Pseudo-scalar Photon Mixing In A Magnetized Medium.

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Abstract
Axions are pseudo-scalar particles, those arise because of breaking of Peccei Queen (PQ) symmetry. Axions have a tree level coupling to two photons. As a consequence there exists a tree level coupling of axion to photon in a magnetic field. However, in an external magnetic field, there exists a new loop induced, axion photon vertex, that gives rise to axion photon coupling. The strength of the tree level axion photon coupling in magnetic field is known to be model dependent. However in a magnetic field, the new loop induced coupling has some interesting features. This note discusses the new axion photon vertex in a magnetized medium and the corrections arising from there. The magnitude of the correction to axion photon coupling, because of magnetized vacuum and matter is estimated in this note. While making this estimate we note that the form of the axion photon vertex is related to the axial polarization tensor. This vertex is shown to satisfy the Ward identity. The coupling is shown to have a momentum dependent piece in it. Astrophysical importance of this extra modification is also pointed out.

I. INTRODUCTION

Axions play an important role in the conceptual aspects of particle physics today. They are believed to be associated with spontaneous breaking of global Chiral symmetry \( U(1)_PQ \) (Peccei Queen symmetry), postulated to provide an elegant solution to strong CP problem \([1,2]\). They are ultralight pseudo-scalar field \([3]\). In the Weinberg-Wilczek-Peccei-Queen model (original), the symmetry breaking scale was assumed to be around weak scale, \( f_w \). Although the original model, associated with the spontaneous breakdown of the global PQ symmetry at the Electro Weak scale (EW) \( f_w \), is excluded experimentally, modified versions of the same with their associated axions are still of interest; where the symmetry breaking scale is assumed to lie between the EW scale and \( 10^{12} \) GeV. Since the breaking scale of the PQ symmetry, \( f_a \), is much larger than the electroweak scale \( f_a \gg f_w \), the resulting axion turns out to be very weakly interacting (coupling constant \( \sim f_a^{-1} \)), very light (\( m_a \sim f_a^{-1} \)) and is often called “the invisible axion model”.

Till date, it remains elusive to experimental confirmation, however there have been some efforts to constrain its parameters through various cosmological or astrophysical considerations. For instance, cosmological observational constraints put bounds on its mass (such that the universe is not over closed). Through such arguments, the allowed range for the axion mass \( m_a \) has turned out to be \([4-7]\),

\[
10^{-5} \text{eV} \lesssim m_a \lesssim 10^{-2} \text{eV}. \tag{I.1}
\]

Apart from the one mentioned above, there are astrophysical considerations too that constrain axion coupling to photons or fermions. For instance, if they exist, being very weakly interacting particle, axions can drain away energy from stellar interiors. Since they are produced through processes like, \( e^+ + e^- \rightarrow \gamma + a \) or the cross channel reaction, \( e^- + \gamma \rightarrow e^- + a \) or \( \gamma_{plasmon} \rightarrow \gamma + a \), \( \gamma + \gamma \rightarrow a \)

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etc.; for a given coupling constant and mass, one can estimate the rate at which the axions would draw away energy from the stars. The bounds are placed by fact that, for a given energy budget of a star, the amount of energy drained by axion emission should be less than its observed luminosity. Apart from the ones mentioned above, there also have been experimental search for solar axions, through the conversion of an axion into a photon in a cavity, in presence of an external magnetic field. These searches also have placed some bound on the axion photon coupling. Incidentally its worth noting that since most of the astrophysical objects are associated with magnetic field the same process (or the reverse of it) can take place even in astrophysical environments too.

The coupling of axion to photon is realized through a term in the Lagrangian of the following form,

\[ \mathcal{L} = \frac{1}{M} a \mathbf{E} \cdot \mathbf{B}. \]  

(I.2)

Where \( a \) is the axion and \( M \) is the axion coupling mass scale. The experimental bound on \( M \) coming from the study of solar axions is set to be, \( M > 1.7 \times 10^{9–11} \text{GeV} \) [8]. It may be worth pointing out here that, though it is usually believed that, \( M > 1.7 \times 10^{9–11} \text{GeV} \), but this is a model dependent number. It should be noted here that the PQ symmetry breaking scale parameter \( f_a \) is proportional to \( M \). A detailed survey of various astrophysical bounds on the parameters of axion models and constraints on them, can be found in Ref. [9].

Since most of the bounds on axion parameters arise from astrophysical and cosmological studies where medium and a magnetic field are present, it becomes important to seek the modification of the axion coupling to photon, in presence of a medium or magnetic field or both. Particularly in some astrophysical situations where the magnetic component, along with medium (usually referred as magnetized medium) dominates. Examples being, the Active Galactic Nuclei (AGN), Quasars, Supernova, the Coalescing Neutron Stars or Nascent Neutron Stars, etc., to name a few. Apart from the ones discussed before, lately there seem to be some observational signature for possible existence of astrophysical objects, (called Magnetars ), with magnetic field strength, \( B \sim 10^{15–17} \text{G} \), i.e., significantly above the critical, Schwinger value \( B_c = m_e^2/e \approx 4.41 \times 10^{13} \text{G} \) [10,11]. However we would like to emphasize that even normal astrophysical objects are always associated with magnetic field, though the strength of the field may not be as strong as \( B_c \). Thus justifying the role of magnetized medium on axion properties. Also recently there has been an attempt to describe the observed faintness of Type Ia Supernova remnants, based on axion photon oscillation in the magnetic field of the intergalactic medium.

In view of these interesting physical applications of axion physics in astrophysics as well as cosmology it seems timely to find out the effect of medium and magnetic field, on the couplings of axions to photons.

In this note we would investigate the matter induced photon axion coupling in a magnetized medium. Where the particle in the plasma will be considered to be mostly electrons, though it could be any fermion. Since the temperature in these astrophysical objects are not too large (of the order of hundred MeV or so at the most) this seems to be a reasonable approximation. Of course our formulas are general enough to be extended to any temperature and density.

The organization of this document is as follows, in section II we would discuss about the physics of axion photon coupling and the model dependent uncertainties that enter in the axion photon coupling parameter \( M \) given in eqn. [I.2]. In the same section we would also try to give a brief overview of the existing axion models and their type of coupling with fermions. In the next section (i.e. section II), following Schwinger’s approach [12], we would elaborate on the details of the magnetized propagators. As would be discussed later, matter induced Axion photon coupling in a magnetized medium has two contributions in it, one coming from the magnetized vacuum and the other from the magnetized medium. Sections III and IV would deal with the details of those contributions. Finally at the end we would conclude by justifying our results through a general analysis and order of magnitude estimation of the
modifications one is getting from the presence of magnetized plasma. Finally we would like to conclude by pointing out the possible applications of our result.

II. AXION MODELS.

After the brief motivation of section [I], we would like to review the relevant details for the popular axion models, since they would be useful for the medium induced modifications to axion photon coupling. Usually there are three types of invisible axion models found in the literature. (a) the Dine-Fischler-Srednicki-Zhitnitskii, usually referred as DFSZ [15] model, (b) The Kim-Shifman-Vainshtein-Zakharov (KSVZ) model [16] and lastly (c) the variant invisible axion model (VIA) [17]. The DFSZ axion has two doublets, \( \phi_i (i = 1, 2) \) and one singlet \( \chi \) Higgs fields; the KSVZ model contains one \( \phi \) and one \( \chi \) Higgs fields along with a super heavy exotic quark that is singlet under SU(2)xU(1). The VIA model has some similarity to that of DFSZ model, except for the fact that it carries an extra Higgs singlet whose phase is identified with the axion. The most widely discussed axion model in the literature are DFSZ and KSVZ models. Apart from what was mentioned at the beginning, the main difference between them comes out on the basis of their tree level couplings with leptons and quarks. The KSVZ model does not have any coupling to electrons at the tree level, however the same might be generated via radiative correction through photon-photon-axion vertex [18]; and is higher order in coupling constant hence not interesting.

The \( U(1)_{PQ} \) transformation rules for the Higgs fields DFSZ model are given by [15],

\[
\phi_u \rightarrow e^{i\alpha X_u} \phi_u, \quad \phi_d \rightarrow e^{i\alpha X_d} \phi_d \quad \text{and} \quad \chi \rightarrow e^{i\alpha X} \chi. \tag{II.3}
\]

Where \( u \) and \( d \) are the generation indices and \( \phi_u \) couples only to up type quarks and similarly \( \phi_d \). The Higgs field \( \chi \) interacts with \( \phi_u \) and \( \phi_d \) through the potential term. The transformation laws for the fermions under the same transformation are fixed by demanding that the Yukawa interactions would remain invariant. In the passing it may noted that, following [19], one can assume for convenience that the left handed doublets transform like singlets under \( U(1)_{PQ} \), thus fixing the PQ charges of the right handed fermions through their coupling to the specifically assigned Higgs field.

Electromagnetic coupling of axions to photons were derived by Kaplan [20] and Sredeniki [21], [22], using current algebra techniques. In order to make the PQ current color anomaly free, a linear combination of chiral currents for light quark are usually subtracted from the same to define the new anomaly free axial vector current, given by,

\[
j_\mu^a = j_\mu^{PQ} - \frac{A_{PQ}}{1+z} \left( \bar{u} \gamma_\mu \gamma_5 u + z \bar{d} \gamma_\mu \gamma_5 d \right). \tag{II.4}
\]

Here, \( A_{PQ} \) is the color anomaly of the PQ charge, defined in terms of the generators of the group \( SU(3)_c \) i.e., \( \lambda_a (a = 1 \text{ to } 8) \) in the following way, \( \delta_{ab} A_{PQ}^c = \text{Tr} \left[ \lambda_a \lambda_b X_f \right] \). Where, the trace is understood to be taken over all the Weyl fermions and \( X_f \) are their generation specific PQ charge.

We note for the sake of completeness that, four divergence of Eqn. [II.4] yields,

\[
\partial^\mu j_\mu^a = \frac{e^2}{16\pi^2} F_{\mu\nu} F^{\mu\nu} \left[ \text{Tr} [X_f Q_f^2] - A_{PQ}^c \frac{2 (4 + z)}{3 (1 + z)} \right] - \frac{A_{PQ}}{1+z} \left( 2i m_u \bar{u} \gamma_5 u + 2i m_d \bar{d} \gamma_5 d \right). \tag{II.5}
\]

The Lagrangian describing the axion-fermion (to be considered as lepton for the estimates made in this note) interaction is given by [23],

\[
\mathcal{L}_{af} = \frac{1}{f_a} j_\mu^a \partial^\mu a = \frac{g_{af}}{m_f} \sum_f (\bar{\Psi}_f \gamma_\mu \gamma_5 \Psi_f) \partial^\mu a, \tag{II.6}
\]
where \( g'_{af} = X_f m_f / f_a \) is a dimensionless Yukawa coupling constant, \( X_f, ( \text{the model-dependent factors} ) \) are the PQ charges of different generations of quarks and leptons [9], as given in eqn. [II.4]. Lastly \( m_f \) is the fermion’s mass. In eqn. [II.6] the sum over \( f \) stands for sum over all the fermions, from each family. Although, in places instead of using the Lagrangian given by, [II.6], the following form for axion fermion Lagrangian has also been employed,

\[
\mathcal{L}_{af} = -2i g'_{af} \sum_f (\bar{\Psi}_f \gamma_5 \Psi_f) a,
\]

but, the correctness using [II.6] has been pointed out by Raffelt and Seckel in [24]. For our calculations we would however use [II.6], with \( g_{af} = X_f / f_a \).

The necessity for going through the details of the model becomes more apparent, if one recalls the way bounds were derived for \( M \), in eqn. [I.2]. The use of the form of that Lagrangian leaves much scope for the uncertainty in estimating the actual PQ symmetry breaking scale — coming from model dependent assignments of PQ charges of the fermion multiplets. To elaborate this point we note that, the effective photon photon axion coupling is obtained via the electromagnetic anomaly generated in the processes of making the PQ current free of anomalous divergence coming from the strong interaction sector. Thus, the axion photon photon Lagrangian as obtained from eqn. [II.5] [23], is given by:

\[
\mathcal{L}_{a\gamma\gamma} = \frac{e^2}{32\pi^2 f_a} \left[ A_{PQ}^{em} - A_{PQ}^c \right] a F \tilde{F}. \tag{II.8}
\]

In eqn. [II.8], following are the definition employed, \( A_{PQ}^{em} = \text{Tr}[Q^2 f] \cdot X_f \) and \( z \) is defined to be the ratio of the masses of two light quarks, i.e \( z = \frac{m_u}{m_d} \) and \( Q_f \) is the fermion electric charge, in units of electronic charge. The axion photon mixing Lagrangian in an external magnetic field turns out to be,

\[
\mathcal{L}_{a\gamma} = -g_{a\gamma\gamma} \frac{e^2}{32\pi^2 a} F \tilde{F}^{\text{ext}}, \quad \text{When } g_{a\gamma\gamma} = \frac{1}{f_a} \left[ A_{PQ}^{em} - A_{PQ}^c \right] \frac{2(4+z)}{3(1+z)}. \tag{II.9}
\]

The mixing strength, \( M \) as had already been given in eqn.[I.2] turns out to be, \( M = \frac{32\pi^2}{e^2 g_{a\gamma\gamma}} \). It is now rather easy to see that as one takes into account the effect of matter contribution (as shown in Fig.[1]), there will be additional contributions to \( g_{a\gamma\gamma} \) given in [II.9] and hence to \( M \). Though in principle (with out matter effects) the choice of PQ charges can make coupling constant \( g_{a\gamma\gamma} \) extremely small, but as we would show in this paper that as one takes into account the magnetized matter effects, there are additional modifications to the vertex, generating momentum dependent coupling in addition to the details of the model and the details of the PQ charge assignments. We would try to establish the same in this note.

### III. AXION PHOTON VERTEX IN A MAGNETIZED MEDIUM.

#### A. Fermion propagator in a magnetized medium.

We are interested in physical processes in an external background magnetic field. Without any loss of generality, the same is taken to be in the \( z \)-direction and would be denoted as \( B \). Any charged fermion propagator in such an external magnetic field, in Schwinger’s approach [12–14] is given by:

\[
i S^V_{\psi}(p) = \int_0^{\infty} ds \ e^{\Phi(p,s)} G(p,s), \tag{III.10}
\]

with \( \Phi \) and \( G \) defined in the following way:
The quantity $Q_f$ stands for the charge of the respective fermions, in units of electronic charge. Also the following useful relation should be noted,

$$\sigma_z = i\gamma_1\gamma_2 = -\gamma_0\gamma_3\gamma_5,$$

and we have used,

$$e^{ieQ_fB\sigma_z} = \cos(eQ_fBS) + i\sigma_z \sin(eQ_fBS).$$

It should be noted that, the metric convention for this propagator is $(+,-,-,-)$. To be more specific, according to the convention we follow,

$$\frac{p}{\parallel} = \gamma_0 p^0 + \gamma_3 p^3$$
$$\frac{p}{\perp} = \gamma_1 p^1 + \gamma_2 p^2$$
$$p^2 = p_0^2 - p_3^2$$
$$p_r^2 = p_1^2 + p_2^2.$$

Of course in the range of integration indicated in Eq. (III.10) $s$ is always positive and hence $|s|$ equals $s$ and by virtue of the $\epsilon$ prescription the exponent damps out at $s$ equal to infinity. Following standard prescriptions of thermal field theory, in the presence of a background medium, the above propagator is modified to [25]:

$$iS(p) = iS_B^V(p) + S_B^n(p),$$

where

$$S_B^n(p) = -\eta_F(p) \left[ iS_B(p) - \overline{S_B^V(p)} \right],$$

and

$$\overline{S_B^V(p)} = \gamma_0 S_B^V(p)\gamma_0,$$

for a fermion propagator, such that

$$S_B^V(p) = -\eta_F(p) \int_{-\infty}^{\infty} ds \, e^{\Phi(p,s)} G(p, s).$$

The information about the medium is carried by, $\eta_F(p)$ that in turn carries the information about the distribution function for the fermions and the anti-fermions:

$$\eta_F(p) = \Theta(p \cdot u_f(p, \mu, \beta)) + \Theta(-p \cdot u_f(-p, -\mu, \beta),$$

$f_f$ denotes the Fermi-Dirac distribution function:

$$f_f(p, \mu, \beta) = \frac{1}{e^{\beta(fu - \mu)} + 1},$$

and $\Theta$ is the step function given by:

$$\Theta(x) = 1, \text{ for } x > 0,$$
$$= 0, \text{ for } x < 0.$$
IV. EXPRESSION FOR PHOTON AXION VERTEX IN PRESENCE OF UNIFORM BACKGROUND MAGNETIC FIELD AND MATERIAL MEDIUM.

In order to estimate the loop induced $\gamma - a$ coupling, we would start with the Lagrangian given by Eqn. [II.6]. The effective vertex for the $\gamma - a$ coupling can be written as (for the sake of brevity, we define the notation, $p' = p + k$):

$$\Gamma_\nu(k) = (-i g_{af} e Q_f)k^\mu \int \frac{d^4p}{(2\pi)^4} \text{Tr}[\gamma_\mu\gamma_5iS(p)\gamma_\nu iS(p')].$$ (IV.21)

The effective vertex given by [IV.21], is computed from the diagram given in [Fig.1]. One can easily recognize that, eqn. [IV.21], has the following structure, $\Gamma_\nu(k) = k^\mu \Pi^{A}_{\mu\nu}(k)$. Where $\Pi^{A}_{\mu\nu}$, is the axial polarization tensor, comes from the axial coupling of the axions to the leptons and it is:

$$i\Pi^{A}_{\mu\nu}(k) = (-1)(-i)^2(g_{af} e Q_f)\int \frac{d^4p}{(2\pi)^4} \text{Tr}[\gamma_\mu\gamma_5S^V_B(p)\gamma_\nu iS^V_B(p') + \gamma_\mu\gamma_5S^V_B(p)\gamma_\nu iS^A_B(p')]$$.

The axial polarization tensor, $\Pi^{A}_{\mu\nu}$, would in general have contributions from pure magnetic field background, as well as magnetic field plus medium (in the text, the same might as well be referred as magnetized medium). The Pure magnetic field contribution (i.e the contribution devoid of any thermal phases space factors) and the one with magnetized medium effects, are to be found in the following expression,

$$i\Pi^{A}_{\mu\nu}(k) = (-1)(-i)^2(g_{af} e Q_f)\int \frac{d^4p}{(2\pi)^4} \text{Tr}\left[\gamma_\mu\gamma_5S^V_B(p)\gamma_\nu iS^V_B(p') + \gamma_\mu\gamma_5S^V_B(p)\gamma_\nu iS^V_B(p') + \gamma_\mu\gamma_5S^V_B(p)\gamma_\nu iS^V_B(p')\right].$$ (IV.23)

The pure magnetic field contribution has already been estimated in [26–30]; however we have rechecked and verified the same according to our convention and would report about it in the next section. Following that, the thermal part would be the one, we would be dealing with. Incidentally since we are dealing with the dispersive effects, the contributions coming from the absorptive part of the diagram are ignored (i.e we have ignored the appearance of the terms in $\text{Tr}[\gamma_\mu\gamma_5S^V_B(p)\gamma_\nu iS^V_B(p')]$ in side the loop integral). This is justified further by the fact that we are interested in photons far below the threshold of pair production.

A. Contribution From Magnetized Vacuum.

The VA response function in a magnetic field $\Pi^{A\mu}$ has been calculated in Ref. [26–30]. However since our conventions are different; also since there are some discrepancies in the results stated there (mostly typographical though), we have redone the calculations and our result reads,

$$\Pi^{A\mu}_{\mu\nu}(k) = \frac{i g_{af} e Q_f}{(4\pi)^2} \int_0^{\infty} dt e^{i\phi_0} \left\{ \left( 1 - \frac{v^2}{2}k^2_\| - 2m^2_\| \right) \bar{F}^{\mu\nu} - (1 - v^2)k_{\|\nu} \left( \bar{F}k\right)_\mu \right\} + R\left[k_{\nu\perp}(k\bar{F})_\mu + k_{\mu\perp}(k\bar{F})_\nu\right],$$ (IV.24)

Where, $R = \left[ 1 - v \sin Z v \sin Z - \cos Z \cos Z v \right] \sin^2 Z$ and $\phi_0 = e^{i\phi_0} \left[ \frac{1 - v^2}{4}k^2_\| - m^2 - \frac{\cos Z v - \cos Z k^2_\|}{2Z \sin Z} \right]$. (IV.25)

and $\bar{F}^{\mu\nu} = \frac{1}{2}e^{\mu\nu\sigma}F_{\rho\sigma}$ with $e^{0123} = 1$ is the dual of the field-strength tensor, with $Z = eQ_fBt$. Therefore, following Eqn. [IV.21], The photon axion vertex in a purely magnetized vacuum, would be given by,
The expression for $\Gamma^{\nu}(k)$ turns out to be,

$$\Gamma^{\nu}(k) = i g_{a f}(e Q f)^2 \left[ 4 i (k F)^{\nu} + \int_0^\infty dt \int_{-1}^{+1} dv e^{i \phi_0} \left\{ \left( - (1 - v^2) k^2 \right) (k F)^{\nu} + R \left[ k_{\nu \mu} (k F)^{\nu} \right] \right\} \right]$$

This result is not gauge invariant. However, one may integrate the first term under the integral by parts \[28,29\] to arrive at,

$$\Gamma^{\nu}(k) = i g_{a f}(e Q f)^2 \left[ 4 i - \int_0^\infty dt \int_{-1}^{+1} dv e^{i \phi_0} \left( 1 - v^2 \right) k^2 \right] (k F)^{\nu}$$

As is evident, terms proportional to $R$ in eqn. (IV.29) cancels out against each other leaving the vertex gauge invariant. Incidentally, in our way of writing $(k F)^{\nu} = k_{\mu} F^{\mu \nu}$ and $(F k)^{\nu} = F^{\nu \mu} k_{\mu}$. So the Effective Lagrangian giving rise to axion photon coupling in a magnetized medium would therefore be given by,

$$L_{B \gamma}^B = a A^\nu \Gamma^{\nu}(k)$$

It should be noted that, $a$ in Eqn.[IV.30], denotes the axion field.

In the limit of $\omega << m_f$, axion photon vertex can be written as:

$$L_{a \gamma}^B = - \frac{1}{32 \pi^2} g_{a f}(e Q f)^2 a F_{\mu \nu} \tilde{F}^{\mu \nu} \left[ 4 + \frac{k^2}{e B 6 h} \right]$$

Equation [IV.31], describes the effective axion photon interaction.

V. CONTRIBUTION FROM THE MAGNETIZED MEDIUM

Having estimated the effective axion photon vertex in a purely magnetic environment, we would now estimate the same in a magnetized medium. As before, one can do that by using the form of the fermion propagator in a magnetic field in presence of a thermal medium, as given by expressions(III.10) and (III.18); on doing that, we arrive at:
and polynomial in powers of the external magnetic field (with even and odd powers of \(R\))

In addition to being just even and odd in powers of \(R\), the traces we obtain (details for discrete symmetry arguments; and that is the reason, why we have treated them separately. Calculating these properties may come very useful while analyzing, the structure of axion photon coupling, using \(B\) of the chemical potential (the same would be clear as we go along) i.e., under charge conjugation, \(k\)

We denote the pieces even and odd orders in the external magnetic field in \(R\) to even and odd orders in magnetic field (with even and odd powers of \(B\)), i.e,

\[
R_{\mu\nu}(p,p',s,s') = R^{(E)}_{\mu\nu}(p,p',s,s') + R^{(O)}_{\mu\nu}(p,p',s,s')
\]

In the sections below we would evaluate them one after another.

**A. \(R_{\mu\nu}\) to even and odd orders in magnetic field**

We denote the pieces even and odd magnetic field in \(R_{\mu\nu}\) as \(R^{(E)}_{\mu\nu}\) and \(R^{(O)}_{\mu\nu}\). In addition to being just even and odd in powers of \(eQfB\), they also are odd and even in powers of the chemical potential (the same would be clear as we go along) i.e., under charge conjugation, \(\mathcal{B} & \mathcal{\mu} \leftrightarrow (-\mathcal{B}) & (-\mathcal{\mu})\), they both behave differently. More over their parity structures are also different. These properties may come very useful while analyzing, the structure of axion photon coupling, using discrete symmetry arguments; and that is the reason, why we have treated them separately. Calculating the traces we obtain (details for \(R^{(O)}\) is provided in the appendix, in any case \(R^{(E)}_{\mu\nu}\) on contraction with \(k^\mu\) vanish and if that can be established on general grounds, one does not have to evaluate it.),

\[
R^{(E)}_{\mu\nu}(p) = 4i\eta_-(p) \left[ \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} (1 + \tan(eQfBs) \tan(eQfBs')) \right] + \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} \sec^2(eQfBs') + \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} \sec^2(eQfBs) + \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} \sec^2(eQfBs) \right] \]

and

\[
R^{(O)}_{\mu\nu}(p) = 4i\eta_+(p) \left[ -m^2 \varepsilon_{\mu\nu\alpha\beta} (\tan(eQf Bs) + \tan(eQfBs')) \right] + \left\{ (g_{\mu\nu} p^{\alpha+} p^{\alpha+} - g_{\mu\nu} p^{\alpha+} p^{\alpha+} + g_{\nu\beta} p^{\beta+} p^{\beta+}) \right\} (eQfBs) + \left\{ (g_{\mu\nu} p^{\alpha+} p^{\alpha+} - g_{\mu\nu} p^{\alpha+} p^{\alpha+} + g_{\nu\beta} p^{\beta+} p^{\beta+}) \right\} (eQfBs) + \left\{ (g_{\mu\nu} p^{\alpha+} p^{\alpha+} - g_{\mu\nu} p^{\alpha+} p^{\alpha+} + g_{\nu\beta} p^{\beta+} p^{\beta+}) \right\} (eQfBs) + \left\{ (g_{\mu\nu} p^{\alpha+} p^{\alpha+} - g_{\mu\nu} p^{\alpha+} p^{\alpha+} + g_{\nu\beta} p^{\beta+} p^{\beta+}) \right\} (eQfBs) \]

FIG. 1. One-loop diagram for the effective axion electromagnetic vertex.
We would further express Eqn.\[V.35\]
as a sum of two pieces, $R_{\mu\nu}^{(c)} = 4i\eta_+(p) \left[ -2m^2\varepsilon_{\mu\nu12}(\tan(eQ_fB_s) + \tan(eQ_fB_{s'})) \right] + R_{\mu\nu}^{(o)}(p, p', s, s')$. Where $R_{\mu\nu}^{(o)}(p, p', s, s')$ is defined as follows,

$$R_{\mu\nu}^{(o)} = 4i\eta_+(p) \left[ m^2\varepsilon_{\mu\nu12}(\tan(eQ_fB_s) + \tan(eQ_fB_{s'})) \right] + \left\{ (g_{\mu\nu}p_{\mu}^\perp p_{\nu}' + g_{\mu\nu}p_{\mu}'^\perp p_{\nu}^\perp) + (g_{\mu\nu}p_{\mu}^\perp p_{\nu}' + g_{\mu\nu}p_{\mu}'^\perp p_{\nu}^\perp) \sec^2(eQ_fB_s) \right\} \tan(eQ_fB_s) + \left\{ (g_{\mu\nu}p_{\mu}^\perp p_{\nu} + g_{\mu\nu}p_{\mu}' p_{\nu}^\perp + g_{\mu\nu}p_{\mu}' p_{\nu}^\perp) + (g_{\mu\nu}p_{\mu}^\perp p_{\nu} + g_{\mu\nu}p_{\mu}' p_{\nu}^\perp) \sec^2(eQ_fB_s) \right\} \tan(eQ_fB_{s'}) \right]. \quad (V.36)$$

Here

$$\eta_+(p) = \eta_F(p) + \eta_F(-p) \quad (V.37)$$
$$\eta_-(p) = \eta_F(p) - \eta_F(-p) \quad (V.38)$$

they carry the informations about the thermal nature of the medium. Moreover, it should be noted that, according to the convention followed in the text,

$$a_{\mu\perp} b^\perp = a_0 b^3 + a_3 b^0.$$  

In the steps below it will be shown that, as Eqn.\[V.36\] is contracted with $k^\alpha$, it would vanish. Hence, only the non-vanishing contribution would be proportional to the integral over the four momentum and the parameters $p$, $s$ and $s'$ respectively of,

$$4i\eta_+(p)e^{\phi(p,s)+\phi(p',s')} \left[ -2m^2\varepsilon_{\mu\nu12}(\tan(eQ_fB_s) + \tan(eQ_fB_{s'})) \right].$$

In the rest frame of the medium, $p \cdot u = p_0$. Thus the distribution function does not depend on the spatial components of $p$. In this case we can write the expressions of $R_{\mu\nu}^{(c)}$ and $R_{\mu\nu}^{(o)}$ using the relations derived earlier [31] inside the integral sign, as

$$p^\beta_\perp \equiv -\frac{\tan(eQ_fB_{s'})}{\tan(eQ_fB_s) + \tan(eQ_fB_{s'})} k^\beta_\perp \quad (V.39)$$
$$p'^\beta_\perp \equiv \frac{\tan(eQ_fB_s)}{\tan(eQ_fB_s) + \tan(eQ_fB_{s'})} k^\beta_\perp \quad (V.40)$$
$$p^2_\perp \equiv \frac{1}{\tan(eQ_fB_s) + \tan(eQ_fB_{s'})} \left[ -ieQ_fB \right.$$
$$\left. + \frac{\tan^2(eQ_fB_{s'})}{\tan(eQ_fB_s) + \tan(eQ_fB_{s'})} k^2_\perp \right] \quad (V.41)$$
$$p'^2_\perp \equiv \frac{1}{\tan(eQ_fB_s) + \tan(eQ_fB_{s'})} \left[ -ieQ_fB \right.$$
$$\left. + \frac{\tan^2(eQ_fB_s)}{\tan(eQ_fB_s) + \tan(eQ_fB_{s'})} k^2_\perp \right] \quad (V.42)$$
$$m^2 \equiv \left( i \frac{d}{ds} + (p^2_\parallel - \sec^2(eQ_fB_s)p^2_\perp) \right) \quad (V.43)$$

Coming back to $R_{\mu\nu}^E$, eqn.\[V.34\] can further be simplified using Eqn.\[V.39,V.40\] and one arrives at,
Here throughout we have omitted terms such as $\varepsilon$, we have
$q$ Noting that it is possible to write, which is zero.
we arrive at the expression
\[ \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} \sec^2(eQ_j B s') \]
\[ + \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta-} \sec^2(eQ_j B s) \]
(VI.44)
The last term in Eqn.[V.34]has vanished because, on using the identities as mentioned above, i.e.
Eqn.[V.39,V.40], it turns out to be proportional to $\varepsilon_{\mu\nu\alpha\beta} k^{\alpha+} k^{\beta+}$, hence zero. The $\overset{\star}{=}$ symbol signifies that the relations above are not proper equations, the equality holds only inside the momentum integrals in Eq.(V.32).

**VI. EVALUATIONS IN EVEN AND ODD POWERS IN MAGNETIC FIELD.**

A. Evaluation of $\Pi^{\alpha\beta(E)}_{\mu\nu}$ to even orders in the external field

The axial polarization tensor, even in powers of the external field ( denoting by $\Pi^{\alpha\beta(E)}_{\mu\nu}$), is given by:
\[ \Pi^{\alpha\beta(E)}_{\mu\nu} = -(-ie)^2 (1) \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} R^{(E)}_{\mu\nu}(p,p',s,s'). \] (VI.45)

Using Eq.(V.44) in the rest frame of the medium, we have:
\[ R^{(E)}_{\mu\nu} \overset{\circ}{=} 4i\eta-(p_0) \left[ \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} (1 + \tan(eQ_j B s) \tan(eQ_j B s')) \right. \]
\[ + \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta-} \sec^2(eQ_j B s') + \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} \sec^2(eQ_j B s) \left. \right]. \] (VI.46)

Noting that it is possible to write, $q^\alpha p_\alpha = q^{\alpha+} p_{\alpha+} + q^{\alpha-} p_{\alpha-}$ Eq.(VI.46) can be written as,
\[ R^{(E)}_{\mu\nu} \overset{\circ}{=} 4i\eta-(p_0) \left[ \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} - \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta-} - \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} (1 + \tan(eQ_j B s) \tan(eQ_j B s')) \right. \]
\[ + \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta-} \sec^2(eQ_j B s') + \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} \sec^2(eQ_j B s) \left. \right]. \] (VI.47)

Here throughout we have omitted terms such as $\varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+}$, since by the application of Eq.(V.39) we have
\[ \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} = \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} p^{\beta+} + \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} k^{\beta+} \overset{\star}{=} \frac{\tan(eQ_j B s')}{\tan(eQ_j B s')} \times \varepsilon_{\mu\nu\alpha\beta} k^{\alpha+} k^{\beta+} \]
which is zero.

After rearranging the terms appearing in Eq.(VI.47), and by the application of Eqs.(V.39) and (V.40) we arrive at the expression
\[ R^{(E)}_{\mu\nu} \overset{\circ}{=} 4i\eta-(p_0) \left[ \varepsilon_{\mu\nu\alpha\beta} p^{\alpha+} k^{\beta+} (1 + \tan(eQ_j B s) \tan(eQ_j B s')) + \varepsilon_{\mu\nu\alpha\beta} k^{\alpha+} k^{\beta+} \right. \]
\[ \left. \times \tan(eQ_j B s) \tan(eQ_j B s') \right]. \] (VI.48)

Because of the presence of terms like $\varepsilon_{\mu\nu\alpha\beta} k^{\beta+}$ and $\varepsilon_{\mu\nu\alpha\beta} k^{\alpha+}$ as we contract $R^{(E)}_{\mu\nu}$ by $k^{\alpha}$, it vanishes. In view of this result it is tempting to conclude that, axion coupling to photon, in a magnetized medium, cannot be even in powers of external magnetic field. With all possibility this would be forbidden by the discrete symmetries i.e CPT. However, apart from the discrete symmetry arguments, there is an
alternative. recalling the fact that each power of \( eQ_f B \) actually denotes insertion of very soft photon (i.e all components of the four vector, \( k^\lambda \to 0 \)). In this case, the coupling of the dynamic photon along with the axion to even number of soft photon insertions, makes the power of the electromagnetic vertex odd. Since axion has spin zero, in order to match the total spin of the system, the sum over photon spins should add up to zero, which is impossible with odd number of photons. Hence this term should vanish in principle. Which is the result one arrives at after explicit computation.

**B. The Result for Odd Orders In Field Strength**

As has already been noted that, to odd orders in field strength one has \( R_{\mu \nu}^{(O)} \) and it can be expressed as a sum over two terms, one proportional to \( m^2 \) and the other is \( R_{\mu \nu}^{(o)} \), given by,

\[
R_{\mu \nu}^{(o)} = 4i\eta_+ (p) \left[ m^2 \varepsilon_{\mu \nu 12} (\tan (eQ_f B s) + \tan (eQ_f B s')) + \left\{ (g_{\mu \nu \alpha} p_{\nu \alpha} - g_{\mu \nu} p_{\alpha} p_{\alpha} + g_{\alpha \beta} p_{\beta} p_{\beta}) \right\} \tan (eQ_f B s) \\
+ \left\{ (g_{\mu \nu \alpha} p_{\nu \alpha} - g_{\mu \nu} p_{\alpha} p_{\alpha} + g_{\alpha \beta} p_{\beta} p_{\beta}) \sec^2 (eQ_f B s') \right\} \tan (eQ_f B s') \right] .
\]  

(VI.49)

In this subsection, we would outline the proof that Eqn. [VI.49] on contraction with \( k^\mu \) vanishes, for all \( \nu \). The proof follows in two steps, first one demonstrates that for \( \nu = 1 \) or \( 2 \) it goes to zero. Followed by that one shows that the same vanishes for \( \nu = 0 \), or \( 3 \) as well. Denoting \( \nu = 1 \) or \( \nu = 2 \) collectively as , \( \nu_\perp \), we show in appendix B,

\[
k^\mu R_{\mu \nu_\perp}^{(o)} (k) = 0 .
\]  

(VI.50)

Next we claim that, for \( \nu = 0 \) or \( 3 \), denoted collectively as \( \nu_\parallel \) (i.e the longitudinal components); \( k^\mu R_{\mu \nu_\parallel}^{(o)} \) would vanish. One can verify, with a bit of algebra, that the same can be written as a sum over two pieces. That is,

\[
k^\mu R_{\mu \nu_\parallel}^{(o)} = k^\mu R_{\mu \nu_\parallel}^{(o,1)} + k^\mu R_{\mu \nu_\parallel}^{(o,2)}
\]

Apart from the uninteresting overall constants and integrations over the variables \( p, p' \), \( s \) and \( s' \) the pieces \( k^\mu R_{\mu \nu_\parallel}^{(o,1)} \) are found to be proportional to,

\[
k^\mu R_{\mu \nu_\parallel}^{(o,1)} = p_{\nu_\parallel} \left[ k^2 + 2(k \cdot p) \right] \left( \tan (eQ_f B s) + \tan (eQ_f B s') \right) - k^2_\perp \tan (eQ_f B s) - \tan (eQ_f B s') \right] .
\]

(VI.51)

The fact that this piece vanishes exactly on general symmetry grounds of the integrals was first noted in [33] and also in [32], so we skip the details till appendix C. The required elaboration is provided there.

The left over pieces, not considered yet, is \( k^\mu R_{\mu \nu_\parallel}^{(o,2)} \). That is given by,

\[
k^\mu R_{\mu \nu_\parallel}^{(o,2)} = -4i\eta_+ (p)k^3 \left[ (p_\parallel^2 - m^2) \left( \tan (eQ_f B s) + \tan (eQ_f B s') \right) - k^2_\perp \tan^2 (eQ_f B s') \right] \times \frac{\sec^2 (eQ_f B s)}{\tan (eQ_f B s) + \tan (eQ_f B s')} \right] .
\]

(VI.52)

Similarly,
\[ k^\mu R^{(\alpha,2)}_{\mu3} = 4i\eta_+(p)k_0 \left[ (p_{\parallel}^2 - m^2) (\tan(eQ_f B s) + \tan(eQ_f B s')) - k_\perp^2 \tan^2(eQ_f B s') \right] \times \frac{\sec^2 eQ_f B s}{\tan(eQ_f B s) + \tan(eQ_f B s')} \]  

(VI.53)

In appendix D we have shown that Eqns [VI.52] and [VI.53] vanishes. Thus the proof that, \( k^\mu R^{(\alpha)}_{\mu\nu}(p, p', s, s') = 0 \) is complete.

VII. TO THE EFFECTIVE INTERACTION.

So using the results of the previous section, the effective axion photon vertex can be written down. Noting that,

\[ k^\mu R^{(Q)}_{\mu\nu} = -8im^2\eta_+(p) [k^\mu \varepsilon_{\mu\nu12}(\tan(eQ_f B s) + \tan(eQ_f B s'))] , \]  

(VII.54)

the vertex function \( \Gamma_{\nu}(k) \) (using Eqn. [V.32]) turns out to be,

\[ \Gamma_{\nu}(k) = (g_{af} eQ_f) (8m^2k^\mu \varepsilon_{\mu\nu12}) \int \frac{d^4p}{(2\pi)^4} \eta_+(p) \int_{-\infty}^{\infty} \frac{d^4s'}{(2\pi)^4} e^{\Phi(p,s)+\Phi(p',s')} \times [\tan(eQ_f B s) + \tan(eQ_f B s')] \]  

(VII.55)

Since the perpendicular components of the momentum integrals are Gaussian, one can carry them out without any difficulty with the following result,

\[ \Gamma_{\nu}(k) = (g_{af} eQ_f) (8m^2k^\mu \varepsilon_{\mu\nu12}) \int \frac{d^2p_{\parallel}}{(2\pi)^2} \eta_+(p) \int_{-\infty}^{\infty} \frac{ds}{-\infty} \int_0^{\infty} ds' e^{is(p_{\parallel}^2 - m^2) + is'(p_{\parallel}^2 - m^2)} \times \frac{-ieQ_f B [\tan(eQ_f B s) + \tan(eQ_f B s')] - i\frac{k_\perp^2}{4\pi} [\tan(eQ_f B s) + \tan(eQ_f B s')] e^{-i\frac{eQ_f B s}{2}}}{[\tan(eQ_f B s) + \tan(eQ_f B s')]^{\frac{-i}{4\pi}}} \]  

(VII.56)

Exact evaluation of [VII.56] is extremely difficult, however it is possible to get some analytical results under certain approximations. In the long wavelength limit and for \( \omega < m_f \) one can get an analytical estimate of the same. However for energies above the Pair Production Threshold (PPT), the one has to be very careful in evaluating the vertex functions. In the passing we note that, in the limit of \( m > \mu \) Eqn. [VII.56], can be evaluated analytically using Poisson summation formula and Laguerre polynomials. However in the limit of vanishing \( |k_\perp| \), the leading order result can easily be recovered from eqn. [VII.56] by setting \( |k_\perp| \to 0 \) in the same. The issue of evaluation of the same for all values of \( \omega \) and \( k \) would be taken up in a separate publication. Since, in this work, we would like to focus our attention to photon energies below pair production threshold. In what follows, from now on, we assume photon energy to be below PPT and would take the long wave length limit, i.e \( |k_\perp| \to 0 \) and try to estimate the magnitude of the contribution of the vertex function.

\[ \lim_{|k_\perp| \to 0} \Gamma_{\nu}(k) = (g_{af} eQ_f) (8m^2k^\mu \varepsilon_{\mu\nu12}) \frac{-ieQ_f B}{4\pi} \int \frac{d^2p_{\parallel}}{(2\pi)^2} \eta_+(p) \int_{-\infty}^{\infty} \frac{ds}{-\infty} \int_0^{\infty} ds' e^{is(p_{\parallel}^2 - m^2) + is'(p_{\parallel}^2 - m^2)} \times \frac{-i\frac{k_\perp^2}{(p_{\parallel}^2 - m^2)}}{(p_{\parallel}^2 - m^2)} \]  

(VII.57)

The last step in Eqn.[VII.57]follows from the definition of the delta function preceded by subsequent integration over \( s' \).
\[
\int_{-\infty}^{\infty} e^{is(p^2-m^2)} ds = 2\pi \delta(p^2 - m^2)
\]

\[
\int_{0}^{\infty} e^{is(p^2-m^2)} ds' = \frac{-i}{(p^2 - m^2)}
\]

(VII.58)

In the rest frame of the medium p.u = p₀, in this frame, the thermal factor, \(\eta_+(p\mu)\) turns out to be, \(\eta_+(p\mu) = n_F(|p\mu|, \mu) + n_F(|p\mu|, -\mu)\) \([33]\). (Here \(n_F(x)\), the thermal distribution function, is defined as \(n_F(x, \mu) = \frac{1}{e^{(x-\mu)/T}+1}\), as is the case for Fermi distribution fn.). using the same and the delta function constraint, one can simplify Eqn. [VII.57] further, to arrive at,

\[
\Gamma_\nu(k) = -(g_{af}(eQf)^2) \left(4m^2k^\mu \varepsilon_{\mu\nu12}\right) \mathcal{E} \int \frac{d^2p_\parallel}{(2\pi)^2} \left[n_F(|p\mu|, \mu) + n_F(|p\mu|, -\mu)\right] \delta(p^2 - m^2) \frac{1}{(k_\parallel^2 + 2(p \cdot k)_\parallel)}
\]

\[
= -16(g_{af}(eQf)^2) \left(\frac{k^\mu \tilde{F}^\mu\nu}{16\pi^2}\right) \Lambda(k_i^2, k \cdot u, \beta, \mu)
\]

(VII.59)

Where \(\Lambda(k_i^2, k \cdot u, \beta, \mu)\) is,

\[
\Lambda(k_i^2, k \cdot u, \beta, \mu) = \int d^2p_\parallel \left[n_F(|p\mu|, \mu) + n_F(|p\mu|, -\mu)\right] \left(m^2 \delta(p^2 - m^2)\right) \frac{1}{(k_\parallel^2 + 2(p \cdot k)_\parallel)}
\]

(VII.60)

Therefore the effective Lagrangian for axion photon interaction would be given by,

\[
\mathcal{L}_{\gamma a}^{\mu, \nu, \beta} = A^\nu \Gamma_\nu(k) = -16 \left(g_{af}(eQf)^2\right) \frac{F_{\mu\nu} \tilde{F}^{\mu\nu}}{16\pi^2} \Lambda(k_i^2, k \cdot u, \beta, \mu).
\]

(VII.61)

Equation [VII.61] is the expression for axion photon coupling in a magnetized medium, in the limit, \(|k_\perp| \to 0\). It is important to note that, this equation depends on many physical parameters, e.g., the temperature of the medium (\(\beta = 1/T\)), number density of the fermions (which in turn is related to \(\mu\)), mass of the particles in the loop (\(m\)), energy and longitudinal momentum of the photon (i.e. \(k_\parallel\)) and of course the symmetry breaking scale (which is included in the coupling constant, \(g_{af}\)).

Since the basic purpose of this exercise is to find out the effect of the medium on the mixing of axions to photons, we would have to choose some energy scale for the parameters entering into the statistical factors. Since most of the studies in literature deals with optical photons, we would assume the following scale, i.e \(m_f \gg k^2\) and finally \(\omega > k_3\). We would like to point out that, \(\omega \equiv k_0\) and \(k_3\) would henceforth be denoted as \(k\). Armed with these we can now turn to evaluate, Eqn. [VII.60].In the limit as mentioned above.

\[
\Lambda(k_i^2, k \cdot u, \beta, \mu) = -\int_{-\infty}^{\infty} dp \left[n_F(E_p, \mu) + n_F(E_p, -\mu)\right] \left(m^2 \cdot \frac{2k_i^2}{4\omega^2 E_p^2}\right).
\]

(VII.62)

We would like to point out that, while coming to the last step in equation [VII.62], we first had performed the the \(p_0\) integral, then had expanded the resulting denominator, in powers of \(\frac{2k_i^2}{2\pi E_p}\), while retaining the leading order term in powers of \(\frac{k_i^2}{2\pi E_p}\). In the expressions above \(E_p = \sqrt{(p^2 + m^2)}\).

\[
\Lambda(k_i^2, k \cdot u, \beta, \mu) = -\left(\frac{k_i^2}{\omega}\right) \int_{-\infty}^{\infty} dp \left[n_F(E_p, \mu) + n_F(E_p, -\mu)\right] \left(m^2 \cdot \frac{2k_i^2}{4\omega^2 E_p^2}\right).
\]

(VII.63)
From now on we denote $\Lambda = - \int_{-\infty}^{\infty} dp \left[ \frac{n_F(E_p, \mu) + n_F(E_p, -\mu)}{2E_p} \right] \frac{m^2}{2E_p}$. If we neglect pieces proportional to $k^2/\omega^2$, it’s easy to see that $\Lambda(k^2, k \cdot u, \beta, \mu) \equiv \Lambda = - \int_{0}^{\infty} dp \left[ \frac{n_F(E_p, \mu) + n_F(E_p, -\mu)}{2E_p} \right] \frac{m^2}{2E_p}$. Other wise (for small $k\parallel k\perp$), $\Lambda(k^2, k \cdot u, \beta, \mu) = \left( \frac{k^2}{\omega} \right) \Lambda$.

A. Limiting Case: $m \gg \mu$.

Analytical evaluation of eqn. [VII.63] for arbitrary values of chemical potential $\mu$ and temperature is an extremely difficult task. However, as has already been mentioned, under some approximations it may be possible to get an analytical form for the same. One such approximation is $m \gg \mu$. This is reasonable in most of the astrophysical and all of cosmological situations, barring the core of the Supernova or active galactic nuclei where this approximation may not hold. In this approximation, i.e $m \gg \mu$, thermal Fermi-Dirac factors can be expanded in a series to give,

$$n_F(E_p, \mu) + n_F(E_p, -\mu) = 2 \sum_{n=0}^{\infty} (-1)^n \cosh \left[ (n + 1) \beta \mu \right] e^{-(n+1)\beta E_p}. \tag{VII.64}$$

In order to evaluate, Eqn. [VII.63], we would substitute Eqn. [VII.64] in the same eqn., to arrive at,

$$\Lambda = - \sum_{n=0}^{\infty} (-1)^n \cosh \left[ (n + 1) \beta \mu \right] \int dp \frac{e^{-(n+1)\beta E_p}}{E_p} \frac{m^2}{E_p^2}. \tag{VII.65}$$

The evaluation of Eqn.[VII.65] is a bit difficult, however, expression appears a bit less formidable, if one identifies, $\sqrt{s}$ with $E_p$ and uses the following integral transform,

$$s^{-\frac{1}{2}} e^{-\alpha \sqrt{s}} = \pi^{-\frac{1}{4}} \int_{0}^{\infty} e^{-us - \frac{\alpha^2}{4u}} \frac{du}{\sqrt{u}} \tag{VII.66}$$

to convert Eqn.[VII.65] close to a Gaussian. Upon using this transform, the $p$ dependent part of the integrand goes over to,

$$2 \int dp \frac{e^{-p^2 u}}{p^2 + m^2} = \pi \left[ 1 - \Phi(m \sqrt{u}) \right] e^{um^2}. \tag{VII.67}$$

Here the function $\Phi(m \sqrt{u})$ stands for error function. Fortunately there exists another integral representation for the right hand side in Eqn. [VII.67], given by,

$$2 \sqrt{\frac{\pi}{m}} \int dt e^{-t^2 - 2tm \sqrt{u}} = \frac{\pi}{m} \left[ 1 - \Phi(m \sqrt{u}) \right] e^{um^2}. \tag{VII.68}$$

Use of this expression makes the expressions manageable. Using eqn.[VII.68], and expanding the left hand side in powers of $u$ and then performing the $t$ and $u$ integration one arrives at,

$$\Lambda = - \frac{m^2}{2m} \sum_{n=0}^{\infty} (-1)^n \cosh \left[ (n + 1) \beta \mu \right] \sum_{l=0}^{\infty} \frac{(-1)^l (2m)^l}{l!} \int_{0}^{\infty} \frac{du}{\sqrt{u} \sqrt{(\sqrt{u})}} e^{-m^2 u - \frac{(n+1)\beta}{\sqrt{u}}}. \tag{VII.69}$$

(1) It should noted that this $s$ is different from the proper time parameter $s$ introduced in the fermionic Schwinger propagators.
In Eqn.[VII.69], u integration would result in giving modified Bessel function and following that, the t integration, can be performed using the formula provided below,

$$\int_0^\infty x^{\nu-1}e^{-\mu x^p} = \frac{1}{p}\mu^{-\nu/p}\Gamma\left(\frac{\nu}{p}\right). \quad (VII.70)$$

to arrive at the following expression for \( \Lambda \).

$$\Lambda = -\frac{1}{2} \sum_{n=0}^\infty (-1)^n \cosh[(n+1)\beta\mu] \sum_{l=0}^\infty \frac{(-1)^{(2l+1)}l!\Gamma(l+\frac{1}{2})}{l!(n+\beta m\mu)^{\frac{l+1}{2}}K_{\frac{l+1}{2}}[(n+1)m\beta]}, \quad (VII.71)$$

One can use the expression [VII.71] in equation [VII.61] and estimate the contribution of magnetized medium to photon axion mixing. For simple cases, e.g. in high temperature (more relevant for the cosmological and astrophysical scenario) or low temperature (intergalactic regions) it should be possible to use different expansions for the modified Bessel functions and estimate the size of the corrections to arrive at the analytical results.

**B. Limit:** \( m \leq \mu \leq \infty \)

In certain physical situations matter density can become extremely high, i.e the highly degenerate Fermi system. Examples are nascent neutron stars or the core of the core collapse supernova where matter density can be \( 10^{15} \frac{gm}{cc} \) or more and temperature can vary between several tens of MeV to few KeV. Keeping this physical situation in mind we would try to estimate the contribution of \( \Lambda \) for various values of chemical potential. Noting that the \( a_\gamma \) effective Lagrangian can be written in the form

$$L_{\gamma a}^{\mu,\beta} = \frac{aF}{64\pi^2}g_{m f}(eQ_f)^2 \left(\frac{k}{\omega}\right)^2 \left[ \int_0^\infty dp \frac{1}{(p^2+1)^{\frac{3}{2}}} \left\{ \frac{1}{e^{\beta m(p^2+1)^{\frac{3}{2}}}-\beta\mu+1} + (\mu \leftrightarrow -\mu) \right\} \right]$$

$$= \frac{aF}{32\pi^2}g_{m f}(eQ_f)^2 \left(\frac{k}{\omega}\right)^2 16 \cdot \tilde{\Lambda} \quad (VII.72)$$

The integral inside square bracket in Eqn. [VII.72] is being denoted by \( \tilde{\Lambda} \) where \( \tilde{\Lambda} = -2\Lambda \). It is worth noting that inside the integral, the function inside the braces for \( m \ll \mu \) and limit \( T \to 0 \) can be approximated by \( \Theta\left(\frac{\mu}{m} - p\right) \). In this limit we can write,

$$\lim_{T \to 0} \tilde{\Lambda} \approx \int_0^{\frac{\mu}{m}} \frac{dp}{(p^2+1)^{3/2}} = \sin \left[ \tan^{-1} \left(\frac{\mu}{m}\right) \right] = \frac{\mu}{\sqrt{1 + \left(\frac{\mu}{m}\right)^2}} \quad (VII.73)$$

In the limit \( \mu \gg m \), the right hand side of Eqn. (VII.73) \( \sim 1 \).

Eqn. [VII.72] has also been estimated numerically, for \( m \leq \mu \leq 6 \) when the temperature is held fixed at, \( T = 10^{-3} \) in units of fermion mass. The behavior of the function \( \tilde{\Lambda} \) in this range can be seen in [Fig.2] which saturates at 0.98 providing good agreement between analytical and numerical result.

The important part of this analysis is, for sufficiently large values of matter density and low temperature the value of \( \tilde{\Lambda} \) is seen to saturate at a value not exceeding one. Though for smaller values of matter density it ranges between half to unity.
VIII. DISCUSSION AND CONCLUSION

As has already been mentioned at the beginning, the purpose of this note is to find out the modifications in the axion photon vertex in the presence of an external magnetic field and matter. Field induced part of the photon axion vertex has been worked out earlier in [34], however the effect of magnetized matter wasn’t taken into account. In this note we have considered the effect of both external magnetic field and medium on the vertex. Moreover we have also written down the explicit formula for the vertex involving the PQ charges of the fermions. In the light of these estimates, it is possible to write down the axion photon mixing Lagrangian, for low frequency photons in an external magnetic field, in the following way:

\[ L_{total}^{a\gamma} = L_{a\gamma}^{vac} + L_{a\gamma}^{B} + L_{a\gamma}^{B,\mu,\beta}. \]  (VIII.74)

Where \( L_{a\gamma}^{vac} \) is the usual axion photon mixing term in vacuum, resulting form the electromagnetic anomaly, and is given by [23],

\[ L_{a\gamma}^{vac} = -g_{a\gamma\gamma} \frac{e^{2}}{32\pi^{2}} a\tilde{F}. \]

\[ L_{a\gamma}^{B} = -\frac{1}{32\pi^{2}} \left[ 4 + \frac{4}{3} \left( \frac{k_{\perp}}{m} \right)^{2} \right] \sum f g_{af}(eQ_{f})^{2} a\tilde{F}. \]

\[ L_{a\gamma}^{B,\mu,\beta} = \frac{16}{32\pi^{2}} \left( \frac{k_{\perp}}{\omega} \right)^{2} (\tilde{\Lambda}) \sum f g_{af}(eQ_{f})^{2} a\tilde{F}. \]  (VIII.75)

To remind ourselves, in the equations above, \( z = \frac{m_{u}}{m_{d}} \) where \( m_{u} \) and \( m_{d} \) stands for the masses of the light quarks. The anomaly factors are given by the following relations, \( A_{PQ}^{em} = Tr(Q_{f}^{2})X_{f} \) and \( \delta_{ab}A_{PQ}^{em} = Tr_{a}X_{a}(X_{f}) \) (where the trace is over the fermion species). Hence, in the limit of \( |k_{\perp}| \to 0 \) and \( \omega << m_{f} \), using eqn. [VIII.75], one can write the total axion photon effective Lagrangian:

\[ L_{a\gamma}^{Total} = - \left[ g_{a\gamma\gamma} + \left( 4 + \frac{4}{3} \left( \frac{k_{\perp}}{m} \right)^{2} \right) \sum f g_{af}(Q_{f})^{2} - 16 \left( \frac{k_{\perp}}{\omega} \right)^{2} (\tilde{\Lambda}) \sum f g_{af}(Q_{f})^{2} \right] \frac{e^{2}}{32\pi^{2}} a\tilde{F}. \]  (VIII.76)

In order to describe the axion photon interaction, for static magnetic field and real photon, the effective Lagrangian employed in the literature is usually given by the following relation,

\[ \frac{1}{M}a\tilde{E}B^{ext}. \]  (VIII.77)

Where the parameter \( M \), defines an energy scale in terms of inverse of the axion photon photon coupling constant in vacuum, i.e \( M \equiv \frac{32\pi^{2}}{e^{2}g_{a\gamma\gamma}} \). In principle this factor is a model dependent quantity, and depending on the particular choice of PQ charges, \( g_{a\gamma\gamma} \) can vary between zero to hundred. This is one of the model dependent uncertainty. On the other hand, in some experiments, attempts have been made to give a bound on \( M \) thus \( g_{a\gamma\gamma} \) through the detection of solar axions in a magnetic cavity haloscope. The astrophysical bounds on \( M \) come from the constraint that the stars don’t lose energy through axion photon conversion at a rate faster than the generation of energy. As a result of these investigations, the present day bound on the energy scale is believed to be, \( M \leq 10^{12} GeV \). On the other hand in this note, we have been able to point out that apart from the model dependent uncertainties, there is also medium dependent uncertainties that can introduce some variation on the bounds on \( M \). The amount of this variation would depend on the kind of environment one is interested in. The details of its implications for various models and physical situations are extremely interesting by their own right [35–44]; however these points are beyond the scope of this work.
FIG. 2. $\bar{\Lambda}$ vs. $\mu/T$ for relativistic degenerate medium. The parameters are as follows: $m \leq \mu \leq 6$ when the temperature is held fixed at, $T = 10^{-3}$ in units of fermion mass.

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APPENDIX

Appendix A : The Equations.

Contribution from magnetized material part to $VA$ response function comes through the pure thermal propagators (in the external magnetic field). Eqn.[IV.23] reads:

$$i\Pi_{\mu\nu}^{A,B}(k) = (-ie^2)(-1) \int \frac{d^4p}{(2\pi)^4} Tr \left[ \gamma_\mu \gamma_5 S_B^\eta(p) \gamma_\nu iS_B^V(p') + \gamma_\mu \gamma_5 iS_B^V(p) \gamma_\nu S_B^\eta(p') \right]$$  \hspace{1cm} (H.78)

Eqn.[H.78] has two pieces in it. In what follows we would try to demonstrate that except the thermal weight factors (i.e the $\eta_F(p)$'s) the traces of both the terms come out to be the same.

To begin with, we would concentrate on the first piece. In terms of the functions $G(p, s)$ and $G(p', s')$ the same can be written as,

$$\text{Tr} \left[ \gamma_\mu \gamma_5 S_B^\eta(p) \gamma_\nu iS_B^V(p') \right] = -\text{Tr} \left[ \gamma_\mu \gamma_5 \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} G(p, s) \gamma_\nu \int_{0}^{\infty} ds' e^{\Phi(p',s')} G(p', s') \right] \eta_F(p)$$

$$= -\int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_{0}^{\infty} ds' e^{\Phi(p',s')} \text{Tr} \left[ \gamma_\mu \gamma_5 G(p, s) \gamma_\nu G(p', s') \right] \eta_F(p)$$  \hspace{1cm} (H.79)
We would like to note the appearance of the integration variables, \( p, p' \) and \( s, s' \).

The second term in the right hand side of eqn. (H.78) needs some formal mathematical manipulations. Its better to do the same in two steps. The first step would be shift the integration variable \( p \to p - k \). And following this the next step would be to change the integration variable \( p \to -p \). As a result of this manipulation one arrives at,

\[
\text{Tr} \left[ \gamma_{\mu} \gamma_{5} i S_{B}^{V}(p) \gamma_{\nu} S_{B}^{0}(p') \right] = \text{Tr} \left[ \gamma_{\mu} \gamma_{5} i S_{B}^{V}(-p') \gamma_{\nu} S_{B}^{0}(-p) \right]
\]

\[
= -\text{Tr} \left[ \gamma_{\mu} \gamma_{5} \int_{0}^{\infty} ds' e^{\Phi(-p', s')} G(-p', s') \gamma_{\nu} \int_{-\infty}^{\infty} ds e^{\Phi(-p, s)} G(-p, s) \right] \eta_{F}(-p)
\]

\[
= -\int_{-\infty}^{\infty} ds e^{\Phi(p, s)} \int_{0}^{\infty} ds' e^{\Phi(p', s')} \text{Tr} \left[ \gamma_{\mu} \gamma_{5} G(-p', s') \gamma_{\nu} G(-p, s) \right] \eta_{F}(-p) \quad \text{(H.80)}
\]

The complete expression is given by,

\[
i\Pi_{\mu\nu}^{A_{0}, 0}(k) = -(-i e)^{2}(-1) \int \frac{d^{4}p}{(2\pi)^{4}} \int_{-\infty}^{\infty} ds e^{\Phi(p, s)} \int_{0}^{\infty} ds' e^{\Phi(p', s')}
\]

\[
\times \left[ \text{Tr} \left[ \gamma_{\mu} \gamma_{5} G(p, s) \gamma_{\nu} G(p', s') \right] \eta_{F}(p) + \text{Tr} \left[ \gamma_{\mu} \gamma_{5} G(-p', s') \gamma_{\nu} G(-p, s) \right] \eta_{F}(-p) \right] \quad \text{(H.81)}
\]

\[
= -(-i e)^{2}(-1) \int \frac{d^{4}p}{(2\pi)^{4}} \int_{-\infty}^{\infty} ds e^{\Phi(p, s)} \int_{0}^{\infty} ds' e^{\Phi(p', s')} R_{\mu\nu}(p, p', s, s') \quad \text{(H.82)}
\]

where \( R_{\mu\nu}(p, p', s, s') \) contains the trace part. It is worth noting that the two pieces appearing in the second line of Eqn. [H.82] are related to each other by the following transformation \( (p \to -p') \) and \( (s \to s') \). So once we compute the trace of one of them, the other can easily be obtained by interchanging the variables, \( p \to -p' \) and \( s \to s' \).

Hence, in the following we would evaluate just one of them. If we introduce the following short hand notations,

\[
\tan(eQ_{f}B s) = T_{s} \quad \text{and} \quad \sec^{2}(eQ_{f}B s) = S_{s}
\]

\[
\tan(eQ_{f}B s') = T_{s'} \quad \text{and} \quad \sec^{2}(eQ_{f}B s') = S_{s'}
\]

\[
\Gamma_{\mu\delta} = \gamma_{\mu} \gamma_{5} \quad \text{(H.83)}
\]

The expression for, \( G(p, s) \) can further be written as,

\[
G(p, s) = (1 + i \sigma_{z} \tan(eQ_{f}B s))(\hat{p}_{\parallel} + m) - \sec^{2}(eQ_{f}B s)\hat{p}_{\perp}
\]

\[
= (1 + i \sigma_{z} T_{s})(\hat{p}_{\parallel} + m) - S_{s} \hat{p}_{\perp} \quad \text{(H.84)}
\]

After doing a bit of algebra, the Dirac trace part of the first term of Eqn. [H.81], odd in powers of \( eQ_{f}B \) turns out to be (written in terms of short-hand notations introduced in Eqn.[H.83]),

\[
i\Pi_{\mu\nu}^{(A_{0})^{-1}} = -(-i)^{2}(g_{\mu\nu} eQ_{f})(-1) \int \frac{d^{4}p}{(2\pi)^{4}} \int_{-\infty}^{\infty} e^{\Phi(p, s)} ds \int_{0}^{\infty} ds' e^{\Phi(p', s')}
\]

\[
\times i \left[ \Gamma_{\mu\delta} \left( \sigma_{z} \hat{p}_{\parallel} \gamma_{\nu} \hat{p}_{\parallel}' T_{s} + \hat{p}_{\parallel} \gamma_{\nu} \sigma_{z} \hat{p}_{\parallel}' T_{s}' \right) + m^{2} (\Gamma_{\mu\delta} \gamma_{\nu} T_{s} + \Gamma_{\mu\gamma} \gamma_{\nu} \sigma_{z} T_{s}) \right]
\]

\[
+ \Gamma_{\mu\delta} \sigma_{z} \hat{p}_{\parallel} \gamma_{\nu} \sigma_{z} \hat{p}_{\parallel}' T_{s} S_{s}' + \Gamma_{\mu\gamma} \gamma_{\nu} \sigma_{z} \hat{p}_{\parallel}' T_{s}' S_{s} \right] \eta_{F}(p) \quad \text{(H.85)}
\]

In order to simplify things a bit, we note that, using the defn of \( \sigma_{z} \), (i.e \( \sigma_{z} = -\gamma_{0} \gamma_{3} \gamma_{5} \)) one can write,

\[
\sigma_{z} \hat{p}_{\parallel} = i \gamma_{1} \gamma_{2} \hat{p}_{\parallel} = \gamma_{0} \gamma_{3} \hat{p}_{\parallel} = \gamma_{0} \gamma_{3} \left( \gamma_{0} p^{0} + \gamma_{3} p^{3} \right) \gamma_{5} = -\left( \gamma_{0} p^{0} + \gamma_{3} p^{3} \right) \gamma_{5}
\]

\[
= -\gamma_{\alpha_{\perp}} p^{\perp} \quad \text{where} \quad (\alpha_{\parallel} \text{ could be, either 0 or 3)}. \quad \text{(H.87)}
\]
We would also like to point out that, depending on the value of \( \alpha_i \), \( \alpha_i^{\dagger} \) would take its complimentary value, i.e. if \( \alpha_i = 0 \) then \( \alpha_i^{\dagger} = 3 \) etc. We also note in the passing that in this notation,

\[
\sigma_z \hat{p}_p^{\dagger} = -\gamma_{\alpha_1} \gamma_5 \sigma_z \hat{p}_p^{\dagger} \gamma_5.
\]

\[
\sigma_z \hat{p}_z = -\gamma_{\alpha_1} \gamma_5 \sigma_z \hat{p}_z \gamma_5
\]

and finally,

\[
\text{Tr}[\Gamma_{\mu_5} \sigma_z \gamma_\nu] = -4 \epsilon_{\mu_\nu 12}
\]

\[
\text{Tr}[\Gamma_{\mu_5} \gamma_\nu \sigma_z] = -4 \epsilon_{\mu_\nu 12}.
\]

Given these equations, it would be easier to evaluate the Dirac traces appearing in Eqn. [H.86]. We would start with the piece proportional to \( m^2 \). It reads,

\[
m^2 (\Gamma_{\mu_5} \sigma_z \gamma_\nu T_s + \Gamma_{\mu_5} \gamma_\nu \sigma_z T_s) = -4m^2 \epsilon_{\mu_\nu 12} (T_s + T'_s).
\]

It is worth noting that, the sign of the piece proportional to \( m^2 \) is negative, more over the piece is antisymmetric with respect to the indices \( \mu \) and \( \nu \). Next we would like to evaluate the trace of the pieces having parallel components of momentums \( p \) and \( p' \), i.e.:

\[
\text{Tr} \left[ \Gamma_{\mu_5} \left( \sigma_z p^{\dagger}_p \gamma_\nu \gamma_5 \hat{p}_p^{\dagger} T_s + \gamma_\nu \sigma_z p^{\dagger}_p \hat{p}_p^{\dagger} T'_s \right) \right] = \text{Tr} \left[ \gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta \gamma_\epsilon \gamma_\zeta \gamma_\eta \gamma_\theta \gamma_\iota \gamma_\kappa \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\omega \gamma_\chi \gamma_\psi \gamma_\varphi \gamma_\omega \gamma_\chi \gamma_\psi \gamma_\varphi \gamma_\omega \gamma_\chi \gamma_\psi \gamma_\varphi \right]
\]

\[
= 4 \left( \left( g_{\mu_\alpha_1} p^{\dagger}_p p^{\dagger}_\mu_\nu - g_{\mu_\nu_1} p^{\dagger}_\nu_\mu \right) T_s + \left( g_{\nu_\alpha_1} p^{\dagger}_p p^{\dagger}_\nu_\mu + g_{\nu_\mu_1} p^{\dagger}_\nu_\mu \right) T'_s \right).
\]

It should be noted that unlike the momentum dependent piece, this part is symmetric in the indices \( \mu \) and \( \nu \). In other words these terms are insensitive to the position of \( \gamma_5 \). That is to say, inside the trace, if we would have pushed the \( \gamma_5 \) matrix to the \( \mu \) vertex, these pieces would remain insensitive. Its also important to notice that these pieces are symmetric with respect to the exchange of the variables \( (p, s) \leftrightarrow (p', s') \). The last trace that needs to be evaluated is,

\[
\text{Tr} \left[ \Gamma_{\mu_5} \left( \sigma_z p^{\dagger}_p \gamma_\nu \gamma_5 \hat{p}_p^{\dagger} T_s + \gamma_\nu \sigma_z p^{\dagger}_p \hat{p}_p^{\dagger} T'_s \right) S_s' \right] = 4 \left( g_{\mu_\alpha_1} p^{\dagger}_p p^{\dagger}_\mu_\nu + g_{\nu_\alpha_1} p^{\dagger}_\nu_\mu \right) T_s S'_s
\]

\[
+ g_{\mu_\alpha_1} p^{\dagger}_p p^{\dagger}_\nu_\mu + g_{\nu_\alpha_1} p^{\dagger}_\nu_\mu \right) T'_s S_s\right)
\]

As can be verified, Eqn. [H.93] is symmetric is the indices \( \mu \& \nu \), more over as before it remains the same under the transformation of variables \( (p, s) \leftrightarrow (p', s') \). On the other hand the symmetry under the exchange of \( (p, s) \leftrightarrow (p', s') \) can be exploited to evaluate the second trace in Eqn. (H.81). Since the momentum dependent terms all appear in even powers of \( p \) and \( p' \), these equations are insensitive to the following transformation \( (p, p') \leftrightarrow (-p', -p) \). Hence under the transformation

\[
\{p, s\} \leftrightarrow \{-p', s'\},
\]

the traces would remain invariant. Therefore the Dirac trace of both these pieces are the same and hence upon using the definitions of, \( T_s, T'_s, S_s \) and \( S'_s \) as given in Eqn.[H.83], we recover equation [V.35].

**APPENDIX**

**Appendix B: Proof Of** \( k^\mu R^{(o)}_{\mu \nu \perp} = 0 \)

The expression for \( R^{(o)}_{\mu \nu \perp}(k) \) as follows from Eqn. [VI.49], turns out to be,
\[ R^{(o)}_{\mu\nu\perp} = 4i\eta_+(p) \left\{ \begin{array}{l} -g_{\mu\nu\perp} k_{\alpha\parallel} p^{\alpha\perp} + g_{\mu\alpha\parallel} p^\alpha p_{\nu\perp} \sec^2(eQ_f B s') \tan(eQ_f B s) \\ + \left\{ -g_{\mu\nu\perp} p_{\alpha\parallel} k^{\alpha\parallel} + g_{\mu\alpha\parallel} p^\alpha p_{\nu\perp} \sec^2(eQ_f B s') \tan(eQ_f B s) \right\} \end{array} \right\} \tag{H.95} \]

Contracting \( R^{(o)}_{\mu\nu\perp} \) with \( k^\mu \), we get,

\[ k^\mu R^{(o)}_{\mu\nu\perp} = 4i\eta_+(p) \left\{ \begin{array}{l} -k_{\nu\perp} k_{\alpha\parallel} p^{\alpha\parallel} \tan(eQ_f B s) - k_{\nu\perp} k_{\alpha\parallel} p^\alpha \tan^2(eQ_f B s) \\ + \left\{ -k_{\nu\perp} p_{\alpha\parallel} k^{\alpha\parallel} + k_{\alpha\parallel} p^\alpha p_{\nu\perp} \right\} \tan(eQ_f B s') \tan(eQ_f B s) \right\} \tag{H.96} \]

In getting Eqn. [H.96] the following definition was used,

\[ a_\mu \parallel b^\mu = a_0 b^3 + a_3 b^0. \tag{H.97} \]

It can be easily verified that, if \( a \) and \( b \) are the same, Eqn.[H.97] vanishes. This fact was additionally exploited while arriving at, Eqn.[H.96]. In particular, if we use relations, [V.40, V.39], for \( p_{\nu\perp} p_{\nu\perp} \) as appearing in Eqn.[H.96], the same simplifies further to

\[ k^\mu R^{(o)}_{\mu\nu\perp} = 4i\eta_+(p) \left\{ \begin{array}{l} -k_{\nu\perp} k_{\alpha\parallel} p^{\alpha\parallel} \tan(eQ_f B s) - k_{\nu\perp} k_{\alpha\parallel} p^\alpha \tan^2(eQ_f B s) \\ + \left\{ -k_{\nu\perp} p_{\alpha\parallel} k^{\alpha\parallel} + k_{\alpha\parallel} p^\alpha p_{\nu\perp} \right\} \tan(eQ_f B s') \tan(eQ_f B s) \right\} = 0. \tag{H.98} \]

So we have proved that, \( k^\mu R^{(o)}_{\mu\nu\perp} = 0. \)

**APPENDIX**

**Appendix C: Proof Of** \( k^\mu R^{(0,1)}_{\mu\nu\perp} = 0 \)

In the text, we claimed that the contribution coming from Eqn. [VI.51] must vanish. Here, we justify that claim. Vanishing of Eqn.[VI.51], actually amounts to showing that the following integral,

\[ C_{1\nu\parallel} = \int \frac{d^4 p}{(2\pi)^4} \eta_+(p) \int_{-\infty}^{\infty} ds^\phi(p,s) \int_0^{\infty} d\phi^\phi(p',s') p_{\nu\parallel} \left( k_{\nu\parallel} + 2(k \cdot p) \right) \left( \tan(eQ_f B s) + \tan(eQ_f B s') \right) - k_{\perp}^2 \left( \tan(eQ_f B s) - \tan(eQ_f B s') \right) = 0. \tag{H.99} \]

We begin by defining two new parameters

\[ \xi = \frac{1}{2} eQ_f B (s + s'), \]
\[ \zeta = \frac{1}{2} eQ_f B (s - s'). \tag{H.100} \]

and noting, ( following [31] ), that:
\[ ie B \frac{d}{d\zeta} e^{\Phi(p, s) + \Phi(p', s')} = e^{\Phi(p, s) + \Phi(p', s')} \left( k^2 + 2(p, k) - p'^2 \sec^2(\xi - \zeta) + p^2 \sec^2(\xi + \zeta) \right). \] (H.101)

Using Eqns. [V.41] and [V.42] in Eqn. [H.101] and little bit of algebra, one can further show that,

\[ C_{1\nu}^- = ieQ_{\nu} B \int \frac{d^4p}{(2\pi)^4} \rho_{\nu}(p) \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \frac{d}{d\zeta} F(\xi, \zeta), \] (H.102)

where

\[ F(\xi, \zeta) = \left( \tan eQ_{\nu} Bs + \tan eQ_{\nu} B's' \right) e^{\Phi(p, s) + \Phi(p', s')} \] (H.103)

with \(s\) and \(s'\) related to \(\xi\) and \(\zeta\) through Eq. (H.100). We can now change the integration variables to \(\xi\) and \(\zeta\) to arrive at (ignoring the unimportant constants in front),

\[ C_{1\nu}^- \propto \int \frac{d^4p}{(2\pi)^4} \rho_{\nu}(p) \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \Theta(\xi - \zeta) \frac{d}{d\zeta} F(\xi, \zeta) \]
\[ = \int \frac{d^4p}{(2\pi)^4} \rho_{\nu}(p) \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \left[ \frac{d}{d\zeta} \left\{ \Theta(\xi - \zeta) F(\xi, \zeta) \right\} - \delta(\xi - \zeta) F(\xi, \zeta) \right] \]
\[ = -\int \frac{d^4p}{(2\pi)^4} \rho_{\nu}(p) \int_{-\infty}^{\infty} d\xi F(\xi, \zeta), \] (H.104)

since the other term vanishes at the limits. In this integrand, \(\zeta = \xi\), which means \(s' = 0\). Looking back at the definition of \(F\), we find

\[ F(\xi, \xi) = \exp \left\{ \Phi(p, \frac{2\xi}{eQ_{\nu} B}) \right\} \tan 2\xi. \] (H.105)

This is an even function of \(p\), whereas \(\left( p_{\nu} \times \eta_+(p) \right)\) is odd. Thus, the expression vanishes on integration over \(p\).

**APPENDIX**

**Appendix D: Proof Of \( k^\mu R^{(0,2)}_{\mu\nu} = 0 \)**

In this appendix we show that, \( k^\mu R^{(0,2)}_{\mu\nu} = 0 \). The two components of \(\nu\) are 0 and 3. Respective components those need evaluation, are,

\[ k^\mu R^{(0,2)}_{\mu\nu} = -4i\eta_+(p)k^3 \left[ \left( p^2 - m^2 \right) (\tan(eQ_{\nu} Bs) + \tan(eQ_{\nu} B's')) - k^2 \sec^2(eQ_{\nu} B) \right] \]
\[ \times \left( \frac{\sec^2 eQ_{\nu} Bs}{\tan(eQ_{\nu} Bs) + \tan(eQ_{\nu} B's')} \right) \] (H.106)

Similarly,

\[ k^\mu R^{(0,2)}_{\mu3} = 4i\eta_+(p)k_0 \left[ \left( p^2 - m^2 \right) (\tan(eQ_{\nu} Bs) + \tan(eQ_{\nu} B's')) - k^2 \sec^2(eQ_{\nu} B) \right] \]
\[ \times \left( \frac{\sec eQ_{\nu} Bs}{\tan(eQ_{\nu} Bs) + \tan(eQ_{\nu} B's')} \right) \] (H.107)

Expressions inside the square bracket of Eqns. [H.106] or [H.107] are the same, introducing compact notation.
\[ \Pi(p, k) = 4i\eta_+(p) \left[ \left( p_\parallel^2 - m^2 \right) \left( \tan(eQ_j B) + \tan(eQ_j B') \right) - k_\perp^2 \tan^2(eQ_j B') \right] \times \left( \frac{\sec^2(eQ_j B)}{\tan(eQ_j B) + \tan(eQ_j B')} \right) \] (H.108)

Actually the expression defined by \( \Pi(p, k) \), appears inside the integral over the loop momentum variable \( p \) as well as the parameters \( s \) and \( s' \). Calling the integrals over \( p, s \) and \( s' \) of \( \Pi(p, k) \) as \( \hat{\Pi}(p, k) \), the same is:

\[ \hat{\Pi}(p, k) = \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} dsds' e^{\Phi(p, s)} e^{\Phi(p', s')} 4i\eta_+(p) \left[ \left( p_\parallel^2 - m^2 \right) \left( \tan(eQ_j B) + \tan(eQ_j B') \right) - k_\perp^2 \tan^2(eQ_j B') \right] \times \left( \frac{\sec^2(eQ_j B)}{\tan(eQ_j B) + \tan(eQ_j B')} \right) \] (H.109)

Though not obvious from appearance, but we would show that Eqn. [H.109] vanishes identically. To show that, we note that it is possible to write,

\[ \int_{-\infty}^{\infty} ds e^{is(p_\parallel^2 - m^2)} \left( p_\parallel^2 - m^2 \right) \left( \tan(eQ_j B) + \tan(eQ_j B') \right) = \int_{-\infty}^{\infty} ds \left\{ \frac{d}{d\epsilon} e^{is(p_\parallel^2 - m^2)} \right\} \times e^{-i\tan(eQ_j B)\frac{p_\perp^2 - \epsilon|s|}{\sec^2(eQ_j B)} (\tan(eQ_j B) + \tan(eQ_j B'))} \] (H.110)

Upon partial integration of Eqn. [H.110], i.e.,

\[ \int_{-\infty}^{\infty} ds \left\{ \frac{d}{d\epsilon} e^{is(p_\parallel^2 - m^2)} \right\} \times e^{-i\tan(eQ_j B)\frac{p_\perp^2 - \epsilon|s|}{\sec^2(eQ_j B)} (\tan(eQ_j B) + \tan(eQ_j B'))} = -\int_{-\infty}^{\infty} ds e^{is(p_\parallel^2 - m^2)} \frac{d}{d\epsilon} \left\{ e^{-i\tan(eQ_j B)\frac{p_\perp^2 - \epsilon|s|}{\sec^2(eQ_j B)} (\tan(eQ_j B) + \tan(eQ_j B'))} \right\} \times \left( \tan(eQ_j B) + \tan(eQ_j B') \right) \] (H.111)

It should be noted that, while deriving eqn. [H.111] a total derivative term in has been thrown away, since the integrand vanishes at the boundary (virtue of the \( \epsilon \) prescription). Of the terms on the right hand side of Eqn. [H.111],

\[ \int_{-\infty}^{\infty} ds e^{is(p_\parallel^2 - m^2)} \frac{d}{d\epsilon} \left\{ e^{-i\tan(eQ_j B)\frac{p_\perp^2 - \epsilon|s|}{\sec^2(eQ_j B)} (\tan(eQ_j B) + \tan(eQ_j B'))} \right\} (\tan(eQ_j B) + \tan(eQ_j B')) = (-1) \int_{-\infty}^{\infty} ds e^{is(p, s)} p_\perp^2 \sec^2(eQ_j B) (\tan(eQ_j B) + \tan(eQ_j B')) \] (H.112)

and,

\[ \int_{-\infty}^{\infty} ds e^{is(p_\parallel^2 - m^2)} \frac{d}{d\epsilon} \left\{ e^{-i\tan(eQ_j B)\frac{p_\perp^2 - \epsilon|s|}{\sec^2(eQ_j B)} (\tan(eQ_j B) + \tan(eQ_j B'))} \right\} \times \left( \tan(eQ_j B) + \tan(eQ_j B') \right) = -ieQ_j B \int_{-\infty}^{\infty} e^{\Phi(p, s)} \sec^2(eQ_j B) ds. \] (H.113)

The expression for \( \Phi(p, s) \) is the same as provided earlier. Therefore using Eqns. [H.112] and [H.113],

\[ \int_{-\infty}^{\infty} e^{\Phi(p, s)} (p_\parallel^2 - m^2)(\tan(eQ_j B) + \tan(eQ_j B')) = \int_{-\infty}^{\infty} ds e^{\Phi(p, s)} p_\perp^2 \sec^2(eQ_j B) (\tan(eQ_j B) + \tan(eQ_j B')) \] (H.114)
Where in the last step of Eqn. [H.114], one has used the identity given by Eqn. [V.41]. It can now be easily verified, that on substituting Eqn. [H.114] in Eqn. [H.109] the pieces proportional to $k^2_{\perp}$ compensate each other. So the proof that, $\kappa \mu R^{(0,2)}_{\mu \nu} = 0$ is complete.

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