Knot Floer homology and fixed points

Yi NI
Department of Mathematics, Caltech, MC 253-37
1200 E California Blvd, Pasadena, CA 91125
Email: yini@caltech.edu

Abstract

If \( K \) is a fibered knot in a closed, oriented 3–manifold \( Y \) with fiber \( F \), and \( \widehat{HF}(Y, K, [F], g(F) - 1; \mathbb{Z}/2\mathbb{Z}) \) has rank \( r \), then the monodromy of \( K \) is freely isotopic to a diffeomorphism with at most \( r - 1 \) fixed points. This generalizes earlier work of Baldwin–Hu–Sivek and Ni.

We also clarify a misleading formula in Cotton-Clay’s computation of the symplectic Floer homology of mapping classes of surfaces.

1 Introduction

Knot Floer homology, defined by Ozsváth–Szabó [17] and Rasmussen [19], is a powerful knot invariant with many applications. One important property of knot Floer homology is that the Seifert genus \( g(K) \) of a knot \( K \subset S^3 \) can be read from the knot Floer homology [18]:

\[
g(K) = \max \{ a \mid \widehat{HF}(S^3, K, a) \neq 0 \}.\]

It is also known that \( K \) is fibered if and only if \( \widehat{HF}(S^3, K, g(K)) \cong \mathbb{Z} \) [5,14].

It is an interesting question to ask what topological property about \( K \) can be deduced from other summands of \( \widehat{HF}(S^3, K) \). In [1], Baldwin–Hu–Sivek proved that if a hyperbolic knot \( K \subset S^3 \) has the same knot Floer homology as the torus knot \( T(5, 2) \), then the monodromy of \( K \) is freely isotopic to a pseudo-Anosov map without fixed point. It is observed in [15] that their argument can be applied to hyperbolic fibered knot in any closed 3–manifold. In fact, the argument in [15] implies the following upper bound on the number of fixed points.

**Theorem 1.1.** Let \( Y \) be a closed, oriented 3–manifold, and \( K \subset Y \) be a hyperbolic fibered knot with fiber \( F \) and monodromy \( \varphi \). If

\[
\text{rank} \widehat{HF}(Y, K, [F], g(F) - 1) = r,
\]

then \( \varphi \) is freely isotopic to a pseudo-Anosov map with at most \( r - 1 \) fixed points.
The main theorem in this paper removes the hyperbolic condition in the above result. For technical reasons, we will use coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

**Theorem 1.2.** Let $Y$ be a closed, oriented 3–manifold, and $K \subset Y$ be a fibered knot with fiber $F$ and monodromy $\varphi$. If 
\[
\text{rank} \widehat{HF}(Y, K, [F], g(F) - 1; \mathbb{F}) = r,
\]
then $\varphi$ is freely isotopic to a diffeomorphism with at most $r - 1$ fixed points.

The proof of Theorem 1.2 uses the same strategy as in [1,15]. It also follows from unpublished work of Ghiggini and Spano [6].

One important ingredient of the proof is Cotton-Clay’s computation of the symplectic Floer homology of a mapping class [2], which is based on the work of Jiang–Guo [10] and Gautschi [4]. There was, however, a gap in the statement of his computation. Below, we will state the theorem in a less confusing way. The meaning of the terms in the formula, as well as the explanation of the mistake, will be given in Section 3. This gap does not affect the argument in [1] and [15].

**Theorem 1.3.** Let $\hat{\varphi} : \Sigma \to \Sigma$ be a perturbed standard form map in a reducible mapping class $h \in \Gamma$. Then
\[
HF_{\ast}(h) = HF_{\ast}(\hat{\varphi}) \cong H_{\ast}(\Sigma, \partial \Sigma; \mathbb{F})
\]
\[
\oplus \bigoplus_p \left( H_{\ast}(\Sigma, \partial \Sigma; \mathbb{F}) \oplus \mathbb{F}^{\text{rank}(\hat{\varphi}|_{\Sigma_0})} \right)
\]
\[
\oplus \bigoplus_q \left( H_{\ast}(\Sigma, \partial \Sigma; \mathbb{F}) \oplus \mathbb{F}^{\text{rank}(\hat{\varphi}|_{\Sigma_1})} \right)
\]
\[
\oplus \mathbb{F}^{\lambda(\hat{\varphi}|_{\Sigma_1})} \oplus \mathbb{F}^{2n_\varphi} \oplus CF_{\ast}(\hat{\varphi}|_{\Sigma_2}).
\]  
(1)

The difference between (1) and the formula in [2, Theorem 4.16] is that there is an extra term $\mathbb{F}^{2n_\varphi}$ in (1), where $n_\varphi$ is the number of annuli on which the restriction of $\hat{\varphi}$ is a flip-twist map. See also Remark 3.1 for another interpretation of [2, Theorem 4.16] which yields the correct result.

This paper is organized as follows. In Section 2 we review some results in Nielsen fixed point theory. In Section 3 we review Cotton-Clay’s work on symplectic Floer homology of mapping classes, and explain how we get Theorem 1.3. In Section 4 we use the zero-surgery formula in Heegaard Floer homology to prove Theorem 1.2.

**Acknowledgements.** The author was partially supported by NSF grant number DMS-1811900. We are grateful to John Baldwin, Andrew Cotton-Clay, Ko Honda, and Steven Sivek for helpful comments.

## 2 Nielsen theory of surfaces

In this section, we will give a brief introduction to Nielsen theory following Jiang [9]. Then we will review Jiang–Guo’s work [10] on the Nielsen problem.
for surface diffeomorphisms.

**Definition 2.1.** Let $X$ be a topological space, $f : X \to X$ be a continuous map. We say two fixed points $a_0, a_1 \in \text{Fix}(f)$ are *Nielsen equivalent*, if there exists a path $\gamma$ from $a_0$ to $a_1$, such that $\gamma$ is homotopic to $f(\gamma)$ rel $\partial$. A *Nielsen class* is the collection of all fixed points which are Nielsen equivalent to a given fixed point.

If $C$ is a Nielsen class, then $C$ is an open set in $\text{Fix}(f)$ [9, Theorem 1.12].

**Definition 2.2.** Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^n$ be a continuous map, $a \in \text{Fix}(f)$ be an isolated fixed point. Let $B_\varepsilon(a) \subset U$ be a small open ball about $a$ satisfying $B_\varepsilon(a) \cap \text{Fix}(f) = \{a\}$. Then the *fixed point index* of $a$, denoted by $\text{index}(f,a)$, is the degree of the map $\varphi : \partial B_\varepsilon(a) \to S^{n-1}$ defined by

$$x \mapsto \frac{x - f(x)}{||x - f(x)||}.$$

The above definition can be extended to the case when $a$ is an isolated fixed point of a map $f : X \to X$ with $X$ being a manifold, since the nature of the definition is local. Moreover, for any open set $U \subset X$ such that $U \cap \text{Fix}(f)$ is compact, one can define the *fixed point index* $\text{index}(f,U)$. One way to define the index is to approximate $f$ with a generic smooth map $g$ in $U$ with only isolated fixed points, and define $\text{index}(f,U)$ to be the sum of the indices of fixed points of $g$.

Now let $X$ be a compact manifold, $f : X \to X$ be a map. Given a Nielsen class $C$, since $C$ is an isolated set of fixed points, we can choose an open set $U \subset X$ such that $U \cap \text{Fix}(f) = C$. We can define the *fixed point index* $\text{index}(f,C) := \text{index}(f,U)$.

**Theorem 2.3** (Lefschetz–Hopf). *The total fixed point index* $\text{index}(f,X)$ *is equal to the Lefschetz number* $\Lambda(f)$.\footnote{See [8, Theorem 1.12].}

**Definition 2.4.** A Nielsen class $C$ is *essential* if $\text{index}(f,C) \neq 0$. The *Nielsen number* $N(f)$ of a map $f$ is the number of essential Nielsen classes of $f$.

The Nielsen number $N(f)$ is a homotopy invariant of the map $f$. It gives a lower bound to the number of fixed points of any map in the homotopy class of $f$. In many cases, this lower bound is sharp. The following theorem of Jiang–Guo [10] establishes the sharpness in the case of surface diffeomorphisms in a given isotopy class. (The original theorem is stated for possibly non-orientable surfaces.)

**Theorem 2.5** (Jiang–Guo). *Let* $\Sigma$ *be a compact, oriented, connected surface, $f : \Sigma \to \Sigma$ *be an orientation-preserving homeomorphism. Then* $f$ *is isotopic to a diffeomorphism* $g$ *which has exactly* $N(f)$ *fixed points.*
3 The symplectic Floer homology of a mapping class

In this section, we will review Cotton-Clay’s computation of the symplectic Floer homology of a mapping class. We will work over \( F = \mathbb{Z}/2\mathbb{Z} \).

Let \( \Sigma \) be a closed, oriented, connected surface with an area form \( \omega \) and \( g(\Sigma) \geq 2 \). Let \( \phi \) be an area-preserving diffeomorphism of \( \Sigma \), such that all fixed points of \( \phi \) are non-degenerate, and \( \phi \) satisfies a monotonicity condition. The symplectic Floer chain complex \( CF^{symp}_*(\phi) \) is an \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space over \( F \), which is freely generated by the fixed points of \( \phi \). The differential \( \partial \) counts holomorphic disks satisfying certain conditions. A fixed point \( y \) appears in \( \partial x \) for another fixed point \( x \), only if \( x \) and \( y \) are Nielsen equivalent. So \( CF^{symp}_*(\phi) \) naturally splits as a direct sum over all Nielsen classes.

Seidel [20] proved that \( HF^{symp}_*(\phi) \) only depends on the mapping class \( h \) of \( \phi \), so one can define
\[
HF^{symp}_*(h) = HF^{symp}_*(\phi)
\]
to be the symplectic Floer homology of the mapping class \( h \). The symplectic Floer homology of mapping classes was extensively studied in [2, 4].

By Thurston’s classification of surface automorphisms [21], we may find a diffeomorphism \( \phi \) in a mapping class \( h \), which is in the standard form in the following sense: There exists a finite union \( N \) of disjoint non-contractible annuli, such that \( \phi(N) = N \). Moreover, the following conditions are satisfied:

1. For any component \( A \) of \( N \), there exists a positive integer \( \ell \), such that \( \phi^\ell(A) = A \) and \( \phi^\ell|_A \) is a twist map or a flip-twist map. A twist map on \([0, 1] \times S^1\) has the form
\[
(q, p) \mapsto (q, p - f(q)),
\]
and a flip-twist map on \([0, 1] \times S^1\) has the form
\[
(q, p) \mapsto (1 - q, -p + f(q)),
\]
where \( f : [0, 1] \to \mathbb{R} \) is a strictly monotonic smooth map. The (flip-)twist map is positive or negative if \( f \) is increasing or decreasing, respectively.

2. In the first condition, if \( \ell = 1 \) and \( \phi|_A \) is a twist map, then \( \phi \) has no fixed point in \( \text{int}(A) \). That is, \( \text{int}(f) \subset [0, 1] \). We also require that parallel twist regions are twisted in the same direction.

3. For any component \( S \) of \( \Sigma \setminus \text{int}(N) \), there exists a positive integer \( \ell \), such that \( \phi^\ell(S) = S \) and \( \phi^\ell|_S \) is either periodic or pseudo-Anosov.

There are 3 types of fixed points of \( \phi \), listed in [2 Subsection 4.3]. Type I consists of points in components \( S \) of \( \Sigma \setminus \text{int}(N) \) which are fixed by \( \phi \) pointwise. Such components are called fixed components. This type is further divided into Type Ia (\( \chi(S) < 0 \)) and Ib (\( \chi(S) = 0 \)). Type II consists of fixed points in periodic components (Type Ia) and flip-twist regions (Type Ib). Type III consists of fixed points in pseudo-Anosov components. This type is further divided into
4 sub-types: IIIa (fixed points not associated with any singular points or punctures), IIIb (fixed points associated with unrotated singular points), IIIc (fixed points associated with rotated singular points), IIId (fixed points associated with unrotated punctures).

In [2, Subsection 4.5], \( \phi \) is further perturbed to a diffeomorphism \( \tilde{\phi} \) for which one will compute the symplectic Floer homology.

The subsurface \( \Sigma \setminus \text{int}(N) \) can be divided into three parts: Let \( \Sigma_0 \) be the collection of fixed components, \( \Sigma_1 \) be the collection of (non-fixed) periodic components, and \( \Sigma_2 \) be the collection of pseudo-Anosov components with punctures.

There are two types of boundary components of \( \Sigma_0 \) defined in [2, Subsection 4.5]: \( \partial_+ \Sigma_0 \) and \( \partial_- \Sigma_0 \).

Let \( \Sigma_a \) be the collection of fixed components which do not meet any pseudo-Anosov components. Let \( \Sigma_{b,p} \) be the collection of fixed components which meet one pseudo-Anosov component at a boundary with \( p \) prongs. Let \( \Sigma_{c,q} \) be the collection of fixed components which meet at least two pseudo-Anosov components such that the total number of prongs over all the boundaries is \( q \).

Let \( \Lambda(\tilde{\phi}|\Sigma_1) \) be the Lefschetz number of \( \tilde{\phi}|\Sigma_1 \). Let \( CF_{\text{symp}}^*(\tilde{\phi}|\Sigma_2) \) be the symplectic Floer chain complex for \( \tilde{\phi}|\Sigma_2 \) on the component \( \Sigma_2 \). More precisely, \( CF_{\text{symp}}^*(\tilde{\phi}|\Sigma_2) \) is freely generated by the fixed points of \( \tilde{\phi}|\Sigma_2 \), except that when a pseudo-Anosov component abuts a fixed component, we include the boundary fixed points as part of the fixed component and not as part of the pseudo-Anosov component.

In [2, Theorem 4.16], Cotton-Clay stated the following formula for the symplectic Floer homology of a reducible mapping class. Let \( \tilde{\phi} : \Sigma \to \Sigma \) be a perturbed standard form map in a reducible mapping class \( h \). Then

\[
HF_{\text{symp}}^*(h) = HF_{\text{symp}}^*(\tilde{\phi}) \cong H_{s(\text{mod~}2)}(\Sigma_0, \partial_+ \Sigma_0; \mathbb{F}) \\
\quad \oplus \bigoplus_p \left( H_{s(\text{mod~}2)}(\Sigma_{b,p}, \partial_+ \Sigma_{b,p}; \mathbb{F}) \oplus \mathbb{F}^{p-1} | \pi_0(\Sigma_{b,p}) \right) \\
\quad \oplus \bigoplus_q \left( H_{s(\text{mod~}2)}(\Sigma_{c,q}, \partial_+ \Sigma_{c,q}; \mathbb{F}) \oplus \mathbb{F}^{q | \pi_0(\Sigma_{c,q})} \right) \\
\quad \oplus \mathbb{F}^{\Lambda(\tilde{\phi}|\Sigma_1)} \oplus CF_{\text{symp}}^*(\tilde{\phi}|\Sigma_2).
\]

However, there is an omission in the above formula. It happened in [2, Lemma 4.15], where the author stated that “… the Floer homology chain complex \( (CF_*(\phi), \partial_J^* \) splits into a sum of chain complexes \( (C_i, \partial_i) \) for each component of \( \Sigma \setminus N \).” The problem is, if \( \phi|_A \) is a flip-twist map for some component \( A \) of \( N \), there will be two Type IIb fixed points in \( A \), and each of these two fixed points is the only fixed point in its Nielsen class as argued in [2, Section 4.3]. So there is a contribution of a rank 2 summand from \( A \subset N \), which is not included in the statements of [2, Lemma 4.15 and Theorem 4.16].

In fact, in the statement of [2, Theorem 1.6], the author stated that “… \( HF_*(h) \) splits into summands for each component of \( \Sigma \setminus C \)”, where \( C = \partial N \) is a collection
of simple closed curves preserved by $h$. So the contribution of the flip-twist annuli is implicitly included in this statement.

To get the correct formula, we should include the flip-twist annuli. Let $n_f$ be the number of annuli on which the restriction of $\hat{\phi}$ is a flip-twist map. Then there is a summand $F^{2n_f}$ in $CF_*(\hat{\phi})$, on which the differential is zero. So $HF_*(\hat{\phi})$ should have such a summand. Other summands of $HF_*(\hat{\phi})$ can be obtained in the same way as in [2]. This finishes the proof of Theorem 1.3.

It is not hard to see

$$2n_f = \Lambda(\hat{\phi}|_N).$$

So the extra summand may also be written as $F^{\Lambda(\hat{\phi}|_N)}$.

**Remark 3.1.** According to Cotton-Clay [3], one can adjust the statements of [2, Lemma 4.15 and Theorem 4.16] so that (2) still gives the correct result. One way is to define $\Sigma_1$ to be the disjoint union of (non-fixed) periodic regions and the flip-twist annuli. The other way is, when one defines the standard form map, if $\phi|_A$ is a flip-twist map, one decomposes $A$ as the union of a 2–periodic annuli and two twist annuli, hence $\Sigma_1$ will include a 2–periodic annulus for each flip-twist annulus. Nonetheless, in this paper we still choose (1) as the formula for symplectic Floer homology since it is more direct.

**Corollary 3.2.** Let $\Sigma$ be a closed, oriented, connected surface, $\phi : \Sigma \to \Sigma$ be an orientation-preserving homeomorphism, and $h$ be the mapping class of $\phi$. Then

$$N(\phi) \leq \text{rank} HF^{\text{symp}}_*(h).$$

**Proof.** Let $\hat{\phi}$ be a standard form map isotopic to $\phi$. For the terms in the last row of (1), the generators correspond to Type II and Type III fixed points, except that when a pseudo-Anosov component abuts a fixed component, we include the Type IIIId boundary fixed points as part of the fixed component and not as part of the pseudo-Anosov component. As summarized in the proof of [2, Lemma 4.15], for Type IIa, Type IIb, Type IIIa and Type IIIc fixed points, each fixed point is the only fixed point in its Nielsen class. Moreover, for Type IIIb fixed points and Type IIIId fixed points which do not abut a Type Ia component, the Nielsen class consists of fixed points of the same index. It is clear that the Nielsen classes corresponding to Type II and Type III fixed points are all essential. So the rank of the last row of (1) is greater than or equal to the number of corresponding essential Nielsen classes.

Let $S$ be a component of $\Sigma_0$, $S^o$ be $S$ with an interior point removed, and $C$ be a possibly empty collection of boundary components of $S$. Let $U$ be an open tubular neighborhood of $S$, then

$$\text{index}(\hat{\phi}, U) = \chi(S).$$

When $\chi(S) = 0$, $S$ itself is the collection of all fixed points in a Nielsen class by [2, Corollary 4.9], so this Nielsen class is inessential. When $\chi(S) < 0$, it is elementary to check

$$\text{rank } H_*(S, C; \mathbb{F}) > 0 \text{ and } \text{rank } H_*(S^o, C; \mathbb{F}) > 0.$$
So $S$ contributes at least 1 to the total rank of $[1]$, while it contributes at most 1 to $N(\hat{\phi})$.

Combining the above analysis, we get our inequality. \hfill \Box

4 Proof of the main theorem

In this section, we will prove Theorem 1.2. We will use $F$-coefficients for Heegaard Floer homology, although all the purely Heegaard Floer results hold true for any coefficients.

We will need the following proposition, which partially generalizes [15, Proposition 3.1]. The concept of left-veering and right-veering diffeomorphisms can be found in [8].

**Proposition 4.1.** Let $Y$ be a closed, oriented 3–manifold, and $K \subset Y$ be a fibered knot with fiber $F$. Let $s \in \text{Spin}^c(Y)$ be the underlying Spin$^c$ structure for the open book decomposition of $Y$ with binding $K$ and page $F$, and suppose that $HF^+(Y,s) = 0$. Let $\hat{F} \subset Y_0(K)$ be the closed surface obtained by capping off $\partial F$ with a disk, and let $t_k \in \text{Spin}^c(Y_0(K))$ be the Spin$^c$ structure satisfying $t_k|_{Y \setminus K} = s|_{Y \setminus K}$, $\langle c_1(t_k), \hat{F} \rangle = 2k$, $k \in \mathbb{Z}$.

Suppose that the monodromy of $K$ is neither left-veering nor right-veering. If $g(F) \geq 3$, then

$$\text{rank } HF^+(Y_0(K), t_{g-2}) = \text{rank } \widehat{HFK}(Y, K, [F], g(F) - 1) - 2.$$  

**Proof.** In [16, Section 9], Ozsváth and Szabó proved an exact sequence

$$\cdots \to HF^+(Y_0(K), [t_k]) \to HF^+(Y_n(K), s_k) \to HF^+(Y, s) \to \cdots,$$  

(3)

where $s_k$ is a certain Spin$^c$ structure on $Y_n(K)$, and $[t_k]$ consists of all Spin$^c$ structures $t$ on $Y_0(K)$ satisfying that

$$t|_{Y \setminus K} = s|_{Y \setminus K}, \quad \langle c_1(t), \hat{F} \rangle = 2nm, \quad \text{for some } m \in \mathbb{Z}.$$  

Let

$$C = CFK^\infty(Y, K, s, [F])$$

be the knot Floer chain complex. When $n \gg 0$,

$$HF^+(Y_0(K), [t_k]) = HF^+(Y_0(K), t_k),$$

and

$$HF^+(Y_n(K), s_k) \cong H_*(A^+_i) := H_*(C\{i \geq 0 \text{ or } j \geq k\}).$$

Since $HF^+(Y, s) = 0$, from (3) and the above two equalities we get

$$HF^+(Y_0(K), t_{g-2}) \cong H_*(A^+_{g-2}).$$
By the exact triangle

\[ H_*(A_{g-2}^+) \rightarrow H_*(C\{i \geq 0\}) \]

and the fact that \( H_*(C\{i \geq 0\}) \cong HF^+(Y, s) = 0 \), we get that

\[ H_*(C\{i < 0, j \geq g - 2\}) \cong H_*(A_{g-2}^+) \]

It follows that

\[ HF^+(Y_0(K), t_{g-2}) \cong H_*(C\{i < 0, j \geq g - 2\}) \]

Using \([15, Lemma 3.2]\), we get our conclusion.

**Proof of Theorem 1.2.** Without loss of generality, we may assume the monodromy \( \varphi \) is not left-veering. We use the same argument as in \([15]\).

As in \([15, Lemma 4.1]\), let \( L \subset Z = S^1 \times S^2 \) be a hyperbolic fibered knot with fiber \( G \), such that the Spin\(^c\) structure of the open book decomposition with binding \( L \) and page \( G \) is non-torsion. Let \( L' \) be the \((2n + 1, 2)\)-cable of \( L \) for a sufficiently large integer \( n \). Let \( E \) be the fiber of the new fibration of \( Z \setminus L' \), then \( E \) is the union of two copies of \( G \) and one copy of \( T \) which is the fiber of the \((2n + 1, 2)\)-cable space. Let \( \psi : E \rightarrow E \) be the monodromy of \( L' \), then \( \psi \) swaps the two copies of \( G \), and its restriction on \( T \) is a periodic map of period \( 4n + 2 \) which has no fixed point. Hence \( \psi \) has no fixed point. The monodromy \( \psi \) is right-veering. By \([7]\),

\[ \text{rank} HF^K(Z, L', [E], g - 1) = 1. \]

Consider the connected sum \( K \# L' \subset Y \# Z \). Let \( \sigma \) be its monodromy, and \( g' \) be the genus of the fiber \( F \) of this knot complement. By the Künneth formula,

\[ \text{rank} HF^K(Y \# Z, K \# L', [F \# E], g - 1) = r + 1. \]

Moreover, since \( \sigma |_{E} \) is left-veering and \( \sigma |_{F} \) is not left-veering, \( \sigma \) is neither left-veering nor right-veering. It follows from Proposition 4.1 and \( 4 \) that

\[ \text{rank} HF^+_((Y \# Z)_0(K \# L'), [F \# E], g' - 2) = r - 1. \]

The manifold \( (Y \# Z)_0(K \# L') \) is a surface bundle over \( S^1 \). Its fiber \( P \) is a closed surface which is the union of \( F \) and \( E \). Let \( \tilde{\sigma} \) be the monodromy, then \( \tilde{\sigma}|_F = \varphi \).

As argued in \([1, Theorem 3.5]\), using work of Lee–Taubes \([13]\), Kutluhan–Lee–Taubes \([12]\), Kronheimer–Mrowka \([11]\), one sees that

\[ HF^+_((Y \# Z)_0(K \# L'), [F \# E], g' - 2) \cong HF_{sym}^*(P, \tilde{\sigma}). \]
So
\[ \text{rank} HF^\text{sym}(P, \hat{\sigma}) = r - 1. \]

By Corollary 3.2, we have \( N(\hat{\sigma}) \leq r - 1. \) Since \( \hat{\sigma}|_P \) has no fixed point, and \( F \) is an essential surface in \( P \),
\[ N(\hat{\sigma}|_F) = N(\hat{\sigma}) \leq r - 1. \]

Now our conclusion follows from Theorem 2.5. \( \square \)

References

[1] John A. Baldwin, Ying Hu, and Steven Sivek, Khovanov homology and the cinquefoil (2021), preprint, available at https://arxiv.org/abs/2105.12102.
[2] Andrew Cotton-Clay, Symplectic Floer homology of area-preserving surface diffeomorphisms, Geom. Topol. 13 (2009), no. 5, 2619–2674.
[3] Ralf Gautschi, Floer homology of algebraically finite mapping classes, J. Symplectic Geom. 1 (2003), no. 4, 715–765.
[4] Paolo Ghiggini, Knot Floer homology detects genus-one fibred knots, Amer. J. Math. 130 (2008), no. 5, 1151–1169.
[5] Paolo Ghiggini and Gilberto Spano, in preparation.
[6] Matthew Hedden, On knot Floer homology and cabling, Algebr. Geom. Topol. 5 (2005), 1197–1222.
[7] Ko Honda, William H. Kazez, and Gordana Matić, Right-veering diffeomorphisms of compact surfaces with boundary, Invent. Math. 169 (2007), no. 2, 427–449.
[8] Yi Ni, Knot Floer homology detects fibred knots, Invent. Math. 170 (2007), no. 3, 577–608.
[9] Yi-Jen Lee and Clifford Henry Taubes, Periodic Floer homology and Seiberg-Witten-Floer cohomology, J. Symplectic Geom. 10 (2012), no. 1, 81–164.
[10] Yi Ni, A note on knot Floer homology and fixed points of monodromy (2021), preprint, available at https://arxiv.org/abs/2106.03884.
[11] Jacob Andrew Rasmussen, Floer homology and knot complements, ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)--Harvard University.
[12] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and three-manifold invariants: properties and applications, Ann. of Math. (2) 159 (2004), no. 3, 1159–1245.
[13] Holomorphic disks and knot invariants, Adv. Math. 186 (2004), no. 1, 58–116.
[14] Holomorphic disks and genus bounds, Geom. Topol. 8 (2004), 311–334.
[15] Jacob Andrew Rasmussen, Floer homology and knot complements, ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)--Harvard University.
[16] Paul Seidel, Symplectic Floer homology and the mapping class group, Pacific J. Math. 206 (2002), no. 1, 219–229.
[17] William P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. (N.S.) 19 (1988), no. 2, 417–431.