On Time-Space Noncommutativity for Transition Processes and Noncommutative Symmetries

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ABSTRACT

We explore the consequences of time-space noncommutativity in the quantum mechanics of atoms and molecules, focusing on the Moyal plane with just time-space noncommutativity \([\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \theta_{0i} \neq 0, \theta_{ij} = 0\). Space rotations and parity are not automorphisms of this algebra and are not symmetries of quantum physics. Still, when there are spectral degeneracies of a time-independent Hamiltonian on a commutative space-time which are due to symmetries, they persist when \(\theta_{0i} \neq 0\); they do not depend at all on \(\theta_{0i}\). They give no clue about rotation and parity violation when \(\theta_{0i} \neq 0\). The persistence of degeneracies for \(\theta_{0i} \neq 0\) can be understood in terms of invariance under deformed noncommutative “rotations” and “parity”. They are not spatial rotations and reflection. We explain such deformed symmetries. We emphasize the significance of time-dependent perturbations (for example, due to time-dependent electromagnetic fields) to observe noncommutativity. The formalism for treating transition processes is illustrated by the example of nonrelativistic hydrogen atom interacting with quantized electromagnetic field. In the tree approximation, the \(2s \rightarrow 1s + \gamma\) transition for hydrogen is zero in the commutative case. As an example, we show that it is zero in the same approximation for \(\theta_{0i} \neq 0\). The importance of the deformed rotational symmetry is commented upon further using the decay \(Z^0 \rightarrow 2\gamma\) as an example.

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1 Introduction

Different approaches to fundamental physics suggest space-time noncommutativity. It arises naturally in quantum gravity when one considers the localization of points in space-time [1]. It arises in string theory as well in a certain limit [2].

Formulation of quantum physics on noncommutative space-times does not present conceptual problems if time commutes with spatial coordinates. That is not the case with time-space noncommutativity. It was the work of Doplicher et al [1] which systematically developed unitary quantum field theories (QFT’s) with time-space noncommutativity. Their ideas were later adapted to quantum mechanics by Balachandran et al [5]. It appears that we now have the tools for doing consistent quantum physics with time-space noncommutativity.

An important task is the extraction of observable consequences of noncommutative space-times. An extensive literature already exists on this subject for space-space noncommutativity (see [6] for a review and references), but that is not the case for time-space noncommutativity. In this paper, we make a beginning in this regard.

After reviewing previous work on quantum physics with time-space noncommutativity, we consider certain implications of the spectral map theorem of that work. The theorem states that if the Hamiltonian has no explicit time-dependence, its spectra for commutative and noncommutative space-times are identical, provided only that spatial coordinates commute for the latter. For Moyal space-times $A_\theta(\mathbb{R}^{d+1})$, where

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \mathbb{1}, \quad \theta_{\mu\nu} \text{ are real constants},$$

that means that

$$\theta_{ij} = 0, \quad i, j \in [1, d] \quad (1.1)$$

if $i, j$ denote the spatial and 0 the time operators. Now even with (1.1), spatial rotations are not automorphisms of $A_\theta(\mathbb{R}^{d+1})$ if $\theta_{0i} \neq 0$. Nor is parity an automorphism if $d$ is odd. Nevertheless for time-independent Hamiltonians invariant under rotations or parity for $\theta_{\mu\nu} = 0$, the spectral theorem implies as a corollary that their energy degeneracies due to symmetries remain intact when $\theta_{0i} \neq 0$. Energy spectra thus give no clue on noncommutative symmetry breakdown if the Hamiltonian is time-independent. This is a surprising result. We explain its conceptual reasons and emphasize the importance of time-dependent phenomena for observing time-space noncommutativity.

Next we develop the formulism for calculating transition processes using the example of hydrogen atom interacting with electromagnetic field. As an application, we consider the

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\footnote{We use the term “parity” as the reflection $\hat{x}_i \rightarrow -\hat{x}_i$ of all spatial coordinates when $d$ is odd. This is the conventional definition. But we can also define it as a fixed element of the orthogonal group $O(d)$ not connected to identity, such as the reflection of a coordinate perpendicular to $\tilde{\theta}_0 = (\theta_{01}, ..., \theta_{0d})$. We note that this particular reflection is an automorphism of our algebra.}

\footnote{For $d$ even, parity is total reflection in the plane perpendicular to $\tilde{\theta}_0$.}
$2s \to 1s + \gamma$ transition in hydrogen. It is forbidden by parity if $\theta_{\mu\nu} = 0$, but being a time-dependent process, can occur if $\theta_{\mu\nu} \neq 0$. We explicitly show that it is nevertheless zero in the tree approximation.

The paper concludes with comments on possible effects of time-space noncommutativity for processes such as $Z^0 \to 2\gamma$ which vanish by Poincaré invariance and Bose symmetry in the commutative case [3,4].

2 Quantum Mechanics on Noncommutative Space-Time

This section is a short review of earlier work on the subject [5].

In the model we consider, $d = 3$ and spatial coordinates commute,

$$[\hat{x}_i, \hat{x}_j] = 0 \ , \ i, j \in [1, 2, 3]$$

(2.2)

while there is time-space noncommutativity:

$$[\hat{x}_0, \hat{x}_i] = i\theta s_i .$$

(2.3)

Here $s_i$ is a fixed unit vector. We conveniently orient it in the 1-direction. We thus assume that

$$[\hat{x}_0, \hat{x}_1] = i\theta, \ [\hat{x}_0, \hat{x}_{2,3}] = 0 .$$

(2.4)

As $\theta \to -\theta$ when $\hat{x}_0 \to -\hat{x}_0$ (or $\hat{x}_1 \to -\hat{x}_1$), we further assume without loss of generality that $\theta \geq 0$.

The algebra generated by $\hat{x}_\mu$ with the relations (2.2) and (2.4) will be denoted by $A_\theta(\mathbb{R}^4)$.

In the algebraic approach to quantum physics, the quantum mechanical Hilbert space $\mathcal{H}$ is built from elements of $A_\theta(\mathbb{R}^4)$ itself. Observables are self-adjoint operators on this Hilbert space. In cases of interest here, their domain contains $A_\theta(\mathbb{R}^4) \cap \mathcal{H}$.

Now the algebra itself acts in two distinct ways on $A_\theta(\mathbb{R}^4)$, namely by the left- and right-regular representations $A^{L,R}_\theta(\mathbb{R}^4)$. For each $\hat{a} \in A_\theta(\mathbb{R}^4)$, we have $\hat{a}^{L,R} \in A^{L,R}_\theta(\mathbb{R}^4)$ where

$$\hat{a}^L \hat{a} = \hat{a} \hat{a} , \ \hat{a}^R \hat{a} = \hat{a} \hat{a} , \ \hat{a} \in A_\theta(\mathbb{R}^4) .$$

We can also associate the adjoint action $ad \hat{a}$ to $\hat{a}$:

$$ad \hat{a} \hat{a} = \hat{a}^L - \hat{a}^R , \ ad \hat{a} \hat{a} = [\hat{a}, \hat{a}] .$$

Many observables of physical interest are obtained from $A^{L,R}_\theta(\mathbb{R}^4)$. In particular the momentum $\hat{P}_1$ in 1-direction and the generator $\hat{P}_0$ of time translations are given by

$$\hat{P}_1 = -\frac{1}{\theta} ad \hat{x}_0 , \ \hat{P}_0 = -\frac{1}{\theta} ad \hat{x}_1 .$$
As a preparation to construct the quantum Hilbert space, we next introduce an inner product on $A_\theta(\mathbb{R}^4)$.

Consider

$$\hat{\alpha} = \frac{1}{(2\pi)^2} \int d^4p \tilde{a}(p)e^{ip\hat{x}}e^{ip_0\hat{x}_0}.$$  \hfill (2.5)

Its symbol $\alpha$ is a function $\mathbb{R}^4 \to \mathbb{C}$. We define it by

$$\alpha(x) = \frac{1}{(2\pi)^2} \int d^4p \tilde{a}(p)e^{ip_0\hat{x}_0}e^{ip\hat{x}}.$$  \hfill (2.6)

For the Moyal symbol $\alpha_M$ of $\hat{\alpha}$, we would have written $e^{ip\hat{x}}e^{ip_0\hat{x}_0}$ in the RHS of Eq.(2.5), then $\alpha_M$ is the RHS of Eq.(2.6). Thus our $\alpha \neq \alpha_M$.

Using the symbol, we can define the positive map $S : \hat{\alpha} \to \mathbb{C}$ by

$$S(\hat{\alpha}) = \int d^3x \alpha(\vec{x}, x_0).$$

The importance of $S$ is that it helps us to introduce an inner product $(\cdot, \cdot)$ on $A_\theta(\mathbb{R}^4)$:

$$(\hat{\alpha}, \hat{\beta}) = S(\hat{\alpha}^* \hat{\beta}) = \int d^3x \alpha(\vec{x}, x_0)\beta(\vec{x}, x_0).$$  \hfill (2.7)

The physical Hilbert space $\mathcal{H}$ is the (completion of the) subspace of $A_\theta(\mathbb{R}^4)$ subject to the Schrödinger equation (or constraint). Thus let $\hat{H}$ be a Hamiltonian, Hermitian in the above inner product. Then if $\hat{\psi} \in A_\theta(\mathbb{R}^4) \cap \mathcal{H}$,

$$(\hat{P}_0 - \hat{H})\hat{\psi} = 0.$$  \hfill (2.8)

One can show that for vectors of $\mathcal{H}$, the above inner product has no nontrivial null vectors and is also independent of $x_0$.

The Hamiltonian is time-independent if

$$[\hat{P}_0, \hat{H}] = 0.$$  \hfill (2.9)

In that case the general solution of the Schrödinger constraint is

$$\hat{\psi} = e^{-i\hat{H}\hat{x}_0^R}\hat{\varphi}(\vec{x}).$$

Here 1) $\hat{\varphi}$ is time independent, $[\hat{P}_0, \hat{\varphi}(\vec{x})] = 0$, and 2) square-integrable, $(\hat{\varphi}, \hat{\varphi}) < \infty$. We regard it as an element of $A_\theta(\mathbb{R}^4)$. Then $\hat{H}$ and $\hat{x}_0^R$ act on it in Eq.(2.9).

We can easily check that $\hat{\psi}$ fulfills (2.8). Let $H$ be a time-independent Hamiltonian in conventional quantum physics with $\theta_0 = 0$. It can be

$$H = \frac{\hat{p}^2}{2m} + V(\vec{x}).$$
Let ϕE be its eigenstates regarded as functions of \( \vec{x} \):

\[
H \varphi_E(\vec{x}) = E \varphi_E(\vec{x}) .
\]

We can associate the Hamiltonian \( \hat{H} = H(\hat{\vec{P}}, \hat{\vec{x}}) \) to \( H \) for \( \theta_{0i} \neq 0 \). Then according to the spectral theorem, \( \hat{H} \) and \( H \) have identical spectra while the eigenvectors of \( \hat{H} \) are \( \hat{\varphi}_E = \varphi_E(\hat{\vec{x}}) \exp(-iE\hat{\vec{x}}_0) \):

\[
\hat{H}\hat{\varphi}_E = E\hat{\varphi}_E , \quad (\hat{P}_0 - \hat{H})\hat{\varphi}_E = 0 .
\]

Proof is by inspection. It is important that \( \hat{\varphi}_E \) fulfills the Schrödinger constraint.

We refer to [5] for discussion of time-dependent Hamiltonians.

3 On Symmetries

i) Commutative Rotations

In commutative quantum physics where \( \theta_{0i} = 0 \), spatial rotations are generated by angular momentum operators \( L_i \) where

\[
L_i = \varepsilon_{ijk}x_jp_k , \quad p_k = -i\frac{\partial}{\partial x_k} ,
\]

where \( \varepsilon_{ijk} \) is the Levi-Civita symbol with \( \varepsilon_{123} = +1 \). Spatial coordinates rotate under the \( SO(3) \) group generated by \( L_i \), whereas time is a rotational scalar:

\[
[L_i, x_j] = i\varepsilon_{ijk}x_k , \quad [L_i, x_0] = 0 .
\]

Moments too rotate like \( \vec{x} \):

\[
[L_i, p_j] = i\varepsilon_{ijk}p_k .
\]

These equations let us identify the \( SO(3) \) group generated by \( L_i \) with spatial rotations.

ii) Noncommutative Rotations

For the algebra \( \mathcal{A}_\theta(\mathbb{R}^4) \) as well, there exist operators \( \hat{L}_i \) which generate \( SO(3) \):

\[
\hat{L}_i = \varepsilon_{ijk}\hat{x}_j\hat{\vec{P}}_k , \quad \hat{\vec{P}}_1 = -\frac{1}{\theta} ad \hat{x}^0 , \quad \hat{\vec{P}}_a = -i\frac{\partial}{\partial \hat{x}_a} (a = 2, 3) , \quad [\hat{L}_i, \hat{L}_j] = i\varepsilon_{ijk}\hat{L}_k .
\]

The coordinates \( \hat{x}_i^L \) and momenta \( \hat{P}_i \) respond to \( \hat{L}_i \) as they should to infinitesimal rotations:

\[
[L_i, \hat{x}_j^L] = i\varepsilon_{ijk}\hat{x}_k^L , \quad [L_i, \hat{P}_j] = i\varepsilon_{ijk}\hat{P}_k .
\]

\footnote{For a different approach to noncommutative space-time symmetries, see [4].}
But still, we cannot regard $\hat{L}_i$ as generating spatial rotations as it affects $\hat{x}_0^L$ as well:

$$[\hat{L}_i, \hat{x}_0^L] = i\theta \epsilon_{iik} \hat{P}_k.$$  \hspace{1cm} (3.11)

We should expect this result as the algebra $A_\theta(\mathbb{R}^4)$ does not admit spatial rotations as automorphisms:

$$[\hat{x}_0^L, R_{ij} \hat{x}_j] \neq i\theta \text{ for all } R \in SO(3).$$

Now suppose that the Hamiltonian $H$ for $\theta_0i = 0$ is time-independent and invariant under rotations. It may have eigenstates $\varphi^{(n)}_E, n \in [1,...,N]$ degenerate in energy and carrying a representation of the symmetry group $SO(3)$. Then by the spectral theorem, $\hat{H}$ for $\theta_0i \neq 0$ also has this energy degeneracy and eigenstates

$$\hat{\psi}^{(n)}_E = \varphi^{(n)}_E(\vec{x})e^{-iE \hat{x}_0},$$

$$\hat{H}\hat{\psi}^{(n)}_E = E\hat{\psi}^{(n)}_E.$$  

Here we have represented $\varphi^{(n)}_E$ as a function of spatial coordinates.

The mechanical reason for the persistence of degeneracies for $\theta_0i \neq 0$ is thus clear. But can we locate an underlying noncommutative symmetry?

We consider $H$ invariant under rotations:

$$[L_i, H](\vec{x}, \vec{p}) = 0.$$ 

Then

$$[\hat{L}_i, \hat{H}] = [L_i, H](\vec{x}, \vec{p})|_{x_i=\hat{x}_i^L, p_i=\hat{P}_i} = 0,$$

$$[\hat{L}_i, \hat{P}_0] = 0.$$  

Thus the group $SO(3)$ generated by $\hat{L}_i$ preserves $\hat{H}$ and the Schrödinger constraint: it is a noncommutative symmetry group. Furthermore its action on energy eigenstates is something familiar:

$$\hat{L}_i \hat{\psi}^{(n)}_E = \left(L_i \varphi^{(n)}_E\right)(\vec{x})|_{x_i=\hat{x}_i^L}e^{-iE \hat{x}_0}.$$  

In this way we see that the noncommutative $SO(3)$ can explain spectral degeneracies even though this $SO(3)$ is not the spatial rotation group.

The noncommutative $SO(3)$ is not a symmetry if the Hamiltonian $H$ for $\theta_0i = 0$, although commuting with $L_i$, has explicit time-dependence:

$$H = H(x_0, \vec{x}, \vec{p}),$$

$$[L_i, H] = 0.$$ 

In that case

$$\hat{H} = \hat{H}(\hat{x}_0^L, \vec{x}, \vec{P}).$$
and 
\[ [\hat{L}_i, \hat{H}] \neq 0 \]
because of Eq. (3.11).

Thus effects of noncommutativity on spatial rotations are revealed only by time-dependent \( \hat{H} \).

**iii) Noncommutative Parity and its Action as a Symmetry**

If \( H \) is a time-independent Hamiltonian for \( \theta_0 = 0 \) which is invariant under parity \( P \),
\[ PH(\vec{x}, \vec{p})P^{-1} = H(-\vec{x}, -\vec{p}) = H(\vec{x}, \vec{p}) , \]
it so happens that there is a deformed noncommutative parity \( \hat{P} \) which is a noncommutative symmetry. But it affects time \( \hat{x}_0^L \) and is not properly spatial reflection. Still it is a valid symmetry and good for explaining energy degeneracies.

\( \hat{P} \) can be constructed as follows. Let \( \mathcal{P}_\theta \) be the plane perpendicular to \( \vec{\theta}_0 \):
\[ \vec{x} \in \mathcal{P}_\theta \iff \vec{x} \cdot \vec{\theta}_0 = 0 . \]
It is spanned by an orthonormal basis \( \vec{e}^{(a)} \), \( a = (1, 2) \):
\[ \vec{e}^{(a)} \cdot \vec{e}^{(b)} = \delta_{ab} , \quad \vec{\theta}_0 \cdot \vec{e}^{(a)} = 0 . \]
We can write
\[ \vec{x} = \vec{e}^{(a)}(\vec{x} \cdot \vec{e}^{(a)}) + \frac{\vec{\theta}_0}{|\vec{\theta}_0|}(\vec{\theta}_0 \cdot \vec{x}) , \]
\[ |\vec{\theta}_0| = \left| \left( \sum_i \theta_{0i}^2 \right)^{1/2} \right| . \]

Let \( K \) be the operator of reflection of just \( \vec{x} \cdot \vec{e}^{(1)} \) in the commutative case:
\[ K \vec{x} \cdot \vec{e}^{(1)} K^{-1} = -\vec{x} \cdot \vec{e}^{(1)} , \]
\[ K \vec{x} \cdot \vec{e}^{(2)} K^{-1} = \vec{x} \cdot \vec{e}^{(2)} , \quad K \vec{x} \cdot \vec{\theta}_0 K^{-1} = \vec{x} \cdot \vec{\theta}_0 . \]
Then commutative parity \( P \) is \( R_{\vec{e}^{(1)}}(\pi)K \) where \( R_{\vec{e}^{(1)}}(\pi) \) is rotation by \( \pi \) around \( \vec{e}^{(1)} \)-axis.

As remarked earlier, \( \hat{K} \), the noncommutative version of \( K \), is an automorphism of \( \mathcal{A}_0(\mathbb{R}^4) \).

The noncommutative version \( \hat{R}_{\vec{e}^{(1)}}(\pi) \) of \( R_{\vec{e}^{(1)}}(\pi) \) is well-defined as well: it is an element of the \( SO(3) \) group with generators \( \hat{L}_i \). The noncommutative parity is thus
\[ \hat{P} = \hat{R}_{\vec{e}^{(1)}}(\pi)\hat{K} . \]
We have
\[ \hat{P}\hat{x}_i^L\hat{P}^{-1} = -\hat{x}_i^L , \quad \hat{P}\hat{P}_i\hat{P}^{-1} = -\hat{P}_i . \]
\( \hat{P} \) affects \( \hat{x}_0^L \) because \( \hat{R}_{\pi(1)}(\pi) \) does so. Hence it cannot be regarded as just total spatial reflection. However it does not affect \( \hat{x}_0^R \).

\( \hat{P} \) does not depend on the choice of the axis \( \vec{e}^{(1)} \) in the plane perpendicular to \( \vec{e}_0 \). For example if

\[
\hat{P}' = \hat{R}_{\pi(2)}(\pi)\hat{K}',
\]

where \( \hat{K}' \) reflects \( \vec{x} \cdot \vec{e}^{(2)} \), then

\[
\hat{P}' = \hat{P}.
\]

The proof of Eq.(3.12) is as follows. Let

\[
U = \hat{P}'^{-1}\hat{P}.
\]

Then

\[
U\hat{x}_i^L U^{-1} = \hat{x}_i^L, \quad U\hat{P}_i U^{-1} = \hat{P}_i, \quad U\hat{x}_i^R U^{-1} = \hat{x}_i^R.
\]

Hence

\[
U\hat{x}_0^L U^{-1} = U[-\theta \hat{P}_1 + \hat{x}_0^R]U^{-1} = \hat{x}_0^L.
\]

Thus since conjugation by \( U \) affects no operator, we can identify \( \hat{P}' \) with \( \hat{P} \).

Since \( \hat{R}_{\pi(1)}(\pi) \) and \( \hat{K} \) commute with \( \hat{P}_0 \), so does \( \hat{P} \). It follows as before that if \( \hat{P} \) commutes with \( \hat{H} \) and \( \hat{H} \) is time-independent, \( \hat{P} \) commutes with \( \hat{H} \) and also preserves the Schrödinger constraint.

Thus degeneracies due to parity in commutative quantum physics are preserved intact in noncommutative quantum physics if \( \hat{H} \) is time-independent.

But \( \hat{P} \) is not spatial reflection as it affects coordinate time: \( \hat{P}\hat{x}_0^L \hat{P}^{-1} \neq \hat{x}_0^L \). It is rather a ‘noncommutative’ or ‘deformed’ parity.

### 4 Forbidden Transition in Hydrogen Atom: A Quantum Field Theory Example.

We saw that energy levels of time-independent Hamiltonians in quantum mechanics cannot reveal effects of noncommutativity. We can examine time-dependent processes such as transitions between levels to see the effects of the latter. Other alternatives are interference phenomena \[8\]. We focus on the former here.

As an example of the transition induced by the noncommutative \( \theta \)-parameter, we examine the one-photon transition \( 2s \rightarrow 1s + \gamma \) in hydrogen. (For discussion of the effects of space-space noncommutativity in hydrogen atom, see \[9\].) It is forbidden for \( \theta = 0 \) in the absence of electron spin effects, but can occur for \( \theta \neq 0 \). It would be thus a genuine \( \theta \)-effect. But the amplitude vanishes at tree level for the same reason (standard rotational invariance) that it vanishes for \( \theta = 0 \). The one-loop amplitude is sensitive to the breakdown of standard
rotational symmetry and is not zero. However it is extremely small even when θ is of the order of \((TeV)^{-2}\). This transition in hydrogen is not a realistic process to detect or bound θ, as backgrounds, including the magnetic \(2s \rightarrow 1s + γ\) transition, will overwhelm effects of θ. So we do not present the one-loop calculation here.

In the final section, we speculate on more realistic processes to detect θ.

\textit{i) On Relative Coordinates.}\footnote{This section is based on work by A.P.B. with Sachin Vaidya.}

We have to clarify a conceptual issue before beginning the study of the process \(2s \rightarrow 1s + γ\).

In conventional physics, when \(θ = 0\) and there are \(N\) non-identical particles moving in \(\mathbb{R}^3\), the configuration space is \(\mathbb{R}^{3N}\) \cite{10}. The corresponding coordinate function \(\hat{x}^{(m)}\) \((m = 1, \ldots, N)\) of the \(m\)th particle is defined by

\[
\hat{x}^{(m)}_j(\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(N)}) = x^{(m)}_j.
\]

There is only one time operator \(\hat{x}^0\) common to all particles.

The \(θ \neq 0\) generalization of this space-time algebra has the same commutator of \(\hat{x}^0\) with all \(\hat{x}^{(m)}\) (see Eq. (2.3)),

\[
[\hat{x}^0, \hat{x}^{(m)}_j] = iθs_i
\]

while spatial coordinate functions have vanishing commutators:

\[
[\hat{x}^{(m)}_i, \hat{x}^{(n)}_j] = 0, \ m,n \in [1, 2, \ldots, N], \ i,j \in [1, 2, 3].
\]

It follows that relative spatial coordinates commute with \(\hat{x}^0\):

\[
[\hat{x}^0, \hat{x}^{(m)}_i - \hat{x}^{(n)}_i] = 0.
\]

The algebra generated by \(\hat{x}^0, \hat{x}^{(m)}_i - \hat{x}^{(n)}_i\) is not sensitive to \(θ\). It is just a commutative algebra.

We can describe the situation in another way. Let us associate the \(θ \neq 0\) space-time algebra of \(\hat{x}^0, \hat{x}\) to one of the particles, say 1, chosen at random. Then the space-time algebra of \(m\)th particle is obtained by spatial translations: It has generators \(\hat{x}^0, \hat{x} + \hat{a}^{(m)}_i\), where \(\hat{a}^{(m)}_i\) is the relative coordinate \(\hat{x}^{(m)}_i - \hat{x}^{(1)}_i\).

For this reason, spatial rotations act in a standard way, with angular momenta \(L_i = -i(\hat{a}_i \wedge \nabla)_i, \ \nabla_i = \frac{\partial}{\partial a_i}\) on relative coordinates. Now suppose the commutative Hamiltonian has rotational symmetry. Then its noncommutative version restricted to relative coordinate-time algebra would also have that symmetry, unless technical problems like factor-ordering interfere. We need a nontrivial presence of the noncommutative algebra of the center of mass before \(θ\)-effects show up.

For such reasons, the \(θ\)-effect does not show up in the process \(2s \rightarrow 1s + γ\) at tree level. But it does show up at one loop.
ii) The Fields and the Hamiltonian.

Quantum field theory (QFT) gives a conceptually clean approach to study our process.

For \( \theta \neq 0 \), the second quantized (free) photon field has the mode expansion
\[
\hat{A}_i(\vec{x}, \hat{x}_0) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_{\alpha} \left( a(\vec{k}, \alpha) e^{i\vec{k} \cdot \vec{x}} e^{-i\omega \hat{x}_0} + e^{i\omega \hat{x}_0} e^{-i\vec{k} \cdot \vec{x}} a(\vec{k}, \alpha)^\dagger \right),
\]
\[
[a(\vec{k}, \alpha), a(\vec{k}', \beta)^\dagger] = \delta^3(\vec{k}' - \vec{k}) \delta_{\alpha\beta} \text{ etc.},
\]
\[
\omega = |\vec{k}|,
\]
\( \alpha \) denoting photon polarization. We work in radiation gauge:
\[
\vec{k} \cdot \vec{e}^{(\alpha)}(\vec{k}) = 0.
\]
\( \vec{e}^{(\alpha)}(\vec{k}) \) is a discrete level and \( \vec{k} \) come from center-of-mass motion.

As for the hydrogen atom, we denote the electron-proton relative coordinate by \( \vec{a} \). As we discussed, it commutes with \( \hat{x}_0 \). If the electron and proton have masses \( m \) and \( M \) and coordinates \( \vec{x}^{(e)} \) and \( \vec{x}^{(p)} \), the center of mass coordinate is
\[
\vec{x} = \frac{m \vec{x}^{(e)} + M \vec{x}^{(p)}}{m + M}.
\]
It has the commutator
\[
[\hat{x}_0, \hat{x}_i] = i\theta \delta_{ii}
\]
with \( \hat{x}_0 \).

The hydrogen atom bound state wave functions for energies \( E_n \) can be denoted by \( \phi_n \) and continuum wave functions of energy \( E \) by \( \phi_E \). (We ignore spin effects. \( n \) is a discrete level and \( E \) is energy.) They are functions of \( \vec{a} \) and have the normalization
\[
(\phi_n, \phi_m) = \int d^3a \ \bar{\phi}_n(a) \phi_m(a) = \delta_{nm},
\]
\[
(\phi_E, \phi_{E'}) = \delta(E - E'), \text{ etc.}
\]

The second-quantized (non-relativistic) hydrogen field \( \Psi \) is given by
\[
\Psi(\vec{x}, \vec{a}, \hat{x}_0) = \int \frac{d^3p}{(2\pi)^{3/2}} \left[ \sum_n a_n(\vec{p}) \phi_n(\vec{a}) e^{-iE_n \hat{x}_0} + \int dE \ a_E(\vec{p}) \phi_E(\vec{a}) e^{-iE \hat{x}_0} \right] e^{i\vec{p} \cdot \vec{x}} e^{-i\frac{p^2}{2(m + M)} \hat{x}_0},
\]
\[
[a_n(\vec{p}), a_m(\vec{p}')^\dagger] = \delta_{nm} \delta^3(\vec{p} - \vec{p}') \ ,
\]
\[
[a_E(\vec{p}), a_{E'}(\vec{p}')^\dagger] = \delta(E - E') \delta^3(\vec{p} - \vec{p}') \text{ etc.}
\]
The labels \( \vec{p}, \vec{p}' \) and the factor \( e^{i\vec{p} \cdot \vec{x} - i\frac{p^2}{2(m + M)} \hat{x}_0} \) come from center-of-mass motion.

For purposes of illustration, it is enough to couple the photon field just to the electron. The single-particle interaction Hamiltonian linear in \( \hat{A} \) is then
\[
-e \left( \vec{P} \cdot \vec{A}(\hat{x}^{(e)}) + \vec{A}(\hat{x}^{(e)}) \cdot \vec{P} \right) = -2e \vec{A}(\hat{x}^{(e)}) \cdot \vec{P}, \quad \vec{P}_i = -i \frac{\partial}{\partial \hat{x}_i^{(e)}}
\]
in view of (4.14). Here \( \hat{x}^{(e)} = \hat{x} + (1 - \mu)\hat{a} \) and \( \mu = \frac{m}{m + M} \) is the reduced mass. The QFT interaction Hamiltonian is thus

\[
H_I = -2e \int d^3 a \, S \left( \hat{\Psi}^\dagger \hat{A}(\hat{x} + (1 - \mu)\hat{a}) \cdot \hat{P} \hat{\Psi} \right),
\]

where the positive map \( S \) refers to the algebra of \( \hat{x}_\mu \) and it is to be evaluated at some time \( x_0 \) (the value of \( x_0 \) does not affect final answers).

The free Hamiltonian \( H_0 \) is the sum of those for hydrogen and photon. The interaction representation \( S \)-matrix is

\[
S = T \exp \left( -i \int_{-\infty}^{\infty} d\tau \, U_0^{-1}(\tau)H_IU_0(\tau) \right),
\]

\[U_0(\tau) = \exp(-i\tau H_0).\]

When the 2s level at rest decays into 1s + photon \( \gamma \), the momenta of 1s and \( \gamma \) being \( \pm \vec{k} \) and photon helicity being \( \lambda \), the first order transition matrix element is

\[
T^{(1)} = -i \int_{-\infty}^{\infty} d\tau \, \langle 1s(\vec{k}), \lambda(-\vec{k}) | e^{ix\vec{H}_0} H_I e^{-ix\vec{H}_0} | 2s \rangle. \tag{4.15}
\]

We now isolate the integral involving relative coordinate here and show that it vanishes.

As the final state involves photon of momentum \( -\vec{k} \) and helicity \( \lambda \), the component

\[
a(\vec{k}, \lambda)\epsilon_\mu^{(\lambda)}(\vec{k})e^{-i\omega\hat{x}_0 - i\vec{k} \cdot \hat{\varepsilon}_\mu^{(\lambda)}} = a(\vec{k}, \lambda)\epsilon_\mu^{(\lambda)}(-\vec{k})e^{-i\omega\hat{x}_0 - i\vec{k} \cdot (\vec{x} + (1 - \mu)\hat{a})} \tag{4.16}
\]

of \( \hat{A}_i \) is picked out in the matrix element of (4.15). It gets multiplied by \( \hat{P}_i \hat{\Psi} \). But the 1s state has momentum \( \vec{k} \) so that \( \hat{P}_i \hat{\Psi} \) has factor \( k_i \). Since \( \vec{\varepsilon}(\vec{k}) \cdot \vec{k} = 0 \) the entire matrix element vanishes:

\[
T^{(1)} = 0.
\]

As the \( S \)-matrix has been presented in the second-quantized formalism, the process \( 2s \rightarrow 1s + \gamma \) can be investigated beyond the tree approximation.

5 Discussion: The Decay \( Z^0 \rightarrow 2\gamma \)

As we saw in previous sections, selection rules from rotational symmetry for \( \theta_{0i} = 0 \) are not in general respected in scattering and decay processes when \( \theta_{0i} \neq 0 \). One candidate for such a process is the decay of a massive vector particle into two photons, such as \( Z^0 \rightarrow 2\gamma \). Though one can easily write an effective Lagrangian density \( L_{\text{int}} \) for this process, the resulting amplitude is zero if \( \theta_{0i} = 0 \). For example, let us consider \( L_{\text{int}} \sim F_{\mu\nu}G^{\rho\sigma}(*F)^\rho_\sigma \), where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \),
\[ G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \text{ and } (\ast F)_{\mu\nu} \text{ is dual of } F_{\mu\nu}. \] 
\[ A_\mu \text{ and } B_\mu \text{ are massless and massive vector fields respectively.} \]

The decay amplitude is then proportional to

\[ \mathcal{A} \sim \varepsilon^{\mu\rho\eta\gamma}k_\eta \epsilon'_\gamma(k_\mu \epsilon'\nu - k'_\nu \epsilon_\mu)(p_\rho \varepsilon_\rho - p_\eta \varepsilon_\nu), \] 

(5.17)

where \( k_\mu, \epsilon_\mu \) and \( k'_\mu, \epsilon'_\mu \) are momentum and polarization of photons and \( p_\mu, \varepsilon_\mu \) those of the massive vector particle. Calculation shows that this is zero upon using the transversality conditions on the polarization vectors. This result holds in general.

The consideration of the above process with time-space noncommutativity requires a better understanding of quantum theory when \( \theta_{0i} \neq 0 \). The work under progress indicates that it would occur when \( \theta_{0i} \neq 0 \).

In conclusion, in this paper we have considered the effects of time-space noncommutativity due to deformation of the rotation symmetry and parity in the case of nonzero \( \theta_{0i} \). It is argued that many processes that are forbidden in the commutative case become allowed by this deformation. Our point is supported by the explicit calculation of the decay rate of the transition \( 2s \rightarrow 1s + 2\gamma \) in hydrogen atom. Comments on the processes like \( Z^0 \rightarrow 2\gamma \) have also been made.

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**Another choice is \( L_{int} \sim F_{\mu\nu}G^{\mu\rho}F_{\rho}^{\nu} \). But this is zero just due to the antisymmetry of \( F_{\mu\nu} \) and \( G_{\mu\nu} \).**
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