MAXIMAL FUNCTION CHARACTERIZATIONS OF MUSIELAK-ORLICZ-HARDY SPACES ASSOCIATED TO NON-NEGATIVE SELF-ADJOINT OPERATORS SATISFYING GAUSSIAN ESTIMATES

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Abstract. Let $L$ be a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ whose heat kernels have the Gaussian upper bound estimates. Assume that the growth function $\phi : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty)$ satisfies that $\phi(x, \cdot)$ is an Orlicz function and $\phi(\cdot, t) \in A_\infty(\mathbb{R}^n)$ (the class of uniformly Muckenhoupt weights). Let $H_{\phi, L}(\mathbb{R}^n)$ be the Musielak-Orlicz-Hardy space introduced via the Lusin area function associated with the heat semigroup of $L$. In this article, the authors obtain several maximal function characterizations of the space $H_{\phi, L}(\mathbb{R}^n)$, which, especially, answer an open question of L. Song and L. Yan under an additional mild assumption satisfied by Schrödinger operators on $\mathbb{R}^n$ with non-negative potentials belonging to the reverse Hölder class, and second-order divergence form elliptic operators on $\mathbb{R}^n$ with bounded measurable real coefficients.

1. Introduction. The real-variable theory of Hardy spaces on the $n$-dimensional Euclidean space $\mathbb{R}^n$, initiated by Stein and Weiss [30] and then systematically developed by Fefferman and Stein [13], plays an important role in various fields of analysis (see, for example, [13, 29]). It is well known that the Hardy space $H^p(\mathbb{R}^n)$, with $p \in (0, 1]$, is a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$; for example, the classical Riesz transform is bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$. Moreover, the Hardy space $H^p(\mathbb{R}^n)$ is essentially related to the Laplace operator $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$. However, in many settings, these classical function spaces are not applicable; for example, the Riesz transforms $\nabla L^{-1/2}$ may not be bounded.
from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $L$ is a second-order divergence form elliptic operator with complex bounded measurable coefficients (see [16]). Motivated by this, the study for the real-variable theory of various function spaces, especially, the Hardy-type spaces, associated with different differential operators, has inspired great interests in recent years; see, for example, [1, 2, 8, 9, 11, 15, 16, 20, 28, 31] for the case of Hardy spaces, [24, 27] for the case of weighted Hardy spaces and [4, 7, 18, 19, 32, 33, 34] for the case of (Musielak-)Orlicz Hardy spaces.

Let $L$ be a second-order divergence form elliptic operator on $\mathbb{R}^n$ with bounded measurable complex coefficients. The Hardy space $H^1_L(\mathbb{R}^n)$ associated with $L$ was characterized by Hofmann and Mayboroda [16] in terms of the molecule, the Lusin area function, the non-tangential maximal function $(N_L(f), N_{L, p}(f))$ or the radial maximal function $(R_L(f), R_{L, p}(f))$, respectively, associated with its heat semigroup or its Poisson semigroup generated by $L$. Meanwhile, the same equivalent characterizations of the Orlicz-Hardy space associated with $L$ as those of $H^1_L(\mathbb{R}^n)$ were independently obtained in [18]. Recall that, for any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the non-tangential maximal function $N_L(f)$ and the radial maximal function $R_L(f)$, associated with the heat semigroup of $L$, are defined by

$$N_L(f)(x) := \sup_{(y,t) \in \Gamma(x)} \left\{ \frac{1}{t^n} \int_{B(y,t)} \left| e^{-t^2 L}(f)(z) \right|^2 \, dz \right\}^{1/2},$$

and

$$R_L(f)(x) := \sup_{t \in (0, \infty)} \left\{ \frac{1}{t^n} \int_{B(x,t)} \left| e^{-t^2 L}(f)(z) \right|^2 \, dz \right\}^{1/2},$$

respectively, here and hereafter, for any $x \in \mathbb{R}^n$,

$$\Gamma(x) := \{(y, t) \in \mathbb{R}^n \times (0, \infty) : |x - y| < t\},$$

and, for any $(y, t) \in \mathbb{R}^n \times (0, \infty)$,

$$B(y, t) := \{z \in \mathbb{R}^n : |y - z| < t\}.$$
**radial maximal function** $f^+_L$, associated with the heat semigroup of $L$, are defined by

$$f^+_L(x) := \sup_{(y,t) \in \Gamma(x)} e^{-t^2 L}(f)(y)$$

and

$$f^+_L(x) := \sup_{t \in (0,\infty)} e^{-t^2 L}(f)(x),$$

respectively. Furthermore, the non-tangential maximal function $f^+_{L,P}$ and the radial maximal function $f^+_{L,P}$, associated with the Poisson semigroup of $L$, are defined via a similar way. Observe that the maximal functions in (1.3) and (1.4) are different from those in (1.1) and (1.2). The main reason for adding the averaging for the spatial variable in (1.1) and (1.2) is that we need to compensate for the lack of pointwise estimates of the heat semigroup and the Poisson semigroup in that case (see [16] for more details). Recall that, when $L := -\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$, its heat semigroup and its Poisson semigroup have pointwise upper bound estimates (see, for example, [15, (8.4)]).

From now on, in the remainder of the whole article, we _always assume_ that $L$ is a densely defined linear operator on $L^2(\mathbb{R}^n)$. In some cases, we also need $L$ to satisfy either or both of the following two assumptions:

**Assumption 1.1.** $L$ is non-negative and self-adjoint.

**Assumption 1.2.** The kernels of the semigroup $\{e^{-tL}\}_{t>0}$, denoted by $\{K_t\}_{t>0}$, are measurable functions on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfy the Gaussian upper bound estimates, namely, there exist positive constants $C$ and $c$ such that, for all $t \in (0,\infty)$ and $x, y \in \mathbb{R}^n$,

$$|K_t(x,y)| \leq \frac{C}{t^{n/2}} \exp \left\{ -\frac{|x-y|^2}{ct} \right\}. \quad (1.5)$$

The typical examples of the operators $L$, satisfying both Assumptions 1.1 and 1.2, include the Schrödinger operator $L := -\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the second-order divergence form elliptic operator $L := -\text{div}(AV)$ with $A := \{a_{ij}\}_{i,j=1}^n$ satisfying that, for all $i, j \in \{1, \ldots, n\}$, $a_{ij}$ is a real measurable function on $\mathbb{R}^n$ and there exists a constant $\lambda \in (0,1)$ such that, for all $i, j \in \{1, \ldots, n\}$ and $x, \xi \in \mathbb{R}^n$,

$$a_{ij}(x) = a_{ji}(x) \quad \text{and} \quad \lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2.$$

Denote by $S(\mathbb{R}^n)$ the _space of all Schwartz functions_ on $\mathbb{R}^n$. Let $p \in (0,1]$, $\alpha \in (0,\infty)$, $\phi \in S(\mathbb{R})$ be an even function and $\phi(0) = 1$. Recently, the characterizations of $H^p_v(\mathbb{R}^n)$ in terms of the non-tangential maximal function $\phi^*_{L,\alpha}(f)$ or the grand maximal function $G^*_{L}(f)$ were obtained by Song and Yan [28] via some essential improvements of techniques due to Calderón [6]. Recall that, for any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the _non-tangential maximal function_ $\phi^*_{L,\alpha}(f)$ is defined by setting

$$\phi^*_{L,\alpha}(f)(x) := \sup_{|y-x| < \alpha t, t \in (0,\infty)} \left| \phi(t\sqrt{L})(f)(y) \right| \quad (1.6)$$

(see (2.1) below for the definition of $\phi(t\sqrt{L})$) and the _grand maximal function_ $G^*_{L}(f)$ is defined by setting

$$G^*_{L}(f)(x) := \sup_{\phi \in A} \sup_{|x-y| < t, t \in (0,\infty)} \left| \phi(t\sqrt{L})(f)(y) \right|, \quad (1.7)$$
where
\[ A := \left\{ \phi \in S(\mathbb{R}) : \phi \text{ is even with } \phi(0) \neq 0 \text{ and} \right. \]
\[ \int_{\mathbb{R}} (1 + |x|)^N \sum_{0 \leq k \leq N} \left| \frac{d^k \phi(x)}{dx^k} \right|^2 dx \leq 1 \]
with \( N \) being a large positive integer. It is easy to see that, when \( \phi(x) := e^{-x^2} \) for all \( x \in \mathbb{R} \) and \( \alpha := 1 \), the maximal function \( \phi_{L,\alpha}^*(f) \) in (1.6) coincides with the maximal function \( f_L^* \) in (1.3).

Let the operator \( L \) satisfy both Assumptions 1.1 and 1.2. In this article, motivated by [4, 28, 34], we characterize the Musielak-Orlicz-Hardy space associated with \( L \) via the non-tangential maximal function in (1.6) or the grand maximal function in (1.7). Under an additional assumption for \( L \) (see Assumption 1.11 below for the details), which is satisfied by Schrödinger operators on \( \mathbb{R}^n \) with non-negative potentials belonging to the reverse Hölder class and second-order divergence form elliptic operators on \( \mathbb{R}^n \) with bounded measurable real coefficients, we obtain the equivalent characterization of the Musielak-Orlicz-Hardy space associated with \( L \) in terms of the radial maximal function in (1.4). As a special case, under the additional mild Assumption 1.11 for \( L \), we give an answer to the open question in [28, Remark 3.4], namely, whether or not the Hardy space \( H^p_L(\mathbb{R}^n) \), with \( p \in (0, 1] \), can be characterized via the radial maximal function in (1.4).

To state the main results of this article, we now describe the Musielak-Orlicz function considered in this article. Recall that a function \( \Phi : [0, \infty) \to [0, \infty) \) is called an \textit{Orlicz function} if it is non-decreasing, \( \Phi(0) = 0, \Phi(t) > 0 \) for any \( t \in (0, \infty) \) and \( \lim_{t \to \infty} \Phi(t) = \infty \) (see, for example, [25, 26]). We point out that, different from the classical definition of Orlicz functions, the \textit{Orlicz functions in this article may not be convex}. Moreover, \( \Phi \) is said to be of \textit{upper} (resp. \textit{lower}) \textit{type} \( p \) for some \( p \in (0, \infty) \) if there exists a positive constant \( C \) such that, for all \( s \in [1, \infty) \) (resp. \( s \in [0, 1] \)) and \( t \in [0, \infty) \),

\[ \Phi(st) \leq Cs^p \Phi(t). \]

For a given function \( \varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty) \) such that, for any \( x \in \mathbb{R}^n, \varphi(x, \cdot) \) is an Orlicz function, \( \varphi \) is said to be of \textit{uniformly upper} (resp. \textit{lower}) \textit{type} \( p \) for some \( p \in (0, \infty) \) if there exists a positive constant \( C \) such that, for all \( x \in \mathbb{R}^n, t \in [0, \infty) \) and \( s \in [1, \infty) \) (resp. \( s \in [0, 1] \)),

\[ \varphi(x, st) \leq Cs^p \varphi(x, t). \]

Let
\[ I(\varphi) := \inf \{ p \in (0, \infty) : \varphi \text{ is of uniformly upper type } p \} \quad (1.8) \]
and
\[ i(\varphi) := \sup \{ p \in (0, \infty) : \varphi \text{ is of uniformly lower type } p \} . \quad (1.9) \]
In what follows, \( I(\varphi) \) and \( i(\varphi) \) are, respectively, called the \textit{uniformly critical upper type index} and the \textit{uniformly critical lower type index} of \( \varphi \). Observe that \( I(\varphi) \) and \( i(\varphi) \) may not be attainable, namely, \( \varphi \) may not be of uniformly upper type \( I(\varphi) \) or uniformly lower type \( i(\varphi) \) (see, for example, [4, 17, 22, 34] for some examples). Moreover, it is easy to see that, if \( \varphi \) is of uniformly upper type \( p_0 \in (0, \infty) \) and lower type \( p_1 \in (0, \infty) \), then \( p_0 \geq p_1 \) and hence \( I(\varphi) \geq i(\varphi) \).
Definition 1.3. Let \( \varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty) \) satisfy that \( \varphi(\cdot, t) \) is measurable for all \( t \in [0, \infty) \). Then \( \varphi \) is said to satisfy the uniformly Muckenhoupt condition for some \( q \in [1, \infty) \), denoted by \( \varphi \in \mathcal{A}_q(\mathbb{R}^n) \), if, when \( q \in (1, \infty) \),

\[
\mathcal{A}_q(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|^q} \int_B \varphi(x, t) \, dx \left\{ \int_B [\varphi(y, t)]^{1-q} \, dy \right\}^{q-1} < \infty
\]
or

\[
\mathcal{A}_1(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x, t) \, dx \left\{ \text{ess sup}_{y \in B} [\varphi(y, t)]^{-1} \right\} < \infty,
\]

where the first suprema are taken over all \( t \in (0, \infty) \) and the second ones over all balls \( B \subset \mathbb{R}^n \).

The function \( \varphi \) is said to satisfy the uniformly reverse Hölder condition for some \( q \in (1, \infty) \), denoted by \( \varphi \in \mathcal{R}_q(\mathbb{R}^n) \), if, when \( q \in (1, \infty) \),

\[
\mathcal{R}_q(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|} \int_B [\varphi(x, t)]^q \, dx \right\}^{1/q} \left\{ \frac{1}{|B|} \int_B \varphi(x, t) \, dx \right\}^{1-1/q} < \infty
\]
or

\[
\mathcal{R}_\infty(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset \mathbb{R}^n} \left\{ \text{ess sup}_{y \in B} \varphi(y, t) \right\} \left\{ \frac{1}{|B|} \int_B \varphi(x, t) \, dx \right\}^{-1} < \infty,
\]

where the first suprema are taken over all \( t \in (0, \infty) \) and the second ones over all balls \( B \subset \mathbb{R}^n \).

Recall that, in Definition 1.3, \( \mathcal{A}_p(\mathbb{R}^n) \), with \( p \in [1, \infty) \), and \( \mathcal{R}_q(\mathbb{R}^n) \), with \( q \in (1, \infty) \), were respectively introduced in [22] and [34]. Furthermore, let

\[
\mathcal{A}_\infty(\mathbb{R}^n) := \bigcup_{q \in [1, \infty)} \mathcal{A}_q(\mathbb{R}^n).
\]

We now recall the notions of the critical indices for \( \varphi \in \mathcal{A}_\infty(\mathbb{R}^n) \) as follows:

\[
q(\varphi) := \inf \{ q \in [1, \infty) : \varphi \in \mathcal{A}_q(\mathbb{R}^n) \} \tag{1.10}
\]

and

\[
r(\varphi) := \sup \{ q \in (1, \infty) : \varphi \in \mathcal{R}_q(\mathbb{R}^n) \}. \tag{1.11}
\]

Recall also that, if \( q(\varphi) \in (1, \infty) \), then, by [17, Lemma 2.4(iii)], we know that \( \varphi \not\in \mathcal{A}_q(\varphi)(\mathbb{R}^n) \) and there exists \( \varphi \not\in \mathcal{A}_1(\mathbb{R}^n) \) such that \( q(\varphi) = 1 \) (see, for example, [21]). Similarly, if \( r(\varphi) \in (1, \infty) \), then, by [32, Lemma 2.3(iv)], we know that \( \varphi \not\in \mathcal{R}_r(\varphi)(\mathbb{R}^n) \) and there exists \( \varphi \not\in \mathcal{R}_\infty(\mathbb{R}^n) \) such that \( r(\varphi) = \infty \) (see, for example, [10]).

Now we recall the notion of growth functions from Ky [22].

Definition 1.4. A function \( \varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty) \) is called a growth function if the following hold true:

(i) \( \varphi \) is a Musielak-Orlicz function, namely,
   (a) \( \varphi(x, \cdot) \) is an Orlicz function for all \( x \in \mathbb{R}^n \);
   (b) \( \varphi(\cdot, t) \) is a measurable function for all \( t \in [0, \infty) \).

(ii) \( \varphi \in \mathcal{A}_\infty(\mathbb{R}^n) \).

(iii) The function \( \varphi \) is of uniformly lower type \( p \) for some \( p \in (0, 1] \) and of uniformly upper type 1.
For a Musielak-Orlicz function $\varphi$ as in Definition 1.3, a measurable function $f$ on $\mathbb{R}^n$ is said to be in the Musielak-Orlicz space $L^{\varphi}(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n} \varphi(x, |f(x)|) \, dx < \infty$. Moreover, for any $f \in L^{\varphi}(\mathbb{R}^n)$, the quasi-norm $\|f\|_{L^{\varphi}(\mathbb{R}^n)}$ of $f$ is defined by setting

$$
\|f\|_{L^{\varphi}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi \left( x, \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.
$$

Clearly, for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$,

$$
\varphi(x, t) := \omega(x) \Phi(t)
$$

is a growth function if $\omega \in A_\infty(\mathbb{R}^n)$ and $\Phi$ is an Orlicz function of lower type $p$ for some $p \in (0, 1)$ and upper type $1$. Here and hereafter, $A_q(\mathbb{R}^n)$ with $q \in [1, \infty]$ denotes the class of Muckenhoupt weights (see, for example, [14]). A typical example of such functions $\Phi$ is $\Phi(t) := t^p$, with $p \in (0, 1]$, for all $t \in [0, \infty)$ (see, for example, [17, 22, 23, 34] for more examples of such $\Phi$). Another typical example $\varphi$ of growth functions is given by setting

$$
\varphi(x, t) := \frac{t}{\ln(e + |x|) + \ln(e + t)}
$$

for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$; more precisely, $\varphi \in A_1(\mathbb{R}^n)$, $\varphi$ is of uniformly upper type $1$ (indeed, $I(\varphi) = 1$, which is attainable) and $i(\varphi) = 1$ which is not attainable (see [22] for the details).

Now we recall the definition of the Musielak-Orlicz-Hardy space $H_{\varphi, L}(\mathbb{R}^n)$ introduced in [4, 34].

**Definition 1.5.** Let $L$ be a densely defined linear operator on $L^2(\mathbb{R}^n)$ satisfying Assumptions 1.1 and 1.2, and $\varphi$ as in Definition 1.4. For $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Lusin area function, $S_L(f)$, associated with $L$ is defined by setting

$$
S_L(f)(x) := \int_{\Gamma(x)} \left| t^2 L^{-t^2 L}(f)(y) \right|^2 \frac{dy \, dt}{ln \pi + t^2}.
$$

A function $f \in L^2(\mathbb{R}^n)$ is said to be in the set $H_{\varphi, L}(\mathbb{R}^n)$ if $S_L(f) \in L^2(\mathbb{R}^n)$; moreover, define $\|f\|_{H_{\varphi, L}(\mathbb{R}^n)} := \|S_L(f)\|_{L^2(\mathbb{R}^n)}$. Then the Musielak-Orlicz-Hardy space $H_{\varphi, L}(\mathbb{R}^n)$ is defined to be the completion of $H_{\varphi, L}(\mathbb{R}^n)$ with respect to the quasi-norm $\| \cdot \|_{H_{\varphi, L}(\mathbb{R}^n)}$.

Moreover, we recall the following definitions of $(\varphi, q, M)_{L}$-atoms and atomic Musielak-Orlicz-Hardy spaces $H_{\varphi, L, at}(\mathbb{R}^n)$ introduced in [4, Definitions 5.2 and 5.3], respectively.

**Definition 1.6.** Let $L$ and $\varphi$ be as in Definition 1.5. Assume that $q \in (1, \infty]$, $M \in \mathbb{N}$ and $B \subset \mathbb{R}^n$ is a ball.

(I) Let $D(L^M)$ be the domain of $L^M$. A function $\alpha \in L^q(\mathbb{R}^n)$ is called a $(\varphi, q, M)_{L}$-atom associated with the ball $B$ if there exists a function $b \in D(L^M)$ such that

(i) $\alpha = L^M b$;

(ii) for all $j \in \{0, 1, \ldots, M\}$, $\text{supp}(L^j b) \subset B$;

(iii) $\|(r_B^j L^j b)\|_{L^q(\mathbb{R}^n)} \leq r_B^{2M} |B|^{1/q} \|\chi_B\|_{L^q(\mathbb{R}^n)}^{-1}$, where $r_B$ denotes the radius of $B$ and $j \in \{0, 1, \ldots, M\}$. 


(II) For $f \in L^2(\mathbb{R}^n)$,

$$f = \sum_j \lambda_j \alpha_j$$

(1.14)

is called an atomic $(\varphi, q, M)_L$-representation of $f$ if, for any $j$, $\alpha_j$ is a $(\varphi, q, M)_L$-atom associated with the ball $B_j \subset \mathbb{R}^n$, the summation (1.14) converges in $L^2(\mathbb{R}^n)$ and $(\{\lambda_j\}_j) \subset \mathbb{C}$ satisfies that

$$\sum_j \varphi(B_j, |\lambda_j||\chi_{B_j}|^{-1}_{L^q(\mathbb{R}^n)}) < \infty.$$ 

Let

$$\mathbb{H}^{M, q}_{\varphi, L, \text{at}}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n): f \text{ has an atomic } (\varphi, q, M)_L\text{-representation}\},$$

equipped with the quasi-norm

$$\|f\|_{H^{M, q}_{\varphi, L, \text{at}}(\mathbb{R}^n)} := \inf \left\{ \Lambda \left(\{\lambda_j\}_j\right): \sum_j \lambda_j \alpha_j \text{ is a } (\varphi, q, M)_L\text{-representation of } f \right\},$$

where

$$\Lambda \left(\{\lambda_j\}_j\right) := \inf \left\{ \lambda \in (0, \infty): \sum_j \varphi(B_j, |\lambda_j| |\chi_{B_j}|^{-1}_{L^q(\mathbb{R}^n)}) \leq 1 \right\}$$

and the infimum is taken over all the atomic $(\varphi, q, M)_L$-representations of $f$ as above. Then the atomic Musielak-Orlicz-Hardy space $H^{M, q}_{\varphi, L, \text{at}}(\mathbb{R}^n)$ is defined as the completion of the set $\mathbb{H}^{M, q}_{\varphi, L, \text{at}}(\mathbb{R}^n)$ with respect to the quasi-norm

$$\|\cdot\|_{H^{M, q}_{\varphi, L, \text{at}}(\mathbb{R}^n)}.$$ 

Now we introduce Musielak-Orlicz-Hardy spaces via maximal functions associated with the operator $L$.

**Definition 1.7.** Let $L$ and $\varphi$ be as in Definition 1.5.

(i) Assume that $\phi \in \mathcal{S}(\mathbb{R})$ is an even function with $\phi(0) = 1$ and $\alpha \in (0, \infty)$. For any $f \in L^2(\mathbb{R}^n)$, let $\phi_{L, \alpha}^* (f)$ be as in (1.6). A function $f \in L^2(\mathbb{R}^n)$ is said to be in the set $\mathbb{H}^{\phi, \alpha}_{\varphi, L, \max}(\mathbb{R}^n)$ if $\phi_{L, \alpha}^*(f) \in L^\varphi(\mathbb{R}^n)$; moreover, define

$$\|f\|_{H^{\phi, \alpha}_{\varphi, L, \max}(\mathbb{R}^n)} := \|\phi_{L, \alpha}^*(f)\|_{L^\varphi(\mathbb{R}^n)}.$$ 

Then the Musielak-Orlicz-Hardy space $H^{\phi, \alpha}_{\varphi, L, \max}(\mathbb{R}^n)$ is defined to be the completion of $\mathbb{H}^{\phi, \alpha}_{\varphi, L, \max}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{\phi, \alpha}_{\varphi, L, \max}(\mathbb{R}^n)}$.

Specially, if $\phi(x) := e^{-x^2}$ for all $x \in \mathbb{R}$ and $\alpha := 1$, denote $\phi_{L, \alpha}^*(f)$ simply by $f_L^*$ and, in this case, denote the space $H^{\phi, \alpha}_{\varphi, L, \max}(\mathbb{R}^n)$ simply by $H_{\varphi, L, \max}(\mathbb{R}^n)$.

(ii) For any $f \in L^2(\mathbb{R}^n)$, let $G^*_L(f)$ be as in (1.7). Then the Musielak-Orlicz-Hardy space $H^{A}_{\varphi, L, \max}(\mathbb{R}^n)$ is defined via replacing $\phi_{L, \alpha}^*(f)$ by $G^*_L(f)$ in the definition of the space $H^{\phi, \alpha}_{\varphi, L, \max}(\mathbb{R}^n)$.

Then the first main result of this article reads as follows.
Theorem 1.8. Let $L$ be a densely defined linear operator on $L^2(\mathbb{R}^n)$ satisfying Assumptions 1.1 and 1.2, and $\varphi$ as in Definition 1.4. Assume that $r(\varphi)$, $I(\varphi)$, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (1.11), (1.8), (1.10) and (1.9), and $[r(\varphi)]'$ denotes the conjugate exponent of $r(\varphi)$.

(i) For any $q \in ([r(\varphi)]'I(\varphi), \infty)$, $M \in \mathbb{N} \cap \left(\frac{nq(\varphi)}{2r(\varphi)}, \infty\right)$ and $\alpha \in (0, \infty)$, the spaces

$$H^M_{\varphi, L, \max}(\mathbb{R}^n), \ H^\alpha_{\varphi, L, \max}(\mathbb{R}^n) \text{ and } H^A_{\varphi, L, \max}(\mathbb{R}^n)$$

coincide with equivalent quasi-norms.

(ii) For any $q \in ([r(\varphi)]'I(\varphi), \infty)$, $M \in \mathbb{N} \cap \left(\frac{nq(\varphi)}{2r(\varphi)}, \infty\right)$ and $\alpha \in (0, \infty)$, the spaces

$$H^M_{\varphi, L}(\mathbb{R}^n), \ H^M_{\varphi, L, \at}(\mathbb{R}^n), \ H^\alpha_{\varphi, L, \max}(\mathbb{R}^n)$$

and $H^A_{\varphi, L, \max}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

The following chains of inequalities give the strategy of the proof of Theorem 1.8(i): For all $f \in H^\alpha_{\varphi, L, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, it holds true that

$$\|f\|_{H^\alpha_{\varphi, L, \max}(\mathbb{R}^n)} \gtrsim \|f\|_{H^M_{\varphi, L, \at}(\mathbb{R}^n)} \gtrsim \|f\|_{H^M_{\varphi, L, \max}(\mathbb{R}^n)} \gtrsim \|f\|_{H^A_{\varphi, L, \max}(\mathbb{R}^n)}, \quad (1.15)$$

where $q \in ([r(\varphi)]'I(\varphi), \infty)$ and the implicit positive constants are independent of $f$.

We establish the first inequality in (1.15) via borrowing some ideas from the proof of [28, Theorem 1.4]. By the definitions of the spaces $H^M_{\varphi, L, \at}(\mathbb{R}^n)$, $H^\alpha_{\varphi, L, \max}(\mathbb{R}^n)$ and $H^A_{\varphi, L, \max}(\mathbb{R}^n)$, we find that the second and the fourth inequalities in (1.15) are obvious. Moreover, we obtain the third inequality in (1.15) by establishing a pointwise estimate for $\alpha^*_L$, where $\alpha$ is a $(\varphi, q, M)_L$-atom (see (2.26) below for the details). Furthermore, (ii) of Theorem 1.8 is obtained via (i) and the fact that the spaces $H^M_{\varphi, L}(\mathbb{R}^n)$ and $H^M_{\varphi, L, \at}(\mathbb{R}^n)$, with $q \in ([r(\varphi)]'I(\varphi), \infty)$, coincide with equivalent quasi-norms, which was established in [4, Theorem 5.4].

Let

$$L_A := -(\nabla - iA)^2 + V \quad (1.16)$$

be a magnetic Schrödinger operator on $\mathbb{R}^n$ with $n \geq 3$, where $A \in L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and the potential $V$ belongs to the Kato class, namely,

$$\sup_{x \in \mathbb{R}^n} \lim_{r \to 0} \int_{B(x, r)} \frac{|V(y)|}{|x - y|^{n-2}} \, dy = 0.$$

Moreover, the Kato norm of $V$ is defined by setting

$$\|V\|_K := \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(y)|}{|x - y|^{n-2}} \, dy.$$

For the potential $V$, let

$$V_+ := \max\{V, 0\} \quad \text{and} \quad V_- := \min\{V, 0\}.$$

Under the assumption that $L_A$ is as in (1.16) with $V_+$ belonging to the Kato class and $\|V_+\|_K < \pi^{n/2}/(n/2 - 1)$, it was showed in [5] that $L$ satisfies Assumptions 1.1 and 1.2. Thus, as a corollary of Theorem 1.8, we have the following several equivalent characterizations of the Musielak-Orlicz-Hardy space $H^\alpha_{\varphi, L_A}(\mathbb{R}^n)$ associated with $L_A$. 
Corollary 1.9. Let \( L_A \) be as in (1.16), with \( V_+ \) belonging to the Kato class and \( \|V_-\|_\mu < \pi^{n/2}/\Gamma(n/2 - 1) \), and \( \varphi \) as in Definition 1.4.

(i) Assume that \( q, M \) and \( \alpha \) are as in Theorem 1.8(i). Then the spaces

\[
H^{M,q}_{\varphi, L_A, \ast}(\mathbb{R}^n), \quad H^{\phi,\alpha}_{\varphi, L_A, \ast}(\mathbb{R}^n) \quad \text{and} \quad H^{A}_{\varphi, L_A, \ast}(\mathbb{R}^n)
\]

coincide with equivalent quasi-norms.

(ii) Assume that \( q, M \) and \( \alpha \) are as in Theorem 1.8(ii). Then the spaces

\[
H^{M,q}_{\varphi, L_A}(\mathbb{R}^n), \quad H^{\phi,\alpha}_{\varphi, L_A, \ast}(\mathbb{R}^n), \quad H^{A}_{\varphi, L_A, \ast}(\mathbb{R}^n)
\]

and \( H^{A}_{\varphi, L_A, \ast}(\mathbb{R}^n) \) coincide with equivalent quasi-norms.

Remark 1.10. We point out that the equivalences of \( H^{M,\infty}_{\varphi, L, at}(\mathbb{R}^n) \), \( H^{\phi,\alpha}_{\varphi, L, at}(\mathbb{R}^n) \), and \( H^{A}_{\varphi, L, at}(\mathbb{R}^n) \) in Theorem 1.8(i) or of \( H^{M,\infty}_{\varphi, L_A, \ast}(\mathbb{R}^n) \) and \( H^{A}_{\varphi, L_A, \ast}(\mathbb{R}^n) \) in Corollary 1.9(i) were obtained, respectively, in [28, Theorem 1.4] and Proposition 3.3 when \( \varphi(x, t) := t^p \), with \( p \in (0, 1) \), for all \( x \in \mathbb{R}^n \) and \( t \in [0, \infty) \). Moreover, Theorem 1.8 and Corollary 1.9 are new even when \( \varphi \) is as in (1.12) or (1.13).

Let \( L \) satisfy Assumption 1.2 and \( \varphi \) be as in Definition 1.4. For any \( f \in L^2(\mathbb{R}^n) \), let \( f^\mu_L \) be as (1.4). Then the Musielak-Orlicz-Hardy space \( H^{\phi,\alpha}_{\varphi, L, rad}(\mathbb{R}^n) \) is defined via replacing \( \phi^\alpha_{L,\ast}(f) \) by the radial maximal function \( f^\mu_L \) in the definition of the space \( H^{\phi,\alpha}_{\varphi, L, \ast}(\mathbb{R}^n) \).

By the definitions of the spaces \( H^{\varphi, L, \ast}(\mathbb{R}^n) \) and \( H^{\varphi, L, rad}(\mathbb{R}^n) \), we know that the continuous inclusion \( H^{\varphi, L, \ast}(\mathbb{R}^n) \subset H^{\varphi, L, rad}(\mathbb{R}^n) \) holds true. It is a natural question whether or not the continuous inclusion \( H^{\varphi, L, rad}(\mathbb{R}^n) \subset H^{\varphi, L, \ast}(\mathbb{R}^n) \) holds true.

To answer this question, we need to introduce another assumption for the operator \( L \) as follows:

Assumption 1.11. There exist positive constants \( C \) and \( \mu \in (0, 1] \) such that, for all \( t \in (0, \infty) \) and \( x, y_1, y_2 \in \mathbb{R}^n \),

\[
|K_t(y_1, x) - K_t(y_2, x)| \leq \frac{C}{t^{n/2}} \frac{|y_1 - y_2|^\mu}{t^{\mu/2}}. \tag{1.17}
\]

We point out that there are lots of operators on \( \mathbb{R}^n \) satisfying Assumption 1.11; for example, Schrödinger operators on \( \mathbb{R}^n \) with non-negative potentials belonging to the reverse Hölder class (see, for example, [12]) and second-order divergence form elliptic operators on \( \mathbb{R}^n \) with bounded measurable real coefficients (see, for example, [3]).

Theorem 1.12. Let \( L \) be a densely defined linear operator on \( L^2(\mathbb{R}^n) \) satisfying Assumptions 1.2 and 1.11, and \( \varphi \) as in Definition 1.4. Then the spaces \( H^{\varphi, L, \ast}(\mathbb{R}^n) \) and \( H^{\varphi, L, rad}(\mathbb{R}^n) \) coincide with equivalent quasi-norms.

Remark 1.13. We show Theorem 1.12 via borrowing some ideas from the proofs of [2, Proposition 21] and [33, Lemma 5.3]. Under the additional assumption that \( L \) satisfies Assumption 1.11, Theorem 1.12 gives an answer to the open question stated in [28, Remark 3.4] by taking \( \varphi(x, t) := t^p \), with \( p \in (0, 1) \), for all \( x \in \mathbb{R}^n \) and \( t \in [0, \infty) \).

As a corollary of Theorems 1.8 and 1.12, we have the following conclusion.
Corollary 1.14. Let $L$ be a densely defined linear operator on $L^2(\mathbb{R}^n)$ satisfying Assumptions 1.1, 1.2 and 1.11, and $\varphi$ as in Definition 1.4. Assume that $q$, $M$ and $\alpha$ are as in Theorem 1.8(ii). Then the spaces $H_{\varphi,L}(\mathbb{R}^n)$, $H_{\varphi,L}^{M,q}$, $H_{\varphi,L}^r$, $H_{\varphi,L}^{r,\alpha}$, $H_{\varphi,L,\max}(\mathbb{R}^n)$, $H_{\varphi,L,\rad}(\mathbb{R}^n)$ and $H_{\varphi,L}^A(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

The layout of this article is as follows. Sections 2 and 3 are, respectively, devoted to the proofs of Theorems 1.8 and 1.12.

Finally we make some conventions on notation. Throughout the whole article, we always denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. We also use $C(\gamma, \beta, \ldots)$ to denote a positive constant depending on the indicated parameters $\gamma$, $\beta$, $\ldots$. The symbol $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

2. Proof of Theorem 1.8. In this section, we give out the proof of Theorem 1.8.

To this end, we first recall some auxiliary conclusions.

For a non-negative self-adjoint operator $L$ on $L^2(\mathbb{R}^n)$, denote by $E_L$ the spectral measure associated with $L$. Then, for any bounded Borel function $F : [0, \infty) \to \mathbb{C}$, the operator $F(L) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined by the formula

$$F(L) := \int_0^\infty F(\lambda) \, dE_L(\lambda).$$

(2.1)

Then we have the following lemma, which was obtained in [15, Lemma 3.5]. In what follows, $C_c^\infty(\mathbb{R})$ denotes the set of all infinitely differentiable functions on $\mathbb{R}$ with compact supports.

Lemma 2.1. Assume that the densely defined linear operator $L$ on $L^2(\mathbb{R}^n)$ satisfies Assumptions 1.1 and 1.2. Let $\phi \in C_c^\infty(\mathbb{R})$ be even and $\text{supp}(\phi) \subset (-1, 1)$. Denote by $\Phi$ the Fourier transform of $\phi$. Then, for any $k \in \mathbb{Z}_+$, the kernels $\{K_{(t^2L)^k\Phi(t\sqrt{L})}\}_{t > 0}$ of the operators $\{(t^2L)^k\Phi(t\sqrt{L})\}_{t > 0}$ satisfy that there exists a positive constant $C$, depending on $n$, $k$ and $\Phi$, such that, for all $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$,

$$\text{supp} \left( K_{(t^2L)^k\Phi(t\sqrt{L})} \right) \subset \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t \}$$

and

$$|K_{(t^2L)^k\Phi(t\sqrt{L})}(x, y)| \leq Ct^{-n}.$$
Lemma 2.2. Let \( \varphi \) be as in Definition 1.4.

(i) There exists a positive constant \( C \) such that, for all \( (x, t_j) \in \mathbb{R}^n \times [0, \infty) \) with \( j \in \mathbb{N} \),
\[
\varphi \left( x, \sum_{j=1}^{\infty} t_j \right) \leq C \sum_{j=1}^{\infty} \varphi(x, t_j).
\]

(ii) For all \( (x, t) \in \mathbb{R}^n \times [0, \infty) \), let
\[
\tilde{\varphi}(x, t) := \int_0^t \frac{\varphi(x, s)}{s} \, ds.
\]
Then \( \tilde{\varphi} \) is equivalent to \( \varphi \), namely, there exists a positive constant \( C \) such that, for all \( (x, t) \in \mathbb{R}^n \times [0, \infty) \),
\[
C^{-1} \varphi(x, t) \leq \tilde{\varphi}(x, t) \leq C \varphi(x, t).
\]

(iii) If \( p \in (1, \infty) \) and \( \varphi \in \mathcal{A}_p(\mathbb{R}^n) \), then there exists a positive constant \( C \) such that, for all measurable functions \( f \) on \( \mathbb{R}^n \) and \( t \in [0, \infty) \),
\[
\int_{\mathbb{R}^n} [\mathcal{M}(f)(x)]^p \varphi(x, t) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \varphi(x, t) \, dx.
\]

(iv) If \( \varphi \in \mathcal{A}_p(\mathbb{R}^n) \) with \( p \in [1, \infty) \), then there exists a positive constant \( C \) such that, for all balls \( B_1, B_2 \subset \mathbb{R}^n \) with \( B_1 \subset B_2 \) and \( t \in (0, \infty) \),
\[
\frac{\varphi(B_2, t)}{\varphi(B_1, t)} \leq C \left[ \frac{B_2}{B_1} \right]^p.
\]

Moreover, to show Theorem 1.8, we need to prove the following conclusion.

Proposition 2.3. Let the densely defined linear operator \( L \) on \( L^2(\mathbb{R}^n) \) satisfy Assumptions 1.1 and 1.2, and \( \varphi \) be as in Definition 1.4. Assume that \( \psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}) \) are even functions satisfying \( \psi_1(0) = \psi_2(0) = 1 \), and \( \alpha_1, \alpha_2 \in (0, \infty) \). For \( i \in \{0, 1\} \), \( f \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), let
\[
\psi^*_i L, \alpha_i(f)(x) := \sup_{|x-y|<\alpha_i t, t \in (0, \infty)} \left| \psi_i(t\sqrt{L})(y) \right|.
\]
Then there exists a positive constant \( C \), depending on \( n, \varphi, \psi_1, \psi_2, \alpha_1 \) and \( \alpha_2 \), such that, for all \( f \in L^2(\mathbb{R}^n) \),
\[
\left\| \psi^*_i L, \alpha_i(f) \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \psi^*_2 L, \alpha_2(f) \right\|_{L^p(\mathbb{R}^n)}. \tag{2.2}
\]

Specially, for any even function \( \phi \in \mathcal{S}(\mathbb{R}) \) satisfying \( \phi(0) = 1 \) and \( \alpha \in (0, \infty) \), there exists a positive constant \( C \), depending on \( n, \varphi, \psi_1 \) and \( \alpha_2 \), such that, for all \( f \in L^2(\mathbb{R}^n) \),
\[
C^{-1} \left\| f^*_L \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \phi L, \alpha(f) \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| f^*_L \right\|_{L^p(\mathbb{R}^n)}.
\]

Proof. We first show that, for any \( \psi \in \mathcal{S}(\mathbb{R}) \) and \( p \in (q(\varphi), \infty) \), there exists a positive constant \( C \), depending on \( n, \varphi, \psi \) and \( p \), such that, for all \( 0 < \alpha_2 \leq \alpha_1 \) and \( f \in L^2(\mathbb{R}^n) \),
\[
\left\| \psi^*_{L, \alpha_1}(f) \right\|_{L^p(\mathbb{R}^n)} \leq C \left[ \frac{\alpha_1}{\alpha_2} \right]^{\alpha_2} \left\| \psi^*_L, \alpha_2(f) \right\|_{L^p(\mathbb{R}^n)}. \tag{2.3}
\]

For any \( \lambda \in (0, \infty) \), let
\[
E_\lambda := \{ x \in \mathbb{R}^n : \psi^*_L, \alpha_2(f)(x) > \lambda \}
\]
and
\[
E^*_\lambda := \left\{ x \in \mathbb{R}^n : \mathcal{M}(\chi_{E^*_\lambda})(x) > \tilde{C}/(\alpha_1/\alpha_2)^n \right\},
\]
where \(\mathcal{M}\) denotes the Hardy-Littlewood maximal operator on \(\mathbb{R}^n\) and \(\tilde{C} \in (0, 1)\) is a positive constant. By \(\varphi \in A_{\infty}(\mathbb{R}^n)\) and the definition of \(q(\varphi)\), we know that, for any \(p \in (q(\varphi), \infty)\), \(\varphi \in A_p(\mathbb{R}^n)\), which, combined with Lemma 2.2(iii), implies that, for all \(t \in (0, \infty)\),
\[
\int_{E^*_\lambda} \varphi(x, t) \, dx \lesssim \frac{(\alpha_1/\alpha_2)^n}{C_p} \int_{E^*_\lambda} \varphi(x, t) \, dx,
\]
where the implicit positive constant depends on \(n, p\) and \(\varphi\).

Furthermore, we claim that \(\psi_{L, \alpha_1}^*(f)(x) \leq \lambda\) for all \(x \notin E^*_\lambda\). Indeed, let \(x \in [\mathbb{R}^n \setminus E^*_\lambda]\) and fix any given \((y, t) \in \mathbb{R}^{n+1} := \mathbb{R}^n \times (0, \infty)\) satisfying \(|y - x| < \alpha_1 t\). Then we claim that \(B(y, \alpha_2 t) \notin E^*_\lambda\). If this is not true, then
\[
\mathcal{M}(\chi_{E^*_\lambda})(x) \geq \frac{|B(y, \alpha_2 t)|}{|B(y, \alpha_1 t)|} = \left( \frac{\alpha_2}{\alpha_1} \right)^n > \frac{\tilde{C}}{(\alpha_1/\alpha_2)^n},
\]
which gives a contradiction to the assumption that \(x \notin E^*_\lambda\), and hence the claim holds true. By this claim, we conclude that there exists \(z \in B(y, \alpha_2 t)\) such that \(\psi_{L, \alpha_2}^*(f)(z) \leq \lambda\), which implies that
\[
|\psi_{L}(f)(y)| \leq \psi_{L, \alpha_2}^*(f)(z) \leq \lambda,
\]
where \(\psi_{L}(f)(y) := \psi(t\sqrt{L})(f)(y)\). From this and the choice of \((y, t)\), we deduce that, for all \(x \notin E^*_\lambda\), \(\psi_{L, \alpha_1}^*(f)(x) \leq \lambda\), which, together with Lemma 2.2(ii), Fubini's theorem and (2.4), further implies that
\[
\int_{\mathbb{R}^n} \varphi \left( x, \frac{\psi_{L, \alpha_1}^*(f)(x)}{\lambda} \right) \, dx \sim \int_{\mathbb{R}^n} \int_0^{\psi_{L, \alpha_1}^*(f)(x)} \varphi(x, t) \frac{dt}{t} \, dx
\]
\[
\sim \int_0^{\infty} \int_{\{x \in \mathbb{R}^n : \psi_{L, \alpha_1}^*(f)(x) > t\}} \varphi(x, t) \frac{dx \, dt}{t}
\]
\[
\lesssim \int_0^{\infty} \int_{E^*_\lambda} \varphi(x, t) \frac{dx \, dt}{t} \lesssim \left( \frac{\alpha_1}{\alpha_2} \right)^{np} \int_0^{\infty} \int_{E^*_\lambda} \varphi(x, t) \frac{dt}{t} \, dx
\]
\[
\sim \left( \frac{\alpha_1}{\alpha_2} \right)^{np} \int_{\mathbb{R}^n} \varphi \left( x, \frac{\psi_{L, \alpha_2}^*(f)(x)}{\lambda} \right) \, dx,
\]
where the implicit positive constants depend on \(n, p, \psi\) and \(\varphi\). By this, we know that, for all \(\lambda \in (0, \infty)\),
\[
\int_{\mathbb{R}^n} \varphi \left( x, \frac{\psi_{L, \alpha_1}^*(f)(x)}{\lambda} \right) \, dx \lesssim \left( \frac{\alpha_1}{\alpha_2} \right)^{np} \int_{\mathbb{R}^n} \varphi \left( x, \frac{\psi_{L, \alpha_2}^*(f)(x)}{\lambda} \right) \, dx,
\]
which further implies that (2.3) holds true.

Let \(\psi := \psi_1 - \psi_2\). Via (2.3), we find that, to prove (2.2), it suffices to show that
\[
\|\psi_{L, 1}^*(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\psi_{L, 2, 1}^*(f)\|_{L^p(\mathbb{R}^n)},
\]
where the implicit positive constant depends on \(n, \psi_1, \psi_2\) and \(\varphi\). Now we prove (2.5). Let \(\Psi(x) := x^{2k} \Phi(x)\) for all \(x \in \mathbb{R}^n\), where \(\Phi\) is as in Lemma 2.1 and \(k \in \mathbb{N}\).
with \( k > nq(\varphi)/[2i(\varphi)] \). By the spectral calculus, we know that there exists a constant \( C(\psi, \psi_2) \), depending on \( \Psi \) and \( \psi_2 \), such that
\[
f = C(\psi, \psi_2) \int_0^\infty \Psi(s\sqrt{t}) \psi_2(s\sqrt{t})(f) \frac{ds}{s},
\]
which further implies that, for any \( t \in (0, \infty) \),
\[
\psi(t\sqrt{L})(f) = C(\psi, \psi_2) \int_0^\infty \left[ \psi(t\sqrt{L}) \Psi(s\sqrt{L}) \right] \psi_2(s\sqrt{L})(f) \frac{ds}{s}.
\]
Denote by \( K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})} \) the kernel of \( \psi(t\sqrt{L})\Psi(s\sqrt{L}) \). Then, for all \( \lambda \in (0, \infty) \) and \( x \in \mathbb{R}^n \), we have
\[
\sup_{|w|<t, t \in (0, \infty)} \left| \psi(t\sqrt{L})(f)(x-w) \right| \\
\sim \sup_{|w|<t, t \in (0, \infty)} \left| \int_0^\infty K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z) \psi_2(s\sqrt{L})(f)(z) \frac{dz \, ds}{s} \right| \\
\lesssim \sup_{|w|<t, t \in (0, \infty)} \int_{\mathbb{R}^n_+} \left| K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z) \right| \left[ 1 + \frac{|x-z|}{s} \right]^\lambda \frac{dz \, ds}{s} \\
\times \left| \psi_2(s\sqrt{L})(f)(z) \right| \left[ 1 + \frac{|x-z|}{s} \right]^{-\lambda} \\
\lesssim \sup_{z \in \mathbb{R}^n, s \in (0, \infty)} \left| \psi_2(s\sqrt{L})(f)(z) \right| \left[ 1 + \frac{|x-z|}{s} \right]^{-\lambda} \frac{dz \, ds}{s}, \quad (2.6)
\]
Moreover, from [28, (3.4)], it follows that, for any \( x \in \mathbb{R}^n \) and \( \lambda \in (0, 2k) \),
\[
\sup_{|w|<t, t \in (0, \infty)} \int_{\mathbb{R}^n_+} \left| K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z) \right| \left[ 1 + \frac{|x-z|}{s} \right]^{\lambda} \frac{dz \, ds}{s} \lesssim 1,
\]
where the implicit positive constant depends on \( n, \Psi, \psi \) and \( \lambda \), which, combined with (2.6), implies that, for all \( x \in \mathbb{R}^n \),
\[
\sup_{|w|<t, t \in (0, \infty)} \left| \psi(t\sqrt{L})f(x-w) \right| \\
\lesssim \sup_{z \in \mathbb{R}^n, s \in (0, \infty)} \left| \psi_2(s\sqrt{L})(f)(z) \right| \left[ 1 + \frac{|x-z|}{s} \right]^{-\lambda}, \quad (2.7)
\]
where the implicit positive constant depends on \( n, \Psi, \psi \) and \( \lambda \). Furthermore, let \( \chi \) be the characteristic function of \([0, 1]\). Then, for all \( \lambda, s \in (0, \infty) \), we have
\[
(1+s)^{-\lambda} \leq \sum_{k=1}^{\infty} 2^{-k} \chi \left( \frac{s}{2^{k/\lambda}} \right),
\]
which further implies that, for all \( x \in \mathbb{R}^n \),
\[
\sup_{z \in \mathbb{R}^n, s \in (0, \infty)} \left| \psi_2(s\sqrt{L})(f)(z) \right| \left[ 1 + \frac{|x-z|}{s} \right]^{-\lambda} \\
\leq \sum_{k=1}^{\infty} 2^{-k} \sup_{z \in \mathbb{R}^n, s \in (0, \infty)} \left| \psi_2(s\sqrt{L})(f)(z) \right| \chi \left( \frac{|x-z|}{s2^{k/\lambda}} \right).
\[
\begin{align*}
&= \sum_{k=1}^{\infty} 2^{-k} \sup_{|x-z| < 2^{k}/\lambda, s \in (0, \infty)} |\psi_2(s\sqrt{L})(f)(z)| \\
&= \sum_{k=1}^{\infty} 2^{-k} \psi_2^*(f)(x).
\end{align*}
\]

(2.8)

Let \( \lambda \in (nq(\varphi)/i(\varphi), 2k) \). Then, by \( \lambda > nq(\varphi)/i(\varphi) \) and the definitions of \( q(\varphi) \) and \( i(\varphi) \), we conclude that there exist \( p_0 \in (0, i(\varphi)) \) and \( q \in (q(\varphi), \infty) \) such that \( \lambda > nq/p_0 \), \( \varphi \) is of uniformly lower type \( p_0 \) and \( \varphi \in A_q(\mathbb{R}^n) \), which, together with (2.8), Lemma 2.2(i) and (2.3), further implies that, for all \( \mu \in (0, \infty) \),

\[
\begin{align*}
&\int_{\mathbb{R}^n} \varphi(x) \sup_{z \in \mathbb{R}^n, s \in (0, \infty)} |\psi_2(s\sqrt{L})(f)(z)| \left[ 1 + \frac{|x-z|}{s} \right]^{-\lambda} \mu \, dx \\
&\leq \int_{\mathbb{R}^n} \varphi(x) \sum_{k=1}^{\infty} 2^{-k} \psi_2^*(f)(x) \mu \\
&\lesssim \sum_{k=1}^{\infty} 2^{-kp_0} \int_{\mathbb{R}^n} \varphi(x, \psi_2^*(f)(x)) \mu \\
&\lesssim \sum_{k=1}^{\infty} 2^{-k[p_0-nq/\lambda]} \int_{\mathbb{R}^n} \varphi(x, \psi_2^*(f)(x)) \mu \\
&\sim \int_{\mathbb{R}^n} \varphi(x, \psi_2^*(f)(x)) \mu,
\end{align*}
\]

where the implicit positive constants depend on \( n, \psi \) and \( \varphi \). From this, it follows that

\[
\left\| \sup_{z \in \mathbb{R}^n, s \in (0, \infty)} |\psi_2(s\sqrt{L})(f)(z)| \left[ 1 + \frac{|x-z|}{s} \right]^{-\lambda} \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| \psi_2^*(f) \right\|_{L^p(\mathbb{R}^n)},
\]

which, combined with (2.7), further implies that

\[
\left\| \psi_2^*(f) \right\|_{L^p(\mathbb{R}^n)} = \left\| \sup_{|u|<t, t \in (0, \infty)} |\psi(t\sqrt{L})(f)(\cdot - w)| \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim \sup_{z \in \mathbb{R}^n, s \in (0, \infty)} |\psi_2(s\sqrt{L})(f)(z)| \left[ 1 + \frac{|x-z|}{s} \right]^{-\lambda} \left\| \psi_2^*(f) \right\|_{L^p(\mathbb{R}^n)} \\
\lesssim \left\| \psi_2^*(f) \right\|_{L^p(\mathbb{R}^n)},
\]

(2.9)

where the implicit positive constants depend on \( n, \psi, \Psi, \lambda \) and \( \varphi \). This finishes the proof of (2.5) and hence Proposition 2.3. \( \square \)

Furthermore, to show Theorem 1.8, we also need the following atomic characteriza-
tion of the Musielak-Orlicz-Hardy space \( H_{\varphi, L}(\mathbb{R}^n) \) obtained in [4, Theorem 5.4].

**Proposition 2.4.** Let the densely defined linear operator \( L \) on \( L^2(\mathbb{R}^n) \) satisfy Assumptions 1.1 and 1.2, and \( \varphi \) be as in Definition 1.4. Assume that \( M \in \mathbb{N} \cap (nq(\varphi), \infty) \) and \( q \in ([r(\varphi)](0, \infty)) \), where \( q(\varphi) \), \( i(\varphi) \), \( r(\varphi) \) and \( I(\varphi) \) are, respectively, as in (1.10), (1.9), (1.11) and (1.8). Then the spaces \( H_{\varphi, L}(\mathbb{R}^n) \) and \( H_{\varphi, L, at}(\mathbb{R}^n) \) coincide with equivalent quasi-norms.
Now we give out the proof of Theorem 1.8 via Propositions 2.3 and 2.4.

Proof of Theorem 1.8. We first show (i) of Theorem 1.8. To this end, we begin with proving that

\[ H^{\phi, \alpha}_{\varphi, \lambda, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset H^{M, \infty}_{\varphi, \lambda, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \]  \hspace{1cm} (2.10)

To prove (2.10), via Proposition 2.3, it suffices to show that, for any \( f \in [H^{\phi, \alpha}_{\varphi, \lambda, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)] \), \( f \in H^{M, \infty}_{\varphi, \lambda, \max}(\mathbb{R}^n) \) and

\[ \|f\|_{H^{M, \infty}_{\varphi, \lambda, \max}(\mathbb{R}^n)} \lesssim \|f\|_{H^{\phi, \alpha}_{\varphi, \lambda, \max}(\mathbb{R}^n)}, \]  \hspace{1cm} (2.11)

where the implicit positive constant depends on \( n, M \) and \( \varphi \). Let \( \Psi(x) := x^2 M \Phi(x) \) for all \( x \in \mathbb{R} \), where \( \Phi \) is as in Lemma 2.1. Then, by the spectral calculus, we know that there exists a constant \( C(\Phi) \) such that

\[ f = C(\Phi) \int_0^\infty \Psi(t\sqrt{L}) t^2 L e^{-t^2 L}(f) \frac{dt}{t} \]

holds true in \( L^2(\mathbb{R}^n) \). For \( x \in \mathbb{R} \), let

\[ \eta(x) := \begin{cases} C(\Phi) \int_1^\infty t^2 x^2 \Psi(tx) e^{-t^2 x^2} \frac{dt}{t}, & x \neq 0, \\ 1, & x = 0. \end{cases} \]

Then \( \eta \in S(\mathbb{R}) \) is an even function and, for any \( a, b \in \mathbb{R} \),

\[ \eta(ax) - \eta(bx) = C(\Phi) \int_a^b t^2 x^2 \Psi(tx) e^{-t^2 x^2} \frac{dt}{t}, \]

which further implies that

\[ C(\Phi) \int_a^b \Psi(t\sqrt{L}) t^2 L e^{-t^2 L}(f) \frac{dt}{t} = \eta(a \sqrt{L})(f) - \eta(b \sqrt{L})(f). \]

For any \( f \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), let

\[ \mathcal{M}_L(f)(x) := \sup_{|x-y| < 5 \sqrt{nt}, t \in (0, \infty)} \left[ \left| t^2 L e^{-t^2 L}(f)(y) \right| + |\eta(t\sqrt{L})(f)(y)| \right]. \]

Then, from Proposition 2.3, it follows that

\[ \|\mathcal{M}_L(f)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{H^{\phi, \alpha}_{\varphi, \lambda, \max}(\mathbb{R}^n)}, \]  \hspace{1cm} (2.12)

where the implicit positive constant depends on \( n, M, \varphi \) and \( \Phi \) as in Lemma 2.1. Let

\[ \hat{O} := \{(x, t) \in \mathbb{R}^{n+1} : B(x, 4 \sqrt{nt}) \subset O\} \]

and, for any \( i \in \mathbb{Z} \),

\[ O_i := \{x \in \mathbb{R}^n : \mathcal{M}_L(f)(x) > 2^i\}. \]

For each \( i \in \mathbb{Z} \), denote by \( \{Q_{i,j}\}_{j \in \mathbb{N}} \) the Whitney decomposition of \( O_i \). For each \( i \in \mathbb{Z} \) and \( j \in \mathbb{N} \), let

\[ \tilde{Q}_{i,j} := \{(y, t) \in \mathbb{R}^{n+1} : y + 3t \varpi \in Q_{i,j}\}, \]

here and hereafter, \( \varpi := (1, \ldots, 1) \in \mathbb{R}^n \). It is easy to prove that, for all \( i \in \mathbb{Z} \),

\[ \hat{O}_i \subset \bigcup_j \tilde{Q}_{i,j}. \]

Indeed, for any \( (y^o, t^o) \in \hat{O}_i, B(y^o, 4 \sqrt{nt^o}) \subset O_i \). Let \( \tilde{y}^o := y^o + 3t^o \). Then

\[ \tilde{y}^o \in B(y^o, 4 \sqrt{nt^o}) \subset O_i, \]
which implies that there exists $Q_{i,j_0} \subset O_i$ such that $y^0 \in Q_{i,j_0}$. By this, we conclude that $(y^0, t^0) \in \widehat{Q}_{i,j_0}$ and hence $O_i \subset \bigcup_j \widehat{Q}_{i,j}$.

Furthermore, notice that, for any $j_1 \neq j_2$, $\widehat{Q}_{i,j_1} \cap \widehat{Q}_{i,j_2} = \emptyset$ and

$$\mathbb{R}^{n+1} = \bigcup_i \widehat{O}_i = \bigcup_i (\widehat{O}_i \setminus \widehat{O}_{i+1}) = \bigcup_i \bigcup_j T_{i,j},$$

where $T_{i,j} := (\widehat{Q}_{i,j} \cap (\widehat{O}_i \setminus \widehat{O}_{i+1}))$. Thus, it holds true that

$$f = \sum_{i,j} C(\psi) \int_0^\infty \Phi(t\sqrt{L}) \left( \chi_{T_{i,j}} t^2 L e^{-t^2 L} f \right) \frac{dt}{t} =: \sum_{i,j} \lambda_{i,j} \alpha_{i,j}, \quad (2.13)$$

where $\lambda_{i,j} := 2^j \| \chi_{Q_{i,j}} \|_{L^\infty(\mathbb{R}^n)}$ and $\alpha_{i,j} := L^M b_{i,j}$ with

$$b_{i,j} := \frac{C(\psi)}{\lambda_{i,j}} \int_0^\infty t^{2M} \Phi(t\sqrt{L}) \left( \chi_{T_{i,j}} t^2 L e^{-t^2 L} f \right) \frac{dt}{t}.$$ 

Now we prove that the summation (2.13) converges in $L^2(\mathbb{R}^n)$. Indeed, it is well known that, for any $f \in L^2(\mathbb{R}^n)$,

$$\left\{ \int_{\mathbb{R}^{n+1}} \left| t^2 L e^{-t^2 L} f (y) \right|^2 \frac{dy}{t} \right\}^{1/2} \lesssim \| f \|_{L^2(\mathbb{R}^n)}$$

(see, for example, [15, (3.14)]), which, together with (2.13), implies that

$$\left\| \sum_{|i| > N_1, |j| > N_2} \lambda_{i,j} \alpha_{i,j} \right\|_{L^2(\mathbb{R}^n)} \sim \left\| \sum_{|i| > N_1, |j| > N_2} \int_{\mathbb{R}^{n+1}} K(t^2 L)^M \Phi(t\sqrt{L}) \chi_{T_{i,j}} (y,t) t^2 L e^{-t^2 L} f (y) \frac{dy}{t} \right\|_{L^2(\mathbb{R}^n)} \lesssim \sup_{\| g \|_{L^2(\mathbb{R}^n)} \leq 1} \sum_{|i| > N_1, |j| > N_2} \int_{T_{i,j}} \left| \left| t^2 L e^{-t^2 L} f (y) \right| \frac{dy}{t} \right|^{1/2} \rightarrow 0,$$

as $N_1 \rightarrow \infty$ and $N_2 \rightarrow \infty$. Thus, the summation (2.13) converges in $L^2(\mathbb{R}^n)$.

Now we claim that there exists a positive constant $\overline{C}$, depending on $n$, $M$, $\Phi$ and $\varphi$, such that, for all $i$ and $j$, $\overline{C}^{-1} \alpha_{i,j}$ is a $(\varphi, \infty, M)_{L^\infty}$-atom associated with the ball $30B_{i,j}$, where $B_{i,j}$ denotes the ball with the center being the same as $Q_{i,j}$ and the radius $r_{B_{i,j}} := \sqrt{n} \ell(Q_{i,j})/2$. Here and hereafter, $\ell(Q_{i,j})$ denotes the side length of $Q_{i,j}$. Once this claim is proved, by Lemma 2.2(iv), we then know that, for all $\lambda \in (0, \infty)$,

$$\sum_{i,j} \varphi \left( 30B_{i,j}, \lambda \| \chi_{30B_{i,j}} \|_{L^\infty(\mathbb{R}^n)} \right) \lesssim \sum_{i,j} \varphi \left( Q_{i,j}, \frac{2^i}{\lambda} \right) \sim \sum_{i} \varphi \left( O_i, \frac{2^i}{\lambda} \right), \quad (2.14)$$
Moreover, similar to the proof of [22, Lemma 5.4] (see also [34, Lemma 3.4]), we conclude that
\[
\sum_{i} \varphi \left( O_i, \frac{2^i}{\lambda} \right) \lesssim \int_{\mathbb{R}^n} \varphi \left( x, \frac{M_L(f)(x)}{\lambda} \right) \, dx,
\]
which, combined with (2.12) and (2.14), further implies that (2.11) holds true.

Now we prove the above claim. We first show that, for any \( k \in \{0, 1, \ldots, M\}, \)
\[
\supp (L^k b_{i,j}) \subset 30Q_{i,j} \subset 30B_{i,j}.
\]
(2.15)
From the definition of \( T_{i,j} \), it follows that, if \( (y, t) \in T_{i,j} \), then \( B(y, 4\sqrt{t} \ell) \subset O_i \). Let \( \tilde{y} := y + 3\ell \).
Then \( \tilde{y} \in Q_{i,j} \) and \( B(\tilde{y}, \sqrt{\ell} \ell) \subset O_i \). Moreover, by the fact that \( Q_{i,j} \) is the Whitney cube of \( O_i \), we know that \( 5Q_{i,j} \cap O_i^c \neq \emptyset \) and hence \( \ell \leq 3\ell(Q_{i,j}) \), which, together with \( y + 3\ell \in Q_{i,j} \), further implies that \( y \in 20Q_{i,j} \). Furthermore, from Lemma 2.1, we deduce that
\[
\supp \left( K_{(t^2L^k)^\Phi(t\sqrt{\ell})} \right) \subset \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\},
\]
which, combined with \( y \in 20Q_{i,j} \), implies that (2.15) holds true. To finish the proof of the above claim, it remains to prove that, for any \( k \in \{0, 1, \ldots, M\}, \)
\[
\|[(30r_{Q_{i,j}})^2L]^k b_{i,j}\|_{L^\infty(\mathbb{R}^n)} \leq \tilde{C}[(30r_{Q_{i,j}})^{2M}\|x_{\mathbb{R}^n}\|_{L^\infty(\mathbb{R}^n)}]^{-1}.
\]
(2.16)
By [28, (3.12)], we find that, for any \( k \in \{0, 1, \ldots, M - 1\} \) and \( x \in \mathbb{R}^n \),
\[
\left| \int_{\mathbb{R}^n} K_{t^{2M}L^k\Phi(t\sqrt{\ell})} (x, y) \chi_{T_{i,j}} (y, t) t^2 L e^{-t^2 L} (f)(y) \, dy dt \right| \lesssim 2^k [t(Q_{i,j})]^{2(M - k)},
\]
where the implicit positive constant depends on \( n \), \( M \) and \( \Phi \), which further implies that
\[
\|[(30r_{Q_{i,j}})^2L]^k b_{i,j}\|_{L^\infty(\mathbb{R}^n)} = \lambda_{i,j}^{-1} (30r_{Q_{i,j}})^{2k} C(\Phi) \left| \int_{\mathbb{R}^n} K_{t^{2M}L^k\Phi(t\sqrt{\ell})} (\cdot, y) \chi_{T_{i,j}} (y, t) t^2 L e^{-t^2 L} (f)(y) \, dy dt \right| \lesssim 2^k [t(Q_{i,j})]^{2(M - k)},
\]
(2.17)
Furthermore, it follows, from [28, (3.13)], that, for any \( x \in \mathbb{R}^n \),
\[
\left| \int_{0}^{\infty} \int_{\mathbb{R}^n} K_{\Phi(t\sqrt{\ell})} (x, y) \chi_{T_{i,j}} (y, t) t^2 L e^{-t^2 L} (f)(y) \, dy dt \right| \lesssim 2^i,
\]
where the implicit positive constant depends on \( n \) and \( \Phi \), which implies that
\[
\|[(30r_{Q_{i,j}})^2L]^M b_{i,j}\|_{L^\infty(\mathbb{R}^n)} = \lambda_{i,j}^{-1} (30r_{Q_{i,j}})^{2M} C(\Phi) \left| \int_{\mathbb{R}^n} K_{\Phi(t\sqrt{\ell})} (\cdot, y) \chi_{T_{i,j}} (y, t) t^2 L e^{-t^2 L} (f)(y) \, dy dt \right| \lesssim 2^i \leq \tilde{C}[(30r_{Q_{i,j}})^{2M}\|x_{\mathbb{R}^n}\|_{L^\infty(\mathbb{R}^n)}]^{-1}.
\]
By this and (2.17), we conclude that (2.16) holds true, which completes the proof of the above claim and hence (2.10).
Now we prove that, for any $M \in \mathbb{N} \cap (nq(\varphi)/2i(\varphi), \infty)$ and $q \in ([r(\varphi)]'I(\varphi), \infty]$,
\[
\left[H^M,q_{\varphi,L,at}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] \subset \left[H^A_{\varphi,L,\max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right].
\]  
(2.18)

For any $\phi \in \mathcal{A}$ and $x \in \mathbb{R}$, let
\[
\tilde{\psi}(x) := [\phi(0)]^{-1}\phi(x) - e^{-x^2}.
\]
Repeating the proof of [28, (3.4)], we know that, for any $\lambda \in (0, 2M)$, there exists a positive constant $C$, depending on $n$, $\Psi$ and $\lambda$, such that, for all $\phi \in \mathcal{A}$ and $x \in \mathbb{R}^n$,
\[
\sup_{|\varphi| < t, r \in (0, \infty)} \int_{\mathbb{R}^{n+1}} K_{\tilde{\varphi}(t, \sqrt{T})} \tilde{\varphi}(x - w, z) \left[1 + \frac{|x - z|}{s}\right]^{\lambda} \, dz \, ds \leq C,
\]
where $\Psi$ is as in (2.6). Via this estimate and repeating the proof of (2.9), we find that
\[
\left\|\sup_{\phi \in \mathcal{A}} \tilde{\psi}_{L,1}^*(f)\right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)},
\]
where the implicit positive constant depends on $n$, $\Psi$ and $\varphi$, which, combined with the fact that $\mathcal{G}_L^*(f) \lesssim \sup_{\phi \in \mathcal{A}} \tilde{\psi}_{L,1}^*(f) + f_L^*$ and Lemma 2.2(i), further implies that
\[
\|\mathcal{G}_L^*(f)\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)}.  
\]
(2.19)

Via (2.19), we find that, to finish the proof of (2.18), it suffices to show that, for any $\lambda \in \mathbb{C}$ and $(\varphi, q, M)_{L}\text{-atom } \alpha$ associated with the ball $B := B(x_B, r_B)$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$,
\[
\int_{\mathbb{R}^n} \varphi(x, \lambda|\alpha_L^*(x)|) \, dx \lesssim \varphi(B, |\lambda| \|\chi_B\|^{-1}_{L^\infty(\mathbb{R}^n)}),
\]
(2.20)

where the implicit positive constant depends only on $n$ and $\varphi$. Indeed, let $f \in \left[H^M,q_{\varphi,L,at}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right]$. Then there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of $(\varphi, q, M)_{L}\text{-atoms}$, associated with the balls $\{B_j\}_j$, such that
\[
f = \sum_j \lambda_j \alpha_j \text{ in } L^2(\mathbb{R}^n) \text{ and } \|f\|_{H^M,q_{\varphi,L,at}(\mathbb{R}^n)} \sim A(\{\lambda_j \alpha_j\}_j),
\]

which, together with (2.20), further implies that, for all $\lambda \in (0, \infty)$,
\[
\int_{\mathbb{R}^n} \varphi(x, \frac{f_L^*(x)}{\lambda}) \, dx \lesssim \sum_j \int_{\mathbb{R}^n} \varphi \left(x, \frac{|\lambda_j| (\alpha_j)_L^*(x)}{\lambda} \right) \, dx \lesssim \sum_j \varphi(B_j, \frac{|\lambda_j|}{\lambda \|\chi_{B_j}\|_{L^\infty(\mathbb{R}^n)}}).
\]

From this and (2.19), it follows that $f \in \left[H^A_{\varphi,L,\max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right]$ and
\[
\|f\|_{H^A_{\varphi,L,\max}(\mathbb{R}^n)} \lesssim \|f\|_{H^M,q_{\varphi,L,at}(\mathbb{R}^n)}.
\]

Now we prove (2.20). By (1.5), we conclude that, for all $x \in \mathbb{R}^n$,
\[
\alpha_L^*(x) \lesssim M(\alpha)(x),
\]
(2.21)

where $M$ denotes the Hardy-Littlewood maximal operator on $\mathbb{R}^n$. Moreover, from $q \in ([r(\varphi)]'I(\varphi), \infty]$, it follows that there exists $p_1 \in [I(\varphi), 1]$ such that $\varphi$ is of uniformly upper type $p_1$ and $\varphi \in \mathcal{R}_{\mathcal{H}(q/p_1)}(\mathbb{R}^n)$, which, combined with (2.21),
Hölder’s inequality, the boundedness of \( M \) on \( L^3(\mathbb{R}^n) \) and Lemma 2.2(iv), further implies that

\[
\begin{align*}
\int_{4B} \varphi(x, |\alpha_L(x)|) \, dx & \lesssim \int_{4B} \varphi(x, |M(\alpha)(x)|) \, dx \\
& \lesssim \int_{4B} \varphi \left( x, |\alpha||\chi_B|^{-1/3}_{L^\infty(\mathbb{R}^n)} \right) \left[ 1 + M(\alpha)(x)||\chi_B||_{L^\infty(\mathbb{R}^n)} \right]^{p_1} \, dx \\
& \lesssim \varphi \left( 4B, |\alpha||\chi_B|^{-1/3}_{L^\infty(\mathbb{R}^n)} \right) + ||\chi_B||_{L^\infty(\mathbb{R}^n)} ||M(\alpha)||_{L^\infty(4B)} \\
& \quad \times \left\{ \int_{4B} \varphi \left( x, |\alpha||\chi_B|^{-1/3}_{L^\infty(\mathbb{R}^n)} \right) \left[ \frac{r}{4B} \right] \, dx \right\}^{\frac{\beta}{\gamma}} \\
& \lesssim \varphi \left( B, |\alpha||\chi_B|^{-1/3}_{L^\infty(\mathbb{R}^n)} \right).
\end{align*}
\]

(2.22)

For any \( x \in \mathbb{R}^n \setminus (4B) \), let

\[
\begin{align*}
\alpha_1^*(x) & := \sup_{|x-y|<t, \, t \in (0, r_B)} \left| e^{-t^2 L}(\alpha)(y) \right|, \\
\alpha_2^*(x) & := \sup_{|x-y|<t, \, t \in (r_B, |x-x_B|/4]} \left| e^{-t^2 L}(\alpha)(y) \right| \\
\alpha_3^*(x) & := \sup_{|x-y|<t, \, t \in [|x-x_B|/4, \infty)} \left| e^{-t^2 L}(\alpha)(y) \right|.
\end{align*}
\]

and

For any \( t \in (0, |x-x_B|/4] \), \( z \in B \) and \( y \in \mathbb{R}^n \) satisfying \(|x-y|<t\), we find that

\[
|y-z| \geq |x-z| - |x-y| \geq |x-x_B| - r_B - t \geq \frac{|x-x_B|}{2}, \tag{2.23}
\]

which, together with (1.5), implies that, for any \( s \in (0, \infty) \),

\[
\begin{align*}
\alpha_1^*(x) & \lesssim \sup_{|x-y|<t, \, t \in (0, r_B]} \int_B \left( \frac{t}{t+t+|z-y|} \right)^{n+s} |\alpha(z)| \, dz \\
& \lesssim \frac{r_B^{n+s}}{|x-x_B|^{n+s}} ||\alpha||_{L^1(\mathbb{R}^n)} \lesssim \frac{r_B^{n+s}}{|x-x_B|^{n+s}} ||\chi_B||_{L^\infty(\mathbb{R}^n)}^{-1}. \tag{2.24}
\end{align*}
\]

Moreover, from \( \alpha = L^M b \), (2.23) and (1.5), it follows that, for any \( s \in (0, 2M) \),

\[
\begin{align*}
\alpha_2^*(x) & \lesssim \sup_{|x-y|<t, \, t \in (r_B, |x-x_B|/4]} \left| (t^2 L)^M e^{-t^2 L}(b)(y) \right| \\
& \lesssim \sup_{|x-y|<t, \, t \in (r_B, |x-x_B|/4]} \left| t^{-2M} \int_B \left( \frac{t}{t+|z-y|} \right)^{n+s} |b(z)| \, dz \right| \\
& \lesssim \sup_{t \in (r_B, |x-x_B|/4]} \left| t^{s-2M} |x-x_B|^{-n-s} ||b||_{L^1(\mathbb{R}^n)} \right| \\
& \lesssim \frac{r_B^{n+s}}{|x-x_B|^{n+s}} ||\chi_B||_{L^\infty(\mathbb{R}^n)}^{-1}. \tag{2.25}
\end{align*}
\]

Furthermore, by \( \alpha = L^M b \) and (1.5), we conclude that, for any \( s \in (0, 2M) \),

\[
\alpha_3^*(x) \lesssim \sup_{|x-y|<t, \, t \in (|x-x_B|/4, \infty)} \left| t^{-2M} \int_B \left( \frac{t}{t+|z-y|} \right)^{n+s} |b(z)| \, dz \right|.
\]
Let $s > nq(s)/i(s)$, we deduce that there exist $p_0 \in (0, i(s))$ and $q \in (q(s), \infty)$ such that $s > nq(p_0)$ and $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$, which, together with (2.26) and Lemma 2.2(iv), implies that

$$\int_{\mathbb{R}^n \setminus (4B)} \varphi(x, |\lambda|a_L^s(x)) \, dx = \sum_{j=2}^{\infty} \int_{S_j(B)} \varphi(x, |\lambda|a_L^s(x)) \, dx$$

$$\lesssim \sum_{j=2}^{\infty} 2^{-j(n+s)p_0} \varphi(S_j(B), |\lambda|\|\chi_B\|_{L^{q'}(\mathbb{R}^n)}^{-1})$$

$$\lesssim \sum_{j=2}^{\infty} 2^{-j[(n+s)p_0-nq]} \varphi(B, |\lambda|\|\chi_B\|_{L^{q'}(\mathbb{R}^n)}^{-1})$$

$$\lesssim \varphi(B, |\lambda|\|\chi_B\|_{L^{q'}(\mathbb{R}^n)}^{-1}).$$

By this and (2.22), we conclude that (2.20) holds true, which completes the proof of (2.18).

By the definitions of the spaces $H^s_{\varphi, L, \max}(\mathbb{R}^n)$ and $H^{A}_{\varphi, L, \max}(\mathbb{R}^n)$, we know that

$$[H^s_{\varphi, L, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)] \subset [H^{s}_{\varphi, L, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)].$$

Moreover, from the definition of the space $H^{M, q}_{\varphi, L, \at}(\mathbb{R}^n)$, with $q \in ([r(\varphi)])'I(\varphi), \infty)$, and Proposition 2.4, it follows that, for any $q \in ([r(\varphi)])'I(\varphi), \infty)$,

$$[H^{M, \infty}_{\varphi, L, \at}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)] \subset [H^{M, q}_{\varphi, L, \at}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)],$$

which, combined with (2.10), (2.18) and (2.27), further implies that, for any $q \in ([r(\varphi)])'I(\varphi), \infty)$,

$$[H^{M, q}_{\varphi, L, \at}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = [H^s_{\varphi, L, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)] = [H^{A}_{\varphi, L, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)]$$

with equivalent quasi-norms, which, together with the fact that $H^{M, q}_{\varphi, L, \at}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $H^{s}_{\varphi, L, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $H^{A}_{\varphi, L, \max}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ are, respectively, dense in the spaces $H^{M, q}_{\varphi, L, \at}(\mathbb{R}^n)$, $H^{s}_{\varphi, L, \max}(\mathbb{R}^n)$ and $H^{A}_{\varphi, L, \max}(\mathbb{R}^n)$, and a density argument, implies that the spaces $H^{M, q}_{\varphi, L, \at}(\mathbb{R}^n)$, $H^{s}_{\varphi, L, \max}(\mathbb{R}^n)$ and $H^{A}_{\varphi, L, \max}(\mathbb{R}^n)$ coincide with equivalent quasi-norms. This finishes the proof of Theorem 1.8(i).

Furthermore, (ii) of Theorem 1.8 is deduced from (i) and Proposition 2.4, which completes the proof of Theorem 1.8. \qed
3. Proof of Theorem 1.12. In this section, we show Theorem 1.12. We first introduce some notation.

Let \( f \in L^2(\mathbb{R}^n) \). For all \( t \in (0, \infty) \) and \( x \in \mathbb{R}^n \), let
\[
 u(x, t) := e^{-tL}(f)(x).
\] (3.1)

For all \( \varepsilon \in (0, \infty) \), \( N \in \mathbb{N} \) and \( x \in \mathbb{R}^n \), define
\[
 u_{\varepsilon, N}(x) := \sup_{|y-x|<\sqrt{t}<\varepsilon^{-1}, t \in (0, \infty)} |u(y, t)| \left( \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right)^N (1 + \varepsilon|y|)^{-N} \] (3.2)
and
\[
 U_{\varepsilon, N}(x) := \sup_{|y_1-x|<\sqrt{t}<\varepsilon^{-1}} \left[ \frac{\sqrt{t}}{|y_1-y_2|} \right]^{\mu} |u(y_1, t) - u(y_2, t)| \\
\times \left[ \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right]^N (1 + \varepsilon|y_1|)^{-N}, \] (3.3)
where \( \mu \) is as in (1.17) and \( y, y_1, y_2 \in \mathbb{R}^n \).

Lemma 3.1. Let \( L \) be a densely defined linear operator on \( L^2(\mathbb{R}^n) \) satisfying Assumptions 1.2 and 1.11, and \( \varphi \) as in Definition 1.4. Then there exists a positive constant \( C \), depending on \( n \) and \( \varphi \), such that, for all \( u \) as in (3.1), \( \varepsilon \in (0, \infty) \) and \( N \in \mathbb{N} \),
\[
 \int_{\mathbb{R}^n} \varphi(x, U_{\varepsilon, N}(x)) \, dx \leq C \int_{\mathbb{R}^n} \varphi(x, u_{\varepsilon, N}(x)) \, dx,
\] (3.4)
where \( u_{\varepsilon, N} \) and \( U_{\varepsilon, N} \) are, respectively, as in (3.2) and (3.3).

Proof. For any \( \alpha \in (0, \infty) \), measurable function \( v : \mathbb{R}^{n+1}_+ \to \mathbb{C} \) and \( x \in \mathbb{R}^n \), let
\[
 v_\alpha(x) := \sup_{|y-x|<\alpha \sqrt{t}, t \in (0, \infty)} |v(y, t)|.
\]
Assume that \( u \) is as in (3.1). Fix \( x \in \mathbb{R}^n \). For any \( y_1, y_2 \in \mathbb{R}^n \) and \( t \in (0, \infty) \) satisfying \( |y_1 - x| < \sqrt{t} \) and \( |y_2 - x| < \sqrt{t} \), let
\[
 v(y_1, t) := u(y_1, t) \left[ \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right]^N (1 + \varepsilon|y_1|)^{-N} \chi(zt),
\]
where \( \chi \) denotes the characteristic function of \([0, 1)\). Then \( v_1^* = u_{\varepsilon, N} \). By the semigroup property of \( \{e^{-tL}\}_{t>0} \), we know that
\[
 |u(y_1, t) - u(y_2, t)| = \left| \int_{\mathbb{R}^n} [K_{t/2}(y_1, z) - K_{t/2}(y_2, z)] u(z, t/2) \, dz \right|
\leq I_0 + \sum_{k=1}^{\infty} I_k, \] (3.5)
where
\[
 I_0 := \int_{B(y_1, \sqrt{t})} |K_{t/2}(y_1, z) - K_{t/2}(y_2, z)| |u(z, t/2)| \, dz
\]
and, for each \( k \in \mathbb{N} \),
\[
 I_k := \int_{B(y_1, 2^k \sqrt{t}) \setminus B(y_1, 2^{k-1} \sqrt{t})} |K_{t/2}(y_1, z) - K_{t/2}(y_2, z)| |u(z, t/2)| \, dz.
\]
Furthermore, from (1.5), (1.17) and the semigroup property of \(\{e^{-tL}\}_{t>0}\), it follows that
\[
\int_{B(y_1,2^k\sqrt{t})\setminus B(y_1,2^k-1\sqrt{t})} \left| K_t(y_1,z) - K_t(y_2,z) \right| \, dz \lesssim \left[ \frac{|y_1-y_2|}{\sqrt{t}} \right]^\mu e^{-\beta 2^{2k}},
\]
where \(\mu\) is as in (1.17) and \(\beta\) is a positive constant determined by \(c\) in (1.5), which, combined with (3.5) and the fact that \((1+\varepsilon|z|)^N \leq (1+\varepsilon|y_1|)^N(1+2^k)^N\) if \(|y_1-z| < 2^k \sqrt{t}\) and \(\varepsilon\sqrt{t} < 1\), further implies that
\[
\left[ \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right]^N \frac{|u(y_1,t) - u(y_2,t)|}{(1+\varepsilon|y_1|)^N} \lesssim \left[ \frac{|y_1-y_2|}{\sqrt{t}} \right]^\mu \left[ v_1^*(x) + \sum_{k=1}^{\infty} e^{-\beta 2^{2k}} (1+2^k)^N v_{2k+2}^*(x) \right].
\]

By this, we conclude that
\[
U_{\varepsilon,N}^*(x) \lesssim v_4^*(x) + \sum_{k=1}^{\infty} e^{-\beta 2^{2k}} (1+2^k)^N v_{2k+2}^*(x). \tag{3.6}
\]

Moreover, repeating the proof of (2.3), we know that there exists \(p_1 \in (q(\varphi),\infty)\) such that, for any \(\alpha \in (1,\infty)\),
\[
\int_{\mathbb{R}^n} \varphi(x,v_{\alpha}^*(x)) \, dx \lesssim \alpha^{np_1} \int_{\mathbb{R}^n} \varphi(x,v_4^*(x)) \, dx,
\]
where the implicit positive constant depends on \(n, p_1\) and \(\varphi\), which, together with (3.6), Lemma 2.2(i) and \(v_1^* = u_{\varepsilon,N}^*\), further implies that (3.4) holds true. This finishes the proof of Lemma 3.1. \(\square\)

Now we prove Theorem 1.12 by using Lemma 3.1.

**Proof of Theorem 1.12.** By the definitions of \(H_{\varphi,L,\text{max}}(\mathbb{R}^n)\) and \(H_{\varphi,L,\text{rad}}(\mathbb{R}^n)\) and the fact that \(H_{\varphi,L,\text{max}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\) and \(H_{\varphi,L,\text{rad}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\) are, respectively, dense in \(H_{\varphi,L,\text{max}}(\mathbb{R}^n)\) and \(H_{\varphi,L,\text{rad}}(\mathbb{R}^n)\), we know that, to show Theorem 1.12, it suffices to show that
\[
[H_{\varphi,L,\text{rad}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)] \subset [H_{\varphi,L,\text{max}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)]. \tag{3.7}
\]

Let \(f \in [H_{\varphi,L,\text{rad}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)]\) and \(u\) be as in (3.1). By (1.5), we conclude that \(f_{L}^+ \lesssim \mathcal{M}(f)\), which, combined with the fact that, for all \(\varepsilon \in (0,\infty)\) and \(N \in \mathbb{N}\), \(u_{\varepsilon,N}^+ \lesssim f_{L}^+\), implies that \(u_{\varepsilon,N}^* \lesssim \mathcal{M}(f)\). From this and the boundedness of \(\mathcal{M}\) on \(L^2(\mathbb{R}^n)\), we deduce that, for all \(\varepsilon \in (0,\infty)\) and \(N \in \mathbb{N}\), \(u_{\varepsilon,N}^* \in L^2(\mathbb{R}^n)\). Define
\[
G_{\varepsilon,N} := \{ x \in \mathbb{R}^n : U_{\varepsilon,N}^*(x) \leq Eu_{\varepsilon,N}(x) \},
\]
where \(E\) is a positive constant determined later. By (3.4), we know that
\[
\int_{\mathbb{R}^n} \varphi(x,U_{\varepsilon,N}^*(x)) \, dx \leq C \int_{\mathbb{R}^n} \varphi(x,u_{\varepsilon,N}^*(x)) \, dx,
\]
where \(C\) is as in (3.4). Let \(p_0 \in (0,i(\varphi))\) be a uniformly lower type of \(\varphi\). Take \(E \in (1,\infty)\) large enough such that
\[
\frac{C}{EP_0} \int_{\mathbb{R}^n} \varphi(x,u_{\varepsilon,N}^*(x)) \, dx \leq \frac{1}{2} \int_{\mathbb{R}^n} \varphi(x,u_{\varepsilon,N}^*(x)) \, dx,
\]
which, together with the definition of $G_{\varepsilon,N}$ and the uniformly lower type $p_0$ property of $\varphi$ and the increasing property of $\varphi$ about the variable $t$, implies that
\[
\int_{\mathbb{R}^n \setminus G_{\varepsilon,N}} \varphi \left( x, u^*_\varepsilon,N(x) \right) \, dx \leq \int_{\mathbb{R}^n \setminus G_{\varepsilon,N}} \frac{U^*_\varepsilon,N(x)}{E} \, dx
\]
\[
\leq \frac{C}{E^{p_0}} \int_{\mathbb{R}^n} \varphi \left( x, u^*_\varepsilon,N(x) \right) \, dx
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^n} \varphi \left( x, u^*_\varepsilon,N(x) \right) \, dx. \tag{3.8}
\]

From (3.8), it follows that
\[
\int_{\mathbb{R}^n} \varphi \left( x, u^*_\varepsilon,N(x) \right) \, dx \leq 2 \int_{G_{\varepsilon,N}} \varphi \left( x, u^*_\varepsilon,N(x) \right) \, dx. \tag{3.9}
\]

For all $x \in \mathbb{R}^n$, let $M_r(f^+_L)(x) := \{M([f^+_L]^r)(x)\}^{1/r}$ with $r \in (0,1)$. Then, for almost every $x \in G_{\varepsilon,N}$, we have
\[
u^*_\varepsilon,N(x) \lesssim M_r(f^+_L)(x), \tag{3.10}
\]
where the implicit positive constant depends on $n$, $\mu$ and $E$. Indeed, let $x \in G_{\varepsilon,N}$ such that $\nu^*_\varepsilon,N(x) < \infty$. By the definition of $\nu^*_\varepsilon,N$, we know that there exist $y \in \mathbb{R}^n$ and $t \in (0,\infty)$ such that $|y - x| < \sqrt{t} < \varepsilon^{-1}$ and
\[
|u(y,t)| \left( \frac{\sqrt{t}}{\sqrt{t + \varepsilon}} \right)^N (1 + \varepsilon |y|)^{-N} \geq \frac{1}{2} \nu^*_\varepsilon,N(x). \tag{3.11}
\]

Since $x \in G_{\varepsilon,N}$, if $|z_1 - x| < \sqrt{t} < \varepsilon^{-1}$ and $|z_2 - x| < \sqrt{t} < \varepsilon^{-1}$, then, from the definition of $U^*_\varepsilon,N(x)$ and (3.11), it follows that
\[
\left( \frac{\sqrt{t}}{|z_1 - z_2|} \right)^\mu |u(z_1, t) - u(z_2, t)| \left( \frac{\sqrt{t}}{\sqrt{t + \varepsilon}} \right)^N (1 + \varepsilon |z_1|)^{-N}
\]
\[
\leq 2E|u(y,t)| \left( \frac{\sqrt{t}}{\sqrt{t + \varepsilon}} \right)^N (1 + \varepsilon |y|)^{-N}. \tag{3.12}
\]

Let
\[
E_t := \left\{ w \in \mathbb{R}^n : |w - y| < \frac{\sqrt{t}}{2C_1} \right\},
\]
where $C_1 := (4E)^{1/\mu}/2$. Obviously, $C_1 \geq 1$. Taking $z_1 := y$ and $z_2 \in E_t$, by (3.12), we find that
\[
\left( \frac{\sqrt{t}}{|y - z_2|} \right)^\mu |u(y,t) - u(z_2, t)| \leq 2E|u(y,t)|.
\]

From this and the choice of $C_1$, it follows that $|u(z_2, t)| \geq \frac{1}{2}|u(y,t)|$. Thus, we have
\[
|u(z_2, t)| \geq \frac{1}{2} |u(y,t)| \geq \frac{1}{2} |u(y,t)| \left( \frac{\sqrt{t}}{\sqrt{t + \varepsilon}} \right)^N (1 + \varepsilon |y|)^{-N}
\]
\[
\geq \frac{1}{4} \nu^*_\varepsilon,N(x),
\]
which further implies that
\[
[M_r(f^+_L)(x)]^r \geq \frac{1}{|B(x, \sqrt{t})|} \int_{B(x, \sqrt{t})} [f^+_L(z)]^r \, dz.
\]
where the implicit positive constants depend on $M$.

**Definition of $M$**

Thus, (3.10) holds true.

Let $q \in (q(\varphi), \infty)$, $p_0 \in (0, i(\varphi))$ and $r_0 \in (0, 1)$ such that $r_0q < p_0$. Then $\varphi$ is of uniformly lower type $p_0$ and $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$. For any $\gamma \in (0, \infty)$ and $g \in L^q_{\text{loc}}(\mathbb{R}^n)$, let

$$g = g_{\mathcal{X}} \{ x \in \mathbb{R}^n : |g(x)| \leq \gamma \} + g_{\mathcal{X}} \{ x \in \mathbb{R}^n : |g(x)| > \gamma \} =: g_1 + g_2.$$

It is easy to see that

$$\{ x \in \mathbb{R}^n : M(g)(x) > 2\gamma \} \subset \{ x \in \mathbb{R}^n : M(g_2)(x) > \gamma \},$$

which, combined with Lemma 2.2(iii), implies that, for all $t \in (0, \infty)$,

$$\int_{\{ x \in \mathbb{R}^n : M(g_2)(x) > \gamma \}} \varphi(x, t) \, dx \leq \int_{\mathbb{R}^n} \varphi(x, t) \, dx \leq \frac{1}{\gamma^q} \int_{\mathbb{R}^n} [M(g_2)(x)]^q \varphi(x, t) \, dx \lesssim \frac{1}{\gamma^q} \int_{\mathbb{R}^n} g_2(x)^q \varphi(x, t) \, dx \lesssim \frac{1}{\gamma^q} \int_{\{ x \in \mathbb{R}^n : |g(x)| > \gamma \}} g(x)^q \varphi(x, t) \, dx,$$

and

$$\int_{\{ x \in \mathbb{R}^n : M_{r_0}(f^+_L)(x) > \gamma \}} \varphi(x, t) \, dx \leq \frac{1}{\gamma^{r_0q}} \int_{\{ x \in \mathbb{R}^n : [f^+_L(x)]^{r_0} > \gamma^{r_0q} \}} [f^+_L(x)]^{r_0q} \varphi(x, t) \, dx \lesssim \sigma_{f^+_L, t} \left( \frac{\gamma}{21/r_0} \right) + \frac{1}{\gamma^{r_0q}} \int_{21/r_0}^{\infty} r_0^{-q} s^{-q-1} \sigma_{f^+_L, t}(s) \, ds,$$

here and hereafter,

$$\sigma_{f^+_L, t}(\gamma) := \int_{\{ x \in \mathbb{R}^n : f^+_L(x) > \gamma \}} \varphi(x, t) \, dx.$$

Let

$$J^+_{f^+_L} := \int_{\mathbb{R}^n} \varphi(x, f^+_L(x)) \, dx.$$

Then, by (3.9), (3.10), (3.14), Lemma 2.2(ii), Fubini’s theorem and the uniformly lower type $p_0$ property of $\varphi$ and the fact that $\varphi$ is increasing for the variable $t$, we conclude that

$$\int_{\mathbb{R}^n} \varphi(x, u^*_z, N(x)) \, dx \lesssim \int_{G_{z, N}} \varphi(x, u^*_z, N(x)) \, dx \lesssim \int_{G_{z, N}} \varphi(x, M_{r_0}(f^+_L)(x)) \, dx \lesssim \int_{\mathbb{R}^n} \varphi(x, M_{r_0}(f^+_L)(x)) \, dx \sim \int_{\mathbb{R}^n} \int_0^{M_{r_0}(f^+_L)(x)} \frac{\varphi(x, t)}{t} \, dt \, dx$$
which, combined with the arbitrariness of \( \lambda \) implies that (3.7) holds true. This finishes the proof of Theorem 1.12.

From this, we deduce that by the Fatou lemma, we have

\[ f \]

which, together with the fact that, for any \( \lambda \in (0, \infty) \), \( (f/\lambda)_{L}^{+} = f_{L}^{+}/\lambda \) and \( (f/\lambda)_{L}^{-} = f_{L}^{-}/\lambda \), implies that

\[ f \]

From this, we deduce that

\[ f \]

which, combined with the arbitrariness of \( f \in [H_{\varphi, L, \text{rad}}(\mathbb{R}^{n}) \cap L^{2}(\mathbb{R}^{n})] \), further implies that (3.7) holds true. This finishes the proof of Theorem 1.12.

\[ \square \]

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