REMARKS ON THE FUNCTIONAL EQUATION $f(x + 1) = g(x)f(x)$ AND A UNIQUENESS THEOREM FOR THE GAMMA FUNCTION

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Abstract. The topic of gamma type functions and related functional equation $f(x + 1) = g(x)f(x)$ has been seriously studied from the first half of the twentieth century till now. Regarding unique solutions of the equation the asymptotic condition $\lim_{x \to \infty} \frac{g(x + w)}{g(x)} = 1$, for each $w > 0$, is considered in many serial papers including R. Webster’s article (1997). Motivated by the topic of limit summability of real functions (introduced by the author in 2001), we first show that the asymptotic condition in the Webster’s paper can be reduced to the convergence of the sequence $\frac{g(n + 1)}{g(n)}$ to 1, and then by removing the initial condition $f(1) = 1$ we generalize it. On the other hand, Rassias and Trif proved that if $g$ satisfies another appropriate asymptotic condition, then the equation admits at most one solution $f$, which is eventually log-convex of the second order. We also show that without changing the asymptotic condition for this case, a uniqueness theorem similar to the Webster’s result is also valid for eventually log-convex solutions $f$ of the second order. This result implies a new uniqueness theorem for the gamma function which states the log-convexity condition in the Bohr-Mollerup Theorem can be replaced by log-concavity of order two. At last, some important questions about them will be asked.

1. Introduction and Preliminaries

The gamma function $\Gamma(x)$ was introduced by Euler in 18th century and improved by Legendre, Gauss and Weierstrass. In 1922, Bohr and Mollerup proved that the Gamma function is unique solution of the equation $f(x + 1) = xf(x)$ with $f(1) = 1$ $(x > 0)$ if the log-convexity condition is considered. Regarding generalizing this equation the following functional equations were studied:

$$f(x + 1) = g(x)f(x), \quad f(x + 1) = g(x) + f(x); \quad x > 0.$$ 

In 1949, Krull [4] studied these equations and obtained many important results about them. In [10], Webster studied the equation $f(x + 1) = g(x)f(x)$ and its special solution namely gamma type function. In order to introduce ultra exponential and infra logarithm functions, the author proposed the topic of limit summability of functions (in 2001, [3]) and showed that the subject “gamma type functions” is its sub-topic. In the topic, domain of the function does not need to be $\mathbb{R}^+$ or $(a, \infty)$, and it is enough to be defined in positive integers. We can state a summary of limit summability as follows.

Let $f$ be a real or complex function with domain $D_f \supseteq \mathbb{N}^* = \{1, 2, 3, \cdots\}$. Put

$$\Sigma_f = \{x | x + \mathbb{N}^* \subseteq D_f\},$$

and then for any $x \in \Sigma_f$ and $n \in \mathbb{N}^*$ set

$$R_n(f, x) = R_n(x) := f(n) - f(x + n),$$

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The function $f$ is called limit summable at $x_0 \in \Sigma_f$ if the functional sequence $\{f_{\sigma_n}(x)\}$ is convergent at $x = x_0$. The function $f$ is called limit summable on the set $S \subseteq \Sigma_f$ if it is limit summable at all the points of $S$.

Now, put

$$f_\sigma(x) = f_{\sigma_n}(x) := xf(n) + \sum_{k=1}^{n} R_k(x).$$

Therefore $D_{f_\sigma} = \{x \in \Sigma_f | f \text{ is limit summable at } x\}$, and $f_{\sigma_n} = f_\sigma$ is the same limit function of $f_{\sigma_n}$ with domain $D_{f_\sigma}$.

The function $f$ is called limit summable if it is summable on $\Sigma_f$, $R(1) = 0$ and $D_f \subseteq D_{f_1}$, where $D_f = D_{f_1} - 1$. In this case the function $f_\sigma$ is referred to as the limit summand function of $f$. If $f$ is limit summable, then $D_{f_\sigma} = D_f - 1$ and

$$f_\sigma(x) = f(x) + f_\sigma(x-1) ; \forall x \in D_f$$

Therefore, if $f$ is limit summable then its limit summand function $f_\sigma$ satisfies the well-known difference functional equation $\varphi(x) - \varphi(x-1) = f(x)$ (e.g., see [2, 3, 4]). Hence,

$$f_\sigma(m) = f(1) + \cdots + f(m) = \sum_{j=1}^{m} f(j) ; \forall m \in \mathbb{N}^*.$$

If $f$ is limit summable then one may use the notation $\sigma_\ell(f(x))$ instead of $f_{\sigma}(x)$.

Very often if a real function $f$ is limit summable on an interval of length 1 and the sequence $R_n(f, 1)$ is convergent, then $f$ is limit summable on whole $\Sigma_f$. One can see some criteria for existence of unique solutions of the above functional equation in [3]. For example, if $0 < b \neq 1$ and $0 < a < 1$, then the real function $f(x) = a^x + \log_b x$ is limit summable and

$$f_\sigma(x) = \frac{a}{a-1}(a^x - 1) + \log_b \Gamma(x+1).$$

Now, let come back to gamma type functions satisfying the equation. The next theorem is one of the main theorems about it.

**Theorem A** ([10, R.J. Webster]). Let the function $g : \mathbb{R}^+ \to \mathbb{R}^+$ have the property

$$\lim_{x \to \infty} \frac{g(x+w)}{g(x)} = 1 , \quad \text{for all } w > 0. \quad (1.1)$$

Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an eventually log-convex function satisfying the functional equation $f(x+1) = g(x)f(x)$ for $x > 0$ and the initial condition $f(1) = 1$. Then $f$ is uniquely determined by $g$ through the equation

$$f(x) = \lim_{n \to \infty} \frac{g(n)\cdots g(1)g(n)^x}{g(n+x) \cdots g(x)} ; \quad x > 0. \quad (1.2)$$

In equation (1.2) the function $f$ is referred as the “gamma type function of $g$”.

2. Results for log-convex and log-concave solutions

Now, we prove that the asymptotic condition of Theorem A can be weak such that the same consequent holds.

**Lemma 2.1.** The asymptotic condition in Theorem A can be replaced by $\frac{g(n+1)}{g(n)} \to 1$ as $n \to \infty$. 

ON THE FUNCTIONAL EQUATION \( f(x + 1) = g(x)f(x) \)

Proof. Let \( M \) be a number that \( f \) is log-convex on \((M, \infty)\) and fix an arbitrary non-integer number \( x > M \) (it is clear for positive integers). With do attention the hypothesis of the theorem and using the log-convexity of \( f \) on \([n, n + 1, n + 1 + x - \lfloor x \rfloor, n + 2]\) where \( n \) is every integer with \( n > \max\{M, \lfloor x \rfloor\} \),

one can write

\[
(2.1) \quad \left( \frac{f(n + 1)}{f(n)} \right)^{x - \lfloor x \rfloor} \leq \frac{f(n + 1 + x - \lfloor x \rfloor)}{f(n + 1)} \leq \left( \frac{f(n + 2)}{f(n + 1)} \right)^{x - \lfloor x \rfloor}
\]

\[
(2.2) \quad f(t + n) = g(t + n - 1) \cdots g(t)f(t) ; \forall t > 0
\]

\[
(2.3) \quad f(n + 1) = g(n) \cdots g(1)
\]

Combining (2.1), (2.2) and (2.3) and putting \( k = n - \lfloor x \rfloor \) we find that

\[
\left( \frac{g(n)}{g(k)} \right)^x \frac{g(n) \cdots g(k + 1)}{g(n) \cdots g(n)} \leq \frac{f(x)g(x + k) \cdots g(x)}{g(k) \cdots g(1)g(k)^x} \leq \frac{g(n) \cdots g(k + 1)}{g(n + 1) \cdots g(n + 1)}.
\]

Since \( \frac{g(n + 1)}{g(n + 2)} \to 1 \) as \( n \to \infty \), for every integers \( i, j \), by letting \( n \to \infty \) in the above last inequalities we arrive at (1.1).

In the next corollary, we remove the initial condition and get a generalization of Theorem A.

Corollary 2.2. (A generalization of the Webster’s theorem) Let the function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) have the property \( \frac{g(n + 1)}{g(n)} \to l \) as \( n \to \infty \). Suppose that \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is a function satisfying the functional equation \( f(x + 1) = g(x)f(x) \) for \( x > 0 \). If one of the following conditions hold:

(a) \( f \) is eventually log-convex and \( 0 < l \leq 1 \),

(b) \( f \) is eventually log-concave and \( l \geq 1 \),

then the function \( \frac{1}{f(1)}f \) is uniquely determined by \( g \) through the equation

\[
(2.4) \quad \frac{1}{f(1)}f(x) = l^{x - \frac{x}{2} - \frac{1}{2}} \lim_{n \to \infty} \frac{g(n) \cdots g(1)g(n)^x}{g(n + x) \cdots g(x)} = l^{x - \frac{x}{2} - \frac{1}{2}} \lim_{n \to \infty} \left( g(n)^x \prod_{k=1}^{n} \frac{g(k)}{g(x + k)} \right) ; \quad x > 0.
\]

Therefore, there exists a constant \( c \) such that

\[
(2.5) \quad f(x + 1) = c l^{x - \frac{x}{2} - \frac{1}{2}} e^{(\log g)_{\ast}(x)} ; \quad x > 0.
\]

Proof. If (a) holds, then putting \( c := f(1), G(x) := l^{-x}g(x) \) and \( F(x) := c^{-1}L \frac{1}{x}f(x) \) we conclude that \( \frac{G(n + 1)}{G(n)} \to 1 \) as \( n \to \infty \), \( F \) is eventually log-convex and

\[
F(x + 1) = G(x)F(x) ; \quad x > 0 , \quad F(1) = 1.
\]

Therefore, Lemma 2.1 implies that

\[
F(x) = \frac{1}{G(x)} \lim_{n \to \infty} \left( G(n)^x \prod_{k=1}^{n} \frac{G(k)}{G(x + k)} \right) = \frac{l^{x - \frac{x}{2} - \frac{1}{2}}}{g(x)} \lim_{n \to \infty} \left( g(n)^x \prod_{k=1}^{n} \frac{g(k)}{g(x + k)} \right)
\]

\[
= \frac{l^{x - \frac{x}{2} - \frac{1}{2}}}{g(x)} \lim_{n \to \infty} e^{x \log g(n) + \sum_{k=1}^{n} \left( \log g(k) - \log g(x + k) \right)} = \frac{l^{x - \frac{x}{2} - \frac{1}{2}}}{g(x)} \lim_{n \to \infty} e^{(\log g)_{\ast}(x)}.
\]

Hence we arrive at (2.4).
Example 2.3. Let $a > 0$ be a constant real number and consider the functional equation
\[ f(x + 1) = x a^x f(x); \quad x > 0. \]
If $a > 1$, then all eventually log-convex solutions of the equation are of the form
\[ f(x) = c a^{\frac{x^2 - x}{2}} \Gamma(x), \]
for all real constants $c$.

Remark 2.4. In [10] Webster denotes the class of all eventually log-concave functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ with the asymptotic property (1.1), by $G$ and shows that if $g \in G$ then the gamma type functional equation has a unique solution and $f$ is eventually log-convex. We remark that $G$ is indeed the class of all eventually log-concave functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ with the property $\frac{g(n+1)}{g(n)} \to 1$ as $n \to \infty$ (consider the log-concavity of $g$ on $[m, m+1, x, x+w, n, n+1]$ where $m, n$ are integers such that $m+1 < x$ and $x+w < n$). Hence, in this case (1.1) and convergence of the sequence are equivalent (in the related theorems).

3. RESULTS FOR LOG-CONVEX AND CONCAVE SOLUTIONS OF ORDER TWO

Let $I$ be an interval of the real line and $f : I \to \mathbb{R}$ a function. Define the divided difference of $f$ at the points $x_0 < x_1 < \cdots < x_{n+1}$ in $I$ by
\[ [x_0; f] = f(x_0), \quad [x_0, x_1, \cdots, x_k; f] = \frac{[x_1, \cdots, x_k; f] - [x_0, \cdots, x_{k-1}; f]}{x_k - x_0}; \quad k \geq 1. \]
The function $f$ is called convex of order $n$ or log-convex if for any system $x_0 < x_1 < \cdots < x_{n+1}$ of points in $I$ it holds that $[x_0, x_1, \cdots, x_{n+1}; f] \geq 0$, and it is $n$-concave if $-f$ is $n$-convex. A positive function $f$ is said to be eventually log-convex of order $n$ or log-convex of order $n$ from a number on, if $I$ contains a subinterval that is unbounded above and on which the restriction of $f$ is convex of order $n$ (analogously for the concave case). If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is eventually log-convex of order 2, then there is a number $M$ such that for every $u < v < w < z$ in $(M, +\infty)$ the following inequality holds
\[ \left( \frac{f(w)}{f(v)} \right)^{z-u} \leq \left( \frac{f(z)}{f(u)} \right)^{w-v}. \tag{3.1} \]
A similar inequality holds for log-concave functions of order 2. Also, if $f$ is three times differentiable on $I$, then $f$ is convex (resp. concave) of order two if and only if $f'''(x) \geq 0$ (resp. $f'''(x) \leq 0$) for all $x \in I$. Since
\[ (\log \Gamma(x))'' = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}, \quad (\log \Gamma(x))''' = -2 \sum_{k=0}^{\infty} \frac{1}{(x+k)^3}; \quad x > 0, \]
then the gamma function is log-convex and also 2-log-concave. More information about $\Gamma(x)$ and log-convex functions can be seen in [5, 6, 8, 9].

In [7], Rassias and Trif use the log-convexity of order two and replace the asymptotic condition (1.1) by
\[ \lim_{x \to \infty} \frac{g(x+r)}{x^r g(x)} = a^r, \quad \text{for some } a > 0 \text{ and all } r > 0. \tag{3.2} \]
for uniqueness conditions to the solutions of the Gamma-type functional equation. Of course, their theorem gives a unique solution $f$ rather than the gamma type function of $g$, and it is usable for some equations such as $f(x + 1) = \Gamma(x)f(x)$ but not $f(x + 1) = xf(x)$. Now, by using the techniques from [3], we prove that the same condition (1.1) can be used alongside log-convexity of order two for
uniqueness conditions on the equation, and then we obtain a new uniqueness theorem for the gamma function.

**Theorem 3.1.** Let \( g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be a given function satisfying (1.1). If \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is an eventually log-convex (or log-concave) function of order two satisfying the gamma type functional equation \( f(x + 1) = g(x)f(x) \), for \( x > 0 \), with the initial condition \( f(1) = 1 \), then \( f \) is uniquely determined by \( g \) through the equation (1.2).

**Proof.** Let \( M \) be a number that \( f \) is log-convex of order two on \((M, \infty)\) and fix an arbitrary \( x > M \).

Then, for every integer \( n > M \) the property \([n, n + 1, n + 1 + x, n + 2 + [x]; \log f] \geq 0 \) implies

\[
\left( \frac{f(n)}{f(n + 2 + [x])} \right)^x \leq \left( \frac{f(n + 1)}{f(n + 1 + x)} \right)^{2+[x]} = \left( \frac{1}{f(x)} \cdot \frac{g(n) \cdots g(1)}{g(n + x) \cdots g(x)} \right)^{2+[x]}.
\]

Thus

\[
f(x) \leq \frac{g(n) \cdots g(1)}{g(n + x) \cdots g(x)} \frac{g(n + 1 + [x]) \cdots g(n)}{g(n) \cdots g(1)} \left( \frac{g(n) \cdots g(1) g(n)^x}{g(n + x) \cdots g(x)} \right)^{2+[x]}.
\]

Also, by applying \([n + [x], n + x, n + 1 + [x], n + 2 + [x]; \log f] \geq 0 \) for the case \( x \) is non-integer, we conclude that

\[
\left( \frac{f(n + [x])}{f(n + 2 + [x])} \right)^{1-[x]} \leq \left( \frac{f(n + x)}{f(n + [x] + 1)} \right)^2 = \left( \frac{g(n + x - 1) \cdots g(x) f(x)}{f(n + [x] + 1)} \right)^2,
\]

hence

\[
f(x) \geq \frac{g(n) \cdots g(1) g(n)^x}{g(n + x) g(n + x - 1) \cdots g(x)} \frac{g(n + [x]) \cdots g(n + 1) g(n + x)}{g(n)^x} \left( \frac{1}{g(n + [x] + 1) g(n + [x])} \right)^{1-[x]}
\]

\[
= \frac{g(n) \cdots g(1) g(n)^x}{g(n + x) \cdots g(x)} \cdot \frac{g(n + [x])}{g(n)} \cdots \frac{g(n + 1)}{g(n)} \left( \frac{g(n + x)}{g(n)} \right)^{\{x\}} \left( \frac{g(n + x)}{g(n + 1 + [x])} \cdot \frac{g(n + x)}{g(n + [x])} \right)^{1-[x]}.
\]

Therefore, the equation (1.2) holds. Note that if \( f \) is log-concave of order two, then considering the equation \( f(x + 1)^{-1} = g(x)^{-1} f(x)^{-1} \) and the above argument, one can get the result. \( \square \)

**Remark 3.2.** In view of the above proof, one find that if the sequence \( \frac{g(n + 1)}{g(n)} \) converges to 1 then

\[
f(x) \leq \lim_{n \to \infty} \frac{g(n) \cdots g(1) g(n)^x}{g(n + x) \cdots g(x)} ; \quad x > 0,
\]

and the equality holds if (1.1) is satisfied. Hence, we remains the following question.

**Question I.** If the asymptotic condition (1.1) in Theorem 3.1 is replaced by \( \frac{g(n + 1)}{g(n)} \to 1 \) as \( n \to \infty \), then does the equality (1.2) hold again?

### 3.1. A new uniqueness theorem for the gamma function.

It is easy to see that the difference functional equation \( f(x + 1) = x f(x) \) for \( x > 0 \) (for the case \( g(x) = x \) ), with the initial condition \( f(1) = 1 \) has infinitely many solutions, having the property \( f(n) = (n - 1)! \), for every positive integer \( n \). In 1922, Bohr and Mollerup proved that the Gamma function \( \Gamma(x) \) is its unique solution if the log-convexity condition is considered. Thereafter, several uniqueness conditions were proved by using monotonicity, geometrically convexity and so on (e.g., see [1, 2, 6, 8]). In [5] (2015), Matkowski proved some other uniqueness theorems for it by using the Jensen convexity and some other conditions. Now, as an interesting result of Theorem 3.1 we conclude that the condition log-convexity in the Bohr-Mollerup Theorem can be replaced by log-concavity of order two.
Corollary 3.3. The gamma function $\Gamma(x)$ is the only function $f$ that has the three properties
(a) $f(x+1) = xf(x)$ for $x > 0$;
(b) $f(1) = 1$;
(c) $f$ is log-concave of the second order.

Remark 3.4. Since the Gamma function is $n$-log-convex (resp. $n$-log-concave) for every odd (resp. even) positive integer $n$, then we can replace the third condition (c) in the Bohr-Mollerup Theorem by each of the following conditions:

(c$_1$) $f$ is $n$-log-convex for every odd positive integers $n$,
(c$_2$) $f$ is $n$-log-concave for every even positive integers $n$.

Hence the next question arises:

Question II. Can the third condition (c) be replaced by one of the next conditions?:

(c$_3$) $f$ is $n$-log-convex for some odd positive integer $n$,
(c$_4$) $f$ is $n$-log-concave for some even positive integer $n$,
(c$_5$) $f$ is $n$-log-convex and $m$-log-concave for some odd positive integer $n$ and even positive integer $m$,
(c$_6$) $f$ is $n$-log-convex and $(n+1)$-log-concave for some odd positive integer $n$.

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