On the asymptotic analysis of the Dirac-Maxwell system in the nonrelativistic limit

Philippe Bechouche, Norbert J. Mauser and Sigmund Selberg
Wolfgang Pauli Institute, c/o Inst. f. Math.,
Universität Wien,
Strudlhofgasse 4, A-1090 Wien

Abstract

We deal with the “nonrelativistic limit”, i.e. the limit $c \to \infty$, where $c$ is the speed of light, of the nonlinear PDE system obtained by coupling the Dirac equation for a 4-spinor to the Maxwell equations for the self-consistent field created by the “moving charge” of the spinor. This limit, sometimes also called “Post-Newtonian” limit, yields a Schrödinger-Poisson system, where the spin and the magnetic field no longer appear. However, our splitting of the 4-spinor into two 2-spinors preserves the symmetry of “electrons” and “positrons”; the latter obeying a Schrödinger equation with “negative mass” in the limit. We rigorously prove that in the nonrelativistic limit solutions of the Dirac-Maxwell system on $\mathbb{R}^{1+3}$ converge in the energy space $C([0, T]; H^1)$ to solutions of a Schrödinger-Poisson system, under appropriate (convergence) conditions on the initial data.

We also prove that the time interval of existence of local solutions of Dirac-Maxwell is bounded from below by $\log(c)$. In fact, for this result we only require uniform $H^1$ bounds on the initial data, not convergence.

Our key technique is “null form estimates”, extending the work of Klainerman and Machedon and our previous work on the nonrelativistic limit of the Klein-Gordon-Maxwell system.

1 Introduction

In this paper we study the behavior of solutions to the Dirac-Maxwell (abbr. $DM$) system in the limit $c \to \infty$, where $c$ is the speed of light. Coupled to the Coulomb gauge condition, this system has the form

$$(i\gamma^\mu \partial_\mu - M + g\gamma^\mu A_\mu) \psi = 0, \quad \partial^\nu F_{\nu\mu} = J_\mu / c, \quad \partial^j A_j = 0. \quad (1.1)$$

AMS Subject Classification: 35Q40, 35L70.
Here the unknowns are the spinor field $\psi(t,x) \in \mathbb{C}^4$, regarded as a column vector, and the electromagnetic potential $A_\mu(t,x) \in \mathbb{R}$, $\mu = 0,1,2,3$. Further, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and

$$J^\mu = c \left< \gamma^0 \gamma^\mu \psi, \psi \right>_{\mathbb{C}^4}$$

is the 4-current density. On the Minkowski spacetime $\mathbb{R}^{1+3}$ we use relativistic coordinates $x^0 = ct \in \mathbb{R}$, $x = (x^1, x^2, x^3) \in \mathbb{R}^3$. $\partial_\mu$ stands for $\frac{\partial}{\partial x^\mu}$. Thus, $\partial_0 = \frac{1}{c} \partial_t$, where $\partial_t = \frac{\partial}{\partial t}$. We also write $\nabla = (\partial_1, \partial_2, \partial_3)$, $\Delta = \partial^2_0 + \partial^2_1 + \partial^2_2 + \partial^2_3$ and $|\nabla|^2 = (-\Delta)^s/2$ for $s \in \mathbb{R}$. Indices are raised and lowered using the metric $(\eta_{\mu\nu}) = \text{diag}(-1,1,1,1)$. The Einstein summation convention is in effect. Thus, repeated greek indices $\mu, \nu, \ldots$ are summed over 0, 1, 2, 3, and repeated roman indices $j, k, \ldots$ over 1, 2, 3. For example, $\Delta = \partial_j \partial^j$. We denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ the standard inner product on $\mathbb{C}^n$.

The physical constants are $M = m_0 c^2/h$, $g = e/\hbar c$, where $m_0$ is the spinor’s rest mass, $h$ is the Planck constant and $e$ is the unit charge. By $\gamma^\mu$, $\mu = 0,1,2,3$, we denote the $4 \times 4$ Dirac matrices, given in $2 \times 2$ block form by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},$$

where the Pauli matrices $\sigma^j$ are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

The following related matrices occur frequently:

$$\alpha^j := \gamma^0 \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \quad S^m := i\gamma^k \gamma^l = \begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix},$$

where $(k,l,m)$ is any cyclic permutation of $(1,2,3)$. Note the identities

$$\alpha^j \alpha^k = -\alpha^k \alpha^j + 2\delta^{jk} I = \delta^{jk} I + i\epsilon^{jkl} S_l. \quad (1.2)$$

The first equation in (1.1) is the Dirac equation. Multiplying it on the left by $\gamma^0$ and taking the imaginary part of its $\mathbb{C}^4$ inner product with $\psi$ yields the conservation law $\partial_\mu J^\mu = 0$. Thus, the “charge” is conserved:

$$\int \langle \psi, \psi \rangle_{\mathbb{C}^4} \, dx = \|\psi(t)\|_{L^2}^2 = \text{const.} \quad (1.3)$$

The second equation in (1.1) is the Maxwell equation. We split $A_\mu$ into its temporal part $A_0$, the electric potential, and its spatial part $A = (A^1, A^2, A^3)$, the magnetic potential. Hence the electric field is given by $E = \nabla A_0 - \partial_0 A$ and the magnetic field by $B = \nabla \times A$, and the second equation in (1.1) is seen to be equivalent to the Maxwell system in classical form, with charge density $\rho = J^0/c$ and current density $(J^k)_{k=1,2,3}$. 

2
The third equation in (1.1) is the Coulomb gauge condition \( \text{div } A = 0 \). The reason for this choice of gauge condition will be explained later. It is equivalent to \( \mathcal{P}A = A \), where \( \mathcal{P} \) is the projection onto divergence free vector fields in \( \mathbb{R}^3 \).

The second and third equations in (1.1) are then seen to be equivalent to

\[
\Delta A_0 = J^0 / c, \quad \left( c^{-2} \partial_t^2 - \Delta \right) A = c^{-1} \mathcal{P}(J_k)_{k=1,2,3}
\]

provided the initial data of \( A \) are divergence free. Thus, when properly rescaled (see [1], [20]), the system (1.1) is conveniently expressed in terms of a small dimensionless parameter

\[
\varepsilon \approx \frac{1}{c}
\]
as follows:

\[
\begin{align*}
&i \partial_t \psi^\varepsilon = -i \varepsilon^{-1} \alpha^j \partial_j \psi^\varepsilon + \varepsilon^{-2} \gamma^0 \psi^\varepsilon - A_j^\varepsilon \alpha^j \psi^\varepsilon - A_0^\varepsilon \psi^\varepsilon, \\
&\Delta A_0^\varepsilon = \rho^\varepsilon, \\
&\Box_\varepsilon A^\varepsilon = \varepsilon \mathcal{P} J^\varepsilon,
\end{align*}
\]

where we have put in superscripts to emphasize the dependence on \( \varepsilon \). Here

\[
\Box_\varepsilon = \varepsilon^2 \partial_t^2 - \Delta
\]

and

\[
\rho^\varepsilon = \langle \psi^\varepsilon, \psi^\varepsilon \rangle_{C^4}, \quad J^\varepsilon = \varepsilon^{-1} \left\{ \langle \alpha^k \psi^\varepsilon, \psi^\varepsilon \rangle_{C^4} \right\}_{k=1,2,3}.
\]

We consider the Cauchy problem for (1.4) with “finite energy” initial data

\[
\psi^\varepsilon|_{t=0} = \psi_0^\varepsilon \in H^1, \quad (A^\varepsilon, \partial_t A^\varepsilon)|_{t=0} = (a_0^\varepsilon, a_1^\varepsilon) \in \mathcal{P} \dot{H}^1 \times \mathcal{P} \dot{L}^2.
\]

We prove three types of results for this system as \( \varepsilon \to 0 \). First, local well-posedness (abbr. l.w.p.) with a logarithmic lower bound on the existence time. Second, convergence in the nonrelativistic limit if the initial datum of \( \psi \) converges. Third, we prove some more precise results on the asymptotic behavior of the Dirac spinor under various smallness assumptions on its “positron part”. These results are described in detail in the next three subsections.

1.1 Local existence

There are two issues here: (i) l.w.p. for \( \varepsilon \) fixed, and (ii) the nature of the \( \varepsilon \)-dependence of the local existence time as \( \varepsilon \to 0 \).

Concerning (i), the main difficulty is that one cannot directly estimate the bilinear term \( A_j \alpha^j \psi \) in the Dirac equation, due to the failure of the endpoint Strichartz estimate for the wave equation in \( 1 + 3 \) dimensions. The crucial fact proved here is that when the Dirac equation is squared, the bilinear terms resulting from this dangerous term can all be expressed in terms of null bilinear forms, provided the Coulomb gauge condition is used, and this enables us to prove l.w.p. of \( DM \) in the energy space (1.6), a result entirely analogous to...
that of Klainerman and Machedon [12] for the Klein-Gordon-Maxwell (KGM) system. (The square of the Dirac eq. is similar to the Klein-Gordon eq., but contains some additional bilinear terms due to the presence of spin.)

Bournaveas [5] proved l.w.p. of DM in the space $\psi(t), A(t) \in H^{1/2+\delta} \times H^{1+\delta}$ for $\delta > 0$, but this result does not take into account the null structure; in fact, by using the special structure of the equations and the so-called Wave-Sobolev spaces, the result can be improved to $\psi(t), A(t) \in H^s \times H^1$ for $1/4 < s \leq 1$; this is proved in an upcoming paper by the third author. It is worth pointing out that these results are all independent w.r.t. $s$, since the regularity of $A(t)$ is kept fixed. Thus, e.g., the l.w.p. in $H^{1/2} \times H^1$ does not imply l.w.p. in $H^1 \times H^1$ (or vice versa).

The question of global existence and uniqueness for DM remains largely open (but see Georgiev [9] for a small data result), however, we prove—and this brings us to the second issue mentioned above—that as $\varepsilon \to 0$ the local existence time goes to infinity, subject to the initial assumptions

$$\|\psi^\varepsilon_0\|_{H^1} = O(1), \quad \|a^\varepsilon_0\|_{H^1} + \varepsilon \|a^\varepsilon_1\|_{L^2} = O\left(\frac{1}{\varepsilon^\Lambda}\right) \quad \text{as} \quad \varepsilon \to 0, \quad (1.7)$$

where

$$0 < \Lambda < \frac{1}{2} \quad (1.8)$$

will be kept fixed throughout the paper. (The upper bound $1/2$ is explained by the factor $\varepsilon^{1/2}$ appearing in the $L^2$ bilinear estimates discussed in Sect. [3].)

**Theorem 1.1. (H \textsuperscript{1} l.w.p. of DM.)** The initial value problem (1.4), (1.6) is locally well posed for fixed $\varepsilon$, with an existence time $T_\varepsilon > 0$ depending only on $\varepsilon$ and the size of the norms of the data. Moreover, if (1.7) holds, then

$$T_\varepsilon \geq c_0 \log \frac{1}{\varepsilon} \quad \text{as} \quad \varepsilon \to 0, \quad (1.9)$$

where $c_0 > 0$ is a universal constant, and we have

$$\|\psi^\varepsilon(t)\|_{H^1} = O(1), \quad \|A^\varepsilon(t)\|_{H^1} + \varepsilon \|\partial_t A^\varepsilon(t)\|_{L^2} = O\left(\frac{1}{\varepsilon^\Lambda}\right) \quad (1.10)$$

uniformly in every finite time interval as $\varepsilon \to 0$.

In order to control the evolution as $\varepsilon \to 0$, it is crucial to have estimates which are sufficiently strong w.r.t. powers of $\varepsilon$. To this end we employ analytical techniques used in our earlier paper [2], where the nonrelativistic limit of KGM was considered. The analysis of DM is more involved, however, due to the additional terms that come up in “squared Dirac” compared to the usual Klein-Gordon (KG) equation. In particular, we prove some new bilinear spacetime estimates which are needed to control these extra terms.

\[2\] Even this is not optimal (the scale invariant space is $L^2 \times H^{1/2}$), and in view of the recent work of Machedon and Sterbenz [18] on KGM one may indeed hope to do better.

\[3\] This in contrast to the situation for KGM; see [12]. The crucial point is that KGM has a positive Hamiltonian, unlike DM.
Once we have obtained closed estimates for the system—sufficiently strong w.r.t. powers of $\varepsilon$—we use a bootstrap argument to prove existence in a short time interval depending only on the $L^2$ norm of $\psi_0^\varepsilon$, provided $\varepsilon$ is sufficiently small, depending on the size of $\left|\text{L}^3\right|$. On account of the conservation of charge $\left|\text{L}^3\right|$ for the Dirac equation we can then iterate this argument to obtain the long time result.

We stress the fact that no convergence assumption is made on the data in the above theorem—all we need is the uniform bound $\left|\text{L}^7\right|$. However, if we do assume that $\psi_0^\varepsilon$ converges in $H^1$, then we can pass to the nonrelativistic limit, which we discuss next.

1.2 Nonrelativistic limit

The nonrelativistic limit of the linear Dirac equation with a given time-dependent electromagnetic potential was treated in [11] (earlier papers, see e.g. [6], dealt only with the static case, i.e. time-independent potential). There are also some results on the nonlinear Dirac and Klein-Gordon equations in the literature, see e.g. [22], but for the coupled nonlinear Dirac-Maxwell and Klein-Gordon-Maxwell systems there are no results previous to our work (i.e. the present paper as well as [2, 3]) and the completely independent work of Masmoudi and Nakanishi [21].

The most marked difference between our work and that of Masmoudi and Nakanishi is that our estimates are strong enough to give uniform (w.r.t. $\varepsilon$) bounds for the solutions of DM assuming only the initial boundedness condition $\left|\text{L}^7\right|$—no convergence assumption is necessary. This, of course, is crucial as far as proving Theorems $\left|\text{L}^3\right|$ and $\left|\text{L}^7\right|$ is concerned. By contrast, in [21] the convergence assumption is essential because uniform estimates are obtained only on an arbitrarily small time interval, and in order to push the result to a larger time interval they must use bounds on solutions of the limiting Schrödinger-Poisson system.

Let us now state our result. We split the Dirac spinor into its upper and lower components:

$$\psi^\varepsilon = \begin{pmatrix} \tilde{\chi}^\varepsilon \\ \tilde{\eta}^\varepsilon \end{pmatrix}$$  \hspace{1cm} (1.11)

where $\tilde{\chi}$ and $\tilde{\eta}$ are 2-spinors, i.e. column vectors in $\mathbb{C}^2$. Before one can pass to the limit $\varepsilon \to 0$, the rest energy must be subtracted, which for the upper "positive energy" component means multiplication by $e^{it/\varepsilon^2}$ and for the lower "negative energy" component multiplication by $e^{-it/\varepsilon^2}$.

**Theorem 1.2. (Nonrelativistic limit of DM.)** Consider the solution of (1.4), (1.6) obtained in Theorem 1.1, with data satisfying:

(i) $v_0 := \lim_{\varepsilon \to 0} \psi_0^\varepsilon$ exists in $H^1$,

(ii) $\|a_0^\varepsilon\|_{H^1} + \varepsilon \|a_1^\varepsilon\|_{L^2} = O \left(\frac{1}{\varepsilon^2}\right)$ as $\varepsilon \to 0$. 

5
Denote the upper and lower 2-spinors of $v_0$ by $v_0^+$ and $v_0^-$ respectively, and let $(u, v_+, v_-)$ be the solution of the Schrödinger-Poisson system\footnote{This system is globally well posed for $L^2$ data.}

$$\Delta u = n, \quad n = |v_+|^2 + |v_-|^2, \quad \left(i\partial_t \pm \frac{\Delta}{2}\right)v_\pm + uv_\pm = 0, \quad (1.12)$$

with initial data $v_\pm|_{t=0} = v_0^\pm$. Then as $\varepsilon \to 0$,

\begin{align*}
\psi^\varepsilon &= e^{-it/\varepsilon^2} \begin{pmatrix} v_+ \\ 0 \end{pmatrix} + e^{+it/\varepsilon^2} \begin{pmatrix} 0 \\ v_- \end{pmatrix} + o(1) \quad \text{in} \quad H^1, \quad (1.13a) \\
A^\varepsilon_0 &= u + o(1) \quad \text{in} \quad \dot{H}^1, \quad (1.13b) \\
\rho^\varepsilon &= n + o(1) \quad \text{in} \quad L^p, \quad 1 \leq p \leq 3, \quad (1.13c)
\end{align*}

uniformly in every finite time interval. Moreover, the relativistic current density converges as follows: Let

$$J^0 = \text{Im} \langle \nabla v_+, v_+ \rangle_{C^2} - \text{Im} \langle \nabla v_-, v_- \rangle_{C^2} + \frac{1}{2} \nabla \times \langle \tilde{\sigma} v_+, v_+ \rangle_{C^2} - \frac{1}{2} \nabla \times \langle \tilde{\sigma} v_-, v_- \rangle_{C^2} \quad (1.14)$$

where $\langle \nabla v_\pm, v_\pm \rangle$ and $\langle \tilde{\sigma} v_\pm, v_\pm \rangle$ are the vectors with components $\langle \partial^j v_\pm, v_\pm \rangle$ and $\langle \sigma^j v_\pm, v_\pm \rangle$, respectively, for $j = 1, 2, 3$. Then

$$J^\varepsilon \rightharpoonup J^0 \quad \text{in} \quad \left[ C^1_c(\mathbb{R}_t \times \mathbb{R}^3_x) \right]' \quad \text{weak} \ast \quad (1.15)$$

as $\varepsilon \to 0$.

The first line in r.h.s. (1.14) is the conserved current associated to the limiting system \footnote{This system is globally well posed for $L^2$ data.}, whereas the second line consists of the well-known divergence-free additional terms due to the interaction spin-magnetic field \footnote{This system is globally well posed for $L^2$ data.}.

We can improve the convergence rate to $O(\varepsilon)$ by strengthening the initial assumptions. Thus, we shall prove:

**Theorem 1.3.** Strengthen the hypotheses of Theorem 1.2 by assuming

$$\|\psi_0^\varepsilon\|_{H^2} = O(1), \quad \|\nabla a^\varepsilon_0\|_{H^1} + \varepsilon \|a^\varepsilon_0\|_{H^1} = O\left(\frac{1}{\varepsilon^2}\right)$$

and

$$\psi_0^\varepsilon = \begin{pmatrix} v_0^+ \\ 0 \end{pmatrix} + O(\varepsilon) \quad \text{in} \quad H^1$$

as $\varepsilon \to 0$. Moreover, assume $v_0^+ \in H^5$. Then

$$\psi^\varepsilon = e^{-it/\varepsilon^2} \begin{pmatrix} v_+ \\ 0 \end{pmatrix} + O(\varepsilon) \quad \text{in} \quad H^1 \quad \text{as} \quad \varepsilon \to 0 \quad (1.16)$$

uniformly in every finite time interval. Furthermore, the convergence in (1.13b) and (1.13c) is also $O(\varepsilon)$.\footnote{This system is globally well posed for $L^2$ data.}
Remark 1.4. The hypotheses are not strong enough to guarantee strong convergence of the current density $J^\varepsilon$ locally uniformly in time. In fact, a simple counterexample is given by the initial datum

$$\psi_0^\varepsilon = \begin{pmatrix} v_0^\varepsilon \\ (\varepsilon v_0^+)^+ \end{pmatrix}.$$  

Then $J^\varepsilon$ initially has vector components $2 \text{Re} \langle \sigma^j v_0^+ v_0^+ \rangle$, which does not agree with the weak limit $J^0$ given by (1.14).

It is instructive to compare the last theorem to the formal derivation of the nonrelativistic limit usually reproduced in physics textbooks, the basic premise of which is a smallness assumption on the lower component $\eta^\varepsilon$ of the spinor. The idea is to define

$$\phi^\varepsilon = \begin{pmatrix} \chi^\varepsilon \\ \eta^\varepsilon \end{pmatrix} := e^{it/\varepsilon^2} \psi^\varepsilon. \quad (1.17)$$

Then (1.16) can be restated

$$\chi^\varepsilon = v_+ + O(\varepsilon), \quad \eta^\varepsilon = O(\varepsilon) \quad \text{in} \quad H^1 \quad \text{as} \quad \varepsilon \to 0. \quad (1.18)$$

The Dirac equation (1.4a) gives

$$iD_0 \chi^\varepsilon = -i\sigma^j D_j \eta^\varepsilon, \quad iD_0 \eta^\varepsilon = -i\sigma^j D_j \chi^\varepsilon - \frac{2}{\varepsilon} \eta^\varepsilon, \quad (1.19)$$

where we write $D_0 = \varepsilon \partial_t - i\varepsilon A_0^\varepsilon$ and $D_j = \partial_j - i\varepsilon A_j^\varepsilon$. Thus,

$$\eta^\varepsilon = -\varepsilon \frac{1}{2} i\sigma^j \partial_j \chi^\varepsilon - \varepsilon^2 \frac{1}{2} \{ i\partial_t \eta^\varepsilon + A_0^\varepsilon \eta^\varepsilon + A_j^\varepsilon \sigma^j \chi^\varepsilon \}, \quad (1.20)$$

and substituting this in the first equation in (1.19) gives, after some algebra,

$$i\partial_t \chi^\varepsilon = \frac{1}{2} \left( \varepsilon \nabla + \varepsilon \mathbf{A}^\varepsilon \right)^2 \chi^\varepsilon - A_0^\varepsilon \chi^\varepsilon - \frac{1}{2} \varepsilon \mathbf{B}_j^\varepsilon \sigma^j \chi^\varepsilon - r^\varepsilon, \quad (1.21)$$

where

$$r^\varepsilon = \frac{1}{2} \sigma^j D_j (\partial_t \eta^\varepsilon - iA_0^\varepsilon \eta^\varepsilon) \quad (1.22)$$

and $\mathbf{B}^\varepsilon = \nabla \times \mathbf{A}^\varepsilon$. Then by formal considerations of magnitude, in particular assuming $\partial_t \eta^\varepsilon = O(1)$, one obtains a Schrödinger equation in the limit $\varepsilon \to 0$. It is possible to make this argument rigorous, but it has a fundamental weakness which limits its usefulness, namely that $\partial_t \eta^\varepsilon$ can be no better than $O(1/\varepsilon)$ unless one adds a further constraint on the initial data. In fact, it is clear from (1.20) that $\partial_t \eta^\varepsilon = O(1)$ in $L^2$ initially if and only if the constraint

$$\eta^\varepsilon = -\varepsilon \frac{1}{2} i\sigma^j \partial_j \chi^\varepsilon + O(\varepsilon^2) \quad (1.23)$$

5Here we break the symmetry of the signs in (1.13a), i.e. between “electrons” and “positrons”, but this is not important since the lower component is in any case expected to vanish.
holds in $L^2$ at time $t = 0$, assuming the data (1.6) are $O(1)$.

However, the constraint (1.23) is not needed in Theorem 1.3, the reason being that instead of the simple splitting into upper and lower components as in (1.11), we apply the eigenspace projections of the “free Dirac operator”

$$Q^s = -i\varepsilon \alpha^k \partial_k + \gamma^0.$$  

As in (1) we use the spectral decomposition

$$Q^s = \lambda^s \Pi^s_+ - \lambda^s \Pi^s_-$$

where

$$\lambda^s = \sqrt{1 - \varepsilon^2 \Delta}, \quad \Pi^s_\pm = \frac{1}{2} (I \pm [\lambda^s]^{-1} Q^s).$$  (1.24)

Since the positive and negative eigenvalues $\pm \lambda^s$ correspond to positive and negative energies of a free Dirac particle, the spectral decomposition is related to electrons and positrons (7). The formal limit $\varepsilon \to 0$ of $\Pi^s_\pm$ yields the operators

$$\Pi^0_\pm = \frac{1}{2} (I \pm \gamma^0), \quad \Pi^0_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi^0_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. $$  (1.25)

The following basic lemma shows that $\Pi^0_\pm$ is the leading order term in a series expansion of $\Pi^s_\pm$ in powers of $\varepsilon$, and moreover that (1.23) is basically equivalent to $\Pi^s_\pm \psi^s = O(\varepsilon^2)$, a condition which resurfaces in the next subsection.

**Lemma 1.5.** For all $s \in \mathbb{R}$, $\Pi^s_\pm$ is bounded from $H^s \to H^s$ uniformly in $\varepsilon$. Moreover,

$$\Pi^s_\pm = \Pi^0_\pm + \varepsilon \mathcal{R}^s_1$$

$$= \Pi^0_\pm + \varepsilon \frac{1}{2} \alpha^k \partial_k + \varepsilon^2 \mathcal{R}^s_2$$  (1.26)

where $\mathcal{R}^s_\pm$ denotes an operator bounded from $H^s \to H^{s-j}$ uniformly in $\varepsilon$.

**Proof.** This follows immediately from

$$\Pi^s_\pm - \Pi^0_\pm = \mp \frac{1}{2} [\lambda^s]^{-1} i \varepsilon \alpha^k \partial_k \mp \frac{1}{2} (1 - [\lambda^s]^{-1}) \gamma^0$$  (1.27)

and the fact that the Fourier symbol of $1 - [\lambda^s]^{-1}$ satisfies the inequalities

$$0 \leq 1 - \frac{1}{\sqrt{1 + \varepsilon^2 |\xi|^2}} \leq \min \left\{ 1, \varepsilon |\xi|, \varepsilon^2 |\xi|^2 \right\}$$  (1.28)

where $\xi$ is the Fourier variable corresponding to $x$. \hfill \Box

Before moving on, we prove that the initial data assumption (6) in Theorem 1.2 implies something stronger, namely the convergence of $\Pi^s_\pm \psi^s_0$. 8
Lemma 1.6. If
\[ \lim_{\varepsilon \to 0} \psi_0^\varepsilon = v_0 = \begin{pmatrix} v_0^+ \\ v_0^- \end{pmatrix} \]
exists in \( H^1 \), then
\[ \lim_{\varepsilon \to 0} \Pi_+^\varepsilon \psi_0^\varepsilon = \begin{pmatrix} v_0^+ \\ 0 \end{pmatrix} \quad \text{and} \quad \lim_{\varepsilon \to 0} \Pi_-^\varepsilon \psi_0^\varepsilon = \begin{pmatrix} 0 \\ v_0^- \end{pmatrix} \]
in \( H^1 \).

Proof. It suffices to prove \((\Pi_+^\varepsilon - \Pi_0^\varepsilon) \psi_0^\varepsilon \to 0 \) in \( H^1 \). But the proof of Lemma 1.5 shows that the Fourier symbol of \( \Pi_+^\varepsilon - \Pi_0^\varepsilon \) is bounded in absolute value by \( \min \{ 1, \varepsilon |\xi| \} \). Thus \((\Pi_+^\varepsilon - \Pi_0^\varepsilon) \psi_0^\varepsilon \to 0 \) in \( H^1 \), and by Plancherel’s theorem and dominated convergence, \((\Pi_+^\varepsilon - \Pi_0^\varepsilon) v_0 \to 0 \) in \( H^1 \). \( \square \)

1.3 Semi-nonrelativistic limit

As in [1], by the “semi-nonrelativistic limit” we understand the approximation of the upper component of the Dirac equation by the Pauli equation for a 2-spinor, which reads
\[ \imath \partial_t \chi_P^\varepsilon = \frac{1}{2} \left( \imath \nabla + \varepsilon A^\varepsilon \right)^2 \chi_P^\varepsilon - A_0^\varepsilon \chi_P^\varepsilon - \frac{1}{2} \varepsilon B^\varepsilon \sigma^j \chi_P^\varepsilon \] (1.30a)
with initial condition
\[ \chi_P^\varepsilon \big|_{t=0} = \chi_{P0} \in H^1. \] (1.30b)

Note that the naive “upper and lower components” approach in (1.19)–(1.22) can give at best an \( O(\varepsilon) \) approximation to the Pauli equation, assuming the initial constraint (1.23), which as remarked is essentially equivalent to \( \Pi_\varepsilon \psi_\varepsilon = O(\varepsilon) \).

In contrast, by using the Dirac projections \( \Pi_\varepsilon^\pm \) instead of just \( \Pi_0^\pm \), we can prove an \( O(\varepsilon^2) \) approximation, with the same initial constraint. In fact, we have the following result:

**Theorem 1.7.** Consider the solution of (1.4), (1.6) obtained in Theorem 1.1. Define \( \chi^\varepsilon \) as in (1.17) and let \( \chi_P^\varepsilon \) be the solution of the Pauli equation (1.30). Assume the initial conditions
(i) \( \| \psi_0^\varepsilon \|_{H^5} = O(1), \quad \| \nabla a_0^\varepsilon \|_{H^4} + \varepsilon \| a_0^\varepsilon \|_{H^4} = O(1) \),
(ii) \( \| \Pi_\varepsilon \psi_0^\varepsilon \|_{H^1} = O(\varepsilon^2) \),
as \( \varepsilon \to 0 \). Then if
\[ \| \chi^\varepsilon - \chi_P^\varepsilon \|_{H^1} = O(\varepsilon^2) \] (1.31)
holds at time \( t = 0 \), it also holds uniformly in every finite time interval. For the current density we then have
\[ J^\varepsilon = J_P^\varepsilon + \frac{1}{2} \nabla \times (\bar{\sigma} \chi_P^\varepsilon, \chi_P^\varepsilon)_{E^2} + O(\varepsilon) \quad \text{in} \quad L^1_T, \] (1.32)
uniformly in every finite time interval, where

\[ J_\varepsilon = \text{Im} \langle (\nabla - i\varepsilon A^\varepsilon)\chi^\varepsilon_P, \chi^\varepsilon_P \rangle_{L^2} \]  

is the current density of the Pauli equation.

The remainder of this paper is organized as follows: In the following section we square the Dirac equation and reinterpret it in terms of the projections \( \Pi^{\pm}_\varepsilon \psi^\varepsilon \) of the spinor, and we prove that the main bilinear terms can be expressed in terms of null forms. Then in Sect. 3 we discuss the linear and bilinear spacetime estimates of Strichartz type that are used in this paper. The proofs of those estimates that are not already in the literature can be found in Sect. 12. In Sect. 4 we define the function spaces that we use, and recall their main properties. The main estimates for the nonlinear terms are proved in Sect. 5, which is the heart of the paper. Then in Sects. 7–11 these estimates are applied to prove the main theorems.

To close this section we introduce some notational conventions which will be in effect throughout:

- For function spaces we use the following notation. If \( X \) is a Banach space of functions on \( \mathbb{R}^{3+1} \), we denote by \( L^p_t X \) the space with norm
  \[ \|u\|_{L^p_t X} = \left( \int_{-\infty}^{\infty} \|u(t,\cdot)\|_X^p \, dt \right)^{1/p}, \]
  with the usual modification if \( p = \infty \). The localization of this norm to a time slab \( S_T = [0,T] \times \mathbb{R}^3 \) is denoted \( \|u\|_{L^p_t X(S_T)} \).

- In estimates, we use the notation \( \lesssim \) to mean \( \leq \) up to multiplication by a positive constant \( C \) independent of \( \varepsilon \). Moreover, in estimates over a time slab \( S_T \), \( C \) is also understood to be independent of \( T \).

- For exponents, we use the convenient shorthand \( p^+ \) (resp. \( p^- \)) for \( p + \zeta \) (resp. \( p - \zeta \)) with \( \zeta > 0 \) sufficiently small, independently of \( \varepsilon \). The notation \( \infty^- \) stands for a sufficiently large, positive exponent.

- We denote by \( f(x) \mapsto \hat{f}(\xi) \) and \( u(t,x) \mapsto \hat{u}(\tau,\xi) \) the Fourier transforms on \( \mathbb{R}^3 \) and \( \mathbb{R}^{1+3} \), respectively. As in [2] we split functions \( f(x) \) into their low (\( |\xi| \lesssim 1/\varepsilon \)) and high (\( |\xi| \gtrsim 1/\varepsilon \)) frequency parts,
  \[ f = f_{\text{low}} + f_{\text{high}}, \]  

  corresponding to a smooth partition of unity in Fourier space.

2 Preliminaries

As already mentioned, our approach to the Dirac equation is to square it and apply techniques similar to those used for \( KGM \) in [2]. It is therefore convenient
to work with the “KG splitting”

\[
\psi_{\pm} = \left( \frac{\tilde{\chi}_{\pm}}{\tilde{\eta}_{\pm}} \right) := \frac{1}{2} \left\{ \psi_{\text{e}} \pm \varepsilon^2 [\lambda^\varepsilon]^{-1} (i\partial_t \psi_{\text{e}} + A_{0}^\varepsilon \psi_{\text{e}}) \right\}
\]  

(2.1)
as used in [2]. In order to compare this to the Dirac projections (1.24), observe that if \( \psi_{\text{e}} \) solves the Dirac equation (1.4a), then

\[
\psi_{\pm} = \Pi_{\pm} \psi_{\text{e}} \mp \frac{1}{2} \varepsilon^2 [\lambda^\varepsilon]^{-1} (A_{0}^\varepsilon \alpha^i \psi_{\text{e}}).
\]  

(2.2)

But using the estimate

\[
||| \lambda^\varepsilon |||_{H^r} \leq \varepsilon^{-r} ||f||_{H^{r-r}} \quad \text{for } 0 \leq r \leq 1,
\]  

(2.3)

followed by Hölder’s inequality and Sobolev embedding, we see that the right-hand side (r.h.s) of (2.2) is \( O(\varepsilon^{1-\Lambda}) \) in \( H^1 \) at time \( t \) if the bound (1.10) in Theorem 1.1 holds. As far as proving Theorem 1.2 is concerned, it is therefore immaterial whether we use \( \psi_{\pm} \) or \( \Pi_{\pm} \psi_{\text{e}} \).

Let us now restate the system (1.4) in terms of the splitting (2.1) of the spinor. First we subtract the rest energy, defining

\[
\phi_{\pm} = \left( \frac{\tilde{\chi}_{\pm}}{\tilde{\eta}_{\pm}} \right) := e^{\pm it/\varepsilon^2} \psi_{\pm}.
\]  

(2.7)

Thus

\[
\psi_{\pm} = \psi_{\pm}^+ + \psi_{\pm}^- = e^{-it/\varepsilon^2} \phi_{\pm}^+ + e^{it/\varepsilon^2} \phi_{\pm}^-.
\]  

(2.8)

**Lemma 2.1.** In terms of the splitting (2.8), defined via (2.1) and (2.7), the Dirac equation (1.4a) is equivalent to a system of two equations

\[
L_{\pm}^\varepsilon \phi_{\pm} = -A_{0}^\varepsilon \phi_{\pm} + \frac{1}{2} e^{it/\varepsilon^2} R^\varepsilon, \quad L_{\pm}^\varepsilon \phi_{\pm} = -A_{0}^\varepsilon \phi_{\pm} - \frac{1}{2} e^{-it/\varepsilon^2} R^\varepsilon,
\]  

(2.9)

provided the constraint (2.2) is satisfied at time \( t = 0 \), or equivalently that the Dirac equation is satisfied at \( t = 0 \). Here

\[
L_{\pm}^\varepsilon = i \partial_t \mp \frac{\lambda^\varepsilon - 1}{\varepsilon^2}
\]  

(2.10)
and \( R^\varepsilon \) is given by
\[
\lambda^R = \varepsilon \left\{ 2i A^\varepsilon \cdot \nabla + i \text{div} A^\varepsilon + i E^\varepsilon_j \alpha_j^3 - B^\varepsilon_j S^j \right\} \psi^\varepsilon \\
+ \varepsilon^2 (A^\varepsilon)^2 \psi^\varepsilon - [\lambda^\varepsilon, (\psi^+ - \psi^-)].
\] (2.11)

Further, \([\cdot, \cdot]\) denotes the commutator and
\[
E = (E^1, E^2, E^3) := \nabla A_0 - \varepsilon \partial_t A, \quad B = (B^1, B^2, B^3) := \nabla \times A.
\] (2.12)

**Proof.** Squaring the Dirac equation (1.4a) yields (cf. [7, Sect. 70])
\[
\varepsilon^2 \left( i \partial_t + A^\varepsilon_0 \right)^2 + (\nabla - i \varepsilon A^\varepsilon)^2 - \varepsilon^2 - 2 \varepsilon \langle \chi^\varepsilon_+, \chi^\varepsilon_- \rangle + \varepsilon B^\varepsilon_j S^j \right\} \psi^\varepsilon = 0.
\] (2.13)

Applying \( i \partial_t + A^\varepsilon_0 \) to both sides of (2.1) and making use of (2.13) and
\[
\varepsilon^2 (i \partial_t + A^\varepsilon_0) \psi^\varepsilon = \lambda^\varepsilon (\psi^+ - \psi^-),
\] which follows from (2.1), one easily obtains (2.9). Reversing these steps, one finds that (2.9) implies the squared Dirac equation (2.13). But the latter implies the Dirac equation, since we assume that (2.2) holds initially, which amounts to saying that the Dirac equation is satisfied initially.

Let us make a brief, heuristic comparison of (2.9) with the expected limit (1.12). As it turns out, \( R^\varepsilon \) vanishes in the limit, so (2.9) tends to the Schrödinger equation in (1.12). In fact, the Fourier symbol of \((\lambda^\varepsilon - 1)/\varepsilon\) is
\[
h^\varepsilon(\xi) := \frac{|\xi|^2}{1 + \varepsilon^2 |\xi|^2} \sim \begin{cases} |\xi|^2/2 & \text{for } |\xi| \lesssim 1/\varepsilon, \\ |\xi|/\varepsilon & \text{for } |\xi| \gtrsim 1/\varepsilon, \end{cases}
\] (2.14)

so \( L^\pm \) tends to the Schrödinger operator \( i \partial_t \pm \Delta/2 \) as \( \varepsilon \to 0 \). Moreover, the charge and current densities (1.5) are given in terms of the fields (2.7) by
\[
\rho^\varepsilon = \langle \chi^\varepsilon_+, \chi^\varepsilon_- \rangle + \langle \chi^\varepsilon_+, \eta^\varepsilon_+ \rangle + \langle \eta^\varepsilon_-, \eta^\varepsilon_- \rangle + 2 \text{Re} \left\{ e^{-2i t/\varepsilon} \langle \chi^\varepsilon_-, \chi^\varepsilon_- \rangle + e^{2 i t/\varepsilon} \langle \eta^\varepsilon_+, \eta^\varepsilon_- \rangle \right\},
\] (2.15)
\[
J^\varepsilon = \frac{2}{\varepsilon} \text{Re} \left\{ \langle \sigma^j \chi^\varepsilon_+, \eta^\varepsilon_+ \rangle + \langle \sigma^j \chi^\varepsilon_-, \eta^\varepsilon_- \rangle + e^{2 i t/\varepsilon} \langle \sigma^j \chi^\varepsilon_-, \eta^\varepsilon_+ \rangle + e^{-2 i t/\varepsilon} \langle \sigma^j \chi^\varepsilon_+, \eta^\varepsilon_- \rangle \right\}_{j=1,2,3}.
\] (2.16)

We expect [cf. (2.6)] that \( \chi^\varepsilon_-, \eta^\varepsilon_+ \to 0 \). Thus, in r.h.s. (2.15) only the first and fourth terms are of importance, and \( \Delta A^\varepsilon_0 = \rho^\varepsilon \) tends to the Poisson equation in (1.12).

For later use we note the estimate
\[
0 \leq |\xi|/\varepsilon - h^\varepsilon(\xi) \lesssim \varepsilon^{-2}.
\] (2.17)
This reduces to $r - \alpha(r) \lesssim 1$, where
\begin{equation}
\alpha(r) := \frac{r^2}{1 + \sqrt{1 + r^2}}.
\end{equation}
But $r - \alpha(r) = r + 1 - s = 1 - \frac{1}{r+s}$, where $s = \sqrt{1 + r^2}$.

We now turn to the problem of obtaining closed estimates for the modified DM system \[(2.9), (1.4b), (1.4c).\] A serious obstacle to estimating the bilinear terms in \[(2.11)\] is the failure of the endpoint Strichartz estimate for the wave equation in dimension $1 + 3$. The salient feature of the Coulomb gauge, however, is that these problematic terms can be expressed in terms of the null bilinear forms
\begin{equation}
Q_0(u, v) = \partial_0 u \partial_0 v - \nabla u \cdot \nabla v, \quad Q_{\alpha\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v, \quad (2.19)
\end{equation}
where $\partial_0$ denotes $\varepsilon \partial_t$ and $0 \leq \alpha < \beta \leq 3$. These bilinear forms enjoy better regularity properties than generic products of derivatives.

We emphasize that in the following result $\psi$ does not have to solve the Dirac equation.

**Lemma 2.2. (Null structure.)** Given a potential $\{A_\mu(t,x)\}$ satisfying the Coulomb condition $\text{div} A = 0$, let $E$ and $B$ be defined as in \[(2.12)\], and consider the bilinear operator
\begin{equation}
\psi \rightarrow \{2iA \cdot \nabla + iE_j (a^j, \psi) \} \psi
\end{equation}
appearing in \[(2.11)\]. We have the following identities:
\begin{equation}
2A \cdot \nabla \psi = -Q_{jk}(|\nabla|^{-1} a^{jk}, \psi)
\end{equation}
and
\begin{equation}
\{i(E_j - \partial_j A_0) a^j - B_j S^j\} \psi
= Q_{jk}(|\nabla|^{-1} \varepsilon \partial_t a^{jk}, U) - Q_{jk}(|\nabla|^{-1} \partial_t a^{jk}, \alpha^j U)
+ Q_0(A_j, \alpha^j U) + Q_{0j}(A_k, \alpha^j \alpha^k U) - \frac{i}{2} Q_{jk}(A_m, \varepsilon^{jkl} S_l \alpha^m U)
\end{equation}
where
\[ a_{jk} = R_j A_k - R_k A_j, \quad R_j = |\nabla|^{-1} \partial_j \]
and $U = U(\psi)$ is the 4-spinor defined by
\[ \Box_{\varepsilon} U = -i (\varepsilon \partial_0 + \alpha^j \partial_j) \psi, \quad U|_{t=0} = 0, \quad i\varepsilon \partial_t U|_{t=0} = \psi_0. \]
Here $\psi_0$ denotes $\psi|_{t=0}$.
Proof. The identity (2.20) goes back to the work of Klainerman and Machedon [12] on KGM, so we concentrate on the new identity (2.21). Define 
\[ \partial_{\pm} = \varepsilon \partial_t \pm \alpha^j \partial_j \]
and observe that
\[ (\partial_- A_j) \alpha^j = -(E_j - \partial_j A_0) \alpha^j - iB_j S^j, \quad (2.23) \]
where we used the second identity in (1.2) and the assumption \( \text{div} \mathbf{A} = 0 \). By the first identity in (1.2), 
\[ \partial_+ \partial_- = \Box \varepsilon. \]
Thus (2.22) implies that \( w = \psi - i\partial_- U \) satisfies \( \partial_+ w = 0 \) with \( w|_{t=0} = 0 \), whence
\[ \psi = i\partial_- U. \]
Apply (2.23) to this and use
\[ \partial_+ (\alpha^j U) = \alpha^j \partial_- U + 2\partial^j U \quad \text{(by 1.2)} \]
to rewrite l.h.s. (2.21) as
\[ (\partial_- A_j) \partial_+ (\alpha^j U) - 2(\partial_- A_j) \partial^j U. \]
To the last term we apply the identity (2.20); this we can do since \( \text{div} \partial_\mu \mathbf{A} = 0 \) for \( \mu = 0, 1, 2, 3 \) by assumption. To the first term we apply the following general formula, obtained using the second identity in (1.2),
\[ (\partial_- \phi)(\partial_+ U) = Q_0(\phi, U) + Q_{0j}(\phi, \alpha^j U) - \frac{i}{2} Q_{jk}(\phi, \varepsilon^{jkl} S_l U), \]
where \( \phi \) is a function and \( U \) a 4-spinor. This last formula is due to Klainerman and Machedon [14]. \( \square \)

3 Bilinear spacetime estimates

The main technical tools used in this paper are spacetime estimates of Strichartz type for solutions of the free initial value problems
\[ \Box \varepsilon u = 0, \quad u|_{t=0} = f, \quad \partial_t u|_{t=0} = 0, \]
\[ L_{\pm}^j v = 0, \quad v|_{t=0} = g, \quad (3.1) \]
on \( \mathbb{R}^{1+3} \). Let us first describe the new \( L^2 \) product estimates that are proved in this paper, and then we recall the estimates proved in [2].

Let \( \mu \) and \( \lambda \) be dyadic numbers of the form \( 2^j, j \in \mathbb{Z} \). Denote by \( \Delta_\mu \) the Littlewood-Paley operator given by
\[ (\Delta_\mu f)^*(\xi) = \beta(\xi/\mu) \hat{f}(\xi), \]
where \( \beta \) is a bump function supported in \( |\xi| \sim 1 \) such that \( \sum_{j \in \mathbb{Z}} \beta(\xi/2^j) = 1 \) for \( \xi \neq 0 \). We write \( f_\mu = \Delta_\mu f \) and similarly for \( g, u, v \). Thus \( f = \sum_\mu f_\mu \) etc.

We shall prove the following:
Theorem 3.1. The solutions $u, v$ of (3.1) satisfy the following dyadic spacetime estimates:

(i) $\| \Delta \mu (u_\lambda v_\lambda) \|_{L^2_{t,x}} \lesssim \varepsilon^{1/2} \mu \| f_\lambda \|_{L^2_x} \| g_\lambda \|_{L^2_x}$ if $\mu \lesssim \lambda \lesssim 1/\varepsilon$.

(ii) $\| \Delta \mu (u_\lambda v_\lambda) \|_{L^2_{t,x}} \lesssim \varepsilon^{1/2} \mu^{1/2} \lambda^{1/2} \| f_\lambda \|_{L^2_x} \| g_\lambda \|_{L^2_x}$ if $\mu \lesssim \lambda, \lambda \gg 1/\varepsilon$.

(iii) $\| u_\mu v_\lambda \|_{L^2_{t,x}} \lesssim \varepsilon^{1/2} \min(\mu, \lambda) \| f_\mu \|_{L^2_x} \| g_\lambda \|_{L^2_x}$ for all $\mu, \lambda$.

See [8, Thm. 12.1] for the analogous estimates in the case where $u$ and $v$ both solve the wave equation.

By decomposing the product $uv$ into dyadic pieces, then applying Theorem 3.1 and finally exploiting the orthogonality properties in Fourier space to sum up, one obtains the following corollary. (The complete argument can be found in [8, Sect. 12].)

Corollary 3.2. The solutions $u, v$ of (3.1) satisfy

$$\| \nabla^{-\sigma} (uv) \|_{L^2_{t,x}} \leq C_{s_1, s_2} \varepsilon^{1/2} \| f \|_{\dot{H}^{s_1}} \| g \|_{\dot{H}^{s_2}}$$

provided that

$s_1, s_2 < 1, \quad \sigma < \frac{1}{2}, \quad s_1 + s_2 + \sigma = 1.$

Estimates of this type for the case where $u$ and $v$ both solve the free wave equation were first investigated by Klainerman and Machedon. The case $(s_1, s_2, \sigma) = (0, 1, 0)$ is excluded, a fact related to the false endpoint case of the Strichartz estimates for the wave equation in 1 + 3 dimensions. However, by assuming a little extra regularity one can easily sum the dyadic pieces and one obtains the following nonsharp bilinear estimate.

Corollary 3.3. The solutions $u, v$ of (3.1) satisfy

$$\| uv \|_{L^2_{t,x}} \leq C_{\delta} \varepsilon^{1/2} \| f \|_{L^2} \| g \|_{H^{1+\delta}}$$

for all $\delta > 0$.

Proof. It suffices to prove the sharp estimate

$$\| uv_\lambda \|_{L^2_{t,x}} \lesssim \varepsilon^{1/2} \| f \|_{L^2_x} \| g_\lambda \|_{L^2_x}.$$  \hspace{1cm} (3.2)

Write $u = \sum_{\mu} u_\mu$ and consider the cases $\mu \lesssim \lambda$ and $\mu \gg \lambda$. In the first case,

$$\left\| \left( \sum_{\mu \lesssim \lambda} u_\mu \right) v_\lambda \right\|_{L^2_x} \lesssim \sum_{\mu \lesssim \lambda} \| u_\mu v_\lambda \|_{L^2_x} \lesssim \varepsilon \left( \sum_{\mu \lesssim \lambda} \| f_\mu \|_{L^2_x} \right) \| g_\lambda \|_{L^2_x},$$

where we used Theorem 3.1(iii) to get the last inequality. In the second case we have, by orthogonality in Fourier space,

$$\left\| \left( \sum_{\mu \gg \lambda} u_\mu \right) v_\lambda \right\|^2_{L^2_x} \lesssim \sum_{\mu \gg \lambda} \| u_\mu v_\lambda \|^2_{L^2_x},$$

and by Theorem 3.1(iii) we dominate this by $\varepsilon \| f \|_{L^2}^2 \| g_\lambda \|^2_{L^2}$.
Here we could also take $f$ in $H^{1+\delta}$ and $g \in L^2$, but we shall not need this. However, for null bilinear forms one can get the sharp result (i.e. $\delta = 0$). Thus, we recall the following, proved in [2, Proposition 4]:

$$\|Q_{ij}(u,v)\|_{L^2_{t,x}} \lesssim \epsilon^{1/2} \|f\|_{\dot{H}^{1/2}} \|g\|_{\dot{H}^1},$$

(3.3)

where $Q_{ij}$ is given by (2.3). We remark that this is the analogue of an estimate for two solutions of the free wave equation proved by Klainerman and Machedon.

Since we will prove part (ii) of Theorem 3.1 by a reduction to linear Strichartz estimates, let us recall these (for + 3 dimensions). We say that a pair $(q,r)$ of Lebesgue exponents is wave admissible if $(q, r) \neq (2, \infty)$ and $1/q + 1/r \leq 1/2$, and sharp wave admissible if the last inequality is an equality.

For the free wave $u$ in (3.1) one has the well-known estimate

$$\|u\|_{L^2_t L^\infty_x} \leq C_{s} \epsilon^{1/2} \|f\|_{\dot{H}^s},$$

(3.4)

for wave admissible $(q, r)$ and $s = 3/2 - 3/r - 1/q$. As proved in [16], this can be improved if the Fourier support of $f$ is small. Thus, if $\hat{f}$ is supported in a cube with side length $\sim \mu$ and at distance $\sim \lambda$ from the origin, where $\mu \ll \lambda$, then

$$\|u\|_{L^2_t L^\infty_x} \leq C_{s} \epsilon^{1/2} \left(\frac{\mu}{\lambda}\right)^{1/2 - 1/r} \|f\|_{\dot{H}^s},$$

(3.5)

for $q, r, s$ as above.

For $v$ satisfying (3.1), we have, as proved in [2, Proposition 1],

$$\|v\|_{L^\infty_t L^2_x} \leq C_{s} \left(\|g_{\text{low}}\|_{\dot{H}^{1/q}} + \epsilon^{1/2} \|g_{\text{high}}\|_{\dot{H}^{2/q}}\right)$$

(3.6)

for sharp wave admissible $(q, r)$. (Then one can use Sobolev embedding to obtain estimates for all wave admissible pairs.) In order to prove Theorem 3.1 (ii), we need the analogue of (3.6) in this context. Thus, we shall prove:

**Proposition 3.4.** Let $v$ be as in (3.1), and suppose $\hat{g}$ is supported in a cube with side length $\sim \mu$ and at distance $\sim \lambda$ from the origin, where $\mu \ll \lambda$. Then

$$\|v\|_{L^\infty_t L^2_x} \leq C_{s} \left(\frac{\mu}{\lambda}\right)^{1/2 - 1/r} \left(\|g_{\text{low}}\|_{\dot{H}^{1/q}} + \epsilon^{1/2} \|g_{\text{high}}\|_{\dot{H}^{2/q}}\right)$$

(3.7)

for sharp wave admissible $(q, r)$.

Finally, recalling the basic heuristic that $L^2$ behaves like a Schrödinger operator at low frequencies, it is not surprising that we have the following Schrödinger type estimates, proved in [2]. We say that a pair $(q, r)$ is Schrödinger admissible if $q, r \geq 2$ and $2/q + 3/r = 3/2$.

**Proposition 3.5.** Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be any two Schrödinger admissible pairs. Then for the solution of $L^2 v = F$ with data $v|_{t=0} = f$ we have

$$\|v_{\text{low}}\|_{L^2_t L^2_x(S_T)} + \|v_{\text{low}}\|_{L^\infty_t L^2_x(S_T)} \lesssim \|f_{\text{low}}\|_{L^2} + \|F_{\text{low}}\|_{L^2_t L^2_x(S_T)},$$

where $\frac{1}{q} + \frac{1}{\tilde{q}} = 1$ and $\frac{1}{r} + \frac{1}{\tilde{r}} = 1$.
4 Function spaces

We shall use the following spaces of functions on $\mathbb{R}^{1+3}$ with weighted norms defined in Fourier space:

- $H^{s,\theta}_{\epsilon}$ with norm $\| \hat{u}(\tau,\xi) \|_{L^2_{\tau,\xi}}$.
- $\dot{H}^{s,\theta}_{\epsilon}$ with norm $\| \hat{u}(\tau,\xi) \|_{L^2_{\tau,\xi}}$.
- $H^{s,\theta}_{\epsilon}$ with norm $\| u \|_{H^{s,\theta}_{\epsilon}} + \epsilon \| \partial_t u \|_{\dot{H}^{s,\theta}_{\epsilon}}$.
- $\dot{H}^{s,\theta}_{\epsilon}$ with norm $\| u \|_{\dot{H}^{s,\theta}_{\epsilon}} + \epsilon \| \partial_t u \|_{\dot{H}^{s,\theta}_{\epsilon}}$.
- $X^{s,\theta}_{\tau=\pm h_{\epsilon}(\xi)}$ with norm $\| \hat{u}(\tau,\xi) \|_{L^2_{\tau,\xi}}$ and $h_{\epsilon}$ as in (2.14).

Here $\langle \cdot \rangle$ stands for $1 + |\cdot|$. These spaces are by now standard, and we will recall their main properties without proofs. For more details and further references to the literature, the reader may consult e.g. [24], [15].

It will be convenient to introduce the notation

$$
U^{\pm}_{\epsilon}(t) = e^{it(\tau^{\pm} - 1)/\epsilon^2} = e^{ith_{\epsilon}(|\nabla|)}, \\
S(t) = e^{it\Delta/2}, \\
W^{\pm}_{\epsilon}(t) = e^{it|\nabla|/\epsilon}
$$

for the propagators associated to, respectively, the operators $L^\pm_{\epsilon}$ defined in Lemma [24] the Schrödinger operator and the wave operator.

(i) **Superposition principle.** A fundamental property of the so-called “Wave Sobolev space” $H^{s,\theta}_{\epsilon}$ is that any function in this space can be written as a superposition ($H^s$-valued integral over the real line) of solutions of the free wave equation with initial data in $H^s$. (See [15] Proposition 3.4] for the precise statement.) This, in effect, replaces Duhamel’s principle in the framework of the Wave Sobolev spaces, and it has the following simple but extremely useful consequence (see [15]):

**Transfer Principle.** Suppose $T$ is a multilinear operator $(f_1(x),\ldots,f_k(x)) \mapsto T(f_1,\ldots,f_k)(x)$ acting in $x$-space. If $T$ satisfies an estimate

$$
\| T(W^{\pm}_{\epsilon}(\pm t)f_1,\ldots,W^{\pm}_{\epsilon}(\pm t)f_k) \|_{L^2_{t}L^2_{x}} \leq C_{1/q}^{1/q} \| f_1 \|_{H^{s,1}} \cdots \| f_k \|_{H^{s,k}},
$$

for all combinations of signs, then

$$
\| T(u_1,\ldots,u_k) \|_{L^2_{t}L^2_{x}} \leq C_{0}^{1/q} \| u_1 \|_{H^{s,1}} \cdots \| u_k \|_{H^{s,k}}
$$

holds for all $u_j \in H^{s,j,\theta}_{\epsilon}$, provided $\theta > 1/2$. Moreover, the same statement holds with $H^s$ and $H^{s,j,\theta}_{\epsilon}$ replaced by their homogeneous counterparts.
to estimates from the previous section, we have, for \( \theta > 0 \), that the Wave Sobolev spaces are related to the free wave equation. Thus, we have a superposition principle and hence a transfer principle for these spaces as well. To be precise, in the above Transfer Principle, one can replace any one of the \( W^s(\pm t) \) by \( U^c(\pm t) \) and correspondingly \( H^{s,\theta}_c \) by \( X^{s,\theta}_{\tau=\pm h_c(\xi)} \). Applying this to estimates from the previous section, we have, for \( \theta > 1/2 \),

\[
\|u_{\text{high}}\|_{L^p_t L^q_x} \lesssim \varepsilon^{1/2+1/p} \|u_{\text{high}}\|_{X^{1,\theta}_{\tau=\pm h_c(\xi)}} \quad \text{for sharp wave adm. (} q, r \text{),} \tag{4.2}
\]

\[
\|u_{\text{low}}\|_{L^p_t L^q_x} \lesssim \|u_{\text{low}}\|_{X^{0,\theta}_{\tau=\pm h_c(\xi)}} \quad \text{for Schr"odinger adm. (} q, r \text{),} \tag{4.3}
\]

\[
\|u_{\text{low}}\|_{L^2_t L^{\infty}_x} \lesssim \|u_{\text{low}}\|_{X^{1,\theta}_{\tau=\pm h_c(\xi)}}. \tag{4.4}
\]

Here (4.3) follows from Proposition 3.5 with \( F = 0 \). By Sobolev embedding we reduce (4.4) to the case \((q, r) = (2, 6)\) of (4.3). Finally, (4.2) holds by virtue of (4.4) and the trivial estimate

\[
\|\hat{f}_{\text{high}}\|_{H^s} \lesssim \varepsilon^{s} \|\hat{f}_{\text{high}}\|_{H^{s+\varepsilon}} \quad \text{for} \quad \varepsilon > 0. \tag{4.5}
\]

(ii) **Embeddings.** The most basic embeddings are

\[
H^{s,\theta}_c, X^{s,\theta}_{\tau=\pm h_c(\xi)} \hookrightarrow C_b(\mathbb{R}; H^s), \quad \dot{H}^{s,\theta}_c \hookrightarrow C_b(\mathbb{R}; \dot{H}^s), \tag{4.6}
\]

which hold uniformly in \( \varepsilon \) for any \( \theta > 1/2 \). Also uniform in \( \varepsilon \) are

\[
L^p_t L^2_x \hookrightarrow H^{0,\theta}_c, \quad X^{1,\theta-1}_{\tau=\pm h_c(\xi)} \quad \text{for} \quad \frac{1}{2} < p \leq 2, \quad \frac{1}{2} < \theta < 1. \tag{4.7}
\]

In fact, the dual statement \( H^{1,\theta-1}_c \hookrightarrow L^p_t L^2_x \) follows by interpolation between the trivial case \( p = 2, \theta = 1 \) and (4.6). We shall also need

\[
\|e^{\pm i \tau \varepsilon / \varepsilon^2} u\|_{X^{0,\theta-1}_{\tau=\pm h_c(\xi)}} \lesssim \varepsilon^{-2(1-\theta)} \|u\|_{H^{0,\theta-1}_c}. \tag{4.8}
\]

This is obvious if \( \hat{u}(\tau, \xi) \) is supported in \( ||\tau| - |\xi|/\varepsilon|| \lesssim \varepsilon^{-2} \); then we can in fact replace the left hand side by \( \|u\|_{L^2} \). On the other hand, if \( \hat{u}(\tau, \xi) \) is supported in \( ||\tau| - |\xi|/\varepsilon|| \gg \varepsilon^{-2} \), then (4.8) follows from (2.17).

(iii) **Time cut-off.** In view of (4.7), we can localize to any finite time slab

\[
S_T = [0, T] \times \mathbb{R}^3.
\]

The restriction space \( H^{s,\theta}_c(S_T) \) is complete when equipped with the norm

\[
\|u\|_{H^{s,\theta}_c(S_T)} := \inf \left\{ \|v\|_{H^{s,\theta}_c} : v = u \text{ on } S_T \right\}. \tag{4.9}
\]
Norms on the other restriction spaces are similarly defined. When \( \theta \leq 1/2 \) the embeddings \( 10 \) fail, but since \( H^{s,\theta}_{t} \) etc. are spaces of tempered distributions, it still makes sense to restrict them to the interior of \( S_{T} \), and we will use the same notation \( H^{s,\theta}_{t}(S_{T}) \) etc. for these spaces. Taking the inf over all extensions produces a seminorm in this case.

The idea behind the following “cut-off lemmas” originates in the work of Bourgain \( 4 \) on the Schrödinger and KdV equations, and was developed further by Kenig-Ponce-Vega \( 10 \) in their work on KdV and by Klainerman-Machedon \( 13 \) and the last author \( 28 \) for the wave equation. In fact, the argument given in \( 10 \) applies to \( X^{s,\theta} \) spaces in general, and in particular proves the following.

**Lemma 4.1.** Suppose \( L^{2}_{t,x}v = F \) on the interior of \( S_{T} \) with \( v|_{t=0} = f \). Let \( \theta > 1/2 \). Then for \( 0 \leq T \leq 1 \),

\[
\|v\|_{X^{s,\theta}_{t=\pm \varepsilon t(x)}(S_{T})} \leq C_{\theta} \left( \|f\|_{H^{s}} + \|F\|_{X^{s,\theta-1}_{t=\pm \varepsilon t(x)}(S_{T})} \right)
\]

where \( C_{\theta} \) is independent of \( T \) and \( \varepsilon \).

By rescaling \( x \to \varepsilon x \) we reduce the next result to the case \( \varepsilon = 1 \), which in turn follows from estimates proved in \( 13 \).

**Lemma 4.2.** Suppose \( \Box_{\pm}u = F \) on the interior of \( S_{T} \) with \( (u, \partial_{t}u)|_{t=0} = (f, g) \). Let \( \theta > 1/2 \). Then for \( 0 \leq T \leq 1 \),

\[
\|u\|_{H^{s,\theta}_{t}(S_{T})} \leq C_{\theta} \left( \|f\|_{H^{s}} + \varepsilon \|g\|_{L^{2}} + \frac{1}{\varepsilon} \|F\|_{H^{0,\theta-1}_{t}(S_{T})} \right)
\]

where \( C_{\theta} \) is independent of \( T \) and \( \varepsilon \). Also, for \( M \in \mathbb{N} \) large enough,

\[
\|u\|_{H^{s,\theta}_{t}(S_{T})} \leq C_{\theta} \varepsilon^{-M} \left( \|f\|_{H^{s}} + \|g\|_{H^{-1,s}} + \|F\|_{H^{0,\theta-1}_{t}(S_{T})} \right).
\]

The last inequality is not sharp w.r.t. \( \varepsilon \), but it will only be used in a situation where powers of \( \varepsilon \) are not important. In order to estimate the Dirac current density we shall need the following “integration by parts”-version of Lemma 4.2.

**Lemma 4.3.** Suppose \( \Box_{\pm}u = e^{it/\varepsilon^{2}}F \) on the interior of \( S_{T} \) with vanishing data. Let \( 1/2 < \theta < 1 \). Then

\[
\|u\|_{H^{s,\theta}_{t}(S_{T})} \leq \varepsilon \|\partial_{t}F\|_{L^{2}(S_{T})} + \|F_{\text{ext}}\|_{L^{2}_{t}H^{s}} + \varepsilon \|\partial_{t}\|_{L^{2}_{t,\varepsilon}}^{\theta} F_{\text{ext}}\|_{L^{2}_{t,\varepsilon}} + \frac{\|F_{\text{ext}}\|_{L^{2}_{t,\varepsilon}}}{\varepsilon^{2\theta - 1}}
\]

for all \( 0 \leq T \leq 1 \) and all extensions \( F_{\text{ext}} \) of \( F \) to all of \( \mathbb{R}^{1+3} \). Here \( \langle \partial_{t}\rangle^{\theta} \) is the multiplier with Fourier symbol \( (1 + |\tau|)^{\theta} \).

**Proof.** Let us denote \( F_{\text{ext}} \) simply by \( F \). Write

\[
e^{it/\varepsilon^{2}}F = (\varepsilon^{2}/i) \partial_{t} \left[ e^{it/\varepsilon^{2}}F \right] - (\varepsilon^{2}/i)e^{it/\varepsilon^{2}} \partial_{t} F
\]

and \( u = u_1 + u_2 \) accordingly. By (4.10), \( \|u_2\|_{\dot{H}_x^{1,\theta}(S_T)} \lesssim \varepsilon \|\partial_t F\|_{L^2(S_T)}. \) Now define \( G = e^{\mu/\varepsilon^2} F. \) Split \( G = G_1 + G_2 \) by a partition of unity in Fourier space such that

\[
\hat{G}_1(\tau, \xi) \text{ is supported in } |\tau| \lesssim |\xi| / \varepsilon,
\]

\[
\hat{G}_2(\tau, \xi) \text{ is supported in } |\tau| \gg |\xi| / \varepsilon,
\]

and write \( u_1 = u_{1,1} + u_{1,2} \) accordingly. That is, \( \Box_u u_{1,j} = (\varepsilon^2 / i) \partial_t G_j \) on the interior of \( S_T \) with vanishing data. By (4.10), \( \|u_{1,1}\|_{\dot{H}_x^{1,\theta}(S_T)} \lesssim \varepsilon \|\partial_t G_1\|_{L^2}, \) but using Plancherel’s theorem and the assumptions on the Fourier support,

\[
\|\partial_t G_1\|_{L^2} \lesssim \frac{1}{\varepsilon} \|\nabla|G_1|\|_{L^2} \leq \frac{1}{\varepsilon} \|\nabla\|G\|_{L^2} = \frac{1}{\varepsilon} \|\nabla|F|\|_{L^2},
\]

whence \( \|u_{1,1}\|_{\dot{H}_x^{1,\theta}(S_T)} \lesssim \|F\|_{L^2 H^{1}}. \) Finally, to estimate \( u_{1,2} \) we first observe that it has an extension to all of \( \mathbb{R}^{1+3} \) defined in Fourier space by

\[
\tilde{u}_{1,2}(\tau, \xi) = \frac{1}{1 - \varepsilon^2 \tau^2 + |\xi|^2} [(\varepsilon^2 / i) \partial_t G_2] \hat{(\tau, \xi)}.
\]

Thus \( |\tilde{u}_{1,2}(\tau, \xi)| \sim \frac{1}{|\tau||\hat{G}_2(\tau, \xi)|, \) and since

\[
\|u_{1,2}\|_{\dot{H}_x^{1,\theta}(S_T)} \leq \|u_{1,2}\|_{\dot{H}_x^{1,\theta}(\mathbb{R}^{1+3})} \lesssim \varepsilon (|\tau| + |\xi|) (|\tau| - |\xi| / \varepsilon)^\theta \tilde{u}_{1,2}(\tau, \xi)\|_{L^2_{\tau,\xi}},
\]

we conclude that \( \|u_{1,2}\|_{\dot{H}_x^{1,\theta}(S_T)} \) is dominated by

\[
\varepsilon \|(\tau)^\theta \hat{G}_2(\tau, \xi)\|_{L^2_{\tau,\xi}} \lesssim \varepsilon \|(\tau + 1/\varepsilon^2)^\theta \hat{F}(\tau, \xi)\|_{L^2_{\tau,\xi}} \lesssim \varepsilon \|(\partial_t)^\theta F\|_{L^2} + \frac{\|F\|_{L^2}}{\varepsilon^{2\theta - 1}}.
\]

This ends the proof of the lemma.  

\section{5 Main estimates}

Here we prove the main \textit{a priori} estimates for the nonlinear terms in the modified \textit{DM} system, in terms of the following spacetime norms.

\begin{definition}
For \( T > 0 \) we define

\begin{itemize}
  \item \( X_T^\varepsilon = \varepsilon^\Lambda \|A^\varepsilon\|_{\dot{H}_x^{1,\theta}(S_T)}, \)
  \item \( Y_T^\varepsilon = \sum_{\pm} \|\phi_{\pm}^\varepsilon\|_{X_{r=\pm \Lambda, \xi}(S_T)^{1,\theta}}, \)
  \item \( Z_T^\varepsilon = \sum_{\pm} \|(\phi_{\pm}^\varepsilon)_{low}\|_{L^2_{\tau}L^2_{\xi} \cap L^2_{\tau}L^\infty_{\xi}(S_T)} \)
\end{itemize}

for \( \theta > 1/2 \) sufficiently close to \( 1/2, \) independently of \( \varepsilon, \) but depending on the fixed parameter \( \Lambda. \) In fact, the relevant condition is

\[
\Lambda + 1 - 2\theta > 0, \tag{5.1}
\]

which we assume from now on.
We also need the following initial data norms.

**Definition 5.2.** For initial data \( \text{(1.6)} \) we set

- \( X_0^\varepsilon = \varepsilon^A (\|a_0\|_{H^1} + \varepsilon \|a_1\|_{L^2}) \),
- \( Y_0^\varepsilon = \|\psi_0^\varepsilon\|_{H^1} \),
- \( Z_0^\varepsilon = \|\psi_0^\varepsilon\|_{L^2} \).

In order to simplify the notation we drop the superscript \( \varepsilon \) on the fields \( \phi, \psi, A \) etc. in the remainder of this section. We assume \( 0 \leq T \leq 1 \) in the estimates that follow, and we write

\[
P_T^\varepsilon = P (X_T^\varepsilon + Y_T^\varepsilon)
\]

where \( P(x) = x + x^N \) for a sufficiently large \( N \in \mathbb{N} \), independent of \( \varepsilon \).

### 5.1 Estimates for \( A_0 \)

Split \( \psi = \psi_{\text{low}} + \psi_{\text{high}} \) and write

\[
A_0 = A_0' + A_0''
\]  
(5.2)

where \( A_0' \) corresponds to “low-low” interactions:

\[
\Delta A_0' = \langle \psi_{\text{low}}, \psi_{\text{low}} \rangle.
\]

Then

\[
\|\Delta A_0'\|_{L^p_t L^{(3/2)+} x(S_T)} \lesssim (Z_T^\varepsilon)^2 \quad \text{for} \; 1 \leq p < 2,
\]  
(5.3)

\[
\|\Delta A_0''\|_{L^p_t L^{(3/2)+} x(S_T)} \lesssim \varepsilon^\frac{1}{2} (Y_T^\varepsilon)^2 \quad \text{for} \; 1 \leq p < 2,
\]  
(5.4)

\[
\|\Delta A_0'\|_{L^r_t L^2 x(S_T)} \lesssim (Z_T^\varepsilon)^2 \quad \text{for} \; 1 \leq r \leq \frac{3}{2},
\]  
(5.5)

\[
\|\Delta A_0''\|_{L^r_t L^2 x(S_T)} \lesssim \varepsilon (Y_T^\varepsilon)^2 \quad \text{for} \; 1 \leq r \leq \frac{3}{2},
\]  
(5.6)

\[
\|\Delta A_0\|_{L^r_t L^2 x(S_T)} \lesssim (Y_T^\varepsilon)^2 \quad \text{for} \; 2 \leq r \leq 6,
\]  
(5.7)

\[
\|\Delta A_0\|_{L^r_t L^2 x(S_T)} \lesssim (Y_T^\varepsilon)^2 \quad \text{for} \; 1 \leq r \leq 3.
\]  
(5.8)

Here (5.8) follows from Hölder’s inequality and Sobolev embedding, while (5.7) reduces to

\[
\|\psi\|_{L^{1+} t L^{3/2} x(S_T)} \lesssim Y_T^\varepsilon \quad \text{for} \; 2 \leq r \leq 6.
\]

By Sobolev embedding and the Transfer Principle, the latter reduces to the \( L^1_t L^{3/2} x \) Strichartz estimate (3.6). Let us now prove (5.3) and (5.4); the proofs of (5.5) and (5.6) are similar. Write

\[
\|\langle \psi, \psi \rangle\|_{L^{(3/2)+} x(S_T)} \leq \|\psi\|_{L^2}^+ \|\psi\|_{L^2}^+ \lesssim \|\psi\|_{L^2}^+ \|\psi\|_{L^2}^+.
\]  
(5.9)
For $\psi = \psi_{\text{low}}$ the $L^2_\xi$ norm of this is clearly dominated by r.h.s. (5.3). On the other hand, if at least one $\psi_{\text{high}}$ is present, then we dominate by r.h.s. (5.4) using the $H^1 \hookrightarrow L^6_\xi$ Sobolev embedding and the estimate (4.5).

We will also need the embeddings

$$\|f\|_{L^\infty_x} \lesssim \|\Delta f\|_{L^3_x}^{3/2} + \|\Delta f\|_{L^3_x}^{-1},$$

$$\|\nabla f\|_{L^\infty_x} \lesssim \|\Delta f\|_{L^3_x}^{3/2} + \|\Delta f\|_{L^3_x}^{-1}.$$  \hspace{1cm} (5.10)

### 5.2 Estimates for the remainder term

For the remainder term $R^\varepsilon$ given by (2.11) we shall prove (cf. Lemma 4.1)

$$\|e^{\pm i t/\varepsilon^2} E_{\varepsilon}^{\pm 1} \psi\|_{X_{\varepsilon}^{1/2,1}((ST))} \lesssim \varepsilon^{(1/2-A)^-} P_T,$$  \hspace{1cm} (5.11)

$$\|R^\varepsilon\|_{L^2_x(S_T)} \lesssim \varepsilon P_T.$$  \hspace{1cm} (5.12)

Using (2.11), (2.3), (5.2) and (4.7) we dominate l.h.s. (5.11) by a sum of terms

$N_1 = \|e^{\pm i t/\varepsilon^2} \{i(E_j - \partial_j A_0)\alpha^j \psi - B_j S^j \psi\}\|_{X_{\varepsilon}^{0,\sigma,-1}((ST))},$

$N_2 = \|A \cdot \nabla \psi\|_{L^2_x(S_T)},$

$N_3 = \|\varepsilon(\partial_j A_0)\alpha^j \psi\|_{L^1_x H^1_x(S_T)},$

$N_4 = \frac{1}{\varepsilon} \|[A_0', \lambda^\varepsilon - 1] \psi\|_{L^2_x(S_T)},$

$N_5 = \frac{1}{\varepsilon} \|[A_0', \lambda^\varepsilon - 1] \psi\|_{L^2_x(S_T)},$

$N_6 = \|\varepsilon (A)^2 \psi\|_{L^2_x(S_T)}.$

All these terms appear also in the KGM case (see [2]), with the notable exception of $N_1$. The latter is however the most interesting (and difficult) term, so we consider it first. Write \( \{i(E_j - \partial_j A_0)\alpha^j - B_j S^j\} \psi = \sum I_\mu \) where

$$I_\mu = \{i(E_j - \partial_j A_0)\alpha^j - B_j S^j\} \Delta_\mu \psi$$

and the sum is over all dyadic numbers $\mu$ of the form $2^j$, $j \in \mathbb{Z}$. Here $\Delta_\mu$ is the Littlewood-Paley operator defined in Sect. 3. We split into the cases

(i) $\mu \leq 1/\varepsilon$,

(ii) $\mu > 1/\varepsilon$.

**Case (i).** By (4.7), we can reduce to proving

$$\left\| \sum_{\mu \leq 1/\varepsilon} I_\mu \right\|_{L^2_x(S_T)} \lesssim \varepsilon^{(1/2-A)^-} X_T^Y,$$  \hspace{1cm} (5.13)

but this follows from Corollary 5.3 via the Transfer Principle.
Case (ii). Using (4.7) we write

\[ \|e^{\pm it/\varepsilon^2} I_\mu \|_{X^0,\theta-1(S_T)} \lesssim \|I_\mu\|_{L^2(S_T)} \|e^{\pm it/\varepsilon^2} I_\mu\|_{X^0,\theta-1(S_T)} \]

where \(0 < \sigma \ll 1\) will be chosen later. Proceeding as in case (i), but using the sharp estimate (3.2), we obtain

\[ \|I_\mu\|_{L^2(S_T)} \lesssim \varepsilon^{1/2-A} X_T Y_T^\varepsilon. \]  

We claim there exist \(\zeta > 0\) and \(M \in \mathbb{N}\), both independent of \(\varepsilon\) and \(\mu\), such that

\[ \|e^{\pm it/\varepsilon^2} I_\mu \|_{X^0,\theta-1(S_T)} \lesssim \varepsilon^{-M} \mu^{-\zeta} P_T^\varepsilon. \]

(5.16)

Granting this for the moment, we see that by choosing \(\sigma\) sufficiently small in (5.14), depending on \(M\), we get

\[ \sum_{\mu > 1/\varepsilon} \|e^{\pm it/\varepsilon^2} I_\mu\|_{X^0,\theta-1(S_T)} \lesssim \varepsilon^{(1/2-A)^-} P_T^\varepsilon \]

as desired. Let us prove the claim. On account of Lemma 2.2

\[ I_\mu = Q_{jk}(|\nabla|^{-1} \varepsilon \partial_\alpha a^{jk}, \Delta_\mu U) - Q_{jk}(|\nabla|^{-1} \partial_\alpha a^{jk}, \alpha^l \Delta_\mu U) + Q_{0j}(A_j, \alpha^k \Delta_\mu U) - Q_{0j}(A_k, \alpha^l \alpha^k \Delta_\mu U) - i Q_{jk}(A_m, \varepsilon^{kl} S^l_i \alpha^m \Delta_\mu U) \]

where \(a^{jk}, U\) are as in Lemma 2.2. But since \(\psi\) solves the Dirac equation,

\[ \Box_\varepsilon U = -\varepsilon^{-1} \gamma^0 \psi + \varepsilon A_j \alpha^j \psi + \varepsilon A_0 \psi \]  

(5.17)

Now we appeal to the following null form estimate.

**Theorem 5.3.** Let \(1/2 < \theta < 1\). Then

\[ \|Q(u, v)\|_{H^{0,\theta-1}_\varepsilon} \leq C_\theta \|u\|_{H^{1,\theta}_\varepsilon} \|v\|_{H^{1+\theta}_\varepsilon} \]

holds on \(\mathbb{R}^{1+3}\) for all null forms \(Q\) in (2.19). Moreover, if \(Q = Q_{ij}\), then the norm \(\|u\|_{H^{1,\theta}_\varepsilon}\) in the r.h.s. can be replaced by \(\|u\|_{H^{1,\theta}_\varepsilon}\).

By a standard procedure we reduce this to well-known bilinear estimates for the homogeneous wave equation; the proof can be found in Sect. 13.

Applying this estimate, and recalling (4.8), we reduce (5.16) to proving

\[ \|\Delta_\mu U\|_{H^{2-2\zeta,1}_\varepsilon(S_T)} \leq C_\zeta \varepsilon^{-M} \mu^{-\zeta} P_T^\varepsilon \]

(5.18)

for \(\zeta > 0\) such that \(1 + \theta < 2 - 2\zeta\). Clearly, it suffices to show

\[ \|U\|_{H^{2-\zeta,1}_\varepsilon(S_T)} \leq C_\zeta \varepsilon^{-M} P_T^\varepsilon, \]

23
but using (4.11) and (5.17) we reduce this to
\[
\| A_j \alpha\psi \|_{L^2_t H^{1-\xi} (S_T)} \leq C\varepsilon^{1/2-\Lambda} X_T^\varepsilon Y_T^\varepsilon, \tag{5.19}
\]
\[
\| A_0 \psi \|_{L^2_t H^1 (S_T)} \lesssim (Y_T)^3. \tag{5.20}
\]
The former follows from Corollary 3.2 and the Transfer Principle, while the latter reduces to (5.8) using Leibniz’ rule, Hölder’s inequality and (5.10). This concludes the estimate for \( N_1 \).

It remains to estimate the terms \( N_2, \ldots, N_6 \). Use Lemma 2.2 and (3.3) via the Transfer Principle to see that
\[
N_2 \lesssim \varepsilon^{1/2-\Lambda} X_T^\varepsilon Y_T^\varepsilon.
\]
Next, by Leibniz’ rule, Hölder’s inequality, (5.10) and (5.7),
\[
N_3 \lesssim (Y_T)^3.
\]
To the term \( N_4 \) we apply we apply the commutator estimate
\[
\| \Delta^{-1} (fg), \lambda^\varepsilon - 1 \|_{L^2_t} \leq \varepsilon^2 C_\rho \| f \|_{H^{1+\rho}} \| g \|_{H^{1+\rho}} \| h \|_{H}, \quad \text{(for all } \rho > 0)\]
proved in [2, Lemma 9]. Thus
\[
N_4 \lesssim \varepsilon^{1-} (Y_T^\varepsilon)^3.
\]
In \( N_5 \) we simply expand the commutator and apply the estimate
\[
\| (\lambda^\varepsilon - 1) f \|_{L^2_t} \lesssim \varepsilon \| f \|_{H^1}, \tag{5.21}
\]
which follows from (2.14). Thus
\[
N_5 \lesssim \left( \| A''_0 \|_{L^2_t L_\infty (S_T)} + \| \nabla A''_0 \|_{L^2_t L_2 (S_T)} \right) \| \psi \|_{L^\infty_t L^1_x (S_T)};
\]
so in view of (5.11), (5.16) and (5.18), \( N_5 \) satisfies the same bound as \( N_4 \). Finally, by Hölder’s inequality and the \( H^1 \hookrightarrow L^6_x \) Sobolev embedding,
\[
N_6 \lesssim \varepsilon^{1-2\Lambda} (X_T^\varepsilon)^2 Y_T^\varepsilon.
\]
This concludes the proof of (5.11).

Now consider the estimate (5.12). Using (2.24) with \( r = 0^+ \) we get
\[
\| R^e \|_{L^2 (S_T)} \lesssim \varepsilon^{1-} \left( \tilde{N}_1 + N_2 + N_4 + N_5 + N_6 \right) + N_3,
\]
where \( N_2, N_3, N_4 \) and \( N_5 \) are as before, whereas
\[
\tilde{N}_1 = \| i(E_j - \partial_j A_0)\alpha^3 \psi - B_j S^3 \psi \|_{L^2_t H^{0-} (S_T)},
\]
\[
\tilde{N}_5 = \frac{1}{\varepsilon} \| [A''_0, \lambda^\varepsilon - 1] \psi \|_{L^2_t H^{0-} (S_T)}.
\]
Write $\tilde{N}_1 \leq \tilde{N}_{1.1} + \tilde{N}_{1.2}$ corresponding to $\psi = \psi_{\text{low}} + \psi_{\text{high}}$. For the low frequency case we apply the nonsharp bilinear Strichartz estimate in Corollary 5.3 via the Transfer Principle, to get
\[ \tilde{N}_{1.1} \lesssim \varepsilon((1/2-\Lambda)^-X^{7\gamma}Y^{7\gamma}. \]
By Sobolev embedding and Hölder’s inequality,
\[ \tilde{N}_{1.2} \lesssim \left( \| \nabla A \|_{L^\infty L^2} + \varepsilon \| \partial_t A \|_{L^\infty L^2} \right) \| \psi_{\text{high}} \|_{L^2_\xi + L^\infty_\xi}, \]
where the pair $(2^+, \infty^-)$ is chosen to be sharp wave admissible. Applying the Strichartz estimate (4.2) we then obtain the same estimate for $\tilde{N}_{1.2}$ as for $\tilde{N}_{1.1}$.

Next, we apply the nonsharp bilinear Strichartz estimate in Corollary 3.3 via the Transfer Principle, to get
\[ \tilde{N}_5 \lesssim \left( \| A_0 \|_{L^2_\xi L^\infty_\xi(S_T)} + \| \nabla A_0 \|_{L^2_\xi L^2_\xi(S_T)} \right) \| \psi_{\pm} \|_{L^\infty_\xi H^1(S_T)} \lesssim \varepsilon(Y_T)^3, \]
where we used (5.12) to get the last inequality. This ends the proof of (5.12).

5.3 Estimates for the current density
Split $\psi = \psi_{\text{low}} + \psi_{\text{high}}$ and write
\[ J = J' + J'', \]
where $J'$ corresponds to “low-low” interactions:
\[ \varepsilon J' = \{ \langle \alpha^k \psi_{\text{low}}, \psi_{\text{low}} \rangle \}_{k=1,2,3}. \]
By (4.2) and (4.3),
\[ \| u_{\text{high}} \|_{L^2_T L^\infty_x} \lesssim \| u_{\text{high}} \|_{L^\infty_T L^2_x} \| v_{\text{low}} \|_{L^2_T L^\infty_x} \lesssim \varepsilon \| u_{\text{high}} \|_{X_{r=\pm \varepsilon h}(\xi)} \| v_{\text{low}} \|_{X_{r=\pm \varepsilon h}(\xi)}, \]
\[ \| u_{\text{high}} \|_{L^2_T} \lesssim \| u_{\text{high}} \|_{L^4_T} \| v_{\text{high}} \|_{L^4_T} \lesssim \varepsilon^{3/2} \| u_{\text{high}} \|_{X_{r=\pm \varepsilon h}(\xi)} \| v_{\text{high}} \|_{X_{r=\pm \varepsilon h}(\xi)}, \]
whence
\[ \| J'' \|_{L^2(S_T)} \lesssim (Y_T)^2. \] (5.22)
In order to estimate $J'$ we expand it as in (4.10). Thus, we write
\[ J' = (J'_1) + (J'_2), \]
where
\[ \varepsilon(J')_1 = 2 \Re \left\{ e^{-2it/\varepsilon^2} \langle \sigma J(\chi_{\text{low}}), (\eta_{\text{low}}) \rangle \right\}_{j=1,2,3}. \]
whereas $(J')_2$ consists of products containing at least one of the fields $(\chi_{\text{low}})$ or $(\eta_{\text{low}})$, which we expect to be small. The latter we estimate, just to take one of these terms,
\[ \| \langle \sigma^j(\chi_{\text{low}}), (\eta_{\text{low}}) \rangle \|_{L^2(S_T)} \leq \| (\chi_{\text{low}}) \|_{L^2_T L^\infty_x(S_T)} \| (\eta_{\text{low}}) \|_{L^\infty_T L^2_x(S_T)}. \]

25
Thus, using \(4.14\) and \(2.6\),
\[
\| (J')_2^\prime \|_{L^2(S_T)} \lesssim (Y_T^2)^2 + \varepsilon^{1 - \Lambda} P_T^\varepsilon. \tag{5.23}
\]
To that part of \(A\) which corresponds to \((J')_1\) we are going to apply Lemma \(4.3\).

Hence we want to estimate
\[
M_T^\varepsilon := \varepsilon \| \partial_t F \|_{L^2(S_T)} + \| F_{\text{ext}} \|_{L^2_t H^1} + \varepsilon \left\| (\partial_t)^\theta F_{\text{ext}} \right\|_{L^2_t, s} + \frac{\| F_{\text{ext}} \|_{L^2_t, s}}{\varepsilon^{2\theta - 1}} \tag{5.24}
\]
where \(F_{\text{ext}}\) is an extension of
\[
F = \{ \langle \sigma^j (\chi^\pm), (\eta^\mp)_{\text{low}} \rangle \}_{j=1,2,3} \tag{5.25}
\]
from \(S_T\) to all of \(\mathbb{R}^{1+3}\). To choose this extension, let
\[
\phi^\prime_\pm = \left( \frac{\chi^\prime_\pm}{\eta^\prime_\pm} \right) \in X_{r=\pm h_\varepsilon(\xi)}^{1,\theta}
\]
be arbitrary extensions of \(\phi_\pm\) and define \(F_{\text{ext}}\) by \(5.29\) with \(\chi^+\) and \(\eta^-\) replaced by their respective extensions. From now on we denote \(F_{\text{ext}}\) simply by \(F\). We claim that
\[
M_T^\varepsilon \lesssim (Y_T)^2 + \varepsilon^{(1/2)^\prime} P_T^\varepsilon + \varepsilon^{1-2\theta} \| \phi^\prime_+ \|_{X_{r=\pm h_\varepsilon(\xi)}^{1,\theta}} \| \phi^\prime_- \|_{X_{r=\mp h_\varepsilon(\xi)}^{1,\theta}} \tag{5.26}
\]
If this holds, then taking the \(\inf\) over all extensions yields
\[
M_T^\varepsilon \lesssim \varepsilon^{1-2\theta} P_T^\varepsilon. \tag{5.27}
\]
Let us prove \(5.26\). First,
\[
\varepsilon \| \partial_t F \|_{L^2(S_T)} \leq \frac{1}{\varepsilon} \| (\lambda^\varepsilon - 1) \chi^+ \|_{L^\infty_t L^2_x(S_T)} \| (\eta^-)_{\text{low}} \|_{L^2_t L^\infty_x(S_T)}
+ \varepsilon \| L^\varepsilon_+ \chi^+ \|_{L^2(S_T)} \| (\eta^-)_{\text{low}} \|_{L^\infty(S_T)} + (\ldots) \tag{5.28}
\]
where \((\ldots)\) stands for symmetric terms. Here we used \(5.21\), \(4.14\) and
\[
\| L^\varepsilon_+ \chi^+ \|_{L^2(S_T)} \lesssim (Y_T)^3 + \varepsilon P_T^\varepsilon,
\]
\[
\| (\eta^-)_{\text{low}} \|_{L^\infty(S_T)} \lesssim \varepsilon^{-(1/2)^\prime} Y_T^\varepsilon.
\]
The former was obtained from \(2.10\), \(5.12\) and \(5.8\), while the latter follows from \(4.6\) and Sobolev embedding. Second, we write
\[
\| F \|_{L^2_t H^1} \lesssim \| \chi^\prime_+ \|_{L^\infty_t H^1} \| (\eta^\prime_-)_{\text{low}} \|_{L^2_t L^\infty_x} + \| (\chi^\prime_+)_\text{low} \|_{L^2_t L^\infty_x} \| \eta^\prime_- \|_{L^\infty_t H^1}
\lesssim \| \phi^\prime_+ \|_{X_{r=\pm h_\varepsilon(\xi)}^{1,\theta}} \| \phi^\prime_- \|_{X_{r=\mp h_\varepsilon(\xi)}^{1,\theta}}, \tag{5.29}
\]
where we used $\| \cdot \|_{L^2_{t,x}}$ and $\| \cdot \|_{H^1_{t,x}}$. This estimate can of course also be used for the term $\| F \|_{L^2_{t,x}}$. Third,
\begin{equation}
\varepsilon \| \langle \partial_t \rangle^\theta F_{\text{ext}} \|_{L^2_{t,x}} \lesssim \varepsilon^{1-\theta} \|\phi'_{\tau} \|_{X^{1,\theta}_{\tau=+h_\varepsilon(\xi)}} \|\phi'_{\tau} \|_{X^{1,\theta}_{\tau=-h_\varepsilon(\xi)}},
\end{equation}
where we used the following:

**Lemma 5.4.** $\| \langle \partial_t \rangle^\theta (u_{\text{low}}v_{\text{low}}) \|_{L^2_{t,x}} \lesssim \varepsilon^{-\theta} \| u_{\text{low}} \|_{X^{1,\theta}_{\tau=+h_\varepsilon(\xi)}} \| v_{\text{low}} \|_{X^{1,\theta}_{\tau=-h_\varepsilon(\xi)}}$.

**Proof.** To simplify the notation, let us write $u, v$ instead of $u_{\text{low}}, v_{\text{low}}$ here. W.l.o.g. we assume $\hat{u}(\tau, \xi), \hat{v}(\tau, \xi) \geq 0$. Then using Plancherel's theorem,
\begin{equation}
\| \langle \partial_t \rangle^\theta (uv) \|_{L^2} \lesssim \| u T_{-}^\theta v \|_{L^2} + \varepsilon^{-\theta} \| u |\nabla|^\theta v \|_{L^2} + \| v T_{+}^\theta u \|_{L^2} + \varepsilon^{-\theta} \| v |\nabla|^\theta u \|_{L^2}.
\end{equation}
Here $T_{\pm}^\theta$ is the multiplier with symbol $(\tau \mp h_\varepsilon(\xi))^\theta$, and we used (2.14). Write
\begin{equation}
\| u T_{-}^\theta v \|_{L^2} \leq \| u \|_{L^\infty T^2} \| T_{-}^\theta v \|_{L^2_{t,x}} \lesssim \| u \|_{X^{1,\theta}_{\tau=+h_\varepsilon(\xi)}} \| v \|_{X^{1,\theta}_{\tau=-h_\varepsilon(\xi)}}
\end{equation}
using (4.6) and Sobolev embedding. Next,
\begin{equation}
\| u |\nabla|^\theta v \|_{L^2} \leq \| u \|_{L^\infty T^2} \| |\nabla|^\theta v \|_{L^2_{t,x}} \lesssim \| u \|_{X^{1,\theta}_{\tau=+h_\varepsilon(\xi)}} \| v \|_{X^{1,\theta}_{\tau=-h_\varepsilon(\xi)}}
\end{equation}
where (4.6) with $(q, r) = (2, 6)$ was used.

Finally, combining (6.2a), (6.2b) and (5.30), we get (6.26).

### 6 Iteration scheme and local existence

For fixed $\varepsilon$ we shall prove the following local existence theorem:

**Theorem 6.1.** For fixed $\varepsilon$, the Dirac-Maxwell-Coulomb system (1.4) is locally well posed for initial data in the space $X^{1,\theta}_{\varepsilon}$. The existence time $T > 0$ only depends on $\varepsilon$ and the size of the norms of the data, and the solution is in the space
\begin{equation}
\psi_\varepsilon \in H^{1,\theta}_\varepsilon(ST), \quad A_\varepsilon \in H^{1,\theta}_\varepsilon(ST), \quad A_0^\varepsilon \in C([0,T]; \dot{H}^1),
\end{equation}
for all $1/2 < \theta < 1$. Moreover, the solution is unique in this regularity class, and we have
\begin{equation}
\phi^\varepsilon_\pm \in X^{1,\theta}_{\tau=\pm h_\varepsilon(\xi)}(ST),
\end{equation}
where $\phi^\varepsilon_\pm$ is defined by (2.1) and (2.7).

We shall prove this by Picard iteration. In order to simplify the notation we drop the superscript $\varepsilon$ on the fields $\psi, A_\varepsilon$ etc. and introduce instead a superscript $(m)$ to denote the $m$-th iterate of a field. For (1.4) we use the iteration scheme
\begin{equation}
\{ i\varepsilon \partial_t + i\alpha_j \partial_j - (1/\varepsilon)\gamma^0 \} \psi^{(m+1)} = -\varepsilon A^{(m)}_\gamma \alpha_j \psi^{(m)} - \varepsilon A_0^{(m)} \psi^{(m)},
\end{equation}
\begin{equation}
\Delta A_0^{(m)} = \rho^{(m)},
\end{equation}
\begin{equation}
\Box \psi^{(m+1)} = \varepsilon P J^{(m)}.
\end{equation}
with initial data as in (6.3a), where $\rho^{(m)}$ and $J^{(m)}$ are given by (6.3b) with $\psi$ replaced by its iterate $\psi^{(m)}$. Note that $A_0$ is not really iterated; (6.2b) simply defines $A_0^{(m)}$ in terms of $\psi^{(m)}$. Observe also that for all $m$,

$$\text{div } A^{(m)} = 0,$$

since $w = A^{(m)} - PA^{(m)}$ satisfies $\Box w = 0$ with vanishing initial data.

By convention we start the iteration at $m = -1$ and set all iterates identically equal to zero there. Then the iterates $\psi^{(0)},A^{(0)}$ are just solutions of the free Dirac and wave equations with data (1.6). Define (cf. (2.1) and (2.7))

$$\psi^{(m+1)}_\pm = \frac{1}{2} \left\{ \psi^{(m+1)}_\pm + \varepsilon^2 |\chi|^2 \left( i \partial_t \psi^{(m+1)} + A^{(m)}_0 \psi^{(m)} \right) \right\}, \quad (6.3a)$$

$$\phi^{(m)}_\pm = \left( \chi^{(m)}_\pm \right) := e^{\pm it/\varepsilon^2} \psi^{(m)}_\pm. \quad (6.3b)$$

Proceeding as in the proof of Lemma 2.1 one finds

$$L^{(m)} \phi^{(m+1)}_\pm = -A^{(m)}_0 \phi^{(m)}_\pm + \frac{1}{2} e^{\pm it/\varepsilon^2} R^{(m)}, \quad (6.4)$$

where

$$\chi^2 R^{(m)} = \varepsilon B^{(m)} + \varepsilon^2 C^{(m)} - \left[ A^{(m)}_0, \chi \right] \left( \psi^{(m)}_+ - \psi^{(m)}_- \right),$$

$$B^{(m)} = \left\{ 2iA^{(m)}_\mu \cdot \nabla + iE^{(m)}_j \alpha^j - B^{(m)}_j \right\} \psi^{(m)}, \quad (6.5)$$

$$C^{(m)} = \left\{ A^{(m)}_j A^{(m-1)}_k \alpha^j \alpha^k + A^{(m)}_j A^{(m-1)}_0 A^{(m-1)}_0 - A^{(m)}_0 A^{(m-1)}_j \alpha^j \right\} \psi^{(m-1)},$$

and $E^{(m)}_j, B^{(m)}_j$ are given by (2.12) with $A_\mu$ replaced by $A^{(m)}_\mu$.

We now turn to the proof of Theorem 6.1. By standard arguments, this reduces to proving closed estimates for the iterates in the space (6.1). Set

$$B^{(m)}_T = \| \psi^{(m)} \|_{\tilde{H}^1(x)(S_T)} + \| A^{(m)} \|_{\tilde{H}^1(x)(S_T)},$$

and denote by $B_0$ the norm of the data (6.6). Then it suffices to prove

$$B^{(m+1)}_T \leq C P(B_0) + C T^\delta P \left( B^{(m)}_T + B^{(m-1)}_T \right), \quad (6.6)$$

for some constants $C, \delta > 0$ and a polynomial $P$ with $P(0) = 0$. Here $C$ and $P$ may depend on $\varepsilon$, but since the latter is fixed we do not indicate this explicitly. In what follows, $C, \delta$ and $P$ may change from line to line. (Observe also that since all the nonlinear terms in $D_M$ are in fact multilinear, the same arguments then give estimates for a difference of two iterates.) By Lemma 4.2

$$\| A^{(m+1)} \|_{\tilde{H}^1(x)(S_T)} \leq C B_0 + C T^\delta \| \psi^{(m)} \|_{L^4(S_T)}^2, \quad (6.7)$$

28
where the $T^\delta$ comes from Hölder’s inequality in time. Now apply the Strichartz estimate (5.10) via the Transfer Principle to see that $\|\psi^{(m)}\|_{L^8(S_T)} \lesssim B_T^{(m)}$. In order to estimate $\psi^{(m+1)}$ we use the splitting (2.11) and the embedding
\[
\|u\|_{H^s} \lesssim \varepsilon^{-2s} \|u\|_{X^s \equiv \pm h_\varepsilon(t)},
\]
which holds in view of (2.14). Thus, we write
\[
\|\psi^{(m+1)}\|_{H^{s,\#}(S_T)} \leq C \sum_i \|\phi_i^{(m+1)}\|_{X^{i,\#} = \pm h_\varepsilon(t)}(S_T).
\]
Using Lemma 4.2 and (4.7), we bound $\|\phi_i^{(m+1)}\|_{X^{i,\#} = \pm h_\varepsilon(t)}(S_T)$ by
\[
CP(B_0) + CT^\delta \|A_0^{(m)}\|_{L_T^\infty H^1(S_T)} + C \|e^{\pm it/\varepsilon^2 R^{(m)}}\|_{X^{0,\#-1} = \pm h_\varepsilon(t)}(S_T) .
\]
For the third term in (6.9) we can apply the estimates proved in Sect. 5.2. For the second term in (6.9) we can apply the estimates proved in Sect. 5.2. In fact, we claim that the proof of (5.11) gives
\[
\|\Delta A_0^{(m)}\|_{L_T^\infty L^\#_T(S_T)} \lesssim \|\psi^{(m)}\|_{H^{2,\#}(S_T)}^2 \quad \text{for } 1 \leq r \leq 3.
\]
and
\[
\|\psi_\pm^{(m)}(t)\|_{H^s} \lesssim \|\psi^{(m)}(t)\|_{H^1} + \varepsilon \|A^{(m-1)}(t)\|_{H^1} \|\psi^{(m-1)}(t)\|_{H^1}.
\]
The latter is just the analogue of (2.6) for the iterates.

For the third term in (6.9) we can apply the estimates proved in Sect. 5.2. For $N_1$ and $N_2$ we only have to observe that the bilinear estimates in Corollaries 3.2 and 3.3 as well as the null form estimate (3.11) are valid also in the case where both $u$ and $v$ solve the homogeneous wave equation, so we can apply the Transfer Principle for the $H^{s,\#}$ spaces instead of $X^{s,\#} = \pm h_\varepsilon(t)$. Note also that (5.11) must be replaced by
\[
\Box \varepsilon \psi = -\varepsilon^{-1} \gamma^0 \psi^{(m)} + \varepsilon A^{(m-1)}_j \alpha_j^\varepsilon \psi^{(m-1)} + \varepsilon A_0^{(m-1)} \psi^{(m-1)},
\]
The estimate for $N_6$ requires no change. Finally, the estimates for the terms involving $A_0$ can be simplified, since we do not care about powers of $\varepsilon$ here. Thus, in $N_5$ we can replace $H^1$ by $L^\infty_T$, by giving up the $\varepsilon$, and then the estimate reduces to (6.10). Finally, the commutator terms $N_4$ and $N_5$ are replaced by a single term, since we do not need to split $A_0$ according to (5.2). We simply expand the commutator and proceed as in the estimate for $N_5$, reducing to (6.10) and (6.11). This concludes the proof of Theorem 6.1.
7 Uniform $H^1$ bounds and long time existence

We shall prove:

**Theorem 7.1.** Consider the solution $(ψ_ε^+, A_μ^ε)$ of (1.4), (1.6) from Theorem 6.1, existing up to a time $T_ε > 0$ and belonging to the space (6.1) over this time interval. There exist

(i) a time $T^* > 0$ depending only on $\sup_{ε > 0} \|ψ_0^ε\|_{L^2}$,

(ii) constants $C, M, ε_0 > 0$ independent of $ε$,

such that if

$$X_0^ε + Y_0^ε ≤ B \text{ for all } ε$$

then

$$X_T^ε + Y_T^ε ≤ CB \text{ for } ε < \frac{ε_0}{1 + (CB)^M} \text{ and } 0 ≤ T ≤ \min(T^*, T_ε).$$

Moreover, there is a polynomial $P$ with $P(0) = 0$, independent of $ε$, such that

$$Z_T^ε ≤ CZ_0^ε + εP(X_0^ε + Y_0^ε),$$

for $T, ε$ as in (7.2).

We claim that this result, together with Theorem 6.1, implies Theorem 1.1. To see this, first observe that the bound in (7.2) implies, on account of (4.6),

$$\|ψ(T)\|_{H^1} + ε^A \left\{ \|A^ε(T)\|_{H^1} + ε \|∂_t A^ε(T)\|_{L^2} \right\} ≤ C'B$$

for some constant $C' > C$ independent of $ε$. Thus Theorem 6.1 implies $T_ε ≥ T^*$ for $ε$ as in (7.2). In view of the conservation of charge (1.3) for the Dirac equation, we can iterate this argument any number of times, obtaining

$$T_ε ≥ NT^* \text{ and } X_{NT^*}^ε + Y_{NT^*}^ε ≤ (C')^N B \text{ for } ε < \frac{ε_0}{1 + (C')^MN B^M}$$

for all $N ∈ \mathbb{N}$. This proves Theorem 1.1.

We shall prove Theorem 7.1 using the iteration scheme from Sect. 6. In order to simplify the notation we drop the superscript $ε$ on the fields $ψ, A_μ$ etc. as well as on the $XYZ$-norms in Definitions 5.1 and 5.2 in the remainder of this section, and introduce instead a superscript $(m)$ to denote the $m$-th iterate of a field. We denote by $X_0^m$ etc. the norms in Definition 5.1 with the respective fields replaced by their $m$-th iterate. Then we have:
Proposition 7.2. There exist $C, \gamma, \delta > 0$ and a polynomial $P$ with $P(0) = 0$, all independent of $\varepsilon$, such that the estimates

\[ X_T^{(m+1)} \leq CX_0 + \varepsilon \gamma P_T^{(m)}, \quad (7.4a) \]
\[ Y_T^{(m+1)} \leq CY_0 + CT^\delta \left( Z_T^{(m)} \right)^2 Y_T^{(m)} + \varepsilon P_T^{(m)}, \quad (7.4b) \]
\[ Z_T^{(m+1)} \leq CZ_0 + CT^\delta \left( Z_T^{(m)} \right)^2 Z_T^{(m)} + \varepsilon P_T^{(m)}, \quad (7.4c) \]

hold for $T \leq 1$ and $m \geq -1$, where

\[ P_T^{(m)} := \begin{cases} 
  P(X_0 + Y_0) & \text{for } m = -1, \\
  P(X_T^{(m)} + X_T^{(m-1)} + Y_T^{(m)} + Y_T^{(m-1)}) & \text{for } m \geq 0.
\end{cases} \quad (7.5) \]

In fact, these estimates hold for (recall (5.1))

\[ \gamma \leq \Lambda + 1 - 2\theta. \quad (7.6) \]

The proof is deferred to the end of this section.

Corollary 7.3. There exist $C, \delta > 0$ and a polynomial $Q$, all independent of $\varepsilon$, such that if $\gamma > 0$ is sufficiently small depending on $\Lambda$, and $T, \varepsilon > 0$ are taken so small that

\[ 2CT^\delta \left[ 2C \| \psi_0 \|_{L^2} + 1 \right]^2 \leq 1, \quad 2\varepsilon^{7/2}Q(X_0 + Y_0) \leq 1, \quad (7.7) \]

then

\[ X_T^{(m)} \leq CX_0 + \varepsilon^{7/2}(X_0 + Y_0), \quad (7.8a) \]
\[ Y_T^{(m)} \leq CY_0 + \varepsilon^{7/2}(X_0 + Y_0), \quad (7.8b) \]
\[ Z_T^{(m)} \leq CZ_0 + \varepsilon^{1-\gamma/2}(X_0 + Y_0), \quad (7.8c) \]

for $m \geq 0$.

Proof. This is a simple induction. Since $P(0) = 0$ in Proposition 7.2, there is a polynomial $Q(r)$ such that

\[ P(4|C + 1|r) \leq rQ(r) \quad \text{for } r \geq 0. \]

Then

\[ P_T^{(m)} \leq Q(X_0 + Y_0) \cdot (X_0 + Y_0) \quad (7.9) \]

holds for $m = -1$, in view of the definition (7.5). Hence (7.8) for $m = 0$ follows from (7.7) and the fact that the iterates at $m = -1$ all vanish. Now assume (7.8) holds for $0 \leq m \leq m_0$. Then (7.8) holds for such $m$, and using (7.7) and (7.9) we obtain (7.8) for $m = m_0 + 1$. \qed
We are now in a position to prove Theorem 7.1. Indeed, from the proof of Theorem 6.1, we know that the iterates $\phi_{X}^{(m)}$ converge in the $Y$-norms. We can therefore pass to the limit $m \to \infty$ in Corollary 6.8. Thus, from (6.8a), (6.8b) we get (7.2), and from (7.8a) we get

$$Z_T \leq 2CZ_0 + 1.$$  \hfill (7.10)

Substituting the latter into the second term in the r.h.s. of (7.4a) in the limit $m \to \infty$, we then obtain (7.8). This proves Theorem 7.1.

**Proof of Proposition 7.2.** By the estimates in Sect. 5.2,

$$\|e^{i\pm t/2} R^{(m)} \|_{X_{t = \pm \Lambda, \psi}^{\pm}(S_T)} \lesssim \varepsilon^{1/2 - \Lambda} P_T^{(m)},$$  \hfill (7.11)

$$\|R^{(m)}\|_{L^2(S_T)} \lesssim \varepsilon P_T^{(m)},$$  \hfill (7.12)

the only difference being that (5.17) must be replaced by (7.12). Then (7.4a) follows from the equation (6.4) by applying Lemmas 4.1 and 4.3, the embedding (4.7), and the estimates proved in Sect. 5.3. However, instead of the estimate (2.6), which was used to prove (5.23), we use the analogous estimate for the iterates:

$$\|\Pi^0_{\psi, \mp} \|_{L_T^2} + \|\Pi^0_{\psi, \pm} \|_{L_T^2} \lesssim \varepsilon \|\psi^{(m)}\|_{H^1} + \varepsilon^2 \|A^{(m-1)}\|_{H^1} \|\psi^{(m-1)}\|_{H^1}.$$  \hfill (7.11)

Next, applying Lemma 4.1 to the equation (6.4) and using the embedding (4.7) and the estimate (7.11), as well as (2.6) at $t = 0$, we reduce (7.4a) to proving

$$\|A_0^{(m)} \phi_{\pm}^{(m)} \|_{L_T^{-1} H^1(S_T)} \lesssim T^d \left[Z_T^{(m)}\right]^2 Y_T^{(m)} + \varepsilon^{1/2} [Y_T^{(m)}].$$

But this follows from Leibniz’ rule, Hölder’s inequality, (5.11) and (5.13)–(5.0), in view of (6.2a). The factor $T^d$ comes from applying Hölder’s inequality in time. Finally, consider (7.4a). Apply Proposition 3.5 to (7.4) and use (2.5) at $t = 0$ to get

$$Z_T^{(m+1)} \lesssim Z_0 + \varepsilon^{2-\Lambda} X_T^{(m)} Y_T^{(m)} + \sum_{\pm} \left\| A_0^{(m)} \phi_{\pm}^{(m)} \right\|_{L_T^{1+2} L_T^2(S_T)} + \left\| P^{(m)} \right\|_{L_T^{1+2} L_T^2(S_T)}.$$  \hfill (7.12)

The last term is covered by (7.12). On account of (5.5) and (5.6),

$$\left\| A_0^{(m)} \phi_{\pm}^{(m)} \right\|_{L_T^{1+2} L_T^2(S_T)} \lesssim T^d \left[Z_T^{(m)}\right]^2 \left\| \phi_{\pm}^{(m)} \right\|_{L_T^{1+2} L_T^2(S_T)} + \varepsilon [Y_T^{(m)}].$$

Then (7.4a) follows, in view of

$$\left\| \phi^{(m)}_{\pm} \right\|_{L_T^{1+2} L_T^2(S_T)} \lesssim Z_T^{(m)} + \varepsilon Y_T^{(m)},$$

which holds by (4.5).
8 Higher order bounds

Here we prove bounds for higher order derivatives. For \( m = 0, 1, 2, \ldots \), set (cf. Definitions 5.1 and 5.2)

- \( X_T^x[m] = \varepsilon^\Lambda \sum_{\mid \alpha \mid \leq m} \| \partial_x^\alpha A^x \|_{\mathcal{H}^{1+\delta}(S_T)} \),
- \( Y_T^x[m] = \sum_{\mid \alpha \mid \leq m} \sum_{\pm} \| \partial_x^\alpha \phi_x^\pm \|_{X^{1+\delta,1}(S_T)} \),
- \( X_0^x[m] = \varepsilon^\Lambda (\| \nabla A_0^x \|_{H^m} + \varepsilon \| A_0^x \|_{H^m}) \),
- \( Y_0^x[m] = \| \psi_0^x \|_{H^{m+1}} \).

The local well-posedness of \( DM \) in these norms for \( m = 0 \) was established in Sect. 6, and a standard argument shows that higher regularity persists, i.e. if the interval of existence \( 0 \leq T < T_\varepsilon \) is so small that (7.7) holds. Arguing as in the paragraph following Theorem 7.1, we iterate this argument to cover the full time interval \( [0, T_\varepsilon] \). Here we concentrate on proving bounds which are uniform in \( \varepsilon \). Thus, we shall prove:

**Proposition 8.1.** If
\[
X_0^x[m] + Y_0^x[m] = O(1)
\]
then
\[
X_T^x[m] + Y_T^x[m] = O(1)
\]
for \( 0 \leq T \leq T_\varepsilon \), where \( T_\varepsilon \) is the existence time from Theorem 7.1.

We claim there exist \( C, \delta, \gamma > 0 \) and polynomials \( Q, P_m \)—all independent of \( \varepsilon \)—such that for \( 0 \leq T \leq 1 \) and \( m \geq 1 \),

\[
X_T^x[m] \leq CX_0^x[m] + \varepsilon^\gamma Q (X_0^x + Y_0^x) \cdot \{ X_T^x[m] + Y_T^x[m] \}
\]
\[
+ P_m (X_T^x[m-1] + Y_T^x[m-1]),
\]
\[
Y_T^x[m] \leq C \{ Y_0^x[m] + \varepsilon X_0^x[m]Y_0^x[m] \} + CT^\delta \{ 1 + 2C \| \psi_0 \|_{L^2} \}^2 Y_T^x[m]
\]
\[
+ \varepsilon^\gamma Q (X_0^x + Y_0^x) \cdot \{ X_T^x[m] + Y_T^x[m] \}
\]
\[
+ P_m (X_T^x[m-1] + Y_T^x[m-1]).
\]

Granting this for the moment, let us prove Proposition 8.1 by induction on \( m \). The case \( m = 0 \) of (8.2) was proved in Sect. 7. Adding up the inequalities (8.3) and (8.4), we see that if (8.2) holds for \( m-1 \), then it also holds for \( m \), provided \( T, \varepsilon > 0 \) are so small that (7.7) holds. Arguing as in the paragraph following Theorem 7.1, we iterate this argument to cover the full time interval \( [0, T_\varepsilon] \).

To prove the claim, we apply \( \sum_{\mid \alpha \mid \leq m} \partial_x^\alpha \) to the system, and imitate the proof of the estimates in Proposition 7.2 for \( m = 0 \). We single out the top order terms where \( m \) derivatives fall on one of the fields \( A, \psi \) or \( \phi_x^\pm \); these are estimated exactly like in the case \( m = 0 \). All other terms are lumped together and yield the term
\[
P_m (X_T^x[m-1] + Y_T^x[m-1]).
\]
We skip the straightforward but tedious details of this argument.
9 Estimates for the small component

In this section we prove that if the “positron part” \( \Pi_+ \psi^\varepsilon \) is small initially, then it stays small uniformly in every finite time interval, where “small” means either \( O(\varepsilon) \) or \( O(\varepsilon^2) \). Here is the precise result:

**Proposition 9.1.** (i) Assume (8.1) holds for some \( m \geq 0 \). Then if

\[
\| \Pi_+ \psi^\varepsilon \|_{H^m} = O(\varepsilon)
\]

holds at time \( t = 0 \), it also holds uniformly in every finite time interval.

(ii) Now replace (8.1) by the stronger condition

\[
\| \psi_0^\varepsilon \|_{H^{m+1}} = O(1), \quad \| \nabla a_0^\varepsilon \|_{H^m} + \varepsilon \| a_1^\varepsilon \|_{H^m} = O(1),
\]

as \( \varepsilon \to 0 \). Then if

\[
\| \Pi_+ \psi^\varepsilon \|_{H^{m-1}} = O(\varepsilon^2)
\]

holds at time \( t = 0 \), it also holds uniformly in every finite time interval.

Let us interpret this result in terms of \( \eta^\varepsilon \), the lower component of \( e^{it/\varepsilon^2} \psi^\varepsilon \), as in (1.17). We claim that (9.1) is equivalent to

\[
\| \eta^\varepsilon \|_{H^m} = O(\varepsilon)
\]

while (9.3) is equivalent to (9.4) and

\[
\| \partial_t \eta^\varepsilon \|_{H^{m-1}} = O(1).
\]

The equivalence of (9.1) and (9.4) follows from (1.26), since

\[
\| \psi^\varepsilon \|_{H^{m+1}} = O(1)
\]

on account of Proposition 8.1. To prove the rest of the claim, note that by (1.27), (9.3) is equivalent to

\[
\sigma^j \partial_j \eta^\varepsilon = O(\varepsilon), \quad \eta^\varepsilon + i\varepsilon \frac{1}{2} \sigma^j \partial_j \chi^\varepsilon = O(\varepsilon^2) \quad \text{in} \ H^{m-1}
\]

where \( \chi^\varepsilon \) is the upper component of \( e^{it/\varepsilon^2} \psi^\varepsilon \), as in (1.17). But by the second equation in (1.19),

\[
i\varepsilon^2 \partial_t \eta^\varepsilon = \eta^\varepsilon + i\varepsilon \frac{1}{2} \sigma^j \partial_j \chi^\varepsilon + O(\varepsilon^2) \quad \text{in} \ H^{m-1}
\]

where we used the fact, proved below, that if (8.1) and (9.2) hold initially, then

\[
\| \nabla A^\varepsilon \|_{H^m} + \varepsilon \| \partial_t A^\varepsilon \|_{H^m} = O(1)
\]

uniformly in every finite time interval.
Proof of Proposition 9.1(i). In view of (2.2)–(2.4), we can replace \( \Pi_\varepsilon^\pm \psi_\varepsilon \) by \( \phi_\varepsilon \) in (9.1), and by (4.5) it suffices to consider the low frequency part \( (\phi_\varepsilon)_\text{low} \). Set (cf. Definition 5.1)
\[
\tilde{Z}_T^\varepsilon [m] = \sum_{|\alpha| \leq m} \left\| \partial_\alpha^\varepsilon (\phi_\varepsilon)_\text{low} \right\|_{L^2_t L^2_x \cap L^\infty_x(S_T)}.
\]
Then recalling Proposition 8.1 and using induction on \( m \), it suffices to prove that there exist constants \( C, \delta > 0 \) and polynomials \( P_m \)—all independent of \( \varepsilon \)—such that for \( 0 \leq T \leq 1 \) and \( m \geq 0 \),
\[
\tilde{Z}_T^\varepsilon [m] \leq C \left\| \phi_{\varepsilon -} (t = 0) \right\|_{H^m} + CT^\delta \left\{ C \left\| \psi_0^\varepsilon \right\|_{L^2} + 1 \right\}^2 \tilde{Z}_T^\varepsilon [m] + \left\{ \varepsilon + \tilde{Z}_T^\varepsilon [m - 1] \right\} P_m \left( X_T^\varepsilon [m] + Y_T^\varepsilon [m] \right),
\]
where by convention \( \tilde{Z}_T^\varepsilon [-1] = 0 \). But this estimate follows by a straightforward modification of the proof of the estimate for the \( Z \)-norm in Proposition 7.2, taking into account the bound (7.10).

Let us now prove (9.9), assuming it holds initially. In view of Lemma 4.2, this reduces to proving \( \left\| J_{\varepsilon -} \right\|_{L^2_t H^m(S_T)} = O(1) \). Split \( J = J' + J'' \) as in Sect. 5.3. To estimate \( J'' \), we proceed as in the proof of (5.22), taking into account the higher order bound (9.6). On the other hand, since \( J' \) has vector components \( \frac{1}{2} \text{Re} \langle \sigma^j (\chi^\varepsilon)_\text{low}, (\eta^\varepsilon)_\text{low} \rangle \), we have
\[
\left\| \partial_\alpha J' \right\|_{L^2} \leq \frac{1}{\varepsilon} \sum_{\beta + \gamma = \alpha} c_{\alpha, \beta} \left\| \partial_\beta^\varepsilon (\chi^\varepsilon)_\text{low} \right\|_{L^2_t L^\infty_x} \left\| \partial_\gamma^\varepsilon (\eta^\varepsilon)_\text{low} \right\|_{L^\infty_t L^2_x},
\]
and the r.h.s. is \( O(1) \) for \( |\alpha| \leq m \) on account of (9.4), (4.4) and (8.2). \( \square \)

Proof of Proposition 9.1(ii). Here we break with our earlier notation, writing
\[
\psi_{\pm}^\varepsilon = \Pi_{\pm}^\varepsilon \psi_\varepsilon, \quad \phi_\varepsilon = e^{it/\varepsilon^2} \psi_\varepsilon, \quad \phi_{\pm}^\varepsilon = e^{it/\varepsilon^2} \psi_{\pm}^\varepsilon.
\]
Then from the Dirac equation,
\[
i \partial_t \phi_\varepsilon + \frac{\lambda_\varepsilon - 1}{\varepsilon^2} \phi_\varepsilon + \Pi_+^\varepsilon (A^\varepsilon \phi_\varepsilon) = 0, \quad (9.10)
\]
\[
i \partial_t \phi_\varepsilon - \frac{\lambda_\varepsilon + 1}{\varepsilon^2} \phi_\varepsilon + \Pi_-^\varepsilon (A^\varepsilon \phi_\varepsilon) = 0, \quad (9.11)
\]
where \( A^\varepsilon = A_\varepsilon^0 \sigma^j + A_\varepsilon^0 \). Thus,
\[
\phi_\varepsilon = (\lambda_\varepsilon + 1)^{-1} \varepsilon^2 \left\{ -i \partial_t \phi_\varepsilon - \Pi_-^\varepsilon (A^\varepsilon \phi_\varepsilon) \right\}, \quad (9.12)
\]
so we reduce (9.3) to proving
\[
\left\| \partial_t \phi_\varepsilon \right\|_{H^{m-1}} = O(1), \quad (9.13)
\]
\[
\left\| \Pi_-^\varepsilon (A^\varepsilon \phi_\varepsilon) \right\|_{H^{m-1}} = O(1). \quad (9.14)
\]
The latter follows readily from (9.9) and (9.6), since $\Pi^\varepsilon_-$ is uniformly bounded. For later use we also note that (9.10) implies
\[\|\partial_t \phi^\varepsilon_+\|_{H^{m-1}} = O(1),\] (9.15)
since the symbol of $\frac{\lambda^\varepsilon - 1}{2}$ is bounded by $|\xi|^2$.

To prove (9.13) we proceed as in [1, Sect. 4]. Consider first the case $m = 1$. Take a time derivative of (9.11), then take the imaginary part of its inner product with $\partial_t \phi^\varepsilon_-$ and integrate in $x$. Making use of the self-adjointness of $\lambda^\varepsilon$, $\Pi^\varepsilon_-$ and $A^\varepsilon$, and the fact that $(\Pi^\varepsilon_-)^2 = \Pi^\varepsilon_-$, we then obtain
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t \phi^\varepsilon_-\|^2_{L^2} \leq \|A^\varepsilon \partial_t \phi^\varepsilon_+\|_{L^2} \|\partial_t \phi^\varepsilon_-\|_{L^2} + \|\partial_t A^\varepsilon \cdot \phi^\varepsilon\|_{L^2} \|\partial_t \phi^\varepsilon_-\|_{L^2} \\
\lesssim (\|\nabla A^\varepsilon\|_{H^1} \|\partial_t \phi^\varepsilon_+\|_{L^2} + \|\nabla \partial_t A^\varepsilon\|_{L^2} \|\phi^\varepsilon\|_{H^1}) \|\partial_t \phi^\varepsilon_-\|_{L^2}.
\]
Thus, dividing by $\|\partial_t \phi^\varepsilon_-\|_{L^2}$ and integrating in time, and using the fact that (9.13) holds at time $t = 0$ (this follows from (9.11) and the initial assumptions), we reduce (9.13) for $m = 1$ to proving that the terms inside the parentheses in the last inequality above are all $O(1)$ locally uniformly in time. But this follows from the bounds (9.6), (9.9) and (9.15). Here we use also the fact that $\partial_t A^\varepsilon_0$ enjoys the same bounds as $\nabla A^\varepsilon_0$, in view of the equation
\[\Delta \partial_t A^\varepsilon_0 = -\text{div} J^\varepsilon\]
which follows from (1.4b) and the conservation law $\partial_t \rho^\varepsilon + \text{div} J^\varepsilon = 0$.

Following [1] we now proceed by induction on $m$, starting at $m = 1$. Thus, we apply $\partial_t \partial^\alpha x$, where $|\alpha| \leq m - 1$, to the equation (9.11), and we take the imaginary part of its inner product with $\partial_t \partial^\alpha x \phi^\varepsilon_-$ and integrate in $x$. Then by a straightforward modification of the argument for $m = 1$, we reduce (9.13) to the $O(1)$ bounds (9.6), (9.9) and (9.15), as well as (9.13) at the previous induction step. We omit the details.

10 Nonrelativistic limit

We first prove Theorem 1.2 then we discuss the modifications needed to prove Theorem 1.3.

Proof of (1.13b). This can be restated:
\[e^{it/\varepsilon^2} \Pi^\varepsilon_0 \psi \to \begin{pmatrix} v_x \\ 0 \end{pmatrix}, \quad e^{-it/\varepsilon^2} \Pi^\varepsilon_0 \psi \to v_- = \begin{pmatrix} 0 \\ v_- \end{pmatrix}\] in $H^1$ as $\varepsilon \to 0$ (10.1)
locally uniformly in time. We claim it suffices to prove
\[e^{it/\varepsilon^2} \Pi^\varepsilon_+ \psi \to \begin{pmatrix} v_x \\ 0 \end{pmatrix}, \quad e^{-it/\varepsilon^2} \Pi^\varepsilon_- \psi \to v_- = \begin{pmatrix} 0 \\ v_- \end{pmatrix}\] in $H^1$ as $\varepsilon \to 0$. (10.2)
To prove the claim, write $\Pi_+^0 \psi^\varepsilon = \Pi_0 \psi^\varepsilon = r^\varepsilon$. By the orthogonality between $\Pi_0^0$ and $\Pi_0$, we get

$$\Pi_0^0 r^\varepsilon = -\Pi_+^0 \Pi_-^e \psi^\varepsilon, \quad \Pi_0^0 r^\varepsilon = \Pi_-^e \psi^\varepsilon.$$  

But if (10.2) holds, then the right hand sides converge to zero in $H^1$. Thus $r^\varepsilon = o(1)$ in $H^1$ and we have proved that (10.2) implies (10.1). In the remainder of the proof we skip the superscript $\varepsilon$ on the fields, to simplify the notation.

Using (2.2) and (2.4) we reduce (10.2) to proving

$$\phi_+ \longrightarrow \begin{pmatrix} v_+ \\ 0 \end{pmatrix}, \quad \phi_- \longrightarrow \begin{pmatrix} 0 \\ v_- \end{pmatrix} \text{ in } H^1 \text{ as } \varepsilon \rightarrow 0, \quad (10.3)$$

uniformly in any given time interval $[0,T]$. By (the proof of) Theorem 1.1 the solution exists in this time interval for all sufficiently small $\varepsilon > 0$, and

$$X_T^\varepsilon + Y_T^\varepsilon = O(1), \quad \|e^{\pm i t/\varepsilon^2} R^\varepsilon\|_{L_t^{1,s-1} (R^3)} = o(1) \quad (10.4)$$

as $\varepsilon \rightarrow 0$. Note that (10.3) holds at time $t = 0$, by Lemma 1.6. Thus, it suffices to prove that there exist $K, \delta > 0$, depending on $T$ and $X_T^\varepsilon + Y_T^\varepsilon$, but independent of $\varepsilon$, such that for every time interval $I = [t_0, t_1] \subset [0,T]$,

$$f(I) \leq Kf(\{t_0\}) + K |I|^\delta f(I) + o(1) \quad (10.5)$$

as $\varepsilon \rightarrow 0$, where

$$f(I) = \left\| \phi_+ - \begin{pmatrix} v_+ \\ 0 \end{pmatrix} \right\|_{L_t^{\infty} H^1(I \times R^3)} + \left\| \phi_- - \begin{pmatrix} 0 \\ v_- \end{pmatrix} \right\|_{L_t^{\infty} H^1(I \times R^3)}. \quad (10.6)$$

W.l.o.g. we assume $I = [0,T]$, and we only estimate the first term in (10.6). Write

$$\phi_+ (t) = U^\varepsilon(t) \phi_0^+ + \int_0^t U^\varepsilon(t-s) \left[ L_+^\varepsilon \phi_+(s) \right] ds,$$

$$v_+(t) = S(t)v_0^+ - \int_0^t S(t-s) [(uv_+)(s)] ds,$$

where $\phi_0^+, v_0^+$ are the data of $\phi_+, v_+$ and $U^\varepsilon(t), S(t)$ are given by (1.1). Thus

$$\phi_+ (t) - \begin{pmatrix} v_+ \\ 0 \end{pmatrix} (t) = U^\varepsilon(t) \left[ \phi_0^+ - \begin{pmatrix} v_0^+ \\ 0 \end{pmatrix} \right] + [U^\varepsilon(t) - S(t)] \begin{pmatrix} v_0^+ \\ 0 \end{pmatrix}$$

$$+ \int_0^t U^\varepsilon(t-s) \left[ (uv_+)(s) + L_+^\varepsilon \phi_+(s) \right] ds + \int_0^t [S(t-s) - U^\varepsilon(t-s)] (uv_+)(s) ds \quad (10.7)$$

$$= I_1 + I_2 + I_3 + I_4.$$
Clearly,
\[ \|I_1\|_{L^\infty_t H^1(S_T)} \lesssim \left\| \phi_0^+ - \begin{pmatrix} v_0^+ \\ 0 \end{pmatrix} \right\|_{H^1}. \]

As in [2, Sect. 5],
\[ \|I_j\|_{L^\infty_t H^1(S_T)} = o(1) \quad \text{for } j = 2, 4, \]
using the dominated convergence theorem and the fact that
\[ \|\nabla u\|_{L^3_x} + \|u\|_{L^\infty_x} \lesssim \|v_+\|_{H^1} + \|v_-\|_{H^1} < \infty \quad (10.8) \]
uniformly in every finite time interval. It remains to consider \( I_3 \). By Lemma [5.1] and the embeddings [4.6] and [4.7],
\[ \|I_3\|_{L^\infty_t H^1(S_T)} \lesssim \left\| \begin{pmatrix} v_+ \\ 0 \end{pmatrix} - A_0 \phi_+ \right\|_{L^2_t H^1(S_T)} + \|e^{\pm it/\varepsilon^2} R^\varepsilon\|_{X^1(\tau = 2 \varepsilon/|\xi|)(S_T)}. \quad (10.9) \]

The second term on the r.h.s. is \( o(1) \) by (10.4), and the first term is bounded by
\[ T^{1/2} \left( \left\| u \left\{ \begin{pmatrix} v_+ \\ 0 \end{pmatrix} - \phi_+ \right\} \right\|_{L^\infty_t H^1(S_T)} + \|(u - A_0)\phi_+\|_{L^\infty_t H^1(S_T)} \right). \]

But using Leibniz’ rule, Hölder’s inequality and Sobolev embedding, it is easy to see that the terms inside the parentheses are dominated by \( Kf(I) \), where \( K \) depends on the size of \( X^\varepsilon_T + Y^\varepsilon_T \) and (10.8).

**Proof of (1.13b) and (1.13c).** Using Sobolev embedding we reduce (1.13b) to (1.13c). To prove the latter, observe that (10.3) implies
\[ \chi_+ \to v_+, \quad \chi_- \to 0, \quad \eta_+ \to 0, \quad \eta_- \to v_- \quad \text{in } H^1 \text{ as } \varepsilon \to 0, \quad (10.10) \]
locally uniformly in time. Thus (1.13c) follows immediately from (2.15) using Hölder’s inequality and Sobolev embedding.

**Proof of (1.15).** Multiply (2.16) by a \( C^1 \) compactly supported test function \( G(t,x) \) and integrate in \( t,x \). W.l.o.g. assume \( G \) is real-valued. The integrals corresponding to the last two terms in r.h.s. (2.16) are \( O(\varepsilon) \) in absolute value. To see this, integrate by parts in time and use
\[ \left| \int f g dx \right| \leq \|f\|_{H^{-1}} \|g\|_{H^1} \quad (10.11) \]
and the bound, locally uniform in time,
\[ \|\partial_t \phi_\pm\|_{H^{-1}} = O(1). \quad (10.12) \]

The latter is easily reduced to the uniform bounds for \( X^\varepsilon_T + Y^\varepsilon_T \), using Lemma [2.1] Sobolev embedding and Hölder’s inequality.
Next, fix $1 \leq j \leq 3$ and consider

$$I_\pm := \frac{2}{\varepsilon} \text{Re} \int \langle \sigma^j \chi_\pm, \eta_\pm \rangle G \, dt \, dx.$$ 

In view of (10.10) and (2.6),

$$I_- = \frac{2}{\varepsilon} \text{Re} \int \langle \sigma^j \chi_-, v_- \rangle G \, dt \, dx + o(1). \quad (10.13)$$

By (2.2), (2.4) and (10.4),

$$\frac{1}{\varepsilon} \left( \chi_0 \right) = e^{-it/\varepsilon^2} \frac{1}{\varepsilon} \Pi^0_+ \Pi^-_\psi + O(\varepsilon^2) \quad \text{in} \quad L^2_x \quad (10.14)$$

locally uniformly in time. But by (1.28),

$$\Pi^0_+ \Pi^-_\psi = i \frac{1}{2} [\lambda^\varepsilon]^{-1} \left( \sigma^k \partial_k \tilde{\eta} \right) + \frac{1}{2\varepsilon} \left( 1 - [\lambda^\varepsilon]^{-1} \right) \left( \tilde{\chi}_0 \right). \quad (10.15)$$

In view of (1.13) and the bound (1.29), it follows that

$$e^{-it/\varepsilon^2} \frac{1}{\varepsilon} \Pi^0_+ \Pi^-_\psi$$

= $i \frac{1}{2} [\lambda^\varepsilon]^{-1} \left( \sigma^k \partial_k v_- \right) + \frac{1}{2\varepsilon} \left( 1 - [\lambda^\varepsilon]^{-1} \right) \left( e^{-2it/\varepsilon^2} v_0 \right) + o(1) \quad (10.16)$$

in $L^2_{\psi}$. Moreover, by dominated convergence,

$$[\lambda^\varepsilon]^{-1} \sigma^k \partial_k v_- = \sigma^k \partial_k v_- + o(1) \quad \text{in} \quad H^{-1}. \quad (10.17)$$

Using (10.13)–(10.16) and either Hölder’s inequality or (10.11), we conclude that

$$I_- = \text{Re} \int i \langle \sigma^j \sigma^k \partial_k v_-, v_- \rangle G \, dt \, dx + I_-' + o(1)$$

where

$$I_-' = \frac{1}{\varepsilon} \text{Re} \int e^{-2it/\varepsilon^2} \langle \sigma^j \left( 1 - [\lambda^\varepsilon]^{-1} \right) v_+ \rangle G \, dt \, dx.$$ 

But the latter is $O(\varepsilon)$ in absolute value (integrate by parts in time and use the analogue of (10.12) for $v_+\pm$). Using (1.2) we finally conclude that

$$I_- = \int \left\{ - \text{Im} \left( \partial_j v_-, v_- \right) - \frac{1}{2} e^{ikl} \partial_k \langle \sigma l v_-, v_- \rangle \right\} G \, dt \, dx + o(1).$$

A similar calculation can be done for $I_+$, and this proves (10.15). \hfill \Box

Next, we prove Theorem 1.3. By hypothesis, (9.4), or equivalently (9.1), holds initially and therefore also uniformly in every finite time interval, by Proposition 9.4. Next observe that since (1.10) holds initially, we have

$$\Pi^0_+ \psi^\varepsilon = \psi^\varepsilon_+ + O(\varepsilon) \quad \text{in} \quad H^1$$

39
locally uniformly in time. In fact, this follows from (2.2)–(2.4), since (8.2) holds with \( m = 1 \). We conclude that it suffices to prove (1.16) with \( \psi^\varepsilon \) replaced by \( \chi^\varepsilon \). We proceed as in the proof of Theorem 1.7, but now the remainder term in (10.6) must be improved from \( o(1) \) to \( O(\varepsilon) \), and \( f(I) \) is given by the first term in r.h.s. (10.6). Again we reduce to estimating the terms \( I_1, \ldots, I_4 \) as given by (10.7).

The term \( I_1 \) is estimated exactly as before, but is now \( O(\varepsilon) \) since (1.16) is assumed to hold initially. Using the fact that \( U(t) - S(t) = \varepsilon^2 R_4(t) \), where \( R_4(t) \) is bounded from \( H^5 \rightarrow H^5 \) uniformly in \( \varepsilon \) and \( 0 \leq t \leq T \), and the assumption that the initial datum of \( v_+ \) is in \( H^5 \), we find that

\[
\|I_j\|_{L^\infty_t H^1(S_T)} = O(\varepsilon^2) \quad \text{for} \quad j = 2, 4.
\]

For \( I_3 \) we use again (10.9), but now the last term is \( O(\varepsilon) \), as follows from the proof of (5.12) taking into account the fact that (8.2) holds with \( m = 1 \). The first term in r.h.s. (10.9) is estimated exactly as before. This proves (1.16), and then it follows immediately that (1.13d) and (1.13c) are also improved to \( O(\varepsilon) \).

### 11 Semi-nonrelativistic limit

Here we prove Theorem 1.7. The initial assumptions (i), (ii) imply, as proved in Sects. 8 and 9, that (8.2) holds with \( m = 4 \), while (9.1)–(9.9) hold with \( m = 2 \).

We write

\[
\phi^\varepsilon = \left( \begin{array}{c} \chi^\varepsilon \\ \eta^\varepsilon \end{array} \right) := e^{it/\varepsilon^2} \psi^\varepsilon, \quad \phi^\varepsilon_\pm := e^{it/\varepsilon^2} \psi^\varepsilon_\pm,
\]

with \( \psi^\varepsilon_\pm \) defined as in (2.2). Also, we denote by \( \chi^\varepsilon_\pm \) the upper component of \( \phi^\varepsilon_\pm \).

Observe that

\[
\Pi^\varepsilon_\pm \psi^\varepsilon = \psi^\varepsilon_\pm + O(\varepsilon^2) \quad \text{in} \quad H^1,
\]

in view of (2.2)–(2.4) and the bounds (9.6) and (9.9) for \( m = 2 \). On account of (1.39), we may therefore replace \( \chi^\varepsilon \) in (1.31) by \( \chi^\varepsilon_\pm \). By Lemma 2.1,

\[
\left( i\partial_t - \frac{\lambda^\varepsilon - 1}{\varepsilon^2} \right) \chi^\varepsilon_+ + A_0^\varepsilon \chi^\varepsilon_+ = \tilde{R}_c,
\]

where

\[
\tilde{R}_c = \frac{1}{2} \varepsilon^2 \chi^\varepsilon \cdot \nabla \chi^\varepsilon - \frac{1}{2} \varepsilon B^c_j \sigma^j \chi^\varepsilon + \frac{1}{2} \varepsilon^2 (A^c)^2 \chi^\varepsilon - \frac{1}{2} (1 - [\lambda^\varepsilon]^{-1}) \varepsilon \left\{ 2i \chi^\varepsilon \cdot \nabla \chi^\varepsilon \cdot B^c_j \sigma^j \chi^\varepsilon \right\} + \frac{1}{2} [\lambda^\varepsilon]^{-1} \varepsilon \left\{ i E^c_j \sigma^j \eta^\varepsilon \right\}
\]

Recalling the bound (1.29) on the symbol of \( 1 - [\lambda^\varepsilon]^{-1} \) and using the fact that (8.2), (9.1) and (9.9) hold with \( m = 2 \), we conclude that

\[
\tilde{R}_c = \frac{1}{2} \varepsilon^2 \chi^\varepsilon \cdot \nabla \chi^\varepsilon - \frac{1}{2} \varepsilon B^c_j \sigma^j \chi^\varepsilon + \frac{1}{2} \varepsilon^2 (A^c)^2 \chi^\varepsilon + O(\varepsilon^2) \quad \text{in} \quad H^1,
\]
locally uniformly in time. Then, since (8.2) holds with $m = 4$ and $\lambda = 4\varepsilon - 1\varepsilon^2 = \Delta + \varepsilon^2 R_4$, where $R_4$ is bounded from $H^{s+4} \to H^s$ uniformly in $\varepsilon$, we further conclude that

$$i\partial_t \chi_+^\varepsilon = \frac{1}{2} (i\nabla + \varepsilon A^\varepsilon)^2 \chi_+^\varepsilon - A_0^\varepsilon \chi_+^\varepsilon - \frac{1}{2} \varepsilon B_j^\varepsilon \sigma^j \chi_+^\varepsilon + \varepsilon^2 r^\varepsilon$$

(11.1)

where $r^\varepsilon = O(1)$ in $H^1$ locally uniformly in time. Comparing (11.1) to the Pauli equation 1.30 via the energy inequality for the self-adjoint “Pauli operator”,

$$P^\varepsilon = \frac{1}{2} (i\nabla + \varepsilon A^\varepsilon)^2 - \frac{1}{2} \varepsilon B_j^\varepsilon \sigma^j$$

one finds that

$$f(I) \leq f(\{t_0\}) + K |I| f(I) + O(\varepsilon^2)$$

as $\varepsilon \to 0$, where

$$f(I) = \| \chi_+^\varepsilon - \chi_P^\varepsilon \|_{L^p_{x,t} H^1(I \times \mathbb{R}^3)}$$

for time intervals $I = [t_0, t_1] \subset [0, T]$, and where $K$ depends on $T$ but not on $\varepsilon$. In fact, $K$ depends on the $O(1)$ bounds in 9.6 and 9.9, which hold for $m = 2$ as we recall. We conclude that $f(I) = O(\varepsilon^2)$, and this proves 1.31.

Observe that 1.23 holds in $H^1$ locally uniformly in time, in view of 1.20 and the fact that 9.4, 9.5 and 9.9 hold for $m = 2$. Substituting 1.23 into

$$J^\varepsilon = \varepsilon^{-1} \{ 2 \text{Re} (\sigma^k \chi_+^\varepsilon, \eta^\varepsilon) \}_{k=1,2,3}$$

and using 1.31 yields 1.32.

12 Proofs of the spacetime estimates

Here we prove Theorem 3.1 and Proposition 3.4.

Proof of Proposition 3.4 Let $Q$ be a cube with side length $\sim \mu$ centered at $\xi_0$, where $|\xi_0| \sim \lambda$, and let $\chi_Q(\xi)$ be a smooth cut-off function equal to 1 on $Q$. For example, we can take

$$\chi_Q(\xi) := \eta \left( \frac{\xi - \xi_0}{\mu} \right),$$

(12.1)

where $\eta$ is a smooth bump function equal to 1 on a neighborhood of the origin. Then by the $TT^*$ method, we reduce 3.1 to the decay estimate

$$|K_{\varepsilon,Q}(t,x)| \lesssim \begin{cases} \mu |t|^{-1} & \text{for } \lambda \lesssim 1/\varepsilon, \\ \varepsilon \mu \lambda |t|^{-1} & \text{for } \lambda \gg 1/\varepsilon, \end{cases}$$

(12.2)

for the convolution kernel

$$K_{\varepsilon,Q}(t,x) := \int_{\mathbb{R}^3} e^{i x \cdot \xi} e^{i h_\varepsilon(\xi)} \chi_Q(\xi) d\xi,$$
with $h_\epsilon$ given by (2.14). In view of the scaling identity

$$K_{\epsilon,Q}(t,x) = \epsilon^{-3} K_{1,\epsilon Q}(\epsilon^{-2} t, \epsilon^{-1} x),$$

it suffices to prove (12.2) for $\epsilon = 1$. To simplify the notation we write $K_Q$ instead of $K_{1,Q}$. Thus,

$$K_Q(t,x) = \int_0^\infty \int_{S^2} e^{ix \cdot \omega} e^{it\alpha(r)} \chi_Q(r\omega) r^2 d\sigma(\omega) dr$$

(12.3)

where $\sigma$ is surface measure on $S^2$ and $\alpha$ is given by (2.18). Note that

$$\alpha'(r) = \frac{r}{\sqrt{1 + r^2}}$$

and

$$\alpha''(r) = \frac{1}{(1 + r^2)^{3/2}}.$$  

(12.4)

We split the problem into the following cases:

(i) $\lambda \lesssim 1$ and $|x| \gtrsim \lambda |t|$,

(ii) $\lambda \lesssim 1$ and $|x| \ll \lambda |t|$,

(iii) $\lambda \gg 1$ and $|x| \gtrsim |t|$,

(iv) $\lambda \gg 1$ and $|x| \ll |t|$.

Rewrite (12.3) as

$$K_Q(t,x) = \int_{S^2} b(\omega) d\sigma(\omega),$$

where

$$b(\omega) = \int_0^\infty \frac{d}{dr} \left[ e^{i(t\alpha(r)+x \cdot \omega)} \right] \frac{\chi_Q(r\omega)r^2}{i(t\alpha'(r)+x \cdot \omega)} dr.$$  

Lemma 12.1. $|a(r,x)| \lesssim (r |x|)^{-1} \chi_I(r)$, where $\chi_I$ is the characteristic function of an interval $I$ of length $\sim \mu$ and centered at a distance $\sim \lambda$ from the origin.

Proof. The statement about the $r$-support of $a(r,x)$ is obvious, and the decay statement follows from the fact that

$$\left| \int_{S^2} e^{ix \cdot \omega} \chi_Q(r\omega) d\sigma(\omega) \right| \lesssim 1/|x|$$

for all smooth functions $\gamma$ such that $|\gamma| \leq 1$. But this fact is easily proved by passing to spherical coordinates and rescaling.

Thus

$$|K_Q(t,x)| \lesssim \int_I \left( \frac{r}{|x|} \right) dr \sim \mu \lambda/|x|,$$

and this covers the cases (i) and (ii) above.

To handle the remaining cases we write (12.3) as $K_Q(t,x) = \int_{S^2} b(\omega) d\sigma(\omega)$, where

$$b(\omega) = \int_0^\infty \frac{d}{dr} \left[ e^{i(t\alpha(r)+x \cdot \omega)} \right] \frac{\chi_Q(r\omega)r^2}{i(t\alpha'(r)+x \cdot \omega)} dr.$$  

42
Integrate by parts and write
\[- \frac{d}{dr} \left( \frac{\chi_Q(r\omega)r^2}{i(t\alpha'(r) + x \cdot \omega)} \right) = \frac{\chi_Q(r\omega)r^2t\alpha''(r)}{i(t\alpha'(r) + x \cdot \omega)} - \frac{d}{dr} \left( \frac{\chi_Q(r\omega)r^2}{i(t\alpha'(r) + x \cdot \omega)} \right).\]

Correspondingly we split \(b = b_1 + b_2\). Observe that the \(r\)-support of \(\chi_Q(r\omega)\) is contained in an interval \(I\) of length \(\sim \mu\) and centered at a distance \(\lambda\) from the origin, while the \(\omega\)-support is contained in a set given by
\[\angle(\omega, \omega_0) \lesssim \mu/\lambda (12.5)\]
for some \(\omega_0 \in S^2\). Moreover, in view of (12.1) we have
\[|\frac{d}{dr} \chi_Q(r\omega)| \lesssim 1/\mu. \quad (12.6)\]

Now consider case (iv). Then on account of (12.4) we have \(\alpha'(r) \sim 1\) and \(\alpha''(r) \sim \lambda^{-3}\) for \(r \in I\), so \(|t\alpha'(r) + x \cdot \omega| \gtrsim |t|\). Thus
\[|b_1(\omega)| \lesssim (1/\lambda |t|) \int_I dr \lesssim \mu/\lambda |t|,\]
which is more than good enough. Next, using (12.6) we have
\[|b_2(\omega)| \lesssim (1/|t|) \int_I (r + r^2/\mu) \, dr \lesssim (\lambda \mu + \lambda^2)/|t| \lesssim \lambda^2/|t|.\]

But integrating this over the region (12.5) on \(S^2\) gives us a bound \(\mu^2/|t|\), which again is more than good enough.

Finally, consider case (ii). Then \(\alpha'(r) \sim \lambda\) and \(\alpha''(r) \sim 1\) for \(r \in I\), so \(|t\alpha'(r) + x \cdot \omega| \gtrsim \lambda |t|\). Thus
\[|b_1(\omega)| \lesssim (1/|t|) \int_I dr \lesssim \mu/|t|\]
and
\[|b_2(\omega)| \lesssim (1/\lambda |t|) \int_I (r + r^2/\mu) \, dr \lesssim (\mu + \lambda)/|t| \lesssim \lambda/|t|.\]

Taking into account (12.5) we thus get the desired bound, and this concludes the proof of Proposition 3.4.

**Proof of Theorem 3.1(ii).** If \(\mu \sim \lambda\), this reduces to part (iii) of the theorem, so we may assume \(\mu \ll \lambda\) (and \(\lambda \gg 1/\varepsilon\)). But then by an orthogonality argument (see, e.g., the proof of the analogous estimate in Theorem 12.1 of [8]) we reduce to proving
\[\|uv\|_{L^2_{t,x}} \lesssim \varepsilon^{1/2} \mu^{1/2} \lambda^{1/2} \|f\|_{L^2} \|g\|_{L^2}\]
in the case where the Fourier transforms of \(f, g\) are supported in (diametrically opposite) cubes with side length \(\sim \mu\) and at distance \(\sim \lambda\) from the origin. But this follows from Hölder’s inequality and the estimates (3.5) and (3.7) with \((q, r) = (4, 4)\).
Proof of Theorem 3.1. If \( \mu \sim \lambda \), this reduces to part (ii) of the theorem, so we may assume \( \mu \ll \lambda \lesssim 1/\varepsilon \). By orthogonality, we reduce to proving
\[
\|uv\|_{L^2_{t,x}} \lesssim \varepsilon^{1/2} \mu \|f\|_{L^2_t} \|g\|_{L^2_t} \tag{12.7}
\]
in the case where \( \hat{f}, \hat{g} \) are supported in opposite cubes \( Q, -Q \) with side length \( \sim \mu \) and at distance \( \sim \lambda \) from the origin. By rescaling \( t \to t/\varepsilon \) we further reduce to proving (12.7) without the \( \varepsilon^{1/2} \) in the right hand side, and with \( u, v \) given by
\[
[u(t)]^\alpha(\xi) = e^{it\xi} \hat{f}(\xi), \quad [v(t)]^\alpha(\xi) = e^{it\xi} \hat{g}(\xi). \tag{12.8}
\]
Here \( \alpha \) is given by (2.18). Then by a standard Cauchy-Schwarz argument, see e.g. [2, Sect. 3.4], we finally reduce to proving that
\[
\int \chi(\eta, \eta \in Q) \cap \{\eta, \eta - \xi \in Q\}(\eta) \delta(\tau - \eta) \pm k(\xi - \eta) \, d\eta \lesssim \mu^2 \tag{12.9}
\]
where
\[
k(\rho) := \varepsilon^{-1} \alpha(\varepsilon\rho). \tag{12.10}
\]
(Here and in what follows we use the notation \( \chi_A \) for the characteristic function of a set \( A \).) Then in view of (12.4) there is an absolute constant \( c_0 \) such that
\[
|k(\rho)| \leq c_0 < 1 \quad \text{for all} \quad \rho \lesssim 1/\varepsilon, \quad 0 < \varepsilon < 1. \tag{12.11}
\]

Denote by \( I_{\pm}(\tau, \xi) \) the integral in (12.9). In polar coordinates \( \eta = r\omega, r > 0, \omega \in S^2 \), we have
\[
I_{\pm}(\tau, \xi) = \int_{S^2} a_{\pm}(\tau, \xi; \omega) \, d\sigma(\omega),
\]
where
\[
a_{\pm}(\tau, \xi; \omega) := \int_0^\infty \chi_{\{r, r\omega \in Q\} \cap \{r, r\omega - \xi \in Q\}}(r) \delta(\tau - r \pm k(\xi - r\omega)) \, r^2 \, dr.
\]
Observe that the \( \omega \)-support of \( a \) is contained in a set given by (12.5), so it suffices to prove that \( a_{\pm} \lesssim \lambda^2 \). Observe also that in the integral defining \( a_{\pm} \), the variable \( r \) is restricted to an interval \( I \) of length \( \sim \mu \) and centered at a distance \( \lambda \) from the origin.

We shall use the following fact: If \( f : \mathbb{R} \to \mathbb{R} \) is differentiable with \( |f'(r)| > 0 \), and \( f \) has a zero at \( r_0 \), then
\[
\delta(f(r)) \, dr = \delta(r - r_0) \, dr / |f'(r_0)|. \tag{12.12}
\]
Take
\[
f(r) := \tau - r \pm k(|\xi - r\omega|), \tag{12.13}
\]
for fixed \( \tau, \xi, \omega \). Then for \( r \) such that \( r\omega - \xi \in Q \),
\[
|f'(r)| = 1 \mp k'(|\xi - r\omega|) \frac{\xi - r\omega}{|\xi - r\omega|} \geq 1 - c_0 \gtrsim 1, \tag{12.14}
\]
where we used (12.11) and the assumption \( \lambda \lesssim 1/\varepsilon \). On account of (12.12) and (12.14), we then get \( a_{\pm} \lesssim \lambda^2 \) as desired. This concludes the proof of part (i) of Theorem 3.1. \( \square \)
Proof of Theorem 3.1(iii). This reduces to proving

\[
\int \chi_{\{y:|\eta|<\mu\}}(\eta) \delta(\tau - |\eta| \pm k(|\xi - \eta|)) \, d\eta \lesssim \left[ \min(\mu, \lambda) \right]^2,
\]

(12.15)

for \( k \) defined by (12.10). Let us denote the above integral by \( I_\pm(\tau, \xi) \). Passing to polar coordinates we have \( I_\pm(\tau, \xi) = \int_{S^2} a_\pm(\tau, \xi; \omega) \, d\sigma(\omega) \), where now

\[
a_\pm(\tau, \xi; \omega) := \int_0^\infty \chi_{\{r:r<\mu\}}(r) \delta(\tau - r \pm k(|\xi - r\omega|)) \, r^2 \, dr.
\]

We split into the cases

(a) \( \lambda \lesssim 1/\varepsilon \),
(b) \( \lambda \gg 1/\varepsilon \).

Case (a). Then in view of (12.12) and (12.14) with \( a \) support of (12.12), thus arriving at the identity

\( (12.15) \), Let us denote the above integral by \( I_\pm(\tau, \xi) \). Passing to polar coordinates we have \( I_\pm(\tau, \xi) = \int_{S^2} a_\pm(\tau, \xi; \omega) \, d\sigma(\omega) \), where now

\[
a_\pm(\tau, \xi; \omega) := \int_0^\infty \chi_{\{r:r<\mu\}}(r) \delta(\tau - r \pm k(|\xi - r\omega|)) \, r^2 \, dr.
\]

We split into the cases

(a) \( \lambda \lesssim 1/\varepsilon \),
(b) \( \lambda \gg 1/\varepsilon \).

Case (a). Then in view of (12.12) and (12.14) with \( f(r) \) given by (12.13), we have \( a_\pm \lesssim \mu^2 \). Now integrate over \( S^2 \), taking into account the fact that on the support of \( a_\pm \),

\[
\angle(\omega, \xi) = \angle(\eta, \xi) \lesssim \lambda/\mu \quad \text{if} \quad \mu \gg \lambda.
\]

(12.16)

Case (b). By rotational symmetry we may assume \( \xi = (|\xi|, 0, 0) \). Now parametrize the sphere \( S^2 \) by

\[
(y, \theta) \mapsto \omega = \left( y, \sqrt{1 - y^2} \, \vec{n}(\theta) \right), \quad \vec{n}(\theta) = (\cos \theta, \sin \theta).
\]

Then surface measure \( d\sigma(\omega) \) on \( S^2 \) becomes \( dy \, d\theta \). Again we use (12.12) with \( f(r) \) given by (12.13). Observe that \( f \) depends implicitly on \( y \) but not on \( \theta \). Denote by \( A = A(\tau, \xi) \) the set of \( y \in (-1, 1) \) such that \( f(r) \) given by (12.13) has a zero \( r_0 = r_0(y) > 0 \). Since \( |f'(r)| > 0 \), the implicit function theorem guarantees that \( A \) is open and \( r_0 : A \to (0, \infty) \) is a smooth function. Differentiating \( f(r_0(y)) = 0 \) gives

\[
0 = f'(r_0) r_0'(y) \mp k'(|\xi - r_0\omega|) \frac{r_0 |\xi|}{|\xi - r_0\omega|},
\]

(12.17)

where we used \( \xi \cdot \partial_y \omega = \xi_1 = |\xi| \) and \( \omega \cdot \partial_y \omega = 0 \).

Let us suppress the subscript and write \( r(y) \) instead of \( r_0(y) \) from now on. Solving (12.17) for \( r'(y) \) and using the fact that \( f'(r) < 0 \), we see that \( \partial r/\partial y \) is either strictly negative or strictly positive, depending on whether we have the + sign or the − sign in (12.16). The function \( r(y) \) is therefore a change of variables.

With this information in hand, we solve (12.17) for \( f'(r) \) and substitute into (12.12), thus arriving at the identity

\[
\int F(\eta) \delta(\tau - |\eta| \pm k(|\xi - \eta|)) \, d\eta = \int \int F(r\omega) \frac{r |\xi - r\omega|}{|\xi| k'(|\xi - r\omega|)} \frac{|\partial r|}{|\partial y|} \, dy \, d\theta.
\]

45
Changing variables $y \rightarrow r$ finally gives
\[
\int F(\eta) \delta(\tau - |\eta|) \pm k(|\xi - \eta|) \, d\eta = \int \int F(r\omega) \frac{r|\xi - r\omega|}{|\xi| k'(|\xi - r\omega|)} \, dr \, d\theta, \quad (12.18)
\]
where $\omega$ is now a function of $r$ and $\theta$. We apply this with
\[
F(\eta) := \chi_{\{r:|\eta| \sim \mu\} \cap \{r:|\xi - \eta| \sim \lambda\}}(\eta).
\]
Since $\lambda \gg 1/\varepsilon$, we see from (12.3) that $k'(|\xi - r\omega|) \sim 1$, whence
\[
F(r\omega) \frac{r|\xi - r\omega|}{|\xi| k'(|\xi - r\omega|)} \sim \frac{\mu \lambda}{|\xi|} F(r\omega). \quad (12.19)
\]

We now split into the subcases
\[\begin{align*}
(b1) \quad & \mu \ll \lambda, \\
(b2) \quad & \mu \sim \lambda, \\
(b3) \quad & \mu \gg \lambda.
\end{align*}\]

Case (b2). In this case we can prove the estimate in Theorem 3.1(iii) directly, by applying Hölder’s inequality followed by the linear Strichartz estimate (3.6) with $(q, r) = (4, 4)$. (This works because we are at high frequency, i.e. $\gg 1/\varepsilon$.)

Case (b1). Then $|\xi| \sim \lambda$, so the desired estimate follows readily from (12.19) and (12.18).

Case (b3). Then $|\xi| \sim \mu$, so (12.19) and (12.18) imply
\[
I_{\pm}(\tau, \xi) \lesssim \lambda \int \int \chi((r:|r| \sim \mu) \cap (r:|\xi - r\omega| \sim \lambda})(r) \, dr \, d\theta.
\]

Recall that $\omega$ is now a function of $(r, \theta)$. However, $|\xi - r\omega|$ is independent of $\theta$, so by a slight abuse of notation we will simply write $\omega = \omega(r)$ and integrate out $\theta$, leaving us with
\[
\lambda \int \chi((r:|\omega| \sim \mu) \cap (r:|\xi - r\omega| \sim \lambda))(r) \, dr.
\]
Clearly it suffices to prove that the support of the integrand is contained in an interval of length $\sim \lambda$. Let us assume there is no such interval, and obtain a contradiction. Fix a point $r_0$ in the support, and write
\[
\tau = r_0 + \kappa
\]
for a general point $r$ in the support. In view of our assumption, $\kappa$ varies on a scale $\gg \lambda$. Thus, if we can show that
\[
|\xi - r\omega(r)|^2 = a + \kappa^2 + O(\lambda \kappa + \lambda^2), \quad (12.20)
\]
for some constant $a$, it follows that $|\xi - r\omega(r)|$ also varies on a scale $\gg \lambda$, and we have the contradiction we seek, since $|\xi - r\omega(r)| \sim \lambda$ on the support.

To prove (12.20), write

$$|\xi - r\omega|^2 = |\xi|^2 + (r^2 - 2r |\xi|) + 2r(1 - \omega_1) |\xi|.$$ 

On account of (12.16) we have $1 - \omega_1 \lesssim (\lambda/\mu)^2$, so the last term on the right hand side is $O(\lambda^2)$. For the second term we calculate

$$r^2 - 2r |\xi| = (r^2_0 - 2r_0 |\xi|) + 2(r_0 - |\xi|)\kappa + \kappa^2.$$ 

But

$$|r_0 - |\xi|| \leq |r_0\omega(r_0) - |\xi|| \sim \lambda,$$

so we conclude that (12.20) holds. This ends the proof of Theorem 3.1.

13 Proof of Theorem 5.3

As remarked, by a standard procedure this reduces to some well-known bilinear estimates for the homogeneous wave equation. The first observation is that by rescaling $x \to \varepsilon x$ we can reduce to the case $\varepsilon = 1$. Thus we suppress the subscript on $H^s,\theta$ etc. from now on.

Some notation: For $s \in \mathbb{R}$, let $D^s$, $D^s_+$ and $D^s_-$ be the Fourier multipliers

$$(D^s u)^\sim = |\xi|^s \hat{u}, \quad (D^s_+ u)^\sim = (|\tau| + |\xi|)^s \hat{u}, \quad (D^s_- u)^\sim = ||\tau| - |\xi||^s \hat{u}.$$ 

The notation $u \preceq v$ means $|\hat{u}| \lesssim |\hat{v}|$. We are concerned with bilinear operators $B(u,v)$ of the form

$$[B(u,v)]^\sim (\tau,\xi) = \int b(\tau - \lambda, \xi - \eta; \lambda, \eta)\hat{u}(\tau - \lambda, \xi - \eta)\hat{v}(\lambda, \eta) d\lambda d\eta,$$

where $b(\tau; \xi, \lambda, \eta)$ is the symbol of $B$. The symbols of the null forms $Q_0, Q_{ij}$ and $Q_{ij0}$ are, respectively,

$$q_0(\tau,\xi,\lambda,\eta) = \tau\lambda - \xi \cdot \eta, \quad (13.1a)$$

$$q_{ij}(\tau,\xi,\lambda,\eta) = -\xi_i \eta_j + \xi_j \eta_i, \quad (13.1b)$$

$$q_{ij0}(\tau,\xi,\lambda,\eta) = -\tau \eta_j + \lambda \xi_j. \quad (13.1c)$$

Since we rely on estimates for the absolute values of these symbols, and since all norms involved only depend on the absolute value of the Fourier transform, we may assume $\hat{u}, \hat{v} \geq 0$ henceforth.

For $s \in \mathbb{R}$, let $R^s$ be the bilinear operator with symbol $r^s$, where

$$r^s(\tau,\xi,\lambda,\eta) = \begin{cases} 
|\tau| + |\xi| - |\xi + \eta| & \text{if } \tau \lambda \geq 0, \\
|\xi + \eta| - ||\xi| - |\eta|| & \text{if } \tau \lambda < 0.
\end{cases}$$

47
Lemma 13.1. The following estimates hold:

and

\( \tau \)

This follows easily from the triangle inequality, if one keeps track of the signs of the symbols (13.1). First, by [8, Lemma 13.2] we have

Proof. All these statements reduce to estimates on the absolute values of the symbols (13.1). First, by [8, Lemma 13.2] we have

\[ Q_{ij}(u, v) \lesssim R^{1/2}(D_u, D^{1/2}u) + R^{1/2}(D^{1/2}u, v) \]

(13.4a)

\[ Q_{ij}(u, v) \lesssim R^{(1/2)}(D_u, D^{(1/2)}v) + R^{(1/2)}(D^{(1/2)}v, Dv), \]

(13.4b)

\[ Q_{ij}(u, v) \lesssim [r.h.s. (13.4a)] + Du \cdot D_u v + D_u \cdot Lv, \]

(13.4c)

\[ Q_{ij}(u, v) \lesssim [r.h.s. (13.4a)] + D_v u \cdot D_u v + D_u \cdot Lv. \]

(13.4d)

Lemma 13.1. The following estimates hold:

\[ Q_{ij}(u, v) \lesssim R^{1/2}(D_u, D^{1/2}v) + R^{1/2}(D^{1/2}u, v) \]

which derives from an estimate for the homogeneous wave equation via the Transfer Principle; see [15] for the details. We also need

\[ R^s(u, v) \lesssim D^s(uv) + (D^s_u u + uD^s_v v) \]

(13.3)

This follows easily from the triangle inequality, if one keeps track of the signs of the symbols (13.1). First, by [8, Lemma 13.2] we have

Proof. We shall need the estimate for \( r > 1/2 \),

\[ \|R^{1/2}(u, v)\|_{L^2} \lesssim \|u\|_{H^{0.5}} \|v\|_{H^{3/2}}. \]

(13.2)

which derives from an estimate for the homogeneous wave equation via the Transfer Principle; see [15] for the details. We also need

\[ R^s(u, v) \lesssim D^s(uv) + (D^s_u u + uD^s_v v) \]

(13.3)

This follows easily from the triangle inequality, if one keeps track of the signs of the symbols (13.1). First, by [8, Lemma 13.2] we have

Proof. All these statements reduce to estimates on the absolute values of the symbols (13.1). First, by [8, Lemma 13.2] we have

\[ |q_{ij}(\tau, \xi; \lambda, \eta)| \leq |\xi \times \eta| \leq |\xi|^{1/2} |\eta|^{1/2} |\xi + \eta|^{1/2} [r(\tau, \xi; \lambda, \eta)]^{1/2}, \]

where \( r \) is the symbol of \( R \) as defined above. Then (13.4a) and (13.4b) follow, in view of the fact that

\[ r(\tau, \xi; \lambda, \eta) \leq 2 \min(|\xi|, |\eta|). \]

(13.5)

To prove (13.4c), write

\[ q_{0j}(\tau, \xi; \lambda, \eta) = (\epsilon_1 |\xi| - \tau)\eta_j + (\lambda - \epsilon_2 |\eta|)\xi_j - \epsilon_1 (|\xi| \eta_j - \epsilon_1 \epsilon_2 |\eta| |\xi|), \]

where \( \epsilon_1 \) and \( \epsilon_2 \) are the signs of \( \tau \) and \( \lambda \), respectively. That is, \( \epsilon_1 \tau = |\tau| \) and \( \epsilon_2 \lambda = |\lambda| \). Now take absolute values and use the fact (see [8, Lemma 13.2]) that

\[ ||\xi|| \eta_j \leq |\eta| |\xi_j| \leq |\xi|^{1/2} |\eta|^{1/2} |\xi + \eta|^{1/2} [r(\tau, \xi; \lambda, \eta)]^{1/2} \]

holds for all \( \tau, \xi, \lambda, \eta \). (The sign in the left hand side is independent of the signs of \( \tau, \lambda \).) This proves (13.4c). The proof of (13.4d) is similar. Write

\[ q_{0j}(\tau, \xi; \lambda, \eta) = (\tau - \epsilon_1 |\xi|)\lambda + (\lambda - \epsilon_2 |\eta|)\epsilon_1 |\xi| + \epsilon_1 \epsilon_2 |\eta| |\xi| - \xi \cdot \eta. \]

Then use (see [8, Lemma 13.2])

\[ ||\eta|| |\xi| - \xi \cdot \eta| \leq (|\xi| + |\eta|)r(\tau, \xi; \lambda, \eta) \]

and (13.5).
Finally, we need the estimate (here $s_1, s_2, \theta_1, \theta_2 \geq 0$)

$$
\|uv\|_{L^2} \lesssim \|u\|_{H^{s_1, \theta_1}} \|v\|_{H^{s_2, \theta_2}} \quad \text{for} \quad s_1 + s_2 > \frac{3}{2}, \quad \theta_1 + \theta_2 > \frac{1}{2}.
$$

(13.6)

See [15, Proposition A.1] for the simple proof of this fact.

We are now ready to prove Theorem 5.3. By interpolation, we reduce to

$$
\|Q(u, v)\|_{L^2} \lesssim \|u\|_{H_{s_1, \theta_1}} \|v\|_{H_{s_2, \theta_2}}
$$

(13.7)

$$
\|Q(u, v)\|_{H^{0, (-1/2)-}} \lesssim \|u\|_{H_{s_1, \theta_1}} \|v\|_{H_{s_2, \theta}^{(1/2)+}}
$$

(13.8)

where $\|u\|_{H_{s_1, \theta}}$ in the right hand side can be replaced by $\|u\|_{H_{s_1, \theta}}$ if $Q = Q_{ij}$.

Proof of (13.7). First observe that for the last two terms in the right hand sides of (13.4c) and (13.4d), the estimate reduces to special cases of (13.6), since we can always replace $D_-$ by $D_\theta D_{1-\theta}$. Thus, it only remains to prove the estimate for the right hand side of (13.4a), but this reduces to (13.2).

Proof of (13.8). First consider $Q_{ij}$. Applying (13.3) to (13.4b), we reduce to

$$
\|uv\|_{L^2} \lesssim \|u\|_{H^0, (1/2)+} \|v\|_{H^{1/2}+},
$$

(13.9a)

$$
\|uv\|_{L^2} \lesssim \|u\|_{H^{1/2}, (1/2)+} \|v\|_{H^{1/2}+},
$$

(13.9b)

$$
\|uv\|_{H^0, (-1/2)-} \lesssim \|u\|_{L^2} \|v\|_{H^{1/2}+},
$$

(13.9c)

$$
\|uv\|_{H^0, (-1/2)-} \lesssim \|u\|_{H^{1/2}, (1/2)+} \|v\|_{H^{1/2}+},
$$

(13.9d)

$$
\|uv\|_{H^0, (-1/2)-} \lesssim \|u\|_{H^0, (1/2)+} \|v\|_{H^{1/2}+},
$$

(13.9e)

$$
\|uv\|_{H^0, (-1/2)-} \lesssim \|u\|_{H^{1/2}, (1/2)+} \|v\|_{H^{1/2}+},
$$

(13.9f)

Via duality and the Transfer Principle, these reduce to the estimates in Corollaries 3.2 and 3.3 which are valid in the case where $u, v$ are both solutions of the homogeneous wave equation, as remarked in Sect. 3.

It remains to consider the second and third terms in the right hand sides of (13.4c) and (13.4d). For the second term we can apply (13.6) directly, while for the third term we replace $D_-$ by $D_{(-1/2)+} D_{(+1/2)+}$, thus reducing to (13.9d).

Acknowledgment.

Financial support by the Austrian START project “Nonlinear Schrödinger and quantum Boltzmann equations” (FWF Y137-TEC) of N.J.M. and by the European network HYKE (HPRN-CT-2002-00282) as well as by the OeAD (“acciones integradas”) is acknowledged.
References

[1] Bechouche, P., Mauser, N.J., Poupaud, F. (1998): (Semî)-nonrelativistic limits of the Dirac equation with external time-dependent electromagnetic field. Comm. Math. Phys. 197, no. 2, 405–425

[2] Bechouche, P., Mauser, N.J., Selberg, S. (2002): Nonrelativistic limit of Klein-Gordon-Maxwell to Schrödinger-Poisson. Submitted to Amer. J. Math.

[3] Bechouche, P., Mauser, N.J., Selberg, S. (2002): Derivation of Schrödinger-Poisson as the nonrelativistic limit of Klein-Gordon-Maxwell. To appear in the Proceedings of the conference Hyperbolic Equations 2002 (Caltech, March 2002).

[4] Bourgain, J. (1993): Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I: Schrödinger equations, and II: The KdV equation. Geom. Funct. Anal. 3, 107–156 and 209–262

[5] Bournaveas, N. (1996): Local existence for the Maxwell-Dirac equations in three space dimensions. Comm. Partial Differential Equations 21, no. 5-6, 693–720

[6] Cirincione, R., Chernoff, P.R. (1981): Dirac and Klein-Gordon equations: convergence of solutions in the nonrelativistic limit. Comm. Math. Phys. 79, no. 1, 33–46

[7] Dirac, P.A.M. (1958): Principles of Quantum Mechanics. 4th ed., Oxford University Press, London

[8] Foschi, D., Klainerman, S. (2000): Homogeneous $L^2$ bilinear estimates for wave equations. Ann. Scient. ENS 4$^{e}$ serie, 23, 211–274

[9] Georgiev, V. (1991): Small amplitude solutions of the Maxwell-Dirac equations. Indiana Univ. Math. J. 40, no. 3, 845–883

[10] Kenig, C., Ponce, G., Vega, L. (1994): The Cauchy problem for the KdV equation in Sobolev spaces of negative indices. Duke Math. J. 71, 1–21

[11] Klainerman, S., Machedon, M. (1993): Space-time estimates for null forms and the local existence theorem. Comm. Pure Appl. Math., 46, 1221–1268

[12] Klainerman, S., Machedon, M. (1994): On the Maxwell-Klein-Gordon equation with finite energy. Duke Math. J. 74, 19–44

[13] Klainerman, S., Machedon, M. (1995): Smoothing estimates for null forms and applications. Duke Math. J. 81, 99–133

[14] Klainerman, S., Machedon, M.: Personal communication
[15] Klainerman, S., Selberg, S. (2002): Bilinear estimates and applications to nonlinear wave equations. Commun. Contemp. Math. 4, no. 2, 223–295

[16] Klainerman, S., Tataru, D. (1999): On the optimal local regularity for Yang-Mills equations in \( \mathbb{R}^{4+1} \). J. Amer. Math. Soc., 12, 93–116

[17] Landau, L.D., Lifschitz, E.M. (1971): *Quantenmechanik*. Vol. III, 2nd ed., Akademie-Verlag, Berlin

[18] Machedon, M., Sterbenz, J. (2002): Optimal local well-posedness of the Maxwell-Klein-Gordon equations in 3 + 1 dimensions. Preprint.

[19] Mauser, N.J. (2000): Semi-relativistic approximations of the Dirac equation: first and second order corrections. Trans. Theor. Stat. Phys., 29, 122–137

[20] Masmoudi, N., Mauser, N.J. (2001): The selfconsistent Pauli equation. Mathematische Monatshefte 132, 19–24

[21] Masmoudi, N., Nakanishi, K.: From Maxwell-Klein-Gordon and Maxwell-Dirac to Poisson-Schrödinger (?). To appear in Int. Math. Res. Notices

[22] Najman, B. (1992): The nonrelativistic limit of the nonlinear Dirac equation. Ann. Inst. Henri Poincaré Anal. Non Lineaire 9, 3–12

[23] Selberg, S. (2002): On an estimate for the wave equation and applications to nonlinear problems. Differential and Integral Equations 2, 213–236

[24] Tao, T. (2001): Multilinear weighted convolution of \( L^2 \) functions, and applications to nonlinear dispersive equations. Amer. J. Math. 123, no. 5, 839–908