QUADRATIC GRÖBNER BASES OF TWINNED ORDER POLYTOPES

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ABSTRACT. Let $P$ and $Q$ be finite partially ordered sets on $[d] = \{1, \ldots, d\}$, and $\mathcal{O}(P) \subset \mathbb{R}^d$ and $\mathcal{O}(Q) \subset \mathbb{R}^d$ their order polytopes. The twinned order polytope of $P$ and $Q$ is the convex polytope $\Delta(P, Q) \subset \mathbb{R}^d$ which is the convex hull of $\mathcal{O}(P) \cup (-\mathcal{O}(Q))$. It follows that the origin of $\mathbb{R}^d$ belongs to the interior of $\Delta(P, Q)$ if and only if $P$ and $Q$ possess a common linear extension. It will be proved that, when the origin of $\mathbb{R}^d$ belongs to the interior of $\Delta(P, Q)$, the toric ideal of $\Delta(P, Q)$ possesses a quadratic Gröbner basis with respect to a reverse lexicographic order for which the variable corresponding to the origin is smallest. Thus in particular if $P$ and $Q$ possess a common linear extension, then the twinned order polytope $\Delta(P, Q)$ is a normal Gorenstein Fano polytope.

INTRODUCTION

In [5], from a viewpoint of Gröbner bases, the centrally symmetric configuration (7) of the order polytope (8) of a finite partially ordered set is studied. In the present paper, a far-reaching generalization of [5] will be discussed.

Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite partially ordered sets (posets, for short) with $|P| = |Q| = d$. A subset $I$ of $P$ is called a poset ideal of $P$ if $p_i \in I$ and $p_j \in P$ together with $p_j \leq p_i$ guarantee $p_j \in I$. Thus in particular the empty set $\emptyset$ as well as $P$ itself is a poset ideal of $P$. Write $\mathcal{J}(P)$ for the set of poset ideals of $P$ and $\mathcal{J}(Q)$ for that of $Q$. A linear extension of $P$ is a permutation $\sigma = i_1 i_2 \cdots i_d$ of $[d] = \{1, \ldots, d\}$ for which $i_a < i_b$ if $p_{i_a} < p_{i_b}$.

Let $e_1, \ldots, e_d$ stand for the canonical unit coordinate vectors of $\mathbb{R}^d$. Then, for each subset $I \subset P$ and for each subset $J$ of $Q$, we define $\rho(I) = \sum_{i \in I} e_i$ and $\rho(J) = \sum_{i \in J} e_i$. In particular $\rho(\emptyset)$ is the origin $0$ of $\mathbb{R}^d$. Define $\Omega(P, Q) \subset \mathbb{Z}^d$ as

$$\Omega(P, Q) = \{ \rho(I) : \emptyset \neq I \in \mathcal{J}(P) \} \cup \{ -\rho(J) : \emptyset \neq J \in \mathcal{J}(Q) \} \cup \{0\}$$

and write $\Delta(P, Q) \subset \mathbb{R}^d$ for the convex polytope which is the convex hull of $\Omega(P, Q)$. We call $\Delta(P, Q)$ the twinned order polytope of $P$ and $Q$. In other words, the twinned order polytope $\Delta(P, Q)$ of $P$ and $Q$ is the convex polytope which is the convex hull of $\mathcal{O}(P) \cup (-\mathcal{O}(Q))$, where $\mathcal{O}(P) \subset \mathbb{R}^d$ is the order polytope of $P$ and $-\mathcal{O}(Q) = \{-\beta : \beta \in \mathcal{O}(Q)\}$. One has $\dim \Delta(P, Q) = d$. Since $\rho(P) = \rho(Q) = e_1 + \cdots + e_d$, it follows that the origin $0$ of $\mathbb{R}^d$ cannot be a vertex of $\Delta(P, Q)$. In fact, the set of vertices of $\Delta(P, Q)$ are $\Omega(P, Q) \setminus \{0\}$. 

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This paper is organized as follows. In Section 1, a basic fact that the origin of \( \mathbb{R}^d \) belongs to the interior of \( \Delta(P, -Q) \) if and only if \( P \) and \( Q \) possess a common linear extension (Lemma 1.1). We then show, in Section 2, that, when the origin of \( \mathbb{R}^d \) belongs to the interior of \( \Delta(P, -Q) \), the toric ideal of \( \Delta(P, -Q) \) possesses a quadratic Gröbner basis with respect to a reverse lexicographic order for which the variable corresponding to the origin is smallest (Theorem 2.1). Thus in particular if \( P \) and \( Q \) possess a common linear extension, then the twinned order polytope \( \Delta(P, -Q) \) is a normal Gorenstein Fano polytope (Corollary 2.2). Finally, we conclude this paper with a collection of examples in Section 3. We refer the reader to [4] for fundamental materials on Gröbner bases and toric ideals.

1. Linear extensions

Let \( P \) and \( Q \) be finite posets with \( |P| = |Q| = d \). In general, the origin \( 0 \) of \( \mathbb{R}^d \) may not belong to the interior of the twinned order polytope \( \Delta(P, -Q) \) of \( P \) and \( Q \). It is then natural to ask when the origin of \( \mathbb{R}^d \) belongs to the interior of \( \Delta(P, -Q) \).

**Lemma 1.1.** Let \( P = \{p_1, \ldots, p_d\} \) and \( Q = \{q_1, \ldots, q_d\} \) be finite posets. Then the following conditions are equivalent:

(i) The origin of \( \mathbb{R}^d \) belongs to the interior of \( \Delta(P, -Q) \);

(ii) \( P \) and \( Q \) possess a common linear extension.

**Proof.** ((i) \( \Rightarrow \) (ii)) Suppose that the origin \( 0 \) of \( \mathbb{R}^d \) belongs to the interior of \( \Delta(P, -Q) \). Since \( \Omega(P, -Q) \setminus \{0\} \) is the set of vertices of \( \Delta(P, -Q) \), the existence of an equality

\[
0 = \sum_{\emptyset \neq I \in \mathcal{I}(P)} a_I \cdot \rho(I) + \sum_{\emptyset \neq J \in \mathcal{J}(Q)} b_J \cdot (-\rho(J)),
\]

where each of \( a_I \) and \( b_J \) is a positive real numbers, is guaranteed. Let

\[
\sum_{\emptyset \neq I \in \mathcal{I}(P)} a_I \cdot \rho(I) = \sum_{i=1}^{d} a_i^* e_i, \quad \sum_{\emptyset \neq J \in \mathcal{J}(Q)} b_J \cdot \rho(J) = \sum_{i=1}^{d} b_i^* e_i,
\]

where each of \( a_i^* \) and \( b_i^* \) is a positive rational number. Since each \( I \) is a poset ideal of \( P \) and each \( J \) is a poset ideal of \( Q \), it follows that \( a_i^* > a_j^* \) if \( p_i < p_j \). Let \( \sigma = i_1 i_2 \cdots i_d \) be a permutation of \([d]\) for which \( i_a < i_b \) if \( a_i^* > a_j^* \). Then \( \sigma \) is a linear extension of \( P \). Furthermore, by using \((\Pi)\), one has \( a_i^* = b_i^* \) for \( 1 \leq i \leq d \). It then turn out that \( \sigma \) is also a linear extension of \( Q \), as required.

((ii) \( \Rightarrow \) (i)) Let \( \sigma = i_1 i_2 \cdots i_d \) be a linear extension of each of \( P \) and \( Q \). Then \( I_k = \{p_{i_1}, \ldots, p_{i_k}\} \) is a poset ideal of \( P \) and \( J_k = \{q_{i_1}, \ldots, q_{i_k}\} \) is a poset ideal of \( Q \) for \( 1 \leq k \leq d \). Hence

\[
\pm e_{i_1}, \pm (e_{i_1} + e_{i_2}), \ldots, \pm (e_{i_1} + \cdots + e_{i_d})
\]
belong to $\Omega(P, -Q)$. Let $\Gamma \subset \mathbb{R}^d$ denote the convex polytope which is the convex hull of $[2]$. Since $\dim \Gamma = d$ and since the origin of $\mathbb{R}^d$ belongs to the interior of $\Gamma$, it follows that the origin of $\mathbb{R}^d$ belongs to the interior of $\Delta(P, -Q)$, as desired. \hfill \square

2. Quadratic Gröbner bases

Let, as before, $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite partially ordered sets. Let $K[t, t^{-1}, s] = K[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}, s]$ denote the Laurent polynomial ring in $2d + 1$ variables over a field $K$. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$, then $t^\alpha$ is the Laurent monomial $t_1^{\alpha_1} \cdots t_d^{\alpha_d}$. In particular $t^0 = s$. The toric ring of $\Omega(P, -Q)$ is the subring $K[\Omega(P, -Q)]$ of $K[t, t^{-1}, s]$ which is generated by those Laurent monomials $t^\alpha$ with $\alpha \in \Omega(P, -Q)$. Let

$$K[x, y, z] = K[\{x_I\}_{\emptyset \not\in I \in \mathcal{J}(P)} \cup \{y_J\}_{\emptyset \not\in J \in \mathcal{J}(Q)} \cup \{z\}]$$

denote the polynomial ring in $|\Omega(P, -Q)|$ variables over $K$ and define the surjective ring homomorphism $\pi : K[x, y, z] \to K[\Omega(P, -Q)]$ by setting

- $\pi(x_I) = t^{\rho(I)}s$, where $\emptyset \not\in I \in \mathcal{J}(P)$;
- $\pi(y_J) = t^{-\rho(J)}s$, where $\emptyset \not\in J \in \mathcal{J}(Q)$;
- $\pi(z) = s$.

The toric ideal $I_{\Omega(P, -Q)}$ of $\Omega(P, -Q)$ is the kernel of $\pi$.

Let $<$ denote a reverse lexicographic order on $K[x, y, z]$ satisfying

- $z < x_I$ and $z < y_J$;
- $x_{I'} < x_I$ if $I' \subset I$;
- $y_{J'} < y_J$ if $J' \subset J$,

and $\mathcal{G}$ the set of the following binomials:

(i) $x_I x_{I'} - x_{I \cap I'} x_{I \cup I'}$;
(ii) $y_J y_{J'} - y_{J \cap J'} y_{J \cup J'}$;
(iii) $x_I y_J - x_{I \setminus \{p_i\}} y_{J \setminus \{q_i\}}$,

where

- $x_\emptyset = y_\emptyset = z$;
- $I$ and $I'$ belong to $\mathcal{J}(P)$ and $J$ and $J'$ belong to $\mathcal{J}(Q)$;
- $p_i$ is a maximal element of $I$ and $q_i$ is a maximal element of $J$.

Theorem 2.1. Work with the same situation as above. Suppose that $P$ and $Q$ possess a common linear extension. Then $\mathcal{G}$ is a Gröbner basis of $I_{\Omega(P, -Q)}$ with respect to $<$.

Proof. It is clear that $\mathcal{G} \subset I_{\Omega(P, -Q)}$. In general, if $f = u - v$ is a binomial, then $u$ is called the first monomial of $f$ and $v$ is called the second monomial of $f$. The initial monomial of each of the binomials (i) – (iii) with respect to $<$ is its first monomial. Let $\text{in}_<(\mathcal{G})$ denote the set of initial monomials of binomials belonging to $\mathcal{G}$. It follows from $[6]$ (0.1) that, in order to show that $\mathcal{G}$ is a Gröbner basis
of $I_{\Omega(P,-Q)}$ with respect to $<$, what we must prove is the following: (♣) If $u$ and $v$ are monomials belonging to $K[x,y,z]$ with $u \neq v$ such that $u \not\in \langle \text{in}_{<}(G) \rangle$ and $v \not\in \langle \text{in}_{<}(G) \rangle$, then $\pi(u) \neq \pi(v)$.

Let $u$ and $v$ be monomials belonging to $K[x,y,z]$ with $u \neq v$. Write

$$u = z^\alpha x_{I_1}^{\xi_1} \cdots x_{I_a}^{\xi_a} y_{J_1}^{\nu_1} \cdots y_{J_b}^{\nu_b}, \quad v = z^\alpha' x_{I'_1}^{\xi'_1} \cdots x_{I'_a}^{\xi'_a} y_{J'_1}^{\nu'_1} \cdots y_{J'_b}^{\nu'_b},$$

where

- $\alpha \geq 0$, $\alpha' \geq 0$;
- $I_1, \ldots, I_a, I'_1, \ldots, I'_a \in \mathcal{J}(P) \setminus \{\emptyset\}$;
- $J_1, \ldots, J_b, J'_1, \ldots, J'_b \in \mathcal{J}(Q) \setminus \{\emptyset\}$;
- $\xi_1, \ldots, \xi_a, \nu_1, \ldots, \nu_b, \xi'_1, \ldots, \xi'_a, \nu'_1, \ldots, \nu'_b > 0$,

and where $u$ and $v$ are relatively prime with $u \not\in \langle \text{in}_{<}(G) \rangle$ and $v \not\in \langle \text{in}_{<}(G) \rangle$. Especially either $\alpha = 0$ or $\alpha' = 0$. Let, say, $\alpha = 0$. Thus

$$u = z^0 x_{I_1}^{\xi_1} \cdots x_{I_a}^{\xi_a} y_{J_1}^{\nu_1} \cdots y_{J_b}^{\nu_b}, \quad v = x_{I'_1}^{\xi'_1} \cdots x_{I'_a}^{\xi'_a} y_{J'_1}^{\nu'_1} \cdots y_{J'_b}^{\nu'_b}.$$  

By using (i) and (ii), it follows that

- $I_1 \subset I_2 \subset \cdots \subset I_a, I_1 \neq I_2 \neq \cdots \neq I_a$;
- $J_1 \subset J_2 \subset \cdots \subset J_b, J_1 \neq J_2 \neq \cdots \neq J_b$;
- $I'_1 \subset I'_2 \subset \cdots \subset I'_a, I'_1 \neq I'_2 \neq \cdots \neq I'_a$;
- $J'_1 \subset J'_2 \subset \cdots \subset J'_b, J'_1 \neq J'_2 \neq \cdots \neq J'_b$.

Furthermore, by virtue of [2], it suffices to discuss $u$ and $v$ with $(a,a') \neq (0,0)$ and $(b,b') \neq (0,0)$.

Let $A_i$ denote the power of $t_i$ appearing in $\pi(x_{I_i}^{\xi_i} \cdots x_{I_a}^{\xi_a})$ and $A'_i$ the power of $t_i$ appearing in $\pi(x_{I'_i}^{\xi'_i} \cdots x_{I'_a}^{\xi'_a})$. Similarly let $B_i$ denote the power of $t_i^{-1}$ appearing in $\pi(y_{J_i}^{\nu_i} \cdots y_{J_b}^{\nu_b})$ and $B'_i$ the power of $t_i^{-1}$ appearing in $\pi(y_{J'_i}^{\nu'_i} \cdots y_{J'_b}^{\nu'_b})$.

Since $P$ and $Q$ possess a common linear extension, after relabeling the elements of $P$ and $Q$, we assume that if $p_r < p_s$ in $P$, then $r < s$, and if $q_{r'} < q_{s'}$ in $Q$, then $r' < s'$.

Let $1 \leq j_* \leq d$ denote the biggest integer for which one has $A_{j*} \neq A'_{j*}$. Since $I_a \neq I'_a$, the existence of $j_*$ is guaranteed. Let $j_* = d$ and, say, $A_d > A'_d$. Then $p_d \in I_a$. Since $p_d$ is a maximal element of $P$ and $q_d$ is that of $Q$, by using (iii), it follows that $q_d$ cannot belong to $J_h$. Hence $\pi(u) \neq \pi(v)$, as desired.

Let $j_* < d$ and $A_{j_*} > A'_{j_*}$. Let $1 \leq e \leq a$ denote the integer with $p_{j_*} \in I_e$ and $p_{j_*} \not\in I_{e-1}$. We claim that $p_{j_*}$ is a maximal element of $I_e$. To see why this is true, let $p_{j_*} < p_h$ in $I_e$. Then $j_* < h$. Since both $p_{j_*}$ and $p_h$ belong to each of $I_e, I_{e+1}, \ldots, I_a$, it follows that $A_{j_*} = A_h$. Now, since $p_{j_*} < p_h$, one has $A'_{j_*} \geq A'_h$. Hence $A_h = A_{j_*} > A'_{j_*} \geq A'_h$. However, the definition of $j_*$ says that $A_h = A'_h$, a contradiction. Hence $p_{j_*}$ is a maximal element of $I_e$. 

Now, suppose that $\pi(u) = \pi(v)$. Then $B_d = B'_d, \ldots, B_{j_*+1} = B'_{j_*+1}$ and $B_{j_*} > B'_{j_*}$. Then the above argument guarantees the existence of $J_{e'}$ for which $q_{j_*}$ is a maximal element of $J_{e'}$. The fact that $p_{j_*}$ is a maximal element of $I_{e'}$ and $q_{j_*}$ is that of $J_{e'}$ contradicts (iii). As a result, one has $\pi(u) \neq \pi(v)$, as desired. 

Theorem 2.1 is a far-reaching generalization of [5, Theorem 2.2]. We refer the reader to [5] and [8] for basic materials on normal Gorenstein Fano polytopes. As in [5, Corollary 2.3] and [8, Corollary 1.3], it follows that

**Corollary 2.2.** If $P$ and $Q$ possess a common linear extension, then the twinned order polytope $\Delta(P, -Q)$ is a normal Gorenstein Fano polytope.

### 3. Examples

We conclude this paper with a collection of examples. It is natural to ask, if, in general, the toric ideal of $I_{\Omega(P, -Q)}$ possesses a quadratic Gr"obner basis with respect to a reverse lexicographic order as in Theorem 2.1.

**Example 3.1.** In general, a toric ideal $I_{\Omega(P, -Q)}$ may not possess a quadratic Gr"obner basis with respect to a reverse lexicographic order $<$ introduced as above. Let $P = \{p_1, \ldots, p_5\}$ and $Q = \{q_1, \ldots, q_5\}$ be the following finite posets:

![Diagram of posets P and Q](image)

Since $p_1 < p_3$ and $q_3 < q_1$, it follows that no linear extension of $P$ is a linear extension of $Q$. Then a routine computation guarantees that, for any reverse lexicographic order as in Theorem 2.1, the binomial

$$ (3) \quad x_{\{2\}}x_{\{1,2,3,4\}}y_{\{1,2,3,4,5\}} - x_{\{2,4\}}y_{\{4,5\}}z $$

belongs to the reduced Gr"obner basis of $I_{\Omega(P, -Q)}$ with respect to $<$. However, the toric ideal $I_{\Omega(P, -Q)}$ is generated by quadratic binomials. The $S$-polynomial of the binomials

$$ x_{\{2,4\}}x_{\{1,2,3\}} - x_{\{2\}}x_{\{1,2,3,4\}}, \quad x_{\{1,2,3\}}y_{\{1,2,3,4,5\}} - y_{\{4,5\}}z $$

belonging to a system of generators of $I_{\Omega(P, -Q)}$ coincided with the binomial (3).

**Conjecture 3.2.** Let $P$ and $Q$ be arbitrary finite posets with $|P| = |Q| = d$. Then the toric ideal $I_{\Omega(P, -Q)}$ is generated by quadratic binomials.

Let $\delta(\Delta(P, -Q))$ denote the $\delta$-vector ([8, p. 79]) of $\Delta(P, -Q)$. It then follows that, if $P$ and $Q$ possess a common linear extension, then $\delta(\Delta(P, -Q))$ is symmetric and unimodal.
Example 3.3. Let $P = \{p_1, \ldots, p_d\}$ be a chain and $Q = \{q_1, \ldots, q_d\}$ an antichain. Then the $\delta$-vector of $\Delta(P, -Q)$ is

\[
\begin{align*}
\delta_2 &= (1, 3, 1), \\
\delta_3 &= (1, 7, 7, 1), \\
\delta_4 &= (1, 15, 33, 15, 1), \\
\delta_5 &= (1, 31, 131, 131, 31, 1), \\
\delta_6 &= (1, 63, 473, 883, 473, 63, 1).
\end{align*}
\]

It seems likely that $\delta(\Delta(P, -Q))$ coincides with the Pascal-like triangle [1, pp. 11–12] with $r = 1$.

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