D-branes, Quivers, and ALE Instantons

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Effective field theories in type I and II superstring theories for D-branes located at points in the orbifold $\mathbb{C}^2/\mathbb{Z}_n$ are supersymmetric gauge theories whose field content is conveniently summarized by a ‘quiver diagram,’ and whose Lagrangian includes non-metric couplings to the orbifold moduli: in particular, twisted sector moduli couple as Fayet-Iliopoulos terms in the gauge theory.

These theories describe D-branes on resolved ALE spaces. Their spaces of vacua are moduli spaces of smooth ALE metrics and Yang-Mills instantons, whose metrics are explicitly computable. For $U(N)$ instantons, the construction exactly reproduces results of Kronheimer and Nakajima.

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1. Introduction

D-branes \([1,2]\) are explicit realizations of RR charged BPS states in superstring theory. Witten \([3]\) proposed that a 5-brane in type I string theory is the zero size limit of the gauge 5-brane solution of \([4]\), built around a conventional gauge theory instanton. Furthermore, the moduli space of instantons is realized as an ADHM hyperkahler quotient.

ALE spaces are interesting because they describe the blowups of K3 singularities, and because the metrics and Yang-Mills instantons are explicitly computable.

We show that placing 5-branes at an orbifold fixed point produces an effective field theory whose vacua are points in instanton moduli space on the resolved ALE space. As in Witten’s work, the \(N = 1\) supersymmetry of the \(d = 6\) D-brane world-volume theory leads to a hyperkahler quotient description of the space of vacua. A new element of the construction is a direct identification between the NS-NS gravitational moduli and Fayet-Iliopoulos terms in the world-volume theory, which provides a very simple way for moduli which blow up the orbifold to couple to the world-volume theory. The results justify a rather surprising claim: by adding these couplings, we get an exact description of D-branes moving on the resolved ALE space.

The simplest case is \(U(N)\) instanton moduli space, for which an ADHM construction has been developed by Kronheimer and Nakajima. \([5]\) This construction describes both instanton moduli spaces and the actual metric on the ALE space.

To get this we want to start with type \(\mathbb{I}\) theory, but as is well known the theory with \(U(N)\) gauge group is anomalous. A simple way around this is to work with a well-defined theory containing \(p\) and \(p - 4\) branes with \(p < 9\). The resulting construction is identical to that of Kronheimer and Nakajima.

In a companion paper \([6]\) the type I quivers defined in this paper are used to construct \(SO(w)\) and \(USp(w)\) instantons on ALE spaces.

1.1. Overview

The body of the paper is the explicit construction of the world-volume Lagrangian for a set of D-branes located at the fixed point in the orbifold \(\mathbb{C}^2/\mathbb{Z}_n\), followed by a discussion and mathematical interpretation of the space of vacua. Section 2 reviews the closed string spectrum and gravitational moduli, and properties of the smooth ALE produced by blowing up the orbifold singularity.
The D-brane world-volume theory will be a supersymmetric gauge theory – for 5-branes, a $d = 6$, ${\mathcal{N}} = 1$ theory, and for $p < 5$ essentially its dimensional reduction (but not precisely – see section 6). Its spectrum is derived by imposing the point group and (for type I) twist projections on $U(N)$ gauge theory. Much of the work here is a careful analysis of the consistency conditions on the combined point/twist group (in section 3) and its representations (in sections 4 and 5). The results are easy to state by using quiver diagrams (introduced in subsection 4.2), and are given in figures 1-11 in sections 4 and 5.

The gauge Lagrangian is augmented by various couplings to the closed string bulk and twisted fields, derived in sections 6, 7 and the appendix. In section 7 we show that twisted moduli couple in the world-volume Lagrangian as Fayet-Iliopoulos terms, using the argument of [7]: they are supersymmetry partners of a scalar required for $U(1)$ anomaly cancellation, and also by world-sheet computation. Combining this with the quiver diagrams, we have the complete D-flatness conditions which determine the space of vacua. As is well known, these conditions are a physical realization of the hyperkähler quotient construction (subsections 6.1 and 6.5).

We proceed to compare these results with the work of Kronheimer and Kronheimer and Nakajima in sections 8 and 9, and show that these theories, derived by working in the orbifold limit, in fact describe a finite region in moduli space. In section 8 we show this for the ALE space itself, in a theory containing a single D-brane (and its orbifold images), and in section 9 for the moduli space of $U(N)$ instantons on the ALE space.

Section 10 contains conclusions.

2. Closed strings on ALE spaces

2.1. ALE spaces and orbifolds of $\mathbb{C}^2$

Here we briefly review a few properties of ALE spaces and define notation. For more information see [8] [9][10][11]. An ALE space or gravitational instanton $\mathcal{M}_\Gamma$ is a 4-manifold with anti-self-dual (hyperkähler ) metric asymptotic to $R^4/\Gamma$, where $\Gamma \in SU(2)$ is a discrete subgroup. When $\Gamma = \mathbb{Z}_n$ an explicit description of the gravitational instanton $X_n$ is available in the form of the multi-center Eguchi-Hanson gravitational instanton [8]

$$ds^2 = V^{-1}(dt + \vec{A} \cdot d\vec{x})^2 + V \, dx^2$$

$$V = \sum_{i=1}^{n} \frac{1}{|\vec{x} - \vec{x}_i|}$$

$$-\tilde{\nabla} V = \tilde{\nabla} \times \vec{A}$$

(2.1)
Here \( t \) is an angular coordinate, \( \vec{x}_i, \vec{x}_i \) are points in \( \mathbb{R}^3 \). Euclidean motions on the \( n \) vectors \( \vec{x}_i \) produce equivalent metrics, while otherwise inequivalent \( \vec{x}_i \) produce inequivalent metrics. The moduli space of such instantons is therefore the \( 3n - 6 \)-dimensional (for \( n > 2 \)) configuration space of \( n \) points in \( \mathbb{R}^3 \). In the limit \( \vec{x}_i \to 0 \), or, equivalently \( \vec{x} \to \infty \) the metric (2.1) is easily seen to degenerate to the metric on the orbifold \( \mathbb{C}^2/\Gamma \).

The coordinates in (2.1) degenerate along line segments between the \( \vec{x}_i \). In fact, for generic \( \vec{x}_i \) the manifold \( X_n \) is smooth and has nontrivial topology: \( \Gamma \) is associated with a simply-laced Dynkin diagram \( D_\Gamma \) with \( r_\Gamma \) nodes and Cartan matrix \( C_\Gamma \) in a well-known way and this appears in the homology: \( H_2(\mathcal{M}_\Gamma, \mathbb{Z}) \cong \mathbb{Z}^{r_\Gamma} \), and the intersection form \(-C_\Gamma\) identifies it with the root lattice of \( D_\Gamma \). \( \square \) A choice of ordering of the \( \vec{x}_i \) corresponds to a choice of simple roots. The cohomology group \( H^2(\mathcal{M}_\Gamma, \mathbb{Z}) \) is identified with the weight lattice and is spanned by a basis of anti-selfdual normalizable two-forms. The three covariantly constant self-dual symplectic forms \( \vec{\omega} \) are not normalizable.

For \( \Gamma = \mathbb{Z}_n \) we may choose a basis \( \Sigma_i \) for \( H_2(\mathcal{M}_\Gamma, \mathbb{Z}) \) corresponding to \( \vec{x}_i \vec{x}_{i+1} \). The periods of the three symplectic forms \( \vec{\omega} \) are [9]:

\[
\int_{\Sigma_i} \vec{\omega} = \vec{x}_{i+1} - \vec{x}_i \equiv \vec{\zeta}_i \quad (2.2)
\]

It is often convenient to let the indices take values modulo \( n \). We also often write formulae with respect to a choice of complex structure. Then the three symplectic forms become the Kähler form \( \omega^R \) and the holomorphic \((2,0)\) form \( \omega^C \). When we wish to emphasize the dependence of the manifold on the gravitational moduli we will write \( \mathcal{M}_\Gamma = X_n(\vec{\zeta}) \). The “global Torelli theorem,” [10] asserts that the periods and asymptotic behavior determine the metric uniquely. Moreover, if \( \psi \) is any automorphism of the root lattice then \( X(\psi(\vec{\zeta})) \cong X(\vec{\zeta}) \). In particular \( X(-\vec{\zeta}) \cong X(\vec{\zeta}) \). Finally, if \( \vec{\zeta} \cdot \alpha = 0 \) for a root \( \alpha \) then \( X(\vec{\zeta}) \) is singular since the 2-cycle associated to \( \alpha \) has zero volume.

There is a third point of view on ALE spaces. We may regard \( \mathbb{C}^2/\mathbb{Z}_n \) as the affine algebraic variety \( X^a + Y Z = 0 \) in \( \mathbb{C}^3 \). The singularity at the origin has a smooth resolution by an algebraic variety \( \mathbb{C}^2/\widetilde{\mathbb{Z}}_n \). From this point of view the nontrivial spheres \( \Sigma_i \) constitute the exceptional divisor of the blow-up. When \( \zeta^C = 0 \) the ALE space is biholomorphic to \( \mathbb{C}^2/\mathbb{Z}_n \), otherwise it is just diffeomorphic. This last point of view makes contact with the physical picture of resolving an orbifold singularity by turning on blowup modes.

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1. We will denote the extended Dynkin diagram and Cartan matrix by \( \tilde{D}_\Gamma \) and \( \tilde{C}_\Gamma \) respectively.
2.2. Sigma model on ALE

Let \((z^1, z^2)\) be complex coordinates on \(\mathbb{C}^2\), with world-sheet supersymmetry partners \((\psi^1, \psi^2)\) (left moving) and \((\tilde{\psi}^1, \tilde{\psi}^2)\) (right moving). We let \(\Gamma = \mathbb{Z}_n\) act with fixed point \(z = 0\), as
\[
g(z^1, z^2) = (\xi z^1, \xi^{-1} z^2). \tag{2.3}
\]
where \(\xi = \exp 2\pi i/n\). It will be useful to exhibit the original rotational symmetry \(SO(4)\) as \(SU(2)_L \times SU(2)_R\) by writing
\[
Z \equiv \begin{pmatrix} z^1 & -\bar{z}^2 \\ z^2 & \bar{z}^1 \end{pmatrix} = Z^{A\alpha}; \quad Z \rightarrow g_L Z g_R. \tag{2.4}
\]
and embedding the twist in \(SU(2)_L\). Then \(\bar{\omega} = -\frac{1}{4} \text{tr} \bar{\sigma} dZ d\bar{Z}\). The unbroken \(SU(2)_R\) acts on the sphere of complex structures of this hyperkähler manifold. The sigma model on ALE target has \((4,4)\) supersymmetry, and contains \(SU(2)_{k=1}\) current algebras on both left and right;
\[
\Psi \equiv \begin{pmatrix} \psi^1 & -\tilde{\psi}^2 \\ \psi^2 & -\tilde{\psi}^1 \end{pmatrix}; \quad \tilde{J} = \text{tr} \Psi \bar{\sigma} \bar{\Psi}^+. \tag{2.5}
\]
The orbifold sigma model may be perturbed by exactly marginal fields in the twisted sectors to obtain a family of sigma models with target \(X_n(\bar{\zeta})\). The \(N = (4,4)\) supersymmetry survives and the somewhat special holonomy \((2.3)\) in \(SU(2)_L\) becomes generic.

When the sigma model is used as part of a “compactification” of a string theory then, since \(\mathcal{M}_\Gamma\) has \(SU(2)\) holonomy the transverse \(d = 6\) field theories for type I, IIa and IIb strings on \(\mathbb{R}^6 \times \mathcal{M}_\Gamma\) have \((0,1)\), \((1,1)\) and \((0,2)\) supersymmetry respectively. The unbroken \(SU(2)_R\) becomes the \(SU(2)_R\) of \(d = 6\) supersymmetry. The left and right current algebras \((2.2)\) do not lead to symmetries of the string theory, but only of the low energy limit (and at leading order in \(\lambda\) and \(\alpha'\)). They produce independent left and right \(SU(2)_R\)’s in type IIa, while in IIb they sit in a \(USp(4)\) not manifest on the world-sheet. In type I theory, the two \(d = 6\) supersymmetries are related as \(\bar{\epsilon} = \epsilon\), and the left and right \(SU(2)_R\)’s are also related, leaving their diagonal subgroup unbroken. We will now list the massless closed string spectrum for the three theories under consideration in terms of their quantum numbers under
\[
[SU(2) \times SU(2)]_{\text{littlegroup}} \times SU(2)^{\text{diag}}_R; \tag{2.3}
\]
their Kaluza-Klein origin, and their orbifold realization. Although this is standard and straightforward it will be useful to have a summary of these states.

\[\footnote{We are not really discussing compactification since the ALE space is noncompact. Thus, the 6d theory will have a continuous spectrum of particles. We will concentrate on the modes which would be part of a massless spectrum if the ALE space were compactified. For example, one may imagine that the ALE space serves as a local description of a singularity in a K3 manifold.} \]
2.3. Massless spectrum: IIa

We assume the \( SU(2)_L \) holonomy on the ALE space space is generic. Under the decomposition of transverse Lorentz groups \([SU(2) \times SU(2)] \times SU(2)_L \times SU(2)_R \subset SO(8)\) we have the decompositions:

\[
8_v = (2, 2; 1, 1) + (1, 1; 2, 2) \\
8_s = (2, 1; 2, 1) + (1, 2; 1, 2) \\
8_c = (2, 1; 1, 2) + (1, 2; 2, 1)
\]

\(N = (1, 1)\) representations are most conveniently summarized by the \([SU(2) \times SU(2)]_{\text{littlegroup}} \times SU(2)_R\) content of the bosonic fields. The untwisted sector contains the \((1, 1)\) gravity multiplet

\[
\text{NS-NS: } (3, 3; 1) + (3, 1; 1) + (1, 3; 1) + (1, 1; 1) \\
\text{R-R: } (2, 2; 1) + (2, 2; 3)
\]

with a \((1, 1)\) matter multiplet:

\[
\text{NS-NS: } (1, 1; 1) + (1, 1; 3) \\
\text{R-R: } (2, 2; 1)
\]

The triplet of scalars \((1, 1; 3)\) is obtained from the KK reduction of \(B\) along the three SD symplectic forms \(\vec{\omega}\).

In addition there are \((n - 1)\) \(\mathcal{N} = (1, 1)\) matter multiplets associated to two-cycles \(\Sigma_k\) of (2.2). In the NS sector the state \((1, 1; 1)\) is obtained by KK reduction \(b_k^{(0)} = \int_{\Sigma_k} B\). The triplet \((1, 1; 3)\) states are associated to the independent complex structure and Kähler deformations which change \(\vec{\zeta}_k\). We denote the associated scalar fields by \(\vec{\phi}_k\). The RR vectors come from KK reduction: \(\delta C_k^{(1)} = \int_{\Sigma_k} 10 C^{(3)}\). Due to the many occurrences of RR differential forms in various dimensions we have adopted the notation \(dC^{(q)}\) to denote a \(q\)-form field in \(d\)-dimensions.

When \(\vec{\zeta} \to 0\) \(X_n(\vec{\zeta})\) reduces to an orbifold and one can write the vertex operators for the above states explicitly. Of particular interest are the fields \(\vec{\phi}_k\) which will come from the NS-NS twisted sectors.

We denote states and fields in the sector twisted by \(z_1(2\pi) = \xi^j z_1(0)\) as (for example) \(\vec{\phi}_j\). It will turn out (in section 8) that this twisted sector basis is Fourier dual to the basis of two-cycles. Since \(\det g = 1\) the lowest dimension NS-NS twist field in each sector has \((h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})\). Taking the twist to act as (2.4) on both \(\psi^i\) and \(\bar{\psi}^i\) gives us the massless fields

\[
\vec{\phi}_k^{AB}(p) \left( \begin{array}{c} \bar{\psi}_{1 - 1/2 + k/n} \\ -\psi_{1 - 1/2 + k/n} \end{array} \right)^A \otimes \left( \begin{array}{c} \bar{\psi}_{1/2 + k/n} \\ \psi_{1/2 + k/n} \end{array} \right)^B |p; k; NS, NS\rangle, 1 \leq k < n/2
\]

\[
\vec{\phi}_k^{AB}(p) \left( \begin{array}{c} \psi_{1 - 1/2 + (n-k)/n} \\ -\bar{\psi}_{1 - 1/2 + (n-k)/n} \end{array} \right)^A \otimes \left( \begin{array}{c} \bar{\psi}_{1/2 + (n-k)/n} \\ \psi_{1/2 + (n-k)/n} \end{array} \right)^B |p; k; NS, NS\rangle, n/2 \leq k \leq n
\]
Here \( \psi \)'s are worldsheet fermions, on which tilde denotes right-mover. The spacetime fields \( \tilde{\phi}^{AB}_k \) are complex fields satisfying the reality condition

\[
\tilde{\phi}^{AB}_k = \epsilon^{AC} \epsilon^{BD} (\bar{\tilde{\phi}}^{CD}_{n-k})^* .
\]  

The result for \( k = n/2 \) is obtained by quantizing the Clifford algebra of zero modes, choosing a ground state and applying the GSO projection. Choosing the ground state to be annihilated by the imaginary parts of all fermions in (2.7), it will be the limit \( k \to n/2 \) of (2.7), again satisfying (2.8). Together with the twisted RR sectors we get the bosons of \( n - 1 \) matter multiplets as described above.

Since the \( SU(2)_L \) holonomy is nongeneric for \( \vec{\zeta} = 0 \) there will be additional matter multiplets in the 3 of \( SU(2)_L \), which can massless in the orbifold limit. This produces an extra 3 multiplets for \( \mathbb{Z}_2 \) and 1 extra multiplet for \( \mathbb{Z}_n, n > 2 \). Note that these are ‘bulk’ modes and thus non-normalizable on \( \mathcal{M}_\Gamma \).

2.4. Massless spectrum: IIB

Repeating the above discussion for the IIB string we have the \( \mathcal{N} = (2, 0) \) gravity multiplet:

- NS-NS: \((3, 3; 1) + (1, 3; 1)\)
- R-R: \((1, 3; 1) + (1, 3; 3)\)

In the untwisted sector there are two matter multiplets. The first, containing the self-dual projection \( B^+ \) of \( B_{\mu \nu} \) and the dilaton is:

- NS-NS: \((3, 1; 1) + (1, 1; 1)\)
- R-R: \((1, 1; 1) + (1, 1; 3)\)

The \((1, 1; 3)\) RR states come from KK reduction of the two-form \( 10 C^{(2)}(x, y) = 6 C^{(0)}_a(y) \omega^a(x) + \cdots \) along \( \tilde{\omega} \).

The second matter multiplet containing the internal volume and the reduction of \( B \) along \( \tilde{\omega} \) is

- NS-NS: \((1, 1; 1) + (1, 1; 3)\)
- R-R: \((3, 1; 1) + (1, 1; 1)\)

As in the IIA theory there are \((n - 1)\) matter multiplets associated to the 2-cycles \( \Sigma_k \).

The NS-NS states \((b^{(0)}_k, \tilde{\phi}_k)\) are obtained exactly as in the IIA case. The RR fields in \((3, 1; 1) + (1, 1; 1)\) are obtained from projection of the 10d RR forms along \( \Sigma_k \):

\[
\int_{\Sigma_k} 10 C^{(2)} = 6 C^{(0)}_k \quad \int_{\Sigma_k} 10 C^{(4)} = 6 C^{(2)}_k
\]  

In the orbifold limit the NS-NS states are obtained exactly as in (2.7).
2.5. Massless spectrum: I

Making an orientation projection on the Ibstring gives the massless closed string sector of the type I string. The untwisted sector gives a \((1, 0)\) gravity multiplet, a tensor multiplet \(((3, 1; 1) + (1, 1; 1))\), a hypermultiplet, and additional hypermultiplets on orbifolds.

Applying the \(\Omega\) projection to the states (2.7) in a twisted sector gives a (linear) hypermultiplet: Using (2.5) and (2.7), we see that the NS-NS scalars form a \((1, 1; 3)\). These are the metric moduli which change \(\vec{\zeta}_k\). The fourth scalar in the \((1, 1; 1)\) is the RR state \(C_k^{(0)} = \int_{\Sigma_k} C^{(2)}\).

3. Adding Dirichlet 5-branes.

We define D-branes on orbifolds of \(\Phi^2\) by first defining D-brane configurations on \(\Phi^2 \times \mathbb{R}^6\) and then extending the action of the orbifold point group to the open string sectors. If \(x\) is an allowed endpoint for open strings, all of its images under the point group must also be allowed endpoints – thus each D-brane will be represented by the set of its images under the point group. Such a formalism has recently been discussed by Gimon and Polchinski [12], and we review and add to their results here.

A \(D_p\)-brane relates the two supersymmetries of type II theory as \(\tilde{\epsilon} = \Gamma_D \epsilon\), where \(\Gamma_D = \epsilon_{\mu_1...\mu_{p+1}} \Gamma^{\mu_1} \ldots \Gamma^{\mu_{p+1}}\) and \(\epsilon_{\mu_1...\mu_{p+1}}\) is the \(p+1\)-dimensional volume form \(\epsilon_{\mu_1...\mu_{p+1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}}\). Thus the maximal supersymmetry in the world-volume theory after orbifold projection will be \(N = 1\) in \(d = 6\).

One can add several D-branes and preserve this supersymmetry if the conditions \(\tilde{\epsilon} = \Gamma_D \epsilon\) are compatible. The theories we consider of parallel \(p\)-branes contained within \(p + 4\)-branes are such a case.

We first treat the subsector of \(N\) 5-branes, each filling \(\mathbb{R}^6\) and located at a point \(x\) in \(\Phi^2\). Each open string sector is labelled by a Chan-Paton index \(i\) at each end. Let \(S\) be the set of these \(N\) indices and \(V \equiv \mathbb{C}^N\). In type II theory each index \(i \in S\) will label a single D-brane, whose position will be \(x(i)\). Let \(A_\mu(x)\) be a \(n \times n\) hermitian matrix gauge field related to an open string state as

\[
|A\rangle = A_\mu(x)^{ij} \psi^\mu |0_{NS}; i, j\rangle. \tag{3.1}
\]

For \(p < 9\), let \(X_I\) be the \(N \times N\) matrix of scalars produced by dimensional reduction.

To define the orbifold, we must define an action \(\gamma\) of the point group \(G_1\) on \(S\), correlated with the positions of the D-branes in space: \(g(x(i)) = x(\gamma(g)(i))\). The resulting theory will be a truncation of the original super Yang-Mills theory and we will describe
this in terms of the action of the point group on the gauge fields $A$ and scalars $X$ of this theory. The action must preserve the inner product $\text{tr} \ A^\dagger B$, and the string joining interaction $AB$. Thus $\gamma(g)$ must act as

$$g : A_\mu(x) \rightarrow \gamma(g) \ A_\mu(x') \gamma(g)^{-1}$$

(3.2)

with $\gamma(g)$ unitary. The action on fields with a vector index in $\mathbb{C}^2$ also includes a rotation on the space indices,

$$g : X^I(x) \rightarrow R(g)^I_J \ \gamma(g)X^J(x')\gamma(g)^{-1}.$$  

(3.3)

Fields surviving the orbifold projection are invariant under the action (3.2) (3.3) and hence the unbroken gauge symmetry will be the commutant of this representation in $U(N)$.

We now extend the Chan-Paton indices $i \in \mathcal{S}$ to label the entire set of D-branes. In all cases, the action (3.2) applies for fields with indices transverse to the orbifold, while (3.3) applies to fields with vector indices in the orbifold.

The theory now contains “DN” open string sectors with one end on a $p + 4$-brane and the other on a $p$-brane. For $p = 5$ these will produce massless hypermultiplets, whose scalars transform in the doublet of $SU(2)_R$. As discussed in [3][13][14] the fields carry an $SU(2)_R$ index $A$ from quantization of the zeromodes of $\psi^{6,7,8,9}$ in addition to a $p+4$-brane index $M$ and a $p$-brane index $m$. This gives scalar fields $h^{Am}_M$ for strings oriented from the $p+4$ to the $p$-brane and $\tilde{h}^{AM}_m$ for strings oriented the other way. The two orientations are related by a reality condition:

$$\epsilon^{AB}(h^{Bm}_M(x))^* = (\tilde{h}^{AM}_m(x)).$$

(3.4)

The point group does not act on $SU(2)_R$, so invariant DN fields satisfy:

$$h^A(x) = \gamma(g)h^A(x')\gamma(g)^{-1}.$$  

(3.5)

Defining a type I theory requires introducing 9-branes, and giving the action of the orientation reversal $\Omega$. This acts on the fields of a general $p$-brane theory as

$$A_\mu(x) = -\gamma(\Omega)A^\dagger_\mu(x)\gamma(\Omega)^{-1}$$

$$X^I(x) = \gamma(\Omega)X^{I,\dagger}(x)\gamma(\Omega)^{-1}$$

$$\epsilon^{AB}(h^{Bm}_M(x))^* = (\tilde{h}^{A}_M)^{tr}(x) = \alpha i(\gamma(\Omega))_{mm'}h^{Am'}_{M'}(x)(\gamma(\Omega)^{-1})^{M'M}.$$  

(3.6)

where $\alpha = \pm 1$. The relative minus sign between $A$ and $X$ is determined by standard world-sheet considerations, while the $\pm i$ in the action on $h$ was explained by Gimon and Polchinski [12]. $\gamma(\Omega)$ must also be unitary. We may absorb $\alpha$ into the definition of $\gamma(\Omega)_{mm'}$. 


3.1. A remark on consistency conditions

We now examine the consistency conditions on the matrices $\gamma(g), \gamma(\Omega)$. We will restrict attention to algebraic consistency conditions, and not consider consistency conditions following from tadpole cancellation. Such conditions are generally of the form $0 = \int dH = \sum \text{(sources)}$ where the integral is zero on a compact space, and one justification for this neglect is that we are working with a non-compact space. Configurations which do not cancel the tadpole are sensible configurations with non-zero charge.

This is not completely satisfactory as there are configurations which cannot be interpreted this way. For example, the original type I anomaly cancellation which required $SO(32)$ (on $\mathbb{R}^{10}$) is phrased as a cancellation between 9-brane and non-orientable closed string tadpoles. These produce a zero-form on the right which is not a source of a physical field. One might also be interested in studying instantons on compact spaces.

To deal with these situations, one can lower the dimensions of the D-branes, so that they occupy a subspace of $\mathbb{R}^6$, and can serve as physical sources. The resulting world-volume theories are essentially dimensional reductions of 5 and 9-brane theories, with the same supersymmetry. In type $\mathbb{I}$ this is easy, while in type I to make complete sense of this one must consider an orientifold of $\mathbb{R}^6$ not containing the $\Omega$ projection, and preserving some supersymmetry, such as the T-dual of type I [1] in four of the six dimensions. One is free to take the D-branes away from the new fixed points.

The conclusion is that for the purpose of studying moduli spaces of D-brane configurations (in up to six dimensions), the tadpoles can be ignored.

3.2. Algebraic consistency conditions

We begin with some general remarks. Consider a string theory in the soliton sector where D-branes are wrapping various supersymmetric cycles $B_i$. The one-string Hilbert space includes sectors associated to pairs of wrapped cycles: $\mathcal{H}_{B,B'}$. These are called “DD sectors” for $B = B'$ and “DN sectors” otherwise. If $B$ has $n$-wrapped D-branes then we associate a vector space $V_B = \mathbb{C}^n$ to the cycle and the Hilbert space is of the form:

$$\mathcal{H}_{B,B'} = \mathcal{H}_B \otimes \mathcal{H}_{B'} \otimes \text{End}(V_B, V_B')$$

for strings oriented from $B$ to $B'$. Here $\mathcal{H}_{B,B'}$ is a chiral conformal field theory specified by boundary conditions (see for example, [15]) while $\text{End}(V_B, V_B')$, the space of all linear transformations, are just the Chan-Paton factors.  

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3 It is argued in [12] that choosing a subspace leads to inconsistent dynamics.
The action of the orbifold group $G_{orb}$ on $\hat{\mathcal{H}}$ is defined by
\[
\hat{U}(g)(\phi \otimes \lambda) = (U_{BB'}(g) \cdot \phi) \otimes \gamma_B(g)\lambda_{BB'}(g)^{-1}
\] (3.8)

In the IIB string we can make a further projection by the orientation operator $\Omega$. The full orbifold group $G^{tot}$ defining the type I theory on an orbifold is then a $\mathbb{Z}_2$ extension of $G_{orb}$:
\[
1 \rightarrow \mathbb{Z}_2 \rightarrow G^{tot} \rightarrow G_{orb} \rightarrow 1
\]

In this paper, we will make the minimal assumption that the extension is trivial $\mathbb{Z}_2 \times G_{orb}$. The action of $\Omega$ then takes the form:
\[
\hat{U}(\Omega)(\phi \otimes \lambda) = (U_{BB'}(\Omega) \cdot \phi) \otimes \gamma_B(\Omega)\lambda_{BB'}(\Omega)^{-1}
\] (3.9)

The algebraic consistency conditions state that the representation $\hat{U}$ must be anomaly-free, that is, it must give a true (not projective) representation of the orbifold group $\mathbb{Z}_2 \times G_{orb}$. Let us consider first the DD sectors. The action of the group on the CP factors is adjoint so the representation on $\mathcal{H}_{BB}$ must be anomaly free. By Schur’s lemma $\gamma_B$ must satisfy the relations of the group up to scalar factors. Specializing to the case $\mathbb{Z}_2 \times \mathbb{Z}_n$ we find the conditions:
\[
\Omega^2 = 1 : \quad \gamma_B(\Omega) = \chi_B(\Omega)\gamma_B(\Omega)^{tr}
\]
\[
\Omega g = g\Omega : \quad \gamma_B(g)\gamma_B(\Omega)\gamma_B(g)^{tr} = \chi_B(g, \Omega)\gamma_B(\Omega)
\]
\[
g^n = 1 : \quad \gamma_B(g)^n = \chi_B(g)1
\] (3.10)

where $\chi_B(\Omega), \chi_B(g, \Omega), \chi_B(g)$ are scalars. The choice of $\gamma$-matrices is of course not unique. First, a unitary change of basis on the Chan-Paton spaces $V_B$ acts by
\[
\gamma(g) \rightarrow U\gamma(g)U^{-1}
\]
\[
\gamma(\Omega) \rightarrow U\gamma(\Omega)U^{tr}
\] (3.11)

Second, we may redefine the matrices $\gamma$ by a scalar factor $\gamma_B \rightarrow \epsilon_B(g)\gamma_B(g)$ etc.

Now let us consider the consistency conditions on the $\chi$-factors. Since $\gamma$ are unitary matrices, all such factors are phases. Consistency requires $\chi_B(\Omega) = \pm 1$ and that $\chi_B(g, \Omega)$ is an $n^{th}$ root of 1. By rescaling $\gamma(g)$ we can set $\chi_B(g) = 1$. This still leaves the freedom of rescaling $\gamma(g)$ by an $n^{th}$ root of unity which changes $\chi_B(g, \Omega) \rightarrow \xi^2\chi_B(g, \Omega)$. Thus we can set:
\[
\chi_B(g, \Omega) = \begin{cases} 1 & \text{(n odd)} \\ 1, \xi & \text{(n even)}. \end{cases}
\] (3.12)

---

4 Note that our conventions for the action on the Chan-Paton factors differ slightly from [12].
Different choices in (3.12) lead to different physics.

There are further consistency conditions on the $\chi$’s following from considerations of the sectors $\hat{\mathcal{H}}_{BB'}$ with $B \neq B'$, i.e., the “DN sectors.” In these sectors two interesting new subtleties can occur. First it can happen that it is not the group $\mathbb{Z}_2 \times G_{\text{orb}}$ but actually a nontrivial extension $\mathcal{G}$ of the orbifold group:

$$1 \to K \to \mathcal{G} \to \mathbb{Z}_2 \times G_{\text{orb}} \to 1$$

which acts separately on the conformal field theory and Chan-Paton factors in such a way that $K$ acts trivially on the product $\hat{\mathcal{H}}$. Second, the group $\mathbb{Z}_2 \times G_{\text{orb}}$ (or an extension of it) can have a projective (=anomalous) action on the separate factors as long as the combined representation $\hat{U}$ is nonanomalous.

An example of the first subtlety has been discussed by Gimon and Polchinski [12]. In $(p+4,p)$ sectors the $\mathbb{Z}_2$ orientation group generated by $\Omega$ is extended to $\mathbb{Z}_4$:

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1$$

when acting on the CFT and Chan-Paton DN factors separately. Indeed, [12] showed that locality of the operator product expansion implies that $\Omega^2$ acts by $-1$ on the CFT space $\mathcal{H}_{p+4,p}$. The requirement that the group $K$, generated by $\Omega^2$, act trivially on $\hat{\mathcal{H}}_{p+4,p}$ then implies:

$$\chi_{p+4}(\Omega) = -\chi_p(\Omega) \quad (3.13)$$

As mentioned above, this is the source of the factor $\alpha i$ in (3.9).

The second subtlety entails the existence of a group cocycle $\epsilon_{BB'} \in H^2(\mathbb{Z}_2 \times G_{\text{orb}}, \mathbb{C}^*)$ in the action on the CFT factor. Cancellation of anomalies then requires

$$\epsilon_{BB'}(g,\Omega)\epsilon_{BB'}(\Omega,g) = \chi_B(g,\Omega)\chi_B^{-1}(g,\Omega) \quad (3.14)$$

In this paper we will restrict attention to the simplest case

$$\chi_B(g,\Omega) = \chi_{B'}(g,\Omega) \quad . \quad (3.15)$$

It would be very interesting to see if the more general possibility (3.14) defines consistent string theories. These would be new discrete parameters needed to specify backgrounds, analogous to [16] [17] [18].

---

5 If we add the condition of tadpole cancellation then in addition $\chi_9(\Omega) = +1$. 11
4. Quiver Diagrams and the DD spectrum of the \( p \)-brane at the fixed point

The general situation is best discussed at a point of maximal symmetry: we locate a set of D-branes at the fixed point, choose an action of the point group, compute the massless spectrum, and then give a geometrical interpretation to the resulting moduli. In this section we consider only the DD sectors.

4.1. Type II

We first discuss a type II theory and a subsector of \( p \)-branes of definite \( p \). This sector is determined by a choice of unitary representation of \( \mathbb{Z}_n \), \( V^{(p)} \). This will be a sum of one-dimensional irreps \( R_i \) on which the generator \( g \) of \( \mathbb{Z}_n \) acts as \( \xi^i \), so the representation is determined by the vector of their multiplicities \( v_i^{(p)} \),

\[
V^{(p)} = \oplus_{i=0}^{n-1} v_i^{(p)} R_i = \oplus_{i=0}^{n-1} V_i^{(p)},
\]

with \( v^{(p)} = \sum_i v_i^{(p)} \). The gauge symmetry \( U(v^{(p)}) \) is broken to

\[
G = \otimes_i U(v_i^{(p)}).
\]

We will use a bi-index notation \( A_{i\alpha, j\beta} \) (with \( 0 \leq i, j \leq n - 1 \) and \( 1 \leq \alpha_i, \beta_i \leq v_i \)) for a matrix in the adjoint of \( U(v) \), and usually abbreviate this to \( A_{i\alpha, j\beta} \). The massless gauge fields then satisfy the projection

\[
A_{i\alpha, j\beta} = \xi^{i-j} A_{i\alpha, j\beta}
\]

leaving \( A_{i\alpha, j\beta} \) with \( i = j \).

The projection (3.3) on the hypermultiplets is just as easy to solve. We assemble \( X^I \) into two scalar components \( X, \bar{X} \) diagonalizing the action of \( R \). Then:

\[
\begin{align*}
X_{i\alpha, j\beta} &= \xi^{i-j+1} X_{i\alpha, j\beta} \\
\bar{X}_{i\alpha, j\beta} &= \xi^{i-j-1} \bar{X}_{i\alpha, j\beta}
\end{align*}
\]

so \( X \) will be “block off-diagonal,” the nonzero blocks being \( X_{i,i+1}, \bar{X}_{i+1,i} \):

\[
X = \begin{pmatrix}
0 & X_{01} & 0 & 0 & \cdots \\
0 & 0 & X_{12} & 0 & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
X_{n-1,0} & 0 & \cdots & 0 \\
0 & 0 & \cdots & \bar{X}_{0,n-1}
\end{pmatrix}
\]

\[
\bar{X} = \begin{pmatrix}
\bar{X}_{10} & 0 & 0 & \cdots \\
0 & \bar{X}_{21} & 0 & \cdots \\
& \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]
Moreover, under the gauge group (4.2) these scalars transform in the representations:

\[ X_{i,i+1} \in \bar{v}_{i+1} \otimes v_i \cong \text{Hom}(V_{i+1}, V_i) \]
\[ 
\bar{X}_{i+1,i} \in \bar{v}_i \otimes v_{i+1} \cong \text{Hom}(V_i, V_{i+1}) \]

(4.6)

Together, these two matrices of scalars comprise a matrix of hypermultiplets.

4.2. Quiver diagrams

The field content of the SYM theory on the p-brane may be summarized using a “quiver diagram.” In these diagrams we associate vector multiplets with vertices and hypermultiplets with links. A vertex will be associated with both a vector space \( V \), and the semisimple component of the gauge group which acts on \( V \). An oriented link from vertex \( V_1 \) to \( V_2 \) represents a complex scalar transforming in the representation \( \bar{V}_1 \otimes V_2 \cong \text{Hom}(V_1, V_2) \). Two links with opposite orientation comprise a single hypermultiplet. Thus, for example, \((X_{i,i+1}, \bar{X}_{i+1,i})\) form hypermultiplets. The field content is summarized in fig. 1.

It is worth remarking that, although this paper focuses on the case \( \Gamma = \mathbb{Z}_n \), in fact most of the results should generalize to arbitrary A-D-E ALE spaces. These other spaces will be obtained from nonabelian orbifolds. In the other cases the diagram fig. 1 will be replaced by the extended Dynkin diagram \( \tilde{D}_\Gamma \).

4.3. Canonical form for \( \gamma(\Omega) \)

The type I effective theory can be derived by further imposing the \( \Omega \) projection. We must find the most general solution of (3.10) up to unitary transformations. We will work in the basis with \( \gamma(g) \) diagonal, and the last condition of (3.10) then requires

\[ (\gamma(\Omega))_{i\alpha,j\beta} = \chi(g, \Omega) \xi^{i+j} (\gamma(\Omega))_{i\alpha,j\beta}. \]

(4.7)

Forcing \( \gamma(\Omega) \) to have nonzero blocks only for \( \chi(g, \Omega) \xi^{i+j} = 1 \). We may still use the freedom to do transformations

\[ \gamma(\Omega) \rightarrow U \gamma(\Omega) U^{tr} \]

(4.8)

with \( U \in \otimes_i U(v_{i}^{(p)}) \) to put \( \gamma(\Omega) \) into canonical form. The unbroken gauge group is then determined from

\[ U\gamma(\Omega)U^{tr}\gamma(\Omega)^{-1} = 1 \]

(4.9)

We first consider the case \( \chi(g, \Omega) = +1 \). The non-zero blocks are

\[ (\gamma(\Omega))_{i\alpha: (n-i)\beta} = \chi(\Omega) (\gamma(\Omega))^{tr}_{(n-i)\alpha:i\beta} \]

(4.10)
Fig. 1: A type II quiver diagram for D-branes transverse to $X_n$, $n = 1, 2, 3, \ldots$. This figure represents the field content of the SYM theory on the transverse 3- or 4-brane. At the vertices we have vectormultiplets in the gauge group indicated, while on the links we have hypermultiplets in representations determined by the fundamental representation at each vertex. Such a diagram with oriented edges will be called a quiver diagram.

$(i = n$ is identified with $i = 0)$. For $i \neq 0$ and $i \neq n/2$ ($n$ even), the condition relates two
different blocks, and we require $v_i = v_{n-i}$. Its general solution can be reduced to

\[
\begin{align*}
(\gamma(\Omega))_{ia:(n-i)\beta} &= \delta_{\alpha,\beta} \quad 0 < i < n/2 \\
(\gamma(\Omega))_{(n-i)\alpha;i\beta} &= \chi(\Omega)\delta_{\alpha,\beta} \quad n/2 < i < n
\end{align*}
\]  

(4.11)

For $i = 0$ or $i = n/2$, the condition relates $(\gamma(\Omega))_{ia,i\beta}$ to its transpose. By a transformation (4.8), this can be reduced to $\delta_{\alpha,\beta}$ if $\chi(\Omega) = +1$, while if $\chi(\Omega) = -1$, $v_0$ (or $v_{n/2}$) must be even and $\gamma(\Omega)$ can be reduced to the canonical skew-symmetric form $\epsilon_{\alpha\beta}$. *

For $\chi(g, \Omega) = \xi$, (4.10) is changed to

\[
(\gamma(\Omega))_{ia,(n+1-i)\beta} = \chi(\Omega) (\gamma(\Omega))_{(n+1-i)\alpha,i\beta}.
\]

(4.12)

For $n$ even, the blocks $i, n + 1 - i$ and $n + 1 - i, i$ related by these conditions are always distinct. Thus we require $v_i = v_{n+1-i}$. Moreover, the blocks can always be diagonalized as in (4.11).

4.4. Type I Quiver diagrams

We now list the unbroken gauge groups for the effective theory on a $p$-brane world-volume surviving after the orbifold and orientation projections. There are five cases to consider:

I. $\chi(\Omega) = +1$, $\chi(g, \Omega) = 1$. $n$ odd.

\[
\gamma(\Omega) = \begin{pmatrix}
1_{v_0} & & & & \\
& 1_{v_2} & & & \\
& & 1_{v_2} & & \\
& & & 1_{v_1} & \\
& & & & 1_{v_1}
\end{pmatrix} \quad (n = 5)
\]

\[
G_1(\vec{v}) \equiv O(v_0) \times [U(v_1) \times U(v_2) \times \cdots U(v_{(n-1)/2})]
\]

\[
= \{(U_0, U_1, \ldots, U_{n-1}) : U_iU_{n-i}^{tr} = 1 \quad 1 \leq i \leq n-1\}
\]

Moreover $V_i = V_{n-i}$ and the conditions on the hypermultiplets are:

\[
(X_{n-i-1,n-i})^{tr} = X_{i,i+1} \in \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_{i+1}})
\]

\[
(X_{n-i+1,n-i})^{tr} = X_{i,i-1}
\]

(4.14)

* To see this for $\chi(\Omega) = +1$, write $\gamma(\Omega) = M + iN$ with $M$ and $N$ real. Using $\gamma(\Omega)^{-1} = \gamma(\Omega)^+$ and the first line of (3.10) one can show that $[M, N] = 0$ and are both symmetric, so can be simultaneously diagonalized by (4.8) with $g$ orthogonal. Finally, (4.8) with $g$ diagonal can be used to reduce the eigenvalues to 1. The argument for $\chi(\Omega) = -1$ is very similar.
Fig. 2: A type I quiver diagram for $X_n$, $n$ odd, $\chi(\Omega) = +1$. Note that the hypermultiplet field content is determined by $V_0 = V^{v_0}$. For $\chi(\Omega) = -1$ replace $O(v) \rightarrow USp(v)$. Double arrows on the edges have been suppressed.

The field content is summarized by the quiver diagram fig. 2. The above conditions may be interpreted as saying that the diagram is symmetrical under reflection about a vertical line through the vertex $V_0$.

I.2. $\chi(\Omega) = -1, \chi(g, \Omega) = 1$. $n$ odd. This is very similar to case I.1. We have a slightly different form for $\gamma(\Omega)$:

$$
\gamma(\Omega) = \begin{pmatrix}
\epsilon_{v_0} & 1_{v_1} \\
1_{v_2} & -1_{v_1}
\end{pmatrix} \quad (n = 5)
$$

$$
G_2(\vec{v}) \equiv USp(v_0) \times [U(v_1) \times U(v_2) \times \cdots U(v_{(n-1)/2})] \\
= \{(U_0, U_1, \ldots, U_{n-1}) : U_i U_{n-i}^{tr} = 1 \quad 1 \leq i \leq n-1\}
$$

We again have $V_i = V_{n-i}$, but the conditions on the hypermultiplets become more...
complicated:

\[ X_{01} = -(X_{n-1,0} \epsilon v_0)^{tr} \]
\[ X_{i,i+1} = (X_{n-i-1,n-i})^{tr} \quad 1 \leq i \leq \frac{n-3}{2} \]
\[ X_{(n-1)/2,(n+1)/2} = -(X_{(n-1)/2,(n+1)/2})^{tr} \]
\[ \tilde{X}_{10} = (\epsilon v_0 \tilde{X}_{0,n-1})^{tr} \]
\[ \tilde{X}_{i+1,i} = (\tilde{X}_{n-i,n-i-1})^{tr} \quad 1 \leq i \leq \frac{n-3}{2} \]
\[ \tilde{X}_{(n+1)/2,(n-1)/2} = -(\tilde{X}_{(n+1)/2,(n-1)/2})^{tr} \]

Again we have a diagram analogous to fig. 2 with reflection symmetry.

Fig. 3: A type I quiver diagram for \( X_n \), \( n \) even, \( \chi(\Omega) = +1 \). For \( \chi(\Omega) = -1 \) replace \( O(v) \rightarrow USp(v) \).

I.3. \( \chi(\Omega) = +1, \chi(g,\Omega) = 1 \). \( n \) even.

\[ \gamma(\Omega) = \begin{pmatrix} 1_{v_0} & 1_{v_1} \\ 1_{v_2} & 1_{v_2} \end{pmatrix} \quad (n = 4) \]

\[ G_3(\vec{v}) \equiv O(v_0) \times [U(v_1) \times U(v_2) \times \cdots \times U(v_{n/2-1})] \times O(v_{n/2}) \]
\[ = \{(U_0, U_1, \ldots, U_{n-1}) : U_i U_n^{tr} U_{n-i} = 1 \quad 0 \leq i \leq n\} \]
(We have simply $O(v_0) \otimes O(v_1)$ for $n = 2$.)

The conditions on the scalar fields are (1.14).

I.4. $\chi(\Omega) = -1, \chi(g, \Omega) = 1, n$ even.

$$
\gamma(\Omega) = \begin{pmatrix}
\epsilon_{v_0} & 1_{v_1} \\
-1_{v_1} & \epsilon_{v_2}
\end{pmatrix} \quad (n = 4)
$$

(4.18)

$$
G_4(v) \equiv USp(v_0) \times [U(v_1) \times U(v_2) \times \cdots \times U(v_{n/2-1})] \times USp(v_{n/2})
= \{(U_0, U_1, \ldots, U_{n-1}) : U_i U_{n-i}^{tr} = 1 \quad 1 \leq i \leq n - 1, i \neq n/2\}
$$

(We have simply $USp(v_0) \otimes USp(v_1)$ for $n = 2$.) The conditions on the scalar fields are:

$$
X_{01} = \epsilon_{v_0} (X_{n-1,0})^{tr}
$$

$$
X_{i,i+1} = (X_{n-i-1,n-i})^{tr} \quad 1 \leq i \leq \frac{n-2}{2}
$$

(4.19)

$$
X_{n/2,(n+2)/2} = -\epsilon_{v_{n/2}} (X_{n-2,n/2})^{tr}
$$

and similarly for $\bar{X}$. The quiver diagram is as in fig. 3. Note that vertices which are fixed by the reflection symmetry have group $O(v)$ or $USp(v)$.

![Quiver diagram](image-url)

**Fig. 4:** A type I quiver diagram for $X_n$, $n$ even, with $\chi(g, \Omega) = \xi$. 

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I.5. $\chi(\Omega) = \pm 1, \chi(g, \Omega) = \xi$, (thus $n$ is even). Here it is more convenient to let the block indices run from 1 to $n$ (modulo $n$). We now have:

$$\gamma(\Omega) = \begin{pmatrix} \chi(\Omega)1_{v_2} & 1_{v_1} \\ \chi(\Omega)1_{v_1} & \end{pmatrix} \quad (n = 4)$$

$$G_5(\vec{\sigma}) \equiv \left[ U(v_1) \times U(v_2) \times \cdots U(v_{n/2-1}) \times U(v_{n/2}) \right]$$

$$= \{(U_1, \ldots, U_n) : U_iU_{n-i+1}^{tr} = 1 \quad 1 \leq i \leq n\}$$

Finally, the conditions on the hypermultiplets are:

$$X_{n,1}^{tr} = \chi(\Omega)X_{n1}$$

$$(X_{n-i,n-i+1})^{tr} = X_{i,i+1} \quad 1 \leq i \leq \frac{n-2}{2}$$

$$(X_{n/2,(n+2)/2})^{tr} = \chi(\Omega)X_{n/2,(n+2)/2}$$

$$\bar{X}_{1,n}^{tr} = \chi(\Omega)\bar{X}_{1,n}$$

$$(\bar{X}_{n+1-i,n-i})^{tr} = \bar{X}_{i+1,i} \quad 1 \leq i \leq \frac{n-2}{2}$$

$$(\bar{X}_{n/2+1,n/2})^{tr} = \chi(\Omega)\bar{X}_{n/2+1,n/2}$$

again, reorienting the diagram as in fig. 4 we have symmetry about the vertical.

5. DD and DN spectrum for $(p, p+4)$ configurations at the fixed point

Let us now consider the above theories for $p = 3$ in $p + 4 = 7$ (in type IIb), $p = 4$ in $p + 4 = 8$ (in type IIa), and $p = 5$ in $p + 4 = 9$ (in type I). In these cases we can use the language of $d = 6, N = 1$ or $d = 4, N = 2$ SYM to describe the spectrum of the theory on the world-volume. Let $w^i = v_i^{(p+4)}$ and $v^i = v_i^{(p)}$. The resulting low energy field content is again nicely summarized by quiver diagrams.

There are three sources of fields in the $p$-brane gauge theory: Restriction of fields from the $(p+4)$-brane, $p$-brane fields, and $(p+4, p)$-sector fields. The fields from the $(p+4)$-brane consist of the restriction of the vectormultiplets $W_i$. The restriction of vector fields in the $(p+4)$-brane which are tangent to $X_n$ gives scalar fields $Y, \bar{Y}$ forming hypermultiplets on the $p$-brane. The $\mathbb{Z}_n$-projection requires these to be in:

$$Y_{i,i+1} \in Hom(W_{i+1}, W_i)$$

$$\bar{Y}_{i+1,i} \in Hom(W_i, W_{i+1})$$

(5.1)
for the $p+4$-brane. These fields comprise the “outer quiver.” Note that the hypermultiplets $Y, \bar{Y}$ only exist for $p + 4 < 9$.

The fields from the $p$-brane theory are described as above by an “inner quiver” with vectormultiplets $V_i$ and hypermultiplets $(X_{i,i+1}, \bar{X}_{i+1,i})$. In addition to this, the inner and outer quivers are joined by “spokes” as in fig. 5 fig. 6. The spokes correspond to the $(p,p + 4)$ and $(p + 4,p)$ fields $h^{AM}_{M}, \tilde{h}^{AM}_{m}$. The transcription to Kronheimer-Nakajima’s notation [5] is

$$J = \tilde{h}^{1} = (h^{2})^{\dagger} \in \text{Hom}(V,W)$$
$$I = (\tilde{h}^{2})^{\dagger} = -h^{1} \in \text{Hom}(W,V)$$

(5.2)

Thanks to the reality condition (3.4) we can work solely with $\tilde{h}^{1}, h^{1}$ and henceforth we drop the index 1. The $\mathbb{Z}_n$ projection makes the matrices $I, J$ block diagonal so that the components are

$$h^{im}_{iM_{i}} \leftrightarrow -I_{i} \in \text{Hom}(W_{i},V_{i})$$
$$\tilde{h}^{im}_{iM_{i}} \leftrightarrow J_{i} \in \text{Hom}(V_{i},W_{i})$$

(5.3)

The complete field content is summarized by the quiver diagrams, e.g., fig. 5, fig. 6 give the diagrams for $X_1, X_3$ respectively.
Fig. 6: A type II quiver diagram for \((4, 8)\) brane configurations on \(X_3\).

5.1. Type I

The conditions for (3.6) on the inner hypermultiplets \((X, \bar{X})\) have been described in detail in the previous section. The conditions on the outer hypermultiplets \((Y, \bar{Y})\) have an extra sign change relative to the condition for \(X\):

\[
Y = -\gamma_9(\Omega) Y^{tr} \gamma_9(\Omega)^{-1}
\]

since they are restrictions of gauge fields. The conditions (3.6) are:

\[
J^{tr} = -i \gamma_5(\Omega) I \gamma_9(\Omega)^{-1}
\]

There are several type I quivers depending on the various unbroken groups \(G_i(\vec{w})\) and \(G_i(\vec{v})\) we associate to the inner and outer quivers. We will consider just two cases

I. \(\chi_9(\Omega) = +1, \chi_5(\Omega) = -1, \chi_9(g, \Omega) = \chi_5(g, \Omega) = 1\).

\[
\begin{align*}
J_0^{tr} &= -i \epsilon_{\nu_0} I_0 \\
J_k^{tr} &= -i I_{n-k} & 0 < k < n/2 \\
J_k^{tr} &= +i I_{n-k} & n/2 < k < n \\
J_{n/2}^{tr} &= -i \epsilon_{\nu_{n/2}} I_{n/2} & n \text{ even}
\end{align*}
\]

The field content is summarized by the quivers shown in fig. 7, fig. 8, fig. 9.
II. $\chi_9(\Omega) = +1, \chi_5(\Omega) = -1, \chi_9(g, \Omega) = \chi_5(g, \Omega) = \xi$.

This case can only occur when $n$ is even. Letting indices run from 1 to $n$ the conditions on the hypermultiplets joining the inner and outer quiver are

$$J^r_j = iI_{n+1-j} \quad j \leq n/2$$
$$J^r_j = -iI_{n+1-j} \quad j > n/2$$ \hspace{1cm} (5.7)

The spectrum summarized by fig. 10 is that worked out by Gimon and Polchinski [12].

6. **World-volume action for $p$-branes transverse to the fixed point**

Having described the world-volume spectrum (at the fixed point) in sections 2,4,5 we
Fig. 8: A type I quiver diagram for \((5, 9)\) brane configurations in \(X_3\)

now proceed to describe the Lagrangian governing the low energy dynamics. There is no simple unified formulae for D-brane actions yet, but several terms are now well-known:

\[
I = I_{BI} + I_{HM} + I_{CS} + I_{susy} + \cdots .
\]  

(6.1)

\(I_{BI}\) is the Born-Infeld action \(\int_{B_p \times \mathbb{R}} \text{Tr} \sqrt{\det(G + F)}\) where \(F = F - B\). Expanding the squareroot gives the Yang-Mills action at leading nontrivial order. \(I_{HM}\) gives the kinetic energies of the hypermultiplets. \(I_{CS}\) is a Chern-Simons coupling found in \([13]\), \(I_{susy}\) contains the supersymmetric completions of the lowest order terms and \(\cdots\) hides our ignorance about higher order terms in the low-energy expansion. In this section we describe in some detail \(I_{CS}\) and \(I_{susy}\) for the 5, 4, 3-brane in the type \(I, \Pi a, \Pi b\) theories.

The Chern-Simons couplings are described in general as follows: Let \(C\) denote the sum of \(p\)-form fields (in ten dimensions). Then, for a flat \(D\)-brane in \(\mathbb{R}^{10}\) we have:

\[
I_{CS} = \int_{B_p \times \mathbb{R}} C \wedge \text{Tr} F^2
\]  

(6.2)

Now let us consider the modifications in the presence of an orbifold. Of course, we retain (5.2) where \(C\) comes from the untwisted sector. The definition of D-branes on an
orbifold correlates the action of a point group element on the world-sheet and Chan-Paton factors, so that the closed string sector twisted by $g$ will couple to an open string boundary with Chan-Paton factors twisted by $\gamma(g)$. Thus we expect extra Chern-Simons couplings:

$$I_{CS} = \int_{\mathbf{B}_p \times \mathbb{R}} \sum_{k=1}^{n-1} p^{+1}C_k \wedge \text{Tr}\gamma(g^k)e^F.$$  \hfill (6.3)

The RR fields $p^{+1}C_k$ are a bispinor field in the $k$-twisted sector, restricted to the $(p+1)$-dimensional world-volume. The existence of these couplings is checked by a vertex operator
calculation in appendix A. Additional couplings to hypermultiplet scalars are obtained by the replacement $\mathcal{F} \rightarrow \mathcal{F} + dX^i b_i$ explained in [13].

As described in the appendix, (6.3) is exact only when the D-branes are coincident with the orbifold fixed point. At non-zero distance $|X|$, we expect this coupling to be suppressed as $\exp \left[ -|X|^2 / \alpha' \right]$. We will neglect this here, obtaining results valid for $|X|^2 << \alpha'$.

6.1. Hyperkähler moment map

Let us now consider the terms in the supersymmetric completion involving only bosonic fields. The completion of $I_{BI} + I_{HM}$ in 6d involves coupling the triplet of $D$-terms for $d = 6, \mathcal{N} = 1$ SYM to the hypermultiplets through the hyperkähler moment map.

Quite generally, in a linear $d = 6, \mathcal{N} = 1$ or $d = 4, \mathcal{N} = 2$ theory the hypermultiplets are described by starting with a complex hermitian vector space $V$ with a unitary action of the gauge group $G$. The hypermultiplets take values in the vector space $V \oplus V^*$. This space is a quaternionic vector space. Indeed, if we choose coordinates: $\{z^\alpha\}_{\alpha=1,\ell}$ for $V$ and dual coordinates $w_\alpha$ for $V^*$ then we define quaternionic coordinates as in subsection 2.2,

$$X_\alpha = \begin{pmatrix} z^\alpha & \bar{w}_\alpha \\ -w_\alpha & \bar{z}_\alpha \end{pmatrix} \quad (6.4)$$

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so that the complex structures $I, J, K$ correspond to right multiplication by $i\sigma_3, i\sigma_2, i\sigma_1$, respectively. Moreover, $G$ acts via

$$\delta_{\Lambda}(z^\alpha; w_\alpha) = \left\{(T_\Lambda)^\alpha_\beta z^\beta; -w_\alpha(T_\Lambda)^\alpha_\beta\right\}$$

(6.5)

where $1 \leq \Lambda \leq \dim G$ is an index labelling a basis for $g$, and $[(T_\Lambda)^\alpha_\beta]^* = -(T_\Lambda)^\beta_\alpha$. This action may be written as:

$$\delta_{\Lambda}X^\alpha = (\tau_\Lambda)^\alpha_\beta X^\beta$$

(6.6)

where we replace $T$ by $\text{Re} T \cdot 1 + \text{Im} T \cdot I$:

$$(\tau_\Lambda)^\alpha_\beta = \left(\begin{array}{c c}
(T_\Lambda)^\alpha_\beta & 0 \\
0 & [(T_\Lambda)^\alpha_\beta]^*
\end{array}\right).$$

(6.7)

In general hypermultiplets take values in a hyperkähler manifold. For the vector space $V \oplus V^*$ we have Kahler and holomorphic symplectic forms

$$\omega^R = \frac{i}{2} \sum [dz^\alpha d\bar{z}_\alpha + dw_\alpha d\bar{w}_\alpha]$$

$$\omega^C = \sum dz^\alpha \wedge dw_\alpha$$

(6.8)

these forms comprise a triplet $\bar{\omega}$. The $G$-action is symplectic with respect to each of these forms and hence we obtain a triplet of Noether charges defining the hyperkähler moment map:

$$\bar{\mu}_{\Lambda} = \frac{1}{2} \text{tr} \bar{\sigma}X_\alpha^\dagger (\tau_\Lambda)^\alpha_\beta X^\beta$$

(6.9)

Explicitly:

$$\mu_{\Lambda}^R = \frac{1}{2} \left[\bar{z}_\alpha (T_\Lambda)^\alpha_\beta z^\beta - w_\alpha (T_\Lambda)^\alpha_\beta \bar{w}^\beta\right] \in \sqrt{-1}\mathbb{R}$$

$$\mu_{\Lambda}^C = w_\alpha (T_\Lambda)^\alpha_\beta z^\beta$$

(6.10)

The completion of $I_{BI} + I_{HM}$ in 6d is:

$$\int_{B_5 \times \mathbb{R}} \sum_{\Lambda} [(\bar{D}_{\Lambda})^2 + \bar{D}_{\Lambda} \cdot \bar{\mu}_{\Lambda}]$$

(6.11)

Here the sum runs over a basis $\Lambda$ for the Lie algebra of the worldbrane gauge group and $\bar{D}_{\Lambda}$ is a triplet of auxiliary D-fields in the vectormultiplet. For $p = 3, 4$ we simply reduce (6.11). In particular, for $p = 3$ we obtain the D and F - auxiliary fields: $D_{\Lambda} = D^r_{\Lambda}, F_{\Lambda} = D^c_{\Lambda}$.

The completion of the Chern-Simons couplings is trickier, in part because they are not obtained by naive dimensional reduction but rather by re-applying (6.2). This is ultimately because the surviving supersymmetry $\tilde{\epsilon} = \Gamma_D \epsilon$ is different for each $p$. We turn to this next.
Here we take $p = 3$. We must reduce the 6d spectrum of section 2.4 to 3+1 dimensions. Let us take coordinates $0, 1, 2, 3$ along the 3-brane $B_3 \times \mathbb{R}$ and coordinates $6, 7, 8, 9$ to describe the ALE space. The twisted sector matter multiplets decompose into a sum of a linear hypermultiplet $(^4C^{(2)}, \vec{\phi}_k)$ and a vector-multiplet of $d = 4, N = 2$.\footnote{Linear hypermultiplets are described in the next section.} The vectormultiplet may be taken to be $((^4C^{(1)}_{k \mu}), ^4C^{(0)}_k, b^{(0)}_k)$ where the vector fields $(^4C^{(1)}_{k \mu})$ and $(^4C^{(1)}_{4\mu})$ are related by $d = 4$ duality.

The twisted sector couplings become

$$
\int_{B_3 \times \mathbb{R}} ^4C^{(0)}_k \text{Tr} \gamma (g^k) F \wedge F + b^{(0)}_k \text{Tr} \gamma (g^k) F \wedge *F \\
+ [(X^4 + iX^5)^4H^+(k) + (X^4 - iX^5)^4H^-(k)] \text{Tr} \gamma (g^k) F + 4^4C^{(2)}_k \text{Tr} \gamma (g^k) F
$$

(6.12)

Here $^4H^{(2)}_{k \pm}$ are the field strengths of $(^4C^{(1)}_{k \mu})$; we have integrated by parts and discarded a term higher order in derivatives.

We now discuss the supersymmetric completion of the terms (6.12). The first two terms contribute to standard couplings of vectormultiplet gauge fields to vectormultiplet scalars. The last term is somewhat more unusual, and gives a coupling between the hypermultiplet scalar $^4C^{(2)}_k$ and vectormultiplets. Its supersymmetric completion involves the NS scalars $\vec{\phi}_k$. As described in more detail in the next section this completion is the $d = 6, N = 1$ Fayet-Iliopoulos term:

$$
\sum_{k=0}^{n-1} \int_{B_3 \times \mathbb{R}} \vec{\phi}_k \text{Tr} \gamma (g^k) \vec{D}
$$

(6.13)

Note, in particular, that the sum on $k$ includes the untwisted sector. The existence of these couplings – which play a crucial role in what follows – may be deduced from worldbrane supersymmetry. As we show in detail in the following section, they may also be predicted from an analysis of anomalies. Finally, it is relatively straightforward to verify their existence by an explicit vertex operator calculation. This is done in appendix A.

Here we take $p = 4$. Now we must reduce the 6d spectrum of section 2 to 5d. The untwisted matter multiplet and $(n - 1)$ twisted matter multiplets again reduce to a vm + hm, now $(^5C^{(1)}, b^{(0)})$ and $(^5C^{(3)}, \vec{\phi}_k)$.
From the untwisted sector we have

$$\int_{\mathcal{B}_4 \times \mathbb{R}} 5C^{(1)}\text{Tr} F \wedge F + 5C^{(3)}\text{Tr} F + \nu^5 C^{(5)}$$  \hspace{1cm} (6.14)$$

while the twisted sector contributes

$$\sum_{k=1}^{n-1} \int_{\mathcal{B}_4 \times \mathbb{R}} 5C^{(1)}_k\text{Tr} \gamma(g^k) F^2 + b^{(0)}_k\text{Tr} \gamma(g^k) F \wedge \ast F + 5C^{(3)}_k\text{Tr} \gamma(g^k) F$$  \hspace{1cm} (6.15)$$

Upon dimensional reduction to 3 + 1 we recognize the standard couplings of vector multiplet scalars to gauge fields governed by a prepotential. The $5C^{(3)}_k F$ coupling reduces to the $4C^{(2)}_k F$ coupling described above, and again supersymmetry will require the Fayet-Iliopoulos coupling (6.13).

6.4. Type I

Each twisted sector gives rise to a hypermultiplet ($6C^{(4)}_k, \vec{\phi}_k$). The 5-brane twisted sector Chern-Simons coupling is

$$\sum_{k=1}^{n-1} \int_{\mathcal{B}_5 \times \mathbb{R}} 6C^{(4)}_k\text{Tr} \gamma(g^k) F.$$  \hspace{1cm} (6.16)$$

and again its partner Fayet-Iliopoulos couplings (6.13) are present.

The couplings to the 9-brane are given by a simple modification to the world-sheet calculations of the appendix. They are similar to (6.12) but localized at the fixed points $x_i$,

$$\int d^{10}x \, \delta^{(4)}(x - x_i) \wedge 6C^{(4)}_k \wedge \text{tr} \gamma(g^k) F,$$  \hspace{1cm} (6.17)$$

Here $6C^{(4)}_k$ is the 4-form dual to the type I scalar RR field in the $k^{th}$ twisted hypermultiplet. There is no such coupling in the untwisted sector.

The NS-NS partner of this term is the same as (6.13), except that there is no contribution from the untwisted sector $k = 0$.

6.5. Hypermultiplet potential energy

Finally, let us work out the potential terms for the scalars in the hypermultiplets. For definiteness we work in the IIb theory. We would like to integrate out the $\vec{D}_A$ in (6.11). Therefore, we must diagonalize the couplings (6.13).
Let $F_j$ be the field strength of the $U(1)$ factor contained in $U(v_j)$, and let \( \tilde{C}_i = 4C^{(2)}_i \) be the RR potential in the $g^i$ twisted sector; then we may rewrite the Green-Schwarz coupling from (6.12) as

\[
\int_{\mathcal{B}_3 \times \mathbb{R}} \sum_{i,j=0}^{n-1} \xi^{i,j} \tilde{C}_i \wedge F_j. \quad (6.18)
\]

The couplings can be diagonalized by doing a discrete Fourier transform on either $F_j$, to produce $\tilde{F}_i$ or on $\tilde{C}_i$ to produce $C_j$. On the one hand, $\tilde{F}_i$ is the $U(1)$ gauge factor in the “$i$’th twisted open string sector,” connecting a D-brane with its image under $\gamma(g)^i$. On the other hand, $C_j$ couples to the $U(1)$ in a single factor $U(v_j)$.

It will turn out (in section 8) that the $C_j$ have a simpler interpretation (they are associated with individual two-cycles), so let us use this basis from now on, and rewrite (6.18) as

\[
\int_{\mathcal{B}_3 \times \mathbb{R}} \sum_j C_j \wedge F_j. \quad (6.19)
\]

$(0,1)$ supersymmetry on the world-volume requires the partners

\[
\int_{\mathcal{B}_3 \times \mathbb{R}} d^4x \sum_j \tilde{\phi}_j \cdot \tilde{D}_j \quad (6.20)
\]

Here $\tilde{D}_j$ is the contribution of D-brane matter to the hyperkähler map for the $U(1)$ in $U(v_j)$.

We are finally able to complete the square to get the hypermultiplet potential energy. Define $\tilde{\phi}_\Lambda$ to be zero for noncentral generators of $G(v)$ and $\tilde{\phi}_\Lambda = \tilde{\phi}_j$ if $T_\Lambda$ is the $U(1)$ generator of $U(v_j)$. Integrating out $\tilde{D}_\Lambda$ we have simply

\[
\sum_\Lambda (\tilde{\mu}_\Lambda - \tilde{\phi}_\Lambda)^2 \quad (6.21)
\]

In what follows we will denote the vev’s of these scalars by

\[
\langle \tilde{\phi}_\Lambda \rangle \equiv \tilde{\zeta}_\Lambda \quad (6.22)
\]

7. Anomalous $U(1)$’s and Fayet-Iliopoulos terms

7.1. $U(1)$ anomaly cancellation

Another way of understanding the presence of the crucial couplings (6.13) is through anomaly cancellation. In $d = 6, \mathcal{N} = 1$, theories with charged $U(1)$ fields are always
This is because supersymmetry correlates the type of supermultiplet, vector or hypermultiplet, with the chirality of the fermions they contain. Since only hypermultiplets can have $U(1)$ charge, all contributions to the $F^4$ anomaly will have the same sign.

Several people (we learned it from John Schwarz) have noticed that the theory of Gimon and Polchinski is an example, and that a $d = 6$ version of the mechanism proposed for $d = 4$ by Dine, Seiberg and Witten will resolve the problem. The idea is that the couplings required for $d = 10$ Green-Schwarz anomaly cancellation, when evaluated with non-zero background gauge field in the internal space, will lead to couplings of the form

$$\int d^D x \ C^{(D-2)} \wedge F,$$

(7.1)

where $F$ is the $U(1)$ field strength and $C^{(D-2)}$ is a $D - 2$-form gauge field. Cancellation of the $F^{(D-2)/2}$ axial anomaly requires a gauge transformation law $C^{(D-2)} \rightarrow C^{(D-2)} + \epsilon F^{(D-2)/2}$. The added term in the world-volume action changes the duality transformation to a scalar $c^{(0)}$ to

$$*_D dc^{(0)} = H^{D-1} + *_D A$$

(7.2)

and hence $c^{(0)}$ has a non-trivial gauge transformation:

$$\delta A_\mu = \partial_\mu \epsilon \quad \delta c^{(0)} = \epsilon$$

(7.3)

and couplings

$$\int d^D x \ (A_\mu - \partial_\mu c^{(0)})^2.$$  

(7.4)

The resulting theory is gauge-invariant and describes a massive vector boson.

The story becomes even more interesting when supersymmetry is taken into account. In $\mathcal{N} = 1$, $d = 4$ supersymmetric Yang-Mills theory, the scalar $c^{(0)}$ is one real component of a chiral superfield $C$ with lowest component $c^{(0)} + i \phi$. Let $V$ be the vector superfield containing $A_\mu$; then a superfield coupling containing (7.4) is

$$\int d^4x \ d^4\theta \ \text{Im} \ \frac{1}{4} (C - \bar{C} - V)^2.$$  

(7.5)

This is gauge invariant if $\delta V = \Lambda - \bar{\Lambda}$ and $\delta C = \Lambda$. It also produces a coupling to $\phi$,

$$- \int d^D x \ \phi \ \text{Im} \ \int d^4 \theta \ V.$$  

(7.6)

Thus $\phi$, the partner to $c^{(0)}$, controls the Fayet-Iliopoulos term.

More general Kähler potentials $K(C - \bar{C} - V)$ are allowed but are better discussed in the language of subsection 7.3.
7.2. Generalization to $\mathcal{N} = 2, d = 4$

$\mathcal{N} = 1, d = 6$ $U(1)$ gauge theories and their dimensional reductions can have three Fayet-Iliopoulos terms $\bar{\zeta} = (\text{Re} \zeta^C, \text{Im} \zeta^C, \zeta^R) = \bar{\sigma} AB \zeta AB$, forming a triplet of $SU(2)_R$. Let us explain this in the (perhaps more familiar) $\mathcal{N} = 2, d = 4$ case, using $\mathcal{N} = 1$ superfield notation and assembling the vector multiplets as $(V, A)$. The superpotential is almost uniquely determined by the gauge group $G$ and matter representation $R$. (Mass terms are allowed but are not relevant for us.) The conditions for a supersymmetric vacuum for $U(1)$ gauge theory with hypermultiplets $(M_i, \bar{M}_i)$ of charge $(q_i, -q_i)$ are*

\[
0 = D = -\zeta^R + \sum_i q_i (M_i^* M_i - \bar{M}_i^* \bar{M}_i) \\
0 = F = \frac{\partial W}{\partial A} \\
W = -\zeta^C A + \sum_i q_i \bar{M}_i M_i A.
\] (7.7)

The term $\zeta^C A$ is an $SU(2)_R$ covariant generalization of the $\mathcal{N} = 1$ FI term.

These equations can also be written in an $SU(2)_R$ covariant form, using ‘quaternionic’ notation (as in (2.4),(6.4))

\[
\bar{M}^{A'}A = \begin{pmatrix} M & \bar{M}^* \\ -\bar{M} & M^* \end{pmatrix}
\] (7.8)

satisfying $(\bar{M}^*)_{A'}A = \epsilon_{A'B'} \epsilon_{AB} \bar{M}^{B'B}$. Then

\[
\bar{\zeta} = \bar{\mu}_j = \frac{1}{2} \bar{\sigma} AB \sum_i q_i \bar{M}_i^{A'} A (\bar{M}_i^*)_{A'B}.
\] (7.9)

As with $\mathcal{N} = 1$, such Fayet-Iliopoulos terms will be $\mathcal{N} = 2$ supersymmetry partners of the anomaly cancelling coupling (7.4). Now $c^{(0)}$ will be one component of a hypermultiplet also containing $\bar{\zeta}$, which we write in terms of $\mathcal{N} = 1$ chiral superfields as $(C, \Phi)$. The $\mathcal{N} = 2$ extension of (7.5) is simply

\[
\int d^4x \ d^4\theta \ \text{Im} \frac{1}{4} (C - C - V)^2 + \Phi \bar{\Phi} + \int d^4x \ d^2\theta \ \Phi A + \text{c.c.}
\] (7.10)

7.3. Linear hypermultiplets

In a general string compactification, the kinetic terms for the moduli need not take the form (7.10). For example, the gravitational moduli for type II on K3 live on a homogeneous space.

* The additional conditions $\partial W/\partial M_i = 0$ are not present in $d = 6$ (where $A$ becomes the 5 and 6 gauge field components), and in any case are not relevant for describing moduli spaces of instantons on the ALE space.
The mechanism just described works for more general kinetic terms. It is best described in terms of a “linear hypermultiplet” whose components are the R-R field strength $H^{(D-1)}$ and the three NS-NS scalars. Its component fields correspond directly to the world-sheet vertex operators, and form a $1 + 3$ of $SU(2)_R$, as we saw for the moduli in section 2.3. A $d = 6, \mathcal{N} = 1$ superfield version of the multiplet is given in [20].

These NS-NS scalars are always equal to the FI terms $\vec{\zeta}$. On the other hand, the relation to the standard hypermultiplet is through dualizing $H^{(D-1)}$, which for a general kinetic term is a non-linear transformation. In this case the $\phi$ and $\Phi|_{\theta=0}$ of the previous section will not be equal to $\vec{\zeta}$.

In terms of $d = 4, \mathcal{N} = 1$ superfields the multiplet becomes a “linear chiral multiplet” $G$ containing a 3-form field strength and a scalar, and a chiral multiplet $\eta$ containing the other scalars. The Lagrangian dual to (7.10) is [21]

$$
\int d^4x \ d^4\theta \ G \ V + \int d^4x \ d^2\theta \ \eta \ A + \text{c. c.} \tag{7.11}
$$

combined with the kinetic term

$$
\int d^4x \ d^4\theta \left( -\frac{1}{2} G^2 + \eta \bar{\eta} \right). \tag{7.12}
$$

A more general kinetic term

$$
\int d^4x \ d^4\theta \ f(G, \eta, \bar{\eta}) \tag{7.13}
$$

will be supersymmetric if $f$ satisfies

$$
\left( \frac{\partial^2}{\partial G^2} + \frac{\partial^2}{\partial \eta \partial \bar{\eta}} \right) f = 0. \tag{7.14}
$$

Dualizing $G$, a transformation described explicitly in [22,21], produces the hypermultiplet form of the theory. All this generalizes to $n$ linear multiplets and provides a general construction of $4n$-dimensional hyperkähler metrics with $U(1)^n$ symmetry.

### 7.4. Application to the D-brane theories

It should now be clear that anomaly cancellation for the $\text{IIb}$ theory requires the couplings (6.12)(6.13). We may also develop the formal realization of this mechanism for a type $\text{IIb}$ compactification on $\mathbb{R}^2/\mathbb{Z}_n$ with 9-branes and 5-branes at the fixed point. Although this theory is anomalous, we will be able to derive the type I result from this by applying the $\Omega$ projection, and the sensible type $\text{II}$ results by dimensional reduction.

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A 5-brane will have couplings to the Ramond-Ramond potentials $C$ of the generalized Green-Schwarz form described in [13], including the term
\[ \int d^6 x \ C^{(4)} \wedge \text{tr} \ F. \] (7.15)

This will serve to cancel the anomalous $U(1)$ contained in $U(v_0)$. However, our orbifold theories contain numerous $U(1)$’s, $n$ for type II, each of which requires its own $c^{(0)}$ for gauge invariance. These come from Ramond-Ramond 4-form potentials in the twisted sectors as above.

The 9-brane $U(1)$ gauge theories are also anomalous, with part of the anomaly from 9-brane matter $Y$, and part from 5-brane matter $H$. The 5-brane couplings are derived from the results of section 6.4 by imposing the $\Omega$ projection. Applying the projection to the 5-brane matter in fact eliminates couplings to the type IIb closed string fields not present in type I; for example the $U(1)$ in $U(v_0)$ is removed, eliminating (7.15), which is consistent with the removal of $C^{(4)}$.

A pair of twist sectors related by $\Omega$ as in section 4.2, e.g. $i$ and $n-i$, will be related to one 9-brane $U(1)$ and one 5-brane $U(1)$. For the anomaly cancellation to work for both $U(1)$’s (6.12) and (6.17) must couple to different linear combinations of $C_i$ and $C_{n-i}$. This is true (and essentially follows from (3.13)).

8. D-flatness equations and the moduli of D-brane ground states

In this section we write the equations determining the collective coordinates of D-branes transverse to the ALE space realized as a blown-up orbifold.

According to (8.21) we must solve the equations
\[ \bar{\mu}_\Lambda = \bar{\zeta}_\Lambda \mod G \] (8.1)

Recall that $\bar{\zeta}_\Lambda$ is only nonzero for central generators of the gauge group.

The equations (8.1) define a special case of a general construction – the hyperkähler quotient. Hyperkähler quotients have been discussed extensively (see for example [11,22]). Quite generally, if $X$ has a hyperkähler metric, then the tangent space admits an action by the quaternions $\mathbb{H}$, and $X$ has 3 covariantly constant symplectic forms which may be assembled into a vector $\bar{\omega}$ in the Lie algebra of the unit quaternions $sp(2)$. If a Lie group $G$ with Lie algebra $\mathfrak{g}$ acts on $X$ preserving the hyperkähler structure then for all $\xi \in \mathfrak{g}$ we have $d\bar{\mu}(\xi) + \iota_\xi \bar{\omega} = 0$. The function $\bar{\mu}(\xi)$ is the Noether charge, and defines
a map $\tilde{\mu} : X \to \mathfrak{g}^* \otimes \text{sp}(2)$. If $\zeta \in \text{Center}(\mathfrak{g})^* \otimes \text{sp}(2)$ then the level set $\tilde{\mu}^{-1}(\zeta)$ is $G$-invariant, and we may take the quotient: $(X/G)_{\zeta} \equiv \tilde{\mu}^{-1}(\zeta)/G$. If we choose a complex structure on $X$ then the hyperkähler moment map equations split naturally into real and complex equations associated with $\omega^R$ and $\omega^C$, the Kähler form and the holomorphic $(2,0)$ symplectic form, respectively. The restriction of the hyperkähler metric to the level set is $G$-invariant and descends to a hyperkähler metric on the quotient. The quotient space will be singular when the group action is not free.

We now specialize the equations to the case of $p = 5, 4, 3$ branes stuck at a fixed point.

8.1. Type II

We begin by considering $X_1 = \Phi^2$. The relevant moment maps are:

$$\mu^C = [X, \bar{X}] + IJ$$
$$\mu^R = [X, X^\dagger] + [\bar{X}, \bar{X}^\dagger] + II^\dagger - J^\dagger J$$

(8.2)

for the $U(v)$ gauge group and

$$\tilde{\mu}^C = [Y, \bar{Y}] + JJ$$
$$\tilde{\mu}^R = [Y, Y^\dagger] + [\bar{Y}, \bar{Y}^\dagger] + JJ^\dagger - I^\dagger I$$

(8.3)

for the $U(w)$ gauge group.

The moduli space of D-brane ground states (which may be identified with the classical moduli of the $d = 6, \mathcal{N} = 1$ SYM theory on the world-volume) is just the hyperkähler quotient:

$$\mu^C = 0, \quad \mu^R = 0 \mod U(v)$$

(8.4)

together with $\tilde{\mu}^C = \tilde{\mu}^R = 0 \mod U(w)$.

Let us now consider the D-branes on an ALE space $X_n(\zeta)$ realized as a blown-up orbifold. As we have seen, the gauge symmetry is broken to

$$U(w) \to U(\bar{w}) \equiv U(w_0) \times \cdots \times U(w_{n-1})$$
$$U(v) \to U(\bar{v}) \equiv U(v_0) \times \cdots \times U(v_{n-1})$$

(8.5)

by the orbifold. The moment maps $\mu^R_i, \mu^C_i$ for the unbroken gauge symmetry are easily obtained by substituting the block-diagonal forms of $X, \bar{X}$ (see (4.5)) into (8.2) to get $\tilde{\mu} = \text{Diag}\{\tilde{\mu}_0, \ldots, \tilde{\mu}_{n-1}\}$ with:

$$\mu^C_i = X_{i,i+1} \bar{X}_{i+1,i} - \bar{X}_{i,i-1} X_{i-1,i} + I_i J_i$$
$$\mu^R_i = X_{i,i+1} \bar{X}_{i+1,i}^\dagger - X_{i-1,i}^\dagger \bar{X}_{i-1,i} + \bar{X}_{i,i-1} \bar{X}_{i+1,i}^\dagger - \bar{X}_{i+1,i} \bar{X}_{i+1,i} + I_i J_i^\dagger - J_i^\dagger J_i$$

(8.6)
The equations \( \mu_i = 0 \mod U(\vec{v}) \), \( \tilde{\mu}_i = 0 \mod U(\vec{w}) \) give the moduli of \( \mathbb{Z}_n \) equivariant D-brane configurations on \( \mathbb{CP}^2 \).

As discussed in the previous section we may – by turning on twist fields in the \( \sigma \)-model of the string theory – resolve the target space. Turning on twist fields induces FI terms in the d=6 SYM theory and the new vacuum equations are consequently

\[
\begin{align*}
\mu_i^C &= \zeta_i^C \\
\mu_i^R &= \zeta_i^R \mod U(\vec{v})
\end{align*}
\]

(8.7)

in addition to similar equations for \( U(\vec{w}) \). The equations \( (8.7) \) (without the \( \tilde{\mu} \) equations) define what is known as a quiver manifold

\[
\mathcal{M}_{\vec{\zeta}}(\vec{v}, \vec{w}) \equiv \{(X, \tilde{X}, I, J) : \tilde{\mu} = \zeta^I / U(\vec{v})
\]

(8.8)

See [19] for an extensive discussion of the properties of these manifolds and for references to the literature. When the action of \( U(\vec{v}) \) is free on the solutions of \( (8.7) \) the dimension of the moduli space is

\[
\dim_R \mathcal{M}_{\vec{\zeta}}(\vec{v}, \vec{w}) = 4\vec{v} \cdot \vec{w} - 2\vec{v} \cdot \tilde{C} \vec{v}
\]

(8.9)

The manifolds are generically smooth and topologically very rich. They do develop important singularities at nongeneric values of \( \tilde{\zeta} \). An extreme case occurs when \( \tilde{\zeta} = 0 \) and the manifolds are extremely singular. Note that, by taking a trace of \( (8.6) \) we see that compatibility of the D-flatness conditions requires

\[
\sum v_i \zeta_i^c = \sum \text{Tr}(I_i J_i) \\
\sum v_i \zeta_i^r = \sum \text{Tr}(I_i I_i^\dagger - J_i J_i^\dagger)
\]

(8.10)

When this condition is violated there is a potential for the gravitational moduli \( \tilde{\zeta}_i \).

For fixed \( Y \) the quiver variety \( \mathcal{M}_{\vec{\zeta}}(\vec{v}, \vec{w}) \) describes the classical moduli of ground states. For \( p = 5, 4 \) this is the same as the quantum moduli. For \( p = 3 \) we can have an interacting SYM theory, but the hyperkahler metric is not corrected by quantum effects in SYM. The situation is less clear when including gravitational interactions. However, \( \mathcal{M}_{\vec{\zeta}}(\vec{v}, \vec{w}) \) is clearly correct to leading order in the string coupling.

In order to understand better the physical significance of the quiver varieties let us take \( \vec{w} = 0 \), i.e., no outer D-branes at all, and consider, moreover, a single D-brane. Such a D-brane must be able to move away from the orbifold fixed point, and when it does so, it is described by a symmetrical configuration of \( n \) images. Hence, a single D-brane transverse to an ALE space is described by \( \vec{v} = \vec{n} \equiv (1, 1, \ldots, 1) \). For such a choice of \( (\vec{v}, \vec{w}) \) the action
of the group $G(\vec{v}) = U(1) \times \cdots \times U(1)$ is not free. At best $G'(\vec{v}) = G(\vec{v})/U(1)_{\text{diagonal}}$ can act freely. Correspondingly, from (8.10), we have the restriction:

$$\sum_{i=0}^{n-1} \zeta_i = 0$$

(8.11)
on the levels. For $\vec{c}$ which are otherwise generic the group $G'(\vec{v})$ in fact does act freely and, by (8.9) the quotient is a smooth four-dimensional hyperkähler manifold. In fact, a theorem of Kronheimer [10] asserts that the quiver variety is the ALE space with periods determined by $\vec{c}$ described in section 2.1:

$$M_{\vec{c}}(\vec{n}, \vec{0}) = X_n(\vec{c})$$

(8.12)

Thus, Kronheimer’s theorem fits in beautifully with the results of D-brane theory: The low energy dynamics of a D-brane transverse to an ALE space may be described both from a 10 dimensional perspective and from a 6-dimensional world-volume perspective. From the 10-dimensional viewpoint the dynamics is clearly given by supersymmetric quantum mechanics with the ALE space $X_n(\vec{c})$ as the target. From the world-brane point of view we have supersymmetric quantum mechanics with target space the vacuum manifold $M_{\vec{c}}(\vec{n}, \vec{0})$ of the $d = 6, \mathcal{N} = 1$ SYM. Kronheimer’s theorem, (8.12) identifies these as the same target.

This is as much as we have any a priori right to expect, but, in fact, much more is true: the full dynamics is described by a sigma model with target given by $M_{\vec{c}}(\vec{n}, \vec{0})$. This suggests a conjecture below.

The interpretation of $M_{\vec{c}}(\vec{v}, \vec{w})$ for other D-brane configurations will be described in the next section.

8.2. Type I

The equations for the type I quivers are the same as the equations described in the previous subsection. The only new point is that the restrictions on the hypermultiplets described in sections four and five restricts the moment maps to take values in the Lie algebras described in subsection 4.4.

For example, consider case I of subsection 5.1. Using the conditions (4.16)(4.21)(5.6) one can check that

$$\epsilon^{\lambda}_{\mu_0} = +\epsilon\mu_0^\Lambda$$

$$\mu_i^\lambda = -\mu_{n-i}^\Lambda$$

$$1 \leq i \leq n - 1, i \neq n/2$$

$$\epsilon^{\lambda}_{\mu_{n/2}} = +\epsilon\mu_{n/2}^\Lambda$$

(8.13)
Thus, the orientation conditions on the fields already guarantee that the moment maps take values in the Lie algebras of the unbroken gauge symmetry. Similarly, in case II of subsection 5.2 one can check that

$$\mu_i^A = -\mu_{n+1-i}^A$$  \hspace{1cm} 1 \leq i \leq n$$

(8.14)
in accord with the relevant gauge groups. Note that a rather peculiar feature of (8.13) is that the equations (8.7) only make sense for

$$\vec{\zeta}_i = -\vec{\zeta}_{n-i}.$$  \hspace{1cm} (8.15)

Further discussion of Type I quivers will be found in [3].

9. Type II D-branes and $U(N)$ gauge instantons on ALE

Let us consider two a priori distinct physical situations. For definiteness we focus on the IIb theory. First we consider $w$ type IIb 7-branes whose world-volume is given by the supersymmetric cycle $B_7 = \mathbb{R}^3 \times X_n(\vec{\zeta})$. Semiclassical supersymmetric groundstates of this 7-brane theory will be described by $U(w)$ instantons. In particular, let us focus on the low energy dynamics of the states given by instantons on $X_n(\vec{\zeta})$. The low energy dynamics of these states is described by supersymmetric quantum mechanics with the target being the moduli of $U(w)$ instantons $M_{\text{inst}}$.

Next, we consider the supersymmetric boundstates of 3-branes $B_3 = \mathbb{R}^3 \times \{P_0\}$ where $P_0 \in X_n(\vec{\zeta})$ is a point with the 7-branes $B_7$. The moduli of these latter boundstates can be given an explicit description using the orbifold construction of this paper. Namely, we consider $w$ 7-branes on $\mathbb{R}^3 \times \mathbb{C}^2/\Gamma$ and $v$ 3-branes on $\mathbb{R}^3 \times \{0\}$. Turning on FI terms $\vec{\zeta}$, restricted by the condition (8.11), resolves the $\sigma$-model on the orbifold to the sigma model on $X_n(\vec{\zeta})$ as discussed in the previous section. At the same time, turning on $\vec{\zeta}$ resolves the moduli of 3-brane collective coordinates: The low energy dynamics of the 3-brane degrees of freedom, for fixed 7-brane degrees of freedom, is governed by supersymmetric quantum mechanics with target $M_{\vec{\zeta}}(\vec{v}, \vec{w})$.

According to the results of [3,13], these two supersymmetric quantum mechanics systems should be the same. Thus we expect that the moduli of instantons on $X_n(\vec{\zeta})$ should be identified with $M_{\vec{\zeta}}(\vec{v}, \vec{w})$. To be a little more precise, if the instanton breaks

$$U(w) \rightarrow U(w_0) \times \cdots \times U(w_{n-1})$$

(9.1)
at infinity in $X_n(\vec{\zeta})$ then we should identify such instantons with the $(3,7)$ branes where the 7-branes also induce the same breaking (9.1). In fact, the identification of $M_{\text{inst}}$ with $M_{\vec{\zeta}}(\vec{v}, \vec{w})$ is known to be correct and is called the Kronheimer-Nakajima theorem. In the next two sections we describe the identification more precisely.\footnote{Moreover, given the case $W = 0$ it is natural to conjecture that in fact the exact dynamics is described as a string theory with the moduli space as target.}

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9.1. Topological classification of instantons on $X_n(\zeta)$.

We must first describe the data needed to specify the components of the moduli spaces. The space of $U(w)$ connections on a complex $w$-dimensional vector bundle $E \to X_n$ of finite action is classified, topologically, by $c_1(E) \in H^2(X_n; \mathbb{Z})$, $ch_2(E)$, and a homomorphism $\rho : \mathbb{Z}_n \to U(w)$. The last piece of data corresponds to the data specifying a flat connection at infinity. If $\rho$ is conjugate to $g \to \text{Diag}\{\xi^j\}$ then we may equivalently say that the unbroken gauge group at infinity is $\prod U(w_j)$. We will denote the space of $U(w)$ instantons on $X_n$ by $\mathcal{M}(X_n(\zeta), U(w); c_1, ch_2, \rho)$.

We will need to be more explicit about the nature of $c_1$. We first introduce a set of line bundles $R_i$. The construction of $X_n$ as a hyperkähler quotient identifies it with a principal $\prod_{i=1}^{n-1} U(1)$ bundle. Choosing the associated bundle with charge 1 for the $i^{th}$ $U(1)$ defines $R_i$. These line bundles carry a natural connection such that $c_1(R_i)$ are harmonic 2-forms forming a basis for $H^2$ dual to the basis $\Sigma_i$ of $H^2(X_n; \mathbb{Z})$. Thus we may expand:

$$c_1(E) = \sum_{i=1}^{n-1} u^i c_1(R_i) \quad (9.2)$$

for some vector $\vec{u} \in \mathbb{Z}^{n-1}$.

It can be shown that

$$c_1(R_i) \cdot c_1(R_j) \equiv \int_X c_1(R_i) \wedge c_1(R_j) = -(C^{-1})_{ij} \quad (9.3)$$

where $C$ is the Cartan matrix. In particular, from (9.3) we see that:

$$ch_2(R_i) = \frac{1}{2} \frac{i(n-i)}{n} \quad (9.4)$$

This can be fractional because $X_n$ is noncompact.

9.2. The Kronheimer-Nakajima Theorem

We now come to the remarkable Kronheimer-Nakajima theorem which gives the isomorphism of hyperkähler manifolds:

$$\mathcal{M}_{\vec{v}, \vec{w}} \cong \mathcal{M}(X_n(-\zeta), U(w); c_1, ch_2, \rho) \quad (9.5)$$

where $(\vec{v}, \vec{w})$ are related to topological quantities $c_1(E), ch_2(E), \rho$ as follows. At in a neighborhood of infinity we may think of the bundle $E_\infty \cong \oplus w_i R_i$ where $R_i$ are flat line bundles on $S^3$ associated to the $i^{th}$ representation of $\mathbb{Z}_n$. The first Chern class is obtained
from (9.2) with $\vec{u} = \vec{w} - \tilde{C}\vec{v}$, where $\tilde{C}$ is the extended Cartan matrix. Finally $ch_2(E)$ is given by:

$$
ch_2(E) = \sum_{i=0}^{n-1} u_i ch_2(R_i) + \frac{1}{n} \dim V
$$

These equations have a beautiful and simple interpretation: The first term comes from magnetic monopoles centered in the different exceptional divisors $\Sigma_i$. Far away from these divisors $X_n(\vec{\zeta})$ appears like $\mathbb{R}^4/\mathbb{Z}_n$. Instantons on this space look like ordinary $\mathbb{Z}_n$-invariant instantons on $\mathbb{R}^4$. Alternatively, D-branes carry instanton charge $1/n$. Some further manipulation leads to a useful alternative formula:

$$
ch_2(E) = v_0 + \sum_{i=1}^{n-1} i(n-i) w_i
$$

Note that if we fix $w_i$ and vary $v$ then only $v_0$ contributes to the instanton number.

Remarks:

1. In fact, from ADHM data we can reconstruct $E$ and the gauge field quite explicitly. Some explicit examples of the construction can be found in [23].
2. The extra minus sign in (9.5) is surprising. However recall that $X_n(\vec{\zeta}) \cong X_n(-\vec{\zeta})$.
3. Thus far we have been assuming the condition (8.11). For general $(\vec{v}, \vec{w})$ this is not a generic choice of $\vec{\zeta}$. Thus, the instanton moduli space will have some singularities. These are associated to zero-scalesize limits.
4. The generalizations of these statements to $SO(w)$ instantons via Type I theories will be discussed in [4].

9.3. Torsion free sheaves

In the previous sections we have interpreted D-brane moduli spaces $\mathcal{M}_{\vec{\zeta}}(\vec{v}, \vec{w})$ under the condition (8.11). In general, when $\vec{w} \neq 0$ this condition is unnecessary. For example, on $X_1 = \mathbb{R}^4$ we can have $\vec{\zeta}_0 \neq 0$ by giving vacuum expectation values to the self dual parts of $B_{mn}$.

Introducing levels with $\sum \vec{\zeta} \neq 0$ changes two aspects of our understanding of the collective coordinate space.

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8 Curiously, the $\beta$ function of $SU(v_k)$ is $\beta(SU(v_k)) = \frac{v_k}{16\pi^2}$. That is, the beta function is the first Chern class! Moreover, it can be shown that $\mathcal{M}_{\vec{\zeta}}$ has regular points only when $u_k \geq 0$ [19]. This coincidence should have a simple physical explanation.

9 For type I it follows from (8.13).
First, $\mathcal{M}_\zeta(\vec{v}, \vec{w})$ has singularities when (8.11) holds but in general is smooth. Indeed one can define a map
\[ \mathcal{M}_\zeta(\vec{v}, \vec{w}) \to \mathcal{M}_0(\vec{v}, \vec{w}) \] (9.8)
which is a resolution of singularities [19] [24] [25]. Physical mechanisms often smooth out the singularities of moduli spaces and this is yet another example.

Second, when $\sum \zeta \neq 0$, $\mathcal{M}_\zeta(\vec{v}, \vec{w})$ can still be interpreted as a moduli space of geometric objects, which generalize Yang-Mills instantons. By the Donaldson-Uhlenbeck-Yau theorem (which continues to hold on the noncompact spaces $X_n(\zeta)$) the moduli of ASD instantons can be identified with the moduli of holomorphic vector bundles. When $\sum \zeta \neq 0$ we must generalize the notion of holomorphic vector bundle to that of torsion free sheaves. Without going into technical details this means – roughly – that the rank of the fiber can change at isolated points. We must introduce extra pointlike degrees of freedom.

For example, on $X_1$, $\mathcal{M}_0(v, w)$ is the moduli of $U(w)$ instantons of instanton number $v$. This space is singular because the scale size of an instanton can shrink to zero. By contrast for $\zeta \neq 0$, $\mathcal{M}_\zeta(v, w)$ is the moduli of rank $w$ torsion free sheaves $S$ with $ch_2(S) = v$ on $\mathbb{P}^2$. As an indication of how different these spaces can be note that for $w = 1$, there are no nontrivial line bundles on $X_1$ so $\mathcal{M}_0(v, 1)$ is a point. However, $\mathcal{M}_\zeta(v, 1)$ is isomorphic to the Hilbert scheme of points $[\mathbb{P}^2][v]$. For details and more precise statements see [24] [25]. This example generalizes to the other ALE spaces. Note that (9.8) may be interpreted as saying that $\mathcal{M}_\zeta(\vec{v}, \vec{w})$ is a smooth hyperkähler compactification of the moduli of instantons.

The generalization from vector bundles to sheaves is significant for several reasons. First, it indicates an important conceptual change since it introduces a wider and more flexible class of geometric objects in string compactification. Second, it has been observed [25] [26] [27] that the appearance of infinite dimensional algebras in the context of gauge theories, discovered by Nakajima [19], necessitates the generalization of vector bundles to sheaves. Moreover, these algebras are related - in ways not yet clearly understood - to duality symmetries in string theory and supersymmetric field theory. Third, several recent calculations [28] [29] [30] [31] [32] involving the counting of D-brane bound states have used the smooth hyperkähler resolution of instanton moduli space. The use of torsion free sheaves as a resolution of singularities of instanton moduli spaces deserves to be understood much better. The extra degrees of freedom somehow resolve the boundary of instanton moduli space, and these degrees of freedom are crucial to the counting of BPS states used in verifying predictions of duality in [28] [29] [31] [32].

Finally, we remark that there is one important qualitative difference between the moduli $\sum \zeta$ and the remaining moduli. Since the self-dual forms on $X_n$ are not normalizable, the modulus $\sum \zeta$ does not fluctuate, in contrast to the remaining degrees of freedom in $\zeta$. 
10. Conclusions

Strings can be sensibly compactified on certain singular spaces, producing completely non-singular effective field theories. Orbifolds were the first example of this – not only are they non-singular CFT’s, but turning on twisted sector moduli resolves the fixed points and produces a smooth manifold, verifying that these are limits of smooth manifolds with singular metric but non-singular physics.

In this work we showed that a straightforward treatment of D-branes on an orbifold reproduces the region of moduli space around this singular point in a simple and mathematically natural way. Furthermore, the D-branes provide descriptions of instanton moduli spaces.

We worked with a non-compact target space, an orbifold containing a single $\mathbb{Z}_n$ singularity, which when resolved becomes an ALE space. Moduli spaces of metrics and instantons on ALE spaces were constructed as hyperkähler quotients by Kronheimer and Nakajima [10,5] and the D-brane theory (for $U(n)$ gauge groups) reproduces their construction exactly. It is straightforward to get an explicit metric on the ALE spaces and moduli spaces from this construction, and indeed this has already been done in [10,5].

The qualitative structure of moduli spaces for compact target spaces obtained by resolving orbifolds, and in particular the enhanced gauge symmetry of the zero instanton size limit, will be determined by the behavior at the orbifold singularities. To find consistent models one must implement the tadpole conditions of [12]. It would be quite amusing if the solutions included models with fixed points of high order. The rank of the enhanced gauge symmetry for a model with $k$ 5-branes at a $\mathbb{Z}_n$ fixed point is roughly $r \sim kn$ (for type II; $r \sim kn/2$ for type I). Taking at face value the possibility of shrinking all instantons in a compactification with $p_1(V) = p_1(TM)$, one might have $k = p_1$, while in six dimensions $\mathbb{Z}_{12}$ orbifolds exist (and even higher order singularities on less symmetric manifolds), so it is conceivable that extremely large gauge groups can be obtained.

In string theory, these results are corrected at all orders in $1/\alpha'$ and the string coupling $\lambda$. By extending the world-sheet computation of the FI couplings as described in the appendix, it may be possible to obtain the exact (in $\alpha'$) relation between twist field moduli $\phi$ and periods $\zeta$, and the exact metric on moduli space. Although we expect corrections in the string coupling as well, using duality to combine these results with the known results for the heterotic string may allow controlling them.

In a sense, this model describes a change of topology on the microscopic level. Many examples of topology change are known, some involving D-branes, but so far the arguments are based on properties of the low-energy effective theory. In our example we have a complete microscopic theory realizing a very simple topology change, the resolution of a singularity. Perhaps other changes of topology can be similarly realized, helping to provide insight into the concepts which must replace metric and topology in a complete theory.
10.1. Reciprocity and T-duality

Finally, let us point out one very interesting direction for future research. The quiver diagrams, figs. 5-11, exhibit an intriguing symmetry: One can switch the inner quiver for the outer quiver. This corresponds to exchanging, say, 5-branes and 9-branes and hence corresponds to T-duality. It follows from the instanton interpretation that there should be a duality between \( U(w) \) instantons of charge \( v \) and \( U(v) \) instantons of charge \( w \), etc. For example, we expect that applying T-duality to configurations of \( (w, v) \) \((9, 5)\)-branes on \( T^4 \times \mathbb{R}^{5,1} \) with the 5-branes transverse to a torus \( T^4 \) gives a stringy realization (and generalization) of the Fourier-Nahm-Mukai transform \[34\] \[35\] \[36\] \[37\] \[38\]. It is straightforward to check that the mapping of the Chern classes \[37\] is precisely the mapping of RR charges predicted by T-duality.

A crucial role in the discussion of instanton reciprocity (and of the proof of completeness of the ADHM construction) is played by the Dirac operator in the field of an instanton. Significantly, the DN fields are valued in the spinor bundle on \( X_n(\tilde{\zeta}) \). Indeed, letting \( m = 1, \ldots, v \) label a basis of zeromodes of the Dirac operator and \( M = 1, \ldots, w \) denote indices with respect to a basis of \( E_\infty \), the asymptotic behavior of a solution of the Dirac equation in the field of an instanton is

\[
\psi^M_{mB}(x) \sim \tilde{h}^{AM} m \frac{x^A x^B}{x^4} \tag{10.1}
\]

in Euclidean coordinates \( x \). We hope the connection \ref{10.1} to fields of a string theory will lead to a deeper understanding of the completeness of the ADHM construction and of instanton reciprocity.

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Appendix A. World-sheet computation of twist field couplings

Here we verify the results (6.2) and (6.20) of section 6, by world-sheet computation. These come from the two-point function

$$\langle V(C_k^{(4)}) V(F) \rangle$$  \hspace{1cm} (A.1)

and the integrated three-point functions

$$\langle V(\phi_k^{CD}) V(X^{A'B'}) \int V(X^{B'B}) \rangle$$ \hspace{1cm} (A.2)

$$\langle V(\phi_k^{CD}) V(H^A) \int V(H^B) \rangle$$

on a disk with boundary on the D-brane, $F$, $\Psi$ or $H$ integrated over the boundary, and $\phi$ in the interior. In fact, we will only check the cross term in (6.21). (The remainder is determined by supersymmetry.)

The amplitude factorizes into three parts: the classical action of the relevant embedding of the disk into the target space, the quantum CFT amplitude, and the contribution of the Chan-Paton factors. If the D-brane is coincident with the fixed point, the embedding is trivial. More generally, the twist operator $V(H)$ will constrain the embedding to the fixed point at the insertion, while the Dirichlet boundary conditions are fixed at the position of the D-brane, so that the embedding will be non-trivial. The classical action will be proportional to the distance squared, leading to an overall factor $A \propto \exp -|X|^2/\alpha'$. However, we will not consider this case in further detail here, instead working in the limit $|X|^2 \ll \alpha'$.

We now explain the definition of Chan-Paton factors in twisted sectors. First, in an untwisted sector, we sum over all orderings of the operators on the boundary, and include the trace over their Chan-Paton matrices $\lambda$ with the same ordering:

$$A = \sum_{\sigma \in S_N} \int_0^{2\pi} d\theta_{\sigma(1)} \int_0^{\theta_{\sigma(1)}} d\theta_{\sigma(2)} \cdots \int_0^{\theta_{\sigma(N-1)}} d\theta_{\sigma(N)} \tr \lambda_{\sigma(1)} \cdots \lambda_{\sigma(N)} \langle V_1(\theta_1) \cdots V_N(\theta_N) \rangle.$$  \hspace{1cm} (A.3)

In a sector twisted by $g$, the contribution of the Chan-Paton factors is modified to be

$$\tr \gamma_g \lambda_{\sigma(1)} \cdots \lambda_{\sigma(N)}$$  \hspace{1cm} (A.4)

In defining the action of the twist on fields such as $X^{A'A}$ with internal Lorentz indices, we have been assuming that the twist can act both on Chan-Paton and Lorentz indices (as
in (3.3)), and that the projection retains the singlet under the combined action. While this
definition is certainly intuitive, we should check that it is consistent. What makes this less
than obvious is that we will be using operators with multi-valued world-sheet correlation
functions. For example, in (A.2), the operator $X$ transforms as an internal vector, so in
circling a $g^n$ twist field, its correlators will have monodromy $\xi^n$. We need to understand
in what sense the twist on the Chan-Paton factors can compensate for this.

The point will be to associate the cuts in a correlation function with the twist $\gamma_g$. We
thus choose a location $\theta_g$ for the twist on the boundary, and define correlators to be
single-valued except on a cut from the twist field to $\theta_g$. For example, to reproduce (A.4)
from the definition (A.3), we should let $\theta_g = 2\pi$.

The sum in (A.3) is now over all orderings of the vertex operators and $\gamma_g$. Since the
Chan-Paton matrices $\lambda(X)$ for the operators $X$ do not commute with $\gamma_g$, all orderings
can contribute to different correlation functions. In our calculations, $\lambda(X_{k,k+1})\gamma(g^n) =
\xi^n\gamma(g^n)\lambda(X_{k,k+1})$, and this difference will be reflected in a phase. After the discrete Fourier
transform of subsection 6.4, it is clear that these two amplitudes couple to the two different
$U(1)$ factors $k$ and $k+1$ under which $X_{k,k+1}$ is charged.

We should check that the particular value of $\theta_g$ does not appear in the final result. If
we change it to $\theta'_g$, we will modify the amplitude by taking the integral over $[\theta_g, \theta'_g]$ away
from one correlator (say $\text{tr} \gamma \lambda_1 \lambda_2$) and adding it, with the appropriate phase produced by
crossing the cut, to another correlator (say $\text{tr} \lambda_1 \gamma \lambda_2$). But the phase produced by the cut
will exactly compensate the phase produced by reordering the Chan-Paton factors, and
this will contribute to the same correlator.

For our purposes, this discussion suffices, assuming that these theories are sensible.
This remains to be proven by checking factorization and other consistency conditions. As
we will see below, the use of multi-valued operators adds new elements to the discussion,
and such a proof is an important open problem.

We proceed to the world-sheet calculation. The $SL(2, \mathbb{R})$ conformal symmetry of
the disk can be used to fix the positions of $V(\phi)$, $V(C)$ and one boundary operator. We
conformally transform the disk to the upper half plane and work on its double $\mathcal{C}$, mapping
the boundary to the real axis. The Dirichlet boundary conditions are encoded in simple
transformation properties for the fermions. We split the operator $V(\phi)$ into its original
chiral part at $z = i$ and the mirror of its anti-chiral part at $z = -i$.

A.1. Chern-Simons couplings

We need to choose a picture for the vertex operators compatible with the total superconformal ghost number $-2$ of the disk. For (A.1), since the $(-1/2, -1/2)$ picture for
$V(C)$ is by far the simplest, we use the $-1$ picture $V(A_\mu) = \psi^\mu e^{ikX} e^{-\phi}$, deriving the equivalent coupling

$$- \int A \wedge H^{(d-1)}$$  \hspace{1cm} \text{(A.5)}$$

Recall that for these we have the worldsheet correlator (for a IIB theory):

$$\langle S_\alpha(z)\psi^\mu(x)(\Gamma^{6789})_{\beta} S_\rho(\bar{z}) \rangle_{\text{IIB}} = \frac{(\Gamma^{6789}C)_{\alpha\beta}}{(z - x)^{1/2}(x - \bar{z})^{1/2}(\bar{z} - \bar{z})^{3/4}}$$  \hspace{1cm} \text{(A.6)}$$

leading to the untwisted coupling $\int \text{Tr} A_\mu H_{\alpha\beta}(\Gamma^{6789}C)_{\alpha\beta}$ where the trace is on Chan-Paton indices. For the twisted case the calculation is essentially the same as for untwisted couplings of this type. The twist eliminates the fermion zero modes in the internal space, (whether or not the brane is transverse), so the R-R boundary state is a bispinor in $d = 6$. $V(A_\mu)$ is trivial in the internal space, so the calculation simply reduces to the untwisted calculation in $d = 6$. This depends trivially on the dimension $p$ of the D-brane (e.g. see [89]).

A.2. FI - couplings

Some but not all of the amplitudes (A.2) are determined by supersymmetry. We do $\langle \phi XX \rangle$, the other one is similar. On symmetry grounds we must find

$$\langle V(\phi^C) V(X^{A'A}) \int V(X^{B'B}) \rangle \sim \epsilon^{A'B'}(\epsilon^{AB} \epsilon^{CD} + \epsilon^{A(C} \epsilon^{D)B}).$$  \hspace{1cm} \text{(A.7)}$$

The coupling $b$ is (6.20), expected by supersymmetry, but the coupling $a$ is unrelated by world-brane supersymmetry and directly couples the invariant $X\bar{X}$ to the singlet in (2.7) present in the type IIB strings (corresponding to the integral of $B$ about a two-cycle). Supersymmetry does not appear to be compatible with $a \neq 0$.

The NS-NS twist field is simplest in the $(-1,-1)$ picture, so we take the $V(X)$’s in the 0 picture. Now we need explicit twist field correlators. Fermi correlators can be computed via bosonization; e.g. write $\psi^{A'} = e^{iH A'}$ and $\bar{\psi}^{A'} = e^{-iH A'}$, (for left movers; resp. $\tilde{H}$ for right movers). Then one twisted massless state is $e^{iH_2 + i\tilde{H}_1} e^{ik(H_1 - H_2 + \tilde{H}_2 - \tilde{H}_1)/n} |0\rangle$. Bose correlators can be done as in [10]. The results we need (on the sphere and up to an overall function of $z_1 - z_2$) are

$$\epsilon_{C'D'} \langle \sigma_k \psi^{C'}(z_1) \psi^{A'A}(x_1) \psi^{B'B}(x_2) \psi^{D'D} \sigma_k^+(z_2) \rangle$$

$$= \epsilon^{A'B'} \epsilon^{AB} \epsilon^{CD} \left[ \frac{(z_1 - x_1)(-)^{A'k/n}}{(2 - z_1)} + \frac{(z_1 - x_2)(-)^{B'k/n}}{(z_2 - x_2)} \right] \times$$

$$\left[ \frac{\epsilon^{AB} \epsilon^{CD} + \epsilon^{AC} \epsilon^{BD}}{(z_1 - z_2)(x_1 - x_2)} + \frac{\epsilon^{AC} \epsilon^{BD}}{(z_1 - x_1)(z_2 - x_2)} + \frac{\epsilon^{AD} \epsilon^{BC}}{(z_1 - x_2)(z_2 - x_1)} \right]$$  \hspace{1cm} \text{(A.8)}$$

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\[ \langle \sigma_k \partial X^{A'}(x_1) \partial X^{B'}(x_2) \sigma_k^\dagger(z_2) \rangle = \epsilon^{A'B'} \epsilon^{AB} \left( \frac{z_1 - x_1}{z_2 - x_1} \right)^{-A'/k} \left( \frac{z_1 - x_2}{z_2 - x_2} \right)^{-B'/k} \times \]
\[ \frac{1}{(x_1 - x_2)^2} \left[ \left( 1 - \frac{k}{N} \right) + \frac{k}{N} \left( \frac{z_1 - x_1}{z_2 - x_1} \right)^{-A'} \left( \frac{z_1 - x_2}{z_2 - x_2} \right)^{-B'} \right]. \] (A.9)

A general correlation function is
\[ \langle V(\phi^{CD}) V(X^{2B}) \int V(X^{1A}) \rangle = 2 \int dx_1 \langle c \gamma \sigma_k \psi \psi \rangle e^{ip_1 X(z)} (i \psi^{1A} p_1 \cdot \psi + \partial X^{1A}) e^{ip_1 X(x_1)} c(i \psi^{2B} p_2 \cdot \psi + \partial X^{2B}) e^{ip_2 X(x_2)} c(\gamma \psi^{2D} \sigma_k^\dagger e^{ip_3 X(z)}). \] (A.10)

As in [41], we will need to take the momenta slightly off-shell: we can take \( \delta \equiv p_1 \cdot p_2, p_3 \cdot p_1 = p_3 \cdot p_2 = -\delta. \)

Evaluating the correlation function at \( z = i, x_2 = 0 \) and \( x_1 = x \) gives
\[ \int_0^\infty dx |x - i|^{-\delta} x^\delta \left( \frac{x - i}{x + i} \right)^{-A'/k} \left( -1 \right)^{-B'/k} \times \]
\[ \left[ \frac{\delta}{x} \left( \frac{\epsilon^{AB} \epsilon^{CD}}{2ix} + \frac{\epsilon^{AC} \epsilon^{BD}}{i(x - i)} - \frac{\epsilon^{AD} \epsilon^{BC}}{i(x + i)} \right) - \frac{\epsilon^{AB} \epsilon^{CD}}{2ix^2} \left( \left( 1 - \frac{k}{N} \right) - \frac{k}{N} \left( \frac{x - i}{x + i} \right)^{-A'} \right) \right]. \] (A.11)

The integral could produce poles as \( x \to 0 \) for \( \delta = 1, 0, \ldots \).

Let us first consider the triplet coupling to \( \bar{\sigma}_C^D \). This reduces the expression in square brackets to \( [2\delta/x(1 + x^2)]. \) Doing the integral one finds integral has a pole at \( \delta = 0 \), cancelling the factor of \( \delta. \) The \( \delta \to 0 \) limit is simply equal to 2, and in particular is independent of \( k/n. \)

The singlet coupling for \( k = 0 \) is also easy: the terms in the brackets combine to \( [\delta(1 + x^2)]. \) The integral is facilitated by changing variables \( x^2 = y, \) producing \( (\delta - 1) \Gamma((\delta - 1)/2) \Gamma(1/2)/\Gamma(\delta/2). \) The prefactor \( (\delta - 1) \) cancels the pole at \( \delta = 1, \) and the result is zero for \( \delta = 0. \)

The computation for \( k \neq 0 \) is substantially more difficult, and we only give the highlights. The double pole cancels as for \( k = 0; \) and there is no single pole (by the definition of the bosonic correlator). Getting the finite part requires doing the integral. We
took the cuts in the $\text{Im} x < 0$ half-plane, and rotated the contour to the positive imaginary axis. The branch point at $x = i$ requires dividing the contour into two parts and special care with the phases. A change of variables $x = it/(1 - t)$ turns these into $t \in [0, \frac{1}{2}]$ and $t \in [\frac{1}{2}, 1]$. Finally, the integrand can be rearranged into the form $\int (1 - 2t)^{-\delta/2}(d/dt)[\ldots]$, which is tractable.

The result is that the two contours combine to zero, with an appropriate choice of relative phase. One way to obtain this choice is to define the correlation function using ‘radial operator ordering.’ This will turn the cut in the $\langle \psi \sigma \sigma \rangle$ correlation functions into a cut at $|x| = 1$ with $\psi(1 + \epsilon) = e^{i \pi k/n} \psi(1 - \epsilon)$.

Without a complete analysis of world-sheet consistency it is not possible to prove that this is the only sensible choice. However, as we mentioned earlier, this coupling would not have been supersymmetric, which is the best argument for its vanishing. We give this calculation more as an illustration of a subtlety in the world-sheet definition of open strings on orbifolds which should be better understood.
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