Higher Order Calculations in Renormalization Group Approach to Matrix Models

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Abstract

We study higher order approximations in the renormalization group approach to matrix models. We use constraint equations on the free energy resulting from a freedom of field redefinitions and obtain the effective beta function for a single coupling constant to the fifth order. The fixed point and the string susceptibility exponent are shown to approach the values obtained in the exact solution as the order of approximations becomes higher.

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Matrix models [1] give a discrete version of two-dimensional quantum gravity. By studying the double scaling limit of the matrix models [2], one can obtain the exact solutions of two-dimensional quantum gravity coupled to minimal conformal field theories with central charges $c \leq 1$. These results are consistent with the results of a continuum version of two-dimensional quantum gravity [3]. For the central charge $c > 1$, we can construct matrix models [4], but cannot obtain exact solutions. Moreover, for those cases we do not know whether there is a continuum theory or not. Brézin and Zinn-Justin [5] have proposed a renormalization group approach to the matrix models in order to understand $c > 1$ cases. One can check the validity of this approach by applying it to $c \leq 1$ theories, since the exact solutions are known for these cases. In ref. [5], they obtained reasonable results for the first order approximation in the case of $c < 1$. The renormalization group approach has been further studied in several papers [6-10].

The purpose of this paper is to study higher order approximations in the renormalization group approach to the matrix models. We consider a one-matrix model corresponding to the case $c = 0$. It was noticed in ref. [9] that one has to use a freedom of field redefinitions to obtain higher order results. The exact beta functions of the vector model were obtained in this way [9]. Although it is difficult to obtain exact results for the matrix models using the renormalization group approach, we can compute beta functions perturbatively in a coupling constant. We derive constraint equations on the free energy resulting from the freedom of field redefinitions. Using these equations we obtain the effective beta function for a single coupling constant to the fifth order. The fixed point of the beta function and the string susceptibility exponent are shown to approach the values obtained in the exact solution of the model as the order of approximations becomes higher.

We shall study the following one-matrix model:

$$Z_N(g) = \int d^{N^2} \phi_N \exp \left[ -\sum_{k \geq 1}^{\infty} \sum_{l \geq k}^{\infty} \frac{g_{k,l}}{k!l!} tr\phi_N^k tr\phi_N^l - N \sum_{m \geq 1}^{\infty} \frac{g_{0,m}}{m} tr\phi_N^m \right],$$

where $\phi_N$ is an $N \times N$ hermitian matrix. The terms $g_{0,m} tr\phi_N^m$ ($m \geq 1$) generate the usual discretizations of random surfaces and the terms $g_{k,l} tr\phi_N^k tr\phi_N^l$ ($k \geq 1, l \geq k$) generate two random surfaces touching each other at one point [11,12]. This model was proposed in ref. [11] to include effects of higher order intrinsic curvature terms. Following ref. [5], by integrating out one row and one column of $(N+1) \times (N+1)$
matrix \( \phi_{N+1} \), we will see that the partition function \( Z_N(g) \) fulfills the equation

\[
Z_{N+1}(g) = \Gamma_N(g) Z_N(g + \delta g),
\]

where \( \Gamma_N(g) \) is a factor which is a function of \( N \) and \( g \). From this relation we have a differential equation

\[
\frac{\partial}{\partial N} \ln Z_N(g) = \ln \Gamma_N(g) + \sum_{k \geq 1} \sum_{l \geq k} \delta g_{k,l} \frac{\partial}{\partial g_{k,l}} \ln Z_N(g)
+ \sum_{m \geq 1} \delta g_{0,m} \frac{\partial}{\partial g_{0,m}} \ln Z_N(g).
\]

(3)

Now, it is convenient to introduce rescaled coupling constants \[9\],

\[
\tilde{g}_{k,l} = \frac{g_{k,l}}{g_{0,2}^{k+l}}, \quad (k \geq 1, \ l \geq k),
\]

(4)

and

\[
\tilde{g}_{0,m} = \frac{g_{0,m}}{g_{0,2}^m}, \quad (m \geq 1).
\]

(5)

By this rescaling, the quadratic term in the potential becomes the standard form \( \frac{1}{2} \text{tr} \phi_N^2 \), i.e., \( \tilde{g}_{0,2} = 1 \). It is clear that we have a relation

\[
Z_N(g) = (g_{0,2})^{-\frac{g_{0,2}^2}{2}} Z_N(\tilde{g}),
\]

(6)

where \( Z_N(g) \) and \( Z_N(\tilde{g}) \) are shorthand notations of \( Z_N(g_{k,l} (k \geq 1, l \geq k), g_{0,m} (m \geq 1)) \) and \( Z_N(\tilde{g}_{k,l} (k \geq 1, l \geq k), \tilde{g}_{0,2} = 1, \tilde{g}_{0,m} (m = 1, 3, \cdots)) \) respectively. We denote the partial derivatives with respect to \( \tilde{g}_{k,l} (k \geq 1, l \geq k) \), \( \tilde{g}_{0,m} (m = 1, 3, \cdots) \) and \( g_{0,2} \) by \( \mid \tilde{g} \), and those with \( g_{k,l} (k \geq 1, l \geq k) \) and \( g_{0,m} (m \geq 1) \) by \( \mid g \). They are related as

\[
\frac{\partial}{\partial g_{0,2}} \mid_g = \frac{\partial}{\partial \tilde{g}_{0,2}} \mid_{\tilde{g}} - \sum_{m=1,3,\ldots} \frac{m \tilde{g}_{0,m}}{2} \frac{\partial}{\partial \tilde{g}_{0,m}} \mid_{\tilde{g}} - \sum_{k \geq 1} \sum_{l \geq k} \frac{k+l}{2} \frac{\partial}{\partial \tilde{g}_{k,l}} \mid_{\tilde{g}},
\]

(7)

\[
\frac{\partial}{\partial g_{k,l}} \mid_g = \frac{1}{g_{0,2}^2} \frac{\partial}{\partial \tilde{g}_{k,l}} \mid_{\tilde{g}} \quad (k \geq 1, l \geq k),
\]

(8)
\[
\frac{\partial}{\partial g_{0,m}} \bigg|_g = \frac{1}{2} \frac{\partial}{\partial \bar{g}_{0,2}} \bigg|_{\bar{g}} (m = 1, 3, \cdots). \tag{9}
\]

If we define the free energy

\[
F(N, \bar{g}) = -\frac{1}{N^2} \ln Z_N(\bar{g}), \tag{10}
\]

the renormalization group equation (3) becomes

\[
\left[ N \frac{\partial}{\partial N} - \sum_{k \geq 1} \sum_{l \geq k} \tilde{\beta}_{k,l}(\bar{g}) \frac{\partial}{\partial \bar{g}_{k,l}} \bigg|_{\bar{g}} - \sum_{m=1,3,\cdots} \tilde{\beta}_{0,m}(\bar{g}) \frac{\partial}{\partial \bar{g}_{0,m}} \bigg|_{\bar{g}} + 2 \right] F(N, \bar{g}) = -\ln g_{0,2} - \frac{1}{N} \ln \Gamma_N + \frac{\delta g_{0,2} N}{2g_{0,2}}, \tag{11}
\]

where the \( \tilde{\beta} \) functions are

\[
\tilde{\beta}_{k,l}(\bar{g}) = N \left( \frac{\delta g_{k,l}}{g_{0,2}} - \frac{k + l}{2} \frac{\delta g_{0,2} \bar{g}_{k,l}}{g_{0,2}} \right) \quad (k \geq 1, \ l \geq k), \tag{12}
\]

\[
\tilde{\beta}_{0,m}(\bar{g}) = N \left( \frac{\delta g_{0,m}}{g_{0,2}^m} - \frac{m}{2} \frac{\delta g_{0,2} \bar{g}_{0,m}}{g_{0,2}} \right) \quad (m = 1, 3, \cdots). \tag{13}
\]

Now we shall consider the simplest case of a cubic potential

\[
g_{0,2} \neq 0, \ g_{0,3} \neq 0, \ g_{0,m} = 0 \ (m = 1, 4, \cdots), \ g_{k,l} = 0 \ (k \geq 1, l \geq k). \tag{14}
\]

The shifts \( \delta g \) in eq. (2) can be obtained exactly as follows. The partition function \( Z_{N+1}(g) \) is defined by

\[
Z_{N+1}(g) = \int d^{(N+1)^2} \phi_{N+1} \exp \left[ -(N + 1) \left( \frac{g_{0,2}}{2} \text{tr} \phi_{N+1}^2 + \frac{g_{0,3}}{3} \text{tr} \phi_{N+1}^3 \right) \right]. \tag{15}
\]

The matrix \( \phi_{N+1} \) is parametrized in terms of a submatrix \( \phi_N \), a complex vector \( v_N \) and a real number \( \alpha \):

\[
\phi_{N+1} = \begin{pmatrix} \phi_N & v_N \\ v_N^* & \alpha \end{pmatrix}. \tag{16}
\]
Since all the terms involving $\alpha$ are of relative order $1/N$ \cite{5}, we can set $\alpha = 0$. Hence we have

$$Z_{N+1}(g) = \int d^N\phi_N d^Nv_N d^Nv_N^*$$

$$\times \exp \left[ -(N+1) \left( \frac{g_{0,2}}{2} tr\phi_N^2 + \frac{g_{0,3}}{3} tr\phi_N^3 + g_{0,2}v_N^*v_N + g_{0,3}v_N^*\phi_N v_N \right) \right] (17)$$

Since the $v_N$ and $v_N^*$ integrals are Gaussian, we can evaluate them exactly. After rescaling of $\phi_N$ we obtain

$$Z_{N+1}(g) = \{-(N+1)g_{0,2}\}^N \int d^N\phi_N \exp \left[ -N \left( \frac{g_{0,2}}{2} tr\phi_N^2 + \frac{g_{0,3}}{3} tr\phi_N^3 \right) \right]$$

$$\times \exp \left[ -\sum_{k \geq 1} \sum_{l \geq k} \frac{\delta g_{k,l}}{kl} tr\phi_N^k tr\phi_N^l - N \sum_{m \geq 1} \frac{\delta g_{0,m}}{m} tr\phi_N^m \right], \quad (18)$$

where

$$\delta g_{k,l} = 0 \quad (k \geq 1, \ l \geq k), \quad \delta g_{0,2} = \frac{1}{N} \left( g_{0,2} - \left( \frac{g_{0,3}}{g_{0,2}} \right)^2 \right),$$

$$\delta g_{0,3} = \frac{1}{N} \left( g_{0,3} + \left( \frac{g_{0,3}}{g_{0,2}} \right)^3 \right), \quad \delta g_{0,m} = -\frac{1}{N} \left( -\frac{g_{0,3}}{g_{0,2}} \right)^m \quad (m = 1, 4, \cdots). \quad (19)$$

The factor in eq. (18) is given by $\Gamma_N(g) = \{-(N+1)g_{0,2}\}^N$. We can also consider cases of other potentials, but then the $v_N$ and $v_N^*$ integrals cannot be evaluated exactly. From eqs. (11) and (19), we have the following renormalization group equation for the cubic potential

$$\left[ N \frac{\partial}{\partial N} - \sum_{m=1,3,\cdots} \tilde{\beta}_{0,m} \frac{\partial}{\partial \tilde{g}_{0,m}} \right] F(N, \tilde{g}) = -\ln g_{0,2} - \frac{1}{N} \ln \Gamma_N + \frac{1}{2} - \frac{1}{2} \tilde{g}_{0,3}^2. \quad (20)$$

where $\tilde{\beta}$ functions are

$$\tilde{\beta}_{0,3}(\tilde{g}_{0,3}) = \frac{1}{2} \tilde{g}_{0,3} - \frac{5}{2} \tilde{g}_{0,3}^3, \quad (21)$$

$$\tilde{\beta}_{0,m}(\tilde{g}_{0,3}) = (-\tilde{g}_{0,3})^m \quad (m = 1, 4, \cdots). \quad (22)$$

We find that even if one starts from the cubic potential, after integrating one low and one column one obtains infinite number of other higher order terms $tr\phi_N^m$. To do higher order calculations, we must take into account these higher order terms.
However, one may be able to put the potential back to a cubic form by appropriate redefinitions of $\phi_N$. Such a freedom of field redefinitions was used in ref. [9] to obtain the exact beta functions in the vector model.

The freedom of field redefinitions can be expressed as constraint equations on the partition function. These constraints were discussed in refs. [13, 14] and were shown to have a Virasoro like structure. In particular they were discussed before taking a continuum limit in ref. [13]. They can be obtained by a change of integration variables

$$\phi_N \rightarrow \phi_N + \delta\phi_N$$  \hspace{1cm} (23)

in the partition function. The transformation considered in ref. [13] are

$$\delta\phi_N = \epsilon\phi_N^{n+1} \hspace{1cm} (n \geq -1),$$  \hspace{1cm} (24)

where $\epsilon$ is an infinitesimal parameter. We call the constraints corresponding to the change of variables (24) as $L_n \hspace{1cm} (n \geq -1)$. In the case of our model the constraints for the free energy $F(N, \tilde{g})$ are

$L_{-1}$ condition:

$$\left. \frac{\partial}{\partial \tilde{g}_{0,1}} \right|_{\tilde{g}} F(N, \tilde{g}) + \tilde{g}_{0,3} = 0,$$  \hspace{1cm} (25)

$L_0$ condition:

$$\left. \frac{\partial}{\partial \tilde{g}_{0,2}} \right|_{\tilde{g}} F(N, \tilde{g}) = 0,$$  \hspace{1cm} (26)

$L_1$ condition:

$$\left. \left[ 3 \frac{\partial}{\partial \tilde{g}_{0,3}} + 4\tilde{g}_{0,3} \frac{\partial}{\partial \tilde{g}_{0,4}} \right] - 2 \frac{\partial}{\partial \tilde{g}_{0,1}} \right|_{\tilde{g}} F(N, \tilde{g}) = 0,$$  \hspace{1cm} (27)

$L_2$ condition:

$$\left. \left[ 4 \frac{\partial}{\partial \tilde{g}_{0,4}} + 5\tilde{g}_{0,3} \frac{\partial}{\partial \tilde{g}_{0,5}} - \frac{\partial}{\partial \tilde{g}_{1,1}} + 6\tilde{g}_{0,3} \frac{\partial}{\partial \tilde{g}_{0,3}} \right] F(N, \tilde{g}) - 2 = 0,$$  \hspace{1cm} (28)
$L_n$ ($n \geq 3$) conditions:

$$\left[ (n + 2) \left. \frac{\partial}{\partial g_{0,n+2}} \right|_{\tilde{g}} + (n + 3) \tilde{g}_{0,3} \left. \frac{\partial}{\partial g_{0,n+3}} \right|_{\tilde{g}} \right] \left( \tilde{g} + (n + 2) \tilde{g}_{0,n+2} \right) - \sum_{a+b=n} \sum_{a \geq 1, b \geq 1} ab \left. \frac{\partial}{\partial g_{a,b}} \right|_{\tilde{g}} - 2n \left. \frac{\partial}{\partial g_{0,n}} \right|_{\tilde{g}} \right] F(N, \tilde{g}) = 0. \quad (29)$$

Here we have set all the coupling constants to be zero except $g_{0,2}$ and $g_{0,3}$ as in eq. (14) after the differentiations.

To do higher order calculations we will need constraints derived from more general transformations of $\phi_N$. Let us consider changes of variables

$$\delta \phi_N = \frac{c}{N} \phi_N^{n+1} \text{tr} \phi_N^p \quad (n = -1, 0, 1, p \geq 1). \quad (30)$$

The constraints $L_{n,p}$ ($n = -1, 0, 1, p \geq 1$) corresponding to these changes of variables are found to be

$L_{-1,p}$ ($p \geq 1$) conditions:

$$\left[ \left. \frac{\partial}{\partial g_{1,p}} \right|_{\tilde{g}} + 2\tilde{g}_{0,3} \left. \frac{\partial}{\partial g_{2,p}} \right|_{\tilde{g}} \right] F(N, \tilde{g}) = 0, \quad (31)$$

$L_{0,p}$ ($p = 1, 3, \ldots$) conditions:

$$\left[ 2 \left. \frac{\partial}{\partial g_{2,p}} \right|_{\tilde{g}} + 3\tilde{g}_{0,3} \left. \frac{\partial}{\partial g_{3,p}} \right|_{\tilde{g}} - \left. \frac{\partial}{\partial g_{0,p}} \right|_{\tilde{g}} \right] F(N, \tilde{g}) = 0, \quad (32)$$

$L_{0,2}$ condition:

$$\left[ 2 \left. \frac{\partial}{\partial g_{2,2}} \right|_{\tilde{g}} + 3\tilde{g}_{0,3} \left. \frac{\partial}{\partial g_{2,3}} \right|_{\tilde{g}} + \frac{3}{2} \tilde{g}_{0,3} \left. \frac{\partial}{\partial g_{0,3}} \right|_{\tilde{g}} \right] F(N, \tilde{g}) - \frac{1}{2} = 0, \quad (33)$$

$L_{1,p}$ ($p \geq 1$) conditions:

$$\left[ 3 \left. \frac{\partial}{\partial g_{3,p}} \right|_{\tilde{g}} + 4\tilde{g}_{0,3} \left. \frac{\partial}{\partial g_{4,p}} \right|_{\tilde{g}} - 2 \left. \frac{\partial}{\partial g_{1,p}} \right|_{\tilde{g}} \right] F(N, \tilde{g}) = 0. \quad (34)$$
We could consider more general transformations of the form (30) with \( n \geq 2, \ p \geq 1 \) or
\[
\delta \phi_N = \epsilon \phi_N^{n+1} \frac{\partial \phi_0^1}{\partial g} \frac{\partial \phi_0^2}{\partial g} \frac{\partial \phi_0^3}{\partial g} \cdots \ (n \geq -1, \ p_1, p_2, p_3, \cdots \geq 1).
\] (35)
However, these transformations induce terms \( \text{tr} \phi^k \text{tr} \phi^k \text{tr} \phi^k \cdots \ (1 \leq k_1 \leq k_2 \leq k_3 \leq \cdots) \) in the potential. So we have to begin with a more general potential than (31), which includes terms of the form \( \text{tr} \phi^k \text{tr} \phi^k \text{tr} \phi^k \cdots \). To the order we will consider in this paper we do not need to use such general transformations.

Let us discuss implications of the above constraint equations. The \( L_{-1} \) condition (23) shows that \( \frac{\partial}{\partial g_{0,1}} |_{\tilde{g}} F \) can be rewritten by \( \frac{\partial}{\partial g_{0,1}} |_{\tilde{g}} F \). This means that the term \( \text{tr} \phi_N \) in the potential can be eliminated by an appropriate field redefinition. The meaning of the \( L_0 \) condition (23) is clear from the definition of the free energy \( F(N, \tilde{g}) \). The \( L_1 \) condition (27) shows that \( \frac{\partial}{\partial g_{0,4}} |_{\tilde{g}} F \) is also written by \( \frac{\partial}{\partial g_{0,1}} |_{\tilde{g}} F \). The \( L_2 \) condition (28), however, shows that we must introduce a new nonlinear term \( \frac{\partial}{\partial g_{0,4}} |_{\tilde{g}} F \) to rewrite \( \frac{\partial}{\partial g_{0,4}} |_{\tilde{g}} F \). Similarly, from the conditions (23), \( \frac{\partial}{\partial g_{0,n}} |_{\tilde{g}} F \) (\( n \geq 6 \)) are not given only by \( \frac{\partial}{\partial g_{0,1}} |_{\tilde{g}} F \) but also by nonlinear terms \( \frac{\partial}{\partial g_{a,b}} |_{\tilde{g}} F \) (\( a+b = n-3 \)). By using \( L_{n,p} \) conditions (31)-(34), these nonlinear terms can be rewritten by \( \frac{\partial}{\partial g_{0,3}} |_{\tilde{g}} F \) and terms higher order in \( \tilde{g}_{0,3} \).

We can now calculate the fixed point \( \tilde{g}_{0,3} \) and the string susceptibility exponent \( \gamma_1 \) using the above results. We assume that \( \tilde{g}_{0,3} \) is small and use a perturbation theory in \( \tilde{g}_{0,3} \). This assumption is justified if a value of the fixed point obtained in this way is small.

We begin with the first order calculation, which taking into account up to and including terms of order \( \tilde{g}_{0,3}^2 \) in the renormalization group equation (20). Since \( \tilde{\beta}_{0,m} \) (\( m \geq 4 \)) are of order \( \tilde{g}_{0,3}^m \), we can ignore them in the renormalization group equation. Using the \( L_{-1} \) condition, we can rewrite \( \tilde{\beta}_{0,1} \) term as the same form as \( \tilde{\beta}_{0,3} \) term. Thus we obtain an effective renormalization group equation with a single beta function \( \tilde{\beta}_{0,3}^{\text{eff}(1)} \)
\[
\left[ N \frac{\partial}{\partial N} - \tilde{\beta}_{0,3}^{\text{eff}(1)} (\tilde{g}_{0,3}) \right] + 2 \right] F(N, \tilde{g}) = - \ln g_{0,2} - \frac{1}{N} \ln \Gamma_N + \frac{1}{2} - \frac{3}{2} \tilde{g}_{0,3}^3, \quad (36)
\]
\[
\tilde{\beta}_{0,3}^{\text{eff}(1)} (\tilde{g}_{0,3}) = - \frac{1}{2} \tilde{g}_{0,3} + \frac{11}{2} \tilde{g}_{0,3}^3. \quad (37)
\]
The fixed point $\tilde{g}^*_{0,3}$ is determined by $\tilde{\beta}_{0,3}^{\text{eff}}(\tilde{g}^*_{0,3}) = 0$ and the string susceptibility exponent is given by $\gamma_1 = \frac{2}{\tilde{\beta}_{0,3}^{\text{eff}}(\tilde{g}^*_{0,3})}$ as in ref. [5]. From eq. (37), we find the fixed point

$$\tilde{g}^{*(1)}_{0,3} = \sqrt{\frac{1}{11}} = 0.301511 \ldots ,$$

(38)

and the string susceptibility exponent

$$\gamma^{(1)}_1 = 2.$$  

(39)

On the other hand, from the exact solution [2] we know that the double scaling limit is achieved near the critical value

$$\tilde{g}^{(\text{exact})}_{0,3} = \sqrt{\frac{1}{12\sqrt{3}}} = 0.2193456 \ldots ,$$

(40)

with the string susceptibility exponent

$$\gamma^{(\text{exact})}_1 = 2.5.$$  

(41)

This first order calculation is essentially equivalent to the first order calculation in ref. [5] for the matrix model with quartic potential.

Next, for the second order approximation we take into account up to and including terms of order $\tilde{g}^4_{0,3}$. Ignoring $\tilde{\beta}_{0,m}$ ($m \geq 5$) and using the $L_{-1}$ and $L_1$ conditions we obtain the effective renormalization group equation

$$\left[ N \frac{\partial}{\partial N} - \tilde{\beta}_{0,3}^{\text{eff}(2)} \frac{\partial}{\partial \tilde{g}_{0,3}} + 2 \right] F(N, \tilde{g}) = -\ln g_{0,2} - \frac{1}{N} \ln \Gamma_N$$

$$+ \frac{1}{2} - \frac{3}{2} g_{0,3}^3 + \frac{1}{2} g_{0,3}^4 ,$$

(42)

$$\tilde{\beta}_{0,3}^{\text{eff}(3)}(\tilde{g}_{0,3}) = -\frac{1}{2} \tilde{g}_{0,3} + \frac{25}{4} \tilde{g}_{0,3}^3 .$$

(43)

Here we have ignored a term of order $\tilde{g}^5_{0,3}$. From eq.(38), we find the fixed point

$$\tilde{g}^{*(2)}_{0,3} = 0.282842 \ldots ,$$

(44)
and the exponent
\[ \gamma_1^{(2)} = 2. \] (45)

At the third order approximation we take into account up to and including terms of order \( \tilde{g}_{0,3}^5 \) in the renormalization group equation \((20)\). By using the \( L_{-1}, L_1, L_2, L_{-1,1} \) and \( L_{-1,2} \) conditions we have the effective renormalization group equation

\[
\left[ N \frac{\partial}{\partial N} - \tilde{\beta}_{0,3}^{\text{eff}(3)} \frac{\partial}{\partial \tilde{g}_{0,3}} - \tilde{\beta}_{2,2}^{\text{eff}(3)} \frac{\partial}{\partial \tilde{g}_{2,2}} \right] F(N, \tilde{g}) + 2 = -\ln g_{0,2} - \frac{1}{N} \ln \Gamma_N + \frac{1}{2} - \frac{3}{2} \tilde{g}_{0,3}^3 + \frac{13}{10} \tilde{g}_{0,3}^4, \] (46)

where
\[
\tilde{\beta}_{0,3}^{\text{eff}(3)}(\tilde{g}_{0,3}) = -\frac{1}{2} \tilde{g}_{0,3} + \frac{137}{20} \tilde{g}_{0,3}^3 - \frac{39}{10} \tilde{g}_{0,3}^5, \] (47)
\[
\tilde{\beta}_{2,2}^{\text{eff}(3)}(\tilde{g}_{0,3}) = \frac{4}{5} \tilde{g}_{0,3}. \] (48)

Although a nonlinear effective beta function \( \tilde{\beta}_{2,2}^{\text{eff}(3)} \) appears in eq. \((43)\), we may neglect it because it is of order \( \tilde{g}_{0,3}^6 \). Then from eq. \((17)\) we find the fixed point
\[ \tilde{g}_{0,3}^{\ast(3)} = 0.276238 \cdots, \] (49)

and the exponent
\[ \gamma_1^{(3)} = 2.09517 \cdots. \] (50)

Similarly we can continue to improve the approximation. At the fourth order, using the \( L_{-1}, L_1, L_2, L_3, L_{-1,1}, L_{-1,2}, L_{0,2} \) and \( L_{0,3} \) conditions and ignoring terms of order \( \tilde{g}_{0,3}^7 \) we obtain the effective renormalization group equation

\[
\left[ N \frac{\partial}{\partial N} - \tilde{\beta}_{0,3}^{\text{eff}(4)} \frac{\partial}{\partial \tilde{g}_{0,3}} \right] F(N, \tilde{g}) + 2 = -\ln g_{0,2} - \frac{1}{N} \ln \Gamma_N + \frac{1}{2} - \frac{3}{2} \tilde{g}_{0,3}^2 + \frac{59}{30} \tilde{g}_{0,3}^4 + \frac{1}{2} \tilde{g}_{0,3}^6, \] (51)

\[
\tilde{\beta}_{0,3}^{\text{eff}(4)}(\tilde{g}_{0,3}) = -\frac{1}{2} \tilde{g}_{0,3} + \frac{147}{20} \tilde{g}_{0,3}^3 - \frac{69}{10} \tilde{g}_{0,3}^5, \] (52)

and the results
\[ \tilde{g}_{0,3}^{\ast(4)} = 0.270249 \cdots, \] (53)
\( \gamma_{1}^{(4)} = 2.15892 \ldots \) (54)

At the fifth order, using the \( L_{-1}, L_{1}, L_{2}, L_{3}, L_{4}, L_{-1,1}, L_{-1,2}, L_{-1,3}, L_{0,2} \) and \( L_{0,3} \) conditions and ignoring terms of order \( \tilde{g}_{0,3}^{8} \), we obtain the effective renormalization group equation

\[
\left[ N \frac{\partial}{\partial N} - \tilde{\beta}_{0,3}^{\text{eff}(5)} \frac{\partial}{\partial \tilde{g}_{0,3}} \right] + 2 \right] \quad F(N, \tilde{g}) = -\ln g_{0,2} - \frac{1}{N} \ln \Gamma_{N}
\]

\[
+ \frac{1}{2} - \frac{3}{2} \tilde{g}_{0,3}^{2} + \frac{443}{210} \tilde{g}_{0,3}^{4} + \frac{1}{2} \tilde{g}_{0,3}^{6},
\]

\[
\tilde{\beta}_{0,3}^{\text{eff}(5)}(\tilde{g}_{0,3}) = -\frac{1}{2} \tilde{g}_{0,3} + \frac{1089}{140} \tilde{g}_{0,3}^{3} - \frac{723}{70} \tilde{g}_{0,3}^{5} - \frac{36}{7} \tilde{g}_{0,3}^{7},
\] (56)

and the results

\[
\tilde{g}_{0,3}^{*}(5) = 0.266948 \ldots ,
\]

\[
\gamma_{1}^{(5)} = 2.25313 \ldots .
\] (58)

Thus, as the order of approximations becomes higher, resulting values of the fixed point and the string susceptibility exponent approach the exact values (40), (41). To carry out much higher order calculations, we have to use constraints corresponding to more general transformations (30) with \( n \geq 2, \ p \geq 1 \) and (35) starting from more general potential than (1).

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11
References

[1] F. David, Nucl. Phys. B257 (1985) 45, 543; J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B257 (1985) 433; V.A. Kazakov, Phys. Lett. 150B (1985) 28.

[2] E. Brézin and V.A. Kazakov, Phys. Lett. 236B (1990) 144; M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635; D.J. Gross and A.A. Migdal, Phys. Rev. Lett. 64 (1990) 127; Nucl. Phys. B340 (1990) 333.

[3] V.G. Knizhnik, A.M. Polyakov and A.A. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819; F. David, Mod. Phys. Lett. A3 (1988) 1651; J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.

[4] E. Brézin and S. Hikami, Phys. Lett. 283B (1992) 203; S. Hikami and E. Brézin, Phys. Lett. 295B (1992) 209; S. Hikami, Phys. Lett. 305B (1993) 327.

[5] E. Brézin and J. Zinn-Justin, Phys. Lett. B288 (1992) 54.

[6] J. Alfaro and P. Damgaard, Phys. Lett. B289 (1992) 342.

[7] V. Periwal, Phys. Lett. B294 (1992) 49.

[8] H.B. Gao, ICTP preprint IC 302/92 (September, 1992).

[9] S. Higuchi, C. Itoi and N. Sakai, preprint TIT/HEP-215 (March, 1993).

[10] C. Ayala, preprint UAB–FT–306 (April, 1993).

[11] S.R. Das, A. Dhar, A.M. Sengupta and S.R. Wadia, Mod. Phys. Lett. A5 (1990) 1041.

[12] L. Alvarez-Gaumé, J.L.F. Barbón and Č. Crnkovič, Nucl. Phys. B394 (1993) 383; G.P. Korchemsky, Mod. Phys. Lett. A7 (1992) 3081; Phys. Lett. B296 (1992) 323.

[13] A. Mironov and A. Morozov, Phys. Lett. B252 (1990) 47.

[14] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6 (1991) 1385; R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. B348 (1991) 435.