Diophantine Approximation on varieties III:
Approximation of non-algebraic points by algebraic points

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1 Introduction

Let $k/\mathbb{Q}$ be a number field with ring of integers $\mathcal{O}_k$, and $X$ a flat quasi projective scheme of finite type over $\text{Spec } \mathcal{O}_k$, equipped with an ample metrized line bundle $\mathcal{L}$. The base extension of $X$ to $k$ will be denoted by $X$. Let further $\sigma : k \to \mathbb{C}$ be some embedding, $X(\mathbb{C}_\sigma)$ the $\mathbb{C}_\sigma$-valued points of $X$ which usually will also be denoted by $X$, and $|\cdot|_{\mathbb{C}}$ any metric on $X(\mathbb{C}_\sigma)$ that induces the usual $\mathbb{C}$-topology.

1.1 Proposition In the above situation, for every $\beta \in X(\bar{k})$ with $\beta \neq \alpha$,

$$ \log |\alpha, \beta| \geq -c_1 \deg \beta c_2 h(\beta), $$

where $c_1$ is positive number depending only on the height of $\alpha$, and $c_2$ is a positive number depending only on the degree of $\beta$.

A proof that uses Liouville's Theorem, and the relation between distance and algebraic distance in Theorem 2.1.1, from [Ma1] is given in the appendix.

The Proposition says that the best approximation of a point $\alpha \in X(\bar{k})$ by another algebraic point $\beta$ is linear in the degree and height of $\beta$. If on the other hand $\theta \in X(\mathbb{C}_\sigma)$, is not algebraic, a possible approximation of $\theta$ by algebraic points is much better. How good the approximation is, depends mainly on the transcendence degree of the field generated by $\theta$. More specifically there is the following Theorem whose proof is the main objectue of this paper.

1.2 Theorem In the above situation, there exists a positive real number $b$, such that for every sufficiently big real number $a$, and any nonalgebraic point $\theta \in X(\mathbb{C}_\sigma)$ with Zariski closure denoted by $\mathcal{Y}$, there exists an infinite subset $M \subset \mathbb{N}$ such that for all $D \in M$ there is an algebraic point $\alpha_D \in \mathcal{Y}(\bar{k})$ fulfilling

$$ \deg(\alpha_D) \leq D^t, \quad h(\alpha_D) \leq a D^t, \quad \log |\alpha_D, \theta| \leq -\frac{1}{\left(\frac{h(\alpha_D)}{a} + 2 \deg X\right)^{\frac{1}{t}}} ba D^{t+1}, $$

where $t$ denotes the relative dimension of $\mathcal{Y}$ over $\text{Spec } \mathcal{O}_k$.

1.3 Corollary For $X = \mathbb{P}^t$, $D \in M$, and every effective cycle $\mathcal{Y}_D$ whose support contains the support of $\alpha_D$ we have

$$ D(X_D, \theta) \leq \log |X_D, \theta| \leq -ba D^{t+1} $$
Proof 1. Follows immediately from the trivial fact \(|X_D, \theta| \leq |\alpha_D, \theta|\) if supp(\(\alpha_D\)) \(\subset\) supp(\(X_D\)), Theorem 2.1.1, and changing \(b\) slightly.

Of course it would be desirable to have an analogue for the lower bound of the approximations that Proposition 1.1 supplies when \(t\) equals zero, i.e., when \(\theta\) is algebraic. Therefore, saying that the approximation given by Theorem 1.2 up to a constant is best possible. However, such a lower estimate is not possible in general due to the existence of transcendental points like Liouville numbers. The best that can be obtained is:

1.4 Conjecture There is a constant \(c\) depending only on \(t\), such that for almost all \(\theta \in \mathcal{X}(\mathbb{C}_\sigma)\) for sufficiently big \(D\) the inequality \(\log |\alpha_D, \theta| \leq -abD^{t+1}\) implies
\[
\deg \alpha_D \geq cD^t \quad \text{or} \quad h(\alpha_D) \geq caD^t.
\]

A proof of this conjecture will be the subject of [Ma3].

There are several applications of this Theorem to algebraic independence theory and other topics in transcendence theory, some of which are alluded to in the outlook at the end of the paper.

The proof of Theorem 1.2 has two steps. In the first step using the Theorem of Minkowski, estimates for the algebraic and arithmetic Hilbert functions, and the arithmetic Bezout Theorem from [Ma1], for each \(D \in \mathbb{N}\), cycles of codimension \(t\) are constructed that have small algebraic distance to \(\theta\). The main idea is to successively intersect properly intersecting cycles of small algebraic distance. If \(\mathcal{Y}\) is such an effective cycle with small algebraic distance, bounden height and degree, has codimension one, and is irreducible, a hypersurface properly intersecting \(\mathcal{Y}_s\) can be obtained if \(\mathcal{Y}\) fulfills certain regularity conditions. Then, the intersection of \(\mathcal{Y}\), and this hypersurface, by the metric Bézout Theorem has also small algebraic distance to \(\theta\).

If \(\mathcal{Y}\) does not fulfill the regularity conditions, one nonetheless can find a cycle \(\mathcal{X}\) that contains \(\mathcal{Y}\) and has good regularity, and work with this cycle. Again either there then is a cycle of codimension one in \(\mathcal{Y}_s\) that or \(\mathcal{X}\) good approximation properties but small degree and height with respect to a certain measure. Since the degree and height can not infinitely decrease, one finds the desired cycle of codimension \(t\). The second step uses the fact that the algebraic distance of a cycle \(X\) not containing \(\theta\) essentially equals the sum of logarithms of certain points on \(X\) to \(\theta\), and if \(X\) is 0-dimensional, this is just the sum of logarithms of distances of the points in \(X\) counted with multiplicity to \(\theta\). It is shown that for \(D\) in an infinite set and \(\mathcal{Y}_D\) a cycle of codimension \(t\) with good approximation properties with respect to \(\theta\), the sizes of these distances are very unequally distributed to the effect that a fixed portion of the algebraic distance actually comes from the logarithm of the distance of the \(\mathbb{C}\)-valued point of \(\mathcal{Y}_D\) closest to \(\theta\). A general version of the Liouville inequality plays an essential role in this step.
The starting point for the proof presented here was the proof of Theorem 1.2 in [RW] for \( t = 1 \). Like that one it depends on a certain metric Bézout Theorem that relates distances and algebraic distances of properly intersecting cycles in projective space with certain multiplicities that are defined by the metric position of the cycles with respect to \( \theta \). This metric Bézout Theorem was proved in [RW] for \( t = 1 \), and generalized to arbitrary dimension in [Ma1]. Some of the new problems that appear in the higher dimensional case can also be solved using these multiplicities.

The other significance difference from the one dimensional case, it the fact that in higher dimensions for each \( D \gg 0 \), one needs to construe several in particular higher codimensional cycles with good approximation that fullfill certain conditions of proper intersection. The key ingredient to attain this, is the concept of locally complete intersections in projective space as introduced and investigated in [CP] and applied to Diophantine Approximation in [Ph]. Only with them it is possible to make good lower bounds on the algebraic and arithmetic Hilbert functions in higher codimension and thus use the Theorem of Minkowski effectively.

Finally the argument, that the construed cycles with good approximation, and bounded height of dimension 1 defined over \( \mathbb{Z} \) for different \( D \)'s eventually differ is also borrowed from [RW]. In the higher dimensionl case however, one can not just use two one dimensional cycles to achieve the result, but must prove that for certain one dimensional cycles \( \alpha_D \), there also is one codimensional cycles, quasi defined over \( \mathbb{Z} \) in a certain sense, that does not contain \( \alpha_D \), and has also good approximation properties. This is the principal part of the proof of the second step mentioned above, and involves a rather complicated combinatorics.

### 2 The algebraic distance

Let \( \mathbb{P}^t_\mathbb{Z} \) be the projective space of dimension \( t \) over \( \text{Spec} \, \mathbb{Z} \), and effective cycles in \( \mathbb{P}^t(\mathbb{C}) \) of codimensions \( p \), and \( q \) respectively. For \( p + q \leq t + 1 \), and \( X, Y \) properly intersecting or in [Ma1] the so called algebraic distance

\[
D(X, Y) \in \mathbb{R},
\]

is defined is known in the literature also as the height pairing of \( X \) and \( Y \) at infinity. Further for \( p + q \geq t + 1 \), and \( X = \theta \) a point not contained in the support of \( Y \), there are several essentially equivalent definitions of the algebraic distance

\[
D(\theta, Y) \in \mathbb{R}.
\]

([Ma1], section 4)

If further \( |\cdot, \cdot| \) denotes the Fubini-Study distance on \( \mathbb{P}^t \) normalized in such a way that the maximal distance of two points is 1, there are the following Theorems for the algebraic distance.
2.1 Theorem I Let $\mathcal{X}, \mathcal{Y}$ be effective cycles intersecting properly, and $\theta$ a point in $\mathbb{P}^t(\mathbb{C}) \setminus (\text{supp}(X_C \cup Y_C))$.

1. There are effectively computable constants $c, c'$ only depending on $t$ and the codimension of $X$ such that

$$\deg(X) \log |\theta, X(\mathbb{C})| \leq D(\theta, X) + c \deg X \leq \log |\theta, X(\mathbb{C})| + c' \deg X,$$

2. If $f \in \Gamma(\mathbb{P}^t, O(D))_{\mathbb{Z}}$, and $\mathcal{X} = \text{div}(f)$ is an effective cycle of codimension one,

$$h(\mathcal{X}) \leq \log \|f_D\|_2 + D\sigma_1,$$

and

$$D(\theta, X) + h(\mathcal{X}) = \log \|(f|\theta)\| + D\sigma_{t-1},$$

where the $\sigma'_s$ are certain constants, and $\|(f|\theta)\|$ is taken to be the norm of the evaluation of $f \in \text{Sym}^D(E) = \Gamma(\mathbb{P}^t, O(D))$ at a vector of length one representing $\theta$.

3. Metric Bézout Theorem For $p + q \leq t + 1$, assume that $\mathcal{X}$, and $\mathcal{Y}$ have pure codimension $p$, and $q$ respectively. There exists an effectively computable positive constant $d$, only depending on $t$, and a map

$$f_{X,Y} : I \to \deg X \times \deg Y$$

from the unit interval $I$ to the set of natural numbers less or equal $\deg X$ times the set of natural numbers less or equal $\deg Y$ such that $f_{X,Y}(0) = (0, 0), f_{X,Y}(1) = (\deg X, \deg Y)$, and the maps $\text{pr}_1 \circ f_{X,Y} : I \to \deg X, \text{pr}_2 \circ f_{X,Y} : I \to \deg Y$ are monotonously increasing, and surjective, fulfilling: For every $T \in I$, and $(\nu, \kappa) = f_{X,Y}(T)$, the inequality

$$\nu \kappa \log |\theta, X + Y| + D(\theta, X.Y) + h(\mathcal{X}, \mathcal{Y}) \leq
\kappa D(\theta, X) + \nu D(\theta, Y) + \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + d \deg X \deg Y$$

holds.

4. In the situation of 3, if further $|\theta, X + Y| = |\theta, X|$, then

$$D(\theta, X.Y) + h(\mathcal{X}, \mathcal{Y}) \leq D(\theta, Y) + \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + d' \deg X \deg Y$$

with $d'$ a constant only depending on $t$.

Proof [Mat], Theorem I.

Part 4 has an easy corollary:
2.2 Corollary

\[ D(\theta, X.Y) + h(\mathcal{X}, \mathcal{Y}) \leq \max(D(\theta, X), D(\theta, Y)) + \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + d' \deg X \deg Y \]

Part three and four have a variant for arbitrary cycles over \( \mathbb{C} \): For \( \mathcal{X}, \mathcal{Y} \) defined over \( \mathbb{Z} \), by \([\text{Ma1}], \text{Scholie 4.3}\),

\[ D(X, Y) = h(\mathcal{X}, \mathcal{Y}) - \deg Y h(\mathcal{X}) - \deg X h(\mathcal{Y}) + c \deg X \deg Y, \]

with \( c \) a constant only depending on \( p, q, \) and \( t \). Replacing this into the formulas of the Theorem gives

2.3 Proposition Let \( Y, \) and \( Z \) be properly effective cycles over \( \mathbb{C} \), and \( \theta \in \mathbb{P}^t(\mathbb{C}) \setminus (\text{supp}Y \cup \text{supp}Z) \).

1. There exists an effectively computable constant \( \bar{d} \), only depending on \( t \), and a map

\[ f_{Y,Z} : I \to \deg X \times \deg Z, \]

with properties as in the Theorem, such that for all \( t \in I \), and Then, with \( \nu, \kappa = f_{Y,Z}(t) \),

\[ \nu \kappa \log |\theta, Y + Z| + D(\theta, Y.Z) + D(Y, Z) \leq \kappa D(\theta, Z) + \nu D(\theta, Y) + \bar{d} \deg Y \deg Z. \]

2. If \( |Z, \theta| \leq |Y, \theta| \), then

\[ D(\theta, Y.Z) + D(Y, Z) \leq D(\theta, Y) + \bar{d} \deg Y \deg Z. \]

3.

\[ D(\theta, Y.Z) + D(Y, Z) \leq \max(D(\theta, Z), D(\theta, Y)) + \bar{d} \deg Y \deg Z. \]

Let now \( \mathcal{Y} \) be defined over \( \mathbb{Z} \), \( f \in \Gamma(\mathbb{P}^t, O(D))_{\mathbb{Z}} \) with \( f|_Y \neq 0 \), define \( f_Y^\perp \) as the orthogonal projection of \( f \) modulo the \( D \)-homogeneous part of the vanishing ideal \( I_Y(D) \) of \( Y \), and \( \mathcal{X} := \text{div} f, Z = Z_{\mathbb{C}} := \text{div} f_Y^\perp \). Then,

\[ h(\mathcal{Y}, \text{div} f) = Dh(\mathcal{Y}) + \int_Y \log |f| \mu \]

\[ = Dh(\mathcal{Y}) + \int_Y \log |f_Y^\perp| \mu \]

\[ = Dh(\mathcal{Y}) + \deg Y \log \int_{\mathbb{P}^t} \log |f_Y^\perp| + D(Y, \text{div} f_Y^\perp) \]

\[ \leq Dh(\mathcal{Y}) + \deg Y \log |f_Y^\perp|_{L^2(\mathbb{P}^t)} + cD \deg Y, \]
by [Ma1], Proposition 6.3, with a constant $c$ only depending on $t$. Further
\[
D(Y, Z) = \int_Y \log |f_Y^\perp| - \deg Y \int_{\mathbb{P}^t} \log |f_Y^\perp| = h(\gamma. \text{div} f) - Dh(\gamma) - \deg Y \log |f|_{L^2(\mathbb{P}^t)}.
\]

By [Ma1],
\[
\int_{\mathbb{P}^t} \log |f_Y^\perp|^t \geq \log |f_Y^\perp|_{L^2(\mathbb{P}^t)} - c_1 D,
\]
and by the proof of [Ma2], Theorem 4.2,
\[
\log |f_Y^\perp|_{L^2(\mathbb{P}^t)} \geq - \frac{h(\gamma)}{\deg Y}D - c_2 D.
\]

with $c_1, c_2$ only depending on $t$. Hence, for every $\kappa$ with $0 \leq \kappa \leq \deg Y$,
\[
\kappa D(Z, \theta) = \kappa \left( \log |\langle f_Y^\perp | \theta \rangle| - \int_{\mathbb{P}^t} \log |f_Y^\perp|^t \right)
\leq \kappa \left( \log |\langle f_Y^\perp | \theta \rangle| - \log |f_Y^\perp|_{L^2(\mathbb{P}^t)} + c_1 D \right)
\leq \kappa \left( \log |\langle f_Y^\perp | \theta \rangle| + \frac{h(\gamma)}{\deg Y}D + c_2 D \right)
\leq \kappa \log |\langle f_Y^\perp | \theta \rangle| + h(\gamma)D + (c_1 + c_2)D \deg Y.
\]

Consequently, the previous Proposition implies

2.4 Proposition With the notations of Theorem 2.1 and $f, f_Y^\perp$ as above,

1. \[h(\gamma. \text{div} f) \leq Dh(\gamma) + \deg Y \log |f_Y^\perp| + \tilde{d} \deg X.\]

2. \[\nu \kappa \log |\theta, Y + Z| + D(\theta, Y.Z) + h(\gamma. \text{div} f) \leq \kappa \log |\langle f_Y^\perp | \theta \rangle| + \nu D(\theta, Y) + 2Dh(\gamma) + \deg Y \log |f_Y^\perp|_{L^2(\mathbb{P}^t)} + \tilde{d} \deg Y.\]

3. If $|Z, \theta| \leq |Y, \theta|$, then \[D(\theta, Y.Z) \leq D(\theta, Y) + 2Dh(\gamma) + \deg Y \log |f_Y^\perp|_{L^2(\mathbb{P}^t)} + \tilde{d} \deg Y \deg Z.\]

If $|Z, \theta| \geq |Y, \theta|$, then \[D(\theta, Y.Z) \leq \log |\langle f_Y^\perp | \theta \rangle| + 2Dh(\gamma) + \deg Y \log |f_Y^\perp|_{L^2(\mathbb{P}^t)} + \tilde{d} \deg Y \deg Z.\]
4. \[ D(\theta, Y, Z) \leq \max(D(\theta, Z), \log |\langle f_Y^- | \theta \rangle|) + 2Dh(\mathcal{Y}) + \deg Y \log |f_Y^-|_{L^2(\mathcal{P}')} + \bar{d} \deg Y \deg Z. \]

One of the Definitions of algebraic distance is as follows: \( X \in Z_{eff}^p(\mathbb{P}^t) \) be an effective of pure codimension one, and \( \mathbb{P}(W) \subset \mathbb{P}^t \) a \( p \)-dimensional projective subspace that intersects \( X \) properly. If \( X.\mathbb{P}(W) = \sum_{x \in \text{supp}(X.\mathbb{P}(W))} n_x x, \)

where \( n_x \) is the intersection multiplicity of \( \mathbb{P}(W) \) and \( X \) at \( x \), define \( \rho_{\mathbb{P}(W)} := \sum_{x \in \text{supp}(X.\mathbb{P}(W))} n_x \log |x, \theta| \), and the algebraic distance

\[ D_{pt} := \inf_{\mathbb{P}(W)} \rho_{\mathbb{P}(W)}. \]

With these notations, and the ones of Theorem 2.1, one further has

2.5 Proposition If \( \mathbb{P}(W) \) is a subspace of \( \mathbb{P}^t \) where \( \rho_{\mathbb{P}(W)} \) attains its infinum, and \( M \) is the set of \( y \in \text{supp}\mathbb{P}(W) \cap Y \) such that \( |Z, \theta| \leq |y, \theta|(|X, \theta| \leq |y, \theta|) \) then

\[ D(Y.Z, \theta) + D(Y, Z) \leq \sum_{y \in M} n_y \log |y, \theta| + \deg X \deg Y. \]

\[ D(X.Y, \theta) + h(X, \mathcal{Y}) \leq \sum_{y \in M} n_y \log |y, \theta| + \deg Y h(X) + \deg X h(\mathcal{Y}) + d \deg X \deg Y. \]

\[ D(\text{div}f.Y, \theta) \leq \sum_{y \in M} n_y \log |y, \theta| + Dh(\mathcal{Y}) + \deg Y \log |f_Y^-|_{L^2(\mathbb{P}')} + d \deg X \deg Y. \]

Proof Reconsider the proof of Theorem 2.13 in [Ma1]. Let \( \mathbb{P}(W) \) be a subspace that intersects \( X \) properly, and contains a point \( z_0 \) with \( |z_0, \theta| = |Z, \theta| \). Then, as in the proof of part 1. in the proof of 2.13 in [Ma1],

\[ D_0((\theta, \theta), X \# Z) \leq \sum_{z \in Z \cap \mathbb{P}(W), y \in Y \cap \mathbb{P}(W)} n_z n_y \log |x \# z, (\theta, \theta)| + \epsilon_3 \deg Z \deg Y \leq \]

\[ \sum_{y \in M} n_{z_0} n_y \log |z_0 \# y, (\theta, \theta)| + \epsilon_3 \deg Z \deg Y, \]

which by [Ma1], Lemma 6.4 is less or equal

\[ \sum_{y \in M} n_y \log |y, \theta| + \epsilon_3 \deg Z \deg Y. \]
As \( D(X, Z) \leq d \deg X \deg Z \) for some \( d \), the first inequality follows. From this the second inequality follows in the same way as par two of the proof of Theorem 2.1.3 in \[Ma1\], again using
\[
D(\mathbb{P}(\Delta), X \# Y) = D(X, Y) + \bar{d} \deg X \deg Y = h(X, Y) - \deg Y h(X) - \deg X h(Y) - t \log 2 \deg X \deg Y + \bar{d} \deg X \deg Y.
\]
The second inequality follows in the same way as Proposition 2.4 above.

2.6 Definition Let \( X \) be a quasiprojective irreducible algebraic variety over a number field \( k \), and \( \sigma : k \rightarrow \mathbb{C} \) some imbedding. A point \( \theta \in X_\sigma(k) \) is called generic if \( \theta \) is contained in no proper algebraic subvariety \( Y \subset X \) defined over \( k \).

Remark: If \( Y, Z \), cycles in \( \mathbb{P}^t(\mathbb{C}) \) are defined over \( \mathbb{Q} \), and \( \theta \in \mathbb{P}^t(\mathbb{C}) \) is a generic point, then automatically \( \theta \notin \text{supp}Y \cup \text{supp}Z \), and the Propositions of this section are applicable. However, Propositions 2.4 and 2.5 will be have to be applied if possibly \( \langle f_\perp Y | \theta \rangle = 0 \). But it is easy to see that Theorem 2.1 and the other Propositions still hold if say \( \theta \in \text{supp} X \setminus \text{supp} Y \) in Theorem 2.1. If \( \nu \kappa \neq 0 \), then both sides in Theorem 2.1.3 are \(-\infty\), and thus the statement trivially holds. If \( \nu \kappa = 0 \), both sides of the inequality are finite. To see that the inequality holds let \( (\theta_n)_{n \in \mathbb{N}} \) be a series of points in \( \mathbb{P}(\mathbb{C}) \) such that \( \theta_n \notin \text{supp} X \cup \text{supp} Y \), \( |\theta_n, X| < |\theta_n, Y| \) for all \( n \), and \( \lim_{n \rightarrow \infty} \theta_n = \theta \). Then, the inequality holds for each \( \theta_n \) instead of \( \theta \) and by continuity also for \( \theta \).

3 Hilbert functions

A subscheme \( \mathcal{X} \) in \( \mathbb{P}^t \) will be called a subvariety if each irreducible component has at least one \( \mathbb{Q} \)-valued point. A global section \( f \in \Gamma(\mathbb{P}^t_\mathcal{O}, \mathcal{O}(D)) \) is called primitive if it cannot be divided by any \( a \in \mathcal{O} \) which is not a unit. Global sections will always be assumed to be primitive. The proper intersection of a subvariety with the divisor of a primitive global section is again a subvariety.

A subvariety \( \mathcal{X} \subset \mathbb{P}^t \) is called a locally complete intersection if \( \text{codim} \mathcal{X} = r \), and there exist global sections \( f_1, \ldots, f_r \), and a Zariski open subset \( U \subset \mathbb{P}^t \) such that
\[
\mathcal{X} = V(f_1) \cap \cdots \cap V(f_r) \cap U,
\]
where \( V(f_i) \) denotes the vanishing set of \( f_i \).

Let \( \mathcal{Y} \subset \mathcal{X} \subset \mathbb{P}^t \) be algebraic subvarieties with \( \mathcal{Y} \) irreducible. \( \mathcal{X} \) is called a complete intersection at \( \mathcal{Y} \), if \( \text{codim} \mathcal{X} = r \), and there are global sections \( f_1, \ldots, f_r \) such that \( \mathcal{X} \) consists of the irreducible components of \( V(f_1) \cap \cdots \cap V(f_r) \) that contain \( \mathcal{Y} \). The same notions can be defined for \( k \)-rational subvarieties of \( \mathbb{P}^t \). There is the easy Lemma:
3.1 Lemma

1. If $X \subset \mathbb{P}^t$ is a locally complete intersection, and $Y \subset X$ is a subvariety of codimension zero, then $Y$ is a locally complete intersection.

2. For any irreducible variety $Y$; if $X$ is a locally complete intersection at $Y$, then $X$ is a locally complete intersection.

3. If $X$ is a locally complete intersection at $Y$, and $Z$ a subvariety that contains $Y$ and intersects $X$ properly, the union $W$ of the components of $X \cap Z$ that contain $Y$ is a locally complete intersection at $Y$.

3.2 Proposition Let $X$ be a subvariety of pure dimension $p$ in $\mathbb{P}^t_k$, and denote by

$$H_X(D) = \dim H^0(X, O(D))$$

the algebraic Hilbert function.

1. 

$$H_X(D) \leq \deg X \left( \frac{D + p}{p} \right)$$

2. If $X$ is locally complete intersection of $p$ hyperplanes of degree $D_1, \ldots, D_p$, then with $\bar{D} := D_1 + \cdots + D_p - (t - p)$, and $D \geq \bar{D}$,

$$H_X(D) \geq \deg X \left( \frac{D - \bar{D} + p}{p} \right).$$

Proof 1. [Ch], Theorem 1.
2. [CP], Corollaire 3.

An arithmetic bundle $\bar{M}$, for the purpose of this paper will just be a free $\mathbb{Z}$ module $M$ with a hermitian metric on $M_\mathbb{C} = M \otimes \mathbb{Z} \mathbb{C}$. The arithmetic degree $\hat{\deg}(\bar{M})$ is then minus the logarithm of the volume of the lattice $M$ in $M_\mathbb{C}$.

Let again $\mathbb{P}_{\mathbb{Z}}^t = \mathbb{P}(\mathbb{Z}^{t+1})$ be projective space of dimension $t$, and

$$E_D := \Gamma(\mathbb{P}^t, O(D)).$$

As $E_D = \text{Sym}^D E_1$, which in turn equals the space of homogeneous polynomials of degree $D$ in $t + 1$ variables, this lattice canonically carries the following metrics:

1. The subspace metric $\text{Sym}^D E_1 \subset E_1^{\otimes D}$.

2. The quotient metric $E_1^{\otimes D} \to \text{Sym}^D E_1$. 
3. The $L^2$-metric
\[ ||f||_{L^2(\mathbb{P}^t)}^2 = \int_{\mathbb{P}^t(\mathbb{C})} |f|^2 \mu^t, \]
where $\mu$ is the Fubini-Study metric on $\mathbb{P}^t$.

Let $X$ be an effective cycle of pure dimension $s$ in $\mathbb{P}^t_{\mathbb{C}}$. Then on
\[ I_X(D) := \{ f \in H^0(\mathbb{P}^t, O(D)) | f|_X = 0 \}, \]
there are the restrictions of the norms $| \cdot |_{\text{sym}}$ and $| \cdot |_{L^2(\mathbb{P}^t)}$, and on
\[ F_X(D) = H^0(X, O(D)), \]
there is the quotient norm $| \cdot |_{QN}$ induced by the canonical quotient map
\[ q_D : E_D \to F_D(X), \]
the $L^2(\mathbb{P}^t)$-norm
\[ | \cdot |_{L^2(\mathbb{P}^t)} : F_D(X) \to \mathbb{R}, \quad f \mapsto \inf_{q_D(f) = f} \int_{\mathbb{P}^t} |f| \mu^t = \int_{\mathbb{P}^t} |g| \mu^t, \]
with $g \in I_X(D)^\perp$, and $q_D(g) = \bar{f}$. and the $L^2(X)$-norm
\[ | \cdot |_{L^2(X)} : F_D(X) \to \mathbb{R}, \quad f \mapsto \int_X |f|^p. \]

By convention $F_X(D)$ always denotes the $\mathbb{Q}$-or $\mathbb{C}$-vector space of global sections, and $F_X(D)$ the corresponding lattice in $F_X(D)$.

3.3 Theorem Let $X$ be an subvariety of dimension $s + 1$ of $\mathbb{P}^t$, and denote by
\[ \hat{H}_X(D) := \deg(F_X(D), | \cdot |_{L^2(\mathbb{P}^t)}), \quad \hat{H}_X(D) := \deg(F_X(D), | \cdot |_{L^2(X)}) \]
the arithmetic Hilbert functions.

1. \[ \hat{H}_X(D) \leq \deg X \left( Dh(X) + \frac{1}{2}(\log \deg X + 2s \log D) \right) \left( \frac{D + s}{D} \right). \]

2. There is a positive constant $c_3$ only depending on $t$, such that
\[ \hat{H}_X(D) \leq \left( Dh(X) + \deg XDc_3 + \deg X \left( \frac{1}{2} \log \deg X + s \log D \right) \right) \left( \frac{D + s}{D} \right). \]

Hence for $c_5 > c_3$, $\deg X$ at most a fixed polynomial in $D$, and $D$ sufficiently large,
\[ \hat{H}_X(D) \leq D \left( \frac{D + s}{D} \right) (h(X) + c_5 \deg X). \]
3. There is a constant $c_4$ only depending on $t$

$$\hat{H}_X(D) \geq \deg(E_D \cap I_D(X)) + \frac{2\sigma}{(t+1)!}D^{t+1} + 2c_4D^t \log D.$$ \hfill (3.1)

For $D$ sufficiently large, this is greater or equal

$$-D\left(\frac{D+s}{D}\right)(h(\mathcal{X}) + c_5 \deg X).$$ \hfill (3.2)

4. There are constants $c_1, c_2 > 0$ and $m \in \mathbb{N}$ only depending on $t$ such that if $\mathcal{X}$ is an irreducible subvariety which is a locally complete intersection of $t-s$ hypersurfaces of degree $D_1, \ldots, D_{t-s}$, then with $\bar{D} := D_1 + \cdots + D_{t-s} - (t-s)$, and $D \geq m \bar{D}$,

$$\hat{H}_X(D) \geq (c_1 h(\mathcal{X}) - c_2 \deg X) D^{s+1}.$$ \hfill (4.1)

**Proof** [Ma2], Theorem 4.1.

### 4 Approximations in Projective Space

#### 4.1 Approximation cycles

An approximation cycle for a point $\theta \in \mathbb{P}_\mathbb{C}^t$ is an effective cycle in $\mathbb{P}_\mathbb{C}^t$ that has small algebraic distance to $\theta$ compared with its degree and height. The precise definition, which with respect to the ratio required between algebraic distance, height, and degree, is to a certain, but inessential degree arbitrary, will be given later. First the general method to construct cycles with good approximation with respect to $\theta$ will be exposed.

##### 4.1.1 Fundamental approximation techniques

**4.1 Theorem (Minkowski)** Let $\bar{M}$ be an arithmetic bundle over $\text{Spec} \mathbb{Z}$, and $K \subset M_{\mathbb{Z}} \mathbb{R}$ any closed convex subset that is symmetric with respect to the origin, and fullfills

$$\log \text{vol}(K) \geq -\deg(\bar{M}) + \text{rk}M \log 2.$$ \hfill (4.1)

Then $K \cap M$ contains a nonzero vector. In particular for $c > 0$, $\text{rk}M \geq 2$, choosing a line $L \subset M_{\mathbb{R}}$ through the origin, and taking $K$ as a rectangular parallelepiped with the edge parallel to $L$ of logarithmic length $c \deg(\bar{M})$, and the edges orthogonal to $L$ with logarithmic length $-\frac{4c}{\text{rk}M} \deg(\bar{M}) + 2 \log 2$ centered at the origin, one sees that there is a non zero lattice point $v \in M$ of logarithmic length

$$\log |v| \leq -\frac{4c}{\text{rk}M} \deg(\bar{M}) + 2 \log \text{rk}M,$$

such that its projection to $L$ has logarithmic length at most $c \deg(\bar{M})$. 

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4.2 Lemma Let $n \in \mathbb{N}$, and $v_1, \ldots, v_n \in \mathbb{C}^n$ be vectors such that for every $i = 1, \ldots, n$ the $i$th component of $v_i$ is nonzero. Then, there are numbers $m_1, \ldots, m_n \in \mathbb{N}$ with $m_i \leq n$ such that each component of

$$v = m_1 v_1 + \cdots + m_n v_n$$

is nonzero.

Proof [Ma2], Lemma 2.6.

For any positive real number $a$, and an effective cycle $\mathcal{X}$ in $\mathbb{P}^t$ define the $a$-size of $\mathcal{X}$ as

$$t_a(\mathcal{X}) := a \deg X + h(\mathcal{X}).$$

If $\mathcal{Y}$ is an irreducible subvariety of $\mathbb{P}^t$, identify the arithmetic bundle $F_\mathcal{Y}(D) = \Gamma(\mathbb{P}^t, O(D))/\Gamma(\mathcal{Y}(D))$, with the arithmetic bundle consisting of lattice in $I_\mathcal{Y}(D)$ consisting of all orthogonal projections of lattice points in $\Gamma(\mathbb{P}^t, O(D))$ or $I_\mathcal{Y}(D)$, and the induced metric on $I_\mathcal{Y}(D)$. If with this convention

$$q_\mathcal{Y} : \Gamma(\mathbb{P}^t, O(D)) \rightarrow F_\mathcal{Y}(D)$$

denotes the quotient map, and $\mathcal{Z}$ is an irreducible subvariety of $\mathcal{Y}$, then, for $f \in \Gamma(\mathbb{P}^t, O(D))$,

$$|q_\mathcal{Z}(f)|_{L^2(\mathbb{P}^t)} \leq |q_\mathcal{Y}(f)|_{L^2(\mathbb{P}^t)} \leq |f|_{L^2(\mathbb{P}^t)}.$$

4.3 Lemma Let $\mathcal{X} \subset \mathbb{P}^t$ be a locally complete intersection of global sections $f_1, \ldots, f_s$ of degree $D_1 \leq \cdots \leq D_s$, $\theta \in \mathbb{P}^t(\mathbb{C})$ a generic point, and define $\tilde{D} := D_1 + \cdots + D_s - s$. Then, with $m, c_1, c_2$ the constants from Theorem 3.3.4, $D \geq \tilde{D}$, $n \geq m$, $b \leq \min(c_1, \frac{1}{2^{t-s} (t-s)!})$, and $a \geq c_2 2^{t+1}(t-s)!$, there is an $f \in \Gamma(\mathbb{P}^t, O(nD))$, and an $\tilde{f} \in \Gamma(\mathbb{P}^t, O(D))$ such that for every irreducible component $X_i$ of $\mathcal{X}$ the quotients $q_{X_i}(f), q_{X_i}(\tilde{f})$ are nonzero, and for every subvariety $Z \subset \mathbb{P}^t$ contained in each irreducible component of $\mathcal{X}$, the quotients $q_Z(f)$ and $q_Z(\tilde{f})$ coincide and further

$$\log |\tilde{f}|_Z = \log |f|_Z \leq 6anD, \quad \text{and} \quad \log |(\tilde{f}|_\theta) - bt_a(\mathcal{X}_{\min})(nD)^{t+1-s},$$

where $\mathcal{X}_{\min}$ denotes an irreducible component of $\mathcal{X}$ with minimal $a$-size.

Proof Assume first $\mathcal{X} = \mathcal{Y}$ is irreducible. By Proposition 3.2,

$$(nD)^t \geq \deg Y(nD)^{t-s} \geq \deg Y t - s \geq H_Y(nD) = \text{rk} F_Y(nD) \geq \deg Y t - s \geq \frac{\deg Y}{2^{t-s} (t-s)!} (nD)^{t-s},$$

(1)
since \( m \geq 2 \) and by Proposition 3.3.4,

\[
- \deg(\tilde{F}_Y(nD)) = \tilde{H}_Y(nD) \leq (-c_1 h(Y) + c_2 \deg Y)(nD)^{t+1-s}.
\] (2)

Let next \( L_\theta \subset F_Y(nD)_C \) be the one dimensional subspace orthogonal to the kernel \( \ker_\theta \) of the evaluation map from \( F_Y(nD)_C \) to the stalk of \( O(nD) \) at \( \theta \), and \( K \) the rectangular parallelepiped with logarithmic length of edge parallel to \( b \),

\[
- bh(Y)(nD)^{t+1-s} - ba \deg Y(nD)^{t+1-s} = -bt_a(Y)(nD)^{t+1-s},
\]

and logarithmic length of edges parallel to \( \ker_\theta \) equal to \( 4a nD \). By the choice of \( a \), and \( b \), and the estimates on the algebraic and arithmetic Hilbert functions above, we have

\[
\log \text{vol}(K) = 4anD(rkF_Y(nD) - 1) - bh(Y)(nD)^{t+1-s} - ba \deg Y(nD)^{t+1-s}
\]

\[
\geq \frac{4a}{2^{t-s+1}(t-s)!} \deg Y(nD)^{t+1-s} - bh(Y)(nD)^{t+1-s}
\]

\[
\geq \left( \frac{4a}{2^{t+1-s}(t-s)!} - ba \right) \deg Y - bh(Y) (nD)^{t+1-s}
\]

\[
\geq (c_2 \deg Y - c_1 h(Y))(nD)^{t+1-s} + \deg Y(nD)^{t-s} \log 2
\]

\[
\geq -\deg(\tilde{F}_Y(nD)) + \text{rk} F_Y(nD) \log 2.
\]

Hence, by the Theorem of Minkowski, \( K \) contains a non zero lattice point \( \tilde{f} \), that is

\[
\log |\tilde{f}| \leq 4anD + \frac{1}{2} \log \text{rk} F_Y(nD) \leq 4anD + \frac{1}{2} \log(nD)^{t} \leq 5anD
\]

for \( D \) sufficiently big, and

\[
\log |\langle \tilde{f}, \theta \rangle| \leq -bh(Y)(nD)^{t+1-s} - ba \deg Y(nD)^{t+1-s} = -bt_a(Y)(nD)^{t+1-s}.
\]

Choose \( f \in \Gamma(\mathbb{P}^t, O(nD))_2 \) as a representative of \( \tilde{f} \). Then for \( \mathcal{Z} \subset \mathcal{Y} \), clearly \( q_Z(f) = q_Z(\tilde{f}) \), and

\[
\log |f_Z^2| \leq |f_Y^2| = |\tilde{f}| \leq 5anD,
\]

proving the Lemma in case \( \mathcal{X} \) is irreducible.

Assume now \( \mathcal{X} \) is not irreducible, and let \( \mathcal{X} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_l \) be the decomposition of \( \mathcal{X} \) into irreducible components, and \( \mathcal{Z} \) a variety contained in each \( \mathcal{X}_i, i = 1, \ldots, l \). We have \( l \leq \deg Y \leq (nD)^s \). By the previous argument for each \( i \) there is a nonzero \( \tilde{f}_i \in F_{\mathcal{X}_i}(nD) \), and an \( f_i \in \Gamma(\mathbb{P}^t, O(D))_2 \) such that \( q_{\mathcal{X}_i}(f_i) = \tilde{f}_i \),

\[
\log |(f_i)_Z^2| \leq \log |(f_i)_{\mathcal{X}_i}|^1 \leq 5anD,
\]
and
\[ \log |\langle f_i|\theta \rangle| \leq -bt_a(X_i)(nD)^{t+1-s} \leq -bt_a(X_{\min})(nD)^{t+1-s}. \]

By the previous Lemma there are natural numbers \( k_i, i = 1, \ldots, l \) with \( \log k_i \leq \log l \leq s \log D \) such that
\[ \bar{f} = \sum_{i=1}^{l} k_i \bar{f}_i \]
is nonzero on each \( X_i \), and the same holds for
\[ f = \sum_{i=1}^{l} k_i f_i. \]

Because of \( q_{X_i}(f_i) = \bar{f}_i \), and \( Z \subset X_i \) for every \( i = 1, \ldots, l \), we have \( q_Z(f) = q_Z(\bar{f}) \), hence
\[ \log |\bar{f}_Z^\perp| = \log |\bar{f}_Z^\perp| \leq \log |\bar{f}| \leq \max_{i=1,\ldots,l} \log |\bar{f}_i| + s \log D \leq 5anD + s \log D \leq 6anD, \]
for \( D \) sufficiently large, and
\[ \log |\langle \bar{f}|\theta \rangle| \leq -bt_a(X_{\min})(nD)^{t+1-s} + s \log D, \]
proving the Lemma.

**Remark:** It is easily seen that the Lemma holds with \( \bar{b} = \frac{m}{n}b \) instead of \( b \) if \( n/m \) is still bigger than \( m \). This fact will be assumed when the Lemma is applied.

A strategy for constructing effective cycles of arbitrary codimension of bounded height and degree with small algebraic distance to a given generic point \( \theta \), that is approximation cycles, could now be the following. Use the previous Lemma to find a vector \( f_1 \in \Gamma(\mathbb{P}^t, O(D)) \) of bounded length such that \( |\langle f_1|\theta \rangle| \) is small, define \( X_1 := \text{div} f \), and use Theorem 2.1.2 to derive that \( X_1 \) has small algebraic distance with respect to \( \theta \). It is then easy to prove that some irreducible component \( Y_1 \) of \( X_1 \) is also an approximation cycle. Then, there are two possibilities to go on.

The first one uses the previous Lemma to find an \( f_2 \) of bounded length and degree that is nonzero on \( Y_1 \) and has small \( |\langle f_2|\theta \rangle| \), and the metric Bézout Theorem to prove that \( X_1 := Y_1, \text{div} f \) has small algebraic distance with respect to \( \theta \), and again some irreducible cycle \( Y_2 \) of \( \text{div} f, Y_1 \) will have good approximation with respect to \( \theta \). This possibility is chosen in [Ph], only with the weaker estimate in [3,3,3] for the arithmetic Hilbert function thereby supplying a weaker approximation than possible with the previous Lemma which rests on the estimate [3,3,4]. The problem with this approach is that there is no guarantee that the successive intersections \( Y_3, Y_4, \ldots \) (i.e. from codimension 3 onwards) are locally complete interesections of bounded degree, and hence the previous Lemma is not applicable.
The second possibility is to use the Lemma for $\mathcal{X}_1$ to obtain an $f_2$ intersecting each irreducible component of $\mathcal{X}_1$ properly and having small $|\langle f_2 | \theta \rangle|$; one does then not leave the realm of locally complete intersections. The problem with this approach is that if the irreducible component with minimal $a$-size $\mathcal{X}_{\text{min}}$ of $\mathcal{X}_1$ has very small $a$-size, the estimate on $\log |\langle f_2 | \theta \rangle|$ one obtains is not very good. If $\mathcal{X}_{\text{min}}$ itself has small algebraic distance to $\theta$, this doesn’t matter, because the metric Bézout Theorem will still give a good estimate, but if $\mathcal{X}_{\text{min}}$ has big algebraic distance one is in a trap. To get out, one has to prove that one does not come along a $\mathcal{X}_{\text{min}}$ with big algebraic distance if one applies a somewhat refined procedure, which involves not only constructing approximation cycles of higher codimension from ones of lower codimension in the way just sketched but also, if it should not be possible to construct an approximation cycle of higher codimension, to obtain one which has lower codimension but better approximation properties than any of the ones so far constructed. To prove that this road finally leads to cycles of codimension $t$ involves rather complex combinatorics which will be presented in the next subsection, but first we prove how to construct the approximation cycle of lower codimension with better approximation.

4.4 Lemma Let $r < s < t$ be natural numbers, $\mathcal{Y}, \mathcal{X} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_l$ varieties in $\mathbb{P}^t$ of codimensions $s, r$, with $\mathcal{Y}$ and every component $\mathcal{X}_i, i = 1, \ldots, l$ of $\mathcal{X}$ containing $\mathcal{Y}$. Let further $\mathcal{X}_{\text{min}}$ be the irreducible component of $\mathcal{X}$ with minimal $a$-size, $n$ a natural number with $n << D$, and $b_s, b_{s+1}, b_t$ positive real numbers with $b_{s+1} \leq b_s/c$. Finally $f \in \Gamma(\mathbb{P}^t, O(nD))$, $\bar{f} \in \Gamma(\mathbb{P}^t, O(nD))$ such that $\langle \bar{f}, f \rangle = \bar{f} f \neq 0$, and $\bar{f} = \sum_{i=1}^l \bar{f}_i$, $f = \sum_{i=1}^l f_i$ such that $\bar{f}$, and $f$ are nonzero on each irreducible component of $\mathcal{X}$, and $\bar{f}_i, f_i$ are nonzero on $\mathcal{X}_i$. If these data fulfill the inequalities

$$\log |\bar{f}_i| = \log |\bar{f}_i| \leq 6anD, \quad \log |\langle f_i \rangle_{\mathcal{X}_i}| \leq 6anD, \quad \log |\langle \bar{f}_i | \theta \rangle| \leq -b_s t_a(\mathcal{X}_i) D^{t+1-r},$$

$$D(\mathcal{Y}, \theta) \leq -b_s t_a(\mathcal{Y}) D^{t+1-s}, \quad b_s t_a(\mathcal{Y}) \geq 2b_t t_a(\mathcal{X}_{\text{min}}) D^{s-r},$$

then $t_a(\text{divf.} \mathcal{Y}) \leq 7nt_a(\mathcal{Y})$, and either $D(\text{divf.} \mathcal{Y}, \theta) \leq -b_{s+1} t_a(\text{divf.} \mathcal{Y}) D^{t-s}$, or

$$D(\mathcal{X}_{\text{min}}, \theta) \leq -b_t t_a(\mathcal{X}_{\text{min}}) D^{t+1-r}.$$

Proof The claim on $t_a(\text{divf.} \mathcal{Y})$ follows from the algebraic Bézout Theorem, and Proposition 2.4. For the other claim, make the case distinction

Case 1: $\deg Y \geq \frac{b_s t_a(\mathcal{Y})}{2b_t t_a(\mathcal{X}_{\text{min}}) D^{s-r}}$.

Choose $t$ in Proposition 2.4.2 such that $\mu = \frac{b_s t_a(\mathcal{Y})}{2b_t t_a(\mathcal{X}_{\text{min}}) D^{s-r}} \leq \deg Y$. Then, by the assumptions on $D(\mathcal{Y}, \theta)$, and $|\langle \bar{f}_i | \theta \rangle|$, and the fact $h(\text{divf.} \mathcal{Y}) \geq 0$,

$$\mu \nu \log |\bar{f}_i| + D(\text{divf.} \mathcal{Y}, \theta) \leq \mu \log |\langle \bar{f} | \theta \rangle| + \nu D(\mathcal{Y}, \theta) + 6nD h(\mathcal{Y}) + \deg Y 2anD + dnD \deg Y \leq$$
- \mu \tilde{b}_t a(X_{\min}) D^{t+1-r} - \nu b_s t_a(Y) D^{t+1-s} + 7nt_a(Y) D = \\
\frac{1}{2} b_s t_a(Y) D^{t+1-s} - \nu b_s t_a(Y) D^{t+1-s} + 7nt_a(Y) D = \\
- \frac{1}{2} b_s t_a(Y) D^{t+1-r} - \mu \nu 2\tilde{b}_r t_a(X_{\min}) D^{t+1-r} + 7nt_a(Y) D.

Since \( s \leq t - 1 \), and \( n << D \), either

\[ D(\text{div} f, Y, \theta) \leq -\frac{1}{4} b_s t_a(Y) D^{t+1-s} - \frac{\nu}{2} b_s t_a(Y) D^{t+1-s} \leq -\frac{1}{4} b_s t_a(Y) D^{t+1-s} \leq -b_{s+1} t_a(\text{div} f, Y) D^{t-s}, \]

and the first possibility holds, or

\[ \log |Y, \theta| \leq -\tilde{b}_r t_a(X_{\min}) D^{t+1-r} - \frac{1}{4 \mu \nu} b_s t_a(Y) D^{t+1-s} \leq -\tilde{b}_r t_a(X_{\min}) D^{t+1-r}, \]

which by Theorem 2.1.1, implies, because \( Y \) is contained in \( X_{\min} \)

\[ D(X_i, \theta) \leq \log |X_{\min}, \theta| + c_1 \deg X_{\min} \leq \log |Y, \theta| + c_1 \deg X_{\min} \leq -\tilde{b}_r t_a(X_{\min}) D^{t+1-r} + O(t_a(X_{\min})). \]

and hence \( \mathcal{X} \) the second possibility holds.

**Case 2:** \( \deg Y \leq \frac{b_s t_a(Y)}{2\tilde{b}_r t_a(X_{\min}) D^{s+r}} \). In this case, by Theorem 2.1.1, and the assumption on \( D(Y, \theta) \),

\[ \log |Y, \theta| \leq \frac{1}{\deg Y} D(Y, \theta) \leq -\frac{b_s}{\deg Y} t_a(Y) D^{t+1-r} \leq -2\tilde{b}_r t_a(X_{\min}) D^{t+1-r}, \]

and the second possibility follows in the same way as above.

### 4.1.2 The combinatorics

Let \( c_1, c_2, m \) be the constants from Theorem 3.3.4, \( d \) the constant from 2.4.1, fix a real number \( a >> 0 \) and a number \( N \in \mathbb{N} \), and define the constants

\[ \bar{n}_1 := 1, \quad \bar{n}_s := Nm(2(1 + m))^{s-2}, \quad 2 \leq s \leq t \]

\[ n_1 := \bar{n}_1, \quad n_{s+1} := \bar{n}_{s+1} n_s, \quad 1 \leq s \leq t - 1, \]

\[ a_1 := a, \quad a_{s+1} := 6a \bar{n}_{s+1} n_s + a_s \bar{n}_{s+1} + dn_s \bar{n}_{s+1}, \quad 1 \leq s \leq t - 1, \]

\[ m_s := \bar{n}_{s+1} + 8\bar{n}_s + 1, \quad 1 \leq s \leq t, \]

\[ m := \prod_{i=s}^t m_i, \quad 1 \leq s \leq t, \]

\[ \bar{b}_1 := \frac{N}{(n_1)}, \quad \bar{b}_s := N \min \left( \frac{1}{2^{t+1}}, c_1 \right), \quad 2 \leq s \leq t, \]

\[ b_1 := \bar{b}_1, \quad b_s := \min \left( \frac{b_{s+1}}{16\bar{n}_2}, \min_{r=1,...,s-1} \bar{b}_r \frac{m_{s+1} a_r}{m_r a_r} \right), \quad 2 \leq s \leq t. \]
Observe that all constants \( n_i, \bar{n}_i, \bar{m}_i, m_i b_i, \bar{b}_i \), and all quotients \( a/a_i, a_i/a, i = 1, \ldots, t \) are bounded by a constant only depending on \( t \) and \( N \).
These constants are used to make a number of Definitions that also illustrate their role. For the purpose this paper, only the case \( N = 1 \) is important, but for later applications the general case will be needed. For this reason it may be advisable on first reading to restrict to the case \( N = 1 \).

4.5 Definition and Lemma A chain of irreducible subvarieties

\[ \mathbb{P}^t = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \cdots \supset \mathcal{Y}_s \]

is called a successive \( D \)-intersection (relative to \( N \)) if for each \( i = 1, \ldots, s \) the corresponding variety \( \mathcal{Y}_i \) is an irreducible component of the proper intersection of \( \mathcal{Y}_{i-1} \) with a global section \( f \in \Gamma(\mathbb{P}^t, O(\bar{n}_i D)) \), and has thereby codimension one. For a successive intersection the inequality \( \deg Y_i \leq n_i D^i \) holds. The chain is called a successive \((D, a)\)-intersection if additionally \( h(\mathcal{Y}_s) \leq a_s D^s \).

Proof. Clearly, \( \deg \mathcal{Y}_0 = \deg \mathbb{P}^t = 1 = D^0 \). Further if \( \mathcal{Y}_{i+1} \) is an irreducible component of \( \text{div} f . \mathcal{Y}_i \), with \( f \in \Gamma(\mathbb{P}^t, O(D)) \), then

\[ \deg Y_{i+1} \leq \deg f \deg Y_i \leq \bar{n}_{i+1} D_n D^i = n_{i+1} D^{i+1}. \]

4.6 Definition and Lemma A subvariety \( \mathcal{X} \subset \mathbb{P}^t \) of constant codimension \( s \) is called a locally complete \( D \)-intersection (relative to \( N \)), if \( \mathcal{X} \) is a locally complete intersection of \( f_1, \ldots, f_s \) with \( \deg f_i \leq \bar{n}_i D \). \( \mathcal{X} \) is called a locally complete \( D \)-intersection at \( \mathcal{Y} \) if \( \mathcal{X} \) is a locally complete intersection of \( f_1, \ldots, f_s \) at \( \mathcal{Y} \), with \( \deg f_i \leq \bar{n}_i D \). Further \( \mathcal{X} \) is called a locally complete \((D, a)\)-intersection (at \( \mathcal{Y} \)) if additionally \( h(\mathcal{X}) \leq a_s D^s \). A locally complete \((D, a)\)-intersection \( \mathcal{X} \) (at \( \mathcal{Y} \)) fulfills \( \deg \mathcal{X} \leq n_s D^s \).

4.7 Definition and Lemma For an effective cycle \( \mathcal{X} \) of pure codimension \( r \) the number

\[ t_a(\mathcal{X}) m_r D^{\dim \mathcal{X}} \]

will be called the dimensional \( a \)-size of order \( D \). If \( f \in \Gamma(\mathbb{P}^t, O(\bar{n}_{r+1} D)) \) intersects a subvariety \( \mathcal{Y} \) of \( \mathcal{X} \) that has pure codimension \( s > r \) in \( \mathbb{P}^t \) properly, and fulfills \( \log |f_Y| \leq 6a n_{r+1} D \), and \( \mathcal{Z} \) is any irreducible component of \( \mathcal{Y} . \text{div} f \), then

\[ t_a(\mathcal{Z}) m_{r+1} D^{\dim \mathcal{Z}} \leq t_a(\mathcal{X}, \text{div} f) m_{s+1} D^{\dim(\mathcal{X}, \text{div} f)} < t_a(\mathcal{Y}) m_s D^{\dim \mathcal{Y}}. \]
Proof The first inequality is obvious. Further, by the algebraic Bézout Theorem, and Proposition 2.4.1, with $a > d$,

$$t_a(Z) \leq t_a(Y, \text{div} f) = a \deg(Y, \text{div} f) + h(Y, \text{div} f)$$

$$\leq a \tilde{n}_{r+1} D \deg Y + 6a \tilde{n}_{r+1} D \deg Y + \tilde{n}_{r+1} D h(Y) + d \tilde{n}_{r+1} D \deg Y$$

$$\leq a \tilde{n}_{r+1} D h(Y) + 7a \tilde{n}_{r+1} D \deg Y$$

$$= t_a(Y) \tilde{m}_r D \leq t_a(Y) \tilde{m}_s D,$$

since $\tilde{m}_r < \tilde{m}_s$. Consequently,

$$t_a(Z) m_{s+1} D^{\dim(Y, \text{div} f)} = t_a(Z) \frac{m_s}{\tilde{m}_s} D^{\dim(Y, \text{div} f)} \leq t_a(Y, \text{div} f) \frac{m_s}{\tilde{m}_s} D^{\dim(X, \text{div} f)} < t_a(Y) m_s D^{\dim Y}.$$

The dimensional size of a cycle $Y$ measures the approximation power of the space of global sections on $Y$ with bounded length and degree.

4.8 Definition and Lemma For a given $\theta \in \mathbb{P}^t(\mathbb{C}), a > 0$, and an effective cycle $X$, of pure codimension $s$ in $\mathbb{P}^t$, such that the support of $X$ does not contain $\theta$, define the weighted algebraic distance of $\theta$ to $X$

$$\varphi_{a, \theta}(X) := \frac{D(\theta, X)}{t_a(X)}.$$

The cycle $X$ is called an approximation cycle, or $a$-approximation cycle of order $D$ relative to $N$ for $\theta$, iff

$$\deg X \leq n_s D^s, \quad h(X) \leq a_s D^s, \quad \text{and} \quad \varphi_{a, \theta}(X) \leq -b_s D^{t+1-s},$$

If $X_l, l = 1, \ldots, L$ are effective cycles, then

$$\min_{l=1, \ldots, L} (\varphi_{a, \theta}(X_l)) \leq \frac{1}{L} \varphi_{a, \theta} \left( \sum_{l=1}^L X_l \right).$$

Hence, if a sum of $L$ effective cycles is an approximation cycles, at least one of the summands is likewise. In particular

$$\varphi_{a, \theta}(nX) = \varphi_{a, \theta}(X),$$

for any natural number $n$, and any effective cycle $X$, which in turn implies that cycles with different weighted algebraic distance can not be multiples of the same irreducible variety. If $X$ is an approximation cycle for $\theta$, then

$$\log |\theta, X| \leq -b_s D^{t+1-s} + c,$$

where $c$ is the constant from 2.4.1.
Proof The first claim follows from the additivity, of the degree, height, and algebraic distance, and from the nonpositivity of the algebraic distance, and the non-negativity of the degree, and height. For the second claim, since $h(\mathcal{X}) \geq 0$, Theorem 2.1.1 implies

$$\log |\theta, X| \leq \frac{D(\theta, X)}{\deg X} + c \leq \frac{D(\theta, X)}{a \deg X + h(\mathcal{X})} + c = \varphi_{a, \theta}(\mathcal{X}) + c \leq -b_s D^{t+1-s} + c.$$ 

Denote now by $\tilde{C}_{t,D}$ be the set of $(D,a)$-approximation chains, that is the set of all successive $(D,a)$-intersections

$$\mathcal{C} : \quad \mathbb{P}^t = \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y}_s$$

such that $\mathcal{Y}_s$ is a $(D,a)$-approximation cycle. Denote further by $C_{t,D}$ the set of equivalence classes in $\tilde{D}_{t,D}$ where two chains are said to be equivalent iff they have the same end term. Of course, $C_{t,D}$ is then just the set of approximation cycles that appear as successive intersection. However, some of the following proofs will have a more lucid appearance if one views the set $C_{t,D}$ as a set of equivalence classes of chains of cycles rather than a set of cycles. On the other hand it will not be necessary to distinguish between an element of $C_{D,t}$ and one of its representatives in $\tilde{C}_{D,t}$, and therefore $\tilde{C}_{D,t}$ will not appear henceforth.

On $C_{t,D}$ further define the following relation. For two chains $C : \mathbb{P}^t = \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y}_s$, $\bar{C} : \mathbb{P}^t = \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y}_k$, the relation $C \prec \bar{C}$ holds iff there is a sequence of approximation chains

$$C = C_0, C_1, \ldots, C_{k-1}, C_k = \bar{C},$$

such that for each pair $C_l, \bar{C}_{l+1}, l = 0, \ldots, k-1$

$$t_a(\mathcal{Y}) m_{t+1-\dim \mathcal{Y}} D^{\dim \mathcal{Y}} < t_a(\mathcal{X}) m_{t+1-\dim \mathcal{X}} D^{\dim \mathcal{X}},$$

where $\mathcal{Y}, \mathcal{X}$ are the end terms of $C_l$, and $\bar{C}_{l+1}$ respectively, and one of the following conditions holds

1. $C_{l+1}, C_l$ look like

   $$C_{l+1} : \quad \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y}, \quad C_l : \quad \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y} \supset \mathcal{X}.$$ 

2. $C_{l+1}, C_l$ look like

   $$C_{l+1} : \quad \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y}, \quad C_l : \quad \mathcal{Y}_0 \supset \cdots \supset \mathcal{X},$$

   with $\mathcal{X}$ containing $\mathcal{Y}$. 

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This relation is obviously transitive, and also antireflexive, since for chains $C, \bar{C}$ with end terms $\mathcal{Y}, \bar{\mathcal{Y}}$ the relation $C \prec \bar{C}$ implies

$$t_a(\mathcal{Y}) m_{t+1-\dim \mathcal{Y}}D^{\dim \mathcal{Y}} < t_a(\bar{\mathcal{Y}}) m_{t+1-\dim \bar{\mathcal{Y}}}\bar{D}^{\dim \bar{\mathcal{Y}}},$$

and hence $\mathcal{Y} \neq \bar{\mathcal{Y}}$ implying $C_1 \neq C_2$.

Since for each $D \in \mathbb{N}$, there are only finitely many subvarieties $Z$ with $\deg(Z) \leq n_{\dim Z}D^{\dim Z}$, and $h(Z) \leq a_{\dim Z}D^{\dim Z}$, the set $C_{D,t}$ is finite, and hence there are minimal approximation chains with respect to the relation $\prec$; i.e. there is at least one $\bar{C} \in C_{t,D}$ such that there is no $C \in C_{t,D}$ with $C \prec \bar{C}$.

### 4.1.3 Existence of Approximation cycles

The fundamental Theorem on approximation cycles is the following:

**4.9 Theorem** For each sufficiently big $a$, and very $D >> 0$ the minimal chains in $C_{t,D}$ with respect to the relation $\prec$ have length $t$. In particular, there is a successive $(D,a)$-intersection $\mathcal{Y}_D$ of pure codimension $t$ which is an approximation cycle of order $D$.

**4.10 Lemma** Let $\mathcal{Y} \subset \mathbb{P}^t$ be a successive $(D,a)$-intersection of codimension $s \leq t - 1$, and

$$\mathcal{X} = \overline{f_1 \cap \cdots \cap f_r \cap U}$$

a locally complete $D$-intersection at $\mathcal{Y}$ of codimension $r < s$. Then, there are global sections $g \in \Gamma(\mathbb{P}^t, O(\bar{n}_{r+1}D))_\mathcal{Y}$, $\bar{g} \in \Gamma(\mathbb{P}^t, O(\bar{n}_{r+1}D))$, such that $g_{\mathcal{Y}}^1 = \bar{g}_{\mathcal{Y}}^1 \neq 0$, the restrictions of $f, \bar{f}$ to every irreducible component of $\mathcal{X}$ are nonzero, and

$$\log |g_{\mathcal{Y}}^1| \leq 6a\bar{n}_{r+1}D, \quad \log |\langle \bar{g} \theta \rangle| \leq -\bar{b}_{r+1} t_a(\mathcal{X}_{\min})D^{\dim \mathcal{X}},$$

and the restrictions of $f$, and $\bar{f}$ to every irreducible component of $X$ are non zero.

**Proof** Since $\mathcal{X}$ is a locally complete $(D,a)$-intersection of the $f_1, \ldots, f_r$, we have

$$\bar{D} := \sum_{i=1}^r \deg f_i - r \leq D \sum_{i=1}^r \bar{n}_i = \leq \frac{\bar{n}_{r+1}}{m} D.$$

Since, by definition of a locally complete intersection at $\mathcal{Y}$ the variety $\mathcal{Y}$ is contained in every irreducible component of $\mathcal{X}$, the claim follows from Lemma 4.3 with $n = \bar{n}_{r+1}, b = \bar{b}_{r+1}$.
4.11 Corollary Let $\mathcal{Y}$ be a successive $(D,a)$-intersection of codimension $s \leq t - 1$. Then, there is $\mathcal{X}$ a locally complete $D$-intersection at $\mathcal{Y}$ of codimension $r \leq s$, and global sections $g \in \Gamma(\mathbb{P}^t, O(n_{r+1}D))_\mathbb{Z}$, $\tilde{g} \in \Gamma(\mathbb{P}^t, O(n_{r+1}D))$, such that $g_Y^\perp = \tilde{g}_Y^\perp \neq 0$, and

$$\log |g_Y^\perp| \leq 6a\bar{n}_{r+1}D, \quad \log |\langle \tilde{g} \theta \rangle| \leq -\bar{b}_{r+1}t_a(\mathcal{X}_{\text{min}})D^{\text{dim} \mathcal{X}},$$

and the restrictions of $f$, and $\tilde{f}$ to every irreducible component of $X$ are non zero.

PROOF We proof by complete induction that $\mathcal{Y}$ is contained in a of locally complete $D$-intersection $\mathcal{X}$ at $\mathcal{Y}$ of codimension $r$ at $\mathcal{Y}$, or there is a global section of the form specified in the Corollary. Of course this claim entails the Corollary. Clearly the claim holds for $r = 0$ with $\mathcal{X} = \mathbb{P}^t$. So assume there is a locally complete $D$-intersection $\mathcal{X}$ at $\mathcal{Y}$ of codimension $r$. By the previous Lemma, there are global sections $g \in \Gamma(\mathbb{P}^t, O(n_{r+1}D))_\mathbb{Z}$, $\bar{g} \in \Gamma(\mathbb{P}^t, O(n_{r+1}D))$, such that $g_Y^\perp = \bar{g}_Y^\perp$, and

$$\log |g_Y^\perp| \leq 6a\bar{n}_{r+1}D, \quad \log |\langle \bar{g} \theta \rangle| \leq -\bar{b}_{r+1}t_a(\mathcal{X}_{\text{min}})D^{\text{dim} \mathcal{X}},$$

and the restrictions of $f$, and $\tilde{f}$ to every irreducible component of $X$ are non zero. If $f$ has nonzero restriction to $Y$ the Corollary follows. If the restriction of $f$ to $Y$ is zero, since $\deg g = n_{r+1}$, the union of the irreducible components of $\mathcal{X}\cdot \text{div} f$ that contain $\mathcal{Y}$ by Lemma 3.1 is a locally complete $D$-intersection of codimension $r + 1$ at $\mathcal{Y}$.

4.12 Lemma For every irreducible $(D,a)$-approximation cycle $\mathcal{Y}$ of codimension $s \leq t - 1$, belonging to an approximation chain $\mathcal{C}$, either there is a $(D,a)$-approximation chain

$$\mathcal{C}_1: \quad \mathbb{P}^t = \mathcal{Y}_0 \supset \cdots \supset \mathcal{X},$$

such that $\mathcal{X}$ is a locally complete $D$-intersection at $\mathcal{Y}$, and $\mathcal{C}_1 \prec \mathcal{C}$, or there is a $(D,a)$-intersection chain

$$\mathcal{C}_2: \quad \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y} \supset \mathcal{Z}$$

with $\mathcal{C}_2 \prec \mathcal{C}_1$.

PROOF Let $\mathcal{X} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_t$ be the locally complete $D$-intersection at $\mathcal{Y}$, and $g \in \Gamma(\mathbb{P}^t, O(n_{r+1}D))_\mathbb{Z}$, $\bar{g} \in \Gamma(\mathbb{P}^t, O(n_{r+1}D))_\mathbb{C}$, the global section such that $g_Y^\perp = \bar{g}_Y^\perp \neq 0$, and

$$\log |g_Y^\perp| \leq 6a\bar{n}_{r+1}D, \quad \log |\langle \bar{g} \theta \rangle| \leq -\bar{b}_{r+1}t_a(\mathcal{X}_{\text{min}})D^{\text{dim} \mathcal{X}},$$

from the Corollary.

By the Theorem of Bézout,

$$\deg(Y, \text{div} f) = \deg Y n_{r+1}D \leq n_s n_{r+1}D^{s+1} \leq n_s n_{s+1}D^{s+1} = n_{s+1}D^{s+1}.$$
Since $\mathcal{Y}$ is an approximation cycle, by Definition $h(\mathcal{Y}) \leq a_sD^s$, and Proposition \ref{prop:approximation-cycle} implies
\[
h(\mathcal{Y} \cdot \text{div} f) \leq 2\bar{n}_{r+1}Dh(\mathcal{Y}) + \deg Y6a\bar{n}_{r+1}D + d\bar{n}_{r+1}D \deg Y \leq (\bar{n}_{r+1}a_s + 6a\bar{n}_{r+1}n_s + dn_s\bar{n}_{r+1})D^s \leq a_{s+1}D^{s+1},
\] (4) and for $a > d$ we get
\[
t_a(\mathcal{Y} \cdot \text{div} f) \leq (7a + 7)\bar{n}_{s+1}D \deg Y + 2\bar{n}_{r+1}Dh(\mathcal{Y}) < 8\bar{n}_{s+1}t_a(\mathcal{Y})
\] (5)
Further, every irreducible component $\mathcal{Z}$ of $\mathcal{Y} \cdot \text{div} f$ is a successive $D$-intersection, hence by Lemma \ref{lem:successive-intersection},
\[
t_a(\mathcal{Z})m_{s+1}D^{t-s} < t_a(\mathcal{Y})m_sD^{t+1-s}.
\] (6)
Finally $f_Y^\perp = \bar{f}_Y^\perp$ implies that the restriction of $f$ to $Y$ equals the restriction of $\bar{f}$ to $Y$, and hence $(\mathcal{Y} \cdot \text{div} f)_Y = \text{div} \bar{f}$.

We make several case distinctions:

**Case 1:** $|\text{div} \bar{f}, \theta| \leq |Y, \theta|$. By Proposition \ref{prop:approximation-cycle},
\[
D(\text{div} f.Y, \theta) = D(\text{div} \bar{f}.Y, \theta) \leq D(Y, \theta) + 2\bar{n}_{s+1}Dh(\mathcal{Y}) + \deg Y6a\bar{n}_{s+1}D + dD \deg Y < \]
\[\quad -b_st_a(\mathcal{Y})D^{t+1-s} + 8\bar{n}_{s+1}t_a(\mathcal{Y})D,
\]
for $a > d$. Since $s \leq t - 1$, we have $t + 1 - s \geq 2$, hence for big enough $D$, (5) implies
\[
D(\text{div} f.Y, \theta) = D(\text{div} \bar{f}.Y, \theta) \leq -\frac{b_s}{2}t_a(\mathcal{Y})D^{t+1-s} \leq -\frac{b_s}{16\bar{n}_{r+1}}t_a(\text{div} f.Y)D^{t-s}
\]
by the choice of $b_{s+1}$, and $\bar{n}_{r+1} < \bar{n}_{s+1}$. By Lemma \ref{lem:divisible-approximation-cycle} $\text{div} f.Y$ has an irreducible component $\mathcal{Z}$ with $D(Z, \theta) \leq -b_{s+1}t_a(\mathcal{Z})D^{t-s}$. Because of (4), the estimate on the degree of $Z$, and the inequalities $\deg Z \leq \deg (Y, \text{div} f)$, and $h(Z) \leq h(\mathcal{Y} \cdot \text{div} f)$, $\mathcal{Z}$ is thus an approximation cycle, and because of (6), merging $\mathcal{Z}$ at the end of $\mathcal{C}$ one obtains an approximation chain of the form $\mathcal{C}_2$ with $\mathcal{C}_2 \prec \mathcal{C}$.

**Case 2:** $|\text{div} f^\perp, \theta| \geq |Y, \theta|$.

**Case 2.A:** $t_a(\mathcal{Y}) \leq 2\bar{b}_r t_a(\mathcal{X}_{\min})D^{s-r}/b_s$. This inequality together with Proposition \ref{prop:approximation-cycle}, implies
\[
D(\text{div} f.Y, \theta) \leq -\bar{b}_r t_a(\mathcal{X}_{\min})D^{t+1-r} + 2\bar{n}_{s+1}Dh(\mathcal{Y}) + \deg Y6a\bar{n}_{s+1}D + dD \deg Y \leq 
\[
-\frac{b_s}{2}t_a(\mathcal{Y})D^{t+1-s} + \bar{n}_{s+1}t_a(\mathcal{Y})D.
\]
Repeating the argument of case 1, one obtains again a chain of type $\mathcal{C}_2$ with $\mathcal{C}_2 \prec \mathcal{C}_1$. 

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Case 2.b: $t_a(Y) > 2\bar{b}_r t_a(X_{\min}) D^{s-r}/b_s$. In this case firstly, since $m_r \leq \frac{2\bar{b}_r m_s a_r}{b_s + 1 a_s}$, by the choice of $b_s$,

$$t_a(X_{\min}) m_r D^{t+1-r} \leq t_a(X_{\min}) \frac{2\bar{b}_r m_s a_r}{b_s + 1 a_s} D^{t+1-r} < t_a(Y) \frac{m_s a_r}{a_s} D^{t+1-s} \leq a_r m_s D^{t+1}.$$

Thus,

$$t_a(X_{\min}) \leq a_r \frac{m_s}{m_r} D^r \leq a_r D^r,$$

and by Lemma 4.4 either $X_{\min}$ is an approximation cycle, in which case $X_{\min}$ being a component of a locally complete $D$-intersection would be the end term of a $(D, a)$-approximation chain

$$C_1 : Y_0 \supset \cdots \supset X$$

with $C_1 < C$, or div$f.Y$ is an approximation cycle, and thus as in case 1 contains an irreducible approximation cycle $Z$ gives rise to a chain of kind $C_2$ with $C_2 \prec C$.

**Proof of Theorem 4.9** Let $r < t$, and

$$C : Y_0 \supset \cdots \supset Y_r,$$

be a chain in $C_{t,D}$ with corresponding global sections $f_1, \cdots f_s$. It has to be shown that if $s \neq t$, there is a $C_2$ with $C_2 \prec C$.

By the previous Lemma there either is a $(D, a)$-approximation chain

$$C_1 : Y_0 \supset \cdots \supset X,$$

with $C_1 < C$ or a $(D, a)$-approximation chain

$$C_2 : Y_0 \supset \cdots Y_r \supset Y_{r+1}$$

with $C_2 < C$. The Theorem follows.

**4.13 Corollary of the proof** For every $D \in \mathbb{N}$, and every $s \leq t$ there is an approximation cycle $Y_{D,s}$ of order $D$ and codimension $s$.

**Proof** By the proof, starting with an approximation cycle $Y_1$ of codimension one, there is a series of approximation cycles

$$Y_1, Y_2, \ldots, Y_{k-1}, Y_k$$

such that $Y_k$ has codimension $t$ and for every $l = 1, \ldots, k - 1$ we either have that $Y_{l+1}$ is a subvariety of codimension one in $Y_l$, or $Y_l$ is a subvariety of $Y_{l+1}$. Hence, (7) contains at least one cycle of every codimension between 1 and $t$.  

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4.2 Algebraic points with small distance

Define the sets
\[ R_D := \left\{ \alpha \in \mathbb{P}_t(\mathbb{Q}) \mid \alpha \text{ is the last term in a } \right. \]
\[ \text{successive } (D, a)-\text{intersection} \}. \tag{8} \]

for every \( D \in \mathbb{N} \). For a generic point \( \theta \in \mathbb{P}_t(\mathbb{C}) \) choose a \( \beta_D \in R_D \) which is the end term of a \( (D, a_D) \)-approximation chain which is minimal with respect to the relation. By Theorem 4.9 such a \( \beta_D \) exists for every \( D \in \mathbb{N} \), and by Lemma 4.8 and Theorem 4.9 \( \log |\beta_D, \theta| \leq \varphi_{a, \theta}(\beta_D, \theta) \leq -b, D \), thus
\[ \lim_{D \to \infty} \beta_D = \theta, \]
and
\[ \bar{M} := \{ D \in \mathbb{N} \mid \varphi_{a, \theta}(\beta_D+1) < \varphi_{a, \theta}(\beta_D) \} \tag{9} \]
is an infinite set.

We will still use the constants (3) and further choose a positive number \( \bar{b} \) that is sufficiently small compared with the constants (3). (How small it has to be will become clear within the following proofs, i.e. within the following proofs there will appear finitely many positive expressions in terms of the constants (3) all of which are needed to be bigger than \( \bar{b} \).) Further define
\[ n := \left\lceil \frac{8\bar{n}_t}{b_t} \right\rceil + 1, \tag{10} \]

4.14 Definition Let \( a >> 0, D \in \mathbb{N} \), and \( \bar{D} = nD \). A triple \( (f, \bar{f}, Y) \) consisting of a global sections \( f \in \Gamma(\mathbb{P}_t, O(D))_{\mathbb{Z}}, \bar{f} \in \Gamma(\mathbb{P}_t, O(D))_{\mathbb{C}} \) and a \( \mathbb{Z} \)-irreducible subvariety \( Y \subset \mathbb{P}^t_{\mathbb{Z}} \) of codimension \( t \) with \( t_a(Y) \leq (a_1 + an_t)\bar{D} \) is called a \( (D, a) \)-approximation triple if \( f_Y^\perp = \bar{f}_Y^\perp \neq 0 \), hence \( f_Y = \bar{f}_Y \neq 0 \), and
\[ D(Y, \theta) \leq -b t_a(Y) \bar{D}, \quad \log |f_Y^\perp| \leq 6a \bar{n}_t \bar{D}, \quad \log |\langle \bar{f} | \theta \rangle| \leq -\frac{\bar{b}}{n_0 \epsilon} t_a(Y) \bar{D}, \]

4.15 Proposition If \( (f, \bar{f}, Y) \) is a \( (D, a) \)-approximation triple, then
\[ \log |Y, \theta| \leq -b t_a(Y) \bar{D}, \]
with a constant \( b > 0 \) only depending on \( t \).
4.16 Proposition There is an infinite subset $M \subset \mathbb{N}$ such that for each $D \in M$ there exists a $(D, a)$-approximation triple $(f, \bar{f}, \mathcal{Y})$ with

$$t_a(\mathcal{Y}) \leq ka \tilde{D}^t, \quad \text{hence} \quad \sqrt[t]{\frac{t_a(\mathcal{Y})}{a}} \leq \tilde{D}^\frac{1}{t+1},$$

with a number $k \in \mathbb{N}$ only depending on $t$.

Let’s first see why the two Propositions together imply Theorem 1.2 for the case of projective space. For $D \in M$ from Proposition 4.16 define

$$D_1 := \left[ \left( \frac{t_a(\mathcal{Y}) \tilde{D}}{a} \right)^\frac{1}{t+1} \right].$$

Then, $D_1 \geq \tilde{D}^\frac{1}{t+1}$, and because of the estimate on $t_a(\mathcal{Y})$,

$$\deg Y \leq \frac{t_a(\mathcal{Y})}{a} \leq D_1^t, \quad h(\mathcal{Y}) \leq t_a(\mathcal{Y} \tilde{D}) \leq aD_1^t.$$ 

Further, by Proposition 4.15,

$$\log |Y, \theta| \leq -bt_a(\mathcal{Y}) \tilde{D} = -baD_1^{t+1},$$

and Theorem 1.2 follows.

4.2.1 Using approximation triples

I present two different proofs of Proposition 4.15 one using Proposition 2.4.2, the other Proposition 2.5. Of course these proofs does not deliver two independent proofs of the main Theorem, because the two Propositions were obtained essentially by a single proof.

Proof 1: Let $Z := \text{div} \bar{f}$. Firstly $|Z, \theta| \geq |Y, \theta|$, since otherwise Proposition 2.4.3 for $a \geq d$ would imply

$$0 = D(\text{div} f, Y, \theta) \leq D(Y, \theta) + 2Dh(\mathcal{Y}) + \deg Y \log |f_{\hat{Y}}^\perp| + dD \deg Y \leq$$

$$-b_t t_a(\mathcal{Y}) nD + 7 \tilde{n}_t t_a(\mathcal{Y}) D,$$

which because of $n = \tilde{D}/D \geq 8 \tilde{n}_t/b_t$ would be a contradiction. Next, by Proposition 2.4.2 for $\nu, \kappa$ defined by $f_{Z,Y}$,

$$\nu \kappa \log |\theta, Y| = \nu \kappa \log |\theta, Y + Z|$$

$$\leq \kappa D(\theta, Y) + \nu D \log |(\bar{f}|\theta)| + 6a \tilde{n}_t D \deg Y + 2Dh(\mathcal{Y}) + d_1 D \deg Y$$

$$\leq -\kappa b_t t_a(\mathcal{Y}) \tilde{D} - \nu \bar{b}/n^\sigma t_a(\mathcal{Y}) D + 7D \tilde{n}_t t_a(\mathcal{Y}).$$
Assume first that $\deg Z + \deg Y > \frac{8n_t}{\min(b/n^6, b_t)}$. Then, choosing $t \in [0, 1]$ in Theorem 2.1.3 in such a way that $\nu + \kappa = \frac{8n_t}{\min(b/n^6, b_t)}$, we get

$$\nu \kappa \log |\theta, Y| \leq (\nu + \kappa) \min(\frac{\bar{b}}{n^6}, b_t) t_a(Y) D + 7D t_a(Y) \leq -8\bar{n}_t t_a(Y) D + 7D \bar{n}_t t_a(Y) \leq -\bar{n}_t t_a(Y) D.$$  

As $\nu \kappa \leq \frac{(\nu + \kappa)^2}{4} = \frac{16n^2}{\min(b/n^6, b_t)^2}$, this in turn implies

$$\log |\theta, Y| \leq \frac{\min(\frac{\bar{b}}{n^6}, b_t)^2}{16n^6 \bar{n}_t} t_a(Y) \bar{D},$$

proving the claim with $b = \frac{\min(b/n^6, b_t)^2}{16n^6 \bar{n}_t}$. If, on the other hand $\deg Z + \deg Y \leq \frac{8n_t}{\min(b/n^6, b_t)}$, then by Theorem 2.1.1,

$$\log |Y, \theta| \leq \frac{1}{\deg Y} D(Y, \theta) + c \leq \frac{-\min(\frac{\bar{b}}{n^6}, b_t)}{8\bar{n}_t} b_t t_a(Y) \bar{D},$$

proving the claim with $b = \frac{b_t \min(b/n^6, b_t)}{8\bar{n}_t}$.

**Proof 2:** We have again $|Y, \theta| \leq |Z, \theta|$ with $Z = \text{div} f$ as in the first proof. Let

$$Y_C = \sum_{i=1}^{\deg Y} y_i,$$

with

$$|y_1, \theta| \leq |y_2, \theta| \leq \cdots \leq |y_{\deg Y}, \theta|,$$

and points counted with multiplicities. If $\deg Y \leq m_3 := [8n^6 t_\bar{n}/\bar{b}] + 1$, with $n$ from (10), clearly $\log |Y, \theta| \leq \frac{1}{\deg Y} D(Y, \theta) \leq \frac{-b_t}{m_3 t_a(Y) \bar{D}},$ proving the claim with $b = b_t/m_3$.

If on the other hand $\deg Y \geq m_3$, I claim, that $|y_{m_3}, \theta| \geq |X, \theta|$. Assume the opposite; then, by Proposition 2.4.2, with $t$ such that $f_{X,Y}(t) = (0, m_3)$,

$$0 = D(\theta, Y, Z) \leq m_3 \log |\tilde{f} \theta| + \deg Y 6aD + 2D h(Y) + dD \deg Y \leq -m_3 \frac{\bar{b}}{n^6 t_a(Y) \bar{D}} + 7\bar{n}_t t_a(Y) \bar{D}.$$  

which because of $m_3 \geq \frac{8n^6 \bar{n}_t}{\bar{b}}$ is a contradiction.

Further, by Proposition 2.5

$$\sum_{i=m_3+1}^{\deg Y} \log |y_i, \theta| \geq -2D h(Y) - \deg Y \log |f_{Y}^\perp|_{L^2(\mathbb{P}^r)} - d \deg X \deg Y \geq -7\bar{n}_t D t_a(Y),$$

(11)
and
\[
\deg Y \sum_{i=1}^{m_3} |\theta, y_i| = D_{pt}(Y, \theta) \leq -b_t t_a(Y) \bar{D},
\]
by assumption. Now (11), and (12) together imply
\[
\sum_{i=1}^{m_3} \log |\theta, y_i| = \deg Y \sum_{i=1}^{m_3} \log |\theta, y_i| - \sum_{i=m_3+1}^{\deg Y} \log |\theta, y_i| \\
\leq -b_t t_a(Y) \bar{D} + 7D t_a(Y) \\
= -b_t t_a(Y) n D + 7D \bar{n}_t t_a(Y),
\]
which because of \( n \geq 8 \bar{n}_t / b_t \) is less or equal \(- \frac{1}{n} t_a(Y) \bar{D}\). Hence,
\[
\log |Y, \theta| = \log |y_1, \theta| \leq \frac{1}{m_3} \sum_{i=1}^{m_3} \log |\theta, y_i| \leq - \frac{1}{nm_3} t_a(Y) \bar{D},
\]
proving the proposition with \( b = \frac{1}{nm_3} \).

4.2.2 Existence of approximation triples

To proof Proposition 4.16, we will need to compare approximation chains \( C_1 \in C_{D,t} \), \( C_2 \in C_{\bar{D},t} \) for \( D \neq \bar{D} \).

Let \( D < \bar{D} \), and \( \alpha \bar{D} \) be an irreducible approximation cycle of codimension \( t \) forming the end term of a minimal \((D, a)\)-approximation chain of order \( \bar{D} \). Define
\[
C_{D,t}(\alpha_D) = \{ C : Y_0 \supset \cdots \supset Y_t \in C_{D,t}| \alpha_D \subset Y_t \}.
\]

Clearly, \( C_{D,t}(\alpha_D) \) is always nonempty, because it contains the trivial chain. On \( C_{D,t}(\alpha_D) \) we still have the relation \( \prec \), and again, because of the finiteness of \( C_{D,t}(\alpha_D) \) the relation \( \prec \) restricted to the set \( C_{D,t}(\alpha_D) \) has at least one minimal element in \( C_{D,t}(\alpha_D) \). These minimal elements now usually do not have length \( t \), and if they are not, because they are minimal it is possible to construe global sections with small evaluation at \( \theta \) that are nonzero on \( \alpha_D \) if \( \bar{D} \). The fundamental Lemma for this construction is the following.

4.17 Lemma If \( \alpha_D \) is not the end term of any minimal chain \( C \in C_{D,t} \), or otherwise said \( C_{D,t}(\alpha_D) \) does not contain a minimal element of length \( t \), let \( Y_r, r \leq t - 1 \) be the last term of such a minimal chain in \( C_1 \in C_{D,t}(\alpha_D) \). Then, there is a locally complete \( D \)-intersection \( \mathcal{X} \) at \( \alpha_D \) of codimension \( p \leq r \) that contains \( Y_r \), and global sections \( g \in \Gamma(\mathbb{P}^t, \bar{n}_{p+1} D) \), \( \bar{g} \in \Gamma(\mathbb{P}^t, \bar{n}_{p+1} D) \) such that \( g_{Y_r}^\perp = \bar{g}_{Y_r}^\perp \neq 0 \), and
\[
\log |g_{Y_r}^\perp| \leq 6a \bar{n}_{p+1} D, \quad \log |\langle \bar{g} \rangle| \leq -\bar{b}_t t_a(\mathcal{X}_{\min}) D^{\dim \mathcal{X}},
\]
and the restrictions of \( g \), and \( \bar{g} \) to every irreducible component of \( X \) are non zero. Furthermore, \( g \) has nonzero restriction to \( \alpha_D \), and

\[
t_a(\mathcal{X}_{\text{min}})m_pD^{\dim X} \geq t_a(\mathcal{Y}_r)m_pD^{\dim Y_r}.
\]

**Proof** If

\[
C_1 : \quad \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y}_r
\]

is a minimal chain in \( C_{D,t}(\alpha_D) \), Corollary 4.14 implies the existence of \( X, g, \) and \( \bar{g} \), with the properties stated in the Lemma. If \( g \) had zero restriction to \( \alpha_D \), then \( C_1 \) would not be minimal in \( C_{D,t}(\alpha_D) \), because by the proof of Lemma 4.12 case 1, and case 2.a, \( \text{div} \mathcal{Y}_r \) would again be a \((D,a)\)-approximation cycle containing \( \alpha_D \) and by Lemma 4.8 likewise some irreducible component \( Z \) of \( \text{div} \mathcal{Y}_r \) would be a \((D,a)\)-approximation cycle giving rise to a chain

\[
C_2 : \quad \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y}_r \supset Z
\]

with \( C_2 \in C_{D,t}(\alpha_D) \), and \( C_2 \prec C_1 \) contradicting the minimality of \( C_1 \) in \( C_{D,t}(\alpha_D) \).

To prove the last claim on the dimensional \( a \)-sizes, assume the opposite. Then, by Lemma 4.12 either \( \mathcal{X}_{\text{min}} \) would be a \((D,a)\)-approximation cycle, hence \( C_3 \prec C_1 \), with \( C_3 \) a chain with end term \( \mathcal{X}_{\text{min}} \), and \( C_1 \) were not minimal in \( C_{D,t}(\alpha_D) \), or again, some \( Z \subset \text{div} \mathcal{Y}_r \) would be a \((D,a)\)-approximation cycle containing \( \alpha_D \) in which case again \( C_1 \) were not minimal in \( C_{D,t}(\alpha_D) \).

**Proof of Proposition 4.16** Recall the definition of the sets \( R_D \) and \( \bar{M} \) in (8). We will prove that for each \( D \in \bar{M} \), there is a \( \bar{D} > D \), and a \((\bar{D},a)\) approximation triple \((f, \bar{f}, \mathcal{Y})\). By Theorem 4.9 for each \( D \in \mathbb{N} \), one can choose a \((D,a)\) approximation cycle \( \alpha_D \) which is the end term of minimal \((D,a)\)-approximation chain, and has minimal weighted algebraic distance among all end terms of minimal \((D,a)\)-approximation chains. In particular

\[
\deg \alpha_D \leq n_tD^t, \quad h(\alpha_D) \leq a_tD^t, \quad \varphi_{a,\theta}(\alpha_D) \leq -b_tD.
\]

With \( D \in \bar{M} \) we make the case distinction as to whether \( \varphi_{a,\theta}(\alpha_D) > -n^2b_tD \) for every \( \bar{D} > D \) or \( \varphi_{a,\theta}(\alpha_D) \leq -n^2b_tD \) for some \( \bar{D} > D \), with \( n \) the constant from (10).

**Case 1** \( \varphi_{a,\theta}(\alpha_D) \leq -c_1b_tD \) for some \( \bar{D} > D \).

**4.18 Lemma** In Case 1, for \( \bar{D} \) the smallest number \( > D \) such that \( \varphi_{a,\theta}(\alpha_D) \leq -c_1b_D \), the variety \( \alpha_{\bar{D}} \) is not the end term of any \((\bar{D} - 1,a)\)-approximation chain, i.e. the minimal elements in \( C_{D-1,t}(\alpha_D) \) have length \( \leq t - 1 \).
Proof A: \( \bar{D} > D + 1 \). In this case, by the choice of \( \bar{D} \),
\[
\varphi_{a, \theta}(\alpha_{\bar{D} - 1}) > -c_1 b_t(\bar{D} - 1) > -c_1 b_t \bar{D} \geq \varphi_{a, \theta}(\alpha_D),
\]
in particular \( \alpha_D \neq \alpha_{\bar{D} - 1} \) by Lemma 4.8 and hence \( \alpha_D \) is not the end term of a minimal \((\bar{D} - 1, a)\)-approximation chain by the choice of \( \alpha_{\bar{D} - k} \) as a minimal element in \( C_{\bar{D}, t} \) that has minimal weighted algebraic distance to \( \theta \).

B: \( \bar{D} = D + 1 \). In this case, \( \alpha_{D+1} \) is not the end term of a minimal \((D, a)\)-approximation chain, since \( D \in \bar{M} \).

From on now let \( D = \bar{D} - 1 \) with \( \bar{D} \) from the lemma. This number is not necessarily contained in \( \bar{M} \) but making this replacement for every \( D \) defines another infinite set \( \bar{M}_1 \subset \mathbb{N} \).

For \( D, \bar{D}, \alpha_D \) as from the previous Lemma let \( \mathcal{Y}_r, \mathcal{X}, g, \bar{g} \) be an irreducible variety, a locally complete \( D \)-intersecting and global section with the properties of Lemma 4.17.

Define the subset \( C_{\bar{D}, t}(\alpha_D, \mathcal{Y}_r) \) consisting of the successive \( \bar{D} \)-intersection chains with end term \( \mathcal{Y}_s \) an irreducible subvariety of codimension \( s \) fulfilling
\[
\alpha_D \subset \mathcal{Y}_s \subset \mathcal{Y}_r, \quad h(\mathcal{Y}_s) \leq a_s \bar{D}_s, \quad D(\mathcal{Y}_s, \theta) \leq -b_s t_a(\mathcal{Y}) D^{t+1-s}.
\]

Note that \( \mathcal{Y}_s \) need neither be a \((D, a)\)- nor a \((\bar{D}, a)\)-approximation cycle. Its purpose is to give a good estimate on \( t_a(\alpha_D) \). Nonetheless on \( C_{\bar{D}, t}(\alpha_D, \mathcal{Y}_r) \) one has the relation \( \prec \) being defined in the same way as on \( C_{\bar{D}, t} \) with the condition on dimensional heigh of order \( \bar{D} \) as second condition.

Observe, that although \( \mathcal{Y}_r \) belongs to a minimal chain in \( C_{\bar{D}, t}(\alpha_D) \) with respect to \( \prec \), it need not belong to a minimal chain in \( C_{\bar{D}, t}(\alpha_D, \mathcal{Y}_r) \) because \( \bar{D} > D \). Nonetheless, \( C_{\bar{D}, t}(\alpha_D, \mathcal{Y}_r) \) is again finite, and therby has minimal elements.

4.19 Lemma With the above notation, for every \( \mathcal{Y}_s \) the end term of a minimal chain in \( C_{\bar{D}, t}(\alpha_D, \mathcal{Y}_r) \), we have \( s := \text{codim} \mathcal{Y}_s \geq r \). Further there exists a \( \mathcal{Y}_s \) which is the end term of a minimal chain \( C_2 \in C_{\bar{D}, t}(\alpha_D, \mathcal{Y}_r) \) that fullfills
\[
m_s t_a(\mathcal{Y}_s) \leq t_a(\mathcal{Y}_r) m_r \bar{D}^{s-r}.
\]

For this \( \mathcal{Y}_s \) there is a locally complete \((D, a)\)-intersection \( \mathcal{X} \) of codimension \( q \leq s \), and \( f \in \Gamma(\mathbb{P}^t, O(\bar{n}_{s+1} \bar{D})) \), \( \bar{f} \in \Gamma(\mathbb{P}^t, O(\bar{n}_{s+1} \bar{D})) \) with \( f_{\mathcal{Y}_s}^{\perp} = \bar{f}_{\mathcal{X}}^{\perp} \neq 0 \), and
\[
\log |f_{\mathcal{Y}_s}^{\perp}| \leq 6a_{q+1} \bar{D}, \quad \log |\langle \bar{f} | \theta \rangle| \leq -b_s t_a(\mathcal{X}_{\text{min}}) \bar{D}^{\text{dim} \mathcal{X}},
\]
such that the restrictions of \( f, \bar{f} \) to every irreducible component of \( \mathcal{X} \) are nonzero. If further \( s \leq t - 1 \), then the restricicnt of \( f, \bar{f} \) to \( \alpha_D \) are also non zero.

Proof The inequality \( s \geq r \) for any \( \mathcal{Y}_s \) as in the Lemma is trivial.
Further, since clearly $\mathcal{Y}_s$ is the end term of a chain $C_1 \in C_{\text{div}}(\alpha_p, \mathcal{Y}_r)$, there is a minimal $C_2 \in C_{\text{div}}(\alpha_p, \mathcal{Y}_r)$, with $C_2 \prec C_1$, hence the end term $\mathcal{Y}_s$ of $C_s$ fulfills

$$t_a(\mathcal{Y}_s)m_s D^{t+1-s} < t_a(\mathcal{Y}_r)m_r D^{t+1-r},$$

and the on the dimensional size of $\mathcal{Y}_s$ follows.

By corollary [4.11] there is a locally complete $(D, a)$-intersection $\mathcal{X}$ at $\mathcal{Y}_s$, and global sections $f \in \Gamma(\mathbb{P}^t, O(\bar{n}+1D))\mathbb{C}$, $\bar{f} \in \Gamma(\mathbb{P}^t, O(\bar{n}+1D))\mathbb{C}$ fulfilling the equalities and inequalities in the Lemma, and having nonzero restriction to $\mathcal{Y}_s$, and every irreducible component of $\mathcal{X}$. It remains to be proved that $f$ has nonzero restriction to $\alpha_D$ if $s \leq t-1$. Assume $f$ is zero on $\alpha_D$. Then $\text{div } f \cap \mathcal{Y}_s$ contains $\alpha_D$, and from the Theorem of Bézout, and Proposition 2.4.1, one could deduce a bound on the height containing $\bar{\alpha}$ by Lemma 4.19, $\log_{\mathbb{C}} D$, additionally $\bar{\alpha}$.

4.20 Lemma With $D, D, \alpha_D$ as above, let $\mathcal{Y}_r, \mathcal{X}, g, \bar{g}$ be the subvarieties and global sections existing by Lemma 4.17, and $\mathcal{Y}_s, \mathcal{X}, f, \bar{f}$ the varieties and global section existing by the previous Lemma. Then,

$$t_a(\alpha_D)m_t \bar{D} \leq 2 \frac{m_t}{m_s} \max \left( \frac{\bar{b}_s}{\bar{b}_t}, \frac{\bar{b}_q}{\bar{b}_t} \right) t_a(\mathcal{X}_{min}) D^{\dim \mathcal{X}},$$

and

$$t_a(\alpha_D)m_t \bar{D} \leq 2 \frac{m_t}{b_t} \max \left( \frac{b_s}{b_t}, \frac{m_s}{m_q} \right) t_a(\mathcal{Y}_s) \bar{D}^{\dim \mathcal{Y}_s}.$$ 

If $\mathcal{Y}_s = \alpha_D$, additionally

$$\frac{1}{2} t_a(\alpha_D) D^{t+1-r} \leq t_a(\alpha_D)m_p D^{r-s} D^{t+1-r} \leq 2 m_r \frac{\bar{b}_s}{\bar{b}_t} m_q t_a(\mathcal{X}_{min}) D^{\dim \mathcal{X}}.$$ 

Proof of Proposition 4.16. Case 1, Finish: With the notations of the two previous Lemmas, assume first that $\alpha_p$ is properly contained in $\mathcal{Y}_s$. Then, the restrictions of $f, \bar{f}$ to $\alpha_D$ are non-zero, and we get $D(\alpha_D, \theta) \leq -b_t n^2 t_a(\alpha_D) \bar{D}$. Secondly, by Lemma 4.19 log $|f_{\alpha_D}^\perp| \leq \log |f_{\mathcal{Y}_s}^\perp| \leq 6 a n_{q+1} D$. Finally, again by Lemma 4.19

$$\log |f_{\mathcal{Y}_s}^\perp| \leq -\bar{b}_q t_a(\mathcal{X}_{min}) D^{\dim \mathcal{X}},$$

which by Lemma 4.20 is at most

$$-\frac{1}{2} \frac{b_t}{m_t} \max \left( \frac{1}{\bar{b}_q}, \frac{\bar{b}_t}{\bar{b}_q} \right) t_a(\alpha_D) \bar{D},$$

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Thus, with $\bar{b} \leq \frac{1}{2}b_r \frac{ma}{m_r} \left( \frac{1}{b_r} + \frac{\bar{b}}{m_r} \right)$, we have the three inequalities

$$D(\alpha_D, \theta) \leq -b_t n^2 t_a(\alpha_D) \bar{D}, \quad \log |f^\perp_{\alpha_D}| \leq 6a\bar{n}_{q+1} \bar{D} \leq 6a\bar{n}_t n \bar{D},$$

$$\log |\langle \bar{f} | \theta \rangle| \leq -\bar{b} t_a(\alpha_D) \bar{D} \leq -\frac{\bar{b}}{n^6 t} t_a(\alpha_D) n \bar{D},$$

which are the 3 inequalities in the Definition of approximation triples with $\bar{D}$ replaced by $n \bar{D}$. Since

$$t_a(\alpha_D) \leq \text{deg}_\alpha \alpha_D + h(\alpha_D) \leq (an_t + a_t) \bar{D} \leq (an_t + a_t)(n \bar{D})^t,$$

this proves that $(f, \bar{f}, \alpha_D)$ is an approximation triple of order $n \bar{D}$.

If on the other hand $\alpha_D = \mathcal{Y}_s$, then firstly still $D(\alpha_D, \theta) \leq -b_t n^2 t_a(\alpha_D) \bar{D}$, secondly, by Lemma 4.117 the restriction of $g, \bar{g}$ to $\alpha_D$ is non zero, and $\log |g^\perp_{\alpha_D}| \leq |g_{\bar{Y}}| \leq 6a\bar{n}_{p+1} D$, and thirdly also by Lemma 4.117

$$\log |\langle \bar{g} | \theta \rangle| \leq -\bar{b} p t_a(\mathcal{X}_{\min}) D^{t+1-p},$$

which by Lemma 4.20 and the face $\bar{D} = D + 1$ is at most

$$-\frac{1}{4} \frac{m_r}{m_r} \bar{b}_r t_a(\alpha_D) \bar{D},$$

and with $\bar{b} \leq -\frac{1}{4} \frac{m_r}{m_r} \bar{b}_r$ we have again three inequalities

$$D(\alpha_D, \theta) \leq -b_t n^2 t_a(\alpha_D) \bar{D}, \quad \log |g^\perp_{\alpha_D}| \leq 6a\bar{n}_t n \bar{D}, \quad \log |\langle \bar{g} | \theta \rangle| \leq -\frac{\bar{b}}{n^6 t} t_a(\alpha_D) n \bar{D},$$

and $(g, \bar{g}, \alpha_D)$ is an approximation triple of order $n \bar{D}$, since again $t_a(\alpha_D) \leq (a_t + an_t) \bar{D}^t$.

**Proof of Lemma 4.20** Let $\mathcal{Y}_s, \tilde{\mathcal{X}}, f, \bar{f}$ be the subvarieties and global sections from Lemma 4.117 and

$$C : \mathcal{Y}_0 \supset \cdots \supset \alpha_D$$

the minimal element of $C_{D,t}$ to which $\alpha_D$ belongs. If $|\text{div} \bar{f}, \theta|$ were smaller than $|\alpha_D, \theta|$, Proposition 2.43 would imply

$$0 = D(\text{div} \bar{f}, \alpha_D, \theta) \leq D(\alpha_D, \theta) + 7\bar{n}_{q+1} D t_a(\mathcal{Y}) \leq -b_t n^2 t_a(\mathcal{Y}) \bar{D} + 7\bar{n}_{q+1} D t_a(\mathcal{Y}),$$

which because of $n \geq (8\bar{n}_q / b_t) \leq (8\bar{n}_{q+1}) / b_t$ is impossible. Hence, $|\text{div} \bar{f}, \theta| \geq |\alpha_D, \theta|$. Further $\text{div} f$ intersects $\mathcal{Y}_s, \tilde{\mathcal{X}}$, and each component of $\tilde{\mathcal{X}}, \mathcal{X}$ properly. The first by Lemma 4.19, the second because $\mathcal{Y}_r$ contains $\mathcal{Y}_s$, and is irreducible, and the third by construction. Finally, the proper intersection with every component of $\mathcal{X}$ follows
from the fact that $\mathcal{X}$ is a locally complete intersection at $\mathcal{Y}_r$, and $\text{div}f$ already intersects $\mathcal{Y}_s$ and thereby $\mathcal{Y}_g$ properly.

Now, if $2t_a(\bar{X}_{\min})b_qD_{\text{dim}\mathcal{X}} < t_a(\alpha_D)b_1D$, since $D(\text{div}\bar{f}, \alpha_D, \theta) = 0$, the proof of Lemma 4.4 implies that $\bar{X}_{\min}$ is a $(\bar{D}, a)$-approximation cycle. Being a locally complete $\bar{D}$-intersection, it belongs to a $(\bar{D}, a)$-approximation chain $\mathcal{C}_1$. Since $\bar{X}_{\min}$ contains $\alpha_D$, we get $\mathcal{C}_1 < \mathcal{C}$ contradicting the minimality of $\mathcal{C}_1$ in $C_{\bar{D},t}$ this is a contradiction. Thus,

$$2t_a(\bar{X}_{\min})b_qD_{\text{dim}\mathcal{X}} \geq t_a(\alpha_D)b_1D,$$

and hence

$$2t_a(\bar{X}_{\min})\frac{m_t}{b_t}b_qD_{\text{dim}\mathcal{X}} \geq t_a(\alpha_D)m_tD. \tag{13}$$

**CASE 1:** $t_a(\bar{X}_{\min})m_qD_{\text{dim}\mathcal{X}} \leq t_a(\mathcal{Y}_s)m_sD_{\text{dim}\mathcal{Y}_s}$. In this case from (13) the inequality

$$t_a(\alpha_D)m_tD \leq 2t_a(\bar{X}_{\min})\frac{m_t}{b_t}b_qD_{\text{dim}\mathcal{X}} \leq 2t_a(\mathcal{Y}_s)\frac{m_t}{m_s}m_t \frac{b_q}{b_1}D_{\text{dim}\mathcal{Y}_s} \tag{14}$$

follows.

**CASE 2:** $t_a(\bar{X}_{\min})m_qD_{\text{dim}\mathcal{X}} \geq t_a(\mathcal{Y}_s)m_sD_{\text{dim}\mathcal{Y}_s}$. If also $t_a(\alpha_D)b_1D \geq 2t_a(\mathcal{Y}_s)b_qD_{\text{dim}\mathcal{Y}_s}$, again because of $D(\text{div}\bar{f}, \alpha_D, \theta) = 0$, the proof of Lemma 4.4 implies that $\mathcal{Y}_s$ is an $(\bar{D}, a)$-approximation cycle. Being a successive $\bar{D}$-intersection, it belongs to a $(\bar{D}, a)$-approximation chain $\mathcal{C}_2$ and $\mathcal{Y}_s$ contains $\alpha_D$, which is impossible because $\mathcal{C}$ is minimal in $C_{\bar{D},t}$. Hence,

$$t_a(\alpha_D)b_1D \leq 2t_a(\mathcal{Y}_s)b_qD_{\text{dim}\mathcal{Y}_s},$$

from which together with the assumption

$$t_a(\alpha_D)m_tD \leq t_a(\mathcal{Y}_s)\frac{m_t}{b_1}b_sD_{\text{dim}\mathcal{Y}_s} \leq 2\frac{m_t}{b_t}b_sD_{\text{dim}\mathcal{Y}_s}m_t \frac{b_q}{b_1}t_a(\bar{X}_{\min})D_{\text{dim}\mathcal{X}} \tag{15}$$

follows.

Finally, if $\mathcal{Y}_s = \alpha_D$ assume $2t_a(\bar{X}_{\min})b_pD_{\text{dim}\mathcal{X}} < t_a(\mathcal{Y}_r)b_1D_{\text{dim}\mathcal{Y}_r}$. Then Lemma 4.4 implies that either divg$_{\mathcal{Y}_r}$ is a $(\bar{D}, a)$-approximation cycle, being the end term of an approximation chain $\bar{C}_2 \in C_{\bar{D},t}(\alpha_D)$ with $\bar{C}_2 < \bar{C}_1$, where $\bar{C}_1$ is the chain in $C_{\bar{D},t}(\alpha_D)$ with end term $\mathcal{Y}_r$, or $\bar{X}_{\min}$ is the end term of a chain $\bar{C}_3 \in C_{\bar{D},t}(\alpha_D)$ with $\bar{C}_3 < \bar{C}_1$. Both possibilities contradict the minimality of $\bar{C}_1$ in $C_{\bar{D},t}(\alpha_D)$. Hence,

$$2t_a(\bar{X}_{\min})m_pD_{\text{dim}\mathcal{X}} \geq t_a(\mathcal{Y}_r)\frac{m_p}{b_p}b_1D_{\text{dim}\mathcal{Y}_r} \geq t_a(\mathcal{Y}_s)\frac{m_p}{m_r}b_rD_{\text{dim}\mathcal{Y}_s} \geq t_a(\mathcal{Y}_r)\frac{m_p}{b_p}b_1D_{\text{dim}\mathcal{Y}_r}D^{t+1-r},$$

the last inequality by Lemma 4.19. Since $\mathcal{Y}_s = \alpha_D$, hence $s = t$, the claim follows, if one remembers $\bar{D} = \bar{D} + 1$.

**Proof of Proposition 4.16** Case 2: $\varphi_{a,\theta}(\alpha_D) \geq -n^2b_1\bar{D}$ for every $\bar{D} > D$.  

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In this case, for every $\bar{D} \geq D$ choose $\beta_{\bar{D}}$ as the end term of a minimal $(\bar{D}, a)$-approximation chain which has minimal dimensional $a$-size $t_a(\beta_{\bar{D}})m_t\bar{D}$ among all end terms of minimal $(\bar{D}, a)$-approximation chains. Since $\alpha_D$ was chosen with minimal weighted algebraic distance among the end terms of minimal elements in $C_{D,t}$, 

$$\varphi_{a, \theta}(\beta_{\bar{D}}) \geq \varphi_{a, \theta}(\alpha_D) > -n^2b_1\bar{D}.$$ 

Further, by Theorem 4.9, $\varphi_{a, \theta}(\beta_{\bar{D}}) \leq -b_1\bar{D}$. Hence, $\varphi_{a, \theta}(\beta_{D_2}) < \varphi_{a, \theta}(\beta_{D_1})$ for $D_1, D_2 > D$, and $D_2 \geq n^2D_1$, and by Lemma 4.8, $\beta_{D_2} \neq \beta_{D_1}$. Assume that 

$$t_a(\beta_{\bar{D}})\bar{D} < \frac{1}{n^6t_a(\beta_{n^2\bar{D}})(n^2\bar{D})}$$ 

for all $\bar{D} > D$. Then, for every $l \in \mathbb{N}$, 

$$(a_t + an_t)(n^2D)^{t+1} \geq t_a(\beta_{n^2\bar{D}})n^2D > n^{6t}t_a(\beta_{\bar{D}})D \geq n^{6t}D,$$ 

which for big enough $l$ contradicts the fact that $t + 1 < 3t$. Hence, there is some $\bar{D} > D$ such that 

$$t_a(\beta_{\bar{D}})\bar{D} \geq \frac{1}{n^6t_a(\beta_{n^2\bar{D}})(n^2\bar{D})}.$$ 

(16) 

Let 

$$C_1 : \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y}_s$$ 

be a minimal element in $C_{D,t}(\beta_{n^2\bar{D}})$. Then, $s < t$, because otherwise $\beta_{D+k}$ would be the end term of a minimal element in $C_{D,t}$ contradicting the assumption $\varphi_{a, \theta}(\beta_{\bar{D}}) > -n^2b_1\bar{D}$ together with the inequality $\varphi_{a, \theta}(\beta_{n^2\bar{D}}) \leq -b_1n^2\bar{D}$ holding by Theorem 4.9. 

Let $\gamma_{\bar{D}}$ be the end term of a minimal chain $C_3 \in C_{D,t}$ with $C_3 \prec C_1$. Then, by the choice of $\beta_{\bar{D}}$, and the definition of the relation $\prec$, 

$$t_a(\beta_{\bar{D}})m_t\bar{D} \leq t_a(\gamma_{\bar{D}})m_t\bar{D} < t_a(\mathcal{Y}_s)m_s\bar{D}^{\dim\mathcal{Y}_s},$$ 

and by (16), 

$$\frac{1}{n^6t_a(\beta_{n^2\bar{D}})m_t(n^2\bar{D})} \leq t_a(\beta_{\bar{D}})m_t\bar{D} < t_a(\mathcal{Y}_s)m_s\bar{D}^{\dim\mathcal{Y}_s}. \tag{17}$$ 

By corollary 4.11 there is a locally complete $\bar{D}$-intersection $\mathcal{X}$ of codimension $r \leq s$ containing $\mathcal{Y}_s$, and $g \in \Gamma(\mathbb{P}^r, \mathcal{O}(\bar{D}))_\mathbb{Z}$, $\bar{g} \in \Gamma(\mathbb{P}^r, \mathcal{O}(\bar{D}))_\mathbb{C}$ with $g^\perp_{\mathcal{Y}_s} = \bar{g}^\perp_{\mathcal{Y}_s} \neq 0$, and 

$$\log |g^\perp_{\mathcal{Y}_s}| \leq 6a\bar{\alpha}_{r+1}\bar{D}, \quad \log |\langle \bar{g}, \theta \rangle| \leq -\bar{b}_s t_a(\mathcal{X}_{\min}) \bar{D}^{\dim \mathcal{X}}.$$ 

Because of the minimality of $C_1$ in $C_{D,t}(\beta_{n^2\bar{D}})$, and (17), 

$$-\bar{b}_s t_a(\mathcal{X}_{\min}) \bar{D}^{\dim \mathcal{X}} \leq -\bar{b}_s \frac{m_s}{m_r} t_a(\mathcal{Y}_s)n^2\bar{D} \leq -\bar{b}_s \frac{m_t}{m_r} \frac{1}{n^6t_a(\beta_{n^2\bar{D}})} n^2\bar{D}.$$ 

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Hence, with $\bar{b} \leq \frac{m}{m'}$, we get
\[
\log |\langle g | \theta \rangle| \leq -\frac{\bar{b}}{n^t} t_a(\beta n^2 \bar{D}) n^2 \bar{D}.
\]
Together with
\[
\log |g_{\bar{Y}}| \leq 6a \bar{n}_{r+1} \bar{D} \leq 6a \bar{n}_t \bar{D} \quad \text{and} \quad D(\beta n^2, \theta) \leq -b_t n^2 \bar{D},
\]
and $t_a(\beta n^2 \bar{D}) \leq (a_t + an_t)(n^2 \bar{D})^t$, this proves that $(g, \bar{g}, \alpha_{\bar{D}+k})$ is an approximation triple with $n$ replaced by $n^2$, and $D$ by $\bar{D}$.

4.3 Varieties of higher dimension

To prove that there are also algebraic subvarieties of higher dimension in $\mathbb{P}^t$, that are very close to $\theta$, the general strategy is to use Corollary 1.3, i.e. find subvarieties of bounded height and degree that contain the subvariety $\alpha_D$ of codimension $t$ that fulfills Theorem 1.2. However, it is not as easy to find these subvarieties as it was to find approximation cycles of higher dimension in Corollary 4.13. The reason is that although it is possible for the reasons stated in Corollary 4.13 to find approximation cycles of every codimension, they need not all contain $\alpha_D$ even though $\alpha_D$ belongs to a minimal element in $\mathcal{C} \in \mathcal{C}_{D,t}$. This is because if in constructing $\mathcal{C} \prec \mathcal{C}_0$ with $\mathcal{C}_0$ the empty chain only because in the series $\mathcal{C} = \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_0 = \mathcal{C}_t$ from the definition of $\prec$ there appears an $i$ with $\mathcal{C}_i \prec \mathcal{C}_{i+1}$ such that the end term $\mathcal{C}_i$ contains the end term of $\mathcal{C}_{i+1}$ one has no guarantee that the end terms of $\mathcal{C}_i$, $j = 1, \ldots, i$ will contain $\alpha_D$.

However, it can be proved that for generic points $\theta \in \mathbb{P}^t$ such that case 2 in the proof of Proposition 4.16 applies, one has a series $\mathcal{C} = \mathcal{C}_1, \ldots, \mathcal{C}_t = \mathcal{C}_0$, and the end term of $\mathcal{C}_i$ is a subvariety of codimension 1 in $\mathcal{C}_{i+1}$ for every $k = 1, \ldots, t - 1$. It can further be proved that these generic points are almost all points in $\mathbb{P}^t$ in the sense of Lebesgue measure (see [Ma3]). For a generic points $\theta$ for which case 1 in the proof of Proposition 4.16 applies, the situation is much more complicated. A proof that one still finds subvarieties of higher dimension that have small distance to $\theta$ will also be given in [Ma3].

5 The general case

Proof of Theorem 1.2 2 (arbitrary quasi projective scheme over Spec $\mathbb{Z}$)

Firstly, replacing $\mathcal{X}$ by the algebraic closure of $\{\theta\}$ it may be assumed that $\mathcal{X} = \mathcal{Y}$ is irreducible of relative dimension $t$, and the algebraic closure of $\{\theta\}$. Assume first
that \( \mathcal{L} \) is very ample, and choose global sections \( s_1, \ldots, s_{m+1} \in \Gamma(\mathcal{X}, \mathcal{L}) \) with \( m \) minimal that define an embedding \( i : \mathcal{X} \to \mathbb{P}^n = \mathbb{P}(\mathbb{Z}^{n+1}) \). Then \( O(1)|_{\mathcal{X}} = \mathcal{L} \),

\[
\deg_{O(1)} i(\mathcal{X}) = \deg_L X, \quad h_{O(1)}(i(\mathcal{X})) = h_{\mathcal{L}}(\mathcal{X}),
\]

and for \( \alpha \in \mathcal{X}(\bar{\mathbb{Q}}) \),

\[
\deg_L i(\alpha) = \deg_L (\alpha), \quad h_{O(1)}(i(\alpha)) = h_{\mathcal{L}}(\alpha).
\]

Since we are only interested in algebraic points very close to \( \theta \) we may replace \( \mathcal{X} \) by the closure of \( i(\mathcal{X}) \) in \( \mathbb{P}^n \). Let \( E \cong \mathbb{Z}^{t+1} \subset M, N \cong \mathbb{Z}^{n-t} \) submodules defined by any choice of \( t+1 \) of the \( n+1 \) coordinates such that \( M = E \oplus N \). Then the canonical projection \( M = E \oplus N \to E \) induces maps

\[
\mathbb{P}^n \setminus \mathbb{P}(N) \to \mathbb{P}(E), \quad \varphi : \mathcal{X} \setminus \mathbb{P}(N) \to \mathbb{P}(E).
\]

Since \( \theta \) is contained in no proper subscheme of \( \mathcal{X} \) it is contained in \( \mathcal{X} \setminus \mathbb{P}(N) \), and \( \varphi \) is injective in a certain neighbourhood \( U_\mathcal{C} \) of \( \theta \) in the \( \mathbb{C} \)-topology, because the derivatives of \( \varphi \) with respect to the coordinates of \( N \) can’t vanish at \( \theta \) as they define algebraic subvarieties.

Let \( V_\mathcal{C} := \varphi(U_\mathcal{C}) \). Then, for any \( \alpha \in \mathbb{P}(E)(\bar{\mathbb{Q}}) \cap U_\mathcal{C} \), the set \( \varphi^{-1}(\alpha) \), equals the intersection of \( \mathcal{X}(\bar{\mathbb{Q}}) \) with the projective subspace \( \mathbb{P}_\alpha \subset \mathbb{P}(M) \) corresponding to the submodule \( M_\alpha := N_\mathcal{Q} \oplus \bar{\mathbb{Q}}\hat{\alpha} \), where \( \hat{\alpha} \) is any vector in \( \mathbb{P}(E)(\bar{\mathbb{Q}}) \) representing \( \alpha \). It is easily seen that \( \deg_{O(1)} \mathbb{P}_\alpha = \deg_{O(1)} \alpha \), and \( h_{O(1)}(\mathbb{P}_\alpha)) = h_{O(1)}(\alpha) + \sigma_{n-t} \).

Now, \( \varphi(\theta) \) is contained in no proper algebraic subset of \( \mathbb{P}(E) \) defined over \( \bar{\mathbb{Q}} \), and thus by Theorem 1.2 for the case of projective space, there is a positive number \( b \) such that for every sufficiently big \( a \), there is an infinite subset \( \bar{L} \subset \mathbb{N} \), such that for each \( D \in \bar{L} \), there is an \( \alpha_D \in V_\mathcal{C} \cap \mathbb{P}(E)(\bar{\mathbb{Q}}) \) with

\[
\deg(\alpha_D) \leq D^t, \quad h(\alpha) \leq aD^t, \quad \log |\varphi(\theta), \alpha_D| \leq -abD^{t+1}.
\]

Consider the points

\[
\beta_D := \varphi^{-1}(\alpha_D) = (\mathbb{P}_\alpha \cap \mathcal{X}) \setminus \mathbb{P}(M) \in U \cap \mathcal{X}(\bar{\mathbb{Q}}).
\]

Clearly, \( \deg \beta_D \leq \deg \mathcal{X} \deg \alpha_D \), thus

\[
\deg \beta_D \leq \deg \mathcal{X} D^t.
\]

Further, by the arithmetic Bézout Theorem,

\[
\begin{align*}
\quad h(\beta_D) & \leq \deg \mathbb{P}_{\alpha_D} h(\mathcal{X}) + \deg \mathcal{X} h(\mathbb{P}_{\alpha_D}) + d \deg \mathbb{P}_{\alpha_D} \deg \mathcal{X} \\
& = \deg \alpha_D h(\mathcal{X}) + \deg \mathcal{X} (h(\alpha_D) + \sigma_{n-t}) + d \deg \alpha \deg \mathcal{X} \\
& \leq h(\mathcal{X}) D^t + \deg \mathcal{X} aD^t + d \deg \mathcal{X} D^t.
\end{align*}
\]
Finally, since $U_c$ may be chosen relatively compact, there are constants $C, C' > 0$ such that for $x, y \in U_c$,

$$C|x, y| \leq |\varphi(x), \varphi(y)| \leq C'|x, y|. \quad (18)$$

Since $\varphi$ is bijective on $U_c$, for every $\alpha_d$ close enough to $\varphi(\theta)$, the $\mathbb{C}$-valued points of $\beta_d$ contain a point $\bar{\beta}_d \in U_c \cap X(Q)$, and

$$\log |\theta, \beta_d| \leq \log |\theta, \bar{\beta}_d| \leq \log |\varphi(\theta), \alpha_d| - \log C \leq -baD^{t+1} - \log C.$$

Choosing

$$\bar{D} := \left[ \frac{\sqrt{D}}{\sqrt[3]{h(X)} + 3/2 \deg X} \right],$$

and remembering that $a$ can be chosen to be $2 \geq d$, we get

$$\deg \beta_d \leq \bar{D}^t, \quad h(\beta_d) \leq a\bar{D}^t, \quad \log |\beta_d, \theta| \leq \frac{1}{\left(\frac{h(X)}{a} + 3/2 \deg X\right)^{t+1}} ba\bar{D}^{t+1} - \log C.$$

For $D >> 0$, replacing $3/2 \deg X$ by $2 \deg X$, the series $\beta_d$ thus fullfills all requirements of Theorem 1.2.

If $\mathcal{X}$ is only ample, use the property that there are global sections $s_1, \ldots, s_{m+1}$ of $\mathcal{L}$ on $\mathcal{X}$ that define a finite map $\psi : \mathcal{X} \to \mathbb{P}^m$. By the proof just given the Theorem holds for $\psi(\mathcal{X})$, and thereby also for $\mathcal{X}$.

**Proof of Theorem 1.2 (Arbitrary scheme over an arbitrary number ring)**

To prove the Theorem over an arbitrary ring of integers $\mathcal{O}_k$ in a number field $k$ there are two possibilities: The first is to extend the proofs of the metric Bézout Theorem in [Ma1], the estimates for arithmetic Hilbert functions in [Ma2], and the Propositions in this paper to projective spaces over $\text{Spec} \, \mathcal{O}_K$ which only entails some further effort of a technical kind, and derive the Theorem in the same way. The second possibility is to use Weyl restriction from $\text{Spec} \, \mathcal{O}_k$ to $\text{Spec} \, \mathcal{Z}$ and use that the Theorem has already been proved over $\text{Spec} \, \mathcal{Z}$. I will here choose the second possibility:

So let $\mathcal{O}_k$ be the ring of integers of a number field $k$ of degree $d$ over $k$, let $\mathcal{X}$ be a projective variety of relative dimension $t$ over $\text{Spec} \, \mathcal{O}_k$, and $\theta \in \mathcal{X}(\mathbb{C}_\sigma)$ a generic point, and define

$$\mathcal{X}_1 := R_{\mathcal{O}_k/\mathcal{Z}} \mathcal{X}.$$

It has relative dimension $t$ over $\text{Spec} \, \mathcal{Z}$.

Then, $\mathcal{X}_1(\mathbb{C}) = \mathcal{X}(k \otimes_\mathbb{Q} \mathbb{C})$, hence $\theta$ defines a generic point $\theta_1 \in \mathcal{X}_1(\mathbb{C})$. By the Theorem for schemes over $\text{Spec} \, \mathcal{Z}$ there is a $b > 0$ such that for given $a > 0$ there is
an infinite subset \( M \subset \mathbb{N} \) such that for all \( D \in M \) there is an \( \alpha_D \in \mathcal{X}_1(\bar{\mathbb{Q}}) \) with

\[
\deg \alpha_D \leq D^t, \quad h(\alpha_D) \leq aD^t, \quad \log |\alpha_D, \theta| \leq - \frac{1}{(h(\mathcal{X}) + 2 \deg X)} \log |\alpha_D, \div f|.
\]

The point \( \alpha_D \) canonically induces a point \( \beta_D \in \mathcal{X}(\bar{k}) \), and we have

\[
\deg \beta_D = [k : \mathbb{Q}] \deg \alpha_D, \quad h(\beta_D) = [k : \mathbb{Q}] h(\alpha_D).
\]

Finally,

\[
\log |\beta_D, \theta| \leq \log |\alpha_D, \theta_1| \leq -baD^{t+1}.
\]

### 6 Algebraic independence Criteria and Outlook

As a corollary of Theorem 1.2, we get a new proof or the Philippon criterion for algebraic independence.

#### 6.1 Theorem

For \( n \in \mathbb{N} \) let \( D_n \in \mathbb{N}, T_n, V_n \in \mathbb{R}^>0 \) be such that

\[
\lim_{n \to \infty} \frac{V_n}{D^k T_n} = \infty,
\]

and assume that for each \( n \) there is a set global sections \( \mathcal{F}_n \subset \Gamma(\mathbb{P}^m, O(D_n)) \) with

\[
\log |f| \leq T_n, \quad \forall f \in \mathcal{F}_n,
\]

and

\[
\log |\langle f | \theta \rangle| \leq -V_n, \quad \forall f \in \mathcal{F}_n,
\]

for some \( \theta \in \mathbb{P}^m(\mathbb{C}) \), and

\[
\log \left| \bigcap_{f \in \mathcal{F}_n} V(f), \theta \right| \geq -V_{n-1},
\]

where \( V(f) \) denotes the vanishing set of \( f \). Then the transcendence degree of the field generated by the coordinates of \( \theta \) is at least \( k+1 \).

**Sketch of Proof:** Let \( \mathcal{X} \) be the algebraic closure of \( \{ \theta \} \), denote by \( t \) its dimension. For appropriate \( K_n \) in relation to \( D_n, T_n, V_n \), one uses Theorem 1.2 to find a point \( \alpha_n \) with height and degree bounded in terms of \( K_n \) with logarithmic distance to \( \theta \) smaller than \(-V_{n-1}\). Thus, there is an \( f \in \mathcal{F}_n \) such that \( |\alpha_n, \theta| < |\alpha, \div f| \), implying \( \alpha_n \notin V(f) \) and Theorem 2.4.2 gives an estimate

\[
0 = D(\alpha_n, \div f, \theta)
\]
against the degree and height of $\alpha_n$ and $f$, and $V_n$ which by appropriate choice of $K_n$ can be proved to be less than 0 if $t \le k$.

Actually this proof morover shows that for $\theta_1, \ldots, \theta_{k+1}$ any subset of the coordinates of $\theta$ of cardinality $k + 1$ with $\text{trdeg}_Q Q(\theta_1, \ldots, \theta_{k+1}) = k + 1$ the point $(\theta_1, \ldots, \theta_{k+1})$ forms an $S$-point in the Zariski closure $(\theta_1, \ldots, \theta_{k+1})$ in the sense of Mahler classification. For this reason, the details of the proof will given in [Ma3].

There is a stronger version of the main Theorem 1.2: Under the assumptions of Theorem 1.2, assume that $\theta$ is not a $U$-point in the sense of Mahler classification. Let $a$ be a sufficiently big real number and choose a natural number $L$. Then, there is an infinite subset $\mathcal{M} \subset \mathcal{O}$ depending on $a$ and $L$ such that for every $D \in \mathcal{M}$ there is a zero dimensional $\mathcal{O}_k$-irreducible variety $Z_D$ with $d := \deg Z_D \ge L$ such that with $\alpha_1, \ldots, \alpha_d$ the $k$-valued points of $Z_D$, ordered in such a way that that $|\alpha_1, \theta| \le \cdots \le |\alpha_d, \theta|$, the inequalities

$$\deg Z_D \le D^t, \quad h(Z_D) \le aD^t,$$

and

$$\log |\alpha_1, \theta| \le \cdots \le |\alpha_L, \theta| \le -\frac{b}{L} \left( \frac{h(\alpha)}{a + 2 \deg X} \right)^{\frac{1}{t+1}} D^{t+1},$$

hold, with $b$ a positive real number, again only depending on $t$. To prove this, one has to consider derivatives of the algebraic distance which are related by a derivative metric Bézout Theorem to be proved in the next part of this series of papers.

Theorem 1.2 allows to prove new algebraic independence criteria, some of which also depend on the derivative metric Bézout Theorem. They also involve global sections with small evaluations at some point $\theta$ on a variety, and possibly also small higher derivatives at this point, but unlike the Philippon criterion above will not require lower bounds on the distances of the divisors of these global sections to $\theta$.

### A Proof of Proposition 1.1

The Proposition will be proved by complete induction on $t$. For $t = 1$, and $X = \mathbb{P}^1$, by 2.1.1, and the arithmetic Bézout Theorem,

$$\log |\alpha, \beta| \ge D(\alpha, \beta) - c \deg \alpha \ge h(\alpha, \beta) - \deg \alpha h(\beta) - \deg \beta h(\alpha) - c' \deg \alpha \deg \beta \ge -(c' \deg \alpha + h(\alpha)) \deg \beta - \deg \alpha h(\beta) - c \deg \alpha,$$

and the Proposition holds with $c_1 = c \deg \alpha + c' \deg \alpha + h(\alpha)$, and $c_2 = \deg \alpha$.

Assume now the Proposition is proved for $t - 1$, and consider $X = \mathbb{P}^t$. By [Ma2], Proposition 4.4, with $D = \sqrt{2t \deg \alpha}$ there is a global section $f \in \Gamma(\mathbb{P}^t, O(D))$ with $\log |f| \le \frac{\sqrt{2t h(\alpha)}}{(\deg \alpha)^{\frac{1}{t+1}}}$, such that $\text{div} f$ contains $\alpha$. If $Y := \text{div} f$ also contains $\beta,$
by induction hypothesis, there are \( c_1, c_2 > 0 \) only depending on \( t \), the degree and height of \( \alpha \), and the degree and height of \( \text{div} f \), hence by the choice of \( f \) only on \( t \), and the degree and height of \( \alpha \), such that

\[
\log |\alpha, \beta| \geq -c_1 \deg \beta - c_2 h(\beta).
\]

If \( \mathcal{Y} \) does not contain \( \beta \), then by 2.1 and the arithmetic Bézout Theorem,

\[
\log |\alpha, \beta| \geq \log |\mathcal{Y}, \beta| \geq D(\mathcal{Y}, \beta) - c \deg \mathcal{Y}
\]

\[
\geq h(\mathcal{Y}, \beta) - \deg \mathcal{Y} h(\beta) - h(\mathcal{Y}) \deg \beta - c' \deg \mathcal{Y} \deg \beta
\]

\[
\geq -(c' \deg \mathcal{Y} + h(\mathcal{Y})) \deg \beta - \deg \mathcal{Y} h(\beta) - c \deg \mathcal{Y},
\]

and the Proposition follows with \( c_1 = c' \deg \mathcal{Y} + c \deg \mathcal{Y} + h(\mathcal{Y}) \), and \( c_2 = \deg \mathcal{Y} \), both depending only on \( t \), and the degree and height of \( \alpha \).

Let now \( \mathcal{X} \) be arbitrary of relative dimension \( t \). Embedding \( \mathcal{X} \) into some projective space, and repeating the argument from the proof of the general case of Theorem 1.2, one obtains the \( c_1, c_2 \) depending only on \( t \), the degree and height of \( \alpha \), and the degree and height of \( \mathcal{X} \) such that

\[
\log |\alpha, \beta| \geq -c_1 \deg \beta - c_2 h(\beta).
\]

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