 - T) T^*$. The element $u_{\alpha'}$ is in $U_{C(G, S^{2m-1}) \otimes (1 - e) E_n \oplus (1 - e)}$ because $\alpha' \in \text{Map}(S^{2m-1}, \text{Aut}_c E_{n+1})$. We remark that

$$\pi(u_{\alpha'}) = \sum_{i=1}^{n+1} \pi_0 (1 - T) T^* \pi_0 = v_0 \in C(S^{2m-1}) \otimes \text{End} E_{n+1}.$$ 

We also note $v \in U_{C([0,1] \times S^{2m-1}) \otimes C_n}$ because $v_0 = 1_C(S^{2m-1})$.

**Lemma 3.10.** There exists a unitary $V \in U_{C([0,1] \times S^{2m-1}) \otimes C_n}$ satisfying the following

$$\pi(V) = v, V_0 = 0, V_1 = u_{\alpha'} + 1_C(S^{2m-1}) \otimes e,$$

$$V_t (1 - C(S^{2m-1}) \otimes e) = (1 - C(S^{2m-1}) \otimes e) V_t = (1 - C(S^{2m-1}) \otimes e), t \in [0, 1].$$

In particular, the element $Y_{\alpha'}$ of the form

$$Y_{\alpha'} := \begin{pmatrix} 0 & T & e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}_{2n+4}(B_{\alpha'})$$

is sent to $y_{\alpha'}$ by the quotient map $\pi_{\alpha'}: B_{\alpha'} \to A_{\alpha'}$ where we write $1_C([0,1] \times S^{2m-1}) \otimes T$ and $1_C([0,1] \times S^{2m-1}) \otimes e$ by $T$ and $e$ respectively for simplicity.

**Proof.** Since $v \in U_{C([0,1] \times S^{2m-1}) \otimes C_n}$, one has a unitary lift $V' \in U_{C([0,1] \times S^{2m-1}) \otimes C_n}$ of $v$ with $V_0 = 0$. By Lemma 3.9, we may assume the following

$$V'_t (1_C(S^{2m-1}) \otimes e) = (1_C(S^{2m-1}) \otimes e) V'_t = (1_C(S^{2m-1}) \otimes e), t \in [0, 1]$$

$$V'_0 = 1_C(S^{2m-1}) \otimes E_{n+1}$$

Now we show that we can get the unitary $V$ by a compact perturbation of $V'$. By (8), the element $u_{\alpha'} V'_t$ is a unitary in $U_{C(S^{2m-1}) \otimes (1 - e) E_n \oplus (1 - e)}$ with $\pi(u_{\alpha'} V'_t) = 1_C(S^{2m-1}) \otimes e$. Since $\alpha'$ is homotopic to $0$ in $\text{Map}(S^{2m-1}, \text{End}_c E_{n+1})$, the unitary $u_{\alpha'}$ is in $U_{C(S^{2m-1}) \otimes (1 - e) E_n \oplus (1 - e)}$. Hence we have $u_{\alpha'} + 1_C(S^{2m-1}) \otimes e \in U_{C([0,1] \times S^{2m-1}) \otimes C_n}$. Recall that the map $K_1(C(S^{2m-1}) \times \mathbb{K}) \hookrightarrow K_1(C(S^{2m-1}) \otimes E_{n+1})$ is injective because $\text{Tor}(K_1(C(S^{2m-1})) \otimes \mathbb{K}) = 0$. For every $t \in [0, 1]$, we have $\lambda(t) \in S^1$ with

$$\lambda(0) = \lambda(1) = 1, \lambda(t) \in \text{End}_c E_{n+1}(1 - e) \otimes (1 - e) E_n \oplus (1 - e) E_{n+1} < 0, \lambda(1 - e) \otimes (1 - e) E_n \oplus (1 - e) E_{n+1}$$

by (9) and $\pi(u_{\alpha'} V'_t) = 1_C(S^{2m-1}) \otimes (1 - e) \otimes (1 - e) E_n \oplus (1 - e) E_{n+1}$ and the function $\lambda: [0, 1] \to S^1$ is continuous. Now we get the element

$$V : (\lambda c + 1_C([0,1] \times S^{2m-1}) \otimes (1 - e)) V' \in U_{C([0,1] \times S^{2m-1}) \otimes (1 - e) E_n \oplus (1 - e) E_{n+1}}$$

satisfying the assertion of the lemma. Since $V_t (1_C(S^{2m-1}) \otimes T) = u_{\alpha'} (1_C(S^{2m-1}) \otimes T) = \alpha' \otimes (1_C(S^{2m-1}) \otimes T)$ and $\alpha' (1_C(S^{2m-1}) \otimes e) = 1_C(S^{2m-1}) \otimes e$, direct computation yields

$$\alpha' \otimes \text{id}_{\mathbb{M}_{2n+4}} \begin{pmatrix} 0 & V'_t & e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & T & e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence $Y_{\alpha'}$ is an element of $\mathbb{M}_{2n+4}(B_{\alpha'})$ that is sent to $y_{\alpha'}$ by the quotient map $\pi_{\alpha'}: B_{\alpha'} \to A_{\alpha'}$.
We remark that $Y_{\alpha'}$ is a partial isometry. We have $\theta_{\alpha'}(Y_{\alpha'})\theta_{\alpha'}(Y_{\alpha'})^* = 1 \oplus 0_{n+1} \oplus 0_{1} \oplus 1_{n+1}$ and $\theta_{\alpha'}(Y_{\alpha'})^* \theta_{\alpha'}(Y_{\alpha'}) = 0_{1} \oplus 1_{n+1} \oplus 1 \oplus 0_{n+1}$. Recall that $W$ is a partial isometry with $WW^* = 0_{1} \oplus 1_{n+1} \oplus 1 \oplus 0_{n+1}$ and $W^*W = 1 \oplus 0_{n+1} \oplus 0_{1} \oplus 1_{n+1}$. Therefore the element $W + \theta_{\alpha'}(Y_{\alpha'})$ is a unitary in $U_{S^{2m} + 1}(M)$.

**Theorem 3.11.** Let $m \geq 1$ be a natural number. Let $\alpha'$ be an element of $\text{Map}(S^{2m+1}, \text{Aut}, E_{n+1})$ that is homotopic to $l$ in $\text{Map}(S^{2m-1}, \text{End}, E_{n+1})$. Let $w_{\alpha'}$ be as mentioned above. Then we have $[w_{\alpha'}] = 0$ in $K_1(Q(C(S^{2m}) \otimes \mathbb{K}))$.

**Proof.** Let $Y_{\alpha'}, V_{\alpha'}$ be as before. Recall the following commutative diagram

$$
\begin{array}{ccc}
B_{\alpha'} & \xrightarrow{\pi_{\alpha'}} & A_{\alpha'} \\
\downarrow & & \downarrow \\
M_{\alpha'} & \xrightarrow{\theta_{\alpha'}} & M \\
\downarrow & & \downarrow \\
M & \xrightarrow{\pi} & Q(C(S^{2m}) \otimes \mathbb{K}).
\end{array}
$$

By Lemma 3.8 and Lemma 3.10, we have

$$V_{\alpha'} + V_{\alpha'}^* = \pi \otimes \text{id}_{S^{2m} + 1}(W) + \tau_{\theta_{\alpha'}} \otimes \text{id}_{S^{2m} + 1}(Y_{\alpha'}) = \pi \otimes \text{id}_{S^{2m} + 1}(W + \theta_{\alpha'} \otimes \text{id}_{S^{2m} + 1}(Y_{\alpha'})),$$

so $W + \theta_{\alpha'}(Y_{\alpha'})$ is a unitary lift of $V_{\alpha'} + V_{\alpha'}^*$. Let $P$ be the projection of the form

$$P := (W + \theta_{\alpha'}(Y_{\alpha'}))(1_{n+2} \otimes 0_{n+2})(W + \theta_{\alpha'}(Y_{\alpha'}))^*.$$

We have $\text{ind}[V_{\alpha'}, V_{\alpha'}^*] = \text{ind}[V_{\alpha'}, 1] - \text{ind}[\alpha, 1] = [P]_0 + [1_{C(S^{2m})} \otimes e][0] - [1_{n+2}]_0 \in K_0((C(S^{2m}) \otimes \mathbb{K}) \sim)$ and we show that the index is $0$. Recall $V_i(1_{C(S^{2m-1})} \otimes e) = (1_{C(S^{2m-1})} \otimes e)V_i = (1_{C(S^{2m-1})} \otimes e)$ and $U_i(1_{C(S^{2m-1})} \otimes e) = (1_{C(S^{2m-1})} \otimes e)U_i = (1_{C(S^{2m-1})} \otimes e)$ by Definition 3.5. Direct computation yields

$$P = W(1_{n+2} \otimes 0_{n+2})W^* + \theta_{\alpha'} \otimes \text{id}_{S^{2m} + 1}(Y_{\alpha'}(1_{n+2} \otimes 0_{n+2})Y_{\alpha'}^*) = 0_1 \oplus 1_{n+1} \oplus 0_{n+2} + (1 - e) \oplus 0_{n+1} \oplus 0_1 \oplus 0_{n+1}.$$

Now we have $P = (1 - e) \oplus 1_{n+1} \oplus 0_{n+2}$ and get $\text{ind}[V_{\alpha'}, V_{\alpha'}^*] = [P]_0 + [1_{C(S^{2m})} \otimes e][0] - [1_{n+2}]_0 = 0$. Therefore we have $-n[w_{\alpha'}] = [V_{\alpha'}, V_{\alpha'}^*] = 0$ and this proves the theorem because $\text{Tor}(K_1(Q(C(S^{2m}) \otimes \mathbb{K})), \mathbb{Z}_n) = 0$.

**Corollary 3.12.** For the above $\alpha'$, we have $[\alpha'] = 0$ in $[S^{2m-1}, \text{Aut}, E_{n+1}]$.

**Proof.** Since $[w_{\alpha'}] = 0$, there is a unitary $W_{\alpha'}$ in $U_{\text{M}(C(S^{2m}) \otimes \mathbb{K})}$ that is a lift of $w_{\alpha'}$. Therefore we have $C(S^{2m})$-linear isomorphism $\text{Ad}W_{\alpha'}: B_1 \to B_{\alpha'}$, and $P_1 \cong P_{\alpha'}$ from Lemma 2.12. By Lemma 3.1, we have $[\alpha'] = [0] = 0$.

**Lemma 3.13.** Let $m \geq 1$ be a natural number. Let $\alpha$ be an element in $\text{Map}(S^{2m-1}, \text{Aut}, E_{n+1})$. If $\alpha \sim_h l$ in $\text{Map}(S^{2m-1}, \text{End}, E_{n+1})$, then there exists $\alpha'$ in $\text{Map}(S^{2m-1}, \text{Aut}, E_{n+1})$ satisfying the following:

$\alpha' \sim_h l$ in $\text{Map}(S^{2m-1}, \text{End}, E_{n+1})$, $\alpha \sim_h \alpha'$ in $\text{Map}(S^{2m-1}, \text{Aut}, E_{n+1})$.

**Proof.** It follows from Lemma 2.34 that there is an exact sequence

$$[S^{2m}, B S^1] \to [S^{2m-1}, \text{End}, E_{n+1}] \to [S^{2m-1}, \text{End}, E_{n+1}] \to [S^{2m-1}, B S^1],$$

and by Remark 2.31 the map $[S^{2m-1}, \text{End}, E_{n+1}] \to [S^{2m-1}, U_{E_{n+1}}]$ which sends $[\alpha]$ to $[u_{\alpha}] := [\alpha + \sum_{i=1}^{n+1} \alpha(T_i)T_i^*] \in [S^{2m-1}, U_{E_{n+1}}]$ is an isomorphism. Since $B S^1$ is $K(\mathbb{Z}, 2)$ space, if $m \geq 2$ and $\alpha \sim_h l$ in $\text{Map}(S^{2m-1}, \text{End}, E_{n+1})$, then we have $[\alpha] = 0$ in $[S^{2m-1}, \text{End}, E_{n+1}]$ because $[S^{2m-1}, \text{End}, E_{n+1}] = [S^{2m-1}, \text{End}, E_{n+1}], m \geq 2$. Hence it is sufficient to show the claim in the case of $m = 1$. Let $\alpha$ be an element of $\text{Map}(S^1, \text{Aut}, E_{n+1})$ with $\alpha \sim_h l$ in $\text{Map}(S^1, \text{End}, E_{n+1})$. The computation in Theorem 2.36 yields that there exists $d \in \mathbb{Z}$ with $[u_{\alpha}] = -nd \in [S^1, U_{E_{n+1}}] = \mathbb{Z}$. We define $\rho_d$ by

$$\rho_d := \text{Ad}(zdT_1T_1^* + (1 - T_1T_1^*)) \in \text{Map}(S^1, \text{Aut}, E_{n+1}).$$
By Lemma 2.10, there is a continuous path from \((z^d T_1 T_i + (1 - T_1) T_i^*)\) to \(z^d\) in \(U_{C(S^1) \otimes E_{n+1}}\), and \(\rho_d \sim \text{Ad} z^d = l\) in \(\text{Map}(S^1, \text{Aut } E_{n+1})\). We have \([u_{\rho, \alpha}] = [\rho_d(u_{\alpha})]_1 + [\rho_d]_1\) in \(K_1(C(S^1) \otimes E_{n+1}) = [S^1, U_{E_{n+1}}]\). By Lemma 2.24, it follows that \([\rho_d(u_{\alpha})]_1 = [u_{\alpha}]_1\). Hence we have \([u_{\rho, \alpha}]_1 = -nd + [\rho_d]_1\). The following computations yield \([u_{\rho, \alpha}] = nd: \)

\[
u_{\rho_d} = e + \sum_{i=1}^{n+1} (z^d T_1 T_i + (1 - T_1) T_i^*) (z^d T_1 T_i + (1 - T_1) T_i^*)
\]

\[
= (z^d T_1 T_i + (1 - T_1) T_i^*) (e + \sum_{i=1}^{n+1} T_i (z^d T_1 T_i + (1 - T_1) T_i^*) T_i^*)
\]

\[
= \left(\begin{array}{cccc}
e + \sum_{i=1}^{n+1} T_i (z^d T_1 T_i + (1 - T_1) T_i^*) & 0 & 0 & 0 \\
0 & 1_{n+1}
\end{array}\right)
\]

\[
= \left(\begin{array}{cccc}
T & e & 0 & 0 \\
0_{n+1} & T
\end{array}\right)
\]

Therefore we have \([u_{\rho, \alpha}] = 0\) in \([S^1, U_{E_{n+1}}]\). By Remark 2.31, we have the isomorphism \([S^1, \text{End}_n E_{n+1}] \ni \rho_d \alpha \mapsto [u_{\rho, \alpha}] \in [S^1, U_{E_{n+1}}]\), and \(\alpha' = \rho_d \alpha\) satisfies all assumptions of the lemma.

We show the weak homotopy equivalence.

**Theorem 3.14.** The inclusion map \(\text{Aut } E_{n+1} \to \text{End}_n E_{n+1}\) is a weak homotopy equivalence.

**Proof.** By Lemma 3.3 and Theorem 2.36, we consider only the case of odd homotopy groups. Let \(k\) be an odd number.

First, we show the map \([S^k, \text{Aut } E_{n+1}] \to [S^k, \text{End}_n E_{n+1}]\) is injective. If \(\alpha\) in \(\text{Map}(S^k, \text{Aut } E_{n+1})\) is homotopic to \(l\) in \(\text{Map}(S^k, \text{End}_n E_{n+1})\), we may assume that there exists \(\alpha' \in \text{Map}(S^k, \text{Aut } E_{n+1})\) homotopic to \(\alpha\) by Remark 2.33. From Lemma 3.13, we may assume that \(\alpha' \sim_l l\) in \(\text{Map}(S^k, \text{End}_n E_{n+1})\), and we have \([\alpha'] = [\alpha] = 0\) in \([S^k, \text{Aut } E_{n+1}]\) by Corollary 3.12. Therefore the map \([S^k, \text{Aut } E_{n+1}] \to [S^k, \text{End}_n E_{n+1}]\) is injective.

Second, we show the surjectivity. The following commutative diagram holds

\[
\begin{array}{ccc}
[S^k, \text{Aut } E_{n+1}] & \xrightarrow{\text{Lemma 3.4}} & [S^k, \text{Aut } O_{n+1}] \\
\downarrow & & \downarrow \\
[S^k, \text{End}_n E_{n+1}] & \to & [S^k, \text{End}_n O_{n+1}].
\end{array}
\]

In the case of \(k = 1\), we have \([S^1, \text{End}_n E_{n+1}] = [S^1, \text{End}_n O_{n+1}] = \mathbb{Z}\) because the generators of the both groups are constructed from the canonical gauge actions of \(S^1\) that are of the form \(\lambda_c: T_i \mapsto z T_i\) and \(\lambda_S: S_i \mapsto z S_i\). Therefore the surjectivity follows from Lemma 3.4.

In the case of \(k \geq 3\), the map \([S^k, U_{E_{n+1}}] \to [S^k, \text{End}_n E_{n+1}]/\mathbb{Z}\) is in \(K_1(C(S^k) \otimes E_{n+1}) = \mathbb{Z}\) in Theorem 2.36 is an isomorphism. Therefore the map \(Z = [S^k, \text{End}_n E_{n+1}] \to [S^k, \text{End}_n O_{n+1}] = [S^k, U_{O_{n+1}}] = \mathbb{Z}\) is the quotient by \(n\mathbb{Z}\). Hence the image of the map \([S^k, \text{Aut } E_{n+1}] \to [S^k, \text{End}_n E_{n+1}] = \mathbb{Z}\) contains an element \(nd + 1\) for some \(d \in \mathbb{Z}\) by Lemma 3.4.

On the other hand, we show that the image contains \(n\mathbb{Z}\). For every \(V \in U_{C(S^k) \otimes E_{n+1}}\), there exists \(V' \in U_{C(S^k) \otimes E_{n+1}}\) with \(V'(1_{C(S^k)} \otimes e) = (1_{C(S^k)} \otimes e)V' = (1_{C(S^k)} \otimes e)\) which is homotopic to \(V\) in \(U_{C(S^k) \otimes E_{n+1}}\) by Remark 2.31.

Since the isomorphism \([S^k, U_{E_{n+1}}] \to [S^k, \text{End}_n E_{n+1}]\) sends

\[-n[V]_1 = [1_{C(S^k)} \otimes e + \sum_{i=1}^{n+1} V'(1 \otimes T_i) V'^*(1 \otimes T_i^*)],\]

to \([\text{Ad } V'] = [\text{Ad } V]\), the subset

\([\{[\text{Ad } V'] \in [S^k, \text{Aut } E_{n+1}] \mid V \in U_{C(S^k) \otimes E_{n+1}}\}\]

is mapped onto the subset

\([-n[V]_1 \in K_1(C(S^k) \otimes E_{n+1}) \mid V \in U_{C(S^k) \otimes E_{n+1}}\} = n\mathbb{Z} \subset [S^k, \text{End}_n E_{n+1}] = \mathbb{Z}.

Therefore the image contains \(nd + 1\) and \(n\mathbb{Z}\), and we have the conclusion.

\[\square\]
3.2 An exact sequence of homotopy sets

We have the principal Aut_\* E_n+1-bundle Aut_\* E_{n+1} \xrightarrow{\sim} Aut E_{n+1} \xrightarrow{\sim} B S^1. We denote by f the classifying map of the bundle and denote by r the restriction map Aut E_{n+1} \rightarrow Aut K. In this section, we show the following theorem.

**Theorem 3.15.** Let X be a compact CW-complex. Then we have the following exact sequence of pointed sets where the first four terms give the exact sequence of groups:

\[ H^1(X) \rightarrow K^1(X) \rightarrow [X, Aut E_{n+1}] \xrightarrow{\sim} H^2(X) \xrightarrow{f} [X, BAut E_{n+1}] \xrightarrow{Br_s} [X, BAut E_{n+1}] \xrightarrow{Br_c} H^3(X). \]

It follows that Im \eta \subset Tor(H^3(X), \mathbb{Z}_n) and Im Br_c \subset Tor(H^3(X), \mathbb{Z}_n).

The following lemma is well-known in homotopy theory. We refer to [20, Chap 3, Section 6]

**Lemma 3.16.** Let X be a CW-complex. Let G be a topological group and let H be a subgroup of G such that H \rightarrow G \rightarrow G/H is a principal H-bundle. Suppose that G/H has a homotopy type of a CW-complex. Let f : G/H \rightarrow B H be its classifying map. Then we have an exact sequence of pointed sets:

\[ [X, G] \rightarrow [X, G/H] \xrightarrow{f} [X, B H] \rightarrow [X, B G]. \]

Since B S^1 has a homotopy type of a CW-complex, we can apply the above lemma to Aut_\* E_{n+1} \rightarrow Aut E_{n+1} \rightarrow B S^1.

**Lemma 3.17.** Let X be a CW-complex. The following sequence of pointed sets is exact:

\[ [X, BAut_\* E_{n+1}] \xrightarrow{Br_s} [X, BAut E_{n+1}] \xrightarrow{Br_c} [X, BAut E_{n+1}] \xrightarrow{Br_s} [X, BAut E_{n+1}] \xrightarrow{Br_c} X, BAut_\* K]. \]

Proof. The group Aut_\* K is identified with the group U_{M(K)} by the map taking the implementing unitary \( U_{\alpha} = \sum_{i \in I} \alpha(e_i) e_{ii} \) for \( \alpha \in Aut_{\*+1} K \). Hence it is contractible, and \([X, BAut_{\*} K] = \{ pt \}\). From the commutative diagram below,

\[ \begin{array}{ccc}
[X, BAut E_{n+1}] & \xrightarrow{Br_s} & [X, BAut E_{n+1}] \\
\downarrow & & \downarrow \\
[X, BAut_\* K] & & [X, BAut E_{n+1}] \\
\end{array} \]

\( Br_s \circ Bi_\* \) is trivial. Therefore it is sufficient to prove that for every \( P \in \text{Map}(X, BAut E_{n+1}) \) with trivial associated bundle \( P \times Aut E_{n+1} \) \( Aut K \), the structure group of \( P \) is reduced to \( Aut E_{n+1} \). Let \( P \in \text{Map}(X, BAut E_{n+1}) \) be a principal Aut E_{n+1} bundle with trivial associated bundle \( P \times Aut E_{n+1} \) \( Aut K \). We take an open covering \( \{ U_i \} \) of \( X \) giving a local trivialization of \( P \), and denote by \( \phi_{ji} : U_j \cup U_i \rightarrow Aut E_{n+1} \) the transition function. By the assumption, there exists a map \( h_i : U_i \times Aut K \rightarrow U_i \times Aut K \) that is compatible with the transition functions, and is equivariant with respect to the right multiplication of \( Aut K \). The diagram below holds

\[ \begin{array}{ccc}
U_i \cap U_j \times Aut K & \xrightarrow{h_i} & U_i \cap U_j \times Aut K \\
\downarrow & & \downarrow & \downarrow \\
U_j \cap U_i \times Aut K & \xrightarrow{h_j} & U_j \cap U_i \times Aut K \\
\end{array} \]

We also denote by \( \phi_{ji} \) the map

\[ U_i \cap U_j \times Aut K \ni (x, \alpha) \mapsto (x, r(\phi_{ji}(x))\alpha) \in U_j \cap U_i \times Aut K. \]

We denote \( h_i(x) := Pr_i(h_i(x, id)) \) where \( Pr_i : U_i \times Aut K \rightarrow Aut K \). Since \( h_i \) is equivariant, we have \( h_i^{-1}(x) = h_i(x)^{-1} \). We have \( h_j(x)r(\phi_{ji}(x))h_i^{-1}(x)(\alpha) = e \) because \( h \circ \phi_{ji} \circ h_i^{-1}(x, id) = (x, id) \) for every \( x \in U_j \cap U_i \). If we take an appropriate refinement of \( \{ U_i \} \), we may assume that for every \( i \), there exists \( x_i \in U_i \) satisfying \( ||h_i^{-1}(x_i)(e) - h_i^{-1}(x_i)(e)|| < 1 \), \( x \in U_i \). There is a unitary \( V_i(x) \) that is the sum of partial isometries constructed from the polar decomposition of \( h_i^{-1}(x_i)(e)h_i^{-1}(x_i)(e) \) and \( (1 - h_i^{-1}(x_i)(e))(1 - h_i^{-1}(x_i)(e)) \), and \( V_i(x)h_i^{-1}(x_i)(e)V_i(x)^* = h_i^{-1}(x_i)(e) \) holds. We fix a unitary \( W_i \in U_{2K} \) with \( W_i e W_i^* = h_i^{-1}(x_i)(e) \). Then we have a unitary \( V_i(x) = V_i(x)W_i \in U_{2K} \) with \( V_i(x)W_i = h_i^{-1}(x_i)(e) \). The collection of the map

\[ u_i : U_i \times Aut E_{n+1} \ni (x, \alpha) \mapsto (x, Ad V_i \alpha) \in U_i \times Aut E_{n+1} \]

gives the following:

\[ \begin{array}{ccc}
U_i \cap U_j \times Aut E_{n+1} & \xrightarrow{u_i} & U_i \cap U_j \times Aut E_{n+1} \\
\downarrow & & \downarrow \phi_{ji} \\
U_j \cap U_i \times Aut E_{n+1} & \xrightarrow{u_j} & U_j \cap U_i \times Aut E_{n+1} \\
\end{array} \]
where $\hat{\phi}_{ji}$ is of the form $\hat{\phi}_{ji} : (x, \alpha) \mapsto (x, \Ad V_j(x)^* \phi_{ji}(x) \Ad V_i(x) \alpha)$.

We have the transition function

$$U_j \cap U_i \ni x \mapsto \Ad V_j(x)^* \phi_{ji}(x) \Ad V_i(x) \in \text{Aut}_c E_{n+1}$$

by the computation below:

$$\Ad V_j(x)^* \phi_{ji}(x) \Ad V_i(x)(e) = \Ad V_j(x)^* \phi_{ji}(x) h_j^{-1}(x)(e)$$
$$= \Ad V_j(x)^* h_j^{-1}(x)(e)$$
$$= \Ad V_j(x)^* \Ad V_i(x)(e)$$
$$= e.$$

The two bundles with transition maps $\phi_{ji}(x)$ and $\hat{\phi}_{ji}(x)$ are isomorphic. Therefore the structure group of $\mathcal{P}$ is reduced to $\text{Aut}_c E_{n+1}$. \(\square\)

**Lemma 3.18.** Let $X$ be a compact Hausdorff space. The map $\text{Aut}_c E_{n+1} \ni \alpha \mapsto e + \sum_{i=0}^{n+1} \alpha(T_i) T_i^* \in U_{E_{n+1}}$ induces a group isomorphism $[X, \text{Aut}_c E_{n+1}] \rightarrow [X, U_{E_{n+1}}] = K^1(X)$.

**Proof.** By Remark 2.35 and Theorem 3.14, the map is bijective. So we show that it is a group homomorphism. Let $\alpha$ and $\beta$ be elements of $\text{Map}(X, \text{Aut}_c E_{n+1})$, and we denote $u_{\alpha} = 1_{C(X)} \otimes e + \sum_{i=0}^{n+1} \alpha(T_i) T_i^* \in U_{C(X) \otimes E_{n+1}}$. We show $[u_{\alpha \beta}] = [u_{\alpha}] + [u_{\beta}]$. Since $\alpha$ and $\beta$ fix $e$, direct computation yields

$$u_{\alpha \beta} = \alpha(e + \sum_i \beta(1_{C(X)} \otimes T_i))(1_{C(X)} \otimes T_i^*)(e + \sum_i \alpha(1_{C(X)} \otimes T_i))(1_{C(X)} \otimes T_i^*)$$
$$= \alpha(u_{\beta} u_{\alpha}).$$

By Lemma 2.24, we have $[\alpha(u_{\beta})]_1 = K_1(\alpha)([u_{\beta}]_1) = [u_{\beta}]_1$. \(\square\)

We need the following fact to determine the second cohomology group of $\text{Aut} E_{n+1}$. See Allen Hatcher’s unpublished book [15, Proposition 5.11].

**Proposition 3.19.** Let $X$ be a path connected space with finite homotopy groups. Then its homology group $H_n(X)$ is finite for all $n > 0$.

**Lemma 3.20.** We have the following cohomology groups:

$$H^2(\text{Aut} E_{n+1}) = \mathbb{Z}_n, \quad H^3(\text{BAut} E_{n+1}) = \mathbb{Z}_n.$$

**Proof.** The two spaces $\text{Aut} E_{n+1}$ and $\text{Aut} O_{n+1}$ are path connected and the map $\text{Aut} E_{n+1} \rightarrow \text{Aut} O_{n+1}$ gives

$$\pi_1(\text{Aut} E_{n+1}) \cong \pi_1(\text{Aut} O_{n+1}), \quad i = 0, 1, 2$$
$$\pi_3(\text{Aut} E_{n+1}) \rightarrow \pi_3(\text{Aut} O_{n+1}).$$

So we have

$$H_i(\text{Aut} E_{n+1}) \cong H_i(\text{Aut} O_{n+1}), \quad i = 0, 1, 2$$
$$H_3(\text{Aut} E_{n+1}) \rightarrow H_3(\text{Aut} O_{n+1}).$$

by Whitehead’s theorem (see [5, Corollary 6.69]). By the universal coefficient theorem, we have

$$H^2(\text{Aut} E_{n+1}) \cong \text{free}(H_2(\text{Aut} E_{n+1})) \otimes \text{Tor}(H_1(\text{Aut} E_{n+1}))$$

where $\text{free}(H_2(\text{Aut} E_{n+1}))$ is the free part of the homology group. By Proposition 3.19, the homology groups of $\text{Aut} O_{n+1}$ are finite, and $\text{free}(H_2(\text{Aut} E_{n+1})) = 0$. So we have $H^2(\text{Aut} E_{n+1}) \cong \text{Tor}(H_1(\text{Aut} O_{n+1}))$. Hurewicz’ theorem ( [5, Theorem 6.66]) yields $H_1(\text{Aut} O_{n+1}) = \pi_1(\text{Aut} O_{n+1}) = \mathbb{Z}_n$. Similarly, we have $H^3(\text{BAut} E_{n+1}) = \mathbb{Z}_n$. \(\square\)

Now we prove Theorem 3.15.
Proof of Theorem 3.15. By Lemma 3.16 and the long exact sequence of the principal bundle $Aut_+ E_{n+1} \to Aut E_{n+1} \to BS^1$, we have an exact sequence of pointed sets where the first four terms give the exact sequence of groups:

$$H^1(X) = [X, S^1] \to [X, Aut_+ E_{n+1}] \to [X, Aut E_{n+1}] \to \cdots$$

By Lemma 3.16 and Lemma 3.18, we have the exact sequence:

$$H^1(X) \to K^1(X) \to [X, Aut_+ E_{n+1}] \to [X, Aut E_{n+1}] \to Br_+ \to H^3(X),$$

where we identify $H^3(X)$ with $[X, BAut K]$ because $BAut K$ is the $K(Z, 3)$-space. We identify $[Aut E_{n+1}, B S^1]$ with $H^3(Aut E_{n+1})$. For every $[\alpha] \in [X, Aut E_{n+1}]$, it follows that $\eta_*(\alpha) = \alpha^*(\eta)$, and the element $\alpha^*(\eta)$ is in the image of the map

$$\alpha^*: H^2(Aut E_{n+1}) = [Aut E_{n+1}, B S^1] \ni [\eta] \mapsto [\eta \circ \alpha] \in [X, B S^1] = H^2(X).$$

Therefore we have $\text{Im} \eta_* \subset \text{Tor}(H^2(X), \mathbb{Z}_n)$ from Lemma 3.20. Similar argument yields $\text{Im} Br_+ \subset \text{Tor}(H^3(X), \mathbb{Z}_n)$. □

Acknowledgements

The author would like to show his greatest appreciation to his supervisor Prof. Masaki Izumi who gave many insightful comments and suggestions, and patiently checked his arguments.

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