A global perspective to connections on principal 2-bundles

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Abstract

For a strict Lie 2-group, we develop a notion of Lie 2-algebra-valued differential forms on Lie groupoids, furnishing a differential graded-commutative Lie algebra equipped with an adjoint action of the Lie 2-group and a pullback operation along Morita equivalences between Lie groupoids. Using this notion, we define connections on principal 2-bundles as Lie 2-algebra-valued 1-forms on the total space Lie groupoid of the 2-bundle, satisfying a condition in complete analogy to connections on ordinary principal bundles. We carefully treat various notions of curvature, and prove a classification result by the non-abelian differential cohomology of Breen-Messing. This provides a consistent, global perspective to higher gauge theory.

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A Formulary for calculations in strict Lie 2-algebras

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1 Introduction

We develop a new, global perspective to connections on principal 2-bundles, enjoying various nice analogies to connections on ordinary principal bundles. The underlying principal 2-bundles have Lie groupoids $\mathcal{P}$ as “total spaces”, which are fibred over a smooth manifold $M$ and carry an action of a
strict Lie 2-group $\Gamma$. As a first step we introduce, for $\gamma$ the Lie 2-algebra of $\Gamma$, a theory of $\gamma$-valued differential forms over Lie groupoids, furnishing a differential graded Lie algebra with an adjoint action of the 2-group $\Gamma$. Using this new language, the definition of a connection on a principal $\Gamma$-2-bundle (Definition 5.1.1) becomes so simple that we can repeat it here in almost one line: a connection on $P$ is a $\gamma$-valued 1-form $\Omega \in \Omega^1(P, \gamma)$ such that the equation

$$R^*\Omega = \text{Ad}_{pr_1}^{-1}(pr_1^*\Omega) + pr_1^*\Theta$$

holds over $P \times \Gamma$, where $pr_\pi$ and $pr_\Gamma$ are the projections to the two factors, and $R : P \times \Gamma \rightarrow P$ is the action. Further, $\Theta \in \Omega^1(\Gamma, \gamma)$ is a canonical $\gamma$-valued 1-form that we discover on every Lie 2-group $\Gamma$; a higher-categorical analog of the Maurer-Cartan form of an ordinary Lie group. We remark that above equation is literally the condition that characterizes connections on ordinary principal bundles. Our hope is that due to this analogy with ordinary gauge theory, our new approach conceptually illuminates higher gauge theory and makes it better accessible.

Before we explain the results of this paper in more detail, let us try to overview some steps in the development of principal 2-bundles. The idea of locally trivial fibre 2-bundles in the sense of a total space Lie groupoid has been introduced by Bartels [Bar04] and Baez-Schreiber [BS07]. A characterization of principal fibre 2-bundles has then been developed in terms of transition data, or cocycles, with respect to an open cover, in work of Schreiber and Baez [Sch05, BS07]. This approach leads directly to Giraud’s non-abelian cohomology [Gir71] as the corresponding classifying theory.

The first realization of a principal 2-bundle in the sense used here, has been developed by Wockel [Woc11]. Wockel also describes a classification of principal 2-bundles up to Morita equivalence by non-abelian cohomology. Schommer-Pries embedded Wockel’s principal 2-bundles into a more general framework, and assembled a whole bicategory of principal 2-bundles [SP11].

Different approaches are non-abelian bundle gerbes [ACJ05], which are equivalent to principal 2-bundles as 2-stacks [NW13], and $G$-gerbes [LGSX09], whose equivalence to principal 2-bundles was shown in [GS08]. Here, $\Gamma$ is the automorphism 2-group of a Lie group $G$.

The theory of connections on the before mentioned versions of principal 2-bundles is well-developed in several aspects, in particular by Breen-Messing [BM05], Baez and Schreiber [Sch05, BS07, Sch11], and also Jurco-Sämann-Wolf [JJS16] in terms of transition data, by Aschieri-Cantini-Jurco in terms of bundle gerbes [ACJ05], and by Laurent-Gengoux-Stiénon-Xu in terms of $G$-gerbes [LGSX09]. However, no treatment of connections on Wockel’s principal 2-bundles is available. The present article closes this gap.

One other aspect that is not much discussed is the question of existence of connections on non-abelian gerbes. Only for abelian 2-groups, existence of connections was proved by Murray [Mur96], and [LGSX09] contains a discussion for $G$-gerbes. Using our new approach, we makes at least a small contribution to this question (Theorem 5.2.14). A further motivation for this article is the increasing number of physical applications of connections on non-abelian gerbes, see e.g. [GK, FSS12, Par15, JJS16] for which our new perspective might be useful.

Let us now describe in more detail the results of this paper.

- We define a whole bicategory of principal 2-bundles with connection, whose 1-morphisms are ana-functors (a certain version of a Morita equivalence, a.k.a. Hilsum-Skandalis maps or bibundles) equipped with additional differential form data.
- We detect four different classes of connections, ordered by increasing generality: flat, fake-flat, regular, and fully general. Here, “fake-flat” has precisely the meaning of Breen-Messing [BM05],
and “regular” corresponds to the usual connections from the transition data picture. The fully general connections are not so attractive from the transition data point of view, but arise naturally in our global picture.

- We provide equivalences between our four classes of principal 2-bundles with connection and corresponding versions of non-abelian differential cohomology, established by explicit reconstruction and extraction procedures.

- We prove an existence result for connections (in the fully general sense) on principal $\Gamma$-bundles, under a mild assumption on the $\Gamma$-action. This assumption is, for instance, satisfied in the case of an abelian 2-group, so that our result is an extension of Murray’s existence result [Mur96].

The organization of the paper is straightforward. In Section 2 we recall some important concepts and results from the theory of Lie groupoids and Lie 2-groups. In Section 3 we set up the theory of principal 2-bundles on the basis of [Woc11, NW13]. As a new tool for working with principal 2-bundles we introduce the notion of a transition span for total spaces of principal 2-bundles and anafunctors (Definition 3.1.5). This tool will be used frequently in the forthcoming discussion and subsequent papers. In Section 4 we introduce our new notion of Lie 2-algebra-valued differential forms on Lie groupoids; this section is disconnected from the theory of principal 2-bundles, and might be useful for other purposes. In Section 5 we introduce and study connections on principal 2-bundles. For the convenience of the reader we include as a one-page appendix a formula for calculations in strict Lie 2-algebras.

There are two subsequent papers that continue our new approach: in [Wal] we construct the parallel transport of a connection in a principal 2-bundle, using horizontal lifts of paths and surfaces to its total space. In a second paper we will prove that the theory of connections on principal 2-bundles developed here is equivalent (as 2-stacks) to connections on non-abelian bundle gerbes, and so obtain a fully consistent and complete picture.

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2 Preliminaries

2.1 Lie 2-groups and Lie 2-algebras

We fix the following notation which will be used throughout the present article. If $\mathcal{X}$ is a Lie groupoid, we denote by $s, t : \text{Mor}(\mathcal{X}) \rightarrow \text{Obj}(\mathcal{X})$ the source and target maps, respectively, and by

$$\circ : \text{Mor}(\mathcal{X}) \times \text{Mor}(\mathcal{X}) \rightarrow \text{Mor}(\mathcal{X})$$

the composition. We recall that a (strict) Lie 2-group is a Lie groupoid $\Gamma$ with smooth functors

$$m : \Gamma \times \Gamma \rightarrow \Gamma \quad \text{and} \quad i : \Gamma \rightarrow \Gamma$$

that satisfy strictly the axioms of multiplication and inversion of a group, see e.g. [BC01, BS07]. Lie 2-groups are in one-to-one correspondence with crossed modules (of Lie groups): quadruples $(G, H, t, \alpha)$ consisting of Lie groups $G$ and $H$, a Lie group homomorphism $t : H \rightarrow G$, and a smooth action $\alpha : G \times H \rightarrow H$ of $G$ on $H$ by Lie group homomorphisms, such that

$$\alpha(t(h), x) = hxh^{-1} \quad \text{and} \quad t(\alpha(g, h)) = gt(h)g^{-1}$$
for all $g \in G$ and $h, x \in H$. The correspondence identifies $\text{Obj}(\Gamma) = G$ and $\text{Mor}(\Gamma) = H \ltimes G$ (the semi-direct product with respect to $\alpha$), $s(h,g) = g$ and $t(h,g) = t(h)g$ (the double usage of $t$ is unavoidable), and $(h_2,g_2) \circ (h_1,g_1) = (h_2h_1, g_1)$. We will instantly switch to the crossed module perspective whenever it seems to be convenient.

For a crossed module $(G,H,t,\alpha)$ we denote by $\alpha_g \in \text{Aut}(H)$ the action of a fixed $g \in G$ on $H$. For $h \in H$ we consider the map $\tilde{\alpha}_h : G \longrightarrow H$ defined by $\tilde{\alpha}_h(g) := h^{-1}\alpha(g,h)$, which is not a Lie group homomorphism but still satisfies $\tilde{\alpha}_h(1) = 1$.

Passing to the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of $G$ and $H$, respectively, the differential of $t$ gives a Lie algebra homomorphism $t_* : \mathfrak{h} \longrightarrow \mathfrak{g}$, and the differential of $G \longrightarrow \text{Aut}(H) : g \longrightarrow \alpha_g$ gives a Lie algebra action $\alpha_* : \mathfrak{g} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations, satisfying

\[ \alpha_*(t_*(Y_1), Y_2) = [Y_1, Y_2] \quad \text{and} \quad t_*\alpha_*(X, Y) = [X, t_*(Y)] \]

for all $X \in \mathfrak{g}$ and $Y_1, Y_2 \in \mathfrak{h}$. The collection $\gamma := (\mathfrak{h}, \mathfrak{g}, t_*, \alpha_*)$ forms the Lie 2-algebra of the Lie 2-group $\Gamma$. Apart from the usual adjoint actions of $H$ on $\mathfrak{h}$ and $G$ on $\mathfrak{g}$, we have – for $g \in G$ – the differential of $\alpha_g$, which is a Lie algebra homomorphism $(\alpha_g)_* : \mathfrak{h} \longrightarrow \mathfrak{h}$, and – for $h \in H$ – the differential of $\tilde{\alpha}_h$, which is a linear map $(\tilde{\alpha}_h)_* : \mathfrak{g} \longrightarrow \mathfrak{h}$.

### 2.2 Non-abelian differential cohomology

Non-abelian differential cohomology unifies Giraud’s non-abelian cohomology [Gir71] and differential cohomology, see e.g. [BM05, SW13]. Let $\Gamma$ be a Lie 2-group with crossed module $(G,H,t,\alpha)$. Let $M$ be a smooth manifold and let $U = \{U_i\}_{i \in I}$ be a cover of $M$ by open sets.

**Definition 2.2.1.** A differential $\Gamma$-cocycle for $U$ consists of the following data:

1. On every open set $U_i$, a 1-form $A_i \in \Omega^1(U_i, \mathfrak{g})$ and a 2-form $B_i \in \Omega^2(U_i, \mathfrak{h})$.

2. On every two-fold intersection $U_{ij} := U_i \cap U_j$, a smooth map $g_{ij} : U_{ij} \longrightarrow G$ and a 1-form $\varphi_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$.

3. On every three-fold intersection $U_{ijk} := U_i \cap U_j \cap U_k$, a smooth map $a_{ijk} : U_{ijk} \longrightarrow H$.

The cocycle conditions are the following:

1. Over every two-fold intersection $U_{ij}$,

\[
A_j + t_*(\varphi_{ij}) = \text{Ad}_{g_{ij}}(A_i) - g_{ij}^*\theta
\]

\[
B_j + d\varphi_{ij} + \frac{1}{2}[\varphi_{ij} \wedge \varphi_{ij}] + \alpha_*(A_j \wedge \varphi_{ij}) = (\alpha_{g_{ij}})_*(B_i)
\]

2. Over every three-fold intersection $U_{ijk}$,

\[
g_{ik} = (t \circ a_{ijk}) \cdot g_{jk}g_{ij}
\]

\[
\text{Ad}_{a_{ijk}}^{-1}(\varphi_{ik}) + (\tilde{\alpha}_{a_{ijk}})_*(A_k) = \varphi_{jk} + (\alpha_{g_{jk}})_*(\varphi_{ij}) - a_{ijk}\theta.
\]

3. Over every four-fold intersection $U_{ijkl}$,

\[
a_{ikl}(g_{kl}, a_{ijk}) = a_{ijl}a_{jkl}.
\]
A differential Γ-cocycle is called generalized if Eq. (2.2.2) is not required. Any differential Γ-cocycle furnishes two notions of curvature:

(a) the “3-curvature”
\[
\text{curv}(A_i, B_i) := dB_i + \alpha_*(A_i \wedge B_i) \in \Omega^3(U_i, \mathfrak{h})
\]
(b) the “fake-curvature”
\[
\text{fcurv}(A_i, B_i) := dA_i + \frac{1}{2}[A_i \wedge A_i] - t_*(B_i) \in \Omega^2(U_i, \mathfrak{g})
\]

We say that a differential Γ-cocycle is fake-flat, if \(\text{fcurv}(A_i, B_i) = 0\), and flat, if it is fake-flat and \(\text{curv}(A_i, B_i) = 0\). Two differential Γ-cocycles \((A, B, g, \varphi, a)\) and \((A', B', g', \varphi', a')\) are called equivalent, if the following structure exists:

(a) On every open set \(U_i\), a smooth map \(h_i : U_i \rightarrow G\) and a 1-form \(\phi_i \in \Omega^1(U_i, \mathfrak{h})\).

(b) On every two-fold intersection \(U_i \cap U_j\), a smooth map \(e_{ij} : U_i \cap U_j \rightarrow H\).

The following conditions have to be satisfied:

1. Over every open set \(U_i\),
\[
A'_i + t_*(\phi_i) = \text{Ad}_{h_i}(A_i) - h_i^*\theta
\]
\[
B'_i + \alpha_*(A'_i \wedge \phi_i) + d\phi_i + \frac{1}{2}[\phi_i \wedge \phi_i] = (A_{h_i})_*(B_i)
\]

2. Over every two-fold intersection \(V_i \cap V_j\),
\[
g'_{ij}h_i = t(e_{ij})h_jg_{ij}
\]
\[
\text{Ad}_{e_{ij}}^{-1}(\alpha'_{ij} + (\alpha'_{g_{ij}})_*(\phi_i)) + (\tilde{\alpha}_{e_{ij}})_*(A'_j) = \phi_j + (\alpha_{h_j})_*(\varphi_{ij}) - e_{ij}^*\theta
\]

3. Over every three-fold intersection \(V_i \cap V_j \cap V_k\),
\[
a'_{ijk} \alpha(g'_{jk}, e_{ij})e_{jk} = e_{ik} \alpha(h_k, a_{ijk}).
\]

An equivalence is called generalized if Eq. (2.2.7) is not required.

Remark 2.2.2. A differential Γ-cocycle is called normalized, if \(g_{ii} = 1\) and \(a_{ii} = a_{ij} = a_{ji} = 1\) for all \(i, j \in I\). Conditions \(g_{ii} = a_{ij} = a_{ji} = 1\) are easy to achieve by passing to an equivalent cocycle using \(e_{ii} := a_{ii}\) and \(e_{ij} := 1\) for \(i \neq j\). For condition \(a_{ij} = 1\) we choose a total order on the index set \(I\) of the open cover, eventually after discarding some open sets. Then, we set \(e_{ij} := \begin{cases} 1 & \text{if } i \leq j \\ a_{ij} & \text{if } i > j. \end{cases}\)

Passing to an equivalent cocycle preserves \(g_{ii} = a_{ii} = a_{ij} = a_{ji} = 1\) and achieves \(a_{ij} = 1\). Summarizing, every (generalized/fake-flat) differential Γ-cocycle is equivalent to a normalized (generalized/fake-flat) differential Γ-cocycle. This result slightly improves [SW13, Lemma 4.1.4].

\(\text{A smooth manifold is second-countable, hence Lindelöf; therefore, every open cover has a countable subcover. Countable sets have total orders (independently of the Axiom of Choice).}\)
We denote by $\hat{\mathcal{H}}^1(\mathcal{U}, \Gamma)$ the set of equivalence classes of differential $\Gamma$-cocycles, and define the differential cohomology with values in $\Gamma$ as the direct limit

$$\hat{\mathcal{H}}^1(M, \Gamma) := \lim_{\mathcal{U}} \mathcal{H}^1(\mathcal{U}, \Gamma)$$

over refinements of covers. Analogously we proceed with equivalence classes of fake-flat differential $\Gamma$-cocycles, leading to a subset

$$\hat{\mathcal{H}}^1(M, \Gamma)^{ff} \subseteq \hat{\mathcal{H}}^1(M, \Gamma).$$

Finally we consider generalized equivalence classes of generalized differential $\Gamma$-cocycles, making up a set $\hat{\mathcal{H}}^1(M, \Gamma)^{gen}$, for which the inclusion of differential $\Gamma$-cocycles into generalized ones induces a well-defined map

$$i : \hat{\mathcal{H}}^1(M, \Gamma) \longrightarrow \hat{\mathcal{H}}^1(M, \Gamma)^{gen}.$$  

This map is in general not injective (see Example 2.2.4).

Giraud’s non-abelian cohomology $\mathcal{H}^1(M, \Gamma)$ is defined in the same way as non-abelian differential cohomology, just without the differential forms and without all conditions involving differential forms. Discarding the differential forms defines maps from differential non-abelian cohomology to Giraud’s non-abelian cohomology. These fit into a commutative diagram

$$\begin{array}{ccc}
\hat{\mathcal{H}}^1(M, \Gamma)^{ff} & \xrightarrow{i} & \hat{\mathcal{H}}^1(M, \Gamma)^{gen} \\
p^{ff} \downarrow & & \downarrow p^{gen} \\
\mathcal{H}^1(M, \Gamma) & \xleftarrow{p^{ff}} & \mathcal{H}^1(M, \Gamma)
\end{array}$$

in which $p^{gen}$ is surjective (Corollary 3.3.4), and $p^{ff}$ is in general not surjective (Example 2.2.5). It seems to be an open question whether $p$ is surjective.

**Example 2.2.3.** We consider an ordinary Lie group $G$ and $\Gamma = G_{dis}$, i.e. $\Gamma$ has only identity morphisms. This means that $H$ is the trivial group. A (generalized or not) differential $G_{dis}$-cocycle consists only of the 1-forms $A_i \in \Omega^1(U_i, g)$ and the smooth maps $g_{ij} : U_i \cap U_j \longrightarrow G$, which satisfy the ordinary cocycle condition $g_{ik} = g_{jk} g_{ij}$ and are gauge transformations: $A_j = \text{Ad}_{g_{ij}}(A_i) - g_{ij}^* \theta$. The 3-cocycle is zero, and the fake-curvature is $dA_i + \frac{1}{2}[A_i, A_i]$. A (generalized or not) equivalence consists of smooth maps $h_i : U_i \longrightarrow G$ such that $A'_i = \text{Ad}_{h_i}(A_i) - h_i^* \theta$ and $g_{ij}' h_i = h_j g_{ij}$. We conclude that

$$\hat{\mathcal{H}}^1(M, G_{dis})^{gen} = \hat{\mathcal{H}}^1(M, G_{dis}) = \hat{\mathcal{H}}^1(M, G)$$

and

$$\hat{\mathcal{H}}^1(M, G_{dis})^{ff} = \hat{\mathcal{H}}^1(M, G)^{flat},$$

where $\hat{\mathcal{H}}^1(M, G)$ and $\hat{\mathcal{H}}^1(M, G)^{flat}$ are the ordinary differential cohomology groups that classify principal $G$-bundles with (flat) connections up to connection-preserving isomorphisms.

**Example 2.2.4.** We consider $\Gamma = BU(1)$, i.e. $\Gamma$ has only one object with automorphism group $U(1)$. This means that $H = U(1)$ and $G$ is the trivial group. We identify $\mathfrak{h} = \mathbb{R}$. A differential $BU(1)$-cocycle consists of the 2-forms $B_i \in \Omega^2(U_i)$, the 1-form $\varphi_{ij} \in \Omega^1(U_i \cap U_j)$ and the smooth map $a_{i,j,k} : U_i \cap U_j \cap U_k \longrightarrow U(1)$, which satisfy

$$B_j + d\varphi_{ij} = B_i$$  \hspace{1cm} (2.2.11)

and $\varphi_{ik} = \varphi_{jk} + \varphi_{ij} - a_{ij}^* \theta$ as well as the cocycle condition $a_{i,k,l} a_{i,j,k} = a_{i,j} a_{j,k,l}$. Generalized means that Eq. (2.2.11) is not present. The fake-curvature is zero, and the 3-curvature is $dB_i$. An equivalence consists of 1-forms $\phi_i \in \Omega^1(U_i, \mathfrak{h})$ and smooth maps $e_{ij} : U_i \cap U_j \longrightarrow H$ such that

$$B'_i + d\phi_i = B_i$$  \hspace{1cm} (2.2.12)
and $\varphi'_{ij} + \phi_i = \phi_j + \varphi_{ij} - e^*_ij\theta$ as well as $a'_{ijk}e_{ij}e_{jk} = e_{ik}a_{ijk}$. Generalized means that Eq. (2.2.12) is not present. We conclude that

$$\hat{H}^1(M, BU(1))^{ff} = \hat{H}^1(M, BU(1)) = \hat{H}^3(M)$$

is the ordinary differential cohomology (Deligne cohomology), whereas one can show that

$$\hat{H}^1(M, BU(1))^{gen} = \hat{H}^3(M, \mathbb{Z}).$$

Finally, we prove a lemma about the maps

$$j : \hat{H}^1(M, G) \longrightarrow \hat{H}^1(M, \Gamma) \quad \text{and} \quad j^{flat} : \hat{H}^1(M, G)^{flat} \longrightarrow \hat{H}^1(M, \Gamma)^{ff}$$

induced by the inclusion $G_{dis} \subseteq \Gamma$ and the identifications of Example 2.2.3. We will re-interpret the lemma in global language in Example [5.2.11] and Corollary [5.3.3].

**Lemma 2.2.5.** Let $f : M \longrightarrow N$ be a smooth map with $\text{rk}(T_x f) \leq 1$ for all $x \in M$. Then, the pullback along $f$ factors through $j$ and $j^{flat}$; i.e., there exist maps

$$j_G : \hat{H}^1(N, \Gamma)^{gen} \longrightarrow \hat{H}^1(M, G) \quad \text{and} \quad j_G^{flat} : \hat{H}^1(N, \Gamma)^{ff} \longrightarrow \hat{H}^1(M, G)$$

such that the following three diagrams commute:

$$
\begin{array}{ccc}
\hat{H}^1(M, G) & \xrightarrow{j} & \hat{H}^1(M, \Gamma) \\
\downarrow{j_G} & & \downarrow{j^*} \\
\hat{H}^1(N, \Gamma)^{gen} & & \hat{H}^1(N, \Gamma) \\
\downarrow{j_G^{flat}} & & \downarrow{j^*} \\
\hat{H}^1(N, \Gamma)^{ff} & & \hat{H}^1(M, G)^{flat} \\
\end{array}
$$

**Proof.** Consider a generalized differential $\Gamma$-cocycle $\xi = (A, B, g, \varphi, a)$ with respect to an open cover $\mathcal{V} = \{V_i\}_{i \in I}$. By Remark 2.2.2 we can assume that it is normalized. Let $X := f(M) \subseteq N$. By [Sar65, Theorem 2] and [Chu63, Proposition 1.3] the Lebesgue covering dimension of $X$ is bounded by 2. Thus, there exists a refinement $\{W_i\}_{i \in I}$ of $\mathcal{V}$, such that no point $x \in X$ is contained in more than two open sets. The restriction of the given $\Gamma$-cocycle to this refinement is again denoted by $(A, B, g, \varphi, a)$. Let $\{\psi_i\}_{i \in I}$ be a smooth partition of unity subordinated to the open cover $\{W_i\}_{i \in I}$. We define

$$\phi_i := \sum_{j \in I} \psi_j(a_{g^{-1}})_*(\varphi_{ij}) \in \Omega^1(W_i, \mathfrak{h}).$$

The cocycle conditions Eqs. (2.2.3) and (2.2.4) imply

$$\phi_j - (a_{g_{ij}})_*(\phi_i) = -\varphi_{ij} + \sum_{k \in I} \psi_k(a_{g^{-1}})_*(\tilde{\alpha}_{a_{ij,k}})_*(A_k) + a_{ijk}^*\theta. \quad (2.2.13)$$

Now we change the $\xi$ using equivalence data $\phi_i, h_i := 1$ and $e_{ij} := 1$, and obtain a new representative $(A', B', g', \varphi', a')$, with $g'_{ij} = g_{ij}$ due to Eq. (2.2.8) and then

$$\varphi'_{ij} = \phi_j - (a_{g'_{ij}})_*(\phi_i) + \varphi_{ij} = \sum_{k \in I} \psi_k(a_{g'_{ij-1}})_*(\tilde{\alpha}_{a_{ij,k}})_*(A_k) + a_{ijk}^*\theta. \quad (2.2.14)$$

due to Eqs. (2.2.9) and (2.2.13). Then we pullback under the map $f : M \longrightarrow N$. By construction, the pullback cover has no non-trivial 3-fold intersections, so that the map $a$ disappears due to its
normalization. In particular, we obtain \( \varphi_{ij} = 0 \) due to Eq. (2.2.14). Further, the pullback of a 2-form under \( f \) vanishes. Thus, we have
\[
f^*(A', B', g', \varphi', a') = (\tilde{A}, 0, \tilde{g}, 0, 1).
\]
This is a differential G_{dx}-cocycle on \( M \), i.e. \( j_G(\xi) := (\tilde{A}, \tilde{g}) \) is a differential G-cocycle, and the diagram on the left commutes. If \( \xi \) is non-generalized, the same construction goes through, noticing that above change of representative is also a non-generalized equivalence. Hence, the diagram in the middle commutes, too. Finally, if \( \xi \) is fake-flat, then \((\tilde{A}, \tilde{g})\) is flat; this defines \( j^\text{flat}_G \) and makes the third diagram commutative. \( \square \)

### 2.3 The bicategory of Lie groupoids

We recall that there is a bicategory with objects Lie groupoids, 1-morphisms anafunctors, and 2-morphisms smooth transformations, as explained in detail in [NW13, Section 2.3] on the basis of [Ler, Met]. This bicategory is equivalent to the bicategory of differentiable stacks [Pro96].

We recall some basic notation and terminology. A right action of a Lie groupoid \( \mathcal{X} \) on a smooth manifold \( M \) consists of smooth maps \( \alpha : M \longrightarrow \text{Obj}(\mathcal{X}) \) and \( \circ : M_{\alpha} \times_{\mathcal{X}} \text{Mor}(\mathcal{X}) \longrightarrow M \) such that
\[
(x \circ g) \circ h = (x \circ (g \circ h)) \quad \text{and} \quad \alpha(x \circ g) = s(g)
\]
for all possible \( g, h \in \text{Mor}(\mathcal{X}) \) and \( x \in M \). The map \( \alpha \) is called anchor. A left action of \( \mathcal{X} \) on \( M \) is a right action of the opposite Lie groupoid. A smooth map \( f : M \longrightarrow M' \) between manifolds with \( \mathcal{X} \)-actions is called \( \mathcal{X} \)-equivariant if
\[
\alpha' \circ f = \alpha \quad \text{and} \quad f(x \circ g) = f(x) \circ g.
\]
A principal \( \mathcal{X} \)-bundle over \( M \) is a smooth manifold \( P \) with a surjective submersion \( \pi : P \longrightarrow M \) and a right \( \mathcal{X} \)-action that respects the projection \( \pi \), such that
\[
P_{\alpha} \times_{\mathcal{X}} \text{Mor}(\mathcal{X}) \longrightarrow P \times_M P : (p, g) \longrightarrow (p, p \circ g)
\]
is a diffeomorphism. We refer to [NW13, Section 2.2] for some examples.

**Definition 2.3.1.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Lie groupoids.

(a) An anafunctor \( F : \mathcal{X} \longrightarrow \mathcal{Y} \) is a smooth manifold \( F \), called “total space”, a left action of \( \mathcal{X} \) on \( F \), and a right action of \( \mathcal{Y} \) on \( F \) such that the actions commute and the left anchor \( \alpha_l : F \longrightarrow \text{Obj}(\mathcal{X}) \) together with the right action of \( \mathcal{Y} \) is a principal \( \mathcal{Y} \)-bundle over \( \text{Obj}(\mathcal{X}) \).

(b) A transformation between anafunctors \( f : F \longrightarrow F' \) is a smooth map \( f : F \longrightarrow F' \) which is \( \mathcal{X} \)-equivariant, \( \mathcal{Y} \)-equivariant, and preserves the anchors.

**Remark 2.3.2.**

(a) The composition of anafunctors is defined as follows. Let \( \mathcal{X} \), \( \mathcal{Y} \) and \( \mathcal{Z} \) be Lie groupoids, and \( F : \mathcal{X} \longrightarrow \mathcal{Y} \) and \( G : \mathcal{Y} \longrightarrow \mathcal{Z} \) be anafunctors. The composition \( G \circ F : \mathcal{X} \longrightarrow \mathcal{Z} \) is the anafunctor with total space
\[
(F_{\alpha} \times_{\alpha_l} G)/\mathcal{Y},
\]
where the quotient identifies \( (f, \eta \circ g) \sim (f \circ \eta, g) \) for all \( \eta \in \text{Mor}(\mathcal{Y}) \) with \( \alpha_l(f) = t(\eta) \) and \( \alpha_l(g) = s(\eta) \). The anchors are \( (f, g) \mapsto \alpha_l(f) \) and \( (f, g) \mapsto \alpha_l(g) \), and the actions are \( \gamma \circ (f, g) := (\gamma \circ f, g) \) and \( (f, g) \circ \gamma := (f, g \circ \gamma) \).
(b) An anafunctor \( F : \mathcal{X} \to \mathcal{Y} \) is called \textit{weak equivalence}, if there exists an anafunctor \( G : \mathcal{Y} \to \mathcal{X} \) together with transformations \( G \circ F \cong \text{id}_\mathcal{X} \) and \( F \circ G \cong \text{id}_\mathcal{Y} \).

(c) It is straightforward to check that an anafunctor \( F : \mathcal{X} \to \mathcal{Y} \) is a weak equivalence if and only if \( \alpha_s : F \to \text{Obj}(\mathcal{Y}) \) is a principal \( \text{Mor}(\mathcal{X}) \)-bundle. In this case, one can obtain an inverse anafunctor \( F^{-1} : \mathcal{Y} \to \mathcal{X} \) with the same smooth manifold \( F \), but the anchors exchanged and the actions exchanged and inverted.

For Lie groupoids \( \mathcal{X} \) and \( \mathcal{Y} \), anafunctors \( F : \mathcal{X} \to \mathcal{Y} \) and transformations form a groupoid \( \text{Ana}^\infty(\mathcal{X}, \mathcal{Y}) \). The composition defined in Remark 2.3.2 (a) extends to a functor

\[
\text{Ana}^\infty(\mathcal{X}, \mathcal{Y}) \times \text{Ana}^\infty(\mathcal{Y}, \mathcal{Z}) \to \text{Ana}^\infty(\mathcal{X}, \mathcal{Z}).
\]

Equipped with this functor as the composition, Lie groupoids, anafunctors, and transformations form a bicategory [Pro96]. Weak equivalences are the 1-isomorphisms in this bicategory.

\textbf{Remark 2.3.3.}

(a) For Lie groupoids \( \mathcal{X} \) and \( \mathcal{Y} \), we denote by \( \text{Fun}^\infty(\mathcal{X}, \mathcal{Y}) \) the category of smooth functors and smooth natural transformations. Anafunctors generalize smooth functors in terms of a full and faithful functor

\[
\text{Fun}^\infty(\mathcal{X}, \mathcal{Y}) \to \text{Ana}^\infty(\mathcal{X}, \mathcal{Y}).
\]

Indeed, if \( \phi : \mathcal{X} \to \mathcal{Y} \) is a smooth functor, we define an anafunctor with total space \( \text{Obj}(\mathcal{X}) \times_t \text{Mor}(\mathcal{Y}) \), anchors \( \alpha_s(x, g) := x \) and \( \alpha_r(x, g) := s(g) \), and actions \( f \circ (x, g) := (t(f), \phi(f) \circ g) \) and \( (x, g) \circ f := (x, g \circ f) \). Likewise, a smooth natural transformation \( \eta : \phi \Rightarrow \phi' \) defines a transformation \( f_\eta : F \to F' \) by \( f_\eta(x, g) := (\eta(x), \phi(x) \circ g) \).

(b) A smooth functor \( \phi : \mathcal{X} \to \mathcal{Y} \) induces a weak equivalence under [a] if and only if it is \textit{smoothly essentially surjective}, i.e., the map

\[
s \circ \text{pr}_2 : \text{Obj}(\mathcal{X}) \times_t \text{Mor}(\mathcal{Y}) \to \text{Obj}(\mathcal{Y})
\]

is a surjective submersion, and it is \textit{smoothly fully faithful}, i.e., the diagram

\[
\begin{array}{ccc}
\text{Mor}(\mathcal{X}) & \xrightarrow{\phi} & \text{Mor}(\mathcal{Y}) \\
\downarrow s \times t & & \downarrow s \times t \\
\text{Obj}(\mathcal{X}) \times \text{Obj}(\mathcal{X}) & \xrightarrow{\phi \times \phi} & \text{Obj}(\mathcal{Y}) \times \text{Obj}(\mathcal{Y})
\end{array}
\]

is a pullback diagram; see [Lev] Lemma 3.34, [Met] Proposition 60.

(c) The following result will be used in Section 5.4 and might be useful in other situations. Suppose \( F : \mathcal{X} \to \mathcal{Y} \) is an anafunctor, \( \phi : \mathcal{X} \to \mathcal{Y} \) is a smooth functor, and \( F_\phi \) is the anafunctor associated to \( \phi \) via [a]. Then, a transformation \( f : F_\phi \Rightarrow F \) is the same as a smooth map \( \tilde{f} : \text{Obj}(\mathcal{X}) \to F \) satisfying

\[
\alpha_s(\tilde{f}(x)) = x \quad , \quad \alpha_r(\tilde{f}(x)) = \phi(x) \quad \text{and} \quad \alpha \circ \tilde{f}(x) \circ \beta = \tilde{f}(t(\alpha)) \circ \phi(\alpha) \circ \beta
\]

for all \( x \in \text{Obj}(\mathcal{X}) \), and appropriate \( \alpha \in \text{Mor}(\mathcal{X}) \) and \( \beta \in \text{Mor}(\mathcal{Y}) \). The correspondence between \( f \) and \( \tilde{f} \) is established by \( f(x) = \tilde{f}(x, \phi(id_x)) \).
2.4 Lie 2-group actions and equivariance

Next we discuss a version of the bicategory of Lie groupoids that is equivariant under the action of a Lie 2-group $\Gamma$, on the basis of [NW13]. It will be used in order to model the 2-group action on the total space groupoid of a principal 2-bundle.

**Definition 2.4.1.**

(a) A smooth right action of $\Gamma$ on a Lie groupoid $X$ is a smooth functor $R : X \times \Gamma \to X$ such that $R(p, 1) = p$ and $R(\rho, \text{id}_1) = \rho$ for all $p \in \text{Obj}(X)$ and $\rho \in \text{Mor}(X)$, and such that the diagram

$$
\begin{array}{ccc}
X \times \Gamma \times \Gamma & \xrightarrow{id \times m} & X \times \Gamma \\
R \times \text{id} & \downarrow & \downarrow R \\
X \times \Gamma & \xrightarrow{m} & X
\end{array}
$$

of smooth functors is commutative (strictly, on the nose).

(b) If $X$ and $Y$ are Lie groupoids with smooth right $\Gamma$-actions, then a $\Gamma$-equivariant anafunctor $F : X \to Y$ is an anafunctor $F$ together with a smooth action $\rho : F \times \text{Mor}(\Gamma) \to F$ of the Lie group $\text{Mor}(\Gamma)$ on its total space $F$ that preserves the anchors in the sense that the diagrams

$$
\begin{array}{ccc}
F \times \text{Mor}(\Gamma) & \xrightarrow{\alpha_l \times t} & F \\
\alpha_l \times s & \downarrow & \downarrow \alpha_r \\
\text{Obj}(X) \times \text{Obj}(\Gamma) & \xrightarrow{R} & \text{Obj}(X)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F \times \text{Mor}(\Gamma) & \xrightarrow{\alpha_l \times t} & F \\
\alpha_l \times s & \downarrow & \downarrow \alpha_r \\
\text{Obj}(Y) \times \text{Obj}(\Gamma) & \xrightarrow{R} & \text{Obj}(Y)
\end{array}
$$

are commutative, and is compatible with the $X$- and $Y$-actions in the sense that the identity

$$
\rho(\chi \circ f \circ \eta, \gamma_l \circ \gamma \circ \gamma_r) = R(\chi, \gamma_l) \circ \rho(f, \gamma) \circ R(\eta, \gamma_r)
$$

holds for all appropriate $\chi \in \text{Mor}(X)$, $\eta \in \text{Mor}(Y)$, $f \in F$, and $\gamma_l, \gamma, \gamma_r \in \text{Mor}(\Gamma)$.

(c) If $F_1, F_2 : X \to Y$ are $\Gamma$-equivariant anafunctors, a transformation $f : F_1 \to F_2$ between total spaces is $\Gamma$-equivariant if the map $f : F_1 \to F_2$ of the Lie group $\text{Mor}(\Gamma)$ is $\Gamma$-equivariant.

**Remark 2.4.2.**

(a) The structure of Definition 2.4.1 forms a bicategory. For instance, the composition $G \circ F$ of $\Gamma$-equivariant anafunctors $F : X \to Y$ and $G : Y \to Z$ with actions $\rho : F \times \text{Mor}(\Gamma) \to F$ and $\tau : G \times \text{Mor}(\Gamma) \to G$, respectively, inherits a $\text{Mor}(\Gamma)$-action induced by

$$
(F_{\alpha_r} \times \alpha_l) G \times \text{Mor}(\Gamma) \to (F_{\alpha_r} \times \alpha_l) G : ((f, g), \gamma) \mapsto (\rho(f, \gamma), \tau(g, \text{id}_{s(\gamma)})) \quad (2.4.1)
$$

On the inverse $F^{-1}$ of a weak equivalence $F : X \to Y$, $\text{Mor}(\Gamma)$ acts through inverses (with respect to the groupoid composition of $\Gamma$).
Consider a smooth functor \( \phi : X \to Y \) between Lie groupoids with \( \Gamma \)-actions that is equivariant in the sense that the diagram

\[
\begin{array}{ccc}
X \times \Gamma & \xrightarrow{\phi \times \text{id}} & Y \times \Gamma \\
\downarrow R & & \downarrow R \\
X & \xrightarrow{\phi} & Y
\end{array}
\]

is strictly commutative. Then, the associated anafunctor \( F = \text{Obj}(X) \times \phi \times \text{id} \to \text{Mor}(Y) \) is \( \Gamma \)-equivariant under the \( \text{Mor}(\Gamma) \)-action on \( F \) given by

\[
\rho : F \times \text{Mor}(\Gamma) \to F : (x, \eta, \gamma) \mapsto (R(x, t(\gamma)), R(\eta, \gamma)).
\]

Likewise, if \( \beta : \phi_1 \to \phi_2 \) is a smooth natural transformation between \( \Gamma \)-equivariant smooth functors, and satisfies the equivariance condition

\[
\beta(R(x, g)) = R(\beta(x), \text{id}_g)
\]

for all \( x \in \text{Obj}(X) \) and \( g \in G \), then the associated transformation is \( \Gamma \)-equivariant. Finally, if \( F : X \to Y \) is a \( \Gamma \)-equivariant anafunctor and \( \phi : X \to Y \) is a \( \Gamma \)-equivariant smooth functor, then the transformation \( f : F \to F \) induced from a smooth map \( \tilde{f} : \text{Obj}(X) \to \text{Obj}(Y) \) as in Remark 2.3.3(c) is \( \Gamma \)-equivariant if and only if that map satisfies \( \tilde{f}(R(x, g)) = \tilde{f}(x) \cdot \text{id}_g \) for all \( x \in \text{Obj}(X) \) and \( g \in G \).

(c) In [NW13, Section 6.1] a more abstract definition of \( \Gamma \)-equivariant anafunctors was given, and it is shown in [NW13, Appendix A] it is equivalent to the one given here.

3 Principal 2-bundles

3.1 The bicategory of principal 2-bundles

In this section we review the notion of a principal 2-bundle for a strict Lie 2-group \( \Gamma \) on the basis of [Woc11, NW13].

Let \( M \) be a smooth manifold, \( M_{\text{dis}} \) the Lie groupoid with objects \( M \) and only identity morphisms, and let \( \mathcal{P} \) be a Lie groupoid. We say that a smooth functor \( \pi : \mathcal{P} \to M_{\text{dis}} \) is a surjective submersion functor, if \( \pi : \text{Obj}(\mathcal{P}) \to M \) is a surjective submersion. Let \( \pi : \mathcal{P} \to M_{\text{dis}} \) be a surjective submersion functor, and let \( \mathcal{Q} \) be a Lie groupoid equipped with some smooth functor \( \chi : \mathcal{Q} \to M_{\text{dis}} \). Then, the fibre product \( \mathcal{P} \times_M \mathcal{Q} \) is defined to be the full sub-Lie groupoid of \( \mathcal{P} \times \mathcal{Q} \) over the submanifold \( \text{Obj}(\mathcal{P}) \times_M \text{Obj}(\mathcal{Q}) \subseteq \text{Obj}(\mathcal{P}) \times \text{Obj}(\mathcal{Q}) \).

**Definition 3.1.1.** A principal \( \Gamma \)-2-bundle over \( M \) is a Lie groupoid \( \mathcal{P} \), a surjective submersion functor \( \pi : \mathcal{P} \to M_{\text{dis}} \), and a smooth right action \( R \) of \( \Gamma \) on \( \mathcal{P} \) that preserves \( \pi \), such that the smooth functor

\[
\tau := (\text{pr}_1, R) : \mathcal{P} \times \Gamma \to \mathcal{P} \times_M \mathcal{P}
\]

is a weak equivalence.

We collect the following four facts about a principal 2-bundle \( \mathcal{P} \) over \( M \). The first fact comes from the condition that the bundle projection \( \pi : \mathcal{P} \to M_{\text{dis}} \) is a surjective submersion functor:
Lemma 3.1.2. Every point \( x \in M \) has an open neighborhood \( x \in U \subseteq M \) supporting a section, i.e. a smooth map \( s : U \to \text{Obj}(\mathcal{P}) \) such that \( \pi \circ s = \text{id}_U \).

As a functor \( \pi : \mathcal{P} \to M_{dis} \) necessarily sends every morphism \( \rho \) of \( \mathcal{P} \) to an identity morphism, we obtain the second fact:

Lemma 3.1.3. Every morphism \( \rho \) of \( \mathcal{P} \) is “vertical” in the sense that \( \pi(s(\rho)) = \pi(t(\rho)) \).

The next two lemmata give a precise formulation of the way in which the \( \Gamma \)-action on \( \mathcal{P} \) is “fibrewise free and transitive”. On the level of morphisms, one uses that \( \tau \), as a weak equivalence, is “smoothly fully faithfully” (Remark 2.3.3 (b)), which implies the following:

Lemma 3.1.4. Suppose \( \rho_1, \rho_2 \in \text{Mor}(\mathcal{P}) \) and \( g_1, g_2 \in G \) such that \( s(\rho_2) = R(s(\rho_1), g_1) \) and \( t(\rho_2) = R(t(\rho_1), g_2) \). Then, there exists a unique \( h \in H \) such that \( R(\rho_1, (h, g_1)) = \rho_2 \) and \( t(h)g_1 = g_2 \). Moreover, if \( \rho_1, \rho_2 \) and \( g_1, g_2 \) depend smoothly on a parameter \( x \in X \), where \( X \) is a smooth manifold, then \( h \) also depends smoothly on this parameter.

On the level of objects the situation is more complicated. We introduce the following terminology, which will be an important tool throughout this article:

Definition 3.1.5. A transition span of \( \mathcal{P} \) over a subset \( U \subseteq \text{Obj}(\mathcal{P}) \times_M \text{Obj}(\mathcal{P}) \) is a smooth map \( \sigma : U \to \text{Mor}(\mathcal{P}) \), such that there exists a smooth map \( g : U \to G \) with

\[
\begin{align*}
  t(\sigma(x,y)) &= x \quad \text{and} \quad s(\sigma(x,y)) = R(y,g(x,y))
\end{align*}
\]

for all \( (x,y) \in U \). The map \( g \) is called a transition function for \( \sigma \).

Thus, a transition span is a combination of morphisms of \( \mathcal{P} \) and the \( G \)-action, that allows to pass from one object \( x \) of \( \mathcal{P} \) to another object \( y \) in the same fibre. The following lemma describes the way in which this passage is “fibrewise free and transitive”.

Lemma 3.1.6. Every open and contractible subset \( U \subseteq \text{Obj}(\mathcal{P}) \times_M \text{Obj}(\mathcal{P}) \) supports transition spans. If \( \sigma \) and \( \sigma' \) are transition spans over an arbitrary open subset \( U \subseteq \text{Obj}(\mathcal{P}) \times_M \text{Obj}(\mathcal{P}) \) with transition functions \( g \) and \( g' \), respectively, then there exists a unique smooth map \( h : U \to H \) such that \( R(\sigma', (h, g^{-1}g)) = \sigma \) and \( g(t(h)) = g' \).

Proof. We define \( P := \text{Mor}(\mathcal{P}) \times G \) equipped with the map

\[
\chi : P \to \text{Obj}(\mathcal{P}) \times_M \text{Obj}(\mathcal{P}) : (\rho, g) \mapsto (t(\rho), R(s(\rho), g^{-1})).
\]

We define a \( \Gamma \)-action on \( P \) with anchor \( \alpha(\rho, g) := g \) and

\[
(\rho, g) \circ (h, g') := (R(\rho, (\alpha(g^{-1}, h), g^{-1}g')), g').
\]

(3.1.1)

By \cite[Lemma 7.1.1]{NW13}, \( P \) is a principal \( \Gamma \)-bundle over \( \text{Obj}(\mathcal{P}) \times_M \text{Obj}(\mathcal{P}) \). It is easy to check that writing a section \( \tilde{\sigma} : U \to P \) of \( P \) as a pair \( (\sigma, g) \) with \( \sigma : U \to \text{Mor}(\mathcal{P}) \) and \( g : U \to G \) establishes a bijection between sections of \( P \) and pairs of transition spans and transition functions. Now, the fact that a \( \Gamma \)-bundle has local sections over contractible open sets implies the existence of transition spans. Existence and uniqueness of the map \( h \) follow from Lemma 3.1.4. \( \quad \square \)
Remark 3.1.7. Transition spans split the tangent spaces to \( \text{Mor}(\mathcal{P}) \) into a “horizontal” part (the image of the transition span) and a “vertical” part (the orbit of the \( \Gamma \)-action). Indeed, let \( \rho \in \text{Mor}(\mathcal{P}) \), and let \( \sigma : U \longrightarrow \text{Mor}(\mathcal{P}) \) be a transition span defined on an open neighborhood \( U \subseteq \text{Obj}(\mathcal{P}) \times_{\text{M}} \text{Obj}(\mathcal{P}) \) of \( u := (t(\rho), s(\rho)) \), together with a transition function \( g : U \longrightarrow G \). Then, there exists a unique \( h \in H \) with \( \rho = R(\sigma(u), (h, g(u))) \), and a splitting
\[
T_p \text{Mor}(\mathcal{P}) = \text{TR}_{h,g(u)}(T_u U) \oplus \text{TR}_{\sigma(u)}(T_h H \oplus T_g(u)G).
\]
(3.1.2)

Here \( R_\rho : \text{Mor}(\Gamma) \longrightarrow \text{Mor}(\mathcal{P}) \) is the action on the fixed \( \rho \), and \( R_{h,g} : \text{Mor}(\mathcal{P}) \longrightarrow \text{Mor}(\mathcal{P}) \) is the action by fixed \( (h, g) \in \text{Mor}(\Gamma) \). In order to see Eq. (3.1.2), let a tangent vector \( \tilde{v} \in T_p \text{Mor}(\mathcal{P}) \) be represented by a smooth curve \( \tilde{\rho} \) in \( \text{Mor}(\mathcal{P}) \) with \( \tilde{\rho}(0) = \rho \). Let \( \gamma_1, \gamma_2 \) be the curves in \( \text{Obj}(\mathcal{P}) \) defined by \( \gamma_1 := t \circ \tilde{\rho} \) and \( \gamma_2 := s \circ \tilde{\rho} \), and let \( v_1, v_2 \) be the corresponding tangent vectors. The pair \( (\gamma_1, \gamma_2) \) is a curve in \( \text{Obj}(\mathcal{P}) \times_{\text{M}} \text{Obj}(\mathcal{P}) \), and we can assume that it is in \( U \). We apply Lemma 3.1.4 to the families \( \rho_1(t) := \gamma_1(t), \gamma_2(t) \) and \( \rho_2(t) := \tilde{\rho}(t) \), as well as \( g_1(t) := g(\gamma_1(t), \gamma_2(t)) \) and \( g_2(t) := 1 \). Hence, there exists a unique smooth curve \( \eta \) in \( H \) through \( h := \eta(0) \), with \( t \circ \eta \cdot g = 1 \) and
\[
\tilde{\rho}(t) = R(\gamma_1(t), \gamma_2(t)), (\eta(t), g(\gamma_1(t), \gamma_2(t)))).
\]
(3.1.3)

In particular, \( \rho = R(\sigma(u), (h, g(u))) \) at \( t = 0 \). Taking derivatives in Eq. (3.1.3) yields
\[
\tilde{v} = T_u R_{h,g(u)}T\sigma(v_1, v_2) + T_h g(u)R_{\sigma(u)}(w, T_u g(v_1, v_2)),
\]
where \( w \in T_h H \) is the tangent vector to the curve \( \eta \). This is the claimed splitting.

Next we discuss the bicategory of principal \( \Gamma \)-2-bundles, starting with the 1-morphisms.

Definition 3.1.8. A 1-morphism between principal \( \Gamma \)-2-bundles is a \( \Gamma \)-equivariant anafunctor \( F : \mathcal{P}_1 \longrightarrow \mathcal{P}_2 \) such that \( \pi_2 \circ F = \pi_1 \).

One can show that every 1-morphism between principal \( \Gamma \)-2-bundles is automatically invertible [NW13, Corollary 6.2.4]; in particular, every 1-morphism is a weak equivalence. Similar to Definition 3.1.5, we say that a transition span of \( F \) over a subset \( U \subseteq \text{Obj}(\mathcal{P}_1) \times_{\text{M}} \text{Obj}(\mathcal{P}_2) \) is a smooth map \( (\sigma : U \longrightarrow F) \) such that there exists a smooth map \( g : U \longrightarrow G \) with
\[
\alpha_1(\sigma(x_1, x_2)) = x_1 \quad \text{and} \quad \alpha_r(\sigma(x_1, x_2)) = R(x_2, g(x_1, x_2))
\]
for all \( (x_1, x_2) \in U \). The map \( g \) is called a transition function for \( \sigma \).

Lemma 3.1.9. Every open and contractible subset \( U \subseteq \text{Obj}(\mathcal{P}_1) \times_{\text{M}} \text{Obj}(\mathcal{P}_2) \) supports transition spans. Moreover, if \( g_1 \) and \( g_2 \) are transition spans over an arbitrary open subset \( U \subseteq \text{Obj}(\mathcal{P}_1) \times_{\text{M}} \text{Obj}(\mathcal{P}_2) \) with transition functions \( g_1 \) and \( g_2 \), respectively, then there exists a unique smooth map \( h : U \longrightarrow H \) such that \( g_1 = (t \circ h) \cdot g_2 \) and \( g_2 = \rho(g_1^{-1}, h, g_1^{-1}g_2) \), where \( \rho \) is the \( \text{Mor}(\Gamma) \)-action on \( F \).

Proof. We define \( Q := F \times G \) equipped with the map
\[
\chi : Q \longrightarrow \text{Obj}(\mathcal{P}_1) \times_{\text{M}} \text{Obj}(\mathcal{P}_2) : (f, g) \longmapsto (\alpha_l(f), R(\alpha_r(f), g^{-1}))).
\]
We define a \( \Gamma \)-action on \( Q \) with anchor \( \alpha(f, g) := g \) and
\[
(f, g) \circ (h, g') := (\rho(f, (\alpha(g^{-1}, h), g^{-1}g'))), g'),
\]
- 13 -
By [NW13] Lemma 7.1.4, $Q$ is a principal $\Gamma$-bundle over $\text{Obj}(\mathcal{P}_1) \times_M \text{Obj}(\mathcal{P}_2)$. Writing a section $\tilde{\sigma} : U \to Q$ of $Q$ as $\tilde{\sigma} = (\sigma, g)$ for smooth maps $\sigma : U \to F$ and $g : U \to G$ establishes a bijection between sections of $Q$ and pairs of transition spans and transition functions. Now the claims follows from the general properties of principal bundles. 

Remark 3.1.10. Transition spans split the tangent spaces to $F$ into a “horizontal” part (the image of the transition span) and a “vertical part” (the orbit of the $\text{Mor}(\Gamma)$-action). Indeed, let $f \in F$ and $U \subseteq \text{Obj}(\mathcal{P}_1) \times_M \text{Obj}(\mathcal{P}_2)$ be an open neighborhood of $u := (\alpha_1(f), \alpha_r(f))$, with a transition span $\sigma : U \to F$ with transition function $g$. Then, there exists a unique $h \in H$ such that $\tilde{f} = \rho(\sigma(u), (h, g(u)^{-1}))$ and

$$T_{\tilde{f}}F = T\rho_{h,g(u)}^{-1}(T\sigma(T_uU)) \oplus T\rho_{\sigma(u)}(T_{h,g(u)}^{-1}\text{Mor}(\Gamma)). \tag{3.1.4}$$

Here $\rho_{h,g} : F \to F$ is the action on $F$ with fixed $(h, g) \in \text{Mor}(\Gamma)$, and $\rho_f : \text{Mor}(\Gamma) \to F$ the action on a fixed $f \in F$. In order to see Eq. (3.1.4), let a tangent vector $v \in T_f F$ be represented by a smooth curve $\phi$ in $F$ with $\phi(0) = f$. Let $\gamma_1, \gamma_2$ be the curves defined by $\gamma_1 := \alpha_1 \circ \phi$ and $\gamma_2 := \alpha_r \circ \phi$, and let $v_1, v_2$ be the corresponding tangent vectors. The pair $(\gamma_1, \gamma_2)$ is a curve in $\text{Obj}(\mathcal{P}_1) \times_M \text{Obj}(\mathcal{P}_2)$, and we can assume that it is in $U$. Since $\alpha_1(\phi(t)) = \alpha_1(\sigma(\gamma_1(t), \gamma_2(t)))$ and $\alpha_1 : F \to \text{Obj}(\mathcal{P}_1)$ is a principal $\mathcal{P}_2$-bundle, there exists a unique curve $\tilde{\rho}$ in $\text{Mor}(\mathcal{P}_2)$ such that $\phi(t) = \sigma(\gamma_1(t), \gamma_2(t)) \circ \tilde{\rho}(t)$. Comparing $\tilde{\rho}(t)$ with $\text{id}_{\gamma_2(t)}$ via Lemma 3.1.4, we obtain a unique curve $\eta$ in $H$ such that

$$\phi(t) = \rho(\sigma(\gamma_1(t), \gamma_2(t)), (\eta(t), g(\gamma_1(t), \gamma_2(t))^{-1}).$$

Putting $h := \eta(0)$ and letting $w \in T_h H$ be the tangent vector of $\eta$, we obtain by taking derivatives

$$v = T_{\eta(u)}\rho_{h,g(u)}^{-1}(T_u\sigma(v_1, v_2)) + T_{h,g(u)}^{-1}\rho_{\sigma(u)}(w, T_u g^{-1}(v_1, v_2)).$$

This is the claimed splitting.

Our final definition is:

Definition 3.1.11. A 2-morphism between 1-morphisms is a $\Gamma$-equivariant transformation.

Principal $\Gamma$-2-bundles over $M$ form a bigroupoid that we denote by $2\text{-}\text{Bun}_\Gamma(M)$. Moreover, the assignment

$$M \to 2\text{-}\text{Bun}_\Gamma(M)$$

is a stack over the site of smooth manifolds [NW13 Theorem 6.2.1].

### 3.2 Classification by non-abelian cohomology

The following result has been obtain in [Woc11] and [NW13] Theorems 5.3.2 & 7.1:

Theorem 3.2.1. Principal $\Gamma$-2-bundles over $M$ are classified by Giraud’s non-abelian cohomology, in terms of a bijection

$$h_0(2\text{-}\text{Bun}_\Gamma(M)) \cong H^1(M, \Gamma),$$

where $h_0$ denotes the set of isomorphism classes of a bicategory.
In the remainder of this section we give a new direct proof of Theorem 3.2.1 which we will later refine to a situation “with connections”. Given a $\Gamma$-cocycle $(g, a)$ with respect to an open cover $\mathcal{U} = \{U_i\}_{i \in I}$, we define a Lie groupoid $\mathcal{P}_{(g, a)}$ with

$$\text{Obj}(\mathcal{P}_{(g, a)}) := \coprod_{i \in I} U_i \times G \quad \text{and} \quad \text{Mor}(\mathcal{P}_{(g, a)}) := \coprod_{i, j \in I} (U_i \cap U_j) \times H \times G.$$  

Target map and source map are defined by

$$s(i, j, x, h, g) := (j, x, g) \quad \text{and} \quad t(i, j, x, h, g) := (i, x, g)(x)^{-1}t(h)g,$$

and the composition is

$$(i, j, x, h_2, g_2) \circ (j, k, x, h_1, g_1) := (i, k, x, a_{ijk}(x)\alpha(g_{jk}(x), h_2)h_1, g_1).$$

A smooth functor $\pi : \mathcal{P}_{(g, a)} \longrightarrow M_{\text{dis}}$ is defined by $\pi(i, x, g) := x$. A smooth $\Gamma$-action is defined by

$$R((i, x, g), g') := (i, x, g'g) \quad \text{and} \quad R((i, j, x, h, g), (h', g')) := (i, j, x, h\alpha(g, h'), g'g).$$

We have three things to check:

1.) $\mathcal{P}_{(g, a)}$ is a principal $\Gamma$-2-bundle. The main part is to show that the functor $\tau$ in Definition 3.1.1 is a weak equivalence; this is straightforward and can be done using Remark 2.3.3 (b).

2.) If another $\Gamma$-cocycle $(g', a')$ is equivalent to $(g, a)$ via equivalence data $(h, e)$, then a smooth functor $\phi : \mathcal{P}_{(g, a)} \longrightarrow \mathcal{P}_{(g', a')}$ is defined by

$$\phi(i, x, g) := (i, x, h_i(x)g) \quad \text{and} \quad \phi(i, j, x, h, g) := (i, j, x, e_{ij}(x)\alpha(h_j(x), h), h_j(x)g).$$

The conditions for a functor are straightforward to check using Eqs. (2.2.8) and (2.2.10) for $(h, e)$. It is also straightforward to check that $\phi$ is $\Gamma$-equivariant. Thus, by Remark 2.4.2 (b) $\phi$ induces a 1-isomorphism $F : \mathcal{P}_{(g, a)} \longrightarrow \mathcal{P}_{(g', a')}$. 

3.) If $\mathcal{U'}$ is a refinement of $\mathcal{U}$, and $(g', a')$ is the refined $\Gamma$-cocycle, then the evident smooth functor $\phi : \mathcal{P}_{(g', a')} \longrightarrow \mathcal{P}_{(g, a)}$ is obviously $\Gamma$-equivariant, and hence a 1-isomorphism.

Now we have defined a “reconstruction” map

$$H^2(M, \Gamma) \longrightarrow \text{h}_0(\text{2-Bun}_\Gamma(M)). \quad (3.2.1)$$

In order to show that it is surjective, we extract a $\Gamma$-cocycle from a given principal $\Gamma$-2-bundle $\mathcal{P}$ over $M$. We make the following choices:

1.) A cover $\{U_i\}_{i \in I}$ of $M$ by contractible open sets with contractible double intersections, together with smooth sections $s_i : U_i \longrightarrow \text{Obj}(\mathcal{P})$.

2.) Transition spans $\sigma_{ij} : U_i \cap U_j \longrightarrow \text{Mor}(\mathcal{P})$ along $(s_i, s_j) : U_i \cap U_j \longrightarrow \text{Obj}(\mathcal{P}) \times_M \text{Obj}(\mathcal{P})$, together with transitions functions $g_{ij} : U_i \cap U_j \longrightarrow G$.

The sections $s_i$ exist due to Lemma 3.1.2 and the transition spans exist due to Lemma 3.1.6. Over a triple intersection $U_i \cap U_j \cap U_k$ it is straightforward to check that $\sigma_{ijk} := \sigma_{ij} \circ R(\sigma_{jk}, \text{id}_{g_{ij}})$ is a transition.
span along \((s_i, s_k)\) with associated transition function \(g_{jk}g_{ij}\). Comparing with the transition span \(\sigma_{ik}\) using Lemma 3.1.6, we obtain a unique smooth map \(a_{ijk} : U_i \cap U_j \cap U_k \rightarrow H\) satisfying
\[
t(a_{ijk})g_{jk}g_{ij} = g_{ik} \quad \text{and} \quad \sigma_{ijk} = R(\sigma_{ik}, (\alpha(g_{ik}^{-1}, a_{ijk}), g_{jk}^{-1}g_{jk}g_{ij})).
\] (3.2.2)

The first equation is the cocycle condition of Eq. (2.2.3). We compute the expression
\[\sigma_{ij} \circ R(\sigma_{jk}, \id_{g_{ij}}) \circ R(\sigma_{kl}, \id_{g_{jk}g_{ij}})\]
in two ways: (1) by substituting using Eq. (3.2.2) \(\sigma_{kl}\) and then \(\sigma_{ijl}\), leading to
\[
R(\sigma_{il}, (\alpha(g_{il}^{-1}, a_{ijl}, g_{il}^{-1}g_{ijl}g_{ij}), g_{il}^{-1}g_{ijl}g_{ij}g_{ij})).
\]
and (2) by substituting \(\sigma_{ijk}\) and then \(\sigma_{ikl}\), leading to
\[
R(\sigma_{il}, (\alpha(g_{il}^{-1}, a_{ikl}\alpha(g_{kl}, a_{ijk})), g_{il}^{-1}g_{ijl}g_{ij}g_{ij})).
\]

Equating the two results, the uniqueness in Lemma 3.1.4 implies \(a_{ijl}a_{ikl} = a_{ikl}\alpha(g_{kl}, a_{ijk})\), this is cocycle condition Eq. (2.2.4). There is a smooth \(\Gamma\)-equivariant functor \(\phi : \mathcal{P}(g,a) \rightarrow \mathcal{P}\) defined by
\[
\phi(i, x, g) := R(s_i(x), g) \quad \text{and} \quad \phi(i, j, x, h, g) := R(\sigma_{ij}(x), (\alpha(g_{ij}(x)^{-1}, h), g_{ij}(x)^{-1}g)).
\]

It induces a 1-isomorphism and hence shows that the map of Eq. (3.2.1) is surjective.

It remains to show that the map of Eq. (3.2.2) is injective. We consider two \(\Gamma\)-cocycles \((g, a)\) and \((g', a')\) with respect to the same open cover, and a 1-morphism \(F : \mathcal{P}(g,a) \rightarrow \mathcal{P}(g', a')\). Note that \(\mathcal{P}(g,a)\) has the sections \(s_i(x) := (i, x, 1)\), and the transition spans \(\sigma_{ij}(x) = (i, j, x, 1, g_{ij}(x))\) with transition functions \(g_{ij}\). Similarly \(\mathcal{P}(g', a')\) has sections \(s_i'\) and transition spans \(\sigma_{ij}'\). We choose transition spans \(\sigma_i : U_i \rightarrow F\) of \(F\) along \((s_i, s_i') : U_i \rightarrow \Obj(\mathcal{P}) \times_M \Obj(\mathcal{P})\) with transition functions \(h_i : U_i \rightarrow G\). Then, we obtain two transition spans of \(F\) along \((s_i, s_i') : U_i \cap U_j \rightarrow F\), namely \(\sigma_i \circ R(\sigma_{ij}', \id_{h_i})\), with transition function \(g_{ij}'h_i\), and \(\sigma_{ij} \circ R(\sigma_{ij}, \id_{g_{ij}})\), with transition function \(h_{ij}g_{ij}\). By Lemma 3.1.9 there exists a unique smooth map \(e_{ij} : U_i \cap U_j \rightarrow H\) such that
\[
t(e_{ij})h_{ij} = g_{ij}'h_i
\] (3.2.3)
and
\[
\sigma_{ij} \circ \rho(\sigma_{ij}, \id_{h_i}) = \rho(\sigma_i \circ R(\sigma_{ij}', \id_{h_i}), (\alpha(h^{-1}g_{ij}'^{-1}, e_{ij}), h^{-1}g_{ij}'^{-1}h_{ij}g_{ij})).
\] (3.2.4)

Now we compute \(\sigma_{ij} \circ R(\sigma_{jk}, \id_{g_{ij}}) \circ \rho(\sigma_{ik}, \id_{g_{ij}g_{ij}})\) in two different ways: (1) by substituting \(\sigma_{jk}'\) via Eq. (3.2.3), and then \(\sigma_{ik}'\) via Eq. (3.2.2), and (2) by substituting \(\sigma_{ik}\) via Eq. (3.2.2) and then \(\sigma_{ik}'\) via Eq. (3.2.4). Comparing the two results using that the right \(\mathcal{P}'\)-action on \(F\) is free and the uniqueness of Lemma 3.1.4 we can then conclude
\[
a_{ij}g_{ik}^{-1}a_{ijk} = e_{ik}a(h_{ik}, a_{ijk}).
\] (3.2.5)

Eqs. (3.2.3) and (3.2.5) show that the \(\Gamma\)-cocycles \((g, a)\) and \((g', a')\) are equivalent. This concludes the proof of Theorem 3.2.1.

*Remark 3.2.2.* If \(\Gamma\) is smoothly separable, there is a classifying space for principal \(\Gamma\)-2-bundles, namely the classifying space of the geometric realization of \(\Gamma\), \(B[\Gamma]\). Thus, there is a bijection
\[
\text{ho}(2\text{-}\mathcal{B}un_{\Gamma}(M)) \cong [M, B[\Gamma]]
\]
with the set of homotopy classes of continuous maps from \(M\) to \(B[\Gamma]\), see [BS09] and [NW13, Section 4].
4 Lie 2-algebra-valued differential forms on Lie groupoids

4.1 The differential graded Lie algebra of forms

We start with an arbitrary Lie 2-algebra \( \gamma = (h, g, t_*, \alpha_*) \) in the formalism described in Section 2.1. We give a short motivation for our Definition 4.1.1 below.

We regard \( \gamma \) as a cochain complex \( (\gamma^\bullet, t^\bullet) \) with \( \gamma^{-1} := h, \gamma^0 := g \) and \( \gamma_k = 0 \) for all \( k \neq 0, -1 \). The only non-trivial differential is \( t_{-1} := t^* \).

If \( P \) is a Lie groupoid, its nerve is a simplicial manifold \( P^\bullet \) with \( P^0 := \text{Obj}(P), P^1 := \text{Mor}(P), \) and \( P^k := \text{Mor}(P)_s \times \text{Mor}(P)_t \times ... \times \text{Mor}(P)_t \) the manifolds of \( k \) composable morphisms. We denote the alternating sum over the pullbacks along the face maps by \( \Delta_j : \Omega^*(P_j) \longrightarrow \Omega^*(P_{j+1}) \), satisfying \( \Delta_{j+1} \circ \Delta_j = 0 \). For instance, \( \Delta_0 = t^* - s^* \), and \( \Delta_1 = \text{pr}_1^* - c^* + \text{pr}_2^* \) where \( c : P_2 \longrightarrow P_1 \) is the composition.

We form the total complex \( T^p := \bigoplus_{p=i+j+k} \Omega^i(P_j, \gamma^k) \) and equip it with the differential \( \tau_{ijk} := d^i + (-1)^i \Delta_j + (-1)^i+j \Delta_k : T^p \longrightarrow T^{p+1} \).

We want to discard all elements of form degree smaller than the total degree ("virtual forms"), because these components do not naturally appear in the context of connections on principal 2-bundles. Hence, we decompose \( T^p = T_{\text{true}}^p \oplus T_{\text{virt}}^p \) with \( T_{\text{true}}^p \) consisting of those summands with form degree \( i \geq p \). The differential \( \tau \) does not restrict to one on \( T_{\text{true}}^p \). Thus, we also decompose

\[
\tau|_{T_{\text{true}}^p} = D \oplus D_{\text{virt}}
\]

with \( D : T_{\text{true}}^p \longrightarrow T_{\text{true}}^{p+1} \) and \( D_{\text{virt}} : T_{\text{true}}^p \longrightarrow T_{\text{virt}}^{p+1} \) and define

\[
\Omega^p(P, \gamma) := \ker(D_{\text{virt}}) \subseteq T_{\text{true}}^p.
\]

By construction, we obtain a well-defined differential \( D : \Omega^p(P, \gamma) \longrightarrow \Omega^{p+1}(P, \gamma) \) satisfying \( D^2 = 0 \). Spelling everything out, we obtain the following definition:

**Definition 4.1.1.** A \( p \)-form \( \Psi \in \Omega^p(P, \gamma) \) consists of three components \( \Psi = (\Psi^a, \Psi^b, \Psi^c) \), which are ordinary differential forms

\[
\Psi^a \in \Omega^p(P_0, g), \quad \Psi^b \in \Omega^p(P_1, h) \quad \text{and} \quad \Psi^c \in \Omega^{p+1}(P_0, h),
\]

such that \( \Delta \Psi^a = t_*(\Psi^b) \) and \( \Delta \Psi^b = 0 \). The differential of a \( p \)-form \( \Psi \) is the \( (p + 1) \)-form \( D\Psi \) with components

\[
(D\Psi)^a = d\Psi^a - (-1)^p t_*(\Psi^c) \\
(D\Psi)^b = d\Psi^b - (-1)^p \Delta(\Psi^c) \\
(D\Psi)^c = d\Psi^c.
\]
Next we define a graded product. Suppose \( \Psi \in \Omega^k(\mathcal{P}, \gamma) \) and \( \Phi \in \Omega^l(\mathcal{P}, \gamma) \). We define
\[
[\Psi \wedge \Phi] \in \Omega^{k+l}(\mathcal{P}, \gamma)
\]
in the following way:
\[
\begin{align*}
[\Psi \wedge \Phi]^a &= [\Psi^a \wedge \Phi^a] \\
[\Psi \wedge \Phi]^b &= \Psi^b \wedge \Phi^b + \alpha_s(s^s \Psi^a \wedge \Phi^b) - (-1)^{kl} \alpha_s(s^s \Phi^a \wedge \Psi^b) \\
[\Psi \wedge \Phi]^c &= \alpha_s(\Psi^a \wedge \Phi^c) - (-1)^{kl} \alpha_s(\Phi^a \wedge \Psi^c).
\end{align*}
\]
Well-definedness of this definition, bilinearity and the following lemma are straightforward to check.

**Lemma 4.1.2.** The graded product equips \( \gamma \)-valued differential forms with the structure of a differential graded Lie algebra, i.e. for \( \Psi \in \Omega^k(\mathcal{P}, \gamma) \), \( \Phi \in \Omega^l(\mathcal{P}, \gamma) \) and \( \Xi \in \Omega^i(\mathcal{P}, \gamma) \) we have:

(a) It is graded-anti-commutative,
\[
[\Phi \wedge \Psi] = -(-1)^{kl}[\Psi \wedge \Phi].
\]
(b) It satisfies a graded Jacobi identity,
\[
(-1)^{ki}[\Psi \wedge \Phi \wedge \Xi] + (-1)^{li}[\Xi \wedge \Psi \wedge \Phi] + (-1)^{kl}[\Phi \wedge \Xi \wedge \Psi] = 0.
\]
(c) It satisfies a Leibnitz rule with respect to the differential,
\[
D[\Psi \wedge \Phi] = [D\Psi \wedge \Phi] + (-1)^k[\Psi \wedge D\Phi].
\]

**Example 4.1.3.** We consider \( \Gamma = G_{\text{dis}} \) for a Lie group \( G \).

(a) A \( p \)-form \( \Psi \in \Omega^p(\mathcal{P}, \gamma) \) only has the component \( \Psi^a \in \Omega^p(\text{Obj}(\mathcal{P}), \mathfrak{g}) \) subject to the condition \( \Delta \Psi^a = 0 \). Differential and Lie bracket are the ordinary ones.
(b) If \( \mathcal{P} = X_{\text{dis}} \) we obtain \( \Omega^p(X_{\text{dis}}, \gamma) = \Omega^p(X, \mathfrak{g}) \) as differential graded Lie algebras.
(c) If \( \mathcal{P} = M/\!/K \) is the action groupoid associated to the action of a Lie group \( K \) on a smooth manifold \( M \), we obtain \( K \)-basic \( \mathfrak{g} \)-valued forms on \( M \),
\[
\Omega^p(M/\!/K, \gamma) = \Omega^p(M, \mathfrak{g})_{K\text{-basic}},
\]
as differential graded Lie algebras.

**Example 4.1.4.** We consider \( \Gamma = BA \), for an abelian Lie group \( A \) with Lie algebra \( \mathfrak{a} \).

(a) A \( p \)-form \( \Psi \in \Omega^p(\mathcal{P}, \gamma) \) has components \( \Psi^b \in \Omega^p(\text{Mor}(\mathcal{P}), \mathfrak{a}) \) and \( \Psi^c \in \Omega^{p+1}(\text{Obj}(\mathcal{P}), \mathfrak{a}) \), subject to the condition \( \Delta \Psi^b = 0 \). The differential is given by \( (D\Psi)^b = d\Psi^b - (-1)^p \Delta(\Psi^c) \) and \( (D\Psi)^c = d\Psi^c \).
(b) For the action groupoid \( \mathcal{P} = */\!/K \) we obtain “multiplicative” forms on \( K \):
\[
\Omega^p(*/\!/K, \gamma) = \{ \psi \in \Omega^p(K, \mathfrak{a}) \mid \forall g_1, g_2 \in K : \psi_{g_1} \cdot \psi_{g_2} = \psi_{g_1 g_2} \}.
\]

**Example 4.1.5.** Every Lie 2-group \( \Gamma \) supports a canonical, \( \gamma \)-valued “Maurer-Cartan” 1-form \( \Theta \in \Omega^1(\Gamma, \gamma) \). Its non-trivial components are \( \Theta^a \in \Omega^1(G, \mathfrak{g}) \) and \( \Theta^b \in \Omega^1(H \times G, \mathfrak{h}) \), given by
\[
\Theta^a := \theta^G \in \Omega^1(G, \mathfrak{g}) \quad \text{and} \quad \Theta^b := (\alpha_{\text{pr}_G})_*(\text{pr}_H^* \theta^H) \in \Omega^1(H \times G, \mathfrak{h}),
\]
where \( \theta^G \) and \( \theta^H \) are the ordinary Maurer-Cartan forms of the Lie groups \( G \) and \( H \), respectively. \( \Theta \) satisfies the Maurer-Cartan equation,
\[
D\Theta + \frac{1}{2}[\Theta \wedge \Theta] = 0. \tag{4.1.1}
\]
Remark 4.1.6. If \( \phi : \mathcal{P} \to \mathcal{Q} \) is a smooth functor between Lie groupoids, then we obtain an obvious pullback map

\[ \phi^* : \Omega^k(\mathcal{Q}, \gamma) \to \Omega^k(\mathcal{P}, \gamma) \]

defined component-wise. It is linear and commutes with \( D \) and the wedge product. One can show that if \( \eta : \phi_1 \to \phi_2 \) is a smooth natural transformation and \( \Phi \in \Omega^p(\mathcal{Q}, \gamma) \) is closed, then there exists \( \Phi_\eta \in \Omega^{p-1}(\mathcal{P}, \gamma) \) such that a “homotopy formula” holds,

\[ \phi_2^* \Phi - \phi_1^* \Phi = D \Phi_\eta. \]

We extend the pullback in Section 4.3 to anafunctors.

4.2 Adjoint action

If \( X \) is a smooth manifold equipped with a smooth map \( g : X \to G \), then we have an adjoint action \( \text{Ad}_g : \Omega^p(X, \gamma) \to \Omega^p(X, \gamma) \) on ordinary differential forms defined at each point \( x \in X \) by

\[ \text{Ad}_g(\varphi)_x := \text{Ad}_{g(x)}(\varphi_x). \]

We generalize this to \( \gamma \)-valued differential forms on a Lie groupoid \( \mathcal{P} \), where \( \gamma \) is now the Lie 2-algebra of a Lie 2-group \( \Gamma \).

Suppose we have a smooth functor \( F : \mathcal{P} \to \Gamma \). We write \( F_0 : \text{Obj}(\mathcal{P}) \to \text{Obj}(\Gamma) \), as well as \( F_G := \text{pr}_G \circ F : \text{Mor}(\mathcal{P}) \to \text{Mor}(\Gamma) \) and \( F_H := \text{pr}_H \circ F : \text{Mor}(\mathcal{P}) \to \text{H} \) on the level of morphisms. We define a linear map

\[ \text{Ad}_F : \Omega^k(\mathcal{P}, \gamma) \to \Omega^k(\mathcal{P}, \gamma) \]

in the following way:

\[
\begin{align*}
\text{Ad}_F(\Psi)^a &= \text{Ad}_{F_0}(\Psi^a) \\
\text{Ad}_F(\Psi)^b &= \text{Ad}_{F_G}((\alpha_{F_G})_*(\Psi^b)) + (\tilde{\alpha}_{F_H}^-)_*(\text{Ad}_{F_G}(s^*\Psi^a)) \\
\text{Ad}_F(\Psi)^c &= (\alpha_{F_0})_*(\Psi^c)
\end{align*}
\]

Well-definedness and the following lemma are straightforward to check.

Lemma 4.2.1. The adjoint action has the following properties:

(a) It is an action in the sense that

\[ \text{Ad}_1 = \text{id} \quad \text{and} \quad \text{Ad}_{F \cdot F'} = \text{Ad}_F \circ \text{Ad}_{F'}, \]

where \( 1 \) is the constant functor with values \( 1 \in \text{Obj}(\Gamma) \) and \( \text{id}_1 \in \text{Mor}(\Gamma) \), and \( F \cdot F' \) is the point-wise product of two \( \Gamma \)-valued functors. In particular, \( \text{Ad}_{F^{-1}} = \text{Ad}_{F^{-1}} \).

(b) It respects the Lie bracket:

\[ \text{Ad}_F([\Psi \wedge \Phi]) = [\text{Ad}_F(\Psi) \wedge \text{Ad}_F(\Phi)]. \]

(c) Its derivative is given by

\[ D(\text{Ad}^{-1}_F(\Psi)) = \text{Ad}^{-1}_F(D\Psi) - [F^*\Theta \wedge \text{Ad}^{-1}_F(\Psi)], \]

where \( \Theta \in \Omega^1(\Gamma, \gamma) \) is the Maurer-Cartan form of Example 4.1.5.

Example 4.2.2. Let \( m : \Gamma \times \Gamma \to \Gamma \) be the multiplication of \( \Gamma \), and let \( \Theta \in \Omega^1(\Gamma, \gamma) \) be the Maurer-Cartan form on the Lie 2-group \( \Gamma \), see Example 4.1.5. We have

\[ m^*\Theta = \text{Ad}^{-1}_{pr_2}(pr_1^*\Theta) + pr_2^*\Theta. \]
4.3 Pullback along anafunctors

We want to pull back differential forms along anafunctors. Not to much surprise, this pullback will be relative to additional structure on the anafunctor.

**Definition 4.3.1.** Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be an anafunctor and let $\Phi \in \Omega^p(\mathcal{Y}, \gamma)$. A **$\Phi$-pullback** on $F$ is a pair $\nu = (\nu_0, \nu_1)$ of a $p$-form $\nu_0 \in \Omega^p(F, \mathfrak{h})$ and a $(p+1)$-form $\nu_1 \in \Omega^{p+1}(F, \mathfrak{h})$ that satisfy over $F_{\sigma} \times \gamma_{\Phi}$ the conditions

\[
\begin{align*}
\rho^*_a \nu_0 &= \text{pr}_F^* \nu_0 + \text{pr}_{\text{Mor}(\mathcal{Y})}^* \Phi^b, \\
\rho^*_a \nu_1 &= \text{pr}_F^* \nu_1 + \text{pr}_{\text{Mor}(\mathcal{Y})}^* \Delta \Phi^c,
\end{align*}
\]

where $\rho_a$ denotes the right $\mathcal{Y}$-action.

**Lemma 4.3.2.** Let $\Phi \in \Omega^p(\mathcal{Y}, \gamma)$. Every anafunctor $F : \mathcal{X} \rightarrow \mathcal{Y}$ admits a $\Phi$-pullback, and the set of all $\Phi$-pullbacks on $F$ is an affine space.

**Proof.** We choose an open cover $\mathcal{V} = \{V_a\}_{a \in A}$ of $\text{Obj}(\mathcal{X})$ together with local trivializations $F\vert_{V_a} \cong V_a \times \gamma_{\Phi}$ of $F$, as a principal $\mathcal{Y}$-bundle over $\text{Obj}(\mathcal{X})$. We denote by $l_a : F\vert_{V_a} \rightarrow \text{Mor}(\mathcal{Y})$ the projection, which is equivariant in the sense that $l_a(f \circ \eta) = l_a(f) \circ \eta$ for all $f \in F\vert_{V_a}$ and $\eta \in \text{Mor}(\mathcal{Y})$. Let $\{\phi_a\}$ be a partition of unity subordinate to $\mathcal{V}$. Define

\[
\nu_0 := \sum_{a \in A} \alpha^*_a \phi_a \cdot l_a^* \Phi^b \quad \text{and} \quad \nu_1 := \sum_{a \in A} \alpha^*_a \phi_a \cdot l_a^* \Delta \Phi^c.
\]

Using the equivariance of $l_a$, the two conditions of Definition 4.3.1 are easy to check. We have an action of $\Omega^p(\text{Obj}(\mathcal{X}), \mathfrak{h}) \oplus \Omega^{p+1}(\text{Obj}(\mathcal{X}), \mathfrak{h})$ on the set of $\Phi$-pullbacks, where $(\kappa_0, \kappa_1)$ takes a $\Phi$-pullback $\nu$ to $(\nu_0 + \alpha^*_a \kappa_0, \nu_1 + \alpha^*_a \kappa_1)$. It is straightforward to see that this action is free and transitive. \quad \square

**Lemma 4.3.3.** Suppose $\nu$ is a $\Phi$-pullback on $F$. Then there exists a unique $\Psi \in \Omega^p(\mathcal{X}, \gamma)$ such that

\[
\begin{align*}
t_*(\nu_0) &= \alpha^*_a \Psi^a - \alpha^*_a \Phi^a, \\
\rho^*_a \nu_0 &= \text{pr}_{\text{Mor}(\mathcal{X})}^* \nu_0 + \text{pr}_F^* \nu_0, \\
\nu_1 &= \alpha^*_a \Psi^c - \alpha^*_a \Phi^c.
\end{align*}
\]

**Proof.** Uniqueness is clear because the pullbacks of $\Psi^a$, $\Psi^b$, and $\Psi^c$ along surjective submersions $(\alpha_1$ and $\text{pr}_{\text{Mor}(\mathcal{X})})$ are fixed. For existence, consider $\psi^a := t_*(\nu_0) + \alpha^*_a \Phi^a \in \Omega^p(F, \mathfrak{g})$. Over $F \times_{\text{Obj}(\mathcal{X})} F$ we have an identity $\text{pr}_1^* \psi^a = \text{pr}_2^* \psi^a$. Since differential forms form a sheaf, there exists $\Psi^a \in \Omega^p(\text{Obj}(\mathcal{X}), \mathfrak{g})$ such that $\alpha^*_a \Psi^a = \psi^a$. This shows Eq. (4.3.3). Next we consider an open subset $U \subseteq \text{Obj}(\mathcal{X})$ with a section $\sigma : U \rightarrow F$ against $\alpha_1$. On $V := s^{-1}(U) \subseteq \text{Mor}(\mathcal{X})$ we have a smooth map

\[
\tilde{\sigma} : V \rightarrow \text{Mor}(\mathcal{X}) \times_{\alpha_1} F : v \mapsto (v, \sigma(s(v))).
\]

We define $\psi^b := \tilde{\sigma}^*(\rho^*_a \nu - \text{pr}_F^* \psi^a) \in \Omega^p(V, \mathfrak{h})$. EQ. (4.3.1) implies that $\psi^b$ is independent of the choice of $\sigma$. Hence we obtain $\Psi^b \in \Omega^p(\text{Mor}(\mathcal{X}), \mathfrak{h})$ such that $\Psi^b|_V = \psi^b$. It is straightforward to check the conditions $\Delta \Psi^a = t_*(\psi^b)$ and $\Delta \Psi^b = 0$. The definition of $\psi^b$ implies Eq. (4.3.1). Finally, we consider $\psi^c := \alpha^*_a \Phi^c + \nu_1 \in \Omega^{p+1}(F, \mathfrak{h})$. EQ. (4.3.2) implies over $F \times_{\text{Obj}(\mathcal{X})} F$ an identity $\text{pr}_1^* \psi^c = \text{pr}_2^* \psi^c$. Thus, there exists $\Psi^c \in \Omega^{p+1}(\text{Obj}(\mathcal{X}), \mathfrak{h})$ such that $\alpha^*_a \Psi^c = \psi^c$. This shows Eq. (4.3.3). \quad \square
We write $F_{\nu}^* \Phi := \Psi$ for the unique $p$-form of Lemma 4.3.3. The following two lemmata are straightforward to deduce from the definitions. The first describes the compatibility of the pullback with the Lie algebra structure on differential forms.

**Lemma 4.3.4.** Let $F : \mathcal{X} \to \mathcal{Y}$ be an anafunctor. We consider differential forms $\Phi \in \Omega^k(\mathcal{Y}, \mathcal{V})$ and $\Phi' \in \Omega^k(\mathcal{Y}, \mathcal{V})$, a $\Phi$-pullback $\nu$ on $F$, a $\Phi'$-pullback $\nu'$ on $F$, and $s \in \mathbb{R}$.

(a) $\nu + \nu' := (\nu_0 + \nu'_{0}, \nu_1 + \nu'_{1})$ is a $(\Phi, \Phi')$-pullback on $F$, and $F_{\nu + \nu'}^*(\Phi + \Phi') = F_{\nu}^* \Phi + F_{\nu'}^* \Phi'$.

(b) $s \nu := (s\nu_0, s\nu_1)$ is a $(s\Phi)$-pullback on $F$, and $F_{s\nu}^*(s\Phi) = sF_{\nu'}^* \Phi$.

(c) $D\nu := (d\nu_0 - (-1)^k\nu_1, d\nu_1)$ is a $D\Phi$-pullback on $F$, and $D(F_{\nu}^* \Phi) = F_{D\nu}^*(D\Phi)$.

(d) $[\nu \wedge \nu']$ defined by

$$[\nu \wedge \nu']_0 := [\nu_0 \wedge \nu'_{0}] - (-1)^k \alpha_r(\alpha_s^* F_{\nu}^* \wedge \nu_0),$$

$$[\nu \wedge \nu']_1 := [\nu_0 \wedge \nu_1] + (-1)^k [\nu_1 \wedge \nu'_{0}] + [\nu_0 \wedge \alpha^*_s F_{\nu}^*] + (-1)^k [\alpha_s^* F_{\nu}^* \wedge \nu_0]$$

$$+ \alpha_r(\alpha_s^* F_{\nu}^* \wedge \nu_1) - (-1)^k \alpha_r(\alpha_s^* F_{\nu}^* \wedge \nu_1)$$

is a $[\Phi \wedge \Phi']$-pullback on $F$, and $F_{[\nu \wedge \nu']}^*([\Phi \wedge \Phi']) = [F_{\nu}^* \Phi \wedge F_{\nu'}^* \Phi']$.

The second lemma describes the compatibility of the pullback with the structure of anafunctors: composition, inversion, and transformations (see Remark 2.3.2).

**Lemma 4.3.5.**

(a) Suppose $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{Z}$ are anafunctors. If $\Phi \in \Omega^k(\mathcal{Z}, \mathcal{V})$, $\nu$ is a $\Phi$-pullback on $G$, and $\nu'$ is a $(G^* \Phi)$-pullback on $F$, then the forms $\tilde{\nu}_0 := pr_0^* \nu_0 + pr_0^* G_0$ and $\tilde{\nu}_1 := pr_1^* \nu_1 + pr_1^* G_1$ on $F^* \alpha^* \wedge \alpha^*_r G$ descend to the total space $(F^* \alpha^* \wedge \alpha^*_r G)/\mathcal{Y} \circ G$, and $\nu' \circ \nu := (\tilde{\nu}_0, \tilde{\nu}_1)$ is a $\Phi$-pullback on $G \circ F$ such that

$$(G \circ F)^*_{\nu' \circ \nu} \Phi = F_{\nu'}^*(G_{\nu}^* \Phi).$$

(b) Suppose an anafunctor $F : \mathcal{X} \to \mathcal{Y}$ is a weak equivalence. If $\Phi \in \Omega^k(\mathcal{Y}, \mathcal{V})$, $\nu$ is a $\Phi$-pullback on $F$, and $\Psi := F_{\nu}^* \Phi$, then $-\nu := (-\nu_0, -\nu_1)$ is a $\Psi$-pullback on the inverse anafunctor $F^{-1}$, and $(F_{-\nu}^{-1})^* \Psi = \Phi$.

(c) Suppose $f : F \to G$ is a transformation between anafunctors $F, G : \mathcal{X} \to \mathcal{Y}$. If $\Phi \in \Omega^k(\mathcal{Y}, \mathcal{V})$ and $\nu$ is a $\Phi$-pullback on $G$, then $f^* \nu := (f^* \nu_0, f^* \nu_1)$ is a $\Phi$-pullback on $F$ such that $F_{f^* \nu}^* \Phi = G_{\nu}^* \Phi$.

**Remark 4.3.6.**

(a) If $\phi : \mathcal{X} \to \mathcal{Y}$ is a smooth functor, then the anafunctor $F$ associated to $\phi$ has a canonical $\Phi$-pullback $\nu$ such that $F_{\nu}^* \Phi = \phi^* \Phi$. More concretely, recall that the total space of $F$ is $F = \text{Obj}(\mathcal{X}) \times_{\mathcal{V}} \text{Mor}(\mathcal{Y})$. Then, the canonical $\Phi$-pullback $\nu$ is given by $\nu_0 := pr_0^* \Phi^b$ and $\nu_1 := -pr_1^* s^* \Phi^c + pr_1^* \phi^* \Phi^c$.

(b) Using the affine space structure of Lemma 4.3.2, the canonical $\Phi$-pullback $\nu$ on $F$ can be shifted by $\kappa = (\kappa_0, \kappa_1)$, where $\kappa_0 \in \Omega^k(\mathcal{X}, \mathcal{H})$ and $\kappa_1 \in \Omega^{k+1}(\mathcal{X}, \mathcal{H})$, namely to $\nu^\kappa := (\nu_0 + pr_1^* \kappa_0, \nu_1 + pr_1^* \kappa_1)$. For the shifted pullback $\nu^\kappa$ we find

$$(F_{\nu}^* \Phi)^0 = \phi^* \Phi^0 + \Delta \kappa_0 , \quad (F_{\nu}^* \Phi)^1 = \phi^* \Phi^1 + \Delta \kappa_0 , \quad (F_{\nu}^* \Phi)^2 = \phi^* \Phi^2 + \kappa_1 , \quad (F_{\nu}^* \Phi)^3 = \phi^* \Phi^3 + \kappa_1 .$$

(4.3.6)
(c) If $\eta : \phi \Rightarrow \phi'$ is a smooth natural transformation, then the induced transformation $f : F \Rightarrow F'$ between the corresponding anafunctors in general does not preserve the canonical $\Phi$-pullbacks $\nu$ and $\nu'$, i.e., $f^*\nu' \neq \nu$. However, if $\kappa$ and $\kappa'$ are shifts for the canonical $\Phi$-pullbacks, then $f$ satisfies $f^*(\nu^\kappa) = \nu^\kappa$ if and only if $\eta^*\Phi^b = \kappa_0 - \kappa'_0$ and $\eta^*\Delta\Phi^c = \kappa_1 - \kappa'_1$.

(d) Suppose $F' : \mathcal{X} \rightarrow \mathcal{Y}$ is an anafunctor equipped with a $\Phi$-pullback $\nu' = (\nu'_0, \nu'_1)$ and $f : F \Rightarrow F'$ is the transformation induced by a smooth map $\tilde{f} : \text{Obj}(\mathcal{X}) \rightarrow F'$ via Remark 2.3.3 (c). Then, we have $f^*\nu' = \nu^\kappa$, where the shift $\kappa = (\kappa_0, \kappa_1)$ is defined by $\kappa_0 := \tilde{f}^*\nu'_0$ and $\kappa_1 := \tilde{f}^*\nu'_1$.

5 Connections on principal 2-bundles

In this section, $M$ is a smooth manifold and $\Gamma$ is a Lie 2-group, with associated crossed module $(G, H, t, \alpha)$ and Lie 2-algebra $\gamma$.

5.1 Connections and curvature

Let $\mathcal{P}$ be a principal $\Gamma$-2-bundle over $M$.

\textbf{Definition 5.1.1.} A connection on $\mathcal{P}$ is a $\gamma$-valued 1-form $\Omega \in \Omega^1(\mathcal{P}, \gamma)$ such that

$$R^*\Omega = \text{Ad}_{pr_\gamma}^{-1}(pr_\gamma^*\Omega) + pr_\Gamma^*\Theta$$

over $\mathcal{P} \times \Gamma$, where $pr_\gamma$ and $pr_\Gamma$ are the projections to the two factors, and $\Theta$ is the Maurer-Cartan form on $\Gamma$.

Let us spell out explicitly all structure and conditions that are packed into Definition 5.1.1. A connection $\Omega$ consists of the following components:

$$\Omega^a \in \Omega^1(\text{Obj}(\mathcal{P}), g) \quad , \quad \Omega^b \in \Omega^1(\text{Mor}(\mathcal{P}), h) \quad \text{and} \quad \Omega^c \in \Omega^2(\text{Obj}(\mathcal{P}), h).$$

These satisfy the following conditions:

$$R^*\Omega^a = \text{Ad}_{g_0}^{-1}(p_0^*\Omega^a) + g_0^*\theta \quad \text{over} \; \text{Obj}(\mathcal{P}) \times \text{Obj}(\Gamma) \quad (5.1.1)$$

$$R^*\Omega^b = (a_{g_1})_* (\text{Ad}_{h}^{-1}(p_1^*\Omega^b) + (\tilde{a}_h)_*)(p_1^*s^*\Omega^c) + h^*\theta \quad \text{over} \; \text{Mor}(\mathcal{P}) \times \text{Mor}(\Gamma) \quad (5.1.2)$$

$$R^*\Omega^c = (a_{g_0})_* (p_0^*\Omega^c) \quad \text{over} \; \text{Obj}(\mathcal{P}) \times \text{Obj}(\Gamma). \quad (5.1.3)$$

Here we have used the maps $p_0, g_0$ defined on $\text{Obj}(\mathcal{P}) \times \text{Obj}(\Gamma)$ by $p_0(p, g) := p$ and $g_0(p, g) := g$, as well as the maps $h, p_1, g_1$ defined on $\text{Mor}(\mathcal{P}) \times \text{Mor}(\Gamma)$ by $h(p, (h, g)) := h$, $p_1(p, (h, g)) := \rho$, and $g_1(\rho, (h, g)) := g$.

\textbf{Remark 5.1.2.} For $g \in G$ we have a functor $R_g : \mathcal{P} \rightarrow \mathcal{P} : p \mapsto R(p, g)$ and Definition 5.1.1 implies $R_g^*\Omega = \text{Ad}_g^{-1}(\Omega)$. Similarly, for $p \in \text{Obj}(\mathcal{P})$ we have a functor $R_p : \mathcal{P} \rightarrow \mathcal{P} : g \mapsto R(p, g)$, and we obtain $R_p^*\Omega = \Theta$. The two equations

$$R_g^*\Omega = \text{Ad}_g^{-1}(\Omega) \quad \text{and} \quad R_p^*\Omega = \Theta \quad (5.1.4)$$

are well-known for connections in ordinary principal bundles, where they are in fact equivalent to the combined equation analogous to the one in Definition 5.1.1. For connections on principal 2-bundles, however, Eq. (5.1.4) is not equivalent to Definition 5.1.1 (consider, e.g., $\Gamma = BU(1)$).
Definition 5.1.3. The 2-form
\[
\text{curv}(\Omega) := D\Omega + \frac{1}{2}[\Omega \wedge \Omega] \in \Omega^2(\mathcal{P}, \gamma)
\]
is called the curvature of \(\Omega\).

Thus, the curvature has the following components:
\[
\begin{align*}
\text{curv}(\Omega)^a &= d\Omega^a + \frac{1}{2}[\Omega^a \wedge \Omega^a] + t_*(\Omega^c) \\
\text{curv}(\Omega)^b &= \Delta\Omega^c + d\Omega^b + \frac{1}{2}[\Omega^b \wedge \Omega^b] + \alpha_*(s^*\Omega^a \wedge \Omega^b) \\
\text{curv}(\Omega)^c &= d\Omega^c + \alpha_*(\Omega^a \wedge \Omega^c).
\end{align*}
\] (5.1.5, 5.1.6, 5.1.7)

The following statement can be deduced abstractly from the definitions, Lemmas 4.2.1 and 4.1.2 and Eq. (4.1.1).

Theorem 5.1.4.

(a) The curvature satisfies a Bianchi identity: \(D\text{curv}(\Omega) = [\text{curv}(\Omega) \wedge \Omega]\).
(b) The curvature is \(R\)-invariant: \(R^*\text{curv}(\Omega) = \text{Ad}_{\text{pr}_2^{-1}}(\text{pr}_2^*\text{curv}(\Omega))\).

Definition 5.1.5. A connection \(\Omega\) on a principal \(\Gamma\)-2-bundle \(\mathcal{P}\) is called:

(a) flat if \(\text{curv}(\Omega) = 0\).
(b) fake-flat if \(\text{curv}(\Omega)^a = 0\) and \(\text{curv}(\Omega)^b = 0\).
(c) regular if every point \((p, q) \in \text{Obj}(\mathcal{P}) \times_M \text{Obj}(\mathcal{P})\) has an open neighborhood \(U\) supporting a transition span \(\sigma\) with \(\sigma^*\text{curv}(\Omega)^b = 0\).

Remark 5.1.6. It follows directly from the definition of a connection that connections form a convex subset of \(\Omega^1(\mathcal{P}, \gamma)\). However, regular, fake-flat, or flat connections do in general not form a convex subset.

Remark 5.1.7. Obviously, flat implies fake-flat and fake-flat implies regular. Conversely, if \(\Omega\) is regular and \(\text{curv}(\Omega)^a = 0\), then \(\Omega\) is fake-flat. Indeed, the vanishing of \(\text{curv}(\Omega)^a\) implies via Theorem 5.1.4 (b)
\[
R^*\text{curv}(\Omega)^b = (\alpha_{g^{-1}})_*(\text{Ad}_{h^{-1}}(\text{pr}_2^*\text{curv}(\Omega)^b)).
\] (5.1.8)

For \(\rho \in \text{Mor}(\mathcal{P})\) consider the splitting of \(T_{\rho}\text{Mor}(\mathcal{P})\) of Remark 3.1.7 with respect to a transition span \(\sigma\) of \(\mathcal{P}\) with \(\sigma^*\text{curv}(\Omega)^b = 0\). EQ. (5.1.8) implies that \(\text{curv}(\Omega)^b\) vanishes when applied to at least one “vertical” tangent vector, i.e. one in the image of \(T\text{Mor}(\Gamma)\). EQ. (5.1.8) implies further that
\[
\sigma^*R_{h,g}^*\text{curv}(\Omega)^b = (\alpha_{g^{-1}})_*(\text{Ad}_{h^{-1}}(\sigma^*\text{curv}(\Omega)^b)) = 0,
\] (5.1.9)
where \(R_{h,g} : \text{Mor}(\mathcal{P}) \longrightarrow \text{Mor}(\mathcal{P})\) is the action with fixed \((h, g) \in \text{Mor}(\Gamma)\). Hence, regularity of \(\Omega\) implies that \(\text{curv}(\Omega)^b\) vanishes when applied to two “horizontal” tangent vectors. In summary, the splitting into horizontal and vertical parts gives \(\text{curv}(\Omega)^b = 0\).
Example 5.1.8. The trivial principal $\Gamma$-2-bundle over $M$ is defined by $\mathcal{J} := M_{dis} \times \Gamma$, with the action by multiplication on the second factor. Using the formula of Example 4.2.2 for the Maurer-Cartan form $\Theta$ one can check that $\Omega := pr_1^*\Theta$ is a connection on $\mathcal{J}$; it is flat due to Eq. (4.1.1). More generally, for $\Psi \in \Omega^1(M_{dis}, \gamma)$
\[
\Omega_\Psi := \text{Ad}_{pr_2}^{-1}(pr_M^*\Psi) + pr_1^*\Theta.
\]
is a connection on $\mathcal{J}$. With Lemmas 4.2.1 and 4.1.2 and Eq. (4.1.1) one can check that
\[
\text{curv}(\Omega_\Psi) = \text{Ad}_{pr_M}^{-1}(pr_M^*(\text{D}\Psi + \frac{1}{2}[\Psi \wedge \Psi])).
\]
(5.1.10)
This has the following implications:

- $\Omega_\Psi$ is regular. To see this, we get from Eq. (5.1.10) the formula
\[
\text{curv}(\Omega_\Psi)^b = (\alpha_{g_1})_*(\tilde{(\delta h)}_*(pr_M^*(\text{D}\Psi + \frac{1}{2}[\Psi \wedge \Psi])))^a.
\]
Consider the global transition span $\sigma((x, g_1), (x, g_2)) := (\text{id}_x, g_1)$. Since $h \circ \sigma = 1$ we have $\tilde{(\delta h)}_* = 0$ and thus $\text{curv}(\Omega_\Psi)^b = 0$.

- $\Omega_\Psi$ is fake-flat if and only if $(\text{D}\Psi + \frac{1}{2}[\Psi \wedge \Psi])^a = 0$.

- $\Omega_\Psi$ is flat if and only if $\text{D}\Psi + \frac{1}{2}[\Psi \wedge \Psi] = 0$.

Finally, it is easy to check that the assignment $\Psi \mapsto \Omega_\Psi$ establishes a bijection between $\Omega^1(M_{dis}, \gamma)$ and the set of connections on $\mathcal{J}$. In particular, every connection on $\mathcal{J}$ is regular.

Example 5.1.9. Let $\Gamma = G_{dis}$, and $\Omega$ be a connection on a principal $\Gamma$-2-bundle. By Example 4.1.3 (a) we have $\Omega^b = \Omega^c = 0$, and the only condition for $\Omega$ is $R^*\Omega^a = \text{Ad}_{y_0}^{-1}(p_0^*\Omega^a) + g_0^*\theta$. The curvature components are $\text{curv}(\Omega)^a = d\Omega^a + \frac{1}{2}[\Omega^a \wedge \Omega^a]$, while $\text{curv}(\Omega)^b = 0$ and $\text{curv}(\Omega)^c = 0$. Thus, every connection is automatically regular, and flat is equivalent to fake-flat.

Example 5.1.10. Let $\Gamma = BU(1)$, and $\Omega$ be a connection on a principal $\Gamma$-2-bundle $\mathcal{P}$. By Example 4.1.4 (a) we have $\Omega^a = 0$, $\Omega^b \in \Omega^p(\mathcal{P})$ subject to the condition $\Delta \Omega^b = 0$, and $\Omega^c \in \Omega^{p+1}(\mathcal{P}_0)$. Definition 5.1.1 implies only that $R^*\Omega^b = p_1^*\Omega^b + h^*\theta$. The non-trivial curvature-components are
\[
\text{curv}(\Omega)^b = \Delta \Omega^c + d\Omega^b \quad \text{and} \quad \text{curv}(\Omega)^c = d\Omega^c.
\]
Thus, with Remark 5.1.7 regular is equivalent to fake-flat.

Example 5.1.11. Consider an ordinary principal $G$-bundle $P$ over $M$. Let $P/\!\!/H$ be the action groupoid for the right $H$-action on $P$ induced along $t : H \to G$. That is, Obj($P/\!\!/H$) = $P$ and Mor($P/\!\!/H$) = $P \times H$, source and target maps are $s(p, h) = p$ and $t(p, h) = pt(h)$, and composition is $(p_2, h_2) \circ (p_1, h_1) = (p_1, h_1 h_2)$. We define a functor
\[
R : P/\!\!/H \times \Gamma \to P/\!\!/H
\]
by $R(p, g) := pg$ and $R((p, h), (h', g)) := (pg, \alpha(g^{-1}, hh'))$. It is straightforward to check that this is a $\Gamma$-action. This construction is compatible with pullbacks, and is functorial in the sense that bundle morphisms $P \to P'$ induce smooth $\Gamma$-equivariant functors $P/\!\!/H \to P'/\!\!/H$. That $P/\!\!/H$ is a principal
\( \Gamma \)-2-bundle can now be seen from the criteria of [NW13 Prop. 6.2.8]: if \( U \subseteq M \) is open and supports a section into \( P \), we obtain the required 1-morphism

\[
\mathcal{J} \cong (U \times G)/H \cong P|_U / H \cong (P//H)|_U.
\]

Next we assume that \( \omega \in \Omega^1(P, \mathfrak{g}) \) is a connection on \( P \). We put

\[
\Omega^a := \omega, \quad \Omega^b := (\tilde{\alpha}_{pr_1})_*(pr_1^*\omega) + pr_1^*\theta \quad \text{and} \quad \Omega^c := 0.
\]

It is straightforward to check that this is a connection on \( P//H \), and to compute its curvature via Eqs. (5.1.5) to (5.1.7):

\[
\text{curv}(\Omega)^a = \psi, \quad \text{curv}(\Omega)^b = (\tilde{\alpha}_{pr_1})_*(pr_1^*\psi) \quad \text{and} \quad \text{curv}(\Omega)^c = 0,
\]

where \( \psi := d\omega + \frac{1}{2}[\omega \wedge \omega] \in \Omega^2(P, \mathfrak{g}) \) is the ordinary curvature of the connection \( \omega \). The connection \( \Omega \) is always regular: consider over \( P \times_M P \) the (global) transition span with \( \sigma(p, p') := (p, 1) \) and \( g : P \times_M P \to G \) defined such that \( p'g(p, p') = p \). Obviously, we have \( \sigma^*\text{curv}(\Omega)^b = 0 \). Further, and even more obviously, we have

\[
\Omega \text{ is flat} \iff \Omega \text{ is fake-flat} \iff \omega \text{ is flat}.
\]

### 5.2 Connection-preserving morphisms

We recall from Section 4.3 that pulling back a \( \gamma \)-valued differential form along an anafunctor requires the choice of a pullback (see Definition 4.3.1).

**Definition 5.2.1.** If \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are principal \( \Gamma \)-bundles over \( M \) equipped with connections \( \Omega_1 \) and \( \Omega_2 \), respectively, then an \( \Omega_2 \)-pullback \( \nu \) on a 1-morphism \( F : \mathcal{P}_1 \to \mathcal{P}_2 \) is called connection-preserving, if \( \Omega_1 = F^*_\gamma \Omega_2 . \) If \( F' : \mathcal{P}_1 \to \mathcal{P}_2 \) is a second 1-morphism equipped with an \( \Omega_2 \)-pullback \( \nu' \), then a 2-morphism \( f : F \Rightarrow F' \) is called connection-preserving, if \( f^*\nu' = \nu \).

Next we introduce two further conditions for an \( \Omega_2 \)-pullback on \( F \). Firstly, we require compatibility with the \( \Gamma \)-equivariance of \( F \).

**Definition 5.2.2.** Let \( F : \mathcal{P}_1 \to \mathcal{P}_2 \) be a \( \Gamma \)-equivariant anafunctor and let \( \Omega_2 \in \Omega^1(\mathcal{P}_2, \gamma) \) be a 1-form. An \( \Omega_2 \)-pullback \( \nu = (\nu_0, \nu_1) \) on \( F \) is called connective, if over \( F \times \text{Mor}(\Gamma) \) we have

\[
\begin{align*}
\rho^*\nu_0 &= (\alpha_{g^{-1}})_*(\text{Ad}^{-1}_h(p_{\mathcal{F}}^*\nu_0)) + (\tilde{\alpha}_h)_*(pr_1^*\alpha^*_\gamma \Omega^c_2 + h^*\theta) \quad (5.2.1) \\
\rho^*\nu_1 &= (\alpha_{g^{-1}})_*(\text{Ad}^{-1}_h(p_{\mathcal{F}}^*\nu_1)) + (\tilde{\alpha}_h)_*(pr_1^*(pr_{\mathcal{F}}^*\alpha^*_\gamma \Omega^c_2)), \quad (5.2.2)
\end{align*}
\]

where \( \rho : F \times \text{Mor}(\Gamma) \to F \) is the \( \text{Mor}(\Gamma) \)-action, and the maps \( g, h, pr_{\mathcal{F}} \) are defined by \( g(f, (h', g')) := g', h(f, (h', g')) := h' \) and \( pr_{\mathcal{F}}(f, (h', g')) := f \).

**Proposition 5.2.3.** Connective \( \Omega_2 \)-pullbacks form a convex subset in the affine space of all \( \Omega_2 \)-pullbacks. If the \( G \)-action on \( \text{Obj} (\mathcal{P}_1) \) is free and proper, then connective \( \Omega_2 \)-pullbacks exist.

**Proof.** Convexity follows immediately. By Lemma 4.3.2 there exists an \( \Omega_2 \)-pullback \( \nu \), but in general it will not be connective. We can write any element \( (h, g) \in \text{Mor}(\Gamma) \) as \( (h, g) = (h, t(h)^{-1}) \cdot (1, t(h)g) \). The action of elements of the form of the first factor can be rewritten as

\[
\rho(f, (h, t(h)^{-1})) = f \circ R(\text{id}_{\alpha_{r_t}(f)}, (h, t(h)^{-1})),
\]

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using the compatibility between the $\text{Mor}(\Gamma)$-action and the right $P_2$-action. From the conditions on $\Omega_2$-pullbacks and the fact that $\Omega_2$ is a connection, one can deduce that $\nu$ satisfies Eqs. \([5.2.1]\) and \([5.2.2]\) for group elements of the form $(h, t(h)^{-1})$. Thus, we can reduce the problem of finding connective $\Omega_2$-pullbacks to the problem of finding an $\Omega_2$-pullback that satisfies Eqs. \([5.2.1]\) and \([5.2.2]\) with respect to the $G$-action induced along the identity map $\text{id}_\Gamma : G \rightarrow \text{Mor}(\Gamma)$ of $\Gamma$. We first treat $\nu_0$. Consider the 1-form

$$\varphi := (\text{id}_F \times \text{id}_\Gamma)\rho^*\nu_0 - (\alpha_{\text{pr}_G})_*(\text{pr}_F\nu_0) \in \Omega^1(F \times G, \mathfrak{h})$$

which measures the failure in Eq. \([5.2.1]\). The 1-form $\varphi$ satisfies a cocycle condition over $F \times G \times G$, namely

$$(\text{id}_F \times m)^*\varphi = (\rho \times \text{id}_G)^*\varphi + \text{pr}_1^*\varphi.$$ \hspace{1cm} (5.2.3)

That $\Omega_2$ is a connection implies that $\varphi$ descends along $\alpha_1 \times \text{id} : F \times G \rightarrow \text{Obj}(\mathcal{P}_1) \times G$. Thus, we have a unique $\psi \in \Omega^1(\text{Obj}(\mathcal{P}_1) \times G, \mathfrak{h})$ such that $(\alpha_1 \times \text{id})^*\psi = \varphi$. The cocycle condition Eq. \((5.2.3)\) implies an analogous condition for $\psi$ over $\text{Obj}(\mathcal{P}_1) \times G \times G$. Our assumptions for the $G$-action guarantee that $\text{Obj}(\mathcal{P}_1)$ is a principal $G$-bundle over the quotient $X := \text{Obj}(\mathcal{P}_1)/G$. We obtain diffeomorphisms $\text{Obj}(\mathcal{P}_1) \times G^k \cong \text{Obj}(\mathcal{P}_1)^k\chi$, under which $\psi$ becomes a cocycle in the complex

$$\Omega^1(X) \rightarrow \Omega^1(\text{Obj}(\mathcal{P}_1)) \rightarrow \Omega^1(\text{Obj}(\mathcal{P}_1)^2\chi) \rightarrow \Omega^1(\text{Obj}(\mathcal{P}_1)^3\chi) \rightarrow \cdots$$

This complex is exact [Mur96], so that there exists $\kappa_0 \in \Omega^1(\text{Obj}(\mathcal{P}_1))$ such that $\psi = R^*\kappa_0 - \text{pr}_{\text{Obj}(\mathcal{P}_1)}\kappa_0$. Treating $\nu_1$ the very same way, we obtain $\kappa_1 \in \Omega^2(\text{Obj}(\mathcal{P}_1))$. It is then straightforward to check that the shifted pullback $\nu' := \nu + \alpha_1^*(\kappa_0, \kappa_1)$ is an $\Omega_2$-pullback and satisfies Eqs. \([5.2.1]\) and \([5.2.2]\) with respect to the induced $G$-action.

Secondly, we introduce a notion of fake-curvature for pullbacks of 1-forms, and then impose flatness conditions.

**Definition 5.2.4.** Let $F : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ be an anafunctor, let $\Omega_2 \in \Omega^1(\mathcal{P}_2, \gamma)$ be a 1-form, and let $\nu = (\nu_0, \nu_1)$ be an $\Omega_2$-pullback on $F$.

(a) The 2-form

$$\text{fcurv}(\nu) := d\nu_0 + \frac{1}{2}[\nu_0 \wedge \nu_0] + \alpha_*(\alpha_1^*\Omega_2^2 \wedge \nu_0) + \nu_1 \in \Omega^2(F, \mathfrak{h})$$

is called the fake-curvature of $\nu$, and $\nu$ is called fake-flat, if $\text{fcurv}(\nu) = 0$.

(b) $\nu$ is called regular, if every point $(p_1, p_2) \in \text{Obj}(\mathcal{P}_1) \times_M \text{Obj}(\mathcal{P}_2)$ has an open neighborhood $U$ supporting a transition span $\sigma$ of $F$ with $\sigma^*\text{fcurv}(\nu) = 0$.

(c) The 3-form

$$\text{curv}(\nu) := d\nu_1 + [\nu_0 \wedge \nu_1] + \alpha_*(\alpha_1^*\Omega_2^2 \wedge \nu_1) \in \Omega^3(F, \mathfrak{h})$$

is called the curvature of $\nu$, and $\nu$ is called flat, if $\text{curv}(\nu) = 0$ and $\text{fcurv}(\nu) = 0$.

**Remark 5.2.5.** Fake-curvature and curvature in Definition 5.2.4 satisfy

$$D\nu + \frac{1}{2}[\nu \wedge \nu] = (\text{fcurv}(\nu), \text{curv}(\nu))$$

with respect to the operations on pullbacks introduced in Lemma 4.3.4. In particular, $\nu' := (\text{fcurv}(\nu), \text{curv}(\nu))$ is a $\text{curv}(\Omega_2)$-pullback on $F$, and we have $\text{curv}(\Omega_1) = \text{pr}_1^*(\text{curv}(\Omega_2))$. 

\hspace{1cm} - 26 -
Remark 5.2.6. Fake-flat obviously implies regular. Conversely, if Ω is a fake-flat connection on P2, and ν is regular and connective, then ν is fake-flat. In order to see this, we first derive using connectivity of ν the following transformation rule of the fake-curvature under the Mor(Γ)-action on F:  
\[ \rho^* f\text{curv}(\nu) = (\alpha_{g^{-1}})_* (\text{Ad}_h^{-1} (\text{pr}_F^* f\text{curv}(\nu)) + (\tilde{\alpha}_h)_* (\text{pr}_F^* \alpha_*^* f\text{curv}(\Omega_2)^a)) \]  
(5.2.4)  
Fake-flatness of Ω and Eq. (5.2.4) imply  
\[ \rho^* f\text{curv}(\nu) = (\alpha_{g^{-1}})_* (\text{Ad}_h^{-1} (\text{pr}_F^* f\text{curv}(\nu))). \]  
(5.2.5)  
For f ∈ F consider the splitting of T_f F of Remark 5.1.10 with respect to a transition span σ with σ^* f\text{curv}(ν) = 0. EQ. (5.2.5) implies that f\text{curv}(ν) vanishes when applied to at least one “vertical” tangent vector, i.e. one in the image of TMor(Γ). EQ. (5.2.5) implies further that  
\[ \sigma^* \rho_{\nu_1}^* f\text{curv}(\nu) = (\alpha_{g^{-1}})_* (\text{Ad}_h^{-1} (\text{F}^* f\text{curv}(\nu))). \]  
(5.2.6)  
Hence, regularity of ν means that f\text{curv}(ν) vanishes when applied to two “horizontal” tangent vectors. In summary, the splitting into horizontal and vertical parts gives f\text{curv}(ν) = 0.

Remark 5.2.7. If Ω is fake-flat and F : P1 → P2 admits a connective Ω2-pullback, then F also admits a connective and fake-flat Ω2-pullback. In order to see this, we derive the following transformation law under the right P2-action ρe, for the fake-curvature of a Ω2-pullback ν = (ν0, ν1):  
\[ \rho^* f\text{curv}(\nu) = \text{pr}_F^* f\text{curv}(\nu) + \text{pr}_{\text{Mor}(P_2)}^* \text{curv}^b(\Omega_2). \]  
Since Ω is fake-flat, f\text{curv}(ν) descends to a 2-form \( \kappa \in \Omega^2(\text{Obj}(P_1), h) \) with \( \alpha_{g}^* \kappa = f\text{curv}(\nu) \). The shifted pullback \( \nu^s = (\nu, \nu_1 - \alpha_{g}^* \kappa_1) = (\nu_0, \nu_1 - f\text{curv}(\nu)) \) is then obviously fake-flat. For connectivity of the new pullback \( \nu^s \), we only have to check Eq. (5.2.2), i.e.  
\[ \rho^* \nu^s_1 = (\alpha_{g^{-1}})_* (\text{Ad}_h^{-1} (\text{pr}_F^* \nu^s_1) + (\tilde{\alpha}_h)_* (\text{pr}_F^* \alpha_*^* \Omega_2^s))). \]  
This follows directly from the connectivity of ν and Eq. (5.2.5).

We recall that the composition of anafunctors comes along with a composition of pullbacks on anafunctors, see Lemma 4.3.5 (a).

Proposition 5.2.8. The composition of pullbacks on 1-morphisms between principal Γ-2-bundles preserves the following conditions for pullbacks: “connection-preserving”, “connective”, “regular”, and “fake-flat”.

Proof. That the composite pullback is connection-preserving is the content of Lemma 4.3.5 (a). Connectivity is straightforward to check using the formulas for the composite pullback (Lemma 4.3.5 (a)) and the definition of the Mor(Γ)-action on the composite anafunctor (Lemma 4.3.1). Concerning fake-flatness, one can deduce the formula  
\[ f\text{curv}(\nu' \circ \nu) = \text{pr}_{F_2}^* f\text{curv}(\nu') + \text{pr}_{F_1}^* f\text{curv}(\nu) \]  
(5.2.6)  
for the fake-curvature of the composite pullback ν' ◦ ν on a composite F' ◦ F of anafunctors F : P1 → P2 and F' : P2 → P3. This shows that fake-flat pullbacks compose to fake-flat ones. For regularity, consider \( p_i \in \text{Obj}(P_i) \) for \( i = 1, 2, 3 \) all projecting to the same point \( x \in M \). Choose an open neighborhood \( U_{12} \) of \( (p_1, p_2) \) with a transition span \( \sigma_{12} \) of F such that \( \sigma_{12}^* f\text{curv}(\nu) = 0 \), and an
open neighborhood \(U_{23}\) of \((p_2, p_3)\) with a transition span \(\sigma_{23}\) of \(F\) such that \(\sigma_{23}^* \text{curv}(\nu') = 0\). Choose further an open neighborhood \(V\) of \(x\) with a section \(\tau : V \to \text{Obj}(\mathcal{P}_2)\) such that \(\tau(x) = p_2\). For \(x_i \in \text{Obj}(\mathcal{P}_1)\) such that \(\pi_i(x_i) \in V\) we write \(x_i^r := \tau(\pi_i(x_i))\). We define the open neighborhood

\[
U := \{(x_1, x_3) \in \text{Obj}(\mathcal{P}_1) \times M \mid \pi_1(x_1), \pi_3(x_3) \in V, (x_1, x_1^r) \in U_{12}, (x_3^r, x_3) \in U_{23}\}
\]

of \((p_1, p_3)\). We find a smooth map \(\sigma : U \to F_{\alpha_r} \times_{\alpha_l} G\) defined by

\[
\sigma(x_1, x_3) \mapsto (\sigma_{12}(x_1, x_1^r), \sigma_{23}(x_3^r, x_3) \cdot \text{id}_{g_{12}(x_1, x_1^r)}),
\]

where \(g_{12} : U_{12} \to G\) is a transition function for \(\sigma_{12}\). This is a transition span of \(F' \circ F\), and Eq. (5.2.6) implies that \(\sigma^* \text{curv}(\nu' \circ \nu) = 0\). This shows that \(\nu' \circ \nu\) is regular.

Now we are in position to set up three bicategories of principal \(\Gamma\)-2-bundles with connection:

(a) A bicategory \(2\text{-Bun}_1^\nabla(M)\) consisting of principal \(\Gamma\)-2-bundles with connections, 1-morphisms with connective, connection-preserving pullbacks, and connection-preserving 2-morphisms.

(b) A bicategory \(2\text{-Bun}_1^{\nabla^\text{reg}}(M)\) consisting of principal \(\Gamma\)-2-bundles with regular connections, 1-morphisms with regular, connective, connection-preserving pullbacks, and connection-preserving 2-morphisms.

(c) A bicategory \(2\text{-Bun}_1^{\nabla^\text{ff}}(M)\) consisting of principal \(\Gamma\)-2-bundles with fake-flat connections, 1-morphisms with fake-flat, connective, connection-preserving pullbacks, and connection-preserving 2-morphisms.

We show in the next section that these three bicategories are classified by the three versions of non-abelian differential cohomology we have described in Section 2.2. Moreover, it is straightforward to see that they form presheaves of bicategories,

\[
2\text{-Bun}_1^\nabla, \quad 2\text{-Bun}_1^{\nabla^\text{reg}} \quad \text{and} \quad 2\text{-Bun}_1^{\nabla^\text{ff}}
\]

over the category of smooth manifolds, i.e., there are consistent pullback 2-functors along smooth maps. We will show in a separate paper that these presheaves are in fact 2-stacks.

Remark 5.2.9. In each of the three bicategories of principal \(\Gamma\)-2-bundles, every 1-morphism in invertible. In order to see this, we recall from [NW13 Corollary 6.2.4] that every 1-morphism \(F : \mathcal{P}_1 \to \mathcal{P}_2\) between the underlying principal \(\Gamma\)-2-bundles is invertible; in particular, a weak equivalence. By Lemma 4.3.5 (b), every connection-preserving pullback \(\nu\) on \(F\) induces a connection-preserving pullback \(-\nu\) on \(F^{-1}\). It is straightforward to check that \(-\nu\) inherits each of the properties “connective”, “regular” and “fake-flat” from \(\nu\).

Remark 5.2.10. Suppose \(\phi : \mathcal{P}_1 \to \mathcal{P}_2\) is a smooth functor between principal \(\Gamma\)-2-bundles that is strictly equivariant with respect to the \(\Gamma\)-actions. Suppose \(\Omega_1\) and \(\Omega_2\) are connections on \(\mathcal{P}_1\) and \(\mathcal{P}_2\), respectively. The canonical \(\Omega_2\)-pullback \(\nu\) on the associated anafunctor \(F := \text{Obj}(\mathcal{P}_1) \circ_{\alpha_l} \text{Mor}(\mathcal{P}_2)\) (see Remark 4.3.6 (a)) has the following properties:

(a) It is connection-preserving if and only if \(\phi^* \Omega_2 = \Omega_1\) (because \(\phi^* \Omega_2 = F^* \Omega_2\)).

(b) It is always connective (a straightforward calculation).

(c) Its fake-curvature is \(\text{fcurv}(\nu) = \text{pr}_{\text{Mor}(\mathcal{P}_2)}^* \text{curv}(\Omega_2)\); hence it is fake-flat if \(\Omega_2\) is fake-flat.
(d) It is regular if $\Omega_2$ is regular. To see this, suppose $\sigma : U \to \mathcal{P}_2$ is a transition span. Define $U' := (\phi \times \text{id})^{-1}(U)$ and $\tau(x_1, x_2) := (x_1, \sigma(\phi(x_1), x_2))$. This is a transition span of $F$, and from (e) we get $f^\sigma \text{curv}(\nu) = (\phi \times \text{id})^*(\mathcal{A}^* \text{curv}(\Omega_2))$.

If the canonical $\Omega_2$-pullback $\nu$ is shifted by $\kappa = (\kappa_0, \kappa_1)$ in the sense of Remark 4.3.6 (b) then the shifted pullback $\nu^\kappa$ has the following properties:

(e) It is connection-preserving if and only if

$$\Omega_1^\kappa = \phi^* \Omega_2^\kappa + t_\kappa(\kappa_0), \quad \Omega_1^b = \phi^* \Omega_2^b + \Delta \kappa_0 \quad \text{and} \quad \Omega_1^c = \phi^* \Omega_2^c + \kappa_1;$$

this is a reformulation of Eq. 4.3.6.

(f) It is connective if $\kappa_0$ and $\kappa_1$ are $R$-invariant in the sense that $R^* \kappa_i = (\alpha_{pr_{-1}})_(pr^*_1 \kappa_i)$ over $\text{Obj}(\mathcal{P}_1) \times G$, for $i = 0, 1$; this is a straightforward calculation.

(g) Its fake-curvature is $f\text{curv}(\nu^\kappa) = f\text{curv}(\nu) + pr^*_1 (d \kappa_0 + \frac{1}{2}[\kappa_0 \land \kappa_0] + \alpha_*(\phi^* \Omega_2^\kappa \land \kappa_0) + \kappa_1)$. Hence, using (c) and (d) we see that $\nu^\kappa$ is regular/fake-flat if

$$d \kappa_0 + \frac{1}{2} [\kappa_0 \land \kappa_0] + \alpha_*(\phi^* \Omega_2^\kappa \land \kappa_0) + \kappa_1 = 0 \quad (5.2.7)$$

and $\Omega_2$ is regular/fake-flat.

Example 5.2.11. We take up Example 5.1.11 where we associated a principal $\Gamma$-2-bundle $P/H$ over $M$ with regular connection $\Omega$ to each ordinary principal $G$-bundle $P$ over $M$ with connection $\omega$. As remarked there, every bundle morphism $\varphi : P \to P'$ induces a $\Gamma$-equivariant smooth functor $\tilde{\varphi} : P/H \to P'/H$, acting as $\varphi$ on the level of objects, and acting on $[p, h] \mapsto (\varphi(p), h)$ on the level of morphisms. If $\varphi$ is connection-preserving, then it is obvious from the definition of the induced connections $\Omega$ and $\Omega'$ that $\tilde{\varphi}^* \Omega' = \Omega$. Hence, by Remark 5.2.10 the canonical pullback on the induced anafunctor is connective and connection-preserving, it is regular because $\Omega'$ is regular, and it is flat if $\omega$ is flat. Summarizing, we have constructed functors

$$\text{Bun}_G^\Gamma(M) \to 2\text{-Bun}_{\Gamma}^\text{un}(M) \quad \text{and} \quad \text{Bun}_G^\Gamma(M)^\text{flat} \to 2\text{-Bun}_{\Gamma}^\text{fr}(M).$$

These functors consistently realize ordinary gauge theory as a special case of higher gauge theory. Using Lemma 2.2.3 we will provide in Corollary 5.3.6 a sufficient condition for being in the image of these functors.

Next we discuss how connections can be “induced” along 1-morphisms between principal $\Gamma$-2-bundles.

Proposition 5.2.12. Let $F : \mathcal{P}_1 \to \mathcal{P}_2$ be a 1-morphism between principal $\Gamma$-bundles, $\Omega_2$ be a connection on $\mathcal{P}_2$, $\nu$ be a connective $\Omega_2$-pullback on $F$, and $\Omega_1 := F^* \nu \Omega_2 \in \Omega^1(\mathcal{P}_1, \gamma)$.

(a) $\Omega_1$ is a connection (and $\nu$ is connection-preserving).

(b) If $\nu$ and $\Omega_2$ are fake-flat, then $\Omega_1$ is fake-flat.

(c) If $\nu$ and $\Omega_2$ are flat, then $\Omega_1$ is flat.

(d) If $\nu$ and $\Omega_2$ are regular, then $\Omega_1$ is regular.
Proof. For [a] we have to verify Eqs. (5.1.1) to (5.1.3). It is straightforward to check using Eq. (5.2.2) over $F \times G$ the relation

$$(\alpha_l \times \text{id})^* R^* \Omega^b_1 = (\alpha_{pr^{g^{-1}}_l})^* (pr^*_F \alpha_l^* \Omega^b_1).$$

Since $\alpha_l$ is a surjective submersion this implies $R^* \Omega^b_1 = (\alpha_{g^{-1}})^* (\Omega^b_1)$, which is Eq. (5.1.3). Similarly one shows Eq. (5.1.1) using Eq. (5.2.1). In order to prove Eq. (5.1.2) we work over a open subset $V \subseteq \text{Mor}(\mathcal{P}_1) \times \text{Mor}(\Gamma)$ that admits a smooth map $f: V \longrightarrow F$ such that $\alpha_l(f(\chi, (h, g))) = R(s(\chi), g)$. Denoting by $\rho_l$ the left action of $\mathcal{P}_1$ on $F$, we have from Eq. 4.3.4

$$R^* \Omega^b_1|_V = (R \times f)^* \rho^*_l \nu_0 - f^* \nu_0.$$ 

Rewriting $\rho_l$ in terms of the Mor($\Gamma$)-action $\rho$ on $F$ using the conditions of Definition 2.3.1 and then using Eq. (5.2.1) yields Eq. 5.1.2. This shows [a]. For [b] we have from Remark 5.2.6

$$\text{curv} (\Omega_1) = F^*_\nu \text{curv} (\Omega_2),$$

where $\nu = (\text{curv}(\nu), \text{curv}(\nu))$. By Lemma 4.3.3 the component $\text{curv} (\Omega_1)^b$ is uniquely determined by

$$\text{pr}^*_F \text{Mor}(\mathcal{P}_1) (\text{curv} (\Omega_1)^b) = \rho^*_l \text{curv} (\nu) - \text{pr}^*_F \text{curv} (\nu)$$

over $\text{Mor}(\mathcal{P}_1) \times_{\alpha_l} F$, where $\rho_l$ is the left action and $\text{pr}_F$ is the projection. The component $\text{curv} (\Omega_1)^b$ is determined by $\alpha_l^* \text{curv} (\Omega_1)^a = t_0 (\text{curv} (\nu)) + \alpha_l^* \text{curv} (\Omega_2)^b$. This shows [b]. For [c] we only have to look at the component $\text{curv} (\Omega_1)^c$, which is according to Lemma 4.3.3 determined by $\alpha_l^* \text{curv} (\Omega_1)^c = \text{curv} (\nu) + \alpha_l^* \text{curv} (\Omega_2)^c$; this shows the claim.

For Proposition 5.2.12 (d) we set $Y_k := \text{Obj}(\mathcal{P}_k) \times_M \text{Obj}(\mathcal{P}_k)$ for $k = 1, 2$, consider $Z := Y_1 \times_M Y_2$ and the projection $pr_{12}: Z \longrightarrow Y_1$. For a given point $(p_1, p_2') \in Y_1$ let $U \subseteq Y_1$ be an open neighborhood with a section $s: U \longrightarrow W$. We put $(p_1, p_1', p_2, p_2') := s(p_1, p_1')$. We choose a transition span $\sigma_2$ of $\mathcal{P}_2$ defined in a neighborhood of $(p_2, p_2') \in Y_2$, as well as transition spans $\tau$ of $F$ in a neighborhood of $(p_1, p_2)$, with transition function $h$, and $\tau'$ in a neighborhood of $(p_1', p_2')$. We claim that there exists a transition span $\sigma_1$ of $\mathcal{P}_1$ with transition function $g_1$ defined on $U_1$ such that over $W$ we have

$$\tau' = \rho (\sigma_1^{-1} \circ \tau \circ R(\sigma_2, \text{id}_h), (1, g_1^{-1})).$$

In order to see this, we choose some transition span of $\mathcal{P}_1$ on $U_1$ with some transition function $g_1$. Then one can prove that the right hand side of Eq. 5.2.9 defines a transition span of $F$ along $pr_{24}: W \longrightarrow \text{Obj}(\mathcal{P}_1) \times_M \text{Obj}(\mathcal{P}_2)$. By Lemma 5.4.9 the difference between this transition span and $\tau' \circ pr_{24}$ defines a smooth map $x: W \longrightarrow H$. Acting with $x \circ s: U \longrightarrow H$ on $\sigma_1$ yields a new transition span of $\mathcal{P}_1$ with the claimed property.

By eventually passing to smaller open sets, we can assume by regularity of $\tau$ that $\tau^* \text{curv}(\nu) = \tau^* \text{curv}(\nu) = 0$, and by regularity of $\Omega_2$ that $\sigma_2^* \text{curv}(\Omega_2)^b = 0$. Now we rewrite Eq. 5.2.9 as

$$\sigma_1^{-1} \circ \tau = \rho (\tau', (1, g_1)) \circ R(\sigma_2^{-1}, \text{id}_h).$$

Eq. 5.2.10 gives

$$\sigma_1^* \text{curv}(\Omega_1)^b = -(\sigma_1^{-1} \circ \tau)^* \text{curv}(\nu) + \tau^* \text{curv}(\nu).$$

Using the expression of Eq. 5.2.10 and then simplifying with Eqs. (5.1.1) and (5.2.1) yields zero. \(\square\)

Remark 5.2.13. Pullbacks on 1-morphisms can be “induced” in the following way: let $F, F': \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be 1-morphisms between principal $\Gamma$-2-bundles equipped with connections $\Omega_1$ and $\Omega_2$, respectively. If $\nu'$ is an $\Omega_2$-pullback on $F'$, and $f: F \longrightarrow F'$ is a 2-morphism, then $\nu := f^* \nu'$ is a pullback of $F$, and each of the properties “connection-preserving”, “connective”, “regular”, and “fake-flat” holds for $\nu$ if and only if it holds for $\nu'$. 

− 30 −
Lemma 5.3.1. The connection Eq. (2.2.1) implies $\Delta\Omega^a = 0$. Thus, we have a 1-form $\Omega \in \Omega^1(P_{(g,a)}, \gamma)$. In order to show that this 1-form is a connection, we check Eqs. (5.1.1) to (5.1.3), which is straightforward. A bit tedious, but straightforward to check is the following.

**5.3 Classification by non-abelian differential cohomology**

Let $(g, a, A, B, \varphi)$ be a generalized differential $\Gamma$-cocycle with respect to an open cover $U = \{U_i\}_{i \in I}$. We define on the reconstructed principal $\Gamma$-2-bundle $P_{(g,a)}$ (see Section 5.2) a connection $\Omega$, via

$$
\Omega^a|_{U_i \times G} := \text{Ad}_{\text{pr}_G}(A_i) + \text{pr}_G^*\theta
$$

$$
\Omega^b|_{(U_i \cap U_j) \times H \times G} := (\alpha_{\text{pr}_H})(\text{Ad}_{\text{pr}_H}^{-1}(\varphi_{ij}) + (\tilde{\alpha}_{\text{pr}_H})_*(A_j) + \text{pr}_H^*\theta)
$$

$$
\Omega^c|_{U_i \times G} := -(\alpha_{\text{pr}_G}^{-1})_*(B_i).
$$

Cocycle condition Eq. (2.2.1) implies $\Delta\Omega^a = t_*(\Omega^b)$, and cocycle conditions Eqs. (2.2.1) and (2.2.4) imply $\Delta\Omega^b = 0$. Thus, we have $1$-form $\Omega \in \Omega^1(P_{(g,a)}, \gamma)$. In order to show that this $1$-form is a connection, we check Eqs. (5.1.1) to (5.1.3), which is straightforward. A bit tedious, but straightforward to check is the following.

**Lemma 5.3.1.** The curvature of the connection $\Omega$ has the following components:

$$
\text{curv}(\Omega)^a|_{U_i \times G} = \text{Ad}_{\text{pr}_G}^{-1}(\text{curv}(A_i, B_i))
$$

$$
\text{curv}(\Omega)^b|_{(U_i \cap U_j) \times H \times G} = (\alpha_{\text{pr}_H}^{-1})_*(\text{Ad}_{\text{pr}_H}^{-1}(\varphi_{ij})) + \text{d}\varphi_{ij} + \frac{1}{2}[\varphi_{ij} \wedge \varphi_{ij}]
$$

$$
B_j - (\alpha_{\text{pr}_G}^{-1})_*(B_i)) + (\tilde{\alpha}_{\text{pr}_H})_*(\text{curv}(A_j, B_j))
$$

$$
\text{curv}(\Omega)^c|_{U_i \times G} = -(\alpha_{\text{pr}_G}^{-1})_*(\text{curv}(A_i, B_i)).
$$

Next we verify that above “reconstruction” of principal $\Gamma$-2-bundles with connection from local data is well-defined in non-abelian differential cohomology.

**Lemma 5.3.2.** The construction of the pair $(P_{(g,a)}, \Omega)$ induces a well-defined map

$$
r_{\text{gen}} : \hat{H}^1(M, \Gamma)^{\text{gen}} \longrightarrow h_0(2\text{-Bun}_\Gamma(M)).
$$

**Proof.** We have to check two things: (a) Invariance under generalized equivalences of cocycles, and (b) Invariance under refinement of open covers. For (a), let another generalized differential $\Gamma$-cocycle $(g', a', A', B', \varphi')$ be equivalent to $(g, a, A, B, \varphi)$ via generalized equivalence data $(h, e, \phi)$. Then we have the functor $\phi : P_{(g,a)} \longrightarrow P_{(g',a')}$ defined in Section 5.2 given by

$$
\phi(i, x, g) := (i, x, h_i(x)g) \quad \text{and} \quad \phi(i, j, x, h, g) := (i, j, x, e_{ij}(x)\alpha(h_j(x), h), h_j(x)g).
$$
Let $F$ be the corresponding anafunctor, and let $\nu$ be the canonical $\Omega$-pullback on $F$. We define $\kappa = (\kappa_0, \kappa_1)$ with $\kappa_0 \in \Omega^1(\text{Obj}(\mathcal{P}_{(g,a)}, \mathfrak{h}))$ and $\kappa_1 \in \Omega^2(\text{Obj}(\mathcal{P}_{(g,a)}, \mathfrak{h}))$ by

$$
\kappa_0|_{U_i \times G} := (\alpha_{pr_G^{-1} \mu}^{-1} \tau_i)_*(\phi_i) \quad \text{and} \quad \kappa_1|_{U_i \times G} := (\alpha_{pr_G^{-1} \mu})_*((\alpha_{h^{-1}})_*(B_j^r) - B_i^r).
$$

The shifted pullback $\nu^c$ on $F$ is connective (the $R$-invariance in Remark 5.2.10 (1) is easy to check) and connection-preserving: using the equivalence condition of Eq. (2.2.6) one can deduce $\Omega^c = \phi^*\Omega^a + t_*\kappa_0$; using the equivalence conditions of Eqs. (2.2.7), (2.2.8) and (2.2.10) one can deduce $\Omega^b = \phi^*\Omega^b + \Delta\kappa_1$, and $\Omega^c = \phi^*\Omega^c + \kappa_1$ holds by definition of $\kappa_1$. Hence $F$ and $\nu^c$ form a 1-morphism in the bicategory $2\text{-Bun}_F^\nu(M)$.

For (b), let $\mathcal{U}'$ be a refinement of $\mathcal{U}$, $(g', a', A', B', \varphi')$ be the refined $\Gamma$-cocycle, and $\Omega'$ be the corresponding connection on $\mathcal{P}_{(g',a')}$. Then, the evident smooth functor $\phi : \mathcal{P}_{(g',a')} \to \mathcal{P}_{(g,a)}$ obviously satisfies $\phi^*\Omega^a = \Omega$. According to Remarks 5.2.10 (a) and 5.2.10 (b) it induces a 1-morphism in the bicategory $2\text{-Bun}_F^\nu(M)$. In both cases, Remark 5.2.10 guarantees that these 1-morphisms are automatically 1-isomorphisms. \hfill \square

**Lemma 5.3.3.** The construction of the pair $(\mathcal{P}_{(g,a)}, \Omega)$ induces a well-defined map

$$
r : \hat{H}^1(M, \Gamma) \longrightarrow h_0(2\text{-Bun}_F^\nu(M)).
$$

**Proof.** If a differential $\Gamma$-cocycle $(g, a, A, B, \varphi)$ is non-generalized, then the additional cocycle condition of Eq. (2.2.2) reduces the expression for $\text{curv}(\Omega)^b$ of Lemma 5.3.1 to

$$
\text{curv}(\Omega)^b|_{(U_i \cap U_j) \times H \times G} = (\alpha_{pr_G^{-1} \mu})_*((\alpha_{pr_H})_*((\text{curv}(A_j, B_j)))�. \quad (5.3.1)
$$

Using the canonical transition spans $\sigma_{ij}(x) = (i, j, x, 1, g_{ij}(x))$ of $\mathcal{P}_{(g,a)}$, we see that $\sigma_{ij}^{*}\text{curv}(\Omega)^b = 0$. Thus, the connection $\Omega$ is regular. Next we modify the parts (a) (Invariance under equivalences of cocycles) and (b) (Invariance under refinement of open covers) of the proof of Lemma 5.3.2. Concerning (a), we suppose another (non-generalized) differential $\Gamma$-cocycle $(g', a', A', B', \varphi')$ is equivalent to $(g, a, A, B, \varphi)$ via non-generalized equivalence data $(h, e, h)$. Then, the additional equivalence condition of Eq. (2.2.7), together with the cocycle condition of Eq. (2.2.6), implies

$$
\kappa_1 = -\alpha_*(\phi^*\Omega^a \land \kappa_0) - d\kappa_0 - \frac{1}{2}[\kappa_0 \land \kappa_0].
$$

Using the formula of Remark 5.2.10 (g) this means that $\text{curv}(\nu^c) = \text{curv}(\nu)$, so that Remark 5.2.10 (d) implies that $\nu^c$ is regular. Concerning (b), Remark 5.2.10 (d) implies directly that the canonical pullback on the obvious smooth functor is regular. \hfill \square

Equipped with the maps $r^{\text{gen}}$ and $r$ we are in position to state the following result.

**Theorem 5.3.4.** Principal $\Gamma$-2-bundles with connections are classified up to 1-isomorphism by differential non-abelian cohomology, in the sense that we have a commutative diagram

$$
\begin{array}{ccc}
\hat{H}^1(M, \Gamma)^{\text{ff}} & \longrightarrow & \hat{H}^1(M, \Gamma) \\
\downarrow & & \downarrow r \\
\text{h}_0(2\text{-Bun}_F^\nu(M)^{\text{ff}}) & \longrightarrow & \text{h}_0(2\text{-Bun}_F^\nu(M)) \\
\end{array}
$$

in which all vertical arrows are bijections.

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Proof. First of all, we note that the restriction of \( r \) to the subset \( \hat{H}^1(M, \Gamma)^{ff} \) results in fake-flat connections \( \Omega \). Indeed, one can see from Lemma \[ 5.3.1 \] and Eq. \[ 5.3.1 \] that the vanishing of \( \text{curv}(A_i, B_i) \) implies the vanishing of \( \text{curv}(\Omega)^a \) and \( \text{curv}(\Omega)^b \). This shows that the diagram exists and is commutative.

In order to show surjectivity of its vertical maps we explain how to extract differential \( \Gamma \)-cocycles. Suppose \( P \) is a principal \( \Gamma \)-2-bundle over \( M \) with connection \( \hat{\Omega} \). We proceed relative to the choices made in Section \[ 3.2 \] i.e. a cover \( \{ U_i \}_{i \in I} \) of \( M \) by contractible open sets with contractible double intersections, smooth sections \( s_i : U_i \rightarrow P \), and transition spans \( \sigma_{ij} : U_i \cap U_j \rightarrow \text{Mor}(P) \) along \( (s_i, s_j) : U_i \cap U_j \rightarrow \text{Obj}(P) \times_M \text{Obj}(P) \), together with transitions functions \( g_{ij} : U_i \cap U_j \rightarrow G \). As described in Section \[ 3.2 \] these choices induce smooth maps \( a_{ijk} : U_i \cap U_j \cap U_k \rightarrow H \) in such a way that \( (g, h) \) is a \( \Gamma \)-cocycle. Now we add the required differential forms:

(a) over each open set \( U_i \), the differential forms

\[
A_i := s_i^* \hat{\Omega}^a \in \Omega^1(U_i, g) \quad \text{and} \quad B_i := -s_i^* \hat{\Omega}^c \in \Omega^2(U_i, h).
\]

(b) Over each double intersection \( U_i \cap U_j \) the differential form

\[
\varphi_{ij} := (\alpha_{g_{ij}})_*(\sigma_{ij}^* \hat{\Omega}^b) \in \Omega^1(U_i \cap U_j, h).
\]

We show the following:

1.) The collection \((g, h, A, B, \varphi)\) is a generalized differential \( \Gamma \)-cocycle.

Over \( U_i \cap U_j \), we pull back Eq. \[ 5.1.1 \] along \((s_i, g_{ij}^{-1})\), resulting in \((s_i, g_{ij}^{-1})^* R^* \hat{\Omega}^a = \text{Ad}_{g_{ij}}(A_i) - g_{ij}^* \hat{\theta} \). On the left, we rewrite \( R(s_i, g_{ij}^{-1}) = t(R(\sigma_{ij}, \text{id}_{g_{ij}^{-1}})) \) and obtain

\[
(\sigma_{ij}, \text{id}_{g_{ij}^{-1}})^* R^* t^* \hat{\Omega}^a = (\sigma_{ij}, \text{id}_{g_{ij}^{-1}})^* R^* (s^* \hat{\Omega}^a + t_*(\hat{\Omega}^b)) = s_j^* \hat{\Omega}^a + t_*((\sigma_{ij}, \text{id}_{g_{ij}^{-1}})^* R^* \hat{\Omega}^b) = A_j + t_*(\varphi_{ij})
\]

where the last step uses Eq. \[ 5.1.2 \]. This is the cocycle condition Eq. \[ 2.2.1 \].

Over \( U_i \cap U_j \cap U_k \) the left hand side becomes

\[
(\sigma_{ij} \circ R(\sigma_{jk}, \text{id}_{g_{jk}}))^* \hat{\Omega}^b = \sigma_{jk}^* \hat{\Omega}^b + (\alpha_{g_{jk}^{-1}})_*(\sigma_{jk}^* \hat{\Omega}^b) = (\alpha_{g_{jk}^{-1}})_*(\sigma_{ij}^* \hat{\Omega}^b + \varphi_{ij} + \varphi_{jk}).
\]

The right hand side is, using the first half of Eq. \[ 5.2.2 \] and Eq. \[ 5.1.1 \],

\[
(\alpha_{g_{ij}^{-1}} g_{jk}^{-1})_* (\text{Ad}_{a_{ijk}^{-1}}(\varphi_{ik}) + (\hat{\theta}_{a_{ijk}})_*(A_k) + a_{ijk}^* \hat{\theta}).
\]

This implies the cocycle condition Eq. \[ 2.2.4 \]. This shows that \((g, a, A, B, \varphi)\) is a generalized differential \( \Gamma \)-cocycle.

2.) If \( \hat{\Omega} \) is regular, then \((g, a, A, B, \varphi)\) is non-generalized.

In fact, we have to pass to a smaller open cover constructed as follows. Consider a cover \( V \) of \( X := \text{Obj}(P) \times_M \text{Obj}(P) \) consisting of open sets \( V \subseteq X \) supporting transition spans \( \sigma \) with \( \sigma^* \text{curv}(\hat{\Omega})^b = 0 \); these open sets exist because \( \hat{\Omega} \) is regular. The inverse image of \( V \) under the map \((s_i, s_j) : U_i \cap U_j \rightarrow X \) is an open cover \( V_{ij} \) of \( U_i \cap U_j \). On paracompact spaces (such as
smooth manifolds), every such “hypercover of height 1” can be refined by an ordinary open cover. Restriction of sections $s_i$ to this refinement guarantees the existence of transition spans $\sigma_{ij}$ along $(s_i, s_j)$ with $\sigma_{ij}^b \text{curv} (\Omega)^b = 0$, for all double intersections.\footnote{This is not totally trivial, but drops a bit out of the context. First of all, in our situation \cite[Lemma 7.2.3.5]{Lurie09} states the following: there exists an open cover $W = \{ W_i \}_{i,j}$ of $M$ and a refinement map $r : J \rightarrow I$ with $W_i \subseteq U_{r(i)}$, such that for $i, j \in J$ with $r(i) \neq r(j)$ there exists an open set $W \in \mathcal{V}_{ij}$ with $W_i \cap W_j \subseteq W$. Let $s_i := s_{r(i)}|_{W_i}$ denote the restricted sections. Consider a double intersection $W_i \cap W_j$.}

We obtain the generalized $\Gamma$-cocycle $(g, a, A, B, \varphi)$ as described before, but over $U_i \cap U_j$ we obtain using Eq. \eqref{2.2.3.2}:

$$0 = (\alpha_{g_{ij}}) \ast (\sigma_{ij} \cdot \text{curv}(\Omega)^b) = -(\alpha_{g_{ij}}) \ast (B_i) + B_j + d\varphi_{ij} + \frac{1}{2} [\varphi_{ij} \wedge \varphi_{ij}] + \alpha_*(A_j \wedge \varphi_{ij});$$

this is exactly the required cocycle condition Eq. \eqref{2.2.2.2}.

3.) If $\tilde{\Omega}$ is fake-flat, then $(g, a, A, B, \varphi)$ is fake-flat. Indeed, $f\text{curv}(A_i, B_i) = s_i^* \text{curv}(\tilde{\Omega})^a$.

4.) The cocycle $(g, a, A, B, \varphi)$ is as a preimage of $\mathcal{P}$ under $r^{\mathcal{P}}$.

We consider the smooth $\Gamma$-equivariant functor $\phi : \mathcal{P}_{(g, a)} \rightarrow \mathcal{P}$ of Section \ref{3.2} defined by

$$\phi(i, x, g) := R(s_i(x), g) \quad \text{and} \quad \phi(i, j, x, h, g) := R(\sigma_{ij}(x), (\alpha(g_{ij}(x)^{-1}, h), g_{ij}(x)^{-1}g)).$$

It is straightforward to check that $\phi^* \tilde{\Omega} = \Omega$ (where $\Omega$ is the connection reconstructed from $(A, B, \varphi)$ as described at the beginning of this subsection). Hence, the canonical $\tilde{\Omega}$-pullback on the induced anafunctor is connection-preserving and by Remark \ref{5.2.10} also guarantees that the canonical pullback is regular if $\tilde{\Omega}$ is regular, and fake-flat if $\tilde{\Omega}$ is fake-flat. Thus, we also obtain 1-morphisms in the bicategories $2-\text{Bun}_{\tilde{\mathcal{V}}}(M)$ and $2-\text{Bun}_{\mathcal{V}}(M)$. By Remark \ref{5.2.10} these are automatically 1-isomorphisms. This shows surjectivity of all three vertical maps.

In order to prove injectivity of the vertical maps in our diagram, we consider two generalized differential $\Gamma$-cocycles $(g, a, A, B, \varphi)$ and $(g', a', A', B', \varphi')$ with respect to the same open cover, and the reconstructed principal $\Gamma$-2-bundles $\mathcal{P} := \mathcal{P}_{(g, a)}$ and $\mathcal{P'} := \mathcal{P}_{(g', a')}$ equipped with the reconstructed connections $\Omega$ and $\tilde{\Omega}'$, respectively. We suppose that there is a 1-morphism $F : \mathcal{P} \rightarrow \mathcal{P'}$ equipped with a connection-preserving, connective pullback $\nu$. We have to show that the two cocycles are generalized equivalent. Following the treatment in Section \ref{3.2} we have transition spans $\sigma_{ij} := (i, j, x, 1, g_{ij}(x))$ of $\mathcal{P}$ with transition functions $g_{ij}$, and analogous transition spans $\sigma'_{ij}$ of $\mathcal{P'}$ with transition functions $g'_{ij}$. We choose transition spans $\sigma_i : U_i \rightarrow F$ of $F$ along $(s_i, s_i') : U_i \rightarrow \text{Obj}(\mathcal{P}) \times_M \text{Obj}(\mathcal{P'})$ with transition functions $h_i : U_i \rightarrow G$. In Section \ref{3.2} we have then derived Eq. \eqref{3.2.4}:

$$\sigma_{ij} \circ \rho(\sigma_j, \text{id}_{g_{ij}}) = \rho(\sigma_i \circ R(\sigma'_{ij}, \text{id}_{h_i}), (\alpha(h_i^{-1}g_{ij}^{-1}, e_i), h_i^{-1}g'_{ij}^{-1}h_jg_{ij}^{-1}));$$

which is an equality between transition spans of $F$, and proved that $(h, e)$ satisfies the equivalence conditions of Eqs. \eqref{2.2.8} and \eqref{2.2.10}. In order to upgrade these results to the differential setting, we
set $\phi_i := (\alpha_{h_i})_*(\sigma_i^*\nu_0) \in \Omega^1(U_i, h)$ and pull back $\nu_0$ separately along both sides of (5.3.2). We get, on the left hand side,

$$(\alpha_{g_{ij}^{-1}h_{ij}^{-1}})_*((\alpha_{h_j})_*(\varphi_{ij} + \phi_j)),$$

and, on the right hand side,

$$(\alpha_{g_{ij}^{-1}h_{ij}^{-1}})_*\left(\text{Ad}_{e_{ci,j}}^{-1}(\alpha_{g_{ij}'})_*(\phi_i) + \varphi_{ij}' + (\alpha_{ci,j})_*(A_j') + \theta_{ci,j}\right).$$

Thus, equality of both formulas gives the equivalence condition of Eq. (2.2.9). This shows that $(h, e, \phi)$ forms a generalized equivalence, which proves the injectivity of $P$. For the injectivity of $r$ we need to show that above equivalence is non-generalized, under the assumption that the pullback $\nu$ on $F$ is regular. With this assumption, we can assume in above constructions that the transition spans $\sigma_i$ satisfy $\sigma_i^*\text{fcurv}(\nu) = 0$. It is straightforward to check that

$$0 = (\alpha_{h_i})_*(\sigma_i^*\text{fcurv}(\nu)) = B_i' + \alpha_*(A_i' \wedge \phi_i) + d\phi_i + \frac{1}{2}[\phi_i \wedge \phi_i] - (\alpha_{h_i})_*(B_i).$$

This is the additional equivalence condition of Eq. (2.2.9) that characterized non-generalized equivalence data.

**Corollary 5.3.5.** The map $p^{gen} : \hat{H}^1(M, \Gamma)^{gen} \longrightarrow H^1(M, \Gamma)$ of Section 2.2 is surjective.

**Proof.** Let $(g, a)$ be a $\Gamma$-cocycle and $\mathcal{P}_{(g, a)}$ be the reconstructed principal $\Gamma$-2-bundle. Since the $G$-action on $\text{Ob}(\mathcal{P}_{(g, a)})$ is essentially the multiplication of $G$ on itself, it is free and proper. By Theorem 5.3.4 there exists a connection on $\mathcal{P}_{(g, a)}$. Extracting local data yields a generalized differential $\Gamma$-cocycle with image equivalent to $(g, a)$.

**Corollary 5.3.6.** Suppose $f : M \longrightarrow N$ is a smooth map with rank at most one, and $\mathcal{P}$ is a principal $\Gamma$-bundle with (any/regular/fake-flat) connection $\Omega$ over $N$. Then, there exists a principal $G$-bundle $P$ with (some/some/flat) connection over $M$ and an isomorphism $P//H \cong f^*\mathcal{P}$ in $\text{2-Bun}^\Omega(M) / 2\text{-Bun}_G^\Omega(M) / 2\text{-Bun}_F^\Omega(M)$.

**Proof.** The functors

$$\text{Bun}_G^\Omega(M) \longrightarrow 2\text{-Bun}_G^\Omega(M) \quad \text{and} \quad \text{Bun}_G^\Omega(M)^\text{flat} \longrightarrow 2\text{-Bun}_F^\Omega(M)$$

of Example 5.2.11 induce the maps $j$ and $j^{\text{flat}}$ considered in Section 2.2 under the isomorphism of Theorem 5.3.4. Then, the claim follows from Lemma 2.2.3.

### 5.4 Local connection forms and gauge transformations

In this subsection we show that principal $\Gamma$-2-bundles with connection are locally modelled by so-called $\Gamma$-connections and gauge transformations. A $\Gamma$-connection on $M$ is a pair $(A, B)$ consisting of a 1-form $A \in \Omega^1(M, \mathfrak{g})$ and a 2-form $B \in \Omega^2(M, \mathfrak{h})$. We define curvature and fake-curvature of a $\Gamma$-connection by

$$\text{curv}(A, B) := dB + \alpha_*(A \wedge B) \in \Omega^2(M, \mathfrak{h})$$

$$\text{fcurv}(A, B) := dA + \frac{1}{2}[A \wedge A] - t_*(B) \in \Omega^2(M, \mathfrak{g})$$
and call it *fake-flat*, if \( \text{curv}(A, B) = 0 \), and *flat*, if it is fake-flat and \( \text{curv}(A, B) = 0 \). There is a bijection between the set of \( \Gamma \)-connections on \( M \) and \( \Omega^1(M_{\text{dis}}, \gamma) \), under which \((A, B)\) corresponds to \( \Psi \in \Omega^1(M_{\text{dis}}, \gamma) \) with \( \Psi^a = A, \Psi^b = 0, \) and \( \Psi^c = -B \). In particular, every \( \Gamma \)-connection \((A, B)\) defines a connection \( \Omega_{A,B} := \Omega_{\Psi} \) on the trivial principal \( \Gamma \)-bundle \( \mathbb{I} \), see Example 5.1.8. Explicitly, we have

\[
\Omega^a_{A,B} = \text{Ad}_{g^{-1}}(A) + g^*\theta, \quad \Omega^b_{A,B} = (\alpha_{g^{-1}})_*(\tilde{\alpha}_h)_*(A) + h^*\theta, \quad \Omega^c_{A,B} = -(\alpha_{g^{-1}})_*(B) \tag{5.4.1}
\]

where \( g \) and \( h \) denote the projections. Recasting the results of Example 5.1.8 in the language of \( \Gamma \)-connections, we obtain the following.

**Lemma 5.4.1.** The assignment \((A, B) \mapsto \Omega_{A,B}\) establishes a bijection between the set of \( \Gamma \)-connections on \( M \) and the set of connections on the trivial principal \( \Gamma \)-bundle \( \mathbb{I} \). Moreover, the conditions “fake-flat” and “flat” are invariant under this bijection.

We will also use the notation \( j_{A,B} \) for the trivial principal \( \Gamma \)-bundle equipped with the connection \( \Omega_{A,B} \). A *gauge transformation* between \( \Gamma \)-connections \((A, B)\) and \((A', B')\) on \( M \) is a pair \((g, \varphi)\) of a smooth map \( g : M \to G \) and a 1-form \( \varphi \in \Omega^1(M, \mathfrak{h}) \) such that

\[
\text{Ad}_g(A) - g^*\theta = A' + t_*(\varphi) \tag{5.4.2}
\]
\[
B' + d\varphi + \frac{1}{2}[\varphi \wedge \varphi] + \alpha_*(A' \wedge \varphi) = (\alpha_g)_*(B). \tag{5.4.3}
\]

We define a smooth, \( \Gamma \)-equivariant functor \( \phi_g : \mathbb{J} \to \mathbb{J} \) by setting

\[
\phi_g(x, g') := (x, g(x)g') \quad \text{and} \quad \phi_g(x, h, g') := (x, \alpha(g(x), h), g(x)g').
\]

The definition of \( \phi_g \) implies the composition law \( \phi_{g_2} \circ \phi_{g_1} = \phi_{g_2 g_1} \). By Remark 2.4.2 (b) \( \phi_g \) is a 1-morphism in \( \text{2-Bun}_1(M) \). We promote \( \phi_g \) to a setting with connections in the following way. We denote by \( \Omega := \Omega_{A,B} \) and \( \Omega' := \Omega_{A',B'} \) the connections on \( \mathbb{I} \) corresponding to the \( \Gamma \)-connections \((A, B)\) and \((A', B')\). It is easy to see that in general \( \phi_g^* \Omega' \neq \Omega \). More precisely, we have

\[
\phi_g^* \Omega^a = \Omega^a - \text{Ad}_{g pr_M}^{-1}(t_*(pr_M^*\varphi))
\]
\[
\phi_g^* \Omega^b = \Omega^b - (\alpha_{pr_M^{-1}pr_{\mathfrak{g}^{-1}pr_{\mathfrak{g}^{-1}g^{-1}}}})_*(pr_M^*\varphi) + (\alpha_{pr_M^{-1}pr_{\mathfrak{g}^{-1}g^{-1}}})_*(pr_M^*\varphi)
\]
\[
\phi_g^* \Omega^c = \Omega^c + (\alpha_{pr_M^{-1}pr_{\mathfrak{g}^{-1}g^{-1}}})_*(\alpha_*(A' \wedge \varphi) + d\varphi + \frac{1}{2} [\varphi \wedge \varphi]).
\]

In other words, if \( F_g \) is the anafunctor associated to the smooth functor \( \phi_g \), then the canonical \( \Omega' \)-pulback \( \nu \) on \( F_g \) is *not* connection-preserving. However, using the affine space structure of Lemma 4.3.2 we shift \( \nu \) to a new \( \Omega' \)-pulback \( \nu^\varphi \) by a pair \((\varphi_0, \varphi_1)\) of forms \( \varphi_0 \in \Omega^1(M \times G, \mathfrak{h}) \) and \( \varphi_1 \in \Omega^2(M \times G, \mathfrak{h}) \), which we obtain from the given 1-form \( \varphi \) by the following formulae:

\[
\varphi_0 := (\alpha_{pr_M^{-1}pr_{\mathfrak{g}^{-1}g^{-1}}})_*(pr_M^*\varphi) \tag{5.4.4}
\]
\[
\varphi_1 := -(\alpha_{pr_M^{-1}pr_{\mathfrak{g}^{-1}g^{-1}}})_*(pr_M^*(d\varphi + \frac{1}{2} [\varphi \wedge \varphi] + \alpha_*(A' \wedge \varphi))). \tag{5.4.5}
\]

**Lemma 5.4.2.** The shifted \( \Omega' \)-pulback \( \nu^\varphi \) on \( F_g \) has the following properties:

(a) *It is connection-preserving, i.e. \((F_g)^{\nu^\varphi}_* \Omega' = \Omega."

(b) *It is connective.*

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(c) It is regular.

(d) It is fake-flat if \((A, B)\) and \((A', B')\) are fake-flat.

Proof. In order to compute \((F_\gamma)_*\Omega'\) we use above calculation of \(\phi^*\Omega\) and then incorporate the shift using Remark 4.3.6 (c) this yields \(\eta\). For \(\nu\) we note that the forms \(\varphi_0\) and \(\varphi_1\) are \(R\)-invariant; hence \(\nu^\varphi\) is connective by Remark 5.2.10 (f). Claims \((c)\) and \((d)\) follow from Remark 5.2.10 (g) by verifying Eq. 5.2.7,

\[
d\varphi_0 + \frac{1}{2} [\varphi_0 \wedge \varphi_0] + \alpha (\phi^* \Omega^a \wedge \varphi_0) + \varphi_1 = 0;
\]

this is straightforward. \(\Box\)

We use the notation \(F_{g, \varphi}\) for the anafunctor associated to \(\phi_g\) equipped with the \(\Omega'\)-pullback \(\nu^\varphi\).

By Lemma 5.4.2 this is a 1-morphism

\[
F_{g, \varphi} : \mathcal{J}_{A, B} \longrightarrow \mathcal{J}_{A', B'}
\]

in \(2\text{-}\text{Bun}^{\Omega_\nu}_\Gamma(M)\), and in case of fake-flat \(\Gamma\)-connections in \(2\text{-}\text{Bun}^{\Omega'}_\Gamma(M)\). Next we discuss the situation that we have \(\Gamma\)-connections \((A, B)\) and \((A', B')\), and two gauge transformations \((g_1, \varphi_1)\) and \((g_2, \varphi_2)\). A gauge 2-transformation is a smooth map \(a : M \longrightarrow H\) such that

\[
g_2 = t(a)g_1 \quad \text{and} \quad \text{Ad}_{a}^{-1}(\varphi_2) + (\Delta a)_{\ast}(A') = \varphi_1 - a^*\theta.
\]

We define the smooth map

\[
\eta_a : \text{Obj}(\mathcal{J}) \longrightarrow \text{Mor}(\mathcal{J}) : (x, g) \longrightarrow (\text{id}_x, (a(x), g_1(x)g)).
\]

Lemma 5.4.3. This is a smooth \(\Gamma\)-equivariant natural transformation \(\eta_a : \phi_{g_1} \Longrightarrow \phi_{g_2}\), and induces a 2-morphism \(\eta_a : F_{g_1, \varphi_1} \Longrightarrow F_{g_2, \varphi_2}\) in \(2\text{-}\text{Bun}^{\Omega_\nu}_\Gamma(M)\).

Proof. Source and target conditions as well as naturality are straightforward to check. For \(\Gamma\)-equivariance, we check according to Remark 2.4.2 (b)

\[
\eta_a(x, gg') = (\text{id}_x, (a(x), g_1(x)g g')) = (\text{id}_x, (a(x), g_1(x)g) \cdot (1, g')) = R(\eta_a(x, g), \text{id}_{g'}).\]

Concerning the connections, we use Remark 4.3.6 (d) and hence have to verify the conditions \(\eta_a^\alpha : \kappa_{1,0} - \kappa_{2,0}\) and \(\eta_a^\Delta : \kappa_{1,1} - \kappa_{2,1}\), where \(\kappa_i = (\kappa_{1,0}, \kappa_{1,1})\) are the shifts induced from \(\varphi_i\) via Eqs. 5.4.4 and 5.4.5. The first condition follows immediately from the definitions, and the second conditions can be checked using Eq. 5.4.3. \(\Box\)

\(\Gamma\)-connections, gauge transformations, and gauge 2-transformation form bicategories \(\text{Con}^{\Gamma}_M(M)\) and \(\text{Con}^{\Omega'}_M(M)\), see [SW11]. The assignments \(M \longrightarrow \text{Con}^{\Gamma}_M(M)\) and \(M \longrightarrow \text{Con}^{\Omega'}_M(M)\) form presheaves of bicategories.

Proposition 5.4.4. The assignments

\[
(A, B) \longmapsto \mathcal{J}_{A, B}, \quad (g, \varphi) \longmapsto F_{g, \varphi} \quad \text{and} \quad a \longmapsto \eta_a
\]

form strict 2-functors

\[
L^{\Omega'} : \text{Con}^{\Omega'}_M(M) \longrightarrow 2\text{-}\text{Bun}^{\Omega'}_\Gamma(M) \quad \text{and} \quad L : \text{Con}^{\Gamma}_M(M) \longrightarrow 2\text{-}\text{Bun}^{\Omega_\nu}_\Gamma(M).
\]

Moreover, \(L^{\Omega'}\) and \(L\) are natural in \(M\); i.e., they form morphisms of presheaves.
Proof. Well-definedness is the content of Lemmas 5.4.2 and 5.4.3. It remains to check that the composition of gauge transformations, and horizontal and vertical composition of gauge 2-transformations are respected. This is straightforward and left as an exercise.

In the remainder of this subsection we prove two statements about the 2-functors \( L \) and \( L^H \), see Propositions 5.4.5 and 5.4.8. We recall that a section in a principal \( \Gamma \)-bundle \( \mathcal{P} \) is a smooth map \( s : M \to \text{Obj}(\mathcal{P}) \) such that \( \pi \circ s = 1_M \). Since \( \pi : \text{Obj}(\mathcal{P}) \to M \) is a surjective submersion, every point \( x \in M \) has an open neighborhood \( U \subseteq M \) that supports a section. Associated to a section \( s \) is a smooth \( \Gamma \)-equivariant functor \( T : \mathcal{J} \to \mathcal{P} \) defined by

\[
T := R \circ (s \times \text{id}_\Gamma) : M_{dis} \times \Gamma \to \mathcal{P},
\]

called the trivialization associated to \( s \). We equip \( T \) with the canonical \( \Omega \)-pullback, which is connective by Remark 5.2.10. Thus, \( \Psi := T^*\Omega \) is a connection on \( \mathcal{J} \) by Proposition 5.2.12 (a) and thus \( \Psi = \Omega_{A,B} \) for a unique \( \Gamma \)-connection \( (A,B) \) on \( M \); see Lemma 5.4.1. By definition of \( \Psi \), \( T \) is connection-preserving. If \( \Omega \) is regular, then the canonical \( \Omega \)-pullback is regular (Remark 5.2.10 (d)), and if \( \Omega \) is flat-flat, then \( \Psi \) is flat-flat (Proposition 5.2.12 (b)), \( (A,B) \) is flat-flat (Lemma 5.4.1), and the canonical \( \Omega \)-pullback is flat-flat (Remark 5.2.10 (c)). This shows the following result.

**Proposition 5.4.5.** Let \( \mathcal{P} \) be a principal \( \Gamma \)-bundle over \( M \) with connection \( \Omega \). For every point \( x \in M \) there exists an open neighborhood \( U \subseteq M \) and a

(a) \( \Gamma \)-connection \( (A,B) \) on \( U \) such that \( \mathcal{J}_{A,B} \cong \mathcal{P} |_U \) in \( 2\text{-Bun}_\Gamma^\mathcal{V}(U) \).

(b) \( \Gamma \)-connection \( (A,B) \) on \( U \) such that \( \mathcal{J}_{A,B} \cong \mathcal{P} |_U \) in \( 2\text{-Bun}_\Gamma^{\mathcal{V} reg}(U) \), if \( \Omega \) is regular.

(c) fake-flat \( \Gamma \)-connection \( (A,B) \) on \( U \) such that \( \mathcal{J}_{A,B} \cong \mathcal{P} |_U \) in \( 2\text{-Bun}_\Gamma^\mathcal{V}(U) \), if \( \Omega \) is fake-flat.

**Remark 5.4.6.** Propositions 5.4.5 (b) and 5.4.5 (c) imply that the sheaf morphisms \( L \) and \( L^H \) are surjective, i.e., essentially surjective on stalks. As 2-functors, \( L \) and \( L^H \) are in general not essentially surjective, regardless to which kind of subset \( U \subseteq M \) one restricts them.

Suppose \( F : \mathcal{J}_{A,B} \to \mathcal{J}_{A',B'} \) is a 1-morphism in \( 2\text{-Bun}_\Gamma^\mathcal{V}(M) \), and denote by \( \nu = (\nu_0, \nu_1) \) its \( \Omega_{A',B'} \)-pullback. Consider the smooth map

\[
s : M \to \text{Obj}(\mathcal{J}_{A,B}) \times_M \text{Obj}(\mathcal{J}_{A',B'}) : x \mapsto ((x,1),(x,1)).
\]

We assume that \( F \) admits a transition span \( \sigma : M \to F \) along \( s \) with \( \sigma^*f_{\text{curv}}(\nu) = 0 \). Let \( g : M \to G \) be a transition functor for \( \sigma \), and let \( \varphi := (\alpha_g)_* (\sigma^*\nu_0) \in \Omega^1(M,g) \).

**Lemma 5.4.7.** The pair \((g, \varphi)\) is a gauge transformation between \((A,B)\) and \((A',B')\). Moreover, there exists a connection-preserving 2-morphism \( F_{g,\varphi} \cong F \).

**Proof.** We have to verify Eqs. 5.4.2 and 5.4.3. The first is a direct calculation using Eq. (4.3.3). For the second, it is straightforward to deduce from the definition of \( f_{\text{curv}}(\nu) \) and Eqs. (4.3.3) and (4.3.5) that

\[
(\alpha_g)_* (\sigma^*f_{\text{curv}}(\nu)) = - (\alpha_g)_* (B + B' + d\varphi + \frac{1}{2}[\varphi \wedge \varphi] + \alpha_*(A' \wedge \varphi)).
\]

Thus, the assumption \( \sigma^*f_{\text{curv}}(\nu) = 0 \) implies Eq. (6.4.3). Next consider the smooth map \( \tilde{\kappa} : M \times G \to F \) defined by \( \tilde{\kappa}(x,g') := \sigma(x) \cdot \text{id}_{g'} \). It is easy to check that it satisfies the conditions of Remark 2.3.3 (c) namely

\[
\alpha t(\tilde{\kappa}(x,g')) = (x,g') \quad \alpha r(\tilde{\kappa}(x,g')) = \phi_g(x,g') \quad \alpha \circ \tilde{\kappa}(x,g') \circ \beta = \tilde{\kappa}(t(\alpha)) \circ \phi_g(\alpha) \circ \beta,
\]

\[
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\]
where \( x \in M \), \( g' \in G \) and \( \alpha, \beta \in M \times \text{Mor}(\Gamma) \). Hence it induces a transformation \( \kappa : F_{g',\phi} \to F \). According to Remark 2.4.2 (b) \( \kappa \) is \( \Gamma \)-equivariant if \( \kappa \) satisfies \( \kappa(x, g'(g'')) = \hat{\kappa}(x, g') \cdot \text{id}_{g''} \); this follows immediately from its definition. This shows that \( \kappa \) is a 2-morphism \( \kappa : F_{g',\phi} \to F \). It remains to check that it is connection-preserving. According to Remark 4.3.6 (d) this is the case if the given pullback \( \nu \) on \( F \) pulls back along \( \kappa \) to the shift \( \varphi \) of the canonical pullback on \( F_g \); i.e.

\[
\kappa^* \nu_0 = \varphi_0 \quad \text{and} \quad \kappa^* \nu_1 = \varphi_1,
\]

where \( \varphi_0 \) and \( \varphi_1 \) are defined in Eqs. (b.4.4) and (b.4.5). The first equation follows from just the definitions and the fact that \( \nu \) is connective. For the second equation one shows first using connectivity of \( \nu \) and the vanishing of \( \sigma^* \text{curv}(\nu) \) that

\[
\kappa^* \nu_1 = - (\alpha_{\text{pr}^{-1}_G})_* \left( \sigma^* (\text{dv}_0 + \frac{1}{2} [\nu_0 \land \nu_0] + \alpha_* (\text{Ad}_g^{-1}(A') \land \nu_0) + \alpha_* (g^* \theta \land \sigma^* \nu_0) \right).
\]

From there it is straightforward to deduce Eq. (b.4.5). \( \square \)

**Proposition 5.4.8.**

(a) Every point \( x \in M \) has an open neighborhood \( U \subseteq M \) such that \( L(U) \) is fully faithful.

(b) For every contractible open set \( U \subseteq M \) the 2-functor \( L^{\tilde{U}}(U) \) is fully faithful.

**Proof.** Suppose \((A, B)\) and \((A', B')\) are \( \Gamma \)-connections on \( M \). We consider the Hom-functor

\[
L : \mathcal{H}om_{\text{Cont}(M)}((A, B), (A', B')) \to \mathcal{H}om_{\mathcal{2Bun}^{\infty}_\Gamma(M)}(\mathcal{J}_A, \mathcal{J}_{A', B'}).\n\]

That a 2-functor is fully faithful means that the Hom-functor is an equivalence of categories. We want to show using Lemma 5.4.7 that it is essentially surjective. By Lemma 5.4.9 we can first choose any contractible open neighborhood \( U \subseteq M \) of \( x \) to guarantee the existence of a transition span \( \sigma : U \to F \) along \( s \). If \( \nu \) is regular, then by definition of regularity we can restrict to a smaller open subset such that \( \sigma^* \text{curv}(\nu) = 0 \). If \( \nu \) is fake-flat, the second restriction is unnecessary; thus Lemma 5.4.7 applies to both cases.

It remains to show that the Hom-functor is fully faithful; there is no difference between the regular and the fake-flat case. We have to analyze the assignment of the natural transformation \( \eta_a \) to a gauge 2-transformation \( a : (g_1, \varphi_1) \to (g_1, \varphi_1) \). From the definition of \( \eta_a \) in Eq. (b.4.7) we conclude immediately that the assignment is injective. For surjectivity, suppose \( \eta : F_{g_1, \varphi_1} \to F_{g_2, \varphi_2} \) is a connection-preserving 2-morphism. We can assume that it is induced from a smooth \( \Gamma \)-equivariant natural transformation \( \eta \). Its components form a smooth map \( \tilde{\eta} : M \times G \to M \times H \times G \). Source and target matching, naturality, and \( \Gamma \)-equivariance imply that \( \tilde{\eta}(x, g) = (x, a(x), g_1(x)g) \) for a smooth map \( a : M \to H \) satisfying

\[
t(a(x)) = g_2(x)g_1(x)^{-1}. \tag{5.4.8}
\]

That \( \eta \) is connection-preserving implies by Remark 4.3.6 (d) that \( \eta^* \Omega^b_{A', B'} = \kappa_{1,0} - \kappa_{2,0} \), where \( \kappa_{i,0} = (\alpha_{\text{pr}^{-1}_G(g_1(x)-)})*(\varphi_i|_x) \) according to Eq. (5.4.4). Then we get

\[
\varphi_1 - \text{Ad}_a^{-1}(\varphi_2) = (\alpha_{g_1})_* (s^* \kappa_{1,0} - s^* \kappa_{2,0}) = (\alpha_{g_1})_* s^* \eta^* \Omega^b = (\tilde{\alpha}_a)_*(A) + a^* \theta, \tag{5.4.9}
\]

where \( s : M \to M \times G : x \mapsto (x, 1) \). Eqs. (5.4.8) and (5.4.9) show that \( a : M \to H \) is a gauge 2-transformation. Obviously, \( \eta_a = \eta \). This shows that the Hom-functor is full. \( \square \)
A Formulary for calculations in strict Lie 2-algebras

The formulas presented here are valid for a strict Lie 2-algebra consisting of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, a Lie algebra homomorphism $t_* : \mathfrak{h} \longrightarrow \mathfrak{g}$, and a bilinear map $\alpha_* : \mathfrak{g} \times \mathfrak{h} \longrightarrow \mathfrak{h}$. The axioms are:

\[
\begin{align*}
\alpha_*(\{X_1, X_2\}, Y) &= \alpha_*(X_1, \alpha_*(X_2, Y)) - \alpha_*(X_2, \alpha_*(X_1, Y)) \\
\alpha_*(X, [Y_1, Y_2]) &= [\alpha_*(X, Y_1), Y_2] + [Y_1, \alpha_*(X, Y_2)], \\
\alpha_*(t_*(Y_1), Y_2) &= [Y_1, Y_2] \\
t_*(\alpha_*(X, Y)) &= [X, t_*(Y)]
\end{align*}
\]

Formulas involving the adjoint action:

\[
\begin{align*}
t_* \circ \text{Ad}_h &= \text{Ad}_{t_*(h)} \circ t_* \\
\text{Ad}_{a}(\alpha_*(X,Y)) &= \alpha_*(\text{Ad}_{a}(X), \text{Ad}_{a}(Y))
\end{align*}
\]

Formulas involving the map $\alpha_g : H \longrightarrow H$ defined by $\alpha_g(h) := \alpha(g, h)$:

\[
\begin{align*}
\text{Ad}_{g} \circ t_* &= t_* \circ (\alpha_g)_* \\
\text{Ad}_{\alpha(g,h)} \circ (\alpha_g)_* &= (\alpha_g)_* \circ \text{Ad}_h \\
(\alpha_t(h))_* &= \text{Ad}_h \\
(\alpha_*(\text{Ad}_g(X), Y) &= (\alpha_g)_*(\alpha_*(X, (\alpha_g^{-1})_*(Y))) \\
(\alpha_g)_*(\alpha_*(X,Y)) &= \alpha_*(\text{Ad}_g(X), (\alpha_g)_*(Y))
\end{align*}
\]

Formulas involving the map $\hat{\alpha}_h : G \longrightarrow H$ defined by $\hat{\alpha}_h(g) := h^{-1}\alpha(g, h)$:

\[
\begin{align*}
(\hat{\alpha}_{h_1 h_2})_* &= \text{Ad}_{h_2}^{-1} \circ (\hat{\alpha}_{h_1})_* \circ (\hat{\alpha}_{h_2})_* \\
(\hat{\alpha}_{\alpha(g,h)})_* &= (\alpha_g)_* \circ (\hat{\alpha}_h)_* \circ \text{Ad}_g^{-1} \\
(\hat{\alpha}_{h^{-1}})_* &= -\text{Ad}_h \circ (\hat{\alpha}_h)_* \\
t_*(((\hat{\alpha}_h)_*(X)) &= \text{Ad}_{t_*(h)}^{-1}(X) - X \\
t_*((\hat{\alpha}_h)_*(X)) &= [X, \alpha_*(X, (\hat{\alpha}_h)_*(Y)) + \alpha_*(X, (\hat{\alpha}_h)_*(Y)) - \alpha_*(X, (\hat{\alpha}_h)_*(X))
\end{align*}
\]

Formulas involving the exterior derivative:

\[
\begin{align*}
d\alpha_*(\omega \wedge \eta) &= \alpha_*(d\omega \wedge \eta) + (-1)^{\text{deg}(\omega)} \alpha_*(\omega \wedge d\eta) \\
d(\alpha_g)_*(\varphi) &= (\alpha_g)_*(d\varphi) + \alpha_*(g^*\theta \wedge (\alpha_g)_*(\varphi)) \\
d\text{Ad}_{g}^{-1}(\varphi) &= \text{Ad}_{g}^{-1}(d\varphi) - [g^*\theta \wedge \text{Ad}_{g}^{-1}(\varphi)] \\
d(\hat{\alpha}_h)_*(\varphi) &= (\hat{\alpha}_h)_*(d\varphi) + (-1)^{\text{deg}(\varphi)} \alpha_*(\varphi \wedge h^*\theta) - [h^*\theta \wedge (\hat{\alpha}_h)_*(\varphi)]
\end{align*}
\]

References

[ACJ05] P. Aschieri, L. Cantini, and B. Jurco, “Nonabelian bundle gerbes, their differential geometry and gauge theory”. Commun. Math. Phys., 254:367–400, 2005. [arxiv:hep-th/0312154]

[Bar04] T. Bartels, 2-bundles and higher gauge theory. PhD thesis, University of California, Riverside, 2004. [arxiv:math/0410328]
[SP11] C. Schommer-Pries, “Central extensions of smooth 2-groups and a finite-dimensional string 2-group”. *Geom. Topol.*, 15:609–676, 2011. [arxiv:0911.2483]

[SW11] U. Schreiber and K. Waldorf, “Smooth functors vs. differential forms”. *Homology, Homotopy Appl.*, 13(1):143–203, 2011. [arxiv:0802.0663]

[SW13] U. Schreiber and K. Waldorf, “Connections on non-abelian gerbes and their holonomy”. *Theory Appl. Categ.*, 28(17):476–540, 2013. [arxiv:0808.1923]

[Wal] K. Waldorf, “Parallel transport in principal 2-bundles”. Preprint. [arxiv:1704.08542]

[Woc11] C. Wockel, “Principal 2-bundles and their gauge 2-groups”. *Forum Math.*, 23:565–610, 2011. [arxiv:0803.3692]

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