ARITHMETIC OF PELL SURFACES

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Abstract. We define a group structure on the primitive integer points \((A, B, C)\) of the algebraic variety \(Q_0(B, C) = A^n\), where \(Q_0\) is the principal binary quadratic form of fundamental discriminant \(\Delta\) and \(n \geq 2\) is fixed. A surjective homomorphism is given from this group to the \(n\)-torsion subgroup of the narrow ideal class group of the quadratic number field \(\mathbb{Q}(\sqrt{\Delta})\).

1. Introduction

A classical technique for constructing quadratic number fields with class number divisible by \(n\) is studying integral solutions of the equation

\[X^2 - \Delta Y^2 = 4Z^n, \quad \gcd(X, Z) = 1, \quad \Delta \text{ a fundamental discriminant.}\]

For each integral point \((X, Y, Z)\) we can form the ideal \(a = (X + Y\sqrt{\Delta}, Z)\) in the ring of integers of \(\mathbb{Q}(\sqrt{\Delta})\); the ideal \(a\) has norm \(|Z|\) and satisfies \(a^n = (\frac{X + Y\sqrt{\Delta}}{2})\), hence generates an ideal class of order dividing \(n\).

It seems that P. Joubert [6] was the first to observe that a class of prime order \(n\) in the group of binary quadratic forms with negative discriminant \(\Delta\) implies the solvability of the equation \((1.1)\); Joubert used techniques from the theory of complex multiplication for exploiting this observation. Nagell [11] later used \((1.1)\) for proving the existence of infinitely many complex quadratic number fields with class number divisible by \(n\). By extending Nagell’s approach, Yamamoto [13] was able to prove the existence of infinitely many real quadratic number fields with class number divisible by \(n\).

The same approach was further extended by various authors; we mention in particular Craig [4].

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In this article, we interpret (1.1) as an affine surface and show that a certain subset $S_n(Z)$ of the integral points on (1.1) can be given a group structure in such a way that

- a) the integral points on the hyperplane $Z = 1$, which lie on the Pell conic $X^2 - \Delta Y^2 = 4$, form a subgroup with respect to the classical group structure on Pell conics (see [7, 8, 9]);

- b) there is a surjective group homomorphism $S_n(Z) \rightarrow \text{Cl}^+(K)[n]$ to the $n$-torsion of the narrow class group of $K = \mathbb{Q}(\sqrt{\Delta})$.

These results explain the success of Yamamoto’s approach, and at the same time raise a few new problems that we do not yet fully understand. The rational points on the surface lying on the hyperplane $Y = 1$ form (the affine part of) a hyperelliptic curve $E : X^2 = 4Z^n + \Delta$; in the case $n = 3$, this is an elliptic curve. Although the integral points on $E$ do not form a group in general, it was observed by Buell [2, 3] and Soleng [12] that the integral points on $E$ (and, more generally, certain rational points satisfying some technical conditions) give ideal classes of order dividing 3 in such a way that the map from $E$ to the 3-class group respects the group law on the elliptic curve, i.e., that collinear points get mapped to classes whose product is trivial. Bölling [11] has extended these results to the hyperelliptic curves lying on the surface (1.1).

![Figure 1. $S_3$ for $\Delta > 0$ with cross sections $Z = 1$ on the left, a Pell conic, and $Y = 1$ on the right, an elliptic curve.](image)

Although we will see below that the group law is best understood by using ideals in quadratic number fields, the explicit addition formulas are tied closely to the composition of binary quadratic forms. For this reason, we replace the equation (1.1) of the surface by $Q_0(X, Y) = Z^n$, where $Q_0$ is the principal form with discriminant $\Delta$ defined below.

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1. The group law on Pell surfaces was discovered by the first author, as was the homomorphism to the class group.
For a brief introduction to the composition of binary quadratic forms via Bhargava’s cubes see Lemmermeyer [10]; more details along more classical lines can be found in Flath [5].

2. Primitive Points on Pell Surfaces

Let \( \Delta \) be a fundamental discriminant (the discriminant of a quadratic number field). The principal form with discriminant \( \Delta \) is defined by

\[
Q_0(x, y) = \begin{cases} 
  x^2 - my^2 & \text{if } \Delta = 4m, \\
  x^2 + xy - my^2 & \text{if } \Delta = 4m + 1.
\end{cases}
\]

By \( Q = (a, b, c) \) we denote the binary quadratic form \( ax^2 + bxy + cy^2 \).

Such a form \( Q \) represents an integer \( d \) if \( Q(x, y) = d \) for some integers \( x, y \); it is said to represent \( d \) primitively if, in addition, \( \gcd(x, y) = 1 \).

An integral point \( (A, B, C) \) on the Pell surface

(2.1) \[ S_n : Q_0(B, C) = A^n \]

is called primitive if \( \gcd(B, C) = 1 \). The set of primitive points on \( S_n \) will be denoted by \( S_n(Z) \).

Now consider Eqn. (1.1) and map a point \( (A, B, C) \) on the Pell surface (2.1) to a point \( (X, Y, Z) \) on (1.1) by setting

\[
(X, Y, Z) = \begin{cases} 
  (2B, C, A) & \text{if } \Delta = 4m, \\
  (2B + C, C, A) & \text{if } \Delta = 4m + 1.
\end{cases}
\]

This clearly gives a bijection between the integral points on these surfaces. In addition, Yamamoto’s condition \( \gcd(X, Z) = 1 \) is easily seen to be equivalent to the primitivity of \( (A, B, C) \), that is, to \( \gcd(B, C) = 1 \).
3. The Group Law

Let $\mathcal{O}$ denote the ring of integers of the quadratic number field $\mathbb{Q}(\sqrt{\Delta})$. There is a natural map $\pi_0 : S_n(\mathbb{Z}) \to \mathcal{O}$ defined by $\pi_0(A, B, C) = B + C\omega$, where

$$\omega = \frac{\sigma + \sqrt{\Delta}}{2},$$

and $\sigma \in \{0, 1\}$ is defined by $\Delta = 4m + \sigma$. The elements in the image of $\pi_0$ have the property that their norms are $n$-th powers: $N(\pi_0(A, B, C)) = Q_0(B, C) = A^n$.

Consider the set $\mathcal{O}^*$ of nonzero elements in $\mathcal{O}$ and its subset $\mathbb{N}^n$ of nonzero natural numbers. The set $\mathcal{O}^*/\mathbb{N}^n$, using $\mathbb{N}^n$ to refer to positive integers which are $n$-th powers, is a group with respect to multiplication: the neutral element is $1_{\mathbb{N}^n}$, the inverse of $\alpha_{\mathbb{N}^n}$ is $1_{\mathbb{N}^n}/|N(\alpha)|_{\mathbb{N}^n}$ (the element $|N(\alpha)|_{\mathbb{N}^n}$ is, up to sign, simply the conjugate $\alpha'$ of $\alpha$, and so belongs to $\mathcal{O}^*$). The norm map induces a group homomorphism $N : \mathcal{O}^*/\mathbb{N}^n \to \mathbb{Z}^*/\mathbb{Z}^n$, where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ and $\mathbb{N}$ denote the monoids of nonzero and of positive integers, respectively.

Observe that if $\alpha, \beta \in \mathcal{O}^*$ are primitive (this means that $p \nmid \alpha$ for all primes $p \in \mathbb{N}$) and $\alpha_{\mathbb{N}^n} \cdot \beta_{\mathbb{N}^n} = \gamma_{\mathbb{N}^n}$, then in general $\gamma$ cannot be chosen to be primitive. An example is provided by $\alpha = 3 + \sqrt{3}$ and $\beta = \sqrt{3}$, where $\gamma = 3 + 3\sqrt{3}$; here $\gamma_{\mathbb{N}^n}$ is not generated by a primitive element for any $n \geq 2$. On the other hand we shall prove below

**Proposition 3.1.** The cosets of primitive elements in the kernel of the norm map $N : \mathcal{O}^*/\mathbb{N}^n \to \mathbb{Z}^*/\mathbb{Z}^n$ form a subgroup $\Pi_n$ of $\mathcal{O}^*/\mathbb{N}^n$.

This fact allows us to prove that there is a bijective map $\pi : S_n(\mathbb{Z}) \to \Pi_n$ given by $\pi(A, B, C) = (B + C\omega)_{\mathbb{N}^n}$; using this bijection we can make $S_n(\mathbb{Z})$ into an abelian group. The situation is summed up by the following diagram:

$$\begin{array}{ccc}
S_n(\mathbb{Z}) & \xrightarrow{\pi} & \Pi_n \\
\cong & \downarrow & \\
1 & \longrightarrow & \ker N \longrightarrow \mathcal{O}^*/\mathbb{N}^n \xrightarrow{N} \mathbb{Z}^*/\mathbb{Z}^n.
\end{array}$$

**Theorem 3.2.** The map $\pi : S_n(\mathbb{Z}) \to \Pi_n$ is bijective; thus $S_n(\mathbb{Z})$ becomes an abelian group by transport of structure.

**Proof.** Injectivity: assume that there are elements $(A, B, C), (A', B', C') \in S_n(\mathbb{Z})$ with $\pi(A, B, C) = \pi(A', B', C')$. Then there exist $a, b \in \mathbb{N}$ with $(B + C\omega)a^n = (B' + C'\omega)b^n$, and the primitivity of $B + C\omega$ and $B' + C'\omega$...
implies that \(a^n\) and \(b^n\) must be units. Since \(a, b \in \mathbb{N}\), this implies \(a^n = b^n = 1\).

Surjectivity: assume that \(\alpha = B + C\omega\) is primitive with \(\alpha \mathbb{N}^n \subset \Pi_n\). Then \(N\alpha = A^n\) for some number \(A \in \mathbb{Z}^n\) implies \(Q_0(B, C) = A^n\), hence \((A, B, C) \in S_n(\mathbb{Z})\) with \(\pi(A, B, C) = \alpha\). □

Observe that the neutral element of \(S_n(\mathbb{Z})\) is the point \((A, B, C) = (1, 1, 0)\), and that the inverse of \((A, B, C)\) is given by

\[-(A, B, C) = \begin{cases} (A, B + \sigma C, -C) & \text{if } A > 0, \\
(A, -B - \sigma C, C) & \text{if } A < 0. \end{cases}\]

Observe also that the integral points on the Pell conic \(Q_0(T, U) = 1\), which correspond to the points \((1, T, U)\) on the Pell surface, inherit their classical group structure since \((T_1 + U_1\omega)(T_2 + U_2\omega) = T_3 + U_3\omega\), where \((T_3, U_3) = (T_1T_2 + mU_1U_2, T_1U_2 + T_2U_1)\) if \(\Delta = 4m\) and \((T_3, U_3) = (T_1T_2 + mU_1U_2, T_1U_2 + T_2U_1 + U_1U_2)\) if \(\Delta = 4m + 1\). In fact, since the elements \(\alpha_j = T_j + U_j\omega\) have norm 1, the element \(\alpha_3 = \alpha_1\alpha_2\) is always primitive.

For proving Prop. 3.1 we use the following characterization of primitive elements:

**Lemma 3.3.** Let \(\alpha \in \mathcal{O}^*\) be a nonzero element of the order \(\mathcal{O}\).

a) \(\alpha\) is primitive if and only if \((\alpha) + (\alpha') = \mathfrak{d}\) for some ideal \(\mathfrak{d}\) dividing the product of all ramified primes.

b) If \(N\alpha = a^n\) for some \(n \geq 2\), then \(\alpha\) is primitive if and only if \((\alpha) + (\alpha') = (1)\).

**Proof.** Assume first that \(\alpha\) is primitive, let \(\mathfrak{p}\) be an unramified prime ideal, and set \((\alpha) + (\alpha') = \mathfrak{d}\). If we had \(\mathfrak{p} \mid \mathfrak{d}\), then \(\mathfrak{p} \mid (\alpha)\) and \(\mathfrak{p}' \mid (\alpha')\). Since \(\mathfrak{p}\) is unramified, the prime \(p\) below \(\mathfrak{p}\) either splits (and then \((p) = \mathfrak{pp}'\)), or \(\mathfrak{p} = (p)\) is inert. In both cases we deduce that \(p \mid (\alpha)\), which contradicts our assumption that \(\alpha\) be primitive.

Conversely, assume that \(\mathfrak{d}\) divides the product of all ramified primes. If \(p \mid \alpha\) for some prime \(p \in \mathbb{N}\), then \(p \mid \alpha'\), hence \(p \mid \mathfrak{d}\). This shows that \((\alpha) + (\alpha')\) is divisible either by an unramified prime ideal or by the square of a ramified prime ideal. This proves the first statement.

For proving b), assume first that \((\alpha) + (\alpha') = (1)\); then \((\alpha)\) is primitive by what we have already proved.

Finally, if \(N\alpha = a^n\) and \(\alpha\) is primitive, then \(\mathfrak{d}\) is a product of ramified prime ideals. But if \(p\mid\alpha\) for some ramified prime ideal \(\mathfrak{p}\) above \(p\), then \(p\mid \alpha\alpha' = a^n\), and this is impossible for \(n \geq 2\). □
Lemma 3.4. Let α be a primitive element. If αN^n ∈ ker N, then (α) = a^n is an n-th ideal power. The converse holds if α is totally positive.

Proof. The claim is trivial for n = 1; assume therefore that n ≥ 2.

If α is primitive and Nα = a^n, Lemma 3.3 implies that α and α' are coprime. Now (α)(α') = a^n implies that (α) = a^n is an n-th ideal power.

Now assume that (α) = a^n. Then Nα = ±A^n for some positive integer A, and since α is totally positive, we have Nα > 0. □

Proof of Prop. 3.1. Assume that α and β are primitive elements representing cosets in the kernel of the norm map. Write αβ = γa^n with γ ∈ O^* and with a ≥ 1 maximal. We have to show that γ is primitive.

Assume not; then p | γ for some rational prime p. Since p ∤ α and p ∤ β (by the primitivity of these elements), the prime p cannot be inert, and there is a prime ideal p above p with p | (α) and p' | (β). Since α and β are n-th ideal powers, we must have p^{kn}|(α) and p'^{kn}|(β), and this implies p^{kn} = (pp')^{kn}|γa^n. By the maximality of a we must have p^{kn}|a^n, and this implies p ∤ γ.

Thus Π_n is closed under multiplication; since the inverse of αN^n is ±αN^n (with the sign chosen in such a way that α · (±α') > 0), the set Π_n forms a subgroup of ker N. □

Remark. The points with A = 1 on S_n form a subgroup of S_n(Z); such points (1, B, C) correspond to units B + Cω ∈ O, and the group law is induced by the usual multiplication of units. This shows that the group law on the Pell conic Q_0(B, C) = 1 coincides with the standard group law on these curves.

4. The homomorphism S_n(Z) → Cl^+(K)[n]

We have already remarked that the set S_n(Z) was used to extract information on the n-torsion of the class group Cl(K) of the quadratic number field K = Q(√Δ). Given a point (A, B, C) ∈ S_n(Z), we know that α = B + Cω is an n-th ideal power: (α) = a^n. Sending α to the narrow ideal class of a we get a map c : S_n(Z) → Cl^+(K)[n] from S_n(Z) to the group of ideal classes (in the strict sense) in K whose order divides n:

Proposition 4.1. The map
c : S_n(Z) → Cl^+(K)[n]

is a surjective group homomorphism.
Proof. Proving that \( c \) is a group homomorphism is easy: let \( P_1 = (A_j, B_j, C_j) \in S_n(\mathbb{Z}) \) with \( P_1 \oplus P_2 = P_3 \), and put \( \alpha_j = B_j + C_j\omega \). Then \( (\alpha_j) = a_j^n \), and \( \alpha_1^{N\mathbb{N}} \cdot \alpha_2^{N\mathbb{N}} = \alpha_3^{N\mathbb{N}} \) for some \( \alpha_3 \) that differs from \( \alpha_1 \alpha_2 \) by the \( n \)-th power of some positive integer \( a \). This implies that \( a_1^n a_2^n = a_3^n \cdot a^n \), hence \( c(P_1)c(P_2) = c(P_1 \oplus P_2) \) as claimed.

For proving that \( c \) is onto, consider the narrow ideal class \([a] \in \text{Cl}^+(K)[n]\) for some ideal \( a \) coprime to the discriminant. Then \( a^n = (\alpha) \) for some \( \alpha = B + C\omega \). We claim that we can choose \( a \) in such a way that \( \alpha \) is primitive: in fact, let \( p \) be a prime dividing \( B \) and \( C \). If \( p \) is inert, then \( a = pb \), and replacing \( a \) by \( b \) does not change the ideal class. If \( (p) = p\mathfrak{P} \) is split, then we must have \( p \mid a \) and \( \mathfrak{P} \mid a \), so again \( a = pb \). Since \( a \) is coprime to the discriminant, ramified prime ideals do not divide \( a \). Since \( (\alpha) \) is principal in the strict sense, we have \( A^n = N\alpha > 0 \); writing \( \alpha = B + C\omega \) we find \( (A, B, C) \in S_n(\mathbb{Z}) \) as claimed.

Observe that \( S_n^+(\mathbb{Z}) \), the subset of all \((A, B, C) \in S_n(\mathbb{Z}) \) with \( A > 0 \), forms a subgroup of \( S_n(\mathbb{Z}) \), and that the proof above shows that the natural map \( S_n^+(\mathbb{Z}) \to \text{Cl}^+(K)[n] \) is surjective.

It is in general difficult to tell whether a point \((A, B, C) \in S_n(\mathbb{Z}) \) gives rise to an element of exact order \( n \) or not, or more generally, whether two points generate independent elements. In the following, we shall briefly recall the criterion used by Yamamoto.

To this end, we introduce a natural homomorphisms between the groups \( S_n(\mathbb{Z}) \):

**Proposition 4.2.** Assume that \( m \mid n \); then there is a group homomorphism

\[
\iota_{m \to n} : S_m(\mathbb{Z}) \to S_n(\mathbb{Z}).
\]

In order to avoid a problematic case, we let \( S_1(\mathbb{Z}) \) denote the set of all primitive points \((A, B, C) \) such that \( \gcd(A, \Delta) = 1 \); equivalently, \( B + C\omega \) is primitive and not divisible by any ramified prime ideal.

Proof. Assume that \((A, B, C) \in S_n(\mathbb{Z}) \). With \( \alpha = B + C\omega \) we have \( (\alpha) = a^m \); setting \( n = km \), we find \( (\alpha^k) = a^n \), hence \( N(\alpha^k) = (A^n)^k = A^n \). Observe that \( \alpha^k \) is primitive if \( \alpha \) is, except possibly when \( m = 1 \) and \( a \) is divisible by a ramified prime.

Setting \( \alpha^k = B' + C'\omega \), we have \((A, B', C') \in S_n(\mathbb{Z}) \). Since the map \( \iota_{m \to n} \) sending \((A, B, C) \) to \((A, B', C') \) is compatible with the group structure (in fact: if \((B_1 + C_1\omega)a_1^m \cdot (B_2 + C_2\omega)a_2^m = (B_3 + C_3\omega)a_3^m \), then raising this equation to the \( k \)-th power shows that \((B'_1 + C'_1\omega)a_1^m \cdot (B'_2 + C'_2\omega)a_2^m = (B'_3 + C'_3\omega)a_3^m \)), the claim follows. \( \square \)
As an example, consider the surface $B^2 + BC + 6C^2 = A^3$; using the point $(6, 1, -1)$ on $S_1(Z)$ we find $(1 - \omega)^3 = -11 + 5\omega$, which gives us the point $(6, -11, 5) \in S_3(Z)$.

It is desirable to have criteria for deciding whether a point $P \in S_n(Z)$ is actually a newpoint, i.e., does not come from $S_m(Z)$ for some proper divisor $m$ of $n$.

**Proposition 4.3.** Assume that $\Delta < -4$. If $P = (A, B, C) \in S_n(Z)$ and $n = mp$ for some odd prime $p$, then $P = \iota_{m \to n}(Q)$ for some $Q \in S_m(Z)$ implies that $2B + \sigma C$ is a $p$-th power modulo $q$ for every prime $q \mid A$.

**Proof.** Let $\alpha = B + C\omega$ and $(\alpha) = \alpha^n$. If $P = \iota_{m \to n}(Q)$ for some $Q \in S_m(Z)$, then $\alpha^m = (\beta)$ for $\beta = b + c\omega$ and $Q = (A, b, c)$. Thus $\alpha = \pm \beta^p = (\pm \beta)^p$ is a $p$-th power. Let $q$ be a prime dividing $A$; then $(q) = q\Omega^q$ splits in $k$, and we have $\beta \in \Omega^q$ and $\beta^q \in q$.

If $\Delta = 4m$, then $b \equiv c\sqrt{m} \mod q$, hence $\beta = b + c\sqrt{m} \equiv 2b \mod q$, $\alpha = B + C\sqrt{m} \equiv 2B \mod q$, and so $2B \equiv \alpha = \beta^p \equiv (2b)^p \mod q$. This implies $2B \equiv (2b)^p \mod q$ as claimed.

Now assume that $\Delta = 4m + 1$. Then $b + c\omega' \in q$ shows that $b + c \equiv c\omega \mod q$ (since $\omega \omega' = 1$), hence $2B + C \equiv B + C\omega = (b + c\omega)^p \equiv (2b + c)^p \mod q$.

This criterion is not very strong; it does not detect that the points $(2, 1, 1)$ or $(3, 1, 2)$ on $S_3 : B^2 + BC + 6C^2 = A^3$ are newpoints. On the other hand, $(13, 37, 6)$ must be a newpoint since $80 = 2 \cdot 37 + 6$ is not a cube modulo 13.

### 5. Explicit Formulas

Let us now make the group law on $S_n(Z)$ explicit by deriving addition formulas

\[(A_1, B_1, C_1) \oplus (A_2, B_2, C_2) = (A_3, B_3, C_3).\]

From the definition of the group law it is clear that such addition formulas must involve computations of greatest common divisors. The following lemma contains the technical part of the proof:

**Lemma 5.1.** For points $(A_j, B_j, C_j) \in S_n(Z), \ j \leq 3$, we set $\alpha_j = B_j + C_j\omega$. Let $\mathfrak{d} = (\alpha_1, \alpha_2^2)$; then $\mathfrak{d} = \varepsilon^n$ is an $n$-th ideal power, and with $e = N\varepsilon$, we have

\[\text{gcd}(B_1B_2 + mC_1C_2, B_1C_2 + B_2C_1 + \sigma C_1C_2) = e^n.\]

Conversely, the gcd on the left hand side of (5.2) is an $n$-th power, and if (5.2) holds, then $(\alpha_1, \alpha_2^2) = \varepsilon^n$ for an ideal $\varepsilon$ with norm $e$. 
Proof. Since \((\alpha_j) = \alpha_j^n\), the ideal \(d\) must be an \(n\)-th power. From \(e \mid a_1\) and \(e \mid a'_2\) we deduce that \((e^n) = (e')^n \mid (\alpha_1 \alpha_2)\), and now (5.3)

\[
\alpha_1 \alpha_2 = (B_1 + C_1 \omega)(B_2 + C_2 \omega) = B_1 B_2 + C_1 C_2 m + (B_1 C_2 + B_2 C_1 + \sigma C_1 C_2) \omega
\]

implies \(e^n \mid \gcd(B_1 B_2 + C_1 C_2 m, B_1 C_2 + B_2 C_1 + \sigma C_1 C_2)\).

If, conversely, \(p\) is a prime dividing \(d = \gcd(B_1 B_2 + C_1 C_2 m, B_1 C_2 + B_2 C_1 + \sigma C_1 C_2)\), then the primitivity of \(P_j\) implies that \((p) = pp'\) must be split in \(K = \mathbb{Q}(\sqrt{\Delta})\). If, say, \(p \mid \alpha_1\), then the primitivity of \(\alpha_1\) shows that we must have \(p' \mid \alpha_2\) and therefore \(p' \mid p_1'. \) Thus if \(p^m\) is the exact power of \(p\) dividing \(d\), then \(p^m\) is the exact power of \(p\) dividing \(\alpha_1\), and the fact that \((\alpha_1)\) is an \(n\)-th ideal power shows that \(m\) must be a multiple of \(n\). This implies that

- \(d = e^n\) must be an \(n\)-th power,
- \((e) = ee'\) is the norm of an ideal \(e\),
- and \(e^n \mid (\alpha_1, \alpha'_2)\).

This completes the proof. \(\square\)

Now we can present the explicit formulas for adding points on \(S_n(\mathbb{Z})\):

**Proposition 5.2.** For \((A_1, B_1, C_1), (A_2, B_2, C_2) \in S_n(\mathbb{Z})\) we have the addition formula (5.1), where

\[
A_3 = \frac{A_1 A_2}{e^2}, \quad B_3 = \frac{B_1 B_2 + mC_1 C_2}{e^n}, \quad C_3 = \frac{B_1 C_2 + B_2 C_1 + \sigma C_1 C_2}{e^n},
\]

with \(e\) as in (5.2).

Proof. The group law is defined via \(\alpha_1 N^n \cdot \alpha_2 N^n = \alpha_3 N^n\), where \(\alpha_3\) is required to be primitive. Equation (5.3) and Lemma 5.1 show that \(\alpha_3 = \alpha_1 \alpha_2 / e^n\). Taking norms yields \(A_3^e = A_1^e A_2^e / e^{2n}\), and this proves the claim. \(\square\)
6. From Points to Forms

Since there is a bijection between ideal classes and equivalence classes of binary quadratic forms, we can also describe the group law in terms of forms. It turns out that the geometric aspects of the description of the group law on $\mathcal{S}_n(\mathbb{Z})$ in terms of forms adds a lot to our understanding of the arithmetic of Pell surfaces and Pell conics. For this reason, we will now construct a map sending primitive points on $\mathcal{S}_n(\mathbb{Z})$ to primitive quadratic forms with discriminant $\Delta$.

Given $(A, B, C) \in \mathcal{S}_n(\mathbb{Z})$, consider the form
\[
\tilde{Q}_P = (A, 2B + \sigma C, A^{n-1}).
\]

In order to get positive definite forms if $\Delta < 0$ we now agree to replace $\mathcal{S}_n(\mathbb{Z})$ by $\mathcal{S}_n(\mathbb{Z})^+$ in this case. It is easily checked that $\text{disc } \tilde{Q}_P = \Delta C^2$; moreover, Dirichlet composition immediately shows that $\tilde{Q}_P$ is the principal form (with discriminant $\Delta C^2$). For constructing a form with discriminant $\Delta$, we have to “underive” $\tilde{Q}_P$. This process replaces a form $(a, b, c)$ with discriminant $\Delta C^2$ by an equivalent form $(a', b'C, c'C^2)$, and then maps it to $Q_P = (a', b', c')$, which is a primitive form with discriminant $\Delta$. Mapping $P \in \mathcal{S}_n(\mathbb{Z})$ to the equivalence class of the form $Q_P$ turns out to be a homomorphism $\mathcal{S}_n(\mathbb{Z}) \rightarrow \text{Cl}^+(\Delta)[n]$.

Underiving $\tilde{Q}_P$ is accomplished by changing the middle coefficient modulo $2A$ in such a way that it becomes a multiple of $C$. For motivating the following lemma, consider the equation $2B + 2Ak = 2\beta C$; dividing through by $2$ and reducing mod $A$ yields $\beta C \equiv B \mod A$, and this congruence has a unique solution. In this way we find

Lemma 6.1. Given a point $P = (A, B, C) \in \mathcal{S}_n(\mathbb{Z})$ with $B^2 - 4AC = \Delta C^2$, let $\beta$ be an integer satisfying the congruence $\beta \equiv B \mod A$; then $\beta^2 \equiv \Delta \mod A$. Define a quadratic form $Q_P = (A, 2\beta + \sigma, \gamma)$ with $\gamma = Q_0(\beta, 1)/A$. Then $Q_P$ is a primitive form with discriminant $\Delta$, and $Q_P$ is positive definite if $\Delta < 0$.

Proof. The claim concerning $\beta$ follows easily from $\beta^2 \equiv \frac{B^2}{C^2} \equiv \frac{B^2 - 4AC}{C^2} = \Delta \mod A$.

Assume now that $\Delta = 4m$, and set $A = (A, 2B, A^{n-1})$. From $\beta \equiv \frac{B}{C} \mod A$ we see that there is an integer $k$ with $\beta C = B + Ak$. Setting $S = (\begin{smallmatrix} k & k \\ 0 & 1 \end{smallmatrix})$ we find $Q' = Q|_S = (A, 2B', C')$ with $2B' = 2B + 2Ak = 2\beta C$; the integer $C'$ is determined by $(2B')^2 - 4AC' = \Delta C^2$, which gives $C' = \frac{\beta^2 - m}{A} C^2$. Setting $\gamma = \frac{\beta^2 - m}{A}$, the form $Q_1 = (A, 2\beta, \gamma)$ is primitive, has discriminant $\Delta$, and the fact that $A > 0$ implies that $Q_1$ is positive definite if $\Delta < 0$. 
The proof in the case $\Delta = 4m + 1$ is analogous; here we find $\gamma = \frac{\beta^2 + \beta - m}{A}$. \hfill \Box

Sending $P \mapsto \mathbf{Q}_P$ defines a map $b : S_n(\mathbb{Z}) \longrightarrow \text{Cl}^+(\Delta)$ between two abelian groups; we already know that the corresponding map to the ideal class group is a homomorphism, and of course the same holds for form classes. We will check the details below; now let us determine the kernel of $b$. To this end, recall how we constructed $b$: to a point $(A,B,C) \in S_n(\mathbb{Z})$ we have attached a quadratic form $\widetilde{Q}_P = (A,2B + \sigma C, A^{n-1})$ with discriminant $\Delta C^2$; this form $\widetilde{Q}_P$ is equivalent to a form $Q'_P = (A,(2\beta + \sigma)C, \gamma C^2)$, and underiving $Q'_P$ gave us $Q_P = (A,2\beta + \sigma, \gamma)$.

Now a point $P = (A,B,C) \in S_n(\mathbb{Z})$ is in the kernel if and only if $Q_P \sim Q_0$, which happens if and only if $Q_P$ represents 1. Multiplying $Q_P(x,y) = 1$ through by $C^2$ this shows that $Q'_P(Cx,y) = C^2$ (conversely, this equation implies $Q_P(x,y) = 1$). But $Q'_P$ represents $C^2$ properly if and only if the equivalent form $\widetilde{Q}_P$ does. Thus we have shown

**Proposition 6.2.** The kernel of the map $b : S_n(\mathbb{Z}) \longrightarrow \text{Cl}^+(\Delta)$ consists of all points $(A,B,C) \in S_n(\mathbb{Z})$ with the following property: there exist coprime integers $T, U$ such that $AT^2 + (2\beta + \sigma)TU + A^{n-1}U^2 = C^2$.

For deciding whether the point $(2,1,1)$ on $S_3 : B^2 + BC + 6C^2 = A^3$ is in the kernel of $b$ we have to look at $2T^2 + 3TU + 4U^2 = 1$. This equation has solutions if and only if the form $(2,3,4)$ with discriminant $-23$ represents 1, hence is equivalent to the principal form $Q_0$. This is not the case, since $(2,3,4) \sim (2,-1,6)$. We may also multiply the original equation through by 8 and complete squares; this gives $(4T + 3U)^2 + 23U^2 = 8$. This equation is clearly unsolvable in integers, but has rational solutions, such as $(T,U) = (0,\frac{1}{2})$, for example; this implies that we do not have a chance to show the unsolvability of the equation using congruences or $p$-adic methods.

**The map $b : S_n(\mathbb{Z}) \longrightarrow \text{Cl}^+(\Delta)$ is a homomorphism.** Consider a point $P = (A,B,C)$ on $S_n(\mathbb{Z})$. We know that $\alpha = B + C\omega = a^n$ for some ideal $a$ in the maximal order of $K$. For finding the form attached to $a$ we have to find an oriented $\mathbb{Z}$-basis $\{A,b + \omega\}$ of $a$. Let $c$ be an integer such that $cC \equiv 1 \mod A$; then $(A,B+C\omega) = (A,cB+cC\omega) = (A,\beta + \omega)$, where $\beta$ denotes an integer in the residue class $cB \equiv \frac{B}{C} \mod A$. It is easy to see that $\{A,\beta + \omega\}$ has the desired
properties; the form attached to $a$ then is
\[ Q_a(x, y) = \frac{N(Ax + (\beta + \omega)y)}{A} = (A, 2\beta + \sigma, \gamma), \]
where $\gamma = N(\beta + \omega)/A = Q_0(\beta, 1)/A$. In particular, $Q_a = Q_P$.

The map sending $a$ to $Q_a$ is known to induce an isomorphism between
the ideal class group of $\mathbb{Q}(\sqrt{\Delta})$ and the strict class group of forms with
discriminant $\Delta$.

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