New perspectives on exponentiated derivations, the formal Taylor theorem, and Faà di Bruno’s formula

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Abstract. We discuss certain aspects of the formal calculus used to describe vertex algebras. In the standard literature on formal calculus, the expression \((x + y)^n\), where \(n\) is not necessarily a nonnegative integer, is defined as the formal Taylor series given by the binomial series in nonnegative powers of the second-listed variable (namely, \(y\)). We present a viewpoint that for some purposes of generalization of the formal calculus including and beyond “logarithmic formal calculus,” it seems useful, using the formal Taylor theorem as a guide, to instead take as the definition of \((x + y)^n\) the formal series which is the result of acting on \(x^n\) by a formal translation operator, a certain exponentiated derivation. These differing approaches are equivalent, and in the standard generality of formal calculus or logarithmic formal calculus there is no reason to prefer one approach over the other. However, using this second point of view, we may more easily, and in fact do, consider extensions in two directions, sometimes in conjunction. The first extension is to replace \(x^n\) by more general objects such as the formal variable \(\log x\), which appears in the logarithmic formal calculus, and also, more interestingly, by iterated-logarithm expressions. The second extension is to replace the formal translation operator by a more general formal change of variable operator. In addition, we note some of the combinatorics underlying the formal calculus which we treat, and we end by briefly mentioning a connection to Faà di Bruno’s classical formula for the higher derivatives of a composite function and the classical umbral calculus. Many of these results are extracted from more extensive papers [R1] and [R2], to appear.

1. Introduction

Our subject is certain aspects of the formal calculus used, as presented in [FLM], to describe vertex algebras, although we do not treat any issues concerning “expansions of zero,” which is at the heart of the subject. An important basic result which we describe in detail is the formal Taylor theorem and this along with some variations is the topic we mostly consider. It is well known, and we recall the simple argument below, that if we let \(x\) and \(y\) be independent formal variables, then the formal exponentiated derivation \(e^{y \frac{d}{dx}}\), defined by the expansion, \(\sum_{k \geq 0} y^k \left( \frac{d}{dx} \right)^k / k!\), acts on a (complex) polynomial \(p(x)\) as a formal translation in \(y\). That is, we have

\[
e^{y \frac{d}{dx}} p(x) = p(x + y).
\]
Formulas of this type, where one shows how a formal exponentiated derivation acts as a formal translation over some suitable space, such as polynomials, are the content of the various versions of the formal Taylor theorem. In the standard literature on formal calculus, the expression \((x + y)^n\) is defined as a formal Taylor series given by the binomial series in nonnegative powers of the second-listed variable. This notational convention is called the “binomial expansion convention,” as in [FLM]; cf. [LL]. (Such series expansions often display interesting underlying combinatorics, as we discuss below.) We note that there are really two issues in this notational definition. One is the relevant “expansion” of interest, which is easy but substantial mathematically. The other is purely a “convention”, namely, deciding which listed variable should be expanded in nonnegative powers. Of course, one needs such a definition before even stating a formal Taylor theorem since one needs to know how to define what we mean when we have a formal function whose argument is \((x + y)\). The issue of how to define \(\log(x + y)\) for use in the recently developed logarithmic formal calculus is parallel. This issue originally arose in [M], where the author introduced logarithmic modules and logarithmic intertwining operators. In that context it was necessary to handle nonnegative integral powers of the logarithmic variables. In fact, the definition given there was
\[
\log(x + y) = e^{y \frac{d}{dx}} \log x,
\]
where \(\log x\) is a formal variable such that \(\frac{d}{dx} \log x = 1/x\) (see Section 1.3 and in particular Proposition 1.5 in [M]). The logarithmic calculus was then further developed in detail in Section 3 of [HLZ], where it was used in setting up some necessary language to handle the recently developed theory of braided tensor categories of non-semisimple modules for a vertex algebra. Actually, in [HLZ] the authors proved a more general formal Taylor theorem than they strictly needed, one involving general complex powers. We discuss this issue of the generality of exponents below. In [HLZ], the authors used a more standard approach which, as we have been discussing, is to define the relevant expressions \(p(x + y)\) via formal analytic expansions and to then prove the desired formal Taylor theorem. We argue that, in fact, for certain purposes it is more convenient to use formulas of the form (1.1) as the definition of \(p(x + y)\), as was done in [M] in the important special case mentioned above where \(p(x + y) = \log(x + y)\), whenever we extend beyond the elementary case of polynomials, but most especially if one wishes to extend beyond the logarithmic formal calculus.

Actually, the necessary structure is contained in the “automorphism property,” which for polynomials \(p(x)\) and \(q(x)\) says that
\[
e^{y \frac{d}{dx}} (p(x)q(x)) = \left( e^{y \frac{d}{dx}} p(x) \right) \left( e^{y \frac{d}{dx}} q(x) \right).
\]
The various formal Taylor theorems may then be interpreted as representations of the automorphism property which specialize properly in the easy polynomial case. We note that from this point of view the “expansion” part of the binomial expansion convention is not a definition but a consequence. (The “convention” part, which tells which listed variable should be expanded in the direction of nonnegative powers is, of course, retained in both approaches as the choice of notational convention.)

Whenever it was necessary to formulate more general formal Taylor theorems, such as in [HLZ], it was heuristically obvious that they could be properly formulated in the standard approach but as soon as one generalizes beyond the case of
the logarithmic calculus then there may be some tedious details to work out. It is hoped that the approach presented here may in the future make such generalization more efficient. In particular, we show how to generalize to a space that involves formal logarithmic variables iterated an arbitrary number of times as an example to show how this approach may be applied to desired generalizations.

We noted that the traditional approach to proving generalized formal Taylor theorems via formal analytic expansions may be tedious, and while narrowly speaking this is true, it is also true that these expansions are themselves interesting. Indeed, once we have firmly established the algebra of the automorphism property and the formal Taylor theorem relevant to any given context we may calculate formal analytic expansions. If there is more than one way to perform this calculation we may equate the coefficients of the multiple expansions and find a combinatorial identity. We record certain such identities, which turn out to involve the well-known Stirling numbers of the first kind and thereby recover and generalize an identity similarly considered in Section 3 of [HLZ], which was part of the motivation for this paper.

We are sometimes also interested in exponentiating derivations other than simply \( \frac{d}{dx} \). For instance, in [M] and [HLZ] the authors needed to consider the operator \( e^{yx} \frac{d}{dx} \). Such exponentiated derivations were considered in [FLM], and in fact much more general derivations appearing in the exponent have been treated at length in [H], but we shall only consider a couple of very special cases like those mentioned already. We present what we call “differential representations,” which help us to transfer formulas involving one derivation to parallel formulas for a second one which can be interpreted as a differential representation of the first. The automorphism property holds true for all derivations, but the formal Taylor theorem becomes a parallel statement telling us that another formal exponentiated derivation acts as a formal change of variable other than translation. For example, for a polynomial \( p(x) \), one may easily show that

\[
e^{yx} \frac{d}{dx} p(x) = p(xe^y).
\]

There is additional very interesting material which the automorphism property, the formal Taylor theorem and the notion of differential representation lead to. For instance, it turns out that certain of the basic structures of the classical umbral calculus, which was studied by G.C. Rota, D. Kahaner, A. Odlyzko and S. Roman ([Rot2], [Rot1], [Rot3] and [Rom]), and certain aspects of the exponential Riordan group, which was studied by L.W. Shapiro, S. Getu, W.-J. Woan and L. C. Woodson ([Sh1] and [Sh2]), may be naturally formulated and recovered in a similar context to the one we are considering. In this paper we only indicate this connection in a brief comment. Such material is treated in [R2].

In Section 2 we give an expository review of the traditional formulation of formal Taylor theorems. In Section 3 we reformulate the material of the previous section from the point of view that formal Taylor theorems may be regarded as representations of the automorphism property. In Section 4 we consider a relation between the formal translation operator and a second formal change of variable operator. In Section 5 we record some underlying combinatorics recovering, in particular, a classical identity involving Stirling numbers of the first kind, which was rediscovered in [HLZ]. Finally, in Section 6 we briefly show a connection to Faà di Bruno’s classical formula for the higher derivatives of a composite function.
following a proof given in [FLM], as well as a related connection to the umbral calculus.

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2. The formal Taylor theorem: a traditional approach

We begin by recalling some elementary aspects of formal calculus (cf. e.g. [FLM]). We write \( \mathbb{C}[x] \) for the algebra of polynomials in a single formal variable \( x \) over the complex numbers; we write \( \mathbb{C}[[x]] \) for the algebra of formal power series in one formal variable \( x \) over the complex numbers, and we also use obvious natural notational extensions such as writing \( \mathbb{C}[x][[y]] \) for the algebra of formal power series in one formal variable \( y \) over \( \mathbb{C}[x] \). Further, we shall frequently use the notation \( e^w \) to refer to the formal exponential expansion, where \( w \) is any formal object for which such expansion makes sense. For instance, we have the linear operator \( e^{yD} : \mathbb{C}[x] \to \mathbb{C}[x][[y]] \):

\[
e^{yD} = \sum_{n \geq 0} \frac{y^n}{n!} \left( \frac{d}{dx} \right)^n.
\]

**Proposition 2.1.** (The “automorphism property”) Let \( A \) be an algebra over \( \mathbb{C} \). Let \( D \) be a derivation on \( A \). That is, \( D \) is a linear map from \( A \) to itself which satisfies the product rule:

\[
D(ab) = (Da)b + a(Db) \text{ for all } a \text{ and } b \text{ in } A.
\]

Then

\[
e^{yD}(ab) = (e^{yD}a) (e^{yD}b).
\]

**Proof.** Notice that

\[
D^n ab = \sum_{n=0}^{r} \binom{r}{n} D^{r-n} a D^n b.
\]

Then divide both sides by \( n! \) and sum over \( y \) and the result follows. \( \square \)

**Proposition 2.2.** (The polynomial formal Taylor theorem) For \( p(x) \in \mathbb{C}[x] \), we have

\[
e^{yD}p(x) = p(x + y).
\]

**Proof.** By linearity we need only check the case where \( p(x) = x^m, m \) a non-negative integer. We simply calculate as follows:
\[
e^{y \frac{d}{dx}} x^m = \sum_{n \geq 0} \frac{y^n}{n!} \left( \frac{d}{dx} \right)^n x^m
= \sum_{n \geq 0} \frac{y^n}{n!} (m)(m-1) \cdots (m-(n-1))x^{m-n}
= \sum_{n \geq 0} \binom{m}{n} x^{m-n} y^n
= (x+y)^m.
\]

Here, so far, we are, of course, using only the simplest, combinatorially defined binomial coefficients, \( \binom{m}{n} \) with \( m, n \geq 0 \). We observe that the only “difficult” point in the proof is knowing how to expand \( (x+y)^m \) as an element in \( \mathbb{C}[x][[y]] \). In other words, the classical binomial theorem is at the heart of the proof of the polynomial formal Taylor theorem as well as at the heart of the proof of the automorphism property.

In order to extend the polynomial formal Taylor theorem to handle the case of Laurent polynomials, we extend the binomial notation to include expressions \( \binom{m}{n} \) with \( m < 0 \) and we also recall the binomial expansion convention:

**Definition 2.1.** We write
\[
(x+y)^m = \sum_{n \geq 0} \binom{m}{n} x^{m-n} y^n, \quad m \in \mathbb{Z},
\]
where we assign to \( \binom{m}{n} \) the algebraic (rather than combinatorial) meaning: for all \( m \in \mathbb{Z} \) and \( n \) nonnegative integers
\[
\binom{m}{n} = \frac{(m)(m-1) \cdots (m-(n-1))}{n!}.
\]

**Remark 2.1.** In the above version of the binomial expansion convention we may obviously generalize to let \( m \in \mathbb{C} \).

With our extended notation, as the reader may easily check, the above proof of Proposition 2.2 exactly extends to give:

**Proposition 2.3.** (The Laurent polynomial formal Taylor theorem) For \( p(x) \in \mathbb{C}[x, x^{-1}] \), we have
\[
e^{y \frac{d}{dx}} p(x) = p(x+y).
\]

**Notation 2.1.** We write \( \mathbb{C}\{[x]\} \) for the algebra of finite sums of monomials of the form \( cx^r \) where \( c \) and \( r \in \mathbb{C} \).

As the reader may easily check, the above proof of Proposition 2.2 exactly extends even further to give:

**Proposition 2.4.** (The generalized Laurent polynomial formal Taylor theorem) For \( p(x) \in \mathbb{C}\{[x]\} \), we have
\[
e^{y \frac{d}{dx}} p(x) = p(x+y).
\]
Remark 2.2. There is an alternate approach to get the generalized Laurent polynomial formal Taylor theorem, an approach which has the advantage that no additional calculation is necessary in the final proof. The argument is simple. For $r \in \mathbb{C}$, we need to verify that

$$e^{\frac{dy}{dx}} x^r = (x + y)^r.$$ 

Now simply notice that both expressions lie in

$$\mathbb{C}x^r[1/x][[y]]$$

with coefficients being polynomials in $r$. But the polynomials on matching monomials agree for $r$ a nonnegative integer and so they must be identical. An argument in essentially this style appeared in [HLZ] to prove a logarithmic formal Taylor theorem (Theorem 3.6 of [HLZ]).

We now extend our considerations to a logarithmic case.

Definition 2.2. Let $\log x$ be a formal variable commuting with $x$ and $y$ such that $\frac{dx}{dx} \log x = x^{-1}$.

We shall need to define expressions involving $\log(x + y)$. In parallel with (2.1) we shall define $(\log(x + y))^r$, $r \in \mathbb{C}$, by its formal analytic expansion:

Notation 2.2. We write

$$(\log(x + y))^r = \left(\log x + \log \left(1 + \frac{y}{x}\right)\right)^r,$$

where we make a second use of the symbol “log” to mean the usual formal analytic expansion, namely

$$\log(1 + X) = \sum_{i \geq 0} \frac{(-1)^{i-1}}{i} X^i,$$

and where we expand (2.2) according to the binomial expansion convention.

Remark 2.3. We note that (2.2) is a special case of the definition used in the treatment of logarithmic formal calculus in [HLZ]. Our special case avoids the complication of the generality, treated in [HLZ], of (uncountable, non-analytic) sums over $r \in \mathbb{C}$.

Remark 2.4. The reader will need to distinguish from context which use of “log” is meant.

Proposition 2.5. (The generalized polynomial logarithmic formal Taylor theorem) For $p(x) \in \mathbb{C}[[x, \log x]]$, we have

$$e^{\frac{dy}{dx}} p(x) = p(x + y).$$

Proof. By linearity and the automorphism property, we need only check the case $p(x) = (\log x)^r$, $r \in \mathbb{C}$. We could proceed by explicitly calculating

$$e^{\frac{dy}{dx}}(\log x)^r,$$

but this is somewhat involved. Instead we argue as in Remark 2.2 to reduce to the case $r = 1$. Even without explicitly calculating $e^{\frac{dy}{dx}}(\log x)^r$, it is not hard to see that it is in

$$\mathbb{C}[r](\log x)^r \mathbb{C}[(\log x)^{-1}, x^{-1}][[y]].$$
When we expand (2.2) we find that it is also in
\[ C[r][\log x]^r \mathbb{C}[(\log x)^{-1}, x^{-1}][[y]]. \]
Thus we only need to check the case for \( r \) a positive integer. A second application of the automorphism property now shows that we only need the case where \( r = 1 \).

This case is not difficult to calculate:
\[
e^{y \frac{d}{dx}} \log x = \log x + \sum_{i \geq 1} \frac{y^i}{i!} \left( \frac{d}{dx} \right)^i \log x
= \log x + \sum_{i \geq 1} \frac{y^i}{i!} \left( \frac{d}{dx} \right)^{i-1} x^{-1}
= \log x + \sum_{i \geq 1} \frac{y^i}{i!} (-1)^{i-1} x^{-i}
= \log x + \log \left( 1 + \frac{y}{x} \right).
\]

Remark 2.5. Although we are working in a more special case than that considered in [HLZ], the argument presented in the proof of Proposition 2.5 could be used as a replacement for much of the algebraic proof of Theorem 3.6 in [HLZ] as long as one is not concerned with calculating explicit formal analytic expansions and checking the corresponding combinatorics. These two approaches are very similar, however, the difference only being how much work is left implicit. In the next section we shall take a different point of view altogether.

3. The formal Taylor theorem from a different point of view

From the examples in Section 2 we see a common strategy for formulating a formal Taylor theorem:

1) Pick some reasonable space (e.g., \( \mathbb{C}[x] \), \( \mathbb{C}\{[x, \log x]\} \)) on which \( \frac{d}{dx} \) acts in a natural way. The space need not be an algebra, but in this paper we shall only consider this case.

2) Choose a plausible formal analytic expansion of relevant expressions involving \( x + y \) (e.g., \( (x + y)^r \), \( r \in \mathbb{C}, \log(x + y) \)).

3) Consider the equality \( e^{y \frac{d}{dx}} p(x) = p(x + y) \) and either directly expand both sides to show equality or if necessary use a trick like in Remark 2.2.

Step 2 is necessarily anticipatory and dependent on formal analytic expressions. Therefore it seems natural to replace Step 2 by simply defining expressions involving \( x + y \) in terms of the operator \( e^{y \frac{d}{dx}} \). Then the formal Taylor theorem is trivially true, being viewed now as a (plausible) representation of the underlying structure of the automorphism property. We redo the previous work from this point of view.

Proposition 3.1. (The polynomial formal Taylor theorem) For \( p(x) \in \mathbb{C}[x] \), we have
\[
e^{y \frac{d}{dx}} p(x) = p(x + y).
\]
Proof. We have by the automorphism property:
\[
e^{y \frac{d}{dx}} p(x) = p \left( e^{y \frac{d}{dx}} x \right) = p(x + y).
\]
\[\square\]

Now for the replacement step:

Definition 3.1. We write
\[(x + y)^r = e^{y \frac{d}{dx}} x^r \quad \text{for} \quad r \in \mathbb{C}.
\]

Remark 3.1. Of course, Definition 3.1 is equivalent to Definition 2.1 together with Remark 2.1. This definition immediately leads to the most convenient proofs of certain “expected” basic properties, instead of needing to wait (as is often done) to prove a formal Taylor theorem to officially obtain these proofs. For example, we have:
\[
(x + y)^{r+s} = e^{y \frac{d}{dx}} x^{r+s}
= e^{y \frac{d}{dx}} (x^r x^s)
= \left( e^{y \frac{d}{dx}} x^r \right) \left( e^{y \frac{d}{dx}} x^s \right)
= (x + y)^r (x + y)^s.
\]

Proposition 3.2. (The generalized Laurent polynomial formal Taylor theorem)
For \( p(x) \in \mathbb{C}\{[x]\} \),
\[e^{y \frac{d}{dx}} p(x) = p(x + y).\]

Proof. This is trivial. \[\square\]

We also have this example of the replacement step:

Definition 3.2. We write
\[(\log(x + y))^r = e^{y \frac{d}{dx}} (\log x)^r \quad \text{for} \quad r \in \mathbb{C}.
\]

Proposition 3.3. (The generalized polynomial logarithmic formal Taylor theorem) For \( p(x) \in \mathbb{C}\{[x, \log x]\} \),
\[e^{y \frac{d}{dx}} p(x) = p(x + y).\]

Proof. The result follows by considering the trivial cases \( p(x) = x^r \) and \( p(x) = (\log x)^r \) for \( r \in \mathbb{C} \) and applying the automorphism property. \[\square\]

The formal analytic expansions are now viewed as calculations rather than definitions or conventions. So, for instance, we may calculate the expansions (2.1) and (2.2) as consequences rather than viewing them as definitions.

4. More general formal changes of variable

There are other formal Taylor-like theorems involving, for instance, the exponentiated derivation, \( e^{y \frac{d}{dx}} \). To recover such results we could repeat a complete parallel set of reasoning beginning with the automorphism property applied to the desired derivation. However, instead of starting over from the beginning, we show how to “lift” them from the formal Taylor theorems we have already proved. This
sort of method has the added benefit of showing relationships between different derivations instead of obtaining isolated results.

To proceed properly we need to look at one more extension of the formal Taylor theorem. To this end we let \( \ell_n(x) \) be formal commuting variables for \( n \in \mathbb{Z} \). We define an action of \( \frac{d}{dx} \), a derivation, on

\[
\mathbb{C}[\ldots, \ell_{-1}(x)^{\pm 1}, \ell_0(x)^{\pm 1}, \ell_1(x)^{\pm 1}, \ldots]
\]

(which for short we denote by \( \mathbb{C}[\ell^{\pm 1}] \)) by

\[
\frac{d}{dx} \ell_{-n}(x) = \prod_{i=-1}^{-n} \ell_i(x),
\]

\[
\frac{d}{dx} \ell_n(x) = \prod_{i=0}^{n-1} \ell_i(x)^{-1},
\]

and \( \frac{d}{dx} \ell_0(x) = 1, \)

for \( n > 0 \). Secretly, \( \ell_n(x) \) is the \((-n)\)-th iterated exponential for \( n < 0 \) and the \( n\)-th iterated logarithm for \( n > 0 \) and \( \ell_0(x) \) is \( x \) itself. We make the following, by now typical, definition in order to obtain a formal Taylor theorem.

**Definition 4.1.** Let

\[
\ell_n(x + y) = e^{y \frac{d}{dx}} \ell_n(x) \quad \text{for} \quad n \in \mathbb{Z}.
\]

This gives:

**Proposition 4.1.** *(The iterated exponential/logarithmic formal Taylor theorem)* For \( p(x) \in \mathbb{C}[\ell^{\pm 1}] \) we have:

\[
e^{y \frac{d}{dx}} p(x) = p(x + y).
\]

**Proof.** The result follows from the automorphism property. \( \square \)

Now consider the substitution map

\[
\phi : \mathbb{C}[\ell^{\pm 1}] \to \mathbb{C}[\ell^{\pm 1}]
\]

and its inverse defined by

\[
\phi(\ell_n(x)) = \ell_{n+1}(x) \quad \text{for} \quad n \in \mathbb{Z}
\]

(and \( \phi^{-1}(\ell_n(x)) = \ell_{n-1}(x) \quad \text{for} \quad n \in \mathbb{Z} \)).

**Proposition 4.2.** We have

\[
\phi \circ \frac{d}{dx} = \ell_0(x) \frac{d}{dx} \circ \phi
\]

and \( \phi^{-1} \circ \ell_0(x) \frac{d}{dx} = \frac{d}{dx} \circ \phi^{-1}. \)

This proposition makes clear that, on the appropriate space, \( e^{y \frac{d}{dx}} \) and \( e^{y \ell_0(x) \frac{d}{dx}} \) are simply shifted (in terms of the subscripts of \( \ell_n(x) \)) versions of each other.

**Proof.** Since \( \frac{d}{dx} \) and \( \ell_0(x) \frac{d}{dx} \) are derivations we need only check the action on \( \ell_n(x) \) \( n \in \mathbb{Z} \). The verification is routine calculation. For instance:
For \( n > 1 \)

\[
\ell_0(x) \frac{d}{dx} \phi \ell_{-n}(x) = \ell_0(x) \frac{d}{dx} \ell_{-n+1}(x) = \ell_0(x) \prod_{i=-1}^{-n+1} \ell_i(x) \\
= \prod_{i=0}^{-n+1} \ell_i(x) \\
= \phi \prod_{i=-1}^{-n} \ell_i(x) \\
= \phi \frac{d}{dx} \ell_{-n}(x).
\]

\[\square\]

We then have the following two examples of the “lifting” process referred to in the introduction to this section:

\[
e^{y \ell_0(x)} \frac{d}{dx} \ell_0(x) = \phi \circ e^{y \ell_0} \phi^{-1}(\ell_0(x)) \\
= \phi \circ e^{y \ell_0} \ell_{-1}(x) \\
= \phi \circ \sum_{n \geq 0} \frac{y^n}{n!} \ell_{-1}(x) \\
= \ell_0(x)e^y,
\]

and

\[
e^{y \ell_0(x)} \frac{d}{dx} \ell_1(x) = \phi \circ e^{y \ell_0} \phi^{-1}(\ell_1(x)) \\
= \phi \circ e^{y \ell_0} \ell_0(x) \\
= \phi(\ell_0(x) + y) \\
= \ell_1(x) + y,
\]

which translate respectively to the following identities in more standard logarithmic notation:

\[
e^{y \ell_0} \frac{d}{dx} x = xe^y \\
e^{y \ell_0} \frac{d}{dx} \log x = \log x + y.
\]

**Remark 4.1.** Of course, these examples can be obtained much more easily without resorting to this method but in more involved examples this approach is very useful (see e.g. [R1]).

**Remark 4.2.** Although we do not give a precise definition here, it is maps like \( \phi \) that we call differential representations. For more on these differential representations see [R1] and [R2].

5. Some combinatorics

The original (algebraic) proof in [HLZ] of the logarithmic formal Taylor theorem used formal analytic expansions (in fact, so did the statement). We have
bypassed those expansions in our approach, but they are themselves of some interest. For instance, the original proof relied on a combinatorial identity arising from equating the coefficients of two different formal analytic expansions.

We shall not get into the details of calculating formal analytic expansions here, but instead, merely briefly state some results to give the reader some idea of the material involved. It is possible to calculate the following three formal analytic expressions for $\ell_n(x+y)^r$, where we fix $r \in \mathbb{C}$ (see [R1]):

$$e^{y\frac{d}{dx}}\ell_n(x)^r = \ell_n(x+y)^r = \sum_{j_0,\ldots,j_n \geq 0} \left( \prod_{i=0}^{n-1} \binom{j_i}{j_{i+1}} \right) (-1)^{j_0+j_n} \frac{j_n!}{j_0!} \cdot \left( \frac{r}{j_n} \right) \ell_n(x)^r \left( \prod_{i=0}^{n} \ell_i(x)^{-j_i} \right) y^{j_0}$$

$$= \sum_{k \geq 0} \frac{y^k}{k!} \sum_{1 \leq j_n \leq \ldots \leq j_0 = k} j_n! \left( \frac{r}{j_n} \right) (-1)^{j_0+j_n} \cdot S(j_n,\ldots,j_0) \ell_n(x)^r - j_n \ell_{n-1}^r \cdots \ell_0^r$$

$$= \sum_{k \geq 0} \frac{y^k}{k!} \sum_{j_0+j_1+\cdots+j_n = k} j_n! \left( \frac{r}{j_n} \right) \cdot \left( \prod_{i=0}^{n-1} \binom{j_i}{\alpha_i+1} \right) \ell_n(x)^r \left( \prod_{i=0}^{n} \ell_i(x)^{-\alpha_i} \right),$$

where

$$\alpha_i = \sum_{l=i}^{n} j_l,$$

$$\binom{k}{j} = \frac{k!}{j! \cdot j_1! \cdots j_j!},$$

$$(m;n) = (-1)^m \sum_{0 \leq i_1 < i_2 < \cdots < i_m \leq m+n-1} i_1 i_2 \cdots i_m,$$

and where $S(j_n,\ldots,j_0)$ is given by the following recursion:

$$S(j_n,\ldots,j_0) = S(j_n-1,\ldots,j_0-1) + (j_n-1)S(j_n,\ldots,j_0-1)$$

$$\vdots$$

$$+ (j_0-1)S(j_n,\ldots,j_1,1),$$

along with the initial conditions,

$$S(j_n,\ldots,j_1,1) = \begin{cases} 1 & j_n = j_{n-1} = \cdots = j_1 = 1 \\ 0 & \text{otherwise.} \end{cases}$$
Equating coefficients yields the identity

\[ S(j_n, j_{n-1}, \ldots, j_0) = \prod_{i=0}^{n-1} \left[ \frac{j_i}{j_{i+1}} \right], \]

for \( j_0 \geq j_1 \geq \cdots \geq j_n \geq 1 \) and \( 0 \leq s \leq j_n \). When we specialize to the \( n = 1 \) case (the single logarithm case), we get

\[ (m; n) = (-1)^m \binom{m+n}{n}, \]

and, more generally,

\[ S(m, n) = \binom{n}{m} = \frac{n!}{m!} \sum_{i_1+\cdots+i_m=n} \frac{1}{i_1 \cdots i_m} \]

(5.1)

\[ = \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-m} \leq n-1} i_1 \cdots i_{n-m}, \]

where \( S(m, n) \) satisfies a standard recurrence for the Stirling numbers of the first kind.

**Remark 5.1.** Most of the identity (5.1) appeared in Section 3 of [HLZ] in the course of a “traditional-style” algebraic proof of a logarithmic formal Taylor theorem. See also Remark 3.8 in [HLZ], where this identity was observed to solve a problem posed by D. Lubell in the Problems and Solutions section of the American Mathematical Monthly [Lu]. The reappearance of this classical identity in this context was one of the original motivations for the present paper.

**Remark 5.2.** We also note that while the formal analytic expansions presented in this section could serve as definitions they would obviously be unwieldy.

### 6. Faà di Bruno and umbral calculus

There are some interesting variants of the notion of differential representation. In fact, one such variant appears implicitly in the proof of Faà di Bruno’s formula in Proposition 8.3.4 of [FLM], an argument which is essentially the basis for proving the (highly-nontrivial) “associativity” property of lattice vertex operator algebras in a setting based on arbitrary rational lattices; see Sections 8.3 and 8.4 of [FLM]. We present a special case of this argument next.
Let \( x, y, z \) be formal commuting variables. Let \( f(x), g(x) \in \mathbb{C}[x] \). Then

\[
e^y \frac{d}{dx} f(g(x)) = f(g(x) + (g(x) + y - g(x))) \\
= e^{(g(x) + y - g(x))}\frac{d}{dx}f(z)|_{z=g(x)} \\
= \sum_{n \geq 0} \frac{f^{(n)}(z)(g(x) + y - g(x))^n}{n!} |_{z=g(x)} \\
= \sum_{n \geq 0} \frac{f^{(n)}(g(x))(g(x) + y - g(x))^n}{n!} \\
= \sum_{n \geq 0} \frac{f^{(n)}(g(x)) \left( e^y \frac{d}{dx} g(x) - g(x) \right)^n}{n!} \\
= \sum_{n \geq 0} \frac{f^{(n)}(g(x)) \left( \sum_{m \geq 1} \frac{g^{(m)}(x)}{m!} \right)^n}{n!}.
\]

(6.1)

Motivated by this, we consider the algebra \( \mathbb{C}[y_0, y_1, y_2, \ldots, x_1, x_2, \ldots] \) where \( y_i, x_j \) for \( i \geq 0 \) and \( j \geq 1 \) are commuting formal variables. Let \( D \) be the unique derivation on \( \mathbb{C}[y_0, y_1, y_2, \ldots, x_1, x_2, \ldots] \) satisfying the following:

\[
Dy_i = y_{i+1}x_1 \quad i \geq 0 \\
Dx_j = x_{j+1} \quad j \geq 1.
\]

Then this question of calculating \( e^y \frac{d}{dx} f(g(x)) \) is seen to be essentially equivalent to calculating

\[
e^{zD}y_0,
\]

where we “secretly,” loosely speaking, identify \( \frac{d}{dx} \) with \( D \), \( f^{(n)}(g(x)) \) with \( y_n \) and \( g^{(m)}(x) \) with \( x_m \) (and \( y \) with \( z \)). The reader may note that we are now really dealing with, among other things, a certain sort of completion of the original problem, so that one may, for instance, wish to view \( f(x) \) as a formal power series and \( g(x) \) as a formal power series with zero constant term, and indeed we note that it was in this generality (and with even more general derivations) that the above argument was carried out in \[FLM\]. For a detailed description of this material, we refer the reader to \[R2\].

Before proceeding, we note that we may write an intermediate step of (6.1) as

\[
e^y \frac{d}{dx} \phi(f(z)) = \phi \left( e^{(g(x) + y - g(x))}\frac{d}{dx}f(z) \right),
\]

where \( \phi : \mathbb{C}[z] \rightarrow \mathbb{C}[x] \) substitutes \( g(x) \) for \( z \). That is, we have a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}[z] & \xrightarrow{e^{y \frac{d}{dx} - g(x)}} & \mathbb{C}[x, y, z] \\
\downarrow \phi & & \downarrow \phi \\
\mathbb{C}[x] & \xrightarrow{e^y \frac{d}{dx}} & \mathbb{C}[x, y].
\end{array}
\]
This commutative diagram shows how $\phi$ may be regarded as a sort of “global (i.e., exponentiated) differential representation.” In our new setting we might consider looking at “nonglobal” differential representations of $D$. This turns out to be too restrictive, but a suitably loosened version of this question turns out to lead to interesting results.

Let $\phi_B$ be the substitution which sends $y_j$ to 1 for $j \geq 0$ and sends $x_i$ to $xB_i$ for $i \geq 1$, where $B_i \in \mathbb{C}$ for $i \geq 1$ is a fixed, arbitrary sequence subject to the requirement that $B_1 \neq 0$.

**Proposition 6.1.** There is a unique linear map $D_B : \mathbb{C}[x] \to \mathbb{C}[x]$ which satisfies the condition

$$(6.2) \quad D_B^n \phi_B(y_0) = \phi_B(D^n(y_0)) \quad n \geq 0.$$

**Proof.** It is easy to see that $\phi_B D^n(y_0)$ is a polynomial of degree $n$ whose leading term is $B_1^n x^n$, where we recall that this coefficient is nonzero. Thus each required equality in turn (indexing by $n$) may be solved to obtain an equation of the form $D_B x^{n-1} = r(x)$, where $r(x)$ is a polynomial of degree $n$. Of course this recursive process solves for and completely determines $D_B x^n$ for all $n \geq 0$. □

The maps $D_B$ are what have been called umbral shifts, as in [Rom]. For more on the connection to classical umbral calculus see [R2].

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