Multi-Boson Correlations Using Wave-Packets

J. Zimányi¹ and T. Csörgő¹,²

¹MTA KFKI RMKI, H-1525 Budapest 114, POB. 49, Hungary
²Department of Physics, Columbia University, 538 W 120-th Street, New York, NY 10027

Abstract

Brooding over bosons, wave packets and Bose–Einstein correlations, we present a generic quantum mechanical system that contains arbitrary number of bosons characterized by wave-packets and that can undergo a Bose-Einstein condensation either by cooling, or increasing the number density of bosons, or by increasing the overlap of the multi-boson wave-packet states, achieved by changing the size of the single-particle wave-packets. We show that the \( n \)-particle correlations may mimic coherent or chaotic behavior for certain limiting wave-packet sizes. Effects of complete \( n \)-particle symmetrization are included. The resulting weights which fluctuate between 1 and \( n! \) are summed up with the help of a formal analogy between the considered wave-packet system and an already explored multi-boson plane-wave system. We solve the model analytically in the highly condensed and in the rare gas limiting cases, numerically in the intermediate cases. The relevance of the model to multi-pion production in high energy heavy ion physics as well as to the Bose-Einstein condensation of atomic vapors is discussed. As a by-product, a new class of probability distribution functions is obtained and the critical density for the onset of pion-lasing is derived. The multiplicity dependence of single-particle momentum distributions are predicted that could be utilized in future event-by-event measurements.

1 Introduction

The study of the statistical properties of quantum systems has a long history with important recent developments. In high energy physics, quantum statistical correlations are studied in order to infer the space-time dimensions of the elementary particle reactions. The measured characteristic length scales are on the \( 10^{-15} \) m scale, the time scales are of the order of \( 10^{-23} \) sec. In high energy heavy ion collisions hundreds of bosons are created in the present CERN SPS reactions when \( Pb + Pb \) reactions are measured at 160 AGeV laboratory bombarding energy. At the planned RHIC accelerator, thousands of pions could be produced in a unit rapidity interval. If the number of bosons in a unit value of phase-space is large enough the bosons may condense into the same quantum state and a pion laser could be created. Similarly to this process, when a large number of bosonic atoms are collected in a magnetic trap and cooled down to increase their density in phase-space, the bosonic nature of the atoms reveals itself in the formation of a Bose-Einstein condensate which can be considered as a macroscopic quantum state. This condensation mechanism may provide the key to the formation of atomic lasers in condensed matter physics. Thus it is essential to understand how the quantum statistical properties of a dilute bosonic gas change if the density of the quanta increases. In terms of multi-particle wave packet states this corresponds to the increase of the overlap of the single-particle wave-packets and this is indeed the key mechanism in our picture to provide the Bose-Einstein condensation effects. Due to the fact...
that the quanta in high energy physics are created by well localized sources, we would like to avoid
the plane-wave approximation and utilize wave-packets instead. In case of the Bose condensation of
atomic vapors, the presence of the magnetic trap prohibits the application of the outgoing plane-
waves. We hope that the wave packet formalism, presented herewith, may have certain applications
in that area of condensed matter physics as well, due to the finiteness of the wave-packet sources.
However, we will not explore this possibility in details herewith, since our main motivation for
considering the problem of quantum statistical correlations for many hundreds of overlapping wave-
packets stems from the experimental program in high energy heavy ion physics, which itself has
been motivated by a search for a new phase of the strongly interacting matter. This new phase is
referred to as the Quark-Gluon Plasma (QGP), where the quark and gluon constituents of strongly
interacting particles are expected to be liberated from the confining forces of the strong interaction.
See refs. [3, 4] for recent reviews on the experimental and theoretical developments in the quest for
the QGP.

Bose - Einstein correlations have already been investigated in a large number of papers for various
reactions of high energy physics (see for example the list in refs. [5, 6, 7]). In particular, some models
assume that pions are created at a given point in space with prescribed momenta that can only
be approximately valid in a quantal description. Such models correspond to certain semi-classical
Monte Carlo simulations of high energy reactions, e.g. [9, 10, 11, 12]. The Bose-Einstein correlations
are turned on only after the particle production is completed in refs. [9, 10, 13] with the help
of a weighting method that utilizes the Wigner-function formalism [14, 15, 16]. In the latter, the
production of the pions is given in space-time as well as in momentum space, since the hadronic string
model associates a momentum space fragmentation with a space-time fragmentation, see ref. [17].

There are different approaches incorporating both quantum mechanics and source models for pions,
such as the Wigner function formalism, developed by Pratt [14], and its different generalizations,
for example the relativistic Wigner function formalism of ref. [15].

In the present paper we will utilize wave packets in order to resolve the clear contradiction of
the classical assumptions in some models of Bose-Einstein correlations with the tenets of quantum
mechanics. Although certain properties of the wave-packets have been already utilized in previous
publications [13, 14, 15], these were usually applied on a phenomenological level, utilizing e.g.
minimal wave packets on the single particle level to mimic the effects of the uncertainty relations.
However, in our case, we shall consider the problems arising from the overlap between different
wave-packets, we shall discuss, how a Poisson multiplicity distribution is gradually transformed to
a multiplicity distribution of a Bose-Einstein condensate for a suitable choice of the wave packet
size. (*Note that this type of Bose-Einstein condensation is different from the text-book example of
Bose-Einstein condensation in statistical mechanics, where the zero-momentum plane-wave modes
are populated with a macroscopic amount of quanta. Here the wave-packet state with zero mean
momentum will be populated with a very large number of particles.* Due to the optical coherence of
the bosons in the zero mean momentum wave-packet state, this system is sometimes referred to as
a laser, c.f. the term “pion-laser” introduced by S. Pratt in ref. [1]).

We shall also see how the single-particle spectrum and the two-particle correlation functions shall be
drastically changed if the overlap of the multi-particle wave-packets reaches the critical size necessary
to create the Bose-Einstein condensate.

In high energy heavy ion reactions the system at break-up is considered as a pion source, which in
general can only be described by a density matrix. One may describe a density operator by a weighted
sum of projection operators. We will follow this method, and specify the various pion sources in
two steps. In section 2 we summarize the generic formulae of the density matrix formalism and
summarize the definitions of the physical quantities under investigation. At this level, projectors to
single-particle wave-packet states shall be introduced. These shall be constructed from single-particle wave-
packet states in subsection 2.1. In subsection 2.2 we specify a density matrix that includes induced
emission on the level of multi-particle wave-packets. We reformulate the multi-particle wave-packet
density matrix model in terms of new variables in such a manner that an effective multi-particle
plane-wave model is obtained, with source parameters explicitly depending on the wave-packet size.
We utilize of the ring-algebra of multi-particle symmetrization – discovered by S. Pratt [1] and
described in details in ref. [23] – to reduce the effective plane-wave model to a set of recurrence
relations in subsection 4.1. We solve these recurrences analytically: A subset of the recurrence equations is solved analytically in the case of arbitrary boson density in section 5. Complete analytic solutions are obtained for the generating function of the multiplicity distribution. Analytic results for the single particle inclusive momentum distributions and the two-particle inclusive correlation functions are presented in subsections 5.1 and 5.2 for a rare gas limiting case and a highly condensed Bose-gas limit, respectively. The onset of the wave-packet version of Bose-Einstein condensation is studied analytically in section 5.3. The analytical results presented in these sections are new results not only for the multi-particle wave-packet systems but also for the multi-particle symmetrization of plane-wave systems. In section 6 we present numerically evaluated single-particle spectra and two-particle correlation functions for the generic case. Finally we summarize and conclude.

2 Formulae for a generic case

The density matrix of a generic quantum mechanical system is prescribed as a sum of density matrixes with different number of bosons,

\[
\hat{\rho} = \sum_{n=0}^{\infty} p_n \hat{\rho}_n, \quad (1)
\]

technically the density matrix of the whole system is normalized as

\[
\text{Tr} (\hat{\rho}) = 1, \quad (2)
\]

similarly to the density matrix characterizing systems with a fixed particle number \( n \), which satisfy the normalization condition

\[
\text{Tr} (\hat{\rho}_n) = 1, \quad (3)
\]

from which it follows that

\[
\sum_{n=0}^{\infty} p_n = 1. \quad (4)
\]

The multiplicity distribution is prescribed in general by the set of \( \{p_n\}_{n=0}^{\infty} \). Later on we shall utilize the Poissonian multiplicity distribution for the case when multi-particle symmetrization effects can be neglected.

The density matrix describing a system with a fixed number of bosons is given by

\[
\hat{\rho}_n = \int d\alpha_1...d\alpha_n \rho_n(\alpha_1,...,\alpha_n) |\alpha_1,...,\alpha_n\rangle \langle \alpha_1,...,\alpha_n|, \quad (5)
\]

where the states \( |\alpha_1,...,\alpha_n\rangle \) denote \( n \)-particle wave-packet boson states, which shall be prescribed in more details in the next subsection. The parameters \( \alpha_i \) characterize a single-particle wave-packet, and the multi-particle wave-packet states are properly normalized to unity:

\[
|\alpha_1,...,\alpha_n\rangle \langle \alpha_1,...,\alpha_n| = 1. \quad (6)
\]

Thus the normalization condition for the \( n \)-particle density matrix can alternatively be written as

\[
\int d\alpha_1...d\alpha_n \rho_n(\alpha_1,...,\alpha_n) = 1. \quad (7)
\]

The expectation value of an observable, represented by an operator \( \hat{O} \) is given by

\[
< \hat{O} > = \text{Tr} \left( \hat{\rho} \hat{O} \right) \quad (8)
\]

The \( i \)-particle inclusive number distributions are given by

\[
N_i(k_1,...,k_i) = \text{Tr} \left( \hat{\rho} a^\dagger(k_1)...a^\dagger(k_i)a(k_i)...a(k_1) \right) \quad (9)
\]
which is normalized to the $i$-th factorial moment of the momentum distribution as
\[
\int d^3k_1 \ldots d^3k_i \rho_i(k_1, \ldots, k_i) = \langle n(n-1)(n-i+1) \rangle. \tag{10}
\]
These inclusive number distributions can be built up from the exclusive number distributions of $N_i^i(k_1, \ldots, k_m)$ for all $m = i, i+1, \ldots$ describing the $i$-particle number distribution for fixed $m$ multiplicity events as
\[
N_i(k_1, \ldots, k_i) = \sum_{m=i}^{\infty} p_i N_i^i(k_1, \ldots, k_i) \tag{11}
\]
where
\[
N_i^i(k_1, \ldots, k_i) = \text{Tr} \left( \hat{\rho}_m a_i^i(k_1) a_i^i(k_2) \ldots a_i^i(k_i) \right) \tag{12}
\]
and
\[
\int d^3k_1 \ldots d^3k_i N_i^i(k_1, \ldots, k_i) = m(n-m+1) \tag{13}
\]
The $i$-particle probability distributions are defined as
\[
P_i(k_1, \ldots, k_i) = \frac{1}{\langle n(n-1)(n-i+1) \rangle} N_i(k_1, \ldots, k_i) \tag{14}
\]
which are normalized to unity. For a fixed multiplicity of $m$ the $i$-particle probability distributions are defined as
\[
P_i^i(k_1, \ldots, k_i) = \frac{1}{m(m-1) \ldots (m-i+1)} N_i^i(k_1, \ldots, k_i) \tag{15}
\]
which are also normalized to unity. The $i$-particle inclusive correlation functions are defined as
\[
C_i(k_1, \ldots, k_i) = \prod_{j=1}^{i} P_i(k_j) \tag{16}
\]
while for a fixed multiplicity $m$ the $i$-particle exclusive correlation functions are defined as
\[
C_i^i(k_1, \ldots, k_i) = \prod_{j=1}^{i} P_i^i(k_j) \tag{17}
\]
Let us note, that another definition of the $i$-particle (exclusive/inclusive) correlation functions has been discussed recently in ref. [20], which will be denoted by $C_i^{N.m}(k_1, \ldots, k_i)$ and $C_i^N(k_1, \ldots, k_i)$, respectively. These are defined as
\[
C_i^{N,m}(k_1, \ldots, k_i) = \prod_{j=1}^{i} N_i^{i,m}(k_j), \tag{18}
\]
\[
C_i^N(k_1, \ldots, k_i) = \prod_{j=1}^{i} N_i(k_j) \tag{19}
\]
and they are related to the probability correlations by a normalization factor,
\[
C_i^{N,m}(k_1, \ldots, k_i) = \frac{m(m-1) \ldots (m-i+1)}{m^i} C_i^i(k_1, \ldots, k_i), \tag{20}
\]
\[
C_i^N(k_1, \ldots, k_i) = \frac{\langle m(m-1) \ldots (m-i+1) \rangle}{\langle m \rangle^i} C_i(k_1, \ldots, k_i). \tag{21}
\]
These relations indicate that for large values of $m$ the difference between the exclusive correlation functions of eqs. (18) and (17) is small, but the definition may matter much more when the inclusive correlation functions are evaluated from eqs. (19) or (18).

The one- and two pion distributions are calculated as

\[ N_2(k_1, k_2) = \text{Tr} \left( \hat{\rho} a^\dagger(k_1)a^\dagger(k_2)a(k_2)a(k_1) \right) \]

\[ N_2^2(k_1, k_2) = \sum_n p_n N_2^{(n)}(k_1, k_2) \]

\[ N_2^{(n)}(k_1, k_2) = \text{Tr} \left( \hat{\rho}_n a^\dagger(k_1)a^\dagger(k_2)a(k_2)a(k_1) \right) \]  \hspace{1cm} (22)

Similarly,

\[ N_1(k_1) = \sum_n p_n N_1^{(n)}(k_1) \]

\[ N_1^{(n)}(k_1) = \text{Tr} \left( \hat{\rho}_n a^\dagger(k_1)a(k_1) \right) \]  \hspace{1cm} (23)

We also have the following general relations

\[ \int d^3 k_2 \, N_2^{(n)}(k_1, k_2) = (n-1) N_1^{(n)}(k_1) \]

\[ \int d^3 k_1 \, N_1^{(n)}(k_1) = n \]

\[ \int d^3 k_1 \, d^3 k_2 \, N_2(k_1, k_2) = < n(n-1) > \]

\[ \int d^3 k_1 \, N_1(k_1) = < n > \]  \hspace{1cm} (24)

The two-particle inclusive Bose-Einstein correlation function is then given by the expression

\[ C(k_1, k_2) = C_2(k_1, k_2) = \frac{< n >^2}{< n(n-1) > \, N_2(k_1, k_2)} = \frac{P_2(k_1, k_2)}{P_1(k_1)P(k_2)} \]  \hspace{1cm} (25)

The general considerations presented above provide the framework for including the effects arising from multi-particle Bose-Einstein symmetrization using wave packets. Before specifying how the $n$-particle density matrixes $\hat{\rho}_n$ are related to the $n$-particle wave-packets, let us first introduce single-particle wave-packet states. The simplest case, when a single particle is emitted from a single source, shall be specified in subsection 2.1 and shall be extended for many particle wave-packet states emitted from many sources in subsection 2.2. We end this subsection by specifying a density matrix for which one can overcome analytically the difficulty related to the over-completeness of single-particle wave-packet states which shall complicate the normalization of multi-particle wave-packet states. The higher-order symmetrizations are reduced to an equivalent plane-wave problem in section 3, that is solved analytically in section 4 in certain limiting cases. Numerical results are shown in the subsequent section 5 to illustrate the new effects in the generic case.

### 2.1 Single-particle wave-packets

The building block for the definition of states will be a single particle wave packet, which is described as follows. The operator $\hat{a}^\dagger(x)$ creates a boson (e.g. a pion) at point $x$. This creation operator $\hat{a}^\dagger(x)$ can be decomposed as

\[ \hat{a}^\dagger(x) = \int \frac{d^3 p}{(2\pi)^2} \exp(i p x) \hat{a}^\dagger(p) . \]  \hspace{1cm} (26)

The creation and annihilation operators in space and in momentum space obey the standard commutation relations

\[ [\hat{a}(x), \hat{a}^\dagger(x')] = \delta(x - x') \]  \hspace{1cm} (27)
Let us define a wave packet creation operator as a linear combination of the creation operators in space,
\[
|\alpha_i = \alpha_i^\dagger |0\rangle,
\]
which is normalized properly as
\[
\langle \alpha_i | \alpha_i \rangle = 1.
\]
Although these wave packets are normalized to unity they are not orthogonal; in fact they form an over-complete set \([21]\). The commutator for the wave-packet creation and annihilation operators is given by the overlap as:
\[
\langle \alpha_i | \alpha_j \rangle = \langle \alpha_i | \alpha_j \rangle,
\]
as can be verified directly from the definition of the wave-packet creation and annihilation operators. The overlap between two wave-packets is given explicitly by
\[
\langle \alpha_i | \alpha_j \rangle = \left( \frac{2\sigma_i \sigma_j}{\sigma_i^2 + \sigma_j^2} \right)^{3/2} \exp \left( -\frac{(\sigma_i^2 \xi_i - \sigma_j^2 \xi_j)^2}{2(\sigma_i^2 + \sigma_j^2)} - \frac{(\pi_i - \pi_j)^2}{2(\sigma_i^2 + \sigma_j^2)} \right) \times \exp \left( i(\pi_i - \pi_j) \left( \frac{\sigma_i^2 \xi_i + \sigma_j^2 \xi_j}{\sigma_i^2 + \sigma_j^2} \right) \right).
\]
The coordinate space representation of a wave packet is given by
\[
\langle x | \alpha \rangle = \frac{1}{(\pi/\sigma^2)^{3/4}} \exp \left[ -\frac{(x - \xi)^2 \sigma^2}{2} - i\pi x \right],
\]
and the momentum space representation can be obtained using the expansion \([26]\) as
\[
\langle p | \alpha \rangle = \frac{1}{(\pi \sigma^2)^{3/4}} \exp \left[ -\frac{(p - \pi)^2}{2\sigma^2} - i\pi (p - \pi) \right].
\]
Up to this point all quantities were defined at a given time \(t_0\), which may be considered as break-up time. The notation \(|\alpha; t_0\rangle \equiv |\alpha, t_0\rangle\) reflects this fact. Assuming that after the break-up the wave packets move without interaction, the time evolution of the wave packet state is given as
\[
|\alpha; t = e^{iH(t-t_0)} |\alpha\rangle = \int d^3p \langle p | \alpha \rangle \exp[i\omega(p)(t-t_0)] \hat{a}^\dagger(p) \langle 0 | \rangle.
\]
We introduce the notation
\[
\omega(\alpha, p) = \langle p | \alpha; t \rangle = g(p, \pi) \exp[i\omega(p)(t-t_0) - i(\pi - p)\xi],
\]
with
\[
g(p, \pi) = \frac{1}{(\pi \sigma)^{3/4}} \exp \left[ -\frac{(p - \pi)^2}{2\sigma^2} \right].
\]
The pure state in eq. \([34]\) describes a wave packet with mean momentum \(\pi_i\), mean position \(x_i\) at time \(\tau_i\), with a spread of \(\sigma = \Delta p\) in momentum space and a corresponding \(\sigma_x = \Delta R = \hbar/\sigma\) spread in configuration space. Although the location and the momentum of a pion cannot be specified simultaneously with an arbitrary precision, the values of the parameters \(\pi, \xi, \sigma\) are not subject to any quantum mechanical restriction. For example, these values of the wave packet parameters can be identified with the classical pion production points in the phase space, as calculated from a Monte Carlo or a hydrodynamical model.
2.2 Building multi-particle wave-packet states

The basic building block for the following considerations is the operator $\alpha_i^\dagger(t)$ which is defined by

$$\alpha_i^\dagger(t) = \alpha_i^\dagger \exp[i\omega(p)(t - t_0)]$$

(39)

where the index $i$ refers to the different possible values of the center of the wave packets in coordinate and momentum space. For simplicity we may assume that all the wave packets are emitted at the same instant. Thus we indicate $(\pi, \xi, \sigma, t_0)$ by $\alpha_i$. Later on when we evaluate the recurrence relations we shall also assume that each wave packet has the same size, $\sigma = \sigma_i$.

Let us assume that there are $M$ boson sources in the system. If one boson can be emitted from any of these sources then the state vector can be written as

$$\langle \{\alpha\}; 1; t \rangle = \langle \alpha_1, ..., \alpha_M; 1; t \rangle \propto \sum_{i=1}^M \alpha_i^\dagger(t) \langle 0 \rangle.$$  

(40)

This state is a linear combination of one boson states and is not normalized. The normalization constant is to be determined from the condition $\langle \{\alpha_i\}; 1; t \rangle \langle \{\alpha_i\}; 1; t \rangle = 1$.

The $n$-boson states can in turn be given as

$$\langle \{\alpha_i\}; n; t \rangle = \langle \alpha_1, ..., \alpha_n; n; t \rangle \propto \left( \sum_{i=1}^M \alpha_i^\dagger(t) \right)^n \langle 0 \rangle.$$  

(41)

In the usual treatment $M = n$. For example, if $M = n = 2$, the two-particle state is specified as

$$\langle \alpha_1, \alpha_2; 2; t \rangle \propto [\alpha_1^\dagger(t) + \alpha_2^\dagger(t)][\alpha_1^\dagger(t) + \alpha_2^\dagger(t)]\langle 0 \rangle.$$  

(42)

When using this type of states one usually introduces a phase-averaging method to get back the Bose-Einstein enhancement of pions at small relative momenta [22]. A different possible $n$-boson state is given by

$$\langle \{\alpha_i\}; \{M_i\}; n; t \rangle = \langle \alpha_1, ..., \alpha_L; M_1, ..., M_L; n; t \rangle \propto \prod_{i=1}^L \langle \alpha_i^\dagger(t) \rangle^{M_i} \langle 0 \rangle.$$  

(43)

where

$$M_1 + M_2 + ... + M_L = n,$$  

(44)

and $\alpha_i \neq \alpha_j$ for $i \neq j$.

We restrict our investigation to this latter case only. In eq. (43) there are exactly $n$ creation operators acting one after the other on the vacuum, and the identical creation operators are grouped together. The most general case is that similar wave-packets can be created repeatedly and that it is not necessary to create two similar wave-packets one after the other, but a different wave packet can be created in between. Because of this reason, the general form of eq. (43) can be written as

$$\langle \alpha_1, ..., \alpha_n; t \rangle \propto \prod_{i=1}^n \alpha_i^\dagger(t)\langle 0 \rangle$$  

(45)

where similar values are allowed for arbitrary subsets of $\{\alpha_i\}_{i=1}^n$. As a special case, it is possible that all bosons are created in the same wave-packet state.

The norm of the state defined in eq. (45) can be calculated using the fact that

$$\langle 0 \rangle \langle a(p_1) ... a(p_n) a^\dagger(p'_1) ... a^\dagger(p'_n) \rangle = \sum_{\sigma(n)} \prod_{i=1}^n [a(p_{\sigma_i}), a^\dagger(p'_{\sigma_i})] = \sum_{\sigma(n)} \prod_{i=1}^n \delta(p_i - p'_{\sigma_i}),$$  

(46)
where the summation over $\sigma^{(n)}$ indicates the summation over the permutations of the first $n$ positive entire numbers and $\sigma_i$ denotes the permuted value of $i$ in a given permutation. (Note that the subscript $\sigma_i$ which stands for the permuted value of $i$ should not be confused with the normal-size $\sigma_i$, which stands for the width of the $i$-th wave packet in the momentum space.) Similarly to the plane wave case, one has

$$
\langle 0 | \alpha_1 \ldots \alpha_n \alpha_n^\dagger \ldots \alpha_1^\dagger | 0 \rangle = \sum_{\sigma^{(n)}} \prod_{i=1}^{n} |\alpha_i, \alpha_{\sigma_i}^\dagger| = \sum_{\sigma^{(n)}} \prod_{i=1}^{n} \langle \alpha_i | \alpha_{\sigma_i} \rangle
$$

The $n$ boson states, normalized to unity, are thus given as

$$
|\alpha_1, \ldots, \alpha_n\rangle = |\alpha_1, \ldots, \alpha_n; t\rangle = \frac{1}{\sqrt{\sum_{\sigma^{(n)}} \prod_{i=1}^{n} \langle \alpha_i | \alpha_{\sigma_i} \rangle}} \alpha_n^\dagger(t) \ldots \alpha_1^\dagger(t) |0\rangle.
$$

Let us introduce the notation

$$
\gamma_{i,j} = \langle \alpha_i | \alpha_{\sigma_j} \rangle
$$

which can be further simplified if the width of the wave-packets are identical, $\sigma_i = \sigma_j$. This simplifying condition is in principle not necessary, but we shall apply it in the forthcoming because this assumption will simplify certain recurrence relations. If all the wave packets have the same width, one has

$$
\gamma_{i,j} = \int d^3p w(p, \alpha_i) w(p, \alpha_j)
= \exp(-\frac{(\pi_i - \pi_j)^2}{4\sigma^2}) \exp\left(-\frac{(\xi_i - \xi_j)^2\sigma^2}{4}\right) \times 
\exp(i\pi_i \xi_j - i\pi_j \xi_i) \exp(i(\pi_i + \pi_j)(\xi_i - \xi_j)/2).
$$

Note, that $\gamma_{i,i} = 1$ and $\gamma_{i,j} = \gamma_{j,i}^*$. The summation over the permutations $\sigma^{(n)}$ can be decomposed as summing over transpositions. Since any permutation in $\sigma^{(n)}$ can be built up from the product of at most $n - 1$ transpositions $(i,j)$, the summation over the permutations can be written as a sum over the partial sums where a partial sum contains all the terms which belong to a class of a given $k$ with $0 \leq k \leq n - 1$, where $k$ is the minimal number of transpositions necessary to build up a given permutation:

$$
\sum_{\sigma^{(n)}} \prod_{i=1}^{n} \gamma_{i,\sigma_i} = 1 + \sum_{(i,j)} |\gamma_{i,j}|^2 + \sum_{(i,j);(j,k)} \gamma_{i,j} \gamma_{j,k} \gamma_{k,i} + \sum_{(i,j);(k,l)} |\gamma_{i,j}|^2 |\gamma_{k,l}|^2 + \ldots
$$

For $n = 2$ and $n = 3$ the normalization factors are given, respectively, by the inverse square root of the quantities

$$
1 + |\gamma_{1,2}|^2 \quad \text{and} \quad 1 + |\gamma_{1,2}|^2 + |\gamma_{2,3}|^2 + |\gamma_{1,3}|^2 + \gamma_{1,2} \gamma_{2,3} \gamma_{3,1} + \gamma_{1,3} \gamma_{3,2} \gamma_{2,1}
$$

In the next section we shall introduce a new type of density matrices using the above defined set of normalized wave-packet states. Later on we shall provide the analytical solutions to these models.

### 3 A New Type of Solvable Wave-Packet Density Matrix

In the formulae displayed up to now the normalization factor always appears as e.g. in eq. (51). For a reaction like Pb + Pb at SPS in a single central collision event one has approximately 500 - 600 $\pi^-$. Thus for this case we have the the sum over all possible permutation of the 600 pions, and each term in the sum is the product of 600 functions, depending on 1200 vector parameters. And
this expression is in the denominator. Later on, this normalization factor will be multiplied with the distribution function of these parameters, and than one should integrate over these parameters. It is clear, that it is practically impossible to perform this task. A very interesting numerical simulation has been reported recently about a very reasonable numerical approximation to evaluate the \(n\)-boson symmetrization effects with the help of a Monte-Carlo algorithm \[24\]. However, the efficiency of such algorithms decreases as \(1/(n!)\) thus it is practically impossible at the moment to go beyond 5-th or 6-th order explicit symmetrizations.

There is one special density matrix, however, for which one can overcome such a difficulty even in an explicit analytical manner. Namely, if one assumes, that we have a system, in which the emission probability of a boson is increased if there is an other emission in the vicinity. This would be an effect similar to the induced emission. Such an \(n\)-boson density matrix, eqs. \((5-7)\), may have the form:

\[
\rho_n(\alpha_1, ..., \alpha_n) = \frac{1}{N(n)} \left( \prod_{i=1}^{n} \rho_1(\alpha_i) \right) \left( \sum_{\sigma^{(n)}} \prod_{k=1}^{n} \langle \alpha_k | \alpha_{\sigma_k} \rangle \right)
\]  

The coefficient of proportionality, \(N(n)\), can be determined from the condition that the density matrix is normalized to unity,

\[
N(n) = \int \prod_{i=1}^{n} d\alpha_i \rho_1(\alpha_i) \sum_{\sigma^{(n)}} \prod_{k=1}^{n} \gamma_{k,\sigma_k} = \sum_{\sigma^{(n)}} \int \prod_{i=1}^{n} d\alpha_i \rho_1(\alpha_i) \langle \alpha_k | \alpha_{\sigma_k} \rangle
\]

The density matrix given by eq. \((54)\) corresponds to the expectation that the creation of a boson has a larger probability in a state, which is already filled by another boson. The above model not only describes such a source but it makes possible to continue the calculation as well, given that the last term of the density matrix cancels the normalization factor of the overlapping wave-packet states.

The induced emission, that is implicitly built in to the above definition \((54)\) of this density matrix, can be made much more transparent, if one evaluates the ratio \(\rho_n(\alpha_1, ..., \alpha_n)/[\prod_{j=1}^{n} \rho_1(\alpha_j)]\) for two special cases: one when each of the \(n\) particles are emitted with the same wave-packet parameter \(\alpha_i\) (maximal overlap) and the other, when the overlap between the wave packets of any pair from the \(n\) can be negligible. If \(n\) particles are emitted in the same wave-packet state, one has

\[
\frac{\rho_n(\alpha_1, ..., \alpha_1)}{[\rho_1(\alpha_1)]^n} = \frac{n!}{N(n)}
\]

while if the overlap between any of the wave packets is negligible, we obtain

\[
\frac{\rho_n(\alpha_1, ..., \alpha_n)}{\prod_{j=1}^{n} \rho_1(\alpha_j)} = \frac{1}{N(n)} \quad \text{(no overlap)}.
\]

In general, the overlap of the wave-packets determines the magnitude of the enhancement of the density matrix:

\[
\frac{\rho_n(\alpha_1, ..., \alpha_n)}{\prod_{j=1}^{n} \rho_1(\alpha_j)} = \frac{\sum_{\sigma^{(n)}} \prod_{k=1}^{n} \langle \alpha_k | \alpha_{\sigma_k} \rangle}{N(n)} \quad \text{(overlap)}.
\]

Thus the density matrix given in eq. \((54)\) describes a quantum-mechanical wave-packet system with induced emission, and the amount of the induced emission is controlled by the overlap of the \(n\) wave-packets, yielding a weight in the range of \([1, n!]\). Although it is very difficult numerically to operate with such a wildly fluctuating weight, we shall show that the special form of our density matrix yields a set of recurrence relations that can be evaluated numerically in an efficient manner. The essence of the method is the reduction of the problem to the already discovered “ring” - algebra of permanents for plane-wave outgoing states \[4\].
We have a larger freedom to choose the form of $\rho_1(\alpha)$, the density matrix describing the distributions of the parameters of wave packets. However, one can perform the calculations analytically when choosing Gaussian forms. For simplicity, we do not discuss the fluctuations of the wave-packet sizes in the forthcoming, although a fluctuating wave-packet size with Gaussian random distribution would lead to a similar mathematical structure of the solution.

Thus for the single-particle density matrix we assume the following form:

$$\rho_1(\alpha) = \rho_x(\xi) \rho_p(\pi) \delta(t - t_0)$$

$$\rho_x(\xi) = \frac{1}{(2\pi R^2)^{3/2}} \exp(-\xi^2/(2R^2))$$

$$\rho_p(\pi) = \frac{1}{(2\pi mT)^{3/2}} \exp(-\pi^2/(2mT))$$ (59)

Note, that this choice corresponds to a non-relativistic, non-expanding static source at rest in the frame where the calculations are performed.

This completes the specification of the model. If one is interested only in the numerical results, we recommend to jump directly to Section 6, where multiplicity distributions, particle spectra and correlations are plotted from a numerical evaluation of the model. For those more theoretically interested, we present in the next sections the analytic solution of this model. We may comment that these solutions are not easily obtainable, however, the algebraic beauty of the structure of the equations well compensates for the difficulty.

4 Algebraic Evaluation of the Model

In this section we make algebraic manipulations that are necessary to reduce the multi-particle wave-packet problem to an analytically solvable plane-wave problem.

The $n$ particle distribution for the system containing $n$ bosons is given by the expression:

$$N_n^{(n)}(k_1, k_2, ..., k_n) = \frac{1}{N(n)} \int \prod_{m=1}^{n} d\alpha_m \rho(\alpha_m) \sum_{\sigma(\alpha)} \prod_{i=1}^{n} w^*(k_i, \alpha_i) w(k_i, \alpha_{\sigma_i})$$ (60)

Since the indices run through all possible values, we can exchange the index of $k$ and $\alpha$ in the second $w$.

$$N_n^{(n)}(k_1, k_2, ..., k_n) = \frac{1}{N(n)} \int \prod_{m=1}^{n} d\alpha_m \rho(\alpha_m) \sum_{\sigma(\alpha)} \prod_{i=1}^{n} w^*(k_i, \alpha_i) w(k_{\sigma_i}, \alpha_i)$$ (61)

One more rewriting becomes possible by using the identity $(\prod_{i=1}^{n} A_i)(\prod_{j=1}^{n} B_j) = \prod_{i=1}^{n}(A_iB_i)$:

$$N_n^{(n)}(k_1, k_2, ..., k_n) = \frac{1}{N(n)} \prod_{i=1}^{n} \int d\alpha_i w^*(k_i, \alpha_i) \rho(\alpha_i) w(k_{\sigma_i}, \alpha_i)$$ (62)

Let us introduce the auxiliary quantity

$$\mathcal{P}(k_i, k_j) = \int d\alpha_i \rho(\alpha_i) w^*(k_i, \alpha_i) w(k_j, \alpha_i)$$ (63)

where the overline and the two arguments distinguish this auxiliary quantity from the single-particle density matrix $\rho(\alpha)$, introduced earlier.

Using these notations Eq. (62) can be rewritten as:

$$N_n^{(n)}(k_1, k_2, ..., k_n) = \frac{1}{N(n)} \prod_{i=1}^{n} \mathcal{P}(k_i, k_j)$$ (64)
With this notation the “source function” will be defined as

\[ S_n(k_1, \ldots, k_n) = \sum_{\sigma^{(n)}} \prod_{i=1}^{n} \rho(k_i, k_{\sigma_i}) \]  

(65)

and further:

\[ N_n^{(n)}(k_1, k_2, \ldots, k_n) = \frac{1}{N(n)} S_n(k_1, \ldots, k_n) \]  

(66)

Using Eq. (64) and the definitions (12,13), the one and the two particle distributions are cast in the form

\[ N_1^{(n)}(k_1) = \frac{n}{N(n)} \int d^3k_i S_n(k_1, \ldots, k_n) \]  

(67)

\[ N_2^{(n)}(k_1, k_2) = \frac{n(n-1)}{N(n)} \int \prod_{i=3}^{n} d^3k_i S_n(k_1, \ldots, k_n) \]  

(68)

From this it is clear that the constant of normalization \( N(n) \) can be expressed as

\[ N(n) = \int d^3k_i S_n(k_1, \ldots, k_n). \]  

(69)

At this point one can realize, that the above equations have algebraic structure similar to those discussed in Refs. [1,23], thus we can use their method to perform the summation over the permutations.

Up to this point the expressions in this subsection refer to events with fixed pion multiplicity. For an average over many events, one has to treat the multiplicity distributions too. Following Refs. [1,23] we introduce the auxiliary parameter, \( n_0 \), which correspond to the mean multiplicity of a source containing classical particles (not symmetrized system). If the system is characterized by a Boltzmann distribution, then this \( n_0 \) is a function of chemical potential, the temperature and the volume (characterized by the radius, \( R \)).

Let us assume a Poissonian multiplicity distribution for the case when the Bose-Einstein effects are switched off (denoted by \( p_n^{(0)} \)), i.e. let us assume that

\[ p_n^{(0)} = \frac{n_0^n}{n!} \exp(-n_0), \]  

(70)

Let us define

\[ G_1(p, q) = \frac{n_0}{1!} \rho(p, q) \]  

(71)

\[ G_2(p, q) = \frac{n_0^2}{2!} \int \rho(p, p_1) \rho(p_1, q) \]  

(72)

\[ G_i(p, q) = \frac{n_0^i}{i!} \int \prod_{j=1}^{i} \rho(p, p_j) \rho(p_j, p_{j+1}) \ldots \rho(p_{i-1}, p_i) \]  

(73)

With these definitions we find that the case of multi-particle wave packets can be considered as a formally equivalent plane-wave system due to the complete formal analogy between the above equations and those of ref. [23].

The above mentioned problem was shown to be solvable with the help of the so-called ring algebra discovered first by S. Pratt [1]. Utilizing that algebra, a set of recurrence relations were obtained that contained some effective parameters of the \( T \) matrix which were assumed to have a Gaussian form. In our case, the width of the single-pion wave packet and the width of the distributions of the wave-packet centers are taken into account explicitly, thus a wave-packet size appears as one of the parameters of the recurrence given below. We shall also present certain analytic solutions to these recurrence relations and investigate them numerically as well.
4.1 Recurrence relations

Having performed the reduction of the multi-particle wave-packet problem to the multi-particle plane-wave problem, we now can utilize the reduction of the multi-particle plane-wave problem to a set of recurrence relations that were given first in refs. [23, 1].

The multiplicity distribution (with the inclusion of the symmetrization effects), the one- and two-particle distribution for a system containing \( n \) pions can be expressed with the help of the three definitions that relate these observables to the elements of the recurrence relations, four recurrence relations and four initial conditions (one for each recurrence relation). Two further equations are necessary to define certain auxiliary variables that are related to the recurrence relations and the observables. Thus one ends up a set of 14 equations, as given below. The observables are defined as

\[
p_n = \frac{\omega_n}{\sum_{k=0}^{\infty} \omega_k}
\]

\[
P_1^{(n)}(k) = \frac{1}{n \omega_n} \sum_{\omega_{n-1}} G_i(k, k),
\]

\[
P_2^{(n)}(k_1, k_2) = \frac{1}{n(n-1) \omega_n} \sum_{l=2}^{n} \sum_{m=1}^{l-1} \omega_{n-1} \left[ G_m(k_1, k_1)G_{l-m}(k_2, k_2) + G_m(k_1, k_2)G_{l-m}(k_2, k_1) \right],
\]

in order to relate these definitions to the recurrence relations, one introduces two auxiliary quantities as

\[
G_n(p, q) = n_0^3 h_n \exp(-a_n(p^2 + q^2) + g_n pq),
\]

\[
C_n = \frac{1}{n} \int d^3 p G_n(p, p) = h_n \frac{n_0}{n} \left( \frac{\pi}{2a_n - g_n} \right)^{3/2}.
\]

The physical interpretation of the quantities \( \omega_n \) and \( C_n \) shall be discussed later. The recurrence relations correspond to the solution of the ring-algebra [23, 1] are given as

\[
\omega_n = \frac{1}{n} \sum_{l=1}^{n} l C_l \omega_{n-l}
\]

\[
h_{n+1} = h_1 h_n \frac{\pi^{3/2}}{(a_1 + a_n)^{3/2}}
\]

\[
a_{n+1} = a_1 - \frac{g_1^2}{4(a_1 + a_n)}
\]

\[
g_{n+1} = \frac{g_1 g_n}{2(a_1 + a_n)}
\]

The initial conditions or the starting elements of the recurrence are

\[
\omega_0 = 1,
\]

\[
h_1 = \frac{1}{[\pi \sigma_T^2]^{3/2}},
\]

\[
a_1 = \frac{1}{2 \sigma_T^2} + \frac{R_{eff}^2}{2},
\]

\[
g_1 = R_{eff}^2,
\]

which yield the following value for \( C_1 \):

\[
C_1 = n_0.
\]
In the initial conditions eq. (84) the following notation is introduced:

\[
\sigma_T^2 = \sigma^2 + 2mT, \\
R_{\text{eff}}^2 = R^2 + \frac{mT}{\sigma^2\sigma_T^2}.
\]  

(88)  

(89)

or using the spatial spread, \(\sigma_x\), and the effective temperature \(T_{\text{eff}}\), that characterizes the system before symmetrization effects are taken into account, one can write

\[
T_{\text{eff}} = T + \frac{\sigma^2}{2m} = T + 1/(2m\sigma_x^2), \\
R_{\text{eff}}^2 = R^2 + \frac{T}{2T_{\text{eff}}}. 
\]  

(90)  

(91)

Observe that the above recurrence relations correspond to the pion laser model of S. Pratt when a replacement \(R \rightarrow R_{\text{eff}}\) and \(T \rightarrow T_{\text{eff}}\) is performed. The parameters of the \(T\) matrix in S. Pratt’s model are thus interpreted here as effective parameters and expressed with the help of the radius of the source of the wave-packets, the temperature of the source of the wave-packets and the width of the wave-packets in our multi-particle wave-packet model.

Thus the reduction of the multi-particle wave-packet model to an equivalent plane-wave model is completed. These recurrence relations were studied numerically but analytic solutions to these equations were not given before. Although numerical investigation of the presented recurrence relations were reported e.g. in refs. [1, 23], to apply these kind of numerical investigations for equations were not given before. Although numerical investigation of the presented recurrence completed. These recurrence relations were studied numerically but analytic solutions to these

Thus the reduction of the multi-particle wave-packet model to an equivalent plane-wave model is completed. These recurrence relations were studied numerically but analytic solutions to these equations were not given before. Although numerical investigation of the presented recurrence relations were reported e.g. in refs. [1, 23], to apply these kind of numerical investigations for equations were not given before. Although numerical investigation of the presented recurrence completed. These recurrence relations were studied numerically but analytic solutions to these

Thus the reduction of the multi-particle wave-packet model to an equivalent plane-wave model is completed. These recurrence relations were studied numerically but analytic solutions to these equations were not given before. Although numerical investigation of the presented recurrence relations were reported e.g. in refs. [1, 23], to apply these kind of numerical investigations for equations were not given before. Although numerical investigation of the presented recurrence completed. These recurrence relations were studied numerically but analytic solutions to these

Thus the reduction of the multi-particle wave-packet model to an equivalent plane-wave model is completed. These recurrence relations were studied numerically but analytic solutions to these equations were not given before. Although numerical investigation of the presented recurrence relations were reported e.g. in refs. [1, 23], to apply these kind of numerical investigations for equations were not given before. Although numerical investigation of the presented recurrence completed. These recurrence relations were studied numerically but analytic solutions to these

Thus the reduction of the multi-particle wave-packet model to an equivalent plane-wave model is completed. These recurrence relations were studied numerically but analytic solutions to these equations were not given before. Although numerical investigation of the presented recurrence relations were reported e.g. in refs. [1, 23], to apply these kind of numerical investigations for equations were not given before. Although numerical investigation of the presented recurrence completed. These recurrence relations were studied numerically but analytic solutions to these
These simple and beautiful relations will be useful when evaluating the inclusive number distributions analytically as well as numerically. The probability distributions can be obtained from eqs. \ref{eq:14}. The importance of the proper normalization of the two-particle inclusive correlation functions was emphasized recently in ref. \cite{20}. We are now in the position that a formula of general validity can be derived that involves no approximation and can be obtained without any reference to the details of the source model that determines the functions $G_i(k_1, k_2)$. As long as the $n$-particle exclusive number-distributions have the form of eqs. \ref{eq:56} - \ref{eq:60}, the probability distributions are given by the functional equations of eq. \ref{eq:76} and the above eqs. \ref{eq:94} - \ref{eq:95} follow. In such cases it is possible to introduce the auxiliary quantity

$$G(k_1, k_2) = \sum_{i=1}^{\infty} G_i(k_1, k_2). \quad \text{(96)}$$

(In the Wigner-function formalism, this quantity corresponds to the on-shell Fourier-transform of the emission function $S(x, k)$.) With this notation, both kind of two-particle inclusive correlation functions, eqs. \ref{eq:14} can be evaluated as

$$C_2^N(k_1, k_2) = 1 + \frac{G(k_1, k_2)G(k_2, k_1)}{G(k_1, k_1)G(k_2, k_2)}, \quad \text{(97)}$$

$$C_2(k_1, k_2) = \frac{\langle n \rangle^2}{\langle n(n-1) \rangle} \left( 1 + \frac{G(k_1, k_2)G(k_2, k_1)}{G(k_1, k_1)G(k_2, k_2)} \right). \quad \text{(98)}$$

Thus we find, in agreement with ref. \cite{20}, that in general a non-trivial pre-factor appears before the two-particle inclusive momentum distribution $C_2(k_1, k_2)$, that can be transformed away if the correlation function is defined as the ratio of the number of counts, eq. \ref{eq:13}. Alternatively, this result can be interpreted in another manner: when measuring the inclusive correlation function as the ratio of the detection probabilities, eq. \ref{eq:14}, an overall normalization constant has to be introduced that will not be arbitrary but will have to be equal to $\langle n \rangle^2/\langle n(n-1) \rangle$, the inverse of the second normalized factorial moment of the multiplicity distribution.

In the next sub-section we re-write these recurrence relations into dimensionless forms and solve analytically three of them completely and present a formal solution for the fourth recurrence too. We present the leading order complete solution of the model both in the rare gas and in the very dense Bose-gas limiting cases.

### 5 Analytic Solutions

When one assumes a Gaussian form for the wave-packets and for the distribution functions of the parameters of the wave-packets, the functional equations for $G_n$-s are reduced to the set of recurrence relationships presented in the previous section. The recurrence equations for the exclusive distributions were known since their discovery in 1993 (except for the inclusion of wave-packet sizes into the definition of $R_{\text{eff}}$ and $T_{\text{eff}}$). Thus, a number of properties of these recurrences were explored but as far as we know, only numerically. In particular, it has been observed that at a critical density $n_c$ the stimulated emission of bosons over-compensates for the decrease of the unsymmetrized $p_n^{(0)}$ probabilities and a lasing effect or coherent behavior appears. The condensation is characterized by the divergence of $p_n$ (symmetrized) probabilities with increasing values of $n$, by an appearance of a low-momentum peak in the single-particle spectrum and by a decreasing intercept of the correlation function: $C(k, k) < 2$. However, no analytical results were known about the spectrum and the correlations at the condensation point as well as in other limiting cases, only the exclusive correlation functions were evaluated numerically but the evaluation of the inclusive correlations was not performed.

Note also, that until now $n_0$ was just the mean number of bosons before the symmetrization, and the parameters $n_0$, $R_{\text{eff}}$ and $T_{\text{eff}}$ were just interpreted in terms of the theoretical input values of the correlated system. It seemed that the actual single-particle spectra and correlation function can only be determined by numerically solving the recurrence relations. The invention of the recurrence
relation in ref. [1] was already a huge step forward, since the number of steps that are required for the solution of the \( n \)-particle symmetrization in general increase as \( n! \propto \exp[n(\ln n - 1)] \) which increases faster than any polynomial - or a non-polynomially (NP) hard problem. The NP-hard problems are very difficult and inefficient to handle numerically for large values of \( n \). S. Pratt reduced this difficult, NP-hard case to a set of recurrence relations, where the number of necessary steps to evaluate the observables increases only as slowly as \( n^2 \) and thus the problem is solvable even for large values of \( n \approx 1000 \) within a few minutes on the current computers.

Here, we would like to present the first analytical solution for the multi-particle symmetrization of the wave-packets — no numerical evaluation of the recurrences will be necessary, and the time needed to compute the result will be independent of \( n \). The solution of the problem is possible in the rare and the dense gas limiting cases, while in the general case the multiplicity distribution shall be given in terms of its combinants.

Before presenting the analytical solution of the recurrence equations, let us re-formulate the recurrences in eqs. (80 - 82) for new, dimensionless quantities. Note that eq. (79) is already referring to dimensionless quantities thus there is no need to reformulate it.

Let us introduce the following dimensionless variables:

\[
A_n = \sigma^2_T (a_1 + a_n) \quad (99)
\]

\[
H_n = h_n/h_1 = h_n/(\pi \sigma^2_T)^{3/2} \quad (100)
\]

\[
G_n = \sigma^2_T g_n \quad (101)
\]

\[
x = R_{eff}^2 \sigma^2_T \quad (102)
\]

Note that the variables indicated by the upper-case \( A_n, H_n \) and \( G_n \) are essentially the dimensionless versions of the variables \( a_n, h_n \) and \( g_n \). The quantity \( x \) corresponds to a dimensionless measure of the phase-space available for a single quanta. Extremely rare gas corresponds to the limit \( x \to \infty \) while a very dense Bose-gas corresponds to the \( x \to 0 \) limiting case. Note also that the dimensionless variable \( x \) should not be confused with the vector \( \mathbf{x} \) indicating a position in space.

The recurrence relations (80 - 82) can be re-written for the dimensionless variables as

\[
A_{n+1} = A_1 - \frac{x^2}{4} \frac{1}{A_n}, \quad (103)
\]

\[
H_{n+1} = H_n \frac{1}{A_n}, \quad (104)
\]

\[
G_{n+1} = G_n \frac{x}{2A_n}, \quad (105)
\]

The initial conditions for these recurrences read as

\[
A_1 = 1 + x, \quad (106)
\]

\[
H_1 = 1, \quad (107)
\]

\[
G_1 = x. \quad (108)
\]

The dimensionless recurrences of eqs. (103-105) can be solved exactly with the help of the auxiliary quantity

\[
Y_n = \prod_{i=1}^{n} A_i, \quad (109)
\]

since the solution of eqs. (103-105) is given as

\[
H_{n+1} = \frac{1}{Y_n^{3/2}} \quad (110)
\]

\[
G_{n+1} = \frac{x^{n+1}}{2^n Y_n} \quad (111)
\]
where the initial conditions of eqs. (107-108) are already taken into account. The remaining recurrence relation, eq. (103) is easy to solve if one rewrites this as

\[ Y_{n+1} = (1 + x)Y_n - \frac{x^2}{4} Y_{n-1}, \]  

(112)

that can be further re-formulated for an arbitrary value of the parameter \( \gamma \neq 1 + x \) as

\[ Y_{n+1} - \gamma Y_n = (1 + x - \gamma) \left[ Y_n - \frac{x^2}{4(1 + x - \gamma)} Y_{n-1} \right] \]  

(113)

and this equations becomes solvable if the value of the parameter \( \gamma \) is chosen such that

\[ \gamma = \frac{x^2}{4(1 + x - \gamma)}. \]  

(114)

This equation has the following two roots:

\[ \gamma_{\pm} = \frac{1}{2} \left( 1 + x \pm \sqrt{1 + 2x} \right) \]  

(115)

Note that these roots satisfy the following useful algebraic relations:

\[ \gamma_+ + \gamma_- = 1 + x, \]  

(116)

\[ \gamma_+ \gamma_- = x^2/4, \]  

(117)

\[ \gamma_+^{(1/2)} - \gamma_-^{(1/2)} = 1. \]  

(118)

One may choose any of \( \gamma_+ \) or \( \gamma_- \) to solve the third recurrence equation by re-scaling eq. (112) and introducing

\[ y_{n+1}^\pm = Y_{n+1} - \gamma_{\mp} Y_n \]  

(119)

One finds that

\[ y_{n+1}^\pm = \gamma_{\pm} y_{n}^\pm, \]  

\[ y_{1}^\pm = \gamma_{\pm}^2. \]  

(120)

(121)

thus the solution for the third recurrence relation can be written as

\[ y_{n}^\pm = \gamma_{\pm}^{n+1}, \]  

(122)

which yields the solution for \( Y_n \) as

\[ Y_n = \frac{\gamma_{\pm}^{n+1} - \gamma_{\mp}^{n+1}}{\gamma_{\pm} - \gamma_{\mp}} \]  

(123)

that can be substituted back to eqs. (110-111) to get an explicit solution. Then the variables that have dimensions can be also calculated easily, and their solution in terms of \( x = R_{eff}^2 \sigma_T^2 \) is especially simple if the variables defined in eq. (113) are used:

\[ a_n = \frac{\gamma_+ - \gamma_-}{2\sigma_T^2} \left[ \gamma_+ + \gamma_- \right] \gamma_+^{3/2} - \gamma_-^{3/2} \]  

(124)

\[ h_n = \frac{1}{\left[ \pi \sigma_T^2 \right]^{3/2}} \left[ \gamma_+ - \gamma_- \right]^{(3/2)} \]  

(125)

\[ g_n = \frac{2}{\sigma_T^2} (\gamma_+ - \gamma_-) \left( \gamma_+ \gamma_- \right)^{(n/2)} \gamma_+^{n} - \gamma_-^{n} \]  

(126)
and these solutions can be utilized to evaluate the quantity $C_n$ from the first part of eq. (78) as

$$C_n = \frac{n_0^n}{n \left[ \gamma_+^{(n/2)} - \gamma_-^{(n/2)} \right]^3} = t^n \frac{1}{[1 - u^n]^3},$$  \hspace{1cm} (127)

$$t = \frac{n_0}{\gamma_+^{(3/2)}},$$  \hspace{1cm} (128)

$$v = \sqrt{\frac{\gamma_-}{\gamma_+}},$$  \hspace{1cm} (129)

$$0 \leq v < 1.$$  \hspace{1cm} (130)

Note at this point, that the behavior of $p_n$ for large values of $n$ is controlled by $nC_n$, a quantity that can have a kind of critical behavior in the present model. Namely, the large $n$ behavior of $nC_n$ depends on the ratio of $t = n_0/\gamma_+^{(3/2)}$, since for large values of $n$, we always have $v^n = (\gamma_-/\gamma_+)^{(n/2)} << 1$. One may introduce a critical value of $n_0$, indicated as

$$n_c = \gamma_+^{(3/2)} = \left[ \frac{1 + x + \sqrt{1 + 2x}}{2} \right]^{(3/2)},$$  \hspace{1cm} (131)

and one may observe at this point that if $n_0 < n_c$, one has $\lim_{n \to \infty} nC_n = 0$, if $n_0 > n_c$, one obtains $\lim_{n \to \infty} nC_n = \infty$ and finally $\lim_{n \to \infty} nC_n = 1$ if $n_0 = n_c$. We shall return to the interpretation of this critical value of $n_0$ later on. We shall see that the quantities $C_n$ correspond to the combinants of the multiplicity distribution and $n_0 = n_c$ critical value corresponds to the divergence of the mean boson multiplicity.

Let us now proceed further with the solution of the model equations. The only remaining unsolved recurrence is eq. (79) which can be formally solved in a general manner, as follows. Let us introduce the generating function of the $\omega_n$-s and as $G_{\omega}(z)$ and let us define an auxiliary $F(z)$ as follows:

$$G_{\omega}(z) = \sum_{n=0}^{\infty} \omega_n z^n,$$  \hspace{1cm} (132)

$$F(z) = \sum_{n=0}^{\infty} (n + 1) z^n,$$  \hspace{1cm} (133)

and observe that

$$F(z)G_{\omega}(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (k + 1) C_{k+1}\omega_{n-k} z^n,$$  \hspace{1cm} (134)

$$\frac{1}{n} \sum_{k=0}^{n-1} (k + 1) C_{k+1}\omega_{n-k-1} z^n = \sum_{n=1}^{\infty} \omega_n z^n,$$  \hspace{1cm} (135)

and this last equation can be re-written with the help of eq. (79) as

$$\frac{1}{n} \sum_{k=0}^{n-1} (k + 1) C_{k+1}\omega_{n-k-1} z^n = \sum_{n=1}^{\infty} \omega_n z^n,$$  \hspace{1cm} (135)

and this last equation can be re-written with the help of eq. (79) as

$$F(z)G_{\omega}(z) = \frac{d}{dz} G_{\omega}(z),$$  \hspace{1cm} (137)

$$G_{\omega}(z) = G_{\omega}(0) \exp \left( \int_0^z dt F(t) \right),$$  \hspace{1cm} (138)

since the expansion coefficients of $F(z)$-s are known. The last step in the formal solution of eq. (79) is the introduction of the generating function for the probability distribution $p_n$ as

$$G(z) = \sum_{n=0}^{\infty} p_n z^n$$  \hspace{1cm} (139)

17
which can be expressed with the generating function of the $\omega_n$ distribution with the help of eq. (74) as

$$G(z) = \frac{G_\omega(z)}{G_\omega(1)}$$  \hspace{1cm} (140)$$

which yields the formal solution for the generating function of the probability distribution of $p_n$ as follows:

$$G(z) = \exp \left( \sum_{n=1}^{\infty} C_n (z^n - 1) \right),$$  \hspace{1cm} (141)$$

where the quantities $C_n$-s are the so called combinants of the probability distribution of $p_n$ and in our case their explicit form is known for any set of model parameters, as given by eqs. (127,115,102,88,89). Note that this generating function depends on two variables, the parameter $n_0$ that controls the phase space density and the parameter $x = R^2 \sigma^2$, eq. (102), that yields the available dimensionless phase-space volume. Alternatively, one may introduce another set of variables by defining $t = n_0/n_c$ and $v = \sqrt{\gamma_-/\gamma_+}$, $0 \leq v < 1$, to obtain a more transparent form of the probability generating function:

$$G(z; t, v) = \exp \left( \sum_{n=1}^{\infty} \frac{t^n}{n} \frac{1}{1 - v^n} (z^n - 1) \right).$$  \hspace{1cm} (142)$$

We find that this multiplicity generating function is not corresponding to known discrete probability generating functions after inspecting a few standard textbooks on statistics that include a large number of probability generating functions, e.g. ref [25]. Thus we may assume that we have found a new type of probability generating function. We shall investigate its properties not only numerically but also analytically in certain simple limiting cases. Especially, we show that there is a critical value for $n_0$ at which a wave-packet version of Bose-Einstein condensation occurs that influences drastically the $p_n$ distribution too. We shall also show that this distribution reduces to known type of distributions in certain limiting cases.

The combinants were introduced to statistics in refs. [27] and some of their properties were discussed recently in refs. [28]-[31]. The auxiliary quantity $\omega_n$ is the probability ratio

$$\omega_n = \frac{p_n}{p_0}. \hspace{1cm} (143)$$

(Note that in our model $p_0 = \exp(-\sum_n C_n) \neq 0$). The combinants up to a given order $n$ can be expressed as a combination [29] of the first $n$ probability ratios as

$$C_q = \omega_q - \frac{1}{q} \sum_{n=1}^{q-1} n C_n \omega_{q-n}, \hspace{1cm} (144)$$

hence their name. It is outside the scope of the present paper to study in detail the properties of the combinants and their relationships to other characteristics of the multiplicity distributions, like cumulants and factorial moments, and scaling laws of count probabilities. For a more detailed discussion on these general topics, we recommend refs. [28]-[31].

Let us note, however, a general property of a multiplicity distribution that is given in terms of its combinants. Any probability generating function that is given in terms of its combinants that are non-negative can be re-written as a convoluted Poisson distribution, or compound Poisson distribution if $p_0 > 0$ as

$$G(z) = \exp(\overline{C}(H(z) - 1)), \hspace{1cm} (145)$$

where $\overline{C} = \sum_{n=1}^{\infty} C_n$ can be interpreted as the mean multiplicity of the Poisson-distributed clusters (or clans). For completeness, we note that the probability generating function for a simple Poisson
distribution is $G_R(z) = \exp[\gamma(z - 1)]$. The particles within a single cluster are distributed according to the probability distribution

$$P_N = \frac{C_n}{\sum_{n=1}^{\infty} C_n} \quad (146)$$

and there is always at least one particle in any cluster i.e. $P_{N=0} = 0$.

Another interesting property of this multiplicity distribution is that the generating function can be written as

$$G(z) = \prod_{n=1}^{\infty} \exp(C_n(z^n - 1)) \quad (147)$$

i.e. as an infinite product of generating functions of Poisson distributions of particle singlets, doublets, triplets etc, having a mean of $C_1$, $C_2$, $C_3$ etc respectively. The corresponding multiplicity distribution can therefore be expressed in terms of Poisson distributions of particle $n$-tuples with means of $C_N$ in a multiple convoluted manner $[27, 29]$. This is a general property of any distribution that is given in terms of its combinants.

Due to this property, the mean multiplicity can be expressed in terms of combinants as

$$\langle n \rangle = \sum_n np_n = \sum_{i=1}^{\infty} iC_i \quad (148)$$

We see that the finiteness of the mean multiplicity is related to the vanishing values of $iC_i$ for large $i$. This limit, on the other hand, can only be reached if $n_0 < n_c$. Thus one finds that $\langle n \rangle < \infty$ if $n_0 < n_c$ and $\langle n \rangle = \infty$ if $n_0 \geq n_c$. Thus $n_c$ can be interpreted as a critical value for the parameter $n_0$. We shall see that divergence of the mean multiplicity $\langle n \rangle$ is related to condensation of the wave-packet modes with the highest multiplicities.

In general, the factorial cumulant moments, $f_q$-s of the probability distribution can be expressed with the help of the combinants in a relatively simple and straightforward manner similarly to the results of refs. $[30, 32]$:

$$f_q = \frac{d^q}{dz^q} \ln G(z)|_{z=1} = \sum_{i=q}^{\infty} i \cdot (i-1) \cdot \ldots \cdot (i-q+1) \cdot C_i \quad (149)$$

that are nothing else than the factorial moments within a single cluster multiplied with the average cluster multiplicity $[30]$.

Note, however, that the multiplicity distribution we find can not only be considered as an infinite convolution of Poisson distribution of particle singlets, pairs triplets etc but can be re-written also as an infinite convolution of Bose-Einstein (or Negative Binomial) distributions with coupled parameters.

In the exponent of $G(z)$ one may apply a negative binomial expansion of the terms

$$\frac{1}{(\gamma_+^{n/2} - \gamma_-^{n/2})^3} = \frac{1}{n^c} \sum_{k=0}^{\infty} (k+1)(k+2) \left( \frac{\gamma_-}{\gamma_+} \right)^{nk/2} \quad (150)$$

and with the help of this transformation, the generating function can be equivalently written as another infinite product:

$$G(z) = \prod_{k=0}^{\infty} G_k(z)^{\frac{(k+1)(k+2)}{2}}, \quad (151)$$

$$G_k(z) = \frac{1}{[1 + \pi_k(1-z)]}, \quad (152)$$

$$\pi_k = t(v^2)^k = \left( \frac{n_0}{n_c} \right)^k \left( \frac{\gamma_-}{\gamma_+} \right)^k = 2^{3/2} n_0 \frac{(1 + x - \sqrt{1 + 2x})^k}{(1 + x + \sqrt{1 + 2x})^{k+3/2}} \quad (153)$$
This expression indicates that the probability distribution \( p_n \) can also be considered as an infinite convolution of generalized Bose-Einstein distributions. The mean multiplicities of the single modes increase in a geometrical series. For any single mode of this convolution, the probability distribution can be written as

\[
p_n^{(k)} = \frac{\pi_k^n}{(\pi_k + 1)^{(n+1)}},
\]

and the mean multiplicity for that given mode can alternatively be written as

\[
\frac{\pi_k}{\pi_0} = \left( \frac{\pi_1}{\pi_0} \right)^k
\]

Thus the probability distribution can be considered both as a superposition of Poisson distributed independent clusters as well as an infinite convolution of Bose-Einstein distributions with coupled mean multiplicities.

It is especially interesting to note, that in the dense Bose-gas limit, the \( k = 0 \) term dominates eq. (151), while in the rare Bose-gas limit each factor contributes with a similar weight from the infinite product in this equation. In contrast, one may consider the Poisson cluster decomposition as given by eqs. (145, 141) and in this type of decomposition, the first factor contributes to the rare gas limiting case but in the dense gas limiting case all factors are important.

Let us make a few further remarks about the structure of the probability distribution \( p_n \) in this class of models. Although this multiplicity distribution is already given in terms of its generating function by eq. (142), and in terms of its combinants, \( C_n \) as given by eq. (127), a more explicit form for the multiplicities \( p_n \) can also be given. In general, if the combinants are known, the probability ratios \( \omega_n = p_n/p_0 \) can always be expressed as

\[
\frac{p_n}{p_0} = \omega_n = \sum_{k=1}^{n_1} \frac{1}{k!} \left[ \sum_{n_2=1}^{n_1-1} \sum_{n_3=1}^{n_2-1} \ldots \sum_{n_k=1}^{n_{k-1}} C_{n-n_2} C_{n_2-n_3} \ldots C_{n_{k-1}-n_k} C_{n_k} \right]
\]

Although this equation is seemingly more complicated than the recurrence given for \( \omega_n \) in eq. (74), it is still useful since the explicit form for \( C_n \) can be inserted from eq. (127) to find that

\[
\omega_n = t^{n_1} \sum_{k=1}^{n_1} \frac{1}{k!} \left\{ \sum_{n_2=1}^{n_1-1} \sum_{n_3=1}^{n_2-1} \ldots \sum_{n_k=1}^{n_{k-1}} \left[ \prod_{j=1}^{k-1} \left( 1 - v^{n_j-n_{j+1}} \right) \right]^{-1/3} \left( 1 - v^{n_k} \right)^{-1/3} \right\},
\]

where one can see that the \( t = n_0/n_c \) dependence of \( \omega_n \propto p_n \) is simple. This can be utilized to simplify the general evaluation of the \( \omega_n \) probability ratios as

\[
\omega_n = t^n F_n(v)
\]

\[
F_0(v) = 1
\]

\[
F_n(v) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{(1-v^j)^3} F_{n-j}(v).
\]

This is the simplest form that we can find for probability distribution in the general case (the probabilities are given by eq. (74) ).

We investigate the probability generating function \( G(z) \) as well as the multiplicities and the particle spectra in more details in the rare and in the dense gas limiting cases in the next sub-sections.

### 5.1 Analytic Results for Rare Bose Gas

In this section we consider large source sizes or large effective temperatures, i.e. we examine the combinants and the generating function of the probability distribution in the \( x >> 1 \) limiting case.
Since the multiplicity distribution was obtained analytically in terms of the combinsants, let us evaluate these to leading order in $1/x < < 1$. One obtains that

$$
\gamma_+ \simeq \left( \frac{x}{2} \right)^n + n \left( \frac{x}{2} \right)^{n-\frac{1}{2}},
$$

(161)

$$
\gamma_- \simeq \left( \frac{x}{2} \right)^n - n \left( \frac{x}{2} \right)^{n-\frac{1}{2}},
$$

(162)

$$
\gamma_+^n + \gamma_-^n \simeq 2 \left( \frac{x}{2} \right)^n,
$$

(163)

$$
\gamma_+^n - \gamma_-^n \simeq 2 n \left( \frac{x}{2} \right)^{n-\frac{1}{2}}.
$$

(164)

These relationships can be substituted to the solution of the recurrence relations, eqs. (125-126) to obtain that

$$
h_n = \frac{1}{(\pi \sigma_T^2)^{(3/2)}} \frac{1}{n^{3/2}} \left( \frac{x}{2} \right)^{\frac{3}{2}(1-n)},
$$

(165)

$$
g_n = \frac{x}{n \sigma_T^2} = \frac{R_{eff}^2}{n},
$$

(166)

$$
C_n = \frac{n_0^n}{n^4} \left( \frac{2}{x} \right)^{\frac{3}{2}(1-n)}.
$$

(167)

Note that the determination of $a_n$ is a bit tricky, since a finite sub-leading part remains that is difficult to determine as the leading orders cancel. However, there is an important constraint between $C_n$ and $a_n$ as given by eq. (78). This constraint is satisfied by

$$
a_n = \frac{n}{2 \sigma_T^2} + \frac{R_{eff}^2}{2n}.
$$

(168)

Inserting these to the definition (77) one obtains that

$$
G_n(k_1, k_2) = \frac{n_0^n}{(n \pi \sigma_T^2)^{3/2}} \left( \frac{2}{x} \right)^{\frac{3}{2}(n-1)} \exp \left[ - \frac{n}{2 \sigma_T^2} (k_1^2 + k_2^2) - \frac{R_{eff}^2}{2n} (k_1 - k_2)^2 \right].
$$

(169)

Both the inclusive and the exclusive momentum distributions will be built up from the auxiliary quantities $G_n(k_1, k_2)$, and the leading order behavior is given by $G_1(k_1, k_2)$ in this rare gas limiting case, with first order corrections from $G_2(k_1, k_2)$.

Eq. (167) indicates, that the leading order result for the combinatorial in the $x >> 1$ limiting case is

$$
C_n = \delta_{1,n} n_0
$$

(170)

and the first sub-leading correction is given by

$$
C_n = \delta_{1,n} n_0 + \delta_{2,n} \frac{n_0^2}{2(2x)^{3/2}}
$$

(171)

Thus, to leading order, the probability distribution can be considered as a Poisson distribution of singlets with a sub-leading correction that yields a convolution of Poisson-distributed doublets. Similarly to this, the Poisson-doublets will modify the single-particle momentum distribution as well as the two-particle correlation function. We consider these modifications in two manner: i) by evaluating the leading order corrections only and ii) by summing up certain sub-leading corrections.
5.1.1 Very rare gas

In the rare gas limiting case, the leading order Poisson distribution can be utilized to solve the model completely as follows:

\[
\omega_n = \frac{n_0^n}{n!}, \quad \text{(172)}
\]

\[
p_n = \frac{n_0^n}{n!} \exp(-n_0), \quad \text{(173)}
\]

\[
G_n(p, q) = \delta_{1,n} \frac{n_0}{(\pi \sigma T)^{3/2}} \exp \left( \frac{R_{eff}^2}{2}(p - q)^2 - \frac{p^2 + q^2}{2mT_{eff}} \right), \quad \text{(174)}
\]

\[
P_1^{(n)}(k) = P_1^{(1)}(k) = \frac{1}{(2\pi mT_{eff})^{3/2}} \exp \left( -\frac{k^2}{2mT_{eff}} \right), \quad \text{(175)}
\]

\[
P_1^{(1)}(k) = \frac{1}{(2\pi mT_{eff})^{3/2}} \exp \left( -\frac{k^2}{2mT_{eff}} \right), \quad \text{(176)}
\]

\[
C_2(k_1, k_2) = 1 + \exp \left( -\frac{R_{eff}^2}{2}(k_1 - k_2)^2 \right). \quad \text{(177)}
\]

Thus the very rare gas limiting case is very simple and solvable completely in an analytical manner. The multiplicity distribution is a Poisson distribution, the mean multiplicity coinciding with \(n_0\). The stimulated emission does not influence the probability distribution for the “unsymmetrized” factor \(n\) modification to the single-particle momentum distributions, and, apart from an overall normalization factor \(N_c\), the two-particle inclusive correlation function takes a Gaussian form with an effective radius parameter \(R_{eff}\). For large mean multiplicities, the overall normalization factor approaches unity since \(\lim_{n_0 \to \infty} N_c = 1\).

To our best knowledge, no similar evaluation of the two-particle inclusive correlations from the recurrence relations of S. Pratt was performed before, only exclusive correlation functions were evaluated numerically. Thus, this is the first time as far as we know that the exclusive momentum distributions are summed up and the inclusive particle spectra and the inclusive correlations are determined from the plane-wave model, too. Our method seems to be powerful enough to do certain analytical calculations even for systems where the numerical evaluation of the inclusive distributions were too tedious to perform earlier. Let us also stress that these results correspond to a multi-particle wave-packet system.

One may argue that the solution presented above is rather trivial and that not much is learned about the nature of the symmetrization effects here since they cancel from the leading order results in the rare Bose-gas limiting case. But the above simple results became very useful when one determines the next to leading order results in the rare gas limit, and it is necessary to know what is the reference point and what kind of changes happen if the leading order contribution from the symmetrization is taken into account.

5.1.2 The rare gas limiting case

Let us evaluate now the leading order corrections to the Poisson limiting case. This corresponds to keeping the first order corrections in \(1/x\). The probability generating function reads as

\[
G(z) = \exp(n_0(z-1) + C_2(z^2 - 1)) \approx \exp(n_0(z-1)) \left( 1 + \frac{n_0^2}{2(2x)^{(3/2)}}(z^2 - 1) \right) \quad \text{(180)}
\]

which yields the following multiplicity distribution:

\[
p_n = \frac{n_0^n}{n!} \exp(-n_0) \left[ 1 + \frac{n(n-1) - n_0^2}{2(2x)^{(3/2)}} \right]. \quad \text{(181)}
\]
We see that the leading order Poisson multiplicity distribution is modified in such a manner that
the corrected multiplicity distribution is depleted in the small \( n \) region, where \( n(n - 1) < n_0^2 \) and
enhanced in the high multiplicity region, where \( n(n - 1) > n_0^2 \), since
\[
\frac{p_n}{p_0,n} = 1 + \frac{n(n - 1) - n_0^2}{2(2n)^{(3/2)}}.
\]
(182)
This shift to large multiplicities is more enhanced for smaller values of \( x \) i.e. for higher phase-space
density. The mean multiplicity can be calculated as
\[
\langle n \rangle = \frac{d}{dz} G(z)|_{z = 1} = n_0 + \frac{n_0^2}{(2n)^{(3/2)}}
\]
(183)
which increases in this leading order calculation quadratically with increasing values of \( n_0 \) due to
the influence of Poisson-doublets. Note also the strong rise of the mean multiplicity with decreasing
values of \( x \) i.e. with increasing phase-space density of the wave-packets.
Evaluation of similar leading order corrections to the spectra and the correlations is straight-forward.
The \( n \)-particle exclusive and inclusive number-distributions have simpler form than the correspond-
ing probabilities, hence we indicate here these quantities:
\[
\begin{align*}
N_1^{(n)}(k) &= \frac{n}{(\pi \sigma_T^2)^{(3/2)}} \exp \left( -\frac{k^2}{\sigma_T^2} \right) + \frac{n(n - 1)}{(2x)^{(3/2)}} \left\{ \frac{2}{\pi \sigma_T} \right\}^{(3/2)} \exp \left( -\frac{2}{\sigma_T^2} k^2 \right) - \left\{ \frac{1}{\pi \sigma_T^2} \right\}^{(3/2)} \exp \left( -\frac{1}{\sigma_T^2} k^2 \right) \\
N_1(k) &= \frac{n_0}{(\pi \sigma_T^2)^{(3/2)}} \exp \left( -\frac{k^2}{\sigma_T^2} \right) + \frac{n_0^2}{(\pi \sigma_T^2 x)^{(3/2)}} \exp \left( -\frac{2}{\sigma_T^2} k^2 \right)
\end{align*}
\]
(184, 185, 186)
Note that these quantities are properly normalized according to eqs. (24), where the mean multi-
plicity is given by eq. (183).
The interpretation of these results is straight-forward: \( i) \) The single-particle inclusive momentum
distribution is enhanced at low momentum.
The enhancement is proportional to \( \frac{n_0^2}{x^{3/2}} = \frac{n_0^2}{(2mT_{eff}R_{eff}^2)^{(3/2)}} \) which is the mean number density
of pairs in the phase-space volume that is available for a single particle in the rare gas limit.
\( ii) \) The single-particle exclusive momentum distribution of rank \( n \) is enhanced at low momentum,
the enhancement being proportional to the number of pairs, \( n(n - 1) \). This predicts a characteristic
dependence in the exclusive momentum distribution as a function of \( n \), that is due to the Bose-
Einstein symmetrization. Such dependences could be studied in the event by event analysis to be
performed by the NA49 Collaboration, for example.
The evaluation of the Bose-Einstein correlation functions in this limiting case is rather involved and
will be published separately [33].

Let us turn our attention to the opposite limiting case, the ultra-dense Bose-gas of wave-packets.

5.2 Analytic Results for Highly Condensed Bose Gas

This limiting case corresponds either to a small effective radius parameter or a low effective tempera-
ture of the multi-boson wave-packet system. Formally, it corresponds to the \( x = 2mT_{eff}R_{eff}^2 << 1 \)
limiting case. In this case, one can determine the leading order expression for the combinatorial of the
probability distribution as
\[
\begin{align*}
\gamma_+ &= 1 + x + \mathcal{O}(x^2), \\
\gamma_- &= 0 + \mathcal{O}(x^2), \\
n_c &= n_c(x) \simeq (1 + x)^{3/2} \simeq 1 + \frac{3}{2} x \\
C_n &= \frac{1}{n} \left( \frac{n_0}{n_c} \right)^n \simeq \frac{n_0^n}{n} - \frac{3x}{2} n_0^n + \mathcal{O}(x^2).
\end{align*}
\]
(187, 188, 189, 190)
Thus the probability generating function can be written in the condensed Bose-gas limiting case as

\[ G(z) \simeq \frac{1}{(1 - \bar{n}(1 - z))} \]

\[ \bar{n} = \frac{n_0}{n_c(x) - n_0} \]

which corresponds to a Bose-Einstein distribution in the \( x << 1 \) limiting case with a mean multiplicity that diverges if \( n_0 \) reaches the value of \( n_c \simeq (1 + 3x/2) \). In this \( x << 1 \) limiting case, the multiplicity distribution shall be given by a Poisson distribution, and the model can be completely solved analytically as follows:

\[ p_n = \frac{\bar{n}^n}{(\bar{n} + 1)^{n+1}}, \]

\[ \omega_n = \frac{n_0}{n_c} = \frac{\bar{n}^n}{(\bar{n} + 1)^n}, \]

\[ G_n(p, q) = \frac{\bar{n}^n}{(\bar{n} + 1)^{n+1}} G_1(p, q) = \frac{\bar{n}^n}{(\bar{n} + 1)^{n+1}} \frac{1}{(\pi \sigma_T^2)^{3/2}} \exp \left(-\frac{p^2 + q^2}{2\sigma_T^2}\right), \]

\[ P_{1}^{(n)}(k) = P_{1}^{(1)}(k) = \frac{1}{(2\pi m T_{eff})^{3/2}} \exp \left(-\frac{k^2}{2m T_{eff}}\right), \]

\[ P_{1}(k) = P_{1}^{(n)}(k) = \frac{1}{(2\pi m T_{eff})^{3/2}} \exp \left(-\frac{k^2}{2m T_{eff}}\right), \]

\[ P_2(k_1, k_2) = P_1(k_1) P_1(k_2), \]

\[ C_2(k_1, k_2) = 1. \]

Note that this solution is obtained by summing up leading order contributions in the small \( x \) region. In this limiting case, corresponding to \( x = 2m R_{eff} T_{eff} << 1 \), essentially all the wave-packets remain condensed in the same wave-packet state as reflected by the shape of the single-particle inclusive spectrum and by the vanishing enhancement of the inclusive Bose-Einstein correlations. The mean multiplicity diverges as \( n_0 \) approaches \( n_c \simeq 1 + \frac{3x}{2} \). This corresponds to a very small wave-packet system where stimulated emission can dominate the particle emission even for very small values of the parameter \( n_0 \). The multiplicity distribution is a Bose-Einstein distribution, the parameter \( n_0 \) can be interpreted as a measure of \( p_n/p_{n+1} < 1 \) in units of \( n_c \), the inclusive and exclusive single-particle distributions coincide and there is no enhancement in the two-particle inclusive correlation function. Due to this reason, the solution corresponds to a completely coherent behavior, when all the particles are emitted in the same wave-packet state. It is interesting to observe, that the coherent behavior in this picture (defined by the vanishing enhancement in the two-particle inclusive correlation function) appears simultaneously with a thermalized Bose-Einstein distribution of the multiplicity distribution for \( k_1 \neq k_2 \). Thus the effective coherence appears due to the negligibly small effective source-size \( R_{eff} \rightarrow 0 \), or a very small effective temperature \( T_{eff} \rightarrow 0 \). In these limiting cases, the model corresponds to a thermalized source prepared in such a manner that all the wave-packets are created in the same state and their overlap is maximal. This is the reason why thermal behavior and coherence can appear simultaneously in this picture. It is interesting to note that the single-particle inclusive and exclusive momentum distributions coincide and are described by a Boltzmann distribution with an effective temperature \( T_{eff} = T + \frac{\sigma^2}{2m} \) that picks up a contribution from the temperature of the source and from the momentum-width of the wave-packets. This shape of the momentum distribution is forced by the Gaussian ansatz for the momentum-space representation of the wave-packets and the Gaussian distribution of the mean momentum of the packets.

In this and the previous sub-sections, analytical solutions to the multi-particle wave-packet system are presented. These analytic solutions yield new result not only in the study of multi-particle wave-
Let us first determine the critical boson multiplicity for the onset of the Bose-Einstein condensation condition is satisfied, the mean multiplicity has to diverge. 

Thus the ratio \( n_0 / n_c \) of the wave-packets into the same wave-packet state. From which is to be satisfied in the large pion density at which the condensation of wave-packets starts to appear. Thus the ratio \( n_0 / n_c \) above the critical multiplicity \( n_{C} \) the decrease of the unsymmetrized emission probabilities and a coherent, laser-like behavior occurs. Thus the condensation point is defined as the parameter \( n_{C} \) and if \( n_{C} = 0 \) but \( n_{C} > n_{c} \), \( \lim_{n \to \infty} p_{n} = 0 \) but above the critical multiplicity \( n_{0} > n_{c} \), \( \lim_{n \to \infty} p_{n} = \infty \) i.e. stimulated emission over-compensates the decrease of the unsymmetrized emission probabilities and a coherent, laser-like behavior occurs. Thus the ratio \( n_0 / n_c \) controls the competition between the stimulated emission and the decrease of the production probabilities of the unsymmetrized emission, and \( n_c \) can be interpreted as a critical pion density at which the condensation of wave-packets starts to appear.

5.3 Other Results for the Critical Multiplicity

The critical behavior, related to the divergence of the mean multiplicity in the original plane-wave system was discussed first in ref. [1]. In this pioneering work, S. Pratt published a formula for the onset of the condensation, determining \( n_c \), the critical number of the unsymmetrized pions from the inspection of certain ring-diagram. Although Figure 1 of ref. [1] was evaluated with the correct expression [34], this expression appeared in the paper in a misprinted manner, eq. (9) of ref. [1], where \( \eta_{c} = n_{c} \) and \( \Delta = mT \). Note that the steps to derive eq. (9) were not given in ref. [1] and the misprinted formula depends also on \( p_0 = \mathbf{p}^2 / (2m) \). Our result on the critical density for the condensation we define can be written as

\[
\eta_{c} = \left[ \frac{1}{2} + \Delta^2 R^2 + \sqrt{\Delta^2 R^2 + \frac{1}{4}} \right]^{(3/2)},
\]

(200)

which corresponds to the correct formula of eq. (200) when going to the wave-packet system from the plane-wave system with the help of the replacements \( \eta_{c} \to n_{c}, \Delta = mT \to mT_{eff} \) and \( R \to R_{eff} \). On Figure (1) we indicate the critical pion multiplicity as a function of wave-packet size.

It is important to discuss how the critical value of the parameter \( n_0 \) can be determined. We were interested in finding that limiting case, when the explosion of the (symmetrized) \( p_{n} \) probabilities just happens. We call this the Bose-Einstein condensation point and we define it physically as the point where a strong stimulated emission is build-up, which corresponds to the onset of the lasing mode. Thus the condensation point is defined as

\[
np_{n}(n_{c}) = (n + 1)p_{n+1}(n_{c})
\]

(202)

which is to be satisfied in the large \( n \) limit only. It is clear from this definition, that when this condition is satisfied, the mean multiplicity has to diverge.

Let us first determine the critical boson multiplicity for the onset of the Bose-Einstein condensation of the wave-packets into the same wave-packet state. From \( np_{n} = (n + 1)p_{n+1} \) in the large \( n \) limit it follows that \( n\omega_{n} = (n + 1)\omega_{n+1} \). From this it follows that \( nC_{n} = (n + 1)C_{n} \) in the large \( n \) limiting case. Comparing this to eq. (113), one finds that at the onset of Bose-Einstein condensation, as noted before, happens at

\[
n_{0} = n_{c} = \left[ \frac{1 + x + \sqrt{1 + 2x}}{2} \right]^{(3/2)}
\]

(203)

where \( x = R_{eff}^{2} \sigma_{T}^{2} = 2mT_{eff}R_{eff}^{2} \) is a dimensionless measure of the size of phase-space cell occupied on average by a single quanta. This expression is identical to eq. (201). One may determine this critical \( n_{c} \) parameter of the distribution as a function of \( \{ R_{eff}, T_{eff} \} \) or as a function of \( \{ R, T, \sigma \} \). Thus the parameter \( n_{0} \) can be compared to its critical value \( n_{c} \) and if \( n_{0} < n_{c}, \lim_{n \to \infty} p_{n} = 0 \) but above the critical multiplicity \( n_{0} > n_{c} \), \( \lim_{n \to \infty} p_{n} = \infty \) i.e. stimulated emission over-compensates the decrease of the unsymmetrized emission probabilities and a coherent, laser-like behavior occurs. Thus the ratio \( n_{0} / n_{c} \) controls the competition between the stimulated emission and the decrease of the production probabilities of the unsymmetrized emission, and \( n_{c} \) can be interpreted as a critical pion density at which the condensation of wave-packets starts to appear.

25
6 Numerical Results for Multi-Particle Wave-Packets

In this section we present some numerical results about the effects of multi-particle symmetrizations on the multiplicity distribution, on the single-particle exclusive momentum distribution and on the two-particle exclusive correlation functions. Although these results are more readily interpretable when one can rely on the analytical insights gained in the previous section, a qualitative understanding of the multi-particle symmetrization effects on these observables can be obtained from a direct inspection of these figures. Let us investigate first the modification of the multiplicity distribution, \( p_n \), due to the overlap of the wave-packets.

In Figs. 2 and 3 the multiplicity distribution, eq. (60) is plotted for various wave packet sizes. The “multiplicity parameter”, \( n_0 \), was fixed to \( n_0 = 600 \). These Figures clearly demonstrate, that the distribution function approaches a divergent region for wave packet sizes in the vicinity of \( \sigma_x = 4 \text{ fm} \) for this set of parameters. This is due to the fact that the overlap between the wave-packets is controlled by the wave-packet size and if the overlap reaches a critical magnitude, stimulated emission of wave-packets in similar wave-packet states leads to a multiplicity distribution that diverges for large values of \( n \). As the size of the overlap of the multi-particle wave-packets is increased, the multiplicity distribution becomes more and more different from the Poissonian limit, and it is shifted to larger multiplicities in accordance with our leading order analytical result given by eq. (181).

Investigating the single-particle exclusive momentum distributions, our analytical results about the development of a low-momentum enhancement is confirmed also numerically, as can be seen on Figures 4 and 5. It is also worth mentioning that the modification of the slope parameter of the high momentum part of the distribution is significant only when the spatial width of the wave-packets is small. This happens since the effective slope parameter, \( T_{\text{eff}} \) is given by eq. (91) and the contribution of the wave-packets to this quantity is large only if the momentum-width of the packets is large i.e. their spatial width is small.

The effect of the wave packet size on the two-particle exclusive correlation functions is shown on Figs. 6 and 7.

The virtual size of the source, \( R_v \), is assumed to be \( R_v = 1/|\Delta k|_{1/2} \), where \(|\Delta k|_{1/2} \) is the momentum difference, at which the correlation \( C(\Delta k) - 1 \) drops to half of its intercept value, \( C(0) - 1 \). Our first expectation would be, that \( R_v \) is a monotone increasing function of \( \sigma_x \). However, as Fig. 8 shows, this expectation is not fulfilled. This effect is even more pronounced in Fig. 9. The explanation of this behavior can be given as follows. From Figs. 3 and 4. we have seen, that the system approaches the formation of a condensate at \( \sigma_x = 4.0 \text{ fm} \). At this size, the overlap between the wave packets within the “virtual volume” is maximal. This case corresponds to the smallest “virtual” radius parameter of the exclusive correlation function and to the smallest value of its intercept parameter \( \lambda = C(k,k) - 1 \).

It is also clear, that the condensation effects the low momentum pions. Thus as we changed the mean pion momentum from 100 MeV to 50 MeV, we included more and more from the condensed part, which is responsible for the strange behavior.

Note that the “virtual” radius parameter first decreases than increases with increasing values of the wave-packet size \( \sigma_x \). Similarly, the intercept parameter \( \lambda = C_2(k,k) \) first decreases then it increases back to its conventional value of 1, for increasing size of the wave-packets. As we have seen in the ultra-dense limiting case, \( \lambda = 0 \) is also possible for high densities.

7 Summary, Conclusions and Outlook

In this paper a consequent quantum mechanical description of multi-boson systems is presented, using properly normalized projector operators to overlapping multi-particle wave-packet states for bosons. Hundreds of overlapping wave packets lead to a difficult problem. We found, however, a possible configuration, for which an exact solution is obtained. In our description no phase averaging is used. The mathematical problem imposed by the large number of source points is dealt with an algebraic procedure, similar to that invented by S.Pratt in ref. 1. The effects arising from the multi-particle symmetrization and from the finite width of the wave packet, \( \sigma_x \), are shown for a system that has a radius \( R \) and a freeze-out temperature \( T \) similar to a fireball that could be formed.
at mid-rapidity in Pb + Pb collisions at CERN SPS. The effects that depend on the wave-packet size are as follows:

1). The critical density of pions, at which the condensation appears, is a function of $\sigma_x$, and one finds that a pion-laser is formed at around $\sigma_x = 4$ fm, when assuming $n_0 = 600$, $R = 11$ fm and $T = 120$ MeV.

2). The multiplicity distribution shifts toward higher and higher multiplicities as $\sigma_x$ is increased towards 4 fm. In a range of wave-packet widths that include 4 fm, the multiplicity distribution explodes, i.e. we approach the condensation point. As $\sigma_x$ is further increased, the average multiplicity decreases back again to normal values.

3). The single particle momentum distribution shows a very large inverse slope parameter for small $\sigma_x$, say $\sigma_x = 1$ fm. This is a trivial effect: in the Fourier transform of a narrow wave packet there is a large contribution from high momentum components, which shows up in the single particle momentum distribution. Further, as $\sigma_x$ approaches the 4 fm value, the low momentum part in the spectrum becomes more and more enhanced. This is understandable, if we keep in mind, that the Bose condensation is approached first for the low momentum part of the spectrum. As $\sigma_x$ is further increased, the spectrum approaches again a Boltzmann distribution. This can be understood, if we take into account, that for larger $\sigma_x$ the virtual volume increases and thus the density of pions within this volume decreases.

4) The two particle correlation function shows an interesting behavior. Two features are to be emphasized: i) The virtual radius first decreases and later increases with increasing $\sigma_x$. This effect is larger as the mean momentum of the two pions decreases. ii) The intercept value, $C(|k_1 - k_2| = 0)$, decreases as $\sigma_x$ is increased from 1 fm to 4 fm, and increases again as $\sigma_x$ is further increased. These effects are caused by the increase and decrease of pion density within the "virtual volume", which is influenced by the distance to onset of the divergence of the mean multiplicity, i.e. the onset of the laser (or condensation) mode.

Finally, we may conclude, that we have found interesting effects for the investigated system. The construction of a clear connection of these effects to the onset of Bose-Einstein condensation as it is known in statistical physics, is to be made. Further, the problem, whether our approach is a useful one for the Bose-Einstein condensation of atomic systems, remains also for future investigations. Such studies may be promising since the atoms have to be trapped with the help of a magnetic trap to cool them below the critical temperature. The usual theoretical description uses the non-linear Schrödinger equation where a potential is created from $|\psi|^2$. Because of the trap, plane wave states cannot be utilized. Thus the wave-packet description, presented in this paper may have a relevance also in the field of Bose-Einstein condensation of atomic vapors.

8 Acknowledgments

Cs. T. would like to thank M. Gyulassy, S. Hegyi, G. Vahtang and X. N. Wang for stimulating discussions. The present study was, in part, supported by the National Science Foundation (USA) – Hungarian Academy of Sciences Grant INT 8210278, and by the National Scientific Research Fund (OTKA,Hungary) Grant (No. F4019, W01015107 and T024094), by the USA - Hungarian Joint Fund grant MAKA 378/93 and by an Advanced Research Award from the Fulbright Foundation. The authors would like to thank these sponsoring organizations for their aid.
Appendix A

Here we collect some formulas for another type of factorization of the $n$-particle density matrix. For the $n$-particle density matrix we also may assume, that it is factorizable in a simple way:

$$\rho_n(\alpha_1, ..., \alpha_n) = \prod_{i=1}^{n} \rho(\alpha_i)$$

(204)

This choice of the density matrix is one of the possibilities. In the body of the paper we study another case where the integrations can be carried out much easier, referred to as “model A” in the forthcoming. In this appendix, we explore the consequences of “model B”, where the integrals are much more difficult and the calculation can be performed only in some approximate manner.

Although in this general case one obtains quite complicated integrals when evaluating the momentum distribution, because of the overlapping of the wave packets, we can still proceed a little further with these expressions. Both in case of model A and B, these integrals for the $m$-pion momentum distribution for fixed multiplicity $n$ contain the factor given by Eq. (48).

In case of model B, the two-particle momentum distribution is given as

$$N_{B,2}^{(n)}(k_1, k_2) = \int \prod_{m=1}^{n} d\alpha_m \rho(\alpha_m) \times$$

$$\sum_{\sigma(n)} \sum_{i \neq j=1}^{n} w^*(k_1, \alpha_i) w^*(k_2, \alpha_j) w(k_1, \alpha_{\sigma_i}) w(k_2, \alpha_{\sigma_j}) \prod_{l=1, l \neq i, j}^{n} \gamma_{l, \sigma_l}$$

$$\sum_{\sigma(n)} \prod_{k=1}^{n} \gamma_{k, \sigma_k}$$

(205)

while in case model A, the integrals become somewhat simpler due to the cancelation of the extra factor in the density matrix with the denominator of the expectation value in eq. (48). For model B, the single particle distribution reads as

$$N_{B,1}^{(n)}(k_1) = \int \prod_{m=1}^{n} d\alpha_m \rho(\alpha_m) \frac{\sum_{\sigma(n)} \sum_{i=1}^{n} w^*(k_1, \alpha_i) w(k_1, \alpha_{\sigma_i}) \prod_{l=1, l \neq i}^{n} \gamma_{l, \sigma_l}}{\sum_{\sigma(n)} \prod_{k=1}^{n} \gamma_{k, \sigma_k}}$$

(206)

In case of model B, the integrals defining the spectrum and the two-particle distributions can also be reduced to a superposition of integrals, which contain only Gaussian factors, using an expansion of the denominator in the integrand. When all $\alpha_i$-s are equal, the denominator reaches its maximum with maximum value of $n!$. Thus one can expand it in the absolutely convergent series:

$$\sum_{\sigma(n)} \prod_{k=1}^{n} \gamma_{k, \sigma_k} = \frac{1}{n!} \frac{1}{(1 - x)} = \frac{1}{n!} \sum_{l=0}^{\infty} x^l$$

(207)

where we introduced the quantity $0 \leq x < 1$ with the definition

$$x = \sum_{\sigma(n)} \frac{1}{n!} \left( 1 - \prod_{k=1}^{n} \gamma_{k, \sigma_k} \right)$$

(208)

This expansion is absolutely convergent in terms of $x$ in its domain. Since $x^n$ contains linear combinations of different powers of $\gamma$ factors, which are themselves Gaussian factors, their powers are also Gaussian and since all the remaining factors in the integrals were Gaussian, we are left with Gaussian integrals. However, these results are rather complicated even in the $n = 2$ case and we do not include them in the present work. Due to the greater analytical simplicity of model A, we have explicitly evaluated integrals only for this latter case, as was given in the body of the paper.
References

[1] S. Pratt, Phys. Lett. B 301 (1993) 159.
[2] M. Anderson, J. Enscher, M. Matthews, C. Wieman and E. Cornell, Science, 269 (1995) 198.
[3] Proceedings of the Quark Matter’96 conference, (ed. by P. Braun-Munzinger, H.J. Specht, R. Stock and H. Stöcker), Nucl. Phys. A610 (1996), 1c - 565c.
[4] Proceedings of the Strangeness in Hadronic Matter’96 conference (ed. by T. Csörgő, P. Lévai and J. Zimányi) Heavy Ion Physics 4 (1996) 1 - 440.
[5] B. Lörstad, Int. J. Mod. Phys. A12 (1989) 2861.
[6] W. A. Zajc, in “Particle Production in Highly Excited Matter”, NATO ASI series B303 (Plenum Press, 1993, ed. H. Gutbrod and J. Rafelski) 435.
[7] U. Heinz, Nucl. Phys. A610 (1996) 264c.
[8] F. B. Yano, S. E. Koonin, Phys. Lett. B78 (1978) 555.
[9] T. Csörgő, J. Zimányi, J. Bondorf, H. Heiselberg, Phys. Lett. B222 (1989) 115.  
  T. Csörgő, J. Zimányi, J. Bondorf, H. Heiselberg, S. Pratt, Phys. Lett. B241 (1990) 301.
[10] K. Werner, Phys. Lett. B219 (1989) 111.
[11] M. Gyulassy, in Proc. 8th Balaton Conf. on Nucl. Phys., ed. Z. Fodor (KFKI, Budapest, 1987).
[12] T. Sjöstrand, Comp. Phys. Comm. 39 (1986) 347.
[13] J. P. Sullivan et al, Phys. Rev. Lett. 70 (1993) 3000;  
  D. E. Fields et al, Phys. Rev. C52 (1995) 986.
[14] S. Pratt, Phys. Rev. Lett. 53 1219 (1984).
[15] S. S. Padula, M. Gyulassy and S. Gavin, Nucl. Phys. B329 (1990) 357.
[16] S. Pratt, T. Csörgő and J. Zimányi, Phys. Rev. C42 (1990) 2646.
[17] B. Andresson, G. Gustafson, G. Ingelman and T. Sjöstrand, Phys. Rep. 97 (1983) 33.
[18] S. Chapman and U. Heinz, Phys. Lett. B340 (1994) 250.
[19] M. Bijayima, N. Suzuki, G. Wilk, Z. Wlodarczyk, Phys. Lett. B386 (1996) 297.
[20] D. Miskowiec and S. Voloshin, nucl-ex/9704006
[21] J. R. Klauder and E. C. G. Sudarshan, Fundamentals of Quantum Optics, Benjamin, New York, 1968, p. 115.
[22] M. Gyulassy, S. Kaufmann, L. Wilson, Phys. Rev. C20 (1979) 2267.
[23] W. Q. Chao , C. S. Gao, Q.H. Zhang, J. Phys. G. Nucl. Part. Phys. 21 (1995) 847.
[24] J. G. Cramer, Event Simulation of Higher Order Bose-Einstein and Coulomb Correlations,  
  University of Washington preprint, submitted to Phys. Rev. C.
[25] N. L. Johnson, S. Kotz and A. W. Kemp, Univariate Discrete Distributions, (John Wiley and Sons, second edition, 1992)
[26] J. Zimányi, T. Csörgő, B. Lukács, M. Rhoades-Brown and N.L. Balazs, Proceedings of the  
  Workshop on Relativistic Heavy Ion Physics at Present and Future Accelerators, preprint  
  KFKI - 1991 - 28/A, p. 91.;  
  J. Zimányi, talk given at the HBT’96 Workshop at ECT*, Trento, Italy, September 1996.
[27] M. Gyulassy and S. K. Kaufmann, Phys. Rev. Lett. 40 (1978) 298; S. K. Kaufmann and M. Gyulassy, J. Phys. A 11 (1978) 1715
[28] S. Hegyi, Phys. Lett. B309 (1993) 443-450
[29] S. Hegyi, Phys. Lett. B318 (1993) 642-647
[30] S. Hegyi, Phys. Lett. B327 (1994) 171-178
[31] S. Hegyi, hep-ph/9608479, Phys. Lett. B in press
[32] M. LeBellac, Acta Phys. Pol. B4 (1973) 901
[33] T. Csörgő and J. Zimányi, hep-ph/9705433, to be published
[34] S. Pratt, private communication
**Figure Captions**

**Fig. 1.** Solid line stands for the critical pion multiplicity distribution, \( n_c \), as a function of the wave packet sizes \( \sigma_x \), as indicated by solid line. The other parameters have the following values: \( T = 120 \text{ MeV}, R = 11 \text{ fm}, n_0 = 600 \).

**Fig. 2.** Pion multiplicity distribution for wave packet sizes \( \sigma_x = 1.0, 2.0 \) and 2.5 fm, as indicated by solid, dashed and dotted line, respectively. The other parameters have the following values: \( T = 120 \text{ MeV}, R = 11 \text{ fm}, n_0 = 600 \).

**Fig. 3.** Pion multiplicity distribution for wave packet sizes \( \sigma_x = 4.0, 7.0 \) and 10.0 fm, as indicated by solid, dashed and dotted lines, respectively. The other parameters have the following values: \( T = 120 \text{ MeV}, R = 11 \text{ fm}, n_0 = 600 \). At a critical size of the wave-packets, the overlap between the packets will be sufficiently large to start an “alavanche” of induced emissions, characterized by a \( p_n \) distribution which increases with increasing values of \( n \). (Such distributions can be normalized if they are truncated at large values of \( n \)). As the spatial width of the wave-packets is increased further, they become more and more similar to plane-waves in momentum space and the overlap of the wave-functions is decreased. The multiplicity distribution is shifted back towards the Poissonian limit.

**Fig. 4.** Exclusive single particle momentum distribution, \( I_{1x}^{(600)}(k) \) for a sub-set of events with a fixed multiplicity of \( n = 600 \). Eq. (15) is plotted for wave packet sizes \( \sigma_x = 1.0, 2.0 \) and 2.5 fm, as indicated by solid, dashed and dotted lines, respectively. The other parameters have the following values: \( T = 120 \text{ MeV}, R = 11 \text{ fm}, n_0 = 600 \). The x-axis is scaled with \( k^2 \). Note that the effective slope parameter changes with the variation of the wave-packet size. As the critical overlap is approached, the low momentum peak becomes more and more pronounced in the spectrum.

**Fig. 5.** Exclusive single particle momentum distribution, \( I_{1x}^{(600)}(k) \) as given by eq. (15) for wave packet sizes \( \sigma = 4.0, 7.0 \) and 10.0 fm, as indicated by solid, dashed and dotted lines, respectively. The other parameters have the following values: \( T = 120 \text{ MeV}, R = 11 \text{ fm}, n_0 = 600 \). The x-axis is scaled with \( k^2 \). The slope parameter at high momentum is hardly changed when \( \sigma_x \) is varied in this range, but the low momentum enhancement is strong only if the overlap of the wave-packets approaches the critical value.

**Fig. 6.** Two-particle exclusive correlation function, as given by eq. (17), for wave packet sizes \( \sigma_x = 1.0, 2.5 \) and 10.0 fm is plotted with solid, dashed and dotted lines, respectively. The mean momentum is fixed to \( |k_1 + k_2|/2 = 100 \text{ MeV} \). The relative momentum is parallel to the mean momentum, corresponding to the \( \text{out} \) direction for spherically symmetric systems. The x-axes is scaled with the relative momentum, \( |k_1 - k_2| = \Delta k_{\text{out}} \). The number of pions was fixed to \( n = 700 \). The “virtual radii” are \( R_v(\sigma_x = 1.0 \text{ fm}) = 13.2 \text{ fm}, R_v(\sigma_x = 2.0 \text{ fm}) = 11.6 \text{ fm}, R_v(\sigma_x = 2.5 \text{ fm}) = 15.2 \text{ fm} \).

**Fig. 7.** Two-particle exclusive correlation function, \( C_2^{(600)}(k_1, k_2) \) is plotted, as given by eq. (17), for wave packet sizes \( \sigma_x = 1.0, 2.5, 4.0 \) and 15.0 fm with solid, dashed, dotted and dense-dotted lines, respectively. The mean momentum is fixed to \( |k_1 + k_2|/2 = 50 \text{ MeV} \). The two momenta are parallel (\( \text{out} \) component). The x-axis is scaled with the momentum difference, \( |k_1 - k_2| \). The actual number of pions was set to \( n = 800 \). The virtual radii are \( R_v(\sigma_x = 1.0 \text{ fm}) = 12.33 \text{ fm}, R_v(\sigma_x = 2.5 \text{ fm}) = 8.57 \text{ fm}, R_v(\sigma_x = 4.0 \text{ fm}) = 8.22 \text{ fm}, R_v(\sigma_x = 15.0 \text{ fm}) = 17.4 \text{ fm} \).
Figure 1:

Figure 2:
Figure 3:

Figure 4:
Figure 5:

Figure 6:
Figure 7: