HARDY INEQUALITIES IN TRIEBEL–LIZORKIN SPACES

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Abstract. We prove an inequality of Hardy type for functions in Triebel–Lizorkin spaces. The distance involved is being measured to a given Ahlfors $d$-regular set in $\mathbb{R}^n$, with $n - 1 < d < n$. As an application of the Hardy inequality, we consider boundedness of pointwise multiplication operators, and extension problems.

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1. Introduction

In the paper we study inequalities of Hardy type

\[
\left( \int_{\mathbb{R}^n} \frac{|f(x)|^p}{\text{dist}(x, S)^{sp}} \, dx \right)^{1/p} \lesssim \|f\|_{\mathcal{F}_{sp}^{pq}(\mathbb{R}^n)}
\]

for functions in Triebel–Lizorkin spaces $\mathcal{F}_{sp}^{pq}(\mathbb{R}^n)$ with vanishing trace on $S$. The set $S \subset \mathbb{R}^n$, $n \geq 2$, under consideration is supposed to be an (Ahlfors) $d$-regular set with $n - 1 < d < n$. We also consider applications to extension problems.

Often Hardy inequalities are formulated in the form

\[
\left( \int_{\Omega} \frac{|f(x)|^p}{\text{dist}(x, \partial \Omega)^{sp}} \, dx \right)^{1/p} \lesssim \|f\|_{\mathcal{F}_{sp}^{pq}(\Omega)}
\]

where $\Omega$ is an appropriate domain in $\mathbb{R}^n$. In case of a $C^\infty$ bounded domain, inequality (1.2) is known to hold for all $f \in \mathcal{F}_{sp}^{pq}(\Omega)$ with $1 \leq p, q < \infty$ and $0 < s < 1/p$, see [30, p. 58] and [27].

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When \( s \geq 1/p \), it is natural to require that \( f \) can be extended by zero, \([30, \text{Proposition } 5.7]\). We postpone the details on the connection between (1.1) and (1.2) to §4.

Inequality (1.2) can be considered as one of the fractional counterparts of the classical Hardy inequality (recall that in case of a \( C^\infty \) bounded domain, the Sobolev space \( W^{k,p}(\Omega) \) coincides with the Triebel–Lizorkin space \( F^k_{p,q}(\Omega) \), \( 1 < p < \infty \) and \( k \in \mathbb{N} \)). While classical Hardy inequalities have been studied under rather general geometric assumptions on the domain (more precisely on its complement, we refer to \([21, 32, 18]\)), the fractional analogs were mostly considered on smooth and Lipschitz domains, see \([27, 10]\). In more irregular geometric settings, fractional Hardy type inequalities have been considered in \([11]\) and \([30, \text{Proposition } 16.5]\). Also, Hardy type inequalities for certain Triebel–Lizorkin spaces on ‘E-thick’ domains \( \Omega \) can be obtained via so called refined localisation spaces \([31, \text{Theorem } 2.18, \text{Proposition } 3.10]\).

Let us turn to a discussion of the objects of this paper. Throughout, we focus on the case of smoothness \( s > (\pi - d)/p \) and \( S \) being a \( d \)-set in \( \mathbb{R}^n \), \( \pi - 1 < d < \pi \).

Our motivation to consider inequality (1.1) arises from the work \([15]\), where we obtained a description of the traces of Besov–Triebel–Lizorkin spaces on \( d \)-sets. This description is given in terms of a local polynomial approximations \([4]\) and, as it turns out, the related tools and porosity based arguments easily adapt to the study of Hardy inequalities, and their consequences. Let us remark that we widely use the fact that \( d \)-sets in \( \mathbb{R}^n \) with \( d < \pi \) are porous.

The main result, Theorem 3.7, states that Hardy inequality (1.1) is valid if the trace of \( f \) on \( S \) is zero pointwise \( H^d \) almost everywhere. Hardy inequalities are related to the question whether the characteristic function \( \chi_\Omega \) of a domain is a pointwise multiplier in \( F^s_{p,q}(\mathbb{R}^n) \). Our auxiliary result, Proposition 4.1, states that inequality,

\[
\|f \chi_\Omega\|_{F^s_{p,q}(\mathbb{R}^n)} \leq \|f\|_{F^s_{p,q}(\mathbb{R}^n)} + \left( \int_\Omega \frac{|f(x)|^p}{\text{dist}(x, \partial \Omega)^sp} \, dx \right)^{1/p},
\]

holds for a domain \( \Omega \) in \( \mathbb{R}^n \) whose boundary is a \( d \)-set with \( \pi - 1 < d < \pi \). As a corollary of Theorem 3.7 and Proposition 4.1, we obtain that \( \chi_\Omega \) is a pointwise multiplier in the subspace \( \{f \in F^s_{p,q}(\mathbb{R}^n) : \text{Tr}_{\partial \Omega} f = 0\} \), where \( \text{Tr}_{\partial \Omega} f \) denotes the trace of \( f \) on \( \partial \Omega \), see Corollary 4.2. For further results on the pointwise multipliers \( \chi_\Omega \) of irregular domains, we refer to \([12, 24, 28]\).

As an application, we study the problem of extending a smooth function from its boundary trace. More precisely, for \( f \in F^s_{p,q}(\mathbb{R}^n) \) with \( s > (\pi - d)/p \) and \( k = [s]+1 \), we define an extension

\[
\text{Ext}_{k,\Omega} f(x) := \begin{cases} f(x), & \text{if } x \in \Omega; \\ \sum_{Q \in \mathcal{W}_{\partial \Omega}} q_Q(x)(\text{Pr}_{k-1,a(Q)} \circ \text{Tr}_{\partial \Omega}(f))(x), & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}
\]

Let us emphasise that the operator \( \text{Pr}_{k-1,a(Q)} \) is a polynomial projector on \( L^1(\alpha(Q) \cap \partial \Omega) \), where \( \alpha(Q) \) is a cube, which is centered in \( \partial \Omega \) and is close to the given Whitney cube \( Q \in \mathcal{W}_{\partial \Omega} \). Observe that, if \( f \) has zero boundary values in \( \partial \Omega \), then \( \text{Ext}_{k,\Omega} f = f \chi_\Omega \).

In Theorem 4.9, we prove boundedness of the operator \( \text{Ext}_{1,\Omega} \) on \( F^s_{p,q}(\mathbb{R}^n) \). The proof is based on restriction and extension theorems for \( d \)-sets \([15]\), as well as the multiplier result mentioned above. To our knowledge, Theorem 4.9 is not previously known in its generality. However, the operator \( \text{Ext}_{1,\Omega} \) has been considered in connection with extension results for first order Sobolev
spaces $W^{1,p}(\mathbb{R}^n)$, [13]. The operators $\text{Ext}_{k+1,\Omega}$ with $k \in \{1, 2, \ldots\}$ share common properties with certain extension operators of Calderón [7], which extend any $f \in W_0^{k,p}(\Omega)$ by zero outside of a given Lipschitz domain $\Omega$. Our extension approach is also related to gluing Sobolev functions with matching traces, [3].

The outline of the paper is as follows. In sections §2.1, §2.2, §2.3, and §2.4 we treat the preliminaries: notation, definitions of function spaces, $d$-sets, and porosity, respectively. In §3.1, we prove a special Hardy inequality, namely the one for $F_{p,p}(\mathbb{R}^n)$ with $s \in (0, 1)$. We include this model result to demonstrate the somewhat technical proof of our main theorem in a transparent manner. The main theorem is proven in §3.2. In section §4, we consider applications to pointwise multipliers and extension problems.

2. Preliminaries

2.1. Notation and auxiliary results. In this paper, unless otherwise specified, we tacitly assume that $n \geq 2$ and that the distances in $\mathbb{R}^n$ are measured in terms of the supremum norm. At the same time, by $B(x, r)$ we denote the open Euclidean ball, centered at $x \in \mathbb{R}^n$ with radius $r > 0$.

For a measurable set $E$, with a finite and positive measure, we write

$$\int_E f(x) \, dx = \frac{1}{|E|} \int_E f(x) \, dx.$$  

We write $x_E$ for the characteristic function of a set $E$.

By $Q = Q(x_Q, r_Q)$ we denote a closed cube in $\mathbb{R}^n$ centered at $x_Q \in \mathbb{R}^n$ with the side length $\ell(Q) = 2r_Q$ and sides parallel to the coordinate axes. By $tQ$, $t > 0$, we mean a cube centered at $x_Q$ with the side length $t\ell(Q)$. We denote by $D$ the family of closed dyadic cubes in $\mathbb{R}^n$, and $D_i$ stands for the family of those dyadic cubes whose side length is $2^{-i}$, $i \in \mathbb{Z}$.

The support of a function $f : \mathbb{R}^n \to \mathbb{C}$ is denoted by $\text{supp} f$, and it is the closure of the set $\{x : f(x) \neq 0\}$ in $\mathbb{R}^n$.

The notation $a \lesssim b$ means that an inequality $a \leq cb$ holds for some constant $c > 0$ the exact value of which is not important. The symbols $c$ and $C$ are used for various positive constants; they may change even on the same line.

Recall also an inequality of Hardy type for sums.

2.1. Lemma. Let $0 < p < \infty$ and $a_j \geq 0$, $j = 0, 1, \ldots$. Then

$$\sum_{j=0}^{\infty} 2^{\sigma j} \left(\sum_{i=j}^{\infty} a_i\right)^p \lesssim \sum_{j=0}^{\infty} 2^{\sigma j} a_i^p \quad \text{for } \sigma > 0.$$  

For $p > 1$ this lemma follows from Leindler’s result in [20]. The case $0 < p \leq 1$ follows by applying the inequality $(\sum_{i=j}^{\infty} a_i)^p \leq \sum_{i=j}^{\infty} a_i^p$.

2.2. Function spaces in $\mathbb{R}^n$. We recall definition of fractional order Sobolev spaces in $\mathbb{R}^n$. For $1 \leq p < \infty$ and $s \in (0, 1)$ we let $W^{s,p}(\mathbb{R}^n)$ denote the collection of functions $f$ in $L^p(\mathbb{R}^n)$ with

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{ps+n}} \, dx \, dy\right)^{1/p} < \infty.$$  

(2.2)
There are several equivalent characterizations for the fractional Sobolev spaces and their natural extensions, Triebel–Lizorkin spaces (see e.g. [1], [2], [26] and [29] for the general theory).

In this paper, we mostly use definitions based on the local polynomial approximation approach. Let \( f \in L^u_{\text{loc}}(\mathbb{R}^n) \), \( 1 \leq u \leq \infty \), and \( k \in \mathbb{N}_0 \). Following [4], we define the \textit{normalized local best approximation} of \( f \) on a cube \( Q \) in \( \mathbb{R}^n \) by

\[
\mathcal{E}_k(f, Q)_1^u(\mathbb{R}^n) := \inf_{P \in \mathcal{P}_k} \left( \frac{1}{|Q|} \int_Q |f(x) - P(x)|^u \, dx \right)^{1/u}.
\]

Here and below \( \mathcal{P}_k \), \( k \geq 0 \), denotes the space of polynomials on \( \mathbb{R}^n \) of degree at most \( k \). Let \( Q_1 \subset Q_2 \) be two cubes in \( \mathbb{R}^n \). Then

\[
\mathcal{E}_k(f, Q_1)_1^u(\mathbb{R}^n) \leq \left( \frac{r_{Q_2}}{r_{Q_1}} \right)^{n/u} \mathcal{E}_k(f, Q_2)_1^u(\mathbb{R}^n).
\]

This property is referred as \textit{the monotonicity of local approximation}.

The following definition of Triebel–Lizorkin spaces with positive smoothness can be found in [26]. Let \( s > 0 \), \( 1 \leq p < \infty \), \( 1 \leq q \leq \infty \), and \( k \) be an integer such that \( s < k \). For \( f \in L^u_{\text{loc}}(\mathbb{R}^n) \), \( 1 \leq u \leq \min\{p, q\} \), set for all \( x \in \mathbb{R}^n \),

\[
g(x) := \left( \int_0^1 \left( \mathcal{E}_k(f, Q(x, t))_1^u(\mathbb{R}^n) \right)^q \frac{dt}{t} \right)^{1/q}, \quad \text{if} \ q < \infty,
\]

and \( g(x) := \sup\{t^{-s} \mathcal{E}_k(f, Q(x, t))_1^u(\mathbb{R}^n) : 0 < t \leq 1\} \) if \( q = \infty \). The function \( f \) belongs to Triebel–Lizorkin space \( F^s_{pq}(\mathbb{R}^n) \) if \( f \) and \( g \) are both in \( L^p(\mathbb{R}^n) \). The Triebel–Lizorkin norms

\[
\|f\|_{F^s_{pq}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}
\]

are equivalent if \( s < k \) and \( 1 \leq u \leq \min\{p, q\} \). In particular, if \( q \geq p \), then we can set \( u = p \).

Triebel–Lizorkin spaces include fractional Sobolev spaces as a special case: if \( 1 < p < \infty \) and \( 0 < s < 1 \), then \( F^s_{pp}(\mathbb{R}^n) \) coincides with \( W^{s,p}(\mathbb{R}^n) \), [8, Theorem 6.7] and [29, pp. 6–7].

2.3. \textbf{d-sets and inequalities.} Recall definition of an Ahlfors \( d \)-regular set, or, briefly, \( d \)-set.

2.4. \textbf{Definition.} Let \( 0 < d \leq n \). A closed set \( S \subset \mathbb{R}^n \) is a \( d \)-set if there is a constant \( C > 1 \) such that \( C^{-1}r^d \leq \mathcal{H}^d(Q(w, r) \cap S) \leq Cr^d \) for every \( w \in S \) and \( 0 < r \leq 1 \).

Note that if \( S \) is a \( d \)-set, then

\[
c^{-1}r^d \leq \mathcal{H}^d(Q(w, r) \cap S) \leq cr^d
\]

for every \( w \in S \) and \( 0 < r \leq R \), where \( R \) is any fixed positive number and the constant \( c \geq 1 \) depends on parameters \( R, C, d, n \).

We write \( L^p(S) \) for the space of \( p \)-integrable functions on a \( d \)-set \( S \) with respect to the natural Hausdorff measure \( \mathcal{H}^d|_S \).

The following is a Remez type theorem, [5].
2.5. **Theorem.** Let \( S \subset \mathbb{R}^n \) be a \( d \)-set, \( n-1 < d \leq n \). Suppose that \( Q = Q(x_Q, r_Q) \) and \( Q' = Q(x_{Q'}, r_{Q'}) \) are cubes in \( \mathbb{R}^n \) such that \( x_{Q'} \in S \), \( Q' \subset Q \), and
\[
0 < r_Q \leq R r_{Q'} \leq R^2
\]
for some \( R > 1 \). Then, for every polynomial \( p \) of degree at most \( k \), we have
\[
\left( \frac{1}{|Q|} \int_Q |p|^r \, dx \right)^{1/r} \leq C \left( \frac{1}{\mathcal{H}^d(Q' \cap S)} \int_{Q' \cap S} |p|^u \, d\mathcal{H}^d \right)^{1/u},
\]
where \( 1 \leq u, r \leq \infty \) and the constant \( C \) depends on \( S, R, n, u, r, k \).

We also need the following energy type estimate.

2.6. **Lemma.** Let \( S \subset \mathbb{R}^n \) be a \( d \)-set, \( n-1 < d < n \). Suppose that \( 1 < r < \infty \) and \( 0 < \omega < 1 \) satisfy \( \omega r > n - d \). Then, for every point \( x \in \partial \Omega \) and \( 0 < t \leq T \in \mathbb{R} \),
\[
(2.7) \quad \left( \int_{Q(x,t)} \left( \int_{Q(y,t) \cap S} \frac{d\mathcal{H}^d(z)}{|y-z|^{n-\omega}} \right)^{r'} \, dy \right)^{1/r'} \lesssim t^{d+\omega-n/r},
\]
where \( 1/r + 1/r' = 1 \).

**Proof.** Let us begin with two auxiliary estimates. The first one,
\[
\sup_{y \in \mathbb{R}^n} \int_{Q(y,t) \cap S} \frac{d\mathcal{H}^d(z)}{|y-z|^\sigma} \lesssim t^{d-\sigma}, \quad 0 < \sigma < d,
\]
follows easily from the layer cake representation and the definition of a \( d \)-set, [16, p. 104].

The second one,
\[
\sup_{x \in S} \int_{Q(x,t)} \text{dist}(y, S)^\mu \, dy \lesssim t^\mu, \quad d < \mu < n,
\]
is closely related to Aikawa dimension of \( S \), and its proof can be found in [19, Lemma 2.1].

Denote the left hand side of (2.7) by LHS. Choose an auxiliary parameter \( \beta > 0 \) such that
\[
0 < n - \omega - \beta < d, \quad \beta r' < n - d.
\]
The assumptions of the lemma ensure that this can be done (e.g. \( \beta = (n - d)/r' - \varepsilon \) with suitable \( \varepsilon > 0 \)).

By the estimates above and the choice of \( \beta \),
\[
\text{LHS} \lesssim \left( \int_{Q(x,t)} \text{dist}(y, S)^{-\beta r'} \left( \int_{Q(y,t) \cap S} \frac{d\mathcal{H}^d(z)}{|y-z|^{n-\omega-\beta}} \right)^{r'} \, dy \right)^{1/r'} \lesssim t^{d-n+\omega+\beta+(n-\beta r')/r'}.
\]
A simplification of exponents finishes the proof. \( \square \)
2.4. **Porous sets and Whitney decomposition.** In this paper we widely use the fact that \(d\)-sets with \(d < n\) are porous. The treatment here follows parts of [15].

2.8. **Definition.** A set \(S \subset \mathbb{R}^n\) is **porous** (or \(\kappa\)-porous) if for some \(\kappa \geq 1\) the following statement is true: For every cube \(Q(x, r)\) with \(x \in \mathbb{R}^n\) and \(0 < r \leq 1\) there is \(y \in Q(x, r)\) such that \(Q(y, \kappa r) \cap S = \emptyset\).

2.9. **Remark.** The observation that \(d\)-sets with \(d < n\) are porous was already done in [17]. See also Proposition 9.18 in [30] which gives this fact as a special case.

Here we recall a reverse Hölder type inequality involving porous sets, Theorem 2.11. For a set \(S\) in \(\mathbb{R}^n\) and a positive constant \(\gamma > 0\) we denote
\[
C_{S, \gamma} = \{Q \in \mathcal{D} : \gamma^{-1} \text{dist}(x_Q, S) \leq \ell(Q) \leq 1\}.
\]
This is the family of dyadic cubes that are relatively close to the set. Such families arise naturally while treating the ‘extension by zero’ -problem, we refer to later Proposition 4.1.

2.11. **Theorem.** Suppose that \(S \subset \mathbb{R}^n\) is porous. Let \(p, q \in (1, \infty)\) and \(\{a_Q\}_{Q \in C_{S, \gamma}}\) be a sequence of non-negative scalars. Then
\[
\left\| \sum_{Q \in C_{S, \gamma}} \chi_Q a_Q \right\|_p \leq c \left( \sum_{Q \in C_{S, \gamma}} \left( \chi_Q a_Q \right)^q \right)^{1/q} \left\| \right\|_p.
\]
Here the constant \(c\) depends on \(n, p, \gamma\) and the set \(S\).

The proof of this theorem is a consequence of maximal-function techniques, we refer to [15].

Supposing that \(S\) is a non-trivial closed set in \(\mathbb{R}^n\), its complement has a Whitney decomposition, see e.g. [25]. That is, there is a family \(\mathcal{W}_S\) of dyadic cubes whose interiors are pairwise disjoint and \(\mathbb{R}^n \setminus S = \bigcup_{Q \in \mathcal{W}_S} Q\). Furthermore, if \(Q \in \mathcal{W}_S\), then
\[
\text{diam}(Q) \leq \text{dist}(Q, S) \leq 4 \text{diam}(Q).
\]
It is easy to check that the standard construction, usually given in terms of the Euclidean metric, admits this modification where we use the uniform metric instead.

Let us write \(\ell(Q) \preceq 2^{-i}\) if \(Q\) is a cube in \(\mathbb{R}^n\) and \(2^{-i}/5\kappa \leq \text{diam}(Q) \leq 2^{-i}\).

2.13. **Lemma.** Suppose that \(S \subset \mathbb{R}^n\) is \(\kappa\)-porous. Let \(x \in S\) and \(i \in \mathbb{N}\). Then there is \(Q \in \mathcal{W}_S\) such that \(\ell(Q) \preceq 2^{-(i+1)}\) and \(Q \subset Q(x, 2^{-i})\).

**Proof.** By \(\kappa\)-porosity of \(S\), there is \(y \in Q(x, 2^{-i-1})\) such that \(Q(y, 2^{-(i-1)}/\kappa) \subset \mathbb{R}^n \setminus S\). Let \(Q \in \mathcal{W}_S\) be a cube containing the point \(y\). Then
\[
2^{-i-1}/\kappa - \text{diam}(Q) \leq \text{dist}(y, S) - \text{diam}(Q) \leq \text{dist}(Q, S).
\]
On the other hand,
\[
\text{dist}(Q, S) \leq \text{dist}(y, S) \leq \|x - y\|_\infty \leq 2^{-i-1}.
\]
By (2.12), \(2^{-i-1}/5\kappa \leq \text{diam}(Q) \leq 2^{-i-1}\), that is, \(\ell(Q) \preceq 2^{-(i+1)}\).

It is also easy to see that \(Q \subset Q(x, 2^{-i})\). \(\Box\)
3. HARDY INEQUALITIES

In §3.2 we prove our main result, Theorem 3.7. Surprisingly, the proof of a restriction theorem [15, Theorem 4.8] can be modified to yield a proof of this Hardy type inequality. This proof is still technical, and for this reason in §3.1 we consider an illustrative special case whose proof is analogous but involves less technicalities. This simpler proof is based on [11], the main difference being that the energy type estimate in Lemma 2.6 replaces certain capacitary considerations. In order to make this paper self contained, we repeat some of the arguments in the aforementioned papers.

We say that a locally integrable function \( f \) is strictly defined at \( x \) if the limit
\[
\overline{f}(x) = \lim_{r \to 0^+} \int_{Q(x,r)} f(y) \, dy
\]
exists. Observe that \( f \) is strictly defined at its Lebesgue points and, by the Lebesgue differentiation theorem, \( f = \overline{f} \) a.e in \( \mathbb{R}^n \).

Let \( S \) be a \( d \)-set in \( \mathbb{R}^n \), \( n - 1 < d < n \). At those points \( x \in S \), in which \( \overline{f}(x) \) exists, we define the trace of the function \( f \) on \( S \) by
\[
\text{Tr}_S f(x) := \overline{f}(x).
\]
Assume that \( f \) belongs to the Triebel–Lizorkin space \( F_{pq}^s(\mathbb{R}^n) \) with \( s > (n - d)/p \), \( 1 < p < \infty \), \( 1 \leq q \leq \infty \). Then the trace of \( f \) is defined \( \mathcal{H}^d \)-almost everywhere on \( S \). In fact, under these assumptions, the exceptional set for the Lebesgue points of \( \overline{f} \) in \( \mathbb{R}^n \) has zero \( d \)-dimensional Hausdorff measure. In order to see this, apply the trivial embeddings \( F_{pq}^s(\mathbb{R}^n) \subset F_{p2}^{-\varepsilon}(\mathbb{R}^n) \) with \( 0 < \varepsilon < s - (n - d)/p \) and use [1, Theorem 1.2.4, Theorem 6.2.1].

3.1. THE CASE OF \( 0 < s < 1 \) AND \( q \leq p \). As a corollary of Theorem 3.1, we obtain Hardy type inequalities for Triebel–Lizorkin spaces \( F_{pq}^s(\mathbb{R}^n) \), \( 0 < s < 1 \), \( 1 \leq q \leq p \). Indeed, this follows from boundedness of relations \( F_{pq}^s(\mathbb{R}^n) \subset F_{p2}^{-\varepsilon}(\mathbb{R}^n) \subset W^{s,p}(\mathbb{R}^n) \), [29, pp. 6–7]. Recall also (2.2).

3.1. Theorem. Suppose that \( S \) is a \( d \)-set in \( \mathbb{R}^n \) and \( (n - d)/p < s < 1 \), where \( n - 1 < d < n \) and \( 1 < p < \infty \). Let \( f \in W^{s,p}(\mathbb{R}^n) \) be such that \( \text{Tr}_S f = 0 \) pointwise \( \mathcal{H}^d \) almost everywhere. Then \( f \) satisfies the Hardy type inequality,
\[
\left( \int_{\mathbb{R}^n} \frac{|f(x)|^p}{\text{dist}(x,S)^{sp}} \, dx \right)^{1/p} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^n)}.
\]

The proof is based upon [11].

We need some preparations. Let us first recall fractional inequalities for functions defined on cubes, Lemma 3.2 and Lemma 3.3. Both inequalities are invariant under scaling and translation. The following lemma follows from the proof of [14, Theorem 4.10].

3.2. Lemma. Let \( \sigma \in (0,1) \) and \( \tau \in (1,\infty) \). Let \( u \in L_{1\text{loc}}^1(Q) \) for a cube \( Q \) in \( \mathbb{R}^n \), \( n \geq 2 \). Then
\[
|u(x) - u_{B_Q}| \leq c_{n,\sigma,\tau} \int_Q \frac{g_Q(y)}{|x - y|^{n - \sigma}} \, dy
\]
if \( x \in Q \) is a Lebesgue point of \( u \). Here \( B_Q := B(x_Q, \ell(Q)/c_n) \), \( c_n > 2 \), and the function \( g_Q \) is defined by \[
g_Q(y) = \left( \int_Q \frac{|u(y) - u(z)|^r}{|y - z|^{n + sr}} \, dz \right)^{1/r}.
\]

The following fractional Sobolev–Poincaré inequality is a consequence of [14, Remark 4.14].

3.3. **Lemma.** Let \( Q \) be a cube in \( \mathbb{R}^n \), \( n \geq 2 \). Suppose that \( p, r \in [1, \infty) \), and \( s \in (0, 1) \) satisfy \( 0 \leq \frac{1}{r} - \frac{1}{p} < \frac{s}{n} \). Then, for every \( u \in L^r(Q) \),

\[
\int_Q |u(x) - u_Q|^p \, dx \leq c|Q|^{1 + ps/n - p/r} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^r}{|x - y|^{n + sr}} \, dy \, dx \right)^{p/r}.
\]

Here the constant \( c > 0 \) is independent of \( Q \) and \( u \).

The last lemma we need is the following.

3.4. **Lemma.** Assume that \( S \) is a non-trivial closed set in \( \mathbb{R}^n \), \( n \geq 2 \), and let \( W_S \) be a Whitney decomposition of \( \mathbb{R}^n \setminus S \). Suppose that \( 1 < s < \infty \) and \( \kappa \geq 1 \). Then

\[
\sum_{Q \in W_S} |Q|^{2} \left( \int_{Q} \int_{Q} |g(x, y)| \, dx \, dy \right)^{s} \leq c_n s \kappa^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |g(x, y)|^s \, dx \, dy
\]

for every \( g \in L^s(\mathbb{R}^n \times \mathbb{R}^n) \).

**Proof.** Throughout this proof, we denote \( m = 2n \). Recall a (non-centered) maximal function of a locally integrable function \( f : \mathbb{R}^m \to [-\infty, \infty] \),

\[
\mathcal{M}f(x) = \sup_{x \in Q} \mathcal{F}_{Q} \int_{Q} |f(y)| \, dy.
\]

The supremum is taken over all cubes in \( \mathbb{R}^m \) containing \( x \in \mathbb{R}^m \).

Let us rewrite the left hand side of inequality (3.5) as

\[
\text{LHS} = \kappa^n \sum_{Q \in W_S} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} X_{Q}(z) X_{Q}(w) \left( \int_{Q} \int_{Q} |g(x, y)| \, dx \, dy \right)^{s} \, dz \, dw.
\]

By definition of maximal function,

\[
\kappa^{-n} \text{LHS} \leq \sum_{Q \in W_S} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} X_{Q}(z) X_{Q}(w) \{\mathcal{M}g(z, w)\}^s \, dz \, dw
\]

\[
\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{\mathcal{M}g(z, w)\}^s \, dz \, dw.
\]

The boundedness of maximal operator on \( L^s(\mathbb{R}^m) \) yields the right hand side of inequality (3.5). □
Proof of Theorem 3.1. Fix a number \( r \in (1, p) \) such that \( \omega = n(1/p - 1/r) + s \in (0, s) \) and
\[
n - d < \omega r < sp, \quad 0 < 1/r - 1/p < \omega/n.
\]
For each \( Q \in W_S \) with \( \ell(Q) \leq 1 \), we write \( Q^* = 12Q \). We let \( B_{Q^*} = B(x_Q, \ell(Q^*)/c_n) \) be given by Lemma 3.2. The proof proceeds with an application of inequality (3.6)
\[
\int_Q |f(x) - f_{B_{Q^*}}|^p \, dx + |Q||f_{B_{Q^*}}|^p
\]
\[
\leq |Q^*|^{1+pw/n-p/r} \left( \int_{Q^*} \int_{Q^*} \frac{|f(x) - f(y)|^r}{|x-y|^{n+\omega r}} \, dy \, dx \right)^{p/r}.
\]
Let us postpone the proof of this inequality for the time being, and finish with the main line of the argument first. By property (2.12) of Whitney cubes and inequality (3.6),
\[
I := \int_{\mathbb{R}^n} \frac{|f(x)|^p}{\text{dist}(x, S)^sp} \, dx
\]
\[
\leq \sum_{Q \in W_S} \text{diam}(Q)^{-sp} \int_Q |f(x)|^p \, dx
\]
\[
\leq \|f\|_{L^p(\mathbb{R}^n)}^p + \sum_{Q \in W_S \atop \ell(Q) \leq 1} |Q^*|^{1+p(\omega/n+1/r-s/n)} \left( \int_{Q^*} \int_{Q^*} \frac{|f(x) - f(y)|^r}{|x-y|^{n+\omega r}} \, dy \, dx \right)^{p/r}.
\]
The definition of \( \omega \) implies that
\[
1 + p(\omega/n + 1/r - s/n) = 2.
\]
Thus, Lemma 3.4 with \( s = p/r > 1 \) yields
\[
I \leq \|f\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{ps+n}} \, dx \, dy \leq \|f\|_{W^{s,p}(\mathbb{R}^n)}^p.
\]
Here we also used the identity \( p(n + \omega r)/r = ps + n \).

It remains to verify inequality (3.6). Using the facts that \( B_{Q^*} \subset Q^* \) and the measures are comparable, this bound for the integral term is a consequence of Lemma 3.3. Hence, it is enough to estimate the remaining term \( |Q||f_{B_{Q^*}}|^p \).

Let us fix \( y_Q \in S \) such that \( \text{dist}(y_Q, Q) = \text{dist}(Q, S) \). Denote
\[
E_Q = Q(y_Q, \ell(Q)) \cap S,
\]
and define
\[
g_{Q^*}(y) = \left( \int_{Q^*} \frac{|f(y) - f(z)|^r}{|y-z|^{n+\omega r}} \, dz \right)^{1/r}.
\]
Recall that $\mathcal{H}^d$ almost every point $x \in S$ is both a Lebesgue point of $\bar{f}$, and satisfies $\bar{f}(x) = 0$. Thus, by the fact that $E_Q \subset Q^*$ and Lemma 3.2 applied with $\bar{f}$,

$$|f_{B_Q^*}| \cdot \mathcal{H}^d(E_Q) = \int_{E_Q} |\bar{f}(x) - f_{B_Q^*}| \, d\mathcal{H}^d(x) \leq \int_{Q^*} \int_{E_Q} \frac{d\mathcal{H}^d(x)}{|x-y|^{n-\omega}} g_{Q^*}(y) \, dy.$$ 

Observe that $Q^* \subset Q(y_Q, t)$ and $E_Q \subset Q(y, t)$ for every $y \in Q^*$, where $t = 2 \text{diam}(Q^*)$. Thus, by Hölder’s inequality, followed by Lemma 2.6 with $x = y_Q$, we obtain

$$|f_{B_Q^*}| \mathcal{H}^d(E_Q) \leq \|g_{Q^*} \chi_{Q^*}\|_r \cdot t^{d+\omega-n/r}.$$ 

Since $t^d \leq \mathcal{H}^d(E_Q)$, the upper bound in (3.6) for $|Q||f_{B_Q^*}|^p$ follows. \hfill $\Box$

### 3.2. The general case.

The following theorem is our main result.

#### 3.7. Theorem. Let $S$ be a $d$-set in $\mathbb{R}^n$, $n-1 < d < n$. Suppose that $1 < p < \infty$, $1 \leq q \leq \infty$, and $s > (n-d)/p$. Let $f \in F^s_{pq}(\mathbb{R}^n)$ be such that $\text{Tr}_S f = 0$ pointwise $\mathcal{H}^d$ almost everywhere. Then $f$ satisfies the Hardy type inequality,

$$\left( \int_{\mathbb{R}^n} \frac{|f(x)|^p}{\text{dist}(x, S)^{sp}} \, dx \right)^{1/p} \leq c \|f\|_{F^s_{pq}(\mathbb{R}^n)}.$$ 

The constant $c > 0$ depends on $n$, $d$, $p$, $s$, and $S$.

We will need several preparations.

Let $Q^j_1$ denote a cube $Q(x, 2^{-j})$ with $x \in S$ and $j \in \mathbb{Z}$. Let $P_{Q^j_1}$ be a projection from $L^p(Q^j_1)$ to $P_{k-1}$ with $k = [s] + 1$ such that $E_k(f, Q^j_1)_{L^p(\mathbb{R}^n)}$ is equivalent to

$$\left( \int_{Q^j_1} |f - P_{Q^j_1} f|^p \, dy \right)^{1/p}.$$ 

For the construction of these projections, we refer to [23, Proposition 3.4] and [9]. We use the following properties of these polynomial projections:

i) If $Q' \subset Q$ are cubes as above and $|Q'| \geq c|Q|$, then for every $z \in Q'$

$$|P_Q f(z) - P_{Q'} f(z)| \leq C(c, n, k) \int_Q |f - P_Q f| \, dy;$$

ii) $\lim_{j \to \infty} P_{Q^j_1} f(z) = f(z)$ at every Lebesgue point $z \in S$ of $f$.

Let $\mathcal{W}_S$ denote the Whitney decomposition of $\mathbb{R}^n \setminus S$. To every $Q = Q(x_Q, r_Q)$ in $\mathcal{W}_S$, assign the cube $a(Q) := Q(a_Q, r_Q/2)$, where $a_Q \in S$ is such that $|x_Q - a_Q|_\infty = \text{dist}(x_Q, S)$. Then

$$\mathcal{H}^d(a(Q) \cap S) \geq c r_Q^d,$$

(3.8) if $\text{diam}(Q) \leq 1$.

This follows from Definition 2.4. The constant $c > 0$ depends on $S$. 

3.9. Lemma. Let $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, be a function for which $\mathcal{H}^d$-almost every point in a given $d$-set $S$ is a Lebesgue point, $n - 1 < d < n$. Then, for every $i \in \mathbb{N}$, we have

\[
\left\{ \sum_{Q \in \mathcal{W}_S \cap D_{i+5}} \int_{32a(Q) \cap S} |f - P_{32a(Q)}f|^p \, d\mathcal{H}^d \right\}^{1/p} \\
\leq 2^{id/p} \sum_{j=1}^{\infty} \|E_k(f, Q(\cdot, 2^{-j}))_{L^p(\mathbb{R}^n)}\|_{L^p(S)}.
\]

Proof. We claim that, for every $x \in 32a(Q) \cap S$, where $Q \in \mathcal{W}_S \cap D_{i+5}$,

\[
\lambda_Q := \left\{ \int_{32a(Q) \cap S} |f - P_{32a(Q)}f|^p \, d\mathcal{H}^d \right\}^{1/p} \\
\leq E_k(f, Q^i_{x})_{L^p(\mathbb{R}^n)} + \sum_{j=i+1}^{\infty} \left( \int_{Q^j_{x} \cap S} E_k(f, Q^j_{z})_{L^p(\mathbb{R}^n)} \, d\mathcal{H}^d(z) \right)^{1/p}.
\]

In order to prove (3.11), we first estimate

\[
\lambda_Q^i \lesssim \int_{Q^j_{x} \cap S} |f - P_{Q^i_{x}}f|^p \, d\mathcal{H}^d + \int_{32a(Q) \cap S} |P_{Q^i_{x}}f - P_{32a(Q)}f|^p \, d\mathcal{H}^d,
\]

where we used the first part of inclusion $32a(Q) \subset Q^{i+1}_{x} \subset Q^i_{x}$. These inclusions and property i) allow us to bound the second term on the right hand side by a constant multiple of $E_k(f, Q^i_{x})_{L^p(\mathbb{R}^n)}$.

Consider the first term on the right hand side of (3.12), and let $z \in Q^j_{x} \cap S$ be a Lebesgue point of $f$. Then, by property ii),

\[
|f(z) - P_{Q^i_{x}}f(z)| \leq |P_{Q^i_{x}}f(z) - P_{Q^{i+1}_{x}}f(z)| + \sum_{j=i+1}^{\infty} |P_{Q^i_{x}}f(z) - P_{Q^{j+1}_{x}}f(z)|.
\]

Since $z \in Q^j_{x} \subset Q^i_{x}$, the property i) and Hölder’s inequality give

\[
|P_{Q^i_{x}}f(z) - P_{Q^{j+1}_{x}}f(z)| \leq \left( \int_{Q^j_{x}} |f - P_{Q^i_{x}}f|^p \, dy \right)^{1/p} \lesssim E_k(f, Q^i_{x})_{L^p(\mathbb{R}^n)}.
\]

Similarly $|P_{Q^j_{x}}f(z) - P_{Q^{j+1}_{x}}f(z)| \lesssim E_k(f, Q^j_{x})_{L^p(\mathbb{R}^n)}$ for $j \in \{i+1, \ldots\}$. We have shown that

\[
|f(z) - P_{Q^i_{x}}f(z)| \lesssim E_k(f, Q^i_{x})_{L^p(\mathbb{R}^n)} + \sum_{j=i+1}^{\infty} E_k(f, Q^j_{x})_{L^p(\mathbb{R}^n)}
\]

where $z \in Q^{i+1}_{x} \cap S$ is a Lebesgue point of $f$. Since $\mathcal{H}^d$-almost every point in $S$ is a Lebesgue point of $f$, we can average the last inequality over the set $Q^{i+1}_{x} \cap S$. This gives

\[
\left( \frac{1}{|Q^{i+1} \cap S|} \int_{Q^{i+1} \cap S} |f(z) - P_{Q^i_{x}}f(z)|^p \, d\mathcal{H}^d(z) \right)^{1/p}.
\]
Using Fubini's theorem and Definition 2.4, we get

\[
\left\lfloor \int_\mathbb{R}^{n+1} \mathcal{E}_k(f, Q^j_x)_{L^p(\mathbb{R}^n)} d\mathcal{H}^d(z) d\mathcal{H}^d(x) \right\rfloor^{1/p} \leq 2^{id/p} \left\lfloor \sum_{Q \in \mathcal{W}_S \cap D_{i+5}} \lambda_Q^p \chi_{32a(Q)}(x) d\mathcal{H}^d(x) \right\rfloor^{1/p}.
\]

This concludes the proof of (3.11).

The family \( \{32a(Q) : Q \in \mathcal{W}_S \cap D_{i+5}\} \) has a bounded overlapping property. Hence, by (3.8) and (3.11), the left hand side of (3.10) can be estimated by a constant multiple of

\[
2^{id/p} \left( \sum_{Q \in \mathcal{W}_S \cap D_{i+5}} \lambda_Q^p \chi_{32a(Q)}(x) d\mathcal{H}^d(x) \right)^{1/p} + 2^{id/p} \left( \int_\mathbb{R}^{n+1} \mathcal{E}_k(f, Q^j_x)_{L^p(\mathbb{R}^n)} d\mathcal{H}^d(z) d\mathcal{H}^d(x) \right)^{1/p}.
\]

Using Fubini’s theorem and Definition 2.4, we get

\[
\int_\mathbb{R}^{n+1} \mathcal{E}_k(f, Q^j_x)_{L^p(\mathbb{R}^n)} d\mathcal{H}^d(z) d\mathcal{H}^d(x) \leq 2^{id} \int_\mathbb{R} \mathcal{E}_k(f, Q^j_x)_{L^p(\mathbb{R}^n)} X_{Q^j_x} d\mathcal{H}^d(x) d\mathcal{H}^d(z) \leq 2^{id} \int_\mathbb{R} \mathcal{E}_k(f, Q^j_x)_{L^p(\mathbb{R}^n)} d\mathcal{H}^d(z) \leq \int_\mathbb{R} \mathcal{E}_k(f, Q^j_x)_{L^p(\mathbb{R}^n)} d\mathcal{H}^d(z).
\]

Collecting the estimates above, we obtain inequality (3.10). □

For \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) and \( s > 0 \) denote

\[
f^x_{s,p}(x) = \sup_{0 < r \leq 1} r^{-s} \mathcal{E}_k(f, Q(x, r))_{L^p(\mathbb{R}^n)}, \quad x \in \mathbb{R}^n, \ k = [s] + 1.
\]

3.13. **Lemma.** Let \( S \) be a d-set in \( \mathbb{R}^n \), \( n - 1 < d < n \). Suppose that \( 1 < p < \infty \), \( s > 0 \), \( k = [s] + 1 \) and \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \). Then, for every \( i \in \{2, 3, \ldots\} \),

\[
\|\mathcal{E}_k(f, Q(\cdot, 2^{-i}))_{L^p(\mathbb{R}^n)}\|_{L^p(S)} \leq c 2^{-i(s-(n-d)/p)} \left( \int_{\bigcup_{Q \in \mathcal{W}_S : f(Q) = (Q)_{2^{-i-4}}} f^x_{s,p}(x) dx \right)^{1/p},
\]

where the constant \( c \) depends on \( s, p, n \) and \( S \).

**Proof.** Let \( \mathcal{F} = \{Q(x, 2^{-i-3}) : x \in S\} \). By the 5r-covering theorem, see e.g. [22, p. 23], there are disjoint cubes \( Q_m = Q(x_m, 2^{-i-3}) \in \mathcal{F} \), \( m = 1, 2, \ldots \) (if there are only a finite number of cubes we change the indexing), such that \( S \subset \bigcup_{m=1}^{\infty} 5Q_m \). Hence

\[
I := \int_S \mathcal{E}_k(f, Q(x, 2^{-i}))_{L^p(\mathbb{R}^n)} d\mathcal{H}^d(x)
\]
\[ I \leq \sum_{m=1}^{\infty} \int_{3Q_m \cap S} \mathcal{E}_k(f, Q(x, 2^{-i}))_{L^p(\mathbb{R}^n)}^p dH^d(x). \]

Notice that, if \( x \in 5Q_m = Q(x_m, (5/8)2^{-i}) \), then \( Q(x, 2^{-i}) \subset Q(x_m, 2^{-i+1}) \). By (2.3),
\[ \mathcal{E}_k(f, Q(x, 2^{-i}))_{L^p(\mathbb{R}^n)} \leq c \mathcal{E}_k(f, Q(x_m, 2^{-i+1}))_{L^p(\mathbb{R}^n)}. \]

Using the observation above and Definition 2.4 we can continue as follows:
\[ I \leq c \sum_{m=1}^{\infty} \mathcal{H}^d(5Q_m \cap S) \mathcal{E}_k(f, Q(x_m, 2^{-i+1}))_{L^p(\mathbb{R}^n)}^p \leq c2^{-id} \sum_{m=1}^{\infty} \mathcal{E}_k(f, Q(x_m, 2^{-i+1}))_{L^p(\mathbb{R}^n)}^p. \]

By Remark 2.9 and Lemma 2.13, for every \( Q_m \), there is \( R_m \in \mathcal{W}_S \) such that \( \ell(R_m) \leq 2^{-i+4} \) and \( R_m \subset Q_m \). If \( x \in R_m \), then \( Q(x_m, 2^{-i+1}) \subset Q(x, 2^{-i+2}) \) and therefore
\[ (3.14) \quad \mathcal{E}_k(f, Q(x_m, 2^{-i+1}))_{L^p(\mathbb{R}^n)} \leq c \mathcal{E}_k(f, Q(x, 2^{-i+2}))_{L^p(\mathbb{R}^n)} \leq c2^{-i} f^p_{s,p}(x). \]

Since \( \ell(R_m) \leq 2^{-i+4} \), we have \( |R_m| \geq c2^{-in} \). By using this and (3.14), we get
\[ I \leq c2^{i(n-d)} \sum_{m=1}^{\infty} |R_m| \mathcal{E}_k(f, Q(x_m, 2^{-i+1}))_{L^p(\mathbb{R}^n)}^p \leq c2^{i(n-d)-sp} \sum_{m=1}^{\infty} \int_{R_m} f^p_{s,p}(x) \, dx \leq c2^{i(n-d)-sp} \int_{\cup \{Q \in \mathcal{W}_S : \ell(Q) \leq 2^{-i-4}\}} f^p_{s,p}(x) \, dx. \]

Taking the \( p' \)-th roots yields the required estimate. \( \square \)

We are now ready for the proof of Theorem 3.7.

Proof. Since \( F^s_{pq}(\mathbb{R}^n) \subset F^s_{pq}(\mathbb{R}^n) \) boundedly, it suffices to verify that
\[ H := \left( \int_{\mathbb{R}^n} \frac{|f(x)|^p}{\text{dist}(x, S)^{sp}} \, dx \right)^{1/p} \leq \|f\|_{L^p(\mathbb{R}^n)} + \|f^p_{s,p}\|_{L^p(\mathbb{R}^n)}, \]
recalling the definition of \( F^s_{pq}(\mathbb{R}^n) \) in §2.2. By properties of Whitney cubes \( Q \in \mathcal{W}_S \), we can bound \( H^p \) by a constant multiple of
\[ \|f\|_{L^p(\mathbb{R}^n)}^p + \sum_{Q \in \mathcal{W}_S} \text{diam}(Q)^{n-sp} \int_{Q} |f(x) - P_{32a(Q)}f(x)|^p + |P_{32a(Q)}f(x)|^p \, dx. \]
By properties of the projection operator and monotonicity (2.3) of local approximations,

\[ \sum_{Q \in \mathcal{W}_S \cap (Q) \leq 2^{-7}} \text{diam}(Q)^{n-sp} \int_Q |f(x) - P_{32a(Q)}f(x)|^p \, dx \]

\[ \leq \sum_{Q \in \mathcal{W}_S \cap (Q) \leq 2^{-7}} \text{diam}(Q)^{n-sp} E_k(f, 32a(Q))^p_{L_p(\mathbb{R}^n)} \leq \|f^s_p\|^p_{L_p(\mathbb{R}^n)}. \]

Recall that \( Q \subset 32a(Q) \), and their measures are comparable. By a Remez-type inequality, see Theorem 2.5, and the fact that \( \mathcal{H}^d \) almost every point \( x \in S \) satisfies \( f(x) = 0 \),

\[ \sum_{Q \in \mathcal{W}_S \cap (Q) \leq 2^{-7}} \text{diam}(Q)^{n-sp} \int_Q |P_{32a(Q)}f(x)|^p \, dx \]

\[ \leq \sum_{i=2}^{\infty} 2^i (sp-n) \sum_{Q \in \mathcal{W}_S \cap \mathcal{D}_{l+5}} \int_{32a(Q) \cap S} |\bar{f}(z) - P_{32a(Q)}\bar{f}(z)|^p \, d\mathcal{H}^d(z) =: \Sigma. \]

Recall that \( \mathcal{H}^d \) almost every point is a Lebesgue point of \( \bar{f} \). Hence, by Lemma 3.9, Lemma 3.13, and Lemma 2.1 with \( \sigma = sp - (n-d) > 0 \),

\[ \Sigma \leq \sum_{i=2}^{\infty} 2^i (sp-(n-d)) \left( \sum_{i=1}^{\infty} 2^{-j(s-(n-d)/p)} \left( \int_{\cup \{Q \in \mathcal{W}_S : \ell(Q) \geq 2^{-i-4}\}} f^s_{p_0}(x)^p \, dx \right)^{1/p} \right)^p \]

\[ \leq \sum_{i=2}^{\infty} \int_{\cup \{Q \in \mathcal{W}_S : \ell(Q) \geq 2^{-i-4}\}} f^s_{p_0}(x)^p \, dx. \]

Let us denote \( U_i := \cup \{\text{int } Q : Q \in \mathcal{W}_S \text{ and } \ell(Q) \geq 2^{-i-4}\} \), and choose \( k_0 \in \mathbb{N} \) such that \( 2^{-k_0} < 1/5\kappa \). Then we claim that

\[ (3.16) \quad U_i \cap U_{i'} = \emptyset, \quad \text{if } i \neq i' \text{ and } i, i' \equiv k \mod k_0. \]

To verify this claim, let \( i > i' \) be such that \( i, i' \equiv k \mod k_0 \). Then \( i - i' \geq k_0 \). In particular, if \( Q \in \mathcal{W}_S \), \( \ell(Q) \geq 2^{-i-4} \), and \( Q' \in \mathcal{W}_S \), \( \ell(Q') \geq 2^{-i'-4} \), then

\[ \text{diam}(Q) \leq 2^{-i-4} \leq 2^{-k_0}2^{-i'-4} < 2^{-i'-4}/5\kappa \leq \text{diam}(Q'). \]

It follows that \( Q \neq Q' \). Since the interiors of Whitney cubes are pairwise disjoint, we find that \( \text{int } Q \cap \text{int } Q' = \emptyset \). Hence, \( (3.16) \) holds.

From \( (3.16) \) it follows that

\[ \sum_{i=2}^{\infty} \int_{\cup \{Q \in \mathcal{W}_S : \ell(Q) \geq 2^{-i-4}\}} f^s_{p_0}(x)^p \, dx = \sum_{k=0}^{k_0-1} \sum_{i \geq 2} \int_{U_i} f^s_{p_0}(x)^p \, dx \]

\[ \leq k_0 \|f^s_{p_0}\|^p_{L_p}. \]
Combining the estimates (3.15) and (3.17), we find that \( \Sigma \leq c \| f_{x,p} \|_p^p \). \( \square \)

4. Extension problems

As an application of Hardy type inequality, we study certain extension problems.

4.1. Extension by zero. First we study the problem of zero extension. For instance, Corollary 4.2 shows that the characteristic function \( \chi_{\Omega} \) of a domain whose boundary is a \( d \)-set, \( n - 1 < d < n \), is a pointwise multiplier in the subspace \( \{ f \in F_{pq}(\mathbb{R}^n) : \text{Tr}_{d\partial\Omega} = 0 \} \) if \( s > (n - d)/p \).

4.1. Proposition. Let \( \Omega \) be a domain in \( \mathbb{R}^n \) whose boundary is porous (in particular, it suffices that \( \partial\Omega \) is a \( d \)-set with \( d < n \)). Let \( f \in F_{pq}(\mathbb{R}^n) \), \( 1 < p < \infty \), \( 1 \leq q < \infty \), \( s > 0 \). Then

\[
\| f\chi_{\Omega} \|_{F_{pq}(\mathbb{R}^n)} \leq \| f \|_{F_{pq}(\mathbb{R}^n)} + \left( \int_{\Omega} \frac{|f(y)|^p}{\text{dist}(y, \partial\Omega)^s} dy \right)^{1/p}.
\]

The implied constant depends on \( p, q, s, n, d \), and \( \partial\Omega \).

Proof. For convenience, denote \( \tilde{f} = f\chi_{\Omega} \). By [26, Theorem 2.2.2] or [15, Remark 3.4], the norm \( \| \tilde{f} \|_{F_{pq}(\mathbb{R}^n)} \) is equivalent to the quantity

\[
\left\| \left( \sum_{j=0}^{\infty} 2^{jsq} \mathcal{E}_k(\tilde{f}, \mathcal{Q}(\cdot, 2^{-j}))_{L^1(\mathbb{R}^n)}^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} + \| \tilde{f} \|_{L^p(\mathbb{R}^n)}, \quad k = [s] + 1.
\]

The second summand is clearly controlled, and we focus on the first one. By monotonicity of local approximations (2.3), if \( j \in \mathbb{N}_0 \),

\[
\mathcal{E}_k(\tilde{f}, \mathcal{Q}(x, 2^{-j}))_{L^1(\mathbb{R}^n)} \leq \sum_{Q \in \mathcal{D}_j} \chi_Q(x) \mathcal{E}_k(\tilde{f}, 4Q)_{L^1(\mathbb{R}^n)}, \quad x \in \mathbb{R}^n.
\]

Next we split the summation on the right hand side in two parts, depending on whether or not \( Q \in \mathcal{C} := \mathcal{C}_{\partial\Omega, \gamma} \) with \( \gamma = 5 \), recall definition (2.10).

Observe that \( 4Q \cap \partial\Omega = \emptyset \) if \( Q \in \mathcal{D}_j \setminus \mathcal{C} \) with \( j \geq 0 \). Thus, for such cubes, we have either \( 4Q \subset \Omega \) or \( 4Q \subset \mathbb{R}^n \setminus \overline{\Omega} \). In both cases,

\[
\mathcal{E}_k(\tilde{f}, 4Q)_{L^1(\mathbb{R}^n)} \leq \mathcal{E}_k(f, 4Q)_{L^1(\mathbb{R}^n)}.
\]

Hence,

\[
\left\| \left( \sum_{j=0}^{\infty} 2^{jsq} \left( \sum_{Q \in \mathcal{D}_j \setminus \mathcal{C}} \chi_Q(x) \mathcal{E}_k(f, 4Q)_{L^1(\mathbb{R}^n)} \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq \| f \|_{F_{pq}(\mathbb{R}^n)}.
\]

The last step follows from monotonicity of local approximations.
In order to estimate the remaining term, associated with cubes \( Q \in \mathcal{C} \), we use Theorem 2.11. Note also that cubes in \( D_j \) have mutually disjoint interiors. Thus, we have

\[
A := \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} \left( \sum_{Q \in D_j \cap \mathcal{C}} \chi_{Q} \mathcal{E}_k(\tilde{f}, 4Q)_{L^1(\mathbb{R}^n)}^{q} \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} 
\leq \left( \int_{\mathbb{R}^n} \sum_{Q \in \mathcal{C}} \chi_{Q}(x) \ell(Q)^{-sp} \mathcal{E}_k(\tilde{f}, 4Q)_{L^1(\mathbb{R}^n)}^{p} \, dx \right)^{1/p}.
\]

By the following inequality,

\[
\mathcal{E}_k(\tilde{f}, 4Q)_{L^1(\mathbb{R}^n)}^{p} \leq \int_{4Q} |\tilde{f}|^p \, dx = \frac{1}{|4Q|} \int_{4Q \cap \Omega} |f|^p \, dx,
\]

and definition of family \( \mathcal{C} \), we obtain

\[
A^p \leq \int_{\Omega} \left\{ \sum_{Q \in \mathcal{C}} \ell(Q)^{-sp} \chi_{4Q}(y) \right\} |f(y)|^p \, dy \leq \int_{\Omega} \frac{|f(y)|^p}{\text{dist}(y, \partial\Omega)^{sp}} \, dy.
\]

This completes the proof. \( \square \)

The following is a consequence of Theorem 3.7 and Proposition 4.1. See also Remark 2.9.

### 4.2. Corollary

Suppose that \( \Omega \) is a domain in \( \mathbb{R}^n \) whose boundary is a \( d \)-set, \( n - 1 < d < n \). Suppose also that \( 1 < p < \infty \), \( 1 \leq q < \infty \), and \( s > (n - d)/p \). Let \( f \in \text{F}^s_{pq}(\mathbb{R}^n) \) be such that \( \text{Tr}_\partial \Omega f = 0 \) pointwise \( \mathcal{H}^d \) almost everywhere. Then

\[
\|f\chi_\Omega\|_{\text{F}^s_{pq}(\mathbb{R}^n)} \leq \|f\|_{\text{F}^s_{pq}(\mathbb{R}^n)},
\]

where the implied constant depends on \( p, q, s, n, d \), and \( \partial\Omega \).

An application of the corollary is Theorem 4.4. In order to formulate this theorem, we recall some notation which is common in the literature on function spaces on domains, \([29, 30]\).

Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( 1 \leq p < \infty \), \( 1 \leq q \leq \infty \), and \( s > 0 \). Then

\[
\text{F}^s_{pq}(\Omega) = \{ f \in L^p(\Omega) : \text{there is a } g \in \text{F}^s_{pq}(\mathbb{R}^n) \text{ with } g|_\Omega = f \}
\]

\[
\|f\|_{\text{F}^s_{pq}(\Omega)} = \inf \|g\|_{\text{F}^s_{pq}(\mathbb{R}^n)},
\]

where the infimum is taken over all \( g \in \text{F}^s_{pq}(\mathbb{R}^n) \) such that \( g|_\Omega = f \) pointwise a.e. As usual, we also denote

(4.3) \[
\text{F}^s_{pq}(\Omega) = \{ f \in L^p(\Omega) : \text{there is a } g \in \text{F}^s_{pq}(\mathbb{R}^n) \text{ with } g|_\Omega = f \text{ and } \text{supp } g \subset \overline{\Omega} \}
\]

\[
\|f\|_{\text{F}^s_{pq}(\Omega)} = \inf \|g\|_{\text{F}^s_{pq}(\mathbb{R}^n)},
\]

where the infimum is taken over all \( g \) admitted in (4.3).

Finally, \( \text{F}^s_{pq}(\Omega) \) is a completion of \( \text{C}^\infty_0(\Omega) \) in \( \text{F}^s_{pq}(\Omega) \).
4.4. **Theorem.** Let $\Omega$ be a domain whose closure $\overline{\Omega}$ is an $n$-set, and whose boundary $\partial \Omega$ is a $d$-set with $n - 1 < d < n$. Suppose that $1 < p < \infty$, $1 \leq q < \infty$ and $s > (n - d)/p$. Then

$$F_{pq}^s(\Omega) \subset F_{pq}^s(\Omega),$$

and this inclusion is bounded.

4.5. **Remark.** Let us first clarify the role of assumptions in the theorem. Suppose that $f \in F_{pq}^s(\Omega)$ and $g \in F_{pq}^s(\mathbb{R}^n)$ is any extension of $f$. Recall that $\mathcal{H}^d$ almost every point in $\partial \Omega$ is a Lebesgue point of $g$. By Lebesgue differentiation theorem, and the assumption that $\overline{\Omega}$ is an $n$-set,

$$\text{Tr}_{\partial \Omega} g(x) = \lim_{r \to 0^+} \frac{1}{|Q(x, r) \cap \Omega|} \int_{Q(x, r) \cap \Omega} f(y) \, dy$$

in the Lebesgue points $x \in \partial \Omega$ of $g$. Here we also used the fact that $\partial \Omega$ has zero $n$-measure. To state the conclusion otherwise, any extension of $f$ has the same trace $\mathcal{H}^d$ a.e. on $\partial \Omega$, and this trace coincides a.e. with the interior trace given above.

**Proof of Theorem 4.4.** Suppose that $f \in F_{pq}^\circ(\Omega)$, then there is a sequence $f_j \in C_c^\infty(\Omega)$ such that $f_j \to f$ in $F_{pq}^\circ(\Omega)$. By definition, there are functions $g$ and $\{g_j\}_{j \in \mathbb{N}}$ belonging to $F_{pq}^s(\mathbb{R}^n)$ for which $g|_{\Omega} = f$ and $G_j|_{\Omega} = f - f_j$, $j \in \mathbb{N}$. Moreover, we can suppose that $\|g\|_{F_{pq}^s(\mathbb{R}^n)} \leq 2\|f\|_{F_{pq}^s(\Omega)}$ and $\|G_j\|_{F_{pq}^s(\mathbb{R}^n)} \leq 2\|f - f_j\|_{F_{pq}^s(\Omega)}$ for all $j$.

Since $(g - G_j)|_{\Omega} = f_j$, and the trace on the boundary is independent of the extension, the trace of $g - G_j$ on $\partial \Omega$ vanishes. Thus,

$$\|\text{Tr}_{\partial \Omega} g\|_{L^p(\partial \Omega)} = \lim_{j \to \infty} \|\text{Tr}_{\partial \Omega} g - \text{Tr}_{\partial \Omega} (g - G_j)\|_{L^p(\partial \Omega)}.$$

By linearity and boundedness of the trace operator, [15, Theorem 4.8],

$$\|\text{Tr}_{\partial \Omega} g\|_{L^p(\partial \Omega)} = \lim_{j \to \infty} \|\text{Tr}_{\partial \Omega} G_j\|_{L^p(\partial \Omega)} \leq \lim_{j \to \infty} \|G_j\|_{F_{pq}^s(\mathbb{R}^n)} = 0.$$

We have shown that $g$ has zero trace on $\partial \Omega$. By Corollary 4.2,

$$\|g\chi_{\Omega}\|_{F_{pq}^s(\mathbb{R}^n)} \leq \|g\|_{F_{pq}^s(\mathbb{R}^n)} \leq 2\|f\|_{F_{pq}^s(\Omega)},$$

which is a sufficient estimate since $(g\chi_{\Omega})|_{\Omega} = f$, and the support of $g\chi_{\Omega}$ is contained in $\overline{\Omega}$. \hfill $\Box$

4.6. **Remark.** Theorem 4.4 is related to the following result due to Caetano, [6, Corollary 2.7]. Let $\Omega$ be a bounded domain such that $\partial \Omega$ is a $d$-set for some $d < n$. Then

$$F_{pq}^\circ(\Omega) = F_{pq}^s(\Omega), \quad s < (n - d)/p.$$  

(4.7) This identification fails if $p > 1$, $s > (n - d)/p$, and $\Omega$ is an $n$-set whose boundary is a $d$-set with $d < n$. [6, Proposition 3.7]. Theorem 4.4 gives a partial counterpart of identification (4.7) in case of $s > (n - d)/p$. 

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4.2. Extension from the boundary trace. We close this paper by considering an extension of a smooth function \( f \) from its trace \( \text{Tr}\partial\Omega f \) on the boundary of a given domain \( \Omega \). In the complement of the domain, an extension is defined in terms of polynomial projections of \( \text{Tr}\partial\Omega f \). Throughout, we assume that the boundary \( \partial\Omega \) is a d-set with \( n - 1 < d < n \).

To each cube \( Q = Q(x_Q, r_Q) \in \mathcal{W}_3\Omega \) in the Whitney decomposition of \( \mathbb{R}^n \setminus \partial\Omega \), we assign a nearby cube \( a(Q) := Q(a_Q, r_Q/2) \), where \( a_Q \in \partial\Omega \) is such that \( \|x_Q - a_Q\|_{\infty} = \text{dist}(x_Q, \partial\Omega) \).

Let \( \{P_\beta\}_{|\beta| \leq k} \) be an orthonormal basis of \( \mathcal{P}_k \), \( k \geq 0 \), with respect to the inner product

\[
\langle p, q \rangle = \int_{a(Q) \cap \partial\Omega} pq \, d\mathcal{H}^d, \quad p, q \in \mathcal{P}_k.
\]

Observe that the zero set of \( p \in \mathcal{P}_k \setminus \{0\} \) has Hausdorff dimension at most \( n - 1 \). Hence, the formula gives an inner product. Define a linear operator \( \text{Pr}_{k,a(Q)} : L^1(a(Q) \cap \partial\Omega) \to \mathcal{P}_k \) by setting

\[
\text{Pr}_{k,a(Q)} f := \sum_{|\beta| \leq k} \langle f, P_\beta \rangle P_\beta = \sum_{|\beta| \leq k} \left( \int_{a(Q) \cap \partial\Omega} f P_\beta \, d\mathcal{H}^d \right) P_\beta
\]

if \( \text{diam}(Q) \leq \Delta := 16000 \), and \( \text{Pr}_{k,a(Q)} f = 0 \) otherwise.

Let \( \{\varphi_Q : Q \in \mathcal{W}_3\Omega\} \) be a smooth partition of unity, subordinate to the Whitney decomposition \( \mathcal{W}_3\Omega \). Then, in particular,

\[
\chi_{\mathbb{R}^n \setminus \partial\Omega} = \sum_{Q \in \mathcal{W}_3\Omega} \varphi_Q \quad \text{and} \quad \text{supp } \varphi_Q \subset (9/8)Q, \text{ if } Q \in \mathcal{W}_3\Omega.
\]

For \( f \in \mathcal{F}_{pq}(\mathbb{R}^n) \) with \( s > (n - d)/p \) and \( k = [s] + 1 \), we define

\[
(4.8) \quad \text{Ext}_{k,\Omega} f(x) := \begin{cases} 
  f(x), & \text{if } x \in \Omega; \\
  \sum_{Q \in \mathcal{W}_3\Omega} \varphi_Q(x)[\text{Pr}_{k-1,a(Q)} \circ \text{Tr}\partial\Omega(f)](x), & \text{if } x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

Observe that (4.8) induces a linear operator \( \text{Ext}_{k,\Omega} \). We emphasise that the values of \( \text{Ext}_{k,\Omega} f \) outside of \( \Omega \) depend only on the trace of \( f \) on the boundary—loosely speaking, we are extending from the boundary trace.

4.9. Theorem. Suppose that \( \Omega \) is a domain whose boundary is a d-set, \( n - 1 < d < n \). Suppose that \( 1 < p < \infty, 1 \leq q < \infty, s > (n - d)/p \), and \( k = [s] + 1 \). Then

\[
\text{Ext}_{k,\Omega} \in \mathcal{L}(\mathcal{F}_{pq}(\mathbb{R}^n)).
\]

That is, the extension operator is a bounded linear operator on \( \mathcal{F}_{pq}(\mathbb{R}^n) \), and the operator norm depends on \( p, q, s, n, d, \) and \( \partial\Omega \).

Proof. Define

\[
\text{Ext}_{k,\partial\Omega} f := \sum_{Q \in \mathcal{W}_3\Omega} \varphi_Q[\text{Pr}_{k-1,a(Q)} \circ \text{Tr}\partial\Omega(f)].
\]

By restriction and extension theorems for d-sets, [15, Theorem 4.8] and [15, Theorem 6.7], we see that \( \text{Ext}_{k,\partial\Omega} \) is a bounded linear operator on \( \mathcal{F}_{pq}(\mathbb{R}^n) \). Moreover, the function \( g := f - \text{Ext}_{k,\partial\Omega} f \)
is such that \( \text{Tr}_{\partial \Omega} g = 0 \) pointwise in \( H^d \) almost everywhere in \( \partial \Omega \), [15, Proposition 5.5]. Therefore, by Corollary 4.2 and the boundedness of \( \text{Ext}_{k, \partial \Omega} \),

\[
\| \text{Ext}_{k, \partial \Omega} f \|_{F^s_{pq}(\mathbb{R}^n)} = \| g \chi_{\Omega} + \text{Ext}_{k, \partial \Omega} f \|_{F^s_{pq}(\mathbb{R}^n)} \lesssim \| f \|_{F^s_{pq}(\mathbb{R}^n)}.
\]

This is the desired norm estimate. \( \square \)

4.10. **Remark.** The operator \( \text{Ext}_{1, \Omega} \) is considered \([13]\) for studying the extension problem on spaces \( W^{1,p}(\Omega) \).

The following is a corollary of Theorem 4.9 and Remark 4.5.

4.11. **Corollary.** Let \( \Omega \) be a domain whose closure \( \overline{\Omega} \) is an \( n \)-set, and whose boundary \( \partial \Omega \) is a \( d \)-set with \( n - 1 < d < n \). Suppose that \( 1 < p < \infty \), \( 1 \leq q < \infty \), \( s > (n - d)/p \), and \( k = \lfloor s \rfloor + 1 \). Fix any bounded extension operator \( \text{Ext}_{k, \partial \Omega} : F^s_{pq}(\Omega) \to F^s_{pq}(\mathbb{R}^n) \) (possibly non-linear). Then, the formula

\[
E^s_{pq} := f \mapsto \text{Ext}_{k, \partial \Omega}(N^s_{pq} f)
\]

defines a bounded linear extension operator \( F^s_{pq}(\Omega) \to F^s_{pq}(\mathbb{R}^n) \), which is independent of the chosen extension operator \( N^s_{pq} \).

4.12. **Remark.** Let us clarify that, by definitions, there is always a bounded non-linear extension operator \( N^s_{pq} : F^s_{pq}(\Omega) \to F^s_{pq}(\mathbb{R}^n) \) for which \( N^s_{pq} f|_{\partial \Omega} = f \) for every \( f \in F^s_{pq}(\Omega) \).

4.13. **Remark.** We observe the following independence on the microscopic parameter: the induced extension operator \( E^s_{pq} \), see Corollary 4.11, has the property that, for every \( 1 \leq r \leq \infty \),

\[
\| E^s_{pq} f \|_{F^r_p(\mathbb{R}^n \setminus \overline{\Omega})} \lesssim \| f \|_{F^s_{pq}(\Omega)}.
\]

This follows from restriction and extension theorems, [15, Theorem 4.8] and [15, Theorem 6.7]. In particular, we use the fact that the trace space \( B^{s-(n-d)/p}_{pp}(\partial \Omega) \) of \( F^s_{pq}(\mathbb{R}^n) \) is independent of the microscopic parameter \( q \).

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