From Vacuum Fluctuations to Radiation: Accelerated Detectors and Black Holes.

S. Massar
Service de Physique Théorique, Université Libre de Bruxelles, Campus Plaine, C.P. 225, Bd du Triomphe, B-1050 Brussels, Belgium

R. Parentani
Laboratoire de Physique Théorique de l’Ecole Normale Supérieure, 24 rue Lhomond 75.231 Paris CEDEX 05, France

Abstract  The energy and particle fluxes emitted by an accelerated two level atom are analysed in detail. It is shown both perturbatively and non perturbatively that the total number of emitted photons is equal to the number of transitions characterizing thermal equilibrium thereby confirming that each internal transition is accompanied by the emission of a Minkowski quantum. The mean fluxes are then decomposed according to the final state of the atom and the notion of conditional flux is introduced. This notion is generalized so as to study the energy content of the vacuum fluctuations that induce the transitions of the accelerated atom. The physical relevance of these conditional fluxes is displayed and contact is made with the formalism of Aharonov et al. The same decomposition is then applied to isolate, in the context of black hole radiation, the energy content of the particular vacuum fluctuations which are converted into on mass shell quanta. It is shown that initially these fluctuations are located around the light like geodesic that shall generate the horizon and have exponentially large energy densities. Upon exiting from the star they break up into two pieces. The external one is red shifted and becomes an on mass shell quantum, the other, its ”partner”, ends up in the singularity. We avail ourselves of this analysis to study back reaction effects to the production of a single quantum.
1 Introduction

The history of Hawking radiation and of uniformly accelerated detectors in Minkowski vacuum have to some extent evolved in parallel. Indeed Hawking’s discovery\cite{1} that black holes emit particles at temperature $1/8\pi M$ ($M$ being the mass of the black hole) closely preceded and inspired Unruh’s discovery\cite{2} that a uniformly accelerated detector will thermalize at a temperature $a/2\pi$ ($a$ being the acceleration of the detector). The global structure of the geometry (the presence of horizons), the singular behavior of the modes and the mechanism of particle emission (excitation of the detector) are in close analogy in both problems.

In this article we exploit this isomorphism to obtain a quantum description of the fluctuations of the energy density in both problems. To this end, we first describe the energy content of the field configurations correlated to excitations of an accelerated system. We then apply the same analysis to obtain the energy distribution of field configurations (the vacuum fluctuations) which get converted into Hawking quanta. The main point of this analysis is that it goes beyond the description of the mean (averaged) energy of the field (see ref.\cite{3} for a review): we compute the energy of the fluctuations around the mean and evaluate specific gravitational back reaction effects induced by these vacuum fluctuations.

In Part 2, following the work of references \cite{2}\cite{4}-\cite{10}, we analyze the mean energy density emitted by a uniformly accelerated detector. We emphasize the difference between global quantities (the total energy, the number of quanta) and local quantities (the energy density) and we exhibit the essential role of transients in ensuring that global properties are respected (such as the positivity of the total energy). Our results generalize previous perturbative\cite{10} and non perturbative\cite{7} analyses. In particular, we show that the total mean number of Minkowski photons emitted is equal to the mean number of internal transitions of the two level atom which have occurred during the interacting period.

In Part 3, we decompose the mean flux analyzed in Part 2. It is written as a sum of two terms\cite{4}\cite{5}\cite{10}. The first term is the energy density if the atom has made a transition and the second term is the energy density has not made a transition. From these energy densities we introduce the central notion of conditional energy emitted. This notion is then generalized so as to investigate others correlations. We decompose the mean vacuum energy (which is zero) into the energy densities correlated to a future transition (or to the absence of a future transition) of the two level atom. These conditional energy densities describe the vacuum fluctuations which shall give rise to transitions of the atom.

This description follows the treatment of ref.\cite{11} wherein the mean current density carried by created pairs in an external electric field was decomposed into conditional current densities so as to describe the current carried by a specific pair of quanta.

As this approach to isolate certain vacuum fluctuations based on conditional values of operators is rather new and leads to peculiar results such as complex energy densities we shall dwell on it somewhat. In Section \ref{3.1} we show how the non diagonal matrix elements which describe these vacuum fluctuations can be obtained by decomposing the mean
value of the energy density. Their physical relevance are then revealed by introducing an additional quantum system coupled to $T_{\mu \nu}$. Indeed, these complex densities have a natural interpretation in Quantum Mechanics only. In perturbation theory, one finds that the modification of the wave function of the additional system is controlled by these matrix elements. (In particular, in the black hole situation, the first order response of the metric to the creation of a particular Hawking photon is controlled by these matrix elements.) This was discussed by Aharonov and collaborators [12] who considered the additional system to be a measuring device as in the von Neumann approach to measurement theory [13]. For the sake of completeness, at the end of this article, in Part 5, we show how the approach of Aharonov et al. can be adapted and generalized to the present cases where we are more concerned by backreaction effects than measurability problems.

The analysis of vacuum fluctuations is applied to black hole radiation in Part 4. We obtain the energy density of the vacuum fluctuations which are converted by the time dependent geometry of a collapsing star into Hawking photons. The properties of these energy densities give rise to specific quantum gravitational back reaction effects which are not present in the mean theory. At the present time, the only mathematically consistent treatment of the backreaction is the semiclassical theory (see refs. [14][15][16]). In this approximation, the external field remains purely classical and only the mean value of the matter current operator acts on it as a source. The quantum properties of the matter, i.e. its fluctuations and correlations, are completely ignored. But it is the fluctuations which are problematic. Indeed it has been stressed by t’Hooft [17] and Jacobson [18] that Hawking’s derivation of black hole radiation is no longer valid as soon as gravitational interactions are taken into account because it makes appeal to the structure of the vacuum on exponentially small scales. In the present work this is seen in particularly vivid fashion. Indeed we shall show that if a particle is emitted by the black hole a time $u$ after collapse with asymptotic frequency $\lambda$, then there was a vacuum fluctuation inside the star of energy density $O(\omega^2)$ (where $\omega = \lambda e^{u/4M}$) located on a distance scale of order $\omega^{-1}$. After a time $u = O(4M \ln M)$ for a typical $\lambda = O(M^{-1})$, $\omega$ is greater than a Planck frequency and the free field theory of Hawking is unjustified. This aspect of the fluctuations has been presented together with F. Englert in ref. [19].

There have been two attitudes in the literature to confront this situation. The first is to try to guess what is the physics at the Planck scale near the horizon and how the Hawking radiation emerges therefrom [21]-[25]. The second has been to use Einstein equations to investigate how back reaction effects modify the production of Hawking photons [26][27]. The present article places itself in this latter vein. We show at the end of Section 4.3 how our results can be applied to study some simple back reaction effects. We first evaluate the change in probability of finding a specific Hawking photon due to a modification of the background metric. We then show how the creation of a particular photon modifies the probability of finding subsequent photons. In both cases, one sees explicitly that the change in probability is entirely controlled by the conditional value of the energy momentum tensor. The more difficult problem of the self interaction of a Hawking photons with itself as it is created necessitates the analysis of loop corrections and we hope to report on it in a subsequent paper.
2 The Energy Emitted by an Accelerated Atom

2.1 Introduction

It is now well known that a uniformly accelerated two level atom thermalizes in Minkowski vacuum at temperature $a/2\pi$. But it is much more complicated to obtain a complete description of the fluxes emitted by this thermalized atom.

As first pointed out by Grove, when thermal equilibrium is reached there is no net emission of energy. His argument is the following. The accelerator feels the effect of a thermal bath. So first consider the inertial two level atom in thermal equilibrium. The time independence of the Hamiltonian and the stationarity of the state of the atom (in the thermodynamic sense) guarantee no mean flux, since each absorbed photon is re-emitted with the same energy. This argument is immediately applicable to the accelerator since the fact that $a = \text{constant}$ implies that his physics is translationally invariant in his proper time $\tau$ (i.e. invariance under boosts). Since Minkowski vacuum is also an eigenstate of the boost operator, the total Rindler energy is conserved.

A detailed picture of the steady state emerges from the following consideration. Focus on the ground state of the accelerator which excites by absorbing a Rindler quanta (a rindleron) coming in from its left. Then the field configurations to its right is depleted of this rindleron. Since this rindleron carried positive energy, its removal can be described as the emission of negative energy to the right. In equilibrium there is also to be considered the process of disexcitation corresponding to the emission of positive energy to the right. The Einstein relation guarantees that the two cancel. This implies no net energy flux.

However the accelerated atom is in Minkowski vacuum, hence all perturbations of the radiation state lead to the production of Minkowski quanta. Then, how do we reconcile the (certainly) positive energy of these produced quanta with the absence of radiated energy in thermal equilibrium? The answer lies in a global treatment of the radiation field which also takes into account the transients due to switching on and off the detector (or equivalently the transients which occur when the detector passes from an inertial trajectory to the accelerated one).

That transients may have a global content which depends on the whole history also occurs in the problem of the classical electromagnetic field emitted by a uniformly accelerated charge in 3 dimensional Minkowski space (see ref. [28]). We point out that this situation is particular to uniformly accelerated systems and makes explicit appeal to the exponential Doppler shift between accelerated and inertial reference frames. Indeed (anticipating some notations introduced in the next sections) one can express the total Minkowski energy emitted (in V-modes) as

$$E_M = \int_0^{+\infty} dV \langle T_{VV} \rangle = \int_{-\infty}^{+\infty} dv \langle T_{vv} \rangle \frac{dv}{dV} = C_i e^{-a\tau_i} - C_f e^{-a\tau_f}$$

(1)

where the first integral is limited to the domain $V = t + z > 0$ because the accelerator remains in the quadrant $z > |t|$. In the second equation we have used the Rindler coordinate $v$ related to $V$ by $dv/dV = e^{-av}$; and in the last equation we have made
appeal to Grove’s theorem which states that the integrand vanishes everywhere except at the endpoints \( v_i (= \tau_i) \) and \( v_f (= \tau_f) \); \( C_i \) and \( C_f \) depend on the exact form of the transients at \( \tau_i,f \).

Equation (1) can also be written as an integral over the rates of absorption and of emission of Rindler photons taking into account that a transition at time \( \tau \) is accompanied by the emission of a Doppler shifted Minkowski photon of frequency \( \omega(\tau) = m dv/dV = me^{-a\tau} \) where \( m \) is the resonant frequency of the atom. Hence the energy emitted is

\[
E_M = \int_{\tau_i}^{\tau_f} R \omega(\tau) d\tau = \frac{Rm}{a}(e^{-a\tau_i} - e^{-a\tau_f}) + (C'_i e^{-a\tau_i} - C'_f e^{-a\tau_f}) \tag{2}
\]

Here \( R \) is the number of transitions per unit proper time. The constants \( C'_i \) and \( C'_f \) depend on the detailed way the interaction is turned on and off at \( \tau_i,f \). So the integral eq. (2) is of the same form as eq. (1). These two expressions are compatible because all the photons in eq. (2) interfere in such a way that their energy is only found at the edges of the interaction period as in eq. (1).

This part is organized as follow. Section 2.2 is devoted to recalling the main properties of Rindler quantization. In Section 2.3 we present the model of the accelerated two level atom. In Section 2.4 we obtain formal expressions for the energy emitted. We decompose the flux according to the final state of the atom thereby introducing the central notion of conditional energy emitted. The properties of these conditional fluxes as well as their physical meaning shall be displayed in Part 3. As a warm up, we present in Sections 2.5 - 2.7 the various properties of the mean flux insisting on the role of the transients in guaranteeing that global properties are respected. The mean fluxes as the atom thermalizes are discussed in Section 2.5 and then in thermal equilibrium in Section 2.6. Finally in Section 2.7 we use the exact solvable model of ref. [6] to extend the previous perturbative results to all order in \( g \).

## 2.2 The Rindler Quantization in 1+1 Dimensions

In this section, we review the relevant properties of the Rindler quantization of the scalar field in Minkowski space time in 1+1 dimensions. The conformal invariance of the massless scalar field is best exploited by using the light like coordinates \( U, V \) defined by

\[
U = t - z \quad V = t + z
\]

Whereupon the Klein-Gordon equation takes the form \( \partial_U \partial_V \phi = 0 \) and any solution can be written as

\[
\phi(U,V) = \phi(U) + \phi(V) \tag{4}
\]

From now on we shall drop the right moving piece and consider the ”V” term only. It is obvious that all conclusions shall be equally valid for the right movers.
The second quantized field can be decomposed into the orthonormal complete basis of Minkowski modes

\[ \phi(V) = \int_0^\infty d\omega \left( a_\omega \varphi_\omega(V) + a_\omega^\dagger \varphi_\omega^*(V) \right) \]  

(5)

\[ \varphi_\omega(V) = \frac{e^{-i\omega V}}{\sqrt{4\pi \omega}} \]

(6)

Minkowski vacuum \( |0_M \rangle \) is the state annihilated by all the \( a_\omega \)'s. The propagator (Wightman function) in Minkowski vacuum is

\[ G_+(V,V') = \langle 0_M | \phi(V) \phi(V') | 0_M \rangle = \int_0^\infty d\omega \varphi_\omega(V) \varphi_\omega^*(V') = -\frac{1}{4\pi} \log(V - V' - i\epsilon) \]

(7)

The (normal ordered) Hamiltonian of the field is (for left movers)

\[ H_M = \int_{-\infty}^{+\infty} dVT_{VV} = \int_0^\infty d\omega (a_\omega^\dagger a_\omega) \]

(8)

where

\[ T_{VV} = \partial_V \phi \partial_V \phi \]

(9)

Therefore Minkowski vacuum is the ground state of the Hamiltonian \( H_M \): \( H_M | 0_M \rangle = 0 \).

The uniformly accelerated observer will be taken to be in the right (R) Rindler quadrant \( U < 0, V > 0 \). In this quadrant one defines Rindler coordinates \( \rho, \tau \) by

\[ \begin{cases} 
  t = \rho \sinh a\tau \\
  x = \rho \cosh a\tau 
\end{cases} \]

(10)

The accelerated observer follows the trajectory \( \rho = 1/a \) (where \( a \) is its acceleration) and its proper time is \( \tau \) (see figure 1). In this quadrant one introduces also the light like Rindler coordinates \( u, v \) defined by

\[ \begin{cases} 
  u = \tau - a^{-1} \ln a\rho \\
  v = \tau + a^{-1} \ln a\rho 
\end{cases} \]

(11)

These are related to the light like Minkowski coordinates \( U, V \) by

\[ \begin{cases} 
  U = -a^{-1} e^{-au} \\
  V = a^{-1} e^{av} 
\end{cases} \]

(12)

The coordinates eq. [11] may be extended to the left (L) Rindler quadrant by the analytic continuation \( \tau \to \tau \pm i\pi/a \). One introduces in L the light like coordinates \( u_L, v_L \) given by

\[ \begin{cases} 
  U = a^{-1} e^{au_L} \\
  V = -a^{-1} e^{-av_L} 
\end{cases} \]

(13)
The natural basis of quantization the uniformly accelerated observer would choose is the Rindler basis which consists of plane waves in the variables $u,v$ (Rindler modes). This is because $u,v$ are related to its proper time $\tau$ as $U,V$ are related to the Minkowski time $t$. The Rindler $v$ modes are thus given, in strict analogy with eq. (6), by

$$\varphi_{\lambda,R}(v) = \frac{e^{-i\lambda v}}{\sqrt{4\pi\lambda}}$$

(14)

Since the $\varphi_{\lambda,R}$ constitute a complete set in $R$ $(V > 0)$ only, they cannot be related to the Minkowski basis by a unitary transformation. One must also introduce Rindler modes living in the left quadrant. But one finds that the Bogoljubov transformation relating the Minkowski modes to the Rindler modes is singular at $V = 0$ and care must be taken to define it as a limit if the Minkowski properties of the theory are to be satisfied. To this end it is useful to first introduce an alternative basis of positive frequency Minkowski modes, eigenmodes of $iaV\partial V$ $(= i\partial_v$ for $V > 0)$, and defined for all $V$:

$$\varphi_{\lambda,M} (V) = \int_{0}^{\infty} d\omega \gamma_{\lambda,\omega} \varphi_{\omega} (V) = a_{\lambda,M} \varphi_{\lambda,M} (V) + a_{\lambda,M}^\dagger \varphi_{\lambda,M}^* (V)$$

(15)

where

$$\gamma_{\lambda,\omega} = \left( \frac{1}{\Gamma(i\lambda/a)} \sqrt{\frac{a\pi}{\lambda\sinh\pi\lambda/a}} \right) \frac{1}{\sqrt{2\pi\alpha}} \left( \frac{\omega}{\alpha} \right)^{i\lambda/a} e^{-\omega\epsilon}$$

(16)

The first factor in $\gamma_{\lambda,\omega}$ is a pure phase introduced for convenience. The factor $e^{-\omega\epsilon}$ is the crux of the construction. It defines the integral eq. (15), regularizes the modes $\varphi_{\lambda,M}(V)$ at $V = 0$ and ensures the correct Minkowski properties of the theory. For instance it gives the correct pole prescription at $V = V'$ of the propagator eq. (7) when expressed in terms of the modes $\varphi_{\lambda,M}$ as $G_+(V,V') = \int_{-\infty}^{+\infty} d\lambda \varphi_{\lambda,M}(V)\varphi_{\lambda,M}^*(V')$. The limit $\epsilon \to 0$ is to be taken at the end of all calculations.

The annihilation and creation operators corresponding to the modes $\varphi_{\lambda,M}$ are $a_{\lambda,M}$ and $a_{\lambda,M}^\dagger$:

$$\phi(V) = \int_{-\infty}^{+\infty} d\lambda \left( a_{\lambda,M} \varphi_{\lambda,M}(V) + a_{\lambda,M}^\dagger \varphi_{\lambda,M}^*(V) \right)$$

(17)

The right and left Rindler modes $\varphi_{\lambda,R}(V)$ and $\varphi_{\lambda,L}(V)$ can now be defined by the linear unitary transformation

$$\begin{cases}
\varphi_{\lambda,M} = \alpha_{\lambda} \varphi_{\lambda,R} + \beta_{\lambda} \varphi_{\lambda,L} \\
\varphi_{-\lambda,M} = \beta_{\lambda} \varphi_{\lambda,R}^* + \alpha_{\lambda} \varphi_{\lambda,L}^*
\end{cases} \quad \lambda > 0$$

(18)
with
\[ \alpha_\lambda = \frac{e^{\pi \lambda/2a}}{\sqrt{e^{\pi \lambda/a} - e^{-\pi \lambda/a}}} \quad , \quad \beta_\lambda = \frac{e^{-\pi \lambda/2a}}{\sqrt{e^{\pi \lambda/a} - e^{-\pi \lambda/a}}} \quad \text{and} \quad \alpha_\lambda^2 - \beta_\lambda^2 = 1 \] (19)

In the limit \( \epsilon \to 0 \) the Rindler modes take the familiar form (see eq. (14))
\[ \varphi_{\lambda,R}(V) = \theta(V)(aV)^{-i\lambda/a}/\sqrt{4\pi \lambda} = e^{-i\lambda v}/\sqrt{4\pi \lambda} \]
\[ \varphi_{\lambda,L}(V) = \theta(-V)(-aV)^{i\lambda/a}/\sqrt{4\pi \lambda} = e^{-i\lambda v_L}/\sqrt{4\pi \lambda} \] (20)

For finite \( \epsilon \) they differ from these limiting forms only when \( V \leq \epsilon \). The Rindler destruction and creation operators \( a_{\lambda,R}, a_{\lambda,L}^\dagger \) and \( a_{\lambda,L}, a_{\lambda,L}^\dagger \) associated to these modes are related to the Minkowski operators \( a_{\lambda,M} \) by the following Bogoljubov transformation
\[
\begin{align*}
   a_{\lambda,R} &= \alpha_\lambda a_{\lambda,M} + \beta_\lambda a_{-\lambda,M}^\dagger \\
   a_{\lambda,L} &= \alpha_\lambda a_{-\lambda,M} + \beta_\lambda a_{\lambda,M}^\dagger
\end{align*}
\] (21)
(by virtue of the orthonormal character of the two sets of modes \( \varphi_{\lambda,R}, \varphi_{\lambda,L} \) and \( \varphi_{\lambda,M} \) as well as \( \alpha_\lambda^2 - \beta_\lambda^2 = 1 \)). One immediately deduces that the mean number of Rindler quanta present in Minkowski vacuum is given by the Bose Einstein distribution
\[ <0_M|a_{\lambda,R}^\dagger a_{\lambda,R}|0_M> = \frac{1}{e^{2\pi \lambda/a} - 1}\delta(\lambda - \lambda') \] (22)

It is useful to introduce the generator of boosts \( H_R \) since it generates translations in \( \tau \) and is therefore the Hamiltonian for the accelerated observer.
\[ H_R = \int_{-\infty}^{+\infty} dV aVT_{VV} \]
\[ = \int_{-\infty}^{+\infty} dvT_{vv} - \int_{-\infty}^{+\infty} dvLT_{vLvL} \]
\[ = \int_{0}^{+\infty} d\lambda \lambda(a_{\lambda,R}^\dagger a_{\lambda,R} - a_{\lambda,L}^\dagger a_{\lambda,L}) \]
\[ = \int_{0}^{+\infty} d\lambda \lambda(a_{\lambda,M}^\dagger a_{\lambda,M} - a_{-\lambda,M}^\dagger a_{-\lambda,M}) \] (23)

One sees that quanta in L carry negative “Rindler energy”. This is because \( \tau \) (defined by \( \text{arctanh}(t/z) \)) goes backwards in time in L (\( d\tau/dt < 0 \) in L). Furthermore since Minkowski vacuum is invariant under boosts it is annihilated by the Rindler energy operator:
\[ H_R|0_M> = 0 \] (24)

Using the above Bogoljubov transformation it is easy to show that
\[ |0_M> = \prod_\lambda \frac{1}{\alpha_\lambda} e^{-\beta_\lambda a_{\lambda,L}^\dagger a_{\lambda,R}^\dagger}|0_{RL}> \] (25)

where \( |0_{RL}> = |0_R> \otimes |0_L> \) is Rindler vacuum in both the right(R) and left(L) quadrants. Thus, from eq. (23), one sees that the pairs of Rindler quanta (whose mean
number is given by eq. (22) present in Minkowski vacuum carry zero Rindler energy. One sees also that upon tracing over the left quanta the reduced density matrix in the right Rindler quadrant is an exact thermal distribution of right rindlerons [3].

Eq. (25) will be useful in the next chapters since it provides an easy way of calculating the probability \( |\langle \psi_R | 0_M \rangle|^2 \) of finding in Minkowski vacuum a state ( \(|\psi_R \rangle\) containing a given number of Rindler quanta.

### 2.3 The Uniformly Accelerated Two Level Atom

The situation we consider is a uniformly accelerated two level atom coupled to the massless field \( \phi \) introduced in the previous section. The trajectory of the two level atom is given by eq. (10) with \( \rho = a^{-1} \):

\[
t_a(\tau) = a^{-1}\sinh a\tau, \quad x_a(\tau) = a^{-1}\cosh a\tau
\]

\[
V_a(\tau) = a^{-1}e^{a\tau}, \quad U_a(\tau) = -a^{-1}e^{-a\tau}
\]

(26)

The time integral of the interaction hamiltonian is

\[
\int dt dx \ H_{\text{int}}(t, x) = gm \int d\tau \left[ \left( f(\tau)e^{-i m\tau} A + f^*(\tau)e^{i m\tau} A^\dagger \right) \phi(t_a(\tau), x_a(\tau)) \right]
\]

(27)

where \( g \) is a dimensionless coupling constant that shall be taken for simplicity small enough that second order perturbation theory be valid, \( m \) is the difference of energy between the ground and the excited state of the atom, \( A \) is the lowering operator that induces a transition from the excited state to the ground state of the atom and \( f(\tau) \) is a dimensionless function that governs when and how the interaction is turned on and off. The factor \( m \) on the r.h.s. of eq. (27) is introduced for dimensional reasons. In addition, we shall assume that the \( V \)-part of the \( \phi \) field only is coupled to the atom. This is a legitimate truncation owing to the chiral character of the field in 1 + 1 dimensions. For simplicity of notation it is convenient to rewrite eq. (27) as

\[
\int dt dx \ H_{\text{int}}(t, x) = gm \left[ A\phi_m^\dagger + A^\dagger\phi_m \right]
\]

(28)

where

\[
\phi_m = \int_{-\infty}^{+\infty} d\tau e^{im\tau} f^*(\tau)\phi(\tau)
\]

(29)

We shall be most interested in the situation where \( f(\tau) = 1 \) inside a long interval \( \tau_i < \tau < \tau_f \) and \( f(\tau) \) tends to zero outside this interval. Then in the limit \( \tau_f - \tau_i = T \to \infty \) with \( g^2 T \) finite (i.e. in the Golden Rule limit) the concept of a transition rate emerges and is due to the resonance of the Minkowski vacuum fluctuations with the fixed Rindler frequency \( m \) [30]. The fundamental reason why we need to work with an explicit switch function is that we want finite energy momentum densities everywhere including the horizon. We shall show that this is possible only for sufficiently rapidly decreasing \( f(\tau) \) (which amounts to deal only with wave packets of Rindler modes which are well defined in the ultraviolet –see eqs. (15, 16)).
When $e^{-im\tau}f(\tau)$ contains no negative frequency in its Fourier transform with respect to $\tau$, eq. (28) defines a Lee model: were it inertial it would only respond to the presence of Minkowski particles. For a Lee model $f(\tau)$ must necessarily decrease less rapidly than an exponential when $\tau \to \pm \infty$. Alas this condition is too strong and will lead to singularities on the horizon in the uniformly accelerating situation. Hence we shall be obliged to consider non Lee models which can spontaneously excite. However by choosing $f(\tau)$ such that the negative frequency part of $e^{-im\tau}f(\tau)$ is exponentially small the spontaneous excitations are exponentially. Moreover the spontaneous excitations occurs only at switch on and switch off transitory periods but do not occur during the steady regime when $f(\tau) = 1$, i.e. these excitations do not contribute to rates.

Let us consider first the situation in which both the atom and the field are initially in their ground state. The state $|\psi_- >$ at $t = -\infty$ is thus

$$|\psi_-(t = -\infty) > = |0_M >| - >$$  \hspace{1cm} (30)

where $| - > (| + >)$ designates the ground (excited) state of the atom. At $t = +\infty$, when the interaction has been switched off, the state can again be expressed in terms of the uninteracting states and it is given, to order $g^2$, by

$$|\psi_-(t = +\infty) > = e^{-i\int dtdxH_{int}}|0_M >| - >$$

$$= |0_M >| - >$$

$$-igm\int_{-\infty}^{+\infty} d\tau f'(\tau)e^{+im\tau}\phi(\tau)|0_M >| + >$$

$$-g^2m^2\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' f(\tau)e^{-im\tau}\phi(\tau)f^*(\tau')e^{+im\tau'}\phi'(\tau')|0_M >| - >$$

$$= |0_M >| - >$$

$$-igm\phi_m|0_M >| + > - \frac{1}{2}g^2m^2\phi_m^\dagger \phi_m|0_M >| - >$$

$$-g^2m^2D|0_M >| - >$$  \hspace{1cm} (31)

where

$$D = \frac{1}{2}\int d\tau_2 \int d\tau_1 \epsilon(\tau_2 - \tau_1)e^{-im\tau_2}f(\tau_2)\phi(\tau_2)e^{+im\tau_1}\phi^*(\tau_1)\phi(\tau_1)$$  \hspace{1cm} (32)

and where $\epsilon(\tau_2 - \tau_1) = \theta(\tau_2 - \tau_1) - \theta(\tau_1 - \tau_2)$. We have split the $g^2$ term in two pieces in order to isolate the steady regime part of the interaction. The latter is given by the term proportional to $\phi_m^\dagger \phi_m$ whereas the $D$ term is concerned with the transitory periods associated with the switch on and off. Indeed to order $g^2$ the $D$ term does not contribute to the energy density emitted in the steady state regime (i.e. its energy density scales like $g^2/T$ rather than like $mg^2$). Furthermore it carries no Minkowski nor Rindler energy. This is proven in the Appendix. We shall therefore drop this term in the sequel. To get a flavor of this, the reader can already verify that $D$ does not contribute to the probability to remain in the ground state. Only the term proportional to $\phi_m^\dagger \phi_m$ does so.

The probability $P_e$ for the two level atom to get excited is, in second order perturbation theory

$$P_e = g^2m^2 < 0_M|\phi_m^\dagger \phi_m|0_M >$$  \hspace{1cm} (33)
Had we coupled the atom to both the $U$ and $V$ parts of $\phi$, the probability would have been twice $P_e$. When $f(\tau)$ is equal to 1 between $\tau_i$ and $\tau_f$ and $\tau_f - \tau_i = T \to +\infty$ while $g^2 m T$ remains finite, the operator $\phi_{m}^{\dagger} \phi_{m}$ appearing in eq. (33) becomes the counting operator for rindlerons of energy $m (= a_{m,R} a_{m,R})$ multiplied by $\pi/m$. A direct golden rule calculation then shows that the probability for the uniformly accelerated atom to get excited is

$$
P_e = \frac{1}{2} g^2 m T N_m \tag{34}
$$

where $N_m = 1/(e^{2\pi m/a} - 1)$ is the mean number of Rindler quanta present in Minkowski vacuum, see eq. (22). This proves that the atom maintained on an accelerated trajectory reacts to the mean number of Rindler quanta as the same atom, put on an inertial trajectory, would have reacted to the mean number of Minkowski quanta.

If the initial state is the product of Minkowski vacuum and the excited state for the atom

$$
|\psi_+(t = -\infty) > = |0_M > |+ > \tag{35}
$$

then $|\psi_+ >$ at $t = +\infty$ is

$$
|\psi_+(t = +\infty) > = |0_M > |+ > - igm \phi_{m}^{\dagger} |0_M > |> - \frac{1}{2} g^2 m^2 \phi_{m}^{\dagger} \phi_{m} |0_M > |+ > + g^2 m^2 D |0_M > |+ > \tag{36}
$$

where the operator $D$ which appears is the same as in eq. (31). The probability to be found in the ground state at $t = +\infty$ (i.e. the probability of disexcitation) in the golden rule limit is

$$
P_d = g^2 m^2 <0_M | \phi_{m}^{\dagger} \phi_{m} |0_M > = (1/2) g^2 m T (N_m + 1) \tag{37}
$$

Hence, at equilibrium, by Einstein’s famous argument, the probabilities $P_+, P_-$ to be in the excited or ground states are given by

$$
\frac{P_+}{P_-} = \frac{P_e}{P_d} = \frac{N_m}{N_m + 1} = e^{-2\pi m/a} \tag{38}
$$

that is a thermal distribution at temperature $T = a/2\pi$.

2.4 The Mean Fluxes to order $g^2$

We investigate now the properties of the flux emitted by the accelerated atom and its relations with the transition probabilities $P_e$ and $P_d$. Since the field is massless, the conservation law reads

$$
\partial_{\tau} T_{VV} = 0 \tag{39}
$$

Thus since the interaction, eq. (27), occurs only on the accelerated trajectory (eq. (26)), one has in the past of this line, or to the right of it by virtue of eq. (39),

$$
\langle T_{VV}(V, U < U_a(\tau)) \rangle = 0 \tag{40}
$$
The mean value means that the initial state can be any combination of $|\psi_-\rangle$ and $|\psi_+\rangle$. (When the initial state does contain Minkowski quanta, the mean value eq. (11) should by understood as the modification of the mean induced by the coupling eq. (28).)

We consider first the situation where the atom is in its ground state at $t = -\infty$ and the field in Minkowski vacuum (i.e. the state of the system is $|\psi_+\rangle$). After the interaction is switched off, the mean flux emitted on $\mathcal{T}^+$ (i.e. on $U = +\infty$) or equivalently to the left of the trajectory by virtue of eq. (39) is, to order $g^2$, |
\langle T_{VV} \rangle_{\psi_-} = \langle T_{VV} \rangle_{\psi_+} + P_e \langle T_{VV} \rangle_e + P_g \langle T_{VV} \rangle_g \quad (42)
\end{align*}

where we have defined
\begin{align*}
\langle T_{VV} \rangle_e &= g^2 m^2 < 0_M | \phi_m^\dagger T_{VV} \phi_m | 0_M > / P_e
\langle T_{VV} \rangle_g &= -g^2 m^2 \text{ Re} \left[ < 0_M | T_{VV} \phi_m^\dagger \phi_m | 0_M > \right] / P_g \quad (43)
\end{align*}

where $P_e$ and $P_g$ are the probabilities to find the atom in the excited or ground state at $t = +\infty$. $P_e$ is given in eq. (34) and $P_g = 1 - P_e$.

The interpretation of the two quantities $\langle T_{VV} \rangle_e$ and $\langle T_{VV} \rangle_g$ is clear when one recalls their origin. $\langle T_{VV} \rangle_e$ comes from the square of the second term of eq. (31) (linear in $g$) whereas $\langle T_{VV} \rangle_g$ comes from an interference between the first term of eq. (31) (unperturbed) and the third term (in which the interaction has acted twice). Hence $\langle T_{VV} \rangle_e$ is the energy emitted if the atom is found excited at $t = +\infty$ and $\langle T_{VV} \rangle_g$ is the energy emitted if the atom is found in the ground state at $t = +\infty$. These fluxes have been normalized so as to express the r.h.s. of eq. (12) as the probability of finding the atom in a final state times the energy emitted if that final state is realized. Thus $\langle T_{VV} \rangle_g$ and $\langle T_{VV} \rangle_e$ are the conditional ”mean” energy emitted. (The word ”mean” is understood here in its quantum sense, i.e. as the average over repeated realizations of the same situation: the same initial state $|\psi_-\rangle$ and the same final state of the atom —see Part 5 for further comments on this point).

Similarly, when the initial state of the system is $|\psi_+\rangle$, the mean energy emitted is
\begin{align*}
\langle T_{VV} \rangle_{\psi_+} &= \langle T_{VV} \rangle_{\psi_+} = g^2 m^2 < 0_M | \phi_m T_{VV} \phi_m | 0_M > - g^2 m^2 \text{ Re} \left[ < 0_M | T_{VV} \phi_m^\dagger \phi_m | 0_M > \right] \quad (44)
\end{align*}

where we have used eq. (36) and dropped the $\mathcal{D}$ term as well. As in eq. (12), we rewrite this flux as
\begin{align*}
\langle T_{VV} \rangle_{\psi_+} = P_d \langle T_{VV} \rangle_d + P_h \langle T_{VV} \rangle_h \quad (45)
\end{align*}
where \( P_d \) is the disexcitation probability given in eq. (37) and where \( P_h \) is the probability to be found in the excited state at \( t = +\infty \), hence \( P_h = 1 - P_d \). The conditional fluxes \( \langle T_{VV} \rangle_d \) and \( \langle T_{VV} \rangle_h \) are given by

\[
\langle T_{VV} \rangle_d = g^2 m^2 < 0_M | \phi_m T_{VV} \phi_m^\dagger | 0_M >/P_d \\
\langle T_{VV} \rangle_h = -g^2 m^2 \text{ Re} \left[ < 0_M | T_{VV} \phi_m \phi_m^\dagger | 0_M > \right] /P_h
\]

These two quantities are interpreted as the energy emitted when the atom is found in the ground state (disexcitation \( d \)) or in the excited state at \( t = +\infty \) knowing that the atom was prepared in the excited state at \( t = -\infty \).

When equilibrium is reached, the state of the atom is a thermal superposition. The probabilities of finding it in the excited or ground state are those given in eq. (38) with \( P_+ + P_- = 1 \). Since Minkowski vacuum is a thermal distribution of Rindler quanta, one can approximate the state at equilibrium by

\[
|\psi_{\text{therm.}}> = P_- |\psi_-> + P_+ |\psi_+>
\]

This neglects the dressing of the states due to high orders in \( H_{int} \) but gives correctly the properties of the fluxes. The conclusions obtained using this naive state will be proven to be true to all orders in \( g^2 \) in Section 3.3.

In the state eq. (47) the energy flux is given given by the weighted sum of \( \langle T_{VV} \rangle_- \) and \( \langle T_{VV} \rangle_+ \):

\[
\langle T_{VV} (V) \rangle_{\text{therm.}} = P_- \langle T_{VV} \rangle_- + P_+ \langle T_{VV} \rangle_+
\]

(This stems from the fact that the energy momentum operator changes the photon number by an even number and that the interaction hamiltonian changes the photon number by an odd number while changing the state of the atom). Hence all the matrix elements of \( T_{VV} \) we shall want to calculate can be expressed in terms of \( \langle T_{VV} \rangle_i \) where \( i \) stands for \( e, g, d \) and \( h \).

At this point, we remark that each each of these matrix elements \( \langle T_{VV} \rangle_i \) are acausal, for instance they are non vanishing in the left Rindler quadrant \( V < 0, U > 0 \). This will be discussed and interpreted in Section 3.3. However the mean energies \( \langle T_{VV} \rangle_j \) (where \( j = +, -, \text{ therm.} \)) are causal. This follows from the following very general argument. If \( V, U \) is separated from the trajectory of the atom \( U_a(\tau), V_a(\tau) \) by a space like distance then \( T_{VV}(V, U) \) commutes with \( H_{int} \) and one can rewrite the mean value as

\[
\langle T_{VV}(V, U) \rangle_j = < \psi_j | e^{+i \int dt H_{int}} T_{VV}(V, U) e^{-i \int dt H_{int}} | \psi_j > \\
= < \psi_j | T_{VV}(V, U) e^{+i \int dt H_{int}} e^{-i \int dt H_{int}} | \psi_j > \\
= < \psi_j | T_{VV}(V, U) | \psi_j >= 0
\]

Furthermore, by virtue of the conservation eq. (39), the zone where \( \langle T_{VV}(V, U) \rangle_j \) vanishes can be extended to regions which are not space likely separated from the trajectory but to which \( V \) modes are unable to propagate. We note that a similar causality argument applies in regions where \( \langle T_{VV}(V, U) \rangle_j \neq 0 \) to guarantee that it only depends on \( H_{int}(\tau) \) for \( \tau \)'s such that \( V(\tau) < V \), i.e. that it only depends on the form of \( H_{int}(\tau) \) in the past light cone of \( (V, U) \).
We also point out here a global property of the matrix elements $\langle T_{VV} \rangle_{e,g}$ which will have important consequences in the sequel. Namely that the Minkowski energy carried by $\langle T_{VV} \rangle_e$ is strictly positive

$$\int_{-\infty}^{+\infty} dV \langle T_{VV} \rangle_e = g^2 m^2 <0_M|\phi_m^\dagger H_M\phi_m|0_M>/P_e > 0$$

(50)

since it is the expectation value of $H_M$ (defined in eq. (8)) in a state which is not Minkowski vacuum. On the other hand the Minkowski energy carried by $\langle T_{VV} \rangle_g$ vanishes identically

$$\int_{-\infty}^{+\infty} dV \langle T_{VV} \rangle_g = -g^2 m^2 \text{Re} \left[ <0_M|H_M\phi_m^\dagger\phi_m|0_M> \right] /P_g = 0$$

(51)

since $H_M|0_M> = 0$. The same results are also true for $\langle T_{VV} \rangle_d$ and $\langle T_{VV} \rangle_h$.

In preparation for the next sections, we remark that all the matrix elements $\langle T_{VV} \rangle_i (i = e, g, d, h)$ can all be expressed in terms of the following two functions (once the $D$ term is dropped)

$$C_+(V) = <0_M|\phi(V)\phi_m^\dagger|0_M> = \int d\tau G_+(V,V_a(\tau))e^{-im\tau} f(\tau)$$

$$C_-(V) = <0_M|\phi(V)\phi_m|0_M> = \int d\tau G_+(V,V_a(\tau))e^{+im\tau} f^*(\tau)$$

(52)

where $G_+(V,V')$ is given in eq. (8). Indeed using eqs. (43) and (46), one has

$$\langle T_{VV} \rangle_e = 2\left(\frac{g^2 m^2}{P_e}\right) (\partial_V C_-(\partial_V C_+^\dagger))$$

$$\langle T_{VV} \rangle_d = 2\left(\frac{g^2 m^2}{P_d}\right) (\partial_V C_+ (\partial_V C_+^\dagger))$$

$$\langle T_{VV} \rangle_g = 2\left(\frac{g^2 m^2}{P_g}\right) \text{Re} \left[ (\partial_V C_-)(\partial_V C_+) \right] = \left(\frac{P_h}{P_g}\right)\langle T_{VV} \rangle_h$$

(53)

For these matrix elements of $T_{VV}$ not to be singular the functions $\partial_V C_+(V)$ and $\partial_V C_-(V)$ must be regular. The function $\partial_V C_+(V)$ can be expressed as

$$\partial_V C_+(V) = -\frac{1}{4\pi} \int d\tau \frac{1}{V-a^{-1}e^{a\tau} - i\epsilon} f(\tau)e^{-im\tau}$$

(54)

It can be singular only for $V = 0$ where it takes the form

$$\partial_V C_+(V) = -\frac{1}{4\pi} \int d\tau \frac{1}{-a^{-1}e^{a\tau} - i\epsilon} f(\tau)e^{-im\tau} \simeq \frac{a}{4\pi} \int d\tau e^{-a\tau} f(\tau)e^{-im\tau}$$

(55)

The last integral is finite if and only if $f(\tau)$ decreases for $\tau \to -\infty$ quicker than $e^{a\tau}$. Similarly if we had considered right movers, the condition for finiteness on the future horizon would have been sufficient rapid decrease of $f$ for $\tau \to +\infty$. Putting all together
the condition to not have singularities on the horizons is that \( f(\tau) \) decreases faster than \( e^{-|\tau|} \). This can be rewritten as

\[
\int d\tau \left| \frac{dt}{d\tau} |f(\tau)| = \int dt |f(\tau(t))| < \infty
\]

i.e. the interaction of the atom with the field must last a finite Minkowski time.

In order to obtain explicit expressions for the functions \( C_\pm \) we reexpress \( f(\tau)e^{-im\tau} \) in terms of its Fourier transform

\[
f(\tau)e^{-im\tau} = \int_{-\infty}^{+\infty} d\lambda \frac{c_\lambda}{2\pi} e^{-i\lambda\tau}
\]

The normalization is

\[
\int d\tau |f(\tau)|^2 = \int d\lambda \frac{|c_\lambda|^2}{2\pi} = T = \text{total time of interaction}
\]

The regularity condition eq. (56) is equivalent to having \( c_\lambda \) be an analytic function of \( \lambda \) in the strip \(-a < \text{Im}\lambda < a\). Hence we will not consider Lee models since they have \( c_\lambda = 0 \) for \( \lambda < 0 \) and thus singularities on the horizons. But in order that the behaviour of the uniformly accelerated two level atom be physically unambiguous, it is necessary that \( c_\lambda \) be peaked around \(+m\) (the contribution of the negative frequency components of \( c_\lambda \) should be negligible) and the golden rule probability of transition eq. (34) be recovered. For this to be the case \( T \) must satisfy \( T >> m^{-1} \) and \( T >> a^{-1} \).

The first of these conditions is that \( f(\tau) \) be spread over a distance at least equal to the inverse frequency \( m^{-1} \) (the time–energy uncertainty condition). The second condition, which corresponds to \( T \) being greater than the euclidean tunneling time \( 2\pi a^{-1} \) [30], is required for the probability \( P_e \) to be linear in time and proportional to the Bose distribution \( N_m \). Both these conditions arise in the standard textbook calculation for the probability of transition in a thermal bath

\[
P_e = 4g^2m^2 \int d\omega \rho(\omega) \frac{\sin^2(m-\omega)T/2}{(m-\omega)^2}
\]

where \( \rho \) is the density of states (available photon states for disexcitation or the density of photons present for excitation). It is legitimate to replace \( [\sin((m-\omega)T/2]/(m-\omega)]^2 \) by \( \pi T\delta(m-\omega)/2 \) in the integrand provided \( T^{-1}d\ln\rho/d\omega << 1 \). For disexcitation in vacuum this yields \( T >> m^{-1} \). In a thermal bath at inverse temperature \( \beta \), the Boltzmann distribution yields the additional condition \( T >> \beta \), hence \( T >> a^{-1} \) in the uniformly accelerated case.

We now express the operator \( \phi_m \) and the functions \( C_\pm \) in terms of \( c_\lambda \):

\[
\phi_m = \int_0^\infty d\lambda \frac{a_{\lambda,R}}{\sqrt{4\pi|\lambda|}} c^*_\lambda + \int_{-\infty}^0 d\lambda \frac{a_{\lambda,R}^\dagger}{\sqrt{4\pi|\lambda|}} c^*_\lambda
\]

\[
= \int_{-\infty}^{+\infty} d\lambda c^*_\lambda \frac{1}{\sqrt{4\pi\lambda}(e^{\pi\lambda/a} - e^{-\pi\lambda/a})} (e^{\pi\lambda/2a} a_{\lambda,M} + e^{-\pi\lambda/2a} a_{\lambda,M}^\dagger)
\]
where we have used eq. (15) for the expression of \( \varphi_{\lambda,M}(V) \). And \( \tilde{n}_\lambda = 1/(e^{2\pi\lambda/a} - 1) \) is equal to (see eq. (19))

\[
\tilde{n}_\lambda = N_\lambda = \frac{\beta^2}{\lambda} \quad \text{for } \lambda > 0
\]

\[
\tilde{n}_\lambda = -(N_{|\lambda|} + 1) = -a^2_{|\lambda|} \quad \text{for } \lambda < 0.
\]

The probability \( P_e \) to excite can also be written in terms of \( c_\lambda \)

\[
P_e = g^2 m^2 \int_{-\infty}^{+\infty} d\lambda \frac{|c_\lambda|^2}{4\pi \lambda} \tilde{n}_\lambda
\]

As one picture is worth a thousand words we take a particular form for \( c_\lambda \) such that all the integrals above are gaussian and can be evaluated explicitly

\[
c_\lambda = D \lambda m \frac{e^{-(\lambda-m)^2T^2/2}}{m} (1 - e^{-2\pi\lambda/a})
\]

where \( D \) is a normalization constant taken such as to verify eq. (58).

We shall give throughout the text the exact expressions followed by the approximate expressions which are valid when \( T \) satisfies the condition discussed above \( T >> m^{-1} \) and \( T >> a^{-1} \) as these last are physically relevant and are particularly easy to read and understand. The approximate expressions are preceded by the symbol \( \simeq \). We conclude this section by the value of the switch off function \( f \) (see figure 2)

\[
f(\tau) = \frac{D}{\sqrt{2\pi T}} e^{-\tau^2/2T^2} \left[ (1 - i \frac{\tau}{mT^2}) - e^{-2\pi m/a e^{i2\pi \tau/aT^2} e^{2\pi^2/a^2T^2}} (1 - i \frac{\tau}{mT^2} - \frac{\pi}{amT^2}) \right]
\]

\[
\simeq \pi^{-1/4} e^{-\tau^2/2T^2} \left[ 1 + N_m (1 - e^{i2\pi \tau/aT^2}) \right]
\]

where the constant \( D \) takes the form \( D \simeq 2^{1/2}\pi^{1/4} T(N_m + 1) \). Eq. (64) shows the almost gaussian character of the switch off function whose width is \( T \). The plateau of the gaussian gives a good approximation of the steady state regime which we which to study.

2.5 Fluxes and Particles to Order \( g^2 \) During Thermalisation

We briefly sketch the main results of this section. During thermalization a steady flux of negative Rindler energy is emitted: \( \langle T_{\nu\nu} \rangle_{\psi_-} \simeq -g^2 m^2 N_m/2 \). This is understood
from the isomorphism [5] with the thermal bath: as the atom gets exited it absorbs energy from the thermal bath, thus the minus sign. The transcription of this flux to Minkowski quanta is more subtle. Oscillatory tails in the Rindler flux are enhanced by the jacobian that converts from Rindler to Minkowski energy with the net result that positive Minkowski energy is emitted. In the Minkowski description the origin of the steady negative flux is due to a ”repolarization” of the atom corresponding to the fact that the probability of finding the atom in its exited level decreases with time. This repolarization is similar (CPT conjugate) with that which occurs when negative energy is absorbed by an inertial detector [34].

We shall discuss both the adiabatic switch on and off presented in the previous section (to reveal the oscillatory tails) and a sudden switch on and off (to display the properties in the stationary regime).

We start with the adiabatic switch on and off. In terms of the function $c_\lambda$ introduced in eq. (63) the mean energy radiated by the two level atom initially in its ground state is

$$\langle T_{vv}(v) \rangle_{\psi_-} = -g^2 m^2 \int d\lambda \int d\lambda' c_\lambda c^*_{\lambda'} \frac{1}{(4\pi)^2} (\tilde{n}_\lambda + \tilde{n}_{\lambda'}) e^{-i(\lambda - \lambda')v}$$

$$\simeq -\frac{g^2 m^2}{2} N_m \frac{e^{-v^2/T^2}}{\pi^{1/2}} [(N_m + 1) \cos(2\pi v/aT^2) - N_m]$$  \hspace{1cm} (65)

As announced it carries negative Rindler energy:

$$\int_{-\infty}^{+\infty} dv \langle T_{vv}(I+) \rangle_{\psi_-} = -\frac{g^2 m^2}{4\pi} \int_{-\infty}^{+\infty} d\lambda |c_\lambda|^2 \tilde{n}_\lambda$$

$$\simeq -\frac{1}{2} g^2 m^2 N_m T = -mP_e$$  \hspace{1cm} (66)

which is equal to the probability to be found excited times the absorbed Rindler energy $-m$.

The total Minkowski energy radiated is

$$\langle H_M \rangle_e = \int_0^{+\infty} dV \langle TVV(V) \rangle_{\psi_-} = \int_{-\infty}^{+\infty} dv e^{av} \langle T_{vv}(v) \rangle_{\psi_-}$$

$$\simeq +\frac{1}{2} g^2 m^2 N_m T e^{av_0} (1 + 2N_m) = +mP_e e^{av_0} (1 + 2N_m)$$  \hspace{1cm} (67)

where $e^{av_0}$ is the mean Doppler effect associated with the window function $f(\tau)$. We define it by

$$\int dv e^{-av} e^{-v^2/T^2} \cos(2\pi v/aT^2) = -e^{av_0} T/\pi^{1/2}$$  \hspace{1cm} (68)

The Minkowski energy is positive (as it should be), whereas the Rindler energy is negative. The flip in sign is due to the effect of the transients around $v = aT^2$. Indeed whereas the transients are negligible upon computing the Rindler energy eq. (66), upon computing the Minkowski energy they are enhanced by the jacobian $dv/dV$ and give rise to the flip in sign. (Note that this sign flip of the Minkowski energy versus the Rindler energy can be conceived as arising from the imaginary part of the saddle point
of eq. (68): \( v_{sp} = -aT^2/4 + i\pi/a \) and is therefore on the same footing as that the flip of frequency which leads to a non vanishing \( \beta \) coefficient, see [30]. The additional factor \( 1 + 2N m \) in eq. (67) and the difference with eq. (2) by a factor \( Ta \) comes from the inherent ambiguity in defining \( e^{a\tau_0} \) as the mean Doppler shift associated to the switch function \( f(\tau) \).

We insist on the fact that the total Minkowski energy radiated can also be expressed as

\[
\langle H_M \rangle_e = P_e \int_{-\infty}^{+\infty} dV \langle T_{VV} \rangle_e
\]

because of eq. (51). So the Minkowski energy can be conceived as coming from \( \langle T_{VV} \rangle_e \) only. When expressed in this fashion the integrand in eq. (69) is strictly positive but located essentially in the region \( V < 0 \) (this is shown in Part 3). (We remark that eq. (2) can be viewed as a rewriting of eq. (69) but with the integration taken in the quadrant \( V < 0 \)). The \( \langle T_{VV} \rangle_g \) term restores causality and localizes all the energy in the transients.

Another case of interest is the golden rule limit for which \( c_\lambda = 2\pi \delta(\lambda - m) \) corresponding to \( f(\tau) = 1 \) for all \( \tau \). In this case their is a constant negative flux for all \( V > 0 \). The transients are located on the past horizon \( V = 0 \) where they consist of a singular positive flux [7]. Rather than this case we now analyse the case where the time dependent coupling is \( f(\tau) = \theta(\tau)\theta(T - \tau) \). With this time dependence the transients are singular and will not be studied (this divergent behavior is already present for an inertial detector with the same switch function and has nothing to do with the presence of a horizon). On the contrary, the steady part is easily computed and corresponds exactly to the intermediate values \((-aT^2 << \tau << aT^2\)) found in the adiabatic situation described above in eq. (63). The reason for which we shall now belabour this case is that it shows explicitly the relations between the flux emitted and the transition rate (not only the relation between the probability and the total Rindler energy as in eq. (66)).

The probability of spontaneous emission is given by

\[
P_e(T) = g^2m^2 \int_0^T d\tau_1 \int_0^T d\tau_2 e^{-im(\tau_2 - \tau_1)} \langle \phi(\tau_2)\phi(\tau_1) \rangle
\]

\[
\simeq \frac{1}{2} g^2 m N_m T
\]

The second line contains the golden rule result valid when \( aT \to \infty \) with \( g^2T \) finite. It is useful to introduce the rate of transition, the derivative of \( P_e(T) \):

\[
\dot{P}_e(T) = \frac{dP_e(T)}{dT} = g^2m^2 2 \text{ Re} \left[ \int_0^T d\tau e^{-im(T - \tau)} \langle \phi(T)\phi(\tau) \rangle \right]
\]

\[
\simeq \frac{1}{2} g^2 m N_m
\]

This rate is related to the (steady part of) the stress energy tensor. Indeed one finds

\[
\langle T_{vv}(v = T) \rangle_{\psi_-} = g^2m^2 2 \text{ Re} \left[ \int_0^T d\tau_2 \int_{\tau_2}^{\tau_1} d\tau_1 e^{-im(\tau_2 - \tau_1)} \langle [\phi(\tau_2), T_{vv}(T)] - \phi(\tau_1) \rangle \right]
\]
\[ g^2 m^2 2 \text{Re} \left[ \int_0^T d\tau e^{-im(T-\tau)} \langle i \partial_\tau \phi(T) \phi(\tau) \rangle \right] \]
\[ = -m \dot{P}_e(T) + g^2 m^2 2 \text{Re} \left[ ie^{-imT} \langle \phi(T) \phi(0) \rangle \right] \]  
(72)

The first equality follows straightforwardly from the expansion of the evolution operator \( e^{-i \int H_{int} d\tau} \) in \( g^2 \). The second equality is obtained using the commutator relation: \( [\phi(\tau_2), T_{vv}(\tau_1)] = i \partial_\tau \phi(\tau_1 - \tau_2) \). The third equality follows by integration by parts. The final result contains a steady part proportional to \(-m \dot{P}_e(T)\) which tends to \(-\frac{1}{2}g^2 m^2 N_m\) in the golden rule limit and an oscillatory term (which is exponentially damped if a slight mass is given to \( \phi \)). The steady piece simply indicates that to an increase of the probability to make a transition corresponds the absorption of the necessary Rindler energy to provoke this increase.

We now turn to the Minkowski description of this steady piece. We first rewrite these expressions in terms of the Minkowski basis \( e^{-i \omega V/\sqrt{4\pi\omega}} \) (see eq. (6)). The probability of transition eq. (70) reads
\[ P_e(T) = g^2 m^2 \int_0^\infty d\omega \left| \int_0^T d\tau e^{-im\tau} \frac{e^{-i\omega e^{\tau}}}{\sqrt{4\pi\omega}} \right|^2 \]
\[ = \int_0^\infty d\omega \ P_{e,\omega}(T) \]  
(73)
Similarly the transition rate eq. (71) becomes
\[ \dot{P}_e(T) = g^2 m^2 \int_0^\infty d\omega \ 2 \text{Re} \left[ \int_0^T d\tau e^{-im(T-\tau)} \frac{e^{-i\omega(e^{\tau}-e^{\tau})}}{4\pi\omega} \right] \]
\[ = \int_0^\infty d\omega \ \dot{P}_{e,\omega}(T) \]  
(74)
And the total Minkowski energy is given by
\[ \langle H_M(T) \rangle_e = \int_{-\infty}^{+\infty} d\omega e^{\omega T} \langle T_{vv} \rangle_{\psi_-} = \int_0^\infty d\omega \ \omega P_{e,\omega}(T) \]  
(75)
(Where in the first equality the integral is only over region of positive \( V \) since by causality the mean energy is unaffected in the other quadrant, see eq. (49)). The second equality follows from the diagonal character of the energy operator \( H_M \) see eqs. (50), (51). The positivity of \( \langle H_M(T) \rangle_e \) is manifest since all the \( P_{e,\omega}(T) \) are positive definite. Nevertheless the time derivative of \( \langle H(T) \rangle_e \) is negative, within the steady regime,
\[ \frac{d\langle H_M(T) \rangle_e}{dT} = \int_0^\infty d\omega \ \omega \dot{P}_{e,\omega}(T) \]
\[ = e^{a(T)} \langle T_{vv}(v(T)) \rangle_{\psi_-} \]
\[ = -m e^{a(T)} \left[ \dot{P}_e(T) + \text{oscillatory “damped” term} \right] \]  
(76)
\( d\langle H_M \rangle/dT \) negative implies thus that, for large \( \omega \) (since \( \dot{P}_e(T) > 0 \)), some \( \dot{P}_{e,\omega} \) are negative. This corresponds to a "repolarization" since all the \( P_{e,\omega} \) are positive definite and vanish for \( \tau \leq 0 \). This repolarization is exactly the inverse process of the absorption of negative energy by an atom described in [34].
2.6 Fluxes and Particles to Order $g^2$ at Equilibrium

Before studying the equilibrium situation it behoves us first to consider the flux emitted by an atom that makes a transition from excited to ground state.

The mean energy emitted when the initial state is $|\psi_+\rangle$ is

$$\langle T_{vv} \rangle_{\psi_+} = \frac{g^2 m^2}{4\pi} \int d\lambda' c_{\lambda'} c^*_{\lambda} \frac{1}{(4\pi)^2} (\tilde{n}_\lambda + \tilde{n}_{\lambda'} + 2) e^{-i(\lambda - \lambda')v}$$

$$\simeq \frac{g^2 m^2}{2\sqrt{\pi}} (N_m + 1) e^{-v^2/T^2} [1 - N_m \{\cos(2\pi v/aT^2) - 1\}]$$  \hspace{1cm} (77)

and the total Rindler energy radiated is

$$\int dv \langle T_{vv}(v) \rangle_{\psi_+} = \frac{g^2 m^2}{4\pi} \int d\lambda |c_{\lambda}|^2 (\tilde{n}_\lambda + 1)$$

$$\simeq \frac{1}{2} g^2 m^2 (N_m + 1)T = mP_d$$  \hspace{1cm} (78)

In the example for which the time dependent coupling is $f(\tau) = \theta(\tau)\theta(T - \tau)$, the relation between the derivative of the probability $\dot{P_d}(T)$ and the flux $\langle T_{vv} \rangle_{\psi_+}$ is

$$\langle T_{vv}(T) \rangle_{\psi_+} = +m\dot{P_d}(T) + \text{oscillatory } "damped" \text{ term}$$  \hspace{1cm} (79)

The sign in front of $\dot{P_d}(T)$ is now positive (contrary to the one in eq. (72)). Disexcitation consists in emitting the energy stored in the atom.

The total Minkowski energy emitted is

$$\int_0^{+\infty} dV \langle T_{VV} \rangle_{\psi_+} \simeq \frac{g^2 m^2}{2} (N_m + 1)T e^{\alpha_0}(2N_m + 1) = mP_d e^{\alpha_0}(2N_m + 1)$$  \hspace{1cm} (80)

For the disexcitation, the integrated Rindler and Minkowski energies have the same sign and are related by the mean Doppler shift $e^{\alpha_0}$ times $(2N_m + 1)$.

We now turn to the thermal equilibrium situation. We recall that the energy radiated is the sum of the fluxes emitted when the atom is initially in its ground state and when the atom is initially in its exited state weighted by their initial probabilities. Hence one has

$$\langle T_{vv} \rangle_{\text{therm.}} = P_- \langle T_{vv} \rangle_{\psi_-} + P_+ \langle T_{vv} \rangle_{\psi_+}$$

$$\simeq -mP_- \dot{P}_e + mP_+ \dot{P_d} = 0$$  \hspace{1cm} (81)

The steady fluxes cancel exactly each other because at thermal equilibrium $P_{\pm}$ satisfy eq. (38). This is Grove theorem in $g^2 \frac{3}{2} \frac{9}{2}$. Only the oscillatory transients remain. They read

$$\langle T_{vv} \rangle_{\text{therm.}} = \frac{g^2 m^2}{2N_m + 1} \int d\lambda \int d\lambda' c_{\lambda} c^*_{\lambda'} \frac{1}{(4\pi)^2} [N_m(\tilde{n}_\lambda + \tilde{n}_{\lambda'} + 2) - (N_m + 1)(\tilde{n}_{\lambda} + \tilde{n}_{\lambda'})] e^{-i(\lambda - \lambda')v}$$

$$\simeq \frac{g^2 m^2}{\sqrt{4\pi}} N_m (N_m + 1) e^{-v^2/T^2} [1 - \cos(2\pi v/aT^2)]$$  \hspace{1cm} (82)
To illustrate the positive transients, we have plotted $\langle T_{vv}\rangle_{\text{therm.}}$ in figure 3.

The total Rindler energy emitted is

$$\int dv \langle T_{vv}\rangle_{\text{therm.}} = \frac{g^2 m^2}{4\pi} \frac{1}{2N_m + 1} \int d\lambda |c_\lambda|^2 (N_m - \tilde{n}_\lambda) \approx \frac{g^2 m^2}{2} N_m (N_m + 1) \frac{\pi^2}{a^2 T^2}$$

(83)

It tends to zero as the time of interaction $T$ tends to $\infty$ i.e. as $c_\lambda$ tends to a $\delta$ function. (In this case, the two level atom tends a Lee model. This can be seen in eq. (83) where the negative frequencies are exponentially suppressed.)

However, the total Minkowski energy increases with the interaction time $T$ and is given by

$$\int_0^{+\infty} dV \langle T_{VV}\rangle_{\text{therm.}} = P_- \int_0^{+\infty} dV \langle T_{VV}\rangle_{\psi_-} + P_+ \int_0^{+\infty} dV \langle T_{VV}\rangle_{\psi_+} \approx m (P_- \hat{P}_e + P_+ \hat{P}_{d,e}) T e^{\alpha_0} (2N_m + 1)$$

(84)

The Minkowski energy of the two fluxes coincide, by virtue of eq. (83) and sum up. This result is what one might have "naively" guessed: The total energy is the integral over the interacting period of the rate of transition times the varying Doppler shift times the energy gap $m$.

We now go to all order in $g$ to prove that this emission of Minkowski quanta is not an artefact of the second order perturbation theory. The forthcoming section can be skipped by the reader mainly interested by the study of vacuum fluctuations and the black hole problem. He can go to Part 3 directly.

### 2.7 Fluxes and Particles to All Order in $g$

We use the exactly solvable model (RSG), used by Raine, Sciama and Grove [6]–[9], to prove that one does recover, to all order in $g$, that every quantum jump of the accelerated oscillator, in thermal equilibrium in Minkowski vacuum, leads to the emission of a Minkowski quantum. Hence the rate of production of the Minkowski quanta is simply the rate of internal transitions of the oscillator. But, as in second order perturbation theory, these quanta interfere and their energy content is found at the edges of the interacting period only. This is due to the complete neglection of the recoils of the oscillator. (Upon taking into account the recoils by giving the oscillator a finite mass, i.e. by quantizing the position of its center of mass, one proves that the Minkowski quanta no longer interfere after a short time (a few $1/a$)). We conclude this section by giving a model independent proof that the stationary thermal Rindler equilibrium corresponds to a production of Minkowski quanta.

We first recall the main properties of the RSG model and then analyse the particle content of the emitted fluxes.

This system consists of a massless field coupled to a harmonic oscillator maintained in constant acceleration. Its action is

$$S = \int dt dx \left[ \frac{1}{2} \left[ (\partial_t \phi)^2 - (\partial_x \phi)^2 \right] \right]$$
\[ + \int d\tau \left[ \frac{1}{2} \left( (\partial_\tau q)^2 - m^2 q^2 \right) + e(\partial_\tau q)\phi \right] \delta^2(X^\mu - X^\mu_a(\tau)) \]  \tag{85}

where \( X^\mu(\tau) \) is the accelerated trajectory eq. (26) and \( e = g\sqrt{2m} \) is a rescaled coupling constant. Since this action is quadratic, the Heisenberg equations are identical to the classical Euler Lagrange ones. They read:

\[ \partial_u \partial_v \phi = \frac{e}{4} \theta(V) \delta(\rho - 1/a) \partial_\tau q \]  \tag{86}

\[ \partial^2_\tau q + m^2 q = -e \partial_\tau \phi(X^\mu(\tau)) \]  \tag{87}

The left part of the field (i.e. for \( V < 0 \)) is, by causality, identically free. And, for \( V > 0 \), on the left of the accelerated oscillator trajectory, the \( v \)-part of the field only is scattered. There the general solution is

\[ \tilde{\phi}(u, v) = \phi(u) + \phi(v) + \frac{e}{2} \tilde{q}(v) \]  \tag{88}

\[ \tilde{q}(v) = q(v) + i \int_{-\infty}^{+\infty} d\lambda \psi_\lambda e^{-i\lambda v} [\phi_{\lambda,R,v} + \phi_{\lambda,R,u}] \]  \tag{89}

where \( \phi(u) \) and \( \phi(v) \) are the homogeneous free solutions of eq. (86); where the operator \( \phi_{\lambda,R,v} \) is defined by

\[ \phi_{\lambda,R,v} = \frac{d^2}{dv^2} \phi(v) \]

\[ = \frac{1}{\sqrt{4\pi|\lambda|}} \left[ \theta(\lambda) a_{\lambda,R} + \theta(-\lambda) a^\dagger_{-\lambda,R} \right] \]  \tag{90}

(a similar equation defines \( \phi_{\lambda,R,u} \)); where \( \psi_\lambda \) is given by

\[ \psi_\lambda = \frac{e\lambda}{m^2 - \lambda^2 - ie^2\lambda/2} \]  \tag{91}

and where \( q(v) \) is a solution of

\[ \partial^2_\tau q + m^2 q + \frac{e^2}{2} \partial_\tau q = 0 \]  \tag{92}

The two independent solutions of eq. (92) are exponentially damped as \( \tau \) increases. Being interested by the properties at equilibrium, we drop \( q(v) \) from now on. Then, the remaining part of \( \tilde{q}(v) \) is a function of the free field only. Hence, in Fourier transform, eq. (88) reads

\[ \tilde{\phi}_{\lambda,R,u} = \phi_{\lambda,R,u} \]

\[ \tilde{\phi}_{\lambda,R,v} = \phi_{\lambda,R,v}(1 + \frac{i}{2} \psi_\lambda) + (\frac{i}{2} \psi_\lambda) \phi_{\lambda,R,u} \]  \tag{93}

The second term in eq. (93) mixes \( u \) and \( v \) modes. It encodes the static Rindler polarization cloud (see [7] [8]) which accompanies the oscillator and carries neither Minkowski
nor Rindler energy. In order to simplify the following equations, we drop it and multiply the other scattered term by two for unitary reason -see below. (By a simple and tedious algebra, one can explicitly verify that this modification does not affect the main properties of the emitted fluxes). Then eq. (93) becomes

$$\tilde{\phi}_{\lambda,R,v} = \phi_{\lambda,R,v}(1 + i\epsilon\psi_\lambda)$$ (94)

It is useful, for future discussions, to introduce explicitly the scattered operators $\tilde{a}_{\lambda,R}$, and the scattered modes $\tilde{\phi}_{\lambda,R}(v)$

$$\tilde{a}_{\lambda,R} = <\phi_{\lambda,R}|\tilde{\phi}| = a_{\lambda,R}(1 + i\epsilon\psi_\lambda)$$ (95)

$$\tilde{\phi}_{\lambda,R}(v) = -\left[a_{\lambda,R}^\dagger,\tilde{\phi}(v)\right] = (1 + i\epsilon\psi_\lambda)\varphi_{\lambda,R}(v)$$ (96)

whereupon the scattered field operator $\tilde{\phi}(v)$ may be written as

$$\tilde{\phi}(v) = \int_0^\infty d\lambda \left[a_{\lambda,R}\varphi_{\lambda,R} + h.c.\right]$$

$$= \int_0^\infty d\lambda \left[a_{\lambda,R}\tilde{\varphi}_{\lambda,R} + h.c.\right]$$ (97)

It is now straightforward to obtain the scattered Green function and its Rindler energy content. If the initial (Heisenberg) state is Minkowski vacuum the $v$-part of the scattered Green function is, for $V, V' > 0$,

$$\tilde{G}_+(v, v') = <0_M|\tilde{\phi}(v)\tilde{\phi}(v')|0_M >$$

$$= \int_0^\infty d\lambda |1 + i\epsilon\psi_\lambda|^2 \left(\beta^2\varphi_{\lambda,R}^*(v)\varphi_{\lambda,R}(v') + \alpha^2\varphi_{\lambda,R}(v)\varphi_{\lambda,R}^*(v')\right)$$

$$= G_+(v, v')$$ (98)

where $G_+(v, v')$ is the unperturbed Minkowski Green function and where we have availed ourselves of the identity (see eq. (91))

$$|1 + i\epsilon\psi_\lambda|^2 = 1$$ (99)

This unitary relation expresses the conservation of the number of Rindler particles. Indeed there is no mixing of positive and negative frequencies in eq. (93); in other words, the $\beta$-term of the "Bogoljubov" transformation eq. (95) vanishes.

The identity of the Green functions in eq. (98) proves that, once the the steady regime is established, no flux is, in the mean, emitted. This is Grove theorem [5][6].

We now examine how this stationary scattering of Rindler modes is perceived in Minkowski terms. The Minkowski scattered modes $\tilde{\varphi}_{\lambda,M}$ are given by

$$\tilde{\varphi}_{\lambda,M} = -\left[a_{\lambda,R}^\dagger,\tilde{\phi}(V)\right]$$

$$= \varphi_{\lambda,M}(1 + i\epsilon\alpha_\lambda^2\psi_\lambda) - i\epsilon\alpha_\lambda\beta_\lambda\psi_\lambda\varphi_{\lambda,M}^*$$

$$= \tilde{\alpha}_\lambda\varphi_{\lambda,M} + \tilde{\beta}_\lambda\varphi_{\lambda,M}^*$$ (100)
\[ \tilde{\varphi}_{-\lambda,M} = \varphi_{-\lambda,M}(1 - ie\beta^2 \psi_{-\lambda}) - ie\alpha \beta \psi_{-\lambda} \varphi^*_{\lambda,M} \]
\[ = \tilde{\alpha}_{-\lambda} \varphi_{-\lambda,M} + \tilde{\beta}_{-\lambda} \varphi^*_{\lambda,M} \quad (101) \]

where \(0 < \lambda < \infty\) and where we have introduced the scattered Bogoljubov coefficients:

\[ \tilde{\alpha}_{\lambda} = 1 + ie\alpha^2 \psi_{\lambda} \]
\[ \tilde{\beta}_{\lambda} = -ie\alpha \beta \psi^*_{\lambda} \]
\[ \tilde{\alpha}_{-\lambda} = 1 + ie\beta^2 \psi_{\lambda} \]
\[ \tilde{\beta}_{-\lambda} = -ie\alpha \beta \psi_{\lambda} \quad (102) \]

One verifies that the unitary relation is satisfied: \(|\tilde{\alpha}_{\lambda}|^2 - |\tilde{\beta}_{\lambda}|^2 = 1\). The fact that the \(\tilde{\beta}\) are different from zero indicates that each couple of jumps of the oscillator (the absorption and subsequent emission of a Rindler quantum) leads, in Minkowski vacuum, to the production of two Minkowski quanta. The member \(\varphi_{-\lambda,M}\) is emitted when the oscillator absorbs a rindleron and jumps into a higher level and the other one, \(\varphi_{\lambda,M}\), is emitted during the inverse process. This is manifest in the mean energy flux:

\[ \langle \tilde{T}_{VV} \rangle = \lim_{V' \to V} \partial_V \partial_{V'} \langle [\tilde{\phi}(V) \tilde{\phi}(V') - \phi(V) \phi(V')] \rangle \]
\[ = 2 \int_{-\infty}^{+\infty} d\lambda |\tilde{\beta}_{\lambda}|^2 |\partial_V \varphi_{\lambda,M}|^2 + Re \left[ \tilde{\alpha}_{\lambda} \tilde{\beta}^*_{\lambda} \partial_V \varphi_{\lambda,M} \partial_{V'} \varphi_{-\lambda,M} \right] \quad (103) \]

whereupon the total Minkowski energy is

\[ \langle \tilde{H}_M \rangle = \int_{-\infty}^{+\infty} dV \langle \tilde{T}_{VV} \rangle \]
\[ = \int_{-\infty}^{+\infty} d\lambda \lambda (|\tilde{\beta}_{\lambda}|^2 + |\tilde{\beta}_{-\lambda}|^2) \int_{-\infty}^{+\infty} dV \frac{1}{2\pi a^2 |V + ie|^2} \quad (104) \]

since the integral of the second term vanishes.

Exactly as in second order perturbation theory, there is a steady regime during which all the emitted quanta interfere destructively leaving no contribution to the mean flux (see eq. (103)). But all non diagonal matrix elements will be sensitive to the created pairs. This is also the case for the the total energy eq. (104) since being diagonal in \(\omega\) it ignores the destructive interferences (the second term of eq. (103) whose role is to make the mean flux vanishing during the steady regime).

In order to prove that eq. (104) corresponds to a steady production of Minkowski quanta during the whole interacting period \(\Delta \tau = T\) (infinite in eq. (104)) we evaluate how many quanta are produced. (Contrary to the energy, the total number of Minkowski quanta is a scalar under the Lorentz group, hence not affected by the exponentially growing Doppler shift present in the energy)

\[ \langle \tilde{N}(\Delta \tau) \rangle = \int_{0}^{+\infty} d\omega \langle 0_M | a_{\omega}^\dagger a_{\omega} | 0_M \rangle \]
\[ = \int_{0}^{+\infty} d\omega \langle 0_M | \tilde{a}_{\omega}^\dagger \tilde{a}_{\omega} | 0_M \rangle \]
\[ = \int_{0}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\lambda |\gamma_{\lambda,\omega}(\Delta \tau)|^2 |\tilde{\beta}_{\lambda}|^2 \quad (105) \]
where $|\tilde{0}_M\rangle$ is the scattered (Schrödinger) state\footnote{The simplest way to obtain this state is to find the scattering operator $U$ such that $\tilde{a}_{\lambda,M} = U^\dagger a_{\lambda,M} U$ where $\tilde{a}_{\lambda,M} = \langle \varphi_{\lambda,M}\tilde{\phi} \rangle$. Then $|0_M\rangle = U|0_M\rangle$.}. The $\tilde{a}_\omega$ are related to the $\tilde{a}_{\lambda,M}$ by (see eq. (13))

$$\tilde{a}_\omega = \int_0^\infty d\omega \, \gamma_{\lambda,\omega}(\Delta \tau)\tilde{a}_{\lambda,M} \tag{106}$$

where $\gamma_{\lambda,\omega}(\Delta \tau)$ takes into account the time dependence of the coupling. As shown in [30][31] $\gamma_{\lambda,\omega}(\Delta \tau)$ is non vanishing only for the $\omega$ which enter into resonance with the oscillator frequency $m$ during the interaction period $\tau_i < \tau < \tau_f = \tau_i + T$. When these frequencies belong to

$$\omega_i = me^{-a\tau_i} < \omega < me^{-a\tau_f} = \omega_f \tag{107}$$

$\gamma_{\lambda,\omega}(\Delta \tau)$ may be replaced by $\gamma_{\lambda,\omega}$ (given in eq. (16)). Hence $\tilde{N}(\Delta \tau)$ reads

$$\langle \tilde{N}(\Delta \tau) \rangle = \int_{\omega_i}^{\omega_f} d\omega \frac{d\omega}{2\pi a \omega} \int_{-\infty}^{+\infty} d\lambda \, |\tilde{\beta}_\lambda|^2$$

$$= \frac{\Delta \tau}{2\pi} \int_{-\infty}^{+\infty} d\lambda \, |\tilde{\beta}_\lambda|^2 \tag{108}$$

The total energy emitted obtained from eq. (108) is

$$\langle \tilde{H}_M(\Delta \tau) \rangle = \int_{\omega_i}^{\omega_f} \frac{d\omega}{2\pi a} \int_{-\infty}^{+\infty} d\lambda \, |\tilde{\beta}_\lambda|^2$$

$$= \int_{\tau_i}^{\tau_f} \frac{d\tau}{2\pi} e^{-a\tau} m \int_{-\infty}^{+\infty} d\lambda \, |\tilde{\beta}_\lambda|^2$$

$$= \int_{\tau_i}^{\tau_f} \frac{d\tau}{2\pi} \frac{1}{a^2 V^2} m \int_{-\infty}^{+\infty} d\lambda \, |\tilde{\beta}_\lambda|^2 \tag{109}$$

in perfect agreement with eq. (104) if the frequency width of the oscillator in small compared to $m$. The rate of production (eq. (108) divided by $\Delta \tau$) is (small width limit) $e^2 \alpha^2 m^2 / 2\pi$ which is the rate of jumps for an inertial oscillator in a bath at temperature $a/2\pi$. Therefore the number of Minkowski quanta produced by the thermalized oscillator equals the number of internal jumps.

We now generalize these results to an arbitrary linear coupling. We believe that it can be generalized, using the same type of argumentation, to nonlinear couplings as well. The proof goes as follow. Any scattering of Rindler quanta by an accelerated system which leads to a thermal equilibrium during a time much larger than $1/a$ can be described as in eq. (95) by

$$\tilde{a}_{\lambda,R} = S_{\lambda\lambda'} a_{\lambda',R} \tag{110}$$

where repeated indices are summed (or integrated) over and where the summation over $\lambda'$ includes both $u$ and $v$-modes (as in eq. (93)). The matrix $S$ satisfy the unitary relation

$$S_{\lambda\lambda'} S_{\lambda'\lambda''}^t = \delta_{\lambda\lambda''} \tag{111}$$
which express the conservation of the number of Rindler quanta since $S_{\lambda,\lambda'}$ mixes positive Rindler frequencies only. It is convenient to introduce the matrix $T$ (from now on we do not write the indices)

$$S = 1 + iT$$

which satisfies

$$2\text{Im} T = TT^\dagger$$

We introduce also the vector operator $b = (a_{\lambda,R}; a_{\lambda,L}; a_{\lambda,R}^\dagger; a_{\lambda,L}^\dagger)$. Then eq. (110) can be written as

$$\tilde{b} = S b$$

where $S$ has the following block structure

$$S = \begin{pmatrix}
1 + iT & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 - iT^\dagger & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

since the $u$ and $v$-modes on the left quadrant are still free. On the other hand, the Bogoljubov transformation eq. (18) reads in this notation

$$c = Bb$$

where $c = (a_{\lambda,M}; a_{-\lambda,M}; a_{\lambda,M}^\dagger; a_{-\lambda,M}^\dagger)$ and where $B$ is

$$B = \begin{pmatrix}
\alpha & 0 & 0 & -\beta \\
0 & \alpha & -\beta & 0 \\
0 & -\beta & \alpha & 0 \\
-\beta & 0 & 0 & \alpha
\end{pmatrix}$$

the diagonal matrices (in $\lambda$) $\alpha$ and $\beta$ being taken real. The scattered Minkowski operators are given by

$$\tilde{c} = BSB^{-1}c = \left(S + B \left[ S, B^{-1} \right] \right) c = S_M c$$

Since $S$ and $B$ do not commute, $S_M$ has non diagonal elements which encode the production:

$$S_M = \begin{pmatrix}
\tilde{\alpha}_1 & 0 & 0 & -\tilde{\beta}_1 \\
0 & \tilde{\alpha}_2 & \tilde{\beta}_1^\dagger & 0 \\
0 & \beta_2^\dagger & \beta_1^\dagger & 0 \\
-\tilde{\beta}_2 & 0 & 0 & \tilde{\alpha}_2^\dagger
\end{pmatrix}$$

where the $\tilde{\alpha}$ $\tilde{\beta}$ are given in terms of $T$ by (see eq. (102))

$$\tilde{\alpha}_1 = 1 + i\alpha T \alpha$$
$$\tilde{\beta}_1 = -i\alpha T \beta$$
$$\tilde{\alpha}_2 = 1 + i\beta T^\dagger \beta$$
$$\tilde{\beta}_2 = i\beta T \alpha$$

QED
3 The Conditional Values of the Energy Momentum Tensor

3.1 Introduction

In Part 2 we analysed the mean energy radiated by the atom and showed, in Section 2.4, how it can be decomposed into two contributions \( \langle T_{VV}(U,V) \rangle_e \) and \( \langle T_{VV}(U,V) \rangle_g \) according to the final state of the two level atom. These correspond to the energy emitted if the atom is found in its excited or ground state at \( t = +\infty \). This decomposition is valid for all points \( U > U_a(V) \) where \( U_a(V) \) is the trajectory of the atom, i.e. in the future of the atom since for our massless field energy flows along \( V = constant \).

In this Part, we generalize this decomposition of the energy density into energy densities correlated to the final state of the atom for all \( U, V \). When \( U < U_a(V) \) this describes the energy momentum of the field configurations which will give rise to excitations of the atom. In order to explicitize this proposition, we proceed as follow. First, the generalization is introduced formally by introducing projectors that specify the state of the atom at \( t = +\infty \). We shall then see that this construction gives rise to nondiagonal matrix elements of \( T_{VV} \) which are complex. We need therefore to discuss their physical meaning. To this end one introduces an additional quantum system coupled to the operator \( T_{VV} \). The picture that emerges is then clear: to first order in the coupling, the modification of the wave function of the additional system is governed by these non diagonal matrix elements. This is presented in summary fashion in Section 3.2 and in more detail in Part 5 which is entirely devoted to a general discussion of the procedure leading to these conditional values of operators. In the present Part, we discuss mainly the properties of these conditional energy densities.

In section 3.3, the conditional values of the energy distribution correlated to the transitions of the accelerated atom are described and interpreted. In Section 3.4 we discuss a generalization of these conditional values which does no longer refer to the transitions of the atom and which finds important application in the black hole problem.

3.2 The Conditional Energy Correlated to a Transition of the Accelerated Atom

In Section 2.4, we had rewritten the mean energy emitted on the left of the accelerated trajectory (i.e. \( U > U_a(V) \)) as

\[
\langle T_{VV} \rangle_{\psi_-} = P_e \langle T_{VV} \rangle_e + P_g \langle T_{VV} \rangle_g
\]  

(121)

using the expression, eq. (11), for the \( |\psi_- \rangle \) at \( t = \infty \) and the probability of transition \( P_e \) given in eq. (14). In order to generalize this decomposition, we first rewrite it in Heisenberg representation by introducing the projectors \( \Pi_+ = |+><+| \) and \( \Pi_- = |-><-| \) onto the excited and ground state of the atom.

In Heisenberg representation, the state of the system is \( |\psi_- \rangle = |0_M \rangle |-> \) and the
projector is a time dependent operator given by

$$\Pi_+(t) = e^{i\int_{-\infty}^{t} dt \Pi(t)} \Pi_+(0)$$

whereupon the probability to be found in the excited state at \( t = +\infty \) is written as

$$P_e = \langle \psi_- | \Pi_+(t = +\infty) | \psi_- \rangle$$

Similarly, the probability to be found in the ground state at \( t = +\infty \) is

$$P_g = \langle \psi_- | \Pi_-(t = +\infty) | \psi_- \rangle$$

The conservation of probability \( P_e + P_g = 1 \) is realized through the completeness of the projectors \( \Pi_+(t) + \Pi_-(t) = I \).

The conditional energies can be now obtained by decomposing the mean using the projectors \( \Pi_{\pm}(t) \)

$$\langle T_{VV}(U, V) \rangle_{\psi_-} = \langle \psi_- | [\Pi_+(+\infty) + \Pi_-(+\infty)] T_{VV}(U, V) | \psi_- \rangle$$

$$= P_e \frac{\langle \psi_- | \Pi_+(+\infty) T_{VV}(U, V) | \psi_- \rangle}{\langle \psi_- | \Pi_+(+\infty) | \psi_- \rangle} + P_g \frac{\langle \psi_- | \Pi_-(+\infty) T_{VV}(U, V) | \psi_- \rangle}{\langle \psi_- | \Pi_-(+\infty) | \psi_- \rangle}$$

$$= P_e \langle T_{VV}(U, V) \rangle_e + P_g \langle T_{VV}(U, V) \rangle_g$$

When \( U > U_a(V) \), the matrix elements \( \langle T_{VV}(U, V) \rangle_e \) and \( \langle T_{VV}(U, V) \rangle_g \) are the expressions obtained less formally in eq. (122). When \( U < U_a(V) \), these matrix elements are the desired expressions of the energy density if the atom shall be found at \( t = +\infty \) in the excited (ground) state.

The normalization in eq. (125) is chosen so that the mean value is expressed as the probability of making a transition times the conditional value exactly like in the usual conditional probabilities. We shall see in the sequel that this decomposition into probability to end up in the excited or ground state times the conditional value automatically occurs in physical processes.

Two important properties of the conditional values for \( U < U_a(V) \) should be noted. First

$$\langle T_{VV}(U < U_a(V), V) \rangle_e = -\frac{P_g}{P_e} \langle T_{VV}(U < U_a(V), V) \rangle_g$$

since \( \langle T_{VV}(U, V) \rangle_{\psi_-} \) vanishes identically for \( U < U_a(V) \) because the interaction with the accelerated atom has not yet perturbed Minkowski vacuum. Secondly, \( \langle T_{VV} \rangle_e \) is complex. This can be seen from the explicit expression

$$\langle T_{VV}(U < U_a(V), V) \rangle_e = \frac{1}{P_e} \langle \psi_- | e^{i\int dt \Pi(t)} \Pi_+ e^{-i\int dt \Pi(t)} T_{VV}(U, V) | \psi_- \rangle$$

$$= \frac{g^2 m^2}{P_e} \langle 0_M | \phi_m^+ T_{VV}(U, V) | 0_M \rangle \frac{g^2 m^2}{P_e} C_+(V) C_-(V)$$

where we have used eq. (31) and eq. (52).
Both the real and imaginary part of \( \langle T_{VV} \rangle_e \) have physical meaning and intervene in physical processes. To prove this fact one should introduce an additional quantum system because these matrix elements have meaning only in quantum mechanics.

For definiteness, we take the additional system to be a quantum oscillator sitting at \( x = x_0 \) and coupled to \( T_{VV} \) by the interaction Hamiltonian

\[
\int dt H_{osc.} = \int dt g^{VV}(t)p(t)T_{VV}(t, x_0)
\]

where \( p(t) \) is the momentum conjugate to the position \( q(t) \) of the oscillator and \( g^{VV}(t) \) is a switch function with the correct Lorentz variance. The initial state of the oscillator is \( |osc.\rangle \). The state of the entire system (i.e. field + two level atom + oscillator) is thus \( |\Psi > = |\psi > - |osc.\rangle \).

Then to first order in \( g^{VV} \), in the interacting picture, the mean position of the oscillator at \( t = \infty \) is given by

\[
\langle q(t = \infty) \rangle_{\psi_-} = \langle osc.|q(t = \infty)|osc.\rangle + \int dt g^{VV}(t)< osc.| - i[q(t = +\infty), p(t)]_-|osc.\rangle \langle T_{VV}(t, x_0)|\psi_- \rangle
\]

(129)

That is, the mean change of the position is driven by the mean value of \( T_{VV}(t, x_0) \) in the state \( |\psi > \). It corresponds to the classical response of \( q(t) \) to a driving force.

But, one can also investigate the correlations among the oscillator state and the atom by asking more detailed questions such that: what is the "mean" (for the use of this word see discussion before eq. (14)) position of the oscillator when the two level atom is found in its excited state? The answer is the conditional value of \( q \) obtained by decomposing the mean according to the final state of the atom at \( t = \infty \)

\[
\langle q(t = +\infty) \rangle_{\psi_-} = P_e\langle q(t = +\infty)\rangle_e + P_g\langle q(t = +\infty)\rangle_g
\]

(130)

where the conditional value \( \langle q(t = +\infty)\rangle_e \) is given by

\[
\langle q(t = +\infty)\rangle_e = \frac{< \Psi_-|\Pi_+(\infty)q(\infty)|\Psi_- >}{< \Psi_-|\Pi_+(\infty)|\Psi_- >}
\]

(131)

To first order in \( g^{VV} \), this conditional position of the oscillator is

\[
\langle q(t = +\infty)\rangle_e = < osc.|q(t = -\infty)|osc.\rangle + \int dt g^{VV}(t)< osc.| - i[q(t = +\infty), p(t)]_-|osc.\rangle \text{ Re}[\langle T_{VV}(t, x_0)\rangle_e]
\]

\[
+ \int dt g^{VV}(t)< osc.| - i\{q(t = +\infty), p(t)\}_+|osc.\rangle \text{ Im}[\langle T_{VV}(t, x_0)\rangle_e]
\]

(132)

Hence both the real and imaginary part of the conditional value of \( T_{VV} \) control the modification of the mean conditional position. Note that \( \text{Re}\langle T_{VV}\rangle_e \) for \( U > U_a(V) \) and
for $U < U_a(V)$ enter exactly in the same way in the integrals giving rise to $\langle q(t = +\infty) \rangle_e$ as the mean value $\langle T_{VV}(t, x_0) \rangle_{\psi -}$ drove the mean $q$ in eq. (129). The imaginary part of $\langle T_{VV} \rangle_e$ appears in an unusual way through an anticommutator which depends explicitly on the state of the oscillator. In quantum mechanics therefore, by coupling an additional system to the operator $T_{VV}$, one can isolate, in a well defined manner, the energy content of the emitted particle correlated to a transition of the atom as well as the energy content of the vacuum fluctuations that shall induce the transition of the atom at later times.

This procedure wherein an external quantum system is introduced to reveal the physical significance of matrix elements like $\langle T_{VV} \rangle_e$, e.g., will be displayed in more details in Part 5 and put in parallel with the treatment of Aharonov et al. [12]. The same procedure will also be used in the black hole situation when evaluating the conditional value of the metric correlated to a particular final state of the radiation.

### 3.3 The Properties of the Conditional Energy

Having indicated by an example how both the real and imaginary parts of $\langle T_{VV} \rangle$ intervene in physical processes we now display the properties of the conditional values. Since $\langle T_{VV}(t, z) \rangle_g = (\langle T_{VV} \rangle_{\psi -} - P_e \langle T_{VV} \rangle_e)/P_g$ we shall discuss $\langle T_{VV} \rangle_e$ only.

In order to obtain exact expressions for this matrix elements, we use again the $c_\lambda$ introduced in eq. (63). We give now the three expressions for $\langle T_{VV} \rangle_e$: three because one finds different expressions for $V > 0, U < U_a(V)$ and for $V > 0, U > U_a(V)$ (i.e. before or after the interaction occurs), and for $V < 0$ all $U$’s.

\begin{equation}
\langle T_{vv}(U < U_a, V > 0) \rangle_e = \frac{g^2 m^2}{P_{e,v}} \int d\lambda \int d\lambda' c_\lambda c'_{\lambda'} \frac{1}{(4\pi)^2} \left[ \bar{n}_\lambda (\bar{n}_{\lambda'} + 1) \right] e^{-i(\lambda'-\lambda)v}
\end{equation}

\begin{equation}
= \frac{m(N_m + 1)}{2\sqrt{\pi} T C_0} (1 - \frac{i v + 2\pi}{mT^2}) (1 + \frac{iv}{mT^2}) e^{-(v+i\pi/a)^2/T^2}
\end{equation}

\begin{equation}
\approx \frac{m(N_m + 1)}{2\sqrt{\pi} T} e^{-(v-i\pi/a)^2/T^2}
\end{equation}

\begin{equation}
(133)
\end{equation}

\begin{equation}
\langle T_{vv}(U > U_a, V > 0) \rangle_e = \frac{g^2 m^2}{P_{e,v}} \int d\lambda c_\lambda \frac{1}{4\pi} \left[ \bar{n}_\lambda e^{-i\lambda v} \right]^2
\end{equation}

\begin{equation}
= \frac{mN_m}{2\sqrt{\pi} T C_0} \left| 1 - \frac{i v + 2\pi/a}{mT^2} \right|^2 e^{-\frac{v^2}{4\pi} e^{3\pi^2/a^2T^2}}
\end{equation}

\begin{equation}
\approx \frac{mN_m}{2\sqrt{\pi} T} e^{-\frac{v^2}{4\pi}}
\end{equation}

\begin{equation}
(134)
\end{equation}

\begin{equation}
\langle T_{vv}(U, V < 0) \rangle_e = \frac{g^2 m^2}{P_{e,v}} \int d\lambda c_\lambda \frac{1}{4\pi} \left[ \bar{n}_\lambda e^{\pi\lambda/a - i\lambda v_L} \right]^2
\end{equation}

\begin{equation}
= \frac{m(N_m + 1)}{2\sqrt{\pi} T C_0} \left| 1 - \frac{iv_L + \pi/a}{mT^2} \right|^2 e^{-\frac{v_L^2}{4\pi}}
\end{equation}

\begin{equation}
\approx \frac{m(N_m + 1)}{2\sqrt{\pi} T} e^{-\frac{v_L^2}{4\pi}}
\end{equation}

\begin{equation}
(135)
\end{equation}
where the last two equalities in eqs. (133, 134, 135) represent the exact expressions if $c_\lambda$ is given by eq. (63) and the approximate expressions valid for $T >> m^{-1}, T >> a^{-1}$. $C_0$ is a constant equal to

$$C_0 = (N_m + 1)^{-1} \left[ (1 - \frac{\pi}{amT^2}) - e^{-2\pi m/a} e^{3a^2/2} (1 - \frac{2\pi}{amT^2}) \right] \approx 1 \quad (136)$$

These fluxes are presented in figure 4.

We now present the complementary Rindler and Minkowski properties of these conditional values of $T_{\nu\nu}$.

The Rindler description is that used by a uniformly accelerated observer in the same quadrant as the two level atom. It is best understood by making appeal to the isomorphism of the state of the field in the right Rindler quadrant with an inertial thermal bath.

By getting excited the two level atom has selected that the thermal bath contains at least one particle in the mode created by $\phi_m^\dagger$. Furthermore since energy flows along the lines $v = cst$, $\langle T_{\nu\nu}(U < U_a, V > 0) \rangle_e$ is centered around $v = 0$ with at spread $\Delta v = T$. It carries a Rindler energy obtained by integrating eq. (133)

$$\int dv \langle T_{\nu\nu}(U < U_a, V > 0) \rangle_e = \frac{\int d\lambda |c_\lambda|^2 \tilde{n}_\lambda (\tilde{n}_\lambda + 1)}{\int d\lambda |c_\lambda|^2 1^{1/2} \tilde{n}_\lambda} \simeq m(N_m + 1) \quad (137)$$

The factor $N_m + 1$ takes correctly into account the Bose statistics of the field since eq. (137) corresponds to evaluating $\langle n^2 \rangle / \langle n \rangle$ in a thermal distribution.

Then, by getting excited the two level atom absorbs one quantum and the residual energy on the future of the accelerated trajectory $U = U_a(V)$ is (see eq. (134))

$$\int dv \langle T_{\nu\nu}(U > U_a, V > 0) \rangle_e = \frac{\int d\lambda |c_\lambda|^2 \tilde{n}_\lambda}{\int d\lambda |c_\lambda|^2 1^{1/2} \tilde{n}_\lambda} \simeq mN_m \quad (138)$$

We now consider what is "seen" by a uniformly accelerated in the left Rindler quadrant (i.e. what is the nature of the correlations between the transition of the atom and an additional system uniformly accelerated in the left Rindler quadrant). Before the strict correlations between the left and right quadrants, see eq. (25), one expects that to the $(N_m + 1)$ Rindler quanta on the right correspond $(N_m + 1)$ Rindler quanta on the left. This can be obtained formally by considering the Rindler energy operator $H_R$. Since Minkowski vacuum $|0_M \rangle$ is annihilated by $H_R$ (see eq. (24), the Rindler energy in the left quadrant is equal to the energy in the right quadrant. Indeed integrating eq. (135) and using the relation $\tilde{n}_\lambda (\tilde{n}_\lambda + 1) = \tilde{n}_\lambda^2 e^{2\pi \lambda/a}$ yields

$$\int dv_L \langle T_{\nu L \nu L}(U < U_a, V < 0) \rangle_e = \int dv \langle T_{\nu\nu}(U < U_a, V > 0) \rangle_e$$

The symmetry between the left and the right Rindler quadrants results in $\langle T_{\nu\nu}(U, V < 0) \rangle_e$ being centered around $v_L = 0$ with the same width $\Delta v_L = T$. Thus $\langle T_{\nu\nu}(U, V < 0) \rangle_e$ carries also the Rindler energy of $N_m + 1$ Rindler quanta and is almost exactly the symmetric of $\langle T_{\nu V}(U < U_a(V), V > 0) \rangle_e$ except for small transient oscillations present for
\[ V > 0 \] (see the explicit expressions eqs. (133, 135) and figure 4). We also note that \( \langle T_{VV}(U, V < 0) \rangle_e \) is real whereas \( \langle T_{VV}(U < U_a(V), V > 0) \rangle_e \) is complex. This results from causality and can be proven in complete generality by making appeal to a reasoning similar to that in eq. (49). It will have important consequences in the black hole problem, in Section 4.3.

The Minkowski description, i.e. that used by an inertial observer, is best understood by rewriting the conditional value of \( T_{VV} \) in terms of the \( \varphi_{\lambda,M}(V) \) modes, eq. (15),

\[
\langle T_{VV}(U < U_a, V) \rangle_e = \frac{1}{a^2 V^2} \frac{g^2 m^2}{P_e} \int d\lambda \int d\lambda' c^*_\lambda c_{\lambda'} \frac{1}{4\pi} \sqrt{\lambda \lambda' (n_{\lambda} + 1)} \varphi_{\lambda,M}^* \varphi_{\lambda',M}^* (1 + \frac{i}{maT^2} \ln(-aV - i\epsilon) - \frac{\pi}{maT^2})
\]

\[
\times (1 - \frac{i}{maT^2} \ln(-aV - i\epsilon) - \frac{\pi}{maT^2}) e^{-[\ln(-aV - i\epsilon)]^2/a^2T^2}
\]

The \( i\epsilon \) defines \( \ln(-aV - i\epsilon) \) as \( \ln|aV| \) for \( V < 0 \) and as \( \ln|aV| - i\pi \) for \( V > 0 \). Upon taking the limit \( \epsilon \to 0 \) no singularity occurs. In fact \( \langle T_{VV}(U < U_a, V) \rangle_e \) given in eq. (140) vanishes for \( V = 0 \). This is an accident due to the particular form of \( c_{\lambda} \) chosen in eq. (63) (it has zero’s for \( \lambda = ina, n = ..., -1, 0, 1, ... \)). But, from the expression for \( \mathcal{C}_{\pm}(V = 0) \) given in eq. (55), it results that the generic behaviour of \( T_{VV} \) is to stay finite as \( V \to 0 \). In more physical terms this corresponds to saying that the Minkowski vacuum fluctuation that induces the transition straddles the horizon with no clear cut separation between the pieces in the left and right quadrants.

Notice how the \( i\epsilon \) prescription which encodes the analyticity of the modes \( \varphi_{\lambda,M} \) in the lower half complex plane now encodes the vanishing of the integral

\[
\int_{-\infty}^{+\infty} dV \langle T_{VV}(U < U_a, V) \rangle_e = 0
\]

by contour integration. The vanishing of this integral can also be seen to result from \( |0_M \rangle \) being the ground state of \( H_M \) (in similar fashion to the vanishing of eq. (51)). In other words the total Minkowski energy does not fluctuate and is always equal to its eigenvalue zero: vacuum fluctuations carry no energy. Notice also how the \( i\epsilon \) encodes the above mentioned slight asymmetry between the left and right quadrants: \( \langle T_{VV}(U < U_a, V) \rangle_e \) is real and positive for \( V < 0 \) whereas it is complex and oscillates for \( V > 0 \).

In view of the vanishing of the total Minkowski energy (eq. (141)) and of the positivity in the region \( V < 0 \), the energy in the region \( V > 0 \) must integrate to an exactly compensating real and negative value. This is not in contradiction with the positivity of Rindler energy in the right quadrant, eq. (137), since the expressions for the Rindler and the Minkowski energy differ by the jacobian \( dv/dV = 1/aV \). The oscillations of \( T_{vv} \) for \( V > 0 \) that occur in eq. (133) as \( v \to -\infty \) (which are negligible in the Rindler description) are dramatically enhanced by the jacobian in such a way that the Minkowski energy in the right quadrant becomes negative, c.f. eq. (68).

For \( U > U_a \), all \( V \), after the atom has made a transition, the Minkowski energy takes
the form

\[
\langle T_{VV}(U > U_a, V) \rangle_e = \frac{1}{a^2 V^2} \frac{g^2 m^2}{P_{e,v}} \int d\lambda c_\lambda \sqrt{\frac{\lambda n_\lambda}{4\pi}} \rho_{-\lambda,M}^2
\]

\[
= \frac{1}{a^2 V^2} \frac{m(N_m + 1)}{2\sqrt{\pi TC_0}} |1 - \frac{i}{maT^2} \ln(-aV - i\epsilon) - \frac{\pi}{maT^2}|^2
\]

\[|e^{-[\ln(-aV - i\epsilon)]^2/a^2T^2}|^2 |e^{-im\ln(-aV - i\epsilon)/a}|^2\]  \quad (142)

It is manifestly real and positive. This is as it should be since we are calculating the mean value of the energy in a state that contains one Minkowski quantum. In particular the integral \( \int dV \langle T_{VV}(U > U_a, V) \rangle_e \) is strictly positive (see eq. (140)). The time evolution has transformed the conditional value eq. (140) which was complex and carried no energy into a real conditional value carrying positive energy. (The change in time of the conditional values is further discussed in Part 5).

Notice how the \( i\epsilon \) prescription in eq (142) encodes the asymmetry between the left quadrant (proportional to \( N_m + 1 \)) and the right quadrant (proportional to \( N_m \)). By absorbing the positive Rindler energy \( m \), the two level atom has reduced the negative Minkowski energy on the right thereby converting a vacuum fluctuation into a quantum. This is summarized in figure 5.

### 3.4 The Energy Correlated to a Rindler State

Up to now we have considered the final state of the atom to isolate certain field configurations (i.e. those correlated to transitions of the atom). These field configurations can also be isolated without making appeal to the accelerated atom through the introduction of projection operators acting directly on field states, decomposing thereby the mean value. To make contact with the previous section we work in Minkowski vacuum and consider projections onto states containing certain Rindler quanta.

We first write unity as

\[
I = \sum_i \Pi_i \quad (143)
\]

where \( \Pi_i \) are a complete set of projectors. The mean value of \( T_{VV} \) is then decomposed as

\[
\langle 0_M | T_{VV} | 0_M \rangle = \langle 0_M | \sum_i \Pi_i T_{VV} | 0_M \rangle
\]

\[
= \sum_i P_i \frac{\langle 0_M | \Pi_i T_{VV} | 0_M \rangle}{\langle 0_M | \Pi_i | 0_M \rangle} \quad (144)
\]

where \( P_i = \langle 0_M | \Pi_i | 0_M \rangle \) is the probability to be in the eigenspace of \( \Pi_i \). As in Section 3.2 the matrix element

\[
\langle T_{VV} \rangle_{\Pi_i} = \frac{\langle 0_M | \Pi_i T_{VV} | 0_M \rangle}{\langle 0_M | \Pi_i | 0_M \rangle} \quad (145)
\]
is the conditional value of $T_{VV}$ if the final state is in the eigenspace of $\Pi_i$.

In this section we shall study the properties of some typical conditional values in preparation for black hole physics. The physical relevance of these matrix elements will be displayed in Part 4 and Part 5.

We first consider the projector

$$\Pi_{\lambda,R}\lambda,L = \hat{a}_{\lambda,R}^\dagger \hat{a}_{\lambda,L}^\dagger |0_{RL} \rangle < 0_{RL} |a_{\lambda,R} a_{\lambda,L}$$

which projects onto the state containing one pair of rindlerons of Rindler energy $\lambda$. By availing oneself of the identity

$$\hat{a}_{\lambda,R}^\dagger \hat{a}_{\lambda,L}^\dagger |0_{RL} \rangle = 1_{\lambda} \alpha_{\lambda} \beta_{\lambda} \phi_{-\lambda,M}(V) \phi_{\lambda,M}(V') + <0_{RL} |\phi(V)\phi(V')|0_{M} >$$

it is straightforward to obtain

$$\langle \phi(V)\phi(V') \rangle_{\Pi_{\lambda,R}\lambda,L} = \frac{<0_{M} |\Pi_{\lambda,R}\lambda,L,\phi(V)\phi(V')|0_{M} >}{<0_{M} |\Pi_{\lambda,R}\lambda,L|0_{M} >} = \frac{<0_{RL} |\alpha_{\lambda} a_{\lambda,M} \phi(V)\phi(V')|0_{M} >}{<0_{RL} |\alpha_{\lambda} a_{\lambda,L}|0_{M} >}$$

It decomposes into two terms. The first depends on the quantum number $\lambda$ and is the contribution of the pair of rindlerons selected by the projector $\Pi_{\lambda,R}\lambda,L$. It carries an energy density equal to

$$\lim_{V' \to V} \frac{\partial}{\partial V} \frac{2}{\alpha_{\lambda}} \phi_{-\lambda,M}^*(V) \phi_{\lambda,M}^*(V') = \frac{\lambda}{2\pi a^2} \frac{1}{(V+i\epsilon)^2}$$

The second term is independent of $\lambda$ and appears because except for the mode $\lambda$, Rindler vacuum has been selected (indeed if the projector $\Pi_{0RL} = |0_{RL}><0_{RL}|$ is used only the second term appears). It is convenient to rewrite this term as the sum of the expectation value of $T_{VV}$ in Minkowski vacuum\(^6\) plus their difference

$$\langle T_{VV} \rangle_{0_{RL}} = \frac{<0_{RL} |T_{VV}|0_{M} >}{<0_{RL} |0_{M} >} = \frac{<0_{M} |T_{VV}|0_{M} >}{<0_{M} |0_{M} >} + \lim_{V' \to V} \frac{\partial}{\partial V} \frac{1}{\alpha_{\lambda}} \phi_{-\lambda,M}^*(V) \phi_{\lambda,M}^*(V')$$

\(^6\) In this section and the following one we shall explicitly write the vacuum expectation value of the energy momentum tensor $<0_{M} |T_{VV}|0_{M} >$ even though it vanishes. It is kept only to facilitate the transcription of these results to the black hole problem where the vacuum expectation of the energy is non trivial and must be renormalized carefully.
The Rindler interpretation is obtained by considering the Rindler energy density $T_{vv}$. Then eq. (149) gives the energy density of the selected rindleron $\lambda/2\pi$, the jacobian being $(dV/dv)^2 = a^2 V^2$. The second term, eq. (150), is the Rindler vacuum energy which is minus the thermal energy density at a temperature $a/2\pi$ (Minkowski vacuum contains a thermal distribution of rindlerons). The energy in the left quadrant is identical to that in the right quadrant since there is a complete symmetry between the two.

The Minkowski interpretation is completely different. Since the Hamiltonian $H_M$ is diagonal in $\omega$ and annihilates Minkowski vacuum $H_M|0_M> = 0$, both eq. (149) and eq. (150) contain zero Minkowski energy. Indeed, the pole prescription at the horizon $V = 0$ ensures that their integrals over the entire domain of $V$ vanish as in eq. (141).

The projector $\Pi_{\lambda,R;\lambda,L}$ cannot be used in the black hole situation since we have no access to the region beyond the horizon nor does it mimic the existence of a detector confined to the right Rindler quadrant. Hence we introduce a projector which selects the presence of a rindleron of energy $\lambda$ in the right quadrant while tracing over the state of the field in the left quadrant:

$$\Pi_{\lambda,R} = I_L \otimes a^\dagger_{\lambda,R}|0_R> <0_R|a_{\lambda,R}$$

where $I_L$ is the identity operator restricted to the left quadrant and $|0_R>$ is Rindler vacuum in the right quadrant. The corresponding conditional value of $T_{VV}$ is given by

$$\langle T_{VV}\rangle_{\Pi_{\lambda,R}} = \frac{<0_M|\Pi_{\lambda,R}T_{VV}|0_M>}{<0_M|\Pi_{\lambda,R}|0_M>}$$

It leads back to eq. (148) because of the EPR correlations between the two quadrants: if there is a rindleron on the right then their necessarily also is a rindleron on the left (its partner) with the opposite Rindler energy. This partner follows from eq. (25) where the operators $a^\dagger_{\lambda,R}$ and $a^\dagger_{\lambda,L}$ appear in product only. In the black hole problem the equivalent EPR correlations will mean that to each outgoing Hawking photon there corresponds an ingoing partner on the other side of the horizon.

An even less restrictive projector specifies only partially the state of the field in the right quadrant. One chooses that the final state contains one rindleron on the right in the mode $\lambda$ while tracing over all other right rindlerons and over all left rindlerons. The resulting projector is

$$\bar{\Pi}_{\lambda,R} = I_L \otimes \prod_{\lambda' \neq \lambda} I_{\lambda',R} \otimes |1_{\lambda,R}> <1_{\lambda,R}|$$

where $I_{\lambda,L}$ is the identity operator restricted to the mode $\lambda$ in the left quadrant and $|1_{\lambda,R}>$ is the one particle state restricted to the right mode $\lambda$. The corresponding conditional value of $T_{VV}$ is

$$\langle T_{VV}\rangle_{\bar{\Pi}_{\lambda,R}} = \frac{<0_M|\bar{\Pi}_{\lambda,R}T_{VV}|0_M>}{<0_M|\bar{\Pi}_{\lambda,R}|0_M>} = \frac{\lambda}{2\pi a^2 (V + i\epsilon)^2} = \frac{1}{4\pi a^2} <0_M|T_{VV}|0_M> + \frac{1}{2\pi a^2}$$

The first term is the energy of the rindleron $\lambda$ already obtained in eq. (148) and eq. (152). The second term is simply the Minkowski vacuum expectation value since no further
specification is imposed on the final state. This is why the probability $< 0_M | \tilde{\Pi}_{\lambda,R} | 0_M >$ to be in the eigenspace of $\tilde{\Pi}_{\lambda,R}$ is finite. This is to be opposed to the probabilities encountered previously (the denominators of eq. (148) and eq. (152)) which vanish because all the Rindler modes have been specified to be in their Rindler ground state. (In physically realistic situations only nonvanishing probabilities will occur. This was indeed the case for the accelerated two level atom coupled to the field).

Nevertheless the conditional values eq. (152) and eq. (154) are related by a unitary relation similar to eq. (144): by taking a set of orthogonal projectors like $\Pi_{\lambda,R}$ whose combined eigenspace is equal to the eigenspace of $\tilde{\Pi}_{\lambda,R}$ and summing the corresponding conditional values multiplied by the relative probabilities that they occur, eq. (154) is recovered. In order to realize this unitary relation one must select the presence of two, three, any number of rindlerons. The corresponding conditional values of $T_{VV}$ are easily obtained and the contribution of each individual selected particle is found to be independent (if the particles are orthogonal) of the selection performed on the other particles. In other words, for a free field the vacuum fluctuations of orthogonal particles are independent of each other.

We finally consider the selection of a wave packet. Instead of the projector eq. (151), we define:

$$\Pi_{v_0,\lambda_0,R} = I_L \otimes a^\dagger_{v_0,\lambda_0,R} | 0_R > < 0_R | a_{v_0,\lambda_0,R}$$

where $a_{v_0,\lambda_0,R} = \int_0^{+\infty} d\lambda f(\lambda) a_{\lambda,R}$ is the destruction operator of a wave packet of right rindlerons centered around $v = v_0$ and $\lambda = \lambda_0$. The state $\Pi_{v_0,\lambda_0,R} | 0_M >$ is

$$\Pi_{v_0,\lambda_0,R} | 0_M > = \left( \int_0^{+\infty} d\lambda f^*(\lambda) a^\dagger_{\lambda,R} \right) \left( \int_0^{+\infty} d\lambda' - \frac{\beta_{\lambda'}}{\alpha_{\lambda'}} f(\lambda') a^\dagger_{\lambda',L} \right) | 0_{RL} >$$

where the EPR correlated wave packet in the left Rindler quadrant appears explicitly. Note the asymmetry of the wave packets: the induced wave packet in the left quadrant contains the factor $\beta_{\lambda'}/\alpha_{\lambda'}$ since it originates from the EPR correlations in eq. (25). This asymmetry plays a fundamental role when analysing the flux emitted by the accelerated detector and the black hole. It is responsible of the fact that $\text{Im} \langle T_{VV}(U < U_a) \rangle_e$ vanishes for $V < 0$ only, see eq. (133). The conditional value of $T_{VV}$ associated with this wave packet is

$$\frac{< 0_M | \Pi_{v_0,\lambda_0,R} T_{VV} | 0_M >}{< 0_M | \Pi_{v_0,\lambda_0,R} | 0_M >} = 2 \left[ \int_0^{+\infty} d\lambda f(\lambda) f^*(\lambda') \partial_{\lambda'} \varphi^*_{\lambda,M} \partial_{\lambda'} \varphi^*_{\lambda',M} \right]^{-1} + \frac{< 0_{RL} | T_{VV} | 0_M >}{< 0_{RL} | 0_M >}$$

The role of $f(\lambda)$ in this equation is very similar to that of $c_\lambda$ in Section 3.3 (where $c_\lambda$ was the Fourier transform of the coupling to the atom, see eq. (57)). However since $f(\lambda)$ in eq. (157) contains no negative frequencies, eq. (157) is singular on the future horizon (see discussion after eq. (54)).
4 Black Hole Radiation

4.1 The Kinematics of the Collapse and the Scattered Modes

We work in the background metric of a spherically symmetric collapsing star of mass \( M \). Outside the star the geometry is described by the Schwarzschild metric

\[
ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2d\Omega^2
\]

\[
v, u = t \pm r^*
\]

\[
r^* = r + 2M \ln \frac{r - 2M}{2M}
\]

The specific collapse we consider is produced by a spherically symmetric shell of pressureless massless matter. Inside the shell space is flat and the metric reads

\[
ds^2 = d\tau^2 - dr^2 - r^2d\Omega^2
\]

\[
v, U = \tau \pm r
\]

where \( v \) is the same coordinate in eq. (158) and eq. (159) since on \( I^- (u = -\infty) \) space time is flat on both sides. The collapsing shell, taken to be thin, follows the geodesic \( v = v_S \). The connection between the two metrics is obtained by imposing the continuity of \( r \) along the shell’s trajectory

\[
dU = du(1 - \frac{2M}{r(u, v_S)}) = du(1 - \frac{4M}{v_S - U})
\]

Then by choosing \( v_S = 4M \) one gets

\[
du = -\frac{dU}{U} (4M - U)
\]

\[
u(U) = U - 4M \ln\left(\frac{-U}{4M}\right)
\]

In the static space time outside the star, the Klein-Gordon equation for a mode of the form \( \phi_{l,m} = \frac{1}{\sqrt{4\pi r^2}} \tilde{Y}_{lm}(\theta, \varphi) \psi_l(t, r) \) reads

\[
\left[\partial_t^2 - \partial_r^2 - (1 - \frac{2M}{r}) \left[\frac{l(l + 1)}{r^2} + m^2 + \frac{2M}{r^3}\right]\right] \psi_l(t, r) = 0
\]

Near the horizon \( r - 2M \ll 2M \), it becomes the wave equation for a massless field in 1 + 1 dimensions. By considering only the s-wave sector of a massless field and dropping the residual ”quantum potential” \( 2M(r - 2M)/r^4 \) the conformal invariance holds everywhere, inside as well as outside the star. From now on we shall work in this
simplified context and only discuss briefly the differences with the more realistic four dimensional case.

The Heisenberg state is chosen to be the initial vacuum i.e. vacuum with respect to the modes which have positive \( v \)-frequency on \( I^- \). These modes are reflected at \( r = 0 \) and read

\[
\varphi_{\omega,0,0}(v,u) = \frac{1}{4\pi r \sqrt{\omega}} \left( e^{-i\omega v} - e^{-i\omega U(u)} \right)
\]  

(163)

Hence, for \( u > 4M \) (or even on both sides of the horizon for \( -M < U < M \)) the state of the field tends exponentially quickly (in \( u \)) to Unruh vacuum\(^2\), i.e. vacuum with respect to the modes

\[
\exp(-i\omega v) \quad \text{and} \quad \exp\left(\frac{i\omega}{4M}e^{\frac{u}{r}}\right)
\]  

(164)

The Schwarzschild \( u \)-modes \( \chi_{\lambda}(u) = e^{-i\lambda u}/(4\pi r \sqrt{\lambda}) \) are needed to analyse the particle content of the scattered modes \( \varphi_{\omega} \) on \( I^+ \). In terms of \( U \) they take the form

\[
\chi_{\lambda}(u) = \theta(-U) \frac{1}{4\pi r \sqrt{\lambda}} \left( \frac{-U}{4M} \right)^{i\lambda M} e^{-i\lambda U}
\]  

(165)

The exact Bogoljubov coefficients between \( \varphi_{\omega} \) and \( \chi_{\lambda} \) are given by

\[
\alpha_{\omega,\lambda} = \langle \varphi_{\omega}, \chi_{\lambda} \rangle = \frac{1}{4\pi} \sqrt{\frac{\omega}{\lambda}} \Gamma(1 + i4M\lambda)[4M(\omega - \lambda)]^{-i4M\lambda} e^{\pm 2\pi M\lambda}
\]  

(166)

where the \( \pm \) is to be understood as + if \( \omega > \lambda \) and − if \( \omega < \lambda \). The expression for \( \beta_{\omega,\lambda} \) is obtained by taking \( \lambda \) into \(-\lambda\). The asymptotic Bogoljubov coefficients (relating Kruskal modes to Schwarzschild modes) are identical to the coefficients relating Minkowski modes to Rindler modes (see eq. (13)) in the limit \( \omega \to +\infty \) which corresponds to resonance at late times \( u \to +\infty \) (see eq. (107)). In this limit the black hole emits uncorrelated (see eq. (26)) quanta at the Hawking temperature \( 1/(8\pi M) \) since

\[
|\beta_{\omega,\lambda}/\alpha_{\omega,\lambda}|^2 = e^{-8\pi M\lambda}.
\]

Having described the kinematics of the collapse we now turn to the description of the energy content of the emitted quanta. The new difficulty lies in the renormalization of the energy momentum tensor which must be carried out in curved space times. We therefore turn to this point.

### 4.2 Matrix Elements in Curved Space-Time

Wald has proposed a set of eminently reasonable conditions that a renormalized energy momentum operator should satisfy\(^3\). By an argument similar to Wald’s (or simply by verifying that it is in accord with his axioms), it is possible to deduce that \( T_{\mu\nu(\text{ren})}(x) \) can be written in the following way

\[
T_{\mu\nu(\text{ren})}(x) = T_{\mu\nu}(x) - t_{\mu\nu(S)}(x)I
\]  

(167)

where \( T_{\mu\nu}(x) \) is the bare energy momentum tensor. The subtraction term \( t_{\mu\nu(S)}(x) \) is an (infinite) conserved c-number function only of the geometry at \( x \). It can be
understood\[36\] as the (infinite) ground state energy of the ”local inertial vacuum”: that state which most resembles Minkowski vacuum at \(x\). Numerous techniques have been developed to calculate \(t_{\mu\nu(S)}\) and we refer the reader to \[3\] for a review.

In a state, say the Heisenberg vacuum \(|0\rangle\), the expectation value of \(T_{\mu\nu}\) takes the form

\[
<T_{\mu\nu}(x)|0\rangle = <0|T_{\mu\nu}(x)|0\rangle - t_{\mu\nu(S)}(x) \tag{168}
\]

where both terms on the r.h.s. are infinite but their difference is finite.

If in addition one specifies the final state (i.e. on \(I^+\)) to be \(\Pi|0\rangle\), where \(\Pi\) is a projector (or more generally the self adjoint operator), the renormalized conditional value of \(T_{\mu\nu}\) reads

\[
\langle T_{\mu\nu}\rangle_\Pi = \frac{<0|\Pi T_{\mu\nu(ren)}|0\rangle}{<0|\Pi|0\rangle} \tag{169}
\]

Inserting eq. (167) into this expression yields

\[
\langle T_{\mu\nu}(x)\rangle_\Pi = \frac{<0|\Pi T_{\mu\nu}(x)|0\rangle}{<0|\Pi|0\rangle} - t_{\mu\nu(S)}(x) \tag{170}
\]

Then by expressing \(T_{\mu\nu}(x)\) in terms of the operators which annihilate the Heisenberg vacuum one obtains

\[
\langle T_{\mu\nu}(x)\rangle_\Pi = \int_0^\infty d\omega \int_0^\infty d\omega' <0|\Pi a_\omega^\dagger a_{\omega'}^\dagger|0\rangle \hat{T}_{\mu\nu}(x) [\varphi_\omega^* \varphi_{\omega'}^*] + <0|T_{\mu\nu(ren)}(x)|0\rangle \tag{171}
\]

where \(\hat{T}_{\mu\nu}(x)\) is the classical differential operator which acting on the waves \(\varphi_\omega^*\), eq. (163), gives their energy density.

The renormalized conditional value contains two contributions. The first term, the fluctuating part, depends on the particle content of the state specified by \(\Pi\). Contrariwise, the second one is the mean energy density of the Heisenberg vacuum eq. (168) obtained when no specification on the final state is added.

The formula eq. (171) warrants a few additional comments. First notice that their are parts of \(\langle T_{\mu\nu}\rangle_\Pi\) that are entirely contained in the subtraction. Most notably there is the trace anomaly and those components of the energy momentum tensor which are related to it by energy conservation (in two dimensions they are \(T_{uu,v}\) and \(T_{vv,u}\)). These parts are independent of \(\Pi\) or, expressed differently, do not fluctuate.

An additional (and related) feature concerns the absence of correlations between \(T_{uu}\) and \(T_{vv}\). Not only shall this give rise to the particular structure of vacuum fluctuations that extend back to \(I^-\), but it also implies that on the horizon the in-going flow and the out-going flow fluctuate independently (for instance the specification of an outgoing particle on \(I^+\) does not affect \(T_{vv}\) outside the star and in particular on the horizon \(r = 2M\)). This last effect disappears partially when considering the potential barrier that occurs in the wave equation eq. (162).

### 4.3 The Conditional Value of the Energy Density

In the absence of specification of the final state, \(<0|T_{\mu\nu(ren)}|0\rangle\) describes the mean energy content carried by Hawking quanta. We remind the reader that \(<0|T_{\mu\nu(ren)}(x)|0\rangle\)
is regular on the future horizon $U = 0$ and that Hawking radiation can be conceived as the matter response that gives regular mean energy densities on the horizon. We refer to ref. [3] for further discussion of the mean flux.

In order to describe the fluctuations around the mean flux, one inquires into the conditional energy density when the final state is a particular out-state which arises in the rewriting of the Heisenberg state $|0\rangle$ into states with definite energy $\lambda$ on $I^+$, see eq. (25). However in the collapsing geometry, an external observer does not have access to the region of space time beyond the horizon, hence the specifications of the final state that he can perform are restricted to an incomplete ($U < 0$) region of space time and are therefore incomplete as well.

A possible specification is, for instance to use the projector, see eq. (151),

$$\Pi_\lambda = I_L \otimes a^\dagger_\lambda |B_R\rangle < B_R|a_\lambda$$

where $I_L$ is the identity operator restricted to the inaccessible region $U > 0$ and $|B_R\rangle$ is Boulware vacuum in the region $U < 0$. $a^\dagger_\lambda$ creates a Schwarzschild outgoing photon.

The corresponding conditional energy reads, see eqs. (148), (150) and (171),

$$\langle T_{\mu\nu}(x)\rangle_{\Pi_\lambda} = \frac{2}{\alpha\beta_\lambda} \hat{T}_{\mu\nu}(x) \left[ \varphi^*_{\lambda,K}\varphi^*_{-\lambda,K} \right] + \left[ \frac{<B|T_{\mu\nu}(x)|0>}{<B|0>} - <0|T_{\mu\nu}(x)|0> \right] + <0|T_{\mu\nu(\text{ren})}(x)|0>$$

where the modes $\varphi_{\lambda,K}$ are defined from the modes $e^{-i\omega U/\sqrt{4\pi\omega}}$, eq. (163), as the Minkowski modes eq. (13) are defined from the Minkowski plane waves eq. (8).

The first two terms arise from the specification of the final state whereas the third term is the mean energy. The first term is equal to the energy of the photon $\lambda$. The second one is the difference of energy between Boulware vacuum and the Heisenberg vacuum. This term appears, as in eq. (150), because one has specified that, apart from $\lambda$, there is no other photon emitted. This is why this term is singular on the horizon and why the probability to be in the eigenstate of $\Pi_\lambda$ vanishes in the absence of backreaction (in the semiclassical approximation, it is of order $e^{-M^2}$ where $M^2$ is approximately the total number of photons emitted).

A more reasonable specification because it has a finite probability of occurring consists in tracing over all the photons except the photon $\lambda$ which is imposed to be present (in the Rindler problem this corresponds to the projector eq. (153)). Then the conditional energy is simply, see eqs. (154) and (171),

$$\langle T_{\mu\nu}(x)\rangle_{\tilde{\Pi}_\lambda} = \frac{2}{\alpha\beta_\lambda} \hat{T}_{\mu\nu}(x) \left[ \varphi^*_{\lambda,K}\varphi^*_{-\lambda,K} \right] + <0|T_{\mu\nu(\text{ren})}(x)|0>$$

Having traced over all the other photons, the second term of eq. (173) is absent in eq. (174). Nevertheless this term can also be constructed as the sum of conditional values that specify completely the state times the probability that they occur (in similar manner to the the unitarity relation eq. (144)). In this way the difference of energy between the Heisenberg vacuum $|0\rangle$ and Boulware vacuum $|B\rangle$ is realized as the sum over all possible radiated photons times the thermal probabilities that they occur.
Finally we consider an inertial two level atom at large distance from the black hole. The specification of the state of the radiation is carried out indirectly by requiring, as in Section 3.2, that the atom get excited. In this case also the final radiation state is partially specified, since the detector is coupled to a finite set of modes. One finds that the conditional energy contains again two terms

$$\langle T_{\mu \nu} \rangle_{\Pi_e} = \langle T_{\mu \nu} \rangle_e + <0|T_{\mu \nu(\text{ren})}|0>$$

where the fluctuating part, in terms of the $\phi_m$ operators (eq. (29)) is, see eq. (127),

$$\langle T_{\mu \nu} \rangle_e = \frac{g^2 m^2}{P_e} <0|\phi_m^\dagger \phi_m : T_{\mu \nu} : |0>$$

where $:T_{\mu \nu}:$ is the energy momentum operator normal ordered with respect to the in operators defining Heisenberg vacuum $|0>$. We shall display the properties of $\langle T_{\mu \nu} \rangle_e$ in the next section.

We note already that the specification of the final radiation state by the correlations to a transition of the two level atom gives rise to finite energy densities on the horizon only if the coupling to the field decreases faster than $e^{-u/4M}$; c.f. discussion after eq. (157) and (54). Therefore, the specification of a mode (eq. (174)) or a wave packet made out of positive $\lambda$ frequencies only gives rise inevitably to singular energy densities on the horizon.

It is also interesting to speculate about the nature of the in-going vacuum fluctuations when they cross the future horizon $U = 0$. They could be analysed by selecting the presence of ingoing quanta near the horizon. A ”natural” set of modes to select near the horizon are Kruskal $\nu$-modes. One is therefore led to consider the Kruskal vacuum fluctuations in Schwarzschild vacuum, which is similar to considering Minkowski fluctuations in Rindler vacuum. If space time were the full Schwarzschild manifold, these would present a singularity on the past horizon that could be smoothed out using wave packets. Since space time is not the full Schwarzschild manifold (there is no past horizon) the star’s surface will play the role of past horizon and one expects large energy densities in the outermost layers of the star.

4.4 From Vacuum Fluctuations to Black Hole Radiation

We now turn to the the fluctuating part of the conditional energy correlated to the transition of the two level atom, i.e. the term $\langle T_{\mu \nu} \rangle_e$ of eq. (175).

For s-waves, when one neglects the residual potential of the d’Alembertian eq. (162), $\langle T_{\mu \nu} \rangle_e$ is completely independent of the geometry. Furthermore, the conformal invariance of the field makes the mapping of the results obtained in the Rindler problem to the present problem straightforward. Let us choose therefor the time dependent coupling $f(t)$ of our detector given by the gaussian switch off of Section 3.2, see eq. (57) and eq. (63). More precisely, the coupling is such that the atom, of resonance frequency $\lambda = m$ will be excited around the retarded time $u = u_0$. Hence its Fourier components are, see eq. (83)

$$c_\lambda = \int dt f(t)e^{-imt}e^{i\lambda t} = D\frac{\lambda}{m}e^{i\lambda u_0}e^{-(\lambda-m)^2T^2/2}(1-e^{-2\pi \lambda/a})$$

(177)
The spread in time is \( \Delta t = \Delta u = T \) and \( u_0 \) is taken well inside the region \( u > 0 \), see eq. (161), where the isomorphism of the scattered waves and the Kruskal modes is achieved.

If the two level atom is found excited after the switch off, due to the vanishing of the modes at \( r = 0 \), see eq. (163), and the light-like character of the propagation, the correlated radiation state eschews from a spherically symmetric vacuum fluctuation on \( I^- \). This fluctuation is located in a region

\[
|v - v_\infty| = |\Delta U| = |\Delta u e^{-u_0/4M}| \simeq T e^{-u_0/4M}
\]

(178)

where \( v = v_\infty \) (= 0 in our collapse) is the light ray that shall become the future horizon \( U = 0 \). Indeed this localization is furnished by the \( v \) dependence of the conditional energy density on \( I^- \) which reads (see eqs. (176) and (140))

\[
\langle T_{vv}(I^-, v) \rangle_e = \frac{g^2 m^2}{P_e} \int d\lambda d\lambda' c^*_\lambda c_{\lambda'} \frac{1}{4\pi\sqrt{\lambda\lambda'}} \sqrt{\tilde{n}_{\lambda'}(\tilde{n}_\lambda + 1)} T_{\mu\nu} \left[ \phi^*_{\lambda,K}\phi_{-\lambda',K} \right]
\]

\[
\simeq \frac{1}{4\pi r^2} \frac{16M^2}{v^2} \frac{m}{2\sqrt{\pi T}} (N_m + 1) \exp \left[ -\left( \frac{4M}{T} \ln\left( -\frac{v - i\epsilon}{4M} \right) + u_0 \right) \right]
\]

(179)

Where the width of the gaussian factor gives eq. (178). As in eq. (140), the \( \epsilon \) specification of the log ensures that the total energy carried by this fluctuation vanishes, see eq. (141). Thus we see that the analysis of the fluctuations by an inertial observer near \( I^- \) is isomorphic with what was called the Minkowski interpretation in Part 4. As in the accelerated case, the energy density is enhanced by the jacobian \( du/dU = e^{u/4M} \) centered around \( u = u_0 \) which appears here as \( 1/v^2 \) when the reflection at \( r = 0 \) is taken into account. Hence after a \( u \)-time of the order of \( 4M \ln M \), the energy density in \( T_{vv} \) (rescaled by \( 4\pi r^2 \)) become “transplanckian” and located within a distance \( \Delta v \) much smaller than the Planck length (If one does not rescale \( T_{\mu\nu} \) the transplanckian energies only exist in a region of finite \( r \) which nevertheless increases exponentially with \( u_0 \)). The dramatic consequences that these transplanckian energies might introduce are discussed in ref. [19]. In that article it is argued that the nonlinearity of general relativity cannot accommodate these densities and that a taming mechanism must exist if Hawking radiation does exist.

After issuing from \( I^- \), the vacuum fluctuation contracts until it reaches \( r = 0 \) and then reexpands along \( U = \text{const} \) lines. Upon crossing the surface of the star in a region \( \Delta U \simeq T e^{-u_0/4M} \) centered on the horizon, it separates into a piece (the partner) that falls into the singularity, carrying a negative Schwarzschild energy equal to \( -m(1 + N_m) \) (see eqs. 135 and 139), and a piece carrying positive energy equal to \( m(1 + N_m) \) that keeps expanding and escapes to \( I^+ \) to constitute the quantum that induces the transition of the atom (see eq. 133 and figure 4). The analysis performed by an inertial asymptotic observer near the detector, on \( I^+ \), is isomorphic with the Rindler interpretation of Section 3.3 since the gravitational red shift replace exactly the role of the Doppler accelerated one in the accelerated place. For instance, one finds readily that the total energy carried by the fluctuating \( \langle T_{uu}(I^+, u) \rangle_e \) is indeed \( m(1 + N_m) \) as in eq. (137).

Similarly, if the two level atom is found in its ground state after the switch off, its wave function is correlated to the absence of the Hawking photon specified by \( c_\lambda \). In
that case, one would find near $\mathcal{I}^-$ a vacuum fluctuation whose energy content is exactly
the opposite of the previously considered case (times $P_g/P_e$). Near $\mathcal{I}^+$ it would therefore
contain a negative energy flux of total energy $-m(N_m+1)P_e/(1-P_e)$ encoding the fact
that their are quanta absent from the thermal flux emitted by the black hole.

If more realistically, we take a two-level atom coupled locally to the field (i.e. coupled
to all the modes $l>0$), it will select particles coming out of the black hole in its direction.
Then the picture that emerges is essentially the same as for an $s$-wave except that on $\mathcal{I}^-$
the vacuum fluctuation is localized on the antipodal point of the detector. The created quantum and its partner, are on the same side and not antipodal (with respect to each
other) because they have opposite energy.

We now turn to the description in the intermediate regions in order to interpolate
between the descriptions between $\mathcal{I}^-$ and $\mathcal{I}^+$. One possible interpolation consists in
using a set of static observers at constant $r$. Then the "Rindler" description would be
used everywhere outside the star. However a difficulty arises in this scheme if one really
considers a set of material "fiducial" detectors at constant $r$. For upon interacting
with the field and thermalizing at the local temperature $\sqrt{\frac{r}{(r-2M)^{3/2}}} \frac{1}{8\pi M}$ the detectors will
emit large amounts of ultraviolet Kruskal "real" quanta (see Section 3.3 wherein it is
shown how the accelerated atom transforms vacuum fluctuations into "real" quanta).
The backreaction of these on mass shell quanta cannot be neglected and, as already
stated, cannot be evaluated owing to the transplanckian energy they carry.

An alternative interpolation consists in giving the value of $T_{\mu\nu}$ in the local inertial
coordinate system (Riemann normal coordinates). This stems from the idea that local
physics should be describe locally in such a coordinate system. This approach has
been used in defining the subtraction necessary to renormalize the energy momentum
tensor $T_{\mu\nu}$. In the two dimensional model the local inertial coordinates are easy
to construct. Since $\tilde{u} = r(u,v)$ is an affine parameter along the geodesics $v = constant$
, a natural way to represent the outgoing flux outside the star is as

\[
T_{uu}(\tilde{u}) = \left( \frac{du(r,v)}{dr} \right)^2 T_{uu}(u(r,v))
\]

This is represented in a Penrose diagram in figure 6 and Eddington-Finkelstein coor-
dinates in figure 7. The inertial coordinate $\tilde{u}$ will come up very naturally in the next
section.

4.5 The Gravitational Back Reaction

Up to now in this Part we have presented the properties of the conditional values of
$T_{\mu\nu}$. We now investigate how these matrix elements intervene in physical processes. An
example of the role of these matrix elements was given in section 3.2. Here we shall
specifically discuss gravitational back reaction effects to black hole radiation.

A simple way to understand the role of the conditional values of $T_{\mu\nu}$ is to imagine a
change in the background geometry $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$. This change modifies both the
mean values of the flux as well as more detailed properties such as the probability to
find a specific photon on $\mathcal{I}^+$. We focus on this later change.
The new probability can be computed, in the interacting picture, around the background $g_{\mu\nu}$ with the Hamiltonian given by

$$\int d^4x H_{\text{int}} = -\frac{1}{2} \int d^4x \sqrt{-g} \delta g^{\mu\nu} T_{\mu\nu}$$

(181)

The new probability is then, see eq. (144),

$$P_{g+\delta g} = \langle 0 | e^{i \int d^4x H_{\text{int}}} \Pi e^{-i \int d^4x H_{\text{int}}} | 0 \rangle$$

(182)

where $\Pi$ specifies the state of the radiation field on $I^+$. To first order in $\delta g_{\mu\nu}$ the relative change in probability is

$$\frac{P_{g+\delta g} - P_g}{P_g} = \frac{\langle 0 | \Pi (-i \int d^4x H_{\text{int}}) | 0 \rangle}{\langle 0 | \Pi | 0 \rangle} + \text{c.c.}$$

$$= \int d^4x \sqrt{-g} \delta g^{\mu\nu} \text{Im} \langle \langle T_{\mu\nu} \rangle_\Pi \rangle$$

(183)

It is thus the imaginary part of $\langle \langle T_{\mu\nu} \rangle_\Pi \rangle$ only which controls the change in probability induced by $\delta g_{\mu\nu}$. Furthermore, since the background part of conditional energies (the second term of eq. (171)) is by construction real, only the fluctuating part, which depends explicitly of the selected quantum, contributes to $P_{g+\delta g} - P_g$.

To illustrate how the various properties of the fluctuating part of $\langle \langle T_{\mu\nu} \rangle_\Pi \rangle$ intervene in such an expression, let us take the simple example wherein $\delta g_{\mu\nu}$ is due to the infall of an additional light like shell of mass $\delta m$ at time $v = v'$ with $v' \geq v_S$. Then for $v_S < v < v'$ the metric, eq. (158) and eq. (161)), is unchanged

$$ds^2 = \left(1 - \frac{2M}{r} \right) dv^2 - 2 dv dr - r^2 d^2 \Omega$$

(184)

whereas for $v > v'$ it is

$$ds^2 = \left(1 - \frac{2M + 2\delta m}{r} \right) dv^2 - 2 dv dr - r^2 d^2 \Omega$$

(185)

where we have used for obvious convenience the Eddington Finkelstein coordinates $v$ and $r$.

The change in the action $S = \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ is for $s$-waves

$$\delta S = -\int d^4x H_{\text{int}} = \int_{v'}^{+\infty} dv \int_0^\infty dr \, 4 \pi r^2 (\delta m/r) \partial_v \phi \partial_r \phi$$

$$= \delta m \int_{v'}^{+\infty} dv \int_0^\infty dr \, 4 \pi r T_{\tilde{\mu} \tilde{\nu}}(r, v)$$

(186)

where $T_{\tilde{\mu} \tilde{\nu}}$ is given by eq. (180). As emphasized at the end of section 4.4 it is the energy momentum in Riemann normal coordinates which appears automatically in such problems since the response to a local change in the geometry is local as well.

In order to compute the change in probability due to $H_{\text{int}}$, we first recall that the imaginary part of $\langle T_{\tilde{\mu} \tilde{\nu}} \rangle_\Pi$ vanishes on the other side of the horizon, for $r < 2M$ (see
discussion after eq. (139). From eq. (180) we understand that this is dictated by causality: a change in $g_{\mu\nu}$ in the region $r < 2M$ cannot affect the probability to find a specific Hawking photon on this side.

In the case when $v' = v_S$, one simply has a shell of mass $M + \delta m$. In this case we know exactly the change in probability since the probability of finding a photon of frequency $\lambda$ is

$$P_{M+\delta m} = |\beta_{\omega,\lambda}|^2 / |\alpha_{\omega,\lambda}|^4 = e^{-8\pi\lambda(M+\delta m)} (1 - e^{-8\pi\lambda(M+\delta m)})$$  \hspace{1cm} (187)

hence

$$\frac{\delta P}{P} = -8\pi\lambda\delta m \left( \frac{1 - 2e^{-8\pi\lambda M}}{1 - e^{-8\pi\lambda M}} \right)$$  \hspace{1cm} (188)

Thus in this case the transplanckian character of the energy density is washed out by the integration in eq. (180). This can be verified explicitly by evaluating the integral near $r = 2M$ and $v = v'$. There, one can replace $dr$ by $-dU/2$ whereupon by making appeal to the vanishing of the integrals $\int dU\langle T_{UU}\rangle_{\Pi}$ and $\int du\text{Im}\langle T_{uu}\rangle_{\Pi}$ one finds that the integral at fixed $v$ $\int dr 4\pi r \text{Im}\langle T_{u\bar{u}}\rangle_{\Pi}$ does not scale like $e^{u/4M}$.

Furthermore at large $r$ the integral, eq. (183), vanishes once more since $\int du\text{Im}\langle T_{uu}\rangle_{\Pi}$ vanishes. Hence if the additional mass crosses the photon trajectory when it is on mass shell at $r >> 4M$ there is no modification of the probability of creating the photon.

We have thus obtained a local description of the quantum matter response to a modification of the classical background geometry. To address the quantum gravitational back reaction to the creation of a Hawking photon one should treat $\delta g_{\mu\nu}$ as a quantum operator [17][11]. For spherically symmetric radiation and spherical symmetric gravitational fields the relation between $\delta g_{\mu\nu}$ and $T_{\mu\nu}$ is a constraint, i.e. $\delta g_{\mu\nu}$ is completely fixed by $T_{\mu\nu}$. One can then envisage the backreaction as an iterative scheme which ultimately should be treated self consistently.

The first step in this procedure is very simple. It consists in taking $\delta g_{\mu\nu}$ to be the "position coordinate" of the additional system (the oscillator) introduced in Section 2.5. Then, as for the oscillator, see eq. (132), the mean value of $\delta g_{\mu\nu}$ is obtained by integrating Einstein’s equations with the mean energy momentum tensor as a source. This corresponds to the linear approximation to the semiclassical solution [10]. But one can also evaluate the "mean" conditional value of $\delta g_{\mu\nu}$. This conditional change in the metric is obtained by integrating Einstein’s equations with $\langle T_{\mu\nu}\rangle_{\Pi}$ as source. Since the total energy carried by the conditional value of $T_{\mu\nu}$ vanishes from $I^-$ till the emergence of the fluctuation from the star after reflection on $r = 0$, $\langle \delta g_{\mu\nu}\rangle_{\Pi}$, the conditional value of $\delta g_{\mu\nu}$, will vanish outside the interval $\Delta v$ eq. (178) centered around $v_S$. Within that interval the precise shape of $\langle \delta g_{\mu\nu}\rangle_{\Pi}$ will depend on the particular choice of selected wave packet by the projector $\Pi$. On the contrary, outside the star, for $r > 4M$ and $u > u_0$ (i.e. in the middle of the two members of the pair), $\langle \delta g_{\mu\nu}\rangle_{\Pi}$ will encode the mass loss $\omega$ and in fact describes a new classical (real valued) Schwarzschild space where the mass is $M - \omega$.

7See however the different ways in which $\text{Re}\langle T_{uu}\rangle_{\Pi}$ and $\text{Im}\langle T_{uu}\rangle_{\Pi}$ enter in eq. (132). This might lead to interesting effects.
The next step consists in taking into account the effect of $\delta g_{\mu\nu}$ on the production of the Hawking photon itself. This gravitational self interaction can be encoded in an interaction hamiltonian of the form $H_{\text{int}} = T_{\mu\nu}D^{\mu\nu\alpha\beta}T_{\alpha\beta}$ where $D$ is the linearized gravitation propagator. To calculate $\delta P/P$ due to this self interaction one must confront the infinities which arise in matter loops. This will not be done here.

However there are some simple question wherein $H_{\text{int}}$ does come in which do not involve loops. One such question is the effect of the creation of a first Hawking photon on subsequent ones. Suppose one calculates $P_{\Pi_{\lambda_1,u_1;\lambda_2,u_2}}$ where $\Pi_{\lambda_1,u_1;\lambda_2,u_2}$ specifies the presence on $I^+$ of a photon of frequency $\lambda_1$ located near $u_1$ and another photon of frequency $\lambda_2$ located near $u_2$. Then to first order in $H_{\text{int}}$, the relative change in the probability to find the two photons is

$$\frac{\delta P_{\Pi_{\lambda_1,u_1;\lambda_2,u_2}}}{P_{\Pi_{\lambda_1,u_1}}P_{\Pi_{\lambda_2,u_2}}} = 2 \text{Im} \left[ \frac{\langle 0|\Pi_{\lambda_1,u_1;\lambda_2,u_2}H_{\text{int}}|0 >}{\langle 0|\Pi_{\lambda_1,u_1;\lambda_2,u_2}|0 >} \right]$$

(189)

where, in the absence of gravitational coupling, the probability of finding two photons factorizes into the individual probabilities. In this expression once more there are infinite loops. However upon taking derivatives the quantity $(d/d\lambda_1)(d/d\lambda_2)(\delta P/P)$ is finite. The effects of particle $\lambda_1$, $u_1$ on particle $\lambda_2$, $u_2$ are isolated from other effects. This finite quantity can then be expressed in terms of the products of the one-particle conditional values

$$\langle T_{\mu\nu}(x)\Pi_{\lambda_1,u_1} D^{\mu\nu\alpha\beta}(x,y)\langle T_{\alpha\beta}(y)\Pi_{\lambda_2,u_2}$$

$$\langle \partial_{\mu}\phi(x)\partial_{\alpha}\phi(y)\Pi_{\lambda_1,u_1} D^{\mu\nu\alpha\beta}(x,y)\langle \partial_{\nu}\phi(x)\partial_{\beta}\phi(y)\Pi_{\lambda_2,u_2}$$

(190)

In conclusion we have shown that the conditional values of $T_{\mu\nu}$ enter into tree graphs. In order to understand the role of the transplanckian frequencies in Hawking radiation and how they are tamed by quantum gravity one should confront loops and the infinities they involve. We hope to report on this in subsequent work.

5 Post Selection, Weak Measurement

5.1 Introduction

In section 3.2 we showed succinctly how the conditional values of $T_{VV}$ control the first order perturbation onto an additional system. The aim of this section is to present a self contained discussion devoted this result. We shall work in complete generality and make no explicit reference to the accelerated system nor to black hole radiation.

This analysis of the conditional values (i.e. non diagonal matrix element of an operator) was first carried out by Aharonov et al. [12] in the context of measurement theory. In essence they studied the first order backreaction onto an additional system which they took to be a measuring device. But the formalism is more general. In the case of pair production the additional system could be the external electric or gravitational field which is now described quantum mechanically. Moreover this formalism can be used to
study the self interaction of the pairs without introducing the additional system. This is because, when the first order (or weak) approximation is valid, the backreaction takes a simple and universal form governed by a c-number, the conditional value of the operator which controls the interaction (called by Aharonov et al. the “weak value”).

In section 5.2 we implement the specification of the final state(s) in a rather formal way by acting with projection operators which select the desired final state(s). Aharonov et al. call this specification of the final state, a “post-selection”.

In section 5.3 we show how post-selection may be realized operationally following the rules of quantum mechanics by coupling the system to be studied (S) an additional system in a metastable state (the ”post–selector” PS) which will make a transition only if the system is in the required final state(s). The conditional value of an operator obtained in this manner changes as time goes by from an off diagonal matrix element to an expectation value, thereby making contact with more familiar physics. This extended formalism finds important application when considering the physics of the accelerated detector since the accelerated detector itself plays the role of post selector.

5.2 Conditional (or Weak) Values

The approach developed by Aharonov et al.\(^{[12]}\) for studying pre- and post-selected ensembles consists in performing at an intermediate time a ”weak measurement” on S.

The system to be studied (S) is in the state \(|\psi_i>\) at time \(t_i\) (or more generally is described by a density matrix \(\rho_i\)). The unperturbed time evolution of this pre-selected state can be described by the following density matrix

\[
\rho_S(t) = U_S(t,t_i)|\psi_i><\psi_i| U_S(t_i,t) \tag{191}
\]

where \(U_S = \exp(-iH_S t)\) is the time evolution operator for the system S. The post-selection at time \(t_f\) consists in specifying that the system belongs to a certain subspace, \(\mathcal{H}_S^0\), of \(\mathcal{H}_S\). Then the probability to find the system in this subspace at time \(t_f\) is

\[
P_{\text{in}}^0 = Tr_S \left[ \pi_S^0 \rho(t_f) \right] = Tr_S \left[ \pi_S^0 U_S(t_f,t_i)|\psi_i><\psi_i| U_S(t_i,t_f) \right] \tag{192}
\]

where \(\pi_S^0\) is the projection operator onto \(\mathcal{H}_S^0\) and \(Tr_S\) is the trace over the states of system S. In the special cases wherein the specification of the final state is to be in a pure state \(|\psi_f>\) (i.e. \(\pi_S^0 = |\psi_f><\psi_f|\)) then the probability is simply given by the overlap

\[
P_f = |<\psi_f| U_S(t_f,t_i)|\psi_i>|^2 \tag{193}
\]

Following Aharonov et al. we introduce an additional system, called the ”weak detector” (WD), coupled to S. The interaction hamiltonian between S and WD is taken to be of the form \(H_{S-WD}(t) = \epsilon f(t) A_S B_{WD}\) where \(\epsilon\) is a coupling constant, \(f(t)\) is a c-number function, \(A_S\) and \(B_{WD}\) are hermitian operators acting on S and WD respectively.

Then to first order in \(\epsilon\) (the coupling is weak), the evolution of the coupled system S and WD is given by

\[
\rho(t_f) = |\Psi(t_f)> <\Psi(t_f)|
\]
where
\[ |\Psi(t_f)\rangle = \left[U_S(t_f, t_i)U_{WD}(t_f, t_i) - i\epsilon \int_{t_i}^{t_f} dt U_S(t_f, t)U_{WD}(t_f, t)f(t)A_S B_{WD} \times U_S(t, t_i)U_{WD}(t, t_i)\right]|\psi_i\rangle |WD\rangle \] (194)

where \( U_S \) and \( U_{WD} \) are the free evolution operators for \( S \) and \( WD \) and \(|WD\rangle\) is the initial state of \( WD \). Upon post-selecting at \( t = t_f \) that \( S \) belongs to the subspace \( \mathcal{H}_S^0 \) and tracing over the states of the system \( S \), the reduced density matrix describing the \( WD \) is obtained
\[ \rho_{WD}(t_f) = Tr_S[\Pi_0^0 \rho(t_f)] \] (195)

To first order in \( \epsilon \), it takes a very simple form
\[ \rho_{WD}(t_f) \simeq P_{\Pi_0^0} |\Psi_{WD}(t_f)\rangle <\Psi_{WD}(t_f)| \]

where
\[ |\Psi_{WD}(t_f)\rangle = \left[U_{WD}(t_f, t_i) - i\epsilon \int_{t_i}^{t_f} dt U_{WD}(t_f, t)f(t)\langle A_S(t)\rangle_{\Pi_0^0} B_{WD} U_{WD}(t, t_i)\right]|WD\rangle \] (196)

where \( P_{\Pi_0^0} \) is the probability to be in subspace \( \mathcal{H}_S^0 \) and
\[ \langle A_S(t)\rangle_{\Pi_0^0} = \frac{Tr_S[\Pi_0^0 U_S(t_f, t)A_S U_S(t, t_i)|\psi_i\rangle <\psi_i|U_S(t_i, t_f)]}{Tr_S[\Pi_0^0 U_S(t_f, t_i)|\psi_i\rangle <\psi_i|U_S(t_i, t_f)]} \] (197)

is a c-number called the weak or conditional value of \( A \). If one specifies completely the final state, \( \Pi_0^0 = |\psi_f\rangle <\psi_f| \) then the result of Aharonov et al. obtains:
\[ \langle A_S(t)\rangle_{\psi_f} = \frac{<\psi_f|U_S(t_f, t)A_S U_S(t, t_i)|\psi_i\rangle}{<\psi_f|U_S(t_f, t_i)|\psi_i>}. \] (198)

The principal feature of the above formalism is its independence on the internal structure of the \( WD \). The first order backreaction of \( S \) onto \( WD \) is universal: it is always controlled by the c-number \( \langle A_S(t)\rangle_{\Pi} \), the ”weak value of \( A \”). Therefore if \( S \) is coupled to itself by an interaction hamiltonian, the backreaction will be controlled by the weak value of \( H_{int} \) in first order perturbation theory. For instance the modification of the probability that the final state belongs to \( \mathcal{H}_S^0 \) is given by the imaginary part of \( \langle H_{int}\rangle_{\Pi_0^0} \). Indeed
\[ P'_{\Pi_0^0} = Tr_S[\Pi_0^0 (1 - i\int dt H_{int})\rho_i (1 + i\int dt H_{int})] = P_{\Pi_0^0} (1 - 2 \text{ Im} \langle H_{int}\rangle_{\Pi_0^0}) \] (199)

The weak value of \( A \) is complex. By performing a series of measurements on \( WD \) and by varying the coupling function \( f(t) \), the real and imaginary part of \( \langle A_S(t)\rangle_{\Pi} \) could in principle be determined. Here the word ”measurement” must be understood in its usual
quantum sense: the average over repeated realizations of the same situation. This means that the weak value of $A_S$ should also be understood as an average. The fluctuations around $\langle A_S(t) \rangle_\Pi$ are encoded in the second order terms of eq. (194) which have been neglected in eq. (194).

To illustrate the role of the real and imaginary parts of $\langle A_S(t) \rangle_\Pi$, we recall the example of Aharonov et al. consisting of a weak detector which has one degree of freedom $q$, with a gaussian initial state $\langle q | WD \rangle = e^{-q^2/2\Delta^2}$, $-\infty < q < +\infty$ (see also the example of Section 3.2). The unperturbed hamiltonian of $WD$ is taken to vanish (hence $U_{WD}(t_1,t_2) = 1$) and the interaction hamiltonian is $H_{S-WD}(t) = \epsilon \delta(t - \tilde{t}) p A_S$ where $p$ is the momentum conjugate to $q$. Then after the post-selection the density matrix of $WD$ is given by $|WD(t_f)\rangle = (1 - i\epsilon p \langle A_S(\tilde{t}) \rangle_\Pi) |WD\rangle$, see eq. (196), where, to first order in $\epsilon$, $|WD(t_f)\rangle$ is

$$|WD(t_f)\rangle = \left(1 - i\epsilon p \langle A_S(\tilde{t}) \rangle_\Pi \right) |WD\rangle$$

whereupon the conditional value of $q$ and $p$ are

$$\langle q \rangle_\Pi = \epsilon \text{Re} \langle A_S(\tilde{t}) \rangle_\Pi$$

$$\langle p \rangle_\Pi = \epsilon \text{Im} \langle A_S(\tilde{t}) \rangle_\Pi / \Delta$$

Thus the real part of $\langle A_S(\tilde{t}) \rangle_\Pi$ induces a translation of the center of the gaussian, the imaginary part a change in the momentum. Their effect on the $WD$ is therefore measurable. The validity of the first order approximation requires $\epsilon|\langle A_S(\tilde{t}) \rangle_\Pi| / \Delta << 1$.

It is instructive to see how unitarity is realized in the above formalism. Take $\Pi_S^0$ to be a complete orthogonal set of projectors acting on the Hilbert space of $S$. Denote by $P_j$ the probability that the final state of the system belong to the subspace spanned by $\Pi_S^j$ and by $\langle A_S(t) \rangle_{\Pi_S^j}$ the corresponding weak value of $A$. Then the mean value of $A_S$ is

$$\langle \psi_i | A_S(t) | \psi_i \rangle = \sum_j P_j \langle A_S(t) \rangle_{\Pi_S^j}$$

Thus the mean backreaction if no post-selection is performed is the average over the post-selected backreactions (in the linear response approximation). Notice that the imaginary parts of the weak values necessarily cancel since the l.h.s. of eq. (202) is real. Equation (202) is the short cut used in the main text to obtain with minimum effort the weak values.

### 5.3 Physical Implementation of Post Selection

Up to now the postselection has been implemented by inserting by hand the projector $\Pi_S^0$. Such a projection may be realized operationally by introducing an additional quantum system, a "post-selector" ($PS$), coupled in such a way that it will make a transition if and only if the system $S$ is in the required final state. Then by considering only that subspace of the final states in which $PS$ has made the transition, a post-selected state is specified. This quantum description of the post-selection is similar in spirit to
the measurement theory developed in ref. [13]: by introducing explicitly the measuring device in the Hamiltonian the collapse of the wave function ceases to be a necessary concomitant of measurement theory.

We shall consider the very simple model of a PS having two states, initially in the ground state, and coupled to the system by an interaction of the form

\[ H_{S-PS} = \lambda g(t)(a^\dagger Q_S + aQ_S^\dagger) \] (203)

where \( \lambda \) is a coupling constant, \( g(t) \) a time dependent function, \( a^\dagger \) the operator that induce transitions from the ground state to the excited state of the PS, \( Q_S \) an operator acting on the system \( S \). The postselection is performed at \( t = t_f \) and consists in finding the PS in the excited state.

For simplicity we shall work to second order in \( \lambda \) (although in principle the interaction of PS with \( S \) need not be weak). In the interaction representation, the wave function of the combined system \( S + WD + PS \) is to order \( \epsilon \) and to order \( \lambda^2 \)

\[ \mathcal{T} e^{-i \int dt (H_{S-WD(t)} + H_{S-PS(t)})} |\psi_i⟩|WD⟩|0_{PS}⟩ = \]

\[ \left[ 1 - i \int dt (H_{S-WD(t)} + H_{S-PS(t)}) \right] - \frac{1}{2} \int dt \int dt' T[H_{S-PS}(t)H_{S-PS}(t')] - \int dt \int dt' T[H_{S-PS}(t)H_{S-PS}(t')] \] \[ + \frac{1}{2} \int dt \int dt' \int dt'' T[H_{S-PS}(t)H_{S-PS}(t')H_{S-WD}(t'')] |\psi_i⟩|WD⟩|0_{PS}⟩ \] (204)

where \( |0_{PS}⟩ \) is the ground state of PS and \( \mathcal{T} \) is the time ordering operator. The probability of finding the PS in the excited state at \( t = t_f \) is, at order \( \lambda^2 \),

\[ P_e = \lambda^2 |\psi_i⟩ \int dt g(t)Q_S^\dagger \int dt' g(t')Q_S |\psi_i⟩ \] (205)

Upon imposing that the PS be in its excited state at \( t = t_f \) the resulting wave function is, to order \( \epsilon \) and \( \lambda^2 \),

\[ \left[ -i \int dt \lambda g(t)Q_S(t) \right] \int dt' T[\epsilon f(t)A_S(t)B_{WD}(t)\lambda g(t')Q_S(t')] |\psi_i⟩|WD⟩|0_{PS}⟩ \] (206)

Making a density matrix out of the state (206), tracing over the states of \( S \) and PS yields the reduced density matrix \( \rho_{WD} = P_e|Ψ_{WD}⟩⟨Ψ_{WD}| \) of WD, where, to order \( \epsilon \) \( |Ψ_{WD}⟩ \) is

\[ |Ψ_{WD}⟩ = \left[ 1 - i\epsilon \int_{t_i}^{t_f} dt f(\tilde{t})B_{WD}(\tilde{t})⟨A_S(\tilde{t})|_e⟩ |WD⟩ \right] \] (207)

and

\[ ⟨A_S(\tilde{t})⟩_e = \frac{⟨ψ_i| \int dt g(t)Q_S^\dagger(t) \int dt' g(t')T[A_S(\tilde{t})Q_S(t')] |ψ_i⟩}{⟨ψ_i| \int dt g(t)Q_S^\dagger(t) \int dt' g(t')Q_S(t') |ψ_i⟩} \] (208)
Note how the weak value of $A_S$ results from the quantum mechanical interference of the two terms in eq. (206).

There are several important cases when the time ordering in eq. (208) simplifies. If $g(t)$ is non-vanishing only after $t = \tilde{t}$, i.e. WD interacts with S before S interacts with PS, then $\langle A_S(\tilde{t}) \rangle_e$ takes a typical (for a weak value) asymmetric form

$$
\langle A_S(\tilde{t}) \rangle_e = \frac{\langle \psi_i | \int dt g(t) Q_S(t) \int dt' g(t') Q_S(t') A_S(\tilde{t}) | \psi_i \rangle}{\langle \psi_i | \int dt g(t) Q_S(t) \int dt' g(t') Q_S(t') | \psi_i \rangle}
$$

(209)

If in addition $g(t) = \delta(t - t_f)$ and $Q_S = \Pi^0_S$, eq. (197) is recovered using $(\Pi^0_S)^2 = \Pi^0_S$ and $t_f > \tilde{t}$. This is expected since in this case the post-selector has simply gotten correlated to the system in the subspace $\mathcal{H}_S^0$.

If on the other hand $g(t)$ is non-vanishing only before $t = \tilde{t}$, i.e. S interacts with PS before WD interacts with S, then the time ordering operator becomes trivial once more and eq. (208) takes the form

$$
\langle A_S(\tilde{t}) \rangle_e = \frac{\langle \psi_i | \int dt g(t) Q_S^\dagger(t) A_S(\tilde{t}) \int dt' g(t') Q_S(t') | \psi_i \rangle}{\langle \psi_i | \int dt g(t) Q_S^\dagger(t) \int dt' g(t') Q_S(t') | \psi_i \rangle}
$$

(210)

This is by construction the expectation value of $A_S$ if the PS has made a transition. It is necessarily real contrary to eq. (209).

Finally, the weak value of $A_S$ if the PS has not made a transition can also be computed. Once more the two cases discussed in eqs (209) and (210) are particularly simple: if $g(t)$ is non-vanishing only after $t = \tilde{t}$ one finds

$$
\langle A_S(\tilde{t}) \rangle_d = \frac{1}{1 - P_e} \left( \langle \psi_i | A_S | \psi_i \rangle - \lambda^2 \text{Re} \left[ \langle \psi_i | \int dt g(t) Q_S^\dagger(t) \int dt' g(t') Q_S(t') A_S(\tilde{t}) | \psi_i \rangle \right] \right)
$$

(211)

On the other hand if $g(t)$ is non-vanishing only before $t = \tilde{t}$ one finds

$$
\langle A_S(\tilde{t}) \rangle_d = \frac{1}{1 - P_e} \left( \langle \psi_i | A_S(\tilde{t}) | \psi_i \rangle - \frac{1}{2} \lambda^2 \text{Re} \left[ \langle \psi_i | A_S(\tilde{t}) \int dt \int dt' T g(t) Q_S^\dagger(t) g(t') Q_S(t') | \psi_i \rangle \right] \right)
$$

(212)

These are related to the mean value of $A_S$ and to eq. (208) through the unitary relation eq. (202): if $g(t)$ is non-vanishing only after $t = \tilde{t}$

$$
P_e \langle A_S(\tilde{t}) \rangle_e + (1 - P_e) \langle A_S(\tilde{t}) \rangle_d = \langle \psi_i | A_S(\tilde{t}) | \psi_i \rangle
$$

(213)

and if $g(t)$ is non-vanishing only before $t = \tilde{t}$

$$
P_e \langle A_S(\tilde{t}) \rangle_e + (1 - P_e) \langle A_S(\tilde{t}) \rangle_d = \langle \psi_i | e^{i \int dt H_{S-PS}} A_S(\tilde{t}) e^{-i \int dt H_{S-PS}} | \psi_i \rangle = \langle \psi_i | A_S(\tilde{t}) | \psi_i \rangle + \lambda^2 \langle \psi_i | \int dt g(t) Q_S^\dagger(t) A_S(\tilde{t}) \int dt' g(t') Q_S^\dagger(t') | \psi_i \rangle
$$

$$
- \lambda^2 \text{Re} \langle \psi_i | A_S(\tilde{t}) \int dt \int dt' T g(t) Q_S^\dagger(t) g(t') Q_S^\dagger(t') | \psi_i \rangle
$$

(214)
Thus \( \langle V \rangle (23) \) annihilates Minkowski vacuum. Therefore by virtue of 1., the Rindler energy in the following properties.

\section*{Acknowledgements}
The authors would like to thank R. Brout, F. Englert, S. Popescu and Ph. Spindel for very helpful discussions.

\section*{6 Appendix: The \( \mathcal{D} \) Term}

We recall the this term arises from the following decomposition of the second \( g^2 \) Born term in \( |\psi_-(t = +\infty)\rangle = e^{-i \int dt dx H_{int}} |0_M\rangle - > \):

\[
-g^2 m^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\tau} d\tau' f(\tau) e^{-im\tau} \phi(\tau) f^*(\tau') e^{+im\tau'} \phi(\tau')|0_M\rangle - > = -g^2 m^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' f(\tau) f^*(\tau') e^{-im(\tau-\tau')} \phi(\tau) \phi(\tau') [1 + \epsilon(\tau - \tau')] |0_M\rangle - > = -g^2 m^2 \frac{1}{2} \phi^*_m \phi_m |0_M\rangle - > - g^2 m^2 \mathcal{D} |0_M\rangle - > \tag{215}
\]

where

\[
\mathcal{D} = \frac{1}{2} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \int_{-\infty}^{\infty} d\tau \ f(\tau_2) f^*(\tau_1) \epsilon(\tau_2 - \tau_1) e^{-im(\tau_2 - \tau_1)} \phi(\tau_2) \phi(\tau_1) \tag{216}
\]

and where \( \epsilon(\tau_2 - \tau_1) = \theta(\tau_2 - \tau_1) - \theta(\tau_1 - \tau_2) \).

To explicitize the role of the \( \mathcal{D} \) term, it is appropriate to compute the energy density carried by it when the initial state is \( |0_M\rangle - > \). One finds, to order \( g^2 \),

\[
\langle T_{VV}(V) \rangle_{\mathcal{D}} = -g^2 m^2 \text{Re} |<0_M | T_{VV}(V) \mathcal{D} |0_M\rangle |
\]

\[
= -g^2 m^2 | <0_M | [T_{VV}(V), \mathcal{D}]_+ |0_M\rangle |
\]

\[
= -g^2 m^2 \int_{-\infty}^{\infty} d\tau_2 \int_{-\infty}^{\infty} d\tau_1 f(\tau_2) f^*(\tau_1) \epsilon(\tau_2 - \tau_1) e^{-im(\tau_2 - \tau_1)}
\]

\[
<0_M | [T_{VV}(V), \phi(\tau_2) \phi(\tau_1)]_+ |0_M\rangle \tag{217}
\]

where we have used the antihermitian property of \( \mathcal{D} : \mathcal{D}^\dagger = -\mathcal{D} \). \( \langle T_{VV}(V) \rangle_{\mathcal{D}} \) enjoys the following properties.

1. Being a commutator, \( \langle T_{VV}(V) \rangle_{\mathcal{D}} \) is causal (see eq. (13)), and vanishes in the left quadrant \( V < 0 \) contrary to \( \langle T_{VV}(V) \rangle_e \) and \( \langle T_{VV}(V) \rangle_g \).

2. \( \langle T_{VV}(V) \rangle_{\mathcal{D}} \) carries no Minkowski energy since the hamiltonian \( H_M \), eq. (8), annihilates Minkowski vacuum.

3. \( \langle T_{VV}(V) \rangle_{\mathcal{D}} \) carries no Rindler energy since \( H_R \) (the boost generator given in eq. (23)) annihilates Minkowski vacuum. Therefore by virtue of 1., the Rindler energy in the right quadrant (\( V > 0 \)) vanishes

\[
\int_{-\infty}^{+\infty} dv \langle T_{hv}(v) \rangle_{\mathcal{D}} = 0 \tag{218}
\]

Thus \( \langle T_{hv}(v) \rangle_{\mathcal{D}} \) is, at most, an energy density repartition.
4. When \( f(\tau) = 1 \) for all \( \tau \), \( \langle T_{vv}(v) \rangle_D \) vanishes identically. To prove this one evaluates the commutator in eq. (217) and one finds

\[
\langle T_{vv}(v) \rangle_D = 2g^2m^2 \int_{-\infty}^{+\infty} d\tau_2 \, \epsilon(\tau_2 - v) \Re \left[ f(\tau_2)f^*(v)e^{-im(\tau_2-v)}\langle 0_M|\phi(\tau_2)i\partial_v\phi(v)|0_M \rangle \right]
\]

(219)

where we have used the commutation relation

\[
[T_{vv}(v), \phi(\tau_2)\phi(\tau_1)] = -2i\delta(\tau_1 - v)\phi(\tau_2)\partial_v\phi(v) - 2i\delta(\tau_2 - v)\partial_v\phi(v)\phi(\tau_1)
\]

(220)

and the antisymmetric character of \( \epsilon(\tau_2-\tau_1) \). Since the expectation value \( \langle 0_M|\phi(\tau_2)\phi(v)|0_M \rangle \) is evaluated along the accelerated trajectory eq. (26), it is a function of \( \tau_2-v \) only. Therefore the integrand of eq. (219) is an odd function of \( \tau_2-v \) and the integral vanishes. Hence \( \langle T_{vv}(v) \rangle_D \) is an energy repartition which is concerned only with the transients induced by the switch on and off effects.

5. When \( f(\tau) \) is a slowly varying function with respect to both \( 1/m \) and \( 1/a \) (c.f. the discussion associated with eq. (23)), \( \langle T_{vv}(v) \rangle_D \) is smaller than the contribution of \( \Re[\langle T_{vv}(v)\phi_m\phi^\dagger_m \rangle] \) by a factor \( 1/aT \) except near the edges of the interaction period where \( f(\tau) \) almost vanishes. This can be seen by developing \( f(\tau_2) \) given in eq. (219) in a series around \( \tau_2 = v \) and evaluating the magnitude of the first non vanishing term, i.e. one treats the variations of the switch off function \( f(\tau) \) as an adiabatic effect. One finds that indeed the \( D \) is smaller than \( \Re[\langle T_{vv}(v)\phi_m\phi^\dagger_m \rangle] \) except when \( \tau > aT^2 \).

References

[1] S.W. Hawking, Nature 248, 30 (1974)
Commun. math. Phys. 43, 199 (1975).

[2] W.G. Unruh, Phys. Rev. D14 (1976) 287.

[3] N.D. Birrel and P.C.W. Davies, Quantum Fields in Curved Space, Cambridge University Press (1982).

[4] W. G. Unruh and R. M. Wald, Phys. Rev. D 29 (1984) 1047

[5] P. G. Grove, Class. Quantum Grav. 3 (1986) 801

[6] D. Raine, D. Sciama and P. Grove, Proc. R. Soc. A435 (1991)

[7] W. G. Unruh, Phys. Rev. D 46 (1992) 3271

[8] F. Hinterleitner, Ann. Phys. (N.Y.) 226 (1993) 165

[9] S. Massar, R. Parentani and R. Brout, Class. Quantum Grav. 10 (1993) 385

[10] J. Audreutsch and R. Müller, Phys. Rev. D 49 (1994) 4056; Phys. Rev D 49 (1994) 6566; Phys. Rev A 50 (1994) 1755
[11] R. Brout, S. Massar, S. Popescu, R. Parentani and Ph. Spindel, “Quantum Source of the Back Reaction on a Classical Field’ Preprint ULB-TH 93/16, UMH-MG 93/03, (1993).
[12] Y. Aharonov, D. Albert, A. Casher and L. Vaidman, Phys. Lett. A 124 199 (1987), Y. Aharonov and L. Vaidman, Phys. Rev. A 41 11 (1990).
[13] J. von Newmann, Mathematical Foundations of Quantum Mechanics, Princeton University Press, Princeton (1955)
[14] J.M. Bardeen, Phys. Rev. Letters 46 (1981) 382.
[15] R. Parentani and T. Piran, Phys. Rev. Lett. 73 (1994) 2805
[16] S. Massar, The semi classical back reaction to black hole evaporation preprint ULB-TH 94/19, gr-qc/9411039
[17] G. ’t Hooft, Nucl. Phys. B 256 (1985) 727
[18] T. Jacobson, Phys. Rev. D44 (1991) 1731, D48 (1993) 728.
[19] F. Englert, S. Massar and R. Parentani, Class. Quantum. Grav. 11 (1994) 2919
[20] L. Susskind, Phys. Rev. D 49 (1994) 6606
[21] L. Susskind, “Some speculations about Black Hole Entropy in String Theory” Preprint RU-93-44, hep-th/9309145 (1993).
[22] L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. D 48 (1993) 3743
[23] C. R. Stephens, G. ’t Hooft and B. F. Whiting, Class. Quant. Grav. 11 (1994) 621
[24] G. ’t Hooft, Horizon Operator Approach to Black Hole Quantization preprint THU-94/02 (1994) gr-qc/9402037
[25] F. Englert, “Operator weak values and black hole complementarity”, preprint ULB-TH 03/95, to be published in the annals of the Oskar Klein Centenary Symposium
[26] K. Schoutens, H. Verlinde, E. Verlinde, “Black Hole Evaporation and Quantum Gravity” Preprint CERN-TH.7142/94, PUPT-1441, (1994), hep-th/ 9401081.
[27] P. Kraus and F. Wilczek, Nucl. Phys. B433 (1995) 403
[28] D. G. Boulware, Annals of Physics 124 (1980) 169
[29] R. Parentani, Class. Quantum Grav. 10 (1993) 1409.
[30] R. Parentani and R. Brout, Int. J. Mod. Phys. D1, 169 (1992).
[31] R. Brout, S. Massar, R. Parentani and Ph. Spindel, *A Primer for Black Hole Quantum Physics* (1995) ULB-TH 95/02, UMH-MG 95/01 and LPTENS 95/03 submitted to *Phys. Rep.*

[32] R. Parentani, *The Recoils of the Accelerated Atom and the Decoherence of its Fluxes*, preprint (1995) LPTHENS 95/02

[33] R. Wald, *Commun. Math. Phys.* **45** (1975) 9.

[34] P. G. Grove in *The Origin of Structure in the Universe* edited by E. Gunzig and P. Nardone, Kluwer Academic Publishers (Netherlands) 1993

[35] R. Wald, *Commun. Math. Phys.*, **54**,1 (1977), *Phys. Rev. D*, **17**, 1477 (1978),

[36] S. Massar, R. Parentani and R. Brout, *Class. Quantum Grav.* **10** (1993) 2431

[37] S. Massar, *Int. J. Mod. Phys. D* **3** (1994) 237
Figure Captions

Figure 1.
The Minkowski coordinates $t, z$ and $U, V$. The left (L) and right (R) Rindler quadrants. The Rindler coordinates $\tau, \rho$ in R and the trajectory of a uniformly accelerated atom.

Figure 2.
The absolute value of the switch function $f(\tau)$ given in eq. (64) for $m = 2a$ and $T = 3a^{-1}$. $\tau$ is given in units of $a^{-1}$.

Figure 3.
The mean Rindler energy density $\langle T_{vv}(v) \rangle_{therm.}$ emitted to order $g^2$ at thermal equilibrium is represented for $m = 2a$ and $T = 3a^{-1}$. $v$ is given in units of $a^{-1}$ and $T_{vv}$ in arbitrary units since the flux is proportional to the coupling $g$. One sees the vanishing of the flux in the steady regime and the positivity of the transients. In the Minkowski description they are enenhanced by the jacobian $dV/dv$ to make the total Minkowski energy emitted positive.

Figure 4.
The conditional value $\langle T_{vv} \rangle_e$ if the two level atom is initially in its ground state and ends up in its excited state. The parameters are the same as in Figure 2 and 3: $m = 2a$ and $T = 3a^{-1}$. The $v$ axis is given in units of $a^{-1}$ and $T_{vv}$ in units of $a^2$. For $U < U_a, V > 0$, $\langle T_{vv} \rangle_e$ is complex and oscillates. The real part has a central positive bump which encodes that their is a rindleron carrying positive energy which will induce the transition of the atom. For $V < 0$, $\langle T_{vv} \rangle_e$ is real and positive. It describes the partner of the rindleron which will be absorbed by the atom. The oscillations of $\langle T_{vv}(U < U_a, V > 0) \rangle_e$ are such that the total Minkowski energy of the vacuum fluctuation vanishes. For $U > U_a, V > 0$, $\langle T_{vv} \rangle_e$ is positive and of order $N_m$. In order to represent it we have had to change the vertical scale.

Figure 5.
A schematic picture of the energy fluxes $\langle T_{vv} \rangle_e$. We have represented in dark grey the regions where $\langle T_{vv} \rangle_e$ is $O(N_m + 1)$ and in light grey the regions where it is $O(N_m)$.

Figure 6.
The local description of a vacuum fluctuation giving rise to a Hawking photon emitted around $u = u_0$ is represented in a Penrose diagram. The shaded areas correspond to the regions where $T_{\tilde{u}\tilde{u}}(\tilde{u})$ is non vanishing. $v = v_S$ is the trajectory of the collapsing spherically symmetric shell of massless matter.

Figure 7.
The same as in figure 6 drawn in Eddington-Finkelstein coordinates.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502024v1
This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502024v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502024v1
This figure "fig2-2.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502024v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502024v1
This figure "fig2-3.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9502024v1