Canonically conjugate variables for the
periodic Camassa-Holm equation

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Abstract

The Camassa-Holm shallow water equation is known to be Hamiltonian with respect to two compatible Poisson brackets. A set of conjugate variables is constructed for both brackets using spectral theory.

2000 Mathematical Subject Classification: 35Q35, 37K10

Keywords: Camassa-Holm equation, Poisson bracket, canonically conjugate variables.

1 Introduction

In 1976 Flaschka and McLaughlin observed on particular examples of the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions that variables arising naturally from spectral theory and algebraic geometry have “nice” symplectic properties. It was surprising because a priori it is not clear why there should exist a relation between the Lax pairs and the Hamiltonian formalism of the corresponding equations. The same phenomenon was later observed on numerous examples. This led Novikov and Veselov to the theory of algebro-geometric Poisson brackets on the universal bundle of hyperelliptic curves. Later Krichever and Phong developed a unified construction of the symplectic forms arising in the $N = 2$ Yang-Mills theories and soliton equations.
The goal of this paper is to prove analogues of some results of Flaschka and McLaughlin [1] for the Camassa-Holm equation, also known as the shallow water equation:

\[ v_t - v_{xxt} + 3vv_x - 2v_xv_{xx} - vv_{xxx} = 0. \]  

(1)

The Camassa-Holm equation is known to be bi-Hamiltonian [6]. To describe two compatible Poisson structures, it is better to use the function \( m = (1 - D^2)v \) instead of \( v \). Here and later \( D \) and \( D' \) denote the derivative with respect to \( x \). Let \( J = mD + Dm \) and \( K = \frac{1}{2}D(1 - D^2) \). Then (in the periodic case, i.e. \( v(x + 1) = v(x) \)) the two compatible Poisson brackets are given by the formulae

\[ \{ A, B \}_1 = \int_0^1 \frac{\partial A}{\partial m} J \frac{\partial B}{\partial m} \, dx \]

and

\[ \{ A, B \}_2 = \int_0^1 \frac{\partial A}{\partial m} K \frac{\partial B}{\partial m} \, dx. \]

The Camassa-Holm equation is Hamiltonian with respect to both brackets, it can be rewritten as

\[ m_t + \{ m, H_2 \}_1 = 0 \quad \text{or} \quad m_t + \{ m, H_3 \}_2 = 0, \]

where \( H_2 = \frac{1}{2} \int_0^1 (v^2 + (v')^2) \, dx \) and \( H_3 = \int_0^1 (v^3 + v(v')^2) \, dx \).

The Camassa-Holm equation can be expressed as a compatibility condition of two equations [6]. Following [7], we will write these equations as

\[ \psi'' = \frac{1}{4} \psi - \lambda m \psi, \]  

(2)

\[ \psi_t = - \left( v + \frac{1}{2\lambda} \right) \psi' + \frac{1}{2} (v') \psi. \]

The Camassa-Holm equation has some particular properties. Firstly, it is necessary to consider not only smooth \( m \) but also distributions. Indeed, even for the traveling wave solution \( u(x, t) = ce^{-|x-ct|} \) we have \( m = 2c\delta(x - ct) \). It creates some difficulties since solutions \( \psi \) of equation (2) are not smooth. The corresponding solutions of (1) are not classical but weak solutions, see [8, 9] for a discussion. Secondly, the dynamics is not linear on the Jacobian of the spectral curve, it is necessary to consider some covering of the spectral curve. It implies that one should use an analogue of the Abel map using meromorphic
differentials to linearize the dynamics. As a result, one has to use piecewise meromorphic functions to write down algebro-geometric solutions. One can find more details in [7, 10]. The algebro-geometric solutions are also studied in [11].

Since the application of the available general theories from the papers [2, 3, 4, 5] is not obvious because of these particularities, we use the methods similar to those of Flaschka and McLaughlin [1].

Let us now recall some results of [1]. The Korteweg-de Vries equation

\[ u_t - 6uu_x + u_{xxx} = 0 \]

is known to be Hamiltonian with respect to the Poisson bracket given (in the periodic case, i.e. \( u(x + 1) = u(x) \)) by the formula

\[ \{A, B\}_{KdV} = \int_0^1 \frac{\partial A}{\partial m} D \frac{\partial B}{\partial m} dx. \]

Let us consider the spectral problem for the Schrödinger operator

\[ -y'' + uy = \lambda y. \tag{3} \]

Let \( y_2(x, \lambda) \) be a solution of (3) normalized by the conditions \( y_2(0, \lambda) = 0, \ y_2'(0, \lambda) = 1 \). Auxiliary eigenvalues \( \mu_i \) are solutions of the equation \( y_2(1, \mu) = 0 \). The functions \( y_2(x, \mu_i) \) are Floquet solutions of the spectral problem (3), i.e. there exists Floquet multipliers \( \rho_i \) such that \( y_2(x + 1, \mu_i) = \rho_i y_2(x, \mu_i) \).

Flaschka and McLaughlin proved [1] that \( \mu_i \) and \( f_j = -2 \log |\rho_j| \mu_j^2 \) are conjugate variables, i.e.

\[ \{\mu_i, \mu_j\}_{KdV} = 0, \quad \{\mu_i, f_j\}_{KdV} = \delta_{ij}, \quad \{f_i, f_j\}_{KdV} = 0. \]

In this paper we prove analogues of this result for the Camassa-Holm equation. Let \( \mu_i \) and \( \rho_j \) be now auxiliary eigenvalues and corresponding Floquet multipliers for the spectral problem (3). Using the theory of the spectral problem developed by Constantin and McKean [7] we prove the following theorems.

**Theorem 1.** The variables \( \mu_i \) and \( f_j = -\frac{\log |\rho_j|}{\mu_j^2} \) are conjugate with respect to the first bracket \( \{,\}_1 \).

**Theorem 2.** The variables \( \mu_i \) and \( g_j = -\frac{\log |\rho_j|}{\mu_j^3} \) are conjugate with respect to the second bracket \( \{,\}_2 \).

The plan of the paper is as following. In Section 2 we recall necessary for us results of Constantin and McKean concerning the theory of the spectral problem [2]. Then in Section 3 we prove Theorems 1 and 2.
2 Spectral theory related to the Camassa-Holm equation

**Definition** We say that a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is piecewise-smooth if the following conditions hold:

1. The function \( f \) is continuous.
2. For any finite interval \([b, c]\) there exists a finite number of points \( b \leq a_1 < \ldots < a_n \leq c \) such that
   
   (a) \( f \) is a smooth function on \([b, c]\) except points \( a_i \);
   
   (b) left and right limits \( f'_-(a_i) \) and \( f'_+(a_i) \) of the derivative at the points \( a_i \) exist.

In the rest of this paper we assume that \( m \) can be written as a smooth function \( m_s \) plus a linear combination of delta-functions

\[
m(x, t) = m_s(x, t) + \sum p_n(t)\delta(x - q_n(t))
\]

such that for any \( t \) the set \( \{q_n(t)\} \) has no accumulation points.

This assumption on \( m \) is reasonable since on the one hand this includes both the case of a smooth \( m \) and the case of multipeakons \( \sum p_n(t)\delta(x - q_n(t)) \) which are of particular interest. On another hand, for such \( m \) the Cauchy problem for ODE (2) always has a unique solution in the class of piecewise-smooth functions defined above. Moreover, many usual properties hold, for example the Wronskian of two solutions is a constant.

Let us recall some results about the spectral problem (2) which will be useful for us. Our basic source is paper of Constantin and McKean [7]. We consider the periodic case, so \( v(x) = v(x + 1) \) and, respectively, \( m(x + 1) = m(x) \).

Let \( y_1(x, \lambda) \) and \( y_2(x, \lambda) \) be a fundamental set of solutions of (2) defined by normalization

\[
y_1(0, \lambda) = 1, \quad y'_1(0, \lambda) = 0, \\
y_2(0, \lambda) = 0, \quad y'_2(0, \lambda) = 1.
\]

Any solution \( \psi \) of (2) can be written as linear combination of \( y_1 \) and \( y_2 \):

\[
\psi(x, \lambda) = \psi(0, \lambda)y_1(x, \lambda) + \psi'(0, \lambda)y_2(\lambda).
\]
It follows that we have the formula
\[
\begin{pmatrix}
\psi(x, \lambda) \\
\psi'(x, \lambda)
\end{pmatrix} =
\begin{pmatrix}
y_1(x, \lambda) & y_2(x, \lambda) \\
y_1'(x, \lambda) & y_2'(x, \lambda)
\end{pmatrix}
\begin{pmatrix}
\psi(0, \lambda) \\
\psi'(0, \lambda)
\end{pmatrix}.
\tag{5}
\]

We will denote the matrix from (5) by \( U(x, \lambda) \).

A solution \( \psi \) of (2) is said to be a Floquet solution if there exist a number \( \rho \) called a Floquet multiplier such that
\[
\psi(x + 1, \lambda) = \rho \psi(x, \lambda).
\]

It follows from (5) that a Floquet solution is an eigenvector of \( U(1, \lambda) \) and \( \rho \) is an eigenvalue of \( U(1, \lambda) \). The determinant of \( U(x, \lambda) \) is a Wronskian and it is easy to see from the definition of \( y_1 \) and \( y_2 \) that it is equal to 1. Hence, we obtain the following equation for \( \rho \):
\[
\rho^2 + 2\Delta(\lambda)\rho + 1 = 0,
\tag{6}
\]
where
\[
\Delta(\lambda) = \frac{1}{2} \text{tr} \, U(1, \lambda) = \frac{1}{2} (y_1(1, \lambda) + y_2'(1, \lambda)).
\]

The case \( \rho = \pm 1 \) corresponds to periodic/antiperiodic solutions.

The corresponding eigenvalues \( \lambda_1^\pm \) define a spectral curve.

Let us consider auxiliary eigenvalues \( \mu_i \) defined as solutions of the equation \( y_2(1, \mu) = 0 \). Since \( m(x) \) is periodic, \( y_2(x + 1, \mu_i) \) is a solution of (2) for \( \lambda = \mu_i \). Using (4) we see that it is proportional to \( y_2(x, \mu_i) \) and the proportionality constant is \( y_2'(1, \mu_i) \). Thus \( y_2(x + 1, \mu_i) = y_2'(1, \mu_i)y_2(x, \mu_i) \) and this means that \( y_2(x, \mu_i) \) is a Floquet solution with the Floquet multiplier \( \rho_i = y_2'(1, \mu_i) \).

Now let us consider the equation (6) for \( \lambda = \mu_i \). We found one root of this equation \( \rho_i \). But there is another root \( \bar{\rho}_i = 1/\rho_i \). Let \( y(x, \mu_i) \) be the corresponding Floquet solution normalized by the condition \( y(0, \mu_i) = 1 \), this normalization is possible since \( y \) and \( y_2 \) are linearly independent.

The disposition of spectra is very similar to the KdV case: the periodic/antiperiodic eigenvalues \( \lambda_1^\pm \) define gaps containing each only one auxiliary eigenvalue \( \mu_i \).

Flaschka and McLaughlin [1] used in their proofs identities with Wronskians, but these identities are not useful in the case of the Camassa-Holm equation. It is the identity from the following lemma that will be our main tool. It was used in [7] to calculate Poisson brackets.
Lemma. Let \( \psi \) and \( \varphi \) be solutions (not necessarily different) of the spectral problem (2) for the same \( \lambda \). Then we have the following identity:

\[
\lambda J \psi \varphi = K \varphi \psi,
\]

where \( J = mD + Dm \) and \( K = \frac{1}{2}D(1 - D^2) \) are the operators used to define Poisson brackets above.

Proof is a direct calculation, one can write down \( K \varphi \psi \) and then eliminate all second and third derivatives of \( \psi \) and \( \varphi \) using (2) and the derivative of (2) with respect to \( x \).

3 Spectral theory and conjugate variables

Let us consider the auxiliary eigenvalues \( \mu_i \). It is easy to see from the spectral problem (2) that \( \mu_i \neq 0 \). Thus we can define variables \( f_j = -\log |\rho_j| \mu_j^2 \) and \( g_j = -\log |\rho_j| \mu_j^3 \). It should be remarked that we use \( |\rho_j| \) instead of \( \rho_j \) only to obtain real-valued \( f_j \) and \( g_j \). If we consider the complex case, we can drop the absolute value signs, the commutation relations will be the same.

Theorem 1. The variables \( \mu_i \) and \( f_j \) are conjugate with respect to the first bracket:

\[
\{\mu_i, \mu_j\}_1 = 0, \quad \{\mu_i, f_j\}_1 = \delta_{ij}, \quad \{f_i, f_j\}_1 = 0.
\]

Proof. Let us start by calculating \( \frac{\partial \mu_i}{\partial m} \). We have

\[
y''_2(x, \mu_i) = \frac{1}{4}y_2(x, \mu_i) - \mu_i m y_2(x, \mu_i). \tag{7}
\]

Let us now write for simplicity \( y_2 \) instead of \( y_2(x, \mu_i) \) and \( \mu \) instead of \( \mu_i \). The variation of (7) equals to

\[
\delta y''_2 = \frac{1}{4} \delta y_2 - \delta \mu m y_2 - \mu \delta m y_2 - \mu m \delta y_2. \tag{8}
\]

Note that under variation \( \mu \) remains an auxiliary eigenvalue. Let us multiply the previous identity by \( y_2 \) and integrate. On the L.H.S. we should integrate twice by parts and use the identities

\[
y_2(0, \mu) = y_2(1, \mu) = 0, \quad \delta y_2(0, \mu) = \delta y_2(1, \mu) = 0
\]
and (7). We obtain on the L.H.S.

\[ \int_0^1 \delta y_2 (\frac{1}{4} y_2 - \mu m y_2) \, dx. \]

Cancelling the same integrals on the L.H.S. and on the R.H.S. we obtain

\[ 0 = -\delta \mu \int_0^1 m y_2^2 \, dx - \int_0^1 \delta m \mu y_2^2. \quad (9) \]

Let us now remark that \( \int_0^1 m y_2^2 \, dx \neq 0 \). Indeed, let us multiply (7) by \( y_2 \) and integrate, we obtain

\[ \int_0^1 y'' y_2 = \frac{1}{4} \int_0^1 y_2^2 - \mu \int_0^1 m y_2^2. \]

Using integration by parts we see that

\[ \mu \int_0^1 m y_2^2 = \int_0^1 \left[ \left( \frac{y_2}{2} \right)^2 + (y'_2)^2 \right] \, dx \neq 0. \]

As we remarked before, auxiliary eigenvalue \( \mu \neq 0 \), hence \( \int_0^1 m y_2^2 \, dx \neq 0 \). Thus, we obtain from (9) that

\[ \frac{\partial \mu_i}{\partial m} = -A_i \mu_i y_2^2(x, \mu_i), \]

where \( A_i = \left( \int_0^1 m y_2^2(x, \mu_i) \, dx \right)^{-1} \).

Let us now calculate \( \frac{\partial \rho_i}{\partial m} \). Let us multiply (8) by \( y(x, \mu_i) \) (this is another Floquet solution for \( \mu_i \) defined in the previous section, we will write simply \( y \) instead of \( y(x, \mu_i) \)), subtract \( y'' = \frac{1}{4} y - \mu m y \) multiplied by \( \delta y_2 \) and finally integrate. On the L.H.S. we have

\[ \int_0^1 (\delta y'_2 y - y'' \delta y_2) \, dx = \int_0^1 (\delta y'_2 y - y' \delta y_2)' \, dx = (\delta y'_2 y - y' \delta y_2)|_0^1. \]

Remember that \( \rho_i = y'_2(1, \mu_i) \) and \( y(1, \mu_i) = \tilde{\rho}_i y(0, \mu_i) = \tilde{\rho}_i = \frac{1}{\rho_i} \). We see that the L.H.S. is equal to \( \frac{\delta y_i}{\rho_i} = \delta \log |\rho_i| \). On the R.H.S. we obtain

\[ \int_0^1 (-\delta \mu m y_2 y - \mu \delta m y_2 y) \, dx. \]

Using the expression for \( \delta \mu \), we obtain

\[ \frac{\partial \log \rho_i}{\partial m} = A_i B \mu_i y_2^2(x, \mu_i) - \mu_i y_2(x, \mu_i) y(x, \mu_i), \]

7
where $B_i = \int_0^1 m y_2(x, \mu_i) y(x, \mu_i) \, dx.$

Let us now calculate brackets. We will do it using the lemma from previous section. Let us prove that $\{\mu_i, \mu_j\}_1 = 0.$ It is clear if $i = j,$ so let us suppose that $i \neq j.$ We have

$$\{\mu_i, \mu_j\}_1 = \int_0^1 \frac{\partial \mu_i}{\partial m} \frac{\partial y_i}{\partial m} \, dx = A_i A_j \mu_i \mu_j \int_0^1 y_2(x, \mu_i) J y_2(x, \mu_j) \, dx.$$ 

Let us now remark that if we have two functions $f$ and $g$ such that $f, f', g$ and $g'$ are equal to zero at points 0 and 1 then

$$\int_0^1 f J g \, dx = - \int_0^1 g J f \, dx, \quad (10)$$

and

$$\int_0^1 f K g \, dx = - \int_0^1 g K f \, dx. \quad (11)$$

Using these identities and the lemma we have

$$\mu_i \mu_j \int_0^1 y_2^2(\mu_i) J y_2^2(\mu_j) \, dx = \mu_i \int_0^1 y_2^2(\mu_i) K y_2^2(\mu_j) \, dx =$$

$$= -\mu_i \int_0^1 y_2(\mu_j) K y_2(\mu_i) \, dx = -\mu_i^2 \int_0^1 y_2^2(\mu_j) J y_2^2(\mu_i) \, dx =$$

$$= \mu_i^2 \int_0^1 y_2^2(\mu_j) J y_2^2(\mu_j) \, dx.$$ 

Since $i \neq j$ and $\mu_i \neq 0$ it follows that $\int_0^1 y_2^2(\mu_i) J y_2^2(\mu_j) \, dx = 0.$ This implies that $\{\mu_i, \mu_j\}_1 = 0.$

We will now prove that $\{\mu_i, \log |\rho_j|\}_1 = -\mu_i^2 \delta_{ij},$ it will imply that $\{\mu_i, f_j\}_1 = \delta_{ij}.$ The proof that $\{\mu_i, \log |\rho_j|\}_1 = 0$ if $i \neq j$ is analogous to the proof that $\{\mu_i, \mu_j\}_1 = 0.$ Let us find $\{\mu_i, \log |\rho_i|\}_1.$ This is equal to

$$-A_i^2 B_i \mu_i^2 \int_0^1 y_2^2(\mu_i) J y_2^2(\mu_i) \, dx + A_i \mu_i^2 \int_0^1 y_2^2(\mu_i) J y_2(\mu_i) g(\mu_i) \, dx.$$ 

Using the identity (10), we can see that the first term is equal to zero. Let us drop the index $i$ for simplicity. We have

$$A \mu^2 \int_0^1 y_2^2 J y_2 y \, dx = A \mu^2 \int_0^1 y_2^2 (mD + Dm) y_2 y \, dx =$$
\[ = A \mu^2 \left[ \int_0^1 y_2^2 m(y_2 y') \, dx + \int_0^1 y_2^2 (my_2 y')' \, dx \right] = \]
\[ = A \mu^2 \left[ \int_0^1 y_2 m(y_2 y') \, dx - \int_0^1 my_2 y_2^2 y' \, dx \right] = \]
\[ = A \mu^2 \int_0^1 my_2^2 (y_2 y' - y_2 y') \, dx. \]

The expression \( y' y_2 - y y_2' \) is a Wronskian \( W(y, y_2) \). It is a constant, not depending on \( x \). Let us calculate it for \( x = 1 \). Since \( y(1, \mu_i) = \tilde{\rho}_i y(0) = \frac{1}{\rho_i} = \frac{1}{y_2(1, \mu_i)} \), we obtain \( y_2 y' - y_2 y = -1 \). This implies (remember the definition of \( A \))

\[ A \mu^2 \int_0^1 my_2^2 (y_2 y' - y_2 y) \, dx = -A \mu^2 \int_0^1 my_2^2 \, dx = -\mu^2. \]

Hence, we obtain \( \{ \mu_i, \log |\rho_j| \}_1 = -\mu_i^2 \delta_{ij} \).

If we prove that \( \{ \log |\rho_i|, \log |\rho_j| \}_1 = 0 \), it will imply that \( \{ f_i, f_j \}_1 = 0 \). But the proof that \( \{ \log |\rho_i|, \log |\rho_j| \}_1 = 0 \) is analogous to previous calculations.

This finishes the proof. \( \square \)

**Theorem 2.** The variables \( \mu_i \) and \( g_j \) are conjugate with respect to the second bracket:

\[ \{ \mu_i, \mu_j \}_2 = 0, \quad \{ \mu_i, g_j \}_2 = \delta_{ij}, \quad \{ g_i, g_j \}_2 = 0. \]

**Proof** is analogous to the proof of Theorem 1. \( \square \)

**Acknowledgments**

The author is very grateful to the Centre de Recherches Mathématiques (CRM) for its hospitality. The author would like to thank Prof. A. Broer and Prof. P. Winternitz for useful discussions.

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