Testing Differential Privacy with Dual Interpreters

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Applying differential privacy at scale requires convenient ways to check that programs computing with sensitive data appropriately preserve privacy. We propose here a fully automated framework for testing differential privacy, adapting a well-known “pointwise” technique from informal proofs of differential privacy. Our framework, called DPCheck, requires no programmer annotations, handles all previously verified or tested algorithms, and is the first fully automated framework to distinguish correct and buggy implementations of PrivTree, a probabilistically terminating algorithm that has not previously been mechanically checked.

We analyze the probability of DPCheck mistakenly accepting a non-private program and prove that, theoretically, the probability of false acceptance can be made exponentially small by suitable choice of test size.

We demonstrate DPCheck’s utility empirically by implementing all benchmark algorithms from prior work on mechanical verification of differential privacy, plus several others and their incorrect variants, and show DPCheck accepts the correct implementations and rejects the incorrect variants.

We also demonstrate how DPCheck can be deployed in a practical workflow to test differentially privacy for the 2020 US Census Disclosure Avoidance System (DAS).

Additional Key Words and Phrases: Differential privacy, testing, symbolic execution

1 INTRODUCTION

Differential privacy, the highest standard for privacy-preserving data analysis, is seeing increasing real-world deployment [Apple 2017; Microsoft 2017; N. Dajani et al. 2017, etc.]. Differential privacy offers precise guarantees, is robust to arbitrary post-processing, and gives a quantitative estimate of privacy loss. However, differentially private algorithms often require subtle reasoning for their proofs of privacy. Even experts get these proofs wrong [Lyu et al. 2017].

A number of recent efforts have focused on partly or fully automating the process of certifying differential privacy for sophisticated algorithms such as the sparse vector technique [Dwork and Roth 2014]. For example, Zhang and Kifer’s LightDP [Zhang and Kifer 2017a] uses a lightweight dependent type system along with some programmer annotations; Ding et al. [2018] propose a fully automated statistical testing framework that attempts to disprove differential privacy of queries using statistical evidence; Albarghouthi and Hsu [2017] demonstrate a completely automated proof synthesis system with a specialized program logic; and Wang et al. [2019] use a proof technique
called “Shadow Execution” to improve upon LightDP and further reduce the amount of programmer annotations required and increase the performance of verification.

We propose a novel framework, called DPCheck, that requires no annotations, handles key benchmark algorithms from previous work, and moreover can accept correct implementation, and reject faulty variants of the PrivTree [Zhang et al. 2016] algorithm, a challenging algorithm for automatic differential privacy verification acknowledged by authors of LightDP [Zhang and Kifer 2017b]. PrivTree was beyond the scope of previous work, owing to a probabilistic main loop that terminates eventually with probability 1 but is not guaranteed to terminate in any bounded number of iterations. Our key insight is that we can combine information from instrumented and symbolic executions of a program to construct privacy proofs for specific executions, then combine these proofs from a large number of executions to give a statistical guarantee of random differential privacy [Hall et al. 2013]. The symbolic interpreter uses information gathered by the instrumented interpreter to build a simple static analysis of the program’s privacy properties, making automated detection of privacy leaks, even for challenging algorithms, feasible.

Following a short review of differential privacy in Section 2 and an overview of DPCheck’s syntax and semantics in Section 3 and Section 4, we offer these main contributions:

1. We present a testing strategy for differential privacy adapted from the pointwise proof technique (Section 5).
2. We prove that the testing strategy always correctly accepts a class of well-behaved differentially private programs, and prove that, in principle, this strategy’s probability of incorrectly accepting ill-behaved (non-private) programs decreases exponentially as test size increases, for a class of ill-behaved programs defined in the framework of random differential privacy (Section 6). We then give an overview of the testing strategy implementation for DPCheck in Haskell (Section 5.4).
3. We demonstrate the effectiveness of the testing strategy by showing that it can detect non-private variants with common programming mistakes, and published mistakes made by experts on the sophisticated benchmark algorithms, and that it accepts correct implementations of all these algorithms. In particular, DPCheck is the first automated framework that can distinguish correct and incorrect variants of PrivTree (Section 7.1). These benchmark algorithms’ differential privacy proofs cover a wide range of complexity, demonstrating DPCheck can analyze both simple and sophisticated differentially private programs.
4. We present a practical workflow that uses DPCheck to re-implement and test the core differential privacy mechanisms in the Disclosure Avoidance System (DAS) [Petti and Flaxman 2019] designed for 2020 US Census; we also show statistical evidence that our re-implemented core mechanism behaves the same as the unmodified DAS (Section 7.2).
5. We implement DPCheck as an embedded language in Haskell and discuss a type-driven optimization adapted from Torlak and Bodik [2014] to speed up symbolic execution, which improves testing time for some our benchmark algorithms (Section 8).

Section 9 enumerates some limitations, and Sections 10 and 11 discuss related and future work.

2 BACKGROUND

A discrete distribution over values of type $\tau$ is a function of type $\tau \mapsto [0, 1]$ mapping each value in $\tau$ to an associated probability. We write $\bigcirc \tau$ for the set of discrete distributions over values in type $\tau$. An event $E$ is a subset of $\tau$. The support of a discrete distribution $\mu$ is the subset of $\tau$ whose values have non-zero probability: $\text{supp}(\mu) = \{x \in \tau \mid \mu(x) > 0\}$.

**Definition 1.** Let $\mu : \bigcirc \tau$ be a discrete distribution. We call $\mu$ a sub-distribution if the sum of probabilities over its support is at most 1: $\sum_{v \in \text{supp} \mu} \mu(v) \leq 1$. 
Sub-distributions are useful for describing the semantics of randomized programs because they naturally model non-termination through the “missing probability.” In what follows, we write just “distribution” to mean sub-distribution.

Differential privacy is a relational property of randomized programs. Informally, a program is differentially private if it produces similar distributions when run on similar inputs. The exact similarity relation on inputs depends on what private information we care about protecting. For example, a program $f$ may be counting the number of patients diagnosed with some disease in a medical database; to conform to regulations, we must not leak the diagnosis of any particular patient. More precisely, the distribution of outputs should be nearly the same if the diagnosis of any single patient changes in the input database. For this example, an appropriate similarity relation on inputs is “at most one patient’s data may be different between the two input databases,” or, more generally:

**Definition 2.** Two multisets have database distance $k$ if at most $k$ items must be added or removed to make the two contain exactly the same items.

Another common similarity relation is the $L_1$-distance between two vectors of numbers.

**Definition 3.** The $L_1$ distance between vectors $x_1$ and $x_2$ is the sum of the coordinate-wise distances between corresponding elements of the two vectors.

Finally, some algorithms’ notion of similar inputs is vectors with bounded coordinate-wise distance.

**Definition 4.** Vectors $x_1$ and $x_2$ have coordinate-wise distance $k$ if $|x_1[i] - x_2[i]|$ is bounded by $k$ for each coordinate $i$.

Here is the fundamental definition of differential privacy:

**Definition 5.** A randomized program $f : \tau \mapsto \sigma$ is $(\epsilon, \delta)$-differentially private if, for all similar inputs $(x_1, x_2) \in \tau \times \tau$, the probability of any event $E \subseteq \sigma$ satisfies the inequality

$$P_{f(x_1)}[E] \leq e^\epsilon P_{f(x_2)}[E] + \delta.$$  (1)

If the support of the probability distributions is countable, we can simplify the definition of $(\epsilon, 0)$-differential privacy using a pointwise inequality on the probability difference:

**Definition 6.** A randomized program $f : \tau \mapsto \sigma$, for some countable domain $\sigma$, is $(\epsilon, 0)$-differentially private if, for all similar inputs $(x_1, x_2) \in \tau \times \tau$, the probability of any singleton event $v \in \sigma$ satisfies

$$P_{f(x_1)}\{v\} \leq e^\epsilon P_{f(x_2)}\{v\}.$$  

The $\epsilon$ and $\delta$ in the definition of differential privacy are “privacy parameters.” We can interpret them as a quantitative measure of how much privacy is lost when a sample is observed from the output distribution. As $\epsilon$ increases, the multiplicative bound on the difference in probabilities of output events becomes looser, increasing an attacker’s confidence in distinguishing two executions of $f$ on similar inputs. Experts recommend picking small $\epsilon$ values (e.g., 1.0) for meaningful privacy protection [Hsu et al. 2014]. On the other hand, $\delta$ bounds the probability of “catastrophic failure”—failure to provide any privacy at all. It should generally be very small.

DPCheck only guarantees to accept programs that achieve $(\epsilon, 0)$-differential privacy. However, nonzero $\delta$s will play a role in our analysis of false negatives—tests in which DPCheck fails to detect a non-$(\epsilon, 0)$-differentially private program.

An important tool for writing differentially private algorithms is the Laplace distribution. It is commonly defined as a continuous distribution, but rigorous proofs of differential privacy using
continuous distributions require sophisticated measure theory [Sato et al. 2019]. To simplify the required foundations, we follow previous work on program semantics and differential privacy [Albarghouthi and Hsu 2017; Hsu 2017; Reed and Pierce 2010; Wang et al. 2019; Zhang and Kifer 2017b] and assume a discretized, countable support over the reals for all representable numbers. We write $\omega$ for the constant gap\(^1\) between consecutive values—the granularity of the discretized domain. In this work, we assume all real values are drawn from this discretized domain with granularity $\omega$.

The discretized Laplace distribution is formally a two-sided geometric distribution [Ghosh et al. 2009]. It is parameterized by a center $c$ and a parameter $\alpha \in [0, 1]$. Ghosh et al. [2009] show the two-sided geometric distribution shares the important privacy properties of the continuous Laplace distribution. The continuous Laplace distribution is parameterized by a center $c$ and a width parameter $w$ that controls how centered the distribution is around $c$. Ghosh et al. [2009] also show a straightforward translation between the two-sided geometric distribution parameter $\alpha$ and the corresponding parameter $w$ for an equivalent discretized Laplace distribution. We will exclusively use the width parameter $w$ to parameterize discrete Laplace distributions in this work.

Each rectangle’s area in the graph to the right represents the probability assigned to the point at the center of the base of the rectangle; each rectangle has width exactly $\omega$—the granularity of the set of representable numbers. We write $\text{lap}_{c,w}$ for the discretized Laplace distribution with center $c$ and noise width $w$.

### 3 TYPES AND SYNTAX

DPCheck is a testing framework built around a probabilistic programming language embedded in the functional language Haskell. We will refer to the embedded language itself as DPCheck and to the rest of our system as the “testing framework.”

The DPCheck language offers a simple, purely functional, notation for differentially private programming. DPCheck provides base types $\text{Bool}$, $\text{Int}$, and $\text{Double}$, as well as three container types: tuples $(\tau, \sigma)$, lists $[\tau]$, and maps $\text{Map} \tau \sigma$. Finally, DPCheck supports probabilistic programming through a distribution monad $\circ$: the type $\circ \tau$ represents a (sub-)distribution over values of type $\tau$. Unlike a number of previously proposed functional languages for differential privacy [Gaboardi et al. 2013; Reed and Pierce 2010, etc.], DPCheck’s type system does not track privacy: this is the job of the testing framework.

We embed DPCheck inside Haskell, using methods developed by Svenningsson and Axelsson [2013, 2015] for the Feldspar language [Axelsson et al. 2010]. Their key insight is a technique for combining “deep” and “shallow” representations of programs, where a deep representation for a program is an abstract syntax tree, while a shallow representation maps language constructs directly to their semantics. Here, the deeply represented parts of the language can be used by DPCheck’s symbolic interpreter for static analysis, while the shallowly represented parts save engineering on things like surface syntax, standard libraries, and compilation by borrowing from the host language.

DPCheck’s deep representation in Haskell uses values of an indexed datatype $\text{Expr}$. For example, a Haskell term of type $\text{Expr} \text{Bool}$ represents a DPCheck program that yields values of type $\text{Bool}$ when evaluated. The type index ($\text{Bool}$) allows us to borrow Haskell’s typechecker to rule out ill-formed programs statically.

DPCheck’s most important syntactic forms are:

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\(^1\) We present DPCheck and its properties using this idealized set of representable reals, but our implementation relies on floating point numbers. This discrepancy and its impact on testing are discussed in Section 9.
These are Haskell constructors allowing us to build DPCheck programs that perform arithmetic using Add, compare numeric values using Lt, branch on boolean conditions using If, sample from Laplace distributions using Laplace, sequence probabilistic computations using Bind, and create point mass distributions using Return. The Bind and Return constructors endow DPCheck with monadic structure [Moggi 1989], allowing computations that return distributions to be coded in a natural style. Additionally, the purity of Haskell automatically rules out programs with potentially non-private side-effects (such as using the current system time to calculate an argument to Literal), because the types of the constructors only allow effect manipulation related to probability distributions.

As an example, we consider implementing the ReportNoisyMax algorithm in DPCheck. The ReportNoisyMax algorithm returns the index of the largest noisy value. Assuming an input list of [1, 2, 10] to ReportNoisyMax, and the noise values are 0.8, −1.2 and −0.9, ReportNoisyMax returns the index of the largest value from the noised list [1.8, 0.8, 9.1]. We can define ReportNoisyMax with the following Haskell syntax and a Haskell library combinator mapM that applies a function uniformly over a list:

```haskell
rnm :: [Expr Double] → Expr (O Int)
rnm (x:xs) = do
  xNoised ← lap x 1.0
  xsNoised ← mapM (λy. lap y 1.0) xs
  rnmAux xsNoised 0 0 xNoised
rnm [] = error "rnm received empty input"

rnmAux :: [Expr Double] → Expr Int → Expr Int → Expr Double → Expr (O Int)
rnmAux [] _ maxIdx _ = return maxIdx
rnmAux (x:xs) lastIdx maxIdx currMax = do
  let thisIdx = lastIdx + 1
  if (x > currMax)
    (rnmAux xs thisIdx thisIdx x)
  (rnmAux xs thisIdx maxIdx currMax)
```

The left pointing arrow ← is a syntactic sugar of the Bind constructor for sequencing probabilistic computations. Intuitively, the syntax x ← m represents a program fragment that runs the probabilistic computation m, giving the result of that computation a name x, and allows x to be used in following computations.

Conceptually, executing DPCheck programs through Haskell is a two-stage process: Haskell itself becomes a host language for constructing embedded DPCheck programs, which are then executed in a testing framework hosted by Haskell.

2The width parameter to the Laplace constructor has type Double instead of Expr Double. This implies the width parameter must be a statically chosen constant. This restriction simplifies the testing framework implementation, and it does not rule out any algorithms in our evaluation.

3In the current implementation, it is still possible to escape Expr’s restrictions by using unsafe language features to subvert Haskell’s type system. We could fix this by requiring DPCheck programmers to only use Safe Haskell [Terei et al. 2012], ruling out such subversions completely. In this paper, we assume the programmer is not adversarial and only wants to test programs that she genuinely believes are differentially private.
executed using an eval function (see Appendix E). This arrangement allows programmers to use convenient Haskell syntax and libraries when writing DPCheck programs.

An unusual aspect of DPCheck’s syntax design is that we rely on recursion in Haskell for—in effect—generating iterative DPCheck code. This design relies critically on Haskell’s support for lazy evaluation. Iteration through host-level recursion, if implemented in a strictly evaluated host language, would restrict the kinds of algorithms that can be represented, since a strict host-language would fully construct the syntax tree before sending it off to an interpreter. This implies all loops built with strict host-language recursion would be fully unrolled immediately. Some programs may have unbounded loops in some control flow paths, and such programs will cause divergence under this scheme. Fortunately, since Haskell is lazily evaluated, its runtime does not construct a value until the value’s contents are required. This means the Haskell code that constructs DPCheck syntax does not run until that syntax is required by the interpreter. In our implementation, we rely on Haskell’s lazy evaluation to automatically interleave DPCheck code execution and (potentially infinite) loop unrolling.

When the function \(rnm\) is applied to an input list \(xs\), we obtain a closed DPCheck program with type \(Expr (\bigcirc Int)\). This program contains data that represents DPCheck commands to be interpreted. The initial commands are Laplace sampling commands that add noise to the input list. These commands are generated with \(\lambda y. Laplace 1.0\), which builds the \(Laplace\) nodes in a DPCheck program’s syntax tree, and \(map\), which applies this function for all items in a list. We then loop over the noised data, keeping track of the maximum value and its index seen so far through If commands, and return the index at the end with the helper function \(rnmAux\).

4 SEMANTICS
We can straightforwardly encode DPCheck’s evaluation semantics using a function \(eval :: Expr r \rightarrow r\) (see Appendix E for details). Recall that a differentially private program whose syntax tree has type \(Expr (\bigcirc a)\) is necessarily probabilistic. Calling eval on such a program results in a distribution (value) of type \(\bigcirc a\), and we can sample from this distribution to acquire a concrete output value.

Besides this evaluation semantics, the testing framework also uses symbolic execution [King 1976] of DPCheck programs. During symbolic execution, concrete samples from Laplace distributions are replaced by symbols, allowing us to explore all possible control flows throughout a DPCheck program, even when branch conditions depend on sampled values. The symbolic execution process records the branch conditions along each control flow path; we call these recorded boolean conditions path conditions. As an example, if we run \(rnm\) on an input list of length two, then the symbolic execution process will create two symbolic values \(s_0\) and \(s_1\), one for each noised sample value. On one path, we explore under the assumption that the branch condition \(s_0 > s_1\) evaluates to True, and this results, say, in the output value 0; on the other path, we explore under the assumption that \(s_0 > s_1\) evaluates to False, and this results in the output value 1. So, in this example, symbolically executing \(rnm\) on this input list of length two yields two possible output values 0 and 1, with path conditions \(s_0 > s_1 = True\) and \(s_0 > s_1 = False\).

5 TESTING DIFFERENTIAL PRIVACY
To show how DPCheck tests differential privacy, we first review how proofs of differential privacy are constructed (5.1), with ReportNoisyMax as an example (5.2). Then we discuss how to adapt the same basic ideas to testing (5.3).

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4Although the ordinary DPCheck interpreter handles infinite syntax trees thanks to laziness, this doesn’t help the symbolic interpreter, because it tries to explore all paths. However, we can use auxiliary information to stop potentially infinite symbolic execution as soon as no further control flow path exploration is necessary. See Section 9.
5.1 Background: apRHL

Differential privacy experts often approach proofs in a “pointwise” fashion using Definition 6: Given two executions on similar inputs, first demonstrate that, no matter what concrete value the first execution of \( \text{rnm} \) yields, the second execution can also produce the same result; then show that the multiplicative difference in the total probability of all executions that lead to these identical outputs is bounded by the algorithm’s prescribed privacy parameter.

Barthe et al. [2016a] formalized the pointwise proof technique in a program logic for differential privacy called apRHL (approximate, relational Hoare logic). A key innovation of apRHL is its notion of “approximate lifting,” which abstracts over relations between distributions. Approximate lifting allows us to use a deterministic relation to simultaneously couple all samples from one distribution with samples from the other distribution, effectively reducing probabilistic reasoning to deterministic reasoning. The details of apRHL are beyond the scope of this paper ([Hsu 2017] is a readable introduction); here we just sketch the parts on which our testing strategy is most immediately based.

An apRHL judgment for differential privacy has the form \( \vdash c \sim_{(\epsilon, \delta)} \Phi \Rightarrow \Psi \), where \( c \) is a randomized program.\(^5\) It describes a relation between the output distributions on related executions of \( c \). Here, \( \Phi \) is a pre-condition on the free variables of \( c \) between related executions, and \( \Psi \) is a deterministic relation on the output samples of \( c \) that, behind the scenes, is lifted into the corresponding relation between the output distributions through apRHL’s approximate lifting machinery. The parameters \( (\epsilon, \delta) \) represent the “cost” of establishing the post-condition relation \( \Psi \). When \( \Psi \) asserts that the related outputs of \( c \) are equal, which implies differential privacy for the program \( c \), the parameters \( (\epsilon, \delta) \) are exactly the privacy cost.

For example, to state that \( \text{rnm} \) is \((2, 0)\)-differentially private using an apRHL judgment, we first encode the similarity relation of the inputs in the precondition. Two similar inputs of \( \text{rnm} \) must have coordinate-wise distance bounded by 1, which we encode with the assertion \( \forall i \left| x_{s_1}[i] - x_{s_2}[i] \right| < 1 \). Then, to encode differential privacy, we assert the outputs of two executions on similar inputs are identical in the postcondition:

\[
\vdash (out_1 \leftarrow \text{rnm} x_{s_1}) \sim_{(2, 0)} (out_2 \leftarrow \text{rnm} x_{s_2}) : \left( \forall i \left| x_{s_1}[i] - x_{s_2}[i] \right| < 1 \right) \Rightarrow (out_1 = out_2).
\]

Two key proof rules that formalize the pointwise proof technique are called Lap-Gen and PW-Eq. The Lap-Gen rule allows us to connect Laplace samples in two executions with a deterministic relation through apRHL’s approximate lifting theory. In particular, this deterministic relation allows us to assume (in the postcondition) that the samples drawn from the Laplace distribution on the two runs are at a fixed distance \( k \) apart.

\[
\text{Lap-Gen} \quad e = \left| k + e_1 - e_2 \right| / w
\]

\[
\vdash (x_1 \leftarrow \text{lap} e_1 w) \sim_{(\epsilon, 0)} (x_2 \leftarrow \text{lap} e_2 w) : \Phi \Rightarrow (x_1 + k = x_2)
\]

Readers encountering couplings for the first time may worry that this proof rule looks too good to be true, since it allows us to assume related samples are always at a deterministic distance \( k \) apart. What makes this work is that we are not considering some particular pair of samples; rather, we are relating the entire support from two Laplace distributions simultaneously. Furthermore, establishing

\(^5\)More generally, a relation can be established between syntactically different program fragments \( c_1 \) and \( c_2 \) instead of a single program \( c \). This general form of apRHL judgment is useful for intermediates steps of proofs; we ignore this refinement here for the sake of simplicity.

Proc. ACM Program. Lang., Vol. 4, No. OOPSLA, Article 165. Publication date: November 2020.
such a relation in the post-condition does not come for free: we are allowed to choose any \( k \), but the privacy cost also depends on \( k \).

This deterministic rule abstracts away direct reasoning over the related Laplace distributions. In Section 5.2, we will present an example that applies Lap-Gen to prove \( \text{rnm} \) is \((2, 0)\)-differentially private. We will refer to these \( k \) values used in Lap-Gen rules as “shift values.”

The PW-EQ rule is a formal description of the pointwise proof technique: if we can show that, for each possible output value \( r \in \tau \), one execution returning \( r \) implies that the other execution also returns \( r \), then these pointwise facts together constitute a complete differential privacy proof.\(^6\)

\[
\text{PW-EQ} \\
e_1, e_2 :: \tau \\ \forall r \in \tau, \vdash c_1 \sim (\varepsilon, 0) \quad c_2 : \Phi \implies (e_1 = r \rightarrow e_2 = r) \\
\vdash c_1 \sim (\varepsilon, 0) \quad c_2 : \Phi \implies (e_1 = e_2)
\]

The variables \( e_1 \) and \( e_2 \) here represent the output values of \( c_1 \) and \( c_2 \) respectively. The power of PW-EQ is that, for each \( r \in \tau \), we are allowed to choose a different shift value for any application of the Lap-Gen proof rule that may appear in the subderivation for this \( r \). To prove \( \text{rnm} \) is \((2, 0)\)-differentially private, we will apply exactly this strategy: given any output \( r \) from one execution of \( \text{rnm} \), we choose some sequence of shift values so that the noisy max also occurs at \( r \) in the second execution, forcing the second execution to also return \( r \).

With these two proof rules, the process of proving differential privacy for ReportNoisyMax reduces to a simpler goal: for each possible output of ReportNoisyMax, find a sequence of shift values to relate the Laplace samples between two executions so that their output values are identical.

Section 5.3 will show how we use the ideas behind the Lap-Gen and PW-EQ rules in designing our testing strategy for differential privacy. But first, a concrete example.

### 5.2 An Example Privacy Proof

Let’s consider a (paraphrase of a) proof by Dwork and Roth [2014] of \((2, 0)\)-differential privacy for \( \text{rnm} \) when the coordinate-wise distance (Definition 4) between the inputs is bounded by 1. We present this proof by sketching applications of Lap-Gen and PW-EQ, and emphasizing steps in the proof that will become key ingredients in our testing algorithm by putting boxes around equations. This proof can be carried out formally in apRHL, but we present it informally to avoid getting bogged down in details.

**Theorem 7.** ReportNoisyMax is \((2, 0)\)-differentially private.

**Proof.** The implementation of ReportNoisyMax creates a list of noised values based on its input. Let \( r \) be any possible output of running \( \text{rnm} \). Let \( \text{argmax} \) be the function that returns the index of the largest value in a list. When \( \text{rnm} \) runs on an list of input values, it first adds Laplace noise to each of these values, then iterates over the noised values, keeping track of the index of the maximum value seen so far, and finally returns that index. Write \( q_s^1 \) and \( q_s^2 \) for the two intermediate lists of noised values from \( q_{s1} \) and \( q_{s2} \). Then, if \( r \) is the result of running \( \text{rnm} \) on \( q_{s1} \), it is easy to see that \( r = \text{argmax} q_s^1 \).

We next apply the pointwise proof technique described in Section 5.1. Since we assumed one execution of \( \text{rnm} \) returned index \( r \), we need to show some control flow through the run of \( \text{rnm} \) on \( q_{s2} \) yields the same \( r \). We demonstrate such a control flow by carefully choosing the shift values

\(^6\)This rule may appear to be showing—counterintuitively—that different executions of a randomized program produce identical results. Of course, any two concrete executions of a randomized program will almost certainly produce different results. But we are not actually reasoning here about some particular pair of executions. Rather, we are reasoning simultaneously about all pairs of coupled executions.
introduced in the Lap-Gen proof rule. Concretely, we need to ensure \( r = \arg\max q_s' \), under some choice of the shift values that connects each \( q_s'[i] \) and \( q_s[i] \).

We proceed by applying Lap-Gen to each pair of noised values with the following choice of shift values, coupling each \( q_s'[i] \) with \( q_s[i] \) such that \( q_s'[i] = q_s'[i] + \text{shift}_i \):

\[
\text{shift}_i = \begin{cases} 
1 & \text{if } i == r \\
q_s[i] - q_s'[i] & \text{otherwise}
\end{cases}
\]  

(2)

This choice implies that the maximum value in the second noised array also occurs at index \( r \); we know that \(-1 < q_s[i] - q_s'[i] < 1\) for all \( i \), since \( q_s[i] \) and \( q_s'[i] \) are similar inputs. In particular, \( q_s[i] - q_s'[i] < 1 \). Adding \( q_s'[i] \) to both sides of this inequality yields

\[
q_s'[i] + (q_s[i] - q_s'[i]) < q_s'[i] + 1.
\]

Since \( q_s'[i] = q_s'[i] + (q_s[i] - q_s'[i]) \) if \( i \neq r \) and \( q_s'[r] = q_s'[r] + 1 \), it follows that \( r = \arg\max q_s' \).

We have shown so far that, for any output index \( r \) from the first execution, it is possible for \( r_{nm} \) to produce the same result \( r \) on the similar input. Next we need to calculate the \( \epsilon \) privacy cost incurred by using Lap-Gen for connecting \( q_s' \) with \( q_s' \), and prove \( \epsilon \) is at most 2.

Consider the cost of an application of Lap-Gen between a given pair of \( i \)th sample values \( q_s'[i] \) and \( q_s'[i] \) in the two runs; call this \( \text{cost}_i \). Under apRHL, the total privacy cost is bounded by the sum of the \( \text{cost}_i \)'s, so we need to show \( \sum_i \text{cost}_i < 2 \).

To bound the sum, let us first consider the cost values for indices \( i \neq r \). We know \( q_s'[i] \) is a sample from the distribution \( \text{Lap}_{q_s[i]},1.0 \) and similarly \( q_s'[i] \) is a sample from \( \text{Lap}_{q_s[i]},1.0 \). Using Lap-Gen, we can conclude

\[
\text{cost}_i = \left| \left( (q_s'[i] + q_s[i] - q_s'[i]) - q_s[i] \right) - q_s'[i] \right| = 0.
\]

On the other hand, if \( i = r \), let \( s = q_s'[r] - q_s[r] \). We again use Lap-Gen to conclude:

\[
\text{cost}_r = \left| (q_s'[r] - q_s[r]) - (q_s'[r] - q_s[r]) \right| = |s - (q_s'[r] + 1 - q_s[r])| \quad \text{by eq. (2) and assumption of } s
\]

\[
= |s - (q_s[r] + s + 1 - q_s[r])| \quad \text{by assumption of } s
\]

\[
< 1 + 1 = 2 \quad \text{by the triangle inequality}
\]

(3)

Since cost values are 0 for indices \( i \neq r \) and less than 2 when \( i = r \), the total cost is less than 2. That is, \( r_{nm} \) is \((2,0)\)-differentially private. \( \square \)

### 5.3 From Proving to Testing

The key steps in the above proof are:

1. Assume an unknown but fixed output \( r \) from one execution.
2. Select a sequence of shift values (eq. (2)) to connect the samples from one execution with samples from the other execution.
3. Show that the second execution, whose Laplace samples are fixed through the shift values, leads to the same output \( r \).
4. Compute the total privacy cost of this pair of executions as the sum of the individual cost values induced by the chosen sequence of shift values and show this total is less than the prescribed \( \epsilon = 2 \) (eq. (3)).
In order to convert this to a testing procedure, we need to check that, for a run of \( r_{nm} \) on \( q_{s1} \), there exists a dual execution of \( r_{nm} \) on \( q_{s2} \) leading to the same output, such that the Laplace samples in these two executions can be connected through a sequence of shift values, while keeping the total privacy cost induced by the shift values under 2.0. However, we need to be careful about how many distinct sequences of shift values are allowed for constructing dual executions. The PW-EQ rule allows us to choose one sequence of shift values for each unique output \( r \) that \( r_{nm} \) may return. Thus, for testing, we must group the runs of \( r_{nm} \) on \( q_{s1} \) by their output, and then try to find a single sequence of shift values for each group, such that this single sequence of shift values leads to corresponding executions of \( r_{nm} \) on \( q_{s2} \) with the same output.

With this in mind, the first step in testing is to run \( r_{nm} \) on \( q_{s1} \) a large number of times and group the runs by their final output values.

Next we need to find the shift values. In steps 2 and 3 of the proof above, we used expert insight to select shift values that allowed us to show that the dual execution must result in the same output \( r \). When testing, we begin with hypothetical (symbolic) shift values and hope to work out their concrete values later: we create one symbolic shift value for each \( q_{s1}[i] \) per group, and pair \( q_{s1}[i] \) with a \( q_{s2}[i] \) through the following equation

\[
q_{s2}[i] = q_{s1}[i] + \text{shift}_i. \tag{4}
\]

This equation is exactly how \( \text{Lap-Gen} \) allows us to connect two Laplace samples. (Throughout this discussion, we highlight the important symbolic formulas created in the testing steps by boxing them.) This also induces a symbolic cost using \( \text{Lap-Gen} \), where

\[
\text{cost}_i = \frac{|(q_{s2}[i] - q_{s2}[i]) - (q_{s1}[i] - q_{s1}[i])|}{w} = \frac{|q_{s1}[i] + \text{shift}_i - q_{s2}[i]|}{w}. \tag{5}
\]

Here \( q_{s1}[i] \) is the concrete center supplied to the \( i \)th call to the sampling instruction in the first execution, \( \text{shift}_i \) is a fresh symbolic variable, \( q_{s2}[i] \) is the concrete center used in the \( i \)th sampling instruction in the second execution, and \( w \) is the (known constant) parameter controlling the width of the Laplace distributions used in \( r_{nm} \). Thus, both \( q_{s2}[i] \) and \( \text{cost}_i \) can be represented as symbolic expressions coupled with concrete execution traces using the formulas above if we know what \( q_{s1}[i] \) and \( q_{s2}[i] \) are.

To capture \( q_{s1} \) and gather many independent samples of \( q_{s1}' \) for testing, we repeatedly run \( r_{nm} \) on \( q_{s1} \) with a special interpreter that instruments Laplace sample instructions, recording the center, width, and returned sample value for each. We group each unique output \( r \) together with all sequences of Laplace sample and parameters that lead to the output \( r \) into a bucket.

For each bucket \( bkt \), we need to show there exists a coupling that connects all sampled traces in \( bkt \) to the Laplace samples from running on \( q_{s2} \), such that this coupling leads the dual execution on \( q_{s2} \) to the same result \( r \) from \( bkt \). To find such a coupling, we symbolically execute \( r_{nm} \) with symbolic samples \( q_{s2}' \), observing all possible outputs of \( r_{nm} \) along all of the control flow paths. Among these paths, some will return \( r \). We gather all of the path conditions from control flow paths that lead to \( r \). These path conditions are then used to constrain the shift values from eq. (4), so that the coupled dual execution only takes the paths that yield the same output \( r \). We use \( \Phi_r \) to denote the symbolic formula that encodes the path conditions, and \( \Omega_r \) to denote the symbolic formula that encodes the shift equations (eq. (4)). These two symbolic formulas form the testing analog of step 2 and 3 from the proof.
Finally, to bound the total privacy cost (as in step 4 of the proof), we create another symbolic formula using the symbolic expressions for each $cost_i$:

$$\sum_i cost_i \leq 2$$  \hspace{2cm} (6)

Together, the boxed symbolic formulas constitute a query that can be dispatched to an off-the-shelf SMT solver—we use Z3 [De Moura and Bjørner 2008]. If the solver returns a satisfying model for these constraints, then we know the distributions produced by running $rnm$ on this particular pair of $q_s_1$ and $q_s_2$ likely satisfy 2-differential privacy. Of course, this does not guarantee differential privacy, because we used sampled traces of Laplace calls to describe properties of a potential proof for differential privacy, instead of universally quantifying over all possible samples in one execution, as in the proof.

At the core of both the proof and the SMT formula is the relation between $q_s_1'$ and $q_s_2'$. The proof demonstrates that there exist shift values such that the related Laplace samples satisfy the privacy cost bounds. The testing process also checks for the existence of such shift values; however, it has a chance of admitting programs that are not $(\epsilon, 0)$-differentially private, because testing does not produce a complete proof of differential privacy—it only checks whether dual executions satisfying differential privacy exist on some set of sampled traces. There are two important test parameters—number of pairs of randomly generated similar inputs, and number of sampled traces collected on a given pair of similar input—that can be independently tuned to make tests more difficult to pass for faulty programs. Intuitively, as we increase both test parameters, it should be less and less likely that our testing process accepts a faulty program. We call programs that are faulty but slip past DPCheck’s testing framework false negatives.

A non-$(\epsilon, 0)$-differentially private program may be faulty for two reasons: 1) it may have a non-zero $\delta$ failure probability, and 2) there may exist similar inputs $(x_1, x_2)$ for which the program produces distributions that do not satisfy definition 6 with the given $\epsilon$. Note that the definitions of differential privacy (5 and 6) require the relation on output distributions in definition 6 to hold on all similar inputs. Although DPCheck’s testing framework can check that eq. (1) holds on a pair of fixed similar inputs by checking the existence of shift values and dual executions, it can never exhaustively check that eq. (1) holds on a potentially infinite set of similar pairs of inputs.

Instead, DPCheck gives probabilistic guarantees using the framework of random differential privacy [Hall et al. 2013]. In the next section, we review random differential privacy and use it to state our main theoretical testing guarantee (Theorem 12).

### 5.4 Implementation

We now describe the testing process more concretely, showing type signatures of Haskell functions that implement key testing steps.

DPCheck’s testing framework takes as inputs a program under test, $\text{prog} :: \sigma \rightarrow \text{Expr (\sigma \tau)}$, a generator, $\text{gen} :: \text{Gen (\sigma, \sigma)}$, that produces pairs of similar inputs for $\text{prog}$, and a privacy parameter $\epsilon$. It then checks, for a large number of $(x_1, x_2)$ pairs produced by $\text{gen}$, that the distributions produced by running $\text{prog} x_1$ and $\text{prog} x_2$ satisfy eq. (1). If it ever finds one that does not, it rejects $\text{prog}$; otherwise it validates $\text{prog}$ as likely to be $(\epsilon, 0)$-differentially private. (We discuss what “likely” means more formally in Section 6. We also show how much time DPCheck takes to reject buggy benchmark programs in Figure 3 in the appendices.) The more tests a program passes, the more likely the program really is $(\epsilon, 0)$-differentially private. Our experiments on benchmark algorithms show that DPCheck rejects many faulty programs within 10 seconds, but it may take significantly

---

7 A value of type $\text{Gen } \tau$ is a function that takes a seed and produces a pseudo-random value of type $\tau$.  

Proc. ACM Program. Lang., Vol. 4, No. OOPSLA, Article 165. Publication date: November 2020.
longer testing time to reject algorithms that only demonstrate privacy violations on larger inputs (see Appendix D for detailed benchmark study).

To verify that the distributions produced by a particular pair of similar inputs are related, we need to construct a coupling between the Laplace distribution samples used by \( \text{prog}_1 \) and \( \text{prog}_2 \) using the methods described in Section 5. We first acquire concrete sample values from some large number (call it \( N \)) of runs of \( \text{prog}_1 \).\(^8\) We denote the output from the \( i \)th run by \( r_i \) and the sample trace from each of these runs by \( tr_i \) :: Trace, where

\[
\begin{align*}
\text{type SampleInfo} &= (\text{Double}, \text{Double}, \text{Double}) \\
\text{type Trace} &= [\text{SampleInfo}]
\end{align*}
\]

and where the following projections extract sample, center, and width values from a SampleInfo:

\[
\begin{align*}
\text{sample}, \text{center}, \text{width} &: \text{SampleInfo} \to \text{Double}
\end{align*}
\]

With the collected outputs \( r_1, r_2, \ldots, r_N \) and traces \( tr_1, tr_2, \ldots, tr_N \), we perform a “bucketing” process so that all traces that lead to the same output value \( r \) are grouped together.

\[
\begin{align*}
\text{type Buckets} \; r &= \text{Map} \; r \; \text{[Trace]} \\
\text{bucket} &: [(r, \text{Trace})] \to \text{Buckets} \; r
\end{align*}
\]

A value of this map type represents a collection of buckets; a particular key-value pair (of an output value with its associated list of traces) is a single bucket.

Next, we perform symbolic execution on \( \text{prog}_2 \). To do this, we first perform a simple program transformation: \( \text{streamline} :: \text{Expr} \; (\tau) \to [\text{Expr} \; (\tau)] \). This transformation repeatedly replaces a program containing \( \text{If} \) commands with two new programs in which the \( \text{If} \) command is replaced by sequencing \( \text{Assert} \; \text{cond} \) with the commands in the true branch and sequencing \( \text{Assert} \; (\neg \text{cond}) \) with commands in the false branch.

\[
\begin{align*}
data \; \text{Expr} \; a \; \text{where} \\
\begin{align*}
\text{Assert} &: \text{Expr} \; \text{Bool} \to \text{Expr} \; (\tau) \\
\text{Sequence} &: \text{Expr} \; (\tau) \to \text{Expr} \; (\tau \; a) \to \text{Expr} \; (\tau \; a)
\end{align*}
\end{align*}
\]

This simplistic approach produces \( 2^n \) straight-line programs in the worst case, where \( n \) is the number of \( \text{If} \) statements; we will discuss a type-driven optimization adapted from [Torlak and Bodik 2014] for speeding up symbolic execution in Section 8.

Note that \( \text{streamline} \) would actually diverge on infinite syntax trees if Haskell were not lazy. Our symbolic interpreter uses information gathered by instrumented implementation to cut off infinite symbolic execution as soon as possible. We use this early cutoff trick in the evaluation of PrivTree; the trick and some directions for generalizing it are discussed in more detail in Section 9.

The programs resulting from this transformation are free of conditional branches, instead explicitly encoding path conditions using \( \text{Assert} \) nodes. We next take these transformed programs and run symbolic execution guided by the trace buckets from the instrumented executions above.

Consider a particular set of executions that lead to the output value \( r \), and let the associated trace bucket contain the traces \( tr_{i_1}, tr_{i_2}, \ldots, tr_{i_k} \), where \( i_1, i_2, \ldots, i_k \) are the indices of instrumented runs that produced \( r \). We then search for paths that produce the same output \( r \) and build eq. (4) between the concrete sampled traces and the symbolic Laplace samples. For each trace \( tr_{ik} \), we pair it with symbolic Laplace samples as follows: on the \( j \)th call to the Laplace sampling instruction during symbolic execution, we create a fresh symbolic value \( \text{lap}_{ik}[j] = \text{sample}(tr_{ik}[j]) + \text{shift}_j \). Let

\(^8\)For invalidating incorrect algorithms, we can start with small \( N \) (such as 50) and keep increasing \( N \) by a factor of 10 until the bug is discovered. For validating correct algorithms, \( N \) should be chosen according to the lower bound on \( m \) in Theorem 12. However, currently the computational cost of running tests with large \( N \) makes validation prohibitively slow. We discuss this issue in Section 9.
ψ_k = ∨_j lap_k [j] = sample (tr_k [j]) + shift, and Ψ_r = ∨_k ψ_k, and let Φ_r encode the disjunction of the path conditions for all control flow paths that lead to the output r.

The final formula Ψ_r ∧ Φ_r ∧ (∑_n cost_n ≤ ε) asserts that prog produces the same output r within the prescribed privacy cost ε. We perform the same process for each unique output r observed from instrumented executions. If these formulas are all satisfiable, we consider this test a passing test case, and we do not reject the claim of (ε, 0)-differential privacy. On the other hand, if, for some output r, Z3 tells us that the formula is not satisfiable, then the program under test does not have a point-wise proof of ε-differential privacy using the proof template discussed in Section 5. This is not a disproof of differential privacy, since the proof template we are using is not complete (there are algorithms whose privacy proofs do not follow this pattern), but it is at least a signal that should prompt us to look at the program skeptically.

The code below sketches the testing process on an input procedure to test prog, a pair of neighboring inputs x1 and x2, an expected privacy parameter eps, and the number of sampled traces to draw for testing ntraces.

\begin{verbatim}
expectDP prog x1 x2 eps ntraces = do
  buckets ← instrumentedExec ntraces prog x1
  constraints ← symbolicExec buckets streamlineProgs
  solverResults ← runSolver eps constraints
  expect (all isOk solverResults)
\end{verbatim}

A distinctive element of our design is the combination of instrumented execution and symbolic execution. An alternative would be to run symbolic execution on both inputs using relational symbolic execution [Farina et al. 2017], then universally quantify over the Laplace samples in one execution, as demonstrated in the analysis of rnm from Section 5. Using relational symbolic execution would bring more confidence to the differential privacy property of the program under test, since the satisfying model from the SMT solver will serve as a formal proof of eq. (1) for the pair of output distributions. However, this approach produces more complex symbolic formulas that may significantly slow down Z3. To strike a balance between execution time and confidence gained from a passing test, we choose to combine instrumented execution and symbolic execution.

6 ASYMPTOTIC PRIVACY GUARANTEES

In this section, we introduce random differential privacy (RDP), and use RDP to quantitatively define “well-behaved” and “ill-behaved” programs. We present Theorem 12, which gives an upper bound on the probability of DPCheck falsely accepting ill-behaved programs. However, DPCheck has a scaling bottleneck that prevents us from applying Theorem 12 to produce meaningful guarantees on the probability of DPCheck falsely accepting ill-behaved programs. However, DPCheck has a scaling bottleneck that prevents us from applying Theorem 12 to produce meaningful guarantees on the probability of DPCheck falsely accepting ill-behaved programs.

To introduce random differential privacy, let us first consider an alternative view of the privacy parameters ε and δ.

\textbf{Definition 8.} Let μ_1, μ_2 :: ∅ τ be two distributions with identical support. Define a function f :: τ → ℝ. Let f (v) = ln \frac{μ_1 (v)}{μ_2 (v)}. The privacy loss random variable is a distribution pv (μ_1, μ_2) ∈ ∅ ℝ defined as: pv (μ_1, μ_2) (x) = ∑_v∈supp (μ_1) s.t. f (v) = x μ_1 (v).

Informally, this distribution can also be expressed as f (v), where the random variable v ∼ μ_1.

\textbf{Definition 9 ([Kasiviswanathan and Smith 2014]).} Two distributions μ_1 and μ_2 are (ε, δ)-pointwise indistinguishable if the probability mass of pv (μ_1, μ_2) in the interval [−ε, ε] is at least 1 − δ.
Note that \((\epsilon, \delta)\)-pointwise indistinguishability implies eq. (1) in the definition of differential privacy (Definition 5). Furthermore, the proof template we introduced in Section 5 constructs proofs of \((\epsilon, 0)\)-pointwise indistinguishability.

**Definition 10** ([Hall et al. 2013]). Assume a fixed distribution over similar inputs \(I :: \bigcirc (\sigma \times \sigma)\). A randomized program \(f :: \sigma \mapsto \bigcirc \tau\) is \((\epsilon, \delta, \alpha)\)-random differentially private if, with probability at least \(1 - \alpha\), sampling similar inputs \((x_1, x_2)\) from \(I\) leads to \((\epsilon, \delta)\)-pointwise indistinguishable distributions \(f(x_1)\) and \(f(x_2)\).

We can give some intuition for Definition 10 by considering visual graphs of the privacy loss random variable under given similar inputs. First, assume a distribution \(I\) of similar inputs. Let us sample nine pairs of similar inputs from \(I\) and draw their privacy loss random variables as a graph centered at 0 on the horizontal axis. We shade each graph with blue if its area is at least \(1 - \delta\) in the interval \([-\epsilon, \epsilon]\), and red otherwise.

Under Definition 9, the two distributions are \((\epsilon, \delta)\)-pointwise indistinguishable if the shaded area in the interval \([-\epsilon, \epsilon]\) is at least \(1 - \delta\). Using this visual criterion, the definition of \((\epsilon, \delta, \alpha)\)-random differential privacy says that, if we repeatedly sample similar inputs from \(I\) and inspect the corresponding graph of the privacy loss random variable, then with probability at least \(1 - \alpha\), we will see a graph whose shaded area is at least \(1 - \delta\). The example graphs show seven privacy loss random variable distributions whose shaded area in the interval \([-\epsilon, \epsilon]\) is at least \(1 - \delta\) (colored in blue), and two whose shaded area is less than \(1 - \delta\) (colored in red). Visually, if we extend this grid of graphs with privacy loss random variables derived from more and more sampled similar inputs from \(I\), then the parameter \(\alpha\) bounds the fraction of red graphs in the entire grid.

We say a program \(f\) is \(\epsilon\)-well-behaved if \(f\) is \((\epsilon, 0)\)-differentially private and if \(f\)'s path conditions are exactly necessary and sufficient for proving \((\epsilon, 0)\)-differential privacy through the pointwise proof technique.

**Lemma 11.** If a program \(f\) is \(\epsilon\)-well-behaved, then \(f\) is never rejected by DPCheck's testing framework when tested with any \(\epsilon' \geq \epsilon\).

The proof of Lemma 11 can be found in Appendix A.

We might hope that all \((\epsilon, 0)\)-differentially private programs are \(\epsilon\)-well-behaved, but this does not hold in general because DPCheck assumes that proofs of differential privacy for these programs have a particular structure (Section 5): the path conditions for these programs must be the necessary and sufficient conditions for its privacy properties. In Section 7 we will see the ReportNoisyMaxWithGap algorithm, whose optimal \(\epsilon\) is rejected by DPCheck because its path conditions are sufficient but not necessary. (DPCheck does accept ReportNoisyMaxWithGap with a non-optimal \(\epsilon\), for which its path conditions are both necessary and sufficient.)

Conversely, assume a fixed distribution \(I\) of similar inputs and a program \(f\). We say that \(f\) is \((\epsilon, \delta, \alpha)\)-ill-behaved if, given fixed \(\epsilon\), all valid random differential privacy parameters \((\epsilon, \delta_f, \alpha_f)\) for \(f\) satisfy \(\delta_f > \delta\) and \(\alpha_f > \alpha\). Intuitively, an \((\epsilon, \delta, \alpha)\)-ill-behaved program has a high probability of two kinds of failure—its “catastrophic failure” probability is at least \(\delta\) when executed on “good” similar inputs from \(I\), and there is at least an \(\alpha\) probability of draws from \(I\) yielding “bad” similar inputs such that, when \(f\) runs on these inputs, there is a greater than \(\delta\) chance that \(f\) will induce more privacy cost than \(\epsilon\).
Theorem 12. Given a fixed distribution $I$ over similar inputs, a positive integer $k$, an $(\epsilon + k\omega, \delta, \alpha)$-ill-behaved program $f$, and a positive value $\theta$, if

(1) $f$ makes at most $k$ calls to the Laplace sampling instructions in one execution,
(2) $f$ has at most $n$ output buckets (Section 5), and
(3) DPCheck failed to reject $f$, because DPCheck discovered $\text{shift}_i$ values (used in eq. (4)) that are valid for the sampled execution traces,

then, as long as we had run at least $d$ tests with independently sampled inputs and with at least $m$ sampled traces in each test, the probability of such a failure invalidating the claim of $(\epsilon, 0)$-differential privacy for $f$ is at most $e^{-d(\theta + \alpha)}$, as long as $m \geq \frac{1}{\delta} (\theta + nk \ln 2 + nk \ln \frac{C_2 - C_1}{\omega})$ where $C_1 = \min_i \text{shift}_i$ and $C_2 = \max_i \text{shift}_i$.

The proof can be found in Appendix B.

In practice, the $\text{shift}_i$ values are bounded by machine limits. Even if we take $C_1$ as the smallest double-precision floating point number, $C_2$ as the largest, and $\omega$ as the smallest gap between two double-precision floats, the factor $\ln \frac{|C_2 - C_1|}{\omega} = \ln \frac{1.798 \times 10^{308}}{2^{-52}}$ is smaller than 747, a small requirement on the number of sampled traces even in this extreme case.

Note that there is a non-zero gap of $k\omega$ between the tested privacy level $\epsilon$ for $f$ and the level $(\epsilon + k\omega, \delta, \alpha)$ to which $f$ is ill-behaved. This means DPCheck is only guaranteed to catch bugs with high probability if $f$’s behavior differs enough from the claimed levels of differential privacy. However, since $\omega$ is the granularity of the discretized domain, the value $k\omega$ is typically very small. (For example, the gap between two double-precision floating point numbers in the interval $[0, 1)$ is $2^{-52}$.) Conversely, if $f$’s behavior is not very far from its claimed level of differential privacy, then we cannot give any guarantees about the probability of falsely accepting $f$.

Although we presented the testing strategy through the example algorithm $rnm$, the testing framework requires no special input particular to $rnm$. In fact, we can use the testing strategy described here to check the differential privacy property of many other algorithms, as long as these algorithms’ differential privacy proofs follow the general template listed in steps 1 to 4 and their program control flow conditions are neccessary and sufficient for differential privacy. In Appendix D, we describe a variety of algorithms tested using this strategy; we also present the ReportNoisyMaxWithGap algorithm, whose optimal privacy cost cannot be established using this strategy, though it can still be validated as differentially private with a non-optimal $\epsilon$. In Section 7, we discuss the details of a practical workflow that applies DPCheck to develop, test, and integrate core differential privacy mechanisms with an existing software system designed for the 2020 US Census.

7 EVALUATION

We seek to answer the following questions:

1. How expressive is DPCheck’s testing strategy?
2. Can DPCheck assist implementations of real-world systems that use differential privacy?

To answer these questions, we first used DPCheck to distinguish private and non-private variants of 10 differential privacy benchmark algorithms from the literature. Second, we used DPCheck in a practical workflow to re-implement and test the core differential privacy mechanism from the Disclosure Avoidance System (DAS) for the 2020 US Census [Petti and Flaxman 2019]. To save space, we concentrate on the latter experiment and give just a brief summary of the former; further details can be found in Appendix C.
7.1 Benchmark Algorithms (Summary)

We used DPCheck to implement a suite of benchmark algorithms from the literature. For each algorithm, we built both a correct implementation and several non-differentially-private variants. We expected DPCheck to accept the correct implementation according to Lemma 11 and to detect and reject all non-differentially-private variants. The results are shown in Figure 1. We place a ✓ in the “Correct” row if DPCheck accepts the correct implementation under the algorithm’s optimal privacy cost, and we put a ✓ in the “Buggy” row if DPCheck rejects all non-differentially private variants of this algorithm. For the ReportNoisyMaxWithGap (rnmGap) algorithm, we write ✓∗ to indicate that DPCheck does not accept its correct implementation with the optimal privacy cost but does accept with twice the optimal privacy cost. Details can be found in Appendix C.

We also compare the coverage over these ten benchmark algorithms between DPCheck and related frameworks in Figure 1. For each related framework, ✓ indicates that the framework in the corresponding row has successfully analyzed the algorithm in the corresponding column. A ✗ represents that the framework in the corresponding row cannot be used to test or verify the algorithm in the corresponding column. A gray ✓? indicates that the authors of this framework have not presented an evaluation of the algorithm in the corresponding column either in its publication or its released software artifact, but that we believe the framework has enough expressive power to handle this algorithm. Similarly, a gray ✗? indicates our belief that the algorithm in the corresponding column is beyond the capabilities of the framework.

The table shows that DPCheck correctly accepts private implementations and rejects faulty variants of all previous studied benchmark algorithms. Additionally, it is the first framework able to distinguish between correct and faulty variants of PrivTree.

PrivTree is challenging for automatic verification of differential privacy for at least two reasons. The first is that PrivTree terminates probabilistically, i.e., the probability of PrivTree not terminating after \( n \) iterations of its main loop diminishes as \( n \) increases. The second reason is that the privacy analysis used in the PrivTree’s privacy proof involves intermediate privacy costs that depend on input values [Zhang and Kifer 2017a].

The first characteristic poses issues for static analyses (including DPCheck’s symbolic interpreter), as we cannot statically know how many iterations PrivTree will run. Fortunately, as discussed in Section 5, our symbolic interpreter only needs to produce trees that match those observed in the instrumented execution. The trees produced by PrivTree contain strictly more nodes as loop iteration counts increase. Thus, our symbolic interpreter can stop searching for matching trees once it realizes that all future iterations will produces trees that cannot match those observed in the instrumented executions.
The second characteristic is a serious issue for tools aimed at automatically generating proofs of differential privacy. Since such tools need to reason over all possible input values, intermediate privacy costs that depend on inputs must be represented by expressions over these unknown inputs. For PrivTree, these intermediate privacy cost expressions involve non-linear arithmetic, an undecidable theory that can only be solved in a best-effort way by SMT solvers.

By contrast, DPCheck’s testing framework chooses a pair of concrete input values and evaluates PrivTree over these inputs. This allows the testing framework to represent intermediate privacy costs with much simpler symbolic expressions.

The two most challenging algorithms for existing validation and verification frameworks are PrivTree and ReportNoisyMaxWithGap. StatDP and DP-Finder cannot process the complex output type of PrivTree—tree data structures, since both of these tools rely on heuristics that are designed to detect DP violations on numerical outputs. LightDP and ShadowDP, the two type-system based tools, fail at PrivTree due to the unbounded probabilistically terminating main loop. We believe PrivTree is out-of-scope for the proof synthesis framework for the same reason—the proof synthesis framework would not be able to analyze an unbounded probabilistically terminating main loop.

We believe StatDP and DP-Finder’s implementations could be improved with additional heuristics to handle the output datatype of ReportNoisyMaxWithGap, while LightDP and ShadowDP require changes to their type systems. The proof synthesis artifact is unavailable, but since it is based on apRHL, and apRHL is expressive enough to prove differential privacy for ReportNoisyMaxWithGap, we believe the proof synthesis framework can handle ReportNoisyMaxWithGap.

7.2 Disclosure Avoidance System

Every ten years, the US Census Bureau conducts a national survey to count the total population in the United States. This survey, referred to as the Decennial Census, provides critical information for the Federal Government to adjust allocation of funds, as well as representation in the US House of Representatives, where each state gets a number of delegates proportional to its population. For the 2020 Census, the US Census Bureau developed an open-source Disclosure Avoidance System (DAS) to aggregate raw survey data into population counts. DAS applies differential privacy to protect the privacy of survey participants.

DAS aims to produce differentially private population counts for each geographical unit within each of the six geographical levels in the US: the whole nation, individual states, counties, “census tracts,” “block groups,” and single city blocks [Petti and Flaxman 2019]. This process would be straightforward if the only requirement were differential privacy: just count the population in each geographical unit and add appropriately sampled noise to each count. However, a census report produced through this idealized process would contain inconsistent counts due to the added noise: for example, the population count in a state might well be different from the sum of the population counts from all counties within the state, and there might even be negative counts for some geographical units where the precise count before adding noise was small. The Census Bureau has a list of data requirements that rules out these inconsistencies, and the final report produced by DAS must satisfy these requirements [Petti and Flaxman 2019].

To address these issues, DAS applies a so-called “TopDown Algorithm.” The TopDown Algorithm consists of 2 phases. The first phase calculates precise (and secret) counts for all geographical units, then adds appropriately sampled noise to produce noisy public counts.

The second phase iterates over the geographical hierarchy, ordered from coarsest (nation) to finest (block). Each step takes the noisy counts from two adjacent levels and perturbs them using constrained optimization. The constrained optimization process perturbs noisy counts to rule out inconsistencies, while the optimization objective keeps the overall perturbation of noisy counts as small as possible.
Since the outputs from the first phase already satisfy differential privacy, the outputs from the second phase do too (because differential privacy is robust to postprocessing [Dwork et al. 2006]). The second phase does not introduce any additional randomly sampled noise.

DAS is an interesting target for differential privacy testing due to the social importance of DAS’s privacy properties. Furthermore, by applying DPCheck on a piece of large, real-world software artifact like DAS, we gain insight on how DPCheck can assist in developing real-world software systems that interact with sensitive data through differential privacy in the future.

For this evaluation, we manually re-implemented, in DPCheck, the core privacy mechanism that calculates the scale of noise distributions and releases the noisy counts for each geographical unit. We used the DPCheck testing framework to check that this mechanism is, indeed, differentially private, and verified DPCheck can reject faulty variants of this mechanism. The faulty variants were edited from the correct implementation to simulate common programming mistakes—sampling noise with wrong parameters, and iterating over the input list with off-by-one errors. Finally, we mechanically extracted the DPCheck code to Python3 code (by pretty-printing, essentially).

Figure 2 shows the DPCheck code that re-implements the core privacy mechanism. The function geometricFixedSens takes a precise count trueAnswer, a parameter sens that bounds the difference of trueAnswer between similar inputs, and the amount of privacy budget allocated for adding noise to this value eps. Here, trueAnswer corresponds to the accurate and secret count of population in a geographical unit. The value of sens measures how much the precise count of this geographical unit can change between two similar inputs of the Census data; it is determined manually by Census scientists. The value of eps is also determined by Census scientists, to provide a suitable level of privacy protection. From sens and eps, we can calculate the appropriate α parameter of the
two-sided geometric distribution and add sampled noise to trueAnswer to produce a differentially private noisy answer.

The function loopGeometricFixedSens takes a list of input values in the form

\[(\text{trueAnswer}_1, (\text{sens}_1, \text{eps}_1)), \ldots, (\text{trueAnswer}_n, (\text{sens}_n, \text{eps}_n))\]

and creates an output list of the same size that contains the noisy answers for each trueAnswer_i in the list. The privacy guarantee of this procedure is that it is \((\sum_i \text{eps}_i, 0)\)-differentially private.

We try to experimentally detect violations of differential privacy from loopGeometricFixedSens by randomly generating similar list inputs, running DPCheck’s testing framework with the generated lists, and checking that each test reports no violations. We repeat this entire procedure in a non-terminating loop running on a cloud virtual machine that stores all test logs and raises alarms for any test failure. The size of the input lists increase with each passing test, up to 100.

We observed that the tests occasionally fail due to a tiny over-use of the \(\epsilon\)-privacy budget—in the range \(10^{-12}\) to \(10^{-15}\). We believe this was caused by rounding errors from the floating point operations that calculate noise distribution parameters in geometricFixedSens. We also observed that if we relax the privacy parameter to \(\sum_i \text{eps}_i + 10^{-12}\), then all of our test cases passed. This suggests that the total privacy cost incurred by running loopGeometricFixedSens is a small value plus the sum of intended \(\epsilon_i\) values, due to rounding in floating point operations.

Indeed, we expect the original version of DAS to also have this property, since its Python3 implementation also uses floating-point arithmetic. We confirmed this conjecture by intercepting the raw inputs to the core privacy mechanism in the original version of DAS, converting these inputs into 128-bit floating point numeric representations, which preserves much more precision than the 64-bit floating point numbers used in DAS, and calculating the total privacy cost in 128-bit floats. We then compared this higher-precision total privacy cost with the total privacy cost reported by the original version of DAS. This comparison reveals that the total privacy cost is around \(1.03 \times 10^{-11}\) more than the reported total privacy cost. Of course, because the extra privacy cost is extremely small, it does not significantly degrade the privacy protections provided by DAS.

Finally, to confirm that our re-implemented privacy mechanism behaves the same as the original version, we extract loopGeometricFixedSens to Python3 and replace the original core mechanism with the extracted code (shown in Appendix H). We then apply the following test setup to compare the behavior between two versions of DAS. For each trial:

1. Run both versions of DAS 500 times.
2. Assume that both groups of outputs come from the same distribution (since we assume both versions of DAS exhibit identical behavior); run statistical test to check if there is evidence to reject this assumption.
3. Record the \(p\)-value from the hypothesis test.

If our null hypothesis—that both versions of DAS behave identically—is true, then we should observe that the recorded \(p\)-values follow a uniform distribution on the interval [0, 1] [Murdoch et al. 2008]. Accordingly, we perform one final hypothesis test on the recorded \(p\)-values to search for evidence that suggests otherwise.

To run DAS, we need census data as inputs. The US Census Bureau tested DAS’s functionality using 1940 Census data [Ruggles et al. 2020]. With the 1940 Census data, each DAS run takes roughly 6 hours on our test machine. Since we need to perform 500 runs on each version of DAS per trial, and perform many trials to record enough \(p\)-values, we cannot afford to run DAS on the full 1940 Census dataset.

Instead, we subsample around 1 percent of the 1940 Census dataset and perform our trials over this smaller dataset. On our subset of the 1940 Census data, each run takes around 10 minutes to
finish and produces a vector of 287509 counts for the geographical units contained in the smaller dataset. We also parallelize the trials with 12 machines to speed up the entire test process.

Since our null hypothesis is that the two versions of DAS produces the same output distribution, we need a statistical test that can invalidate our null hypothesis based on the observations of the output vectors. We use multivariate permutation testing [Chung and Romano 2016] for this task. The multivariate permutation test takes two groups of samples as inputs. In our case, the two groups are each 500 vectors, one produced by our modified version of DAS and the other produced by the original version of DAS. The test randomly swaps vectors between these two groups and compares a test statistic derived from the difference of sample mean vectors from them before and after swapping. Intuitively, if both groups of samples truly come from the same distribution, then the test statistic should not change much due to swapping.

Each run of the permutation test produces a \( p \)-value. Tests that consistently produce very small \( p \)-values are evidence invalidating our null hypothesis that two versions of DAS have identical behavior. Here, we are checking for the lack of such evidence: when our assumption is indeed true, the \( p \)-values are samples drawn uniformly at random in the interval \([0, 1]\). We perform a final Kolmogorov-Smirnov test [Massey 1951] to check if there is evidence for rejecting the hypothesis that \( p \)-values are drawn from a uniform distribution. This final test produces a \( p \)-value of 0.68, signaling a lack of evidence to reject the hypothesis that the recorded \( p \)-values are sampled from a uniform distribution. We can also plot a histogram of the observed \( p \)-values (Section 7.2), which visually shows the collected \( p \)-values. These results produce no evidence for rejecting our null hypothesis that the two versions of DAS indeed behave identically.

To summarize our evaluation of DPCheck on the DAS workflow: we successfully re-implemented the privacy mechanism used by DAS, tested its privacy properties using DPCheck’s testing framework, extracted the DPCheck implementation into Python3 code, re-integrated this extracted privacy mechanism with the rest of DAS, and confirmed statistically that this modified version of DAS behaves the same as the original. This case study demonstrates that we can develop and test core differential privacy mechanisms in DPCheck and then integrate these core procedures with large software systems through mechanized code extraction.

8 OPTIMIZATIONS

Speeding up symbolic execution. We described a program transformation, streamline, that turns DPCheck programs into straight-line programs in Section 5.4. This transformation produces exponentially many straight-line programs that each need to be analyzed, placing a bottleneck on the size of inputs we can test. To mitigate this blowup in some cases, we implemented an optimized symbolic interpreter that does not require streamline and instead applies a type-driven state-merging algorithm developed by Torlak and Bodik [2014] This massively speeds up DPCheck’s symbolic execution and allows us to scale the generation of symbolic formulas to much larger input sizes. However, these formulas are more complex than those produced by the simpler method, and solving them is likely NP-complete. After comparing the end-to-end testing time with and without this optimization, we observed that the only algorithms that saw a speedup are the ones whose intermediate states merge well (such as ReportNoisyMax), while algorithms such as SparseVector, whose intermediate states do not merge well, performed worse than with the original approach.
DPCheck’s testing framework supports both versions, so that users may take advantage of the state-merging optimizations when appropriate.

**Bucketing Double results.** As described in Section 5, the testing process involves a “bucketing” step that groups sampled traces with the same outputs. This step is easy for algorithms that yield a small number of different outputs, as we only need to perform equality tests to group the sampled traces. However, DPCheck is not limited to such algorithms. For example, SmartSum and SparseVectorGap both yield Doubles, which are computed using values sampled from Laplace distributions. It is highly unlikely that any two runs of SmartSum or SparseVectorGap will produce the same Laplace samples, even if they follow the same control flow path. So we cannot simply use equality tests to bucket the outputs.

One solution is to restrict the output types of algorithms so that they only contain a small number of possible values, but this severely limits the kinds of algorithms that can be tested with DPCheck.

Instead, DPCheck chooses a heuristic that trades off some completeness for allowing programmers to test algorithms that may return sampled Double values. At test time, DPCheck’s instrumented interpreter attaches a distribution provenance to each sampled value and to results of arithmetic expressions that involve sample values.

For example, if \(x_1\) and \(x_2\) are two independent samples from the Laplace distribution with center 0 and width 1, then \(x_1\) and \(x_2\) have distribution provenance \(\text{lap}_{0,1}^1\) and \(\text{lap}_{0,1}^2\), and an expression such as \(x_1 \cdot x_2\) has distribution provenance \(\text{lap}_{0,1}^1 \cdot \text{lap}_{0,1}^2\). The superscript allows us to distinguish the distribution provenance of \(x_1 \cdot x_2\) from \(x_1 \cdot x_1\) and thus recognize that these are two different distributions.

DPCheck’s testing framework then buckets output sample values based on the equality between distribution provenance structures when the output values are not equal due to independent sampling. Since most algorithms that do return sampled values only sample from a handful of possible output distributions, this heuristic significantly cuts down the number of output buckets for such algorithms.

This heuristic does not sacrifice soundness with respect to the PW-Eq proof rule. PW-Eq allows us to construct one pointwise proof for each pair of equal output values. Here, DPCheck’s heuristic still adheres to this quota; indeed, it goes a step further by posing an even more stringent quota that only allows one pointwise proof for each pair of equal output distributions. In our evaluation, this heuristic allows us to test several benchmark algorithms that return sampled Double values. Attaching distribution provenance values to sample values introduces some interpretation overhead, but this overhead is not a bottleneck in DPCheck’s testing performance in our evaluation.

**9 LIMITATIONS**

**Gap from Theoretical Guarantees.** Due to scaling issues, our current testing framework does not allow us to run tests large enough to give meaningful \((\epsilon, \delta, \alpha)\)-random differential privacy guarantees through Theorem 12. For example, to achieve guarantees with \(\delta = 10^{-5}\), by Theorem 12 we know we need at least \(10^{15}\) samples. Our current evaluation uses between 500 to 5000 sampled traces per test iteration—i.e. the theoretical lower limit on sampled traces is at least 20 times larger than our current test parameter. Furthermore, the size of the formula to be solved by Z3 grows linearly with the number of sampled traces we use. Since the time it takes to solve these formulas grows exponentially in terms of formula size, testing with \(10^5\) sampled traces may conservatively slow down testing time by around \(k^{20}\) for some base exponent \(k\). Given these scaling issues, DPCheck’s testing framework is more useful for catching differential privacy bugs than for validation.
Numerical Implementation Issues. We study DPCheck’s testing guarantees by assuming a countably infinite discretized domain, where consecutive points in the discretized domain are exactly $\omega$ apart. But DPCheck’s implementation uses double-precision floating point numbers. This mismatch is known to lead to privacy leaks [Ding et al. 2019; Mironov 2012]. It can be remedied by using fixed-precision numbers.

Improving Symbolic Execution on Probabilistically Terminating Programs. PrivTree is the only probabilistically terminating program we have tested in the evaluation. To avoid infinitely unrolling the main loop of PrivTree during its symbolic execution, we explicitly abort the computation when the outputs from symbolic execution cannot possibly be matched with those observed in instrumented execution. This is a rather ad-hoc treatment that introduces an unnecessary abort instruction in DPCheck. However, it is possible to generalize the underlying principle behind our current treatment of probabilistically terminating programs. First, the programmer needs to identify a metric over the program’s output, and must ensure that this metric is monotonically increasing as more iterations of the program are executed. Next, we can find the maximum value of this metric in the outputs among the instrumented executions, and cut off a potentially infinite unrolling in the symbolic execution when this loop metric exceeds the maximum value observed from instrumented executions.

10 RELATED WORK

The diverse body of related research on programming-language approaches to differential privacy can be roughly grouped into these categories.

Axiomatic Systems. Many languages have been designed explicitly for implementing differentially private programs. Notable examples are: (1) Fuzz [Reed and Pierce 2010], a functional programming language with a linear type system for tracking functional sensitivity and $\epsilon$-differential privacy, (2) DFuzz [Gaboardi et al. 2013], a dependently typed Fuzz that allows index-refinement types for more precise tracking of $\epsilon$-differential privacy, (3) AdaptiveFuzz [Winograd-Cort et al. 2017], a multi-stage functional programming language that supports Adaptive Composition [Rogers et al. 2016] of differential privacy, (4) and Duet [Near et al. 2019], a functional programming language that extends Fuzz to approximate differential privacy.

These languages are designed with specialized type systems that internalize differential privacy proofs of useful mechanisms. DPCheck is also a language for programming differential privacy, but it is our choice to not use a specialized type system for differential privacy in DPCheck. The datatypes in DPCheck are only simple types that help programmers avoid common mistakes such as multiplying a number by a list. By only using a standard type system, we keep DPCheck’s testing design applicable to conventional programming languages and idioms.

Mechanized Proof Systems. There has been a line of work on developing type and proof systems for the purpose of building machine-checkable proofs for differential privacy. Examples include: (1) LightDP [Zhang and Kifer 2017b], a dependently typed language that uses annotations to automate differential privacy proofs, (2) ShadowDP [Wang et al. 2019], an improvement over LightDP that handles more sophisticated algorithms, with less annotation, and in less time, (3) apRHLP [Barthe et al. 2016b], a program logic built on probability distribution coupling theory for manual proofs of differential privacy, (4) work by Albarghouthi and Hsu [2017] on a system that borrows from apRHLP to automatically synthesize differential privacy proofs, and (5) work by Barthe et al. [2019] on a system that automatically proves and disproves differential privacy for a language restricted to finite domains.
The goal of these systems is to mechanically certify privacy of small programs with complex proofs. They all rely on general typing (or proof) rules that can capture the proofs of mechanisms such as ReportNoisyMax and SparseVector. LightDP, ShadowDP, and Albarghouthi and Hsu [2017]'s work all perform static analysis with the help of a solver; for privacy mechanisms with intermediate steps that depend on input values (e.g., PrivTree), the underlying solver would likely yield inconclusive results due to arithmetic complexities [Zhang and Kifer 2017b]. Furthermore, these systems often carry the burden of proving termination of the program under analysis. When faced with probabilistically terminating programs (e.g. PrivTree), they fail to produce useful analysis. DPCheck achieves greater expressiveness compared to these systems by considering the privacy properties of particular runs of a program, rather than trying to prove the program’s privacy property. This testing-based approach avoids the arithmetic challenge described by [Zhang and Kifer 2017b], since much of the arithmetic complexity is evaluated away early, and allows DPCheck to gracefully test probabilistically-terminating programs by limiting its symbolic exploration through instrumentation traces.

Barthe et al. [2019]'s work is unique in that the authors restrict the problem of automatic verification to a finite domain. In their model language, all data types contain a finite number of elements. This restriction allows the authors to craft complete decision procedures that prove or disprove differential privacy. DPCheck does not restrict its datatypes to finite domains, and its testing framework is not complete. Supporting programs with unusual differential privacy proof structures remains an important direction of our future research.

Since DPCheck tests a program for differential privacy, and cannot prove differential privacy. DPCheck by itself may not be sufficient for critical applications. When absolute guarantee of differential privacy is required, a more manual verification with a mechanized proof system is still necessary.

Statistical Testing. studies how to invalidate hypotheses about probability distributions; its techniques have also been applied to detect violations of differential privacy. Representative work includes: (1) StatDP [Ding et al. 2018], a framework for statistical testing of differential privacy, (2) DP-Finder [Bichsel et al. 2018], a framework that detects violations of differential privacy through code transformation, a careful sampling technique and objective optimization, and (3) work by Wilson et al. [2019], a SQL toolkit for differential privacy.

DPCheck is very similar to StatDP and DP-Finder in their goal of automatic testing of differential privacy, but they are very different in how they achieve this goal.

StatDP repeatedly runs the program under test, constructs two histograms approximating the two output distributions on similar inputs, and compares these two histograms using statistical tests to detect violations of differential privacy.

DP-Finder applies a novel sampling technique to construct a formula that approximates the privacy loss random variable, and then infers a lower bound of \( \epsilon \) through objective optimization on this approximation formula.

DPCheck also repeatedly runs a program under test on one of the two similar inputs, using an instrumented interpreter that collects traces. Both StatDP and DP-Finder place restrictions on the shape of outputs from programs under tests, because both frameworks apply heuristics to detect output events that likely indicates a violation of differential privacy. DPCheck adapts a general proof technique into a testing strategy, so that we only require equality tests on outputs of the program under test.

Wilson et al. [2019] developed an extension to the PostgreSQL database that checks differential privacy properties of SQL queries. They also applied histogram-based statistical testing to validate the correctness of their implementation of the extension.
11 FUTURE WORK

Addressing the scalability of DPCheck’s testing framework remains the most important avenue for improvement. Section 8 introduced a type-driven optimization to remove one of two significant scaling bottlenecks; this optimization speeds up symbolic execution, but checking satisfiability for the resulting formulas remains a serious bottleneck. In practice, this bottleneck prohibits DPCheck’s testing framework to run tests large enough for validation at a high confidence level.

To reduce this gap between theory and implementation, we plan on improving both sides. On the theory side, Theorem 12 only gives a very crude lower bound on the number of sampled traces required for a given confidence level; we believe it can be improved by more careful analysis. On the implementation side, (1) we can develop domain specific solver heuristics for the kinds of formulas that DPCheck generates, and (2) we can develop specification-based testing for DPCheck, approaching validation through programmer annotations of shift values rather than using Z3 to synthesize them.

Other avenues for improvement also remain. In particular, we hope to (1) harden the current implementation using fixed-precision instead of floating point numbers, (2) improve the current ad-hoc treatment of probabilistically terminating programs as discussed in Section 9, and (3) incorporate more sophisticated relations on samples to increase expressiveness.

12 ACKNOWLEDGMENTS

We are grateful to Danfeng Zhang, Daniel Winograd-Cort, Justin Hsu, and the Penn PLClub for discussion and comments, and we thank the anonymous reviewers for their detailed feedback. This work was supported in part by the National Science Foundation under grants CNS-1065060 and CNS-1513694.

REFERENCES

Aws Albarghouthi and Justin Hsu. 2017. Synthesizing Coupling Proofs of Differential Privacy. Proc. ACM Program. Lang. 2, POPL, Article 58 (Dec. 2017), 30 pages. https://doi.org/10.1145/3158146

Apple. 2017. Apple Differential Privacy Whitepaper. https://images.apple.com/privacy/docs/Differential_Privacy_Overview.pdf

E. Axelsson, K. Claessen, G. Dévai, Z. Horváth, K. Keijzer, B. Lyckegård, A. Persson, M. Sheeran, J. Svenningsson, and A. Vajdax. 2010. Feldspar: A domain specific language for digital signal processing algorithms. In Eighth ACM/IEEE International Conference on Formal Methods and Models for Codesign (MEMOCODE 2010). 169–178.

Gilles Barthe, Rohit Chadhya, Vishal Jagannath, A. Prasad Sistla, and Mahesh Viswanathan. 2019. Automated Methods for Checking Differential Privacy. arXiv:cs.CR/1910.04137

Gilles Barthe, Noémie Fong, Marco Gaboardi, Benjamin Grégoire, Justin Hsu, and Pierre-Yves Strub. 2016a. Advanced Probabilistic Couplings for Differential Privacy. Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security - CCS ’16 (2016). https://doi.org/10.1145/2976749.2978391

Gilles Barthe, Marco Gaboardi, Benjamin Grégoire, Justin Hsu, and Pierre-Yves Strub. 2016b. Proving Differential Privacy via Probabilistic Couplings. In Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science (LICS ’16). ACM, New York, NY, USA, 749–758. https://doi.org/10.1145/2933575.2934554

Benjamin Bichsel, Timon Gehr, Dana Drachsler-Cohen, Petar Tsankov, and Martin Vechev. 2018. DP-Finder: Finding Differential Privacy Violations by Sampling and Optimization. In Proceedings of the 2018 ACM SIGSAC Conference on Computer and Communications Security (CCS ’18). Association for Computing Machinery, New York, NY, USA, 508–524. https://doi.org/10.1145/3243734.3243863

T.-H. Hubert Chan, Elaine Shi, and Dawn Song. 2011. Private and Continual Release of Statistics. ACM Trans. Inf. Syst. Secur. 14, 3, Article 26 (Nov. 2011), 24 pages. https://doi.org/10.1145/2043621.2043626

EunYi Chung and Joseph P Romano. 2016. Multivariate and multiple permutation tests. Journal of econometrics 193, 1 (2016), 76–91.

Koen Claessen and John Hughes. 2000. QuickCheck: A Lightweight Tool for Random Testing of Haskell Programs. In Proceedings of the Fifth ACM SIGPLAN International Conference on Functional Programming (ICFP ’00). ACM, New York, NY, USA, 268–279. https://doi.org/10.1145/351240.351266
S. Petti and A. Flaxman. 2019. Differential privacy in the 2020 US census: what will it do? Quantifying the accuracy/privacy tradeoff [version 1; peer review: 1 approved with reservations]. Gates Open Research 3, 1722 (2019). https://doi.org/10.12688/gatesopenres.13089.1

Jason Reed and Benjamin C. Pierce. 2010. Distance Makes the Types Grow Stronger: A Calculus for Differential Privacy. In Proceedings of the 15th ACM SIGPLAN International Conference on Functional Programming (ICFP ’10). ACM, New York, NY, USA, 157–168. https://doi.org/10.1145/1863543.1863568

Ryan M. Rogers, Aaron Roth, Jonathan Ullman, and Salil Vadhan. 2016. Privacy Odometers and Filters: Pay-as-you-Go Composition. In Advances in Neural Information Processing Systems 29. D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett (Eds.). Curran Associates, Inc., 1921–1929. http://papers.nips.cc/paper/6170-privacy-odometers-and-filters-pay-as-you-go-composition.pdf

Steven Ruggles, Sarah Flood, Ronald Goeken, Josiah Grover, Erin Meyer, Jose Pacas, and Matthew Sobek. 2020. IPUMS USA: Version 10.0 [dataset]. https://doi.org/10.18128/D010.V10.0

T. Sato, G. Barthe, M. Gaboardi, J. Hsu, and S. Katsumata. 2019. Approximate Span Liftings: Compositional Semantics for Relaxations of Differential Privacy. In 2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS). 1–14. https://doi.org/10.1109/LICS.2019.8785668

Josef Svenningsson and Emil Axelsson. 2013. Combining Deep and Shallow Embedding for EDSL. In Proceedings of the 2012 Conference on Trends in Functional Programming - Volume 7829 (TFP 2012). Springer-Verlag New York, Inc., New York, NY, USA, 21–36. https://doi.org/10.1007/978-3-642-40447-4_2

Josef Svenningsson and Emil Axelsson. 2015. Combining deep and shallow embedding of domain-specific languages. Computer Languages, Systems & Structures 44 (2015), 143 – 165. https://doi.org/10.1016/j.cl.2015.07.003 SI: TFP 2011/12.

David Terei, Simon Marlow, Simon Peyton Jones, and David Mazières. 2012. Safe Haskell. In Proceedings of the 2012 Haskell Symposium (Haskell ’12). ACM, New York, NY, USA, 137–148. https://doi.org/10.1145/2364506.2364524

Emina Torlak and Rastislav Bodik. 2014. A Lightweight Symbolic Virtual Machine for Solver-aided Host Languages. SIGPLAN Not. 49, 6 (June 2014), 530–541. https://doi.org/10.1145/2666356.2594340

Yuxin Wang, Zeyu Ding, Guanhong Wang, Daniel Kifer, and Danfeng Zhang. 2019. Proving Differential Privacy with Shadow Execution. In Proceedings of the 40th ACM SIGPLAN Conference on Programming Language Design and Implementation (PLDI 2019). ACM, New York, NY, USA, 655–669. https://doi.org/10.1145/3314221.3314619

Royce J. Wilson, Celia Yuxin Zhang, William Lam, Damien Desfontaines, Daniel Simmons-Marengo, and Bryant Gipson. 2019. Differentially Private SQL with Bounded User Contribution. arXiv:cs.CR/1909.01917

Daniel Winograd-Cort, Andreas Haeberlen, Aaron Roth, and Benjamin C. Pierce. 2017. A Framework for Adaptive Differential Privacy. Proc. ACM Program. Lang. 1, ICFP, Article 10 (Aug. 2017), 29 pages. https://doi.org/10.1145/3110254

Danfeng Zhang and Daniel Kifer. 2017a. LightDP: Towards Automating Differential Privacy Proofs. SIGPLAN Not. 52, 1 (Jan. 2017), 888–901. https://doi.org/10.1145/3093333.3009884

Danfeng Zhang and Daniel Kifer. 2017b. LightDP: Towards Automating Differential Privacy Proofs. In Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages (POPL 2017). ACM, New York, NY, USA, 888–901. https://doi.org/10.1145/3009837.3009884

Jun Zhang, Xiaokui Xiao, and Xing Xie. 2016. PrivTree: A Differentially Private Algorithm for Hierarchical Decompositions. In Proceedings of the 2016 International Conference on Management of Data (SIGMOD ’16). ACM, New York, NY, USA, 155–170. https://doi.org/10.1145/2882903.2882928
A PROOF OF LEMMA 12

DPCheck’s testing framework only rejects programs for which it cannot construct valid dual executions. Here, we show that $f$ always has a valid dual execution, so that it is never rejected by the testing framework.

If the program $f$ is $\epsilon$-well-behaved, then we know there is a pointwise proof for its $(\epsilon, 0)$-differential privacy. In particular, for any similar inputs $(x_1, x_2)$ for $f$, consider a run of $f$ on $x_1$ that outputs some output $r$. Take the application of PW-Eq in the privacy proof of $f$, and inspect the applications of the Lap-Gen rule for that output $r$.

This sequence of applications of Lap-Gen specifies a sequence of shift values for this run of $f$ applied to $x_1$. Consider a dual execution of $f$ on $x_1$ where the Laplace samples are constructed from this sequence of shift values specified by the applications of Lap-Gen. Since $f$’s path conditions are necessary and sufficient for proving its $(\epsilon, 0)$-differential privacy, we know the constructed Laplace samples in dual execution must follow a control flow path that leads to the same output $r$. So DPCheck would not reject this $f$.

□

B PROOF OF THEOREM 13

Given the assumptions of Theorem 12, we know there are at most $n$ possible output buckets. For each of output bucket, our testing process is trying to find a valid shift vector, such that the ensemble of the $n$ shift vectors form valid dual executions.

DEFINITION 13. Given vectors $\tilde{o}$ and $\tilde{q}$ in $\mathbb{R}^n$, we say that $\tilde{q}$ is an $\omega$-approximation of $\tilde{o}$ if each coordinate $\tilde{q}_i$ is in the discretized domain, and $|\tilde{o}_i - \tilde{q}_i| \leq \omega$.

However, recall that DPCheck’s semantics assumes a discretized subset of the reals with granularity $\omega$ for all representable numbers. Our testing process uses Z3, which operates with rational semantics. To ensure that each coordinate of the shift vector is in the discretized ring, we must use the $\omega$-approximations of the rational shift vectors that Z3 finds in this analysis. The associated privacy costs of these $\omega$-approximations will be larger than the original prescribed $\epsilon$ privacy cost, since each coordinate has been perturbed by up to $\omega$. Taking the cost equation for Lap-Gen into account, the total privacy cost could increase by a maximum of $k\omega$. So, in fact, the testing process is trying to find evidence of $(\epsilon + k\omega, 0)$-differential privacy for the program under test, where $k$ is the maximum size of a shift vector among the $n$ output buckets. We will let $\epsilon' = \epsilon + k\omega$ in the remainder of the proof.

Now, on a pair of sampled inputs, by assumption, we know $\delta$ lower-bounds the probability that the instrumented run draws the “bad” sampled traces, for which we cannot construct dual executions whose output distributions satisfy eq. (1) with $(\epsilon', 0)$. So, as long as any bad sampled trace appears, the testing process will successfully reject this faulty program. We are considering the opposite case, where the testing process fails to reject this faulty program.

By assumptions, each “good” sampled trace appears with probability at most $1 - \delta$. So the probability that all $m$ sampled traces are good is at most $(1 - \delta)^m$. For these “good” sampled traces, the testing process is trying to discover $n$ shift vectors (one for each output bucket), that are evidence of valid dual executions.

Again, by assumptions, each shift value is in $[C_1, C_2]$ with $C_1 = \min_i \text{shift}_i$ and $C_2 = \max_i \text{shift}_i$. This bound is only discovered ex post, after DPCheck has already failed to reject the algorithm $f$. For our theoretical analysis, we can assume DPCheck starts the search of shift values in a small interval, and doubling the search space each time it fails to discover any valid shift values until it succeeds.

So the search process covers $\frac{C_2 - C_1}{\omega} + \frac{C_2 - C_1}{2\omega} + \frac{C_2 - C_1}{4\omega} + \cdots = \frac{2(C_2 - C_1)}{\omega}$ different values in each coordinate. And, since there are at most $k$ coordinates in each shift vector, and there are $n$ shift
vectors in total, the total number of possible shift vectors is

$$2^{nk} \left( \frac{C_2 - C_1}{\omega} \right)^{nk}.$$ 

If the testing process does find some set of $n$ shift vectors that lead to valid dual executions for the $m$ sampled traces, then this test fails to reject the program under test. Let $\mathbb{P}[\text{FAIL}(x_1, x_2)]$ denote the total probability that the testing process fails to reject the program under test on the sampled similar inputs $(x_1, x_2)$. Then, we know

$$\mathbb{P}[\text{FAIL}(x_1, x_2)] \leq (1 - \delta) \cdot 2^{nk} \left( \frac{C_2 - C_1}{\omega} \right)^{nk}.$$

If we want $e^{-\delta m} \cdot 2^{nk} \left( \frac{C_2 - C_1}{\omega} \right)^{nk} \leq e^{-\theta}$ for some $\theta$, then we can derive a lower bound for $m$

$$e^{-\delta m} \cdot 2^{nk} \left( \frac{C_2 - C_1}{\omega} \right)^{nk} \leq e^{-\theta}$$

$$-\delta m + nk \ln 2 + \ln \left( \frac{C_2 - C_1}{\omega} \right)^{nk} \leq -\theta$$

$$\theta + nk \ln 2 + nk \ln \frac{C_2 - C_1}{\omega} \leq \delta m$$

$$\frac{1}{\delta} \left( \theta + nk \ln 2 + nk \ln \frac{C_2 - C_1}{\omega} \right) \leq m. \quad (7)$$

Now, let’s consider the probability of failing to reject $f$ if we repeat the test with $d$ pairs of sampled inputs

$$(x_1^1, x_2^1), \ldots, (x_1^d, x_2^d).$$

We know that as long as $m$ satisfies the lower bound in eq. (7), then the probability $\mathbb{P}[\text{FAIL}(x_1^i, x_2^i)]$ (on any pair of similar inputs) of failing to reject $f$ is at most $e^{-\theta}$.

By assumptions, we know that under the distribution $I$ for similar inputs, with probability at least $\alpha$, the similar inputs have no valid proof of eq. (1) under privacy budget $\epsilon$, which means our testing process can immediately reject $f$ if such a pair of inputs are generated. So we know that with probability at most $1 - \alpha$, we may get a pair $(x_1^i, x_2^i)$ that the testing process cannot immediately reject. So, on a single pair of similar inputs, the probability of failure to reject is at most

$$(1 - \alpha)e^{-\theta} + \alpha \cdot 0 = (1 - \alpha)e^{-\theta}.$$ 

Now, for $d$ pairs of similar inputs drawn independently at random from $I$, the probability of failure to reject any of them $\mathbb{P}[\text{FAIL}(x_1^1, x_2^1), \ldots, \text{FAIL}(x_1^d, x_2^d)]$ is at most

$$\left( (1 - \alpha)e^{-\theta} \right)^d = (1 - \alpha)^d e^{-d\theta} \leq e^{-\alpha d} e^{-d\theta} = e^{-d(\theta + \alpha)}.$$ 

This is the bound on failing to reject a faulty $f$, as long as each test uses more than $m$ sampled traces, where $m$ is lower bounded by eq. (7).  \(\square\)
C EVALUATION OF PRIVACY MECHANISMS

We implemented a number of algorithms from related work on automated verification and testing [Albarghouthi and Hsu 2017; Ding et al. 2018; Wang et al. 2019; Zhang and Kifer 2017b], as well as the recently proposed ReportNoisyMaxWithGap and SparseVectorWithGap algorithms from [Ding et al. 2019] and the PrivTree algorithm. PrivTree, in particular, is cited by authors of LightDP [Zhang and Kifer 2017b] as a challenging example. The programs we evaluated can be found in Appendix D.

We ran DPCheck’s testing framework on the correct implementation of each of these algorithms and confirmed that it is accepted. We also implemented non-differentially private variants of each algorithm and measured the time DPCheck’s testing framework takes to discover the failure. We used the QuickCheck [Claessen and Hughes 2000] randomized testing library to build test input generators for DPCheck’s privacy testing framework. We manually wrote one generator for each type of similarity relation introduced in Section 2.

We ran 100 tests on correct algorithms and verify they all pass. For catching bugs, we run only 50 tests and verify the bug is caught within 50 tests. The test count 100 is chosen to run the test suite for as long as possible on differentially private programs without overloading our test environment. The test count 50 is empirically chosen to give the testing framework enough chances to observe failures on all our non-differentially-private benchmarks.

The experiments were run on a virtual test machine with a 2-Core CPU clocked at 2.3GHz, and 7GB of RAM. The results are summarized in Figure 3. For SparseVector, we implemented all four of the non-differentially-private variants studied in [Lyu et al. 2017]. For the other algorithms, we implemented one non-differentially private variant. Among these non-private variants, we try to mimic typical programming mistakes, such as using a wrong variable with name similar to the correct one, using incorrect width parameter to Laplace sampling instruction, off-by-one errors, and using the wrong arithmetic operator. The rest of this section reports in more detail on the two most interesting benchmarks.

ReportNoisyMaxWithGap. ReportNoisyMax returns the index of the largest value in an input list, after adding some random noise to each of the original values. In addition the returning the index of

| Bug     | $\epsilon$ | TTB  | Std. dev. | ITB  | Std. dev. |
|---------|-------------|------|-----------|------|-----------|
| nc      | 1.0         | 0.8s | 0.7s      | 1.9  | 1.6       |
| nm      | 1.0         | 2.3s | 0.5s      | 1.0  | 0.0       |
| ns      | 1.0         | 0.5s | 0.1s      | 1.0  | 0.0       |
| ps      | 1.0         | 2.0s | 0.5s      | 1.0  | 0.0       |
| pt      | 2.58        | 0.8s | 0.4s      | 1.0  | 0.0       |
| rnm     | 2.0         | 16.7s| 16.3      | 6.4  | 5.8       |
| rnmGap  | 4.0         | 8.9  | 12.9s     | 1.4  | 0.6       |
| ss      | 1.0         | 29.8s| 27.7s     | 11.0 | 9.8       |
| sv3     | 1.0         | 8.7s | 3.9s      | 1.0  | 0.0       |
| sv4     | 1.0         | 27.1s| 15.8s     | 2.3  | 1.3       |
| sv5     | 1.0         | 7.8s | 8.2s      | 1.8  | 1.1       |
| sv6     | 1.0         | 218.8s| 173.4s    | 4.1  | 3.0       |
| svGap   | 1.0         | 1899.1s| 1750.2s  | 2.1  | 1.3       |

Fig. 3. Mean time and mean iteration to bug discovery, and their standard deviation on incorrect implementations. (The value $\epsilon = 2.58$ is derived by instantiating PrivTree’s privacy cost formula [Zhang et al. 2016] with all parameters set to 1.0.)
largest noised value, ReportNoisyMaxWithGap also releases the numerical gap between the noised max value and noised runner-up value. Surprisingly, this additional information does not increase the privacy cost compared to the original ReportNoisyMax algorithm [Ding et al. 2019]. We ran the testing framework on a correct implementation of ReportNoisyMaxWithGap with its optimal $\epsilon = 2.0$. However, this claim was incorrectly rejected by the testing framework. We manually inspected the generated symbolic formula for ReportNoisyMaxWithGap, and realized that its path condition essentially requires releasing the index of the noisy runner-up value. This path condition is sufficient but not necessary for proving $(2.0, 0)$-differential privacy for ReportNoisyMaxWithGap. Our manual analysis also revealed that this path condition should lead to $(4.0, 0)$-differential privacy for ReportNoisyMaxWithGap. We ran another test on ReportNoisyMaxWithGap with $\epsilon = 4.0$, and this time the testing framework correctly accepted the algorithm as private.

PrivTree. PrivTree [Zhang et al. 2016] is a differentially private algorithm for building spatial decomposition trees that approximate occupied regions of space. We implemented a one-dimensional version of the PrivTree algorithm over the unit interval.

PrivTree is challenging for automatic checking for at least two reasons. The first is that PrivTree has an unbounded loop that terminates with probability 1—the probability of PrivTree not terminating after $n$ iterations of its main loop vanishes as $n$ increases, such that the main loop eventually surely terminates, but the exact number of iterations cannot be statically computed. The second is that the privacy analysis for PrivTree involves intermediate privacy costs that depend on input values [Zhang and Kifer 2017a].

The first characteristic poses issues for general static analysis, as we cannot statically unroll PrivTree’s main loop. DPCheck’s symbolic interpreter is also susceptible to this challenge. However, for our testing strategy in Section 5 to work, the symbolic interpreter only needs to produce trees that match those observed in the instrumented execution. With each additional iteration of PrivTree, the size of the spatial decomposition tree strictly increases. Thus, our symbolic interpreter can eagerly cut off the rest of the infinite search once it realizes that all future iterations will produce trees that do not match those observed in the instrumented executions.

The second characteristic poses issues for tools that generate proofs of privacy. As such tools need to reason over all possible similar inputs, the input values must be universally quantified and unknown at proof-generation time. So intermediate privacy costs that depend on input values are arithmetic expressions over variables representing these unknown inputs. For PrivTree, these intermediate privacy cost expressions involve non-linear arithmetic, an undecidable theory that can only be solved in a best-effort way by SMT solvers.

By contrast, our testing framework repeatedly chooses pairs of concrete inputs to PrivTree. This means that the intermediate privacy costs can be represented with much simpler symbolic formula, making automatic privacy analysis for PrivTree feasible with Z3.

D EVALUATED PROGRAMS

D.1 ReportNoisyMax

We presented the source code of ReportNoisyMax in Section 3. The privacy analysis in Section 5 shows that this algorithm is $(2.0, 0)$-differentially private.

To test ReportNoisyMax, we need to generate inputs whose coordinate-wise distance (Definition 4) is bounded by 1. We implement such a generator manually using QuickCheck [Claessen and Hughes 2000], and repeat DPCheck’s tests for as long as allowed by the virtual test machine. All of our experiments on ReportNoisyMax so far have yielded successful results.

We also implemented an incorrect version that passes the first value from the input list to $\text{rnmAux}$ without adding Laplace noise. The mistake is highlighted (in bold and red) below:
testing Differential Privacy with Dual Interpreters

DPCheck’s testing framework catches this mistake very quickly since the un-noised first input to \texttt{rnmAux} immediately invalidates ReportNoisyMax’s privacy analysis for many similar input values.

\subsection{SparseVector}

The SparseVector algorithm takes a list of real-valued inputs as the private input and returns the indices of all the large elements of the input. Two input lists are again considered similar if their coordinate-wise distance is bounded by 1.

In addition to the private input list, SparseVector takes two additional, non-private parameters: \( n :: \text{Int} \) and \( \text{threshold} :: \text{Double} \). SparseVector produces a list of boolean values where each True represents an above-threshold input value. The parameter \( n \) bounds the total number of Trues \texttt{SparseVector} may emit before the rest of the computation is truncated. As an example, if the input is a three-element list and \( n \) is 1, then \texttt{SparseVector} may return any of \([\text{True}], [\text{False}, \text{True}], \) and \([\text{False}, \text{False}, \text{True}]\).

We implement \texttt{SparseVector} in DPCheck as follows:

\begin{verbatim}
sv xs n thresh = do
let width = 4.0 * fromIntegral n
thresh' ← lap thresh 2.0
xs' ← mapM (λx. lap x width) xs
svAux xs' n thresh' nil
svAux [] _ _ acc = return acc
svAux (x:xs) n thresh acc
| n <= 0 = return acc
| otherwise = do
let recur n' acc' =
svAux xs' n' thresh acc'
if (x > thresh)
(recur (n-1) (snoc acc True))
(recur n (snoc acc False))
\end{verbatim}

Here, \texttt{nil} is an empty list, and \texttt{snoc} is a function that takes a DPCheck list and a list element, and returns a new list with the element appended to the end of the supplied one.

Lyu et al. [2017] studied six published variants of SparseVector, among which only two actually satisfy differential privacy with the intended privacy parameter. The SparseVector implementation shown here is the a correct version proposed by Lyu et al., named Algorithm 1 in [Lyu et al. 2017]. This variant is \((1,0)\)-differentially private. DPCheck’s testing framework correctly rejects the four incorrect variants and accepts the two correct variants. We re-used the generators for ReportNoisyMax to generate input lists for SparseVector, and we used QuickCheck to repeat the testing process for as long as allowed by our virtual test machine.

\subsection{PrefixSum}

The PrefixSum algorithm takes a list of numbers and returns a list of the same length, where each \( i \)th value in the list is the sum of the values at index \( 0,1,\ldots,i \). Two inputs to PrefixSum are considered similar if they have the same length and their \( L_1 \)-distance (Definition 3) is bounded by 1. The PrefixSum algorithm achieves \((1,0)\)-differential privacy by adding Laplace noise with
width 1 to each of the values in the input list before summing. The DPCheck code implementing PrefixSum is shown here:

```haskell
ps xs = do
    xs' ← mapM (λx. lap x 1.0) xs
    return (psAux (reverse xs') nil)

psAux [] acc = acc
psAux (x:xs) acc =
    psAux xs (cons (sum (x:xs)) acc)
```

The `reverse` :: [a] → [a] function comes from Haskell’s standard library; it returns a list in the reversed order. (Since the helper function `psAux` accumulates the prefix sums in the reverse order, we also need to reverse the noised list of input values.)

We use another manually written generator for $L_1$-distance-bounded pairs of lists to test PrefixSum, and we similarly repeat the testing process until at least 100 test cases pass.

### D.4 SmartSum

The SmartSum algorithm, also known as the Binary Mechanism, is a sophisticated improvement over PrefixSum that provides the same $(1, 0)$-differential privacy guarantees, but releases noised sums with much smaller asymptotic error [Chan et al. 2011]. Data analysts can benefit from the accuracy improvement with no sacrifice in privacy guarantees by using SmartSum instead of PrefixSum.

The core idea behind SmartSum is to build a binary tree of partial sums from the input values, instead of summing up each prefix. We defer the code listing of SmartSum to Appendix F, where we also show a buggy variant that we created unintentionally along the way. We reuse the generator for PrefixSum to test SmartSum. DPCheck’s testing framework accepts the correct implementation and rejects our incorrect one.

### D.5 ReportNoisyMaxWithGap

Ding et al. [2019] recently proposed novel variants of the ReportNoisyMax and SparseVector algorithms that release more information about the input data without increasing their privacy cost.

For ReportNoisyMaxWithGap, the extra information released is the numerical gap between the largest noised value and the second largest noised value. The implementation is very similar to ReportNoisyMax:

```haskell
rnmGap [] =
    error "rnmGap received empty input"

rnmGap [] =
    error "rnmGap received only one input"

rnmGap (x:y:xs) = do
    x' ← lap x 1.0
    y' ← lap y 1.0
    xs' ← mapM (λx. lap x 1.0) xs
    if (x' > y')
        (rnmGapAux xs' 1 0 x' y')
    (rnmGapAux xs' 1 1 y' x')

rnmGapAux [] _ maxIdx
    currMax currRunnerUp =
    return (maxIdx, currMax - currRunnerUp)

rnmGapAux (x:xs) lastIdx maxIdx
    currMax runnerUp = do
        let thisIdx = lastIdx + 1
        let recur = rnmGapAux xs thisIdx
        return (thisIdx, max(currMax + runnerUp, recur))

rnmGapAux (x:xs) maxIdx
    currMax currRunnerUp = do
        let thisIdx = maxIdx + 1
        let recur = rnmGapAux xs thisIdx
        return (thisIdx, max(currMax + currRunnerUp, recur))
```

Proc. ACM Program. Lang., Vol. 4, No. OOPSLA, Article 165. Publication date: November 2020.
if (x > currMax)
  (recur thisIdx x currMax)
(if (x > runnerUp)
  (recur maxIdx currMax x)
  (recur maxIdx currMax runnerUp))

The algorithm keeps track of the runner-up at each iteration in addition to the current maximum value and its index in the input list, and eventually returns the index of the largest value, and the difference between the maximum and the runner-up. We reuse the generator for ReportNoisyMax to test ReportNoisyMaxWithGap with the same privacy parameter $\epsilon = 2.0$.

However, DPCheck’s testing framework incorrectly rejects this claim. We manually inspected the generated symbolic formula to investigate the cause of rejection. The path conditions DPCheck collects effectively requires the algorithm to also release the index of the runner-up value, and this is equivalent to running ReportNoisyMax twice, once to find out the index of the largest value, and a second time to find out the largest in the remaining values. Running ReportNoisyMax twice in such a way induces a privacy parameter $\epsilon = 2.0 \times 2 = 4.0$. We check this conjecture by testing ReportNoisyMaxWithGap again with $\epsilon = 4.0$, and DPCheck indeed accepts this claim.

This is a case where DPCheck fails to accept the optimal privacy parameter. The authors of [Ding et al. 2019] use an analysis that does not require releasing the index of the second largest noised value, but DPCheck can only make use of facts collected from path conditions. In this case, the path conditions are overly strict, which doubles the privacy bound required by DPCheck to check the differential privacy property.

D.6 SparseVectorWithGap

SparseVectorWithGap [Ding et al. 2019] is an improvement over SparseVector that releases the numeric gap between noised input values and the noised threshold, when the noised input value is above the noised threshold. We implement SparseVectorWithGap using the following DPCheck program:

```haskell
svGap xs n thresh = do
  thresh' ← lap thresh 2.0
  let width = 4.0 * fromIntegral n
  xs' ← mapM (λx. lap x width) xs
  svGapAux xs' n thresh' nil
svGapAux [] _n _thresh acc =
  return acc
svGapAux (x:xs) n thresh acc |
| n <= 0 = return acc
| otherwise = do
  let recur n' acc' = svGapAux xs n' thresh acc'
  if (x > thresh)
    (do let acc' =
      snoc acc (just (x - thresh))
      recur (n-1) acc')
    (recur n (snoc acc nothing))
```

Compared to SparseVector, instead of returning a list of boolean values, SparseVectorWithGap returns a list of optional values. Each nothing value represents the absence of a value, and each just x represents the existence and the value of x. Our implementation of SparseVectorWithGap yields just gap instead of True for each above-threshold noised value and its gap between the noised threshold; it uses nothing instead of False for below-threshold noised values. The parameter n again bounds the number of above-threshold optionals SparseVectorWithGap can emit before the rest of the computation is cut short.
Unlike ReportNoisyMaxWithGap, SparseVectorWithGap’s differential privacy property is accepted by DPCheck’s testing framework. As the path conditions collected in a symbolic execution of svGap are the same as that those collected in a symbolic execution of sv, DPCheck’s testing framework has no issue accepting SparseVectorWithGap’s privacy claims.

D.7 PrivTree
PrivTree [Zhang et al. 2016] is a differentially private algorithm for building spatial decomposition trees that approximate occupied regions of space. We implemented a one-dimensional version of the PrivTree algorithm over the unit interval. The input is a list of points on the unit interval, represented by a list `xs :: [Double]`. Two input lists are similar if their database distance is bounded by 1 (Definition 2). Our implementation of PrivTree outputs a distribution over spatial decomposition trees over the unit interval, represented by the type `Map Node ()`. A node in the tree is an interval `Node = (Double, Double)`, representing a sub-interval of the unit interval. In our implementation of PrivTree, the final output spatial decomposition tree is a collection of leaf nodes in the tree, and the internal nodes of the decomposition tree are implicitly represented by their constituent sub-intervals.

For example, if the input list is `[0.1, 0.3]`, then PrivTree may output a tree with leaf nodes `(0.0, 0.25)`, `(0.25, 0.5)` and `(0.5, 1.0)`. The first two leaf nodes are occupied by the input points, while the last leaf node is not occupied; is created when we split the root node (the unit interval) into `(0.0, 0.5)` and `(0.5, 1.0)`. A naive attempt at building such spatial decomposition trees is to take the textbook QuadTree algorithm [Finkel and Bentley 1974] and use Laplace noise to turn it into a differentially private algorithm. This naive approach would maintain a queue of spatial sub-regions to analyze. On each iteration, the algorithm would use the Laplace distribution to obtain a noisy count of nodes in the sub-region, then decide whether to split the sub-region by comparing the noisy count with a pre-determined threshold value. Zhang et al. [2016] argue that this method has two significant drawbacks: 1) to ensure the final privacy parameter has a finite bound, we must also bound the maximum depth of the spatial tree built by this procedure, and 2) it is difficult to pick a threshold value that leads to accurate spatial decomposition trees.

The PrivTree algorithm solves these issues by removing requirements of both the depth bound and the pre-determined threshold. To save space, we defer the implementation of PrivTree to Appendix G.

PrivTree is a challenging algorithm for automatic verification for at least two reasons. The first is that PrivTree terminates probabilistically, i.e., the probability of PrivTree not terminating after $n$ iterations of its main loop diminishes as $n$ increases. The second one is that the privacy cost analysis used in the PrivTree’s privacy proof involves intermediate privacy costs that depend on input values [Zhang and Kifer 2017a].

The first characteristic poses issues for static analysis, as we cannot statically know how many iterations PrivTree will run. DPCheck’s symbolic interpreter is also susceptible to this challenge. However, recall from Section 5 that the symbolic interpreter only needs to produce trees that match those observed in the instrumented execution. PrivTree would only need to run more iterations if it decides to split the current sub-region, which means the final tree will contain more and more leaf nodes as the number of iterations increases. Thus, our symbolic interpreter can eagerly cut off the rest of the (infinite) search once it realizes that all future iterations will produces trees that do
not match those observed in the instrumented executions\textsuperscript{9}, saving DPCheck’s testing framework from infinite unrolling.

The second characteristic poses issues for tools aimed at automatically generating proofs of differential privacy. As such tools need to reason over all possible input values, these input values must be universally quantified and unknown at proof-generation time, which means intermediate privacy costs that depend on input values must be represented by expressions over the unknown input values. For PrivTree, these intermediate privacy cost expressions involve non-linear arithmetic, an undecidable theory that can only be solved in a best-effort way by SMT solvers.

On the other hand, DPCheck’s testing framework chooses a pair of concrete input values and evaluates PrivTree over these inputs. This allows the testing framework to represent these intermediate privacy costs through the much simpler generic shift relation introduced in Section 5.

DPCheck’s testing framework accepts the correct implementation of PrivTree over the unit interval. We also implemented a buggy version similar to the naive approach but not placing a depth bound on the tree created. DPCheck’s testing framework correctly rejects this buggy variation.

\textbf{D.8 NoisySum, NoisyCount and NoisyMean}

The NoisySum, NoisyCount, and NoisyMean algorithms are simple mechanisms that aggregate private data. We show their implementations below.

\begin{bash}
noisySum :: [Expr Double] → Expr (Circle Double)
noisySum xs = lap (sum xs) 1.0

noisyCount :: [Expr Double] → Expr Double → Expr (Circle Double)
noisyCount xs threshold = do
  let c = length (filter (>= threshold) xs)
lap (fromIntegral c) 1.0

noisyMean xs clipBound
  | clipBound < 0 = error "simpleMean: received clipBound < 0"
  | otherwise = do
    s ← clippedSum xs 0 clipBound
    noisedS ← lap s 1.0
    let count = fromIntegral (lit (length xs))
    noisedC ← lap count 1.0
    return (noisedS, noisedC)

clippedSum [] acc clipBound = acc
\end{bash}

\textsuperscript{9}This ad-hoc early cut-off is not hardcoded in the symbolic interpreter, but represented by \texttt{abort} instructions in the source code. We will discuss how to generalize the ad-hoc cut-off in Section\textsuperscript{9}.
clippedSum (x:xs) acc clipBound = 
  ifM (x >= clipBound)
    (clippedSum xs (acc + clipBound))
  (ifM (x < (-clipBound))
    (clippedSum xs (acc - clipBound))
    (clippedSum xs (acc + x)))

The NoisyMean algorithm’s similarity relation also bounds two lists’ database distance bounded with 1, while the parameter clipBound is public data. NoisyMean is \((\text{clipBound} + 1.0)\)-differentially private. A critical intermediate step in NoisyMean is to clip each input value into the range \([-\text{clipBound}, \text{clipBound}]\) before summing. This step is necessary because extreme outliers will lead to violations of differential privacy.
E IMPLEMENTATION OF eval

\[
\begin{align*}
\text{return} & :: a \to \mathbb{O} a \\
(\gg=) & :: \mathbb{O} a \to (a \to \mathbb{O} b) \to \mathbb{O} b \\
\text{laplace} & :: \text{Double} \to \text{Double} \to \mathbb{O} \text{Double} \\
\text{eval} & :: \text{Expr} a \to a \\
\text{eval} (\text{Literal} a) & = a \\
\text{eval} (\text{Return} a) & = \text{return} (\text{eval} a) \\
\text{eval} (\text{Bind} a f) & = \text{eval} a \gg= (\text{eval} . f . \text{Literal}) \\
\text{eval} (\text{Laplace} c w) & = \text{laplace} (\text{eval} c) w \\
\text{eval} (\text{If} \ \text{cond} a b) & = \text{if} (\text{eval} \ \text{cond}) (\text{eval} a) (\text{eval} b) \\
\text{eval} (\text{Add} a b) & = (\text{eval} a) + (\text{eval} b) \\
\text{eval} (\text{Lt} a b) & = (\text{eval} a) < (\text{eval} b) \\
\text{eval} (\text{Loop} \ \text{acc} \ \text{pred} \ \text{iter}) & = \text{eval} (\text{Loop} \ \text{acc} \ \text{pred} \ \text{iter}) = \\
& \quad \text{runLoop} (\text{eval} \ \text{acc}) (\text{eval} . \ \text{pred} . \ \text{Lit}) (\text{eval} . \ \text{iter} . \ \text{Lit}) \\
\text{runLoop} & :: \text{Monad} m \Rightarrow \\
& \quad a \to (a \to \text{bool}) \to (a \to m a) \to m a \\
\text{runLoop} \ \text{acc} \ \text{pred} \ \text{iter} & = \text{do} \\
& \quad \text{if} \ \text{pred} \ \text{acc} \\
& \quad \text{then} \ \text{do} \\
& \quad \quad \text{acc'} \leftarrow \text{iter} \ \text{acc} \\
& \quad \quad \text{runLoop} \ \text{acc'} \ \text{pred} \ \text{iter} \\
& \quad \text{else} \\
& \quad \quad \text{return} \ \text{acc}
\end{align*}
\]

Fig. 4. Source code for eval

F SMARTSUM

\[
\begin{align*}
\text{smartSum} \ x s & = \text{smartSumAux} \ x s 0 0 0 0 \text{nil} \\
\text{smartSumAux} \ [] & = \quad \quad \quad \text{results} = \text{return} \ \text{results} \\
\text{smartSumAux} \ (x:xs) & \text{next} \ n \ i \ \text{sum} \ \text{results} = \text{do} \\
& \quad \text{let} \ \text{sum'} = \text{sum} + x \\
& \quad \quad \text{if}_- \ ((\text{mod} \ (i+1) \ 2) \ == \ 0) \\
& \quad \quad \quad \text{(do} \ n' \leftarrow \text{lap} \ (n + \text{sum'}) \ 1.0 \\
& \quad \quad \quad \quad \text{smartSumAux} \ xs \ n' \ n' \ (i+1) \ 0 \ \text{(snoc} \ \text{results} \ n') \\
& \quad \quad \quad \text{)} \\
& \quad \quad \text{(do} \ \text{next'} \leftarrow \text{lap} \ (\text{next} + x) \ 1.0 \\
& \quad \quad \quad \text{smartSumAux} \ xs \ \text{next'} \ n \ (i+1) \ \text{sum'} \ \text{snoc} \ \text{results} \ \text{next'})
\end{align*}
\]

Fig. 5. Source code for SmartSum
smartSumBuggy xs = smartSumAuxBuggy xs 0 0 0 0 nil

smartSumAuxBuggy [] _ _ _ results = return results
smartSumAuxBuggy (x:xs) next n i sum results = do
  let sum' = sum + x
  if_ ((mod (i + 1) 2) == 0)
    (do n' ← lap (n + sum') 1.0
        smartSumAuxBuggy xs n' n' (i+1) sum' (snoc results n')) -- bug
    (do next' ← lap (next + x) 1.0
        smartSumAuxBuggy xs next' n (i+1) sum' (snoc results next'))

Fig. 6. Source code for SmartSumBuggy

G PRIVTREE

privTree xs = privTreeAux xs [rootNode] (S.singleton rootNode)
  emptyTree

privTreeAux points queue leafNodes tree
  | length leafNodes > k_PT_MAX_LEAF_NODES
    -- to avoid infinite unrolling in symbolic execution
    = abort "unreachable code: there are too many leaf nodes"
  | otherwise
    = case queue of
      [] → return tree
      (thisNode:more) → do
        let biasedCount =
            countPoints points thisNode - depth thisNode * k_PT_DELTA
        biasedCount' ←
          if (biasedCount > (k_PT_THRESHOLD - k_PT_DELTA))
            (return biasedCount)
          (return $ k_PT_THRESHOLD - k_PT_DELTA)
        noisedBiasedCount1 ← lap biasedCount' k_PT_LAMBDA
        let updatedTree = updatePT thisNode () tree
        if (noisedBiasedCount1 > k_PT_THRESHOLD)
          (do let (left, right) = split thisNode
              let leafNodes' =
                S.insert right
                (S.insert left (S.delete
                  thisNode leafNodes))
              privTreeAux points (more++[left,right])
              leafNodes' updatedTree)
          (privTreeAux points more leafNodes updatedTree)

Fig. 7. Source code for PrivTree

In the implementation of PrivTree, we keep track of the current set of leafNodes, and cut off the rest of the computation when there are more leaf nodes than we have observed in the instrumented execution.

H EXTRACTED PYTHON3 CODE

Proc. ACM Program. Lang., Vol. 4, No. OOPSLA, Article 165. Publication date: November 2020.
def loop_geometric(true_answer_sens_eps):
    """
    :param true_answer_sens_eps: an array-like of (true_answer, (sensitivity, epsilon))
    :return: a list of (noised_answer, variance)
    """
    loop_acc = (true_answer_sens_eps, [])
    loop_cond = not [] == loop_acc[0]
    while loop_cond:
        if loop_acc[0]:
            uncons_head = loop_acc[0][0]
            uncons_tail = (loop_acc[0])[1:]
            uncons_result = (uncons_head, uncons_tail)
        else:
            uncons_result = None
        if loop_acc[0]:
            uncons_head1 = loop_acc[0][0]
            uncons_tail1 = (loop_acc[0])[1:]
            uncons_result1 = (uncons_head1, uncons_tail1)
        else:
            uncons_result1 = None
        if loop_acc[0]:
            uncons_head2 = loop_acc[0][0]
            uncons_tail2 = (loop_acc[0])[1:]
            uncons_result2 = (uncons_head2, uncons_tail2)
        else:
            uncons_result2 = None
        x = (prim_symmetric_geometric(uncons_result[0][0], (np.exp((0.0 - uncons_result1[0][1][1]) / uncons_result2[0][1][0])
        if loop_acc[0]:
            uncons_head3 = loop_acc[0][0]
            uncons_tail3 = (loop_acc[0])[1:]
            uncons_result3 = (uncons_head3, uncons_tail3)
        else:
            uncons_result3 = None
        if loop_acc[0]:
            uncons_head4 = loop_acc[0][0]
            uncons_tail4 = (loop_acc[0])[1:]
            uncons_result4 = (uncons_head4, uncons_tail4)
        else:
            uncons_result4 = None
        if loop_acc[0]:
            uncons_head5 = loop_acc[0][0]
            uncons_tail5 = (loop_acc[0])[1:]
            uncons_result5 = (uncons_head5, uncons_tail5)
        else:
            uncons_result5 = None
        if loop_acc[0]:
            uncons_head6 = loop_acc[0][0]
            uncons_tail6 = (loop_acc[0])[1:]
            uncons_result6 = (uncons_head6, uncons_tail6)
        else:
            uncons_result6 = None
        if loop_acc[0]:
            uncons_head7 = loop_acc[0][0]
            uncons_tail7 = (loop_acc[0])[1:]
            uncons_result7 = (uncons_head7, uncons_tail7)
        else:
            uncons_result7 = None
        if loop_acc[0]:
            uncons_head8 = loop_acc[0][0]
            uncons_tail8 = (loop_acc[0])[1:]
            uncons_result8 = (uncons_head8, uncons_tail8)
        else:
uncons_tail8 = (loop_acc[0])[1:]
uncons_result8 = (uncons_head8, uncons_tail8)
else:
    uncons_result8 = None

if loop_acc[0]:
    uncons_head9 = loop_acc[0][0]
    uncons_tail9 = (loop_acc[0])[1:]
    uncons_result9 = (uncons_head9, uncons_tail9)
else:
    uncons_result9 = None

loop_acc = (uncons_result3[1], loop_acc[1] + [(x, 2.0 * (np.exp
((0.0 - uncons_result4[0][1][1]) / uncons_result5[0][1][0])) /  
((1.0 - (np.exp((0.0 - uncons_result6[0][1][1]) / uncons_result7[0][1][0])) * (1.0 - (np.exp((0.0 -
uncons_result8[0][1][1]) / uncons_result9[0][1][0]))) invisible))])

loop_cond = not [] == loop_acc[0]
x1 = loop_acc
return x1[1]