Matrix ansatz for the fluctuations of the current in the ASEP with open boundaries

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Abstract
The asymmetric simple exclusion process (ASEP) is one of the most extensively studied models in non-equilibrium statistical mechanics. The macroscopic particle current produced in its steady state is directly related to the breaking of detailed balance, and is therefore a physical quantity of particular interest. In this paper, we build a matrix product ansatz which allows us to access the exact statistics of the fluctuations of that current for finite sizes, as well as the probabilities of configurations conditioned on the mean current. We also show how this ansatz can be used for the periodic ASEP and how it translates into the language of the XXZ spin chain.

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(Some figures may appear in colour only in the online journal)

1. Introduction
In the study of systems of many interacting particles, one of two situations might arise. If the system is isolated, or if the interaction with its environment allows it (is invariant under the reversing of time, for instance), one could observe an equilibrium state in which the probability of any configuration is simply related to the energy of that configuration via the Gibbs–Boltzmann law. In principle, from that information, and the expression of those energies, one could calculate any equilibrium quantity that might take ones fancy. Obviously, those calculations could still be extremely difficult, but the general framework provided by the Gibbs–Boltzmann law systematically solves the first part of the problem. In the other situation, where no equilibrium state can be observed, there is no such framework. The system might reach a non-equilibrium steady state, where the probabilities of the configurations do not depend on time, but there is no way to calculate those probabilities a priori, and one has to solve the whole dynamical equation that describes the evolution of the system to obtain the desired quantities.
There have been, however, many attempts to generalize the concept of Boltzmann weight to non-equilibrium systems. The best candidates to play the role of the energy in these systems are large deviation functions, which contain information on the probabilities of rare events or configurations, in the limit of some large extensive quantity (usually time or system size) [2,4–9]. Those large deviation functions have for instance helped uncover some very general symmetries, called 'fluctuation theorems', that are verified by systems however far from equilibrium [12,13]. The study of large deviation functions is therefore an important task to statistical physicists.

Many of those systems that display non-equilibrium steady states describe the transport of carriers (e.g. of mass, electrical charge or thermal energy), that may interact with each other, through some domain (a one-dimensional channel, for example), and driven by an external field in the bulk of the system and/or unbalanced reservoirs at its boundaries. One may think, for instance, of a metallic wire conducting electrons between two masses at different potentials, or an artery conducting blood cells between two organs at different pressures. Because of those driving forces, the system exhibits a non-vanishing macroscopic current in its steady state. That current is related to the microscopic entropy production that comes from the breaking of detailed balance and time reversal invariance, and which is characteristic of non-equilibrium states. (In some cases, the relation between the macroscopic current and microscopic entropy production is a very simple one, as one can see in [12].)

One of the simplest examples of driven particle models, and one of the most extensively studied, is the asymmetric simple exclusion process (ASEP). It consists of a one-dimensional lattice, the sites of which hold particles that jump forwards and backwards stochastically. The particles interact via hard-core repulsion, so that there can be only one particle on a given site at a given time (hence exclusion). They jump preferentially to one direction, which accounts for the driving force in the bulk of the system (and makes it asymmetric). Each side of the system is connected to a particle reservoir characterized by a fixed density. The ASEP has many qualities which explain the extent to which it has been studied in the past 20 years or so [1,7,8,11,14–19,31,53]. First of all, it is simple in its definition, and has the mathematical property of being integrable, which makes it a good candidate for analytical calculations and exact solutions. This integrability property implies that the methods used to treat the model are usually not general and transposable (except to other integrable systems), but the actual results could give us valuable insights into the general behaviour of generic non-equilibrium systems. Moreover, the ASEP has connections with many and various other systems, such as ribosomes travelling on a m-RNA strand [4,20–22] (which is what the ASEP was originally meant to describe), random polymer models [23], growing interfaces [10,24], pedestrian and car traffic [52], quantum spin chains [49], random matrices [25–27] and even pure combinatorics [29,30].

In this paper, we add our own effort to the long history of results on the steady state of the ASEP and the fluctuations of the current it exhibits, by solving the long-standing problem of obtaining the distribution of those fluctuations in the open case. Because the system is out of equilibrium, the choice of the boundary conditions matters greatly, much as for systems with long-range interactions. The statistical ensembles are not equivalent, and fixing the number of particles (as in the periodic case) or not (as in the open case) makes the results, as well as the methods applicable to their acquisition, significantly different. For instance, the full current fluctuations for the periodic TASEP (where the 'T' stands for 'totally', i.e. the particles jump in only one direction) were obtained by the Bethe ansatz in [35], and the same method was used many years later for the periodic ASEP in [36], but that method turned out to be inapplicable to the open case, precisely because of the non-conservation of the total number of particles. The Bethe ansatz could also have led to the distribution of the steady state probabilities in
the periodic case, had it not been trivially flat, but the same could not be done in the open case (we now know that the Bethe ansatz, or some variation of it, can be used to access the steady state of the open ASEP [40], but not the fluctuations of the current; more details on this are given in the conclusion), and another method was devised, namely the ‘matrix ansatz’ [1, 33], by which some recursion relations [32] that were found in the weights of configurations of systems of different sizes are expressed through algebraic relations. The first extension of that method was used to obtain the diffusion constant (second cumulant of the current) in the open ASEP [34]. During the last couple of years, the author and collaborators managed to generalize that method further to calculate all the cumulants in the open case, first for the TASEP [42], and then for the ASEP [43]. In both cases, the results are exact for any values of the parameters and for any finite size of the system. However, at the time, large portions of the proof were guessed rather than fully worked out (more precisely, the formulation used was that of section 4.3, and the justification for the ansatz was an inelegant version of equations (23), (25) and (26), checked only for the TASEP in the case of [42], which was much too ad hoc to be of interest), so that the results were given as a conjecture along with numerical evidence to support them. In this paper, we give the complete algebraic proof of the validity of our ansatz, and explain how it gives us access not only to the fluctuations of the current, but also to any spatial observable conditioned on the mean current. We also show how the ansatz extends to the periodic case, and to the spin-$\frac{1}{2}$ XXZ chain with non-diagonal boundary terms.

The layout of the paper is as follows: in section 2, we define the model, and do a quick review of the matrix ansatz for the steady state; in section 3, we recall a few well-known results as a preliminary to our calculations: we restate the problem of finding the fluctuations of the current as that of finding the first eigenvalue of a deformed Markov matrix, and we define the ‘s-ensemble’ as the distribution of the principal eigenstate of that same matrix; section 4 constitutes the core of this paper and contains our new results: we present our perturbative matrix ansatz, along with its proof (the technical portions of which are carried out in the appendices), an alternative formulation (which was used in [43]) and its equivalent for the periodic case and the XXZ chain; finally, in section 5, we give a quick overview of how the explicit calculations of the cumulants of the current were carried out in [42, 43] using our ansatz.

2. Definition of the model and steady state properties

2.1. The open ASEP

The open ASEP in continuous time is defined as follows. Let us consider a chain of $L$ sites, numbered 1 through $L$, each of which can hold at most one particle. The occupation of site $i$ is denoted $\tau_i$ (equal to 0 for an empty site, and 1 for an occupied site). Those particles can jump stochastically to the right with rate $p = 1$ and to the left with rate $q < 1$, provided that the receiving site is empty. In addition, each end of the chain is connected to a reservoir of particles, so that particles may enter the system at site 1 with rate $\alpha$ or at site $L$ with rate $\delta$, and exit the system from site 1 with rate $\gamma$ and from site $L$ with rate $\beta$ (figure 1) (the rate $p$ can be set to 1 without any loss of generality).

At any time $t$, the system can be described by the probability vector $\langle|P_t\rangle\rangle$, the entries $P_t(C)$ of which give the probability of being in the configuration $C = (\tau_i)_{1..L}$ at time $t$. This probability depends only on the initial condition $\langle|P_0\rangle\rangle$ and verifies the master equation:

$$\frac{d}{dt}\langle|P_t\rangle\rangle = M\langle|P_t\rangle\rangle$$

(1)
Figure 1. Dynamical rules for the ASEP with open boundaries. The rate of forward jumps has been
normalized to 1. Backward jumps occur with rate $q < 1$. All other parameters are arbitrary. The
jumps shown in green are allowed by the exclusion constraint. Those shown in red and crossed out
are forbidden.

solved by

$$||P_t|| = e^{tM}||P_0||,$$

where the Markov matrix $M$ is given by

$$M = M_0 + \sum_{i=1}^{L-1} M_i + M_L$$

with $M_i$ containing the jumping rates between sites $i$ and $i + 1$, and $M_0$ and $M_L$ corresponding
to the couplings with the left and right reservoirs (see equation (10) with $\mu$ set to 0 for the
explicit expression of those matrices).

Each non-diagonal entry of $M$ contains one of the aforementioned rates, and is non-zero
only if the initial and final configurations differ by the jumping of exactly one particle. The
diagonal entries contain (minus) the rate at which the system leaves a given configuration,
which is also the sum of the rates from that configuration to any other, so that the sum of each
column of $M$ is 0 (i.e. the evolution of the system conserves probability and $M$ is a stochastic
matrix). From the Perron–Frobenius theorem, we know that the largest eigenvalue of $M$
is 0, and that, for large times, $||P_t||$ converges to the corresponding right eigenvector $||P^\ast||$ (the
corresponding left eigenvector is the vector with all entries equal to 1, and will be denoted by
$\langle\langle 1||$).

2.2. Matrix ansatz for the steady state

The exact expression of the steady state $||P^\ast||$ can be written in the form of a matrix product
state (also called matrix ansatz) [33]

$$P^\ast(C) = \frac{1}{Z_L} \langle W| \prod_{i=1}^{L} (\tau_i D + (1 - \tau_i) E) |V\rangle,$$

where the matrices $D$ and $E$, and the vectors $|W\rangle$ and $|V\rangle$, are defined by the following algebraic
relations:

$$DE - qED = (1 - q)(D + E)$$

$$\langle W| (\alpha E - \gamma D) = (1 - q) |W\rangle$$

$$\langle \beta D - \delta E | V\rangle = (1 - q) |V\rangle,$$

and $Z_L = \langle W|(D + E)^L |V\rangle$.

Let us remark here that throughout this paper, we use the standard bra/ket notation for
vectors from the space on which matrices $D$ and $E$ act (e.g. $|V\rangle$), and the doubled bra/ket
notation for vectors from the configurations space (e.g. $||P^\ast||$).
As stated in (4), the weight of a given configuration $C = (\tau_i)_{i \leq L}$ in $||P^\tau||$ can be written as the product of $L$ matrices $D$ or $E$, contracted between two vectors. The $i$th matrix in the product corresponds to the occupation of the $i$th site in $C$: it is $D$ if $\tau_i = 1$ and $E$ if $\tau_i = 0$. The first algebraic relation in (5) encodes a recursion between the steady state probabilities of systems of successive sizes. Combined with the two other relations, it allows, in principle, to compute explicit expressions of any of those probabilities, although that computation would be extremely impractical. However, it can be used, in a most elegant manner, to compute the mean values of certain observables, like the particle current passing through the system, or the local density [32], which are more physically relevant than the probability of a single configuration. For instance, for a system of size $ML$, the stationary current $J$ is given by $J = (1-q)Z_{L-1}/Z_L$, and can be expressed in terms of certain orthogonal polynomials [29, 38]. Those calculations are especially easy to carry out in the simpler case of the TASEP [33] (for which $q = \gamma = \delta = 0$), and even more so in the case $\alpha = \beta = 1$ [30].

In order to access the fluctuations of the current, and not only its mean value, we need to find the steady state probabilities as a function of the time-integrated current as well as of the configuration. That is what we propose to do in this paper.

3. Current-counting Markov matrix and the s-ensemble

Suppose that we want to keep track of the number of particles that jump over the bond that links sites $i$ and $i+1$ through the evolution of the system, starting from some initial state $||P_0||$. One easy way to do this is to multiply the off-diagonal entries of $M_i$ by a fugacity $e^{\pm \mu_i}$, so that every time those jumping rates are used in the evolution of the system, the weight of the corresponding history gets multiplied by $e^{\pm \mu_i}$ if the jump was made forwards, or $e^{-\mu_i}$ if it was made backwards. After a time $t$, the weight of any history $C(t)$ carries an extra weight $e^{J(C(t))t\mu}$, where $J(C(t))$ is the (algebraic) number of particles that jumped from site $i$ to site $i+1$, which is precisely the time-integrated current that went through that bond. One can then access the moments of that current simply by taking derivatives with respect to these currents, at time $t$. Taking a $k$th order derivative in any

$$
||P_t(\{\mu_i\})|| = e^{M_{\mu_t}}||P_0||
$$

which is the Laplace transform of the joint probabilities of the configurations and the time-integrated currents, with respect to these currents, at time $t$. Taking a $k$th order derivative in any

$$
M_0(\mu_0) = \begin{bmatrix}
-a & q e^{-\mu_0} \\
-a & -q
\end{bmatrix},
M_i(\mu_i) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -q e^{\mu_i} & 0 \\
0 & q e^{-\mu_i} & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
M_L(\mu_L) = \begin{bmatrix}
-a & q e^{-\mu_L} \\
-b & -b
\end{bmatrix}.
$$

(It is implied that $M_0$ acts as written on site 0 in the basis $\{0, 1\}$ and as the identity on the other sites, and the same goes for $M_L$ on site $L$; similarly, $M_i$ is expressed by its action on sites $i$ and $i+1$ in the basis $\{00, 01, 10, 11\}$ and acts as the identity on the rest of the system.)

By using this deformed Markov matrix in the time evolution of $||P_t||$ instead of the usual one, one obtains

$$
||P_t(\{\mu_i\})|| = e^{M_{\mu_t}}||P_0||
$$

where

$$
M_0(\mu_0) = \begin{bmatrix}
-a & q e^{-\mu_0} \\
-a & -q
\end{bmatrix},
M_i(\mu_i) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -q e^{\mu_i} & 0 \\
0 & q e^{-\mu_i} & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
M_L(\mu_L) = \begin{bmatrix}
-a & q e^{-\mu_L} \\
-b & -b
\end{bmatrix}.
$$

(7)
of the $\mu_i$s and then projecting this vector onto $\langle 1| \rangle$ yields the $k$th moment of the corresponding current, up to a normalization.

In the long time limit, the matrix $e^{tM_{(\mu)}}$ converges to the projector onto its principal eigenvector $|P_{(\mu)}\rangle$, with the eigenvalue $e^{E(\mu)}$ where $E(\mu)$ is the largest eigenvalue of $M_{(\mu)}$, so that

$$\langle 1|P_{(\mu)}\rangle \sim e^{E(\mu)}$$

and $E(\mu)$ is therefore identified as the generating function of the cumulants of the instantaneous currents $J_i/t$, which is the Laplace transform of the large deviation function of those same currents, as explained by the Donsker–Varadhan theory of temporal large deviations $[2, 50]$. That is the main quantity that we want to calculate. Our problem thus reduces to that of finding the largest eigenvalue of the current-counting matrix $M_{(\mu)}$. The corresponding eigenvector $|P_{(\mu)}\rangle$ also holds important information, as we will see below.

Let us first make things a little simpler by noting that one can go from any set $\{\mu_i\}$ to any other set $\{\mu'_i\}$ by a matrix similarity, as long as $\sum_{i=0}^{d} \mu_i = \sum_{i=0}^{d} \mu'_i = \mu$ [12]. This means that the eigenvalues of $M_{(\mu)}$ only depend on $\mu$, regardless of how the fugacities are distributed: the currents through each of the bonds are all exactly equivalent. In particular, there is a set for which $\mu = \lambda \log \left( \frac{\bar{e}^{qL}}{\bar{e}^{qS}} \right)$, where $\lambda$ is the quantity conjugate to the entropy production in the system. This is an easy way to prove the Gallavotti–Cohen symmetry for the current in this system, and it shows that the current and its fluctuations are simply proportional to the entropy production. That entropy production being non-zero is the defining characteristic of a non-equilibrium system.

All this being said, we can now consider, without any loss of generality, the case where only the first bond (between the leftmost reservoir and the first site) is marked: $\mu_0 = \mu$, $\mu_i \neq 0 = 0$, so that the individual jump matrices we will work with are given by

$$M_0(\mu) = \begin{bmatrix} -\alpha & \gamma e^{-\mu} \\ \alpha e^{\mu} & -\gamma \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -q & 1 \\ 0 & q & -1 \end{bmatrix}$$

$$M_L = \begin{bmatrix} -\delta & \beta \\ \delta & -\beta \end{bmatrix}$$

and that

$$|P_{(\mu)}\rangle = e^{M_{(\mu)}}|P_0\rangle \sim e^{E(\mu)}|P_{(\mu)}\rangle$$

with

$$E(\mu) = \sum_{k=1}^{\infty} E_k \frac{\mu^k}{k!}$$

being the exponential generating series of the cumulants of the current.

The large deviation function for the instantaneous current, $G(j) \sim - \frac{1}{t} \log[P(J/t = j)]$, is related to $E(\mu)$ by

$$G(j) = \mu j - E(\mu), \quad \frac{d}{d\mu} E(\mu) = j.$$  

Moreover, one can show [39] that the principal right eigenvector $|P_{(\mu)}\rangle$ of $M_{\mu}$ and its principal left eigenvector $\langle \tilde{P}_{(\mu)} |$ hold, respectively, the probabilities of observing configurations coming from the steady state and the probabilities of having come from configurations while in the steady state, all conditioned on having observed a mean current $j = \frac{d}{d\mu} E(\mu)$:

$$P_{\mu}(\mathcal{C}) = P\left( |\mathcal{C}, t\rangle | P^* , t = -\infty \right) \land j = \frac{d}{d\mu} E(\mu)$$

$$\tilde{P}_{\mu}(\mathcal{C}) = P\left( |\mathcal{C}, t\rangle | P^* , t = +\infty \right) \land j = \frac{d}{d\mu} E(\mu)$$
so that the product of the two is the probability of observing a configuration at any time, conditioned on the mean current (up to a normalization)

\[ P^\mu(C)\tilde{P}^\mu(C) = P\left( C \left| j = \frac{d}{d\mu} E(\mu) \right. \right). \]  

(16)

The ensemble defined by these probabilities, with \( \mu \) as a parameter, is sometimes called the ‘s-ensemble’ [51] (the reason being that \( \mu \) is often denoted \( s \)), and contains all the information needed to build the joint large deviation functions of the current and any spatial observables (i.e. depending only on \( C \)). We will now construct a matrix ansatz that holds the exact expressions of those probabilities and of \( E(\mu) \), as series in \( \mu \), up to arbitrary orders.

4. Perturbative matrix ansatz for the s-ensemble

In this section, we will show that we can define two transfer matrices \( T^\mu \) and \( U^\mu \), such that

\[ [M^\mu, U^\mu T^\mu] = 0 \]  

(17)

\[ T_0 = [1, C, C = |1\rangle\langle 1|] \]  

(18)

\[ U_0(|1\rangle = |P^*\rangle, \]  

(19)

where the weights of \( T^\mu \) and \( U^\mu \) are expressed as products of matrices between two vectors, much as in (4). Moreover, equation (17) for \( \mu = 0 \) allows us to recover the original matrix ansatz, as we will show below, and to define the matrix product states that were used in [42, 43] to obtain our previous results.

Those three relations will be used to prove that the transfer matrix \( U^\mu T^\mu \) is almost a projector onto the leading eigenstate of \( M^\mu \), and that when applied repeatedly, the precision in orders of \( \mu \) of the projection increases. In other terms:

\[ (U^\mu T^\mu)^k \sim |P^\mu\rangle\langle P^\mu| + O(\mu^k) \]  

(20)

up to a multiplicative constant of order 1.

We will then use that relation to show the main results of this paper, namely that

\[ |P^\mu\rangle = \frac{1}{Z_L^{(k)}} (U^\mu T^\mu)^k |P^*\rangle + O(\mu^{k+1}) \]  

(21)

\[ \langle \tilde{P}^\mu | = \frac{1}{Z_L^{(k)}} \langle 1 | (U^\mu T^\mu)^k + O(\mu^{k+1}), \]  

(22)

where \( Z_L^{(k)} = \langle 1 | (U^\mu T^\mu)^k |P^*\rangle \), and that

\[ E(\mu) = \frac{\langle 1 | M^\mu (U^\mu T^\mu)^k |P^*\rangle}{\langle 1 | (U^\mu T^\mu)^k |P^*\rangle} + O(\mu^{k+2}). \]  

(23)

Those results hold for any integer \( k \), so that, in essence, we have complete exact expressions for \( |P^\mu\rangle \), \( \langle \tilde{P}^\mu | \) and \( E(\mu) \), expanded as infinite series in \( \mu \).

4.1. Definitions and commutation relations

Let us consider two matrices \( d = D - 1 \) and \( e = E - 1 \), where \( D \) and \( E \) are defined as in (5), and a matrix \( A^\mu \), such that

\[ de = qed = (1 - q) \]

\[ eA^\mu = e^\mu A^\mu e \]

\[ A^\mu d = e^\mu d A^\mu. \]

(24)
The first of these relations, which is a simple consequence of (5), defines the algebra of a $q$-deformed harmonic oscillator [54], of which $e$ is the creation operator and $d$ is the annihilation operator.

Let us also define two more boundary vectors ($\tilde{V}$ and $\tilde{W}$) by

\[
[\beta(1 - d) - \delta(1 - e)][V] = 0
\]

\[\langle \tilde{W} | \alpha(1 - e) - \gamma(1 - d) \rangle = 0
\]

and let us recall that

\[
[\beta(1 + d) - \delta(1 + e)][V] = (1 - q)[V]
\]

\[\langle W | \alpha(1 + e) - \gamma(1 + d) \rangle = (1 - q)|W|.
\]

By writing

\[X_{0,0} = X_{1,1} = 1, \quad X_{1,0} = d, \quad X_{0,1} = e,
\]

we can finally define the weights of $T_{\mu}$ and $U_{\mu}$ between configurations $C' = (\tau_j')_{1..L}$ and $C = (\tau_j)_{1..L}$:

\[
U_{\mu}(C, C') = \frac{1}{Z_{L}} |W| A_{\mu} \prod_{i=1}^{L} X_{\tau_i, \tau_i'} |V|
\]

with $Z_{L} = |W|(2 + d + e)^L |V|$ and

\[
T_{\mu}(C, C') = \langle \tilde{W} | A_{\mu} \prod_{i=1}^{L} X_{\tau_i, \tau_i'} |\tilde{V}|
\]

Those weights are entirely determined by the algebra defined above. Specifically, one can obtain the weights of $U_{\mu}$ by using (24) and (26), and those of $T_{\mu}$ by using (24) and (25).

We may note that the matrix $A_{\mu}$ is set between the left boundary vector and the first matrix because it is the bond between the left reservoir and the first site that is marked. For a general set of weights $\{|\mu\}$, we would have to add matrices $A_{\mu}$ between $X_{\tau_i, \tau_i'}$ and $X_{\tau_{i+1}, \tau_{i+1}}$ in both of the products above, so that

\[
U_{\{|\mu\}}(C, C') = \frac{1}{Z_{L}} |W| A_{\{|\mu\}} \prod_{i=1}^{L} X_{\tau_i, \tau_i'} |V|
\]

\[
T_{\{|\mu\}}(C, C') = \langle \tilde{W} | A_{\{|\mu\}} \prod_{i=1}^{L} X_{\tau_i, \tau_i'} |\tilde{V}|
\]

In appendix A, we derive equation (17), using a method closely related to the matrix ansatz for multispecies ASEP on a ring [37], and which makes use of the so-called hat matrices. That equation is the main point to our ansatz: the transfer matrix $U_{\mu} T_{\mu}$ that we built has the same eigenvectors as $M_{\mu}$, so that we can try to extract the information we need from it instead of $M_{\mu}$.

For $\mu = 0$, one particular solution to (24) and (25) is $d = e = A_0 = 1$, so that for any $C'$ and $C$, we have $T_0(C, C') = |V|/|V|$, which we can set to 1. This proves (18). Furthermore, projecting $U_0$ onto $||1\rangle$ means summing over all configurations $C'$ in (28), so that

\[
\langle ||1\rangle |U_0||1\rangle \rangle = \sum_{C'} U_0(C, C') = \langle V | A_0 \prod_{i=1}^{L} (X_{\tau_i,0} + X_{\tau_i,1}) |V\rangle
\]

We can set $A_0$ to 1, and remark that for $\tau_i = 0$, we have $(X_{\tau_i,0} + X_{\tau_i,1}) = 1 + e = E$ and that for $\tau_i = 1$, we have $(X_{\tau_i,0} + X_{\tau_i,1}) = d + 1 = D$, so that this expression is exactly that of $P^*(C)$ as given in (4), which proves (19).
Using relations (17)–(19) together, at \( \mu = 0 \), we obtain
\[
[M, U_0 T_0] = 0 = (M|P^\mu\rangle\langle 1|) - (|P^\mu\rangle\langle 1|M).
\] (33)

Since we know that \( \langle\langle 1|M = 0 \) (because \( M \) is a stochastic matrix), this implies that
\( M|P^\mu\rangle = 0 \), which yields the original matrix ansatz (4) [33].

This alternative proof of (4) relies on the fact that the transfer matrix \( U_\mu T_\mu \) is a projector in the limit \( \mu \to 0 \). It would be interesting to determine whether for other situations with matrix product states, one can generically find a transfer matrix that commutes with a deformation of the dynamics of the system and is a projector in the non-deformed limit. One could, for instance, look at the ASEP in discrete time with different versions of the update [48], or at the multispecies ASEP on a ring [37].

### 4.2. Validation of the perturbative matrix ansatz

To prove (21) and (23), we use the relations derived above. Since, for \( \mu = 0 \), the matrix \( U_0 T_0 \) is the projector onto the principal eigenspace of \( M \), one can write, for an infinitesimal \( \mu \):
\[
U_\mu T_\mu = \Lambda_\mu (|P^\mu\rangle\langle \tilde{P}_\mu| + r_\mu),
\] (34)
where \( \Lambda_\mu \sim 1 + \mathcal{O}(\mu) \) is the largest eigenvalue of \( U_\mu T_\mu \), and \( r_\mu \sim \mathcal{O}(\mu) \) is the part of \( U_\mu T_\mu \) that is orthogonal to its principal eigenspace, and has eigenvalues of order \( \mu \). In other words, \( U_\mu T_\mu \) is almost a projector, with an error \( r_\mu \) of order \( \mu \).

Since \( r_\mu (|P^\mu\rangle) = 0 \) and \( \langle\langle \tilde{P}_\mu| r_\mu = 0 \), one has that
\[
(U_\mu T_\mu)^k = (\Lambda_\mu)^k (|P^\mu\rangle\langle \tilde{P}_\mu| + r_\mu^k),
\] (35)
so that the difference from the projector onto \( |P^\mu\rangle\langle \tilde{P}_\mu| \) is now \( r_\mu^k \sim \mathcal{O}(\mu^k) \).

Let us now remark that the parts of \( |P^\mu\rangle \) and \( \langle\langle 1| \) which are not in the kernel of \( r_\mu \) are of order \( \mu \), so that both \( r_\mu^k |P^\mu\rangle \) and \( \langle\langle 1|\mu^k \) are of order \( \mu^{k+1} \). It follows that \( (U_\mu T_\mu)^k |P^\mu\rangle \) is proportional to \( |P^\mu\rangle \) with an error of order \( \mu^{k+1} \) (and the same goes for \( \langle\langle \tilde{P}_\mu| \rangle \), which proves (21).

Equation (23) is then proven by simply applying \( M_\mu \) to \( (U_\mu T_\mu)^k |P^\mu\rangle \):
\[
\langle\langle 1| (U_\mu T_\mu)^k |P^\mu\rangle \rangle = E(\mu) (\Lambda_\mu)^k \langle\langle 1| (P^\mu) \rangle\langle \tilde{P}_\mu| P^\mu \rangle + \langle\langle 1| M_\mu r_\mu^k |P^\mu\rangle \rangle
\] (36)
\[
\langle\langle 1| (U_\mu T_\mu)^k |P^\mu\rangle \rangle = (\Lambda_\mu)^k \langle\langle 1| (P^\mu) \rangle\langle \tilde{P}_\mu| P^\mu \rangle + \langle\langle 1| r_\mu^k |P^\mu\rangle \rangle,
\] (37)
where \( \langle\langle 1| (M_\mu r_\mu^k |P^\mu\rangle \rangle \) is of order \( \mu^{k+2} \) because \( \langle\langle 1| M_\mu \) is of order \( \mu \) and \( r_\mu^k \) is of order \( \mu^{k+1} \), and \( \langle\langle 1| r_\mu^k |P^\mu\rangle \rangle \) is of order \( \mu^{k+2} \) for the reason given above. The ratio of those two equations is therefore equal to \( E(\mu) \) up to order \( \mu^{k+2} \).

### 4.3. Formulation as a matrix product

The formulation we gave here of the perturbative matrix ansatz in terms of transfer matrices is quite different from that which was given in [43]. They are of course equivalent, which is what we will show in this section.

The main point that needs to be made here is that, unlike the Markov matrix, which is a sum of elementary matrices, the transfer matrices \( U_\mu \) and \( T_\mu \) are products of the elementary matrices \( X_{\tau_{i,j}} \), so that the product of those transfer matrices can be seen as a tensor network, the tensors being of order 4: \( (X_{\tau_{i,j}})_{i,j} \), where \( i \) and \( j \) are the internal indices of \( X \) (figure 2).
Figure 2. One of the tensor networks that add up to $U_{\mu}T_{\mu}|\!\!\!P^\ast\rangle\rangle$ when expanded in terms of the intermediate configurations at each step of the product. Rows represent the transfer matrices $U_{\mu}$ and $T_{\mu}$, whereas columns represent $E_k$ or $D_k$. See the detailed explanation below.

A consequence of this is that the object $(U_{\mu}T_{\mu})^{k}|\!\!\!P^\ast\rangle\rangle$ can be written in terms of the columns of this tensor network instead of the rows. Let us therefore define, by recursion (and denote the product between successive rows by a tensor product $\otimes$)

$$D_{k+1} = (1 \otimes 1 + d \otimes e) \otimes D_k + (1 \otimes d + d \otimes 1) \otimes E_k$$

$$E_{k+1} = (1 \otimes 1 + e \otimes d) \otimes E_k + (e \otimes 1 + 1 \otimes e) \otimes D_k$$

$$A_{\mu}^{(k+1)} = A_{\mu} \otimes A_{\mu} \otimes A_{\mu}^{(k)}$$

with $D_0 = D$, $E_0 = E$ and $A_{\mu}^{(0)} = 1$, and

$$|V_{k+1}\rangle = |V\rangle \otimes |\tilde{V}\rangle \otimes |V_k\rangle$$

$$|W_{k+1}\rangle = |W\rangle \otimes |\tilde{W}\rangle \otimes |W_k\rangle,$$

with $|V_0\rangle = |V\rangle$ and $|W_0\rangle = |W\rangle$.

In this formalism, equation (21) becomes

$$\langle\langle C|P_{\mu}\rangle\rangle = \frac{1}{Z_L} \langle W_k | A_{\mu}^{(L)} \prod_{i=1}^{L} (\tau_i D_k + (1 - \tau_i) E_k) |V_k\rangle \rangle + O(\mu^{k+1}).$$

To give a simple explicit example, one of the tensor networks from the expansion of $U_{\mu}T_{\mu}|\!\!\!P^\ast\rangle\rangle$ for $L = 5$, namely $U_{\mu}(C, C')T_{\mu}(C', C'')P^\ast(C'')$, with $C = [1, 0, 0, 1, 1], C' = [1, 0, 1, 1, 1]$ and $C'' = [0, 1, 0, 0, 1]$ is shown in figure 2. The blue rectangle corresponds to $T_{\mu}(C', C'')$, the black one to $P^\ast(C'')$ and the red rectangle is one of the elements in $D_1$, namely $X_{1,1} \otimes X_{1,0} \otimes E$. Summing over the second and third indices in any column gives $E_1$ or $D_1$, depending on whether the first (upper) index is 0 or 1.

While the transfer matrix formulation (21) is better suited to the algebraic proof of the ansatz, this matrix product formulation (41) is useful for doing explicit calculations, like those of the cumulants of the current (see section 5). Naturally, all the calculations done here using the transfer matrix formulation can be done with these matrix products (and were, for the most part, done that way in order to obtain our previous results [42, 43]), but are much more cumbersome and convoluted. Let us also note that the matrices $E_k$ and $D_k$ are related to the ones used in the matrix ansatz solution of the multispecies periodic ASEP [37], although for the moment we have no understanding of why that is the case.
4.4. Periodic case and XXZ spin chain

The same ansatz can be applied to the periodic case with just one alteration: instead of projecting the matrix products between boundary vectors, one has to take a trace. Moreover, only one transfer matrix $T_{\mu}^{\text{per}}$ needs to be defined. This can be written as

$$T_{\mu}^{\text{per}}(C, C') = \text{Tr}\left[A_\mu L \prod_{i=1}^{L} X_{\tau_i, \tau'_i}\right]$$

if the marked bond is the one between sites $L$ and 1. One can then show that $[M_\mu, T_{\mu}^{\text{per}}] = 0$ and the rest follows, by replacing $U_\mu T_\mu$ by $T_{\mu}^{\text{per}}$ in every equation, using the steady state $|1_N\rangle\rangle$ (with coefficient 1 for all configurations with $N$ particles), and making only one tensor product per order in (38). See appendix B for all the derivations related to the periodic case.

In appendix C, we show how a special choice of the parameters $\{\mu_i\}$ allows our ansatz to be applied to the spin-1/2 XXZ chain with non-diagonal boundaries. We also note that the structure of our construction is strikingly similar to that used in [47] to solve the Lindblad equation for the XXZ chain.

5. Calculating the cumulants of the current (a quick overview)

The principal interest of the ansatz we constructed is that it allows us to calculate the expected values of some observables without having to diagonalize $M_\mu$ explicitly. The question is, then, how we can use formula (23) to obtain explicit expressions of the cumulants of the current. As one can see in [42, 43], we did manage to perform that calculation. However, a certain amount of guesswork was used, and we believe that there is a simpler and more compact way to do it than the one we used. For that reason, we will not expose here the full detail of those (rather tedious) calculations, but rather a quick overview of the principles behind it. We hope to be able to do the full calculation in an elegant and self-sufficient way in the near future, and possibly to do the same for observables other than the cumulants of the current (as we said earlier, we should have access to any spatial observable in the s-ensemble).

5.1. Expressing $E(\mu)$ and $\mu$ as parametric infinite series

The main point of our reasoning from here on is that we expect the solution to have the same structure as in the periodic case [35], i.e. we expect to be able to write $E(\mu)$ and $\mu$ as two infinite logarithmic series in a parameter $B$ which goes to 0 with $\mu$:

$$E(\mu) = -\sum_{k=1}^{\infty} D_k \frac{B^k}{k}$$

$$\mu = -\sum_{k=1}^{\infty} C_k \frac{B^k}{k}$$

From calculations using our ansatz in the periodic TASEP, and comparing them to the results from the Bethe ansatz [35], we were able to determine that this parameter $B$ is proportional to $\left(\frac{1-e^{-\mu}}{\Lambda_\mu}\right)$, where we recall that $\Lambda_\mu$ is the largest eigenvalue of $(U_\mu T_\mu)$ (which is the one associated with $|1_N\rangle\rangle$, and goes to 1 when $\mu$ goes to 0).

Luckily, there is a way to write $|1_N\rangle\rangle$ as a series in $\left(\frac{1-e^{-\mu}}{\Lambda_\mu}\right)$, from which we could obtain the formulae we are looking for. Let us first define

$$\widetilde{U_\mu T_\mu} = \frac{1}{(1-e^{-\mu})} (U_\mu T_\mu - |1_N\rangle\rangle\langle 1|)$$

(45)
which is finite for $\mu \to 0$. We can now write (by taking formally the limit $k \to \infty$ in (21))

$$
\langle\langle P_\mu \rangle\rangle = \frac{U_\mu T_\mu}{\Lambda_\mu} \langle\langle P_\mu \rangle\rangle = \frac{\langle\langle P^* \rangle\rangle \langle\langle |1| + (1 - e^{-\mu})U_\mu T_\mu \rangle\rangle}{\Lambda_\mu} \langle\langle P_\mu \rangle\rangle
$$

(46)
or equivalently

$$
\Lambda_\mu \langle\langle P_\mu \rangle\rangle = \left(1 - \frac{1 - e^{-\mu}}{\Lambda_\mu} U_\mu T_\mu\right)^{-1} \langle\langle P^* \rangle\rangle = \sum_{k=0}^{\infty} \left(U_\mu T_\mu\right)^k \langle\langle P^* \rangle\rangle \left(\frac{1 - e^{-\mu}}{\Lambda_\mu}\right)^k
$$

(47)

which is a well-defined series in $\frac{(1-e^{-\mu})}{\Lambda_\mu}$.

From this, we obtain

$$
E(\mu) = \langle\langle |1| M_\mu \langle\langle P_\mu \rangle\rangle \rangle = \sum_{k=0}^{\infty} \frac{\langle\langle 1\rangle\langle\langle |1| M_\mu (U_\mu T_\mu)^k |P^* \rangle\rangle}{(1 - e^{-\mu})} \left(\frac{1 - e^{-\mu}}{\Lambda_\mu}\right)^{k+1}
$$

(48)

where $\frac{\langle\langle 1\rangle\langle\langle |1| M_\mu (U_\mu T_\mu)^k |P^* \rangle\rangle}{(1 - e^{-\mu})}$ is finite for $\mu \to 0$ because $\langle\langle |1| M_\mu \sim \mu \rangle\rangle$.

We also obtain, tautologically (since $\langle\langle |1| P_\mu \rangle\rangle = 1$),

$$
\mu = -\log[1 - (1 - e^{-\mu})\langle\langle |1| P_\mu \rangle\rangle] = -\log \left[ 1 - \sum_{k=0}^{\infty} \frac{\langle\langle 1\rangle\langle\langle |1| (U_\mu T_\mu)^k |P^* \rangle\rangle}{(1 - e^{-\mu})} \left(\frac{1 - e^{-\mu}}{\Lambda_\mu}\right)^{k+1} \right],
$$

(49)

which we can then expand in $\frac{(1-e^{-\mu})}{\Lambda_\mu}$.

5.2. Inferring the final formulae

There is a major difference between expressions (49) and (44) (or between (48) and (43)); the coefficients $C_k$ and $D_k$ should not depend on $\mu$, but $\frac{\langle\langle 1\rangle\langle\langle |1| M_\mu (U_\mu T_\mu)^k |P^* \rangle\rangle}{(1 - e^{-\mu})}$, for instance, does. From this point on, the calculations become less precise. The reasoning is as follows:

- we postulate that the coefficients $C_k$ and $D_k$ should be a somewhat identifiable part of, respectively, $\langle\langle 1\rangle\langle\langle |1| (U_\mu T_\mu)^k |P^* \rangle\rangle$ and $\langle\langle 1\rangle\langle\langle |1| (U_\mu T_\mu)^k |P^* \rangle\rangle$;
- we then calculate the equivalent terms for the periodic TASEP (which is the simplest case to which our ansatz applies), namely $\langle\langle 1_\infty \rangle\langle\langle |1_\infty| (T_{\mu}^{\text{per}})^k |1_\infty) \rangle\rangle$ and $\langle\langle 1_\infty \rangle\langle\langle |1_\infty| (M_{\mu}^{\text{per}} (T_{\mu}^{\text{per}})^k |1_\infty) \rangle\rangle$, for $k = 2$, using the matrix product formalism and the $q$-deformed oscillator algebra (which, for $q = 0$, becomes a random walk on $\boxplus$);
- we compare the result with the coefficients from [35], and identify in which part of the calculation they emerge;
- we isolate the corresponding part of $\langle\langle 1\rangle\langle\langle |1| (U_\mu T_\mu)^k |P^* \rangle\rangle$ and $\langle\langle 1\rangle\langle\langle |1| (M_{\mu} (U_\mu T_\mu)^k |P^* \rangle\rangle$, and inject it in (44) and (43);
- we check numerically (i.e. do exact calculations on small sizes and orders of $\mu$) that our conjecture is correct.

The results of this calculation for the open TASEP can be found in [42]. The generalization to the open ASEP [43] comes from applying the same reasoning to the periodic ASEP [36];
in that case, we also checked our results against DMRG calculations for low-order cumulants ($E_2$ to $E_4$) and larger sizes of the system (up to 100 sites).

6. Conclusion

In this paper, we define and expose the algebraic proof of a matrix ansatz which gives access to the principal eigenvalue and eigenvectors of the current-counting Markov matrix of the open ASEP, for any size and any value of the parameters. Using this ansatz, the author and collaborators were able to obtain exact expressions for the cumulants of the current in the open TASEP [42] and the open ASEP [43], which had been an open question for many years in the field of non-equilibrium statistical physics.

Much remains to be done on this subject, and we believe that this ansatz still has a lot to offer. For instance, as argued in section 3, the principal eigenvectors of $M_\mu$ hold the probabilities of observing a given configuration conditioned on the current flowing through the system; in other words, it should allow us to analyse the best profiles (in the sense of most probable) to produce a given atypical current. The question of finding the optimal path to produce a rare event is an important one, notably in the context of complex chemical reactions, and much work has been done to find algorithms that produce this optimal path [44]. Obtaining an exact analytical result on that type of problem could provide valuable insight or help devise simpler algorithms.

Another situation where our method could be of use is the symmetric exclusion process, for which many results are known for the large size limit, and have been obtained using a coarse-grained description of the system named ‘macroscopic fluctuation theory’ [45] and the related ‘additivity principle’ [7, 8], but only a few exist for finite sizes [46]. The limit $q \to 1$ cannot be taken directly in our results, but the present ansatz can still be applied to the symmetric case with a few crucial alterations. However, we have yet to analyse that case in detail, which we intend to do in the near future.

There remains also the question of determining how specific or general the method we have applied here could be. We have shown that by defining the transfer matrix $U_\mu T_\mu$ which commutes with $M_\mu$, and then taking $\mu$ to 0, one can retrieve the original matrix ansatz [33]. It would be interesting to know whether the same procedure can be applied to other models with matrix product eigenstates, and if there is anything general or generalizable to it. We would also like to have a clear idea of the physical significance of that transfer matrix, which we used as a mere calculation tool, but might be interesting in itself. Some preliminary results show that it is closely related to the algebraic Bethe ansatz, and we are confident that we will soon be able to demonstrate the precise nature of that relation. Up until now, the algebraic Bethe ansatz was known to be applicable to the open ASEP with a current-counting fugacity only at a finite set of points of its phase diagram [40] (i.e. such that the size, the boundary rates and $\mu$ are related by a constraint that has only a finite number of solutions), which is not enough to obtain the exact full statistics of the current (one would need an infinite number of values of $\mu$ for given values of the other parameters). In [19], that method was used for systems of large size, where the set of special points collapses onto a curve, to obtain the generating function of the cumulants of the current at the large size limit, but only in the case where that function is best behaved (that is, inside the high- and low-density phases). A clear understanding of the relation between the Bethe ansatz and our method could allow us to study not only the fluctuations of the mean current, but also the relaxation towards that current, and, in the most optimistic of scenarios, relate it to the results on Tracy–Widom distributions for that same relaxation in systems of infinite size [27, 28].
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Appendix A. Derivation of equation (17) for the open ASEP

The derivation of the commutation relation (17) can be carried out in two steps: first, we
express the commutator of $M_i$ (for $1 \leq i \leq L$) with either $U_\mu$ or $T_\mu$ (the two results are similar)
using (24), and show that for both the sum of those commutators cancels out except for two
terms, related to each of the boundaries. Secondly, we check that those boundary terms, as
they appear in the commutator of $M_\mu$ with the product $U_\mu T_\mu$, cancel out as well, using (25)
and (26).

For convenience, we will here write $U_\mu$ and $T_\mu$ as

$$U_\mu = \frac{1}{Z_L} \langle W | A_\mu \prod_{i=1}^L X^{(i)} | V \rangle$$

$$T_\mu = \langle \tilde{W} | A_\mu \prod_{i=1}^L X^{(i)} | \tilde{V} \rangle$$

with

$$X^{(i)} = \left[ \begin{array}{c} e \\ d \end{array} \right].$$

Let us therefore consider $[M_i, U_\mu]$ and $[M_i, T_\mu]$. The elementary matrix $M_i$ acts only on sites
$i$ and $i+1$, so that we only have to consider the commutation with the part of the matrix
products that corresponds to those sites. In both $U_\mu$ and $T_\mu$, that part is

$$X^{(i)} X^{(i+1)} = \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ d & 1 & e & e \\ d & e & 1 & e \\ d & d & e & 1 \end{array} \right].$$

$$M_i = \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & q & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

We now calculate

$$[M_i, X^{(i)} X^{(i+1)}] = \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ (1-q)d & de-q & 1-q ed & (1-q)e \\ (q-1)d & q-de & q ed-1 & (q-1)e \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$- \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & q(ed-1) & 1-ed & 0 \\ 0 & q(1-de) & de-1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$= \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ (1-q)d & de-q ed & (1-q)ed & (1-q)e \\ (q-1)d & (q-1)de & q ed-de & (q-1)e \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$= (1-q) \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ d & 1 & ed & e \\ -d & -de & -1 & -e \\ 0 & 0 & 0 & 0 \end{array} \right].$$

by using the first relation in (24) to go from the third line to the fourth.
Let us now define the ‘hat’ matrices $\hat{X}$ by $\hat{X}_{\tau, \tau'} = (-1)^{\tau} \frac{(1-q)}{2} X_{\tau, \tau'}$:

$$\hat{X}^{(i)} = \frac{(1-q)}{2} \begin{bmatrix} 1 & e & e & e \\ d & 1 & ed & e \\ -d & -de & -1 & -e \\ -dd & -d & -d & -1 \end{bmatrix} X$$  \hfill (A.6)$$

(where we denote by $\cdot$ the product in the two-dimensional space corresponding to the occupation number on one site).

If we replace the first or the second $X$ in $X^{(i)}X^{(i+1)}$ by $\hat{X}$, we obtain

$$\hat{X}^{(i)}X^{(i+1)} = \frac{(1-q)}{2} \begin{bmatrix} 1 & e & e & e \\ d & 1 & ed & e \\ -d & -de & -1 & -e \\ -dd & -d & -d & -1 \end{bmatrix} X^{(i+1)}$$  \hfill (A.7)$$

and we finally find that

$$[M_i, X^{(i)}X^{(i+1)}] = \hat{X}^{(i)}X^{(i+1)} - X^{(i)}\hat{X}^{(i+1)}.$$  \hfill (A.8)$$

The relation equivalent to this one in the matrix product formalism (i.e. after making multiple tensor products) can be related to the one found in [37] and used to derive the matrix ansatz for the multispecies periodic ASEP.

Putting this relation back in $U_\mu$ or $T_\mu$ and summing over $i$ cancels out all the terms except for those containing $\hat{X}^{(1)}$ and $\hat{X}^{(L)}$, because all the other $\hat{X}^{(i)}$ appear exactly twice (once in $[M_i, U_\mu]$ and once in $[M_{i+1}, U_\mu]$) with opposite signs. Ultimately, we obtain

$$\sum_{i=1}^{L-1} M_i, U_\mu = \frac{1}{Z_L} \langle W | A_\mu \hat{X}^{(1)} \prod_{i=2}^{L} X^{(i)} | V \rangle - \frac{1}{Z_L} \langle W | A_\mu \prod_{i=1}^{L-1} X^{(i)} \hat{X}^{(L)} | V \rangle$$  \hfill (A.9)$$

which we may write as

$$\sum_{i=1}^{L-1} M_i, U_\mu = \hat{U}_{\mu}^{(1)} - \hat{U}_{\mu}^{(L)}$$  \hfill (A.11)$$

so that

$$\sum_{i=1}^{L-1} M_i, U_\mu T_\mu = \hat{U}_{\mu}^{(1)} T_\mu - \hat{U}_{\mu}^{(L)} T_\mu + U_\mu \hat{T}_{\mu}^{(1)} - U_\mu \hat{T}_{\mu}^{(L)}.$$  \hfill (A.12)$$

We now need to check that $[M_0(\mu), U_\mu T_\mu] = -\hat{U}_{\mu}^{(1)} T_\mu - U_\mu \hat{T}_{\mu}^{(1)}$ and $[M_L, U_\mu T_\mu] = \hat{U}_{\mu}^{(L)} T_\mu + U_\mu \hat{T}_{\mu}^{(L)}$. As before, $M_0$ acts only on site 1, so that only $X^{(1)}$ is affected, and the same goes for $M_L$ and site $L$.

Let us recall

$$M_0(\mu) = \begin{bmatrix} -\alpha & \gamma e^{-\mu} \\ \alpha & -\gamma \end{bmatrix}, \quad M_L = \begin{bmatrix} -\delta & -\beta \\ \delta & -\beta \end{bmatrix}.$$  \hfill (A.14)$$
We calculate
\[
[M_0(\mu), X^{(1)}] = \begin{bmatrix} \gamma e^{-\mu d} - \alpha e^\nu e \frac{(\gamma - \alpha)e}{(\alpha - \gamma)d} \end{bmatrix}.
\]
\[
[M_L, X^{(L)}] = \begin{bmatrix} \beta d - \delta e \frac{(\beta - \delta)e}{(\delta - \beta)d} \end{bmatrix}.
\]
By projecting these equations on, respectively, \(\langle \hat{W} | A_\mu \rangle\) and \(|\hat{V}\rangle\), we obtain
\[
[M_0(\mu), \langle \hat{W} | A_\mu X^{(1)} \rangle] = \langle \hat{W} | (\gamma d - \alpha e)A_\mu (\alpha - \gamma)d (e e - \gamma e^\nu d) A_\mu \rangle
\]
\[
[M_L, A_\mu X^{(L)} \langle \hat{V} \rangle] = \begin{bmatrix} \beta d - \delta e \frac{(\beta - \delta)e}{(\delta - \beta)d} \end{bmatrix} \langle \hat{V} \rangle
\]
where we used the second and third relations in (24) to get rid of the \(\mu\)s. We naturally find the same expressions for \(|\hat{W}\rangle\) and \(|\hat{V}\rangle\).

We can then use relations (25) and (26) to simplify those four equations. We obtain
\[
[M_0(\mu), \langle W | A_\mu X^{(1)} \rangle] = (\alpha - \gamma) \langle W | A_\mu \begin{bmatrix} 1 & -e \\ d & -1 \end{bmatrix} \rangle + (1 - q) \langle W | A_\mu \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \rangle
\]
\[
[M_0(\mu), \langle \hat{W} | A_\mu X^{(1)} \rangle] = (\alpha - \gamma) \langle \hat{W} | A_\mu \begin{bmatrix} -1 & e \\ d & 1 \end{bmatrix} \rangle
\]
\[
[M_L, A_\mu X^{(L)} |\hat{V}\rangle] = (\beta - \delta) \begin{bmatrix} -1 & e \\ -d & 1 \end{bmatrix} |\hat{V}\rangle + (1 - q) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} |\hat{V}\rangle
\]
so that on site 1:
\[
[M_0(\mu), \langle W | A_\mu X^{(1)} \rangle \cdot \langle \hat{W} | A_\mu X^{(1)} \rangle] = (\alpha - \gamma) \langle W | A_\mu \begin{bmatrix} 1 & -e \\ d & -1 \end{bmatrix} \rangle \cdot \langle \hat{W} | A_\mu X^{(1)} \rangle
\]
\[
+ (1 - q) \langle W | A_\mu \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \rangle \cdot \langle \hat{W} | A_\mu X^{(1)} \rangle
\]
\[
+ (\langle W | A_\mu X^{(1)} \rangle \cdot (\alpha - \gamma) \langle \hat{W} | A_\mu \begin{bmatrix} -1 & e \\ d & 1 \end{bmatrix} \rangle)
\]
(1.18)

Seeing that \(\begin{bmatrix} 1 & -e \\ d & -1 \end{bmatrix} = X \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\) and that \(\begin{bmatrix} -1 & e \\ -d & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot X\), the first and third parts of the right-hand side of this equation cancel out.

What is more, we can write \((1 - q) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} X^{(1)} = \frac{(1 - q) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} X + (1 - q) X \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}{2}\) so that
\[
(1 - q) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} X^{(1)} = -\hat{X}^{(1)} \cdot X^{(1)} - X^{(1)} \cdot \hat{X}^{(1)},
\]
(1.19)
which means precisely that \([M_0(\mu), U_\mu T_\mu] = -\hat{U}_\mu^{(1)} T_\mu - U_\mu \hat{T}_\mu^{(1)}\). The exact same calculations on the other boundary lead to \([M_L, U_\mu T_\mu] = U_\mu^{(L)} T_\mu + U_\mu \hat{T}_\mu^{(L)}\), and this concludes the proof.

Appendix B. Derivation of equation (17) for the periodic case

The periodic case is much simpler than the open one. Equation (17) takes the form
\[
[M_\mu, T_\mu^{\text{per}}] = 0
\]
(1.1)
In this section, we explain how our construction for the open ASEP can be translated for the spin-1/2 XXZ chain with non-diagonal boundary conditions [49] and hint at a possible relation to the recent solution of the XXZ chain with a Lindblad boundary drive [47].

We have

\[ M_\mu = \begin{bmatrix} A_\mu & A_\mu e^{-\mu} & eA_\mu & eA_\mu e^{-\mu} \\ A_\mu d & A_\mu d & A_\mu d & A_\mu d \\ dA_\mu d & dA_\mu d & dA_\mu d & dA_\mu d \\ dA_\mu d & dA_\mu d & dA_\mu d & dA_\mu d \end{bmatrix}, \quad M_{L, \mu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & e^{\mu} & 0 \\ 0 & q^{-\mu} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

so that

\[ [M_{L, \mu}, X^{(L)} A_\mu X^{(1)}] = \hat{X}^{(L)} A_\mu X^{(1)} - X^{(L)} A_\mu \hat{X}^{(1)}. \]

We have

\[ X^{(j)} A_\mu X^{(j+1)} = \begin{bmatrix} A_\mu & A_\mu e^{-\mu} & eA_\mu & eA_\mu e^{-\mu} \\ A_\mu d & A_\mu d & A_\mu d & A_\mu d \\ dA_\mu d & dA_\mu d & dA_\mu d & dA_\mu d \\ dA_\mu d & dA_\mu d & dA_\mu d & dA_\mu d \end{bmatrix}, \quad M_{L, \mu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & e^{\mu} & 0 \\ 0 & q^{-\mu} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

and we need to check that \([M_{L, \mu}, X^{(L)} A_\mu X^{(1)}] = \hat{X}^{(L)} A_\mu X^{(1)} - X^{(L)} A_\mu \hat{X}^{(1)}.\]

Let us note that this same calculation can in fact be used anywhere in the bulk of the open system in order to validate the ansatz using equation (30).

Appendix C. Spin-1/2 XXZ chain with nondiagonal boundaries

In this section, we explain how our construction for the open ASEP can be translated for the spin-1/2 XXZ chain with non-diagonal boundary conditions [49] and hint at a possible relation to the recent solution of the XXZ chain with a Lindblad boundary drive [47].
Let us first define the bulk Hamiltonian of the XXZ spin chain of length $L$:

$$H_b = \frac{1}{2} \sum_{k=1}^{L-1} h_i$$

with $h_i$ acting as

$$h_i = \begin{bmatrix} \Delta & 0 & 0 & 0 \\ 0 & -\Delta & 1 & 0 \\ 0 & 1 & -\Delta & 0 \\ 0 & 0 & 0 & \Delta \end{bmatrix}$$

on sites $i$ and $i+1$ (in the same basis as for equation (7)), and as the identity on the rest of the chain.

Let us then consider the von Neumann equation for the density operator $\rho$ with the XXZ Hamiltonian with boundary terms $h_0$ and $h_L$ acting only on sites 0 and $L$:

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho] \quad \text{with} \quad H = h_0 + H_b + h_L.$$  \hspace{1cm} (C.3)

We will now show how our matrix product construction can be used to find a solution of the stationary equation $\frac{\partial \rho}{\partial t} = 0$.

Let us write the deformed Markov matrix $M_{\{\mu\}}$ for the special choice of weights defined by

$$\mu_0 = \frac{1}{2} \log \left( \frac{\nu}{\alpha} \right) + i\theta_0, \quad \mu_j = \frac{1}{2} \log (q), \quad \mu_L = \frac{1}{2} \log \left( \frac{\delta}{\beta} \right) + i\theta_L,$$

which is on the line $\mu = \frac{1}{2} \log \left( \frac{\nu \delta}{\alpha \beta q L} \right) + i\theta$ and for which $M_{\{\mu\}}$ is Hermitian. The deformed local matrices become

$$M_0(\mu_0) = \begin{bmatrix} -\alpha & \sqrt{\alpha \gamma} e^{-i\theta_0} & 0 & 0 \\ \sqrt{\alpha \gamma} e^{i\theta_0} & -\gamma & 0 & 0 \\ 0 & 0 & -\beta & \sqrt{\beta \delta} e^{i\theta_L} \\ 0 & 0 & \sqrt{\beta \delta} e^{-i\theta_L} & -\beta \end{bmatrix},$$

$$M_i(\mu_i) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & \sqrt{\gamma} & 0 \\ 0 & \sqrt{\gamma} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_L(\mu_L) = \begin{bmatrix} -\delta & \sqrt{\beta \delta} e^{-i\theta_L} & 0 & 0 \\ -\sqrt{\beta \delta} e^{i\theta_L} & -\beta & 0 & 0 \\ 0 & 0 & -\beta & \sqrt{\beta \delta} e^{i\theta_L} \\ 0 & 0 & \sqrt{\beta \delta} e^{-i\theta_L} & -\beta \end{bmatrix}.$$  \hspace{1cm} (C.5)

It is straightforward to check that, in this case, we have $M_{\{\mu\}} = \sqrt{q} H + \epsilon$, where $\epsilon$ is a constant, with the boundary matrices being equal to

$$h_0 = \frac{1}{2\sqrt{\gamma}} \begin{bmatrix} 1-q+\alpha+\gamma & 2\sqrt{\alpha \gamma} e^{-i\theta_0} \\ 2\sqrt{\alpha \gamma} e^{i\theta_0} & 1-q+\alpha-\gamma \end{bmatrix},$$

$$h_L = \frac{1}{2\sqrt{\beta}} \begin{bmatrix} 1-q+\beta-\delta & 2\sqrt{\beta \delta} e^{-i\theta_L} \\ 2\sqrt{\beta \delta} e^{i\theta_L} & 1-q+\beta+\delta \end{bmatrix}.$$  \hspace{1cm} (C.6)

The transfer matrix $U_{\{\mu\}}$ defined in (30) is therefore a solution to equation (C.3). It might not be a suitable density matrix, as its eigenvalues might not be positive, but we may in any case define one by

$$\rho = \frac{U T (U T)^\dagger}{\text{Tr}[U T (U T)^\dagger]}$$  \hspace{1cm} (C.7)

(where we omitted to write the dependence in $\{\mu\}$).

We can also rewrite expressions (30) in a way better suited to this situation:

$$U (C, C') = \frac{1}{Z_L} \langle \phi | \prod_{i=1}^{L} Y_{n, \vec{r}_i} | \psi \rangle$$  \hspace{1cm} (C.8)
\[ T(C, C') = \langle \phi | \prod_{i=1}^{L} Y_{\tau_i, \tau'_i} | \psi \rangle. \] (C.9)

with
\[ Y = \begin{bmatrix} N & S_+ \\ S_+ & N \end{bmatrix} \] (C.10)

and
\[ N = A_{\mu_i}, \]
\[ S_+ = A_{\mu_i} e^{A_{\mu_i}}, \]
\[ S_- = A_{\mu_i} d^{A_{\mu_i}} \]
\[ \langle \phi | = \langle W | A_{(\mu_0 - \frac{\omega}{2})} \]
\[ | \psi \rangle = A_{(\mu_L - \frac{\omega}{2})} | V \rangle \]
\[ \langle \tilde{\phi} | = \langle \tilde{W} | A_{(\mu_0 - \frac{\omega}{2})} \]
\[ | \tilde{\psi} \rangle = A_{(\mu_L - \frac{\omega}{2})} | \tilde{V} \rangle. \] (C.17)

Matrices $N$, $S_+$ and $S_-$ satisfy a special parametrization of the $U_q[SU(2)]$ algebra [54]:
\[ [S_+, S_-] = \left( \frac{1}{\sqrt{q}} - \sqrt{q} \right) N^2 \] (C.18)
\[ S_+ N = \frac{1}{\sqrt{q}} NS_+ \] (C.19)
\[ NS_- = \frac{1}{\sqrt{q}} S_- N. \] (C.20)

It was surprising to find that this solution has a structure almost identical to that of the Lindblad master equation found in [47], where $Y$ is denoted $\Omega$ and $\tilde{Y}$ is denoted $\Xi$. In that case, it seems that $\rho$ can be written in the simpler form $\rho = \frac{U (U^* \rho U^*)}{T(r_U)}$, where the algebraic relations satisfied by the boundary vectors $\langle \phi \rangle$ and $| \psi \rangle$ might be different from ours. It would be interesting to understand the precise relation between those two a priori very different situations.

References

[1] Blythe R A and Evans M R 2007 Nonequilibrium steady states of matrix-product form: a solver’s guide J. Phys. A: Math. Theor. 40 R333
[2] Touchette H 2009 The large deviation approach to statistical mechanics Phys. Rep. 478 1
[3] Derrida B 2007 Non-equilibrium steady states: fluctuations and large deviations of the density and of the current J. Stat. Mech. P07023
[4] Schmittmann B and Zia R K P 1995 Statistical mechanics of driven diffusive systems Phase Transitions and Critical Phenomena vol 17 ed C Domb and J L Lebowitz (San Diego, CA: Academic)
[5] Schütz G M 2011 Phase Transitions and Critical Phenomena vol 19 ed C Domb and J L Lebowitz (San Diego, CA: Academic)
[6] Spohn H 1991 Large Scale Dynamics of Interacting Particles (New York: Springer)
[7] Bodineau T and Derrida B 2004 Current fluctuations in nonequilibrium diffusive systems: an additivity principle Phys. Rev. Lett. 92 180601
[8] Bodineau T and Derrida B 2005 Distribution of currents in non-equilibrium diffusive systems and phase transitions Phys. Rev. E 72 066110
[9] Derrida B, Lebowitz J L and Speer E R 2002 Exact free energy functional for a driven diffusive open stationary nonequilibrium system Phys. Rev. Lett. 89 030601
[10] Takeuchi K A and Sano M 2010 Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals Phys. Rev. Lett. 104 230601
[11] Katz S, Lebowitz J L and Spohn H 1984 Phase transitions in stationary nonequilibrium states of model lattice systems J. Stat. Phys. 34 497
[12] Lebowitz J L and Spohn H 1999 A Gallavotti–Cohen type symmetry in the large deviation functional for stochastic dynamics J. Stat. Phys. 95 333
de Gier J and Essler F H L 2006 Exact spectral gaps of the asymmetric exclusion process with open boundaries J. Stat. Mech. P12011
[13] Evans D J, Cohen E D G and Morriss G P 1993 Probability of second law violations in shearing steady flows Phys. Rev. Lett. 71 2401
Gallavotti G and Cohen E D G 1995 Dynamical ensembles in nonequilibrium statistical mechanics Phys. Rev. Lett. 74 2694
[14] Schütz G M and Domany E 1993 Phase transitions in an exactly soluble one-dimensional exclusion process J. Stat. Phys. 72 277
[15] Derrida B 1998 An exactly soluble non-equilibrium system: the asymmetric simple exclusion process Phys. Rep. 301 65
[16] Varadhan S R S 1996 The complex story of simple exclusion Itô’s Stochastic Calculus and Probability Theory vol 385 (Tokyo: Springer)
[17] Gorissen M and Vanderzande C 2011 Finite size scaling of current fluctuations in the totally asymmetric exclusion process J. Phys. A: Math. Theor. 44 115005
[18] Derrida B, Douçot B and Roche P-E 2004 Current fluctuations in the one-dimensional symmetric exclusion process with open boundaries J. Phys. 115 717
[19] de Gier J and Essler F H L 2011 Current large deviation function for the open asymmetric simple exclusion process Phys. Rev. Lett. 107 010602
[20] Krapivsky P L, Redner S and Ben-Naim E 2010 A Kinetic View of Statistical Physics (Cambridge: Cambridge University Press)
[21] Chou T, Mallick K and Zia R K P 2011 Non-equilibrium statistical mechanics: from a paradigmatic model to biological transport Rep. Prog. Phys. 74 116601
[22] Adams D A, Schmittmann B and Zia R K P 2009 Far-from-equilibrium transport with constrained resources J. Stat. Mech. P06009
[23] Sasamoto T and Spohn H 2010 The one-dimensional KPZ equation: an exact solution and its universality Phys. Rev. Lett. 104 230602
Amir G, Corwin I and Quastel J 2011 Probability distribution of the free energy of the continuum directed random polymer in 1+1 dimensions Commun. Pure Appl. Math. 64 466
Ferrari P L 2010 From interacting particle systems to random matrices J. Stat. Mech. P10016
[24] Kardar M, Parisi G and Zhang Y-C 1986 Dynamic scaling of growing interfaces Phys. Rev. Lett. 56 889
Halpin-Healy T and Zhang Y-C 1995 Kinetic roughening, stochastic growth, directed polymers and all that Phys. Rep. 254 215
Ferrari P L 2010 From interacting particle systems to random matrices J. Stat. Mech. P10016
[25] Sasamoto T 2007 Fluctuations of the one-dimensional asymmetric exclusion process using random matrix techniques J. Stat. Mech. P07007
Johansson K 2000 Shape fluctuations and random matrices Commun. Math. Phys. 209 437
Kriecherbauer T and Krug J 2010 A pedestrian’s view on interacting particle systems, KPZ universality, and random matrices J. Phys. A: Math. Theor. 43 403001
Corwin I 2012 The Kardar–Parisi–Zhang equation and universality class Random Matrices, Theory Appl. 1 1130001
[26] Tracy C A and Widom H 2009 Total current fluctuations in ASEP J. Math. Phys. 50 095204
Corteel S and Williams L 2011 Tableaux combinatorics for the asymmetric exclusion process and Askey–Wilson polynomials Duke Math. J. 159 385–415
[27] Viennot X G 2007 Canopy of binary trees, Catalan tableaux and the asymmetric exclusion process FPSAC 2007: 19th Int. Conf. on Formal Power Series and Algebraic Combinatorics (arXiv:0905.3081)
[28] Krug J 1991 Boundary-induced phase transitions in driven diffusive systems Phys. Rev. Lett. 67 1882
Derrida B, Domany E and Mukamel D 1992 An exact solution of a one-dimensional asymmetric exclusion model with open boundaries J. Stat. Phys. 69 667
Derrida B, Evans M R, Hakim V and Pasquier N 1993 Exact solution of a 1D asymmetric exclusion model using a matrix formulation J. Phys. A: Math. Gen. 26 1493
[31] Kriecherbauer T and Krug J 2010 A pedestrian’s view on interacting particle systems, KPZ universality, and random matrices J. Stat. Mech. P07007
[32] Johansson K 2000 Shape fluctuations and random matrices Commun. Math. Phys. 209 437
Kriecherbauer T and Krug J 2010 A pedestrian’s view on interacting particle systems, KPZ universality, and random matrices J. Phys. A: Math. Theor. 43 403001
Corwin I 2012 The Kardar–Parisi–Zhang equation and universality class Random Matrices, Theory Appl. 1 1130001
[33] Tracy C A and Widom H 2009 Total current fluctuations in ASEP J. Math. Phys. 50 095204
Corteel S and Williams L 2011 Tableaux combinatorics for the asymmetric exclusion process and Askey–Wilson polynomials Duke Math. J. 159 385–415
[34] Viennot X G 2007 Canopy of binary trees, Catalan tableaux and the asymmetric exclusion process FPSAC 2007: 19th Int. Conf. on Formal Power Series and Algebraic Combinatorics (arXiv:0905.3081)
[35] Krug J 1991 Boundary-induced phase transitions in driven diffusive systems Phys. Rev. Lett. 67 1882
Derrida B, Domany E and Mukamel D 1992 An exact solution of a one-dimensional asymmetric exclusion model with open boundaries J. Stat. Phys. 69 667
Derrida B, Evans M R, Hakim V and Pasquier N 1993 Exact solution of a 1D asymmetric exclusion model using a matrix formulation J. Phys. A: Math. Gen. 26 1493
[34] Derrida B, Evans M R and Mallick K 1995 Exact diffusion constant of a one-dimensional asymmetric exclusion model with open boundaries J. Stat. Phys. 79 833
[35] Derrida B and Lebowitz J L 1998 Exact large deviation function in the asymmetric exclusion process Phys. Rev. Lett. 80 209
[36] Prolhac S 2010 Tree structures for the current fluctuations in the exclusion process J. Phys. A: Math. Theor. 43 105002
[37] Prolhac S, Evans M R and Mallick K 2009 The matrix product solution of the multispecies partially asymmetric exclusion process J. Phys. A: Math. Theor. 42 165004
[38] Sasamoto T 1999 One-dimensional partially asymmetric simple exclusion process with open boundaries: orthogonal polynomials approach J. Phys. A: Math. Gen. 32 7109
[39] Simon D 2009 Construction of a coordinate Bethe ansatz for the asymmetric simple exclusion process Phys. Rev. Lett. 80 209
[40] de Gier J and Essler F H L 2005 Bethe ansatz solution for a defect particle in the asymmetric exclusion process J. Phys. A: Math. Gen. 32 4833
[41] Derrida B and Appert C 1999 Universal large deviation function of the Kardar–Parisi–Zhang equation in one dimension J. Stat. Phys. 94 1
[42] Lazarescu A and Mallick K 2011 An exact formula for the statistics of the current in the TASEP with open boundaries J. Phys. A: Math. Theor. 44 315001
[43] Gorissen M, Lazarescu A, Mallick K and Vanderzande C 2012 Exact current statistics of the ASEP with open boundaries Phys. Rev. Lett. 109 170601
[44] Giardina C, Kurchan J and Peliti L 2006 Direct evaluation of large-deviation functions Phys. Rev. Lett. 96 120603
[45] Bertini L, De Sole A, Gabrielli D, Jona-Lasinio G and Landim C 2005 Macroscopic current fluctuations in stochastic lattice gases Phys. Rev. Lett. 94 030601
[46] Appert-Rolland C, Derrida B, Lecomte V and Van Wijland F Universal cumulants of the current in diffusive systems on a ring Phys. Rev. E 78 021122
[47] Karevski D, Popkov V and Schütz G M 2012 Exact matrix product solution for the boundary-driven Lindblad XXZ-chain arXiv:1211.7010
[48] Rajewsky N, Santen L, Schadschneider A and Schreckenberg M 1998 The asymmetric exclusion process: comparison of update procedures J. Stat. Phys. 92 154
[49] Sandow S 1994 Partially asymmetric exclusion process with open boundaries Phys. Rev. E 50 2660
[50] Deuschel J D and Stroock D W 1989 Large Deviations (Boston, MA: Academic)
[51] Jack R L and Sollich P 2010 Large deviations and ensembles of trajectories in stochastic models Prog. Theor. Phys. Suppl. 184 304317
[52] Evans M R, Rajewsky N and Speer E R 1999 Exact solution of a cellular automaton for traffic J. Stat. Phys. 95 45
[53] Depken M and Stinchcombe R 2004 Exact joint density-current probability function for the asymmetric exclusion process Phys. Rev. Lett. 93 040602
[54] Chaichian M and Demichev A 1996 Introduction to Quantum Groups (Singapore: World Scientific)