Symmetries of Gromov-Witten Invariants

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Abstract

The group $(\mathbb{Z}/n\mathbb{Z})^2$ is shown to act on the Gromov-Witten invariants of the complex flag manifold. We also deduce several corollaries of this result.

1 Introduction

The aim of this paper is to present certain symmetry properties of the Gromov-Witten invariants for type $A$ complex flag manifolds.

Recall that the cohomology ring of the complex flag manifold $F_l n$ has an additive basis of Schubert classes $\sigma_w$, which are indexed by permutations $w$ in the symmetric group $S_n$. For permutations $u, v, w \in S_n$, the Schubert number $c_{u,v,w}$ is the structure constant of the cohomology ring in the basis of Schubert classes:

$$\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{u,v,w} \sigma_{w o w},$$

where $w_0$ is the longest permutation in $S_n$. Equivalently,

$$c_{u,v,w} = \int \sigma_u \cdot \sigma_v \cdot \sigma_w$$

is the intersection number of Schubert varieties. Thus these numbers are nonnegative integers symmetric in $u, v,$ and $w$. They generalize the famous Littlewood-Richardson coefficients. If $\ell(u) + \ell(v) + \ell(w) \neq \frac{n(n-1)}{2}$ then the Schubert number $c_{u,v,w}$ is zero for an obvious degree reason.

A long standing open problem is to find an algebraic or combinatorial construction for the coefficients $c_{u,v,w}$ that would imply their nonnegativity. A possible approach to this problem could be in its generalization to the quantum cohomology ring of the flag manifold $F_l n$. The structure constants of this ring are certain polynomials whose coefficients are the Gromov-Witten invariants $\langle \sigma_u, \sigma_v, \sigma_w \rangle_{(d_1,\ldots,d_n-1)}$. The Schubert number $c_{u,v,w}$ is a special case of the Gromov-Witten invariants: $c_{u,v,w} = \langle \sigma_u, \sigma_v, \sigma_w \rangle_{(0,\ldots,0)}$. These invariants are
defined as numbers of certain rational curves in $Fl_n$. The geometric definition of the Gromov-Witten invariants implies their nonnegativity.

In this paper we establish cyclic symmetries of the Gromov-Witten invariants that could not be detected in their full generality on the “classical” level of the Schubert numbers $c_{u,v,w}$. Several related results for the $c_{u,v,w}$ when $u$ is a Grassmannian permutation were, however, found by Bergeron and Sottile, see [2, Theorems 1.3.4, 1.3.4]. In case of the Gromov-Witten invariants we do not need to restrict the rule to Grassmannian permutations. Similar symmetries of the Gromov-Witten invariants for Grassmannian varieties were found in [1].

2 Gromov-Witten invariants

Let $Fl_n$ denote the manifold of complete flags of subspaces in the complex $n$-dimensional linear space $\mathbb{C}^n$. One can also define the flag manifold as $Fl_n = GL_n(\mathbb{C})/B$, where $B$ is the Borel subgroup of upper triangular matrices in the general linear group. The flag manifold is a compact smooth complex manifold. For a permutation $w \in S_n$, the Schubert variety $X_w$ is the closure of the Schubert cell $B_-wB/B$ in $Fl_n$, where $B_-$ is the subgroup of lower triangular matrices and $w$ is viewed as a permutation matrix in $GL_n$. The Schubert classes $\sigma_w \in H^*(Fl_n, \mathbb{Z})$, indexed by permutations $w \in S_n$, are defined as the Poincaré duals of the homology classes $[X_w]$ of Schubert manifolds. They form an additive $\mathbb{Z}$-basis of the cohomology ring $H^*(Fl_n, \mathbb{Z})$. Moreover, $\sigma_w \in H^{2l}(Fl_n, \mathbb{Z})$, where $l = \ell(w)$ is the length of permutation $w$, i.e., its number of inversions.

Recently, attention has been drawn to the (small) quantum cohomology ring $QH^*(Fl_n, \mathbb{Z})$ of the flag manifold. The definition of quantum cohomology can be found, for example, in [5]. Here we briefly outline several notions and results.

As a vector space, the quantum cohomology of $Fl_n$ is the usual cohomology tensored with the polynomial ring in $n-1$ variables:

$$QH^*(Fl_n, \mathbb{Z}) \cong H^*(Fl_n, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \ldots, q_{n-1}].$$ (1)

The Schubert classes $\sigma_w$, thus, form a $\mathbb{Z}[q_1, \ldots, q_{n-1}]$-basis of the quantum cohomology ring.

The multiplication in $QH^*(Fl_n, \mathbb{Z})$ (quantum product) is a commutative $\mathbb{Z}[q_1, \ldots, q_{n-1}]$-linear operation. It is therefore sufficient to specify the quantum product of any two Schubert classes. To avoid confusion with the multiplication in the usual cohomology ring, we will use “*” to denote the quantum product. The quantum product $\sigma_u \ast \sigma_v$ of two Schubert classes can be expressed in the basis of the Schubert classes as

$$\sigma_u \ast \sigma_v = \sum_{w \in S_n} C_{u,v,w} \sigma_{w_0},$$ (2)

where $C_{u,v,w} \in \mathbb{Z}[q_1, \ldots, q_{n-1}]$ and $w_0 = \left( \frac{1}{n} \ 2 \ \ldots \ n \right)$ is the longest permutation in $S_n$. 

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The coefficient of $q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ in the polynomial $C_{u,v,w}$ is the Gromov-Witten invariant $\langle \sigma_u, \sigma_v, \sigma_w \rangle_{(d_1, \ldots, d_{n-1})}$. The Gromov-Witten invariants are defined geometrically as numbers of certain rational curves in $\mathbb{F} l_n$. (See [5] or [3] for details.) Let us summarize the main properties of these invariants. It will be more convenient for us to work with the polynomials $C_{u,v,w}$.

1. **(Nonnegativity)** All coefficients of the $C_{u,v,w}$ are nonnegative integers.

2. **($S_3$-symmetry)** The polynomials $C_{u,v,w}$ are invariant with respect to permuting $u$, $v$, and $w$.

3. **(Degree condition)** The polynomial $C_{u,v,w}$ is a homogeneous polynomial of degree $\frac{1}{2}(\ell(u) + \ell(v) + \ell(w) - \frac{n(n-1)}{2})$.

4. **(Classical limit)** The Schubert number $c_{u,v,w}$ is the constant term of the polynomial $C_{u,v,w}$.

5. **(Associativity)** The operation \( \ast \) defined by (2) via the polynomials $C_{u,v,w}$ is associative.

The first four properties are clear from geometric definitions. It was conjectured in [3] that nonnegativity, associativity, degree condition, and classical limit condition uniquely determine the Gromov-Witten invariants.

The conditions 3 and 4 immediately imply the following statement.

**Proposition 1** We have

\[
C_{u,v,w} = \begin{cases} 
0 & \text{if } \ell(u) + \ell(v) + \ell(w) < \frac{n(n-1)}{2}, \\
0 & \text{if } \ell(u) + \ell(v) + \ell(w) - \frac{n(n-1)}{2} \text{ is odd}, \\
c_{u,v,w} & \text{if } \ell(u) + \ell(v) + \ell(w) = \frac{n(n-1)}{2}, \\
?? & \text{otherwise.}
\end{cases}
\]

In [3] we gave a method for calculation of the Gromov-Witten invariants. Among several approaches presented in that paper, one is based on the quantum analogue of Monk’s formula.

For $1 \leq i < j \leq n$, let $s_{ij}$ be the transposition in $S_n$ that permutes $i$ and $j$. Then $s_i = s_{i,i+1}$ is an adjacent transposition. Also, let $q_{ij}$ be a shorthand for the product $q_i q_{i+1} \cdots q_{j-1}$.

**Proposition 2** [3, Theorem 1.3] (quantum Monk’s formula) For $w \in S_n$ and $1 \leq k < n$, the quantum product of the Schubert classes $\sigma_{s_k}$ and $\sigma_w$ is given by

\[
\sigma_{s_k} \ast \sigma_w = \sum \sigma_{w_{s_{ab}}} + \sum q_{cd} \sigma_{w_{s_{cd}}},
\]

where the first sum is over all transpositions $s_{ab}$ such that $a \leq k < b$ and $\ell(w_{s_{ab}}) = \ell(w) + 1$, and the second sum is over all transpositions $s_{cd}$ such that $c \leq k < d$ and $\ell(w_{s_{cd}}) = \ell(w) - 2(d-c) + 1$.
Remark 3 The two-dimensional Schubert classes $\sigma_{sk}$ generate the quantum cohomology ring. Thus formula (3) uniquely determines the multiplicative structure of $\text{QH}^*(Fl_n, \mathbb{Z})$ and, therefore, the Gromov-Witten invariants.

3 Cyclic symmetry

Let $o = (1, 2, \ldots, n)$ be the cyclic permutation in $S_n$ given by

$$o(i) = i + 1, \text{ for } i = 1, \ldots, n - 1, \quad o(n) = 1.$$ 

Recall that $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$ for $i < j$. We also define $q_{ij} = q_{ji}^{-1}$ for $i > j$ and $q_{ii} = 1$.

Theorem 4 For any $u, v, w \in S_n$ we have

$$C_{u,v,w} = q_{ij} C_{u,o^{-1}v,ow}, \quad (4)$$

where $i = v^{-1}(1)$ and $j = w^{-1}(n)$.

The $S_3$-invariance of the $C_{u,v,w}$ under permuting $u$, $v$, and $w$ implies a more general statement.

For $w \in S_n$ and $1 \leq a \leq n$, define the following Laurent monomials in the $q_i$

$$Q_{w,a} = \prod_{i : w(i) \geq n-a+1} q_{ii}, \quad Q_{w,-a} = \prod_{j : w(j) \leq a} (q_{1j})^{-1},$$

and let $Q_{w,0} = 1$.

Corollary 5 For any $u, v, w \in S_n$ and $-n \leq a, b, c \leq n$ such that $a + b + c = 0$, we have

$$C_{u,v,w} = Q_{u,a} Q_{v,b} Q_{w,c} C_{o^a u, o^b v, o^c w}. \quad (5)$$

In many cases Corollary 5 and Proposition 1 allow us to reduce the polynomials $C_{u,v,w}$ to the Schubert numbers $c_{u,v,w}$.

Corollary 6 For $u, v, w \in S_n$, let us find a triple $-n \leq a, b, c \leq n$, $a + b + c = 0$, for which the expression

$$\ell_{a,b,c} = \ell(o^a u) + \ell(o^b v) + \ell(o^c w)$$

is as small as possible. If $\ell_{a,b,c} < \frac{n(n-1)}{2}$ then $C_{u,v,w} = 0$. If $\ell_{a,b,c} = \frac{n(n-1)}{2}$ then $C_{u,v,w} = Q_{u,a} Q_{v,b} Q_{w,c} c_{o^a u, o^b v, o^c w}$.

Remark 7 (Reduction of Gromov-Witten invariants) The Gromov-Witten invariants have the following stability property. If $u, v, w \in S_n$ are three permutations such that $u(n) = v(n) = n$ and $w(n) = 1$ then $C_{u,v,w} = C_{u',v',w'}$, where
u′, v′, w′ ∈ Sn−1 are permutations obtained from u, v, w by removing the last entry (and subtracting 1 from all entries of w).

For a triple of permutation u, v, w ∈ Sn such that u(n) + v(n) + w(n) ≡ 1 (mod n), we can use the relation (5) to transform the triple to the above case when we can use the stability property. This shows that 1/n of all Gromov-Witten invariants for Fln can be reduced to the Gromov-Witten invariants of Fln−1. Analogously, we can reduce the problem to a lower level for a triple of permutations u, v, w ∈ Sn such that u(1) + v(1) + w(1) ≡ 2 (mod n).

Remark 8 (New rules for multiplication of Schubert classes) Suppose that a rule is known for the quantum multiplication of an arbitrary Schubert class by certain Schubert class σu. Theorem 4 immediately produces a new rule for the quantum multiplication by σωu, where a ∈ Z. For example, we get for free a rule for σω*a σw. Quantum Monk’s formula (3) can be extended to a rule for σω*sk * σw. More generally, quantum Pieri’s formula [6, Corollary 4.3] extends to an explicit rule for σω*u * σw, where u is a permutation of the form u = sksk+1 ··· sk+l or u = sksk−1 ··· sk−l.

4 Twisted cyclic shift

Let Tij, 1 ≤ i < j ≤ n, be the Z[q1, . . . , qn−1]-linear operators that act on the quantum cohomology ring QH∗(Fln, Z) by

\[ T_{ij} : σ_w \mapsto \begin{cases} 
σ_{ws_{ij}} & \text{if } ℓ(ws_{ij}) = ℓ(w) + 1, \\
q_j σ_{ws_{ij}} & \text{if } ℓ(ws_{ij}) = ℓ(w) - 2(j - i) + 1, \\
0 & \text{otherwise.}
\end{cases} \] (6)

Then quantum Monk’s formula (3) can be written as:

\[ σ_{sk} * σ_w = \sum_{i ≤ k < j} T_{ij}(σ_w). \] (7)

The operators Tij satisfy certain simple quadratic relations. The formal algebra defined by these relations was studied in [4] and [6].

Let us also define the twisted cyclic shift operator O that acts on the quantum cohomology ring QH∗(Fln, Z), linearly over Z[q1, . . . , qn−1], by

\[ O : σ_w \mapsto q^{(w)} σ_{ωw}, \]

where q(w) = qrn with r = w−1(n).

Proposition 9 For any 1 ≤ i < j ≤ n, the operators Tij and O commute:

\[ T_{ij} O = OT_{ij}. \]

The following lemma clarifies the conditions in the right-hand side of (6). Its proof is a straightforward observation.
Lemma 10 Let \( w \in S_n \) and \( 1 \leq i < j \leq n \). Then

1. \( \ell(ws_{ij}) = \ell(w) + 1 \) if and only if for all \( i \leq k \leq j \) we have
   \[
   w(k) \geq w(j) \geq w(i) \quad \text{or} \quad w(j) \geq w(i) \geq w(k);
   \]

2. \( \ell(ws_{ij}) = \ell(w) - \ell(s_{ij}) = \ell(w) - 2(j - i) + 1 \) if and only if for all \( i \leq k \leq j \)
   we have
   \[
   w(i) \geq w(k) \geq w(j).
   \]

Proof of Proposition 9 — The crucial observation is that, for fixed \( i \leq k \leq j \),
the set of permutations \( w \) such that

\[
 w(k) \geq w(j) \geq w(i) \quad \text{or} \quad w(j) \geq w(i) \geq w(k) \quad \text{or} \quad w(i) \geq w(k) \geq w(j)
\]
is invariant under the left multiplications of \( w \) by the cycle \( o \). This fact, together
with Lemma 10, implies that \((T_{ij}O)(\sigma_w)\) is nonzero if and only if \( T_{ij}(\sigma_w) \) is nonzero.
Assume that \( T_{ij}(\sigma_w) \neq 0 \) and consider three cases:

I. Neither \( w(i) \) nor \( w(j) \) is equal to \( n \). Then either of the conditions in
the right-hand side of (3) is satisfied for \( w \) if and only if the same condition is
satisfied for \( ow \). Also \( q(w) = q(ws_{ij}) \). Thus \((T_{ij}O)(\sigma_w) = (OT_{ij})(\sigma_w)\).

II. We have \( w(j) = n \). Then \( w(i) < w(j) \) and \( ow(i) > ow(j) \). Thus \( \ell(ws_{ij}) = \ell(w) + 1 \) and \( \ell(ows_{ij}) = \ell(ow) - 1 \). Thus \( T_{ij}(\sigma_w) = \sigma_{ws_{ij}} \) and \( T_{ij}(\sigma_{ow}) = q_{ij}\sigma_{ows_{ij}} \).
Also we have \( q(w) = q_{jn} \) and \( q(ws_{ij}) = q_{in} \). Therefore, \((T_{ij}O)(\sigma_w) = q_{ij}q_{jn}\sigma_{ows_{ij}} = q_{in}\sigma_{ows_{ij}} = (OT_{ij})(\sigma_w)\).

III. We have \( w(i) = n \). Then \( w(i) > w(j) \) and \( ow(i) < ow(j) \). Thus \( \ell(ws_{ij}) = \ell(w) - \ell(s_{ij}) \) and \( \ell(ows_{ij}) = \ell(ow) + 1 \). Thus \( T_{ij}(\sigma_w) = q_{ij}\sigma_{ws_{ij}} \) and \( T_{ij}(\sigma_{ow}) = \sigma_{ows_{ij}} \). Also we have \( q(w) = q_{jn} \) and \( q(ws_{ij}) = q_{jn} \). Therefore, \((T_{ij}O)(\sigma_w) = q_{jn}\sigma_{ows_{ij}} = q_{ij}q_{jn}\sigma_{ows_{ij}} = (OT_{ij})(\sigma_w)\).

\( \Box \)

Corollary 11 For any \( w \in S_n \), the operator of quantum multiplication by the
Schubert class \( \sigma_w \) commutes with the operator \( O \).

Proof — Proposition 3 and quantum Monk’s formula (6) imply that the operator
of quantum multiplication by a two-dimensional Schubert class \( \sigma_w \) commutes
with the twisted cyclic shift operator \( O \). By Remark 3, for any \( w \in S_n \),
the operator of quantum multiplication by \( \sigma_w \) commutes with \( O \).

This also proves Theorem 4, because it is equivalent to Corollary 6.

\( \Box \)

5 Transition graph

The Bruhat order \( Br_n \) is the partial order on the set of all permutations in \( S_n \)
given by the following covering relation: \( u \rightarrow w \) if \( w = u s_{ab} \) and \( \ell(w) = \ell(u) + 1 \).
In other words, \( u \rightarrow w \) if \( \sigma_w \) appear in the expansion of \( \sigma_{sk} \cdot \sigma_u \) for some
\( 1 \leq k < n \) (the product in the usual cohomology ring).
The analogue of the Bruhat order for the quantum cohomology ring is the following transition graph. The transition graph $T_n$ is the directed graph on the set of permutations in $S_n$. Two permutations are connected by an edge $u \to w$ in $T_n$ if $w = u s_{ab}$ and either $\ell(w) = \ell(u) + 1$ or $\ell(w) = \ell(u) - \ell(s_{ab})$. We will label the edge $u \to u s_{ab}$ by the pair $(a, b)$. Equivalently, two permutations are connected by the edge $u \to w$ in $T_n$ whenever $\sigma_w$ appear in the expansion of the quantum product $\sigma_k * \sigma_u$ for some $1 \leq k < n$.

Proposition 9 implies the cyclic symmetry of the transition graph:

**Corollary 12** The transition graph $T_n$ is invariant under the cyclic shift: $w \mapsto o w$, for $w \in S_n$.

![Figure 1: Bruhat order $Br_3$.](image)

Figures 1 and 2 show the Bruhat order $Br_3$ and the transition graph $T_3$. The transition graph $T_3$ is obtained by adding several new edges to $Br_3$, which makes the picture symmetric with respect to the cyclic group $\mathbb{Z}/3\mathbb{Z}$. The generator $o$ of the cyclic group rotates the graph $T_3$ by $180^\circ$ clockwise.
Figure 2: Transition graph $Tr_3$.

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