Holographic Characterisation of Locally Anti-de Sitter Spacetimes

Alex McGill

Abstract. It is shown that an \((n+1)\)-dimensional asymptotically anti-de Sitter solution of the Einstein-vacuum equations is locally isometric to pure anti-de Sitter spacetime near a region of the conformal boundary if and only if the boundary metric is conformally flat and (for \(n \neq 4\)) the boundary stress–energy tensor vanishes, subject to (i) sufficient (finite) regularity in the metric and (ii) the satisfaction of a conformally invariant geometric criterion on the boundary region. A key tool in the proof is the Carleman estimate of Chatzikaleas and Shao (Commun Math Phys, 2022)—a generalisation of previous work by the author with McGill and Shao in (Class Quant Gravity 38(5), 2020)—which is applied to prove a unique continuation result for the Weyl curvature at the conformal boundary given vanishing to sufficiently high order over the boundary region.

Contents

1. Introduction 2138
   1.1. Overview 2138
   1.2. aAdS Spacetimes and the Correspondence Problem 2139
   1.3. The Rigidity Result 2141
   1.4. Background and Proof Outline 2142
      1.4.1. Unique Continuation for Wave Equations 2142
      1.4.2. The Weyl Curvature 2144
   1.5. Organisation 2145
   1.6. Acknowledgements 2145
2. Asymptotically AdS Spacetimes 2146
3. The Mixed Tensor Calculus 2148
   3.1. The Formalism 2148
   3.2. Conversion Formulae 2150
1. Introduction

1.1. Overview

Asymptotically anti-de Sitter (aAdS) spacetimes play a central role in the conjectured *AdS/CFT correspondence* [15], which posits a duality between the bulk gravitational theory in such spacetimes and *conformal field theories* (CFT) living on their lower-dimensional boundaries. In this sense, it is a realisation of the more general *holographic principle*.

In order to provide a rigorous positive statement of this conjecture in the classical setting, one could seek to prove a $1-1$ correspondence between $(n+1)$-dimensional aAdS solutions of the Einstein-vacuum equations with a (conveniently normalised) negative cosmological constant $\Lambda$,

$$Rc - \frac{1}{2} Rs \cdot g + \Lambda \cdot g = 0, \quad \Lambda := -\frac{n(n-1)}{2}.$$  \hspace{1cm} (1.1)

and suitably defined data which characterise a particular boundary CFT. This was achieved in the *Riemannian* and *stationary Lorentzian* cases in the works [2,3,7], respectively. The series of articles [12,13,16] initiated a programme whose goal is to study the *non-stationary Lorentzian* setting by considering uniqueness properties of wave equations on fixed aAdS backgrounds near the boundary. This article provides a first application of the methods developed in this programme to vacuum spacetimes. Specifically, subject to the satisfaction of a geometric criterion on the boundary (identified in [6]), we will provide necessary and sufficient conditions on the boundary data for a given vacuum aAdS spacetime to be locally isometric to the pure AdS spacetime near the boundary.

The preceding discussion raises the following questions which will be informally addressed in the remainder of this section:

- In what sense does an aAdS spacetime have a boundary?
- What constitutes appropriate boundary data in this setting?

---

1Here, $Rc$ and $Rs$ represent the $g$-Ricci and scalar curvatures, respectively.
1.2. aAdS Spacetimes and the Correspondence Problem

Pure AdS spacetime is the maximally symmetric solution of (1.1). It has the representation \((\mathcal{M}_0, g_0)\), where

\[
\mathcal{M}_0 = \mathbb{R}^{n+1}, \quad g_0 = (1 + r^2)^{-1} dr^2 - (1 + r^2) dt^2 + r^2 \cdot \hat{\gamma}.
\]  

(1.2)

Here, we have covered the manifold using polar coordinates; \(\hat{\gamma}\) is the unit round metric. Under the coordinate transformation

\[
r = \frac{1}{4}\rho^{-1}(2 + \rho)(2 - \rho), \quad \rho \in (0, 2],
\]  

(1.3)

one finds that

\[
g_0 = \rho^{-2} \left[ d\rho^2 - dt^2 + \hat{\gamma} - \frac{1}{2} \rho^2 (dt^2 + \hat{\gamma}) + \frac{1}{16} \rho^4 (-dt^2 + \hat{\gamma}) \right].
\]  

(1.4)

In particular, one may attach a timelike conformal boundary

\[(\mathcal{I}, g) \simeq (\mathbb{R} \times S^{n-1}, -dt^2 + \hat{\gamma}), \]

(1.5)

to \(\rho^2 g_0\) at \(\rho = 0\).

An asymptotically anti-de Sitter (aAdS) spacetime region \((\mathcal{M}, g)\) is a manifold of the form

\[
\mathcal{M} = (0, \rho_0] \times \mathcal{I}, \quad \rho_0 > 0,
\]  

(1.6)

for smooth manifolds \(\mathcal{I}\). On such \(\mathcal{M}\), it is natural to consider the so-called vertical tensor fields, which only have components in directions tangential to \(\mathcal{I}\). In other words, a vertical tensor field on an aAdS region \(\mathcal{M}\) can be viewed as a \(\rho \in (0, \rho_0]\)-parametrised family of tensor fields on \(\mathcal{I}\). They thus admit an intuitive and consistent notion of ‘boundary limit’ as \(\rho \downarrow 0\).

On these manifolds, we consider metrics of the form

\[
g = \rho^{-2} \left( d\rho^2 + g_{ab} dx^a dx^b \right),
\]  

(1.7)

where \(g\) is a vertical tensor field \((x^a\) are \(\rho\)-transported coordinates on \(\mathcal{I}\)) and there exists some Lorentzian metric \(g\) on \(\mathcal{I}\) such that \(g \to g\) as \(\rho \downarrow 0\). \((\mathcal{I}, g)\) is then the conformal boundary associated with \((\mathcal{M}, g, \rho)\) and the form (1.7) is referred to as Fefferman–Graham gauge [9]. Accordingly, we refer to this class of spacetimes as FG-aAdS segments.

Remark 1.1. A standard definition of asymptotically AdS spacetimes is that they are conformally compact solutions of (1.1) (see, for example, [20]). It can be shown that such spacetimes admit timelike conformal boundaries and curvature tensors matching that of pure AdS to leading order near the boundary. Moreover, they can always be expressed in Fefferman–Graham gauge near the boundary [10]. As such, imposing this gauge does not lead to any loss of generality—see [13, Appendix A].
In this article, we will restrict our attention to vacuum solutions, i.e. to those FG-aAdS segments whose metrics solve (1.1). In [18], it was demonstrated that such segments admit partial expansions at the conformal boundary \( \mathcal{I} \) even for vertical metrics of only finite regularity:

\[
g = \begin{cases} 
\sum_{k=0}^{n-1} \rho^{2k} g^{(2k)} + \rho^n g^{(n)} + \rho^n r & \text{n odd}, \\
\sum_{k=0}^{n-2} \rho^{2k} g^{(2k)} + \rho^n \log \rho g^{(*)} + \rho^n g^{(n)} + \rho^n r & \text{n even},
\end{cases}
\]

(1.8)

where \( r \) is a vertical tensor field with vanishing boundary limit. All terms below order \( n \), as well as the trace and divergence of \( g^{(n)} \), are formally determined via (1.1) by \( g^{(0)} = g \), i.e. there exist functions \( F^{(2k)}(g, \partial g, \ldots, \partial^{2k} g) \) and \( F^{(*)}(g, \partial g, \ldots, \partial^{n} g) \) such that

\[
F^{(2k)} = \mathcal{F}^{(2k)}, \quad 0 < 2k < n,
\]

\[
F^{(*)} = \mathcal{F}^{(*)},
\]

(1.9)

as well as functions \( \mathcal{F}^{(n)}_{\text{tr}}(g, \partial g, \ldots, \partial^{n} g) \) and \( \mathcal{F}^{(n)}_{\text{div}}(g, \partial g, \ldots, \partial^{n+1} g) \) such that

\[
\text{tr}_g g^{(n)} = \mathcal{F}^{(n)}_{\text{tr}}, \quad \text{div}_g g^{(n)} = \mathcal{F}^{(n)}_{\text{div}},
\]

(1.10)

All terms beyond order \( n \) in (1.8) are determined by \( g^{(0)} \) and the trace/divergence-free part of \( g^{(n)} \). In other words, they constitute the free data in this expansion. This identifies pairs \( (g^{(0)}, g^{(n)}) \) satisfying (1.10) as appropriate data that one could prescribe on \( \mathcal{I} \) in an attempt to characterise the bulk spacetime.

An important observation is that there exist coordinate transformations \( (\rho, x^a) \rightarrow (\tilde{\rho}, \tilde{x}^a) \) preserving the Fefferman–Graham gauge (1.7), i.e.

\[
g = \tilde{\rho}^{-2} \left( d\tilde{\rho}^2 + \tilde{g}_{ab} d\tilde{x}^a d\tilde{x}^b \right),
\]

(1.11)

but altering the coefficients in the partial expansion of the corresponding vertical metric \( \tilde{g} \) (now given in terms of \( \tilde{\rho} \)). In particular, such a coordinate transformation induces a conformal transformation of \( g^{(0)} \); for the higher-order coefficients, the effect is more complicated [14,19]. Since the underlying physical theory is invariant under such transformations, it therefore only makes sense for us to speak of gauge-equivalent classes of boundary data \( [g^{(0)}, g^{(n)}] \) in the above sense.

A naive way to rigorously formulate the above-described AdS/CFT problem in the classical setting would be as a boundary Cauchy problem for (1.1) with data \( [g^{(0)}, g^{(n)}] \) prescribed on \( \mathcal{I} \). However, in analogy with the boundary Cauchy problem for the wave equation on a cylinder, one in fact expects this problem to be ill-posed given general (i.e. possibly non-analytic) boundary data. Nevertheless, it remains apposite to ask whether solutions, if they exist, remain unique:

**Question 1.2.** Assuming that a solution of (1.1) exists, is it uniquely determined in the bulk (up to isometry) by \( [g^{(0)}, g^{(n)}] \)?

---

2In the context of AdS/CFT, this corresponds to the expectation value of the boundary CFT stress–energy tensor [8].
1.3. The Rigidity Result

Let us consider the above question in the case of pure AdS. In [21], the authors proved that if a vacuum FG-aAdS segment \((n > 2)\) is locally isometric to pure AdS\(^3\) then its metric is given near the conformal boundary by

\[
g = \rho^{-2} \left( d\rho^2 + g_{ab} dx^a dx^b \right), \quad g_{ab} = g_{ab} - p_{ab} \cdot \rho^2 + \frac{1}{4} g^{cd} p_{ac} p_{bd} \cdot \rho^4,
\]

where \(g\) is conformally flat and \(p\) is the \(g\)-Schouten tensor

\[
p := \frac{1}{n-2} \left( R_{\text{c}} - \frac{1}{2(n-1)} R_{\text{s}} \cdot g \right),
\]

in which \(R_{\text{c}}\) and \(R_{\text{s}}\) are the \(g\)-Ricci and scalar curvatures, respectively.

It is then natural to ask if the converse statement is true, i.e. if a vacuum FG-aAdS segment \((\mathcal{M}, g)\) has boundary data \([g^{(0)}, g^{(n)}]\) in which \(g\) is conformally flat and

\[
g^{(n)} = \begin{cases} \frac{1}{2} g^{-1} p p, & n = 4, \\ 0, & n \neq 4, \end{cases}
\]

must \((\mathcal{M}, g)\) be locally isometric to pure AdS near \(\mathcal{I}\)? In this article, we provide a positive answer to this question subject to a geometric condition on the boundary (identified in [6] as a conformally invariant generalisation of the analogous condition in [16]):

**Definition 1.3** ([6]). Suppose that \((\mathcal{M}, g)\) is a vacuum FG-aAdS segment and \(\mathcal{D} \subseteq \mathcal{I}\) is open with compact closure. One says that \(\mathcal{D}\) satisfies the *generalised null convexity criterion (GNCC)* iff there exists a constant \(c > 0\) and a smooth function \(\eta \in C^4(\overline{\mathcal{D}})\) such that

\[
[D^2 \eta + \eta \cdot p] (3, 3) > c \cdot \eta \cdot h(3, 3) \quad \text{in} \quad \mathcal{D},
\]

\[
\eta > 0 \quad \text{in} \quad \mathcal{D},
\]

\[
\eta = 0 \quad \text{on} \quad \partial \mathcal{D},
\]

for all \(g\)-null \(3\), where \(D^2 \eta\) is the Hessian of \(\eta\) with respect to \(g\) and \(h\) is an arbitrary but fixed Riemannian metric on \(\mathcal{I}\).

The main result of this article may now be stated as follows\(^4\):

**Theorem 1.4.** Fix \(n > 2\). Suppose \((\mathcal{M}, g)\) is an \((n+1)\)-dimensional vacuum FG-aAdS segment for which \(g\) is sufficiently (finitely) regular and the GNCC is satisfied on \(\mathcal{D} \subseteq \mathcal{I}\). Then, \((\mathcal{M}, g)\) is locally isometric to pure AdS in the bulk near \(\mathcal{D}\) if and only if \((\mathcal{M}, g)\) has boundary data for which the following hold on \(\mathcal{D}\):

\[
g^{(0)}\text{is conformally flat},
\]

---

\(^{3}\)Not necessarily globally isometric to pure AdS; the boundary topology could differ from that given in (1.5).

\(^{4}\)The rigorous statement of this result in Theorem 5.9 makes precise sense of ‘near’ using the definition provided in (5.5).
\[ g^{(n)} = \begin{cases} \frac{i}{2} g^{-1} p p, & n = 4, \\ 0, & n \neq 4, \end{cases} \quad (1.19) \]

It is straightforward to verify that the GNCC is satisfied for sufficiently long timespans of the standard AdS conformal boundary \((\mathcal{I}, g) = (\mathbb{R} \times S^{n-1}, -dt^2 + \gamma)\). If one fixes this, Theorem 1.4 implies that the vacuum aAdS spacetime is isometric to pure AdS near such boundary segments if and only if \(g^{(n)}\) satisfies (1.19). The forthcoming work \cite{17} aims to extend this to a global rigidity statement for pure AdS by applying the hyperbolic positive mass theorem to a maximal hypersurface of the spacetime whose existence follows from the imposition of an additional (though weak) global assumption. One could view this as complementary to the classical rigidity results of \cite{4,22}, which proved that pure AdS is the unique static vacuum aAdS spacetime with the standard AdS conformal boundary; in our case, the staticity assumption is exchanged for a weaker global assumption plus the boundary (i.e. ‘holographic’) assumption on \(g^{(n)}\).

1.4. Background and Proof Outline

1.4.1. Unique Continuation for Wave Equations. The series of articles \cite{12,13,16} considered tensorial solutions to wave equations of the form

\[ (\Box_g + \sigma)u = G(u, \nabla u), \quad \sigma \in \mathbb{R}, \quad (1.20) \]

on fixed aAdS backgrounds.\(^5\) Specifically, these articles aimed to determine if a solution \(u\) of (1.20) with vanishing boundary data on \(\mathcal{I}\) must necessarily vanish in the interior. This is what is known as a unique continuation problem.

A positive answer to this problem was provided in the first of these articles \cite{12}. However, this rested on the assumptions that:

1. The boundary metric is static.
2. \(u\) vanishes at a sufficiently fast rate along a sufficiently long timespan on \(\mathcal{I}\).

The first assumption constitutes a significant restriction on the class of spacetimes for which the result may be applied. Furthermore, it was not known if the second assumption was strictly necessary.

The subsequent article \cite{13} weakened assumption (1) by generalising to spacetimes with boundary metrics of only bounded ‘non-stationarity’. However, this still left open the question of whether assumption (2) could be weakened (or even removed).

Most recently, \cite{16} further weakened assumption (1) by permitting spacetimes with general time functions; the new assumption was then formulated as the (gauge-dependent) null convexity criterion (NCC)\(^6\):

\textbf{Definition 1.5} (\cite{16}). Suppose \((\mathcal{M}, g)\) is a vacuum FG-aAdS segment for which \((\mathcal{I}, g)\) is foliated by a global time function (i.e. one that splits \(\mathcal{I}\) into the form

\(^5\)\(G(u, \nabla u)\) represents lower-order, possibly nonlinear, terms.

\(^6\)It is demonstrated in \cite{6} that the GNCC implies the NCC.
We say that the null convexity criterion is satisfied on \((I, g)\) if the bounds
\[
p(Z, Z) \geq C(Zt)^2, \quad |\mathcal{D}^2_{ZZ}t| \leq B(Zt)^2,
\]
hold for some constants \(0 \leq B < C\), where \(Z\) is any vector field on \(I\) satisfying \(g(Z, Z) = 0\) and \(\mathcal{D}\) is the \(g\)-Levi-Civita connection.

This article also rigorously justified the necessity of assumption (2) by linking it to the trajectories of near-boundary null geodesics. Specifically, upper and lower bounds (depending on the constants \(B, C\) featuring in (1.21)) were proved on the ‘time of return’ of such geodesics to \(I\). Since counterexamples to unique continuation were constructed in [1] using geometric optics methods with these geodesics,\(^7\) the bounds show that there is:

- A maximum timespan across which such counterexamples may be constructed.
- A minimum timespan across which we must assume vanishing of our field in order to eliminate the possibility of such counterexamples existing. Crucially, this minimum timespan matches the one in assumption (2).

FG-aAdS segments are constructed in such a way that it is natural to view tensor fields as mixed, containing vertical components that are treated using \(g\) and spacetime components that are treated using \(g\).\(^8\) Accordingly developed a mixed covariant formalism which makes sense of higher-order derivatives acting on vertical tensor fields. To this end, an extension \(\mathcal{D}\) of the \(g\)-Levi-Civita connection \(\mathcal{D}\) is constructed\(^8\) so as to permit covariant derivatives of vertical tensor fields in all \(M\)-directions. A ‘mixed’ connection \(\nabla\) is then defined in such a way that it acts as the standard \(g\)-Levi-Civita connection \(\nabla\) on the spacetime components and \(\mathcal{D}\) on the vertical components of a given tensor field; the vertical wave operator \(\square_g\) is defined as the \(g\)-trace of \(\nabla^2\).

Working in this framework, the article studied vertical tensor field solutions \(u\) of wave equations of the form
\[
(\square_g + \sigma)u = G(u, \mathcal{D}u), \quad \sigma \in \mathbb{R}.
\]

The main result was as follows:

**Theorem 1.6** [16]. Assume the following:

- \((\mathcal{M}, g)\) is a FG-aAdS segment satisfying the null convexity criterion (1.21).
- There is some \(p > 0\) such that \(G\) in (1.22) satisfies the bound
  \[
  |G(u, \mathcal{D}u)|^2 \lesssim \rho^{4+p} |\mathcal{D}u|^2 + \rho^{-3p} |u|^2.
  \]
- \(u\) is a solution of (1.22) for which—for sufficiently large \(\kappa\) depending on \(\sigma, g, t\) and the rank of \(u\)—the limit
  \[
  \rho^{-\kappa} u \to 0, \quad \rho \searrow 0,
  \]

---

\(^7\)These counterexamples were only constructed for the case when the mass \(\sigma\) is the specific value that makes the corresponding Klein–Gordon equation conformally invariant. In the upcoming work [11], the author intends to extend the construction to the non-conformal case.

\(^8\)In exactly such a way that it acts as a tensor derivation and is \(g\)-compatible.
holds in $C^1$ over a sufficiently long timespan $\{t_0 < t < t_1\} \subseteq I$ determined by $B,C$ from (1.21).

Then, $u \equiv 0$ in some interior neighbourhood of $\{t_0 < t < t_1\} \subseteq I$.\(^9\)

The key tool in the proof of this unique continuation statement was a novel Carleman estimate. [6] provided a version of this estimate adapted to the GNCC.

**Remark 1.7.** The NCC is gauge-dependent in the sense that there exist coordinate transformations preserving the Fefferman–Graham gauge but altering whether or not the criterion is satisfied. The key contribution of [6] was to generalise this to a gauge-invariant criterion and to derive corresponding Carleman estimates with which similar unique continuation statements can be proved. It is this form of the Carleman estimate that we work with.

### 1.4.2. The Weyl Curvature.

The Weyl curvature is the traceless part of the Riemann tensor; if it vanishes for a solution of (1.1), then the solution has constant curvature (i.e. it is maximally symmetric) and is thus locally isometric to pure AdS since (1.1) involves a negative cosmological constant [5]. The key observation, proved here in Proposition 4.5, is that the Weyl curvature of a vacuum FG-aAdS segment satisfies the wave equation

$$\Box g + 2n) W_{\alpha \beta \gamma \delta} = 4 W^\lambda_\alpha \mu [\partial_\lambda \partial_\mu] - W^\lambda_\mu \gamma \delta W_{\alpha \beta \lambda \mu}. \tag{1.25}$$

This can be decomposed into a system of vertical wave equations for the vertical tensor fields

$$W^0_{abcd} := \rho^2 W_{abcd}, \quad W^1_{abc} := \rho^2 W_{pabc}, \quad W^2_{ab} := \rho^2 W_{papb}, \quad (1.26)$$

that, together, fully determine the spacetime Weyl curvature. Crucially, the lower-order nonlinearities in these equations satisfy (1.23). Theorem 1.6 may therefore be applied to demonstrate that the spacetime Weyl curvature identically vanishes near the boundary given vanishing to sufficiently high order on approach to the boundary for $W^0, W^1$ and $W^2$. In particular, these ‘vanishing rates’ should be understood in the context of Fefferman–Graham expansions for the vertical components of the Weyl curvature, as given in Corollary 4.11.

[18, Proposition 2.25] uses (1.1) to relate each of the vertical components of the Weyl curvature to vertical metric quantities:

$$W^0_{abcd} = R_{abcd} + \frac{1}{2} \mathcal{L}_\rho g_a[c \mathcal{L}_\rho g_d]b + \rho^{-1} \left( g_a[c \mathcal{L}_\rho g_d]b - g_b[c \mathcal{L}_\rho g_d]a \right), \tag{1.27}$$

$$W^1_{abc} = D_{[b} \mathcal{L}_\rho g_a[c], \quad (1.28)$$

$$W^2_{ab} = -\frac{1}{2} \mathcal{L}_\rho^2 g_{ab} + \frac{1}{2} \rho^{-1} \mathcal{L}_\rho g_{ab} + \frac{1}{4} g^{cd} \mathcal{L}_\rho g_{ac} \mathcal{L}_\rho g_{bd}. \tag{1.29}$$

Moreover, we have boundary expansions for the right-hand sides in which the leading-order terms feature the $g$-Weyl and Cotton tensors $\mathcal{W}$ and $\mathcal{C}$, whose

\(^9\)In general, one also requires a compact support assumption for $u$ on level sets of $(\rho, t)$. In our case this need not be of concern since we assume $I$ has compact cross-sections.
vanishing follows from the condition that $g$ is conformally flat\textsuperscript{10}:

$$W^0 = W + o(1), \quad W^1 = \frac{\rho}{2(n-2)} C + o(\rho), \quad W^2 = o(1). \quad (1.30)$$

This yields a base level of vanishing for each of $W^0$, $W^1$ and $W^2$ which we improve by making use of the equations obtained by expressing the second Bianchi identity in terms of vertical objects (see Proposition (4.13)). Iteratively substituting the partial Fefferman–Graham expansions of $W^0$, $W^1$ and $W^2$ into these equations and applying the condition on $g^{(n)}$, we obtain

$$W^0 = o(\rho^{n-2}), \quad W^1 = o(\rho^{n-1}), \quad W^2 = o(\rho^{n-2}). \quad (1.31)$$

Once this is obtained, it is possible to iteratively integrate the vertical Bianchi equations; at each iteration, degrees of vertical regularity are exchanged for additional orders of vanishing. The process is continued until the vanishing rate required for unique continuation is obtained.

1.5. Organisation

- In Sect. 2, we formally define FG-aAdS segments. Vertical tensor fields are introduced and a notion of boundary limits for such objects is established.
- In Sect. 3, we introduce the mixed tensor calculus which will enable us to make sense of a wave operator acting on a vertical tensor field in a consistent way. We also present formulae used to convert spacetime equations into their mixed counterparts.
- In Sect. 4, we study FG-aAdS segments whose metrics solve (1.1). The spacetime Weyl curvature for such segments satisfies a wave equation; we use the tools developed in Sect. 3 to decompose this into a system of wave equations for each of the vertical components of the spacetime Weyl curvature.
- In Sect. 5, we state the Carleman estimate from [6] in a form suited to our purposes. We then prove some preliminary results that are combined to prove the main result of this article. It is demonstrated in Proposition 5.10 that the conditions (1.18) and (1.19) are invariant under coordinate transformations preserving the Fefferman–Graham gauge. In other words, if one representative of a gauge-equivalent class of boundary data satisfies these conditions, then the same must be true for all other representatives of the class.

1.6. Acknowledgements

The author would like to thank Arick Shao for his support via discussions on a number of topics and the provision of notes regarding the computations involved in the proofs of Propositions 3.7 and 3.9.

\textsuperscript{10}See Proposition 4.12 for a precise statement and proof of these leading-order expressions.
2. Asymptotically AdS Spacetimes

We begin by recalling some basic definitions from [16] concerning the spacetime manifolds on which we will be working and the natural tensor fields to consider on them.

**Definition 2.1.** An aAdS region is a manifold of the form
\[ M := (0, \rho_0) \times \mathcal{I}, \quad \rho_0 > 0, \] (2.1)
in which \( \mathcal{I} \) is a smooth \( n \)-dimensional manifold for some \( n \in \mathbb{N} \). Given an aAdS region \( M \), \( \rho \) denotes the coordinate function on \( M \) projecting onto the \( (0, \rho_0) \)-component and \( \partial_\rho \) denotes the \( M \)-lift of the canonical vector field \( d\rho \) on \((0, \rho_0)\).

**Definition 2.2.** The vertical bundle \( V^k_{l} M \) of rank \((k, l)\) over \( M \) is the manifold of all rank \((k, l)\) tensors on level sets of \( \rho \) in \( M \):
\[ V^k_{l} M = \bigcup_{\sigma \in (0, \rho_0)} T^k_{l}\{\rho = \sigma\}. \] (2.2)
Sections of \( V^k_{l} M \) are called vertical tensor fields of rank \((k, l)\).

**Definition 2.3.** We adopt the following notational conventions and natural identifications:
- Italicized font (as in \( g \)) denotes tensor fields on \( M \).
- Serif font (as in \( g \)) denotes vertical tensor fields. Any vertical tensor field \( A \) can be uniquely identified with a tensor field on \( M \) by demanding that the contraction of any component of \( A \) with \( \partial_\rho \) or \( d\rho \) identically vanishes.
- Fraktur font (as in \( g \)) denotes tensor fields on \( \mathcal{I} \). If \( \mathfrak{A} \) is a tensor field on \( \mathcal{I} \), then \( \mathfrak{A} \) will also denote the vertical tensor field on \( M \) obtained by extending \( \mathfrak{A} \) as a \( \rho \)-independent family of tensor fields on \( \mathcal{I} \).

**Definition 2.4.** Let \( M \) be an aAdS region, and let \( A \) be a vertical tensor field.
- Given any \( \sigma \in (0, \rho_0) \), \( A|_\sigma \) denotes the tensor field on \( \mathcal{I} \) obtained from restricting \( A \) to the level set \( \{\rho = \sigma\} \) and then identifying \( \{\rho = \sigma\} \) with \( \mathcal{I} \).
- The \( \rho \)-Lie derivative of \( A \), denoted \( \mathcal{L}_\rho A \), is defined to be the vertical tensor field satisfying
\[ \mathcal{L}_\rho A|_\sigma = \lim_{\sigma' \to \sigma} (\sigma' - \sigma)^{-1}(A|_{\sigma'} - A|_\sigma), \quad \sigma \in (0, \rho_0). \] (2.3)

Next, we establish coordinate system conventions on \( \mathcal{I} \) and \( M \):

**Definition 2.5.** Suppose \( M \) is an aAdS region and \((U, \varphi)\) is a coordinate system on \( \mathcal{I} \). We write \( \varphi_\rho := (\rho, \varphi) \) to denote the corresponding lifted coordinates on \((0, \rho_0) \times U \) and adopt the following notational conventions:
- Latin indices \( a, b, c, \ldots \) denote \( \varphi \)-coordinate components.
- Greek indices \( \alpha, \beta, \mu, \nu, \ldots \) denote \( \varphi_\rho \)-coordinate components.

---

11 A vertical tensor field of rank \((k, l)\) on an aAdS region \( M \) can be equivalently viewed as a \( \rho \in (0, \rho_0) \)-parameterised family of rank \((k, l)\) tensor fields on \( \mathcal{I} \).
Definition 2.6. Suppose $\mathcal{M}$ is an aAdS region. A coordinate system $(U, \varphi)$ on $\mathcal{I}$ is called compact iff $\bar{U}$ is a compact subset of $\mathcal{I}$ and $\varphi$ extends smoothly to an open neighbourhood of $\bar{U}$.

We now define a notion of magnitude for vertical tensor fields with respect to a given coordinate system and use this to make sense of boundary limits in a natural way:

Definition 2.7. Let $\mathcal{M}$ be an aAdS region and fix some $M \geq 0$. Furthermore, let $A$ and $\mathfrak{A}$ be a rank $(k, l)$ vertical tensor field and a rank $(k, l)$ tensor field on $\mathcal{I}$, respectively.

- Given a compact coordinate system $(U, \varphi)$ on $\mathcal{I}$, we define
  \[ |A|_{M, \varphi} := \sum_{m=0}^{M} \sum_{a_1, \ldots, a_m} |\partial_{a_1} \ldots \partial_{a_m} A_{b_1 \ldots b_k c_1 \ldots c_l}|. \]  
  \[ (2.4) \]

- We write $A \rightarrow^M \mathfrak{A}$ iff given any compact coordinate system $(U, \varphi)$ on $\mathcal{I}$,
  \[ \lim_{\sigma \searrow 0} \sup_{\{\sigma\} \times U} |A - \mathfrak{A}|_{M, \varphi} = 0. \]  
  \[ (2.5) \]

- $A$ is weakly locally bounded iff for any compact coordinate system $(U, \varphi)$ on $\mathcal{I}$,
  \[ \sup_{U} \int_{0}^{\rho_0} |A|_{0, \varphi} |d\sigma| < \infty. \]  
  \[ (2.6) \]

- We additionally define a local uniform norm of $A$: \[ \|A\|_{M, \varphi} := \sup_{(0, \rho_0] \times U} |A|_{M, \varphi}. \]  
  \[ (2.7) \]

- $A$ is locally bounded in $C^M$ iff for any compact coordinate system $(U, \varphi)$ on $\mathcal{I}$,
  \[ \|A\|_{M, \varphi} < \infty. \]  
  \[ (2.8) \]

Now, let us use the above-defined notion of a boundary limit to rigorously define the class of metrics we are interested in.

Definition 2.8. $(\mathcal{M}, g)$ is called a FG-aAdS segment iff the following hold:

- $\mathcal{M}$ is an aAdS region and $g$ is a Lorentzian metric on $\mathcal{M}$.
- There exists a rank $(0, 2)$ vertical tensor field $\mathfrak{g}$ such that
  \[ g := \rho^{-2}(d\rho^2 + \mathfrak{g}). \]  
  \[ (2.9) \]

- There exists a Lorentzian metric $\mathfrak{g}$ on $\mathcal{I}$ such that
  \[ \mathfrak{g} \rightarrow^0 \mathfrak{g}. \]  
  \[ (2.10) \]

Given such a FG-aAdS segment,

- We refer to the form (2.9) for $g$ as the Fefferman–Graham gauge condition.

\[ ^{12} \] Each $\partial_{a_i}$ denotes a $\varphi$-coordinate derivative.
• $(\mathcal{I}, g)$ is the conformal boundary associated with $(\mathcal{M}, g, \rho)$.\footnote{As noted in Sect. 1, there exist coordinate transformations preserving (2.9) but altering $g$. Such transformations induce conformal transformations of $g$, so it only makes sense to speak of the conformal boundary metric up to a conformal factor.}

We also define a suitable regularity class for the vertical metrics considered in this article. This will enable us to apply the results of [18], which rigorously derived Fefferman–Graham expansions for the metric in the finitely regular setting; later on, we will also consider the corresponding expansions for the Weyl curvature. This will be crucial in the proof of our main result, where we will need to connect conditions on the boundary data to vanishing rates of the Weyl curvature (i.e. to the coefficients of its Fefferman–Graham expansion).

Definition 2.9. We say that a FG-aAdS segment $(\mathcal{M}, g)$ is $k$--regular if $g$ is locally bounded in $C^{k+2}$ and $\mathcal{L}_\rho g$ is weakly locally bounded.

For the sake of clarity, let us define some further notational conventions before continuing.

Definition 2.10. Given a FG-aAdS segment $(\mathcal{M}, g)$,

• $g^{-1}$, $\nabla$, $\nabla^\#$, $R$, $Rc$ and $Rs$, respectively, denote the metric dual, Levi-Civita connection, gradient, Riemann curvature, Ricci curvature and scalar curvature with respect to $g$.

• $g^{-1}$, $D$, $D^\#$, $R$, $Rc$ and $Rs$, respectively, denote the above objects with respect to $g$.

• $g^{-1}$, $\bar{D}$, $\bar{D}^\#$, $\bar{R}$, $\bar{Rc}$ and $\bar{Rs}$, respectively, denote the above objects with respect to $\bar{g}$.

3. The Mixed Tensor Calculus

3.1. The Formalism

In this section (which again follows the presentation of [16]), our aim is to make sense of a $g$-wave operator acting on a vertical tensor field—in such a way that it is compatible with standard covariant operations. Our first step in this direction is to construct connections on the vertical bundles which permit covariant derivatives of vertical tensor fields in all directions along $\mathcal{M}$.

Definition 3.1. We denote multi-indices by $\bar{\mu} := \mu_1 \ldots \mu_k$. Additionally, we write

• $\hat{\mu}_i[\alpha]$ to denote $\bar{\mu}$ with the $i$th component replaced with an $\alpha$-component.

• $\hat{\mu}_{i,j}[\alpha, \beta]$ to denote $\bar{\mu}$ with the $i$th and $j$th components replaced with $\alpha$ and $\beta$ components, respectively.

Proposition 3.2. Let $(\mathcal{M}, g)$ be a FG-aAdS segment. There exists a unique connection $\bar{D}$ on $\nabla_b^k \mathcal{M}$ such that the following hold for rank $(k, l)$ vertical tensor fields $A$ with respect to any coordinate system $(U, \varphi)$ on $\mathcal{I}$:

$$\bar{D}_c \bar{A}^a_b = D_c A^a_b,$$

\hspace{1cm} (3.1)
The connection $\bar{\nabla}$ extends the vertical Levi-Civita connections $\nabla$ to permit covariant derivatives of vertical fields in the $\rho$-direction. In order to construct the $g$-wave operator for vertical tensor fields in this spirit, we must first define some further tensorial objects on $\mathcal{M}$.

**Definition 3.3.** Let $(\mathcal{M}, g)$ be a FG-aAdS segment. The mixed bundle of rank $(\kappa, \lambda; k,l)$ over $\mathcal{M}$ is given by

$$T^\kappa_\lambda V^k_l \mathcal{M} := T^\kappa_\lambda \mathcal{M} \otimes V^k_l \mathcal{M}. \quad (3.6)$$

Sections of $T^\kappa_\lambda V^k_l \mathcal{M}$ are called mixed tensor fields of rank $(\kappa, \lambda; k,l)$. Furthermore, the bundle connection $\nabla$ on the mixed bundle $T^\kappa_\lambda V^k_l \mathcal{M}$ is defined as the tensor product connection of $\nabla$ on $T^\kappa_\lambda \mathcal{M}$ and $\bar{\nabla}$ on $V^k_l \mathcal{M}$.

**Proposition 3.4.** Let $(\mathcal{M}, g)$ be a FG-aAdS segment. Then:
- For any vector field $X$ on $\mathcal{M}$ and mixed tensor fields $A$ and $B$,
  $$\nabla_X (A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B. \quad (3.7)$$
- For any vector field $X$ on $\mathcal{M}$,
  $$\nabla_X g = 0, \quad \nabla_X g^{-1} = 0, \quad \nabla_X \bar{g} = 0, \quad \nabla_X \bar{g}^{-1} = 0. \quad (3.8)$$

**Proof.** See [16, Proposition 2.28].

Generally speaking, the mixed connections $\nabla$ behave like $\nabla$ on spacetime components and $\bar{\nabla}$ on vertical components. The properties demonstrated in Proposition 3.4 are analogous to the properties of covariant derivatives that enable the standard integration by parts formulae; we are thus able to extend these directly to mixed tensor fields.

We now define higher covariant derivatives in the context of mixed bundles:

**Definition 3.5.** Let $(\mathcal{M}, g)$ be a FG-aAdS segment and $A$ be a mixed tensor field of rank $(\kappa, \lambda; k,l)$. 

\[
\bar{D}_\rho A^a_b = \mathcal{L}_\rho A^a_b + \frac{1}{2} \sum_{i=1}^{k} g^{a_i c} \mathcal{L}_\rho g_{cd} A^{a[d]}_{b} - \frac{1}{2} \sum_{j=1}^{l} g^{cd} \mathcal{L}_\rho g_{bjc} A^a_{b[j]}.
\] 

(3.2)

Furthermore, for any vector field $X$ on $\mathcal{M}$,

1. The following holds for vertical tensor fields $A$ and $B$:
   $$\bar{D}_X (A \otimes B) = \bar{D}_X A \otimes B + A \otimes \bar{D}_X B. \quad (3.3)$$

2. The following holds for vertical tensor fields $A$ and tensor contraction operations $C$:
   $$\bar{D}_X (CA) = C(\bar{D}_X A). \quad (3.4)$$

3. The connection $\bar{D}$ is $g$-compatible:
   $$\bar{D}_X g = 0, \quad \bar{D}_X g^{-1} = 0. \quad (3.5)$$

**Proof.** See [16, Proposition 2.23].

The connections $\bar{D}$ extend the vertical Levi-Civita connections $\nabla$ to permit covariant derivatives of vertical fields in the $\rho$-direction. In order to construct the $g$-wave operator for vertical tensor fields in this spirit, we must first define some further tensorial objects on $\mathcal{M}$.
• The mixed covariant differential of $\mathbf{A}$ is the mixed tensor field $\nabla A$, of rank $(\kappa, \lambda + 1; k, l)$, that maps each vector field $X$ on $\mathcal{M}$ to $\nabla_X A$.
• The mixed Hessian $\nabla^2 A$ is defined as the mixed covariant differential of $\nabla A$.
• The wave operator $\square A$ is the $g$-trace of $\nabla^2 A$.

3.2. Conversion Formulae
In this article, we will need to convert equations for spacetime quantities to corresponding equations for vertical quantities. In this subsection, we present a systematic method for doing so. The schematic notation and the computations involved in the proofs for this section were assembled by Arick Shao and kindly shared with the author for the present article.

Let us begin by fixing schematic notations for asymptotic error terms:

Definition 3.6. Let $(\mathcal{M}, g)$ be a FG-aAdS segment, fix an integer $M \geq 0$ and let $h \in C^\infty(\mathcal{M})$. Then, $\mathcal{O}_M(h)$ refers to any vertical tensor field $a$ satisfying

$$|a|_{M, \varphi} \lesssim \varphi h,$$

(3.9)

for any compact coordinate system $(U, \varphi)$ on $\mathcal{I}$.

Furthermore, given a vertical tensor field $B$, $\mathcal{O}_M(h; B)$ refers to any vertical tensor field $A$ that is expressible in the form

$$A = \sum_{k=1}^{N} \mathcal{C}_k (a_k \otimes B_k^*),$$

(3.10)

where $N \geq 0$ and for each $1 \leq k \leq N$,

- $B_k^*$ is $B$ composed with some permutation of its components.
- $a_k$ is a vertical tensor field satisfying $a_k = \mathcal{O}_M(h)$.
- $\mathcal{C}_k$ is a composition of zero or more contractions and $g$-metric contraction operations.

Next, we establish some commutation identities for vertical tensor fields:

Proposition 3.7. Let $(\mathcal{M}, g)$ be a FG-aAdS segment, fix $M \geq 2$ and assume

$$g = \mathcal{O}_M(1), \quad \mathcal{L}_\rho g = \mathcal{O}_{M-2}(\rho).$$

(3.11)

Then, the following commutation identities hold for any vertical tensor field $A$ and $p \in \mathbb{R}$:

$$\bar{D}_\rho (DA) = D(\bar{D}_\rho A) + \mathcal{O}_{M-2}(\rho; DA) + \mathcal{O}_{M-3}(\rho; A)$$

(3.12)

$$\square (p^2 A) = p^2 \square A + 2p p^{p+1} \bar{D}_\rho A - p(n - p) \rho^p A + \mathcal{O}_{M-2}(\rho^2; \rho^p A).$$

(3.13)

Proof. See Appendix 5.3.

We now fix some further notation so as to be able to express the conversion formulae in a compact form:
Definition 3.8. Suppose $A$ is a tensor field on $\mathcal{M}$ of rank $(0, r_1 + r_2)$, where $r_1, r_2 \geq 0$, and let $\mathbf{A}$ be the corresponding rank $(0, r_2)$ vertical tensor field defined with respect to any coordinates $(U, \varphi)$ on $\mathcal{I}$ by

$$A_{\bar{a}} := A_{\bar{\rho}\bar{a}},$$  

where the multi-index $\bar{\rho} := \rho \ldots \rho$ represents $r_1$ copies of $\rho$, while $\bar{a} := a_1 \ldots a_{r_2}$. Then:

- For any $1 \leq i \leq r_1$, the rank $(0, r_2 + 1)$ vertical tensor field $A^\rho_i$ is given by
  $$\left(A^\rho_i\right)_{b\bar{a}} := A_{\bar{\rho}i[b]\bar{a}},$$

- For any $1 \leq j \leq r_2$, the rank $(0, r_2 - 1)$ vertical tensor field $A^x_j$ is given by
  $$\left(A^x_j\right)_{\bar{a}j} := A_{\bar{\rho}a[j],\rho},$$

- For any $1 \leq i, j \leq r_1$ with $i \neq j$, the rank $(0, r_2 + 2)$ vertical tensor field $A^\rho_{i,j}$ is given by
  $$\left(A^\rho_{i,j}\right)_{c\bar{b}\bar{a}} := A_{\bar{\rho}i,j[c,b]\bar{a}},$$

- For any $1 \leq i, j \leq r_2$ with $i \neq j$, the rank $(0, r_2 - 2)$ vertical tensor field $A^v_{i,j}$ is given by
  $$\left(A^v_{i,j}\right)_{\bar{a}i,j} := A_{\bar{\rho}a[i,j],\rho},$$

- For any $1 \leq i \leq r_1$ and $1 \leq j \leq r_2$, the rank $(0, r_2)$ vertical tensor field $A^\rho_{i,j}$ is given by
  $$\left(A^\rho_{i,j}\right)_{b\bar{a}} := A_{\bar{\rho}[b\bar{a},i,j],\rho}.$$

We are now in a position to state the conversion formulæ:

Proposition 3.9. Let $(\mathcal{M}, g)$ be a FG-aAdS segment and assume (3.11) holds for some $M \geq 2$. Let $A$ be a tensor field on $\mathcal{M}$ of rank $(0, r_1 + r_2)$, where $r_1, r_2 \geq 0$, and let $\mathbf{A}$ be the associated rank $(0, r_2)$ vertical tensor field defined by (3.14).

Then, the following identities hold with respect to any coordinates $(U, \varphi)$ on $\mathcal{I}$:

\[ \nabla_\rho A_{\bar{\rho}\bar{a}} = \rho^{-r_1-r_2} \mathbf{D}_\rho (\rho^{-r_1-r_2} A_{\bar{a}}) , \]  

\[ \nabla_c A_{\bar{\rho}\bar{a}} = \mathbf{D}_c A_{\bar{a}} + (\rho^{-1} \delta^b_c - \frac{1}{2} g^{bd} \mathcal{L}_\rho g_{dc}) \sum_{i=1}^{r_1} (A^\rho_{i})_{b\bar{a}} \]

\[ - \sum_{j=1}^{r_2} (\rho^{-1} g_{ca_j} - \frac{1}{2} \mathcal{L}_\rho g_{ca_j} ) (A^v_{j})_{\bar{a}j} \]

\[ = \mathbf{D}_c A_{\bar{a}} + \rho^{-1} \sum_{i=1}^{r_1} (A^\rho_{i})_{c\bar{a}} - \rho^{-1} \sum_{j=1}^{r_2} g_{ca_j} (A^v_{j})_{\bar{a}j} \]

\[ + \sum_{i=1}^{r_1} \Theta_{M-2}(\rho; A^\rho_{i})_{c\bar{a}} + \sum_{j=1}^{r_2} \Theta_{M-2}(\rho; A^v_{j})_{c\bar{a}} , \]
\[ \Box A_{\bar{\rho}\bar{a}} = \rho^{-r_1 - r_2} \Box (\rho^{r_1 + r_2} A)_{\bar{a}} + 2\rho \sum_{i=1}^{r_1} g_{\bar{b}c} D_b (A_{i\bar{c}}^\rho)_{\bar{a}} \]

\[ - 2\rho \sum_{j=1}^{r_2} D_j (A_j^\rho)_{\bar{a}} - (nr_1 + r_2) A_{\bar{a}} - 2 \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} (A_{1i,j}^{\rho\nu})_{\bar{a}} \bar{a}_j \]

\[ + 2 \sum_{1 \leq i < j \leq r_1} g_{\bar{b}c} (A_{1i,j}^{\rho\nu})_{bc\bar{a}} + 2 \sum_{1 \leq i < j \leq r_2} g_{a_i a_j} (A_{i,j}^{\nu\nu})_{\bar{a}i\bar{a}j} \]

\[ + \sum_{i=1}^{r_1} \theta_{M-2}(\rho^3; DA_{\bar{a}}^\nu)_{\bar{a}} + \sum_{j=1}^{r_2} \theta_{M-2}(\rho^3; DA_{j}^\nu)_{\bar{a}} + \sum_{i=1}^{r_1} \theta_{M-3}(\rho^3; A_{i}^{\nu\nu})_{\bar{a}} \]

\[ + \sum_{j=1}^{r_2} \theta_{M-3}(\rho^3; A_{j}^{\nu\nu})_{\bar{a}} + \theta_{M-2}(\rho^2; A_{i,j}^{\nu\nu})_{\bar{a}} + \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} \theta_{M-2}(\rho^2; A_{i,j}^{\nu\nu})_{\bar{a}} \]  

\[ \tag{3.22} \]

\textbf{Proof.} See Appendix 5.3. \hfill \Box

\section{4. Vacuum Spacetimes}

\subsection{4.1. The Metric}

\textbf{Definition 4.1.} An \((n + 1)\)-dimensional FG-aAdS segment \((\mathcal{M}, g)\) is called a \textit{vacuum FG-aAdS segment} iff it satisfies \eqref{1.1}.

The following boundary limits were derived in [18, Theorem 3.3]:

\textbf{Theorem 4.2.} Fix \(n > 2\) and \(M_0 \geq n + 2\). Suppose that \((\mathcal{M}, g)\) is an \((n + 1)\)-dimensional, \(M_0\)-regular vacuum FG-aAdS segment. Then,

\[ g \rightarrow M_0 g, \quad g^{-1} \rightarrow M_0 g^{-1}, \quad \tag{4.1} \]

and, for \(0 \leq k < n\), there exists tensor fields \(g^{(k)}\) on \(\mathcal{I}\) such that

\[ \mathcal{L}_\rho^k g \rightarrow M_0^{-k} k! g^{(k)}, \quad \rho \mathcal{L}_\rho^{k+1} g \rightarrow M_0^{-k} 0, \quad \tag{4.2} \]

where

- \(g^{(0)} = g\).
- \(g^{(2)} = -p\), where \(p\) is the \(g\)-Schouten tensor:

\[ p = \frac{1}{n - 2} \left( \mathcal{Rc} - \frac{1}{2(n - 1)} \mathcal{Rs} \cdot g \right). \quad \tag{4.3} \]

- \(g^{(k)} = 0\) if \(n\) is odd.\(^{14}\)

Furthermore, there exist tensor fields \(g^{(*)}\) and \(g^{(\dagger)}\)\(^{15}\) on \(\mathcal{I}\) such that

\[ \rho \mathcal{L}_\rho^{n+1} g \rightarrow M_0^{-n} n! g^{(*)}, \quad \mathcal{L}_\rho^n g - n! (\log \rho) g^{(*)} \rightarrow M_0^{-n} n! g^{(\dagger)}, \quad \tag{4.4} \]

where \(g^{(*)} = 0\) if \(n\) is odd.

\(^{14}\)See [18, Definition 3.2].

\(^{15}\)\(g^{\dagger}\) is in \(C^{M_0-n}\) on \(\mathcal{I}\).
Note that the above result implies that any vacuum FG-aAdS segment is a ‘strongly’ FG-aAdS segment, as defined in [16, Definition 2.13]; this was the condition required for the Carleman estimate (and hence the unique continuation result) of that article to hold.

Using the above limits, the following precise statement of the Fefferman–Graham expansion (1.8) for sufficiently regular vacuum FG-aAdS segments was given in [18, Theorem 3.6]¹⁷:

**Corollary 4.3.** Fix \( n > 2 \) and \( M_0 \geq n + 2 \). Suppose that \((\mathcal{M}, g)\) is an \((n+1)\)-dimensional, \(M_0\)-regular vacuum FG-aAdS segment. Let \( g^{(k)} \) for \( 0 \leq k < n \) and \( g^{(\ast)} \) be as in Theorem 4.2. Then, there exists a \( C^{M_0-n} \) tensor field \( g^{(n)} \) on \( I \) and a vertical tensor field \( r \) such that

\[
g = \begin{cases} 
\sum_{k=0}^{\frac{n-1}{2}} \rho^{2k} g^{(2k)} + \rho^n g^{(n)} + \rho^n r, & n \text{ odd}, \\
\sum_{k=0}^{\frac{n-2}{2}} \rho^{2k} g^{(2k)} + \rho^n \log \rho g^{(\ast)} + \rho^n g^{(n)} + \rho^n r, & n \text{ even}, 
\end{cases}
\]

where the ‘remainder’ \( r \) satisfies

\[
r \to M_0 - n. \quad (4.6)
\]

**Remark 4.4.** The above shows that if \((\mathcal{M}, g)\) is a \(M_0\)-regular vacuum FG-aAdS segment, then (3.11) holds for \( M = M_0 \).

### 4.2. The Weyl Curvature

In this section, we show that the Weyl curvature of a vacuum segment satisfies a wave equation which can be decomposed into a system of equations for its vertical components. Moreover, we present the Fefferman–Graham expansions of these components—originally derived in [18]—which will enable us to connect their vanishing rates as \( \rho \searrow 0 \) to the boundary data. This will later be used to demonstrate that the conditions stated on the boundary data in our main result are sufficient to obtain the conditions required for unique continuation to hold for the system of vertical equations.

**Proposition 4.5.** Suppose that \((\mathcal{M}, g)\) is an \((n+1)\)-dimensional vacuum FG-aAdS segment. Let \((U, \varphi)\) be a coordinate system on \( I \). Then, the following hold with respect to \( \varphi \)-coordinates:

\[
Rc_{\alpha\beta} = -n \cdot g_{\alpha\beta}, \quad Rs = -n(n+1), \\
W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}, \quad (4.7)
\]

where \( W \) is the g-Weyl curvature. Furthermore, the Weyl curvature satisfies the wave equation

\[
(\Box_g + 2n) W_{\alpha\beta\gamma\delta} = 4W_{\alpha\beta}^{\lambda\mu} [\delta] W_{\lambda\beta\mu[\gamma]} - W_{\alpha\beta}^{\lambda\mu} W_{\lambda\beta\mu[\gamma]} W_{\delta\mu\lambda[\gamma]} \cdot (4.8)
\]

¹⁶Similar expansions were also derived for \( R \) and \( D_L \rho \tilde{g} \). For the sake of brevity, we do not reproduce these here.

¹⁷That is, the \( g \)-traceless part of the \( g \)-Riemann curvature.
Proof. The first two identities in (4.7) follow by taking the trace of (1.1). The third identity is a substitution of these expressions into the Weyl curvature,

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{2}{n-1} (g_{\alpha[\gamma} R_{\delta]\beta} - g_{\beta[\gamma} R_{\delta]\alpha}) + \frac{2}{n(n-1)} R_s \cdot g_{\alpha[\gamma} g_{\delta]\beta}. $$

(4.9)

Next, consider the Bianchi equation

$$\nabla_{[\mu} W_{\alpha\beta] \gamma\delta} = 0, \quad (4.10)$$

and its trace

$$\nabla^{\mu} W_{\alpha\beta\delta\mu} = 0. \quad (4.11)$$

Taking the divergence of (4.10) gives

$$\Box g W_{\alpha\beta\gamma\delta} + g^{\mu\nu} \nabla_{\nu} \nabla_{\gamma} W_{\alpha\beta\delta\mu} + g^{\mu\nu} \nabla_{\nu} \nabla_{\delta} W_{\alpha\beta\mu\gamma} = 0. \quad (4.12)$$

The covariant derivatives are permuted using

$$[\nabla_{\nu}, \nabla_{\gamma}] W_{\alpha\beta\delta\mu} = -R^{\lambda}_{\alpha\nu\gamma} W_{\lambda\beta\delta\mu} - R^{\lambda}_{\beta\nu\gamma} W_{\alpha\lambda\delta\mu} - R^{\lambda}_{\delta\nu\gamma} W_{\alpha\beta\lambda\mu} - R^{\lambda}_{\mu\nu\gamma} W_{\alpha\beta\delta\lambda}, \quad (4.13)$$

to give

$$\Box g W_{\alpha\beta\gamma\delta} + 2 \Box_{[\gamma] \nabla^{\mu} W_{\alpha\beta] \delta\mu} - 2 g^{\mu\nu} \left( R^{\lambda}_{\alpha\nu[\gamma} W_{\lambda\beta]\delta]\mu} + R^{\lambda}_{\beta\nu[\gamma} W_{\alpha\lambda\delta]\mu} + R^{\lambda}_{\mu\nu[\gamma} W_{\alpha\beta]\delta]\lambda}\right) = 0, \quad (4.14)$$

in which the second term vanishes by (4.11). For the remaining terms, one uses (4.7) to replace the Riemann curvature with the Weyl curvature and the metric. After simplifying, one finds

$$\Box g W_{\alpha\beta\gamma\delta} + 2n W_{\alpha\beta\gamma\delta} + 2 W^{\lambda\mu}_{\alpha\gamma} W_{\lambda\beta\mu\gamma} + W^{\lambda\mu}_{\gamma\delta} W_{\alpha\beta\lambda\mu} - 2 W^{\lambda\mu}_{\alpha\delta} W_{\lambda\beta\mu\gamma} = 0, \quad (4.15)$$

from which (4.8) follows immediately. \qed

Our goal is to use the mixed covariant formalism to convert the spacetime wave Eq. (4.8) into a system of wave equations for each of the independent vertical components of the Weyl curvature, defined below.

**Definition 4.6.** Suppose \((\mathcal{M}, g)\) is a FG-aAdS segment and \((U, \varphi)\) is a coordinate system on \(\mathcal{M}\). With respect to \(\varphi^\rho\)-coordinates, we define the following independent vertical components of the Weyl curvature:

$$W^0_{abcd} := \rho^2 W_{abcd}, \quad W^1_{abc} := \rho^2 W_{pabc}, \quad W^2_{ab} := \rho^2 W_{pab}. \quad (4.16)$$

**Remark 4.7.** Since the spacetime Weyl curvature is trace-free, one has that

$$W^2_{ab} = g^{\rho\theta} W_{pab} = -g^{cd} W_{cadb} = -g^{cd} W^0_{cadb}. \quad (4.17)$$

In other words, \(-W^2\) is a \(g\)-trace of \(W^0\).
**Definition 4.8.** Suppose \((M, g)\) is a FG-aAdS segment and \((U, \varphi)\) is a coordinate system on \(\mathcal{I}\). With respect to \(\varphi_\rho\)-coordinates, the g-traceless part of \(W^0\) is

\[
\hat{W}^0_{abcd} := W^0_{abcd} + \frac{2}{n-2} \left( g_{a[c} W^2_{b]d} + g_{b[d} W^2_{c]a} \right). \tag{4.18}
\]

In [18] it was demonstrated that the vertical components of the Weyl curvature for a vacuum FG-aAdS segment can be expressed in terms of \(g\), \(g^{-1}\), \(\mathcal{L}_\rho g\) and \(\mathcal{R}\):

**Proposition 4.9 [18, Proposition 2.25].** Suppose \((M, g)\) is a vacuum FG-aAdS segment and \((U, \varphi)\) is a coordinate system on \(\mathcal{I}\). Then, the following relations hold with respect to \(\varphi\)-coordinates:

\[
\begin{align*}
W^0_{abcd} &= R_{abcd} + \frac{1}{2} \mathcal{L}_\rho g_{a[c} \mathcal{L}_\rho g_{b]d} + \rho^{-1} \left( g_{a[c} \mathcal{L}_\rho g_{b]d} - g_{b[d} \mathcal{L}_\rho g_{c]a} \right), \tag{4.19} \\
W^1_{cab} &= D_{[b} \mathcal{L}_\rho g_{a]c}, \tag{4.20} \\
W^2_{ab} &= -\frac{1}{2} \mathcal{L}^2_{gab} + \frac{1}{2} \rho^{-1} \mathcal{L}_\rho g_{ab} + \frac{1}{4} g^{cd} \mathcal{L}_\rho g_{ac} \mathcal{L}_\rho g_{bd}. \tag{4.21}
\end{align*}
\]

This was proved by considering the Gauss and Codazzi equations on level sets of \(\rho\) and then using (4.7) to exchange the Riemann curvature for the Weyl curvature.

The limits proved in [18, Theorem 3.6] may be straightforwardly applied to derive similar limits for derivatives of the vertical components of the Weyl curvature, and hence to write down Fefferman–Graham expansions similar to (4.5):

**Theorem 4.10.** Fix \(n > 2\) and \(M_0 \geq n + 2\). Suppose that \((M, g)\) is an \((n + 1)\)-dimensional, \(M_0\)-regular vacuum FG-aAdS segment. Then, for \(0 \leq k < n - 2\) and \(i = 0, 1, 2\), there exist tensor fields \(\mathcal{W}^k_i\) on \(\mathcal{I}\) such that

\[
\begin{align*}
\mathcal{L}_\rho W^0 &\rightarrow M_0 - k - 2 \quad k! \mathcal{W}^{(k)}_0, \quad \rho \mathcal{L}^{k+1}_\rho W^0 \rightarrow M_0 - k - 2 \quad 0, \quad 0 \leq k < n - 2, \tag{4.22} \\
\mathcal{L}_\rho W^1 &\rightarrow M_0 - k - 2 \quad k! \mathcal{W}^{(k)}_1, \quad \rho \mathcal{L}^{k+1}_\rho W^1 \rightarrow M_0 - k - 2 \quad 0, \quad 0 \leq k < n - 1, \tag{4.23} \\
\mathcal{L}_\rho W^2 &\rightarrow M_0 - k - 2 \quad k! \mathcal{W}^{(k)}_2, \quad \rho \mathcal{L}^{k+1}_\rho W^2 \rightarrow M_0 - k - 2 \quad 0, \quad 0 \leq k < n - 2, \tag{4.24}
\end{align*}
\]

where \(\mathcal{W}^{(k)}_0 = \mathcal{W}^{(k)}_2 = 0\) if \(k\) is odd and \(\mathcal{W}^{(k)}_1 = 0\) if \(k\) is even. Furthermore, there exist tensor fields \(\mathcal{W}^{(s)}_1\) and \(\mathcal{W}^{(t)}_1\) on \(\mathcal{I}\) such that

\[
\begin{align*}
\rho \mathcal{L}^{n-1}_\rho W^0 &\rightarrow M_0 - n \quad (n - 2)! \mathcal{W}^{(s)}_0, \\
\mathcal{L}^{n-2}_\rho W^0 - (n - 2)! (\log \rho) \mathcal{W}^{(s)}_0 &\rightarrow M_0 - n \quad (n - 2)! \mathcal{W}^{(t)}_0, \tag{4.25} \\
\rho \mathcal{L}^{n-1}_\rho W^1 &\rightarrow M_0 - n - 1 \quad (n - 1)! \mathcal{W}^{(s)}_1, \\
\mathcal{L}^{n-2}_\rho W^1 - (n - 1)! (\log \rho) \mathcal{W}^{(s)}_1 &\rightarrow M_0 - n - 1 \quad (n - 1)! \mathcal{W}^{(t)}_1. \tag{4.26}
\end{align*}
\]
where $\mathcal{W}^{(3)}_i = 0$ if $n$ is odd.

**Corollary 4.11.** Fix $n > 2$ and $M_0 \geq n + 2$. Suppose $(\mathcal{M}, g)$ is an $(n + 1)$-dimensional, $M_0$-regular vacuum FG-aAdS segment. Then, with respect to any compact coordinate system on $\mathcal{I}$,

$$W^0 = \begin{cases} \sum_{k=0}^{\frac{n-3}{2}} \rho^{2k} \cdot \mathcal{W}^{(2k)}_0 + \rho^{n-2} \cdot \mathcal{W}^{(n-2)}_0 + \rho^{n-2} \cdot r_0, & \text{n odd,} \\ \sum_{k=0}^{\frac{n-1}{2}} \rho^{2k} \cdot \mathcal{W}^{(2k)}_0 + \rho^{n-2} \log \rho \cdot \mathcal{W}^{(n-2)}_0 + \rho^{n-2} \cdot r_0, & \text{n even,} \end{cases}$$

$$W^1 = \begin{cases} \sum_{k=0}^{\frac{n-3}{2}} \rho^{2k+1} \cdot \mathcal{W}^{(2k+1)}_1 + \rho^{n-1} \cdot \mathcal{W}^{(n-1)}_1 + \rho^{n-1} \cdot r_1, & \text{n odd,} \\ \sum_{k=0}^{\frac{n-1}{2}} \rho^{2k+1} \cdot \mathcal{W}^{(2k+1)}_1 + \rho^{n-1} \log \rho \cdot \mathcal{W}^{(n-1)}_1 + \rho^{n-1} \cdot r_1, & \text{n even,} \end{cases}$$

$$W^2 = \begin{cases} \sum_{k=0}^{\frac{n-3}{2}} \rho^{2k} \cdot \mathcal{W}^{(2k)}_2 + \rho^{n-2} \cdot \mathcal{W}^{(n-2)}_2 + \rho^{n-2} \cdot r_2, & \text{n odd,} \\ \sum_{k=0}^{\frac{n-1}{2}} \rho^{2k} \cdot \mathcal{W}^{(2k)}_2 + \rho^{n-2} \log \rho \cdot \mathcal{W}^{(n-2)}_2 + \rho^{n-2} \cdot r_2, & \text{n even,} \end{cases}$$

where:

- $\mathcal{W}^{(n-2)}_0$, $\mathcal{W}^{(n-1)}_1$ and $\mathcal{W}^{(n-2)}_2$ are $C^{M_0-n}$, $C^{M_0-n-1}$ and $C^{M_0-n}$ $\mathcal{I}$-tensor fields, respectively.
- $r_0$, $r_1$ and $r_2$ are vertical tensor fields for which
  $$r_0 \rightarrow M_0-n \ 0, \quad r_1 \rightarrow M_0-n-1 \ 0, \quad r_2 \rightarrow M_0-n \ 0. \quad (4.31)$$

In fact, one can use Proposition 4.9 to compute the leading-order terms in each of these boundary expansions:

**Proposition 4.12.** Fix $n > 2$ and $M_0 \geq n + 2$. Suppose $(\mathcal{M}, g)$ is a $(n + 1)$-dimensional, $M_0$-regular vacuum FG-aAdS segment. Then,

$$\mathcal{W}^{(0)}_{0\ abcd} = \mathcal{W}^{(3)}_{abcd}, \quad \mathcal{W}^{(1)}_{1\ abcd} = \frac{1}{n-2} \mathcal{C}^{abc}, \quad \mathcal{M}^{(0)}_{2\ ab} = 0, \quad (4.32)$$

where $\mathcal{W}$ and $\mathcal{C}$ are the g-Weyl curvature and Cotton tensors, respectively, and

$$n = 3 : \quad \mathcal{W}^{(1)}_{2\ ab} = -\frac{3}{2} \mathcal{G}^{(3)}_{ab}, \quad (4.33)$$

$$n = 4 : \quad \mathcal{W}^{(2)}_{2\ ab} = -4\mathcal{G}^{(3)}_{ab}, \quad \mathcal{W}^{(2)}_{2\ ab} = -3\mathcal{G}^{(4)}_{ab} - 4\mathcal{G}^{(4)}_{ab} + \mathcal{G}^{cd}p_{ac}p_{bd}, \quad (4.34)$$

$$n > 4 : \quad \mathcal{W}^{(2)}_{2\ ab} = -4\mathcal{G}^{(4)}_{ab} + \mathcal{G}^{cd}p_{ac}p_{bd}, \quad (4.35)$$

$$n > 6 : \quad \mathcal{W}^{(4)}_{2\ ab} = -12\mathcal{G}^{(6)}_{ab} - 2\mathcal{G}^{cd}\left(p_{ac}\mathcal{G}^{(4)}_{bd} + \mathcal{G}^{(4)}_{ac}p_{bd}\right) + \mathcal{G}^{ce}\mathcal{G}^{df}p_{ef}p_{ac}p_{bd}. \quad (4.36)$$

**Proof.** Substituting the appropriate expansions into the right-hand sides of (4.19) and (4.20) yield

$$\mathcal{W}^{(0)}_{0\ abcd} = \mathcal{R}_{abcd} + 2\mathcal{G}_{[a|c}p_{d]|b} - 2\mathcal{G}_{[c|p_{d]}a} = \mathcal{W}_{abcd},$$

$$\mathcal{W}^{(1)}_{1\ abcd} = 2\mathcal{D}_{[c|p_{d]a} = \frac{1}{n-2} \mathcal{C}^{abc}.$$
as required. To prove the remaining expressions, we first require the leading-order terms in the boundary expansion of \( g^{-1} \). These are obtained by taking derivatives of
\[
g^{ab} g_{bc} = \delta^a_c, \tag{4.38}
\]
and computing boundary limits by applying Theorem 4.2. In particular, one finds that \(^{18}\)
\[
g^{-1} \to \rho \to M_0^{-}\ g^{-1}, \quad \mathcal{L}_\rho g^{-1} \to M_0^{-1} \ 0, \quad \mathcal{L}_\rho^2 g^{-1} \to M_0^{-2} \ 2 \ g^{-1} \rho g^{-1} . \tag{4.39}
\]
Substituting this along with the boundary expansions of \( \mathcal{L}_\rho g \) and \( \mathcal{L}_\rho^2 g \) (on a dimensional case-by-case basis) into the right-hand side of (4.21) yields the required expressions.

For our forthcoming analysis to work, it will be necessary to supplement the vertical decomposition of the wave Eq. (4.8) with the vertical decomposition of the second Bianchi identity for the spacetime Weyl curvature. This yields a series of first-order equations involving only vertical objects:

**Proposition 4.13.** Fix \( n > 2 \) and \( M_0 \geq n + 2 \). Suppose \( (\mathcal{M}, g) \) is an \((n+1)\)-dimensional, \( M_0 \)-regular vacuum FG-aAdS segment. If \((U, \varphi)\) is a coordinate system on \( \mathcal{I} \), then the following vertical Bianchi equations hold for the vertical Weyl fields with respect to \( \varphi_\rho \)-coordinates:
\[
\rho^n \bar{\nabla}_\rho W_{abcd}^0 = 2\rho D_{[a} W_{b]c}^1 - 2 g_{[c[a} W_{b]d}^2 + 2 g_d [a W_{b]c}^2 \\
+ \rho g^{ef} \mathcal{L}_\rho g_{[e[a} W_{f]b]}^0 + \rho \mathcal{L}_\rho g_{[c[a} W_{b]d]} - \rho \mathcal{L}_\rho g_{d[a} W_{b]c}^2, \tag{4.40}
\]
\[
\rho^{n-2} \bar{\nabla}_\rho \left( \rho^{-(n-2)} W_{abc}^1 \right) = -g^{de} D_d W_{eabc} - g^{de} \left( \mathcal{L}_\rho g_{[e[a} W_{b]c]} + \mathcal{L}_\rho g_{[e[a} W_{b]c]}^1 \right), \tag{4.41}
\]
\[
\rho \bar{\nabla}_\rho \left( \rho^{-1} W_{abc}^1 \right) = 2 D_{[b} W_{c]a}^2 - g^{de} \left( \mathcal{L}_\rho g_{b[e} W_{a]c]} + \mathcal{L}_\rho g_{b[e} W_{a]c]}^1 \right), \tag{4.42}
\]
\[
\rho^{n-2} \bar{\nabla}_\rho \left( \rho^{-(n-2)} W_{ed}^2 \right) = -2 g^{ac} D_{[a} W_{b]d}^1 \\
- g^{ac} \left( g^{ef} \mathcal{L}_\rho g_{[e[a} W_{f]b]}^0 + \mathcal{L}_\rho g_{[c[a} W_{b]d]} - \frac{1}{2} \mathcal{L}_\rho g_{d[a} W_{b]c}^2 \right). \tag{4.43}
\]

**Remark 4.14.** One can apply (3.2) to (4.40)-(4.43) to express them in terms of \( \mathcal{L}_\rho \)-derivatives:
\[
\rho \mathcal{L}_\rho W_{abcd}^0 = 2\rho D_{[a} W_{b]c}^1 - 2 g_{[c[a} W_{b]d}^2 + 2 g_d [a W_{b]c}^2 .
\]

\(^{18}\)Additional limits are computed for the case when \( g \) is conformally flat in Corollary 5.7.
The wave equation satisfied by the spacetime Weyl curvature can now be decomposed into a series of wave equations involving only vertical objects as follows:

**Proposition 4.15.** Fix $n > 2$ and $M_0 \geq n + 2$. Suppose $(\mathcal{M}, g)$ is an $(n + 1)$-dimensional, $M_0$-regular vacuum FG-aAdS segment. Then,

\[
\begin{align*}
\Box_g \mathring{W}^0 &= \mathcal{O}_{M_0-2}(\rho^2; \mathring{W}^0) + \mathcal{O}_{M_0-3}(\rho^3; W^1) + \mathcal{O}_{M_0-2}(\rho^2; W^2) \\
&\quad + \mathcal{O}_{M_0-2}(\rho^3; \mathring{\Box} W^1), \\
(\Box_g + (n-1)) W^1 &= \mathcal{O}_{M_0-3}(\rho^3; \mathring{W}^0) + \mathcal{O}_{M_0-2}(\rho^2; W^1) + \mathcal{O}_{M_0-3}(\rho^3; W^2) \\
&\quad + \mathcal{O}_{M_0-2}(\rho^3; \mathring{\Box} W^1), \\
(\Box_g + 2(n-2)) W^2 &= \mathcal{O}_{M_0-2}(\rho^2; \mathring{W}^0) + \mathcal{O}_{M_0-3}(\rho^3; W^1) + \mathcal{O}_{M_0-2}(\rho^2; W^2) \\
&\quad + \mathcal{O}_{M_0-2}(\rho^3; \mathring{\Box} W^1).
\end{align*}
\]

**Proof.** (4.8) gives

\[
\rho^2 \Box_g W_{papb} = -2n W_{ab}^2 + \rho^2 \cdot Q_{ab}^2
\]
\[ \rho^2 \Box_g W_{pabc} = -2nW^1_{ab} + \rho^2 \cdot Q^1_{abc} \]  
\[ \rho^2 \Box_g W_{abcd} = -2nW^0_{abcd} + \rho^2 \cdot Q^0_{abcd} \]  

in which \( Q^0, Q^1, Q^2 \) are vertical tensor fields consisting only of terms quadratic in \( \hat{W}^0, W^1, W^2 \), the precise form of which are found in Appendix 5.3; in particular, one finds

\[ Q^2 = \mathcal{O}_{M_0-2} \left( 1; \hat{W}^0 \right) + \mathcal{O}_{M_0-3} (\rho; W^1) + \mathcal{O}_{M_0-2} \left( 1; W^2 \right), \]  
\[ Q^1 = \mathcal{O}_{M_0-3} (\rho; \hat{W}^0) + \mathcal{O}_{M_0-2} (1; W^1) + \mathcal{O}_{M_0-3} (\rho; W^2), \]  
\[ Q^0 = \mathcal{O}_{M_0-2} \left( 1; \hat{W}^0 \right) + \mathcal{O}_{M_0-3} (\rho; W^1) + \mathcal{O}_{M_0-2} (1; W^2). \]

Using (3.22) to replace the left-hand side of (4.54) with vertical objects, one obtains

\[ \rho^2 \Box_g W_{pab} = \rho^2 \Box_g (\rho^2 W^2_{ab}) + 2\rho g^{cd} \left( D_c W^1_{bda} + D_c W^1_{adb} \right) 
- 2(n - 1)W^2_{ab} + 2g^{cd} W^0_{cadb} 
+ \mathcal{O}_{M_0-2} (\rho^2; \hat{W}^0) + \mathcal{O}_{M_0-2} (\rho^2; W^2) 
+ \mathcal{O}_{M_0-3} (\rho^3; W^1) + \mathcal{O}_{M_0-2} (\rho^3; \hat{D}W^1). \]  

(3.13) is used to extract the \( \rho^2 \) factor from the \( \Box_g \)-terms:

\[ \rho^{-2} \Box_g (\rho^2 W^2_{ab}) = \Box_g W^2_{ab} + 4\rho \hat{D}_\rho W^2_{ab} - 2(n - 2)W^2_{ab} + \mathcal{O}_{M_0-2} (\rho^2; W^2). \]  

Finally, one applies (4.17) and (4.43) to (4.60) in order to deal with the terms involving \( W^0 \) and \( \hat{D}W^1 \). This yields (4.53), as required.

Using (3.22) to replace the left-hand side of (4.55) with vertical objects, one obtains

\[ \rho^2 \Box_g W_{pabc} = \rho^2 \Box_g (\rho^2 W^1_{abc}) + 2\rho g^{de} D_d W^0_{eabc} 
- 2\rho \left( D_c W^2_{ac} - D_c W^2_{ab} \right) 
- (n + 3)W^1_{ab} + 2(W^1_{abc} + W^1_{cab} + W^1_{bca}) 
+ \mathcal{O}_{M_0-2} (\rho^2; W^1) + \mathcal{O}_{M_0-3} (\rho^3; \hat{W}^0) + \mathcal{O}_{M_0-3} (\rho^3; W^2) 
+ \mathcal{O}_{M_0-2} (\rho^3; \hat{D}W^0) + \mathcal{O}_{M_0-2} (\rho^3; \hat{D}W^2). \]  

As before, (3.13) is used to extract the \( \rho^2 \) factor from the \( \Box_g \)-terms. In (4.62), one uses the first Bianchi identity

\[ W^1_{abc} + W^1_{cab} + W^1_{bca} = 0, \]

along with (4.42) and (4.41) to deal with the terms involving \( \hat{D}W^0 \) and \( \hat{D}W^2 \). This yields (4.52), as required.

Using the definition of \( \hat{W}^0 \) and the fact that \( \hat{D} \) is compatible with \( g \),

\[ \Box_g W^0_{abcd} = \Box_g W^0_{abcd} + \frac{1}{n - 2} \left( g_{ac} \Box_g W^2_{bd} - g_{ad} \Box_g W^2_{bc} + g_{bd} \Box_g W^2_{ac} - g_{bc} \Box_g W^2_{ad} \right). \]  

(4.64)
Equation (4.53) is used to replace the $\Box_g W^2$ terms on the right-hand side with lower-order terms:

$$\Box_g \hat{W}^0_{abcd} = \Box_g W^0_{abcd} - 2 \left( g_{ac} W^2_{bd} - g_{ad} W^2_{bc} + g_{bd} W^2_{ac} - g_{bc} W^2_{ad} \right)$$

$$+ \mathcal{O}_{M_0-2}(\rho^2; \hat{W}^0) + \mathcal{O}_{M_0-3}(\rho^3; W^1) + \mathcal{O}_{M_0-2}(\rho^2; W^2) + \mathcal{O}_{M_0-2}(\rho^3; \bar{D}W^1).$$

(4.65)

Next, (3.13) and (3.22) imply that

$$\Box_g W^0_{abcd} = \rho^{-2} \Box_g (\rho^2 W^0_{abcd}) - 4 \rho \bar{D}_\rho W^0_{abcd} + 2(n-2) W^0_{abcd} + \mathcal{O}_{M_0-2}(\rho^2; W^0)$$

$$= \rho^{-2} \Box_g (\rho^2 W^0_{abcd}) + 2 \rho \left( D_a W^1_{bcd} - D_b W^1_{acd} + D_c W^1_{abd} - D_d W^1_{abc} \right)$$

$$+ 4 W^0_{abcd} - 2 \left( g_{ac} W^2_{bd} - g_{ad} W^2_{bc} + g_{bd} W^2_{ac} - g_{bc} W^2_{ad} \right)$$

$$- 4 \rho \bar{D}_\rho W^0_{abcd} + 2(n-2) W^0_{abcd} + \mathcal{O}_{M_0-2}(\rho^2; W^0).$$

(4.66)

Finally, one applies (4.56) and (4.40) to exchange the $\Box_g W$ and $\bar{D}W^1$ terms, respectively, for terms involving $W^0$, $\bar{D}_\rho W^0$ and $g \cdot W^2$. This yields (4.51), as required.

5. Local AdS Rigidity via the Boundary Data

In this section, we present a series of preliminary results before proving our main result, Theorem 5.9. Most of the work here is required to prove the ‘backward’ statement—namely, that the stated conditions on the boundary data imply that a given aAdS spacetime is locally isometric to AdS.

To achieve this, we begin in Sect. 5.1 by stating a form of the Carleman estimate from [6] that may be applied to the vertical components of the Weyl curvature, which satisfy the vertical wave equations derived in the previous section. This enables us to identify sufficient vanishing conditions on the Weyl curvature for unique continuation to hold, which would yield $W = 0$ (and thus that our spacetime is locally isometric to AdS) in a near-boundary bulk region. Section 5.2 is dedicated to identifying assumptions on the boundary data that imply these sufficient vanishing conditions; Sect. 5.3 combines all of this to prove our main result.

5.1. The Carleman Estimate

In order to state the Carleman estimate, we require a notion of the ‘size’ of a vertical tensor field. To achieve this, we make use of a Riemannian metric (later chosen to be the one featuring in the GNCC).

**Definition 5.1.** Given a rank-$(k, l)$ vertical tensor field $A$ and an arbitrary Riemannian metric $\mathfrak{h}$ on $\mathcal{F}$ (viewed as a $\rho$-independent vertical tensor field), we define the $\mathfrak{h}$-norm $|A|_\mathfrak{h}^2$ in terms of coordinates on $\mathcal{F}$ by

$$|A|_\mathfrak{h}^2 := \Pi_{i=1}^k \mathfrak{h}_{ai} c_i \cdot \Pi_{j=1}^l \mathfrak{h}_{bj} d_j \cdot A^{a_1 \cdots a_k}_{b_1 \cdots b_l} A_{c_1 \cdots c_k}^{d_1 \cdots d_l}.$$  

(5.1)

The Carleman estimate of [6, Theorem 5.11] may now be stated in a form suitable for our purposes as follows.
Theorem 5.2. Let \((\mathcal{M}, g)\) be a vacuum FG-aAdS segment and fix a Riemannian metric \(\mathfrak{h}\) on \(\mathcal{I}\). Furthermore, assume that

- \(\mathcal{D} \subseteq \mathcal{I}\) is open with compact closure and satisfies the GNCC as stated in Definition 1.3, with \(\mathfrak{h}\) as above and \(\eta \in C^4(\mathcal{D})\) satisfying (1.15)–(1.17).
- \(f : (0, \rho_0] \times \mathcal{D} \rightarrow \mathbb{R}\) is defined by \(f := \rho/\eta\).
- \(k, l \geq 0\) are fixed integers.
- \(\sigma \in \mathbb{R}\).

Then, there exist constants \(C_0 \geq 0\), \(C_1 > 0\) (depending on \(g, \mathcal{D}, k, l\)) such that

- for any \(\kappa \in \mathbb{R}\) satisfying
  \[2\kappa \geq n - 1 + C_0, \quad \kappa^2 - (n - 2)\kappa + \sigma - (n - 1) - C_0 \geq 0,\]
- and for any constants \(f_*, \lambda, p > 0\) satisfying
  \[0 < f_* \ll 1, \quad \lambda \gg |\kappa| + |\sigma|, \quad 0 < 2p < 1,\]

the following Carleman estimate holds for any rank-\((k, l)\) vertical tensor field \(u\) on \(\mathcal{M}\) for which \(u\) and \(\tilde{\nabla}u\) vanish on \(\mathcal{M} \cap \{f = f_*\}\):

\[
\int_{\mathcal{D}(f_*)} e^{-2\lambda p^{-1}f_*} f^{n-2p-2\kappa} \cdot \left| (\tilde{\Box} + \sigma) u^2 \mathfrak{h} \right| d\mathfrak{h} + C_1 \lambda^3 \limsup_{\rho' \searrow 0} \int_{\mathcal{D}(f_*) \cap \{\rho = \rho'\}} \left| \tilde{\nabla}_\rho (\rho^{-\kappa} u)^2 \mathfrak{h} \right| + \left| \nabla (\rho^{-\kappa} u)^2 \mathfrak{h} \right| + \left| \rho^{-\kappa-1} u \right|^2 d\rho' \\
\geq \lambda \int_{\mathcal{D}(f_*)} e^{-2\lambda p^{-1}f_*} f^{n-2-2\kappa} \cdot \left( \rho^4 \left| \tilde{\nabla}_\rho u \right|^2 \mathfrak{h} + \rho^4 \left| \nabla u \right|^2 \mathfrak{h} + f^{2p} \left| u^2 \mathfrak{h} \right| \right) d\rho,
\]

where \(\mathcal{D}(f_*)\) is the region

\[\mathcal{D}(f_*) := [(0, \rho_0] \times \mathcal{D} \cap \{f < f_*\}.\]

Remark 5.3. Note that [6, Assumption 2.4] is automatically satisfied due to our regularity assumptions and the fact that (1.1) is satisfied.

5.2. Preliminary Results

We begin by applying our Carleman estimate to prove unique continuation for the Weyl curvature given sufficiently fast vanishing along a suitable boundary region.

Proposition 5.4. In addition to the assumptions of Theorem 5.2, fix \(n > 2\), \(M_0 \geq n + 2\) and suppose \((\mathcal{M}, g)\) is \(M_0\)-regular. Let \(C_i\) be the corresponding constant featuring in Theorem 5.2 and define

\[\kappa_i := \max \left\{ \frac{1}{2} [n - 1 + C_0], \frac{1}{2} [n - 2 + \sqrt{n^2 - 4i(n - i) + 4C_0}] \right\}, \quad i = 0, 1, 2.\]

Suppose that, on \(\mathcal{D}\),

\[
\tilde{\nabla}_0^0 = \mathcal{O}_1 (\rho^{\kappa_0+2}), \quad \tilde{\nabla}_0 \tilde{\nabla}_0^0 = \mathcal{O}_0 (\rho^{\kappa_0+1}), \quad \tilde{\nabla}_0 \tilde{\nabla}_1^0 = \mathcal{O}_0 (\rho^{\kappa_1+1}), \quad \tilde{\nabla}_0 \tilde{\nabla}_1^1 = \mathcal{O}_0 (\rho^{\kappa_1+1}).
\]

(5.7)
\[ W^2 = \mathcal{O}_1 (\rho^{\kappa_2 + 2}) , \quad \bar{\nabla}_\rho W^2 = \mathcal{O}_0 (\rho^{\kappa_2 + 1}) . \]  

Then, the spacetime Weyl curvature identically vanishes on \( \mathcal{D}(\frac{1}{2} f_*) \).

Remark 5.5. The following bounds hold for each of the above-defined \( \kappa_i \):

\[ \kappa_0 \geq n - 1, \quad \kappa_1 \geq n - 2, \quad \kappa_2 \geq n - 3. \]  

Proof. Take some smooth cut-off function

\[ \bar{\chi} : [0, f_+] \rightarrow [0, 1], \quad \bar{\chi}(s) = \begin{cases} 1, & 0 \leq s \leq \frac{1}{2} f_* , \\ 0, & \frac{3}{4} f_* \leq s . \end{cases} \]

Let \( \chi := \bar{\chi} \circ f \) and, for convenience, define

\[ w^0 := \hat{W}^0 , \quad w^1 := W^1 , \quad w^2 := W^2 . \]  

Let \( \sigma_i := i(n - i) \) for \( i = 0, 1, 2 \). Then,

\[ (\square_g + \sigma_i) (\chi \cdot w^i) = \chi (\square_g + \sigma_i) w^i + \chi'' \bar{\nabla}_\rho \bar{\nabla}_\rho f w^i + \chi' (2 \bar{\nabla}_\rho \bar{\nabla}_\rho w^i + \square_g f \cdot w^i) , \]  

where \( \chi' \) denotes the derivative of \( \chi \) with respect to \( f \). Using that \( \chi', \chi'' \) are supported in \([ \frac{1}{2} f_*, \frac{3}{4} f_* ] \) and applying the relations derived in [6, Section 2.2], one finds

\[ \sum_{i=0}^{2} \left| (\square_g + \sigma_i) (\chi \cdot w^i) \right| \leq \begin{cases} \sum_{i=0}^{2} \left( |(\square_g + \sigma_i) w^i|^2_{b} + \rho f^2 \left| \bar{\nabla}_\rho w^i \right|^2_{b} \\ + \rho^2 f^2 |Dw^i|^2_{b} + f^2 |w^i|^2_{b} \right) , & \frac{1}{2} f_* \leq f \leq \frac{3}{4} f_* , \\ \sum_{i=0}^{2} |(\square_g + \sigma_i) w^i|^2_{b} , & 0 \leq f \leq \frac{1}{2} f_* . \end{cases} \]

(5.13)

\( (4.51), (4.52) \) and \( (4.53) \) in addition to the fact that \( f \simeq 1 \) in \([ \frac{1}{2} f_*, \frac{3}{4} f_* ] \) implies that

\[ \sum_{i=0}^{2} \left| (\square_g + \sigma_i) (\chi \cdot w^i) \right| \leq \begin{cases} \sum_{i=0}^{2} \left( \rho^2 |\bar{\nabla}_\rho w^i|^2_{b} + \rho^2 |Dw^i|^2_{b} + |w^i|^2_{b} \right) , & \frac{1}{2} f_* \leq f \leq \frac{3}{4} f_* , \\ \sum_{i=0}^{2} \left( \rho^6 |Dw^i|^2_{b} + \rho^4 |w^i|^2_{b} \right) , & 0 \leq f \leq \frac{1}{2} f_* . \end{cases} \]

(5.14)

We define regions

\[ \Omega_i := \mathcal{D} \left( \frac{1}{2} f_* \right) , \]  

\[ \Omega_\epsilon := \mathcal{D} \left( \frac{3}{4} f_* \right) \cap \left\{ f > \frac{1}{2} f_* \right\} , \]  

(5.15)

(5.16)
and sum the Carleman estimates (5.4), as applied to \( w^i := \chi \cdot w^i \). The left-hand side \( L \) of the sum of these Carleman estimates can be estimated by

\[
L \lesssim \sum_{i=0}^{2} \int_{\Omega} e^{-2\lambda p - 1} f^p f^{n-2-2\kappa_i - p} \left[ \rho^2 |\bar{\mathcal{D}}_\rho w^i|_b^2 + \rho^2 |Dw^i|_b^2 + |w^i|_b^2 \right] dg \\
+ \sum_{i=0}^{2} \int_{\Omega_i} e^{-2\lambda p - 1} f^p f^{n-2-2\kappa_i - p} \left[ \rho^6 |Dw^i|_b^2 + \rho^4 |w^i|_b^2 \right] dg \\
+ C_1 \lambda^3 \sum_{i=0}^{2} \limsup_{\rho_+ \searrow 0} \int_{\mathcal{D}(f_+) \cap \{\rho = \rho_+\}} \left[ |\bar{\mathcal{D}}_\rho (\rho^{-\kappa_i} w^i)|_b^2 + |\mathcal{D}(\rho^{-\kappa_i} w^i)|_b^2 \right]
\]

where

\[
\|w^i\|_{2} + \|\bar{\mathcal{D}}_\rho w^i\|_{b} + \|Dw^i\|_{b} + \|w^i\|_{b} \geq L_1 + L_2 + L_3.
\]

\[\text{L is bounded below by}\]

\[
L \gtrsim \lambda \sum_{i=0}^{2} \int_{\Omega} e^{-2\lambda p - 1} f^p f^{n-2-2\kappa_i} (\rho^4 |\bar{\mathcal{D}}_\rho w^i|_b^2 + \rho^4 |Dw^i|_b^2 + f^{2p} |w^i|_b^2) dg,
\]

\[\text{into which } L_2 \text{ can be absorbed. By (5.2), the vanishing assumptions (5.7) and the fact that } |\bar{\partial}_\rho \chi| + |\partial_{\alpha} \chi| \lesssim \rho^{-1}, \text{ we also have that } L_3 \to 0 \text{ as } \rho_+ \searrow 0.
\]

The \( e^{-2\lambda p - 1} f^p f^{n-2-2\kappa_i} \) factors can be bounded above in \( \Omega_e \) and bounded below in \( \Omega_i \):

\[
e^{-2\lambda p - 1} f^p f^{n-2-2\kappa_i} \begin{cases} 
\leq e^{-2\lambda p - 1} \left( \frac{L_2}{2} \right)^p \left( \frac{L_2}{2} \right)^{n-2-2\kappa_i}, & \text{in } \Omega_e \\
\geq e^{-2\lambda p - 1} \left( \frac{L_2}{2} \right)^p \left( \frac{L_2}{2} \right)^{n-2-2\kappa_i}, & \text{in } \Omega_i.
\end{cases}
\]

Hence, for large \( \lambda \),

\[
\sum_{i=0}^{2} \int_{\Omega} \left| w^i \right|_b^2 + \rho^2 |\bar{\mathcal{D}}_\rho w^i|_b^2 + \rho^2 |Dw^i|_b^2 + \left| w^i \right|_b^2 \geq \lambda \sum_{i=0}^{2} \int_{\Omega_i} f^{2p} \left| w^i \right|_b^2 dg
\]

\[\text{The left-hand side of (5.20) is bounded above by}\]

\[
\lesssim \sum_{i=0}^{2} \int_{\Omega_e} \left| \rho^{-\kappa_i} w^i \right|_b^2 + |\bar{\mathcal{D}}_\rho (\rho^{-\kappa_i} w^i)|_b^2 + |\mathcal{D}(\rho^{-\kappa_i} w^i)|_b^2 dg,
\]

and so is finite by the vanishing assumptions (5.7). As a result, taking \( \lambda \to \infty \) in (5.20) yields \( w^0, w^1, w^2 \equiv 0 \) (i.e. \( \bar{\mathcal{W}}^0, \mathcal{W}^1, \mathcal{W}^2 \equiv 0 \)) on \( \Omega_i := \mathcal{D}(\frac{1}{2} f_+) \). In other words, the full spacetime Weyl curvature \( W \equiv 0 \) on \( \mathcal{D}(\frac{1}{2} f_+) \) as required. \( \square \)

With this result in hand, our objective is to identify sufficient conditions on the boundary data to obtain the vanishing rates in the vertical components of the Weyl curvature required for unique continuation to hold. First, we show that conformal flatness of \( g \) yields vanishing up to—but not including—the order at which \( g^{(n)} \)-dependency enters the coefficients in the expansions (4.28)–(4.30).
Proposition 5.6. Fix $n > 2$ and $M_0 \geq n + 2$. Suppose $(\mathcal{M}, g)$ is an $(n + 1)$-dimensional, $M_0$-regular vacuum FG-aAdS segment. If $g$ is conformally flat then

\begin{align*}
W^0 &= \rho^{n-2} \cdot \mathcal{M}^{(n-2)}_0 + \rho^{n-2} \cdot r_0, \quad r_0 \rightarrow M_0-n 0, \quad \text{(5.22)} \\
W^1 &= \rho^{n-1} \cdot \mathcal{M}^{(n-1)}_1 + \rho^{n-1} \cdot r_1, \quad r_1 \rightarrow M_0-n-1 0, \quad \text{(5.23)} \\
W^2 &= \rho^{n-2} \cdot \mathcal{M}^{(n-2)}_2 + \rho^{n-2} \cdot r_2, \quad r_2 \rightarrow M_0-n 0. \quad \text{(5.24)}
\end{align*}

Proof. Recall the leading-order expressions for $W^0$, $W^1$ and $W^2$ in Proposition 4.12. If $n = 3$, then $\mathcal{W}$ identically vanishes and conformal flatness of $g$ implies that $\mathcal{C}$ vanishes. Corollary 4.11 then yields (5.22)–(5.24) as required.

If $n > 3$, then conformal flatness of $g$ implies that $\mathcal{W}$ vanishes, which in turn implies that $\mathcal{C}$ vanishes. Hence,

\begin{equation}
\mathcal{M}^{(0)}_0 = \mathcal{M}^{(1)}_1 = \mathcal{M}^{(2)}_2 = 0. \quad \text{(5.25)}
\end{equation}

If $n = 4$, then this implies

\begin{align*}
W^0 &= \rho^2 \log \rho \cdot \mathcal{M}^{(\star)}_0 + \rho^2 \cdot \mathcal{M}^{(2)}_0 + \rho^2 \cdot r_0, \quad \text{(5.26)} \\
W^1 &= \rho^3 \log \rho \cdot \mathcal{M}^{(\star)}_1 + \rho^3 \cdot \mathcal{M}^{(3)}_1 + \rho^3 \cdot r_1, \quad \text{(5.27)} \\
W^2 &= \rho^2 \log \rho \cdot \mathcal{M}^{(\star)}_2 + \rho^2 \cdot \mathcal{M}^{(2)}_2 + \rho^2 \cdot r_2. \quad \text{(5.28)}
\end{align*}

Given (5.28), the left-hand side of the vertical Bianchi Eq. (4.47) reads

\begin{equation}
\rho^2 \mathcal{L}_\rho (\rho^{-2} \cdot W^2) = \rho \cdot \mathcal{M}^{(\star)}_2 + o(\rho), \quad \text{(5.29)}
\end{equation}

where $o(\rho)$ denotes a vertical tensor field $t$ for which $\rho^{-1} \cdot t \rightarrow 0$. The factors of $g^{-1}$ and $\mathcal{L}_\rho g$ present in the right-hand side of (4.47) are $O(1)$ and $O(\rho)$, respectively; given (5.26) and (5.27), the right-hand side thus only contains $O(\rho^3 \log \rho)$ terms. Collecting strictly order $\rho$ terms in (4.47), one finds

\begin{equation}
\mathcal{M}^{(\star)}_2 = 0, \quad \text{(5.30)}
\end{equation}

which yields (5.24) as required.

Given (5.26), the left-hand side of (4.44) reads

\begin{equation}
\rho \mathcal{L}_\rho W^0 = \rho^2 (1 + 2 \log \rho) \cdot \mathcal{M}^{(\star)}_0 + 2 \rho^2 \cdot \mathcal{M}^{(2)}_0 + o(\rho^2). \quad \text{(5.31)}
\end{equation}

As above, the factors of $g^{-1}$ and $\mathcal{L}_\rho g$ present in the right-hand side of (4.44) are $O(1)$ and $O(\rho)$, respectively; given (5.26) and (5.24), the right-hand side thus only contains $O(\rho^2)$ terms. Collecting strictly order $\rho^2 \log \rho$ terms in (4.44), one finds

\begin{equation}
\mathcal{M}^{(\star)}_0 = 0, \quad \text{(5.31)}
\end{equation}

which yields (5.22) as required. Note that it was essential for us to derive (5.24) before completing this step; otherwise, the right-hand side of (4.44) would still contain strictly order $\rho^2 \log \rho$ terms.

Finally, given (5.27), the left-hand side of (4.46) reads

\begin{equation}
\rho \mathcal{L}_\rho (\rho^{-1} \cdot W^1) = \rho^2 (1 + 2 \log \rho) \cdot \mathcal{M}^{(\star)}_1 + 2 \rho^2 \cdot \mathcal{M}^{(3)}_1 + o(\rho^2). \quad \text{(5.32)}
\end{equation}
Once more, the factors of $g^{-1}$ and $\mathcal{L}_p g$ present in the right-hand side of (4.46) are $\mathcal{O}(1)$ and $\mathcal{O}(\rho)$, respectively. Given (5.27) and (5.24), the right-hand side thus only contains $\mathcal{O}(\rho^2)$ terms; collecting strictly order $\rho^2 \log \rho$ terms in (4.46), one finds

$$\mathcal{W}_1^{(s)} = 0,$$

which yields (5.23) as required. Again, it was crucial to derive (5.24) before completing this step to remove all order $\rho^2 \log \rho$ terms from the right-hand side of (4.44). This completes the proof of the $n = 4$ case.

For $n > 4$, (5.25) yields

$$W^0 = \begin{cases} \sum_{k=1}^{n/2} \rho^{2k} \cdot \mathcal{W}_0^{(2k)} + \rho^{n-2} \cdot \mathcal{W}_0^{(n-2)} + \rho^{n-2} \cdot r_0, & n \text{ odd}, \\ \sum_{k=1}^{n/2} \rho^{2k} \cdot \mathcal{W}_0^{(2k)} + \rho^{n-2} \log \rho \cdot \mathcal{W}_0^{(s)} + \rho^{n-2} \cdot \mathcal{W}_0^{(n-2)} + \rho^{n-2} \cdot r_0, & n \text{ even}, \end{cases}$$

(5.33)

$$W^1 = \begin{cases} \sum_{k=1}^{n/2} \rho^{2k+1} \cdot \mathcal{W}_1^{(2k+1)} + \rho^{n-1} \cdot \mathcal{W}_1^{(n-1)} + \rho^{n-1} \cdot r_1, & n \text{ odd}, \\ \sum_{k=1}^{n/2} \rho^{2k+1} \cdot \mathcal{W}_1^{(2k+1)} + \rho^{n-1} \log \rho \cdot \mathcal{W}_1^{(s)} + \rho^{n-1} \cdot \mathcal{W}_1^{(n-1)} + \rho^{n-1} \cdot r_1, & n \text{ even}, \end{cases}$$

(5.34)

$$W^2 = \begin{cases} \sum_{k=1}^{n/2} \rho^{2k} \cdot \mathcal{W}_2^{(2k)} + \rho^{n-2} \cdot \mathcal{W}_2^{(n-2)} + \rho^{n-2} \cdot r_2, & n \text{ odd}, \\ \sum_{k=1}^{n/2} \rho^{2k} \cdot \mathcal{W}_2^{(2k)} + \rho^{n-2} \log \rho \cdot \mathcal{W}_2^{(s)} + \rho^{n-2} \cdot \mathcal{W}_2^{(n-2)} + \rho^{n-2} \cdot r_2, & n \text{ even}. \end{cases}$$

(5.35)

Given (5.35), the left-hand side of the vertical Bianchi Eq. (4.47) reads

$$\begin{cases} \sum_{k=1}^{n-3} (2k + 2 - n)\rho^{2k-1} \cdot \mathcal{W}_2^{(2k)} + o(\rho^{n-2}), & n \text{ odd}, \\ \sum_{k=1}^{n-4} (2k + 2 - n)\rho^{2k-1} \cdot \mathcal{W}_2^{(2k)} + \rho^{n-3} \cdot \mathcal{W}_2^{(s)} + o(\rho^{n-2}), & n \text{ even}. \end{cases}$$

(5.36)

Given (5.33) and (5.34), the right-hand side thus only contains $\mathcal{O}(\rho^3)$ terms. Collecting strictly order $\rho$ terms in (4.47), one finds

$$\mathcal{W}_2^{(2)} = 0.$$

(5.37)

In other words, for $n = 5$ we have (5.24) as required and, for $n > 5$,

$$W^2 = \begin{cases} \rho^4 \log \rho \cdot \mathcal{W}_2^{(s)} + \rho^4 \cdot \mathcal{W}_2^{(4)} + \rho^4 \cdot r_2, & n = 6, \\ \sum_{k=2}^{n-3} \rho^{2k} \cdot \mathcal{W}_2^{(2k)} + \rho^{n-2} \cdot \mathcal{W}_2^{(n-2)} + \rho^{n-2} \cdot r_2, & n > 6 \text{ odd}, \\ \sum_{k=2}^{n-3} \rho^{2k} \cdot \mathcal{W}_2^{(2k)} + \rho^{n-2} \log \rho \cdot \mathcal{W}_2^{(s)} + \rho^{n-2} \cdot \mathcal{W}_2^{(n-2)} + \rho^{n-2} \cdot r_2, & n > 6 \text{ even}. \end{cases}$$

(5.38)

Given (5.33), the left-hand side of (4.44) reads

$$\begin{cases} \sum_{k=1}^{n-3} (2k)\rho^{2k} \cdot \mathcal{W}_0^{(2k)} + (n - 2)\rho^{n-2} \cdot \mathcal{W}_0^{(n-2)} + o(\rho^{n-2}), & n \text{ odd}, \\ \sum_{k=1}^{n-4} (2k)\rho^{2k} \cdot \mathcal{W}_0^{(2k)} + \rho^{n-2} [1 + (n - 2) \log \rho] \cdot \mathcal{W}_0^{(s)} + (n - 2)\rho^{n-2} \cdot \mathcal{W}_0^{(n-2)} + o(\rho^{n-2}), & n \text{ even}. \end{cases}$$

(5.39)
Given (5.33), (5.34) and (5.38) (or (5.24) if \( n = 5 \)), the right-hand side thus only contains

\[
\begin{cases}
\mathcal{O}(\rho^3) \text{ terms if } n = 5, \\
\mathcal{O}(\rho^4 \log \rho) \text{ terms if } n = 6, \\
\mathcal{O}(\rho^4) \text{ terms if } n > 6.
\end{cases}
\] (5.40)

Collecting strictly order \( \rho^2 \) terms in (4.44), one hence finds

\[
\mathcal{W}_0^{(2)} = 0.
\] (5.41)

In other words, for \( n = 5 \), we have (5.22) as required and, for \( n > 5 \),

\[
\begin{align*}
W^0 &= \begin{cases}
\rho^5 \log \rho \cdot \mathcal{W}_0^{(r)} + \rho^4 \cdot \mathcal{W}_0^{(4)} + \rho^4 \cdot r_0, & n = 6, \\
\sum_{k=2}^{n-2} \rho^{2k} \cdot \mathcal{W}_0^{(2k)} + \rho^{n-2} \cdot \mathcal{W}_0^{(n-2)} + \rho^{n-2} \cdot r_0, & n > 6 \text{ odd}, \\
\sum_{k=2}^{n-2} \rho^{2k} \cdot \mathcal{W}_0^{(2k)} + \rho^{n-2} \cdot \log \rho \cdot \mathcal{W}_0^{(r)} + \rho^{n-2} \cdot \mathcal{W}_0^{(n-2)} + \rho^{n-2} \cdot r_0, & n > 6 \text{ even}.
\end{cases}
\end{align*}
\] (5.42)

Given (5.34), the left-hand side of (4.46) reads

\[
\begin{align*}
\sum_{k=1}^{n-3} (2k) \rho^{2k} \cdot \mathcal{W}_1^{(2k+1)} + (n-2) \rho^{n-2} \cdot \mathcal{W}_1^{(n-1)} + \mathcal{O}(\rho^{n-2}), & n \text{ odd,} \\
\sum_{k=1}^{n-1} (2k) \rho^{2k} \cdot \mathcal{W}_1^{(2k+1)} + \rho^{n-2} [1 + (n-2) \log \rho] \cdot \mathcal{W}_1^{(r)} \\
+ (n-2) \rho^{n-2} \cdot \mathcal{W}_1^{(n-1)} + \mathcal{O}(\rho^{n-2}), & n \text{ even.}
\end{align*}
\] (5.43)

Given (5.34) and (5.38) (or (5.24) if \( n = 5 \)), the right-hand side thus only contains

\[
\begin{cases}
\mathcal{O}(\rho^3) \text{ terms if } n = 5, \\
\mathcal{O}(\rho^4 \log \rho) \text{ terms if } n = 6, \\
\mathcal{O}(\rho^4) \text{ terms if } n > 6.
\end{cases}
\] (5.44)

Collecting strictly order \( \rho^2 \) terms in (4.46), one hence finds

\[
\mathcal{W}_1^{(3)} = 0.
\] (5.45)

In other words, for \( n = 5 \), we have (5.23) as required and, for \( n > 5 \),

\[
\begin{align*}
W^1 &= \begin{cases}
\rho^5 \log \rho \cdot \mathcal{W}_1^{(r)} + \rho^5 \cdot \mathcal{W}_1^{(5)} + \rho^5 \cdot r_1, & n = 6, \\
\sum_{k=2}^{n-2} \rho^{2k+1} \cdot \mathcal{W}_1^{(2k+1)} + \rho^{n-1} \cdot \mathcal{W}_1^{(n-1)} + \rho^{n-1} \cdot r_1, & n > 6 \text{ odd,} \\
\sum_{k=2}^{n-2} \rho^{2k+1} \cdot \mathcal{W}_1^{(2k+1)} + \rho^{n-1} \log \rho \cdot \mathcal{W}_1^{(r)} + \rho^{n-1} \cdot \mathcal{W}_1^{(n-1)} + \rho^{n-1} \cdot r_1, & n > 6 \text{ even.}
\end{cases}
\end{align*}
\] (5.46)

This completes the proof for the \( n = 5 \) case. For \( n > 5 \), one may now iterate this process to derive vanishing of successively higher-order coefficients; at the \( k^{th} \) iteration, substitute the updated expansions of \( W^0, W^1 \) and \( W^2 \) into (4.47), then (4.44) and (4.46), collecting order \( \rho^{2k+1}, \rho^{2k+2} \text{ and } \rho^{2k+2} \text{ terms, respectively} \) (or order \( \rho^{2k+1}, \rho^{2k+2} \log \rho \text{ and } \rho^{2k+1} \log \rho \text{ if dealing with log coefficients}).

Note that this process cannot be continued to derive vanishing of \( \mathcal{W}_2^{(n-2)} \) (and hence \( \mathcal{W}_0^{(n-2)}, \mathcal{W}_1^{(n-1)} \) too), since this coefficient is always eliminated upon substitution of (5.24) into (4.47). \( \square \)
This result enables us to deduce more information about the expansion of \( g \) itself.

**Corollary 5.7.** Fix \( n > 2 \) and \( M_0 \geq n + 2 \). Suppose \((\mathcal{M}, g)\) is an \((n + 1)\)-dimensional, \( M_0 \)-regular vacuum FG-aAdS segment. If \( g \) is conformally flat, then

\[
g = \begin{dcases}
g - \rho^2 \mathbf{p} + \rho^n g^{(n)} + \rho^n r, & n \leq 4, \\
g - \rho^2 \mathbf{p} + \frac{1}{4} \rho^4 g^{-1} \mathbf{p} \mathbf{p} + \rho^n g^{(n)} + \rho^n r, & n > 4,
\end{dcases}
\]

where the remainder \( r \) satisfies

\[
r \rightarrow M_0 - n \ 0. \tag{5.48}
\]

Furthermore, there exist tensor fields \( \mathfrak{g}(k) \) on \( \mathcal{I} \) such that

\[
\mathcal{L}_p g^{-1} \rightarrow M_0 - k \mathfrak{g}(k), \quad 0 \leq k < n, \tag{5.49}
\]

where \( \mathfrak{g}(k) \) vanishes for \( k \) odd and, for \( 0 < 2l < n \),

\[
\mathfrak{g}_{(2l)}^{ab} = \frac{l + 1}{2l} \mathfrak{g}^{ac} \mathfrak{g}^{bc} \mathfrak{g}^{cd} \cdots \mathfrak{g}^{c_2 c_1} \mathfrak{g}^{c_2 b} (\mathfrak{p}_{c_1 c_2} \mathfrak{p}_{c_3 c_4} \cdots \mathfrak{p}_{c_{2l-1} c_{2l}}), \tag{5.50}
\]

\[
= \frac{l + 1}{2l} \mathfrak{g}^{ac} \mathfrak{p}_{cd} \mathfrak{g}_{(2l-2)}^{db}. \tag{5.51}
\]

**Proof.** Theorem 4.3 gave the near-boundary expansion of \( g \) and established, furthermore, that \( g^{(2)} = - \mathbf{p} \). To prove (5.47), we must therefore show that

1. When \( n > 4 \), \( g^{(4)} = \frac{1}{4} g^{-1} \mathbf{p} \mathbf{p} \).
2. When \( n > 6 \), \( g^{(k)} = 0 \) for \( 6 \leq k < n \).
3. When \( n \geq 4 \), \( g^{(*)} = 0 \).

We address each of these statements in turn:

1. This follows from (4.35) and (5.24) (i.e. \( \mathcal{W}^{(2)}_2 = 0 \)).
2. We proceed via induction. (4.36) combined with (5.24) (i.e. \( \mathcal{W}^{(4)}_2 = 0 \)) provides the base case. Fix \( 6 \leq 2K < n - 2 \) and assume that \( g^{(j)} = 0 \) for all \( 6 \leq j \leq 2K \), i.e.

\[
g = g - \rho^2 \cdot \mathbf{p} + \frac{1}{4} \rho^4 \cdot g^{-1} \mathbf{p} \mathbf{p}
+ \left\{ \begin{array}{ll}
\sum_{k=K+1}^{n} \rho^{2k} g^{(2k)} + \rho^n g^{(n)} + \rho^n r, & n \text{ odd,} \\
\sum_{k=K+1}^{n} \rho^{2k} g^{(2k)} + \rho^n \log \rho g^{(*)} + \rho^n g^{(n)} + \rho^n r, & n \text{ even.}
\end{array} \right. \tag{5.52}
\]

Substituting this into the relation

\[
g_{ab} \cdot g^{bc} = \delta^c_a, \tag{5.53}
\]
and applying the inductive assumption to match coefficients order-by-order, one finds that

\[
\begin{align*}
g_{ab}^{(0)} &= g_{ab}^{(1)} = g_{ac}^{(2)} g_{bd}^{(2)} p_{cd} \\
g_{ab}^{(2l)} &= g_{bc}^{(2l-2)} - p_{ab}^{(2l-2)} g_{bc}^{(2l-4)} = 0, &4 \leq 2l \leq 2K,
\end{align*}
\]  

(5.54)

where \( g_{ab} := (g^{-1})_{ab} \). One may straightforwardly verify that this is satisfied by the ansatz (5.50), which in turn satisfies the recurrence relation (5.51).

Substituting (5.24) into the left-hand side of (4.21) and collecting strictly order \( \rho^{2K} \) terms, one finds

\[
0 = -2K(K+1)g_{ab}^{(2K+2)} + g_{ac}^{cd} p_{ac} p_{bd} \\
- \frac{1}{2} g_{(2K-4)}^{cd} (p_{ae} p_{ef} p_{df} + g_{ae} p_{ef} p_{bd}) \\
+ \frac{1}{4} g_{(2K-6)}^{ef} p_{ae} p_{cf} g^{gh} p_{bg} p_{dh}. 
\]  

(5.55)

Applying the relations (5.51), one finds that all but the first of the above terms cancel. Hence,

\[
g_{ab}^{(2K+2)} = 0, \tag{5.56}
\]

which closes the inductive argument.

(3) Substituting (5.24) and the properties derived above in parts (1) and (2) into the left-hand side of (4.21) and collecting strictly order \( \rho^{n-2} \log \rho \) terms, one finds

\[
g_{ab}^{(*)} = 0, \tag{5.57}
\]

as required.

\[
W_0 = \rho^{n-2} \cdot r_0, \quad r_0 \rightarrow M_0 - n 0, \\
W_1 = \rho^{n-1} \cdot r_1, \quad r_1 \rightarrow M_0 - n - 1 0, \\
W_2 = \rho^{n-2} \cdot r_2, \quad r_2 \rightarrow M_0 - n 0. 
\]

(5.58)

(5.59)

(5.60)

if and only if \((\mathcal{M}, g)\) has boundary data \((g^{(0)}, g^{(n)})\) for which

\[
g^{(0)}\text{is conformally flat}, \quad \text{and} \tag{5.61}
\]
\[ g^{(n)} = \begin{cases} \frac{1}{4} g^{-1} \cdot p \cdot p, & n = 4, \\ 0, & n \neq 4, \end{cases} \]  
\tag{5.62}

where \( p \) is the \( g^{(0)} = g \)-Schouten tensor.

Proof. We begin by proving the reverse direction of this statement. Given (5.61), we obtain a simplified expansion of \( g \) via Corollary 5.7. Substituting this along with (5.24) into (4.21) and collecting order \( \rho^{n-2} \) terms, we find

\[ \mathcal{W}_{2ab}^{(n-2)} = -\frac{1}{2} n(n-2)g^{(n)}_{ab} \]

\[ + \begin{cases} 0, n \text{ odd}, \\ p_{ac}p_{bd}g^{cd}, n=4, \\ p_{ac}p_{bd}g^{cd}_{(n-2)} - g^{ef}p_{ac}p_{be}p_{df}g^{cd}, n=6, \\ p_{ac}p_{bd}g^{cd}_{(n-4)} - \frac{1}{2} g^{ef}(p_{ac}p_{be}p_{df} + p_{ae}p_{cf}p_{bd}) g^{cd}_{(n-6)} + \frac{1}{2} g^{ef}g^{gh}p_{ae}p_{cf}p_{bg}p_{dh}g^{cd}_{(n-8)}, & n \geq 8. \end{cases} \]  
\tag{5.63}

The relations (5.51) result in each of the \( n \geq 6 \) terms in the brace vanishing identically. The above hence reduces to

\[ \mathcal{W}_{2ab}^{(n-2)} = \begin{cases} -\frac{1}{2} n(n-2)g^{(n)}_{ab}, & n \neq 4, \\ -4g^{(4)}_{ab} + g^{cd}p_{ac}p_{bd}, & n = 4, \end{cases} \]  
\tag{5.64}

from which we see that (5.62) implies

\[ \mathcal{W}_{2}^{(n-2)} = 0. \]  
\tag{5.65}

We have thus used this condition to overcome the barrier to the iterative process identified in the proof of Proposition 5.6; substituting the updated expansions (4.28), (4.29) and (4.30) into the vertical Bianchi Eq. (4.44) and collecting strictly order \( \rho^{n-2} \) terms, one finds

\[ \mathcal{W}_{0}^{(n-2)} = 0. \]  
\tag{5.66}

Finally, substituting the updated expansions (4.28), (4.29) and (4.30) into the vertical Bianchi Eq. (4.46) and collecting strictly order \( \rho^{n-2} \) terms, one finds

\[ \mathcal{W}_{1}^{(n-1)} = 0. \]  
\tag{5.67}

This yields (5.58), (5.59) and (5.60) as required.

For the forward direction, recall the leading-order expressions for \( \mathcal{W}^{0} \) and \( \mathcal{W}^{1} \) found in Proposition 4.12. If \( n = 3 \), then the vanishing of \( \mathcal{W}_{1}^{(1)} = \mathcal{C} \) guaranteed by (5.59) implies that \( g = g^{(0)} \) is conformally flat. If \( n > 3 \), then the vanishing of \( \mathcal{W}_{0}^{(0)} = \mathcal{W} \) guaranteed by (5.58) implies that \( g = g^{(0)} \) is conformally flat.

We hence have (5.61) and, as above, can apply Corollary 5.7 to substitute the simplified expansion of \( g \) into (4.21) to obtain (5.64). The vanishing of \( \mathcal{W}_{2}^{(n-2)} \) guaranteed by (5.60) thus implies the condition (5.62) as required.

\( \square \)
5.3. The Main Result

**Theorem 5.9.** Fix $n > 2$.\(^{19}\) Suppose $(\mathcal{M}, g)$ is an $(n+1)$-dimensional vacuum FG-aAdS segment for which $\mathcal{D} \subseteq \mathcal{I}$ is open with compact closure and satisfies the GNCC as given by Definition 1.3. There exist constants $M_0 \geq n+2$ and $f_* > 0$ such that if $(\mathcal{M}, g)$ is $M_0$-regular, then it is locally isometric to pure AdS on $\mathcal{D}(f_*^2)$ if and only if $(\mathcal{M}, g)$ has boundary data $(g^{(0)}, g^{(n)})$ for which the following hold on $\mathcal{D}$:

$$g^{(0)} \text{ is conformally flat, and}$$

$$g^{(n)} = \begin{cases} \frac{1}{4} g^{-1} \cdot p \cdot p, & n = 4, \\ 0, & n \neq 4, \end{cases}$$

where $p$ is the $g$-Schouten tensor.

**Proof.** Solutions of (1.1) are locally isometric to AdS if and only if

$$R_{\alpha\beta\gamma\delta} = -g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma},$$

or, equivalently,

$$W_{\alpha\beta\gamma\delta} = 0.$$\(^{5.71}\)

Vanishing of the spacetime Weyl curvature implies vanishing of $W^0$, $W^1$ and $W^2$ to arbitrarily high order at $\mathcal{I}$. By Proposition 5.8, the conditions (5.68) and (5.69) follow.

For the other direction, let us write $o_K(\rho^L)$ to denote a vertical tensor field $t$ for which

$$\rho^{-L} \cdot t \rightarrow K 0.$$\(^{5.72}\)

In terms of this notation, (5.68) and (5.69) imply

$$W^0 = o_{M_0-n}(\rho^{n-2}), \quad W^1 = o_{M_0-n-1}(\rho^{n-1}), \quad W^2 = o_{M_0-n}(\rho^{n-2}),$$

by Proposition 5.8.

Substituting (5.73) into (4.47), one has

$$\mathcal{L}_\rho (\rho^{-(n-2)} W^2) = \rho^{-(n-2)} \cdot o_{M_0-n-2}(\rho^{n-1}) + \rho^{-(n-2)} \cdot o_{M_0-n}(\rho^{n-1})$$

$$= o_{M_0-n-2}(\rho),$$\(^{5.74}\)

where a degree of vertical regularity has been lost due to the vertical derivative present in the right-hand side of (4.47). Given (5.73), $\rho^{-(n-2)} W^2$ has vanishing boundary limit so we may integrate (5.74) from the boundary to deduce

$$\rho^{-(n-2)} \cdot W^2 = o_{M_0-n-2}(\rho^2) \implies W^2 = o_{M_0-n-2}(\rho^n).$$\(^{5.75}\)

Substituting (5.73) and (5.75) into (4.44), one has

---

\(^{19}\)If $n = 2$, the Weyl curvature identically vanishes. Hence, in this case, every vacuum FG-aAdS segment is locally isometric to pure AdS and the result in Theorem 5.9 is trivial.

\(^{20}\)That is, in the bulk ‘near’ $\mathcal{D}$; see (5.5).

\(^{21}\)See, for example, [5].
\[ \mathcal{L}_\rho W^0 = o_{M_0-n-2}(\rho^{n-1}) + \rho^{-1} \cdot o_{M_0-n-2}(\rho^n) + \rho^{-1} \cdot o_{M_0-n}(\rho^n) \]
\[ = o_{M_0-n-2}(\rho^{n-1}). \]  
(5.76)

Integrating this from the boundary, we deduce
\[ W^0 = o_{M_0-n-2}(\rho^n). \]  
(5.77)

Substituting (5.73) and (5.75) into (4.46), one has
\[ \mathcal{L}_\rho (\rho^{-1}W^1) = \rho^{-1} \cdot o_{M_0-n-3}(\rho^n) + \rho^{-1} \cdot o_{M_0-n-1}(\rho^n) \]
\[ = o_{M_0-n-3}(\rho^{n-1}). \]  
(5.78)

Again, a degree of vertical regularity has been lost due to the vertical derivative present in the right-hand side of (4.46). Integrating from the boundary, we deduce
\[ \rho^{-1} \cdot W^1 = o_{M_0-n-3}(\rho^n) \implies W^1 = o_{M_0-n-3}(\rho^{n+1}). \]  
(5.79)

Iterating the above process (that is, integrating (4.47) followed by (4.44) and (4.46) using the improved rates (5.75), (5.77) and (5.79)) \(i\) times, we deduce
\[ W^0 = o_{M_0-n-2i}(\rho^{n-2+2i}), \quad W^1 = o_{M_0-n-1-2i}(\rho^{n-1+2i}), \]
\[ W^2 = o_{M_0-n-2i}(\rho^{n-2+2i}). \]  
(5.80)

At each iteration, additional orders of vanishing are exchanged for degrees of vertical regularity. By Proposition 5.4, for unique continuation to hold it will suffice for \(n - 2 + 2i \geq \kappa_{\text{max}} + 2\), where
\[ \kappa_{\text{max}} := \begin{cases} \frac{1}{2} (n - 2 + \sqrt{n^2 + 4C_0}), & C_0 \leq n + 1, \\ \frac{1}{2} (n - 1 + C_0), & C_0 > n + 1. \end{cases} \]  
(5.81)

In other words, we must iterate \(i = \lceil \frac{1}{2} (\kappa_{\text{max}} - n + 4) \rceil\) times. If we choose \(M_0 = \lceil \kappa_{\text{max}} \rceil + 6\), then this yields
\[ W^0 = o_2(\rho^{\kappa_{\text{max}}+2}), \quad W^1 = o_1(\rho^{\kappa_{\text{max}}+3}), \quad W^2 = o_2(\rho^{\kappa_{\text{max}}+2}), \]  
(5.82)

which in turn implies (5.7), (5.8) and (5.9) as required.

In Sect. 1, it was noted that there exist coordinate transformations \((\rho, x^a) \to (\bar{\rho}, \bar{x}^a)\) preserving the Fefferman–Graham gauge, i.e.
\[ g = \bar{\rho}^{-2} \left( d\bar{\rho}^2 + \bar{g}_{ab} d\bar{x}^a d\bar{x}^b \right), \]  
(5.83)

but altering the coefficients in the corresponding partial expansion of \(\bar{g}\). Since the underlying theory should be invariant under these transformations, we only speak of *gauge-equivalent* classes of boundary data \([g^{(0)}, g^{(n)}]\). Accordingly, one should verify that if the conditions on the boundary data in Theorem 5.9 hold in *one* Fefferman–Graham coordinate system, then they hold in *all* Fefferman–Graham coordinate systems. This is proven in the following proposition.

---

22Note that if we had not first derived (5.75), then one would only have \(\mathcal{L}_\rho W^0 = o_{M_0-n-2}(\rho^{n-3})\) and we would not have been able to obtain this improvement via integration.

23Again, it was crucial to derive (5.75) first.
Proposition 5.10. The conditions (5.68) and (5.69) are invariant under coordinate transformations preserving the Fefferman–Graham gauge.

Proof. As in [14, 19], though now only working in finite regularity, we consider a general ansatz for the coordinate transformations \((\rho, x) \rightarrow (\tilde{\rho}, \tilde{x})\) of the form

\[
\rho = \tilde{\rho} \cdot e^{-\sigma(\tilde{x})} + \tilde{\rho}^2 \cdot a_{(2)}(\tilde{x}) + \tilde{\rho}^3 \cdot a_{(3)}(\tilde{x}) + \tilde{\rho}^3 \cdot s, \quad (5.84)
\]

\[
x^b = \tilde{x}^b + \tilde{\rho} \cdot b_{(1)}(\tilde{x}) + \tilde{\rho}^2 \cdot b_{(2)}(\tilde{x}) + \tilde{\rho}^2 \cdot t^b, \quad (5.85)
\]

where \(\sigma(\tilde{x}) \in C^{M+1}\) on \(\mathscr{F}\) and \(s\) and \(t\) are vertical (with respect to \(\tilde{\rho}\)) tensor fields satisfying

\[
s \rightarrow M_0 - 1 \ 0, \quad t \rightarrow M_0 - 1 \ 0. \quad (5.86)
\]

One may determine the coefficients \(a_{(k)}\) and \(b_{(k)}\) order-by-order by imposing the condition that the Fefferman–Graham gauge is preserved, i.e. that there exists some vertical what respect to \(\tilde{\rho}\) tensor field \(g\) such that

\[
g = \tilde{\rho}^{-2} \left[d\tilde{\rho}^2 + \tilde{g}_{ab} d\tilde{x}^a d\tilde{x}^b\right]. \quad (5.87)
\]

At leading order, one finds

\[
a_{(2)} = 0, \quad a_{(3)} = -\frac{1}{4} e^{-3\sigma} g^{bc} \mathcal{D}_b \mathcal{D}_c \sigma, \quad (5.88)
\]

\[
b_{(1)} = 0, \quad b_{(2)} = \frac{1}{2} e^{-2\sigma} g^{bc} \mathcal{D}_c \sigma. \quad (5.89)
\]

One may use this to compute

\[
\frac{\partial}{\partial \tilde{\rho}} = \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} + \frac{\partial \tilde{x}^b}{\partial \tilde{\rho}} \frac{\partial}{\partial \tilde{x}^b}, \quad \frac{\partial}{\partial \tilde{x}^a} = \frac{\partial}{\partial \tilde{x}^a} \frac{\partial}{\partial \tilde{\rho}} + \frac{\partial x^b}{\partial \tilde{x}^a} \frac{\partial}{\partial x^b}, \quad (5.90)
\]

and thus derive (to leading order) the transformations of the vertical components of the Weyl curvature under the above change of coordinates. For example, one finds

\[
\frac{\partial \rho}{\partial \tilde{x}^a} = \mathcal{O}_{M_0 - 2}(\rho), \quad \frac{\partial x^b}{\partial \tilde{x}^a} = \delta^b_a + \mathcal{O}_{M_0 - 2}(\rho^2), \quad (5.91)
\]

which implies that

\[
\tilde{W}_{abcd}^0 = (\rho e^\sigma + \mathcal{O}_{M_0 - 2}(\rho^2))^2 \cdot W \left( \mathcal{O}_{M_0 - 2}(\rho) \frac{\partial}{\partial \rho} + (1 + \mathcal{O}_{M_0 - 2}(\rho^2)) \frac{\partial}{\partial x^a} \right) \frac{\partial}{\partial x^b},
\]

\[
\mathcal{O}_{M_0 - 2}(\rho) \frac{\partial}{\partial \rho} + (1 + \mathcal{O}_{M_0 - 2}(\rho^2)) \frac{\partial}{\partial x^c},
\]

\[
\mathcal{O}_{M_0 - 2}(\rho) \frac{\partial}{\partial \rho} + (1 + \mathcal{O}_{M_0 - 2}(\rho^2)) \frac{\partial}{\partial x^d}
\]

\[
= e^{2\sigma} W_{abcd}^0 + \mathcal{O}_{M_0 - 2}(\rho^2; W^0) + \mathcal{O}_{M_0 - 2}(\rho; W^1) + \mathcal{O}_{M_0 - 2}(\rho^2; W^2). \quad (5.92)
\]

Similarly, one finds

\[
\tilde{W}_{abc}^1 = e^\sigma W_{abc}^1 + \mathcal{O}_{M_0 - 2}(\rho; W^0) + \mathcal{O}_{M_0 - 2}(\rho^2; W^1) + \mathcal{O}_{M_0 - 2}(\rho; W^2), \quad (5.93)
\]
\[ W_{ab} = W_{ab}^{0} + \mathcal{O}_{M_0 - 2}(\rho^2; W^0) + \mathcal{O}_{M_0 - 2}(\rho; W^1) + \mathcal{O}_{M_0 - 2}(\rho^2; W^2). \]  

By Proposition 5.8, the conditions (5.68) and (5.69) imply (5.58)–(5.60). Substituting this into the above, one may deduce the existence of vertical tensor fields \( \tilde{r}_0, \tilde{r}_1 \) and \( \tilde{r}_2 \) for which

\[ \tilde{W}^0 = \rho^{n-2} \cdot \tilde{r}_0, \quad \tilde{r}_0 \to M_0 - n 0, \]  
\[ \tilde{W}^1 = \rho^{n-1} \cdot \tilde{r}_1, \quad \tilde{r}_1 \to M_0 - n 0, \]  
\[ \tilde{W}^2 = \rho^{n-2} \cdot \tilde{r}_2, \quad \tilde{r}_2 \to M_0 - n 0. \]

Since Proposition 5.8 also applies with respect to the new coordinate system, one may conclude that

\[ \tilde{g}^{(0)} \text{ is conformally flat, and} \]  
\[ \tilde{g}^{(n)} = \begin{cases} \frac{1}{4} \tilde{g}^{-1} \cdot \tilde{p} \cdot \tilde{p}, & n = 4, \\ 0, & n \neq 4, \end{cases} \]  

as required. \( \square \)

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A. Proofs of Conversion Formulae

Given a mixed tensor field \( A \) of rank \((\kappa, \lambda; k, l)\), Propositions 3.2 and 3.4 give the following formula for the mixed derivative in terms of \( \varphi \)- and \( \varphi \)-coordinates:

\[
\bar{\nabla}_\gamma A^\alpha_\beta \bar{a}_\bar{b} = \partial_\gamma (A^\alpha_\beta \bar{a}_\bar{b}) + \sum_{i=1}^{\kappa} \Gamma_{\gamma \delta}^\alpha_i A^\hat{\alpha}_\hat{\delta} \hat{\beta} \bar{a}_\bar{b} - \sum_{j=1}^{\lambda} \Gamma_{\gamma \beta j}^\alpha A^\hat{\alpha}_j \hat{\beta} \bar{a}_\bar{b} \\
+ \sum_{i=1}^{k} \bar{\Gamma}_{\gamma d}^{a_i} A^\alpha_\beta \hat{a}_d \bar{b} - \sum_{j=1}^{l} \bar{\Gamma}_{\gamma b_j}^{d} A^\alpha_\beta \hat{b}_j \bar{d} ,
\]  

(A.1)

where the Christoffel symbols

\[ \nabla_\alpha \partial_\beta := \Gamma_{\alpha \beta \gamma} \partial_\gamma, \quad \bar{\nabla}_\alpha \partial_\bar{b} := \Gamma_{\alpha \bar{a} \bar{c}} \partial_\bar{c}, \]  

(A.2)
are given by
\[
\Gamma^\alpha_{\rho\rho} = -\rho^{-1}\delta^\alpha_\rho, \quad \Gamma^\rho_{\alpha\rho} = 0,
\]
\[
\Gamma^c_{\alpha\rho} = -\rho^{-1}\delta^c_\alpha + \frac{1}{2}g^{cd}\mathcal{L}_\rho g_{ad},
\]
\[
\Gamma^\rho_{\alpha\beta} - \Gamma^c_{\rho\alpha} = -\rho^{-1}\delta^c_\alpha,
\]
\[
\Gamma^\rho_{\alpha\beta} = \rho^{-1}g_{\alpha\beta} - \frac{1}{2}\mathcal{L}_\rho g_{\alpha\beta}, 
\]
\[
\Gamma^c_{\alpha\beta} - \Gamma^c_{\alpha\beta} = 0. \quad (A.3)
\]

**A.1. Proposition 3.7**

Suppose that \( A \) has rank \((k,l)\). (3.2) gives, with respect to any coordinates \((U, \varphi)\) on \( \mathcal{I} \),

\[
\mathcal{D}_\rho D_c A^\alpha_{\ b} = \mathcal{L}_\rho D_c A^\alpha_{\ b} - \frac{1}{2}g^{de} \mathcal{L}_\rho g_{cd} D_e A^\alpha_{\ b} + \frac{1}{2} \sum_{i=1}^k g^{a_i d} \mathcal{L}_\rho g_{de} D_c A^a_{\ b},
\]

\[
- \frac{1}{2} \sum_{j=1}^l g^{de} \mathcal{L}_\rho g_{b_j d} D_c A^\alpha_{\ b},
\]

\[
D_c(\mathcal{D}_\rho A)^\alpha_{\ b} = D_c(\mathcal{L}_\rho A)^\alpha_{\ b} + \frac{1}{2} \sum_{i=1}^k D_c (g^{a_i d} \mathcal{L}_\rho g_{de} A^a_{\ b},
\]

\[
- \frac{1}{2} \sum_{j=1}^l D_c (g^{de} \mathcal{L}_\rho g_{b_j d} A^\alpha_{\ b}). \quad (A.4)
\]

The difference of these equations reads

\[
\mathcal{D}_\rho D_c A^\alpha_{\ b} = D_c(\mathcal{D}_\rho A)^\alpha_{\ b} + \mathcal{L}_\rho D_c A^\alpha_{\ b} - D_c(\mathcal{L}_\rho A)^\alpha_{\ b} - \frac{1}{2}g^{de} \mathcal{L}_\rho g_{cd} D_e A^\alpha_{\ b}
\]

\[
- \frac{1}{2} \sum_{i=1}^k g^{a_i d} \mathcal{L}_\rho g_{cd} D_e A^a_{\ b} + \frac{1}{2} \sum_{j=1}^l g^{de} D_c \mathcal{L}_\rho g_{b_j d} A^\alpha_{\ b}. \quad (A.5)
\]

We apply the commutation formula contained in [18, Proposition 2.27] to obtain

\[
\mathcal{D}_\rho D_c A^\alpha_{\ b} = D_c(\mathcal{D}_\rho A)^\alpha_{\ b} - \frac{1}{2}g^{de} \mathcal{L}_\rho g_{cd} D_e A^\alpha_{\ b}
\]

\[
+ \frac{1}{2} \sum_{i=1}^k g^{a_i d}(D_c \mathcal{L}_\rho g_{cd} - D_d \mathcal{L}_\rho g_{ce}) A^a_{\ b},
\]

\[
- \frac{1}{2} \sum_{j=1}^l g^{de}(D_b j \mathcal{L}_\rho g_{cd} - D_d \mathcal{L}_\rho g_{cb}) A^\alpha_{\ b}. \quad (A.6)
\]

Combining (A.6) with (3.11) yields the identity (3.12), as required.

Next, recalling Definition 3.5 of \( \Box \), we expand (partially in \( \varphi_\rho \)-coordinates)

\[
\Box(\rho^p A) = g^{\alpha\beta} \nabla_\alpha (\rho^p \Box_\beta A + pp^{-1} \Box_\beta \rho \cdot A)
\]

\[
= \rho^p \Box_\alpha A + 2pp^{-1}g^{\alpha\beta} \nabla_\alpha \rho \Box_\beta A + p(p-1)\rho^{p-2}g^{\alpha\beta} \nabla_\alpha \rho \nabla_\beta \rho A
\]

\[
+ pp^{-1} \Box_\rho A.
\]
The formulae in (A.3) imply that
\[ \Box \rho = -\rho^2 \Gamma^\rho_{\rho \rho} - \rho^2 g^{ab} \Gamma^\rho_{ab} \]
\[ = -(n-1)\rho + \frac{1}{2} \rho^2 g^{ab} \mathcal{L}_\rho g_{ab}. \]  

(A.8)

Substituting (A.8) into (A.7) yields
\[ \Box (\rho^p A) = \rho^p \Box A + 2p \rho^{p+1} \tilde{\Box}_\rho A - p(n-p)\rho^p A + \frac{1}{2} p \rho^p g^{ab} \mathcal{L}_\rho g_{ab} \tilde{A}. \]  

(A.9)

Combining (A.9) with (3.11) immediately implies (3.13), as required.

\[ \square \]

\[ \textbf{A.2. Proposition 3.9} \]

Assume all indices are with respect to \( \varphi \)- and \( \varphi_\rho \)-coordinates and let \( \Gamma \) and \( \tilde{\Gamma} \) be the corresponding Christoffel symbols defined (A.3). For future convenience, we set \( l := r_1 + r_2 \) and define (via local coordinates) the vertical tensor fields
\[ k_{ac} := \rho^{-1} g_{ac} - \frac{1}{2} \mathcal{L}_\rho g_{ac}, \quad k^b_a := \rho^{-1} \delta^b_a - \frac{1}{2} g^{bc} \mathcal{L}_\rho g_{ac}. \]  

(A.10)

By the definitions of \( \nabla \) and \( \tilde{\nabla} \), we have
\[ \nabla_\rho A_{\hat{p} \hat{a}} = \partial_\rho (A_{\hat{p} \hat{a}}) - \sum_{i=1}^{r_1} \Gamma^\rho_{\hat{p} \rho} A_{\hat{p} \hat{a}} - \sum_{j=1}^{r_2} \Gamma^b_{\rho a_j} A_{\hat{p} \hat{a},[b]}, \]
\[ \tilde{\nabla}_\rho A_{\hat{a}} = \partial_\rho (A_{\hat{a}}) - \sum_{j=1}^{r_2} \tilde{\Gamma}^b_{\rho a_j} A_{\hat{a},[b]}. \]

Subtracting these equations and applying (3.14) and (A.3) yields
\[ \nabla_\rho A_{\hat{p} \hat{a}} = \tilde{\nabla}_\rho A_{\hat{a}} - \sum_{i=1}^{r_1} \Gamma^\rho_{\hat{p} \rho} A_{\hat{a}} - \sum_{j=1}^{r_2} (\Gamma^b_{\rho a_j} - \tilde{\Gamma}^b_{\rho a_j}) A_{\hat{a},[b]} \]
\[ = \tilde{\nabla}_\rho A_{\hat{a}} + r_1 \rho^{-1} A_{\hat{a}} + r_2 \rho^{-1} A_{\hat{a}} \]
\[ = \tilde{\nabla}_\rho A_{\hat{a}} + (r_1 + r_2) \rho^{-1} A_{\hat{a}}, \]
from which (3.20) follows.

Similarly, the definitions of \( \nabla \) and \( \tilde{\nabla} \) imply
\[ \nabla_{\hat{c}} A_{\hat{p} \hat{a}} = \partial_{\hat{c}} (A_{\hat{p} \hat{a}}) - \sum_{i=1}^{r_1} \Gamma^b_{\hat{c} \rho} A_{\hat{p} [\hat{a} \hat{b}] \hat{a}} - \sum_{j=1}^{r_2} \Gamma^\beta_{\hat{c} a_j} A_{\hat{p} \hat{a},[\beta]}, \]
\[ \tilde{\nabla}_{\hat{c}} A_{\hat{a}} = \partial_{\hat{c}} (A_{\hat{a}}) - \sum_{j=1}^{r_2} \tilde{\Gamma}^b_{\hat{c} a_j} A_{\hat{a},[b]}. \]

Subtracting the above equations and recalling (3.15), (3.16), and (A.3), we obtain
\[ \nabla_{\hat{c}} A_{\hat{p} \hat{a}} = \tilde{\nabla}_{\hat{c}} A_{\hat{a}} - \sum_{i=1}^{r_1} \Gamma^b_{\hat{c} \rho} A_{\hat{p} ([\hat{a} \hat{b}] \hat{a})} - \sum_{j=1}^{r_2} \Gamma^\beta_{\hat{c} a_j} A_{\hat{p} \hat{a},[\beta]} \]
\[ = \tilde{\nabla}_{\hat{c}} A_{\hat{a}} - \sum_{i=1}^{r_1} \Gamma^b_{\hat{c} \rho} A_{\hat{p} \hat{a}} - \sum_{j=1}^{r_2} \Gamma^\beta_{\hat{c} a_j} A_{\hat{p} \hat{a},[\rho]} \]  

(A.11)
\[ = \bar{\mathcal{D}}_a A_{\bar{a}} + \sum_{i=1}^{r_1} k^b_c (A^\rho_i)_{\bar{b} \bar{a}} - \sum_{j=1}^{r_2} k_{ca_j} (A^\nu_j)_{\bar{a}_j}. \quad (A.12) \]

Combining the above with (3.11) and (A.10) yields (3.21).

For (3.22), we start by computing \( \rho \)-derivatives. By (A.1),

\[ \nabla_{\rho \rho} A_{\bar{b} \bar{a}} = \partial_\rho (\nabla_\rho A_{\bar{b} \bar{a}}) - \Gamma^\rho_{\rho \rho} \nabla_\rho A_{\bar{b} \bar{a}} = \sum_{i=1}^{r_1} \Gamma^\rho_{\rho \rho} \nabla_\rho A_{\bar{b} \bar{a}} - \sum_{j=1}^{r_2} \Gamma^b_{\rho a_j} \nabla_\rho A_{\bar{b} \bar{a},[j]}, \]

\[ = \partial_\rho (\nabla_\rho A_{\bar{b} \bar{a}}) + (r_1 + 1) \rho^{-1} \nabla_\rho A_{\bar{b} \bar{a}} - \sum_{j=1}^{r_2} \Gamma^b_{\rho a_j} \nabla_\rho A_{\bar{b} \bar{a},[j]}, \quad (A.13) \]

Similarly, for the corresponding mixed derivatives, we apply (A.1) and compute

\[ \rho^{-l} \nabla_{\rho \rho} (\rho A^l_{\bar{a}}) = \rho^{-l} \partial_\rho [\bar{D}_\rho (\rho A^l_{\bar{a}})] - \Gamma_{\rho \rho}^\rho \rho^{-l} \bar{D}_\rho (\rho A^l_{\bar{a}}) - \sum_{j=1}^{r_2} \Gamma_{\rho a_j}^b \rho^{-l} \bar{D}_\rho (\rho A^l_{\bar{a},[j]}) \]

\[ = \partial_\rho [\rho^{-l} \bar{D}_\rho (\rho A^l_{\bar{a}})] + (l + 1) \rho^{-1} \rho^{-l} \bar{D}_\rho (\rho A^l_{\bar{a}}) \]

\[ - \sum_{j=1}^{r_2} \Gamma_{\rho a_j}^b \rho^{-l} \bar{D}_\rho (\rho A^l_{\bar{a},[j]}), \quad (A.14) \]

where we also applied (A.3) and the properties contained in Proposition 3.4. Subtracting (A.14) from (A.13), while applying both (A.3) and (3.20), we obtain that

\[ \nabla_{\rho \rho} A_{\bar{b} \bar{a}} = \rho^{-l} \nabla_{\rho \rho} (\rho A^l_{\bar{a}}) + (r_1 - l) \rho^{-1} \rho^{-l} \bar{D}_\rho (\rho A^l_{\bar{a}}) \]

\[ - \sum_{j=1}^{r_2} (\Gamma_{\rho a_j}^b - \Gamma_{\rho a_j}^b) \rho^{-l} \bar{D}_\rho (\rho A^l_{\bar{a},[j]} \]

\[ = \rho^{-l} \nabla_{\rho \rho} (\rho A^l_{\bar{a}}) + (r_1 - l) \rho^{-1} \rho^{-l} \bar{D}_\rho (\rho A^l_{\bar{a}}) + r_2 \rho^{-l} \rho^{-l} \bar{D}_\rho (\rho A^l_{\bar{a}}) \]

\[ = \rho^{-l} \nabla_{\rho \rho} (\rho A^l_{\bar{a}}). \quad (A.15) \]

Next, we apply (A.1) to compute

\[ \nabla_{bc} A_{\bar{b} \bar{a}} = \partial_b (\nabla_c A_{\bar{b} \bar{a}}) - \Gamma^c_{bc} \nabla_\alpha A_{\bar{b} \bar{a}} - \sum_{i=1}^{r_1} \Gamma^c_{\bar{b} \bar{d}} \nabla_c A_{\bar{b} \bar{d},[\bar{a}]} - \sum_{j=1}^{r_2} \Gamma^{\bar{c}}_{\bar{b} \bar{a_j}} \nabla_c A_{\bar{b} \bar{a}_j,[\bar{a}]} \]

\[ \rho^{-l} \nabla_{bc} (\rho A^l_{\bar{a}}) = \partial_b (\nabla_c A_{\bar{a}}) - \Gamma^c_{bc} \rho^{-l} \bar{D}_\alpha (\rho A^l_{\bar{a}}) - \sum_{j=1}^{r_2} \Gamma^{\bar{c}}_{\bar{b} \bar{a_j}} A_{\bar{c} \bar{a}_j,[\bar{a}]} \]

Subtracting the two equations and recalling (A.3) yields

\[ \nabla_{bc} A_{\bar{b} \bar{a}} = \rho^{-l} \nabla_{bc} (\rho A^l_{\bar{a}}) + \partial_b (\nabla_c A_{\bar{b} \bar{a}} - \bar{D} c A_{\bar{a}}) - \Gamma^c_{bc} [\nabla_\alpha A_{\bar{b} \bar{a}} - \rho^{-l} \bar{D}_\alpha (\rho A^l_{\bar{a}}) \]

\[ - \sum_{i=1}^{r_1} \Gamma^c_{\bar{b} \bar{d}} \nabla_c A_{\bar{b} \bar{d},[\bar{a}]} - \sum_{j=1}^{r_2} \Gamma^{\bar{c}}_{\bar{b} \bar{a}_j} \nabla_c A_{\bar{b} \bar{a}_j,[\bar{a}]} - \sum_{j=1}^{r_2} \Gamma^{\bar{c}}_{\bar{b} \bar{a}_j} \nabla_c A_{\bar{b} \bar{a}_j,[\bar{a}]}, \]

\[ := \rho^{-l} \nabla_{bc} (\rho A^l_{\bar{a}}) + I_1 + I_2 + I_3 + I_4 + I_5. \quad (A.16) \]
To simplify the upcoming computations, we define, for all $1 \leq i \leq r_1$ and $1 \leq j \leq r_2$, the vertical tensor fields $z, z^c_i, z^v_j$—of ranks $(0, r_2 + 1), (0, r_2 + 2), (0, r_2)$, respectively—via the index formulae

$$z_{c\bar{a}} := \nabla_{c} A_{\rho \bar{a}} - \breve{D}_{c} A_{\bar{a}},$$

$$(z^c_i)_{cb\bar{a}} := \nabla_{c} A_{\rho [b|\bar{a}} - \breve{D}_{c} (A_{\rho}^c)_b \bar{a},$$

$$(z^v_j)_{c\bar{a}j} := \nabla_{c} A_{\rho \bar{a} [\rho] - \breve{D}_{c} (A_{\rho}^v)_{\bar{a}}}, \quad (A.17)$$

Applying (3.20), (A.3), and (A.17) to the term $I_2$ from (A.16), we obtain

$$I_2 = -k_{bc}[\nabla_{\rho} A_{\rho \bar{a}} - \rho^{-1} \breve{D}_{\rho} (\rho^l A)^{\bar{a}}] - \Gamma_{bc}^{d}(\nabla_{d} A_{\rho \bar{a}} - \breve{D}_{d} A_{\bar{a}})$$

$$= -\Gamma_{bc}^{d}(\nabla_{d} A_{\rho \bar{a}} - \breve{D}_{d} A_{\bar{a}}).$$

From (A.16), the first part of (A.17), and the above, we see that

$$I_1 + I_2 + I_5 = \breve{D}_{b} z_{c\bar{a}}. \quad (A.18)$$

Similarly, for $I_3$ and $I_4$, we again apply (A.3) and (A.17):

$$I_3 = \sum_{i=1}^{r_1} k_{d b}^{d} \breve{D}_{c} (A_{i}^{\rho})_{d \bar{a}} + \sum_{i=1}^{r_1} k_{d b}^{d} (z^c_i)_{cd\bar{a}},$$

$$I_4 = -\sum_{j=1}^{r_2} k_{a_j b} \breve{D}_{c} (A_{j}^{v})_{a_j} - \sum_{j=1}^{r_2} k_{a_j b} (z^v_j)_{c\bar{a} j}. \quad (A.19)$$

Now, recalling (3.21), along with (A.17), we deduce

$$z_{c\bar{a}} = \sum_{i=1}^{r_1} k_{c e}^{e} (A_{i}^{\rho})_{e\bar{a}} - \sum_{j=1}^{r_2} k_{a_j c} (A_{j}^{v})_{a_j},$$

$$(z^c_i)_{cd\bar{a}} = \sum_{1 \leq j \leq r_1} k_{c e}^{e} (A_{i,j}^{\rho})_{e d \bar{a}} - \sum_{j=1}^{r_2} k_{a_j c} (A_{i,j}^{v})_{d a_j} - k_{cd} A_{\bar{a}},$$

$$(z^v_j)_{c\bar{a} j} = \sum_{i=1}^{r_1} k_{c e}^{e} (A_{i,j}^{\rho,v})_{e a_j} + k_{c e} A_{a_j [e]} - \sum_{1 \leq i \leq r_2} k_{a,i c} (A_{i,j}^{v,v})_{a_i,j}. \quad (A.20)$$

Combining (3.11), (A.10), (A.18), and the above, we conclude that

$$I_1 + I_2 + I_5 = \sum_{i=1}^{r_1} [k_{c e}^{e} \breve{D}_{b} (A_{i}^{\rho})_{e\bar{a}} + \breve{D}_{b} k_{d b}^{d} (A_{i}^{\rho})_{e\bar{a}}]
- \sum_{j=1}^{r_2} [k_{a_j c} \breve{D}_{b} (A_{j}^{v})_{a_j} + \breve{D}_{b} k_{a_j c} (A_{j}^{v})_{a_j}]
= \rho^{-1} \sum_{i=1}^{r_1} \breve{D}_{b} A_{i}^{\rho} c\bar{a} - \rho^{-1} \sum_{j=1}^{r_2} g_{a_j c} \breve{D}_{b} (A_{j}^{v})_{a_j} + \sum_{i=1}^{r_1} \breve{G}_{M-2}(\rho; \breve{D} A_{i}^{\rho})_{cb\bar{a}}
+ \sum_{j=1}^{r_2} \breve{G}_{M-2}(\rho; \breve{D} A_{j}^{v})_{cb\bar{a}} + \sum_{i=1}^{r_1} \breve{G}_{M-2}(1; A_{i}^{\rho})_{cb\bar{a}}.$$
\[ + \sum_{j=1}^{r_2} \mathcal{O}_{M-2}(1; A_j^v)_{cb\bar{a}}. \]  

 Similar computations using (A.19) yield

\[ I_3 = \sum_{i=1}^{r_1} k^d_b D_c(A_i^\rho)_{d\bar{a}} + 2 \sum_{1 \leq i < j \leq r_1} k^d_b k^e_c (A_{i,j}^{\rho, v})_{cd\bar{a}} \]
\[ - \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} k^d_b k_{a,j,c} (A_{i,j}^{\rho, v})_{d\bar{a} j} - r_1 k^d_b k_{dc} A_{\bar{a}} \]
\[ = \rho^{-1} \sum_{i=1}^{r_1} \bar{D}_c(A_i^\rho)_{b\bar{a}} - r_1 \rho^{-2} g_{bc} A_{\bar{a}} + 2 \rho^{-2} \sum_{1 \leq i < j \leq r_1} (A_{i,j}^{\rho, v})_{cb\bar{a}} \]
\[ - \rho^{-2} \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} g_{a,j,c} (A_{i,j}^{\rho, v})_{b\bar{a} j}, \]
\[ + \sum_{i=1}^{r_1} \mathcal{O}_{M-2}(\rho; \bar{D}A_i^v)_{cb\bar{a}} + \mathcal{O}_{M-2}(1; A)_{cb\bar{a}} + \sum_{1 \leq i < j \leq r_1} \mathcal{O}_{M-2}(1; A_{i,j}^{\rho, v})_{cb\bar{a}} \]
\[ + 2 \rho^{-2} \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} \mathcal{O}_{M-2}(1; A_{i,j}^{\rho, v})_{cb\bar{a} j}, \]

and

\[ I_4 = - \sum_{j=1}^{r_2} k_{a,j,b} \bar{D}_c(A_j^v)_{a\bar{j}} - \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} k_{a,j,b} k^e_c (A_{i,j}^{\rho, v})_{e\bar{a} j} - \sum_{j=1}^{r_2} k_{a,j,b} k^e_c A_{\bar{a} j[c]} \]
\[ + 2 \sum_{1 \leq i < j \leq r_2} k_{a,c} k_{a,b} (A_{i,j}^{\rho, v})_{a\bar{i} j} \]
\[ = -\rho^{-1} \sum_{j=1}^{r_2} g_{a,j,b} \bar{D}_c(A_j^v)_{a\bar{j}} - \rho^{-2} \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} g_{a,j,b} (A_{i,j}^{\rho, v})_{c\bar{a} j} - \rho^{-2} \sum_{j=1}^{r_2} g_{a,j,b} A_{\bar{a} j[c]} \]
\[ + 2 \rho^{-2} \sum_{1 \leq i < j \leq r_2} g_{a,c} g_{a,b} (A_{i,j}^{\rho, v})_{a\bar{i} j} + \sum_{j=1}^{r_2} \mathcal{O}_{M-2}(\rho; DA_j^v)_{cb\bar{a}} \]
\[ + \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} \mathcal{O}_{M-2}(1; A_{i,j}^{\rho, v})_{cb\bar{a}} + \mathcal{O}_{M-2}(1; A)_{cb\bar{a}} + \sum_{1 \leq i < j \leq r_2} \mathcal{O}_{M-2}(1; A_{i,j}^{\rho, v})_{cb\bar{a}}. \]

Finally, combining (A.16), (A.21), and the above, we obtain

\[ g^{bc} \nabla_{bc} A_{\bar{a}} = \rho^{-1} g^{bc} \bar{\nabla}_{bc} (\rho A)_{\bar{a}} + 2 \rho^{-1} \left[ \sum_{i=1}^{r_1} g^{bc} \bar{D}_b(A_i^v)_{c\bar{a}} \right. \]
\[ - \sum_{j=1}^{r_2} D_{a,j}(A_j^v)_{\bar{a} j} - (nr_1 + r_2) \rho^{-2} A_{\bar{a}} \]
\[ + 2 \rho^{-2} \left[ \sum_{1 \leq i < j \leq r_1} g^{bc} (A_{i,j}^{\rho, v})_{cb\bar{a}} - \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} (A_{i,j}^{\rho, v})_{a_j\bar{a} j} \right], \]
In this section, we derive the precise forms of the vertical tensor fields $Q$, Appendix B. (3.22) now follows from (A.15), the above, and the fact that

\[ \square A = \rho^2 (\nabla_{\rho A} + g^{bc} \nabla_{bc} A), \]

\[ \rho^{-1} \square (\rho^1 A) = \rho^2 [\rho^{-l} \nabla_{\rho \rho} (\rho^1 A) + g^{bc} \rho^{-l} \nabla_{bc} (\rho^1 A)]. \]

**Appendix B. Q⁰, Q¹ and Q²**

In this section, we derive the precise forms of the vertical tensor fields $Q^0$, $Q^1$, $Q^2$ appearing in Proposition 4.15. Let $Q_{\alpha\beta\gamma\delta}$ denote the right-hand side of (4.8):

\[ Q_{\alpha\beta\gamma\delta} := 4W^{\lambda}_{\alpha}[\delta] W^{\rho}_{\lambda\beta\mu}[\gamma] - W^{\lambda\mu}_{\gamma\delta} W^{\rho\gamma\delta}_{\alpha\beta\lambda\mu} \]

\[ = g^{\lambda\nu} g^{\mu\sigma} (4W_{\nu\alpha\sigma}[\delta] W^{\rho}_{\lambda\beta\mu}[\gamma] - W^{\rho\nu\sigma}_{\lambda\gamma\delta} W^{\rho\gamma\delta}_{\alpha\beta\lambda\mu}) \]

\[ = \rho^4 (4W_{\rho\alpha\rho}[\delta] W^{\rho}_{\lambda\beta\rho}[\gamma]) + \rho^4 g^{\epsilon\nu} (4W_{\rho\alpha f}[\delta] W^{\rho}_{\beta\epsilon\gamma}[\gamma] - W^{\rho\epsilon\nu}_{\beta\gamma\delta} W^{\rho\gamma\delta}_{\alpha\beta\rho\epsilon}) + \rho^4 g^{\epsilon\nu} g^{\rho\gamma} (4W_{f\alpha h}[\delta] W^{\rho}_{\epsilon\beta\gamma}[\gamma] - W^{\rho h\gamma}_{f\gamma\delta} W^{\rho\gamma\delta}_{\alpha\beta\epsilon\rho}) \]

where we have applied the Fefferman–Graham gauge condition (1.7). $Q^0$, $Q^1$ and $Q^2$ are given by

\[ Q^0_{abcd} = Q_{abcd}, \quad Q^1_{bcd} = Q_{\rho c d}, \quad Q^2_{bd} = Q_{\rho b d}. \] (B.3)

In particular, one finds for $Q^0$:

\[ Q^0_{abcd} = 4W^2_{a[d} W^2_{c]b} + 2g^{\epsilon\nu} \left[ 2W^1_{[a|f_a w^1_{c]eb} + 2W^1_{a|f} W^1_{b|c} - W^1_{f c d} W^1_{e a b} \right] \]

\[ + g^{\epsilon\nu} g^{\rho\gamma} \left[ 2 \left( \hat{W}^0_{f a h d} - \frac{2}{n-2} (g_{f[h} W^2_{d]a} + g_{a|d} W^2_{h]f}) \right) \right. \]

\[ \left. \hat{W}^0_{e b g c} + \frac{2}{n-2} \left( g_{[g[e} W^2_{c]b} + g_{b[e} W^2_{g]c} \right) \right) \]

\[ - 2 \left( \hat{W}^0_{f a h c} - \frac{2}{n-2} \left( g_{f[h} W^2_{c]a} + g_{a|c} W^2_{h]f} \right) \right) \]

\[ \left( \hat{W}^0_{e b g d} + \frac{2}{n-2} \left( g_{[g[e} W^2_{d]b} + g_{b[d} W^2_{g]e} \right) \right) \]
\[
- \left( \hat{W}_{fhd}^0 - \frac{2}{n-2} \left( g_{f[c} W_{d]h}^2 + g_{h[d} W_{c]f}^2 \right) \right) \\
\left( \hat{W}_{abeg}^0 + \frac{2}{n-2} \left( g_{a[e} W_{g]b}^2 + g_{b[g} W_{e]a}^2 \right) \right)
\]
\[
= \mathcal{O}_{M_0-2} \left( ; \hat{W}^0 \right)_{abcd} + \mathcal{O}_{M_0-3} \left( \rho; W^1 \right)_{abcd} + \mathcal{O}_{M_0-2} \left( 1; W^2 \right)_{abcd},
\]

for $Q^1$:
\[
Q^1_{bcd} = -2 g^{ef} \left[ W_{fcd}^1 W_{be}^2 + 2 W_{f[d}^2 W_{e]c}^1 \right] \\
- g^{ef} g^{gh} \left[ 2 W_{fhd}^1 \left( \hat{W}_{ebgc}^0 - \frac{2}{n-2} \left( g_{e[g} W_{c]b}^2 + g_{b[d} W_{g]c}^2 \right) \right) \right] \\
- 2 W_{fhc}^1 \left( \hat{W}_{ebgd}^0 - \frac{2}{n-2} \left( g_{e[g} W_{d]b}^2 + g_{b[d} W_{g]c}^2 \right) \right) \\
+ \left( \hat{W}_{fhd}^0 + \frac{2}{n-2} \left( g_{f[c} W_{d]h}^2 + g_{h[d} W_{c]f}^2 \right) \right) W_{beg}^1
\]
\[
= \mathcal{O}_{M_0-3} \left( \rho; \hat{W}^0 \right)_{bcd} + \mathcal{O}_{M_0-2} \left( 1; W^1 \right)_{bcd} + \mathcal{O}_{M_0-3} \left( \rho; W^2 \right)_{bcd},
\]

and, finally, for $Q^2$:
\[
Q^2_{bd} = -2 g^{ef} W_{f[d}^2 W_{eb}^2 + g^{ef} g^{gh} \left[ 2 W_{fhd}^1 W_{geb}^1 - W_{dfh}^1 W_{beg}^1 \right] \\
- W_{fh}^2 \left( \hat{W}_{ebgd}^0 - \frac{2}{n-2} \left( g_{e[g} W_{d]b}^2 + g_{b[d} W_{g]c}^2 \right) \right) \\
= \mathcal{O}_{M_0-2} \left( 1; \hat{W}^0 \right)_{bd} + \mathcal{O}_{M_0-3} \left( \rho; W^1 \right)_{bd} + \mathcal{O}_{M_0-2} \left( 1; W^2 \right)_{bd},
\]
in which we have applied (4.28), (4.29) and (4.30).

References

[1] Alinhac, S., Baouendi, M.S.: A non uniqueness result for operators of principal type. Math. Z. 220, 561–568 (1995)
[2] Anderson, M.T., Herzlich, M.: Unique continuation results for Ricci curvature and applications. J. Geom. Phys. 58(2), 179–207 (2008)
[3] Biquard, O.: Continuation unique à partir de l’infini conforme pour les métriques d’Einstein. Math. Res. Lett. 15(6), 1091–1099 (2008)
[4] Boucher, W., Gibbons, G.W., Horowitz, G.T.: Uniqueness theorem for anti-de Sitter spacetime. Phys. Rev. D 30(12), 2447–2451 (1984)
[5] Carroll, S.M.: Spacetime and Geometry. Cambridge University Press, Cambridge (2004)
[6] Chatzikaleas, A., Shao, A.: A gauge-invariant unique continuation criterion for waves in asymptotically anti-de sitter spacetimes. Commun. Math. Phys. (2022)
[7] Chruściel, P.T., Delay, E.: Unique continuation and extensions of Killing vectors at boundaries for stationary vacuum space-times. J. Geom. Phys. 61(8), 1249–1257 (2011)

[8] De Haro, S., Skenderis, K., Solodukhin, S.N.: Holographic Reconstruction of Spacetime and Renormalization in the AdS/CFT Correspondence. Commun. Math. Phys. 217(3), 595–622 (2001)

[9] Fefferman, C., Graham, C. R.: Conformal invariants. Élie Cartan et les mathématiques d’aujourd’hui–Lyon, 25–29 juin 1984 S131 (1985), pp. 95–116

[10] Graham, R. C.: Volume and area renormalizations for conformally compact Einstein metrics. In: Proceedings of the 19th Winter School “Geometry and Physics”. Palermo: Circolo Matematico di Palermo, 2000, pp. 31–42

[11] Guisset, S.: Construction of counter-examples to the wave equation unique continuation problem with a critically singular potential. In preparation

[12] Holzegel, G., Shao, A.: Unique continuation from infinity in asymptotically anti-de Sitter spacetimes. Commun. Math. Phys. 347(3), 723–775 (2016)

[13] Holzegel, G., Shao, A.: Unique continuation from infinity in asymptotically anti-de Sitter spacetimes II: Non-static boundaries. Commun. Partial Differ. Equ. 42(12), 1871–1922 (2017)

[14] Imbimbo, C., et al.: Diffeomorphisms and holographic anomalies. Class. Quantum Gravity 17(5), 1129–1138 (2000)

[15] Maldacena, J.: Int. J. Theor. Phys. 38(4), 1113–1133 (1999)

[16] McGill, A., Shao, A.: Null geodesics and improved unique continuation for waves in asymptotically anti-de Sitter spacetimes. Class. Quant. Gravity 38.5 (2020)

[17] McGill, A., Shao, A.: Holographic AdS Rigidity. In preparation

[18] Shao, A.: The near-boundary geometry of Einstein-vacuum asymptotically anti-de Sitter spacetimes. Class. Quant. Gravity 38(3) (2020)

[19] Skenderis, K.: Asymptotically Anti-de Sitter spacetimes and their stress energy tensor. Int. J. Mod. Phys. A 16(05), 740–749 (2001)

[20] Skenderis, K.: Lecture notes on holographic renormalization. Class. Quantum Gravity 19(22), 5849 (2002)

[21] Skenderis, K., Solodukhin, S.N.: Quantum effective action from the AdS/CFT correspondence. Phys. Lett. B 472(3–4), 316–322 (2000)

[22] Wang, X.: On the uniqueness of the AdS spacetime. Acta Math. Sinica 21(4), 917–922 (2005)

Alex McGill
School of Mathematical Sciences
Queen Mary University of London
London
UK
e-mail: a.mcgill@qmul.ac.uk

Communicated by Mihalis Dafermos.
Received: September 9, 2022.
Accepted: February 13, 2023.