Blow up of mild solutions of a system of partial differential equations with distinct fractional diffusions

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Abstract
We give a sufficient condition for blow up of positive mild solutions to an initial value problem for a nonautonomous weakly coupled system with distinct fractional diffusions. The proof is based on the study of blow up of a particular system of ordinary differential equations.

Key words: Blow up, weakly coupled system, mild solution, nonautonomous initial value problem, fractal diffusion.

Mathematics Subject Classification: Primary 35K55, 35K45; Secondary 35B40, 35K20.

1 Introduction: statement of the result and overview

Let $i \in \{1, 2\}$ and $j = 3 - i$. In this paper we study blow up of positive mild solutions of
\[
\frac{\partial u_i(t,x)}{\partial t} = g_i(t) \Delta_{\alpha_i} u_i(t,x) + h_i(t) u_j^{\beta_i}(t,x), \quad t > 0, \quad x \in \mathbb{R}^d, \quad (1)
\]
\[u_i(0,x) = \varphi_i(x), \quad x \in \mathbb{R}^d,
\]
where $\Delta_{\alpha_i} = (-\Delta)^{\alpha_i/2}$, $0 < \alpha_i \leq 2$, is the $\alpha_i$-Laplacian, $\beta_i \geq 1$ are constants, $\varphi_i$ are non negative, not identically zero, bounded continuous functions and $h_i, g_i : (0, \infty) \to [0, \infty)$ are continuous functions.

If there exist a solution $(u_1, u_2)$ of (1) defined in $[0, \infty) \times \mathbb{R}^d$, we say that $(u_1, u_2)$ is a global solution, on the other hand if there exists a number $t_e < \infty$ such that $(u_1, u_2)$ is unbounded in $[0, t_e] \times \mathbb{R}^d$, for each $t > t_e$, we say that $(u_1, u_2)$ blows up in finite time.

The associated integral system of (1) is
\[
u_i(t,x) = \int_{\mathbb{R}^d} p_i(G_i(t), y-x) \varphi_i(y)dy \quad (2)
\]
\[+ \int_0^t \int_{\mathbb{R}^d} p_i(G_i(s,t), y-x) h_i(s) u_j^{\beta_i}(s,y) dyds.
\]

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Here \( p_i(t, x) \) denote the fundamental solution of \( \frac{\partial}{\partial t} - \Delta_{\alpha_i} \) and

\[
G_i(s, t) = \int_s^t g_i(r) \, dr, \quad 0 \leq s \leq t,
\]

where \( G_i(t) = G_i(0, t) \). We say that \((u_1, u_2)\) is a mild solution of (1) if \((u_1, u_2)\) is a solution of (2).

The main result is:

**Theorem 1** Assume that \( \beta_i \beta_j > 1 \) and

\[
\lim_{t \to \infty} G_i(t) = \infty.
\]

Let \( a \in \{1, 2\} \) such that

\[
\alpha_a = \min\{\alpha_1, \alpha_2\} \quad \text{and} \quad b = 3 - a.
\]

Define

\[
f_i(t) = h_i(t) \left( \frac{G_b(t)}{(G_j(t)^{\alpha_j/\alpha_j} + G_b(t))^{\beta_i}} \right)^{d/\alpha_b}, \quad t > 0.
\]

Then the positive solution of (2) blows up in finite time if

\[
\int_0^\infty F(s) \, ds = \infty,
\]

where

\[
F(t) = \left( f_i(t)^{1/(\beta_i+1)} f_j(t)^{1/(\beta_j+1)} \right)^{(\beta_i+1)(\beta_j+1)/(\beta_i+\beta_j+2)}.
\]

It is well known that a classical solution is a mild solution. Therefore, if we give a sufficient condition for blow up of positive solutions to (2) then we have a condition for blow up of classical solutions to (1).

**Corollary 2** Moreover, assume that \( \rho_i > 0, \sigma_i > -1 \) and

\[
\frac{\partial \rho_i}{\partial s} + \frac{\sigma_i (1+\beta_i) + \sigma_j (1+\beta_j)}{\beta_i + \beta_j + 2} + 1 \geq \frac{d}{\beta_i + \beta_j + 2} \left[ \beta_i (\beta_j + 1) \max \left\{ \frac{\rho_i}{\alpha_i}, \frac{\rho_j}{\alpha_j} \right\} + \beta_j (\beta_i + 1) \max \left\{ \frac{\rho_i}{\alpha_i}, \frac{\rho_j}{\alpha_j} \right\} \right],
\]

then each (classical) solution to

\[
\frac{\partial u_i(t, x)}{\partial t} = \rho_i t^{\rho_i-1} \Delta_{\alpha_i} u_i(t, x) + t^{\sigma_i} u_j^{\beta_i}(t, x), \quad t > 0, \quad x \in \mathbb{R}^d,
\]

\[
u_i(0, x) = \varphi_i(x), \quad x \in \mathbb{R}^d.
\]

blow up in finite time.

\(2\)
In applied mathematics it is well known the importance of the study of equations like (1). In fact, for example, they arise in fields like molecular biology, hydrodynamics and statistical physics [13]. Also, notice that generators of the form $g_i(t) \Delta_{\alpha_i}$ arise in models of anomalous growth of certain fractal interfaces [8].

There are many related works. Here are some of them:

- When $\alpha_1 = \alpha_2 = 2$, $\rho_1 = \rho_2 = 1$, $\sigma_1 = \sigma_2 = 0$ and $\varphi_1 = \varphi_2$ in (9), Fujita [3] showed that if $d < \alpha_1/\beta_1$, then for any non-vanishing initial condition the solution of (9) is infinite for all $t$ large enough.

- When $\alpha_1 = \alpha_2$, $\rho_1 = \rho_2$, $\sigma_1 = \sigma_2$ and $\varphi_1 = \varphi_2$ in (9), Pérez and Villa [11] showed that if $\sigma_1 + 1 \geq d(\beta_1 - 1)/\alpha_1$, then the solutions of (9) blow up in finite time.

- When $\alpha_1 = \alpha_2 = 2$ and $\rho_1 = \rho_2 = 1$ in (9), Uda [15] proved that all positive solutions of (9) blow up if $\max \left\{ \frac{(\sigma_1 + 1)\beta_1 + \sigma_1 + 1}{\beta_1\beta_2 - 1}, \frac{(\sigma_1 + 1)\beta_2 + \sigma_2 + 1}{\beta_1\beta_2 - 1} \right\} \geq \frac{d}{2}$.

- When $\alpha_1 = \alpha_2$, $g_1(t) = g_2(t) = t^{\rho - 1}$, $\rho > 0$, and $h_1(t) = h_2(t) = 1$ in (1), Pérez [10] proved that every positive solution blows up in finite time if $\min \left\{ \frac{\alpha_1}{\beta_1}, \frac{\alpha_1}{\beta_2} \right\} > d$.

- When $\rho_1 = \rho_2 = 1$ and the nonlinear terms in (9) are of the form $h(t, x) u^{\beta_i}$, $h(t, x) = O(t^\sigma |x|^{\gamma})$, Guedda and Kirane [5] also studied blow up.

Other related results (when $\alpha_1 = \alpha_2 = 2$) can be found, for example in [1], [2], [6], [9] and references therein.

It is worth while to mention that Guedda and Kirane [5] observed that to reduce the study of blow up of (1) to a system of ordinary differential equations we must have a comparison result between $p_i(t, x)$ and $p_j(t, x)$. Therefore, the goal of this paper is to use the comparison result given in [7] (Lemma 2.4) to follow the usual approach, see among others [14] or [4].

When $\alpha_1 = \alpha_2 = 2$, $\rho_1 = \rho_2 = 1$ and $\sigma_1 = \sigma_2 = 0$ the Uda condition (10), the Pérez condition (11) and the condition (8) become

$$d \leq \frac{2(\max\{\beta_1, \beta_2\} + 1)}{\beta_1\beta_2 - 1} = C_U,$$ (10)

$$d < \frac{2}{\max\{\beta_1, \beta_2\} - 1} = C_A,$$ (11)

$$d \leq \frac{\beta_1 + \beta_2 + 2}{\beta_1\beta_2 - 1} = C_V,$$ (12)

respectively. Since $C_A \leq C_V \leq C_U$ we see that the Uda condition (10) is the best. Also, from this we see that $C_V$, given in (12), is not the optimal bound.
(critical dimension), but we believe that it is the best we can get by constructing a convenient subsolution of the solution of (2). In fact, the condition (8) coincides with the condition for blow up given by Pérez and Villa [11].

The paper is organized as follows. In Section 1 we prove the existence of local solutions for the equation (2). In Section 2 we give some preliminary results and discuss a sufficient condition for blow up of a system of ordinary differential equations, finally in Section 3 we prove the main result and its corollary.

2 Local existence

The existence of local solutions for the weakly coupled system (2) follows from the fixed-point theorem of Banach. We begin introducing some normed linear spaces. By \( L^\infty (\mathbb{R}^d) \) we denote the space of all real-valued functions essentially bounded defined on \( \mathbb{R}^d \). Let \( \tau > 0 \) be a real number that we will fix later. Define

\[
E_\tau = \{(u_1, u_2) : [0, \tau] \to L^\infty (\mathbb{R}^d) \times L^\infty (\mathbb{R}^d), \|\| (u_1, u_2) \|\| < \infty \},
\]

where

\[
\|\| (u_1, u_2) \|\| = \sup_{0 \leq t \leq \tau} \{\|u_1(t)\|_\infty + \|u_2(t)\|_\infty\}.
\]

Then \( E_\tau \) is a Banach space and the sets

\[
P_\tau = \{(u_1, u_2) \in E_\tau, u_1 \geq 0, u_2 \geq 0\},
\]

\[
B_\tau = \{(u_1, u_2) \in E_\tau, \|\| (u_1, u_2) \|\| \leq 1\},
\]

are closed subspaces of \( E_\tau \).

**Theorem 3** There exists a \( \tau = \tau (\varphi_1, \varphi_2) > 0 \) such that the integral system (2) has a local solution in \( B_\tau \cap P_\tau \).

**Proof.** Define the operator \( \Psi : B_\tau \cap P_\tau \to B_\tau \cap P_\tau \), by

\[
\Psi (u_1, u_2) (t, x)
= \left( \int_{\mathbb{R}^d} p_1 (G_1 (t), y - x) \varphi_1 (y) dy, \int_{\mathbb{R}^d} p_2 (G_2 (t), y - x) \varphi_2 (y) dy \right)
+ \left( \int_0^t \int_{\mathbb{R}^d} p_1 (G_1 (s, t), y - x) h_1 (s) u_2^{\beta_1} (s, y) dy ds, \right.
\]

\[
\left. \int_0^t \int_{\mathbb{R}^d} p_2 (G_2 (s, t), y - x) h_2 (s) u_1^{\beta_2} (s, y) dy ds \right).
\]
We are going to show that $\Psi$ is a contraction, therefore $\Psi$ has a fix point. Let $(u_1, u_2), (\tilde{u}_1, \tilde{u}_2) \in B_\tau \cap P_\tau$ with $u_i(0) = \tilde{u}_i(0)$,

$$ |||\Psi (u_1, u_2) - \Psi (\tilde{u}_1, \tilde{u}_2) ||| $$

$$ = \left| \left| \left( \int_0^t \int_{\mathbb{R}^d} p_1 (G_1 (s, t), y - x) h_1 (s) \left[ u_2^{\beta_1} (s, y) - \tilde{u}_2^{\beta_1} (s, y) \right] dyds, \right. \right. \right.$$

$$ \left. \left. - \int_0^t \int_{\mathbb{R}^d} p_2 (G_2 (s, t), y - x) h_2 (s) \left[ u_1^{\beta_2} (s, y) - \tilde{u}_1^{\beta_2} (s, y) \right] dyds \right) \right| $$

$$ \leq \sum_{i=1}^2 \sup_{t \in [0, \tau]} \int_0^t \int_{\mathbb{R}^d} p_i (G_i (s, t), y - x) h_i (s) \left\| u_i^{\beta_i} (s) - \tilde{u}_i^{\beta_i} (s) \right\|_\infty dyds. $$

Let $w, z > 0$ and $p \geq 1$, then

$$ |w^p - z^p| \leq p (w \vee z)^{p-1} |w - z|. $$

Using the previous elementary inequality we get

$$ \left| u_j^{\beta_j} (s, x) - \tilde{u}_j^{\beta_j} (s, x) \right| \leq \beta_i (u_j (s, x) \vee \tilde{u}_j (s, x))^{\beta_i - 1} |u_j (s, x) - \tilde{u}_j (s, x)| \leq \beta_i \| u_j - \tilde{u}_j \|_\infty, $$

from this we deduce

$$ |||\Psi (u_1, u_2) - \Psi (\tilde{u}_1, \tilde{u}_2) ||| \leq \sum_{i=1}^2 \sup_{t \in [0, \tau]} \int_0^t h_i (s) \beta_i \| u_i (s) - \tilde{u}_i (s) \|_\infty ds $$

$$ \leq \left( \sum_{i=1}^2 \beta_i \int_0^\tau h_i (s) ds \right) \| (u_1, u_2) - (\tilde{u}_1, \tilde{u}_2) \|. $$

Since $\lim_{t \to 0} \int_0^t h_i (s) ds = 0$, we can choose $\tau > 0$ small enough such that $\Psi$ is a contraction. 

3 Preliminary results

We begin with:

**Lemma 4** For any $s, t > 0$ and any $x, y \in \mathbb{R}^d$, we have

(i) $p_i (ts, x) = t^{-d/\alpha} p_i (s, t^{-1/\alpha} x)$. 

(ii) $p_i (t, x) \geq (\frac{t}{s})^{d/\alpha} p_i (s, x)$, for $t \geq s$. 

(iii) $p_i (t, \frac{1}{t} (x - y)) \geq p_i (t, x) p_i (t, y)$, if $p_i (t, 0) \leq 1$ and $\tau \geq 2$. 

(iv) There exist constants $c_i \in (0, 1]$ such that

$$ p_i (t, x) \geq c_i p_i (t^{\alpha_i/\alpha}, x), $$

where $b$ is as in (4).
Proof. For (i)-(iii) see Section 2 in [14] and for (iv) see Lemma 2.4 in [7].

Lemma 5 Let $u_i$ be a positive solution of (2), then

$$u_i(t_0, x) \geq c_i(t_0) p_b \left( 2^{-\alpha_i} G_i(t_0)^{\alpha_b/\alpha_i}, x \right), \quad \forall x \in \mathbb{R}^d, \quad (14)$$

where

$$c_i(t_0) = c_i 2^{-d} \int_{\mathbb{R}^d} p_b \left( G_i(t_0)^{\alpha_b/\alpha_i}, 2y \right) \varphi_i(y) dy$$

and $t_0 > 1$ is large enough such that

$$p_b \left( G_i(t_0)^{\alpha_b/\alpha_i}, 0 \right) \leq 1. \quad (15)$$

Proof. By (i) of Lemma 4 and (3) there exist $t_0$ large enough such that

$$p_b \left( G_i(t_0)^{\alpha_b/\alpha_i}, 0 \right) = G_i(t_0)^{-d/\alpha_i} p_b(1, 0) \leq 1. \quad (16)$$

Using (iii) and (i) of Lemma 4 we get

$$p_b \left( G_i(t_0)^{\alpha_b/\alpha_i}, y - x \right) \geq p_b \left( G_i(t_0)^{\alpha_b/\alpha_i}, 2x \right) p_b \left( G_i(t_0)^{\alpha_b/\alpha_i}, 2y \right) \geq 2^{-d} p_b \left( 2^{-\alpha_i} G_i(t_0)^{\alpha_b/\alpha_i}, x \right) p_b \left( G_i(t_0)^{\alpha_b/\alpha_i}, 2y \right). \quad (17)$$

From (2), (iv) of Lemma 4 and the previous inequality we conclude

$$u_i(t_0, x) \geq \left( c_i 2^{-d} \int_{\mathbb{R}^d} p_b \left( G_i(t_0)^{\alpha_b/\alpha_i}, 2y \right) \varphi_i(y) dy \right) p_b \left( 2^{-\alpha_i} G_i(t_0)^{\alpha_b/\alpha_i}, x \right).$$

Getting the desired result. \hfill \blacksquare

Observe that the semigroup property implies

$$u_i(t + t_0, x) = \int_{\mathbb{R}^d} p_i \left( G_i(t_0, t + t_0), y - x \right) u_i(t_0, y) dy + \int_0^t \int_{\mathbb{R}^d} p_i \left( G_i(s + t_0, t + t_0), y - x \right) h_i(s + t_0) u_j^\delta(s + t_0, y) dy ds. \quad (17)$$

Let

$$\bar{u}_i(t) = \int_{\mathbb{R}^d} p_b \left( G_b(t, x) \right) u_i(t, x) dx, \quad t \geq 0. \quad (18)$$

Lemma 6 If $\pi_i$ blow up in finite time, then $u_i$ also does.

Proof. Let $t_0$ be given in Lemma 4. Take $t_0 < t_j < \infty$ the explosion time of $\pi_j$. From (3) we can choose $t > t_j$ large enough such that

$$G_i(t_j + t_0, t + t_0) > 2^{\alpha_i} G_b(t_j + t_0)^{\alpha_i/\alpha_b}.$$
Thus, for each $0 \leq s \leq t_j$,
\[
\int_{s+t_0}^{t+j_0} g_i (r) \, dr \geq \int_{t+j_0}^{s+t_0} g_i (r) \, dr \\
> 2^{\alpha_i} \left( \int_0^{t+j_0} g_b (r) \, dr \right)^{\alpha_i / \alpha_b} \geq 2^{\alpha_i} \left( \int_0^{s+t_0} g_b (r) \, dr \right)^{\alpha_i / \alpha_b},
\]

hence
\[
\tau_i = \frac{G_i (s+t_0, t+j_0)^{1/\alpha_i}}{G_b (s+t_0)^{1/\alpha_b}} \geq 2.
\]

On the other hand, (16) implies
\[
p_b (G_b (s+t_0), 0) \leq p_b (G_b (t_0), 0) = G_b (t_0)^{-d/\alpha} p_b (1, 0) \leq 1.
\]

Using (i) and (iii) of Lemma 4 we get
\[
p_b \left( G_i (s+t_0, t+j_0)^{\alpha_i/\alpha}, y-x \right) = \tau_i^{-d} p_b \left( G_b (s+t_0), \frac{1}{\tau_i} (y-x) \right) \geq \tau_i^{-d} p_b \left( G_b (s+t_0), x \right) p_b \left( G_b (s+t_0), y \right).
\]

From (17), (iv) of Lemma 4 and Jensen’s inequality we deduce that
\[
u_i (t+t_0, x) \geq c_i \int_0^{t+j} h_i (s+t_0) \int_{\mathbb{R}^d} p_b \left( G_i (s+t_0, t+j_0)^{\alpha_i/\alpha}, y-x \right) u_j (s+t_0, y)^{\beta_j} \, dy \, ds \\
\geq c_i \tau_i^{-d} \int_0^{t+j} h_i (s+t_0) p_b \left( G_b (s+t_0), x \right) \overline{\nu}_j (s+t_0)^{\beta_j} \, ds.
\]

Then $u_i (t+t_0, x) = \infty$. The definition (18) of $\overline{\nu}_i$ implies that $\overline{\nu}_i$ blows up in finite time, and working as before we conclude that $u_j$ also blows up in finite time.

In what follows by $c$ we mean a positive constant that may change from place to place.

The following result is interesting in itself.

**Proposition 7** Let $v_i, f_i : [t_0, \infty) \to \mathbb{R}$ be continuous functions such that
\[
v_i (t) \geq k + k \int_{t_0}^{t} f_i (s) v_j (s)^{\beta_j} \, ds, \quad t \geq t_0,
\]

where $k > 0$ is a constant. Then $v_i$ blow up in finite time if
\[
\int_{t_0}^{\infty} \left( f_i (s)^{1/(\beta_i+1)} f_j (s)^{1/(\beta_j+1)} \right)^{(\beta_i+1)(\beta_j+1)/(\beta_i+\beta_j+2)} \, ds = \infty.
\]
**Proof.** Consider the system

\[ z_i(t) = \frac{k}{2} + k \int_{t_0}^{t} f_i(s) z_j(s)^{\beta_i} \, ds, \quad t \geq t_0. \tag{19} \]

Let \( N_i = \{ t > t_0 : z_i(s) < v_i(s), \ \forall s \in [0, t] \} \). It is clear that \( N_i \neq \emptyset \). Let \( e_i = \sup N_i \). Without loss of generality suppose that \( e_i \geq e_j \). If \( e_i < \infty \), then the continuity of \( v_j - z_j \), yields

\[ 0 = (v_j - z_j)(e_j) \geq \frac{k}{2} + k \int_{t_0}^{e_j} f_j(s) \left[ v_i(s)^{\beta_j} - z_i(s)^{\beta_j} \right] \, ds \geq \frac{k}{2}. \]

Therefore \( z_i(t) \leq v_i(t) \), for each \( t \geq t_0 \).

Define

\[ Z(t) = \log \frac{z_i(t)}{z_j(t)}, \quad t \geq t_0. \tag{20} \]

Then, by (19)

\[
Z'(t) = \frac{f_i(t) z_j(t)^{\beta_i}}{z_i(t)} + \frac{f_j(t) z_i(t)^{\beta_j}}{z_j(t)} = \frac{(f_i(t)^{1/(\beta_i+1)} z_j(t))^{\beta_i+1} + (f_j(t)^{1/(\beta_j+1)} z_i(t))^{\beta_j+1}}{z_i(t) z_j(t)}.
\]

From Proposition 1 (p.259) of [12] we see that for each \( x, y > 0 \),

\[ y^{\beta_i+1} + x^{\beta_j+1} \geq c(xy)^{(\beta_i+1)/(\beta_i+\beta_j+2)}. \]

Using this and (20) we obtain

\[
Z'(t) \geq c \left( f_i(t)^{1/(\beta_i+1)} f_j(t)^{1/(\beta_j+1)} \right)^{(\beta_i+1)/(\beta_i+\beta_j+2)} \times (z_i(t)^{\beta_i} z_j(t))^{(\beta_i-1)/(\beta_i+\beta_j+2)} = cF(t) \exp \left( \frac{\beta_i-1}{\beta_i+\beta_j+2} Z(t) \right),
\]

where \( F \) is like (7). Consider the equation

\[ H'(t) = cF(t) \exp (cH(t)), \quad t > t_0, \quad H(t_0) = 2 \log \frac{k}{2}. \]

whose solution is

\[ H(t) = \log \left( e^{-cH(t_0)} - c^2 \int_{t_0}^{t} F(s) \, ds \right)^{-1/c}. \]

Since \( H \leq Z \) then the result follows from (3).  \[ \blacksquare \]
4 Blow up results

We begin with the:

**Proof of Theorem** [1] From (17) and (13)

\[
    u_i(t + t_0, x) \geq \int_{\mathbb{R}^d} c_i p_b \left( G_i \left( t_0, t + t_0 \right)^{\alpha_b / \alpha_i}, y - x \right) u_i(t_0, y) dy \\
    + \int_0^t h_i(s + t_0) \int_{\mathbb{R}^d} c_i p_b \left( G_i \left( s + t_0, t + t_0 \right)^{\alpha_b / \alpha_i}, y - x \right) u_j^\beta_i (s + t_0, y) dy ds.
\]

Multiplying by \( p_b \left( G_b \left( t + t_0 \right), x \right) \) and integrating with respect to \( x \) we get

\[
    \bar{u}_i(t + t_0) \geq c_i \int_{\mathbb{R}^d} p_b \left( G_i \left( t_0, t + t_0 \right)^{\alpha_b / \alpha_i} + G_b \left( t + t_0 \right), y \right) u_i(t_0, y) dy \\
    + c_i \int_0^t h_i(s + t_0) \int_{\mathbb{R}^d} p_b \left( G_i \left( s + t_0, t + t_0 \right)^{\alpha_b / \alpha_i} + G_b \left( t + t_0 \right), y \right) \\
    \times u_j^\beta_i \left( s + t_0, y \right) dy ds.
\]

the property (ii) of Lemma [4] and Jensen’s inequality, rendering

\[
    \bar{u}_i(t + t_0) \geq c_i \int_{\mathbb{R}^d} p_b \left( G_i \left( t_0, t + t_0 \right)^{\alpha_b / \alpha_i} + G_b \left( t + t_0 \right), y \right) u_i(t_0, y) dy \\
    + c_i \int_0^t \left( \frac{G_b (s + t_0)}{G_i \left( s + t_0, t + t_0 \right)^{\alpha_b / \alpha_i} + G_b \left( t + t_0 \right)} \right)^{d/\alpha_b} \\
    \times h_i(s + t_0) \left( \bar{u}_i \left( s + t_0 \right) \right)^{\beta_i} ds.
\]

Moreover, (14) and that \( G_i(s, \cdot) \) is increasing implies

\[
    \bar{u}_i(t + t_0) \geq c_i c_i(t_0)p_b \left( 1, 0 \right) \left( 2G_i(t + t_0)^{\alpha_b / \alpha_i} + 2G_b \left( t + t_0 \right) \right)^{-d/\alpha_b} \\
    + c_i \int_0^t h_i(s + t_0) \left( \frac{G_b(s + t_0)}{2G_i(t + t_0)^{\alpha_b / \alpha_i} + 2G_b \left( t + t_0 \right)} \right)^{d/\alpha_b} \left( \bar{u}_i \left( s + t_0 \right) \right)^{\beta_i} ds.
\]

Let

\[
    v_i(t + t_0) = \bar{u}_i \left( t + t_0 \right) \left( G_i \left( t + t_0 \right)^{\alpha_b / \alpha_i} + G_b \left( t + t_0 \right) \right)^{d/\alpha_i}, \ t_0,
\]

then

\[
    v_i(t + t_0) \geq c + c \int_0^t f_i(t + t_0)v_j \left( s + t_0 \right)^{\beta_i} ds,
\]

where \( f_i \) is defined in [5]. The result follows from Proposition [7] and Lemma [9].

**Proof of Corollary** [2] Let

\[
    f_i(t) = \frac{t^{\sigma_i + d \rho_b / \alpha_b}}{(t^{\rho_j \alpha_b / \alpha_j} + t^{\rho_b})^{d \beta_i / \alpha_b}}.
\]
then
\[ F(t) = \frac{t^{\theta_1}}{(t^{\theta_2} + t^{\theta_3})^{\theta_4}(t^{\theta_5} + t^{\theta_6})^{\theta_6}} \]
where
\[ \theta_1 = \frac{d\rho_b}{\alpha_b} + \frac{\sigma_i(1 + \beta_j) + \sigma_j(1 + \beta_i)}{2 + \beta_i + \beta_j}, \]
\[ \theta_2 = \frac{\beta_j\alpha_b}{\alpha_j}, \quad \theta_3 = \rho_b, \quad \theta_4 = \frac{d\beta_i(\beta_j + 1)}{\alpha_b(2 + \beta_i + \beta_j)}, \]
\[ \theta_5 = \frac{\beta_i\alpha_b}{\alpha_i}, \quad \theta_6 = \frac{d\beta_i(\beta_i + 1)}{\alpha_b(2 + \beta_i + \beta_j)}. \]

Using the elementary inequality
\[ (t^{\theta_2} + t^{\theta_3})^{\theta_4}(t^{\theta_5} + t^{\theta_6})^{\theta_6} \leq (2t^{\max\{\theta_2,\theta_3\}})^{\theta_4}(2t^{\max\{\theta_5,\theta_6\}})^{\theta_6}, \quad t > 1, \]
the result follows.

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