The Gauge Hierarchy Problem
and
Higher Dimensional Gauge Theories

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Abstract

We report on an attempt to solve the gauge hierarchy problem in the framework of higher dimensional gauge theories. Both classical Higgs mass and quadratically divergent quantum correction to the mass are argued to vanish. Hence the hierarchy problem in its original sense is solved. The remaining finite mass correction is shown to depend crucially on the choice of boundary condition for matter fields, and a way to fix it dynamically is presented. We also point out that on the simply-connected space \(S^2\) even the finite mass correction vanishes.

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1. Introduction

The Standard Model of electroweak theories has been very successful in describing observed phenomena. However, a serious problem arises when it is regarded as the effective low energy theory of a certain more fundamental theory. Suppose that the Standard Model is valid up to a physical momentum cutoff $\Lambda$, where it should be replaced by a more fundamental theory. When the cutoff is huge compared with the weak scale, $\Lambda \gg M_W$, the huge mass scale $\Lambda$ may potentially disturb the low energy ($\sim M_W$) physics. This is the so-called gauge hierarchy problem. It has played crucial role in particle physics; the attempts at solving the problem have led to theories beyond the Standard Model, such as supersymmetric models [1], models with dynamical gauge symmetry breaking [2], and an idea that the Higgs particle should be regarded as a pseudo-Nambu-Goldstone boson [3].

There are actually two kinds of gauge hierarchy problem: (i) When the fundamental theory is of the type of grand unified theory, the problem arises already at the tree level. For instance, in the $SU(5)$ GUT, in order to keep the mass of doublet component of the 5-plet Higgs to the weak scale, while that of color triplet partner to the GUT scale, one must fine-tune the parameters in the scalar potential to the precision of $(M_W/M_{GUT})^2 \sim 10^{-26}$: the triplet-doublet mass splitting problem. (ii) Even if the mass of the Higgs doublet is tuned to be small at the tree level, the mass-squared will suffer from huge quantum corrections proportional to $\Lambda^2$: the problem of “quadratic divergence”. Again one has to fine-tune the bare mass parameter to the precision $\sim 10^{-26}$. We note that the SUSY theory, an attractive candidate for the “new physics”, does not immediately solve the first problem, though it solves the second one successfully.

The hierarchy problem is a clue to the search for the candidates for new physics. Hence to exhaust possibilities of the solution to the hierarchy problem is highly desirable. The guiding principle in the investigation in this direction is the concept of the “naturalness” by ’t Hooft [4]; the smallness of some physical quantity is naturally ensured provided the symmetry of the theory is enhanced when the quantity vanishes. In this paper we shall investigate a new possibility to solve the hierarchy problem where the smallness of the Higgs mass is naturally guaranteed by the local gauge symmetry of a higher dimensional gauge theory.

The idea is the following. Let $A_M (M = 0 \text{ to } D - 1)$ be a gauge field in the $D$ dimensional space-time, and decompose it into 4-dimensional and extra space components,

$$A_\mu (\mu = 0, \ldots, 3), \quad A_m (m = 4, \ldots, D - 1).$$

After the compactification of the extra space, the extra space component $A_m$ behaves as a set of scalar fields in our 4-dimensional world, which we regard as our Higgs fields. Apparently, the naturalness condition stated above is satisfied, since when the mass of $A_m$ vanishes the local gauge symmetry with respect to the extra space coordinates arises. Thus the second problem of quadratic divergence should be solved. Moreover at the tree level the local gauge
invariance automatically forbids the presence of the Higgs mass, and the problem (i) is also readily solved once a viable (GUT-type) model is constructed. By taking a toy model we will confirm this assertion, finding that a mechanism works to eliminate the ultraviolet quadratic divergence thanks to the local gauge invariance. We generally get a finite mass correction, and it depends on the global property of the compactified space. It is widely understood that the supersymmetry plays an essential role for the vanishing quadratic divergence [5], even in higher dimensional supersymmetric Yang-Mills theories. In our view, it is worthwhile clarifying the question whether the divergence can also be eliminated by the mechanism advocated here relying on the gauge symmetry.

Another interesting aspect of studying higher dimensional gauge theories is the possibility for the non-trivial topology of the compactified extra space to affect physics, especially via the boundary conditions (b.c. for short) imposed on fields along the directions of \(y_m\). Casimir effect in QED is a typical example. Quantum corrections have been studied in theories with compactified space [6], [7], [8]. In particular, it has been pointed out that in the space-time of \(M^3 \times S^1\) a photon would propagate faster than the speed of light (!), i.e. \(m_{\gamma}^2 < 0\), a consequence of the periodic b.c. of fermion field with respect to the direction \(S^1\) [6], [7]. This result suggests the importance of quantum fluctuations in the compactified extra space, and that the b.c. of the fluctuating fields does affect the physics. In fact, we will discuss that the remaining finite quantum correction to the Higgs mass-squared crucially depends on the choice of the b.c. It will also be argued that, since the compactified space \(S^4\) is nonsimply-connected space, the b.c. may be reduced to the effect of constant background of \(A_m\), a sort of Aharonov-Bohm (A-B) effect, and therefore can be fixed dynamically [10].

A remark is in order concerning the tower of massive Kaluza-Klein modes. The invariance under local gauge transformations with gauge parameters depending on the coordinates \(y^m\) is essential for our mechanism to work. To keep this gauge symmetry, the tower of massive modes have to be taken into account. This is because, after a gauge transformation, e.g. of a fermion field,

\[
\psi \rightarrow \psi' = e^{i\epsilon^a T^a} \psi,
\]

with a gauge parameter \(\epsilon^a(y^m)\) depending only on the extra space coordinate \(y_m\), \(\psi'\) necessarily depends on \(y^m\) and hence contains higher modes (Fourier modes in the case of \(S^1\)) in \(y^m\). Our proposal of considering the local gauge symmetry with respect to the coordinate \(y^m\) differs from the usual view in which only massless modes (zero modes) are assumed to take part in the compactified theory.

By local gauge transformations in compactified gauge theories one usually refers to local gauge transformation with respect to the 4-dimensional space-time coordinates \(x^\mu\). In the present work local gauge transformations mean gauge transformations which depend on the higher dimensional coordinates \(x^\mu, y^m\). We focus more specifically on the gauge transforma-
tions Eq.(2) which depend on the coordinate $y^m$.

2. A toy model and the quantum correction to the Higgs mass

We begin by showing the mechanism for the disappearance of the quadratically divergent quantum correction to the Higgs mass in a toy model: $D + 1$ dimensional QED in the space-time $M^D \times S^1$, the product of $D$-dimensional Minkowski space-time and a circle with a radius $R$, whose coordinates are $x^\mu$ and $y$, respectively. We present the computation for an arbitrary dimension $D$, and the case of our main interest, $M^4 \times S^1$, is obtained by setting $D = 4$. The field contents are the $D + 1$ dimensional photon $A_M$ and an electron $\psi$ with mass $m$. The gauge field $A_M$ decomposes into $D$-dimensional and extra-space components, $A_M = (A_\mu, A_y)$. Only zero modes are considered as external (real) states, while massive modes have to be taken account of in the intermediate states. The latter will play essential role in the quantum corrections.

At the tree level the Higgs mass $m_H$ vanishes, i.e. $m_H = m_{A_y} = 0$, due to the local gauge invariance. Thus the first hierarchy problem is readily solved. Our next task is to calculate the quantum correction to $m_H^2$, to see whether the quadratic divergence disappears, as we naively expect from the fact that the photon mass remains zero under quantum corrections in ordinary QED, again due to the gauge invariance.

We begin by recapitulating the $m_H^2$ at the lowest order in the limit of $R \to \infty$, i.e., in $M^{D+1}$ space-time. As is well-known, $m_H^2$ vanishes in this limit:

$$m_H^2 = \left(\frac{2^{D+1}}{D + 1}\right)e^2L \int \frac{d^{D+1}k}{(2\pi)^{D+1}} \left\{ (1 - D) \frac{1}{k^2 - m^2} + \frac{2m^2}{(k^2 - m^2)^2} \right\}$$

$$= \left(\frac{2^{D+1}}{D + 1}\right) \frac{e^2L}{(4\pi)^{D+1}} \left\{ (1 - D) + \frac{2m^2}{D} \frac{\partial}{\partial m^2} \right\} \Gamma \left(\frac{1}{2} - \frac{D}{2}\right) (m^2)^{\frac{D-1}{2}}$$

$$= 0,$$  \hspace{1cm} (3)

where $L \equiv 2\pi R$ and the D-dimensional charge $e$ is related to the original charge $e^{(D+1)}$ as $e = L^{-1/2} e^{(D+1)}$.

Next let us calculate $m_H^2$ in the manifold of our interest, $M^D \times S^1$ with a finite radius $R$ of the circle. The requirement that physical quantities should be single-valued functions of space-time coordinates demands that the gauge field $A_M$ should be single-valued. However, the electron may have an arbitrary b.c., as only the product $\bar{\psi}\psi$ is of physical relevance:

$$\psi(x_\mu, y + 2\pi R) = e^{i\alpha}\psi(x_\mu, y),$$  \hspace{1cm} (4)

where $\alpha$ is an arbitrary phase. The Higgs mass-squared $m_H^2$ calculated for the b.c. is given
as

\[ m^2_H = i e^2 2^{D+1} \int \frac{d^Dk}{(2\pi)^D} \sum_n \left\{ -\frac{1}{(\frac{2\pi n + \alpha}{L})^2 + \rho^2} + 2\rho^2 \frac{1}{(\frac{2\pi n + \alpha}{L})^2 + \rho^2} \right\}, \]  

(5)

where \( \rho^2 \equiv -k^\mu k_\mu + m^2 \). It should be noted that the discrete momentum in y-direction gets a constant shift proportional to \( \alpha \) due to the arbitrarily chosen non-trivial b.c. Eq.(4): \( k_y = \frac{2\pi n + \alpha}{L} \) (n : integer).

The above expression for \( m^2_H \) is superficially highly divergent. However, a finite expression is obtained by subtracting zero from Eq.(5), i.e. by subtracting the contribution for \( L \to \infty \), Eq.(3). By utilizing a relation

\[ \sum_n \left\{ \frac{-1}{(2\pi n + \alpha L)^2 + \rho^2} \right\} = \int_0^\infty dk \frac{1}{2\pi} \frac{1}{k^2 + \rho^2}, \]

we arrive at (after Wick rotation)

\[ m^2_H = -i e^2 2^{D+1} \int \frac{d^Dk}{(2\pi)^D} \left( 1 + \rho \frac{\partial}{\partial \rho} \right) \left( \frac{L}{2\rho} \right) \left[ \frac{\sinh(\rho L)}{\cosh(\rho L) - \cos \alpha} - 1 \right], \]

(6)

where a dimensionless variable \( s = kL \) has been introduced. As expected, no divergence appears in the quantum correction to \( m^2_H \), for arbitrary dimension D; the integral is super-convergent, i.e. for larger \( k \) the integrand behaves as \( -2(\cos \alpha)k^{D-1}e^{-\sqrt{k^2 + m^2 L}} \), where \( e^{-\sqrt{k^2 + m^2 L}} \) is analogous to the Boltzman factor in the case of finite temperature QED where \( \psi \) has an anti-periodic b.c. \( (\alpha = \pi) \) in the time direction. Let us note that if we take only the zero mode \( n = 0 \) into account in Eq.(5), the calculation reduces to that of the pseudo-scalar 2-point function in the ordinary 4-dimension (for \( D = 4 \)) and therefore the result diverges quadratically.

We thus have solved the second hierarchy problem of quadratic divergence, and the gauge hierarchy problem in its original sense has been solved. We note, however, there remains a finite correction to \( m^2_H \). In the usual argument the gauge hierarchy problem is concerned with U.V. behavior of the theory. By contrast, the appearance of finite \( m^2_H \) depends on the global nature of the extra space, i.e., whether it is infinite or compact, and is controlled by the I.R. nature of the theory.

When \( L \) is relatively large, such I.R. effect is expected to be small, roughly of the order \( (1/L)^2 \). Thus for \( L \geq 1/(1\text{TeV}) \) [11] the hierarchy problem is totally solved. It is noteworthy that such large (length) scale compactification has recently attracted revived interest in
Figure 1: The $m_H^2$ measured in the unit of $e^2/(2\sqrt{2}\pi^2L^2)$ as a function of $mL$ in the case of periodic b.c..

the context of GUT and/or superstring [12]. For much smaller L, say of $O(1/M_{GUT})$ or $O(1/M_{pl})$, such “I.R.” effect is expected to become large, in general. To be precise, $m_H^2$ also depends on m and the boundary condition.

In order to see how $m_H^2$ behaves as a function of L and m, we first study a specific case of periodic boundary condition, i.e. $\alpha = 0$, where $m_H^2$ is simply given by

$$m_H^2 = -\frac{1}{2^{D+2}\pi^D\Gamma(D/2)}e^2L^{-D+2}\int_0^\infty ds \frac{s^{D-1}}{\cosh(s^2+(mL)^2-1)}.$$  \hspace{1cm}(8)

Fig.1 shows, for the case of $D = 4$, the behavior of $m_H^2$ as a function of $mL$(electron mass measured in the unit of $1/L$). As is seen there $m_H^2$ indicates an exponential damping (for large $mL$), essentially due to the aforementioned “Boltzman factor”. Thus, if $mL$ is as large as $\sim 30$, the gauge hierarchy is understood naturally as $M_W^2 \sim (1/L)^2e^{-mL}$ (geometrical hierarchy). For smaller values of $mL$ the $m_H^2$ suffers from a large correction of $O(1/L^2)$.

3. The effect of boundary condition and its dynamical determination

Next we will investigate how the boundary condition (b.c.)$\alpha$ for $\psi$ affects $m_H$. In Fig.2 we plot the result of Eq.(7) as a function of both $\alpha$ and $mL$. It is interesting to note that $m_H^2$ even changes its sign as $\alpha$ varies. In fact, we find from Eq.(7) that $m_H^2 < 0$ for $\alpha = 0$ and $m_H^2 > 0$ for $\alpha = \pi$ irrespective of $mL$. Specifically, $m_H^2 = 0$ is realized at $\alpha \simeq 0.46\pi$ for the case of $m = 0$.

So far the b.c. has been put by hand, and we feel it uncomfortable that the physical quantity $m_H$ depends on the b.c.. We will now attempt to attribute some physical meaning to the phase $\alpha$ and will study the possibility of dynamically fixing it. In order to consider the physical relevance of the phase $\alpha$ it is worthwhile noting that if we wish we can always
rotate away the phase $\alpha$ by re-phasing of $\psi$, since the theory has local U(1) gauge symmetry:

$$\psi(x_\mu, y) \rightarrow \psi'(x_\mu, y) = U \psi(x_\mu, y),$$

$$U = e^{-i \frac{y}{L} \alpha} \in U(1).$$

(9) (10)

The field $\psi'$ obeys the periodic b.c.:

$$\psi'(x_\mu, y + L) = \psi'(x_\mu, y).$$

(11)

On the other hand, $A_y$ gets a constant shift as

$$A_y \rightarrow A'_y = A_y - \left( \frac{1}{e} \right) \frac{\alpha}{L},$$

(12)

i.e., only the zero-mode of $A_y$ is affected. Thus, without loss of generality, we may always assume that the field $\psi$ obeys the periodic b.c., while $A_y$ may have a constant background, or a vacuum expectation value, $\langle A_y \rangle$. We have already seen in the Eq.(8) that $m^2_H$ is negative for periodic b.c. This clearly suggests that the real vacuum state is not at $\langle A_y \rangle = 0$; we are led to consider nonvanishing $\langle A_y \rangle$.

Naively thinking, such constant shift of the higher dimensional gauge field has no physical effect. In fact if $A_y = 0$ and $F'_{\mu\nu} = 0$, corresponding to the naive vacuum $\langle A_y \rangle = 0$, then the transformed $A'_y = - \left( \frac{1}{e} \right) \frac{\alpha}{L}$ again satisfies $F'_{\mu\nu} = 0$. For the Minkowski space $M^5$, therefore, the non-vanishing zero-mode is just of pure gauge, and a theory with $\langle A_y \rangle \neq 0$ is gauge-equivalent to the case of $\langle A_y \rangle = 0$.  

Figure 2: The 3D plot of $m^2_H$ (in the unit of $e^2/(2\sqrt{2}\pi^2 L^2)$) as a function of $a$ and $mL$, with $a$ standing for $\alpha$. 
The above argument has to be modified, once \( M^5 \) is replaced by \( M^4 \times S^1 \). What is crucial here is the property that the compact space \( S^1 \) is a nonsimply-connected space. To see how a contant \( A_y \) can be physical, let us write \( A_y \) as

\[
A_y(x_\mu, y) = A^c_y + A^q_y(x_\mu, y),
\]

where \( A^c_y \) denotes the constant background field (zero-mode) or \( A^c_y = \langle A_y \rangle \), while \( A^q_y \) denotes quantum fluctuation around the background. Solving the Dirac equation in the presence of \( A^c_y \), we get a spinor \( \psi \) with non-trivial b.c.:

\[
\psi(x_\mu, y + L) = \psi(x_\mu, y)e^{ie \oint A^c_y dy},
\]

even if we assume that \( \psi \) satisfies periodic b.c. in the absence of \( A^c_y \). The Wilson-loop along the \( S^1 \), \( e^{\oint A^c_y dy} = e^{A^c_y L} \) (corresponding to \( \alpha \)) is a gauge invariant quantity, and may be understood as \( e^\Phi \) with \( \Phi \) being the “magnetic flux” inside the \( S^1 \) (we do not have to worry about what the “inside” of \( S^1 \) means). This is a sort of Aharonov-Bohm effect. Since \( e^\Phi \) has gauge invariant physical meaning as the magnetic flux, the theory should be different if the value of \( A^c_y \) is different.

In particular, the vacuum energy will depend on \( A^c_y \), which in turn means that \( A^c_y \) and equivalently \( \alpha \) can be fixed dynamically as the value which realizes the ground state. Namely we can calculate the vacuum energy density \( V(A^c_y) \) as a function of \( A^c_y \) (“gauge potential”) and the vacuum state is realized as the minimum point of \( V(A^c_y) \). \( V(A^c_y) \) is obtained by evaluating \( \text{Tr} \ln(i\gamma^M D_M) \), where the covariant derivative is due to \( A^c_y \): \( D_y = \partial_y - ieA^c_y \), just as in the case of the calculation of the Coleman-Weinberg potential in the background field method. The result is

\[
V(A^c_y) = -\frac{L^{-D}}{2^{\frac{D+1}{2}}\pi^{\frac{D+1}{2}}\Gamma(D/2)} \int_0^\infty ds \ s^{D-1} \times \ln \left[ 1 - 2\cos(eA^c_y L)e^{-\sqrt{s^2+(mL)^2}} + e^{-2\sqrt{s^2+(mL)^2}} \right].
\]

The divergent contribution for \( L \to \infty \), which is independent of \( A^c_y \) and has no physical significance, has been subtracted in Eq.(15). \( V(A^c_y) \) is plotted as a function of \( \alpha = eA^c_y L \) (for \( D = 4 \) and \( m = 0 \)) in Fig.3. In this approach \( m^2_H \) is given as the coefficient of the term quadratic in \( A^c_y \):

\[
m^2_H = \frac{d^2 V}{dA^c_y^2}.
\]

When \( eA^c_y L \) is identified with \( \alpha \), this definition of \( m^2_H \) coincides with the direct calculation of the 2-point function, Eq.(7). As is shown in Fig.3, the value \( \alpha \simeq 0.46\pi \) corresponds to the point where \( \frac{d^2 V}{dA^c_y^2} = 0 \), but it does not correspond to the stationary point with \( \frac{dV}{dA^c_y} = 0 \). The minimum of \( V(A^c_y) \) is at \( \alpha = \pi \), where \( m^2_H > 0 \). Thus, unfortunately \( m^2_H = 0 \) is not achieved dynamically, in the toy model \( M^4 \times S^1 \).
Calculations of the vacuum energy under constant background gauge fields have previously been made by Hosotani in $M^3 \times S^1$ with the fermion mass $m = 0$ \cite{10}. His purpose was to study the gauge symmetry breaking due to the VEV of non-Abelian gauge fields. After this pioneering work many papers have appeared to discuss the Hosotani mechanism in various models \cite{13}. Our purpose was not to discuss the spontaneous gauge symmetry breaking ($\langle A_y \rangle$ does not break $U(1)$) but to discuss $m_H$.

4. The Higgs mass on $S^2$

A natural question to be raised is why non-vanishing gauge boson mass $m_{A_y}^2 = m_H^2 \neq 0$ became possible, without contradicting the local gauge invariance. One may naively argue that if $m_{A_y}^2$ is present the mass term is not invariant under a gauge transformation, $A_y \rightarrow A_y + c$ ($c$ is a constant). We, however, have shown that $A_y$ and $A_y + c$ are no longer gauge equivalent to each other, unless $c = (2\pi/eL)n$ ($n$ is an integer), since $A_y$ and $A_y + c$ correspond to different values of magnetic flux in the sense of the A-B effect. This is why the vacuum energy depends on $A_y^2$, generating the “curvature” $\frac{d^2V}{dA_y^2} = m_{A_y}^2$.

Thus it may be reasonable to expect that the local gauge invariance does work to guarantee $m_H^2 = 0$, provided the compactified space is a simply-connected manifold, not allowing the penetration of magnetic flux. We will illustrate this by calculating $m_H^2$ in the case where the compactified space is $S^2$. For simplicity, we ignore 4-dimensional space-time and calculate $m_H^2$ just on $S^2$, taking scalar QED theory as an example. We assume that the matter scalar is massless.

According to the method we took above, we calculate the vacuum energy density under the background photon field $A_m$. This generates the effective action for $A_m$, i.e. $I =$
\[ \int d^2 y \sqrt{g} L_{\text{eff}} = \text{Tr} \ln \{(D_m + i e A_m)(D^m + i e A^m)\} \] and \( D^m \) is covariant under the general coordinate transformation \((g_{mn}, \text{ the metric of } S^2)\). The \( \text{Tr} \ln \) is obtained by path-integration over the scalar field under the background \( A_m \). On the other hand, gauge and general coordinate invariance imply that the term in \( L_{\text{eff}} \), which are quadratic in \( A_m \), should be written in a form

\[ L_{\text{eff}} = a (\partial_m A_n - \partial_n A_m)^2 + b g_{mn} A^m A^n, \]

where the kinetic term takes the ordinary form so that it respects local gauge symmetry, as the coefficient \( a \) is a dimensionless parameter and should not be affected by whether \( R \) (the radius of \( S^2 \)) is \( \infty \) or finite. To obtain the mass-squared \( b \), which should be identified with \( \frac{1}{2} m_H^2 \), we take a specific choice of background, \( A_0 = 0 \) and \( A_\phi = \text{constant} \) (\( (\theta, \phi) \) are angular coordinates on \( S^2 \)). Then a term disappears and

\[ b = \frac{1}{2} m_H^2 = \text{Tr} \{ -e^2 g_{\phi\phi}(\Delta_{S^2})^{-1} + 2e^2 \partial_\phi^2 (\Delta_{S^2})^{-2} \} / (\int d^2 y \sqrt{g} g_{\phi\phi}), \]

where the numerator comes from the second derivative of the \( \text{Tr} \ln \) with respect to \( A_\phi \), and \( \Delta_{S^2} \equiv g^{mn} D_m \partial_n \) is the laplacian on \( S^2 \). The first term of the numerator can be evaluated by taking the trace with respect to configuration space coordinates as

\[ - e^2 \left( \int d^2 y \sqrt{g} g_{\phi\phi} \right) \cdot G(0, 0) \]

in terms of the propagator \( G(y, y') \) (\( \Delta_{S^2} G(y, y') = \delta(y - y') \)),

\[ G(y, y') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ -\frac{1}{l(l+1)} \right\} Y^m_l(\theta', \phi') Y^m_l(\theta, \phi). \]

The second term is simply given as the sum of eigenvalues of \( \partial_\phi^2 (\Delta_{S^2})^{-2} \), i.e.

\[ -m^2 a^4 / [l(l+1)]^2. \]

We thus get

\[ m_H^2 = 2e^2 \sum_{l,m} \frac{1}{l(l+1)} |Y^m_l(0,0)|^2 - 4e^2 a^4 \left\{ \sum_{l,m} \frac{m^2}{l(l+1)[l]} \right\} / (\int d^2 y \sqrt{g} g_{\phi\phi}). \]

Since \( \sum_l |Y^m_l(0,0)|^2 = \frac{2l+1}{4\pi} \), \( \sum_l m^2 = \frac{1}{3} l(l+1)(2l+1) \) and \( \int d^2 y \sqrt{g} g_{\phi\phi} = \frac{8\pi}{3} a^4 \), we finally get

\[ m_H^2 = 2e^2 \frac{2l+1}{2\pi} \sum_l - \frac{2e^2}{2\pi} \sum_l \frac{2l+1}{l(l+1)} = 0. \]

To be consistent with our naive expectation, the Higgs mass-squared \( m_H^2 \) identically vanishes irrespective of the size of the compactified space \( S^2 \). It needs some further study to see whether the vanishing \( m_H \) is still realized when 4-dimensional space-time is added to \( S^2 \).

5. Concluding remarks

We studied the possibility to solve the gauge hierarchy problem thanks to the local gauge symmetry in the framework of higher-dimensional gauge theories. We first took a toy model,
i.e., QED in $M^D \times S^1$ space-time. Because of the local gauge symmetry, at the tree level the Higgs mass-squared $m_H^2$ vanishes automatically and the ultraviolet quadratic divergence in quantum correction to $m_H^2$ also cancels out when non-zero (massive) modes are all summed up in the internal loop. We, however, found a finite correction to $m_H^2$. When periodic boundary condition (b.c.) is taken for the fermion, small $m_H$ of the weak scale was argued to be realized either by a relatively large compactification (length) scale, say $R \sim 1/(1\text{TeV})$ or so, or by a heavy fermion of mass $m > 1/R$. The $m_H^2$ was shown to be very sensitive to the b.c. and to vanish for a specific choice of the b.c..

We also discussed that the b.c. may be regarded as the consequence of a sort of Aharonov-Bohm effect, and can be fixed dynamically. Unfortunately, it turned out that such fixed b.c. does not lead to $m_H^2 = 0$. Finally we investigated the case where the compactified space is $S^2$. The A-B effect, which is known to play an essential role to yield the finite $m_H^2$, is not allowed for the simply-connected space $S^2$, and our calculation shows that $m_H^2$ identically vanishes.

The analysis given in this paper are all in toy models. In order to make our mechanism viable, further discussions are obviously necessary to clarify e.g. whether the mechanism also works in realistic GUT models and how a Higgs field belonging to fundamental representation is realized starting from the adjoint repr. of gauge field.

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