AN OPTIMAL ALGORITHM FOR PRODUCT STRUCTURE IN PLANAR GRAPHS

Prosenjit Bose\textsuperscript{b} Pat Morin\textsuperscript{y} Saeed Odak\textsuperscript{\sigma}

Abstract. The Product Structure Theorem for planar graphs (Dujmovi´c et al. JACM, 67(4):22) states that any planar graph is contained in the strong product of a planar 3-tree, a path, and a 3-cycle. We give a simple linear-time algorithm for finding this decomposition as well as several related decompositions. This improves on the previous $O(n \log n)$ time algorithm (Morin. Algorithmica, 85(5):1544–1558).

1 Introduction

For two graphs $G$ and $X$, the notation $G \subseteq X$ denotes that $G$ is isomorphic to some subgraph of $X$. The following planar product structure theorems have recently been used as a key tool in resolving a number of longstanding open problems on planar graphs, including queue number [7], nonrepetitive chromatic number [9], adjacency labelling [8], universal graphs [10], $p$-centered colouring [5], and vertex ranking [4].\textsuperscript{1}

Theorem 1 (Dujmovi´c et al. [7], Ueckerdt, Wood, and Yi [16]). For any planar graph $G$, there exists:

(a) a planar graph $H$ of treewidth at most 3 and a path $P$ such that $G \subseteq H \Box P \Box K_3$ [7];
(b) a planar graph $H$ of treewidth at most 4 and a path $P$ such that $G \subseteq H \Box P \Box K_2$; and
(c) a planar graph $H$ of treewidth at most 6 and a path $P$ such that $G \subseteq H \Box P$ [16].

In each of the applications of Theorem 1, the proofs are constructive and lead to algorithms whose running-time is dominated by the time required to compute the relevant decomposition. The proofs of each part of Theorem 1 are constructive and lead to $O(n^2)$ time algorithms as observed already by Dujmovi´c et al. [7]. Morin [14] later showed that there exists an $O(n \log n)$ time algorithm to find the decomposition in Theorem 1.a. In the current note, we show that there exists a linear time algorithm for finding each of the three decompositions guaranteed by Theorem 1. This immediately gives an $O(n)$-time algorithm for each of the following problems on any $n$-vertex planar graph $G$:

- computing an $O(1)$-queue layout of $G$ [7];
- nonrepetitively vertex-colouring $G$ with $O(1)$ colours [9];
- assigning $(1 + o(1)) \log n$-bit labels to the vertices of $G$ so that one can determine from the labels of vertices $v$ and $w$ whether or not $v$ and $w$ are adjacent in $G$ [8];

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\textsuperscript{b}School of Computer Science, Carleton University
\textsuperscript{\sigma}Department of Computer Science and Electrical Engineering, University of Ottawa
\textsuperscript{1}In this paper, we will not be working directly with treewidth or the strong graph product ($\Box$), so we omit their definitions.
• mapping the vertices of $G$ into a universal graph $U_n$ that has $n^{1+o(1)}$ vertices and edges so that any pair of vertices that are adjacent in $G$ maps to a pair of vertices that are adjacent in $U_n$ [10];
• colouring the vertices of $G$ with $O(p^3 \log p)$ colours so that each connected subgraph $H$ of $G$ contains a vertex whose colour is unique in $H$ or contains vertices of at least $p + 1$ different colours [5]; and
• colouring the vertices of $G$ with $O(\log n/\log \log \log n)$ integers so that the maximum colour that appears on any path $P$ of length at most $\ell$ appears at exactly one vertex of $P$ (for any fixed $\ell \geq 2$) [4].

The remainder of this paper is organized as follows: Section 2 presents some necessary background and notation. Section 3 reviews the proof of Theorem 1.a. Section 4 describes the algorithms for finding the decompositions in Theorem 1.b and Theorem 1.c.

## 2 Preliminaries

Throughout this paper we use standard graph theory terminology as used in the textbook by Diestel [6]. All graphs discussed here are simple and finite. For a graph $G$, $V(G)$ and $E(G)$ denote the vertex and edge sets of $G$, respectively. We use the terms *vertex* and *node* interchangeably, though we typically refer to the vertices of some primary graph $G$ of interest and refer to the nodes of some auxiliary graph (such as a spanning tree) related to $G$. We say that a subgraph $G'$ of a graph $G$ *spans* a set $S \subseteq V(G)$ if $S \subseteq V(G')$.

**Quotient Graphs.** Given a graph $G$ and a partition $P$ of $V(G)$, the quotient graph $G/P$ is the graph with vertex set $V(G/P) := P$ and in which two nodes $X, Y \in V(G/P)$ are adjacent if $G$ contains at least one edge $xy$ with $x \in X$ and $y \in Y$.

**Embeddings, Planar Graphs, and (Near-)Triangulations.** An embedding $\psi$ of a graph $G$ associates each vertex $v$ of $G$ with a point $\psi(v) \in \mathbb{R}^2$ and each edge $vw$ of $G$ with a simple open curve $\psi(vw) : (0,1) \to \mathbb{R}^2$ whose endpoints\(^2\) are $\psi(v)$ and $\psi(w)$. We do not distinguish between such a curve $\psi(vw)$ and the point set $\{\psi(vw)(t) : 0 < t < 1\}$. We let $\psi(V(G)) := \{\psi(v) : v \in V(G)\}$, $\psi(E(G)) := \bigcup_{vw \in E(G)} \psi(vw)$, and $\psi(G) := \psi(V(G)) \cup \psi(E(G))$. An embedding $\psi$ of $G$ is *plane* if $\psi(vw) \cap \psi(V(G)) = \emptyset$ and $\psi(vw) \cap \psi(xy) = \emptyset$ for each distinct pair of edges $vw, xy \in E(G)$. A graph $G$ is *planar* if it has a plane embedding. A *triangulation* is an edge-maximal planar graph.

If $\psi$ is a plane embedding of a planar graph $G$, then we call the pair $(G, \psi)$ an *embedded graph* and we will not distinguish between a vertex $v$ of $G$ and the point $\psi(v)$ or between an edge $vw$ of $G$ and the curve $\psi(vw)$. Similarly, we will not distinguish between $G$ and the point set $\psi(G)$. Any cycle in an embedded graph defines a Jordan curve. For such a cycle $C$, $\mathbb{R}^2 \setminus C$ has two components, one bounded and the other unbounded. We will refer to the bounded component as the *interior* of $C$ and the unbounded component as the *exterior* of $C$. If $G$ is an embedded triangulation, then the subgraph of $G$ consisting of all edges and vertices of $G$ contained in the closure of the interior of $C$ is called a *near-triangulation*.

Each component of $\mathbb{R}^2 \setminus G$ is a *face* of $G$ and we let $F(G)$ denote the set of faces of $G$. If

\(^2\)The endpoints of an open curve $\psi : (0,1) \to \mathbb{R}^2$ are the two points $\lim_{\epsilon \downarrow 0} \psi(\epsilon)$ and $\lim_{\epsilon \downarrow 0} \psi(1-\epsilon)$.\)
G is 2-connected then, for any face \( f \in F(G) \), the set of vertices and edges of G contained in the boundary of \( f \) forms a cycle. We may therefore treat a face \( f \) of a 2-connected graph G as a component of \( \mathbb{R}^2 \setminus G \) or as the cycle of G on the boundary of \( f \), relying on context to distinguish between the two usages. Note that every embedded graph contains exactly one face—the outer face—that is unbounded.

**Duals and Cotrees.** The dual \( G^* \) of an embedded graph G is the graph with vertex set \( V(G^*) := F(G) \) and edge set \( E(G^*) := \{fg \in E^{\mathcal{F}(G)} : E(f) \cap E(g) \neq \emptyset\} \).\(^3\) If T is a spanning tree of G then the cotree \( \overline{T} \) of (G, T) is the graph with vertex set \( V(\overline{T}) := V(G^*) \) and edge set \( E(\overline{T}) := \{ab \in E(G^*) : E(a) \cap E(b) \setminus E(T) \neq \emptyset\} \). It is well known that, if G is connected, then \( \overline{T} \) is a spanning tree of \( G^* \).

For our purposes, a binary tree is a rooted tree of maximum degree 3 whose root has degree at most 2 and in which each child \( v \) of a node \( u \) is either the unique left child or the unique right child of \( u \). If G is a triangulation and we root \( \overline{T} \) at any face \( f_0 \in F(G) \) that contains an edge of \( T \), then \( \overline{T} \) is a binary tree, with the classification of left and right children determined by the embedding of G.\(^4\)

**Paths and Distances.** A path in G is a (possibly empty) sequence of vertices \( v_0, \ldots, v_r \), with the property that \( v_{i-1}, v_i \in E(G) \), for each \( i \in \{1, \ldots, r\} \). The endpoints of a path \( v_0, \ldots, v_r \), are the vertices \( v_0 \) and \( v_r \). The length of a non-empty path \( v_0, \ldots, v_r \) is the number, \( r \), of edges in the path.

**Trees, Depth, Ancestors, and Descendants.** Let T be a tree rooted at a vertex \( v_0 \in V(T) \). For any vertex \( w \in V(T) \), \( P_T(w) \) denotes the path in T from \( w \) to \( v_0 \). For any \( w_0 \in V(T) \), any prefix \( w_0, \ldots, w_r \) of \( P_T(w_0) \) is called an upward path in T; \( w_0 \) is the lower endpoint of this path and \( w_r \) is the upper endpoint. The T-depth of a node \( w \in V(T) \) is the length of the path \( P_T(w) \). The second node in \( P_T(v) \) (if any) is the T-parent of \( v \). A vertex \( a \in V(T) \) is a T-ancestor of \( w \in V(T) \) if \( a \in V(P_T(w)) \). If \( a \) is a T-ancestor of \( w \) then \( w \) is a T-descendant of \( a \).

**Lowest Common Ancestors.** For any two vertices \( v, w \in V(T) \), the lowest common ancestor \( \text{lca}_T(v, w) \) of \( v \) and \( w \) is the node \( a \) in \( P_T(v) \cap P_T(w) \) having maximum T-depth. The lowest common ancestor problem is a well-studied data structuring problem that asks to preprocess a given \( n \)-vertex rooted tree so that one can quickly return \( \text{lca}_T(v, w) \) for any two nodes \( v, w \in V(T) \). A number of optimal solutions to this problem exist that, after \( O(n) \) time preprocessing using \( O(n) \) space, can answer queries in \( O(1) \) time [1–3, 11, 13, 15]. The most recent work in this area includes simple and practical data structures that achieve this optimal performance [1, 2, 11].

**Reconstructing Binary Tree Models.** Let T be a binary tree and \( S \subseteq V(T) \). An upward path \( v_0, \ldots, v_r \) in a binary tree \( T \) is S-non-branching if \( v_i \) has degree 2 and \( v_i \notin S \) for each \( i \in \{1, \ldots, r-1\} \). For any binary tree \( T \) and set \( S \subseteq V(T) \), the model \( T' \) of \( T \) with respect to \( S \) is the binary tree obtained by replacing each maximal S-non-branching path \( v_0, \ldots, v_r \)

\(^3\) For a set \( S, \binom{S}{2} \) denotes the \( \binom{|S|}{2} \)-element set \( \binom{S}{2} := \{x, y : x, y \in S, x \neq y\} \).

\(^4\) There is a small ambiguity here when \( T \) contains two edges of \( f_0 \), in which case the unique child of \( f_0 \) in \( \overline{T} \) can be treated as the left or right child of \( f_0 \).
with the edge \(v_0v_r\); if \(v_{r-1}\) is the left (respectively, right) child of \(v_r\) then \(v_0\) becomes the left (respectively, right) child of \(v_r\).

**Lemma 2.** Let \(T\) be a binary tree, let \(S = \{x_1, \ldots, x_d\} \subseteq V(T)\), and let \(T_0\) be the minimal subtree of \(T\) that spans \(S\). Then there exists an algorithm that, given an \(O(1)\)-query time lowest common ancestor data structure for \(T\), computes the model \(T'_0\) of \(T_0\) with respect to \(S\) in \(O(d^2)\) time.

**Proof.** The proof is by induction on \(|S|\). The base case \(|S| = 1\) is trivial, since then \(T'_0 = T_0\) is the tree with one node, which is the unique element in \(S\).

If \(|S| \geq 2\), then the first step is to determine the root \(r\) of \(T_0\), which must also be the root of \(T'_0\). This is easily done by first setting \(r := x_1\) and then repeatedly setting \(r := \text{lca}_T(r, x_i)\) for each \(i \in \{1, \ldots, d\}\). This step takes \(O(d)\) time.

If \(r\) has no left child in \(T\), then we can immediately apply induction on \(S \setminus \{r\}\) and make the right child of \(r\) in \(T'_0\) the root of the model obtained by induction. The case in which \(r\) has no right child can be handled similarly. If \(r\) has both a left child \(r_1\) and a right child \(r_2\), then the next step is to partition \(S \setminus \{r\}\) into a set \(S_1\) of descendants of \(r_1\) and a set \(S_2\) of descendants of \(r_2\). For each \(x \in S \setminus \{r\}\) there are only two possibilities for \(\text{lca}_T(r_1, x)\)

1. If \(\text{lca}_T(r_1, x) = r_1\) then \(x \in S_1\).
2. If \(\text{lca}_T(r_1, x) = r\) then \(x \in S_2\).

Therefore, using \(O(d)\) lowest common ancestor queries, we can determine the root \(r\) of \(T'_0\) and partition \(S \setminus \{r\}\) into sets \(S_1\) and \(S_2\) that define the left and right subtrees of \(r\). We can now recurse on \(S_1\) to obtain a tree with root \(r'_1\) and recurse on \(S_2\) to obtain a tree with root \(r'_2\). We make \(r'_1\) the left child of \(r\) and \(r'_2\) the right child of \(r\) to obtain the model \(T'_0\) of \(T_0\).

The running-time of this algorithm obeys the recurrence

\[
T(d) \leq O(d) + T(d_1) + T(d_2),
\]

where \(d_1 + d_2 \leq d\) and \(d_1, d_2 \leq d - 1\). This recurrence resolves to \(T(d) \in O(d^2)\).

\[\square\]

## 3 Tripod Decompositions

Let \(G\) be an \(n\)-vertex triangulation and let \(T\) be a spanning tree of \(G\). For a face \(uvw\) of \(G\), a \((G, T)\)-tripod \(Y\) with crotch \(uvw\) is the vertex set of three disjoint (and each possibly empty) upward paths (the legs of \(Y\)) whose lower endpoints are \(u\), \(v\), and \(w\). A \((G, T)\)-tripod decomposition is a partition of \(V(G)\) into \((G, T)\)-tripods. Dujmović et al. [7] proved the following result:

**Theorem 3.** Let \(G\) be a triangulation and \(T\) be a spanning tree of \(G\). Then there exists a \((G, T)\)-tripod decomposition \(\mathcal{Y}\) such that \(G/\mathcal{Y}\) has treewidth at most 3.

It is straightforward to verify that Theorem 3 implies Theorem 1.a by first triangulating the given graph and then taking \(T\) to be a breadth-first spanning tree of the resulting triangulated graph [7, Observation 35].

### 3.1 Tripod Decompositions from Face Orderings

We now describe how a \((G, T)\)-tripod decompositions can be obtained from a sequence of distinct faces of \(G\). Throughout this section (and for the remainder of the paper):
• $G$ is an embedded triangulation with outer face $f_0$ and
• $T$ is a spanning-tree of $G$ rooted at a vertex $v_0 \in V(f_0)$.

For any subgraph $f$ of $G$, we define $Y_T(f) := f \cup \bigcup_{v \in V(f)} P_T(v)$. In words, $Y_T(f)$ is the subgraph of $G$ that includes all the vertices and edges of $f$ and all the vertices and edges of each path from each vertex of $f$ to the root of $T$.

Let $\mathcal{F} := f_0, \ldots, f_r$ be a sequence of distinct faces of $G$ whose first element is the outer face $f_0$. Let $G_{-1}$ denote the graph with no vertices and, for each $i \in \{0, \ldots, r\}$, define the graph $G_i := \bigcup_{j=0}^i Y_T(f_j)$ and let $Y_i := V(G_i) \setminus V(G_{i-1})$. Let $\mathcal{G}_\mathcal{F} := G_0, \ldots, G_r$ and let $\mathcal{Y}_\mathcal{F} := Y_0, \ldots, Y_r$.

Informally, we require that each of the legs of each tripod $Y_i$ have a foot on a different vertex of $G_{i-1}$ and that the tripods $Y_1, \ldots, Y_r$ cover all the vertices and edges of $G$. Formally, we say that the sequence $\mathcal{F}$ is proper if, for each $i \in \{1, \ldots, r\}$, and each distinct $\nu, \omega \in V(f_i)$, $V(Y_T(\nu) \cap G_{i-1}) \neq V(Y_T(\omega) \cap G_{i-1})$. The sequence $\mathcal{F}$ is complete for $G$ if $G_r = G$. Note that, if $\mathcal{F}$ is complete, then $\{Y_0, \ldots, Y_r\}$ is a tripod decomposition of $G$.

From the preceding definitions it follows that, if $\mathcal{F}$ is proper, then $G_i$ is 2-connected for each $i \in \{0, \ldots, r\}$. For any $i \in \{0, \ldots, r\}$, consider any face $f$ of $G_i$, that we now treat as a cycle in $G$. An easy proof by induction shows that, for any $j \in \{0, \ldots, i\}$, the induced graph $f[Y_j]$ is connected. We are interested in keeping the number of tripods in $Y_0, \ldots, Y_i$ that contribute to $V(f)$ as small as possible, which motivates our next definition.

The sequence $\mathcal{F}$ is good if the resulting sequence of graphs $\mathcal{G}_\mathcal{F} := G_0, \ldots, G_r$ and tripods $\mathcal{Y}_\mathcal{F} := Y_0, \ldots, Y_r$ satisfy the following condition: For each $i \in \{0, \ldots, r\}$ and each face $f$ of $G_i$,

$$|\{\ell \in \{0, \ldots, i\} : V(f) \cap Y_\ell \neq \emptyset\}| \leq 3.$$ 

In words, each face of each graph $G_i$ has vertices from at most three tripods of $Y_0, \ldots, Y_i$ on its boundary. Even more, the vertices of $f$ can be partitioned into at most three paths where the vertices of each path belong to a single tripod. Dujmović et al. [7] prove Theorem 3 by proving the next lemma.

**Lemma 4.** Let $G$ be a triangulation with a vertex $v_0$ on its outer face $f_0$ and let $T$ be a spanning tree of $G$ rooted at $v_0$. Then there exists a sequence $\mathcal{F} := f_0, \ldots, f_r$ of distinct faces of $G$ that is proper, good, and complete.

**Remark 5.** Lemma 4 is stated in terms of sequences only for convenience and could be rephrased in terms of partial orders. Indeed, consider the partial order $<$ defined as follows: For each $i \in \{1, \ldots, r\}$ let $f'_i$ be the face of $G_{i-1}$ that contains $f_i$; then $f'_\ell < f_i$ for each $\ell \in \{0, \ldots, i-1\}$ such that $V(f'_\ell) \cap Y_i \neq \emptyset$. It is straightforward to check that any linearization of this partial order will result in the same tripod decomposition $\mathcal{Y}_\mathcal{F} := \{Y_0, \ldots, Y_r\}$.

Dujmović et al. [7] prove Lemma 4 by giving a recursive algorithm that constructs the face sequence $\mathcal{F}$. For a face $f$ of $G_i$, define the set $I_f := \{\ell \in \{0, \ldots, i\} : V(f) \cap Y_\ell \neq \emptyset\}$. They begin with the outer face $f_0$ of $G$. To find the face $f_i$, $i > 0$, they consider some face $f \not\in \{f_0, \ldots, f_{i-1}\}$ of $G_{i-1}$ and use Sperner’s Lemma to show that there is an appropriate face.

\[\text{In all of our examples, the subgraph } f \text{ will always be a single edge or single face of } G.\]
Figure 1: Each face \( f \) in \( G_{i-1} \) is bounded by at most three tripods \( Y_a, Y_b, \) and \( Y_c \) and the tripod \( Y_i \) is chosen so that it connects each of these.

\( f_i \) of \( G \) (called a Sperner triangle) that is contained in \( f \). In particular, \( f_i \) is chosen so that the three upward paths in \( Y_F(f_i) \) lead back to each of the (at most 3) tripods in \( \{ Y_j : j \in I_f \} \). See Figure 1.

This proof leads to a divide-and-conquer algorithm: After finding \( f_i \), the algorithm recursively decomposes each of the near-triangulations that are bounded by the at most three new faces in \( S_i := F(G_i) \setminus F(G_{i-1}) \setminus \{ f_i \} \). The Sperner triangle \( f_i \) can easily be found in time proportional to the number of faces of \( G \) in the interior of \( f \). However, because the resulting recursion is not necessarily balanced, a straightforward implementation of this yields an algorithm with \( \Theta(n^2) \) worst-case running time.

Morin [14] later showed that, using an appropriate data structure for \( T \), this approach can be implemented in such a way that the resulting algorithm runs in \( O(n \log n) \) time. Essentially, Morin’s algorithm works by finding the Sperner triangle \( f_i \) in time proportional to the minimum number of faces of \( G \) contained in any of the faces in \( S_i \). In the next section, we will show that, by using a lowest common ancestor data structure for the cotree \( \overline{T} \) along with Lemma 2, the Sperner triangle \( f_i \) can be found in constant time, yielding an \( O(n) \) time algorithm.

By now, our presentation of this material differs somewhat from that in [7, 16]. Therefore, we now pause to explain how Lemma 4 implies Theorem 3.a. To do this, we show that there exists a chordal graph \( H \) whose largest clique has size at most 4 and that contains \( G/Y_F \). We construct the graph \( H \) so that for each \( i \in \{0,\ldots,r\} \) and each face \( f \) of \( G_i \), \( H \) contains a clique on \( \{ Y_j : j \in I_f \} \). To accomplish this, for each \( i \in \{1,\ldots,r\} \) we let \( f \) be the face of \( G_{i-1} \) that contains \( f_i \) and we form a clique on \( \{ Y_i \} \cup \{ Y_j : j \in I_f \} \). Inductively, the elements of \( \{ Y_j : j \in I_f \} \) already form a clique, so this operation is equivalent to attaching \( Y_i \) to all the vertices of an existing clique of size at most 3. Therefore, this results in a chordal graph \( H \) whose largest clique has size at most 4 and therefore \( H \) has treewidth at most 3 [12].
An $O(n)$-Time Algorithm

Refer to Figure 2 for an illustration of the following (probably well-known) baby version of Sperner’s Lemma:

**Lemma 6.** Let $N$ be a near-triangulation with outer face $v_0,\ldots, v_r$ and colour each vertex of $N$ red or blue in such a way that $v_0,\ldots, v_\ell$ are coloured red for some $\ell \in \{0, \ldots, r-1\}$ and $v_{\ell+1},\ldots, v_r$ are coloured blue. Then there exists a path $w_0,\ldots, w_k$ in $N^*$ such that

1. $w_0$ is the inner face of $N$ with $v_0 v_r$ on its boundary;
2. $w_k$ is the inner face of $N$ with $v_\ell v_{\ell+1}$ on its boundary; and
3. for each $i \in \{1,\ldots, k\}$, the single edge in $E(w_{i-1}) \cap E(w_i)$ has an endpoint of each colour.

**Proof.** If $w_0 = w_k$, the lemma is immediately true, so assume $w_0 \neq w_k$. Say that an edge of $N$ is bichromatic if one of its endpoints is red and the other is blue. Any edge that is not bichromatic is monochromatic. The outer face $f_0$ of $N$ has exactly two bichromatic edges $v_0 v_r$ and $v_\ell v_{\ell+1}$ and any inner face of $N$ has either zero or two bichromatic edges. Consider the subgraph $H$ of $N^*$ obtained removing each edge $f g \in E(N^*)$ such that the edge in $E(f) \cap E(g)$ is monochromatic. Every vertex in $H$ has degree 0 or 2, so each connected component of $H$ is either an isolated vertex or a cycle. The face $f_0$ has degree 2 so it is contained in a cycle $C$ of $H$. The two neighbours of $f_0$ in $H$ are $w_0$ and $w_k$. Therefore $C$ contains a path $w_0,\ldots, w_k$ that satisfies the conditions of the lemma.

The next lemma, which is the main new insight in this paper, allows us to use Lemma 2 to find Sperner triangles in constant time.

**Lemma 7.** Let $G$ be a triangulation with a vertex $v_0$ on its outer face $f_0$; let $T$ be a spanning tree of $G$ rooted at $v_0$; let $\overline{T}$ be the cotree of $(G, T)$ rooted at $f_0$; let $f_0, \ldots, f_{i-1}$ be a good proper sequence of faces of $G$ that yields a sequence $G_F := G_0, \ldots, G_{i-1}$ of graphs and a sequence $Y_F := Y_0, \ldots, Y_{i-1}$ of tripods; let $f \notin \{f_0, \ldots, f_{i-1}\}$ be a face of $G_{i-1}$, and let $S \subseteq F(G)$ contain exactly the (at most three) faces $g \in F(G)$ such that
(i) \( g \) is contained in the interior of \( f \);
(ii) \( g \) contains an edge \( vw \in E(f) \) with \( v \in Y_a \) and \( w \in Y_b \) for some distinct \( a, b \in I_f \).

Let \( \mathcal{T}_0 \) be the minimal subtree of \( \mathcal{T} \) that spans \( S \). Then, if \( S \) is non-empty and \( f_i \in V(\mathcal{T}_0) \) is such that each component of \( \mathcal{T}_0 − f_i \) contains at most one element of \( S \), Then \( f_0, \ldots, f_i \) is good.

**Proof.** Let \( N \) be the near-triangulation consisting of all vertices and edges of \( G \) contained in the closure of the interior of \( f \). Recall that \( I_f := \{ j \in [0, \ldots, i−1] : Y_j \cap V(f) \neq \emptyset \} \). Since \( f_0, \ldots, f_i \) is good, \( |I_f| \leq 3 \). Since \( S \) is non-empty \( |I_f| \geq 2 \). For each \( j \in I_f \), colour each vertex \( v \) of \( N \) with the colour \( j \) if the first vertex of \( P_\mathcal{T}(v) \) in \( V(f) \) is contained in \( Y_j \). Say that an edge or face of \( N \) is monochromatic, bichromatic, or trichromatic if it contains vertices of one, two, or three colours, respectively.

\( E(f) \) contains exactly \( |I_f| \) bichromatic edges. Since each element of \( S \) is an inner face of \( N \) that contains a bichromatic edge of \( f \), \( |S| \leq |I_f| \leq 3 \). Let \( X \) be the subgraph of \( N^* \) that contains an edge \( fg \in E(N^*) \) if and only if \( f \) and \( g \) are inner faces of \( N \) and the edge in \( E(f) \cap E(g) \) is bichromatic. We claim that \( X \) is a subgraph of \( \mathcal{T} \). In order to show this, we need only argue that each edge \( uv \) of \( T \) in the interior of \( f \) is monochromatic. Consider any \( uv \in E(N) \setminus E(f) \) where \( u \) is the \( T \)-parent of \( v \). If \( v \notin V(f) \) then, by definition, \( v \) has the same colour as \( u \), so \( uv \) is monochromatic. The case where \( v \in V(f) \) and \( u \notin V(f) \) can not occur since \( v \in V(f) \) implies that \( P_\mathcal{T}(v) \subseteq G_{i−1} \), but \( u \notin V(G_{i−1}) \). Similarly, the case in which \( u \in V(f) \) and \( v \in V(f) \) can not occur since this implies that \( P_\mathcal{T}(v) \subseteq G_{i−1} \), but \( uv \notin E(G_{i−1}) \).

Next we claim that all the elements of \( S \) are in a single connected component of \( X \). If \( |I_f| = 2 \), then this follows immediately from **Lemma 6**. If \( |I_f| = 3 \), then let \( \{a,b,c\} := I_f \) and consider a pair \( g_1, g_2 \in S \) where (without loss of generality) \( g_1 \) contains a bichromatic edge of \( f \) with colours \( a \) and \( b \) and \( g_2 \) contains a bichromatic edge of \( f \) with colours \( b \) and \( c \). By treating \( a \) and \( c \) as a single colour we may again apply **Lemma 6** to conclude that \( g_1 \) and \( g_2 \) are in the same component of \( X \).

Refer to **Figure 3(a)**. Therefore \( X \) is a subgraph of \( \mathcal{T} \) that has a component containing all the elements of \( S \). Therefore \( X \) contains \( \mathcal{T}_0 \). By choice, \( \mathcal{T}_0 \) contains a path from \( f_i \) to each \( g \in S \) and each of these paths is disjoint except for their shared starting location \( f_i \).

Refer to **Figure 3(b)**. Now, consider the embedded graph \( X_0 \) obtained as follows: For

![Figure 3: The proof of Lemma 7](image-url)
each $g \in V(T_0)$, place a vertex on the center of each bichromatic edge of $g$ and, if $g$ is trichromatic, then place a vertex in the center of $g$. Next,

1. add an edge joining the center of each trichromatic triangle to each of the centers of its bichromatic edges; and
2. add an edge (embedded as a straight line segment) joining the centers of each pair of bichromatic edges that are on a common bichromatic face $g \in V(T_0)$.

The graph $X_0$ is a tree of maximum-degree 3 that has $|I_f|$ leaves. (Each leaf in $X_0$ is the center of a bichromatic edge in $E(f)$). With the exception of these three leaves, every point in the embedding of $X_0$ is contained in the interior of $f$.

Refer to Figure 3(c). Now treat $X_0$ as a point set and consider the point set $f'$ obtained by removing $X_0$ from the closure of $f$. Now $f'$ has $|I_f|$ connected components and each vertex of $f_i$ is in a different component. Each of the components of $f'$ contains vertices of $Y_j$ for exactly one $j \in I_f$; call this the colour of the component. Since no edge of $T$ crosses $X_0$, the colour of each vertex in $f_i$ is equal to the colour the component of $f'$ that contains it.

Finally, to see that $f_0, \ldots, f_i$ is good first observe that we need only be concerned with the at most three faces in $F(G_f) \setminus F(G_{f-1}) \setminus \{f_i\}$ and each of these shares a bichromatic edge with $f_i$. If $g$ is a face in $F(G_f) \setminus F(G_{f-1}) \setminus \{f_i\}$ with $E(g) \cap E(f_i) = \{uv\}$ and $uv$ is coloured with $a$ and $b$, then $V(g) \cap Y_j = \emptyset$ for any $j \in \{0, \ldots, i\} \setminus \{a, b, i\}$. This completes the proof.

**Theorem 8.** There exists an $O(n)$ time algorithm that, given any $n$-vertex triangulation $G$ and any rooted spanning tree $T$ of $G$, produces a $(G,T)$-tripod decomposition $\mathcal{Y}$ such that $\text{tw}(G/\mathcal{Y}) \leq 3$.

**Proof.** Let $v_0$ be the root of $T$ and let $f_0$ be a face of $G$ incident to $v_0$ that contains an edge of $T$ incident to $v_0$. In a preprocessing step, we compute the cotree $\overline{T}$ of $(G,T)$ and construct a lowest common ancestor data structure for $\overline{T}$ in $O(n)$ time that allows us to compute $\text{lca}_\overline{T}(f,g)$ for any two faces $f, g \in F(G)$ in $O(1)$ time.

After this preprocessing, we construct the good sequence $f_0, \ldots, f_r$ recursively. Conceptually, during any recursive invocation, the input is a near-triangulation $N$ bounded by a cycle $C$ in $G$ whose vertices belong to at most three tripods computed in previous steps. Each vertex of $G$ starts initially unmarked and we mark a vertex once we have placed it in a tripod. The precise input to a recursive invocation is defined as follows:

1. If $C$ intersects three tripods then the input consists of the three inner faces $g_1, g_2,$ and $g_3$ of $N$ that contain bichromatic edges of $C$. Lemma 7 characterizes the face $f_i$ in terms of the minimum subtree $T_0$ of $\overline{T}$ that contains $g_1, g_2,$ and $g_3$. Indeed, $f_i$ is either the unique degree-3 node of $T_0$ (if $g_1, g_2,$ and $g_3$ are all leaves of $T_0$) or $f_i$ is the unique node among $g_1, g_2,$ or $g_3$ that has degree 2. By Lemma 2 we can construct the model $T_0'$ of $T_0$ in constant time and find the node $f_i$.

2. If $C$ intersects two tripods, then the input consists of two inner faces $g_1, g_2,$ of $N$ with bichromatic edges of $C$ on their boundary. In this case, we let $f_i := g_1$ or $f_i := g_2,$ either choice satisfies our requirements.
3. If $C$ intersects only one tripod, then the input consists of any inner face $g_1$ of $N$ that contains an edge in $E(f)$. In this case $f_i := g_1$ satisfies our requirements.

Once we have found the Sperner triangle $f_i$, we can compute the tripod $Y_i$ and mark its vertices by following the path in $T$ from each vertex of $f_i$ to its nearest marked ancestor in $T$. This takes $O(1 + |Y_i|)$ time. Once we have done this, we have also found the at most three bichromatic edges of $G_i$ that are needed to perform the at most three recursive invocations on the near triangulations whose outer faces coincide with each of the new faces in $F(G_i) \setminus F(G_{i-1}) \setminus \{f_i\}$.

After setting $f_0$, the initial recursive call falls into the third case above, so its input is any of the three inner faces that shares an edge with the outer face, $f_0$. Each recursive invocation adds a new face $f_i$ to the good face sequence $f_0,\ldots,f_r$ and takes $O(1 + |Y_i|)$ time. Since $Y_0,\ldots,Y_r$ is a partition of $V(G)$, the running time of this algorithm is therefore $\sum_{i=0}^r O(1 + |Y_i|) = O(n)$. \qed

5 Variations

In this section we show that there are $O(n)$ time algorithms for computing the decompositions in Theorem 1.b and Theorem 1.c. In the same way that Theorem 1.a follows from the tripod decomposition of Theorem 3, Theorem 1.b follows from a bipod decomposition given by Theorem 10 and Theorem 1.c follows from a monopod decomposition given by Theorem 11.

5.1 Bipod Decompositions

We begin with the decomposition in Theorem 1.b, which was communicated to us by Vida Dujmović, and has not appeared before. This decomposition is obtained by selecting a proper sequence $E := e_0,\ldots,e_k$ of distinct edges of $G$, which define a sequence of graphs $G'_E := G_{e_0},\ldots,G_k$ where $G_i := \bigcup_{j=0}^i P_T(e_j)$ and a sequence of bipods $I'_E := \Lambda_0,\ldots,\Lambda_k$ where $\Lambda_i = V(G_i) \setminus V(G_{i-1})$. We call $E$ good if, for each $i \in \{0,\ldots,k\}$ and each face $f \in F(G_i)$, $V(f)$ has a non-empty intersection with at most 4 bipods in $\Lambda_0,\ldots,\Lambda_i$.

Exactly the same argument used in Section 3.1 to show that $G/Y_E$ is contained in a chordal graph of maximum clique size 4 also shows that if $E$ is a good edge sequence that produces a bipod partition $I'_E$ of $V(G)$, then $G/I'_E$ is contained in a chordal graph of maximum clique size 5, so $G/I'_E$ has treewidth at most 4.

We now explain why a good edge sequence $e_0,\ldots,e_r$ exists.\textsuperscript{6} As before, we set $f_0$ to be any face of $G$ such that $E(f_0)$ contains an edge of $T$ incident to the root $v_0$ of $T$. The edge $e_0$ is any edge of $E(f_0) \setminus E(T)$. Next we take special care to ensure that $G_i$ is biconnected for $i \geq 1$. In particular, if $G_0$ contains only two edges of $f_0$, then we take $e_1$ to be the edge of $f_0$ that does not appear in $G_0$. Otherwise, we choose $e_1$ using the general strategy for choosing $e_i$, described next.

Refer to Figure 4. Now we may assume that $G_{i-1}$ is biconnected. To choose the edge $e_i$, we consider any face $f \in F(G_{i-1}) \setminus F(G)$. Inductively, $V(f)$ contains vertices from at most

\textsuperscript{6}The existence of this edge sequence is more easily proven using Sperner’s Lemma, but we want a proof that lends itself to a linear time algorithm.
four bipods in \( \Lambda_0, \ldots, \Lambda_{i-1} \). Let \( I_f := \{ j \in [0, \ldots, i - 1] : \Lambda_j \cap V(f) \neq \emptyset \} \). If \(|I_f| < 4\) then we can select \( e_i \) to be any edge in the interior \( f \). Therefore, we focus on the case \(|I_f| = 4\). As before we colour vertices in the near triangulation \( N \) using colours in the set \( I_f \); we let \( S \) be the set of inner faces in \( N \) that contain a bichromatic edge in \( E(f) \); and let \( T_0' \) be the minimal subtree of \( T \) that spans \( S \). The same argument in the proof of Lemma 7 shows that every node of \( T_0 \) is contained in \( f \).

Claim 9, below, shows that \( T_0 \) contains an edge \( xy \) such that each component of \( T_0 - xy \) contains at most two elements of \( S \). It is straightforward to verify that, if we choose \( e_i \) to the be the edge in \( E(x) \cap E(y) \) then we obtain a graph \( G_i \) in which each of the two new faces containing vertices from \( \Lambda_i \) contains vertices from at most three bipods in \( \{ \Lambda_j : j \in I_f \} \), as required.

Figure 4: Choosing the next \( e_i \) in a good edge sequence.

Claim 9. \( T_0 \) contains an edge \( xy \) such that each component of \( T_0 - xy \) contains at most two nodes of \( S \).

Proof. Direct each edge \( xy \) of \( T_0 \) in the direction \( x \rightarrow y \) if the component of \( T_0 - xy \) that contains \( y \) contains three or more nodes of \( S \). It is sufficient to show that this process leaves some edge \( xy \) of \( T_0 \) undirected. Assume for the sake of contradiction that every edge of \( T_0 \) is directed. Then some node \( x \) of \( T_0 \) has only incoming edges. Certainly \( x \) does not have degree 1 in \( T_0 \).

If \( x \) has degree 2 in \( T_0 \) then \( T_0 \) contains two subtrees \( T_1 \) and \( T_2 \) that have only the node \( x \) in common and such that \( |V(T_1) \cap S| \geq 3 \) and \( |V(T_2) \cap S| \geq 3 \), which implies that \( |S| \geq 3 + 3 - 1 > 4 \), a contradiction.

Suppose therefore that \( x \) has degree 3 in \( T_0 \). Each face in \( S \) contains an edge in \( E(f) \), so each face in \( S \) has degree at most 2 in \( T_0 \). Therefore \( x \notin S \). Therefore \( T_0 - x \) contains three components \( T_1, T_2, T_3 \) such that each pair of components contains at least 3 elements of \( S \). But this implies that \( |S| \geq (3 \times 3)/2 > 4 \), a contradiction. \( \square \)

Algorithmically, using Lemma 2, we can construct the model \( T_0' \) of \( T_0 \) in constant time given the set \( S \). The model \( T_0' \) will also contain an edge \( a \beta \) such that each component of \( T_0' - a \beta \) contains at most two nodes in \( S \). We claim that \( E(a) \) contains an edge that makes a suitable choice for \( e_i \), and this edge can be found in constant time. Indeed, the edge \( a \beta \)
in $\overline{T}_0'$ corresponds to a path $\alpha, x_1, \ldots, x_k, \beta$ in $\overline{T}_0$ and the unique edge in $E(\alpha) \cap E(x_1)$ is a suitable choice for $e_i$.

The rest of the details of the algorithm are similar to those given in the proof of Theorem 8: Each subproblem is a near-triangulation $N$ bounded by a cycle $C$ and the input that defines the subproblem consists of the (at most four) faces $S \subseteq F(N)$ incident to bichromatic edges of $C$.

**Theorem 10.** There exists an $O(n)$ time algorithm that, given any $n$-vertex triangulation $G$ and any rooted spanning tree $T$ of $G$, produces a $(G, T)$-bipod decomposition $I$ such that $\text{tw}(G/I) \leq 4$.

### 5.2 Monopod Decompositions

Finally we consider the decomposition described in Theorem 1.c. This decomposition is obtained from a tripod decomposition $Y := Y_0, \ldots, Y_r$, obtained by a sequence $F := f_0, \ldots, f_r$ of faces of $G$ in the same manner described in Section 3.1. However in this setting, the sequence $f_0, \ldots, f_r$ is good if, for each $i \in \{0, \ldots, r\}$ and each face $f$ of $G_i := \bigcup_{j=0}^{i} Y_T(f_j)$, $V(f)$ contains vertices from at most 5 legs of tripods in $Y_0, \ldots, Y_i$. Under these conditions, Ueckerdt et al. [16] are able to show that the monopod decomposition $I$ obtained by splitting each tripod $Y_i$ into three upward paths yields a quotient graph $G/I$ of treewidth at most 6.

As before we focus on the extreme case when $V(f)$ contains vertices from exactly 5 legs of tripods. Refer to Figure 5. Following the same strategy used for the previous two decompositions, the set $S$ in this case has size at most 5 and the face $f_i$ corresponds to a node of $\overline{T}_0$ such that each component of $\overline{T}_0 - f_i$ contains at most 2 nodes in $S$. (This is always possible because $\lfloor 5/2 \rfloor = 2$.) Again, a suitable choice for $f_i$ can be found in the model $\overline{T}_0'$ of $T_0$ in constant time.

![Figure 5: The selection of a tripod by Ueckerdt et al. [16]](image)

**Theorem 11.** There exists an $O(n)$ time algorithm that, given any $n$-vertex triangulation $G$ and any rooted spanning tree $T$ of $G$, produces a $(G, T)$-monopod decomposition $I$ such that $\text{tw}(G/I) \leq 6$.

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7In the degenerate case where $C$ has no bichromatic edges, the input is any face of $N$ incident to an edge of $C$. 

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