Positive matrices partitioned into a small number of Hermitian blocks

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Abstract
Positive semidefinite matrices partitioned into a small number of Hermitian blocks have a remarkable property. Such a matrix may be written in a simple way from the sum of its diagonal blocks: the full matrix is a kind of average of copies of the sum of the diagonal blocks. This entails several eigenvalue inequalities. The proofs use a decomposition lemma for positive matrices, isometries with complex entries, and the Pauli matrices.

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1 Introduction

Positive matrices partitioned into blocks frequently occur as basic tools or for their own interest in matrix analysis and mathematical physics. For instance, to define the geometric mean of two $n \times n$ matrices $A, B \in \mathbb{M}_n^+$, the space of positive semidefinite matrices, one consider the class of block-matrices

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix}$$

(1.1)

belonging to $\mathbb{M}_{2n}^+$ (hence $X$ is Hermitian). The geometric mean of $A$ and $B$ is then characterized as the largest possible $X$ such that (1.1) is positive. Positive matrices $H$ of the form (1.1) enjoy a remarkable property: for all symmetric norms

$$\|H\| \leq \|A + B\|.$$  

(1.2)

This says that we have a majorisation between $H$ and the sum of its diagonal block,

$$\sum_{j=1}^{k} \lambda_j(H) \leq \sum_{j=1}^{k} \lambda_j(A + B), \quad k = 1, \ldots, 2n$$  

(1.3)

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where \( \lambda_j(T) \) is the \( j \)th eigenvalue of \( T \in \mathbb{M}^+_m \) (in decreasing order, with the convention \( \lambda_j(T) = 0 \) for \( j > m \)).

Another typical example of positive matrices written in blocks are formed by tensor products. Indeed, the tensor product \( A \otimes B \) of \( A \in \mathbb{M}_\beta \) with \( B \in \mathbb{M}_n \) can be identified with an element of \( \mathbb{M}_\beta(\mathbb{M}_n) = \mathbb{M}_{\beta n} \). Starting with positive matrices in \( \mathbb{M}^+_\beta \) and \( \mathbb{M}^+_n \) we then get a matrix in \( \mathbb{M}^+_{\beta n} \) partitioned in blocks in \( \mathbb{M}_n \). In quantum physics, sums of tensor products of positive semi-definite (with trace one) occur as so-called separable states. In this setting, the sum of the diagonal block is called the partial trace (with respect to \( \mathbb{M}_\beta \)). Hiroshima \([6]\) proved a beautiful extension of (1.2)-(1.3):

**Theorem 1.1.** Let \( H = [A_{s,t}] \in \mathbb{M}^+_{n \times n} \) be partitioned into \( \alpha \times \alpha \) Hermitian blocks in \( \mathbb{M}_n \) and let \( \Delta = \sum_{s=1}^\alpha A_{s,s} \) be its partial trace. Then, we have

\[
\|H\| \leq \|\Delta\|
\]

for all symmetric norms.

This result seems to be not so well-known among matrix-functional analysts. We recently rediscovered it and actually obtained a stronger decomposition theorem \([4]\). For small partitions, when \( \alpha \in \{2, 3, 4\} \), special proofs are available that differ from the general case in two related ways. First, these proofs are simpler (especially for \( \alpha = 2 \)) but also yield a much sharper decomposition than the one we can obtain in the general case. Secondly, and this is rather surprising, even though we consider a positive matrix with real entries, its decomposition involves some complex matrices. This note is concerned with these special decompositions for small partitions.

In the next section we will see what can be said for \( \alpha = 2 \). This situation was already implicitly covered in some proofs given in our first note \([3]\). Section 3 deals with the case \( \alpha = 4 \) (and as a byproduct the case \( \alpha = 3 \)).

## 2 Two by two blocks

For partitions of positive matrices, the diagonal blocks play a special role. This is apparent in a rather striking decomposition due to the two first authors \([2]\):

**Lemma 2.1.** For every matrix in \( \mathbb{M}^+_{n+m} \) partitioned into blocks, we have a decomposition

\[
\begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} = U \begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix} U^* + V \begin{bmatrix}
0 & 0 \\
0 & B
\end{bmatrix} V^*
\]

for some unitaries \( U, V \in \mathbb{M}_{n+m} \).

This decomposition turned out to be an efficient tool and it also plays a major role below. A proof and several consequences can be found in \([2]\) and \([1]\). Of course, \( \mathbb{M}_n \) is the algebra of \( n \times n \) matrices with real or complex entries, and \( \mathbb{M}^+_n \) is the positive
part. That is, $M_n$ may stand either for $M_n(R)$, the matrices with real entries, or for $M_n(C)$, those with complex entries. The situation is different in the next statement, where complex entries seem unavoidable.

**Theorem 2.2.** Given any matrix in $M_{2n}^+(C)$ partitioned into blocks in $M_n(C)$ with Hermitian off-diagonal blocks, we have
\[
\begin{bmatrix} A & X \\ X & B \end{bmatrix} = \frac{1}{2} \{ U(A + B)U^* + V(A + B)V^* \}
\]
for some isometries $U, V \in M_{2n,n}(C)$.

Here $M_{p,q}(C)$ denote the space of $p$ rows and $q$ columns matrices with complex entries, and $V \in M_{p,q}(C)$ is an isometry if $p \geq q$ and $V^*V = I_q$. Even for a matrix in $M_{2n}^+(R)$, it seems essential to use isometries with complex entries.

Theorem 2.2 is implicit in [3]. We detail here how it follows from Lemma 2.1.

**Proof.** Taking the unitary matrix
\[
W = \frac{1}{\sqrt{2}} \begin{bmatrix} -iI & iI \\ I & I \end{bmatrix},
\]
where $I$ is the identity of $M_n$, then
\[
W^* \begin{bmatrix} A & X \\ X & B \end{bmatrix} W = \frac{1}{2} \begin{bmatrix} A + B & * \\ * & A + B \end{bmatrix}
\]
where $*$ stands for unspecified entries. By Lemma 2.1 there are two unitaries $U, V \in M_{2n}$ partitioned into equally sized matrices,
\[
U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}
\]
such that
\[
\frac{1}{2} \begin{bmatrix} A + B & * \\ * & A + B \end{bmatrix} = \frac{1}{2} \left\{ U \begin{bmatrix} A + B & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & A + B \end{bmatrix} V^* \right\}.
\]
Therefore
\[
\frac{1}{2} \begin{bmatrix} A + B & * \\ * & A + B \end{bmatrix} = \frac{1}{2} \left\{ \tilde{U}(A + B)\tilde{U}^* + \tilde{V}(A + B)\tilde{V}^* \right\}
\]
where
\[
\tilde{U} = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} \quad \text{and} \quad \tilde{V} = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}
\]
are isometries. The proof is complete by assigning $W\tilde{U}, W\tilde{V}$ to new isometries $U, V$, respectively. \qed
Theorem 2.2 yields (1.2). As a consequence of this inequality we have a refinement of a well-known determinantal inequality,

$$\det(I + A) \det(I + B) \geq \det(I + A + B)$$

for all $A, B \in \mathbb{M}_n^+$.

**Corollary 2.3.** Let $A, B \in \mathbb{M}_n^+$. For any Hermitian $X \in \mathbb{M}_n$ such that $H = \begin{bmatrix} A & X \\ X & B \end{bmatrix}$ is positive semi-definite, we have

$$\det(I + A) \det(I + B) \geq \det(I + H) \geq \det(I + A + B).$$

Here $I$ denotes both the identity of $\mathbb{M}_n$ and $\mathbb{M}_{2n}$. Note that equality obviously occurs in the first inequality when $X = 0$, and equality occurs in the second inequality when $AB = BA$ and $X = A^{1/2}B^{1/2}$.

**Proof.** The left inequality is a special case of Fisher’s inequality,

$$\det X \det Y \geq \det \begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$$

for any partitioned positive semi-definite matrix. The second inequality follows from (1.2). Indeed, the majorisation $S \prec T$ in $\mathbb{M}_n^+$ entails the trace inequality

$$\text{Tr} f(S) \geq \text{Tr} f(T)$$

(2.1)

for all concave functions $f(t)$ defined on $[0, \infty)$. Using (2.1) with $f(t) = \log(1 + t)$ and the relation $H \prec A + B$ we have

$$\det(I + H) = \exp \text{Tr} \log(I + H) \geq \exp \text{Tr} \log(I + ((A + B) \oplus 0_n)) = \det(I + A + B).$$

\[ \square \]

Theorem 2.2 says more than the eigenvalue majorisation (1.3). We have a few other eigenvalue inequalities as follows.

**Corollary 2.4.** Let $H = \begin{bmatrix} A & X \\ X & B \end{bmatrix} \in \mathbb{M}_{2n}^+$ be partitioned into Hermitian blocks in $\mathbb{M}_n$. Then, we have

$$\lambda_{1+2k}(H) \leq \lambda_{1+k}(A + B)$$

for all $k = 0, \ldots, n - 1$. 

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Proof. Together with Theorem 2.2, the alleged inequalities follow immediately from a simple fact, Weyl’s theorem: if $Y, Z \in \mathbb{M}_m$ are Hermitian, then

$$
\lambda_{r+s+1}(Y + Z) \leq \lambda_{r+1}(Y) + \lambda_{s+1}(Z)
$$

for all nonnegative integers $r, s$ such that $r + s \leq m - 1$. \qed

**Corollary 2.5.** Let $S, T \in \mathbb{M}_n$ be Hermitian. Then,

$$
\|T^2 + ST^2S\| \leq \|T^2 + TS^2T\|
$$

for all symmetric norms, and

$$
\lambda_{1+2k}(T^2 + ST^2S) \leq \lambda_{1+k}(T^2 + TS^2T)
$$

for all $k = 0, \ldots, n - 1$.

Proof. The nonzero eigenvalues of $T^2 + ST^2S = [T \quad ST][T \quad ST]^*$ are the same as those of

$$
[T \quad ST]^*[T \quad ST] = \begin{bmatrix}
T^2 & TST \\
TST & TS^2T
\end{bmatrix}.
$$

This block-matrix is of course positive and has its off-diagonal blocks Hermitian. Therefore, the norm inequality follows from (1.2), and the eigenvalue inequalities from Corollary 2.4. The norm inequality was first observed in [5]. \qed

### 3 Quaternions and 4-by-4 blocks

Theorem 2.2 refines Hiroshima’s theorem in case of two by two blocks. Some interesting new eigenvalue inequalities are obtained. How to get a similar result for partitions into a larger number of blocks? The question whether a positive block-matrix $H$ in $\mathbb{M}_{3n}^+$,

$$
H = \begin{bmatrix}
A & X & Y \\
X & B & Z \\
Y & Z & C
\end{bmatrix}
$$

with Hermitian off-diagonal blocks $X, Y, Z$, can be decomposed as

$$
H = \frac{1}{3}\{U\Delta U^* + V\Delta V^* + W\Delta W^*\}
$$

where $\Delta = A + B + C$ and $U, V, W$ are isometries, is a difficult one. However, we will give a rather satisfactory answer by considering direct sums. We have been unable to find any direct proof for partitions in 3-by-3 blocks. The key idea was then to introduce quaternions and to deal with 4-by-4 partitions. This approach leads to the following theorem.
Theorem 3.1. Let $H = [A_{s,t}] \in \mathbb{M}_{\beta n}^+(\mathbb{C})$ be partitioned into Hermitian blocks in $\mathbb{M}_n(\mathbb{C})$ with $\beta \in \{3, 4\}$ and let $\Delta = \sum_{s=1}^{\beta} A_{s,s}$ be the sum of its diagonal blocks. Then,

$$H \oplus H = \frac{1}{4} \sum_{k=1}^{4} V_k (\Delta \oplus \Delta) V_k^*$$

for some isometries $V_k \in \mathbb{M}_{2\beta,2n}(\mathbb{C})$, $k = 1, 2, 3, 4$.

Note that, for $\alpha = \beta \in \{3, 4\}$, Theorem 3.1 considerably improves Theorem 1.1. Indeed, Theorem 3.1 implies the majorisation $\|H \oplus H\| \leq \|\Delta \oplus \Delta\|$ which is equivalent to the majorisation of Theorem 1.1 $\|H\| \leq \|\Delta\|$.

Likewise for Theorem 2.2, we must consider isometries with complex entries, even for a full matrix $H$ with real entries. In [4] we will develop a real approach for real matrices. The isometries are then with real coefficients, but the proof is more intricate and the result is not so simple since it requires direct sums of sixteen copies: we obtain a decomposition of $\oplus^{16} H$ in term of $\oplus^{16} \Delta$.

Before turning to the proof, we recall some facts about quaternions.

The algebra $\mathbb{H}$ of quaternions is an associative real division algebra of dimension four containing $\mathbb{C}$ as a sub-algebra. Quaternions $q$ are usually written as

$$q = a + bi + cj + dk$$

with $a, b, c, d \in \mathbb{R}$ and $a + bi \in \mathbb{C}$. The quaternion units $1, i, j, k$ satisfy

$$i^2 = j^2 = k^2 = ik = -1.$$ 

The algebra $\mathbb{H}$ can be represented as the real sub-algebra of $\mathbb{M}_2$ consisting of matrices of the form

$$\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

by the identification map

$$a + bi + cj + dk \mapsto \begin{pmatrix} a + bi & ic - d \\ ic + d & a - ib \end{pmatrix}.$$ 

The quaternion units $1, i, j, k$ are then represented by the matrices (related to the Pauli matrices),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(3.1)

that we will use in the following proof of Theorem 3.1.

We will work with matrices in $\mathbb{M}_{8n}$ partitioned in 4-by-4 blocks in $\mathbb{M}_{2n}$.

Proof. It suffices to consider the case $\beta = 4$, the case $\beta = 3$ follows by completing $H$ with some zero columns and rows.
First, replace the positive block matrix $H = [A_{s,t}]$ where $1 \leq s, t, \leq 4$ and all blocks are Hermitian by a bigger one in which each block is counted two times:

$$G = [G_{s,t}] := [A_{s,t} \oplus A_{s,t}].$$

Thus $G \in M_{8n}(\mathbb{C})$ is written in 4-by-4 blocks in $M_{2n}(\mathbb{C})$. Then perform a unitary congruence with the matrix

$$W = E_1 \oplus E_2 \oplus E_3 \oplus E_4$$

where the $E_i$ are the analogues of quaternion units, that is, with $I$ the identity of $M_n(\mathbb{C})$,

$$E_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad E_2 = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & iI \\ iI & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Note that $E_s E_t^*$ is skew-Hermitian whenever $s \neq t$. A direct matrix computation then shows that the block matrix

$$\Omega := W G W^* = [\Omega_{s,t}]$$

has the following property for its off-diagonal blocks: For $1 \leq s < t \leq 4$

$$\Omega_{s,t} = -\Omega_{t,s}.$$

Using this property we compute the unitary congruence implemented by

$$R_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and we observe that $R_2 \Omega R_2^*$ has its four diagonal blocks $(R_2 \Omega R_2^*)_{k,k}$, $1 \leq k \leq 4$, all equal to the matrix $D \in M_{2n}(\mathbb{C})$,

$$D = \frac{1}{4} \sum_{s=1}^{4} A_{s,s} \oplus A_{s,s}.$$

Let $\Gamma = D \oplus 0_{6n} \in M_{8n}$. Thanks to the decomposition of Lemma 2.1 there exist some unitaries $U_i \in M_{8n}(\mathbb{C})$, $1 \leq i \leq 4$, such that

$$\Omega = \sum_{i=1}^{4} U_i \Gamma U_i^*.$$

That is, since $\Omega$ is unitarily equivalent to $H \oplus H$, and $\Gamma = W D W^*$ for some isometry $W \in M_{8n,2n}(\mathbb{C})$,

$$H \oplus H = \sum_{s=1}^{4} V_k D V_k^*$$

for some isometries $V_k \in M_{8n,2n}(\mathbb{C})$. Since $D = \frac{1}{4} \Delta \oplus \Delta$, the proof is complete. $\square$
In the same vein as in Section 2, we have the following consequences.

**Corollary 3.2.** Let \( H = [A_{s,t}] \in \mathbb{M}_{\beta n}^+ \) be written in Hermitian blocks in \( \mathbb{M}_n \) with \( \beta \in \{3,4\} \) and let \( \Delta = \sum_{s=1}^{\beta} A_{s,s} \) be the sum of its diagonal blocks. Then,
\[
\prod_{s=1}^{\beta} \det(I + A_{ss}) \geq \det(I + H) \geq \det \left( I + \sum_{s=1}^{\beta} A_{ss} \right).
\]

**Corollary 3.3.** Let \( H = [A_{s,t}] \in \mathbb{M}_{\beta n}^+ \) be written in Hermitian blocks in \( \mathbb{M}_n \) with \( \beta \in \{3,4\} \) and let \( \Delta = \sum_{s=1}^{\beta} A_{s,s} \) be the sum of its diagonal blocks. Then,
\[
\lambda_{1+4k}(H) \leq \lambda_{1+k}(A + B)
\]
for all \( k = 0, \ldots, n - 1 \).

**Corollary 3.4.** Let \( T \in \mathbb{M}_n \) be Hermitian and let \( \{S_i\}_{i=1}^{\beta} \in \mathbb{M}_n \) be commuting Hermitian matrices with \( \beta \in \{3,4\} \). Then,
\[
\left\| \sum_{i=1}^{\beta} S_i T^2 S_i \right\| \leq \left\| \sum_{i=1}^{\beta} TS_i^2 T \right\|
\]
for all symmetric norms, and
\[
\lambda_{1+4k} \left( \sum_{i=1}^{\beta} S_i T^2 S_i \right) \leq \lambda_{1+k} \left( \sum_{i=1}^{\beta} TS_i^2 T \right)
\]
for all \( k = 0, \ldots, n - 1 \).

The proofs of these corollaries are quite similar of those of Section 2. We give details only for the norm inequality of Corollary 3.4.

**Proof.** We may assume that \( \beta = 4 \) by completing, if necessary with \( S_4 = 0 \). So, let \( T \in \mathbb{M}_n^+ \) and let \( \{S_i\}_{i=1}^{4} \) be four commuting Hermitian matrices in \( \mathbb{M}_n \). Then
\[
H = XX^* = \begin{bmatrix} TS_1 \\ TS_2 \\ TS_3 \\ TS_4 \end{bmatrix} = \begin{bmatrix} S_1T & S_2T & S_3T & S_4T \end{bmatrix}
\]
is positive and partitioned into Hermitian blocks with diagonal blocks \( TS_i^2 T \), \( 1 \leq i \leq 4 \). Thus, from Theorem 3.1, for all symmetric norms,
\[
\|H \oplus H\| \leq \left\| \sum_{i=1}^{4} TS_i^2 T \right\| \oplus \left\| \sum_{i=1}^{4} TS_i^2 T \right\|
\]
or equivalently
\[ \| H \| \leq \left\| \sum_{i=1}^{4} T S_i T \right\| \]

Since \( H = XX^* \) and \( X^*X = \sum_{i=1}^{4} S_i T^2 S_i \), the norm inequality of Corollary 3.4 follows.

\[ \square \]

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