A global fractional Caccioppoli-type estimate for solutions to nonlinear elliptic problems with measure data

by

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Abstract. We prove a global fractional differentiability result via the fractional Caccioppoli-type estimate for solutions to nonlinear elliptic problems with measure data. This work is inspired by the recent paper [B. Avelin, T. Kuusi and G. Mingione, Arch. Ration. Mech. Anal. 227 (2018), 663–714], devoted to the local fractional regularity for solutions to nonlinear elliptic equations with measure right-hand side, of type $-\text{div} A(\nabla u) = \mu$ in the limiting case. Being a contribution to recent results of identifying function classes in which solutions to such problems could be defined, our aim is to establish a global regularity result in weighted fractional Sobolev spaces, where the weights are powers of the distance function to the boundary of the smooth domain.

1. Introduction and main results. In this study, we are interested in the following Dirichlet problem with measure data:

$$
\begin{align*}
-\text{div} A(\nabla u) &= \mu & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega.
\end{align*}
$$

Here, $\Omega$ is an open bounded domain of $\mathbb{R}^n$ ($n \geq 2$); $\mu$ is a Borel measure in $\Omega$ with finite mass; and the nonlinearity $A$ is a Carathéodory vector field defined on $\Omega \times \mathbb{R}^n$ and satisfying the following ellipticity and growth assumptions:

$$
\begin{align*}
|A(z)| + |\partial_z A(z)||z|^2 + \kappa^2 &\leq c_A(|z|^2 + \kappa^2)^{(p-1)/2}, \\
c_A^{-1}(|z|^2 + \kappa^2)^{(p-2)/2} |\zeta|^2 &\leq (\partial_z A(z)\zeta, \zeta),
\end{align*}
$$

for all $z, \zeta \in \mathbb{R}^n$ and $x \in \Omega$. In (1.2), we only consider $p > 2 - 1/n$, and $c_A > 1$ is the given ellipticity constant; $\kappa \in [0, 1]$ represents the degeneracy parameter to distinguish between two cases of problem (1.1) in our study. In particular, $\kappa = 0$ for the degenerate case and $\kappa > 0$ the non-degenerate case.

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Moreover, when \(2 - 1/n < p < 2\), we further impose the following symmetry condition on \(\mathcal{A}\):

\[
\partial_z \mathcal{A} \text{ is symmetric, i.e., } \partial_{z_j} \mathcal{A}_i = \partial_{z_i} \mathcal{A}_j, \quad \forall i, j \in \{1, \ldots, n\}.
\]

(1.3)

We notice that a significant case is \(\mathcal{A}(z) = |z|^{p-2}z\), which gives rise to the \(p\)-Laplace operator \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\). And although we only consider general degenerate equations of the type (1.1), it is straightforward to derive from our results regularity results for the \(p\)-Laplace equation.

In recent years, a great deal of effort has gone into investigating nonlinear elliptic/parabolic equations involving measure data. Apart from their theoretical interest, these equations have also entered several models describing numerous phenomena in applied sciences, for instance non-Newtonian fluids, electrorheological fluids, flows in porous medium, dislocation and image restoration problems, etc. Together with studies focused on the existence and uniqueness of solutions to (1.1), the integrability and differentiability properties of solutions also attracted a lot of attention. Starting from the fact that when \(p = 2\), the equation \(-\Delta u = \mu\) satisfies

\[
\mu \in L^q_{\text{loc}}(\Omega) \implies \nabla u \in W^{1,q}_{\text{loc}}(\Omega), \quad 1 < q < \infty,
\]

which is no longer true for \(q = 1\), the fractional Sobolev spaces were studied to get the maximal regularity estimates. For instance, in the recent paper by Avelin et al. [1], it is proved that

\[
\mu \in L^1_{\text{loc}}(\Omega) \implies \nabla u \in W^{\sigma, 1}_{\text{loc}}(\Omega), \quad 0 < \sigma < 1.
\]

(1.4)

Moreover, in the same paper, the authors prove an important regularity result for local SOLA solutions to problem (1.1) when \(2 - 1/n < p \leq 2\), which we re-state for the readers’ convenience:

**Theorem 1.1** (Limiting case of Calderón–Zygmund theory, [1]). Let \(p > 2 - 1/n\), \(\Omega\) be an open subset in \(\mathbb{R}^n\) and \(\mu\) be a Borel measure in \(\Omega\) with finite mass. Assume that the operator \(\mathcal{A}\) satisfies assumptions (1.2)–(1.3) and \(u \in W^{1,\max\{1,p-1\}}_{\text{loc}}(\Omega)\) is a SOLA solution to (1.1). Then for any \(\sigma \in (0, 1)\),

\[
\mathcal{A}(\nabla u) \in W^{\sigma, 1}_{\text{loc}}(\Omega).
\]

Moreover, there is a constant \(C = C(c_{\mathcal{A}}, \sigma, n, p) > 0\) such that

\[
\int_{B_{R/2}} \int_{B_{R/2}} \frac{|\mathcal{A}(\nabla u(x)) - \mathcal{A}(\nabla u(y))|}{|x - y|^{n+\sigma}} \, dx \, dy \\ \leq \frac{C}{R^\sigma} \int_{B_R} |\mathcal{A}(\nabla u(x))| \, dx + \frac{C}{R^\sigma} \left[ \frac{\|\mu\|_{(B_R)}}{R^{n-1}} \right]
\]

for every ball \(B_R \subset \Omega\).
It is worth noting that for nonlinear elliptic problems with measure data, weak solutions may not be unique (see a counterexample in [19]). Therefore, a rather extensive literature is devoted to various definitions of solutions to such equations, where the existence and uniqueness are possible: entropy solutions [3], renormalized solutions [9], SOLA solutions [4, 5] (see Definition 2.1). In this article we shall adopt the concept of SOLA solutions when dealing with the measure data problem (1.1), whose definition will be specified in Section 2.2 below.

For $p > 2 - 1/n$, an impressive result $\mathcal{A}(\nabla u) \in W^{\sigma,1}_{\text{loc}}(\Omega)$ comes along with (1.6) in Theorem 1.1: a complete linearization effect of the equation with respect to fractional differentiability of weak solutions. In addition, one concludes that for the nonlinear problem (1.1), the results obtained are exactly the same as in the linear case $-\Delta u = \mu$ via fundamental solutions as in (1.4). Moreover, there have been a number of contributions on local fractional regularity for solutions to measure data problems, such as the differentiability for $\nabla u$ when $p = 2$ in [16]; the Calderón–Zygmund type estimates for $\mathcal{A}(\nabla u)$ in the scale of Besov or Triebel–Lizorkin spaces when $p \geq 2, n = 2$ obtained in [2]; results for the vectorial case in [14]; some results concerning global gradient estimates in Lorentz or Morrey spaces in [17, 20–22] with the singular range of $p$, i.e. $1 < p \leq 2 - 1/n$; and many further interesting results in [6, 7, 11–13, 18, 23, 24] for those readers interested in pursuing further studies of this issue.

Motivated by [1], in this work we will study the limiting case of Calderón–Zygmund estimates in Theorem 1.1 up to the boundary with a smoothness assumption on $\partial \Omega$. Restricting to the case $p > 2 - 1/n$, we construct an appropriate function class to achieve a global regularity result that corresponds to the ones proved in [1, Theorem 1.2]. More precisely, the use of weighted fractional Sobolev spaces equipped with weights chosen as a power of the distance to a point at the boundary (see Definition 2.5 below) enables us to apply local results of Theorem 1.1 to a set of sufficiently small balls in $\Omega$.

We now state our main results in the following two theorems.

**Main Theorem 1.2.** Let $p > 2 - 1/n, \sigma \in (0, 1), \Omega$ be an open bounded and smooth domain in $\mathbb{R}^n$ and $\mu$ be a Borel measure in $\Omega$ with finite mass. Assume that the operator $\mathcal{A}$ satisfies assumptions (1.2)–(1.3) and $u \in W^{1,\max\{1,p-1\}}(\Omega)$ is a SOLA solution to (1.1). Then for all $\alpha, \beta > 0$ satisfying $\alpha + \beta > \sigma$, one has

\[(1.7) \quad \mathcal{A}(\nabla u) \in W^{\sigma,1}(\Omega; \alpha, \beta).\]

Moreover, there is a constant $C = C(c_A, \sigma, n, p, \alpha, \beta, \text{diam}(\Omega)) > 0$ such that
\[
\int_\Omega \int_\Omega d(x)^\alpha d(y)^\beta \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x - y|^{n+\sigma}} \, dx \, dy \leq C \left( \int_\Omega |A(\nabla u(x))| \, dx + |\mu|(\Omega) \right),
\]

where \( d(x) := \text{dist}(x, \partial \Omega) \).

**Main Theorem 1.3.** Let \( p \geq 2, \sigma \in (0, 1) \), \( \Omega \) be an open bounded and smooth domain in \( \mathbb{R}^n \) and \( \mu \) be a Borel measure in \( \Omega \) with finite mass. Assume that the operator \( A \) satisfies assumptions (1.2) and \( u \in W^{1,p-1}(\Omega) \) is a SOLA solution to (1.1). Then for all \( \alpha, \beta > 0 \) satisfying \( \alpha + \beta > \sigma \) and \( \frac{1}{p-1} \leq \gamma \leq 1 \) one has \( E(\nabla u) \in W^{\gamma\sigma,1/\gamma}(\Omega; \alpha, \beta) \), where the function \( E : \mathbb{R}^n \to [0, \infty) \) is defined by
\[
E(\xi) = (|\xi| + \kappa)^{\gamma p - \gamma - 1}\xi, \quad \xi \in \mathbb{R}^n.
\]
In particular, there is a constant \( C = C(c_A, \sigma, n, p, \alpha, \beta, \gamma) > 0 \) such that
\[
[\mathbb{E}(\nabla u)]_{W^{\sigma,1}(\Omega; \alpha, \beta)} \leq C \left( \int_\Omega |A(\nabla u(x))| \, dx + |\mu|(\Omega) \right)^\gamma.
\]

We remark that when \( \gamma = 1 \), one can obtain (1.8) from (1.10). Moreover, for \( \gamma = \frac{1}{p-1} \), one has
\[
\nabla u \in W^{\sigma,1/p-1} \quad \text{for every} \quad \sigma \in (0, 1).
\]

The remainder of our paper is organized as follows. In the next section we introduce some preliminaries and function spaces, focusing on the concept of weighted fractional Sobolev space. In Section 3, we establish some gradient estimates near the boundary of the domain. We end up with a section devoted to proving our main results, and to deducing a global fractional Caccioppoli-type inequality for solutions to the measure data problem (1.1).

### 2. Preliminaries and function spaces

**2.1. Basic notation.** The general constants, varying from line to line, will always be denoted by \( C \), and the dependencies of \( C \) will be highlighted by writing e.g. \( C = C(n, p, \sigma, c_A) \). We write \( B_\rho(\xi) \) for the ball of radius \( \rho \) centered at \( \xi \in \Omega \); we write \( B_\rho \) if the center is irrelevant. Further, for \( 1 \leq q < \infty \), \( L^q(\Omega) \) denotes the usual Lebesgue space; and \( W^{s,q}(\Omega) \) stands for the Sobolev space. Finally, for a given measurable subset \( \mathcal{O} \subset \mathbb{R}^n \), the integral average of \( \varphi \in L^1(\mathcal{O}) \) will be denoted by
\[
(\varphi)_{\mathcal{O}} = \frac{1}{|\mathcal{O}|} \int_\mathcal{O} \varphi(\xi) \, d\xi = \frac{1}{|\mathcal{O}|} \int_\mathcal{O} \varphi(\xi) \, d\xi,
\]
where \( |\mathcal{O}| \) stands for the Lebesgue measure of \( \mathcal{O} \) in \( \mathbb{R}^n \).
2.2. The notion of solution: SOLA. In a natural way, one defines distributional solutions to (1.1):

**Definition 2.1 (Distributional solution).** A function \( u \in W^{1,1}_0(\Omega) \) is said to be a very weak solution to (1.1) if \( A(\nabla u) \in L^1(\Omega; \mathbb{R}^n) \) and for all \( \varphi \in C_c^\infty(\Omega) \),

\[
\int_\Omega \langle A(\nabla u), \nabla \varphi \rangle \, dx = \int_\Omega \varphi \, d\mu.
\]

This type of distributional solution usually exists. However, according to the counterexample by Serrin [19], the problem of uniqueness of such solutions presents itself. For this reason, many possible definitions have been proposed such as the notions of entropy solutions, SOLA (Solutions Obtained as Limits of Approximations), renormalized solutions, etc. (see [4, 5, 8, 16] and references therein concerning nonlinear measure data problems). For our purposes, we restrict ourselves to the notion of SOLA, whose definition is recalled below.

**Definition 2.2 (Local SOLA, [1, 4, 5]).** A function \( u \in W^{1,1}_{loc}(\Omega) \) is a local SOLA to (1.1) under assumptions (1.2) if one can find a sequence \( \{u_k\} \subset W^{1,p}_{loc}(\Omega) \) of functions satisfying

\[
-\text{div}(A(x, \nabla u_k)) = \mu_k \in L^\infty_{loc}(\Omega)
\]

such that \( u_k \rightharpoonup u \) weakly in \( W^{1,1}_{loc}(\Omega) \), where the sequence \( \{\mu_k\} \) converges weakly in the sense of measures and satisfies

\[
\limsup_k |\mu_k|(B) \leq |\mu|(B)
\]

for any ball \( B \Subset \Omega \).

**Remark 2.3.** By the arguments in [4] and [1, Proposition 2.2] with \( p > 2 - \frac{1}{n} \), if \( u \in W^{1,1}_{loc}(\Omega) \) is a local SOLA to problem (1.1) and \( \{u_k\} \) is the sequence of approximating solutions in Definition 2.2, then \( \{u_k\} \) converges strongly to \( u \) in \( W^{1,q}_{loc}(\Omega) \) for any \( q < \min \{p, \frac{n(p-1)}{n-1}\} \). In particular, this implies that there exists a very weak solution \( u \in W^{1,\max\{1,p-1\}}_{loc}(\Omega) \) to (1.1).

2.3. Function spaces. Let us first recall the classical definition of fractional Sobolev spaces in the sense of Gagliardo (see [1, 10] for instance).

**Definition 2.4 (Gagliardo’s fractional Sobolev space).** Let \( \Omega \) be a general open set in \( \mathbb{R}^n \) with \( n \geq 2 \) and let \( s \in (0,1) \) be a fractional exponent. Then, for any \( 1 \leq q < \infty \), the fractional Sobolev space \( W^{s,q}(\Omega) \) is

\[
W^{s,q}(\Omega) = \left\{ v \in L^q(\Omega) : \frac{|v(x) - v(y)|}{|x - y|^{n/q + s}} \in L^q(\Omega \times \Omega) \right\},
\]
and this is a Banach space when endowed with the Gagliardo-type norm
\[ \|v\|_{W^{s,q}(\Omega)} = \left[ \int_{\Omega} |v(x)|^q \, dx + \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^q}{|x-y|^{n+sq}} \, dx \, dy \right]^{1/q}. \]

In what follows, we will denote by
\[ [v]_{W^{s,q}(\Omega)} := \left[ \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^q}{|x-y|^{n+sq}} \, dx \, dy \right]^{1/q} \]
the Gagliardo seminorm of \( v \). The space \( W^{s,q}(\Omega) \) is the interpolated space between \( L^q(\Omega) \) and \( W^{1,q}(\Omega) \).

There are some well-known Sobolev embedding theorems in the case of fractional spaces. For instance, by [10, Proposition 2.1], \( W^{t,q}(\Omega) \subseteq W^{s,q}(\Omega) \) for all \( t \in (s, 1) \).

On the other hand, with an additional assumption on the regularity of the boundary of \( \Omega \), in particular for bounded Lipschitz domains \( \Omega \) (see [10, Proposition 2.2]), one has \( W^{1,q}(\Omega) \subseteq W^{s,q}(\Omega) \).

Let us also recall the fractional Sobolev embedding. If \( \Omega \subset \mathbb{R}^n \) is a domain with \( C^{0,1} \)-boundary and \( sq < n \), then \( W^{s,q}(\Omega) \hookrightarrow L^{\frac{nq}{n-sq}}(\Omega) \), with continuous embedding. In other words, if \( \Omega \) is a bounded Lipschitz domain and \( sq < n \), then one can find a constant
\[ C = C(n, q, s, \text{diam}(\Omega), |\partial\Omega|_{0,1}) > 0 \]
such that for \( v \in L^{\frac{nq}{n-sq}}(\Omega) \),
\[ \|v\|_{L^{\frac{nq}{n-sq}}(\Omega)} \leq C \|v\|_{W^{s,q}(\Omega)}. \]
Moreover, a Poincaré-type inequality in fractional Sobolev spaces can be given (see [15, Section 4] for the detailed proof). It states that there exists a positive constant \( C = C(n, q, s) > 0 \) such that
\[ \int_{B_R} |v - (v)_{B_R}|^q \, dx \leq CR^s \int_{B_R} \int_{B_R} \frac{|v(x) - v(y)|^q}{|x-y|^{n+sq}} \, dx \, dy \]
for all \( v \in W^{s,q}(B_R) \) and every ball \( B_R \Subset \Omega \).

In this paper, we will consider weighted Gagliardo fractional Sobolev spaces associated to a power of the distance to a point at the boundary of the domain. More precisely, in order to study the fractional order regularity for gradients of solutions to the measure data problem (1.1), we employ
Definition 2.5 (A weighted Gagliardo fractional Sobolev space). Let $\Omega$ be an open bounded Lipschitz domain in $\mathbb{R}^n$. For any $q \in [1, \infty)$, $s \in (0, 1)$ and $\alpha, \beta \geq 0$, we define the weighted Gagliardo fractional Sobolev space by

$$W^{s,q}(\Omega; \alpha, \beta) = \left\{ v \in L^q(\Omega) : d(x)^{\alpha/q}d(y)^{\beta/q}\frac{|v(x) - v(y)|}{|x - y|^{n/q + s}} \in L^q(\Omega \times \Omega) \right\},$$

endowed with the natural norm

$$(2.5) \quad \|v\|_{W^{s,q}(\Omega; \alpha, \beta)} = \left[ \int_{\Omega} |v(x)|^q \, dx + \int_{\Omega} \int_{\Omega} d(x)^{\alpha}d(y)^{\beta}\frac{|v(x) - v(y)|^q}{|x - y|^{n + sq}} \, dx \, dy \right]^{1/q}.$$  

Here $d(x) = \text{dist}(x, \partial \Omega)$ denotes the distance from $x$ to the boundary of $\Omega$.

When $\alpha = \beta$, we simply write $W^{s,q}(\Omega; \alpha)$ instead of $W^{s,q}(\Omega; \alpha, \alpha)$. Similar to non-weighted spaces, we also consider the weighted Gagliardo seminorm of $v \in W^{s,q}(\Omega; \alpha, \beta)$,

$$(2.6) \quad [v]_{W^{s,q}(\Omega; \alpha, \beta)} := \left[ \int_{\Omega} \int_{\Omega} d(x)^{\alpha}d(y)^{\beta}\frac{|v(x) - v(y)|^q}{|x - y|^{n + sq}} \, dx \, dy \right]^{1/q}.$$  

It is clear from the definition that

$$[v]_{W^{s,q}(\Omega; \alpha, \beta)} \leq (\text{diam}(\Omega))^{(\alpha + \beta)/q}\|v\|_{W^{s,q}(\Omega)},$$

and therefore

$$W^{s,q}(\Omega) \subset W^{s,q}(\Omega; \alpha, \beta).$$  

3. Gradient estimates near the boundary. By some important properties of weighted fractional Sobolev spaces discussed in Section 2, we now establish several gradient estimates near the boundary. In this section, we always suppose that $p > 2 - 1/n$, $\sigma \in (0, 1)$ and $\Omega$ is an open bounded and smooth domain in $\mathbb{R}^n$. Moreover, the operator $A$ satisfies assumptions [1.2]–[1.3] and $u \in W^{1,\max\{1, p-1\}}(\Omega)$ is considered as a SOLA solution to (1.1).

For a given $0 < R_0 < \text{diam}(\Omega)/2$, let us consider

$$\Omega_0 := \{ x \in \Omega : 0 < d(x) \leq R_0/2 \}$$

to be the set of points near the boundary of $\Omega$. We first decompose

$$\Omega_0 = \bigcup_{k=1}^{\infty} \Omega_k,$$

where

$$\Omega_k := \{ x \in \Omega : r_{k+1} < d(x) \leq r_k \},$$

with $r_k = 2^{-k}R_0$, for every $k \in \mathbb{N}^*$.  

LEMMA 3.1. For all $\alpha, \beta > 0$ satisfying $\alpha + \beta > \sigma$, there exists a constant $C = C(n, \sigma, \alpha, \beta) > 0$ such that

$$\sum_{|k-j|\geq 2} \int_{\Omega_k} \int_{\Omega_j} d(x)\alpha d(y)\beta |A(\nabla u(x)) - A(\nabla u(y))| |x - y|^{n+\sigma} \, dx \, dy \leq C \int_{\Omega_0} |A(\nabla u(x))| \, dx \sum_{j=1}^{\infty} r_j^{\alpha+\beta-\sigma}. \tag{3.1}$$

Proof. Denote the LHS of (3.1) by $(\mathbb{I}_1)$. For any $x \in \Omega_k$ and $y \in \Omega_j$ with $|k-j| \geq 2$, it is easy to check that

$$|x - y| \geq \max \{r_k/4, r_j/4\} \geq (r_k + r_j)/8,$$

and this allows us to derive

$$\int_{\Omega_k} \int_{\Omega_j} d(x)\alpha d(y)\beta |A(\nabla u(x))| |x - y|^{n+\sigma} \, dx \, dy \leq r_k^\alpha r_j^\beta \int_{\Omega_k} \left( \int_{\{ |\xi| \geq (r_k+r_j)/8 \}} \frac{1}{|\xi|^{n+\sigma}} \, d\xi \right) |A(\nabla u(x))| \, dx \leq 8^\sigma \frac{r_k^\alpha r_j^\beta}{(r_k+r_j)\sigma} \int_{\Omega_k} \left( \int_{\{ |\xi| \geq 1 \}} \frac{1}{|\xi|^{n+\sigma}} \, d\xi \right) |A(\nabla u(x))| \, dx \leq C(n, \sigma) \frac{r_k^\alpha r_j^\beta}{(r_k+r_j)\sigma} \int_{\Omega_k} |A(\nabla u(x))| \, dx. \tag{3.2}$$

The last inequality in (3.2) comes from the fact that $\int_{\{ |\xi| \geq 1 \}} \frac{1}{|\xi|^{n+\sigma}} \, d\xi$ is finite since $n+\sigma > n$. Applying this estimate in $(\mathbb{I}_1)$, one has

$$\sum_{|k-j|\geq 2} \int_{\Omega_k} \int_{\Omega_j} d(x)\alpha d(y)\beta |A(\nabla u(x)) - A(\nabla u(y))| |x - y|^{n+\sigma} \, dx \, dy \leq \sum_{|k-j|\geq 2} \left( \int_{\Omega_k} \int_{\Omega_j} d(x)\alpha d(y)\beta |A(\nabla u(x))| |x - y|^{n+\sigma} \, dx \, dy \right) \leq C(n, \sigma)((\mathbb{I}_{11}) + (\mathbb{I}_{12})), \tag{3.3}$$

where

$$(\mathbb{I}_{11}) := \sum_{k-j \geq 2} \frac{r_k^\alpha r_j^\beta}{(r_k+r_j)\sigma} \int_{\Omega_k} |A(\nabla u(x))| \, dx \quad + \sum_{k-j \geq 2} \frac{r_j^\alpha r_k^\beta}{(r_k+r_j)\sigma} \int_{\Omega_j} |A(\nabla u(y))| \, dy,$$
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\( (I)_{12} := \sum_{j-k \geq 2} \frac{r_k^\alpha r_j^\beta}{(r_k + r_j)^\sigma} \int_{\Omega_k} |A(\nabla u(x))| \, dx \)
\[ + \sum_{j-k \geq 2} \frac{r_j^\alpha r_k^\beta}{(r_k + r_j)^\sigma} \int_{\Omega_j} |A(\nabla u(y))| \, dy. \]

Since \( r_k \leq r_j \) for all \( k \geq j + 2 \), one has

\( (I)_{11} = \sum_{j=1}^{\infty} r_j^{\beta-\sigma} \sum_{k=j+2}^{\infty} \frac{r_k^\alpha}{(2^{j-k}+1)^\sigma} \int_{\Omega_k} |A(\nabla u(x))| \, dx \)
\[ + \sum_{j=1}^{\infty} r_j^{\alpha-\sigma} \int_{\Omega_j} |A(\nabla u(y))| \, dy \sum_{k=j+2}^{\infty} \frac{r_k^\beta}{(2^{j-k}+1)^\sigma} \]
\[ \leq \sum_{j=1}^{\infty} r_j^{\alpha+\beta-\sigma} \sum_{k=j+2}^{\infty} \int_{\Omega_k} |A(\nabla u(x))| \, dx \]
\[ + \sum_{j=1}^{\infty} r_j^{\beta-\sigma} \int_{\Omega_j} |A(\nabla u(y))| \, dy \sum_{k=j+2}^{\infty} r_k^\alpha \]
\[ \leq \int_{\Omega_0} |A(\nabla u(x))| \, dx \sum_{j=1}^{\infty} r_j^{\alpha+\beta-\sigma} + C(\alpha) \sum_{j=1}^{\infty} r_j^{\alpha+\beta-\sigma} \int_{\Omega_j} |A(\nabla u(y))| \, dy \]
\[ \leq C(\alpha) \int_{\Omega_0} |A(\nabla u(x))| \, dx \sum_{j=1}^{\infty} r_j^{\alpha+\beta-\sigma}, \]

and similarly

\( (I)_{12} = \sum_{j-k \geq 2} \left( \frac{r_k^\alpha r_j^\beta}{(r_k + r_j)^\sigma} \int_{\Omega_k} |A(\nabla u(x))| \, dx \right) \)
\[ + \frac{r_j^\alpha r_k^\beta}{(r_k + r_j)^\sigma} \int_{\Omega_j} |A(\nabla u(y))| \, dy \)
\[ \leq C(\beta) \int_{\Omega_0} |A(\nabla u(x))| \, dx \sum_{j=1}^{\infty} r_j^{\alpha+\beta-\sigma}; \]

these can be substituted into (3.3) to obtain

\[ (I)_{1} \leq C(n, \sigma, \alpha, \beta) \int_{\Omega_0} |A(\nabla u(x))| \, dx \sum_{j=1}^{\infty} r_j^{\alpha+\beta-\sigma}, \]

which allows us to get (3.1).
Lemma 3.2. For all $\alpha, \beta > 0$ satisfying $\alpha + \beta > \sigma$, there exists a constant $C = C(n, p, c_A, \sigma, R_0) > 0$ such that

$$\sum_{k=1}^{\infty} \int_{\Omega_k} \int_{\Omega_k} d(x)^{\alpha} d(y)^{\beta} \frac{|\mathcal{A}(\nabla u(x)) - \mathcal{A}(\nabla u(y))|}{|x-y|^{n+\sigma}} \, dx \, dy$$

$$\leq C \left( \int_{\Omega_0} |\mathcal{A}(\nabla u(x))| \, dx + |\mu|(\Omega_0) \right) \sum_{j=1}^{\infty} r_j^{\alpha+\beta-\sigma}.$$  

Proof. Denote the LHS of (3.5) by $I_2$. Notice that $\Omega_k$ can be covered by $N_k \sim |\partial \Omega|/r_k$ balls with radius $r_k$ and centered at $z^k_l \in \Omega_k$ for $l \in \{1, \ldots, N_k\}$, so

$$\Omega_k \subset \bigcup_{l=1}^{N_k} B_{r_k}(z^k_l) = \bigcup_{z^k_l \in Q_k} B_{r_k}(z^k_l),$$

where

$$Q_k := \{z^k_l \in \Omega_k : l \in \{1, \ldots, N_k\}\}.$$  

From the geometric features of each set $Q_k$, one decomposes the integral over $\Omega_k \times \Omega_k$ as follows:

$$\sum_{z^k_l, z^k_j \in Q_k} \int_{B_{r_k}(z^k_l) \setminus B_{r_k}(z^k_j)} d(x)^{\alpha} d(y)^{\beta} \frac{|\mathcal{A}(\nabla u(x)) - \mathcal{A}(\nabla u(y))|}{|x-y|^{n+\sigma}} \, dx \, dy$$

$$\leq \sum_{z^k_l \in Q_k} \sum_{z^k_j \in Q_k} \int_{B_{r_k}(z^k_l) \setminus B_{r_k}(z^k_j)} d(x)^{\alpha} d(y)^{\beta} \frac{|\mathcal{A}(\nabla u(x)) - \mathcal{A}(\nabla u(y))|}{|x-y|^{n+\sigma}} \, dx \, dy$$

$$+ \sum_{z^k_l \in Q_k} \sum_{z^k_j \in Q_k} \int_{B_{r_k}(z^k_l) \setminus B_{r_k}(z^k_j)} d(x)^{\alpha} d(y)^{\beta} \frac{|\mathcal{A}(\nabla u(x)) - \mathcal{A}(\nabla u(y))|}{|x-y|^{n+\sigma}} \, dx \, dy,$$

where $Q_{k,z^k_i}$ contains the centers that are very close to $z^k_l$, namely

$$Q_{k,z^k_i} := \{z^k_l \in Q_k : B_{3r_k/2}(z^k_l) \cap B_{3r_k/2}(z^k_i) \neq \emptyset\}.$$  

The nice feature here is that the cardinality of $Q_{k,z^k_i}$ is finite and depends only on $n$ and $R_0$. This means that there exists a constant $C(n, R_0) > 0$ such that $|Q_{k,z^k_i}| \leq C(n, R_0)$. Moreover, it is not difficult to check that

$$B_{r_k}(z^k_j) \subset B_{4r_k}(z^k_j)$$  

for all $z^k_j \in Q_{k,z^k_i}$.
Therefore, we can estimate the first term on the RHS of (3.6) as

\begin{equation}
\sum_{z_i^k \in Q_k} \sum_{z_j^k \in Q_k} \int \int d(x)^\alpha d(y)^\beta \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x - y|^{n+\sigma}} \, dx \, dy 
\end{equation}

\begin{equation}
\leq C(n, R_0) \sum_{z_i^k \in Q_k} \int \int d(x)^\alpha d(y)^\beta \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x - y|^{n+\sigma}} \, dx \, dy.
\end{equation}

Applying (3.10), we obtain

\begin{equation}
\sum_{z_i^k \in Q_k} \sum_{z_j^k \in Q_k} \int \int d(x)^\alpha d(y)^\beta \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x - y|^{n+\sigma}} \, dx \, dy 
\end{equation}

\begin{equation}
\leq r_k^{\alpha+\beta} \sum_{z_i^k \in Q_k} \int \int \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x - y|^{n+\sigma}} \, dx \, dy 
\end{equation}

\begin{equation}
\leq C(n, p, c_A, \sigma) r_k^{\alpha+\beta-\sigma} \left( \int_{B_{8r_k}(z_i^k)} |A(\nabla u(x))| \, dx + r_k \left[ \mu(B_{8r_k}) \right] \right).
\end{equation}

The estimates (3.7) and (3.8) yield

\begin{equation}
\sum_{z_i^k \in Q_k} \sum_{z_j^k \in Q_k} \int \int d(x)^\alpha d(y)^\beta \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x - y|^{n+\sigma}} \, dx \, dy 
\end{equation}

\begin{equation}
\leq C(n, p, c_A, \sigma, R_0) r_k^{\alpha+\beta-\sigma} \left( \sum_{z_i^k \in Q_k} \int_{B_{8r_k}(z_i^k)} |A(\nabla u(x))| \, dx \right.
\end{equation}

\begin{equation}
+ r_k \sum_{z_i^k \in Q_k} \left| \mu(B_{8r_k}) \right|.
\end{equation}

Note that there is a constant \( C = C(n) > 0 \) such that

\[ \sum_{z_i^k \in Q_k} \chi_{B_{8r_k}(z_i^k)}(\xi) \leq C \chi_{\Omega_0}(\xi), \quad \forall \xi \in \Omega. \]

Then, for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \),

\begin{equation}
\sum_{z_i^k \in Q_k} \int \chi_{B_{8r_k}(z_i^k)}(\xi) f(\xi) \, d\xi = \sum_{z_i^k \in Q_k} \int \chi_{B_{8r_k}(z_i^k)}(\xi) f(\xi) \, d\xi \leq C \int_{\Omega_0} f(\xi) \, d\xi.
\end{equation}

Applying (3.10) to (3.9), one concludes that

\begin{equation}
\sum_{z_i^k \in Q_k} \sum_{z_j^k \in Q_k} \int \int d(x)^\alpha d(y)^\beta \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x - y|^{n+\sigma}} \, dx \, dy 
\end{equation}

\begin{equation}
\leq C(n, p, c_A, \sigma, R_0) r_k^{\alpha+\beta-\sigma} \left( \int_{\Omega_0} |A(\nabla u(x))| \, dx + r_k \left[ \mu(\Omega_0) \right] \right).
\end{equation}
On the other hand, for any $x \in B_{r_k}(z^k_i)$ and $y \in B_{r_k}(z^k_j)$ with $z^k_i \in Q_k$, $z^k_j \in Q_k \setminus Q_{k,z^k_i}$, we have $|x - y| \geq r_k$. By the same argument as in (3.2), one also has

$$
\sum_{z^k_j \in Q_k \setminus Q_{k,z^k_i}} \int_{B_{r_k}(z^k_j)} \int_{B_{r_k}(z^k_j)} d(x)^\alpha d(y)^\beta \frac{|A(\nabla u(x))|}{|x - y|^{n+\sigma}} dx dy

\leq r_k^{\alpha+\beta} \int_{B_{r_k}(z^k_i)} \left( \sum_{z^k_j \in Q_k \setminus Q_{k,z^k_i}} \int_{B_{r_k}(z^k_j)} \frac{1}{|x - y|^{n+\sigma}} dy \right) |A(\nabla u(x))| dx

\leq r_k^{\alpha+\beta-\sigma} \int_{B_{r_k}(z^k_i)} \left( \int_{|\xi| \geq 1} \frac{1}{|\xi|^{n+\sigma}} d\xi \right) |A(\nabla u(x))| dx

\leq C(n, \sigma)r_k^{\alpha+\beta-\sigma} \int_{B_{r_k}(z^k_i)} |A(\nabla u(x))| dx.
$$

Taking into account the above inequality, we may estimate the last term in (3.6) as

$$
(3.12)
\sum_{z^k_i \in Q_k} \sum_{z^k_j \in Q_k \setminus Q_{k,z^k_i}} \int_{B_{r_k}(z^k_i)} \int_{B_{r_k}(z^k_i)} d(x)^\alpha d(y)^\beta \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x - y|^{n+\sigma}} dx dy

\leq C(n, \sigma)r_k^{\alpha+\beta-\sigma} \sum_{z^k_i \in Q_k} \int_{B_{r_k}(z^k_i)} |A(\nabla u(x))| dx

\leq C(n, \sigma, R_0)r_k^{\alpha+\beta-\sigma} \int_{\Omega_0} |A(\nabla u(x))| dx.
$$

Substituting (3.11) and (3.12) into (3.6), one gets

$$
(3.13)
(\mathbb{I})_2 = \sum_{k=1}^{\infty} \int_{\Omega_k} \int_{\Omega_k} d(x)^\alpha d(y)^\beta \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x - y|^{n+\sigma}} dx dy

\leq C(n, p, c_A, \sigma) \left( \int_{\Omega_0} |A(\nabla u(x))| dx \sum_{k=1}^{\infty} r_k^{\alpha+\beta-\sigma} + |\mu|(\Omega_0) \sum_{k=1}^{\infty} r_k^{\alpha+\beta-\sigma+1} \right)

\leq C(n, p, c_A, \sigma, R_0) \left( \int_{\Omega_0} |A(\nabla u(x))| dx + |\mu|(\Omega_0) \right) \sum_{k=1}^{\infty} r_k^{\alpha+\beta-\sigma}.
$$

**Lemma 3.3.** For all $\alpha, \beta > 0$ satisfying $\alpha + \beta > \sigma$, there exists a constant $C = C(n, p, c_A, \sigma, R_0) > 0$ such that
(3.14) \[ \sum_{|k-j|=1}^{\infty} \int_{\Omega_k} \int_{\Omega_j} d(x)^{\alpha} d(y)^{\beta} \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x-y|^{n+\sigma}} \, dx \, dy \]
\[ \leq C \left( \int_{\Omega_0} |A(\nabla u(x))| \, dx + |\mu|_{(\Omega_0)} \right) \sum_{k=1}^{\infty} r_k^{\alpha+\beta-\sigma}. \]

**Proof.** Let (I)₃ be the LHS of (3.14). Notice that
\[ (I)_3 = 2 \sum_{k=1}^{\infty} \int_{\Omega_k} \int_{\Omega_{k+1}} d(x)^{\alpha} d(y)^{\beta} \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x-y|^{n+\sigma}} \, dx \, dy \]
\[ \leq 2 \sum_{k=1}^{\infty} \int_{P_k} \int_{P_k} d(x)^{\alpha} d(y)^{\beta} \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x-y|^{n+\sigma}} \, dx \, dy, \]
where
\[ P_k := \Omega_k \cup \Omega_{k+1} = \{ x \in \Omega : r_k/d(x) < d(x) \leq r_k \}. \]
In a similar fashion as for (I)₂ in Lemma 3.2 we may decompose P_k just as \( \Omega_k \) in (3.6) and perform the same computation to observe that
\[ (3.15) \quad (I)_3 \leq C(n, p, c_A, \sigma, R_0) \left( \int_{\Omega_0} |A(\nabla u(x))| \, dx + |\mu|_{(\Omega_0)} \right) \sum_{k=1}^{\infty} r_k^{\alpha+\beta-\sigma}. \]

4. Proofs of main theorems. By applying the local results of Theorem 1.1 we now prove the main theorems, where the fractional regularity for the solutions to (1.1) up to the boundary of a smooth domain \( \Omega \) will be established in weighted fractional Sobolev spaces.

**Proof of Theorem 1.2** For simplicity of notation, set
\[ T(x, y) := d(x)^{\alpha} d(y)^{\beta} \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x-y|^{n+\sigma}}, \quad x, y \in \Omega, \ x \neq y. \]
The integral of \( T \) over \( \Omega \times \Omega \) can be split as follows:
\[ \int_{\Omega} \int_{\Omega} T(x, y) \, dx \, dy = \int_{\Omega_0} \int_{\Omega_0} T(x, y) \, dx \, dy + 2 \int_{\Omega_0} \int_{\Omega_0 \setminus \Omega_0} T(x, y) \, dx \, dy \]
\[ + \int_{\Omega_0 \setminus \Omega_0} \int_{\Omega_0 \setminus \Omega_0} T(x, y) \, dx \, dy \]
\[ =: (\mathbb{II}) + 2(\mathbb{III}) + (\mathbb{III}). \]
Terms (\( \mathbb{II} \)) and (\( \mathbb{III} \)), containing integrals over the interior domain \( \Omega \setminus \Omega_0 \), can be estimated by applying the local inequality (1.6) of Theorem 1.1. Now, let us rewrite (\( \mathbb{II} \)) as
\[ (4.1) \quad (\mathbb{II}) = \sum_{k,j=1}^{\infty} \int_{\Omega_k} \int_{\Omega_j} T(x, y) \, dx \, dy \]
One can find

\[ \sum_{|k-j| \geq 2} \int_{\Omega_k \Omega_j} \mathbb{T}(x, y) \, dx \, dy + \sum_{|k-j| = 1} \int_{\Omega_k \Omega_j} \mathbb{T}(x, y) \, dx \, dy \]

\[ + \sum_{k=1}^{\infty} \int_{\Omega_k} \int_{\Omega_k} \mathbb{T}(x, y) \, dx \, dy \]

\[ =: (\mathbb{I})_1 + (\mathbb{I})_3 + (\mathbb{I})_2. \]

We estimate the terms on the RHS of (4.1) by applying Lemmas 3.1–3.3.

One can find \( C = C(n, p, c_A, \sigma, \alpha, \beta, R_0) > 0 \) such that

\[ \text{(4.2)} \quad (\mathbb{I}) \leq C \left( \int_{\Omega} |A(\nabla u(x))| \, dx + |\mu|(\Omega) \right) \sum_{k=1}^{\infty} r_k^{\alpha+\beta-\sigma}. \]

Finally, the assumption \( \alpha + \beta > \sigma \) allows us to find

\[ \sum_{k=1}^{\infty} r_k^{\alpha+\beta-\sigma} = C R_0^{\alpha+\beta-\sigma} \quad \text{with} \quad C = \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^{(\alpha+\beta-\sigma)k} < \infty, \]

which leads to the desired result (1.8) from (4.2).

\[ \text{[Proof of Theorem 1.3]} \]

For all \( \xi, \zeta \in \mathbb{R}^n \), we have the elementary inequalities

\[ \text{(4.3)} \quad \left| \mathbb{E}(\xi) - \mathbb{E}(\zeta) \right|^{1/\gamma - 1} = \left| (|\xi| + \kappa)^{\gamma p - \gamma - 1} \xi - (|\zeta| + \kappa)^{\gamma p - \gamma - 1} \zeta \right|^{1/\gamma - 1} \]

\[ \leq \left( (|\xi| + \kappa)^{\gamma p - \gamma} + (|\zeta| + \kappa)^{\gamma p - \gamma} \right)^{1/\gamma - 1} \]

\[ \leq C(p, \gamma)(|\xi| + |\zeta| + \kappa)^{(p-1)(1-\gamma)}, \]

and

\[ \text{(4.4)} \quad \left| \mathbb{E}(\xi) - \mathbb{E}(\zeta) \right|^{1/\gamma} = \left| (|\xi| + \kappa)^{\gamma p - \gamma - 1} \xi - (|\zeta| + \kappa)^{\gamma p - \gamma - 1} \zeta \right| \]

\[ \leq C(p, \gamma)(|\xi| + |\zeta| + \kappa)^{\gamma p - \gamma - 1} |\xi - \zeta|. \]

Combining (4.3) and (4.4) leads to

\[ |\mathbb{E}(\xi) - \mathbb{E}(\zeta)|^{1/\gamma} \leq C(p, \gamma)(|\xi| + |\zeta| + \kappa)^{p-2} |\xi - \zeta|, \]

and together with assumption (1.2) allows us to arrive at

\[ \text{(4.5)} \quad |\mathbb{E}(\xi) - \mathbb{E}(\zeta)|^{1/\gamma} \leq C(p, \gamma) |\mathcal{A}(\xi) - \mathcal{A}(\zeta)|. \]

On the other hand, thanks to (4.5) and Theorem 1.2, one has

\[ [\mathbb{E}(\nabla u)]_{W^{\gamma,1/\gamma}(\Omega;\alpha,\beta)} = \left[ \int_{\Omega} \int_{\Omega} \mathbb{E}(\nabla u(x)) - \mathbb{E}(\nabla u(y)) \right]_{1/\gamma}^{\gamma} \]

\[ \leq C(p, \gamma) \left[ \int_{\Omega} \int_{\Omega} \mathbb{E}(\nabla u(x)) - \mathbb{E}(\nabla u(y)) \right]_{1/\gamma}^{\gamma} \]

\[ \leq C(c_A, \sigma, n, p, \alpha, \beta, \gamma) \left( \int_{\Omega} |\mathcal{A}(\nabla u(x))| \, dx + |\mu|(\Omega) \right)^{\gamma}. \]
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