Nonlinear interaction of electromagnetic field with quantum plasma

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Abstract

The analysis of nonlinear interaction of transversal electromagnetic field with quantum collisionless plasma is carried out. Formulas for calculation electric current in quantum collisionless plasma at any temperature are deduced. It has appeared, that the nonlinearity account leads to occurrence of the longitudinal electric current directed along a wave vector. This second current is orthogonal to the known transversal classical current, received at the classical linear analysis. The case of degenerate electronic plasma is considered. It is shown, that for degenerate plasmas the electric current is calculated under the formula, not containing quadratures. In this formula we have allocated known Kohn’s singularities (W. Kohn, 1959).

Key words: collisionless plasmas, Schrödinger equation, Dirac, Fermi, degenerate plasma, electrical current.

PACS numbers: 52.25.Dg Plasma kinetic equations, 52.25.-b Plasma properties, 05.30 Fk Fermion systems and electron gas

Introduction

Dielectric permeability in quantum plasma was studied by many authors [1] – [11]. Dielectric permeability is one of the major plasma characteristics.

This quantity is necessary for the description of skin-effect [12], for the analysis surface plasmons [13], for descriptions of process of propagation...
and attenuation of the transversal plasma oscillations [8], for studying of the mechanism of penetration electromagnetic waves in plasma [7], and for the analysis of other problems in the plasma physics [14] – [19].

Let us notice, that for the first time in work [1] the formula for calculation of longitudinal dielectric permeability into quantum plasma has been deduced. Then the same formula has been deduced and in work [2].

In the present work formulas for calculation electric current into quantum collisionless plasma at any temperature (at any degrees of degeneration of the electronic gas) are deduced.

Here the approach developed by Klimontovich Silin [1] is generalized.

At the solution of Schrödinger equation we consider and in expansion of distribution Wigner function, and in Wigner—Vlasov integral expansion the quantities proportional to square of potential of an external electromagnetic field.

It has appeared, that electric current expression consists of two summands. The first summand, linear on vector potential, is known classical expression of an electric current. This electric current is directed along vector potential electromagnetic field. The second summand represents itself an electric current, which is proportional to the square vector potential of electromagnetic fields. The second current it is perpendicular to the first and it is directed along the wave vector. Occurrence of the second current comes to light the spent account nonlinear character interactions of an electromagnetic field with quantum plasma.

For the case of degenerate quantum plasma expression of the electric current, not containing quadratures, is received. At the deducing of this expression Landau’ rule for calculation singular integrals is used. At use of this rule calculation these integrals containing a pole on the real axis, it is carried out by means of integration on infinitesimal semi-circles in the bottom semi-plane with the centre in this pole.
1. Kinetic equation for Wigner function

Let us consider Shrödinger equation which has been written down for a particle in an electromagnetic field on a density matrix $\rho$:

$$i\hbar \frac{\partial \rho}{\partial t} = H\rho - H^*\rho.$$  \hspace{1cm} (1.1)

Here $H$ is the Hamilton operator, $H^*$ is the complex conjugate operator to $H$, $H^*'$ is the complex conjugate operator to $H$, which operates on the shaded spatial variables $\mathbf{r}'$.

The operator of Hamilton of the free particle which is in the field of the scalar potential $U$ and in the field of vector potential $\mathbf{A}$, looks like:

$$H = \frac{(\mathbf{p} - (e/c)\mathbf{A})^2}{2m} + eU =$$

$$= \frac{\mathbf{p}^2}{2m} - \frac{e}{2mc}(\mathbf{p}\mathbf{A} + \mathbf{A}\mathbf{p}) + \frac{e^2}{2mc^2}\mathbf{A}^2 + eU.$$ \hspace{1cm} (1.2)

Here $\mathbf{p}$ is the momentum operator, $\mathbf{p} = -i\hbar \nabla$, $e$ is the electron charge, $m$ is the electron mass, $c$ is the speed of light.

We will rewrite the operator of Hamilton (1.2) in the explicit form

$$H = -\frac{\hbar^2}{2m} \Delta + \frac{ie\hbar}{2mc} \left(2\mathbf{A}\nabla + \nabla\mathbf{A}\right) + \frac{e^2}{2mc^2}\mathbf{A}^2 + eU.$$ \hspace{1cm} (1.3)

The complex conjugate to the $H$ operator $H^*$ according to (1.3) looks like

$$H^* = -\frac{\hbar^2}{2m} \Delta - \frac{ie\hbar}{2mc} \left(2\mathbf{A}'\nabla + \nabla\mathbf{A}'\right) + \frac{e^2}{2mc^2}\mathbf{A}'^2 + eU.'$$

Operators $H$ and $H^*'$, calculated on the density matrix, look like

$$H\rho = -\frac{\hbar^2}{2m} \Delta \rho + \frac{ie\hbar}{2mc} \left(2\mathbf{A}\nabla \rho + \rho \nabla \mathbf{A}\right) + \frac{e^2}{2mc^2}\mathbf{A}^2 \rho + eU \rho$$ \hspace{1cm} (1.4)

and

$$H^*\rho = -\frac{\hbar^2}{2m} \Delta' \rho - \frac{ie\hbar}{2mc} \left(2\mathbf{A}'\nabla' \rho + \rho \nabla' \mathbf{A}\right) + \frac{e^2}{2mc^2}\mathbf{A}'^2 \rho + eU' \rho.$$ \hspace{1cm} (1.5)
Operators $\nabla$ and $\Delta$ in (1.4) and (1.5) take action on not shaded spatial variable matrixes of density, i.e. $\nabla = \nabla_{\mathbf{R}}$, $\Delta = \Delta_{\mathbf{R}}$. In the operator $H^{*'}$ it is necessary to replace operators $\nabla = \nabla_{\mathbf{R}}$ and $\Delta = \Delta_{\mathbf{R}}$ on operators $\nabla' \equiv \nabla_{\mathbf{R}'}$ and $\Delta' \equiv \Delta_{\mathbf{R}'}$ accordingly.

Besides, here the following designations are entered

$$A' \equiv A(\mathbf{R}', t), \quad U' \equiv U(\mathbf{R}', t).$$

Let’s find the right part of the equation (1.1), i.e. the difference equalities (1.4) and (1.5) : $H\rho - H^{*'}\rho$. According to equalities (1.4) and (1.5) it is had

$$H\rho - H^{*'}\rho = -\frac{\hbar}{2m}(\Delta\rho - \Delta'\rho) +$$

$$+ \frac{i\hbar}{2mc}[2(A\nabla\rho + A'\nabla'\rho) + \rho(\nabla A + \nabla'A)] +$$

$$+ \frac{e^2}{2mc^2}[A^2(\mathbf{R}, t) - A^2(\mathbf{R}', t)] + e[U(\mathbf{R}, t) - U(\mathbf{R}', t)]\rho.$$

Connection between the density matrix $\rho(\mathbf{r}, \mathbf{r}', t)$ and Wigner’ function $f(\mathbf{r}, \mathbf{p}, t)$ is given by inverse and direct Fourier transformation

$$f(\mathbf{r}, \mathbf{p}, t) = \int \rho(\mathbf{r} + \frac{\mathbf{a}}{2}, \mathbf{r} - \frac{\mathbf{a}}{2}, t)e^{-i\mathbf{p}\mathbf{a}/\hbar}d^3\mathbf{a},$$

$$\rho(\mathbf{R}, \mathbf{R}', t) = \frac{1}{(2\pi\hbar)^3} \int f(\frac{\mathbf{R} + \mathbf{R}'}{2}, \mathbf{p}, t)e^{i\mathbf{p}(\mathbf{R} - \mathbf{R}')/\hbar}d^3\mathbf{p}.$$

Wigner’ function is analogue of distribution function for quantum systems. It is widely used in the diversified questions physicists.

Substituting representation of the density matrix through Wigner’ function in Shrödinger equation on the matrix of density (1.1), we receive

$$i\hbar \frac{\partial \rho}{\partial t} = H \left\{ \frac{1}{(2\pi\hbar)^3} \int f(\frac{\mathbf{R} + \mathbf{R}'}{2}, \mathbf{p}', t)e^{i\mathbf{p}'(\mathbf{R} - \mathbf{R}')/\hbar}d^3\mathbf{p}' \right\}.$$
\[-H^{\ast\prime} \left\{ \frac{1}{(2\pi\hbar)^3} \int f\left( \frac{\mathbf{R} + \mathbf{R'}}{2}, \mathbf{p}', t \right) e^{ip'(\mathbf{R}-\mathbf{R}')/\hbar} d^3p' \right\}. \]

We will take advantage of the equalities written above. Thus the right part the previous equation we will present in an explicit form and as a result it is received following equation

\[
\frac{i\hbar}{\partial t} = \frac{1}{(2\pi\hbar)^3} \int \left\{ -\frac{i\hbar}{m} \mathbf{p}' \nabla f + \frac{ie\hbar}{2mc} [\text{div} \mathbf{A}(\mathbf{R}, t) + \text{div} \mathbf{A}(\mathbf{R'}, t)] f + \frac{ie\hbar}{2mc} [\mathbf{A}(\mathbf{R}, t) + \mathbf{A}(\mathbf{R'}, t)] \nabla f - \frac{e}{mc} [\mathbf{A}(\mathbf{R}, t) - \mathbf{A}(\mathbf{R'}, t)] \mathbf{p}' f + \frac{e^2}{2mc^2} [\mathbf{A}^2(\mathbf{R}, t) - \mathbf{A}^2(\mathbf{R'}, t)] f + e [U(\mathbf{R}, t) - U(\mathbf{R'}, t)] f \right\} e^{ip'(\mathbf{R}-\mathbf{R}')/\hbar} d^3p'. \quad (1.6)
\]

We put in equation (1.6)

\[
\mathbf{R} = \mathbf{r} + a, \quad \mathbf{R}' = \mathbf{r} - a/2.
\]

Then in this equation

\[
f\left( \frac{\mathbf{R} + \mathbf{R'}}{2}, \mathbf{p}', t \right) e^{ip'(\mathbf{R}-\mathbf{R}')/\hbar} = f(\mathbf{r}, \mathbf{p}', t) e^{ip'a/\hbar}.
\]

We multiply the equation (1.6) on \( e^{-ip'a/\hbar} \) and we integrate on \( a \). Then we divide both parts of the equation on \( i\hbar \). As a result we receive

\[
\frac{\partial f}{\partial t} = \int \int \left\{ -\frac{\mathbf{p}'}{m} \nabla f + \frac{e}{2mc} [\mathbf{A}(\mathbf{r} + \frac{a}{2}, t) + \mathbf{A}(\mathbf{r} - \frac{a}{2}, t)] \nabla f + \frac{ie}{mch} [\mathbf{A}(\mathbf{r} + \frac{a}{2}, t) - \mathbf{A}(\mathbf{r} - \frac{a}{2}, t)] \mathbf{p}' f + \frac{e}{2mc} [\text{div} \mathbf{A}(\mathbf{r} + \frac{a}{2}, t) + \text{div} \mathbf{A}(\mathbf{r} - \frac{a}{2}, t)] f - \frac{ie^2}{2mc^2\hbar} [\mathbf{A}^2(\mathbf{r} + \frac{a}{2}, t) - \mathbf{A}^2(\mathbf{r} - \frac{a}{2}, t)] f - \right\}.
\]
\[-\frac{ie}{\hbar} \left[ U(r + \frac{a}{2}, t) - U(r - \frac{a}{2}, t) \right] f \right\} e^{i(p' - p)a/\hbar} d^3 a d^3 p' \over (2\pi\hbar)^3}. \quad (1.7)\]

In the left part of the equation (1.7) is \( f = f(r, p, t) \), under integral is \( f = f(r, p', t) \).

We consider the integral

\[
\int \int p' (\nabla f) e^{i(p' - p)a/\hbar} d^3 a d^3 p' \over (2\pi\hbar)^3 = \nabla \int \int p' f e^{i(p' - p)a/\hbar} d^3 a d^3 p' \over (2\pi\hbar)^3 =
\]

\[
= \nabla \int p' f \delta(p' - p) d p' = p \nabla f(r, p, t).
\]

Two other equalities are similarly checked

\[
\int \int \frac{e}{mc} A(r, t)(\nabla (f(r, p', t))) e^{i(p' - p)a/\hbar} d^3 a d^3 p' \over (2\pi\hbar)^3 =
\]

\[
= \frac{e}{mc} A(r, t) \nabla f(r, p, t),
\]

and

\[
\int \int \frac{e}{mc} (\text{div} A(r, t)) f(r, p', t) e^{i(p' - p)a/\hbar} d^3 a d^3 p' \over (2\pi\hbar)^3 =
\]

\[
= \frac{e}{mc} (\text{div} A(r, t)) f(r, p, t).
\]

Then the equation (1.6) we can rewrite in the form

\[
\frac{\partial f}{\partial t} + \frac{1}{m}(p - \frac{e}{c} A) \nabla f - \frac{e}{mc} (\text{div} A(r, t)) f(r, p, t) = W[f]. \quad (1.8)
\]

In equation (1.8) symbol \( W[f] \) is the Wigner—Vlasov’ integral (functional), defined by equality

\[
W[f] = \int \int \left\{ \frac{e}{2mc} \left[ A(r + \frac{a}{2}, t) + A(r - \frac{a}{2}, t) - 2A(r, t) \right] \nabla f +
\right. \]

\[
+ \frac{ie}{mch} \left[ A(r + \frac{a}{2}, t) - A(r - \frac{a}{2}, t) \right] p' f +
\]

\[
\left. \frac{e}{mc} \left[ A(r + \frac{a}{2}, t) + A(r - \frac{a}{2}, t) - 2A(r, t) \right] \nabla f \right\}.
\]
Energy of a particle is equal
\[ E = E(r, p, t) = \frac{1}{2m} (p - \frac{eA(r, t)}{c})^2 + eU(r, t). \]

Therefore velocity of a particle \( \mathbf{v} \) is equal
\[ \mathbf{v} = \mathbf{v}(r, p, t) = \frac{\partial E}{\partial p} = \frac{1}{m} (\mathbf{p} - \frac{e}{c} \mathbf{A}), \]
besides,
\[ \nabla \mathbf{v} = -\frac{e}{mc} \operatorname{div} \mathbf{A}. \]

Hence, the left part of the equation (1.9) is equal
\[ \frac{\partial f}{\partial t} + \frac{1}{m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right) \nabla f - f \frac{e}{mc} \operatorname{div} \mathbf{A} = \frac{\partial f}{\partial t} + \mathbf{v} \nabla f + f \nabla \mathbf{A} = \frac{\partial f}{\partial t} + \nabla (\mathbf{v} f). \]

Therefore the equation (1.9) can be rewritten in the form, which is standard form for the transport theory
\[ \frac{\partial f}{\partial t} + \nabla (\mathbf{v} f) = W[f]. \] (1.10)

Let \( f^{(0)} \) is the locally equilibrium Fermi–Dirac’ distribution,
\[ f^{(0)} = \left[ 1 + \exp \left( \frac{\mathcal{E} - \mu}{k_B T} \right) \right]^{-1}. \]

Here \( k_B \) is the Boltzmann constant, \( T \) is the plasma temperature, \( \mu \) is the chemical potential of electronic gas.
In an explicit form locally equilibrium function of distribution can be presented in the form

\[
f^{(0)}(\mathbf{r}, \mathbf{p}, t) = \left\{1 + \exp\left[\frac{\mathbf{p} - (e/c) \mathbf{A}(\mathbf{r}, t)}{2mk_B T} + \frac{eU(\mathbf{r}, t) - \mu}{k_B T}\right]\right\}^{-1}.
\]

Let us enter dimensionless electron velocity \(C(\mathbf{r}, t)\), scalar potential \(\phi(\mathbf{r}, t)\) and chemical potential \(\alpha\)

\[
C(\mathbf{r}, t) = \frac{v(\mathbf{r}, \mathbf{p}, t)}{v_T} = \frac{1}{p_T} \left[\mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}, t)\right],
\]

\[
\phi(\mathbf{r}, t) = \frac{eU(\mathbf{r}, t)}{k_B T}, \quad \alpha = \frac{\mu}{k_B T},
\]

where \(v_T = \frac{1}{\sqrt{\beta}}\) is the thermal electron velocity, \(\beta = \frac{m}{2k_B T}\).

Now locally equilibrium function can be presented by electron velocity as

\[
f^{(0)}(\mathbf{r}, \mathbf{p}, t) = \left[1 + \exp\left(\frac{mv^2(\mathbf{r}, \mathbf{p}, t)}{2k_B T} + \frac{eU(\mathbf{r}, t) - \mu}{k_B T}\right)\right]^{-1},
\]

or, in dimensionless parameters,

\[
f^{(0)}(\mathbf{r}, \mathbf{p}, t) = \frac{1}{1 + \exp \left[ C^2(\mathbf{r}, \mathbf{p}, t) + \phi(\mathbf{r}, t) - \alpha \right]}.
\]  \hspace{1cm} (1.11)

We denote \(\chi = \alpha - \phi\). Then

\[
f^{(0)} = \frac{1}{1 + e^{C^2 - \chi}}.
\]

The quantity \(\chi\) it is defined from the law of preservation of number of particles

\[
\int f d\Omega_F = \int f^{(0)} d\Omega_F.
\]

Here \(d\Omega_F\) is the quantum measure for electrons,

\[
d\Omega_F = \frac{2d^3p}{(2\pi\hbar)^3}.
\]

Let us notice, that in case of constant potentials \(U = \text{const}\), \(A = \text{const}\) the equilibrium function of distribution (1.11) is the solution of the equation (1.10).
Let us find the electron concentration (numerical density) $N$ and average electron velocity $u$ in the equilibrium condition. These macroparameters are defined as follows

$$N(r, t) = \int f(r, p, t) d\Omega_F,$$

$$u(r, t) = \frac{1}{N(r, t)} \int v(r, p, t) f(r, p, t) d\Omega_F.$$

For calculation of these macroparameters in the equilibrium condition it is necessary to put $f = f^{(0)}$, where $f^{(0)}$ is defined by equality (1.11). To designate these macroparameters in an equilibrium condition let us be through $N^{(0)}(r, t)$ and $u^{(0)}(r, t)$.

Let us carry out replacement of the variable of integration

$$p - \frac{e}{c} A(r, t) = p'$$

in these previous equalities. Then, passing to integration in spherical coordinates, for numerical density in an equilibrium condition we receive

$$N^{(0)} = \frac{m^3 v_T^3}{\pi^2 h^3} f_2(\alpha - \phi),$$

(1.12)

where

$$f_2(\alpha - \phi) = \int_0^{\infty} \frac{x^2 dx}{1 + \exp(x^2 + \phi - \alpha)} = \int_0^{\infty} x^2 f_F(\alpha - \phi) dx.$$

In the same way, as for numerical density, for average velocity in an equilibrium condition it is received

$$u^{(0)}(r, t) = \frac{1}{N^{(0)}} \int v(r, p, t) f^{(0)}(r, p, t) d\Omega_F,$$

or, in explicit form,

$$u^{(0)}(r, t) = \frac{2}{N^{(0)}(2\pi h)^3} \int \frac{[p - (e/c) A] d^3p}{1 + \exp \left[ \frac{(p - (e/c) A)^2}{2k_B T m} + \frac{eU - \mu}{k_B T m} \right]}.$$
After the same replacement of variables \( p - (e/c)A(\mathbf{r}, t) = p' \) we receive

\[
\mathbf{u}^{(0)}(\mathbf{r}, t) = \frac{2}{N^{(0)}(2\pi \hbar)^3} \int \frac{p' \, d^3p'}{1 + \exp \left[ \frac{p'^2}{2k_B T m} + \frac{eU - \mu}{k_B T m} \right]} = 0. \tag{1.13}
\]

So, electron velocity in an equilibrium condition agree to (1.13) it is equal to zero.

Let us notice, that numerical electron density and their average velocity satisfy to the usual equation of a continuity

\[
\frac{\partial N}{\partial t} + \text{div}(N \mathbf{u}) = 0. \tag{1.14}
\]

For conclusion of the equation of continuity (1.14) it is necessary integrate the kinetic the equation (1.10) on the quantum measure for electrons \( d\Omega_F \) and use definition of numerical density and average velocity. Then it is necessary to take advantage of the law of preservation of number of particles and check up, that integral on a quantum measure \( d\Omega_F \) from Wigner—Vlasov’ integral is equal to zero. Really, we have

\[
\int W[f] \frac{d^3p}{(2\pi \hbar)^3} = 2 \int \int \left\{ \cdots \right\} e^{i\mathbf{p}'/\hbar} \delta(\mathbf{a}) \, d^3\mathbf{a} \, d^3p' =
\]

\[
= 2 \int \left\{ \cdots \right\} \bigg|_{\mathbf{a}=0} d^3p' \equiv 0,
\]

for, as it is easy to check up simple substitution,

\[
\left\{ \cdots \right\} \bigg|_{\mathbf{a}=0} \equiv 0.
\]

Here the symbol \( \left\{ \cdots \right\} \) means the same expression, as in the right part of a parity (1.9), i.e.

\[
\left\{ \cdots \right\} = \left\{ \frac{e}{2mc} \left[ A(\mathbf{r} + \frac{\mathbf{a}}{2} , t) + A(\mathbf{r} - \frac{\mathbf{a}}{2} , t) - 2A(\mathbf{r}, t) \right] \nabla f + \right.
\]

\[
\left. \cdots \right|_{\mathbf{a}=0}
\]
\[ + \frac{ie}{mch} \left[ \mathbf{A}(r + \frac{a}{2}, t) - \mathbf{A}(r - \frac{a}{2}, t) \right] p' f + \]

\[ + \frac{e}{2mc} \left[ \text{div} \mathbf{A}(r + \frac{a}{2}, t) + \text{div} \mathbf{A}(r - \frac{a}{2}, t) - 2 \text{div} \mathbf{A}(r, t) \right] f - \]

\[ - \frac{ie^2}{2mc^2 \hbar} \left[ \mathbf{A}^2(r + \frac{a}{2}, t) - \mathbf{A}^2(r - \frac{a}{2}, t) \right] f - \]

\[ - \frac{ie}{\hbar} \left[ U(r + \frac{a}{2}, t) - U(r - \frac{a}{2}, t) \right] f \right \} e^{i(p' - p)a/\hbar}. \]  

(1.9')

Let us notice, that the left part of the kinetic equation (1.10) takes standard form for the transport theory at the following calibration condition

\[ \text{div} \mathbf{A}(r, t) = 0. \]  

(1.15)

Thus, i.e. in case of calibration (1.15), the kinetic equation (1.10) becomes the following form

\[ \frac{\partial f}{\partial t} + \mathbf{v} \nabla f = W[f], \]  

(1.16)

in which the Wigner–Vlasov integral is equal

\[ W[f] = \int \int \left\{ \frac{e}{2mc} \left[ \mathbf{A}(r + \frac{a}{2}, t) + \mathbf{A}(r - \frac{a}{2}, t) - 2\mathbf{A}(r, t) \right] \nabla f + \right. \]

\[ + \frac{ie}{mch} \left[ \mathbf{A}(r + \frac{a}{2}, t) - \mathbf{A}(r - \frac{a}{2}, t) \right] p' f - \]

\[ - \frac{ie^2}{2mc^2 \hbar} \left[ \mathbf{A}^2(r + \frac{a}{2}, t) - \mathbf{A}^2(r - \frac{a}{2}, t) \right] f - \]

\[ - \frac{ie}{\hbar} \left[ U(r + \frac{a}{2}, t) - U(r - \frac{a}{2}, t) \right] f \right \} e^{i(p' - p)a/\hbar} \frac{d^3a}{(2\pi \hbar)^3}. \]  

(1.17)
2. Kinetic equation and its solution

Let us consider, that the scalar potential in the equation (1.17) is equal to zero \( U(\mathbf{r}, t) \equiv 0 \).

Vector potential we take as orthogonal to direction of a wave vector \( \mathbf{k} \):

\[
\mathbf{kA}(\mathbf{r}, t) = 0. \tag{2.1}
\]

in the form of the running harmonious wave

\[
\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 e^{i(\mathbf{k}\mathbf{r} - \omega t)}. 
\]

Let us consider the field of vector potential small enough. It assumption allows to simplify the equation, not neglecting components, square-law (quadratic) on vector potential of electromagnetic field.

Then the equation (1.17) becomes simpler

\[
\frac{\partial f}{\partial t} + \mathbf{v} \nabla f = W[f]. \tag{2.2}
\]

In the equation (2.2) locally equilibrium distribution Fermi—Dirac’ becomes simpler

\[
f^{(0)} = f^{(0)}(\mathbf{r}, \mathbf{p}, t) = \left[1 + \exp \left(C^2(\mathbf{r}, \mathbf{p}, t) - \alpha \right) \right]^{-1}. \tag{2.3}
\]

Wigner—Vlasov’ integral (1.17) becomes simpler now also and has the following form

\[
W[f] = \int\int \left\{ \frac{e}{2mc} \left[ \mathbf{A}(\mathbf{r} + \frac{\mathbf{a}}{2}, t) + \mathbf{A}(\mathbf{r} - \frac{\mathbf{a}}{2}, t) - 2\mathbf{A}(\mathbf{r}, t) \right] \nabla f(\mathbf{r}, \mathbf{p}', t) + \right. \\
+ \frac{ie}{m\hbar} \left[ \mathbf{A}(\mathbf{r} + \frac{\mathbf{a}}{2}, t) - \mathbf{A}(\mathbf{r} - \frac{\mathbf{a}}{2}, t) \right] \mathbf{p}' f(\mathbf{r}, \mathbf{p}', t) - \\
- \frac{ie^2}{2mc^2\hbar} \left[ \mathbf{A}^2(\mathbf{r} + \frac{\mathbf{a}}{2}, t) - \mathbf{A}^2(\mathbf{r} - \frac{\mathbf{a}}{2}, t) \right] f(\mathbf{r}, \mathbf{p}', t) \right\} e^{i(\mathbf{p}' - \mathbf{p})a/\hbar} d^3a d^3p'. \tag{2.4}
\]
We note that
\[ A(r + \frac{a}{2}, t) + A(r - \frac{a}{2}, t) - 2A(r, t) = A(r, t) \left[ e^{ika/2} + e^{-ika/2} - 2 \right], \]
\[ A(r + \frac{a}{2}, t) - A(r - \frac{a}{2}, t) = A(r, t) \left[ e^{ika/2} - e^{-ika/2} \right], \]
\[ A^2(r + \frac{a}{2}, t) - A^2(r - \frac{a}{2}, t) = A^2(r, t) \left[ e^{ika} - e^{-ika} \right]. \]

Substituting these equalities in integral (2.4), we find that
\[ W[f] = \frac{e}{2mc} A(r, t) \times \]
\[ \times \int \int \left[ e^{ika/2} + e^{-ika/2} - 2 \right] \nabla f(r, p', t) e^{i(p'-p)a/\hbar} \frac{d^3a d^3p'}{(2\pi\hbar)^3} + \]
\[ \frac{ie}{mch} A(r, t) \int \int \left[ e^{ika/2} - e^{-ika/2} \right] p' f(r, p', t) e^{i(p'-p)a/\hbar} \frac{d^3a d^3p'}{(2\pi\hbar)^3} - \frac{ie^2}{2mc^2\hbar} A^2(r, t) \int \int \left[ e^{ika} - e^{-ika} \right] f(r, p', t) e^{i(p'-p)a/\hbar} \frac{d^3a d^3p'}{(2\pi\hbar)^3}. \]

Internal integrals in two last summand accordingly are equal
\[ \frac{1}{(2\pi\hbar)^3} \int \left\{ \exp \left( i \left[ p' - p + \frac{\hbar k}{2} \right] \frac{a}{\hbar} \right) + \exp \left( i \left[ p' - p - \frac{\hbar k}{2} \right] \frac{a}{\hbar} \right) - 2 \exp \left( i \left[ p' - p \right] \frac{\hbar k}{2} \right) \right\} d^3a = \]
\[ = \delta(p' - p + \frac{\hbar k}{2}) + \delta(p' - p - \frac{\hbar k}{2}) - 2\delta(p' - p), \]
\[ \frac{1}{(2\pi\hbar)^3} \int \left\{ \exp \left( i \left[ p' - p + \frac{\hbar k}{2} \right] \frac{a}{\hbar} \right) - \exp \left( i \left[ p' - p - \frac{\hbar k}{2} \right] \frac{a}{\hbar} \right) \right\} d^3a = \]
\[ = \delta(p' - p + \frac{\hbar k}{2}) - \delta(p' - p - \frac{\hbar k}{2}), \]
and
\[
\frac{1}{(2\pi\hbar)^3} \int \left\{ \exp \left( i \left[ p' - p + \frac{k\hbar}{\hbar} \right] \frac{a}{\hbar} \right) - \exp \left( i \left[ p' - p - \frac{k\hbar}{\hbar} \right] \frac{a}{\hbar} \right) \right\} d^3a = \\
= \delta(p' - p + \hbar k) - \delta(p' - p - \hbar k).
\]

Now the Wigner–Vlasov integral is expressed through one-dimensional integral

\[
W[f] = A(r, t) \frac{e}{2mc} \times \\
\times \int \left[ \delta(p' - p + \frac{\hbar k}{2}) + \delta(p' - p - \frac{\hbar k}{2}) - 2\delta(p' - p) \right] \nabla f(r, p', t) d^3p' + \\
+ A(r, t) \frac{ie}{mch} \int \left[ \delta(p' - p + \frac{\hbar k}{2}) - \delta(p' - p - \frac{\hbar k}{2}) \right] p' f(r, p', t) d^3p' - \\
- A^2(r, t) \frac{ie^2}{2mc^2\hbar} \int \left[ \delta(p' - p + \hbar k) - \delta(p' - p - \hbar k) \right] f(r, p', t) d^3p'.
\]

It is necessary to take advantage of properties Dirac delta–function. We receive, that

\[
W[f] = A(r, t) \frac{e}{2mc} \left[ \nabla f(r, p - \frac{\hbar k}{2}, t) + \nabla f(r, p + \frac{\hbar k}{2}, t) - 2\nabla f(r, p, t) \right] + \\
+ A(r, t) \frac{ie}{mch} \left[ (p - \frac{\hbar k}{2}) f(r, p - \frac{\hbar k}{2}, t) - (p + \frac{\hbar k}{2}) f(r, p + \frac{\hbar k}{2}, t) \right] - \\
- A^2(r, t) \frac{ie^2}{2mc^2\hbar} \left[ f(r, p - \hbar k, t) - f(r, p + \hbar k, t) \right].
\]

Let us transform the received expression of Wigner–Vlasov integral

\[
W[f] = A(r, t) \frac{e}{2mc} \left[ \nabla f(r, p - \frac{\hbar k}{2}, t) + \nabla f(r, p + \frac{\hbar k}{2}, t) - 2\nabla f(r, p, t) \right] + \\
+ A(r, t) \frac{ie}{mch} \left[ (p - \frac{\hbar k}{2}) f(r, p + \frac{\hbar k}{2}, t) - (p + \frac{\hbar k}{2}) f(r, p - \frac{\hbar k}{2}, t) \right] - \\
- A^2(r, t) \frac{ie^2}{2mc^2\hbar} \left[ f(r, p + \hbar k, t) - f(r, p - \hbar k, t) \right].
\]
\[ + A(r, t) \frac{ie}{mch} \{ p \left[ f(r, p - \frac{\hbar k}{2}, t) - f(r, p + \frac{\hbar k}{2}, t) \right] - \]

\[ - \frac{\hbar k}{2} \left[ f(r, p - \frac{\hbar k}{2}, t) + f(r, p + \frac{\hbar k}{2}, t) \right] \} - \]

\[ - A^2(r, t) \frac{ie^2}{2mc^2\hbar} \left[ f(r, p - \hbar k, t) - f(r, p + \hbar k, t) \right]. \]

Let us enter following designations

\[ f_\pm \equiv f(r, p \mp \frac{\hbar k}{2}, t), \quad f_{++} \equiv f(r, p - \hbar k, t), \quad f_{--} \equiv f(r, p + \hbar k, t). \]

Then Wigner—Vlasov integral (2.4) can be rewritten more shortly

\[ W[f] = A(r, t) \frac{e}{2mc} \left( \nabla f_+ + \nabla f_- - 2\nabla f \right) + \]

\[ + A(r, t) \frac{ie}{mch} \left( f_+ - f_- \right) - A^2(r, t) \frac{ie^2}{2mc^2\hbar} \left( f_{++} - f_{--} \right). \quad (2.5) \]

Further it is more convenient to use dimensionless velocity \( C \) in the form

\[ C = \frac{\mathbf{v}}{v_T} = \frac{\mathbf{P}}{p_T} - \frac{e}{c p_T} A(r, t) \equiv \mathbf{P} - \frac{e}{c p_T} \mathbf{A}(r, t), \]

where \( \mathbf{P} = \frac{\mathbf{P}}{p_T} \) is the dimensionless momentum.

Let here and more low expression \( \mathbf{PA} \) means scalar product.

In quadratic (square-law) approximation on vector potential \( \mathbf{A}(r, t) \) Wigner’s function \( f \) in the first summand in Wigner—Vlasov integral it is necessary to replace on locally equilibrium Fermi—Dirac’s distribution (2.3) \( f^{(0)}(r, \mathbf{P}, t) \), i.e. we consider that \( f = f^{(0)}(r, \mathbf{P}, t) \), where

\[ f^{(0)} = f^{(0)}(r, \mathbf{P}, t) = \left[ 1 + \exp \left( C^2(r, \mathbf{P}, t) - \alpha \right) \right]^{-1}. \]

In the third summand it is necessary to replace Wigner’s function \( f \) on absolute Fermian, i.e. we consider that \( f = f_F(P) \), where
\[ f_F(P) = \frac{1}{1 + \exp(P^2 - \alpha)}, \quad \alpha = \text{const}. \]

In the second summand it is necessary to replace Wigner’s function \( f \) on its linear approximation found in our work [16] i.e. we consider that

\[ f = f^{(0)} - \mathbf{PA}(r, t) \left[ \frac{2e}{cPT} g(P) + \frac{ev_T}{\hbar} \frac{f^+_F - f^-_F}{\omega - v_T kP} \right], \]

where

\[ g(P) = \frac{e^{P^2 - \alpha}}{(1 + e^{P^2 - \alpha})^2} = -\frac{\partial}{\partial P^2} f_F(P). \]

Let us notice, that in linear approximation

\[ f^{(0)} = f_F(P) + g(P) \frac{2e}{cPT} \mathbf{PA}(r, t). \]

Hence, Wigner’s function \( f \) is represented in the form

\[ f = f_F(P) - \frac{ev_T}{\hbar} \mathbf{PA}(r, t) \frac{f^+_F - f^-_F}{\omega - v_T kP}. \]

Thus the Wigner–Vlasov integral (2.5) will look like

\[
W[f] = \frac{e}{2mc} \mathbf{A}(r, t) \left[ \nabla f^{(0)+} + \nabla f^{(0)-} - 2\nabla f^{(0)} \right] + \\
+ \frac{iEv_T}{\hbar} \mathbf{PA}(r, t) \left[ f^+_F - f^-_F - \frac{ev_T}{\hbar} \mathbf{PA}(r, t) \frac{(f^+_F - f^-_F)^+ - (f^+_F - f^-_F)^-}{\omega - v_T kP} \right] - \\
- \mathbf{A}^2(r, t) \frac{ie^2}{2mc^2 \hbar} \left( f^{(0)+}_F - f^{(0)-}_F \right). \tag{2.6}
\]

Here

\[ f^{(0)\pm} = f^{(0)\pm}(r, P, t) = \frac{1}{1 + \exp \left[ \left( P \mp \frac{\hbar k}{2PT} - \frac{e}{cPT} \mathbf{A}(r, t) \right)^2 - \alpha \right]}, \]

\[ f^{(0)} = f^{(0)}(r, P, t) = \frac{1}{1 + \exp \left[ \left( P - \frac{e}{cPT} \mathbf{A}(r, t) \right)^2 - \alpha \right]}, \]
\[ f^{\pm}_F \equiv f^{\pm}_F(P) = \frac{1}{1 + \exp \left[ \left( P \mp \frac{\hbar k}{p_T} \right)^2 - \alpha \right]}, \]

\[ f^{++}_F \equiv f^{++}_F(P) = \frac{1}{1 + \exp \left[ \left( P - \frac{\hbar k}{p_T} \right)^2 - \alpha \right]}, \]

\[ f^{--}_F \equiv f^{--}_F(P) = \frac{1}{1 + \exp \left[ \left( P + \frac{\hbar k}{p_T} \right)^2 - \alpha \right]}, \]

and \( p_T = mv_T \) is the thermal electron momentum.

It is possible to present these designations more shortly

\[ f^{(0)} = f^{(0)}(r, C, t) = \frac{1}{1 + e^{C^2_{\pm} - \alpha}}, \]

\[ f^{\pm} \equiv f_{F}(P_{\pm}) = \frac{1}{1 + e^{P^2_{\pm} - \alpha}}, \]

\[ f^{++} \equiv f_{F}(P^{++}) = \frac{1}{1 + e^{P^{++2} - \alpha}}, \]

\[ f^{--} \equiv f_{F}(P^{--}) = \frac{1}{1 + e^{P^{--2} - \alpha}}. \]

Here

\[ C^2_{\pm} = \left( P \mp \frac{\hbar k}{2p_T} - \frac{e}{c p_T} A(r, t) \right)^2, \]

\[ P^2_{\pm} = \left( P \mp \frac{\hbar k}{2p_T} \right)^2 = \left( P_x \mp \frac{\hbar k_x}{2p_T} \right)^2 + \left( P_y \mp \frac{\hbar k_y}{2p_T} \right)^2 + \left( P_z \mp \frac{\hbar k_z}{2p_T} \right)^2, \]

or

\[ P^2_{\pm} = \left( p_x \mp \frac{\hbar k_x}{2} \right)^2 + \left( p_y \mp \frac{\hbar k_y}{2} \right)^2 + \left( p_z \mp \frac{\hbar k_z}{2} \right)^2 \]

Besides,

\[ P^{++} = \left( P - \frac{\hbar k}{p_T} \right)^2 = \left( P_x - \frac{\hbar k_x}{p_T} \right)^2 + \left( P_y - \frac{\hbar k_y}{p_T} \right)^2 + \left( P_z - \frac{\hbar k_z}{p_T} \right)^2, \]
or
\[ P_{++}^2 = \frac{(p_x - \hbar k_x)^2 + (p_y - \hbar k_y)^2 + (p_z - \hbar k_z)^2}{p_T^2}. \]

Analogous,
\[ P_{--}^2 = \left( P + \frac{\hbar k}{p_T} \right)^2 = \left( P_x + \frac{\hbar k_x}{p_T} \right)^2 + \left( P_y + \frac{\hbar k_y}{p_T} \right)^2 + \left( P_z + \frac{\hbar k_z}{p_T} \right)^2, \]
or
\[ P_{--}^2 = \frac{(p_x + \hbar k_x)^2 + (p_y + \hbar k_y)^2 + (p_z + \hbar k_z)^2}{p_T^2}. \]

Let us show, that the first summand in Wigner–Vlasov integral (2.6) equally to zero. According to statement problem vector potential of electromagnetic field varies along an axis \( x \). Hence, gradient of locally Fermi–Dirac' equilibrium distribution is proportional to a vector \( k \): \( \nabla f^{(0)} \pm \sim k, \nabla f^{(0)} \sim k \). Therefore
\[
A(r, t) \left[ \nabla f^{(0)} + \nabla f^{(0)} - 2\nabla f^{(0)} \right] \sim A k = 0.
\]

Thus, the Wigner–Vlasov integral is equal
\[
W[f] = \frac{iev_T}{\hbar} \text{PA}(r, t) \left[ f_F^+ - f_F^- - \frac{ev_T}{\hbar} \text{PA}(r, t) \frac{(f_F^+ - f_F^-) + (f_F^+ - f_F^-)}{\omega - v_T k P} \right] -
- A^2(r, t) \frac{ie^2}{2mc^2\hbar} \left( f_F^{++} - f_F^{--} \right). \quad (2.7)
\]

We note, that
\[ f_F^{++} = f_F^{--} = f_F = f_F(P). \]

Therefore, the Wigner–Vlasov integral (2.7) is equal
\[
W[f] = \frac{iev_T}{\hbar} \text{PA}(r, t) \left[ f_F^+ - f_F^- - \frac{ev_T}{\hbar} \text{PA}(r, t) \frac{f_F^{++} + f_F^{--} - 2f_F}{\omega - v_T k P} \right] -
- A^2(r, t) \frac{ie^2}{2mc^2\hbar} \left( f_F^{++} - f_F^{--} \right). \quad (2.8)
\]
Let us return to the solution of the equation (2.2). We will search for Wigner’ function in the form, quadratic (square-law) concerning vector potential $\mathbf{A}(\mathbf{r}, t)$:

$$f = f^{(0)} - \mathbf{P}\mathbf{A}(\mathbf{r}, t) \left[ \frac{2e}{\hbar c} g(P) + \frac{ev_T}{\hbar} \left( \frac{f_F^+ - f_F^-}{\omega - v_T k P} \right) \right] + [\mathbf{A}(\mathbf{r}, t)]^2 h(P) =$$

$$= f_F(P) - \frac{ev_T}{\hbar} \mathbf{P}\mathbf{A}(\mathbf{r}, t) \frac{f_F^+ - f_F^-}{\omega - v_T k P} + [\mathbf{A}(\mathbf{r}, t)]^2 h(P),$$

where $h(P)$ is the new unknown function.

We receive equation, from which we find

$$[\mathbf{A}(\mathbf{r}, t)]^2 h(P) = \frac{(ev_T)^2}{2(\hbar c)^2} [\mathbf{P}\mathbf{A}(\mathbf{r}, t)]^2 \frac{f_{F^+} + f_{F^-} - 2f_F}{(\omega - v_T k P)^2} +$$

$$+ \frac{e^2}{4mc^2\hbar} [\mathbf{A}(\mathbf{r}, t)]^2 \frac{f_{F^+} - f_{F^-}}{\omega - v_T k P}.$$

By means of last two equalities let us construct Wigner’ function in the second approximation on the vector field $\mathbf{A}(\mathbf{r}, t)$:

$$f = f^{(0)} - \mathbf{P}\mathbf{A}(\mathbf{r}, t) \left[ \frac{2e}{\hbar c} g(P) + \frac{ev_T}{\hbar} \left( \frac{f_F^+ - f_F^-}{\omega - v_T k P} \right) \right] +$$

$$+ \frac{(ev_T)^2}{2(\hbar c)^2} [\mathbf{P}\mathbf{A}(\mathbf{r}, t)]^2 \frac{f_{F^+} + f_{F^-} - 2f_F}{(\omega - v_T k P)^2} +$$

$$+ \frac{e^2}{4mc^2\hbar} [\mathbf{A}(\mathbf{r}, t)]^2 \frac{f_{F^+} - f_{F^-}}{\omega - v_T k P}. \quad (2.9)$$

This function represents quadratic (square-law) decomposition of distribution function on vector potential $\mathbf{A}(\mathbf{r}, t)$. 
3. Density of electric current in quantum plasmas

By definition, the density of electric current is equal

\[ j(r, t) = e \int v(r, p, t)f(r, p, t) \frac{2d^3p}{(2\pi\hbar)^3}. \] (3.1)

In our work [16] it is shown, that density of electric current in an equilibrium condition (the calibrating current) is equal to zero

\[ j^{(0)}(r, t) = e \int v(r, p, t)f^{(0)}(r, p, t) \frac{2d^3p}{(2\pi\hbar)^3} = 0. \]

Hence, for the density of electric current we receive the following expression

\[
j(r, t) = e \int v(r, p, t)\left[ -PA(r, t) \left[ \frac{2e}{cp_T}g(P) + \frac{ev_T}{\hbar} f_F^+ - f_F^- \right] + \frac{(ev_T)^2}{2(\hbar c)^2 [PA(r, t)]^2 \left[ f_F^{++} + f_F^{--} - 2f_F^- \right]} (\omega - v_T kP)^2 + \frac{e^2}{4mc^2\hbar} [A(r, t)]^2 \left[ f_F^{++} - f_F^{--} \right] \frac{2d^3p}{(2\pi\hbar)^3} \right].
\]

Substituting in equality (3.1) explicit expression for velocity

\[ v(r, P, t) = \frac{p}{m} - \frac{eA(r, t)}{mc} = \frac{p_T P}{m} - \frac{eA(r, t)}{mc} = v_T P - \frac{eA(r, t)}{mc}. \]

and, leaving linear and quadratic (square-law) expressions concerning vector potential of the field, we receive

\[
j(r, t) = -\frac{2e^2p_T^4}{(2\pi\hbar)^3mc} \int P[PA]\left[ \frac{2}{p_T^2}g(P) + \frac{v_T}{\hbar} \frac{f_F^+ - f_F^-}{\omega - v_T kP} \right] d^3P + \frac{2e^3p_T^3}{(2\pi\hbar)^3mc^2} \int A[PA]\left[ \frac{2}{p_T^2}g(P) + \frac{v_T}{\hbar} \frac{f_F^+ - f_F^-}{\omega - v_T kP} \right] d^3P +
\]
\[ + \frac{2e^3p_T^4}{(2\pi \hbar)^3mc^2\hbar} \int P \left[ \frac{mv_T^2}{2\hbar} [\mathbf{PA}]^2 \frac{f_F^{++} + f_F^{--} - 2f_F}{(\omega - v_Tk\mathbf{P})^2} + \frac{A^2 f_F^{++} - f_F^{--}}{4 \omega - v_Tk\mathbf{P}} \right] d^3P. \] 

(3.2)

The first summand in (3.2) is linear expression of the density of electric current, found, in particular, in our previous work [16]. Other summands are the quadratic (square-law) amendments caused by vector potential of electromagnetic field.

4. Linear part of density of electric current

Linear part of density of electric current

\[ j_{\text{linear}}(\mathbf{r}, t) = \]

\[ = - \frac{2e^2p_T^4}{(2\pi \hbar)^3mc} \int P [\mathbf{PA}] \left[ \frac{2}{p_T} g(P) + \frac{v_T}{\hbar} \frac{f_F^+ - f_F^-}{\omega - v_Tk\mathbf{P}} \right] d^3P. \]

(4.1)

we represent in invariant form.

We note, that

\[ (\mathbf{P} \pm \frac{\hbar \mathbf{k}}{2p_T})^2 = P^2 \pm \frac{\hbar}{p_T} \mathbf{P}k + \left( \frac{\hbar}{2p_T} \right)^2 k^2. \]

We take the unity vector \( \mathbf{e}_1 = \frac{\mathbf{A}}{A} \), directed along the vector \( \mathbf{A} \). Then equality (4.1) it is possible to write down in the form

\[ j_{\text{linear}}(\mathbf{r}, t) = - \frac{2e^2p_T^4 A(\mathbf{r}, t)}{(2\pi \hbar)^3mc} \int (\mathbf{Pe}_1) [\mathbf{PA}(\mathbf{r}, t)] \mathbf{P} \times 
\]

\[ \times \left[ \frac{2}{p_T} g(P) + \frac{v_T}{\hbar} \frac{f_F^+ - f_F^-}{\omega - v_Tk\mathbf{P}} \right] d^3P. \]

(4.2)

Take into account symmetry value of the first integral (4.2) will not change, if vector \( \mathbf{e}_1 \) to replace with any other unity vector \( \mathbf{e}_2 \), perpendicular to the vector \( \mathbf{k} \), i.e.

\[ \mathbf{e}_2 = \frac{\mathbf{A} \times \mathbf{k}}{|\mathbf{A} \times \mathbf{k}|} = \frac{\mathbf{A} \times \mathbf{k}}{Ak}, \]
where \( \mathbf{A} \times \mathbf{k} \) is the vector product.

Let us develop the vector \( \mathbf{P} \) by three orthogonal directions \( \mathbf{e}_1, \mathbf{e}_2 \) and \( \mathbf{n} = \frac{\mathbf{k}}{k} \):

\[
\mathbf{P} = (\mathbf{Pn})\mathbf{n} + (\mathbf{Pe}_1)\mathbf{e}_1 + (\mathbf{Pe}_2)\mathbf{e}_2.
\]

By means of this decomposition it is received

\[
(\mathbf{PA})\mathbf{P} = A(\mathbf{Pe}_1)\mathbf{P} =
A(\mathbf{Pe}_1)(\mathbf{Pn})\mathbf{n} + A(\mathbf{Pe}_1)^2\mathbf{e}_1 + A(\mathbf{Pe}_1)(\mathbf{Pe}_2)\mathbf{e}_2.
\]

Substituting this decomposition in (4.2), and, considering, that integrals on odd functions on the symmetric interval are equal to zero, we receive

\[
j_{\text{linear}}(\mathbf{r}, t) = -\frac{2e^2p_T^4A(\mathbf{r}, t)}{(2\pi\hbar)^3mc} \int (\mathbf{Pe}_1)^2 \left[ \frac{2}{p_T}g(P) + \frac{v_T}{\hbar} \frac{f^+_F - f^-_F}{\omega - v_Tk}\right]. \tag{4.3}
\]

In view of symmetry quantity of integral will not change, if the vector \( \mathbf{e}_1 \) to replace with any other unity vector \( \mathbf{e}_2 \), which is perpendicular to the vector \( \mathbf{k} \). Therefore

\[
\int [(\mathbf{e}_1\mathbf{P})^2 + (\mathbf{e}_2\mathbf{P})^2]d^3P = \int [(\mathbf{e}_1\mathbf{P})^2 + (\mathbf{e}_2\mathbf{P})^2]d^3P =
= \frac{1}{2} \int [(\mathbf{e}_1\mathbf{P})^2 + (\mathbf{e}_2\mathbf{P})^2]d^3P.
\]

Let us notice, that the square of length of a vector \( \mathbf{P} \) is equal

\[
P^2 = (\mathbf{Pe}_1)^2 + (\mathbf{Pe}_2)^2 + (\mathbf{Pn})^2,
\]

whence

\[
(\mathbf{e}_1\mathbf{P})^2 + (\mathbf{e}_2\mathbf{P})^2 = P^2 - \frac{(\mathbf{Pk})^2}{k^2} = P^2 - (\mathbf{Pn})^2 = P^2 - P^2_\perp,
\]

where \( P^2_\perp \) is the projection of vector \( \mathbf{P} \) on a straight line, which is perpendicular to planes \( \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \).

From here for current density it is received the following expression

\[
j_{\text{linear}}(\mathbf{r}, t) = -\frac{e^2p_T^4A(\mathbf{r}, t)}{(2\pi\hbar)^3mc} \int \left[ \frac{2}{p_T}g(P) + \frac{v_T}{\hbar} \frac{f^+_F(\mathbf{P}) - f^-_F(\mathbf{P})}{\omega - v_Tk}\right] P^2_\perp d^3P.
\]
5. Density of longitudinal electric current

Let us show, that vector potential of the electromagnetic field besides transversal current generates also a longitudinal electric current. From the formula (3.2) it is visible, that the density of longitudinal current is defined last summand.

Let us return to expression (3.2), square-law on the vector potential of the electromagnetic field.

Vector potential of the field we will direct along an axis \( y \)

\[
\mathbf{A}(\mathbf{r}, t) = A(\mathbf{r}, t)\mathbf{e}_y, 
\]

and we direct wave vector \( \mathbf{k} \) along an axis \( x \):

\[
\mathbf{k} = k\mathbf{e}_x.
\]

Then

\[
\mathbf{A}(\mathbf{r}, t) = \mathbf{A}(x, t) = \{0, A_y e^{i(kx - \omega t)}, 0\}, \quad k\mathbf{P} = kP_x,
\]

\[
\mathbf{PA}(\mathbf{r}, t) = A(\mathbf{r}, t)P_y, \quad \mathbf{A}(\mathbf{PA}) = A^2(\mathbf{r}, t)P_y\mathbf{e}_y,
\]

\[
[\mathbf{PA}]^2 = A^2(\mathbf{r}, t)P_y^2, \quad \mathbf{A}^2 = A^2(\mathbf{r}, t).
\]

According to (3.2) the density of longitudinal current equals

\[
j_{\text{long}}(\mathbf{r}, t) = \frac{e^3 p_T^4 A^2(\mathbf{r}, t)}{2(2\pi\hbar)^3 mc^2\hbar} \times \\
\times \int \mathbf{P} \left[ 2 \frac{mv_T^2}{\hbar} P_y^2 \frac{f^++ f^- - 2f_F}{(\omega - v_TkP_x)^2} + \frac{f^+ - f^-}{\omega - v_TkP_x} \right] d^3\mathbf{P}. \quad (5.1)
\]

This vector has only the one nonzero first component, therefore we can write down, that

\[
j_{\text{long}}(\mathbf{r}, t) = j_{x,\text{long}}(x, t)\{1, 0, 0\},
\]

where the quantity of current density is defined by expression

\[
J^\text{long}_x(x, t) = \frac{e^3 p_T^4 A^2(\mathbf{r}, t)}{2(2\pi\hbar)^3 mc^2\hbar} \times \\
\times \int \mathbf{P} \left[ 2 \frac{mv_T^2}{\hbar} P_y^2 \frac{f^++ f^- - 2f_F}{(\omega - v_TkP_x)^2} + \frac{f^+ - f^-}{\omega - v_TkP_x} \right] d^3\mathbf{P}. \quad (5.1)
\]

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\[
\times \left[ \frac{2mv_T^2}{h} \int \frac{f_{F}^{++} + f_{F}^{--} - 2f_{F}^{P}P_x^2d^3P}{(\omega - v_TkP_x)^2} + \int \frac{f_{F}^{++} - f_{F}^{--}}{\omega - v_TkP_x}P_xd^3P \right].
\]

(5.2)

Let us consider the first integral from (5.2). We will calculate the internal integrals in a plane \((P_y, P_z)\), passing to polar coordinates

\[
\int f_{F}^{++}P_y^2dP_ydP_z = \int_0^{2\pi} \int_0^\infty \frac{\cos^2\varphi \rho^3 d\varphi d\rho}{1 + \exp\left((P_x - \frac{\hbar k}{p_T})^2 + \rho^2 - \alpha\right)} =
\]

\[
= \pi \int_0^\infty \frac{\rho^3 d\rho}{1 + \exp\left((P_x - \frac{\hbar k}{p_T})^2 + \rho^2 - \alpha\right)} =
\]

\[
= \pi \int_0^\infty \rho \ln(1 + \exp\left(-\left(P_x - \frac{\hbar k}{p_T}\right)^2 - \rho^2 + \alpha\right))d\rho.
\]

Similarly, other integrals are equal

\[
\int f_{F}^{--}P_y^2dP_ydP_z = \int_0^{2\pi} \int_0^\infty \frac{\cos^2\varphi \rho^3 d\varphi d\rho}{1 + \exp\left((P_x + \frac{\hbar k}{p_T})^2 + \rho^2 - \alpha\right)} =
\]

\[
= \pi \int_0^\infty \rho \ln(1 + \exp\left(-\left(P_x + \frac{\hbar k}{p_T}\right)^2 - \rho^2 + \alpha\right))d\rho,
\]

and

\[
\int f_{F}P_y^2dP_ydP_z = \int_0^{2\pi} \int_0^\infty \frac{\cos^2\varphi \rho^3 d\varphi d\rho}{1 + \exp\left(P_x^2 + \rho^2 - \alpha\right)} =
\]

\[
= \pi \int_0^\infty \rho \ln(1 + \exp(-P_x^2 - \rho^2 + \alpha))d\rho.
\]

We enter designations

\[
l_{1}^{++}(P_x, \alpha) = \int_0^\infty \rho \ln(1 + e^{-(P_x - \frac{\hbar k}{p_T})^2 - \rho^2 + \alpha})d\rho,
\]
\[ l_1^-(P_x, \alpha) = \int_0^\infty \rho \ln(1 + e^{-(P_x + \frac{\hbar k}{pT})^2 - \rho^2 + \alpha}) d\rho, \]

\[ l_1(P_x, \alpha) = \int_0^\infty \rho \ln(1 + e^{-P_x^2 - \rho^2 + \alpha}) d\rho. \]

Hence, the first integral from (5.2) is equal

\[
\int \frac{f_F^{++} + f_F^{--} - 2f_F}{(\omega - v_T k P_x)^2} P_y P_z d^3 P =
\]

\[
= \pi \int_{-\infty}^{\infty} \frac{l_1^{++}(\tau, \alpha) + l_1^{--}(\tau, \alpha) - 2l_1(\tau, \alpha)}{(\omega - v_T k \tau)^2} \tau d\tau.
\]

We consider the second integral from (5.2)

\[
\int \frac{f_F^{++} - f_F^{--}}{\omega - v_T k P_x} P_x d^3 P =
\]

\[
= \int_{-\infty}^{\infty} \frac{P_x dP_x}{\omega - v_T k P_x} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dP_y dP_z}{1 + \exp\left(\left(\frac{P_x - \frac{\hbar k}{pT}}{P_y} + P_x^2 + P_z^2 - \alpha\right)\right)} -
\]

\[
- \int_{-\infty}^{\infty} \frac{P_x dP_x}{\omega - v_T k P_x} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dP_y dP_z}{1 + \exp\left(\left(\frac{P_x + \frac{\hbar k}{pT}}{P_y} + P_x^2 + P_z^2 - \alpha\right)\right)}.
\]

Internal double integrals are equal

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dP_y dP_z}{1 + \exp\left(\left(\frac{P_x + \frac{\hbar k}{pT}}{P_y} + P_x^2 + P_z^2 - \alpha\right)\right)} =
\]

\[
= \pi \ln\left(1 + e^{-\left(P_x + \frac{\hbar k}{pT}\right)^2 + \alpha}\right).
\]

Hence, the second integral is equal

\[
\int \frac{f_F^{++} - f_F^{--}}{\omega - v_T k P_x} P_x d^3 P =
\]
\[= \pi \int_{-\infty}^{\infty} \ln\left(1 + e^{-\left(\frac{\hbar k}{p_T}\right)^2 + \alpha}\right) - \ln\left(1 + e^{-\left(\frac{\hbar k}{p_T}\right)^2 + \alpha}\right) \frac{d\tau}{\omega - v_T k \tau} = \]

\[= \pi \int_{-\infty}^{\infty} \ln\left(1 + e^{-\left(\frac{\hbar k}{p_T}\right)^2 + \alpha}\right) - \ln\left(1 + e^{-\left(\frac{\hbar k}{p_T}\right)^2 + \alpha}\right) \frac{d\tau}{\omega - v_T k \tau}.\]

Thus, the quantity of the generated longitudinal current in quantum plasma is equal

\[j_{long}^x(x, t) = \frac{\pi e^3 p_T^4 A^2(x, t)}{2(2\pi \hbar)^3 mc^2 \hbar} \times \]

\[\times \left[2 \frac{mv_T^2}{\hbar} \int_{-\infty}^{\infty} \frac{l_{1+}^{++}(\tau, \alpha) + l_{1-}^{--}(\tau, \alpha) - 2l_1(\tau, \alpha)}{(\omega - v_T k \tau)^2} \frac{d\tau}{d\tau} + \right. \]

\[+ \int_{-\infty}^{\infty} \ln\left(1 + e^{-\left(\frac{\hbar k}{p_T}\right)^2 + \alpha}\right) - \ln\left(1 + e^{-\left(\frac{\hbar k}{p_T}\right)^2 + \alpha}\right) \frac{d\tau}{\omega - v_T k \tau}. \]

\[\text{(5.3)}\]

At calculation integrals from (5.3) it is necessary to take advantage known Landau’ rule, having included in an integration contour semicircle with radius \(\varepsilon\) (laying in the bottom semiplane) concerning of the pole \(x_0 = \omega/(v_T k)\), with the subsequent transition to the limit at \(\varepsilon \to 0\). The first integral from (5.3) it is necessary to integrate preliminary in parts.

Landau’ rule completely is equivalent to following reception. The pole is shifted from the real axis in the top semiplane in the point \(x_0 = \omega/(v_T k) + i\varepsilon\). Then we integrate in parts and we pass to limit at \(\varepsilon \to 0\). Not to repeat this reception each time, let us carry out this calculation in a general view

\[\int_{a}^{b} \frac{\varphi(\tau)d\tau}{(\tau - x)^2} = \lim_{\varepsilon \to 0} \int_{a}^{b} \frac{\varphi(\tau)d\tau}{[\tau - (x + i\varepsilon)]^2} = \]
\[
= \lim_{\varepsilon \to 0} \left[ -\frac{\varphi(\tau)}{\tau - (x + i\varepsilon)} \right]_a^b + \int_a^b \frac{\varphi'(\tau)d\tau}{\tau - (x + i\varepsilon)} = \\
= -\frac{\varphi(\tau)}{\tau - x} \bigg|_a^b + i\pi \varphi'(x) + \int_a^b \frac{\varphi'(\tau)d\tau}{\tau - x}.
\]

Last integral is understood in sense of a principal value (symbol V.P. we do not write).

To the same result it is possible to come and as follows

\[
= \lim_{\varepsilon \to 0} \int_a^b \frac{\varphi(\tau)d\tau}{(\tau - x)^2} = \int_a^b \frac{\varphi(\tau)d\tau}{[\tau - (x + i\varepsilon)]^2} = \\
= \frac{d}{dx} \left[ \lim_{\varepsilon \to 0} \int_a^b \frac{\varphi(\tau)d\tau}{\tau - (x + i\varepsilon)} \right] = \frac{d}{dx} \left[ i\pi \varphi'(x) + \int_a^b \frac{\varphi'(\tau)d\tau}{\tau - x} \right] = \\
= i\pi \varphi'(x) - \frac{\varphi(\tau)}{\tau - x} \bigg|_a^b + \int_a^b \frac{\varphi'(\tau)d\tau}{\tau - x}.
\]

We will designate

\[
f(\tau, \alpha) = l_{1+}^+(\tau, \alpha) + l_{1-}^-(\tau, \alpha) - 2l_1(\tau, \alpha)
\]

and

\[
\varphi(\tau, \alpha) = \ln \frac{1 + \exp \left[ -\left( \tau - \frac{\hbar k}{p_T} \right)^2 + \alpha \right]}{1 + \exp \left[ -\left( \tau + \frac{\hbar k}{p_T} \right)^2 + \alpha \right]}
\]

Let us transform equality (5.3) in the form

\[
j_x^{\text{long}}(x, t) = \frac{\pi e^3 p_T^4 A^2(x, t)}{2(2\pi\hbar)^3 mc^2 \hbar} \left[ \frac{2m}{\hbar k^2} \int_{-\infty}^t \frac{\tau f(\tau, \alpha)d\tau}{(\tau - x_0)^2} - \frac{1}{v_T k} \int_{-\infty}^t \frac{\varphi(\tau, \alpha)d\tau}{\tau - x_0} \right],
\]

where

\[
x_0 = \frac{\omega}{v_T k}.
\]
Thus, according to the previous remark the generated density of longitudinal current is equal

\[ j_{\text{long}}(x, t) = \pi e^3 p_F^4 A^2(x, t) \frac{2m}{2\pi \hbar^2} \int_{-\infty}^{\infty} \frac{\tau f(\tau, \alpha)'}{\tau - x_0} + \]

\[ + i\pi \left. \frac{\partial(\tau f(\tau, \alpha))}{\partial \tau} \right|_{\tau = x_0} - \frac{1}{v_T k} \int_{-\infty}^{\infty} \varphi(\tau, \alpha) d\tau \]. \tag{5.4}

6. Degenerate plasmas

Let’s consider the case of degenerate plasmas.

In the formula (5.2) we will carry out replacement of one variable of integration

\[ P_x \rightarrow \frac{v_F}{v_T} P_x, \]

where \( v_F \) is the electron velocity on Fermi’ surface.

Let us receive following expression for quantity of electric current density into quantum plasmas

\[ j_{\text{long}}(x, t) = \frac{e^3 p_F^4 A^2(x, t)}{2(2\pi \hbar)^3 m^2 c^2 \hbar} \times \]

\[ \int \frac{f_F^{++} + f_F^{--} - 2f_F}{\omega - v_F k P_x} P_x P_y^2 d^3 P + \int \frac{f_F^{++} - f_F^{--}}{\omega - v_F k P_x} P_x d^3 P \]. \tag{6.1}

In the formula (6.1) \( p_F = mv_F \) is the electron momentum on Fermi’ surface,

\[ f_F = \frac{1}{1 + \exp \frac{\mathcal{E}_F P^2 - \mu}{k_B T}}, \]

where \( \mathcal{E}_F \) is the electron energy on Fermi’ surface,

\[ f_F^{\pm \pm} = \frac{1}{1 + \exp \frac{\mathcal{E}_F \left[ \left( \frac{P_x \mp \hbar k}{p_F} \right)^2 + P_y^2 + P_z^2 \right] - \mu}{k_B T}}. \]
Let us notice, that in the limit of zero absolute temperature we have

\[ \lim_{T \to 0} \mu = \varepsilon_F, \quad \varepsilon_F = \frac{mv_F^2}{2}, \]

Hence, in limit of zero temperature absolute Fermi–Dirac’ distribution passes in Fermi’s absolute distribution for degenerate plasmas

\[ \lim_{T \to 0} f_F = \lim_{T \to 0} \frac{1}{1 + \exp \left( \frac{\varepsilon_F P^2 - \mu}{k_B T} \right)} = \Theta(\varepsilon_F(1 - P^2)) = \Theta(1 - P^2) = \Theta. \]

Here \( \Theta(x) \) is the Heaviside function (unit step),

\[ \Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0 \end{cases}. \]

Similarly,

\[ \lim_{T \to 0} f_F^{++} = \Theta \left[ 1 - \left( P_x - \frac{\hbar k}{p_F} \right)^2 - P_y^2 - P_z^2 \right] = \Theta^{++}, \]

\[ \lim_{T \to 0} f_F^{--} = \Theta \left[ 1 - \left( P_x + \frac{\hbar k}{p_F} \right)^2 - P_y^2 - P_z^2 \right] = \Theta^{--}. \]

Thus, in the limit of zero temperature the formula (6.1) will be transformed to the form

\[ j_{x_{\text{long}}}^{\text{long}}(x, t) = \frac{e^3 p_F^4 A^2(x, t)}{2(2\pi\hbar)^3 m^2 c^2 \hbar} \times \]

\[ \times \left[ 2 \frac{mv_F^2}{\hbar} \int \frac{\Theta^{++} + \Theta^{--} - 2\Theta}{(\omega - v_F k P_x)^2} P_x P_y d^3 P + \int \frac{\Theta^{++} - \Theta^{--}}{\omega - v_F k P_x} P_x d^3 P \right]. \]

(6.2)

We note that

\[ (\omega - kv_F P_x)^2 = (kv_F)^2 \left( P_x - \frac{\omega}{kv_F} \right)^2, \]

\[ \omega - kv_F P_x = -kv_F \left( P_x - \frac{\omega}{kv_F} \right). \]
By means of these equalities we will transform the formula (6.2) to the following form

\[ j_{x}^{\text{long}}(x, t) = \frac{e^{3}p_{F}^{2}A^{2}(x, t)}{(2\pi\hbar)^{3}mc^{2}q^{2}} \times \]

\[ \times \left[ \int \frac{\Theta^{++} + \Theta^{--} - 2\Theta}{(P_{x} - \omega/kv_{F})^{2}} P_{x}P_{y}^{2}d^{3}P - \frac{q}{2} \int \frac{\Theta^{++} - \Theta^{--}}{P_{x} - \omega/kv_{F}} P_{x}d^{3}P \right]. \quad (6.3) \]

In (6.3) Fermi’s wave number \( k_{F} \) and dimensionless wave number \( q \) is entered.

We consider the first integral from (6.3). We have

\[ \int \frac{\Theta^{++} + \Theta^{--} - 2\Theta}{(P_{x} - \omega/kv_{F})^{2}} P_{x}P_{y}^{2}d^{3}P = \]

\[ = \int \Theta \left[ 1 - \left( \frac{P_{x} - \hbar k}{p_{F}} \right)^{2} - P_{y}^{2} - P_{z}^{2} \right] \frac{P_{y}^{2}d^{3}P}{(P_{x} - \omega/kv_{F})^{2}} + \]

\[ + \int \Theta \left[ 1 - \left( \frac{P_{x} + \hbar k}{p_{F}} \right)^{2} - P_{y}^{2} - P_{z}^{2} \right] \frac{P_{x}P_{y}^{2}d^{3}P}{(P_{x} - \omega/kv_{F})^{2}} \]

\[ - 2 \int \frac{\Theta(1 - P^{2})}{(P_{x} - \omega/kv_{F})^{2}} P_{x}P_{y}^{2}d^{3}P = J_{+} + J_{-} - 2J_{0}. \]

Here the following designations are entered

\[ J_{+} = \int \Theta \left[ 1 - \left( \frac{P_{x} - \hbar k}{p_{F}} \right)^{2} - P_{y}^{2} - P_{z}^{2} \right] \frac{P_{x}P_{y}^{2}d^{3}P}{(P_{x} - \omega/kv_{F})^{2}} \]

\[ = \int \frac{P_{x}P_{y}^{2}d^{3}P}{(P_{x} - \omega/kv_{F})^{2}}, \]

\[ \left( P_{x} - \frac{\hbar k}{p_{F}} \right)^{2} + P_{y}^{2} + P_{z}^{2} < 1 \]

\[ J_{-} = \int \Theta \left[ 1 - \left( \frac{P_{x} + \hbar k}{p_{F}} \right)^{2} - P_{y}^{2} - P_{z}^{2} \right] \frac{P_{x}P_{y}^{2}d^{3}P}{(P_{x} - \omega/kv_{F})^{2}} = \]
\[ \frac{P_x P_y^2 d^3 P}{(P_x - \omega/k v_F)^2}, \]
\[ \left( P_x + \frac{\hbar}{p_F} \right)^2 + P_y^2 + P_z^2 < 1, \]
\[ J_0 = \int \frac{\Theta(1 - P^2)}{(P_x - \omega/k v_F)^2} P_x P_y^2 d^3 P = \]
\[ = \int \frac{P_x P_y^2 d^3 P}{(P_x - \omega/k v_F)^2}, \]
\[ P_x^2 + P_y^2 + P_z^2 < 1. \]

In the integral $J_+$ we carry out replacement of variable of integration $P_x - \frac{\hbar}{p_F} \to P_x$, and in integral $J_-$ we replace $P_x + \frac{\hbar}{p_F} \to P_x$. As result we receive, that

\[ J_+ = \int_{P^2 < 1} \frac{P_x (P_x + \frac{\hbar}{p_F}) P_y^2 d^3 P}{\left[ P_x + \frac{\hbar}{p_F} - \omega \frac{1}{k v_F} \right]^2}, \]
\[ J_- = \int_{P^2 < 1} \frac{P_x (P_x - \frac{\hbar}{p_F}) P_y^2 d^3 P}{\left[ P_x - \frac{\hbar}{p_F} - \omega \frac{1}{k v_F} \right]^2}, \]
\[ J_0 = \int_{P^2 < 1} \frac{P_x P_y^2 d^3 P}{\left[ P_x - \omega \frac{1}{k v_F} \right]^2}. \]

Let us reduce these integrals to the repeated

\[ J_+ = \int_{-1}^{1} \frac{(P_x + \frac{\hbar}{p_F}) dP_x}{\left[ P_x + \frac{\hbar}{p_F} - \omega \frac{1}{k v_F} \right]^2} \int_{P_y^2 + P_z^2 < 1 - P_x^2} P_y^2 dP_y dP_z, \]
\[ J_- = \int_{-1}^{1} \frac{(P_x - \frac{\hbar}{p_F}) dP_x}{\left[ P_x - \frac{\hbar}{p_F} - \omega \frac{1}{k v_F} \right]^2} \int_{P_y^2 + P_z^2 < 1 - P_x^2} P_y^2 dP_y dP_z, \]
\[ J_0 = \int_{-1}^{1} \frac{P_x dP_x}{\left[ P_x - \frac{\omega}{k v_F} \right]^2} \int_{P_y^2 + P_z^2 < 1 - P_x^2} P_y^2 dP_y dP_z. \]

We will calculate internal double integrals, passing to the polar coordinates

\[ P_y = \rho \cos \varphi, \quad P_z = \rho \sin \varphi, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \rho \leq \sqrt{1 - P_x^2}. \]

We receive, that

\[ \int_{P_y^2 + P_z^2 < 1 - P_x^2} P_y^2 dP_y dP_z = \int_0^{2\pi} \cos^2 \varphi d\varphi \int_0^{\sqrt{1 - P_x^2}} \rho^3 d\rho = \frac{\pi}{4} (1 - P_x^2)^2. \]

Therefore, we receive the following one-dimensional integrals

\[ J_+ = \frac{\pi}{4} \int_{-1}^{1} \frac{(P_x + \frac{\hbar k}{p_F})(1 - P_x^2)^2 dP_x}{\left[ P_x + \frac{\hbar k}{p_F} - \frac{\omega}{k v_F} \right]^2}, \]

\[ J_- = \frac{\pi}{4} \int_{-1}^{1} \frac{(P_x - \frac{\hbar k}{p_F})(1 - P_x^2)^2 dP_x}{\left[ P_x - \frac{\hbar k}{p_F} - \frac{\omega}{k v_F} \right]^2}, \]

\[ J_0 = \frac{\pi}{4} \int_{-1}^{1} \frac{P_x (1 - P_x^2)^2 dP_x}{\left[ P_x - \frac{\omega}{k v_F} \right]^2}. \]

So, the first integral from (6.3) equals

\[ \int \frac{\Theta^{++} + \Theta^{--} - 2\Theta}{(P_x - \omega/k v_F)^2} P_x P^2 d^3P = \frac{\pi}{4} \int_{-1}^{1} \frac{(P_x + \frac{\hbar k}{p_F})(1 - P_x^2)^2 dP_x}{\left[ P_x + \frac{\hbar k}{p_F} - \frac{\omega}{k v_F} \right]^2} + \]
\[+ \frac{\pi}{4} \int_{-1}^{1} \left( P_x - \frac{\hbar k}{p_F} \right) (1 - P_x^2)^2 dP_x - 2 \frac{\pi}{4} \int_{-1}^{1} \frac{P_x (1 - P_x^2)^2 dP_x}{\left( P_x - \frac{\omega}{kv_F} \right)^2}.
\]

The second integral from (6.3) is calculated similarly

\[\int \frac{\Theta^{++} - \Theta^{--}}{P_x - \omega/kv_F} P_x d^3 P =
\]

\[= \pi \int_{-1}^{1} \frac{(1 - P_x^2) \left( P_x + \frac{\hbar k}{p_F} \right) dP_x}{P_x + \frac{\hbar k}{p_F} - \frac{\omega}{kv_F}} - \pi \int_{-1}^{1} \frac{(1 - P_x^2) \left( P_x - \frac{\hbar k}{p_F} \right) dP_x}{P_x - \frac{\hbar k}{p_F} - \frac{\omega}{kv_F}}.
\]

Hence, the generated longitudinal electric current is equal

\[j_{x}^{\text{long}}(x, t) = \frac{e^3 p_F^2 A^2(x, t)}{32\pi^2 \hbar^3 mc^2 q^2} \left\{ \left[ \int_{-1}^{1} \frac{(1 - \tau^2)^2 (\tau + \tau_0) d\tau}{(\tau + \tau_0 - x_0)^2} + \right.ight.

\[+ \left. \int_{-1}^{1} \frac{(1 - \tau^2)^2 (\tau - \tau_0) d\tau}{(\tau - \tau_0 - x_0)^2} \right] - 2 \int_{-1}^{1} \frac{(1 - \tau^2) \tau d\tau}{(\tau - x_0)^2} \right\} - 2q \left[ \int_{-1}^{1} \frac{(1 - \tau^2) (\tau + \tau_0) d\tau}{\tau + \tau_0 - x_0} - \int_{-1}^{1} \frac{(1 - \tau^2) (\tau - \tau_0) d\tau}{\tau - \tau_0 - x_0} \right]. \tag{6.4}
\]

In (6.4) the following designations are accepted

\[\tau = P_x, \quad \tau_0 = \frac{\hbar k}{p_F}, \quad x_0 = \frac{\omega}{v_F k}.
\]

First two integrals from (6.4) we will present as one equality

\[J^\pm = \int_{-1}^{1} \frac{(1 - \tau^2)^2 (\tau \pm \tau_0) d\tau}{(\tau \pm \tau_0 - x_0)^2} = \int_{-1}^{1} \frac{[(1 - \tau^2)^2 (\tau \pm \tau_0)]' d\tau}{\tau \pm \tau_0 - x_0} +
\]

\[+ i\pi [(1 - \tau^2)^2 (\tau \pm \tau_0)] \bigg|_{\tau = x_0 \mp \tau_0} \times \left\{ \begin{array}{ll} 1, & |x_0 \mp \tau_0| < 1 \\ 0, & |x_0 \mp \tau_0| > 1 \end{array} \right\}. \tag{6.5}
\]
In the same way the third integral is equal
\[ J^0 = \int_{-1}^{1} \frac{\tau(1 - \tau^2)^2}{(\tau - x_0)^2} d\tau = \int_{-1}^{1} \frac{[\tau(1 - \tau^2)]'}{\tau - x_0} + \]
\[ + i\pi [\tau(1 - \tau^2)^2]' \bigg|_{\tau = x_0} \times \left\{ \begin{array}{ll} 1, & |x_0| < 1 \\ 0, & |x_0| > 1 \end{array} \right\}. \] (6.6)

Let us calculate in an explicit form integral from (6.6)
\[ J^0 = \int_{-1}^{1} \frac{[\tau(1 - \tau^2)^2]'}{\tau - x_0} = 10x_0^2 - \frac{26}{3}x_0 + \]
\[ + (5x_0^4 - 6x_0^2 + 1) \left[ \ln \left| \frac{x_0 - 1}{x_0 + 1} \right| + \left\{ \begin{array}{ll} i\pi, & |x_0| < 1 \\ 0, & |x_0| > 1 \end{array} \right\} \right]. \] (6.7)

Let us pass to calculation of integrals from equality (6.5). We have
\[ \int_{-1}^{1} \frac{[(1 - \tau^2)^2(\tau + \tau_0)]'}{\tau + \tau_0 - x_0} = \]
\[ = \int_{-1}^{1} \frac{1 - 4\tau_0\tau - 6\tau^2 + 4\tau_0\tau^3 + 5\tau^4}{\tau + \tau_0 - x_0} d\tau. \]

After replacement of a variable of integration
\[ u = \tau + \tau_0 - x_0, \quad u_0 = -1 + \tau_0 - x_0, \quad u_1 = 1 + \tau_0 - x_0, \]
we receive
\[ \int_{u_0}^{u_1} \left[ 1 - 4\tau_0(u - \tau_0 + x_0) - 6(u - \tau_0 + x_0)^2 + \right. \]
\[ + 4\tau_0(u - \tau_0 + x_0)^3 + 5(u - \tau_0 + x_0)^4 \right] \frac{du}{u} = \]
\[ = \int_{u_0}^{u_1} \left\{ 1 - 4\tau_0(x_0 - \tau_0) + 6(x_0 - \tau_0)^2 + 4\tau_0(x_0 - \tau_0)^3 + 5(x_0 - \tau_0)^4 \right\} + \]
\[
+ \left[ -4\tau_0 - 12(x_0 - \tau_0) + 12\tau_0(x_0 - \tau_0)^2 + 20(x_0 - \tau_0)^3 \right] u + \\
+ \left[ -6 + 12\tau_0(x_0 - \tau_0) + 30(x_0 - \tau_0)^2 \right] u^2 + \left[ 4\tau_0 + 20(x_0 - \tau_0) \right] u^3 + 5u^4 \right\} \frac{du}{u}.
\]

As a result of rectilinear calculations we receive, that

\[
J^+ = \left[ 5(x_0 - \tau_0)^4 + 4\tau_0(x_0 - \tau_0)^3 - 6(x_0 - \tau_0)^2 - 4\tau_0(x_0 - \tau_0) + 1 \right] \times
\]
\[
\times \left[ \ln \frac{x_0 - \tau_0 - 1}{x_0 - \tau_0 + 1} + \left\{ \begin{array}{c} i\pi, \quad |x_0 - \tau_0| < 1 \\ 0, \quad |x_0 - \tau_0| > 1 \end{array} \right\} + \frac{2}{3} [5(x_0 - \tau_0) + 4\tau_0] + \\
+ 2[5(x_0 - \tau_0)^3 + 4\tau_0(x_0 - \tau_0)^2 - 6(x_0 - \tau_0) - 4\tau_0]. \quad (6.8)
\]

Let us calculate the second integral from (6.5). We have

\[
J^- = \left[ 5(x_0 + \tau_0)^4 - 4\tau_0(x_0 + \tau_0)^3 - 6(x_0 + \tau_0)^2 + 4\tau_0(x_0 + \tau_0) + 1 \right] \times
\]
\[
\times \left[ \ln \frac{x_0 + \tau_0 - 1}{x_0 + \tau_0 + 1} + \left\{ \begin{array}{c} i\pi, \quad |x_0 + \tau_0| < 1 \\ 0, \quad |x_0 + \tau_0| > 1 \end{array} \right\} + \frac{2}{3} [5(x_0 + \tau_0) - 4\tau_0] + \\
+ 2[5(x_0 + \tau_0)^3 - 4\tau_0(x_0 + \tau_0)^2 - 6(x_0 + \tau_0) + 4\tau_0]. \quad (6.9)
\]

We notice that from equalities (6.8) and (6.9) follows that

\[
J^+ \bigg|_{\tau_0=0} = J^- \bigg|_{\tau_0=0} = J^0.
\]

Besides, we will notice, that

\[
5x_0^4 - 6x_0^2 + 1 = (x_0^2 - 1)(5x_0^2 - 1),
\]

\[
5(x_0 - \tau_0)^4 + 4\tau_0(x_0 - \tau_0)^3 - 6(x_0 - \tau_0)^2 - 4\tau_0(x_0 - \tau_0) + 1 = \\
= [(x_0 - \tau_0)^2 - 1][5(x_0 - \tau_0)^2 + 4\tau_0(x_0 - \tau_0) - 1],
\]
\[5(x_0 + \tau_0)^4 - 4\tau_0(x_0 + \tau_0)^3 - 6(x_0 + \tau_0)^2 + 4\tau_0(x_0 + \tau_0) + 1 =
\]
\[= [(x_0 + \tau_0)^2 - 1][5(x_0 + \tau_0)^2 - 4\tau_0(x_0 + \tau_0) - 1].\]

The first square bracket from (6.4) thanking (6.7) – (6.9) is equal

\[J^+ + J^- - 2J^0 = 28x_0\tau_0^2 +
\]
\[+[(x_0 - \tau_0)^2 - 1][5(x_0 - \tau_0)^2 + 4\tau_0(x_0 - \tau_0) - 1]\times
\]
\[\times \left[ \ln \left| \frac{x_0 - \tau_0 - 1}{x_0 - \tau_0 + 1} \right| + \left\{ \begin{array}{ll}
i\pi, & |x_0 - \tau_0| < 1 \\
0, & |x_0 - \tau_0| > 1
deletedtext\right. \right] +
\]
\[+[(x_0 + \tau_0)^2 - 1][5(x_0 + \tau_0)^2 - 4\tau_0(x_0 + \tau_0) - 1]\times
\]
\[\times \left[ \ln \left| \frac{x_0 + \tau_0 - 1}{x_0 + \tau_0 + 1} \right| + \left\{ \begin{array}{ll}
i\pi, & |x_0 + \tau_0| < 1 \\
0, & |x_0 + \tau_0| > 1
deletedtext\right. \right] -
\]
\[-2(x_0^2 - 1)(5x_0^2 - 1)\left[ \ln \left| \frac{x_0 - 1}{x_0 + 1} \right| + \left\{ \begin{array}{ll}
i\pi, & |x_0| < 1 \\
0, & |x_0| > 1
deletedtext\right. \right].\]

The remained two integrals from equality (6.3) are calculated or by means of Landau’ rule or the same as it has been stated above. The first of these integrals is calculated as follows

\[I^+ \equiv \int_{-1}^{1} \frac{(1 - \tau^2)(\tau + \tau_0)}{\tau - (x_0 - \tau_0)} d\tau = \lim_{\varepsilon \to 0} \int_{-1}^{1} \frac{(1 - \tau^2)(\tau + \tau_0)}{\tau - (x_0 + i\varepsilon - \tau_0)} d\tau +
\]
\[+i\pi[1 - (x_0 - \tau_0)^2]x_0\left\{ \begin{array}{ll}
1, & |x_0 - \tau_0| < 1 \\
0, & |x_0 - \tau_0| > 1
deletedtext\right. \right] +
\]
\[+ \int_{-1}^{1} \frac{(1 - \tau^2)(\tau + \tau_0)}{\tau - (x_0 - \tau_0)} d\tau,
\]
and last integral is understood in sense of a principal value. This integral is equal
\[
\int_{-1}^{1} \frac{(1 - \tau^2)(\tau + \tau_0)}{\tau - (x_0 - \tau_0)} d\tau = \frac{4}{3} - 2\tau_0(x_0 - \tau_0) - 2(x_0 - \tau_0)^2 +
\]
\[+ \left[ x_0 - \tau_0(x_0 - \tau_0)^2 - (x_0 - \tau_0)^3 \right] \ln \left| \frac{x_0 - \tau_0 - 1}{x_0 - \tau_0 + 1} \right|.
\]

The second integral is calculated similarly. We have
\[
I^- \equiv \int_{-1}^{1} \frac{(1 - \tau^2)(\tau - \tau_0)}{\tau - (x_0 + \tau_0)} d\tau = \lim_{\varepsilon \to 0} \int_{-1}^{1} \frac{(1 - \tau^2)(\tau - \tau_0)}{\tau - (x_0 + i\varepsilon + \tau_0)} d\tau =
\]
\[= i\pi [1 - (x_0 + \tau_0)^2] x_0 \begin{cases} 1, |x_0 + \tau_0| < 1 \\ 0, |x_0 + \tau_0| > 1 \end{cases} +
\]
\[+ \int_{-1}^{1} \frac{(1 - \tau^2)(\tau - \tau_0)}{\tau - (x_0 + \tau_0)} d\tau,
\]

and last integral is understood in sense of a principal value. This integral is equal
\[
\int_{-1}^{1} \frac{(1 - \tau^2)(\tau - \tau_0)}{\tau - (x_0 + \tau_0)} d\tau = \frac{4}{3} + 2\tau_0(x_0 - \tau_0) - 2(x_0 - \tau_0)^2 +
\]
\[+ \left[ x_0 + \tau_0(x_0 + \tau_0)^2 - (x_0 + \tau_0)^3 \right] \ln \left| \frac{x_0 + \tau_0 - 1}{x_0 + \tau_0 + 1} \right|.
\]

We notice that
\[-(x_0 - \tau_0)^3 - \tau_0(x_0 - \tau_0) + x_0 = x_0[1 - (x_0 - \tau_0)^2],
\]
\[-(x_0 + \tau_0)^3 + \tau_0(x_0 + \tau_0) + x_0 = x_0[1 - (x_0 + \tau_0)^2].
\]

Now we will find the difference \(I^+ - I^-\). We have
\[
I^+ - I^- = 4x_0\tau_0 +
\]
$$+ x_0 [1 - (x_0 - \tau_0)^2] \left[ \ln \left| \frac{x_0 - \tau_0 - 1}{x_0 - \tau_0 + 1} \right| + \left\{ \begin{array}{ll} i\pi, & |x_0 - \tau_0| < 1 \\ 0, & |x_0 - \tau_0| > 1 \end{array} \right\} \right] -$$

$$- x_0 [1 - (x_0 + \tau_0)^2] \left[ \ln \left| \frac{x_0 + \tau_0 - 1}{x_0 + \tau_0 + 1} \right| + \left\{ \begin{array}{ll} i\pi, & |x_0 + \tau_0| < 1 \\ 0, & |x_0 + \tau_0| > 1 \end{array} \right\} \right].$$

It is necessary to find density electric current under the formula

$$j_{x,\text{long}}(x, t) = \frac{e^3 p_F^2 A^2(x, t)}{32\pi^2 \hbar^3 m c^2 q^2} \left[ (J^+ + J^- - 2J^0) - 2q(I^+ - I^-) \right].$$

Substituting the found integrals in this equality, we receive

$$j_{x,\text{long}}(x, t) = \frac{e^3 p_F^2 A^2(x, t)}{32\pi^2 \hbar^3 m c^2 q^2} \left[ 20x_0 \tau_0^2 +$$

$$+ [(x_0 - \tau_0)^2 - 1] \left[ 5(x_0 - \tau_0)^2 + 4\tau_0(x_0 - \tau_0) + 2x_0 \tau_0 - 1 \right] \times$$

$$\times \left[ \ln \left| \frac{x_0 - \tau_0 - 1}{x_0 - \tau_0 + 1} \right| + \left\{ \begin{array}{ll} i\pi, & |x_0 - \tau_0| < 1 \\ 0, & |x_0 - \tau_0| > 1 \end{array} \right\} \right] +$$

$$+ [(x_0 + \tau_0)^2 - 1] \left[ 5(x_0 + \tau_0)^2 - 4\tau_0(x_0 + \tau_0) - 2x_0 \tau_0 - 1 \right] \times$$

$$\times \left[ \ln \left| \frac{x_0 + \tau_0 - 1}{x_0 + \tau_0 + 1} \right| + \left\{ \begin{array}{ll} i\pi, & |x_0 + \tau_0| < 1 \\ 0, & |x_0 + \tau_0| > 1 \end{array} \right\} \right] -$$

$$- 2(x_0^2 - 1)(5x_0^2 - 1) \left[ \ln \left| \frac{x_0 - 1}{x_0 + 1} \right| + \left\{ \begin{array}{ll} i\pi, & |x_0| < 1 \\ 0, & |x_0| > 1 \end{array} \right\} \right].$$

(6.10)

We will remind that $q = \frac{k}{k_F}$ is the dimensionless wave number, where $k_F$ is the wave Fermi’ number, $k_F = \frac{p_F}{\hbar}$, $p_F = m v_F$ is the electron momentum on Fermi’ surface,

$$\tau_0 = \frac{\hbar k}{mv_F} = \frac{k}{k_F} = q, \quad x_0 = \frac{\omega}{kv_F} = \frac{\omega}{v_F k_F} \cdot \frac{1}{q} = \frac{\Omega}{q},$$

where

$$\Omega = \frac{\omega}{v_F k_F}.$$

Now it is possible to present the formula (6.10) in the form
\[ j_{x,\text{long}}(x, t) = \frac{e^3 p_F^2 A^2(x, t)}{32\pi^2 \hbar^3 mc^2 q^2} \left[ 20\Omega q + \left( \frac{\Omega}{q} - q \right)^2 - 1 \right] \left[ \frac{5\Omega^2}{q^2} - 4\Omega + q^2 - 1 \right] \times \]
\[ \times \left[ \ln \left| \frac{\Omega - q^2 - q}{\Omega - q^2 + q} \right| + \left\{ \begin{array}{ll} i\pi, & |\Omega - q^2| < |q| \\ 0, & |\Omega - q^2| > |q| \end{array} \right\} \right] + \left[ \frac{\Omega}{q} + q \right]^2 - 1 \times \]
\[ \left[ \frac{5\Omega^2}{q^2} + 4\Omega + q^2 - 1 \right] \left[ \ln \left| \frac{\Omega + q^2 - q}{\Omega + q^2 + q} \right| + \left\{ \begin{array}{ll} i\pi, & |\Omega + q^2| < |q| \\ 0, & |\Omega + q^2| > |q| \end{array} \right\} \right] - \]
\[ -2 \left( \frac{\Omega^2}{q^2} - 1 \right) \left( \frac{5\Omega^2}{q^2} - 1 \right) \left[ \ln \left| \frac{\Omega - q}{\Omega + q} \right| + \left\{ \begin{array}{ll} i\pi, & |\Omega| < |q| \\ 0, & |\Omega| > |q| \end{array} \right\} \right] \]  \hspace{1cm} (6.11)

Let us allocate at equality (6.11) on the right the real and imaginary parts

\[ j_{x,\text{long}}(x, t) = \frac{e^3 p_F^2 A^2(x, t)}{32\pi^2 \hbar^3 mc^2} \frac{R(\Omega, q) + i\pi S(\Omega, q)}{q^2}. \]  \hspace{1cm} (6.12)

In expression (6.12) the following designations are entered

\[ R(\Omega, q) = 20\Omega q + \left( \frac{\Omega}{q} - q \right)^2 - 1 \left[ \frac{5\Omega^2}{q^2} - 4\Omega + q^2 - 1 \right] \times \]
\[ \times \ln \left| \frac{\Omega - q^2 - q}{\Omega - q^2 + q} \right| + \left( \frac{\Omega}{q} + q \right)^2 - 1 \left[ \frac{5\Omega^2}{q^2} + 4\Omega + q^2 - 1 \right] \times \]
\[ \times \ln \left| \frac{\Omega + q^2 - q}{\Omega + q^2 + q} \right| - 2 \left( \frac{\Omega^2}{q^2} - 1 \right) \left( \frac{5\Omega^2}{q^2} - 1 \right) \ln \left| \frac{\Omega - q}{\Omega + q} \right|, \]

and

\[ S(\Omega, q) = \left[ \left( \frac{\Omega}{q} - q \right)^2 - 1 \right] \left[ \frac{5\Omega^2}{q^2} - 4\Omega + q^2 - 1 \right] \left\{ \begin{array}{ll} 1, & |\Omega - q^2| < |q| \\ 0, & |\Omega - q^2| > |q| \end{array} \right\} + \]
\[ + \left[ \left( \frac{\Omega}{q} + q \right)^2 - 1 \right] \left[ \frac{5\Omega^2}{q^2} + 4\Omega + q^2 - 1 \right] \left\{ \begin{array}{ll} 1, & |\Omega + q^2| < |q| \\ 0, & |\Omega + q^2| > |q| \end{array} \right\} - \]
\[ -2 \left( \frac{\Omega^2}{q^2} - 1 \right) \left( \frac{5\Omega^2}{q^2} - 1 \right) \left\{ \begin{array}{ll} 1, & |\Omega| < |q| \\ 0, & |\Omega| > |q| \end{array} \right\}. \]
7. Conclusions

In the present work the account of nonlinear character of interaction of electromagnetic field with quantum plasma is considered. It has appeared, that the effect of nonlinearity of an electromagnetic field affects in generating of an electric current, transversal to a direction of electromagnetic fields.

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