WEIGHING THE MILKY WAY

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Abstract. We describe an experiment to measure the mass of the Milky Way galaxy. The experiment is based on calculated light travel times along orthogonal directions in the Schwarzschild metric of the Galactic center. We show that the difference is proportional to the Galactic mass. We apply the result to light travel times in a 10cm Michelson type interferometer located on Earth. The mass of the Galactic center is shown to contribute $10^{-6}$ to the flat space component of the metric. An experiment is proposed to measure the effect.

Acknowledgement 1. We acknowledge the technical advice of M. Bocko, and A. Marnell for help with graphics. A.B. thanks St. John Fisher College for its hospitality and the United States Fulbright Scholarship Board for the award of a Fulbright Fellowship.

1. Introduction

What is the mass of the Galaxy? We answer this question by computing, in a model metric of the Galaxy, the travel time of light beams in a Michelson type interferometer located on Earth’s equator. We show that the light travel time is proportional to the mass of the Galaxy. The result suggests that an experiment can measure the mass of the Galaxy.

We start with a metric in Boyer-Lindquist coordinates, representing on the equatorial plane, the axially symmetric field of a rotating matter distribution with the same symmetry

\[
\text{(1.1)} \quad ds^2 = g_{tt} dt^2 + 2g_{t\phi} dtd\phi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2
\]

We will calculate and compare the light travel times for beams traveling along the radial, polar and azimuthal axes. The interferometer is small compared to the scale factors of the field.

1.1. Light beam traveling along the $\phi$–direction. In this section we calculate the travel time of a beam of light traveling along the $\phi$–direction. The beam is located on Earth’s equator ($\theta = \pi/2$), a coordinate distance $R$ from the field center. The interferometer orbits about the field center with a constant coordinate speed $v = R \Omega$, where $\Omega$ is the coordinate angular speed. The trajectory of light is a null geodesic, furthermore, $dr = d\theta = 0$:

\[
\text{(1.2)} \quad ds^2 = 0 = g_{tt} dt^2 + 2g_{t\phi} dtd\phi + g_{\phi\phi} d\phi^2
\]

Key words and phrases. 4.20.-q, 4.80.-y, 98.35.Jk.
We neglect the difference between a null geodesic and an arc of an equatorial circle (valid when the coordinate arm length of the interferometer \( l \ll R \)). The world line of the light beam between the two mirrors is:

\[
(1.3) \quad t_{\pm} = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{tt}} (\phi - \phi_0)
\]

We choose \( \phi_0 = 0 \). The positive sign is used when the light beam is traveling in the same sense as the interferometer. The world line of the end mirror of the equatorial arm is:

\[
(1.4) \quad t = \frac{\phi - \Phi}{\Omega}
\]

where \( \Phi \) is the angle subtended by the interferometer at the field center. The light ray reaches the mirror when

\[
(1.5) \quad \frac{-g_{t\phi} + \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{tt}} \phi = \frac{\phi - \Phi}{\Omega}
\]

Solving for the azimuth gives

\[
(1.6) \quad \phi_r = \frac{\Phi g_{tt}}{g_{t\phi} \Omega + g_{tt} - \Omega \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}
\]

and the coordinate time

\[
(1.7) \quad t_r = -\Phi \frac{g_{t\phi} - \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{t\phi} \Omega + g_{tt} - \Omega \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}
\]

The world line of the reflected ray is

\[
(1.8) \quad t = t_r + \frac{-g_{t\phi} - \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{tt}} (\phi - \phi_r)
\]

and of the beam splitter is

\[
(1.9) \quad t = \frac{\phi}{\Omega}
\]

The ray returns to the beam splitter when

\[
(1.10) \quad \frac{\phi}{\Omega} = t_r + \frac{-g_{t\phi} - \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{tt}} (\phi - \phi_r)
\]

or when the azimuth is
\[
\phi_t = \frac{g_{tt}t_r + g_{t\phi}\phi_r + \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}\phi_r}}{g_{tt} + g_{t\phi}\Omega + \Omega\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}
\]

and coordinate time
\[
\phi_t = \frac{g_{tt}t_r + g_{t\phi}\phi_r + \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}\phi_r}}{g_{tt} + g_{t\phi}\Omega + \Omega\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}
\]

Substituting the expressions for \(\phi_r\) and \(t_r\) gives
\[
t_l = 2\Phi \frac{\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{tt} + 2g_{t\phi}\Omega + \Omega^2g_{\phi\phi}}
\]

In terms of proper time of the interferometer this is equivalent to
\[
\tau_c = \frac{1}{c} \sqrt{g_{tt} + 2g_{t\phi}\Omega + \Omega^2g_{\phi\phi}t_l}
\]

1.2. Light travel time along the \(r\)-direction. We repeat the calculation for a light beam traveling inward from the beam-splitter at \(r = R\) to the end mirror a coordinate distance \(l\) at \(r = R'\), where \(R > R'\). In this case \(d\theta = 0\). The coordinate length of the interferometer arm is \(l\). The world line of the mirror is
\[
t = \frac{\phi}{\Omega}
\]
\[
r = R'
\]

The world line of the light ray is now
\[
t = \int_R^{R'} \left\{ \frac{-g_{t\phi} \frac{d\phi}{dr} - \sqrt{g_{t\phi}^2 \left( \frac{d\phi}{dr} \right)^2 - g_{tt} \left( g_{rr} + g_{\phi\phi} \left( \frac{d\phi}{dr} \right)^2 \right)}}{g_{tt}} \right\} dr
\]

The negative sign of the radical is used because the light beam is traveling inwards. Assume for the moment that \(\frac{d\phi}{dr}\) is small enough to be kept to first order only. The expression simplifies to:
\[
t \approx \int_R^{R'} \left( \frac{g_{t\phi}}{g_{tt}} \frac{d\phi}{dr} - \sqrt{-\frac{g_{rr}}{g_{tt}}} \right) dr
\]

In the example under consideration we can approximate the space trajectory of the light beam as a straight line starting from the origin at \((R,0,0)\) and passing through \((R',0,\phi_i)\). We can relate \(r\) with \(\phi\).
\begin{equation}
 r = \frac{R}{\cos \phi + \tan \xi \sin \phi} \tag{1.20}
\end{equation}

\(\xi\) is fixed by imposing

\begin{equation}
 R' = \frac{R}{\cos \phi_i + \tan \xi \sin \phi_i} \tag{1.21}
\end{equation}

This gives

\begin{equation}
 \tan \xi = \frac{R - R' \cos \phi_i}{R' \sin \phi_i} \approx \frac{R - R'}{R' \phi_i} \text{ for } \phi_i \ll 1 \tag{1.22}
\end{equation}

\begin{equation}
 r = \frac{R}{\cos \phi + \frac{R - R' \cos \phi}{R' \phi_i} \sin \phi} \tag{1.23}
\end{equation}

\begin{equation}
 r \approx R - R\phi \tan \xi + \left( \frac{1}{2} R \tan^2 \xi \right) \phi^2 + O(\phi^3) \approx R - R\phi \tan \xi \tag{1.24}
\end{equation}

\begin{equation}
 \frac{d\phi}{dr} \approx - \frac{1}{R \tan \xi} \approx - \frac{1}{R} \frac{R'}{R - R'} \phi_i \tag{1.25}
\end{equation}

The expression for \(t\) approximates to:

\begin{equation}
 t \approx \frac{1}{R} \frac{R'}{R - R'} \phi_i \int_{R}^{R'} \frac{g_{t\phi}}{g_{tt}} dr - \int_{R}^{R'} \sqrt{-g_{rr}} dr \tag{1.26}
\end{equation}

The metric elements are

\begin{align}
 g_{tt} &= c^2 \left( 1 - \frac{\alpha}{r} \right) \tag{1.27} \\
 g_{rr} &= - \left( 1 - \frac{\alpha}{r} \right)^{-1} \tag{1.28} \\
 g_{\theta\theta} &= - r^2 \tag{1.29} \\
 g_{\phi\phi} &= -r^2 \sin^2 \theta \tag{1.30} \\
 g_{t\phi} &= \frac{\alpha}{c} r \sin^2 \theta \tag{1.31}
\end{align}

where \(\alpha\) is the Schwarzschild radius and \(a\) is the specific angular momentum of the field source.

The integral for \(t\) is:

\begin{equation}
 t \approx \frac{1}{R} \frac{R'}{R - R'} \phi_i \frac{\alpha a}{c} \int_{R}^{R'} \frac{1}{r} dr - \frac{1}{c} \int_{R}^{R'} \left( 1 + \frac{\alpha}{r} \right) dr \tag{1.32}
\end{equation}

\begin{equation}
 t_i \approx \frac{R - R'}{c} + \frac{\alpha}{c} \left( 1 - \frac{R' \phi_i}{R (R - R')} a \right) \log \frac{R}{R'} \tag{1.33}
\end{equation}

Recalling that \(\phi_i = \Omega t_i\)
(1.34) \[ t_i \simeq \frac{R - R'}{c} + \frac{\alpha}{c} \left( 1 - \frac{R' \Omega t_i}{R(R - R') a} \right) \log \frac{R}{R'} \]

(1.35) \[ t_i \simeq \frac{R - R'}{c} + \frac{\alpha}{c} \left( 1 - \frac{R' \Omega t_i a}{R c} \right) \log \frac{R}{R'} \]

Under conditions where the length of the interferometer arm \( l \ll R \)

(1.36) \[ t_i \simeq \frac{l}{c} \left[ 1 + \frac{\alpha}{R} \left( 1 - \frac{a}{c} \right) \right] \]

This expression reproduces the Shapiro time delay \[ 3 \] with a small correction due to the angular momentum of the field source.

For the return trip to the center of the interferometer,

(1.37) \[ r = \frac{R'}{\cos \phi - \tan \zeta \sin \phi} \]

In this case both \( \phi \) and \( t \) are zero at the starting point i.e., the end mirror. Once again

(1.38) \[ \tan \zeta = \frac{R \cos \phi_e - R'}{R \sin \phi_e} \simeq \frac{R - R'}{R \phi_e} \]

(1.39) \[ \frac{d\phi}{dr} \simeq \frac{1}{R' \tan \zeta} \]

The time for the return trip (using now a positive sign for the radical in (1.18)), \( t_e \simeq t_i \). The round-trip takes

(1.40) \[ t_r = t_i + t_e \simeq 2 \frac{l}{c} \left[ 1 + \frac{\alpha}{R} \left( 1 - \frac{a}{c} \right) \right] \]

We note that the corrections add in the round-trip travel time.

In terms of proper time

(1.41) \[ \tau_r = \frac{1}{c} \sqrt{g_{tt} + 2g_{t\phi} \Omega + g_{\phi\phi} \Omega^2 t_r} \]

(1.42) \[ \tau_r \simeq \sqrt{1 - \frac{\alpha}{R} + 2 \frac{\alpha a R \Omega}{R^2 c} - \left( \frac{R \Omega}{c} \right)^2 \frac{1}{2c} \left[ 1 + \frac{\alpha}{R} \left( 1 - \frac{a}{c} \right) \right]} \]

We define three dimensionless parameters:

(1.43) \[ \mu \equiv \frac{\alpha}{R} \]

(1.44) \[ \kappa \equiv \frac{a}{R} \]

(1.45) \[ \beta \equiv \frac{\Omega R}{c} \]

Keeping terms which are of lowest order in the parameters \( \mu, \kappa \) and \( \beta \), the radial proper time of flight is
There is a general relativistic contribution of $\simeq \alpha / R$.

Substituting the metric elements in the expression for $\tau_e$ and carrying out the calculations to the same order (with $l \simeq R \Phi$)

\begin{align}
\tau_e & \simeq \frac{2 r_1 \Phi}{c} \sqrt{\frac{(\mu \kappa)^2 + (1 - \mu)}{1 - \mu + 2 \mu \kappa \beta - \beta^2}} \\
\tau_e & \simeq \frac{2 l c}{c} \left[ 1 + \frac{1}{2} \beta^2 \right]
\end{align}

Note that there is no first order general relativistic contribution. However, in the difference in the time of flight between the two arms

$$\delta \tau_{re} = \tau_r - \tau_e \simeq \frac{l}{c} \left( \mu - 2 \beta^2 - \frac{5}{4} \mu^2 \right)$$

there is a general relativistic contribution.

1.3. **Light beam traveling along the $\theta$–direction.** The light ray travels from the beam splitter along the local meridian, either towards the North or South. In this case $dr \simeq 0$. The travel time from the beam splitter to the end mirror is

$$t = -\Phi \int_0^1 \frac{g_{\phi \phi}}{g_{tt}} \frac{d\phi}{d\chi} d\chi \pm \Phi \int_0^1 \sqrt{\frac{1}{g_{tt}} \left( \frac{d\Phi}{d\chi} \right)^2 - g_{tt} \left( g_{\theta \theta} + g_{\phi \phi} \left( \frac{d\phi}{d\chi} \right)^2 \right)} d\chi$$

where the angle $\chi$ is complementary to $\theta$. The integral limits are chosen as 0 and $\Phi$ instead of $\chi_1$ and $\chi_2$ because $\Phi$ is the angle subtended by the interferometer arms. The $g_{\mu \nu}$’s maintain the values they have at the equator because we approximate the rays as straight lines on a sphere of radius $R$. The rays have a space trajectory

$$\phi = k \chi$$

The time of flight is

$$t_N = -\frac{g_{\phi \Phi}}{g_{tt}} k \Phi + \frac{1}{g_{tt}} \sqrt{\frac{g_{\phi \phi}}{g_{\phi \phi}} k^2 - g_{tt} \left( g_{\theta \theta} + g_{\phi \phi} k^2 \right)} \Phi$$

$$t_N \simeq \frac{1}{g_{tt}} \sqrt{-g_{tt} g_{\phi \Phi} k} - \frac{g_{\phi \phi}}{g_{tt}} k \Phi$$

Also

$$\Omega t_N = k \Phi$$

$$k = \frac{\Omega}{\Phi} t_N$$
Thus

\[ t_N = \sqrt{-g_{tt}g_{\theta \theta} \Phi} \]

The reverse path, calculated with a \((-\cdot\) sign outside the radical in (1.51), is just the same i.e., \(t_S = t_N\), thus the total time of flight is

\[ t_M = t_S + t_N = 2\sqrt{-g_{tt}g_{\theta \theta} \Phi} \]

(1.57)

\[ t_M \simeq \frac{l}{c} \left[ 1 + \frac{1}{2} \mu + \frac{3}{8} \mu^2 \right] \]

(1.58)

In terms of proper time

\[ \tau_m = t_M \sqrt{1 - \mu + 2 \mu \kappa \beta - \beta^2} \]

(1.59)

\[ \tau_m \simeq \frac{l}{c} \left[ 1 - \frac{1}{2} \beta^2 \right] \]

(1.60)

Notice again the absence of general relativistic contribution; this holds true for light travel time on the tangent or \(\theta - \phi\) plane.

The differences in proper times are

\[ \delta \tau_{rm} = \tau_r - \tau_m = \frac{l}{c} \mu \left[ 1 - \frac{5}{4} \mu \right] \]

(1.61)

\[ \delta \tau_{em} = \tau_e - \tau_m = \frac{l}{c} \beta \]

(1.62)

Expressions (1.61) and (1.62) suggest the experiment.

It is worth emphasizing that the differences are in proper time; the result is independent of any coordinate system. Indeed a careful calculation using isotropic or harmonic Schwarzschild coordinates confirms this assertion as does a calculation using an expansion about a point.

These results are not really new; they conform to other relativistic time delay phenomena such as the radar-echo experiment of Shapiro as well as the clock corrections needed for GPS navigation satellites.

1.4. Numerical values. For the Milky Way Galaxy (assuming it is a homogeneous disk):

\[ \frac{\alpha}{R} \simeq \frac{10^{16} m}{2.8 \times 10^{20} m} \simeq 10^{-4} \text{ to } 10^{-6}; \]

(1.63)

\[ \beta \simeq 10^{-3} \]

(1.64)

\[ a = \frac{R^2 \Omega}{2c} = \frac{R v_p}{2c} \sim (10^{14} \text{ to } 10^{18}) m \]

(1.65)

\[ \frac{a}{R} \sim 10^{-6} \text{ to } 10^{-2} \]

(1.66)

for peripheral velocity \(v_p \sim 600 \text{ km/s}\).

For an interferometer located on Earth of length 10cm \(\delta \tau_{rm} \simeq 6 \times 10^{-15}\) seconds.

This corresponds to an apparent increase in length due to the Galactic center of \(\simeq 1000 \text{ Å} \text{ in the radial arm.} \)

Corresponding figures for the Sun-Earth system are:
(1.67) \[ \frac{\alpha}{R} = \frac{10^3m}{10^{11}m} \simeq 10^{-8}; \]
(1.68) \[ \beta \simeq 10^{-4} \]

And at Earth’s surface:

(1.69) \[ \frac{\alpha}{R} = \frac{10^{-2}m}{6 \times 10^6m} \simeq 10^{-8}; \]
(1.70) \[ \beta \simeq 10^{-6} \]

The Galaxy is the major source of metric perturbation in the vicinity of Earth. Note that the Galactic influence may need to be built into the clock rate adjustment in the GPS navigation system; currently only the much smaller Earth’s effect is built-in.

Why is there a measurable effect proportional to the potential \( \alpha/r \), when one can always define a new set of coordinates to make \( \alpha/r = 0 \) at \( r \)? The answer lies in the extended reach of the interferometer; it is not possible, except in flat space, to make the potential zero everywhere over an extended region. The interferometer measures a potential average over a 10cm \( \times \) 10cm \( \times \) 10cm space-like region. This is not zero; and cannot be made zero everywhere within the region using a single transformation.

An alternative explanation is this: Two null vectors are transported simultaneously, in closed paths, along two orthogonal axes; when they return to the origin there is an angle defect between them, which is, as expected, a first order effect in \( \alpha/r \).

1.5. Experiment, Apparatus and Noise. As the interferometer rotates with Earth the output measures alternately, \( \delta \tau_{rm} \) and \( \delta \tau_{em} \). The signal appears as alternating bright and dark regions in the combined beam. The interferometer is located at the equator (assumed only to simplify the discussion), with the two arms aligned West-East and North-South. As the interferometer rotates with Earth, the North-South arm maintains its relative alignment (this is not strictly true but we make this assumption to simplify the discussion) with the Galactic center while the orientation of West-East arm alternates between the radial and azimuthal directions every 12 hours. The signal is sinusoidal, modulated with a period of 12 hours; it can be recovered using a homodyne detector. This is a null experiment in the sense that the appropriate phase of the dynamic output is zero in the absence of any general relativistic effect, the output measures only deviations from a flat metric.

It is worth emphasizing that the computed general relativistic fringe shift is for each round-trip of the light beam. The fringe shift is the result of transporting, in a closed loop, a null vector in curved space. Each subsequent round-trip adds to the fringe shift from the previous cycle. By contrast a fringe shift in flat space is not cumulative; it is static.

Each round-trip, which takes \( 0.67 \times 10^{-9} \) seconds, contributes 1/5 of a fringe shift. Thus the fringe shift is wiped out every 5 round-trips or in \( 3 \times 10^{-9} \) seconds. In order to observe the fringe shift one needs to strobe the output signal with a frequency which is the inverse of \( 0.67 \times 10^{-9} \) seconds or 1.5 GHz.
Anisotropic Lorentz contraction of the interferometer arms \((1.47)\) will occur with a magnitude \(\propto (1/2) \beta^2 \approx 10^{-7}\). This is smaller than the expected signal but also 90° out of phase with it, hence distinguishable.

Although we have assumed that the length of both interferometer arms is exactly equal, this will not be so in practice, nor is it necessary. Any remnant inequality will show up as a zero-offset in the sinusoidal signal.

A time delay of \(10^{-16}\) secs. corresponds to \(\approx 1/5\) of a fringe shift. Is this measurable in the presence of noise from sources such as mechanical/thermal noise, photodetector noise; mechanical distortions such as sag and tilt, seismic noise, laser intensity fluctuations, laser wavelength/frequency fluctuations etc.\(\quad [1]\)? The effect we are looking for is independent of wavelength, so neither laser wavelength nor intensity fluctuations nor drift will affect the result if the apparatus is kept in a vacuum environment. However, the interference condition does depend on wavelength, so the laser wavelength needs to be stable to \(\ll 1/5\) of a wavelength.

Among the noise sources listed, the most relevant is thermal stability since we are looking for a change in apparent length. Low-expansion substances such as Zerodur or sapphire would be suitable platform materials. With expansion coefficients of \(\simeq 10^{-6}\) a length of 10cm can be kept to within \(\ll \pm 1000\) Å by controlling the temperature to within \(\pm 10^{-4}\)K. This is a modest challenge at room temperatures.

Vibration isolation is effective with increasing resonance frequencies; this also makes sapphire or Zerodur suitable materials because of their mechanical stiffness.

Integration intervals over several days or weeks will require reference oscillators stable over long periods. Oscillators are available with fractional frequency stabilities of \(\pm 10^{-16}\) over months.

One may ask why this relatively large effect was not detected in experiments going back to that of Michelson and Morley. We have studied published accounts\(\quad [2]\) and have concluded, tentatively, that the apparatuses and methods used were neither sensitive to nor designed to measure what we are proposing.

1.6. **Proposed experiments.** Two experiments are proposed, and a recommendation:

1) Measure mass of Galaxy (the part within the Galactic center and Solar system)

2) Possible improvement in the uncertainties in the Shapiro time delay using the effect of the Sun on the interferometer.

3) An immediate recommendation: Make clock rate corrections to satellite based navigation systems.

The second would be a challenge; it may require a cryogenic environment where temperature stabilities of \(\pm 10^{-6}\)K are possible. This, combined with an expansion coefficient of sapphire at 4K which is estimated to be \(\sim 10^{-11} K^{-1}\), may reduce thermal stability to \(\sim 10^{-17}\)\(\quad [4]\).

The Galactic center would provide a strong background for the second experiment. It may be possible to extract the solar signal by monitoring the interferometer output during months of June and December when the Sun, Earth and Galactic center are aligned, then during March and September when they form a right angle triangle. The two signals, with different amplitudes, will have a relative phase which will progress from 0° to 90° between December and March.

It would appear that the Galaxy can be weighed with a table-top device which is sensitive to metric perturbations over cosmic distances.
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