Phase transitions and entanglement properties in spin-1 Heisenberg clusters with single-ion anisotropy

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Abstract
The incipient quantum phase transitions of relevance to nonzero fluctuations and entanglement in Heisenberg clusters are studied in this paper by exploiting negativity as a measure in bipartite and frustrated spin-1 anisotropic Heisenberg clusters with bilinear–biquadratic exchange, single-ion anisotropy and magnetic field. Using the exact diagonalization technique, it is shown that quantum critical points signaled by qualitative changes in behavior of magnetization and particle number are ultimately related to microscopic entanglement and collective excitations. The plateaus and peaks in spin and particle susceptibilities define the conditions for a high/low-density quantum entanglement and various ordered phases with different spin (particle) concentrations.

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1. Introduction

Entanglement properties of a few spins or electrons can display the general features of large thermodynamic systems, and different measures of entanglement have been defined for understanding quantum phase transitions (QPTs) [1–3]. A finite-spin system is important in the context of molecular magnetism and spin pairing. A new line of research points to a connection between the local entanglement in one-dimensional (1D) correlated systems and the existence of QPTs and scaling [4, 5] relevant to quantum critical points (QCPs). Furthermore, such a link can be exploited to unveil a fundamental connection between the QCPs in finite-size small and large clusters [6, 7] and macroscopic systems [2]. Particle and spin density fluctuations, extending the essential properties of entanglement beyond the conventional framework, have been introduced with explicit reference to the phase transitions in canonical and grand canonical ensembles signaled by a critical behavior in terms of the energy gaps and susceptibilities [8, 9]. The quantum gas of clusters at equilibrium gives an unprecedented opportunity to explore exactly these ideas for quantum dynamics of spin fluctuations [10]. While the basic features of entanglement in spin-1/2 systems are now fairly well understood [11], entanglement properties of larger-spin fermions (or bosons) are less known due to the lack of good operational measures of high spin entanglement [12]. A general classical spin-1 Blume–Emery–Griffiths (BEG) model [13] has proved to be useful for the description of liquid–gas, liquid–crystal phase transitions, tricritical and λ points, and spontaneous phase separation [14–17]. The integer spin Heisenberg model exhibits a characteristic spin gap and very rich phase diagrams [18, 19]. Exact calculations of thermodynamic and entanglement properties in finite-size clusters can give an appealing alternative to get insights into the general features of bipartite and frustrated systems [20]. Some analytical and numerical studies of entanglement and negativity in the bilinear–biquadratic spin-1 Heisenberg model on dimerized bipartite and frustrated systems have been performed as reported in [22–25]. The entanglement with a bilinear–biquadratic Hamiltonian has been considered for the case of two spins (qubit) in the absence of a crystal field [26]. One of the interesting problems concerning entanglement is to study the effect of uniaxial single-ion anisotropy and magnetic field on negativity. An exact calculation of entanglement versus longitudinal crystal field and biquadratic coupling for analyzing the variation of negativity versus parameters of the spin-1 system has not been attempted either.
for the ferromagnetic or the antiferromagnetic Heisenberg model even for small bipartite and frustrated clusters. The aim of this work is to discuss, in a general framework, how microscopic entanglement in the two- and three-qubit context can be related to QCPs characterized by plateaus in peak behavior of the spin revealed in saddle point singularities on model parameters. We provide a reinterpretation of the spin and particle susceptibilities near QCPs in terms of quantum entanglement in a physically transparent way. Here, we adopt negativity to measure the ground state entanglement for spin-1 systems, to reveal QPTs in terms of negativity. We have two main goals in this paper: the first is to provide a global view of the most general spin-1 Heisenberg model that has not been highlighted so far in minimal clusters; the second is to show that quantum entanglement exhibits the existence of characteristic plateaus in negativity related to QPTs. In this paper, we perform exact calculations of entanglement and response functions in the spin-1 Heisenberg model with bilinear–biquadratic exchange interactions in longitudinal crystal and magnetic fields. The basic principles for the calculation of negativity are introduced in section 3. The ground state magnetic and entanglement properties in the spin-1 Heisenberg model for ferromagnetic and antiferromagnetic couplings are given in section 4.1. The negativity analyses in the absence and presence of a magnetic field are given in sections 4.2 and 4.3, respectively. The effect of nonlinear interaction is studied in section 4.4. The results for frustrated trimers are presented in section 4.5. Our conclusions are given in section 5.

2. The spin-1 Heisenberg model

We consider the spin-1 isotropic Heisenberg model in the presence of a magnetic field $B < 0$,

$$H = \sum_{i=1}^{N} [J (\vec{S}_i \vec{S}_{i+1}) + K (\vec{S}_i \vec{S}_{i+1})^2] + D \sum_{i=1}^{N} (S^z_i)^2 + B \sum_{i=1}^{N} S^z_i. \tag{1}$$

The linear $J$ and nonlinear $K$ terms are the exchange and quadrupolar interactions. Here we implemented the longitudinal crystal field $D$, which describes a uniaxial single-ion anisotropy. In what follows, the crystal field significantly changes the results on the entanglement. Note that an effective spin Hamiltonian (1) can be derived from the Bose–Hubbard model in the strong coupling limit. The local spin vector $\vec{S}_i$ for each site has the components of the spin-1 operators

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

Unless otherwise specified, we will consider periodic boundary conditions, such that $\vec{S}_N+1 = \vec{S}_1$, where $N$ is the total number of lattice sites. The sum over lattice sites for the crystal field term with $(S^z_i)^2$ in (1) can be reduced to the spin concentration (particle number),

$$\sum_{i=1}^{N} (S^z_i)^2 = P = P_0,$$

where $P_0$ is the number of lattice sites with $S^z_i = 0$. Note that the axial anisotropy in many respects is analogous to the chemical potential $D = -\mu$.

3. Definitions and basics

At thermal equilibrium, the state of the system is determined by the density matrix

$$\hat{\rho}(T) = \frac{e^{-H/k_BT}}{Z} = \sum_{i} \frac{e^{-E_i/k_BT}}{Z} |\psi_i\rangle \langle \psi_i|, \tag{3}$$

where $E_i$ are the eigenvalues of the $i$th quantum many-body eigenstate and the partition function is $Z = \sum_i e^{-E_i/\beta}$ with $\beta = 1/k_BT$ ($k_B = 1$). The many-body entanglement is described by the density operator in [20, 27–29]. For the spin-1 system, the degree of pairwise entanglement, measured in terms of the negativity $Ne$, can be employed to evaluate the thermal state of concern [30]. The negativity of a state $\rho$ is defined as

$$Ne = \sum_{i} |\mu_i|, \tag{4}$$

where $\mu_i$s are negative eigenvalues of $\rho^T_i$ and $T_i$ denotes the partial pairwise transpose with respect to the first system, i.e., for a bipartite system in state $\rho$ it is defined as

$$\langle i_1, j_2 | \rho^T_i | k_1, l_2 \rangle \equiv \langle k_1, j_2 | \rho | i_1, l_2 \rangle, \tag{5}$$

for any orthonormal but fixed basis. Definition (4) is equivalent to

$$Ne = \frac{\| \rho^{T_i} \| - 1}{2}, \tag{6}$$

where $\| \rho^{T_i} \|$ is trace norm of $\rho$ ( $\rho = Tr \sqrt{\rho \rho}$). For entangled states negativity vanishes, while $Ne > 0$ gives a computable measure of thermal entanglement. As a thermodynamical characterization we have used the responses of the thermodynamical potential with respect to $D$ and $B$, which are as follows:

$$P = \langle (S^z_i)^2 \rangle = \frac{\partial F}{\partial D}, \quad \langle S^z \rangle = \frac{\partial F}{\partial B}. \tag{7}$$

Here $F$ is the free energy $F = -T \ln Z$ and $\langle \cdots \rangle$ indicates averaging performed within a canonical ensemble. The responses for the first derivatives of the thermodynamic potential with respect to $D$ and $B$ provide exact expressions for particle $\chi_D$ and spin $\chi_B$ susceptibilities:

$$\chi_D = \frac{\partial P}{\partial D}, \quad \chi_B = \frac{\partial \langle S^z \rangle}{\partial B}. \tag{8}$$
4. Results

4.1. Entanglement and magnetic properties of spin-1 isotropic Heisenberg dimers

In this section, we consider the Hamiltonian in the case of $N = 2$, namely the Heisenberg model. In the two-qubit case, we diagonalize the Hamiltonian, and obtain the eigenvalues

$$
E_1 = -2(B - J - K - D), \quad E_2 = -2(J - K - D),
$$
$$
E_3 = 2(B + J + K + D), \quad E_4 = -B - 2J + 2K + D,
$$
$$
E_5 = B - 2J + 2K + D, \quad E_6 = -B + 2J + 2K + D,
$$
$$
E_7 = B + 2J + 2K + D, \quad E_8 = -J + 5K + D - \lambda_0,
$$
$$
E_9 = -J + 5K + D + \lambda_0,
$$

and corresponding eigenvectors

$$
|\psi_1\rangle = |-1, -1\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|-1, 1\rangle - |1, -1\rangle),
$$
$$
|\psi_3\rangle = |1, 1\rangle, \quad |\psi_4\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle - |0, 1\rangle),
$$
$$
|\psi_5\rangle = \frac{1}{\sqrt{2}}(|0, 1\rangle - |1, 0\rangle), \quad |\psi_6\rangle = \frac{1}{\sqrt{2}}(|-1, 0\rangle + |0, 1\rangle),
$$
$$
|\psi_7\rangle = \frac{1}{\sqrt{2}}(|0, 1\rangle + |1, 0\rangle),
$$
$$
|\psi_8\rangle = \frac{1}{\sqrt{2 + \lambda_1^2}}(|-1, 1\rangle + \lambda_1 |0, 0\rangle + |1, -1\rangle),
$$
$$
|\psi_9\rangle = \frac{1}{\sqrt{2 + \lambda_2^2}}(|-1, -1\rangle + \lambda_2 |0, 0\rangle + |1, 1\rangle),
$$

where

$$
\lambda_0 \equiv \sqrt{9(J - K)^2 - 2(J - K)D + D^2}, \quad \lambda_1 \equiv \frac{J - K - D - \lambda_0}{2(J - K)},
$$
$$
\lambda_2 = \frac{J - K - D + \lambda_0}{2(J - K)}.
$$

and $|i, j\rangle (i = -1, 0, 1$ and $j = -1, 0, 1)$ are the eigenvectors of $S_i^z S_j^z$. According to the Schmidt theorem, $|\psi_5\rangle$ and $|\psi_7\rangle$ are not entangled and the maximum entangled states can be only $|\psi_2\rangle$ or $|\psi_9\rangle$. The partial transpose density matrix of the thermal state $\rho (T)$ at equilibrium is

$$
\rho^{\Omega} = \frac{1}{Z} \begin{pmatrix}
\omega^- & 0 & 0 & 0 & \chi^- & 0 & 0 & 0 & \Xi^-
0 & \chi^+ & 0 & 0 & 0 & \Omega & 0 & 0 & 0
0 & 0 & \Xi^+ & 0 & 0 & 0 & \Omega & 0 & 0
0 & 0 & 0 & \chi^+ & 0 & 0 & 0 & \Omega & 0
\chi^- & 0 & 0 & 0 & \Lambda & 0 & 0 & 0 & \zeta^-
0 & \Omega & 0 & 0 & 0 & \chi^+ & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & \Xi^+ & 0 & 0 & 0
0 & 0 & 0 & 0 & \Omega & 0 & 0 & \zeta^+ & 0
0 & 0 & 0 & 0 & \Xi^- & 0 & 0 & 0 & \omega^+
\end{pmatrix},
$$

where

$$
\omega^\pm = e^{2(\pm B - D - J - K)/T}, \quad \chi^\pm = \frac{1}{2} e^{-B + 2(J + K) + D/T} (1 \pm e^{4J/T}),
$$
$$
\Xi^\pm = \frac{1}{2} e^{2(J - D - K)/T} + e^{J - 2K + D/T} \left( \lambda_0 \cosh \frac{2\phi}{\lambda_0} + (J - K - D) \sinh \frac{2\phi}{\lambda_0} \right),
$$
$$
\Omega = \left( 2e^{J - 5K - D}/T \right) \left( \lambda_0 \cosh \frac{2\phi}{\lambda_0} - (J - K - D) \sinh \frac{2\phi}{\lambda_0} \right),
$$
$$
\Lambda = \left( e^{J - 5K - D}/T \right) / \left( \lambda_0 \cosh \frac{2\phi}{\lambda_0} - (J - K - D) \sinh \frac{2\phi}{\lambda_0} \right).
$$

Here the partition function is

$$
Z = e^{-2(D + 1)/T} \left( 2e^{(D + 3K)/T} \left( 1 + e^{4J/T} \right) \cosh \left( \frac{B}{T} \right) + e^{(4J + 3K)/T} \right) + 2e^{3K/T} \cosh \left( \frac{B}{T} \right) + 2e^{(D + 3J)/T} \cosh \left( \frac{\lambda_0}{T} \right).
$$

4.2. Spin-1 in zero magnetic field

Here we analyze the effects of crystal field on the ground state entanglement in the spin-1 Heisenberg model (1) at zero field ($B = 0$). The (quadrupole) particle number and negativity plots in figures 1(a) and (b) are both asymmetric as a function of $D$ for ferromagnetic $J > 0$ and antiferromagnetic $J < 0$ couplings. The monotonic behavior of $P$ versus $D$ in figure 1(a) signals the smooth character of the phase transition. Note that this smooth bosonic behavior of spin concentration $P$ versus $D$ contrasts with the (step-like) abrupt fermionic change in the electron number as a function of the chemical potential in [7]. At infinitesimal $T \to 0$, the variation of negativity (figure 1(b)) versus $D$ for the antiferromagnetic case ($J > 0$) is non-monotonic. So, for $D = 0$, we have the highest possible entanglement and $\psi_8$ is a ground state of the system. The system for $J < 0$ displays two distinct phases: one separable and the other entangled. For the positive $D$ region the negativity for $J > 0$ is more than that for $J < 0$. For $D = 0$ the ground state is fivefold degenerate (i.e. it is a mixture of $\psi_1, \psi_3, \psi_6, \psi_7$ and $\psi_9$ states) with zero negativity. At infinitesimal $D \to +0$, the system is entangled in the pure state, $\psi_8$. For $D < 0$ the ground state at $J < 0$ is double degenerate with a mixture of $\psi_1$ and $\psi_3$ states.

Note that these states are separable (can be factorized), and therefore, according to definition, these quantum states are without entanglement. Thus, the entanglement in the $D < 0$ region can be used to detect quantum correlations in the antiferromagnetic case, which are absent for ‘classical’ ferromagnetics.

The negativity is a non-monotonic function with one maximum at $D = 0$ for $J > 0$ and in close proximity to the origin at $J < 0$. The magnetic and quadrupole susceptibilities, i.e. $\chi_B$ and $\chi_D$, allow us to distinguish the ordered and disordered phases in the case of broken symmetry at QPTs. Figure 2 shows the pure (extremal) and mixed (non-extremal) quantum states. The disentangled dark region in the ferromagnetic case corresponds to the plateau-like behavior in zero (spin).
magnetic susceptibility $\chi_0 = \frac{\partial (S^z)}{\partial B} |_{B=0}$ versus the $J$ and $D$ planes in figure 3(a) at $J < 0$.

The high-density magnetic (spin) susceptibility in the white sector corresponds to the low density of negativity in figure 2. The strong enhancement of negativity along the line $D = 0$ is relevant to the observed peaks in the particle susceptibility. $\chi_D = \frac{\partial (S^z)^2}{\partial D}$ in figure 3(a). The various regions

seen for (density) negativity in figure 2 are reproduced in the density of quadrupole susceptibility in figure 3(b) versus $D$ and $J$. Similarly, the phase diagram in the $K-J$ space in the absence of $D$ and $B$ fields shows the degree of entanglement and phases due to the effect of nonlinearity on the eigenvalues and eigenvectors in (10). For example, the $J = K$ line separates the maximum entangled and non-entangled phases for ferromagnetic coupling, while the $J = 3K$ line is broader between the entangled and new less-entangled phases for antiferromagnetic coupling.

We find that for $K > 0$, the line $J = 0$ as before separates non-entangled and maximum entangled phases. The maximum entanglement, which exists for $J < 0$ at $K < J$ and for $J > 0$ at $J > 3K$, corresponds to the observed condition for Bose condensation of unpolarized Na atoms on the optical lattice [21].

4.3. Effects of magnetic field

The magnetic field $B$ partially removes the ground state degeneracy, and in figure 4, one can see the presence of new phase boundaries. The entanglement properties of the excited states are independent of those in the ground states. Also we found that the pairwise entanglement decreases from ground state to excited states, i.e. the more excited the system is, the less the entanglement. In the ferromagnetic case for the $D = 0$ and $B = 0$ point, there is a maximum entangled state. In figures 4(a), for $J = -1$ and in figure 4(b) for $J = 1$, the energies are measured with respect to $|J|$, which is set to 1. When $D < |B|$ the system is in the $\psi_1$ or $\psi_3$ state, which is non-entangled. For fixed $B$ the two consecutive phase transitions take place by increasing $D$ at $D = |B|$ and $D = \sqrt{1 + B^2} + B^2 |B| = 1$, into $\psi_6$ and $\psi_5$ ground states correspondingly. For the antiferromagnetic case, the phase diagram is more complex. The negativity contains the triple point at $|B| = \frac{3}{8}$ and $D = -\frac{4}{7}$, which implies the presence of various phases, possible coexistence or phase separation in the spin-1 system. When $D < -\frac{4}{7}$, the line $|B| = -\frac{2}{\sqrt{5}}B$ separates $\psi_8$ and $\psi_{1,4}$, i.e. the maximum entangled phase from the non-entangled one. For $D > -\frac{4}{7}$, there are three phases in the ground state: non-entangled state at $D < |B| - 4$; the maximum entanglement at between $1 + \sqrt{B^2 + 2|B|} - 7$ and $1 - \sqrt{B^2 + 2|B|} - 7$; non-saturated entanglement for $\psi_{4,5}$ states. In figure 4, there is no entanglement beyond some critical field $B_c$ restricting the black region. Also note that entanglement increases with $D$. 

![Figure 1](image1.png)  
**Figure 1.** The density variation of (a) the particle number $P$ and (b) the negativity $Ne$ versus $D$ for antiferromagnetic, $J = 1$ (dashed), and ferromagnetic, $J = -1$ (solid), cases.

![Figure 2](image2.png)  
**Figure 2.** The density of the negativity $Ne$ versus that of $J$ and $D$. The crystal field $D$ enhances the entanglement at $J < 0$. 

![Figure 3](image3.png)  
**Figure 3.** The densities for (a) zero field (magnetic) spin susceptibility and (b) particle susceptibility versus $J$ and $D$. 

![Figure 4](image4.png)  
**Figure 4.** The density variation of (a) the particle number $P$ and (b) the negativity $Ne$ versus $D$ for antiferromagnetic, $J = 1$ (dashed), and ferromagnetic, $J = -1$ (solid), cases.
Figure 4. The density plot for negativity dependences on crystal field $D$ and magnetic field $B$ at $T = 0$ for (a) $J = -1$ and (b) $J = 1$. For both (antiferromagnetic and ferromagnetic) cases there are more than three phases, which indicates the possible existence of triple or tricritical points.

Figure 5. The density plots for (a) particle number $P$, (b) (quadrupole) particle susceptibility, (c) magnetic susceptibility and (d) negativity versus $B$ and $D$, when $K = 2$ in the antiferromagnetic case $J = 1$.

Positive $D$ values favor the larger entanglement, while $D < 0$ shows the tendency toward non-entangled states with larger total spin. The ground state diagrams, figures 2 and 4, exhibit quantum critical behavior on the borderlines between various states with continuously varying QCPs separating antiferromagnetically ordered distinct phases from the non-entangled state (spin–liquid phase). These critical lines, similar to continuous QCPs, can be used for the
classification of the many-body ground states of interacting spins and quadrupole moment in multidimensional parameter space. Dynamic interactions between the spins strongly renormalize various parameters in the effective Hamiltonian and, therefore, spin and quadrupole momentum have properties different from a quasi-particle description. As in QCP [2], various states along quantum critical boundaries here are necessarily separated by second-order phase transitions with various entanglement and susceptibility. The quantum critical (lines) boundaries appear to be useful for understanding the formation of various thermodynamic phases in the ground state. These continuous boundaries in the thermodynamic phase diagram at infinitesimal temperature coincide with the corresponding QCPs, derived from the peaks of magnetic susceptibilities in agreement with our preliminary analysis (see also [8, 9]). The boundaries between the various phases are also useful for understanding the behavior of thermal negativity for departure to non-zero temperatures.

The distances along the magnetic field in figures 4 between various phases define the stable magnetic phases with distinct spin gap configurations, characterized by different spin concentrations and diverging susceptibilities along the boundaries. For example, the negativity for white areas reaches the maximum (saturated) value, while there are also different distinct areas with partial (unsaturated) entanglement.

These density plots can be used to determine QCPs and the boundaries for various QPTs. This result for finite-size clusters can have important consequences on the physics of QPTs [2], where so far the usual method for detecting a phase transition is by looking at the scaling in the thermodynamic systems. The competition among the different phases can lead to complex behavior with the two triple points. The density of negativity is an efficient indicator of QPTs. In figure 5(a), we find the new spin phase boundaries from the black to the gray region with jump 1/2 and from the gray region into the two white ones with the same jump. The white middle line \( B = 0 \) in figure 5(c) corresponds to the classical effect in the \( J < 0 \) case (without change in entanglement). On the other hand, the continuous lines seen in the same plot at \( J > 0 \) correspond to QPTs (observable also in negativity).

4.4. Effect of the quadrupole term

Here we display the effect of nonlinear interactions between the spins. The variation of \( P \) versus \( D \) is shown in figure 6(a) for two quadruple interactions \( K > 0 \) for the antiferromagnetic case with \( J = 1 \). At \( K = 2 \), an opposite spin pairing gap is opened at \( P = 1/2 \). Such a density profile, showing finite leap near \( P = 1/2 \), resembles the Mott–Hubbard (MH) plateau behavior for the number of particles versus chemical potential in the Hubbard clusters. This is indicative of a possible opposite spin pairing instability [7]. Therefore, the cluster behaves at large \( K \) as an MH-like insulator in contrast to the spin liquid-like behavior with the zero gap, shown at \( K = 0 \) in figure 6(b).

As can be seen from the density plot, the magnetic field and quadrupole interaction \( (K) \) make the phase structure in the antiferromagnetic case richer. Our analysis shows that the negativity in the \( D–K \) space for ferromagnetic coupling is always less than that for the antiferromagnetic case.

**Figure 6.** The density variation of (a) particle number \( P \) and (b) negativity \( Ne \) versus \( D \) for \( K = 0 \) (solid) and \( K = 2 \) (dashed) for the antiferromagnetic case \( (J = 1) \) in the zero field \( B = 0 \).

**Figure 7.** The density plot of negativity via \( J \) and \( D \) at \( K = B = 0 \) in the three-site cluster.

4.5. Non-bipartite clusters

Finally, we display the results on negativity versus \( D \) and \( J \) for frustrated three-site clusters in figure 7 at rather low \( T = 0.01 \). This picture for the ferromagnetic case resembles the corresponding figure 2. However, there is an apparent difference in behavior for the region \( J > 0 \), where there are two extra continuous borderlines.
5. Conclusion

In this paper, we have adopted the concept of entanglement to analyze the behavior of small-size spin-1 Heisenberg clusters. We used negativity as a computable measure of entanglement to perform extensive calculations of the negativity and response functions. The critical fields and intrinsic parameters beyond which entanglement disappears were calculated. We found regions where the quantum entanglement can be increased more rapidly by increasing both, $D$ and $K$. The negativity can determine the borders between ordered phases with excess correlations, above the classical ones. The observed plateaus and peaks in spin and particle behaviors and susceptibilities can be considered as a possible universal method for the simultaneous detection of quantum and classical phase transitions. The (density) plots are a convenient (topographic map) tool for the observation of quantum phases and quantum transitions. The states with vanishing classical correlations but existing quantum correlations in entanglement open up new opportunities for phase transitions that are detectable only through correlation in behavior of entanglement and thermodynamic properties. Our studies of QCPs in small-size spin clusters appear to be generic to large thermodynamic systems. The exact diagonalization is completely unbiased for the study of QPTs and QCPs in strongly correlated spin and electron systems [8]. Although the exact studies have limitations (since the computations grow exponentially with cluster size), we do not find a minimal critical length in clusters below which a quantum critical behavior disappears. The spin-1 boson Hubbard-like model at certain conditions can be mapped onto the spin-1 Heisenberg model. Then these studies can also be useful for the analysis of spontaneous phase separation and the transition from Mott insulator to quantum superfluid in spin-1 Bose–Hubbard models on optical lattices.

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