Counterpart synchronization of duplex networks with delayed nodes and noise perturbation

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Abstract. In the real world, many complex systems are represented not by single networks but rather by sets of interdependent ones. In these specific networks, nodes in one network mutually interact with nodes in other networks. This paper focuses on a simple representative case of two-layer networks (the so-called duplex networks) with unidirectional inter-layer couplings. That is, each node in one network depends on a counterpart in the other network. Accordingly, the former network is called the response layer and the latter network is the drive layer. Specifically, synchronization between each node in the drive layer and its counterpart in the response layer (counterpart synchronization (CS)) in these kinds of duplex networks with delayed nodes and noise perturbation is investigated. Based on the LaSalle-type invariance principle, a control technique is proposed and a sufficient condition is developed for realizing CS of duplex networks. Furthermore, two corollaries are derived as special cases. In addition, node dynamics within each layer can be varied and topologies of the two layers are not necessarily identical. Therefore, the proposed synchronization method can be applied to a wide range of multiplex networks. Numerical examples are provided to illustrate the feasibility and effectiveness of the results.

Keywords: network dynamics, nonlinear dynamics, peer-to-peer networks
1. Introduction

Complex networks abound in almost every aspect of science and technology. Examples include the Internet, the World Wide Web, social networks, metabolic networks, food webs, and networks of citations between papers, among many others [1–3]. Synchronization is one of the most common phenomena in nature where interacting nodes can reach a coherent state, and it has been extensively investigated and discussed during the past two decades [4–10]. For example, Pecora et al [7] used the master stability function (MSF) approach to analyze the stability of the synchronous state in coupled systems, Huang et al [8] classified synchronization into five categories based on the MSF approach, and Wang and Chen investigated synchronization in small-world networks [9] and scale-free [10] networks.

Much of the literature, including the examples mentioned above, are primarily focused on synchronization within single networks that do not interact with other networks. However, many real-world networks often interact with and depend on each other. For example, people in a society interact with each other via their family relationships, friendships, or formal work-related acquaintanceships [11]. Countries in the global economic system also interact via various international relations. Transportation depends on air traffic networks, railway networks and road traffic networks. Obviously, in describing and dealing with such problems, the multiplex network representation would be more appropriate than the single network. Not surprisingly, multiplex networks have attracted enormous attention in the past few years in various fields of application. For example, Xiong et al [12] analysed the correlation between the information diffusion process and the opinion evolution process, and found an obvious interaction between the two processes. Liu et al [13] investigated preferred degree networks and their interactions, and found two very similar networks to have dramatically different behaviours.

Counterpart synchronization (CS) describes how the individuals in one network behave coherently with their counterparts in other associated networks, and so it...
Counterpart synchronization of duplex networks represents harmonious coexistence of nodes in multiplex networks. The proposed CS comes from the concept of synchronization for coupled drive–response chaotic systems proposed in [14]. Since 1990, researchers have paid significant attention to synchronization control of two chaotic systems [15, 16] and its applications in the fields of information science, secure communication, biological systems, etc [17–19]. In [14], a driver system generates a signal sent over a channel to a response system, which uses this signal to synchronize itself with the driver system. So in the drive–response systems, the response system is influenced by the behavior of the drive system, but the latter is independent of the former. Many natural phenomena reflect synchronization of drive and response networks. A typical example is synchronization between two networks of clock neurons in the adult drosophila brain. In these networks, the drive network includes the self-sustained lateral neurons, which communicate with each other, and the response network includes the damped dorsal neurons, which receive neurotransmitters from the drive network without interaction within it [20].

This sort of synchronization in a duplex network can also be understood as the so-called outer synchronization between two networks, and has attracted wide attention. For example, Wu et al. investigated generalized outer synchronization between two different complex dynamical networks by employing nonlinear control [21]. Louzada et al. [22] investigated the synchronization of interconnected oscillator networks with time delay between networks, and discovered a breathing synchronization regime at which two groups with different frequencies in each network can achieve synchronization. Dong et al. [23] investigated finite-time outer synchronization with on–off coupling and the two networks realized outer synchronization for fixed on–off rate with the large enough coupling strength. In [24], the authors offered an understanding of how the multilayer nature of the network affected processes and dynamics by using diverse relationships (layers) between its constituents and redefining the basic structural measures. In [25], the authors analyzed synchronization oscillators indirectly coupled through an inhomogeneous medium layer and found a novel regime of synchronization, in which intra-layer coherence does not require inter-layer coherence between the two layers of the multiplex.

In particular, this synchronization has been widely applied in topology identification of complex networks. To give some examples, Wu [26] and Zhao et al. [27] employed complete outer synchronization to identify topologies for weighted complex networks; Zhang et al. [28] et al. adopted generalized outer synchronization to recover network structures.

Time delays are unavoidable in complex networks due to finite information processing and propagation speeds. They exist extensively in the real world, for example in communication networks, gene regulatory networks, and electrical power grids. Time delays greatly influence the behaviors of dynamical systems. Much of the literature is focused on the synchronization and control of complex networks with coupling delay among different nodes [29, 30].

Noise is another important factor affecting the behaviors of dynamical systems. First, noise is omnipresent in real-world complex networks. Furthermore, when signals collected from the drive network are transmitted to the response networks, there inevitably exist perturbations, such as loss of information, during the transmission process [31]. That is to say, noise is more likely to exist in response networks which receive signals...
from the driver. Therefore, it is more practical to consider noise in the response layer. Generally, noise is harmful. However, the presence of noise sometimes plays a positive role [32], such as in inducing synchronization [33] and in facilitating topology identification of complex networks [34, 35].

An adaptive controller is designed to synchronize the coupled nodes. The adaptive control method has many advantages. First, when the system parameters are unpredictable, the adaptive synchronization mechanism can achieve synchronization without any calculation of parameters. It can also adjust the strength of controllers according to the error between two systems. Thus it can greatly reduce control cost [36]. Furthermore, adaptive parameters can adjust themselves according to some formal updating laws, which are designed under control purposes according to the characteristics of the considered system [37]. In [38], the authors used two connected memristors to realize an adaptive law for the consensus and synchronization of a network; the approach was an effective strategy since the coupling configuration consisted of only two components. In [39], the authors proposed a hierarchy of decentralized adaptive pinning strategies for controlling a network onto a desired trajectory, and the coupling and control gains were adaptive during the network evolution process.

Motivated by the discussions above, we investigate CS of duplex networks with delayed nodes and noise perturbation. Based on the LaSalle-type invariance principle for stochastic differential delay equations, we design adaptive controllers to synchronize nodes of the response layer to their counterparts in the drive layer, and put forward some sufficient conditions for guaranteeing CS.

The rest of this paper is organized as follows. The modeling of duplex networks and some preliminaries are introduced in section 2. Sufficient conditions for CS in duplex networks are presented in section 3. In section 4, two numerical examples are provided to illustrate the feasibility and effectiveness of our method. Finally, some conclusions are drawn in section 5.

**Notation:** Some necessary notation used throughout the paper is introduced. $\mathbf{x}^\top$ (or $\mathbf{A}^\top$) denotes the transpose of a vector $\mathbf{x}$ (or a matrix $\mathbf{A}$), $\|\mathbf{x}\|_2$ is the Euclidean-norm of $\mathbf{x}$, $\otimes$ represents the Kronecker product, $\mathbb{R}^n$ is the $n$-dimensional real space, $I_n \in \mathbb{R}^{n \times n}$ represents an identity matrix of order $n$, and $C^0[a,b]$ $(a, b \in \mathbb{R}, a < b)$ represents the $n$th-order continuously differentiable function space in $[a, b]$.

## 2. Modeling and preliminaries

Consider a duplex network consisting of $N$ nodes in each layer, as shown in figure 1. For convenience, take the upper layer as the drive layer, and the lower layer (which is dependent on signals from the drive layer) as the response layer. We are concerned about the impact of delayed nodes and noise caused by control input. Thus a drive layer consisting of $N$ linearly coupled nodes is described by

$$
d\mathbf{x}_i(t) = \left[ f_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t - \tau(t))) + \sum_{j=1}^{N} c_{ij} \mathbf{G} \mathbf{x}_j(t) \right] dt, \quad i = 1, 2, \ldots, N,
$$

(1)
and the response layer with control input is given by

\[ \frac{dy_i(t)}{dt} = f_i(t, y(t), y(t - \tau(t))) + \sum_{j=1}^{N} d_{ij} \Gamma_j(t) y_j(t) + u_i(t) \]  
\[ + \sigma_i(t, e_i(t), e_i(t - \tau(t)))dw(t), \quad i = 1, 2, ..., N. \]  

Here, \( x_i(t) = (x_{i1}, ..., x_{in})^T \in \mathbb{R}^n \) and \( y(t) = (y_1, ..., y_n)^T \in \mathbb{R}^n \) are state vectors, \( u_i(t) \) is the control input for node \( i \), \( f_i : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuously differentiable function determining the dynamical behavior of node \( i \), \( \Gamma = (\gamma_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \) is the inner coupling matrix, and \( C = (c_{ij})_{N \times N} \in \mathbb{R}^{N \times N} \) is the coupling configuration matrix representing the coupling strength and the topological structure of network (1), with \( c_{ij} \) being defined as follows: if there is a link from node \( j \) to node \( i \) \( (i \neq j) \), \( c_{ij} \neq 0 \); otherwise, \( c_{ij} = 0 \). The diagonal elements of matrix \( C \) is \( c_{ii} = - \sum_{j=1,j \neq i}^{N} c_{ij} \) for \( i = 1, 2, ..., N \). \( D = (d_{ij})_{N \times N} \in \mathbb{R}^{N \times N} \) is the coupling configuration matrix of network (2), which has the same meaning as that of \( C \). \( \tau(t) \) denotes the time delay of nodes, \( e_i(t) = y_i(t) - x_i(t) \). The noise term in network (2) is utilized to describe the perturbation caused by the control input process influenced by environmental fluctuations [40]. In particular, \( \sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m} \) is called the noise intensity matrix, \( w(t) = (w_1(t), ..., w_m(t))^T \) is an \( m \)-dimensional Brownian motion defined on a complete probability space \( (\Omega, \mathcal{F}, P) \) with a natural filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). This type of stochastic perturbation is omnipresent because of the random uncertainties from environment during the process of signal transmission [41]. Throughout this paper, we make the following assumptions:

**Assumption 2.1.** \((H1)\) The noise intensity function \( \sigma(t, x, y) \) \((i = 1, 2, ..., N)\) satisfies the Lipschitz condition and there exists positive constants \( p, q \) such that

\[ \text{trace}(\sigma_i^T \sigma_i) \leq p x^T x + q y^T y. \]  

Moreover \( \sigma(t, 0, 0) \equiv 0 \).
Assumption 2.2. \((H2)\) There exists a positive constant \(M\) such that
\[
\|f(t, x(t), x(t - \tau(t))) - f(t, y(t), y(t - \tau(t)))\| \\
\leq M[\|x(t) - y(t)\|^2 + \|x(t - \tau(t)) - y(t - \tau(t))\|^2]^\frac{1}{2}.
\] (4)

Assumption 2.3. \((H3)\) \(\tau(t)\) is a differentiable function with
\[
0 \leq \tau(t) \leq \mu < 1.
\] (5)

Obviously, this assumption is ensured if the delay \(\tau(t)\) is constant.

Our purpose is to design proper controllers so that the noise-perturbed response layer (2) can reach CS with the drive layer (1). For this purpose, some necessary concepts and a lemma of stochastic differential equations are presented.

Consider the following \(n\)-dimensional stochastic differential delay equation:
\[
dz(t) = \phi(t, z(t), z(t - \tau))dt + \varphi(t, z(t), z(t - \tau))dw
\] (6)
on \(t \geq 0\) with an initial value \(\xi \in C_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)\), where \(C_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)\) represents the family of all \(\mathcal{F}_0\)-measurable bounded \(C([-\tau, 0], \mathbb{R}^n)\)-valued random variables, the measurable functions \(\phi, \varphi: [0, +\infty] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n\) satisfy the locally Lipschitz condition and the linear growth condition. It is known that equation (6) has a unique solution for any initial value \(\xi\) that is denoted by \(z(t, \xi)\) on \(t \geq -\tau\).

Let \(C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)\) denote the family of all non-negative functions \(V(t, z)\) on \(\mathbb{R}_+ \times \mathbb{R}^n\), which are continuously once differentiable in \(t\) and twice differentiable in \(z\). For each \(V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)\), the diffusion operator \(\mathcal{L} V\) associated with (6) acting on \(C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)\) is defined by
\[
\mathcal{L} V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial z} \cdot \phi + \frac{1}{2} \text{trace} \left[ \varphi^T \frac{\partial^2 V}{\partial^2 z} \cdot \varphi \right].
\] (7)

where \(\partial V/\partial z = (\partial V/\partial z_1, ..., \partial V/\partial z_n), \partial^2 V/\partial^2 z = (\partial^2 V/\partial z_i \partial z_j)_{n \times n} \).

Lemma 2.1. (A Lasalle-type invariance theorem for stochastic differential equations [42]). Assume that both \(\phi(t, u, v)\) and \(\varphi(t, u, v)\) are locally bounded in \((u, v)\) while uniformly bounded in \(t\). Assume also that there are functions \(V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+), \gamma \in L^1(\mathbb{R}_+, \mathbb{R}_+), \) and \(\omega_1, \omega_2 \in C(\mathbb{R}^n, \mathbb{R}_+)\) such that
\[
\mathcal{L} V(t, u, v) \leq \gamma(t) - \omega_1(u) + \omega_2(v), \quad \forall (t, u, v) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n,
\]
\[
\omega_1(u) \geq \omega_2(u), \quad \forall u \in \mathbb{R}^n
\]
and
\[
\lim_{\|u\| \to +\infty} \inf_{0 \leq t < \infty} V(t, u) = \infty.
\]

Then \(\text{Ker}(\omega_1 - \omega_2) = \emptyset\) and for every initial value \(\xi \in C_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)\), the solution \(z(t, \xi)\) of equation (6) has the following property:
\[
\lim_{t \to \infty} \text{dist}\{z(t; \xi), \text{Ker}(\omega_1 - \omega_2)\} = 0 \text{ a.s.}
\]

Moreover, if \(\text{Ker}(\omega_1 - \omega_2) = 0\), then for every \(\xi \in C_{\mathcal{F}_0}^\mu(\mathbb{R}^n)\), \(\lim_{t \to \infty} z(t; \xi) = 0 \text{ a.s.}\)

## 3. Sufficient conditions for CS of duplex networks

In this section, we will first give the definition of counterpart synchronization in duplex networks.

**Definition 3.1.** The duplex network formed by the drive layer (1) and the response layer (2) is said to almost surely achieve CS if

\[
e_i(t) = \lim_{t \to \infty} \mathbb{E}\|y(t) - x_i(t)\| = 0, \quad i = 1, 2, ..., N.
\]

With the network models and the definition given previously, we arrive at the following main theorem.

**Theorem 3.1.** Let (H1), (H2) and (H3) hold. The response layer (2) can almost surely achieve CS with the drive layer (1) with the following control scheme:

\[
u_i(t) = \sum_{j=1}^{N} b_{ij}(t) \Gamma y_j(t) - g_i(t)e_i(t), \quad i = 1, 2, ..., N,
\]

\[
g_i(t) = k_i \|e(t)\|^2, \quad b_{ij}(t) = -e_i(t) \Gamma y_j(t), \quad i = 1, 2, ..., N,
\]

where \(k_i > 0 \quad (i = 1, 2, ..., N)\) are arbitrary constants, and \(b_{ij}(t), g_i(t) \quad (i, j = 1, 2, ..., N)\) are adaptive parameters updating with network dynamics.

**Proof.** Since \(e_i(t) = y_i(t) - x_i(t)\), the dynamics of the synchronization error between counterparts in layers (1) and (2) can be written as follows:

\[
de_i(t) = \left[ f_i(t, y_i(t), y_i(t - \tau(t))) - f_i(t, x_i(t), x_i(t - \tau(t))) + \sum_{j=1}^{N} (d_{ij} \Gamma y_j(t) - c_{ij} \Gamma x_j(t)) \right] dt + \sigma_i(t, e_i(t), e_i(t - \tau(t))) \, dw(t), \quad i = 1, 2, ..., N.
\]

Consider the following Lyapunov functional:

\[
V = \sum_{i=1}^{N} e_i(t)^\top e_i(t) + \sum_{i=1}^{N} \sum_{j=1}^{N} (b_{ij}(t) + d_{ij} - c_{ij})^2 + \sum_{i=1}^{N} \frac{1}{k_i}(g_i(t) - \bar{g})^2 + \int_{t-\tau(t)}^{t} \frac{M}{1 - \mu} \sum_{i=1}^{N} e_i(\theta)^\top e_i(\theta) d\theta,
\]

where \(\bar{g}\) is a sufficiently large positive constant to be determined. Thus the diffusion operator \(\mathcal{L}\) defined in (7) onto the function \(V\) along with the error system (11) is: 

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\[ \mathcal{L} V = 2 \sum_{i=1}^{N} e_i^T(t) \left[ f_i(y(t), y(t - \tau(t))) - f_i(x(t), x(t - \tau(t))) + \sum_{j=1}^{N} (d_{ij} \Gamma y_j(t) - c_{ij} \Gamma x_j(t)) \right] \\
\quad + \sum_{j=1}^{N} b_{ij}(t) \Gamma y_j(t) - g_i(t)e_i(t) \right] - \sum_{i=1}^{N} \sum_{j=1}^{N} (b_{ij}(t) + d_{ij} - c_{ij}) e_i^T(t) \Gamma y_j(t) \\
\quad + 2 \sum_{i=1}^{N} (g_i(t) - \bar{g}) e_i^T(t)e_i(t) + \frac{M}{1 - \mu} \sum_{i=1}^{N} e_i^T(t)e_i(t) \\
\quad - \frac{M(1 - \bar{g})}{1 - \mu} \sum_{i=1}^{N} e_i^T(t)e_i(\tau(t))e_i(t - \tau(t)) + \sum_{i=1}^{N} \text{trace}(\sigma_i^T \sigma_i). \] (13)

With the well-known inequality \( 2\mathbf{x}^T \mathbf{y} \leq \mathbf{x}^T \mathbf{x} + \mathbf{y}^T \mathbf{y} \) and Assumption (H2), one obtains

\[ 2e_i^T(t)[f_i(t, y(t), y(t - \tau(t))) - f_i(x(t), x(t - \tau(t)))] \leq e_i^T(t)e_i(t) + M[e_i^T(t)e_i(t) + e_i(t - \tau(t))^T e_i(t - \tau(t))]. \]

Let \( \mathbf{e}(t) = (e_1^T(t), e_2^T(t), ..., e_N^T(t))^T \), then

\[ \mathcal{L} V \leq (1 + M) \sum_{i=1}^{N} e_i^T(t)e_i(t) + M \sum_{i=1}^{N} e_i^T(t - \tau(t))e_i(t - \tau(t)) + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} e_i^T \Gamma e_j(t) \\
\quad - 2 \sum_{i=1}^{N} g_i e_i^T(t)e_i(t) + \frac{M}{1 - \mu} \sum_{i=1}^{N} e_i^T(t)e_i(t) - \frac{M(1 - \bar{g})}{1 - \mu} \sum_{i=1}^{N} e_i^T(t - \tau(t))e_i(t - \tau(t)) \\
\quad + \sum_{i=1}^{N} \text{trace}(\sigma_i^T \sigma_i) \\
\quad = \left( 1 + M + \frac{M}{1 - \mu} \right) e_i^T(t)e_i(t) + Me_i^T(t - \tau(t))e_i(t - \tau(t)) + 2e_i^T(t)Pe_i(t) - 2g_i e_i^T(t)e_i(t) \]

\[ - \frac{M(1 - \bar{g})}{1 - \mu} e_i^T(t - \tau(t))e_i(t - \tau(t)) + pe_i^T(t)e_i(t) + qe_i^T(t - \tau(t))e_i(t - \tau(t)) \]

\[ \leq \left( 1 + M + \frac{M}{1 - \mu} + 2\lambda_{\max} \left( \frac{P^T + P}{2} \right) - 2\bar{g} \right) e_i^T(t)e_i(t) \]

\[ + \frac{M(\bar{\sigma}(t) - \mu)}{1 - \mu} e_i^T(t - \tau(t))e_i(t - \tau(t)) + pe_i^T(t)e_i(t) + qe_i^T(t - \tau(t))e_i(t - \tau(t)), \] (14)

where \( P = C \otimes \Gamma \).

From Assumption (H3), one has \( \frac{\bar{\sigma}(t) - \mu}{1 - \mu} \leq 0 \), which results in

\[ \mathcal{L} V \leq \left( 1 + 2\lambda_{\max} \left( \frac{P^T + P}{2} \right) + \frac{2M - \mu M}{1 - \mu} + p - 2\bar{g} \right) e_i^T(t)e_i(t) \]

\[ + qe_i^T(t - \tau(t))e_i(\tau(t)) \]

\[ \triangleq -\omega_1(e_i(t)) + \omega_2(e_i(t - \tau(t))). \] (15)

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Let
\[
\hat{g} > g^* \triangleq \frac{1}{2} \left[ 1 + 2\lambda_{\max} \left( \frac{P' + P}{2} \right) + \frac{2M - \mu M}{1 - \mu} + p + q \right],
\]
(16)
one obtains \( \omega_1(e) > \omega_2(e) \) for any \( e \neq 0 \). Moreover, \( \lim_{\|e\| \to +\infty} \inf_{0 \leq t \leq \infty} V = \infty \). From lemma 2.1, one obtains \( \lim_{t \to \infty} e(t; \xi) = 0 \) a.s. for any initial data \( \xi \in C_{\tau}^0([-\tau, 0], \mathbb{R}^n) \). This means that CS of the duplex network (1) and (2) can be almost surely achieved for almost every initial data. This completes the proof.

**Remark 3.1.** In the duplex, the drive layer (1) and the response layer (2) may have different topologies. In addition, the configuration matrices \( C \) and \( D \) are not necessarily symmetric or irreducible, which means that the intra-layer topologies can be undirected or directed, and they may also contain isolated nodes and disconnected clusters. Therefore, the control scheme can be applied to a wide range of duplex networks with unidirectional couplings.

**Remark 3.2.** It is obvious that when CS between the two layers (1) and (2) is almost surely realized, one has \( e_i(t) \to 0 \) as \( t \to \infty \) for \( i = 1, 2, ..., N \). Furthermore, it renders \( \hat{g}_i(t) \to 0 \) and \( \hat{b}_i(t) \to 0 \) for \( i, j = 1, 2, ..., N \). This means that \( g_i(t) \) and \( b_i(t) \) will almost surely become constant.

Based on theorem 3.1, one can easily derive the following corollaries:

**Corollary 3.1.** Assume that (H1) (H2) and (H3) hold. If the two layers have identical configuration matrices (\( C = D \)), then the drive layer (1) and response layer (2) can almost surely reach CS through the following simplified adaptive control:
\[
u_i(t) = -g_i(t)e_i(t), \quad \hat{g}_i(t) = k_i\|e_i(t)\|.
\]

**Corollary 3.2.** Assume that (H1) (H2) and (H3) hold. If there is no noise perturbation, the duplex network (1) and (2) can reach CS through the following adaptive control:
\[
u_i(t) = \sum_{j=1}^N b_{ij}(t)\Gamma y_j(t) - g_i(t)e_i(t), \quad \hat{g}_i(t) = k_i\|e_i(t)\|^2, \quad \hat{b}_i(t) = -e_i^\mathsf{T}\Gamma y_j(t).
\]

4. Numerical simulations

In this section, two examples are given to illustrate the feasibility and effectiveness of the proposed synchronization scheme.

**Example 4.1.** Consider a duplex network, each layer being composed of 5 nodes. The chaotic Lü system with various parameters is taken as node dynamics, with the \( i \)-th (\( i = 1, 2, ..., 5 \)) node in both layers being described by
\[
x_i = f_i(t, x_i(t), x_i(t - \tau(t))) = \begin{pmatrix}
(36 + i \cdot 0.1)(x_{i2}(t) - x_{i1}(t)) \\
-x_{i1}(t - \tau)x_{i3}(t - \tau) + 20x_{i2}(t) \\
x_{i1}(t - \tau)x_{i2}(t - \tau) - 3x_{i3}(t)
\end{pmatrix} + \begin{pmatrix}
0 \\
x_{i1} \\
x_{i2}
\end{pmatrix} + \begin{pmatrix}
-x_{i1}(t - \tau)x_{i3}(t - \tau) \\
x_{i1}(t - \tau)x_{i2}(t - \tau)
\end{pmatrix},
\]
(17)
\[ H x(t) + G(x(t - \tau)). \]  

Since the Lü system is chaotic, it is bounded in a certain region [43]. Thus there exists a positive constant \( R \) such that \( \| y_k \| \leq R \) and \( \| z_k \| \leq R \) for \( k = 1, 2, 3 \). Therefore, one has

\[
\| G(y) - G(z) \| = \sqrt{[z_3(y_1 - z_1) + y_1(y_3 - z_3)]^2 + [y_1(y_2 - z_2) + z_2(y_1 - z_1)]^2} \\
\leq \sqrt{2} R \| y - z \|. 
\]  

That is to say, Assumption (H2) is satisfied with \( M = \sqrt{2} R \) for \( i = 1, 2, ..., 5 \).

The configuration matrices \( \mathbf{C} \) and \( \mathbf{D} \) for the drive layer and response layer are given as

\[
\mathbf{C} = \begin{pmatrix}
-6 & 2 & 0 & 3 & 1 \\
3 & -4 & 1 & 0 & 0 \\
0 & 1 & -4 & 3 & 0 \\
3 & 0 & 3 & -7 & 1 \\
1 & 0 & 0 & 2 & -3
\end{pmatrix}
\quad \text{and} \quad
\mathbf{D} = \begin{pmatrix}
-3 & 0 & 2 & 0 & 1 \\
0 & -6 & 1 & 3 & 2 \\
3 & 1 & -4 & 0 & 0 \\
0 & 1 & 0 & -3 & 2 \\
1 & 2 & 0 & 1 & -4
\end{pmatrix},
\]  

respectively. The inner coupling matrix is taken as \( \Gamma = [110; 010; 001] \) and node delay is \( \tau = 0.003 \). Take \( \sigma(t, e, e(t - \tau)) = \sigma_0 \text{diag}(e_{11}(t) - e_{11}(t - \tau), e_{12}(t) - e_{12}(t - \tau), e_{13}(t) - e_{13}(t - \tau)) \), \( \sigma_0 = 1 \) for \( i = 1, 2, ..., 5 \), then \( \sigma(t, e, e(t - \tau)) \) satisfies the Lipschitz condition and the linear growth condition. That is, \( \text{trace}(\sigma_t^T \sigma_t) \leq 2\sigma_0^2 e_{11}(t) e_{11}(t) + 2\sigma_0^2 e_{12}(t - \tau) e_{12}(t - \tau) \). Meanwhile, assume that \( w(t) = [w_1(t), w_2(t), w_3(t)] \) is a three-dimensional Brownian motion. The initial values of the \( i \)-th nodes in the drive and response layers are set to be \( (x_{1i}(0), x_{2i}(0), x_{3i}(0)) = (1 + 0.3i, -0.6 + 0.3i, 0.3 + 0.3i) \), and \( (y_{1i}(0), y_{2i}(0), y_{3i}(0)) = (1 - \sin i, 1 - 0.3 \cos i, -0.3i), t \in [-\tau, 0] \) \( (i = 1, 2, ..., 5) \), respectively. The initial values of adaptive gains \( g_i(t) \) \( (i = 1, 2, ..., 5) \) and adaptive parameters \( b_{ij}(i, j = 1, 2, ..., 5) \) are chosen randomly in \( (0,1) \).

Figure 2 shows the CS error of the duplex network (1) and (2). The left panel shows \( e_i(t) \), while the right panel shows the total synchronization error \( \| e \| = \sqrt{\sum_{i=1}^5 \sum_{j=1}^3 (y_{ij}(t) - x_{ij}(t))^2} \). It is obvious that CS is almost surely achieved once the proposed control scheme is employed. Figure 3 further displays the adaptive feedback gains \( g_i(t) \) \( (i = 1, 2, ..., 5) \) and adaptive parameters \( b_{ij}(i, j = 1, 2, ..., 5) \) varying with time. It is seen that all the parameters reach constant values, which is consistent with remark 3.2.

**Example 4.2.** Synchronization in neuronal networks is one of the pressing challenges in neuroscience in recent years, and the Hindmarsh–Rose model [44] has become popular for analysis of neuronal activity and has also been extensively investigated. For example, Fang et al explored chaotic synchronization of nearest-neighbor diffusive coupling Hindmarsh–Rose neural networks in noisy environments [45], and Zhou et al discussed the model by using impulsive pinning control [46]. We will discuss the synchronization between two coupled Hindmarsh–Rose neuronal networks. The Hindmarsh–Rose model can be described by a three-dimensional nonlinear differential equations as follows [44]:

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\[ \dot{x}_i = f(t, x_i(t), x_i(t - \tau(t))) = \begin{pmatrix} x_{i2}(t) - x_{i3}(t) - x_{i1}(t - \tau)^3 + 3x_{i1}(t - \tau)^2 + I \\ 1 - x_{i2}(t) - 5x_{i3}(t - \tau)^2 \\ \mu(4x_{i1}(t) + \bar{x}) - x_{i3}(t) \end{pmatrix} \]

Take \( I = 3, \bar{x} = 1.56, \mu = 0.006, \sigma_0 = 1, \tau = 0.1 \). Assumption (H2) is satisfied [40]. The inner coupling matrix \( \Gamma = [1 \ 0; 0 \ 1; 0 \ 0 \ 1] \). The intra-layer topologies, the noise term and initial states of nodes are taken as the same as those in the previous example. Figure 4 shows CS errors between two unidirectionally connected Hindmarsh–Rose networks. Figure 5 further presents the updated feedback gains \( g_i(t) (i = 1, 2, ..., 5) \) and adaptive parameters \( b_{ij}(t) (i, j = 1, 2, ..., 5) \). It is clearly seen that the numerical simulations perfectly match the theoretical results.

Figure 2. The CS error between the drive layer (1) and the response layer (2) formed by Lü oscillators. Left: \( e_{ij}(t) \) varying with time \( t \); right: the total synchronization error.

Figure 3. The adaptive feedback gains \( g_i(t) (i = 1, 2, ..., 5) \) (left) and parameters \( b_{ij}(t) (i, j = 1, 2, ..., 5) \) updating according to (10) (right).
5. Conclusions

In this paper, counterpart synchronization (CS) of duplex networks with delayed nodes and noise perturbation has been investigated. Based on the LaSalle-type invariance principle for stochastic differential equations, a sufficient condition guaranteeing CS with the proposed control scheme has been provided. Numerical examples have also been presented to illustrate the effectiveness of the method. The proposed method will find applicability in a wide range of practical duplex networks.

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