Integrability and duality in two-dimensional QCD

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Abstract

We consider bosonized QCD$_2$, and prove that after rewriting the theory in terms of gauge-invariant fields, there exists an integrability condition that is valid for the quantum theory as well. Furthermore, performing a duality-type transformation we obtain an appropriate action for the description of the strong coupling limit, which is still integrable. We also prove that the model displays a complicated set of constraints, restricting the dynamics of part of the theory, but which are necessary to maintain the positive metric Hilbert space.
1. Introduction

In contrast with the case of the Schwinger model, quantum chromodynamics of massless fermions in 1+1 dimensions cannot be solved in terms of free fields. Several methods have been used in such a case, some of them giving useful results. We mention here the $1/N$ expansion introduced by ’t Hooft\(^1\), from which one obtains some information about the spectrum of the theory, and the computation of the exact fermion determinant\(^2\) in terms of a Wess–Zumino–Witten model\(^3,4\), by which one arrives at an equivalent bosonic action\(^2,5\). Several authors made efforts in the direction of solving such a difficult model\(^6\), but an exact solution is still missing (see [7] for an extensive review).

Working in the light cone gauge ($A_− = 0$) and formulating the problem in terms of light cone variables, ’t Hooft obtained a non-linear equation for the fermion self energy, from which he could obtain the above-mentioned information about the spectrum of the theory. This procedure is however ambiguous, as pointed out by Wu\(^8\), and implies a tachyon for small bare fermion masses (therefore also in the massless fermions case), see also [9]. This situation clearly requires that a non-perturbative and explicitly gauge-invariant approach should be used in order to obtain information based on firm grounds. Some authors speculated that this situation was a sign of a possible non-trivial phase structure of the theory\(^26\).

The model has also been extensively studied, especially in the absence of fermions, in relation to string theory. Indeed, string theory should be a model describing bound states in strong interactions. Therefore Wilson loops should there be described by the string approach as well\(^10\). In this sense it is natural to integrate out the fermions obtaining gauge-invariant objects, which describe mesonic bound states with an infinite string attached to it. In such a case, the construction of fields such as $\Sigma = UV$ (see sec. 2) or $\tilde{g} = UgV$, describing gauge-invariant bound states, is natural after all. The original content of the (gauge-dependent) fermion fields can still be recovered from the source terms. Those are kept as a bookkeeping concerning the translation between original gauge-dependent fields and the bosonic formulation, which is non-local. In this sense, we also recall the construction of bilinear fermion fields\(^11\) in the light cone gauge, which arises also from a WZW-type action.

The fact that QCD\(^2\) can be studied using non-perturbative methods is by itself a non-trivial statement. In the light cone gauge this can be motivated by the fact that bilinears in the fermion fields form a $W_\infty$ algebra\(^11\), which underlies integrable theories\(^12\). Here, we shall see that one obtains higher-conservation laws, building an affine Lie algebra. We use methods based on the Polyakov–Wiegman identity\(^2\) extensively used in refs. [13] and [14], and the consequent duality transformations\(^14\), which seem to have a widespread application in such a class of models\(^14,15\). Due to the underlying gauge invariance, imposition of the BRST condition in order to obtain the spectrum will be of crucial importance\(^14,16\).

Having in mind the above motivations, we first rewrite in section 2 the problem in terms of bosonic matrix variables. Fermionic Green functions can be obtained from the sources, which have been kept in the process. However, the path integration is performed in terms of bosonic fields. We found a set of fields in terms of which the partition function factorizes, as a product of two conformal theories, and a third model corresponding to
an off-critical perturbation of the WZW theory. Later we verify that the different sectors interact non-trivially due to the constraint structure. However we proceed with a semi-classical reasoning, computing the equation of motion of the latter field, proving that it corresponds to an integrability condition of the theory. Notice that due to the fermionic integration, there are already, at this point, corrections of the order of $\hbar$. We verify that there is a non-trivial change of variables that leads to a dual formulation of the theory, in terms of the fields appropriately to describing the strong coupling limit. Using the Poisson bracket structure in section 4, we verify that the higher conservation laws obey a Kac–Moody algebra. We also argue that the quantization of such higher conservation laws can be done by means of the introduction of renormalization factors for the current contribution. In section 5 we discuss the constraints arising from the structure of the gauge interaction, and subsequently (section 6) we obtain also second class constraints. In section 7 we discuss the consequence of the constraints for the dual theory, and later we argue about the Regge behaviour of the spectrum$^1$. We still discuss the possibility of a conformally-invariant type solution for the current, ending with some further conclusions.

2. Bosonization of two-dimensional QCD

We shall consider the QCD$_2$ Lagrangian, given by the expression

$$\mathcal{L} = -\frac{1}{4} \text{tr} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} i \mathcal{D} \psi ,$$

where $D_{\mu} = \partial_{\mu} - ieA_{\mu}$. We work out some results in the path integral formulation, but at a later stage consider also the canonical quantization, forcing us to use both Euclidean and Minkowski spaces alternatively. The conventions are given in the appendix.

The partition function is

$$Z[\eta, \bar{\eta}, i_\mu] = \int D\psi D\bar{\psi} D A_{\mu} \ e^{-\int d^2z \mathcal{L} - \int d^2z (\bar{\eta}\psi + \psi \bar{\eta} + i_\mu A_{\mu})} ,$$

where $\eta, \bar{\eta}$ are the external sources for the fermions $\bar{\psi}, \psi$, and $i_\mu$ the external source for the gauge field $A_{\mu}$.

In order to obtain the bosonized version of the theory one has to rewrite the fermionic determinant $\det i \mathcal{D}$ as a bosonic functional integral. To achieve this we consider the change of variables$^2$

$$\bar{A} = \frac{i}{e} V \overline{V}^{-1} ,$$

$$A = \frac{i}{e} U^{-1} \partial U .$$

The fermionic determinant is given up to factors of the free Dirac operator, to be discussed later in connection with the ghost system, by the expression

$$\det i \mathcal{D} = e^{\Gamma[U,V]} ,$$

2
where the $\Gamma[g]$ is the Euclidean Wess–Zumino–Witten (WZW) functional, given by the expression

$$\Gamma[g] = \frac{1}{8\pi} \int d^2z \partial_\mu g^{-1} \partial_\mu g - \frac{i}{4\pi} \epsilon^{\mu\nu} \int dr \int d^2z \hat{g}^{-1} \hat{g}^{-1} \partial_\mu \hat{g} \hat{g}^{-1} \partial_\nu \hat{g} \ ,$$  

(2.5)

where $\hat{g}(r, z, \overline{z})$ is the usual extension of $g(z, \overline{z})$ to a space having the Euclidean two-dimensional space as a boundary, and $\hat{g}(1, z, \overline{z}) = g(z, \overline{z})$, $\hat{g}(0, z, \overline{z}) = 1$. The WZW functional obeys the Polyakov–Wiegman identity

$$\Gamma[UV] = \Gamma[U] + \Gamma[V] + \frac{1}{4\pi} \text{tr} \int d^2z U^{-1} \partial U V \partial V^{-1} \ ,$$  

(2.6)

which will be used extensively in this work. We find the seeked bosonized formulation, taking advantage of the invariance of the Haar measure, and write the above fermion determinant as

$$\det i \mathcal{D} \equiv e^{-W(A)} = \int \mathcal{D} g e^{-S_F[A, g]} \ ,$$  

(2.7a)

where $S_F[A, g]$ is the equivalent of the fermionic action in terms of the bosonic variables and gauge field and reads

$$S_F[A, g] = \Gamma[UgV] - \Gamma[UV] = \Gamma[g] + \frac{1}{4\pi} \int d^2z \left[ e^2 A_\mu A_\mu - e^2 Ag \overline{A}g^{-1} - ieAg \overline{A}g^{-1} - ie \overline{A}g^{-1} \right] \ .$$  

(2.7b)

The above equation was obtained by a repeated use of the Polyakov–Wiegman identity (2.6), respecting local gauge transformations. The external sources have been used to redefine the integration over the fermionic field as

$$\overline{\psi} i \mathcal{D} \psi + \overline{\eta} \psi + \overline{\psi} \eta = (\overline{\psi} + \overline{\eta}(i \mathcal{D})^{-1}) i \mathcal{D} (\psi + (i \mathcal{D})^{-1} \eta) - \overline{\eta}(i \mathcal{D})^{-1} \eta \ .$$  

(2.8)

The change of variables (2.3) leads to a non-trivial Jacobian, but fortunately it is also the exponential of the WZW functional, that is

$$\mathcal{D} \mathcal{A} \mathcal{D} \overline{\mathcal{A}} = e^{c_V \Gamma[UV]} \mathcal{D} U \mathcal{D} V \ ,$$  

(2.9)

where $c_V$ is the quadratic Casimir, and a definite regularization respecting vector gauge invariance has been chosen. As we stressed after (2.3), this is again written up to a factor containing the free Dirac operator.

The non-linearity in the gauge field interaction can also be disentangled by means of the identity (see notation in the appendix)

$$e^{\frac{i}{2} \int d^2z \text{tr} F_{\mu\nu}^2} = \int \mathcal{D} E \ e^{\int d^2z \left[ \frac{i}{4} \text{tr} E^2 + \frac{i}{2} \text{tr} EF_{zz} \right]} \ ,$$  

(2.10)
where \( E \) is a matrix-valued field. Taking into account the previous set of information we arrive at

\[
Z[\eta, \eta, i_\mu] = \int DEDUDVDg \times \\
\times e^{-\Gamma[UgV]+(cV+1)\Gamma[UV]+\int d^2z \text{tr} [\frac{1}{2}E^2 + \frac{1}{2}EF_{zz}] - \int d^2z i_\mu A_\mu + \int d^2z d^2w \eta(z)(i\partial)^{-1}(z,w)\eta(w)} .
\]

(2.11)

If we were considering massive fermions, we should include a term \( m \text{tr} (g + g^{-1}) \) in the effective action. However we shall avoid such a complication and only consider the massless case. We also have to deal with gauge fixing. In fact, the process of introducing ghosts is standard, and we suppose that the procedure is included above, until it is necessary to explicitly take into account the ghost degrees of freedom, which will be the case upon consideration of the spectrum, when the BRST condition has to be used. Up to that point our manipulations do not explicitly depend on the gauge fixing/ghost system, and we proceed without it (or else, keeping it at the back of our minds).

Defining the gauge invariant field \( \tilde{g} = UgV \), and using the invariance of the Haar measure, that is \( Dg = D\tilde{g} \), we see that the \( \tilde{g} \) field decouples (always up to BRST condition, see later) and we are left with

\[
Z[\eta, \eta, i_\mu] = \int D\tilde{g}e^{-\Gamma[\tilde{g}]} \int DEDUDVD(\text{ghosts}) \times \\
\times e^{(cV+1)\Gamma[UV]+\int d^2z \text{tr} [\frac{1}{2}E^2 + \frac{1}{2}EF_{zz}] - S_{\text{ghosts}} - \int d^2z i_\mu A_\mu + \int d^2z d^2w \eta(z)(i\partial)^{-1}(z,w)\eta(w)} ,
\]

(2.12)

where \( A_\mu \) variables are given in terms of the \( UV \) variables.

The presence of the gauge field strength \( F_{z\bar{z}} \) hinders further developments in the way it is presented above. However, in terms of the \( U \) and \( V \) variables we can write

\[
\text{tr} EF_{z\bar{z}} = \frac{i}{e} \text{tr} UEU^{-1}\partial(\Sigma\bar{\partial}\Sigma^{-1}) ,
\]

(2.13)

where \( \Sigma = UV \). This will permit a complete separation of some variables. Indeed, \( \Sigma \) is a more natural candidate to represent the physical degrees of freedom, since \( U \) and \( V \) are not separately gauge-invariant. In the way it is written in eq. (2.13), we can redefine \( E \) taking advantage once more of the invariance of the Haar measure, in such a way that the effective action only depends on the combination \( \Sigma = UV \), while \( U \) and \( V \) appear separately only in the source terms, which are gauge-dependent, as they should be, that is,

\[
A = \frac{i}{e} U^{-1}\partial U ,
\]

(2.14)

\[
\overline{A} = \frac{i}{e} (U^{-1}\Sigma)\bar{\partial}(\Sigma^{-1}U) .
\]

If we eventually choose the light cone gauge, \( U = 1 \), \( \overline{A} = \frac{i}{e} \Sigma\bar{\partial}\Sigma^{-1} \) and \( A = 0 \). From the structure of (2.13), it is natural to redefine variables as \( \tilde{E}' = UEU^{-1} \), \( \tilde{D}E = \tilde{D}\tilde{E}' \). Notice that already at this point the \( E \) redefinition implies, in terms of the gauge potential, an
infinite gauge tail, which captures the possible gauge transformations. It is also convenient to make the rescaling $\tilde{E}' = 2ie(c_V + 1)\tilde{E}$, with a constant Jacobian. In terms of the field $\tilde{E}$, consider the change of variables

$$\partial\tilde{E} = \frac{1}{4\pi} \beta^{-1} \partial \beta , \quad D \tilde{E} = e^{c_V \Gamma[\beta]} D \beta .$$

(2.15)

We use the identity (2.6) to transform the $\beta \Sigma$ interaction into terms that can be handled in a more appropriate fashion. Writing both steps separately we have

$$Z[\eta, \eta', \mu] = \int D\tilde{g} e^{-\Gamma[\beta]} D\tilde{U} D(\text{ghosts}) e^{-S_{\text{ghosts}}} \int D\Sigma D\tilde{E} e^{(c_V + 1)\Gamma[\Sigma]} \times$$

$$\times e^{(c_V + 1)\Gamma[\beta \Sigma] - \Gamma[\beta]} - \frac{2e^2(c_V + 1)^2}{(4\pi)^2} \int d^2 z [\theta^{-1}(\beta^{-1} \partial \beta)]^2 - \int d^2 z i_{\mu} A_{\mu} + \int d^2 z d^2 w \tilde{\eta}(z)(i\partial)^{-1}(z,w) \eta(w) .$$

(2.16)

in such a way that after substitution of (2.15) into (2.16) and using (2.6) for $\Gamma[\beta \Sigma]$, we arrive at

$$Z[\eta, \eta', \mu] = \int D\tilde{g} e^{-\Gamma[\beta]} D\tilde{U} D(\text{ghosts}) e^{-S_{\text{ghosts}}} \int D\Sigma D \beta \times$$

$$\times e^{(c_V + 1)\Gamma[\beta \Sigma] - \Gamma[\beta]} - \frac{2e^2(c_V + 1)^2}{(4\pi)^2} \int d^2 z [\theta^{-1}(\beta^{-1} \partial \beta)]^2 - \int d^2 z i_{\mu} A_{\mu} + \int d^2 z d^2 w \tilde{\eta}(z)(i\partial)^{-1}(z,w) \eta(w) .$$

(2.17)

We define the (massive) parameter

$$\lambda = \frac{c_V + 1}{2\pi} e ,$$

(2.18)

and the field $\tilde{\Sigma} = \beta \Sigma$, in terms of which the partition function reads

$$Z[\eta, \eta', \mu] = \int D\tilde{g} e^{-\Gamma[\beta]} D\tilde{U} D(\text{ghosts}) e^{-S_{\text{ghosts}}} \int D\tilde{\Sigma} e^{(c_V + 1)\Gamma[\tilde{\Sigma}]} \times$$

$$\times \int D\beta e^{-\Gamma[\beta]} - \frac{2}{c_V} \int d^2 z \theta^{-1}(\beta^{-1} \partial \beta)^2 e - \int d^2 z i_{\mu} A_{\mu} + \int d^2 z d^2 w \tilde{\eta}(z)(i\partial)^{-1}(z,w) \eta(w) ,$$

(2.19)

where now $A = \frac{i}{\lambda} U^{-1} \partial U , \overline{A} = \frac{i}{\lambda} (U^{-1} \beta^{-1} \tilde{\Sigma}) \overline{\partial} (\tilde{\Sigma}^{-1} \beta U)$.

Up to source terms, and the BRST constraints to be discussed later, the above generating functional factorizes in terms of a conformal theory for $\tilde{g}$, representing a gauge-invariant fermionic bound states degrees of freedom, a second conformal field theory for $\tilde{\Sigma}$, representing some gauge condensate, and an off-critically perturbed conformal field theory for the $\beta$-field, which describes also a gauge field condensate, in view of the change of variables (2.15). The conformal field theory representing $\tilde{\Sigma}$ has an action with a negative sign (see (2.19)). Therefore we have to carefully take into account the BRST constraints in order to arrive at a positive metric Hilbert space. Since we will study first the $\beta$-degrees of freedom, we leave this problem for a later section.
We consider the perturbed WZW action
\[ S = \Gamma[\beta] + \frac{1}{2} \lambda^2 \text{tr} \int d^2 z \left[ \partial^{-1} (\beta^{-1} \partial \beta) \right]^2 , \]
\[ = \Gamma[\beta] + \frac{1}{2} \lambda^2 \Delta(\beta) . \]  

\[ 3.1 \]

We will look for the Euler–Lagrange equations for \( \beta \). It is not difficult to find the variations:
\[ \delta \Gamma[\beta] = \left[ -\frac{1}{4\pi} \partial (\beta^{-1} \partial \beta) \right] \beta^{-1} \delta \beta , \]
\[ \delta \Delta(\beta) = 2 \left[ \partial^{-1} (\beta^{-1} \partial \beta) + \left[ \partial^{-2} (\beta^{-1} \partial \beta), (\beta^{-1} \partial \beta) \right] \right] \beta^{-1} \delta \beta . \]

\[ 3.2 \]

Collecting the terms, we find it useful to define the current components
\[ J^\beta = \beta^{-1} \partial \beta \]
\[ \overline{J}^\beta = 4\pi \lambda^2 \partial^{-2} J^\beta = 4\pi \lambda^2 \partial^{-2} (\beta^{-1} \partial \beta) \]
\[ 3.3 \]

which summarize the \( \beta \) equation of motion as a zero curvature condition given by
\[ [\mathcal{D}, \overline{\mathcal{D}}] = [\partial - J^\beta, \overline{\partial} - \overline{J}^\beta] = \overline{\partial} J^\beta - \partial \overline{J}^\beta - [\overline{J}^\beta, J^\beta] = 0 \]
\[ 3.4 \]

This is the integrability condition for the Lax pair\(^{17}\)
\[ \mathcal{D}_\mu M = 0 \quad \text{with} \quad \mathcal{D}_\mu = \partial_\mu - J^\beta_\mu \]
\[ 3.5 \]

where \( J^\beta = J_1^\beta + iJ_2^\beta , \overline{J}^\beta = J_1^\beta - iJ_2^\beta \) and \( M \) is the monodromy matrix. This is not a Lax pair as in usual non-linear sigma models\(^{18}\), where \( J^\beta_\mu \) is a conserved current, and we obtain a conserved non-local charge from (3.4), as well as higher local and non-local conservation laws, derived from an extension of (3.4) in terms of an arbitrary spectral parameter\(^{18}\).

However, to a certain extent, the situation is simpler in the present case, due to the rather unusual form of the currents (3.3), which permits us to write the commutator appearing in (3.4) as a total derivative, in such a way that in terms of the current \( \overline{J}^\beta \), we have
\[ \partial \left\{ 4\pi \lambda^2 \overline{J}^\beta - \partial \overline{\partial} \overline{J}^\beta + [\overline{J}^\beta, \partial \overline{J}^\beta] \right\} = 0 \]
\[ 3.6 \]

Therefore the quantity
\[ \overline{J}^\beta(\overline{z}) = 4\pi \lambda^2 \overline{J}^\beta(z, \overline{z}) - \partial \overline{\partial} \overline{J}^\beta(z, \overline{z}) + [\overline{J}^\beta(z, \overline{z}), \partial \overline{J}^\beta(z, \overline{z})] \]
\[ 3.7 \]
does not depend on $z$, and it is a simple matter to derive an infinite number of conservation laws from the above (see later in the Minkowskian formulation).

This means that two-dimensional QCD is an integrable system! Moreover, it corresponds to an off-critical perturbation of the WZW action. If we write $\beta = e^{i\phi} \sim 1 + i\phi$, we verify that the perturbing term corresponds to a mass term for $\phi$. Later we will discuss in more detail this issue in the large-$N$ limit (for the $SU(N)$ theory). The next natural step is to obtain the algebra obeyed by (3.7), and its representation. However, there is a difficulty presented by the non-locality of the perturbation. We now introduce a further auxiliary field defining a dual action, local in all fields, and representing the low-energy scales of the theory, and later we return to the problem of finding the algebra obeyed by (3.7).

Consider the $\Delta$-term of the action (3.2). We write the quadratic term in (3.1), introducing the integral over a Gaussian field $\partial C$ as

$$e^{-\frac{1}{2} \lambda^2 \Delta} = \int D\bar{C} e^{\int d^2 z \frac{1}{2} \text{tr} (\partial \bar{C})^2 - \lambda \text{tr} \int d^2 z \bar{C} (\beta^{-1} \partial \beta)} ,$$

where the left-hand side is readily obtained by completing the square in the r.h.s.

Indeed, at this point we have two choices. We can turn to Minkowski space, and proceed with the canonical quantization of the action (3.1) with the non-local term substituted in terms of the $\bar{C}$ field-dependent expression obtained in the exponent of the integrand of the r.h.s. of eq. (3.8). Before that, motivated by the presence of the auxiliary vector field $\bar{C}$, we make again a change of variables of the type

$$\bar{C} = \frac{1}{4\pi \lambda} W \partial W^{-1} ,$$

$$D\bar{C} = e^{c_V \Gamma[W]} DW ,$$

(3.9)

together with the now very frequently used identity (2.6) in order to find a dual action. We have for the $\beta$-partition function the expression

$$Z = \int D\beta D\bar{C} e^{-\Gamma[\beta] + c_V \Gamma[W] - \frac{1}{4\pi} \int d^2 z W \partial W^{-1} \beta^{-1} \partial \beta + \frac{1}{2(4\pi \lambda)^2} \int d^2 z (\partial W \bar{W} W^{-1})^2} ,$$

(3.10)

from which we can separate the contribution $-\Gamma[\beta W] \equiv -\Gamma[\bar{\beta}]$; after such manœuvre we are left with

$$Z = \int D\bar{\beta} e^{-\Gamma[\bar{\beta}]} \int D\bar{W} e^{(c_V + 1) \Gamma[W] + \frac{1}{2(4\pi \lambda)^2} \int d^2 z [\partial (W \bar{W} W^{-1})]^2} .$$

(3.11)

The dual action has now a coupling constant corresponding to the inverse of the initial charge. Therefore (3.11) is appropriate to the study of a strongly coupled limit. Notice that the procedure is, in a sense, familiar to the one used to obtain a dual action, where a non-dynamical field is introduced, and one eliminates the original fields by integration leaving the so-called dual formulation. See refs. [13], [14] and [15] for further details. We separate a further WZW-conformal piece, and we are left with a local massive action for $W$. 7
The drawback is the fact that now $W$ itself has an action with a negative sign. Naïvely it describes also massive excitations, although a complete description of the spectrum can only be obtained after disentangling the non-linear relations and imposing the BRST conditions.

For the sources, we now have to replace $A$ in (2.19) by $\frac{i}{2}(U^{-1}W\tilde{\beta}^{-1}\tilde{\Sigma})\partial(\tilde{\Sigma}^{-1}\tilde{\beta}W^{-1}U)$. We notice also here that we have dual descriptions of two-dimensional QCD. In the first, valid in the perturbative region, for high energies, we find out a non-local perturbation of the WZW action. In terms of $W$ the perturbation is local, but at the price of a negative sign in the naïve kinetic term in the $W$ action, which is appropriate to describing the low energy (strong coupling) regime of the theory. In spite of such different complementary descriptions, both models are integrable. In the weak coupling regime we found the conservation laws (3.3 – 3.6). In the case of the $W$-theory, it is not difficult to find the equations of motion, and again derive similar relations for the quantity

$$I^W(z) = \frac{1}{4\pi}(c_V + 1)J^W(z, \bar{z}) + \frac{1}{2(4\pi\lambda)^2}\partial\bar{\partial}J^W(z, \bar{z}) + \frac{1}{2(4\pi\lambda)^2}[J^W(z, \bar{z}), \partial J^W](z, \bar{z}) ,$$

(3.12)

with $J^W = W\bar{\partial}W^{-1}$ and $\partial I^W = 0$, i.e. $I^W$ does not depend on $z$.

Therefore, after finding isomorphic higher charges for both formulations, we are motivated to find their corresponding algebras, and later quantize them.

4. Higher conservation laws and corresponding algebras

To obtain the algebra obeyed by the previously found conserved charges, it is easier to go to Minkowski space, proceed with the canonical quantization\textsuperscript{19}, obtaining first the Poisson algebra, and later the constraints and the quantum commutators of the model. In fact, from the computation of the fermion determinant, we have an effective bosonic action that already takes into account some quantum corrections, namely the fermionic loops have been summed up. Therefore, the Poisson brackets already have quantum corrections arising from fermionic loops. This fact minimizes the possibilities of anomalies in the full quantum definition of the charges\textsuperscript{20}. As a matter of fact, we shall see that quantum corrections are restricted to the introduction of renormalization constants.

From the conventions described in the appendix we find the Minkowski space action

$$S = -(c_V + 1)\Gamma_M[W] + \frac{1}{2(4\pi\lambda)^2}\int d^2x \left[\partial_{\pm}(W\partial_{\pm}W^{-1})\right]^2 ,$$

(4.1)

with the Minkowski space WZW functional given by

$$\Gamma_M[W] = \frac{1}{8\pi} \text{tr} \int d^2x \partial_{\mu}W^{-1}\partial_{\mu}W + \frac{1}{4\pi}e^{\mu\nu}\text{tr} \int_0^1 dr \int d^2x \dot{W}W^{-1}\dot{\bar{W}}\dot{W}^{-1}\partial_{\mu}\dot{W} \dot{W}^{-1}\partial_{\nu}\dot{W} ,$$

(4.2)
Due to the presence of higher derivatives in the above action, it is convenient to introduce an auxiliary field and rewrite it in the equivalent form

$$S = -(c_V + 1)\Gamma_M[W] + \mathrm{tr}\left\{ \frac{1}{2} d^2 x \left[ -B^2 + \frac{1}{2\pi\lambda} \partial_+ B \partial_- W W^{-1} \right] \right\}, \quad (4.3)$$

where (4.1) is obtained by completing the square in the $B$-term in (4.3). The momentum canonically conjugated to the variable $W$ is

$$\Pi_{ij}^W = \frac{\partial S}{\partial \partial_0 W_{ij}} = -\frac{1}{4\pi} (c_V + 1) \partial_0 W_{ij}^{-1} - \frac{1}{4\pi} (c_V + 1) A_{ji} + \frac{1}{4\pi\lambda} (W^{-1} \partial_+ B)_{ji} , \quad (4.4)$$

where the first term is obtained from the principal sigma-model term in the WZW action, the second arises from the pure WZW term, and the third one from the interaction with the auxiliary field. It is convenient to separate the WZW contribution $A_{ij}$ to the momentum, since the new variable $\hat{\Pi}_{ij}^W$ is local in the original fields. The treatment of the WZW term (second in the right hand side above) follows closely the one introduced in [19], see also [7]. An explicit form for $A_{ij}$ cannot be obtained in terms of local fields, but we only need its derivatives, which are not difficult to obtain, that is

$$F_{ij;kl} = \frac{\delta A_{ij}}{\delta W_{lk}} - \frac{\delta A_{kl}}{\delta W_{ji}} = \partial_1 W_{il}^{-1} W_{kj}^{-1} - W_{il}^{-1} \partial_1 W_{kj}^{-1} , \quad (4.5)$$

in terms of which we have the Poisson bracket relation

$$\{ \hat{\Pi}_{ij}^W(x), \hat{\Pi}_{kl}^W(y) \} = -\frac{c_V + 1}{4\pi} \left( \frac{\delta A_{lk}}{\delta W_{ij}} - \frac{\delta A_{ji}}{\delta W_{kl}} \right) = \frac{c_V + 1}{4\pi} \left( \partial_1 W_{ij}^{-1} W_{ij}^{-1} - \partial_1 W_{ij}^{-1} W_{ij}^{-1} \right) \delta(x^1 - y^1) . \quad (4.6)$$

The momentum associated with the $B$ field is

$$\Pi_{ij}^B = -\frac{1}{4\pi\lambda} (W \partial_- W^{-1})_{ji} . \quad (4.6)$$

We can now list the relevant field operators appearing in the definition of the conservation law (3.12), which we rewrite in Minkowski space as

$$I_W^\mu = \frac{1}{4\pi} (c_V + 1) J_W^\mu - \frac{1}{4\pi\lambda} \partial_+ \partial_- J_W^\mu - \frac{1}{(4\pi\lambda)^2} [J_W^\mu, \partial_+ J_W^\mu] , \quad (4.7)$$

$$\partial_+ I_W^\mu = 0 . \quad (4.7)$$

In terms of phase-space variables, they are

$$J_W^\mu = W \partial_- W^{-1} = -4\pi\lambda \bar{\Pi}_B , \quad J_W^\mu = -4\pi\lambda \partial_+ \bar{\Pi}_B = 4\pi\lambda B , \quad (4.8)$$

$$\partial_+ \partial_- J_W^\mu = (4\pi\lambda)^2 [W \bar{\Pi}^W - (c_V + 1)\lambda \bar{\Pi}_B] - (4\pi\lambda)\lambda (c_V + 1) W' W^{-1} - 8\pi\lambda B' , \quad (4.8)$$
where the tilde means a transposition of the matrix indices. It is straightforward to compute the Poisson algebra. We have

$$\{ I^W_{ij}(t, x), I^W_{kl}(t, y) \} = [I^W_{ij} \delta_{kl} - I^W_{kl} \delta_{ij}] \delta(x^1 - y^1) - \alpha \delta^{il} \delta^{kj} \delta'(x^1 - y^1)$$  \hspace{1cm} (4.9)

where $\alpha = \frac{1}{2\pi} (c_V + 1)$. The current itself is a realization of the Kac–Moody algebra, since

$$\{ I^W_{ij}(t, x), J^W_{+kl}(t, y) \} = (J^W_{+kj} \delta_{il} - J^W_{+il} \delta_{kj}) \delta(x^1 - y^1) + 2 \delta_{il} \delta_{kj} \delta'(x^1 - y^1)$$

$$\{ J^W_{ij}(t, x), J^W_{+kl}(t, y) \} = 0$$  \hspace{1cm} (4.10)

We thus obtain a Kac–Moody algebra for $I^W_-$, and $J^W_+$ is a representation of such an algebra, with a central extension. We shall return to this discussion later, after consideration of the quantization of the charge.

The Hamiltonian density can also be computed, and we arrive at the phase-space expression

$$H^W = \tilde{\Pi}^W W' + 4\pi \lambda \tilde{\Pi}^W \tilde{\Pi}^B W - \tilde{\Pi}^B B - 4\pi \lambda^2 (c_V + 1)(\tilde{\Pi}^B)^2$$

$$- 2(c_V + 1)\lambda \tilde{\Pi}^B W' W^{-1} + \frac{1}{4\pi} (c_V + 1)(W' W^{-1})^2 + \frac{1}{2} B^2$$  \hspace{1cm} (4.11)

where $B' = \partial_t B$, $W' = \partial_t W$; the above Hamiltonian can be rewritten in a quadratic form in terms of the currents, although in such a case we have also velocities, due to the appearance of the time derivatives:

$$H^W = \alpha (J^W_1)^2 - \frac{1}{(4\pi \lambda)^2} \left[ \partial_+ J^W_+ J^W_+ + J^W_- \partial_- \partial_+ J^W_- - (\partial_+ J^W_-)^2 \right]$$  \hspace{1cm} (4.12)

where $J^W_1 = \frac{1}{2}(J^W_+ - J^W_-)$ and $J^W_+ = W \partial_+ W^{-1}$. At this point we can compare the model with its $\beta$-formulation. In this case we have the action

$$S = \Gamma_M[\beta] + \lambda \text{tr} \int d^2 x \, C_- \beta^{-1} \partial_+ \beta + \frac{1}{2} \text{tr} \int d^2 x \, (\partial_+ C_-)^2$$  \hspace{1cm} (4.13)

where $C_-$ is the Minkowski space counterpart of $\overline{C}$ (see eq. (3.8)).

The canonical quantization proceeds straightforwardly, and the relevant phase-space expressions are obtained for $\overline{J}^\beta$ in (3.3), which in Minkowski space, due to the $C_-$ equation of motion, reads

$$\overline{J}^\beta_\alpha = 4\pi \lambda^2 \partial_+^2 (\beta^{-1} \partial_+ \beta) = 4\pi \lambda C_-$$  \hspace{1cm} (4.14a)

$$\overline{\Pi}_- = \partial_+ C_-$$  \hspace{1cm} (4.14b)

while the $\beta$-momentum is given by

$$\tilde{\overline{\Pi}}_{ji}^\beta = \frac{1}{4\pi} \partial_0 \beta^{-1} + \lambda (C_- \beta^{-1})_{ji}$$  \hspace{1cm} (4.14c)
where the hat above $\Pi^\beta$ means that we neglected the WZW contribution as before\(^{25}\), and as a consequence

$$\left\{ \tilde{\Pi}^\beta_{ji}(t,x), \tilde{\Pi}^\beta_{lk}(t,y) \right\} = -\frac{1}{4\pi} \left( \partial_1 \beta^{-1}_{jk} \beta^{-1}_{li} - \partial_1 \beta^{-1}_{li} \beta^{-1}_{jk} \right) \delta(x - y) \quad (4.14d)$$

From the definition of the canonical momentum associated with $C_-$ we have

$$\partial_+ J_-^\beta = 4\pi \lambda \Pi_- \quad (4.15)$$

The conserved charge is (from (3.6) we change $\partial \to -\partial_-, \bar{\partial} \to \partial_+, J_-^\beta \to J_-^\beta$):

$$I_-^\beta = 4\pi \lambda^2 J_-^\beta + \partial_+ \partial_- J_-^\beta + [J_-^\beta, \partial_+ J_-^\beta] \quad (4.16)$$

therefore the situation is analogous to the one we found previously by interchanging the $(B, \Pi_B)$ phase-space variables with $(\Pi_-, C_-)$ (noticing the exchanged order).

At this point the Hamiltonian might be computed. However we will postpone it to a later section, since we will have to compute it in terms of more appropriate currents, making the problem easier to formulate in terms of the constraints that are hidden in the gauge transformation properties.

We now come to the point where we are urged to consider the quantization of the symmetry current (4.16). Let us consider the problem in the $\beta$-language, since the short-distance expansion depends on the high-energy behaviour of the theory; therefore, since the only massive scale is the coupling constant, we have to consider the weak coupling limit. The weak coupling limit is better described by the $\beta$-action. In such a case, we need the short-distance expansion of the current $J_-^\beta = 4\pi \lambda^2 \partial_+^{-2}(\beta^{-1} \partial_+ \beta)$ with itself. Since the short distance expansion is compatible with the weak coupling limit, where the theory is conformally invariant, Wilson expansions can be dealt with as usually.

We consider the short distance expansion

$$[J_-^\beta(x), \partial_+ J_-^\beta(y)] \bigg|_{y = x + \epsilon} = \sum_n a^{(n)}(\epsilon) \mathcal{O}^{(n)}(x) \quad (4.18)$$

aiming at a classification of $\mathcal{O}^{(n)}(x)$ according to its own dimension\(^{20}\). It is in fact easier to start out of local objects, that is in terms of $J_+^\beta$:

$$J_+^\beta = \frac{1}{4\pi \lambda^2} \partial^2_+ J_-^\beta = \beta^{-1} \partial_+ \beta \quad (4.19)$$

and later act with antiderivative operators. Therefore we analyse the auxiliary operator product expansion

$$[J_+^\beta(x), J_+^\beta(y)] \bigg|_{y = x + \epsilon} = \sum_n a^{(n)}_+(\epsilon) \mathcal{O}^{(n)}_+(x) \quad (4.20)$$
Let us suppose that the theory is local. The fact that the interaction contains an antiderivative will be taken into account subsequently. Along such premises the problem is very simple, and has been solved long ago \(^{20}\), with the classification for possible operators \( \mathcal{O}^{(n)}_{J^\beta_+}(x) \):

1. \( \dim \mathcal{O}^{(n)} = 0 \), no operator ,
2. \( \dim \mathcal{O}^{(n)} = 1 \), \( \mathcal{O}^{(1)}_{J^\beta_+}(x) = \beta^{-1 \partial \beta} \),
3. \( \dim \mathcal{O}^{(n)} = 2 \), \( \mathcal{O}^{(2)}_{J^\beta_+}(x) = \partial(\beta^{-1 \partial \beta}) \),
4. \( \dim \mathcal{O}^{(n)} \geq 3 \), operators with finite coefficients .

Therefore we find

\[
[J^\beta_+(x), J^\beta_+(y)] = a^{(1)}(\epsilon)J^\beta_+(x) + a^{(2)}(\epsilon)\partial J^\beta_+(x) ,
\]

where \( \dim a^{(1)}(\epsilon) = 1 \), therefore \( a^{(1)}(\epsilon) \) is linearly divergent, that is, \( a^{(1)}(\epsilon) \sim 1/\epsilon \), while \( \dim a^{(2)} = 0 \), and \( a^{(2)}(\epsilon) \) is logarithmically divergent, that is \( a^{(2)}(\epsilon) \sim \ln \epsilon \). Acting on the above Wilson expansion with \( \partial^{-2} \), we obtain the new expansion

\[
[\partial^{-2}J^\beta_+(x), J^\beta_+(y)] = a^{(1)}(\epsilon)\partial^{-2}J^\beta_+(x) + a^{(2)}(\epsilon)\partial^{-1}J^\beta_+(x) ,
\]

where \( \tilde{a}^{(2)}(\epsilon) \) is \( a^{(2)}(\epsilon) \) plus a possible \( \partial^{-1}a^{(1)}(\epsilon) \) correction. We now act with \( \partial^{-1} \), obtaining

\[
[\partial^{-2}J^\beta_+(x), \partial^{-1}J^\beta_+(y)] = \partial^{-1}\tilde{a}^{(2)}(\epsilon)\partial^{-2}J^\beta_+(x) + \text{finite ,}
\]

since \( \partial^{-1}a^{(2)} \) is finite.

Let us now discuss the effect of the non-local term in the action. Its local version is given by an expression containing the \( C_- \) field (see [19]). Such a field has dimension zero, and its interaction contains a \( \lambda \) factor in the Lagrangian, namely

\[
\mathcal{L}_{C_-} = \frac{1}{2} (\partial_+ C_-)^2 + \lambda C_- \beta^{-1} \partial_+ \beta .
\]

Therefore \( C_- \) comes in a Wilson expansion accompanied by a \( \lambda \)-factor, and has, effectively, dimension 1. Moreover, it cannot appear alone, since by Lorentz transformation it acquires a factor that is the inverse of the one required for the current, since it is the \((-\)) component of a vector, while the current is a \((+\)) component. Therefore it can appear at most with a logarithmically divergent coefficient in the \( J^\beta_+(x)J^\beta_+(x + \epsilon) \) expansion, and is irrelevant to the present problem, due to the subsequent manipulations.

This discussion leads to the definition of the cut-off charges

\[
Q^f_\delta = \int dx^- f(x^-) \left\{ Z_\delta J^\beta_-(x) + [J^\beta_-(x), \partial J^\beta_-(x + \delta)] \right\} ,
\]

with renormalized charge \( Q^f \) and renormalization constant \( Z_\delta \), respectively given by

\[
Q^f = \lim_{\delta \to 0} Q^f_\delta \quad \text{and} \quad Z_\delta = 1 - \partial^{-1}a^{(1)}(\delta) .
\]
Such a charge is finite, and
\[
\frac{dQ^f_\delta}{dx^+} = \lim_{\delta \to 0} \int dx^- f(x^-) \left\{ Z_\delta \partial_+ J^\beta_- + \partial_+ [\partial^{-1} a^{(1)} J^\beta_- + \mathcal{N}[J^\beta_-, \partial_+ J^\beta_-]] \right\} \to 0 , \tag{4.27}
\]
where \( \mathcal{N} \) is a normal product prescription rendering the product in \([J^\beta_-, \partial_+ J^\beta_-]\) finite.

The infinite constant can also be interpreted as a charge renormalization. Due to the renormalization of the higher charge, we cannot give an interpretation to the field operator \( I^\beta_{ij} \) by itself, but only to an arbitrary linear combination involving the charge and the current. In any case, since \( I^\beta_{ij} \) is a right-moving field operator, it is natural to assume, in view of the Poisson algebra (4.9), that it obeys an algebra given by\(^{21,22}\)
\[
I^\beta_{ij}(z) I^\beta_{kl}(\bar{w}) = (I^\beta_{kj} \delta_{il} - I^\beta_{il} \delta_{kj}) (w, \bar{w}) \frac{1}{z - \bar{w}} - \alpha \frac{\delta_{il} \delta^{kj} (z - \bar{w})}{(z - \bar{w})^2} , \tag{4.28}
\]

For \( J^\beta_{-ij} \), we are forced into a milder assumption. Indeed, since \( J^\beta_{-ij} \) is a representation of such an algebra with a central extension and since it commutes with itself, the equation \( \partial_+ J^\beta_{-ij} = 0 \) would be too simple to realize the whole problem we are considering. In such a case we would be left with unequal time commutators for the last equation (4.10). But in any case, since \( J^\beta_{-ij} \) is a right-moving field operator, the equal time requirement in the first equation (4.10) is also superfluous, and we get an operator product algebra of the type
\[
I^\beta_{-ij}(z) J^\beta_{-kl}(\bar{w}, \bar{w}) = (J^\beta_{-kj} \delta_{ij} - J^\beta_{-il} \delta_{kj}) (w, \bar{w}) \frac{1}{z - \bar{w}} + 2 \frac{\delta_{il} \delta^{kj} (z - \bar{w})}{(z - \bar{w})^2} , \tag{4.29}
\]
where once again we turned to the Euclidian variables. The consequence is the fact that holomorphic derivatives of the current are indeed primary fields. However the second equation in (4.10) cannot be taken at arbitrary times, since \( J^\beta_- \) depends on both \( x^+ \) and \( x^- \). Moreover, if \( J^\beta_- \) were purely right-moving, the last equation would imply, for unequal times, that it is a trivial operator.

Some conclusions may be drawn for \( J^\beta_- \). As we stressed above, \( \partial_+ J^\beta_- \) cannot be zero\(^*\), in the full quantum theory; however, in view of (4.29), we conclude that left(-) derivatives of this current are primary fields\(^{22}\), since
\[
I^\beta_{ij} \partial^n_+ J^\beta_-^{kl} = \frac{\partial^n_+ J^\beta_-^{kj} \delta_{il} - \partial^n_+ J^\beta_-^{il} \delta_{kj}}{\bar{z} - \bar{w}} \tag{4.30}
\]

Therefore, we expect a Kac–Moody algebra for \( I^\beta(z) \), and \( \partial^n I^\beta \) should be primary fields, depending on parameters \( \mathbf{z} \).

Such an underlying Kac–Moody structure is the most unexpected result in this paper, since it arose out of a non-linear relation obeyed by the current, which can be traced back

\(^*\) In the case \( J^\beta_- \) is left-moving, we expect further modifications of the commutators. See discussion in the conclusions.
to an integrability condition of the model. Moreover, the theory has an explicit mass term – although free massive fermionic theories as well as some off-critical perturbations of conformally invariant theories in two dimensions may contain affine Lie symmetry algebras.

The current itself is now a realization of such algebra in its right-moving sector. Indeed, we have derived the algebra (4.28) from the Poisson structure.

5. The GKO construction

Gauged WZW theories provide a Lagrangian realization of the GKO construction\textsuperscript{16,23}. Deleting the “mass term” $\lambda$ in (2.19) we have a gauged WZW theory as explicated in (2.7). The WZW functional is invariant under a $G \times G$ symmetry transformation given by

$$g(z, \bar{z}) \rightarrow \mathcal{G}(z)g(z, \bar{z})G(z).$$

(5.1)

In general one can gauge the anomaly-free vector subgroup $H \subset G \times G$ by means of the addition of the term

$$\frac{1}{4\pi} \text{tr} \int d^2 x \left[ e^2 A_+ A_- - e^2 a_+ g A_- g^{-1} + i e A_- g^{-1} \partial_+ g + i e A_+ g \partial_- g^{-1} \right].$$

(5.2)

In the QCD\textsubscript{2} case, $H$ corresponds to $G$.

Such a gauging procedure introduces constraints in the theory, as discussed by Karabali and Schnitzer\textsuperscript{16}. In order to understand this point in more detail, we have to consider the effect of the ghost sector. In general, ghosts are introduced considering a gauge-fixing function $\mathcal{F}(A)$, and introducing a factor

$$\det \left( \frac{\partial \mathcal{F}}{\partial A_\mu} \frac{\partial A_\mu}{\partial \epsilon} \right) \delta (\mathcal{F}(A))$$

(5.3)

in the partition function, where $\epsilon$ is the gauge parameter. However, if we are to render explicit the conformal content of the theory, it is more useful here to represent all possible chiral determinants in terms of ghost integrals, so that the reparametrization invariance is also explicit and one can later verify that the gauge-fixing procedure, as outlined above, and which is more frequently used in the gauge field literature, is trivial in the sense that one is led to a unit Faddeev–Popov determinant.

Therefore ghosts are introduced by writing determinants in terms of ghost systems and decoupling them from the gauge fields by a chiral rotation, a procedure which is possible in two-dimensional space-time. This is equivalent to writing all determinants as

$$\det D = e^{cV \Gamma[U]}(\det \partial)^{cV}, \quad \det \overline{D} = e^{cV \Gamma[U]}(\det \overline{\partial})^{cV}$$

(5.4)
and substitute the free Dirac determinant in terms of ghosts as
\[
\begin{align*}
(\det \partial)^{cV} &= \int D\bar{b} Dc e^{-\text{tr} \int d^2x \bar{b} \partial c}, \\
(\det \bar{\partial})^{cV} &= \int Db D\bar{c} e^{-\text{tr} \int d^2x b \bar{\partial} \bar{c}}.
\end{align*}
\]

(5.5)

In fact the determinant of the Dirac operator does not factorize as in (5.4) because of the regularization ambiguity. At every step, one has to assure vector current conservation. Such determinants cancel out by changing some of variables (as in (2.15)) but do not cancel in (2.19), from which we are led to the contribution
\[
\int D\bar{b} Db Dc D\bar{c} e^{-\text{tr} \int d^2x (b \partial (\bar{c} + \bar{b} \partial))}.
\]

(5.6)

Although decoupled at the Lagrangian level, such terms are essential due to constraints arising in the zero total conformal charge sector, leading to BRST constraints on physical states. Such constraints are obtained in a system of interacting conformally invariant sectors \((g, \Sigma, b, \bar{b}, c, \bar{c})\) described by the partition function
\[
\mathcal{Z} = \int Dg D\Sigma Db D\bar{b} Dc D\bar{c} e^{-k \Gamma[g] + (cV + k) \Gamma[\Sigma] - \text{tr} \int d^2x (b \partial (\bar{c} + \bar{b} \partial))},
\]

(5.7)

One can couple the system to external gauge fields \(A^{\text{ext}}\) and \(\overline{A}^{\text{ext}}\) as above, or equivalently by means of the minimal substitution\(^{16}\) \(\partial \to D^{\text{ext}} = \partial - ie A^{\text{ext}}\) and \(\bar{\partial} \to \overline{D}^{\text{ext}} = \overline{\partial} - ie \overline{A}^{\text{ext}}\) with \(A^{\text{ext}} = U^{-1}_{\text{ext}} \partial U_{\text{ext}}\), and \(\overline{A}^{\text{ext}} = V^{-1}_{\text{ext}} \bar{\partial} V_{\text{ext}}\). The interaction of the fields from the WZW theory with such external gauge fields is equivalently obtained from (2.7b), that is\(^{16}\)
\[
-k \Gamma[g, A] = -k \Gamma[U_{\text{ext}} g V_{\text{ext}}] + k \Gamma[U_{\text{ext}} V_{\text{ext}}] = (cV + k) \Gamma[U_{\text{ext}} \Sigma V_{\text{ext}}] - (cV + k) \Gamma[U_{\text{ext}} V_{\text{ext}}] \]
\[
-\text{tr} \int d^2x [b \overline{D}^{\text{ext}} c + \overline{b} D^{\text{ext}} \bar{c}] = -\text{tr} \int d^2x [b V_{\text{ext}} \overline{\partial} (\overline{V}^{-1}_{\text{ext}} c) + \overline{b} U^{-1}_{\text{ext}} \partial (U_{\text{ext}} \bar{c})],
\]

(5.8)

where \(k\) is the central charge. In our preceding discussion \(k = 1\).

In the first two cases, the invariance of the Haar measure permits to change variables as
\[
\begin{align*}
\tilde{g} &= U_{\text{ext}} g V_{\text{ext}} , \quad D\tilde{g} = Dg , \\
\tilde{\Sigma} &= U_{\text{ext}} \Sigma V_{\text{ext}} , \quad D\tilde{\Sigma} = D\Sigma ,
\end{align*}
\]

(5.9)

while in the last case one can do a chiral rotation, leaving back the free ghost system and a WZW term \(cV \Gamma[U_{\text{ext}} V_{\text{ext}}]\). Therefore, the \(\Gamma[U_{\text{ext}} V_{\text{ext}}]\) term cancels due to the balance of central charges and the partition function does not depend on the external gauge fields. This implies, in particular, that the functional derivative of the partition function with respect to the external gauge fields vanish, therefore
\[
\frac{\delta \mathcal{Z}(A^{\text{ext}}, \overline{A}^{\text{ext}})}{\delta A^{\text{ext}}} \bigg|_{A^{\text{ext}} = 0} = 0 = \frac{\delta \mathcal{Z}(A^{\text{ext}}, \overline{A}^{\text{ext}})}{\delta \overline{A}^{\text{ext}}} \bigg|_{A^{\text{ext}} = 0} ,
\]

(5.10)
which are equivalent, due to the minimal coupling, to the set of constraints

$$\langle k g^{-1} \partial g - (c_V + k) \Sigma^{-1} \partial \Sigma - 4\pi [b, c] \rangle = 0 = \langle J_g + J_\Sigma + J_{\text{ghost}} \rangle,$$  \hspace{1cm} (5.11a)

as well as

$$\langle k g^{-1} \partial g - (c_V + k) \Sigma^{-1} \partial \Sigma - 4\pi [b, c] \rangle = 0 = \langle J_g + J_\Sigma + J_{\text{ghost}} \rangle.$$

Each above current satisfies a Kac–Moody algebra with a corresponding central charge. One can build up a BRST charge $Q$ as

$$Q = \sum : c_{-n} (J_g^i + J_\Sigma^i) : - \frac{1}{2} i f^{ijk} \sum : c_{-n}^i b_{-m}^j c_{n+m}^k :, \hspace{1cm} (5.12)$$

where the $i, j, k$ indices refer to the adjoint representation of the symmetry group, $f^{ijk}$ the structure constants, and the mode expansion of the fields read

$$c^i = \sum c_{n}^i z^{-n},$$

$$b^i = \sum b_{n}^i z^{-n-1},$$

$$J_g^i, \Sigma = \sum (J_g^i, \Sigma)_n z^{-n-1}.$$

The above charge is nilpotent: $Q^2 = \frac{1}{2} \{Q, Q\} = 0$. This implies that the above system is a set of first-class constraints (indeed a similar set of constraints originates for $\overline{Q}$ resp. $\overline{J}, \overline{b}, \overline{\tau}$).

The stress tensor can be computed in terms of such currents, and we have three contributions, namely $T(z) = T_g(z) + T_\Sigma(z) + T_{\text{ghost}}(z)$, with the respective central charges $c_g = c(g, k) = \frac{2kd_g}{c_{g+2k}},$ $c_\Sigma = c(H, -k - c_H),$ and $c_{\text{ghost}} = -2d_H,$ where one supposes here that $\Sigma$ takes values in $H \in G$. The total central charge, $c_{\text{tot}} = c_g + c_\Sigma + c_{\text{ghost}}$ coincides with that obtained from the GKO construction for coset space conformal theories, and the total energy tensor decomposes in terms of the GKO stress energy tensor and a residual piece, $T'$, with zero central charge.

Representations of $T'$ are thus trivial, and the gauged WZW model is equivalent to the GKO construction of $G/H$ conformal field theories. The physical subspace is generated by a product of matter and ghost sectors, obeying the equation

$$Q|_{\text{phys}} = 0.$$

This solves also the problem of the sector with negative central charge, which should not be considered separately, being coupled through the BRST condition. Had we not such condition one would expect problems concerning negative metric states. Therefore one cannot consider each sector separately.

In the case of the inclusion of $QCD_2$ in such a scheme, we shall see that there are further constraints. Although the new constraints seem to be of the first-class type when considered alone, there is a combination that is second class due to the cancellation of
the ghost contribution. Therefore, in the case of QCD$_2$ we have to deal with a Dirac quantization procedure of second-class constraints!\(^{24}\)

However, we shall see that several interesting properties, characteristic of the model, as well as part of the conformal structural relations, still hold true, and QCD$_2$ problem can be understood as an integrable perturbation of a (very simple) GKO construction of coset space conformal field theory. We will have a GKO construction of a very simple coset model to an off-critical perturbation of a WZW theory by means of second-class constraints.

### 6. Coupling to external gauge fields and constraints

Consider the Minkowskian effective action

\[
S_{\text{eff}} = \Gamma[\tilde{g}] - (c_V + 1)\Gamma[\Sigma] + \Gamma[\beta] - \frac{1}{2}\lambda^2 \int d^2 x [\partial^{-1}_+ (\beta^{-1}_- \partial_+) ]^2 + S_{\text{ghosts}} \quad . \tag{6.1}
\]

Let us start with by first coupling the fields $(\tilde{g}, \Sigma, \text{ghosts})$ to external gauge fields

\[
A^\text{ext}_- = \frac{i}{e} V^\text{ext}_- \partial^- \tilde{V}^\text{ext}_- \quad , \quad A^\text{ext}_+ = \frac{i}{e} U^\text{ext}_+ \partial^+ U^\text{ext}_+ \quad . \tag{6.1a}
\]

We find

\[
S_{\text{eff}}(A) = \Gamma[U^\text{ext}_- \tilde{g} V^\text{ext}_- ] - (c_V + 1)\Gamma[U^\text{ext}_- \Sigma V^\text{ext}_- ] + \Gamma[U^\text{ext}_- \text{ghosts} V^\text{ext}_- ] + [1 - (c_V + 1) + c_V] \Gamma[U^\text{ext}_- V^\text{ext}_- ] \quad . \tag{6.2}
\]

Invariance of the Haar measure, and vanishing of the total central charge (i.e. vanishing coefficient of the last term above) tell us that the action does not depend on the external gauge fields. Nevertheless, the action can also be written as

\[
S_{\text{eff}}(A) = S_{\text{eff}}(0) - \frac{1}{4\pi} A^\text{ext}_+ \left[i e \tilde{g} \partial_+ \tilde{g}^{-1} - i e (c_V + 1) \Sigma \partial_- \Sigma^{-1} + J_- (\text{ghost}) \right] - \frac{1}{4\pi} A^\text{ext}_- \left[i e \tilde{g}^{-1} \partial_- \tilde{g} - i e (c_V + 1) \Sigma^{-1} \partial_+ \Sigma + J_+ (\text{ghost}) \right] + O(A^2) \quad . \tag{6.3}
\]

Functionally differentiating the partition function once with respect to $A^\text{ext}_+$ and separately with respect to $A^\text{ext}_-$, and putting $A^\text{ext}_\pm = 0$ we find the constraints

\[
i \tilde{g} \partial_- \tilde{g}^{-1} - i (c_V + 1) \Sigma \partial_- \Sigma^{-1} + J_- (\text{ghosts}) \sim 0 \\
i \tilde{g}^{-1} \partial_+ \tilde{g} - i (c_V + 1) \Sigma^{-1} \partial_+ \Sigma + J_+ (\text{ghosts}) \sim 0 \quad . \tag{6.4}
\]

leading to two BRS charges $Q^{(\pm)}$ as discussed by [16], which are nilpotent. Therefore we find two first-class constraints.
The field $A^{ext}_+$ can also be coupled to the field $\beta$ instead of $\tilde{g}$, since the system $(\beta, \Sigma, \text{ghosts})$ has also vanishing central charge. In such a case we have to disentangle the non-local interaction considering instead of the third and fourth terms in (6.1) the $\beta$-action

$$S(\beta) = \Gamma[\beta] + \int d^2x \frac{1}{2}(\partial_+ C_-)^2 + \int d^2x \lambda C_- \beta^{-1} \partial_+ \beta \ .$$  

(6.5)

We make the minimal substitution $\partial_+ \to \partial_+ - ie A^{ext}_+$, repeating the previous arguments for the $(\beta, \Sigma, \text{ghosts})$ system, and we now arrive at the constraint

$$\beta \partial_- \beta^{-1} + 4\pi \lambda \beta C_- \beta^{-1} - i(\kappa_V + 1)\Sigma \partial_- \Sigma^{-1} + J_-(\text{ghost}) \sim 0 \ .$$  

(6.6)

One could naively expect that, repeating the previous arguments the system has a new set of first-class constraints. But if we instead consider the equivalent system of the first set, together with the difference of the $(-)$ currents, namely

$$\Omega_{ij} = (\beta \partial_- \beta^{-1})_{ij} + 4\pi \lambda (\beta C_- \beta^{-1})_{ij} - (\tilde{g} \partial_- \tilde{g}^{-1})_{ij} \ ,$$  

(6.7)

one readily verifies that the above constraint cannot lead to a nilpotent BRST charge due to the absence of ghosts. Therefore, it must be treated as a second-class constraint, defining the field $C_-$. The Poisson algebra obeyed by $\Omega_{ij}$ is

$$\{\Omega_{ij}(t, x), \Omega_{kl}(t, y)\} = (\tilde{\Omega}_{il} \delta_{kj} - \tilde{\Omega}_{kj} \delta_{il})(t, x) \delta(x - y) + 2\delta_{il} \delta_{kj} \delta'(x - y)$$

$$\tilde{\Omega} = \tilde{g} \partial_- \tilde{g}^{-1} + \beta \partial_- \beta^{-1} + 4\pi \lambda \beta C_- \beta^{-1} \ .$$  

(Notice the change of sign in $\tilde{\Omega}$). Using the above, we can thus define the undetermined velocities, and no further constraint is generated.

The fact that the theory possesses second-class constraints is very annoying, since these cannot be realized by the usual cohomology construction. Therefore, instead of building a convenient Hilbert space, one has to modify the dynamics, since the usual relation between Poisson brackets and commutators is replaced by the relation between Dirac brackets and commutators.

Nevertheless, as we will see, several nice structures unravelled so far remain untouched after such a harsh mutilation. Indeed, we shall see that there is a rather deep separation between the “right” currents, obeying equations analogous to those written so far, and the “left” currents, which will obey a modified dynamics, due to the second-class constraints.

As a consequence of the definition of the canonical momenta, eq. (4.14c), the constraints have a phase-space formulation as

$$\Omega_{ij} = 4\pi (\beta \Pi^\beta)_{ij} + \partial_1 \beta \beta^{-1} - 4\pi (\tilde{g} \Pi^\beta)_{ij} - \partial_1 \tilde{g} \tilde{g}^{-1} \ ,$$  

(6.9)

which has been used to compute (6.8). Notice that the structure of the right-hand side of the phase-space expression is rather simple. Indeed, the $C_-$ field just redefines the momentum associated with $\beta$, and the above constraints are analogous to those appearing
in the description of non-Abelian chiral bosons, that is WZW theory with a constraint on a chiral current. It follows that the Poisson algebra is very simple. Indeed, one finds

\[\{\Omega_{ij}(y), \Omega_{kl}(x)\} = 16\pi\delta_{il}\delta_{kj}\delta'(x^1 - y^1) + 4\pi[(4\pi\beta\tilde{\Pi} + \beta'\beta^{-1} + 4\pi\tilde{\gamma}\frac{\partial}{\partial x} + \beta' g^{-1})_{ij} \delta_{kl} - \\
- (4\pi\beta\tilde{\Pi} + \beta'\beta^{-1} + 4\pi\tilde{\gamma}\frac{\partial}{\partial x} + \beta' g^{-1})_{il} \delta_{kj}\] \delta(x^1 - y^1) = 16\pi\delta_{il}\delta_{kj}\delta'(x^1 - y^1) + 8\pi[j_{-ij} \delta_{il} - j_{-il} \delta_{kj}] \delta(x^1 - y^1) ,

where \(j_-= 4\pi\beta\tilde{\Pi} + \beta'\beta^{-1}\) satisfies the Poisson algebra

\[\{j_{-ij}, j_{-kl}\} = 8\pi\delta_{il}\delta_{kj}\delta'(x - y) + 4\pi(j_{-jk} \delta_{il} - j_{-il} \delta_{kj}) \delta(x - y) .
\]

The above expression defines also the Q-matrix

\[Q_{ij;kl} = \{\Omega_{ij}(x), \Omega_{kl}(y)\} \bigg|_{equal \ time}\ ,\]

which is not a combination of constraints, therefore no further constraint is generated by the Dirac algorithm. The inverse of the Dirac matrix is not difficult to compute and we have the expression

\[(Q^{-1})_{ij;kl} = \frac{1}{32\pi}\delta_{il}\delta_{kj}\epsilon(x) + \\
\quad + \frac{1}{64\pi}(\delta_{il}j_{jk} - \delta_{jk}j_{li})|x| + \\
\quad + \frac{1}{128\pi}(\delta_{ia}j_{ib} - \delta_{ib}j_{ia})(\delta_{al}j_{bk} - \delta_{bk}j_{la})\frac{1}{2}x^2\epsilon(x) + \\
\quad + \frac{1}{256\pi}(\delta_{ia}j_{jb} - \delta_{ib}j_{ia})(\delta_{ca}j_{bd} - \delta_{cd}j_{ab})(\delta_{ce}j_{dk} - \delta_{de}j_{ck})\frac{1}{3}x^3\epsilon(x) + \cdots ,\]

where \(x\) is the space component of \(x^\mu\).

The next step consists in replacing the Poisson brackets by Dirac brackets. Thus we have to compute the Poisson brackets of the relevant quantities with the constraints. We use

\[\{A, B\}_{DB} = \{A, B\}_{PB} - \{A, \Omega_{\alpha}\}_{PB}Q^{-1}_{\beta\alpha}\{\Omega_{\beta}, B\}_{PB} .\]

We will see that functions of

\[J_+^\beta = \beta^{-1}\partial_+ \beta = -4\pi\tilde{\Pi} \beta + \beta^{-1}\beta' + 4\pi\lambda C_-
\]

commute with \(\Omega_{\alpha}\), and their Dirac brackets coincide with their Poisson brackets.

Canonical quantization through the Dirac formulation of the \(\beta\) sector is achieved by the formulae (4.14a, b, c), (4.15), from which we obtain

\[\frac{1}{4\pi}\beta^{-1}\partial_\pm \beta = -\tilde{\Pi} \beta \pm \frac{1}{4\pi}\beta^{-1}\beta' + \lambda C_- .\]
It is useful, in view of (6.8), to consider the combination

\[ \frac{1}{4\pi} \partial_- \beta \beta^{-1} = -\beta \tilde{\Pi} - \frac{1}{4\pi} \beta' \beta^{-1} + \lambda \beta C_- \beta^{-1} , \]  

(6.19)

or also, aiming at the expression of the constraint (6.8), which contains the \( C_- \) field, we have

\[ \beta \partial_- \beta^{-1} + 4\pi \lambda C_- \beta^{-1} = -4\pi \beta \tilde{\Pi} + \beta' \beta^{-1} . \]  

(6.20)

Thus, in terms of phase-space variables the constraint is given by (6.9). Using the above phase-space expressions we find

\[
\begin{align*}
\{ J_-^\beta, \Omega \} &= 0 \\
\{ [J_-^\beta, \partial_+ J_-^\beta], \Omega \} &= \{ [C_-, \Pi_-], \Omega \} = 0 .
\end{align*}
\]

(6.21)

For \( \{ \partial_+ \partial_- J_-^\beta, \Omega \} \) we first have to compute

\[
\begin{align*}
\partial_+ \partial_- J_-^\beta &= \partial_+^2 J_-^\beta - 2(\partial_+ J_-^\beta)' \\
&= 4\pi \lambda^2 \beta^{-1} \partial_+ \beta - 2(\Pi_-)' .
\end{align*}
\]

(6.22)

We use the fact that \( \{ \Pi_-', \Omega \} = 0 \) and we are left with

\[ \beta^{-1} \partial_+ \beta = -4\pi \tilde{\Pi} \beta + \beta^{-1} \beta' + 4\pi \lambda C_- . \]  

(6.23)

Using now \( \{ C_-, \Omega \} = 0 \) we just have to consider

\[ j_{+,ij} = \left( -4\pi \tilde{\Pi} \beta + \beta^{-1} \beta' \right)_{ij} . \]  

(6.24)

However, since \( \{ j_+, j_- \} = 0 \) we have \( \{ j_+, \Omega \} = 0 \)! As a conclusion, the Dirac algebra is the same as the Poisson algebra!! This is a non-trivial result, because it holds in spite of the fact that due to (6.9) the Dirac algebra obeyed by \( \tilde{\Pi}^\beta \) and \( \beta \) changes drastically, especially if we take into account the expression of the inverse Dirac matrix (6.13), which is non-local and has an infinite number of terms!
7. BRST constraints in the dual case

In the duality transformation relating the $\beta$ and the $W$ fields, we also find interesting relations arising out of the constraint structure of the theory. First let us perform a more detailed analysis of the ghost structure. Back to the transformations defined by (3.9) we have the factor $(\det \partial_+ \det \partial_-)^{cv}$ left out, which contributes as

$$Z^{gh'} = \int DB'_+ DB'_- DC'D\Sigma' e^{-tr\int d^2x (b'_+ \partial_- c' + b'_- \partial_+ \Sigma')}.$$  \hspace{1cm} (7.1)

The coupling of a subset of fields to an external gauge potential written in the form (6.1a), as described in section 6, can be made, and as usual. If such a set has a vanishing total central charge, the partition function does not depend on the gauge potential, and we are led to constraints again. With the partition function written in the $W$ language as in (4.3), and taking into account all appropriate ghosts, we have various self-commuting constraints. Some of them, such as

$$J_{\tilde{g}} - (c_V + 1)J_\Sigma + J_{\text{ghost}} \sim 0 ,$$

$$J_{\tilde{\beta}} - (c_V + 1)J_\Sigma + J_{\text{ghost}} \sim 0 ,$$

are the same as before, with the advantage that now $\tilde{\beta}$ is a pure WZW field, so that it can be simply identified with $\tilde{g}$, without further consequences. However, further constraints involving also the $W$ field arise, such as

$$J^\beta_+ - (c_V + 1)J^W_+ + J_{\text{ghost}} \sim 0 ,$$  \hspace{1cm} (7.2)

in such a way that we have, as a consequence, the non-trivial second-class constraint

$$J^\Sigma_+ - J^W_+ \sim 0 ,$$  \hspace{1cm} (7.4)

or, more explicitly,

$$(c_V + 1)\Sigma^{-1} \partial_+ \Sigma - (c_V + 1)W^{-1} \partial_+ W + \frac{1}{\lambda} W^{-1} \partial_+ BW = 0 .$$

Above, we proceeded as in the $\beta$ formulation, but with the interaction of the $A^\text{ext}_-$ field with the $W$, while in the (dual) $\beta$ case we considered $A^\text{ext}_+$. The phase-space expression is given in the formula

$$\Omega^{W,\Sigma} = -\hat{\Pi}^{W} W + \frac{1}{4\pi} W^{-1} W' + \hat{\Pi}^{\Sigma} \Sigma - \frac{1}{4\pi} \Sigma^{-1} \Sigma' \sim 0 ,$$  \hspace{1cm} (7.5)

and resembles the $\beta$-formulation (see (6.9)). However, as intriguing as it might appear, if we now substitute the $B$ field from the constraint (7.4) back into the action we find a non-local term. This means that while in the $\beta$-formulation, which is non-local at the
beginning, we end up with a local action after substituting back the constraint, in the W-action formulation, which is local at the beginning, we end up with a non-local action; another feature of duality in both formulations.

Keeping the Dirac algebra in mind, we substitute back the configuration-space constraints into the action, maintaining the phase-space structure. In such a case, using (6.7), and (2.6), we redefine \( \beta g \equiv P, \beta = Pg^{-1} \), and find the effective action

\[
S = \Gamma[P] - \frac{1}{2\pi} g^{-1} \partial_+ gg^{-1} \partial_- g - \frac{1}{4\pi} P^{-1} \partial_+ PP^{-1} \partial_- P + \frac{1}{4\pi} P^{-1} \partial_+ Pg^{-1} \partial_- g + \frac{1}{4\pi} P^{-1} \partial_- Pg^{-1} \partial_+ g + \frac{1}{4\pi} \partial_- gg^{-1} \partial_+ PP^{-1} - \frac{1}{2\pi} \partial_- gg^{-1} Pg^{-1} \partial_+ gg^{-1} P + \frac{1}{2(4\pi \lambda)^2} \left[ \partial_+ \{ g P^{-1} g \partial_- (g^{-1} P g^{-1}) \} \right]^2.
\]

(7.6)

The equation of motion/conservation law (3.6) still holds, as previously proved. From action (7.6) we can find the equations of motion. Notice that the final action is a WZW theory off the critical point, a principal \( \sigma \)-model, and current–current-type interactions between them.

For the dual formulation a further interesting structure arises. The constraint is now

\[
\partial_+ B = -\lambda (c_V + 1) W\Sigma^{-1} \partial_+ (\Sigma W^{-1})
\]

(7.7)

Similarly to the above, we use (7.7) and (2.6) to introduce \( S = W\Sigma \), replacing the \( W \)-field. Interesting enough, it is now the dual formulation that is non-local due to the presence of the \( B \)-field. We arrive again at the WZW theory for \( S \), a principal \( \sigma \)-model term for \( \Sigma \), current–current-type interactions, and principal \( \sigma \)-model terms for \( S \). The latter are such that the (wrong) sign of the principal \( \sigma \) term in \( \Gamma[S] \) changes, and we arrive at the WZW model with a relative minus sign, or \( \Gamma[S^{-1}]! \).

However, the standard procedure to deal with the constraints is to substitute the phase-space expressions in the Hamiltonian. But in such a case, the constraint (6.9) does not depend upon \( C_- \), and leads just to a connection between the right-moving current of the \( g \) sector, the left-moving current being untouched by such a relation! Therefore, still in the present case, where we witnessed the appearance of second-class constraints, their main role was to assure the positive metric requirement, as we have seen by means of the change of sign of the WZW action in the dual formulation.

8. Spectrum

Having recognized the role played by the \( \beta \)-action, we pass to discuss the spectrum of the theory. The first and huge step towards understanding the model was taken by 't Hooft, who used the Bethe–Salpeter equation in the large-\( N \) limit to prove that the bound states
form a Regge trajectory. By adopting the light cone gauge \((A_\sigma = 0)\) and formulating the Feynman rules in terms of light cone coordinates, the non-linear integral equation

\[
\Sigma(p) = -\frac{ie^2}{2\pi^2} \int \frac{dk_-}{k_-^2} \int \frac{d(k_+ + p_-)}{M^2 + (k + p)^2 + (k_+ + p_-)\Sigma(k + p) - i\epsilon}, \tag{8.1}
\]

was obtained for the fermion self-energy \(\Sigma(p)\), and the \(ie\)-description was used in order to perform the \(dk_+dk_-\) integrals. The infrared problems are very serious (as we readily see from the above equations). Using an infrared cut-off \(\lambda\) in order to restrict the \(k_-\) integration to \(k_- \geq \lambda\), one can perform the integral above. Since \(k_-\) scales as a boost, the procedure turns out to be Lorentz-invariant. The quark poles are pushed to infinity as \(\lambda \to \infty\) displaying in a clear fashion the confining properties of the theory. The solution is \(\Sigma(p) = \frac{e^2}{\pi} \frac{1}{p_-}\), and this leads\(^1\) to the above-mentioned Regge behaviour.

However, the procedure has been subjected to some criticism. In particular, Wu\(^8\) pointed out that the principal value prescription is ambiguous due to its non-commutative nature. Moreover, the above solution for the self-energy function implies a tachyon for small bare electron mass, as explicited in

\[
S_F(p) = \frac{\not{p} + M + \frac{e^2}{2\pi p_-} \gamma \not{p}}{p^2 - M^2 + \frac{e^2}{\pi}}. \tag{8.2}
\]

In particular, the massless theory has such a tachyon pole!

Wu performed a Wick rotation, working in the Euclidean space, and after rotating back to the Minkowski space found a different result for the fermion self-energy, namely

\[
\Sigma(p) = \frac{1}{p_-} \left\{ M^2 - p^2 - \sqrt{(M^2 - p^2)^2 + \frac{4e^2}{\pi}p^2} \right\}^{1/2}. \tag{8.3}
\]

The anomalous branch cut reflects the fact that all rainbow graphs have been tested for the Schwinger model. However, the complex light cone gauge involves a non-unitarity transformation and the relation between the results remained unclear. By all means, there are works indicating that in the axial gauge \(n^\mu A_\mu = 0, n^\mu n_\mu = -1\), it is inconsistent to use principal value prescription\(^9\).

Some authors have even speculated that QCD\(_2\) may exist in two distinct phases\(^{26}\). In the large-\(N\) regime (weak coupling) the gluons remain massless, since fermion loops do not contribute. Such is the ’t Hooft phase. There would exist also a Higgs phase, as in \(U(1)\) gauge interaction (Schwinger model) where the gauge field acquires a mass via the well-known Higgs mechanism. In this case the \(SU(N)\) symmetry would be broken to the maximal Abelian subgroup of \(SU(N)\).

Here we do not intend to provide a definite answer to such a complex question, but some directions may be outlined from the computation we performed. Indeed, we have an appropriate formulation to deal separately with both regimes: the weak-coupling regime described by the \(\beta\)-action may be discussed perturbatively. We will see that in the large-\(N\)
limit the relevant mass parameter is the one defined by 't Hooft, and we arrive at a possibility of computing the exact mass spectrum, once the complicated constraint structure is disentangled.

In order to understand the question concerning the spectrum, we first have to know which is the mass of the simplest excitation, or the mass parameter characterizing the theory. We thus consider the action

\[ S[\beta] = \Gamma[\beta] + \frac{1}{2} \lambda^2 \int d^2 x [\partial_+^{-1}(\beta^{-1}\partial_+ \beta)]^2 , \]  

and write a background-quantum splitting for the \( \beta \)-field as

\[ \beta = \beta_0 e^{i \xi} , \]  

after which we have the background-quantum splitting of the action up to second order in the quantum field \( \xi \). However, we have to be careful since, in the large-\( N \) limit, the second term is the zeroth-order Lagrangian, from which we suppose that the \( \xi \) field acquires a mass \( \mu^2 \) to be computed. The WZW term splits as

\[ \Gamma[\beta] = \Gamma[\beta_0] + \frac{1}{2} \int d^2 x \beta_0^{-1} \partial_\mu \beta_0 \xi \partial_\nu \xi (g^{\mu \nu} + \epsilon^{\mu \nu}) . \]  

Using the fact that \( \Gamma[\beta] \) is at the critical point, it is not difficult to compute the \( \beta_0^{-1} \partial_\mu \beta_0 \) two-point function at the one-loop order. We have the zeroth-order contribution from the second term in (8.4), and the one-loop contribution, which leads to the result:

\[ \beta^{-1} \partial_+ \beta \frac{\lambda^2}{p^2} \beta^{-1} \partial_+ \beta - N \frac{p_\mu p_\nu}{p^2} (g^{\mu \rho} + \epsilon^{\mu \rho})(g^{\nu \sigma} + \epsilon^{\nu \sigma}) F(p) \beta^{-1} \partial_\rho \beta \beta^{-1} \partial_\sigma \beta , \]  

where

\[ F(p) = \frac{1}{2\pi} \sqrt{\frac{p^2 - 4\mu^2}{p^2}} \ln \frac{\sqrt{-p^2 + 4\mu^2} + \sqrt{-p^2}}{\sqrt{-p^2 + 4\mu^2} - \sqrt{-p^2}} - \frac{1}{\pi} . \]  

For \( p^2 = \mu^2 \), we find

\[ \beta_0^{-1} \partial_+ \beta_0 \beta_0^{-1} \partial_+ \beta_0 \frac{1}{p^2} (\lambda^2 - 4N\mu^2 F(\mu^2)) . \]

The zero of the two-point function contribution to the action is at

\[ \mu^2 = f e^2 N = f \left( e^{t'Hooft} \right)^2 , \]  

where \( f \) is a numerical constant, in accordance with 't Hooft’s results.

The fact that the second term in (8.4) has an extra factor of \( N \) arises from the fact that the fermion loops are suppressed by a factor \( 1/N \). Since the fermion loops contribute with a WZW functional, while the \( \lambda \) term stems from the gauge field self-interaction (see
eqs. (2.17-19)) the factors of $N$ are correct. Moreover, it is exactly the given assignment that is compatible with the planar expansion. Finally, we have to quote the fact that 't Hooft’s analysis for the bound state $\psi\gamma_+\psi$ leads to a Bethe–Salpeter equation compatible with the previous results, the methods following closely 't Hooft’s analysis.

A more detailed information about the spectrum of the theory can be obtained from the Hamiltonian formulation. From the action

$$S = \Gamma[\beta] + \int d^2x \frac{1}{2}(\partial_+ C_-)^2 + \int d^2x \lambda C_- \beta^{-1} \partial_+ \beta$$

we obtain the canonical momenta

$$\tilde{\Pi}^\beta = \frac{1}{4\pi} \partial_0 \beta^{-1} + \lambda C_- \beta^{-1}$$

and the Hamiltonian density

$$H = \frac{1}{2} \Pi_- (\Pi_- - 2 C') - 2\pi (\tilde{\Pi}^\beta)^2 - \frac{1}{8\pi} (\beta^{-1} \beta')^2 + 4\pi \tilde{\Pi}^\beta \beta C_- - 2\pi \lambda^2 C_-^2 - \lambda \beta^{-1} \beta' C_-.$$  

The important currents are

$$J_+^\beta = \beta^{-1} \partial_+ \beta = -4\pi \tilde{\Pi}^\beta + \beta^{-1} \beta' + 4\pi \lambda C_-$$

$$j_-^\beta = \beta \partial_- \beta^{-1} + 4\pi \lambda \beta C_- \beta^{-1} = 4\pi \beta \tilde{\Pi}^\beta + \beta' \beta^{-1},$$

in terms of which the Hamiltonian density reads (notice that $j_-^\beta$ is not related to $J_-^\beta$, eqs. (3.3) and (4.14a))

$$H_\beta = -\frac{1}{16\pi} \left[ (J_+^\beta)^2 + (j_-^\beta)^2 \right] - \frac{1}{2} \lambda J_+^\beta C_- + \pi \lambda^2 C_-^2 + \frac{1}{2} \Pi (\Pi_- - 2 C').$$

From the previously discussed constraint structure (6.9), the current $j_-^\beta$ is related to the free right-moving fermion current $j_-^\beta = g \partial_- g^{-1}$, and we will drop it in the discussion of the spectrum for $\beta$. Moreover, from the Sugawara construction of the Virasoro algebra, in terms of the Kac–Moody generators, we know that the Sugawara piece $H_+ = \frac{1}{16\pi} (J_+^\beta)^2$ acquires a factor $(c_V + 1)^{-1} SU(N) (N + 1)^{-1}$ in the quantum theory. The $C_-^2$ terms are not known, since the $C_-$ equation of motion is not easily solvable. Nevertheless, in terms of $C_-$ and its conjugate momentum the Hamiltonian is quadratic. If we take it for granted that the zero-mode term is just the squared momentum, and moreover neglecting the $C_- J_+$ interaction, the Hamiltonian eigenstates have masses obeying the Regge behaviour

$$m^2 \sim n\mu^2.$$  

Corrections to this equation can be obtained using a large-$N$ expansion for the field $C_-$, a procedure which is at least possible upon considering the large-$N$ limit of (8.11).

25
9. Conclusions

We have reached several aims in the present work. The first concerns the issue of obtaining a bosonized version of QCD$_2$. In fact, this problem has been solved long ago$^{2,3,5}$. Here we use those well-known methods in order to rephrase this problem in terms of perturbations of a set of WZW models. Therefore features concerning integrability of the original theory are rendered much clearer, and an approach based on higher symmetries might be envisaged$^{27,28,29}$.

However QCD$_2$ is a very complex theory. From the different results obtained by ’t Hooft on the one hand, and by Wu on the other hand, several authors were led to support the idea that QCD$_2$ presents two phases, an unbroken weak coupling limit, as described by ’t Hooft, with a mesonic spectrum described by a Regge trajectory, and a Higgs phase, corresponding to the break of $SU(N)$ symmetry to the maximal Abelian subgroup of $SU(N)$. There is no sign of such spontaneous breakdown for vector-like theories, but this may happen in chiral gauge theories as a consequence of the vacuum polarization. In order to be able to deal with such a problem, dual formulations valid for different regimes must be available. In this direction we found the alternatives presented by the $\beta$ and $W$ fields, the first being an alternative for the weak coupling limit, where we gave arguments to support ’t Hooft’s proof of the Regge behaviour. However the $W$-field formulation is more involved, and we could not draw any result based on firm ground. In any case, the integrability of both formulations seems to be assured by the existence of higher conservation laws, which are in fact very similar in both cases. Nevertheless, a proof of the quantum integrability has only been possible in the $\beta$ formulation. Whether this is just a missing technical detail or a sign of some new physics cannot be decided but by speculation.

The integrability of the theory is one of the strongest points in this work. Several signs have been pointing, in the literature, to the possible integrability of non-abelian gauge theories in two dimensions. Gorsky and Nekrasov$^{30}$ have studied the large-$N$ Calogero-type Hamiltonian systems and found interesting relations with two-dimensional Yang-Mills theory. More recently Fadeev and Korchemski$^{31}$ found that the Lipatov model$^{32}$ is described by the spin zero limit of a spin system, which in turn is integrable. Our integrability conditions eqs. (3.4-7) and (3.12) are at the core of the integrability of the model, proving it. It would be interesting to translate such conditions to Calogero-type Hamiltonian systems, as well as to the Lipatov model, or still to Verlindes’ high–energy description of strong interactions$^{33}$.

We have to point out that gauged WZW models contain a rather non-trivial set of constraints. As pointed out by [16], although there are non-interacting subsets of fields at the Lagrangian level, the BRST constraints couple them. Such a coupling is essential for the maintenance of positivity of Hilbert space, due to some wrong sign of a part of the WZW actions. In the present case some combinations of the constraints are second class, and the Dirac procedure has to be used in full detail. However, as it turns out, there is a decoupling between the non-trivial sector described by the perturbed (off-critical) WZW theory, and the constrained sector and the integrability condition turns out to fulfil the same algebra for the Dirac as well as for the Poisson algebra. It turns out that such is the Kac–Moody algebra, and one component of the current is a realization of the Kac–Moody
algebra.

Using such splitting between the off-critical $J_+$ current and the constraint $j_-$ current, we can write the Hamiltonian in a convenient way, and relate the squared momentum eigenstates to the Sugawara Hamiltonian eigenstates, supporting the Regge behaviour obtained by ’t Hooft in the large–$N$ limit. The same method does not seem to work in the $W$ formulation.

There is also a solution to our problem, which is compatible with the classical structure of the current algebra, mentioned after eq. (4.28), namely

$$\partial_+ J_- = 0 \ , \quad (9.1)$$

for both $J^\beta$ and $J^W$. If this is the case, we have to modify the algebraic structure and impose (9.1) as a constraint, which in terms of the canonical fields reads

$$\Omega_1 = - \frac{1}{\lambda (c \nu + 1)} B \sim 0 \quad (9.2)$$

in the $W$ case (for the $\beta$ case one must change $B \to \Pi_-$).

The time independence of such a constraint leads to a secondary constraint

$$\Omega_2 = j_- - (c \nu + 1)\lambda \Pi_B - \frac{1}{2\pi \lambda} B' \sim \partial_+ \partial_- J^W \sim 0 \quad (9.3)$$

There are no further constraints, and the Dirac $Q$-matrix is

$$Q_{ij;kl} = \{ \Omega_{ij}, \Omega_{kl} \}^{-1} = \begin{pmatrix}
(j_{-i} \delta_{kj} - j_{-k} \delta_{ij}) \delta(x^1 - y^1) & \delta_{ij} \delta_{kj} \delta(x^1 - y^1) \\
-\delta_{il} \delta_{kj} \delta(x^1 - y^1) & 0
\end{pmatrix} \ . \quad (9.4)$$

In particular, for the conservation relations involving $J_-$, it leads to a Kac–Moody algebra. Moreover, using the constraints we identify $I_-$ with the current itself! Such a semi–classical reasoning misses the central term. The presence of a Kac–Moody algebra in the QCD$_2$ would be sufficiently astonishing, and we are not able, at the moment, to foresee either its consequences, or even whether such a possibility can indeed be realized in the present model, or speculate whether it may be so in some conformally invariant phase. We intend to delve deeper into the algebraic structure of the model in a subsequent publication.
Appendix

In Minkowski space,

\begin{equation}
    x^\mu = (x^0, x^1), \quad \partial_\pm = \partial_0 \pm \partial_1, \quad x^\pm = x^0 \pm x^1. \tag{A.1}
\end{equation}

In Euclidean space

\begin{equation}
    x_\mu = (x_1, x_2), \quad \overline{\partial} = \partial_1 - i \partial_2 \equiv \partial^-_E, \quad \partial = \partial_1 + i \partial_2 \equiv \partial^+_E, \quad z = x_1 - ix_2, \quad \overline{z} = x_1 + ix_2. \tag{A.2}
\end{equation}

In order to translate from one space to the other, we have \( x_2 = ix_0 \), implying (notice the important (-) sign!)

\begin{equation}
    \overline{\partial} \longleftrightarrow -\partial_-, \quad \partial \longleftrightarrow \partial_+. \tag{A.3}
\end{equation}

Notice also that

\begin{equation}
    \frac{\partial}{\partial \overline{z}} = \frac{1}{2} \overline{\partial}, \quad \frac{\partial}{\partial z} = \frac{1}{2} \partial. \tag{A.4}
\end{equation}

With these conventions,

\begin{equation}
    F_{\mu\nu}F^{\mu\nu} = 2F_{12}^2 = -\frac{1}{2}F_{\overline{z}z}^2 \quad \text{and} \quad F_{z\overline{z}} = -iF_{12} + iF_{21} = -2iF_{12}. \tag{A.5}
\end{equation}

Path integrals are always performed in Euclidean space, while in the canonical quantization we use the Minkowski version.

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