EXTENSIONS OF NONNATURAL HAMILTONIANS

C. M. Chanu* and G. Rastelli*

The concept of extended Hamiltonian systems allows a geometric interpretation of several integrable and superintegrable systems with polynomial first integrals of a degree depending on a rational parameter. Until now, the extension procedure has been applied only in the case of natural Hamiltonians. We give several examples of application to nonnatural Hamiltonians, such as the Hamiltonian of a system of two point-vortices, the Hamiltonian of the Lotka–Volterra model, and some Hamiltonians quartic in the momenta. We effectively obtain extended Hamiltonians in some cases, fail in other cases, and briefly discuss the reasons for these results.

Keywords: finite-dimensional Hamiltonian system, constant of motion, superintegrable system

DOI: 10.1134/S0040577920090019

1. Introduction

Given a Hamiltonian \( L \) with \( N \) degrees of freedom, we can use the extension procedure to construct Hamiltonians \( H \) with \( N+1 \) degrees of freedom admitting \( L \) itself as a first integral with all its possible constants of motion plus a characteristic first integral dependent on a rational parameter \( k \) and functionally independent from the existing constants of motion of \( L \). In particular, it follows that the extension of a maximally superintegrable Hamiltonian is again maximally superintegrable, admitting \( 2(N+1)-1 \) independent constants of motion. For Hamiltonians whose configuration manifold is Riemannian or pseudo-Riemannian, the existence of the characteristic constant of motion is connected with the structure of a warped manifold of the configuration manifold of \( L \), and the extension procedure is completely characterized in geometric terms.

In [1]–[7], we gave several example extensions of natural integrable and superintegrable Hamiltonians on Riemannian and pseudo-Riemannian manifolds, including anisotropic and harmonic oscillators and three-body Calogero and Tremblay–Turbiner–Winternitz systems. In all these examples, \( L \) and \( H \) are natural Hamiltonians, and the characteristic first integral is polynomial in the momenta of some degree depending on \( k \).

But the extension procedure only assumes that \( L \) is a regular function on some cotangent bundle \( T^*M \). Until now, we always regarded \( L \) as a natural Hamiltonian, and the extended Hamiltonian itself is hence natural. Here, we apply the extension procedure to functions \( L \) that are not quadratic in the momenta, and the extended Hamiltonian is consequently not natural. The construction of an extended Hamiltonian requires determining a certain function \( G \) that is well defined on all \( T^*M \) up to some lower-dimensional subset of singular points. The extended Hamiltonian is a polynomial in \( p_u \) and \( L \), and its characteristic

*Dipartimento di Matematica, Università di Torino, Turin, Italy, e-mail: claudiamaria.chanu@unito.it, giovanni.rastelli@unito.it (corresponding author).

Prepared from an English manuscript submitted by the authors; for the Russian version, see Teoreticheskaya i Matematicheskaya Fizika, Vol. 204, No. 3, pp. 321–331, September, 2020. Received January 3, 2020. Revised April 7, 2020. Accepted April 8, 2020.
first integral is a polynomial in $p_u$, $L$, $G$, and $X_LG$ (the derivative of $G$ with respect to the Hamiltonian vector field of $L$), and its global definition hence ultimately depends on $G$ and $X_LG$. Therefore, we focus our analysis on determining $G$ in the different cases and on studying its global behavior on $T^*M$.

Our aim is to understand the extension procedure applied to systems that are more general than those determined by natural Hamiltonians, not necessarily associated with Riemannian structures. In particular, we can analyze obstructions to the extension procedure due to non-globally defined solutions or multivalued solutions of the extension conditions.

Because we intend this work as a preliminary study of possible applications of the extension procedure to nonnatural Hamiltonians, we do not pretend to obtain complete and general results here and focus only on some meaningful examples.

This paper is organized as follows. In Sec. 2, we recall the fundamentals of the theory of extended Hamiltonians. In Sec. 3, we consider extensions of functions quartic in the momenta and find examples of extended Hamiltonians by analogy with the quadratic Hamiltonian case. The analysis becomes more subtle in Sec. 4, where we try to extend functions that are not polynomial in the momenta, as for the Hamiltonian of a system of two point-vortices. In this case, the global definition of the extended Hamiltonian and its characteristic first integral can become an issue: the extension is possible only for some values of the constant of motion $L$ in some cases, while the extension is always possible in other cases. In Sec. 5, we conclude with some examples where we cannot find any properly globally defined extended Hamiltonian.

2. Extensions of Hamiltonian systems

Let $L(q^i, p_i)$ be a Hamiltonian with $N$ degrees of freedom defined on the cotangent bundle $T^*M$ of an $N$-dimensional manifold $M$. We say that $L$ admits extensions if there exists $(c, c_0) \in \mathbb{R}^2 - \{(0, 0)\}$ such that there exists a nonzero solution $G(q^i, p_i)$ of

$$X_L^2(G) = -2(cL + c_0)G,$$

(2.1)

where $X_L$ is the Hamiltonian vector field of $L$.

If $L$ admits extensions, then for any solution $\gamma(u)$ of the ordinary differential equation

$$\gamma' + c\gamma^2 + C = 0$$

(2.2)

depending on an arbitrary constant parameter $C$, we say that any Hamiltonian $H(u, q^i, p_u, p_i)$ with $N+1$ degrees of freedom of the form

$$H = \frac{1}{2}p_u^2 - k^2\gamma'L + k^2c_0\gamma^2 + \frac{\Omega}{\gamma^2}, \quad k = \frac{m}{n}, \quad m, n \in \mathbb{N} - \{0\}, \quad \Omega \in \mathbb{R},$$

(2.3)

is an extension of $L$.

We introduced extensions of Hamiltonians in [2] and studied them because they admit first integrals polynomial in the momenta generated via a recursive algorithm. Moreover, the degree of the first integrals is related to the choice of $m$ and $n$. Indeed, for any $m, n \in \mathbb{N} - \{0\}$, we consider the operator

$$U_{m,n} = p_u + \frac{m}{n^2}\gamma X_L.$$

(2.4)

We have the following proposition.
Proposition 1 [5]. For \( \Omega = 0 \), Hamiltonian (2.3) is in involution with the function
\[
K_{m,n} = U_{m,n}^m(G_n) = \left( p_u + \frac{m}{n^2} \gamma(u)X_L \right)^m(G_n),
\] (2.5)
where \( G_n \) is the \( n \)th term of the recurrence relation
\[
G_1 = G, \quad G_{n+1} = X_L(G)G_n + \frac{1}{n}G X_L(G_n),
\] (2.6)
starting from any solution \( G \) of (2.1).

For \( \Omega \neq 0 \), the recursive construction of a first integral is more complicated. We construct a function depending on two strictly positive integers \( s \) and \( r \):
\[
\overline{K}_{2s,r} = (U_{2s,r}^2 + 2\Omega\gamma^{-2})^s(G_r),
\] (2.7)
where the operator \( U_{2s,r}^2 \) is defined according to (2.4) as
\[
U_{2s,r}^2 = \left( p_u + \frac{2s}{r^2} \gamma(u)X_L \right)^2
\]
and \( G_r \), as in (2.5), is the \( r \)th term of recurrence relation (2.6) with the solution \( G_1 = G \) of (2.1). For \( \Omega = 0 \), functions (2.7) reduce to (2.5) and can hence also be calculated if the first index is odd.

Theorem 1 [6]. For any \( \Omega \in \mathbb{R} \), Hamiltonian (2.3) satisfies \( \{ H, K_{m,n} \} = 0 \) for \( m = 2s \) and \( \{ H, \overline{K}_{2m,2n} \} = 0 \) for \( m = 2s + 1 \).

We call \( K \) and \( \overline{K} \) of the respective forms (2.5) and (2.7) characteristic first integrals of the corresponding extensions. It was proved in [2], [6] that the characteristic first integrals \( K \) or \( \overline{K} \) are functionally independent of \( H, L \), and any first integral \( I(p_u, q^i) \) of \( L \). This means that the extensions of (maximally) superintegrable Hamiltonians are (maximally) superintegrable Hamiltonians with one additional degree of freedom (also see [4]). In particular, any extension of a one-dimensional Hamiltonian is maximally superintegrable.

We give the explicit expression for the characteristic first integrals [5], [6]. For \( r \leq m \), we have
\[
U_{m,n}^r(G_n) = P_{m,n,r}G_n + D_{m,n,r}X_L(G_n)
\] (2.8)
with
\[
P_{m,n,r} = \sum_{j=0}^{[r/2]} \binom{r}{2j} \left( \frac{m}{n} \gamma \right)^{2j} p_u^{r-2j}(-2)^j(cL + c_0)^j,
\] (2.9)
\[
D_{m,n,r} = \frac{1}{n} \sum_{j=0}^{[(r-1)/2]} \binom{r}{2j+1} \left( \frac{m}{n} \gamma \right)^{2j+1} p_u^{r-2j-1}(-2)^j(cL + c_0)^j,
\]
where \([ \cdot ]\) denotes the integer part and \( D_{1,n,1} = \gamma/n^2 \).

The expansion of first integral (2.7) is
\[
\overline{K}_{2m,n} = \sum_{j=0}^{m} \binom{m}{j} \frac{(2\Omega)^j}{\gamma^2} U_{2m,n}^{2(m-j)}(G_n),
\] (2.10)
where \( U_{2m,n}^0(G_n) = G_n \) and
\[
G_n = \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} (-2(cL + c_0))^k G^{2k+1}(X_L G)^{n-2k-1}.
\] (2.11)
Remark 1. It was proved in [3] that ordinary differential equation (2.2) defining $\gamma$ is a necessary condition for obtaining a characteristic first integral of form (2.5) or (2.7). Depending on $c$ and $C$, the explicit form of $\gamma(u)$ is given (up to constant translations of $u$) by

$$
\gamma = \begin{cases} 
-Cu, & c = 0, \\
\frac{C_\kappa(cu)}{S_\kappa(cu)}, & c \neq 0.
\end{cases}
$$

(2.12)

where $\kappa = C/c$ is the ratio of the constant parameters in (2.2) and $S_\kappa$ and $C_\kappa$ are the trigonometric tagged functions (also see [5] for a summary of their properties)

$$
S_\kappa(x) = \begin{cases} 
\sin \frac{\sqrt{\kappa} x}{\sqrt{|\kappa|}}, & \kappa > 0, \\
x, & \kappa = 0, \\
\sinh \frac{\sqrt{|\kappa|} x}{\sqrt{|\kappa|}}, & \kappa < 0.
\end{cases}
$$

$$
C_\kappa(x) = \begin{cases} 
\cos \sqrt{\kappa} x, & \kappa > 0, \\
1, & \kappa = 0, \\
\cosh \sqrt{|\kappa|} x, & \kappa < 0.
\end{cases}
$$

Therefore, we have

$$
\gamma' = \begin{cases} 
-C, & c = 0, \\
-\frac{c}{S_\kappa^2(cu)}, & c \neq 0.
\end{cases}
$$

(2.13)

Remark 2. The global definition of the characteristic first integral of the extended Hamiltonian is ultimately determined by the definition of $G$ and its derivative $X_L G$. If these objects are defined globally, then the characteristic first integral is also defined globally.

It is clear from the brief exposition above that if we know a solution $G$ of (2.1), then the extensions of a function $L$ on $T^*M$ are completely determined if $G$ is regular and well defined on $T^*M$. The fundamental step in applying the extension procedure is therefore to determine $G$. In all existing examples of extended Hamiltonians, the function $L$ is always a polynomial quadratic in the momenta. The examples include anisotropic harmonic oscillators and the Tremblay–Turbiner–Winternitz and Post–Winternitz systems. For several of these systems there exists a quantization theory based on the Kuru–Negro factorization in shift and ladder operators adapted to extended Hamiltonians [8].

To generalize the extension procedure to nonnatural Hamiltonians, we focus our research here on determining functions $G$ that solve (2.1), leaving a deeper analysis of the resulting extended systems to other works. Below, we consider some examples of nonnatural Hamiltonians and natural Hamiltonians in a noncanonical symplectic or Poisson structure. Because the forms of the extended Hamiltonian and the characteristic first integral are completely determined if $L$ and $G$ are known, we are only concerned with determining and analyzing $G$ and $X_L G$.

3. Extensions of quartic Hamiltonians

Hamiltonians of degree four in the momenta were considered in [9]. These Hamiltonians are written in Andoyer projective variables and allow a unified representation of several mechanical systems, such as the harmonic oscillator, the Kepler system, and rigid body dynamics, corresponding to different choices of the parameters. We here consider some toy model Hamiltonians of a degree up to four in the momenta.

Example 3.1. Let

$$
L = p^4 + f_1(q)p^3 + f_2(q)p^2 + f_3(q)p + V(q).
$$

(3.1)
We can extend $L$ if we know global solutions $G$ of (2.1). If we assume that $G = G(q)$, then $G = C_1 q + C_2$ is a solution of (2.1) if $L$ has the form

$$L = \frac{(16 C_1 p^2 + 8 C_1 f p + 2 c C_1 q^2 + 4 c C_2 q + C_1 f^2 + 8 C_1 C_3)^2}{256 C_1^4} - \frac{c_0}{c}, \quad (3.2)$$

where $C_i$ are real constants and $f(q)$ is an arbitrary function. Consequently, this system admits the most general extension with both positive and negative $c$.

**Example 3.2.** We now consider

$$L = p^4 + f(q)p^2 + V(q). \quad (3.3)$$

If we assume that $G = g(q)p$ and find the coefficients of the monomials in $p$ in (2.1), then we obtain two solutions. One solution has the form

$$V = \frac{1}{4} \left( \frac{1}{16} q^2 + \frac{1}{8 c_1} c C_2 q - \frac{1}{2 c} C_4 (C_1 q + C_2)^2 + C_4 \right)^2 - \frac{c_0}{c}, \quad (3.4)$$

$$g = C_1 q + C_2, \quad f = \frac{1}{16} q^2 c + \frac{1}{8 c_1} c C_2 q - \frac{1}{2 c} C_4 (C_1 q + C_2)^2 + C_4,$$

which substituted in $L$ with $c \neq 0$ gives

$$L = \frac{1}{1024 C_1^2 (C_1 q + C_2)^4} (32 p^2 (C_1 q^2 + 2 C_2 C_2 q + C_1 C_2^2) + C_4 q^4 + 4 C_2^2 C_2 q^3 +$$

$$+ 16 C_3^2 C_1 q^2 + 5 C_1 C_2^2 C_2 q + (32 C_1^2 C_2 C_4 + 2 C_3^2 c) q + 16 C_1 C_2^2 C_4 - 8 C_3 t)^2 - \frac{c_0}{c}, \quad (3.5)$$

and the other also with $c \neq 0$ has the form

$$V = \frac{C_4}{(C_1 q + C_2)^4} - \frac{q}{(C_1 q + C_2)^4} \cdot 1024 c C_1^3 (-C_1 C_3 q^2 - 8 C_2 C_1 C_3 q^2 - 2 C_2^2 C_3 q^2 -$$

$$- 56 C_1^2 C_3^2 q^2 + (70 C_1 C_3^2 - 1024 c_1 C_1 - 32 C_1 C_3^2) q^3 +$$

$$+ (4096 c_1 C_3^2 C_2 - 128 C_1 C_3 C_2) q^2 - 56 C_2^2 C_3 q^2 +$$

$$+ (-28 C_1 C_3^3 C_2 + 6144 c_1 C_3^2 C_2 - 192 C_1 C_3^2 C_3 q - 8 C_2^2 C_3 q^2 + 4096 c_1 C_1 C_3^2 - 128 C_1 C_3 C_2 q^2)^2)$$

$$g = C_1 q + C_2, \quad f = \frac{1}{16 C_1^2 (C_1 q + C_2)^2} c + \frac{C_3}{(C_1 q + C_2)^2},$$

where the $C_i$ are constants. It is interesting that in the last case, the expression for $L$ is not a perfect square plus a constant, as in (3.2) and (3.5).

**Example 3.3.** We now assume that $L$ has the form

$$L = \left( p_1^2 + \frac{1}{(q_1)^2} p_2^2 + V(q_1, q_2) \right)^2, \quad (3.6)$$

which is the square of a natural Hamiltonian on $\mathbb{R}^2$, and we seek functions $V$ that admit the existence of non-trivial solutions $G$ of (2.1). We again assume that $G = G(q^1, q^2)$ and collect like powers of $(p_1, p_2)$ in (2.1).
The requirement that the coefficients of the momenta vanish identically, together with the assumption that \( c_0 = 0 \), gives the solution

\[
G = (C_2 \sin q^2 + C_3 \cos q^2)q^1 + C_1,
\]

\[
V = -\frac{c}{8} \frac{(C_3 \sin q^2 - C_2 \cos q^2)^2}{C_3^2 (\tan q^2 - C_2)^2} (q^1)^2 + \frac{c}{4} \frac{(C_3 \sin q^2 - C_2 \cos q^2)C_1}{C_3 (\tan q^2 - C_2)} q^1 + F((C_3 \sin q^2 - C_2 \cos q^2)q^1),
\]

where \( F \) is an arbitrary function.

In all these examples, the elements of the extension procedure are polynomial in the momenta, and the extended Hamiltonian and its characteristic first integral are therefore globally defined in the same way as in the case of a natural Hamiltonian.

4. Extensions of the Hamiltonian of a system of two point-vortices

The dynamics of two point-vortices \( z_j = x_j + iy_j \) of intensity \( k_j \), \( j = 1, 2 \), in the plane \((x, y)\) is described in canonical coordinates \((Y_i = k_i y_i, X_i = x_i)\) by the Hamiltonian

\[
L = -\alpha k_1 k_2 \log \left( (X_1 - X_2)^2 + \left( \frac{Y_1}{k_1} - \frac{Y_2}{k_2} \right)^2 \right),
\]

where \( \alpha = 1/8\pi \) and \( k_i \) are real numbers [10]. If \( k_2 \neq -k_1 \), then the functions \((k_1 z_1 + k_2 z_2, L)\) are independent first integrals of the system, i.e., three real-valued functions. If \( k_2 = -k_1 \), then these functions give only two real independent first integrals.

The coordinate transformation

\[
\tilde{X}_1 = \frac{X_1 - X_2}{2}, \quad \tilde{X}_2 = \frac{X_1 + X_2}{2}, \quad \tilde{Y}_1 = Y_1 - Y_2, \quad \tilde{Y}_2 = Y_1 + Y_2,
\]

is canonical and transforms \( L \) into

\[
L = -\alpha k_1 k_2 \log \left( 4 \tilde{X}_1^2 + \left( \frac{\tilde{Y}_1 + \tilde{Y}_2}{2 k_1} + \frac{\tilde{Y}_1 - \tilde{Y}_2}{2 k_2} \right)^2 \right).
\]

We can extend \( L \) if we know global solutions \( G \) of (2.1). We consider two cases.

Example 4.1. If \( k_1 = k_2 = k > 0 \), then the Hamiltonian becomes

\[
L = -\alpha k^2 \log \left( 4 \tilde{X}_1^2 + \tilde{Y}_2^2 \right).
\]

For \( c = 0 \), using Maple to compute \( G \), we obtain

\[
G = \left( \frac{\tilde{Y}_1 + 2ik\tilde{X}_1}{\sqrt{Q_1}} \right)^{Q_1 \sqrt{2c}/4\alpha k^3} F_1 + \left( \frac{\tilde{Y}_1 + 2ik\tilde{X}_1}{\sqrt{Q_1}} \right)^{-Q_1 \sqrt{2c}/4\alpha k^3} F_2,
\]

where

\[
Q_1 = k^2 e^{-L/\alpha k^2} = 4\tilde{X}_1^2 + \tilde{Y}_2^2
\]
and $F_i$ are arbitrary functions of $L$. The function $G$ is not single-valued in the general case, but it is single-valued, for example, if $Q_1 \sqrt{2c_0/4\alpha k^3}$ is an integer.

We now consider $X_L G$. Because $Q_1$ depends on the canonical coordinates only via $L$, the function $Q_1$ and the exponents in $G$ behave as constants under the differential operator $X_L$. Hence, the exponents in $G$ remain integer if they are integer in $G$ and $X_L G$ is well defined on $T^*M$. Therefore, both $H$ and its characteristic first integral are globally well defined for integer values of $Q_1 \sqrt{2c_0/4\alpha k^3}$. In this case, we have an example where the possibility of finding an extension depends on the system parameters and, in particular, on the values of the constant of motion $L$.

**Example 4.2.** If $k_2 = -k_1 = -k$, $k > 0$, then the Hamiltonian is

$$L = \alpha k^2 \log \left( 4 \tilde{X}_1^2 + \frac{\tilde{Y}_2^2}{k^2} \right).$$

For $c = 0$, using Maple to compute $G$, we obtain

$$G = F_1 \sin \left( \frac{\sqrt{2c_0} Q_2 \tilde{X}_2}{2 \alpha k^2 Y_2} \right) + F_2 \cos \left( \frac{\sqrt{2c_0} Q_2 \tilde{X}_2}{2 \alpha k^2 Y_2} \right),$$

(4.2)

where $Q_2 = k^2 e^{L/\alpha k^2}$ and $F_i$ are arbitrary first integrals of $L$. It is obvious that this real or complex function $G$ is always globally defined, as is $X_L G$, up to lower-dimensional sets, and this allows an effective extension of the Hamiltonian $L$. We note that in this case, the extended Hamiltonian has four independent constants of motion.

5. Hamiltonians with no known extension

The extension procedure can be applied in any Poisson manifold, not only in symplectic manifolds with a canonical symplectic structure, as in the examples above. Indeed, if $\pi$ is the symplectic form or the Poisson bivector determining the Hamiltonian structure of the system of the Hamiltonian $L$ in the coordinates $(x^1, \ldots, x^n)$, then the symplectic or Poisson structure of the extended manifold in the coordinates $(u, p_u, x^1, \ldots, x^n)$ is given by

$$\Pi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \pi \end{pmatrix}.$$

We recall that the Hamiltonian vector field of $L$ on a Poisson manifold with the Poisson vector $\pi$ is $\pi dL$.

Below, we consider two cases of Hamiltonian systems for which we cannot find extended Hamiltonians. In both cases, the obstruction to the extension is in the nonglobal definition of the known solutions of (2.1).

5.1. The Lotka–Volterra system. It is well known that the Lotka–Volterra predator–prey system

$$\dot{x} = ax - bxy, \quad \dot{y} = dxy - gy,$$

(5.1)

where $a$, $b$, $d$, and $g$ are real constants, can be represented in a Hamiltonian form (see [11]), for example, with the Poisson bivector

$$\pi = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \quad A = -x^1 + g y^1 + a e^{-by - dx},$$

(5.2)
and the Hamiltonian

\[ L = x^{-a} y^{-a} e^{d x + b y}. \]  

(5.3)

Because the manifold is symplectic, there is only one degree of freedom, and the existence of the Hamiltonian itself makes the system superintegable. Equation (2.1) with \( c = 0 \) has a solution \( G \) of the form

\[ G = F_1(L)e^{-B} + F_2(L)e^B, \]  

(5.4)

where

\[ B = -\frac{\sqrt{-2C_0}}{a} \int \left[ t \left(W\left( -\frac{b}{a} e^{\frac{a}{g/a}x^g/a y^e^{d(t-x)-b y)/a}} + 1 \right) \right) \right]^{-1} dt \]  

(5.5)

and the Lambert \( W \)-function is defined by \( z = W(z)e^{W(z)} \) for \( z \in \mathbb{C} \). If we set

\[ F_1 = \frac{1}{2} \left( \alpha + \frac{\beta}{t} \right), \quad F_2 = \frac{1}{2} \left( \alpha - \frac{\beta}{t} \right), \]

where \( \alpha \) and \( \beta \) are real constants, then

\[ G = \alpha \cos B - \beta \sin B. \]

But the Lambert \( W \) function is multivalued in \( \mathbb{C} - \{0\} \) even if its variable is real (in this case, it is defined only for \( z \geq -1/e \) and double-valued for \(-1/e < z < 0 \)). Therefore, such a \( G \) cannot provide a globally defined first integral and does not determine an extension of \( L \).

By comparison, the Hamiltonian of the one-dimensional harmonic oscillator admits iterated extensions where \( c = 0 \) and \( c_0 \) is always equal to the elastic parameter of the first oscillator. We thus obtain the \( n \)-dimensional, \( n \in \mathbb{N} \), anisotropic oscillator with parameters having rational ratios, which is therefore always superintegrable [5]. This is not the case for the Lotka–Volterra system, where the periods of the closed trajectories in \((x, y)\) are not all equal, in contrast to the case of harmonic oscillators.

5.2. The Euler system. It is well known that the Euler rigid-body system is described by the Hamiltonian

\[ L = \frac{1}{2} \left( \frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3} \right) \]

on the Poisson manifold with the coordinates \((m_1, m_2, m_3)\) and the Poisson bivector

\[ \pi = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}. \]

The \((m_i)\) are the components of the angular momentum in the moving frame, and they are conjugate momenta of the three components of the principal axes along one fixed direction. A Casimir of \( \pi \) is \( M = m_1^2 + m_2^2 + m_3^2 \). The system has two functionally independent constants of the motion: \( L \) and one of the components of the angular momentum in the fixed frame.

A solution of Eq. (2.1) can be found using the Kuru–Negro ansatz [8]

\[ X_L G = \pm \sqrt{-2(cL + c_0)} G, \]  

(5.6)

which also yields solutions of Eq. (2.1). A solution of (5.6) is

\[ G = f \cdot \exp \left[ \mp \frac{I_1 I_2 I_3}{\sqrt{I_2(I_1 - I_3)} X_2} \sqrt{-2(cL + c_0)} X_1 \right] \left( m_1 \sqrt{\frac{I_2(I_1 - I_3)}{I_1}} X_1, \sqrt{\frac{I_2(I_1 - I_3)}{I_1}} X_2 \right), \]

1108
where \( f \) is an arbitrary function of the first integrals \( L \) and \( M \),

\[
X_1 = I_1 I_2 (M - 2I_3 L), \quad X_2 = I_1 I_3 (2I_2 L - M),
\]

and \( F(\phi, k) \) is the incomplete elliptic integral of the first kind

\[
F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},
\]

which is a multivalued function (the inverse of the Jacobi sinus amplitudinis function \( \text{sn} \)). The function \( G \), possibly complex-valued and with lower-dimensional singular sets depends essentially on \( m_1 \) because all other arguments in it are either constants or first integrals. Because our function \( G \) is not single-valued, we cannot construct an extended Hamiltonian from \( L \) in this case.

6. Conclusions

From the discussed examples, we see that extended Hamiltonians can be obtained not only from natural Hamiltonians but also from nonnatural Hamiltonians \( L \). The case where \( L \) is quartic in the momenta is very similar to the previously studied quadratic cases, and the extension procedure does not encounter new problems. For the Hamiltonian of a system of two point-vortices, global solutions of (2.1) can be obtained corresponding to particular choices of the parameters in \( L \). In the remaining examples, we cannot obtain globally defined solutions of (2.1) and hence cannot construct extensions in those cases.

In future works, we could study the extended Hamiltonians obtained here in more detail and could seek global solutions in the cases of the Lotka–Volterra model and rigid-body dynamics or the reasons for their nonexistence.

Conflicts of interest. The authors declare no conflicts of interest.

REFERENCES

1. C. Chanu, L. Degiovanni, and G. Rastelli, “Polynomial constants of motion for Calogero-type systems in three dimensions,” J. Math. Phys., 52, 032903 (2011).
2. C. Chanu, L. Degiovanni, and G. Rastelli, “First integrals of extended Hamiltonians in \( n+1 \) dimensions generated by powers of an operator,” SIGMA, 7, 038 (2011).
3. C. Chanu, L. Degiovanni, and G. Rastelli, “Generalizations of a method for constructing first integrals of a class of natural Hamiltonians and some remarks about quantization,” J. Phys.: Conf. Ser., 343, 012101 (2012); arXiv:1111.0030v2 [nlin.SI] (2011).
4. C. Chanu, L. Degiovanni, and G. Rastelli, “Superintegrable extensions of superintegrable systems,” SIGMA, 8, 070 (2012).
5. C. Chanu, L. Degiovanni, and G. Rastelli, “Extensions of Hamiltonian systems dependent on a rational parameter,” J. Math. Phys., 55, 122703 (2014); arXiv:1310.5690v1 [math-ph] (2013).
6. C. Chanu, L. Degiovanni, and G. Rastelli, “The Tremblay–Turbiner–Winternitz system as extended Hamiltonian,” J. Math. Phys., 55, 122701 (2014); arXiv:1404.4825v1 [math-ph] (2014).
7. C. Chanu, L. Degiovanni, and G. Rastelli, “Extended Hamiltonians, coupling-constant metamorphosis, and the Post–Winternitz system,” SIGMA, 11, 094 (2015).
8. C. M. Chanu and G. Rastelli, “Extended Hamiltonians and shift, ladder functions and operators,” Ann. Phys., 386, 254–274 (2017); arXiv:1705.09519v1 [math-ph] (2017).
9. S. Ferrer and F. Crespo, “Parametric quartic Hamiltonian model: A unified treatment of classic integrable systems,” J. Geom. Mech., 6, 479–502 (2014).
10. T. Kambe, Elementary Fluid Mechanics, World Scientific, Singapore (2007).
11. Y. Nutku, “Hamiltonian structure of the Lotka–Volterra equations,” Phys. Lett. A, 145, 27–28 (1990).