PERFECTION OF GROUP SCHEMES
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Abstract. We initiate a systematic study of the perfection of affine group schemes of finite type over fields of positive characteristic. The main result intrinsically characterises and classifies the perfections of reductive groups, and obtains a bijection with the set of classifying spaces of compact connected Lie groups topologically localised away from the characteristic. We also study the representations of perfectly reductive groups. We establish a highest weight classification of simple modules, the decomposition into blocks, and relate extension groups to those of the underlying abstract group.

Introduction

For a (group) scheme over a field \( k \) of characteristic \( p > 0 \), its ‘perfection’ is defined as the inverse limit over the Frobenius homomorphism. In this paper we study the perfection of group schemes and their representation theory. We place particular emphasis on reductive groups. We obtain an intrinsic characterisation (‘perfectly reductive groups’) and give a classification in terms of root data ‘with \( p \) inverted’. We also give a highest weight classification of simple modules for perfectly reductive groups, establish the block decomposition, and make a first step towards the study of multiplicity questions. Finally, we prove that perfectly reductive groups and the classifying spaces of compact Lie groups localised away from \( p \) are classified by the same data. This result is the ‘perfect analogue’ of the fact that reductive groups over algebraically closed fields and compact Lie groups are both classified by root data.

The motivations for the current work were three-fold, and came from several directions. Before describing the structure of this paper in more detail, we outline these motivations and possible future directions.

Perfect representability. Sometimes functors on rings in characteristic \( p \) are only well-behaved on perfect algebras (that is, algebras for which the Frobenius homomorphism is an isomorphism). A prominent example is the functor of Witt vectors and its relatives. An important observation (see for instance [BS, BD, Zh]) is that if a functor on the category of perfect commutative \( k \)-algebras can be represented by a scheme, it determines the scheme ‘up to perfection’. An example of such a setting is the Witt vector affine Grassmannian, which plays a prominent role in recent advances in the Langlands program.

It is important that passage to the perfection gives an isomorphism of étale topoi. In particular, constructions built via étale sheaves (like étale cohomology, or categories of perverse sheaves in the étale topology) are insensitive to passage to the perfection. This fact plays an important role in [Zh], where a mixed characteristic analogue of the geometric Satake equivalence is obtained. Similarly, it plays an important role in [BD], where the ‘Serre dual’ of a unipotent group is shown to be the perfection of a unipotent group, and its character sheaves are studied.

By the above, also the étale homotopy type of a (simplicial) scheme only depends on its perfection. In [Fr], Friedlander used this homotopy type to construct interesting maps
between topological localisations of classifying spaces of compact Lie groups, based on (exceptional) isogenies in positive characteristic. This is one of the main results we rely on to establish our bijection between perfectly reductive groups and localised classifying spaces.

**Fractal representation theory.** For a (reduced) group scheme defined over $\mathbb{F}_p$, the Frobenius twist realises its category of representations as a full subcategory of itself. This self-similarity induces a fractal-like structure. For example, Figure 1, shows a classic picture of the non-zero weight spaces of simple modules for $SL_2$ in characteristic 3. (For the reader unfamiliar with this picture, it may be helpful to note that it simply depicts the (non)-vanishing behaviour of Pascal’s triangle modulo $p$, see for instance [Wi, §1,].) This picture is fractal-like, but not genuinely fractal: one can ‘zoom out’ but one cannot ‘zoom in’ indefinitely since the Frobenius homomorphism is not an isomorphism. By passing to the perfection, one gets a genuine fractal. One dream (not realised in the current paper) is to use this fractal structure to say something about important open questions in representation theory like dimensions and characters of simple modules.

Much of the difficulty in the representation theory of reductive groups in characteristic $p$ remains after perfecting. We do observe two interesting simplifications. Firstly, the complexities of the block decomposition disappear after passage to the perfection (see Theorem 5.1.4). Secondly, perfect representation theory appears to provide the correct setting for results of [CPSV]: for a perfectly reductive group, extensions computed inside algebraic representations agree with extensions computed as abstract groups (see Theorem 5.2.2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fractal.png}
\caption{Characters of simple modules for $SL_2$ in characteristic $p = 3$.}
\end{figure}

**Tensor categories in characteristic $p$.** Over fields of characteristic zero, a famous theorem of Deligne classifies those tensor categories which admit a fibre functor to super vector spaces as precisely those of moderate growth [D]. It is a fascinating open problem to find an analogue of this problem in characteristic $p$, with many potential applications to modular representation theory. Recently, this problem was solved in [CEO] for tensor categories with exact Frobenius functor. An important technical tool was a limit procedure in [CEO, §6] which, by restriction to representation (tensor) categories of affine group schemes, generalises
perfection of group schemes. Remarkably, the ‘perfection’ of a Frobenius exact tensor category of moderate growth essentially returns the representation category of a perfect group scheme. In other words, up to perfection, all Frobenius exact tensor categories of moderate growth arise from (perfect) group schemes. We consider this as further evidence for their importance.

Structure of the paper. Motivated by the above considerations, we initiate a systematic study of perfecting group schemes. The paper is organised as follows:

In Section 1 we investigate the purely combinatorial problem of describing root data over the ring $\mathbb{Z}[1/p]$, for later use in our classification results. In Section 2, we derive some general results on perfect schemes. In Section 3 we start studying the perfections of group schemes. We study perfect subgroups of perfect groups and their quotients. We also obtain criteria for when two group schemes perfect to isomorphic groups and derive some results on the perfections of the additive and multiplicative groups. In Section 4 we classify perfectly reductive groups by their $\mathbb{Z}[1/p]$-root data. In Section 5 we study the representation theory of perfectly reductive groups. We classify simple modules and realise them as socles of induced modules from Borel subgroups. Then we show that the block decomposition simplifies considerably compared to the non-perfect case, in fact blocks are governed by the root lattice. We also show that extension groups for the perfected groups are given in terms of generic cohomology in the sense of [CPSV]. This actually implies that extensions in the category of (rational) representations over the perfected reductive group can be computed of generic cohomology in the sense of [CPSV]. This actually implies that extensions in the category of (rational) representations over the perfected reductive group can be computed in the category of representations of the abstract group of $\mathbb{F}_{p^r}$-points. In Section 6 we prove that $\mathbb{Z}[1/p]$-root data also classify the localisations away from $p$ of the classifying spaces of compact connected Lie groups. Finally, in Section 7, we present some explicit computations for extension groups, decomposition multiplicities and line bundle cohomology for perfected $SL_2$. We also make explicit the fractal behaviour of perfected representation theory for $SL_2$.

1. Root data over rings

For the entire section, we assume that $D$ is a principal ideal domain of characteristic 0. By a $D$-lattice we understand a finitely generated free $D$-module. Because $D$ is a PID we could replace ‘free’ by ‘projective’, so our definition agrees with standard terminology (e.g. in [CR]). For a lattice $V$, we have the dual lattice $V^* := \text{Hom}_D(V, D)$.

1.1. Reflection groups and root data. We follow the definition of root data of for instance [Gr].

Definition 1.1.1. (0) A reflection $\sigma \in \text{Aut}_D(V)$, for a $D$-lattice $V$, is a non-trivial automorphism that fixes a submodule $V' \subset V$ such that $V/V'$ is free of rank 1.

(1) A $D$-reflection group is a pair $(W, V)$, where $V$ is a $D$-lattice and $W < \text{Aut}_D(V)$ is a subgroup generated by reflections. We say $(W, V)$ is finite if $W$ is a finite group.

(2) A $D$-root datum is a triple $(W, V, \{P_\sigma\})$ where $(W, V)$ is a finite $D$-reflection group and $\{P_\sigma\}$ is a collection of rank one submodules of $V$, indexed by the set $\{\sigma\}$ of reflections in $W$, satisfying

(a) $\text{im}(1 - \sigma) \subset P_\sigma$ and
(b) $w(P_\sigma) = P_{w\sigma w^{-1}}$ for all $w \in W$.

An isomorphism of $D$-reflection groups $(W, V) \sim (W', V')$ is an isomorphism $\varphi : V \to V'$ with $W' = \varphi W \varphi^{-1}$. An isomorphism of $D$-root data $(W, V, \{P_\sigma\}) \sim (W', V', \{P'_\sigma\})$ is such an isomorphism $\varphi : V \to V'$ satisfying $\varphi(P_\sigma) = P'_{\varphi^{-1} \varphi \sigma}$ for each reflection $\sigma \in W$.

1.1.2. A $\mathbb{Z}$-reflection group $(W, V)$ yields a $D$-reflection group $D \otimes (W, V) := (W, D \otimes V)$ via extension of scalars. Similarly, for a root datum $(W, V, \{P_\sigma\})$ over $\mathbb{Z}$, we have the $D$-root datum $D \otimes (W, V, \{P_\sigma\}) := (W, D \otimes V, \{DP_\sigma\})$. 

Lemma 1.1.3. If there exists an embedding $D \hookrightarrow \mathbb{Q}$, then every $D$-root datum is the extension of scalars of a $\mathbb{Z}$-root datum.

Proof. Let $(W, V)$ be a $D$-reflection group. Starting from a $\mathbb{Z}$-lattice in $V$ and acting on it with $W$ shows there exists a finitely generated $\mathbb{Z}W$-submodule $V^0 \subset V$ with $D \otimes \mathbb{Z} V^0 \to V$ an isomorphism, see [CR, Corollary 23.14]. For a $D$-root datum $(W, V, \{P_\sigma\})$, we can then take the $\mathbb{Z}$-root datum $(W, V^0, \{P_\sigma^0\})$, with $P_\sigma^0 := P_\sigma \cap V^0$. \hfill $\square$

Remark 1.1.4.

(1) Lemma 1.1.3 does not imply that root data over $D \subset \mathbb{Q}$ are ‘the same’ as root data over $\mathbb{Z}$, see Example 1.4.2.

(2) The condition $D \subset \mathbb{Q}$ is necessary in Lemma 1.1.3, as one observes by considering dihedral groups as real reflection groups on $\mathbb{R}^2$, or the complex reflection groups generated by a root of unity acting by multiplication on $\mathbb{C}$.

We will use several times the following direct computation.

Lemma 1.1.5. For a $D$-lattice $V$, a fixed $\phi \in V^*$ and $\kappa_1, \kappa_2 \in V$ with $\phi(\kappa_1) = 2 = \phi(\kappa_2)$, we have the reflections $\kappa_i : \lambda \mapsto \lambda - \phi(\lambda) \kappa_i$ of order 2 on $V$. Then

$$(s_1 s_2)^j(\kappa_1) = \kappa_1 + 2j(\kappa_1 - \kappa_2) \quad \text{for all } j \in \mathbb{N}.$$ 

Lemma 1.1.6. Consider a prime $p$ and two $\mathbb{Z}$-root data $(W, V, \{P_\sigma\})$ and $(W', V', \{P'_\sigma\})$. Assume there exists an isomorphism $\varphi : \mathbb{Z}[1/p] \otimes (W, V) \to \mathbb{Z}[1/p] \otimes (W', V')$ of $\mathbb{Z}[1/p]$-reflection groups.

(1) If $p = 2$, then $\varphi$ is actually an isomorphism of $\mathbb{Z}[1/p]$-root data.

(2) If $p > 2$ and the further extension along $\mathbb{Z}[1/p] \to \mathbb{Z}_2 = \lim \mathbb{Z}/2^n$ of $\varphi$ induces an isomorphism of $\mathbb{Z}_2$-root data, then $\varphi$ is an isomorphism of $\mathbb{Z}[1/p]$-root data.

Proof. It is well known and easy to show that for $\mathbb{Z}$-root data, we either have

$$P_\sigma = \text{im}(1 - \sigma) \quad \text{or} \quad \text{im}(1 - \sigma) = 2P_\sigma.$$ 

(1.1)

Hence the additional condition in the definition of an isomorphism of $\mathbb{Z}[1/2]$-root data is trivially satisfied.

Now assume $p > 2$. In this case, (1.1) shows that in $\mathbb{Z}[1/p] \otimes V'$ we have either $\varphi(P_\sigma) = P'_\varphi \varphi^{-1}$, $\varphi(P'_\sigma) = 2P'_\varphi \varphi^{-1}$, or $2\varphi(P'_\sigma) = P'_\varphi \varphi^{-1}$. By assumption, after extension to scalars to $\mathbb{Z}_2$, only the first option is possible. As 2 is not invertible in $\mathbb{Z}_2$, this means that also over $\mathbb{Z}[1/p]$ only the first option was possible. \hfill $\square$

1.2. Real-type root data. The following definition is closer to the classical definition of (reduced) root data.

Definition 1.2.1. A real-type $D$-root datum is a quadruple $(X, R, Y, Y')$, where $X$ and $Y$ are $D$-lattices with subsets $R \subset X$ and $R' \subset Y$, together with

(a) a perfect bilinear pairing $(\cdot, \cdot) : X \times Y \to D$;

(b) a bijection $R \to R'$, $\beta \mapsto \beta'$;

such that:

(1) We have $\langle \alpha, \alpha' \rangle = 2$ for all $\alpha \in R$.

(2) If $\alpha' \in R'$ and $a \in D$, then $a\alpha' \in R'$ if and only if $a \in D^\times$.

(3) There are only finitely many $D^\times$ orbits in $R'$.

(4) For each $\alpha \in R$, the reflection $s_\alpha : \lambda \mapsto \lambda - (\alpha, \lambda)\alpha' \in \text{Aut}_D(Y)$ preserves $R'$. 

(4') For each $\alpha \in R$, the reflection $s_\alpha : \lambda \mapsto \lambda - (\lambda, \alpha')\alpha \in \text{Aut}_D(X)$ preserves $R$.

To a real-type root datum $(X, R, Y, R')$ over $\mathbb{Z}$ we can define a real-type $D$-root datum $(D \otimes X, D^\times R, D \otimes Y, D^\times R')$, with obvious bilinear pairing and bijection $D^\times R \to D^\times R'$ given by $a\lambda \mapsto a^{-1}\lambda'$. 


Remark 1.2.2. For $D = \mathbb{Z}$, Definition 1.2.1 is equivalent to the definition of a ‘donnée radicielle reduite’ in [De, §3.6]. We will show below in Lemma 1.2.5 that a real-type $D$-root datum $(X, R, Y, R^γ)$ also satisfies:

(2') If $α ∈ R$ and $a ∈ D$, then $aa ∈ R$ if and only if $a ∈ D^x$.

(3') There are only finitely many $D^x$ orbits in $R$.

In particular, the definition of real-type root datum is closed under duality.

Example 1.2.3. Consider a $D$-root datum $(W, V, \{P_α\})$. Take a reflection $σ ∈ W$ and a generator $b ∈ P_α$. By condition 1.1.1(2)(a), there exists (a unique) $β ∈ V^*$ such that

$$σ(λ) = λ - β(λ)b,$$

for all $λ ∈ V$. (1.2)

We then define $R^γ ⊂ V$ as the set of generators of the submodules $\{P_α\}$ and $R ⊂ V^*$ as the set of elements $β$ constructed by the above procedure. In particular, we have the defining map $R^γ → R$.

Theorem 1.2.4. Assume that for the $D$-root datum $(W, V, \{P_α\})$, every reflection in $W$ has order 2, then $R^γ → R$ in 1.2.3 is a bijection. The quadruple $(V^*, R, V, R^γ)$ equipped with inverse bijection $R → R^γ$ and evaluation pairing $V^* × V → D$, is a real-type $D$-root datum.

Proof. The assumption $σ^2 = 1$ in (1.2) shows that $β(b) = 2$. Assume now that for two generators $b_1 ∈ P_{σ_1}$ and $b_2 ∈ P_{σ_2}$, we obtain the same $β ∈ V^*$. By finiteness of $W$, Lemma 1.1.5 and the fact that $D$ is of characteristic 0 we find $b_1 = b_2$, hence $R^γ → R$ is indeed a bijection.

By the above paragraph, 1.2.1(1) is satisfied. To establish 1.2.1(2) it suffices to show that we cannot have non-trivial inclusions $P_{σ_1} ⊂ P_{σ_2}$. We can extend scalars along $D → K$, with $K$ the field of fractions. Now $σ_1σ_2$ acts as the identity on $KP_{σ_1} = KP_{σ_2}$, but also as the identity on $KV/KP_{σ_1}$. Since it has finite order and $\text{char} K = 0$, we find $σ_1σ_2 = 1$. Property 1.2.1(3) follows immediately from the fact that $W$ is finite. Property 1.2.1(4) is an immediate consequence of 1.1.1(2)(b).

By letting $w ∈ W$ act on $V^*$ by $w(f) = f ◦ w^{-1}$, we also have a reflection group $(W, V^*)$. We can define $Q_α := Dβ$, for each reflection $σ ∈ W$ with $β$ as in (1.2), since $Q_α$ does not depend on our choice of generator $b ∈ P_α$. It follows immediately that $(W, V^*, \{Q_α\})$ is a $D$-root datum. That 1.2.1(4') is satisfied follows by applying the proof for (4) to $(W, V^*, \{Q_α\})$. □

We conclude this section with some technical results needed later.

Lemma 1.2.5. Consider a real-type $D$-root datum $(X, R, Y, R^γ)$. For $β ∈ R$ and $b ∈ D$, we have $bβ ∈ R$ if and only if $b ∈ D^x$ and then $(bβ)^γ = b^{-1}β^γ$. In particular, $s_βb = s_β$ and conditions (2') and (3') in 1.2.2 hold.

Proof. Set $β_1 := bβ$ and $λ := bβ^γ - β^γ ∈ Y$. Lemma 1.1.5 for $φ = ⟨β, −⟩$ and $κ_1 = bβ^γ$, $κ_2 = β^γ$ implies

$$b(s_βb)^j(β^γ) = bβ^γ + 2jλ, \quad \text{for all } j ∈ \mathbb{N}.$$ Conditions 1.2.1(3) and (4) thus imply $λ = 0$. Finally 1.2.1(2) then implies $b$ is a unit. □

Lemma 1.2.6. Consider a real-type $D$-root datum $(X, R, Y, R^γ)$. If, for $α, β, γ ∈ R$, we have $s_β(α^γ) = γ^γ$, then $s_βs_αs_β = s_γ$ and $s_β(α) = γ$.

Proof. A direct calculation shows

$$s_βs_αs_β(λ) = λ - ⟨s_β(α), λ⟩s_β(α^γ).$$

Set $γ_1 = s_β(α) ∈ R$. Then we can apply Lemma 1.1.5 to $φ = ⟨γ_1, −⟩$ and $κ_1 = γ_1^γ, \quad κ_2 = γ^γ$ (so $s_1 = s_γ$ and $s_2 = s_βs_αs_β$). By 1.2.1(3) and (4), it thus follows that $γ_1 = γ$. □

1.3. Real reflection groups.
1.3.1. **Hypotheses.** Consider a finite-dimensional real vector space $V$ (without fixed inner product/Euclidean structure), a reflection group $W < \text{Aut}_\mathbb{R}(V)$, in the sense of Definition 1.1.1(1), and a fixed finite generating set $T$ of reflections in $W$ such that

- The map from $T$ to the set of hyperplanes in $V$, $s \mapsto H_s := \ker(1 - s)$, is injective.
- We have $wT^w^{-1} \subset T$ for all $w \in W$.

**Theorem 1.3.2.** Under the assumptions in 1.3.1, $W$ is a finite group

**Remark 1.3.3.** If a real reflection group $(W, V)$ is finite, by Weyl’s unitary trick, we can assume it is Euclidean (meaning there is an inner product on $V$ and each reflection in $W$ is orthogonal).

The remainder of this section is devoted to the proof.

1.3.4. All topological references consider the Euclidean topology on $V$. Consider

$$\mathcal{H} = \cup_{s \in T} H_s \subset V$$

and refer to the connected components of the complement of $\mathcal{H}$ in $V$ as chambers. Fix one such chamber $A_0$. Denote by $S \subset T$ the set of reflections $s$ for which the intersection of $A_0$ and $H_s$ cannot be contained in a codimension 2 hyperplane. Our assumptions in 1.3.1 imply that $W$ acts on the (finite) set of chambers.

**Lemma 1.3.5.** The set $S$ is a set of generators for $W$.

**Proof.** First we show that every $W$-orbit in $V$ intersects $A_0$. By continuity it suffices to show that every $W$-orbit in $V \setminus \mathcal{H}$ intersects $A_0$. For $v \in V \setminus \mathcal{H}$, there exists a sequence $A_0, A_1, \ldots, A_l$ of distinct chambers where $v \in A_l$, and for each $0 \leq i < l$ there is $t_i \in T$ such that the intersections of $H_{t_i}$ with $A_i$ and $A_{i+1}$ cannot be contained in a codimension 2 hyperplane.

If $l = 0$ there is nothing to prove, so assume $l > 0$. Then $t_0(A_1) = A_0$, and by assumption $t_0 \in S$. Now $A_0, A'_1 = t_0(A_2), A'_2 = t_0(A_3), \ldots, A'_{l-1} = t_0(A_l)$ forms a chain of distinct chambers as before and $t_0(v) \in A'_{l-1}$. We can thus perform induction on $l$ to deduce the claim.

The proof can now be concluded exactly as in [Bo, V, §3.1 Lemma 2(iii)]. Indeed, we have proved above the conclusion of Lemma 2(i) loc. cit. after which one can follow the proof using our assumption (trivial in the Euclidean context of [Bo]) that $H_t$ determines $t$.

Precisely as in [Bo, V, §3.2 Theorem 1], we then find the following consequence.

**Corollary 1.3.6.** The pair $(W, S)$ is a Coxeter system.

**Proof of Theorem 1.3.2.** The reflections in the Coxeter group $(W, S)$ are by definition the elements of the set $\cup_{w \in W} wS w^{-1}$, which by assumption is included in $T$ and hence finite. By [BB, Corollary 1.4.5] any Coxeter group $(W, S)$ with finitely many reflections is finite.

1.4. **Equivalence of definitions.**

**Theorem 1.4.1.** If there exists an embedding $D \hookrightarrow \mathbb{R}$, then the map in Theorem 1.2.4 is a bijection between the sets of isomorphism classes of $D$-root data and real-type $D$-root data.

**Proof.** Since we have $D \subset \mathbb{R}$, the only roots of unity in $D$ are $\pm 1$ and it follows that every reflection must have order two. Hence the map in Theorem 1.2.4 is defined on every $D$-root datum.

To each $D$-root datum $(X, R, Y, R^\vee)$ we will now associate a real-type $D$-root datum, in a way which is easily seen to be the inverse of the above map. Define the $D$-reflection group $W < \text{Aut}_D(Y)$ generated by $\{s_\alpha \mid \alpha \in R\}$. To a reflection $s_\gamma$, $\gamma \in R$, we associate the corresponding rank one submodule $D\gamma^\vee \subset Y$. 
We show that $W$ is finite by considering the corresponding real reflection group acting on $Y \otimes D \mathbb{R}$. By Lemma 1.2.5, $W$ is generated by a finite (see 1.2.1(2)) set of reflections \( \{s_\alpha | \alpha \in R\} \), such that the reflecting hyperplane \( \ker(1 - s_\alpha) = \ker(\alpha, -) \) determines \( s_\alpha \). Moreover, we claim that for each $\alpha \in R$ and $w \in W$, we have $ws_\alpha w^{-1} = s_\gamma$ for some $\gamma \in R$. Clearly it suffices to consider the case $w = s_\beta$, which is Lemma 1.2.6. We can now apply Theorem 1.3.2.

Now it follows that the triple $(W, Y, \{D\gamma^\vee\})$ is a $D$-root datum. Indeed, by Remark 1.3.3 and [Hu, Proposition 1.14], every reflection in $W$ is equal to $s_\gamma$ for some $\gamma \in R$, and property (a) in 1.1.1(2) is automatic, while (b) follows from Lemma 1.2.6.

Clearly, the bijection in Theorem 1.4.1 exchanges the two notions of extensions of scalars of root data. We conclude this section with some examples of root data which become isomorphic after extension of scalars.

**Example 1.4.2.**

1. The root datum of $SO_{2n+1}$ becomes isomorphic to its dual (the root datum of $SP_{2n}$) after extension of scalars to $D$ if and only if 2 is invertible in $D$.
2. The root datum of $SL_n$ becomes isomorphic to its dual (the root datum of $PGL_n$) after extension of scalars to $D$ if and only if $n$ is invertible in $D$.

**Remark 1.4.3.** We expect that our methods can be used to prove that for $D \subset \mathbb{C}$ (a relevant case would be $D = \mathbb{Z}_p$), there is a similar 1-to-1 correspondence between $D$-root data and the generalisation of real-type $D$-root data where 1.2.1(1) is replaced by the condition that $f_\alpha := 1 - \langle \alpha, \alpha^\vee \rangle$ is a non-trivial root of unity in $D$ with $f_\alpha = f_\alpha$, see for instance [LT, Definition 1.43].

1.5. **Isogenies.** We fix a prime $p$ and define isogenies of $\mathbb{Z}[1/p]$-root data. We will work with real-type root data, but refer to them simply as root data (this is justified by Theorem 1.4.1).

**Definition 1.5.1.** An isogeny $(X, R, Y, R^\vee) \rightarrow (X_1, R_1, Y_1, R_1^\vee)$ of $\mathbb{Z}[1/p]$-root data is an injective morphism $\varphi: X \rightarrow X_1$ of $\mathbb{Z}[1/p]$-modules such that

1. the induced $\varphi^\vee: Y_1 \rightarrow Y$ is also injective;
2. $\varphi$ restricts to a bijection $R \rightarrow R_1$;
3. $\varphi^\vee(\varphi(\alpha)^\vee) = \alpha^\vee$, for all $\alpha \in R$.

**Example 1.5.2.** An isogeny of $\mathbb{Z}$-root data, with respect to some prime $p$, is defined in [St, §1]. It follows immediately that the induction to $\mathbb{Z}[1/p]$ of such an isogeny yields an isogeny of $\mathbb{Z}[1/p]$-root data. Note that Definition 1.5.1 is simpler than the definition in [St, §1], as the powers of $p$ present in [St, §1] are subsumed by (2), since multiplication by $p$ is invertible on $\mathbb{Z}[1/p]$-root data.

Conversely, if for $\mathbb{Z}$-root data $RD_1$ and $RD_2$, there exists an isogeny $\varphi: \mathbb{Z}[1/p] \otimes RD_1 \rightarrow \mathbb{Z}[1/p] \otimes RD_2$, then for some $l \in \mathbb{N}$, the map $p^l \varphi$ restricts to an isogeny $RD_1 \rightarrow RD_2$ in the sense of [St] for all $j \geq l$.

2. **Perfection of schemes**

2.1. **Notation.** We recall some basic set-up of algebraic geometry, see for instance [DG].

Fix a field $k$. Denote by $\text{Alg}_k$ the category of commutative $k$-algebras. We consider the categories (where the first two ‘inclusions’ are fully-faithful embeddings)

$$\text{Alg}_k^{op} \subset \text{Sch}_k \subset \text{Fais}_k \subset \text{Fun}_k. \quad (2.1)$$

Here $\text{Sch}_k$ is the category of $k$-schemes and $\text{Fun}_k$ is the category of functors $\text{Alg}_k \rightarrow \text{Set}$. The category $\text{Fais}_k$ stands for the full subcategory of such functors which are sheaves for the fpqc topology. In other words, a functor $F$ is in $\text{Fais}_k$ if and only if

$$F(A) \rightarrow F(B) \Rightarrow F(B \otimes_A B)$$
is an equaliser for every faithfully flat $A$-algebra $B$, and $F$ commutes with finite products. When $k$ is clear, we will usually leave out the subscript in the above categories.

The inclusion $I : \text{Fais} \hookrightarrow \text{Fun}$ has a left adjoint

\[ S : \text{Fun} \to \text{Fais} \]

which commutes with finite limits (as well as all colimits).

By a subgroup of an affine group scheme $G$ we understand a closed subscheme which inherits a group structure, or in other words an affine group scheme represented by a quotient of the Hopf algebra representing $G$.

### 2.2. Perfection functors.

#### 2.2.1. Frobenius morphisms. For a commutative $\mathbb{F}_p$-algebra $A$, we have the $p$-th power algebra morphism $A \to A, a \mapsto a^p$. For an $\mathbb{F}_p$-scheme $\mathfrak{X}$, we have the morphism $\text{Fr} : \mathfrak{X} \to \mathfrak{X}$ which is the identity on the underlying topological space and given by the $p$-th power map on the sheaf of algebras. For $F \in \text{Fun}_{\mathbb{F}_p}$, we can also define $F \to F$ by evaluation at the $p$-th power morphism. These Frobenius morphisms are compatible with the inclusions (2.1).

For an object $F$ of $\text{Fun}_{\mathbb{F}_p}$ or $\text{Alg}_{\mathbb{F}_p}$, the notation $\lim F$ or $\lim F$ will always be used for the direct or inverse limit along the Frobenius morphism.

An $\mathbb{F}_p$-scheme (or an algebra or functor) is called perfect if the Frobenius map is an isomorphism, see [BS, Definition 3.1].

**Lemma 2.2.2.** The endofunctor of $\text{Fun}_{\mathbb{F}_p}$

\[ F \mapsto F_{\text{perf}} := \lim F \]

restricts to endofunctors of $\text{Fais}$ and $\text{Sch}$. Moreover, for $A \in \text{Alg}_{\mathbb{F}_p}$, we have $(\text{Spec} A)_{\text{perf}} = \text{Spec}(A_{\text{perf}})$ with $A_{\text{perf}} := \lim A$.

**Proof.** It is a standard property that limits exist in a Grothendieck topos and can be computed in the presheaf category, which shows that perfection restricts to $\text{Fais}$. The remaining properties follow from the explicit realisation in Example 2.2.3 below. \qed

**Example 2.2.3.** For an $\mathbb{F}_p$-algebra $A$, set $(X, \mathcal{O}) = \text{Spec} A$. Using the basis of distinguished open subsets it follows easily that $\text{Spec}(A_{\text{perf}}) = (X, \mathcal{O}')$, with $\mathcal{O}'$ the sheafification of the presheaf $U \mapsto \mathcal{O}(U)_{\text{perf}}$.

It then follows that for an arbitrary $\mathbb{F}_p$-scheme $\mathfrak{X} = (X, \mathcal{O})$, the scheme $\mathfrak{X}_{\text{perf}}$ can be realised as $(X, \mathcal{O}')$ with $\mathcal{O}'$ the sheafification of the presheaf $U \mapsto \mathcal{O}(U)_{\text{perf}}$.

**Remark 2.2.4.** In 2.2.3 is essential to take $\mathcal{O}'$ as the direct limit of $\mathcal{O}$ in the category of sheaves (as opposed to presheaves). For example:

1. For an infinite family of $\mathbb{F}_p$-algebras $A_i$, consider the (non-affine) scheme $(X, \mathcal{O}) = \sqcup_i \text{Spec} A_i$. Then, for general $A_i$, we have, by the sheaf axioms and Lemma 2.2.2,

\[ \Gamma(X, \mathcal{O}') = \prod_i (A_i)_{\text{perf}} \neq \Gamma(X, \mathcal{O})_{\text{perf}} = \left( \prod_i A_i \right)_{\text{perf}}. \]

2. Also for non-noetherian affine schemes this phenomenon occurs. Consider $A = \mathbb{F}_p[x_i | i \in \mathbb{N}]/(x_i x_j, i \neq j)$. Consider $U_i$ the distinguished open corresponding to $x_i$. The (disjoint) union $U = \sqcup_i U_i$ is the complement of the origin and as in (1) we find $\mathcal{O}(U)_{\text{perf}} \neq \mathcal{O}'(U)$.

**Remark 2.2.5.** For $\mathbb{F}_p$-algebras $A, B$ we have

\[ \text{Alg}(A_{\text{perf}}, B) \cong \lim_i \text{Alg}(A, B) \cong \text{Alg}(A, \lim_i B). \]

In particular, if $B$ is perfect, we have $\text{Alg}(A_{\text{perf}}, B) \cong \text{Alg}(A, B)$. 
Lemma 2.2.6. Let \( \mathcal{X} \) be an \( \mathbb{F}_p \)-scheme.

1. We have \( \dim \mathcal{X} = \dim \mathcal{X}_{\text{perf}} \) and \( \mathcal{X}_{\text{perf}} \) is quasi-compact (resp. connected) if and only if \( \mathcal{X} \) is quasi-compact (resp. connected).
2. Any radical ideal \( I \) in a perfect \( \mathbb{F}_p \)-algebra \( A \) satisfies \( I^2 = I \).
3. If \( \mathcal{X} \) is perfect, for \( x \in \mathcal{X} \) we have \( \mathcal{T}_{X,x} = 0 \).
4. Perfect schemes are reduced. Moreover, \( -\text{perf} \) sends \( \mathcal{X}_{\text{red}} \to \mathcal{X} \) to an isomorphism.

Proof. Part (1) follows immediately from the fact that the underlying topological spaces of \( \mathcal{X} \) and \( \mathcal{X}_{\text{perf}} \) are the same, see Example 2.2.3. Part (2) is obvious. By (2), it is clear that the Zariski cotangent space is zero, which proves (3). Alternatively, for (3), let \( A \) be a perfect \( \mathbb{F}_p \)-algebra and \( \kappa \) a field. Every algebra morphism \( A \to \kappa [e]/(e^2) \) factors through \( \kappa \to \kappa [e]/(e^2) \). Applying this to \( \text{Spec}(\kappa(x)[e]/(e^2)) \to \mathcal{X} \) shows the claim. Part (4) is immediate. \( \square \)

2.3. Relative version. Fix a perfect field \( k \) of characteristic \( p > 0 \).

2.3.1. For a fixed \( \mathbb{F}_p \)-scheme \( \mathcal{T} \), perfection naturally yields a functor from the category of \( \mathcal{T} \)-schemes to the category of \( \mathcal{T}_{\text{perf}} \)-schemes. Using the canonical morphism \( \mathcal{T}_{\text{perf}} \to \mathcal{T} \), we can also interpret perfection as an endofunctor of the category of \( \mathcal{T} \)-schemes. We will take the latter point of view for \( \mathcal{T} = \text{Spec} k \) (in which case \( \mathcal{T}_{\text{perf}} \to \mathcal{T} \) is an isomorphism) and we will henceforth interpret the perfection functor as an endofunctor of \( \text{Sch}_k \).

2.3.2. Sometimes it will be beneficial to consider an alternative realisation of the perfection of \( k \)-schemes, in which the morphisms in the chain of which we take the limit are morphisms of \( k \)-schemes. For a \( k \)-scheme \( \mathcal{X} \), let \( \mathcal{X}^{(1)} \) denote the extension of scalars of \( \mathcal{X} \) along the Frobenius automorphism \( k \to k \). The morphism \( F \mathcal{X} : \mathcal{X} \to \mathcal{X} \) over \( \mathbb{F}_p \) from 2.2.1 then lifts to a morphism \( F \mathcal{X} : \mathcal{X} \to \mathcal{X}^{(1)} \) of \( k \)-schemes. For instance, for a \( k \)-algebra \( A \) this corresponds to the morphism

\[
A^{(1)} \to A, \quad \lambda \otimes a \mapsto \lambda a^p,
\]

with \( A^{(1)} = k \otimes A \) the extension of scalars along the Frobenius automorphism of \( k \).

By taking iterates of the Frobenius automorphism and its inverse (\( k \) is perfect), we define \( \mathcal{X}^{(i)} \) for \( i \in \mathbb{Z} \). Then we have (over \( k \))

\[
\mathcal{X}_{\text{perf}} \cong \lim_{\to \infty} \mathcal{X}^{(-i)}.
\]

The advantage of the approach in this subsection is that it extends to \( \text{Fun}_k \), by setting

\[
F_{\text{perf}}(A) := \lim_{\to \infty} F(A^{(i)}).
\]

By construction, perfection commutes with limits, for instance products, in \( \text{Fun}_k \).

Proposition 2.3.3. Consider a morphism \( f \) in \( \text{Fais}_k \).

1. If \( f \) is an epimorphism in \( \text{Fais}_k \), then so is \( f_{\text{perf}} \).
2. If \( f \) is an monomorphism in \( \text{Fais}_k \), then so is \( f_{\text{perf}} \).

Proof. For part (1), we can use the criterion from [DG, Corollaire 2.8] to describe that \( f \) is an epimorphism, which carries over to \( f_{\text{perf}} \) by [BS, Lemma 3.4(xii)]. Part (2) is a generality for limits of monomorphisms.

\( \square \)

It is obvious that \( \mathcal{X} \to \mathcal{X}_{\text{perf}} \) loses a lot of information. For instance

\[
(\mathcal{X}_{\text{perf}})_{\text{perf}} \cong \mathcal{X}_{\text{perf}} \cong (\mathcal{X}_{\text{red}})_{\text{perf}}.
\]

A more subtle example is given below.

Example 2.3.4. Assume \( p > 2 \). Consider the algebra \( A = k[x,y]/(y^p - x^2) \) with injective algebra morphism \( A \hookrightarrow k[z] \), given by \( x \mapsto z^p, y \mapsto z^2 \). Then \( \mathcal{X} := \text{Spec} A \) is reduced, but perfection sends \( \mathbb{A}^1_k \to \mathcal{X} \) to an isomorphism.
2.4. Perfect finite type. Recall that \( X \in \text{Sch}_k \) is of finite type (over \( k \)) if the underlying topological space is quasi-compact and for every \( x \in X \) there exists an affine open neighbourhood isomorphic to the spectrum of a finitely generated \( k \)-algebra.

**Lemma 2.4.1.** For a perfect commutative \( k \)-algebra \( A \), the following conditions are equivalent:

(a) There is a finite subset \( S \subset A \) such that the set
\[
S' = \{ x \in A \mid x^{p^n} \in S \text{ for some } n \in \mathbb{N} \}
\]
generates \( A \) as a \( k \)-algebra;
(b) There exists a finitely generated \( k \)-algebra \( A_0 \) with \( A \cong (A_0)_{\text{perf}} \).

If the conditions are satisfied we say that \( A \) is **perfectly finitely generated**.

**Proof.** Exercise. \( \square \)

**Proposition 2.4.2.** For a perfect \( k \)-scheme \( X \), the following are equivalent:

(a) \( X \) is quasi-compact and for every \( x \in X \) there exists an affine open neighbourhood corresponding to a perfectly finitely generated \( k \)-algebra;
(b) There exists scheme \( Y \) of finite type over \( k \) with \( X \cong Y_{\text{perf}} \).

If the conditions are satisfied we say that \( X \) is of **perfect finite type** (over \( k \)).

**Proof.** Clearly (b) implies (a). That (a) implies (b) is proved in [BS, Proposition 3.13]. \( \square \)

2.5. Perfection of line bundle cohomology and quotients. All schemes and functors are assumed to be over \( k \).

**Lemma 2.5.1.** Let \( X \) be a quasi-compact separated scheme over \( k \) and \( L \) a line bundle on \( X \). For the pull-back \( p^*L \) along \( p : X_{\text{perf}} \to X \) and \( i \in \mathbb{N} \), we have
\[
H^i(X_{\text{perf}}, p^*L) \cong \lim_{\to} H^i(X, L^{\otimes p^j}),
\]
where the transition maps are induced from \( L^{\otimes p^j} \to L^{\otimes p^{j+1}} \), \( f \mapsto f^{\otimes p} \).

**Proof.** Since \( X \) is quasi-compact we can take a finite cover \( U \) by affine opens and, since \( X \) is separated, intersections of these opens are again affine. Moreover, the cohomology groups \( H^i(X, -) \) are canonically isomorphic to the Čech cohomology groups \( H^i(U, -) \). It follows that \( H^i(X, -) \) commutes with direct limits.

Now, for a line bundle, we have (as sheaves on the underlying topological space of \( X_{\text{perf}} \) or \( X \)) isomorphisms
\[
p^*L \cong (\lim_{\to} \mathcal{O}_X) \otimes_{\mathcal{O}_X} L \cong \lim_{\to} L^{\otimes p^j}.
\]

The conclusion follows from the combination of the two paragraphs. \( \square \)

2.5.2. For \( X \in \text{Fais} \) and \( G \) a group object in \( \text{Fais} \) acting on \( X \) (on the right), we consider the corresponding co-equalisers in \( \text{Fun} \) and \( \text{Fais} \) of \( X \times G \rightrightarrows X \), and denote them by \( X/0G \) and \( X/1G \). In particular \( X/1G = S(X/0G) \), for the sheafification functor \( S : \text{Fun} \to \text{Fais} \).

It follows from the definitions that \( G_{\text{perf}} \) is again a group object, and acts on \( X_{\text{perf}} \). We create the following commutative diagram in \( \text{Fun} \)
\[
\begin{array}{ccc}
X_{\text{perf}} & \rightarrow & X_{\text{perf}}/0G_{\text{perf}} \\
\downarrow & & \downarrow \\
X_{\text{perf}}/1G_{\text{perf}} & \rightarrow & (X/1G)_{\text{perf}}
\end{array}
\]

(2.3)

The vertical arrows are induced from the adjunction \( S \dashv I \) (either directly or via the action of the perfection functor). The left horizontal arrow is the defining one for the co-equaliser.
The remaining two arrows are uniquely defined from the coequaliser properties applied to the perfection of the morphisms $X/1G \leftarrow X \rightarrow X/0G$.

**Theorem 2.5.3.** Assume that the action of $G$ on $X$ is free, then the morphism from (2.3)

$$X_{\text{perf}}/1G_{\text{perf}} \rightarrow (X/1G)_{\text{perf}}$$

in $\text{Fais}_k$ is an isomorphism.

**Proof.** By Proposition 2.3.3(1), the composite morphism (from top left to bottom right) in (2.3) is an epimorphism in $\text{Fais}$. In particular the lower horizontal arrow is an epimorphism. Since isomorphisms in Grothendieck topoi are precisely morphisms which are both monomorphisms and epimorphisms, it now suffices to show this arrow is also a monomorphism.

Since sheafification sends monomorphisms to monomorphisms, it actually suffices to show that $X_{\text{perf}}/0G_{\text{perf}} \rightarrow (X/0G)_{\text{perf}}$ and $(X/0G)_{\text{perf}} \rightarrow (X/1G)_{\text{perf}}$ are monomorphisms.

Indeed, the first case follows from the fact that for an inverse system of sets $X_i$ with free actions of groups $G_i$,

$$\lim_{\leftarrow} X_i/\lim_{\leftarrow} G_i \rightarrow \lim_{\leftarrow} (X_i/G_i)$$

is injective. By Proposition 2.3.3(2), for the second morphism, it suffices to demonstrate that $X/0G \rightarrow X/1G$ is a monomorphism. This is equivalent to the claim that the presheaf $X/0G$ is separated for the fpqc topology, meaning that

$$X(A)/G(A) \rightarrow X(B)/G(B)$$

is an injection for every faithfully flat $A$-algebra $B$ (and the same for finite products of algebras). This is easily verified for free actions, see for instance [Ja, §I.5.5].

### 3. Perfection of group schemes

Let $k$ be a perfect field of characteristic $p > 0$.

3.1. **Perfection.** For an affine group scheme $G$ over $k$, clearly $G_{\text{perf}}$ is again an affine group scheme over $k$. Note also that, since $k$ is perfect, $G_{\text{red}}$ is a subgroup of $G$.

3.1.1. The group $p^\mathbb{Z}$. We denote by $p^\mathbb{Z} < \mathbb{Z}[1/p]^\times$ the group of powers of $p$, an infinite cyclic group on one generator. Let $G$ be an affine group scheme over $k$, which can be defined over $\mathbb{F}_p$. Then we can choose an isomorphism $\phi : G^{(1)} \cong G$, which yields an automorphism

$$\Phi = \phi_{\text{perf}} \circ \text{Fr} : G_{\text{perf}} \cong G^{(1)}_{\text{perf}} \cong G_{\text{perf}},$$

and a corresponding group homomorphism $p^\mathbb{Z} \rightarrow \text{Aut}(G_{\text{perf}}), p \mapsto \Phi$. Examples are given in 3.4.5.

**Lemma 3.1.2.** For a perfect group scheme $G$, we have $\text{Lie} G = 0$ and $\text{Dist} G = k$.

**Proof.** This is an immediate consequence of Lemma 2.2.6(2) and (3).

**Theorem 3.1.3.** Let $G$ be a perfect affine group scheme over $k$. The following are equivalent:

(a) The scheme $G$ is of perfect finite type, i.e. $k[G]$ is perfectly finitely generated;

(b) The group scheme $G$ is a subgroup of $GL(V)_{\text{perf}}$ for a finite-dimensional vector space $V$;

(c) There exists an affine group scheme $G$ of finite type with $G \cong G_{\text{perf}}$;

(d) There exists a reduced affine group scheme $G$ of finite type with $G \cong G_{\text{perf}}$. 


Proof. Clearly (d) implies (c). Any affine group scheme $G$ of finite type is a subgroup of some $GL(V)$, see e.g. [DM, Corollary 2.5]. It follows immediately that $G_{\text{perf}} < GL(V)_{\text{perf}}$, so (c) implies (b). That (b) implies (a) follows by the observation that $k[G]$ is a quotient of the perfectly finitely generated algebra $\lim k[GL(V)]$.

Finally, we prove that (a) implies (d). Let $S$ be a finite set which perfectly generates $k[G]$ as in Lemma 2.4.1(a). It is possible to replace $S$ by a finite set $S_1 \supseteq S$ which generates a Hopf subalgebra, see [Ab, Lemma 3.4.5]. Let $G$ be the corresponding affine group scheme. By construction $G_{\text{perf}} \cong G$ and, since $k[G]$ is a subalgebra of $k[G]$, it follows that $G$ is reduced. □

**Lemma 3.1.4.** Let $G$ be a perfect affine group scheme and $H$ an affine group scheme. Then $H_{\text{perf}} \rightarrow H$ induces an isomorphism

$$\text{Hom}(G, H_{\text{perf}}) \sim \text{Hom}(G, H),$$

with inverse given by perfection. Moreover, if $H$ is of finite type and $G$ an affine group scheme, then

$$\text{Hom}(G_{\text{perf}}, H_{\text{perf}}) \sim \text{lim} \text{Hom}(G^{(-i)}, H).$$

Proof. This is an immediate application of Remark 2.2.5, or the fact that the perfection functor on $\text{Fp}$-schemes is right adjoint to the inclusion functor for perfect schemes. □

3.1.5. For a subgroup $H$ of an affine group scheme $G$, we denote by $G/H$, when it exists, the equaliser of $G \times H \rightarrow G$ in $\text{Sch}_k$.

Recall from [DG, III, §3 Théorème 5.4] that for $G$ of finite type, the quotient $G/1H$ in $\text{Fais}_k$ is a scheme, of finite type over $k$, so in particular is equal to $G/H$.

**Theorem 3.1.6.** (1) For every perfect subgroup $H$ of an affine group scheme $G$ of perfect finite type, the quotient $G/H$ exists, is of perfect finite type and isomorphic to $G/1H$.

(2) For an affine group scheme $G$, every perfect subgroup of $G_{\text{perf}}$ is the perfection of a subgroup of $G$. More precisely, every perfect (normal) subgroup of $G_{\text{perf}}$ is the perfection of a reduced (normal) subgroup of $G_{\text{red}} < G$.

(3) For an affine group scheme $G$ of finite type with subgroup $H$, the quotient $G_{\text{perf}}/H_{\text{perf}}$ exists and is isomorphic to $(G/H)_{\text{perf}}$ and $G_{\text{perf}}/1H_{\text{perf}}$.

Proof. We will freely use the results from [DG] recalled above. Part (1) is then an immediate consequence of parts (2) and (3).

Now we prove part (2). Take a perfect subgroup $H < G_{\text{perf}}$. We have a commutative square, where $\rightarrow$ denotes the inclusion of a subgroup and $\rightarrow$ denotes a faithfully flat homomorphism

$$
\begin{array}{ccc}
H & \rightarrow & G_{\text{perf}} \\
\downarrow & & \downarrow \\
L & \rightarrow & G_{\text{red}},
\end{array}
$$

where $L$ is just defined to be the image of the composite diagonal homomorphism. By Lemma 3.1.4, perfecting the lower path in the square yields homomorphisms $H \rightarrow L_{\text{perf}} \rightarrow G_{\text{perf}}$ which compose to the original inclusion $H \rightarrow G_{\text{perf}}$. Clearly, $H \rightarrow L_{\text{perf}}$ must be an isomorphism. If $H$ is a normal subgroup, it follows easily that so is $L < G_{\text{red}}$.

Part (3) is an application of Theorem 2.5.3. □

**Remark 3.1.7.** A perfect group scheme can have non-perfect subgroups, for instance

$$\mu_{p^\infty} := \lim \mu_{p^i}, \quad \text{i.e. } k[\mu_{p^\infty}] = k[x^{1/p^\infty}]/(x - 1)$$

is a subgroup of $(\mathbb{G}_m)_{\text{perf}}$. Moreover, $(\mathbb{G}_m)_{\text{perf}}/\mu_{p^\infty} \cong \mathbb{G}_m$. 

For our purposes it is most convenient to define short exact sequences of affine group schemes as those sequences \( N \to G \to Q \) in which \( G \to Q \) is faithfully flat and \( N \) is the kernel of the latter morphism.

**Lemma 3.1.8.** The perfection functor acting on a short exact sequence of affine group schemes
\[
1 \to N \to G \to Q \to 1,
\]
yields a short exact sequence
\[
1 \to N_{\text{perf}} \to G_{\text{perf}} \to Q_{\text{perf}} \to 1.
\]

**Proof.** Faithful flatness is preserved by perfection, see [BS, Lemma 3.4]. Taking inverse limits of affine group schemes always respects kernels, alternatively we can apply Remark 2.2.5. □

**Corollary 3.1.9.** A reduced affine group scheme \( G \) is solvable if and only if \( G_{\text{perf}} \) is solvable.

**Proof.** Lemma 3.1.8 shows that for any solvable affine group scheme, its perfection is again solvable. On the other hand, assume that \( G_{\text{perf}} \) is solvable and \( G \) reduced. Applying iteratively Theorem 3.1.6(2) allows us to construct a finite chain of reduced normal subgroups such that the perfection of the quotients are abelian. A reduced affine group scheme with abelian perfection is clearly abelian itself. □

### 3.2. Isomorphic perfections

We gather some examples and results about affine group schemes with isomorphic perfections.

**Example 3.2.1.**
1. Let \( G \) be a finite group scheme (i.e. \( k[G] \) is finite dimensional). We have \( G_{\text{perf}} \cong G_{\text{red}} \), so in particular, \( G \) is infinitesimal if and only if \( G_{\text{perf}} \) is trivial.
2. Reduced affine group schemes can also become isomorphic after perfection. For instance, if \( q \) is a power of \( p \), then \( (SL_q)_{\text{perf}} \cong (PGL_q)_{\text{perf}} \). This is an example of Lemma 3.2.2 below, or follows from 4.2.3 below and Example 1.4.2. Moreover, the latter results also show that, conversely, \( (SL_n)_{\text{perf}} \cong (PGL_n)_{\text{perf}} \) implies that \( n \) must be a power of \( p \).
3. As follows from Remark 2.2.5, a necessary condition for affine group schemes \( G,H \) to have isomorphic perfections is that there exists an isomorphism \( G(k) \cong H(k) \) as abstract groups.

Recall that an **isogeny** is a faithfully flat homomorphism of affine group schemes with finite kernel \( N \). An isogeny is **infinitesimal** if \( N \) is infinitesimal. An isogeny \( G \to Q \) between reduced and connected affine group schemes is **purely inseparable** if the induced morphism \( k(Q) \to k(G) \) between the fields of fractions is a purely inseparable field extension.

**Lemma 3.2.2.** For an isogeny \( q : G \to Q \), the following conditions are equivalent:

(a) \( q \) is infinitesimal;
(b) The perfection of \( q \) is an isomorphism.

Moreover, assuming that \( G,Q \) are reduced and connected, the above properties imply

(c) \( q \) is purely inseparable.

**Proof.** The equivalence between (a) and (b) is an immediate application of Example 3.2.1(1) and Lemma 3.1.8.

Condition (b) implies that for every \( a \in k[G] \), there is \( i \in \mathbb{N} \) such that \( a^p^i \) is in the image of \( k(Q) \to k(G) \), from which (c) follows immediately. □

**Remark 3.2.3.** If furthermore \( G \) is of finite type, 3.2.2(c) implies (a) and (b).

Indeed, this follows from the equality \( k[Q] = k(Q) \cap k[G] \) inside \( k(G) \). The latter can be observed as follows. For \( f \in k(Q) \cap k[G] \) consider the span \( S \) of \( \{ g(f) | g \in G(k) \} \). Since \( f \in k[G] \), \( S \) is finite dimensional. Clearly the action of \( G(k) \) on \( f \) factors through the
canonical action of $Q(k)$ and by our finite type assumption we have $G(k) \to Q(k)$. Hence the elements $h \in k[Q]$ for which $hS \subseteq k[Q]$ form a non-zero $Q(k)$-invariant ideal $I \subseteq k[Q]$ and hence $I = k[Q]$. So $f \in S \subseteq k[Q]$.

An alternative argument considers the purely inseparable isogeny $G \to G/(\ker q)^0$, which is also étale and therefore an isomorphism.

**Proposition 3.2.4.** The following conditions are equivalent on two reduced affine group schemes $G, H$ of finite type.

(a) $G_{\text{perf}} \cong H_{\text{perf}}$;
(b) There exists $j \in \mathbb{N}$ and an infinitesimal isogeny $G \to H^{(j)}$;
(c) There are $i, j \in \mathbb{N}$ and homomorphisms $\phi : G \to H^{(j)}$, $\psi : H \to G^{(i)}$ for which the following triangles are commutative

\[
\begin{array}{ccc}
G & \xrightarrow{\phi} & H^{(j)} \\
\downarrow{\text{Fr}^{i+j}} & & \downarrow{\psi^{(j)}} \\
G^{(i+j)} & \xrightarrow{\phi^{(i)}} & H^{(i+j)}
\end{array}
\]

**Proof.** That (b) implies (a) is a special case of Lemma 3.2.2.

Now we prove that (c) implies (b). The kernel of the Frobenius homomorphism is infinitesimal, hence the left diagram shows that the kernel of $\phi$ is infinitesimal. Since we assume that $G$ and $H$ are reduced, the Frobenius homomorphism is faithfully flat. The right diagram thus proves that $\phi$ is faithfully flat.

Applying Lemma 3.1.4 to the isomorphism in (a) yields morphisms $\phi : G^{(-j)} \to H$ and $\psi : H^{(-i)} \to G$. Expressing that these induce mutually inverse homomorphisms on the perfected groups then states that there exists $l \in \mathbb{N}$ such that the composition

\[
H^{(-i-j+l)} \xrightarrow{\text{Fr}^l} H^{(-i-j)} \xrightarrow{\psi^{(-j)}} G^{(-j)} \xrightarrow{\phi} H
\]

is $\text{Fr}^{i+j+l}$. Since $\text{Fr}^l$ is faithfully flat, we arrive precisely at the conditions in (c), so (a) implies (c). \qed

3.3. **Tannakian point of view.** For an affine group scheme $G$ we denote by $\text{Rep}G$ its category of (rational) representations which are finite dimensional over $k$. Its category of all representations will be denoted by $\text{Rep}G^\infty \cong \text{IndRep}G$.

3.3.1. Let us interpret $G_{\text{perf}}$ as $\varprojlim G^{(-i)}$ as in 2.3.2. Clearly

\[
\text{Rep}(G_{\text{perf}}) \cong \varprojlim \text{Rep}G^{(-i)},
\]

where the $k$-linear (exact) tensor functors in the chain are given by the pullback along $G^{(-i-1)} \to G^{(-i)}$. These functors fit into commutative diagrams:

\[
\begin{array}{ccc}
\text{Rep}G & \xrightarrow{(V \to V \otimes k[G])} & \text{Rep}G^{(-1)} \\
\downarrow{\sim} & & \downarrow{(V \to V \otimes k[G]^{(-1)})} \\
\text{Rep}G & & \text{Rep}G^{(-1)}
\end{array}
\]

(3.2)

The non-horizontal arrows are only $k$-linear up to twist. By definition, $V \to V \otimes k[G]^{(-1)}$ comes from $k[G] \to k[G]^{(-1)}$ in (2.2). The downwards arrow is given by applying $-(1)$ to both vector space and co-action. Note that we can equivalently realise the $G$-representation $V^{(1)}$ from the bottom right in (3.2) as the subquotient of $\otimes^p V$ given by the image of $\Gamma^p V \to S^p V$. This gives a more palatable definition of the Frobenius twist from the Tannakian point of view.
Diagram (3.2) allows us to realise \( \text{Rep}_{G_{\text{perf}}} \) alternatively as
\[
\text{Rep}(G_{\text{perf}}) \cong \lim_{\leftarrow} \text{Rep}G,
\]
in the spirit of interpretation 2.3.1. Compared to (3.1) we no longer have to work with twists of \( G \), but we do have the drawback that the defining functors are not \( \mathbb{k} \)-linear.

3.3.2. Notation. For \( M \in \text{Rep}G \), we denote by \( M[i] \) the object of \( \lim_{\leftarrow} \text{Rep}G \) where \( M \) is placed in the \( i \)-th copy of \( \text{Rep}G \) in the chain, and use the same notation for the corresponding object in \( \text{Rep}(G_{\text{perf}}) \), via (3.3). Note that every object in \( \text{Rep}(G_{\text{perf}}) \) is of this form and furthermore \( M[i] \cong M^{(1)}[i+1] \).

**Lemma 3.3.3.** [CEO, Remark 6.5] Let \( G \) be an affine group scheme over \( k \).

1. \( G \) is reduced if and only if \( -(1) : \text{Rep}G \rightarrow \text{Rep}G \) is full.

2. \( G \) is perfect if and only if \( -(1) : \text{Rep}G \rightarrow \text{Rep}G \) is an equivalence.

**Proof.** If \( G \) is reduced, the \( p \)-th power map is injective on \( k[G] \) and the fullness in (1) follows.

On the other hand, if \( -(1) \) is not full then (by applying adjunction) there is a \( G \)-representation \( V \) with a vector \( v \in V \) which is not \( G \)-invariant, but for which \( 1 \otimes v \in V^{(1)} \) is \( G \)-invariant. Looking at the \( k[G] \) coaction then provides a non-zero \( f \in k[G] \) with \( f^p = 0 \).

Via diagram (3.2), the functor is an equivalence if and only if \( G^{(-1)} \rightarrow G \) is an isomorphism, which is equivalent to \( G \) being perfect. \( \square \)

**Remark 3.3.4.** As for any direct limit of abelian categories, for objects \( M[i], N[i] \in \text{Rep}(G_{\text{perf}}) \), using notation from 3.3.2, we have
\[
\lim_{\leftarrow} \text{Ext}^l_G(M^{(j)}, N^{(j)}) \cong \text{Ext}^l_{G_{\text{perf}}}(M[i], N[i]).
\]

3.4. **Additive, multiplicative and unipotent groups.** For convenience we let \( k \) be algebraically closed in this section.

3.4.1. To lighten expressions, we introduce the following notation
\[
G_a := (G_a)_{\text{perf}} \quad \text{and} \quad G_m := (G_m)_{\text{perf}}
\]
for the perfection of the additive and multiplicative group of \( k \).

**Proposition 3.4.2.** Assume that \( k \) is algebraically closed.

1. Let \( G \) be a connected affine group scheme of perfect finite type and of dimension 1, then either \( G \cong G_a \) or \( G \cong G_m \).

2. Let \( G \) be a reduced affine group scheme of finite type with \( G_{\text{perf}} \cong G_a \) (resp. \( G_{\text{perf}} \cong G_m \)), then \( G \cong G_a \) (resp. \( G \cong G_m \)).

**Proof.** By [Sp, Theorem 3.4.9], for any connected reduced affine group scheme \( G \) of finite type and of dimension 1, we must have \( G \cong G_a \) or \( G \cong G_m \). This implies part (2), by Lemma 2.2.6(1). Part (1) follows similarly, using characterisation 3.1.3(d).

By a torus we mean an affine group scheme isomorphic to \( G_m^\times \) for some \( n \in \mathbb{N} \). Similarly, **perfect tori** are the affine group schemes isomorphic to \( G_m^\times \) for some \( n \in \mathbb{N} \).

**Lemma 3.4.3.** For a connected reduced affine group scheme \( G \) of finite type, the following are equivalent:

(a) \( G \) is a torus;

(b) \( G_{\text{perf}} \) is a perfect torus.

**Proof.** Clearly (a) implies (b). On the other hand, (b) implies that \( G \) is connected and, since
\[
\text{Rep}G \rightarrow \text{Rep}G_{\text{perf}}
\]
is exact and fully faithful, see Lemma 3.3.3, it follows that \( \text{Rep} G \) is semisimple and pointed (every simple representation has dimension one). That \( G \) is a torus then follows from \([\text{Sp}, 3.2.3 \text{ and } 3.2.7(ii)]\). \( \square \)

Recall that an affine group scheme \( G \) is \textbf{unipotent} if and only if every representation has an invariant vector (equivalently every simple object in \( \text{Rep} G \) is trivial).

**Lemma 3.4.4.** For an affine group scheme \( G \) over \( k \), \( G_{\text{perf}} \) is unipotent if and only if \( G_{\text{red}} \) is unipotent.

**Proof.** By equivalence (3.1), if \( G \) (or \( G_{\text{red}} \)) is unipotent, then so is \( G_{\text{perf}} \). On the other hand, if \( G_{\text{perf}} \) is unipotent, then the fully faithful exact functor from \( \text{Rep} G_{\text{red}} \) to \( \text{Rep} G_{\text{perf}} \) shows that also \( G_{\text{red}} \) is unipotent. \( \square \)

3.4.5. We have a ring isomorphism

\[
\mathbb{Z}[1/p] \cong \text{End}(\mathbb{G}_m), \quad a \mapsto \{\lambda \mapsto \lambda^n\},
\]

(3.4)

where \( \lambda \) stands for an element of \( \mathbb{G}_m(A) = \lim \leftarrow A^\times \) for a commutative \( k \)-algebra \( A \). The restriction to \( p\mathbb{Z} \hra \text{Aut}(\mathbb{G}_m) \) is the homomorphism from 3.1.1.

For \( \mathbb{G}_a \), the latter homomorphism extends to an isomorphism

\[
k^\times \times p\mathbb{Z} \cong \text{Aut}(\mathbb{G}_a), \quad (\kappa, n) \mapsto \{\theta^n : \lambda \mapsto \kappa \lambda^n\}.
\]

(3.5)

3.4.6. For a perfect affine group scheme \( G \), Lemma 3.1.4 shows that characters of \( G \) correspond to homomorphisms \( G \to \mathbb{G}_m \). As the latter formulation carries more structure, we define

\[
X(G) := \text{Hom}(G, \mathbb{G}_m) \cong \text{Hom}(G, \mathbb{G}_m) = X(G).
\]

This is a \( \mathbb{Z}[1/p] \)-module via (3.4). Consequently, we will define cocharacters of \( G \) to be the \( \mathbb{Z}[1/p] \)-module

\[
Y(G) := \text{Hom}(\mathbb{G}_m, G).
\]

We have the obvious bilinear pairing

\[
X(G) \times Y(G) \to \mathbb{Z}[1/p].
\]

(3.6)

It is non-degenerate if \( G \) is a perfect torus.

3.5. **Induction.**

3.5.1. For a homomorphism \( f : H \to G \) of affine group schemes, we will sometimes abbreviate \( f_* := \text{Ind}^G_H \) and \( f^* := \text{Res}^G_H \). This gives an adjoint pair \( f^* \dashv f_* \) of functors between \( \text{Rep}^\infty G \) and \( \text{Rep}^\infty H \).

3.5.2. For a commutative square of group homomorphisms

\[
\begin{array}{ccc}
H & \xrightarrow{f} & G \\
\downarrow a & & \downarrow b \\
A & \xrightarrow{g} & B,
\end{array}
\]

the adjunction morphisms yield a natural transformation

\[
\xi : b^* g_* \Rightarrow f_* a^*.
\]

In particular, for \( M \in \text{Rep}^\infty A \), the morphism \( \xi_M \) is zero if and only if the composite

\[
g^* g_* M \to M \to a_* a^* M
\]

is zero.
3.5.3. Now we consider two affine group schemes of the form \( A = \lim A_i \) and \( B = \lim B_i \) for inverse systems of affine group schemes \( (A_i | i \in \mathbb{N}) \) and \( (B_i | i \in \mathbb{N}) \). We label the homomorphisms \( p_i : A \to A_i, q_i : B \to B_i \) and \( a_{i,j} : A_i \to A_j \) for \( i > j \). Assume also given homomorphisms \( A_i \to B_i \), leading to \( A \to B \).

Proposition 3.5.4.  
(1) For \( M_i \in \text{Rep}^\infty A_i \) and morphisms \( a_{i+1,i}^* M_i \to M_{i+1} \), the evaluations at \( M_i \) of the natural transformations in 3.5.2 lead to an isomorphism
\[
\varinjlim q_i^* \text{Ind}_A^{B_i} M_i \cong \text{Ind}_A^{B_i} \varinjlim p_i^* M_i.
\]

(2) For \( M \in \text{Rep}^\infty A_i \) and \( n \in \mathbb{N} \), we have a canonical isomorphism
\[
\varinjlim q_i^* (R^n \text{Ind}_{A_i}^{B_i} (a_{i,j}^* M)) \cong R^n \text{Ind}_{A_i}^{B_i} (p_i^* M).
\]

Proof. The right-hand side in part (1) is given by the \( A \)-invariants in the vector space
\[
\varinjlim (M_i \otimes k[B_i]).
\]
Since direct limits commute with co-equalisers, this is isomorphic to the direct limit of \( A_i \)-invariants in the above spaces. This proves part (1).

Note that the case \( n = 0 \) in part (2) is the special case of part (1) where we set \( M_i := a_{i,j}^* M \). We can prove the analogue of part (1) for derived functors, which similarly specialises to part (2), as follows. For a chain \( \{M_i\} \) as in part (1), we consider injective hulls in \( \text{Rep}^\infty A_i \), yielding short exact sequences
\[
0 \to M_i \to I_i \to Q_i \to 0.
\]
The defining property of injective modules gives a chain map from the action of \( a_{i+1,i}^* \) on the above sequence and the corresponding sequence for \( i + 1 \). In particular, this makes \( \{I_i\} \) and \( \{Q_i\} \) into chains of representations as in part (1), and we have a short exact sequence
\[
0 \to \varinjlim p_i^* M_i \to \varinjlim p_i^* I_i \to \varinjlim p_i^* Q_i \to 0
\]
in \( \text{Rep}^\infty A \). The claim now follows by induction on \( n \), using long exact sequences in homology if we observe that \( I := \varinjlim p_i^* I_i \) is injective in \( \text{Rep}^\infty A \). Exactness of
\[
\text{Hom}_A(-, I) : \text{Rep} A \to \text{Vec}^\infty
\]
follows from the observation \( \text{Rep} A \cong \varinjlim \text{Rep} A_i \). The latter exactness is sufficient to conclude that \( I \) is injective. Indeed, we can consider an injective hull \( I \subset I' \) and an intermediate module \( I \subset I'' \subset I' \) for which \( I''/I \) is finite dimensional. Now we must have \( I'' \cong I \oplus I''/I \) (apply \( \text{Hom}(-, I) \) to a finite submodule of \( I'' \) which still surjects onto \( I''/I \)) which violates \( \text{soc}I = \text{soc}I'' \) unless \( I'' = I \).

Corollary 3.5.5. Let \( G \) be an affine group scheme with subgroup \( H \), set \( G = H_{\text{perf}} \), \( H = H_{\text{perf}} \) and take \( n \in \mathbb{N} \). For \( M \in \text{Rep} H \) such that \( R^n \text{Ind}_H^G (M[i]) \) is finite dimensional for each \( i \in \mathbb{N} \). For each \( j \in \mathbb{N} \) we have an isomorphism
\[
\varinjlim_{i \geq j} (R^n \text{Ind}_H^G (M^{(i-j)})[i]) \sim R^n \text{Ind}_H^G (M[j]),
\]
with notation as in 3.3.2.

Proof. We start by applying Proposition 3.5.4(2), using the interpretation \( G_{\text{perf}} = \varinjlim G^{(-i)} \), applied to \( M^{(-j)} \in \text{Rep} H^{(-j)} \). By assumption, all the representations appearing in the direct limit in 3.5.4(2) are finite dimensional. After passing from (3.1) to (3.3) this allows us to use the notation from 3.3.2 to rewrite the isomorphism in the desired form.

Remark 3.5.6.  
(1) For group schemes of finite type, we can prove Corollary 3.5.5 alternatively using Theorem 3.1.6(3) and (generalisation from line bundles to general quasi-coherent sheaves with identical proof of) Lemma 2.5.1.

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(2) Assume that $M$ is one-dimensional, and $\text{Ind}_H^G(M^{(j)}) \neq 0$ for all $j$. It follows from 3.5.2 that the morphisms in the directed system for the left-hand side in Corollary 3.5.5 for $n = 0$ are all non-zero.

4. Perfectly reductive groups

In this section we assume that $k$ is algebraically closed of characteristic $p > 0$.

4.1. Definition. Recall that a reductive group over $k$ is a connected reduced (smooth) affine group scheme of finite type which has no non-trivial normal reduced unipotent subgroup.

Theorem 4.1.1. (1) For an affine group scheme $G$, the following are equivalent:

(a) $G$ is a connected affine group scheme of perfect finite type and has no non-trivial perfect normal unipotent subgroups;

(b) $G \cong G_{\text{perf}}$ for a reductive group $G$;

(c) $G$ is an affine group scheme of perfect finite type and, if $G \cong H_{\text{perf}}$ for a reduced affine group scheme of finite type $H$, then $H$ is reductive.

If these conditions are satisfied, we call $G$ perfectly reductive.

(2) For every connected affine group scheme of perfect finite type $H$, there exists a short exact sequence of perfect affine group schemes

$$1 \to U \to H \to Q \to 1,$$

where $U$ is unipotent and $Q$ is perfectly reductive.

Proof. First we show that 1(b) implies 1(a). Set $G = G_{\text{perf}}$ for a reductive group $G$. Let $U \triangleleft G$ be a perfect normal unipotent subgroup. Then by Theorem 3.1.6(2), there exists a reduced normal subgroup $U \triangleleft G$ with $U_{\text{perf}} = U$. By Lemma 3.4.4, $U$ is unipotent, so $U$ is trivial. Consequently $U$ is trivial.

That 1(c) implies 1(b) follows from Theorem 3.1.3.

For $H$ as in (2), we know that $H = H_{\text{perf}}$ for a connected reduced affine group scheme of finite type $H$ by Theorem 3.1.3. By taking the perfection of the short exact sequence corresponding to the unipotent radical $R_uH \triangleleft H$, see [Mi, §6.4.6], we get a short exact sequence as desired in (2) (provided we define, for now, perfectly reductive groups as the perfections of reductive groups), with $U := (R_uH)_{\text{perf}}$, by Lemma 3.1.8.

Since $R_uH$ is reduced, $U$ is trivial if and only if $R_uH$ is trivial, which shows that 1(a) implies 1(c).

Remark 4.1.2. In addition to Theorem 4.1.1(2), we can also observe that every affine group scheme of perfect finite type $G$ admits a short exact sequence $H \to G \to Q$ where $Q$ is a finite abstract group and $H$ is a connected affine group scheme of perfect finite type. This follows from perfecting the classical theory, see [Mi, §2.g].

4.1.3. A perfect Borel subgroup $B$ of a perfectly reductive group is a maximal solvable perfect subgroup. For a reductive group $G$, every perfect Borel subgroup of $G_{\text{perf}}$ is the perfection of a Borel subgroup of $G$ by Theorem 3.1.6(2) and Corollary 3.1.9. In particular, every two perfect Borel subgroups are conjugate.

Similarly, by Theorem 3.1.6(2) and Lemma 3.4.3, every maximal perfect torus $T$ in $G_{\text{perf}}$ is the perfection of a maximal torus $T \triangleleft G$, and so maximal perfect tori are unique up to conjugation.

We henceforth use freely that every such choice $T < B < G_{\text{perf}}$, for a reductive group $G$, can be obtained as the perfection of a corresponding choice $T < B < G$ of Borel subgroup and maximal torus in $G$. 


4.2. **Classification.** We freely use the equivalence between $D$-root data and real-type $D$-root data for $D = \mathbb{Z}$ and $D = \mathbb{Z}[1/p]$ from Theorem 1.4.1.

**Theorem 4.2.1.** There is a canonical bijection between the set of isomorphism classes of perfectly reductive groups over $k$ and the set of isomorphism classes of $\mathbb{Z}[1/p]$-root data.

**Remark 4.2.2.** As in the classical case, this theorem can be extended to cover isogenies. We do this in 4.3 below.

4.2.3. **Idea of the proof.** We will prove that, when characterising a reductive group in terms of its root datum, two reductive groups become isomorphic after perfection if and only if their root data become isomorphic after extension of scalars to $\mathbb{Z}[1/p]$. More explicitly:

Denote by $D$-$RD$ the set of isomorphism classes of $D$-root data. Furthermore, we let $\mathcal{R}e\mathcal{G}r$, resp. $\mathcal{P}e\mathcal{R}e\mathcal{G}r$, denote the set of isomorphism classes of reductive groups, resp. perfectly reductive groups, over $k$. We can exploit the classical bijection between $\mathbb{Z}$-$RD$ and $\mathcal{R}e\mathcal{G}r$, see for instance [De], and include it in the following (commutative) diagram:

$$
\begin{array}{ccc}
\mathcal{R}e\mathcal{G}r & \overset{-\text{perf}}{\longrightarrow} & \mathcal{P}e\mathcal{R}e\mathcal{G}r \\
\downarrow & & \downarrow \\
\mathbb{Z}$-$RD & \longrightarrow & \mathbb{Z}[1/p]\otimes\mathcal{R}D.
\end{array}
$$

The upper surjection is given by the definition in 4.1.1(1)(b) of perfectly reductive groups. The lower surjection comes from Lemma 1.1.3. To prove Theorem 4.2.1, it suffices to show the dashed arrows in (4.1) exist. This is established in the following two propositions.

**Proposition 4.2.4.** If the root data of two reductive groups $G_1, G_2$ extend to isomorphic root data over $\mathbb{Z}[1/p]$, then $(G_1)_{\text{perf}}$ and $(G_2)_{\text{perf}}$ are isomorphic. In particular, the upwards dashed arrow in (4.1) exists.

**Proof.** Let $(X_i, R_i, Y_i, R'_i)$, with $i \in \{1, 2\}$, denote the root datum of $G_i$. Consider an isomorphism $\psi : Z[1/p] \otimes X_1 \cong Z[1/p] \otimes X_2$. Replacing $\psi$ by $p^j \psi$ if necessary (as is allowed by Lemma 1.2.5), we can assume that $\psi$ restricts to an embedding $X_1 \hookrightarrow X_2$. Furthermore, we get a bijection $R_1 \rightarrow R_2$, by associating to $\alpha \in R_1$ the unique element $\alpha' \in R_2$ for which $\psi(\alpha) = p^j \alpha'$ for some $j \in \mathbb{Z}$. Again, after replacing $\psi$ by $p^j \psi$ if necessary, we can assume $j \in \mathbb{N}$. Now it follows quickly that this $X_1 \hookrightarrow X_2$ satisfies the requirements to apply [St, Theorem 1.5], which yields an isogeny $G_1 \rightarrow G_2$.

We can apply the same procedure to $\psi^{-1}$ to obtain an isogeny $G_2 \rightarrow G_1$. As we might need to replace $\psi^{-1}$ again by a composition with multiplication by a power of $p$, our two isogenies will not necessarily be induced by mutually inverse maps $X_1 \hookrightarrow X_2$, but by maps which compose to $p^l$ times the identity for some $l \in \mathbb{N}$. Uniqueness of isogenies in [St, Theorem 1.5] then states that composition of the isogenies between $G_1$ and $G_2$ yield morphisms $\phi^l \circ \text{Fr}^l$, for isomorphisms $\phi : G^{(1)}_i \rightarrow G_i$ as in 3.1.1 (up to possible composition with inner automorphisms). That $(G_1)_{\text{perf}}$ and $(G_2)_{\text{perf}}$ are isomorphic now follows from Proposition 3.2.4. \qed

Establishing the existence of the downwards dashed arrow will take more work. The following lemma can be proved by looking at the Hopf algebra morphisms.

**Lemma 4.2.5.** Consider a reduced affine group scheme $H$ with a homomorphism $\phi : H^{(-i)} \rightarrow G_m$, such that the diagram

$$
\begin{array}{ccc}
G_m \times G_a & \overset{\mu \cdot \phi}{\longrightarrow} & G_a \\
\phi \times \text{id} & \downarrow & \downarrow \\
H^{(-i)} \times G_a & \overset{\text{Fr}^i \times \text{id}}{\longrightarrow} & H \times G_a
\end{array}
$$

holds.
can be completed with dashed arrow to a commutative square. Then \( \phi \) factors through \( \text{Fr}^i : H^{(-i)} \to H \).

**Definition 4.2.6.** Let \( G \) be a perfectly reductive group with maximal perfect torus \( T \). An **rt-pair** is a pair \((x, \alpha) \) of a subgroup inclusion \( x : G_a \to G \) and \( \alpha \in X \) (i.e. \( \alpha : T \to G_m \)), for which the following square is commutative

\[
\begin{array}{ccc}
G_m \times G_a & \xrightarrow{(\lambda, \mu) \mapsto \lambda \mu} & G_g \\
\alpha \times \text{id} \downarrow & & \downarrow x \\
T \times G_a & \xrightarrow{(t, \mu) \mapsto t x (\mu)^{-1}} & G.
\end{array}
\]

(4.2)

If we apply this definition to ordinary reductive groups (we replace every perfect group in (4.2) by its finite type analogue), we get precisely the pairs of inclusions of root subgroups and their corresponding root. Since root subgroups are unique, the inclusion of the root subgroup is unique up to scalar in \( k^\times \cong \text{Aut}(G_a) \).

**Lemma 4.2.7.** Let \( G \) be a perfectly reductive group with maximal perfect torus \( T \).

1. The group \( k^\times \rtimes \mathbb{Z}^2 \) acts on the set of rt-pairs as

\[
(x, \alpha) \xrightarrow{(\kappa, n)} (x \circ (\theta_\kappa^n)^{-1} n \alpha), \quad \text{for all } (\kappa, n) \in k^\times \rtimes \mathbb{Z}^2.
\]

(See 3.4.5 for the definition of \( \theta_\kappa^n \) and 3.4.6 for the action of \( n \in \mathbb{Z}[1/p] \) on \( X \).)

2. If for one \( \alpha : T \to G_m \) we have two rt-pairs \((x, \alpha)\) and \((z, \alpha)\), then \( z = x \circ \theta_1^1 \) for some \( \kappa \in k^\times \).

3. Consider a reductive group \( G \) with \( G_{\text{perf}} \cong G \), with maximal torus \( T \) which perfects to \( T \). Consider an rt-pair \((x, \alpha)\) for which \( x \) is the perfection of some \( x_0 : G_a \to G \), then \( \alpha \) is the perfection of a root homomorphism \( T \to G_m \) and \( x_0 \) is an inclusion of the corresponding root subgroup.

4. For every rt-pair \((x, \alpha)\), there exists \( n \in \mathbb{Z}^2 \) such that \( x \circ \theta_1^n \) is the perfection of the inclusion of a root subgroup \( G_a \hookrightarrow G \) and \( n^{-1} \alpha \) is the perfection of the corresponding root homomorphism \( T \to G_m \).

**Proof.** Part (1) follows from a direct calculation.

For part (3), the homomorphism \( \alpha : T \to G_m \) is induced from \( T^{(-i)} \to G_m \) for some \( i \in \mathbb{N} \), as in Lemma 3.1.4. We find a diagram

\[
\begin{array}{ccc}
G_m \times G_a & \xrightarrow{\text{id} \times \text{id}} & G_g \\
\downarrow & & \downarrow x \\
T^{(-i)} \times G_a & \xrightarrow{\text{Fr}^i \times \text{id}} & T \times G_a \to G
\end{array}
\]

which ‘perfects’ to diagram (4.2). More precisely, after perfecting the above diagram and removing the automorphism which is the perfection of \( \text{Fr}^i \times \text{id} \), we recover (4.2). In particular, we find that the above diagram is commutative. It now follows from Lemma 4.2.5 that \( \alpha \) comes from \( \alpha_0 : T \to G_m \) and it follows immediately that \( \alpha_0 \) is a root.

Now we prove part (4). By Theorem 3.1.6(2) and Proposition 3.4.2(2), there exists a group monomorphism \( G_a \hookrightarrow G \) which perfects to \( x \circ \phi_\kappa^n \) for some \( n \in \mathbb{Z}^2, \kappa \in k^\times \). By identifying \( k^\times \) with \( \text{Aut}(G_a) \), we might as well take \( \kappa = 1 \). The claim about \( \alpha \) now follows from parts (1) and (3).

Finally, we prove part (2). Since roots of reductive groups cannot be multiples of one another, part (4) implies that there is \( n \in \mathbb{Z}^2 \) for which both \( x \circ \phi_\kappa^n \) and \( z \circ \theta_1^n \) are the perfections of inclusions of the same root subgroup. Those inclusions must be the same, up to a scalar in \( k^\times \cong \text{Aut}(G_a) \), from which the claim follows. \( \square \)
Proposition 4.2.8. Consider a reductive group $G$ corresponding to a root datum $RD$. One can extract $\mathbb{Z}[1/p] \otimes RD$ from the group $G_{\text{perf}}$. In particular, the downwards dashed arrow in (4.1) exists.

Proof. We construct a $\mathbb{Z}[1/p]$-root datum $(X, R, Y, R^\vee)$ from $G := G_{\text{perf}}$. It will follow from the construction that $(X, R, Y, R^\vee)$ is isomorphic to $\mathbb{Z}[1/p] \otimes RD$.

First we let $T$ be a maximal perfect torus in $G$. Recall $T$ is the perfection of a maximal torus $T < G$ (and hence unique up to conjugation). We choose such a $T$. We set $X = X(T)$ and $Y = Y(T)$, with (non-degenerate) pairing (3.6). We denote the root datum of $G$ corresponding to $T$ by $(X, R, Y, R^\vee) \cong RD$.

Following Lemma 3.1.4 we find

$$\mathbb{Z}[1/p] \otimes X = \mathbb{Z}[1/p] \otimes \text{Hom}(T, G_m) \cong \lim \text{Hom}(T^{(i)}, G_m) \cong \text{Hom}(T, G_m) = X. \quad (4.3)$$

We define $R \subset X$ as the set of $\alpha$ for which there exists an rt-pair $(x, \alpha)$. It follows from Lemma 4.2.7(1) and (4) that $R \subset p^\infty R$, under $R \subset X \subset X$ from (4.3). That $R \subset R$, so $p^\infty R \subset R$ by Lemma 4.2.7(1), follows from perfecting the root homomorphisms into $G$. In conclusion, $R = p^\infty R$.

Having defined $(X, R, Y)$, we now define the injection $-^\vee : R \to Y$ completing the root datum. Choose $\alpha \in R \subset R$. Denote by $H$ the minimal perfect subgroup of $G$ which contains the images of the two morphisms $G_\alpha \to G$ corresponding to $\alpha, -\alpha$. That the images only depend on $\alpha, -\alpha$ follows from Lemma 4.2.7(2). That there exists such a minimal $H < G$ follows from the noetherian property of $G$ and Theorem 3.1.6(2). The latter also shows that $H$ must be isomorphic to $(SL_2)_{\text{perf}}$ or $(PGL_2)_{\text{perf}}$ and that $T \cap H$ is a (maximal) torus in $H$. We use this to define $G_m \to T$, determined up to $\text{Aut}(G_m) = \mathbb{Z}[1/p]^\times$, either as the inclusion of this maximal torus of $(SL_2)_{\text{perf}}$ or by similarly restricting the homomorphism $(SL_2)_{\text{perf}} \to (PGL_2)_{\text{perf}} \to G$. Finally, we can then define $\alpha^\vee : G_m \to T$ as the unique such morphism for which composition with $\alpha : T \to G_m$ yields $2 \in \text{End}(G_m)$. By construction, $\alpha^\vee$ is defined independently of $G$, but clearly corresponds to the direct definition via $G = G_{\text{perf}}$. For $p^\infty \alpha$, we set $(p^\infty \alpha)^\vee = p^{-1} \alpha^\vee$, which extends the definition to $R = p^\infty R$. \qed

Remark 4.2.9. An alternative to the proof of Proposition 4.2.8 is given by the proof of (a) $\Rightarrow (b) \Rightarrow (c)$ in Theorem 6.1.1. However, the latter uses deep results about 2-compact groups and étale homotopy types of reductive groups, hence it is preferable to have this direct proof.

4.3. Isogenies. We establish a connection between isogenies of perfectly reductive groups and our notion of isogenies of $\mathbb{Z}[1/p]$-root data from Section 1.5.

Theorem 4.3.1. Consider two perfectly reductive groups $G_1$ and $G$ with perfect maximal tori $T_1, T$ and the corresponding $\mathbb{Z}[1/p]$-root data $(X_1, R_1, Y_1, R_1^\vee)$ and $(X, R, Y, R^\vee)$. There is a bijection between the sets of

(a) Isogenies $(X, R, Y, R^\vee) \to (X_1, R_1, Y_1, R_1^\vee)$;

(b) Equivalence classes of isogenies $G_1 \to G$ which send $T_1$ to $T$, where two isogenies are equivalent if one is obtained from the other by composition with an inner automorphism effected by an element of $T_1(k) = T_1(k)$.

Proof. Let $G, G_1$ denote reductive groups which perfect to $G, G_1$, with maximal tori $T, T_1$ which perfect to $T, T_1$, and denote their root data by $(X, R, Y, R^\vee)$ and $(X_1, R_1, Y_1, R_1^\vee)$.

We will prove both sets (a) and (b) are in bijection with the set of

(c) Equivalence classes of pairs $(\theta, i)$ of an isogeny $\theta : (X, R, Y, R^\vee) \to (X_1, R_1, Y_1, R_1^\vee)$ (in the sense of [St, §1]) and $i \in \mathbb{Z}$, where the equivalence relation is generated by $(\theta, i) \sim (p\theta, i - 1)$.### PERFECTION 21
We can alternatively construct the modules $\lambda$ for $\phi$ via Lemma 3.1.4. Its kernel is an affine group scheme of finite type which perfects to a finite group scheme. It must therefore be a finite group scheme and $G_1^{(-i)} \to G$ is an isogeny. This principle allows us to establish a bijection between the set of isogenies $G_1 \to G$ and the set of equivalence classes of isogenies $G_1^{(-i)} \to G$, with equivalence generated by the condition that $\phi : G_1^{(-i)} \to G$ be equivalent to $\phi \circ \text{Fr} : G_1^{(-i-1)} \to G$.

The above connection between isogenies $G_1 \to G$ and equivalence classes of isogenies $G_1 \to G$ allows us to use the classical Isogeny Theorem [St, 1.5] to establish the bijection between (b) and (c). □

5. Perfected representation theory

Let $k$ be an algebraically closed field of characteristic $p$.

5.1. Simple and induced modules and block structure. Let $G$ be a perfectly reductive group, $B$ a perfect Borel subgroup and $T < B$ a maximal perfect torus. We consider the set $R^+ \subset R$ of positive roots, which are the ones for which the corresponding $G_\alpha \to G$ does not land in $B$ (i.e. we let $B$ be the negative Borel).

We set $X = X(T)$ and $X_+ \subset X$ the subset of $\lambda \in X$ which satisfy $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in R^+$. We have a canonical bijection $\{ \lambda \mapsto k\lambda \}$ between $X$ and the set of isomorphism classes of simple $B$-representations (which are all one-dimensional).

**Theorem 5.1.1.**

1. The representation

$$\nabla(\lambda) := \text{Ind}^G_B k\lambda \in \text{Rep}^\infty G$$

is zero if $\lambda \not\in X_+$. If $\lambda \in X_+$, it has simple socle, which we denote by $L(\lambda)$.

2. The above association $\lambda \mapsto L(\lambda)$ is a bijection between $X_+$ and the set of isomorphism classes of simple representations in $\text{RepG}$.

**Proof.** We choose a reductive group $G$ with $G_{\text{perf}} \cong G$ and maximal torus and Borel subgroup $T < B < G$ which perfect to $T$ and $B$. We will use the notation in 3.3.2. By equivalence (3.3), every simple object in $\text{RepG}$ is of the form $L(\mu)[i]$ for some $\mu \in X_+$ and $i \in \mathbb{N}$. Since $L(\mu)[1] \cong L(p\mu)$, see [Ja, II.3], we can define unambiguously the simple object $L(p^{-i}\mu) := L(\mu)[i]$ for $i \in \mathbb{N}$ and $\mu \in X_+$. This clearly gives a bijection between $X_+$ and the set of isomorphism classes of simple objects in $\text{RepG}$.

By Corollary 3.5.5, we have

$$\nabla(\lambda) \cong \lim_{\rightarrow} \nabla(p^i\lambda)[i], \quad (5.1)$$

where the chain of which we take the limit starts at $i$ where $p^i\lambda \in X \subset X$. It follows now from [Ja, II.2] that $\nabla(\lambda) = 0$ whenever $\lambda \not\in X_+$.

Now consider $\lambda \in X_+$. The morphisms in the direct system for (5.1) are not zero by Remark 3.5.6(2). Hence, the defining morphisms $\nabla(p^i\lambda)[1] \to \nabla(p^{i+1}\lambda)$ are unique (up to scalar) and injective since $\nabla(p^i\lambda)[1]$ has socle $L(p^{i+1}\lambda)$ by Remark 5.1.3 below and $\nabla(p^{i+1}\lambda)$ is the injective hull of this socle in a Serre subcategory containing $\nabla(p^i\lambda)[1]$. In particular, for $\lambda, \mu \in X_+$, we have

$$\text{Hom}_G(L(\mu), \nabla(\lambda)) \cong \lim_{\rightarrow} \text{Hom}_G(L(p^i\mu), \nabla(p^i\lambda)) \cong k\delta_{\lambda, \mu}.$$ 

We can therefore identify $L(\mu)$ with the socle of $\nabla(\mu)$. □

**Remark 5.1.2.**

1. We can alternatively construct the modules $\nabla(\lambda)$ as the global sections of line bundles on $G/B \cong (G/B)_{\text{perf}}$. 

(2) It follows for instance from the proof of Theorem 5.1.1 that
\[ \nabla(\lambda) : L(\mu) = \nabla(p^j \lambda) : L(p^j \mu), \]
for all \( \lambda, \mu \in X_+ \) and \( j \in \mathbb{Z} \).

In fact, the proof even shows that
\[ \nabla(\lambda) : L(\mu) = \lim_{m \to \infty} \nabla(p^m \lambda) : L(p^m \mu). \]

The sequence on the right-hand side is monotone increasing, but (except for \( G \) of rank 1) we do not know whether it is bounded. We hope to return to the question of whether multiplicities in \( \text{Rep}G \) can be infinite in the future.

**Remark 5.1.3.** Let \( G \) be a reductive group. The canonical morphism
\[ \text{Ext}^2_G(M,N) \to \text{Ext}^2_G(M^{(1)},N^{(1)}) \]
is injective. This is proved in [Ja, II.10.14] if \( (p-1)p \in X \) (e.g. \( p \neq 2 \)).

If \( p \notin X \), we can extend \( X \subset X' \subset \mathbb{Q} \otimes X \) by taking the lattice maximal \( X' \) for which \( \langle - , \alpha^\vee \rangle \) still takes values in \( \mathbb{Z} \) for every \( \alpha^\vee \in R^\vee \). In particular \( p \in X' \). Taking the appropriate \( Y' \subset Y \) yields a new root datum \( (X', R, Y', R^\vee) \) corresponding to a reductive group \( G' \). By [St, Theorem 1.5], there is an isogeny \( G' \to G \). The claim then reduces to the previous case via an obvious commutative square.

As in any category of finite dimensional modules over a coalgebra over a field, the blocks in \( \text{Rep}G \) are determined by the first extensions between simple objects.

**Theorem 5.1.4.** For \( \lambda, \mu \in X_+ \), the simple representations \( L(\lambda) \) and \( L(\mu) \) are in the same block of \( \text{Rep}G \) if and only if \( \lambda - \mu \in \mathbb{Z}[1/p]R = ZR \).

**Proof.** We resume the notation and conventions from the proof of Theorem 5.1.1. By Remarks 3.3.4 and 5.1.3, \( L(\lambda) \) and \( L(\mu) \) are in the same block if and only if there exists \( j \in \mathbb{N} \) for which \( p^j \lambda, p^j \mu \in X \) and \( L(p^j \lambda) \) and \( L(p^j \mu) \) are in the same block of \( \text{Rep}G \).

Assume first that \( L(\lambda) \) and \( L(\mu) \) are in the same block. By the above, \( p^j \lambda - p^j \mu \in ZR \) for some \( j \), from which \( \lambda - \mu \in \mathbb{Z}[1/p]R = \mathbb{Z}[1/p]R \) follows.

Now assume that \( \lambda - \mu \in \mathbb{Z}[1/p]R \). Then there exists \( j \in \mathbb{N} \) such that

(i) \( p^j \lambda, p^j \mu \in X \),
(ii) \( p^j \lambda - p^j \mu \in p\mathbb{Z}R \),
(iii) \( \langle \alpha^\vee, p^j \lambda + \rho \rangle \notin p\mathbb{Z} \) for some \( \alpha \in R \).

Indeed, for (iii) it suffices to take \( \alpha \) simple and \( j \) such that \( p^j \lambda \in pX \).

From (ii), it follows that \( p^j \lambda \) and \( p^j \mu \) are in the same \( (p\text{-shifted}) \) orbit of \( p\mathbb{Z}R \rtimes W \), so by (iii) and [Ja, II.7.2(2)], \( L(p^j \lambda) \) and \( L(p^j \mu) \) are in the same block of \( \text{Rep}G \) from which the conclusion follows.

**Remark 5.1.5.** Theorem 5.1.4 can be explained by the observation that the orbits of the affine Weyl group \( W \rtimes p\mathbb{Z}R \) on \( X \) describe most of the block decomposition in \( \text{Rep}G \), and the orbits of \( W \rtimes p\mathbb{Z}[1/p]R \) on \( X \) coincide with those of \( \mathbb{Z}[1/p]R \).

5.1.6. Let \( n \) be the length of the longest element \( w_0 \) of the Weyl group (i.e. the dimension of \( G/B \)). We also set
\[ X_{++} = \{ \lambda \in X \mid \langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \mathbb{R}^+ \} \subset X_+. \]

Another class of \( G \)-representations which seems of interest are
\[ W(\lambda) := \text{R}^0\text{Ind}_B^G k_{w_0}(\lambda) \cong \lim_{i \to \infty} \Delta(p^i \lambda - 2\rho)[i], \quad \lambda \in X_{++} \]
where the isomorphism is an instance of Corollary 3.5.5, using [Ja, II.5.11, Remark (1)].

5.2. **Generic cohomology.** In this section we show how the result from [CPSV] can be formulated very elegantly in terms of perfected groups.
5.2.1. Let $G$ be an affine group scheme over $k$, which is the extension of scalars of an affine group scheme over $\mathbb{F}_p$, which we also denote by $G$. Set $\mathbb{F} = \mathbb{F}_p$. We can consider the forgetful functor from rational $G$-representations to representations over $k$ of the abstract group $G(\mathbb{F})$ (for instance using the homomorphism $G(\mathbb{F}) \to G(k)$), which induces comparison morphisms

$$\text{Ext}^i_G(M, N) \to \text{Ext}^i_{kG(\mathbb{F})}(M, N).$$

**Theorem 5.2.2.** Let $G$ be a perfectly reductive group, then the morphism

$$\text{Ext}^i_G(M, N) \to \text{Ext}^i_{kG(\mathbb{F})}(M, N)$$

is an isomorphism for all $M, N \in \text{Rep} G$ and $i \in \mathbb{N}$.

We start the proof by pointing out a technical generality.

**Remark 5.2.3.** Consider a chain of finite abstract groups $\{H_n \mid n \in \mathbb{N}\}$ and set $H = \varprojlim H_n$. For two finite dimensional $H$-representations $M, N$, the canonical morphism

$$\text{Ext}^i_{kH}(M, N) \to \varprojlim \text{Ext}^i_{kH_n}(M, N)$$

is an isomorphism. Indeed, using group cohomology, $\text{Ext}^i_{kH}(M, N)$ is the cohomology of the inverse limit of chain complexes with cohomology $\text{Ext}^i_{kH_n}(M, N)$. Since all vector spaces involved are finite-dimensional, the Mittag-Leffler property leads to the conclusion.

**Proof of Theorem 5.2.2.** Let $G$ be a reductive group with perfection $G$ and recall that $G(\mathbb{F}) = G(\mathbb{F})$ and $\text{Rep} G \cong \varprojlim \text{Rep} G$. Without loss of generality we assume that $M, N$ factor over the natural map $G \to G$. By Remark 3.3.4, we have

$$\text{Ext}^i_G(M, N) \cong \varprojlim \text{Ext}^i_G(M^{(a)}, N^{(a)}).$$

It is proved in [CPSV] that the directed system in the above limit stabilises and moreover, for large enough $a$ and $q$, the morphism $\text{Ext}^i_G(M^{(a)}, N^{(a)}) \to \text{Ext}^i_{G(q)}(M, N)$ is an isomorphism. Hence also the inverse system in

$$\varprojlim \text{Ext}^i_{kG(\mathbb{F}_q)}(M, N) \cong \text{Ext}^i_{kG(\mathbb{F})}(M, N)$$

stabilises and we find the isomorphism in the theorem.

Strictly speaking, [CPSV] only deals with semisimple groups. However, for a general reductive group $G$, we have a short exact sequence $N \to G \to G/N$ with $G/N$ semisimple and $N$ a torus. Since both $N$ and $N(\mathbb{F}_q) \cong C_{q-1}^{\times r}$ have semisimple representation theory over $k$, the result extends easily, for instance via a collapsing Hochschild-Serre spectral sequence. \hfill $\square$

**Remark 5.2.4.** The comparison morphism is not always an isomorphism for perfect groups, for instance

$$\text{Ext}^1_{G_a}(k, k) \to \text{Ext}^1_{kG}(k, k)$$

is the canonical inclusion

$$k\mathbb{Z} \hookrightarrow k\hat{\mathbb{Z}} \cong k\text{Gal}(\mathbb{F} : \mathbb{F}_p).$$

**Question 5.2.5.** The formulation of Theorem 5.2.2 suggests the question of whether the monomorphism (by Theorem 5.2.2)

$$\text{Ext}^i_G(M, N) \to \text{Ext}^i_{kG(k)}(M, N)$$

is also an isomorphism. This is equivalent to the question of whether the epimorphism

$$\text{Ext}^i_{kG(k)}(M, N) \to \text{Ext}^i_{kG(\mathbb{F})}(M, N)$$

is an isomorphism for all $M, N \in \text{Rep} G$ and $i \in \mathbb{N}$. Note that (5.2) does not involve any perfection.
Example 5.2.6. The question in 5.2.5 has an affirmative answer for \( G = G_m \). Consider the short exact sequence

\[
1 \rightarrow G_m(\mathbb{F}) \rightarrow G_m(k) \rightarrow Q \rightarrow 1.
\]

By the Lyndon-Hochschild-Serre spectral sequence, showing (5.2) is an isomorphism can be quickly reduced to showing the group cohomology \( H^i(Q, k) \) is zero for \( i > 0 \). Now the group structure on \( Q \) extends (uniquely) to a \( Q \)-vector space (since \( G_m(\mathbb{F}) < G_m(k) \) is the group of roots of unity and \( k \) is algebraically closed), so we only need to show \( H^i(Q, k) = 0 \) for \( i > 0 \). The case \( i = 1 \) is obvious. One can compute directly that \( H^i(Q, -) = 0 \) for \( i > 1 \) (or via \( BQ \), see [Su, (10) on p42]), hence \( H^i(Q, -) = 0 \) for \( i > 2 \). Finally,

\[
H^2(Q, k) \cong \text{Ext}^1_\mathbb{Z}(Q, k)
\]

must be an abelian group admitting both the structure of a \( Q \)-vector space as well as a \( k \)-vector space, hence it is zero.

We conclude with an example showing that (5.2) being an isomorphism is also something which should not be expected to hold outside of reductive groups.

Example 5.2.7. If instead of a reductive group, we consider \( G = G_a \), as well as \( i = 1, M = N = k \) in (5.2), we obtain the morphism between spaces of group homomorphisms

\[
\text{End}(k^+) \rightarrow \text{Hom}(\mathbb{F}^+, k^+),
\]

induced by restriction along \( F \hookrightarrow k \). This is not a bijection as soon as \( F \neq k \).

6. Localisation of classifying spaces

Fix a prime \( p \).

6.1. Main result. Following [Su], by a simple space we mean a connected topological space having the homotopy type of a CW complex and abelian fundamental group which acts trivially on the homotopy and homology of the universal covering space. Let \( F \) be a connected topological group of homotopy type of a CW complex (below we will consider more specifically complex Lie groups), then its classifying space \( BF \) is a simple space (note that \( \pi_1(BF) = \pi_0(F) \) is trivial). Following [Su, Chapter 2], to each simple space \( X \) we can associate a (simple) space \( X_{\ell} \) the localisation of \( X \) away from \( \ell \). Note that loc. cit. \( X_{\ell} \) is denoted by \( X_{\ell} \) where \( \ell \) is the set of all primes different from \( p \).

Theorem 6.1.1. Let \( G, H \) be split (connected) reductive groups over \( \mathbb{Z} \). The following are equivalent:

(a) The perfections of \( G_k \) and \( H_k \) are isomorphic for \( k = \mathbb{F}_p \);

(b) The localisations \( BG(\mathbb{C})_{\frac{1}{p}} \) and \( BH(\mathbb{C})_{\frac{1}{p}} \) are homotopy equivalent;

(c) The root data of \( G \) and \( H \) become isomorphic after extension to \( \mathbb{Z}[1/p] \).

Proof of (c) \( \Leftrightarrow \) (a) \( \Rightarrow \) (b). The equivalence of (a) and (c) is already established in 4.2.3.

Assume that the perfections of \( G_k \) and \( H_k \) are isomorphic. By Proposition 3.2.4, after replacing \( H_k \) with an (isomorphic) Frobenius twisted version, there is an infinitesimal isogeny \( G_k \rightarrow H_k \). By Lemma 3.2.2, this isogeny is purely inseparable. It then follows from [Fr, Theorem 1.6] that \( BG(\mathbb{C})_{\frac{1}{p}} \) and \( BH(\mathbb{C})_{\frac{1}{p}} \) are homotopy equivalent.

The rest of this chapter is devoted to the proof of (b) \( \Rightarrow \) (c).
6.2. Some useful facts.

(a) For a complex reductive group $F$ and a maximal compact subgroup $K < F$ (the corresponding compact connected Lie group), the homomorphism $K \to F$ is a homotopy equivalence and hence $BK \simeq BF$. We will therefore henceforth replace $BF$ by $BK$.

(b) For a simple space $X$, the defining map $X \to X_{[1/p]}$ see [Su, Chapter 2], induces an isomorphism

$$H_\ast(X; \mathbb{Z}) \otimes \mathbb{Z}[1/p] \overset{\sim}{\to} H_\ast(X_{[1/p]}; \mathbb{Z}).$$

(c) For a commutative ring $D$ in which $p$ is invertible, by (b) and the universal coefficient theorem, we have a natural isomorphism $H^\ast(X_{[1/p]}; D) \cong H^\ast(X; D)$. In particular, a map $X_{[1/p]} \to Y_{[1/p]}$ induces a graded algebra morphism $H^\ast(Y; D) \to H^\ast(X; D)$.

(d) For a flat morphism $A \to B$ of commutative algebras and a topological space $X$ for which $H_\ast(X; \mathbb{Z})$ are all finitely generated, $H^\ast(X; A) \otimes_A B \to H^\ast(X; B)$ is an isomorphism.

(e) For a torus $T \cong (S^1)^r$, $H^\ast(BT; \mathbb{Z})$ is a polynomial ring with $r$ generators in degree 2. Consequently, for any commutative ring $R$, $H^\ast(BT; R) \cong H^\ast(BT; \mathbb{Z}) \otimes R$ is a polynomial ring and we have a bijection between $R$-linear morphisms $\theta : H^2(BT; R) \to H^2(BT; R)$ and graded $R$-algebra morphisms $\theta : H^\ast(BT; R) \to H^\ast(BT; R)$.

(f) For a compact connected Lie group $G$, with maximal torus $T$, and a commutative ring $R$, consider canonical isomorphisms

$$R \otimes X(T) \cong R \otimes H^1(T; \mathbb{Z}) \cong R \otimes H^2(BT; \mathbb{Z}) \cong H^2(BT; R).$$

The Weyl group $W$ thus acts $R$-linearly on $H^2(BT; R)$. Moreover the image of $H^\ast(BG; R) \to H^\ast(BT; R)$

takes values in the algebra of $W$-invariants, see [Bor, §27].

(g) For a prime $q$ and a connected CW complex $Y$, we denote by $Y_{[1]}$ the profinite completion at $q$ of $Y$, see [Su, Chapter 3]. If $q \neq p$, then the universality of $X \to X_{[1/p]}$ in the definition in [Su, Chapter 2] shows that the latter map induces a homotopy equivalence $X_q \simeq (X_{[1/p]}).$

(h) For $G$ a compact connected Lie group, $BG$ satisfies the requirement in (c), i.e. the homology groups $H_i(BG; \mathbb{Z})$ are finitely generated. One can observe this for instance via induction on $i$ using the Serre fibration $G \to EG \to BG$. Note that $EG$ is contractible, $BG$ is simply connected and $G$ is a finite cell complex. The Leray-Serre spectral sequence thus implies that the trivial group can be obtained, starting from $H_i(BG; \mathbb{Z})$ by a finite iteration of taking kernels of morphisms to finitely generated groups (subquotients of $H_i(BG; H_0(G))$, with $a < i$). Consequently, $H_i(BG; \mathbb{Z})$ must also be finitely generated.

6.3. Some results of Adams and Mahmud. We reformulate some results of Adams and Mahmud in the form we will need.

**Theorem 6.3.1** (Adams - Mahmud). Let $G$ and $G'$ be two compact connected Lie groups, with maximal tori $T, T'$ and Weyl groups $W, W'$.

1. For a map $f : (BG)_{[1/p]} \to (BG')_{[1/p]}$, there exists a $\mathbb{Z}[1/p]$-linear morphism

$$\theta : H^2(BT'; \mathbb{Z}[1/p]) \to H^2(BT; \mathbb{Z}[1/p])$$

yielding a commutative diagram of graded algebra homomorphisms

$$
\begin{array}{ccc}
H^\ast(BG; \mathbb{Z}[1/p]) & \overset{\sim}{\leftarrow} & H^\ast(BG'; \mathbb{Z}[1/p]) \\
\downarrow & & \downarrow \\
H^\ast(BT; \mathbb{Z}[1/p]) & \overset{\bar{\theta}}{\leftarrow} & H^\ast(BT'; \mathbb{Z}[1/p]),
\end{array}
$$

where $\bar{\theta}$ is the graded algebra morphism induced by $\theta$.
Proof. For the first statement of part (1), consider the morphism $H^2(BT'; D) \to H^2(BT; D)$ obtained from $f$ via (c), or equivalently via (d) from the map displayed in part (1). Then [AM, Theorem 1.5(a)] implies existence of a morphism $H^2(BT'; D) \to H^2(BT; D)$ yielding the commutative diagram in part (1) with $\mathbb{Z}[1/p]$ replaced by $\mathbb{Q}$. That the latter is induced from a morphism $H^2(BT'; \mathbb{Z}[1/p]) \to H^2(BT; \mathbb{Z}[1/p])$ follows from [AM, Theorem 1.5(b)] and the discussion after [AM, Lemma 1.2]. That the diagram over $\mathbb{Z}[1/p]$ is commutative follows from faithful flatness of $\mathbb{Z}[1/p] \to \mathbb{Q}$.

The case $D = \mathbb{Q}$ of part (2) is a reformulation of [AM, Theorem 1.7]. The proof loc. cit. works for any field of characteristic zero. The case of integral domains follows from extension of scalars to the field of fractions, using (d).

The second statement of part (1) now follows from part (2) and fact (f), by using $\theta_1 = \theta$ and $\theta_2 = w\theta$. \hfill \Box

Corollary 6.3.2. With notation as in Theorem 6.3.1, assume that $f$ is a homotopy equivalence. Then the morphism

$$t : \mathbb{Z}[1/p] \otimes X(T') \to \mathbb{Z}[1/p] \otimes X(T)$$

obtained from $\theta$ via the isomorphisms in (f), induces an isomorphism of $\mathbb{Z}[1/p]$-reflection groups $(W', \mathbb{Z}[1/p] \otimes X(T')) \to (W, \mathbb{Z}[1/p] \otimes X(T))$.

6.4. Conclusion of the proof of Theorem 6.1.1.

Proof of (b) $\Rightarrow$ (c). Assume first that $p = 2$. Then the result follows from Corollary 6.3.2 and Lemma 1.1.6(1). Similarly, for $p > 2$, by Lemma 1.1.6(2) it is sufficient to prove that the extension of scalars along $\mathbb{Z}[1/p] \to \mathbb{Z}_2$ of $t$ yields an isomorphism of $\mathbb{Z}_2$-root data. This allows us to resort to the established theory of 2-compact groups, see [AG].

Starting from a homotopy equivalence $f$ as in Theorem 6.3.1, we have our $\theta$ from 6.3.1(1) which induces the isomorphism of reflection groups in Corollary 6.3.2, and using (g) we have a homotopy equivalence $f_2$. We consider the diagram

$$
\begin{array}{ccc}
(BG)_{\hat{2}} & \xrightarrow{f_2} & (BG')_{\hat{2}} \\
\downarrow & & \downarrow \\
(BT)_{\hat{2}} & \to & (BT')_{\hat{2}}.
\end{array}
$$

Now $(BT)_{\hat{2}} \to (BG)_{\hat{2}}$ is a maximal torus of the 2-compact group $(BG)_{\hat{2}}$, in the sense of [Gr, Theorem 2.2]. By uniqueness of such maximal tori, see loc. cit., there exists a homotopy equivalence corresponding to the dashed arrow in the above diagram so that the diagram is commutative up to homotopy.
By [Su, Theorem 3.9], this induces
\[ \phi : H^2(BT'; \mathbb{Z}_2) \to H^2(BT; \mathbb{Z}_2) \]
yielding a commutative diagram
\[
\begin{array}{ccc}
H^*(BG; \mathbb{Z}_2) & \rightarrow & H^*(BG'; \mathbb{Z}_2) \\
\downarrow & & \downarrow \\
H^*(BT; \mathbb{Z}_2) & \leftarrow & H^*(BT'; \mathbb{Z}_2). \\
\end{array}
\]
By uniqueness in Theorem 6.3.1(2) applied to \( D = \mathbb{Z}_2 \), we may assume that \( \phi \) is actually induced from \( \theta \) by extension of scalars \( \mathbb{Z}[1/p] \to \mathbb{Z}_2 \). Finally, the \( \mathbb{Z}_2 \)-root data of the 2-compact group \( (BG)_{\mathbb{Z}_2} \), as defined in [AG], is obtained from the map \( BT_{\mathbb{Z}_2} \to BG_{\mathbb{Z}_2} \) and by construction yields the extension of scalars along \( \mathbb{Z}_2 \) of the classical root datum of \( G \). The homotopy equivalence \( (BG)_{\mathbb{Z}_2} \simeq (BG')_{\mathbb{Z}_2} \) with commutative diagram therefore indeed implies that our isomorphism of \( \mathbb{Z}[1/p] \)-reflection groups extends to an isomorphism of \( \mathbb{Z}_2 \)-root data.

\[ \square \]

7. Perfected \( SL_2 \)

Let \( k \) be an algebraically closed field of characteristic \( p \). For \( \lambda \) in \( \mathbb{N} \) or \( \mathbb{N}[1/p] := \mathbb{Z}[1/p] \cap \mathbb{R}_{\geq 0} \), we consider its \( p \)-adic expansion \( \lambda = \sum_i \lambda_ip^i \) with \( 0 \leq \lambda_i < p \).

7.1. Fractal.

7.1.1. For \( i,j \in \mathbb{N} \), denote by \( \binom{i}{j} \), the zero coefficient of the \( p \)-adic expansion of the binomial coefficient. By convention, \( \binom{i}{j} = 0 \) if \( j > i \). For \( i,j \in \mathbb{N}[1/p] \), we set
\[
\binom{i}{j} := \binom{pl^i}{pl^j} \in \{0,1,\cdots,p-1\},
\]
for some \( l \in \mathbb{N} \) for which \( pl^i,pl^j \in \mathbb{N} \). By Lucas’ theorem, this definition does not depend on the choice of \( l \).

7.1.2. We can describe the weight spaces in simple \( (SL_2)_{\text{perf}} \)-modules by
\[
\dim_k L(n)_{n-2j} = \begin{cases} 
1 & \text{if } \binom{n}{j} \neq 0 \\
0 & \text{if } \binom{n}{j} = 0,
\end{cases}
\]
for \( n,j \in \mathbb{N}[1/p] \). In particular, the set
\[
F := \{(n,i) \mid \dim_k L(n)_i \neq 0\} \subset \mathbb{Z}[1/p]^2 \subset \mathbb{R}^2
\]
is a fractal. Concretely, \((a,b) \in F \) if and only if \( lp^ia,lp^jb \in F \) for all \( l \in \mathbb{Z} \).

The integer points of the fractal \( F \) for \( p = 3 \) are displayed in Figure 1.

7.2. First extensions. In this section and the next we will apply the short exact sequence
\[
\nabla(\kappa)^{(1)} \rightarrow \nabla(pk) \rightarrow \nabla(\kappa - 1)^{(1)} \otimes L(p - 2), \quad \kappa \in \mathbb{N} = X_+,
\]
see [Pa, equation (3)], as well as the isomorphism
\[ \nabla(pk - 1) \cong \nabla(\kappa - 1)^{(1)} \otimes L(p - 1), \quad \kappa \in \mathbb{Z}_{>0}. \]
Proposition 7.2.1. For \( \lambda, \mu \in X_+ = \mathbb{N}[1/p] \), we have

\[
\dim \text{Ext}^1(L(\lambda), \nabla(\mu)) = \begin{cases} 
1 & \text{if there is } i \in \mathbb{Z} \text{ with } \lambda_i + \mu_i = p - 2 \text{ and } \\
\lambda - p^i \lambda_i & = \mu - p^i \mu_i + p^{i+1} , \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
\dim \text{Ext}^1(L(\lambda), L(\mu)) = \begin{cases} 
1 & \text{if there is } i \in \mathbb{Z} \text{ with } \lambda_i + \mu_i = p - 2, \\
|\lambda_{i+1} - \mu_{i+1}| = 1 \text{ and } & \lambda_j = \mu_j \text{ for } j \notin \{i, i+1\}, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. Recall from equation (5.1) that \( \nabla(\lambda) \) is a direct limit of pullbacks to \( G_{\text{perf}} \) of costandard \( G \)-modules. Since \( \text{Ext}^1(L(\lambda), -) \) commutes with direct limits in the second argument, we can use Remark 3.3.4 to conclude

\[
\text{Ext}^1(L(\lambda), \nabla(\mu)) = \lim \text{Ext}^1(L(p^i \lambda), \nabla(p^i \mu)).
\]

The transition maps are given by the composite

\[
\text{Ext}^1(L(p^i \lambda), \nabla(p^i \mu)) \to \text{Ext}^1(L(p^i \lambda)^{(1)}, \nabla(p^i \mu)^{(1)}) \to \text{Ext}^1(L(p^{i+1} \lambda), \nabla(p^{i+1} \mu)).
\]

Here, the first map is given by the action of the Frobenius twist, so is injective by Remark 5.1.3. The second map comes from the inclusion \( \nabla(p^i \mu)^{(1)} \to \nabla(p^{i+1} \mu) \). From the description of the cokernel of the inclusion in (7.1) it follows that the second map is also injective for \( p > 2 \). For \( p = 2 \) the second map need not be injective, but one shows easily that the composite is still injective.

Assume \( p > 2 \), the case \( p = 2 \) can be proved similarly. It follows quickly from [Pa, Corollary 6.2], that for \( 0 \leq i < p \)

\[
\text{Ext}^1(L(pa + i), \nabla(pb + i)) \cong \text{Ext}^1(L(a), \nabla(b)),
\]

while for \( 0 \leq i < p - 1 \)

\[
\dim \text{Ext}^1(L(pa + p - 2 - i), \nabla(pb + i)) = \delta_{b+1,a}.
\]

Together with the block decomposition, this allows by iteration to calculate the first set of extensions.

It follows from Remark 3.3.4 that

\[
\text{Ext}^1(L(\lambda), L(\mu)) = \lim \text{Ext}^1(L(p^i \lambda), L(p^i \mu)),
\]

where the transition maps are injective by Remark 5.1.3.

Assume \( p > 2 \), the case \( p = 2 \) can be proved similarly. By [Pa, Theorem 4.3], we have for \( 0 \leq i < p \)

\[
\text{Ext}^1(L(pa + i), L(pb + i)) \cong \text{Ext}^1(L(a), L(b)),
\]

while for \( 0 \leq i < p - 1 \)

\[
\text{Ext}^1(L(pa + i), L(pb + p - 2 - i)) \cong \text{Hom}(L(a), L(b) \otimes L(1)).
\]

On the other hand, we have

\[
\dim \text{Hom}(L(a), L(b) \otimes L(1)) = \begin{cases} 
1 & \text{if } b_0 = 0 \text{ and } a = b + 1, \\
1 & \text{if } 0 < b_0 < p - 1 \text{ and } a = b \pm 1, \\
1 & \text{if } b_0 = p - 1 \text{ and } a = b - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

The cases \( b_0 < p - 1 \) follow immediately from the Steinberg tensor product theorem. If \( b_0 = p - 1 \), we know by parity that the space is zero unless \( a_0 < p - 1 \) in which case we can use symmetry between \( a \) and \( b \) to reduce to the already known cases. □
Remark 7.2.2. Equation (7.3) shows that \( \text{RepSL}_2 \to \text{Rep}(SL_2)_{\text{perf}} \) yields isomorphisms on first extensions between simple objects for \( p > 2 \). This is not true for \( p = 2 \).

7.3. Costandard modules. We describe the multiplicities of the simple modules in \( \nabla(\lambda) \).

By Remark 5.1.2(2) it is sufficient to consider \( \lambda \in \mathbb{N} \) (with the case \( \lambda = 0 \) trivial).

**Proposition 7.3.1.** For \( \lambda \in \mathbb{Z}_{>0} \) consider the finite sets
\[
E^0(\lambda) := \{ \nu \in \mathbb{N} \mid |\nabla(\lambda) : L(\nu)| \neq 0 \} \quad \text{and} \quad E^\infty(\lambda) := \{ \nu \in \mathbb{Z}_{>0} \mid |\nabla(\lambda - 1) : L(\nu - 1)| \neq 0 \}.
\]
Then, for all \( \mu \in \mathbb{N}[1/p] \), we have
\[
[\nabla(\lambda) : L(\mu)] = \begin{cases} 1 & \text{if } \mu \in E^0(\lambda) \\ 1 & \text{if } \mu = \nu - \frac{2}{p^i} \text{ with } \nu \in E^\infty(\lambda) \text{ and } i > 0 \\ 0 & \text{otherwise.} \end{cases}
\]

More precisely, \( \nabla(\lambda) \) has a filtration \( 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \) with \( \cup_i M_i = \nabla(\lambda) \) and
\[
M_1 = \nabla(\lambda)[0] \quad \text{and} \quad M_{i+1}/M_i \cong \left( \nabla(\lambda - 1)^{(i)} \otimes L(p^i - 2) \right)[i], \quad \text{for } i > 0.
\]

**Proof.** Recall from the proof of Theorem 5.1.1 that \( \nabla(\lambda) = \varinjlim \nabla(p^i\lambda)[i] \) where every morphism in the chain is injective. The corresponding filtration is the desired one. Indeed the subquotients are given by the cokernel in (7.1) for \( \kappa = p^i\lambda \) on which we can apply iteratively (7.2) and the Steinberg tensor product theorem. \( \square \)

**Example 7.3.2.** Combining Propositions 7.2.1 and 7.3.1 shows:

1. Consider \( 0 < \lambda < p \), then the socle filtration of \( \nabla(\lambda) \) is given by \( \text{soc} \nabla(\lambda) = L(\lambda) \) and
   \[
   \text{soc}^i \nabla(\lambda) = L(\lambda - \frac{2}{p^i}), \quad i > 0.
   \]
2. The socle filtration of \( \nabla(2p - 1) \) is given by \( \text{soc} \nabla(2p - 1) = L(2p - 1), \text{soc}^0 \nabla(2p - 1) = L(2p - 1 - 2/p) \) and
   \[
   \text{soc}^{i+1} \nabla(2p - 1) \cong L(2p - 1 - \frac{2}{p^{i+1}}) \oplus L(1 - \frac{2}{p^i}), \quad i > 0.
   \]

Remark 7.3.3. We can explicitly realise \( \nabla(\lambda) \) as the space of ‘degree \( \lambda \)’ elements in \( k[x^{1/p\nu}, y^{1/p\nu}] \), that is the span of \( \{ x^\mu y^\nu \mid \mu, \nu \in \mathbb{N}[1/p], \mu + \nu = \lambda \} \).

7.4. Line bundle cohomology. We consider the representations \( W(\lambda), \lambda \in \mathbb{N}[1/p] \backslash \{0\} = X_{++}, \) from 5.1.6.

**Proposition 7.4.1.** Recall the finite sets \( E^0, E^\infty \) from Proposition 7.3.1. For \( \lambda \in \mathbb{Z}_{>0} \) and \( \mu \in \mathbb{N}[1/p] \), we have
\[
[W(\lambda) : L(\mu)] = \begin{cases} 1 & \text{if } \mu \in E^0(\lambda - 2), \text{ (with } E^0(-1) := \emptyset) \\ 1 & \text{if } \mu = \nu - \frac{2}{p^i} \text{ with } \nu \in E^\infty(\lambda) \text{ and } i > 0 \\ 0 & \text{otherwise.} \end{cases}
\]

More precisely, \( W(\lambda) \) has a filtration \( 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \) with \( \cup_i M_i = W(\lambda) \) and (with convention \( \Delta(-1) = 0 \))
\[
M_1 = \Delta(\lambda - 2)[0] \quad \text{and} \quad M_{i+1}/M_i \cong \left( \Delta(\lambda - 1)^{(i)} \otimes L(p^i - 2) \right)[i], \quad \text{for } i > 0.
\]

**Proof.** Using Čech cohomology (see proof of Lemma 2.5.1), it follows easily that the morphisms in the directed system in 5.1.6 are injective. The result then follows as in the proof of Proposition 7.3.1, by now using [Pa, (3)] for \( i = p - 2 \). \( \square \)
Remark 7.4.2. It follows that in the Grothendieck group of $\text{Rep}(SL_2)_{\text{perf}}$, we have
$$[\nabla(\lambda)] - [W(\lambda)] = [\Delta(\lambda)[0]] - [\Delta(\lambda - 2)[0]].$$

Example 7.4.3. We have a short exact sequence
$$0 \rightarrow L(1) \rightarrow \nabla(1) \rightarrow W(1) \rightarrow 0.$$  
For $1 < \lambda < p - 1$, we have
$$\nabla(\lambda)/L(\lambda) \cong W(\lambda)/L(\lambda - 2).$$

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