Abstract

We investigate the properties of ideals associated with Kuratowski partitions of non-complete Baire metric spaces. We show that such an ideal can be precipitous.

1 Introduction

In 1935 K. Kuratowski in [11] posed the problem whether a function \( f : X \to Y \), (where \( X \) is completely metrizable and \( Y \) is metrizable), such that each preimage of an open set of \( Y \) has the Baire property, is continuous apart from a meager set.

R. H. Solovay and L. Bukovský independently proved the non-existence of Kuratowski partitions of the unit interval \([0,1]\) for measure and category by forcing methods (and the generic ultrapower), but Bukovský’s proof, (see [1]), is shorter and less complicated than Solovay’s (unpublished results).

In [4] there is shown that this problem is equivalent to the problem of the existence of partitions of completely metrizable spaces into meager sets with the property that the union of each subfamily of this partition has the Baire property. Such a partition is called \( \text{Kuratowski partition} \), (see the next section for a formal definition).

*The author is partially supported by Wroclaw University of Science and Technology grant no. 049U/0032/19.

*Mathematics Subject Classification: Primary 03C25, 03E35, 03E55, 54E52.

Keywords: non-complete Baire metric space, Kuratowski partition, precipitous ideal, K-ideal, game theory.
In paper [10] there was introduced the notion of K-ideals associated with Kuratowski partitions and there were examined their properties. It can be supposed that from a structure of such K-ideal one can ”decode” complete information about Kuratowski partition of a given space. Unfortunately, this is not the case because, as shown in [10], the structure of such an ideal can be almost arbitrary, i.e. it can be the Fréchet ideal, so by [8, Lemma 22.20, p. 425] it is not precipitous, whenever \( \kappa \) is regular. Moreover, as demonstrated in [10], for measurable cardinal \( \kappa \), a \( \kappa \)-complete ideal can be represented by some K-ideal. However if \( \kappa = |F| \) is not measurable cardinal, where \( F \) is Kuratowski partition of a given space, then one can obtain an \( |F| \)-complete ideal which can be the Fréchet ideal or a \( \kappa \)-complete ideal representing some K-ideal or can be a proper ideal of such K-ideal and contains the Fréchet ideal. Thus, for obtaining Kuratowski partition from K-ideal we need to have complete information about the space in which the ideal is considered.

In the presence of above considerations and the statement that ZFC + ”there exists precipitous ideal” is equiconsistent with ZFC + ”there exists measurable cardinal”, (see [6]), the natural question is: under which assumptions K-ideals can be precipitous, see [8, Theorem 22.33, p.432]. Such information can lead us to prove the required implication mentioned above. Our work in this topic (divided into two papers) enlarges results of [6], where the authors proved among others that ZFC + ”there exists a Kuratowski partition of a Baire metric space” is consistent, then ZFC + ”there exists measurable cardinal” is consistent as well, by using forcing methods (i.e. a model of the G-generic ultrapower in Keisler sense, see [2, sec. 6.4] and [8] for details). In this paper we show that K-ideal of non-complete Baire metric space can be precipitous. In [6] there is shown combinatorial proof of the following statement: if a cardinal \( \kappa \) is measurable, then there exists a complete metric Baire space with Kuratowski partition of size \( \kappa \). Thus, it remains to show the converse implication. (As it will be shown in the proof, the assumption of completeness of this space cannot be omitted). This result will be given in a separate paper.

This paper consists of three sections. Section 2 contains definitions and previous results concerning among others a Kuratowski partition, and a precipitous ideal. Section 3 there is presented the main result: if \( X \) is a Baire metric space with Kuratowski partition \( F \) of size \( \kappa \), (\( \kappa \) is regular and uncountable) and \( I_{F} \) is a K-ideal associated with \( F \), then there exists an open set \( U \subseteq X \) such that the K-ideal \( I_{F \cap U} \) is a precipitous ideal on \( \kappa \). Since the proof of this result seems to be a bit complicated, we decided to present the
second proof of this theorem in the game theoretic notion. Namely, we will use the game theoretic characterisation of precipitous ideals, see ⁹. In this place it is worth emphasizing that the notion ”α-favorable”, used in Section 3, comes from Choquet, (see ³, where the reader can also find more information about equivalences of Baire spaces in terms of games). The paper is finished with Section 4 including open problem concerning possibilities of elaborating our results for weakly- and pseudo precipitous ideals.

For definitions and facts not cited here we refer to e.g. ⁵, ¹² (topology) and ⁸ (set theory).

2 Definitions and previous results

2.1. Let $X$ be a topological space. A set $U \subseteq X$ has the Baire property iff there exist an open set $V \subset X$ and a meager set $M \subset X$ such that $U = V \triangle M$, where $\triangle$ means the symmetric difference of sets.

2.2. A partition $\mathcal{F}$ of $X$ into meager subsets of $X$ is called Kuratowski partition iff $\bigcup \mathcal{F}'$ has the Baire property for all $\mathcal{F}' \subseteq \mathcal{F}$. If there exists a Kuratowski partition of $X$ we always denote by $\mathcal{F}$ with the smallest cardinality $\kappa$. Moreover, we enumerate

$$\mathcal{F} = \{ F_\alpha : \alpha < \kappa \}.$$  

Obviously, $\kappa$ is regular. If $\kappa$ was singular, then $cf(\kappa)$ would be the minimal one. By Baire Theorem $\kappa$ is uncountable.

For a given set $U \subseteq X$ the family

$$\mathcal{F} \cap U = \{ F \cap U : F \in \mathcal{F} \}$$

is Kuratowski partition of $U$ as a subspace of $X$.

2.3. With any Kuratowski partition $\mathcal{F} = \{ F_\alpha : \alpha < \kappa \}$, indexed by a cardinal $\kappa$, one may associate an ideal

$$I_\mathcal{F} = \{ A \subset \kappa : \bigcup_{\alpha \in A} F_\alpha \text{ is meager} \}$$

which is called $K$-ideal, (see ¹⁰). Note, that $I_\mathcal{F}$ is a non-principal ideal. Moreover, $[\kappa]^{<\kappa} \subseteq I_\mathcal{F}$ because $\kappa =$
min\{\vert F \vert ; F \text{ is Kuratowski partition of } X \}.

2.4. Let \( I \) be an ideal on \( \kappa \) and let \( S \) be a set with positive measure, i.e. \( S \in P(\kappa) \setminus I \). (For our convenience we use \( I^+ \) instead of \( P(\kappa) \setminus I \).

An \( I \)-partition of \( S \) is a maximal family \( W \) of subsets of \( S \) of positive measure such that \( A \cap B \in I \) for all distinct \( A, B \in W \).

An \( I \)-partition \( W_1 \) of \( S \) is a refinement of an \( I \)-partition \( W_2 \) of \( S \), \( (W_1 \leq W_2) \), iff each \( A \in W_1 \) is a subset of some \( B \in W_2 \).

A functional \( \Phi \) on \( S \) is a collection of functions such that \( \{ \text{dom} (f) : f \in \Phi \} \) is an \( I \)-partition of \( S \) and \( \text{dom} (f) \neq \text{dom} (g) \), whenever \( f \neq g \in \Phi \). This \( I \)-partition will be denoted by \( W_\Phi \). Elements of functionals will be called \( I \)-functions.

We define \( \Phi \leq \Psi \) if
(i) each \( f \in \Phi \cup \Psi \) is a function into the ordinals;
(ii) \( W_\Phi \leq W_\Psi \);
(iii) if \( f \in \Phi \) and \( g \in \Psi \) are such that \( \text{dom} (f) \subseteq \text{dom} (g) \), then \( f(x) < g(x) \)
for all \( x \in \text{dom} (f) \).

If \( I \) is a \( \kappa \)-complete ideal on \( \kappa \) containing singletons, then \( I \) is precipitous iff whenever \( S \in I^+ \) and \( \{ W_n : n < \omega \} \) is a sequence of \( I \)-partitions of \( S \) such that \( W_0 \supseteq W_1 \supseteq \ldots \supseteq W_n \supseteq \ldots \), then there exists a sequence of sets \( X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \supseteq \ldots \) such that \( X_n \in W_n \) for each \( n \in \omega \) and \( \bigcap_{n=0}^{\infty} X_n \neq \emptyset \), (see also [8, p. 424-425]).

The ideal \( I_F \) is an everywhere precipitous ideal if \( I_F \cap U \) is precipitous for each non-empty open set \( U \subseteq X \).

We will need the following characterization of precipitous ideals, (see [8, Lemma 22.19, p. 424-25]).

**Fact 1** ([8]) The following are equivalent
(i) \( I \) is precipitous;
(ii) For no \( S \) of a positive measure is there a sequence of functionals on \( S \) such that \( \Phi_0 > \Phi_1 > \ldots > \Phi_n > \ldots \).

2.5. Let \( FN(\kappa) = \{ f \in X_\kappa : \exists U_f \text{ family of open disjoint sets, } \bigcup U_f \text{ is dense in } X \text{ and } \forall F_\alpha \in F, \forall U \in U_f, f \text{ is constant on } F_\alpha \cap U \} \).
Fact 2 (6) If \( f, g \in FN(\kappa) \), then
(a) \( \{ x : f(x) < g(x) \} \) has the Baire property,
(b) \( \{ x : f(x) = g(x) \} \) has the Baire property.

2.6. Let \( \kappa \) be, as previously, a regular uncountable cardinal and let \( I \) be a non-principal \( \kappa \)-complete ideal on \( \kappa \).

Consider an infinite game \( \mathcal{G}(I) \) played by two players \( \alpha \) and \( \beta \) as follows:
\( \alpha \) moves first by choosing a set \( A_0 \in I^+ \). Then \( \beta \) chooses a set \( B_0 \subseteq A_0 \) such that \( B_0 \in I^+ \). Then \( \alpha \) chooses \( A_1 \subseteq B_0 \) such that \( A_1 \in I^+ \) and so on. Thus, players produce a sequence of sets

\[ A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \supseteq \ldots \]

where \( A_i, B_i \in I^+, i \in \omega \). Player \( \alpha \) wins iff \( \bigcap_{n \in \omega} A_n = \emptyset \). Then we say that the game \( \mathcal{G}(I) \) is \( \alpha \)-favorable.

Fact 3 (9) Let \( \kappa \) be a regular uncountable cardinal and let \( I \) be a non-principal \( \kappa \)-complete ideal on \( \kappa \). Then \( I \) is a precipitous ideal iff \( \mathcal{G}(I) \) is not \( \alpha \)-favorable.

Consider an infinite game \( \mathcal{G}_1(I) \) played by two players \( \alpha \) and \( \beta \). \( \alpha \) starts the game by choosing an \( I \)-function \( h_0 \). Then \( \beta \) answers by choosing a set \( B_0 \subseteq \text{dom}(h_0) \) such that \( B_0 \in I^+ \). Then \( \alpha \) chooses an \( I \)-function \( h_1 \) such that \( \text{dom}(h_1) \subseteq B_0 \) and \( h_1(\xi) < h_0(\xi) \) for all \( \xi \in \text{dom}(h_1) \). Players continue the game as is described above producing a sequence

\[ (h_0, B_0, h_1, B_1, \ldots) \]

of \( I \)-functions \( h_0, h_1, \ldots \) such that \( \text{dom}(h_{n+1}) \subseteq \text{dom}(h_n) \) and \( h_{n+1}(\xi) < h_n(\xi) \) for all \( \xi \in \text{dom}(h_{n+1}) \) and \( B_n \in I^+, (n \in \omega) \). The game \( \mathcal{G}_1(I) \) is \( \alpha \)-favorable if \( \alpha \) can continue the game infinitely. Otherwise \( \beta \) wins.

Fact 4 (9) \( \mathcal{G}(I) \) is \( \alpha \)-favorable iff \( \mathcal{G}_1(I) \) is \( \alpha \)-favorable.

Fact 5 (9) Let \( \kappa \) be a regular uncountable cardinal and let \( I \) be a non-principal \( \kappa \)-complete ideal on \( \kappa \). Then \( I \) is a precipitous ideal iff \( \mathcal{G}_1(I) \) is not \( \alpha \)-favorable.
3 Main result

**Theorem 1** Let $X$ be a Baire metric space with Kuratowski partition $\mathcal{F}$ of cardinality $\kappa$, where $\kappa = \min\{|\mathcal{K}|: \mathcal{K} \text{ is Kuratowski partition of } X\}$. Then there exists an open set $U \subset X$ such that the $\mathcal{K}$-ideal $I_{\mathcal{F} \cap U}$ on $\kappa$ associated with $\mathcal{F} \cap U$ is precipitous.

**Proof.** Let $\mathcal{F} = \{F_\alpha: \alpha < \kappa\}$, (as was fixed in Section 2.2). We will show that there exists an open set $U \subset X$ such that

$$I_{\mathcal{F} \cap U} = \{A \subset \kappa: \bigcup_{\alpha \in A} F_\alpha \cap U \text{ is meager}\}$$

is precipitous.

Suppose that for any open $U \subset X$ the ideal $I_{\mathcal{F} \cap U}$ is not precipitous. Fix a family $\mathcal{U}$ of open and disjoint subsets of $X$ such that $\bigcup \mathcal{U}$ is dense in $X$ and fix $U \in \mathcal{U}$. Then by Fact 1 there exists a sequence of functionals $\Phi_U^0 > \Phi_U^1 > ...$ on some set $S^U \in I_{\mathcal{F} \cap U}^+$. Let $W^U_k = W_{\Phi_U^k}$ be an $I_{\mathcal{F} \cap U}$-partition (defined in Section 2.4). Let $X^U_k \subset S^U$ be such that $X^U_k \cap W^U_k \neq \emptyset$ for any $X^U_k \in W^U_k$, $k \in \omega$.

Each $X^U_k$ is the domain of some $I_{\mathcal{F} \cap U}$-function $h^U_k \in \Phi^U_k$ and if $X^U_k \supseteq X^U_{k+1}$, then $h^U_k(\beta) > h^U_{k+1}(\beta)$ for all $\beta \in X^U_{k+1}$, (see Section 2.4.)

Now, for any $\beta \in X^U_k$ and any $h^U_k \in \Phi^U_k$ define a function $f^U_{k, \beta} \in X^U_k$ such that

1) $\text{dom}(f^U_{k, \beta}) = F_\beta \cap U$,
2) $f^U_{k, \beta}(x) = h^U_k(\beta)$ for any $x \in F_\beta \cap U$.

Then, by properties of functions $h^U_k, k \in \omega$ we have that

$$f^U_{k, \beta}(x) > f^U_{k+1, \beta}(x) \text{ for any } x \in F_\beta \cap U.$$

Now, for any $k \in \omega$ consider a function

$$f_k = \bigcup_{U \in \mathcal{U}} \bigcup_{\beta < k} f^U_{k, \beta}.$$

Then $f_k \in FN(\kappa)$ for any $k \in \omega$. By Fact 2, for any $k \in \omega$ the set

$$V_k = \{x: f_k(x) > f_{k+1}(x)\}$$
has the Baire property. Then for any \( k \in \omega \) the set \( X \setminus V_k \) has also the Baire property and moreover is meager in \( X \).

Indeed. Suppose that there is \( k_0 \in \omega \) for which \( X \setminus V_{k_0} = M \triangle W \) for some meager \( M \) and open \( W \) and such that \( (X \setminus V_{k_0}) \cap W \) is nonempty. Let \( x' \in (X \setminus V_{k_0}) \cap W \). Then \( f_{k_0}(x') \leq f_{k_0+1}(x') \). But \( f_k = \bigcup_{U \in \mathcal{U}} \bigcup_{\beta < \kappa} f^U_{k,\beta} \) and by 2) in the definition of \( f^U_{k,\beta} \) we have that \( h^U_{k_0}(\beta) \leq h^U_{k_0+1}(\beta) \). A contradiction to the properties of \( I_{\mathcal{F}\cap U} \)-functions \( h^U_k \). Thus, \( V_k \) is co-meager for any \( k \in \omega \).

By the Baire Category Theorem, (see e.g. [3, p. 197-198, 277]), there exists \( x_0 \in \bigcap_{k \in \omega} V_k \). Then 
\[
\begin{align*}
f_0(x_0) &> f_1(x_0) > f_2(x_0) > \ldots
\end{align*}
\]
what is impossible since \( f_k(x_0) \) are ordinals. \( \blacksquare \)

**Theorem 2** Let \( X \) be a Baire metric space with Kuratowski partition \( \mathcal{F} \) of cardinality \( \kappa \), where \( \kappa = \min \{|K|: K \text{ is Kuratowski partition of } X\} \), and let \( I_\mathcal{F} \) be a \( K \)-ideal on \( \kappa \) associated with \( \mathcal{F} \). Then the game \( G_1(I_\mathcal{F}\cap U) \) is not \( \alpha \)-favorable for some open set \( U \subset X \).

**Proof.** Let \( \mathcal{F} = \{F_\alpha: \alpha < \kappa\} \), (as was fixed in Section 2.2). We will show that there exists an open set \( U \subset X \) such that the game \( G_1(I_\mathcal{F}\cap U) \) is not \( \alpha \)-favorable, where

\[
I_{\mathcal{F}\cap U} = \{A \subset \kappa: \bigcup_{\alpha \in A} F_\alpha \cap U \text{ is meager}, F_\alpha \in \mathcal{F}\}.
\]

Suppose that for any open set \( U \subset X \) the game \( G_1(I_\mathcal{F}\cap U) \) is \( \alpha \)-favorable.

Fix a family \( \mathcal{U} \) of open and disjoint subsets of \( X \) such that \( \bigcup \mathcal{U} \) is dense in \( X \) and fix \( U \in \mathcal{U} \). \( \alpha \) starts the game \( G_1(I_{\mathcal{F}\cap U}) \) by choosing \( I_{\mathcal{F}\cap U} \)-function \( h^U_0 \). Then \( \beta \) chooses \( B^U_0 \subseteq \text{dom}(h^U_0) \) such that \( B^U_0 \in I_{\mathcal{F}\cap U}^+ \). Then \( \alpha \) chooses \( I_{\mathcal{F}\cap U} \)-function \( h^U_1 \) such that \( \text{dom}(h^U_0) \subseteq B^U_0 \) and \( h^U_1(\beta) < h_0(\beta) \) for all \( \beta \in \text{dom}(h^U_1) \). Then \( \beta \) choose \( B^U_1 \subseteq \text{dom}(h^U_1) \) such that \( B^U_1 \in I_{\mathcal{F}\cap U}^+ \) and so on. Let

\[
(h^U_0, B^U_0, h^U_1, B^U_1, ..., h^U_n),
\]
be a finite sequence obtained in this game, \( (n \in \omega) \).

Let \( T^U \) denotes the set of all such finite sequences. Note that \( T^U \) ordered by extension of sequences is a tree, but our consideration below we will provided for a fixed path of \( T^U \).
Since \( I_{\tau \cap U} \) is not precipitous, \( \alpha \) has a winning strategy. Thus, we have \( h_{k+1}^U(\beta) < h_k^U(\beta) \) for all \( \beta \in \text{dom}(h_{k+1}^U) \) and \( \alpha \) has a legal move for any \( k \in \omega \).

Denote \( X_k^U = \text{dom}(h_k^U) \), for \( U \subset X \) and \( k \in \omega \). Then by our construction

\[
B_k^U := X_{k+1}^U = \{ \beta : h_k^U(\beta) > h_{k+1}^U(\beta) \}.
\]

Now, for each \( \beta \in B_k^U \) and \( h_k^U \in \Phi_k^U \) define exactly one function \( f_{k,\beta}^U \in X_\kappa \) such that

1) \( \text{dom}(f_{k,\beta}^U) = F_\beta \cap U \),
2) \( f_{k,\beta}^U(x) = h_k^U(\beta) \) for any \( x \in F_\beta \cap U \).

Then by properties of functions \( h_k^U \), \( k \in \omega \) we have that

\[
f_{k,\beta}^U(x) > f_{k+1,\beta}^U(x) \text{ for any } x \in F_\beta \cap U.
\]

Since our considerations have been provided for any \( U \in U \), we can take a function \( f_k = \bigcup_{U \in U} \bigcup_{\beta < \kappa} f_{k,\beta}^U \), for any \( k \in \omega \). Then \( f_k \in FN(\kappa) \), \( k \in \omega \).

By Fact 2, for any \( k \in \omega \) the set \( V_k = \{ x : f_k(x) > f_{k+1}(x) \} \) has the Baire property and, moreover, is co-meager, (see the adequate part of the proof of Theorem 1).

By the Baire Category Theorem, (see e.g. [5, p. 197-198, 277]), there exists \( x_0 \in \bigcap_{k \in \omega} V_k \). Then

\[
f_0(x_0) > f_1(x_0) > f_2(x_0) > ...
\]

what is impossible since \( f_k(x) \) are ordinals. Thus, the game \( G_1(I_{\tau \cap U}) \) is not \( \alpha \)-favorable for some open set \( U \subset X \). ■

4 Open problem

In [9] one can find other "types" of precipitous ideals: weakly-precipitous, (equivalent to some game introduced by S. Shelah) and pseudo-precipitous. Since both mentioned notions are consistent to something stronger than "measurable" namely "\( \kappa^+ \)-saturated", the natural question is arisen, whether one can consider theorem adequate for Theorem 1 (and Theorem 2) but for these notions. But such theorems may occur false, because in game characterisation of weakly- and pseudo-precipitousness the assumption that an ideal must be normal is important. Now we cannot explicity state whether the \( K \)-ideal from Theorem 1 is normal. Furthermore, there are models of ZFC in which there are precipitous ideals which are not normal precipitous ideals,
(see e.g. [7]). Summarizing, the following question has been arisen.

**Open Problem.** Considering assumptions given in Theorem 2, does there exist an open set $U \subseteq X$ for which $I_{F \cap U}$ is a normal precipitous ideal? If yes, whether $I_{F \cap U}$ can be pseudo-precipitous (defined in [9])?

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