New hairy black-hole solutions with a dilaton potential

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Abstract

We consider black-hole solutions with a dilaton field possessing a nontrivial potential approaching a constant negative value at infinity. The asymptotic behaviour of the dilaton field is assumed to be slower than that of a localized distribution of matter. A non-Abelian SU(2) gauge field is also included in the total action. The mass of the solutions admitting a power series expansion in \(1/r\) at infinity and preserving the asymptotic anti-de Sitter geometry is computed by using a counterterm subtraction method. Numerical arguments are presented for the existence of hairy black-hole solutions for a dilaton potential of the form

\[ V(\phi) = C_1 \exp(2\alpha_1 \phi) + C_2 \exp(2\alpha_2 \phi) + C_3, \]

special attention being paid to the case of the \(N = 4, D = 4\) gauged supergravity model of Gates and Zwiebach.

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1. Introduction

According to the so-called ‘no-hair’ conjecture, a stationary black hole is uniquely described in terms of a small set of asymptotically measurable quantities. This hypothesis was disproved more than ten years ago, when several authors presented a counterexample within the framework of SU(2) Einstein–Yang–Mills (EYM) theory [1]. Although the new solution was static with vanishing Yang–Mills (YM) charges, it was different from the Schwarzschild black hole and, therefore, not characterized by its total mass (see [2] for a comprehensive review of this topic and an extensive bibliography).

However, much on the literature on hairy black-hole solutions is restricted to the case of an asymptotically flat spacetime. Since asymptotic flatness is not always an appropriate theoretical idealization, and is never satisfied in reality, it may be important to consider other types of asymptotics, in particular, solutions with a cosmological constant \(\Lambda\).

Asymptotically anti-de Sitter (AAdS) black-hole solutions with SU(2) non-Abelian fields have been presented in [3, 4]. The properties of these configurations are strikingly different from those valid in the asymptotically flat case (for example there are stable solutions in which the gauge field has no nodes; also solutions exist for continuous intervals of the parameter space, rather than discrete points). In an unexpected development, it has been shown recently that for \(\Lambda < 0\) even a conformally coupled scalar field can be painted as hair [5].
Much of the discussion on AdS hairy black holes has concerned the case when the matter fields fall off sufficiently fast such that the conserved charges can be written as surface integrals involving only the metric and its derivatives. However, recently it became clear that the usual AdS-invariant boundary conditions do not include all AAdS configurations. Several solutions involving a minimally coupled self-interacting scalar field in the bulk action and preserving the asymptotic AdS symmetry group despite the fact that the standard gravitational mass diverges, have appeared in the literature [6–12]. An exact four-dimensional black-hole solution of gravity with a minimally coupled self-interacting scalar field has been presented in [6] by Martinez, Troncoso and Zanelli (MTZ). Hairy black-hole solutions of $\mathcal{N} = 8$ gauged supergravity in four and five dimensions are described in [8]. Solutions describing a gravitating self-interacting scalar field whose mass saturates the Breitenlohner–Freedman bound are discussed in [11].

In all these cases, the scalar fields drop off so slowly in the asymptotic region, such that they add a nonzero contribution to the conserved charges. The mass of these solutions is computed by using a Hamiltonian method, such that the divergencies from the gravity and scalar parts cancel out, yielding a finite total charge.

In this paper we consider the case of Einstein gravity coupled to a dilaton field $\phi$ with a dilaton potential $V(\phi)$, looking for solutions satisfying a weakened set of boundary conditions at infinity, which implies a diverging ADM mass, despite the fact that the spacetime is still AAdS. To simplify the general picture, we restrict ourselves to the case $D = 4$. Also, since non-Abelian fields usually occur together with the dilaton in the bosonic sector of many gauged supergravity theories, we include an $SU(2)$ non-Abelian field in the action (Abelian solutions of this theory are discussed in [13], however for a particular set of boundary conditions which implies a finite ADM mass). We suppose that the dilaton field approaches asymptotically a constant value $\phi_0$, which corresponds to an extremum of the potential such that $dV/d\phi|_{\phi_0} = 0$ and $V(\phi_0) < 0$. Therefore the configurations present an effective negative cosmological constant.

Since a negative cosmological constant allows for the existence of black holes whose horizon has nontrivial topology, we consider apart from spherically symmetric solutions, topological black holes also. The mass of these solutions is computed by using a counterterm method.

Although we discuss a general case, assuming only the existence of a power series expansion at infinity, numerical results are presented mainly for the case of the $\mathcal{N} = 4$, $D = 4$ gauged supergravity model of Gates and Zwiebach [14].

The paper is structured as follows: in the following section we explain the model and derive the basic equations and the asymptotic form of the solutions. The boundary stress tensor and the associated conserved charge are computed in section 3, by using a counterterm prescription adapted to our case. The counterterm choice is tested for the MTZ black-hole solution. In section 4 we present the numerical results, the case of regular solutions being also briefly discussed. We conclude with section 5 where the results are compiled.

2. General framework and equations of motion

2.1. Basic ansatz and field equations

We start with the following action principle:

\[
I = \int_M d^4x \sqrt{-g} \left( \frac{R}{16\pi G} - \frac{1}{2} e^{2\phi} Tr(F_{MN}F^{MN}) - \frac{1}{2} \partial_M \phi \partial^M \phi - V(\phi) \right)
\]
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\[ -\frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{-h} K, \]  

(1)

where \( G \) is the gravitational constant, \( R \) is the Ricci scalar associated with the spacetime metric \( g_{MN} \). \( F_{MN} = \frac{1}{2} \tau^a F^a_{MN} \) is the gauge field strength tensor defined as

\[ F_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N], \]  

(2)

where the gauge field is \( A_M = \frac{1}{2\pi} \tau^a A^a_M, \tau^a \) being the Pauli matrices. The constant \( a \) governs the coupling of \( \phi \) to the non-Abelian field while \( g \) is the gauge coupling constant. The last term in (1) is the Hawking–Gibbons surface term [15], where \( K \) is the trace of the extrinsic curvature for the boundary \( \partial M \) and \( h \) is the induced metric of the boundary.

The field equations are obtained by varying action (1) with respect to the field variables \( g_{MN}, A_M \) and \( \phi \)

\[ R_{MN} - \frac{1}{2} g_{MN} R = 8\pi G T_{MN} \]

\[ \nabla^2 \phi - a e^{2\phi} Tr(F_{MN} F^{MN}) - \frac{\partial V}{\partial \phi} = 0, \]  

(3)

\[ \nabla_M (e^{2\phi} F^{MN}) - ig e^{2\phi} [A_M, F^{MN}] = 0, \]  

(4)

where the energy–momentum tensor is defined by

\[ T_{MN} = \partial_M \phi \partial_N \phi - \frac{1}{2} g_{MN} \partial_P \phi \partial^P \phi - g_{MN} V(\phi) + 2 e^{2\phi} Tr(F_{MP} F_{NQP} g^{PQ} - \frac{1}{4} g_{MN} F_{PQ} F^{PQ}). \]

Since for a negative cosmological constant topological black holes may appear (whose topology of the event horizon is no longer the two-sphere \( S^2 \)), we consider a general metric ansatz

\[ ds^2 = \frac{dr^2}{H(r)} + r^2 d\Omega_k^2 - \sigma^2(r) H(r) dt^2, \]  

(5)

where \( d\Omega_k^2 = d\theta^2 + f^2(\theta) d\phi^2 \) is the metric on a two-dimensional surface of constant curvature \( 2k \). The discrete parameter \( k \) takes the values 1, 0 and \(-1 \) and implies the form of the function \( f(\theta) \)

\[ f(\theta) = \begin{cases} 
\sin \theta, & \text{for } k = 1 \\
\theta, & \text{for } k = 0 \\
\sinh \theta, & \text{for } k = -1.
\end{cases} \]  

(6)

When \( k = 1 \), the metric takes on the familiar spherically symmetric form, for \( k = -1 \) the \((\theta, \phi)\) sector is a space with constant negative curvature, while for \( k = 0 \) this is a flat surface (see, e.g. the discussion in [16]).

Taking into account the symmetries of the line element (5), we find the expression of the purely magnetic YM ansatz [17]

\[ A = \frac{1}{2g} \left[ \omega(r) \tau_1 d\theta + \left( \frac{d\ln f}{d\theta} \tau_3 + \omega(r) \tau_2 \right) f d\phi \right], \]  

(7)

which gives the YM curvature

\[ F = \frac{1}{2g} \left[ \omega' \tau_1 dr \wedge d\theta + f \omega' \tau_3 dr \wedge d\phi + (w^2 - k) f \tau_3 d\theta \wedge d\phi \right], \]  

(8)

where a prime denotes a derivative with respect to \( r \).
Inserting this ansatz into action (1), the field equations reduce to
\[ \sigma' = \frac{8\pi G\sigma}{r} \left( \frac{e^{2\phi}}{g^2} \phi'^2 + \frac{1}{2} \phi'^2 r^2 \right), \]
\[ r H' = k - H - 8\pi G \left( \frac{e^{2\phi}}{g^2} \left( \frac{\omega'^2}{2r^2} + \frac{(\omega^2 - k)^2}{2r^2} \right) + \frac{r^2}{2} H \phi'^2 + V(\phi) r^2 \right), \]
\[ (\sigma e^{2\phi} H \omega')' = \sigma e^{2\phi} \frac{\omega(\omega^2 - k)}{r^2}, \]
\[ (H r^2 \sigma ') = 2a\sigma e^{2\phi} g^2 \left( \frac{\omega'^2}{2r^2} + \frac{(\omega^2 - k)^2}{2r^2} \right) + \frac{\partial V}{\partial \phi} r^2 \sigma. \]
(9)

2.2. Asymptotic expansion

We assume that the solution of the above equations admits at large \( r \) a power series expansion of the form
\[ \phi = \sum_{i=0}^{\infty} \phi_i r^{-i}, \quad \omega = \sum_{i=0}^{\infty} \omega_i r^{-i}, \quad H = h_0 r^2 + \sum_{i=2}^{\infty} h_i r^{-i+2}, \quad \sigma = \sum_{i=0}^{\infty} \sigma_i r^{-i}. \]
(10)

From the lowest order term in the equation of \( \phi \) we find that \( V''_0 = 0 \) (we shall note \( V^{(k)}_0 = V^{(k)}(\phi_0) \)). We remark that, by using a suitable redefinition, we can always set \( \phi_0 = 0 \), with no loss of generality. The effective cosmological constant is
\[ \Lambda_{\text{eff}} = 8\pi G V_0 = -\frac{3}{\ell^2}. \]
(11)

The generic solution has \( \lim_{r \to \infty} r^2 \phi' \neq 0 \), which, from the field equations, implies the following consistency conditions on the dilaton potential:
\[ V'''_0 = -\frac{2}{\ell^2}, \quad V''_0 = 0. \]
(12)

Note that the scalar field mass \( m^2 = V''_0 \) is larger than the Breitenlohner–Freedman bound \( m^2 = -9/4\ell^2 \).

Assumption (10) leads to the asymptotic expansion at large \( r \)
\[ H = k + \frac{4\pi G \phi_1}{\ell^2} - \frac{2M_0}{r} + \frac{r^2}{\ell^2} + O(1/r^2), \quad \sigma = 1 - \frac{2\pi G \phi_1}{r^2} - \frac{16\pi G \phi_1 \phi_2}{3r^3} + O(1/r^4), \]
\[ \omega = \omega_0 + \frac{\omega_1}{r} + \frac{\ell^2 \omega_0 (\omega_0^2 - k) - 2a \phi_1 \omega}{2r^2} + O(1/r^3), \quad \phi = \phi_0 + \frac{\phi_1}{r} + \frac{\phi_2}{r^2} + O(1/r^3), \]
(13)

where \( M_0, \omega_0, \omega_1, \phi_0, \phi_1, \phi_2 \) are arbitrary constants, which implies the asymptotic form of the metric function
\[ -g_{tt} = k - \frac{2M + 32\pi G \phi_1 \phi_2}{3\ell^2} + \frac{r^2}{\ell^2} + O(1/r^2), \]
\[ g_{rr} = \frac{-\ell^2}{r^2} + \frac{1}{r^2} \left( 4\pi G \phi_1^2 + k \ell^2 \right) + \frac{2M \ell^2}{r^5} + O(1/r^6). \]

For any \( \phi_1, \phi_2 \), this set of asymptotics preserve the full AdS symmetry group. Different from the asymptotically flat case, there are no obvious restrictions on the value of \( \omega_0 \).
3. A computation of mass

In order to compute quantities like the action and mass one usually encounters infrared divergences, associated with the infinite volume of the spacetime manifold. The traditional approach to this problem is to use a background subtraction whose asymptotic geometry matches that of the solutions. However, this approach breaks down when there is no appropriate or obvious background.

In the AdS/CFT inspired counterterm method, this problem is solved by adding additional surface terms to the theory action. These counterterms are built up with curvature invariants of a boundary $\partial M$ (which is sent to infinity after the integration) and thus obviously they do not alter the bulk equations of motion. This yields a well-defined boundary stress tensor and a finite action and mass of the system.

As found in [18], the following counterterms are sufficient to cancel divergences in four dimensions, for vacuum solutions with a negative cosmological constant $\Lambda = -3/\ell^2$:

$$ I^0_{ct} = -\frac{1}{8\pi G} \int_{\partial M} d^3 x \sqrt{-h} \left[ \frac{2}{\ell} + \frac{\ell}{2} R \right], \quad (14) $$

where $R$ is the Ricci scalar for the boundary metric $h$.

Using these counterterms one can construct a divergence-free boundary stress tensor $T_{\mu\nu}$ from the total action $I = I_{\text{bulk}} + I_{\text{surf}} + I^0_{ct}$ by defining

$$ T_{\mu\nu} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h^{\mu\nu}} = \frac{1}{8\pi G} \left( K_{\mu\nu} - K h_{\mu\nu} - \frac{2}{\ell} h_{\mu\nu} + \ell E_{\mu\nu} \right), \quad (15) $$

where $E_{\mu\nu}$ is the Einstein tensor of the boundary metric, $K_{\mu\nu} = -1/2(\nabla_{\mu} n_{\nu} + \nabla_{\nu} n_{\mu})$ is the extrinsic curvature, $n^M$ being an outward pointing normal vector to the boundary.

If $\xi^\mu$ is a Killing vector generating an isometry of the boundary geometry, there should be an associated conserved charge. We suppose that the boundary geometry is foliated by spacelike surfaces $\Sigma$ with metric $\sigma_{ab}$

$$ h_{\mu\nu} \, dx^\mu \, dx^\nu = -N_\Sigma^2 \, dt^2 + \sigma_{ab}(dx^a + N_a^b \, dt)(dx^b + N_b^b \, dt). \quad (16) $$

Thus the conserved charge associated with time translation $\partial/\partial t$ is the mass of spacetime

$$ M = \int_{\Sigma} d^2 x \sqrt{N} \epsilon. \quad (17) $$

Here $\epsilon = \kappa^a u^b T_{ab}$ is the proper energy density while $u^a$ is a timelike unit normal to $\Sigma$.

The presence of the additional matter fields in (1) brings the potential danger of having divergent contributions coming from both the gravitational and matter action [19]. For a $1/r^2$ (or faster) decay of the dilaton field in the asymptotic region, we find that prescription (14) (with $\ell^2 = -3/(8\pi G V_0)$) removes all divergences of the total action, and implies the usual configurations mass $M = \frac{V}{4\pi \ell^2} M_0$, $V$ being the area of the $(\theta, \phi)$ surface ($V = 4\pi$ for $k = 1$).

As expected, a dilaton field which behaves asymptotically as $O(1/r)$ necessarily contributes to the action and its variations in the asymptotic region. The counterterms (14) will not yield in this case a finite action or mass. However, similar to the case of three-dimensional gravity with a minimally coupled scalar field [20], it is still possible to obtain a finite mass by allowing $I_{ct}$ to depend not only on the boundary metric $h_{\mu\nu}$, but also on the scalar field. This means that the quasilocal stress-energy tensor (15) also acquires a contribution coming from the matter field.
Since a polynomial in $\phi$ does not remove the divergencies, we are forced to consider terms containing the normal derivative of $\phi$. Following [20], we find that by adding a counterterm of the form
\[
I^{(\phi)}_{ct} = \frac{1}{3} \int d^3 x \sqrt{-\gamma} \left( \phi n^M \partial_M \phi + \frac{\ell}{4} m^2 \phi^2 \right) \frac{1}{3} \int d^3 x \sqrt{-\gamma} \left( \phi n^M \partial_M \phi - \frac{1}{2\ell} \phi^2 \right) \quad (18)
\]
to expression (14), the divergence disappears.

This yields a supplementary contribution to (15), $T_{ab}^{(\phi)} = \frac{1}{3} g_{ab}(\phi n^M \partial_M \phi - \phi^2 / (2\ell))$. The nonvanishing components of the resulting boundary stress tensor are (here we choose $\partial_M$ to be a three surface of fixed $r$, while $n_M = \sqrt{g_{rr}} \delta r_M$)
\[
T_{\theta \theta} = T_{\phi \phi} = \left( \frac{2}{3} \phi \frac{\phi_2}{\ell} + \frac{M_0}{8\pi G} \right) \frac{1}{r^3} + O\left( \frac{1}{r^4} \right), \quad T_{t t} = \left( -\frac{4}{3} \phi \frac{\phi_2}{\ell} - \frac{2M_0}{8\pi G} \right) \frac{1}{r^3} + O\left( \frac{1}{r^4} \right).
\]

We remark that, to leading order, this stress tensor is traceless as expected from the AdS/CFT correspondence, since even dimensional bulk theories are dual to odd dimensional CFTs which have a vanishing trace anomaly. Employing the AdS/CFT correspondence, this result can be interpreted as the expectation value of the stress tensor in the boundary CFT [21].

The mass of these solutions, as computed from (17) is
\[
M = \sqrt{\ell} \left( \frac{M_0}{4\pi G} + \frac{4\phi_1 \phi_2}{3\ell^2} \right). \quad (19)
\]
This coincides with the mass of the $N = 8$, $D = 4$ gauged supergravity solutions considered in [8], as computed by using a Hamiltonian method (the asymptotics of those solutions is a particular case of (13)). Also, it can be proven that the above counterterm choice yields a finite Euclidean action.

3.1. Testing the counterterms with the MTZ exact solution

Recently Martinez, Troncoso and Zanelli have found an exact black-hole solution of the field equations (3), corresponding to a $F_{\mu \nu} = 0$ truncation of the action (1) and a scalar potential [6]^{4}
\[
V(\phi) = V_0 \left( 1 + 2 \sinh^2 \frac{4\pi G}{3} \phi \right) = -\frac{3}{8\pi G \ell^2} \left( 1 + 2 \sinh^2 \frac{4\pi G}{3} \phi \right). \quad (20)
\]
The metric and the scalar field are given by
\[
d s^2 = \frac{r(r + 2G\mu)}{r + G\mu} \left[ \frac{dr^2}{r^2 - \left( 1 + \frac{G\mu}{r} \right)^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - \left( \frac{r^2}{\ell^2} - \left( 1 + \frac{G\mu}{r} \right)^2 \right) dr^2 \right],
\]
\[
\phi = \sqrt{\frac{3}{4\pi G}} \arctanh \frac{G\mu}{r + G\mu}.
\]
(21)
The only singularities of the curvature and the scalar field occur at $r = 0$ and at $r = -2G\mu$. These singularities are surrounded by an event horizon located at
\[
r_* = \frac{\ell}{2} \left( 1 + \sqrt{1 + 4G\mu / \ell} \right),
\]
3 Note that one can replace the term $\phi n^M \partial_M \phi$ in (18) with a suitable combination of $(n^M \partial_M \phi)^2$ and $m^2 \phi^2$, in which case the matter counterterm reads $I^{(\phi)}_{ct} = -\frac{\ell}{3} \int d^3 x \sqrt{-\gamma} (n^M \partial_M \phi)^2 - m^2 \phi^2$, without changing the expressions for boundary stress-tensor, mass and total action.
4 See also [22, 23] for other recent examples of scalar hairy black holes and solitons.
while the Hawking temperature is \( T_H = \beta^{-1} = (2r_c/\ell - 1)/(2\pi \ell) \). A straightforward computation yields the nonvanishing components of boundary stress tensor
\[
T_\theta^\theta = T_\phi^\phi = \frac{\mu \ell}{8\pi r^3} + O\left(\frac{1}{r^4}\right), \quad T_t^t = - \frac{\mu \ell}{4\pi r^3} + O\left(\frac{1}{r^4}\right).
\]
Thus, from (17) we find a total black-hole mass
\[
M = \frac{\mathcal{V}}{4\pi \mu}, \quad (22)
\]
while the total Euclidean action is
\[
I = \frac{\beta \mathcal{V}}{8\pi G} (2G\mu + \ell), \quad (23)
\]
which, from
\[
S = \beta M - I \quad (24)
\]
gives an entropy which is one quarter of the event horizon area \( A_H \), the first law of thermodynamics being also satisfied.

These results coincide with those found in [6] by using a Hamiltonian formalism.

4. Numerical solutions

To solve the field equation (9), we change to dimensionless variables by using the rescaling
\[
r \to \sqrt{\frac{4\pi G}{g}} r, \quad \phi \to \sqrt{\frac{1}{4\pi G}} \phi, \quad a \to \sqrt{\frac{4\pi G}{a}} a,
\]
and a rescaling of the potential.

4.1. The form of the dilaton potential

Given the inherent difficulties involved in studies of these models, we will restrict ourselves to the case of a potential on the form
\[
V(\phi) = C_1 e^{2\alpha_1 \phi} + C_2 e^{2\alpha_2 \phi} + C_3, \quad (25)
\]
where we suppose \( \alpha_1 \neq \alpha_2 \). This type of potential can be obtained when a higher dimensional theory is compactified to four dimensions, including various supergravity models (see [24] for a recent discussion of these aspects).

For \( -\alpha_1 = \alpha_2 = \alpha, \quad C_1 = -1/8, \quad C_2 = -\xi^2/8, \quad C_3 = -\xi/2 \), this is the dilaton potential appearing in the \( \mathcal{N} = 4 \) gauged supergravity model of Gates and Zwiebach. For \( a = 1 \), action (1) is also a consistent truncation of the bosonic sector of this model. Spherically symmetric BPS regular solutions of this theory with the general asymptotics (13) have been constructed recently in [25].

Although a Liouville-type potential plus a cosmological constant (obtained for \( C_1 \) or \( C_2 = 0 \)) might be an interesting choice, one can prove that such a model does not possess solutions with AdS asymptotics.

For a nonzero \( 1/r \) term in the asymptotic expansion of the dilaton field, conditions (12) impose the following relations between the potential parameters:
\[
-\alpha_1 = \alpha_2 = \alpha, \quad C_2 = C_1 e^{-4\alpha_0}, \quad C_3 = 2C_1 e^{-2\alpha_0} (3\alpha^2 - 1). \quad (26)
\]
By using the scaling properties of the system \( \phi \to \phi + \phi_0, \quad r \to r e^{2\alpha_0} \) we can always set \( \phi_0 = 0 \), resulting in the simple potential
\[
V(\phi) = C (\sinh^2 \alpha \phi + \frac{3}{2} \alpha^2), \quad (27)
\]
where \( C = 4C_1 e^{2\alpha_0 (\alpha - \alpha_0)} \).
We observe that the potential of the GZ model can also be written in this form (for $\xi > 0$ i.e. a negative effective cosmological constant), with $C = -1/2$ and $\alpha = 1$. In this case, following the approach in [25], we find the set of Bogomolnyi equations

$$\phi' = -\frac{r}{2H} F_1 F_2, \quad \omega' = e^{-\phi} \frac{\sigma \omega}{2H} F_2,$$

$$H = \omega^2 + \frac{r^2}{2} F_2^2, \quad e^\phi \omega \sigma = \text{const.},$$

$$\frac{r e^\phi F_1}{2H} = \frac{H'}{2H} + \frac{\sigma'}{\sigma} - \phi'.$$

(28)

(29)

(30)

(with $F_1 = \cosh \phi + e^{\alpha (k - \omega^2) / r^2}$, $F_2 = \sinh \phi + e^{\alpha (k - \omega^2) / r^2}$), equation (30) being a differential consequence of the first four equations. One can verify that these Bogomolnyi equations are compatible with equations (9). Also, there are no black-hole solutions of the above equations.

Numerical arguments for the existence of $k = 1$ regular solutions are presented in [25]. The $k = 0, -1$ configurations preserving any supersymmetry present naked singularities. For example, a $k = 0$ solution of the above equations is (with $c$ an arbitrary constant)

$$ds^2 = \frac{dr^2}{r^2/2 + c^2} + r^2(d\theta^2 + \theta^2 d\phi^2) - r^2 dt^2,$$

$$\phi(r) = \arcsinh \sqrt{2c}, \quad \omega(r) = 0, \quad (31)$$

presenting a naked singularity at $r = 0$.

4.2. Black-hole solutions

We are interested in black-hole solutions having a regular event horizon at $r = r_h > 0$. The field equations implies the following behaviour as $r \to r_h$ in terms of three parameters ($\phi_h$, $\sigma_h$, $\omega_h$):

$$H(r) = \left( k - e^{2\alpha \phi_h} \frac{\omega_h^2 - k}{r_h^2} - 2V(\phi_h) r_h \right) (r - r_h) + O(r - r_h)^2,$$

$$\sigma(r) = \sigma_h + \frac{2\alpha \phi_h}{r_h} \left( e^{2\alpha \phi_h} \omega_h^2 (r_h) + \frac{1}{2} \omega_h^2 r_h^2 \right) (r - r_h) + O(r - r_h)^2,$$

$$\omega(r) = \omega_h + \frac{1}{H'(r_h) r_h^2} \left( 2a e^{2\alpha \phi_h} \left( \frac{\omega_h^2 - k}{r_h^2} \right)^2 + \frac{\partial V}{\partial \phi_h} \right) (r - r_h) + O(r - r_h)^2.$$  

(32)

The condition for a regular event horizon is $H'(r_h) > 0$, which places a bound on $\omega_h$

$$\frac{e^{2\alpha \phi_h}}{r_h^2} \left( \omega_h^2 - k \right)^2 < k - 2V(\phi_h) r_h^2,$$

(33)

and implies the positiveness of the quantity $\omega'(r_h)$. In the $k = -1$ case, (34) implies the existence of a minimal value of $|V(\phi_h)|$, i.e. for a given $r_h$

$$|V(\phi_h)| > \frac{1}{r_h^2} \left( 1 + \frac{e^{2\alpha \phi_h}}{r_h^2} \right).$$
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Figure 1. The functions $H(r)$, $\sigma(r)$, $\omega(r)$ and $\phi(r)$ are plotted for two $k = 1$ typical black-hole solutions.

By going to the Euclidean section (or by computing the surface gravity) one finds the Hawking temperature

$$T_H = \frac{1}{\beta} = \frac{\sigma(r_h)}{4\pi}.$$  \hspace{1cm} (34)

Using the initial conditions on the event horizon (32), the equations were integrated for $a = \alpha = 1$, several values of $C$, and a large set of $\omega_h, \phi_h$. Also, although we present here only the case $r_h = 1$, similar solutions seem to exist for any value of $r_h$. Since equations (9) are invariant under the transformation $\omega \rightarrow -\omega$, only values of $\omega_h \geq 0$ are considered. Nontrivial black-hole solutions with scalar hair exist also in the absence of a gauge field$^5$.

The basic properties of these solutions are very similar to the known EYM-Lambda black holes \cite{3, 4, 17}. For any $\phi_h$ and $C < 0$, solutions appear for continuous intervals of $\omega_h$ (for $C = -1/2$, we find always only one interval). For small $|V_0|$, these intervals are separated by intervals on which there are no solutions with asymptotics (13). As $\omega_h$ approaches some critical value $\omega_c$, the metric function $\sigma(r)$ approaches a zero value on the event horizon. The value of $\omega_c$ increases as $|V_0|$ increases. Also, there are black-hole solutions for which $\omega_0 > 1$ although $\omega_h < 1$. Typical spherically symmetric solutions of the GZ model are presented in figure 1.

For $k = 0$, $-1$, in contrast to the spherically symmetric case, we find only nodeless solutions, for all values of the parameters. This can be analytically proven by integrating the equation for $\omega$, $\sigma e^{2\omega_0} H \omega' = \sigma e^{2\omega_0} \omega (\omega^2 - k)/r^2$ between $r_h$ and $r$; thus obtaining $\omega' > 0$ for every $r > r_h$. For $k = 1$ and $|V_0|$ sufficiently large (i.e. $|V_0| > 0.01$), there also exist solutions for which the gauge function $\omega$ has no nodes. Increasing the value of $|V_0|$, the ratio $\omega_0/\omega_h$ remains close to one for most of the $\omega_h$ interval, and we find nodeless solutions only.

$^5$ For example, for $k = -1$ and the dilaton potential (20) we have found a family of solutions in terms of $\phi_h$, the exact MTZ solution (21) being a particular case.
The asymptotic parameters $\omega_0$, $M_0$, $\phi_1$, the total mass $M$, the Hawking temperature $T_H$ and the value $\sigma_h$ of the metric function $\sigma(r)$ at the event horizon are shown as a function of $\omega(r_h)$ for spherically symmetric and $k = -1$ topological black-hole solutions of the GZ model.

The properties of typical solutions are presented in figure 2 for $k = 1, -1$ and a GZ potential (a similar picture is found for $k = 0$). We observe that the generic solutions we find do not have the usual AdS asymptotics since $\phi_1 = -\lim_{r \to \infty} r^2 \phi' \neq 0$. However, for any $(V_0, \omega_h)$, solutions with $\phi_1 = 0$ exist for a discrete set of $\phi_h$. 
Both the parameter $M_0$ and the total mass $M$ of some $k = -1$ solutions are negative, a common situation in the topological black-hole physics. For $k = 1, 0$ we find $M > 0$ for all boundary conditions we have considered. It can be proven that the entropy of these black holes is one quarter of the event horizon area, as expected. By integrating the Killing identity $\nabla^a \nabla_b K_a = R_{bc} K^c$, for the Killing field $K^a = \delta^a_t$, together with the Einstein equation (here we have set $4\pi G = 1$)

$$\frac{1}{2} R_i^j = \frac{R}{4} - \frac{1}{2} e^{2a\phi} Tr(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{2} \partial_{\mu} \phi \partial^\mu \phi - V(\phi),$$

it is possible to isolate the bulk action contribution at infinity and on the event horizon. The counterterms discussed in section 3 regularize the infrared divergencies, such that the contribution from the asymptotic region to the total action is found to be $V(\beta(M_0 + 4\phi_1\phi_2/(3\ell^2))$, the relation $S = A_H/4$ resulting straightforwardly from (25), (35).

### 4.3. Regular solutions

The existence of regular counterparts of the black-hole solutions is possible in the spherically symmetric case only. For $k \neq 1$, a direct inspection of system (9) reveals the absence of solutions with a regular origin. In this case, it is not possible to take a consistent set of boundary conditions at the origin without introducing a curvature singularity at $r = 0$. This fact has to be attributed to the particular form of the potential term $V(YM) = e^{2a\phi}(\omega^2 - k^2)/(2r^2)$, which is fourth order in the YM function $\omega$, in the reduced Lagrangian of the system. We observe the similarity with the EYM system with $\Lambda > 0$, where the absence of $k \neq 1$ regular configurations has been noted in [17].

For completeness we present here the basic properties of the $k = 1$ regular configurations. The behaviour of regular solutions near the origin is

$$H(r) = 1 - \left(4b^2 e^{2a\phi} + \frac{2}{3} V(\phi_0)\right)r^2 + O(r^3), \quad \sigma(r) = \sigma_0(1 + 4 e^{2a\phi} b^2)r^2 + O(r^3),$$

$$\omega(r) = 1 - br^2 + O(r^4), \quad \phi(r) = \phi_1 + \left(2a e^{2a\phi} b^2 + \frac{dV}{d\phi}(\phi_1)\right)r^2 + O(r^3),$$

(36)

where $b, \sigma_0$ and $\phi_1$ are arbitrary parameters. We note that for BPS solutions of the GZ model, there are only two independent parameters since in this case $b = 1/6 e^{-\phi_0} \sinh \phi_0$.

The overall picture we find is similar to that described in [4] for the EYM-$\Lambda$ system (here we consider again the case of the GZ model). By varying the parameters $b, \phi_0$, a continuum of monopole solutions is obtained. As $b$ increases, the value at the origin of the metric function $\sigma(r)$ decreases and, for some finite values of $b$, a singularity appears. The total mass of the solutions as given by (19) is an increasing function of $b$. For any value of $b$ it seems to be always possible to find a initial value of the scalar field such that $\phi_1 = 0$.

As seen in figure 3, the solution with $b = 0$ (i.e. $\omega(r) \equiv 1$) is not the vacuum AdS spacetime. Thus, rather unexpectedly, for these asymptotics there are regular solutions even without a non-Abelian field. For the GZ model, we find a continuum of (nonsupersymmetric-) scalar solitons, as a function of the value of the scalar field at the origin $\phi_1$. A similar property has been noted recently for a truncation of $\mathcal{N} = 8, D = 4$ gauged supergravity [10].

The expansion as $r \to \infty$ is valid also for these regular solutions. For the GZ model, the coefficients of BPS solutions in asymptotics (13) satisfy the relations $\phi_2 = k - \omega_0^2$, $M = \phi_1(\omega_0^2 - k)$, $\omega_0 = \omega_0(\phi_1$ (we recall that for $k \neq 1$, these describe configurations with a
The asymptotic parameters $\omega_0, M_0, \phi_1$, the total mass $M$ and the value $\sigma_0$ of the metric function $\sigma(r)$ at the origin are shown as a function of $b$ for typical spherically symmetric regular solutions of the GZ model.

We note also the intriguing expression for the total mass of BPS configurations
\[ M_{\text{BPS}} = \frac{1}{3} Q_D Q_M, \]
with $Q_D = -\phi_1, Q_M = k - \omega_0^2$.

5. Conclusions

If the gravitating matter fields do not fall off sufficiently fast at infinity, the asymptotic behaviour of the metric can be different from that in pure gravity. By relaxing the standard asymptotic conditions for AAdS solutions, it is possible to preserve the original symmetries at infinity, while the conserved charges are modified by including matter field terms.

The aim of this paper was to consider this situation for a theory including, apart from an $SU(2)$ non-Abelian field, a dilaton field with a nontrivial potential, playing the role of a cosmological term. We have found that the addition of the scalar potential greatly increases the wealth of possible solutions, preserving at the same time all features familiar from the EYM-Λ case. For solutions admitting an asymptotic power series expansion, we have proposed a counterterm choice which gives finite mass and Euclidean action. The results we have found are in agreement with those obtained via the Hamiltonian method.

Numerical solutions have been presented mainly for the case of a consistent truncation of $N = 4, D = 4$ GZ gauged supergravity model. Both regular and black-hole solutions have been presented, the solutions with a finite ADM mass constituting a discrete set. By using the relations in [26], we can uplift these configurations to $D = 11$, which may suggest a holographic interpretation for them. As observed in [8], according to the general AdS/CFT correspondence, there should be a dual CFT corresponding to each choice of boundary conditions.
New hairy black-hole solutions with a dilaton potential

Similar to the EYM-Λ case, we expect some of the solutions with no nodes in the non-Abelian magnetic field to be stable against linear perturbations. Also, it is possible to relax the asymptotic assumptions (10), allowing a generic noninteger decay at infinity. In this case, apart from $V_0 = -3/(8\pi G\ell^2)$, $V'_0 = 0$ there are no other restrictions on the dilaton potential and we find the asymptotic behaviour (with $m^2 = V''_0 < 0$)

$$H = k - \frac{2M_0}{r} + \frac{r^2}{\ell^2} + f(m, \ell, \phi_1) r^{\lambda_+ - \lambda_- - 1} + \cdots, \quad \sigma = \exp \left(2\pi G\lambda_- \phi_1^2 / r^{2\lambda_-} \right) + \cdots,$$

$$\phi = \frac{\phi_1}{r^{\lambda_-}} + \frac{\phi_2}{r^{\lambda_+}} + \cdots, \quad \omega = \omega_0 + \frac{\omega_1}{r} + \cdots,$$

(38)

where

$$\lambda_\pm = \frac{3 \pm \sqrt{9 + 4m^2\ell^2}}{2}.$$

$$f(m, \ell, \phi_1) = -4\pi G\phi_1^2 m^2 + \frac{\lambda_+^2}{\lambda_+ - \lambda_-} \ell^2.$$

(39)

Both regular and black-hole solutions with these asymptotics are likely to exist. It can be proven that the counterterm choice (18) gives a finite mass and action also in this case, yielding very similar results to those derived in this paper for $\lambda_- = 1$, $\lambda_+ = 2$.

It would be interesting to generalize these solutions to higher dimensions and to find the general matter counterterms expression.

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