In these lectures we sketch a rapid survey of recent theoretical advances in the study of frustrated quantum magnets with a special emphasis on two dimensional magnets. One dimensional problems are only very briefly discussed: this field has been extraordinarily flourishing during the past twenty years both experimentally and theoretically. In contrast the understanding of two dimensional quantum magnets is much more limited. The number of unexplained experimental results is probably extremely large, unfortunately the theoretical tools to deal with exotic quantum phases in 2 dimensions are still rather limited. In the absence of a significant amount of exact results, the picture drawn in the following sections is based on the comparison of various approaches to $SU(2)$ spins: series expansions, exact diagonalizations, Quantum Monte-Carlo calculations and the hints got from large-N generalizations.

From these comparisons we suggest that one could expect at least 4 kinds of different low temperature physics in two-dimensional systems:

- semi-classical Néel like phases
- and three kinds of purely quantum phases.

These 4 phases are the subject of the first 4 sections of this paper: their properties are summarized in Table I. The quantum phases appear in situations where there are competing interactions, a high degree of frustration and a rather low coordination number. In section V we discuss the very rich magnetic phase diagram of frustrated quantum magnets, which can exhibit hysteretic metamagnetic transitions and magnetic plateaus at zero and at rational values of the reduced magnetic field.

### Table I: The four 2-dimensional phases described in the four first sections.

| Phases                  | G.-S. Symmetry Breaking | Order Parameter | First excitations                        |
|-------------------------|-------------------------|-----------------|------------------------------------------|
| Semi-class. Néel order  | $SU(2)$                 | Staggered Magnet. | Gapless magnons                          |
|                         | Space Group             |                 |                                          |
|                         | Time Reversal           |                 |                                          |
| Valence Bond Crystal    | Space Group             | dimer-dimer LRO or S=0 plaquettes LRO | all excitations are gapped Confined spinons thermally activated $C_v$ and $\chi$ |
|                         |                         |                 |                                          |
| R.V.B. Spin Liquid      | topological degeneracy | No local order parameter | all excitations are gapped Deconfined spinons thermally activated $C_v$ and $\chi$ |
| (Type I)                |                         |                 |                                          |
| R.V.B. Spin Liquid      | topological degeneracy | No local order parameter | Singlet excitations are gapless Triplet excitations are gapped Deconfined spinons $T=0$ entropy thermally activated $\chi$ $C_v$ insensitive to magn. field at low $T$ |
| (Type II)               |                         |                 |                                          |
of the magnetization. Chiral phases, and the possible relation between magnetization plateaus and Hall conductance plateaus specifically studied in some other lectures of this School are briefly discussed.

To complete Table I let us underline that Néel ordered magnets are characterized by a unique energy scale (essentially given by its Curie-Weiss temperature $\theta_{CW}$, which is directly related to the coupling constant of the Hamiltonian). All the quantum phases studied up to now display a second energy scale which can be an order or two order of magnitude lower than $\theta_{CW}$ and mainly associated with a spin gap. It is the range of temperatures and energies under study in this lectures.

I. SEMI-CLASSICAL GROUND-STATES

A. Heisenberg problem

In most of the cases the ground-state of the antiferromagnetic spin-1/2 Heisenberg Hamiltonian

$$\mathcal{H} = 2J \sum_{<ij>} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (1)$$

with $J > 0$ and the sum limited to first neighbors, is Néel ordered at $T = 0$. In 3 dimensions, when increasing the temperature the Néel ordered phase gives place to a paramagnet through a 2nd order phase transition ($T_N \sim O(J)$). In 2 dimensions the ordered phase only exists at $T = 0$ ($\text{Mermin Wagner theorem}$). In one dimension, even at $T = 0$, there is only quasi-long range order with algebraically decreasing correlations (for an introduction to spin systems see [2, 3, 4]).

Néel ground-states break the continuous $SU(2)$ symmetry and support low lying excitations which are the Goldstone modes associated with this broken symmetry. These excitations, called magnons, are $\Delta S^z = 1$ bosons that can be pictured as long wave-length twists of the order parameter (the staggered magnetization). They form isolated branches of excitations described by their dispersion relation $\omega(k)$. $\omega(k)$ vanishes in reciprocal space at the set of $k$ vectors $\{k\}$ characteristic of the long range order: i.e. the wave vector $k = 0$ and the peaks in the structure function

$$S(q) = \int d^D r \, e^{i q \cdot r} \mathbf{S}_0 \cdot \mathbf{S}_r,$$

(2)

where $D$ is the lattice dimensionality.

The geometry of the Néel order parameter can in general be determined through a classical minimization of Eq. 1. It is a two-sublattice up-down order (noted $ud$ in the following) in the case of the square lattice, a three-sublattice with magnetization at 120 degrees from each other in the triangular case.

This classical approach neglects the quantum fluctuations: the classical solution $ud$ is an eigenstate of the Ising Hamiltonian but not of the Heisenberg Hamiltonian, the extra $X - Y$ terms of the Heisenberg Hamiltonian induce pairs of spin-flips and reduce the value of the staggered magnetization. For a moderate reduction, the spin-wave calculation is usually a good approximation to take this quantum effect into account. Introduction of quantum fluctuations

- renormalizes the ground-state energy through the zero point fluctuations
- decreases the sublattice magnetization by a contribution

$$\delta \propto \int \frac{g(k)}{\omega(k)} d^D k$$

(3)

where $\omega(k)$ is the dispersion law and $g(k)$ is a smooth function of $k$, depending on the lattice, which is non zero in the whole Brillouin zone. It is a straightforward exercise to show that $\omega(k) \propto k$ for small $k$. Eq. 3 indicates the absence of long range order (LRO) in $D = 1$ dimension through the breakdown of the spin-wave approximation (divergence of the integral).

The reduction of the order parameter with respect to the classical saturation value $M/M_O$ increases when the coordination number decreases; it is larger on triangular based lattices than on bipartite ones (see Table I). This last feature is sometimes called “geometrical frustration”. In the framework of the Heisenberg model this should be understood in the following sense: the ground-state of the Heisenberg problem on the triangular lattice is less stable than the ground-state of the same problem on the square lattice (and bipartite lattices), both in the classical regime (where $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle_{sq} = -0.25$ and $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle_{tri} = -0.125$), as well as in the $SU(2)$ quantum regime ($\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle_{sq} = -0.3346$, whereas $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle_{tri} = -0.1796$).

The ground-state of the spin 1/2 Heisenberg model (Eq. 1) does not have Néel order on the following lattices:
TABLE II: Energy per bond and sublattice magnetization in the ground-state of the spin-1/2 Heisenberg Hamiltonian on various lattices. The sq-hex-dod. is a bipartite lattice formed with squares, hexagons and dodecagons.

| Lattices          | Coordination number | $2 < S_i S_j >$ per bond | $M/M_{cl}$ |
|-------------------|---------------------|---------------------------|------------|
| dimer             | 1                   | -1.5                      |            |
| 1 D Chain         | 2                   | -0.886                    | 0          |
| honeycomb         | 3                   | -0.726                    | 0.44       |
| sq-hex-dod.       | 3                   | -0.721                    | 0.63       |
| square            | 4                   | -0.669                    | 0.60       |
| one triangle      | 2                   | -0.5                      |            |
| kagomé            | 4                   | -0.437                    | 0          |
| triangular        | 6                   | -0.363                    | 0.50       |

B. Semi-classical ground-states with competing interactions

Another way to frustrate Néel order and try to destabilize it, consists in adding competing interactions. The archetype of such a problem is the $J_1 - J_2$ model:

$$\mathcal{H} = 2J_1 \sum_{<ij>} S_i S_j + 2J_2 \sum_{<<ij>>} S_i S_j,$$

where the first sum runs on first neighbors and the second on second neighbors. As an example let us consider the square lattice case. The nearest neighbor antiferromagnetic coupling favors a $(\pi, \pi)$ order, which can be destabilized by a 2nd neighbor $J_2$ coupling $> J_2c \sim 0.35$. For much larger $J_2$, the system recovers some Néel long range ordering. With a simple classical reasoning one would then expect an order parameter with a 4 sublattice geometry and an internal degeneracy $^3$. As a consequence of quantum or thermal fluctuations a collinear configuration $((\pi, 0) \text{ or } (0, \pi))$ is selected via the mechanism of “order through disorder” $^3$: the collinear order is stabilized by fluctuations because it is a state of larger symmetry with a larger reservoir of soft fluctuations nearby. In the classical problem the stabilization is entropy driven (whence the name of the mechanism), in the quantum problem the softer fluctuations induce a smaller zero point energy and thus a stabilization of the more symmetric state. In all these cases, the first excitations are gapless $\Delta S^2 = 1$ magnons.

C. One-dimensional problems

As already mentioned, the Heisenberg problem in that case (Bethe Chain) is critical. Its low-lying excitations are free spin-1/2 excitations called spinons (first discovered by Fadeev). Due to quantum selection rules they appear in pairs, the spectrum of excitation is thus a continuum with two soft points (0 and $\pi$). A frustrating $J_2$ coupling, larger than the critical value $J_2^c \sim 0.2411$, opens a gap in the spectrum and the ground-state becomes dimerized (and two fold degenerate) with long range order in singlet-singlet correlations. Majumdar and Gosh $^3$ have shown that the ground-state of this problem for $J_2 = 0.5$ is an exact product of singlet wave-functions $^3$. The low-lying excitations

$^2$ It should be remarked that in the last two cases, the divergence of Eq. $^3$ comes from the existence of a zero mode on the whole Brillouin zone. This zero mode is due to a local degeneracy of the classical ground-state $^3$.

$^3$ Notice that the extra degeneracy is a global one and not a local one as in the kagomé problem.
FIG. 1: A simple “columnar” valence-bond crystal on the square lattice: fat links indicate the pair of sites where the spins are combined in a singlet $|↑↓⟩ − |↓↑⟩$. A realistic VBC will also include fluctuations of these singlet positions so that the magnetic correlation $⟨\vec{S}_i \cdot \vec{S}_j⟩$ will be larger than $-\frac{3}{4}$ along fat links and will not be zero along the dashed ones.

are pairs of solitons $[21]$, forming a continuum. For specific values of the coupling there appears an isolated branch of singlets around the point $\pi/2$ $[22]$.

II. VALENCE BOND CRYSTALS

A. Effect of a frustrating interaction on a Néel state

Coming back to the $J_1 − J_2$ model of Eq. 4, we examine the possibility of destroying Néel ground-states by a frustrating interaction. At the point of maximum frustration ($J_2 \sim 0.5J_1$) between the two Néel phases ((π, π) and (π, 0)) the system can lower its energy by forming $S = 0$ valence bonds between nearest neighbors. This maximizes the binding energy of the bonds involved in singlets and decreases the frustrating couplings between different singlets. But contrary to Néel states, nearest neighbors not involved in singlets do not contribute to the stabilization of this phase: so the presence of such a phase between two different Néel phases is not mandatory and should be studied in each case. The nature of the arrangements of singlets is also an open question: the coverings could have long range order or not. We call the phases with long range order in the dimer arrangements: Valence Bond Crystals (noted in the following VBC). The simplest picture of a Valence Bond Crystal ground-state is given in Fig. 1. The other kinds of dimer covering ground-states will be studied in the next sections.

B. Ground-state properties of the Valence Bond Crystals

• The ground-state of a VBC is a pure $S = 0$ state: it does not break $SU(2)$ rotational invariance and has only short range spin-spin correlations,

• It has long range dimer-dimer or $S = 0$ plaquettes-plaquettes correlations $^4$,

• In the thermodynamic limit, it breaks the symmetries of the lattice: translational symmetry and possibly rotational symmetry.

Remarks:

Such a ground-state is highly reminiscent of the dimerized ground-state appearing in the $J_1 − J_2$ chain for $J_2 > J_2^c$. But, as explained below, the spectrum of excitations of the two-dimensional VBC is expected to be different from the one-dimensional case.

On a finite sample the exact ground-state of a VBC cannot be reduced to the symmetry breaking configuration A of Fig. 1: it is the symmetric superposition of configuration A and of its three transforms (B, C and D) in the symmetry operations of the lattice group. In the thermodynamic limit, the 4 different superpositions of the A, B, C, and D configurations are degenerate, allowing the consideration of a symmetry breaking configuration like A as “a thermodynamic ground-state” of the system. In the general case of a VBC, the degeneracy of the thermodynamic ground-state is finite and directly related to the dimension of the lattice symmetry group. On a finite square sample,

$^4$ A $S = 0$ plaquette is an ensemble of an even number of nearby spins arranged in a singlet state: a 4-spin S=0 plaquette long range order has been suggested for the $J_1 − J_2$ model on the square lattice $[24, 25]$ and a 6-spin S=0 plaquette long range order for the same model on the hexagonal lattice $[13, 24, 26]$. 
The system will have a spin-gap. The nature of the first excitations is not totally settled: it is believed that they should be $S = 1$ bosons (some kinds of optical magnons). The possibility of $S = 1/2$ excitations (spinons) is dismissed on the following basis: the creation of a pair of spins 1/2 by excitation of a singlet is certainly the basic mechanism for producing an excitation. The question is then: can these spins 1/2 be separated beyond a finite distance? In doing so they create a string of misaligned valence bonds, the energy of which increases with its length. It is the origin of an elastic restoring force which binds the two spins 1/2. This simple picture shows the influence of dimensionality; creating a default in the $J_1 - J_2$ chain only affects the neighboring spins of the chain, the perturbation does not depend on the distance between the two spins 1/2, in this case the spinons are de-confined giving rise to a continuum of two-particle excitations [21, 22].

Because of the confinement of spinons in the VBCs, the excitation spectrum of this kind of states will exhibit isolated modes (“optical magnons”) below the continuum of multi-particle excitations.

An open question is whether the effective coupling between the two spinons is preferentially ferromagnetic or antiferromagnetic? This might be connected to the preferred position of the spinons on the same (or different) sublattice(s)? Numerical results on the $J_1 - J_2$ model on the hexagonal lattice plead in favor of an antiferromagnetic coupling between spinons, manifesting itself by a gap in the $S = 0$ sector smaller than in the $S = 1$ sector.

As a consequence of the gap(s), both the spin-susceptibility $\chi$ and the specific heat $C_v$ of the VBCs are thermally activated.

D. A toy model

The simplest 2-dimensional quantum model exhibiting VBC phases is the quantum hard-core dimer (QHCD) model introduced by Rokshar and Kivelson in the context of high $T_c$ super-conductivity [27]. It is not strictly speaking a spin model since the Hamiltonian directly operates on nearest-neighbor dimer coverings. The Hamiltonian is defined by its leading non-zero matrix elements within this subspace. On the square lattice it reads:

$$H = \sum_{\text{Plaquette}} \left[ -J \left( |\mathbf{1}\mathbf{1}\rangle\langle \mathbf{1}\mathbf{1}| + \text{h.c.} \right) + V \left( |\mathbf{1}\mathbf{1}\rangle\langle \mathbf{1}\mathbf{1}| + |\mathbf{1}\mathbf{1}\rangle\langle \mathbf{1}\mathbf{1}| \right) \right]$$

(5)

The connection between this model and a spin model such as the Heisenberg model is in the general case a difficult issue, because it involves the overlap matrix of dimer configurations (we will come back to this point in section IV). However, this model can be usefully seen as an effective description of the low-energy physics of some non Néel phases. The kinetic term ($J > 0$) favors resonances between dimer configurations which differ by two dimers flips around a plaquette. The potential term ($V > 0$) is a repulsion between parallel dimers. A strong negative $V$ favors a Valence

5 These 4 superpositions will have the following momenta: $(0, 0)$, $(0, 0)$, $(\pi, 0)$ and $(0, \pi)$. The two later will be degenerate (because of the $\pi/2$-rotation symmetry) so that 3 nearly-degenerate energies will appear in the finite-size spectrum.

6 Consider the restriction of the Heisenberg Hamiltonian to the subspace of nearest-neighbor dimer coverings. Due to the non-orthogonality between dimer coverings, this effective Hamiltonian is complicated and non-local. However, it can be formally expanded in powers of $x = 1/\sqrt{N}$ where $N$ counts the number of states a spin can have at each site (of course in our case $N = 2$). When only the lowest order ($O(x^4)$) terms are kept, the effective Hamiltonian is given by Eq. [8].
Bond columnar phase (Fig. 1), whereas a strong positive V forbids crystallization in the columnar phase and favors a staggered one (Fig. 3). From our present point of view, both of these phases are gapped VBC. They are separated by a quantum critical point at \( J = V \). At that point the exact ground-state is the equal-amplitude superposition of all nearest-neighbor dimer configurations \( \sigma \), the correlations are algebraic \( 28 \), the gap closes and the system supports \( S = 0 \) (quasi-)Goldstone modes (called “resonons” in ref. \( 27 \)).

Moessner and Sondhi have recently studied the same kind of model on the hexagonal lattice: it has a richer phase diagram of VBC phases, and essentially the same generic physics \( 29 \).

### E. Possible realizations in Spin-1/2 SU(2) models

The \( J_1 - J_2 \) model on the square lattice near the point of maximum frustration has been studied by different approaches: exact diagonalizations, series expansions, Quantum Monte-Carlo and Stochastic Reconfiguration. There is a general agreement that the \( J_1 - J_2 \) model on the square \( 17, 23, 24, 80 \), hexagonal \( 10 \) and checker board lattices \( 31 \) has at least one VBC phase between the two semi-classical Néel phases. Long range order might involve plaquettes of 4 spins (on the square lattice or on the checker board lattice) and even 6 spins on the hexagonal lattice.

Two elements should be noticed:

- All these lattices are bipartite and their coordination number is not extremely large,
- These phases appear when both \( J_1 \) and \( J_2 \) are antiferromagnetic (see next section for a ferromagnetic phase destabilized by an antiferromagnetic coupling).

### F. Large-N limits

Introduced by Affleck and Marston \( 22 \) and Arovas and Auerbach \( 23 \), large-N limits are powerful analytical methods. The \( SU(2) \) algebra of a spin \( S \) at one site can be represented by \( N = 2 \) species of particles \( a_\sigma^\dagger \) (with \( \sigma = \uparrow, \downarrow \)), provided that the total number of particles on one site is constrained to be \( a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow = 2S \). The raising operator \( S^+ \) (resp. \( S^- \)) is simply represented by \( a_\uparrow^\dagger a_\downarrow \) (resp. \( a_\downarrow^\dagger a_\uparrow \)). These particles can be chosen to be fermions (Abrikosov fermions) or bosons (Schwinger bosons). The Heisenberg interaction is a four-body interaction for these particles.

The idea of large-N limits is to generalize the \( SU(2) \) symmetry of the spin–S algebra to an \( SU(N) \) (or \( Sp(N) \)) symmetry by letting the flavors index \( \sigma \) go from 1 to \( N \). The \( SU(N) \) (or \( Sp(N) \)) generalization of the Heisenberg model is solved by a saddle point calculation of the action, which decouples the different flavors. It is equivalent to a mean-field decoupling of the four-body interaction of the physical \( N = 2 \) model.

Whether the large-N limit of some \( N = 2 \) model is an accurate description of the physics of “real” \( N = 2 \) spins is a difficult question but the phase diagrams obtained using these approaches (where the value of the “spin” \( S \) can be varied) are usually coherent pictures of the competing phases in the problem. In particular, these methods can describe both Néel ordered states and Valence-Bond Crystals \( 25 \), as well as short-range Resonating Valence Bonds phases (section \( 11 \) or Valence-Bond Solids \( 24 \). Read and Sachdev \( 25 \) studied the \( J_1 - J_2 - J_3 \) model on the square lattice by a \( Sp(N) \) bosonic representation and predicted a columnar VBC phase (as in Fig. 1) near \( J_2 \approx 0.5 J_1 \).

### G. Experimental realizations of Valence Bond Crystals

In the recent years many experimental realizations of VBCs have been studied. The most studied prototype in 1d is \( CuGeO_2 \): it is not a pure realization of the dimerized phase of a pure \( J_1 - J_2 \) model insofar as a Spin-Peierls instability induces a small alternation in the Hamiltonian. This compound has received a lot of attention both from the experimental and theoretical points of view (see \( 22 \) and references therein). In two dimensions two compounds are 2D VBCs \( CaV_4O_9 \) \( 33, 37, 38, 39, 40, 41 \), \( SrCu_2(BO_3)_2 \). This last compound is a good realization of the Shastry Sutherland model \( 49, 50 \). However, in both case the ground-state is non-degenerate because the Hamiltonian has an integer spin in the unit cell (4 spins 1/2) and the dimerization does not break any lattice symmetry.
III. SHORT-RANGE RESONATING VALENCE-BOND PHASES: TYPE I SRRVB SPIN LIQUID

A. Anderson’s idea

Inspired by Pauling’s idea of “resonating valence bonds” (RVB) in metals, P. W. Anderson [51] introduced in 1973 the idea that antiferromagnetically coupled spins $1/2$ could have a ground-state completely different from the two previous cases.

An RVB state can be viewed as a linear superposition of an exponential number of disordered valence bond configurations where spins are coupled by pairs in singlets (contrary to the Valence Bond Crystal where a finite number of ordered configurations dominate the ground-state wave-function and the physics of the phase). Because of these singlet pairings, many configurations are expected to have a reasonably low energy for an antiferromagnetic Hamiltonian. However, these energies are not necessarily lower that the variational energy of a competing Néel state. What lower the energy of a RVB state with respect to the energy of one particular lattice dimerization, are the resonances between the exponential number of dimer configurations which are energetically very close or degenerate. These resonances are possible because the Hamiltonian (the Heisenberg one for instance) has non-diagonal matrix elements between almost any pair of dimer coverings.

One central question is to characterize the kinds of dimer configurations which have the most important weights in the wave-function. The low-energy physics will crucially depend on the separation between the spins which are paired in singlets 7. Both ideas of short- and long-ranged RVB states have been developed in the literature. Liang et al. have shown that long range order on the square lattice can be recovered with dimer wave functions as soon as the weight of configurations with bonds of length $l$ decreases more slowly than $l^{-5}$ [52]. In this section we concentrate on the case where only short-ranged (i.e. a finite number of lattice spacings but not necessarily first-neighbors) dimer singlets participate significantly in the ground-state wave-function. In such a situation, taken apart the singular case of the Valence Bond Crystals, the short range RVB state has no long range order, it is fully invariant under $SU(2)$ rotations: it is a genuine Spin Liquid 8.

B. Ground-state properties of type I SRRVB Spin Liquids

Consider a spin model with a spin-$1/2$ in the unit cell. Our definition of a short-range RVB state is a wave-function which has the following properties:

(A) it can be written as a superposition of short-ranged dimer configurations,

(B) it has exponentially decaying spin-spin correlations, dimer-dimer correlations and any higher order correlations,

(C) its ground-state displays a subtle topological degeneracy [27, 53, 54, 55, 56, 57, 58].

Let us make some remarks on this definition. First, property (A) does not imply (B). As an example, the equal-amplitude superposition of all nearest-neighbor dimer configurations on the square lattice has algebraic spin-spin correlations [28]. However such equal-amplitude superposition on the triangular lattice was recently shown to satisfy (B) [59]. Property C is specific to systems with half odd integer spins in the crystallographic unit cell.

C. Elementary excitations of type I SRRVB Spin Liquids

If all kinds of correlations decay exponentially with distance over a finite correlation length $\xi$, all symmetry sectors are expected to be gapful 9. The simplest heuristic argument is the following: low-energy excitations are usually obtained by long wavelength deformations of the ground-state order parameter. If the correlation length is finite, elementary excitations will have a size of order $\xi$ (or smaller), which is the largest distance in the problem. Such excitations will therefore have a finite energy (uncertainty principle).

7 The set of all dimer coverings, including singlet dimers of all lengths, is in fact an over-complete basis of the whole $S = 0$ subspace. Without specifying which dimer configurations do enter in the wave-function, a linear superposition of dimer configurations can in fact be any $S = 0$ state, including a state with long-range spin-spin correlations!

8 Some authors use the word spin liquid with a less restrictive meaning for all spin systems exhibiting a spin gap inasmuch as they do not break SU(2). In view of the importance of the LRO in the ground-state and low lying excitations of the Valence Bond Crystals described in the previous section, we prefer the present definitions.

9 and $\chi$ and $C_v$ are thermally activated
Magnons and spinons

A conventional Néel antiferromagnet has elementary excitations called magnons (or spin-waves) which carry an integer spin $\Delta S = \pm 1$. In one dimension, there are free spin-$\frac{1}{2}$ elementary excitations (spinons). In two dimensions, spinons are confined in VBC phases but the possibility of unconfined spinons exists in a short-range RVB phase (naively speaking there are no more elastic forces to bind spinons in disordered dimer coverings). A field-theoretic description (Large $-N$ limit and gauge-theory) of spinon (de-)confinement in spin liquids was carried out by Read and Sachdev [35]. Unconfined spinons are fractional excitations in the sense that they carry a quantum number (total spin) of $\Delta S = \pm \frac{\Delta}{2}$, although they are a generic feature in large-$N$ ($Sp(N)$) approaches to frustrated spin-liquids (see for instance Refs. [35, 62, 63]) at least at mean-field level.

D. The hard core quantum dimer model on the triangular lattice

The QHCD model was originally introduced to look for an RVB phase. As explained in the previous section, it turns out that on the square lattice it displays only (gapped or critical) VBC phases. It has been recently generalized to the triangular lattice by Moessner and Sondhi [59]. Contrary to the original square lattice case, the QHCD model on the triangular lattice provides a short-range RVB spin-liquid with a finite correlation length at zero temperature. This phase survives in a finite interval of parameter $V_c \leq V \leq J$. The triangular version of the QHCD model is probably the simplest microscopic model which exhibits a short-range RVB phase, and quasi particle deconfinement.

E. Realizations of a type I Spin Liquid in $SU(2)$ spin models

Since the pioneering work of Anderson, there has been an intense theoretical activity on the RVB physics, specially after Anderson [64] made the proposition that such an insulating phase could be closely related to the mechanism of high-$T_c$ superconductivity. Short-range RVB states are certainly a stimulating theoretical concept, their realization in microscopic spin models is a more complicated issue. Up to now, two-dimensional models which could exhibit a short-range RVB phase are still few: Ising-like models in a transverse magnetic field [65] which are closely related to QHCD models by duality [61, 66], a quasi 1d model [67], a spin-orbital model [68], and the two short-range RVB phases in $SU(2)$ models, that we will now present.

The multiple-spin exchange model

The multiple-spin exchange model (called MSE in the following) was first introduced by Thouless [69] for the nuclear magnetism of three-dimensional solid He$^3$ [70] and by Herring [71] for the Wigner crystal. It is an effective Hamiltonian which governs the spin degrees of freedom in a crystal of fermions. The Hamiltonian is a sum of permutations which exchange the spin variables along rings of neighboring sites. It is now largely believed that MSE interactions on the triangular lattice also describe the magnetism of solid He$^3$ mono-layers adsorbed on graphite [72, 73, 74] and that it could be a good description of the two dimensional Wigner crystal of electrons [75]. In the He$^3$ system, exchange terms including up to 6 spins are present [72]. Here we will only focus on 2- and 4-spin interactions which constitute the minimal MSE model where a short-range RVB ground-state is predicted from exact diagonalizations [56]. The Hamiltonian reads:

$$H = J_2 \sum_{ij} P_{ij} + J_4 \sum_{ijkl} (P_{ijkl} + P_{lkji})$$

The first sum runs over all pairs of nearest neighbors on the triangular lattice and $P_{ij}$ exchanges the spins between the two sites $i$ and $j$. The second sum runs over all the 4-sites plaquettes and $P_{ijkl}$ is a cyclic permutation around the plaquette. The 2-spin exchange is equivalent to the Heisenberg interaction since $P_{ij} = 2 \mathbf{S}_i \cdot \mathbf{S}_j + 1/2$, but the four-spin term contains terms involving 2 and 4 spins and makes the model a highly frustrated one.

As $J_2 < 0$ and $J_2/J_4 \simeq -2$ in low-density solid He$^3$ films, the point $J_2 = -2, J_4 = 1$ has been studied by means of exact diagonalizations up to $N = 36$ sites [56]. These data point to a spin-liquid with a short correlation length and
a spin gap. No sign of a VBC could be found. In addition, a topological degeneracy which characterizes short-range RVB states was observed. From the experimental point of view, early specific heat and spin susceptibility measurements are not inconsistent with a spin-liquid phase in solid He$_3$ films. Indeed very low temperature magnetization measurements suggest a small but non-zero spin gap in low-density mono-layers.

An explanation of the origin of this short-range RVB phase in the MSE model can be guessed from the analogy between multiple-spin interactions and QHCD models. From the analysis of QHCD models we understand that RVB phases are possible when VBC are energetically unstable. Columnar VBC are stabilized by strong parallel dimer attraction and staggered VBC appear when the repulsion between these parallel dimers is strong. In between, an RVB phase can arise. From this point of view, tuning the dimer-dimer interactions is of great importance and the four-spin interaction of model plays this role.

A type I SRRVB phase on the hexagonal lattice

A second RVB phase has recently been found on the hexagonal lattice in a highly frustrating regime where the three first neighbors are coupled ferro-magnetically whereas the six second neighbor couplings are antiferromagnetic. For weak second neighbor coupling, the ground-state is a ferromagnet. Increasing the second neighbor coupling leads to an instability toward a short range RVB phase which has all the above-mentioned properties (except the topological degeneracy, because there are two spins 1/2 in the unit cell). This kind of phase has possibly been observed years ago by L.P. Regnault and J. Rossat-Mignod in BaCo$_2$(AsO$_4$)$_2$.

These two RVB phases appear in the vicinity of a ferromagnetic phase: is it or is it not an essential ingredient to form an RVB phase? One might argue that this feature helps in favoring RVB phases against VBC ones, because the plausible VBCs would have large elementary plaquettes and would be very sensitive to resonances between the different forms of plaquettes, thus disrupting long range order. It should also be noticed that the first neighbor coupling being ferromagnetic, the short range RVB phase will predominantly form on second neighbors, and thus on a triangular lattice. So the properties of the dimer coverings on the triangular lattice might at the end be the essential ingredient to have a RVB phase!

F. Chiral spin-liquid

The definition of a short-range RVB spin-liquid proposed in section excludes any long range order. However, a state with broken time-reversal symmetry and chiral long range order could accommodate all the other properties of a spin-liquid (the chiral observable is the triple product of three spins: see ref. for various equivalent definitions). Such a chiral phase would have a doubly degenerate ground-state in the thermodynamic limit. Inspired from Laughlin’s fractional quantum Hall wave functions, Kalmeyer and Laughlin have build a spin-$1/2$ state on the triangular lattice which exhibit some chiral long-range order (see also ). This complex wave function is directly obtained from the bosonic $m = 2$ ($\nu = 1/4$) Laughlin wave-function. Such a state is a spin singlet with unconfined spinon excitations which have anyonic statistics (right in between Bose and Fermi). This chiral liquid was initially proposed for the triangular-lattice Heisenberg antiferromagnet but the later turned out to be Néel long-range ordered. Wen et al discussed the properties expected for a chiral spin-liquid, its excitations and some possible mean-field descriptions. To our knowledge, there is no example of a microscopic $P$– and $T$–symmetric spin model which exhibits a chiral spin-liquid phase. The possibility of realizing a chiral phase in the presence of an external magnetic field (which explicitly breaks the time-reversal invariance) is discussed in paragraph V D.

IV. TYPE II SRRVB PHASES: “THE KAGOMÉ -LIKE MAGNETS”

On triangular based lattices, de-stabilization of the coplanar 3-sublattice Néel order either by an increase of the quantum fluctuations (through a decrease of the coordination number when going from the triangular lattice to the kagomé one) or by adding competing interactions (4-spin exchange processes) leads to an unexpected situation where the degeneracy of the exponential number of short range dimer coverings is only marginally lifted by quantum resonances, giving rise to a quantum system with a continuum of singlet excitations adjacent to the ground-state. This

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$^{10}$ In Ref. an RVB state is selected by introducing defects in the lattice in order to destabilize the competing VBC states.
property has first been shown by exact diagonalizations of the Heisenberg Hamiltonian on the kagomé lattice \cite{13}, then for the MSE Hamiltonian on the triangular lattice \cite{81}.

A. Description of the ground-state and of the first excitations in the \( S = 0 \) sector

The ground-state is a trivial superposition of an exponential number of singlets, like in any RVB ground-state described in the previous section. But contrary to the situations described in the previous sections there is no gap above the ground-state in the singlet sector.

Mambrini and Mila \cite{82} have shown that the qualitative properties of the ground-state and the first excitations are well described in the restricted basis of nearest neighbor couplings: to this extent, this second spin-liquid is a real short-range RVB state (indeed dressing these states with longer dimer coverings improves quantitatively the energy, but does not change the picture).

To understand the mechanism of (non)formation of the gap in the kagomé spin-liquid, it is interesting to compare Mambrini’s results to the earlier work of Zeng and Elser \cite{83}. This comparison shows that the non orthogonality of the dimer basis is an essential ingredient to produce the continuum of singlets adjacent to the ground-state. The above-mentioned QHCD model which implicitly truncates the expansion in the overlaps of dimers is by the fact unable to describe such a phase. On the other hand, taking into account the non orthogonality of dimer configurations would generate a QHCD model involving an infinite expansion of n-dimers kinetic and potential terms. In this basis the effective Hamiltonian describing the original Heisenberg problem has an infinite range of exponentially decreasing matrix elements.

This system has a \( T = 0 \) residual entropy in the singlet sector \( (\sim \ln(1.15) \text{ per spin}) \) \cite{13, 84}. The \( S = 0 \) excitations cannot be described as Goldstone modes of a quasi long range order in dimers (similar to the critical point of the R.K. QHCD model of subsection 2.4). In such a system, the density of states would increase as \( \exp(N^{\alpha/(\alpha+D)}) \), where \( D \) is the dimensionality of the lattice, and \( \alpha \) the power index of the dispersion law of the excitations \( (\epsilon(k) = |k|^\alpha) \), whereas it increases as \( \exp(N) \) in the present case (this represents a large numerical difference \cite{85}).

B. Excitations in the \( S \neq 0 \) sectors

The magnetic excitations are probably gapped: this assumption is a weak one. The spin gap if it exists is small (of the order of \( J/20 \)).

In each \( S \neq 0 \) sector the density of low lying excitations increase exponentially with the system size as the \( S=0 \) density of states, but with extra prefactors (as for example a \( N \) prefactor in the case of the \( S = 1/2 \) sector \cite{13, 82, 84, 86}).

The elementary excitations are de-confined spinons \cite{13}. An excitation in the \( S = 1/2 \) sector could be seen as a dressed spin-1/2 in the sea of dimers \cite{84}. The picture of Uemura et al \cite{87} drawn from the analysis of muon data on \( SrCrGaO \) is perfectly supported by exact diagonalizations results. The analytic description of such a phase remains a challenge. The fermionic \( SU(N) \) description \cite{88, 89} might give a good point of departure: here the \( S = 0 \) ground-state is indeed degenerate in the saddle point approximation. But the difficulty of the analysis of this degenerate ground-state in a \( 1/N \) expansion remains to be solved! A recent attempt to deal with such problems in a dynamical mean field approach looks promising \cite{90}.

C. Experimental realizations

No perfect \( S = 1/2 \) kagomé antiferromagnet has been up to now synthesized.

An organic composite \( S = 1 \) system has been studied experimentally \cite{91}: it displays a large spin gap ( of the order of the supposed-to-be coupling constant and thus much larger than what is expected on the basis of the \( S = 1/2 \) calculations). It is difficult to claim that it is an experimental manifestation of an even-odd integer effect, because the ferromagnetic binding of the spins 1/2 in a spin 1 is not so large that the identities of the underlying compounds could not play a role. (From a theoretical point of view, it would be extremely interesting to have exact spectra of a spin-1 kagomé antiferromagnet: if topological effects are essential to the physics of the spin-1/2 kagomé one might expect completely different spectra for the spin-1 system.)

A spin-3/2 bilayer of kagomé planes, the \( SrCrGaO \) oxide has been extensively studied \cite{92}. It displays some features that could readily be explained in the present framework of the spin-1/2 theoretical model:

- Dynamics of the low lying magnetic excitations \cite{87}
FIG. 3: Schematic view of the sublattice magnetization vectors when the external magnetic field is increased in an two-sublattice Heisenberg antiferromagnet.

- Vanishing elastic scattering at low temperature [93]
- Very low sensitivity of the low-T specific heat to very large magnetic fields [94, 95]

but some features (essentially the anomalous spin glass behavior [96, 97]) remain to be explained in a consistent way.

There are also a large number of magnetic compounds with a pyrochlore lattice (corner sharing tetrahedra). At the classical level such Heisenberg magnets have ground-states with a larger degeneracy than the kagomé problem [9]. They are expected to give spin-liquids [8], and indeed some of them display no frozen magnetization [98]. Whether the Heisenberg nearest neighbor problem for spin-1/2 has the same generic properties on the pyrochlore lattice and on the kagomé lattice is still an open question. Contrary to some expectations [99], the Heisenberg problem on the checker-board lattice (a 2-dimensional pyrochlore) has a VBC ground-state [31, 100]. Nevertheless it is up to now totally unclear if the checker-board problem is a correct description of the 3d pyrochlores, there are even small indications that this could be untrue [31].

V. MAGNETIZATION PROCESSES

In this section we discuss some aspects of the behavior of Heisenberg spin systems in the presence of an external magnetic field. Frustrated magnets either with a semi-classical ground-state or in purely quantum phases exhibit a large number of specific magnetic behaviors: metamagnetism, magnetization plateaus. In the first subsection we describe the free energy patterns associated with these various behaviors. In subsection 5.2, we discuss a classical and a quantum criterion for the appearance of a magnetization plateau. In the following subsections we described two mechanisms recently proposed to explain the formation of a plateau.

A. Magnetization curves and free energy patterns

The simplest quantum antiferromagnet in 2D is the spin-$-\frac{1}{2}$ Heisenberg model on the square lattice:

$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - B \sum_i S_i^z$$

The system has two sublattices with opposite magnetizations in zero field and at zero temperature. The full magnetization curve of this model has been obtained by numerical and analytical approaches [101]. The sublattice magnetizations gradually rotate toward the applied field direction as the magnetic field is increased (Fig. 3). At some finite critical field $B_{\text{sat}}$ the total magnetization reaches saturation$^{11}$.

We define $e(m)$ as the energy per site $e = E/N$ of the system, as a function of the net magnetization $m = M/M_{\text{sat}}$. From this zero field information, we get the full magnetization curve $m(B)$ by minimizing $e(m) - mB$, that is $B = \frac{\partial e}{\partial m}$. In the square-lattice antiferromagnet case discussed above, $e(m)$ is almost quadratic $e(m) = \frac{m^2}{2\chi_0}$ and the corresponding magnetization is almost linear (Fig. 4).

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$^{11}$ As in most models, the saturation field can be computed exactly by comparing the energy of the ferromagnetic state $(E_F/N = J/2 - B/2)$ with the (exact) energy of a ferromagnetic magnon $(E(\mathbf{k} \neq 0) = E_F + B + J(\cos(k_x) + \cos(k_y) - 2))$. The magnon energy is minimum in $k_{\text{min}} = (\pi, \pi)$. The saturation field is obtained when $E_F = E(k_{\text{min}})$, that is $B_{\text{sat}} = 4J$. The calculation is unchanged for an arbitrary value of the spin $S$ and one finds $B_{\text{sat}} = 8JS$. 


FIG. 4: Linear response obtained with $e \simeq \frac{m^2}{2N}$ as in an AF system with collinear LRO.

FIG. 5: $m = 0$ magnetization plateau due to a linear $e(m) = \Delta m + o(m)$.

When the system is more complicated, because of frustration for instance, the magnetization process can be more complex. In particular, magnetization plateaus or metamagnetic transitions can occur. For instance, the addition of a second-neighbor coupling on the square-lattice antiferromagnet opens a plateau at one half of the saturated magnetization $\text{sat}$. It is useful to translate these anomalies of the magnetization curve into properties of $e(m)$.

A plateau at $m_0 = 0$ (also called spin gap) is equivalent to $\frac{\partial e}{\partial m}\big|_{m_0=0} > 0$ (Fig. 3). At the field where the magnetization starts growing the transition can be critical or first order. In one dimension, exact results on the Bose condensation of a dilute gas of interacting magnons lead to $m \sim \sqrt{\delta B}$ for integer spin chains [103]. In that case $e(m) = \Delta m + \alpha m^3 + o(m^3)$ and the system is critical at $B = \Delta$.

A plateau at finite magnetization $m_0 > 0$ for $B \in [B_1, B_2]$ is a discontinuity of $\frac{\partial e}{\partial m}$ (Fig. 4). Such a behavior can arise in a frustrated one (see next subsection). The vanishing susceptibility when $B$ is inside the plateau comes from the fact that magnetic excitations (which increase or decrease the total magnetization) are gapped when the magnetic field lies in the interval $[B_1, B_2]$. As before, the system can be critical at the “edges” of the plateaus. In the following subsections we examine the origins of such plateaus in classical and quantum spin systems.

A metamagnetic transition is a discontinuity of the magnetization as a function of the applied field, it is a first order phase transition and is equivalent to a concavity in $e(m)$ (Fig. 5). As in any first order transition, this can give rise to an hysteretic behavior in experiments. Such a behavior is highly probable in frustrated magnets [73, 76] for essentially two reasons:

- Due to the frustration, different configurations of spins corresponding to phases with different space symmetry breakings are very near in energy,

FIG. 6: A magnetization plateau originates from a discontinuity in the slope of $e(m)$. The simplest example of quantum Heisenberg model with such a plateau is the triangular-lattice antiferromagnet (see Fig. 10 and text), a more sophisticated example is the behavior of the same model on a kagomé lattice, which equally displays a magnetization plateau at $M/M_{\text{sat}} = 1/3$. 
FIG. 7: Metamagnetic transition associated to a concavity in the $e(m)$ curve.

FIG. 8: Collinear spin structure of type $uuud$ on the triangular lattice. This state with $M/M_{\text{sat}} = \frac{1}{3}$ is realized in the classical and quantum MSE model under magnetic field (see text and Fig. 9).

- Impurities in magnetic compounds pin the existing structures and hinder the first order phase transitions, at variance with their role in the standard liquid-gas transition.

As a last general, but disconnected, remark let us underline that the magnetic studies of quantum frustrated magnets could also help in solving elusive questions relative to the existence of an exotic $H=0$ ground-state. We have already seen that the extraordinary thermo-magnetic behavior of $SrCrGaO_4$ could be a signature of a type II Spin Liquid [94, 95]. Years ago L.P. Regnault and J. Rossat-Mignod studied $BaCo_2(AsO_4)_2$: the knowledge of its magnetic phase diagram convinced them that something queer was going on in this material at $H = 0$ and in fact exact diagonalizations now point to a type I Spin Liquid! In fact in very large magnetic fields the magnets become increasingly classical and a semi-classical approach is justified: any deviation from a semi-classical behavior when $H$ is decreasing is thus an important indication of a possible quantum exotic ground-state!

B. Magnetization plateaus

Classical spins: collinearity criterion

Magnetization plateaus are often believed to be a purely quantum-mechanical phenomenon which is sometimes compared in the literature to Haldane phases of integer spin chains. This is certainly not always true since some classical spin models have magnetization plateaus at zero temperature. For instance, as shown by Kubo and Momo [104], the MSE model on the triangular lattice has a large range of parameters where magnetization plateaus at $M/M_{\text{sat}} = \frac{1}{3}$ and $M/M_{\text{sat}} = \frac{1}{2}$ appear at zero temperature [12]. The ground-state of the system at $M/M_{\text{sat}} = \frac{1}{3}$ is the so-called $uud$ structure where two sublattices have spins pointing “up” along the field axis and the third one has down spins. At $M/M_{\text{sat}} = \frac{1}{2}$ the ground-state is of $uuud$ type (Fig. 8).

In fact, we show below that if a classical spin system exhibits a magnetization plateau then all the spins are necessarily collinear to the magnetic field. This restricts possible spin configurations to states of the $u^{n-p}d^p$ kind with $n - p$ spins “up” and $p$ spins “down” in the unit cell. As a consequence, the total magnetization per site must be of the form: $M/M_{\text{sat}} = 1 - 2p/n$ where $n$ is the size of the unit cell, and $p$ an integer. This commensuration between the magnetization value at a plateau and the total spin in the unit cell will turn out to hold equally for quantum spins.

The proof of this classical collinearity condition is sketched below. We consider the function $e(m)$ (which is defined without external field). Let us suppose that the ground state $\Psi_0$ at $m_0$ is not a collinear configuration. We will show that $e(m)$ has a continuous derivative in $m_0$, i.e. no plateau occurs at $m_0$. From a non-collinear state, one can chose

\[12\] As discussed below, these plateaus are also present in the quantum $S = \frac{1}{2}$ model [73, 74, 103].
FIG. 9: Magnetization curve of the MSE model at $J_2 = -2$ and $J_4 = 1$ computed numerically \cite{105, 106} for finite-size samples with $N$ up to 28 sites. The magnetization plateau at $M/M_{\text{sat}} = \frac{1}{2}$ (due to the $uuud$ state of Fig. 8) clearly appears even at finite temperature (full line \cite{106}).

an angle $\theta$ to deform $\Psi_0$ into a new configuration $\Psi(\theta)$ with a different magnetization

$$m(\theta) = m_0 + a\theta + \mathcal{O}(\theta^2)$$

and $a \neq 0$. This can be done, for instance, by rotating the spins of a sublattice whose magnetization is not collinear to the field \footnote{If $\Psi_0$ was a collinear state, the deviation of magnetization would be smaller: $m(\theta) = m_0 + b\theta^2 + \mathcal{O}(\theta^3)$ and the argument would fail.}. Here we just require that the spins can be rotated of an infinitesimal angle, as it is the case for three-component classical spins. $\Psi(\theta)$ is a priori no longer the ground state and its energy is:

$$e(\Psi(\theta)) = e(m_0) + a\theta + \mathcal{O}(\theta^2)$$

No assumption on $\alpha$ is needed. As a variational state of magnetization $m(\theta)$, $\Psi(\theta)$ has an energy $e(\Psi(\theta))$ larger or equal to the ground state energy at magnetization $m(\theta)$:

$$e(m_0) + a\theta + \mathcal{O}(\theta^2) \geq e(m(\theta))$$

With Eq. (8) we see that:

$$e(m_0) + \frac{\alpha}{a} (m(\theta) - m_0) + \mathcal{O}((m(\theta) - m_0)^2) \geq e(m(\theta))$$

If we assume that no metamagnetic transition occurs in the neighborhood of $m_0$, then $e(m)$ is convex about $m_0$ and Eq. (9) insures the differentiability of $e(m)$ in $m = m_0$. We have $\frac{\partial e}{\partial m}|_{m = m_0} = \frac{\partial e}{\partial m}|_{m = m_0}$ and no plateau occurs at $m_0$.

This proof is not restricted to rotation invariant interactions. We have only made the (extremely weak) assumption that the energy is a continuous and differentiable function of the spins directions (Eq. (8)). If the couplings are written $J^x\vec{S}_i\cdot\vec{S}_j + J^y\vec{S}_i\cdot\vec{S}_j + J^z\vec{S}_i\cdot\vec{S}_j$, a plateau must correspond to a collinear state with respect to the magnetic field direction, whatever the easy-plane or easy-axis might be. This property remains also true if multiple-spin interactions are present.

Fluctuations

Zero-temperature magnetization plateaus will of course be smeared out at very high temperature. However, in some cases, thermal fluctuations can enhance magnetizations plateaus. In the case of the triangular-lattice classical Heisenberg antiferromagnet the magnetization is perfectly linear at zero temperature for isotropic Heisenberg interactions. It is thanks to thermal fluctuations that a $uud$ structure is stabilized and that the susceptibility is reduced.
The number of sites in the plateaus satisfy then numerous studies of magnetization plateaus in one-dimensional systems [114, 115, 116] confirmed that all the interacting bosons, his results applies to quantum magnets and reduces to gap is only possible when the number of particle in the unit cell is an integer. Since spins itinerant particles on lattice models when the number of particles is a conserved quantum number. It states that a state produced by LSM has the same magnetization $s$ to make to establish the result is the following: the system has periodic boundary conditions (it is a boundary conditions in the thermodynamic limit.

Quantum fluctuations can play a very similar role: the quantum $S = \frac{1}{2}$ Heisenberg model on the triangular lattice has an exact magnetization plateau at $M/M_{\text{sat}} = \frac{1}{3}$ at zero temperature [108, 109, 110]. Again, the ground-state at $M/M_{\text{sat}} = \frac{1}{3}$ is a uud-like state. This uud plateau has a simple origin in the Ising limit [108, 110] $(H = J^z \sum_{(i,j)} S^z_i S^z_j)$ but survives up to the isotropic point $J^x = J^z$. What should be stressed is that the introduction of quantum fluctuations through couplings in the $xy$ plane renormalizes the magnetic field interval where the susceptibility vanishes but the total magnetization at the plateau remains exactly $\frac{1}{3}$. The sublattice magnetizations of the two $u$ sublattices and the $d$ sublattice are reduced from their classical values to $M_u = 1 - \eta$ and $M_d = -(1 - 2\eta)$ but the total magnetization is unaffected $M = (2M_u + M_d) = \frac{1}{3}M_{\text{sat}}$. The same phenomenon was observed numerically for the $uud$ plateau of the the MSE model on the triangular lattice which also exists in the classical limit [104] and survives quantum fluctuations in the spin-$\frac{1}{2}$ model [73, 81, 105] without change in the magnetization value. In the following paragraphs we discuss the origin of this robustness of the magnetization value at a plateau.

Quantum spins and Oshikawa’s criterion

The Lieb-Schultz-Mattis (LSM) theorem [111] proves that in one dimension and in the absence of magnetic field the ground-state is either degenerate or gapless if the spin $S$ at each site is half-integer. The proof relies on the construction of a low energy variational state which is orthogonal to the ground state. Oshikawa, Yamanaka and Affleck [112] realized that this proof can be readily extended to magnetized states as long as $n(S - s^z)$ is not an integer, where $s^z$ is the magnetization per site and $n$ the period of the ground state 14. As a magnetization plateau requires a spin gap, this suggests that plateaus can only occur when $n(S - s^z)$ is an integer. However the low-energy state produced by LSM has the same magnetization $s^z$ as the ground state, - it is a non-magnetic excitation-, and does not exclude a gap to magnetic excitations when $n(S - s^z)$ is not a integer. In fact arguments based on the bosonization technique were provided by Oshikawa et al. [112] (see also Ref. [113]) to support the hypothesis that $n(S - s^z) \in \mathbb{Z}$ is indeed a necessary condition to have a magnetization plateau in one-dimensional systems. Since then numerous studies of magnetization plateaus in one-dimensional systems [114, 115, 116] confirmed that all the plateaus satisfy $n(S - s^z) \in \mathbb{Z}$. We will see below that this criterion appears also to apply in higher dimension.

The LSM construction does not give a low energy state in dimension higher than one but Oshikawa [5] developed a different approach to relate the magnetization value to the number of spins in the unit cell. His result applies to itinerant particles on lattice models when the number of particles is a conserved quantum number. It states that a gap is only possible when the number of particle in the unit cell is an integer. Since spins $S$ are exactly represented by interacting bosons, his results applies to quantum magnets and reduces to $n(S - s^z) \in \mathbb{Z}$. The hypothesis that one has to make to establish the result is the following: the system has periodic boundary conditions (it is a $d$-dimensional torus) and the gap, if any, does not close when adiabatically inserting one fictitious flux quantum inside the torus 15. In the spin language, this amounts to say that the spin gap does not close when twisting the boundary conditions from 0 to $2\pi$ 5. This hypothesis can only be checked numerically 5 but it is completely consistent with the idea that the gapped system is a liquid with a finite correlation length and which physical properties do not depend on boundary conditions in the thermodynamic limit.

14 $n$ can be larger than what is prescribed by the Hamiltonian if the ground-state spontaneously breaks the translation symmetry.
15 The number of sites in the $(d - 1)$-dimensional section of the torus is supposed to be odd.
All known examples of magnetization plateaus indeed do satisfy Oshikawa's criterion. It is interesting to notice the close resemblance between the requirement that \( n(S - s^z) \) is an integer and the collinearity condition discussed in paragraph \( \frac{16}{18} \). Let us assume that we can represent one spin \( S \) with 2S classical spins of length \( \frac{1}{4} \) and require that these spins are in a collinear configuration. If the unit cell has \( n \) sites it contains \( n' = 2Sn \) small spins, \( p \) of which are down \( (\mu^{n' - p}d^n) \). For one unit cell the magnetization is \( M = n'/2 - p \) so that the magnetization per site is \( s^z = M/n = S - p/n \). The classical criterion reads \( n(s^z - S) = -p \in \mathbb{Z} \) which is the same condition as Oshikawa's result. In spite of this striking analogy, the classical collinearity condition is an exact result valid in any dimension and it does not involve a topological property (as periodic boundary conditions on the sample) unlike the quantum version. Ultimately indeed it rests on the property that the discretization of a spin \( S \) only involves spin-1/2 units.

C. Magnetization plateaus as crystal of magnetic particles

Magnetization plateaus can often be understood simply if the system has some unit cell where spins are strongly coupled (exchange \( J_1 \)) and weak bonds \( J_2 << J_1 \) between the cells. In this case, magnetization plateaus are governed by the quantized magnetization of a single cell, and the plateaus can be continuously connected to the un-coupled cells limit \( J_2 = 0 \). A perturbative calculation in powers of \( J_2/J_1 \) will in general predict the plateau to be stable over a finite interval of \( J_2 \). This situation appears, for instance, in one-dimensional ladder systems \( [15] \). In 2D, an example is the \( M/M_{\text{sat}} = \frac{1}{2} \) plateau predicted \( [17] \) in the 1/5-depleted square lattice realized in \( \text{CaV}_4\text{O}_9 \). This lattice is made of coupled square plaquettes and the \( M/M_{\text{sat}} = \frac{1}{2} \) plateau comes from the magnetization curve of a single plaquette (which magnetization can of course take only three values: \( M/M_{\text{sat}} = 0, \frac{1}{2}, \text{and } 1 \)).

However, in a translation invariant 2D system, with a single spin in the unit cell, the mechanism for the appearance of plateaus is not so simple. The \( M/M_{\text{sat}} = \frac{1}{2} \) plateau of the triangular-lattice Heisenberg antiferromagnet cannot be understood in such a strong-coupling picture. The magnetization plateaus predicted at magnetizations smaller than \( \frac{1}{2}M_{\text{sat}} \) in the 1/5-depleted square lattice mentioned above cannot either be understood within such a picture.

Totsuka \( [13] \) and Momoi and Totsuka \( [18] \) have associated magnetization plateaus with the crystallization of “magnetic particles”. In their picture the zero-field ground-state is a vacuum which is populated by bosonic particles when the external magnetic field is turned on. Depending on the model and its zero-field ground-state these bosons can represent different microscopic degrees of freedom: a spin flip on a single site, an \( S^z = 1 \) state on a link or a magnetic state of a larger number of spins. These bosons obey an hard-core constraint and carry one quantum of magnetization which cannot be understood within such a picture.

As for the antiferromagnetic interactions (such as \( J\vec{S}_i \cdot \vec{S}_j \)), they generate kinetic as well as repulsive interactions terms for these bosons. On general grounds, Momoi and Totsuka expect this gas of interacting bosons to be either a Bose condensate (superfluid) or a crystal (or charge-density wave). On one hand, the finite compressibility of the superfluid leads to a continuously varying magnetization as a function of magnetic field \( [16] \). On the other hand, the underlying lattice will make the crystal incompressible (density fluctuations are gapped) which precisely corresponds to a magnetization plateau. Such a crystalline arrangement of the magnetic moments is also consistent with the quasi-classical picture provided by \( u^{n' - p}d^n \)-like states.

In this approach, the densities (i.e magnetizations) at which the bosons crystallize (i.e form plateaus) is mainly determined by the range and strength of the boson-boson repulsion and the geometry of the lattice. In some toy models where the kinetic terms for the bosons vanishes (hopping is forbidden by the lattice geometry), this allows to demonstrate rigorously the existence of magnetization plateaus \( [19] \). Looking at the spatial structures which minimizes the repulsion (and thus neglecting kinetic terms in the boson Hamiltonian) gives useful hints of the possible plateaus. Since these structures are stabilized by the repulsive interactions, they can be compared to a Wigner crystal (except that particles are bosons, not fermions).

The magnetization curve of the quasi 2D oxide \( \text{SrCu}_2(\text{BO}_3)_2 \) has been measured at very low temperatures up to 57 Tesla \( [20, 21] \), which corresponds to \( M/M_{\text{sat}} \approx \frac{1}{2} \). The magnetization curve displays a large spin gap and plateaus at \( \frac{1}{4} \) and \( \frac{3}{4} \). A small one at \( \frac{1}{3} \) is also reported. This compound is the first experimental realization of the Shastry-Sutherland model \( [22] \) (see Fig. \( [11] \)). In this system magnetic excitations have a very small dispersion. This can be understood from the geometry of this particular lattice \( [12] \) and has been confirmed experimentally by inelastic neutron scattering experiments \( [13] \). This low kinetic energy makes the bosons quasi-localized objects and explains their ability to crystallize \( [22, 24] \) and give magnetization plateaus.

\( [16] \) In some cases, the superfluid component is believed \( [18] \) to coexist with a crystal phase between magnetization plateaus (supersolid).
D. More exotic states - analogy with quantum Hall effect

A recent piece of work on magnetization plateaus in two-dimensional spin-$\frac{1}{2}$ magnets by Misguich, Jolicoeur and Girvin [125] establishes a connection between this phenomenon and quantized plateau of the (integer) quantum Hall effect. Although both phenomena show up as quantized plateaus in a 2D interacting system, they are apparently not directly related: magnetization plateaus involve spins on a lattice whereas the quantum Hall effect appears with fermions in the continuum with plateaus in their transverse Hall conductance $\sigma_{xy}$. The link between the two problems appears when the spins are represented as fermions attached to one quantum of fictitious magnetic flux, as explained below.

A spin-$\frac{1}{2}$ model is equivalent to hard-core bosons \(^{17}\). It is possible to map exactly these bosons to a model of fermions in the presence of a fictitious gauge field \(^{18}\). The idea is that the hard-core constraint will be automatically satisfied by the Pauli principle and the fictitious flux quantum attached to each fermion will transmute the Fermi statistics into a bosonic one \([126, 127, 128]\).

In this framework a down spin is an empty site and an up spin is a composite object made of one (spinless) fermion and one vortex in the fictitious gauge field centered on a neighboring plaquette. This vortex can be simply pictured as an infinitely thin solenoid piercing the plane through a plaquette adjacent to the site of the fermion. As two of these fermion+flux objects are exchanged adiabatically, the $-1$ factor due to the Pauli principle is exactly compensated by the factor $-1$ of the Aharonov-Bohm effect of one charge making half a turn around a flux quantum. Consequently, these objects are bosons and represent faithfully the spin-$\frac{1}{2}$ algebra. Technically, the flux attachment is performed by adding a Chern-Simons term in the Lagrangian of the model, the role of which is to enforce the constraint that each fictitious flux is tied to one fermion.

At this stage the spin problem is formulated as fermions interacting with a Chern-Simons gauge theory. A mean field approximation which is not possible in the original spin formulation is now transparent: the gauge field can be replaced by its static mean value \([129]\). Since this static flux $\Phi$ per plaquette comes from the flux tubes initially attached to each fermion, it is proportional to the fermion density $\Phi = 2 \langle c^+ c \rangle$. Because of this flux, each energy band splits into sub-bands with a complicated structure. When this magnetic field is spatially uniform the band structure has a fractal structure as a function of the flux which is called a Hofstadter “butterfly” \([125]\).

The mean-field ground-state is obtained by filling these lowest energy sub-bands with fermions until their density satisfies $\Phi = 2 \pi \langle c^+ c \rangle$. For a given flux, one can then integrate over the density of states to get the ground-state energy. This energy as a function of the flux (or the fermion density) is equivalent to $e(m)$ since $2m + \frac{1}{2} = \langle c^+ c \rangle$ and one can compute the magnetization curve. Magnetization plateaus open when some particular band-crossings appear in the Hofstadter spectrum \([125]\).

This approximation scheme was applied to the Shastry-Sutherland model (Fig. 11) and a good quantitative agreement \([125]\) was found with the experimental results of Onizuka et al. \([120]\) on SrCu$_2$(BO$_3$)$_2$. In particular, the magnetization plateaus at $M/M_{\text{sat}} = 0$, $\frac{1}{4}$ and $\frac{1}{2}$ were reproduced. The $M/M_{\text{sat}} = \frac{1}{4}$ due to the $uuu$ state on the triangular-lattice antiferromagnet has also been described with this technique \([125]\).

This method is indeed very similar to the Chern-Simons approach to the fractional quantum Hall effect \([130]\) in

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\(^{17}\) Raising and lowering operators $S_i^+$ and $S_i^-$ are equivalent to bosonic creation and annihilation operators $b_i^+$ and $b_i$ for bosons satisfying an hard-core constraint $b_i^+ b_i = S_i^2 + \frac{1}{2} \leq 1$.

\(^{18}\) In one dimension, this mapping from spins to fermions is the famous Jordan-Wigner transformation and no extra degree of freedom is required. It is a bit more involved in 2D where introducing a fictitious magnetic field is necessary.
which real fermions (electrons) are represented as hard-core bosons carrying \(m\) (odd integer) flux quanta. In particular, as in the quantum Hall effect, the system is characterized by a quantized response coefficient. In the quantum Hall effect this quantity is the transverse conductance \(\sigma_{xy}\) which relates the electric field to the charge current in the perpendicular direction and in the spin system it relates the spin current with a Zeeman field gradient in the perpendicular direction. In the mean field approximation this transverse spin conductance is obtained from the TKNN integers of the associated Hofstadter spectrum.

As in the quantum Hall effect, the topological nature of the quantized conductance protects the plateaus from Gaussian fluctuations of the (fictitious) gauge field around its mean-field value. If this mean-field theory captures the physics of magnetization plateaus, this topological picture of the quantized magnetization establishes a deep connection with the conductance plateaus in the Quantum Hall Effect.

There could not be now any definitive conclusion, neither on a set of conditions both necessary and sufficient to have a plateau in two-dimensional frustrated systems, nor on the existence of other mechanisms than those presented in sections 5.3 and 5.4. But magnetization plateaus as well as metamagnetic transitions are probably rather ubiquitous properties of frustrated magnets which still deserve both experimental and theoretical studies.

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