A sub-additive inequality for the volume spectrum

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Abstract

Let \((M, g)\) be a closed Riemannian manifold and \(\{\omega_p\}_{p=1}^{\infty}\) be the volume spectrum of \((M, g)\). We will show that \(\omega_{k+m+1} \leq \omega_k + \omega_m + W\) for all \(k, m \geq 0\), where \(\omega_0 = 0\) and \(W\) is the one-parameter Almgren-Pitts width of \((M, g)\). We will also prove the similar inequality for the \(\varepsilon\)-phase-transition spectrum \(\{c_\varepsilon(p)\}_{p=1}^{\infty}\) using the Allen-Cahn approach.

1 Introduction

For a closed Riemannian manifold \((M^{n+1}, g)\), the spectrum of the Laplacian on \((M, g)\), denoted by \(\{\lambda_p\}_{p=1}^{\infty}\), has the following min-max characterization. Let \(H^1(M)\) be the Sobolev space of real valued functions \(\varphi \in L^2(M)\) such that \(\nabla \varphi\), the distributional derivative of \(\varphi\), is also in \(L^2(M)\). Suppose \(V_p\) denotes the set of all \(p\)-dimensional vector subspaces of \(H^1(M)\). Then

\[
\lambda_p = \inf_{V \in V_p} \sup_{f \in V \setminus \{0\}} R(f) \quad \text{where} \quad R(f) = \frac{\int_M |\nabla f|^2}{\int_M f^2}.
\]  

(1.1)

The Rayleigh quotient \(R(f)\) is invariant under the scaling, i.e. for all \(c \in \mathbb{R} \setminus \{0\}\), \(R(cf) = R(f)\). Hence, it descends to a well-defined functional on \(\mathbb{P}(H^1(M))\), the projective space associated to the real vector space \(H^1(M)\).

In [Gro88, Gro03, Gro09], Gromov introduced various non-linear analogues of the spectrum of the Laplacian. In the context of the area functional and the minimal hypersurfaces, the relevant spectrum is the volume spectrum \(\{\omega_p\}_{p=1}^{\infty}\). \(\omega_p\) is defined by a min-max quantity, similar to (1.1). One replaces the vector space \(H^1(M)\) by \(Z_n(M; \mathbb{Z}_2)\), the space of mod 2 flat hypercycles which bound a region in \(M\) and the Rayleigh quotient by the area functional (see Section 2.2 for the precise definition). By the works of Almgren [Alm62] and Marques-Neves [MN18], the space \(Z_n(M; \mathbb{Z}_2)\) is weakly homotopy equivalent to \(\mathbb{R}P^{\infty}\). The cohomology ring \(H^*(Z_n(M; \mathbb{Z}_2), \mathbb{Z}_2)\) is the polynomial ring \(\mathbb{Z}_2[\bar{\lambda}]\) where \(\bar{\lambda} \in H^1(Z_n(M; \mathbb{Z}_2), \mathbb{Z}_2)\). In the definition of \(\omega_p\), instead of considering \(p\)-dimensional vector subspaces of \(H^1(M)\), one considers all the \(S \subset Z_n(M; \mathbb{Z}_2)\) such that \(\bar{\lambda}^p|_S \neq 0\) in \(H^p(S, \mathbb{Z}_2)\).

The connection between the volume spectrum and the minimal hypersurfaces comes from the Almgren-Pitts min-max theory, developed by Almgren [Alm65], Pitts [Pit81], Schoen-Simon

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If $\Pi$ is a homotopy class of maps $X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$, by the Almgren-Pitts min-max theory, the width of $\Pi$ is achieved by the area of a closed, minimal hypersurface (which can have a singular set of Hausdorff dimension $\leq n-7$ if the ambient dimension $n+1 \geq 8$), possibly with multiplicities. The index upper bound of the min-max minimal hypersurfaces was proved by Marques-Neves [MN16] and Li [Li19]. Sharp [Sha17] proved the compactness of the space of minimal hypersurfaces with bounded area and index, which holds in higher dimensions as well [Dey19]. Combining all these, one can conclude that [IMN18, Proposition 2.2] for each $p \in \mathbb{N}$, $\omega_p$ is achieved by the area of a closed, minimal hypersurface with optimal regularity (possibly with multiplicities), whose index is bounded above by $p$; see also [Li19b, Corollary 3.2] where Li gave a different argument. Moreover, if the ambient dimension $3 \leq n + 1 \leq 7$, by the works of Marques-Neves [MN18] and Zhou [Zho19], for a generic (bumpy) metric, $\omega_p$ is realized by the area of a closed, two-sided, minimal hypersurface with multiplicity one, whose index $= p$. When the ambient dimension is 3, this was also proved by Chodosh and Mantoulidis [CM20], in the Allen-Cahn setting.

By the works of Gromov [Gro88, Gro03], Guth [Gut09], Marques and Neves [MN17], there exist positive constants $C_1, C_2$, depending on the metric $g$, such that

$$C_1 p^{\frac{1}{n+1}} \leq \omega_p \leq C_2 p^{\frac{1}{n+1}} \quad \forall p \in \mathbb{N}. \quad (1.2)$$

In [LMN18], Liokumovich, Marques and Neves proved that $\{\omega_p\}_{p=1}^{\infty}$ satisfies the following Weyl law, which was conjectured by Gromov [Gro03].

$$\lim_{p \to \infty} \omega_p(M,g) p^{-\frac{1}{n+1}} = a(n) \Vol(M) p^{\frac{n+1}{n+1}}, \quad (1.3)$$

where $a(n) > 0$ is a constant, which depends only on the ambient dimension.

The above mentioned theorems regarding the asymptotic behaviour of $\{\omega_p\}_{p=1}^{\infty}$, has a number of important applications, which we discuss now. By the works of Marques-Neves [MN17] and Song [Son18], every closed manifold $(M^{n+1}, g)$, $3 \leq n + 1 \leq 7$, contains infinitely many closed minimal hypersurfaces, which confirms a conjecture of Yau [Yau82]. In [IMN18], Irie, Marques and Neves proved that for a generic metric $g$ on $M^{n+1}$, $3 \leq n + 1 \leq 7$, the union of all closed minimal hypersurfaces is dense in $M$. This theorem was quantified in [MNS19] by Marques, Neves and Song, where they proved that for a generic metric $g$, there exists an equidistributed sequence of closed minimal hypersurfaces in $(M, g)$. Recently, Song and Zhou [SZ20] proved the generic scarring phenomena for minimal hypersurfaces, which can be interpreted as the opposite of the equidistribution phenomena. (The arguments in the papers [Son18] and [SZ20] use the Weyl law for the volume spectrum on certain non-compact manifolds with cylindrical ends.) In higher dimensions, Li [Li19b] proved the existence of infinitely many closed minimal hypersurfaces (with optimal regularity) for a generic set of metrics.

The properties of the volume spectrum also turn out to be useful in the context of constant mean curvature (CMC) hypersurfaces. In [ZZ19], Zhou and Zhu developed the min-max
theory for CMC hypersurfaces, which was further extended by Zhou [Zho19]. In particular, they [ZZ19] proved that for all $c > 0$, every closed manifold $(M^{n+1}, g)$, $n + 1 \geq 3$, contains a closed $c$-CMC hypersurface (with optimal regularity). Building on the works in [ZZ19] and [Zho19], we proved in [Dey19b] that the number of closed $c$-CMC hypersurfaces (with optimal regularity) in $(M^{n+1}, g)$ is at least $g_0 c^{-\frac{1}{n+1}}$, where $g_0 > 0$ is a constant, which depends on the metric $g$. To obtain this estimate, we used the lower bound of $\omega_p$, stated in (1.2) and the following inequality.

\[ \omega_{p+1} \leq \omega_p + W \quad \forall p \in \mathbb{N}, \quad (1.4) \]

where $W$ is the one parameter Almgren-Pitts width of $(M, g)$.

In the present article, we will prove a more general sub-additive inequality for the volume spectrum, as stated below.

**Theorem 1.1.** Let $(M, g)$ be a closed Riemannian manifold. Let $\{\omega_p\}_{p=1}^\infty$ be the volume spectrum and $W$ be the one parameter Almgren-Pitts width of $(M, g)$ (see Section 2.2). Then, for all $k, m \geq 0$,

\[ \omega_{k+m+1} \leq \omega_k + \omega_m + W, \quad (1.5) \]

where we set $\omega_0 = 0$.

A similar inequality also holds for the phase-transition spectrum (as stated below in Theorem 1.2), which is the Allen-Cahn analogue of the volume spectrum. The Allen-Cahn min-max theory is a PDE based approach to the min-max construction of minimal hypersurfaces, which was introduced by Guaraco [Gua18] and further extended by Gaspar and Guaraco [GG18]. The regularity of the minimal hypersurfaces, obtained from the Allen-Cahn theory, depends on the previous works by Hutchinson-Tonegawa [HT00], Tonegawa [Ton05], Wickramasekera [Wic14] and Tonegawa-Wickramasekera [TW12]. In [GG18], Gaspar and Guaraco defined the phase-transition spectrum and proved a sub-linear growth estimate for them (which is similar to (1.2)). In [GG19], they proved a Weyl law for the phase-transition spectrum (which is similar to the Weyl law for the volume spectrum (1.3)).

**Theorem 1.2.** Let $(M, g)$ be a closed Riemannian manifold. For $\varepsilon > 0$, let $\{c_\varepsilon(p)\}_{p=1}^\infty$ be the $\varepsilon$-phase-transition spectrum and $\gamma_\varepsilon$ be the one parameter $\varepsilon$-Allen-Cahn width of $(M, g)$ (see Section 2.3). Then, for all $k, m \geq 0$,

\[ c_\varepsilon(k + m + 1) \leq c_\varepsilon(k) + c_\varepsilon(m) + \gamma_\varepsilon, \quad (1.6) \]

where $c_\varepsilon(0) = 0$. Hence, letting $\varepsilon \to 0^+$, we obtain the following inequality for the phase-transition spectrum $\{\ell_p\}_{p=1}^\infty$.

\[ \ell_{k+m+1} \leq \ell_k + \ell_m + \gamma, \]

where $\ell_0 = 0$ and $\gamma$ is the one parameter Allen-Cahn width of $(M, g)$. 

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Acknowledgements. I am very grateful to my advisor Prof. Fernando Codá Marques for many helpful discussions and for his constant support and guidance. The author is partially supported by NSF grant DMS-1811840.

2 Notation and Preliminaries

2.1 Caccioppoli sets

In this subsection, we will briefly recall the notion of the Caccioppoli set; further details can be found in [Sim83] and [AFP00]. An $\mathcal{H}^{n+1}$-measurable set $E \subset (M,g)$ is called a Caccioppoli set if $D\chi_E$, the distributional derivative of the characteristic function $\chi_E$, is a finite Radon measure on $M$. This is equivalent to

$$\sup \left\{ \int_E \text{div} \omega \, d\mathcal{H}^{n+1} : \omega \in \mathcal{X}^1(M), \|\omega\|_\infty \leq 1 \right\} < \infty;$$

where $\mathcal{X}^1(M)$ denotes the space of $C^1$ vector-fields on $M$. Let us use the notation $\mathcal{C}(M)$ to denote the space of all Caccioppoli sets in $M$. If $E \in \mathcal{C}(M)$, there exists an $n$-rectifiable set $\partial E$ such that the total variation measure $|D\chi_E| = \mathcal{H}^n \llcorner \partial E$. Hence, for all $\omega \in \mathcal{X}^1(M)$,

$$\int_E \text{div} \omega \, d\mathcal{H}^{n+1} = \int_{\partial E} \langle \omega, \nu_E \rangle \, d\mathcal{H}^n,$$

where $\nu_E$ is a $|D\chi_E|$-measurable vector-field; $\|\nu_E\| = 1 |D\chi_E|$-a.e. The following proposition can be found in [AFP00]. For the sake of completeness, we also include its proof (following [AFP00]).

Proposition 2.1 ([AFP00, Proposition 3.38]). Let $E \subset M$ be $\mathcal{H}^{n+1}$-measurable and $U \subset M$ be an open set. Let

$$P(E,U) = \sup \left\{ \int_E \text{div} \omega \, d\mathcal{H}^{n+1} : \omega \in \mathcal{X}^1_c(U), \|\omega\|_\infty \leq 1 \right\};$$

where $\mathcal{X}^1_c(U)$ denotes the space of compactly supported $C^1$ vector-fields on $U$. If $E, F \in \mathcal{C}(M)$,

$$P(E \cap F, U) + P(E \cup F, U) \leq P(E, U) + P(F, U). \tag{2.1}$$

Hence, if $E, F \in \mathcal{C}(M)$, $E \cap F$ and $E \cup F$ also belong to $\mathcal{C}(M)$.

Proof. Since $E, F \in \mathcal{C}(M)$, by [MPPP07, Proposition 1.4], there exist $\{f_i\}_{i=1}^\infty, \{g_i\}_{i=1}^\infty \subset C^\infty(U)$ such that $0 \leq f_i, g_i \leq 1$ for all $i$;

$$f_i \to \chi_E\vert_U \text{ in } L^1(U) \text{ and pointwise a.e.;} \quad P(E, U) = \lim_{i \to \infty} \int_U |\nabla f_i| \, d\mathcal{H}^{n+1};$$

$$g_i \to \chi_F\vert_U \text{ in } L^1(U) \text{ and pointwise a.e.;} \quad P(F, U) = \lim_{i \to \infty} \int_U |\nabla g_i| \, d\mathcal{H}^{n+1}.$$
By the dominated convergence theorem,

\[ f_i g_i \to \chi_{E \cap F} |_U \text{ in } L^1(U) \text{ and } f_i g_i - f_i g_i \to \chi_{E \cup F} |_U \text{ in } L^1(U). \]

Therefore,

\[
P(E \cap F, U) + P(E \cup F, U) \\
\leq \liminf_{i \to \infty} \int_U \left( |\nabla (f_i g_i)| + |\nabla f_i + \nabla g_i - \nabla (f_i g_i)| \right) \, d\mathcal{H}^{n+1} \\
\leq \liminf_{i \to \infty} \int_U \left( g_i |\nabla f_i| + f_i |\nabla g_i| + (1 - g_i) |\nabla f_i| + (1 - f_i) |\nabla g_i| \right) \, d\mathcal{H}^{n+1} \\
= P(E, U) + P(F, U).
\]

\[ \square \]

### 2.2 The space of hypercycles and the volume spectrum

For \( l \in \mathbb{N} \), let \( \mathbf{I}_l(M^{n+1}; \mathbb{Z}_2) \) be the space of \( l \)-dimensional flat chains in \( M \) with coefficients in \( \mathbb{Z}_2 \). We will only need to consider \( l = n, n + 1 \). \( \mathcal{Z}_n(M^{n+1}; \mathbb{Z}_2) \) denotes the space of flat chains \( T \in \mathbf{I}_n(M; \mathbb{Z}_2) \) such that \( T = \partial \Omega \) for some \( \Omega \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2) \). If \( T \in \mathbf{I}_n(M; \mathbb{Z}_2) \), \( |T| \) stands for the varifold associated to \( T \) and \( \|T\| \) is the Radon measure associated to \( |T| \). \( \mathcal{F} \) and \( \mathcal{M} \) denote the flat norm and the mass norm on \( \mathbf{I}_l(M; \mathbb{Z}_2) \). When \( l = n + 1 \), these two norms coincide. We will always assume that the spaces \( \mathbf{I}_{n+1}(M; \mathbb{Z}_2) \) and \( \mathcal{Z}_n(M; \mathbb{Z}_2) \) are equipped with the \( \mathcal{F} \) norm. We will also identify \( \mathcal{C}(M) \) with \( \mathbf{I}_{n+1}(M; \mathbb{Z}_2) \), i.e. \( E \in \mathcal{C}(M) \) will be identified with \( [E] \), the current associated with \( E \). Similarly, \( \partial E \) (with \( E \in \mathcal{C}(M) \)) will be identified with \( [\partial E] = \partial[E] \).

In [MN18], Marques and Neves proved that the space \( \mathcal{C}(M) \) is contractible and the boundary map \( \partial : \mathcal{C}(M) \to \mathcal{Z}_n(M; \mathbb{Z}_2) \) is a double cover. Indeed, by the constancy theorem, for \( \Omega_1, \Omega_2 \in \mathcal{C}(M) \), \( \partial \Omega_1 = \partial \Omega_2 \) if and only if either \( \Omega_1 = \Omega_2 \) or \( \Omega_1 = M - \Omega_2 \). Thus \( \pi_1(\mathcal{Z}_n(M; \mathbb{Z}_2)) = \mathbb{Z}_2 \). It was also proved in [MN18] that \( \mathcal{Z}_n(M; \mathbb{Z}_2) \) is weakly homotopy equivalent to \( \mathbb{R}P^\infty \). The cohomology ring \( H^*(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2) \) is the polynomial ring \( \mathbb{Z}_2[\bar{x}] \) where \( \bar{x} \) is the unique non-zero cohomology class in \( H^1(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2) \).

\( X \) is called a cubical complex if \( X \) is a subcomplex of \( [0,1]^N \) for some \( N \in \mathbb{N} \). By [BP02, Chapter 4], every cubical complex is homeomorphic to a finite simplicial complex and vice-versa. In the present article, we choose to work with simplicial complexes.

Let \( X \) be a finite simplicial complex. Suppose \( \Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2) \) is a continuous map. \( \Phi \) is called a p-sweepout if \( \Phi^*(\bar{x}^p) \neq 0 \) in \( H^p(X, \mathbb{Z}_2) \). \( \Phi \) is said to have no concentration of mass if

\[ \lim_{r \to 0^+} \sup \{ \|\Phi(x)\| (B(p,r)) : x \in X, p \in M \} = 0. \]
Let \( P_p \) denote the set of all \( p \)-sweepouts with no concentration of mass. The volume spectrum \( \{\omega_p\}_{p=1}^{\infty} \) is defined by

\[
\omega_p = \inf_{\Phi \in P_p} \sup \{ M(\Phi(x)) : x \in \text{domain of } \Phi \}.
\]

Let us also recall the definition of the one parameter Almgren-Pitts width, denoted by \( W \). Suppose \( S \) is the set of all continuous maps \( \Lambda : [0, 1] \rightarrow C(M) \) such that \( \Lambda(0) = M, \Lambda(1) = \emptyset \) and \( \partial \circ \Lambda : [0, 1] \rightarrow \mathbb{Z}_n(M;\mathbb{Z}_2) \) has no concentration of mass. We define

\[
W = \inf_{\Lambda \in S} \sup \{ M(\partial\Lambda(t)) : t \in [0,1] \}.
\]

### 2.3 The phase-transition spectrum

Here we briefly recall the definition of the phase-transition spectrum, which was originally defined by Gaspar and Guaraco in [GG18]. Let \( p : E \rightarrow B \) be a universal \( \mathbb{Z}_2 \)-principal bundle. Then \( B \) is homotopy equivalent to \( \mathbb{R}P^\infty \); hence \( H^*(B,\mathbb{Z}_2) \) is isomorphic to the polynomial ring \( \mathbb{Z}_2[\xi] \), where \( \xi \in H^1(B,\mathbb{Z}_2) \). Suppose \( X \) is a finite simplicial complex with a free, simplicial \( \mathbb{Z}_2 \) action (i.e. \( \mathbb{Z}_2 \) acts on \( X \) by simplicial homeomorphism) and \( X \) is the quotient of \( \tilde{X} \) by \( \mathbb{Z}_2 \). There exists a continuous \( \mathbb{Z}_2 \)-equivariant map \( \tilde{f} : \tilde{X} \rightarrow E \), which is unique up to \( \mathbb{Z}_2 \)-homotopy (see [Die08, Chapter 14.4]). \( \tilde{f} \) descends to a map \( f : X \rightarrow B \), i.e. if \( \pi : \tilde{X} \rightarrow X \) is the projection map, \( f \circ \pi = p \circ \tilde{f} \). Let \( f^* : H^*(B,\mathbb{Z}_2) \rightarrow H^*(X,\mathbb{Z}_2) \) be the map induced by \( f \). One defines

\[
\text{Ind}_{\mathbb{Z}_2}(\tilde{X}) = \sup \{ p \in \mathbb{N} : f^*(\xi^p) \neq 0 \in H^{p-1}(X,\mathbb{Z}_2) \}.
\]

\( C_p \) denotes the set of all finite simplicial complex \( \tilde{X} \), with free, simplicial \( \mathbb{Z}_2 \) action, such that \( \text{Ind}_{\mathbb{Z}_2}(\tilde{X}) \geq p + 1 \).

Let us recall that

\[
H^1(M) = \{ \varphi \in L^2(M) : \text{the distributional derivative } \nabla \varphi \in L^2(M) \}.
\]

\( H^1(M) \) is an infinite dimensional, separable Hilbert space; hence \( H^1(M) \setminus \{0\} \) is contractible. There is a free \( \mathbb{Z}_2 \) action on \( H^1(M) \setminus \{0\} \) given by \( u \mapsto -u \). Thus \( H^1(M) \setminus \{0\} \), equipped with this \( \mathbb{Z}_2 \) action, is the total space of a universal \( \mathbb{Z}_2 \)-principal bundle (see [Die08, 14.4.12]). If \( \tilde{X} \) is a finite simplicial complex with free, simplicial \( \mathbb{Z}_2 \) action, the set of all continuous \( \mathbb{Z}_2 \)-equivariant maps \( \tilde{X} \rightarrow H^1(M) \setminus \{0\} \) is denoted by \( \Gamma(\tilde{X}) \).

For \( \varepsilon > 0 \), the \( \varepsilon \)-Allen-Cahn functional \( E_\varepsilon : H^1(M) \rightarrow [0, \infty) \) is defined by

\[
E_\varepsilon(u) = \int_M \varepsilon \frac{\left| \nabla u \right|^2}{2} + \frac{W_*(u)}{\varepsilon},
\]
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where \( W_s : \mathbb{R} \to \mathbb{R} \) is a smooth, symmetric double-well potential. More precisely, \( W_s \) is bounded; \( W_s(t) = W_s(-t) \) for all \( t \); \( W_s \geq 0 \) and \( W_s \) has precisely three critical points 0, \( \pm 1 \); \( W_s(\pm 1) = 0 \) and \( W''_s(\pm 1) > 0 \); 0 is a local maximum of \( W_s \). We note that \( E_\varepsilon(u) = E_\varepsilon(-u) \).

The \( \varepsilon \)-phase-transition spectrum \( \{ c_\varepsilon(p) \}_{p=1}^\infty \) can be defined as follows (see [GG18, Lemma 6.2]).

\[
c_\varepsilon(p) = \inf_{\tilde{X} \in \mathcal{C}_p} \left( \inf_{h \in \Gamma(\tilde{X})} \sup_{x \in \tilde{X}} E_\varepsilon(h(x)) \right).
\]

The phase-transition spectrum \( \{ \ell_p \}_{p=1}^\infty \) is defined by

\[
\ell_p = \frac{1}{2\sigma} \lim_{\varepsilon \to 0^+} c_\varepsilon(p),
\]

where \( \sigma = \int_{-1}^{1} \sqrt{W(t)/2} \, dt \). Combining the results in [Gua18], [GG18] and [Dey20], it follows that \( \ell_p = \omega_p \) for all \( p \). Let us also recall the definition of the one parameter Allen-Cahn width \( \gamma \), which was defined by Guaraco [Gua18]. Suppose \( \Gamma = \{ h : [0,1] \to H^1(M) : h \) is continuous, \( h(0) = 1, h(1) = -1 \} \)

(1 denotes the constant function 1). Then

\[
\gamma_\varepsilon = \inf_{h \in \Gamma} \sup_{t \in [0,1]} E_\varepsilon(h(t)); \quad \gamma = \frac{1}{2\sigma} \lim_{\varepsilon \to 0^+} \gamma_\varepsilon.
\]

Again, by the results in [Gua18] and [Dey20], \( \gamma = W \).

### 2.4 Join of two topological spaces

Let \( A \) and \( B \) be two topological spaces. The join of \( A \) and \( B \), denoted by \( A \ast B \), is defined as follows.

\[
A \ast B = \frac{A \times B \times [0,1]}{\sim},
\]

where the equivalence relation ‘\( \sim \)’ is such that

\[
(a,b_1,0) \sim (a,b_2,0) \quad \forall \ b_1, b_2 \in B \quad \text{and} \quad (a_1,b,1) \sim (a_2,b,1) \quad \forall \ a_1, a_2 \in A,
\]

i.e. \( A \times B \times \{0\} \) is collapsed to \( A \) and \( A \times B \times \{1\} \) is collapsed to \( B \). If either \( A \) or \( B \) is path-connected, using van Kampen’s Theorem, one can show that \( A \ast B \) is simply connected.

By the abuse of notation, an element of \( A \ast B \) will be denoted by a triple \((a,b,t)\) with \( a \in A, b \in B, t \in [0,1] \). If \( f : A \to C \) and \( g : B \to D \) are continuous maps, one can define a continuous map \( f \ast g : A \ast B \to C \ast D \) as follows.

\[
(f \ast g)(a,b,t) = (f(a),g(b),t); \quad a \in A, b \in B, t \in [0,1].
\]
Let us recall that the standard \( d \)-simplex \( \Delta^d \) is defined as follows.

\[
\Delta^d = \{ x = (x_1, x_2, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1 + x_2 + \cdots + x_{d+1} = 1 \text{ and } x_i \geq 0 \ \forall i \}.
\]

\( \Delta^p \ast \Delta^q \) can be identified with the simplex \( \Delta^{p+q+1} \). Indeed, \( F : \Delta^p \ast \Delta^q \to \Delta^{p+q+1} \), defined by

\[
F(u, v, t) = ((1 - t)u, tv) \ (\in \mathbb{R}^{p+q+2}),
\]

is a homeomorphism. Hence, join of two simplicial complexes is again a simplicial complex.

If \( \mathcal{X} \) is a topological space, let \( C_d(\mathcal{X}, \mathbb{Z}_2) \) denote the abelian group of singular \( d \)-chains in \( \mathcal{X} \) with coefficients in \( \mathbb{Z}_2 \). For \( p, q \in \mathbb{N}_0 \), we define a bilinear map \( \mathcal{F} : C_p(A, \mathbb{Z}_2) \times C_q(B, \mathbb{Z}_2) \to C_{p+q+1}(A \ast B, \mathbb{Z}_2) \) as follows. If \( \phi : \Delta^p \to A \) and \( \psi : \Delta^q \to B \) are continuous maps,

\[
\mathcal{F}(\phi, \psi) = (\phi \ast \psi) \circ (F^{-1}).
\]

For convenience, if \( (\phi, \psi) \in C_p(A, \mathbb{Z}_2) \times C_q(B, \mathbb{Z}_2) \), let us denote \( \mathcal{F}(\phi, \psi) \) simply by \( \phi \ast \psi \). The following identity can be verified by explicit computation.

\[
\partial(\phi \ast \psi) = (\partial \phi) \ast \psi + \phi \ast (\partial \psi).
\] (2.2)

### 3 Proof of Theorem 1.1

If \( k = m = 0 \), (1.5) reduces to \( \omega_1 \leq W \), which holds by the definitions of \( \omega_1 \) and \( W \). So let us assume that \( k \geq 1 \). The proof of Theorem 1.1 is divided into four parts.

**Part 1.** Let us fix \( \delta > 0 \). We choose a \( k \)-sweepout \( \Phi_1 : X \to Z_n(M; \mathbb{Z}_2) \), with no concentration of mass, such that \( X \) is connected and

\[
\sup_{x \in X} \{ M(\Phi_1(x)) \} \leq \omega_k + \delta. \tag{3.1}
\]

Following the argument of Zhou [Zho19], \( \Phi_1 \) is a \( k \)-sweepout implies that

\[
\Phi_1^* : H^1(Z_n(M; \mathbb{Z}_2), \mathbb{Z}_2) \to H^1(X, \mathbb{Z}_2)
\]

is non-zero. Hence,

\[
(\Phi_1)_* : \pi_1(X) \to \pi_1(Z_n(M; \mathbb{Z}_2)) (= \mathbb{Z}_2)
\]

is onto. Thus \( \ker(\Phi_1)_* \) is an index 2 subgroup of \( \pi_1(X) \). By [Hat02, Proposition 1.36], there exists a double cover \( \rho_1 : \tilde{X} \to X \) such that \( \tilde{X} \) is connected and if

\[
(\rho_1)_* : \pi_1(\tilde{X}) \to \pi_1(X),
\]

then \( \text{im}(\rho_1)_* = \ker(\Phi_1)_* \). Using [Hat02, Proposition 1.33], \( \Phi_1 \) has a lift \( \tilde{\Phi}_1 : \tilde{X} \to \mathcal{C}(M) \) such that \( \partial \circ \tilde{\Phi}_1 = \Phi_1 \circ \rho_1 \). \( \tilde{\Phi}_1 \) is \( \mathbb{Z}_2 \)-equivariant, i.e. if \( T_1 : \tilde{X} \to \tilde{X} \) is the deck transformation,

\[
\tilde{\Phi}_1(T_1(x)) = M - \tilde{\Phi}_1(x) \ \forall \ x \in \tilde{X}.
\]
A sub-additive inequality for the volume spectrum

Similarly, for $m \geq 1$, we can choose an $m$-sweepout $\Phi_2 : Y \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ with no concentration of mass such that $Y$ is connected and

$$\sup_{y \in Y} \{ \mathcal{M}(\Phi_2(y)) \} \leq \omega_m + \delta. \quad (3.2)$$

There exists a double cover $\rho_2 : \tilde{Y} \to Y$ such that $\tilde{Y}$ is connected and $\Phi_2$ has a lift $\tilde{\Phi}_2 : \tilde{Y} \to \mathcal{C}(M)$ such that $\partial \circ \tilde{\Phi}_2 = \Phi_2 \circ \rho_2$. If $T_2 : \tilde{Y} \to \tilde{Y}$ is the deck transformation, then

$$\tilde{\Phi}_2(T_2(y)) = M - \tilde{\Phi}_2(y) \; \forall \; y \in \tilde{Y}.$$ 

On the other hand, if $m = 0$, we simply choose $Y$ to be a singleton set $\{ \bullet \}$ and $\Phi_2(\bullet)$ to be the zero cycle. Consequently, in this case, $\tilde{Y} = \{-1, 1\}$ with $\tilde{\Phi}_2(-1) = \emptyset$ and $\tilde{\Phi}_2(1) = M$. We also choose $\Lambda : [0, 1] \to \mathcal{C}(M)$ such that $\Lambda(0) = M$, $\Lambda(1) = \emptyset$, $\partial \circ \Lambda : [0, 1] \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ has no concentration of mass and

$$\sup_{t \in [0, 1]} \{ \mathcal{M}(\partial \Lambda(t)) \} \leq W + \delta. \quad (3.3)$$

**Part 2.** Let us consider $\tilde{X} \ast \tilde{Y}$. As remarked in Section 2.4, $\tilde{X} \ast \tilde{Y}$ is simply-connected. There exists a free $\mathbb{Z}_2$ action on $\tilde{X} \ast \tilde{Y}$ given by

$$(x, y, t) \mapsto (T_1(x), T_2(y), t). \quad (3.4)$$

$\tilde{X}$ and $\tilde{Y}$ can be equipped with simplicial complex structures such that the maps $T_1 : \tilde{X} \to \tilde{X}$ and $T_2 : \tilde{Y} \to \tilde{Y}$ are simplicial homeomorphisms. Let $\{ \Delta_{1,i} : i = 1, \ldots, I \}$ and $\{ \Delta_{2,j} : j = 1, \ldots, J \}$ be the set of simplices in $\tilde{X}$ and $\tilde{Y}$. As discussed in Section 2.4, we can put a canonical simplicial complex structure on $\tilde{X} \ast \tilde{Y}$, whose simplices are $\{ \Delta_{1,i}, \Delta_{2,j}, \Delta_{1,i} \ast \Delta_{2,j} : i = 1, \ldots, I; \; j = 1, \ldots, J \}$, such that the above action of $\mathbb{Z}_2$ on $\tilde{X} \ast \tilde{Y}$ ((3.4)) is a simplicial action.

One can define a $\mathbb{Z}_2$-equivariant map $\tilde{\Psi} : \tilde{X} \ast \tilde{Y} \to \mathcal{C}(M)$ as follows.

$$\tilde{\Psi}(x, y, t) = \left( \tilde{\Phi}_1(x) \cap \Lambda(t) \right) \bigcup \left( \tilde{\Phi}_2(y) \cap (M - \Lambda(t)) \right). \quad (3.5)$$

This definition is motivated by the work of Marques and Neves [MN18, Proof of Claim 5.3], where they proved that $\mathcal{C}(M)$ is contractible. By Proposition 2.1, R.H.S. of (3.5) indeed belongs to $\mathcal{C}(M)$. Moreover, $\tilde{\Psi}$ is well-defined on $\tilde{X} \ast \tilde{Y}$ as $\Lambda(0) = M$ and $\Lambda(1) = \emptyset$.

**Claim 3.1.** The map $\tilde{\Psi} : \tilde{X} \ast \tilde{Y} \to \mathcal{C}(M)$, defined above, is continuous in the flat topology.

**Proof.** Since $\tilde{\Phi}_1$, $\tilde{\Phi}_2$ and $\Lambda$ are continuous in the flat topology, it is enough to prove that the maps $\mu_1, \mu_2 : \mathcal{C}(M) \times \mathcal{C}(M) \to \mathcal{C}(M)$ defined by

$$\mu_1(E, F) = E \cap F \quad \text{and} \quad \mu_2(E, F) = E \cup F$$

consider the set of all $\mathbb{Z}_2$-equivariant maps $\Psi : \tilde{X} \ast \tilde{Y} \to \mathcal{C}(M)$, continuous in the flat topology.
are continuous. For arbitrary sets $A_1$, $A_2$, $B_1$, $B_2$,

$$(A_1 \cap A_2) \triangle (B_1 \cap B_2) \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2),$$

and

$$(A_1 \cup A_2) \triangle (B_1 \cup B_2) \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2).$$

Here $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of $A$ and $B$. Therefore,

$$\mathcal{F}(\mu_1(E, F) - \mu_1(E', F')) \leq \mathcal{F}(E - E') + \mathcal{F}(F - F'),$$

$$\mathcal{F}(\mu_2(E, F) - \mu_2(E', F')) \leq \mathcal{F}(E - E') + \mathcal{F}(F - F'),$$

which proves the continuity of $\mu_1$ and $\mu_2$. \hfill \square

Let $Z$ denote the quotient of $\tilde{X} \ast \tilde{Y}$ under the above mentioned $\mathbb{Z}_2$ action (3.4) ($Z$ is a simplicial complex as the $\mathbb{Z}_2$ action on $\tilde{X} \ast \tilde{Y}$ is simplicial) and $\rho : \tilde{X} \ast \tilde{Y} \to Z$ be the covering map. Since $\tilde{X}$ is a covering map. Since $\tilde{X} \ast \tilde{Y}$, we have no concentration of mass, from Proposition 2.1, it follows that $\Psi$ also has no concentration of mass.

**Part 3.** We will show that $\Psi : Z \to Z_n(M; \mathbb{Z}_2)$ is a $(k + m + 1)$-sweepout. Since $\tilde{X} \ast \tilde{Y}$ is simply-connected and $Z$ is the quotient of $\tilde{X} \ast \tilde{Y}$ by $\mathbb{Z}_2$, $\pi_1(Z) = \mathbb{Z}_2$; hence $H^1(Z, \mathbb{Z}_2) = \mathbb{Z}_2$.

Let $\lambda$ be the unique non-zero element of $H^1(Z, \mathbb{Z}_2)$ so that

$$\lambda.[\kappa] = 1$$

for all non-contractible loop $\kappa : S^1 \to Z$. 

As discussed in Section 2.4, $\tilde{X}$ naturally sits inside $\tilde{X} \ast \tilde{Y}$, via the map $x \mapsto (x, \overline{y}, 0)$ and $\tilde{Y}$ naturally sits inside $\tilde{X} \ast \tilde{Y}$, via the map $y \mapsto (\overline{x}, y, 1)$. Hence, $X$ and $Y$ can be naturally identified as subspaces of $Z$; let $\iota_1 : X \hookrightarrow Z$ and $\iota_2 : Y \hookrightarrow Z$ be the inclusion maps. By the definition of $\Psi$ in (3.5),

$$\Psi|_X = \Phi_1 \quad \text{and} \quad \Psi|_Y = \Phi_2. \quad (3.7)$$

Thus, recalling the notation from Section 2.2 that $H^*(Z_n(M; \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\overline{\lambda}]$,

$$\Psi^*\overline{\lambda}|_X = \Phi_1^*\overline{\lambda} \quad \text{and} \quad \Psi^*\overline{\lambda}|_Y = \Phi_2^*\overline{\lambda}. \quad (3.8)$$

In particular, this implies $\Psi^*\overline{\lambda}$ is non-zero and hence $\Psi^*\overline{\lambda} = \lambda$. Therefore, to prove that $\Psi : Z \to Z_n(M; \mathbb{Z}_2)$ is a $(k + m + 1)$-sweepout, it is enough to find $c \in H_{k+m+1}(Z, \mathbb{Z}_2)$ such that $\lambda^{k+m+1}c = 1$.

Since $\Phi_1$ is a $k$-sweepout, there exists $\alpha \in H_k(X, \mathbb{Z}_2)$ such that

$$(\Phi_1^*\overline{\lambda})^k \cdot \alpha = 1. \quad (3.9)$$
Let $\alpha = \sum_{i=1}^{I} \alpha_i$, where each $\alpha_i : \Delta^k \to X$ is a singular $k$-simplex and
\[
\partial \left( \sum_i \alpha_i \right) = 0. \tag{3.10}
\]

Similarly, as $\Phi_2$ is an $m$-sweepout, there exists $\beta \in H_m(Y, \mathbb{Z}_2)$ such that
\[
(\Phi_2^* \lambda)^m \cdot \beta = 1. \tag{3.11}
\]

Let $\beta = \sum_{j=1}^{J} \beta_j$, where each $\beta_j : \Delta^m \to Y$ is a singular $m$-simplex and
\[
\partial \left( \sum_j \beta_j \right) = 0. \tag{3.12}
\]

Let $\tau : C_k(X, \mathbb{Z}_2) \to C_k(\tilde{X}, \mathbb{Z}_2)$ be the transfer homomorphism defined as follows (see [Hat02, Proof of Proposition 2B.6]). If $\eta : \Delta^k \to X$ is a continuous map, $\eta$ has exactly two lifts $\tilde{\eta}_1, \tilde{\eta}_2 : \Delta^k \to \tilde{X}$. One defines $\tau(\eta) = \tilde{\eta}_1 + \tilde{\eta}_2$. Let $\tilde{\alpha}_{i,1}, \tilde{\alpha}_{i,2} : \Delta^k \to \tilde{X}$ be the lifts of $\alpha_i$ so that $\tau(\alpha_i) = \tilde{\alpha}_{i,1} + \tilde{\alpha}_{i,2}$. We also choose $\tilde{\beta}_j : \Delta^m \to \tilde{Y}$, which is a lift of $\beta_j$.

**Claim 3.2.**
\[
\partial \rho_\# \left( \sum_{i,j} \tau(\alpha_i) \ast \tilde{\beta}_j \right) = 0 \text{ in } C_{k+m}(Z, \mathbb{Z}_2). \tag{3.13}
\]

**Proof.** By (2.2),
\[
\partial \left( \sum_{i,j} \tau(\alpha_i) \ast \tilde{\beta}_j \right) = \sum_{i,j} \partial \tau(\alpha_i) \ast \tilde{\beta}_j + \sum_{i,j} \tau(\alpha_i) \ast \partial \tilde{\beta}_j. \tag{3.14}
\]

Since $\tau$ is a chain map (i.e. $\tau$ commutes with $\partial$), (3.10) implies
\[
\sum_i \partial \tau(\alpha_i) = 0. \tag{3.15}
\]

Further, by (3.12),
\[
\rho_\# \left( \sum_j \partial \tilde{\beta}_j \right) = 0.
\]

Hence, there exist singular $(m-1)$-simplices $\theta_{r,1}, \theta_{r,2} : \Delta^{m-1} \to \tilde{Y}$ such that $\theta_{r,2} = T_2 \circ \theta_{r,1}$ and
\[
\sum_j \partial \tilde{\beta}_j = \sum_{r=1}^{R} \theta_{r,1} + \theta_{r,2}.
\]

Thus
\[
\sum_{i,j} \tau(\alpha_i) \ast \partial \tilde{\beta}_j = \sum_{i,r} \tilde{\alpha}_{i,1} \ast \theta_{r,1} + \tilde{\alpha}_{i,2} \ast \theta_{r,1} + \tilde{\alpha}_{i,1} \ast \theta_{r,2} + \tilde{\alpha}_{i,2} \ast \theta_{r,2}. \tag{3.16}
\]
However,
\[ \rho \circ (\tilde{\alpha}_{i,1} \ast \theta_{r,1}) = \rho \circ (\tilde{\alpha}_{i,2} \ast \theta_{r,2}) \quad \text{and} \quad \rho \circ (\tilde{\alpha}_{i,2} \ast \theta_{r,1}) = \rho \circ (\tilde{\alpha}_{i,1} \ast \theta_{r,2}). \]  
(3.17)
Therefore, combining (3.14), (3.15), (3.16) and (3.17), we obtain (3.13).

Claim 3.2 implies that
\[ c = \left[ \rho \# \left( \sum_{i,j} \tau(\alpha_i \ast \tilde{\beta}_j) \right) \right] \]  
(3.18)
is a well-defined homology class in \( H_{k+m+1}(Z, \mathbb{Z}_2) \).

**Claim 3.3.** \( \lambda^{k+m+1}.c = 1 \), where \( c \) is as defined in (3.18).

**Proof.** Let us choose \( h : C_1(Z, \mathbb{Z}_2) \to \mathbb{Z}_2 \) such that \( \lambda = [h] \). Suppose \( \{\nu_1, \nu_2, \ldots, \nu_{k+m+2}\} \) are the vertices of \( \Delta^{k+m+1} \). We note that
\[ \rho \#(\tilde{\alpha}_{i,1} * \tilde{\beta}_j) |_{[\nu_1, \ldots, \nu_{k+1}]} = \nu_1 \circ \alpha_i = \rho \#(\tilde{\alpha}_{i,2} * \tilde{\beta}_j) |_{[\nu_1, \ldots, \nu_{k+1}]} . \]  
(3.19)
Similarly,
\[ \rho \#(\tilde{\alpha}_{i,1} * \tilde{\beta}_j) |_{[\nu_{k+2}, \ldots, \nu_{k+m+2}]} = \nu_2 \circ \beta_j = \rho \#(\tilde{\alpha}_{i,2} * \tilde{\beta}_j) |_{[\nu_{k+2}, \ldots, \nu_{k+m+2}]} . \]  
(3.20)
Moreover,
\[ (\tilde{\alpha}_{i,1} * \tilde{\beta}_j) |_{[\nu_{k+1}, \nu_{k+2}]} \]  
is a curve joining \( \tilde{\alpha}_{i,1}(\nu_{k+1}) \) and \( \tilde{\beta}_j(\nu_{k+2}) \);
\[ (\tilde{\alpha}_{i,2} * \tilde{\beta}_j) |_{[\nu_{k+1}, \nu_{k+2}]} \]  
is a curve joining \( \tilde{\alpha}_{i,2}(\nu_{k+1}) \) and \( \tilde{\beta}_j(\nu_{k+2}) \).
Since \( \tilde{\alpha}_{i,2} = T_1 \circ \tilde{\alpha}_{i,1} \),
\[ \rho \#(\tilde{\alpha}_{i,1} * \tilde{\beta}_j) |_{[\nu_{k+1}, \nu_{k+2}]} + \rho \#(\tilde{\alpha}_{i,2} * \tilde{\beta}_j) |_{[\nu_{k+1}, \nu_{k+2}]} \]
is a non-contractible loop in \( Z \). Hence, using (3.19), (3.20) and (3.6), we obtain
\[ h^{k+m+1} \left( \rho \#(\tilde{\alpha}_{i,1} * \tilde{\beta}_j) + \rho \#(\tilde{\alpha}_{i,2} * \tilde{\beta}_j) \right) \]  
\[ = h^k(\nu_1 \circ \alpha_i) h \left( \rho \#(\tilde{\alpha}_{i,1} * \tilde{\beta}_j) |_{[\nu_{k+1}, \nu_{k+2}]} + \rho \#(\tilde{\alpha}_{i,2} * \tilde{\beta}_j) |_{[\nu_{k+1}, \nu_{k+2}]} \right) h^m(\nu_2 \circ \beta_j) \]  
\[ = h^k(\nu_1 \circ \alpha_i) h^m(\nu_2 \circ \beta_j) . \]  
(3.21)
(3.8), (3.9), (3.11) and (3.21) imply that \( \lambda^{k+m+1}.c = 1 \).

**Part 4.** Since \( \Psi : Z \to Z_n(M; \mathbb{Z}_2) \) is a \((k + m + 1)\)-sweepout with no concentration of mass,
\[ \sup_{z \in Z} \{ M(\Psi(z)) \} \geq \omega_{k+m+1} . \]  
(3.22)

The following proposition essentially follows from the proof of Proposition 2.1.
Proposition 3.4. For all \((x, y, t) \in \tilde{X} \ast \tilde{Y}\),
\[
M(\partial \tilde{\Psi}(x, y, t)) \leq M(\Phi_1(x)) + M(\Phi_2(y)) + M(\partial \Lambda(t)).
\]

Proof. By [MPPP07, Proposition 1.4], there exist \(\{u_i\}_{i=1}^{\infty}, \{v_i\}_{i=1}^{\infty}, \{w_i\}_{i=1}^{\infty} \subset C^\infty(M)\) such that \(0 \leq u_i, v_i, w_i \leq 1\) for all \(i\);
\[
u_i \to \chi_{\tilde{\Phi}_1(x)} \text{ in } L^1(M) \text{ and pointwise a.e.;} \quad M(\Phi_1(x)) = \lim_{i \to \infty} \int_M |\nabla u_i| dH^{n+1};
\]
\[
u_i \to \chi_{\tilde{\Phi}_2(y)} \text{ in } L^1(M) \text{ and pointwise a.e.;} \quad M(\Phi_2(y)) = \lim_{i \to \infty} \int_M |\nabla v_i| dH^{n+1};
\]
\[
u_i \to \chi_{\Lambda(t)} \text{ in } L^1(M) \text{ and pointwise a.e.;} \quad M(\partial \Lambda(t)) = \lim_{i \to \infty} \int_M |\nabla w_i| dH^{n+1}.
\]

By the dominated convergence theorem,
\[
u_i w_i + v_i(1 - w_i) \to \chi_{\tilde{\Psi}(x, y, t)} \text{ in } L^1(M).
\]
Therefore,
\[
M(\partial \tilde{\Psi}(x, y, t)) \leq \liminf_{i \to \infty} \int_M \left( |\nabla u_i w_i| + |\nabla v_i (1 - w_i)| \right) dH^{n+1}
\]
\[
\leq \liminf_{i \to \infty} \int_M \left( |\nabla u_i| + |\nabla v_i| + |\nabla w_i| \right) dH^{n+1}
\]
\[
= M(\Phi_1(x)) + M(\Phi_2(y)) + M(\partial \Lambda(t)).
\]

Combining Proposition 3.4, (3.1), (3.2), (3.3) and (3.22), we obtain
\[
\omega_{k+m+1} \leq \omega_k + \omega_m + W + 3\delta.
\]
Since \(\delta > 0\) is arbitrary, this finishes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

If \(k = m = 0\), (1.6) reduces to \(c_\varepsilon(1) \leq \gamma_\varepsilon\), which holds by the definitions of \(c_\varepsilon(1)\) and \(\gamma_\varepsilon\) (see [GG18, Remark 3.6]). So let us assume that \(k \geq 1\). We fix \(\delta > 0\). There exists a connected simplicial complex \(\tilde{X} \in \mathcal{C}_k\) and \(\tilde{h}_1 \in \Gamma(\tilde{X})\) such that
\[
\sup_{x \in \tilde{X}} E_\varepsilon(\tilde{h}_1(x)) \leq c_\varepsilon(k) + \delta. \quad (4.1)
\]
Similarly, for \(m \geq 1\), there exists a connected simplicial complex \(\tilde{Y} \in \mathcal{C}_m\) and \(\tilde{h}_2 \in \Gamma(\tilde{Y})\) such that
\[
\sup_{y \in \tilde{Y}} E_\varepsilon(\tilde{h}_2(y)) \leq c_\varepsilon(m) + \delta. \quad (4.2)
\]
If \( m = 0 \), we choose \( \tilde{Y} = \{-1, 1\} \) and \( \tilde{h}_2(\pm 1) = \pm 1 \). Let \( w : [0, 1] \to H^1(M) \) be such that \( w(0) = 1, w(1) = -1 \) and

\[
\sup_{t \in [0, 1]} E_\varepsilon(w(t)) \leq \gamma_\varepsilon + \delta. \tag{4.3}
\]

Suppose \( \mathcal{T} : H^1(M) \to H^1(M) \) is the following truncation map (see [Gua18, Section 4]).

\[
\mathcal{T}(u) = \min\{\max\{u, -1\}, 1\}.
\]

Then \( \mathcal{T} \) is continuous, \( -1 \leq \mathcal{T}(u) \leq 1 \), \( \mathcal{T}(-u) = -\mathcal{T}(u) \) and \( E_\varepsilon(\mathcal{T}(u)) \leq E_\varepsilon(u) \).

Therefore, without loss of generality, we can assume that

\[
-1 \leq \tilde{h}_1(x), \tilde{h}_2(y), w(t) \leq 1 \tag{4.4}
\]

for all \( x \in \tilde{X}, y \in \tilde{Y} \) and \( t \in [0, 1] \).

For \( a \in \mathbb{R} \), let \( a^+ = \max\{a, 0\} \); \( a^- = \min\{a, 0\} \). The following maps \( \phi, \psi, \theta : H^1(M) \times H^1(M) \times H^1(M) \to H^1(M) \) were defined in [Dey20].

\[
\phi(u_1, u_2, v) = \min\{\max\{u_1, -v\}, \max\{u_2, v\}\};
\psi(u_1, u_2, v) = \max\{\min\{u_1, v\}, \min\{u_2, -v\}\};
\theta(u_1, u_2, v) = \phi(u_1, u_2, v)^+ + \psi(u_1, u_2, v)^-.
\]

**Proposition 4.1 ([Dey20, Proposition 3.12.]).** The map \( \theta \), defined above, has the following properties.

(i) \( \theta \) is continuous.

(ii) \( \theta(-u_1, -u_2, v) = -\theta(u_1, u_2, v) \).

(iii) If \( -1 \leq u_1, u_2 \leq 1 \), \( \theta(u_1, u_2, 1) = u_1 \) and \( \theta(u_1, u_2, -1) = u_2 \).

(iv) \( E_\varepsilon(\theta(u_1, u_2, v)) \leq E_\varepsilon(u_1) + E_\varepsilon(u_2) + E_\varepsilon(v) \).

We define \( \tilde{q} : \tilde{X} \ast \tilde{Y} \to H^1(M) \) as follows.

\[
\tilde{q}(x, y, t) = \theta \left( \tilde{h}_1(x), \tilde{h}_2(y), w(t) \right). \tag{4.5}
\]

By (4.4) and Proposition 4.1, items (i), (iii), \( \tilde{q} \) is a well-defined, continuous map and

\[
\tilde{q}\big|_{\tilde{X}} = \tilde{h}_1; \quad \tilde{q}\big|_{\tilde{Y}} = \tilde{h}_2. \tag{4.6}
\]

(We recall from Part 3 of the proof of Theorem 1.1 that \( \tilde{X} \) and \( \tilde{Y} \) can be naturally identified as subspaces of \( \tilde{X} \ast \tilde{Y} \).)
Claim 4.2. $\tilde{Z} = \tilde{X} \ast \tilde{Y} \in \mathcal{C}_{k+m+1}$ and there exists $\tilde{\zeta} \in \Gamma(\tilde{Z})$ such that

$$E_\varepsilon(\tilde{\zeta}(z)) \leq E_\varepsilon(\tilde{q}(z)) + \delta \quad \forall z \in \tilde{Z}. \quad (4.7)$$

Proof. Let $T_1 : \tilde{X} \to \tilde{X}$ be such that the $\mathbb{Z}_2$ action on $\tilde{X}$ is given by $x \mapsto T_1(x)$ and $T_2 : \tilde{Y} \to \tilde{Y}$ be such that the $\mathbb{Z}_2$ action on $\tilde{Y}$ is given by $y \mapsto T_2(y)$. There exist simplicial complex structures on $\tilde{X}$ and $\tilde{Y}$ such that the maps $T_1$ and $T_2$ are simplicial homeomorphisms. Let $\{\Delta_{1,i} : i = 1, \ldots, I\}$ and $\{\Delta_{2,j} : j = 1, \ldots, J\}$ be the set of simplices in $\tilde{X}$ and $\tilde{Y}$. As discussed in Section 2.4, one can put a canonical simplicial complex structure on $\tilde{Z} = \tilde{X} \ast \tilde{Y}$, whose simplices are $\{\Delta_{1,i}, \Delta_{2,j}, \Delta_{1,i} \ast \Delta_{2,j} : i = 1, \ldots, I; j = 1, \ldots, J\}$. There is a free, simplicial $\mathbb{Z}_2$ action on $\tilde{Z}$ given by

$$z = (x, y, t) \mapsto T(z) = (T_1(x), T_2(y), t).$$

By Proposition 4.1(ii), $\tilde{q}(T(z)) = -\tilde{q}(z)$ for all $z \in \tilde{Z}$. There might exist $z \in \tilde{Z}$ such that $\tilde{q}(z) = 0 \in H^1(M)$. However, as we discuss below, one can perturb $\tilde{q}$ and obtain a new map $\tilde{\zeta} : \tilde{Z} \to H^1(M) \setminus \{0\}$, which is $\mathbb{Z}_2$-equivariant and satisfies

$$E_\varepsilon(\tilde{\zeta}(z)) \leq E_\varepsilon(\tilde{q}(z)) + \delta \quad \forall z \in \tilde{Z}; \quad (4.8)$$

$$\tilde{\zeta}|_{\tilde{X}} = \tilde{q}|_{\tilde{X}} = \tilde{h}_1; \quad \tilde{\zeta}|_{\tilde{Y}} = \tilde{q}|_{\tilde{Y}} = \tilde{h}_2. \quad (4.9)$$

Let $\{e_s, f_s : s = 1, \ldots, S\}$ be the set of all simplices in $\tilde{Z}$, indexed in such a way that $T(e_s) = f_s$ and $s_1 \leq s_2$ implies $\dim(e_{s_1}) \leq \dim(e_{s_2})$. We will define $\tilde{\zeta}$ inductively on the simplices of $\tilde{Z}$. By abuse of notation, a simplex of $\tilde{Z}$ will be identified with its support. For $R > 0$, let

$$B_R = \{u \in H^1(M) : \|u\|_{H^1(M)} < R\}.$$ 

$B_R \setminus \{0\}$, being homeomorphic to $H^1(M) \setminus \{0\}$, is contractible. Let $r > 0$ be such that

- $B_r$ is disjoint from $\tilde{h}_1(\tilde{X})$ and $\tilde{h}_2(\tilde{Y})$ (we note that $\tilde{h}_1(\tilde{X})$ and $\tilde{h}_2(\tilde{Y})$ are compact subsets of $H^1(M)$, which are disjoint from $\{0\}$);

- $|E_\varepsilon(u) - E_\varepsilon(0)| < \delta/2$ for all $u \in B_r$.

Suppose

$$\tilde{Z}_0 = \{z \in \tilde{Z} : \tilde{q}(z) = 0 \in H^1(M)\}.$$ 

Using barycentric subdivision, without loss of generality one can assume that

$$\text{if } \alpha \text{ is a simplex in } \tilde{Z} \text{ and } \alpha \cap \tilde{Z}_0 \neq \emptyset, \text{ then } \tilde{q}(\alpha) \subset B_r. \quad (4.10)$$

If $s$ is such that $\dim(e_s) = 0$, we set

$$\tilde{\zeta}(e_s) = \begin{cases} \tilde{q}(e_s) & \text{if } \tilde{q}(e_s) \neq 0; \\ \text{an arbitrary point in } B_r \setminus \{0\} & \text{if } \tilde{q}(e_s) = 0; \end{cases} \quad (4.11)$$
and \( \tilde{\zeta}(f_s) = -\tilde{\zeta}(e_s) \). Let us assume that we have defined

\[
\tilde{\zeta} : \bigcup_{s \leq l-1} (e_s \cup f_s) \to H^1(M) \setminus \{0\}
\]

in such a way that

1. \( \tilde{\zeta}(T(z)) = -\tilde{\zeta}(z) \) for all \( z \in \bigcup_{s \leq l-1} (e_s \cup f_s) \);
2. \( \tilde{\zeta}|_{e_s \cup f_s} = \tilde{q}|_{e_s \cup f_s} \) if \( s \leq l-1 \) and \( e_s \cap \tilde{Z}_0 = \emptyset \) (which is equivalent to \( f_s \cap \tilde{Z}_0 = \emptyset \));
3. \( (\tilde{\zeta}(e_s) \cup \tilde{\zeta}(f_s)) \subseteq B_r \setminus \{0\} \) if \( s \leq l-1 \) and \( e_s \cap \tilde{Z}_0 \neq \emptyset \) (which is equivalent to \( f_s \cap \tilde{Z}_0 \neq \emptyset \)).

We define \( \tilde{\zeta} : (e_l \cup f_l) \to H^1(M) \setminus \{0\} \) as follows. By the induction hypothesis, \( \tilde{\zeta} \) has already been defined on \( (\partial e_l \cup \partial f_l) \). If \( e_l \cap \tilde{Z}_0 = \emptyset = f_l \cap \tilde{Z}_0 \), we set

\[
\tilde{\zeta}|_{e_l \cup f_l} = \tilde{q}|_{e_l \cup f_l}.
\]

Otherwise, \( \tilde{\zeta} \) is defined on \( e_l \) in such a way that the definition of \( \tilde{\zeta} \) on \( \partial e_l \) matches with its previous definition and \( \tilde{\zeta}(e_l) \subseteq B_r \setminus \{0\} \). Such an extension is possible as \( B_r \setminus \{0\} \) is contractible. For \( z \in f_l \), \( \tilde{\zeta}(z) \) is defined to be equal to \( -\tilde{\zeta}(T(z)) \). Thus we have defined \( \tilde{\zeta} : \tilde{Z} \to H^1(M) \setminus \{0\} \), which is \( \mathbb{Z}_2 \)-equivariant. (4.8) and (4.9) follow from the choice of \( r \) and (4.10).

Suppose \( Z \) and \( \mathcal{H}(M) \) denote the quotients of \( \tilde{Z} \) and \( H^1(M) \setminus \{0\} \), respectively, by \( \mathbb{Z}_2 \). Then \( \tilde{\zeta} \) descends to a map \( \zeta : Z \to \mathcal{H}(M) \). As discussed in Section 2.3, \( H^*(\mathcal{H}(M), \mathbb{Z}_2) = \mathbb{Z}_2[\xi] \). Furthermore, \( \tilde{Z} \in C_{k+m+1} \) if and only if

\[
\zeta^*(\xi^{k+m+1}) \neq 0 \in H^{k+m+1}(Z, \mathbb{Z}_2). \tag{4.12}
\]

However, since \( \tilde{X} \in C_k \), if \( X \) denotes the quotient of \( \tilde{X} \) by \( \mathbb{Z}_2 \), \( \tilde{h}_1 \) descends to a map \( h_1 : X \to \mathcal{H}(M) \) such that

\[
h_1^*(\xi^k) \neq 0 \in H^k(X, \mathbb{Z}_2). \tag{4.13}
\]

Similarly, as \( \tilde{Y} \in C_m \), if \( Y \) denotes the quotient of \( \tilde{Y} \) by \( \mathbb{Z}_2 \), \( \tilde{h}_2 \) descends to a map \( h_2 : Y \to \mathcal{H}(M) \) such that

\[
h_2^*(\xi^m) \neq 0 \in H^m(Y, \mathbb{Z}_2). \tag{4.14}
\]

One can prove (4.12) by using (4.13), (4.14), (4.9) and an argument similar to Part 3 of the proof of Theorem 1.1. \( \square \)

By the above Claim 4.2,

\[
\sup_{z \in \tilde{X} \times \tilde{Y}} E_\xi(\tilde{\zeta}(z)) \geq c_\xi(k + m + 1). \tag{4.15}
\]

On the other hand, by (4.7) and Proposition 4.1(iv), for all \( (x, y, t) \in \tilde{X} \times \tilde{Y} \),

\[
E_\xi(\tilde{\zeta}(x, y, t)) \leq E_\xi(\tilde{q}(x, y, t)) + \delta \leq E_\xi(h_1(x)) + E_\epsilon(h_2(y)) + E_\epsilon(w(t)) + \delta. \tag{4.16}
\]
A sub-additive inequality for the volume spectrum

Combining (4.15), (4.16), (4.1), (4.2) and (4.3), we obtain

\[ c_\varepsilon(k + m + 1) \leq c_\varepsilon(k) + c_\varepsilon(m) + \gamma_\varepsilon + 4\delta. \]

Since \( \delta > 0 \) is arbitrary, this finishes the proof of Theorem 1.2.

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