Localized anomalies in heterotic orbifolds

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Abstract

Recently spatially localized anomalies have been considered in higher dimensional field theories. The question of the quantum consistency and stability of these theories needs further discussion. Here we would like to investigate what string theory might teach us about theories with localized anomalies. We consider the $\mathbb{Z}_3$ orbifold of the heterotic $E_8 \times E_8'$ theory, and compute the anomaly of the gaugino in the presence of Wilson lines. We find an anomaly localized at the fixed points, which depends crucially on the local untwisted spectra at those points. We show that non–Abelian anomalies cancel locally at the fixed points for all $\mathbb{Z}_3$ models with or without additional Wilson lines. At various fixed points different anomalous $U(1)$s may be present, but at most one at a given fixed point. It is in general not possible to construct one generator which is the sole source of the anomalous $U(1)$s at the various fixed points.

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1 Introduction

Recently, there have been various calculations of the structure of anomalies in five dimensional field theories compactified on orbifolds, where it was shown that the anomalies are localized at the orbifold fixed points. The first such calculation was presented in ref. [1] for $S^1/Z_2$, and subsequent works on the orbifold $S^1/Z_2 \times Z'_2$ can be found in refs. [2, 3, 4] using Kaluza–Klein expansions of the five dimensional fermions and gauge fields. In ref. [5] topological arguments taken from [6, 7] were used to obtain the same results for Abelian and non–Abelian gauge theories. It turned out that if there is no anomaly for the zero modes of the theory, then it is possible to choose a regularization scheme or introduce appropriate Chern–Simons terms so as to cancel the local gauge anomaly. A localized U(1) gauge anomaly (or more precisely, a mixed U(1) gravitational anomaly), is associated to quadratically divergent Fayet–Iliopoulos terms [8] localized at the fixed points [2, 4, 5, 9]. In [5, 9] it was shown that those localized Fayet–Iliopoulos terms give rise to an instability, which leads to localization of charged bulk fields to one of the fixed points of the orbifold.

This investigation of localized anomalies and their physical consequences on five dimensional orbifolds in field theory is very interesting, but it leads to the question whether such theories are fully consistent quantum field theories and free of instabilities. This becomes even more pronounced when we go to higher dimensions; some investigations in that direction have been performed in [10, 11]. One strategy of answering these questions would be to consider higher dimensional string theories, that are believed to represent consistent quantum theories including gravity. We could then learn what string theory teaches us about localized anomalies. Local anomalies have been explored before in the context of open string theory [12, 13], (heterotic) M–theory orbifolds [14, 15, 16, 17], heterotic M–theory in five dimensions [18].

In order to have a framework to perform explicit calculations of anomalies, we consider the field theory limit (i.e. the limit of the string tension $\alpha' \rightarrow 0$) of the $E_8 \times E_8'$ heterotic string compactified on an orbifold [19, 20]. For simplicity we take this orbifold to be $T^6/Z_3$. It is well–known, that the string spectrum then contains both so–called untwisted (bulk) and twisted (brane) states. The field theory limit of the untwisted sector gives rise to ten dimensional $N = 1$ supergravity coupled to $E_8 \times E_8'$ super Yang–Mills gauge theory on this orbifold. The twisted states are localized at the fixed points of the orbifold only. Modular invariance determines their spectrum precisely; even when Wilson lines are present [21, 22, 23]. (This is contrary to orbifold field theories, where fixed point states are only required to be compatible with the (super) symmetries present at the fixed points.) In addition, we will relate our localized anomaly results to the well–known situation of four dimensional zero modes. The zero mode limit (i.e. the radii of the orbifold $R_i \rightarrow 0$) leads to an effective $N = 1$ supergravity coupled to gauge and chiral multiplets in four dimensions. In figure 1 we give a schematic overview of the various states and limits of heterotic string theory discussed above.

In a recent paper [24] the structure of localized anomalies in heterotic string models was investigated, raising the question how localized non–Abelian anomalies can be canceled, if the chiral twisted and untwisted zero mode states are not anomaly free locally. This observation partly triggered the work reported in this paper. Here, we give a simpler example of this situation, than in [24], though the essential features are the same: Consider a $Z_3$ orbifold model with gauge shift and single Wilson line that are equal: $v = a_1 = (-2, 1^2, 0^5 | 0^8)$. The main properties of this model are: the unbroken gauge group is $E_8 \times SU(3) \times E_8'$, and there is no untwistedmatter. The twisted states at the 27 fixed points fall into three sets that are distinguished by the Wilson line on the first two torus. Classifying the twisted states according to the fixed points, denoted here by 0,1, 2, on that two torus, we have

\begin{align}
0 & : \ (1, 27)(1)' + 3 \cdot (3, 1)(1)' , & 1 & : \ (1, \overline{27})(1)' + 3 \cdot (\overline{3}, 1)(1)' , & 2 & : \ 3 \cdot (\overline{1}, 1)(1)' .
\end{align}
In this example it is clear that the global four dimensional anomaly cancels; but there seems to be an SU(3) anomaly locally at the fixed points 0 and 1, as there are no chiral untwisted states that can compensate the chiral twisted states at the fixed points. Does this mean that in this model we observe localized non–Abelian anomalies?

In this work we will show that the answer to this question is no. There are no localized non–Abelian gauge anomalies on the heterotic $\mathbb{Z}_3$ orbifold. The situation with Abelian gauge anomalies is more complex and will be clarified in section 4.5.

The remainder of this paper is organized as follows:

Section 2. Field theory description of the heterotic string on $T^6/\mathbb{Z}_3$.

The basic features of the field theory approximation of heterotic $E_8 \times E_8'$ string theory on the orbifold $T^6/\mathbb{Z}_3$ are recalled; in particular, the Hosotani description of Wilson lines in subsection 2.1. The untwisted matter surviving the orbifold twist projection at a given fixed point are determined in subsection 2.2 and its relation to the four dimensional zero mode matter is indicated. The final subsection 2.3 discusses the additional twisted states at the fixed points.

Section 3. Orbifold gauge anomalies.

The core of this work is formed by the computation of the gaugino anomaly in the presence of Wilson lines in subsection 3.1. Using Fujikawa’s method in ten dimensions we show that the anomaly becomes localized at the four dimensional fixed points. The formula for the localized anomalies due to twisted states is exposed next. The reduction to the four dimensional zero mode anomaly concludes this section.

Section 4. Localized heterotic anomalies.

In this section we combine the results of the previous sections to investigate the structure of anomaly cancelation in heterotic string theory. To facilitate this discussion we introduce the concept of fixed point equivalent models. Using this concept it is straightforward to infer that non–Abelian gauge anomalies always cancel locally, even in the presence of Wilson lines. In subsection 4.4 we return to the example (1) of this introduction and explain the resolution to the apparent paradox. The localized anomalous U(1)s have a richer structure as is explained in subsection 4.5. Some intriguing aspects of the localized anomalous U(1)s are illustrated in examples in section 4.6.

Section 5. Conclusion and outlook.

Appendices.

Various more technical issues, the decomposition of the ten dimensional Clifford algebra, the torus gaugino wave functions, and the action of Weyl reflections on shift vectors, are collected.
Figure 1: The figure shows that orbifolding and taking the low energy field theory limit ($\alpha' \to 0$) of the heterotic string theory do not commute. If the limit is taken first, then ten dimensional supergravity coupled to super Yang–Mills is obtained. This theory does not contain the additional twisted string states that arise by putting the string on the orbifold. By taking all radii $R_i \to 0$ the theory becomes an effective $N=1$ supersymmetric field theory in four dimensions.

$\bar{z} = (\bar{z}^i)$ we denote their complex conjugates). These complex coordinates transform as a $3_H$ with respect to the holonomy group $SU(3)_H$, embedded in the $D = (1, 9)$ Lorentz group. (The subscript $H$ is used to distinguish the holonomy $SU(3)_H$ representations, from the representations that are related to the $E_8 \times E_8'$ gauge group.)

The two dimensional tori are defined by the relations $z^i \sim z^i + R_i$ and $z^i \sim z^i + e^{2\pi i \phi_i} R_i$. Here the phases $e^{2\pi i \phi_i}$ are defined in terms of numbers $\phi_i$ that satisfy $3\phi_i \equiv 0$ (where the equivalence relation $a \equiv b$ means $a$ is equal to $b$ modulo an integer.) If we use $i, e^{2\pi i \phi_i} i$ as the basis vectors of these three complex tori with length $R_i$, an integral lattice $\Gamma$ can be defined by $\Gamma = \{m_i i + n_i e^{2\pi i \phi_i} i | m_i, n_i \in \mathbb{Z}\}$.

On each of these complex planes the $\mathbb{Z}_3$–orbifold twist operator $\Theta$ acts as an $SU(3)_H$ element

$$\Theta(x, z^1, z^2, z^3) = (x, e^{2\pi i \phi_1} z^1, e^{2\pi i \phi_2} z^2, e^{2\pi i \phi_3} z^3),$$

so that $\sum_i \phi_i \equiv 0$. The definition of the orbifold–twist is extended to spinors in ten dimensions by requiring that

$$-\Gamma_0 R^I_\Theta \Gamma_0 \Gamma_M R_\Theta = (\Theta^{-1})_M^N \Gamma_N, \quad R^2_\Theta = 1.$$  

For the properties and conventions concerning the Clifford algebra that we are employing, see appendix A. An explicit representation of $R_\Theta$ reads

$$R_\Theta = e^{-2\pi \phi^j \frac{2}{3} \Sigma_j}, \quad \Sigma_j = \frac{1}{2} [\Gamma_{2j+2}, \Gamma_{2j+3}] = \begin{cases} i \mathbb{1} \otimes \mathbb{1} \otimes \sigma_3 & j = 1, \\ i \mathbb{1} \otimes \sigma_3 \otimes \mathbb{1} & j = 2, \\ i \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} & j = 3. \end{cases}$$
Of course, the central importance of the orbifolding lies in the fact that it breaks 3/4 of supersymmetries in the effective four dimensional theory. The requirement for the existence of a spinorial supersymmetry parameter $\epsilon(x)$ which is constant over the orbifold and satisfies $R_0 \epsilon(x) = \epsilon(x)$, can be solved in terms of one four dimensional Majorana spinor. Using the notation $\eta^\alpha = \eta^\alpha_1 \alpha_2 \alpha_3$ for an six dimensional spinor, with two dimensional chiralities $\alpha_i$, as introduced in appendix A we find that

$$R_0 \eta^\alpha = e^{-2\pi i \frac{1}{2} \phi_i \alpha_i} \eta^\alpha. \quad (5)$$

The numbers $\phi_i$ can be chosen such that $\eta^0 = \eta^{+++}$ is invariant, while the other spinors $\eta^i = (\eta^{+--}, \eta^{-++}, \eta^{-+-})$ form a triplet $3_H$ of SU(3)$_H$ and transform with the same phase. (The charge conjugate spinors transform with opposite phases.) A consistent choice for the numbers $\phi_i$, which we use from now on, is such that all phases $\theta = e^{2\pi i \phi_i}$ are equal: $(\phi_1, \phi_2, \phi_3) = \frac{1}{3}(1, 1, -2)$.

In each of the three complex tori we have three fixed points, therefore we have 27 fixed points in total which are collectively denoted by $\zeta_s = (R_1 \zeta_1, R_2 \zeta_2, R_3 \zeta_3)$ with the integers $s = (s_1, s_2, s_3)$, $s_i = 0, 1, 2$. Here we have defined the fixed points $\zeta_0 = 0, \zeta_1 = \frac{1}{3}(2 + \theta)$ and $\zeta_2 = \frac{1}{3}(1 + 2\theta)$ for a unit two dimensional $\mathbb{Z}_3$ orbifold, which satisfy

$$\theta \zeta_0 = \zeta_0, \quad \theta \zeta_1 = \zeta_1 - 1, \quad \theta \zeta_2 = \zeta_2 - 1 - \theta. \quad (6)$$

The delta function on the torus $\delta(z - \Gamma)$, obtained from the delta function $\delta(z)$ on $\mathbb{C}^3$, has the property that

$$\delta((1 - \theta^k)z - \Gamma) = \sum_s \frac{1}{27} \delta(z - \zeta_s - \Gamma), \quad (7)$$

for $k = 1, 2$. The factor $1/27$ arises because of the factor $(1 - \theta^k)$ in front of $z$ in the first delta function.

### 2.1 Super Yang–Mills and supergravity on $T^6/\mathbb{Z}_3$

We now give a brief recapitulation of the field content of the supersymmetric low energy field theory of the heterotic $E_8 \times E_8'$ string in $D = (1, 9)$ dimensional Minkowski space. The Lagrangian for this supergravity theory coupled to super Yang–Mills gauge theory has been first derived in ref. [27, 28, 29] (for a text book introduction, see [30]). The supergravity multiplet contains the metric $G_{MN}$, the anti-symmetric two form $B_{MN}$ and a dilaton $\phi$ as bosonic states, and the left–handed Majorana–Weyl Rarita–Schwinger field $\psi_M$ and a right–handed Majorana–Weyl spinor $\lambda$ as fermionic states. The gauge multiplet consists of gauge fields $A_M$ and left–handed Majorana–Weyl gauginos $\chi$ both in the adjoint of $E_8 \times E_8'$. As we will be primarily concerned with gauge anomalies, we only state the gauge transformation of the gauge multiplet

$$i \ 6_A(x, z) = g(x, z) \left( i A_M(x, z) + \partial_M \right) g^{-1}(x, z), \quad 6_\chi(x, z) = g(x, z) \chi(x, z) g^{-1}(x, z), \quad (8)$$

with the local $E_8 \times E_8'$ gauge group element $g(x, z)$. (For most parts of the discussion below it is irrelevant that the gauge group of the heterotic theory is a direct product of two simple Lie groups.)

Next we describe how the orbifold boundary condition is realized on the gauge multiplet. (The analysis for the supergravity multiplet is analogous.) First the gauge field is discussed, as this is more involved than the situation with the gaugino. Let $H_I$ denote the generators of a Cartan subalgebra of the gauge group $E_8 \times E_8'$, and denote by $E_w$ the other generators with weights (roots) $w$. Assume that these Hermitian generators are normalized such that

$$[H_I, E_w] = w_I E_w, \quad H_I^\dagger = H_I, \quad E_w^\dagger = E_{-w}, \quad e^{2\pi i v^I H_I} E_w e^{-2\pi i v^I H_I} = e^{2\pi i v^I w_I} E_w. \quad (9)$$
The algebra valued gauge fields can be decomposed as \( A_M = A_M^t H_I + A_M^w E_w \), where the sum over the Cartan index \( I \) and the roots \( w \) is understood. Using the collective notation for the generators \( T_A \), the Killing metric is defined as \( \eta_{AB} = \frac{1}{2g} \text{tr}(T_A^I T_B^J) \), (\( g \) denotes the dual Coxeter number of \( E_8 \times E_8' \)) and its inverse is denoted by the indices as superscript. The orbifold-twist acts on the gauge fields as

\[
A_M(x, \Theta z) = (\Theta^{-1})_M^N U A_N(x, z) U^{-1}, \quad U = e^{2\pi i v^I H_I} \quad \text{with} \quad \forall v : 3v^I w_I \equiv 0. \tag{10}
\]

The last condition assures that the matrix \( U \) is a \( \mathbb{Z}_3 \) representation. Because the gauge field can be represented as a one-form \( A = A_M dx^M \), it follows from \( \Box \) that \( A_M \) transforms with the inverse twist.

The gauge fields do not have to be strictly invariant when going round one of the cycles of the torus \( T^6 \). Indeed, they only need to be invariant up to a group transformation. Since the cycles of the torus are non-contractable, the Hosotani mechanism \( \Box \) can be at work by implementing Scherk–Schwarz boundary conditions \( \Box \) using the gauge symmetry. This gives rise to three matrices \( T_i \) with the properties

\[
A_M(x, z + i) = T_i A_M(x, z) T_i^{-1}, \quad A_M(x, z + \theta i) = T_i A_M(x, z) T_i^{-1}, \quad T_i = e^{2\pi i a^I_i H_I}, \tag{11}
\]

with \( \forall w : 3a^I_i w_I \equiv 0 \). For simplicity we have chosen the orbifold and the Hosotani boundary conditions to commute, and reside in the Cartan of \( E_8 \times E_8' \). The consistency condition on the Wilson line coefficients \( a^I_i \) follows directly because \( 1 + \theta + \theta^2 = 0 \).

Instead of working with a gauge field \( A_M \), which is periodic on the torus \( T^6 \) up to a group transformation, one can also work with a strictly periodic gauge field \( \tilde{A}_M \), by performing the following field redefinition

\[
A_M(x, z) = T(z) \tilde{A}_M(x, z) T^{-1}(z), \quad T(z) = e^{2\pi i a^I(z) H_I}, \quad a^I(z) = \frac{z^I (1 - \bar{\theta}) - \bar{z}^I (1 - \theta)}{\theta - \bar{\theta}} \frac{1}{R_i a^I_i}. \tag{12}
\]

Notice that the functions \( a^I(z) \) are real: \( (a^I(z))^* = a^I(z) \). From the shift property \( a^I(z + i) = a^I(z + \theta i) = a^I(z) + a^I_i \), for \( i = 1, 2, 3 \) it follows directly that if \( \tilde{A}_M \) is periodic, \( A_M \) satisfies \( \Box \). However, this field redefinition can be partly undone by a gauge transformation \( \Box \) with gauge group element \( T(z) \)

\[
i A_M(x, z) = T(z) \left( i \tilde{A}_M(x, z) + iB_M + \partial_M \right) T^{-1}(z), \quad B_i = 2\pi \partial_i a^I(z) H_I = \frac{2\pi}{R_i} \frac{1 - \bar{\theta}}{\theta - \bar{\theta}} a^I_i H_I. \tag{13}
\]

Of course, also the conjugate \( B_{\bar{i}} \) is non-vanishing, while \( B_\mu = 0 \). The conclusion of this computation is that a theory with (gauge) field periodic up to group transformations, generated by the Cartan subalgebra of the gauge group, is equivalent to having truely periodic gauge fields which have constant background values \( B_i, B_{\bar{i}} \) in the Cartan subalgebra directions. As we have seen above these so-called Wilson-lines are quantized because of consistency with the orbifold boundary condition.

Next we turn to the gauginos on \( T^6/\mathbb{Z}_3 \). Because of gauge invariance, the gauginos satisfy the fermionic equivalent of the boundary conditions \( \Box \) and \( \Box \)

\[
\chi(x, \Theta z) = U R_\Theta \chi(x, z) U^{-1}, \quad \chi(x, z + i) = T_i \chi(x, z) T_i^{-1}, \tag{14}
\]

with the spinor twist-rotation \( R_\Theta \) given in \( \Box \).
2.2 Untwisted matter at the fixed points and their zero modes

In this subsection we identify all untwisted states that exist at the fixed points. In general this set of states is larger than the set of zero mode untwisted states that we will discuss at the end of this subsection. Again we only focus on the gauge multiplet here. At the fixed point $\mathcal{Z}_s$ the surviving states have to satisfy the condition

$$(\Theta^{-1})_M N A_N(x, \mathcal{Z}_s) U^{-1} = A_M(x, \theta^s \mathcal{Z}_s) = (T_1^{s_1} T_2^{s_2} T_3^{s_3})^{-1} A_M(x, \mathcal{Z}_s) (T_1^{s_1} T_2^{s_2} T_3^{s_3}).$$

(15)

In the first equality we have used the action of the orbifold–twist given in (10). The second equality follows because the functions $a^I(z)$ introduced in (12) have the property that for any integer $k$

$$a^I(\mathcal{Z}_s) - a^I(\theta^k \mathcal{Z}_s) \equiv k s_i a_i^I,$$

(16)

using the definition of the fixed points (13). This equation can also be stated as the projection for both the gauge fields and the gauginos

$$(\Theta^{-1})_M N s U_s A_N(x, \mathcal{Z}_s) U_s^{-1} = A_M(x, \mathcal{Z}_s), \quad U_s = T_1^{s_1} T_2^{s_2} T_3^{s_3} U = e^{2\pi i Q_s},$$

$$U_s R_{\Theta}^s \chi(x, \mathcal{Z}_s) U_s^{-1} = \chi(x, \mathcal{Z}_s), \quad Q_s = v^I_s H_I, \quad v^I_s = v^I + s_i a_i^I.$$

(17)

With the algebra (9) this condition can be worked out for the various components, we get

$$\begin{pmatrix} A^I_{\mu} & A^w_{\mu} \\ A^I_{w} & A^w_{w} \end{pmatrix}(x, \mathcal{Z}_s) = A_M(x, \mathcal{Z}_s) = \begin{pmatrix} A^I_{\mu} & e^{2\pi i v^I_s w_I} A^w_{\mu} \\ e^{2\pi i v^I_s w_I} A^I_{\mu} & e^{2\pi i (v^I_s w_I)} A^w_{w} \end{pmatrix}(x, \mathcal{Z}_s).$$

(18)

And for the gauginos we find the fixed point projections

$$\begin{pmatrix} \chi^I_0 & \chi^w_0 \\ \chi^I_a & \chi^w_a \end{pmatrix}(x, \mathcal{Z}_s) = \chi(x, \mathcal{Z}_s) = \begin{pmatrix} \chi^I_0 & e^{2\pi i v^I_s w_I} \chi^w_0 \\ e^{2\pi i (v^I_s w_I)} \chi^I_0 & e^{2\pi i (v^I_s w_I)} \chi^w_a \end{pmatrix}(x, \mathcal{Z}_s),$$

(19)

where we used $v^I_s$ defined in (17). Because of the Majorana–Weyl condition of the gauginos in ten dimensions, it follows that the four dimensional negative chirality states are not independent, and therefore we did not indicate them here.

The Cartan subalgebra gauge fields always exist at the fixed points. In addition we find, that the gauge fields and gauginos, as well as the complex scalars and chiral fermions that exist at the fixed point $\mathcal{Z}_s$ are determined by the relations:

$$\text{gauge } \text{Ad}_s : \begin{cases} v^I_s w_I \equiv 0 \\ H_I \end{cases}, \text{ matter } \begin{cases} (3_H, R_s : \frac{1}{2} + v^I_s w_I \equiv 0) \\ (\overline{3}_H, \overline{R}_s : \frac{1}{2} + v^I_s w_I \equiv 0) \end{cases}, \text{ at fixed point } \mathcal{Z}_s.$$  

(20)

Here $\text{Ad}_s$ denotes the adjoint representation of gauge fields and gauginos at fixed point $\mathcal{Z}_s$, corresponding to both the Cartan subalgebra and the generators $E_{\mu}$ which commute with $U_s$, defined in (17). The corresponding gauge group is $G_s \subset E_8 \times E_8'$. These gauge multiplets are singlets under the holonomy group $\text{SU}(3)_H$. The untwisted matter at fixed point $\mathcal{Z}_s$ consists of chiral multiplets containing complex scalars $A^w_\omega$ and the left–handed fermions $\chi^w_a$; they form triplets of $\text{SU}(3)_H$. We label this set of chiral multiplets by $R_s$. (Their conjugates, like $A^w_\omega(x, \mathcal{Z}_s) = \exp 2\pi i (\frac{1}{2} + v^I_s w_I) A^w_\omega(x, \mathcal{Z}_s)$, are labeled by $\overline{R}_s$, but they are not independent.) Since the union of the set $\text{Ad}_s$, $R_s$ and $\overline{R}_s$ describes the
full $E_8 \times E_8'$ algebra, and the group $G_s$ has maximal rank (as it contains the full Cartan of $E_8 \times E_8'$), it follows that each $R_i$ labels a representation of the corresponding group $G_s$.

If the condition for the surviving gauge group at the fixed point $\mathfrak{Z}_s$ had been $v^{I}_s w_I = 0$, the group $G_s$ would always have contained two $U(1)$ factor generators

$$q_s = Q_s|_{E_8} = v^{I}_s H_I|_{E_8}, \quad q'_s = Q_s|_{E_8'} = v^{I}_s H_I|_{E_8'},$$

because the heterotic gauge group is a direct product of two $E_8$s. With the notation $|_{E_8}$ and $|_{E_8'}$ we indicate that this expression is projected onto the first and second $E_8$ factor, respectively. The surviving generators $\text{Ad}_s$ have to commute with $U_s$, rather than $Q_s$ (they only have to satisfy $v^{I}_s w_I = 0$, instead of $v^{I}_s w_I = 0$). Therefore, it may happen that the generators $q_s$ and $q'_s$ become part of a non–Abelian factor in $G_s$.

The four dimensional zero modes in the effective four dimensional theory are represented by constant states over the full internal dimension. Therefore they exist at all fixed points simultaneously, because the heterotic gauge group is a direct product of two $E_8$.

First, we would like to stress, that for modular invariance of string theory we have to fulfill another consistency condition for every fixed point $s$, namely

$$\frac{2}{3} (v^{I}_s)^2 \equiv 0.$$  

As we are only interested in the QFT limit $\alpha' \to 0$ it suffices to examine the massless twisted states. However, they are necessarily localized at the fixed points, which follows from the oscillator expansion and monodromies of the corresponding coordinate fields. This implies, that they are unaffected by the zero mode limit $R_i \to 0$, since they do not depend on the size of the tori. Therefore, we can take the massless twisted chiral matter, as computed from string theory, and include it in the field theory analysis of the gauge anomalies presented in the next section.

The twisted massless states are built up by taking the tensor product of left– and right–moving massless states. For the $\mathbb{Z}_3$ orbifold the situation is particularly simple, since there are only two right–moving twisted massless states, the non–degenerate vacua from the $\tilde{NS}$ and the $\tilde{R}$ sector, which we denote by $|0\rangle_{\tilde{NS},\text{tw}}$ and $|0\rangle_{\tilde{R},\text{tw}}$, respectively. For the left–movers the situation is more involved. At a given fixed point $\mathfrak{Z}_s$ the massless twisted states have to fulfill the condition

$$\frac{1}{2} (w^I + v^{I}_s)^2 + N - \frac{2}{3} = 0.$$  

2.3 Twisted heterotic string states

As discussed in the introduction and indicated in figure 1 the existence of additional twisted string states cannot be inferred from ten dimensional supergravity coupled to super Yang-Mills theory. Therefore, we briefly review in this subsection how the zero modes of the twisted string states can be derived. Our presentation is based on [21], where the inclusion of Wilson lines in orbifold models was studied for the first time. (See also [23] for the $\mathbb{Z}_3$ orbifold specifically.)

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The twisted massless states are built up by taking the tensor product of left– and right–moving massless states. For the $\mathbb{Z}_3$ orbifold the situation is particularly simple, since there are only two right–moving twisted massless states, the non–degenerate vacua from the $\tilde{NS}$ and the $\tilde{R}$ sector, which we denote by $|0\rangle_{\tilde{NS},\text{tw}}$ and $|0\rangle_{\tilde{R},\text{tw}}$, respectively. For the left–movers the situation is more involved. At a given fixed point $\mathfrak{Z}_s$ the massless twisted states have to fulfill the condition

$$\frac{1}{2} (w^I + v^{I}_s)^2 + N - \frac{2}{3} = 0.$$  

Here $w^I$ are elements of the $E_8 \times E_8'$ root lattice and $N$ counts (fractional) string excitations. There are two possibilities to obtain massless states: either $N = 0$ or $N = 1/3$. For the latter option, the right–moving vacua have to be exited by the creation operators $\alpha_{-1/3}^i$, and therefore these states form a $3_H$ triplet of $SU(3)_H$. The obtained massless twisted string states

$$\begin{align*}
N = 0 : & \left( |S_s, 0\rangle_{\tilde{N}S, tw}, |S_s, 0\rangle_{\tilde{R}, tw} \right), \\
N = \frac{1}{3} : & \left( \alpha_{-1/3}^i |T_s, 0\rangle_{\tilde{N}S, tw}, \alpha_{-1/3}^i |T_s, 0\rangle_{\tilde{R}, tw} \right),
\end{align*}$$

are labeled by the sets $S_s$ and $T_s$ defined by

$$\text{twisted matter} \begin{cases} 
(1_H, S_s : (w^I + v^I_s)^2 = \frac{4}{3}) \\
(3_H, T_s : (w^I + v^I_s)^2 = \frac{2}{3})
\end{cases}$$

(25)

The complex conjugated states, coming from the inversely twisted sector, are not independent and therefore not considered here.

### 3 Orbifold gauge anomalies

Before we embark on the calculation of the gauge anomaly of the gaugino on the orbifold in the presence of Wilson lines, we would like to make a few comments on some recent calculations of gauge anomalies in five dimensional orbifold theories (see [1] for $S^1/Z_2$, and [2, 3, 4] for $S^1/Z_2 \times Z'_2$), which served as an inspiration and partial guideline for our computation.

Many of these calculations of the anomaly on those orbifolds have been performed using the gauge choice, that sets the fifth component of the gauge field to zero: $A_5 = 0$. This is a point of potential concern [2], since one is investigating violation of gauge invariance in a specific gauge. In two or more internal dimensions (like the six dimensional case studied in this work) the gauge choice putting all internal gauge potentials to zero is inconsistent in general. It is only possible when the corresponding internal field strength vanishes: $A_i = \partial_i \Lambda \Rightarrow F_{ij} = 0$. In appendix B of [4] an argument is presented, that the result for the anomaly is independent of this gauge choice for the five dimensional orbifold models. Using similar ideas as presented there, we show below, that also in our setting, with six internal dimensions, no assumptions need to be made concerning the gauge field in the extra dimensions.

Any appearance of a 10D anomaly is canceled by the standard Green–Schwarz mechanism, and will therefore be disregarded in most parts of this paper.

A final general comment concerning the method we use in the computation below: Instead of explicitly constructing and using the orbifold wave functions, we introduce an orbifold projector, which projects on orbifold consistent states. (This method has also been used in a string theory calculation of the Fayet–Iliopoulos term in type I orbifolds [34].) This allows us to use mode expansions on the torus, rather than on the orbifold, which make computations considerably easier. This method may also be applied to other orbifolds, for example the ones mentioned above.

A related calculation of localized anomalies in six dimensions has recently appeared in [35].

#### 3.1 Gaugino anomalies

We start with the calculation of the gauge anomalies of the gaugino on the orbifold $T^6/Z_3$ by reviewing the standard functional methods [36] to describe their origin. We consider here only anomalies in gauge symmetries, but take into account the effect of the spin connection $\omega(e)$ as a function of the vielbein $e$ in the Dirac operator. The Dirac operator of the gaugino maps the Hilbert space of positive chirality to that of negative chirality. By introducing a non–interacting right–handed fermion $\xi$, a Dirac operator
We employ Fujikawa’s method \cite{41, 42} to regularize this trace by using the heat–kernel
\begin{equation}
\rho \exp \left( - \frac{1}{2} \frac{1}{2} + \frac{\Gamma}{2} \right) \rho \, S[\rho, g, A, e] = S[\rho, g, A, e].
\end{equation}
We have an anomaly if, the effective path integral $Z[A]$ obtained by integrating out the fermions
\begin{equation}
Z[A] = \int \mathcal{D} \Psi \mathcal{D} \Phi e^{it[\Psi, A, e]} \neq \int \mathcal{D} \Phi e^{it[\Psi, A, e]}, \quad A[\Lambda] = \text{Tr}[P_\Theta \Lambda \tilde{\Gamma}],
\end{equation}
is not gauge invariant: $A[g] \neq 0$. In the last equation we have restricted ourselves to infinitesimal
gauge transformations, denoted by $\Lambda$. The trace $\text{Tr}$ here is formal, as it is taken over both spinor and
gauge indices. The Minkowski space delta function can be expanded into plane waves
\begin{equation}
\delta(x - x') = \int \frac{d^4 \omega}{(2\pi)^4} \frac{e^{i \omega x} - e^{i \omega x'}}{i \omega}.
\end{equation}
To show this, notice that by applying this operator on a torus gaugino, a gaugino state is obtained
\begin{equation}
\text{Tr} \Phi \eta \Phi \Psi \eta = 0.
\end{equation}
Before we dive into the details of the gaugino orbifold anomaly, we recall that the Wess-Zumino
consistency condition \cite{37} of the anomaly action
\begin{equation}
\delta \Lambda_1 A[\Lambda_2] - \delta \Lambda_2 A[\Lambda_1] = A[\Lambda_3], \quad \Lambda_3 = [\Lambda_1, \Lambda_2],
\end{equation}
fixes the structure of anomalies in terms of invariant $\Omega_{2n}$ and covariant anomaly polynomials $\Omega_{2n}^{1}$. The construction of these anomaly polynomial forms can be summarized by the descent equations \cite{38}
that hold for any integer $n \geq 0$
\begin{equation}
\Omega_{2n+2} = d\Omega_{2n+1} = \text{ch}(iF) \hat{A}(R), \quad \delta \Lambda \Omega_{2n+2} = 0, \quad \delta \Lambda \Omega_{2n+1} = d\Omega_{2n}^{1}(\Lambda).
\end{equation}
Here ch is the Chern character and $\hat{A}$ is the roof genus of the curvature tensor $R$. Their expressions can be found in \cite{38, 39, 40, 39}, for example. To facilitate our calculation below, we introduce the notation: $\hat{\Omega}_{2n}^{1}$, which is defined such that $\text{tr}_p \hat{\Omega}_{2n}^{1} = \Omega_{2n}^{1} \rho$, where the trace is taken over a $\rho$ representation of a
group $G$.

We turn to the evaluation of the formal trace formula \cite{28} for the anomaly on the orbifold $T^6/Z_3$. We employ Fujikawa’s method \cite{11, 12} to regularize this trace by using the heat–kernel
\begin{equation}
A[\Lambda] = \text{Tr} \left[ P_\Theta \Lambda \tilde{\Gamma} e^{-\beta(\Phi/M)^2} \right] = \frac{1}{3} \int_{M^4 \times T^6} dx d^6 z \text{tr} \left[ P_\Theta \Lambda \tilde{\Gamma} e^{-\beta(\Phi/M)^2} \delta(x - x') \delta(z - z' - x) \right],
\end{equation}
where the limits $x' \rightarrow x$, $z' \rightarrow z$ and $M \rightarrow \infty$ are to be taken once the anomaly expression is well–
defined. The factor $1/3$ takes into account that the volume of the orbifold is 1/3 of that of the torus.
(In the remainder of the calculation we drop the notation $M^4 \times T^6$.) The trace $\text{tr}$ is taken over all
spinor and gauge indices. The Minkowski space delta function can be expanded into plane waves $e^{ipx}$,
while the torus delta function can be replaced by the completeness relation \((74)\) of appendix \(1\). Using the action of the orbifold twist operator \(\Theta\) in the projector \(P_\Theta\), one obtains

\[
\mathcal{A}[\Lambda] = \frac{1}{2} \frac{1}{32} \int d^4 x d^6 z \int \frac{d^4 p}{(2\pi)^4} \sum_{k,\alpha, A, A'} \theta^{-\sigma(\alpha, A) k} e^{-ipx'} \\
\eta^{A\alpha} \text{tr}_G \left[ \gamma_q^{\alpha\dagger} (\theta^{-k} z') \text{tr}_4 \left( \Lambda^7 \bar{\sigma} e^{-iD/M^2} \right) \eta^{\alpha A}(z) \right] e^{ipx} .
\]

(33)

Here \(\text{tr}_G\) denotes the trace over the gauge group. The operator \(\Theta\) acts to the right, and therefore, in particular, on the Dirac operator. To avoid having to indicate the phase \(\theta^k\) all over the place, we have implicitly performed the coordinate transformation \(z \rightarrow \theta^{-k} z\).

We first investigate the case \(k = 0\). The Dirac operator squared reads \(D^2 = D^2 + i \frac{p}{4} F_{MN} [\Gamma_M, \Gamma_N]\). The eigenvalues of the operator \(D^2\) are \(- (p^2 + 4(2\pi)^2 |q_i + b_i A|^2 / R_i^2)\) using the mode functions \(\eta^{\alpha A}(z) e^{ipx}\), where \(b_i A = (1-\theta) a_i^l w_l (T_A) / (\theta - \bar{\theta})\). All internal spinor components are treated equally, therefore \(\frac{1}{2} \sum \eta^\alpha \eta^{\alpha\dagger} = 1\). The \(k = 0\) contribution can be written as

\[
\mathcal{A}_{k=0}(x, z) = \frac{1}{32} \int \frac{d^4 p}{(2\pi)^4} \sum_{A} \text{tr}_{10} \left[ \hat{\Gamma} e^{-\frac{i}{4} F_{MN} [\Gamma_M, \Gamma_N] / M^2} \right]_A
\]

\[
\frac{1}{R_1^2 R_2^2 R_3^2} \left( \frac{2}{|\theta - \bar{\theta}|} \right)^3 \sum_q e^{- (p^2 + 4(2\pi)^2 |q_i + b_i A|^2 / R_i^2) / M^2},
\]

(34)

where we have used the inverse Killing spinors \(\eta^{AA'}\) and the normalization of the wave functions. The trace \(\text{tr}_{10}\) over full ten dimensional spinors is non–vanishing if the exponential inside is expanded to fifth order. The resulting factor \(1/M^{10}\) is partly canceled by a rescaling \(p \rightarrow M p\). In the limit \(M \rightarrow \infty\) the remaining factor \(1/M^6\) is used to replace the sum by an integral:

\[
\frac{1}{M^6} \sum_q F \left( \frac{|q_i + b_i A|^2}{R_i^2 M^2} \right) \rightarrow \int d^3 n d^3 m \ F \left( \frac{|\theta n^i - m^i|^2}{R_i^2 (\theta - \bar{\theta})^2} \right).
\]

(35)

By the change of variables \(P_R^i + iP_I^i = 4\pi (\bar{\theta} n^i - m^i) / (R_i |\bar{\theta} - \theta|)\), this gives a six dimensional Gaussian integral. Hence the \((k = 0)\) anomaly finally reads

\[
\mathcal{A}_{k=0}[\Lambda] = \frac{1}{3} \int_{M^4 \times \mathbb{T}^6 / \mathbb{Z}_3} d^4 x d^6 z \Omega_{10}^1 \Omega_{10}^2 (\Lambda; A, F, R),
\]

(36)

where \(\Omega_{10}^1\) is defined by the descent equations. As the cancelation of this anomaly involves the well–know Green–Schwarz mechanism in ten dimensions, we do not discuss this contribution in the subsequent sections.

Next we turn to the case \(k \neq 0\), where the chiralities are not treated equally. Since we work with a basis that is diagonal with respect to the two–tori chiralities, the six dimensional chirality \(\bar{\sigma}\) is equal to \((-)^\alpha\), see \((36)\). Moreover, the Majorana condition allows us to restrict the sum of the internal chiralities to \((-)^\alpha = +\), removing the double counting and hence the factor \(\frac{1}{2}\). This expression can be evaluated further by substituting the expressions for the mode functions and their conjugates \((35)\), using the scalar completeness relation \((72)\) to remove the sum over the torus momentum \(q\), and that spinors \(\eta^\alpha\) are normalized to unity: \((\eta^\alpha) \eta^{\alpha\dagger} = 1\) (no sum over \(\alpha\) here). We evaluate the trace over the gauge group, which means taking out the \(AA'\)–component of the algebra expression of the \(\bar{\Omega}_{10}^1\) using
the inverse Killing metric $\eta^{\lambda\lambda'}$ we can write this as

$$A[\Lambda] = \frac{1}{32} \sum_{k,a} \int d^4x d^6z \theta^{-\sigma(\alpha, A)k} e^{2\pi i (\alpha, A)\varphi_k} \left[ \bar{\Omega}_4^1(\Lambda)(x, z) \right]^A \delta(z - \theta^{-k}z - \Gamma).$$  (37)

Because of (17) we only find non-vanishing contributions at the fixed points $\mathbf{3}_s$, and in addition we get a factor of $1/27$. The Wilson line functions $a_A$ are then also evaluated at the fixed points $\mathbf{3}_s$, hence the property (13) has to be used. This leads to a modification of the twist phase:

$$\theta^{-\sigma(\alpha, A)k} e^{2\pi i (\alpha, A)\varphi_k} = \theta^{-\sigma(\alpha, A)k} e^{-2\pi i \alpha x a'^T (T_A)},$$  (38)

so that $\sigma(\alpha, A, s) = 3(-\frac{1}{2}\phi_i x \alpha_i + (v' + s_i a'_i) w_I(T_A))$. By introducing the notation $\sigma(\alpha, r) = \sigma(\alpha, A, s)$ for $T_A \in \mathbf{r}$, where $r = \text{Ad}_s, \mathbf{R}_s, \overline{\mathbf{R}}_s$ denote the different possible fixed point representations (20), the anomaly can be rewritten as

$$A[\Lambda] = \frac{1}{32} \sum_{k,\alpha, s} \sum_{r=\text{Ad}_s, \mathbf{R}_s, \overline{\mathbf{R}}_s} \int d^4x d^6z \theta^{-\sigma(\alpha, r)k} \Omega_4^1(\Lambda)(x, z) \left|_{\text{Ad}_s} \frac{1}{27} \delta(z - \mathbf{3}_s - \Gamma).$$  (39)

Observe that the anomaly polynomial $\Omega_4^1(\Lambda)$ is evaluated at the fixed points $\mathbf{3}_s$, therefore the gauge parameter $\Lambda$, the gauge field $A_\mu$ and field strength $F_{\mu\nu}$ are restricted to those fixed points as well, and hence form adjoint representations $\text{Ad}_s$.

The traces over the adjoint representations $\text{Ad}_s$ never give a contribution for the anomaly, while the anomaly due to $\overline{\mathbf{R}}_s$ is opposite to that of $\mathbf{R}_s$: we denote this relative sign as $(\cdot)^r$ defined by $-(-)^r \overline{\mathbf{R}}_s = (-)^r \mathbf{R}_s = +$. We need to be careful to take the multiplicities correctly into account. For this purpose we collect the phase-factor multiplicities in the table below, and compute the sum of these phase-factors

| $r$ | $\sigma(\alpha, r)$ | $\text{multipl.}$ |
|-----|---------------------|------------------|
| $\mathbf{R}_s$ | 0                   | 1          |
| $\overline{\mathbf{R}}_s$ | 3                   | 1          | 0       |
| $\mathbf{R}_s$ | 0                   | 3          |

$$\sum_{k,\alpha, s, r=\mathbf{R}_s, \overline{\mathbf{R}}_s} (-)^r \theta^{-\sigma(\alpha, r)k} = 2 + \theta^2 - 3\theta^2 - \theta + 3 + \theta - 3\theta - \theta^2 = 9.$$  (40)

The first (second) four terms arise from $k = 1$ ($k = 2$). Hence we find the final result for the untwisted (or gaugino) anomaly on the orbifold $T^6/\mathbf{Z}_3$

$$A_{un} = \int_{M^4 \times T^6/\mathbf{Z}_3} d^4x d^6z \sum_s 3 \Omega_4^1(\Lambda; A, F, R) \left|_{\text{Ad}_s} \frac{1}{27} \delta(z - \mathbf{3}_s - \Gamma).$$  (41)

The factor 3 is due to the fact that we use the integral on the orbifold (instead of the torus).

Let us discuss a few issues of the interpretation of this result. We see that the gaugino anomaly (11) takes the form of a four dimensional gauge anomaly localized at the fixed points. Moreover, the states $\mathbf{R}_s$ contributing to the anomaly at fixed point $\mathbf{3}_s$ are precisely those gaugino states that are not projected away at that fixed point, see (20). This confirms the naive argument, that only four dimensional gaugino anomalies can arise at the fixed points, since only there the gaugino may give rise to a four dimensional chiral spectrum. Moreover, the multiplicity 3 in (11) is due to the fact that the chiral gaugino states at the fixed points form a triplet under the holonomy group SU(3)$_H$. In subsection 3.3 we explain that also the factor 1/27 had to be expected.
3.2 Twisted matter anomalies

The situation of the gauge anomalies due to the twisted matter is much less involved than the one of the gauginos. As has been reviewed in subsection 2.3, the twisted matter states are necessarily four dimensional and exist at the fixed points $Z_s$ of the orbifold only. Therefore, the corresponding chiral fermions can only give rise to four dimensional anomalies localized at the fixed points. The total twisted anomaly reads

$$A_{\text{tw}} = \int_{M^4 \times T^6/Z_3} d^4x d^6z \sum_s \left[ \Omega^1_{4s}(\Lambda; A, F, R) + 3\Omega^1_{4s}(\Lambda; A, F, R) \right]_{\text{Ad}}, \delta(z - Z_s - \Gamma),$$

(42)

using the definitions of the representations $S_s$ and $T_s$ given in (26). The representations $T_s$ contribute with a multiplicity 3 to the anomaly, since they are triplets under the holonomy group SU(3)$_H$.

3.3 Zero mode anomalies

Let us conclude this section by comparing the results, obtained in the previous subsections, to the well–known zero mode result for the anomaly of untwisted states on the orbifold $T^6/Z_3$.

For the untwisted anomaly (41) the four dimensional zero mode result can be obtained as follows. As was argued above (22), the zero mode untwisted states are constant over the orbifold, therefore they exist at all fixed points at the same time. All other gauge field states are massive from the four dimensional effective field theory point of view, hence the ten dimensional part $A_{k=0}$ of the anomaly does not contribute to the zero mode anomaly. It follows that the untwisted zero mode anomaly is given by

$$A_{\text{un,zero}} = \int_{M^4} d^4x \sum_s \left[ 3 \Omega^1_{4s}(\Lambda; A, F, R) \right]_{\text{Ad}} = \int_{M^4} d^4x 3\Omega^1_{4r}(\Lambda; A, F, R)_{\text{Ad}}.$$ (43)

It should be stressed here, that the gauge fields that appear in this formula are in the adjoint representation $\text{Ad}$ defined in (22). The expression after the second equality sign is the standard result obtained by computing the zero mode anomaly, taking into account the four dimensional chiral zero modes (22) only. For orbifold models where the spectrum at all fixed points is equal, this formula is easily checked; all 27 fixed points give the same trace over $R_s = R$.

Reversing this reasoning, boils down to the argumentation of ref. [7]. (However, when Wilson lines are present it is not so easy to generalize such arguments to obtain the correct form of the localized anomaly; for this reason we preformed the direct calculation in subsection 3.1.)

Since the twisted states are already four dimensional, the reduction to the zero modes of the gauge fields in the anomaly formula (42) directly gives

$$A_{\text{tw,zero}} = \int_{M^4} d^4x \sum_s \left[ \Omega^1_{4s}(\Lambda; A, F, R) + 3\Omega^1_{4s}(\Lambda; A, F, R) \right]_{\text{Ad}}.$$ (44)

4 Local anomaly cancelation

In the previous section we have collected the various sources of gauge anomalies in the field theory limit of heterotic $E_8 \times E_8$' string theory on $T^6/Z_3$ in the possible presence of multiple Wilson lines. Using the expressions for the localized gauge anomalies (41) and (42), due to the ten dimensional gaugino and four dimensional chiral twisted states, respectively, one can now explicitly check whether the anomalies cancel locally, or not.
As was shown in the previous section, the local four dimensional anomalies for both the untwisted \cite{11} and twisted states \cite{12} are determined by the local four dimensional chiral spectra, given in \cite{20} and \cite{26}, respectively. Therefore the analysis of the cancelation of the localized fixed point anomalies can be performed using the information of the four dimensional chiral spectrum at the fixed points, only. The spectra for both untwisted and twisted states at fixed point $3_s$ are determined by the shift vector $v_s(v,a) = v + s_i a_i$, see \cite{20} and \cite{26}.

4.1 Fixed point equivalent models

Even if one uses the knowledge of the spectrum at the fixed points, the analysis of the localized gauge anomalies may not be very practical, since the number of different models is large due to the possibility of including Wilson lines. To overcome this hurdle, we first develop the concept of fixed point equivalent models, which allows us to systematically analyze the anomalies of a model with arbitrary consistent choice of shift and Wilson lines. Fixed point equivalent models are not only a useful tool to investigate localized anomalies, but give a clear insight in the structure of orbifold models in general.

Consider two $T^6/\mathbb{Z}_3$ orbifold models with shifts $v, v'$ and Wilson lines $a_i, a'_i$. Let $3_s$ and $3_s'$ be fixed points of these orbifold models. As stressed in previous sections, the complete spectra of (untwisted and twisted) states at the fixed points are determined by the local shift vectors $v_s(v,a)$ and $v_{s'}(v',a')$, respectively, and therefore also the gauge anomaly is determined by those shifts. We call these two models fixed point equivalent, if their local gauge shift vectors are equal up to Weyl reflections or lattice shifts; denoted by $v_{s'}(v',a') \simeq v_s(v,a)$. (Properties of this equivalence relation $\simeq$ and Weyl reflections are given in appendix C.)

We can also introduce the notion of equivalent fixed points $3_s$ and $3_s'$ within one model, by requiring that $v_s(v,a) \simeq v_{s'}(v,a)$. In a pure orbifold model, of course, all 27 fixed points are equivalent. If there is one Wilson line, there can exist 3 sets of 9 fixed points that may be inequivalent. With two Wilson lines there are in principle 9 sets of 3 fixed points that may be inequivalent, and so on. With the corners of the triangles in figure 2 the inequivalent fixed points can be distinguished graphically.

Using the concept of fixed point equivalent models, fixed points of different orbifold models can thus be related to each other. In particular, the model with gauge shift $v$ and Wilson lines $a_i$ at the fixed point $3_s$ is equivalent to the pure orbifold theory with $v' = v_s(v,a)$ (and $a'_i = 0$). This naturally leads to a classification of orbifold models in terms of pure orbifold models, which are discussed in the next subsection. This is, of course, very convenient for the investigation of the localized gauge anomalies in arbitrary orbifold models, since we can reduce the problem to a standard case.

4.2 Eight pure $T^6/\mathbb{Z}_3$ orbifold models

Pure orbifold models do not have Wilson lines $v_s(v,0) = v$, therefore the spectra at each fixed point, in both the twisted and untwisted sectors, are identical. Moreover, using \cite{20} and \cite{22} it follows that local and global projections for the matter fields are identical and local gauge groups and zero mode gauge group are the same, i.e. local and global massless spectra coincide: $R_s = R$ and $G_s = G$ for all $s$. This shows that the zero mode anomaly \cite{13} is democratically distributed over all fixed points according to the localized untwisted anomaly \cite{11}. Put differently, by investigating the four dimensional zero mode anomalies, we can directly infer the properties locally at the fixed points. This

\footnote{The concept of fixed point equivalent models is not entirely new \cite{13}, however there this concept was only introduced for the twisted states so as to facilitate the analysis of possible low energy anomalous $U(1)$s.}
agrees with the intuitive expectation, since without Wilson lines, the fixed points are indistinguishable from each other.

Up to the equivalence relation $\simeq$ there are exactly eight pure $\mathbb{Z}_3$ orbifold models (cf. [20], [33]). Since three of those models can be obtained from three others by a simple interchange of $E_8$ and $E_8'$, we only give the spectra of the five fundamental pure orbifold models in table 1. We have indicated the local gauge group, the untwisted and the twisted matter at the fixed points. This table also shows which twisted states are triplets under $SU(3)_H$. Furthermore, in models with $U(1)$ factors in the gauge group $G$, the subscripts indicate the $U(1)$ charges of these matter representations. We will use the model names $E_8$, $E_6$, $E_6'$, $E_6^2$, $E_7$, $E_7'$, $SU(9)$ and $SU(9)'$ to indicate to which model the spectrum at a given fixed point in a given model is equivalent. We use similar terminology to indicate to which standard shift a generic shift is equivalent. In table 2 we summarized how to identify the standard shifts; appendix C explains how these results are obtained.

4.3 Non–Abelian anomalies

We come to our first main conclusion, using the fixed point equivalent model analysis developed in the previous subsections. As has been done for the zero modes in the past, it is not difficult to see from the spectra given in table 1 that none of the eight pure orbifold models has non–Abelian anomalies. Since any $\mathbb{Z}_3$ orbifold with Wilson lines is equivalent to one of the pure orbifold models at a given fixed point, it follows that there are no non–Abelian anomalies at that fixed point.

From this we can derive some interesting results for the anomalies in the effective four dimensional zero mode theory. Indeed, using (43) we obtain

$$\left[3 \Omega^1_4 \mathbf{R} + \sum_s \left(\Omega^1_4 \mathbf{S}_s + 3 \Omega^1_4 \mathbf{T}_s\right)\right]_{\text{Ad}} = \sum_s \left[\frac{1}{27} 3 \Omega^1_4 \mathbf{R}_s + \Omega^1_4 \mathbf{S}_s + 3 \Omega^1_4 \mathbf{T}_s\right]_{\text{Ad}}, \quad (45)$$

We see that the zero mode anomaly is the sum of the gauge anomalies at the fixed points, with four dimensional gauge fields that do not depend on the six internal dimensions. But since we know that there are no non–Abelian anomalies at the fixed points, we conclude that there are no non–Abelian
### Table 1: The local fixed point spectra of five of the eight pure $Z_3$ orbifold models are displayed. (The $E_6'$, $E_7'$ and $SU(9)'$ models are obtained from the $E_6$, $E_7$ and $SU(9)$ models by interchanging the primed and unprimed shift entries.) For these pure orbifold models the four dimensional zero mode spectrum is identical to the local spectrum.

| Model | Shift $v'$ and gauge group $G = G_s$ | Untwisted $(3_H, R = R_s)$ | Twisted $(1_H, S_s)$ $(3_H, T_s)$ |
|-------|-----------------------------------|--------------------------|----------------------------------|
| $E_8$ | $\frac{1}{2}(0^8 \mid 0^8)$ $E_8 \times E_8'$ | $(27, 3)(1)'$ | $(1)(1)'$ |
| $E_6$ | $\frac{1}{3}(-2, 1^2, 0^5 \mid 0^6)$ $E_6 \times SU(3) \times E_6'$ | $(27, 3)(1, 1) + (1, 1)(27, 3)'$ | $(1, 3)(1)'$ |
| $E_6'$ | $\frac{1}{3}(-2, 1^2, 0^5 \mid -2, 1^2, 0^5)$ $E_6 \times SU(3) \times E_6' \times SU(3)'$ | $(1)(64)'_1 + (1)(64)'_2$ | $(1)(64)'_1 + (1)(64)'_2$ |
| $E_7$ | $\frac{1}{3}(0, 1^2, 0^5 \mid -2, 0^7)$ $E_7 \times U(1) \times SO(14)' \times U(1)'$ | $(1)(64)'_1 + (1)(64)'_2$ | $(1)(64)'_1 + (1)(64)'_2$ |
| $SU(9)$ | $\frac{1}{3}(-2, 1^4, 0^3 \mid -2, 0^7)$ $SU(9) \times SO(14)' \times U(1)'$ | $(9)(1)'_1$ | $(9)(1)'_1$ |

### Table 2: The number of zeros (mod 1) of an $E_8$ shift $v$ determines to which standard shift, given in table 1, it is equivalent. If the shift does not have any zeros, we distinguish whether the product of the entries, chosen to be $\pm 1$ (using lattice shifts), is even ($0_+$) or odd ($0_-$. (For details see appendix C.)
gauge anomalies at the four dimensional zero mode level. Moreover, let $Y$ be a $U(1)$ factor generator, then it is not anomalous at the zero mode level, if at each of the fixed points $Y$ is part of a sub-algebra that generates a non-Abelian factor in $G_s$.

This confirms the well-known result for the global four dimensional gauge anomaly: the states subjected to (22) contribute to the untwisted zero mode anomaly (23). The zero mode twisted anomalies (14) are the same as the localized twisted anomalies (12), and therefore determined by the spectrum (26), since the information concerning the localization of the twisted states is ignored.

4.4 Example: revisiting the $v = a_1$ model

In the introduction we have given one simple example of a $\mathbb{Z}_3$ orbifold model where local anomaly cancelation does not seem to hold. However, from the analysis of the previous section, we know that for any orbifold model the non-Abelian anomalies cancel locally. To show exactly at which point the analysis presented in the introduction was too naive, and to illustrate various aspects of the general investigation, we return briefly to that example here.

The $\mathbb{Z}_3$ orbifold model discussed in the introduction has the gauge shift equal to its single Wilson line: $v = a_1 = \frac{1}{3}(-2, 1^2, 0^5 | 0^8)$. The local shifts $v_s$ are given by

$$v + a_1 \simeq -\frac{1}{3}(-2, 1^2, 0^5 | 0^8)$$

$$v + 2a_1 \simeq (0^8 | 0^8)$$

As at the fixed points $\mathbb{Z}_{2s3}^3$ of the first $T^2$ the local shift is trivial, the full spectrum there is equivalent to the $E_6$ model. The shift $v_{0s2s3}$ is equal to the shift of the $E_6$ model, while the shift $v_{1s2s3} = -v_{0s2s3}$ of the fixed points $\mathbb{Z}_{1s2s3}$ gives rise to the complex conjugate states w.r.t. those arising on the fixed point $\mathbb{Z}_{0s2s3}$. For the twisted states one may use table 1 to arrive at the spectrum given in the introduction in (1). We have used the schematic picture, defined in figure 2, to give an overview of the fixed point equivalent models that are induced at the fixed points of the orbifold.

In the reasoning presented in the introduction, we only took the four dimensional zero modes into account. The twisted states zero modes are of course localized at the fixed points, but the untwisted states analysis was too naive in the sense that the collective effects of the massive states of the untwisted sector from the effective four dimensional viewpoint were ignored. These states were taken into account in the direct calculation of the gaugino anomaly presented in section 3.1.

In terms of figure 1 of the introduction the problem may be stated as follows. As argued in section 2.3 the zero mode limit of the twisted states does not remove any states, as the twisted states are already four dimensional. Therefore, we can safely go into the opposite direction of the zero mode limit indicated in that figure. For the four dimensional zero mode untwisted matter, however, we are not allowed to do this, since in the zero mode limit many states of the super Yang–Mills theory coupled to supergravity are removed. The correct way of identifying the untwisted matter states, responsible for the localized gauge anomalies, was discussed in section 2.2 leading to the conditions (20).

The apparent paradox of the introduction is thus resolved by taking into account all chiral states, twisted and untwisted, present at a given fixed point. This is conveniently described by using the fixed point equivalent pure orbifold models discussed in the previous subsection.
Table 3: The $U(1)^3$ and mixed gravitational anomaly traces are computed for the two pure orbifold models that contain $U(1)$ factors. We have included the level $k_q$ with which (48) can be checked straightforwardly.

### 4.5 Anomalous $U(1)$s in heterotic $Z_3$ orbifolds

In the previous subsections we have drawn the conclusion, that in heterotic $Z_3$ orbifold models there are never (localized) non–Abelian anomalies. This followed directly from the analysis of the fixed point equivalent pure orbifold models, where this statement can be proven by direct inspection. In this section we use the same logic to investigate the situation of anomalous $U(1)$s in heterotic orbifold models. Therefore, we start by summarizing the well–known status of anomalous $U(1)$s for pure orbifold models. By employing the fixed point equivalent models, we can use this information to understand the localized anomalous $U(1)$s in orbifold models with Wilson lines, and relate this to the four dimensional zero mode situation.

**Pure orbifold models with an anomalous $U(1)$**

The eight possible pure $Z_3$ orbifold models have been summarized in table 1 of section 4.2. This table tells us, that only four of those models have $U(1)$ factors in their gauge groups: the $E_7$, $E_7'$ and $SU(9)$, $SU(9)'$ model. The latter two have only one $U(1)$ factor, while the former ones have two. The precise forms of these $U(1)$ generators have been identified in (21).

For the $E_7$ and $SU(9)$ models and all three $U(1)$s we calculate the pure gauge and mixed gravitational anomaly locally at a fixed point in table 2 by computing

$$tr_{L_s} Q_s^3 = \frac{1}{27} \left( 3 tr_{R_s} + tr_{S_s} + 3 tr_{T_s} \right) Q_s^3,$$

$$tr_{L_s} Q_s = \frac{1}{27} \left( 3 tr_{R_s} + tr_{S_s} + 3 tr_{T_s} \right) Q_s.$$

As the combination of traces in brackets in these equations will appear often in the discussion below, we use $tr_{L_s}$ to denote the local trace over all representations present at fixed point $s$. Here, we only focus on fermions as sources of possible anomalous $U(1)$ contribution, we investigate the role of the anti–symmetric tensor later. The results are displayed in table 3 using the notation for the $U(1)$ generators introduced in (21). We see that both models have one anomalous $U(1)$.

Apart from the pure and mixed gravitational anomalies there are also mixed non–Abelian anomalies. It turns out that anomalous $U(1)$s at the fixed points are universal in the sense that the following
relation holds [44] (see also the discussion in [43])

\[
\frac{1}{6} k_q s \text{tr}_{L_s}(Q_s^3) = \frac{1}{2} \sum_a Q_s(L_s^{(a)}) I_2(L_s^{(a)}) = \frac{1}{48} \text{tr}_{L_s}(Q_s).
\] (48)

The sum is over the irreducible representations \(L_s^{(a)}\) contained in the fixed point representation \(L_s = \frac{1}{27} 3R_s + S_s + 3T_s\), with the same multiplicity factors. The quadratic indices \(I_2(L_s^{(a)})\) are normalized w.r.t. the simple factors of the local gauge group \(G_s\), and \(Q_s(L_s^{(a)})\) is the U(1) charge of \(L_s^{(a)}\). Because of the inclusion of the level \(k_q = 2q_s^2\) of \(Q_s\), this formula is valid for any normalization of this local U(1) generator.

**Wilson lines and anomalous U(1)s**

The universality relation (48) is consistent with the localized pure and mixed U(1) anomalies, of the ten dimensional gaugino (41) and the twisted states (42), computed before:

\[
A_{U(1)} = \frac{-1}{(2\pi)^3} \sum_s \left\{ \text{str}_{L_s} \left[ \frac{1}{6} \Lambda_1 F_1^2 + \frac{1}{2} \Lambda_1 F^2 \right]_{\text{Ad}_s} + \frac{1}{48} \text{tr}_{L_s} \left[ \Lambda_1 \right]_{\text{Ad}_s} \text{tr} R^2 \right\} \delta(z - 3s - \Gamma) d^6z.
\] (49)

Here we have used the solutions to the descent equations (31); and \(d^6z\) denotes the volume form of the torus. The local trace \(\text{tr}_{L_s}\) over all representations at fixed point \(3_s\) has been defined in (47). We have decomposed the field strength \(F|_{\text{Ad}_s} = F_1|_{\text{Ad}_s} + \tilde{F}|_{\text{Ad}_s}\) at fixed point \(3_s\) in a U(1) part \(F_1\) and a non–Abelian part \(\tilde{F}\), and \(\Lambda_1|_{\text{Ad}_s}\) denotes the infinitesimal U(1) parameter.

In the previous subsection we investigated the pure orbifold models with anomalous U(1) factor groups. Applying the fixed point equivalent model analysis introduced in the previous section, we immediately conclude, that in a \(Z_3\) orbifold model with Wilson lines there is a local anomalous U(1) factor group present at fixed point \(3_s\), if the model at that fixed point is equivalent to one of the pure orbifold models \(E_7, E_7', SU(9)\) or \(SU(9)'.\)

Moreover, the generator of the anomalous U(1) factor is identified by the results of table 3. There the pure and mixed gravitational anomalies are calculated for the generators \(q_s\), or \(q'_s\), given in (21), when the fixed point equivalent model is \(E_7, SU(9)', E_7', SU(9)\), respectively.

Obviously, at each fixed point there is at most one anomalous U(1). However, in the heterotic orbifold models as a whole, there may be many different anomalous U(1) generators, corresponding to anomalous U(1)s at different fixed points. It is therefore possible that a generator of an anomalous U(1), at a given fixed point, is not anomalous at another fixed point, or is even part of a non–Abelian factor.

In the case of different anomalous U(1) factors at various fixed points, it is not possible to define a set of linear combinations of their generators, such that only one linear combination is anomalous at all anomalous fixed points simultaneously, while all perpendicular local U(1) generators are anomaly free at all fixed points.

The next question we address is how to identify the anomalous U(1) in the zero mode theory, if present. To this end we use the relation between the localized fixed point anomaly and the zero mode anomaly, given in (45), for the mixed gravitational gauge anomaly (47):

\[
A_{U(1)}^{\text{grav},\text{zero}} \propto \sum_s \text{tr}_{L_s}(\Lambda) = \sum_s \text{tr}_{L_s}(H_I) \Lambda^I.
\] (50)

(Here we have restricted the gauge parameter \(\Lambda\) to the Cartan subalgebra, since all localized anomalous U(1)s are contained in the Cartan of \(E_8 \times E_8'\).) From this it follows that there can only be an anomalous
U(1) at the zero mode level, when there is at least one localized anomalous U(1); otherwise all local traces $\text{tr}_{L_s}(H_I)$ vanish. The gauge parameters $\Lambda = \Lambda^I H_I$ can be decomposed

$$
\Lambda_{\parallel} = \sum_{I,J} \Lambda^I \sum_s \frac{\text{tr}_{L_s}(H_J) \sum_t \text{tr}_{L_t}(H_J)}{\sum_{K} (\sum_s \text{tr}_{L_s}(H_K))^2} H_J, \quad \Lambda_{\perp} = \Lambda - \Lambda_{\parallel}
$$

(51)

into parallel and perpendicular components with respect to the anomaly

$$
A_{U(1)}^{\text{grav,zero}} \propto \sum_s \text{tr}_{L_s}(\Lambda_{\parallel}), \quad \text{tr}_{L_s}(\Lambda_{\perp}) = 0.
$$

(52)

Hence at the zero mode level one can define one anomalous U(1), such that all perpendicular U(1) generators are anomaly free. This anomalous U(1) is of course canceled by the standard Green–Schwarz mechanism [45]; due to the resulting quadratically divergent Fayet–Iliopoulos D–term [46] this symmetry is spontaneously broken [47, 48, 49].

4.6 Examples with localized U(1)s

Wilson line induced anomalous U(1)s

We close this section with two examples. Contrary to our previous example, discussed in the introduction and section 4.4, we choose the gauge shift of the double $E_6$ embedding and Wilson line

$$
v = \frac{1}{3}(-2, 1^2, 0^2, 0^3 \mid -2, 1^2, 0, 0^4), \quad a_1 = \frac{1}{3}(0, 0^2, 1^2, 0^3 \mid 0, 0^2, -1, 0^4),
$$

(53)

such that U(1) factors arise. The fixed point equivalent pure orbifold models are easily identified using the method exposed in this section, the corresponding local shifts are

$$
v_{0\text{s}s_3} = v + 2a_1 = \frac{1}{3}(-2, 1^2, 0^2, 0^3 \mid -2, 1^2, 0, 0^4),
$$

$$
v_{1\text{s}s_3} = v + a_1 = \frac{1}{3}(-2, 1^2, 1^2, 0^3 \mid -2, 1^2, -1, 0^4),
$$

$$
v_{2\text{s}s_3} = v + 2a_1 = \frac{1}{3}(-2, 1^2, -1^2, 0^3 \mid -2, 1^2, 2, 0^4).
$$

(54)

To show that these local shifts are equivalent to the ones indicated in the picture, one may count the number of zeros of a shift and use table 2. Obviously, also in this case the non–Abelian anomalies cancel. Although the local gauge shifts $v_{1\text{s}s_3}$ and $v_{2\text{s}s_3}$ are equivalent, they are definitely not equal (up to an overall sign). This fact implies that the embedding of SU(9) and SO(14)$\uparrow \times U(1)^\downarrow$ in $E_8$ and $E_8'$, respectively, at the fixed points $3_{1\text{s}s_3}$ and $3_{2\text{s}s_3}$ is different. This can have some important consequences, as we will see throughout this example.

The first place where this can be noticed, is in the identification of the generators of the anomalous U(1)s at the fixed points $3_{1\text{s}s_3}$ and $3_{2\text{s}s_3}$. The general arguments of section 4.5 explained that the candidates for anomalous U(1)s are given by [21]. This means that their generators can be read off from the local shifts [53]. Therefore, the charges of the anomalous U(1)s of the SU(9) models are with respect to the generators (cf. table 2)

$$
q'_1 = 3q'_{1\text{s}s_3} = (0^8 \mid 1^3, 1, 0^4), \quad q'_2 = 3q'_{2\text{s}s_3} = (0^8 \mid 1^3, -1, 0^4),
$$

(55)
such that only twisted states at the fixed points, in addition some untwisted zero mode matter arises. Since the U(1) factors can be found below.) As usual the twisted zero mode states are the same as the twisted (\(q_1\)) and \(q_2\) untwisted (\(q_0\)) representations at that fixed point, see table 1, it follows that this charge is zero as indicated in table 4. Moreover, since the shift vectors \(v \equiv v_1 \times v_2\) and \(v_1\) generate the anomalous U(1) there, the perpendicular part of \(q_2\) with respect to \(q_1\) is one of the generators of SO(14)

Therefore, contrary to our previous example of section 4.4, here the unbroken gauge group at the zero mode level

\[
G = SU(6) \times SU(3) \times U(1) \times SO(8)\times SU(3)\times U(1)_+ \times U(1)_-
\]  

forms a true subgroup \(G = \cap_G\) of the local gauge groups \(G_s\) at the fixed points. (A discussion of the U(1) factors can be found below.) As usual the twisted zero mode states are the same as the twisted states at the fixed points, in addition some untwisted zero mode matter arises. Since the twisted states may now be charged under all the zero mode U(1) factors, we present the full zero mode matter spectrum in table 4. Again it may be checked that there is no non–Abelian anomaly, as is to be expected from the general analysis.

There are three U(1) factors at the zero mode level. From the local shifts, it can be inferred that the U(1) factor in the first E\(_8\) corresponds to the Cartan element

\[
q = (0^3, 1^2, 0^5 | 0^8).
\]

Since the local shift \(v_{0\times2\times3}\) of the fixed points \(3_{0\times2\times3}\) does not have entries in this direction, it follows that twisted states at that fixed point should have net charge zero. As there is just one irreducible representation at that fixed point, see table 1, it follows that this charge is zero as indicated in table 4. Moreover, since the shift vectors \(v_{1\times2\times3}\) and \(v_{2\times2\times3}\) have opposite entries in the direction of \(q\), it follows that the \(q\)–charges of the twisted states at fixed points \(3_{1\times2\times3}\) and \(3_{2\times2\times3}\), are opposite. This is again, in agreement with the twisted spectrum of table 4. It can be confirmed that this U(1) is not anomalous. The reason for this has been discussed in section 4.3 at all fixed points this U(1) generator is part of the set of generators of non–Abelian subgroups of E\(_8\), see the picture in 54.

The two U(1) generators in the E\(_8\) are orthogonal linear combinations of the anomalous generators \(q'_1\) and \(q'_2\) at the two sets of fixed points

\[
q'_+ = \frac{1}{2}(q'_1 + q'_2) = (0^8 | 1^3, 0, 0^4), \quad q'_- = \frac{1}{2}(q'_1 - q'_2) = (0^8 | 0^3, 1, 0^4),
\]

such that only \(q'_+\) is anomalous. This is a special case of the general results given in 51. Also, in this specific example, it is not possible to define one linear combination of these U(1) generators which is responsible for both the localized and global anomalous U(1)s at the same time.

| States | Representation | Spectrum | \((SU(6), SU(3))q(SO(8), SU(3))q'_+q'_-\) |
|--------|----------------|----------|------------------------------------------|
| untwisted | \((3_H, R)\) | \((15, \bar{3})_0(1, 1)_0^0 + (1, 1)_0^3(\bar{3}, \bar{3})_1^0 + (1, 1)_0^3(\bar{3}, \bar{3})_0^2^{	ext{rep}}\) |
| twisted | \((1_H, S_{0\times2\times3})\) | \((1, 3)_0(1, 3)_0^0\) |
|         | \((1_H, S_{1\times2\times3})\) | \((6, 1)_0^3(1, 1)_1^4 + (1, 3)_0^3(1, 1)_1^4\) |
|         | \((1_H, S_{2\times2\times3})\) | \((6, 1)_0^3(1, 1)_1^4 + (1, 3)_0^3(1, 1)_1^4\) |

Table 4: The zero mode matter representations of the model with a shift and a Wilson line, given in \(53\), charged under the zero mode gauge group \(56\).
| States | Representation | Spectrum | (SU(6), SU(3))_q(SO(8), SU(3))^q_{L,R} |
|--------|---------------|----------|----------------------------------------|
| untwisted | (3_H, R) | (3, 6)_2(1, 1)_{0,0} |
| twisted | (1_H, S_{00s_{3}}) | (1, 15)_{0}(1, 1)_{0,0} + (1, 6)_2(1, 1)_{0,0} + (1, 3)_{2}(1, 1)_{0,0} |
| | (3_H, T_{00s_{3}}) | (3, 1)_{0}(1, 1)_{0,0} |
| | (1_H, S_{10s_{3}}) | (1, 3)_{0}(1, 3)_{0,0} |
| | (1_H, S_{20s_{3}}) | (1, 3)_{0}(1, 3)_{0,0} |
| | (1_H, S_{01s_{3}}) | (1, 1)_{4}(8, 1)_{0,0} + (1, 1)_{4}(1, 3)_{0,0} + (1, 1)_{4}(1, 3)_{1,0} + (1, 1)_{4}(1, 1)_{0,0} |
| | (3_H, T_{01s_{3}}) | (1, 1)_{4}(1, 1)_{0,0} |
| | (1_H, S_{11s_{3}}) | (1, 1)_{4}(8, 1)_{0,0} + (1, 1)_{4}(1, 3)_{0,0} + (1, 1)_{4}(1, 3)_{1,0} + (1, 1)_{4}(1, 1)_{0,0} |
| | (3_H, T_{11s_{3}}) | (1, 1)_{4}(1, 1)_{0,0} |
| | (1_H, S_{21s_{3}}) | (1, 1)_{4}(8, 1)_{0,0} + (1, 1)_{4}(1, 3)_{0,0} + (1, 1)_{4}(1, 3)_{1,0} + (1, 1)_{4}(1, 1)_{0,0} |
| | (3_H, T_{21s_{3}}) | (1, 1)_{4}(1, 1)_{0,0} |
| | (1_H, S_{02s_{3}}) | (1, 6)_{4}(1, 1)_{0,0} + (3, 1)_{4}(1, 1)_{0,0} |
| | (1_H, S_{12s_{3}}) | (1, 6)_{4}(1, 1)_{0,0} + (3, 1)_{4}(1, 1)_{0,0} |
| | (1_H, S_{22s_{3}}) | (1, 6)_{4}(1, 1)_{0,0} + (3, 1)_{4}(1, 1)_{0,0} |

Table 5: The zero mode matter representations of the model with a shift and two Wilson lines, given in [59], charged under the zero mode gauge group [62].

All non-trivial equivalent models at the fixed points

\[
\begin{align*}
\nu &= \frac{1}{3} (-2, 1^2, 0, 0^4 ) \quad | \ 0, 0^2, 0, 0^4 ) , \\
\alpha_1 &= \frac{1}{3} ( 0, 0^2, -2, 1^4 ) \quad | \ 0, 0^2, -2, 0^4 ) , \\
\alpha_2 &= \frac{1}{3} ( 0, 0^2, 0, 0^4 ) \quad | \ -2, 1^2, 0, 0^4 ) .
\end{align*}
\]

(59)

Our final example contains two Wilson lines in addition to the gauge shift given in the equation [59], above. This model thus provides an example of the second diagram in figure [2].

This model has the feature that all non-trivial fixed point equivalent models of table [1] appear at
one or more fixed points, as can be seen from the local shift vectors

\begin{align*}
 v_{00s3} &= \frac{1}{3} (-2, 1^2, 0, 0^4 | 0, 0^2, 0, 0^4), & v_{21s3} &= \frac{1}{3} (-2, 1^2, -2, 1^4 | 2, -1^2, -2, 0^4), \\
v_{10s3} &= \frac{1}{3} (-2, 1^2, 0, 0^4 | -2, 1^2, 0, 0^4), & v_{02s3} &= \frac{1}{3} (-2, 1^2, 2, -1^4 | 0, 0^2, 2, 0^4), \\
v_{20s3} &= \frac{1}{3} (-2, 1^2, 0, 0^4 | 2, -1^2, 0, 0^4), & v_{12s3} &= \frac{1}{3} (-2, 1^2, 2, -1^4 | -2, 1^2, 2, 0^4), \\
v_{01s3} &= \frac{1}{3} (-2, 1^2, -2, 1^4 | 0, 0^2, -2, 0^4), & v_{22s3} &= \frac{1}{3} (-2, 1^2, 2, -1^4 | 2, -1^2, 2, 0^4), \\
v_{11s3} &= \frac{1}{3} (-2, 1^2, -2, 1^4 | -2, 1^2, -2, 0^4),
\end{align*}

Using these shifts and table \ref{table:shifts} we composed the figure in \ref{figure:figure}. Since all non–trivial fixed point models arise at the fixed points, both the E_7 and SU(9) equivalent models give rise to anomalous U(1)s at the fixed points. These generators take the form

\begin{align*}
 q_{01s3} &= (1^8 | 0^8), & q_{01s3} &= -q_{02s3} = (0^8 | 0^3, 1, 0^4), \\
 q_{11s3} &= (1^8 | 0^8), & q_{11s3} &= -q_{22s3} = (0^8 | 1^3, 1, 0^4), \\
 q_{21s3} &= (1^8 | 0^8), & q_{21s3} &= -q_{12s3} = (0^8 | -1^3, 1, 0^4).
\end{align*}

Therefore, it is in this case even more complicated to see directly, which linear combination of anomalous U(1) generators at the fixed points, is the generator of the anomalous U(1) at the zero mode level. In the zero mode gauge group

\[ G = SU(6) \times SU(3) \times U(1) \times SO(8)' \times SU(3)' \times U(1)'_+ \times U(1)'_- \]  

we find again three U(1) factors; two of them in the E_8' group. The corresponding shift of these U(1)s are given by

\begin{align*}
 q &= \frac{1}{3} (1^8 | 0, 0^2, 0, 0^4), & q_+' &= \frac{1}{3} (0^8 | 0^3, 1, 0^4), \\
 q_- &= \frac{1}{3} (0^8 | 1^3, 0, 0^4).
\end{align*}

The zero mode matter spectrum is collected in table \ref{table:spectra}. Using \ref{table:spectra} we identify the zero mode anomalous U(1) to be generated by \( q \).

5 Conclusions and outlook

We have investigated the structure of local anomalies of heterotic orbifold models. For this purpose we considered the field theory limit of the heterotic E_8 \times E_8' string compactified on the orbifold \( T^6/Z_3 \). The main results of this endeavor can be summarized as follows:

We have computed the gaugino anomaly in the presence of (shift embedded) Wilson lines that commute with the orbifold boundary conditions, using the Fujikawa method. The result of this calculation \ref{table:gauge} shows that the anomaly becomes localized at the orbifold fixed points, and depends crucially on the local spectra of untwisted states \ref{table:untwisted}. In addition there is a ten dimensional anomaly on the orbifold, which has a normalization factor of 1/3 compared to the anomaly on the torus.

Combining this result with the twisted spectra obtained from string theory, it followed, that there are no non–Abelian anomalies in heterotic \( Z_3 \) orbifold models with shift embedded Wilson lines.
This conclusion can be drawn by direct inspection for all possible models with non–vanishing Wilson lines. However, to understand the local structure of orbifold models with Wilson lines better, we employed the notion of fixed point equivalent models. Two models are said to be fixed point equivalent, if at a given fixed point their defining gauge shifts are equal up to Weyl reflections and lattice shifts. This implies that at those fixed points their twisted and untwisted spectra are isomorphic.

It followed, that any orbifold model of this class at a given fixed point is equivalent to one of the eight pure orbifold models, which are summarized in table I. As none of them suffers from a non–Abelian anomaly, the conclusion of the previous point is confirmed.

By using fixed point equivalent models, it is easy to show that at each fixed point there is at most one anomalous U(1) present, which exists only if its fixed point equivalent model is either E_7, E_7', SU(9) or SU(9)' (see table I). However, in general the anomalous U(1)s at the different fixed points correspond to different generators of the Cartan subgroup of E_8 × E_8'. Because of this, it is not possible to define a single linear combination which is the sole source of the U(1) anomalies, at the various fixed points. At the effective four dimensional zero mode level a single anomalous U(1) can be defined from the local anomalous U(1) generators.

Let us conclude by giving an outlook on possible future directions based on the work presented in this paper:

The method we employed to compute the gaugino gauge anomaly can be applied to a much wider range of calculations on orbifolds. We have used an explicit orbifold projection operator in our computation, to be able to use the mode functions of the torus. This avoids having to work with the more complicated mode functions on the orbifold. It may be checked that for computations on five dimensional orbifolds this technique is also very powerful.

Throughout this paper we have restricted ourselves to heterotic E_8 × E_8' string theory on the simplest six dimensional prime orbifold \( T^6 / \mathbb{Z}_3 \). The calculation of the gaugino anomaly, for example, can be extended to other prime or non–prime orbifolds, of six or other dimensions. This leads to interesting questions, for example, whether non–prime orbifold models could have a more complicated anomaly structure than the one considered here?

Also the convenient concept of fixed point equivalent models will definitely have useful applications to other (heterotic) orbifold models. In addition, this tool can be useful for more phenomenological applications, as it gives insight in the structure of the theory at the fixed points, and its zero modes.

We showed, that the structure of localized anomalous U(1)s in heterotic orbifolds can be quite complicated. Therefore, we will investigate the localized Fayet–Iliopoulos terms for these models in a future publication. In particular we would like to settle the question, whether similar destabilization effects are at work as in supersymmetric five dimensional orbifold models [3, 4].

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A Spinors in relevant dimensions

In this appendix we provide some of the details involving the decomposition of the $D = (1, 9)$ spinor representation when compactified on a six dimensional torus or orbifold. Many details are omitted, they can be found in [50, 51]; here we have focused on the material needed for proper understanding of the discussions in the main text. Because of the orbifold twist action [2], we further decompose the spinor representation on $T^6$ into $T^2$ subrepresentations. As all information concerning the spinor representation is encoded in the Clifford algebra, we start by describing the Clifford algebras in $D = 2; 6; (1, 3);$ and $(1, 9)$ dimensions.

The following properties hold in any of these dimensions. We described them with generic generators $\Gamma_M$ of a $D$-dimensional Clifford algebra. The generators, the chiral operator $\tilde{\Gamma}$ and the Dirac conjugation of a fermion $\psi$ are defined by

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN} 1, \quad \tilde{\Gamma} = \alpha_D \prod \Gamma_M, \quad \Gamma^\dagger_M = \begin{cases} \Gamma_M & \text{Eucl.,} \\ \Gamma_0 \Gamma_M \Gamma_0 & \text{Mink.} \end{cases}, \quad \bar{\psi} = \begin{cases} \psi^\dagger & \text{Eucl.,} \\ \psi^\dagger \Gamma_0 & \text{Mink.} \end{cases},$$

where $\eta_{MN}$ denotes a Minkowski or Euclidean metric with signatures $(-1, 1, \ldots)$ or $(1, 1, \ldots)$, respectively. (The factor $\alpha_D$ is chosen such that the chirality operator $\tilde{\Gamma}$ is Hermitian.) In addition in even dimensions there are two charge conjugation matrices $S^\pm$ with the properties

$$S^{-1}_M S^\pm_M = \pm \Gamma^T_M, \quad S^\dagger = S^{-1} = S^\pm, \quad \psi^{S^\pm} = S^\pm \bar{\psi}^T,$$

and $\psi^{S^\pm}$ denotes the Majorana conjugations with respect to both charge conjugation operators. In the table [6] we give an explicit basis representation of the Clifford algebra generators, the chirality operator and the charge conjugation operators, and give their dimension dependent properties.

Next we exploit the properties indicated in table [6] to discuss how a generic $D = (1, 9)$ dimensional Majorana-Weyl spinor $\psi$ with chirality $\beta = \pm$ can be decomposed. Let $\xi^\alpha$ denote a spinor in two dimensions with chirality $\alpha = \pm$. By taking a tensor product of three of such spinors, a chiral spinor $\eta^\alpha = \eta^{\alpha_1 \alpha_2 \alpha_3}$ in six dimensions is obtained, which satisfies

$$-i \Sigma_i \eta^\alpha = \alpha_i \eta^\alpha, \quad \bar{\sigma} \eta^\alpha = (-)^\alpha \eta^\alpha = \alpha_1 \alpha_2 \alpha_3 \eta^\alpha.$$  \hspace{1cm} (66)

The Majorana-Weyl spinor can then be decomposed as

$$\psi = \frac{1}{\sqrt{2}} \sum_\alpha \eta^{\alpha_1 \alpha_2 \alpha_3} \otimes \psi^{\alpha_1 \alpha_2 \alpha_3} = \frac{1}{\sqrt{2}} \sum_\alpha \xi^{\alpha_1} \otimes \xi^{\alpha_2} \otimes \xi^{\alpha_3} \otimes \psi^{\alpha_1 \alpha_2 \alpha_3}.$$  \hspace{1cm} (67)

According to table [6] the $(1, 9)$ dimensional chirality and the charge conjugation matrices can be written in terms of the $2 \times 2 \times 2$ dimensional representations as

$$\tilde{\Gamma} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \tilde{\gamma}, \quad S_- = (-\sigma_3 \otimes 1 \otimes \sigma_3 \otimes 1)(s_+ \otimes s_+ \otimes s_+ \otimes C_-),$$

hence this gives the chiral and charge conjugation properties of the $D = (1, 3)$ dimensional spinors $\psi^{\alpha_1 \alpha_2 \alpha_3}$:

$$\beta \alpha_1 \alpha_2 \alpha_3 \tilde{\gamma} \psi^{\alpha_1 \alpha_2 \alpha_3} = (-\alpha_1 \alpha_3)(\psi^{-\alpha_1 - \alpha_2 - \alpha_3} C_-) = \psi^{\alpha_1 \alpha_2 \alpha_3}.$$  \hspace{1cm} (69)

Now we choose a chiral basis for the fermions in $D = (1, 3)$ dimensions of chirality $\beta$, i.e. $\alpha_1 \alpha_2 \alpha_3 = \beta = \pm$, and see that the states with opposite chirality are related by the four dimensional Majorana
Table 6: This table summarizes the dimension dependent properties of the Clifford algebra in the dimensions $D = 2; 6; (1,3); (1,9)$. Using tensor products an explicit basis is indicated for the generators of the Clifford algebra, the chirality operator and the charge conjugation matrices. Furthermore, we have indicated which charge conjugation can be used to construct Majorana fermions; only in $D = (1,9)$ a spinor can be both Majorana and Weyl.
condition. The \( D = (1, 3) \) dimensional spinors form a singlet and a triplet under the holonomy group \( SU(3)_H \):

\[
\psi^0 = \psi^{++}, \quad \psi^a = (\psi^{+-}, \psi^{-+}, \psi^{--}).
\]

Notice that it is important to keep track of the ten dimensional chirality, since not all fermions have the same chirality: the gauginos \( \chi \) and the gravitino \( \psi_M \) are left–handed, while the dilatino \( \lambda \) is right–handed.

## B Gaugino mode functions on \( T^6 \) with Wilson lines

This appendix is devoted to the construction of a complete set of torus mode functions for scalars and gauginos, which take the Wilson lines into account. The calculation of the gaugino anomalies, presented in section 3.1, relies heavily on the material developed here.

The mode functions \( \phi_q(z) \), of the torus defined above (2), are the periodic scalar functions on \( \mathbb{C}^3 \)

\[
\phi_q(z + i) = \phi_q(z) \\
\phi_q(z + \theta i) = \phi_q(z) \\
\Rightarrow \phi_q(z) = N_q e^{2\pi i (q_i z^i + q_2 \omega^2)/R_i}, \quad \left( \begin{array}{c} q_i \\ q_2 \end{array} \right) = \frac{1}{\theta - \theta} \left( \begin{array}{c} \bar{\theta} n_i - m_i \\ -\theta n_i + m_i \end{array} \right),
\]

with \( n_i, m_i \in \mathbb{Z} \). The normalization \( N_q \) is chosen such that these wave functions are orthonormal and form a complete set on the torus \( T^6 \)

\[
\int_{T^6} dz \phi_q^\dagger(z) \phi_{q'}(z) = \delta_{q q'}, \quad \sum_q \phi_q(z) \phi_q^\dagger(z^\prime) = \delta(z - z^\prime - \Gamma).
\]

The gaugino can be decomposed as \( \chi(x, z) = \frac{1}{\sqrt{2}} \sum_{\alpha, A, q} \eta^\alpha_{q A}(z) \chi^\alpha_{q A}(x) \) with the help of the mode functions

\[
\eta^\alpha_{q A}(z) = \phi_q(z) T(z) T^{-1}(z) \eta^\alpha = \phi_q(z) e^{2\pi i a_A(z)} T_A \eta^\alpha, \\
\eta^\alpha_{q A}^\dagger(z) = \phi_q^\dagger(z) T(z) T^{-1}(z) \eta^\alpha^\dagger = \phi_q^\dagger(z) e^{-2\pi i a_A(z)} T_A^\dagger \eta^\alpha^\dagger,
\]

using the notation \( a_A(z) = a^I(z) w_I(T_A) \), and that \( w_I(T_A^\dagger) = -w_I(T_A) \) because of the Hermitean conjugation properties of the algebra (9). For \((-)^a = +\) the states \( \chi^\alpha_{q A}(x) \) are left–handed spinors in four dimensions. The factor \( 1/\sqrt{2} \) takes into account that only the positive chiral four dimensional spinors are independent, because the ten dimensional gaugino is Majorana. The gaugino wave functions \( \eta^\alpha_{q A}(z) \) are periodic up to conjugation with \( T_i \), as follows from the definition of \( T(z) \) in (12). The completeness of these gaugino mode functions can be stated as

\[
\frac{1}{2} \sum_{\alpha, q} \eta^\alpha_{q A}(z) \eta^\alpha_{q A}^\dagger(z^\prime) = \delta(z - z^\prime - \Gamma) \mathbb{1}_G \otimes \mathbb{1}_S.
\]

Here \( \mathbb{1}_S \) denotes the identity in spinor space, \( \mathbb{1}_G \) is the identity in the adjoint representation of the gauge group \( E_8 \times E_8' \) and \( \eta^{AA'} \) is the inverse Killing metric defined below (9). The factor \( 1/2 \) takes care of the double counting over charge conjugate states.
C \ E_8 Weyl reflections and classification of \ E_8 \ shifts

Many statements involving the gauge shift vectors in the main text of this article have been made up to Weyl reflections and lattice shifts. In this appendix we define equivalent \ E_8 \ shift vectors, and briefly describe the actions of some generators of the Weyl group of \ E_8 \ on those vectors. Some of the material found in this appendix is taken from \cite{33}.

The \ E_8 \ roots are given as the roots and the weights of one of the spinor representations of \ SO(16):

\[
(±1, ±1, 0^6), \quad \text{and} \quad (±\frac{1}{2}, \ldots, ±\frac{1}{2}) \quad \text{with number of minus signs even.} \quad \tag{75} \]

All permutations of the underlined components give rise to roots of \ SO(16). Let \( \Gamma_8 \) denote the root lattice of \ E_8 \. For all vectors in the lattice \( \Gamma_8 \) the sum over the entries is even, since this holds for the roots (75) that span this lattice. For more details we refer to \cite{51, 52}. Since a gauge shift \( v \) has to fulfill \( 3v^T w_I = 0 \) for all roots \( w \), it follows that \( 3v \in \Gamma_8 \) as the \ E_8 \ root lattice is self-dual.

Two \ E_8 \ gauge shifts \( v \) and \( v' \) are said to be equivalent, \( v \simeq v' \), if

\[
v' = v + u, \quad u \in \Gamma_8 \quad \text{or} \quad v' = W_\alpha(v) = v - (\alpha, v)\alpha. \quad \tag{76} \]

where \( W_\alpha(v) \) is called the Weyl reflection in root \( \alpha \) of \ E_8 \.

In general, the order in which two Weyl reflections are preformed, is relevant

\[
W_\alpha\beta(v) = W_\alpha(W_\beta(v)) = v - (\alpha, v)\alpha - (\beta, v)\beta + (\alpha, \beta)(\beta, v)\alpha. \quad \tag{77} \]

However, if the \ E_8 \ roots \( \alpha, \beta \) are orthogonal \( (\alpha, \beta) = 0 \), clearly, their Weyl reflections do commute.

Next, we describe the action of some Weyl reflections in more detail. The Weyl reflections corresponding to the \ SO(16) \ roots act as

\[
(\ v_1, v_2, v_3, \ldots) \simeq W_{(1,±1,0^6)}(\ v_1, v_2, v_3, \ldots) = (\mp v_2, \mp v_1, v_3, \ldots). \quad \tag{78} \]

Hence we see that by interchanging two shift elements, or replacing two shift elements by minus those elements equivalent shifts are obtained. In particular, if a shift has at least one zero, the sign of all other entries is irrelevent. The action of the spinorial root \( \alpha = \frac{1}{2}(1^8) \) reads

\[
v \simeq W_\alpha(v) = v - \frac{1}{4} \sum_{I=1}^{8} v^I(1^8). \quad \tag{79} \]

Let the shift be of the form \( v = \frac{1}{3}(±, \ldots, ±) \), and \( \Delta = \text{diag}(±, \ldots, ±) \) be a diagonal \( 8 \times 8 \) matrix with entries \( \Delta_I = ±1 \), such that the product of the entries of \( \alpha = 3\Delta v/2 \) is positive. Only then \( \alpha \) is a spinorial root of \ E_8 \, which can be used to Weyl reflect in

\[
v \simeq W_\alpha(v) = (I - \frac{1}{2}(\text{tr} \Delta)\Delta) v = (2v_1, 2v_2, 0^6) \simeq (-v_1, -v_2, 0^6), \quad \tag{80} \]

where we used that \( 3^2(v, \Delta v) = \text{tr} \Delta \) and chose \( \Delta_1 = \Delta_2 = -1, \Delta_{I≠1,2} = 1 \). The final equivalence is valid, since the two vectors always differ by a \ SO(16) \ root \( (±1, ±1, 0^6) \), depending on the signs of \( v_1 \) and \( v_2 \).

Consider two \ E_8 \ roots of the form \( \alpha = (a, a) \) and \( \beta = (a, -a) \), then it is immediate that they are orthogonal. On a shift vector \( v = (r, s) \) that is also split into two 4 component vectors, \( r \) and \( s \), their composition acts as

\[
(r, s) \simeq W_\alpha(r, s) = W_{αβ}(v) = (r - 2(a, r)a, s - 2(a, s)a), \quad \tag{81} \]
such that these vectors do not mix. Next we discuss an interesting application of this formula in the same spirit as \((80)\) above. Let \(v\) be a shift vector which has four or more components not equal to zero (modulo integers). By interchanging the components of this vector, it can be brought to the form \(v = (r, s)\), with \(r = \frac{1}{8}(\pm, \ldots, \pm)\). Furthermore, let \(\delta = \text{diag}(\pm, \ldots, \pm)\) be a diagonal \(4 \times 4\) matrix with entries \(\delta_i = \pm 1\). By taking \(a = 3\delta r/2\), we find that \(\alpha, \beta\) are weights of the spinor representation of \(SO(16)\). Applying \((81)\) and using that \(3^2(r, \delta r) = \text{tr} \delta\) gives

\[
(r, s) \simeq W_a(r, s) = \left( \begin{pmatrix} 1 \quad -\frac{1}{2} (\text{tr} \delta) r \end{pmatrix}, s - \frac{9}{2} (r, \delta s) \delta r \right) = \left( 0^{i-1}, 2r_i, 0^{4-i}, s - \frac{9}{2} (r, \delta s) \delta r \right).
\]

In the final equivalence we took \(\delta_i = -1, \delta_k \neq i = 1\) for a fixed \(i = 1, \ldots, 4\). These examples of equivalence of \(E_8\) shift vectors can be used to determine to which standard shift an arbitrary \(E_8 \times E_8'\) shift \((v, v')\), that satisfies the requirement of modular invariance \((\text{23})\), is equivalent. A set of standard shifts in the two \(E_8\) factors is given in table \((\text{1})\). The results of the following analysis have been summarized in table \((\text{2})\).

First, we bring the entries of \(3v \in \Gamma_8\) to a standard form: If \(3v\) has a half-integer entry, all its entries are half integer, therefore by adding any spinorial root to \(v\) all entries of \(3v\) become integer. Since \((2, 0^7)\) and its permutations are the sums of two roots of \(SO(16)\), we infer that the integer valued entries of \(3v\) can be restricted to \(3v^I = -2, \ldots, 3\). In fact, we may even assume that no entry \(3v^I\) is equal 3: If there are two or more entries equal to 3, then by adding the \(SO(16)\) root with \(-1\) at two of these entries, they become zero. If there is just one entry equal to 3, there is at least one other entry of \(3v^I\) equal to \(\pm 1\), otherwise the sum of entries of \(3v\) is not even, i.e. \(3v \notin \Gamma_8\). Again, by adding an appropriate \(SO(16)\) root we can make the 3 entry 0, and turn the \(\pm 1\) entry into \(\pm 2\). (We have assumed that this procedure has been applied throughout the paper to set all entries \(3v^I \in \{-2, -1, 0, 1, 2\}\).)

Let us first consider the case, that there is at least one entry \(3v^I = 0\). It follows from \((\text{78})\), that signs do not matter anymore; we take them positive. As the sum of entries \(3v^I\) is even, it follows that the number of 1’s is even. By a \(SO(16)\) root any pair of 2’s can be mapped to 1’s, therefore all the entries \(3v^I = 0, 1\) if the number of zeros is even, or with one additional entry of 2 if the number of zeros is odd. However, the number of zeros determines the equivalent standard shift. We find, that if the \(E_8\) shift has 8, 7, 6, 5, or 3 zeros, it is equivalent to the \(E_8\), \(SO(14)\), \(E_7\), \(E_6\), or \(SU(9)\) shifts, respectively, using permutations. This leaves the shifts with 4, 2, 1 and 0 zeros to be considered, which have to be treated separately. If the shift \(3v\) has 4 zero entries, the shift can be represented by \((-1, 1^3, 0^4)\), since the signs of the entries do not matter. Applying \((\text{82})\) with \(r = (-1, 1^3)\), we infer that it is equivalent to the \(SO(14)\) shift. Similarly, if \(3v\) has 2 zero entries, it can be brought to the form \((-1, 1^5, 0, 0)\). Using \((\text{82})\) again shows, that this corresponds to an \(E_6\) shift. If \(3v\) contains only 1 zero, we can put it into the form \(3v = (1^6, -2, 0)\), which is, using \((\text{79})\), equivalent to \(3v = (0^6, -3, -1)\). Adding the root \((0^6, 1, 1)\) to \(v\) we find the \(SO(14)\) shift vector again. If the shift does not have any zeros, all the entries of \(3v\) can be chosen to be \(\pm 1\). This again follows because the sum of all entries of \(3v\) is even, hence the number of \(\pm 2\) is even, so that by adding appropriate \(SO(16)\) roots it can be put in this form. When the product of all these entries is positive, we can employ \((\text{80})\) to show that the shift is equivalent to the \(E_7\) shift. For the negative case, we split the shift into two 4 component vectors, \(r, s\) and use \((\text{82})\), with \(\delta\) chosen such that \((r, \delta s) = 0\), to conclude that the shift is equivalent to the \(SU(9)\) shift.

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