A hydrogen atom on curved noncommutative space

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Abstract
We have calculated the hydrogen atom spectrum on curved noncommutative space defined by the commutation relations $[\hat{x}_i, \hat{x}_j] = i\theta \hat{\omega}_{ij}(\hat{x})$, where $\theta$ is the parameter of noncommutativity. The external antisymmetric field which determines the noncommutativity is chosen as $\omega_{ij}(x) = \varepsilon_{ijk} x_k f(x_i x_j)$. In this case, the rotational symmetry of the system is conserved, preserving the degeneracy of the energy spectrum. The contribution of the noncommutativity appears as a correction to the fine structure. The corresponding nonlocality is calculated:

$$\Delta x \Delta y \geq \frac{\theta^2}{4} |m(f^2)|,$$

where $m$ is a magnetic quantum number.

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1. Introduction
A combination of quantum mechanics with general relativity indicates that at distances of the order of the Planck length the spacetime is nonlocal, and therefore it cannot be described as a differentiable manifold and should be treated as a kind of noncommutative (NC) structure [1]. Consider the NC space realized by the coordinate operators $\hat{x}_i$, $i = 1, \ldots, N$, satisfying the algebra

$$[\hat{x}_i, \hat{x}_j] = i\theta \hat{\omega}_{ij}(\hat{x}),$$

where $\hat{\omega}_{ij}(\hat{x})$ is an operator describing the noncommutativity of the space and $\theta$ is the parameter of noncommutativity. The consistency condition for the algebra (1) implies, see e.g. [2], that the symbol $\omega_{ij}(x)$ of the operator $\hat{\omega}_{ij}(\hat{x})$ should have the form

$$\omega_{ij}(x) = \omega^{(i)}(x) + \omega^{(ij)}(x),$$

where $\omega^{(i)}(x)$ is a Poisson bi-vector, i.e. obeys the Jacobi identity

$$\omega^{ik} \partial_j \omega^{jk} + \omega^{jk} \partial_i \omega^{jk} + \omega^{kl} \partial_j \omega^{lk} = 0,$$

and the term $\omega^{(ij)}(x)$ stands for non-Poisson corrections to $\omega^{(i)}(x)$ of higher order in $\theta$, expressed in terms of $\omega^{(i)}(x)$ and its derivatives, which depend on the specific ordering of the operator $\hat{\omega}_{ij}(\hat{x})$. So, in order to determine the noncommutativity (1), we should define from physical...
considerations the antisymmetric field $\omega^{ij}(x)$ obeying equation (3) and specify the ordering. In what follows, we treat $\omega^{ij}(x)$ as an external field and choose the symmetric Weyl ordering.

This paper aims to illustrate the possible phenomenological consequences of the presence of the noncommutativity of type (1), when the commutator between coordinates is a function of these coordinates. As a starting point to study noncommutativity of the general form, we consider quantum mechanics (QM). A two-dimensional model of position-dependent noncommutativity in QM was proposed in [3]. The particular example was considered in [4], where it was also observed that canonical operators in these models are in general non-Hermitian with respect to standard inner products [5]. For the example of QM and field theory on kappa–Minkowski space, see e.g. [6] and references therein. A hydrogen atom in fuzzy spaces was discussed in [7]. A general form of coordinate-dependent noncommutativity in QM was considered in [8]. Note that the quantum-mechanical scale of energies is rather different from the Planck scale; however, some important properties like the preservation of symmetries and corresponding consequences can be studied in QM.

In particular, it is a well-known fact that the canonical noncommutativity, $\{\hat{x}^i, \hat{x}^j\} = i\theta^{ij}$, where $\theta^{ij}$ is an antisymmetric constant matrix, breaks the rotational symmetry of the hydrogen atom, which removes the degeneracy of the energy levels [9]. This fact leads us to the bounds of noncommutativity in this model. The same logic remains in the field theory; see e.g. [10] and references therein. We will show here that the noncommutativity can be introduced in a way to preserve the symmetries and the corresponding degeneracy. To this end, in section 2, we describe the model of QM with NC coordinates [8] and construct an explicit form of the trace functional on the algebra of the star product corresponding to (1). In section 3, we consider a particle in a central potential on the curved NC space, defined by the external antisymmetric field $\omega^{ij}(x) = \epsilon^{ijk}x_k f(x,x')$. For simplicity, we set $(m_e = c = \hbar = 1)$.

### 2. Quantum mechanics with noncommutative coordinates

To define the QM on the NC spaces of type (1), we will need to introduce two objects, the star product and the trace functional. The existence of the star product for any Poisson bi-vector $\omega^{ij}(x)$ is guaranteed by the formality theorem by Kontsevich [11]; the explicit construction of the star product up to the fifth order in deformation parameter can be found in [2]. In particular, for any two functions, $f$ and $g$, it has the form

$$(f \star g)(x) = f(\hat{x})g(x) = fg + \frac{i\theta}{2} \partial_i f \omega^{ij} \partial_j g$$

$$- \frac{\theta^2}{4} \left[ \frac{1}{2} \omega^{ij} \omega^{kl} \partial_i \partial_k f \partial_j g - \frac{1}{3} \omega^{ij} \partial_j f \partial_k g - \partial_k f \partial_i g \right] + O(\theta^3).$$

The trace functional is determined as $\text{Tr}(f) = \int \Omega(x) f(x)$, where $\Omega(x)$ is an integration measure. The trace should obey the property

$$\text{Tr}(f \star g) - \text{Tr}(fg) = 0,$$

which also guarantee the cyclic property of trace. The existence of a trace functional for a Kontsevich star product related to any Poisson bi-vector $\omega^{ij}$ such that $\text{div}_{\Omega} \omega^{ij} = 0$ was demonstrated in [12]. But to make calculations, one needs an explicit form for it. Suppose that there exists a function $\mu(x)$ such that

$$\partial_i (\mu \omega^{ij}) = 0.$$  

Let $\Omega(x) = d^N x \mu(x)$, so

$$\text{Tr}(f) = \int d^N x \mu(x) f(x).$$

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We should verify if this definition of trace is correct, i.e. if (5) holds true for any two functions $f$ and $g$ vanishing on infinity. In the zero order in $\theta$, equation (5) is just an identity, in the first order it is satisfied due to (6). In the second order in $\theta$, the right-hand side of (5) is

$$\frac{\theta^2}{4} \int d^N x \mu(x) \left[ \frac{1}{2} \omega^{ij} \omega^{kl} \partial_i \partial_k f \partial_j \partial_l g - \frac{1}{3} \omega^{ij} \partial_j \omega^{kl} (\partial_i \partial_k f \partial_l g - \partial_i \partial_k f \partial_l g) \right].$$

Integrating this expression by parts on $f$ and $g$ and using (6) we rewrite it as

$$\frac{\theta^2}{12} \int d^N x f \partial_i (\mu \omega^{ij} \partial_j \omega^{kl}) \partial_k g,$$

where the matrix $\partial_i (\mu \omega^{ij} \partial_j \omega^{kl})$ is symmetric, i.e.

$$\partial_i (\mu \omega^{ij} \partial_j \omega^{kl}) = \partial_i (\mu \omega^{ij} \partial_j \omega^{kl}),$$

due to the Jacobi identity and (6). Expression (9) is different from zero for two arbitrary functions $f$ and $g$. To solve this problem, we use the gauge freedom in the definition of the star product [11]. Let us construct a new star product

$$f \star' g = D^{-1}(Df \star Dg),$$

choosing a gauge operator $D$ in such a way that (5) holds true for this new product. Taking into account (9) and (10), one should look for $D$ in the form

$$D = 1 + \theta^2 b^k \partial_k + O(\theta^3).$$

In this case,

$$f \star' g = f \star g - 2\theta^2 b^k \partial_k f \partial_k g + O(\theta^3).$$

Condition (5) for the star product (13) in the second order gives

$$\frac{\theta^2}{12} \partial_i f \partial_i (\mu \omega^{ij} \partial_j \omega^{kl}) \partial_k g - 2\theta^2 \mu b^k \partial_i f \partial_k g = 0.$$

That is,

$$b^k = \frac{1}{24 \mu} \partial_i (\mu \omega^{ij} \partial_j \omega^{kl}).$$

Note that the star product (13) will be associative if and only if $b^k$ is a symmetric tensor, which in turn is provided by equation (10). We conclude that given a Poisson bi-vector $\omega^{ij}$ and a function $\mu(x)$ obeying (6), the modified star product (13) admits the trace (7).

Now, according to [8] we describe the model of QM on NC spaces of a general form, defined by the algebra (1). The Hilbert space is determined as a space of complex-valued functions which are square-integrable with a measure $\Omega(x)$. The internal product between two states $\psi(x)$ and $\psi(x)$ from the Hilbert space is defined as

$$\langle \psi | \psi \rangle = \text{Tr}(\psi^\ast \star \psi).$$

The action of the coordinate operators $\hat{x}$ on functions $\psi(x)$ from the Hilbert space is defined through the modified star product (13), for any function $V(x)$ one has

$$V(\hat{x})\psi(x) = V(x) \star' \psi(x).$$

In particular, from $\hat{x} \psi = x \star' \psi$, one may see that

$$\hat{x}' = x' + \frac{\theta^2}{2} \omega^{ij} \partial_j - \frac{\theta^2}{2} \omega^{ij} \partial_j \partial_i + \frac{\theta^2}{12 \mu} \partial_i (\mu \omega^{ij} \partial_j \omega^{kl}) \partial_k + O(\theta^3).$$

Definitions (16) and (17) mean that the coordinate operators are self-adjoint with respect to the introduced scalar product: $\langle \hat{x} \psi | \psi \rangle = \langle \psi | \hat{x} \psi \rangle$. The momentum operators $\hat{p}_i$ are fixed from
the condition that they also should be self-adjoint with respect to (16). One of the possibilities is to choose it in the form

\[ \hat{p}_i = -i\hat{\partial}_i - \frac{i}{2} \hat{\partial}_i \ln \mu(x). \]  

(19)

One can easily verify that in this case \( \langle \hat{p}_i \phi | \psi \rangle = \langle \phi | \hat{p}_i \psi \rangle \).

The momentum operators (19) commute, \([\hat{p}_i, \hat{p}_j] = 0\). The commutator between \( \hat{x}^i \), defined in (18), and \( \hat{p}_j \) is

\[ [\hat{x}^i, \hat{p}_j] = i\delta^i_j - \frac{i\partial_j}{2} (\partial_i \omega^{kl}(\hat{x}) \hat{p}_k + i\partial_j (\omega^{kl} \hat{\partial}_l \ln \mu(\hat{x}))) + O(\theta^2). \]  

(20)

So, the complete algebra of commutation relations involving \( \hat{x}^i \) and \( \hat{p}_j \) is a deformation in \( \theta \) of a standard Heisenberg algebra. Finally, it should be noted that all expansions in our approach are formal; we do not discuss the convergence of perturbative series here.

3. Particle in a central potential

Let us consider a particle placed in a central potential \( V(r^2/2) \), where \( r^2 = x^2 + y^2 + z^2 \). The corresponding Hamiltonian reads

\[ \hat{H} = \frac{\hat{p}^2}{2} + V \left( \frac{r^2}{2} \right). \]  

(21)

As was already mentioned in the introduction, the canonical noncommutativity which can be realized by the coordinate operators \( \hat{x}_{can}^i = x^i + i/2\theta^{ij}\partial_j \) violates the rotational symmetry of (21), since \([\hat{L}_i, V(\hat{x}_{can}^2/2)] \neq 0\), where \( L^i = -i\epsilon^{ijk}x_j\hat{\partial}_k \) is the angular momentum operator. And this problem cannot be solved by introducing a noncommutativity as a Drinfeld twist deformation [13]. See also [14] for an example of non-Abelian twist in QM.

The external antisymmetric field \( \omega^{ij}(x) \) can be chosen in a way to preserve a rotational symmetry of the system. Let us suppose that

\[ \omega^{ij} = \epsilon^{ijk}x^k f(r^2), \]  

(22)

where \( f \) is a given function. The corresponding NC algebra is

\[ [\hat{x}^i, \hat{x}^j] = i\theta \epsilon^{ijk}x^k f(r^2). \]  

(23)

The consistency condition for this algebra, \([\hat{x}^i, \ epsilon^{jkl}x^l f(r^2)] + cycl.(ijk) = 0\), is satisfied automatically and no corrections \( \omega_{\epsilon}^{ij}(x) \) are needed to (22) to construct the explicit form of the star product in any order in \( \theta \) using the iterative procedure [2]. Note that the algebra (23) is a generalization of the algebra of a fuzzy sphere [15]. In particular, the rotationally invariant NC space can be obtained as a foliation of fuzzy spheres [16]. Moreover, since \([\hat{x}^2, \hat{x}^2] = 0\), one can introduce new NC coordinates \( \hat{x}_k = \hat{x}_k / f(r^2) \) that satisfy simpler commutation relations \([\hat{x}_k, \hat{x}_l] = i\theta \epsilon_{kij} \hat{x}_j\) and possess simple operator realizations.

One can easily see that any function \( \mu(r^2) \) obeys equation (6):

\[ \partial_i (\mu(r^2) \epsilon^{ijk}x^k f(r^2)) = 0, \]

and can be chosen as a measure to define a trace functional. For simplicity, we set \( \mu(x) = 1 \). So, the momentum operators are just derivatives \( \hat{p}_i = -i\hat{\partial}_i \). According to the previous section, these operators are self-adjoint with respect to the inner product (16), defined on the configuration space \( \mathbb{R}^3 \) with a measure \( \mu(x) = 1 \). However, some QM models with a rotational invariant potentials, like e.g. the hydrogen atom, are well defined on the configuration space \( \mathbb{R}^3 \setminus \{0\} \). In this case, the property of the momentum operator to be self-adjoint should be investigated separately, see e.g., [17] for this discussion in a standard QM.
where the external field is zero at the origin and then grows along the corresponding axis; however, \( m \) is a magnetic quantum number. According to Messiah [18], we use the following expressions for the unperturbed normalized position wavefunction in spherical coordinates:

\[
\psi_\text{1}\langle 0 \rangle = \sqrt{\frac{2}{n a_0}} \frac{2m(n + l)}{2n \alpha_0} \frac{2\alpha_0}{2(n + l)} Y_l^m(\theta, \varphi),
\]

for the Coulomb potential, the leading-order perturbation due to noncommutativity is

\[
\hat{V}_\text{NC} = -\frac{e^2\theta^2}{12} \frac{f^2 L^2}{r^3}.
\]

We use the usual perturbation theory to calculate the leading corrections to the energy levels

\[
\Delta E_n^\text{NC} = \langle \psi_0 | \hat{V}_\text{NC} | \psi_0 \rangle.
\]

According to Messiah [18], we use the following expressions for the unperturbed normalized position wavefunction in spherical coordinates:

\[
|\psi_0\rangle = \sqrt{\frac{2}{n a_0}} \frac{2m(n + l)}{2n \alpha_0} \frac{2\alpha_0}{2(n + l)} Y_l^m(\theta, \varphi),
\]

corresponding to the energy \( E_n = -e^2/2a_0r^2 \), where \( n \) is a principal quantum number and \( a_0 = 1/\alpha \) is the Bohr radius. Integrating over the angular variables \( \theta \) and \( \varphi \), we stay with

\[
\Delta E_n^\text{NC} = -\frac{e^2\theta^2}{12} \frac{f^2 (l + 1)}{r^3},
\]

where \( l \) is the azimuthal quantum number.

In order to calculate the corresponding nonlocality, we first write (23) as

\[
[\hat{\xi}, \hat{\xi}^*] = i\theta f e^{i\theta} x^k + \frac{i\theta^2}{2} f^2 e^{i\theta} L_k + O(\theta^3).
\]

Taking into account that \( \langle \psi_0 | f(r^2) x^k | \psi_0 \rangle = 0 \), due to the integration over the angular variables, we end up with

\[
\Delta x \Delta y \geq \frac{\theta^2}{4} |m(f^2)|,
\]

where \( m \) is a magnetic quantum number.

We will consider two possibilities for the function \( f(r) \): \( f = 1 \) and \( f = r \). In both cases, the external field is zero at the origin and then grows along the corresponding axis; however, in the second case the growth is higher. In the first case, taking into account the fact that

\[
\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{n^3a_0^3} \frac{1}{l(l + 1/2)(l + 1)},
\]

we end up with

\[
\Delta x \Delta y \geq \frac{\theta^2}{4} |m(f^2)|,
\]

where \( m \) is a magnetic quantum number.
the energy level shift is
\[ \Delta E_{NC}^n = -\frac{\theta^2 E_n^2 n}{3a_0 e^2 (l + 1/2)}. \] (32)

This energy correction has the same form as a relativistic kinetic energy contribution to the fine structure. Nonlocality in this case is defined by the relation
\[ \Delta x \Delta y \geq \frac{\theta^2}{4} |m|. \] (33)

Recalling that \( \langle \frac{1}{r} \rangle = \frac{1}{n^2 a_0} \), the energy correction for the second possibility \((f = r)\) is given by
\[ \Delta E_{NC}^n = \frac{\theta^2}{6} E_n l(l + 1). \] (34)

Finally since,
\[ \langle r^2 \rangle = \frac{n^2 a_0^2}{4} (4n^2 - l^2 + 2nl + 1), \]
the corresponding nonlocality is given by
\[ \Delta x \Delta y \geq |m| \theta^2 \frac{n^2 a_0^2}{16} (4n^2 - l^2 + 2nl + 1). \] (35)

We can see that in this case the nonlocality depends on the energy of the system—the greater the energy, the greater the nonlocality.

The alternative approach to the hydrogen atom on rotationally invariant NC space, obtained as a sequence of fuzzy spheres [16], was discussed in [7]. In this paper, the Hamiltonian of the system was constructed in such a way that all the symmetries of the H-atom were conserved, preserving the degeneracy of the energy spectrum over the magnetic \( m \) and the azimuthal \( l \) quantum numbers. While in our construction only the rotational symmetry of the system remains untouchable, the degeneracy over the azimuthal quantum number \( l \) is removed. We see the advantage of our approach to NCQM in the possibility of generalization. It is applicable to the case of the arbitrary NC space (1), see section 2, and not only to fuzzy spaces. However, our approach is essentially a perturbative construction and it does not allow one to recover non-perturbative consequences of the underlying NC space. To study such a consequence, one will need an explicit form of the star product on the considered NC space, which can be found in [19, 20].

4. Conclusions and perspectives

This example shows that the noncommutativity can be introduced in a minimal way in the theory, i.e. one may obtain nonlocality without violating physical observables like the energy spectrum, etc.

The considered model also indicates how to construct a relativistic generalization of QM with NC coordinates [8]. If the external field \( \omega^{\rho\sigma}(x) \), which determines the commutation relations \([\hat{x}^\rho, \hat{x}^\sigma] = i\theta \delta^{\rho\sigma} (\hat{\lambda})\), transforms as a two tensor with respect to a Lorentz group, the operators \( \hat{x}^\rho = x^\rho + i\theta / 2 \omega^{\rho\sigma} \delta_{\sigma} + O(\theta^2) \) and \( \hat{p}_\mu = -i\partial_\mu - i\partial_\mu \mu(x) \) will transform as vectors. This fact may be used to construct relativistic wave equations on curved NC spacetime. For simplicity, one may begin with \((2 + 1)\) dimensions, choosing the external antisymmetric field as \( \omega^{\rho\sigma}(x) = \delta^{\rho\sigma} x_3 f(\hat{\lambda}^2) \).

Alternative models of relativistic QM with noncommutative coordinates were constructed in [21] in \((3 + 1)\) dimensions and in [22] in \((2 + 1)\) dimensions as a relativistic version of
spin noncommutativity [23]. However, in this case, the commutator between coordinates is proportional to the spin, i.e. coordinates do not form an algebra. This leads to difficulties in the definition of the star product and makes the operators non-Hermitian [24].

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References

[1] Doplicher S, Fredenhagen K and Roberts J 1995 Commun. Math. Phys. 172 187
[2] Kupriyanov V G and Vassilevich D V 2008 Eur. Phys. J. C 58 627
[3] Gomes M and Kupriyanov V G 2009 Phys. Rev. D 79 125011
[4] Fring A, Gouba L and Scholtz F G 2010 J. Phys. A: Math. Theor. 43 345401
[5] Bagchi B and Fring A 2009 Phys. Lett. A 373 4307
[6] Harikumar E, Juric T and Meljanac S 2011 Phys. Rev. D 84 085020
Harikumar E, Juric T and Meljanac S 2012 Phys. Rev. D 86 045002
[7] Galikova V and Presnajder P 2012 J. Phys.: Conf. Ser. 343 012096
[8] Kupriyanov V G 2012 General form of quantum mechanics with noncommutative coordinates arXiv:1204.4823 [math-ph]
[9] Chaichian M, Sheikh-Jabbari M M and Tureanu A 2001 Phys. Rev. Lett. 86 2716
[10] Adorno T C, Gitman D M, Shabad A E and Vassilevich D V 2011 Phys. Rev. D 84 085031
[11] Kontsevich M 2003 Lett. Math. Phys. 66 157
[12] Felder G and Soibelman B 2000 Lett. Math. Phys. 53 75
[13] Chakraborty C, Kuznetsova Z and Toppan F 2010 J. Math. Phys. 51 112102
[14] Castro P G, Kullock R and Toppan F 2011 J. Math. Phys. 52 062105
[15] Madore J 1992 Class. Quantum Grav. 9 69
Grosse H, Madore J and Steinacker H 2002 Int. J. Mod. Phys. A 17 2095
[16] Moreno E 2005 Phys. Rev. D 72 045001
[17] Roy U, Gosh S and Bhattacharya K 2008 Rev. Mex. Fis. E 54 160
[18] Messiah A 1999 Quantum Mechanics (New York: Dover)
[19] Presnajder P 2000 J. Math. Phys. 41 2789
[20] Alexanian G, Pinzul A and Stern A 2001 Nucl. Phys. B 600 531
[21] Gomes M, Kupriyanov V G and da Silva A J 2010 Phys. Rev. D 81 085024
[22] Falomir H, Vega F, Gamboa J, Mendez F and Loewe M 2012 Phys. Rev. D 86 105035
[23] Falomir H, Gamboa J, Lopez-Sarrion J, Mendez F and Prisani P A G 2009 Phys. Lett. B 680 384
[24] Ferrari A F, Gomes M, Kupriyanov V G and Stechhahn C A 2013 Phys. Lett. B 718 1475