Geometrical Origin of the Cosmological Constant

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Abstract

We show that the description of the space-time of general relativity as a diagonal four dimensional submanifold immersed in an eight dimensional hypercomplex manifold, in torsionless case, leads to a geometrical origin of the cosmological constant. The cosmological constant appears naturally in the new field equations and its expression is given as the norm of a four-vector $U$, i.e., $\Lambda = 6g_{\mu\nu}U^\mu U^\nu$ and where $U$ can be determined from the Bianchi identities. Consequently, the cosmological constant is space-time dependent, a Lorentz invariant scalar, and may be positive, negative or null. The resulting energy momentum tensor of the dark energy depends on the cosmological constant and its first derivative with respect to the metric. As an application, we obtain the spherical solution for the field equations. In cosmology, the modified Friedmann equations are proposed and a condition on $\Lambda$ for an accelerating universe is deduced. For a particular case of the vector $U$, we find the well known result $\Lambda \propto a(t)^{-2}$ where $a(t)$ is the scale factor.

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1 Introduction

Since the discovery of the accelerating expansion of the universe \[1, 2, 3, 4, 5\], dark energy is often invoked to explain this phenomena and has led to renewed interest in the cosmological constant. The simplest model for dark energy (\(DE\)) is the cosmological constant with the equation of state \(\omega_{DE} = -1\). Nowadays, the cosmological constant problem is one of the most fundamental problems in physics \[6, 7, 8, 9, 10\]. Indeed, many models with variable cosmological constant have been proposed, in some cases it depends on space \[11, 12, 13\], time \[14\] or both of them \[15\]. Other authors have suggested that the cosmological constant can be written as a trace of an energy-momentum tensor, i.e., a Lorentz invariant scalar \[16, 17, 18, 19, 20\].

In this paper, we try to tackle the cosmological constant problem from a geometrical point of view. We will use Crumeyrolle’s results on hypercomplex manifolds where the space-time is considered as a diagonal four dimensional submanifold immersed in an eight dimensional hypercomplex manifold \[21, 22, 23\]. In his work, Crumeyrolle tried to obtain an unified theory as that of Einstein-Schrodinger using a geometric construction in the general case of a nonsymmetric connection, and the applications of this approach have been performed by Clerc \[24, 25, 26\]. A similar approach has been suggested where the tangent bundle of space-time is endowed with a hypercomplex algebraic structure \[27, 28, 29\]. Historically, Einstein was the first who used a complex metric in order to unify gravity and electromagnetism \[30, 31\]. Moffat has also used a nonsymmetric complex metric as an attempt to a new theory of gravity \[32\]. In the last decades, we note that complex and hypercomplex coordinates have been used in both field theory \[33, 34\], and general relativity \[35, 36, 37, 38, 39, 40\]. In the literature, the hypercomplex numbers are also called complex hyperbolic numbers, pseudo-complex numbers, double numbers, paracomplex numbers or split numbers \[33, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50\].

On the other hand, it is important to note that there are also other geometric approaches especially those accounting for the acceleration of the universe by modifying the Einstein-Hilbert action \[51\], and a truly geometric approach to modified gravity without affecting the early universe \[52, 53\]. We note that there is a clear physical principle, discovered from an investigation of the dynamical symmetries of Born-Infeld electrodynamics, motivating the consideration of an eight-dimensional pseudo-complex spacetime and which is related to the maximal acceleration \[54, 55\].

In this work, we apply the results of Crumeyrolle to the torsionless case in order to describe the theory of general relativity. The Ricci scalar in the space-time submanifold is given by 

\[P = R + \Lambda\]

where \(R\) is the Ricci scalar of the Levi-Civita connection and \(\Lambda\) is a scalar function of a four-vector \(U\), and therefore has a geometrical origin due to the immersion of the space-time submanifold in an eight dimensional manifold. It appears as a correction of the Ricci scalar curvature \(R\) used in the general relativity \[24, 25\]. Using a variational principle, the modified Einstein’s equations are obtained where \(\Lambda\) is identified to be the cosmological constant. From the Bianchi identities, a particular form of the four vector \(U\) can be given, this leads to a well known decaying cosmological constant \(\Lambda \propto a(t)^{-2}\) which gives a correction to the time evolutions of the matter and radiation as can be seen from the conservation law \[59\].

The main result of this paper is that the cosmological constant appears naturally in the modified Einstein’s equations as a norm of a four-vector \(U\), \(\Lambda = 6g_{\mu \nu} U^\mu U^\nu\). Consequently the cosmological constant is space-time dependent, Lorentz invariant, and may be positive, negative or null, depending on the nature of the four-vector \(U\). The resulting energy momentum tensor of
the dark energy depends on the cosmological constant and its first derivative with respect to the metric $g^{\mu\nu}$. From the modified Friedmann equations, we deduce a condition on the cosmological constant for an accelerating universe.

This paper is organized as follows: In section II, we recall briefly Cru
eyrolle’s work on which our paper is based. In section III, we derive the modified Einstein’s equations using a variational principle. Then, we derive the spherical solution in section IV and discuss its limit. In section V, the modified Friedmann equations are deduced and the results are discussed. We apply our result in section VI for a special form of the four vector $U$ and deduce a decaying cosmological constant. Finally, the conclusion is given in section VII.

2 Space-time as a diagonal submanifold

In Cru
eyrolle’s work [21, 22, 23], the space-time $V_4$ is considered as a diagonal submanifold of a $C^\infty$ eight dimensional manifold $V_8$ which is the product of two identical four dimensional real manifolds $W_4$

$$V_8 = W_4 \times W_4. \quad (1)$$

The real coordinates $(x^\alpha, x^{\alpha*})$ are called the associated diagonal coordinates.

Suppose that the manifold $V_8$ is endowed with a symmetric non degenerate metric tensor $G_{ij}$, then $V_8$ is seen to have a structure of a pseudo-riemannian manifold. According to $G_{ij}$, the metric tensor $g_{\alpha\beta}$ in $V_4$ is defined by setting in the natural diagonal frames of $V_4$ (intrinsic conditions) [21, 22, 23]

$$\tilde{G}_{\alpha\beta} = \tilde{G}_{\alpha^*\beta^*} = 0, \quad \tilde{G}_{\alpha\beta*} = \tilde{G}_{\beta^*\alpha} = g_{\alpha\beta}. \quad (3)$$

where $\tilde{\;}$ means the restriction in $V_4$.

A generalization of the Ricci theorem is to postulate that for every path of $V_4$, the covariant derivative of the tensor $G_{ij}$ vanishes

$$\nabla_\rho G_{ij} = 0, \quad (4)$$

using conditions (3), the above relations (1) will be written in $V_4$ as

$$\nabla_\rho \tilde{G}_{\alpha\beta} = \nabla_\rho \tilde{G}_{\alpha\beta*} = \nabla_\rho \tilde{G}_{\alpha^*\beta^*} = 0. \quad (5)$$

Consider the connections in the natural diagonal frame bundle of $V_8$ such that [21, 22, 23, 25]

$$\Gamma^i_{jk} = \Gamma^i_{j^*k^*}, \quad \Gamma^i_{jk} = \Gamma^i_{j^*k^*}. \quad (6)$$

By restriction in $V_4$, we obtain

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^{\alpha^*}_{\beta^*\gamma} = \Gamma^\alpha_{\beta^*\gamma^*}, \quad \Gamma^\alpha_{\beta\gamma} = \Gamma^{\alpha^*}_{\beta\gamma} = \Gamma^{\alpha^*}_{\beta^*\gamma^*}. \quad (7)$$

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we can show that the coefficients $\Gamma^i_{jk}$ with even number of asterisks transform as connections, while the others (with odd number of asterisks) transform as tensors in all natural diagonal frame of $V_4$ \[56\].

Then according to Eqs.\[7\], one can define in $V_4$ a connection $\mathcal{L}_\gamma^\alpha$ and a tensor $\Lambda^\alpha_{\beta\gamma}$ by the relations \[25\]

$$
\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\alpha_{\beta\gamma} = \mathcal{L}_\gamma^\alpha, \quad \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} = \Lambda^\alpha_{\beta\gamma}.
$$

The connection $\mathcal{L}_\gamma^\alpha$ is generally nonsymmetric.

Using the relations \[3\] and the properties \[5\], the conditions \[5\] give

$$
\begin{align*}
\partial_\rho g_{\alpha\beta} - \mathcal{L}_\gamma^\gamma g_{\gamma\beta} - \mathcal{L}_\rho^\gamma g_{\alpha\gamma} &= 0, \\
\lambda^\alpha_{\alpha\lambda} g_{\gamma\beta} + \mathcal{L}_\gamma^\gamma g_{\alpha\alpha} &= 0, \\
\lambda^\alpha_{\alpha\lambda} g_{\beta\gamma} + \lambda^\alpha_{\beta\lambda} g_{\alpha\gamma} &= 0,
\end{align*}
$$

The solution of equations (9) and (11) is the antisymmetric tensor $\Lambda^\alpha_{\beta\gamma}$ in $\alpha, \beta$

$$
\Lambda^\alpha_{\beta\gamma} = g^\alpha_{\gamma\beta} \epsilon_{\gamma\beta\rho} U^\rho,
$$

where $\epsilon_{\gamma\beta\rho}$ is the antisymmetric Levi-Civita tensor and $U^\rho$ is an arbitrary 4-vector in $V_4$.

By the immersion of the submanifold $V_4$ in the manifold $V_8$, the curvature form induced in $V_4$ is \[25\]

$$
\hat{\Omega}^i_j = \frac{1}{2} \widetilde{R}^i_{j\lambda\mu} dx^\lambda \wedge dx^\mu,
$$

where $\widetilde{\ }$ means the restriction in $V_4$ (remember that $x^{\mu*} = 0$ in $V_4$).

Then the induced curvature tensor in $V_4$ becomes

$$
\widetilde{R}^i_{j\lambda\mu} = \partial_\lambda \Gamma^i_{j\mu} - \partial_\mu \Gamma^i_{j\lambda} + \Gamma^i_{\rho\lambda} \Gamma^\rho_{j\mu} + \Gamma^i_{\beta\lambda} \Gamma^\alpha_{j\mu} - \Gamma^i_{\rho\alpha} \Gamma^\rho_{j\lambda} - \Gamma^i_{\beta\alpha} \Gamma^\rho_{\lambda\rho},
$$

by contraction and using Eqs.\[8\], one can obtain two independent Ricci tensors in $V_4$ \[25\]

$$
\begin{align*}
P_{\alpha\beta} &= \widetilde{R}^\lambda_{\beta\alpha\lambda} = \partial_\lambda \mathcal{L}_\lambda^{\alpha\beta} - \partial_\lambda \mathcal{L}_{\lambda\beta}^{\alpha} - \mathcal{L}_\rho^{\lambda} \mathcal{L}_{\alpha\beta}^{\rho} - \mathcal{L}_{\alpha\beta}^{\rho} \mathcal{L}_{\lambda\rho}^{\lambda} + \lambda^\lambda_{\rho\lambda} \lambda^\rho_{\beta\alpha} - \lambda^\rho_{\alpha\lambda} \lambda^\lambda_{\rho\beta}, \\
Q_{\alpha\beta} &= \widetilde{R}^\alpha_{\alpha\beta\lambda} = \partial_\lambda \lambda^\alpha_{\beta\lambda} - \partial_\lambda \lambda^\lambda_{\beta\alpha} + \lambda^\rho_{\lambda\rho} \lambda^\alpha_{\beta\lambda} - \lambda^\rho_{\beta\rho} \lambda^\lambda_{\alpha\lambda} + \lambda^\rho_{\rho\lambda} \lambda^\alpha_{\beta\alpha} - \lambda^\rho_{\rho\beta} \lambda^\lambda_{\alpha\alpha}.
\end{align*}
$$

The two other Ricci tensors $\overline{P}_{\alpha\beta}$ and $\overline{Q}_{\alpha\beta}$ can be obtained from $P_{\alpha\beta}$ and $Q_{\alpha\beta}$ using Einstein’s principle of pseudo-hermiticity \[25\], i.e., by changing $\mathcal{L}_\gamma^\alpha$, $\Lambda^\alpha_{\beta\gamma}$, $g_{\alpha\beta}$ by $\mathcal{L}_\gamma^\alpha$, $\Lambda^\alpha_{\beta\gamma}$, $g_{\alpha\beta}$ in $P_{\alpha\beta}$ and $Q_{\alpha\beta}$ respectively

$$
\begin{align*}
\overline{P}_{\alpha\beta} &= \partial_\lambda \mathcal{L}_\lambda^{\alpha\beta} - \partial_\lambda \mathcal{L}_{\alpha\beta}^{\alpha} + \lambda^\rho_{\lambda\rho} \mathcal{L}_{\alpha\beta}^{\rho} - \mathcal{L}_{\beta\rho}^{\rho} \mathcal{L}_{\alpha\lambda}^{\lambda} + \lambda^\lambda_{\rho\lambda} \lambda^\rho_{\alpha\beta} - \lambda^\rho_{\rho\lambda} \lambda^\lambda_{\alpha\alpha}, \\
\overline{Q}_{\alpha\beta} &= \partial_\lambda \lambda^\alpha_{\beta\lambda} - \partial_\lambda \lambda^\lambda_{\beta\alpha} + \lambda^\rho_{\lambda\rho} \lambda^\alpha_{\beta\lambda} - \mathcal{L}_{\beta\rho}^{\rho} \mathcal{L}_{\alpha\lambda}^{\lambda} + \lambda^\rho_{\rho\lambda} \mathcal{L}_{\alpha\beta}^{\rho} - \lambda^\rho_{\rho\beta} \lambda^\lambda_{\alpha\alpha}.
\end{align*}
$$
Using the antisymmetric property of \( \Lambda^{\lambda}_{\alpha\beta} \), two scalar curvatures are obtained from \( P_{\alpha\beta} \) and \( \nabla_{\beta\alpha} \)

\[
P = g^{\alpha\beta} P_{\alpha\beta} = R + \Lambda + 2 \nabla^\alpha S_\alpha, \tag{20}
\]

\[
\nabla_P = g^{\alpha\beta} \nabla_{\beta\alpha} = R + \Lambda, \tag{21}
\]

where \( R \) is the Ricci scalar obtained by contracting the Ricci tensor

\[
R_{\alpha\beta} = \partial_\lambda L^\lambda_{\alpha\beta} - \partial_\beta L^\lambda_{\alpha\lambda} + L^\rho_\alpha L^\lambda_{\rho\lambda} - L^\lambda_{\rho\lambda} L^\rho_\alpha,
\]

and \( S_\alpha = S^\lambda_{\alpha\lambda} \) is the torsion vector. The scalar \( \Lambda \) is defined by \[25\]

\[
\Lambda = g^{\alpha\beta} \Lambda^\sigma_{\alpha\lambda} \Lambda^\lambda_{\beta\sigma}. \tag{22}
\]

In the case \( S_\alpha = 0 \), the scalar \( \Lambda = P - R \) appears as a correction of the curvature, and represents the contribution of a new tensor field \( \Lambda^\rho_{\beta\lambda} \) which results from the immersion of the manifold \( V_4 \) in \( V_8 \) \[25\].

One can also obtain two others scalar curvatures from \( Q_{\alpha\beta} \) and \( \nabla_{\beta\alpha} \)

\[
Q = g^{\alpha\beta} Q_{\alpha\beta} = g^{\alpha\beta} (\Lambda^\rho_{\beta\lambda} S^\lambda_{\rho\alpha} + \Lambda^\rho_{\alpha\lambda} S^\lambda_{\rho\beta}) \tag{23}
\]

\[
\nabla_Q = g^{\alpha\beta} \nabla_{\beta\alpha} = Q. \tag{24}
\]

The general field equations can be obtained from the general action

\[
S = \int \sqrt{-g} \left( \frac{P + \nabla_P}{2} + \theta \frac{Q + \nabla_Q}{2} \right) d^4x, \tag{25}
\]

where \( g = \det g_{\mu\nu} \) and \( \theta \) is a function of coordinates of \( V_4 \).

These results have been obtained for the general case of a nonsymmetric connection by Crumeyrolle and Clerc. In our work, we will consider the simple case of vanishing torsion in \( V_4 \), and write the field equations and discuss the consequences.

### 3 Modified Einstein’s equations

Following the same steps as Crumeyrolle, where the four dimensional space-time of general relativity is considered as a diagonal manifold \( V_4 \) with vanishing torsion, all \( S^\lambda_{\alpha\lambda} = 0 \), immersed in an eight dimensional hypercomplex manifold \( V_8 \). This implies that the connection \[12\] is reduced to the symmetric Levi-Civita connection

\[
\mathcal{L}^\alpha_{\beta\gamma} = \{^\alpha_{\beta\gamma}\} = \frac{1}{2} g^{\alpha\rho} (\partial_\beta g_{\rho\gamma} + \partial_\gamma g_{\beta\rho} - \partial_\rho g_{\beta\gamma}), \tag{26}
\]

and then the scalar curvatures defined in the last section are reduced to

\[
P = \nabla_P = R + \Lambda, \quad Q = \nabla_Q = 0, \tag{27}
\]

where \( R \) is the scalar curvature of the symmetric connection \[26\].
From Eqs. (13) and (22), the scalar \( \Lambda \) can be written in the form

\[
\Lambda = 6 g^{\mu \nu} U_{\mu} U_{\nu} = 6 U^2.
\]  

(28)

In this case the new Einstein-Hilbert action is obtained from the action (25)

\[
S = \int \sqrt{-g} P d^4 x = \int \sqrt{-g} (R + \Lambda) d^4 x.
\]  

(29)

In the absence of ordinary matter, the field equations are obtained using a variational principle (it is important to remember that \( \Lambda \) depends on \( g^{\mu \nu} \))

\[
\delta S = \int \sqrt{-g} d^4 x \left[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - \frac{1}{2} \Lambda g_{\mu \nu} + \frac{\partial \Lambda}{\partial g_{\mu \nu}} \right] \delta g^{\mu \nu} = 0.
\]  

(30)

Let us put \( K_{\mu \nu} = \frac{\partial \Lambda}{\partial g_{\mu \nu}} \), then we obtain the modified Einstein’s equations

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - \frac{1}{2} \Lambda g_{\mu \nu} + K_{\mu \nu} = 0.
\]  

(31)

These equations are the field equations in vacuum because we haven’t introduced any matter term in the action. In this case, the two last terms in Eq. (31) can be considered as a source for dark energy and the scalar \( \Lambda \) is identified to be the cosmological constant, and its expression is given by (28).

Then, we deduce that the cosmological constant \( \Lambda \) is space-time dependent, a Lorentz invariant scalar and its sign depends on the nature of the arbitrary four-vector \( U \); i.e., For a time like vector \( \Lambda > 0 \), and for a space-like vector \( \Lambda < 0 \) and for a light-like vector \( \Lambda = 0 \).

It is important to note that the cosmological constant \( \Lambda \) has a geometrical origin due to the immersion of the space-time submanifold \( V_4 \) in an eight dimensional manifold. It is defined as a trace of the tensor \( \Lambda^\sigma_{\alpha \lambda} \Lambda^\lambda_{\beta \sigma} \), and the tensor \( \Lambda^\sigma_{\alpha \lambda} \) verifies the two equations (10) and (11). In conclusion, the geometrical nature of the vector \( U \) allows to determine the sign of the cosmological constant.

From equations (31), a new energy-momentum tensor for dark energy can be defined by

\[
T^{DE}_{\mu \nu} = \frac{1}{8 \pi G} \left( \frac{1}{2} \Lambda g_{\mu \nu} - K_{\mu \nu} \right),
\]  

(32)

this tensor contains two terms while in the Einstein cosmological model it contains only the term (\( -\frac{\Lambda}{8 \pi G} g_{\mu \nu} \)). In order to determine the terms \( \Lambda \) and \( K_{\mu \nu} \) we have to know the expression of the arbitrary 4-vector \( U \).

The general Bianchi identities in \( V_8 \) are written as [21, 22, 23, 24, 25]

\[
(\partial_{[\nu} R_{ijkl]} - R_{nkli} \Gamma_{jm}^{n} + R_{njkl} \Gamma_{im}^{n}) dx^{k} \wedge dx^{l} \wedge dx^{m} = 0.
\]  

(33)

By restriction in the diagonal submanifold \( V_4 \) (\( x^{\mu} = 0 \)), and contraction one can obtain [25]

\[
\nabla^{\nu} \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \right) = \nabla^{\nu} \left( 2 U_{\nu} U_{\mu} + g_{\mu \nu} U^2 \right) + U^{\sigma} \left( \nabla_{\sigma} U_{\mu} - \nabla_{\mu} U_{\sigma} \right),
\]  

(34)
The appearance of the terms in right hand side of the last equation is due to the immersion, these terms have to vanish in the case of a Levi-Civita connection, then

\[ \nabla^\nu (2U_\nu U_\mu + g_{\mu\nu}U^2) + U^\sigma (\nabla_\sigma U_\mu - \nabla_\mu U_\sigma) = 0, \]  

which can be simplified to

\[ \nabla_\alpha (U^2 U^\alpha) = 0. \]  

In section six, we determine the four vector \( U \) as a solution to this equation and deduce the expression of the cosmological constant.

4 Spherical solution

The static spherically symmetric space-time metric is given by

\[ ds^2 = -e^{2\mu(r)}dt^2 + e^{2\nu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \]  

where \( \mu, \nu \) are functions of \( r \).

Using the field equations in vacuum (31), the general solution for the metric (37) is given by

\[ ds^2 = -\exp\left\{ \int \left[ \frac{4r^2 - r^2K_{rr} - a_1 - a_2}{r (1 + a_1 + a_2)} \right] dr \right\} dt^2 + \frac{1}{1 + a_1 + a_2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \]  

where

\[ a_1 = \frac{1}{2r} \int \Lambda r^2dr; \quad a_2 = \frac{1}{r} \int r^2K_{00}dr. \]  

In the particular case where \( \Lambda \) is constant, i.e. \( \delta\Lambda = 0 \) and \( K_{\mu\nu} = 0 \), the above coefficients are reduced to \( a_1 = \frac{\Lambda}{6}r^2 \) and \( a_2 = 0 \) and the metric (38) is reduced to the de Sitter or anti-de Sitter metric

\[ ds^2 = -\left( 1 + \frac{\Lambda}{6}r^2 \right) dt^2 + \frac{1}{1 + \frac{\Lambda}{6}r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \]  

we note that for a time like vector \( U \), i.e. \( \Lambda > 0 \), the metric (40) corresponds to an anti-de Sitter space. For a space like vector \( U \), i.e. \( \Lambda < 0 \), the metric (40) corresponds to a de Sitter space. While for a light-like vector \( U \), i.e. \( \Lambda = 0 \), corresponds to a flat space.

5 Modified Friedmann equations

Let us introduce the flat Friedmann-Robertson-Walker metric

\[ ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right], \]  

where \( a(t) \) is the scale factor. Using the field equations (31), we obtain the Friedmann equations

\[ \left( \frac{a}{\dot{a}} \right)^2 = \frac{\Lambda}{6} - \frac{K_{00}^0}{3}, \]  

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\[
\left( \frac{\ddot{a}}{a} \right) = \frac{\Lambda}{6} + \frac{K^0_0}{6} + \frac{K^r_r}{2},
\]
\[
\left( \frac{\dot{a}}{a} \right) = \frac{\Lambda}{6} + \frac{K^0_0}{6} + \frac{K^\theta_\theta}{2},
\]
\[
\left( \frac{\dot{a}}{a} \right) = \frac{\Lambda}{6} + \frac{K^0_0}{6} + \frac{K^\phi_\phi}{2}. \tag{43}
\]

The last three equations give

\[ K^r_r = K^\theta_\theta = K^\phi_\phi, \tag{44}\]

then, we obtain the spacial equation

\[
\left( \frac{\ddot{a}}{a} \right) = \frac{\Lambda}{6} + \frac{K^0_0}{6} + \frac{K^r_r}{2}. \tag{45}
\]

From Eq. (42), we obtain the condition

\[ \frac{\Lambda}{6} - \frac{K^0_0}{3} > 0, \tag{46}\]

which gives a positive energy density for the dark energy

\[ \rho = \frac{1}{8\pi G} \left( \frac{1}{2} \Lambda - K^0_0 \right) > 0. \tag{47}\]

We also define the pressure of the dark energy by

\[ p_{DE} = \frac{1}{8\pi G} \left( -\frac{1}{2} \Lambda - K^r_r \right) \tag{48}\]

Integrating equation (42), we obtain

\[ a(t) = a(t_0) \exp \left[ \int \sqrt{\frac{\Lambda}{6} - \frac{K^0_0}{3}} \, dt \right]. \tag{49}\]

This is like the case of the de Sitter space: it describes an empty exponentially expanding space. The differences from the de Sitter case are the presence of the term \(K^0_0\) and here \(\Lambda\) is not constant.

Using Eqs. (12) and (15) we obtain the expression of the deceleration parameter \(q\)

\[ q = -\frac{\ddot{a}}{\dot{a}^2} = \left[ \frac{\Lambda + K^0_0}{6} + \frac{K^r_r}{2} \right] \tag{50}\]

and for the equation of state \(\omega = \frac{p_{DE}}{\rho_{DE}}\), we have

\[ \omega = -\frac{\frac{1}{2} \Lambda + K^r_r}{\frac{1}{2} \Lambda - K^0_0}. \tag{51}\]
for the particular case $K_0^0 = -K_r^r$, we obtain the well known result $\omega = -1$.

We remark that the deceleration parameter $q$ vanishes for

$$\Lambda = -(K_0^0 + 3K_r^r),$$

which gives

$$\omega = -\frac{1}{3}.$$  \hspace{1cm}(53)

For an accelerated universe ($q < 0$), we must have the condition

$$\Lambda > -(K_0^0 + 3K_r^r),$$

which gives with Eq.(46), the well known condition $\omega < -\frac{1}{3}$.

From the conditions (46) and (54) we obtain

$$-\frac{1}{2}\Lambda - K_r^r < 0,$$

which implies that the pressure of the dark energy is negative

$$p_{DE} = \frac{1}{8\pi G} \left(-\frac{1}{2}\Lambda - K_r^r\right) < 0.$$  \hspace{1cm}(56)

At the end, we note that for a decelerating phase ($q > 0$), we have

$$\Lambda < -(K_0^0 + 3K_r^r),$$

which gives with Eq.(46), the condition $\omega > -\frac{1}{3}$. As we saw, what we have given until now is a general frame work, and one has to give an expression for the four vector $U$, which can be obtained from the condition (36).

6 Cosmological constant as a function of the scale factor

As an application, we propose a solution to the equation (36) in the form \cite{22, 25}

$$U_\mu \propto \sqrt{(-g)^q} u_\mu,$$

where $q$ is a constant, $u_\mu$ is a unit vector which verifies $g^{\mu\nu} u_\mu u_\nu = 1$, and $g = \det g_{\mu\nu}$.

From this solution we obtain the cosmological constant

$$\Lambda = 6g^{\mu\nu} U_\mu U_\nu \propto 6(-g)^q$$

Using the flat Friedmann-Robertson-Walker (FRW) metric in cartesian coordinates

$$ds^2 = -dt^2 + a^2(t) \left[dx^2 + dy^2 + dz^2\right],$$

i.e. $g_{\mu\nu} = diag [-1, a^2, a^2, a^2]$, which gives for the determinant

$$(-g) = a^6(t),$$

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and finally the cosmological constant becomes time dependent or scale factor dependent

\[ \Lambda \propto 6a^6(t). \]  

(62)

This is a general expression of the time dependent cosmological constant.

For a purely physical reasons we suggest that \( q = -1/3 \) to obtain the well known formula for the cosmological constant \[58, 59, 60, 61, 62, 63\]

\[ \Lambda(t) \propto 6a^{-2}(t). \]  

(63)

This scale factor dependent cosmological constant may give theoretically an explanation of why it has decayed from a large value in the early universe to have small one today. Particle and cosmology theories suggest that as the universe expanded and cooled different phases took place (symmetry breaking) and the energy density associated to the cosmological constant decreased from an initially large value which is supported by the inflationary scenario \[6, 57\] and attained a small value today. Although we did not give numerical values, the evolving cosmological constant \[63\] is interesting, the energy associated to it is decreasing with an increasing scale factor (the expanding). For physical reasons which motivate a decaying cosmological constant \[63\], we refer the reader to Refs. \[58, 59, 60, 61, 62, 63\], our approach here in deriving this formula is different from those references, in fact we gave a pure geometric origin of the cosmological constant.

Using the the expressions

\[ U_\mu = \sqrt{(-g)^{-1/3}} u_\mu \]  

and \( \Lambda = 6g^{\mu\nu}U_\mu U_\nu = 6(-g)^{-1/3} \) we obtain the tensor

\[ K_{\mu\nu} = \frac{\partial \Lambda}{\partial g^{\mu\nu}} = 6U_\mu U_\nu + 6g^{\alpha\beta} \frac{\partial U_\alpha}{\partial g^{\mu\nu}} U_\beta + 6g^{\alpha\beta} U_\alpha \frac{\partial U_\beta}{\partial g^{\mu\nu}} \]

\[ = \Lambda \left( u_\mu u_\nu + \frac{1}{3} g_{\mu\nu} \right), \]

and the energy momentum tensor \[32\] becomes

\[ T_{\mu\nu}^{DE} = \frac{3}{4\pi G} a^{-2}(t) \left( -\frac{5}{6} g_{\mu\nu} - u_\mu u_\nu \right). \]  

(64)

This gives the density and the pressure

\[ \rho^{DE} = T_0^{0DE} = \frac{11}{8\pi G} a^{-2}(t), \quad p^{DE} = T_i^{iDE} = -\frac{5}{8\pi G} a^{-2}(t). \]  

(65)

We see here that the energy density of the dark energy falls off by the expansion more slowly than those of the ordinary matter and radiation. From this model we conclude that for dark energy dominated epoch (no matter and radiation) the universe takes an accelerating phase, this can be seen from \[65\], i.e; the fact that \( p^{DE} = -\frac{5}{11} \rho^{DE} \) which gives \( \omega_{DE} = -\frac{5}{11} \) or exactly \( -\frac{1}{2} < \omega_{DE} < -\frac{1}{3} \).

Let us see now the Friedmann equations in this case in the absence of ordinary matter and radiation; of course this is almost not natural because our universe contains also matter and radiation but one has also to study the dark energy dominated epoch because as we expect the
ordinary matter and radiation will fall off by the expansion and remains only dark energy. The Friedmann equations become
\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{11}{3} a^{-2}(t) \quad \text{and} \quad \left(\frac{\ddot{a}}{a}\right) = \frac{2}{3} a^{-2}(t),
\]
(66)

These equations give us the behavior of the scale factor, the first equation can be integrated simply giving a linear expanding \( a - a_0 = \sqrt{\frac{11}{3}} (t - t_0) \) where \( a_0 \) is the scale factor at time \( t_0 \). This time evolution of the scale factor is different from that of an exponentially expanding \( a \propto e^{Ht} \) when the cosmological constant is really constant. From the second equation we have \( \ddot{a} = \frac{2}{3} a^{-1}(t) \), which has to vanish in the case of linear expanding, this can be verified only when \( a \to \infty \), but from the linear evolution \( a \propto \sqrt{\frac{11}{3}} t \) this is equivalent to \( t \to \infty \). So we say that the dark energy epoch will be only in the limit \( t \to \infty \) (very late universe) and in that epoch the universe will take a linear expansion (a constant velocity!) rather than an exponential expansion.

Now let us study the behavior of the ordinary matter and radiation. In this case the energy momentum tensor of the matter (radiation) will appear in the Einstein’s field equations as a perfect fluid \( T_{\mu \nu}^m = (\rho + p) u_\mu u_\nu + p g_{\mu \nu} \), then the Friedmann equations become
\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho + \frac{11}{3} a^{-2}(t) \quad \text{and} \quad \left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3} (\rho + 3p) + \frac{2}{3} a^{-2}(t),
\]
(67)
here \( \rho \) and \( p \) may be the density and pressure of matter or radiation or both of them, and the appearance of \( a^{-2}(t) \) is due to \( \rho^{DE} \) and \( p^{DE} \).

The conservation law \( \nabla_\mu T^\mu_\nu = 0 \), where \( T_{\mu \nu} = T_{\mu \nu}^m + T_{\mu \nu}^{DE} \), becomes
\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = -4 \frac{\dot{a}}{a^3}.
\]
(68)
This is the new conservation law with the new term in the right hand side as a consequence of the time dependence of the cosmological constant. For matter \( (p = 0) \) or for radiation \( (p = \frac{1}{3} \rho) \), equation (68) becomes respectively
\[
\dot{\rho}^m + 3 \frac{\dot{a}}{a} \rho^m = -4 \frac{\dot{a}}{a^3} \quad \text{and} \quad \dot{\rho}^r + 4 \frac{\dot{a}}{a} \rho^r = -4 \frac{\dot{a}}{a^3}.
\]
(69)
As in Ref.59, solutions of these equations are of the form
\[
\rho^m = A_1 a^{-3}(t) + A_2 a^{-2}(t) \quad \text{and} \quad \rho^r = B_1 a^{-4}(t) + B_2 a^{-2}(t),
\]
(70)
where \( A_1, A_2, B_1 \) and \( B_2 \) are constants.

We note here that the solutions (70) are different from the standard ones, \( \rho^m \propto a^{-3} \) and \( \rho^r \propto a^{-4} \), by the presence of the additional term \( a^{-2} \) due the time dependence of the cosmological constant. These solutions combined with Friedmann equations (67) will give the time evolution of the scale factor \( a(t) \).
7 Conclusion

In this paper, we have used Crumeyrolle’s results on hypercomplex manifold where the four dimensional space-time of general relativity is considered as a submanifold immersed in an eight dimensional hypercomplex manifold. In the case of symmetric connection, we have seen that the cosmological constant $\Lambda$ appears naturally in Einstein’s equations and its expression is given as a norm of a four-vector $U$. Then, the cosmological constant can be positive, negative or null. A new energy momentum tensor of the dark energy is obtained which depends on the cosmological constant and its first derivatives with respect to the metric. In the first application, the spherical solution of the Einstein’s equations is obtained. In the second, we have obtained the modified Friedmann equation in the standard flat Friedmann-Robertson-Walker metric, and found that the equation of state depends on $\Lambda$ and its first derivative with respect to the metric $\omega = \omega(\Lambda, \frac{\partial \Lambda}{\partial g_{\mu\nu}})$. At the end, a condition on $\Lambda$ was deduced for an accelerating universe which is equivalent to the well known condition $\omega < -\frac{1}{3}$. For a particular case of the four vector $U$, we deduced a decaying cosmological constant $\Lambda \propto a(t)^{-2}$, which in turn modifies the behaviors of ordinary matter and radiation.

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