AN ELEMENTARY PROOF
OF THE IRRATIONALITY OF TSCHAKALOFF SERIES

WADIM ZUDILIN‡ (Moscow)

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To A. B. Shidlovskii on the occasion of his 90th birthday

Abstract. We present a new proof of the irrationality of values of the series
\[ T_q(z) = \sum_{n=0}^{\infty} z^n q^{-n(n+1)/2} \]
in both qualitative and quantitative forms. The proof is based on
a hypergeometric construction of rational approximations to \( T_q(z) \).

1. Introduction. In 1919, L. Tschakaloff introduced the series [10]
\[ T_q(z) = \sum_{n=0}^{\infty} z^n q^{-n(n+1)/2}, \tag{1} \]
convergent in the whole complex \( z \)-plane whenever \( |q| > 1 \), and proved the irrationality
and linear independence of its values at rational non-zero points \( z \) and \( q \) (under certain
assumptions on \( q \)). His method generalized that by O. Szász [9] for a special case
of (1), namely, the function \( \Theta_q(z) = \sum_{n=0}^{\infty} z^n q^{-n^2} = T_{q^2}(z/q) \); at about the same time
F. Bernstein and O. Szász [1] used a continued fraction for \( \Theta_q(z) \) due to Eisenstein
to provide another irrationality proof for its values at certain rational \( q \) and \( z \). These
seem to be the very first results on the arithmetic nature of values of \( q \)-series.

The aim of this note is to give an elementary proof of Tschakaloff’s theorem [10]
and also its quantitative form given by P. Bundschuh in [3, Satz 2].

Theorem. Let \( q = q_1/q_2 \) and \( z \) be non-zero rational numbers, where \( |q| > 1 \) and
\( q_1, q_2 \in \mathbb{Z} \). Suppose that the non-negative number
\[ \gamma = \frac{\log |q_2|}{\log |q_1|} \]
satisfies \( \gamma < \gamma_0 = (3 - \sqrt{5})/2 \). Then the value \( T_q(z) \) is irrational. Moreover, for any
\( \varepsilon > 0 \) there exists a positive constant \( b_0(\varepsilon) \) such that
\[ \left| T_q(z) - \frac{a}{b} \right| > |b|^{-1} - \frac{\sqrt{5} - 1}{2(\gamma_0 - \gamma)} - \varepsilon \tag{2} \]
for all integers \( a \) and \( b \) with \( |b| \geq b_0(\varepsilon) \).

The rational approximations to the Tschakaloff function (1) that we construct in the next section are actually the same as those in [10] and [3]. Our contribution here is to provide an elementary explanation of why these approximations are good enough to obtain the irrationality of \( T_q(z) \). Our proof is inspired by the ideas of L. Gutnik and Yu. Nesterenko [7, Section 1] in their proof that \( \zeta(3) \notin \mathbb{Q} \). This is the famous theorem due to R. Apéry; elementary proofs and interrelations with irrationality results for other mathematical constants may be found in [4] and [8].

2. Proof. For the first paragraph, we shall think of \( q \) as a variable. Let \( n \) be a positive integer and define the polynomial

\[
R(T; q) = R_n(T; q) = (1 - qT)(1 - q^2T) \cdots (1 - q^nT).
\]

Multiplication gives

\[
R(T; q) = \sum_{k=0}^{n} C_k(q)T^k,
\]

where, for \( k = 0, 1, \ldots, n \),

\[
C_k(q) = C_{k,n}(q) \in \mathbb{Z}[q]
\]

is a polynomial in \( q \) with

\[
\text{degree } C_k(q) \leq \frac{n(n + 1)}{2}.
\]

Conditions (4) and (5) imply that, if \( q = q_1/q_2 \), then

\[
q_2^{n(n+1)/2}C_k\left(\frac{q_1}{q_2}\right) \in \mathbb{Z}
\]

for \( k = 0, 1, \ldots, n \) and arbitrary non-zero integers \( q_1 \) and \( q_2 \).

Let \( m = \lfloor \beta n \rfloor \) (here \( \lfloor \cdot \rfloor \) denotes the integer part of a number), where \( \beta = (\sqrt{5} - 1)/2 \) is the positive root of the polynomial \( x^2 + x - 1 \), and introduce the series

\[
I_n = I_n(z; q) = \sum_{t=1}^{\infty} R_n(q^{-t}; q)z^{t+m}q^{-(t+m)(t+m-1)/2},
\]

which converges if \( |q| > 1 \). Using (3) we obtain

\[
I_n = \sum_{t=1}^{\infty} \sum_{k=0}^{n} C_k(q)q^{-(k-1)/2+km}z^{t+m}q^{-(t+m)(t+m-1)/2} \cdot \sum_{t=1}^{\infty} z^{l}q^{-l(l-1)/2}
\]
\[ I_n = \sum_{k=0}^{n} z^{-k} C_k(q) q^{k(k-1)/2+km} \left( \sum_{l=0}^{\infty} z^l q^{-l(l-1)/2} - \sum_{l=0}^{k+m} z^l q^{-l(l-1)/2} \right) \]

\[ = \sum_{k=0}^{n} z^{-k} C_k(q) q^{k(k-1)/2+km} \cdot T_q(z) - \sum_{k=0}^{n} C_k(q) \sum_{l=0}^{k+m} z^{-k-l} q^{k(k-1)/2+km-l(l-1)/2}. \]  

(8)

If \( q = q_1/q_2 \) and \( z = z_1/z_2 \), where \( q_1, q_2, z_1, z_2 \in \mathbb{Z} \setminus \{0\} \), then from (6) and (8) we see that the quantity \( \tilde{I}_n = \tilde{I}_n(z; q) \) defined by

\[ \tilde{I}_n = z_1 z_2 q_1^{m(m-1)/2} q_2^{n(n+1)/2+n(n-1)/2+nm} I_n \]

(9)

is of the form

\[ \tilde{I}_n = B_n \cdot T_q(z) - A_n, \]

(10)

where \( A_n \) and \( B_n \) are integers, determined by (8) and (9). In addition, since equality in (5) is achieved only when \( k = n \), we see that the coefficient of \( T_q(z) \) in (8) has the following asymptotics as \( n \to \infty \) (where \( f(n) \sim g(n) \) means that \( f(n)/g(n) \to 1 \)):

\[ \left| \sum_{k=0}^{n} z^{-k} C_k(q) q^{k(k-1)/2+km} \right| \sim |z|^{-n} |C_n(q)| |q|^{n(n-1)/2+nm} \]

\[ = |z|^{-n} |q|^{n(n+m)}. \]

(11)

In order to evaluate the asymptotic behavior of the sum of the series (7), notice that \( R_n(q^{-t}; q) = 0 \) for \( t = 1, 2, \ldots, n \). Therefore (using \( f(n) = O(g(n)) \) as \( n \to \infty \) to mean that \( |f(n)| \leq C|g(n)| \) for some constant \( C > 0 \) and all \( n \) sufficiently large),

\[ I_n = \sum_{t=n+1}^{\infty} R_n(q^{-t}; q) z^t q^{-t(t+m)(t+m-1)/2} \]

\[ = R_n(q^{-n+1}; q) z^{n+m+1} q^{-n(n+m)(n+m+1)/2} + O(q^{-n(n+m+1)(n+m+2)/2}) \]

\[ = z^{n+m+1} q^{-n(n+m)(n+m+1)/2} (1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-n}) \]

\[ + O(q^{-n(n+m+1)(n+m+2)/2}) \]

\[ \sim z^{n+m+1} q^{-n(m+n)(m+n+1)/2} \quad \text{as} \quad n \to \infty. \]

(12)

In particular, \( I_n \neq 0 \) for all \( n \) sufficiently large.

Finally, since \( |q_1/q_2| = |q| > 1 \) implies that \( |q_1| > 1 \), we may define \( \gamma \) by the relation \( \log |q_2| = \gamma \log |q_1| \), so that \( \gamma \geq 0 \). Assume that \( \gamma < \gamma_0 = (3 - \sqrt{5})/2 \). Then, from (9), (11), (12), and the relation \( m = \lfloor \beta n \rfloor \), for the quantities \( B_n \) and \( \tilde{I}_n \) in (10) we have

\[ \lim_{n \to \infty} \frac{\log |B_n|}{n^2 \log |q_1|} = (1 - \gamma)(1 + \beta) + \gamma(1 + \beta) + \frac{\beta^2}{2} = \frac{\sqrt{5}(\sqrt{5} + 1)}{4} \]

(13)
and
\[
\lim_{n \to \infty} \frac{\log |\tilde{I}_n|}{n^2 \log |q_1|} = -(1 - \gamma) \frac{(1 + \beta)^2}{2} + \gamma(1 + \beta) + \frac{\beta^2}{2} = -\frac{\sqrt{5}(\sqrt{5} + 1)(\gamma_0 - \gamma)}{2(\sqrt{5} - 1)} < 0. \tag{14}
\]

Now let us show that \( T_q(z) \) cannot be rational. Suppose, on contrary, that \( T_q(z) = a/b \) for some integers \( a \) and \( b \neq 0 \). Then from (10)
\[
b\tilde{I}_n = B_n a - A_n b \in \mathbb{Z} \quad (n = 1, 2, \ldots).
\]
Recalling that (12) yields \( I_n \neq 0 \) for \( n \) large, we conclude that \( |b\tilde{I}_n| \geq 1 \). But, by (14), we have \( |b\tilde{I}_n| \to 0 \) as \( n \to \infty \). The contradiction implies that \( T_q(z) \notin \mathbb{Q} \).

We leave to the reader the derivation of estimate (2) from (10), (13), and (14) by letting \( a_n = A_n \) and \( b_n = B_n \) in the following standard lemma (compare [2, Section 11.3, Exercise 3]).

**Lemma.** Let \( \alpha \) be an irrational real number. Suppose that we have a sequence of rational approximations \( a_n/b_n \) to \( \alpha \) (where \( a_n, b_n \in \mathbb{Z} \) for \( n = 1, 2, \ldots \)) such that the sequence \( |b_n| \) tends to infinity with \( n \),
\[
\lim_{n \to \infty} \frac{\log |b_{n+1}|}{\log |b_n|} = 1,
\]
and with some constant \( c > 0 \)
\[
\left| \alpha - \frac{a_n}{b_n} \right| < \frac{1}{|b_n|^{1+c}}
\]
for all \( n \) sufficiently large. Then for any \( \varepsilon > 0 \) there exists a positive constant \( b_0(\varepsilon) \) such that
\[
\left| \alpha - \frac{a}{b} \right| > \frac{1}{b^{1+1/c+\varepsilon}}
\]
for all integers \( a \) and \( b \) with \( b \geq b_0(\varepsilon) \).

**3. Related results.** Although we are able to prove the irrationality of \( T_q(z) \) only under the hypothesis \( \gamma < \gamma_0 = 0.381966 \ldots \), it is expected that this hypothesis can be dropped, i.e., that \( T_q(z) \) is irrational for all \( z \in \mathbb{Q} \setminus \{0\} \) and \( q \in \mathbb{Q} \) with \( |q| > 1 \). This remains an open problem. The earlier method in [9] requires the condition \( \gamma < 1/3 \) (which is worse, since \( 1/3 < \gamma_0 \)) corresponding to the simpler choice \( \beta = 0 \) in our notation. The choice \( \beta = (\sqrt{5} - 1)/2 \) ensures the optimal value of \( \gamma_0 \) in terms of the construction presented here.

The Tschakaloff function (1) might be viewed as “half” of the theta series \( \sum_{n \in \mathbb{Z}} z^{n-1/2} q^{-(n-1/2)^2} \). This viewpoint and Nesterenko’s theorem [6] on the transcendence of certain theta series imply the transcendence of \( T_q(z) \) for \( q \) algebraic, \( |q| > 1 \), and \( z = q^k \) with some \( k \in \mathbb{Z} \), solving this case of the open problem. On the other hand, when \( z \) and \( q \) are multiplicatively independent, no transcendence results are known. This is part of a general problem posed by K. Mahler in [5] for analytic
functions which satisfy functional equations (such as $T_q(z) = 1 + zT_q(z/q)$ for the function (1)), but to which his method from [5] cannot be applied.

The constants $\beta = (\sqrt{5} - 1)/2$ and $\gamma_0 = 1 - \beta$, involved in the proof of the Theorem, are related to the golden mean (or golden section), the positive root of the polynomial $x^2 - x - 1$. It is quite curious that the golden mean and its generalizations (the so-called metallic means) also occur in other irrationality proofs related to Apéry’s theorem [4].

Finally, we mention that a special case of the $q$-binomial theorem implies the following explicit formula for the polynomial (4):

$$C_k(q) = (-1)^k \binom{n}{k} q^{k(k+1)/2}$$

involving the $q$-binomial coefficients

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n - k]_q!} \in \mathbb{Z}[q],$$

where $[0]_q! = 1$ and, for $k = 1, 2, \ldots,$

$$[k]_q! = \frac{(q - 1)(q^2 - 1)(q^3 - 1) \cdots (q^k - 1)}{(q - 1)^k}.$$

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Department of Mechanics and Mathematics
Moscow Lomonosov State University
Vorobyovy Gory, GSP-2
119992 Moscow, RUSSIA
URL: http://wain.mi.ras.ru/
E-mail address: wadim@ips.ras.ru