COHOMOLOGY AT INFINITY
AND THE WELL-ROUNDED RETRACT
FOR GENERAL LINEAR GROUPS

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INTRODUCTION

(0.1). Let $G$ be a reductive algebraic group defined over $\mathbb{Q}$, and let $\Gamma$ be an arithmetic subgroup of $G(\mathbb{Q})$. Let $X$ be the symmetric space for $G(\mathbb{R})$, and assume $X$ is contractible. Then the cohomology (mod torsion) of the space $X/\Gamma$ is the same as the cohomology of $\Gamma$. In turn, $X/\Gamma$ will have the same cohomology as $W/\Gamma$, if $W$ is a “spine” in $X$. This means that $W$ (if it exists) is a deformation retract of $X$ by a $\Gamma$-equivariant deformation retraction, that $W/\Gamma$ is compact, and that $\dim W$ equals the virtual cohomological dimension ($\text{vcd}$) of $\Gamma$. Then $W$ can be given the structure of a cell complex on which $\Gamma$ acts cellularly, and the cohomology of $W/\Gamma$ can be found combinatorially.

Spines have been found for many groups $G$ (see (2.6) below, and [A1] [M-M0] [M-M1]). This paper concerns the case where $G$ is the restriction of scalars of a general linear group over a number field $k$ with ring of integers $\mathcal{O}$. This means $G(\mathbb{Q}) = \text{GL}_n(k)$. We shall take $\Gamma$ to be a subgroup of finite index in $\text{GL}_n(\mathcal{O})$, or more generally in $\text{GL}(\mathcal{P})$, where $\mathcal{P}$ is a projective $\mathcal{O}$-module of rank $n$. In [A3], Ash found a spine $W$ for these $G$, calling $W$ the well-rounded retract. (This generalized [Sou2].) The retract has been used in computations: see [Sou1] [A-G-G] [A-M1] [A-M2] [vG-T]. We remark that $k = \mathbb{Q}$, $\Gamma \subseteq \text{GL}_n(\mathbb{Z})$ for $n = 2, 3, 4$ still provide our main cases of computational interest. For imaginary quadratic fields $k$ and $n = 2$, see [Men] [S-V] [V]. For real quadratic $k$ and $n = 2$, see [B].

(0.2). The present paper extends [A3] to deal with the “cusps” of $X/\Gamma$, in a way which we now describe.

Borel and Serre [B-S] introduced a bordification $\bar{X}$ of $X$ such that $\bar{X}/\Gamma$ is a compactification of $X/\Gamma$. The space $\bar{X}/\Gamma$ has the same homotopy type as $X/\Gamma$, and therefore has the same homotopy type as $W/\Gamma$. The boundary of $\bar{X}/\Gamma$ is a union of finitely many “faces”, one for each equivalence class mod $\Gamma$ of parabolic $\mathbb{Q}$-subgroups of $G$.

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The Borel-Serre boundary gives a geometric approach for distinguishing the cuspidal cohomology within the whole cohomology of \( \Gamma \). A cohomology class on \( \bar{X}/\Gamma \) with coefficients in \( \mathbb{C} \) can be viewed, via the de Rham theorem, as coming from a \( \Gamma \)-invariant automorphic form on \( G(\mathbb{R}) \). If the class has non-zero restriction to the boundary, it is certainly not a cusp form. The converse is not completely true, but the difference between the "interior cohomology" (the span of classes that restrict to zero on the boundary) and the "cuspidal cohomology" (the span of classes whose associated automorphic forms are cuspidal) is fairly well understood, thanks to the theory of Eisenstein series. In many practical examples, like the \( \text{GL}_4(\mathbb{Z}) \) example mentioned below, there is no difference; to compute the cuspidal cohomology in certain degrees, we simply compute the restriction of the cohomology to the boundary.

A related question is the existence of "ghost classes"—those classes which restrict to zero on each face of the boundary, but do not restrict to zero on the whole boundary. Harder has given examples of ghost classes [H]; it is of interest to know how widespread the phenomenon is.

In specific cases, then, one may need to compute the cohomology of the boundary of \( \bar{X}/\Gamma \), the cohomology of individual boundary faces, and the restriction map from \( H^*(\bar{X}/\Gamma) \) to these groups.

This paper builds a technology for these computations, when \( G \) is the general linear group in (0.1) and \( W \) is the well-rounded retract of [A3]. To each Borel-Serre boundary face in \( \bar{X} \), we associate a closed subcomplex \( W' \) of \( W \). Let \( \Gamma' \) be the stabilizer of \( W' \) in \( \Gamma \). We prove that the inclusion \( W'/\Gamma' \to W/\Gamma \) induces the same map on cohomology as the inclusion of the corresponding boundary face into \( \bar{X}/\Gamma \). Our main result, Theorem 9.5, is that there is a spectral sequence converging to \( H^*(\partial \bar{X}/\Gamma) \) whose \( E_1 \) term is the direct sum of the cohomology of the various \( W'/\Gamma' \). There are only finitely many terms in this direct sum. There is a map of spectral sequences, starting with a natural map from \( H^*(W/\Gamma) \) to the \( E_1 \) term, that induces the canonical restriction map \( H^*(\bar{X}/\Gamma) \to H^*(\partial \bar{X}/\Gamma) \) up to canonical isomorphism. There is a dual statement in homology (10.2). We do not assume that \( \Gamma \) is torsion-free. We use (co)homology with coefficients in any fixed abelian group.

We can also find the maps on (co)homology induced by the inclusion of a single boundary face into \( \bar{X}/\Gamma \), without any spectral sequences (10.3).

Each \( W'/\Gamma' \) is a finite cell complex, so its cohomology can be worked out combinatorially. See (6.3) for an example. We emphasize that each \( W' \) is a subcomplex of \( W \), so that we can work with it in terms already available once we have computed \( W \) in any given case.

In a future paper, we will use this method to study the congruence subgroups \( \Gamma_0(p) \) of prime level \( p \) in \( \text{GL}_4(\mathbb{Z}) \). For a range of \( p \), we will compute \( H^5(\Gamma_0(p), \mathbb{C}) \), and will isolate the subspace of classes that restrict to zero on the boundary. We will compute the Hecke eigenvalues of these classes, using either the new techniques of Paul Gunnells, or the new techniques of R. MacPherson and the second author. We will try to determine whether any of the cusp forms we find are functorial lifts from smaller groups such as \( \text{Sp}_4 \) or \( \text{O}_4 \).

(0.3). We have long wished to see spelled out in detail the relationship between the well-rounded retract and the Borel-Serre compactification. The retraction \( r : X \to W \) carries any \( x \in X \) into \( W \) by a combination of geodesic actions depending on \( x \)—see (4.2). Each
Borel-Serre boundary component $e$ in $\bar{X}$ has a tubular neighborhood $\bar{N}$, homeomorphic to $e \times [0,1]^l$ for appropriate $l$, where the fibers $[0,1]^l$ are given by the geodesic action. The retraction $r$ is constant on the interior of each fiber, and carries the interior of $\bar{N}$ onto the corresponding cell complex $W'$.

The main idea of the paper is to extend the retraction $r : X \rightarrow W$ to a continuous $\Gamma$-equivariant map $\bar{r} : \bar{X} \rightarrow \bar{W}$. It is relatively easy to see that the continuous extension exists and is $\Gamma$-equivariant. However, $r = r_1$ really comes from a deformation retraction $r_t : X \times [0,1] \rightarrow X$, and it is harder to prove that $\bar{r}$ comes from a deformation retraction of $\bar{X}$ onto $\bar{W}$. A simple example will illustrate the problem. The map $f_t : \mathbb{R} \times [0,1] \rightarrow \mathbb{R}$ given by $(x,t) \mapsto (1-t) \cdot x$ is a deformation retraction of $\mathbb{R}$ onto $\{0\}$. The compactification $[-\infty, \infty]$ of $\mathbb{R}$ certainly has a deformation retraction onto $\{0\}$. But $f_t$ does not extend to the deformation retraction for $[-\infty, \infty]$.

To fix the problem with $[-\infty, \infty]$, we can first use any deformation retraction onto $[-x_0, x_0]$ for some large $x_0$, and then use the old $f_t$ to retract $[-x_0, x_0]$ to $\{0\}$. We do something similar to prove $\bar{r}$ is a deformation retraction. We use Saper’s recent construction [S] of a “central tile” $X_0 \subset X$ and a deformation retraction $r_{t,S}$ of $\bar{X}$ onto $X_0$. This $X_0$ is a large codimension-zero subset of $X$, which we may take arbitrarily close to $\partial \bar{X}$ in all directions. We show that composing our $r_t$ with Saper’s $r_{t,S}$ gives a deformation retraction $R_t : \bar{X} \times [0,1] \rightarrow \bar{X}$ with $\bar{r} = R_1$.

By what we have said, $\bar{r}$ gives a homotopy equivalence between $\bar{X}/\Gamma$ and $W/\Gamma$. We show it also gives a homotopy equivalence between any Borel-Serre boundary component $\bar{e}/\Gamma'$ and the corresponding cell complex $W'/\Gamma'$. It is then routine to show that $\bar{r}$ gives an isomorphism between a standard spectral sequence for $H^*(\partial \bar{X}/\Gamma)$ and the spectral sequence described in (0.2). This proves our main theorem.

The sections of the paper follow this outline. In Sections 1–4 we summarize basic material on algebraic groups, lattices, the well-rounded retract and retraction, and the Borel-Serre compactification. None of this material is new, but we must recall it to establish our notation, and because we emphasize some less familiar aspects of it—marked lattices, the flag of successive minima, the orthogonal scaling group (which is the geodesic action from a lattice’s point of view), and the connections between $r_t$ and the geodesic action. Section 5 summarizes our notation. In Section 6 we define the sets $W_{\mathcal{F}}$ (called $W'$ in this introduction). In Section 7 we define the tubular neighborhoods $\bar{N}_{\mathcal{F}}$ (called $\bar{N}$ here); this is the technical heart of the paper, where we explore the connections between the well-rounded retraction and the geodesic action. Section 8 summarizes the material from [S] that we need, and contains our results on $R_t$ and $\bar{r}$. In Section 9, we use $\bar{r}$ to prove our main theorem on spectral sequences.

(0.4). The results in this paper could be extended in several ways. First, [A3] defines $W$ for $\text{GL}_n$ of any finite dimensional division algebra over $\mathbb{Q}$, not necessarily commutative. Our results could be generalized to that case, presumably with only technical changes. Second, our arguments work for SL as well as GL, with minor changes (10.5). See (10.4) for comments on $\Gamma$-equivariant cohomology, and (10.6)–(10.7) for other connections with reduction theory.

Parts (1.6), (2.6), and (6.3) present the case $k = \mathbb{Q}$ in more detail. We hope they help
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**SECTION 1—BACKGROUND ON ALGEBRAIC GROUPS**

(1.1). We fix a number field $k$ of degree $d$ over $\mathbb{Q}$, with ring of integers $O$. Let $v$ run through the archimedean places of $k$; the completion $k_v$ is either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{R}_+$ be the positive real numbers.

(1.2). Fix an integer $n \geq 1$. The algebraic group $\text{GL}_n$ is defined over $\mathbb{Q}$, and hence over $k$. Let $G$ be the restriction of scalars of $\text{GL}_n$ from $k$ down to $\mathbb{Q}$.

To see this algebraic group concretely, fix a $\mathbb{Q}$-basis of $k$. Let $k_1$ be any commutative $k$-algebra. A point of $G$ with entries in $k_1$ is an $n \times n$ matrix with non-zero determinant whose $(i,j)$-th entry is a $d$-tuple of coordinates $(x^{(1)}_{ij}, \ldots, x^{(d)}_{ij})$ with $x^{(l)}_{ij} \in k_1$. The $d$-tuple gives an element of $k \otimes_{\mathbb{Q}} k_1$ with respect to the fixed $\mathbb{Q}$-basis of $k$. Addition and multiplication of $d$-tuples are defined using the corresponding operations in $k$. We fix such a realization of $G$ throughout the paper.

(1.3). Let $S = k \otimes_{\mathbb{Q}} \mathbb{R}$. The real and complex embeddings of $k$ induce a canonical isomorphism of $\mathbb{R}$-algebras $S \cong \prod_v k_v$. Clearly $k \hookrightarrow S$ by the canonical diagonal embedding.

Set $G = G(\mathbb{R}) = \text{GL}_n(\mathbb{R})$. Each entry in a matrix of $G$ breaks up as a product over the $v$'s, so that $G \cong \prod_v \text{GL}_n(k_v)$.

(1.4). Let $\mathbb{Q}T$ be the following maximal $\mathbb{Q}$-split torus of $G$: if the fixed basis of $k$ over $\mathbb{Q}$ has 1 as its first element, then $\mathbb{Q}T$ is the subgroup where all the $x^{(m)}_{ij}$ are zero except for $x^{(1)}_{ii}$, $i = 1, \ldots, n$. Write $\mathbb{Q}T = \{\text{diag}(a_1, \ldots, a_n)\}$ with $a_i = x^{(1)}_{ii}$.

The split radical $R_dG$ of $G$ is the subgroup of $\mathbb{Q}T$ in which all the $a_i$ are equal to each other. Let $H$ be the identity component of $R_dG(\mathbb{R})$. These are the positive real homotheties: $h \in H$ acts by a common positive real scalar on each factor $\text{GL}_n(k_v)$ of $G$. We identify $H \cong \mathbb{R}_+^n$.

(1.5). Let $a, b \in S^n$. These break up as a product of vectors $a_v, b_v \in (k_v)^n$. Write $a_v = (a_1, \ldots, a_n)$ with $a_i \in k_v$, and write $b_v$ similarly. For each $v$, let $\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle_v = \sum_{i=1}^n \text{trace}_{k_v}^c(a_i b_i)$, where the bar denotes complex conjugation when $v$ is complex and the identity when $v$ is real. Let $\langle a, b \rangle = \sum_v \langle a_v, b_v \rangle_v$, a positive-definite product on $S^n$. Let $K_v$ be the subgroup of $\text{GL}_n(k_v)$ preserving $\langle , \rangle_v$. Let $K = \prod_v K_v$; this is the largest subgroup of $G$ preserving $\langle , \rangle$, and is a maximal compact subgroup of $G$. Let $\|a\| = \langle a, a \rangle^{1/2}$.

**Definition.** Let $X = (HK) \backslash G$.

In the language of [B-S], $X$ is a homogeneous space for $G$ of type $S-\mathbb{Q}$. It is a symmetric Riemannian space for the Lie group $H \backslash G$. 
Example of $\text{GL}_n(\mathbb{Z})$. Let $k = \mathbb{Q}$. Then $G = \text{GL}_n(\mathbb{R})$, the group of $n \times n$ invertible matrices with real entries. The group $\mathbb{Q}^n$ consists of the diagonal matrices, and $H = \{ \text{diag}(h, \ldots, h) \mid h > 0 \}$. The space $X$ is the symmetric space consisting of the real $n \times n$ symmetric positive-definite matrices mod homotheties. When $n = 2$, $X$ is isomorphic to the upper half-plane (see (2.6)).

Section 2—Marked Lattices. The Well-Rounded Retract

We recall the definition and basic properties of the well-rounded retract for $\text{GL}_n$. Except for the emphasis on marked lattices, the material comes from [A3]; we specialize this to the number field case, and omit the proofs. For the $\text{GL}_n(\mathbb{Z})$ case, see (2.6).

The space $S^n$ of column vectors is an $\mathbb{R}$-vector space. $S$ (and hence $k \subseteq S$) acts on it by coordinate-wise multiplication, and $G = \text{GL}_n(S)$ acts on it on the left. These module structures commute with each other. If $A$ is a subset of $S^n$, the $S$-module it spans in $S^n$ is denoted $S \cdot A$.

A $\mathbb{Z}$-lattice in a vector space $V$ over $\mathbb{Q}$ is a finitely generated subgroup of $V$ which contains a $\mathbb{Q}$-basis of $V$. A $\mathbb{Z}$-lattice in a vector space over $\mathbb{R}$ is a finitely generated subgroup which contains an $\mathbb{R}$-basis that is also a $\mathbb{Z}$-basis.

Fix once and for all a $\mathbb{Z}$-lattice $L_0$ in $k^n$. Let $O_0$ be the subset of $k$ which stabilizes $L_0$. Then $O_0$ is an order in $k$. Let $\Gamma_0$ be the arithmetic group consisting of the $\gamma \in G$ such that $\gamma \cdot L_0 = L_0$. Let $\Gamma$ be an arithmetic subgroup of finite index in $\Gamma_0$.

We shall identify lattices that differ only by a homothety, namely $L$ and $hL$ for $h \in H$. Let $Y$ be the set of all $\mathbb{Z}$-lattices in $S^n$ that are stable under $O_0$ and are isomorphic to $L_0$ as $O_0$-module. Then we may identify $Y$ with $H \backslash G / \Gamma_0$, where for $g \in G$ the lattice $L = gL_0$ corresponds to the coset $g\Gamma_0$. Give $Y$ the topology coming from $H \backslash G / \Gamma_0$.

For any subset $A \subseteq L = gL_0$, the $S$-span $S \cdot A$ is a free $S$-module whose $S$-rank equals the dimension over $k$ of the $k$-vector space spanned by $g^{-1}A$.

The group $K$ acts on $Y$, and in fact

$$K \backslash Y = X / \Gamma_0,$$

since both are $(HK) \backslash G / \Gamma_0$. However, the points of view on the two sides of this equation are different: $K \backslash Y$ is a space of lattices modulo rotations, while $X / \Gamma_0$ is a homogeneous space modulo an arithmetic group. We now make explicit how to go back and forth between the two sides.

Definition. A marked lattice in $S^n$ is a function $f : L_0 \to S^n$ of the form $f(x) = gx$ for $g \in G$. We shall identify marked lattices that differ only by a homothety, i.e. $f$ and $hf$ for $h \in H$.

Let $Y' = H \backslash G$ be the set of marked lattices. A marked lattice $f$ gives rise to an ordinary lattice $L \in Y$ by setting $L = f(L_0)$. This realizes the projection $Y' = H \backslash G \to Y = H \backslash G / \Gamma_0$. On the other hand, $f(x) = gx$ gives a point $(HK)g \in X$, which realizes the projection $Y' = H \backslash G \to X = (HK) \backslash G$. The right action of $g_1 \in G$ on $Y'$ sends $f : x \mapsto gx$. 

to $x \mapsto gg_1x$. The left action of $k_1 \in K$ on $Y'$ sends $f$ to $k_1 \cdot f$. Diagram (2.3.1) shows these spaces.

\begin{equation}
\begin{align*}
Y' &= H\backslash G \\
&\quad \text{(marked lattices)} \\
X &= HK\backslash G \\
&\quad \text{(homogeneous space)} \\
X/\Gamma_0 &= K\backslash Y \\
&\quad \text{(arithmetic quotient)} \\
Y &= H\backslash G/\Gamma_0 \\
&\quad \text{(lattices)}
\end{align*}
\end{equation}

(2.4). Let $f$ be a marked lattice, fixed within its homothety class. Let $L = f(L_0)$. The arithmetic minimum of $f$ is defined to be $m(f) = \min\{\|a\| : a \in L - \{0\}\}$. This number is positive. The set of minimal vectors of $f$ is defined to be $M(f) = \{a \in L : \|a\| = m(L)\}$.

The definitions of $m(f)$ and $M(f)$ use only the image $L$ of $f$, so they descend to functions $m(L)$ and $M(L)$ on $Y$. On the other hand, $m(f)$ and $M(f)$ are $K$-equivariant, since $K$ preserves $\|\ldots\|$.

Unless otherwise specified, we normalize $f$ (resp. $L$) in its homothety class so that $m(f)$ (resp. $m(L)$) equals 1.

(2.5). Definition. A marked lattice $f$ is well-rounded if $M(f)$ spans $S^n$ as $S$-module.

By the $\Gamma_0$-invariance and $K$-equivariance noted in (2.4), we get a definition of well-rounded elements in all the spaces in (2.3.1). For instance, a point in $X$ is well-rounded if and only if, for the class of marked lattices $f$ mod $K$ that represents it, one (and hence each) of the $M(f)$ spans $S^n$ as $S$-module. The definitions in $X/\Gamma_0$ and $Y$ are similar.

Definition. $W$ is the set of well-rounded elements in $X$. We call this the well-rounded retract in $X$. For any arithmetic group $\Gamma \subseteq \Gamma_0$, $W/\Gamma$ is called the well-rounded retract in $X/\Gamma$.

In Section 4, we will explain why these are deformation retracts.

(2.6). Example of $\text{GL}_n(\mathbb{Z})$. As in (1.6), take $k = \mathbb{Q}$ and $G = \text{GL}_n(\mathbb{R})$. If $L_0$ is the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, then $\Gamma_0 = \text{GL}_n(\mathbb{Z})$, and $Y$ is the set of all rank-$n$ lattices in $\mathbb{R}^n$ mod homotheties. The product $\langle , \rangle$ is the standard dot product.

Let’s work out the $\text{GL}_2(\mathbb{Z})$ case in detail. For each marked lattice $f : L_0 \to \mathbb{R}^2$, rotate the image until $f((1,0))$ lies on the positive $x$-axis, and use a homothety to make $\|f((1,0))\| = 1$. This identifies $X$ with the upper half-plane $\mathfrak{h}$ by sending $x = HK \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$ (with $b > 0$) to the point $a + bi \in \mathfrak{h}$. The well-rounded retract $W$ is the tree shown in the figure below. The lower branches break in two infinitely many times as they approach the
The group $G$ is generated by $\text{SL}_2(\mathbb{R})$, $H$, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Under our identification of $X$ with $\mathfrak{h}$, $\text{SL}_2(\mathbb{R})$ acts on $\mathfrak{h}$ by linear fractional transformations as usual, $H$ acts trivially, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts by $a + bi \mapsto -a + bi$. The well-known fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathfrak{h}$ is $F = \{a + bi \in \mathfrak{h} \mid |a| \leq \frac{1}{2}, a^2 + b^2 \geq 1\}$, shown with hatching. A fundamental domain for $\text{GL}_2(\mathbb{Z})$ is the right-hand half of $F$, namely $F^* = \{a + bi \in F \mid a \geq 0\}$, shown with double hatching. The space $X/\Gamma_0$ is homeomorphic to $F^*$. The retract $W/\Gamma_0$ is identified with the bottom arc of $F^*$.

Section 3—The Borel-Serre Compactification

In this section we recall the construction of the Borel-Serre manifold with corners $\bar{X}$, and we summarize facts about flags and parabolic subgroups. The main goal of the section is Proposition 3.6, which relates the geodesic action on $X$ to the idea of rescaling a lattice in directions given by a fixed flag $F$.

(3.1). Recall that $\mathbb{Q}T = \{\text{diag}(a_1, \ldots, a_n)\}$ is a maximal $\mathbb{Q}$-split torus in $G$. The set of $\mathbb{Q}$-roots of $G$ with respect to $\mathbb{Q}T$ is $\{a_i a_j^{-1} \mid 1 \leq i \neq j \leq n\}$. Fix the fundamental system of simple roots $\Delta = \{a_i a_{i+1}^{-1} \mid i = 1, \ldots, n-1\}$.

(3.2). We recall some facts about parabolic $\mathbb{Q}$-subgroups and flags. Let $\{e_1, \ldots, e_n\}$ be the standard $k$-basis of $k^n$. Let $J \subseteq \Delta$. Form a graph with vertices in $\{1, \ldots, n\}$ and with an edge joining vertices $i$ and $i+1$ if and only if $a_i a_{i+1}^{-1} \in J$. The graph has $l = \#(\Delta - J) + 1$ connected components. Say the $j$-th component in order from left to right has $\nu_j$ vertices. Consider the flag $\mathcal{F}_J = \{0 \subseteq V_1 \subseteq \cdots \subseteq V_l = k^n\}$ in which $V_j$ is spanned by $\{e_1, \ldots, e_{\nu_1+\ldots+\nu_j}\}$. The standard parabolic $\mathbb{Q}$-subgroup $P_J$ is the subgroup of $G$ such that $P_J(\mathbb{Q})$ is the stabilizer of $\mathcal{F}_J$ in $G(\mathbb{Q})$. In the coordinates of (1.2), $P_J$ has the usual block upper-triangular form, with diagonal blocks of sizes $\nu_1 \times \nu_1, \ldots, \nu_l \times \nu_l$. Every parabolic $\mathbb{Q}$-subgroup $P$ of $G$ is conjugate by some $g \in G(\mathbb{Q})$ to a unique $P_J$.

Let $P_J = P_J(\mathbb{R})$, and $P = P(\mathbb{R})$ in general.
As a set, \( \bar{X} \) over all \( P \) preserved, \( G \) fiber product corresponding to \( P \). Any flag of the latter form is called a \( Q \)-flag. There is a one-to-one correspondence between the parabolic \( Q \)-subgroups and the \( Q \)-flags they stabilize. We write \( \mathcal{F}' \supseteq \mathcal{F} \) if every \( V_j \) belonging to \( \mathcal{F} \) is also a member of \( \mathcal{F}' \).

For the \( Q \)-flag \( \mathcal{F} = \{ 0 \subseteq V_1 \subseteq \cdots \subseteq V_l = \mathbb{Q}^n \} \), we set \( \# \mathcal{F} = l \). Thus \( \# \mathcal{F} = n \) when \( P \) is a Borel subgroup, \( \# \mathcal{F} = 2 \) for maximal proper \( P \), and \( \# \mathcal{F} = 1 \) when \( P = G \).

\[ \text{(3.3).} \quad \text{Let } qT_J \text{ be the subgroup of } qT \text{ in which the } i \text{-th and } (i + 1) \text{-st diagonal entries are equal whenever } t_it_{i+1}^{-1} \in J. \text{ Let } T_J \text{ be the identity component of } qT_J(\mathbb{R}), \text{ and let } A_J = H \setminus T_J. \]

Let \( P = g^{-1}P_Jg \) for \( g \in G(\mathbb{Q}) \). There is a semi-direct product decomposition \( P = M_P T_P U_P \). Here \( U_P \) is the unipotent radical of \( P \), \( T_P = g^{-1}T_Jg \), and \( M_P T_P \) is the unique Levi subgroup of \( P \) which is stable under the Cartan involution of \( G \) which fixes \( g^{-1}Kg \).

We set \( A_P = H \setminus T_P \).

When \( P \subseteq P' \), \( A_{P'} \) is naturally identified with a subgroup of \( A_P \) by taking the kernel of appropriate roots in \( \Delta \).

\[ \text{(3.4).} \quad \text{We recall the definition of the geodesic action [B-S, §3]. Let } P = g^{-1}P_Jg \text{ for } g \in G(\mathbb{Q}). \text{ Let } Z \text{ be the identity component of the center of } M_PT_P. \text{ Let } y \in X. \text{ There is a } p \in P \text{ with } y = HKgp. \text{ For any } z \in Z, \text{ define } y \circ z = HKgzp. \]

The right side is independent of the choices of \( g \) and \( p \). This action of \( Z \) is the geodesic action on \( X \) associated to \( P \). It is a proper and free action, and it commutes with the ordinary action of \( P \). The group \( A_P \) is contained in \( H \setminus Z \), so it operates on \( X \) by the geodesic action.

\[ \text{(3.5).} \quad \text{We recall the construction of } \bar{X} \text{ [B-S, §§5–7]. The roots in } \Delta - J \text{ determine an isomorphism between } A_J \text{ and } (0, \infty)^{l-1}, \text{ sending the point } \]

\[ \text{diag}(a_1, \ldots, a_1, \ldots, a_l, \ldots, a_l) \quad (a_j > 0, \text{ mod } H) \]

to \( (a_1a_2^{-1}, \ldots, a_{l-1}a_l^{-1}) \in (0, \infty)^{l-1} \). Let \( \bar{A}_J \) be the partial compactification of \( A_J \) corresponding to \( [0, \infty)^{l-1} \) under this isomorphism. The corner \( X(P_J) \) associated to \( P_J \) is the fiber product \( X(P_J) = X \times^{\bar{A}_J} A_J \), where \( A_J \) acts on \( X \) by the geodesic action. Conjugating by an element of \( G(\mathbb{Q}) \), we define \( A_P \) and a corner \( X(P) = X \times^{A_P} A_P \) for any \( P \).

Whenever \( P \subseteq P' \), there is a map \( X(P') \hookrightarrow X(P) \) induced by the inclusion on \( X \); we identify \( X(P') \) with its image under this map. As a set, \( \bar{X} \) is the union of the \( X(P) \) over all \( P \). One puts on \( \bar{X} \) the topology such that the original topology on each corner is preserved, \( \bar{X} \) is Hausdorff, and \( \Gamma \) acts properly on \( \bar{X} \) with a compact quotient.

Let \( e(P) \) be the set corresponding to \( X \times^{A_P} \{ 0 \}^{l-1} \) in the corner \( X(P) = X \times^{A_P} \bar{A}_P \). As a set, \( \bar{X} \) is the disjoint union of \( X \) and the \( e(P) \) for all proper parabolic \( Q \)-subgroups \( P \). We call \( e(P) \) the boundary face corresponding to \( P \).
Let \( \bar{e}(P) \) be the closure of \( e(P) \) in \( X \). We have \( \bar{e}(P) = \bigsqcup_{P' \subseteq P} e(P') \). The sets \( \bar{e}(P_1) \) and \( \bar{e}(P_2) \) are disjoint unless \( P_1 \cap P_2 \) is a parabolic \( \mathbb{Q} \)-subgroup \( P_3 \), in which case \( \bar{e}(P_1) \cap \bar{e}(P_2) = \bar{e}(P_3) \).

Notice that the largest subgroup of \( \Gamma \) which stabilizes \( X(P), e(P), \) and \( \bar{e}(P) = \Gamma \cap P \).

Modding out by the geodesic action of \( A_P \) gives a \((\Gamma \cap P)\)-equivariant bundle map \( q_P : X(P) \to e(P) \). The orbits in \( X(P) \) under \( M_P U_P \) are sections of this bundle called the canonical cross-sections.

**Definition.** Let \( \mathfrak{A}_{\mathcal{F},f} \) be the group of \( \mathbb{R} \)-linear maps \( S^n \to S^n \) which act by positive real homotheties on each summand \( \tilde{V}_j \) separately. Let \( \mathfrak{A}_{\mathcal{F},f} = \mathfrak{A}_{\mathcal{F},f}/H \). We call \( \mathfrak{A}_{\mathcal{F},f} \) the orthogonal scaling group with respect to \( \mathcal{F} \) and \( f \).

Note that every \( \alpha \in \mathfrak{A}_{\mathcal{F},f} \) is \( S \)-linear. The group \( \mathfrak{A}_{\mathcal{F},f} \) acts on \( Y' \) via \( f_1 \mapsto \alpha \cdot f_1 \). However, it does not act on \( X \). (The next proposition will show that \( \mathfrak{A}_{\mathcal{F},f} \) acts on \( Y' = H \backslash G \) as a lift of the action of \( A_P \) on \( X = HK \backslash G \), but this lift is only determined by an arbitrary choice of an element of \( K \). The choice is encoded in \( f \).)

Let \( \Psi : Y' = H \backslash G \to X = HK \backslash G \) be the projection. Let \( y = \Psi(f) \). Let \( \iota : A_P \to \mathfrak{A}_{\mathcal{F},f} \) be the following map: if \( z \in A_P \) with

\[
(3.6.1) \quad z = g^{-1} \operatorname{diag}(a_1, \ldots, a_1, \ldots, a_l, \ldots, a_l)g \quad (a_j > 0),
\]

let \( \iota(z) \) be the element of \( \mathfrak{A}_{\mathcal{F},f} \) which (for each \( j = 1, \ldots, l \)) acts on \( \tilde{V}_j \) by the scalar \( a_j \). It is clear that \( \iota \) is an isomorphism of groups.

**Proposition.** For any \( z \in A_P \), with \( \iota \) as above,

\[
(3.6.2) \quad y \circ z = \Psi(\iota(z) \cdot f).
\]

**Remark.** The proposition is stated in [G, (2.4)], without proof.

**Proof.** We use the notation of (3.4). Write \( y = HKgp \) and \( p = g^{-1} p_j g \) for \( p_j \in P_j \). Consider the marked lattice \( f_1 : x \mapsto gp_{\xi} = p_j g_{\xi} \). Since \( \Psi(f_1) = y = \Psi(f) \), we may choose a \( k_1 \in K \) such that \( f_1 = k_1 \cdot f \).

Write \( z \in A_P \) as \( z = g^{-1} z_j g \) for \( z_j = \operatorname{diag}(a_1, \ldots, a_1, \ldots, a_l, \ldots, a_l) \in A_j \). Then

\[
y \circ z = HKgzp = HKz_j p_j g.
\]
The marked lattice \( f_z : x \to z_j p_j g x \) descends mod \( K \) to \( y o z \).

Let \( \mathcal{I}_j \) be the flag in \( S^n \) whose \( j \)-th member is the coordinate subspace spanned by \( \{ e_1, \ldots, e_{\nu_1+\ldots+\nu_j} \} \). (This \( \mathcal{I}_j \) is essentially \( \mathcal{F}_j \otimes \mathbb{R} \).) The map \( x \mapsto g x \) carries \( \mathcal{F} \) to \( \mathcal{I}_j \), and the further multiplication by \( p_j \) and \( z \) preserves \( \mathcal{I}_j \). Thus for any \( z \in A_P \), \( f_z(v_j) \) is exactly the \( j \)-th member of \( \mathcal{I}_j \). (The idea is that, out of the whole \( K \)-equivalence class of marked lattices representing \( y o z \), \( f_z \) is a representative for which the \( f_z(v_j) \) will be in standard position.) The orthocomplement of \( f_z(v_{j-1}) \) in \( f_z(v_j) \) is the coordinate subspace \( E_j \) spanned by \( \{ e_{\nu_1+\ldots+\nu_{j-1}+1}, \ldots, e_{\nu_1+\ldots+\nu_j} \} \). The \( E_j \) are orthogonal, and \( z_j \) acts by the map \( \alpha_1 : S^n \to S^n \) characterized by the property that it acts on each \( E_j \) by the scalar \( a_j \).

Thus for any \( y \) in \( A_F \), \( f \in A_F \), \( z \in A_P \), \( y o z \) given the composition \( L_0 \overset{f_1}{\to} S^n \overset{\alpha_1}{\to} S^n. \quad \square \)

**Note.** We will always normalize elements of \( A_P \) and \( A_{F, f} \) in their classes modulo \( H \) so that \( a_1 = 1 \).

**3.7.** While \( A_P \) and \( A_{F, f} \) act on different spaces, we use similar notation for the two actions. We write

\[
\rho = (\rho_1, \ldots, \rho_{l-1}) \in (0, \infty)^{l-1}
\]

for a point in either group. The \( \rho_i \) are the coordinates coming from the simple roots in \( J \); that is, \( \rho_j = a_j / a_{j+1} \) for \( j = 1, \ldots, l-1 \). By Proposition 3.6, the point of \( A_{F, f} \) corresponding to \( (\rho_1, \ldots, \rho_{l-1}) \) acts on \( \tilde{V}_j \) by the scalar

\[
(3.7.1) \quad a_j = (\rho_1 \cdots \rho_{j-1})^{-1} \quad (j = 2, \ldots, l).
\]

(Because of our normalizations, the action on \( \tilde{V}_1 \) is by 1.)

**Remark.** The \( \rho_j \)'s are good coordinates for rescaling sublattices within lattices, while the \( a_j \) are the natural coordinates for the geodesic action. Though (3.7.1) is messy, we cannot avoid using it, since part of our goal is to relate lattices and the geodesic action. The inverses in (3.7.1) arise because the well-rounded retraction goes away from the Borel-Serre boundary.

Multiplication in \( A_P \) corresponds to coordinate-wise multiplication of the \( \rho_i \). As in (3.5), the \( \rho_j \) extend to coordinates on \( \hat{A}_P \) with \( (\rho_1, \ldots, \rho_{l-1}) \in [0, \infty)^{l-1} \). Going to infinity in \( X(P) \) means going to zero in the \( \rho_j \) variables. We introduce a partial ordering on \( A_P \) and \( \hat{A}_P \), where \( (\rho_1, \ldots, \rho_{l-1}) \prec (\rho'_1, \ldots, \rho'_{l-1}) \) if and only if \( \rho_j \leq \rho'_j \) for each \( j \).

The action of \( \rho = (\rho_1, \ldots, \rho_{l-1}) \) on \( Y' \) is denoted \( f \mapsto \rho \cdot f \). This descends mod \( \Gamma_0 \) to an action \( L \mapsto \rho \cdot L \) on \( Y' \). Descending mod \( K \) as in Proposition 3.6, \( \rho \cdot x \) denotes the geodesic action on points \( x \) of \( X \) or \( X/ (\Gamma \cap P) \).
Let (4.1). Let \( f \) be a marked lattice, with \( L = f(L_0) \). We normalize \( f \) up to homothety so that \( m(f) = 1 \). If \( V \subseteq k^n \) is a \( k \)-subspace, \( f(V) \otimes \mathbb{Q} \mathbb{R} \) is abbreviated \( f(V) \).

Recall that \( Y' = H \setminus G \) is the space of marked lattices (2.3). For \( i = 1, \ldots, n \), let

\[
Y'_i = \{ f \in Y' \mid \text{rank}_S(S \cdot M(f)) \geq i \}.
\]

Observe that \( Y' = Y'_1 \supset Y'_2 \supset \cdots \), and that \( Y'_n \) is the set of well-rounded elements of \( Y' \). For \( i = 1, \ldots, n-1 \), we will define a deformation retraction \( r^{(i)}_t : Y'_i \times [0, 1] \to Y'_i \) with image \( Y'_i+1 \). The deformation retraction will be equivariant for both the \( K \) and \( \Gamma_0 \) actions. For any \( f \in Y' \), we will also define a \( \mathbb{Q} \)-flag \( M = \{ 0 \subseteq M^{(1)} \subseteq \cdots \subseteq M^{(n)} = k^n \} \) depending on \( f \). (This \( M \) could have equalities, as in (3.8).) The definitions proceed simultaneously by induction on \( i \).

Let \( f_1 = f \), and for \( i > 1 \) let \( f_i = r^{(i-1)}_1(f_{i-1}) \) inductively. Let \( M^{(i)} \) be the \( k \)-span of \( f_i^{-1}(M(f_i)) \). This is a subspace of \( k^n \) of dimension \( \geq i \). If \( f_i \) is in \( Y'_{i+1} \) already, set \( r^{(i)}_t(f_i) = f_i \) for all \( t \in [0, 1] \), and set \( \mu_i = 1 \). Otherwise, consider the \( S \)-linear maps \( \varphi_{\mu} \) obtained by multiplying in \( f_i(M^{(i)}) \) by \( 1 \) and in the orthocomplement \( f_i(M^{(i)})^\perp \) by a scalar \( \mu > 0 \). By [A3, p. 463], there is a unique smallest \( \mu_i \in (0, 1) \) such that

\[
(4.1.1) \quad \varphi_{\mu_i} \cdot f_i \text{ still has arithmetic minimum equal to } 1.
\]

This \( \varphi_{\mu_i} \cdot f_i \) lies in \( Y'_{i+1} \). We set \( r^{(i)}_t(f_i) = \varphi_{\mu_i-1} \cdot f_i \). This is continuous in both the \( f_i \) and \( t \) variables, and defines a homotopy between \( r^{(i)}_0 = (\text{identity on } Y'_i) \) and \( r^{(i)}_1 : Y'_i \to Y'_{i+1} \).

**Definition.** The well-rounded retraction \( r_t \) on \( Y' \) is obtained by composing the deformation retractions \( r^{(i)}_t \) in the order \( i = 1, \ldots, n-1 \). More precisely, \( r_t : Y' \times [0, 1] \to Y' \) is given by

\[
r_t(f) = r^{(i)}_{t(i-1)}(f) \circ r^{(i-1)}_{i-1} \circ \cdots \circ r^{(1)}_1(f),
\]

where \( i \) is determined by the rule \( t \in [\frac{i-1}{n-1}, \frac{i}{n-1}] \) for \( i \in \{1, \ldots, n-1\} \).

Let \( r = r_1 \). We will also call this the well-rounded retraction.

Notice that \( M^{(i+1)} \supseteq M^{(i)} \) by construction. This gives the
Definition. The flag of successive minima for $f$ is the $\mathbb{Q}$-flag $\mathcal{M} = \{0 \subseteq M^{(1)} \subseteq \cdots \subseteq M^{(n)} = k^n\}$.

This $\mathcal{M}$ is an invariant of $f$’s class mod $K$, and is equivariant for $\Gamma_0$. It is a “lower semi-continuous” function of $f$ in the sense that for any $f \in Y'$, there is a neighborhood $U$ of $f$ such that for all $f' \in U$, the flag $\mathcal{M}'$ of successive minima for $f'$ satisfies $\mathcal{M}' \subseteq \mathcal{M}$.

(4.2). Remark. Once the inductive construction of $r$ is completed, we obtain the following interpretation of $r(f)$ as the result of applying a geodesic action to $f$. Let $\mathcal{M}$ be the flag of successive minima for $f$, with irredundant version $\mathcal{F}$. Let $\mu = (\mu_1^{-1}, \ldots, \mu_n^{-1}) \in \mathfrak{A}_{F,f}$ with the $\mu_i$ depending on $f$ as in (4.1). (As in (3.8), $\mu_i^{-1}$ is omitted when $M^{(i)} = M^{(i+1)}$.) Then

$$r(f) = \mu \cdot f.$$ 

(4.3). We now adapt the definition of the well-rounded retraction to the other spaces in (2.3.1). In each case, the retraction will still be denoted $r_t$ for $t \in [0,1]$, with $r = r_1$, and will be called the “well-rounded retraction”.

The whole construction is $\Gamma_0$- and $K$-equivariant. Hence $r_t$ descends to deformation retractions of $Y$ onto the set of well-rounded lattices, of $X$ onto $W$, and of $X/\Gamma$ onto $W/\Gamma$ for any $\Gamma \subseteq \Gamma_0$ of finite index.

We recall the main theorem of [A3].

Theorem. The map $r_t$ is a $\Gamma_0$-equivariant deformation retraction of $X$ onto $W$. The quotient $W/\Gamma_0$ is compact. The dimension of $W$ and $W/\Gamma_0$ is the virtual cohomological dimension of $\Gamma_0$, which equals $\dim X - (n-1)$. The space $W$ has a natural structure as a cell complex on which $\Gamma_0$ acts cell-wise with finite stabilizers of cells. The first barycentric subdivision of the cell structure descends to a finite cell complex structure on $W/\Gamma_0$.

The example of $\text{GL}_2(\mathbb{Z})$ is discussed in (6.3).

Remark. When the class number $h$ of $k$ is $>1$, [A3] defines a whole $(h-1)$-dimensional family of retracts, depending on a parameter called a set of weights. For simplicity, we will use only the trivial set of weights (identically equal to 1). Readers may easily make the needed changes if they wish to use a non-trivial set of weights.

Section 5—Summary of Notational Conventions

The letter $f$ always stands for a marked lattice $f : \mathbf{x} \mapsto gx$ for $g \in G$, modulo the homotheties $H$. We let $L = f(L_0)$, sometimes without explicit mention. We let $L'$ be the lattice for $f'$, etc. As in (3.6), $f(V_j \cap L_0) \otimes_\mathbb{Q} \mathbb{R} \subseteq S^n$ will be written $f(V_j)$. Let $V_j$ be the orthogonal complement of $f(V_{j-1})$ in $f(V_j)$, so that $S^n = V_1 \oplus \cdots \oplus V_l$ as an orthogonal direct sum. As in (2.4), lattices are always scaled by a homothety so that their shortest non-zero vector has length 1.

For the rest of the paper, $P$ denotes the real points of a parabolic $\mathbb{Q}$-subgroup. Let $\mathcal{F} = \{0 \subseteq V_1 \subseteq \cdots \subseteq \cdots \subseteq V_l = k^n\}$ be the $\mathbb{Q}$-flag for $P$ as in (3.2), and $\mathfrak{A} = \mathfrak{A}_{F,f}$ the orthogonal scaling group with respect to $\mathcal{F}$ and $f$ (see (3.6)–(3.7)).
Whenever we mention $P$, the letters $\mathcal{F}$ and $\mathfrak{A}$ will refer to the objects corresponding to $P$, and vice versa. The symbol $P'$ is associated to $\mathcal{F}'$ and $\mathfrak{A}'$, etc.

From Section 6 on, we will often write $f$ for the point that $f$ determines in $X$ or $X/\Gamma$. We will speak of marked lattices $f$ “in” $X$ or $W$, or write $L$ for the point that $L$ determines in $X/\Gamma$. These abuses are justified because our constructions involving marked lattices are $K$-equivariant.

Section 6—The Sets $W_{\mathcal{F}}$

(6.1). Throughout this section, $\mathcal{F} = \{0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = k^n\}$ will be a fixed $\mathbb{Q}$-flag.

Definition. Let $W_{\mathcal{F}}$ be the set of all marked lattices $f$ in $W$ such that for all $j = 1, \ldots, l$,

$$S \cdot (M(L) \cap f(V_j)) = f(V_j).$$

$(W_{\mathcal{F}}$ was called $W'$ in the introduction.) Geometrically, $W_{\mathcal{F}}$ consists of the lattices $L = f(L_0)$ whose intersection with each subspace $f(V_j)$ is well-rounded in that subspace. By the definition of the cell structure on $W$ [A3, pp. 466–7], $W_{\mathcal{F}}$ is a closed subcomplex of $W$ and is stable under $\Gamma \cap P$. A cell in $W_{\mathcal{F}}$ is said to respect $\mathcal{F}$. If $\mathcal{F} = \gamma \mathcal{F}'$ for $\gamma \in \Gamma$, then the images of $W_{\mathcal{F}}$ and $W_{\mathcal{F}'}$ in $W/\Gamma$ are equal. The $\mathbb{Q}$-flags fall into only finitely many equivalence classes mod $\Gamma$, so only finitely many distinct subcomplexes of $W/\Gamma$ arise in this way.

If $\mathcal{M}$ is a $\mathbb{Q}$-flag with possible equalities (3.8), and $\mathcal{F}$ is the irredundant version of $\mathcal{M}$, let $W_{\mathcal{M}} = W_{\mathcal{F}}$.

(6.2). The following is immediate from the definitions in (4.1).

Lemma. If $f$ has flag of successive minima $\mathcal{M}$, then $r(f) \in W_{\mathcal{M}}$.

(6.3). Example of $\text{GL}_2(\mathbb{Z})$. Refer again to the picture in (2.6). For the standard flag $\mathcal{F} = \{0 \subsetneq \mathbb{Q} \cdot (1,0) \subsetneq \mathbb{Q}^2\}$ with associated $P$, the space $W_{\mathcal{F}}$ is the horizontal sequence of arcs forming the top of $W$. The boundary face $e(P)$ in $\overline{X}$ (not shown) is the horizontal line at $b = \infty$. The geodesic action by $A_P$ flows straight up, pushing $X$ upward to converge to $e(P)$. The well-rounded retraction pulls the region above $W$ straight down onto $\overline{W}_\mathcal{F}$. For torsion-free $\Gamma$, $e(P)/(\Gamma \cap P)$ and $W_{\mathcal{F}}/(\Gamma \cap P)$ are homeomorphic to circles, and the well-rounded retraction induces the obvious isomorphism between the cohomology of these spaces. (The situation is more complicated for $n > 2$, when the boundary faces are no longer disjoint.)

Remark. For small $n$, the $W_{\mathcal{F}}/(\Gamma \cap P)$ are all readily computable. See (10.6).

Section 7—Neighborhoods of Infinity in $\overline{X}$

We now prove our main results explaining the connection between the well-rounded retraction $r$, the geodesic action, and the topology of the Borel-Serre boundary. Parts (7.1)–(7.4) give a chain of propositions that allow us to define a tubular neighborhood $\overline{N}_\mathcal{F}$ of $e(P)$ (called $\overline{N}$ and $e$ in the introduction). In (7.5)–(7.6) we prove that $r : X \to W$ has the continuous extension $\overline{r} : \overline{X} \to W$, where (for all $\mathcal{F}$) $\overline{r}$ is constant on the fibers of $\overline{N}_\mathcal{F}$ and carries $\overline{N}_\mathcal{F}$ onto $W_{\mathcal{F}}$. 
(7.1). Fix a marked lattice $f$, with $L = f(L_0)$. As in (4.1), let $M = \{0 \subseteq M^{(1)} \subseteq \cdots \subseteq M^{(n)} = k^n\}$ be the flag of successive minima for $f$. Let $F$ be the irredundant version of $M$, with associated $P$. Let $\mathfrak{A} = \mathfrak{A}_F$. Let $f'' = r(f)$ be the image of $f$ under the well-rounded retraction.

The idea of the following lemma is that if $f$ is close enough to $e(P)$, then the well-rounded retraction carries the whole orthant $\{\rho \in \mathfrak{A} \mid \rho \leq 1\} \cdot f$ to a single point of $W$. The lemma says that, to know that $f$ is “close enough” to $e(P)$, it suffices to know that $f$’s flag of successive minima is the flag $F$ corresponding to $P$ (or at least contains this $F$—see (7.2)).

**Lemma.** Let $f$, $M$, $F$, $P$, and $\mathfrak{A}$ be as above. For any $\rho = (\rho_1, \ldots, \rho_{l-1}) \in \mathfrak{A}$ with $\rho \leq (1, \ldots, 1)$, $f' = \rho \cdot f$ satisfies $r(f') = r(f) = f''$. The common image $f''$ will lie in $W_M = W_F$.

**Proof.** It suffices to prove the lemma when $\rho = (1, \ldots, 1, \rho_j, 1, \ldots, 1)$ for a fixed $\rho_j \leq 1$; the general case follows by applying this for each $j$. Let $i$ be the unique index such that $V_i = M^{(i)}$ and $M^{(i)} \not\subseteq M^{(i+1)}$.

The marked lattices $f$ and $f'$ agree on $L_0 \cap M^{(i)}$, since $\rho$ acts by 1 in that region. For $x \in L_0$ with $x \notin M^{(i)}$, no vector $f'(x)$ is shorter than the corresponding vector $f(x)$, since $\rho$ magnifies lengths in the directions perpendicular to $f(M^{(i)})$ by $\rho_j^{-1} \geq 1$. And $f(x)$ cannot be a minimal vector of $f$, since the minimal vectors lie in $f(M^{(1)})$, which is contained in $f(M^{(i)})$. It follows that the well-rounded retractions for $f$ and $f'$ will proceed exactly the same up through the end of the $r^{(i-1)}$ step—that is, if $\mu_i$ and $\mu_i'$ (for $f$ and $f'$ resp.) are as in (4.1), then

\[(7.1.1) \quad \mu_i = \mu_i' \text{ for all } i < i.\]

Set $g = r_1^{(i-1)} \circ \cdots \circ r_1^{(1)}(f)$ and $g' = r_1^{(i-1)} \circ \cdots \circ r_1^{(1)}(f')$. It follows from (3.7.1) and (7.1.1) that $g' = (1, \ldots, 1, \rho_j^{-1}, 1, \ldots, 1) \cdot g$. The $r_i^{(i)}$ stage of the retraction will, by definition, carry $g$ to

\[(7.1.2) \quad (1, \ldots, 1, \mu_i^{-1}, 1, \ldots, 1) \cdot g,\]

and carry $g'$ to

\[(7.1.3) \quad (1, \ldots, 1, \mu_i'^{-1}, 1, \ldots, 1) \cdot g'.\]

However, by the uniqueness statement in (4.1.1), $\mu_i^{-1}$ must equal $\mu_i'^{-1} \rho_i$. Thus the marked lattices (7.1.2) and (7.1.3) will coincide. The $r^{(i+1)}$ and later stages of the retraction will agree on this common lattice, so the final values $r(f)$ and $r(f')$ are equal. By (6.2), $f'' \in W_M$. □

(7.2). We record a corollary of the previous lemma. The idea is that if $f$’s flag of successive minima refines the flag $F'$ for some $P'$, then $f$ is “close enough” to $e(P')$. After all, if $f$ is close enough to $e(P)$, it should be close enough to any larger Borel-Serre boundary component $e(P')$ that has $e(P)$ in its closure.
Corollary. Let the notation be as in (7.1). Let $F'$ be any $\mathbb{Q}$-flag with $F' \subseteq F$, and let $A'$ be the orthogonal scaling group for $F'$ and $f$. Then for any $\rho \in A'$ with $\rho \leq (1,\ldots,1)$, $\rho \cdot f$ and $f$ map to the same point under the well-rounded retraction. This point lies in $W_{F'}$.

Proof. This is trivial, since $A'$ is naturally a subgroup of $A$, and $W_F \subseteq W_{F'}$. □

(7.3). In (7.1) we took an $f$ close to $e(P)$ and studied the orthant $\{\rho \leq 1\} \cdot f$. Now, instead, we take an arbitrary $f$. We show in (7.3) (actually in the corollary) that there is a $t \in A$ which pushes $f$ so close to $e(P)$ that the orthant $\{\rho \leq t\} \cdot f$ still has the property of (7.1)—namely, $r$ maps it to a single point in $W_F$.

Lemma. Choose any marked lattice $f$. Fix a $\mathbb{Q}$-flag $F = \{0 \subseteq V_1 \subseteq \cdots \subseteq V_l = k^n\}$, and let $A = A_{F,f}$. Then there exists $t = (t_1,\ldots,t_l-1) \in A$, depending on $f$, such that for any $\rho \in A$ with $\rho \leq t$, the flag $M'$ of successive minima for $f' = \rho \cdot f$ satisfies $M' \supseteq F$.

Proof. Let $j \in \{1,\ldots,l-1\}$. Let $d_j = \text{dim}_k V_j$. Let $\mathring{L} = f(V_j \cap L_0)$. This is a $\mathbb{Z}$-lattice in the $S$-submodule $\mathring{V} = f(V_j \otimes \mathbb{Q} \otimes \mathbb{R}) \subseteq S^n$. Perform the well-rounded retraction on $\mathring{L}$ within $\mathring{V}$. As in (4.1), this involves multiplying by constants $\mu_1,\ldots,\mu_{d_j-1} \leq 1$ in various subspaces of $\mathring{V}$. (We must first rescale $f$ by a homothety so that $m(\mathring{L}) = 1$; this does not affect the value of the $\mu_i$.) Let $\beta_j = \mu_1 \cdots \mu_{d_j-1}$. The orthogonal projection of $L = f(L_0)$ onto $f(V_j)^\perp$ is a lattice $L^\dagger$ under the restriction of the inner product to $f(V_j)^\perp$. Let $\alpha_j > 0$ be the length of the shortest non-zero vector of $L^\dagger$.

Let $t_1 = \min(1,\frac{1}{2}(1,\ldots,1) \cdot (1,\ldots,1)^{-1})$, and inductively let $t_j = \min(1,\frac{1}{2}((t_1 \cdots t_{j-1})^{-1}) \cdot (1,\ldots,1)^{-1})$. This means $(t_1 \cdots t_j)^{-1} \geq 2\alpha_j^{-1} \beta_j^{-1}$ for $j = 1,\ldots,l-1$.

Now let $f' = \rho \cdot f$ for $\rho \leq t$. Fix $j$. Let $a$ be any vector of $L$ lying outside $f(V_j \cap L_0)$. Let $b = \rho \cdot a$ be the corresponding vector for $f'$. Notice that $\rho$ multiplies $\mathring{V}_{j+1}$ by $(\rho_1 \cdots \rho_j)^{-1}$, by (3.7.1). Since $\rho_j \leq 1$ for all $j$, $\rho$ multiplies by factors at least as large as $(\rho_1 \cdots \rho_j)^{-1}$ in each summand of $\mathring{V}_{j+1} \oplus \cdots \oplus \mathring{V}_1 = f(V_j)^\perp$.

Imagine performing the well-rounded retraction on $\mathring{L}$, but carrying out all the retractions on $L' = f'(L_0)$ itself; more precisely, consider the lattice

$$\text{(7.3.1)} \quad (1,\ldots,1,\mu_{d_j-1}^{-1},1,\ldots,1) \cdot (1,\ldots,1,\mu_{d_j-2}^{-1},1,\ldots,1) \cdots \cdot (\mu_1^{-1},1,\ldots,1) \cdot L'$$

where the $\mu_{d_j}$ are the constants for $\mathring{L}$. (The $\mu_i^{-1}$ is in the $i$-th position when the minimal vectors of $\mathring{L}$ at the $i$-th step of the retraction span the $i$-th member of the irredundant version of the flag of successive minima for $\mathring{L}$.) Let $b^{(j)}$ be the vector corresponding to $b$
in (7.3.1). Observe, using (3.7.1), that
\[
\|b^{(j)}\| \geq \|(\text{orthogonal projection of } b^{(j)} \text{ onto } f(V_j)\|) \\
= \mu_1 \cdots \mu_{d_j - 1} \|(\text{orthogonal projection of } b \text{ onto } f(V_j)\|) \\
= \beta_j \|(\text{orthogonal projection of } b \text{ onto } f(V_j)\|) \\
\geq \beta_j (\rho_1 \cdots \rho_j)^{-1} \|(\text{orthogonal projection of } a \text{ onto } f(V_j)\|) \\
\geq \beta_j (t_1 \cdots t_j)^{-1} \alpha_j \\
\geq 2 \\
> 1.
\]

This says that the image of $L'$ under the well-rounded retraction on $L$ up through the $r^{(d_j - 1)}$ stage is equal to (7.3.1): no vector outside $\hat{L}$ will affect the stopping times of the retraction on $L'$, since each will have length $> 1$ during the entire process leading to (7.3.1).

In particular, $f(V_j)$ is a member of $\mathcal{M}'$, the flag of successive minima of $L'$, which is what we wanted to prove. □

**Corollary.** With notation as in the lemma, the points $\rho \cdot f$ with $\rho \leq t$ map to a common point of $W_F$ under the well-rounded retraction $r$.

**Proof.** Apply Corollary 7.2 to the marked lattice $t \cdot f$ of the lemma. □

(7.4). We can now construct neighborhoods of the Borel-Serre boundary component $e(P)$ that are well behaved with respect to the well-rounded retraction. Fix a $\mathbb{Q}$-flag $\mathcal{F}$, with its associated $P$.

Recall that the corner $X(P)$ is a fiber bundle with base $e(P)$ and fiber $\hat{A}_P$. By a neighborhood in $X$ near $e(P)$, we mean an open set in $X$ whose intersection with every fiber contains \{a $\in A_P$ | a < a_0\} for some a_0 $\in A_P$ depending on the fiber.

**Proposition.** There is a neighborhood $N_{\mathcal{F}}$ in $X$ near $e(P)$ such that, on the intersection of $N_{\mathcal{F}}$ with a fiber of $X(P)$, the well-rounded retraction $r$ takes a constant value. Furthermore, $r$ carries $N_{\mathcal{F}}$ surjectively to $W_F$.

**Proof.** As in (3.5), let $C_1$ be the canonical cross-section $HK \cdot MPUP$. Let $f_1$ be a marked lattice corresponding to a point in $C_1$. For this $f_1$, let $\mathfrak{A} = \mathfrak{A}_{\mathcal{F}, f_1}$, and let $t_1 \in \mathfrak{A}$ be the point constructed in Lemma 7.3. We define

\[
(7.4.1) \\
N_{\mathcal{F}} = \{\rho_1 \cdot f_1 | \rho_1 < t_1\},
\]

where $f_1$ ranges over all marked lattices representing points of $C_1$, and $t_1$ depends on $f_1$. For any given $f_1$, the map $r$ is constant on the set \{$\rho_1 \cdot f_1$\} $= N_{\mathcal{F}} \cap ($fiber), by Corollary 7.3.

To prove $N_{\mathcal{F}}$ is an open set in $X$, we must show the $t_1$ vary continuously with $f_1$. The $\mu_i$ of (4.1) vary continuously as a function of $f$, by [A3, §3(i)]. Hence the $\beta_j$ of (7.3) vary continuously with $f$. For any continuous family of lattices (not necessarily normalized
by \(m(L) = 1\), the function \(m(L)\) is also continuous in \(L\); applying this to \(L\) (defined in (7.3)), we see the \(\alpha_j\) of (7.3) are continuous in \(f\). Hence the \(t_j\) are continuous in \(f\).

By Corollary 7.2, \(r\) carries \(N_\mathcal{F}\) into \(W_\mathcal{F}\). In the rest of the proof, we prove the map is surjective. Let \(f\) be a marked lattice representing a point in \(W_\mathcal{F}\). Let \(d_j = \dim_k V_j\). Let \(f_1\) be the marked lattice corresponding to the point in \(C_1\) in the \(A_P\)-fiber through \(f\); let \(\rho \cdot f = f_1\). Choose \(t\) for \(f_1\) as in Lemma 7.3. For a sufficiently small \(\tau \in (0, 1)\), \(\tau = (\tau, \ldots, \tau)\) satisfies \(\tau < t \cdot \rho\), implying \(\tau \cdot f \in N_\mathcal{F}\). We will show \(r\) carries \(\tau \cdot f\) back to \(f\).

The lattice for \(\tau \cdot f\) is the lattice for \(f\) scaled by a factor of \(1/\tau^{j-1}\) on \(\tilde{V}_j\). The minimal vectors of \(f\) span their intersection with \(f(V_1)\), by the definition of \(W_\mathcal{F}\). Since \(1/\tau > 1\), the minimal vectors of \(\tau \cdot f\) are exactly the minimal vectors of \(f\) that lie in \(f(V_1)\). Hence the minimal vectors of \(\tau \cdot f\) lie in \(f(V_1)\) and span \(f(V_1)\). This means the steps \(r(1), \ldots, r(d_1-1)\) of the well-rounded retraction are trivial.

The retraction \(r(d_1)\) fixes \(f(V_1)\) and acts by a scalar \(\mu_{d_1} \leq 1\) on all orthogonal directions. Consider what happens as a scalar \(\mu\) descends from 1 to \(\mu_{d_1}\). The marked lattice \(\tau \cdot f\) becomes

\[
(7.4.2) \quad (\mu^{-1}, \tau, \ldots, \tau) \cdot f.
\]

As long as \(1/(\mu^{-1} \tau) > 1\), there are no new minimal vectors. But when \(\mu^{-1} \tau = 1\), the minimal vectors of (7.4.2) are exactly the minimal vectors of \(f\) that lie in \(f(V_2)\). These span \(f(V_2)\) (again by the definition of \(W_\mathcal{F}\)), which has rank bigger than the rank of \(f(V_1)\). So by (4.1.1) for \(i = d_1\), we must have \(\mu_{d_1} = \tau\), and at the end of the \(r(d_1)\) stage we have produced \((1, \tau, \ldots, \tau) \cdot f\). The well-rounded retraction goes on similarly to produce \((1, 1, \tau, \ldots, \tau) \cdot f, \ldots\), and finally \((1, \ldots, 1) \cdot f = f\). \(\square\)

**Definition.** \(\tilde{N}_\mathcal{F} = \{ f = \rho \cdot f_1 \mid \rho_1 \in \bar{\mathcal{A}}_{\mathcal{F}, f_1}, \rho_1 < t_1 \}\), where \(f_1\) ranges over all marked lattices representing points of \(C_1\), and \(t_1\) depends on \(f_1\) as in Lemma 7.3.

It is clear that \(\tilde{N}_\mathcal{F}\) is a tubular neighborhood of \(e(P)\) in \(X(P)\), in the sense that \(\tilde{N}_\mathcal{F}\) is homeomorphic to \(e(P) \times [0, 1]^{l-1}\) with \(e(P)\) included as \(e(P) \times (0, \ldots, 0)\). Also, the set \(\tilde{N}_\mathcal{F}\) is \((\Gamma_0 \cap P)\)-invariant.

**Proposition 7.4.** We now define a map \(\tilde{r} : \tilde{X} \to W\) that continuously extends the well-rounded retraction \(r\). Choose \(x \in \tilde{X}\). If \(x \in X\), let \(\tilde{r}(x) = r(x)\). Otherwise, \(x \in e(P)\) for a unique \(P\) with associated \(\mathcal{F}\). Hence \(x \in \tilde{N}_\mathcal{F}\). Consider the fiber \(\varphi'\) of \(\tilde{N}_\mathcal{F}\) over \(x\). By Proposition 7.4, \(r\) takes a constant value on \(\varphi' \cap X\). Define \(\tilde{r}(x)\) to be this common value \(r(\varphi' \cap X)\).
**Proposition.** The map $\tilde{r}$ is a continuous $\Gamma$-equivariant extension of $r$ to $\bar{X}$. For any $\mathbb{Q}$-flag $F$, it is constant on the $\bar{A}_P$-fibers of $\bar{N}_F$. It is a retraction $\bar{X} \to W$ (i.e. it fixes $W$ pointwise).

**Proof.** It suffices to prove $\tilde{r}$ is constant on $\varphi'$. For then, the continuity of $\tilde{r}$ on $\bar{N}_F$ will follow from the tubular neighborhood structure and the continuity of $r$ on $N_F$. The continuity of $\tilde{r}$ on $\bar{X}$ will follow from its continuity on $X$ and on each $\bar{N}_F$, because these form an open cover of $\bar{X}$. The $\Gamma$-equivariance of $\tilde{r}$ will follow from that of $r$.

Let $y$ lie on the fiber $\varphi'$ of $\bar{N}_F$ over $x$, with $y \neq x$. If $y \in X$, then $\tilde{r}(x) = r(x) = \tilde{r}(y)$ by construction. Next, suppose $y \in e(P_1)$ for some $P_1$ with associated $F_1$. Then $P_1 \supseteq P$. Now $A_{P_1}$ is naturally a subgroup of $A_P$ by (3.3), and $\bar{A}_{P_1} \subseteq \bar{A}_P$. So $\varphi'$ fibers over $e(P_1)$ with fibers given by the action of $\bar{A}_{P_1}$. Let $\varphi'_1$ be the intersection of $\varphi'$ with the $\bar{A}_{P_1}$-fiber of $\bar{N}_{X_1}$ over $y$. Then

$$\begin{align*}
\tilde{r}(y) &= r(\varphi'_1 \cap X) \quad \text{by def. of } \tilde{r}(y) \\
&= r(\varphi' \cap X) \quad \text{since } \varphi'_1 \cap X \subseteq \varphi' \cap X, \text{ and } \\
&= \tilde{r}(x) \quad \text{by def. of } \tilde{r}(x).
\end{align*}$$

This proves $\tilde{r}$ is constant on the $\bar{A}_P$-fibers of $\bar{N}_F$.

Finally, $\tilde{r}$ fixes $W$ pointwise because $r$ does. $\Box$

**Section 8—Extending the Deformation Retraction to the Borel-Serre Boundary**

(8.1). We recall from [S] some facts about Saper’s tiling of $\bar{X}$. Saper constructs a $\Gamma$-equivariant subspace $X_0$ of $X$ called the central tile. This is a codimension-zero submanifold with corners in $X$. The closed boundary faces $\partial^P X_0$ are in one-to-one correspondence with the proper parabolic $\mathbb{Q}$-subgroups $P$; inclusions $P \subseteq P'$ correspond to inclusions $\partial^P X_0 \subseteq \partial^{P'} X_0$, and conversely. Every $y \in \partial X_0$ lies on $\partial^P X_0$ for a unique smallest $P$. There is a $\Gamma$-equivariant piecewise-analytic deformation retraction $r_S : \bar{X} \to X_0$; more precisely, $r_S$ fits into a piecewise-analytic family $r_{t,S}$ for $t \in [0, 1]$ with $r_{0,S} =$ (identity on $\bar{X}$) and $r_{1,S} = r_S$. The retraction $r_S$ is uniquely characterized by

$$\begin{align*}
(8.1.1) \quad r_S(y \circ a) &= y \quad \text{whenever } y \in \partial^P X_0 \text{ and } a \in \bar{A}_P, a \leq 1.
\end{align*}$$

The converse of (8.1.1) also holds: if $y \in \partial^P X_0$, and $P$ is the smallest with this property, then

$$\begin{align*}
(8.1.2) \quad r_S(y') &= y \quad \text{implies } \exists a \in \bar{A}_P, a \leq 1, \text{ such that } y' = y \circ a.
\end{align*}$$

For any arithmetic subgroup $\Gamma$, the quotient $X_0/\Gamma$ is compact. Also, for any $P$ and given any open neighborhood $U$ of $e(P)/\Gamma \cap P$ in $\bar{X}/\Gamma$, we may choose $X_0$ so that

$$\begin{align*}
(8.1.3) \quad \partial^P X_0/(\Gamma \cap P) \subseteq U.
\end{align*}$$

In fact, given any central tile, we may choose enlargements of it whose complements in $\bar{X}$, modulo $\Gamma$, form a fundamental system of neighborhoods of $\partial X/\Gamma$ in $\bar{X}/\Gamma$. 

We now show that our global retraction \( \bar{r} \) of (7.6) is really a deformation retraction.

**Theorem.** \( \bar{r} = R_1 \).

**Proof.** If \( x \in X_0 \), then \( r_S \) does not move \( x \), and \( R_1(x) = r(x) = \bar{r}(x) \). Next, assume \( x \in \bar{X} \) with \( x \notin X_0 \). The retraction \( r_S \) carries \( x \) to a point \( y \), and there is a unique smallest \( P \) such that \( y \) lies on the face \( \partial^P X_0 \). However, by (8.1.2), \( x \) is of the form \( x = y \circ a \) for some \( a \in A_P \), \( a \leq 1 \). By (8.2) part (i), \( y \in \bar{N}_{F''} \) for some \( F'' \supseteq F \). Since \( \bar{A}_P \subseteq \bar{A}_{P''} \), we have \( x \in \bar{N}_{F''} \), and \( x \) lies on the same \( A_{P''} \)-fiber of \( \bar{N}_{F''} \) as \( y \). By (7.6) applied to \( F'' \), we have \( \bar{r}(x) = \bar{r}(y) \). But \( \bar{r}(y) = r(y) = R_1(x) \) by the definition of \( R_t \). \( \square \)

(8.5). Here is a simple consequence of Theorem 8.4.
Corollary. The map \( \bar{r} : X \to W \) is a \( \Gamma \)-equivariant homotopy equivalence. \( X/\Gamma \) and \( W/\Gamma \) have the same homotopy type via the map \( \bar{r} \mod \Gamma \).

We now generalize this corollary to all the boundary components.

Theorem. Let \( P \) be a proper parabolic \( \mathbb{Q} \)-subgroup, with associated \( \mathcal{F} \). Then \( \bar{r} \) induces a \((\Gamma \cap P)\)-equivariant homotopy equivalence of \( \bar{e}(P) \) and \( W_\mathcal{F} \).

Proof. We must define a \((\Gamma \cap P)\)-equivariant homotopy inverse \( \bar{s} \) for \( \bar{r} \mid_{\bar{e}(P)} \). Choose a point \( x \in W_\mathcal{F} \), represented by a marked lattice \( f \). Let \( \tau = (\tau, \ldots, \tau) \in \mathfrak{A} = \mathfrak{A}_\mathcal{F},f \). We define \( \bar{s}(x) \) to be the point of \( X \) corresponding to

\[
\lim_{\tau \to 0^+} \tau \cdot f.
\]

For all \( \tau > 0 \), \( \tau \cdot f \) lies in a single \( A_P \)-fiber of \( X \). Let \( q_P \) be as in (3.5). It is clear that the limit in (8.5.1) is \( q_P(x) \). Thus an equivalent definition is

\[
(8.5.2) \quad s \text{ is the restriction of } q_P \text{ to } W_\mathcal{F}.
\]

By (8.5.2), \( s \) is continuous. Since \( q_P \) commutes with the action of \( P \), \( s \) is \((\Gamma \cap P)\)-equivariant.

We now show that \( \bar{r} \circ s \) is the identity on \( W_\mathcal{F} \). For a sufficiently small \( \tau_0 > 0 \), \( (\tau_0, \ldots, \tau_0) \cdot f \) will lie in \( N_\mathcal{F} \). By (7.6), the value of \( \bar{r} \) on the point \( (\tau, \ldots, \tau) \cdot f \) is the same for all \( \tau \in [0, \tau_0] \). Using (8.5.1), \( \bar{r} \circ s(x) = \bar{r}((\tau_0, \ldots, \tau_0) \cdot f) \). However, the proof of Proposition 7.4 shows \( \bar{r}((\tau_0, \ldots, \tau_0) \cdot f) = f \).

Next, we show \( s \circ \bar{r} \mid_{\bar{e}(P)} \) is homotopic to the identity on \( \bar{e}(P) \). Consider the family of maps

\[
(8.5.3) \quad q_P \circ R_t \quad \text{restricted to } \bar{e}(P).
\]

Here \( R_t \) is from (8.3), and \( q_P \) is extended continuously to a neighborhood of \( \bar{e}(P) \) in \( \bar{X} \) in a natural way \([B-S, (5.5)]\). (A dangerous mistake would be to assume that the canonical cross-sections in \( \bar{X}(P) \) extend over \( \bar{e}(P) \) as cross-sections of \( q_P \).) The function in (8.5.3) is continuous because the image of \( \bar{e}(P) \) under \( R_t \) lies in \( \bar{e}(P) \cup X \), where \( q_P \) is continuous. At \( t = 0 \), (8.5.3) is the identity \( \bar{e}(P) \to \bar{e}(P) \). At \( t = 1 \), for any \( x \in \bar{e}(P), \)

\[
q_P \circ R_1(x) = q_P(\bar{r}(x)) \quad \text{by (8.4)}
\]

\[
= \bar{s}(\bar{r}(x)) \quad \text{by (8.5.2)}.
\]

Thus \( q_P \circ R_t \) affords a \((\Gamma \cap P)\)-equivariant homotopy between \( s \circ \bar{r} \mid_{\bar{e}(P)} \) and the identity. \( \square \)

Remark. Formula (8.5.3) defines a well-rounded retraction within \( \bar{e}(P) \) itself, carrying \( \bar{e}(P) \) onto a subset of \( e(P) \) homeomorphic to \( W_\mathcal{F} \). We know that \( \bar{e}(P)/(\Gamma \cap P) \) is a compactification of a bundle whose base is a locally symmetric space (perhaps with finite quotient singularities) for a group of lower rank, and whose fiber is an arithmetic quotient of \( U_P \). Accordingly, there is a cellular quotient map from \( W_\mathcal{F}/(\Gamma \cap P) \) to the well-rounded retract in the lower-rank locally symmetric space. This map may be constructed by dualizing the techniques of [M2]; see (10.7).
SECTION 9—THE MAIN RESULTS

In (9.5) we establish our main result, as we construct the spectral sequence described in the introduction. The bulk of the proofs resides in (9.2)–(9.4). We set up two spectral sequences, one (9.2.1) computing the cohomology of the Borel-Serre boundary $\bar{X}/\Gamma$, and the other (9.3.1) based on the subcomplexes $W_F/\Gamma \cap P$ in the well-rounded retract $W/\Gamma$. Because the map $\bar{r}$ gives homotopy equivalences on $\bar{X}/\Gamma$ and all its faces, it induces an isomorphism between these spectral sequences, and between their abutments. We set up canonical maps $H^*(\bar{X}/\Gamma) \to (9.2.1)$ and $H^*(W/\Gamma) \to (9.3.1)$. Finally, we show that the map $H^*(W/\Gamma) \to (9.3.1)$ computes the canonical restriction map $H^*(\bar{X}/\Gamma) \to H^*(\partial \bar{X}/\Gamma)$.

(9.1). Let $A^{p,q}$ be any first-quadrant double complex. There are two standard filtrations of the double complex, $\{A^{p,q} | q \geq q_0\}$ and $\{A^{p,q} | p \geq p_0\}$. The spectral sequences these give are called respectively the type I and type II sequences for $A^{p,q}$. Our main results involve type II sequences. The cohomology of the single complex associated to $A^{p,q}$ is called the abutment of (either) spectral sequence.

Throughout the paper, (co)homology groups have coefficients in any fixed abelian group.

(9.2). We set up a Mayer-Vietoris spectral sequence for the cohomology of $\partial \bar{X}/\Gamma$. Let $\Phi_l$ be a set of representatives of the $\Gamma$-equivalence classes of the $\mathbb{Q}$-flags $F = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = k^n\}$ (see (3.2)). There are only finitely many such equivalence classes. Let $\mathcal{E} = \{\bar{e}(P)/(\Gamma \cap P) | F \in \Phi_2\}$ (corresponding to maximal proper parabolic $\mathbb{Q}$-subgroups). This $\mathcal{E}$ is a cover of $\partial \bar{X}/\Gamma$. By (3.5), the non-empty $(p+1)$-fold intersections of distinct members of $\mathcal{E}$ are exactly the sets in $\{\bar{e}(P)/(\Gamma \cap P) | F \in \Phi_{p+2}\}$. This means we are in the correct setting to use the Čech cohomology techniques of [B-T, §15].

For $0 \leq p \leq n-2$, define

$$\mathcal{X}^{p,q} = \bigoplus_{F \in \Phi_{p+2}} C^q(\bar{e}(P)/(\Gamma \cap P)),$$

where $C^q$ denotes the singular $q$-cochains. We make $\mathcal{X}^{p,q}$ into a double complex, where the vertical maps are $(-1)^p$ times the coboundary maps, and the horizontal maps are induced as in Čech cohomology from the inclusions $\bar{e}(P) \leftarrow \bar{e}(P')$ for $F' \supseteq F$, with the alternating sign convention of [B-T, (15.7.1)]. Exactly as in [B-T, (15.7)], we see that the $q$-th row of $\mathcal{X}^{p,q}$ is exact, except at the $p = 0$ position where the kernel equals $C^q(\partial \bar{X}/\Gamma)$. Thus the type I spectral sequence for $\mathcal{X}^{p,q}$ collapses at $E_2$ to $H^*(\partial \bar{X}/\Gamma)$. Therefore, the total complex of $\mathcal{X}^{p,q}$ computes $H^*(\partial \bar{X}/\Gamma)$.

Let $E_{1,X}^{p,q}$ be the type II spectral sequence for $\mathcal{X}^{p,q}$. We have

$$E_{1,X}^{p,q} = \bigoplus_{F \in \Phi_{p+2}} H^q(\bar{e}(P)/(\Gamma \cap P)) \Rightarrow H^{p+q}(\partial \bar{X}/\Gamma).$$

(9.3). For $0 \leq p \leq n-2$, define

$$\mathcal{W}^{p,q} = \bigoplus_{F \in \Phi_{p+2}} C^q(W_F/(\Gamma \cap P)),$$
made into a double complex in the same way as $\mathcal{X}^{p,q}$. (Recall that $W_{\mathcal{F}} \hookrightarrow W_{\mathcal{F}'}$, whenever $\mathcal{F}' \supseteq \mathcal{F}$.) Let $E_{r,W}^{p,q}$ be the type II spectral sequence for $W^{p,q}$. As above,

\[(9.3.1) \quad E_{1,W}^{p,q} = \bigoplus_{\mathcal{F} \in \Phi_{p+2}} H^q(W_{\mathcal{F}}/(\Gamma \cap P)).\]

Let $W^*$ denote the single complex associated to $W^{p,q}$. Thus the abutment of (9.3.1) is $H^{p+q}(W^*)$.

(9.4). We now define a map of double complexes. Consider the single-column double complex

\[\hat{\mathcal{X}}^{p,q} = \begin{cases} 
C^q(\bar{X}/\Gamma) & \text{if } p = 0 \\
0 & \text{if } p > 0.
\end{cases}\]

There is an obvious map $\hat{\mathcal{X}}^{p,q} \to \mathcal{X}^{p,q}$, where for $p = 0$ we map $\omega \in C^q(\bar{X}/\Gamma)$ to the direct sum over $\mathcal{F} \in \Phi_2$ of the restriction of $\omega$ to $C^q(\bar{e}(P)/\Gamma \cap P))$.

**Lemma.** The map $\hat{\mathcal{X}}^{p,q} \to \mathcal{X}^{p,q}$ is a chain map of double complexes.

**Proof.** Let $\omega \in C^q(\bar{X}/\Gamma)$, and let $\tau$ be its image in $\mathcal{X}^{0,q}$. The only thing to check is that the horizontal arrow $\delta : \mathcal{X}^{0,q} \to \mathcal{X}^{1,q}$ carries $\tau$ to 0. But as we have said, the kernel of $\delta$ is precisely the set of elements that come from a global cochain on $\partial \bar{X}/\Gamma$. And $\tau$ does come from a global cochain, namely the restriction of $\omega$ to $\partial \bar{X}/\Gamma$. □

Considering the type I spectral sequences, we see that $\hat{\mathcal{X}}^{p,q} \to \mathcal{X}^{p,q}$ induces on the abutments the canonical restriction map $H^q(\bar{X}/\Gamma) \to H^q(\partial \bar{X}/\Gamma)$.

Next, consider

\[\hat{W}^{p,q} = \begin{cases} 
C^q(W/\Gamma) & \text{if } p = 0 \\
0 & \text{if } p > 0,
\end{cases}\]

made into a double complex the same way $\hat{\mathcal{X}}$ was. There is an obvious map $\psi : \hat{W}^{p,q} \to \mathcal{W}^{p,q}$ carrying $\omega \in C^q(W/\Gamma)$ to the direct sum over $\mathcal{F} \in \Phi_2$ of the restrictions of $\omega$ to $C^q(W_{\mathcal{F}}/(\Gamma \cap P))$. Again, $\psi : \hat{W}^{p,q} \to \mathcal{W}^{p,q}$ is a chain map of double complexes.

(9.5). We now give the main result of the paper.

**Theorem.** (i) We have a commutative diagram

\[
\begin{array}{ccc}
H^*(\bar{X}/\Gamma) & \xrightarrow{\text{restriction}} & H^*(\partial \bar{X}/\Gamma) \\
\cong \downarrow & & \cong \downarrow \\
H^*(W/\Gamma) & \xrightarrow{\psi^*} & H^*(W^*)
\end{array}
\]

where the vertical maps are natural isomorphisms and the top map is the canonical restriction map.
(ii) There is a spectral sequence

\[(9.5.1) \quad E_{p,q}^{1,\mathcal{W}} = \bigoplus_{\mathcal{F} \in \Phi_{p+2}} H^q(W_{\mathcal{F}}/(\Gamma \cap P)) \Rightarrow H^{p+q}(\mathcal{W}^*),\]

and \(\psi\) induces a map of spectral sequences given on the \(E_1\) page by the natural restriction map \(H^q(W/\Gamma) \to E_{0,\mathcal{W}}^{0,q}\).

Here \(X\) is as in (1.5), \(\Gamma\) is any arithmetic subgroup of \(\Gamma_0\) as in (2.2), \(\Phi_l\) is as in (9.2), and \(P\) is the \(\mathbb{R}\)-points of the parabolic \(\mathbb{Q}\)-subgroup associated to \(\mathcal{F}\) as in (3.2). The space \(W_{\mathcal{F}}\), a closed subcomplex of the well-rounded retract \(W\), is defined in (6.1). We use any fixed abelian group of coefficients.

Proof. First we observe that the global deformation retraction \(\bar{r}: \bar{X} \to W\) induces a map of double complexes \(X_{p,q} \to W_{p,q}\). The map \(\bar{r}\) commutes with the vertical maps of these double complexes, because any continuous map commutes with the coboundary maps in singular cohomology. More importantly, \(\bar{r}\) commutes with the horizontal maps because it is defined \(\text{globally}\) on \(\bar{X}\), and because the horizontal maps are induced from inclusion maps on various subspaces of \(\bar{X}\).

It follows that \(\bar{r}\) induces a homomorphism of spectral sequences \(E^{p,q}_{r,X} \to E^{p,q}_{r,W}\). The map on \(E_1\) terms comes from the maps \(H^q(\bar{e}(P)/(\Gamma \cap P)) \to H^q(W_{\mathcal{F}}/(\Gamma \cap P))\) induced for all \(\mathcal{F}\) by \(\bar{r}\). By Theorem 8.5, the latter maps are canonical isomorphisms. By [Br, VII.2.6], the right-hand arrow in (i) is an isomorphism.

Similarly, \(\bar{r}\) induces a map \(\bar{X}^{p,q} \to \bar{W}^{p,q}\). By Corollary 8.4, it gives a canonical isomorphism in cohomology. This is the left vertical arrow in (i).

Part (9.4) defined the maps of double complexes that induce the horizontal arrows in (i), and it explained why the top arrow is the canonical restriction. This proves (i).

The spectral sequence in (ii) is (9.3.1). The assertion about \(\psi\) in (ii) is immediate from the definition of \(\psi\). \(\square\)

Section 10—Practical Comments and Related Results

(10.1) We make some remarks about how to use the spectral sequence (9.5.1) in practice. When we work with the finite cell complex \(W/\Gamma\), we represent each cell by some piece of data. (See (10.6), [M-M0], or [A-G-G].) Since \(W_{\mathcal{F}}\) is a subcomplex of \(W\), it is tempting to imagine representing each cell of \(W_{\mathcal{F}}/(\Gamma \cap P)\) by some piece of the \(W/\Gamma\) data. However, two cells of \(W_{\mathcal{F}}\) may be equivalent mod \(\Gamma\) and yet inequivalent mod \((\Gamma \cap P)\). An example when \(\Gamma = \text{GL}_3(\mathbb{Z})\), \(\mathcal{F} = \{0 \subseteq (\ast, \ast, 0) \subseteq \mathbb{Q}^3\}\), is provided by the two 0-cells of \(W\) whose minimal vectors are given by the columns of the following matrices (together with their negatives):

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]
The cells are $\Gamma$-equivalent. (In the language of [M-M0], their $C$-configurations are both complete quadrilaterals.) But any element of $\Gamma$ preserving $F$ must carry the left-hand matrix to a matrix with three $0$’s in its bottom row.

In general, each $\Gamma$-class of cells of $W_F$ will break up into finitely many $(\Gamma \cap P)$-classes. So a cell in $W_F/(\Gamma \cap P)$ is represented by a piece of data for $W/\Gamma$, plus a finite amount of extra structure depending on $F$. Say one has worked out all the classes in (9.5.1) as explicit cocycles on the cell complexes $W_F/(\Gamma \cap P)$. To compute the $d_1$ and higher differentials, one has to work out how the data-with-extra-structure in $W_F/(\Gamma \cap P)$ is connected with the data-with-extra-structure in $W_{F’}/(\Gamma \cap P)$ for $F’ \supseteq F$. Sheafhom, a suite of object-oriented programs currently under development by the second author, may help to automate such computations.

Say that $\Gamma$ is small enough if, whenever $F, F’$ are distinct $Q$-flags and $W_F \cap W_{F’} \neq \emptyset$, there is no element of $\Gamma$ carrying $F$ to $F’$. When $\Gamma$ is small enough in this sense, we need never worry about the extra structure just described. To see why, assume there were two cells $C_a, C_b \in W_F$ which were not $(\Gamma \cap P)$-equivalent but such that $C_b = \gamma \cdot C_a$ for some $\gamma \in \Gamma$. Set $F’ = \gamma^{-1}F$; we have $F’ \neq F$ since $\gamma \notin P$. Because $C_b$ respects $F$, $C_a$ must respect $F’$. Hence $C_a \in W_F \cap W_{F’}$. Since $\Gamma$ was assumed small enough, the existence of such a $\gamma$ is a contradiction.

When $k = Q$ and $n \leq 4$, one can use the methods of [M1] to show that the principal congruence subgroups $\Gamma(N) \subset \text{GL}_n(\mathbb{Z})$ of prime level $N \geq 3$ are small enough in the above sense. We do not know whether all neat $\Gamma$ are small enough.

(10.2). For convenience, we record the homology version of our main theorem. We use the obvious analogues of the notation of (9.5).

**Theorem.** (i) We have a commutative diagram

$$
\begin{aligned}
H_*(\partial \bar{X}/\Gamma) & \xrightarrow{\text{inclusion}} H_*(\bar{X}/\Gamma) \\
\cong \downarrow & \cong \\
H_*(W_*) & \rightarrow H_*(W/\Gamma)
\end{aligned}
$$

where the vertical maps are natural isomorphisms and the top map is the canonical inclusion map.

(ii) There is a spectral sequence

$$
E_{p,q}^1 = \bigoplus_{F \in \Phi_{p+2}} H_q(W_F/(\Gamma \cap P)) \Rightarrow H_{p+q}(W_*),
$$

and $\psi$ induces a map of spectral sequences given on the $E_1$ page by the natural inclusion map $E_{0,q}^1 \rightarrow H_q(W/\Gamma)$.

This is proved by dualizing the constructions of Section 9.
Consider one Borel-Serre boundary face $\bar{e}(P)/(\Gamma \cap P)$. As we mentioned in the introduction, our methods allow us to compute the canonical maps $H^* (\bar{X}/\Gamma) \to H^* (e(P)/(\Gamma \cap P))$ and $H_* (e(P)/(\Gamma \cap P)) \to H_* (\bar{X}/\Gamma)$ without any spectral sequences. We simply use the cellular map $W_F/(\Gamma \cap P) \to W/\Gamma$.

In the application to $GL_4(\mathbb{Z})$ described near the end of (0.2), we are primarily interested in the degree-five homology. We believe that in degree five, all the homology of the boundary comes from the maximal boundary faces. We could therefore find the image of

$$
\left( \bigoplus_{\mathcal{F} \text{ for maximal faces mod } \Gamma} H_5(W_F/(\Gamma \cap P)) \right) \to H_5(W/\Gamma)
$$

without computing any spectral sequences.

It would very interesting to find the image of the map $H_* (\partial \bar{X}/\Gamma) \to H_* (\bar{X}/\Gamma)$ without having to find $H_* (\partial \bar{X}/\Gamma)$ directly, and without having to work out any spectral sequences.

All our results hold for $\Gamma$-equivariant cohomology. That is, one may replace $H^*((\ldots)/\Gamma)$ with $H^*_F((\ldots))$ in the theorems of Section 9 and throughout the paper. Of course, if we use coefficients in a field of characteristic 0 or of characteristic prime to the order of any torsion element of $\Gamma$, then the two kinds of cohomology are canonically isomorphic. All these results hold for homology.

In the $\Gamma$-equivariant setting, the summands in each column of our spectral sequences would be replaced by $H^*_F((\Gamma \cap P)/(W_F)$, and each of these would have to be computed by a spectral sequence in its own right. We would get a spectral sequence of spectral sequences.

Both $H^*_F(X)$ and $H^*(X/\Gamma)$ are interesting objects from the point of view of number theory and automorphic forms. Both have Hecke operators acting on them. In [A4], the first author has conjectured that any Hecke eigenclass in $H^*_F(X; \mathbb{F}_p)$, for any $p$, should have an attached mod $p$ representation of the absolute Galois group of $\mathbb{Q}$. In [A-M2], we have conjectured the same for the Hecke module $H^*(X/\Gamma; \mathbb{F}_p)$. Both papers provide examples.

The method of this paper can easily be extended to $SL_n$. If $G$ were $SL_n(S)$, then $K\backslash G$ would be the product of the irreducible symmetric spaces for the $SL_n(k_v)$. In the case $G = GL_n(S)$ of this paper, $HK\backslash G$ is the product of the $SL_n(k_v)$ symmetric spaces and a Euclidean factor $H\backslash (\prod_v \mathbb{R}_+^r) \cong \mathbb{R}^{r_1+r_2-1}$, where $r_1, r_2$ are the number of real and complex places of $k$, respectively.

Let $k = \mathbb{Q}$, so that $\Gamma \subseteq GL_n(\mathbb{Z})$ or $SL_n(\mathbb{Z})$, and let $n \leq 4$. In [M1] and the survey article [M-M0], an interpretation of the cell structure on the well-rounded retract $W$ is given. There is a one-to-one correspondence in which each cell corresponds to a configuration of points and lines (called a $C$-configuration) in the projective space $\mathbb{P}^{n-1}(\mathbb{Q})$. It is easy to interpret our $W_F$ in this language.

Again let $k = \mathbb{Q}$ and $n \leq 4$. Let $\bar{X}/\Gamma$ be any Satake compactification of $X/\Gamma$. In [M2], one constructs a cell structure on $\bar{X}/\Gamma$. Intersecting with $X/\Gamma$ gives a “locally-compact cell structure” on $X/\Gamma$ that is dual to the well-rounded retract $W/\Gamma$ inside $X/\Gamma$. 

(see [M1, §2] for definitions and precise statements). There is a cellular map from the barycentric subdivision of the boundary of $\tilde{X}/\Gamma$ to the barycentric subdivision of $W/\Gamma$. This offers an interesting contrast to the present paper.

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