Fractional Quantum Hall Effect, Cranked Harmonic Oscillator, and Classical Periodic Orbits

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Abstract

A two-dimensional harmonic oscillator, when rotated by the oscillator frequency, generates Landau-like levels. A further cranking results in condensates and gaps resembling the fractional quantum Hall effect. For a filling fraction \( \nu = p/q \), with \( q \) odd, the model predicts that the gap is proportional to \( \sqrt{n_e/pq} \), where \( n_e \) is the electron density of the sample. This agrees well with recent experimental data. Qualitative arguments for the success of the model are given.

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In this paper, it is first recalled that a two-dimensional harmonic oscillator, when rotated by the oscillator frequency, generates Landau-like levels. A new result is then proven: that a further cranking of the oscillator gives rise to a succession of quantum gaps and condensates at particular values of the incremental frequency. When expressed as dimensionless fractions, these reduce, for the odd denominators, to the Haldane hierarchy in the fractional quantum Hall effect (FQHE). It is of interest to investigate whether or not the gaps in this model have any relationship to those observed in the FQHE. The phenomenon of FQHE is reasonably well understood, though it continues to draw both experimental and theoretical attention. It has been established that in the FQHE, the electrons in the lowest Landau level are strongly correlated due to the repulsive Coulomb interaction between the pairs, and the quantum gaps arise from this effect. It is shown in this paper that the centrifugal term in the rotating frame gives rise to a repulsive correlation in the two-particle coordinate similar to the Coulomb potential. The result of this equivalence is that the gaps must vary as \( \sqrt{B} \), where \( B \) is the applied external magnetic field. The predictions of the cranking model, at a quantum level, are compared with the latest experimental data. The close connection between the quantum gaps and the classical periodic orbits is emphasized. Finally the implication of the success of the model is discussed.

The integral quantum Hall effect (IQHE) can be explained in the independent particle model in conjunction with the localized states due to impurities. To obtain the gaps by the cranked oscillator in this case, consider the single-electron Hamiltonian \( H = (p - \frac{e}{c}A)^2 / 2M \). Here the vector potential \( A = \frac{1}{2}(B \times r) \), \( e = -|e| \), and the motion is confined in the \( xy \) plane. In this symmetric gauge, \( H \) reduces to the form

\[
H = \frac{1}{2M} (p_x^2 + p_y^2) + \frac{1}{2}M \left( \frac{\omega_c}{2} \right)^2 (x^2 + y^2) + \frac{\omega_c l_z}{2},
\]

where \( \omega_c = \frac{|e|B}{Mc} \), and \( l_z = xp_y - yp_x \). Note that Eq.(1) may be interpreted as the Hamiltonian of a particle in a harmonic oscillator of Larmor frequency \( \omega_c / 2 \), that is itself rotating about the (negative) \( z \)-axis with the same frequency. The collapse of the single-particle states into degenerate Landau levels may be shown graphically by considering an auxiliary Hamiltonian
\[ \hat{H}, \]
\[ \hat{H} = \frac{1}{2M}(p_x^2 + p_y^2) + \frac{1}{2} M \left( \frac{\omega_c}{2} \right)^2 (x^2 + y^2) + \omega l_z = H + \tilde{\omega} l_z, \]

where \( \tilde{\omega} = (\omega - \omega_c/2) \) and \( H \) is defined by Eq. (1). Fig.1 depicts the spectrum of the energy levels as a function of \( \tilde{\omega}/\omega_c \) in the range \(-1/2 \leq \tilde{\omega}/\omega_c \leq 1/2\). The Landau gaps of \( \hbar \omega_c \) appear at \( \tilde{\omega}/\omega_c = 0 \), while the uncranked oscillator spectrum of Larmor frequency \( (\omega_c/2) \) is at \( \tilde{\omega}/\omega_c = -1/2 \). Note that for certain values of parameter \( \tilde{\omega}/\omega_c \), a sequence of well-defined gaps develop. The repeating pattern for \( \tilde{\omega}/\omega_c \geq 0 \) is known as a Farey fan\(^{13} \), and has been studied in the context of number theory and continued fractions. Since the gaps that appear in the range \( 1/2 < \tilde{\omega}/\omega_c \leq 1 \) are symmetric about \( \tilde{\omega}/\omega_c = 1/2 \), the horizontal axis is terminated at 1/2. As Fig.1 shows, the angular momentum states collapsing at the lowest Landau level are all aligned in the same (negative \( z \)) direction, and each value of the angular momentum shows up only once. At the same energy \( E = \frac{1}{2} \hbar \omega_c \), inspection of the level at \( \tilde{\omega}/\omega_c = 1/m \) (\( m \) an integer > 1) reveals that the number of converging single-particle states is exactly a fraction 1/m of the Landau levels. For example, the successive harmonic oscillator states meeting at \( \tilde{\omega}/\omega_c = 1/3 \), with energy \( E = \frac{1}{2} \hbar \omega_c \) have angular momenta 0, -3, -6, etc. in units of \( \hbar \) (see Fig.2). So for every triplet of adjacent states of the Landau level, e.g., \( (0, -1, -2) \), there is one \( (l = 0 \text{ in this case}) \) at \( \tilde{\omega}/\omega_c = 1/3 \). To construct a stable model, assume the “sea” of states below \( E = \frac{1}{2} \hbar \omega_c \) to be all occupied, and the states with \( E > \frac{1}{2} \hbar \omega_c \) to be empty. Then the number of electrons per unit area occupying the level with \( \tilde{\omega}/\omega_c = 1/3 \) is exactly \( \frac{1}{3} n_1 \), where \( n_1 = \frac{|e| B_1}{\hbar c} \) is the degeneracy of a Landau level. Therefore the dimensionless parameter \( \tilde{\omega}/\omega_c \) for nonzero positive values can be identified with the “filling fraction” \( \nu \) of the FQHE. This has a curious implication on the statistics of a daughter state, e.g., at \( \tilde{\omega}/\omega_c = 2/5 \). As shown in Fig.2, one state from each quantum shell of the \( \nu = 1/3 \) mother converges at \( E = \frac{1}{2} \hbar \omega_c \), constituting this daughter at 2/5, just as the mother herself was formed from the collapse of the states from each Landau level. Note that the successive states collapsing at 2/5 have angular momenta 0, -5, -10, etc., so there are only one fifth as many states as in the Landau level, despite the electron filling fraction
being two fifths. For this sequence of daughters, \( \frac{p}{2p+1} \), with \( p = 1, 2, 3, \ldots \), the occupancy is \( p \) times the number of states. Furthermore Fig.2 illustrates that the gaps for this same sequence are governed by the right-angled triangle \( abf \). At the end-point, \( \nu = 1/2 \), however, the degeneracy of the level again matches the occupancy.

Comparison with the experimental data\(^7\) can be made directly by noting, from the triangle \( abf \) in Fig.2, that the model gaps \( \tilde{\Delta} \) are proportional to \((\nu - 1/2)\). Since \( \nu = \frac{\hbar e}{|e|B} n_e \) for a sample with electron density \( n_e \), it follows that \( \tilde{\Delta} \) should vary linearly when plotted against \( 1/B \). This is shown to be the case in Fig.3, where the magnitude of the negative intercept on the \( y \)-axis (arising from the finite width of the levels\(^7\)) plus the experimental \( \Delta \) may be interpreted as \( \tilde{\Delta} \). This plot is to be contrasted with Fig.3 of Ref.7, where the same gaps were plotted as a linear function of \( B \). Both appear to be valid, since the range of \( B \) is limited. Comparison with experiment over a wider range of a variable may be made by combining some theoretical input with the model prediction (as seen in Fig.1) that the gap at \( \nu = p/q \) varies as \( \hbar \omega_c / q \). It is well-known that the energy scale in FQHE is determined by the Coulomb term \( \frac{e^2}{\epsilon l_0} \), rather than \( \hbar \omega_c \), where \( l_0 = \sqrt{\frac{\hbar e}{|e|B}} \) is the magnetic length. For this reason, in Fig.2, the mother-daughter sequence of condensations are shown with the \( y \)-axis for the energy scale in units of \( C \frac{e^2}{\epsilon l_0} \), where \( C \) is a dimensionless constant. This procedure of replacing the energy unit \( \hbar \omega_c \) by something proportional to the Coulomb term will be justified presently from the dynamical model. From Fig.2, it then follows that the FQHE gap at \( \nu = p/q \) is given by

\[
\tilde{\Delta}_{p/q} = \frac{1}{q} C \frac{e^2}{\epsilon l_0} = C \frac{e^2}{\epsilon} \sqrt{\frac{2\pi n_e}{pq}}, \tag{3}
\]

where \( n_e = \nu/(2\pi l_0^2) \). That the gap is proportional to the quantity \( \sqrt{\frac{2\pi n_e}{pq}} \) is another prediction of the model. This is tested in Fig.4, where the experimental data are seen to obey this relation over a wide range of \( \sqrt{\frac{2\pi n_e}{pq}} \). In order to fit the magnitudes of the gaps by Eq.(3), it is found that \( C \approx 0.15 \) for \( \nu = 1/3 \).

It is important to remember that in a single-particle Hamiltonian (like \( \tilde{H} \) of Eq.(2)), quantum gaps and classical periodic orbits are closely linked\(^\ldots\).
motion of a particle are easily obtained from $\tilde{H}$, and may be expressed in a compact form in terms of the variable $z = x + iy$:

$$\ddot{z} = 2i\omega \dot{z} + \left(\omega^2 - \frac{\omega_c^2}{4}\right)z.$$  \hspace{1cm} (4)

For the Landau orbits, $\omega = \omega_c/2$, so the centrifugal term drops out, giving $\ddot{z} = i\omega_c \dot{z}$, and only circular solutions are obtained. The general solution of Eq. (4) is given by $z = Ae^{i(\omega - \omega_c/2)t} + Be^{i(\omega + \omega_c/2)t}$. When the ratio of the normal modes $\tilde{\omega}$ and $(\tilde{\omega} + \omega_c)$ is a rational fraction, the resulting orbit in the $xy$ plane is periodic. The first generation mothers in the present model have the normal modes in the ratio $(\tilde{\omega} + \omega_c)/\omega = (m + 1)$, yielding $\tilde{\omega}/\omega_c = 1/m$, matching with the occurrence of the quantum gaps. More generally, if $\tilde{\omega}/\omega_c$ equals the irreducible fraction $p/q$, a classical periodic orbit of $q$-fold symmetry is obtained. The corresponding quantum gap, from Fig.1, is seen to be $\hbar \omega_c/q$.

It is now appropriate to discuss the rationale behind the cranking model. To see how the Coulomb repulsion between two electrons may be simulated by the centrifugal term, consider two noninteracting particles, each with an effective mass $M^*$, being cranked about the $z$-axis as in Eq. (2). The two-particle Hamiltonian may be easily separated into the relative and the centre-of-mass coordinates. The cranking term is still $\tilde{\omega} l_z$, but now $l_z$ is the relative angular momentum $\frac{1}{2}(r_1 - r_2) \times (p_1 - p_2)$ between the two particles. For a mother $\nu = 1/m$, it was noted that the adjacent collapsing levels differ in their angular momenta by the magnitude $m\hbar$. For such a pair, the additional term $\tilde{\omega} l_z = (\tilde{\omega}/\omega_c)\omega_c l_z = \hbar \omega_c$. For this term to simulate the repulsive Coulomb repulsion, it follows that $\hbar \omega_c = C \frac{e^2}{\ell m}$, as was done in Fig.2. This, of course, is only possible if $\omega_c$ is replaced by $\omega_c^* = \frac{|e|B}{M^*c}$, and $M^* \propto \sqrt{B}$. Furthermore, for the classical motion in relative coordinates, an equation analogous to Eq. (4) is obtained, where $z$ now refers to the relative coordinates. Note that the effective centrifugal term in Eq. (4) is repulsive for $\omega > \omega_c/2$, and is proportional to $z$. On the other hand, the Coulomb potential between two electrons is $\frac{e^2}{\epsilon(x^2 + y^2)^{1/2}}$, and the resulting repulsive force is $\frac{e^2z}{\epsilon(x^2 + y^2)^{3/2}}$. Since the relative distance $(x^2 + y^2)^{1/2}$ is determined by the density of the electrons (which in turn is fixed by the magnetic length), it follows that the Coulomb repulsion between two electrons...
also varies linearly with $z$. Thus the cranking model generates an effective repulsion between adjacent particles that may simulate the Coulomb repulsion by choosing the energy scale suitably.

The many-body wave function in the cranking model at $\nu = 1/m$ may be easily calculated using the harmonic oscillator states. It does not, of course, yield the form of the correlated Laughlin wave function. Apparently, the cranking model condensate for a mother (or a daughter) contains components from every Landau level, which is totally different from the Laughlin state. This may not really be the case since the energy scale $\hbar \omega_c$ is being replaced by the Coulomb term $C e^2/\epsilon_0$ in the cranking model. Nevertheless, it appears that the full complement of the Laughlin correlations is not necessary to obtain the FQHE gaps. The situation is reminiscent of the nuclear (incompressible) fluid in an atomic nucleus, where the Jastrow$^{14}$ correlations are not needed to obtain the shell gaps at the magic numbers. Indeed, the superdeformed nuclei$^{15}$ discovered recently have quantum gaps$^{16-17}$ related to classical periodic orbits similar to the model presented here.

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FIGURES

FIG. 1. The energy spectrum (Eq.(2)) of a cranked two-dimensional harmonic oscillator, shown as a function $\tilde{\omega}/\omega_c$, where $\tilde{\omega} = (\omega - \omega_c/2)$. The angular momenta $l$ of the first few states collapsing at the lowest Landau level are also shown. Note the appearance of gaps at particular values of $\tilde{\omega}/\omega_c$. The pattern (in the range $1/2 < \tilde{\omega}/\omega_c < 1$) is symmetrical about $\tilde{\omega}/\omega_c = 1/2$, and is not shown.

FIG. 2. The mother-daughter sequence of condensed states. The energy scale in the $y$-axis has an overall scaling factor $C$ (see text). One particle state from each Landau level, of alternating parity, condense at $\tilde{\omega}/\omega_c = 1/3$ to form a mother. Similarly, one state from each shell at $1/3$ condense at $2/5$. Note that a similar condensation takes place from the holes at $1/3$, to form a hole-daughter at $2/7$. Only a few converging lines are shown for clarity. In the triangle $abf$, the vertical lines $ab$ and $cd$ show the gaps at $\nu = 1/3$ and $2/5$ respectively.

FIG. 3. The FQHE experimental gaps $\Delta$ (in degrees $K$), taken from Fig.3 of Ref.7, are plotted as a function of $1/B$, where $B$ is the applied external field in Tesla. The data refer to two different samples (squares and diamonds : $n_e = 1.12 \times 10^{11}$/cm$^2$, triangles and inverted triangles : $n_e = 2.3 \times 10^{11}$/cm$^2$). The diamonds and inverted triangles have mother $1/3$, squares and triangles refer to mother $2/3$ and her daughters. The two crosses belong to a different sequence with mother $1/5$.

FIG. 4. The experimental gaps$^7$, divided by $\sqrt{n_e}$, are plotted in arbitrary units as a function of $1/\sqrt{pq}$, where $\nu = p/q$ is an irreducible fraction. All four branches of the data points of Fig.3 fall on a single straight line. The notation is the same as in Fig.3.