A DIVERGENCE-FREE FINITE ELEMENT METHOD FOR THE STOKES PROBLEM WITH BOUNDARY CORRECTION

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Abstract. This paper constructs and analyzes a boundary correction finite element method for the Stokes problem based on the Scott-Vogelius pair on Clough-Tocher splits. The velocity space consists of continuous piecewise quadratic polynomials, and the pressure space consists of piecewise linear polynomials without continuity constraints. A Lagrange multiplier space that consists of continuous piecewise quadratic polynomials with respect to boundary partition is introduced to enforce boundary conditions as well as to mitigate the lack of pressure-robustness. We prove several inf-sup conditions, leading to the well-posedness of the method. In addition, we show that the method converges with optimal order and the velocity approximation is divergence free.

1. Introduction

Boundary correction methods are a broad class of unfitted finite element methods, i.e., methods in which the computational mesh does not conform to the physical domain \( \Omega \). In contrast to, e.g., isoparametric methods, in which a domain is approximated via curved elements, boundary correction methods generally solve a PDE in a polytopal interior domain and transfer boundary conditions in a way such that the scheme still maintains optimal order convergence. This polytopal approximation is, in general, not an \( O(h^2) \) approximation to the physical domain and in particular, the polytope’s vertices are not necessarily on the boundary of \( \Omega \). This approach can be advantageous for, e.g., dynamic problems with moving boundaries, as remeshing is not needed at each time step. Another feature of boundary correction methods, in contrast to other unfitted schemes, is the absence of ‘cut elements’ which may require special quadrature formula and algebraic stabilization. Boundary correction methods were first introduced and analyzed nearly 50 years ago [6] for the Poisson problem, and the technique has been improved and refined recently resulting in practical and robust implementations [12, 28, 24, 2, 3, 4] (see also [7, 17] for variants).

In this article, we construct a boundary correction finite element method for the Stokes problem based on the Scott-Vogelius pair on Clough-Tocher (or Alfeld) splits. The velocity approximation is sought in the space of continuous piecewise quadratic polynomials, whereas the pressure space is approximated by piecewise linear polynomials without continuity constraints. From their definitions, we see that the divergence operator maps the velocity space into the pressure space, and therefore, the scheme yields divergence-free velocity approximations. As far as we are aware this is the first \( H^1 \)-conforming divergence-free finite element method for incompressible flow on unfitted meshes.

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The construction and analysis of divergence-free methods is an active area of research, and many schemes have been proposed [21, 11, 33, 15, 14]. These schemes have several inherent advantages, e.g., exact conservation laws for any mesh size and long-time stability [10]. Another feature of these schemes is pressure-robustness; similar to the continuous setting, modifying the source term in the Stokes problem by a gradient field only affects the pressure approximation. This feature leads to a decoupling in the velocity error, with abstract estimates independent of the viscosity. Thus, divergence-free schemes may be advantageous for high Reynolds number flows and/or flows with large pressure gradients [31, 32, 23]. Except for the recent work [25], where isoparametric methods are introduced and studied, all of these divergence-free methods are applied to PDEs on polytopal domains.

Let us describe the scheme in more detail and briefly summarize the context of our results. The method starts with a background mesh enveloping the domain \( \Omega \), and the computational mesh simply consists of those elements fully contained in \( \bar{\Omega} \). The method is based on a standard Nitsche-based formulation, where the Dirichlet boundary conditions are enforced via penalization. As the computational domain does not conform to \( \Omega \), boundary conditions are corrected via simple applications of Taylor’s theorem to reduce the inconsistency of the scheme.

The procedure described so far is relatively standard for the Poisson problem (cf. [6, 24, 2, 3, 4]), but leads to some pressing issues for the Stokes equations. First, because the computational domain explicitly depends on the mesh parameter \( h \), inf-sup stability of the Stokes pair is not immediately obvious. As explained in [16], the standard proof of inf-sup stability in the continuous setting (which is needed for the discrete result) is based on a decomposition of the computational domain into a finite number of strictly star shaped domains; the number of star shaped domains is generally unbounded as \( h \to 0 \). This issue can be circumvented with pressure-stabilization [24, 3], but at the price of additional consistency errors and poor conservation properties. We address this stability issue by designing the computational mesh such that it inherits a macro element structure and applying the framework developed in [16] for Stokes pairs on unfitted domains. Doing so, we show that the resulting pair is uniformly stable on the unfitted domain with respect to the discretization parameter.

The second difficulty of a boundary correction method for the Stokes problem is its lack of pressure-robustness. This feature is not due to the boundary correction per se, but rather due to the weak enforcement of boundary conditions via penalization. In particular, a divergence-free method for the Stokes problem with weak enforcement of the boundary conditions is not pressure robust. We mitigate the lack of pressure robustness in the scheme by introducing an additional Lagrange multiplier that enforces the boundary conditions of the normal component of the velocity. The Lagrange multiplier space consists of continuous piecewise quadratic polynomials with respect to the boundary partition, and the Lagrange multiplier is an approximation to the pressure (modulo an additive constant) restricted to the computational boundary. The Lagrange multiplier ameliorates the lack of pressure robustness of the method and leads to a weakly coupled velocity error estimate; the velocity error’s dependence on the viscosity is compensated by a higher-order power of the discretization parameter \( h \). We remark that Lagrange multipliers within boundary correction schemes have been proposed and studied in [9, 11] for the Poisson problem.

The rest of the paper is organized as follows. In the next section, we state the Stokes problem, the computational mesh, and the boundary transfer operator. In Section 3, we state the finite element method and show that the scheme yields exactly divergence-free velocity approximations. Section 4 proves several inf-sup conditions and the well-posedness of the method. In Section 5, we prove optimal order convergence provided the exact solution is sufficiently smooth. Finally, in
Section 6 we perform some numerical experiments which verify the theoretical results, and give some concluding remarks in Section 7.

2. Preliminaries

For a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$, we consider the Stokes problem

\begin{align}
-\nu \Delta u + \nabla p &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
u u &= g \quad \text{on } \partial \Omega,
\end{align}

where $\nu > 0$ is the viscosity, assumed to be constant. For simplicity in the presentation, and without loss of generality, we assume that $g = 0$. The extension to non-homogeneous boundary conditions is relatively straightforward [19].

We assume the domain has smooth boundary $\partial \Omega$ with outward unit normal $\mathbf{n}$. We denote by $\phi$ the signed distance function of $\Omega$ such that $\phi(x) < 0$ for $x \in \Omega$ and $\phi(x) \geq 0$ otherwise, so that $\mathbf{n} = \nabla \phi / |\nabla \phi|$ on $\partial \Omega$. For a positive number $\tau$, denote by $\Gamma_\tau = \{ x \in \mathbb{R}^2 : |\phi(x)| \leq \tau \}$ the tubular region around $\partial \Omega$. By [13, Lemma 14.16], there exists $\tau_0 > 0$ such the closest point projection $p : \Gamma_\tau_0 \to \partial \Omega$ is well defined and satisfies $p(x) = x - \phi(x) \mathbf{n}(p(x))$ for all $x \in \Gamma_\tau_0$ [9].

Let $S \subset \mathbb{R}^2$ be a polygon such that $\Omega \subset S$, and let $S_h$ be a quasi-uniform triangulation of $S$ that consists of shape regular triangles. We define the computational mesh as $T_h = \{ T \in S_h : \bar{T} \subset \bar{\Omega} \}$, and set

$\Omega_h = \text{int} \left( \bigcup_{T \in T_h} \bar{T} \right) \subset \Omega$

to be the associated domain. We denote by $T^{ct}_h$ the Clough-Tocher refinement of $T_h$, obtained by connecting the vertices of each $T \in T_h$ to its barycenter. The set of boundary of edges of $T_h$, which is also the set of boundary edges of $T^{ct}_h$, is denoted by $E^B_h$. With an abuse of notation, for a piecewise smooth function $q$ (with respect to $E^B_h$), we write

$$\int_{\partial \Omega_h} q \, ds = \sum_{e \in E^B_h} \int_e q \, ds.$$

We use $n_h$ to denote the outward unit normal with respect to the computational boundary $\partial \Omega_h$. For $K \in T^{ct}_h$, we set $h_K = \text{diam}(K)$ and $h = \max_{K \in T^{ct}_h} h_K$. Likewise, for $e \in E^B_h$, we set $h_e = \text{diam}(e)$.

Remark 2.1. Denote by $S^{ct}_h$ the Clough-Tocher refinement of the background mesh $S_h$. We emphasize that $T^{ct}_h \subset S^{ct}_h$, however,

$$T^{ct}_h \neq \{ K \in S^{ct}_h : \bar{K} \subset \bar{\Omega} \}.$$

In particular, $T^{ct}_h$ inherits the macro-element structure needed to prove the stability of the Scott-Vogelius pair.

2.1. Boundary transfer operator. The main component of boundary correction methods is a well-defined mapping $M : \partial \Omega_h \to \partial \Omega$ that assigns each point on the computational boundary to physical one in order to “transfer” the boundary information on $\partial \Omega$ to $\partial \Omega_h$. With such a mapping in hand, we can define the transfer direction as

$\mathfrak{d}(x) = (M - I)x \quad x \in \partial \Omega_h$, 

and transfer length
\begin{equation}
\delta(x) = |\delta(x)|. \tag{2.2}
\end{equation}

Several choices of the mapping $M$ and corresponding transfer directions have appeared in the literature. A common choice (and arguably the most natural) is to take $M$ to be the closest point projection, i.e., $M = p$. In this case, assuming $\Omega_h$ approximates $\Omega$ well enough, the distance vector $\delta$ defined above coincides (up to a multiplicative constant) with the outward unit normal vector $n$ of the original boundary $\partial \Omega$. In particular, there holds $\delta(x) = \phi(x)n(p(x))$ and $\delta(x) = |\phi(x)|$. Another common choice is to take the transfer direction to be parallel to the outward unit normal of the computational boundary, i.e., $\delta = n$. In this case, we have $\delta(x) \geq |\phi(x)|$ with possible large discrepancies between $\delta(x)$ and $|\phi(x)|$, but it leads to a simpler implementation in the numerical method.

In the definition and analysis of the method below, we do not explicitly define the mapping $M$: rather, our main requirement for the mapping $M$ is to satisfy the Assumption (A) below. In particular, and similar to [6, 28, 22, 11, 2], the stability and convergence analysis only assumes that the transfer distance $\delta(x)$ is sufficiently small relative to the mesh parameter $h$. In the numerical experiments provided in Section 5, we take $M$ to be an approximation to the closest point projection.

Set $d = \delta/\delta$, for $x \in \partial \Omega_h$, and define the boundary transfer operator
\begin{equation}
(S_hv)(x) = v(x) + \delta(x)\frac{\partial v}{\partial d}(x) + \frac{1}{2}(\delta(x))^2\frac{\partial^2 v}{\partial d^2}(x).
\end{equation}
Note that $(S_hv)(x)$ is the second-order Taylor expansion of the function $v$.

**Remark 2.2.** Throughout this paper, the constants $C$ and $c$ (with or without subscripts) denote some positive constants that are independent of the mesh parameter $h$ and the viscosity.

3. A DIVERGENCE–FREE FINITE ELEMENT METHOD

For $D \subset \mathbb{R}^d$, denote by $\mathcal{P}_k(D)$ the space of polynomials of degree $\leq k$ with domain $D$. Analogous vector-valued spaces are denoted in boldface. We define the lowest-order Scott-Vogelius finite element pair with respect to the Clough-Tocher triangulation $T_h^4$:

\begin{align*}
V_h &= \{ v \in H^1(\Omega_h) : \forall K \in T_h^4, \int_{\partial \Omega_h} \langle v \cdot n_h \rangle \, ds = 0 \}, \\
Q_h &= \{ q \in L^2(\Omega_h) : \forall K \in T_h^4, \int_{\partial \Omega_h} \langle q \rangle \, ds = 0 \},
\end{align*}

and the analogous spaces with boundary conditions

\begin{align*}
\hat{V}_h &= V_h \cap H^1_0(\Omega_h), \\
\hat{Q}_h &= Q_h \cap L^2(\Omega_h).
\end{align*}

We further introduce a Lagrange multiplier space

\begin{equation}
X_h = \{ \mu \in C(\partial \Omega_h) : \forall e \in E_h \, \forall e \in E_h \}
\end{equation}

and its variant,

\begin{equation}
\hat{X}_h = \{ \mu \in X_h : \int_{\partial \Omega_h} \mu \, ds = 0 \}.
\end{equation}

We define the bilinear form

\begin{equation}
a_h(u, v) = \nu \left( \int_{\Omega_h} \nabla u : \nabla v \, dx - \int_{\partial \Omega_h} \frac{\partial u}{\partial n_h} \cdot v \, ds + \int_{\partial \Omega_h} \frac{\partial v}{\partial n_h} \cdot (S_hu) \, ds \right),
\end{equation}
where \( \sigma > 0 \) is a penalty parameter.

Remark 3.1. The bilinear form \( a_h(\cdot, \cdot) \) is based on a standard “Nitsche bilinear form” associated with the Laplace operator, but with boundary correction \([26, 30]\). Note that the bilinear form is based on a non-symmetric version of Nitsche’s method due to the positive sign in front of the third term in the bilinear form \( a_h(\cdot, \cdot) \). However, boundary correction methods based on the symmetric version of Nitsche’s method still yield a non-symmetric bilinear form \([6, 24]\). The non-symmetric version allows less restrictions on the penalty parameter \( \sigma \) to ensure stability if the extension direction coincides with the outward unit normal of \( \partial \Omega_h \). In particular, if \( d = n_h \), a standard argument shows that the bilinear form \( a_h(\cdot, \cdot) \) is coercive on \( V_h \) for any \( \sigma > 0 \); cf. Lemma 4.4.

We define two bilinear forms associated with the continuity equations, one without and one with boundary correction:

\[
b_h(v, (q, \mu)) = -\int_{\Omega_h} (\text{div} \, v) q \, dx + \int_{\partial \Omega_h} (v \cdot n_h) \mu \, ds,
\]

\[
b_h^e(v, (q, \mu)) = -\int_{\Omega_h} (\text{div} \, v) q \, dx + \int_{\partial \Omega_h} ((S_h v) \cdot n_h) \mu \, ds.
\]

We consider the method of finding \( (u_h, p_h, \lambda_h) \in V_h \times Q_h \times X_h \) such that

\[
a_h(u_h, v) + b_h(v, (p_h, \lambda_h)) = \int_{\Omega_h} f \cdot v \, dx \quad \forall v \in V_h,
\]

\[
b_h^e(u_h, (q, \mu)) = 0 \quad \forall (q, \mu) \in Q_h \times X_h.
\]

Remark 3.2. The zero mean-value constant defined in the Lagrange multiplier space \( X_h \) mods out constants, and is due to the condition \( \int_{\partial \Omega_h} (v \cdot n_h) \, ds = 0 \) in the definition of the discrete velocity space \( V_h \). If this constraint is not imposed in the Lagrange multiplier space, then in general \( (3.1) \) is ill-posed since

\[
b_h(v, (0, 1)) = 0 \quad \forall v \in V_h.
\]

On the other hand, the constraint \( \int_{\partial \Omega_h} (v \cdot n_h) \, ds = 0 \) is needed to ensure that method \( (3.1) \) yields a divergence-free solution, as the next lemma shows.

Lemma 3.3 (Divergence-free property). If \( (u_h, p_h, \lambda_h) \in V_h \times Q_h \times X_h \) satisfies \( (3.1) \), then \( \text{div} \, u_h \equiv 0 \) in \( \Omega_h \).

Proof. The definition of the Stokes pair \( V_h \times Q_h \) shows \( \text{div} \, u_h \in Q_h \). Then, letting \( q = \text{div} \, u_h \) and \( \mu = 0 \) in \( (3.1b) \) yields

\[
0 = b_h^e(u_h, (\text{div} \, u_h, 0)) = -\|\text{div} \, u_h\|_{L^2(\Omega_h)}^2.
\]

Thus, \( \text{div} \, u_h \equiv 0 \).

4. Stability and Continuity estimates

In our stability and convergence analysis, we make an assumption regarding the distance between the PDE domain \( \Omega \) and the computational domain \( \Omega_h \). To state this assumption, we define for a boundary edge \( e \in E_h^B \),

\[
\delta_e := \max_{x \in e} \delta(x).
\]
We make the assumption
\[ \max_{e \in E_h^B} h_e^{-1} \delta_e \leq c_\delta < 1, \quad \text{for } c_\delta \text{ sufficiently small.} \]

**Remark 4.1.** Assumption (A) essentially states that the distance between \( \partial \Omega \) and \( \partial \Omega_h \) is of order \( h \), i.e., \( \delta = O(h) \) with (hidden) constant sufficiently small. Similar assumptions, in the context of boundary correction methods, are made in, e.g., [6, 27, 24, 2, 3].

We define three \( H^1 \)-type norms on \( V_h \):
\[
\| v \|_{0,h}^2 = \| \nabla v \|_{L^2(\Omega_h)}^2 + \sum_{e \in E_h^B} h_e^{-1} \| S_h v \|_{L^2(e)}^2,
\]
\[
\| v \|_{1,h}^2 = \| \nabla v \|_{L^2(\Omega_h)}^2 + \sum_{e \in E_h^B} h_e^{-1} \| v \|_{L^2(e)}^2,
\]
\[
\| v \|_{0,1,h}^2 = \| v \|_{L^2(\Omega_h)}^2 + \sum_{e \in E_h^B} h_e \| \nabla v \|_{L^2(e)}^2.
\]

In addition, we define a \( H^{-1/2} \)-norm on the Lagrange multiplier space \( X_h \):
\[
\| \mu \|_{-1/2,h}^2 = \sum_{e \in E_h^B} h_e \| \mu \|_{L^2(e)}^2.
\]

Finally, we define the norm on \( Q_h \times X_h \) as
\[
\| (q, \mu) \| := \| q \|_{L^2(\Omega_h)} + \| \mu \|_{-1/2,h}.
\]

**Lemma 4.2.** There holds for all \( v \in V_h \),
\[
\sum_{e \in E_h^B} h_e^{-1} \| S_h v - v \|_{L^2(\Omega_h)}^2 \leq C c_\delta \| v \|_{L^2(\Omega_h)}^2,
\]
\[
\sum_{e \in E_h^B} h_e^{-1} \| S_h v \|_{L^2(e)}^2 \leq C \| v \|_{1,h}^2,
\]
provided that \( c_\delta \) in (A) is sufficiently small. In particular, \( \| \cdot \|_h, \| \cdot \|_{1,h}, \) and \( \| \cdot \|_{h} \) are equivalent on \( V_h \).

**Proof.** By trace and inverse inequalities, the shape-regularity of \( T_h \) and (A), there holds for \( e \in E_h^B \),
\[
h_e^{-1} \int_e |\delta|^{2j} \frac{\partial^j v}{\partial d^j} \| \nabla v \|_{L^2(T_e)}^2 \leq C c_\delta^j \| v \|_{L^2(T_e)}^2 \quad j = 1, 2,
\]
where \( T_e \in T_h \) satisfies \( e \subset \partial T \). It then follows from the definition of \( S_h \) and \( \| \cdot \|_{1,h} \) that
\[
\sum_{e \in E_h^B} h_e^{-1} \| S_h v \|_{L^2(e)}^2 \leq C \sum_{j=0}^2 \int_e |\delta|^{2j} \frac{\partial^j v}{\partial d^j} \| v \|_{L^2(T_e)}^2 \| v \|_{L^2(e)}^2 \leq C \| v \|_{1,h}^2.
\]
This inequality immediately yields \( \| v \|_h \leq C \| v \|_{1,h} \). Moreover, standard arguments involving the trace and inverse inequalities show \( \| v \|_h \leq \| v \|_h \leq C \| v \|_h \) on \( V_h \). Thus, to complete the proof, it suffices to show that \( \| v \|_{1,h} \leq C \| v \|_h \).

To this end, we once again use (4.2) to obtain
\[
\sum_{e \in E_h^B} h_e^{-1} \| v \|_{L^2(e)}^2 \leq 2 \sum_{e \in E_h^B} h_e^{-1} \| S_h v \|_{L^2(e)}^2 + 2 \sum_{e \in E_h^B} h_e^{-1} \| S_h v - v \|_{L^2(e)}^2.
\]
The continuity estimate of (4.5) follows from the definition of the forms, the Cauchy-Schwarz inequality, and (4.2):

\[
\|b_v - b^c_v\|_{L^2(\Omega)} \leq C_c \|\nu\|_{H^{1/2} - 1/2, h}.
\]

This inequality implies \(\|v\|_{1, h} \leq C\|v\|_h\).

4.1. Continuity and coercivity estimates of bilinear forms.

**Lemma 4.3.** There holds

\[
|a_h(v, w)| \leq c_2(1 + \sigma)\|v\|_h\|w\|_h \quad \forall v, w \in V_h + H^3(\Omega_h),
\]

\[
|b_h(v, (q, \mu))| \leq C\|v\|_{1, h}\|(q, \mu)\| \quad \forall (q, \mu) \in Q_h \times \tilde{X}_h,
\]

\[
|b_h(v, (q, \mu)) - b^c_h(v, (q, \mu))| \leq C c_3\|v\|_{1, h}\|(q, \mu)\| \quad \forall v \in V_h, \ (q, \mu) \in Q_h \times \tilde{X}_h.
\]

**Proof.** The proof of the continuity estimate of (4.3) is given in [2] Proposition 1 (with superficial modifications). The continuity estimate of \(b_h(\cdot, \cdot)\) (4.4) follows directly from the Cauchy-Schwarz inequality.

This third estimate (4.5) follows from the definition of the forms, the Cauchy-Schwarz inequality, and (4.2):

\[
|b_h(v, (q, \mu)) - b^c_h(v, (q, \mu))| = \left| \sum_{e \in e^h} \int_e ((v - S_h v) \cdot n_h) \mu ds \right|
\]

\[
\leq C \left( \sum_{e \in e^h} h^{-1}_e \left| \int_e |\mu||\nu||n_h|\right|^2 \right)^{1/2} \|\mu\|_{-1/2, h}
\]

\[
\leq C c_3 \|v\|_{1, h}\|\mu\|_{-1/2, h}.
\]

**□

**Lemma 4.4.** Suppose that Assumption [A] is satisfied for \(c_3\) sufficiently small. Then there exists \(\sigma_0 > 0\) such that, for \(\sigma \geq \sigma_0\),

\[
c_1 \nu \|v\|_{1, h}^2 \leq a_h(v, v) \quad \forall v \in V_h
\]

for \(c_1 > 0\) independent of \(h\) and \(\nu\). If the extension direction \(d\) coincides with the outward unit normal of \(\partial \Omega_h\), i.e., if \(d = n_h\), then coercivity is satisfied for any positive penalty parameter \(\sigma > 0\).

**Proof.** The proof the result under assumption [A] follows exactly from the arguments in [2] Theorem 2 (see also [6] Lemma 6), so the proof is omitted.

If \(d = n_h\), then by definition of the bilinear form \(a_h(\cdot, \cdot)\),

\[
a_h(v, v) = \nu \left( \|\nabla v\|_{L^2(\Omega_h)}^2 + \sum_{e \in e^h} \left( \int_e \frac{\partial v}{\partial n_h} (S_h v - v) ds + \frac{\sigma}{h_e} \|S_h v\|_{L^2(\Omega)}^2 \right) \right)
\]

\[
= \nu \left( \|\nabla v\|_{L^2(\Omega_h)}^2 + \sum_{e \in e^h} \left( \int_e \delta^2 \frac{\partial v}{\partial n_h} ds + \frac{1}{2} \int e \delta^2 \frac{\partial^2 v}{\partial n_h^2} ds + \frac{\sigma}{h_e} \|S_h v\|_{L^2(\Omega)}^2 \right) \right).
\]
We then use the Cauchy-Schwarz inequality, standard trace and inverse estimates, and Assumption (A) to get
\[
\sum_{e \in E_h^b} \int_{e} \beta_e \frac{\partial v}{\partial n} \frac{\partial^2 v}{\partial n^2} \ ds \leq \left( \max_{e \in E_h^b} \beta_e^{-1} \right)^2 \left( \sum_{e \in E_h^b} \beta_e \| \nabla v \|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in E_h^b} \beta_e \| D^2 v \|_{L^2(e)}^2 \right)^{1/2}
\leq C \sigma \| \nabla v \|_{L^2(\Omega_h)}^2.
\]

Thus, we find
\[
a_h(v, v) \geq \nu \left( 1 - C \sigma^2 \right) \| \nabla v \|_{L^2(\Omega_h)}^2 + \sigma \| S_h v \|_{L^2(\Omega_h)}^2 \geq C \nu \| v \|_{1, h}^2
\]
for \( \sigma \) sufficiently small and for \( \sigma > 0 \).

\[\square\]

4.2. Inf-Sup Stability I. In this section we prove the discrete inf-sup (LBB) condition for the Stokes pair \( V_h \times Q_h \) with stability constants independent of \( h \). In the case of a fixed polygonal domain, the LBB stability for this pair is well-known (cf. [1, 29, 14]); however, the extension of these results to the unfitted domain \( \Omega_h \) is not immediate. In particular, the proofs in [1, 29, 14] (directly or indirectly) rely on the Nečas inequality:

\[
c_h \| q \|_{L^2(\Omega_h)} \leq \sup_{v \in H^1_0(\Omega_h) \setminus \{ 0 \}} \frac{\int_{\Omega_h} (\text{div } v) q \ dx}{\| \nabla v \|_{L^2(\Omega_h)}} \quad \forall q \in L^2(\Omega_h)
\]

for some \( c_h > 0 \) depending on the domain \( \Omega_h \). As explained in [16], it is unclear if the constant \( c_h \) in this inequality is independent of \( h \).

Our approach is to simply combine the local stability of the Scott-Vogelius pair with the stability of the \( P_2 \times P_0 \) pair. For a (macro) element \( T \in T_h \), we define the local spaces with boundary conditions

\[
V_0(T) = \{ v \in H^1_0(T) : v|_K \in P_2(K) \ \forall K \subset T, \ K \in T_h^e \},
\]

\[
Q_0(T) = \{ q \in L^2_0(T) : q|_K \in P_1(K) \ \forall K \subset T, \ K \in T_h^e \}.
\]

We state a local surjectivity of the divergence operator acting on these spaces. The proof is found in, e.g., [13].

**Lemma 4.5.** For every \( q \in Q_0(T) \), there exists \( v \in V_0(T) \) such that \( \text{div } v = q \) and \( \| \nabla v \|_{L^2(T)} \leq \beta_T^{-1} \| q \|_{L^2(T)} \). Here, the constant \( \beta_T > 0 \) depends only on the shape-regularity of \( T \).

Next, we state the recent stability result of the \( P_2 \times P_0 \) pair on unfitted domains (cf. [16] Theorem 1, Section 6.3, and Remark 1).

**Lemma 4.6.** Define the space of piecewise constants with respect to the mesh \( T_h \):

\[
\tilde{Q}_h = \{ q \in L^2_0(\Omega_h) : q|_T \in P_0(T) \ \forall T \in T_h \} \subset Q_h.
\]

There exists \( \beta_0 > 0 \) and \( h_0 > 0 \) such that for \( h \leq h_0 \), there holds

\[
\sup_{v \in V_h \setminus \{ 0 \}} \frac{\int_{\Omega_h} (\text{div } v) q \ dx}{\| \nabla v \|_{L^2(\Omega_h)}} \geq \beta_0 \| q \|_{L^2(\Omega_h)} \quad \forall q \in \tilde{Q}_h.
\]

Combining Lemmas 4.5, 4.6 yields the following stability result for the \( V_h \times Q_h \) Stokes pair.
Lemma 4.7. There exists $\beta_1 > 0$ independent of $h$ such that
\[
\sup_{v \in V_h \setminus \{0\}} \frac{\int_{\Omega_h} (\text{div } v) q \, dx}{\|\nabla v\|_{L^2(\Omega_h)}} \geq \beta_1 \|q\|_{L^2(\Omega_h)} \quad \forall q \in \hat{Q}_h.
\]
for $h \leq h_0$.

Proof. The proof essentially follows from Lemmas 4.5-4.6 with the arguments in [1, 29, 14]. We provide the proof for completeness. Let $q \in \hat{Q}_h$, and let $\bar{q} \in \bar{Y}_h$ be its piecewise average, i.e., $\bar{q}|_T = |T|^{-1} \int_T q \, dx$ for all $T \in \mathcal{T}_h$. We then have $(q - \bar{q})|_T \in Q_0(T)$ for all $T \in \mathcal{T}_h$, and therefore, by Lemma 4.5, there exists $v_1|_T \in V_0(T)$ such that $\text{div } v_1|_T = (q - \bar{q})|_T$ and $\|\nabla v\|_{L^2(T)} \leq \beta_T^{-1} \|q\|_{L^2(T)}$. Defining $v_1 \in \bar{V}_h$ by $v_1|_T = v_1|_T \forall T \in \mathcal{T}_h$, we have $\text{div } v_1 = (q - \bar{q})$ in $\Omega_h$ and $\|\nabla v_1\|_{L^2(\Omega_h)} \leq \beta_*^{-1} \|q - \bar{q}\|_{L^2(\Omega_h)}$, where $\beta_* = \min_{T \in \mathcal{T}_h} \beta_T$.

With this result, and by Lemma 4.6 we conclude
\[
\beta_0 \|q\|_{L^2(\Omega_h)} \leq \sup_{v \in V_h \setminus \{0\}} \frac{\int_{\Omega_h} (\text{div } v) \bar{q} \, dx}{\|\nabla v\|_{L^2(\Omega_h)}} \leq \sup_{v \in V_h \setminus \{0\}} \frac{\int_{\Omega_h} (\text{div } v) q \, dx}{\|\nabla v\|_{L^2(\Omega_h)}} + \|q - \bar{q}\|_{L^2(\Omega_h)} \\
\leq (1 + \beta_*^{-1}) \sup_{v \in V_h \setminus \{0\}} \frac{\int_{\Omega_h} (\text{div } v) q \, dx}{\|\nabla v\|_{L^2(\Omega_h)}}.
\]
Thus,
\[
\|q\|_{L^2(\Omega_h)} \leq \|q - \bar{q}\|_{L^2(\Omega_h)} + \|\bar{q}\|_{L^2(\Omega_h)} \leq (\beta_*^{-1} + \beta_0^{-1}(1 + \beta_*^{-1})) \sup_{v \in V_h \setminus \{0\}} \frac{\int_{\Omega_h} (\text{div } v) q \, dx}{\|\nabla v\|_{L^2(\Omega_h)}}.
\]
Setting $\beta_1 = (\beta_*^{-1} + \beta_0^{-1}(1 + \beta_*^{-1}))^{-1}$ completes the proof. □

4.3. Inf-Sup Stability II. The following lemma proves inf-sup stability for the Lagrange multiplier part of the bilinear form $b_h(\cdot, \cdot)$.

Lemma 4.8. There holds
\[(4.6) \quad \sup_{v \in V_h \setminus \{0\}} \frac{\int_{\partial \Omega_h} (v \cdot n) \mu \, ds}{\|v\|_{1,h}} \geq \beta_2 \|\mu\|_{-1/2,h} \quad \forall \mu \in \bar{X}_h.
\]
for some $\beta_2 > 0$ independent of $h$.

Proof. We label the boundary edges as $\{e_j\}_{j=1}^N = \mathcal{E}_h^B$, and denote the boundary vertices by $\{a_j\}_{j=1}^N = \mathcal{V}_h^B$, labeled such that $e_j$ has vertices $a_j$ and $a_{j+1}$, with the convention that $a_{N+1} = a_1$. Define the set of boundary edge midpoints $M_h^B = \{m_j\}_{j=1}^N$ with $m_j = \frac{1}{2}(a_j + a_{j+1})$. Let $n_j$ be the normal vector of $\partial \Omega_h$ restricted to the edge $e_j$, and let $t_j$ be the tangent vector obtained by rotating $n_j$ 90 degrees clockwise. Without loss of generality, we assume that $t|_{e_j}$ is parallel to $a_{j+1} - a_j$. We further denote by $\mathcal{V}_h^C$ the set of boundary corner vertices, i.e., if $a_j \in \mathcal{V}_h^C$, then the outward unit normals $n_j, n_{j-1}$ of the edges touching $a_j$ are linearly independent. The set of flat boundary vertices are defined as $\mathcal{V}_h^F = \mathcal{V}_h^B \setminus \mathcal{V}_h^C$. Note that $n_j = n_{j-1}$ and $t_j = t_{j-1}$ for $a_j \in \mathcal{V}_h^F$. 
Given \( \mu \in \mathcal{X}_h \), we define \( v \in \mathbf{V}_h \) by the conditions

\[
\begin{align*}
(v \cdot n_j)(a_j) &= \mu(a_j), & (v \cdot n_{j-1})(a_j) &= \mu(a_j) & \forall a_j \in \mathbf{V}_h^C, \\
(v \cdot n_j)(a_j) &= \mu(a_j), & (v \cdot t_j)(a_j) &= 0 & \forall a_j \in \mathbf{V}_h^E, \\
(v \cdot n_j)(m_j) &= \mu(a_j), & (v \cdot t_j)(m_j) &= 0 & \forall m_j \in \mathcal{M}_h^B.
\end{align*}
\]

All other (quadratic Lagrange) degrees of freedom of \( v \) are set to zero, i.e., \( v(a) = 0 \) at all interior vertices and interior edge midpoints in \( \mathcal{O}_h \).

Since \( (v \cdot n_j - \mu)|_{\varepsilon_j} \) is a quadratic on each \( \varepsilon_j \in \mathcal{E}_h^B \), and \( v \cdot n_j = \mu \) at three distinct points on \( e_j \), we have that \( v \cdot n_j - \mu|_{\varepsilon_j} = 0 \). Moreover, using quasi uniformity, we have

\[
\int_{\partial V_h} (v \cdot n) \mu \, ds \geq C \| \mu \|^2_{-2, h/\varepsilon}.
\]

It remains to show that \( \| v \|_{1,h} \leq C \| \mu \|_{-1/2,h} \) to complete the proof.

For \( K \in \mathcal{K}_h \), let \( \mathcal{V}_h^B, \mathcal{V}_h^C, \mathcal{V}_h^E, \mathcal{M}_h^B \) be the sets of elements in \( \mathcal{V}_h^B, \mathcal{V}_h^C, \mathcal{V}_h^E, \mathcal{M}_h^B \) contained in \( K \), respectively. By a standard scaling argument and \( (4.7) \), we get (\( m = 0, 1 \))

\[
\| v \|_{H^m(K)}^2 \leq C \sum_{a_j \in \mathcal{V}_h^C \cup \mathcal{M}_h^B} h^{-2m}_{e_j} |v(a_j)|^2
\]

Claim: \( |v(a_j)| \leq Ch|\mu(a_j)| \) for all \( a_j \in \mathcal{V}_h^C \), where \( C > 0 \) is uniformly bounded and independent of \( h, n_j \) and \( n_{j-1} \).

Proof of the claim: Assume that \( \mathcal{V}_h^C \) is non-empty for otherwise the proof is trivial. For \( a_j \in \mathcal{V}_h^C \), we write \( v(a_j) \) in terms of the basis \( \{ t_j, t_{j-1} \} \), use (\( 4.7 \)), and apply some elementary vector identities:

\[
v(a_j) = \frac{1}{t_{j-1} \cdot n_j} (v \cdot n_j)(a_j) t_{j-1} + \frac{1}{t_j \cdot n_{j-1}} (v \cdot n_{j-1})(a_j) t_j
\]

\[
= \mu(a_j) \left( \frac{1}{t_{j-1} \cdot n_j} t_{j-1} + \frac{1}{t_j \cdot n_{j-1}} t_j \right)
\]

\[
= \frac{\mu(a_j)}{t_j \cdot n_{j-1}} \left( t_j - t_{j-1} \right)
\]

Write \( t_j = (\cos(\theta_j), \sin(\theta_j))^\top \). We then compute \( t_j \cdot n_{j-1} = \sin(\theta_j - \theta_j) \), and therefore

\[
\frac{t_j - t_{j-1}}{t_j \cdot n_{j-1}} = \frac{(\cos(\theta_j) - \cos(\theta_{j-1}), \sin(\theta_j) - \sin(\theta_{j-1}))^\top}{\sin(\theta_{j-1} - \theta_j)}.
\]

Since

\[
\lim_{\theta_j \to \theta_{j-1}} \frac{(\cos(\theta_j) - \cos(\theta_{j-1}), \sin(\theta_j) - \sin(\theta_{j-1}))^\top}{\sin(\theta_{j-1} - \theta_j)} = \lim_{\theta_j \to \theta_{j-1}} \frac{(-\sin(\theta_j), \cos(\theta_j))^\top}{-\cos(\theta_{j-1} - \theta_j)} = (\sin(\theta_{j-1}, \cos(\theta_{j-1}))^\top,
\]

we conclude that \( \frac{t_j - t_{j-1}}{t_j \cdot n_{j-1}} \) is bounded for \( t_j \cdot n_{j-1} \ll 1 \), i.e., for “nearly flat boundary vertices”.

This conclusion and the shape regularity of the mesh shows \( \frac{t_j - t_{j-1}}{t_j \cdot n_{j-1}} \leq C \) for some \( C > 0 \) independent of \( h \) and \( \{ n_{j-1}, n_j \} \). This concludes the proof of the claim.
Applying the claim to (4.9) and a scaling argument yields
\[
\|v\|_{H^m(K)}^2 \leq C \sum_{a_j \in \mathcal{V}_h^K \cup \mathcal{M}_h^K} h_e^{4-2m} |\mu(a_j)|^2 \leq C \sum_{e \in \mathcal{E}_h^K} h_e^{3-2m}\|\mu\|_{L^2(e)}^2.
\]
Therefore, by an inverse inequality and shape-regularity of \(T_h^e\),
\[
\|v\|_{1,h}^2 = \|\nabla v\|_{L^2(\Omega_h)}^2 + \sum_{e \in \mathcal{E}_h^K} \frac{1}{h_e} \|v\|_{L^2(e)}^2 \leq C\|\mu\|_{-1/2,h}^2 + C \sum_{K \in T_h^e} h_K^{-2} \|v\|_{L^2(K)}^2 \leq C\|\mu\|_{-1/2,h}^2.
\]
Combining this estimate with (4.8) yields the desired inf-sup condition (4.6).

Remark 4.9. The proof of Lemma 4.8 and in particular the proof of the claim, relies on the continuity properties of the Lagrange multiplier space at nearly flat corner vertices.

4.4. Main Stability Estimates. Combining Lemmas 4.7 and 4.8 yields inf-sup stability for the bilinear form \(b_h(\cdot,\cdot)\). We also show that this result implies inf-sup stability for the bilinear form with boundary correction \(b_h^b(\cdot,\cdot)\).

**Theorem 4.10.** Then there exists \(\beta > 0\) depending only on \(\beta_1\) and \(\beta_2\) such that
\[
\beta\|q\| \leq \sup_{v \in \mathcal{V}_h \setminus \{0\}} \frac{b_h(v,q,\mu)}{\|v\|_{1,h}} \quad \forall (q,\mu) \in \tilde{Q}_h \times \tilde{X}_h.
\]

**Proof.** We use Lemmas 4.7 and 4.8 and follow the arguments in [20, Theorem 3.1].

Fix \((q,\mu) \in \tilde{Q}_h \times \tilde{X}_h\). The statement (4.6) implies the existence of \(v_2 \in \mathcal{V}_h\) such that \(\|v_2\|_{1,h} \leq 1\) and
\[
\int_{\partial \Omega_h} (v_2 \cdot n) \mu \, ds \geq \beta_2 \|\mu\|_{-1/2,h}.
\]
By Lemma 4.7 there exists \(v_1 \in \tilde{V}_h\) satisfying \(\|\nabla v_1\|_{L^2(\Omega_h)} = \|v_1\|_{1,h} \leq 1\) and
\[
-\int_{\Omega_h} (\text{div } v_1) q = \beta_1 \|q\|_{L^2(\Omega_h)}.
\]
Set \(v = cv_1 + v_2\) for some \(c > 0\), so that \(\|v\|_{1,h} \leq (1 + c)\), and
\[
-\int_{\Omega_h} (\text{div } v) q \, dx \geq c\beta_1 \|q\|_{L^2(\Omega_h)} - \|\text{div } v_2\|_{L^2(\Omega_h)} \|q\|_{L^2(\Omega_h)} \\
\geq c\beta_1 \|q\|_{L^2(\Omega_h)} - \sqrt{2} \|\nabla v_2\|_{L^2(\Omega_h)} \|q\|_{L^2(\Omega_h)} \\
\geq c\beta_1 \|q\|_{L^2(\Omega_h)} - \sqrt{2} \|v_2\|_{1,h} \|q\|_{L^2(\Omega_h)} \\
= (c\beta_1 - \sqrt{2}) \|q\|_{L^2(\Omega_h)}.
\]
Because \(v_1|_{\partial \Omega} = 0\), we have
\[
\int_{\partial \Omega_h} (v \cdot n_h) \mu \, ds = \int_{\partial \Omega_h} (v_2 \cdot n_h) \mu \, ds \geq \beta_2 \|\mu\|_{-1/2,h}.
\]
Therefore,
\[
b_h(v,q,\mu) \geq (c\beta_1 - \sqrt{2}) \|q\|_{L^2(\Omega_h)} + \beta_2 \|\mu\|_{-1/2,h}
\]
\[ \beta \| (q, \mu) \| \leq \sup_{v \in V_h \setminus \{0\}} \frac{b_h(v, (q, \mu))}{\|v\|_{1,h}} \quad \forall (q, \mu) \in \bar{Q}_h \times \bar{X}_h. \]

**Proof.** Combining Theorem 4.10 and Lemma 4.3, we have

\[ \beta \| (q, \mu) \| \leq \sup_{v \in V_h \setminus \{0\}} \frac{b_h(v, (q, \mu))}{\|v\|_{1,h}} + C_{c_3} \| (q, \mu) \| \quad \forall (q, \mu) \in \bar{Q}_h \times \bar{X}_h. \]

This result implies (4.11) for \( c_3 \) sufficiently small with \( \beta_c = \beta - C_{c_3} \).

**Theorem 4.12.** Let \((u_h, p_h, \lambda_h) \in V_h \times \bar{Q}_h \times \bar{X}_h\) satisfy (3.1). Then, provided \( c_3 \) in Assumption (A) is sufficiently small, there holds

\[ \nu \| u_h \|_{1,h} + \| (p_h, \lambda_h) \| \leq C \| f \|_{-1,h}, \]

where \( \| f \|_{-1,h} = \sup_{v \in V_h \setminus \{0\}} \frac{\int_{\Omega_h} f \cdot v \, dx}{\|v\|_{1,h}} \). Consequently, there exists a unique solution to (3.1).

**Proof.** Setting \( v = u_h \) in (3.1a), \( (q, \mu) = (p_h, \lambda_h) \) in (3.1b), and subtracting the resulting expressions yields

\[ a_h(u_h, u_h) = \int_{\Omega_h} f \cdot u_h \, dx + \int_{\partial \Omega_h} ((S_h u_h - u_h) \cdot n_h) \lambda_h \, ds. \]

We apply the coercivity result in Lemma 4.4, the Cauchy-Schwarz inequality, and (4.1) to get

\[ \nu c_1 \| u_h \|_{1,h}^2 \leq \| f \|_{-1,h} \| u_h \|_{1,h} + C_{c_3} \| u_h \|_{1,h} \| \lambda_h \|_{-1/2,h}. \]

On the other hand, we use inf-sup stability (4.10) to conclude

\[ \beta \| (p_h, \lambda_h) \|_{-1/2,h} \leq \sup_{v \in V_h \setminus \{0\}} \frac{b_h(v, (p_h, \lambda_h))}{\|v\|_{1,h}} \leq \sup_{v \in V_h \setminus \{0\}} \frac{\int_{\Omega_h} f \cdot v \, dx - a_h(u_h, v)}{\|v\|_{1,h}}. \]

Using the continuity estimate (4.3) gets

\[ \beta \| \lambda_h \|_{-1/2,h} \leq \beta \| (p_h, \lambda_h) \| \leq \| f \|_{-1,h} + c_2 (1 + \sigma) \nu \| u_h \|_{1,h}. \]

Inserting this estimate into (4.13), we obtain

\[ \nu (c_1 - C_{c_3} c_2 \beta^{-1} (1 + \sigma)) \| u_h \|_{1,h} \leq (1 + C_{c_3} \beta^{-1}) \| f \|_{-1,h}. \]

Thus, \( \| u_h \|_{1,h} \leq C_{\nu}^{-1} \| f \|_{-1,h} \) for \( c_3 \) sufficiently small. This, combined with (4.14), yields the desired stability result (4.12). \[ \square \]
In this section, we show that the solution to the finite element method (3.1) converges with optimal order provided the exact solution is sufficiently smooth. As a first step, we derive some consistency estimates for the boundary correction operator and the bilinear form $a_h(\cdot, \cdot)$.

5. Convergence Analysis

5.1. Consistency Estimates. The following lemma bounds the boundary correction operator acting on the exact velocity function. The result is essentially an estimate on the Taylor polynomial remainder and follows directly from the arguments in [4, Proposition 3] (also see [6]). For this reason, its proof is omitted.

**Lemma 5.1.** For any $u \in H^3(\Omega) \cap H^1_0(\Omega)$, there holds

$$
\sum_{e \in \mathcal{E}_h^B} h_e^{-1} \int_e |S_h u|^2 \, ds \leq Ch^4 \|u\|^2_{H^3(\Omega)}.
$$

**Lemma 5.2.** There holds for all $u \in H^3(\Omega) \cap H^1_0(\Omega)$,

$$
- \nu \int_{\Omega_h} \Delta u \cdot v \, dx - a_h(u, v) \leq Ch^2 \|u\|_{H^3(\Omega)} \|v\|_{1,h} \quad \forall v \in V_h.
$$

If $\text{div} u = 0$ in $\Omega$, then

$$
|b_h(u, (q, \mu))| \leq Ch^2 \|u\|_{H^3(\Omega)} \|(q, \mu)\| \quad \forall (q, \mu) \in \hat{Q}_h \times \hat{X}_h.
$$

**Proof.** We integrate-by-parts to write

$$
| - \nu \int_{\Omega_h} \Delta u \cdot v \, dx - a_h(u, v) | = | \nu \sum_{e \in \mathcal{E}_h^B} \int_e \frac{\partial v}{\partial n_h}(S_h u) \, ds + \sum_{e \in \mathcal{E}_h^B} \sigma \frac{1}{h_e} \int_e (S_h u)(S_h v) \, ds |.
$$

Next, we estimate the two terms on the right hand side of the above equality by using the Cauchy-Schwarz inequality, trace and inverse inequalities, along with Lemmas 4.2 and 5.1 as follows:

$$
\left( \sum_{e \in \mathcal{E}_h^B} \int_e \frac{\partial v}{\partial n_h}(S_h u) \, ds \right)^2 \leq \left( \sum_{e \in \mathcal{E}_h^B} h_e \int_e \left| \frac{\partial v}{\partial n_h} \right|^2 \, ds \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \int_e |S_h u|^2 \, ds \right)^{1/2}
$$

$$
\leq Ch^2 \|u\|_{H^3(\Omega)} \|v\|_{1,h},
$$

and

$$
\left( \sum_{e \in \mathcal{E}_h^B} \frac{\sigma}{h_e} \int_e (S_h u)(S_h v) \, ds \right)^2 \leq \sigma \left( \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \int_e |S_h u|^2 \, ds \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h^B} h_e^{-1} \int_e |S_h v|^2 \, ds \right)^{1/2}
$$

$$
\leq Ch^2 \|u\|_{H^3(\Omega)} \|v\|_{1,h}.
$$

Thus, the first estimate (5.1) holds.

Similarly, another use of the Cauchy-Schwarz inequality with Lemma 5.1 yields

$$
|b_h(u, (q, \mu))| = \left| \sum_{e \in \mathcal{E}_h^B} \int_e (S_h u \cdot n_h) \mu \, ds \right| \leq Ch^2 \|u\|_{H^3(\Omega)} \|\mu\|_{-1/2,h},
$$

and this completes the proof. \qed
5.2. Approximation Properties of the Kernel. We define the discrete kernel as

\[ Z_h = \{ v \in V_h : \beta_h^2(v, (q, \mu)) = 0, \ \forall (q, \mu) \in \tilde{Q}_h \times \tilde{X}_h \}. \]

Note that if \( v \in Z_h \), then \( \text{div} \ v = 0 \) in \( \Omega_h \) (cf. Lemma 3.3), and

\[ (5.2) \quad \int_{\partial \Omega_h} ((S_h v) \cdot n_h) \mu \, ds = 0 \quad \forall \mu \in \tilde{X}_h. \]

In this section, we show that the kernel \( Z_h \) has optimal order approximation properties with respect to divergence-free smooth functions. To this end, we define the orthogonal complement of \( Z_h \) as

\[ Z_h^\perp := \{ v \in V_h : (v, w)_{1,h} = 0 \ \forall w \in Z_h \}, \]

where \((\cdot, \cdot)_{1,h}\) is the inner product on \( V_h \) that induces the norm \( \| \cdot \|_{1,h} \).

**Lemma 5.3.** There holds

\[ \beta_e \| w \|_{1,h} \leq \sup_{(q, \mu) \in \tilde{Q}_h \times \tilde{X}_h \setminus \{0\}} \frac{b_h^2(w, (q, \mu))}{\| (q, \mu) \|} \quad \forall w \in Z_h^\perp. \]

**Proof.** The result follows from Corollary 4.11 and standard results in mixed finite element theory (cf. [5], Lemma 12.5.10).

The following theorem states the approximation properties of the discrete kernel.

**Theorem 5.4.** For any \( u \in H^3(\Omega) \cap H^1_0(\Omega) \) with \( \text{div} \ u = 0 \), there holds

\[ (5.3) \quad \inf_{w \in Z_h} \| u - w \|_h \leq C h^2 \| u \|_{H^3(\Omega)}. \]

**Proof.** Let \( v \in V_h \) be arbitrary. By Corollary 4.11, there exists \( y \in Z_h^\perp \) such that

\[ b_h^2(y, (q, \mu)) = b_h^2(u - v, (q, \mu)) \quad \forall (q, \mu) \in \tilde{Q}_h \times \tilde{X}_h, \]

and \( \| y \|_{1,h} \leq C \beta_e^{-1} \| u - v \|_{1,h} \). We then let \( z \in Z_h^\perp \) satisfy

\[ b_h^2(z, (q, \mu)) = -b_h^2(u, (q, \mu)) \quad \forall (q, \mu) \in \tilde{Q}_h \times \tilde{X}_h. \]

Then \( w := v + y + z \in Z_h \), and

\[ \| u - w \|_{1,h} \leq \| u - v \|_{1,h} + \| y \|_{1,h} + \| z \|_{1,h} \]

\[ \leq (1 + C \beta_e^{-1}) \| u - v \|_{1,h} + \| z \|_{1,h}. \]

By Lemmas 5.3 and 5.2,

\[ \beta_e \| z \|_{1,h} \leq \sup_{(q, \mu) \in \tilde{Q}_h \times \tilde{X}_h \setminus \{0\}} \frac{b_h^2(u, (q, \mu))}{\| (q, \mu) \|} \leq C h^2 \| u \|_{H^3(\Omega)}, \]

and so, by Lemma 4.2,

\[ \| u - w \|_h \leq \| u - v \|_h + C \| v - w \|_{1,h} \leq C (\| u - v \|_h + \| u - w \|_{1,h}) \]

\[ \leq C (1 + \beta_e^{-1}) (\| u - v \|_h + \| u - v \|_{1,h} + h^2 \| u \|_{H^3(\Omega)}) \quad \forall v \in V_h. \]

Taking \( v \) to be the nodal interpolant of \( u \), we obtain the desired result. \( \square \)
Theorem 5.5. Suppose that the solution to (2.1) has regularity \((u, p) \in H^3(\Omega) \cap H^1_0(\Omega) \times H^s(\Omega)\) for some \(1 \leq s \leq 3\). Furthermore, without loss of generality, assume that \(p|_{\partial \Omega} \in L^2(\partial \Omega)\). Then,

\[
\|u - u_h\|_{1, h} \leq C(h^2 \|u\|_{H^3(\Omega)} + \nu^{-1} \inf_{\mu \in X_h} \|p - \mu\|_{-1, h}),
\]

\[
\|p - p_h\|_{L^2(\Omega)} \leq C(\nu h^2 \|u\|_{H^3(\Omega)} + \inf_{\mu \in X_h} \|p - \mu\|_{-1, h} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2(\Omega)}),
\]

\[
\|\lambda_h - \mu\|_{-1, h} \leq C(\nu h^2 \|u\|_{H^3(\Omega)} + \|p - \mu\|_{-1, h}) \quad \forall \mu \in X_h,
\]

where \(\hat{\mu} := \mu - \frac{1}{|\partial \Omega|} \int_{\partial \Omega} p ds\). In particular there holds

\[
\|u - u_h\|_{1, h} \leq C(h^2 \|u\|_{H^3(\Omega)} + \nu^{-1} h^s \|p\|_{H^s(\Omega)}),
\]

\[
\|p - p_h\|_{L^2(\Omega)} \leq C(\nu h^2 \|u\|_{H^3(\Omega)} + h^{m(2, s)} \|p\|_{H^{m(2, s)}(\Omega)}).
\]

Proof. Let \(w \in Z_h\) be arbitrary. We then have, for all \(v \in Z_h\) and \(\mu \in X_h\),

\[
a_h(u_h - w, v) = \int_{\Omega_h} f \cdot v - a_h(w, v) - b_h(v, (p_h, \lambda_h))
\]

\[
= -\nu \int_{\Omega_h} \Delta u \cdot v dx - a_h(w, v) - \int_{\partial \Omega_h} (v \cdot n_h)(p - \lambda_h) ds
\]

\[
= -\nu \int_{\Omega_h} \Delta u \cdot v dx - a_h(w, v) - \int_{\partial \Omega_h} (v \cdot n_h)(p - \lambda_h) ds + \int_{\partial \Omega_h} (v \cdot n_h)(\lambda_h - \hat{\mu}) ds,
\]

where \(\hat{\mu} := \mu - \frac{1}{|\partial \Omega|} \int_{\partial \Omega} p ds\).

Therefore by Lemma 5.2 the continuity of \(a_h(\cdot, \cdot)\) (cf. (4.3)), and the Cauchy-Schwarz inequality,

\[
a_h(u_h - w, v) \leq C(\nu h^2 \|u\|_{H^3(\Omega)} + \|p - \mu\|_{-1, h}) \|v\|_{1, h} + a_h(u - w, v) + \int_{\partial \Omega_h} (v \cdot n_h)(\lambda_h - \hat{\mu}) ds
\]

\[
\leq C(\nu h^2 \|u\|_{H^3(\Omega)} + \nu(1 + \sigma) \|u - w\|_{h} + \|p - \mu\|_{-1, h}) \|v\|_{1, h} + \int_{\partial \Omega_h} (v \cdot n_h)(\lambda_h - \hat{\mu}) ds.
\]

We then use (5.2) and (4.1) to obtain

\[
\int_{\partial \Omega_h} (v \cdot n_h)(\lambda_h - \hat{\mu}) ds = \int_{\partial \Omega_h} ((v - S_h v) \cdot n_h)(\lambda_h - \hat{\mu}) ds \leq Cc_h \|v\|_{1, h} \|\lambda_h - \hat{\mu}\|_{-1, h}.
\]

Setting \(v = u_h - w\), applying the coercivity of \(a_h(\cdot, \cdot)\) and Theorem 5.3 we obtain

\[
(5.5) \quad c_1 \nu \|u_h - w\|_{1, h} \leq C(\nu(1 + \sigma) h^2 \|u\|_{H^3(\Omega)} + \|p - \mu\|_{-1, h} + c_1 \|\lambda_h - \hat{\mu}\|_{-1, h}.
\]

for \(w \in Z_h\) satisfying (5.3).

Next, let \(P_h \in \mathcal{Q}_h\) be the \(L^2\)-projection of \(p\) and note that, due to the definitions of the finite element spaces, \(\int_{\Omega_h} (\text{div} \, v)(p - P_h) dx = 0\) for all \(v \in \mathcal{V}_h\). This identity, along with the inf-sup stability estimate given in Theorem 4.10 yields

\[
\beta \|(p_h - P_h, \lambda_h - \hat{\mu})\| \leq \sup_{v \in \mathcal{V}_k \setminus \{0\}} \frac{b_h(v, (p_h - P_h, \lambda_h - \hat{\mu}))}{\|v\|_{1, h}} = \sup_{v \in \mathcal{V}_k \setminus \{0\}} \frac{b_h(v, (p_h - P_h, \lambda_h - \hat{\mu}))}{\|v\|_{1, h}}.
\]

Using Lemma 5.2 we write the numerator as

\[
b_h(v, (p_h - p, \lambda_h - \hat{\mu})) = b_h(v, (p_h, \lambda_h)) - b_h(v, (p, \hat{\mu}))
\]

\[
= \int_{\Omega_h} f \cdot v dx - a_h(u_h, v) + \int_{\partial \Omega_h} \frac{\text{div} \, v}{p} dx - \int_{\partial \Omega_h} (v \cdot n_h) \mu ds.
\]
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
0.8
3
1
10^{-4}
4
0.6
0.8
5
6
0.6
0.4
7
0.4
0.2 0.2
0 0

Figure 1. Left: The domain and mesh with $h = 1/24$. Right: The graph of the error $|u - u_h|$ with exact solution (6.2).

\begin{align}
\notag &\leq C\nu h^2 \|u\|_{H^3(\Omega)} + a_h(u - u_h, v) - \int_{\partial\Omega_h} (v \cdot n_h)(\mu - p) \, ds.
\end{align}

By continuity and the Cauchy-Schwarz inequality,

\begin{align}
\notag &\leq C\nu h^2 \|u\|_{H^3(\Omega)} + c_2 \nu (1 + \sigma)(\|u - w\|_h + \|u_h - w\|_{1,h}) + \|p - \mu\|_{-1/2,h}.
\end{align}

Inserting this estimate into (5.5), we get

\begin{align}
\nu(c_1 - C\beta^{-1} c_2 (1 + \sigma)c_3)\|u_h - w\|_{1,h} \leq C\nu(1 + \sigma) h^2 \|u\|_{H^3(\Omega)} + \|p - \mu\|_{-1/2,h}.
\end{align}

Using the approximation properties of the discrete kernel once again (cf. Theorem 5.4), and for $c_3$ sufficiently small,

\begin{align}
\notag &\|u - u_h\|_{1,h} \leq C(h^2 \|u\|_{H^3(\Omega)} + \nu^{-1} \inf_{\mu \in X_h} \|p - \mu\|_{-1/2,h}).
\end{align}

This establishes the velocity estimate (5.4a).

To obtain the estimate for the pressure approximation (5.4b), we use the triangle inequality and the approximation properties of the $L^2$-projection:

\begin{align}
\notag &\|p - p_h\|_{L^2(\Omega_h)} \leq \|p - P_h\|_{L^2(\Omega_h)} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2(\Omega_h)}.
\end{align}

Inserting (5.6) and (5.7) into the right-hand side yields the desired bound for the pressure. Likewise, combining (5.6) and (5.7) yields (5.4c).

6. Numerical Experiments

In this section we perform simple numerical experiments of the finite element method (3.1) which verify the theoretical rates of convergence established in the previous sections.
In the series of tests, the domain is defined via a level set function \[\Omega = \{x \in \mathbb{R}^2 : \phi(x) < 0\}\], where \(\phi = r - 0.3723423423343 - 0.1 \sin(6\theta)\), with \(r = \sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2}\), and \(\theta = \tan^{-1}((x_2 - 0.5)/(x_1 - 0.5))\). We take \(S = (0, 1)^2\), and the background mesh \(S_h\) to be a sequence of type I triangulations of \(S\), i.e., a mesh obtained by drawing diagonals of a cartesian mesh; cf. Figure 1. For all tests, the Nitsche penalty parameter \(\sigma\) and the background mesh \((6.1)\)

and set \(d = (x - x_\ast)/|x - x_\ast|\) and \(\delta(x) = |x - x_\ast|\). The first equation ensures that \(x_\ast\) is on the boundary \(\partial\Omega\), whereas the second equation states that \(d\) is parallel to the outward unit normal of \(\partial\Omega\) at \(x\).

We choose the data such that the exact solution to the Stokes problem is given by

\[
(6.2) \quad \mathbf{u} = \begin{pmatrix} 2(x_1^2 - x_1 + \frac{1}{4} + x_2^2 - x_2)(2x_2 - 1) \\ 2x_1 - x_1 + \frac{1}{4} + x_2^2 - x_2)(2x_1 - 1) \end{pmatrix}, \quad p = 10(x_1^2 - x_2^2)^2.
\]

Because the exact solution is smooth, Theorem 5.5 predicts the convergence rates

\[
(6.3) \quad \|\nabla (\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega_h)} = \mathcal{O}(h^2 + \nu^{-1}h^3), \quad \|p - p_h\|_{L^2(\Omega_h)} = \mathcal{O}(h^2).
\]

The velocity and pressure errors are plotted in Figure 2 for mesh parameters \(h = 2^{-j}\) \((j = 3, 4, 5, 6, 7)\) and viscosities \(\nu = 10^{-k}\) \((k = 1, 3, 5)\). The results show that, for the moderately sized viscosities \(\nu = 10^{-1}\) and \(\nu = 10^{-3}\), the \(L^2\) and \(H^1\) velocities converge with the optimal order three and two, respectively. We also observe larger velocity errors for viscosity value \(\nu = 10^{-5}\), although, rates of convergence are higher; Figure 2 shows fourth and third order convergence in the \(L^2\) and \(H^1\) norms. This behavior is consistent with the theoretical estimate (6.3). Finally, the numerical experiments show second order convergence for the pressure approximation (with only marginal differences for different viscosity values) and divergence errors comparable to machine epsilon.

7. Concluding Remarks

This paper constructed a uniformly stable and divergence-free method for the Stokes problem on unfitted meshes using a boundary correction approach. While the method is not pressure-robust, a Lagrange multiplier enforcing the normal boundary conditions is included to mitigate the affect of the pressure contribution in the velocity error. Theoretical results and numerical experiments show that the method converges with optimal order.

The presentation is confined to the two dimensional setting, however many of the results extend to 3D as well. For example, the proof of inf-sup stability given in Lemma 4.3 applies mutatis mutandis to the three-dimensional Scott-Vogelius pair. On the other hand, inf-sup stability of the velocity-Lagrange multiplier pairing (cf. Lemma 4.8), and its dependence on the geometry of the computational mesh is less obvious. We plan to address this issue in the near future.
\[ \| \mathbf{u} - \mathbf{u}_h \|_{L^2(\Omega_h)} \]

\[ \| \nabla (\mathbf{u} - \mathbf{u}_h) \|_{L^2(\Omega_h)} \]

\[ \| p - p_h \|_{L^2(\Omega_h)} \]

\[ \| \text{div} \mathbf{u}_h \|_{L^\infty(\Omega_h)} \]

**Figure 2.** Errors for the velocity and pressure for a sequence of meshes on domain \( (6.1) \) and exact solution \( (6.2) \).

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