On Further Generalization of the Rigidity Theorem for Spacetimes with a Stationary Event Horizon or a Compact Cauchy Horizon

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Abstract

A rigidity theorem that applies to smooth electrovac spacetimes which represent either (A) an asymptotically flat stationary black hole or (B) a cosmological spacetime with a compact Cauchy horizon ruled by closed null geodesics was given in a recent work [5]. Here we enlarge the framework of the corresponding investigations by allowing the presence of other type of matter fields. In the first part the matter fields are involved merely implicitly via the assumption that the dominant energy condition is satisfied. In the second part Einstein–Klein-Gordon (EKG), Einstein–[non-Abelian] Higgs (E[nA]H), Einstein–[Maxwell]–Yang-Mills-dilaton (E[M]YMd) and Einstein–Yang-Mills–Higgs (EYMH) systems are studied. The black hole event horizon or, respectively, the compact Cauchy horizon of the considered spacetimes is assumed to be a smooth non-degenerate null hypersurface. It is proven that there exists a Killing vector field in a one-sided neighborhood of the horizon in EKG, E[nA]H, E[M]YMd and EYMH spacetimes. This Killing vector field is normal to the horizon, moreover, the associated matter fields are also shown to be invariant with respect to it. The presented results provide generalizations of the rigidity theorems of Hawking (for case A) and of Moncrief and Isenberg (for case B) and, in turn, they strengthen the validity of both the black hole rigidity scenario and the strong cosmic censor conjecture of classical general relativity.

1 Introduction

There are two seemingly disconnected areas within general relativity each possessing its own ‘rigidity’ theorem. One of these is the determination of the possible asymptotic final states of black holes while the other is the justification of the strong cosmic censor hypothesis of Penrose [21] for closed cosmological models. The relevant rigidity theorems of Hawking [3, 4] and of Isenberg and Moncrief [5, 9] apply to analytic electrovac spacetimes of type A and B, respectively. Since the use of the analyticity assumption is incompatible with the concept of causality the generalization of these rigidity results from the analytic to the smooth setting is not only of pure mathematical interest. The main result of [5] was that for both type A and B electrovac spacetime configurations the existence of a horizon Killing field – alternatively, the rigidity of these spacetimes – can be proven for the case of smooth geometrical setting. Clearly, it is also of obvious interest to know whether these results

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are valid for the case of other type of matter fields. The main purpose of this paper is to show that the rigidity theorems of \[5\] generalize further to various Einstein-matter systems.

Our principal result can be formulated as follows: Consider a smooth spacetime which represents either an asymptotically flat stationary black hole or a cosmological spacetime with compact Cauchy horizon generated by closed null geodesics. Suppose, furthermore, that the black hole event or compact Cauchy horizon of these spacetimes is smooth and non-degenerate. Then it is shown that in the cases of EKG, E[nA]H, E[M]YMd and EYMH systems there exists a smooth Killing vector field defined in a one-sided neighborhood of the horizon. This Killing vector field is normal to the horizon, moreover, the associated matter fields are also shown to be invariant with respect to it.

In case of Yang-Mills (YM) configurations interesting new phenomena get into play. Although the spacetime geometry is found to be as regular as in the case of the other particular matter fields, in general, there is a ‘parallelly propagated’ blowing up of the ‘adapted’ gauge potentials along the generators of the horizon. This irregular behavior, however, is balanced entirely in the associated Lie algebra valued 2-form field and, in turn, in the energy-momentum tensor. In addition, the adapted gauge representations are found to belong to either of the following two characteristically disjoint classes: There are ‘preferred gauge representations’ which turn to be invariant with respect to the ‘candidate horizon Killing vector field’. The second class is formed by non-preferred gauge representations which possess only certain discrete symmetries. Despite of their different characters there is a considerable interplay between these two types of representations. Whenever a YM filed possesses a non-preferred gauge representation the existence of an infinite ‘crystal’ of non-preferred representations can also be shown. Each of these crystals of representations is, however, centered by a preferred gauge representation.

This paper is organized as follows: In the next section, some preliminary notions and results are recalled. Sections 3 and 4 are devoted to the investigation of the geometrical properties of ‘elementary spacetime regions’ which are covering spaces of neighborhoods of sections of the horizons of the considered spacetimes. Section 5 is for the study of EKG, E[nA]H, E[M]YMd and EYMH systems. It is shown there that these systems possess a ‘candidate horizon Killing vector field’. In section 6, first it is demonstrated that the elementary spacetime regions extend to a spacetime possessing a bifurcate null hypersurface. The Lie derivative of the metric and the matter fields with respect to the candidate Killing vector field are shown to vanish on this null hypersurface. Finally, the null characteristic initial value problem is applied to prove our main result while section 7 contains our concluding remarks.

2 Preliminaries

In this section the basic notions and results regarding the set of spacetimes studied in this paper will be recalled. Throughout this paper, unless otherwise stated, a spacetime is considered to be represented by a pair \((M, g_{ab})\) where \(M\) is a smooth paracompact connected orientable manifold while \(g_{ab}\) is a smooth Lorentzian metric of signature \((+,-,-,-)\) on \(M\). It is assumed that \((M, g_{ab})\) is time orientable and also that a time orientation has been chosen.

In the first part of this paper the matter fields will merely be involved implicitly via the assumption that the dominant energy condition is satisfied. Accordingly we shall assume that for all future directed timelike vector \(\xi^a\) the contraction \(T^a_{\ b}\xi^b\) is a future directed timelike or null vector, where \(T_{ab}\) denotes the energy-momentum tensor.

There will be two classes, \(A\) and \(B\), of spacetimes investigated throughout this paper. In short terms, class \(A\) (see Sect. 2.1 of \[6\] for further details) consists of spacetimes which are asymptotically stationary with respect to a smooth Killing vector field \(t^a\). The Killing orbits of \(t^a\) are assumed to be complete and the associated one-parameter group of isometries is denoted by \(\phi_t\). The event
horizon $\mathcal{N}$ is required to be a $\phi_t$-invariant smooth null hypersurface so that the manifold of the null geodesic generators of $\mathcal{N}$ has the topology $S^2$. Finally, whenever matter fields are present they are supposed to be stationary. Correspondingly, whenever the matter fields are represented by tensor fields they are assumed to be $\phi_t$-invariant. In case of gauge fields, however, only the existence of an adapted $\phi_t$-invariant gauge representation is required.

Spacetimes of class $B$ are assumed to possess a compact orientable smooth null hypersurface, $\mathcal{N}$, generated by closed null geodesics. In most of the spacetimes belonging to this class (see e.g. [4, 5, 6]) the null hypersurface $\mathcal{N}$ plays the role of a Cauchy horizon. Since the globally hyperbolic region of these spacetimes possesses compact Cauchy surfaces spacetimes of class $B$ can be considered, and are frequently referred, to be closed cosmological models.

It was shown (see Prop. 3.1 of [7]) that to any spacetime $(M, g_{ab})$ of class $A$ there exists an open neighborhood $\mathcal{V}$ of the event horizon $\mathcal{N}$ such that $(\mathcal{V}, g_{ab})$ is a covering space of a spacetime of class $B$. Therefore to give a simultaneous generalization of both the results of Hawking and of Isenberg and Moncrief it suffices to show the existence of a horizon Killing field for spacetimes of class $B$.

Moreover, for a spacetime of class $B$ the existence of ‘tubular’ spacetime neighborhoods $\mathcal{U}_i$ can be shown. These neighborhoods are fibered by circles and sufficiently small spacetime neighborhoods of the Cauchy horizon $\mathcal{N}$ can always be covered by a finite subset of them. Most importantly, it follows from the analyses of [4] that there exist simply connected ‘elementary spacetime neighborhoods’ $\mathcal{O}_i$ and fiber preserving local isometry mappings $\psi_i : \mathcal{O}_i \rightarrow \mathcal{U}_i$ onto the tubular spacetime neighborhoods $\mathcal{U}_i$ so that the followings hold:

(i) Gaussian null coordinates ($u, r, x^3, x^4$) can be introduced in $\mathcal{O}_i$ such that the coordinate range of $u$ is $(-\infty, \infty)$ whereas the coordinate range of $r$ is $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ and the surface $r = 0$ is the inverse image $\tilde{\mathcal{N}}_i$ of $\mathcal{N}_i = \mathcal{N} \cap \mathcal{U}_i$.

(ii) $k^a = (\partial/\partial u)^a$ can be set to be a future directed null vector field normal to $\tilde{\mathcal{N}}_i$ while $l^a = (\partial/\partial r)^a$ is defined in terms of the affine parameter $r$ measured along the null geodesics starting orthogonally to the 2-dimensional cross sections, $u = \text{const}$, of $\tilde{\mathcal{N}}_i$ with tangent $l^a$ satisfying that $l^a k_a = 1$ throughout $\tilde{\mathcal{N}}_i$.

(iii) The spacetime metric in $\mathcal{O}_i$ takes the form

$$ds^2 = r \cdot f du^2 + 2drdu + 2r \cdot h_A du dx^A + g_{AB} dx^A dx^B$$

(2.1)

where $f, h_A$ and $g_{AB}$ are smooth $u$-periodic functions, with a period $P$, such that $g_{AB}$ is a negative definite $2 \times 2$ matrix. (The uppercase Latin indices take everywhere the values 3, 4.)

(iv) In these coordinates the components of the matter field tensors and also that of the adapted gauge representations are $u$-periodic ‘functions’ with periodicity length $P$. [7]

3 The rigidity of the horizon of a spacetime of class $B$

It is known (see e.g. [4]) that whenever the null convergence condition is satisfied the event horizon of a spacetime belonging to class $A$ is rigid. By making use of the same energy condition, along with the $u$-periodicity, the Cauchy horizon of spacetimes of class $B$ can also shown to be rigid.

Proposition 3.1 Let $(M, g_{ab})$ be a spacetime of class $B$. Then the closed null geodesic generators of $\mathcal{N}$ are expansion and shear free, i.e. for any choice of an elementary neighborhood $(\mathcal{O}_i, g_{ab}|_{\mathcal{O}_i})$ we have that $\partial g_{AB}/\partial u = 0$ throughout $\tilde{\mathcal{N}}_i$.

Although, in most of the cases it does not play any role, unless otherwise stated, we shall tacitly assume that $P$ is the smallest possible positive period of the considered $u$-periodic functions.
Proof Let \((O, g_{ab}|_{O})\) be an elementary neighborhood. Since \(k^a = (\partial/\partial u)^a\) is null on \(\tilde{N}_i\) we have
\[
R_{ab}k^ak^b = 8\pi T_{ab}k^ak^b
\]
there. In Gaussian null coordinates \((3.1)\) reads on \(\tilde{N}_i\) as
\[
\frac{\partial^2 [\ln \sqrt{g}]}{\partial u^2} + f \frac{\partial [\ln \sqrt{g}]}{\partial u} + \frac{1}{4} g^{AC} g^{BD} \left( \frac{\partial g_{AB}}{\partial u} \right) \left( \frac{\partial g_{CD}}{\partial u} \right) + 8\pi T_{ab}k^ak^b = 0,
\]
where \(g := -\det (g_{AB})\) and the \(2 \times 2\) matrix \(g^{AB}\) denotes the inverse of \(g_{AB}\). Since \(g_{AB}\) is negative definite and the null energy condition holds (which follows from the dominant energy condition), both of the last two terms on the l.h.s. of \((3.2)\) have to be greater than or equal to zero. Moreover, since the metric functions \(g_{AB}\) are \(u\)-periodic there exists a point \(u_0\) with \((\partial [\ln \sqrt{g}]/\partial u)(u_0) = 0\). Reading \((3.2)\) as an equation of first order for \(\partial [\ln \sqrt{g}]/\partial u\) along the generators of \(\tilde{N}_i\) we get
\[
\left( \frac{\partial [\ln \sqrt{g}]}{\partial u} \right)(u) = -e^{\frac{1}{2} \int_{u_0}^u f(u') \, du'} \int_{u_0}^u b(u') e^{\frac{1}{2} \int_{u_0}^{u'} f(u'') \, du''} \, du',
\]
where \(b\) stands for the last two terms of the l.h.s. of \((3.2)\). Using again the periodicity, along with the fact that \(b \geq 0\), we find that both \(\partial [\ln \sqrt{g}]/\partial u\) and \(b\) have to vanish identically along the generators of \(\tilde{N}_i\). Thereby, both of the last two terms on the l.h.s. of \((3.2)\) have also to vanish identically on \(\tilde{N}_i\). Hence we obtain that
\[
\left( \frac{\partial g_{AB}}{\partial u} \right)^{\circ} = 0
\]
holds.

Remark 3.1 It follows from \((3.4)\), along with the dominant energy condition, that \(k^a\) is a repeated principal null vector of the Weyl tensor, i.e. \(k_{[a}C_{b]cdef}k^c k^f \equiv 0\) on \(\tilde{N}_i\). To see this note first that in virtue of \((3.4)\) the spin coefficients \(\lambda\) and \(\mu\) must vanish on \(\tilde{N}_i\). Moreover, the dominant energy condition implies that \(R_{ab}k^a X^b = 0\) (see e.g. the first part of the proof of Prop. \((3.4)\) for any vector field \(X^b\) tangent to \(\tilde{N}_i\). Hence, the Ricci spinor components \(\Phi_{22}\) and \(\Phi_{21}\) also have to be zero throughout \(\tilde{N}_i\). Finally, by making use of \((NP.10)\), \((NP.13)\) and \((NP.14)\) [see footnote 6] the vanishing of \(\Psi_3^0\) and \(\Psi_4^0\) can be justified.

Remark 3.2 Contrary to the general expectations (see e.g. section 8.5 of \(\ddagger\)) whenever the dominant energy condition is satisfied \(k_{[a}R_{b]cdef}k^c k^f\) vanish identically along the closed null geodesic generators of a compact Cauchy horizon. Nevertheless, some (other) components of the curvature tensor, in parallelly propagated frames, can blow up there (see e.g. Remark \(\ddagger\)).

Remark 3.3 There is another possible reading of the above result: In a spacetime satisfying both the ‘genericness condition’ and the dominant energy condition there cannot exist a smooth compact Cauchy horizon ruled by closed null geodesics.

\[1\] I wish to thank Helmut Friedrich for this elementary argument demonstrating the vanishing of the functions \(\partial [\ln \sqrt{g}]/\partial u\) and \(b\).

\[2\] As a shorthand way of notation we shall denote by \(\varphi^\circ\) the restriction \(\varphi|_{\tilde{N}_i}\) of a function \(\varphi\) to \(\tilde{N}_i\).
4 Further properties of elementary neighborhoods

By making use the dominant energy condition, along with the u-periodicity, the following can be proven:

**Proposition 4.1** Let \( (O_i, g_{ab}|O_i) \) be an elementary neighborhood of \( \tilde{N}_i \) such that \( (\partial g_{AB}/\partial u)^{\circ} \) is identically zero. Then there always exists a Gaussian null coordinate system so that \( f^{\circ} = -2\kappa^{\circ} \), with a constant \( \kappa^{\circ} \geq 0 \), and that \( \partial h_A/\partial u = 0 \) throughout \( \tilde{N}_i \).

**Proof** Since \( k^a \) is normal to the coordinate basis field \( (\partial/\partial x^A)^a \) on \( \tilde{N}_i \)

\[
R_{ab}k^a\left( \frac{\partial}{\partial x^A} \right)^b = 8\pi T_{ab}k^a\left( \frac{\partial}{\partial x^A} \right)^b \tag{4.1}
\]

there. Moreover, in virtue of the dominant energy condition \( T_{ab}k^b \) has to be a future directed timelike or null vector. On the other hand, \( T_{ab}k^ak^b = 0 \) on \( \tilde{N}_i \). This implies that \( T_{ab}k^b \) must point in the direction of \( k^a \), i.e. \( T_{ab}k^a (\partial/\partial x^A)^b = 0 \) on \( \tilde{N}_i \). Hence, by making use of the fact that \( (\partial g_{AB}/\partial u)^{\circ} = 0 \), we get that in the underlying Gaussian null coordinates \( (4.1) \) reads as

\[
(\frac{\partial f}{\partial x^A} - \frac{\partial h_A}{\partial u})^{\circ} = 0. \tag{4.2}
\]

Integrating this equation with respect to \( u \) and using the u-periodicity and smoothness of \( h_A \) we get that

\[
\frac{\partial}{\partial x^A} \int_0^P f^{\circ}(u, x^3, x^4) \, du = 0, \tag{4.3}
\]

which in particular means that for some constant \( \kappa^{\circ} \geq 0 \)

\[
\int_0^P f^{\circ}(u, x^3, x^4) \, du = -2P\kappa^{\circ} \tag{4.4}
\]

holds throughout \( \tilde{N}_i \). (If \( \kappa^{\circ} \) was smaller than zero then we could achieve \( \kappa^{\circ} \geq 0 \) by the application of the transformation \( (u, r, x^3, x^4) \rightarrow (-u, -r, x^3, x^4) \) along with the simultaneous reversing of the time orientation.)

To prove then that new Gaussian null coordinate systems can be introduced so that \( f^{\circ} = -2\kappa^{\circ} \) holds the argument of Moncrief and Isenberg (see pages 395-398 of [17]) can be applied. Since in these new coordinates \( f^{\circ} \) is constant the relevant form of \( (4.3) \) implies that \( (\partial h_A/\partial u)^{\circ} = 0 \).

**Remark 4.1** According to \( (4.4) \) in any elementary neighborhood the u-periodicity of the metric functions selects a preferred value of \( \kappa^{\circ} \) which is uniquely determined in any associated Gaussian null coordinate system as \( \kappa^{\circ} = -1/2P \int_0^P f^{\circ} \, du \). Then, because \( k^a = (\partial/\partial u)^a \) on \( \tilde{N}_i \) satisfies

\[
k^a\nabla_a k^b = \kappa^{\circ} k^b, \tag{4.5}
\]

we have that the null generators of \( \tilde{N}_i \) – which are complete with respect to the parameter \( u \) – are null geodesically complete if \( \kappa^{\circ} = 0 \), whereas, if \( \kappa^{\circ} \) happens to be nonzero the generators of \( \tilde{N}_i \) are geodesically complete only in one direction.
Remark 4.2 There is a significant consequence of Prop. 4.1 in connection with the ‘zeroth law’ of black hole thermodynamics. It is known that for any static black hole (in an arbitrary covariant metric theory of gravity) the surface gravity has to be constant throughout the event horizon \( [23] \). In view of the above result there is a unique way to introduce a quantity, \( \kappa_o \), which plays the role of surface gravity. The constancy of \( \kappa_o \) throughout \( \mathcal{N} \) is guaranteed by the dominant energy condition.

Remark 4.3 Based on Prop. 4.1 and Prop. 4.2 hereafter, without loss of generality, we shall assume that the Gaussian null coordinate system \((u, r, x^3, x^4)\) associated with an elementary spacetime region \( \mathcal{O}_i \) is so that \( f^o = -2\kappa_o \), moreover, \( h_A \) and \( g_{AB} \) are \( u \)-independent on \( \tilde{\mathcal{N}_i} \).

For particular spacetimes further characterization of the functions \( f, h_A \) and \( g_{AB} \) can also be given. For instance, for the case of an asymptotically flat stationary non-static vacuum black hole it was proven in \([3] \) that the \( r \)-derivatives of the functions \( f, h_A \) and \( g_{AB} \) up to any order are \( u \)-independent on \( \tilde{\mathcal{N}_i} \). Exactly the same property of these functions was proven to be held for the case of electrovac cosmological situations \([17, 10] \), i.e. for spacetimes possessing a compact Cauchy horizon with closed generators and satisfying the Einstein-Maxwell equations. In fact, the \( u \)-independendness of the \( r \)-derivatives of the functions \( f, h_A \) and \( g_{AB} \) up to any order is the very property that can be used to argue, in the case of analytic spacetime configurations, that \( k^a = (\partial/\partial u)^a \) is a Killing vector field in an elementary neighborhood \( \mathcal{O}_i \) with respect to which \( \tilde{\mathcal{N}_i} \) is a Killing horizon.

The purpose of the remaining part of this section is to investigate what are the necessary and sufficient conditions ensuring that the \( r \)-derivatives of the functions \( f, h_A \) and \( g_{AB} \) up to any order are \( u \)-independent on \( \tilde{\mathcal{N}_i} \). Recall that the proof of Moncrief and Isenberg \([17, 10] \) for the electrovac case was based on the detailed application of the coupled Einstein-Maxwell equations. To be able to separate all the conditions which are in certain sense ‘purely geometrical’ from the ones which are related to particular properties of matter fields first we consider the relevant necessary and sufficient geometrical conditions. The corresponding analysis when the effect of particular matter fields are taken into consideration will be presented in the next section.

To prove the main result of this section the use of the Newman-Penrose formalism \([19] \) turned out to be most effective. Thereby, first we recall the relation between the two geometrical settings based on the Gaussian null coordinates and the Newman-Penrose formalism, respectively.

The contravariant form of the metric \((2.1) \) in a Gaussian null coordinate system \((u, r, x^3, x^4)\), covering the elementary region \( \mathcal{O}_i \), reads as

\[
g^{\alpha\beta} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & g^{rr} & g^{rB} & 0 \\
0 & g^{Ar} & g^{AB} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]  

(4.6)

Choosing now real-valued functions \( U, X^A \) and complex-valued functions \( \omega, \xi^A \) on \( \mathcal{O}_i \) such that

\[
g^{rr} = 2(U - \omega \bar{\omega}), \quad g^{rA} = X^A - (\bar{\omega} \xi^A + \omega \xi^A), \quad g^{AB} = -(\xi^A \bar{\xi}^B + \bar{\xi}^A \xi^B),
\]

(4.7)

and setting

\[
l^\mu = \delta^\mu_r, \quad n^\mu = \delta^\mu_u + U \delta^\mu_r + X^A \delta^\mu_A, \quad m^\mu = \omega \delta^\mu_r + \xi^A \delta^\mu_A,
\]

(4.8)

we obtain a complex null tetrad \( \{\ell^a, n^a, m^a, \bar{m}^a\} \). We require that \( U, X^A \), and \( \omega \) vanish on \( \tilde{\mathcal{N}_i} \) such that \( n^a, m^a \) and \( \bar{m}^a \) are tangent to \( \tilde{\mathcal{N}_i} \). In the following we shall consider the derivatives of functions in the direction of the frame vectors above and denote the corresponding operators in \( \mathcal{O}_i \) by

\[
D = \partial/\partial r, \quad \Delta = \partial/\partial u + U \cdot \partial/\partial r + X^A \cdot \partial/\partial x^A, \quad \delta = \omega \cdot \partial/\partial r + \xi^A \cdot \partial/\partial x^A.
\]

(4.9)

\(^4\)Actually, the above argument is ‘local’ in the space directions but, in virtue of \((4.4) \), it is obvious that the value of \( \kappa_o \) has to be the same on overlapping primary neighborhoods.
To simplify the Newman-Penrose equation we fix the remaining gage freedom by assuming that the tetrad \( \{l^a, n^a, m^a, \overline{m}^a\} \) is parallelly propagated along the null geodesics with tangent \( l^a = (\partial / \partial r)^a \). This condition ensures e.g. that for the spin coefficients related to this complex null tetrad \( \kappa = \pi = \varepsilon = 0, \rho = \overline{\rho}, \tau = \overline{\tau} + \beta \) hold in \( \mathcal{O}_i \), and in particular, \( \nu = 0, \gamma = \overline{\gamma} \) and \( \mu = \overline{\mu} \) on \( \tilde{N}_i \).

**Proposition 4.2** Denote by \( g^{\alpha \beta} \) the contravariant components of the spacetime metric in a Gaussian null coordinate system associated with an elementary spacetime region \( (\mathcal{O}_i, g_{ab}|_{\mathcal{O}_i}) \). Then we have that for all values of \( i \in \{1, 2, \ldots, n\} \)

\[
\Delta \left( D^{(i)} \left( \{g^{\alpha \beta}\} \right) \right)^o = 0 \quad (4.10)
\]

if and only if \( \Delta \left( \{(\Phi_{11} + 3\Lambda), \Phi_{02}\} \right)^o = 0 \) and for all values of \( j \in \{0, 1, 2, \ldots, n - 2\} \)

\[
\Delta \left( D^{(j)} \left( \{\Phi_{00}, \Phi_{01}, (\Phi_{11} - 3\Lambda), D(\Phi_{02})\} \right) \right)^o = 0. \quad (4.11)
\]

**Proof** By \( \{4.6\} \) and \( \{4.7\} \), along with the vanishing of the functions \( \omega, X^A, U \) on \( \tilde{N}_i \), we have that \( \{4.10\} \) is satisfied whenever for all values of \( i \in \{1, 2, \ldots, n\} \)

\[
\Delta \left( D^{(i)} \left( \{\xi^A, \omega, X^A, U\} \right) \right)^o = 0. \quad (4.12)
\]

Moreover, by making use of the metric equations – see eqs. (6.10a) - (6.10h) of \( [19] \) – it can be checked that \( \{4.12\} \) holds if and only if for all values of \( i \in \{0, 1, 2, \ldots, n - 1\} \)

\[
\Delta \left( D^{(i)} \left( \{\rho, \sigma, \tau, (\gamma + \tilde{\gamma})\} \right) \right)^o = 0. \quad (4.13)
\]

Furthermore, since \( T^a_{uu} = T^a_{uA} = (\partial g_{AB} / \partial u)^o = 0 \) we have that \( \Phi^o_{22} = \Phi^o_{21} = \lambda^o = \mu^o = 0 \) as well as that \( \Psi^o_3 = \Psi^o_4 = 0 \) (for the later relations see e.g. the argument applied at Remark 3.1).

To see that for \( n = 1 \) \( \{4.13\} \) is equivalent to the \( u \)-independence of \( \Phi_{11} + 3\Lambda \) and \( \Phi_{02} \) on \( \tilde{N}_i \), note first that by the definition of \( \gamma \) and \( \{4.5\} \)

\[
(\gamma + \tilde{\gamma})^o = (n^a n^b \nabla_a b_b)^o = (k^a k^b \nabla_a b_b)^o = -(k^a l^b \nabla_a b_b)^o = -\kappa_o, \quad (4.14)
\]

where \( \kappa_o = const \) throughout \( \tilde{N}_i \). Thereby, along with the fact that \( \tilde{\gamma}^o = \gamma^o \), we get

\[
\delta(\gamma)^o = \tilde{\delta}(\gamma)^o = \Delta(\gamma + \tilde{\gamma})^o = 0. \quad (4.15)
\]

To see that \( \Delta(\tau)^o = 0 \) note that \( \tau = \bar{\alpha} + \beta \) and by \( \{NP.15\} \) \( \{NP.18\} \) and \( \{4.14\} \) we have

\[
\Delta(\alpha)^o = \Delta(\beta)^o = 0. \quad (4.16)
\]

An immediate consequence of \( \{4.13\} \) and \( \{NP.12\} \) is that

\[
\Delta(\Psi^o_2 + 2\Lambda)^o = \Delta(\Phi^o_{11} + 3\Lambda)^o = 0. \quad (4.17)
\]

In virtue of \( \{4.17\} \) and \( \{NP.17\} \) \( \Delta(\rho) \) satisfies

\[
\Delta (\rho)^o + \kappa_o \Delta(\rho)^o = 0 \quad (4.18)
\]

\( ^5 \Delta \left( D^{(j)} \left( \{f_1, f_2, \ldots, f_N\} \right) \right) \) denotes the list of functions resulted by the action of the differential operators \( D \) \( (j) \)-times and \( \Delta \) once on the functions \( f_1, f_2, \ldots, f_N \). In particular, \( \Delta \left( D^{(0)} \left( \{f_1, f_2, \ldots, f_N\} \right) \right) \) is defined to be \( \Delta(\{f_1, f_2, \ldots, f_N\}) \).

\( ^6 \)Throughout this proof the equations referred as \( \{NP.n\} \) and \( \{B,m\} \) are yielded by the substitution of the above gauge fixing relations into the \( n^{th} \) Newman-Penrose equation and the \( m^{th} \) Bianchi identity of the Newman-Penrose formalism as they are listed in the appendix of \( [26] \).
on \( \tilde{N}_i \), with the only periodical solution \( \Delta(\rho)^o = 0 \), if and only if \( \Delta(\Phi_{11} + 3\Lambda)^o = 0 \).

Finally, by making use of (NP.16) it can be shown that \( \Delta(\sigma)^o = 0 \) whenever \( \Delta(\Phi_{02})^o = 0 \).

To see that our statement is true for \( n = 2 \) one can proceed as follows. By (NP.6) we have that
\[
\Delta(D(\gamma + \tilde{\gamma}))^o = 0
\]
if and only if \( \Delta(\Psi_2 + \bar{\Psi}_2 - 2\Lambda + 2\Phi_{11})^o = 0 \). However, in virtue of (4.17) the last equation is equivalent to \( \Delta(\Phi_{11})^o = 0 \) which follows from our assumption that both \( \Delta(\Phi_{11} + 3\Lambda) \) and \( \Delta(\Phi_{11} - 3\Lambda) \) vanish on \( \tilde{N}_i \).

The vanishing of \( \Delta(D(\alpha))^o \), \( \Delta(D(\beta))^o \) and, thereby also, of \( \Delta(D(\tau))^o \) by (NP.3) - (NP.5) and our assumptions are equivalent to the condition \( \Delta(\Psi_1)^o = 0 \). However, by (B.4) we have that on \( \tilde{N}_i \)
\[
\Delta(D(\Psi_1)) + \kappa_\sigma \Delta(\Psi_1) - \tau \Delta(2\Phi_{11} - 3\Psi_2) = 0.
\]
(4.19)

The last term on the l.h.s. of (4.19) is just \( \tau \Delta(\Phi_{11} + 3\Lambda) \) which is identically zero on \( \tilde{N}_i \). Therefore, the only periodical solution of (4.19) is \( \Delta(\Psi_1)^o = 0 \).

It follows immediately from (NP.1) that \( \Delta(D(\rho))^o = 0 \) if and only if \( \Delta(\Phi_{00})^o = 0 \), and similarly, from (NP.2) that \( \Delta(D(\sigma))^o = 0 \) precisely when \( \Delta(\Psi_0)^o = 0 \). By our assumption we have that \( \Delta(\Phi_{00})^o = 0 \) while from (B.2) we get that on \( \tilde{N}_i \)
\[
\Delta(D(\Psi_0)) + 2\kappa_\sigma \Delta(\Psi_0) - \sigma \Delta(3\Psi_2 + 2\Phi_{11}) + \Delta(D(\Phi_{02})) = 0.
\]
(4.20)

Again it follows from our assumptions that the last two terms on the l.h.s. of (4.20) vanish, whence the only \( u \)-periodic solution of (4.20) satisfies \( \Delta(\Psi_0)^o = 0 \).

An immediate further consequence of the \( u \)-independence of \( \alpha, \beta, \gamma, \tau, \rho, \sigma \) and \( \Phi_{00}, \Phi_{01}, \Phi_{11} \) on \( \tilde{N}_i \) are the following: By (B.9) - (B.11) we have
\[
\Delta(D(\{\Phi_{11} + 3\Lambda, \Phi_{22}, \Phi_{21}\}))^o = 0,
\]
(4.21)
and by (NP.7) - (NP.9)
\[
\Delta(D(\{\lambda, \mu, \nu\}))^o = 0.
\]
(4.22)

We can now proceed inductively to show that all the \( D \)-derivatives of \( (\gamma + \tilde{\gamma}), \tau, \rho, \sigma \) are \( u \)-independent on \( \tilde{N}_i \) up to order \( n - 1 \) precisely when the functions \( \Phi_{00}, \Phi_{01}, \Phi_{02}, (\Phi_{11} + 3\Lambda) \) and \( D(\Phi_{02}) \) and their \( D \)-derivatives up to order \( n - 2 \) are \( u \)-independent there. Suppose, as our inductive assumption, that the above statement is satisfied for \( n = \bar{n} \). Then, by an exactly the same type of argument as the one yielded the equations (4.21) and (4.22) it can be proven that
\[
\Delta(D^{(i)}(\{\lambda, \mu, \nu, (\Phi_{11} + 3\Lambda), \Phi_{22}, \Phi_{21}\}))^o = 0,
\]
(4.23)
for any value of \( i \in \{1, 2, ..., \bar{n}\} \). We show now that our inductive assumption holds also for \( n = \bar{n} + 1 \).

First note that by (NP.6) and our inductive hypothesis \( \Delta(D^{(i)}(\gamma + \tilde{\gamma}))^o = 0 \) is equivalent to \( \Delta(D^{(\bar{n} - 1)}(\Psi_2 + \bar{\Psi}_2 - 2\Lambda + 2\Phi_{11}))^o = 0 \), which in virtue of (NP.12) is equivalent to the vanishing of \( \Delta(D^{(\bar{n} - 1)}(\Phi_{11}))^o \). This, however, follows from our inductive assumption, i.e. from the vanishing of the functions \( \Delta(D^{(\bar{n} - 1)}(\Phi_{11} + 3\Lambda)) \) and \( \Delta(D^{(\bar{n} - 1)}(\Phi_{11} - 3\Lambda)) \) on \( \tilde{N}_i \).

By (NP.3) - (NP.5) we have that \( \Delta(D^{(\bar{n}}(\{\alpha, \beta, \tau\}))^o = 0 \) if and only if \( \Delta(D^{(\bar{n} - 1)}(\{\Phi_{01}, \Psi_1\}))^o = 0 \). The first part, i.e. \( \Delta(D^{(\bar{n} - 1)}(\Phi_{01}))^o = 0 \), follows from our inductive hypothesis while to see that \( \Delta(D^{(\bar{n} - 1)}(\Psi_1))^o = 0 \) consider the \( (\bar{n} - 2) \)-times \( D \)-derivatives of (B.1). Using the commutators of \( D, \delta \) and \( \tilde{\delta} \) we get that \( D^{(\bar{n} - 1)}(\Psi_1) \) can be given in terms of \( u \)-independent quantities on \( \tilde{N}_i \).

Similarly, it follows from (NP.1) and (NP.2) that \( \Delta(D^{(\bar{n}}(\{\rho, \sigma\}))^o = 0 \) holds if and only if \( \Delta(D^{(\bar{n} - 1)}(\{\Phi_{00}, \Psi_0\}))^o = 0 \). Again, the first half is just a part of inductive hypothesis while the
vanishing of \( \Delta \left( D^{(\bar{n}-1)} (\Psi_0) \right) \) follows from the fact that on \( \tilde{N}_i \) \( \Delta \left( D^{(\bar{n}-1)} (\Psi_0) \right) \) has to satisfy the equation

\[
\Delta \left( D^{(\bar{n}-1)} (\Psi_0) \right) + \kappa \cdot \text{const} \cdot D^{(\bar{n}-1)} (\Psi_0) + \{ \text{terms independent of } u \} = 0 \quad (4.24)
\]

which is yielded by making use of the \((\bar{n}-1)\)-times D-derivative of (B.2) along with the application of the relevant form of the commutators of D, \( \delta \) and \( \tilde{\delta} \) several times.

Consequently, the spin coefficients \((\gamma + \tilde{\gamma}), \tau, \rho, \sigma\) and their D-derivatives up to order \( \bar{n} \) are independent of \( u \) on \( \tilde{N}_i \) which completes the proof of our inductive argument.

By making use of the relation between the Gaussian null coordinates and the above applied null tetrads, along with the relationship between the components of the energy-momentum tensor, \( T_{ab} \), and the Ricci spinor components \( \Phi_{\alpha\beta} \) and \( \Lambda \), Prop. 4.2 can be rephrased as:

**Corollary 4.1** Let \((O_i, g_{ab} | O_i)\) be an elementary spacetime region. Suppose that the components \( T_{ur}, T_{rr}, T_{rA}, T_{AB} \) along with the r-derivatives of \( T_{rr}, T_{rA}, T_{AB} \) up to order \( n-1 \) are \( u \)-independent on \( \tilde{N}_i \). Then the r-derivatives of the functions \( f, h_A, g_{AB} \) up to order \( n \) are also \( u \)-independent on \( \tilde{N}_i \).

5 Particular Einstein-matter systems

This section is to introduce particular gravity-matter systems, such as EKG, E[H], E[YMn] and EYMH configurations, into our analysis. It is important to emphasize that the dominant energy condition is not needed to be imposed separately because it follows from the particular form of the relevant energy momentum tensors that \( T^a_b k^b \) is proportional to \( k^a \) on \( \tilde{N}_i \) for these systems.

5.1 Einstein–Klein-Gordon–Higgs systems

A Klein-Gordon field is represented by a single real scalar field \( \psi \) satisfying the linear second order hyperbolic equation

\[
\nabla^a \nabla_a \psi + m^2 \psi = 0 \quad (5.1)
\]

and the energy-momentum tensor of the relevant EKG system reads as

\[
T_{ab} = (\nabla_a \psi)(\nabla_b \psi) - \frac{1}{2} g_{ab} \left[ (\nabla_c \psi)(\nabla_c \psi) - m^2 \psi^2 \right]. \quad (5.2)
\]

**Proposition 5.1** Let \((O_i, g_{ab} | O_i)\) be an elementary spacetime region associated with an EKG system. Then the functions \( f, h_A, g_{AB} \) and \( \psi \), along with their r-derivatives up to any order, are \( u \)-independent on \( \tilde{N}_i \).

**Proof** We shall prove our statement by induction. To see that the functions \( f^\circ, h_A^\circ, g_{AB}^\circ \) and \( \psi^\circ \) are \( u \)-independent note that as a consequence of (5.2) \( T_{uu} \geq 0 \) on \( \tilde{N}_i \) which, along with the argument applied in Prop. 4.3 implies that \( T_{uu}^\circ = 0 \) and, in turn,

\[
\left( \frac{\partial \psi}{\partial u} \right)^\circ = \left( \frac{\partial g_{AB}}{\partial u} \right)^\circ = 0. \quad (5.3)
\]

(5.2) and (5.3), implies then that

\[
T_{uA}^\circ = 0, \quad (5.4)
\]
which by the argument of Prop. 4.1 yields that

\[
\left( \frac{\partial f}{\partial u} \right)^\circ = \left( \frac{\partial h_A}{\partial u} \right)^\circ = 0. \tag{5.5}
\]

To show that the first order \(r\)-derivatives of the functions \(f, h_A\) and \(g_{AB}\) are \(u\)-independent on \(\tilde{N}_i\), in virtue of Cor. 4.1, we need to demonstrate that \(T_u, T_{rr}, T_{rA}, T_{AB}\) are \(u\)-independent there. Recall now that, by the relation (5.2), the components \(T_{ab}\) can be given in terms of \(f, h_A, g_{AB}, \psi\) and the first order partial derivatives, \(\partial \psi / \partial x^\delta\), of \(\psi\). It follows from (5.3) and (5.5) that all of these functions but \(\partial \psi / \partial r\) are \(u\)-independent on \(\tilde{N}_i\). The \(u\)-independence of \(\partial \psi / \partial r\) can be justified as follows: Since \(\psi\) satisfies (5.1), we have that, in an arbitrary local coordinate system \((x^1, x^2, x^3, x^4)\),

\[
g^{\alpha\beta} \frac{\partial^2 \psi}{\partial x^\alpha \partial x^\beta} - g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} \frac{\partial \psi}{\partial x^\gamma} + m^2 \psi = 0 \tag{5.6}
\]

holds. In Gaussian null coordinates, i.e. whenever \(x^1 = u, x^2 = r\), this equation reads on \(\tilde{N}_i\) as

\[
\frac{\partial^2 \psi}{\partial u \partial r} + \kappa_\sigma \frac{\partial \psi}{\partial r} + \{ \text{terms independent of } u \} = 0. \tag{5.7}
\]

The \(u\)-derivative of (5.7) is a homogeneous linear ordinary differential equation for \(\partial^2 \psi / \partial u \partial r\) along the generators of \(\tilde{N}_i\) with the only \(u\)-periodic solution \((\partial^2 \psi / \partial u \partial r)^\circ \equiv 0\). Thus the first order \(r\)-derivatives of the functions \(f, h_A, g_{AB}\) and \(\psi\) are \(u\)-independent on \(\tilde{N}_i\).

Assume, now, as our inductive hypothesis that the \(r\)-derivatives of the functions \(f, h_A, g_{AB}\) and \(\psi\) are \(u\)-independent on \(\tilde{N}_i\) up to order \(n\) \(\in\mathbb{N}\). We need to show that \(r\)-derivatives of the functions \(f, h_A, g_{AB}\) and \(\psi\) up to order \(n + 1\) are also \(u\)-independent on \(\tilde{N}_i\). In virtue of Cor. 4.1, the \(u\)-independence of the \(r\)-derivatives \(f, h_A\) and \(g_{AB}\) up to order \(n + 1\) can be traced back to the \(u\)-independence of the \(r\)-derivatives of \(T_{rr}, T_{rA}\) and \(T_{AB}\) up to order \(n\). However, according to (5.2), these derivatives can be given in terms of the \(r\)-derivatives of \(\psi, \partial \psi / \partial x^\alpha\), and \(f, h_A, g_{AB}\) up to order \(n\). Note that, by our inductive assumption, we have that \(\psi, \partial \psi / \partial u, \partial \psi / \partial x^A\) and the functions \(f, h_A, g_{AB}\) possess \(u\)-independent \(r\)-derivatives up to order \(n\). Thereby, the \(u\)-independence of the \(r\)-derivatives of \(\partial \psi / \partial r\) up to the same order have to be shown. To do this, differentiate first (5.6) \(n\)-times with respect to \(r\) and set \(r = 0\). The yielded equation is

\[
\frac{\partial^{n+2} \psi}{\partial u \partial r^{n+1}} + 2 \kappa_\sigma \frac{\partial^{n+1} \psi}{\partial r^{n+1}} + \{ \text{terms independent of } u \} = 0. \tag{5.8}
\]

Then, by differentiating (5.8) with respect to \(u\), we get a homogeneous linear ordinary differential equation for \((\partial^{n+2} \psi / \partial u \partial r^{n+1})\) with the only \(u\)-periodic solution \((\partial^{n+2} \psi / \partial u \partial r^{n+1})^\circ \equiv 0\).

\[\square\]

**Remark 5.1** Essentially the same simple reasoning does apply to the case of a set of self-interacting complex scalar fields, whenever the interaction terms do not contain derivatives of the fields. The above argument also generalize to the following case of [non-Abelian] Higgs fields: Let \(g\) be a Lie algebra associated with a Lie group \(G\). For the sake of definiteness \(G\) will be assumed to be a matrix group and it will also be assumed that there exists a positive definite real inner product, denoted by \((\, , \, )\), on \(g\) which is invariant under the adjoint representation. (The relevant gauge or matrix indices will be suppressed throughout.) Denote by \(\psi : O_r \to g\) the Higgs field and consider the associated matter Lagrangian

\[
\mathcal{L}_{\text{matter}}^{Higgs} = 2 \left[ g^{ab} \left( \nabla_a \psi / \nabla_b \psi \right) - V(\psi) \right], \tag{5.9}
\]

where \(V\) is a sufficiently regular but otherwise arbitrary gauge invariant expression of the field variable \(\psi\). The claim that Prop. 7.7 generalizes to the corresponding \(E[nA]\) \(H\) systems is based on the
following observations: First, since the real inner product, $(\cdot, \cdot)$, is positive definite, by a straightforward modification of the first part of the proof of Prop. 5.1, the vanishing of $T^u_a$ and $T^u_A$ can be justified for an adapted gauge representation $\psi$. Second, because of the ‘special’ choice of the interaction term, the equations for the $r$-derivatives of the various gauge components of $\psi$ decouple so that each equation possesses the fundamental form of (5.7) or (5.8). This way we get the following:

**Corollary 5.1** Let $(\mathcal{O}_i, g_{ab}|\mathcal{O}_j)$ be an elementary spacetime region associated with an $E/nA/H$ system as it was specified above. Then there exists a gauge representation $\psi$ so that the functions $f, h_A, g_{AB}$ and $\psi$, along with their $r$-derivatives up to any order, are $u$-independent on $\mathcal{N}_i$.

### 5.2 Einstein–Maxwell systems

In this subsection we shall consider the case of source free electromagnetic fields. The proof of Prop. 5.2 is an alternative of the argument given in [17] and it is presented to provide a certain level of preparation for the more complicated case of EYM systems.

A source free electromagnetic field, in any simply connected elementary spacetime region, can be represented by a vector potential $A_a$ related to the Maxwell tensor as

$$F_{ab} = \nabla_a A_b - \nabla_b A_a.$$  \hfill (5.10)

The energy-momentum tensor of the related Einstein-Maxwell system reads as

$$T_{ab} = -\frac{1}{4\pi} \left\{ F_c^e F_{b}^e - \frac{1}{4} g_{ab} (F_e^f F^e_f) \right\}$$  \hfill (5.11)

while the Maxwell equations are

$$\nabla^a F_{ab} = 0.$$  \hfill (5.12)

**Proposition 5.2** Let $(\mathcal{O}_i, g_{ab}|\mathcal{O}_i)$ be an elementary spacetime region and consider a source free electromagnetic field $F_{ab}$ on $\mathcal{O}_i$ in Einstein theory. Then there exists a vector potential $A_a$ associated with $F_{ab}$ so that $A_a$ vanishes on $\mathcal{N}_i$, moreover, the $r$-derivatives of the functions $f, h_A$ and $g_{AB}$ and that of the components $A_u, A_r, A_B$ up to any order are $u$-independent on $\mathcal{N}_i$.

**Proof** We shall prove the above statement by induction. To start off pick up an arbitrary, $u$-periodic, vector potential $A'_a$ of $F_{ab}$. We have the ‘gauge freedom’ of adding the gradient $\nabla_a \alpha$ of a function $\alpha$ to $A'_a$. At least the major part of this freedom can be fixed by the specification of the gauge source function $A' = \nabla^a A'_a$. Since the components of $g_{ab}$ and $A'_a$ were assumed to be $u$-periodic $A'$ is also a smooth $u$-periodic function on $\mathcal{O}_i$.

**Lemma 5.1** There exists a smooth $u$-periodic function $\alpha : \mathcal{O}_i \to \mathbb{R}$ such that

$$\partial \alpha / \partial u = -A'_u$$  \hfill (5.13)

holds on $\mathcal{N}_i$, moreover, the gauge source function

$$A := \nabla^a \nabla_a \alpha + A' = 0$$  \hfill (5.14)

on $\mathcal{O}_i$.
In virtue of (5.16) and (5.17) we also have that $\alpha_{(o)} = -\int_{u_0}^{u} A_u^o + \chi$ where $\chi$ is an arbitrarily chosen smooth function of $x^3$ and $x^4$. In virtue of (5.17) $F_{uB}^o = (\partial A_B^o/\partial u - \partial A_u^o/\partial x^B)^o = 0$ (the vanishing of $F_{uB}^o$ is independent of the applied vector potential) we have that

$$\frac{\partial \alpha_{(o)}}{\partial x^B} = -A_B^o + \chi^*, \tag{5.15}$$

where $\chi^*$ is a smooth function of $x^3$ and $x^4$. Since the r.h.s. of (5.13) is $u$-periodic $\alpha_{(o)}$ has also to be $u$-periodic.

To show that there exists a smooth function $\alpha$ on $O_i$ so that both (5.13) and (5.14) are satisfied we shall use the characteristic initial value problem associated with (5.14). Notice first that (5.13) will be immediately satisfied if the relevant solution of (5.14) is ensured to coincide with

$$-\int_{\gamma_i^1}^\gamma u \, du = \alpha_{(o)}(\gamma_i^1) = 0 \quad \text{on} \quad O_i \cap \gamma_i^1.$$ \hspace{1cm} \text{for} \qquad k \in \mathbb{Z} \tag{5.18}

To specify our initial data hypersurface consider first a smooth cross-section $\tilde{\gamma}_i(\gamma_i^1) = u = u_0 \in \mathbb{R}$ of $\tilde{N}_i$. This cross-section divides the boundary of the causal future $J^+[\tilde{\gamma}_i(\gamma_i^1), O_i]$ of $\tilde{\gamma}_i(\gamma_i^1)$ into two connected pieces. Denote by $\tilde{N}_i^+(u_0)$ the component contained by $\tilde{N}_i$ and by $\tilde{N}_i^-(u_0)$ the other part. Since there is no conjugate point to $\tilde{\gamma}_i(u_0)$ along the null generators of the smooth hypersurfaces $\tilde{N}_i^-(u_0)$ and $\tilde{N}_i^+(u_0)$ they comprise a suitable initial data surface $\tilde{\gamma}_i(u_0) = \tilde{N}_i^+(u_0) \cup \tilde{N}_i^-(u_0)$ for (5.14). To specify the initial data extend first the function $\alpha_{(o)}$ from $\tilde{N}_i$ onto $O_i$ by keeping its value to be constant along the integral curves of $\tilde{\gamma}_i = (\partial/\partial u)^a$. Choose then the restriction of this extension $\alpha_{(o)}$ onto $\tilde{\gamma}_i(u_0)$ as our initial data.

Consider now the discrete isometry action $\Psi_1^{(k)} : O_i \to O_i \ [k \in \mathbb{Z}]$, associated with the fiber preserving local isometry action $\psi_i$ (see section 2). By construction for any value of $k \in \mathbb{Z} \ \Psi_1^{(k)}$ maps $\tilde{\gamma}_i(u_0)$ to the surface $\tilde{\gamma}_i(u_0 + kP)$. Moreover, since $\alpha_{(o)}$ is invariant under the action of $\Psi_1^{(k)}$, the initial data specifications $\alpha_{(o)}|_{\tilde{\gamma}_i(u_0)}$ and $\alpha_{(o)}|_{\tilde{\gamma}_i(u_0 + kP)}$ are mapped onto each other by $\Psi_1^{(k)}$. Appealing now to the uniqueness of the solutions to the linear wave equation (5.13) in the domain of dependence of an initial data hypersurface it follows then that the relevant Cauchy developments of $\alpha_{(o)}|_{\tilde{\gamma}_i(u_0)}$ and $\alpha_{(o)}|_{\tilde{\gamma}_i(u_0 + kP)}$ are also mapped onto each other by $\Psi_1^{(k)}$. This, however, in turn implies that the unique smooth solution $\alpha$ of (5.14) with initial data $\alpha_{(o)}|_{\tilde{\gamma}_i(u_0)}$ is $u$-periodic with periodicity length $P$. By making use of this property $\alpha$ can be extended onto the entire of $J^+[\tilde{N}_i] \cap O_i$ so that the extension coincides with $\alpha_{(o)}$ on $\tilde{N}_i$.

An analogous argument applies to the complementary region, $J^-[\tilde{N}_i, O_i]$, which completes then the proof of the second part of our lemma.

Turning back to the proof of Prop. 5.2 consider the vector potential $A_a := A_a^\prime \nabla_a \alpha$. In virtue of (5.13), we have that

$$A_u^o = 0. \tag{5.16}$$

Moreover, we also have that $A = \nabla^a A_a$ vanishes in $O_i$.

From this point one can proceed as follows: By (5.11) and the negative definiteness of $g_{AB}$ one gets that $T_{uu}^o = -\frac{1}{4\pi} (F_{uA}F_{uB}g_{AB})^o \geq 0$ which, along with the argument of Prop. 3.3, implies that

$$F_{uB}^o = \left(\frac{\partial A_B^o}{\partial u} - \frac{\partial A_u^o}{\partial x^B}\right)^o = 0 \quad \text{and} \quad \left(\frac{\partial g_{AB}}{\partial u}\right)^o = 0. \tag{5.17}$$

In virtue of (5.16) and (5.17) we also have that

$$\left(\frac{\partial A_B^o}{\partial u}\right)^o = 0. \tag{5.18}$$

I would like to say thank to the unknown referee who pointed out the incompleteness of the relevant argument contained by the former version of this paper.

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\[7\]
Then (5.17) and (5.11) imply that $T_{uA}^\circ = 0$ which by the argument of Prop. 4.1 yields that
\[
\left( \frac{\partial f}{\partial u} \right)^\circ = \left( \frac{\partial h_A}{\partial u} \right)^\circ = 0.
\] (5.19)

What remains to be shown, to complete the first step of our inductive proof, is the $u$-independence of $A_r$ on $\tilde{N}_i$ which can be demonstrated as follows: In Gaussian null coordinates the ‘$u$’-component of the Maxwell equation (5.12) on $\tilde{N}_i$ takes the form
\[
\frac{\partial F_{ru}}{\partial u} = 0,
\] (5.20)
i.e.,
\[
\frac{\partial}{\partial u} \left( \frac{\partial A_r}{\partial u} - \frac{\partial A_u}{\partial r} \right)^\circ = 0.
\] (5.21)

By differentiating now $\nabla^a A_u = 0$ with respect to $u$ and setting $r = 0$ we get
\[
\frac{\partial}{\partial u} \left( \frac{\partial A_r}{\partial u} + \frac{\partial A_u}{\partial r} \right) + 2\kappa \frac{\partial A_r}{\partial u} = 0.
\] (5.22)

It follows from (5.21) and (5.22) that
\[
\frac{\partial}{\partial u} \left( \frac{\partial A_r}{\partial u} \right) + \kappa \frac{\partial A_r}{\partial u} = 0
\] (5.23)
holds on $\tilde{N}_i$, which, in turn, along with periodicity, implies then that $A_r$ has to be $u$-independent on $\tilde{N}_i$ as we wanted to demonstrate.

To see that the first $r$-derivative of the functions $f, h_A, g_{AB}$, moreover, that of the components $A_u, A_r, A_B$ are $u$-independent on $\tilde{N}_i$ we can proceed as follows: Note first that by the $u$-independentness of $A_r$ on $\tilde{N}_i$ and by (5.21) we also have that $(\partial A_u/\partial r)^\circ$ is $u$-independent.

The $u$-independentness of $(\partial A_B/\partial r)^\circ$ can be shown by making use of the ‘$B$’-component of the Maxwell equation (5.12) which reads on $\tilde{N}_i$ as
\[
\frac{\partial}{\partial u} \left( \frac{\partial F_{ur}}{\partial r} \right) + \{\text{terms independent of } u\} = 0
\] (5.24)
with the only periodic solution $(\partial^2 A_B/\partial u \partial r)^\circ \equiv 0$. It follows then that all the components of $F_{ab}$ are $u$-independent on $\tilde{N}_i$ which, along with (5.11) and the $u$-independentness of the functions $f, h_A$ and $g_{AB}$, implies that each of the components $T_{ur}, T_{rr}, T_{rA}$ and $T_{AB}$ is $u$-independent on $\tilde{N}_i$. This later property, however, in virtue of Cor. 1.1 yields that the first $r$-derivatives of the functions $f, h_A$ and $g_{AB}$ are $u$-independent on $\tilde{N}_i$. Thus only the $u$-independentness of $(\partial A_r/\partial r)^\circ$ remained to be shown which can be done as follows: The first $r$-derivative of the ‘$u$’-component of the Maxwell equation (5.13) gives that on $\tilde{N}_i$
\[
\frac{\partial}{\partial u} \left( \frac{\partial F_{ur}}{\partial r} \right) + \{\text{terms independent of } u\} = 0
\] (5.25)
holds which, along with the $u$-periodicity, implies that
\[
\frac{\partial}{\partial u} \left[ \frac{\partial}{\partial r} \left( \frac{\partial A_r}{\partial u} - \frac{\partial A_u}{\partial r} \right) \right]^\circ = 0.
\] (5.26)
Moreover, the vanishing of the first $r$-derivative of the gauge source function $A$ on $\tilde{N}_i$ gives that

$$\frac{\partial}{\partial u} \left[ \frac{\partial}{\partial r} \left( \frac{\partial A_r}{\partial u} + \frac{\partial A_u}{\partial r} \right) + 4 \kappa_\circ \frac{\partial A_r}{\partial r} \right] \overset{\circ}{=} 0.$$ (5.27)

In virtue of (5.26) and (5.27) we have that both $(\partial^2 A_r/\partial u\partial r)\overset{\circ}{=} 0$ and $(\partial^3 A_r/\partial u\partial r^2)\overset{\circ}{=} 0$ vanish identically.

Assume, now, as our inductive hypothesis that the $r$-derivatives of the functions $f, h_A, g_{AB}$ and that of $A_u, A_r, A_B$ are $u$-independent on $\tilde{N}_i$ up to order $\bar{n} \in \mathbb{N}$. Then, by the vanishing of the $\bar{n}^{th}$ $r$-derivative of $A$ together with the $u$-independentness of $(\partial^{\bar{n}} A_r/\partial r^{\bar{n}})\overset{\circ}{=} 0$ on $\tilde{N}_i$, we also have that

$$\frac{\partial}{\partial u} \left( \frac{\partial^{\bar{n}+1} A_u}{\partial r^{\bar{n}+1}} \right) \overset{\circ}{=} 0.$$ (5.28)

Then the $u$-independentness of $(\partial^{\bar{n}+1} A_B/\partial r^{\bar{n}+1})\overset{\circ}{=} 0$ can be seen as follows: Differentiate the 'B'-component of the Maxwell equation (5.12) $\bar{n}$-times with respect to $r$ and set $r = 0$. This way we get

$$\frac{\partial}{\partial u} \left( \frac{\partial^{\bar{n}+1} A_B}{\partial r^{\bar{n}+1}} \right) + 2 \kappa_\circ \frac{\partial^{\bar{n}+1} A_B}{\partial r^{\bar{n}+1}} + \{\text{terms independent of } u\} = 0$$ (5.29)

which implies, along with the $u$-periodicity, that $(\partial^{\bar{n}+2} A_B/\partial u \partial r^{\bar{n}+1})\overset{\circ}{=} 0$. It follows then that the $r$-derivatives of the components of $F_{ab}$ up to order $\bar{n}$ are $u$-independent on $\tilde{N}_i$ which, in turn, along with our inductive hypotheses, implies that the $r$-derivatives of the components $T_{rr}, T_r A$ and $T_{AB}$ are $u$-independent there. This later property, however, in virtue of Cor.4.1, guarantees the $u$-independentness of the $r$-derivatives of the functions $f, h_A$ and $g_{AB}$ up to order $\bar{n} + 1$. Thus to complete our inductive proof we need only to demonstrate the $u$-independentness of $(\partial^{\bar{n}+1} A_r/\partial r^{\bar{n}+1})\overset{\circ}{=} 0$ which can be done as follows: Differentiate $(\bar{n}+1)$-times the 'u'-component of (5.12) with respect to $r$ and set $r = 0$. The yielded equation is of the form

$$\frac{\partial}{\partial u} \left( \frac{\partial^{\bar{n}+1} F_{ur}}{\partial r^{\bar{n}+1}} \right) + \{\text{terms independent of } u\} = 0,$$ (5.30)

which, along with the $u$-periodicity, implies that

$$\frac{\partial}{\partial u} \left[ \frac{\partial^{\bar{n}+1}}{\partial r^{\bar{n}+1}} \left( \frac{\partial A_r}{\partial u} - \frac{\partial A_u}{\partial r} \right) \right] \overset{\circ}{=} 0.$$ (5.31)

In addition, the vanishing of the $(\bar{n}+1)^{th}$ $r$-derivative of the gauge source function $A$ on $\tilde{N}_i$ yields that

$$\frac{\partial}{\partial u} \left[ \frac{\partial^{\bar{n}+1}}{\partial r^{\bar{n}+1}} \left( \frac{\partial A_r}{\partial u} + \frac{\partial A_u}{\partial r} \right) + 4 \kappa_\circ \frac{\partial^{\bar{n}+1} A_r}{\partial r^{\bar{n}+1}} \right] \overset{\circ}{=} 0.$$ (5.32)

In virtue of the last two equations, (5.31) and (5.32), we have that both $(\partial^{\bar{n}+2} A_r/\partial u \partial r^{\bar{n}+1})\overset{\circ}{=} 0$ and $(\partial^{\bar{n}+3} A_r/\partial u \partial r^{\bar{n}+2})\overset{\circ}{=} 0$ vanish, as we wanted to demonstrate.

### 5.3 Einstein–[Maxwell]–Yang-Mills (–dilaton –Higgs) systems

Let us consider now the case of a YM gauge field which can be represented by a vector potential $A_a$ taking values in the Lie algebra $\mathfrak{g}$ of a matrix group $G$, i.e. $G \subset \text{GL}(N, \mathbb{C})$ for some $N \in \mathbb{N}$. In terms of the vector potential $A_a$ the Lie-algebra-valued 2-form field $F_{ab}$ is given as

$$F_{ab} = \nabla_a A_b - \nabla_b A_a + [A_a, A_b]$$ (5.33)
where \( [ , ] \) denotes the product in \( g \). The energy-momentum tensor of the related EYM system is
\[
T_{ab} = -\frac{1}{4\pi}\left\{ (F_{ac}/F_b^c) - \frac{1}{4}g_{ab}(F_{ef}/F^e_f) \right\},
\]
(5.34)

where \((/)/\) is a positive definite real inner product in \( g \) which is invariant under the adjoint representation. Finally, the field equations of such a YM field read as
\[
\nabla^a F_{ab} + [A^a, F_{ab}] = 0.
\]
(5.35)

We would like to generalize Prop. 5.2 for such an EYM system. In doing this start with an arbitrary \( u \)-periodic adapted gauge potential \( A'_a \) defined on a simply connected elementary spacetime region \( O_i \). It is known that there is a freedom in representing a YM field, i.e. instead of \( A'_a \) we can also use the gauge related field
\[
A_a = m^{-1}(\nabla_a m + A'_a m),
\]
(5.36)

where \( m : O_i \to G \) is an arbitrary smooth function. Then the following can be proven:

**Lemma 5.2** There exists a smooth \( u \)-periodic function \( m : O_i \to G \) such that
\[
A^o_a = \left[ m^{-1}(\partial m/\partial u + A'_a m) \right]^o := a_o
\]
(5.37)
is a \( u \)-independent \( g \)-valued function on \( \tilde{N}_i \), moreover, the gauge source function
\[
A := \nabla^a A_a = \nabla^a [m^{-1}(\nabla_a m + A'_a m)]
\]
(5.38)
vanishes on \( O_i \).

**Proof** Consider first the (matrix) differential equation
\[
\partial m^*/\partial u + A^o_a m^* = 0
\]
(5.39)
where \( m^* \) is a \( G \)-valued function on \( \tilde{N}_i \). The general solution of \((5.39)\) is of the form (see e.g. Theorem 2.2.5 of [3])
\[
m^* = m_{(0)}\exp(-a_o u)
\]
(5.40)
where \( m_{(0)} : \tilde{N}_i \to G \) is a \( u \)-periodic function, with the same period as \( A'_a \), and \( a_o : \tilde{N}_i \to g \) is independent of \( u \). Then, \((5.39)\) and \((5.40)\) implies that
\[
\partial m_{(0)}/\partial u + A^o_a m_{(0)} - m_{(0)} a_o = 0.
\]
(5.41)
It follows then from \((5.36)\) that for any choice of a gauge transformation \( m \) with \( m|_{\tilde{N}_i} = m_{(0)}(u, x^3, x^4) \)
\( A^o_a = a_o \) holds on \( \tilde{N}_i \).

The second part of our statement can be proven by making use of an argument analogous to that of the proof of the second part of lemma 5.1. The only significant difference is that the evolution equation \((5.14)\) for \( \alpha \) has to be replaced by
\[
\nabla^a \nabla_a m + A'm + A'_a \nabla^a m - (\nabla^a m)m^{-1}(\nabla_a m + A'_a m) = 0
\]
(5.42)
and the relevant initial data for \( m \) has to be constructed by making use of the above defined \( m_{(0)} \).
Remark 5.2 Hereafter, without loss of generality, we shall assume that the \( u \)-component of an adapted gauge potential is \( u \)-independent. Note, however, that neither \( a_\circ \) nor its eigenvalues are uniquely determined by (5.32). To see this note that \( m^* = m_{(\circ)} \cdot \tilde{m} \cdot \exp[2\pi i u/P \cdot \text{diag}(k_1, ..., k_N)] \cdot \exp(-\tilde{a}_\circ u) \) is also a solution of (5.32) with

\[
\tilde{a}_\circ = \tilde{m}^{-1}a_\circ \tilde{m} + 2\pi i/P \cdot \text{diag}(k_1, ..., k_N),
\]

where \( \tilde{m} : \tilde{N} \to G \) is \( u \)-independent and the entries \( k_i \) are integers so that the matrices \( \tilde{m}^{-1}a_\circ \tilde{m} \) and \( \text{diag}(k_1, ..., k_N) \) commute. The simplest possible \( u \)-periodic gauge transformation \( m : \mathcal{O}_i \to G \) manifesting this freedom can be given as

\[
m = \tilde{m}(x^3, x^4) \cdot \exp[2\pi i u/P \cdot \text{diag}(k_1, ..., k_N)].
\]

Remark 5.3 In what follows key role will be played by the possible behavior of \( u \)-periodic \( \mathfrak{g} \)-valued functions satisfying differential equations – see e.g. (5.53), (5.54) and (5.64) – of the form

\[
\frac{\partial \mathcal{F}}{\partial u} = c_1 \kappa_\circ \mathcal{F} + c_2 [a_\circ, \mathcal{F}],
\]

with \( c_1 = 0, c_2 = -1 \) or \( c_1 = -1, c_2 = -1/2 \). This equation can also be read as

\[
\frac{\partial \mathcal{F}}{\partial u} = \mathcal{C}(a_\circ; \kappa_\circ, c_1, c_2) \mathcal{F},
\]

where now \( \mathcal{F} \) is considered to be a ‘vector’ whereas \( \mathcal{C}(a_\circ; \kappa_\circ, c_1, c_2) \) is a linear map acting on the corresponding ‘vector space’ given as

\[
\mathcal{C}(a_\circ; \kappa_\circ, c_1, c_2) = c_1 \kappa_\circ \mathcal{E} \otimes \mathcal{E} + c_2 (a_\circ \otimes \mathcal{E} - \mathcal{E} \otimes a_\circ^t).
\]

Here \( \mathcal{E} \), \( \otimes \) and \( a_\circ^t \) denote the unit element of \( G \), the tensor product on \( G \) and the transpose of \( a_\circ \), respectively. The system (5.47) is known (see e.g. (5.3)) to have a \( u \)-periodic solution of period \( P \) if and only if \( 2\pi ik/P \) is an eigenvalue of \( \mathcal{C}(a_\circ; \kappa_\circ, c_1, c_2) \) for some \( k \in \mathbb{Z} \). If \( k = 0 \) then \( \mathcal{C}(a_\circ; \kappa_\circ, c_1, c_2) \) is singular and (5.44) has non-trivial \( u \)-independent solutions. If \( k \in \mathbb{Z} \setminus \{0\} \) then (5.44) has a \( u \)-periodic solutions with smallest positive period \( P/|k| \).

Definition 5.1 Let \( A_\circ \) be an adapted gauge representation of a YM field with \( A_\circ^0 = a_\circ \). Then \( A_\circ \) is called to be a ‘preferred’ gauge representation if neither of the eigenvalues of \( \mathcal{C}(a_\circ; \kappa_\circ, c_1, c_2) \), with \( c_1 = 0, c_2 = -1 \) or \( c_1 = -1, c_2 = -1/2 \), is of the form \( 2\pi ik/P \) for any \( k \in \mathbb{Z} \setminus \{0\} \).

Remark 5.4 The eigenvalues of \( \mathcal{C}(a_\circ; \kappa_\circ, c_1, c_2) \) are the roots of its characteristic polynomial. The characteristic polynomial of \( \mathcal{C}(a_\circ; \kappa_\circ, c_1, c_2) \) is in fact an ‘invariant polynomial’ because its coefficients can be given in terms of \( \kappa_\circ, c_1, c_2 \) and polynomials of the traces of various powers of \( a_\circ \). In addition, (5.54) and (5.55) imply that \( \partial a_\circ / \partial x^B = -[a_\circ, A_B^\circ] \) thereby these traces have to be constant throughout \( \tilde{N}_i \), i.e. the characteristic polynomial of \( \mathcal{C}(a_\circ; \kappa_\circ, c_1, c_2) \) is the simplest possible type with degree \( \text{dim}(G) = N^2 \) and with constant coefficients.

Lemma 5.3 Let \( A_\circ \) be an adapted gauge representation of a YM field with \( A_\circ^0 = a_\circ \). Then there exist \( \tilde{m} : \tilde{N}_i \to G \) and \( k_i \in \mathbb{Z} \) so that the gauge representation \( \tilde{A}_\circ \) yielded by (5.44) is preferred.

\[\text{Remark 5.2 These equations are valid for any } u \text{-periodic gauge representation } A_\circ \text{ having } u \text{-independent } A_\circ^0 = a_\circ.\]
Proof. Denote by $a_\mu^i$ the Jordan normal form of $a_\mu$ and assume that the $G$-valued function $m_\mu$ is so that $a_\mu^i = m_\mu^{-1}a_\mu m_\mu$. Then it follows from $C(m_\mu^{-1}a_\mu m_\mu; \kappa_\mu, c_1, c_2) = (m_\mu \otimes m_\mu)^{-1} \cdot C(a_\mu; \kappa_\mu, c_1, c_2) \cdot (m_\mu \otimes m_\mu)$ that the eigenvalues of $C(a_\mu^i; \kappa_\mu, c_1, c_2)$ and $C(a_\mu^i; \kappa_\mu, c_1, c_2)$ coincide. Denote by $\alpha^i_j$ the eigenvalue of the $i^{th}$ eigenvalue of the $i^{th}$ diagonal block of $a_\mu^i$. Using then the special block structure of $a_\mu^i$, along with the definition (5.47), the eigenvalues of $C(a_\mu^i; \kappa_\mu, c_1, c_2)$ can be seen to be

$$c_{ij} = c_1 \kappa_\mu + c_2 (\alpha^i_j - \alpha^j_j), \tag{5.48}$$

where for distinct eigenvalues $\alpha^i_j$ and $\alpha^j_j$ of $a_\mu^i$ the multiplicity of the eigenvalue $c_{ij}$ is $m(c_{ij}) = m(\alpha^i_j)m(\alpha^j_j)$ whereas the multiplicity of $c_0 = c_1 \kappa_\mu$ is $m(c_0) = N^2 - 2 \sum_{\alpha^i_j \neq \alpha^j_j} m(\alpha^i_j)m(\alpha^j_j)$.

Consider now a $u$-periodic gauge transformation of the form

$$m = \exp \left[ 2\pi i u / P \cdot \text{diag}(k_1', \ldots, k_N', \ldots, k_N') \right], \tag{5.49}$$

where the multiplicity of the eigenvalues of $\text{diag}(k_1', \ldots, k_N', \ldots, k_N')$ is chosen to be the same as that of $a_\mu^i$ so that they commute. According to Remark 5.2 by making use of such a gauge transformation the eigenvalues $\alpha^i_j$ of $a_\mu^i$ can be shifted by adding the values $2\pi i k^i_j / P$ where $k^i_j \in \mathbb{Z}$ denotes the eigenvalue of the $i^{th}$ diagonal block of $\text{diag}(k_1', \ldots, k_N', \ldots, k_N')$. This, in virtue of (5.48), yields a discrete shifting of $c_{ij}$ by adding a term of the form $2\pi i c_2 (k^i_j - k^j_j) / P$ to the eigenvalues $c_{ij}$.

Assume now that $c_{ij'} = 2\pi i k^i_{j'} / P$ for a fixed set of the values of $i', j'$ with $\alpha^i_{j'} \neq \alpha^j_{j'}$ where $k^i_{j'} \in \mathbb{Z} \setminus \{0\}$. Then, by choosing the integers $k^i_j$ in (5.49) so that

$$k^i_j - k^j_j = c_2^{-1} k^i_{j'}, \tag{5.50}$$

all of the eigenvalues $c_{ij'}$ can simultaneously be shifted to be zero. To see that the inhomogeneous linear system (5.50) does really have a solution of the needed type recall that the integers $k^i_{j'}$ are not independent since $c_2[3(\alpha^i_{j'}) - 3(\alpha^j_{j'})] = 2\pi i k^i_{j'}/P$, where $\Im(\alpha)$ denotes the imaginary part of $\alpha$. Thereby, the coefficient and augmented matrices of (5.50) are always of the same rank.

It follows then that (5.44) with $\bar{m} = m_\mu$ and with a string $(k_1, \ldots, k_N)$ built up from suitably chosen integers $k^i_j$, yields a preferred gauge representation $\bar{A}_\mu$ with $\bar{c}_{ij'} = 0$.

Remark 5.5 Consider now the preferred representation $\bar{A}_\mu$ yielded by the above described process with $\bar{c}_{ij'} = 0$. By making use of a suitable gauge transformation of the form (5.44) it can be mapped to a non-preferred representation so that the eigenvalues $c_{ij'} = 2\pi i k^i_{j'}/P$ are freely specifiable. Correspondingly, centered on any preferred gauge representation of the type of $\bar{A}_\mu$ an infinite crystal of non-preferred representations can be built up. On the other hand, those preferred representations for which neither of the eigenvalues $c_{ij}$ with $\alpha^i_j \neq \alpha^j_j$ vanishes are always shifted to another preferred representation. Consequently, there can exist YM fields such that all of their adapted gauge representations are preferred.

Example 5.1 Consider the particular case of $SU(2)$ gauge group. Since $g = su(2)$ consists of the traceless skew-hermitian 2-matrices $a_\mu^i$ must be of the form $a_\mu^i = i \alpha_\mu \text{diag}(1, -1)$ for some $\alpha_\mu \in \mathbb{R}$. Hence, for $\kappa_\mu = 0$, an adapted gauge representation $A_\mu$ with $A_\mu^0 = a_\mu$ is non-preferred whenever $\alpha_\mu = \pi k / P$ for some $k \in \mathbb{Z} \setminus \{0\}$. Correspondingly, in this case there is a single one-dimensional infinite crystal of non-preferred representations centered on the preferred representation $a_\mu = 0$.

Turning back to our general argument assume that $A_\mu$ is a preferred gauge representation, with $A_\mu^0 = a_\mu$, of the considered EYM system. Since the gauge transformation $\bar{m}$ applied in the proof of the above lemma had no $r$-dependence the gauge source function

$$A = \nabla^\alpha A_\mu, \tag{5.51}$$
also vanishes on $O_i$. Moreover, by (5.37)

$$
\left(\frac{\partial A_u}{\partial u}\right)^\circ = 0.
$$

(5.52)

From this point one can proceed as follows: Since $g_{AB}$ is negative definite and $(/)$ is positive definite in virtue of (5.34)

$$
T_{wu}^\circ = -\frac{1}{4\pi} \left[ (F_u A / F_u B) g^{AB} \right]^\circ \geq 0
$$

(5.53)

holds which, according to the argument of Prop. 3.1 implies that

$$
F_{uB}^\circ = \left( \frac{\partial A_B}{\partial u} - \frac{\partial A_u}{\partial x_B} + [A_u, A_B] \right)^\circ = 0 \text{ and } \left( \frac{\partial g_{AB}}{\partial u} \right)^\circ = 0.
$$

(5.54)

Then, in virtue of (5.52) and (5.54)

$$
\frac{\partial}{\partial u} \left( \frac{\partial A_B}{\partial u} \right)^\circ + \left[ a_\circ, \left( \frac{\partial A_B}{\partial u} \right)^\circ \right] = 0.
$$

(5.55)

Any $u$-periodic solution $A_B$ of (5.55) has also to satisfy

$$
\left( \frac{\partial A_B}{\partial u} \right)^\circ = 0.
$$

(5.56)

Note that (5.54) and (5.34) imply that $T_{uA}^\circ = 0$ which by the argument of Prop. 4.1 yields that

$$
\left( \frac{\partial f}{\partial u} \right)^\circ = \left( \frac{\partial h_A}{\partial u} \right)^\circ = 0.
$$

(5.57)

It can now be shown that $A_r$ is $u$-independent on $\tilde{N}_i$. To see this, consider the $'u'$-component of (5.33) in Gaussian null coordinates which on $\tilde{N}_i$ reads as

$$
\frac{\partial F_r}{\partial u} + [a_\circ, F_r] = 0.
$$

(5.58)

In the case of the considered gauge representation any $u$-periodic solution of (5.58) is also $u$-independent on $\tilde{N}_i$ and it has to commute with $a_\circ$. Hence, in particular,

$$
\left( \frac{\partial F_r}{\partial u} \right)^\circ = \frac{\partial}{\partial u} \left[ \frac{\partial A_r}{\partial u} - \frac{\partial A_u}{\partial r} \right] + \left[ a_\circ, \left( \frac{\partial A_r}{\partial u} \right)^\circ \right] = 0.
$$

(5.59)

In addition, by differentiating (3.51) with respect to $u$ and setting $r = 0$, we get

$$
\frac{\partial}{\partial u} \left( \frac{\partial A_r}{\partial u} + \frac{\partial A_u}{\partial r} \right) + 2\kappa_\circ \left( \frac{\partial A_r}{\partial u} \right)^\circ = 0.
$$

(5.60)

Then by (5.59) and (5.60) we have that on $\tilde{N}_i$

$$
2\frac{\partial}{\partial u} \left( \frac{\partial A_r}{\partial u} \right) + 2\kappa_\circ \frac{\partial A_r}{\partial u} + \left[ a_\circ, \left( \frac{\partial A_r}{\partial u} \right) \right] = 0
$$

(5.61)

which for the case of the selected type of gauge representation has the only periodic solution

$$
\left( \frac{\partial A_r}{\partial u} \right)^\circ = 0.
$$

(5.62)
From this point the argument of the proof of Prop. 5.2 can be repeated for the case of a YM field with obvious notational changes and the additional analysis related to the presence of the second term on the l.h.s. of (5.33) and the third term on the r.h.s. of (5.33). As we have seen – compare eqs. (5.53), (5.58) and (5.61) to the corresponding equations applied to prove Prop. 5.2. – these new terms contribute only a single commutator of \(a_o\) and the ‘unknown’ of the relevant equations and ‘\{terms independent of \(u\}\’.

By an inductive argument, it can also be shown that

\[
\left( \frac{\partial^n}{\partial r^n} [A_u, A_r] \right)^o = \left[ a_o, \left( \frac{\partial^n A_r}{\partial r^n} \right)^o \right] + \{terms \ independent \ of \ u\}. \tag{5.63}
\]

Similarly, by induction we get that for the ‘\(u\)’ and ‘\(B\)’-components of the second term on the l.h.s. of (5.33)

\[
\left( \frac{\partial^n}{\partial r^n} [A^a, F_{au}] \right)^o = \left[ a_o, \left( \frac{\partial^n F_{ru}}{\partial r^n} \right)^o \right] + \{terms \ independent \ of \ u\} \tag{5.64}
\]

and

\[
\left( \frac{\partial^n}{\partial r^n} [A^a, F_{uB}] \right)^o = \left[ a_o, \left( \frac{\partial^{n+1} A_B}{\partial r^{n+1}} \right)^o \right] + \{terms \ independent \ of \ u\} \tag{5.65}
\]

hold on \(\tilde{N_i}\). Based on these observations the following can be proven:

**Proposition 5.3** Let \((\mathcal{O}, g_{ab}|_{\mathcal{O}})\) be an elementary spacetime region associated with an E[M]YM system. Then there exists a gauge potential \(A_u: \mathcal{O} \to \mathfrak{g}\) so that the \(r\)-derivatives of the functions \(f, h_A\) and \(g_{AB}\) and also of the components \(A_u, A_r, AB\) up to any order are \(u\)-independent on \(\tilde{N_i}\).

**Remark 5.6** It is also important to know what kind of symmetries are adopted by an EYM system associated with a non-preferred \(u\)-periodic gauge representation. Let \(A_u\) be such a gauge representation so that \(A_u^o = a_o\) is \(u\)-independent. Since \(A_u\) is non-preferred there must exist \(k \in \mathbb{Z} \setminus \{0\}\) so that \(2\pi ik/P\) is an eigenvalue of \(C(a_o; \kappa_0, c_1, c_2)\). Recall now that the equations (5.53), (5.58) and (5.64) possess, instead of the above \(u\)-independent solutions, \(u\)-periodic solutions with smallest positive period \(P/|k|\). These solutions in the succeeding equations always yield periodic ‘forcing terms’. Therefore these equations – and also all the succeeding ones at higher levels of the corresponding hierarchy – possess \(u\)-periodic solutions, with smallest positive period \(P/|k|\). (The last claim follows from Theorem 2.3.6 of \cite{3} since all the relevant functions are ensured to be bounded by their original \(u\)-periodicity.) Consequently, in case of a non-preferred gauge representation only the \(u\)-periodicity of the restrictions of the \(r\)-derivatives of the functions \(f, h_A\) and \(g_{AB}\) and also of the components \(A_u, A_r, AB\) onto \(\tilde{N_i}\) can be demonstrated. Nevertheless, we would like to emphasize that by making use of the ‘discrete shifting freedom’ the value of \(k\) can be adjusted to be an arbitrary integer. Hence, by ranging through all the associated non-preferred representations at least the \(u\)-independentness of the metric functions could be shown even though we had no preferred gauge representation.

**Remark 5.7** In the case of asymptotically flat static spherically symmetric SU(2) YM black holes a gauge potential with vanishing \(a_0\) can always be chosen. However, numerical studies indicated that even in the static case the YM black hole spacetimes need not to be spherically symmetric \cite{7,9}. Moreover, perturbative analyses showed that they need not even to be axially symmetric \cite{4,26,11}. In virtue of Remarks 5.3 and 5.4 it would be important to make it clear what an extent the relevant gauge freedom had been exhausted in arriving to these conclusions. Clearly, by an adaptation of the framework of the present paper analytic studies of the corresponding YM configurations could also be carried out at least in a sufficiently small neighborhood of the horizon.
Remark 5.8 By a straightforward combination of the arguments that apply for the separated cases of a scalar field, a Higgs field and a Yang-Mills field it can be shown that Prop. 5.2 and Prop. 5.3 generalize to E[M]YMd and EYMH systems. To see this consider an E[M]YMd, resp. an EYMH, system given by the Lagrangian $L = R + L_{\text{matter}}$ with

$$L_{[M]YMd}^{[M]YMd} = -e^{2\gamma_d} \psi (F_{ef}/F^{ef}) + 2\nabla^e \psi \nabla_e \psi, \quad (5.66)$$

resp.

$$L_{\text{matter}}^{YM} = -(F_{ef}/F^{ef}) + 2[(D^e \psi/D_e \psi) - V(\psi)], \quad (5.67)$$

where $R$ is the 4-dimensional Ricci scalar, $\psi$ stands for the (real) dilaton field and $\gamma_d$ is the dilaton coupling constant, resp. for the Higgs field with $D_a \psi = \nabla_a \psi - [A_a, \psi]$ and with a sufficiently regular but otherwise arbitrary gauge invariant potential $V(\psi)$, $F_{ab}$ is the YM field strength. The validity of our statement is based on the observations that the energy-momentum tensor for such an E[M]YMd, resp. EYMH, system reads as

$$T_{ab}^{[M]YMd} = -\frac{1}{4\pi} \left[ e^{2\gamma_d} \psi (F_{ac}/F_b^c) - \nabla_a \psi \nabla_b \psi + \frac{1}{4} g_{ab} L_{\text{matter}}^{[M]YMd} \right], \quad (5.68)$$

resp.

$$T_{ab}^{YM} = -\frac{1}{4\pi} \left[ (F_{ac}/F_b^c) - (D_a \psi/D_b \psi) + \frac{1}{4} g_{ab} L_{\text{matter}}^{YM} \right], \quad (5.69)$$

and also that the field equations are

$$\nabla^a \nabla_a \psi + \frac{\gamma_d}{2} e^{2\gamma_d} \psi (F_{ef}/F^{ef}) = 0 \quad (5.70)$$

resp.

$$\nabla^a F_{ab} + [A_a, F_{ab}] + 2\gamma_d F^{ab} \nabla_a \psi = 0, \quad (5.71)$$

$$\nabla^a D_a \psi - [A_a, D_a \psi] + \frac{1}{2} \frac{\partial V}{\partial \psi} = 0 \quad (5.72)$$

resp.

$$\nabla^a F_{ab} + [A_a, F_{ab}] - [\psi, D_b \psi] = 0. \quad (5.73)$$

By making use of these relations, along with a suitable inductive argument, in case of a preferred gauge representation, first the $u$-independence of the $r$-derivatives of gauge potential components – and, in turn, that of $F_{ab}, T_{ab}$ and of $g_{ab}$ – can be demonstrated. Then the same order of $r$-derivative of dilaton, resp. Higgs, field, $\psi$, can also be proven to be constant along the generators of $\hat{N}_i$.

A similar reasoning does apply in the case of a complex Higgs field $\psi$ in the fundamental representation of $G$ where the Higgs part of the Lagrangian is given as

$$L_{\text{matter}}^{Higgs} = 2[(D^e \psi)^* (D_e \psi) - V(\psi^* \psi)], \quad (5.74)$$

with $D_a \psi = \nabla_a \psi - i A_a \psi$.

Corollary 5.2 Let $(O_i, g_{ab}|\sigma_i)$ be an elementary spacetime region associated with an E[M]YMd or an EYMH system as they were specified above. Then there exists a gauge potential $A_a : O_i \rightarrow g$ so that the $r$-derivatives of the functions $f, h_A, g_{AB}$ and $\psi$ and also that of the components $A_u, A_r, A_B$ up to any order are $u$-independent on $\hat{N}_i$.

This Lagrangian with $\gamma_d = 1$ reproduces the usual ‘low energy’ Lagrangian obtained from string theory and it reduces to the pure E[M]YM Lagrangian with $\gamma_d = 0$ and $\psi \equiv \text{const}$. 

20
In virtue of Props. 5.1 - 5.3, Cors. 5.1 - 5.2 and of the above remark we have that in the case of an analytic EKG, E[nA]/H, E[M]/YMd or EYMH system $(\partial/\partial u)^a$ is a Killing vector field in a neighborhood of $\tilde{N}$. In addition, by making use of the argument applied in Remark 3.2 of the local Killing fields induced by the maps $\psi_i: \mathcal{O}_i \rightarrow U_i$ can be shown to patch together to a global Killing field on a neighborhood of $\mathcal{N}$. Thereby we have the following:

**Corollary 5.3** Let $(M, g_{ab})$ be an analytic EKG, E[nA]/H, E[M]/YMd or EYMH spacetime of class $A$ or $B$. Then there exists a Killing vector field $k^a$ in an open neighborhood, $\mathcal{V}$ of $\mathcal{N}$ so that it is normal to $\mathcal{N}$ and the matter fields are also invariant in $\mathcal{V}$.

### 6 Existence of a horizon Killing vector field

This section is to show that Prop. 4.1 and Theor. 4.1 of [5] generalize from Einstein-Maxwell spacetimes to EKG, E[nA]/H and (in parts) also to E[M]/YMd and EYMH configurations. The matter fields of these coupled Einstein-matter systems will simply be denoted by $(0, j_i)$ type tensor fields, $T_{(i)}$. Accordingly, $T_{(i)}$ stands for a single field $\psi$ in case of EKG and E[nA]/H systems whereas $\tilde{T}_{(i)}$ and $\hat{T}_{(i)}$ denote the dilaton or Higgs field $\psi$ and the vector potential $A_a$, respectively, in case of E[M]/YMd or EYMH systems.

**Proposition 6.1** Let $(\mathcal{O}_i, g_{ab} |_{\mathcal{O}_i})$ be an elementary spacetime region associated with either an EKG, E[nA]/H, E[M]/YMd or an EYMH spacetime of class $B$ such that $\kappa_\circ > 0$. Then the followings hold:

(i) There exists an open neighborhood, $\mathcal{O}_i' \setminus \mathcal{N}_1$ in $\mathcal{O}_i$ such that $(\mathcal{O}_i', g_{ab} |_{\mathcal{O}_i'})$ can be extended to a smooth spacetime, $(\mathcal{O}_i', g_{ab}^* |_{\mathcal{O}_i'})$, that possesses a bifurcate null surface, $\tilde{N}^* \equiv \tilde{N}^*$ is the union of two null hypersurfaces, $\tilde{N}_1^*$ and $\tilde{N}_2^*$, which intersect on a 2-dimensional spacelike surface, $S$—such that $\tilde{N}_i$ corresponds to the portion of $\tilde{N}_i^*$ that lies to the future of $S$ and $I^+[S] = \mathcal{O}_i' \cap \mathcal{T}[\tilde{N}_i]$. 

(ii) The fields $\mathcal{T}_{(i)}$ also extend smoothly to tensor fields $\mathcal{T}_{(i)}^{\mathcal{O}}$ on $\mathcal{O}^*$ in the case of EKG, E[nA]/H and EMd spacetimes. In general, preferred gauge potentials of E[M]/YMd or EYMH spacetimes blow up at $\mathcal{N}_2^*$ so they can be smoothly extended only onto $\mathcal{O}^* \setminus \mathcal{N}_2^*$. 

(iii) $k^a = (\partial/\partial u)^a$ extends smoothly from $\mathcal{O}_i'$ to a vector field $k^a*$ on $\mathcal{O}^*$. In addition, $\mathcal{L}_k \mathcal{T}_{(i)}^*$ can be defined everywhere in $\mathcal{O}^*$ for any of the considered systems, moreover, $\mathcal{L}_k g_{ab}^*$ and $\mathcal{L}_k \mathcal{T}_{(i)}^*$ vanish on $\tilde{N}^*$.

**Proof** The justification of the smooth extendibility of the spacetime geometry $g_{ab}$ is almost identical to that of the first part of Prop. 4.1 with the following distinction: To demonstrate that in $\mathcal{O}_i$, (which has now the same properties as $\mathcal{O}_i'$ had in (3)) the spacetime metric $g_{ab}$ can be decomposed as

$$g_{ab} = g_{ab}^{(0)} + \gamma_{ab} \quad (6.1)$$

where, in the Gaussian null coordinates of Prop. 4.1, the components, $g_{ab}^{(0)}$, of $g_{ab}^{(0)}$ are independent of $u$, whereas the components, $\gamma_{ab}$, of $\gamma_{ab}$ and all of their derivatives with respect to $r$ vanish at $r = 0$ we need to refer to Props. 5.1 - 5.3 and Cors. 5.1 - 5.2 instead of eq. (3.2) of [5].

In turning to the proofs of the statements of (ii) and (iii) note first that in virtue of Props. 5.1 - 5.3 and Cors. 5.1 - 5.2 the fields $\mathcal{T}_{(i)}$ can be decomposed as

$$\mathcal{T}_{(i)} = \mathcal{T}_{(i)}^{(0)} + \mathcal{T}_{(i)}^*, \quad (6.2)$$

where the components of $\mathcal{T}_{(i)}^{(0)}$, in Gaussian null coordinates of Prop. 4.1, are independent of $u$, while the components of $\mathcal{T}_{(i)}^*$ and all of their $r$-derivatives vanish at $r = 0$. Then an argument, analogous
to that applied in [2] to show the smooth extendibility of $\gamma_{ab}$, can be used to demonstrate that $\hat{T}_{(j)}$ extends smoothly to $\hat{T}_{(j)}^*$ on $O^*$ so that the components of $\hat{T}_{(j)}^*$ and all of their Kruskal coordinate derivatives are zero on $\hat{N}^*$. In particular, this extension can be done so that $\hat{T}_{(j)}^*$ are invariant under the action of the ‘wedge reflection’ isometry defined by $(U, V) \rightarrow (-U, -V)$ on $(O^*, g^{*(a)}_{ab})$.

Thereby the fields $T_{(j)}$ themselves extend smoothly to $O^*$ whenever the fields $T_{(j)}^*(0)$ do. It is straightforward to show that a $u$-independent scalar field (or a set of scalar fields), associated with a Klein-Gordon, Higgs or dilaton field, represented by $\psi \tilde{N}$ extends smoothly to a field $\psi(0)^* \tilde{N}$ on $O^*$. Hence $\psi$ smoothly extends to

$$\psi^* = \psi(0)^* + \tilde{\psi}^* \quad (6.3)$$

that is constant along the generators of $\tilde{N}^*$. This, along with the fact that $k^a = (\partial/\partial u)^a$ extends smoothly to

$$k^a_* = \kappa_a \left[ U \left( \frac{\partial}{\partial U} \right)^a - V \left( \frac{\partial}{\partial V} \right)^a \right], \quad (6.4)$$

implies that $L_{k^*} \psi^*$ vanishes throughout $\tilde{N}^*$.

Consider now the extendibility of the $k^a_*$-invariant part $A^{(0)}_{a}(0)$ of a preferred gauge representation of a Maxwell-Yang-Mills field. In Gaussian null coordinates $A^{(0)}_{a}(0)$ can be given as

$$A^{(0)}_{a}(0) = A^{(0)}_{a}(0) + r \cdot A^{(0,1)}_{a} + O(a r^2). \quad (6.5)$$

Taking account of the transformation between the Gaussian null coordinates and the generalized Kruskal coordinates (see eqs. (24) and (25) of [2]) we get that there exist smooth functions $A^{(0,1)}_{a}$ with $I = 0, 1$ so that

$$A^{(0)}_{U} = (\kappa_a U)^{-1} a_x + V \left\{ A^{(0,0)}_{U}(x^3, x^4) + U V A^{(0,1)}_{U}(x^3, x^4) \right\} \quad (6.6)$$

$$A^{(0)}_{V} = U \left\{ A^{(0,0)}_{V}(x^3, x^4) + U V A^{(0,1)}_{V}(x^3, x^4) \right\}$$

$$A^{(0)}_{a} = A^{(0,0)}_{a}(x^3, x^4) + U V A^{(0,1)}_{a}(x^3, x^4).$$

Clearly $A^{(0)}_{a}$ extends smoothly to a wedge reflection invariant field $A^{(0)}_{a}$ on $O^*$ in the case of a Maxwell or a Yang-Mills field with vanishing $a_e$. Whenever $a_x \neq 0$ an extension of $A^{(0)}_{a}$ of $A^{(0)}_{a}$ can be defined only on $O^* \setminus N^*_2$ because then the first term of the r.h.s. of the first equation of (6.6) blows up at $U = 0$, i.e. at $N^*_2$.

On contrary to the ‘parallelly propagated’ blowing up of the gauge potential $A^{(0)}_{a}$ at $U = 0$ its Lie derivative, $L_{k} A^{(0)}_{a}(0)$, with respect to $k^a$ is regular on $O_i$ and vanishes on $\hat{N}_i$. To see this recall that $L_{k} A^{(0)}_{a}(0) = k^e \partial_e A^{(0)}_{a}(0) + A^{(0)}_{e} \partial_a k^e$. Moreover, by (6.4) $\partial_a k^e = \kappa_a (\partial^e U \delta U^a - \partial^3 V \delta V^a)$ in $O_i$, which implies that

$$L_{k} A^{(0)}_{a}(0) = \kappa_a \left\{ U \frac{\partial A^{(0)}_{a}}{\partial U} - V \frac{\partial A^{(0)}_{a}}{\partial V} + \left( A^{(0)}_{U} \delta U^a - A^{(0)}_{V} \delta V^a \right) \right\} \quad (6.7)$$

holds. Since the singular terms in (6.7) compensate each other $L_{k} A^{(0)}_{a}(0)$ is well-defined in $O_i$ and vanishes on $\hat{N}_i$. In turn, $L_{k} A^{(0)}_{a}$ can be defined everywhere in $O^*$ and it follows from (6.7) that $L_{k^*} A^{(0)}_{a}$ vanishes whenever either $U = 0$ or $V = 0$, i.e. on $\hat{N}^*$.

The wedge reflection symmetry of the fields $T_{(j)}^*$ ensure that the coupled Einstein-matter fields equations are satisfied everywhere in the common domain of their definition.

\textbf{Remark 6.1} Since $L_{k^*} g^{*(a)}_{ab} = 0$ on $\hat{N}^*$ the bifurcate null surface $\hat{N}^*$ is expansion and shear free. In addition, by an argument analogous to the one applied in Remark 2.4, it can be shown that the vector field $k^a_*$ is a repeated principal null vector field of the Riemann tensor on $\hat{N}^*_2$, i.e. we have that $\Psi_0 = \Psi_1 = \Phi_{00} = \Phi_{01} = 0$ there.
Remark 6.2 It is tempting to conclude that \( \tilde{N}^* \) is completely regular, at least in case of the considered systems. On contrary to this, some components of the curvature tensor can blow up along the generators of \( \tilde{N}^* \) in certain situations. To see this recall that by the relation of the generalized Kruskal coordinates, introduced in [27, 28], and the Gaussian null coordinates we have that on \( \tilde{N}_i \)

\[
R_{UVAV} = \kappa_0^{-1} U \cdot R_{urAr} \\
R_{AVBV} = U^2 \cdot R_{ArBr}.
\]

Consequently, whenever either of the curvature tensor components \( R_{urAr} \) or \( R_{ArBr} \) is not identically zero along one of the generators of \( \tilde{N}_i \) then the corresponding component, \( R_{UVAV} \) or \( R_{AVBV} \), blows up while \( U \) tends to infinity.\(^{10}\) Such a ‘parallelly propagated’ curvature blowing up can be shown to happen along the generators of the event horizon of the ‘naked black hole’ spacetimes – including e.g. certain EMd systems – studied in [8, 9]. The curvature blows up at ‘\( U = \infty \)’, i.e. infinitely far from the bifurcation surface, and this can occur even though \( \tilde{N}_i \) is a Killing horizon.

The remaining part of this section is devoted to the presentation of our main result:

Theorem 6.1 Let \((M, g_{ab})\) be a smooth EKG, \(E[nA]H\), \(E[M]YMd\) or \(EYMH\) spacetime of class B so that the generators of \( \tilde{N}^* \) are past incomplete. Then there exists an open neighborhood, \( V \) of \( \tilde{N}^* \) such that in \( J^+[\tilde{N}] \cap V \) there exists a smooth Killing vector field \( k^a \) which is normal to \( \tilde{N}^* \). Furthermore, the matter fields are also invariant, i.e. \( \mathcal{L}_k \mathcal{T}_{(j)} \) vanish, in \( J^+[\tilde{N}] \cap V \).

Proof The proof of the above statement is almost identical with that of Thsr. 4.1 of [3]. The only distinction is that in showing the existence of a Killing vector field in the domain of dependence of \( \tilde{N}^* = \tilde{N}_i^* \cup \tilde{N}_i^* \) in the extended spacetime \( \tilde{O}^* \) Prop. B.1 of [3] has to be replaced by the following argument:

In view of Prop. 3.1 for EKG, \(E[nA]H\), \(E[M]YMd\) and \(EYMH\) systems \( \mathcal{L}_k g_{ab} \) and \( \mathcal{L}_k \mathcal{T}_{(j)} \) vanish on \( \tilde{N}_i^* \). By referring to Thors. 3.1, 4.1 and Remark 4.1 of [23] it can be shown then that the solution \( K^a \) of

\[
\nabla^c \nabla^c K^a + R^a_{\quad bc} K^c = 0
\]

(6.9)

with initial data \( [K^a] = k^a|_{\tilde{N}} \) is a Killing vector field (at least in a sufficiently small neighborhood of \( \tilde{N}^* \)) in the domain of dependence of \( \tilde{N}^* \) so that the matter fields are also invariant there. The only non-trivial step related to the justification of the above claim is to show the uniqueness of solutions to the coupled linear homogeneous wave equations satisfied by \( \mathcal{L}_k g_{ab}^* \) and \( \mathcal{L}_k \mathcal{T}_{(j)}^* \) in case of \( EYMd \) systems with a preferred gauge representation possessing a p.p. blowing up at \( \tilde{N}_i^* \). The key observation here is that, due to the regularity of \( \mathcal{L}_k A^a_{\nu} \) in \( \tilde{O}^* \) (see Prop. 5.3) and also to the smoothness of \( g_{ab}^* \), the principal parts of the relevant linear homogeneous wave equations are regular. Thereby the energy estimate method of the standard uniqueness argument can also be adapted to the present case.

It follows then that by restricting to \( \partial_{\tilde{O}_i} \) we obtain a Killing field \( K^a \) (with \( \mathcal{L}_K \mathcal{T}_{(j)} = 0 \)) on a one-sided neighborhood of \( \tilde{N}_i \) of the form \( J^+[\tilde{N}_i] \cap \tilde{V}_i \), where \( \tilde{V}_i \) is an open neighborhood of \( \tilde{N}_i \).

From this point the proof is identical to that of Thsr. 4.1 of [3].

\[ \blacksquare \]

In view of Prop. 3.1 of [3] we also have the following

Corollary 6.1 Let \((M, g_{ab})\) be a smooth EKG, \(E[nA]H\), \(E[M]YMd\) or \(EYMH\) spacetime of class A so that the generators of the event horizon \( \tilde{N}_i \) are past incomplete. Then there exists an open neighborhood, \( V \) of \( \tilde{N}_i \) such that in \( J^+[\tilde{N}] \cap V \) there exists a smooth Killing vector field \( k^a \) which is normal to \( \tilde{N}_i \). Furthermore, the matter fields are also invariant in \( J^+[\tilde{N}] \cap V \).

\(^{10}\)The divergence rate of the components \( R_{UVAV} \) and \( R_{AVBV} \) is exactly the reciprocal of the power law found for the strongest possible blowing up of the tidal force tensor components along an incomplete maximal causal geodesics upon approaching to the associated spacetime ‘singularity’ (see e.g. [23] and references therein).
7 Concluding remarks

In virtue of Cor. 6.1 any EKG, E[\mathbb{N}]H, E[M]YMd or EYMH black hole spacetime of class A has to admit a horizon Killing vector field. In the smooth non-static case, the domain on which the existence of this Killing vector field is guaranteed is ‘one-sided’ and it is contained by the black hole region. On the other hand, e.g. in the black hole uniqueness arguments the existence of this Killing vector field in the exterior region is what is relevant. Therefore, further investigations will be needed to show that the horizon Killing vector field ‘extends’ to the domain of outer communication side.

Note that in the case of spacetimes of class B with a compact Cauchy horizon it is completely satisfactory to show the existence of a Killing vector field on the Cauchy development side, $J^+[\mathcal{N}] \cap \mathcal{V}$. According to our results the presence of a compact Cauchy horizon ruled by closed null geodesics is simply an artifact of a spacetime symmetry. In turn, the presented results support the validity of the strong cosmic censor hypotheses by demonstrating the non-genericness of spacetimes possessing such a compact Cauchy horizon.

It is important to emphasize that to have a complete proof of the strong cosmic censorship conjecture for spacetimes with a compact Cauchy horizon the case of non-closed generators also has to be investigated.

Remember that our result concerning the extendibility of an elementary spacetime region is based on the assumption that the horizon is non-degenerate. Therefore in the non-analytic case spacetimes with a degenerate horizon are out of our scope. Obviously, to show the existence of a horizon Killing field in case of a smooth spacetime with geodesically complete horizon would also deserve further attentions.

We would like to emphasize that in sections 3 and 4 no use of the particular form of the Einstein’s equations and the matter field equations was made. Thereby, the results contained by these sections generalize straightforwardly to those covariant metric theories of coupled gravity-matter systems within which the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ satisfies the following generalized form of the dominant energy condition: For all future directed timelike vector $\xi^a$ the contraction $G^{a}_{\ b} \xi^b$ is a future directed timelike or null vector.

In particular, the ‘zeroth law’ of black hole thermodynamics can be shown to be valid (see Remark 4.2) for arbitrary black hole spacetimes of class A. It is important to emphasize that the relevant argument rests only on the use of the above generalized form of the dominant energy condition and the event horizon is not assumed to be a Killing horizon as it is usually done in either of the standard arguments.

It is also of obvious interest to know what sort of event horizon can be associated, in this general setting, with a black hole spacetime of class A. In general, there seems to be no way to show the $u$-independence of the $r$-derivatives of the metric functions up to arbitrary order. Hence, a full generalization of Prop. 6.1 will probably not be available. Nevertheless, the functions $f, h_A$ and $g_{AB}$ have been found to be constant along the generators of $\tilde{\mathcal{N}}$. This, whenever $\kappa_0 \neq 0$ along with a straightforward adaptation of the argument of Prop. 6.1 (see also Prop. 4.1 of [3]), can be used to show that $(\mathcal{O}^\prime_i, g_{ab}|_{\mathcal{O}^\prime_i})$ extends into a $C^0$ spacetime $(\mathcal{O}^*, g^*_{ab})$ that possesses a bifurcate type null hypersurface $\tilde{\mathcal{N}}^*$. The metric $g^*_{ab}$ can be ensured to be smooth on $\mathcal{O}^* \setminus \mathcal{N}^*_2$ although it is guaranteed only to be continuous through $\mathcal{N}^*_2$. This argument applies to an arbitrary gravity-matter system provided that the generalized form of the dominant energy condition is satisfied. Thereby, this result strengthens the conclusion of [21, 22] significantly by demonstrating that the event horizon of a physically reasonable asymptotically flat stationary black hole spacetime is either degenerate or it is of bifurcate type.
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References

[1] O. Brodbeck and M. Heusler: Stationary perturbations and infinitesimal rotations of static Einstein-Yang-Mills configurations with bosonic matter, Phys. Rev. D. 56, 6278-6283 (1997)
[2] P.T. Chruściel and R.M. Wald: Maximal hypersurfaces in asymptotically flat spacetimes, Commun. Math. Phys. 163, 561-604 (1994)
[3] M. Farkas: Periodic motions, Springer-Verlag, New York Inc., New York (1994)
[4] H. Friedrich: On the hyperbolicity of Einstein’s equations and other gauge field equations, Commun. Math. Phys. 100, 525-543 (1985)
[5] H. Friedrich, I. Rácz and R.M. Wald: On the rigidity theorem for spacetimes with a stationary event horizon or a compact Cauchy horizon, Commun. Math. Phys. 204, 691-707 (1999)
[6] S.W. Hawking: Black holes in general relativity, Commun. Math. Phys. 25, 152-166 (1972)
[7] S.W. Hawking and G.F.R. Ellis: The large scale structure of space-time, Cambridge University Press (1973)
[8] G.T. Horowitz and S.F. Ross: Naked black holes, Phys. Rev. D. 56, 2180-2187 (1997)
[9] G.T. Horowitz and S.F. Ross: Properties of naked black holes, Phys. Rev. D. 57, 1098-1107 (1998)
[10] J. Isenberg and V. Moncrief: Symmetries of cosmological Cauchy horizons with exceptional orbits, J. Math. Phys. 26, 1024-1027 (1985)
[11] J. Kannár and I. Rácz: On the strength of spacetime singularities, J. Math. Phys. 33, 2842-2848 (1992)
[12] B. Kleihaus and J. Kunz: Static black-hole solutions with axial symmetry, Phys. Rev. Lett. 79, 1595-1598 (1997)
[13] B. Kleihaus and J. Kunz: Static axially symmetric Einstein-Yang-Mills-dilaton solutions. II. Black-hole solutions, Phys. Rev. D. 57, 6138-6157 (1998)
[14] J.G. Miller: Global analysis of the Kerr-Taub-NUT metric, J. Math. Phys. 14, 486-494 (1973)
[15] V. Moncrief: Infinite-dimensional family of vacuum cosmological models with Taub-NUT (Newman-Unti-Tamburino)-type extensions, Phys. Rev. D. 23, 312-315 (1981)
[16] V. Moncrief: Neighborhoods of Cauchy horizons in cosmological spacetimes with one Killing field, Ann. of Phys. 141, 83-103 (1982)
[17] V. Moncrief and J. Isenberg: Symmetries of cosmological Cauchy horizons, Commun. Math. Phys. 98, 387-413 (1983)
[18] H. Müller zum Hagen: Characteristic initial value problem for hyperbolic systems of second order differential systems, Ann. Inst. Henri Poincaré 53, 159-216 (1990)
[19] E. Newman and R. Penrose: An Approach to Gravitational Radiation by a Method of Spin Coefficients, J. Math. Phys. 3, 566-578 (1962)
[20] R. Penrose: *Singularities an time asymmetry* in *General relativity: An Einstein centenary survey*, eds. S.W. Hawking, W. Israel, Cambridge University Press (1979)

[21] I. Rácz and R.M. Wald: *Extension of spacetimes with Killing horizon*, Class. Quant. Grav. 9, 2643-2656 (1992)

[22] I. Rácz and R.M. Wald: *Global extensions of spacetimes describing asymptotic final states of black holes*, Class. Quant. Grav. 13, 539-553 (1996)

[23] I. Rácz: *On the existence of Killing vector fields*, Class. Quant. Grav. 16, 1695-1703 (1999)

[24] A.D. Rendall: *Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations*, Proc. R. Soc. Lond. A 427, 221-239 (1990)

[25] S.A. Ridgway and E.J. Weinberg: *Static black-hole solutions without rotational symmetry*, Phys. Rev. D. 52, 3440-3456 (1995)

[26] J. Stewart: *Advanced general relativity*, Cambridge University Press (1991)

[27] M.S. Volkov and N. Straumann: *Slowly rotating non-Abelian black-holes*, Phys. Rev. Lett. 79, 1428-1431 (1997)

[28] R.M. Wald: *General relativity*, University of Chicago Press, Chicago (1984)