Extended complex scalar field as quintessence

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Abstract

In this paper, we show the possibility that an extended complex scalar field can be considered as an extended complex quintessence. For this model, we derive the basic equations with a parameter $\eta$ which govern the evolution of the Universe. Our model may contain a complex quintessence model for $\eta = -1$ and a real quintessence model for $\eta = 0$.

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I. INTRODUCTION

The recent astronomy observations on type Ia supernovae suggests the expansion of the Universe at the present is accelerating [1] and measurement of the cosmic microwave background (CMB) indicate that it is spatially flat [2, 3]. Our Universe has a critical energy density which consists of 1/3 the matter density and 2/3 of the dark energy density with negative pressure [4]. One proper candidate for the dark energy is a slowly-evolving real scalar field $\Phi$ with a potential $V$, called quintessence [5, 6]. In this model the energy density and pressure from the scalar quintessence are $\rho_\Phi = \frac{1}{2}\dot{\Phi}^2 + V$ and $P_\Phi = \frac{1}{2}\dot{\Phi}^2 - V$ and the state equation is $P_\Phi = w_\Phi \rho_\Phi$ with $-1 < w_\Phi < -0.5$, the recent study based on the final Cosmic Lens All Sky Survey data suggest a constraint $w_\Phi < -0.55$ [7]. Recently, the quintessence has extended to the complex scalar field $\Phi = \phi e^{i\theta}$, the energy density and pressure being $\rho_\Phi = \frac{1}{2}(\dot{\phi}^2 + \phi^2\dot{\theta}^2) + V$, $P_\Phi = \frac{1}{2}(\dot{\phi}^2 + \phi^2\dot{\theta}^2) - V$. Compared with real quintessence model, complex quintessence possess the extra term of the energy density and pressure, $\frac{1}{2}\phi^2\dot{\theta}^2$, the contributions from the angular-motion. Similarly to the real quintessence model, the field and potential in the complex quintessence model may be reconstructed from the observational data by using Huterer and Turner’s procedure [8].

In this paper, we will consider an extended complex scalar field as a quintessence [11]. We do so by writing scalar field as $\Phi_\eta = \phi e^{i\eta\theta}$, introducing a real parameter $\eta$ into the energy density and pressure, $\rho_\Phi = \frac{1}{2}(\dot{\phi}^2 - \eta\phi^2\dot{\theta}^2) + V$, $P_\Phi = \frac{1}{2}(\dot{\phi}^2 - \eta\phi^2\dot{\theta}^2) - V$ which for $\eta = -1$ and $\eta = 0$ gives those of complex and real quintessence. When $\eta = 1$, our model shows a new quintessence configuration. As will be shown, in the case of extended complex quintessence, all quantities including the potential $V$, the amplitude quintessence field $\phi$, the equation of state $P_\Phi = w_\Phi \rho_\Phi$ will be reconstructed in terms of the observable quantities.

II. EXTENDED COMPLEX SCALAR FIELD AS QUINTESSENCE

Consider the Robertson-Walker Universe which is spatially flat

$$ds^2 = dt^2 - a^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2)$$

with the action

$$s = \int d^4x \sqrt{-g}[-\frac{1}{16\pi G}R - \rho_m + L_\Phi],$$
where $g$ is the matrix determinant of the metric, $R$ is the Ricci scalar, $G$ is the Newton’s constant, $\rho_m$ is the matter density, and $L_\Phi$ is the Lagrangian density for extended complex scalar field $\Phi_\eta$.

$$L_\Phi = \frac{1}{2} \partial^\mu \Phi^*_\eta \partial_\mu \Phi_\eta - V(\Phi^*_\eta \Phi_\eta),$$

(3)

which is formally the same as that, where $\Phi_\eta$ is the extended complex scalar field of the form $\Phi_\eta = Re\Phi_\eta + i_\eta Im\Phi_\eta$ with $i_\eta$ satisfying $i_\eta^2 = \eta$ being imaginary unit, $Re\Phi_\eta$ and $Im\Phi_\eta$ being real and imaginary parts, respectively, and $\Phi^*_\eta = Re\Phi_\eta + i_\eta Im\Phi_\eta$ is called the complex conjugate of $\Phi_\eta$.

It is easily proven that the Lagrangian possesses the $U(1; i_\eta)$ symmetry.

In contrast to complex scalar field, $\Phi_\eta$ has exponential form

$$\Phi_\eta = \phi e^{is\theta} = c(\theta; i_\eta) + i_\eta s(\theta; i_\eta),$$

(4)

where $\phi = \sqrt{(Re\Phi_\eta)^2 - \eta(Im\Phi_\eta)^2}$ and $\theta$, a real parameter, is called also "the amplitude" and "phase angular" of $\Phi_\eta$, respectively. Putting (4) into the Lagrangian (3) we have

$$L_\Phi = \frac{1}{2} (\partial^\mu \phi \partial_\mu \phi - \eta \phi^2 \partial^\mu \theta \partial_\mu \theta) - V(\phi),$$

(5)

which becomes the Lagrangian density (5) in Ref. for $\eta = -1$ and gives the following Lagrangian density for $\eta = 1$

$$L_\Phi = \frac{1}{2} (\partial^\mu \phi \partial_\mu \phi - \phi^2 \partial^\mu \theta \partial_\mu \theta) - V(\phi).$$

(6)

The variation of the action with the Lagrangian density leads to the equations of motion

$$H^2 = \frac{8\pi G}{3} \rho,$$

(7)

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3} (\rho + 3P),$$

(8)

$$\ddot{\phi} + 3H \dot{\phi} + \eta \phi \dot{\theta}^2 + V' = 0,$$

(9)

$$\ddot{\theta} + (2 \frac{\dot{\phi}}{\phi} + 3H) \dot{\theta} = 0,$$

(10)
where a dot and prime denotes derivatives with respect to $t$ and $\phi$, respectively, $H = \dot{a}/a$ is the Hubble parameter, $\rho$ and $P$ are the total density and pressure of the Universe

$$\rho = \rho_m + \rho_\Phi, \quad \rho_\Phi = \frac{1}{2}(\dot{\phi}^2 - \eta \phi^2 \dot{\theta}^2) + V,$$

(11)

$$P = P_m + P_\Phi, \quad P_\Phi = \frac{1}{2}(\dot{\phi}^2 - \eta \phi^2 \dot{\theta}^2) - V$$

(12)

with $\rho_\Phi$ and $P_\Phi$ denote the energy density and pressure from the scalar field $\Phi_\eta$ and potential $V$, called an extended complex scalar quintessence. $\rho_\Phi$ and $P_\Phi$ in (11) and (12) consist of the three terms: the kinetic energy contribution from the amplitude of the extended complex scalar field $\frac{1}{2} \dot{\phi}^2$, the 'kinetic energy' contribution from the 'angular-motion' $\frac{1}{2} \eta \phi^2 \dot{\theta}^2$ and the potential energy $V$.

Note that equation (10) is independent of the parameter $\eta$, for arbitrary $\eta$ there is the solution

$$\dot{\theta} = c a^{-3} \phi^{-2},$$

(13)

where $c$ is an integration constant. According to [9], the term $\phi^2 \dot{\theta}^2$ from the angular-motion in the quintessence energy density $\rho_\Phi$ and pressure $P_\Phi$ may be treated as an effective potential. Clearly, such an interpretation for the term $\phi^2 \dot{\theta}^2$ is improper since $\rho_\Phi$ and $P_\Phi$ takes the form $\rho_\Phi = \frac{1}{2} \dot{\phi}^2 + (V - \eta \phi^2 \dot{\theta}^2)$ and $P_\Phi = \frac{1}{2} \dot{\phi}^2 - (V + \eta \phi^2 \dot{\theta}^2)$. It is more like an effective kinetic energy than a potential energy. Certainly, if the term $\frac{1}{2} \eta \phi^2 \dot{\theta}^2$ is treated as effective kinetic for $\eta = 1$ it will take an negative value, this seem to be unreasonable. For this, the further discussion will be given in Sec.3. Now, let’s turn our attention to the problem of reconstruction equations.

From the relationships

$$a = \frac{1}{z+1},$$

$$r(z) = \int_0^z \frac{dz'}{H(z')},$$

$$H(z) = \frac{\dot{a}}{a} = \frac{1}{(dr/dz)},$$

$$\rho_M = \Omega_M \rho_c = \frac{3 \Omega_M H_0^2}{16 \pi G} (1 + z)^3,$$

(14)

where $z$, $\rho_c$, $\Omega_M$, $H_0$ and $r(z)$ are the redshift, the critical density, the matter energy density fraction and the coordinate distance to an object at redshift $z$, the reconstruction equations of the potential function $V$, the amplitude $\phi$ and phase angular of extended complex
quintessence in terms of the observational quantities are

\[ V = \frac{1}{8\pi G} \left[ \frac{3}{(dr/dz)^2} + (1 + z) \frac{d^2r/dz^2}{(dr/dz)^3} \right] - \frac{3\Omega_M H_0^2}{16\pi G}(1 + z)^3, \]

(15)

\[ \left( \frac{d\phi}{dz} \right)^2 - \frac{c^2}{\phi^2}(1 + z)^4 \frac{dr}{dz} \left(1 + z \right)^2 = \frac{dr/dz}{(1 + z)^2} \left[ -\frac{1}{4\pi G} \frac{(1 + z)d^2r/dz^2}{(dr/dz)^3} \right] - \frac{3\Omega_M H_0^2}{8\pi G}(1 + z)^3, \]

(16)

\[ \frac{d\theta}{dz} = -c(1 + z)^2 H^{-1}(z)\phi^{-2} \]

(17)

where \( \dot{\phi} = \frac{d\phi}{dz} = \frac{d}{dz} \frac{\dot{a}}{a} \) has been used in deriving equation (16). For \( \eta = -1 \), (16) gives equations (25) in Ref. [9], and for \( \eta = 0 \) or \( c = 0 \) it gives the real scalar field equation [8]. Interestingly, for either the real, complex or extended complex scalar quintessence model, the reconstruction potential \( V \) is the same one. For the angular-motion contribution being negligible [9] (compared with the evolving-\( \phi \)), there is

\[ \left( \frac{d\phi}{dz} \right)^2 = \frac{dr/dz}{(1 + z)^2} \left[ -\frac{1}{4\pi G} \frac{(1 + z)d^2r/dz^2}{(dr/dz)^3} \right] - \frac{3\Omega_M H_0^2}{8\pi G}(1 + z)^3, \]

(18)

and for the evolving-\( \phi \) contribution being negligible (compared with the angular-motion contribution) there is

\[ \eta c^2 \frac{c^2}{\phi^2} = \frac{1}{(1 + z)^4} \left[ \frac{1}{4\pi G} \frac{(1 + z)d^2r/dz^2}{(dr/dz)^3} + \frac{3\Omega_M H_0^2}{8\pi G}(1 + z)^3 \right]. \]

(19)

Noting that

\[ \dot{\phi} = -\frac{d\phi}{dz} \frac{\dot{a}}{a^2} = -\frac{(1 + z) d\phi}{(dr/dz) dz}, \]

(20)

for the extended complex quintessence we obtain the pressure of the energy density given in terms of \( z \) as

\[ P_\Phi = -\frac{1}{8\pi G} \left[ \frac{3}{(dr/dz)^2} + 2(1 + z) \frac{d^2r/dz^2}{(dr/dz)^3} \right], \]

(21)

\[ \rho_\Phi = \frac{3}{8\pi G} \left[ \frac{1}{(dr/dz)^2} - \Omega_M H_0^2(1 + z)^3 \right], \]

(22)

and the reconstruction equation of the state

\[ w_\Phi = \frac{1}{3} \left( \frac{2(1 + z)(d^2r/dz^2) + 3(dr/dz)}{3\Omega_M H_0^2(1 + z)^3(dr/dz)^3 - (dr/dz)} \right). \]

(23)

This reconstruction equation [23] is the same one as (17) in Ref. [8].
III. CONCLUSIONS

For non-minimally coupling to the curvature, the Lagrangian density is

\[ L_* = \frac{1}{2} \partial^\mu \Phi_\eta^* \partial_\mu \Phi_\eta - V(\Phi_\eta^* \Phi_\eta) - \frac{1}{2} \xi R \Phi_\eta^* \Phi_\eta, \]  

(24)

for \( i_\eta = i \) which gives the one [10]. In this case, the energy density and pressure are

\[ \rho_\Phi = \frac{1}{2}(\dot{\phi}^2 - \eta \phi^2 \theta^2) + V - \frac{3}{2} \xi \phi^2 \phi + \xi(-\frac{3}{2} \phi^2 + \frac{3}{2} H \phi + H^2 \phi^2 - 3V), \]  

(25)

\[ P_\Phi = \frac{1}{2}(\dot{\phi}^2 - \eta \phi^2 \theta^2) - V + \frac{3}{2} \xi \phi^2 \phi - \xi(2 \dot{\phi}^2 + \frac{1}{2} \phi^2 + \frac{3}{2} H \phi - \phi^2 G_1 - 3V), \]  

(26)

which may be reformulated as

\[ \rho_\Phi = E_{\text{eff}} + V_{\text{eff}}, \]  

(27)

\[ P_\Phi = E_{\text{eff}} - V_{\text{eff}}, \]  

(28)

with

\[ E_{\text{eff}} = \frac{1}{2}(\dot{\phi}^2 - \eta \phi^2 \theta^2) + \frac{1}{2} \xi(-2 \dot{\phi}^2 - 2 \phi^2 + 6H^2 \phi^2), \]  

(29)

\[ V_{\text{eff}} = V - \frac{3}{2} \xi \phi^2 \phi + \frac{1}{2} \xi(2 \dot{\phi}^2 - \phi + 3H \phi - 6V). \]  

(30)

Obviously, for the Lagrangian density, there always are the reconstruction equations

\[ V_{\text{eff}} = \frac{1}{8\pi G} \left[ \frac{3}{(dr/dz)^2} + (1 + z)^2 \frac{d^2r/dz^2}{(dr/dz)^3} \right] - \frac{3\Omega_M H_0^2}{16\pi G} (1 + z)^3, \]  

(31)

\[ E_{\text{eff}} = -\left(1 + z\right) \frac{d^2r/dz^2}{4\pi G} \left[ \frac{1}{(dr/dz)^3} - \frac{3\Omega_M H_0^2}{2} (1 + z)^2 \right], \]  

(32)

In fact, the form of the energy density and pressure (27) and (28) are general and may applied to any model interpreted as dark energy, so does the reconstruction equations for \( E_{\text{eff}} \) and \( V_{\text{eff}} \).

In different model \( E_{\text{eff}} \) and \( V_{\text{eff}} \) take different form means different understanding of the origin of the dark energy. We propose the extended complex quintessence model, in which a real parameter is introduced. For \( \eta = -1 \), it gives the complex quintessence case.
As discussed above, term $\frac{1}{2} \eta \dot{\phi}^2 - \dot{\theta}^2$ appearing in the energy density and the pressure can be treated as an effective kinetic energy. However, for $\eta = 1$ it contributes a negative kinetic energy. Here, we would like to consider the Lagrangian density $L_\Phi$ as a sum of the two parts: $L_\Phi = L_\phi + L_{a\phi}$ with $L_\phi = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi)$ and $L_{a\phi} = -\frac{1}{2} \eta c^2 a^{-6} \dot{\phi}^2$. The first term is equivalent to the Lagrangian density of a real scalar field and the second term may be explained as a Lagrangian density from the coupling of $\phi$ to $a$, noting $\dot{\theta} = ca^{-3} \dot{\phi}^2$.

The Lagrangian density $L_{a\phi}$ contributes the equivalent pressure and energy density in the Universe $P_c = \rho_c = -\frac{1}{2} \eta c^2 a^{-6} \dot{\phi}^2$ and may be reconstructed as

$$L_{a\phi} = -\frac{1}{8 \pi G} \frac{(1 + z) d^2 r / d z^2}{(d r / d z)^3} - \frac{3 \Omega_M H_0^2 (1 + z)^3}{16 \pi G} - (1 + z)^6 \left( \frac{d \phi}{d r} \right)^2. \quad (33)$$

Compared to the previous scalar quintessence model, the extended complex quintessence model has the following characteristic. The so-called slowly-evolving scalar field in the complex or real quintessence model implies only $\dot{\phi}^2 - \dot{\phi}^2 \dot{\theta}^2$ is slowly evolving, here, so, the field $\phi$ could be possibly moving with the kinetic energy over the potential energy, even it may be fastly evolving since $\dot{\phi}^2 \dot{\theta}^2$ is positive term and hasn’t been determined. Another point is, as discussed above, the extended complex quintessence model may applied to the future observational data in a wider range. In addition, for $\eta = 1$ case, there is the possibility of the equation of the state with $w_\Phi < -1$ if the condition $\dot{\phi}^2 - \dot{\phi}^2 \dot{\theta}^2 < 0$ is satisfied. Although the observation data at the present implies the equation of state for dark energy component $w_\Phi = P_\Phi / \rho_\Phi \leq -1$, it must not be the constraint condition on the future Universe since we cannot specify how the Universe has evolved or will evolve.

It is also an important and interesting fact that the reconstruction equation of the potential $V$ doesn’t depend on the scalar quintessence models (real, complex, or extended complex), namely, it is independent of the parameter $\eta$. This feature of the scalar quintessence means that our generalization to the work should be natural.

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This upper limit of $w_\Phi$ is determined by following considerations: providing that for the matter and dark energy components there are the effective equations of state $P_M = w_M \rho_M$ and $P_\Phi = w_\Phi \rho_\Phi$; from (8) to be shown, for the Universe having an accelerated expansion there is $3P < -\rho$, which yields $w_M + 2w_\Phi < -1$, considering that $w_M$ is nonnegative we have $w_\Phi < -\frac{1}{2}$.

Considering the following transformation $\Phi_\eta \rightarrow \Phi'_\eta = e^{i\alpha} \Phi_\eta$ with $\alpha$ being a real constant and $\Phi'_\eta = \Phi_\eta^* e^{-i\alpha}$, then we have $\Phi'_\eta^* \Phi'_\eta = \Phi_\eta^* e^{-i\alpha} e^{i\alpha} = \phi^2$ and $\partial^\mu \Phi'_\eta \partial_\mu \Phi'_\eta = \partial^\mu \Phi_\eta^* \partial_\mu \Phi_\eta$ and therefore $L_\Phi = L_{\Phi'}$. This is called the $U(1; i_\eta)$ symmetry, which reduces to the $U(1)$ symmetry when $i_\eta = i$. 

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