Security analysis of the W-OTS$^+$ signature scheme: Updating security bounds

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Abstract

In this work, we discuss in detail a flaw in the original security proof of the W-OTS$^+$ variant of the Winternitz one-time signature scheme, which is an important component for various stateless and stateful many-time hash-based digital signature schemes. We update the security proof for the W-OTS$^+$ scheme and derive the corresponding security level. Our result is of importance for the security analysis of hash-based digital signature schemes.

Keywords: post-quantum cryptography, hash-based signatures, W-OTS signature.

1 Introduction

Many commonly used cryptographic systems are vulnerable with respect to attacks with the use of large-scale quantum computers. The essence of this vulnerability is the fact that quantum computers would allow solving discrete logarithm and prime factorization problems in polynomial time \cite{1}, which makes corresponding key sharing schemes and digital signatures schemes breakable. At the same time, there exist a number of mathematical operations for which quantum algorithms offer little advantage in speed. The use of such mathematical operations in cryptographic purposes allows developing quantum-resistant (or post-quantum) algorithms, i.e. cryptographic systems that remain secure under the assumption that the attacker has a large quantum computer. There are several classes of post-quantum cryptographic systems, which are based on error-correcting codes, lattices, multivariate quadratic equations and hash functions \cite{2}.

Among existing post-quantum cryptographic systems, hash-based signature schemes \cite{3} attracted significant attention. This is easy to explain since the security of hash-based cryptographic primitives is a subject of extended
research activity, and hash functions are actively used in the existing cryptographic infrastructure. One of the main components of their security is as follows: For hash functions finding a pre-image for a given output string is computationally hard. Up to date known quantum attacks are based on Grover’s algorithm [4], which gives a quadratic speed-up in the brute-force search. Quantum attacks, in this case, are capable to find (i) preimage, (ii) second preimage, and (iii) collision, with time growing sub-exponentially with a length of hash function output. Moreover, the overall performance of hash-based digital signatures makes them suitable for the practical use. Several many-time hash-based digital signatures schemes are under consideration for standardization by NIST [5] and IETF [6,7].

We note that still the cryptographic security of hash-based digital signatures is a subject of ongoing debates, so security proofs for such schemes regularly appear (see e.g. [8–14]). These studies are partially focused on the security of basic building blocks of many-time hash-based digital signatures, which are one-time signature scheme. In particular, a variant of the Winternitz signature scheme, which is known as W-OTS+ is considered. The original security proof for the W-OTS+ scheme is presented in Ref. [8], and the W-OTS+ scheme is used in XMSS(-MT) [7], SPHINCS [9], Gravity SPHINCS [12], and SPHINCS+ [11] hash-based digital signatures. The security of many-time digital signatures obviously depends on the security level of the used one-time signature scheme.

In this work, we study the security of the W-OTS+ signature scheme. We identify security flaws in the original security proof for W-OTS+, which lead to the underestimated level of the security. We modify the security analysis of the W-OTS+ scheme.

The paper is organized as follows. We introduce necessary definitions and notations as well as describe the W-OTS+ scheme in Sec. 2. In Sec. 3 we provide a detailed updated security analysis of the W-OTS+ and discuss its differences from the previous version. We conclude in Sec. 4.

2 Preliminaries

2.1 One-time and many-time hash-based signatures

The Winternitz one-time signature (W-OTS) [15,16] has been introduced as an optimization of the seminal Lamport one-time signature scheme [17]. In order to use such one-time signature in practice several its modifications have been discussed. In particular, the W-OTS+ scheme has received a sig-
significant attention in the view of standardization processes, in which one of the candidates is the XMSS signature that uses the W-OTS+ \[5\].

It order to use hash-based digital signatures in practice one should make them usable for many times. In order to do so it is possible to use Merkle trees. Using a root of the tree one can authenticate public keys of many one-time signature. This idea is used in several many-time hash-based signatures based on the W-OTS+ scheme. The security of many-time digital signatures clearly depends on the security level of the used one-time signature scheme. The original security proof for the W-OTS+ scheme is presented in Ref. \[8\].

We note that there are other modifications of the W-OTS scheme (e.g. see \[6,18\]), however they are beyond the scope of the present paper.

2.2 Definitions and notations

We start our discussion with introducing basic definitions and notations also used in Ref. \[8\]. Let \(x \leftarrow X\) denote an element \(x\) chosen uniformly at random from some the set \(X\). Let \(y \leftarrow \text{Alg}(x)\) denote an output of the algorithm \(\text{Alg}\) processed on the input \(x\). We write \(\log\) instead of \(\log_2\) and denote a standard bitwise exclusive or operation with \(\oplus\), \([\cdot]\) and \([\cdot]\) stand for standard ceiling and floor functions.

**Definition 1** (Digital signature schemes). Let \(\mathcal{M}\) be a message space. A digital signature scheme \(\text{Dss} = (\text{Kg}, \text{Sign}, \text{Vf})\) is a triple of probabilistic polynomial time algorithms:

- \(\text{Kg}(1^n)\) on input of a security parameter \(1^n\) outputs a private key \(sk\) and a public key \(pk\);
- \(\text{Sign}(sk, M)\) outputs a signature \(\sigma\) under secret key \(sk\) for message \(M \in \mathcal{M}\);
- \(\text{Vf}(pk, \sigma, M)\) outputs 1 iff \(\sigma\) is a valid signature on \(M\) under \(pk\);

such that \(\forall (pk, sk) \leftarrow \text{Kg}(1^n), \forall (M \in \mathcal{M}) : \text{Vf}(pk, \text{Sign}(sk, M), M) = 1\).

Consider a signature scheme \(\text{Dss}(1^n)\), where \(n\) is the security parameter. A common definition for the security of \(\text{Dss}(1^n)\), which is known as the existential unforgeability under the adaptive chosen message attack (EU-CMA), is defined using the following experiment.

**Experiment** \(\text{Exp}_{\text{Dss}(1^n)}^{\text{EU-CMA}}(\mathcal{A})\)

\[(sk, pk) \leftarrow \text{Kg}(1^n).\]
\[(M^*, \sigma^*) \leftarrow \mathcal{A}^{\text{sign}(\cdot)}(pk)\].

\[\{(M_i, \sigma_i)\}_{i=1}^q\] be the query answers for \(\text{Sign}(sk, \cdot)\).

Return 1 iff \(Vf(pk, \sigma^*, M^*) = 1\) and \(M^* \notin \{M_i\}_{i=1}^q\).

In our work we consider one-time signatures, so the number of allowed queries \(q\) is set to 1.

Let
\[
\mathbb{Succ}^{\text{EU-CMA}}_{\text{Dss}(1^n)}(\mathcal{A}) = \Pr[\text{Exp}^{\text{EU-CMA}}_{\text{Dss}(1^n)}(\mathcal{A}) = 1]
\]
be the success probability of an adversary \(\mathcal{A}\) in the above experiment.

**Definition 2** (EU-CMA). Let \(t, n \in \mathbb{N}, t = \text{poly}(n)\), \(\text{Dss}(1^n)\) is a digital signature scheme. We call \(\text{Dss}\) EU-CMA-secure if the maximum success probability
\[
\mathbb{InSec}^{\text{EU-CMA}}(\text{Dss}(1^n), t) \defeq \max_{\mathcal{A}} \left\{ \mathbb{Succ}^{\text{EU-CMA}}_{\text{Dss}(1^n)}(\mathcal{A}) \right\} = \text{negl}(n).
\]

We then consider proof of the EU-CMA property for the W-OTS+ scheme on the basis of the assumption that the scheme is constructed with the function family having some particular properties. Let us discuss these required properties in detail.

Consider a function family \(F_n = \{f_k : \{0,1\}^n \rightarrow \{0,1\}^n\}_{k \in \mathcal{K}_n}\), where \(\mathcal{K}_n\) is some set. We assume that it is possible to generate \(k \leftarrow \mathcal{K}_n\) and evaluate each function from \(F_n\) for given \(n\) in \(\text{poly}(n)\) time. Then, we require three basic security properties for \(F_n\): (i) it is one-way (OW), (ii) it has the second preimage resistance (SPR) property, and (iii) it has the undetectability (UD) property.

The success probabilities of an adversary \(\mathcal{A}\) against OW and SPR of \(F_n\) are defined as follows:

\[
\mathbb{Succ}^{\text{OW}}_{F_n}(\mathcal{A}) = \Pr[k \leftarrow \mathcal{K}_n, x \leftarrow \{0,1\}^n, y = f_k(x), x' \leftarrow \mathcal{A}(k, y) : y = f_k(x')]
\] (1)

and

\[
\mathbb{Succ}^{\text{SPR}}_{F_n}(\mathcal{A}) = \Pr[k \leftarrow \mathcal{K}_n, x \leftarrow \{0,1\}^n, x' \leftarrow \mathcal{A}(k, x) : (x \neq x') \land (f_k(x) = f_k(x'))],
\] (2)

respectively. By using these notations, we introduce the basic definitions of OW and SPR.
Definition 3 (One-wayness and second preimage resistance of a function family). We call $F_n$ one-way (second preimage resistant), if the success probability of any adversary $A$ running in time $\leq t$ against the OW (SPR) of $F_n$ is negligible:

$$\text{InSec}^{\text{OW(SPR)}}(F_n; t) \equiv \max_A \{\text{Succ}^{\text{OW(SPR)}}_{F_n}(A)\} = \text{negl}(n).$$

(3)

To define the UD property we first need to introduce a definition of the (distinguishing) advantage.

Definition 4 (Advantage). Given two distributions $X$ and $Y$ we define the advantage $\text{Adv}_{X,Y}(A)$ of an adversary $A$ in distinguishing between these two distributions as follows:

$$\text{Adv}_{X,Y}(A) = |\Pr[1 \leftarrow A(X)] - \Pr[1 \leftarrow A(Y)]|.$$  

(4)

Consider two distributions $D_{\text{UD},U}$ and $D_{\text{UD},F_n}$ over $\{0, 1\}^n \times \mathcal{K}_n$. Sampling of an element $(u, k)$ from the first distribution $D_{\text{UD},U}$ is realized in the following way: $u \leftarrow \{0, 1\}^n$, $k \leftarrow \mathcal{K}_n$. Sampling of an element $(u, k)$ from the second distribution $D_{\text{UD},F_n}$ is realized by sampling $k \leftarrow \mathcal{K}_n$ and $x \leftarrow \{0, 1\}^n$, and then setting $u = f_k(x)$. The advantage of an adversary $A$ against the UD of $F_n$ is defined as the distinguishing advantage between these distributions:

$$\text{Adv}_{F_n}^{\text{UD}}(A) = \text{Adv}_{D_{\text{UD},U},D_{\text{UD},F_n}}(A).$$

(5)

Definition 5 (Undetectability). We call $F_n$ undetectable, if the advantage of any adversary $A$ against the UD property of $F_n$ running in time $\leq t$ is negligible:

$$\text{InSec}^{\text{UD}}(F_n; t) \equiv \max_A \{\text{Adv}_{F_n}^{\text{UD}}(A)\} = \text{negl}(n).$$

(6)

2.3 The W-OTS$^+$ signature scheme

Here we describe the construction of the W-OTS$^+$ signature scheme. First of all, we define basic parameters of the scheme. Let $n \in \mathbb{N}$ be the security parameter, and $m$ be the bit-length of signed messages, that is $\mathcal{M} = \{0, 1\}^m$. Let $w \in \mathbb{N}$ be so-called Winternitz parameter, which determines a base of the representation that is used in the scheme. Let us define the following constants:

$$l_1 = \left\lceil \frac{m}{\log(w)} \right\rceil, \quad l_2 = \left\lfloor \frac{\log(l_1(w - 1))}{\log(w)} \right\rfloor + 1, \quad l = l_1 + l_2.$$  

(7)
By using the described above function family $F_n$, we define a chaining function $c_k^j(x, r)$ for $x \in \{0, 1\}^n$, $r = (r_1, \ldots, r_j) \in \{0, 1\}^{n \times j}$, and $j \geq i \geq 0$ as follows:

$$c_k^0(x, r) = x, \quad c_k^i(x, r) = f_k(c_k^{i-1}(x, r) \oplus r_i) \text{ for } i > 0.$$ (8)

In what follows $r_{a,b}$ is a substiring $(r_a, \ldots, r_b)$ of $r$ if $b > a$ or it is an empty string otherwise.

Now we are ready to define the basic algorithms of the W-OTS+ scheme. Key generation algorithm ($\text{Kg}(1^n)$) consists of the following steps:

1. Sample the values

$$k \leftarrow \mathcal{K}, \quad r = (r_1, \ldots, r_{w-1}) \leftarrow \{0, 1\}^{n \times (w-1)}. \quad (9)$$

2. Sample the secret signing key

$$sk = (sk_1, \ldots, sk_l) \leftarrow \{0, 1\}^{n \times l}. \quad (10)$$

3. Compute the public key as follows:

$$pk = (pk_0, pk_1, \ldots, pk_l) = ((r, k), c_k^{w-1}(sk_1, r), \ldots, c_k^{w-1}(sk_l, r)). \quad (11)$$

Signature algorithm ($\text{Sign}(sk, M, r)$) consists of the following steps:

1. Convert $M$ to the base $w$ representation: $M = (M_1, \ldots, M_l)$ with $M_i \in \{0, \ldots, w-1\}$.

2. Compute the checksum $C = \sum_{i=1}^l (w - 1 - M_i)$ and its base $w$ representation $C = (C_1, \ldots, C_{l_2})$.

3. Set $B = (b_1, \ldots, b_l) = M||C$ as the concatenation of the base $w$ representations of $M$ and $C$.

4. Compute the signature on $M$ as follows:

$$\sigma = (\sigma_1, \ldots, \sigma_l) = (c_k^{b_1}(sk_1, r), \ldots, c_k^{b_l}(sk_l, r)). \quad (12)$$

Verification algorithm ($\text{Vf}(pk, \sigma, M)$) consists of the following steps:

1. Compute $(b_1, \ldots, b_l)$ as it is described in steps 1-3 of the signature algorithm.

2. Do the following comparison:

$$pk_i \leftarrow c_k^{w-1-b_i}(\sigma_i, r_{b_i+1,w-1}), \quad i \in \{1, \ldots, l\}. \quad (13)$$

If the comparison holds for all $i$, return 1, otherwise return 0.
We assume that the runtime of all three algorithm is determined by the evaluation of $f_k$, while time, which is required for other operations, in negligible. Thus, the upper bound on the runtime of $Kg, \text{Sign}, Vf$ is given by the value of $lw$.

3 Security of W-OTS$^+$

3.1 Security proof

In this section we consider the security proof of the W-OTS$^+$ scheme. The general line of our proof coincides with the one from Ref. [8]. However there are important differences, which yield another expression for the resulting security value.

**Theorem 1.** Let $n, w, m \in \mathbb{N}$ and $w, m = \text{poly}(n)$. Let $\mathcal{F}_n = \{f_k : \{0, 1\}^n \to \{0, 1\}^n\}_{k \in \mathcal{K}_n}$ be a one-way, second preimage resistant, and undetectable function family. Then, the insecurity of the W-OTS$^+$ scheme against an EU-CMA attack is bounded by

$$\text{InSec}^{\text{EU-CMA}}(\text{W-OTS}^+(1^n, w, m); t, 1) < lw \cdot \left( w \cdot \text{InSec}^{\text{UD}}(\mathcal{F}_n; \tilde{t}) + \text{InSec}^{\text{OW}}(\mathcal{F}_n; \tilde{t}) + w \cdot \text{InSec}^{\text{SPR}}(\mathcal{F}_n; \tilde{t}) \right)$$

(14)

with $\tilde{t} = t + 3lw + w - 2$, where time is given in number of evaluation function from $\mathcal{F}$.

**Proof.** The proof is by contrapositive. Suppose there exists an adversary $A$ that can produce existential forgeries for W-OTS$^+(1^n, w, m)$ scheme by running an adaptive chosen message attack in time $\leq t$ with the success probability $\varepsilon_A \equiv \text{Succ}^{\text{EU-CMA}}_{\text{W-OTS}(1^n, w, m)}(A)$.

Then we are able to construct an oracle machine $M^A$ that either breaks the OW or SPR of $\mathcal{F}_n$ using the adversary algorithm $A$. Consider a pseudocode description of $M^A$ in Algorithm 1 and block scheme in Fig. 1(a).

The algorithm is based on the following idea. We generate a pair of W-OTS$^+$ keys, and then introduce OW and SPR challenges in the $\alpha$th chain, where the index of the chain $\alpha$, position of the OW challenge $\beta$, and position of the SPR challenge $\gamma$ are picked up at random [see also Fig. 1(b)]. Then we submit a modified public key $pk'$ to $A$. The adversary can ask to provide a signature for some message $M$. If the element $b_\alpha$ calculated from $M$ is less than $\beta$, that is it locates below our challenge $y_e$, then we are not able to generate a signature and we abort. Otherwise, we compute the signature $\sigma$ with respect to our modified public key and give it to $A$. Finally, we obtain
some forged message-signature pair \((M', \sigma')\), and if the forgery is valid then \(\sigma'\) eventually contains the solution for the one of our challenges. Otherwise \(\mathcal{M}^A\) return fail.

**Algorithm 1: \(\mathcal{M}^A\)**

| Input | Security parameter \(n\), function key \(k\), OW challenge \(y_c\) and SPR challenge \(x_c\). |
|-------|-----------------------------------------------------------------|
| Output | A value \(x\) that is either a preimage of \(y_c\) (i.e. \(f_k(x) = y\)) or a second preimage for \(x_c\) under \(f_k \) (i.e. \(f(x_c) = f(x)\) and \(x \neq x_c\)) or fail. |

1. Generate W-OTS\(+\) key pair: \((sk, pk) \leftarrow \text{Kg}(1^n)\)
2. Choose random indices \(\alpha \leftarrow \{1, \ldots, l\}, \beta \leftarrow \{1, \ldots, w - 1\}\)
3. if \(\beta = w - 1\) then
   4. set \(r' = r\)
   5. else
   6. Choose random index \(\gamma \leftarrow \{\beta + 1 \ldots w - 1\}\)
   7. Set \(r' = r\) and replace \(r'_\gamma\) by \(c_k^{\beta - 1}(y_c, r_{\beta+1,w-1}) \oplus x_c\)
8. Obtain modified public key \(pk'\) by setting \(pk'_0 = (r', k)\), \(pk'_i = c_k^{w-1}(sk_i, r')\) for \(1 \leq i \leq l, i \neq \alpha\), and \(pk'_{\alpha} = c_k^{w-1}(y_c, r'_{\beta+1,w-1})\)
9. Run \(\mathcal{A}^\text{Sign}(sk,.)\)(\(pk'\))
10. if \(\mathcal{A}^\text{Sign}(sk,.)\)(\(pk'\)) queries to sign message \(M\) then
11. Compute \(B = (b_1, \ldots, b_l)\) which corresponds to \(M\)
12. if \(b_\alpha < \beta\) then
   13. return fail
14. Generate signature \(\sigma\) of \(M\) with respect to the modified public key:
   1. Run \(\sigma = (\sigma_1, \ldots, \sigma_l) \leftarrow \text{Sign}(M, sk, r')\)
   2. Set \(\sigma_\alpha = c_k^{b_\alpha - \beta}(y_c, r'_{\beta+1,w-1})\)
15. Reply to the query using \(\sigma\)
16. if \(\mathcal{A}^\text{Sign}(sk,.)\)(\(pk'\)) returns valid \((\sigma', M')\) then
17. Compute \(B' = (b'_1, \ldots, b'_l)\) which corresponds to \(M'\)
18. if \(b'_\alpha \geq \beta\) then
   19. return fail
20. else if \(\beta = w - 1\) or \(c_k^{b'_\alpha - \beta}(\sigma'_\alpha, r'_{\beta+1,w-1}) = y_c\) then
   21. return preimage \(c_k^{w-1-b'_\alpha-1}(\sigma'_\alpha, r'_{\beta+1,w-1}) \oplus r_\beta\)
22. else if \(x' = c_k^{\gamma-b'_\alpha-1}(\sigma'_\alpha, r'_{\beta+1,w-1}) \oplus r_\gamma \neq x_c\) and \(c_k^{\gamma-b'_\alpha}(\sigma'_\alpha, r'_{\beta+1,w-1}) = c_k^{\gamma-\beta}(y_c, r_{\beta+1,w-1})\) then
   23. return second preimage \(x' = c_k^{\gamma-b'_\alpha-1}(\sigma'_\alpha, r'_{\beta+1,w-1}) + r'_\gamma\)
   24. else
   25. return fail
26. else
27. return fail

We start with computing the success probability of \(\mathcal{M}^A\) in solving one of the challenges. Let \(\tilde{\mathcal{A}}\) be a probability that Algorithm \(\text{II}\) execution comes to the line 20. More formally, it can be written as follows:

\[
\tilde{\mathcal{A}} = \Pr[b_\alpha \geq \beta \land \text{"Forgery is valid"} \land b'_\alpha < b_\alpha],
\]

where the event “Forgery is valid” stands for \((1 \leftarrow \text{Vf}(pk; \sigma'; M')) \land (M' \neq \)

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We denote the whole event of Eq. (15) as “Forgery is fortunate”.

We then can consider two mutually exclusive cases: either (i) $\beta = w - 1$ or the chain started from $\sigma'_\alpha$ come to $y_c$ at the $\beta$th level, or (ii) $\beta < w - 1$ and the chain started from $\sigma'_\alpha$ does not come to $y_c$ at the $\beta$th level. Let these two case realizing with probabilities $p$ and $(1 - p)$ correspondingly conditioned by the event “Forgery is fortunate”.

In the first case, the adversary $\mathcal{A}$ somehow found a preimage for the $y_c$. The total probability of this event is upper bounded by $\text{InSec}^{\text{OW}}(\mathcal{F}_n; \tilde{t})$, so we can write

$$p \cdot \tilde{\epsilon}_A \leq \text{InSec}^{\text{OW}}(\mathcal{F}_n; \tilde{t}).$$

The time $\tilde{t} = t + 3lw + w - 2$ appears as the upper bound on the total running time of $\mathcal{A}$ plus each of the W-OTS$^+$ algorithms $Kg$, Sign, and $Vf$ plus preparing $\alpha$th chain in $\text{pk}'$ (see line 8 in Algorithm 1).

In the second case, we have a collision somewhere between $(\beta + 1)\text{th}$ and $(w - 1)\text{th}$ level. If the collision appears at the level $\gamma$ we obtain the second preimage of $x_c$. Since the SPR challenge was taken uniformly at random, the value of $r'_\gamma$ remains to be a uniformly random variable, therefore there is no way for $\mathcal{A}$ to detect and intentionally avoid the position $\gamma$. Thus, we obtain the collision at the level $\gamma$ with probability $(w - 1 - \beta)^{-1} > w^{-1}$ conditioned by the event “Forgery is fortunate”. On the other hand, this probability is upper bounded by $\text{InSec}^{\text{SPR}}(\mathcal{F}_n; \tilde{t})$. So we have

$$1 - p \frac{\tilde{\epsilon}_A}{w} < \text{InSec}^{\text{SPR}}(\mathcal{F}_n; \tilde{t}).$$  (17)
Again, the time $\tilde{t} = t + 3lw + w − 2$ appears as the upper bound on the total running time of our algorithm.

By combining Eq. (16) and Eq. (17), we obtain the following expression:

$$\tilde{\epsilon}_A < \text{InSec}^\text{OW}(F_n; \tilde{t}) + w \cdot \text{InSec}^\text{SPR}(F_n; \tilde{t}).$$

(18)

In the remainder of the proof we derive a lower bound for $\tilde{\epsilon}_A$ as the function of $\epsilon_A$. We note that in general $\mathcal{A}$ may behave in a ‘nasty’ way making $\tilde{\epsilon}_A \ll \epsilon_A$ e.g. by always asking to sign ‘bad’ messages with $b_\alpha < \beta$ or avoiding forgeries in ‘good’ positions $b_\alpha' > b_\alpha$. In other words, the algorithm may avoid crossing the point shown in Fig. 1(a). This behaviour of $\mathcal{A}$ means that it can somehow reveal the challenge position from the modified public key $pk'$. We below consider the strategy of using this possible ability of $\mathcal{A}$ to break UD property.

Consider two distributions $D_M$ and $D_{Kg}$ over $\{1, \ldots, w − 1\} \times \{0, 1\}^n \times \{0, 1\}^{n \times (w−1)} \times \mathcal{K}_n$. An element $(\beta, u, r, k)$ is obtained from $D_M$ by generating all subelements $\beta$, $u$, $r$, and $k$ uniformly at random from the corresponding sets. At the same time, an element $(\beta, u, r, k)$ is obtained from $D_{Kg}$ by generating $\beta$, $r$, and $k$ uniformly at random, but setting $u = c_k^{\beta}(x, r)$ with $x \xleftarrow{\$} \{0, 1\}^n$. One can see that $D_{Kg}$ corresponds to the generation of elements in W-OTS$^+$ signature chain from the secret key element up to the $\beta$th level.

Consider a pseudocode of Algorithm 2 of a machine $M^{\mathcal{A}}$ taking the security parameter $n$ and an element from either $D_M$ or $D_{Kg}$ as input. One can see that the operation of $M^{\mathcal{A}}$ is very similar to the operation of $M^\mathcal{A}$.

Given an input $(\beta, u, r, k)$ from $D_M$, $M^{\mathcal{A}}$ sets $y_c = u$ and then works exactly as $M$ up to line 19 of the Algorithm 1 If the event “Forgery is fortunate” happens, then $M^{\mathcal{A}}$ returns 1. Otherwise, it returns 0. So given an input $(\beta, u, r, k)$ from $D_M$, $M^{\mathcal{A}}$ outputs 1 with probability $\tilde{\epsilon}_A$.

Let us consider the behavior of $M^{\mathcal{A}}$ given an input from $D_{Kg}$. In this case $\mathcal{A}$ obtains a fair W-OTS$^+$ public key. The probability that $M^{\mathcal{A}}$ outputs 1 is thus given by

$$\hat{\epsilon}_A \equiv \Pr[b_\alpha \geq \beta \wedge \text{“Forgery is valid”} \wedge b'_\alpha < b_\alpha]$$

$$= \epsilon_A \cdot \Pr[b_\alpha \geq \beta \wedge b'_\alpha < b_\alpha | \text{“Forgery is valid”}]$$

$$\geq \epsilon_A \cdot \Pr[b_\alpha = \beta \wedge b'_\alpha < b_\alpha | \text{“Forgery is valid”}].$$

(19)

Here we used the fact that in the considered case $\Pr[\text{“Forgery is valid”}] = \epsilon_A$. 

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Algorithm 2: \( M' \)

**Input**: Security parameter \( n \), a sample \((\beta, u, r, k)\).

**Output**: 0 or 1.

1. Generate W-OTS\(^+\) key pair: \((sk, pk) \leftarrow Kg(1^n)\) taking bitmasks from \( r \) and a function for chain \( f_k \) instead of random ones.

2. Choose random index \( \alpha \overset{s}{\leftarrow} \{1, \ldots, l\} \).

3. Obtain modified public key \( pk' \) by setting \( pk'_0 = (r, k) \), \( pk'_i = c_{w-1}^k (sk_i, r') \) for \( 1 \leq i \leq l, i \neq \alpha \), and \( pk'_\alpha = c_{w-1-\beta}^k (u, r'_{\beta+1}, w-1) \).

4. Run \( \mathcal{A} \) Sign\((sk, \cdot)\)(\( pk' \))

5. if \( \mathcal{A} \) Sign\((sk, \cdot)\)(\( pk' \)) queries to sign message \( M \) then

6. Compute \( B = (b_1, \ldots, b_l) \) which corresponds to \( M \)

7. if \( b_\alpha < \beta \) then

8. \[ \text{return 0} \]

9. Generate signature \( \sigma \) of \( M \) with respect to the modified public key:

i. Run \( \sigma = (\sigma_1, \ldots, \sigma_l) \leftarrow \text{Sign}(M, sk, r') \)

ii. Set \( \sigma_\alpha = c_{w-1-\beta}^k (y_\alpha, r'_{\beta+1}, w-1) \)

10. Reply to the query using \( \sigma \)

11. if \( \mathcal{A} \) Sign\((sk, \cdot)\)(\( pk' \)) returns valid \( (\sigma', M') \) then

12. Compute \( B' = (b'_1, \ldots, b'_l) \) which corresponds to \( M' \)

13. if \( b'_\alpha \geq \beta \) then

14. \[ \text{return 0} \]

15. else

16. \[ \text{return 1} \]

Then we can write

\[
\Pr[b_\alpha = \beta \land b'_\alpha < b_\alpha | \text{"Forgery is valid"}] = \Pr[b_\alpha = \beta | \text{"Forgery is valid"}] \cdot \Pr[b'_\alpha < b_\alpha | b_\alpha = \beta \land \text{"Forgery is valid"}] = \Pr[b_\alpha = \beta | \text{"Forgery is valid"}] \cdot \frac{X}{l(w - 1)} > \frac{X}{lw}.
\]

and consider each term of the RHS in detail. Let \( X \) be a random variable equal to a number of elements in the requested W-OTS\(^+\) signature \( \sigma \) which lie above the zero level, which is conditioned by the fact the the forgery produced by \( \mathcal{A} \) is valid (if one gives \( \sigma \) to \( \mathcal{A} \)). More formally we define \( X \) as follows:

\[
X = |\{i : 1 \leq i \leq l, b_i > 0\}| \text{ conditioned by "Forgery is valid"}. \tag{21}
\]

Since \( \alpha \) and \( \beta \) are chosen at random from the sets \( \{1, \ldots, l\} \) and \( \{1, \ldots, w - 1\} \) we have

\[
\Pr[b_\alpha = \beta | \text{"Forgery is valid"}] = \frac{X}{l(w - 1)} > \frac{X}{lw}. \tag{22}
\]
Then, since the forged message $M'$ has at least one element in its signature $\sigma'$ which went down through its chain compared to the signature $\sigma$, and this element is certainly among $X$ elements, we have

$$\Pr[b'_\alpha < b_\alpha | b_\alpha = \beta \land \text{“Forgery is valid”}] \geq \frac{1}{X}. \quad (23)$$

Taking together Eqs. (19), (22), (23), and putting the result into Eq. (19) we obtain

$$\widehat{\epsilon}_A > \frac{\epsilon_A}{lw}. \quad (24)$$

By the definition, the advantage of distinguishing $D_M D_{Kg}$ by $\mathcal{M}^{tA}$ is given by

$$\text{Adv}_{D_M, D_{Kg}}(\mathcal{M}^{tA}) = |\widehat{\epsilon}_A - \widehat{\epsilon}_A|. \quad (25)$$

Using the obtained bound (24) and expanding absolute value in Eq. (25) we come to the following upper bound on $\epsilon_A$:

$$\epsilon_A < lw \cdot \left( \text{Adv}_{D_M, D_{Kg}}(\mathcal{M}^{tA}) + \widehat{\epsilon}_A \right). \quad (26)$$

The remaining step is to derive an upper bound of $\text{Adv}_{D_M, D_{Kg}}(\mathcal{M}^{tA})$ using the maximal possible insecurity level of the UD property. For this purpose we employ the hybrid argument method. First, we note that

$$\text{Adv}_{D_M, D_{Kg}}(\mathcal{M}^{tA}) = \sum_{\beta' = 1}^{w-1} \frac{1}{w-1} \text{Adv}_{D^{\beta=\beta'}, D_{Kg}^{\beta=\beta'}}(\mathcal{M}^{tA}), \quad (27)$$

where $D^{\beta=\beta'}_M$ and $D^{\beta=\beta'}_{Kg}$ denote distributions with fixed first subelement $\beta = \beta'$. Expression (27) leads to the fact that there must exist at least one value $\beta^*$ such that

$$\text{Adv}_{D_M^{\beta=\beta^*}, D_{Kg}^{\beta=\beta^*}}(\mathcal{M}^{tA}) \geq \text{Adv}_{D_M, D_{Kg}}(\mathcal{M}^{tA}). \quad (28)$$

Then we define a sequence of distributions $\{\mathcal{H}_i\}_{i=0}^{\beta^*}$ over $\{1, \ldots, w - 1\} \times \{0, 1\}^n \times \{0, 1\}^{n \times (w-1)} \times K_n$, such that an element $(\beta, u, r, k)$ is generated from $\mathcal{H}_i$ by setting

$$\beta = \beta^*, \quad x \leftarrow \{0, 1\}^n, \quad u = c_k^{\beta^*-i}(x, r_{j+1,w-1}), \quad (29)$$

and sampling $r$ and $k$ uniformly at random from the corresponding spaces. One can see that $\mathcal{H}_0$ and $\mathcal{H}_{\beta^*}$ coincide with $D^{\beta=\beta^*}_{Kg}$ and $D^{\beta=\beta^*}_M$, correspondingly. So, Eq. (28) can be rewritten as follows:

$$\text{Adv}_{\mathcal{H}_{\beta^*}, \mathcal{H}_0}(\mathcal{M}^{tA}) \geq \text{Adv}_{D_M, D_{Kg}}(\mathcal{M}^{tA}). \quad (30)$$
The triangular inequality yields the fact that there must exist two consecutive distributions \( \mathcal{H}_{i*} \) and \( \mathcal{H}_{i*+1} \) with \( 0 \leq i^* < \beta^* \) such that
\[
\text{Adv}_{\mathcal{H}_{i*}, \mathcal{H}_{i*+1}}(\mathcal{M}^{A}) \geq \frac{1}{\beta^*} \text{Adv}_{\mathcal{D}, \mathcal{D}_{kg}}(\mathcal{M}^{A}) > \frac{1}{w} \text{Adv}_{\mathcal{D}, \mathcal{D}_{kg}}(\mathcal{M}^{A}).
\]  
(31)

We are ready to construct our final machine \( B^{M^A} \), shown in Algorithm 3, which employs \( M^A \) to break the UD property.

| Algorithm 3: \( B^{M^A} \) |
|---|
| **Input**: Security parameter \( n \), a sample \( (u, k) \). |
| **Output**: 0 or 1. |
| 1. Generate \( r \overset{\xrightleftharpoons{\$}}{\sim} \{0, 1\}^{n(w-1)} \) |
| 2. Input \( n \) and \((\beta^*, e_k, \beta^*+(i^*+1)(u, r_{i^*+1,w-1}, r, k))\) into \( M^A \) |
| 3. return the result from \( M^A \). |

One can see that
\[
\text{Adv}_{\mathcal{D}_{UD}, \mathcal{D}_{UD}, \mathcal{D}_{U}}(B^{M^A}) = \text{Adv}_{\mathcal{H}_{i*}, \mathcal{H}_{i*+1}}(M^A),
\]  
(32)
since the input to \( M^A \) with \((u, k)\) from \( \mathcal{D}_{UD, U} \) is equivalent to a sample from \( \mathcal{H}_{i*+1} \), while this input to \( M^A \) with \((u, k)\) from \( \mathcal{D}_{UD, F} \) is equivalent to a sample from \( \mathcal{H}_{i*} \). Indeed,
\[
e_k^{\beta^*-(i^*+1)}(f_k(x), r_{i^*+1,w-1}) = e_k^{\beta^*-i^*}(x \oplus r_{i^*}, r_{i^*,w-1})
\]  
(33)
and \( x \oplus r_{i^*} \) is indistinguishable from the uniformly random string. At the same time, we have
\[
\text{Adv}_{\mathcal{D}_{UD}, \mathcal{D}_{UD}, \mathcal{D}_{U}}(B^{M^A}) \leq \text{InSec}_{UD}(F_n, \tilde{t}).
\]  
(34)
The runtime bound \( \tilde{t} = t + 3lw + w - 2 \) is obtained as sum of time \( t \) required for \( A \), at most \( 3lw \) calculations of \( f_k \) required in \( Kg, \text{Sign}, \) and \( Vf \) used in \( M^A \), and at most \( w - 2 \) calculations of \( f_k \), while preparing input for \( M^A \) in \( B^{M^A} \) (line 2 in Algorithm 3) and preparing \( \alpha \)th chain in \( M^A \) (line 3 in Algorithm 2) (the total number of \( f_k \) evaluations is given by \( w - 1 - (i^* + 1) \leq w - 2 \)).

By combining together Eqs. (31), (32), and (34) we obtain
\[
\text{Adv}_{\mathcal{D}, \mathcal{D}_{kg}}(M^A) < w \cdot \text{InSec}_{UD}(F_n, \tilde{t}).
\]  
(35)
Then putting this result into Eq. (26) we arrive at
\[
\epsilon_A < lw \cdot (w \cdot \text{InSec}_{UD}(F_n, \tilde{t}) + \tilde{\epsilon}_A)
\]  
(36)
Finally, taking into account Eq. (18) we obtain the desired upper bound. □

**Remark 1.** One can note that the bound \( \tilde{t} = t + 3lw + w - 2 \) can be tightened at least to \( \tilde{t} = t + 3lw \) by firstly choosing the value \( \alpha \) and then removing calculation of \( \alpha \)th chain within \( Kg \) used in \( M^A \) and \( M^A \). However, it has almost no practical value since usually is assumed that \( t \gg 4lw \).
3.2 Difference from the previous version of the proof

Here we point out main differences between our security proof and the original proof from Ref. [8] that contains a slightly different security bound, namely:

\[ \text{InSec}^{\text{EU-CMA}}(\text{W-OTS}^+(1^n, w, m); t, 1) \leq w l \cdot \max \{ \text{InSec}^{\text{OW}}(\mathcal{F}_n; t'), w \cdot \text{InSec}^{\text{SPR}}(\mathcal{F}_n; t') \} + w \cdot \text{InSec}^{\text{UD}}(\mathcal{F}_n; t^*), \quad (37) \]

where \( t' = t + 3lw \) and \( t^* = t + 3lw + w - 1 \).

First of all, during the discussion of \( \mathcal{M}^A \), that is the same in both proofs, it was stated that \( \Pr[b_\alpha = \beta] \geq \frac{1}{w} \), motivated by the fact that \( \beta \) is chosen uniformly at random (see p. 181 of [8]). However, as we discussed in our proof, \( \mathcal{A} \) may reveal the chain containing challenges, and also may always ask to sign a message with \( b_\alpha = 0 \) thus making \( \Pr[b_\alpha = \beta] = 0 \).

In the proof of [8] it is stated that \( \Pr[b'_\alpha < \beta | \text{"Forgery is valid"} \land b_\alpha = \beta] \geq l^{-1} \). This is also may not be correct if \( \mathcal{A} \) is able to reveal the chain containing challenges and, e.g., make forgery only with \( b'_\alpha = \beta \). Actually, accounting a possibility of hostile behavior of \( \mathcal{A} \) forces us to introduce the “Forgery is fortunate” event and bound its probability by employing \( \text{InSec}^{\text{UD}}(\ldots) \).

We note that our treatment also gives a different factor before the term \( \text{InSec}^{\text{UD}}(\ldots) \).

Moreover, in Ref. [8] the obtained bound contains \( \max\{ \text{InSec}^{\text{OW}}(\ldots), w \cdot \text{InSec}^{\text{SPR}}(\ldots) \} \) instead of \( \text{InSec}^{\text{OW}}(\ldots) + w \cdot \text{InSec}^{\text{SPR}}(\ldots) \). Perhaps, it appeared by putting multiples \( p \) and \((1 - p)\) on the opposite side of inequalities corresponded to Eq. (16) and Eq. (17) of the present paper.

Finally, the used different runtime bounds \( t' \) and \( t^* \) for breaking \( \text{OW}/\text{SPR} \) and \( \text{UD} \) of \( \mathcal{F}_n \), however, as it is shown above they can be considered to be the same.

Anyway, as we demonstrate below, both expressions (14) and (37) provide close levels of security. Moreover, we note that the security level of W-OTS\(^+\) used in the security proof of SPHINCS coincides with the derived expression (14) (see \#ots term on page 382 of [9]).

3.3 Security level

Given results of the Theorem 1, we are able to compute the security level against classical and quantum attacks. Following reasoning from Refs. [8,19],
we say the scheme has security level $b$ if a successful attack is expected to require $2^{b-1}$ evaluations of functions from $\mathcal{F}_n$. We calculate lower bound on $b$ by considering the inequality $\text{InSec}^{\text{EU-CMA}}(\text{W-OTS}^+(1^n, w, m); t, 1) \geq 1/2$. We assume that

$$\text{InSec}^{\text{OW}}(\mathcal{F}(n); t) = \text{InSec}^{\text{SPR}}(\mathcal{F}(n); t) = \text{InSec}^{\text{UD}}(\mathcal{F}(n); t) = \frac{t}{2^n} \quad (38)$$

for brute force search attacks with classical computer [3], and

$$\text{InSec}^{\text{OW}}(\mathcal{F}(n); t) = \text{InSec}^{\text{SPR}}(\mathcal{F}(n); t) = \text{InSec}^{\text{UD}}(\mathcal{F}(n); t) = \frac{t}{2^{n/2}} \quad (39)$$

for attack with quantum computer using Grover’s algorithm [4]. We also assume that $t \gg 4lw$, so all runtime bounds used in [14] and (37) are the same: $\tilde{t} \approx t' \approx t^* \approx t$. The results of comparison are shown in Table 1. The new bound is smaller the previous one by $\log \frac{l(2w+1)}{lw+1} \approx 1$ bit for typical parameter values $w = 16$ and $l = 67$.

|                      | Bound from [8]                               | Bound from present work                     |
|----------------------|----------------------------------------------|----------------------------------------------|
| Classical attacks    | $b > n - \log w - \log(lw + 1)$              | $b > n - \log(lw) - \log(2w + 1)$            |
| Quantum attacks      | $b > \frac{n}{2} - \log w - \log(lw + 1)$   | $b > \frac{n}{2} - \log(lw) - \log(2w + 1)$ |

Table 1: Comparison of security levels for the W-OTS$^+$ scheme.

4 Conclusion and outlook

Here we summarize the main results of our work. We have recapped the security analysis of the W-OTS$^+$ signature presented in Ref. [8], and pointed out some of its flaws. Although the updated security level almost coincides with the one from Ref. [8], we believe that our contribution is important for a fair justification of the W-OTS$^+$ security.

We note that a security analysis of the many-times stateless signature scheme SPHINCS$^+$, which uses W-OTS$^+$ a basic primitive and which was submitted to NIST process [11], originally was based on another approach for evaluating the security level [10]. However, it was discovered that the employed security analysis has some critical flaws (see C.J. Peikert official comment on Round 1 SPHINCS$^+$ submission [20]).

Recently, a new approach for the security analysis of hash-based signature was introduced [14]. It suggests a novel property of hash functions, namely the decisional second-preimage resistance, and therefore requires an additional deep comprehensive study.
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