ON AN ESTIMATE IN THE SUBSPACE PERTURBATION PROBLEM

ALBRECHT SEELMANN

ABSTRACT. The problem of variation of spectral subspaces for linear self-adjoint operators under an additive bounded perturbation is considered. The aim is to find the best possible upper bound on the norm of the difference of two spectral projections associated with isolated parts of the spectrum of the perturbed and unperturbed operators.

In the approach presented here, a constrained optimization problem on a specific set of parameters is formulated, whose solution yields an estimate on the arcsine of the norm of the difference of the corresponding spectral projections. The problem is solved explicitly. This optimizes the approach by Albeverio and Motovilov in [Complex Anal. Oper. Theory 7 (2013), 1389–1416]. In particular, the resulting estimate is stronger than the one obtained there.

1. INTRODUCTION AND THE MAIN RESULT

Let $A$ be a self-adjoint possibly unbounded operator on a separable Hilbert space $H$ such that the spectrum of $A$ is separated into two disjoint components, that is,

$$
\text{spec}(A) = \sigma \cup \Sigma \quad \text{with} \quad d := \text{dist}(\sigma, \Sigma) > 0.
$$

Let $V$ be a bounded self-adjoint operator on $H$.

It is well known (see, e.g., [6, Theorem V.4.10]) that the spectrum of the perturbed self-adjoint operator $A + V$ is confined in the closed $\|V\|$-neighbourhood of the spectrum of the unperturbed operator $A$, that is,

$$
\text{spec}(A + V) \subset \mathcal{O}_{\|V\|}(\text{spec}(A)),
$$

where $\mathcal{O}_{\|V\|}(\text{spec}(A))$ denotes the open $\|V\|$-neighbourhood of $\text{spec}(A)$. In particular, if

$$
\|V\| < \frac{d}{2},
$$

then the spectrum of the operator $A + V$ is likewise separated into two disjoint components $\omega$ and $\Omega$, where

$$
\omega = \text{spec}(A + V) \cap \mathcal{O}_{d/2}(\sigma) \quad \text{and} \quad \Omega = \text{spec}(A + V) \cap \mathcal{O}_{d/2}(\Sigma).
$$

Therefore, under condition (1.3), the two components of the spectrum of $A + V$ can be interpreted as perturbations of the corresponding original spectral components $\sigma$ and $\Sigma$ of $\text{spec}(A)$. Clearly, the condition (1.3) is sharp in the sense that if $\|V\| \geq d/2$, the spectrum of the perturbed operator $A + V$ may not have separated components at all.

The effect of the additive perturbation $V$ on the spectral subspaces for $A$ is studied in terms of the corresponding spectral projections. Let $E_A(\sigma)$ and $E_{A+V}(\mathcal{O}_{d/2}(\sigma))$ denote the spectral projections for $A$ and $A + V$ associated with the Borel sets $\sigma$ and $\mathcal{O}_{d/2}(\sigma)$, respectively. It is well known that $\|E_A(\sigma) - E_{A+V}(\mathcal{O}_{d/2}(\sigma))\| \leq 1$ since the corresponding inequality holds for every difference of orthogonal projections in $H$, see, e.g., [1] Section 34]. Moreover, if

$$
\|E_A(\sigma) - E_{A+V}(\mathcal{O}_{d/2}(\sigma))\| < 1,
$$

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then the spectral projections $E_A(\sigma)$ and $E_{A+V}(O_d/2(\sigma))$ are unitarily equivalent, see, e.g., [6, Theorem I.6.32].

In this sense, if inequality (1.4) holds, the spectral subspace $\text{Ran} \ E_{A+V}(O_d/2(\sigma))$ can be understood as a rotation of the unperturbed spectral subspace $\text{Ran} \ E_A(\sigma)$. The quantity

$$\arcsin\left(\|E_A(\sigma) - E_{A+V}(O_d/2(\sigma))\|\right)$$

serves as a measure for this rotation and is called the maximal angle between the spectral subspaces $\text{Ran} \ E_A(\sigma)$ and $\text{Ran} \ E_{A+V}(O_d/2(\sigma))$. A short survey on the concept of the maximal angle between closed subspaces of a Hilbert space can be found in [2, Section 2]; see also [5, 8, Theorem 2.2], [11, Section 2], and references therein.

It is a natural question whether the bound (1.3) is sufficient for inequality (1.4) to hold, or if one has to impose a stronger bound on the norm of the perturbation $V$ in order to ensure (1.4). Basically, the following two arise:

(i) What is the best possible constant $c_{\text{opt}} \in (0, \frac{1}{2}]$ such that

$$\arcsin\left(\|E_A(\sigma) - E_{A+V}(O_d/2(\sigma))\|\right) < \frac{\pi}{2} \quad \text{whenever} \quad \|V\| < c_{\text{opt}} \cdot d ?$$

(ii) Which function $f : [0, c_{\text{opt}}) \to [0, \frac{\pi}{2})$ is best possible in the estimate

$$\arcsin\left(\|E_A(\sigma) - E_{A+V}(O_d/2(\sigma))\|\right) \leq f\left(\frac{\|V\|}{d}\right), \quad \|V\| < c_{\text{opt}} \cdot d ?$$

Both the constant $c_{\text{opt}}$ and the function $f$ are supposed to be universal in the sense that they are independent of the operators $A$ and $V$.

Note that we have made no assumptions on the disposition of the spectral components $\sigma$ and $\Sigma$ other than (1.1). If, for example, $\sigma$ and $\Sigma$ are additionally assumed to be subordinated, that is, $\sup \sigma < \inf \Sigma$ or vice versa, or if one of the two sets lies in a finite gap of the other one, then the corresponding best possible constant in problem (i) is known to be $\frac{1}{2}$, and the best possible function $f$ in problem (ii) is given by $f(x) = \frac{1}{2} \arcsin(2x)$, see, e.g., [7, Lemma 2.3] and [4, Theorem 5.1]; see also [11, Remark 2.9].

However, under the sole assumption (1.1), both problems are still unsolved. It has been conjectured that $c_{\text{opt}} = \frac{1}{2}$ (see [11]; cf. also [1] and [9]), but there is no proof available for that yet. So far, only lower bounds on the optimal constant $c_{\text{opt}}$ and upper bounds on the best possible function $f$ can be given. For example, in [7, Theorem 1] it was shown that

$$c_{\text{opt}} \geq \frac{2}{2 + \pi} = 0.3889845 \ldots$$

and

$$f(x) \leq \arcsin\left(\frac{\pi}{2} \frac{x}{1 - x}\right) < \frac{\pi}{2} \quad \text{for} \quad 0 \leq x < \frac{2}{2 + \pi}.$$  

In [11, Theorem 6.1] this result was strengthened to

$$c_{\text{opt}} \geq \frac{\sinh(1)}{\exp(1)} = 0.4323323 \ldots$$

and

$$f(x) \leq \frac{\pi}{4} \log\left(\frac{1}{1 - 2x}\right) < \frac{\pi}{2} \quad \text{for} \quad 0 \leq x < \frac{\sinh(1)}{\exp(1)}.$$  

Recently, Albeverio and Motovilov have shown in [2, Theorem 5.4] that

$$c_{\text{opt}} \geq c_s = 16 \frac{\pi^6 - 2\pi^4 + 32\pi^2 - 32}{(\pi^2 + 4)^4} = 0.4541692 \ldots$$
Theorem 1. Let $A$ be a self-adjoint operator on a separable Hilbert space $H$ such that the spectrum of $A$ is separated into two disjoint components, that is,

$$\text{spec}(A) = \sigma \cup \Sigma \quad \text{with} \quad d := \text{dist}(\sigma, \Sigma) > 0.$$

Let $V$ be a bounded self-adjoint operator on $H$ satisfying

$$\|V\| < c_{\text{crit}} \cdot d$$

with

$$c_{\text{crit}} = \frac{1 - (1 - \frac{\sqrt{3}}{2})^3}{2} = 3\sqrt{3} \pi^2 - \sqrt{3}\pi + 1 = 0.4548399 \ldots$$

Then, the spectral projections $E_A(\sigma)$ and $E_{A+V}(O_{d/2}(\sigma))$ for the self-adjoint operators $A$ and $A + V$ associated with $\sigma$ and the open $\frac{d}{4}$-neighbourhood $O_{d/2}(\sigma)$ of $\sigma$, respectively, satisfy the estimate

$$(1.10) \quad \arcsin(\|E_A(\sigma) - E_{A+V}(O_{d/2}(\sigma))\|) \leq N\left(\frac{\|V\|}{d}\right) < \frac{\pi}{2},$$

where the function $N: [0, c_{\text{crit}}] \to [0, \pi]$ is given by

$$(1.11) \quad N(x) = \begin{cases} 
\frac{4}{3} \arcsin(\pi x) & \text{for} \quad 0 \leq x \leq \frac{4}{\pi^2 + 4}, \\
\arcsin\left(\frac{2x^2 - 4}{\pi^2 - 4}\right) & \text{for} \quad \frac{4}{\pi^2 + 4} < x < \frac{4\sqrt{\pi^2 - 2}}{\pi^2}, \\
\frac{3}{2} \arcsin\left(\frac{\pi}{2}(1 - \sqrt{1 - 2x})\right) & \text{for} \quad 4\frac{\sqrt{\pi^2 - 2}}{\pi^2} \leq x \leq \kappa, \\
\frac{3}{2} \arcsin\left(\frac{\pi}{2}(1 - \sqrt{1 - 2x})\right) & \text{for} \quad \kappa < x \leq c_{\text{crit}}.
\end{cases}$$

Here, $\kappa \in \left(4\frac{\sqrt{\pi^2 - 2}}{\pi^2}, \frac{2\sqrt{\pi^2 - 2}}{\pi^2}\right)$ is the unique solution to the equation

$$\arcsin\left(\frac{\pi}{2}(1 - \sqrt{1 - 2\kappa})\right) = \frac{3}{2} \arcsin\left(\frac{\pi}{2}(1 - \sqrt{1 - 2\kappa})\right)$$

in the interval $\left(0, \frac{2\sqrt{\pi^2 - 2}}{\pi^2}\right)$. The function $N$ is strictly increasing, continuous on $[0, c_{\text{crit}}]$, and continuously differentiable on $(0, c_{\text{crit}}) \setminus \{\kappa\}$.

Numerical calculations give $\kappa = 0.4098623 \ldots$

The estimate $(1.10)$ in Theorem 1 remains valid if the constant $\kappa$ in the definition of the function $N$ is replaced by any other constant within the interval $\left(4\frac{\sqrt{\pi^2 - 2}}{\pi^2}, \frac{2\sqrt{\pi^2 - 2}}{\pi^2}\right)$, see Remark 2.8.
below. However, the particular choice \( (1.12) \) ensures that the function \( N \) is continuous and as small as possible. In particular, we have \( N(x) = M_*(x) \) for \( 0 \leq x \leq \frac{4}{\pi^2 + 4} \) and

\[
N(x) < M_*(x) \quad \text{for} \quad \frac{4}{\pi^2 + 4} < x \leq c_*,
\]

where \( c_* \) and \( M_* \) are given by \( (1.7) \) and \( (1.9) \) respectively, see Remark \( 2.10 \) below.

From Theorem \( 1 \) we immediately deduce that

\[
c_{\text{opt}} \geq c_{\text{crit}} > c_*,
\]

and

\[
f(x) \leq N(x) < \frac{\pi}{2} \quad \text{for} \quad 0 \leq x < c_{\text{crit}}.
\]

Both are the best respective bounds for the two problems (i) and (ii) known so far.

The paper is organized as follows: In Section 2, based on the triangle inequality for the maximal angle and a suitable a priori rotation bound for small perturbations (see Proposition \( 2.3 \)), we formulate a constrained optimization problem, whose solution provides an estimating function for the maximal angle between the corresponding spectral subspaces, see Definition \( 2.5 \), Proposition \( 2.6 \), and Theorem \( 2.7 \). In this way, the approach by Albeverio and Motovilov in \( [2] \) is optimized and, in particular, a proof of Theorem \( 1 \) is obtained. The explicit solution to the optimization problem is given in Theorem \( 2.7 \) which is proved in Section 3. The technique used there involves variational methods and may also be useful for solving optimization problems of a similar structure.

Finally, Appendix A is devoted to some elementary inequalities used in Section 3.

2. AN OPTIMIZATION PROBLEM

In this section, we formulate a constrained optimization problem, whose solution provides an estimate on the maximal angle between the spectral subspaces associated with isolated parts of the spectrum of the corresponding perturbed and unperturbed operators, respectively. In particular, this yields a proof of Theorem \( 1 \).

We make the following notational setup.

**Hypothesis 2.1.** Let \( A \) be as in Theorem \( 1 \) and let \( V \neq 0 \) be a bounded self-adjoint operator on the Hilbert space \( \mathcal{H} \). For \( 0 \leq t < \frac{1}{2} \), introduce \( B_t := A + td \frac{V}{\|V\|} \), \( \text{Dom}(B_t) := \text{Dom}(A) \), and denote by \( P_t := \mathcal{E}_{B_t}(\mathcal{O}_{d/2}(\sigma)) \) the spectral projection for \( B_t \) associated with the open \( \frac{d}{2} \)-neighbourhood \( \mathcal{O}_{d/2}(\sigma) \) of \( \sigma \).

Under Hypothesis \( 2.1 \) one has \( \|B_t - A\| = td < \frac{d}{2} \) for \( 0 \leq t < \frac{1}{2} \). Taking into account the inclusion \( (1.2) \), the spectrum of each \( B_t \) is likewise separated into two disjoint components, that is,

\[
\text{spec}(B_t) = \omega_t \cup \Omega_t \quad \text{for} \quad 0 \leq t < \frac{1}{2},
\]

where

\[
\omega_t = \text{spec}(B_t) \cap \mathcal{O}_{d/2}(\sigma) \quad \text{and} \quad \Omega_t = \text{spec}(B_t) \cap \overline{\mathcal{O}_{d/2}(\sigma)}.
\]

In particular, one has

\[
(2.1) \quad \delta_t := \text{dist}(\omega_t, \Omega_t) \geq (1 - 2t)d > 0 \quad \text{for} \quad 0 \leq t < \frac{1}{2}.
\]

Moreover, the mapping \( [0, \frac{1}{2}] \ni t \mapsto P_t \) is norm continuous, see, e.g., \( [2] \) Theorem 3.5]; cf. also the forthcoming estimate \( (2.7) \).

For arbitrary \( 0 \leq r \leq s < \frac{1}{2} \), we can consider \( B_s = B_r + (s - r)d \frac{V}{\|V\|} \) as a perturbation of \( B_r \). Taking into account the a priori bound \( (2.1) \), we then observe that

\[
(2.2) \quad \frac{\|B_s - B_r\|}{\delta_r} = \frac{(s - r)d}{\text{dist}(\omega_r, \Omega_r)} \leq \frac{s - r}{1 - 2r} < \frac{1}{2} \quad \text{for} \quad 0 \leq r \leq s < \frac{1}{2}.
\]
Furthermore, it follows from (2.2) and the inclusion (1.2) that \( \omega_s \) is exactly the part of \( \text{spec}(B_s) \) that is contained in the open \( \frac{\delta}{2} \)-neighbourhood of \( \omega_r \), that is,

\[
(2.3) \quad \omega_s = \text{spec}(B_s) \cap C_{\delta_r/2}(\omega_r) \quad \text{for} \quad 0 \leq r \leq s < \frac{1}{2}.
\]

Let \( t \in (0, \frac{1}{2}) \) be arbitrary, and let \( 0 = t_0 < t_1 < \cdots < t_{n+1} = t \) with \( n \in \mathbb{N}_0 \) be a finite partition of the interval \([0, t]\). Define

\[
(2.4) \quad \lambda_j := \frac{t_{j+1} - t_j}{1 - 2t_j} < \frac{1}{2}, \quad j = 0, \ldots, n.
\]

Recall that the mapping \( \rho \) given by

\[
(2.5) \quad \rho(P, Q) = \arcsin(||P - Q||) \quad \text{with} \quad P, Q \quad \text{orthogonal projections in} \ \mathcal{H},
\]

defines a metric on the set of orthogonal projections in \( \mathcal{H} \), see [3], and also [2, Lemma 2.15] and [10]. Using the triangle inequality for this metric, we obtain

\[
(2.6) \quad \arcsin(||P_0 - P_t||) \leq \sum_{j=0}^{n} \arcsin(||P_{t_j} - P_{t_{j+1}}||).
\]

Considering \( B_{t_{j+1}} \) as a perturbation of \( B_{t_j} \), it is clear from (2.2) and (2.3) that each summand of the right-hand side of (2.6) can be treated in the same way as the maximal angle in the general situation discussed in Section II. For example, combining (2.2)–(2.4) with the bound (1.5) yields

\[
(2.7) \quad ||P_{t_j} - P_{t_{j+1}}|| \leq \frac{\pi}{2} \frac{\lambda_j}{1 - \lambda_j} = \frac{\pi}{2} \frac{t_{j+1} - t_j}{1 - t_j - t_{j+1}} \leq \frac{\pi}{2} \frac{t_{j+1} - t_j}{1 - 2t_j}, \quad j = 0, \ldots, n,
\]

where we have taken into account that \( ||P_{t_j} - P_{t_{j+1}}|| \leq 1 \) and that \( \frac{\pi}{2} \frac{\lambda_j}{1 - \lambda_j} \geq 1 \) if \( \lambda_j \geq \frac{2}{2 + \pi} \).

Obviously, the estimates (2.6) and (2.7) hold for arbitrary finite partitions of the interval \([0, t]\). In particular, if partitions with arbitrarily small mesh size are considered, then, as a result of \( \frac{t_{j+1} - t_j}{1 - 2t_{j+1}} \leq \frac{t_{j+1} - t_j}{1 - 2t_j} \), the norm of each corresponding projector difference in (2.7) is arbitrarily small as well. At the same time, the corresponding Riemann sums

\[
\sum_{j=0}^{n} \frac{t_{j+1} - t_j}{1 - 2t_{j+1}}
\]

are arbitrarily close to the integral \( \int_0^t \frac{1}{1 - 2\tau} \, d\tau \). Since \( \frac{\arcsin(x)}{x} \rightarrow 1 \) as \( x \rightarrow 0 \), we conclude from (2.6) and (2.7) that

\[
\arcsin(||P_0 - P_t||) \leq \frac{\pi}{2} \int_0^t \frac{1}{1 - 2\tau} \, d\tau = \frac{\pi}{4} \log\left( \frac{1}{1 - 2t} \right).
\]

Once the bound (1.5) has been generalized to the case where the operator \( A \) is allowed to be unbounded, this argument is an easy and straightforward way to prove the bound (1.6).

Albeverio and Motovilov demonstrated in [2] that a stronger result can be obtained from (2.6). They considered a specific finite partition of the interval \([0, t]\) and used a suitable a priori bound (see [2, Corollary 4.3 and Remark 4.4]) to estimate the corresponding summands of the right-hand side of (2.6). This a priori bound, which is related to the Davis-Kahan \( \sin 2\Theta \) theorem from [5], is used in the present work as well. We therefore state the corresponding result in the following proposition for future reference. It should be noted that our formulation of the statement slightly differs from the original one in [2]. A justification of this modification, as well as a deeper discussion on the material including an alternative, straightforward proof of the original result [2, Corollary 4.3], can be found in [11].
Proposition 2.2 ([11, Corollary 2]). Let $A$ and $V$ be as in Theorem 2.1 If $\|V\| \leq \frac{d}{2}$, then the spectral projections $E_A(\sigma)$ and $E_{A+V}(O_{d/2}(\sigma))$ for the self-adjoint operators $A$ and $A + V$ associated with the Borel sets $\sigma$ and $O_{d/2}(\sigma)$, respectively, satisfy the estimate

$$\arcsin\left(\|E_A(\sigma) - E_{A+V}(O_{d/2}(\sigma))\|\right) \leq \frac{1}{2} \arcsin\left(\frac{\pi}{2} \cdot 2 \|V\| \right) \leq \frac{\pi}{4}.$$ 

The estimate given by Proposition 2.2 is universal in the sense that the estimating function $x \mapsto \frac{1}{2} \arcsin(\pi x)$ depends neither on the unperturbed operator $A$ nor on the perturbation $V$. Moreover, for perturbations $V$ satisfying $\|V\| \leq \frac{d}{\pi^2 + 4} d$, this a priori bound on the maximal angle between the corresponding spectral subspaces is the strongest one available so far, cf. [2, Remark 5.5].

Assume that the given partition of the interval $[0, t]$ additionally satisfies

$$\lambda_j = \frac{t_{j+1} - t_j}{1 - 2t_j} \leq \frac{1}{\pi}, \quad j = 0, \ldots, n. \quad (2.8)$$

In this case, it follows from (2.2), (2.3), (2.6), and Proposition 2.2 that

$$\arcsin(\|P_0 - P\|) \leq \frac{1}{2} \sum_{j=0}^{n} \arcsin(\pi \lambda_j). \quad (2.9)$$

Along with a specific choice of the partition of the interval $[0, t]$, estimate (2.9) is the essence of the approach by Albeverio and Motovilov in [2]. In the present work, we optimize the choice of the partition of the interval $[0, t]$, so that for every fixed parameter $t$ the right-hand side of inequality (2.9) is minimized. An equivalent and more convenient reformulation of this approach is to maximize the parameter $t$ in estimate (2.9) over all possible choices of the parameters $n$ and $\lambda_j$ for which the right-hand side of (2.9) takes a fixed value.

Obviously, we can generalize estimate (2.9) to the case where the finite sequence $(t_j)_{j=1}^{n}$ is allowed to be just increasing and not necessarily strictly increasing. Altogether, this motivates the following considerations.

Definition 2.3. For $n \in \mathbb{N}_0$ define

$$D_n := \left\{ (\lambda_j) \in t^1(\mathbb{N}_0) \mid 0 \leq \lambda_j \leq \frac{1}{\pi} \text{ for } j \leq n \text{ and } \lambda_j = 0 \text{ for } j \geq n + 1 \right\},$$

and let $D := \bigcup_{n \in \mathbb{N}_0} D_n$.

Every finite partition of the interval $[0, t]$ that satisfies condition (2.8) is related to a sequence in $D$ in the obvious way. Conversely, the following lemma allows to regain the finite partition of the interval $[0, t]$ from this sequence.

Lemma 2.4.

(a) For every $x \in \left[0, \frac{1}{2}\right]$ the mapping $[0, \frac{1}{2}] \ni t \mapsto t + x(1 - 2t)$ is strictly increasing.

(b) For every $\lambda = (\lambda_j) \in D$ the sequence $(t_j) \subset \mathbb{R}$ given by the recursion

$$t_{j+1} = t_j + \lambda_j (1 - 2t_j), \quad j \in \mathbb{N}_0, \quad t_0 = 0, \quad (2.10)$$

is increasing and satisfies $0 \leq t_j < \frac{1}{2}$ for all $j \in \mathbb{N}_0$. Moreover, one has $t_j = t_{n+1}$ for $j \geq n + 1$ if $\lambda \in D_n$. In particular, $(t_j)$ is eventually constant.

Proof. The proof of claim (a) is straightforward and is hence omitted. For the proof of (b), let $\lambda = (\lambda_j) \in D$ be arbitrary and let $(t_j) \subset \mathbb{R}$ be given by (2.10). Observe that $t_0 = 0 < \frac{1}{2}$ and that (a) implies that

$$0 \leq t_{j+1} = t_j + \lambda_j (1 - 2t_j) < \frac{1}{2} + \lambda_j \left(1 - 2 \cdot \frac{1}{2}\right) = \frac{1}{2} \quad \text{if} \quad 0 \leq t_j < \frac{1}{2}.$$
Thus, the two-sided estimate $0 \leq t_j < \frac{1}{2}$ holds for all $j \in \mathbb{N}_0$ by induction. In particular, it follows that $t_{j+1} - t_j = \lambda_j (1 - 2t_j) \geq 0$ for all $j \in \mathbb{N}_0$, that is, the sequence $(t_j)$ is increasing. Let $n \in \mathbb{N}_0$ such that $\lambda \in D_n$. Since $\lambda_j = 0$ for $j \geq n + 1$, it follows from the definition of $(t_j)$ that $t_{j+1} = t_j$ for $j \geq n + 1$, that is, $t_j = t_{n+1}$ for $j \geq n + 1$.

It follows from part (b) of the preceding lemma that for every $\lambda \in D$ the sequence $(t_j)$ given by (2.10) yields a finite partition of the interval $[0, t]$ with $t = \max_{j \in \mathbb{N}_0} t_j < \frac{1}{2}$. In this respect, the approach to optimize the parameter $t$ in (2.9) with a fixed right-hand side can now be formalized in the following way.

**Definition 2.5.** Let $W : D \to \mathbb{R}^\infty(\mathbb{N}_0)$ denote the (non-linear) operator that maps every sequence in $D$ to the corresponding increasing and eventually constant sequence given by the recursion (2.10). Moreover, let $M : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ be given by

$$M(x) := \frac{1}{2} \arcsin(\pi x).$$

Finally, for $\theta \in [0, \frac{\pi}{2}]$ define

$$D(\theta) := \left\{ (\lambda_j) \in D \left| \sum_{j=0}^\infty M(\lambda_j) = \theta \right\} \subset D \right.$$  

and

$$(2.11) \quad T(\theta) := \sup \{ \max W(\lambda) \mid \lambda \in D(\theta) \},$$

where $\max W(\lambda) := \max_{j \in \mathbb{N}_0} t_j$ with $(t_j) = W(\lambda)$.

For every fixed $\theta \in [0, \frac{\pi}{2}]$, it is easy to verify that indeed $D(\theta) \neq \emptyset$. Moreover, one has $0 \leq T(\theta) \leq \frac{1}{2}$ by part (b) of Lemma 2.4 and $T(\theta) = 0$ holds if and only if $\theta = 0$. In order to compute $T(\theta)$ for $\theta \in (0, \frac{\pi}{2}]$, we have to maximize $\max W(\lambda)$ over $\lambda \in D(\theta) \subset D$. This constrained optimization problem plays the central role in the approach presented in this work.

The following proposition shows how this optimization problem is related to the problem of estimating the maximal angle between the corresponding spectral subspaces.

**Proposition 2.6.** Assume Hypothesis 2.7. Let $[0, \frac{\pi}{2}] \ni \theta \mapsto S(\theta) \in C, S(\theta) \mapsto [0, \frac{\pi}{2}]$ be a continuous, strictly increasing (hence invertible) mapping with

$$0 \leq S(\theta) \leq T(\theta) \quad \text{for} \quad 0 \leq \theta < \frac{\pi}{2}.$$

Then

$$\arcsin(\|P_0 - P_t\|) \leq S^{-1}(t) \quad \text{for} \quad 0 \leq t < S\left(\frac{\pi}{2}\right).$$

**Proof.** Since the mapping $\theta \mapsto S(\theta)$ is invertible, it suffices to show the inequality

$$(2.12) \quad \arcsin(\|P_0 - P_{S(\theta)}\|) \leq \theta \quad \text{for} \quad 0 \leq \theta < \frac{\pi}{2}.$$ 

Considering $T(0) = S(0) = 0$, the case $\theta = 0$ in inequality (2.12) is obvious. Let $\theta \in (0, \frac{\pi}{2})$. In particular, one has $T(\theta) > 0$. For arbitrary $t$ such that $0 \leq t < T(\theta)$ we choose $\lambda = (\lambda_j) \in D(\theta)$ such that $t < \max W(\lambda) \leq T(\theta)$. Denote $(t_j) := W(\lambda)$. Since $t_j < \frac{1}{2}$ for all $j \in \mathbb{N}_0$ by part (b) of Lemma 2.4, it follows from the definition of $(t_j)$ that

$$\frac{t_{j+1} - t_j}{1 - 2t_j} = \lambda_j \leq \frac{1}{\pi} \quad \text{for all} \quad j \in \mathbb{N}_0.$$

Moreover, considering $t < \max W(\lambda) = \max_{j \in \mathbb{N}_0} t_j$, there is $k \in \mathbb{N}_0$ such that $t_k \leq t < t_{k+1}$. In particular, one has

$$(2.14) \quad \frac{t - t_k}{1 - 2t_k} < \frac{t_{k+1} - t_k}{1 - 2t_k} = \lambda_k \leq \frac{1}{\pi}.$$
Using the triangle inequality for the metric $\rho$ given by (2.5), it follows from (2.2), (2.3), (2.13), (2.14), and Proposition 2.2 that
\[
\arcsin\left(\|P_0 - P_t\|\right) \leq \sum_{j=0}^{k-1} \arcsin\left(\|P_{t_j} - P_{t_{j+1}}\|\right) + \arcsin\left(\|P_k - P_t\|\right)
\leq \sum_{j=0}^{k-1} M(\lambda_j) + M(\lambda_k) \leq \sum_{j=0}^{\infty} M(\lambda_j) = \theta,
\]
that is,
\[
(2.15) \quad \arcsin\left(\|P_0 - P_t\|\right) \leq \theta \quad \text{for all} \quad 0 \leq t < T(\theta).
\]
Since the mapping $[0, \frac{1}{2}] \ni \tau \mapsto P_\tau$ is norm continuous and $S(\theta) < S\left(\frac{\pi}{2}\right) \leq \frac{1}{2}$, estimate (2.15) also holds for $t = S(\theta) \leq T(\theta)$. This shows (2.12) and, hence, completes the proof.

It turns out that the mapping $[0, \frac{\pi}{2}] \ni \theta \mapsto T(\theta)$ is continuous and strictly increasing. It therefore satisfies the hypotheses of Proposition 2.6. In this respect, it remains to compute $T(\theta)$ for $\theta \in [0, \frac{\pi}{2}]$ in order to prove Theorem 1. This is done in Section 3 below. For convenience, the following theorem states the corresponding result in advance.

**Theorem 2.7.** In the interval $\left(0, \frac{\pi}{2}\right)$ the equation
\[
\left(1 - \frac{2}{\pi} \sin \vartheta\right)^2 = \left(1 - \frac{2}{\pi} \sin \left(\frac{2\vartheta}{3}\right)\right)^3
\]
has a unique solution $\vartheta \in \left(\arcsin\left(\frac{2}{3}\right), \frac{\pi}{2}\right)$. Moreover, the quantity $T(\theta)$ given in (2.11) has the representation
\[
(2.16) \quad T(\theta) = \begin{cases} 
\frac{1}{\pi} \sin(2\theta) & \text{for } 0 \leq \theta \leq \arctan\left(\frac{2}{3}\right) = \frac{1}{2} \arcsin\left(\frac{4\pi}{\pi^2 + 4}\right), \\
\frac{2}{\pi^2 + 4} \sin^2 \theta & \text{for } \arctan\left(\frac{2}{3}\right) < \theta < \arctan\left(\frac{4\pi}{3}\right), \\
\frac{1}{\pi} - \frac{1}{\pi} \left(1 - \frac{2}{\pi} \sin \theta\right)^2 & \text{for } \arcsin\left(\frac{2}{3}\right) \leq \theta \leq \vartheta, \\
\frac{1}{\pi} - \frac{1}{\pi} \left(1 - \frac{2}{\pi} \sin \left(\frac{2\vartheta}{3}\right)\right)^3 & \text{for } \vartheta < \theta \leq \frac{\pi}{2}.
\end{cases}
\]
The mapping $[0, \frac{\pi}{2}] \ni \theta \mapsto T(\theta)$ is strictly increasing, continuous on $[0, \frac{\pi}{2}]$, and continuous differentiable on $\left(0, \frac{\pi}{2}\right) \setminus \{\vartheta\}$.

Theorem 1 is now a straightforward consequence of Proposition 2.6 and Theorem 2.7.

**Proof of Theorem 1.** According to Theorem 2.7, the mapping $[0, \frac{\pi}{2}] \ni \theta \mapsto T(\theta)$ is strictly increasing and continuous. Hence, its range is the whole interval $[0, c_{\text{crit}}]$, where $c_{\text{crit}}$ is given by $c_{\text{crit}} = T\left(\frac{\pi}{2}\right) = \frac{1}{2} - \frac{1}{2} \left(1 - \frac{2}{\pi^2 + 4}\right)^3$. Let $N = T^{-1} : [0, c_{\text{crit}}] \rightarrow [0, \frac{\pi}{2}]$ denote the inverse of this mapping.

Obviously, the function $N$ is also strictly increasing and continuous. Moreover, using representation (2.16), it is easy to verify that $N$ is explicitly given by (1.11). In particular, the constant $\kappa = T(\vartheta) = \frac{1}{2} - \frac{1}{2} \left(1 - \frac{2}{\pi} \sin \vartheta\right)^2 \in \left(\frac{4\pi^2 - 2}{\pi^2 + 4}, 2\pi - 1\right)$ is the unique solution to equation (1.12) in the interval $[0, 2\pi - 2\pi^2]$. Furthermore, the function $N$ is continuously differentiable on $[0, c_{\text{crit}}] \setminus \{\kappa\}$ since the mapping $\theta \mapsto T(\theta)$ is continuously differentiable on $\left(0, \frac{\pi}{2}\right) \setminus \{\vartheta\}$.

Let $V$ be a bounded self-adjoint operator on $H$ satisfying $\|V\| < c_{\text{crit}} \cdot d$. The case $V = 0$ is obvious. Assume that $V \neq 0$. Then, $B_t := A + td\frac{V}{\|V\|}$, $\text{Dom}(B_t) := \text{Dom}(A)$, and $P_t := \text{E}_{B_t\{O_{d/2}(\sigma)\}}$ for $0 \leq t < \frac{1}{2}$ satisfy Hypothesis 2.1. Moreover, one has $A + V = B_t$, etc.
with \( r = \frac{\|V\|}{d} < c_{\text{crit}} = T\left(\frac{\pi}{2}\right) \). Applying Proposition 2.6 to the mapping \( \theta \mapsto T(\theta) \) finally gives

\[
(2.17) \quad \arcsin\left(\|E_A(\sigma) - E_{A+V}(O_{d/2}(\sigma))\|\right) = \arcsin\left(\|P_0 - P_r\|\right) \leq N(r) = N\left(\frac{\|V\|}{d}\right),
\]

which completes the proof. \( \Box \)

**Remark 2.8.** Numerical evaluations give \( \vartheta = 1.1286942 \ldots < \arcsin\left(\frac{4\pi}{\pi + r}\right) = 2 \arctan\left(\frac{2}{r}\right) \)
and \( \kappa = T(\vartheta) = 0.4908623 \ldots < \frac{8s^2}{(\pi^2 + 4)^2}. \)

However, the estimate (2.17) remains valid if the constant \( \kappa \) in the explicit representation for the function \( N \) is replaced by any other constant within the interval \( (\frac{4\pi}{\pi + r}, \frac{2\pi}{\pi + 1}) \). This can be seen by applying Proposition 2.6 to each of the two mappings

\[
\theta \mapsto \frac{1}{2} - \frac{1}{2} \left(1 - \frac{2}{\pi} \sin \theta\right)^2 \quad \text{and} \quad \theta \mapsto \frac{1}{2} - \frac{1}{2} \left(1 - \frac{2}{\pi} \sin \left(\frac{2\theta}{3}\right)\right)^3.
\]

These mappings indeed satisfy the hypotheses of Proposition 2.6. Both are obviously continuous and strictly increasing, and, by particular choices of \( \lambda \in D(\theta) \), it is easy to see from the considerations in Section 3 that they are less or equal to \( T(\theta) \), see equation (3.5) below.

The statement of Theorem 2.7 actually goes beyond that of Theorem 1. As a matter of fact, instead of equality in (2.16), it would be sufficient for the proof of Theorem 1 to have that the right-hand side of (2.16) is just less or equal to \( T(\theta) \). This, in turn, is rather easy to establish by particular choices of \( \lambda \in D(\theta) \), see Lemma 3.3 and the proof of Lemma 3.6 below.

However, Theorem 2.7 states that the right-hand side of (2.16) provides an exact representation for \( T(\theta) \), and most of the considerations in Section 3 are required to show this stronger result. As a consequence, the bound from Theorem 1 is optimal within the framework of the approach by estimate (2.9).

In fact, the following observation shows that a bound substantially stronger than the one from Proposition 2.2 is required, at least for small perturbations, in order to improve on Theorem 1.

**Remark 2.9.** One can modify the approach (2.9) by replacing the term \( M(\lambda_j) = \frac{1}{2} \arcsin(\pi \lambda_j) \) by \( N(\lambda_j) \) and relaxing the condition (2.3) to \( \lambda_j \leq c_{\text{crit}} \). Yet, it follows from Theorem 2.7 that the corresponding optimization procedure leads to exactly the same result (2.16). This can be seen from the fact that each \( N(\lambda_j) \) is of the form of the right-hand side of (2.9) (cf. the computation of \( T(\theta) \) in Section 3 below), so that we are actually dealing with essentially the same optimization problem. In this sense, the function \( N \) is a fixed point in the approach presented here.

We close this section with a comparison of Theorem 1 with the strongest previously known result by Albeverio and Motovilov from [1].

**Remark 2.10.** One has \( N(x) = M_+(x) \) for \( 0 \leq x \leq \frac{4}{\pi^2 + 4} \), and the inequality \( N(x) < M_+(x) \) holds for all \( \frac{1}{\pi^2 + 4} < x \leq c_0 \), where \( c_0 \in (0, \frac{1}{2}) \) and \( M_+: [0, c_0] \to [0, \frac{\pi}{2}] \) are given by (1.7) and (1.9), respectively. Indeed, it follows from the computation of \( T(\theta) \) in Section 3 (see Remark 3.10 below) that

\[
x < T(M_+(x)) \leq c_{\text{crit}} \quad \text{for} \quad \frac{4}{\pi^2 + 4} < x \leq c_0.
\]

Since the function \( N = T^{-1}: [0, c_{\text{crit}}] \to [0, \frac{\pi}{2}] \) is strictly increasing, this implies that

\[
N(x) < N\left(T(M_+(x))\right) = M_+(x) \quad \text{for} \quad \frac{4}{\pi^2 + 4} < x \leq c_0.
\]
3. Proof of Theorem 2.7

We split the proof of Theorem 2.7 into several steps. We first reduce the problem of computing \( T(\theta) \) to the problem of solving suitable finite-dimensional constrained optimization problems, see equations (3.1) and (3.3). The corresponding critical points are then characterized in Lemma 3.3 using Lagrange multipliers. The crucial tool to reduce the set of relevant critical points is provided by Lemma 3.4. Finally, the finite-dimensional optimization problems are solved in Lemmas 3.6, 3.8, and 3.9.

Throughout this section, we make use of the notations introduced in Definitions 2.3 and 2.5. In addition, we fix the following notations.

**Definition 3.1.** For \( n \in \mathbb{N}_0 \) and \( \theta \in [0, \frac{\pi}{2}] \) define \( D_n(\theta) := D(\theta) \cap D_n \). Moreover, let

\[
T_n(\theta) := \sup \{ \max W(\lambda) \mid \lambda \in D_n(\theta) \} \quad \text{if} \quad D_n(\theta) \neq \emptyset,
\]

and set \( T_n(\theta) := 0 \) if \( D_n(\theta) = \emptyset \).

As a result of \( D(0) = D_n(0) = \{0\} \subset l^1(\mathbb{N}_0) \), we have \( T(0) = T_n(0) = 0 \) for every \( n \in \mathbb{N}_0 \). Let \( \theta \in (0, \frac{\pi}{2}] \) be arbitrary. Since \( D_0(\theta) \subset D_1(\theta) \subset D_2(\theta) \subset \ldots \), we obtain

\[
T_0(\theta) \leq T_1(\theta) \leq T_2(\theta) \leq \ldots
\]

Moreover, we observe that

\[
(3.1) \quad T(\theta) = \sup_{n \in \mathbb{N}_0} T_n(\theta).
\]

In fact, we show below that \( T_n(\theta) = T_2(\theta) \) for every \( n \geq 2 \), so that \( T(\theta) = T_2(\theta) \), see Lemma 3.9.

Let \( n \in \mathbb{N} \) be arbitrary and let \( \lambda = (\lambda_j) \in D_n \). Denote \((t_j) := W(\lambda)\). It follows from part (b) of Lemma 2.4 that \( \max W(\lambda) = t_{n+1} \). Moreover, we have

\[
1 - 2t_{j+1} = 1 - 2t_j - 2\lambda_j(1 - 2t_j) = (1 - 2t_j)(1 - 2\lambda_j), \quad j = 0, \ldots, n.
\]

Since \( t_0 = 0 \), this implies that

\[
1 - 2t_{n+1} = \prod_{j=0}^{n}(1 - 2\lambda_j).
\]

In particular, we obtain the explicit representation

\[
(3.2) \quad \max W(\lambda) = t_{n+1} = \frac{1}{2} \left( 1 - \prod_{j=0}^{n}(1 - 2\lambda_j) \right).
\]

An immediate conclusion of representation (3.2) is the following statement.

**Lemma 3.2.** For \( \lambda = (\lambda_j) \in D_n \) the value of \( \max W(\lambda) \) does not depend on the order of the entries \( \lambda_0, \ldots, \lambda_n \).

Another implication of representation (3.2) is the fact that \( \max W(\lambda) = t_{n+1} \) can be considered as a continuous function of the variables \( \lambda_0, \ldots, \lambda_n \). Since the set \( D_n(\theta) \) is compact as a closed bounded subset of an \((n + 1)\)-dimensional subspace of \( l^1(\mathbb{N}_0) \), we deduce that \( T_n(\theta) \) can be written as

\[
(3.3) \quad T_n(\theta) = \max \{ t_{n+1} \mid (t_j) = W(\lambda), \lambda \in D_n(\theta) \}.
\]

Hence, \( T_n(\theta) \) is determined by a finite-dimensional constrained optimization problem, which can be studied by use of Lagrange multipliers.
Taking into account the definition of the set \( D_n(\theta) \), it follows from equation (3.3) and representation (3.2) that there is some point \((\lambda_0, \ldots, \lambda_n) \in [0, \frac{1}{\pi}]^{n+1}\) such that
\[
T_n(\theta) = t_{n+1} = \frac{1}{2} \left( 1 - \prod_{j=0}^{n} (1 - 2\lambda_j) \right) \quad \text{and} \quad \sum_{j=0}^{n} M(\lambda_j) = \theta,
\]
where \(M(x) = \frac{1}{2} \arcsin(\pi x)\) for \(0 \leq x \leq \frac{1}{2}\). In particular, if \((\lambda_0, \ldots, \lambda_n) \in (0, \frac{1}{\pi})^{n+1}\), then the method of Lagrange multipliers gives a constant \(r \in \mathbb{R}, r \neq 0\), with
\[
\frac{\partial t_{n+1}}{\partial \lambda_k} = r \cdot M'(\lambda_k) = r \cdot \frac{\pi}{2\sqrt{1 - \pi^2 \lambda_k^2}} \quad \text{for} \quad k = 0, \ldots, n.
\]
Hence, in this case, for every \(k \in \{0, \ldots, n-1\}\) we obtain
\[
\sqrt{1 - \pi^2 \lambda_k^2} = \frac{\partial t_{n+1}}{\partial \lambda_k} = \frac{\prod_{j=0}^{n} (1 - 2\lambda_j)}{\prod_{j=0}^{n} (1 - 2\lambda_j)_{j \neq k}} = 1 - 2\lambda_{k+1}.
\]
This leads to the following characterization of critical points of the mapping \(\lambda \mapsto \max W(\lambda)\) on \(D_n(\theta)\).

**Lemma 3.3.** For \(n \geq 1\) and \(\theta \in \left(0, \frac{\pi}{2}\right)\) let \(\lambda = (\lambda_j) \in D_n(\theta)\) with \(T_n(\theta) = \max W(\lambda)\). Assume that \(\lambda_0 \geq \cdots \geq \lambda_n\). If, in addition, \(\lambda_0 < \frac{\pi}{4}\) and \(\lambda_n > 0\), then either one has
\[
\lambda_0 = \cdots = \lambda_n = \frac{1}{\pi} \sin \left( \frac{2\theta}{n+1} \right),
\]
so that
\[
\max W(\lambda) = \frac{1}{2} - \frac{1}{2} \left( 1 - \frac{2}{\pi} \sin \left( \frac{2\theta}{n+1} \right) \right)^{n+1},
\]
or there is \(l \in \{0, \ldots, n-1\}\) with
\[
\frac{4}{\pi^2 + 4} > \lambda_0 = \cdots = \lambda_l > \frac{2}{\pi^2} > \lambda_{l+1} = \cdots = \lambda_n > 0.
\]
In the latter case, \(\lambda_0\) and \(\lambda_n\) satisfy
\[
\lambda_0 + \lambda_n = \frac{4\alpha^2}{\pi^2 + 4\alpha^2} \quad \text{and} \quad \lambda_0 \lambda_n = \frac{\alpha^2 - 1}{\pi^2 + 4\alpha^2},
\]
where
\[
\alpha = \sqrt{\frac{1 - \pi^2 \lambda_0^2}{1 - 2\lambda_0}} = \sqrt{\frac{1 - \pi^2 \lambda_n^2}{1 - 2\lambda_n}} \in (1, m), \quad m := \frac{\pi}{2} \tan \left( \arcsin \left( \frac{2}{\pi} \right) \right).
\]

**Proof.** Let \(\lambda_0 < \frac{\pi}{4}\) and \(\lambda_n > 0\). In particular, one has \((\lambda_0, \ldots, \lambda_n) \in \left(0, \frac{1}{\pi}\right)^{n+1}\). Hence, it follows from (3.4) that
\[
\alpha := \sqrt{\frac{1 - \pi^2 \lambda_k^2}{1 - 2\lambda_k}}
\]
does not depend on \(k \in \{0, \ldots, n\}\).

If \(\lambda_0 = \lambda_n\), then all \(\lambda_j\) coincide and one has \(\theta = (n+1)M(\lambda_0) = \frac{n+1}{2} \arcsin(\pi \lambda_0)\), that is,
\[
\lambda_0 = \cdots = \lambda_n = \frac{\pi}{2} \sin \left( \frac{2\theta}{n+1} \right).\]
Inserting this into representation (3.2) yields equation (3.5).
Now assume that $\lambda_0 > \lambda_n$. A straightforward calculation shows that $x = \frac{2}{\pi}$ is the only critical point of the mapping

\[(3.10) \quad [0, \frac{1}{\pi}] \ni x \mapsto \frac{\sqrt{1 - \pi^2 x^2}}{1 - 2x},\]

cf. Fig. 1. The image of this point is $(1 - \frac{4}{\pi^2})^{-1/2} = m > 1$. Moreover, 0 and $\frac{4}{\pi^2 + 4}$ are mapped to 1, and $\frac{1}{\pi}$ is mapped to 0. In particular, every value in the interval $(1, m)$ has exactly two preimages under the mapping \((3.10)\), and all the other values in the range $[0, m]$ have only one preimage. Since $\lambda_0 > \lambda_n$ by assumption, it follows from \((3.9)\) that $\alpha$ has two preimages. Hence, $\alpha \in (1, m)$ and $\frac{4}{\pi^2 + 4} > \lambda_0 > \frac{2}{\pi} > \lambda_n > 0$. Furthermore, there is $l \in \{0, \ldots, n - 1\}$ with $\lambda_0 = \cdots = \lambda_l$ and $\lambda_{l+1} = \cdots = \lambda_n$. This proves \((3.6)\) and \((3.8)\).

Finally, the relations \((3.7)\) follow from the fact that the equation $\frac{\sqrt{1 - \pi^2 x^2}}{1 - 2x} = \alpha$ can be rewritten as

\[0 = z^2 - \frac{4\alpha^2}{\pi^2 + 4\alpha^2}z + \frac{\alpha^2 - 1}{\pi^2 + 4\alpha^2} = (z - \lambda_0)(z - \lambda_n) = z^2 - (\lambda_0 + \lambda_n)z + \lambda_0\lambda_n.\]

The preceding lemma is one of the main ingredients for solving the constrained optimization problem that defines the quantity $T_n(\theta)$ in \((3.3)\). However, it is still a hard task to compute $T_n(\theta)$ from the corresponding critical points. Especially the case \((3.6)\) in Lemma 3.3 is difficult to handle and needs careful treatment. An efficient computation of $T_n(\theta)$ therefore requires a technique that allows to narrow down the set of relevant critical points. The following result provides an adequate tool for this and is thus crucial for the remaining considerations. The idea behind this approach may also prove useful for solving similar optimization problems.

**Lemma 3.4.** For $n \geq 1$ and $\theta \in (0, \frac{\pi}{4}]$ let $\lambda = (\lambda_j) \in D_n(\theta)$. If $T_n(\theta) = \max W(\lambda)$, then for every $k \in \{0, \ldots, n\}$ one has

\[\max W((\lambda_0, \ldots, \lambda_k, 0, \ldots)) = T_k(\theta_k) \quad \text{with} \quad \theta_k = \sum_{j=0}^{k} M(\lambda_j) \leq \theta.\]

**Proof.** Suppose that $T_n(\theta) = \max W(\lambda)$. The case $k = n$ in the claim obviously agrees with this hypothesis.

Let $k \in \{0, \ldots, n - 1\}$ be arbitrary and denote $(t_j) := W(\lambda)$. It follows from part (b) of Lemma 2.4 that $t_{k+1} = \max W((\lambda_0, \ldots, \lambda_k, 0, \ldots))$. In particular, one has $t_{k+1} \leq T_k(\theta_k)$ since $(\lambda_0, \ldots, \lambda_k, 0, \ldots) \in D_k(\theta_k)$.

**FIG. 1.** The mapping $[0, \frac{1}{\pi}] \ni x \mapsto \frac{\sqrt{1 - \pi^2 x^2}}{1 - 2x}$. 

The image of this point is $(1 - \frac{4}{\pi^2})^{-1/2} = m > 1$. Moreover, 0 and $\frac{4}{\pi^2 + 4}$ are mapped to 1, and $\frac{1}{\pi}$ is mapped to 0. In particular, every value in the interval $(1, m)$ has exactly two preimages under the mapping \((3.10)\), and all the other values in the range $[0, m]$ have only one preimage. Since $\lambda_0 > \lambda_n$ by assumption, it follows from \((3.9)\) that $\alpha$ has two preimages. Hence, $\alpha \in (1, m)$ and $\frac{4}{\pi^2 + 4} > \lambda_0 > \frac{2}{\pi} > \lambda_n > 0$. Furthermore, there is $l \in \{0, \ldots, n - 1\}$ with $\lambda_0 = \cdots = \lambda_l$ and $\lambda_{l+1} = \cdots = \lambda_n$. This proves \((3.6)\) and \((3.8)\).

Finally, the relations \((3.7)\) follow from the fact that the equation $\frac{\sqrt{1 - \pi^2 x^2}}{1 - 2x} = \alpha$ can be rewritten as

\[0 = z^2 - \frac{4\alpha^2}{\pi^2 + 4\alpha^2}z + \frac{\alpha^2 - 1}{\pi^2 + 4\alpha^2} = (z - \lambda_0)(z - \lambda_n) = z^2 - (\lambda_0 + \lambda_n)z + \lambda_0\lambda_n.\]

The preceding lemma is one of the main ingredients for solving the constrained optimization problem that defines the quantity $T_n(\theta)$ in \((3.3)\). However, it is still a hard task to compute $T_n(\theta)$ from the corresponding critical points. Especially the case \((3.6)\) in Lemma 3.3 is difficult to handle and needs careful treatment. An efficient computation of $T_n(\theta)$ therefore requires a technique that allows to narrow down the set of relevant critical points. The following result provides an adequate tool for this and is thus crucial for the remaining considerations. The idea behind this approach may also prove useful for solving similar optimization problems.

**Lemma 3.4.** For $n \geq 1$ and $\theta \in (0, \frac{\pi}{4}]$ let $\lambda = (\lambda_j) \in D_n(\theta)$. If $T_n(\theta) = \max W(\lambda)$, then for every $k \in \{0, \ldots, n\}$ one has

\[\max W((\lambda_0, \ldots, \lambda_k, 0, \ldots)) = T_k(\theta_k) \quad \text{with} \quad \theta_k = \sum_{j=0}^{k} M(\lambda_j) \leq \theta.\]

**Proof.** Suppose that $T_n(\theta) = \max W(\lambda)$. The case $k = n$ in the claim obviously agrees with this hypothesis.

Let $k \in \{0, \ldots, n - 1\}$ be arbitrary and denote $(t_j) := W(\lambda)$. It follows from part (b) of Lemma 2.4 that $t_{k+1} = \max W((\lambda_0, \ldots, \lambda_k, 0, \ldots))$. In particular, one has $t_{k+1} \leq T_k(\theta_k)$ since $(\lambda_0, \ldots, \lambda_k, 0, \ldots) \in D_k(\theta_k)$.
Assume that \( t_{k+1} < T_k(\theta_k) \), and let \( \gamma = (\gamma_j) \in D_k(\theta_k) \) with \( \max W(\gamma) = T_k(\theta_k) \). Denote \( \mu := (\gamma_0, \ldots, \gamma_k, \lambda_{k+1}, \ldots, \lambda_n, 0, \ldots) \in D_n(\theta_n) \) and \((s_j) := W(\mu)\). Again by part (b) of Lemma \( \ref{lem:3.4} \) one has \( s_{k+1} = \max W(\gamma) > t_{k+1} \) and \( s_{n+1} = \max W(\mu) \leq T_n(\theta_n) \). Taking into account part (a) of Lemma \( \ref{lem:3.4} \) and the definition of the operator \( W \), one obtains that
\[
t_{k+2} = t_{k+1} + \lambda_{k+1}(1 - 2t_{k+1}) < s_{k+1} + \lambda_{k+1}(1 - 2s_{k+1}) = s_{k+2}.
\]
Iterating this estimate eventually gives \( t_{n+1} < s_{n+1} \leq T_n(\theta_n) \), which contradicts the case \( k = n \) from above. Thus, \( \max W((\lambda_0, \ldots, \lambda_k, 0, \ldots)) = t_{k+1} = T_k(\theta_k) \), as claimed. \( \square \)

Lemma \( \ref{lem:3.3} \) states that if a sequence \( \lambda \in D_n(\theta) \) solves the optimization problem for \( T_n(\theta) \), then every truncation of \( \lambda \) solves the corresponding reduced optimization problem. This allows to exclude many sequences in \( D_n(\theta) \) from the considerations once the optimization problem is understood for small \( n \). The number of parameters in \( \ref{lem:3.3} \) can thereby be reduced considerably.

The following lemma demonstrates this technique. It implies that the condition \( \lambda_0 < \frac{1}{2} \) in Lemma \( \ref{lem:3.3} \) is always satisfied except for one single case, which can be treated separately.

**Lemma 3.5.** For \( n \geq 1 \) and \( \theta \in (0, \frac{\pi}{2}) \) let \( \lambda = (\lambda_j) \in D_n(\theta) \) with \( T_n(\theta) = \max W(\lambda) \) and \( \lambda_0 \geq \cdots \geq \lambda_n \). If \( \lambda_0 = \frac{1}{2} \), then \( \theta = \frac{\pi}{2} \) and \( n = 1 \).

**Proof.** Let \( \lambda_0 = \frac{1}{2} \) and define \( \theta_1 := M(\lambda_0) + M(\lambda_1) \leq \theta \). It is obvious that \( \lambda \in D_1(\frac{\pi}{2}) \) is equivalent to \( \theta_1 = \theta = \frac{\pi}{2} \). Assume that \( \theta_1 < \frac{\pi}{2} \). Clearly, one has \( \theta_1 \geq M(\lambda_0) = \frac{\pi}{2} \) and
\[
\lambda_1 = \frac{1}{\pi} \sin(2\theta_1 - \frac{\pi}{2}) = \frac{1}{2} \left( 1 - 2 \sin^2 \theta_1 \right) \in \left( 0, \frac{\pi}{2} \right).
\]
Taking into account representation \( \ref{eq:3.2} \), for \( \mu := (\lambda_0, \lambda_1, 0, \ldots) \in D_1(\theta_1) \) one computes
\[
\max W(\mu) = \frac{1}{2} - \frac{1}{2} (1 - 2\lambda_0)(1 - 2\lambda_1) = (\lambda_0 + \lambda_1) - 2\lambda_0\lambda_1 = \frac{2}{\pi} \sin^2 \theta_1 + \frac{2}{\pi^2} \left( 1 - 2 \sin^2 \theta_1 \right) = \frac{2}{\pi^2} + \frac{2}{\pi} - \frac{4}{\pi^2} \sin^2 \theta_1.
\]
Since \( \arcsin \left( \frac{2}{\pi^2} \right) < \frac{\pi}{4} \leq \theta_1 < \frac{\pi}{2} \), it follows from part (b) of Lemma \( \ref{lem:appendix} \) that
\[
\max W(\mu) < \frac{2}{\pi} \left( 1 - \frac{1}{\pi} \sin \theta_1 \right) \sin \theta_1 = \frac{1}{2} - \left( 1 - \frac{2}{\pi} \sin \theta_1 \right)^2 \leq T_1(\theta_1),
\]
where the last inequality is due to representation \( \ref{eq:3.3} \). This is a contradiction to Lemma \( \ref{lem:3.4} \).

Hence, \( \theta_1 = \theta = \frac{\pi}{2} \) and, in particular, \( \lambda = \mu \in D_1(\frac{\pi}{2}) \).

Obviously, one has \( D_1(\theta) = \left\{ \left( \frac{1}{\pi}, \frac{1}{\pi}, 0, \ldots \right) \right\} \), so that \( \lambda = \left( \frac{1}{\pi}, \frac{1}{\pi}, 0, \ldots \right) \). Taking into account that \( \sin \left( \frac{\pi}{2} \right) = \frac{\sqrt{3}}{2} \), it follows from representations \( \ref{eq:3.2} \) and \( \ref{eq:3.5} \) that
\[
\max W(\lambda) = \frac{1}{2} - \frac{1}{2} \left( 1 - \frac{2}{\pi} \right)^2 \leq \frac{1}{2} - \left( 1 - \frac{\sqrt{3}}{\pi} \right)^3 \leq T_2(\frac{\pi}{2}).
\]
Since \( \max W(\lambda) = T_n(\theta) \) by hypothesis, this implies that \( n = 1 \). \( \square \)

We are now able to solve the finite-dimensional constrained optimization problem in \( \ref{lem:3.3} \) for every \( \theta \in [0, \frac{\pi}{2}] \) and \( n \in \mathbb{N} \). We start with the case \( n = 1 \).

**Lemma 3.6.** The quantity \( T_1(\theta) \) has the representation
\[
T_1(\theta) = \begin{cases} T_0(\theta) = \frac{1}{\pi} \sin(2\theta) & \text{for } 0 \leq \theta \leq \arctan \left( \frac{2}{\pi} \right) = \frac{1}{2} \arcsin \left( \frac{4\pi^{3/2}}{\pi^2 + 1} \right), \\ \frac{2}{\pi} + \frac{2}{\pi^2} \sin^2 \theta & \text{for } \arctan \left( \frac{2}{\pi} \right) < \theta < \arcsin \left( \frac{2}{\pi} \right), \\ \frac{1}{2} - \frac{1}{4} \left( 1 - 2 \sin \theta \right)^2 & \text{for } \arcsin \left( \frac{2}{\pi} \right) \leq \theta \leq \frac{\pi}{2}. \end{cases}
\]

In particular, if \( 0 < \theta < \arcsin \left( \frac{2}{\pi} \right) \) and \( \lambda = (\lambda_0, \lambda_1, 0, \ldots) \in D_1(\theta) \) with \( \lambda_0 = \lambda_1 \), then the strict inequality \( \max W(\lambda) < T_1(\theta) \) holds.

The mapping \( \left[ 0, \frac{\pi}{2} \right] \ni \theta \mapsto T_1(\theta) \) is strictly increasing, continuous on \( [0, \frac{\pi}{2}] \), and continuously differentiable on \( (0, \frac{\pi}{2}) \).
Proof. Since \( T_1(0) = T_0(0) = 0 \), the representation is obviously correct for \( \theta = 0 \). For \( \theta = \frac{\pi}{2} \) one has \( D_1 \left( \frac{\pi}{2} \right) = \left\{ \left( \frac{1}{\pi}, \frac{1}{\pi}, 0, \ldots \right) \right\} \), so that \( T_1 \left( \frac{\pi}{2} \right) = \frac{1}{\pi} - \frac{1}{2} \left( 1 - \frac{2}{\pi} \right)^2 \) by representation (3.2). This also agrees with the claim.

Now let \( \theta \in (0, \frac{\pi}{2}) \) be arbitrary. Obviously, one has \( D_0(\theta) = \left\{ \left( \frac{1}{\pi} \sin(2\theta), 0, \ldots \right) \right\} \) if \( \theta \leq \frac{\pi}{4} \), and \( D_0(\theta) = \emptyset \) if \( \theta > \frac{\pi}{4} \). Hence,

\[
T_0(\theta) = \frac{1}{\pi} \sin(2\theta) \quad \text{if} \quad 0 < \theta \leq \frac{\pi}{4},
\]
and \( T_0(\theta) = 0 \) if \( \theta > \frac{\pi}{4} \).

By Lemmas 3.2, 3.3, and 3.5 there are only two sequences in \( D_1(\theta) \setminus D_0(\theta) \) that need to be considered in order to compute \( T_1(\theta) \). One of them is given by \( \mu = (\mu_0, \mu_1, 0, \ldots) \) with \( \mu_0 = \mu_1 = \frac{1}{\pi} \sin \theta \in (0, \frac{1}{\pi}) \). For this sequence, representation (3.5) yields

\[
\max W(\mu) = \frac{1}{2} - \frac{1}{2} \left( 1 - \frac{2}{\pi} \sin \theta \right)^2 = \frac{2}{\pi} \left( 1 - \frac{1}{2} \sin \theta \right) \sin \theta.
\]

The other sequence in \( D_1(\theta) \setminus D_0(\theta) \) that needs to be considered is \( \lambda = (\lambda_0, \lambda_1, 0, \ldots) \) with \( \lambda_0 \) and \( \lambda_1 \) satisfying

\[
\frac{4}{\pi^2 + 4} > \lambda_0 > \frac{2}{\pi^2} > \lambda_1 > 0 \quad \text{and}
\]

\[
\lambda_0 + \lambda_1 = \frac{4\alpha^2}{\pi^2 + 4\alpha^2}, \quad \lambda_0 \lambda_1 = \frac{\alpha^2 - 1}{\pi^2 + 4\alpha^2},
\]
where

\[
\alpha = \sqrt{1 - \frac{\pi^2 \lambda_0}{1 - 2\lambda_0}} = \sqrt{1 - \frac{\pi^2 \lambda_1^2}{1 - 2\lambda_1}} \in (1, m), \quad m = \frac{\pi}{2} \tan \left( \arcsin \left( \frac{2}{\pi} \right) \right).
\]

It turns out shortly that this sequence \( \lambda \) exists if and only if \( \arctan \left( \frac{2}{\pi} \right) < \theta < \arcsin \left( \frac{2}{\pi} \right) \).

Using representation (3.2) and the relations in (3.13), one obtains

\[
\max W(\lambda) = \frac{1}{2} - \frac{1}{2} (1 - 2\lambda_0)(1 - 2\lambda_1) = (\lambda_0 + \lambda_1) - 2\lambda_0 \lambda_1 = 2 \frac{\alpha^2 + 1}{\pi^2 + 4\alpha^2}.
\]

The objective is to rewrite the right-hand side of (3.15) in terms of \( \theta \).

It follows from

\[
2\theta = \arcsin(\pi \lambda_0) + \arcsin(\pi \lambda_1)
\]

and the relations (3.13) and (3.14) that

\[
\sin(2\theta) = \pi \lambda_0 \sqrt{1 - \pi^2 \lambda_0^2} + \pi \lambda_1 \sqrt{1 - \pi^2 \lambda_1^2} = \pi \alpha \lambda_0 (1 - \lambda_1) + \alpha \pi \lambda_1 (1 - \lambda_0)
\]

\[
= \alpha \pi (\lambda_0 + \lambda_1 - 4\lambda_0 \lambda_1) = \frac{4\alpha \pi}{\pi^2 + 4\alpha^2}.
\]

Taking into account that \( \sin(2\theta) > 0 \), equation (3.17) can be rewritten as

\[
\alpha^2 - \frac{\pi}{\sin(2\theta)} \alpha + \frac{\pi^2}{4} = 0.
\]

In turn, this gives

\[
\alpha = \frac{\pi}{2 \sin(2\theta)} \left( 1 \pm \sqrt{1 - \sin^2(2\theta)} \right) = \frac{\pi}{2} \left( 1 \pm \frac{\cos^2 \theta - \sin^2 \theta}{2 \sin \theta \cos \theta} \right),
\]
that is,

\[
\alpha = \frac{\pi}{2} \tan \theta \quad \text{or} \quad \alpha = \frac{\pi}{2} \cot \theta.
\]

We show that the second case in (3.18) does not occur.

Since \( 1 < \alpha < m < \frac{\pi}{2} \), by equation (3.17) one has \( \sin(2\theta) < 1 \), which implies that \( \theta \neq \frac{\pi}{2} \). Moreover, combining relations (3.13) and (3.14), \( \lambda_1 \) can be expressed in terms of \( \lambda_0 \) alone. Hence, by equation (3.16) the quantity \( \theta \) can be written as a continuous function of the
sole variable \( \lambda_0 \in \left( \frac{\pi}{2}, \frac{4\pi}{n+1} \right) \). Taking the limit \( \lambda_0 \to \frac{4\pi}{n+1} \) in equation (3.16) then implies that \( \lambda_1 \to 0 \) and, therefore, \( \alpha \to \frac{1}{2} \arcsin \left( \frac{4\pi}{\pi^2+4} \right) < \frac{\pi}{4} \). This yields \( \theta < \frac{\pi}{4} \) for every \( \lambda_0 \in \left( \frac{2\pi}{n^2}, \frac{4\pi}{n+1} \right) \) by continuity, that is, the sequence \( \lambda \) can exist only if \( \theta < \frac{\pi}{4} \). Taking into account that \( \alpha \) satisfies \( 1 < \alpha < m = \frac{\pi}{2} \tan \left( \arcsin \left( \frac{2}{\pi} \right) \right) \), it now follows from (3.18) that the sequence \( \lambda \) exists if and only if \( \arctan \left( \frac{2}{\pi} \right) < \theta < \arcsin \left( \frac{2}{\pi} \right) \) and, in this case, one has

\[
(3.19) \quad \alpha = \frac{\pi}{2} \tan \theta.
\]

Combining equations (3.19) and (3.19) finally gives

\[
(3.20) \quad \max W(\alpha) = \frac{1}{2} \frac{\pi^2 - 4}{\pi^2} \sin \theta \cos \theta = \frac{2}{\pi^2} \cos^2 \theta + \frac{1}{2} \sin^2 \theta = \frac{2}{\pi^2} + \frac{\pi^2 - 4}{2\pi^2} \sin^2 \theta
\]

for \( \arctan \left( \frac{2}{\pi} \right) < \theta < \arcsin \left( \frac{2}{\pi} \right) \).

As a result of Lemmas 3.2, 3.3, and 3.5 the quantities (3.11), (3.12), and (3.20) are the only possible values for \( T_1(\theta) \), and we have to determine which of them is the greatest.

The easiest case is \( \theta > \frac{\pi}{4} \) since then (3.12) is the only possibility for \( T_1(\theta) \).

The quantity (3.20) is relevant only if \( \arctan \left( \frac{2}{\pi} \right) < \theta < \arcsin \left( \frac{2}{\pi} \right) \). In this case, it follows from parts (b) and (c) of Lemma A.1 that (3.20) gives the greatest value of the three possibilities and, hence, is the correct term for \( T_1(\theta) \) here.

For \( 0 < \theta < \arctan \left( \frac{2}{\pi} \right) < 2 \arctan \left( \frac{2}{\pi} \right) \), by part (d) of Lemma A.1 the quantity (3.11) is greater than (3.12). Therefore, \( T_1(\theta) \) is given by (3.11) in this case.

Finally, consider the case \( \arcsin \left( \frac{2}{\pi} \right) \leq \theta \leq \frac{\pi}{4} \). Since \( 2 \arctan \left( \frac{2}{\pi} \right) < \arcsin \left( \frac{2}{\pi} \right) \), it follows from part (c) of Lemma A.1 that (3.12) is greater than (3.11) and, hence, coincides with \( T_1(\theta) \).

This completes the computation of \( T_1(\theta) \) for \( \theta \in \left( 0, \frac{\pi}{4} \right) \). In particular, it follows from the discussion of the two cases \( 0 < \theta < \arctan \left( \frac{2}{\pi} \right) \) and \( \arctan \left( \frac{2}{\pi} \right) < \theta < \arcsin \left( \frac{2}{\pi} \right) \) that \( \max W(\mu) \) is always strictly less than \( T_1(\theta) \) if \( 0 < \theta < \arcsin \left( \frac{2}{\pi} \right) \).

The piecewise defined mapping \( [0, \frac{\pi}{4}] \ni \theta \mapsto T_1(\theta) \) is continuously differentiable on each of the corresponding subintervals. It remains to prove that the mapping is continuous and continuously differentiable at the points \( \theta = \arctan \left( \frac{2}{\pi} \right) = \frac{1}{2} \arcsin \left( \frac{4\pi}{\pi^2+4} \right) \) and \( \theta = \arcsin \left( \frac{2}{\pi} \right) \).

Taking into account that \( \sin^2 \theta = \frac{\pi^2 - 4}{\pi^2} \) for \( \theta = \frac{1}{2} \arcsin \left( \frac{4\pi}{\pi^2+4} \right) \), the continuity is straightforward to verify. The continuous differentiability follows from the relations

\[
\frac{\pi^2 - 4}{\pi^2} \sin \theta \cos \theta = \frac{2}{\pi} \left( 1 - \frac{2}{\pi} \sin \theta \right) \cos \theta \quad \text{for} \quad \theta = \arcsin \left( \frac{2}{\pi} \right)
\]

and

\[
\frac{2}{\pi} \cos(2\theta) = \frac{\pi^2 - 4}{2\pi^2} \sin(2\theta) = \frac{\pi^2 - 4}{\pi^2} \sin \theta \cos \theta \quad \text{for} \quad \theta = \frac{1}{2} \arcsin \left( \frac{4\pi}{\pi^2+4} \right),
\]

where the latter is due to

\[
\cot \left( \arcsin \left( \frac{4\pi}{\pi^2+4} \right) \right) = \sqrt{1 - \frac{16\pi^2}{(\pi^2+4)^2}} = \frac{\pi^2 - 4}{4\pi}.
\]

This completes the proof. \( \square \)

So far, Lemma 3.4 has been used only to obtain Lemma 3.5. Its whole strength becomes apparent in connection with Lemma 3.2. This is demonstrated in the following corollary to Lemma 3.6 which states that in 3.6 the sequences with \( l \in \{0, \ldots, n-2\} \) do not need to be considered.

**Corollary 3.7.** In the case (3.6) in Lemma 3.3 one has \( l = n - 1 \).
Proof. The case \( n = 1 \) is obvious. For \( n \geq 2 \) let \( \lambda = (\lambda_0, \ldots, \lambda_n, 0, \ldots) \in D_n(\theta) \) with

\[
\frac{4}{\pi^2 + 4} > \lambda_0 = \cdots = \lambda_1 > \frac{2}{\pi^2} > \lambda_{n+1} = \cdots = \lambda_n > 0
\]

for some \( l \in \{0, \ldots, n-2\} \). In particular, one has \( 0 < \lambda_{n-1} = \lambda_n < \frac{\sqrt{2}}{\pi} \), which implies that

\[
0 < \tilde{\theta} := M(\lambda_{n-1}) + M(\lambda_n) < \arcsin\left(\frac{\sqrt{2}}{\pi}\right).
\]

Hence, it follows from Lemma 3.6 that

\[
\max W((\lambda_{n-1}, \lambda_n, 0, \ldots)) < T_1(\tilde{\theta}).
\]

By Lemmas 3.2 and 3.4 one concludes that

\[
\max W(\lambda) = \max W((\lambda_{n-1}, \lambda_n, 0, \ldots, \lambda_{n-2}, 0, \ldots)) < T_n(\theta).
\]

This leaves \( l = n - 1 \) as the only possibility in (3.6).

We now turn to the computation of \( T_2(\theta) \) for \( \theta \in \left[0, \frac{\pi}{2}\right) \).

Lemma 3.8. In the interval \( \left(0, \frac{\pi}{2}\right) \) the equation

\[
(1 - \frac{2}{\pi} \sin \theta)^2 = \left(1 - \frac{2}{\pi} \sin \left(\frac{2\theta}{3}\right)\right)^3
\]

has a unique solution \( \vartheta \in \left(\arcsin\left(\frac{2}{\pi}\right), \frac{\pi}{2}\right) \). Moreover, the quantity \( T_2(\theta) \) has the representation

\[
T_2(\theta) = \begin{cases} 
T_1(\theta) & \text{for } 0 \leq \theta \leq \vartheta, \\
\frac{1}{2} - \frac{1}{2} \left(1 - \frac{2}{\pi} \sin \left(\frac{2\theta}{3}\right)\right) & \text{for } \vartheta < \theta < \frac{\pi}{2}.
\end{cases}
\]

In particular, one has \( T_1(\theta) < T_2(\theta) \) if \( \theta > \vartheta \), and the strict inequality \( \max W(\lambda) < T_2(\theta) \) holds for \( \theta \in \left(0, \frac{\pi}{2}\right) \) and \( \lambda = (\lambda_0, \lambda_1, \lambda_2, 0, \ldots) \in D_2(\theta) \) with \( \lambda_0 = \lambda_1 > \lambda_2 > 0 \).

The mapping \( \left[0, \frac{\pi}{2}\right) \ni \theta \mapsto T_2(\theta) \) is strictly increasing, continuous on \( \left[0, \frac{\pi}{2}\right) \), and continuously differentiable on \( \left(0, \frac{\pi}{2}\right) \setminus \{\vartheta\} \).

Proof. Since \( T_2(0) = T_1(0) = 0 \), the case \( \theta = 0 \) in the representation for \( T_2(\theta) \) is obvious. Let \( \theta \in \left(0, \frac{\pi}{2}\right) \) be arbitrary. It follows from Lemmas 3.2, 3.3, and 3.5 and Corollary 3.7 that there are only two sequences in \( D_2(\theta) \setminus D_1(\theta) \) that need to be considered in order to compute \( T_2(\theta) \). One of them is \( \mu = (\mu_0, \mu_1, \mu_2, 0, \ldots) \) with \( \mu_0 = \mu_1 = \mu_2 = \frac{1}{2} \sin\left(\frac{2\theta}{3}\right) \). For this sequence representation (3.5) yields

\[
\max W(\mu) = \frac{1}{2} - \frac{1}{2} \left(1 - \frac{2}{\pi} \sin \left(\frac{2\theta}{3}\right)\right)^3.
\]

The other sequence in \( D_2(\theta) \setminus D_1(\theta) \) that needs to be considered is \( \lambda = (\lambda_0, \lambda_1, \lambda_2, 0, \ldots) \), where \( \frac{4}{\pi^2 + 4} > \lambda_0 = \lambda_1 > \frac{2}{\pi^2} > \lambda_2 > 0 \) and \( \lambda_0 \) and \( \lambda_2 \) are given by (3.7) and (3.8). Using representation (3.2), one obtains

\[
\max W(\lambda) = \frac{1}{2} - \frac{1}{2} (1 - 2\lambda_0)^2 (1 - 2\lambda_2).
\]

According to Lemma A.3 this sequence \( \lambda \) can exist only if \( \theta \) satisfies the two-sided estimate

\[
\frac{\pi}{2} \arcsin\left(\frac{2}{\pi}\right) < \theta < \arcsin\left(\frac{12 + \pi^2}{4\pi^2}\right) + \frac{1}{2} \arcsin\left(\frac{12 - \pi^2}{4\pi^2}\right).
\]

However, if \( \lambda \) exists, combining Lemma A.3 with equations (3.22) and (3.23) yields

\[
\max W(\lambda) < \min W(\mu).
\]

Therefore, in order to compute \( T_2(\theta) \) for \( \theta \in \left(0, \frac{\pi}{2}\right) \), it remains to compare (3.22) with \( T_1(\theta) \). In particular, for every sequence \( \lambda = (\lambda_0, \lambda_1, \lambda_2, 0, \ldots) \in D_2(\theta) \) with \( \lambda_0 = \lambda_1 > \lambda_2 > 0 \) the strict inequality \( \max W(\lambda) < T_2(\theta) \) holds.
According to Lemma A.2 there is a unique \( \vartheta \in (\arcsin(\frac{2}{\pi}), \frac{\pi}{2}) \) such that
\[
(1 - \frac{2}{\pi} \sin \theta )^2 < (1 - \frac{2}{\pi} \sin (\frac{2\theta}{3}) )^3 \quad \text{for} \quad 0 < \theta < \vartheta
\]
and
\[
(1 - \frac{2}{\pi} \sin \theta )^2 > (1 - \frac{2}{\pi} \sin (\frac{2\theta}{3}) )^3 \quad \text{for} \quad \vartheta < \theta \leq \frac{\pi}{2}.
\]
These inequalities imply that \( \vartheta \) is the unique solution to equation (3.21) in the interval \((0, \frac{\pi}{2})\). Moreover, taking into account Lemma 3.6, equation (3.22), and the inequality \( \vartheta > \arcsin(\frac{2}{\pi}) \), it follows that \( T_1(\theta) < \max W(\mu) \) if and only if \( \theta > \vartheta \). This proves the claimed representation for \( T_2(\theta) \).

By Lemma 3.6 and the choice of \( \vartheta \) it is obvious that the mapping \([0, \frac{\pi}{2}] \ni \theta \mapsto T_2(\theta)\) is strictly increasing, continuous on \([0, \frac{\pi}{2}]\), and continuously differentiable on \((0, \frac{\pi}{2}) \setminus \{ \vartheta \} \).

In order to prove Theorem 2.7 it remains to show that \( T(\theta) \) coincides with \( T_2(\theta) \).

**Proposition 3.9.** For every \( \theta \in [0, \frac{\pi}{2}] \) and \( n \geq 2 \) one has \( T(\theta) = T_n(\theta) = T_2(\theta) \).

**Proof.** Since \( T'(0) = 0 \), the case \( \theta = 0 \) is obvious. Let \( \theta \in (0, \frac{\pi}{2}) \) be arbitrary. As a result of equation (3.1), it suffices to show that \( T_n(\theta) = T_2(\theta) \) for all \( n \geq 3 \). Let \( n \geq 3 \) and let \( \lambda = (\lambda_j) \in D_n(\theta) \setminus D_{n-1}(\theta) \). The objective is to show that \( \max W(\lambda) < T_n(\theta) \).

First, assume that \( \lambda_0 = \cdots = \lambda_n = \frac{1}{\pi} \sin(\frac{2\theta}{3}) > 0 \). We examine the two cases \( \lambda_0 < \frac{2}{\pi} \) and \( \lambda_0 \geq \frac{2}{\pi} \). If \( \lambda_0 < \frac{2}{\pi} \), then \( 2M(\lambda_0) < \arcsin(\frac{2}{\pi}) \). In this case, it follows from Lemma 3.6 that \( \max W((\lambda_0, \lambda_0, 0, \ldots)) < T_1(\hat{\theta}) \) with \( \hat{\theta} = 2M(\lambda_0) \). Hence, by Lemma 3.4 one has \( \max W(\lambda) < T_n(\theta) \). If \( \lambda_0 \geq \frac{2}{\pi} \), then
\[
(n + 1) \arcsin\left(\frac{2}{\pi}\right) \leq 2(n + 1)M(\lambda_0) = 2\theta \leq \pi,
\]
which is possible only if \( n \leq 3 \), that is, \( n = 3 \). In this case, one has \( \lambda_0 = \frac{1}{\pi} \sin(\frac{2\theta}{3}) \). Taking into account representation (3.5), it follows from Lemma A.4 that
\[
\max W(\lambda) = \frac{1}{2} - \frac{1}{2} \left(1 - \frac{2}{\pi} \sin\left(\frac{\theta}{2}\right)\right)^4 < \frac{1}{2} - \frac{1}{2} \left(1 - \frac{2}{\pi} \sin\left(\frac{2\theta}{3}\right)\right)^3 \leq T_2(\theta) \leq T_n(\theta).
\]
So, one concludes that \( \max W(\lambda) < T_2(\theta) \) again.

Now, assume that \( \lambda = (\lambda_j) \in D_n(\theta) \setminus D_{n-1}(\theta) \) satisfies \( \lambda_0 = \cdots = \lambda_{n-1} > \lambda_n > 0 \). Since, in particular, \( \lambda_{n-2} = \lambda_{n-1} > \lambda_n > 0 \), Lemma 3.8 implies that
\[
\max W((\lambda_{n-2}, \lambda_{n-1}, \lambda_n, 0, \ldots)) < T_2(\hat{\theta}) \quad \text{with} \quad \hat{\theta} = \sum_{j=n-2}^{n} M(\lambda_j).
\]
It follows from Lemmas 3.2 and 3.4 that
\[
\max W(\lambda) = \max W((\lambda_{n-2}, \lambda_{n-1}, \lambda_n, 0, \ldots, \lambda_{n-3}, 0, \ldots)) < T_n(\theta),
\]
that is, \( \max W(\lambda) < T_n(\theta) \) once again.

Hence, by Lemmas 3.2, 3.3 and 3.5 and Corollary 3.7 the inequality \( \max W(\lambda) < T_n(\theta) \) holds for all \( \lambda \in D_n(\theta) \setminus D_{n-1}(\theta) \), which implies that \( T_n(\theta) = T_{n-1}(\theta) \). Now the claim follows by induction.

We close this section with the following observation, which, together with Remark 2.10 above, shows that the estimate from Theorem 1 is indeed stronger than the previously known estimates.
Remark 3.10. It follows from the previous considerations that
\[ x < T(M_\lambda(x)) \text{ for } \frac{4}{\pi^2 + 4} < x \leq c_\lambda, \]
where \( c_\lambda \in (0, \frac{1}{2}) \) and \( M_\lambda : [0, c_\lambda] \to [0, \frac{2}{\pi}] \) are given by (1.7) and (1.9), respectively. Indeed, let \( x \in \left(\frac{4}{\pi^2 + 4}, c_\lambda\right) \) be arbitrary and set \( \lambda := M_\lambda(x) > \frac{1}{2} \arcsin\left(\frac{2\pi}{\pi^2 + 4}\right) \). Define \( \lambda \in D_2(\theta) \) by
\[
\lambda := \left(\frac{4}{\pi^2 + 4}, \frac{1}{\pi^2 + 4}\right) \sin\left(2\theta - \arcsin\left(\frac{4\pi}{\pi^2 + 4}\right)\right), 0, \ldots \quad \text{if } \theta \leq \arcsin\left(\frac{4\pi}{\pi^2 + 4}\right)
\]
and by
\[
\lambda := \left(\frac{4}{\pi^2 + 4}, \frac{4}{\pi^2 + 4}\right) \sin\left(2\theta - 2\arcsin\left(\frac{4\pi}{\pi^2 + 4}\right)\right), 0, \ldots \quad \text{if } \theta > \arcsin\left(\frac{4\pi}{\pi^2 + 4}\right).
\]
Using representation (3.2), a straightforward calculation shows that in both cases one has
\[ x = M_\lambda^{-1}(\lambda) = \max W(\lambda). \]
If \( \lambda = \arcsin\left(\frac{4\pi}{\pi^2 + 4}\right) > \theta \) (cf. Remark 2.8), that is, \( \lambda = \left(\frac{4}{\pi^2 + 4}, \frac{4}{\pi^2 + 4}, 0, \ldots\right) \), then it follows from Lemma 3.3 that \( \max W(\lambda) \leq T_1(\theta) < T_2(\theta) \).

If \( \theta \neq \arcsin\left(\frac{4\pi}{\pi^2 + 4}\right) \), then the inequality \( \max W(\lambda) < T_2(\theta) \) holds since, in this case, \( \lambda \) is none of the critical points from Lemma 3.3.

So, in either case one has \( x = \max W(\lambda) < T_2(\theta) = T(\theta) = T(M_\lambda(x)) \).

APPENDIX A. PROOFS OF SOME INEQUALITIES

Lemma A.1. The following inequalities hold:
(a) \( \frac{2}{\pi^2} + \frac{2\pi-4}{\pi^2} \sin^2 \theta < \frac{2}{\pi} \left(1 - \frac{1}{\pi} \sin \theta\right) \sin \theta \quad \text{for } \arcsin\left(\frac{1}{\pi-1}\right) < \theta < \frac{\pi}{2} \),
(b) \( \frac{1}{\pi} \sin(2\theta) < \frac{2}{\pi^2} + \frac{2\pi-4}{\pi^2} \sin^2 \theta \quad \text{for } \arctan\left(\frac{2}{\pi}\right) < \theta < \frac{\pi}{2} \),
(c) \( \frac{\pi}{2} (1 - \frac{1}{\pi} \sin \theta) \sin \theta < \frac{4}{\pi^2} \frac{\pi-4}{2\pi^2} \sin^2 \theta \quad \text{for } \theta \neq \arcsin\left(\frac{2}{\pi}\right) \),
(d) \( \frac{\pi}{2} (1 - \frac{1}{\pi} \sin \theta) \sin \theta < \frac{\pi}{2} \sin(2\theta) \quad \text{for } 0 < \theta < 2 \arctan\left(\frac{1}{\pi}\right) \),
(e) \( \frac{\pi}{2} (1 - \frac{1}{\pi} \sin \theta) \sin \theta > \frac{\pi}{2} \sin(2\theta) \quad \text{for } 2 \arctan\left(\frac{1}{\pi}\right) < \theta < \pi \).

Proof. One has
\[
\frac{2}{\pi} \left(1 - \frac{1}{\pi} \sin \theta\right) \sin \theta - \left(\frac{2}{\pi^2} + \frac{2\pi-4}{\pi^2} \sin^2 \theta\right)
= -\frac{2(\pi-1)}{\pi^2} \left(\sin^2 \theta - \frac{\pi}{\pi-1} \sin \theta + \frac{1}{\pi-1}\right)
= -\frac{2(\pi-1)}{\pi^2} \left(\sin \theta - \frac{\pi}{2(\pi-1)}\right)^2 < \frac{(\pi-2)^2}{4(\pi-1)^2},
\]
which is strictly positive if and only if
\[
\left(\sin \theta - \frac{\pi}{2(\pi-1)}\right)^2 < \frac{(\pi-2)^2}{4(\pi-1)^2}.
\]
A straightforward analysis shows that the last inequality holds for \( \arcsin\left(\frac{1}{\pi-1}\right) < \theta < \frac{\pi}{2} \), which proves (2).

For \( \theta_0 := \arctan\left(\frac{2}{\pi}\right) = \frac{1}{2} \arcsin\left(\frac{4\pi}{\pi^2 + 4}\right) \) one has \( \sin(2\theta_0) = \frac{4\pi}{\pi^2 + 4} \) and \( \sin^2 \theta_0 = \frac{4}{\pi^2 + 4} \). Thus, the inequality in (b) becomes an equality for \( \theta = \theta_0 \). Therefore, in order to show (b), it suffices to show that the corresponding estimate holds for the derivatives of both sides of the inequality, that is,
\[
\frac{2}{\pi} \cos(2\theta) < \frac{\pi^2 - 4}{2\pi^2} \sin(2\theta) \quad \text{for } \theta_0 < \theta < \frac{\pi}{4}.
\]
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This inequality is equivalent to \( \tan(2\theta) > \frac{4x}{x^2 + 2} \) for \( \theta_0 < \theta < \frac{\pi}{2} \), which, in turn, follows from \( \tan(2\theta) = \frac{2\tan\theta_0}{1 - \tan^2\theta_0} = \frac{4\pi}{\pi^2 - 4} \). This implies (b).

The claim (c) follows immediately from

\[
\frac{2}{\pi^2} + \frac{\pi^2 - 4}{2\pi^2} \sin^2 \theta - \frac{2}{\pi} \left( 1 - \frac{1}{\pi} \sin \theta \right) \sin \theta = \frac{1}{2} \left( \frac{2}{\pi} - \sin \theta \right)^2.
\]

Finally, observe that

\[
(A.1) \quad \frac{1}{\pi} \sin(2\theta) - \frac{2}{\pi} \left( 1 - \frac{1}{\pi} \sin \theta \right) \sin \theta = \frac{2}{\pi} \left( \cos \theta - \frac{1}{\pi} \sin \theta \right) \sin \theta.
\]

For \( 0 < \theta < \pi \), the right-hand side of (A.1) is positive if and only if \( \frac{1 - \cos \theta}{\sin \theta} = \tan(\frac{\theta}{2}) \) is less than \( \frac{1}{2} \). This is the case if and only if \( \theta < 2 \arctan(\frac{1}{2}) \), which proves (l). The proof of claim (c) is analogous.

**Lemma A.2.** There is a unique \( \vartheta \in (\arcsin(\frac{2}{3}), \frac{\pi}{2}) \) such that

\[
\left( 1 - \frac{2}{\pi} \sin \vartheta \right)^2 < \left( 1 - \frac{2}{\pi} \sin \left( \frac{2\vartheta}{3} \right) \right)^3 \quad \text{for} \quad 0 < \vartheta < \vartheta
\]

and

\[
\left( 1 - \frac{2}{\pi} \sin \vartheta \right)^2 > \left( 1 - \frac{2}{\pi} \sin \left( \frac{2\vartheta}{3} \right) \right)^3 \quad \text{for} \quad \vartheta < \theta < \frac{\pi}{2}.
\]

**Proof.** Define \( u, v, w : \mathbb{R} \to \mathbb{R} \) by

\[
u(\theta) := \sin \left( \frac{2\theta}{3} \right), \quad v(\theta) := \frac{\pi}{2} - \frac{\pi}{2} \left( 1 - \frac{2}{\pi} \sin \theta \right)^{2/3}, \quad \text{and} \quad w(\theta) := u(\theta) - v(\theta).
\]

Obviously, the claim is equivalent to the existence of \( \vartheta \in (\arcsin(\frac{2}{3}), \frac{\pi}{2}) \) such that \( w(\theta) < 0 \) for \( 0 < \theta < \vartheta \) and \( w(\theta) > 0 \) for \( \vartheta < \theta < \frac{\pi}{2} \).

Observe that \( u''(\theta) = -\frac{8}{3\pi} \cos \left( \frac{2\theta}{3} \right) < 0 \) for \( 0 \leq \theta \leq \frac{\pi}{2} \). In particular, \( u'' \) is strictly decreasing on the interval \( [0, \frac{\pi}{2}] \). Moreover, \( u'' \) is strictly increasing on \( [0, \frac{\pi}{2}] \), so that the inequality \( u'' \geq u''(0) = -\frac{8}{3\pi} > -\frac{1}{2} \) holds on \( [0, \frac{\pi}{2}] \).

One computes

\[
v^{(1)}(\theta) = \frac{2\pi^{1/3}}{81} \frac{p(\sin \theta)}{\sin(\theta - 2 \sin \theta)^{1/3}} \quad \text{for} \quad 0 \leq \theta \leq \frac{\pi}{2},
\]

where

\[
p(x) = 224 - 72\pi^2 + 27\pi^3 x - (160 + 36\pi^2)x^2 + 108\pi x^3 - 64x^4.
\]

The polynomial \( p \) is strictly increasing on \( [0, 1] \) and has exactly one root in the interval \( (0, 1) \). Combining this with equation (A.2), one obtains that \( v^{(1)} \) has a unique zero in \( (0, \frac{\pi}{2}) \) and that \( v^{(1)} \) changes its sign from minus to plus there. Observing that \( v''''(0) < -\frac{1}{2} \) and \( v'''' \left( \frac{\pi}{2} \right) = 0 \), this yields \( v'''' < 0 \) on \( [0, \frac{\pi}{2}] \), that is, \( v'''' \) is strictly decreasing on \( [0, \frac{\pi}{2}] \). Moreover, it is easy to verify that \( v'''' \left( \frac{\pi}{2} \right) < v''''(0) < -\frac{1}{2} \) on \( [0, \frac{\pi}{2}] \). Since \( v'''' > -\frac{1}{2} \) on \( [0, \frac{\pi}{2}] \) as stated above, it follows that \( w'''' = v'''' - v'''' > 0 \) on \( [0, \frac{\pi}{2}] \), that is, \( w'''' \) is strictly increasing on \( [0, \frac{\pi}{2}] \).

Recall that \( w'' \) and \( w''' \) are both decreasing functions on \( [0, \frac{\pi}{2}] \). Observing the inequality \( w'' \left( \frac{\pi}{2} \right) > w''' \left( \frac{\pi}{2} \right) \), one deduces that

\[
w''(\theta) = w''(\theta) - w''(\theta) \geq w'' \left( \frac{\pi}{2} \right) - w'' \left( \frac{\pi}{3} \right) > 0 \quad \text{for} \quad \theta \in \left( \frac{\pi}{3}, \frac{\pi}{2} \right).
\]

Moreover, one has \( w''(0) < 0 \). Combining this with (A.3) and the fact that \( w'' \) is strictly increasing on \( [0, \frac{\pi}{2}] \), one concludes that \( w'' \) has a unique zero in the interval \( (0, \frac{\pi}{2}) \) and that \( w'' \) changes its sign from minus to plus there. Since \( w''(0) = 0 \) and \( w'' \left( \frac{\pi}{2} \right) = \frac{1}{2} > 0 \), it follows that
that $w$ has a unique zero in $(0, \frac{\pi}{2})$, where it changes its sign from minus to plus. Finally, observing that $w(0) = 0$ and $w(\frac{\pi}{2}) > 0$, in the same way one arrives at the conclusion that $w$ has a unique zero $\vartheta \in (0, \frac{\pi}{2})$ such that $w(\vartheta) < 0$ for $0 < \vartheta < \vartheta$ and $w(\vartheta) > 0$ for $\vartheta < \vartheta < \frac{\pi}{2}$. As a result of $w(\arcsin(\frac{2}{\pi})) < 0$, one has $\vartheta > \arcsin(\frac{2}{\pi})$.

**Lemma A.3.** For $x \in (\frac{2}{\pi}, \frac{4}{\pi + 4})$ let

$$\alpha := \frac{\sqrt{1 - \pi^2 x^2}}{1 - 2x} \quad \text{and} \quad y := \frac{4\alpha^2}{\pi^2 + 4\alpha^2} - x.$$  

Then, $\theta := \arcsin(\pi x) + \frac{1}{2} \arcsin(\pi y)$ satisfies the inequalities

$$\frac{3}{2} \arcsin\left(\frac{2}{\pi}\right) < \theta \leq \arcsin\left(\frac{12 + \pi^2}{8\pi}\right) + \frac{1}{2} \arcsin\left(\frac{12 - \pi^2}{4\pi}\right)$$

and

$$\left(1 - \frac{2}{\pi} \sin\left(\frac{2\theta}{3}\right)\right)^3 < (1 - 2x)^2 (1 - 2y).$$

**Proof.** One has $1 < \alpha < m := \frac{\pi}{2} \tan(\arcsin(\frac{2}{\pi}))$, $y \in (0, \frac{\pi}{2})$, and $\alpha = \sqrt{1 - \pi^2 y^2}$ (cf. Lemma A.3). Moreover, taking into account that $\alpha^2 = \frac{1 - \pi^2 x^2}{1 - 2x}$ by (A.4), one computes

$$y = \frac{4 - (\pi^2 + 4)x}{\pi^2 + 4 - 4\pi^2 x}.$$  

Observe that $\alpha \to 1$ and $y \to 0$ as $x \to \frac{4}{\pi + 4}$. With this and taking into account (A.7), it is convenient to consider $\alpha = \alpha(x)$, $y = y(x)$, and $\theta = \theta(x)$ as continuous functions of the variable $x \in \left(\frac{2}{\pi}, \frac{4}{\pi + 4}\right)$.

Straightforward calculations show that

$$1 - 2y(x) = \frac{\pi^2 - 4}{\pi^2 + 4 - 4\pi^2 x} \cdot (1 - 2x) \quad \text{for} \quad \frac{2}{\pi^2} \leq x \leq \frac{4}{\pi^2 + 4},$$

so that

$$y'(x) = -\frac{(\pi^2 - 4)^2}{(\pi^2 + 4 - 4\pi^2 x)^2} = -\frac{(1 - 2y(x))^2}{(1 - 2x)^2} \quad \text{for} \quad \frac{2}{\pi^2} < x < \frac{4}{\pi^2 + 4}.$$  

Taking into account that $\alpha(x) = \alpha(y(x))$, that is, $\frac{1 - \pi^2 x^2}{1 - \pi^2 y(x)} = \frac{1 - 2x}{1 - 2y(x)}$, this leads to

$$\theta'(x) = -\frac{\pi}{\sqrt{1 - \pi^2 x^2}} + \frac{\pi y'(x)}{2\sqrt{1 - \pi^2 y(x)^2}} = \frac{\pi}{2\sqrt{1 - \pi^2 x^2}} \frac{2 + \frac{1 - 2x}{1 - 2y(x)} \cdot y'(x)}{2\sqrt{1 - \pi^2 x^2}}.$$  

In particular, $x = \frac{12 + \pi^2}{8\pi}$ is the only critical point of $\theta$ in the interval $(\frac{2}{\pi}, \frac{4}{\pi + 4})$ and $\theta'$ changes its sign from plus to minus there. Moreover, using $y\left(\frac{2}{\pi}\right) = \frac{\pi}{2}$ and $y\left(\frac{4}{\pi + 4}\right) = 0$, one has $\theta\left(\frac{2}{\pi}\right) = \frac{3}{2} \arcsin\left(\frac{2}{\pi}\right) < \arcsin\left(\frac{4x}{\pi^2 + 4}\right) = \theta\left(\frac{4}{\pi^2 + 4}\right)$, so that

$$\frac{3}{2} \arcsin\left(\frac{2}{\pi}\right) < \theta(x) \leq \theta\left(\frac{12 + \pi^2}{8\pi}\right) \quad \text{for} \quad \frac{2}{\pi^2} < x < \frac{4}{\pi^2 + 4}.$$  

Since $y\left(\frac{12 + \pi^2}{8\pi}\right) = \frac{12 - \pi^2}{16\pi}$, this proves the two-sided inequality (A.5).

Further calculations show that

$$\theta''(x) = \frac{\pi^4}{2} \frac{p(x)}{(1 - \pi^2 x^2)^{3/2} (\pi^2 + 4 - 4\pi^2 x)^2} \quad \text{for} \quad \frac{2}{\pi^2} < x < \frac{4}{\pi^2 + 4},$$

where $p(x)$ is a polynomial.
where
\[ p(x) = 16 - 4\pi^2 + (48 + 16\pi^2 + \pi^4)x - 8\pi^2(12 + \pi^2)x^2 + 32\pi^4x^3. \]
The polynomial \( p \) is strictly negative on the interval \( \left[ \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right] \), so that \( \theta' \) is strictly decreasing.

Define \( w: \left[ \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right] \to \mathbb{R} \) by
\[ w(x) := (1 - 2x)^2 \cdot (1 - 2y(x)) - \left( 1 - \frac{2}{\pi} \sin \left( \frac{2\theta(x)}{3} \right) \right)^3. \]

The claim (A.6) is equivalent to the inequality \( w(x) > 0 \) for \( \frac{2}{\pi^2} < x < \frac{4}{\pi^2+4} \). Since \( y(\frac{2}{\pi^2}) = \frac{2}{\pi^2} \) and, hence, \( \theta(\frac{2}{\pi^2}) = \frac{2}{\pi} \arcsin(\frac{2}{3}) \), one has \( w(\frac{2}{\pi^2}) = 0 \). Moreover, a numerical evaluation gives \( w(\frac{4}{\pi^2+4}) > 0 \). Therefore, in order to prove \( w(x) > 0 \) for \( \frac{2}{\pi^2} < x < \frac{4}{\pi^2+4} \), it suffices to show that \( w \) has exactly one critical point in the interval \( \left( \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right) \) and that \( w \) takes its maximum there.

Using (A.8) and taking into account that \( \sqrt{1 - \pi^2x^2} = \alpha(x)(1 - 2x) \), one computes
\[
\frac{d}{dx} (1 - 2x)^2 (1 - 2y(x)) = -4(1 - 2x)(1 - 2y(x)) - 2(1 - 2x)^2 y'(x)
= -2(1 - 2x)(1 - 2y(x)) \left( 2 + \frac{1 - 2\pi}{1 - 2y(x)} \cdot y'(x) \right)
= -\frac{4}{\pi} (1 - 2x)^2 (1 - 2y(x)) \alpha(x) \theta'(x).
\]

Hence, for \( \frac{2}{\pi^2} < x < \frac{4}{\pi^2+4} \) one obtains
\[
w'(x) = -\frac{4}{\pi} \theta'(x) \cdot \left( \alpha(x)(1 - 2x)^2 (1 - 2y(x)) - \left( 1 - \frac{2}{\pi} \sin \left( \frac{2\theta(x)}{3} \right) \right)^2 \cos \left( \frac{2\theta(x)}{3} \right) \right)
= \frac{4}{\pi} \theta'(x) \cdot (u(x) - v(x)),
\]
where \( u, v: \left[ \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right] \to \mathbb{R} \) are given by
\[ u(x) := \alpha(x)(1 - 2x)^2 (1 - 2y(x)), \quad v(x) := \left( 1 - \frac{2}{\pi} \sin \left( \frac{2\theta(x)}{3} \right) \right)^2 \cos \left( \frac{2\theta(x)}{3} \right). \]

Suppose that the difference \( u(x) - v(x) \) is strictly negative for all \( x \in \left( \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right) \). In this case, \( w' \) and \( \theta' \) have the same zeros on \( \left( \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right) \), and \( w'(x) \) and \( \theta'(x) \) have the same sign for all \( x \in \left( \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right) \). Combining this with (A.8), one concludes that \( x = \frac{12 + 3\pi^2}{\pi^2} \) is the only critical point of \( w \) in the interval \( \left( \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right) \) and that \( w \) takes its maximum in this point.

Hence, it remains to show that the difference \( u - v \) is indeed strictly negative on \( \left( \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right) \).

Since \( \alpha(\frac{2}{\pi^2}) = \frac{2}{\pi} \tan(\arcsin(\frac{2}{3})) \), \( y(\frac{2}{\pi^2}) = \frac{2}{\pi^2} \), and \( \theta(\frac{2}{\pi^2}) = \frac{2}{\pi} \arcsin(\frac{2}{3}) \), it is easy to verify that \( u(\frac{2}{\pi^2}) = v(\frac{2}{\pi^2}) \) and \( u'(\frac{2}{\pi^2}) = v'(\frac{2}{\pi^2}) < 0 \). Therefore, it suffices to show that \( u' < v' \) holds on the whole interval \( \left( \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right) \).

One computes
\[
u''(x) = \frac{(\pi^2 - 4)q(x)}{(1 - \pi^2x^2)^{3/2}(\pi^2 + 4 - 4\pi^2x^2)^3}
\]
where
\[
q(x) = (128 - 80\pi^2 - \pi^6) + 12\pi^2(\pi^2 + 4)x - 12\pi^2(7\pi^4 + 24\pi^2 + 48)x^2
+ 32\pi^4(5\pi^2 + 12)x^3 + 24\pi^4(\pi^4 + 16)x^4 - 96\pi^6(\pi^2 + 4)x^5 + 128\pi^8x^6.
\]
A further analysis shows that \( q'' \), which is a polynomial of degree 4, has exactly one root in the interval \( \left( \frac{2}{\pi^2}, \frac{4}{\pi^2+4} \right) \) and that \( q'' \) changes its sign from minus to plus there. Moreover, \( q' \) takes
a positive value in this root of $q''$, so that $q' > 0$ on $\left(\frac{\sqrt{13}}{2}, \frac{\sqrt{17}}{2}\right)$, that is, $q$ is strictly increasing on this interval. Since $q\left(\frac{\sqrt{13}}{2}\right) < 0$, one concludes that $q < 0$ on $\left(\frac{\sqrt{13}}{2}, \frac{\sqrt{17}}{2}\right)$. It follows from (A.10) that $q'' < 0$ on $\left(\frac{\sqrt{13}}{2}, \frac{\sqrt{17}}{2}\right)$, so that $u'$ is strictly decreasing. In particular, one has $u' < u'\left(\frac{\sqrt{17}}{2}\right) < 0$ on $\left(\frac{\sqrt{13}}{2}, \frac{\sqrt{17}}{2}\right)$.

A straightforward calculation yields

$$v'(x) = -\frac{2}{\pi}\left(1 - \frac{2}{\pi} \sin\left(\frac{2\theta(x)}{3}\right)\right) \cdot \theta'(x) \cdot r\left(\sin\left(\frac{2\theta(x)}{3}\right)\right),$$

where $r(t) = \frac{4}{\pi} + t - \frac{6}{\pi}t^2$. The polynomial $r$ is positive and strictly decreasing on the interval $[\frac{1}{3}, 1]$. Moreover, taking into account (A.5), one has $\frac{1}{2} < \sin\left(\frac{2\theta(x)}{3}\right) < 1$. Combining this with equation (A.11), one deduces that $v'(x)$ has the opposite sign of $\theta'(x)$ for all $\frac{4}{\pi} < x < \frac{4}{\pi + 1}$.

In particular, by (A.8) it follows that $v'(x) \geq 0$ if $x \geq \frac{12 + \pi^2}{8\pi^2}$. Since $u' < 0$ on $\left(\frac{2\pi}{\pi + 1}, \frac{4\pi}{\pi + 1}\right)$, this implies that $v'(x) > u'(x)$ for $\frac{12 + \pi^2}{8\pi^2} \leq x < \frac{4}{\pi + 1}$. If $\frac{4}{\pi} < x < \frac{12 + \pi^2}{8\pi^2}$, then one has $\theta'(x) > 0$. In particular, $\theta$ is strictly increasing on $\left(\frac{2\pi}{\pi}, \frac{12 + \pi^2}{8\pi^2}\right)$. Recall, that $\theta'$ is strictly decreasing by (A.9). Combining all this with equation (A.11) again, one deduces that on the interval $\left(\frac{2\pi}{\pi}, \frac{4\pi}{\pi + 1}\right)$ the function $-v'$ can be expressed as a product of three positive, strictly decreasing terms. Hence, on this interval $v'$ is negative and strictly increasing. Recall that $u' < u'\left(\frac{2\pi}{\pi}\right) = v'\left(\frac{2\pi}{\pi}\right)$ on $\left(\frac{2\pi}{\pi}, \frac{4\pi}{\pi + 1}\right)$, which now implies that

$$u'(x) < u'\left(\frac{2\pi}{\pi}\right) = v'\left(\frac{2\pi}{\pi}\right) < v'(x) \quad \text{for} \quad \frac{2\pi}{\pi} < x < \frac{12 + \pi^2}{8\pi^2}.$$ 

Since the inequality $u'(x) < v'(x)$ has already been shown for $x \geq \frac{12 + \pi^2}{8\pi^2}$, one concludes that $u' < v'$ holds on the whole interval $\left(\frac{2\pi}{\pi}, \frac{4\pi}{\pi + 1}\right)$. This completes the proof. 

**Lemma A.4.** One has

$$\left(1 - \frac{2}{\pi} \sin\left(\frac{2\theta}{3}\right)\right)^3 < \left(1 - \frac{2}{\pi} \sin\left(\frac{\theta}{2}\right)\right)^4 \quad \text{for} \quad 0 < \theta \leq \frac{\pi}{2}.$$ 

**Proof.** The proof is similar to the one of Lemma A.2. Define $u, v, w: \mathbb{R} \to \mathbb{R}$ by

$$u(\theta) := \sin\left(\frac{\theta}{2}\right), \quad v(\theta) := \frac{\pi}{2} - \frac{\pi}{2}\left(1 - \frac{2}{\pi} \sin\left(\frac{2\theta}{3}\right)\right)^{3/4}, \quad \text{and} \quad w(\theta) := u(\theta) - v(\theta).$$

Obviously, the claim is equivalent to the inequality $w(\theta) < 0$ for $0 < \theta \leq \frac{\pi}{2}$.

Observe that $u'''(\theta) = -\frac{1}{2} \cos\left(\frac{\theta}{2}\right) < 0$ for $0 \leq \theta \leq \frac{\pi}{2}$. In particular, $u'''$ is strictly increasing on $\left[0, \frac{\pi}{2}\right]$ and satisfies $u'''(\theta) > u'''(0) = -\frac{1}{8}$.

One computes

$$v^{(4)}(\theta) = \frac{\pi^{1/4}}{54} p\left(\frac{2\theta}{3}\right) \left(\frac{\pi - 2 \sin\left(\frac{2\theta}{3}\right)}{\pi}\right)^{13/4} \quad \text{for} \quad 0 \leq \theta \leq \frac{\pi}{2},$$

where

$$p(x) = 45 - 16\pi^2 + 4\pi(1 + 2\pi^2)x - (34 + 20\pi^2)x^2 + 44\pi x^3 - 27x^4.$$ 

The polynomial $p$ is strictly increasing on $[0, \frac{\sqrt{2}}{2}]$ and has exactly one root in the interval $\left(0, \frac{\sqrt{2}}{2}\right)$. Combining this with equation (A.12), one obtains that $v^{(4)}$ has a unique zero in the interval $\left(0, \frac{\sqrt{2}}{2}\right)$ and that $v^{(4)}$ changes its sign from minus to plus there. Moreover, it is easy to verify that $v^{(4)}\left(\frac{\sqrt{2}}{2}\right) < v^{(4)}(0) < -\frac{1}{8}$. Hence, one has $u''' < -\frac{1}{8}$ on $\left[0, \frac{\pi}{2}\right]$. Since $u''' \geq -\frac{1}{8}$ on $\left[0, \frac{\pi}{2}\right]$ as stated above, this implies that $w''' = u''' - v''' > 0$ on $\left[0, \frac{\pi}{2}\right]$, that is, $w'''$ is strictly increasing on $[0, \frac{\pi}{2}]$. 


With \( w''(0) < 0 \) and \( w''\left(\frac{\pi}{2}\right) > 0 \) one deduces that \( w'' \) has a unique zero in \((0, \frac{\pi}{2})\) and that \( w'' \) changes its sign from minus to plus there. Since \( w'(0) = 0 \) and \( w'(\frac{\pi}{2}) > 0 \), it follows that \( w' \) has a unique zero in \((0, \frac{\pi}{2})\), where it changes its sign from minus to plus. Finally, observing that \( w(0) = 0 \) and \( w\left(\frac{\pi}{2}\right) < 0 \), one concludes that \( w(\theta) < 0 \) for \( 0 < \theta \leq \frac{\pi}{2} \). □

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A. Seelmann, FB 08 - Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudinger Weg 9, D-55099 Mainz, Germany

E-mail address: seelmann@mathematik.uni-mainz.de