DISCRETE MEAN SQUARE OF THE RIEMANN ZETA-FUNCTION OVER IMAGINARY PARTS OF ITS ZEROS

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Abstract. Assume the Riemann hypothesis. On the right-hand side of the critical strip, we obtain an asymptotic formula for the discrete mean square of the Riemann zeta-function over imaginary parts of its zeros.

1. Introduction

Let $s = \sigma + it$ be a complex variable. In this paper, $T$ always tends to plus infinity.

Let $N(T)$ denote the number of zeros of the Riemann zeta-function $\zeta(s)$ in the region $0 \leq \sigma \leq 1, 0 < t \leq T$. The Riemann-von Mangoldt formula states (Titchmarsh [23, Theorem 9.4]) that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Let $\rho = \beta + i\gamma$ denote a non-real zero of $\zeta(s)$. The Riemann hypothesis (RH) states that $\beta = 1/2$ for all non-real zeros of the Riemann zeta-function. We prove the following two theorems.

Theorem 1. Assume RH. Let $\sigma > 1/2$. Let $A > 0$ be as large as we like. Then there is $\eta = \eta(A) > 0$ such that, for $|t| \leq T^A$,

$$\sum_{0 < \gamma \leq T} |\zeta(s + i\gamma)|^2 = \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re \left( \frac{\zeta'}{\zeta}(s + 1/2) \right) \frac{T}{\pi} + O_{\sigma,A}(T^{1-\eta}).$$

Theorem 2. Assume RH. Let $\sigma > 1/2$. Then, for any $\varepsilon > 0$,

$$\sum_{0 < \gamma \leq T} |\zeta(s + i\gamma)|^2 \ll_{\sigma,\varepsilon} T \log T + |t|^{\varepsilon},$$

uniformly in $t$.

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In Garunkštis and Laurinčikas [14], we use Theorem 2 in order to study the discrete universality of the Riemann zeta-function over the imaginary parts of its zeros. Informally speaking, this means that a wide class of analytic functions can be approximated by shifts $\zeta(s + i\gamma)$. We note that the discrete universality was proposed by Reich [20]. It was developed by Bagchi [1], Sander and Steuding [21]. These authors investigated the approximation of analytic functions by shifts $\zeta(s + i\tau)$, where $\tau$ takes values from arithmetic progression $\{kh : k = 0, 1, 2, \ldots\}$ with $h > 0$ fixed. Instead of arithmetic progressions, Dubickas and Laurinčikas [2] considered the set $\{k^\alpha h : k = 0, 1, 2, \ldots\}$ with $0 < \alpha < 1$ fixed.

The related discrete mean square was considered by Gonek [15]. He proved, assuming RH, that for real $\alpha$, $|\alpha| \leq (1/4\pi) \log(T/2\pi)$,

$$
\sum_{0 < \gamma \leq T} \left| \zeta \left( \frac{1}{2} + i \left( \gamma + \frac{2\pi\alpha}{\log(T/2\pi)} \right) \right) \right|^2 = \left( 1 - \left( \frac{\sin(\pi\alpha)}{\pi\alpha} \right)^2 \right) \frac{T}{2\pi} \log^2 T + O(T \log T).
$$

The error term in the last formula was improved (on RH) by Fujii [11], see also Conrey and Snaith [5, Section 7.3], where this formula is investigated using the ratio conjecture. Ivić [10] obtained that

$$
\sum_{0 < \gamma \leq T} \left| \zeta \left( \frac{1}{2} + i\gamma \right) \right|^2 \ll T \log^2 T \log \log T
$$

unconditionally.

In the next section, we prove Theorems 1 and 2. In Section 3, we discuss several discrete mean square results for Dirichlet $L$-functions.

2. Proofs

In the proofs of Theorems 1 and 2, we will use the approximation of $\zeta(s)$ by a finite sum and the uniform version of Landau’s formula (see Lemmas 3 and 4 below).

**Lemma 3.** Assume RH. Let $\sigma > 1/2$ and $t > 0$. Then, for any given positive number $\delta$, there is $\lambda = \lambda(\delta, \sigma) > 0$ such that

$$
\zeta(s) = \sum_{n \leq t^\delta} \frac{1}{n^s} + r(s),
$$

where $r(s) \ll t^{-\lambda}$.

**Proof.** The lemma follows from Tichmarsh [23, Theorem 13.3]. □
Lemma 4. Assume RH. Let \( x, T > 1 \). Then

\[
\sum_{0 < \gamma \leq T} x^{i\gamma} = -\frac{T}{2\pi} \frac{\Lambda(x)}{\sqrt{x}} + O \left( \sqrt{x} \log(2xT) \log \log(3x) \right)
+ O \left( \frac{\log x}{\sqrt{x}} \min \left( T, \frac{x}{\langle x \rangle} \right) \right) + O \left( \frac{\log(2T)}{\sqrt{x}} \min \left( T, \frac{1}{\log x} \right) \right),
\]

where \( \langle x \rangle \) denotes the distance from \( x \) to the nearest prime power other than \( x \) itself.

Proof. Under RH, the lemma follows immediately from Gonek [16] and [17]. Note that stronger forms of Landau’s formula are obtained by Fujii [8], [9] (under RH), also by Ford and Zaharescu [6]. \( \square \)

The following lemma will be useful.

Lemma 5. Let \( 0 < \lambda < 1/2 \). Let \( t > 0 \) or \( -t > T \). Then

\[
\sum_{0 < \gamma \leq T} (\gamma + t)^{-2\lambda} \ll_{\lambda} T|T + t|^{-2\lambda} \log T \ll_{\lambda} T|T + t|^{-\lambda}
\]

uniformly in \( t \).

Proof. By partial summation and by the Riemann-von Mangoldt formula (1), we have

\[
\sum_{0 < \gamma \leq T} (\gamma + t)^{-2\lambda} = \frac{N(T)}{(T + t)^{2\lambda}} + 2\lambda \int_{0}^{T} \frac{N(x)dx}{(x + t)^{2\lambda + 1}}
\ll \frac{T \log T}{(T + t)^{2\lambda}} + \log T \int_{1}^{T} \frac{x dx}{(x + t)^{2\lambda + 1}}.
\]

Let \( I = \int_{1}^{T} x(x + t)^{-2\lambda - 1} dx \). Then

\[
(2) \quad I = \frac{1}{1 - 2\lambda} \left( (T + t)^{1-2\lambda} - (1 + t)^{1-2\lambda} \right) + \frac{t}{2\lambda} \left( (T + t)^{-2\lambda} - (1 + t)^{-2\lambda} \right).
\]

If \( 0 \leq t \leq T \) then

\[
I \ll (T + t)^{1-2\lambda} + t(1 + t)^{-2\lambda} \ll T(T + t)^{-2\lambda}.
\]
If \(|t| > T\) then, using the formula (2) and Taylor series, we get

\[
I = \frac{t^{1-2\lambda}}{1-2\lambda} \left( \left(1 + \frac{T}{t}\right)^{1-2\lambda} - \left(1 + \frac{1}{t}\right)^{1-2\lambda} \right) + \frac{t^{1-2\lambda}}{2\lambda} \left( \left(1 + \frac{T}{t}\right)^{-2\lambda} - \left(1 + \frac{1}{t}\right)^{-2\lambda} \right)
\]

\[
= \frac{t^{1-2\lambda}}{1-2\lambda} \left( (1-2\lambda) \frac{T}{t} - (1-2\lambda) \frac{1}{t} + O \left( \frac{T^2}{t^2} \right) \right) + \frac{t^{1-2\lambda}}{2\lambda} \left( -2\lambda \frac{T}{t} + 2\lambda \frac{1}{t} + O \left( \frac{T^2}{t^2} \right) \right)
\]

\[
= T^2|t|^{-1-2\lambda} \ll T|t|^{-2\lambda} \ll T(T + t)^{-2\lambda}.
\]

Lemma 5 is proved. □

To prove Theorems 1 and 2, we consider two cases: \(t \geq 0\) and \(t < 0\). These cases correspond to Propositions 6 and 7 below.

**Proposition 6.** Assume RH. Let \(\sigma > 1/2\), \(t \geq 0\), and \(0 < \delta < 1\). Then there is a positive number \(\lambda = \lambda(\delta, \sigma)\) such that

\[
\sum_{0<\gamma \leq T} \left| \zeta(s + \gamma) \right|^2 = \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re \left( \frac{\zeta'(s + 1/2)}{\zeta(s + 1/2)} \right) \frac{T}{\pi} + O \left( T(T + t)^{-\delta(\sigma - 1/2)} + T(T + t)^{-\lambda} + (T + t)^{2\delta} + T^{1/2}(T + t)^{\delta - \lambda} \right).
\]

**Proof.** In view of Lemma 3, we have

\[
\sum_{0<\gamma \leq T} \left| \zeta(s + i\gamma + it) \right|^2 = \sum_{\gamma \leq T} \sum_{n,m \leq (\gamma + t)^\delta} \frac{(m/n)^{i\gamma + it}}{(mn)^\sigma}
\]

\[
+ \sum_{0<\gamma \leq T} \left( |r(s + i\gamma + it)|^2 + 2\Re \sum_{n \leq (\gamma + t)^\delta} \frac{r(s + i\gamma + it)}{n^{s+i\gamma+it}} \right)
\]

\[=: A + R.\]

By Lemmas 3 and 5 and by the Cauchy-Schwarz inequality, there exists a number \(0 < \lambda < 1/2\) such that

\[
R \ll \sum_{0<\gamma \leq T} |r(s + i\gamma + it)|^2 + \left( A \sum_{0<\gamma \leq T} |r(s + i\gamma + it)|^2 \right)^{1/2}
\]

\[\ll T(T + t)^{-\lambda} + A^{1/2} T^{1/2}(T + t)^{-\lambda} \log^{1/2} T.\]
We rewrite the term $A$ in the following way

$$
A = \sum_{\gamma \leq T} \sum_{n \leq (\gamma + t)^\sigma} \frac{1}{n^{2\sigma}} + 2\Re \sum_{\gamma \leq T} \sum_{n < m \leq (\gamma + t)^\delta} \frac{(m/n)^{i\gamma + it}}{(mn)^\sigma} \\
=: A_1 + 2\Re A_2.
$$

By the Riemann-von Mangoldt formula (1), we see that

$$
A_1 = \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + O \left( T(T + t)^{-\delta(2\sigma-1)} \log(T + t) + \log(T) \right). 
$$

The sum $A_2$ requires longer consideration. Changing the order of the summation, we obtain

$$
A_2 = \sum_{\gamma \leq T} \sum_{n < m \leq (\gamma + t)^\delta} \frac{(m/n)^{i\gamma + it}}{(mn)^\sigma} \\
= \sum_{n < m \leq (T + t)^\delta} \sum_{\max(0,m^{1/\delta}-t) < \gamma \leq T} \frac{(m/n)^{i\gamma + it}}{(mn)^\sigma}.
$$

On the right-hand side of the last equality, we will use Lemma 4 for the inner sum. We have

$$
\left\langle \frac{m}{n} \right\rangle \geq \min_{d \in \mathbb{N} \setminus m/n} \left| \frac{m}{n} - d \right| = \min_{d \in \mathbb{N} \setminus m/n} \frac{1}{n} |m - dn| \geq \frac{1}{n}
$$

and

$$
\log \frac{m}{n} = \log \left( 1 + \frac{m - n}{n} \right) \gg \frac{1}{n}.
$$

Then by Lemma 4 for $n < m \leq (T + t)^\delta$, we get

$$
\sum_{\max(0,m^{1/\delta}-t) < \gamma \leq T} (m/n)^{i\gamma} = - \frac{T - \max(0,m^{1/\delta}-t) \Lambda(m/n)}{2\pi} \sqrt{m/n} \\
+ O \left( (T + t)^\delta \log(T + t) \right).
$$
Further,

\[
\sum_{n<m \leq (T+t)^\delta} \frac{(m/n)^it \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} = \sum_{n<p^kn \leq (T+t)^\delta} \frac{p^{kit} \Lambda(p^k)}{(p^k n^2)^\sigma \sqrt{p^k}}
\]

\[
= \sum_{n \leq (T+t)^\delta} \frac{1}{n^{2\sigma}} \sum_{j \leq (T+t)^\delta/n} \frac{j^{it} \Lambda(j)}{j^{\sigma+1/2}}
\]

\[
= -\sum_{n \leq (T+t)^\delta} \frac{1}{n^{2\sigma}} \left(\frac{\zeta'}{\zeta}(\sigma-it+1/2) + \sum_{j>(T+t)^\delta/n} \frac{j^{it} \Lambda(j)}{j^{\sigma+1/2}}\right)
\]

In view of

\[
\sum_{n \leq (T+t)^\delta} \frac{1}{n^{\sigma+1/2}} \ll 1 \quad \text{and} \quad \sum_{n>(T+t)^\delta} \frac{1}{n^{2\sigma}} \ll (T+t)^{\delta(1-2\sigma)},
\]

we get

\[
\sum_{n<m \leq (T+t)^\delta} \frac{(m/n)^it \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}}
\]

\[
= -\zeta(2\sigma) \frac{\zeta'}{\zeta}(\sigma-it+1/2) + O((T+t)^{-\delta(\sigma-1/2)}).
\]

We continue to consider the right-hand side of the formula (8). Reasoning similarly as in (9), we obtain

\[
S := \left| \sum_{n<m \leq (T+t)^\delta} \frac{(m/n)^it \max(0, m^{1/\delta} - t) \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} \right|
\]

\[
= \left| \sum_{n<p^kn \leq (T+t)^\delta} \frac{p^{kit} \max(0, (p^kn)^{1/\delta} - t) \Lambda(p^k)}{(p^k n^2)^\sigma \sqrt{p^k}} \right|
\]

\[
= \left| \sum_{n \leq (T+t)^\delta} \frac{1}{n^{2\sigma}} \sum_{t^{\delta}/n < j \leq (T+t)^\delta/n} \frac{j^{it}((jn)^{1/\delta} - t) \Lambda(j)}{j^{\sigma+1/2}} \right|
\]

\[
\leq \sum_{n \leq (T+t)^\delta} n^{1/\delta-2\sigma} \sum_{t^{\delta}/n < j \leq (T+t)^\delta/n} j^{1/\delta-\sigma-1/2} \Lambda(j).
\]
In light of

\[
\sum_{t^\delta/n<j<(T+t)^\delta/n} j^{1/\delta-\sigma-1/2} \Lambda(j) \ll \frac{(T + t)^{1-\delta(\sigma-1/2)} - t^{1-\delta(\sigma-1/2)}}{n^{1/\delta-\sigma+1/2}} < \frac{T(T + t)^{-\delta(\sigma-1/2)}}{n^{1/\delta-\sigma+1/2}},
\]

we see that

\[(12) \quad S \ll T(T + t)^{-\delta(\sigma-1/2)}.\]

Hence, putting together formulas (7)–(12), we get

\[
2 \Re A_2 = \zeta(2\sigma) \Re \left( \frac{\zeta'}{\zeta}(s + 1/2) \right) \frac{T}{\pi} + O \left( T(T + t)^{-\delta(\sigma-1/2)} \right)
+ O \left( ((T + t)^{2\delta(1-\sigma)} + 1)(T + t)^{\delta}\log(T + t) \right).
\]

By the last formula together with formulas (5) and (6), we have

\[
(13) \quad A = \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re \left( \frac{\zeta'}{\zeta}(s + 1/2) \right) \frac{T}{\pi}
+ O \left( T(T + t)^{-\delta(\sigma-1/2)} + (T + t)^{2\delta} \right).
\]

Then by (11), we see that \( R \ll T(T + t)^{-\lambda/2} + T^{1/2}(T + t)^{\delta-\lambda/2} \). From this and formulas (3), (13), replacing \( \lambda/2 \) by \( \lambda \), we obtain Proposition 6.

\[\square\]

**Proposition 7.** Assume RH. Let \( \sigma > 1/2 \) and let

\[
\sum_{\gamma \leq T} |\zeta(s + i\gamma)|^2 = \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re \left( \frac{\zeta'}{\zeta}(s + 1/2) \right) \frac{T}{\pi} + E.
\]

If 0 < \(-t\) ≤ 2T, then there is a positive number \( \eta = \eta(\sigma) \) such that

\[E \ll T^{1-\eta}.\]

Moreover, let 0 < \( \delta \leq 1/2 \). If \(-t > 2T > 0\), then there is a positive number \( \lambda = \lambda(\delta, \sigma) \) such that

\[E \ll T(-t)^{-\delta(\sigma-1/2)} + T(-t)^{-\lambda} + (-t)^{2\delta} + T^{1/2}(-t)^{\delta-\lambda}.\]

**Proof.** This proof differs from the proof of Proposition 6 in the way that now the number \( \gamma + t \) could be negative and its absolute value could be small.

Let \( \varepsilon > 0 \). First we consider the case 0 < \(-t\) ≤ \( T + T^\varepsilon \). If \( |\gamma + t| \) is small, then, using the bound \( \zeta(s) \ll t^{\varepsilon/3} \) (Titchmarsh [23, formula (14.2.5)]) and the
Riemann-von Mangoldt formula \(1\), we have
\[
\sum_{0 < \gamma \leq T, \gamma + t \leq T^\epsilon} |\zeta(s + i\gamma)|^2 \ll T^{2\epsilon}.
\]

Moreover, we separate the terms with negative \(\gamma + t\) into a different sum. Then
\[
\sum_{0 < \gamma \leq T} |\zeta(s + i\gamma + it)|^2 = \sum_{0 < \gamma \leq -t - T^\epsilon} |\zeta(s + i\gamma + it)|^2 + \sum_{T^\epsilon - t < \gamma \leq T} |\zeta(s + i\gamma + it)|^2 + O(T^{2\epsilon}).
\]

Here and later we assume that the empty sum is equal to zero.

We consider the first sum on the right-hand side of the last equality. Lemma 5 gives that
\[
\sum_{0 < \gamma \leq -t - T^\epsilon} (\gamma + t)^{-2\lambda} \ll T^{1-2\epsilon\lambda} \log T \ll T^{1-\epsilon\lambda}.
\]

Then by Lemma 3, reasoning similarly as in formulas \(3\)–\(7\), we have that there exists a number \(0 < \lambda < 1/2\) such that
\[
\sum_{0 < \gamma \leq -t - T^\epsilon} |\zeta(s + i\gamma + it)|^2 =: \sum_{\gamma \leq -t - T^\epsilon} \sum_{n \leq (-t - \gamma)^\delta} \frac{1}{n^{2\sigma}}
\]
\[
+ 2\Re \left( \sum_{n < m \leq (-t)^\delta} \sum_{\gamma \leq \min(-T^\epsilon, -m^{1/\delta}-t)} \frac{\langle m/n \rangle^{i\gamma+it}}{(mn)^\sigma} \right) + R =: A + R,
\]

with
\[
R \ll A^{1/2} T^{1/2-\epsilon\lambda} \log^{1/2} T + T^{1-\epsilon\lambda}.
\]

By the Riemann-von Mangoldt formula \(1\) and partial summation
\[
A = \zeta(2\sigma) \frac{-t}{2\pi} \log \frac{-t}{2\pi e} + 2\Re \left( \sum_{n < m \leq (-t)^\delta} \sum_{\gamma \leq \min(-T^\epsilon, -m^{1/\delta}-t)} \frac{\langle m/n \rangle^{i\gamma+it}}{(mn)^\sigma} \right)
\]
\[
+ O \left( (T^{1-\epsilon(2\sigma-1)} + T^\epsilon) \log T \right).
\]
We consider the double sum of the last formula. Using Lemma 4 for \( n < m \leq (-t)^\delta \), we find

\[
\sum_{\gamma \leq \min(-T^\varepsilon, m^{-1/\delta}) - t} (m/n)^{i\gamma} = -\min(-T^\varepsilon, m^{-1/\delta}) - t \Lambda(m/n) \frac{2\pi}{\sqrt{m/n}} + O \left( T^\delta \log T \right).
\]

Reasoning similarly as in formulas (9) and (10), we get

\[
t \sum_{n < m \leq (-t)^\delta} \frac{1}{n^{2\sigma}} \sum_{j \leq (-t)^{\delta/n}} \frac{j^{i\Lambda(j)}}{j^{\sigma+1/2}} = -t \zeta(2\sigma) \zeta' \left( \frac{\zeta'}{\zeta}(s+1/2) \right) \frac{-t}{\pi} + O \left( T^{1-\delta(\sigma-1/2)} + 1 \right).
\]

Further,

\[
\sum_{n < m \leq (-t)^\delta} \frac{m^{1/\delta} \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} = \sum_{n \leq (-t)^\delta} \sum_{j \leq (-t)^{\delta/n}} j^{1/\delta - \sigma - 1/2} \Lambda(j) \ll (-t)^{1-\delta(\sigma-1/2)} + 1 \ll T^{1-\delta(\sigma-1/2)} + 1.
\]

By the inequality \( \min(-T^\varepsilon, m^{-1/\delta}) \leq T^\varepsilon + m^{1/\delta} \), formulas (19), (20), and (21) we obtain

\[
2\Re \left( \sum_{n < m \leq (-t)^\delta} \sum_{\gamma \leq \min(-T^\varepsilon, m^{-1/\delta}) - t} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}} \right) = \zeta(2\sigma) \Re \left( \frac{\zeta'}{\zeta}(s + 1/2) \right) \frac{-t}{\pi} + O \left( T^{1-\delta(\sigma-1/2)} + T^{2\delta} + T^\varepsilon \right).
\]

This and the equality (18) yield that

\[
A = \zeta(2\sigma) \frac{-t}{2\pi} \log \frac{-t}{2\pi e} + \zeta(2\sigma) \Re \left( \frac{\zeta'}{\zeta}(s + 1/2) \right) \frac{-t}{\pi} + O \left( T^{1-\delta(\sigma-1/2)} + T^{2\delta} + T^\varepsilon \log T \right).
\]
Then in view of (17) we see that $R \ll T^{1-\epsilon\lambda/2}$. By the last bound together with formulas (16) and (23) we get

$$\sum_{0<\gamma-t-T^\epsilon} |\zeta(\sigma+i\gamma+it)|^2 = \zeta(2\sigma) \frac{-t}{2\pi} \log \frac{-t}{2\pi e} + \zeta(2\sigma) \Re \left( \frac{\zeta'}{\zeta(s+1/2)} \right) \frac{-t}{\pi}$$

(24)$$+ O \left( T^{1-\epsilon\lambda/2} + T^{1-\epsilon\delta(\sigma-1/2)} + T^{2\delta} + T^{\epsilon} \log T \right),$$

where $0 < -t \leq T + T^\epsilon$.

We turn to the next sum on the right-hand side of the formula (14). That is, we consider the sum

$$\sum_{T^\epsilon-t<\gamma\leq T} |\zeta(\sigma+i\gamma+it)|^2.$$

(25)

Note that the last sum is empty if $T - T^\epsilon < -t \leq T + T^\epsilon$. We therefore assume that

$$0 < -t \leq T - T^\epsilon$$

(26)

in the formula (25). By the Riemann-von Mangoldt formula (1),

$$\sum_{T^\epsilon-t<\gamma\leq T} (\gamma + t)^{-2\lambda} \ll \lambda T^{1-2\epsilon\lambda} \log T \ll \lambda T^{1-\epsilon\lambda}.$$

Then

$$\sum_{T^\epsilon-t<\gamma\leq T} |\zeta(\sigma+i\gamma+it)|^2 =: \sum_{T^\epsilon-t<\gamma\leq T} \sum_{n \leq (t+\gamma)^\delta} \frac{1}{n^{2\sigma}}$$

$$+ 2\Re \left( \sum_{n \leq (T+t)^\delta} \sum_{\max(T^\epsilon,m^{1/\delta})-t<\gamma\leq T} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} \right) + R'$$

(27)

$$=: A' + R',$$

where

$$R' \ll A'^{1/2} T^{1/2-\epsilon\lambda} \log^{1/2} T + T^{1-\epsilon\lambda}.$$

Further,

$$A' = \zeta(2\sigma) \left( \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{t}{2\pi} \log \frac{-t}{2\pi e} \right)$$

$$+ 2\Re \left( \sum_{n \leq (T+t)^\delta} \sum_{\max(T^\epsilon,m^{1/\delta})-t<\gamma\leq T} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} \right)$$

$$+ O \left( (T^{1-\epsilon\delta(2\sigma-1)} + T^{\epsilon}) \log T \right).$$

(28)
In view of inequalities (26) we have $T + t \geq T^\varepsilon$. We split the double sum in the equation (28) in the following way.

\begin{equation}
\sum_{n < m \leq (T + t)^\delta} \sum_{\max(T^\varepsilon, m^{1/\delta}) - t < \gamma \leq T} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}} = \sum_{n < m \leq T^\varepsilon} \sum_{T^\varepsilon - t < \gamma \leq T} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}} + \sum_{T^\varepsilon < m \leq (T + t)^\delta} \sum_{n < m^{1/\delta} - t < \gamma \leq T} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}} =: C' + D'
\end{equation}

For $n < m \leq (T + t)^\delta$, Lemma 4 yields

\begin{equation}
\sum_{\max(T^\varepsilon, m^{1/\delta}) - t < \gamma \leq T} \frac{(m/n)^{i\gamma}}{(m/n)^{\sigma}} = \frac{T + t - \max(T^\varepsilon, m^{1/\delta})}{2\pi} \Lambda(m/n) \frac{\Lambda(m/n)}{\sqrt{m/n}} + O(T^\delta \log T).
\end{equation}

Reasoning similarly as in formulas (9) and (10), we get

\begin{equation}
C' = -\frac{T + t - T^\varepsilon}{2\pi} \sum_{n < m \leq T^\varepsilon} \frac{(m/n)^{it} \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + O(T^{2\delta})
= \frac{T + t}{2\pi} \zeta(2\sigma) \frac{\zeta'(\sigma - 1/2)}{\zeta(\sigma)} \Lambda(m/n) \frac{\Lambda(m/n)}{\sqrt{m/n}} + O(T^{2\delta}) + T^\varepsilon.
\end{equation}

We turn to the term $D'$ defined by the formula (29). In view of (30) we obtain

\begin{equation}
D' = -\sum_{T^\varepsilon < m \leq (T + t)^\delta} \sum_{n < m} \frac{T + t - m^{1/\delta}}{2\pi} \frac{(m/n)^{it} \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + O(T^{2\delta}).
\end{equation}

In formula (32) we have that $T + t - m^{1/\delta} < T$. Thus

\begin{equation}
D' \ll \sum_{T^\varepsilon < m \leq (T + t)^\delta} \sum_{n < m} \frac{T \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + O(T^{2\delta})
= \sum_{n < (T + t)^\delta} \sum_{\max(n, T^\varepsilon) < m \leq (T + t)^\delta} \frac{T \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + O(T^{2\delta})
= \sum_{n \leq T^\varepsilon} \sum_{T^\varepsilon < m \leq (T + t)^\delta} \frac{T \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + \sum_{T^\varepsilon < n < (T + t)^\delta} \sum_{n < m \leq (T + t)^\delta} \frac{T \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + O(T^{2\delta})
=: D'_1 + D'_2 + O(T^{2\delta}).
\end{equation}
Reasoning as in formulas (9) and (10), we get

\[ D'_1, D'_2 \ll T^{1-\varepsilon \delta (\sigma - 1/2)}. \]

Formulas (33) and (34) yield the bound

\[ D' \ll T^{1-\varepsilon \delta (\sigma - 1/2)} + T^{2\delta}. \]

Formulas (29), (31), and (35) give

\[ 2\Re \left( \sum_{n<m \leq (T+t)^{6 \max(T^{\varepsilon}, m^{1/8})}} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}} \right) = \zeta(2\sigma) \Re \left( \frac{\zeta'(s + 1/2)}{\zeta(s + 1/2)} \right) \frac{T + t}{\pi} + O \left( T^{1-\varepsilon \delta (\sigma - 1/2)} + T^{2\delta} + T^\varepsilon \right). \]

Note that the error terms in formulas (27), (28), and (36) are the same as in corresponding formulas (16), (18), and (22). Therefore, similarly to the derivation of the formula (24), using (27), (28), and (36), we get

\[ \sum_{|\gamma| \leq T} |\zeta(\sigma + i\gamma + it)|^2 = \zeta(2\sigma) \left( \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{t}{2\pi} \log \frac{-t}{2\pi e} \right) + \zeta(2\sigma) \Re \left( \frac{\zeta'(s + 1/2)}{\zeta(s + 1/2)} \right) \frac{T + t}{\pi} + O \left( T^{1-\varepsilon \lambda/2} + T^{1-\varepsilon \delta (\sigma - 1/2)} + T^{2\delta} + T^\varepsilon \log T \right), \]

where \(0 < -t \leq T - T^\varepsilon\). Note again that the sum (37) is empty for \(T - T^\varepsilon < -t \leq T + T^\varepsilon\).

Next we consider the case \(-t > T + T^\varepsilon\). Lemma 5 gives that

\[ \sum_{0 < \gamma \leq T} (\gamma + t)^{-2\lambda} \ll \lambda \begin{cases} T^{1-\varepsilon \lambda} \log T & \text{if } T + T^\varepsilon < -t \leq 2T, \\ T(-t)^{-2\lambda} \log T & \text{if } -t > 2T. \end{cases} \]

Therefore

\[ \sum_{0 < \gamma \leq T} |\zeta(\sigma + i\gamma + it)|^2 =: \sum_{\gamma \leq T} \sum_{n \leq (-t-\gamma)^{i\lambda}} \frac{1}{n^{2\sigma}} + 2\Re \left( \sum_{n<m \leq (-t)^{6 \max(T^{\varepsilon}, m^{1/8})}} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}} \right) + R'' =: A'' + R'.'. \]
where

\[
R'' \ll \begin{cases}
A^{n/2}T^{1/2} - \varepsilon\log T + T^{1-\varepsilon} & \text{if } T + T^{\varepsilon} < -t \leq 2T, \\
A^{n/2}T^{1/2}(-t)^{-\lambda} \log T + T(-t)^{-\lambda} & \text{if } -t > 2T.
\end{cases}
\]

Further

\[
A'' = \zeta(2\sigma)\frac{T}{2\pi} \log \frac{T}{2\pi e} + 2\Re \sum_{n \leq (-t)^{\delta/2}} \sum_{\gamma \leq \min(T, -t - m^{1/\varepsilon})} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}} + R_{A''},
\]

where

\[
R_{A''} \ll \begin{cases}
T^{-\varepsilon(2\sigma-1)} \log T & \text{if } T + T^{\varepsilon} < -t \leq 2T, \\
(-t)^{-\delta(2\sigma-1)} \log T + T^{\varepsilon} & \text{if } -t > 2T.
\end{cases}
\]

We split the double sum in the equation (40) in the following way.

\[
\sum_{n \leq (-t)^{\delta/2}} \sum_{\gamma \leq \min(T, -t - m^{1/\varepsilon})} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}} = \sum_{n \leq (-t-T)^{\delta}} \sum_{\gamma \leq T} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}} + \sum_{(-t-T)^{\delta} < n \leq (-t)^{\delta}} \sum_{m \leq (-t-n^{1/\varepsilon})} \sum_{\gamma \leq -t - m^{1/\varepsilon}} \frac{(m/n)^{i\gamma + it}}{(mn)^{\sigma}}.
\]

Then

\[
A'' = \zeta(2\sigma)\frac{T}{2\pi} \log \frac{T}{2\pi e} + 2\Re(C'' + D'') + R_{A''}.
\]

Next we consider terms $C''$ and $D''$. By Lemma 4, for $n < m \leq (-t)^{\delta}$, we obtain

\[
\sum_{\gamma \leq \min(T, -t - m^{1/\varepsilon})} (m/n)^{i\gamma} = -\frac{\min(T, -t - m^{1/\varepsilon})}{2\pi} \Lambda(m/n) \sqrt{m/n} + O((-t)^{\delta} \log(-t)).
\]

By the last formula we get

\[
C'' = -\frac{T}{2\pi} \sum_{n \leq (-t-T)^{\delta}} \frac{(m/n)^{it} \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + R_{C''},
\]
where

\[ R_{C''} \ll \begin{cases} T^{2\delta} & \text{if } T + T^e < -t \leq 2T, \\ (-t)^{2\delta} & \text{if } -t > 2T. \end{cases} \]

Following the reasoning used in (9) and (10) we see that

\[
\sum_{n < m \leq (-t-T)^{\delta}} \frac{(m/n)^{it} \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} = -\zeta(2\sigma) \frac{\zeta'}{\zeta}(\sigma + 1/2 - it) + Q,
\]

where

\[ Q \ll \begin{cases} T^{-\varepsilon\delta(\sigma-1/2)} & \text{if } T + T^e < -t \leq 2T, \\ (-t)^{-\delta(\sigma-1/2)} & \text{if } -t > 2T. \end{cases} \]

By this and the expression (45) we get

\[
C'' = \frac{T}{2\pi} \zeta(2\sigma) \frac{\zeta'}{\zeta}(\sigma + 1/2 - it)
\ll \begin{cases} T^{2\delta} + T^{1-\varepsilon\delta(\sigma-1/2)} & \text{if } T + T^e < -t \leq 2T, \\ (-t)^{2\delta} + T(-t)^{-\delta(\sigma-1/2)} & \text{if } -t > 2T. \end{cases}
\]

We turn to the term \( D'' \) defined by the formula (42). In view of (44) we obtain

\[
D'' = -\sum_{(-t-T)^{\delta} < m \leq (-t)^{\delta}} \sum_{n < m} \frac{-t - m^{1/\delta}}{2\pi} \frac{(m/n)^{it} \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} + R_{D''},
\]

where

\[ R_{D''} \ll \begin{cases} T^{2\delta} & \text{if } T + T^e < -t \leq 2T, \\ (-t)^{2\delta} & \text{if } -t > 2T. \end{cases} \]
In formula (47) we have that \(-t - m^{1/\delta} < T\). Thus

\[
D'' \ll \sum_{(-t - T)^{\delta} < m \leq (-t)^{\delta}} \sum_{n < m} \frac{T \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + R_{D''}
\]

\[
= \sum_{n < (-t)^{\delta}} \sum_{\max(n, (-t-T)^{\delta}) < m \leq (-t)^{\delta}} \frac{T \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + R_{D''}
\]

\[
= \sum_{n \leq (-t-T)^{\delta}} \sum_{(-t-T)^{\delta} < m \leq (-t)^{\delta}} \frac{T \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + R_{D''}
\]

\[
+ \sum_{(-t-T)^{\delta} < n < (-t)^{\delta}} \sum_{n < m \leq (-t)^{\delta}} \frac{T \Lambda(m/n)}{(mn)^{\sigma} \sqrt{m/n}} + R_{D''}
\]

\[
=: D''_1 + D''_2 + R_{D''}.
\]

Reasoning as in formulas (9) and (10), we get

\[
D''_1, D''_2 \ll \begin{cases} 
T^{1-\epsilon\delta(\sigma-1/2)} & \text{if } T + T^\epsilon < -t \leq 2T, \\
T(-t)^{-\delta(\sigma-1/2)} & \text{if } -t > 2T.
\end{cases}
\]

Formulas (47), (48), and (49) yield the bound

\[
D'' \ll \begin{cases} 
T^{2\delta} + T^{1-\epsilon\delta(\sigma-1/2)} & \text{if } T + T^\epsilon < -t \leq 2T, \\
(-t)^{2\delta} + T(-t)^{-\delta(\sigma-1/2)} & \text{if } -t > 2T.
\end{cases}
\]

Summarizing the results obtained in (43), (41), (46), and (50) we see that

\[
A'' - \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} - \zeta(2\sigma) \frac{\zeta'}{\zeta}(\sigma + 1/2 - it) \frac{T}{\pi}
\ll \begin{cases} 
T^{2\delta} + T^{1-\epsilon\delta(\sigma-1/2)} & \text{if } T + T^\epsilon < -t \leq 2T, \\
(-t)^{2\delta} + T(-t)^{-\delta(\sigma-1/2)} & \text{if } -t > 2T.
\end{cases}
\]

Then the formula (39) gives that

\[
R'' \ll \begin{cases} 
T^{1-\epsilon\lambda/2} & \text{if } T + T^\epsilon < -t \leq 2T, \\
T(-t)^{-\lambda/2} + T^{1/2}(-t)^{\delta-\lambda/2} & \text{if } -t > 2T.
\end{cases}
\]
In view of formulas (38), (51), and (52) we have

\[ \sum_{0 < \gamma \leq T} |\zeta(\sigma + i\gamma + it)|^2 - \zeta(2\sigma)^2 \frac{T}{2\pi} \log \frac{T}{2\pi e} - \zeta(2\sigma)\Re \left( \frac{\zeta'}{\zeta}(s + 1/2) \right) \frac{T}{\pi} \]

\[ \ll \begin{cases} T^{2\delta} + T^1 - \delta(\sigma-1/2) + T^{1-\varepsilon\lambda/2} & \text{if } T + T^\varepsilon < -t \leq 2T, \\ (-t)^{2\delta} + T(-t)^{-\delta(\sigma-1/2)} + T(-t)^{-\lambda/2} + T^{1/2}(-t)^{\delta-\lambda/2} & \text{if } -t > 2T. \end{cases} \]

From the last formula, replacing \( \lambda/2 \) with \( \lambda \), we obtain Proposition 7 for \( -t > 2T \).

For \( 0 < -t \leq 2T \), Proposition 7 follows by formulas (14), (24), (37), and (53) choosing appropriate constants \( \varepsilon \) and \( \delta \). This finishes the proof.

**Proof of Theorems 1 and 2.** The theorems immediately follow by Propositions 6 and 7.

3. **Concluding remarks**

Laaksonen and Petridis [19] investigated a similar sum to the sum of Theorems 1 and 2. Let \( L(s, \chi_1) \) and \( L(s, \chi_2) \) be the Dirichlet L-functions attached to the primitive Dirichlet characters \( \chi_1 \) and \( \chi_2 \). For some fixed prime \( P \), let

\[ B(s, P) = \prod_{p \leq P} (1 - \chi_1(p)p^{-s})(1 - \chi_2(p)p^{-s}), \]

where the product runs over prime numbers \( p \). Under RH, for fixed \( 1/2 < \sigma < 1 \), they proved that

\[ \sum_{0 < \gamma \leq T} B(\sigma + i\gamma, P)L(\sigma + i\gamma, \chi_1)L(\sigma + i\gamma, \chi_2) \sim N(T) \sum_{n=1}^{\infty} \frac{d_n\chi_2(n)}{n^{2\sigma}}, \]

and

\[ \sum_{0 < \gamma \leq T} B(\sigma + i\gamma, P)L(\sigma + i\gamma, \chi_1)L(\sigma + i\gamma, \chi_2) \sim N(T) \sum_{n=1}^{\infty} \frac{e_n\chi_1(n)}{n^{2\sigma}}, \]

where

\[ B(s, P)L(s, \chi_1) = \sum_{n=1}^{\infty} \frac{d_n}{n^s}, \quad B(s, P)L(s, \chi_2) = \sum_{n=1}^{\infty} \frac{e_n}{n^s}. \]

From this Laaksonen and Petridis [19] derived the result that, under RH, for a positive proportion of non-trivial zeros of \( \zeta(s) \) with \( \gamma > 0 \), the values of the Dirichlet L-functions \( L(\sigma + i\gamma, \chi_1) \) and \( L(\sigma + i\gamma, \chi_2) \) are linearly independent over \( \mathbb{R} \).
The discrete mean value of the Dirichlet $L$-function at nontrivial zeros of another Dirichlet $L$-function were investigated by Garunkštis and Kalpokas [13]. See also Fujii [7,10], Conrey, Ghosh and Gonek [3,4], Steuding [22], and Garunkštis, Kalpokas, and Steuding [12].

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