Multiparty quantum states stabilized by the diagonal subgroup of the local unitary group

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We classify, up to local unitary equivalence, the set of $n$-qubit states that is stabilized by the diagonal subgroup of the local unitary group. We exhibit a basis for this set, parameterized by diagrams of nonintersecting chords connecting pairs of points on a circle, and give a criterion for when the stabilizer is precisely the diagonal subgroup and not larger. This investigation is part of a larger program to partially classify entanglement type (local unitary equivalence class) via analysis of stabilizer structure.

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The desire to measure and classify entanglement for states of $n$-qubit systems has been motivated by potential applications in quantum computation and communication that utilize entanglement as a resource. More deeply, the mystery of entanglement has played a key role in foundational questions about quantum mechanics itself. Because entanglement properties of multi-qubit states are invariant under local unitary transformations, attempts to classify entanglement lead naturally to the problem of classifying local unitary equivalence classes of states.

The results presented in this article arise from the following framework, utilized in [3, 4] for 3-qubit systems and developed further by the authors in [5, 6, 7, 8] for $n$ qubits, for approaching local unitary equivalence classification. The equivalence class of a state—its orbit under the local unitary group action—is a submanifold of Hilbert space. There is a natural diffeomorphism

$$O_\psi \leftrightarrow G/\text{Stab}_\psi$$

between the orbit

$$O_\psi = \{g \ket{\psi} : g \in G\}$$

of a state $\ket{\psi}$ and the set $G/\text{Stab}_\psi$ of cosets of the stabilizer subgroup

$$\text{Stab}_\psi = \{g \in G : g \ket{\psi} = \ket{\psi}\}$$

(termed simply stabilizer hereafter) of the local unitary group

$$G = U(1) \times SU(2)^n.$$  

This duality between orbits and stabilizers provides a means of studying entanglement types (orbits) by analyzing stabilizer subgroups. Focusing on stabilizers affords the additional advantage of exploiting the well-developed theory of Lie groups and their Lie algebras of infinitesimal transformations.

In [7], we showed that for any state $\psi$, there is a disjoint union of the set of qubit labels

$$\{1, 2, \ldots, n\} = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_p \cup \mathcal{R}$$

so that the stabilizer (after an LU transformation, if necessary) has the form

$$\text{Stab}_\psi = \Delta_1 \times \cdots \times \Delta_p \times H, \tag{1}$$

where each $\Delta_j$ is a subgroup isomorphic to $SU(2)$ consisting of elements the form

$$\frac{1}{\sqrt{|\mathcal{R}|}} \times (g, \ldots, g) \times (\text{Id}, \ldots, \text{Id})$$

in qubits $\mathcal{B}_j$ in qubits outside of $\mathcal{B}_j$.

where $g$ ranges over $SU(2)$, and $H$ is a subgroup whose projection into each $SU(2)$ factor of $G$ is trivial in qubits $\cup_j \mathcal{B}_j$ and is 0- or 1-dimensional in the remaining $SU(2)$ factors in qubits in $\mathcal{R}$.

In previous work [5, 6, 7], we have studied stabilizers with maximum possible dimension in the factor $H$ in the decomposition [11] and have shown that states with such a stabilizer have important entanglement properties. This leads naturally to the question of what states have stabilizer

$$\Delta = \{(1, g, g, \ldots, g) : g \in SU(2)\}$$

and what interesting entanglement properties do they have? Evidence that this investigation will be fruitful is the 4-qubit state

$$\ket{M_4} = \frac{1}{\sqrt{6}}[\ket{0011} + \ket{1100} + \omega(\ket{1010} + \ket{0101})]$$

$$+ \omega^2(\ket{1001} + \ket{0110})],$$

where $\omega = \exp(2\pi i/3)$. The state $M_4$ has stabilizer $\Delta$ and has been shown [11, 12] to maximize average two-qubit bipartite entanglement, averaged over all partitions into 2-qubit subsystems. With this background, we pose the problem considered in this article.

**Problem.** Classify, up to local unitary equivalence, the space of states whose stabilizer contains $\Delta$. Among these states, which have stabilizer precisely equal to $\Delta$ and not larger?
Let $V_{\Delta}$ denote the space of states whose stabilizer contains $\Delta$. In the language of representation theory, $V_{\Delta}$ is the trivial subrepresentation Hilbert space $(\mathbb{C}^2)^{\otimes n}$ under the action of $\Delta \cong SU(2)$. In physical terms, regarding qubits as spin-1/2 particles, $V_{\Delta}$ is the space of states with zero total angular momentum [15]. It is known [11] that $V_{\Delta} = 0$ for odd $n$ and that the dimension of $V_{\Delta}$ for $n = 2m$ is the $m$th Catalan number

$$\dim V_{\Delta} = \frac{1}{m+1} \binom{2m}{m}.$$

It is clear that any product of $m$ singlet states $|01\rangle - |10\rangle$ (in any pairs of qubits) is in $V_{\Delta}$. In fact, as we show below, it turns out that all states in $V_{\Delta}$ are linear combinations of such states.

We can represent any product of $m$ singlets by a diagram consisting of $2m$ consecutively labeled points, joined in pairs by chords, no two of which share an endpoint. For example, the 6-qubit state

$$|\psi\rangle = |001011\rangle - |001110\rangle - |011001\rangle - |011100\rangle$$

$$+ |100110\rangle + |100110\rangle + |011100\rangle - |110100\rangle$$

which is the product of singlets in qubit pairs $\{1,3\}, \{2,5\}, \{4,6\}$ is shown in Figure 1.

**FIG. 1:** Diagram for product of 3 singlet pairs specified by chords.

Given a partition $\mathcal{P}$ of $\{1,2,\ldots,2m\}$ into 2-element subsets, let $|s_\mathcal{P}\rangle$ denote the singlet product state with singlet qubit pairs determined by $\mathcal{P}$. We shall say that $\mathcal{P}$ has no intersections if the associated chord diagram has no intersecting chords. Figure 2 illustrates all such states for $m = 2$.

Now we can state the solution to the above Problem. Statement 1 in the Theorem below gives a unique way to write any state whose stabilizer contains $\Delta$, and Statement 3 asserts this representation is unique in its local unitary equivalence class. Statement 2 answers the second question in the Problem above by giving a simple geometric criterion for when a state has its stabilizer precisely equal to $\Delta$.

**Theorem 1.**

1. The set $\{|s_\mathcal{P}\rangle : \mathcal{P} \text{ has no intersections} \}$ is a basis for $V_{\Delta}$.

2. For

$$|\psi\rangle = \sum_{\mathcal{P}} c_{\mathcal{P}} |s_\mathcal{P}\rangle$$

for which $c_{\mathcal{P}} = 0$ if $\mathcal{P}$ has intersections, we have $\text{Stab}_\psi = \Delta$ if and only if the following condition holds.

(*) For every proper subset $S \subset \{1,2,\ldots,2m\}$, there exists a partition $\mathcal{P}$ with $c_{\mathcal{P}} \neq 0$ and some $\{a,b\} \in \mathcal{P}$ with $a \in S$ and $b \notin S$.

3. Two states $\psi, \psi'$ that are local unitary equivalent with $\text{Stab}_\psi = \text{Stab}_\psi' = \Delta$ are in fact equal up to a phase factor.

To prove Theorem 1, we begin with a device for assigning a particular bit string to a partition $\mathcal{P}$ that has no intersections. Given a partition $\mathcal{P}$, we define $I_{\mathcal{P}}$ to the smallest (as a binary number) multi-index that occurs in the expansion of $|s_\mathcal{P}\rangle$ with nonzero coefficient in the computational basis. More generally, given an ordering $\mathcal{O} = (k_1,k_2,\ldots,k_{2m})$ of the qubit labels $\{1,2,\ldots,2m\}$, we define $I^\mathcal{O}_{\mathcal{P}}$ to be the smallest binary number $i_{k_1}i_{k_2}\ldots i_{k_{2m}}$, where $I = (i_1,i_2,\ldots,i_{2m})$ ranges over the multi-indices that occur with nonzero coefficient in the expansion of $|s_\mathcal{P}\rangle$ in the computational basis. It is easy to see how to construct $I_{\mathcal{P}}$. For each $\{a,b\} \in \mathcal{P}$ with $a < b$, assign $i_a = 0$ and $i_b = 1$. Similarly, to construct $I^\mathcal{O}_{\mathcal{P}}$, for each $\{k_a,k_b\} \in \mathcal{P}$ with $a < b$, assign $i_{k_a} = 0$ and $i_{k_b} = 1$. Observe that if $\mathcal{P} \neq \mathcal{P}'$, then $I_{\mathcal{P}} \neq I_{\mathcal{P}'}$ and $I^\mathcal{O}_{\mathcal{P}} \neq I^\mathcal{O}_{\mathcal{P}'}$.

Now we proceed with the proof of Theorem 1.

**Proof of Statement 1.** Since the cardinality of $\{|s_\mathcal{P}\rangle : \mathcal{P} \text{ has no intersections} \}$ is the dimension of $V_{\Delta}$ [12, 13], it suffices to show that the $|s_\mathcal{P}\rangle$ are independent.

Suppose there is a linear relation $\sum_{\mathcal{P}} c_{\mathcal{P}} |s_\mathcal{P}\rangle = 0$ with one or more $c_{\mathcal{P}}$ nonzero. Then there is some partition $\mathcal{P}_0$ with $c_{\mathcal{P}_0} \neq 0$ whose associated smallest multi-index $I_{\mathcal{P}_0}$ is smaller than the associated multi-index for all other partitions with $c_{\mathcal{P}} \neq 0$. The expansion of $\sum_{\mathcal{P}} c_{\mathcal{P}} |s_\mathcal{P}\rangle$ in the

**FIG. 2:** The two nonintersecting 4-qubit chord diagrams and their associated singlet product states.
computational basis contains the term $|I_P\rangle$ with nonzero coefficient, so $c_P$ must be zero. This contradiction implies that there is no linear relation $\sum_P c_P |s_P\rangle = 0$ with nonzero coefficients, and independence is established. \qed

**Proof of Statement 3.** If condition (⋆) in Statement 2 does not hold, then there is a set of qubits $K$ so that every $|s_P\rangle$ occurring in $|\psi\rangle$ is a product of singlets in $K$ times a product of singlets in the complementary set of qubits $\overline{K}$. It follows that $\text{Stab}_\psi$ contains a product $\Delta_1 \times \Delta_2$ of diagonal subgroups in qubits $K, \overline{K}$ that properly contains $\Delta$.

Conversely, suppose that condition (⋆) holds. Since the projection of $\text{Stab}_\psi$ in each $SU(2)$ factor of $G$ is 3-dimensional, we know that $H$ in (11) is trivial and therefore $\text{Stab}_\psi$ is a product $\Delta_1 \times \cdots \times \Delta_p$ for some $p \geq 1$ [7]. It is our aim to show that in fact, $p = 1$. Suppose on the contrary that $p > 1$, and let $K$ be the proper subset of $\{1,2,\ldots,2m\}$ consisting of qubits in which $\Delta_1$ has nontrivial coordinates. Consider the element

$$X = (0, X_1, X_2, \ldots, X_n)$$

in the Lie algebra $K_\psi$ of $\text{Stab}_\psi$, where $X_k = [i\ 0\ 0\ -i]$ for qubits $k \in K$ and $X_k = 0$ for qubits $k \notin K$. Given a multi-index $I = (i_1, i_2, \ldots, i_{2m})$, let

$$\alpha_I = \#\{i_k: i_k = 0\}_{k \in K} - \#\{i_k: i_k = 1\}_{k \notin K}. \quad (3)$$

The action of $X$ on the computational basis vector $|I\rangle$ is the following [3].

$$X |I\rangle = i\alpha_I |I\rangle \quad (4)$$

Let $O$ be an ordering $(k_1, k_2, \ldots, k_{2m})$ of the qubits $\{1,2,\ldots,2m\}$ obtained by choosing $(k_1, \ldots, k_{|K|})$ to be any ordering of the qubits in $K$, and choosing any ordering $(k_{|K|+1}, \ldots, k_{2m})$ of the qubits in $\overline{K}$. Condition (⋆) implies that there exist one or more partitions $P$ with $c_P \neq 0$ and with $P$ having at least one chord with one end in $K$ and the other end in $\overline{K}$. Let $I^P_P$ be the associated smallest string. The number of $0$s in $I^P_P$ in qubits $K$ is the number of chords with initial ends in $K$ and the number of $1$s in $I^P_P$ in $K$ is the number of chords with terminal ends in $K$. The only way to have the number of $0$s equal the number of $1$s is to have all the chords that begin in $K$ also end in $K$. But the choice of $P$ guarantees that this is not the case. By (11), $X$ kills a basis vector $I$ if and only if $\alpha_I = 0$ in (3), so it follows that $X |I^P_P\rangle \neq 0$.

Let $A$ be the set of partitions that have at least one chord with one end in $K$ and the other end in $\overline{K}$, and let $P_0 \in A$ be the partition whose associated smallest string $I^P_P$ is the smallest among all $I^P_P$ for $P$ in $A$. Since it follows that the term $X |I^P_P\rangle \neq 0$ survives in the expansion of $X |\psi\rangle$ in the computational basis. But this contradicts the assumption that $X \in K_\psi$. We conclude that there must be only one $\Delta_1$ factor in $\text{Stab}_\psi$, and we are done. \qed

**Proof of Statement 3.** Let $g = (e^{it}, g_1, \ldots, g_n)$ be a local unitary operator such that $|\psi'\rangle = g|\psi\rangle$. The hypotheses imply that $g \text{Stab}_\psi = \text{Stab}_\psi g$ or $g \text{Stab}_\psi g^{-1} = \text{Stab}_\psi$. Therefore we have

$$g_i h g_i^{-1} = g_j h g_j^{-1}$$

for all $i, j$, $1 \leq i, j \leq n$ and all $h \in SU(2)$, so $g_i^{-1} g_j$ stabilizes $h$ (with respect to the action of $SU(2)$ on itself by conjugation) for all $h \in SU(2)$. Therefore $g_i^{-1} g_j$ must be plus or minus the identity. The same holds for all pairs $g_i, g_j$, and so we have

$$|\psi'\rangle = (\pm e^{it}, g_1, \ldots, g_1) |\psi\rangle = \pm e^{it} |\psi\rangle$$

as claimed. \qed

**Conclusion.** We have described, in terms of a natural basis of combinatorial objects, those states whose stabilizers contain the diagonal subgroup of the local unitary group and have shown that expansions in this basis are unique (up to phase) representatives of their local unitary equivalence class. We have also given a simple geometric condition for when a state written in terms of this basis has its stabilizer subgroup precisely equal to the diagonal subgroup and not larger. Together with previous work, the results of this paper classify local unitary equivalence classes for states whose stabilizers are special cases of the general stabilizer decomposition (11). Natural next steps in this analysis are to classify subgroups of $H$ in (11) and classify the corresponding states that have those subgroups as stabilizers, and to classify states whose stabilizers are products of two or more factors in (11).

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[14] To be precise, these results are stated and proved in terms of the stabilizer Lie subalgebra in [7]. On the group level, the stabilizer may also have a discrete (0-dimensional and therefore finite, since $G$ is compact) factor. We may ignore this technicality for the purposes of the present discussion.
[15] $V_\Delta$ is the zero set of the angular momentum operator

$$J^2 = \left[ \frac{\hbar}{2} \sum_{i=1}^{n} (\sigma_x)_i \right]^2 + \left[ \frac{\hbar}{2} \sum_{i=1}^{n} (\sigma_y)_i \right]^2 + \left[ \frac{\hbar}{2} \sum_{i=1}^{n} (\sigma_z)_i \right]^2$$

where $(\sigma_a)_i$ is the Pauli matrix $\sigma_a$ ($a = x, y, z$) acting on the $j$th qubit. The operators $i \sum (\sigma_x)_i, i \sum (\sigma_y)_i, i \sum (\sigma_z)_i$ form a basis for the Lie algebra of $\Delta$. 