Holographic subregion complexity under thermal quench in Einstein-Maxwell-Axions theory with momentum relaxation

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Abstract

We investigate the evolution of holographic entanglement entropy (HEE) and holographic complexity (HC) under a thermal quench in Einstein-Maxwell-Axion theory (EMA), which is dual to a field theory with momentum relaxation on the boundary. A strip-shaped boundary geometry is utilized to calculate HEE and HC via ‘entropy=surface’ and ‘complexity=volume’ conjecture, respectively. By fixing other parameters we claim that either large enough black hole charge or width of the strip will introduce swallow-tail behaviors in HEE and multi-values in HC due to the discontinuity of the minimum Hubeny-Rangamani-Takayanagi (HRT) surface. Meanwhile, we explore the effects of momentum relaxation on the evolution of HEE and HC. The results present that the momentum relaxation will suppress the discontinuity to occur as it increases. For large enough momentum relaxation the continuity of HEE and HC will be recovered.

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I. INTRODUCTION

Holography [1–3] has provided a close connection among the quantum information, condensed matter and quantum gravity. These connections become more and more important in the realm of theoretical physics and quantum information. From the view of technics, it is also a powerful tool to study many physical quantities in these areas, especially in strongly correlated systems.

Among the physical quantities, entanglement entropy (EE) and complexity are two significant concepts in the theoretical physics and quantum information. Essentially, EE measures the degrees of freedom in a strongly coupled system while the complexity measures the difficulty of turning a quantum state into another state. But it is extremely difficult to evaluate them on the side of the field theory when the degrees of freedom of the system become large. Fortunately, both of them can be evaluated with the help of holography and their elegant geometric duality from gravity side have been provided. Specially, in the holographic framework, it has been proposed that in [4, 5], the EE for a subregion on the dual boundary is proportional to the minimal Hubeny-Rangamani-Takayanagi (HRT) surface in the bulk geometry. Later, two different methods were proposed to evaluate the complexity from geometry. One is the ‘Complexity=Volume’ conjecture (CV), which states that the holographic complexity (HC) is proportional to the volume of a codimension-one hypersurface with the AdS boundary and the HRT surface [6, 7]. While the other is ‘Complexity=Action’ conjecture (CA) in which one identifies the HC with the gravitational action evaluated on the Wheeler-DeWitt patch in the bulk [8, 9]. In this paper, we will follow the CV conjecture and study its evolution under a thermal quench.

In the framework of CV conjecture, some of related quantities and their geometric descriptions have been holographically investigated. For instance, entanglement of purification (EoP), which is an important quantum information quantity for mixed states, has been holographically dual to the minimal entanglement wedge cross section [10, 11] and been generalized in [12–14]. The bit thread formalism for studying EoP was then addressed in [14–17]. Complexity of purification (CoP), which describes the minimum number of gates
needed to purify a mixed state, was holographically explored in [14, 18]. Another interesting quantity is the logarithmic negativity which captures the quantum correlations with the nature of entanglement [19]. It is also a quantum entanglement measure for mixed quantum states and the holographic dual has been studied in [20, 21].

Additionally, the study of HEE and HC will provide us more indirect but effective information to explore the nature of the spacetime, in particular the physics of the black hole horizon and its thermal and entanglement structures. Specifically, it was shown in [22, 23] that the EE is not enough to understand the physics of the black hole horizon via studying the information paradox in black holes. Therefore, the authors proposed the ER=EPR conjecture and argued that the creation of the firewall behind the horizon is essentially a problem of quantum computational complexity [6, 7], where ER and EPR stand for Einstein-Rosen bridges and Einstien-Podolsky-Rosen correlations, respectively. The evolution of the HEE and HC under a thermal quench has been explored in various dynamical backgrounds such as Einstein theory [24–26], Einstein-Born-Infeld theory [27], massive gravity theory [28], and has been further generalized to chaotic system [29] and dS boundary [30]. The evolution of HEE and HC for quantum quench have also been investigated in [31–35] and references therein.

In this paper, we will investigate the evolution of HEE and HC under a thermal quench in four dimensional Einstein-Maxwell gravity coupled with two linear spacial-dependent scalar fields in the bulk, which is called Einstein-Maxwell-Axions (EMA) theory. It was addressed in [36] that the scalar fields in the bulk source a spatially dependent field theory with momentum relaxation, while the linear coefficient of the scalar fields describes the strength of the momentum relaxation. This means that our study will involve in momentum relaxation, which is more closer to the reality. Moreover, the effect of momentum relaxation on the time evolution of the optical conductivity [37] and the equilibrium chiral magnetic effect [38] with Vaidya quench in this model has been studied, which shed light on the quark gluon plasma produced in heavy ion collisions as well as the real-world systems. It is worthwhile to point out that in this model, momentum relaxation in the dual boundary is sourced by the bulk scalar fields in the bulk, but the black hole geometry is homogeneous as we will present soon. The fully inhomogeneous holographic thermalization process with spacial dependent bulk geometry has been studied in [39].

Our paper is organized as follows. In section II, we study the generalized Vaidya-AdS black brane in EMA theory. Then, in section III, we present the holographic setup of HEE and HC for a stripe geometry. We show our results and analyze the effect of momentum relaxation on the evolution of HEE and HC in section IV. Finally, section V contributes to our conclusions and discussions.

II. VAIDYA ADS BLACK BRANES IN EINSTEIN-MAXWELL-AXIONS GRAVITY THEORY

We consider the AdS black branes in EMA gravity proposed in [36]. The action of the four dimensional theory is given by

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( R + \frac{6}{\ell^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \sum_{I=1}^{2} (\partial \psi_I)^2 \right). \tag{1}
\]
By setting the scalar fields to linearly depend on the two dimensional spatial coordinates $x^a$, i.e., $\psi_I = \beta \delta_I a x^a$ where the index $a$ goes $a = 1, 2$, the action admits the charged black brane solution

$$ds^2 = -r^2 f(r) dt^2 + \frac{1}{r^2 f(r)} dr^2 + r^2 (dx^2 + dy^2), \quad A = A_t(r) dt,$$

with

$$f(r) = \frac{1}{\ell^2} - \frac{\beta^2}{2 r^2} - \frac{m}{r^3} + \frac{q^2}{r^4}, \quad A_t = \left(1 - \frac{r_h}{r}\right) \frac{2q}{r_h}. \tag{2}$$

Here, the horizon $r_h$ satisfies $f(r_h) = 0$; $\ell$ describes the radius of AdS spacetime, and for simplicity we will set $\ell = 1$. $m$ and $q$ are the mass and charge of the black brane, respectively, with the relation given by

$$1 - \frac{\beta^2}{2 r_h^2} - \frac{m}{r_h^3} + \frac{q^2}{r_h^4} = 0. \tag{3}$$

It is worthwhile to point out that the scalar fields in the bulk source a spatially dependent boundary field theory with momentum relaxation, which is dual to a homogeneous and isotropic black brane (2). The linear coefficient $\beta$ of the scalar fields is usually considered to describe the strength of the momentum relaxation in the dual boundary theory [36]. A general action with axions terms and the holography has been studied in [40]. We note that the extended thermodynamics of the black brane has also been studied in [41–43]. The Hawking temperature of the black brane reads

$$T = \frac{1}{4\pi} \left. \frac{d (r^2 f(r))}{dr} \right|_{r_h} \tag{4}$$

which is treated as the temperature of the dual boundary field.

With properly chosen coordinate transformation, the above black hole brane (2) can be represented as in the Eddington-Finkelstein coordinates,

$$ds^2 = \frac{1}{z^2} \left[-f(z) dv^2 - 2dv dz + dx^2 + dy^2\right], \tag{5}$$

$$f(z) = 1 - \frac{1}{2} \beta^2 z^2 - mz^3 + q^2 z^4, \tag{6}$$

with

$$dv = dt - \frac{1}{f(z)} dz \quad \text{and} \quad z = \frac{1}{r}. \tag{7}$$

We note that the coordinates $v$ and $t$ coincide on the boundary. Thus, in order to holographically describe the evolution of HEE and HC, one usually frees the mass and charge parameter as smooth functions of $v$ as [44, 45]

$$M(v) = \frac{v_0}{2} \left[1 + \tanh \left(\frac{v}{v_0}\right)\right], \tag{8}$$

$$Q(v) = \frac{q_0}{4} \left[1 + \tanh \left(\frac{v}{v_0}\right)\right], \tag{9}$$

where $v_0$ represents the finite thickness of the falling charged dust shell. Then the related Vaidya AdS black brane is

$$ds^2 = \frac{1}{z^2} \left[-f(v, z) dv^2 - 2dv dz + dx^2 + dy^2\right], \tag{10}$$

with

$$f(v, z) = 1 - \frac{1}{2} \beta^2 z^2 - M(v) z^3 + Q(v) z^4, \tag{11}$$

and

$$A_v = 2Q(v)(1 - z). \tag{12}$$
Now $v$ stands for the ingoing null trajectory, which coincides with the time coordinate $t$ on the conformal boundary. It is easy to check when $v \to \infty$, the above formula denotes a Vaidya-AdS metric (5) while in the limit $v \to -\infty$, it corresponds to a pure AdS spacetime.

Following the strategy of [46], we obtain the above solution (10)-(12) corresponds to the external sources of current and energy-momentum tensor

$$J^{z}_{(ext)} = 2\frac{dQ(v)}{dv}, \tag{13}$$

$$T^{vv}_{(ext)} = z^2\frac{dM(v)}{dv} - 2z^3Q(v)\frac{dQ(v)}{dv}, \tag{14}$$

respectively. It is noticed that in order to probe the time-dependent optical conductivity without translation invariance, more external sources were considered to construct the solution in Vaidya setup of the EMA theory[37]. However, here in our solution, we consider the simple case that only the mass and charge depend on the time but the momentum relaxation coefficient $\beta$ does not vary with time. This is reasonable because as addressed in [38] for five dimensional case that $M$ and $Q$ are integration constants when solving the differential equations while $\beta$ is fixed when sourcing the scalars.

### III. HOLOGRAPHIC SETUP OF HEE AND HC FOR A STRIPE

In this section, we shall address the holographic setup of HEE and HC under a thermal quench in the field theory with momentum relaxation, which is dual to the bulk with axions described in last section. We first consider the subregion with a straight strip geometry

![Figure 1](image1.png)

**FIG. 1:** Geometrical description of the subregion $\mathcal{A}$ (the light blue area) with width $l$ and length $L$. The yellow area indicates the HRT bulk surface $\gamma_{\mathcal{A}}$ while the hypersurface $\Gamma_{\mathcal{A}}$ is the bulk area with boundaries $\mathcal{A}$ and $\gamma_{\mathcal{A}}$.**

described by $\mathcal{A} \equiv \{ x \in \left( -\frac{l}{2}, \frac{l}{2} \right), y \in \left( -\frac{L}{2}, \frac{L}{2} \right) \}$, see Fig.1 where the finite $l$ is the width and $L$ is length of the region $\mathcal{A}$. As proposed in [5] that in the dynamical spacetime, HEE for a subregion $\mathcal{A}$ on the boundary is captured by the HRT bulk surface $\gamma_{\mathcal{A}}$, while the corresponding HC is proportional to the volume of a codimension-one hypersurface $\Gamma_{\mathcal{A}}$ with the boundaries $\mathcal{A}$ and $\gamma_{\mathcal{A}}$. Thus, we will follow the steps in [24] to analytically deduce the...
expressions of HEE via the minimal surface, and HC via CV conjecture of the boundary theory. It is noticed that other probes, two-point correlation functions and HEE with circle profile during thermal quench in this theory have been explored in [47].

Due to the symmetry of the system, one can parameterize the corresponding extremal surface $\gamma_{s\ell}$ in the bulk as

$$v = v(x), \quad z = z(x), \quad z(\pm l/2) = \epsilon, \quad v(\pm l/2) = t - \epsilon,$$

where $\epsilon$ is the UV cut-off. Then the induced metric on the surface is

$$ds^2 = \frac{1}{z^2} \left[ -f(v, z)v'^2 - 2z'v' + 1 \right] dx^2 + \frac{1}{z^2} dy^2,$$

where the prime means taking the derivative to $x$. Then the area of the extremal surface is calculated as

$$\text{Area}(\gamma_{s\ell}) = L \int_{-l/2}^{l/2} \frac{1}{z^2} \left[ -f(v, z)v'^2 - 2z'v' \right] dx.$$

To obtain the HEE, one has to minimize the above area. The trick is to treat the above function as an action with $x$ instead of ‘time’, and so the related Lagrangian and Hamiltonian density are

$$\mathcal{L}_S = \frac{\sqrt{1 - f(v, z)v'^2 - 2z'v'}}{z^2},$$

$$\mathcal{H}_S = \frac{1}{z^2} \sqrt{1 - f(v, z)v'^2 - 2z'v'}.$$

It is obvious that the Hamiltonian does not explicitly depend on the variable $x$, so it is conserved. Beside, the symmetry of the surface implies a turning point $(z_*, v_*)$ at $x = 0$ on the extremal surface $\gamma_{s\ell}$ (see Fig.1 for a fixed $v$). Then we can set

$$v'(0) = z'(0) = 0, \quad z(0) = z_*, \quad v(0) = v_*.$$

Subsequently, the conserved Hamiltonian give us a condition

$$1 - f(v, z)v'^2 - 2z'v' = \frac{z^4}{z^4}.$$

One can take derivative of the Lagrangian (18) with respect to $x$. Combining the derivative equation and the equations of motion for $z(x)$ and for $v(x)$, respectively, we get a group of partial differential equations

$$0 = -4 + 2zv'' + v' \left[ 4f(v, x)v' + 8z' - zv' \partial_z f(v, z) \right],$$

$$0 = 4f(v, z)^2v'^2 + f(v, z) \left[ -4 + 8z'v' + zv'^2 \partial_z f(v, z) \right]$$

$$- z \left[ 2z'' + v' \left( 2z' \partial_z f(v, z) + v' \partial_v f(v, z) \right) \right].$$

By solving the above equations using the boundary conditions (20), one can extract the solutions of $v = \tilde{v}(x), z = \tilde{z}(x)$ for the extremal surface $\gamma_{s\ell}$. Then the area of the extremal surface $\gamma_{s\ell}$ is simplified as

$$\text{Area}(\gamma_{s\ell}) = 2L \int_0^{l/2} \frac{z^2_*}{\tilde{z}(x)^4} dx,$$
which gives the HEE of the subregion on the boundary. It is noted that the surface does not live on a constant time slice for the general \( f(v, z) \), and both \( z_* \) and \( \tilde{z}(x) \) are time dependent.

Using the same profile as HEE, we then further derive the general expression of HC by evaluating the dual volume in the bulk of the background (5), i.e., we should evaluate the volume with the codimension-one extremal surface \( \Gamma_{\mathcal{A}} \) which is bounded by the surface \( \gamma_{\mathcal{A}} \). To this end, we parameterize the extremal bulk region \( \Gamma_{\mathcal{A}} \) enclosed by \( v = \tilde{v}(x), z = \tilde{z}(x) \) via \( z = z(v) \). Thus, the induced metric on \( \Gamma_{\mathcal{A}} \) is

\[
\begin{aligned}
\text{ds}^2 &= \frac{1}{z^2} \left[ -\left( f(v, z) + 2\frac{\partial f}{\partial v} \right) dv^2 + dx^2 + dy^2 \right].
\end{aligned}
\]

Consequently, the volume can be evaluated as

\[
\begin{aligned}
V(\Gamma_{\mathcal{A}}) &= 2L \int_{\nu_*}^{\tilde{v}(l/2)} dv \int_0^{\tilde{x}(v)} dx \left[ -f(v, z) - 2\frac{\partial f}{\partial v} \right]^{1/2},
\end{aligned}
\]

where \( \tilde{x}(v) \) is one coordinate in the codimension-two extremal surface \( \gamma_{\mathcal{A}} \). Using the same trick as before, we treat the above integral function as a Lagrangian, and then the corresponding equation of motion is

\[
\begin{aligned}
0 &= 6f(v, z)^2 + 12z'(v)^2 - 3z(v)z''(v)\partial_z f(v, z) + f(v, z) \left( 18z'(v) - z(v)\partial_z f(v, z) \right) \\
&- z(v) \left( 2z''(v) + \partial_v f(v, z) \right).
\end{aligned}
\]

There are two possible ways to get the solution to the above equation. One is to solve it with the use of the boundary conditions determined by the codimension-two surface \( \gamma_{\mathcal{A}} = (\tilde{v}(x), \tilde{z}(x)) \) and \( \mathcal{A} \). The other is to figure out the solution by seeking \( \tilde{z}(\nu) \) on the boundary \( \gamma_{\mathcal{A}} \). Subsequently, the volume (26) can be further written as

\[
\begin{aligned}
V(\Gamma_{\mathcal{A}}) &= 2L \int_{\nu_*}^{\tilde{v}(l/2)} dv \int_{\gamma(v)}^{\tilde{x}(v)} dx \left[ -f(v, z(v)) - 2\frac{\partial f}{\partial v} \right]^{1/2} \tilde{x}(v),
\end{aligned}
\]

which is dual to HC of the subregion in the boundary.

So far, we have obtained the explicit expressions of HEE (24) and HC (28) of the boundary theory dual to the background (2) under a thermal quench. In the following section we will show our numerical results presenting the evolving behaviors of HEE and HC.

**IV. NUMERICAL RESULTS**

To numerically study the evolution of HEE and HC, we have to solve the equations of motion (22) and (23) with the boundary conditions,

\[
\begin{aligned}
v'(0) = z'(0) = 0, \quad z(0) = z_*, \quad v(0) = v_*, \quad z(l/2) = \epsilon, \quad v(l/2) = t - \epsilon.
\end{aligned}
\]

Conventionally, we are only interested in the finite physical quantities while both HEE and HC evaluated by above conditions are divergent if \( \epsilon \to 0 \). Therefore, we could define the
following finite quantities for HEE and HC
\[
S = \frac{\text{Area}(\gamma_{\mathcal{A}}) - \text{Area}_{\text{AdS}}(\gamma_{\mathcal{A}})}{2l},
\]
\[
C = \frac{V(\Gamma_{\mathcal{A}}) - V_{\text{AdS}}(\Gamma_{\mathcal{A}})}{2l},
\]
where \(\text{Area}(\gamma_{\mathcal{A}})\) and \(V(\Gamma_{\mathcal{A}})\) are defined in equation (24) and (28), respectively. The quantities with subscript \(\text{AdS}\) correspond to the vacuum part dual to the pure AdS geometry.

Next, we present the numerical results for the evolutions of these two quantities with quench. For the sake of the numerical precision, we will set the UV cutoff \(\epsilon = 0.05\), the thickness of the shell \(v_0 = 0.01\) and the mass parameter \(m = 1\) in the calculation.

A. Evolution of HEE and HC in neutral case

To see the role of momentum dissipation plays in the evolution of HEE and HC, we first consider the neutral case, i.e., \(q = 0\) so we can focus on the effect of \(\beta\). In addition, we fix the strip width \(l = 2\) as the first step.

Before presenting the main results of HEE and HC, we first show the evolution of the HRT surface \(\gamma_{\mathcal{A}}\) for \(\beta = 5\) to give an intuitive understanding of the evolution. The left panel in Fig.2 exhibits the evolution in \((x,v,z)\) space, in which \(\gamma_{\mathcal{A}}\) evolves from left to right. In the middle panel of Fig.2, the corresponding projection in \((x,z)\) planer is shown, in which the evolution is from top to bottom. It is obvious that \(\gamma_{\mathcal{A}}\) evolves smoothly from the initial state to the final state. After obtaining the HRT surface \(\gamma_{\mathcal{A}}\), we can then work out the codimension-one surface \(\Gamma_{\mathcal{A}}\), which characterizes the subregion complexity bounded by the HRT surface \(\gamma_{\mathcal{A}}\) as well as subregion \(\mathcal{A}\), which is exhibited in the right of Fig.2. The process is similar with the case in the Einstein gravity in [24].

The evolution of HEE and HC affected by \(\beta\) is shown in Fig.3. Several properties can be read off from the figures. First, the left panel shows that as \(\beta\) goes larger, the turning point \(z^*_s\) is always smaller, and its evolution becomes much more smoother. Then, the corresponding stable HEE in the middle panel for larger \(\beta\) is smaller. In the contrary, the
FIG. 3: Left: The evolution of $z_*$ for different $\beta$. Middle: The evolution of HEE the for different $\beta$. Right: The relation between the HC and $\beta$. Here we have set $l = 2$.

right panel shows that $\beta$ slightly affect the stable value of HC while its peak is explicitly higher.

We then study the effect of $\beta$ on HEE and HC with wide subregion, i.e., $l = 5$. The results are plotted in Fig.4.

For $\beta = 0$, we reproduced the result of four dimensional Schwashiz black hole [24], which is denoted by the blue lines in each plot. For different $\beta$, HEE has swallow-tail behaviors and HC has the multi-valued regions, but only solid lines describe the physical procedure. This phenomena can be explained by the evolution of $\gamma_A$ shown in Fig.5, which is no longer a continuous function and different from the case of the small width strips. The discontinuous evolution, i.e., the dropping behavior is related with the jump in the minimal area surface from phenomena perspective.

FIG. 4: Left: The evolution of $z_*$ for different $\beta$. Middle: The evolution of HEE for different $\beta$. Right: The relation between the HC and $\beta$. Here we have set $l = 5$.

For non-vanishing $\beta$, we find that the effect of momentum relaxation for wider strip is more explicit than that for narrow strip by comparing Fig.3 and Fig.4. More interesting properties can be observed, which are summarized as follows:

- The turning point $z_*$, HEE and HC all tend to be constants as time evolves, and $\beta$ definitely affect their stable values. Specially, as $\beta$ increases, all the stable values become smaller.

- A novel feature is that as $\beta$ increases to be a certain value, the swallow tail behavior in HEE and the multi-valued region in HC disappear as shown in Fig.4. In other words, the evolution of $\gamma_A$ recovers to be continue. It implies that comparing to the case in Einstein gravity, in this model with axion field, the system with wide strip can emerge discontinue evolution of $\gamma_A$. 
Fig. 5: Left: The evolution of HRT surface in \((x, v, z)\) space. Right: The projection of HRT surface in \((x, z)\) planer. Here we have set \(\beta = 0\) and \(l = 5\).

- Fig. 3 shows that for \(l = 2\), the larger \(\beta\) promotes the system to become equilibrium. That is to say, larger \(\beta\) need less time to reach the stable state in thermalization. While for \(l = 5\) in Fig. 4, larger \(\beta\) corresponds to longer time to become stable. Moreover, larger \(\beta\) in this case is related with lower stable value of both HEE and HC, which is also different from that \(\beta\) suppresses the stable HEE but promotes HC for \(l = 2\). This property suggests that if we fix \(q\) in advance, the time that HEE or HC needs to reach the stable values may not monotonically depend on \(\beta\) for fixed \(l\), and vice versa.

- The systems with large \(l\) need more time to reach stable than those with small \(l\). This rule can also be explicitly extracted from Fig. 6 where we present the evolution of HEE and HC with different \(l\) for fixed \(\beta = 1\). This feature is similar as that in Einstein case[24] and Einstein-Born-Infeld case[27]. Theoretically, one can understand it in terms of the entanglement tsunami as studied in [48, 49]. In details, as discussed in [27], the spacetime during tsunami is divided into two parts by the infalling thin shell, i.e., the AdS-Schwarzschild region and the pure AdS region. The tsunami sweeps the AdS-Schwarzschild region but the pure AdS region is not affected. Consequently, the HRT surface is also divided into two parts and the part in the pure AdS region is located at a time slice, while the other part in the Schwarzschild region is not. It is obvious that the tip of the HRT surfaces with large \(l\) stretches deeper into the bulk, so the infalling shell takes longer time to catch this location. Therefore, the location of HRT surfaces with larger \(l\) spend more time to reach its stable state.

B. Evolution of HEE and HC in charged case

In this section, we shall turn on the charge in this model and study the joined effect of \(\beta\) and \(q\) on the evolution of HEE and HC. It is noticed that the evolution of HEE and HC affected by the charge in four dimensional Einstein theory and Einstein-Born-Infeld theory has been studied in [27]. We first repeat their results and then turn on \(\beta\) to see the effect of charge in the model with momentum
relaxation. Our results are shown in Fig.7 for a small width ($l = 2$) and in Fig.8 for a large width ($l = 5$). From the figures, we see that the charged black holes need longer time to become stable, and larger charge corresponds to smaller stable HEE but bigger stable HC. This phenomena are similar as the existed observes in [27].

The shallow tail behaviors and multi-valued regions are clearer than $l = 2$ case.

Comparing Fig.7 and Fig.8, it is obvious that, again, larger size of width brings in swallow-tails in HEE and multi-values in HC due to the discontinue behavior of minimum area.
Moreover, Fig.8 shows that larger $q$ makes the discontinue behavior more evident, which is similar to that found in [27, 28].

In the charged case with $q = 0.5$, we show the evolution of HEE and HC for different $\beta$ in Fig.9 ($l = 2$) and Fig.10 ($l = 5$). For $l = 2$, the evolution is still continue as shown in Fig.9. While in Fig.10, they are discontinue and $\beta$ also suppresses the effect of $l$, so that the swallow tail and multi-values behavior disappear as we increase $\beta$, which is similar to what we observe in neutral case.

![Graphs showing the evolution of HEE and HC for different \( \beta \) and \( l \)](image)

FIG. 9: Left: The evolution of HEE the for different $\beta$. Right: The evolution of HC for different $\beta$. Here we have set $l=2$, $q = 0.5$.

![Graphs showing the evolution of HEE and HC for different \( \beta \) and \( l \)](image)

FIG. 10: Left: The evolution of HEE for different $\beta$. Right: The evolution of HC for different $\beta$. Here we have set $l=5$, $q = 0.5$. Clearly, larger $\beta$ restrains the swallow tails and multi-valued regions.

V. CONCLUSION AND DISCUSSION

In this paper, we studied the evolution of HEE and HC under a thermal quench in EMA theory. In this theory, the black brane solution is homogeneous but the dual boundary theory has momentum dispersion because the spacial dependent axion fields provide the source to break the translation symmetry. We mainly investigated the effects of the momentum relaxation and the charge of the black brane on the evolution of HEE and HC.
In neutral case, when $l$ is tuned large, the evolutions of the HEE and the HC behave from a continuous function into a discontinuous one. For the continuous case, the HEE increases at the first stage and then it arrives at a stable final region, while HC grows until it arrives at a maximum point, and then after that it quickly drops to become a stable final stage. For the discontinuous case, a swallow tail appears in the evolution of HEE while correspondingly, a multi-valued behavior can be seen in HC. This picture is the same as that found in [24, 27]. However, as the momentum relaxation increases, the swallow tail behavior in HEE and the multi-valued region in HC disappear, i.e. the continuous evolutions will be recovered. This denotes that in this model, the system with larger width would emerge discontinuous evolution of $\gamma_{\alpha\beta}$ in contrast to that in Einstein theory. Moreover, large $\beta$ corresponds to smaller stable value of both HEE and HC, while the time that HEE or HC needs to reach the stable values is not monotonically depend on $\beta$ for fixed $l$, and vice versa.

We found that the charge would make the discontinue behaviour in HEE and HC more explicit at large $l$. That’s to say, for bigger charge, the system with narrower size could emerge discontinuity in the evolution. This phenomena has also been observed in [27, 28]. Similar as that occurred in neutral case, when we increase the momentum relaxation, the swallow tail in HEE and the multi-valued region in HC become weaken and finally the continuous evolutions would be recovered.

In all of the evolutions we plotted, it is obvious that at the early stage the behaviour of evolution are almost the same, i.e., they almost does not depend on the parameters including the size the the strip. This is because the growth of complexity stems from the local operator excitations. The possible bound of HEE growth rate for various stages of evolution has been discussed in [49], while the Lloyd bound for HC growth rate $dC(t)/dt \leq 2M/\pi$, where $M$ is total mass of the system at any time $t$, has been addressed in [8]. Thus, it would be very interesting to carefully analyze the growth rate in various stages of the evolution of HEE and HC in our model, and compare them with the related bounds. This work is under progress.

Besides the HEE and HC, our studies can be extended into the related quantities we mentioned in the introduction, i.e., EoP, CoP and logarithmic negativity and their evolution under the thermal quench using similar methods. EoP of mixed state in the dual theory of this model with momentum relaxation has been studied in [50] very recently. The evolutions of EoP in this setup of Einstein gravity can be seen in [51], and it is natural to ask how its evolution will be affected when the momentum dispersion is involved in the system.

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[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] [hep-th/9711200].
[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428, 105 (1998) [hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].

[4] T. Takayanagi, “Entanglement Entropy from a Holographic Viewpoint,” Class. Quant. Grav. 29, 153001 (2012) [arXiv:1204.2450 [gr-qc]].

[5] V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP 0707 (2007) 062 [arXiv:0705.0016 [hep-th]].

[6] D. Stanford and L. Susskind, “Complexity and Shock Wave Geometries,” Phys. Rev. D 90, no. 12, 126007 (2014) [arXiv:1406.2678 [hep-th]].

[7] L. Susskind and Y. Zhao, “Switchbacks and the Bridge to Nowhere,” arXiv:1408.2823 [hep-th].

[8] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle and Y. Zhao, “Holographic Complexity Equals Bulk Action?,” Phys. Rev. Lett. 116, no. 19, 191301 (2016) [arXiv:1509.07876 [hep-th]].

[9] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle and Y. Zhao, “Complexity, action, and black holes,” Phys. Rev. D 93, no. 8, 086006 (2016) [arXiv:1512.04993 [hep-th]].

[10] Barbara M. Terhal, Michal Horodecki, Debbie W. Leung, David P. DiVincenzo, “The entanglement of purification,” J. Math. Phys. 43, 4286–4298 (2002) [arXiv:quant-ph/0202044].

[11] T. Takayanagi and K. Umemoto, “Entanglement of purification through holographic duality,” Nature Phys. 14, no. 6, 573 (2018) [arXiv:1708.09393 [hep-th]].

[12] K. Umemoto and Y. Zhou, “Entanglement of Purification for Multipartite States and its Holographic Dual,” JHEP 1810, 152 (2018) [arXiv:1805.02625 [hep-th]].

[13] P. Liu, Y. Ling, C. Niu and J. P. Wu, “Entanglement of Purification in Holographic Systems,” JHEP 1909 (2019) 071 [arXiv:1902.02243 [hep-th]].

[14] M. Ghodrati, X. M. Kuang, B. Wang, C. Y. Zhang and Y. T. Zhou, “The connection between holographic entanglement and complexity of purification,” JHEP 1909 (2019) 009 [arXiv:1902.02475 [hep-th]].

[15] J. Harper and M. Headrick, “Bit threads and holographic entanglement of purification,” arXiv:1906.05970 [hep-th].

[16] N. Bao, A. Chatwin-Davies, J. Pollack and G. N. Remmen, “Towards a Bit Threads Derivation of Holographic Entanglement of Purification,” JHEP 1907, 152 (2019) [arXiv:1905.04317 [hep-]
[17] D. H. Du, C. B. Chen and F. W. Shu, “Bit threads and holographic entanglement of purification,” arXiv:1904.06871 [hep-th].

[18] C. A. Agon, M. Headrick and B. Swingle, “Subsystem Complexity and Holography,” JHEP 1902, 145 (2019) [arXiv:1804.01561 [hep-th]].

[19] M. B. Plenio, “Logarithmic Negativity: A Full Entanglement Monotone That is not Convex,” Phys. Rev. Lett. 95, 119902 (2005) [arXiv:1512.04993 [hep-th]].

[20] Y. Kusuki, J. Kudler-Flam and S. Ryu, “Derivation of holographic negativity in $\text{AdS}_3/\text{CFT}_2$,” arXiv:1907.07824 [hep-th].

[21] J. Kudler-Flam and S. Ryu, “Entanglement negativity and minimal entanglement wedge cross sections in holographic theories,” Phys. Rev. D 99, no. 10, 106014 (2019) [arXiv:1808.00446 [hep-th]].

[22] L. Susskind, “Computational Complexity and Black Hole Horizons,” Fortsch. Phys. 64, 44 (2016) [arXiv:1403.5695 [hep-th]].

[23] L. Susskind, “Entanglement is not enough,” Fortsch. Phys. 64, 49 (2016) [arXiv:1411.0690 [hep-th]].

[24] B. Chen, W. M. Li, R. Q. Yang, C. Y. Zhang and S. J. Zhang, “Holographic subregion complexity under a thermal quench,” JHEP 1807, 034 (2018) [arXiv:1803.06680 [hep-th]].

[25] R. Auzzi, G. Nardelli, F. I. Schaposnik Massolo, G. Tallarita and N. Zenoni, “On volume subregion complexity in Vaidya spacetime,” arXiv:1908.10832 [hep-th].

[26] Y. Ling, Y. Liu, C. Niu, Y. Xiao and C. Y. Zhang, “Holographic Subregion Complexity in General Vaidya Geometry,” JHEP 1911 (2019) 039 [arXiv:1908.06432 [hep-th]].

[27] Y. Ling, Y. Liu and C. Y. Zhang, “Holographic Subregion Complexity in Einstein-Born-Infeld theory,” Eur. Phys. J. C 79, no. 3, 194 (2019) [arXiv:1808.10169 [hep-th]].

[28] Y. T. Zhou, M. Ghodrati, X. M. Kuang and J. P. Wu, “Evolutions of entanglement and complexity after a thermal quench in massive gravity theory,” Phys. Rev. D 100 (2019) no.6, 066003 [arXiv:1907.08453 [hep-th]].

[29] R. Q. Yang, K. Y. Kim, “Time evolution of the complexity in chaotic systems: concrete examples,” arXiv:1906.02052 [hep-th].

[30] S. J. Zhang, “Subregion complexity in holographic thermalization with dS boundary,” Eur. Phys. J. C 79 (2019) no.8, 715 [arXiv:1905.10605 [hep-th]].
[31] S. Leichenauer and M. Moosa, “Entanglement Tsunami in (1+1)-Dimensions,” Phys. Rev. D 92 (2015) 126004 [arXiv:1505.04225 [hep-th]].

[32] S. Leichenauer, M. Moosa and M. Smolkin, “Dynamics of the Area Law of Entanglement Entropy,” JHEP 1609 (2016) 035 [arXiv:1604.00388 [hep-th]].

[33] Z. Y. Fan and M. Guo, “Holographic complexity under a global quantum quench,” arXiv:1811.01473 [hep-th].

[34] J. Chu, R. Qi and Y. Zhou, “Generalizations of Reflected Entropy and the Holographic Dual,” arXiv:1909.10456 [hep-th].

[35] M. Moosa, “Divergences in the rate of complexification,” Phys. Rev. D 97, no. 10, 106016 (2018) [arXiv:1712.07137 [hep-th]].

[36] T. Andrade and B. Withers, “A simple holographic model of momentum relaxation,” JHEP 1405, 101 (2014) [arXiv:1311.5157 [hep-th]].

[37] A. Bagrov, B. Craps, F. Galli, V. Ker?nen, E. Keski-Vakkuri and J. Zaanen, “Holography and thermalization in optical pump-probe spectroscopy,” Phys. Rev. D 97 (2018) no.8, 086005 [arXiv:1708.08279 [hep-th]].

[38] J. Fernndez-Pends and K. Landsteiner, “Out of equilibrium chiral magnetic effect and momentum relaxation in holography,” arXiv:1907.09962 [hep-th].

[39] V. Balasubramanian et al., “Inhomogeneous Thermalization in Strongly Coupled Field Theories,” Phys. Rev. Lett. 111 (2013) 231602 [arXiv:1307.1487 [hep-th]].

[40] L. Alberte, M. Baggioli, A. Khmelnitsky and O. Pujolas, “Solid Holography and Massive Gravity,” JHEP 1602 (2016) 114 [arXiv:1510.09089 [hep-th]].

[41] L. Q. Fang and X. M. Kuang, “Holographic heat engine with momentum relaxation,” Sci. China Phys. Mech. Astron. 61, 080421 (2018) [arXiv:1710.09054 [hep-th]].

[42] A. Cisterna, S. Q. Hu and X. M. Kuang, “Joule-Thomson expansion in AdS black holes with momentum relaxation,” arXiv:1808.07392 [gr-qc].

[43] S. Q. Hu and X. M. Kuang, “Holographic heat engine in Horndeski model with the k-essence sector,” Sci. China Phys. Mech. Astron. 62 (2019) no.6, 60411 [arXiv:1808.00176 [hep-th]].

[44] D. Galante and M. Schvellinger, “Thermalization with a chemical potential from AdS spaces,” JHEP 1207 (2012) 096 [arXiv:1205.1548 [hep-th]].

[45] V. Balasubramanian et al., “Holographic Thermalization,” Phys. Rev. D 84 (2011) 026010 [arXiv:1103.2683 [hep-th]].
[46] E. Caceres and A. Kundu, “Holographic Thermalization with Chemical Potential,” JHEP 1209 (2012) 055 [arXiv:1205.2354 [hep-th]].

[47] Y. Z. Li, X. M. Kuang, “Probes of holographic thermalization in a simple model with momentum relaxation,” [arXiv:1911.11980 [hep-th]].

[48] H. Liu and S. J. Suh, “Entanglement Tsunami: Universal Scaling in Holographic Thermalization,” Phys. Rev. Lett. 112 (2014) 011601 [arXiv:1305.7244 [hep-th]].

[49] H. Liu and S. J. Suh, “Entanglement growth during thermalization in holographic systems,” Phys. Rev. D 89 (2014) no.6, 066012 [arXiv:1311.1200 [hep-th]].

[50] Y. f. Huang, Z. j. Shi, C. Niu, C. y. Zhang and P. Liu, “Mixed State Entanglement for Holographic Axion Model,” arXiv:1911.10977 [hep-th].

[51] R. Q. Yang, C. Y. Zhang and W. M. Li, “Holographic entanglement of purification for thermofield double states and thermal quench,” JHEP 1901 (2019) 114 [arXiv:1810.00420 [hep-th]].