HOMOTOPY EXPONENTS FOR LARGE $H$-SPACES

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Abstract. We show that $H$-spaces with finitely generated cohomology, as an algebra or as an algebra over the Steenrod algebra, have homotopy exponents at all primes. This provides a positive answer to a question of Stanley.

Introduction

A simply connected space is elliptic if both its rational homotopy and rational homology are finite. Moore’s conjecture, see for example [9], predicts that elliptic complexes have an exponent at any prime $p$, meaning that there is a bound on the $p$-torsion in the graded group of all homotopy groups. Any finite $H$-space is known to be elliptic as it is rationally equivalent to a finite product of (odd dimensional) spheres. Relying on results by James [6] and Toda [11] about the homotopy groups of spheres, the fourth author (re)proved in [10] Long’s result that finite $H$-spaces have an exponent at any prime [7]. He proved in fact a stronger result which holds for example for $H$-spaces for which the mod $p$ cohomology is finite. He also asked whether this would hold for finitely generated cohomology rings. The aim of this note is to give a positive answer to this question and provide a way larger class of $H$-spaces which have homotopy exponents.

Theorem 1.2 Let $X$ be a connected and $p$-complete $H$-space such that $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then $X$ has an exponent at $p$.

This class of $H$-spaces is optimal in the sense that $H$-spaces with a larger mod $p$ cohomology, such as an infinite product of Eilenberg-Mac Lane spaces $K(\mathbb{Z}/p^n, n)$, will not have in general an exponent at $p$. As a corollary, we obtain the desired result. In fact we obtain the following global theorem.

Theorem 1.4 Let $X$ be a connected $H$-space such that $H^*(X; \mathbb{Z})$ is finitely generated as an algebra. Then $X$ has an exponent at each prime $p$.

The methods we use are based on the deconstruction techniques of the third author in his joint work with Castellana and Crespo [3]. Our results on homotopy exponents should also be compared to

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with the computations of homological exponents done with Clément, [4]. Whereas such $H$-spaces always have homotopy exponents, they almost never have homological exponents. The only simply connected $H$-spaces for which the 2-torsion in $H_*(X;\mathbb{Z})$ has a bound are products of mod 2 finite $H$-spaces with copies of the infinite complex projective space $\mathbb{C}P^\infty$ and $K(\mathbb{Z},3)$.

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1. Homotopy exponents

Our starting point is the fact that mod $p$ finite $H$-spaces have always homotopy exponents. The following is a variant of Stanley’s [10, Corollary 2.9]. Whereas he focused on spaces localized at a prime, we will stick to $p$-completion in the sense of Bousfield and Kan, [2]. Since the $p$-localization map $X \to X(p)$ is a mod $p$ homology equivalence, his result implies the following.

Proposition 1.1 (Stanley). Let $p$ be a prime and $X$ be a $p$-complete and connected $H$-space such that $H^*(X;\mathbb{F}_p)$ is finite. Then $X$ has an exponent at $p$.

We will not repeat the proof, but let us sketch the main steps. Let us consider a decomposition of $X$ by $p$-complete cells, i.e. $X$ is obtained by attaching cones along maps from $(S^n)^\wedge_p$. The natural map $X \to \Omega\Sigma X$ factors then through the loop spaces on a wedge $W$ of a finite numbers of such $p$-completed spheres, up to multiplying by some integer $N$: the composite $X \to \Omega\Sigma X \xrightarrow{N} \Omega\Sigma X$ is homotopic to $X \to \Omega W \to \Omega\Sigma X$. The proof goes by induction on the number of $p$-complete cells and the key ingredient here is Hilton’s description of the loop space on a wedge of spheres, [5]. Note that the suspension of a map between spheres is torsion except for the multiples of the identity. This idea to “split off” all the cells of $X$ up to multiplication by some integer is dual to Arlettaz’ way to split off Eilenberg-Mac Lane spaces in $H$-spaces with finite order $k$-invariants, [11 Section 7]. The final step relies on the classical results by James, [6], and Toda, [11], that spheres do have homotopy exponents at all primes.

Theorem 1.2. Let $X$ be a connected and $p$-complete $H$-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then $X$ has an exponent at $p$.

Proof. A connected $H$-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra can always be seen as the total space of an $H$-fibration $F \to X \to Y$ where $Y$ is an $H$-space with finite mod $p$ cohomology and $F$ is a $p$-torsion Postnikov piece whose homotopy groups are finite direct sums of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$, [3 Theorem 7.3]. This is a fibration of $H$-spaces and $H$-maps, so that we obtain another fibration $F^\wedge_p \to X \to Y^\wedge_p$ by $p$-completing it. The base space $Y^\wedge_p$ now satisfies the assumptions of Proposition 1.1. It has therefore an exponent at $p$. The homotopy groups of the fiber $F^\wedge_p$ are finite direct sums of cyclic
groups $\mathbb{Z}/p^n$ and copies of the $p$-adic integers $\mathbb{Z}_p^\wedge$. Thus $F_p^\wedge$ has an exponent at $p$ as well. The homotopy long exact sequence of the fibration allows us to conclude.

We see here how the $p$-completeness assumption plays an important role. The space $K(\mathbb{Z}_p^\infty, 1)$ for example has obviously no exponent at $p$, but its $p$-completion is $K(\mathbb{Z}_p^\wedge, 2) = (\mathbb{C}P^\infty)^\wedge_p$, which is a torsion free space. The mod $p$ cohomology of $K(\mathbb{Z}_p^\infty, 1)$ is a polynomial ring on one generator in degree 2, we must thus also work with $p$-complete spaces to give an answer to Stanley’s question [10, Question 2.10].

**Corollary 1.3.** Let $X$ be a connected and $p$-complete $H$-space such that $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra. Then $X$ has an exponent at $p$.

In fact, when the mod $p$ cohomology is finitely generated, the fiber $F$ in the fibration described in the proof of Theorem 1.2 is a single Eilenberg-Mac Lane space $K(P, 1)$. Thus the typical example of an $H$-space with finitely generated mod $p$ cohomology is the 3-connected cover of a simply connected finite $H$-space ($P$ is $\mathbb{Z}_p^\infty$ in this case). Likewise, the typical example in Theorem 1.2 are highly connected covers of finite $H$-spaces. This explains why such spaces have homotopy exponents!

If one does not wish to work at one prime at a time and prefers to find a global condition which permits to conclude that a certain class of spaces have exponents at all primes, one must replace mod $p$ cohomology by integral cohomology.

**Theorem 1.4.** Let $X$ be a connected $H$-space such that $H^*(X; \mathbb{Z})$ is finitely generated as an algebra. Then $X$ has an exponent at each prime $p$.

**Proof.** Since the integral cohomology groups are finitely generated it follows from the universal coefficient exact sequence (see 8) that the integral homology groups are also finitely generated. Since $X$ is an $H$-space we may use a standard Serre class argument to conclude that so are the homotopy groups. Therefore the $p$-completion map $X \to X_p^\wedge$ induces an isomorphism on the $p$-torsion at the level of homotopy groups. The theorem is now a direct consequence of the next lemma.

**Lemma 1.5.** Let $X$ be a connected space. If $H^*(X; \mathbb{Z})$ is finitely generated as an algebra, then so is $H^*(X; \mathbb{F}_p)$.

**Proof.** Let $u_1, \ldots, u_r$ generate $H^*(X; \mathbb{Z})$ as an algebra. Consider the universal coefficients short exact sequences

$$0 \to H^n(X; \mathbb{Z}) \otimes \mathbb{Z}/p \to H^n(X; \mathbb{F}_p) \xrightarrow{\partial} \text{Tor}(H^{n+1}(X; \mathbb{Z}); \mathbb{Z}/p) \to 0.$$

Since $H^*(X; \mathbb{Z})$ is finitely generated as an algebra it is degree-wise finitely generated as a group and therefore $\text{Tor}(H^*(X; \mathbb{Z}); \mathbb{Z}/p)$ can be identified with the ideal of elements of order $p$ in $H^*(X; \mathbb{Z})$. 
This ideal must be finitely generated since $H^\ast(X;\mathbb{Z})$ is Noetherian. Choose generators $a_1, \ldots , a_s$. Each $a_i$ corresponds to a pair $\alpha_i, \beta \alpha_i$ in $H^\ast(X;\mathbb{F}_p)$, where $\beta$ denotes the Bockstein.

We claim that the elements $\alpha_1, \ldots , \alpha_s$ together with the mod $p$ reduction of the algebra generators, denoted by $\bar{u}_1, \ldots , \bar{u}_r$, generate $H^\ast(X;\mathbb{F}_p)$ as an algebra. Let $x \in H^\ast(X;\mathbb{F}_p)$ and write its image $\partial(x) = \sum \lambda_j a_j$ with $\lambda_j = \lambda_j(u)$ a polynomial in the $u_i$'s. Define now $\bar{\lambda}_j = \lambda_j(\bar{u}) \in H^\ast(X;\mathbb{F}_p)$ to be the corresponding polynomial in the $\bar{u}_i$'s. As the action of $H^\ast(X;\mathbb{Z})$ on the ideal $\text{Tor}(H^\ast(X;\mathbb{Z});\mathbb{Z}/p)$ factors through the mod $p$ reduction map $H^\ast(X;\mathbb{Z}) \rightarrow H^\ast(X;\mathbb{F}_p)$, the element $x - \sum \bar{\lambda}_j \alpha_j$ belongs to the kernel of $\partial$, i.e. it lives in the image of the mod $p$ reduction. It can be written therefore as a polynomial $\bar{\mu}$ in the $\bar{u}_i$'s. Thus $x = \bar{\mu} + \sum \bar{\lambda}_j \alpha_j$. 

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