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CONSTRAINED MINIMUM RIESZ ENERGY PROBLEMS FOR
A CONDENSER WITH INTERSECTING PLATES

P.D. DRAGNEV, B. FUGLEDE, D.P. HARDIN, E.B. SAFF, AND N. ZORII

Abstract. We study the constrained minimum energy problem with an external field relative to the \( \alpha \)-Riesz kernel \( |x - y|^{\alpha - n} \) of order \( \alpha \in (0, n) \) for a generalized condenser \( A = (A_i)_{i \in I} \) in \( \mathbb{R}^n \), \( n \geq 3 \), whose oppositely charged plates intersect each other over a set of zero capacity. Conditions sufficient for the existence of minimizers are found, and their uniqueness and vague compactness are studied. Conditions obtained are shown to be sharp. We also analyze continuity of the minimizers in the vague and strong topologies when the condenser and the constraint both vary, describe the weighted equilibrium vector potentials, and single out their characteristic properties. Our arguments are based particularly on the simultaneous use of the vague topology and a suitable semimetric structure on a set of vector measures associated with \( A \), and the establishment of completeness theorems for proper semimetric spaces. The results remain valid for the logarithmic kernel on \( \mathbb{R}^2 \) and \( A \) with compact \( A_i, i \in I \). The study is illustrated by several examples.

1. Introduction

The purpose of the paper is to study minimum energy problems with an external field (also known in the literature as weighted minimum energy problems) relative to the \( \alpha \)-Riesz kernel \( \kappa_\alpha(x, y) = |x - y|^{\alpha - n} \) of order \( \alpha \in (0, n) \) on \( \mathbb{R}^n \), \( n \geq 3 \), where \( |x - y| \) denotes the Euclidean distance between \( x, y \in \mathbb{R}^n \) and infimum is taken over classes of vector measures \( \mu = (\mu_i)_{i \in I} \) associated with a generalized condenser \( A = (A_i)_{i \in I} \). More precisely, a finite ordered collection \( A \) of closed sets \( A_i \subset \mathbb{R}^n \), \( i \in I \), termed plates, with the sign \( s_i = \pm 1 \) prescribed is a generalized condenser if oppositely signed plates intersect each other over a set of \( \alpha \)-Riesz capacity zero, while \( \mu = (\mu_i)_{i \in I} \) is associated with \( A \) if each \( \mu_i, i \in I \), is a positive scalar Radon measure on \( \mathbb{R}^n \) supported by \( A_i \). Note that any two equally signed plates may intersect each other over a set of nonzero \( \alpha \)-Riesz capacity (or even coincide). In accordance with an electrostatic interpretation of a condenser, we say that the interaction between the components \( \mu^i, i \in I \), of \( \mu \) is characterized by the matrix \( (s_is_j)_{i,j \in I} \), so that
The α-Riesz energy of $\mu$ is defined by

$$\kappa_\alpha(\mu, \mu) := \sum_{i,j \in I} s_i s_j \int \int |x - y|^{\alpha-n} d\mu_i(x) d\mu_j(y).$$

The difficulties appearing in the course of our investigation are caused by the fact that a short-circuit between $A_i$ and $A_j$ with $s_i s_j = -1$ may occur since those plates may have zero Euclidean distance; see Theorem 4.6 below providing an example of a condenser with no α-Riesz energy minimizer. Therefore it is meaningful to ask what kinds of additional requirements on the objects under consideration will prevent this blow-up effect, and secure that a minimizer for the corresponding minimum energy problem does exist. This is shown to hold if we impose a proper upper constraint $\sigma = (\sigma^i)_{i \in I}$ on the vector measures in question (see Section 5 for a formulation of the constrained problem).

Having in mind a further extension of the theory, we formulate main definitions and prove auxiliary results for a general strictly positive definite kernel $\kappa$ on a locally compact space $X$ (Sections 2–5). The approach developed for these $\kappa$ and $X$ is mainly based on the simultaneous use of the vague topology and a suitable semimetric structure on a set $\mathcal{E}_\kappa^+(A)$ of all vector measures of finite energy associated with $A$ (see Section 3.3 for a definition of this semimetric structure). A key observation behind this approach is the fact that since a nonzero positive scalar measure of finite energy does not charge any set of zero capacity, there corresponds to every $\mu \in \mathcal{E}_\kappa^+(A)$ a scalar (signed) Radon measure $R\mu = \sum_{i \in I} s_i \mu^i$ on $X$, and the mapping $R$ preserves the energy (semi)metric (Theorem 3.9), i.e.

$$\|\mu_1 - \mu_2\|_{\mathcal{E}_\kappa^+(A)} = \|R\mu_1 - R\mu_2\|_{\mathcal{E}_\kappa(X)}.$$ 

Here $\mathcal{E}_\kappa(X)$ is the pre-Hilbert space of all scalar Radon measures on $X$ with finite energy. This implies that the semimetric on $\mathcal{E}_\kappa^+(A)$ is a metric if and only if any two equally signed plates intersect each other only in a set of zero capacity\footnote{See Lemma 3.6 and its proof providing an explanation of this phenomenon.} This approach extends that from [40]–[43] where the oppositely charged plates were assumed to be mutually disjoint.

Based on the convexity of the class of vector measures admissible for the problem in question, the isometry between $\mathcal{E}_\kappa(A)$ and its $R$-image, and the pre-Hilbert structure on the space $\mathcal{E}_\kappa(X)$, we analyze the uniqueness of solutions (Lemma 5.4). In view of the above observation, this solution is unique whenever any two equally signed plates intersect each other only in a set of zero capacity; otherwise any two solutions have equal $R$-images.

As for the vague topology, crucial to our arguments is Lemma 5.8 which asserts that if the $\sigma^i$, $i \in I$, are bounded then the admissible measures form a vaguely compact space. Intuitively this is clear since $\sigma^i(X) < \infty$ implies that $\sigma^i(U_\infty) < \varepsilon$ for any sufficiently small neighborhood $U_\infty$ of the point at infinity and $\varepsilon > 0$ small enough. Thus, under the vague convergence of a net of positive scalar measures $\mu^i_s \leq \sigma^i$, $s \in S$, to $\mu^i$ no part of the total mass $\mu^i_s(X)$ can disappear at infinity.
This general approach is further specified for the $\alpha$-Riesz kernels $\kappa_\alpha$ of order $\alpha \in (0, n)$ on $\mathbb{R}^n$. Due to the establishment of completeness results for proper semimetric subspaces of $\mathcal{E}^{+}_{\kappa_\alpha}(A)$ (Theorems 7.3 and 7.4), we have succeeded in working out a substantive theory for the constrained $\alpha$-Riesz minimum energy problems. The theory developed includes sufficient conditions for the existence of minimizers (see Theorems 6.1 and 7.1, the latter referring to a generalized condenser whose plates may be noncompact); sufficient conditions obtained in Theorem 7.1 are shown by Theorem 7.9 to be sharp. Theorems 6.1 and 7.1 are illustrated by Examples 6.3 and 7.2, respectively. We establish continuity of the minimizers in the vague and strong topologies when the condenser $A$ and the constraint $\sigma$ both vary (Theorem 8.1), and also describe the weighted vector potentials of the minimizers and specify their characteristic properties (Theorem 9.2). Finally, in Section 10 we provide a duality relation between non-weighted constrained and weighted unconstrained minimum $\alpha$-Riesz problems for a capacitor ($I = \{1\}$), thereby extending the logarithmic potential result of [15, Corollary 2.15], now for a closed (not necessarily compact) set. For this purpose we utilize the established characteristic properties of the solutions to such extremal problems (see Theorem 9.2 below and [42, Theorem 7.3]).

The results obtained in Sections 6 and 8–10 and the approach developed remain valid for the logarithmic kernel on $\mathbb{R}^2$ and $A$ with compact $A_i$, $i \in I$ (compare with [3]). However, in the case where at least one of the plates of a generalized condenser is noncompact, a refined analysis is still provided as yet only for the Riesz kernels. This is caused by the fact that the above-mentioned completeness results (Theorems 7.3 and 7.4), crucial to our investigation, are substantially based on the earlier result of the fifth named author [38, Theorem 1] which states that, in contrast to the fact that the pre-Hilbert space $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ is incomplete in the topology determined by the $\alpha$-Riesz energy norm [9], the metric subspace of all $\nu \in \mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ such that $\nu^\pm$ are supported by fixed closed disjoint sets $F_i$, $i = 1, 2$, respectively, is nevertheless complete. In turn, the quoted theorem has been established with the aid of Deny’s theorem [11] showing that $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ can be completed by making use of tempered distributions on $\mathbb{R}^n$ with finite energy, defined in terms of Fourier transforms, and this result by Deny seems not yet to have been extended to other classical kernels.

**Remark 1.1.** Regarding methods and approaches applied, assume for a moment that
\[
\kappa|_{A_i \times A_j} \leq M < \infty \text{ whenever } s_is_j = -1,
\]
$\kappa$ being a strictly positive definite kernel on a locally compact space $X$. (For the $\alpha$-Riesz kernels on $\mathbb{R}^n$, (1.1) holds if and only if oppositely signed plates have nonzero Euclidean distance.) If moreover $\kappa$ is perfect [17], then a fairly general theory of unconstrained minimum weighted energy problems over $\mu \in \mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ has been developed in [42, 43] (see Remark 4.5 below for a short survey). The approach developed in [42, 43] substantially used requirement (1.1), which made it possible to extend Cartan’s proof [9] of the strong completeness of the cone $\mathcal{E}^+_{\kappa_\alpha}(\mathbb{R}^n)$ of all positive measures on $\mathbb{R}^n$ with finite Newtonian energy to a perfect kernel $\kappa$ on a locally compact space $X$ and suitable classes of (signed) measures $\mu \in \mathcal{E}_{\kappa_\alpha}(X)$. Theorem 4.6 below, pertaining to the Newtonian kernel, shows that assumption (1.1) is essential not only for the proofs in [42, 43], but also for the validity of the
approach developed therein. Omitting now (1.1), in the present paper we have nevertheless succeeded in working out a substantive theory for the Riesz kernels by imposing instead an appropriate upper constraint on the vector measures under consideration.

While our investigation is focused on theoretical aspects in a very general context, and possible applications are so far outside the frames of the present paper, it is noteworthy to remark that minimum energy problems in the constrained and unconstrained settings for the logarithmic kernel on $\mathbb{R}^n$, also referred to as ‘vector equilibrium problems’, have been considered for several decades in relation to Hermite–Padé approximants [22, 1] and random matrix ensembles [20, 4]. See also [24, 35, 28] and the references therein.

2. Preliminaries

Let $X$ be a locally compact (Hausdorff) space, to be specified below, and $\mathcal{M}(X)$ the linear space of all real-valued scalar Radon measures $\mu$ on $X$, equipped with the vague topology, i.e. the topology of pointwise convergence on the class $C_0(X)$ of all real-valued continuous functions on $X$ with compact support. The vague topology on $\mathcal{M}(X)$ is Hausdorff; hence, a vague limit of any sequence (net) in $\mathcal{M}(X)$ is unique (whenever it exists). These and other notions and results from the theory of measures and integration on a locally compact space, to be used throughout the paper, can be found in [16, 7] (see also [17] for a short survey). We denote by $\mu^+$ and $\mu^-$ the positive and the negative parts in the Hahn–Jordan decomposition of a measure $\mu \in \mathcal{M}(X)$ and by $S_{\mu}^X = S(\mu)$ its support. A measure $\mu$ is said to be bounded if $|\mu|(X) < \infty$ where $|\mu| := \mu^+ + \mu^-$. Let $\mathcal{M}^+(X)$ stand for the (convex, vaguely closed) cone of all positive $\mu \in \mathcal{M}(X)$, and let $\Psi(X)$ consist of all lower semicontinuous (l.s.c.) functions $\psi : X \to (-\infty, \infty]$, nonnegative unless $X$ is compact. The following well known fact (see e.g. [17, Section 1.1]) will often be used.

Lemma 2.1. For any $\psi \in \Psi(X)$, $\mu \mapsto \langle \psi, \mu \rangle := \int \psi \, d\mu$ is vaguely l.s.c. on $\mathcal{M}^+(X)$.  

A kernel $\kappa(x, y)$ on $X$ is defined as a symmetric function from $\Psi(X \times X)$. Given $\mu, \mu_1 \in \mathcal{M}(X)$, we denote by $\kappa(\mu, \mu_1)$ and $\kappa(\cdot, \mu)$ the mutual energy and the potential relative to the kernel $\kappa$, respectively, i.e.

\[
\kappa(\mu, \mu_1) := \int \int \kappa(x, y) \, d\mu(x) \, d\mu_1(y), \\
\kappa(x, \mu) := \int \kappa(x, y) \, d\mu(y), \quad x \in X.
\]

\begin{itemize}
  \item[] When speaking of a continuous numerical function we understand that the values are finite real numbers.  
  \item[] Throughout the paper the integrals are understood as upper integrals [7].  
  \item[] When introducing notation of a numerical value, we assume the corresponding object on the right to be well defined (as a finite number or $\pm \infty$).
\end{itemize}
Observe that $\kappa(x, \mu)$ is well defined provided that $\kappa(x, \mu^+)$ or $\kappa(x, \mu^-)$ is finite, and then $\kappa(x, \mu) = \kappa(x, \mu^+) - \kappa(x, \mu^-)$. In particular, if $\mu \in \mathcal{M}^+(X)$ then $\kappa(\cdot, \mu)$ is defined everywhere and represents a l.s.c. function on $X$, bounded from below (see Lemma 2.1). Also note that $\kappa(\mu, \mu_1)$ is well defined and equal to $\kappa(\mu_1, \mu)$ provided that $\kappa(\mu^+ + \mu_1^+, \mu^- + \mu_1^-)$ or $\kappa(\mu^+, \mu_1^+) + \kappa(\mu^-, \mu_1^-)$ is finite.

For $\mu = \mu_1$ the mutual energy $\kappa(\mu, \mu_1)$ becomes the energy $\kappa(\mu, \mu)$. Let $\mathcal{E}_\kappa(X)$ consist of all $\mu \in \mathcal{M}(X)$ whose energy $\kappa(\mu, \mu)$ is finite, which by definition means that $\kappa(\mu^+, \mu^+)$, $\kappa(\mu^-, \mu^-)$ and $\kappa(\mu^+, \mu^-)$ are all finite, and let $\mathcal{E}_\kappa^+(X) := \mathcal{E}_\kappa(X) \cap \mathcal{M}^+(X)$.

For a set $Q \subset X$ let $\mathcal{M}^+(Q)$ consist of all $\mu \in \mathcal{M}^+(X)$ carried by $Q$, which means that $Q^c := X \setminus Q$ is locally $\mu$-negligible, or equivalently that $Q$ is $\mu$-measurable and $\mu = \mu|_Q$ where $\mu|_Q = 1_Q \cdot \mu$ is the trace (restriction) of $\mu$ on $Q$ [7, Chapter V, Section 5, Example]. (Here $1_Q$ denotes the indicator function of $Q$.) If $Q$ is closed, then $\mu$ is carried by $Q$ if and only if it is supported by $Q$, i.e. $S(\mu) \subset Q$. Also note that if either $X$ is countable at infinity (i.e. $X$ can be represented as a countable union of compact sets [5, Chapter I, Section 9, n° 9]), or $\mu$ is bounded, then the concept of local $\mu$-negligibility coincides with that of $\mu$-negligibility; and hence $\mu \in \mathcal{M}^+(Q)$ if and only if $\mu^*(Q^c) = 0$, $\mu^*(\cdot)$ being the outer measure of a set. Write $\mathcal{E}_\kappa^+(Q) := \mathcal{E}_\kappa(X) \cap \mathcal{M}^+(Q)$, $\mathcal{M}^+(Q, q) := \{\mu \in \mathcal{M}^+(Q) : \mu(Q) = q\}$ and $\mathcal{E}_\kappa^+(Q, q) := \mathcal{E}_\kappa(X) \cap \mathcal{M}^+(Q, q)$, where $q \in (0, \infty)$.

In the rest of this section and throughout Sections 3–5 a kernel $\kappa$ is assumed to be strictly positive definite, which means that the energy $\kappa(\mu, \mu)$, $\mu \in \mathcal{M}(X)$, is nonnegative whenever defined, and it equals 0 only for $\mu = 0$. Then $\mathcal{E}_\kappa(X)$ forms a pre-Hilbert space with the inner product $\kappa(\mu, \mu_1)$ and the energy norm $\|\mu\|_{\mathcal{E}_\kappa(X)} := \|\mu\|_\kappa := \sqrt{\kappa(\mu, \mu)}$ [17]. The (Hausdorff) topology on $\mathcal{E}_\kappa(X)$ defined by $\|\cdot\|_\kappa$ is termed strong.

In contrast to [18, 19] where a capacity has been treated as a functional acting on positive numerical functions on $X$, in the present study we use the (standard) concept of capacity as a set function. Thus the (inner) capacity of a set $Q \subset X$ relative to the kernel $\kappa$, denoted $c_\kappa(Q)$, is defined by

$$c_\kappa(Q) := \left[ \inf_{\mu \in \mathcal{E}_\kappa^+(Q, 1)} \kappa(\mu, \mu) \right]^{-1}$$

(see e.g. [17, 32]). Then $0 \leq c_\kappa(Q) \leq \infty$. (As usual, the infimum over the empty set is taken to be $+\infty$. We also set $1/(+\infty) = 0$ and $1/0 = +\infty$.)

An assertion $\mathcal{U}(x)$ involving a variable point $x \in X$ is said to hold $c_\kappa$-n.e. on $Q$ if $c_\kappa(N) = 0$ where $N$ consists of all $x \in Q$ for which $\mathcal{U}(x)$ fails to hold. We shall use the short form 'n.e.' instead of 'c_\kappa-n.e.' if this will not cause any misunderstanding.

**Definition 2.2.** Following [17], we call a (strictly positive definite) kernel $\kappa$ perfect if every strong Cauchy sequence in $\mathcal{E}_\kappa^+(X)$ converges strongly to any of its vague cluster points.\footnote{It follows from Theorem 2.4 that for a perfect kernel such a vague cluster point exists and is unique.}
Remark 2.3. On \( X = \mathbb{R}^n, \ n \geq 3 \), the Riesz kernel \( \kappa_{\alpha}(x, y) = |x - y|^{\alpha - n}, \ \alpha \in (0, n) \), is strictly positive definite and moreover perfect \([11, 12]\), and hence so is the Newtonian kernel \( \kappa_2(x, y) = |x - y|^{2 - n} [9]\). Recently it has been shown that if \( X = D \) where \( D \) is an arbitrary open set in \( \mathbb{R}^n, \ n \geq 3 \), and \( G^0_D, \ \alpha \in (0, 2] \), is the \( \alpha \)-Green kernel on \( D \) \([27\text{ Chapter IV, Section 5}]\), then \( \kappa = G^0_D \) is strictly positive definite and moreover perfect \([20\text{ Theorems 4.9, 4.11}]\). The restriction of the logarithmic kernel \( -\log |x - y| \) on \( \mathbb{R}^2 \) to the closed disk \( \overline{B}(0, r) := \{ \xi \in \mathbb{R}^2 : |\xi| \leq r < 1 \} \) is perfect as well\(6\).

Theorem 2.4 (see \([17]\]). If the kernel \( \kappa \) is perfect, then the cone \( \mathcal{E}_\kappa^+(X) \) is strongly complete and the strong topology on \( \mathcal{E}_\kappa^+(X) \) is finer than the (induced) vague topology on \( \mathcal{E}_\kappa^+(X) \).

Remark 2.5. When speaking of the vague topology, one has to consider nets or filters in \( \mathfrak{M}(X) \) instead of sequences since the vague topology in general does not satisfy the first axiom of countability. We follow Moore and Smith’s theory of convergence \([29]\), based on the concept of nets (see also \([25]\text{ Chapter 2}) \) and \([16]\text{ Chapter 0} \). However, if \( \varrho(\cdot, \cdot) \) denotes a metric on \( X \), then a countable base \( (\mathcal{V}_k)_{k \in \mathbb{N}} \) of vague neighborhoods of a measure \( \mu_0 \in \mathfrak{M}^+(X) \) can be obtained for example as follows after choosing a countable dense sequence \( \{x_k\}_{k \in \mathbb{N}} \) of points of \( X \):

\[
\mathcal{V}_k = \left\{ \mu \in \mathfrak{M}^+(X) : \int (1 - k \varrho(x_k, x))^+ \, d|\mu - \mu_0|(x) < 1/k \right\}.
\]

(The existence of such \( \{x_k\}_{k \in \mathbb{N}} \) for \( X \) in question is ensured by \([6\text{ Chapter IX, Section 2, n}^8 8, \text{ Proposition 12}] \) and \([6\text{ Chapter IX, Section 2, n}^8 9, \text{ Corollary to Proposition 16}] \).)

Remark 2.6. In contrast to Theorem 2.4, for a perfect kernel \( \kappa \) the whole pre-Hilbert space \( \mathcal{E}_\kappa(X) \) is in general strongly incomplete, and this is the case even for the \( \alpha \)-Riesz kernel of order \( \alpha \in (1, n) \) on \( \mathbb{R}^n, \ n \geq 3 \) (see \([9]\) and \([27\text{ Theorem 1.19}]\)). Compare with \([38\text{ Theorem 1}]\) where the strong completeness has been established for the metric subspace of all (signed) \( \nu \in \mathcal{E}_{\kappa_{\alpha}}(\mathbb{R}^n), \ \alpha \in (0, n) \), such that \( \nu^+ \) and \( \nu^- \) are supported by closed nonintersecting sets \( F_1, F_2 \subset \mathbb{R}^n \), respectively. This result from \([38]\) was proved with the aid of Deny’s theorem \([11]\) stating that \( \mathcal{E}_{\kappa_{\alpha}}(\mathbb{R}^n) \) can be completed by making use of tempered distributions on \( \mathbb{R}^n \) with finite energy, defined in terms of Fourier transforms.

Remark 2.7. The concept of perfect kernel is an efficient tool in minimum energy problems over classes of vector measures on \( X \) with finite energy. Indeed, if \( Q \subset X \) is closed, \( c_\kappa(Q) \in (0, +\infty) \), and \( \kappa \) is perfect, then the problem \((2.1)\) has a unique solution \( \lambda \) \([17\text{ Theorem 4.1}]\); we shall call such \( \lambda \) the (inner) \( \kappa \)-capacitary measure on \( Q \). Later the concept of perfectness has been shown to be efficient in minimum energy problems over classes of vector measures associated with a standard condenser \( A \) in \( X \) (see \([40]–[43]\), where \( \kappa \) and \( A \) were assumed to satisfy \((1.1)\)). See Remark 1.1 above for some details of the approach developed in \([40]–[43]\); compare with the above-mentioned \([38\text{ Theorem 1}]\).

\(6\)Indeed, the restriction of the logarithmic kernel to \( \overline{B}(0, r), \ r < 1, \) is strictly positive definite by \([27\text{ Theorem 1.16}]\). Since it satisfies Frostman’s maximum principle \([27\text{ Theorem 1.6}]\), it is regular according to \([32\text{ Eq. 1.3}]\), and hence perfect by \([31]\) (see also \([17\text{ Theorem 3.4.1}]\)).
where condition (1.1) has not been required. See also Remarks 4.5 and 5.3 below for a short survey of the results obtained in [40]–[43].

3. Vector measures associated with a generalized condenser

3.1. Generalized condensers. Fix a finite set $I$ of indices $i \in \mathbb{N}$ and an ordered collection $A := (A_i)_{i \in I}$ of nonempty closed sets $A_i \subset X$, $X$ being a locally compact space, where each $A_i$, $i \in I$, has the sign $s_i := \text{sign } A_i = \pm 1$ prescribed. Let $I^+$ consist of all $i \in I$ such that $s_i = +1$, and $I^- := I \setminus I^+$. The sets $A_i$, $i \in I^+$, and $A_j$, $j \in I^-$, are termed the positive and the negative plates of the collection $A$. Write

$$A^+ := \bigcup_{i \in I^+} A_i, \quad A^- := \bigcup_{j \in I^-} A_j, \quad A := A^+ \cup A^-, \quad \delta_A := A^+ \cap A^-.$$

**Definition 3.1.** $A = (A_i)_{i \in I}$ is said to be a standard condenser in $X$ if $\delta_A = \emptyset$.

Note that any two equally signed plates of a standard condenser may intersect each other over a set of nonzero capacity (or even coincide).

Fix a (strictly positive definite) kernel $\kappa$ on $X$. By relaxing in the above definition the requirement $\delta_A = \emptyset$, we slightly generalize the notion of standard condenser as follows.

**Definition 3.2.** $A = (A_i)_{i \in I}$ is said to be a generalized condenser in $X$ if $c_\kappa(\delta_A) = 0$.

A (generalized) condenser $A$ is said to be compact if all the $A_i$, $i \in I$, are compact, and noncompact if at least one of the $A_i$, $i \in I$, is noncompact.

In Examples 3.3 and 3.4 below, $X = \mathbb{R}^n$ with $n \geq 3$. Let $B(x, r)$, respectively $\overline{B}(x, r)$, denote the open, respectively closed, $n$-dimensional ball of radius $r$ centered at $x \in \mathbb{R}^n$. We shall also write $S(x, r) := \partial_{\mathbb{R}^n} B(x, r)$.

**Example 3.3.** Let $I^+ := \{1\}$ and $I^- := \{2, 3, 4\}$. Define $A_1 := \overline{B}(\xi_1, 1)$, $A_2 := \overline{B}(\xi_2, 1)$, $A_3 := \overline{B}(\xi_3, 2)$ and $A_4 := \overline{B}(\xi_4, 1)$ where $\xi_1 = (0, 0, \ldots, 0)$, $\xi_2 = (2, 0, \ldots, 0)$, $\xi_3 = (3, 0, \ldots, 0)$ and $\xi_4 = (-2, 0, \ldots, 0)$. Since $\delta_A$ consists of the points $\xi_5 = (-1, 0, \ldots, 0)$ and $\xi_6 = (1, 0, \ldots, 0)$, $A = (A_i)_{i \in I}$ forms a generalized condenser in $\mathbb{R}^n$ for any (strictly positive definite) kernel $\kappa$ on $\mathbb{R}^n$ with the property that $\kappa(x, y) = \infty$ whenever $x = y$.

See Example 6.3 for kernels and constraints under which the constrained minimum energy problem (Problem 5.1) for such a condenser admits a solution (has no short-circuit) despite the two touching points for the oppositely charged plates.

**Example 3.4.** Assume that $n = 3$, $I^+ := \{1\}$, $I^- := \{2\}$, and let

$$A_1 := \{x \in \mathbb{R}^3 : 1 \leq x_1 < \infty, \quad x_2^2 + x_3^2 = \exp(-2x_1^2)\},$$

$$A_2 := \{x \in \mathbb{R}^3 : 2 \leq x_1 < \infty, \quad x_2^2 + x_3^2 = \exp(-2x_1^2)\}.$$
where \( 1 < r_1 < r_2 < \infty \). Then \( A_1 \) and \( A_2 \) form a standard condenser in \( \mathbb{R}^3 \) such that

\[
\text{dist} (A_1, A_2) := \inf_{x \in A_1, \ y \in A_2} |x - y| = 0.
\]

See Example 7.2 for a kernel and constraints under which the constrained minimum energy problem (Problem 5.1) for such a condenser admits a solution (has no short-circuit) despite the touching point at infinity.

3.2. Vector measures associated with a condenser \( A \). Vague topology. In the rest of the paper, fix a generalized condenser \( A = (A_i)_{i \in I} \) in \( X \), and let \( \mathcal{M}^+(A) \) stand
for the Cartesian product $\prod_{i \in I} \mathcal{M}^+(A_i)$. Then $\mu \in \mathcal{M}^+(A)$ is a positive vector measure $(\mu^i)_{i \in I}$ with the components $\mu^i \in \mathcal{M}^+(A_i)$; such $\mu$ is said to be associated with $A$.

**Definition 3.5.** The vague topology on $\mathcal{M}^+(A)$ is the topology of the product space $\prod_{i \in I} \mathcal{M}^+(A_i)$ where each of the factors $\mathcal{M}^+(A_i)$, $i \in I$, is endowed with the vague topology induced from $\mathcal{M}(X)$. Namely, a net $(\mu_s)_{s \in S} \subset \mathcal{M}^+(A)$ converges to $\mu$ vaguely if for every $i \in I$, $\mu^i_s \to \mu^i$ vaguely in $\mathcal{M}(X)$ when $s$ increases along $S$.

As all the $A_i$, $i \in I$, are closed in $X$, $\mathcal{M}^+(A)$ is vaguely closed in $\mathcal{M}^+(X)^{|I|}$ where $|I| := \text{Card} \, I$. Furthermore, since every $\mathcal{M}^+(A_i)$ is Hausdorff in the vague topology, so is $\mathcal{M}^+(A)$ [25, Chapter 3, Theorem 5]. Hence, a vague limit of any net in $\mathcal{M}^+(A)$ belongs to $\mathcal{M}^+(A)$ and is unique (provided that the vague limit exists).

Given $\mu \in \mathcal{M}^+(A)$ and a vector-valued function $u = (u_i)_{i \in I}$ with $u_i : X \to [-\infty, \infty]$ such that each $\int u_i \, d\mu^i$ as well as their sum over $i$ exist (as finite numbers or $\pm \infty$), write

$$
\langle u, \mu \rangle := \sum_{i \in I} \langle u_i, \mu^i \rangle = \sum_{i \in I} \int u_i \, d\mu^i.
$$

Let $\mathcal{E}^+_\kappa(A)$ consist of all $\mu \in \mathcal{M}^+(A)$ such that $\kappa(\mu^i, \mu^j) < \infty$ for all $i \in I$; in other words, $\mathcal{E}^+_\kappa(A) := \prod_{i \in I} \mathcal{E}^+_\kappa(A_i)$. For any $\mu \in \mathcal{E}^+_\kappa(A)$, $\mu^i(\delta_A) = 0$ by [17, Lemma 2.3.1]. Hence each of the $i$-components $\mu^i$, $i \in I$, of $\mu \in \mathcal{E}^+_\kappa(A)$ is carried by $A^+_i$, where

$$
A^+_i := A_i \setminus \delta_A,
$$

though the support of $\mu^i$ may coincide with the whole $A_i$. We thus actually have

$$
\mathcal{E}^+_\kappa(A) = \prod_{i \in I} \mathcal{E}^+_\kappa(A^+_i).
$$

Write $A^+ := A^+ \setminus \delta_A$ and $A^- := A^- \setminus \delta_A$. For any $\mu \in \mathcal{E}^+_\kappa(A)$ define

$$
R\mu := R_A \mu := \sum_{i \in I} s_i \mu^i,
$$

the 'resultant' of $\mu$. Since $A^+_o \cap A^-_o = \emptyset$, $R\mu$ is a (signed) scalar Radon measure on $X$ whose positive and negative parts, carried respectively by $A^+_o$ and $A^-_o$, are given by

$$
(R\mu)^+ := \sum_{i \in I^+} \mu^i \quad \text{and} \quad (R\mu)^- := \sum_{i \in I^-} \mu^i.
$$

If $\mu = \mu_1$ where $\mu, \mu_1 \in \mathcal{E}^+_\kappa(A)$, then $R\mu = R\mu_1$, but not the other way around.

**Lemma 3.6.** For the mapping $\mu \mapsto R\mu$ to be injective it is necessary and sufficient that all the $A_i$, $i \in I$, be mutually essentially disjoint, i.e. with $c_{\kappa}(A_i \cap A_j) = 0$ for all $i \neq j$. 
Lemma 3.7. For any measurable sets \([17, \text{Lemma 2.3.5}]\), this proves the lemma. □

Proof. Since a nonzero positive scalar measure of finite energy does not charge any set of zero capacity \([17]\) Lemma 2.3.1, the sufficiency part of the lemma is obvious. To prove the necessity part, assume on the contrary that there are two equally signed plates \(A_k\) and \(A_\ell\), \(k \neq \ell\), with \(c_\kappa(A_k \cap A_\ell) > 0\). By \([17]\) Lemma 2.3.1, there exists a nonzero positive scalar measure \(\tau \in \mathcal{E}_\kappa^+(A_k \cap A_\ell)\). Choose \(\mu = (\mu^i)_{i \in I} \in \mathcal{E}_\kappa^+(A)\) such that \(\mu^k|_{A_k \cap A_\ell} - \tau \geq 0\), and define \(\mu_m = (\mu^i_m)_{i \in I} \in \mathcal{E}_\kappa^+(A)\), \(m = 1, 2\), where \(\mu^i_1 = \mu^k - \tau\) and \(\mu^i_2 = \mu^i\) for all \(i \neq k\), while \(\mu^i_2 = \mu^i + \tau\) and \(\mu^i_2 = \mu^i\) for all \(i \neq l\). Then \(R\mu_1 = R\mu_2\), but \(\mu_1 \neq \mu_2\). □

We call \(\mu, \mu_1 \in \mathcal{E}_\kappa^+(A)\) \(R\)-equivalent if \(R\mu = R\mu_1\). For \(\mu \in \mathcal{E}_\kappa^+(A)\), let \([\mu]\) consist of all \(\mu_1 \in \mathcal{E}_\kappa^+(A)\) that are \(R\)-equivalent to \(\mu\). Note that \(\mu = 0\) is the only element of \([0]\).

3.3. A semimetric structure on classes of vector measures. To avoid trivialities, for a given (generalized) condenser \(A = (A_i)_{i \in I}\) and a given (strictly positive definite) kernel \(\kappa\) on a locally compact space \(X\) we shall always require that

\[
(3.4) \quad c_\kappa(A_i) > 0 \quad \text{for all} \quad i \in I.
\]

In accordance with an electrostatic interpretation of a condenser, we say that the interaction between the components \(\mu^i, i \in I, \mu \in \mathcal{E}_\kappa^+(A)\) is characterized by the matrix \((s_is_j)_{i,j \in I}\), where \(s_i := \text{sign} A_i\). Given \(\mu, \mu_1 \in \mathcal{E}_\kappa^+(A)\), we define the mutual energy

\[
(3.5) \quad \kappa(\mu, \mu_1) := \sum_{i,j \in I} s_is_j\kappa(\mu^i, \mu^j_1)
\]

and the vector potential \(\kappa^{\mu}_i(\cdot)\) as a vector-valued function on \(X\) with the components

\[
(3.6) \quad \kappa^{\mu}_i(\cdot) := \sum_{j \in I} s_is_j\kappa(\cdot, \mu^j), \quad i \in I.
\]

Lemma 3.7. For any \(\mu \in \mathcal{E}_\kappa^+(A)\), the \(\kappa^{\mu}_i(\cdot), i \in I\), are well defined and finite n.e. on \(X\).

Proof. Since \(\mu^i \in \mathcal{E}_\kappa^+(X)\) for every \(i \in I\), \(\kappa(\cdot, \mu^i)\) is finite n.e. on \(X\) \([17], \text{p. 164}\). Furthermore, the set of all \(x \in X\) with \(\kappa(x, \mu^i) = \infty\) is universally measurable, for \(\kappa(\cdot, \mu^i)\) is l.s.c. on \(X\). Combined with the fact that the inner capacity \(c_\kappa(\cdot)\) is subadditive on universally measurable sets \([17], \text{Lemma 2.3.5}\), this proves the lemma. □

Lemma 3.8. For any \(\mu, \mu_1 \in \mathcal{E}_\kappa^+(A)\) we have

\[
(3.7) \quad \kappa(\mu, \mu_1) = \kappa(R\mu, R\mu_1) \in (-\infty, \infty).
\]

Proof. This is obtained directly from relations \((3.3)\) and \((3.5)\). □

For \(\mu = \mu_1 \in \mathcal{E}_\kappa^+(A)\) the mutual energy \(\kappa(\mu, \mu_1)\) becomes the energy \(\kappa(\mu, \mu)\) of \(\mu\). By the strict positive definiteness of the kernel \(\kappa\), we see from Lemma \(3.8\) that \(\kappa(\mu, \mu), \mu \in \mathcal{E}_\kappa^+(A),\) is always \(\geq 0\), and it is zero only for \(\mu = 0\).
In order to introduce a (semi)metric structure on the cone $E_+^\kappa(A)$, we define
\begin{equation}
\|\mu - \mu_1\|_{E_+^\kappa(A)} := \|R\mu - R\mu_1\|_\kappa \quad \text{for all } \mu, \mu_1 \in E_+^\kappa(A).
\end{equation}
Based on (3.7), we see by straightforward calculation that, in fact,
\begin{equation}
\|\mu - \mu_1\|^2_{E_+^\kappa(A)} = \sum_{i,j \in I} s_is_j\kappa(\mu^i - \mu_1^i, \mu^j - \mu_1^j).
\end{equation}
On account of Lemma 3.6, we are thus led to the following conclusion.

**Theorem 3.9.** $E_+^\kappa(A)$ is a semimetric space with the semimetric defined by either of the (equivalent) relations (3.8) or (3.9), and this space is isometric to its R-image in $E_+(X)$. The semimetric $\|\mu - \mu_1\|_{E_+^\kappa(A)}$ is a metric on $E_+^\kappa(A)$ if and only if all the $A_i$, $i \in I$, are mutually essentially disjoint.

Similar to the terminology in the pre-Hilbert space $E_+(X)$, we therefore call the topology of the semimetric space $E_+^\kappa(A)$ strong. Now [33, Theorem 1 and Corollary 1], mentioned in Remark 2.7 above, can be rewritten as follows.

**Theorem 3.10.** If $A = (A_1, A_2)$ is a standard condenser in $\mathbb{R}^n$, $n \geq 3$, with $s_1s_2 = -1$ and $\kappa_\alpha$ is the Riesz kernel of an arbitrary order $\alpha \in (0, n)$, then the metric space $E_+^{\kappa_\alpha}(A)$ is strongly complete, and the strong topology on this space is finer than the vague topology.

Note that, under the assumptions of Theorem 3.10, $\kappa_\alpha$ may be unbounded on $A_1 \times A_2$ (compare with Remarks 1.1, 2.7, 4.5 and 5.3).

4. **Unconstrained f-weighted minimum energy problem**

For a (strictly positive definite) kernel $\kappa$ on $X$ and a (generalized) condenser $A = (A_i)_{i \in I}$, we shall consider minimum energy problems with an external field over certain subclasses of $E_+^\kappa(A)$, to be defined below. Since the admissible measures in those problems are of finite energy, there is no loss of generality in assuming that each $A_i$ coincides with its $\kappa$-reduced kernel [27, p. 164], which consists of all $x \in A_i$ such that $c_\kappa(A_i \cap U_x) > 0$ for every neighborhood $U_x$ of $x$ in $X$.

Fix a vector-valued function $f = (f_i)_{i \in I}$, where each $f_i : X \to [-\infty, \infty]$ is $\mu$-measurable for every $\mu \in M^+(X)$ and treated as an external field acting on the charges (measures) from $E_+^\kappa(A_i)$. The $f$-weighted vector potential and the $f$-weighted energy of $\mu \in E_+^\kappa(A)$ are defined respectively by
\begin{align}
W_{\kappa,f}^\mu := \kappa_\mu + f, \\
G_{\kappa,f}(\mu) := \kappa(\mu, \mu) + 2\langle f, \mu \rangle.
\end{align}
Let $E_+^{\kappa,f}(A)$ consist of all $\mu \in E_+^\kappa(A)$ with finite $G_{\kappa,f}(\mu)$ (equivalently, with finite $\langle f, \mu \rangle$).

$^8$G$\kappa,f(\cdot)$ is also known as the Gauss functional (see e.g. [32]). Note that when defining G$\kappa,f(\cdot)$, we have used the notation (3.1).
In this paper we shall tacitly assume that either Case I or Case II holds, where

I. For every $i \in I$, $f_i \in \Psi(X)$;

II. For every $i \in I$, $f_i(x) = s_i\kappa(x, \zeta)$ where a (signed) measure $\zeta \in \mathcal{E}_\kappa(X)$ is given.

For any $\mu \in \mathcal{E}_\kappa^+(A)$, $G_{\kappa, f}(\mu)$ is then well defined. Furthermore, if Case II takes place then, by (3.3), (3.7) and (4.2),

$$G_{\kappa, f}(\mu) = \|R\mu\|_{\kappa}^2 + 2 \sum_{i \in I} s_i\kappa(\zeta, \mu^i)$$

and consequently

$$-\infty < -\|\zeta\|_{\kappa}^2 \leq G_{\kappa, f}(\mu) < \infty \text{ for all } \mu \in \mathcal{E}_\kappa^+(A).$$

Also fix a numerical vector $a = (a_i)_{i \in I}$ with $a_i > 0$ and a vector-valued function $g = (g_i)_{i \in I}$ where all the $g_i : X \to (0, \infty)$ are continuous and such that

$$g_{i, \inf} := \inf_{x \in X} g_i(x) > 0.$$

Write

$$\mathfrak{M}^+(A, a, g) := \{ \mu \in \mathfrak{M}^+(A) : \langle g_i, \mu_i \rangle = a_i \text{ for all } i \in I \},$$

$$\mathcal{E}_\kappa^+(A, a, g) := \mathcal{E}_\kappa^+(A) \cap \mathfrak{M}^+(A, a, g).$$

Because of (4.5), we thus have

$$\mu^i(A_i) \leq a_i g_{i, \inf}^{-1} < \infty \text{ for all } \mu \in \mathfrak{M}^+(A, a, g).$$

Since any $\psi \in \Psi(X)$ is lower bounded if $X$ is compact, and it is $\geq 0$ otherwise, we conclude in Case I from (4.6) that there is $M_f \in (0, \infty)$ such that

$$G_{\kappa, f}(\mu) \geq -M_f > -\infty \text{ for all } \mu \in \mathcal{E}_\kappa^+(A, a, g).$$

Also denote

$$\mathcal{E}_{\kappa, f}^+(A, a, g) := \mathcal{E}_\kappa^+(A, a, g) \cap \mathfrak{M}^+(A, a, g),$$

$$G_{\kappa, f}(A, a, g) := \inf_{\mu \in \mathcal{E}_{\kappa, f}^+(A, a, g)} G_{\kappa, f}(\mu).$$

In either Case I or Case II, we then get from (4.4) and (4.7)

$$G_{\kappa, f}(A, a, g) > -\infty.$$

If the class $\mathcal{E}_{\kappa, f}^+(A, a, g)$ is nonempty, or equivalently if

$$G_{\kappa, f}(A, a, g) < \infty,$$

then the following (unconstrained) $f$-weighted minimum energy problem, also known as the Gauss variational problem [21, 32], makes sense.

---

9The notation $\Psi(X)$ has been introduced at the end of the first paragraph in Section 2.
**Problem 4.1.** Does there exist $\lambda_A \in \mathcal{E}^+_{\kappa,f}(A, a, g)$ with $G_{\kappa,f}(\lambda_A) = G_{\kappa,f}(A, a, g)$?

If $I^+ = \{1\}$, $I^- = \emptyset$, $g = 1$, $a = 1$ and $f = 0$, then Problem 4.1 reduces to the minimum energy problem (2.1) solved by [17, Theorem 4.1] (see Remark 2.7 above).

**Remark 4.2.** An analysis similar to that for a standard condenser, cf. [42, Lemma 6.2], shows that requirement (4.9) is fulfilled if and only if $c_\kappa(A_\delta^i) > 0$ for every $i \in I$, where $A_\delta^i$ being defined by (3.2). By (3.4), this yields that (4.9) holds automatically whenever Case II takes place, for the potential of $\zeta \in \mathcal{E}_\kappa(X)$ is finite n.e. on $X$. [17, p. 164].

**Remark 4.3.** If $A$ is a compact standard condenser, the kernel $\kappa$ is continuous on $A^+ \times A^-$, and Case I holds, then the solvability of Problem 4.1 can easily be established by exploiting the vague topology only, since then $\mathcal{M}^+(A, a, g)$ is vaguely compact, while $G_{\kappa,f}(\cdot)$ is vaguely l.s.c. on $\mathcal{E}^+_{\kappa,f}(A)$ (see [32, Theorem 2.30]). However, these arguments break down if any of the $A_i$ is noncompact in $X$, for then $\mathcal{M}^+(A, a, g)$ is no longer vaguely compact.

The purpose of the example below is to give an explicit formula for a solution to Problem 4.1 with a particular choice of $X$, $\kappa$, $A$, $a$, $g$, and $f$. Write $S_r := S(0, r)$.

**Example 4.4.** Let $\kappa_2(x, y) = |x - y|^{2-n}$ be the Newtonian kernel on $\mathbb{R}^n$ with $n \geq 3$, $I^+ = \{1\}$, $I^- = \{2\}$, $g = 1$, $a = 1$, $f = 0$, $A_1 = S_{r_1}$, and $A_2 = S_{r_2}$, where $0 < r_1 < r_2 < \infty$. According to Remark 4.3, a solution to Problem 4.1 exists. Let $\lambda_r$, $0 < r < \infty$, denote the $\kappa_2$-capacitary measure on $S_r$, then by symmetry $\lambda_r$ is the uniformly distributed unit mass over $S_r$. Based on well known properties of the Newtonian potential of $\lambda_r$ [27, Chapter II, Section 3, n° 13] we have

$$c_{\kappa_2}(S_r) = r^{n-2},$$

$\kappa_2(x, \lambda_r) = r^{2-n}$ for all $x \in \overline{B}(0, r)$ and $\kappa_2(x, \lambda_r) = R^{2-n}$ for all $x \in S_R$, $R > r$. Thus

$$\kappa_2(\cdot, \lambda_{r_1} - \lambda_{r_2}) = \begin{cases} r_1^{2-n} - r_2^{2-n} & \text{on } S_{r_1} , \\ 0 & \text{on } S_{r_2} . \end{cases}$$

Application of [39, Proposition 1(iv)], providing characteristic properties of solutions to Problem 4.1 for a standard condenser, shows that $\lambda := (\lambda_{r_1}, \lambda_{r_2})$ solves Problem 4.1 with $X$, $\kappa$, $A$, $a$, $g$, and $f$, chosen above. Hence the corresponding minimum value $G_{\kappa,f}(A, a, g)$ equals $\kappa_2(\lambda, \lambda)$ and

$$\kappa_2(\lambda, \lambda) = \|\lambda_{r_1} - \lambda_{r_2}\|_{\kappa_2}^2 = r_1^{2-n} - r_2^{2-n}.$$
Remark 4.5. Assume that $A$ is still a standard condenser, though now, in contrast to Remark 4.3 its plates may be noncompact in $X$. Under the assumption (1.1), an approach has been worked out in [42, 43], based on both the vague and the strong topologies on $\mathcal{E}_κ^+(A)$, which made it possible to provide a fairly complete analysis of Problem 4.1. In more detail, it has been shown that if the kernel $κ$ is perfect and all the $g_i|_{A_i}, i \in I,$ are bounded, then in either Case I or Case II the requirement

$$c_κ(A) < \infty$$

is sufficient for Problem 4.1 to be solvable for every vector $a$ [42, Theorem 8.1]. However, if (4.14) does not hold then in general there exists a vector $a'$ such that the problem admits no solution [42, 12]. Therefore, it was interesting to give a description of the set of all vectors $a$ for which Problem 4.1 is nevertheless solvable. Such a characterization has been established in [43]. See also footnote 14 below.

Unless explicitly stated otherwise, in all that follows we do not assume (1.1) to hold. Then the results obtained in [42, 43] and the approach developed are no longer valid. In particular, assumption (4.14) does not guarantee anymore that $G_{κ,f}(A,a,g)$ is attained among $μ \in \mathcal{E}_{κ,f}(A,a,g)$. This can be illustrated by the following assertion.

Theorem 4.6. Let $X = \mathbb{R}^n$ with $n \geq 3, I^+ = \{1\}, I^- = \{2\}, g = 1, a = 1, f = 0,$

$$A_1 = \bigcup_{k \geq 2} S(x_k, r_{1,k}), \quad A_2 = \bigcup_{k \geq 2} S(x_k, r_{2,k}),$$

where $x_k = (k,0,\ldots,0)$, $r_{2,k}^2 = k^2$ and $r_{1,k}^{2-n} = k^2 + k^{-q}$ with $q \in (0, \infty)$, and let $κ = κ_2$ be the Newtonian kernel. Then $G_{κ,f}(A,a,g)$ equals 0 and hence cannot be an actual minimum.

Proof. Note that $A = (A_1, A_2)$ forms a standard condenser in $\mathbb{R}^n$ such that (1.1) fails to hold. Let $λ_{k,r}, 0 < r < \infty,$ denote the $κ_2$-capacitary measure on $S(x_k, r)$ (see Remark 2.7).

In the case of the $α$-Riesz kernels of order $1 < α \leq 2$ on $\mathbb{R}^3$, some of the (theoretical) results on the solvability or unsolvability of Problem 4.1 obtained in [42] have been illustrated in [23, 30] by means of numerical experiments.
Then \( \lambda_k = (\lambda_{k,1}, \lambda_{k,2}) \), \( k \in \mathbb{N} \), are admissible for Problem [1.1] and therefore, by (3.7),
\[ 0 \leq G_{\kappa_2}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \leq \kappa_2(\lambda_k, \lambda_k) = \|\lambda_{k,1} - \lambda_{k,2}\|^2_{k_2}, \ k \in \mathbb{N}. \]
According to (4.13), the right-hand side equals
\[ r_{1,k}^{2-n} - r_{2,k}^{2-n} = k^{-q}, \]
and hence it tends to 0 as \( k \to \infty \). This yields that \( G_{\kappa_2}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = 0 \). By the strict positive definiteness of \( \kappa_2 \), \( G_{\kappa_2}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \) cannot therefore be an actual minimum, though \( c_{\kappa_2}(A) < \infty \), which is clear from (4.11) by the countable subadditivity of the inner capacity on the universally measurable sets [17, Lemma 2.3.5].

Using the electrostatic interpretation, which is possible for the Coulomb kernel \(|x - y|^{-1}\) on \( \mathbb{R}^3 \), we say that a short-circuit occurs between the oppositely charged plates \( A_1 \) and \( A_2 \) from Theorem [4.6] which touch each other at the point at infinity. This certainly may also happen for a generalized condenser (see Definition 3.2). Therefore, it is meaningful to ask what kinds of additional requirements on the vector measures under consideration will prevent this phenomenon, and secure that a solution to the corresponding \( f \)-weighted minimum energy problem does exist. The idea below is to impose such an upper constraint on the measures from \( \mathcal{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) \) which would prevent the blow-up effect.

5. Constrained \( f \)-weighted minimum energy problem

5.1. Statement of the problem. Unless stated otherwise, in all that follows \( \kappa, \mathbf{A}, \mathbf{a}, \mathbf{g}, \) and \( f \) are as indicated at the beginning of the preceding section. Let \( \mathcal{C}(\mathbf{A}) \) consist of all \( \sigma = (\sigma^i)_{i \in I} \in \mathcal{M}^+(\mathbf{A}) \) such that \( \langle g_i, \sigma^i \rangle > a_i \) and\(^{13}\)
\[ S_X^\sigma = A_i \text{ for all } i \in I. \]
These \( \sigma \) will serve as constraints for \( \mu \in \mathcal{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) \). Given \( \sigma \in \mathcal{C}(\mathbf{A}) \), write
\[ \mathcal{M}^\sigma(A) := \{ \mu \in \mathcal{M}^+(A) : \mu^i \leq \sigma^i \text{ for all } i \in I \}, \]
where \( \mu^i \leq \sigma^i \) means that \( \sigma^i - \mu^i \) is a positive scalar Radon measure on \( X \), and
\[ \mathcal{M}^\sigma(A, \mathbf{a}, \mathbf{g}) := \mathcal{M}^\sigma(A) \cap \mathcal{M}^+(A, \mathbf{a}, \mathbf{g}), \]
\[ \mathcal{E}_{\kappa, f}^\sigma(A, \mathbf{a}, \mathbf{g}) := \mathcal{E}_{\kappa, f}^+(A) \cap \mathcal{M}^\sigma(A, \mathbf{a}, \mathbf{g}). \]
Note that \( \mathcal{M}^\sigma(A) \) along with \( \mathcal{M}^+(A) \) is vaguely closed, for so is \( \mathcal{M}^+(X) \).

Since \( \mathcal{E}_{\kappa, f}^\sigma(A, \mathbf{a}, \mathbf{g}) \subset \mathcal{E}_{\kappa, f}^+(A, \mathbf{a}, \mathbf{g}) \), we get from (4.8)
\[ -\infty < G_{\kappa, f}(A, \mathbf{a}, \mathbf{g}) \leq G_{\kappa, f}^\sigma(A, \mathbf{a}, \mathbf{g}) := \inf_{\mu \in \mathcal{E}_{\kappa, f}^+(A, \mathbf{a}, \mathbf{g})} G_{\kappa, f}(\mu) \leq \infty. \]

\(^{13}\)Recall that \( S(\mu) = S_X^\mu \) denotes the support of \( \mu \in \mathcal{M}(X) \). For the notation \( \langle g_i, \sigma^i \rangle \) see [3.1], noting that the \( g_i, i \in I \), are continuous on \( X \) and > 0.
Unless explicitly stated otherwise, in all that follows we assume that the class $E^\sigma_{\kappa,f}(A, a, g)$ is nonempty, or equivalently

$$G^\sigma_{\kappa,f}(A, a, g) < \infty. \tag{5.3}$$

Then the following constrained $f$-weighted minimum energy problem, also known as the constrained Gauss variational problem, makes sense.

**Problem 5.1.** Given $\sigma \in \mathcal{C}(A)$, does there exist $\lambda^\sigma_A \in E^\sigma_{\kappa,f}(A, a, g)$ with

$$G_{\kappa,f}(\lambda^\sigma_A) = G^\sigma_{\kappa,f}(A, a, g)?$$

Note that assumption [5.1] in fact causes no restriction on the objects in question since, if it does not hold, then Problem [5.1] reduces to the same problem for the generalized condenser $F := (S_X^i)_{i \in I}$ in place of $A = (A_i)_{i \in I}$, because $E^\sigma_{\kappa,f}(F, a, g) = E^\sigma_{\kappa,f}(A, a, g)$.

**Lemma 5.2.** For (5.3) to hold, it is sufficient that for every $i \in I$, $\langle g_i, \sigma|_{\hat{A}^\delta_i} \rangle > a_i$ where $\hat{A}^\delta_i$ is defined by (4.10), and in addition $\kappa(\sigma^i|_{K}, \sigma^i|_{K}) < \infty$ for any compact $K \subset \hat{A}^\delta_i$.

**Proof.** Noting that the $\hat{A}^\delta_i$, $i \in I$, are universally measurable, we see that for every $i \in I$ there exists a compact set $K_i \subset \hat{A}^\delta_i$ such that $\langle g_i, \sigma|_{K_i} \rangle > a_i$ and $|f_i| \leq M_i < \infty$ on $K_i$. Then $\mu \in E^\sigma_{\kappa,f}(A, a, g)$ with $\mu^i := a_i \sigma^i|_{K_i}/\langle g_i, \sigma^i|_{K_i} \rangle$, $i \in I$, and (5.3) follows.

**Remark 5.3.** Assume for a moment that (1.1) holds. It has been shown by [41] Theorem 6.2 that if, in addition, the kernel $\kappa$ is perfect, all the $g_i|_{A_i}$, $i \in I$, are bounded, and condition (4.14) is satisfied, then in both Cases I and II Problem 5.1 is solvable for any vector $a$ and any constraint $\sigma \in \mathcal{C}(A)$. But if requirement (1.1) is omitted then the approach developed in [41] breaks down, and the quoted result is no longer valid. This can be seen for $\kappa_2$ on $\mathbb{R}^n$, $n \geq 3$, if we restrict ourselves to $\mu \in E^\sigma_{\kappa_2,f}(A, a, g)$ with $\sigma^i := \sum_{k \in \mathbb{N}} \lambda_{k,r_i,k}$, $i = 1, 2$, where $A$, $a$, $g$, $f$ and $\lambda_{k,r_i,k}$ are as chosen in Theorem 4.6 and its proof. Observe that these $\sigma^i$, $i = 1, 2$, are unbounded; compare with Theorem 7.1 below.

Let $E^\sigma_{\kappa,f}(A, a, g)$ (possibly empty) consist of all the solutions to Problem 5.1.

**Lemma 5.4.** If $\lambda$ and $\tilde{\lambda}$ are two elements of $E^\sigma_{\kappa,f}(A, a, g)$ then $\|\lambda - \tilde{\lambda}\|_{E^\sigma_{\kappa}(A)} = 0$, and hence $\lambda$ and $\tilde{\lambda}$ are $R$-equivalent in $E^\sigma_{\kappa}(A)$. Thus a solution to Problem 5.1 is unique (provided it exists) whenever all the $A_i$, $i \in I$, are mutually essentially disjoint.

**Proof.** Since the class $E^\sigma_{\kappa,f}(A, a, g)$ is convex, we conclude from (4.14) and (3.7) that

$$4G^\sigma_{\kappa,f}(A, a, g) \leq G_{\kappa,f}(\lambda + \tilde{\lambda}) = \|R\lambda + R\tilde{\lambda}\|^2_{\kappa} + 4(f, \lambda + \tilde{\lambda}).$$

Actually, the results described in Remarks 4.5 and 5.3 have been obtained in [41]–[43] even for infinite dimensional vector measures.
On the other hand, applying the parallelogram identity in the pre-Hilbert space $\mathcal{E}_\kappa(X)$ to $R\lambda$ and $R\tilde{\lambda}$ and then adding and subtracting $4\langle f, \lambda + \tilde{\lambda} \rangle$ we get
\[ \|R\lambda - R\tilde{\lambda}\|_\kappa^2 = -\|R\lambda + R\tilde{\lambda}\|_\kappa^2 - 4\langle f, \lambda + \tilde{\lambda} \rangle + 2G_{\kappa, f}(\lambda) + 2G_{\kappa, f}(\tilde{\lambda}). \]
When combined with the preceding relation, this yields
\[ 0 \leq \|R\lambda - R\tilde{\lambda}\|_\kappa^2 \leq -4G_{\kappa, f}(\lambda, a, g) + 2G_{\kappa, f}(\lambda) + 2G_{\kappa, f}(\tilde{\lambda}) = 0, \]
which establishes the former assertion of the lemma because of \((3.8)\). On account of Lemma \([3.6]\) this completes the proof.

The following example shows that in general a solution to Problem \([5.1]\) is not unique.

**Example 5.5.** Let $\kappa_2$ be the Newtonian kernel on $\mathbb{R}^n$ with $n \geq 3$, $I = I^+ = \{1, 2\}$, $a = 1$, $f = 0$, $g = 1$, and let $A_1 = A_2$ be the $(n - 1)$-dimensional unit sphere $S := S(0, 1)$. Let $\lambda$ denote the $\kappa_2$-capacitary measure on $S$; then $S(\lambda) = S$. Choose $\sigma^1 = \sigma^2 = 3\lambda$. Then $\lambda = (\lambda, \lambda)$ is obviously a solution to Problem \([5.1]\). Choose compact disjoint sets $K_k \subset S$, $k = 1, 2$, so that $c_2(K_1) = c_2(K_2) > 0$ and define $\lambda_k = \lambda|K_k/2$. Then $\lambda_k = (\lambda - \lambda_k, \lambda + \lambda_k)$, $k = 1, 2$, are admissible measures for Problem \([5.1]\) with the data chosen above such that $R\lambda_k = R\lambda$, $k = 1, 2$, and hence $\kappa_2(\lambda_k, \lambda_k) = \kappa_2(\lambda, \lambda)$, $k = 1, 2$. Thus each of the non-equal, although $R$-equivalent vector measures $\lambda$, $\lambda_1$, and $\lambda_2$ is a solution to Problem \([5.1]\)

5.2. **Auxiliary results.** In view on the definition of the vague topology on $\mathcal{M}^+(A)$, we call a set $\mathcal{F} \subset \mathcal{M}^+(A)$ *vaguely bounded* if for every $i \in I$ and every $\varphi \in C_0(X)$,
\[ \sup_{\mu \in \mathcal{F}} |\mu^i(\varphi)| < \infty. \]

**Lemma 5.6.** If $\mathcal{F} \subset \mathcal{M}^+(A)$ is vaguely bounded, then it is vaguely relatively compact.

**Proof.** It is clear from the above definition that for every $i \in I$ the set
\[ \mathcal{F}^i := \{ \mu^i \in \mathcal{M}^+(A_i) : \mu = (\mu^j)_{j \in I} \in \mathcal{F} \} \]
is vaguely bounded in $\mathcal{M}^+(X)$, and hence $\mathcal{F}^i$ is vaguely relatively compact in $\mathcal{M}(X)$ \([7\, \text{Chapter III, Section 2, Proposition 9}]\). Since $\mathcal{F} \subset \prod_{i \in I} \mathcal{F}^i$, Tychonoff’s theorem on the product of compact spaces \([3\, \text{Chapter I, Section 9, Theorem 3}]\) implies the lemma.

Let $\mathcal{M}^+(A, \leq a, g)$ consist of all $\mu \in \mathcal{M}^+(A)$ with $\langle g_i, \mu^i \rangle \leq a_i$ for all $i \in I$. We also write $\mathcal{M}^\sigma(A, \leq a, g) := \mathcal{M}^\sigma(A) \cap \mathcal{M}^+(A, \leq a, g)$. By \((4.5)\),
\[ \mu^i(A_i) \leq a_i g_i^{-1} < \infty \quad \text{for all} \quad \mu \in \mathcal{M}^+(A, \leq a, g). \]

**Lemma 5.7.** $\mathcal{M}^+(A, \leq a, g)$ and $\mathcal{M}^\sigma(A, \leq a, g)$ are vaguely bounded and closed, and hence they both are vaguely compact. If the condenser $A$ is compact then the same holds for $\mathcal{M}^+(A, \leq a, g)$ and $\mathcal{M}^\sigma(A, \leq a, g)$. 
Proof. It is seen from \([5.4]\) that \(\mathcal{M}^+(A, \leq a, g)\) is vaguely bounded. Hence, by Lemma \(5.6\) for any net \((\mu_s)_{s \in S} \subset \mathcal{M}^+(A, \leq a, g)\) there exists a vague cluster point \(\mu\). Since \(\mathcal{M}^+(A)\) is vaguely closed, we have \(\mu \in \mathcal{M}^+(A)\). Choose a subnet \((\mu_t)_{t \in T}\) of \((\mu_s)_{s \in S}\) converging vaguely to \(\mu\). As \(g_i\) is positive and continuous, we get from Lemma \(2.1\)
\[
(5.5) \quad \langle g_i, \mu^i \rangle \leq \liminf_{t \in T} \langle g_i, \mu_t^i \rangle \leq a_i \quad \text{for all } i \in I.
\]
Thus \(\mu \in \mathcal{M}^+(A, \leq a, g)\), which shows that indeed \(\mathcal{M}^+(A, \leq a, g)\) is vaguely closed and compact. Since \(\mathcal{M}^\sigma(A)\) is vaguely closed in \(\mathcal{M}^+(A)\), the first assertion of the lemma follows. Assume now that \(A\) is compact, and let the above \((\mu_s)_{s \in S}\) be taken from \(\mathcal{M}^+(A, a, g)\). Then \(\nu \mapsto \langle g_i, \nu \rangle\) is vaguely continuous on \(\mathcal{M}^+(A_i)\) for every \(i \in I\), and therefore all the inequalities in \((5.5)\) are in fact equalities. Thus \(\mathcal{M}^+(A, a, g)\) and hence also \(\mathcal{M}^\sigma(A, a, g)\) are vaguely closed and compact. \(\square\)

Lemma \(5.8\). \(\mathcal{M}^\sigma(A, a, g)\) is vaguely compact for any \(\sigma \in \mathcal{C}(A)\) possessing the property\(^{15}\)
\[
(5.6) \quad \langle g_i, \sigma^i \rangle < \infty \quad \text{for all } i \in I.
\]

Proof. Fix a vague cluster point \(\mu\) of a net \((\mu_s)_{s \in S} \subset \mathcal{M}^\sigma(A, a, g)\). By Lemma \(5.7\) such a \(\mu\) exists and belongs to \(\mathcal{M}^\sigma(A, \leq a, g)\). We only need to show that, under requirement \((5.6)\), \(\langle g_i, \mu^i \rangle = a_i\) for every \(i \in I\). Passing to a subnet and changing notations assume that \(\mu\) is the vague limit of \((\mu_s)_{s \in S}\). Consider an exhaustion of \(A_i\) by an upper directed family of compact sets \(K \subset A_i\).\(^{16}\) Since the indicator function \(1_K\) of \(K\) is upper semicontinuous, we get from Lemma \(2.1\) (with \(\psi = \cdot g_i 1_K \in \Psi(X)\)) and \([17\) Lemma 1.2.2]
\[
a_i = \langle g_i, \mu^i \rangle = \lim_{K \uparrow A_i} \langle g_i, 1_K, \mu^i \rangle \geq \lim_{K \uparrow A_i} \limsup_{s \in S} \langle g_i, 1_K, \mu^i_s \rangle
\]
\[
= \lim_{K \uparrow A_i} \liminf_{s \in S} \langle g_i, 1_{A_i \setminus K}, \mu^i_s \rangle.
\]
Hence, the lemma will follow once we show that
\[
(5.7) \quad \lim_{K \uparrow A_i} \liminf_{s \in S} \langle g_i, 1_{A_i \setminus K}, \mu^i_s \rangle = 0.
\]
Since, by \((5.6)\),
\[
\infty > \langle g_i, \sigma^i \rangle = \lim_{K \uparrow A_i} \langle g_i, 1_K, \sigma^i \rangle,
\]
we have
\[
\lim_{K \uparrow A_i} \langle g_i, 1_{A_i \setminus K}, \sigma^i \rangle = 0.
\]
When combined with
\[
\langle g_i, 1_{A_i \setminus K}, \mu^i_s \rangle \leq \langle g_i, 1_{A_i \setminus K}, \sigma^i \rangle \quad \text{for every } s \in S,
\]
this implies \((5.7)\) as desired. \(\square\)

\(^{15}\)For a compact \(A\) relation \((5.6)\) holds automatically, and Lemma \(5.8\) then in fact reduces to Lemma \(5.7\).

\(^{16}\)A family \(\Omega\) of sets \(Q \subset X\) is said to be upper directed if for any \(Q_1, Q_2 \in \Omega\) there exists \(Q_3 \in \Omega\) such that \(Q_1 \cup Q_2 \subset Q_3\).
Lemma 5.9. In Case I the mapping \( \mu \mapsto G_{\kappa,t}(\mu) \) is vaguely l.s.c. on \( \mathcal{E}_{\kappa,t}^+(A) \), and it is strongly continuous if Case II holds.

Proof. This is obtained from Lemma 2.1 and relation (4.3), respectively. □

Definition 5.10. A net \((\mu_s)_{s \in S} \subset \mathcal{E}_{\kappa,t}^+(A,a,g)\) is said to be minimizing in Problem 5.1 if

\[
\lim_{s \in S} G_{\kappa,t}(\mu_s) = G_{\kappa,t}^\sigma(A,a,g).
\]

Let \(M_{\kappa,t}(A,a,g)\) consist of all these nets \((\mu_s)_{s \in S} \); it is nonempty because of (5.3).

Lemma 5.11. For any \((\mu_s)_{s \in S}\) and \((\nu_t)_{t \in T}\) in \(M_{\kappa,t}^\sigma(A,a,g)\) we have

\[
\lim_{(s,t) \in S \times T} \|\mu_s - \nu_t\|_{\mathcal{E}_{\kappa}^+(A)} = 0,
\]

where \(S \times T\) is the upper directed product of the upper directed sets \(S\) and \(T\).

Proof. In the same manner as in the proof of Lemma 5.4 we get

\[
0 \leq \|R \mu_s - R \nu_t\|_\kappa^2 \leq -4G_{\kappa,t}^\sigma(A,a,g) + 2G_{\kappa,t}(\mu_s) + 2G_{\kappa,t}(\nu_t),
\]

which gives (5.9) when combined with (3.8), (5.8) and the finiteness of \(G_{\kappa,t}^\sigma(A,a,g)\). □

Taking here \((\mu_s)_{s \in S}\) and \((\nu_t)_{t \in T}\) to be equal, we arrive at the following conclusion.

Corollary 5.12. Every \((\mu_s)_{s \in S} \in M_{\kappa,t}^\sigma(A,a,g)\) is strong Cauchy in \(\mathcal{E}_{\kappa}^+(A)\).

The result below will be used in subsequent work of the authors.

Theorem 5.13. Let the kernel \(\kappa\) be perfect and let either I+ or I− be empty. If moreover (5.6) holds, then in both Cases I and II Problem 5.1 is solvable for any vector \(a\). Furthermore, every \((\mu_s)_{s \in S} \in M_{\kappa,t}^\sigma(A,a,g)\) converges to any \(\lambda \in \mathcal{E}_{\kappa,t}^+(A,a,g)\) strongly in \(\mathcal{E}_{\kappa}^+(A)\); and hence also vaguely whenever the \(A_i, i \in I\), are mutually essentially disjoint.

Proof. Assume for definiteness that I− = ∅ and fix \((\mu_s)_{s \in S} \in M_{\kappa,t}^\sigma(A,a,g)\). By Lemma 5.8 there is a subnet \((\mu_t)_{t \in T}\) of \((\mu_s)_{s \in S}\) converging vaguely to some \(\mu \in M_{\kappa}^\sigma(A,a,g)\). The net \((\mu_t)_{t \in T}\) belongs to \(M_{\kappa,t}^\sigma(A,a,g)\), for so does \((\mu_s)_{s \in S}\), and hence it is strong Cauchy in the semimetric space \(\mathcal{E}_{\kappa,t}^+(A)\) by Corollary 5.12. Application of relations (3.7) and (3.8) then shows that the net of positive scalar measures \(R \mu_t, t \in T\), is strong Cauchy in the metric space \(\mathcal{E}_{\kappa}^+(A)\), and therefore it is strongly bounded. Furthermore, since \(R \mu_t \to R \mu\) vaguely in \(\mathcal{M}^+(X)\), we have \(R \mu_t \otimes R \mu_t \to R \mu \otimes R \mu\) vaguely in \(\mathcal{M}^+(X \times X)\) [7, Chapter 3, Section 5, Exercise 5]. Applying Lemma 2.1 to \(X \times X\) and \(\psi = \kappa\), we thus get \(R \mu \in \mathcal{E}_{\kappa}^+(A)\). Moreover, \((R \mu_t)_{t \in T}\) converges to \(R \mu\) strongly in \(\mathcal{E}_{\kappa}^+(X)\), which follows from the above in view of the perfectness of \(\kappa\). Hence, by (3.8), \((\mu_t)_{t \in T}\) converges to \(\mu\) strongly in \(\mathcal{E}_{\kappa}^+(A)\).

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\[\text{See e.g. [25] Chapter 2, Section 3.}\]
In either Case I or Case II, we get from (5.8) and Lemma 5.9

\( -\infty < G_{\kappa,f}(\mu) \leq \lim_{t \to T} G_{\kappa,f}(\mu_t) = G_{\kappa,f}(A, a, g) < \infty, \)

where the first inequality is valid by (4.4) and (4.7), while the last one holds by (5.3). This yields that \( \mu \in \mathcal{E}_{\kappa,f}^+(A, a, g) \), and therefore \( G_{\kappa,f}(\mu) \geq G_{\kappa,f}^+(A, a, g) \). It is seen from (5.10) that in fact equality prevails in the last relation, and hence indeed \( \mu \in \mathcal{G}_{\kappa,f}^+(A, a, g) \).

Since a strong Cauchy net converges strongly to any of its strong cluster points (even in the present case of a semimetric space), it follows from the above that \( \mu_s \to \mu \) strongly in \( \mathcal{E}_{\kappa}^+(A) \). Finally, if \( \lambda \) is any other element of the class \( \mathcal{G}_{\kappa,f}^+(A, a, g) \), then also \( \mu_s \to \lambda \) strongly in \( \mathcal{E}_{\kappa}^+(A) \) because \( \| \mu - \lambda \|_{\mathcal{E}_{\kappa}^+(A)} = 0 \) according to Lemma 5.4.

It has been shown that any of the vague cluster points of \( (\mu_s)_{s \in S} \) belongs to \( \mathcal{G}_{\kappa,f}^+(A, a, g) \). Assuming now that the \( A_i, i \in I \), are mutually essentially disjoint, we see from the latter assertion of Lemma 5.4 that then the vague cluster set of \( (\mu_s)_{s \in S} \) reduces to the measure \( \mu \), chosen at the beginning of the proof. As \( \mathcal{M}_+^+(A) \) is Hausdorff in the vague topology, \( \mu_s \to \mu \) also vaguely [5, Chapter I, Section 9, n° 1, Corollary].

6. On the solvability of Problem 5.1 for Riesz kernels. I

Throughout Sections 6–10 let \( n \geq 3, n \in \mathbb{N}, \) and \( \alpha \in (0, n) \) be fixed. On \( X = \mathbb{R}^n \), consider the \( \alpha \)-Riesz kernel \( \kappa(x,y) = \kappa_\alpha(x,y) := |x-y|^{\alpha-n} \) of order \( \alpha \). The \( \alpha \)-Riesz kernel is known to be strictly positive definite and moreover perfect [11, 12], and hence the metric space \( \mathcal{E}_{\kappa}^+(\mathbb{R}^n) \) is complete in the induced strong topology. However, by Cartan [9], the whole pre-Hilbert space \( \mathcal{H}_{\kappa_\alpha}(\mathbb{R}^n) \) for \( \alpha \in (1, n) \) is strongly incomplete (compare with Theorem 3.10 above, as well as with Theorems 7.3, 7.4 and Remark 7.6 below).

From now on we shall write simply \( \alpha \) instead of \( \kappa_\alpha \) if \( \kappa_\alpha \) serves as an index. For example, \( c_\alpha(\cdot) = c_{\kappa_\alpha}(\cdot) \) denotes the \( \alpha \)-Riesz inner capacity of a set.

**Theorem 6.1.** Let a generalized condenser \( A \) be compact and let each of the potentials \( \kappa_\alpha(\cdot, \sigma^i), i \in I, \sigma \in \mathcal{C}(A) \) be given. Then in either Case I or Case II Problem 5.1 is solvable for any given \( a \) and \( g \), and the class \( \mathcal{G}_{\alpha,f}^+(A, a, g) \) of all its solutions is vaguely compact. Furthermore, every minimizing sequence \( \{\mu_k\}_{k \in \mathbb{N}} \in \mathcal{M}_{\alpha,f}^+(A, a, g) \) converges to every \( \lambda \in \mathcal{G}_{\alpha,f}^+(A, a, g) \) strongly in \( \mathcal{E}_{\alpha}^+(A) \); and hence also vaguely provided that all the \( A_i, i \in I \), are mutually essentially disjoint.

Theorem 6.1 is inspired partly by [3] and will be proved in Section 6.1. The following lemma goes back to [33, 15].

**Lemma 6.2.** Let \( A \) be an arbitrary (not necessarily compact) generalized condenser, and let each of the \( \kappa_\alpha(\cdot, \sigma^i), i \in I, \sigma \in \mathcal{C}(A) \) be given. Then for every \( \mu \in \mathcal{M}_{\sigma}(A) \) and every \( i \in I \), \( \kappa_\alpha(\cdot, \mu^i) \) is continuous on \( \mathbb{R}^n \).
Proof. Actually, \( \kappa_\alpha(\cdot, \sigma^i) \) is continuous on all of \( \mathbb{R}^n \) by [27] Theorem 1.7. Since \( \kappa_\alpha(\cdot, \mu^i) \) is l.s.c. and since \( \kappa_\alpha(\cdot, \mu^i) = \kappa_\alpha(\cdot, \sigma^i) - \kappa_\alpha(\cdot, \mu^i) \) with \( \kappa_\alpha(\cdot, \sigma^i) \) continuous and \( \kappa_\alpha(\cdot, \sigma^i - \mu^i) \) l.s.c., \( \kappa_\alpha(\cdot, \mu^i) \) is also upper semicontinuous, hence continuous.

Example 6.3. Let \( A = \{A_i\}_{i \in I} \) be as in Example 3.3 (see Figure 1), and let \( \alpha \in (0, 2) \). Also assume that \( g = 1 \) and that either Case II holds or \( f_i(x) < \infty \) n.e. on \( A_i \), \( i = 1, 2 \).

Let \( \lambda_i \) denote the (unique) \( \kappa_\alpha \)-capacitary measure on \( A_i \) (see Remark 2.5), where the first inequality is valid by (4.4) and (4.7), while the last one holds according to the standing assumption (5.3). Hence, \( \kappa_\alpha(\cdot, \lambda) \) clearly has finite \( \alpha \)-Riesz energy, relation (5.3) holds by Lemma 5.2, and since \( A \) is compact, Problem 5.1 admits a solution according to Theorem 6.1. Thus, no short-circuit occurs between the oppositely charged plates of the condenser \( A \), though they intersect each other over the set \( \delta A = \{\xi_5, \xi_6\} \).

6.1. Proof of Theorem 6.1 Fix any \( \{\mu_k\}_{k \in \mathbb{N}} \in M^\sigma_{\alpha,f}(A, a, g) \); it exists because of the (standing) assumption (5.3). By Lemma 5.7, any of its vague cluster points \( \lambda \) (which exist) belongs to \( M^\sigma(A, a, g) \). As \( M^\sigma(A, a, g) \) is in fact sequentially vaguely compact (see Remark 2.5), one can select a subsequence \( \{\mu_{k_m}\}_{m \in \mathbb{N}} \) of \( \{\mu_k\}_{k \in \mathbb{N}} \) such that

\[
\mu_{k_m} \rightharpoonup \lambda \quad \text{vaguely as} \quad m \to \infty.
\]

Since by Lemma 6.2 each of \( \kappa_\alpha(\cdot, \mu^i_k) \) and \( \kappa_\alpha(\cdot, \lambda) \), \( k \in \mathbb{N}, i \in I \), are continuous and hence bounded on the (compact) set \( A_j, j \in I \), the preceding display yields

\[
\lim_{m \to \infty} \lim_{i \to \infty} \kappa_\alpha(\cdot, \mu^i_{k_m}) = \lim_{m \to \infty} \lim_{i \to \infty} \int \kappa_\alpha(\cdot, \mu^i_{k_m}) d\mu^i_{k_m} = \lim_{m \to \infty} \int \kappa_\alpha(\cdot, \mu^i_{k_m}) d\lambda^i = \int \kappa_\alpha(\cdot, \lambda) d\mu_{k_m} = \kappa_\alpha(\lambda, \lambda) < \infty \quad \text{for all} \quad i, j \in I.
\]

Hence, \( \lambda \in \mathcal{E}^+_{\alpha}(A) \) and moreover

\[
\mu_{k_m} \to \lambda \quad \text{strongly as} \quad m \to \infty.
\]

We assert that this \( \lambda \) solves Problem 5.1. Applying Lemma 5.9 from (5.8), (6.1) and (6.2) we obtain

\[
-\infty < G_{\alpha,f}(\lambda) \leq \lim_{m \to \infty} G_{\alpha,f}(\mu_{k_m}) = G^\sigma_{\alpha,f}(A, a, g) < \infty,
\]

where the first inequality is valid by (4.4) and (4.7), while the last one holds according to the (standing) assumption (5.3). Hence \( \lambda \in \mathcal{E}^+_{\alpha,f}(A, a, g) \), and \( \lambda \in \mathcal{G}^\sigma_{\alpha,f}(A, a, g) \) follows.

Note that the minimizing sequence \( \{\mu_k\}_{k \in \mathbb{N}} \) is strong Cauchy by Corollary 5.12. Since a strong Cauchy sequence converges strongly to any of its strong cluster points, we infer from (6.2) that \( \{\mu_k\}_{k \in \mathbb{N}} \) converges to \( \lambda \) strongly. The same holds for any other \( \nu \in \mathcal{G}^\sigma_{\alpha,f}(A, a, g) \), for \( \|\lambda - \nu\|_{\mathcal{E}^+_{\alpha}(A)} = 0 \) according to Lemma 5.4.

To prove that \( \mathcal{G}^\sigma_{\alpha,f}(A, a, g) \) is vaguely compact, consider a sequence \( \{\lambda_k\}_{k \in \mathbb{N}} \) of its elements. Since it belongs to \( M^\sigma_{\alpha,f}(A, a, g) \), it is clear from what has been shown above that any of the vague cluster points of \( \{\lambda_k\}_{k \in \mathbb{N}} \) belongs to \( \mathcal{G}^\sigma_{\alpha,f}(A, a, g) \).
Assume finally that the $A_i$, $i \in I$, are mutually essentially disjoint. Then, by Lemma 5.4 a solution to Problem 5.1 is unique, which yields that the vague cluster set of the given minimizing sequence $\{\mu_k\}_{k \in \mathbb{N}}$ reduces to the unique measure $\lambda$. Since the vague topology is Hausdorff, $\lambda$ is actually the vague limit of $\{\mu_k\}_{k \in \mathbb{N}}$ [3, Chapter I, Section 9, n° 1]. □

7. On the solvability of Problem 5.1 for Riesz kernels. II

Let $\overline{Q}$ be the closure of $Q \subset \mathbb{R}^n$ in $\mathbb{R}^n := \mathbb{R}^n \cup \{\omega_{\mathbb{R}^n}\}$, the one-point compactification of $\mathbb{R}^n$. The following theorem provides sufficient conditions for the solvability of Problem 5.1 in the case where $A_i$, $i \in I$, are not necessarily compact; compare with Theorem 6.1 above.

Regarding the uniqueness of a solution to Problem 5.1 see Lemma 5.4.

**Theorem 7.1.** Let the set $\overline{A^+} \cap \overline{A^-}$ consist of at most one point, i.e.

\begin{equation}
\text{either } \overline{A^+} \cap \overline{A^-} = \emptyset, \text{ or } \overline{A^+} \cap \overline{A^-} = \{x_0\} \text{ where } x_0 \in \mathbb{R}^n,
\end{equation}

and let the given $g$ and $\sigma \in C(A)$ satisfy (5.6). Then in either Case I or Case II Problem 5.1 is solvable for any given $a$, and the class of all its solutions is vaguely compact. Furthermore, every minimizing sequence converges to every $\lambda \in \mathcal{G}^E_{\alpha}\{A, a, g\}$ strongly in $\mathcal{E}^+_\alpha(A)$; hence also vaguely whenever all the $A_i$, $i \in I$, are mutually essentially disjoint.

Theorem 7.1 is sharp in the sense that it is no longer valid if assumption (5.6) is omitted (see Theorem 7.9 below).

The proof of Theorem 7.1 is given in Section 7.2; it is based on the approach that has been developed in Sections 3 and 5 as well as on Theorems 7.3 and 7.4 below providing strong completeness results for semimetric subspaces of $\mathcal{E}^+_\alpha(A)$, properly chosen. In turn, the proofs of Theorems 7.3 and 7.4 substantially use Theorem 3.10 on the strong completeness of $\mathcal{E}^+_\alpha(A)$ in the case of a standard condenser.

**Example 7.2.** Let $A = (A_i)_{i \in I}$ be as in Example 3.4 (see Figure 2) and let $\alpha = 2$; then $c_2(A_i) < \infty$, $i = 1, 2$ [22, Example 8.2], and hence there exists a (unique) $\kappa_2$-capacitary measure $\lambda_i$ on $A_i$ (see Remark 2.7). Let $g = 1$, and let either Case II hold or $f_i(x) < \infty$ n.e. on $A_i$, $i = 1, 2$. For any $a = (a_1, a_2)$ define $\sigma^i := c_i \lambda_i$, $i = 1, 2$, where $a_i < c_i < \infty$.\footnote{In a particular case of a condenser with two oppositely charged plates some of the results presented in this section have been obtained earlier in [41].} Then $\langle g_i, \sigma^i \rangle = \sigma^i(A_i) = c_i < \infty$, and hence (5.6) is fulfilled. As $\sigma = (\sigma^i)_{i \in I}$ has finite Newtonian energy, (5.3) holds by Lemma 5.2 and Problem 5.1 admits a solution according to Theorem 7.1 which is unique by Lemma 5.4. Thus, no short-circuit occurs between $A_1$ and $A_2$, though these oppositely charged conductors touch each other at the point $\omega_{\mathbb{R}^3}$. Note that, although $c_2(A) < \infty$, Theorem 6.2 from [41] on the solvability of Problem 5.1 for a standard condenser cannot be applied, which is caused by the unboundedness of the Coulomb kernel $\kappa_2$ on $A_1 \times A_2$.\footnote{Under the assumptions of Example 7.2 (5.1) holds since for the given $A_i$, $i = 1, 2$, and $\kappa = \kappa_2$ we have $S^A_{\kappa^3} = A_i$. This is seen from the construction of the $\kappa_2$-capacitary measure described in [27, Theorem 5.1].}
7.1. Strong completeness theorems for semimetric subspaces of $\mathcal{E}_\alpha^+(A)$. The (convex) sets $\mathcal{E}_\alpha^+(A, \leq a, g) := \mathcal{E}_\alpha^+(A) \cap \mathcal{M}^+(A, \leq a, g)$ and $\mathcal{E}_\alpha^+(A, a, g) := \mathcal{E}_\alpha^+(A) \cap \mathcal{M}_\sigma(A, a, g)$, $\sigma \in \mathcal{C}(A)$ being given, can certainly be thought of as semimetric subspaces of $\mathcal{E}_\alpha^+(A)$; their topologies will likewise be called strong.

**Theorem 7.3.** Suppose that a generalized condenser $A$ satisfies condition (7.1). Then for any given $g$ and $a$ the semimetric space $\mathcal{E}_\alpha^+(A, \leq a, g)$ is strongly complete. In more detail, any strong Cauchy sequence $\{\mu_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_\alpha^+(A, \leq a, g)$ converges strongly to any of its vague cluster points. If moreover all the $A_i$, $i \in I$, are mutually essentially disjoint, then the strong topology on the space $\mathcal{E}_\alpha^+(A, \leq a, g)$ is finer than the induced vague topology.

We first outline the scheme of the proof of Theorem 7.3. In view of Lemma 5.7 on the vague compactness of $\mathcal{M}^+(A, \leq a, g)$, we can assume that a strong Cauchy sequence $\{\mu_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_\alpha^+(A, \leq a, g)$ converges vaguely to $\mu \in \mathcal{M}^+(A, \leq a, g)$. It remains to show that $\mu_k \rightarrow \mu$ in the strong topology of $\mathcal{E}_\alpha^+(A)$, which by the isometry between $\mathcal{E}_\alpha^+(A)$ and its $R$-image is equivalent to the assertion $R\mu_k \rightarrow R\mu$ strongly in $\mathcal{E}_\alpha(R^n)$. The difficulty appearing here is the strong incompleteness of $\mathcal{E}_\alpha(R^n)$. However, if $\overline{A^+} \cap \overline{A^-}$ consists of at most $\omega_R^n$, the completeness of the metric space of all $\nu \in \mathcal{E}_\alpha(R^n)$ such that $\nu^\pm$ are supported by $A^\pm$ was shown in [38 Theorem 1]. The remaining case $\overline{A^+} \cap \overline{A^-} = \{x_0\}$, $x_0 \neq \omega_R^n$, is reduced to the case $\overline{A^+} \cap \overline{A^-} = \{\omega_R^n\}$ with the aid of the Kelvin transformation relative to $S(x_0, 1)$.

**Proof.** Fix a strong Cauchy sequence $\{\mu_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_\alpha^+(A, \leq a, g)$. By Lemma 5.7 for any of its vague cluster points $\mu$ (which exist) we have $\mu \in \mathcal{M}^+(A, \leq a, g)$. As $\mathcal{M}^+(A, \leq a, g)$ is sequentially vaguely closed (see Remark 2.5, one can choose a subsequence $\{\mu_{k_m}\}_{m \in \mathbb{N}}$ of $\{\mu_k\}_{k \in \mathbb{N}}$ converging vaguely to the measure $\mu$, i.e.

$$\mu^i \rightarrow \mu^i \text{ vaguely in } \mathcal{M}(X), \ i \in I.$$  

It is obvious that $\{\mu_{k_m}\}_{m \in \mathbb{N}}$ is likewise strong Cauchy in $\mathcal{E}_\alpha^+(A)$.

We proceed by showing that $\kappa_\alpha(\mu, \mu)$ is finite, hence

$$\mu \in \mathcal{E}_\alpha^+(A, \leq a, g),$$

and moreover that $\mu_{k_m} \rightarrow \mu$ strongly as $m \rightarrow \infty$, i.e.

$$\lim_{m \rightarrow \infty} \sup_{X} |\mu_{k_m} - \mu| = 0.$$  

Assume first that $\overline{A^+} \cap \overline{A^-}$ either is empty or coincides with $\{\omega_R^n\}$. Then $A$ forms a standard condenser in $\mathbb{R}^n$ and hence, by (7.2), $R\mu_{k_m} \rightarrow R\mu$ (as $m \rightarrow \infty$) in the vague topology of $\mathcal{M}(\mathbb{R}^n)$. Noting that $\{R\mu_{k_m}\}_{m \in \mathbb{N}}$ is a strong Cauchy sequence in $\mathcal{E}_\alpha(\mathbb{R}^n)$, we conclude from [38 Theorem 1 and Corollary 1] (see also Theorem 3.10 above) that there exists a unique $\eta \in \mathcal{E}_\alpha(\mathbb{R}^n)$ such that

$$R\mu_{k_m} \rightarrow \eta \text{ strongly and vaguely as } m \rightarrow \infty.$$  

As the vague topology on $\mathcal{M}(\mathbb{R}^n)$ is Hausdorff, we thus have $\eta = R\mu$, which in view of (3.7), (3.8) and the last display results in (7.3) and (7.4).
We next proceed by analyzing the case
\begin{equation}
A^+ \cap A^- = \{x_0\}, \text{ where } x_0 \in \mathbb{R}^n.
\end{equation}

Consider the inversion \( I_{x_0} \) with respect to \( S(x_0,1) \); namely, each point \( x \neq x_0 \) is mapped to the point \( x^* \) on the ray through \( x \) which issues from \( x_0 \), determined uniquely by
\[
|x - x_0| \cdot |x^* - x_0| = 1.
\]
This is a self-homeomorphism of \( \mathbb{R}^n \setminus \{x_0\} \); furthermore,
\begin{equation}
|x^* - y^*| = \frac{|x - y|}{|x - x_0||y - x_0|}.
\end{equation}

Extend it to a self-homeomorphism of \( \mathbb{R}^n \) by setting \( I_{x_0}(x_0) = \omega_{\mathbb{R}^n} \) and \( I_{x_0}(\omega_{\mathbb{R}^n}) = x_0 \).

To each (signed) scalar measure \( \nu \in \mathcal{M}(\mathbb{R}^n) \) with \( \nu(\{x_0\}) = 0 \), in particular for every \( \nu \in \mathcal{E}_\alpha(\mathbb{R}^n) \), there corresponds the Kelvin transform \( \nu^* \in \mathcal{M}(\mathbb{R}^n) \) by means of the formula
\[
d\nu^*(x^*) = |x - x_0|^{\alpha-n} \, d\nu(x), \quad x^* \in \mathbb{R}^n
\]
(see [34] or [27, Chapter IV, Section 5, n° 19]). Then, in consequence of (7.6),
\[
\kappa_\alpha(x^*, \nu^*) = |x - x_0|^{n-\alpha} \kappa_\alpha(x, \nu), \quad x^* \in \mathbb{R}^n,
\]
and therefore
\begin{equation}
\kappa_\alpha(\nu^*, \nu_1^*) = \kappa_\alpha(\nu, \nu_1)
\end{equation}
for every \( \nu_1 \in \mathcal{M}(\mathbb{R}^n) \) that does not have an atomic mass at \( x_0 \). It is clear that the Kelvin transformation is additive and that it is an involution, i.e.
\begin{equation}
(\nu + \nu_1)^* = \nu^* + \nu_1^*,
\end{equation}
\begin{equation}
(\nu^*)^* = \nu.
\end{equation}

Write \( A^*_i := I_{x_0}(A_i^+) \cap \mathbb{R}^n \) and sign \( A^*_i := \text{sign} \, A_i = s_i \) for each \( i \in I \); then \( A^* = (A^*_i)_{i \in I} \) forms a standard condenser in \( \mathbb{R}^n \), which is seen from (7.5) in view of the properties of \( I_{x_0} \).

Applying the Kelvin transformation to each of the components \( \nu^i \) of any given \( \nu = (\nu^i)_{i \in I} \in \mathcal{E}_\alpha^+(A) \) we get \( \nu^* := (\nu^*_i)_{i \in I} \in \mathcal{M}(A^*) \). Based on Lemma 3.8, identity (3.8) and relations (7.7)–(7.9), we also see that the \( \alpha \)-Riesz energy of \( \nu^* \) is finite, and furthermore
\begin{equation}
\|\nu_1^* - \nu_2^*\|_{\mathcal{E}_\alpha^+(A^*)} = \|\nu_1 - \nu_2\|_{\mathcal{E}_\alpha^+(A)} \quad \text{for all } \nu_1, \nu_2 \in \mathcal{E}_\alpha^+(A).
\end{equation}

Summarizing the above, because of (7.9), we arrive at the following observation: \textit{the Kelvin transformation is a bijective isometry of } \( \mathcal{E}_\alpha^+(A) \text{ onto } \mathcal{E}_\alpha^+(A^*) \).

Let \( \mu_{k_m}, m \in \mathbb{N}, \) and \( \mu \) be the measures chosen at the beginning of the proof. In view of (5.4) and (7.2), for each \( i \in I \) one can apply [27, Lemma 4.3] to \( \mu_{k_m}^i, m \in \mathbb{N}, \) and \( \mu^i \), and consequently
\begin{equation}
\mu_{k_m}^* \rightharpoonup \mu^* \quad \text{vaguely as } m \to \infty.
\end{equation}
But \( \{ \mu_{km}^\ast \}_{m \in \mathbb{N}} \) is a strong Cauchy sequence in \( \mathcal{E}_\alpha^+(A^\ast) \), which is clear from (7.10). This together with (7.11) implies with the aid of Theorem 3.10 that \( \mu^\ast \in \mathcal{E}_\alpha^+(A^\ast) \) and also that
\[
\lim_{m \to \infty} \| \mu_{km}^\ast - \mu^\ast \|_{\mathcal{E}_\alpha^+(A^\ast)} = 0.
\]

Another application of the above observation then leads to (7.3) and (7.4), as was to be proved.

In turn, (7.4) implies that \( \mu_k \to \mu \) strongly in \( \mathcal{E}_\alpha^+(A) \) as \( k \to \infty \), for \( \{ \mu_k \}_{k \in \mathbb{N}} \) is strong Cauchy and hence converges strongly to any of its strong cluster points. It has thus been established that \( \{ \mu_k \}_{k \in \mathbb{N}} \) converges strongly to any of its vague cluster points, which is the first assertion of the theorem. Assume now that all the \( A_i, i \in I \), are mutually essentially disjoint. Then \( \| \mu_1 - \mu_2 \|_{\mathcal{E}_\alpha^+(A)} \) is a metric (Theorem 3.9), and hence \( \mu \) has to be the unique vague cluster point of \( \{ \mu_k \}_{k \in \mathbb{N}} \). Since the vague topology is Hausdorff, \( \mu \) is actually the vague limit of \( \{ \mu_k \}_{k \in \mathbb{N}} \) [5, Chapter I, Section 9, n° 1].

**Theorem 7.4.** Given \( A, g, \) and \( \sigma \in \mathcal{E}(A) \), assume that (5.6) and (7.1) both hold. Then for every vector \( a \) the semimetric space \( \mathcal{E}_\alpha^+(A, a, g) \) is strongly complete.

**Proof.** Fix a strong Cauchy sequence \( \{ \mu_k \}_{k \in \mathbb{N}} \subset \mathcal{E}_\alpha^+(A, a, g) \). By Lemma 5.8, any of its vague cluster points \( \mu \) (which exist) belongs to \( \mathcal{M}_\sigma^+(A, a, g) \), while according to Theorem 7.3 it has finite \( \alpha \)-Riesz energy and moreover \( \mu_k \to \mu \) strongly as \( k \to \infty \).

**Remark 7.5.** Theorem 7.4 does not remain valid if assumption (5.6) is omitted from its hypotheses. This is seen from the proof of Theorem 7.9 (see Section 7.3 below).

**Remark 7.6.** Since either of the semimetric spaces \( \mathcal{E}_\alpha^+(A, \leq a, g) \) and \( \mathcal{E}_\alpha^+(A, a, g) \) is isometric to its \( R \)-image, Theorems 7.3 and 7.4 have singled out strongly complete topological subspaces of the pre-Hilbert space \( \mathcal{E}_\alpha(R^n) \), whose elements are (signed) Radon measures. This is of independent interest because, by Cartan, \( \mathcal{E}_\alpha(R^n) \) is strongly incomplete.

### 7.2. Proof of Theorem 7.1

Fix any \( \{ \mu_k \}_{k \in \mathbb{N}} \in \mathcal{M}_\alpha^+(A, a, g) \); it exists because of assumption (5.3), and it is strong Cauchy in the semimetric space \( \mathcal{E}_\alpha^+(A, a, g) \) by Corollary 5.12. Since \( \mathcal{M}_\alpha^+(A) \) is sequentially vaguely closed (see Remark 2.5), by Lemma 5.8 there is a subsequence \( \{ \mu_{km} \}_{m \in \mathbb{N}} \) of \( \{ \mu_k \}_{k \in \mathbb{N}} \) converging vaguely to some \( \mu \in \mathcal{M}_\sigma^+(A, a, g) \), while by Theorem 7.4 we actually have \( \mu \in \mathcal{E}_\alpha^+(A, a, g) \) and
\[
\lim_{k \to \infty} \| \mu_k - \mu \|_{\mathcal{E}_\alpha^+(A)} = 0.
\]

Also note that, by relations (5.3), (5.8) and Lemma 5.9,
\[
-\infty < G_{\alpha,f}(\mu) \leq \lim_{m \to \infty} G_{\alpha,f}(\mu_{km}) = G_{\alpha,f}^\sigma(A, a, g) < \infty,
\]
the first inequality being valid according to (4.4) and (4.7). Thus \( \mu \in \mathcal{E}_\alpha^+(A, a, g) \) and therefore \( G_{\alpha,f}(\mu) \geq G_{\alpha,f}^\sigma(A, a, g) \). All this combined implies that \( \mu \in \mathcal{E}_\alpha^+(A, a, g) \).
To verify that $\mathcal{G}^*_\alpha(A, a, g)$ is vaguely compact, fix a sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ of its elements. By Lemma 5.4, it is strong Cauchy in $\mathcal{E}^*_\alpha(A, a, g)$, and the same arguments as above show that the (nonempty) vague cluster set of $\{\lambda_k\}_{k \in \mathbb{N}}$ is contained in $\mathcal{G}^*_\alpha(A, a, g)$.

If $\lambda$ is any element of $\mathcal{G}^*_\alpha(A, a, g)$, then by Lemma 5.4, $\lambda$ belongs to the $R$-equivalence class $[\mu]$, which in view of (7.12) implies that $\mu_k \rightarrow \lambda$ strongly in $\mathcal{E}^*_\alpha(A)$. Assuming now that all the $A_i$, $i \in I$, are mutually essentially disjoint, we see from the last assertion of Theorem 7.3 that $\mu_k \rightarrow \lambda$ also vaguely.

7.3. On the sharpness of Theorem 7.1. The purpose of this section is to show that Theorem 7.1 on the solvability of Problem 5.1 is no longer valid if assumption (5.6) is dropped from its hypotheses.

Definition 7.7 (see [8, Theorem VII.13]). A closed set $F \subset \mathbb{R}^n$ is said to be $\alpha$-thin at $\omega_{\mathbb{R}^n}$ if either $F$ is compact, or the inverse of $F$ relative to $S(0,1)$ has $x = 0$ as an $\alpha$-irregular boundary point (cf. [27, Theorem 5.10]).

Remark 7.8. Alternatively, by Wiener’s criterion of $\alpha$-irregularity of a point, a closed set $F \subset \mathbb{R}^n$ is $\alpha$-thin at $\omega_{\mathbb{R}^n}$ if and only if

$$\sum_{k \in \mathbb{N}} c_\alpha(F_k) < \infty,$$

where $q > 1$ and $F_k := F \cap \{x \in \mathbb{R}^n : q^k \leq |x| < q^{k+1}\}$. Since, by [27, Lemma 5.5], $c_\alpha(F) = \infty$ is equivalent to the relation

$$\sum_{k \in \mathbb{N}} c_\alpha(F_k) = \infty,$$

one can define a closed set $Q \subset \mathbb{R}^n$ with $c_\alpha(Q) = \infty$, but $\alpha$-thin at $\omega_{\mathbb{R}^n}$ (see also [10, pp. 276–277]). In the case $n = 3$ and $\alpha = 2$, such a $Q$ can be given as follows:

$$Q := \{x \in \mathbb{R}^3 : 0 \leq x_1 < \infty, x_2^2 + x_3^2 \leq \exp(-2x_1^r) \text{ with } r \in (0, 1)\};$$

note that $Q$ thus defined has finite $c_\alpha(\cdot)$ if $r$ in its definition is $> 1$ [42, Example 8.2].

Assume for simplicity that $g = 1$, $a = 1$ and $f = 0$. Furthermore, let $0 < \alpha < 2$, $I^+ = \{1\}$, $I^- = \{2\}$, and let $A_2 \subset \mathbb{R}^n$ be a closed set with $c_\alpha(A_2) = \infty$, though $\alpha$-thin at $\omega_{\mathbb{R}^n}$ (see Remark 7.8). Assume moreover that the (open) set $D := A_2^c$ is connected and that $A_1$ is a compact subset of $D$ with $c_\alpha(A_1) > 0$. Given the (standard) condenser $A := (A_1, A_2)$ and a constraint $\sigma \in \mathcal{C}(A)$, let $\mathcal{E}^*_\alpha(A, 1)$ stand for the class of vector measures admissible in Problem 5.1 with those data. The sharpness of condition (5.6) for the validity of Theorem 7.1 is illustrated by the following assertion.

Theorem 7.9. Under the above assumptions there exists a constraint $\sigma \in \mathcal{C}(A)$ with $1 < \sigma^1(A_1) < \infty$ and $\sigma^2(A_2) = \infty$ such that

$$\kappa_\alpha(\nu, \nu) > \inf_{\mu \in \mathcal{E}^*_\alpha(A, 1)} \kappa_\alpha(\mu, \mu) =: w^*_\alpha(A, 1) \text{ for every } \nu \in \mathcal{E}^*_\alpha(A, 1).$$
Crucial for the proof to be given below is to show that for a certain \( \sigma \in \mathcal{C}(A) \), \( w^\sigma_\alpha(A, 1) \) equals the \( G \)-energy of the \( G \)-capacitary measure \( \lambda \) on \( A_1 \), where \( G \) denotes the \( \alpha \)-Green kernel on \( D \). Intuitively this is clear since for any given \( \mu \in \mathcal{E}^+_\alpha(A_1, 1) \), \( \mu - \mu' \) minimizes the \( \alpha \)-Riesz energy among all \( \mu - \nu \), \( \nu \) ranging over \( \mathcal{E}^+_\alpha(A_2) \). (Here and in the sequel \( \mu' \) denotes the \( \alpha \)-Riesz balayage of \( \mu \in \mathcal{M}^+(\mathbb{R}^n) \) onto \( A_2 \), uniquely determined in the frame of the classical approach by [20, Theorem 3.6].) And since \( q := 1/\mu \) over \( \mu \), a constraint \( \sigma \) chooses properly the sequence \( (\lambda, \lambda' + \tau_\ell) \in \mathcal{E}^+_\alpha(A; 1) \), \( \ell \in \mathbb{N} \), is therefore minimizing, but it converges vaguely and strongly to \( (\lambda, \lambda') \) which is not admissible.

**Proof.** Denote by \( G = G^\alpha_D \) the \( \alpha \)-Green kernel on the locally compact space \( D \), defined by
\[
G^\alpha_D(x, y) := \kappa_\alpha(x, y) - \kappa_\alpha(x, y) - \kappa_\alpha(x, y), \quad x, y \in D,
\]
e\( y \) being the unit Dirac measure at a point \( y \) [27] [20]. Since \( c_\alpha(A_1) > 0 \), so is \( c_G(A_1) \) [13] Lemma 2.6], and hence, by the compactness of \( A_1 \), there is \( \lambda \in \mathcal{E}^+_\alpha(A_1, 1) \) with \( G(\lambda, \lambda) = c_G(A_1)^{-1} < \infty \).

In fact, such a \( \lambda \) is unique, for the \( \alpha \)-Green kernel is strictly positive definite [20 Theorem 4.9]. As \( S^\lambda_\beta \) is compact, it is seen from [14] Lemmas 3.5, 3.6] that both \( \lambda \) and \( \lambda' \) have finite \( \alpha \)-Riesz energy and moreover
\[
\|\lambda\|_G = \|\lambda - \lambda'\|_\alpha.
\]

Finally, since \( D^c = A_2 \) is \( \alpha \)-thin at \( \omega_{\mathbb{R}^n} \), from [20 Theorem 3.21] (see also the earlier papers [36] Theorem B] and [37] Theorem 4)) we get
\[
q := \lambda(A_1) - \lambda'(A_2) > 0.
\]

Consider an exhaustion of \( A_2 \) by an increasing sequence of compact sets \( K_\ell \), \( \ell \in \mathbb{N} \). Since \( c_\alpha(A_2) = \infty \), the strict positive definiteness of the \( \alpha \)-Riesz kernel and the subadditivity of \( c_\alpha(\cdot) \) on universally measurable sets yield \( c_\alpha(A_2 \setminus K_\ell) = \infty \) for all \( \ell \in \mathbb{N} \). Hence, for every \( \ell \) one can choose a measure \( \tau_\ell \in \mathcal{E}^+_\alpha(A_2 \setminus K_\ell, q) \) with compact support so that
\[
\lim_{\ell \to \infty} \|\tau_\ell\|_\alpha = 0.
\]

Certainly, there is no loss of generality in assuming \( K_\ell \cup S^\tau_\ell_{\mathbb{R}^n} \subset K_{\ell+1} \).

Choose a constraint
\[
\sigma^1 := \lambda + \delta_1, \quad \sigma^2 := \lambda' + \sum_{\ell \in \mathbb{N}} \tau_\ell + \delta_2,
\]
where $\delta_i$, $i = 1, 2$, is a positive bounded Radon measure whose (closed) support coincides with $A_i$.

We assert that the problem of minimizing $\kappa_\alpha(\mu, \mu)$ over the class $\mathcal{E}_\alpha^\sigma(A, 1)$ with the constraint $\sigma$ thus defined is unsolvable.

It follows from the above that $\{\mu_\ell\}_{\ell \in \mathbb{N}}$ with $\mu_1 = \lambda$ and $\mu_\ell^2 = \lambda^* + \tau_\ell$, $\ell \in \mathbb{N}$, belongs to $\mathcal{E}_\alpha^\sigma(A, 1)$, so that

\begin{equation}
(7.14) \quad \kappa_\alpha(\mu_\ell, \mu_\ell) \geq w_\alpha^\sigma(A, 1) \quad \text{for all } \ell \in \mathbb{N},
\end{equation}

and moreover

\begin{equation}
(7.15) \quad \lim_{\ell \to \infty} \kappa_\alpha(\mu_\ell, \mu_\ell) = \lim_{\ell \to \infty} \|\lambda - \lambda^* - \tau_\ell\|_\alpha^2 = \|\lambda - \lambda^*\|_\alpha^2 = \|\lambda\|_G^2.
\end{equation}

On the other hand, for any $\zeta \in \mathcal{E}_\alpha^\sigma(\mathbb{R}^n)$ the balayage $\zeta'$ is in fact the orthogonal projection of $\zeta$ onto the convex cone $\mathcal{E}_\alpha^\sigma(D^c)$, i.e.

\begin{equation}
\|\zeta - \zeta'\|_\alpha < \|\zeta - \nu\|_\alpha \quad \text{for all } \nu \in \mathcal{E}_\alpha^\sigma(D^c), \nu \neq \zeta'
\end{equation}

(see [19, Theorem 4.12] or [20, Theorem 3.1]). For any $\mu \in \mathcal{E}_\alpha^\sigma(A, 1)$ we therefore obtain

\begin{equation}
(7.16) \quad \kappa_\alpha(\mu, \mu) = \|\mu^1 - \mu^2\|_\alpha^2 \geq \|\mu^1 - (\mu^1)'\|_\alpha^2 = \|\mu^1\|_G^2 \geq \|\lambda\|_G^2,
\end{equation}

which in view of the arbitrary choice of $\mu \in \mathcal{E}_\alpha^\sigma(A, 1)$ yields $w_\alpha^\sigma(A, 1) \geq \|\lambda\|_G^2$. As the converse inequality holds in consequence of (7.14) and (7.15), we actually have

\begin{equation}
(7.17) \quad w_\alpha^\sigma(A, 1) = \|\lambda\|_G^2.
\end{equation}

To complete the proof, assume on the contrary that the extremal problem under consideration is solvable. Then there exists the (unique) $\mu \in \mathcal{E}_\alpha^\sigma(A, 1)$ such that all the inequalities in (7.16) are in fact equalities. But this is possible only provided that both $\lambda = \mu^1$ and $\lambda'(D^c) = 1 = \lambda(A_1)$ hold, which contradicts (7.13).

\section*{8. ON CONTINUITY OF THE MINIMIZERS $\lambda_\alpha^\sigma_A$ WITH RESPECT TO $(A, \sigma)$}

Recall that we are working with the $\alpha$-Riesz kernel $\kappa_\alpha(x, y) = |x - y|^{\alpha - n}$ of order $\alpha \in (0, n)$ on $\mathbb{R}^n$, $n \geq 3$. Given an arbitrary (generalized) condenser $A = (A_i)_{i \in I}$, fix a sequence of (generalized) condensers $A_\ell := (A_\ell^j)_{j \in I}$, $\ell \in \mathbb{N}$, with $\text{sign } A_\ell^j = \text{sign } A_i$ such that

\begin{equation}
A_{\ell+1}^j \subset A_\ell^j \quad \text{and } A_i = \bigcap_{k \in \mathbb{N}} A_i^k \quad \text{for any } i \in I, \quad \ell \in \mathbb{N}.
\end{equation}

Fix also constraints $\sigma = (\sigma^i)_{i \in I} \in \mathcal{C}(A)$ and $\sigma_\ell = (\sigma_\ell^i)_{i \in I} \in \mathcal{C}(A_\ell)$ with the properties that

\begin{equation}
(8.1) \quad \sigma_\ell^i \geq \sigma_{\ell+1}^i \geq \sigma^i \quad \text{for all } \ell \in \mathbb{N} \text{ and } i \in I, \quad \text{and}
\end{equation}

then the following statement on continuity holds.

\footnote{Such a $\delta_i$ can be constructed as follows. Consider a sequence of points $x_j$ of $A_i$ which is dense in $A_i$ and define $\delta_i = \sum_{j \in \mathbb{N}} 2^{-j} \varepsilon x_j$. Actually, the summands $\delta_i$, $i = 1, 2$, are added only in order to satisfy (5.1).}
**Theorem 8.1.** Assume in addition that for a certain \( \ell_0 \in \mathbb{N} \) all the hypotheses of Theorem 6.1 or Theorem 7.1 hold for \( A_{\ell_0} \) and \( \sigma_{\ell_0} \) in place of \( A \) and \( \sigma \). Then

\[
G_{a,f}^{\sigma}(A, a, g) = \lim_{\ell \to \infty} G_{a,f}^{\sigma}(A_{\ell}, a, g).
\]

Fix \( \lambda_a^\sigma \in \mathcal{E}_{a,f}^{\sigma}(A, a, g) \), and for every \( \ell \geq \ell_0 \) fix \( \lambda_a^\sigma_{\ell} \in \mathcal{E}_{a,f}^{\sigma}(A_{\ell}, a, g) \); such solutions to Problem 5.1 with the corresponding data exist. Then for every \( m \geq \ell_0 \) the (nonempty) vague cluster set of the sequence \( \{ \lambda_a^\sigma_{\ell} \}_{\ell \geq m} \) is contained in \( \mathcal{E}_{a,f}^{\sigma}(A, a, g) \). Furthermore, \( \lambda_a^\sigma_{\ell} \to \lambda_a^\sigma \) strongly in \( \mathcal{E}_{a}^+(A_m) \), i.e.

\[
\lim_{\ell \to \infty} \| \lambda_a^\sigma_{\ell} - \lambda_a^\sigma \|_{\mathcal{E}_{a}^+(A_m)} = 0,
\]

and hence also vaguely provided that all the \( A_i, i \in I \), are mutually essentially disjoint.

**Proof.** From the monotonicity of \( A_{\ell} \) and \( \sigma_{\ell} \) we get

\[
\mathcal{E}_{a,f}^{\sigma}(A, a, g) \subset \mathcal{E}_{a,f}^{\sigma_{\ell+1}}_{\ell+1}(A_{\ell+1}, a, g) \subset \mathcal{E}_{a,f}^{\sigma_{\ell}}_{\ell}(A_{\ell}, a, g), \quad \ell \in \mathbb{N},
\]

and therefore

\[
-\infty < G_{a,f}^{\sigma_{\ell_0}}(A_{\ell_0}, a, g) \leq \lim_{\ell \to \infty} G_{a,f}^{\sigma_{\ell}}(A_{\ell}, a, g) \leq G_{a,f}^{\sigma}(A, a, g) < \infty,
\]

where the first inequality is valid by (5.2) with \( A_{\ell_0} \) and \( \sigma_{\ell_0} \) in place of \( A \) and \( \sigma \), respectively, while the last one holds by the (standing) assumption (5.3).

According to Theorem 6.1 and Theorem 7.1, under the stated hypotheses for every \( \ell \geq \ell_0 \) there exists a minimizer \( \lambda_{\ell} := \lambda_a^\sigma_{\ell} \in \mathcal{E}_{a,f}^{\sigma}(A_{\ell}, a, g) \). By (8.4), \( \lim_{\ell \to \infty} G_{a,f}(\lambda_{\ell}) \) exists and

\[
-\infty < \lim_{\ell \to \infty} G_{a,f}(\lambda_{\ell}) \leq G_{a,f}(A, a, g) < \infty.
\]

For an arbitrary fixed \( m \geq \ell_0 \) we also see from (8.3) that

\[
\lambda_{\ell} \in \mathcal{E}_{a,f}^{\sigma_{m}}(A_m, a, g) \quad \text{for all} \quad \ell \geq m.
\]

We next proceed by showing that

\[
\| \lambda_{\ell_0} - \lambda_{\ell_1} \|_{\mathcal{E}_{a}^+(A_m)}^2 \leq G_{a,f}(\lambda_{\ell_0}) - G_{a,f}(\lambda_{\ell_1}) \quad \text{whenever} \quad m \leq \ell_1 \leq \ell_2.
\]

For any \( \tau \in (0, 1) \) we have \( \mu := (1 - \tau)\lambda_{\ell_1} + \tau\lambda_{\ell_2} \in \mathcal{E}_{a,f}^{\sigma_{\ell_1}}(A_{\ell_1}, a, g) \), hence \( G_{a,f}(\mu) \geq G_{a,f}(\lambda_{\ell_1}) \). Evaluating \( G_{a,f}(\mu) \) and then letting \( \tau \to 0 \), we get

\[
-\kappa_a(\lambda_{\ell_1}, \lambda_{\ell_2}) + \kappa_a(\lambda_{\ell_1}, \lambda_{\ell_1}) - \langle f, \lambda_{\ell_1} \rangle + \langle f, \lambda_{\ell_2} \rangle \geq 0,
\]

and (8.6) follows. Noting that, by (8.5), the sequence \( G_{a,f}(\lambda_{\ell}), \ell \geq \ell_0, \) is Cauchy in \( \mathbb{R} \), we see from (8.6) that \( \{ \lambda_{\ell} \}_{\ell \geq m} \) is strong Cauchy in \( \mathcal{E}_{a}^{\sigma_m}(A_m, a, g) \).

According to Lemma 5.8 and Remark 2.5, the set \( \mathcal{E}_{a,f}^{\sigma_m}(A_m, a, g) \) is sequentially vaguely compact. Hence there is a (strong Cauchy) subsequence \( \{ \lambda_{k} \} \) of \( \{ \lambda_{\ell} \}_{\ell \geq m} \) such that

\[
\lambda_{k} \to \lambda \quad \text{vaguely as} \quad k \to \infty,
\]

where \( \lambda \) is the sought for limit point.
where \( \lambda \in \mathcal{M}^{\sigma \tau}(A_m, a, g) \). Since the vague limit is unique, \( \lambda^i \) is carried by \( A^m_i \) for every \( m \geq \ell_0 \), and hence by \( A_i = \bigcap_{m \geq \ell_0} A^m_i \). As the vague limit of the \((positive)\) measures \( \sigma^i_{\ell_k} - \lambda^i_{\ell_k} \) is likewise the positive measure \( \sigma^i - \lambda^i \) (see (8.1) and (8.7)), we altogether get (8.8)

\[
\lambda \in \mathcal{M}^{\sigma \tau}(A, a, g).
\]

Assume first that \( A_{\ell_0} \) and \( \sigma_{\ell_0} \) satisfy the assumptions of Theorem 6.1. Then so do \( A_\ell \) and \( \sigma_\ell \) for every \( \ell \geq \ell_0 \) and hence, according to Lemma 6.2, all the potentials \( \kappa_\alpha(x, \lambda^i_\ell) \) with \( \ell \geq \ell_0 \) and \( i \in I \) are \((bounded\ and\ finitely)\ continuous\) on the compact sets \( A^\ell \). Applying arguments similar to those that have been applied in the proof of Theorem 6.1 we then conclude from (8.7) and (8.8) that \( \lambda \in \mathcal{E}_\alpha^\sigma(A, a, g) \) and moreover

\[
\lambda_{\ell_k} \to \lambda \quad \text{strongly in } \mathcal{E}_\alpha^+(A_m) \quad \text{as } k \to \infty.
\]

In view of Lemma 5.9 applied to \( A_m \) instead of \( A \), we get from (8.5), (8.7), and (8.9)

\[
-\infty < G_{\alpha, f}(\lambda) \leq \lim_{k \to \infty} G_{\alpha, f}(\lambda_{\ell_k}) = \lim_{k \to \infty} G_{\alpha, f}^\sigma(\lambda_{\ell_k}, A_{\ell_k}, a, g) \leq G_{\alpha, f}(\lambda, a, g) < \infty,
\]

the first inequality being valid by (4.4) and (4.7). Thus \( \lambda \in \mathcal{E}_\alpha^\sigma_f(A, a, g) \) (see (8.8)), and hence \( G_{\alpha, f}(\lambda) \geq G_{\alpha, f}^\sigma(A, a, g) \). Combined with the last display, this proves (8.2) and also (8.10)

\[
\lambda \in \mathcal{E}_\alpha^\sigma_f(A, a, g).
\]

Assume now instead that \( A_{\ell_0} \) and \( \sigma_{\ell_0} \) satisfy the assumptions of Theorem 7.1. Applying Theorem 7.3, we infer from what has been obtained above (see (8.7) and (8.8)) that \( \lambda \in \mathcal{E}_\alpha^\sigma(A, a, g) \) and \( \lambda_{\ell_k} \to \lambda \) (vaguely and) strongly in \( \mathcal{E}_\alpha^+(A_m) \). Then in the same way as it has been established just above, we again get (8.2) and (8.10).

It has thus been shown that, under the hypotheses of Theorem 8.1, relation (8.2) holds and \( \{\lambda_{\alpha, f}^{\sigma \tau}\}_{\ell \geq m} \), being strong Cauchy in \( \mathcal{E}_\alpha^+(A_m) \), converges strongly in \( \mathcal{E}_\alpha^+(A_m) \) to any of its vague cluster points \( \lambda \), and also that this \( \lambda \) solves Problem 5.1 for the condenser \( A \) and the constraint \( \sigma \). In the case where all the \( A_i, i \in I \), are mutually essentially disjoint, such a solution is determined uniquely according to Lemma 5.4, so that the vague cluster set of \( \{\lambda_{\alpha, f}^{\sigma \tau}\}_{\ell \geq m} \) reduces to the given \( \lambda \). Hence \( \lambda_{\alpha, f}^{\sigma \tau} \to \lambda \) also vaguely [5, Chapter I, Section 9, n° 1, Corollary].

9. The \( f \)-weighted vector potential of a minimizer \( \lambda \in \mathcal{E}_\alpha^\sigma_f(A, a, g) \)

Theorem 9.2 below establishes a description to the \( f \)-weighted \( \alpha \)-Riesz vector potentials \( W_{\alpha, f}^\lambda = (W_{\alpha, f}^{\lambda, i})_{i \in I} \) (see (4.1)) of the solutions to Problem 5.1 (provided they exist), and it also singles out their characteristic properties.

Lemma 9.1. For \( \lambda \in \mathcal{E}_\alpha^\sigma_f(A, a, g) \) to solve Problem 5.1, it is necessary and sufficient that

\[
\sum_{i \in I} \langle W_{\alpha, f}^{\lambda, i}, \nu^i - \lambda^i \rangle \geq 0 \quad \text{for all } \nu \in \mathcal{E}_\alpha^\sigma_f(A, a, g).
\]
Proof. By direct calculation, for any $\underline{\mu},\nu \in \mathcal{E}_{\alpha,f}^{\sigma}(A,a,g)$ and any $h \in (0,1]$ we obtain
\[
G_{\alpha,f}(h\nu + (1-h)\underline{\mu}) - G_{\alpha,f}(\underline{\mu}) = 2h \sum_{i \in I} \langle W_{\alpha,f}^{\mu_i},\nu_i - \mu_i \rangle + h^2 \|\nu - \underline{\mu}\|_{\mathcal{E}_{\alpha}^+(A)}^2.
\]
If $\underline{\mu} = \lambda$ solves Problem (5.1) then the left hand (and hence the right hand) side of this display is $\geq 0$, for the class $\mathcal{E}_{\alpha,f}^\sigma(A,a,g)$ is convex, which leads to (9.1) by letting $h \to 0$ (after division by $h$). Conversely, if (9.1) holds, then the preceding formula with $\underline{\mu} = \lambda$ and $h = 1$ implies that $G_{\alpha,f}(\nu) \geq G_{\alpha,f}(\lambda)$ for all $\nu \in \mathcal{E}_{\alpha,f}^{\sigma}(A,a,g)$, hence $\lambda \in \mathcal{E}_{\alpha,f}^{\sigma}(A,a,g)$. □

Theorem 9.2. Assume that for every $i \in I$, $\kappa_\alpha(\cdot,\sigma^i)$ is (finitely) continuous on $A_i$ and upper bounded on some neighborhood of $\omega_{\mathbb{R}^n}$, and
\[
\sigma(\cdot)\setminus \hat{A}_i^\delta = 0,
\]
$\hat{A}_i^\delta$ being defined by (4.10). If moreover Case I takes place and
\[
g_i(x) \leq M_i < \infty \text{ for all } x \in \mathbb{R}^n, \ i \in I,
\]
then for any given $\lambda \in \mathcal{E}_{\alpha,f}^{\sigma}(A,a,g)$ the following two assertions are equivalent:

(i) $\lambda \in \mathcal{E}_{\alpha,f}^\sigma(A,a,g)$.

(ii) There exists $(w^i_\lambda)_{i \in I} \in \mathbb{R}^{|I|}$ (where $|I| := \text{Card } I$) such that for all $i \in I$
\[
W_{\alpha,f}^{\lambda_i} \geq w^i_\lambda g_i \quad (\sigma^i - \lambda^i)$-a.e. on $A_i,$
\]
\[
W_{\alpha,f}^{\lambda_i} \leq w^i_\lambda g_i \quad \text{everywhere on } S_{\mathbb{R}^n}^{\lambda_i}.
\]

Proof. Since for every $i \in I$, $\kappa_\alpha(\cdot,\sigma^i)$ is continuous on $A_i$, so is $\kappa_\alpha(\cdot,\mu^i)$, where $(\mu^i)_{i \in I}$ is any measure from $\mathcal{M}^\sigma(A,a,g)$ (see Lemma 6.2). By [27, Theorem 1.7], all these potentials are then continuous on all of $\mathbb{R}^n$. As $\kappa_\alpha(\cdot,\sigma^i)$, and hence $\kappa_\alpha(\cdot,\mu^i)$ is bounded on some neighborhood of $\omega_{\mathbb{R}^n}$, it thus follows that all these potentials are bounded on all of $\mathbb{R}^n$. Another consequence is that
\[
\sigma^i|_K \in \mathcal{E}_{\alpha}^+(K) \quad \text{for any compact } K \subset \mathbb{R}^n,
\]
and hence the measures $\sigma^i, i \in I$, are $c_0$-absolutely continuous.

Suppose first that (i) holds, i.e. $\lambda \in \mathcal{E}_{\alpha,f}^\sigma(A,a,g)$ solves Problem (5.1). To verify (ii), fix $i \in I$. For every $\mu = (\mu^\ell)_{\ell \in I} \in \mathcal{E}_{\alpha,f}^\sigma(A,a,g)$ write $\mu_i := (\mu^\ell_i)_{\ell \in I}$ where $\mu^\ell_i := \mu^\ell$ for all $\ell \neq i$ and $\mu^i_i = 0$; then $\mu_i \in \mathcal{E}_{\alpha,f}^\sigma(A)$. Also define $\tilde{f}_i := f_i + (\kappa_\alpha)_{\hat{A}_i}^\lambda$; then by substituting (3.6) we obtain
\[
\tilde{f}_i(x) = f_i(x) + s_i \sum_{\ell \in I, \ell \neq i} s_{\ell}\kappa_\alpha(x,\lambda^\ell), \ x \in \mathbb{R}^n.
\]
Being of the class $\Psi(\mathbb{R}^n)$, $f_i$ is l.s.c. on $\mathbb{R}^n$ and $\geq 0$. Combined with the properties of $\kappa_\alpha(\cdot,\lambda^\ell), \ell \in I$, established above, this implies that
\[
W_{\alpha,f}^{\lambda_i} := \kappa_\alpha(\cdot,\lambda^i) + \tilde{f}_i,
\]
is l.s.c. on $\mathbb{R}^n$ and lower bounded. Also note that $W^{\lambda_i}_{\alpha_i,\tilde{f}_i}$ is finite on $\tilde{A}^i_\theta$ (see (4.10)).

Furthermore, by (3.5) and (4.2) we get for any $\mu \in \mathcal{E}_{\alpha,\tau}(A, a, g)$ with the additional property that $\mu_i = \lambda_i$ (in particular, for $\mu = \lambda$)

$$G_{\alpha,\tau}(\mu) = G_{\alpha,\tau}(\lambda_i) + G_{\alpha,\tilde{f}_i}(\mu_i).$$

Combined with $G_{\alpha,\tau}(\mu) \geq G_{\alpha,\tau}(\lambda_i)$, this yields $G_{\alpha,\tilde{f}_i}(\mu_i) \geq G_{\alpha,\tilde{f}_i}(\lambda_i)$, and hence $\lambda_i$ minimizes $G_{\alpha,\tilde{f}_i}(\nu)$ where $\nu$ ranges over $\mathcal{E}_{\alpha,\tilde{f}_i}(A_i, a_i, g_i)$. This enables us to show that there exists $w_{\lambda_i} \in \mathbb{R}$ such that

$$W^{\lambda_i}_{\alpha_i,\tilde{f}_i} \geq w_{\lambda_i} g_i \ (\sigma^i - \lambda^i)\text{-a.e. on } A_i,$$

(9.9)

$$W^{\lambda_i}_{\alpha_i,\tilde{f}_i} \leq w_{\lambda_i} g_i \text{ everywhere on } S^{\lambda_i}_{\mathbb{R}^n}.$$

(9.10)

Indeed, (9.9) holds with

$$w_{\lambda_i} := L_i := \sup \{ t \in \mathbb{R} : W^{\lambda_i}_{\alpha_i,\tilde{f}_i} \geq tg_i \ (\sigma^i - \lambda^i)\text{-a.e. on } A_i \}.$$  

In turn, (9.9) with $w_{\lambda_i} = L_i$ implies that $L_i < \infty$, because

$$\tilde{W}^{\lambda_i}_{\alpha_i,\tilde{f}_i}(x) := \frac{W^{\lambda_i}_{\alpha_i,\tilde{f}_i}(x)}{g_i(x)} < \infty$$

for all $x \in \tilde{A}^i_\theta$ (see (4.5)), hence $(\sigma^i - \lambda^i)$-a.e. on $A_i$ by (9.2). Also, $L_i > -\infty$ since in consequence of (9.3), $W^{\lambda_i}_{\alpha_i,\tilde{f}_i}$ is lower bounded on $\mathbb{R}^n$.

We next proceed by establishing (9.10) with $w_{\lambda_i} = L_i$. Assume, on the contrary, that this does not hold. For any $w \in \mathbb{R}$ write

$$A_i^+(w) := \{ x \in A_i : W^{\lambda_i}_{\alpha_i,\tilde{f}_i}(x) > wg_i(x) \},$$

$$A_i^-(w) := \{ x \in A_i : W^{\lambda_i}_{\alpha_i,\tilde{f}_i}(x) < wg_i(x) \}.$$

By the lower semicontinuity of $\tilde{W}^{\lambda_i}_{\alpha_i,\tilde{f}_i}$ on $\mathbb{R}^n$, there is $w_i \in (L_i, \infty)$ such that $\lambda_i(A_i^+(w_i)) > 0$. At the same time, as $w_i > L_i$, (9.9) with $w_{\lambda_i} = L_i$ yields $(\sigma^i - \lambda^i)(A_i^-(w_i)) > 0$. Therefore, one can choose compact sets $K_1 \subset A_i^+(w_i)$ and $K_2 \subset A_i^-(w_i)$ so that

$$0 < \langle g_i, \lambda_i |_{K_1} \rangle < \langle g_i, (\sigma^i - \lambda^i) |_{K_2} \rangle.$$

(9.11)

Write $\tau^i := (\sigma^i - \lambda^i) |_{K_2}$; then $\kappa_i(\tau^i, \tau^i) < \infty$ by (9.6). Since $\langle W^{\lambda_i}_{\alpha_i,\tilde{f}_i}, \tau^i \rangle \leq \langle w_i g_i, \tau^i \rangle < \infty$, we get $\langle \tilde{f}_i, \tau^i \rangle < \infty$ in view of (9.8). Define

$$\theta^i := \lambda^i - \lambda^i |_{K_1} + c_i \tau^i,$$

where $c_i := \langle g_i, \lambda^i |_{K_1} \rangle / \langle g_i, \tau^i \rangle$.  

$$\sigma^i := (\sigma^i - \lambda^i) |_{K_2}$$.  


Noting that \( c_i \in (0,1) \) by (9.11), we see by straightforward verification that \( \langle g_i, \theta^i \rangle = a_i \) and \( \theta^i \leq \sigma^i \), hence \( \theta^i \in \mathcal{E}^{\sigma^i}_{\alpha,i}(A_i, a_i, g_i) \). On the other hand,

\[
\langle W^{\lambda^i}_{\alpha,fi}, \theta^i - \lambda^i \rangle = \langle W^{\lambda^i}_{\alpha,fi} - w_i g_i, \theta^i - \lambda^i \rangle \\
= -\langle W^{\lambda^i}_{\alpha,fi} - w_i g_i, \lambda^i |_{K_i} \rangle + c_i \langle W^{\lambda^i}_{\alpha,fi} - w_i g_i, \tau^i \rangle < 0,
\]

which is impossible in view of the scalar version of Lemma 9.1. The contradiction obtained establishes (9.10).

Substituting (9.7) into (9.8) and then comparing the result obtained with (3.6) and (4.1), we get

\[
W^{\lambda^i}_{\alpha,fi} = W^{\lambda^i}_{\alpha,f}.
\]

Combined with (9.9) and (9.10), this proves (9.4) and (9.5) with \( w^i_\lambda : = w_{\lambda^i}, i \in I \).

Conversely, suppose (ii) holds. On account of (9.12), for every \( i \in I \) relations (9.9) and (9.10) are then fulfilled with \( w^i_{\lambda^i} : = w^i_{\lambda} \) and \( f_i \) defined by (9.7). This yields \( \lambda'(A^i_\lambda(w^i_{\lambda})) = 0 \) and \( (\sigma^i - \lambda')(A^i_\lambda(w^i_{\lambda})) = 0 \). For any \( \nu \in \mathcal{E}^{\sigma}_\alpha(A,a,g) \) we therefore get

\[
\langle W^{\lambda^i}_{\alpha,f}, \nu^i - \lambda^i \rangle = \langle W^{\lambda^i}_{\alpha,fi} - w^i_{\lambda} g_i, \nu^i - \lambda^i \rangle \\
= \langle W^{\lambda^i}_{\alpha,fi} - w^i_{\lambda} g_i, \nu^i |_{A^i_\lambda(w^i_{\lambda})} \rangle + \langle W^{\lambda^i}_{\alpha,fi} - w^i_{\lambda} g_i, (\nu^i - \lambda^i) |_{A^i_\lambda(w^i_{\lambda})} \rangle \geq 0.
\]

Summing up these inequalities over all \( i \in I \), in view of the arbitrary choice of \( \nu \in \mathcal{E}^{\sigma}_\alpha(A,a,g) \) we conclude from Lemma 9.1 that \( \lambda \) is a solution to Problem 5.1. \( \square \)

10. Duality relation between non-weighted constrained and weighted unconstrained minimum \( \alpha \)-Riesz energy problems for scalar measures

We now present an extension of [15, Corollary 2.15] to Riesz kernels. Throughout this section, set \( I = \{1\}, s_1 = +1, g_1 = 1 \) and \( a_1 = 1 \). Fix a closed set \( F = A_1 \) in \( \mathbb{R}^n \) with \( c_0(F) > 0 \) that may coincide with the whole of \( \mathbb{R}^n \) and a constraint \( \sigma \in 2.M(F) \) with \( 1 < \sigma(F) < \infty \). By Theorem 7.1 (see also [13 Theorem 5.1] with \( D = \mathbb{R}^n \)), there exists \( \lambda \in \mathcal{E}^{\sigma}_\alpha(F,1) := 2.M(F) \cap \mathcal{E}^{\sigma}_\alpha(F,1) \) whose \( \alpha \)-Riesz energy is minimal in this class, i.e.

\[
\|\lambda\|^2_\alpha = \inf_{\mu \in \mathcal{E}^{\sigma}_\alpha(F,1)} \|\mu\|^2_\alpha,
\]

and this \( \lambda \) is unique according to Lemma 5.4 (with \( f = 0 \)). Write \( q : = 1/(\sigma(F) - 1) \).

**Theorem 10.1.** Assume in addition that \( 0 < \alpha \leq 2 \) and that \( \kappa_\alpha(\cdot, \sigma) \) is (finitely) continuous and upper bounded on \( F \) (hence, on \( \mathbb{R}^n \) according to [27, Theorems 1.5, 1.7]). Then the measure \( \theta := q(\sigma - \lambda) \) is the solution to (the unconstrained) Problem 4.1 with the external field \( f := -q\kappa_\alpha(\cdot, \sigma) \), i.e. \( \theta \in \mathcal{E}^{\sigma}_\alpha(F,1) := 2.M(F,1) \cap \mathcal{E}^{\sigma}_\alpha(F) \) and

\[
G_{\alpha,f}(\theta) = \inf_{\mu \in \mathcal{E}^{\sigma}_\alpha(F,1)} G_{\alpha,f}(\mu).
\]
Moreover, there exists $\eta \in (0, \infty)$ such that
\begin{align}
W_{\alpha,f}^\theta &= -\eta \text{ on } S(\theta), \quad (10.3) \\
W_{\alpha,f}^\theta &\geq -\eta \text{ on } \mathbb{R}^n, \quad (10.4)
\end{align}
and these two relations \((10.3)\) and \((10.4)\) determine uniquely the solution to Problem 4.1 among the measures of the class $\mathcal{E}^+_{\alpha,f}(F,1)$.

\textbf{Remark 10.2.} The external field $f$ thus defined satisfies Case II with $\zeta = -q\sigma \leq 0$, and this $f$ is lower bounded on $\mathbb{R}^n$, for $\kappa_\alpha(\cdot,\sigma)$ is upper bounded by assumption.

\textbf{Proof.} Under the stated assumptions, relation \((9.5)\) for the solution $\lambda$ to the (constrained) problem \((10.1)\) takes the form $\kappa_\alpha(\cdot,\lambda) \leq w$ on $S(\lambda)$, where $w \in (0, \infty)$. By the Frostman maximum principle, which can be applied because $\alpha \leq 2$, we thus have
\[\kappa_\alpha(\cdot,\lambda) \leq w \text{ on } \mathbb{R}^n.\]
Combined with \((9.4)\), this gives
\[\kappa_\alpha(\cdot,\lambda) = w \text{ on } S(\sigma - \lambda),\]
for $\kappa_\alpha(\cdot,\lambda)$ is (finitely) continuous on $\mathbb{R}^n$ along with $\kappa_\alpha(\cdot,\sigma)$ by Lemma 6.2. In the notations used in Theorem 10.1 these two displays can alternatively be rewritten as \((10.3)\) and \((10.4)\), respectively, with $\eta := qw$. In turn, \((10.3)\) and \((10.4)\) imply that $\theta$, $f$ and $-\eta$ satisfy \cite[Eqs. 7.9, 7.10]{42}, which according to \cite[Theorem 7.3]{42} establishes \((10.2)\). \hfill \square

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