Global well-posedness and regularity of 3D Burgers equation with multiplicative noise

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Abstract: In this paper, we develop low regularity theory for 3D Burgers equation perturbed by a linear multiplicative stochastic force. This method is new and essentially different from the deterministic partial differential equations (PDEs). Our results and method can be widely applied to other stochastic hydrodynamic equations and the deterministic PDEs. As a further study, we establish a random version of maximum principle for random 3D Burgers equations, which will be an important tool for the study of 3D stochastic Burgers equations. As we know establishing moment estimates for highly nonlinear stochastic hydrodynamic equations is difficult. But moment estimates are very important for us to study the probabilistic properties and long-time behavior for the stochastic systems. Here, the random maximum principle helps us to achieve some important moment estimates for 3D stochastic Burgers equations and lays a solid foundation for the further study of 3D stochastic Burgers equations.

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1 Introduction

In this paper, we are concerned with 3D positive viscosity stochastic Burgers equation driven by linear multiplicative noise on the three dimensional torus $T^3 := \mathbb{R}^3 / 2\pi \mathbb{Z}^3$, that is, the considered equation is periodic in the spatial variable $x \in T^3$. To be more precise, fix any $T > 0$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}, (\{B_k(t)\}_{k \in \mathbb{N}})_{t \in [0,T]})$ be a stochastic basis which is given throughout the whole paper, wherein the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is assumed to fulfill the usual conditions and $(\{B_k(t)\}_{k \in \mathbb{N}})_{t \in [0,T]}$ is a sequence of independent, identically distributed (one-dimensional) $(\mathcal{F}_t)_{t \in [0,T]}$-Brownian motions. We use

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\[ du(t, x) = (\nu \Delta u(t, x) - (u \cdot \nabla) u(t, x)) dt + \sum_{k=1}^{\infty} b_k u \circ dB_k(t), \quad \text{on } [0, T] \times \mathbb{T}^3, \quad (1.1) \]

\[ u(0, x) = u_0(x), \quad x = (x_1, x_2, x_3) \in \mathbb{T}^3, \]

for a 3D vector valued random field \( u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x)) \in \mathbb{R}^3, t \in [0, T] \) and \( x \in \mathbb{T}^3 \), where the parameter \( \nu > 0 \) in the system \((1.1)\) stands for the viscosity, \( W(t) := \sum_{k \geq 1} b_k B_k(t), t \in [0, T], b_k \in \mathbb{R} \) with \( \sum_{k \geq 1} b_k^2 < \infty \), and \( \circ \) denotes the Stratonovich integral. To simplify the notations, we set \( \partial_i = \partial x_i, \quad i = 1, 2, 3 \). Moreover, \( \Delta := \partial_1^2 + \partial_2^2 + \partial_3^2 \) is the Laplace operator, \( \nabla := (\partial_1, \partial_2, \partial_3) \) is the gradient operator.

The Burgers equation is the simplest PDE combining both nonlinear propagation effects and diffusive effects, occurring in various areas of applied mathematics, such as gas dynamics, fluid mechanics, nonlinear acoustics, and (more recently) traffic flow. The equation was first introduced by H. Bateman in 1915 (see [7] and also Forsyth [23]) and studied mathematically study by J. M. Burgers (see [12]) in the 1940s. As by now is well known, the Burgers equation is not a suitable model to describe turbulence, due to the fact that it does not perform any chaos even if a force is added to this model. Specifically, all the solutions to the Burgers equation will converge to a unique stationary solution as time tends to infinity. However, adding a random force so that the equation becomes a stochastic Burgers equation, the corresponding result is totally different, see for instance [13] [16] [30] [27]. Moreover, the stochastic Burgers equations have also been applied to study the dynamics of interfaces in the seminal work [33]. One dimensional (i.e., scalar valued) stochastic Burgers equation has been fairly well studied. By an adaptation of the celebrated Hopf-Cole transformation, Bertini, Cancrini, and Jona-Lasinio [6] solve the one dimensional modified Burgers equation with additive space-time white noise, where the nonlinearity in the equation was formulated in terms of Wick product. Moreover, Chan [14] utilises Hopf-Cole transformation to study the scaling limit of Wick ordered KPZ equation involving additive space-time white noise. Later, Da Prato, Debussche, Temam [16] study the Burgers equation based on semigroup property for the heat equation on a bounded domain. In the paper [16], the authors establish the existence of an invariant measure for the corresponding transition semigroup. Weinan E, Khanin, Mazel, and Sinai [22] establish the existence and uniqueness of a stationary distribution and obtain the convergence of stationary distributions in the limit when viscosity tends to zero. In [5], Bakhtin, Cator, and Khanin study the long-term behavior of the Burgers dynamics for the situation where the forcing is a space-time stationary random process. In particular, they construct space-time stationary global solutions for the Burgers equation on the real line and show that they can be viewed as one-point attractors. In [5], Bakhtin consider the Burgers equation with random boundary conditions. Concerning the one-dimensional Burgers equation with viscosity coefficient defined on a bounded domain driven by multiplicative Gaussian noise, Da Prato and Debussche [15] succeed to obtain its global well-posedness. Furthermore, Gyöngy and Nualart [26] extend the results of Da Prato and Debussche to the Burgers equation defined on the whole line. By establishing the exponential tightness, Gourcy [24] show that a large deviation principle holds for the occupation measure of the solutions to the stochastic Burgers equations with small noises, which describes
the exact rate of exponential convergence. When the Gaussian force is replaced by Lévy jumps, Dong and Xu prove its global well-posedness of the strong, weak and mild solutions as well as the ergodicity in [21, 17]. Later, Dong, Xu and Zhang generalize the noise to be $\alpha$-stable process and show the exponential ergodicity in [20]. When the noise is changed to be space-time homogeneous random kick forcing, Bakhtin and Li [4] develop its ergodic theory without any compactness assumptions. Concretely, the authors prove a one force-one solution principle by applying the infinite-volume polymer measures to construct a family of stationary global solutions for this system, and proving that each of those solutions is a one point pullback attractor on the initial conditions with the same spatial average. When the initial condition for the Burgers equation is random, Mohammed and Zhang [37] achieve the global well-posedness. Then in [38], the same authors establish the dynamics and stability for one dimensional stochastic Burgers equations. When the initial condition for the Burgers equation is random, Mohammed and Zhang [37] achieve the global well-posedness. Then in [38], the same authors establish the dynamics and stability for one dimensional stochastic Burgers equations. Wang, Zeng, and Guo [41] obtain the same result for the case that the noise is fractional Brownian motion. For the one dimension stochastic Burgers equation with reflection, Zhang [42] establish the global well-posedness. There are lots of other works which are very interesting and meaningful for the stochastic one dimensional Burgers equations. But here, we do not list them.

For the multidimensional Burgers equations, Kiselev and Ladyzhenskaya [34] prove the existence and uniqueness of solution in the class of functions $L^\infty(0,T;L^\infty(O)) \cap L^2(0,T;H^1_0(O))$. Inspired by [34], Pooley, Robinson [36] prove the global well-posedness for 3D Burgers equations in $H^2$. When the viscosity tends to zero and the initial condition is zero, Bui [11] prove the convergence of solutions to the inviscid Burgers equations on a small time interval. In the higher dimensional inviscid stochastic case, the stationary solution and a stationary distribution were constructed by Gomes, Iturriaga, Khanin and Padilla in [29] based on a very delicate use of the Lagrangian formalism and the Hamilton-Jacobi equation. Based on the stochastic version of Lax formula for solutions to the initial and final value problems for the viscous Hamilton-Jacobi equation, Gomes, Iturriaga, Khanin and Padilla in [25] prove convergence of stationary distributions for the randomly forced multi-dimensional Burgers and Hamilton-Jacobi equations in the limit when viscosity tends to zero. Utilising the maximum principle, Brzeźniak, Goldys and Neklyudov [9] establish the global existence and uniqueness for the mild solutions to multidimensional Burgers equations with additive noise. Furthermore, the asymptotic behavior of solutions to multidimensional stochastic Burgers equations is studied when the viscosity tends to zero. For the multidimensional generalised stochastic Burgers equation in the space-periodic setting, Boritchev [8] prove that if the solution $u$ of this equation is a gradient, then each of Sobolev norms of $u$ averaged in time and in ensemble behaves as a given negative power of the viscosity coefficient $\mu$, which gives the sharp upper and lower bounds for natural analogues of quantities characterising the hydrodynamical turbulence. Moreover, the author establish the existence and uniqueness of stationary measure. Recently, Khanin and Zhang [32] generalized the results of [22] to arbitrary dimensional Burgers equation by using Green bundles method, which is complete different from the approach used by [22].

In [9], multidimensional Burgers equations with additive noise are studied and the existence and uniqueness of global strong solutions are established. A natural question then arises: can one obtain global well-posedness for the more difficult case of multidimensional stochastic Burgers equations driven
by a multiplicative noise? In this paper, we establish the global well-posedness for 3D Burgers equations with linear multiplicative noises in a low regularity functional space, and then extend the result to smooth space. Particularly, we establish a maximum principle for the relevant random 3D Burgers equations (see Theorem 5.2 below) which is a key tool in studying regularities, moment estimates and long-time behavior of solutions of the 3D stochastic Burgers equations we concerned. We would like to point out the difference strategies between [9] and our present paper. In [9], the authors apply maximum principle directly to each component of the velocity vector field. Then, by using a prior estimates in \( L^\infty(\mathbb{T}^3) \), the authors obtain the global well-posedness for the mild solutions to stochastic multidimensional Burgers equations with additive noise, where the noise actually acts only on one coordinate. Here in Lemma 4.1 of our paper, we apply the maximum principle to the Galerkin approximations (2.12). Obviously, there is a big gap between ordinary differential equations (2.12) and stochastic partial differential equations (1.1). To extend the global existence of (2.12) to (1.1), we firstly improve the regularity of the local weak solutions as we do in Lemma 3.1 and then in Theorem 4.1 we adapt a delicate argument of compactness and regularity to remedy the gap between the finite dimensional cases (2.12) and infinite dimensional cases (1.1). Consequently, we extend the noise to all coordinates (hence lift the restriction that the noise acts only on one coordinate in [9]) and we succeed to show the global well-posedness for the stochastic equations with linear multiplicative noise in various functional spaces.

In what follows, let us explicate the essential difficulties we encountered in treating this high nonlinear multidimensional stochastic system without incompressibility as it has been enforced for the stochastic 2D Navier-Stokes equations.

1. The first difficulty is the absence of cancellation property. Due to this, one can not even establish a priori estimates in \( L^2(\mathbb{T}^3) \) to ensure the global existence of weak solutions to 3D stochastic Burgers equations. How to get rid of this difficulty? We adapt the techniques from deterministic partial differential equations (PDEs). That is, we derive energy estimate in a more regular space. By comparison principle, we establish the local existence for the solutions to (1.1), see Proposition 3.1.

2. The next step is to extend the local existence of solutions to the global existence. Then, the second difficulty arises. There is no classical solutions to stochastic partial differential equations. That is to say, we can not adapt the method of deterministic cases that applying the maximum principle directly to multidimensional Burgers equations to obtain the global existence of solutions. How to overcome the difficulty? As we mentioned above, we apply the maximum principle to Galerkin approximations. Here, we should emphasize that, it is the reason we choose the noise to be linear in the equation (1.1). If the noise is nonlinear, there is no classical solutions even to the ordinary differential equations (2.12). More precisely, the nonlinearity of the noise will result in the failure of the maximum principle.

3. After establish the a priori estimates for the Galerkin approximations, it comes to remedy the gap between (2.12) and (1.1). However, it is not obvious. We encounter some further difficulties. In
fact, since the dimension of $T^3$ is three and the initial data $u_0$ belongs to $H^1(T^3)$, then the first difficulty is that we do not have that $u_0 \in \mathbb{L}^{\infty}(T^3)$. Consequently, we can not use the ideal of the maximum principle that utilising the initial data $|u(0)|_{\infty}$ to bound the velocity $|u(t)|_{\infty}$. Thus, in the next step, our effort is to improve the regularity of the initial data $u_0$ (see (4.33) and (4.34) in the proof of Theorem 4.1). On the other hand, in the proof of local existence of solutions to (1.1) (see Proposition 3.1), we get that the obtained Galerkin approximation $v_n(t)$ converges to the local solution $v(t)$ almost everywhere with respect to time $t$ in $H^1(T^3)$. To remedy the gap mentioned above, it is not enough. We want that $v_n(t)$ also converges to $v(t)$ in $H^2(T^3)$, which becomes another difficulty. Hence, we establish Lemma 3.1 to get the convergence in $H^2(T^3)$. The idea of proof of Lemma 3.1 is that given the initial data $u_0 \in H^1(T^3)$, the local solution $v(t)$ of (1.1) should take values in $H^2(T^3)$ after evolving for a while. Then utilising the parabolic structure of 3D Burgers equations and the commutator estimate (see Theorem A.8 in [31]), we establish the convergence of $v_n(t)$ in $H^2(T^3)$. We would like to emphasise that the convergence aforementioned is in the almost everywhere sense. But this convergence is not strong enough for our purpose. Therefore, we utilise Lemma 2.3 to show that the solutions of stochastic Burgers equations are sufficient regularity. Then we find that a priori estimates obtained for the 3D stochastic Burgers equations hold for each time not just almost everywhere, see the estimate (4.38) below. This then paves a way for us to get a rigorous proof of Theorem 4.1.

4. Once the global existence of solutions to equation (1.1) is established, the next naturally interest question is the uniqueness of the solutions. A naive approach is to check the uniqueness in the space $L^2(T^3)$. Noticing that there is no energy estimates in $L^2(T^3)$ and further there is not Poincaré inequality in hand, namely, the inequality $\| \cdot \|_{L^2(T^3)} \leq c \| \nabla \cdot \|_{L^2(T^3)}$ does not hold for equation (1.1). This lead to a new difficulty, since even if we assume that the initial data has zero average, the solutions are not necessary to have zero average at positive time. To overcome this difficulty, we establish Corollary 2.1 where we use semi-norm $\| \cdot \|_s$ (see the definition under (2.2)) and the initial data to dominate the norm $L^2(T^3)$.

5. In Theorem 5.2, we establish a maximum principle for the relevant random 3D Burgers equations. We should emphasize that as there is no classical solutions to stochastic partial differential equations, this method of establishing the random maximum principle is new, much more difficult and essentially different from the PDEs. Our results and method can be applied to the deterministic PDE. This random maximum principle will be an important tool for the study of 3D stochastic Burgers equations. As we know establishing moment estimates for highly nonlinear stochastic hydrodynamic equations is difficult. But the moment estimates are very important for us to study the probabilistic properties and long-time behavior for the stochastic systems. Here, the random maximum principle helps us to derive some important moment estimates for 3D stochastic Burgers equations. We believe these moments estimates lay solid foundation for our further study such as large deviation principle and ergodicity etc.
The paper is organised as follows. Preliminaries are presented in Section 2, the local existence and uniqueness of the solutions to (1.1) are given in Section 3; the global existence and regularities of solutions to (1.1) are established in Section 4. Moment estimates are derived in Section 5 for (1.1) in various functional spaces.

Throughout the paper, we use $c > 0$ for a generic constant with possibly different values at each appearance. Unless a specific description is given, we denote by $c(a) > 0$ a constant which depends on parameter $a$.

## 2 Mathematical preliminaries

### 2.1 Notational conventions

For $1 \leq p \leq \infty$, let $L^p(\mathbb{T}^3)$ be the usual Lebesgue spaces with the norm $|\cdot|_p$. When $p = 2$, we denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbb{T}^3)$. For $s \geq 0$, we introduce an operator $\Lambda^s$ acting on $\mathbb{H}^s(\mathbb{T}^3)$ as follows.

Assuming $f \in \mathbb{H}^s(\mathbb{T}^3)$ with the Fourier series

$$f(x) = \sum_{k \in \mathbb{Z}^3} \hat{f}_k e^{i k \cdot x} \in \mathbb{H}^s(\mathbb{T}^3),$$

we define

$$\Lambda^s f(x) = \sum_{k \in \mathbb{Z}^3} |k|^s \hat{f}_k e^{i k \cdot x} \in L^2(\mathbb{T}^3).$$

Denote by $\| \cdot \|_s$ the seminorm $|\Lambda^s \cdot|_2$, then the Sobolev norm $\| \cdot \|_{\mathbb{H}^s}$ of $\mathbb{H}^s(\mathbb{T}^3)$ is equivalent to $| \cdot |_2 + \| \cdot \|_s$. Hence, we can define the norm on $\mathbb{H}^s$ by

$$\|f\|_{\mathbb{H}^s} = \left( \sum_{k \in \mathbb{Z}^3} (1 + |k|^{2s}) |\hat{f}_k|^2 \right)^{1/2}.$$  

Obviously, for $0 < s_1 \leq s_2$, we have $\|f\|_{s_1} \leq \|f\|_{s_2}$ and $\Lambda^2 = -\Delta$. For $s \in \mathbb{R}^+$, set $\mathbb{H}^s(\mathbb{T}^3) = \{ f \in L^2(\mathbb{T}^3) : \sum_{k \in \mathbb{Z}^3} |k|^2 |\hat{f}_k|^2 < \infty \}$. Then, we have $\mathbb{H}^s(\mathbb{T}^3) \subset \mathbb{H}^s(\mathbb{T}^3)$. Obviously, $| \cdot |_s$ is the seminorm in $\mathbb{H}^s(\mathbb{T}^3)$.

Without loss of generality, we simply take the viscosity parameter $\nu = 1$. In fact, we only need $\nu$ to be any strictly positive number. Recall that through out the paper, we are given a fixed stochastic basis $(\Omega, \mathcal{F}, P, \{ \mathcal{F}_t \}_{t \in [0, T]}, (\{ B_k(t) \}_{t \in [0, T]})_{k \in \mathbb{Z}^3})$ to work with. In the following, we introduce the definitions of weak, mild and strong solutions, which are close to [13].

**Definition 2.1 (Local strong solutions).** Suppose $u_0$ is an $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|_{\mathbb{H}^1}^2 < \infty$.

1. A pair $(u, \tau)$ is a local strong pathwise solution to (1.1) if $\tau$ is a strictly positive random variable and $u(\cdot \wedge \tau)$ is an $\mathcal{F}_t$-adapted process in $\mathbb{H}^1(\mathbb{T}^3)$ so that

$$u(\cdot \wedge \tau) \in L^2(\Omega; C([0, \infty); \mathbb{H}^1(\mathbb{T}^3))), \quad u_t \in L^2(\Omega; L^2_{loc}([0, \infty); \mathbb{H}^2(\mathbb{T}^3)));$$

$$u(\cdot \wedge \tau) \in L^2(\Omega; \mathbb{H}^1(\mathbb{T}^3));$$

$$u(\cdot \wedge \tau) \in L^2(\Omega; \mathbb{H}^2(\mathbb{T}^3));$$

$$u(\cdot \wedge \tau) \in L^2(\Omega; \mathbb{H}^3(\mathbb{T}^3));$$

...
and, for every $t \geq 0$,

$$u(t \wedge \tau) - \int_0^{t \wedge \tau} \Delta u(s)ds + \int_0^{t \wedge \tau} (u \cdot \nabla u)(s)ds = u(0) + \sum_{k=1}^{\infty} \int_0^{t \wedge \tau} b_k u(s) \circ dB_k(s),$$

with equality understood in $H$.

2. Strong pathwise solutions of (1.1) are said to be pathwise unique up to a random positive time $\tau > 0$ if given any pair of solutions $(u^1, \tau), (u^2, \tau)$ which coincide at $t = 0$ on the event $\tilde{\Omega} = \{u^1(0) = u^2(0)\} \subset \Omega$, then

$$P(I_{\tilde{\Omega}}(u^1(t \wedge \tau) - u^2(t \wedge \tau)) = 0; \forall t \geq 0) = 1.$$ 

Definition 2.2 (Maximal and global strong solutions).

(i) Let $\xi$ be a positive random variable. We say the pair $(u, \xi)$ is a maximal pathwise strong solution if $(u, \tau)$ is a local strong pathwise solution for each $\tau < \xi$ and

$$\sup_{r \in [0, \xi)} ||u||_1 = \infty \quad (2.4)$$

almost surely on the set $\{\xi < \infty\}$.

(ii) If $(u, \xi)$ is a maximum pathwise strong solution and $\xi = \infty$ a.s., then we say the solution is global.

Definition 2.3 (Local weak solutions). Suppose $u_0$ is an $H^{1/2}(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable with $u_0 \in \mathbb{E}[||u_0||^2_{H^{1/2}(\mathbb{T}^3)}] < \infty$.

(i) A pair $(u, \tau)$ is a local weak pathwise solution to (1.1) if $\tau$ is a strictly positive random variable and $u(\cdot \wedge \tau)$ is an $\mathcal{F}_t$-adapted process in $H^{1/2}(\mathbb{T}^3)$ so that

$$u(\cdot \wedge \tau) \in L^2(\Omega; C([0, \infty); H^{1/2}(\mathbb{T}^3))), \quad u_{I \leq \tau} \in L^2(\Omega; L^2_{loc}([0, \infty); H^{1/2}(\mathbb{T}^3))); \quad (2.5)$$

and, for every $t \geq 0$,

$$\langle u(t \wedge \tau), \phi \rangle + \int_0^{t \wedge \tau} \langle u(s), \Lambda^2 \phi \rangle ds + \int_0^{t \wedge \tau} \langle (u \cdot \nabla u)(s), \phi \rangle ds = \langle u(0), \phi \rangle + \sum_{k=1}^{\infty} \int_0^{t \wedge \tau} b_k \langle u(s, x), \phi \rangle \circ dB_k(s), \quad (2.6)$$

for all $t \in \mathbb{R}^+$ and $\phi \in D(\Lambda^2)$.

(ii) Weak pathwise solutions of (1.1) are said to be pathwise unique up to a random positive time $\tau > 0$ if given any pair of solutions $(u^1, \tau), (u^2, \tau)$ which coincide at $t = 0$ on the event $\tilde{\Omega} = \{u^1(0) = u^2(0)\} \subset \Omega$, then

$$P(I_{\tilde{\Omega}}(u^1(t \wedge \tau) - u^2(t \wedge \tau)) = 0; \forall t \geq 0) = 1.$$ 

Definition 2.4 (Maximal and global weak solutions).
Let $\xi$ be a positive random variable. We say the pair $(u, \xi)$ is a maximal weak pathwise solution if $(u, \tau)$ is a local weak pathwise solution for each $\tau < \xi$ and

$$\sup_{t \in (0, \xi)} \|u(t, \cdot)|_2 = \infty$$

almost surely on the set $\{\xi < \infty\}$.

(ii) If $(u, \xi)$ is a maximum weak pathwise solution and $\xi = \infty$ a.s., then we say the solution is global.

**Definition 2.5.** Suppose $u_0$ is a $L^p(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable with $\mathbb{E}|u_0|^q < \infty$ with $p > 3$ and $q \geq \frac{3}{2}$. A stochastic process $u$ is said to be a global mild solution to (1.1) if

(i) for arbitrary $T > 0$ and $t \in [0, T]$, $u(t)$ is an $\mathcal{F}_t$ adapted process satisfying $u \in L^q(\Omega; C([0, T]; L^p(\mathbb{T}^3)))$;

(ii) $u$ solves the stochastic 3D Burgers equation in the following sense:

$$u(t) = S(t)u(0) - \int_0^t S(t-s)(u \cdot \nabla u)ds + \sum_{k=1}^{\infty} \int_0^t S(t-s)b_k u \circ dB_k(s), \text{ a.s.},$$

where $u(t) := u(t, \cdot)$ and $(S(t))_{t \in [0, T]}$ denotes the semigroup generated in $L^p(\mathbb{T}^3)$ by the operator $\Delta$. Furthermore, let $u$ and $\tilde{u}$ be two mild solutions to (1.1). If $u(0) = \tilde{u}(0)$ a.s., we have

$$P(u(t) = \tilde{u}(t), \text{ for all } t \geq 0) = 1$$

then we say the mild solution $u$ to (1.1) is unique.

### 2.2 Reformulation of 3D stochastic Burgers equations

Let $\alpha(t) = \exp(-\sum_{k=1}^{\infty} b_k B_k(t))$. Then by the Novikov condition, we know $\exp(-\sum_{k=1}^{\infty} b_k B_k(t) - \frac{1}{2} \sum_{k=1}^{\infty} b_k^2 t)$ is a martingale. Hence, by Doob’s maximum inequality and Novikov’s condition we have

$$\mathbb{E} \sup_{t \in [0, T]} \alpha(t) = \mathbb{E} \sup_{t \in [0, T]} \exp \left( -\sum_{k=1}^{\infty} nb_k B_k(t) \right)$$

$$= \mathbb{E} \sup_{t \in [0, T]} \exp \left( -\sum_{k=1}^{\infty} nb_k B_k(t) - \frac{n^2}{2} \sum_{k=1}^{\infty} b_k^2 t + \frac{n^2}{2} \sum_{k=1}^{\infty} b_k^2 t \right)$$

$$\leq \mathbb{E} \sup_{t \in [0, T]} \exp \left( -\sum_{k=1}^{\infty} nb_k B_k(t) - \frac{n^2}{2} \sum_{k=1}^{\infty} b_k^2 t \right) \exp \left( \frac{n^2}{2} \sum_{k=1}^{\infty} b_k^2 T \right)$$

$$\leq \left( \mathbb{E} \sup_{t \in [0, T]} \exp \left( -\sum_{k=1}^{\infty} 2nb_k B_k(t) - 2n^2 \sum_{k=1}^{\infty} b_k^2 t \right) \right)^{\frac{1}{2}} \exp \left( n^2 \sum_{k=1}^{\infty} b_k^2 T \right)$$

$$\leq 2 \left( \mathbb{E} \exp \left( -\sum_{k=1}^{\infty} 2nb_k B_k(T) - 2n^2 \sum_{k=1}^{\infty} b_k^2 T \right) \right)^{\frac{1}{2}} \exp \left( n^2 \sum_{k=1}^{\infty} b_k^2 T \right)$$

$$= 2 \exp \left( n^2 \sum_{k=1}^{\infty} b_k^2 T \right).$$

(2.8)
for arbitrary $n \geq 1$. Similarly, we also have
\[
\mathbb{E} \sup_{t \in [0,T]} \alpha^{-n}(t) \leq 2 \exp \left( n^2 \sum_{k=1}^{\infty} b_k^2 T \right). \tag{2.9}
\]

Set $v = au$, then equations (1.1) is equivalent to the following
\[
dv(t, x) = \Delta v(t, x)dt - \alpha^{-1}(t)(v \cdot \nabla v)(t, x)dt, \quad \text{on } [0, T] \times \mathbb{T}^3, \tag{2.10}
\]
\[
v(0, x) = u(0, x), \quad x = (x_1, x_2, x_3) \in \mathbb{T}^3. \tag{2.11}
\]

From Definition 2.1 to Definition 2.5, we can define the corresponding solutions to (2.10)-(2.11). For $n \in \mathbb{N}$ let $P_n$ denote the projection on to the Fourier modes of order up to $n$, that is
\[
P_n \left( \sum_{k \in \mathbb{Z}^3} \hat{v}_k e^{i k \cdot x} \right) = \sum_{|k| \leq n} \hat{v}_k e^{i k \cdot x}.
\]

Then we obtain the Galerkin approximation of (2.10)-(2.11) as the following
\[
dv_n(t, x) = \Delta v_n(t, x)dt - \alpha^{-1}(t)P_n[(v_n \cdot \nabla v_n)(t, x)]dt, \quad \text{on } [0, T] \times \mathbb{T}^3, \tag{2.12}
\]
\[
v_n(0, x) = u_n(0, x) = P_n u(0, x), \quad x = (x_1, x_2, x_3) \in \mathbb{T}^3. \tag{2.13}
\]

Since (2.12)-(2.13) is a locally-Lipschitz system of random ODEs, we set $v_n$ to be the unique local solution to (2.12)-(2.13) with $v_n(0, x) \in \mathbb{H}^\frac{1}{2}(\mathbb{T}^3)$. Define
\[
\tau_n = \inf \{ t \in \mathbb{R}^+ : \sup_{0 \leq s \leq t} \| v_n(s) \|_{\mathbb{H}^\frac{1}{2}} = \infty \}.
\]

Obviously, $v_n \in C([0, \tau_n) \times \mathbb{T}^3)$.

For the multidimensional Burgers equations, if the initial data has zero average, the solutions are not necessary to have zero average for positive times. This leads to that, for positive $s$, $\| \cdot \|_s$ is smaller than $\| \cdot \|_{\mathbb{H}^s}$. Hence, $\| \cdot \|_s$ is not equivalent to $\| \cdot \|_{\mathbb{H}^{s}}$ for the multidimensional Burgers equations. This is different from the case of Navier-Stokes equations. Further more, due to the absence of the incompressible property and high nonlinearity of 3D Burgers equations, one can not obtain the a priori estimates in $L^2(\mathbb{T}^3)$. To overcome the difficulty, we need to use estimates in $\mathbb{H}^{s}(\mathbb{T}^3)$ and $L^1(\mathbb{T}^3)$ norm of initial data to dominate the energy in $L^2(\mathbb{T}^3)$, see (2.16) and Lemma 2.1. In fact, one can see that estimate (2.14) in Lemma 2.1 is vital to establish the uniqueness of the solutions to the 3D stochastic Burgers equations, see derivation of (5.21), (5.22), (4.69) and (4.70).

### 2.3 Useful lemmas

**Lemma 2.1.** Let $u, v$ be the local solutions of (2.12) up to a random positive time $\tau > 0$, with initial data $u_0 \in \mathbb{H}^\frac{1}{2}(\mathbb{T}^3)$ and $v_0 \in \mathbb{H}^\frac{1}{2}(\mathbb{T}^3)$, respectively. Let $\xi := u - v$ and $\xi_0 := u_0 - v_0$, then for $t \in [0, \tau]$, we have
\[
\left| \int_{\mathbb{T}^3} (\xi(t) - \xi_0) dx \right| \leq 8 \pi^3 \alpha^{-1} \int_0^\tau \| \xi \|_{\frac{1}{2}} (\| u(s) \|_{\frac{1}{2}} + \| v(s) \|_{\frac{1}{2}}) ds. \tag{2.14}
\]
In particular, taking \( v \equiv 0 \) yields the following
\[
\left| \int_{\mathbb{T}^3} u(x,t) dx \right| \leq 8\pi^3 \int_0^\tau \alpha^{-1}(s)\|u(s)\|_2^2 ds + \left| \int_{\mathbb{T}^3} u_0(x) dx \right|.
\] (2.15)

Proof. For \( k \in \mathbb{Z}^3 \), let \( \hat{u}_k, \hat{v}_k \) and \( \hat{\xi}_k \) be the \( k \)th Fourier coefficients of \( u, v \) and \( \xi \), respectively. In view of the equations of \( u \) and \( v \), we derive the following
\[
\frac{d}{dt} \int_{\mathbb{T}^3} \xi(t,x) dx = -\alpha^{-1} \int_{\mathbb{T}^3} \left( (u \cdot \nabla)\xi(t,x) + (\xi \cdot \nabla)v(t,x) \right) dx
\]
\[
= -8\pi^3 i\alpha^{-1} \sum_{k \in \mathbb{Z}^3} \left\{ \left| \hat{u}_k(t) \cdot k \right| \hat{\xi}_k(t) + \left| \hat{\xi}_k(t) \cdot k \right| \hat{v}_k(t) \right\}.
\]
Hence
\[
\left| \frac{d}{dt} \int_{\mathbb{T}^3} \xi(t,x) dx \right| \leq 8\pi^3 \alpha^{-1} \sum_{k \in \mathbb{Z}^3} \left| \hat{\xi}_k \right| \left\| k \right\| \left( \| \hat{u}_k \| + \| \hat{v}_k \| \right)
\]
\[
\leq 8\pi^3 \alpha^{-1} \left\| \xi(t) \right\|_l \left( \| u(t) \|_l + \| v(t) \|_l \right).
\]
and (2.14) then follows from the integration of the above estimate with respect to \( t \). \( \square \)

In view of Lemma 2.1, we can obtain Corollary 2.1 where the formula (2.16) will play important roles in the proofs of global existence and uniqueness results for solutions to (2.10), (2.11), see Theorem 4.1 and Theorem 4.3.

Corollary 2.1. Let \( v_\tau \) be the solution to (2.12)-(2.13) with \( v_\tau(0) \in \mathbb{H}^{\frac{3}{2}}(\mathbb{T}^3) \) and \( \tau \) being its initial data and existence time, respectively. For any \( s > 0 \) and \( t \in [0, \tau] \), we have
\[
\| v_\tau(t) \|_s \leq \| v_\tau(t) \|_{\mathbb{H}^{\frac{3}{2}}} \leq \| v_\tau(t) \|_s + c \int_0^t \| v_\tau(s) \|_2^2 ds + c |u_0|_1,
\] (2.16)
for some \( c = \max\{ 2(\pi)^{9/2} \alpha^{-1}(t), (2\pi)^{3/2} \} \).

Proof. Define \( \bar{v}_\tau = \bar{v}_\tau(t) = \int_{\mathbb{T}^3} v_\tau(t,x) dx, t \in [0, \tau] \). If we make a decomposition of \( v_\tau \) as in (2.2), then we find that \( \bar{v}_\tau \) is the first component of \( v_\tau \). Hence, for any \( s > 0 \), we have
\[
|v_\tau|_2 \leq |v_\tau - \bar{v}_\tau|_2 + (2\pi)^{3/2} |\bar{v}_\tau| \leq |v_\tau|_s + (2\pi)^{3/2} |\bar{v}_\tau|
\]
\[
\leq |v_\tau|_s + (2\pi)^{9/2} \alpha^{-1}(t) \int_0^t \| v_\tau \|_2^2 ds + (2\pi)^{3/2} |u_0|_1,
\]
where the last inequality follows by Lemma 2.1. Consequently, we have
\[
\| v_\tau(t) \|_s \leq \| v_\tau(t) \|_{\mathbb{H}^{\frac{3}{2}}} \leq \| v_\tau(t) \|_s + c \int_0^t \| v_\tau(s) \|_2^2 ds + c |u_0|_1,
\]
for some \( c = \max\{ (2\pi)^{9/2} \alpha^{-1}(t), (2\pi)^{3/2} \} \). \( \square \)

To prove Theorem 4.1 and Theorem 4.3, we further need the following two classical lemmas from [40] and [35] respectively.
Lemma 2.2. Let $B_0, B, B_1$ be Banach spaces such that $B_0, B_1$ are reflexive and $B_0 \subset B \subset B_1$, where $B_0 \subset B$ stands for compact imbedding. Define, for $0 < T < \infty$,

$$X := \left\{ \frac{dh}{dt} \in L^2([0,T]; B_0), dh \in L^2([0,T]; B_1) \right\}.$$  

Then $X$ is a Banach space equipped with the norm $\|h\|_{L^2([0,T]; B_0)} + |h'|_{L^2([0,T]; B_1)}$. Moreover,

$$X \subset L^2([0,T]; B).$$

Lemma 2.3. Let $V, H, V'$ be three Hilbert spaces such that $V \subset H = H \subset V'$, where $H'$ and $V'$ are the dual spaces of $H$ and $V$ respectively. Suppose $u \in L^2(0,T; V)$ and $u' \in L^2(0,T; V')$. Then $u$ is almost everywhere equal to a function continuous from $[0, T]$ into $H$.

Lemma 2.2 is also called Aubin-Lions Lemma which plays an important role in the proof of Proposition 3.1. One can refer to [40] and other references for the proof of the Lemma 2.2. The Lemma 2.3, a special case of a general result of Lions and Magenes [35], will help us to verify the continuity of the solution to stochastic Burgers equations with respect to time. In fact, this regularity is important for us to establish the global existence of solutions to stochastic equations (2.10)-(2.11). As we know, the maximum principle should be applied to classical solutions to differential equations. But there is no classical solutions to stochastic partial differential equations. Therefore, our ideal is that we apply the maximum principle to random Galerkin approximations. Then, we combine the compactness argument with the regularity of the local solutions to show that the global well-posedness of (2.10)-(2.11) holds, see the proof of Theorem 4.1 for details. For the proof of the Lemma 2.3 one can see [40].

3 Local existence of the solutions to (1.1)

We will use the approach of Galerkin approximations to show the existence of a local strong solution to equation (2.10)-(2.11). In fact, it is sufficient to establish the existence of a local strong solution on time interval $[0, 1]$ as what we do in Proposition 3.1. Because, in view of Proposition 3.1 we can extend the existence time of the local solutions to a more broad time interval than $[0, 1]$ by repeating the proof of Proposition 3.1. Through the iterative extension, we can seek the maximum existence time for the local strong solutions to (2.10)-(2.11). If the maximum existence time equals to infinite almost surely, then the local strong solutions are the global strong solutions.

Proposition 3.1. Suppose $u_0$ is an $\mathbb{H}^1(T^3)$ valued, $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|^2_{\mathbb{H}^1} < \infty$. Then, there exists a unique local strong pathwise solution $v$ to equation (2.10)-(2.11) on the time interval $[0, 1]$ satisfying

$$\sup_{\tau \in [0, \tau^*_1]} \|v(t)\|^2_{\mathbb{H}^1} + \int_0^{\tau^*_1} \|v(t)\|^2 dt < \infty, \mathbb{P} - a.e. \omega \in \Omega,$$

(3.17)

where $\tau^*_1$ is a positive random variable. Moreover, the local strong pathwise solution $v$ to equation (2.10)-(2.11) is Lipschitz continuous with respect to the initial data $u_0$ in $\mathbb{H}^1(T^3)$.  

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Proof. For $t \in (0, \tau_n \wedge 1)$, taking inner product of (2.12) in $L^2([0, t] \times \mathbb{T}^3)$ with $\Lambda^2 v_n$ yields

$$
\|v_n(t)\|_1^2 + 2 \int_0^t \|v_n(s)\|_2^2 ds \\
\leq \|v_n(0)\|_1^2 + \int_0^t \alpha^{-1}(s) \int_{\mathbb{T}^3} |(v_n \cdot \nabla)v_n(s, x)| \times |\Lambda^2 v_n(s, x)| dx ds.
$$

Then by the Hölder inequality, the Sobolev imbedding theorem and the interpolation inequality, we have

$$
\|v_n(t)\|_1^2 + 2 \int_0^t \|v_n(s)\|_2^2 ds \\
\leq \|v_n(0)\|_1^2 + e \int_0^t \|v_n\|_2^2 ds + c(e) \int_0^t \alpha^{-2}(s) |\nabla v_n|_2^2 \|v_n\|_2^2 ds \\
\leq \|v_n(0)\|_1^2 + e \int_0^t \|v_n\|_2^2 ds + c(e) \int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 \|v_n(s)\|_4^2 ds \\
\leq \|v_n(0)\|_1^2 + e \int_0^t \|v_n\|_2^2 ds + c(e) \int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 \|v_n(s)\|_4^2 ds.
$$

Note that

$$
\|v_n(s)\|_4^2 \leq (\|v_n(s)\|_1 + \|v_n(s)\|_2)^4
$$

and

$$
\|v_n(s)\|_2 \leq \|v_n(s)\|_1 + c \int_0^t \|v_n(r)\|_2^2 dr + |u_0|_1,
$$

where $c$ only depends on the domain $\mathbb{T}^3$. In view of the estimates above, we arrive at

$$
\|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \\
\leq \|v_n(0)\|_1^2 + c \int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 \left(\|v_n(s)\|_1 + c \int_0^t \|v_n(r)\|_2^2 dr + |u_0|_1\right)^4 ds \\
\leq \|v_n(0)\|_1^2 + c \int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 \left[\|v_n(s)\|_1^2 + c \left(\int_0^t \|v_n(r)\|_2^2 dr + |u_0|_1\right)^4\right] ds \\
\leq \|v_n(0)\|_1^2 + c \int_0^t \alpha^{-4}(s) \|v_n(s)\|_1^2 \left[\|v_n(s)\|_1^2 + c \left(\int_0^t \|v_n(r)\|_2^2 dr + |u_0|_1\right)^4\right] ds.
$$
Split the last term into two terms, we get

\[
\|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \leq \|v_n(0)\|_1^2 + c \int_0^t \alpha^{-4}(s)\|v_n(s)\|_2^2 ds + c \int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds \left( \int_0^t \|v_n(s)\|_1^2 ds + \|u_0\|_1^4 \right)^4
\]

\[
\leq \|v_n(0)\|_1^2 + c \int_0^t \alpha^{-4}(s)\|v_n(s)\|_2^2 ds + c\|u_0\|_1^4 \int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds
\]

\[
+ c \int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds \left( \int_0^t \|v_n(s)\|_1^2 ds \right)^4
\]

\[
\leq \|v_n(0)\|_1^2 + c \int_0^t \alpha^{-4}(s)\|v_n(s)\|_2^2 ds + c\|u_0\|_1^4 \int_0^t \alpha^{-4}(s)\|v_n(s)\|_1^2 ds
\]

\[
+ c \int_0^t \alpha^{-20}(s)\|v_n(s)\|_1^2 ds + c \int_0^t \|v_n(s)\|_1^4 ds.
\]

At the beginning of the proof, we know that \( t \in [0, 1] \), so we obtain

\[
\|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \leq \|v_n(0)\|_1^2 + \int_0^t \left( c(1 + \alpha^{-4}(s))(1 + \|u_0\|_1^{4/5})\|v_n(s)\|_1^2 \right) ds
\]

\[
+ \int_0^t \left( c(1 + \|u_0\|_1^{4/5})(1 + \alpha^{-4}(s)) \right)^5 ds
\]

Let \( f(s) := c(1 + \alpha^{-4}(s)) \) and \( g(s) := c(1 + \|u_0\|_1^{4/5}(1 + \alpha^{-4}(s)) \). Then

\[
\|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds = \|v_n(0)\|_1^2 + \int_0^t \left( f(s)\|v_n(s)\|_1^2 \right)^5 ds + \int_0^t g^5(s) ds
\]

\[
\leq \|v_n(0)\|_1^2 + \int_0^t \left( f(s)\|v_n(s)\|_1^2 + g(s) \right)^5 ds
\]

\[
\leq \|u_0\|_1^2 + \int_0^t \left( \|v_n(s)\|_1^2 \sup_{s \in [0,1]} f(s) + \sup_{s \in [0,1]} g(s) \right)^5 ds
\]

For simplicity, we set \( A := \sup_{s \in [0,1]} f(s) \) and \( B := \sup_{s \in [0,1]} g(s) \), then we have

\[
\|v_n(t)\|_1^2 + \int_0^t \|v_n(s)\|_2^2 ds \leq \|v_n(0)\|_1^2 + \int_0^t (A\|v_n(s)\|_1^2 + B)^5 ds.
\] (3.18)

By the comparison theorem (see Theorem III-5-1 in page 59 of [28]), it follows that

\[
\|v_n(t)\|_1^2 \leq \frac{A\|u_0\|_1^2 + B}{A(1 - 4At(A\|u_0\|_1^2 + B)^{4/5})^{4/5}} - \frac{B}{A}.
\] (3.19)
Let $\tau$ verify (2.3), we define solution to (2.10)-(2.11). Again, by formula (2.16) and Lemma 2.1, we obtain the estimate (3.17). To $\tau \in v$ that

\[ \tau^* := \frac{1}{4A(\|\alpha_0\|^2_2 + B)^4}. \]

It follows then that one can choose $\tau_0^* = \frac{\tau^*}{\varepsilon} > 0$ such that $\tau_0^*$ is independent of $n \in \mathbb{N}$, which together with (3.18) and (3.19) implies that $v_n$ are uniformly bounded in $L^\infty([0, \tau_0^*]; H^1) \cap L^2([0, \tau_0^*]; H^2)$. From (2.10), by virtue of the Hölder inequality, the Sobolev imbedding theorem and Young’s inequality we have

\[ |\partial_t v_n|_2 \leq \alpha^{-1} |v_n \cdot \nabla v_n|_2 + |\Delta v_n|_2 \]
\[ \leq \alpha^{-1} |v_n|_\infty \|\nabla v_n\|_2 + \|v_n\|_2 \]
\[ \leq c \alpha^{-1} \|v_n\|_3^\varepsilon \|\nabla v_n\|_3^\varepsilon \|v_n\|_1 + \|v_n\|_2 \]
\[ \leq c \alpha^{-1} (\|v_n\|_3^\varepsilon \|\nabla v_n\|_3^\varepsilon \|v_n\|_2^\varepsilon + \|v_n\|_3^\varepsilon \|v_n\|_2^\varepsilon + \|v_n\|\|v_n\|_1) + \|v_n\|_2, \]

where $c$ is independent of dimension $n$ and random time $\tau$. In view of (2.16), we note that

\[ |v_n(t)|_2 \leq c \int_0^t \|v_n(s)\|^2_2 ds + \|v_0\|_1. \]

where $c$ is independent of $n$ and random time $\tau$. Hence, from (3.18)-(3.19), we know $\partial_t v_n$ uniformly bounded in $L^2([0, \tau_0^*]; L^2(\Omega))$. From Lemma 2.2 and Lemma 2.3 we conclude that there exists a subsequence of $v_n$, which is still denoted by $v_n$, such that $v_n$ converges to $v$ in $L^2([0, \tau_0^*]; H^1(\Omega))$ and $v \in C([0, \tau_0^*]; H^1(\Omega))$. Following a standard argument as in [40], one can show that $v$ is the local strong solution to (2.10)-(2.11). Again, by formula (2.16) and Lemma 2.1, we obtain the estimate (3.17). To verify (2.3), we define

\[ \tilde{\tau} = \inf \{ s : \sup_{s \in [0, \tau \wedge \tau_0^*]} \|v(s)\|^2_{H^1} + \int_0^{\tau \wedge \tau_0^*} \|v(s)\|^2_2 ds > 1 \}. \]

Let $\tau_0^* = \tau_0^* \wedge \tilde{\tau}$. This is clearly true that

\[ v(\cdot \wedge \tau_0^*) \in L^2(\Omega; C([0, \infty); H^1(\Omega))), \quad v(t \wedge \tau_0^*) \in L^2(\Omega; L^2_{\text{loc}}([0, \infty); H^2(\Omega))). \]

Finally, to show the pair $(v, \tau_0^*)$ is a local strong pathwise solution to (2.10)-(2.11), it is sufficient to prove that $v(\cdot \wedge \tau_0^*)$ is an $\mathcal{F}_t$-adapted process in $H^1(\Omega)$. This is obvious! Because, from the theory of ordinary differential equations (see Theorem I-1-4 and Theorem I-2-5 of [28]), we conclude that $v(\cdot \wedge \tau_0^*)$ is an $\mathcal{F}_t$-adapted process in $\mathbb{H}^1(\Omega)$. From the argument above, we know there exists a subsequence of $v_n(\cdot \wedge \tau_1^*)$ which converges to $v(\cdot \wedge \tau_1^*)$ in $L^2([0, \tau_1^*]; H^1(\Omega))$. It implies that $v(\cdot \wedge \tau_1^*)$ is an $\mathcal{F}_t$-adapted process in $\mathbb{H}^1(\Omega)$. Up to now, we prove the existence of a local strong pathwise solution $v$ to (2.10)-(2.11). It remains to prove the uniqueness of $(v, \tau_1^*)$.  

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In the following, we will prove the uniqueness of $v$ in $C([0, \tau^*_1]; \mathbb{H}^1(\mathbb{T}^3))$. Let $v_1$ and $v_2$ be two local strong solutions to (2.10)-(2.11). Denote by $\hat{v} := v_1 - v_2$. Then, for $t \in [0, \tau^*_1]$, we have

$$\frac{1}{2} \partial_t \| \hat{v} \|^2 + \| \hat{v} \|^2 \leq - \alpha^{-1} \langle \hat{v} \cdot \nabla v_1, \hat{v} \rangle - \alpha^{-1} \langle v_2 \cdot \nabla \hat{v}, \hat{v} \rangle \leq \alpha^{-1} v_2^0 \| \hat{v} \|^2_{L^2} \| v_1 \|_1 + \varepsilon \| \hat{v} \|^2_1 + c \alpha^{-2} \| v_2 \|^2_{L^2} \| \hat{v} \|^2_2$$

$$\leq \varepsilon \| \hat{v} \|^2_2 + \alpha^{-4} \| \hat{v} \|^2_1 \| v_1 \|_1 + c \alpha^{-2} (\| v_2 \|^2_{L^2} + \int_0^t \| v_2 \|^2_{L^2} ds + |u_0|_1)$$

$$\times (\| v_2 \|_2 + \int_0^t \| v_2 \|^2_{L^2} ds + |u_0|_1) \| \hat{v} \|^2_2, \quad (3.21)$$

which implies via the Gronwall inequality and $\hat{v}(0) = 0$ that $|\hat{v}(t)|_2 = 0, t \in [0, \tau^*_1]$. Taking inner product of (2.12) in $\mathbb{L}^2(\mathbb{T}^3)$ with $(-\Delta v_n)$ and using interpolation inequality further yields,

$$\frac{1}{2} \partial_t \| \hat{v} \|^2 + \| \hat{v} \|^2 \leq \alpha^{-1} \langle \hat{v} \cdot \nabla v_1, \Delta \hat{v} \rangle + \alpha^{-1} \langle v_2 \cdot \nabla \hat{v}, \Delta \hat{v} \rangle \leq \alpha^{-1} \| \hat{v} \|_2 \| \hat{v} \|_1 \| \nabla v_1 \|_1 + \alpha^{-1} \| v_2 \|_{L^2} \| v_2 \|_{L^2} \| \hat{v} \|_1 \| \hat{v} \|_2$$

$$\leq \varepsilon \| \hat{v} \|^2_2 + c(\varepsilon) \alpha^{-4} \| \hat{v} \|^2_1 \| v_1 \|_1 + c(\varepsilon) \alpha^{-2} \| v_2 \|_{L^2} \| v_2 \|_{L^2} \| \hat{v} \|^2_1, \quad (3.22)$$

which implies via the Gronwall inequality and $\hat{v}(0) = 0$ that $|\hat{v}(t)|_1 = 0$ for $t \in [0, \tau^*_1]$. In view of (3.21) and (3.22), the Lipschitz continuity of the local strong solution with respect to the initial data in $\mathbb{H}^1(\mathbb{T}^3)$ follows.

**Proposition 3.2.** Suppose $u_0$ is an $\mathbb{H}^1(\mathbb{T}^3)$ valued, $\mathcal{F}_0$ measurable random variable with $\mathbb{E} \| u_0 \|^2_{L^2} < \infty$. Then, there exists a unique local strong pathwise solution $(u, \tau^*_1)$ to equation (1.1) on the time interval $[0, 1]$.

**Proof.** Since we know $v = au$, then according to Definition 2.1, $(u, \tau^*_1)$ is a local strong pathwise solution to (1.1) if and only if $(u, \tau^*_1)$ satisfies (2.3) and $u(t \wedge \tau^*_1)$ is an $\mathcal{F}_t$-adapted process in $\mathbb{H}^1(\mathbb{T}^3)$.

In fact, by virtue of (2.9), (3.20) and the definition of $\tau^*_1$, yields,

$$\mathbb{E} \sup_{t \in [0, \tau^*_1]} \| u(t) \|^2_{L^2} + \mathbb{E} \int_0^{\tau^*_1} \| u(t) \|^2_{L^2} dt$$

$$\leq \mathbb{E} \sup_{t \in [0, \tau^*_1]} \| v(t) \|^2_{L^2} + \mathbb{E} \int_0^{\tau^*_1} \| v(t) \|^2_{L^2} \alpha^{-2}(t) dt$$

$$\leq \mathbb{E} \sup_{t \in [0, \tau^*_1]} \alpha^{-2}(t) \leq 2 \exp \left(4 \sum_{k=1}^{\infty} b_k^2 T \right) < \infty.$$ 

Noticing both $v$ and $\alpha^{-1}$ are $\mathcal{F}_t$-adapted processes in $\mathbb{H}^1(\mathbb{T}^3)$, so is $u$. Noting the local strong pathwise solution $v$ to (2.10)-(2.11) is unique up to time $\tau^*_1$ and $v = au$, one can easily derive that $u$ is unique up to time $\tau^*_1$. \qed
We should emphasise here that the following Lemma is key to establish the global well-posedness for strong solutions and weak solutions to (1.1) in $C([0, T]; H^1(T^3))$ and $C([0, T]; H^{1/2}(T^3))$, respectively. In fact, the maximum principle is not enough to establish the global well-posedness for 3D stochastic Burgers equations. The local strong solutions should be smoother. Then the following lemma 3.1 will help us to achieve it.

The proof of Lemma 3.1 relies on commutator estimates which is different from the proof of Proposition 3.17 see Theorem A.8 of [31] for more details of commutator estimates.

**Lemma 3.1.** Suppose $u_0$ is an $H^{3/2}(T^3)$ valued, $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|_{H^{3/2}}^2 < \infty$. Then, there exists a unique local strong pathwise solution $v$ to equation (2.10)-(2.11) on $[0, 1]$ satisfying

$$\sup_{t \in [0, T]} \|v(t)\|_{H^1}^2 + \int_0^T \|v(t)\|_{H^2}^2 \, dt < \infty, \mathbb{P} - a.e. \omega \in \Omega,$$

where the positive random variable $\tau^*_1$ is the local existence time for $v$. Moreover, the local strong pathwise solution $v$ to equation (2.10)-(2.11) is Lipschitz continuous with respect to the initial data in $H^{3/2}(T^3)$.

**Proof.** For $t \in (0, \tau_n)$, taking inner product of (2.12) in $L^2([0, t] \times T^3)$ with $\Lambda^3 v_n$ yields

$$\frac{1}{2} \partial_t \|v_n\|_{L^2}^2 + \|v_n\|_{L^2}^2 = -\alpha^{-1} \langle \Lambda^{1/2}(n \cdot \nabla v_n), \Lambda^{3/2} v_n \rangle$$

which implies

$$\|v_n(t)\|_{L^2}^2 + 2 \int_0^t \|v_n(s)\|_{L^2}^2 \, ds \leq \|u_0\|_{L^2}^2 + \varepsilon \int_0^t \|v_n(s)\|_{L^2}^2 \, ds + e \sup_{s \in [0, t]} \alpha^{-2}(s) \int_0^t \int_T \Lambda^{1/2}(n \cdot \nabla v_n) \, dx \, ds. \quad (3.23)$$

In order to bound the last term on the right hand side of (3.23), we will use Theorem A.8 in [31]. Without loss of generality, we assume $t \leq 1$. Then the estimates of the last term follows

$$\int_0^t \int_T |\Lambda^{1/2}(n \cdot \nabla v_n)|^2 \, dx \, ds$$

$$\leq 3 \int_0^t \int_T |v_n \cdot (\Lambda^{3/2} v_n)|^2 \, dx \, ds + 3 \int_0^t \int_T |(\Lambda^{1/2} v_n) \cdot (\Lambda v_n)|^2 \, dx \, ds$$

$$+ \frac{c}{3} \left( \int_0^t \left( \int_T |\Lambda^{3/8} n|^6 \, dx \right)^{2/3} \, ds \right)^{1/2} \left( \int_0^t \left( \int_T |\Lambda^{9/8} n|^3 \, dx \right)^{4/3} \, ds \right)^{1/2}$$

$$=: I_1 + I_2 + I_3 + I_4 \quad (3.24)$$
Using the Holder inequality, the interpolation inequality and Young’s inequality, we get

\[ I_2 \leq \int_0^t |\Lambda v_n(s)|^2 |\Lambda^{1/2} v_n(s)|^2 ds \]
\[ \leq c \int_0^t ||v_n(s)||^2 ||v_n(s)||^2 d\tau \]
\[ \leq e \int_0^t ||v_n(s)||^2 d\tau + c \int_0^t ||v_n(s)||^4 d\tau. \]

In view of the Holder inequality, the Sobolev embedding theorem and (2.16), we have

\[ I_1 \leq c \int_0^t ||v_n||^2 ||v_n||^2 d\tau \leq c \int_0^t ||v_n||^4 d\tau \]
\[ \leq c \int_0^t ||v_n||^4 d\tau + c \int_0^t ||v_n||^2 d\tau \]
\[ \leq c \int_0^t ||v_n||^4 d\tau + c \int_0^t ||v_n(s)||^2 d\tau + c \int_0^t |u_0|^4 d\tau. \]

Utilising the interpolation inequality and (2.16), we then derive

\[ I_4 \leq c \int_0^t |\Lambda^{9/8} v_n|^2 d\tau \leq c \int_0^t ||v_n||^{7/2} ||v_n||^{1/2} d\tau \]
\[ \leq e \int_0^t ||v_n||^2 d\tau + c \int_0^t ||v_n||^{14/3} d\tau \]
\[ \leq e \int_0^t ||v_n(s)||^2 d\tau + c \int_0^t ||v_n(s)||^{14/3} d\tau \]
\[ + c \int_0^t ||v_n(s)||^1 d\tau + c \int_0^t ||v_n(s)||^{28/3} d\tau + c \int_0^t |u_0|^4 d\tau. \]

By the Sobolev imbedding theorem, we obtain

\[ I_3 \leq c \int_0^t ||v_n||^4 d\tau. \]

Combing (3.23), (3.24) and estimates from I_1 to I_4, we arrive at

\[ ||v_n(t)||^2 + \int_0^t ||v_n(s)||^2 d\tau \]
\[ \leq c ||u_0||^2 + c \sup_{s \in [0, 1]} \alpha^{-2}(s) \int_0^t \left( (1 + |u_0|^2) + ||v_n(s)||^2 \right)^{14} d\tau \]

Define \( A = 1 + |u_0|^2 \). Then we have

\[ ||v_n(t)||^2 + \int_0^t ||v_n(s)||^2 d\tau \leq c ||u_0||^2 + c \sup_{s \in [0, 1]} \alpha^{-2}(s) \int_0^t (A + ||v_n(s)||^2)^{14} d\tau. \] (3.25)
Again, by the comparison theorem (see Theorem III-5-1 in page 59 of [28])

\[ ||v_n(t)||_{2} \leq \frac{A + ||u_0||_{2}^{\frac{2}{13}}}{1 - 13c \sup_{s \in [0,1)} \alpha^{-2}(s) (A + ||u_0||_{2})^{13}} - A. \]  

(3.26)

Hence the estimates (3.26) rule out a blowup of \( v_n \) in \( H^{\frac{3}{2}} \) before the time \( \tau_n^{*} = \tau_1^{*} / 2 \), such that \( \tau_n \geq \tau_1 \) for all \( n \). From (3.25) and (3.26), we have uniform bounds for \( v_n \) in \( L^{\infty}([0, \tau_1]; H^{\frac{3}{2}}(\mathbb{T}^3)) \) and in \( L^{2}([0, \tau_1]; H^{\frac{3}{2}}(\mathbb{T}^3)) \). It is easy to show that \( \partial_t v_n \) is uniformly bounded in \( L^{2}([0, \tau_1]; L^{2}(\mathbb{T}^3)) \). By Lemma 2.2 and Lemma 2.3 there exists a subsequence of \( v_n \), which converges to \( v \) in \( L^{2}([0, \tau_1]; H^{\frac{3}{2}}(\mathbb{T}^3)) \) with \( v \in C([0, \tau_1]; H^{\frac{3}{2}}(\mathbb{T}^3)) \). Follow the standard argument one can show that \( v \) is a local solution to (2.10)-(2.11). Following the argument as in Proposition 3.1, there exists a positive random time \( \tau_1^{**} \) such that \( v(\cdot \land \tau_1^{**}) \) is an \( \mathcal{F}_T \)-adapted process in \( H^{\frac{3}{2}}(\mathbb{T}^3) \) so that

\[ v(\cdot \land \tau_1^{**}) \in L^{2}(\Omega; C([0, \infty); H^{\frac{3}{2}}(\mathbb{T}^3))), \quad v_{I \leq \tau_1^{**}} \in L^{2}(\Omega; L^{2}_{loc}([0, \infty); H^{\frac{3}{2}}(\mathbb{T}^3))). \]

Taking a similar argument as in Proposition 3.1, we can show that \( v \) is Lipschitz continuous with respect to the initial data in \( H^{\frac{3}{2}}(\mathbb{T}^3) \).

Following the proof of Proposition 3.2 and utilizing Lemma 3.1 yields the following Lemma 3.2.

**Lemma 3.2.** Suppose \( u_0 \) is an \( H^{\frac{3}{2}}(\mathbb{T}^3) \) valued, \( \mathcal{F}_0 \) measurable random variable with \( \mathbb{E}||u_0||_{H^{\frac{3}{2}}}^{2} < \infty \). Then, there exists a unique local pathwise strong pathwise solution \((u, \tau_1^{**})\) to equation (1.1) on the time interval \([0, 1] \).

**Proposition 3.3.** If we denote by \( \tau^{*} \) and \( \tau^{**} \) the maximum existence times for local solutions in Proposition 3.1 and Lemma 3.1 respectively. Then we have

\[ \mathbb{P}(I_{(\tau^{**}, \infty)}(\tau^{**} - \tau^{*}) \geq 0) = 1. \]

**Proof.** We will prove this result by contradiction. If

\[ \mathbb{P}(I_{(\tau^{**}, \infty)}(\tau^{**} - \tau^{*}) \geq 0) < 1. \]

Then we get that

\[ \mathbb{P}(\tau^{**} < \infty \land (\tau^{**} < \tau^{*})) > 0, \]

(3.27)

which implies that on \((\tau^{**} < \infty) \land (\tau^{**} < \tau^{*})\) we have

\[ \sup_{t \in [0, \tau^{**})} ||v(t)||_{2} = \infty. \]
That is to say, on \((\tau^{**} < \infty) \cap (\tau^{**} < \tau^*)\) we have
\[
\lim_{\tau^*} \limsup_{t \uparrow \tau^{**}} \|v(t)\|_{L^2} = \infty.
\]

It means that given initial data \(u_0 \in H^\frac{3}{2}(T^3)\),
\[
\text{the local solution } v \text{ can not reach time } \tau^{**} \text{ in } H^\frac{3}{2}(T^3) \text{ space.} \quad (3.28)
\]

But by the assumption of this proposition, we know that
\[
v \in C([0, \tau^*); H^1(T^3)) \cap L^2_{loc}([0, \tau^*); H^2(T^3)), \ P - a.e. \omega \in \Omega.
\]

Remembering (3.27), we get that
\[
v(\tau^{**}) \in H^2(T^3)( \subset H^\frac{3}{2}(T^3)), \ \text{on } \ (\tau^{**} < \infty) \cap (\tau^{**} < \tau^*). \quad (3.29)
\]

Obviously, (3.28) is contradictory with (3.29). The result follows. \(\square\)

4 Global well-posedness to (1.1)

The key tool in our study of the regularity of the solutions to 3D stochastic Burgers equations is the maximum principle, stated as Lemma 4.1. The maximum principle will play important roles in establishing the global well-posedness and moment estimates for 3D stochastic Burgers equations, see the proofs of Theorem 4.1, Proposition 5.1 and Proposition 5.3.

**Lemma 4.1.** If \(v_n\) is a solution to the random Burgers equation (2.10)-(2.11) on the time interval \([0, t]\), then
\[
\sup_{s \in [0, t]} |v_n(s)|_{\infty} \leq |v_n(0)|_{\infty}, \ P - a.s. \omega \in \Omega. \quad (4.30)
\]

**Proof.** Let \(\beta > 0\) and set \(f(s, x) := e^{-\beta s}v_n(s, x)\) for all \(s \in [0, t]\) and \(x \in T^3\). Then, multiplying \(v\) on both sides of (2.10) yields
\[
\partial_s|v_n(s)|^2 + \alpha^{-1}(s)v_n(s) \cdot \nabla|v_n(s)|^2 - 2(\Delta v_n \cdot v_n)(s) = 0.
\]

Note that \(|v_n|^2 = |f(s)|^2 e^{2\beta s}\) satisfies
\[
(e^{2\beta s} \partial_s|f(s)|^2 + 2\beta e^{2\beta s}|f(s)|^2) + e^{2\beta s} \alpha^{-1}(s)f(s) \cdot \nabla|f(s)|^2 - 2e^{2\beta s}\Delta f(s) \cdot f(s) = 0,
\]
which implies
\[
\partial_s|f(s)|^2 + 2\beta|f(s)|^2 + e^{2\beta s} \alpha^{-1}(s)f(s) \cdot \nabla|f(s)|^2 - 2\Delta f(s) \cdot f(s) = 0.
\]
On the other hand, since
\[ 2\Delta f(s) \cdot f(s) = \Delta |f(s)|^2 - 2|\nabla f|^2, \]
then we have
\[ \partial_s |f(s)|^2 + 2\beta |f(s)|^2 + \epsilon^{\beta_2} e^{-1}(s) f(s) \cdot \nabla |f(s)|^2 - \Delta |f(s)|^2 + 2|\nabla f|^2 = 0. \quad (4.31) \]
We observe that if \(|f|\) has local maximum at \((t, x) \in (t_0, t) \times \mathbb{R}^3\), then the left hand side of (4.31) is positive unless \(|f(t, x)| \equiv 0\). Therefore,
\[ |f(s)|_\infty \leq |f(0)|_\infty, \]
which implies
\[ |v_n(s)|_\infty \leq \epsilon^{\beta_2} |v_n(0)|_\infty, \quad \text{for } s \in (0, t]. \]
Let \(\beta\) tends to 0, we get the desired result. \(\square\)

In the following, we will use Lemma 4.1 to establish the global existence of the strong solutions to (2.10)-(2.11). As we know, the maximum principle should be applied to classical solutions to differential equations. But there are no classical solutions to stochastic partial differential equations. Therefore, our idea is that we apply the maximum principle to random Galerkin approximations. Then, utilising the compactness argument, we can have a subsequence of the solutions to Galerkin approximations, which converges almost everywhere with respect to time to the solution to (2.10)-(2.11). Through the process of taking limit, we can obtain a priori estimates for the solutions to (2.10)-(2.11). But, note that the a priori estimates only hold almost every for the time. So, what we should do next is using continuity of the local solutions with respect to time to show that the a priori estimates hold for every time. This then completes the proof of the global well-posedness of (2.10)-(2.11).

**Theorem 4.1.** Suppose \(u_0\) is an \(H^1(\mathbb{R}^3)\) valued, \(F_0\) measurable random variable with \(\mathbb{E}\|u_0\|_{H^1}^2 < \infty\). Then, for any \(T > 0\), there exists a unique global strong pathwise solution \(v\) to (2.10)-(2.11) in the sense of Definition 2.1 satisfying
\[ \sup_{t \in [0, T]} \|v(t)\|_{H^1}^2 + \int_0^T \|v(s)\|_{H^1}^2 ds \leq \|u_0\|_{H^1}^2 + c\|v(\epsilon)\|_{H^1}^2 \exp \left\{ c\|v(\epsilon)\|_{H^1}^2 \int_0^T \alpha^{-2}(s) ds \right\} < \infty, \quad (4.32) \]
where (4.32) holds \(P - a.e.\omega \in \Omega\), and \(\epsilon\) is some positive random variable in \((0, T)\). Moreover, the strong pathwise solution \(v\) to equation (2.10)-(2.11) is Lipschitz continuous with respect to the initial data in \(H^1(\mathbb{R}^3)\).

**Proof.** Let \(T^*\) be the maximum existence time for the unique local strong solution \(v\) to (2.10)-(2.11). Let \(\epsilon \in (0, T^*)\). Then taking inner product of (2.12) with \(\Lambda^2 v_n\) in \(L^2(\mathbb{R}^3)\) yields
\[ \partial_t \|v_n\|_1^2 \leq 2\alpha^{-1} \int_{\mathbb{T}^3} (v_n \cdot \nabla) v_n \Delta v_n dx - 2\|v_n\|_2^2 \leq \alpha^{-2} \|v_n\|_\infty^2 \|v_n\|_1^2 + \|v_n\|_2^2. \quad (4.33) \]
For $0 < \epsilon \leq s < t < \tau^*$, by Lemma 4.1, we have
\[
\|v_n(t)\|_{1}^2 + \int_{s}^{t} \|v_n(s)\|_{2}^2 ds \leq c \|v_n(s)\|_{1}^2 \exp \left( c \|v_n(s)\|_{1}^2 \int_{0}^{t} \alpha^{-2}(r) dr \right). \tag{4.34}
\]

From the proofs of Proposition 3.1 and Lemma 3.1, there exists a subsequence of $v_n$ converging to $v$ in both $L^2([0, \tau]; H^{1}(\mathbb{T}^3))$ and $L^2((\epsilon, \tau]; H^{\frac{3}{2}}(\mathbb{T}^3))$ for arbitrary $\tau < \tau$. It implies that we can choose a subsequence of $v_n$ still denoted by itself such that
\[
v_n(t) \rightharpoonup v(t) \text{ in } H^{1}(\mathbb{T}^3) \text{ almost every with respect to } t \in [0, \tau], \tag{4.35}
\]
and
\[
v_n(s) \rightharpoonup v(s) \text{ in } H^{\frac{3}{2}}(\mathbb{T}^3) \text{ almost every with respect to } s \in [\epsilon, t). \tag{4.36}
\]

Letting $n$ tend to infinite in (4.34) via (4.35) and (4.36) yields
\[
\|v(t)\|_{1}^2 \leq c \|v(s)\|_{1}^2 \exp \left( c \|v(s)\|_{1}^2 \int_{0}^{t} \alpha^{-2}(s) ds \right), \tag{4.37}
\]
where (4.37) holds for $t$ and $s$ almost everywhere in $[0, \tau)$ and $[\epsilon, t)$ respectively. Keeping in mind that $v \in C([0, \tau); H^{1}(\mathbb{T}^3)) \cap C((\epsilon, \tau); H^{\frac{3}{2}}(\mathbb{T}^3))$, then from (4.37) we get that
\[
\|v(t)\|_{1}^2 \leq \|v(t)\|_{2}^2 \exp \left( c \|v(t)\|_{2}^2 \int_{0}^{T} \alpha^{-2}(s) ds \right), \text{ for arbitrary } t \in [0, \tau) \tag{4.38}
\]
which implies
\[
\sup_{t \in [0, T]} \|v(t)\|_{1}^2 \leq \|v(t)\|_{2}^2 \exp \left( c \|v(t)\|_{2}^2 \int_{0}^{T} \alpha^{-2}(s) ds \right), \text{ for arbitrary } T \in [0, \infty). \tag{4.39}
\]

Similar to the arguments as in (4.35) and (4.36), we can choose $s \in \left( \frac{\epsilon}{2}, \epsilon \right)$ such that there exists a subsequence $(v_{n'}_{n \in \mathbb{N}'})$ of $(v_n)_{n \in \mathbb{N}}$ with $\mathbb{N}' \subset \mathbb{N}$ satisfying
\[
v_{n'}(s) \rightharpoonup v(s) \text{ in } H^{1}(\mathbb{T}^3), \text{ for almost everywhere } s \in \left( \frac{\epsilon}{2}, \epsilon \right), \tag{4.40}
\]
and
\[
v_{n'}(s) \rightharpoonup v(s) \text{ in } H^{\frac{3}{2}}(\mathbb{T}^3), \text{ for almost everywhere } s \in \left( \frac{\epsilon}{2}, \epsilon \right). \tag{4.41}
\]

From (4.34), for $t \in (0, \tau^*)$, we arrive at
\[
\sup_{n' \in \mathbb{N}'} \int_{s}^{t} \|v_{n'}(s)\|_{2}^2 ds \leq c \sup_{n'' \in \mathbb{N}''} \int_{s}^{t} \|v_{n''}(s)\|_{2}^2 \exp \left( c \|v_{n''}(s)\|_{2}^2 \int_{0}^{t} \alpha^{-2}(r) dr \right) < \infty, \text{ i.e. a.e. } \omega \in \Omega. \tag{4.42}
\]

In view of (4.42) there exists a subsequence $(v_{n''})_{n'' \in \mathbb{N}''}$ of $(v_{n'})_{n' \in \mathbb{N}'}$ with $\mathbb{N}'' \subset \mathbb{N}'$ such that $v_{n''}$ converges to $v$ weakly in $L^2\left( \left[ \frac{\epsilon}{2}, T \right]; H^{\frac{3}{2}}(\mathbb{T}^3) \right)$. That is,
\[
v_{n''} \rightharpoonup v, \text{ in } L^2\left( \left[ \frac{\epsilon}{2}, T \right]; H^{\frac{3}{2}}(\mathbb{T}^3) \right), \text{ as } n'' \to \infty, \tag{4.43}
\]
where $\to$ stands for weak convergence. Let $\phi$ be the test function in $L^2([0, T]; H^2(\mathbb{T}^3))$ with $\int_0^T \|\phi(s)\|^2_2 ds \leq 1$. By virtue of (4.43), we obtain

$$
\int_s^t \langle \Delta v(r), \Delta \phi \rangle dr = \lim_{n'' \to \infty} \int_s^t \langle \Delta v_{n''}(r), \Delta \phi \rangle dr
$$

$$
\leq c \lim_{n'' \to \infty} \|v_{n''}(s)\|_1 \exp\left(c \|v_{n''}(s)\|^2_{H^1} \int_0^t \alpha^{-2}(r) dr\right)
$$

$$
= c\|v(s)\|_1 \exp\left(c \|v(s)\|^2_{H^1} \int_0^t \alpha^{-2}(r) dr\right) < \infty, \mathbb{P} - \text{a.e.} \quad (4.44)
$$

(4.44) implies that

$$
\int_s^t \|v(r)\|^2_{L^2} dr \leq c\|v(s)\|_1^2 \exp\left(c \|v(s)\|^2_{H^1} \int_0^t \alpha^{-2}(r) dr\right) < \infty, \mathbb{P} - \text{a.e.} \quad (4.45)
$$

For simplicity, we set

$$
\int_s^t \|v(r)\|^2_{L^2} dr = h_1(s) \text{ and } c\|v(s)\|_1^2 \exp\left(c \|v(s)\|^2_{H^1} \int_0^t \alpha^{-2}(r) dr\right) := h_2(s).
$$

Then

$$
h_1(s) \leq h_2(s), \text{ for almost everywhere } s \in \left(\frac{\epsilon}{2}, \epsilon\right). \quad (4.46)
$$

From Proposition 3.1 and Lemma 3.1 we know that

$$
v \in C\left([0, \tau); H^1(\mathbb{T}^3) \cap L^2([0, \tau^*); H^2(\mathbb{T}^3)) \right) \text{ and } v \in C\left(\left[\frac{\epsilon}{2}, \tau\right); H^1(\mathbb{T}^3) \cap L^2\left(\left[\frac{\epsilon}{2}, \tau^*\right); H^2(\mathbb{T}^3)\right)\right).
$$

It means $h_i(s), i = 1, 2$, is continuous in $[\frac{\epsilon}{2}, \epsilon]$. Furthermore, we note (4.46) holds. Consequently, we get that $h_1(\epsilon) \leq h_2(\epsilon)$. That is,

$$
\int_{\epsilon}^{t} \|v(r)\|^2_{L^2} dr \leq c\|v(\epsilon)\|_1^2 \exp\left(c \|v(\epsilon)\|^2_{H^1} \int_0^t \alpha^{-2}(r) dr\right) < \infty, \mathbb{P} - \text{a.e.} \quad (4.47)
$$

In Proposition 3.1 we have establish the existence of local strong solutions to (2.10)-(2.11). Hence, we define

$$
t^* = \inf \left\{ t \in [0, \tau) \mid \int_0^t \|v(r)\|^2_{L^2} dr \geq \|u_0\|^2_{H^1} \right\}.
$$

Then we have

$$
\int_0^{t^*} \|v(r)\|^2_{L^2} dr \leq \|u_0\|^2_{H^1}, \mathbb{P} - \text{a.e. } \omega \in \Omega. \quad (4.48)
$$

We can choose $\epsilon$ small enough such that $\epsilon \leq t^*, \mathbb{P} - \text{a.e. } \omega \in \Omega$. Combining (4.47) and (4.48) yields,

$$
\int_0^t \|v(r)\|^2_{L^2} dr \leq \|u_0\|^2_{H^1} + c\|v(\epsilon)\|_1^2 \exp\left(c \|v(\epsilon)\|^2_{H^1} \int_0^t \alpha^{-2}(r) dr\right) < \infty, \mathbb{P} - \text{a.e. } \omega \in \Omega. \quad (4.49)
$$

where $t \in [0, \tau^*)$. Therefore, (4.32) follows from (4.49). In view of the definition of global strong solutions (see Definition 2.2), the global existence of $v$ to (2.10) is achieved by (4.32). The uniqueness is given in Proposition 3.1. \(\square\)
In view of Theorem 4.1 and $v = \alpha u$, it is clearly true that

**Theorem 4.2.** Suppose $u_0$ is an $H^1(T^3)$ valued, $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|^2_{H^1} < \infty$. Then, there exists a unique global strong pathwise solution $u$ to (1.1) in the sense of Definition 2.1.

The following theorem states the uniqueness and global existence for the weak solutions to (2.10)-(2.11). Our idea is that if the initial data $u_0 \in H^1(T^3)$, taking advantage of the parabolic structure of the Burgers equation we know that for arbitrary positive constant $\epsilon$ and $t \in (0, \epsilon)$, the local weak solution $v(t) \in H^2(T^3) \subset H^1(T^3)$. Then the global existence of the strong solutions to (2.10)-(2.11) can be applied to the case of weak solutions.

**Theorem 4.3.** Suppose $u_0$ is an $H^1(T^3)$ valued, $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|^2_{H^1} < \infty$. Then, there exists a unique global weak pathwise solution $v$ to (2.10)-(2.11) in the sense of Definition 2.4. Moreover, the weak pathwise solution $v$ to equation (2.10)-(2.11) is Lipschitz continuous with respect to the initial data in $H^1(T^3)$.

**Proof.** Let $z$ be the periodic solution to the linear heat equation with initial data $u_0$, then $z_n := P_n z$ satisfies

$$\partial_t z_n - \Delta z_n = 0, \quad z_n(0) = P_n u_0. \quad (4.50)$$

Let $\hat{v}_n := v_n - z_n$, then $\hat{v}_n$ satisfies

$$\partial_t \hat{v}_n + \alpha^{-1} P_n [(v_n \cdot \nabla) v_n] - \Delta \hat{v}_n = 0, \quad \hat{v}_n(0) = 0. \quad (4.51)$$

Let $\tau_n$ be the maximal existence time for $v_n$ to (2.12)-(2.13). Then for $t \in [0, \tau_n)$, taking inner product of (4.50) in $H^1$ yields,

$$\frac{1}{2} \partial_t |z_n(t)|^2 + |z_n(t)|^2 = 0. \quad (4.52)$$

which implies

$$|z_n(t)|^2 + 2 \int_0^t |z_n(s)|^2 ds = 2 |P_n u_0|^2_{H^1}. \quad (4.53)$$

Multiplying (4.50) with $\Delta z_n(t)$, taking integration with respect to spatial variables and time yields

$$|z_n(t)|^2 + 2 \int_0^t |z_n(s)|^2 ds = 2 |P_n u_0|^2_{H^1}. \quad (4.54)$$

Combining (4.52) and (4.53) we obtain

$$\sup_{s \in [0,t]} |z_n(s)|^2_{H^1} + 2 \int_0^t |z_n(s)|^2_{H^1} \leq 2 |P_n u_0|^2_{H^1}. \quad (4.54)$$
Multiplying (4.51) with $\Lambda\hat{v}_n$ and integrating over $T^3$ gives
\[
\|\hat{v}_n(s)\|^2 + 2\int_0^s \|\hat{v}_n(s)\|^2 ds \\
\leq \int_0^s \alpha^{-1}(s)|v_n(s)|_0|\nabla v_n(s)|_2|\Lambda\hat{v}_n(s)|_3 ds \\
\leq c\int_0^s \alpha^{-1}(s)||v_n(s)||_{L^4(||v_n(s)||_{L^1}||\hat{v}_n(s)||_{2})} ds,
\]
(4.55)
where the last inequality follows by the Sobolev imbedding theorem. To estimate (4.55), by Lemma 2.1 we have
\[
\alpha^{-1}(s)||v_n(s)||_{L^4}||v_n(s)||_{L^1} \\
\leq c\alpha^{-1}(s)(||v_n(s)||_{L^1} + \int_0^s ||v_n(s)||_{L^2} ds + |u_0|_1)||v_n(s)||_{L^1} \\
\leq c\alpha^{-1}(s)(||z_n(s)||_{L^2} + ||\hat{v}_n(s)||_{L^2}) + c\alpha^{-1}(s)(||z_n(s)||_{L^1} + ||\hat{v}_n(s)||_{L^1}) \left(\int_0^s ||v_n(s)||_{L^2} ds + |u_0|_1\right).
\]
(4.56)
In view of (4.55), (4.56) and Young’s inequality we have
\[
\alpha^{-1}(s)(||z_n(s)||_{L^2} + ||\hat{v}_n(s)||_{L^2}) ||\hat{v}_n(s)||_{L^2} \\
\leq \varepsilon||\hat{v}_n(s)||_{L^2}^2 + c(\varepsilon)\alpha^{-2}(s)(||z_n(s)||_{L^1}^4 + \alpha^{-1}(s)||\hat{v}_n(s)||_{L^2}^2 ||\hat{v}_n(s)||_{L^2}^2) \\
\leq \varepsilon||\hat{v}_n(s)||_{L^2}^2 + c(\varepsilon)\alpha^{-2}(s)(||z_n(s)||_{L^1}^4 + \alpha^{-1}(s)||\hat{v}_n(s)||_{L^2}^2 ||\hat{v}_n(s)||_{L^2}^2, \\
(4.57)
\]
and
\[
\alpha^{-1}(s)(||z_n(s)||_{L^1} + ||\hat{v}_n(s)||_{L^1}) \left(\int_0^s ||v_n(s)||_{L^2} ds + |u_0|_1\right)||\hat{v}_n(s)||_{L^2} \\
\leq \frac{1}{2}\alpha^{-2}(s)||\hat{v}_n(s)||_{L^2}^2 (||z_n(s)||_{L^1}^2 + ||\hat{v}_n(s)||_{L^2}^2) \\
+ \alpha^{-2}(s)||\hat{v}_n(s)||_{L^2} \left(\int_0^s ||v_n(s)||_{L^2} ds + |u_0|_1\right)^2 \\
\leq \varepsilon||\hat{v}_n(s)||_{L^2}^2 + c(\varepsilon)\alpha^{-4}(s)(||z_n(s)||_{L^1}^4 + \frac{1}{2}\alpha^{-2}(s)||\hat{v}_n(s)||_{L^2}^2 ||\hat{v}_n(s)||_{L^2}^2 \\
+ c(\varepsilon)\alpha^{-2}(s) \left(\int_0^s ||v_n(s)||_{L^2} ds + |u_0|_1\right)^4 \\
\leq \varepsilon||\hat{v}_n(s)||_{L^2}^2 + c(\varepsilon)\alpha^{-4}(s)(||z_n(s)||_{L^1}^4 + c(\varepsilon)\alpha^{-2}(s)||\hat{v}_n(s)||_{L^2}^2 ||\hat{v}_n(s)||_{L^2}^2 \\
+ c(\varepsilon)\alpha^{-4}(s) \left(\int_0^s ||v_n(s)||_{L^2} ds + |u_0|_1\right)^4.
\]
(4.58)
From (4.55)-(4.58), we have
\[
\begin{align*}
&\sup_{s \in [0, t]} |\hat{v}_n(s)|^2 + 2 \int_0^t |\hat{v}_n(s)|^2 ds \\
&\leq c \int_0^t \alpha^{-3}(s)|z_n(s)|^4 ds + c \sup_{s \in [0, t]} |\hat{v}_n(s)|^2 \int_0^t \alpha^{-2}(s)|\hat{v}_n(s)|^2 ds \\
&+ c \int_0^t \alpha^{-4}(s)\left( \int_0^s |\nu_n|_{L^2}^2 ds + |u_0|_1 \right)^4 ds + c \int_0^t \alpha^{-2}(s)|z_n(s)|^4 ds \\
&+ \sup_{s \in [0, t]} |\hat{v}_n(s)|^2 \int_0^t \alpha^{-1}(s)|\hat{v}_n(s)|^2 ds \\
&\leq e \sup_{s \in [0, t]} |\hat{v}_n(s)|^2 + c(e) \left( \int_0^t (\alpha^{-2}(s) + 1)|\hat{v}_n(s)|^2 ds \right)^2 \\
&+ c \int_0^t \alpha^{-4}(s)\left( \int_0^s |\nu_n|_{L^2}^2 ds + |u_0|_1 \right)^4 ds + c \int_0^t (1 + \alpha^{-4}(s))|z_n(s)|^4 ds.
\end{align*}
\]

Hence, we have
\[
\begin{align*}
&\sup_{s \in [0, t]} |\hat{v}_n(s)|^2 + \int_0^t |\hat{v}_n(s)|^2 ds \\
&\leq ct \sup_{s \in [0, t]} \alpha^{-3}(s)|u_0|_1^4 + c \left( 1 + \sup_{s \in [0, t]} \alpha^{-4}(s) \right) \int_0^t |z(s)|_{L^2}^4 ds \\
&+ ct \sup_{s \in [0, t]} \alpha^{-4}(s)\left( \int_0^s |z(s)|_{L^2}^2 ds \right)^4 + ct^5 \sup_{s \in [0, t]} \alpha^{-4}(s)|\hat{v}_n(s)|^8 \\
&+ c \sup_{s \in [0, t]} \left( 1 + \alpha^{-4}(s) \right)\left( \int_0^s |\hat{v}_n(s)|_{L^2}^2 ds \right)^2
\end{align*}
\]

To simplify the notations in (4.60), we introduce
\[
\begin{align*}
h(t) := c \sup_{s \in [0, t]} \left( 1 + \alpha^{-4}(s) \right) \int_0^s |\hat{v}_n(s)|_{L^2}^2 ds + ct^5 \sup_{s \in [0, t]} \alpha^{-4}(s)|\hat{v}_n(t)|^6 \\
\text{and} \\
I(t) := ct \sup_{s \in [0, t]} \alpha^{-4}(s)|u_0|_1^4 + c \left( 1 + \sup_{s \in [0, t]} \alpha^{-4}(s) \right) \int_0^t |z(s)|_{L^2}^4 ds + ct \sup_{s \in [0, t]} \alpha^{-4}(s)\left( \int_0^s |z(s)|_{L^2}^2 ds \right)^4.
\end{align*}
\]

Then (4.60) is given by
\[
\sup_{s \in [0, t]} |\hat{v}_n(s)|^2 + \int_0^t |\hat{v}_n(s)|^2 ds \leq I(t) + h(t)\left( \sup_{s \in [0, t]} |\hat{v}_n(s)|^2 + \int_0^t |\hat{v}_n(s)|_{L^2}^2 ds \right).
\]

In the following, we will find a uniform lower bound for the maximal existence time $\tau_n$, then we can show that the local solutions exist. Set
\[
\tau_n^* := \sup \{t \in [0, \tau_n) : h(t) \leq 1/2 \}.
\]
Since \( h \) is continuous, we have \( \tau_n^* < \tau_n \). Obviously, \( h(t) \uparrow \infty \) as \( t \uparrow \tau_n \) and \( h(\tau_n) = \frac{1}{2} \). It is easy to see that \( I(t) \) is continuous, increasing and positive except at \( t = 0 \). Denote

\[
\kappa := \sup \left\{ t \in [0, \infty) : I(t) < \min \left( \frac{1}{8c} \sup_{s \in [0, t]} (1 + \alpha^{-4}(s)), \frac{1}{32cr^5} \sup_{s \in [0, t]} \alpha^{-4}(s)^{1/3} \right) \right\}.
\]

Obviously, \( \kappa > 0 \) and is independent of \( n \). We will show that \( \tau_n \geq \kappa \) for all \( n \). Suppose, for contradiction, that \( \tau_n^* < \kappa \) for some \( n \), then by (4.60),

\[
\sup_{s \in [0, \tau_n]} \|\hat{v}_n(s)\|_2^2 + \int_0^{\tau_n} \|\hat{v}_n(s)\|_2^2 \, ds \leq 2I(\tau_n^*),
\]

which implies

\[
h(\tau_n) := c \sup_{s \in [0, \tau_n]} (1 + \alpha^{-4}(s)) \int_0^{\tau_n} \|\hat{v}_n(s)\|_2^2 \, ds + c(\tau_n^*)^5 \sup_{s \in [0, \tau_n]} \alpha^{-4}(s)\|\hat{v}_n(\tau_n^*)\|_1^6 < \frac{1}{2}.
\]

This results in a contradiction. So, we obtain that \( \tau_n \geq \kappa \) for all \( n \). Furthermore, we have that

\[
\sup_{s \in [0, \kappa]} \|\hat{v}_n(s)\|_2^2 + \int_0^{\kappa} \|\hat{v}_n(s)\|_2^2 \, ds \leq 2I(\kappa). \tag{4.62}
\]

Therefore, \((\hat{v}_n)_{n=1}^\infty\) is uniformly bounded in \( L^2([0, \kappa]; \H^{\frac{3}{2}}(\T^3)) \) and \( L^\infty([0, \kappa]; \H^{\frac{3}{2}}(\T^3)) \). From (4.51), let \( \varphi \in \H^{\frac{1}{2}}(\T^3) \), we have

\[
\langle \partial_t \hat{v}_n, \varphi \rangle = \alpha^{-1} \langle P_n(v_n \cdot \nabla) v_n(s), \varphi \rangle - \langle \Lambda^{3/2} \hat{v}_n, \Lambda^{1/2} \varphi \rangle \\
\leq \alpha^{-1} \|\varphi\|_3 \|\nabla v_n(s)\|_2 \|v_n(s)\|_6 + \|\hat{v}_n(s)\|_{\H^{\frac{1}{4}}} \|\varphi\|_{\H^{\frac{1}{4}}} \\
\leq c(\alpha^{-1} \|\varphi\|_{\H^{\frac{1}{2}}} \|v_n(s)\|_{\H^{\frac{3}{2}}} \|v_n(s)\|_{\H^{\frac{3}{2}}} + \|\hat{v}_n(s)\|_{\H^{\frac{1}{2}}} \|\varphi\|_{\H^{\frac{1}{2}}},
\]

where the last inequality follows from the interpolation inequality. Therefore,

\[
\int_0^\kappa \langle \partial_t \hat{v}_n(s), \varphi \rangle^2 \, dt \leq c \int_0^\kappa \alpha^{-2}(s) \|\varphi\|_{\H^{\frac{1}{2}}}^2 \|v_n(s)\|_{\H^{\frac{3}{2}}}^2 \|v_n(s)\|_{\H^{\frac{3}{2}}}^2 \, ds \\
+ 2 \int_0^\kappa \|\hat{v}_n(s)\|_{\H^{\frac{1}{2}}}^2 \|\varphi\|_{\H^{\frac{1}{2}}}^2 \, ds. \tag{4.63}
\]

From (4.62), (4.63) and (2.16), we have \( \partial_t \hat{v}_n \in L^2([0, \kappa]; \H^{\frac{1}{2}}(\T^3)) \). In view of Lemma 2.2 and Lemma 2.3, we have \((v_n)_{n \geq 1}\) converges to \( v \) in \( L^2([0, \kappa]; \H^{\frac{1}{2}}(\T^3)) \) and \( v \in C([0, \kappa]; \H^{\frac{3}{2}}(\T^3)) \cap L^2([0, \kappa]; \H^{\frac{5}{2}}(\T^3)) \). Obviously, following a standard argument (see [40]), we see that \( v \) satisfies (2.6). Taking a similar argument as in Proposition 3.1, one can prove that \( v \) satisfies (2.5) and is an \( \mathcal{F}_t \)-adapted process in \( \H^{\frac{3}{2}}(\T^3) \). So far, we prove that \( v \) is the local weak pathwise solution to (2.10)-(2.11) according to Definition .

Let \( \tau_v \) be the maximum existence time of \( v \). In order to prove the global existence of \( v \), in view of Definition 2.4, it is sufficient to show that

\[
\mathbb{P}\{\tau_v < \infty\} = 0.
\]
Let us prove it by contradiction. In deed, if we assume
\[ \mathbb{P}(\tau_v < \infty) > 0, \]  
then for arbitrary \( \epsilon \in (0, \kappa) \), we have
\[ \mathbb{P} \left( \sup_{t \in [\epsilon, \tau_v]} \|v(t)\|_{\mathbb{H}^1} = \infty | \tau_v < \infty \right) = 1. \]  
By the local existence of weak solutions established above, we know that
\[ v(\epsilon) \in \mathbb{H}^2(T^3) \subset \mathbb{H}^1(T^3). \]
If we regard \( v(\epsilon) \in \mathbb{H}^1(T^3) \) as the initial data of (2.10)-(2.11), by Theorem 4.1, we know that the unique global strong solution exists on \([\epsilon, \infty)\), \( \mathbb{P} \)-a.e. \( \omega \in \Omega \). More precisely, for arbitrary \( T > 0 \), we have
\[ \mathbb{P} \left( \sup_{t \in [\epsilon, T]} \|v(t)\|_{\mathbb{H}^1} < \infty \right) = 1, \]
or
\[ \mathbb{P} \left( \sup_{t \in [\epsilon, T]} \|v(t)\|_{\mathbb{H}^1} = \infty \right) = 0. \]  
We will apply (4.66) to (4.65) to derive a contradiction. In deed,
\[ \mathbb{P} \left( \sup_{t \in [\epsilon, \tau_v]} \|v(t)\|_{\mathbb{H}^1} = \infty, \tau_v < \infty \right) = \mathbb{P} \left( \sup_{t \in [\epsilon, \tau_v]} \|v(t)\|_{\mathbb{H}^1} = \infty, \bigcup_{n=1}^{\infty} (\tau_v < n) \right) \leq \sum_{n=1}^{\infty} \mathbb{P} \left( \sup_{t \in [\epsilon, n]} \|v(t)\|_{\mathbb{H}^1} = \infty \right) \leq \sum_{n=1}^{\infty} \mathbb{P} \left( \sup_{t \in [\epsilon, n]} \|v(t)\|_{\mathbb{H}^1} = \infty \right) = 0. \]  
By (4.67) we know
\[ \mathbb{P} \left( \sup_{t \in [\epsilon, \tau_v]} \|v(t)\|_{\mathbb{H}^1} = \infty | \tau_v < \infty \right) = \frac{\mathbb{P} \left( \sup_{t \in [\epsilon, \tau_v]} \|v(t)\|_{\mathbb{H}^1} = \infty, \tau_v < \infty \right)}{\mathbb{P}(\tau_v < \infty)} = 0, \]
which is contradict with (4.65). Therefore, we arrive at \( P(\tau_v < \infty) = 0 \), which implies the global existence of the weak pathwise solutions.

Let \( v_1 \) and \( v_2 \) be weak pathwise solutions to equation (2.10). Then we denote by \( v \) the difference of \( v_1 \) and \( v_2 \), i.e., \( v = v_1 - v_2 \). Taking inner product of the equation satisfied by \( v \) with \( \Lambda v \) in \( L^2(T^3) \) yields,
\[ \|v(t)\|_2^2 + 2 \int_0^t \|v(s)\|_2^2 \, ds \leq c \int_0^t \alpha^{-1}(s)\|v_1(s)\|_6 \|v(s)\|_2 |\Lambda v(s)|_3 \, ds + c \int_0^t \alpha^{-1}(s)\|v(s)\|_6 \|\nabla v_1(s)\|_3 \|\Lambda v(s)\|_2 \, ds \leq c \int_0^t \alpha^{-1}(s)\|v_1(s)\|_6 \|v(s)\|_1 \|\Lambda v(s)\|_2 \, ds + c \int_0^t \alpha^{-1}(s)\|v(s)\|_6 \|v_1(s)\|_2 \|\Lambda v(s)\|_1 \, ds \leq K_1 + K_2, \]  
(4.68)
where in the last inequality, we have used $|\nabla v_1(s)|_3 \leq c\|v_1(s)\|_{\frac{3}{2}}$. Since by the interpolation inequality, the Sobolev inequality and Young’s inequality, we have

$$K_1 \leq c \int_0^t \alpha^{-1}(s)\|v_1(s)\|_{H^2}^2 \|v(s)\|_{\frac{3}{2}}^2 ds + c \int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds + \varepsilon \int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds,$$

and

$$K_2 \leq c \int_0^t \alpha^{-1}(s)\|v(s)\|_{H^0}^2 \|v(s)\|_{\frac{3}{2}} ds + c \int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds$$

From (4.68) and estimates of $K_1$ and $K_2$, we have

$$\|v(t)\|_{\frac{3}{2}}^2 + 2 \int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds$$

$$\leq c \int_0^t \alpha^{-4}(s)\|v_1(s)\|_{H^2}^4 \|v(s)\|_{\frac{3}{2}}^2 ds + c \int_0^t \alpha^{-2}(s)\|v(s)\|_{\frac{3}{2}}^2 \|v_1(s)\|_{\frac{3}{2}}^2 ds$$

$$+ c \int_0^t \alpha^{-1}(s)\|v_1(s)\|_{\frac{3}{2}}^2 + \|v_2(s)\|_{\frac{3}{2}}^2 ds \int_0^t \|v(s)\|_{\frac{3}{2}}^2 ds. \quad (4.69)$$

From (2.16), we know that

$$\int_0^t \alpha^{-4}(s)\|v_1(s)\|_{H^2}^4 ds$$

$$\leq \int_0^t \alpha^{-4}(s)\|v_1(s)\|_{\frac{3}{2}}^2 \|v_1(s)\|_{\frac{3}{2}}^2 ds + c \sup_{s \in [0,t]} \alpha^{-4}(s)\left( \int_0^t \|v_1(s)\|_{\frac{3}{2}}^2 ds + |u_0| \right)^4 \quad (4.70)$$

for arbitrary $t > 0$. Therefore, by (4.69), (4.70) and the Gronwall inequality, we obtain that $\|v(t)\|_{\frac{3}{2}} = 0$ for arbitrary $t > 0$. Then in view of (2.16) that

$$\|v(t)\|_{\frac{3}{2}} \leq \|v(t) - \bar{v}(t)\|_{\frac{3}{2}} + (2\pi)\frac{1}{2} |\bar{v}(t)| \leq \|v(t) - \bar{v}(t)\|_{\frac{3}{2}} + (2\pi)\frac{1}{2} |\bar{v}(t)|$$

$$\leq \|v(t)\|_{\frac{3}{2}} + \|v(t)\|_{\frac{3}{2}} + (2\pi)\frac{1}{2} |\bar{v}(t)| = \|v(t)\|_{\frac{3}{2}} + (2\pi)\frac{1}{2} |\bar{v}(t)|$$

$$\leq \|v(t)\|_{\frac{3}{2}} + (2\pi)\frac{9}{2} \int_0^t \|v(s)\|_{\frac{3}{2}} \left( \|v_1(s)\|_{\frac{3}{2}} + \|v_2(s)\|_{\frac{3}{2}} \right) ds = 0. \quad (4.71)$$

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where \( \bar{v} = \int_{T^3} v(x) dx \) and \( \| \bar{v}(t) \|^2_{L^1} = 0 \). Hence, in view of (4.71) we arrive at that, for each \( t \geq 0 \), \( \bar{v}(t, x) = 0 \) a.e. \( x \in T^3 \). Moreover, from (4.69) and (4.71), we can establish the Lipschitz continuity of \( \bar{v} \) with respect to the initial data in \( H^{\frac{1}{2}}(T^3) \). The uniqueness of the weak solutions to (2.10)-(2.11) follows from the continuous dependence of \( \bar{v} \) with respect to the initial data.

In view of Theorem 4.3 and \( \nu = \alpha u \), it is clearly true that

**Theorem 4.4.** Suppose \( u_0 \) is an \( H^{\frac{1}{2}}(T^3) \) valued, \( \mathcal{F}_0 \) measurable random variable with \( \mathbb{E}\|u_0\|^2_{H^{\frac{1}{2}}} < \infty \). Then, there exists a unique global weak pathwise solution \( u \) to (1.1) in the sense of Definition 2.2.

Following the method of Theorem 4.1 in [9], we have

**Theorem 4.5.** For any \( \mathcal{F}_0 \)-adapted initial data \( u_0 \in L^p(T^3) \) with \( \mathbb{E}\|u_0\|_p^q < \infty \) satisfying \( p > 3 \) and \( q \geq p \), there exists a unique global mild solution \( u \) to (1.1).

**Theorem 4.6.** Suppose \( u_0 \) is an \( H^{\frac{3}{2}}(T^3) \) valued, \( \mathcal{F}_0 \) measurable random variable with \( \mathbb{E}\|u_0\|^2_{H^{\frac{3}{2}}} < \infty \). Then, for any \( T > 0 \), there exists a unique global strong pathwise solution \( v \) to (2.10)-(2.11) satisfying \( v \in C([0, T]; H^{\frac{3}{2}}(T^3)) \cap L^2([0, T]; H^{\frac{5}{2}}(T^3)) \) and \( v \) is Lipschitz continuous with respect to the initial data in \( H^{\frac{3}{2}}(T^3) \).

**Proof.** We have established the unique local strong solution \( v \) to (2.10)-(2.11) with initial data \( u_0 \in H^{\frac{3}{2}}(T^3) \) in Lemma 2.1. We denote by \( \tau^{**} \) the maximum existence time for \( v \). To prove the global existence of \( v \), it suffices to show that \( \mathbb{P}(\tau^{**} < \infty) = 0 \).

We will prove the result by contradiction. That is, we assume \( \mathbb{P}(\tau^{**} < \infty) > 0 \). By Proposition 3.3, we get that

\[
\mathbb{P}\left( I_{\tau^{**} < \infty} (\tau^{**} - \tau^*) \geq 0 \right) = 1, \tag{4.72}
\]

where \( \tau^* \) is the maximum existence time for local strong solution \( v \) to (2.10)-(2.11) with initial data \( u_0 \in H^{\frac{3}{2}}(T^3) \). (4.72) implies that

\[
\mathbb{P}\left( (\tau^{**} - \tau^*) \geq 0 | \tau^{**} < \infty \right) = 1. \tag{4.73}
\]

As we know, in Theorem 4.1, we have proven the global well-posedness for (2.10)-(2.11) given the initial data \( u_0 \in H^{\frac{3}{2}}(T^3) \). That is to say

\[
\mathbb{P}(\tau^* < \infty) = 0. \tag{4.74}
\]
Based on (4.74), we will recalculate (4.73). That is

\[ \mathbb{P}(\tau^{**} - \tau^{*} \geq 0 | \tau^{**} < \infty) = \frac{\mathbb{P}(\tau^{**} - \tau^{*} \geq 0, \tau^{**} < \infty)}{\mathbb{P}(\tau^{**} < \infty)} = \frac{\mathbb{P}(\tau^{**} - \tau^{*} \geq 0, \bigcup_{k=1}^{\infty} (\tau^{**} < k))}{\mathbb{P}(\tau^{**} < \infty)} \leq \frac{1}{\mathbb{P}(\tau^{**} < \infty)} \sum_{k=1}^{\infty} \mathbb{P}(\tau^{**} - \tau^{*} \geq 0, \tau^{**} < k) \leq \frac{1}{\mathbb{P}(\tau^{**} < \infty)} \sum_{k=1}^{\infty} \mathbb{P}(\tau > \tau^{*}) = 0. \] (4.75)

Obviously, (4.75) is contradict with (4.73). The contradiction implies that \( \mathbb{P}(\tau^{**} < \infty) = 0 \). The Lipschitz continuity with respect to the initial data in \( \mathbb{H}^3(T^3) \) is proved in Lemma 2.1 or we can follow the argument in Proposition 3.1 to establish it. □

Remark 1. In view of Theorem 4.6, repeating the argument in Proposition 3.1 one can choose a subsequence of \( v_n \), which is still denoted by \( v_n \), such that \( v_n \) is uniformly bounded in \( L^\infty([0,T]; \mathbb{H}^3(T^3)) \) and converges to \( v \) in \( L^2([0,T]; \mathbb{H}^3(T^3)) \).

Noticing the argument above and \( v = \alpha u \), we arrive at

Theorem 4.7. Suppose \( u_0 \) is an \( \mathbb{H}^m(T^3) \) valued, \( \mathcal{F}_0 \) measurable random variable with \( \mathbb{E} \| u_0 \|_{\mathbb{H}^m(T^3)}^2 < \infty, m \geq 1 \). Then, for any \( T > 0 \), there exists a unique global strong pathwise solution \( u \) to (1.1) satisfying \( u \in C([0,T]; \mathbb{H}^m(T^3)) \cap L^2([0,T]; \mathbb{H}^{m+1}(T^3)) \).

5 Moment estimates to (1.1)

Multidimensional Burgers equations have been used in studying turbulence and in modelling the large scale structure of the universe, see for instance [2, 8, 29, 39]. Hence, the ergodicity is an important topic for multidimensional stochastic Burgers equations. The first step of study ergodicity is to consider the existence of the invariant measure which obligates us to establish the moment estimates for 3D stochastic Burgers equations. This section is devoted to moment estimates for 3D stochastic Burgers equations with multiplicative noise.

Proposition 5.1. For any \( \mathcal{F}_0 \)-adapted initial value \( u_0 \in \mathbb{H}^3(T^3) \) satisfying \( \mathbb{E} \| u_0 \|_{\mathbb{H}^3(T^3)}^q < \infty, q \geq 1 \), and \( \delta \) is an arbitrary small positive constant. Then for any \( T > 0 \), the unique global strong solution \( u \) to (1.1) satisfies

\[ \mathbb{E} \sup_{t \in [0,T]} |u(t)|_p^q \leq c \exp cT, \] (5.76)

where the constant \( c \) is independent of \( T \).
Proof. From Lemma 4.1 and the Sobolev imbedding theorem, we know

\[
\sup_{t\in[0,T]} |v_n(t)|_p \leq \sup_{t\in[0,T]} |v_n(t)|_\infty \leq \sup_{n\in\mathbb{N}} |v_n(0)|_\infty \leq c||v_n(0)||_1 \leq c|v(0)||_1 = c||u(0)||_1, \tag{5.77}
\]

where the constant c is independent of T. Hence, there exists a subsequence of \(v_n\) still denoted by \(v_n\) such that

\[
v_n \rightarrow^* v \quad \text{in } L^\infty([0,T];\mathbb{L}^p(T^3)), \mathbb{P} - a.e. \omega \in \Omega, \tag{5.78}
\]

where v is the strong solution to (2.10)-(2.11), see Theorem 4.1.

Let \(\phi \in L^1([0,T];\mathbb{L}^q(D))\) with \(\frac{1}{p} + \frac{1}{q} = 1\) and \(\int_0^T |\phi(s)||_q ds \leq 1\). Then we have that

\[
\int_0^T \int_D \phi(t,x)v(t,x)dxdt = \lim_{n\to\infty} \int_0^T \int_D \phi(t,x)v_n(t,x)dxdt \\
\leq \int_0^T |\phi(s)||_q ds \sup_{t\in[0,T]} |v_n(t)|_p \\
\leq ||u(0)||_1. \tag{5.79}
\]

Hence, for arbitrary \(m \geq 1\), from (5.79) we obtain that

\[
\mathbb{E} \sup_{t\in[0,T]} |v(t)|_p^q \leq c\mathbb{E}||u(0)||_1^q < \infty. \tag{5.80}
\]

For positive constants \(p'\) and \(q'\) satisfying \(\frac{1}{p'} + \frac{1}{q'} = 1\), we have

\[
\mathbb{E} \sup_{t\in[0,T]} |u(t)|_p^q \leq \left(\mathbb{E} \sup_{t\in[0,T]} |v(t)|_p^{pq'}\right)^{\frac{1}{q'}} \left(\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t)\right)^{\frac{1}{q'}} \\
\leq c\left(\mathbb{E}||u(0)||_1^{pp'}\right)^{\frac{1}{q'}} \left(\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t)\right)^{\frac{1}{q'}}. \tag{5.81}
\]

Let \(Q = qq'\), \(\sum_{j=1}^{\infty} b_j^2 = b\) and \(W(t) = \sum_{j=1}^{\infty} b_j B_j(t), t \in [0, T]\). Then \(W(t) \sim N(0, bt)\), where \(N(0, bt)\) denotes the normal distribution with mean 0 and variance \(bt\). In the following, we will compute

\[
\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t). \tag{5.82}
\]

By the Doob’s maximal inequality,

\[
\mathbb{E} \sup_{t\in[0,T]} \alpha^{-qq'}(t) = \mathbb{E} \sup_{t\in[0,T]} \left(\exp W(t)\right)^Q \\
\leq \left(\frac{Q}{Q-1}\right)^Q \mathbb{E} \exp QW(T) \\
= \left(\frac{Q}{Q-1}\right)^Q \frac{1}{\sqrt{2\pi bT}} \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2bT}) Qx dx \\
= \sqrt{2\pi} \left(\frac{Q}{Q-1}\right)^Q \exp(bt Q^2/2). \tag{5.83}
\]

Hence

\[
\mathbb{E} \sup_{t\in[0,T]} |u(t)|_p^q \leq \left(\mathbb{E}||u(0)||_1^{pp'}\right)^{\frac{1}{q'}} 2^{\frac{qq'}{qq'-1}} \exp(bt Q^2/2). \tag{5.82}
\]

\[\square\]
Remark 2. From (5.81) and (5.82) in the proof of Proposition 5.1, if we only estimate $E|u(t)|_p^p$, then we should estimate $E^{\exp QW(t)}$. By the Novikov condition, we get that

\[
E^{\exp QW(t)} = E^{\exp QW(t) - 2^{-1} Q^2 \sum_{k=1}^{\infty} b_k^2 t + 2^{-1} Q^2 \sum_{k=1}^{\infty} b_k^2 t} = \exp(2^{-1} Q^2 \sum_{k=1}^{\infty} b_k^2 t).
\]

From the above argument, we find that the bound for moment estimates of 3D stochastic Burgers equations has exponential growth. As we know, in order to establish the existence of invariant measure for stochastic hydrodynamics equations, the common method is to use Krylov-Bogoliubov procedure. However, this procedure obligate us to show that the moment estimates for the corresponding stochastic hydrodynamic equations should have linear growth but not the exponential growth. Hence, the existence of an invariant measure for multidimensional stochastic Burgers equations is a difficult problem.

In the following, we try to obtain the moment estimates for the strong solutions to (1.1). That is, we will estimate $E \lim_{t \in [0, T]} \|u(t)\|_{L^2}^2$, where $u$ is the strong solution to (1.1). But, please remember that the maximum principle is a critical tool to establish the energy estimates for the multidimensional stochastic Burgers equations, that is, we need to estimate of $u$ in $L^\infty(\mathbb{T}^3)$ space to reduce the powers of nonlinear terms, see the proof of Theorem 5.4. Hence, we should obtain the moment estimates in $L^\infty(\mathbb{T}^3)$ space before establishing the estimates in $H^1(\mathbb{T}^3)$ space. That is what we do in Proposition 5.3. However, before we establish Proposition 5.3, we need develop a maximum principle for the 3D random Burgers equations. That is our Theorem 5.2.

**Theorem 5.2.** For any $\mathcal{F}_0$–adapted initial value $u_0 \in H^1(\mathbb{T}^3)$ with $E\|u_0\|_{H^1}^2 < \infty$. Then for any $T > 0$, the unique global strong solution $v$ to (2.10)-(2.11) satisfies

\[
\sup_{t \in [0, T]} |v(t)|_{\infty} \leq |v(0)|_{\infty} = |u(0)|_{\infty}, \mathbb{P} - a.e. \omega \in \Omega. \tag{5.83}
\]

**Proof:** In view of Remark 1 there exists a subsequence of solutions $v_n$ to (2.12) and (2.13), which is still denoted by $v_n$ such that

\[
v_n(t) \to v(t) \text{ in } L^2([0, T]; H^1(\mathbb{T}^3)).
\]

Then we can choose a subsequence of $v_n$ still denoted by $v_n$ satisfying

\[
v_n(t) \to v(t) \text{ in } L^\infty(\mathbb{T}^3) \text{ for almost every } t \in [0, T].
\]

Let $\varphi \in L^1(\mathbb{T}^3)$ with $|\varphi| \leq 1$, we have

\[
\langle v(t), \varphi \rangle = \lim_{n \to \infty} \langle v_n(t), \varphi \rangle \leq \lim_{n \to \infty} |v_n(t)|_{\infty} \leq \lim_{n \to \infty} |v_n(0)|_{\infty} \leq |v(0)|_{\infty}, \tag{5.84}
\]
where the constant \( c \) is independent of \( T \) and the second inequality follows by Lemma 4.1 (5.34) implies that

\[
\sup_{t \in [0,T]} |v(t)|_\infty \leq |v(0)|_\infty = |u(0)|_\infty.
\]

\( \square \)

**Proposition 5.3.** For any \( \mathcal{F}_0 \)-adapted initial value \( u_0 \in H^\frac{1}{2}(\mathbb{T}^3) \) satisfying \( \mathbb{E}|u_0|^{q+\delta} < \infty \), \( q \geq 1 \) and \( \delta \) is an arbitrary small positive constant. Then, for any \( T > 0 \), the unique global strong solution \( u \) to (1.7) satisfies

\[
\mathbb{E} \sup_{t \in [0,T]} |u(t)|_\infty^q \leq c \exp cT, \tag{5.85}
\]

where the constant \( c \) is independent of \( T \).

**Proof.** Let \( u \) and \( v \) be the unique strong solutions to equations (1.1) and (2.10) respectively. Then note that \( u = \alpha^{-1}v \). Hence for \( q \geq 1 \) we have

\[
\sup_{t \in [0,T]} |u(t)|_\infty^q \leq \sup_{t \in [0,T]} |v(t)|_\infty^q \sup_{t \in [0,T]} \alpha^{-q}(t) \leq |u(0)|_\infty^q \sup_{t \in [0,T]} \alpha^{-q}(t). \tag{5.86}
\]

For positive constants \( p' \) and \( q' \) satisfying \( \frac{1}{p'} + \frac{1}{q'} = 1 \), we have

\[
\mathbb{E} \sup_{t \in [0,T]} |u(t)|_\infty^q \leq \left( \mathbb{E} \sup_{t \in [0,T]} |v(t)|_\infty^{qq'} \right)^{\frac{1}{q'}} \left( \mathbb{E} \sup_{t \in [0,T]} \alpha^{-qq'}(t) \right)^{\frac{1}{q'}} \leq c \left( \mathbb{E}|u(0)|^{qq'} \right)^{\frac{1}{q'}} \left( \mathbb{E} \sup_{t \in [0,T]} \alpha^{-qq'}(t) \right)^{\frac{1}{q'}}. \tag{5.87}
\]

Finally the estimate (5.85) follows by (5.82) and (5.87). \( \square \)

Next, we aim to obtain \( \mathbb{E} \lim_{t \in [0,T]} |u(t)|^2_{H^\frac{3}{2}} < \infty \) for the strong solution \( u \) of (1.1). Notice that, the initial data \( u_0 \in H^\frac{1}{2}(\mathbb{T}^3) \), hence, we prefer to prove \( \mathbb{E} \lim_{t \in [0,T]} |u(t)|^2_{H^\frac{3}{2}} < \infty \). But, it is difficult! Due to the high nonlinearity of 3D stochastic Burgers equations, we can only establish the logarithmic moments in \( H^1(\mathbb{T}^3) \), see (5.88) of Theorem 5.4 below. We need techniques from logarithmic moments to reduce the powers arising from the nonlinear term. In Remark 5 below, we will illustrate that \( \mathbb{E} \sup_{t \in [0,T]} \log(1 + |u(t)|^2_{H^\frac{3}{2}}) < \infty \) remains unknown.

**Theorem 5.4.** For any \( \mathcal{F}_0 \)-adapted initial value \( u_0 \in H^\frac{1}{2}(\mathbb{T}^3) \) satisfying \( \mathbb{E}|u_0|^{2+\sigma} < \infty \), and \( \sigma \) is an arbitrary small positive constant. Then, for any \( T > 0 \), the unique global strong solution \( u \) to (1.7) satisfies

\[
\mathbb{E} \sup_{t \in [0,T]} \log(1 + |u(t)|^2_{H^1}) \leq c \exp cT, \tag{5.88}
\]

where the constant \( c \) is independent of \( T \).
Proof. Taking a similar argument as in the proof of Lemma 2.1 yields,
\[
\frac{1}{2} \partial_t \|v\|_1^2 + \|v\|_2^2 \leq \alpha^{-1}(v \cdot \nabla v, A^2 v) \leq c\|v\|_2^2 + c(\varepsilon)\alpha^{-2}\|v\|_m^2\|v\|_1^2.
\]
Then by the maximum principle for random Burgers equations Theorem 5.2, we have
\[
\partial_t \log(\|v\|_1^2 + 1) + \frac{\|v\|_2^2}{\|v\|_1^2 + 1} \leq c(\alpha^{-2} + c\|u(0)\|_{L^\infty}^2), \tag{5.89}
\]
where \( p > 1, q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). By the Gronwall inequality, we have
\[
\mathbb{E} \sup_{t \in [0,T]} \log(\|v(t)\|_1^2 + 1) \leq \mathbb{E} \log(\|v(0)\|_1^2 + 1) + c\mathbb{E} \sup_{t \in [0,T]} \alpha^{-2}\|v(t)\|_{L^q}^2 + c\mathbb{E} \|u(0)\|_{L^\infty}^{2q} \leq c \exp cT,
\]
where the constant \( c \) is independent of \( T \). Note that \( \log(\|u(t)\|_1^2 + 1) \leq \log(\|v(t)\|_1^2 + 1) + \log(\alpha^{-2}(t) + 1), \) the result follows.

Remark 3. If the dimension of stochastic Burgers equation (1.1) is two, then we can replace the condition \( u_0 \in H^1 \) in Proposition 5.7, Proposition 5.3, and Theorem 5.4 with \( u_0 \in H^1 \). That is we can obtain the logarithmic moments for the strong solutions, i.e.,
\[
\mathbb{E} \sup_{t \in [0,T]} \log(\|u(t)\|_1^2 + 1) \leq c \exp cT,
\]
provided \( u_0 \in H^1(\mathbb{T}^3) \) and \( \mathbb{E}\|u_0\|_{H^1}^{2+\sigma} < \infty \). On the other hand, if the stochastic equation is three dimensional, then we should assume \( u_0 \in H^\frac{5}{2} \) instead. However, it is difficult to obtain the logarithmic moments \( \mathbb{E} \sup_{t \in [0,T]} \log \left( 1 + \|u(t)\|_{H^\frac{5}{2}}^2 \right) \) due to the high non-linearity and the incompressibility of the 3D stochastic Burgers equation. Probably, the vector valued chain rule for fractional derivatives would be helpful for this problem, see the vector valued chain rule in Theorem A.6 of [31].

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