Linear mappings as self-similarities of mathematical models of quasicrystals

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Abstract. In this paper an overview of the so-called cut-and-project method is presented together with new results on the construction of a cut-and-project set with a given linear diagonalizable self-similarity $A$. Such a construction is illustrated on a two-dimensional mathematical quasicrystal related to the Pisot number $1 + 2 \cos \frac{2\pi}{7}$. This model is described in detail and the associated Voronoi tiling is discussed. Moreover, it is shown that there exists a connection between the planar quasicrystal and higher-dimensional quasicrystal with 7-fold symmetry.

1. Introduction
Quasicrystals were discovered by Shechtman in 1982 as a new phase of matter, whose diffraction pattern was discrete but revealed a non-crystallographic symmetry [1]. Quasicrystalline materials are thus ordered but not in a periodic way as it is the case for classical crystals. The first observed quasicrystalline alloys had symmetry of order 10, later also octagonal and dodecagonal quasicrystals have been found. Such symmetries are intrinsically connected to quadratic irrational numbers. The coordinates of atomic positions belong to the quadratic extension fields $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{3})$, respectively. Recently, a colloid with symmetry of order 18 has been discovered [2], whose symmetry is given by a cubic irrational [3]. A discussion about possibility of existence of a 7-fold material can be found in [4].

A mathematical model of quasicrystals is an aperiodic discrete set of points in the space. A survey on recent developments in mathematical modeling of quasicrystals can be found in [5, 6]. One of the suitable models is obtained as a projection from a regular lattice to a subspace of lower dimension (the so-called physical space). Actually only a cutted part of the lattice is projected, whence the name of such construction: the cut and project method. The choice of the cut is directed by a projection to the complement of the physical space – the inner space. Specific choices of the lattice lead to models with different properties. In particular, one can obtain models with the desired rotational symmetries, see e.g. [7].

In our contribution, we are interested in models which are closed under chosen linear mappings. When such mappings are rotations or scalings, one speaks about symmetries or self-similarities. Such models have been studied both for particular cases and in more generality, see e.g. [8, 9, 10, 11, 12, 13]. We extend the notion of self-similarity to any linear mapping. In this contribution we provide an overview of our recent results on such generalized self-similarities of cut-and-project sets [14]. Moreover, we focus on the example of a planar cut-and-project set.
defined using the cubic Pisot-cyclotomic number $\beta$, root of the polynomial $x^3 - 2x^2 - x + 1$, which is connected to the 7-fold symmetry. A one-dimensional model based on this irrational number is studied in [15]. The example was chosen in the pursuit of the most simple case of a construction which – though not being closed under any non-trivial rotation or scaling – nevertheless reveals multitude of linear mappings as generalized self-similarities. These are described in Propositions 3.1 and 3.2.

For the study of physical properties of quasicrystalline materials, e.g. electron conductivity or inter-atomic interactions, the knowledge of local configurations of particles in the material is essential. In a mathematical model of quasicrystals, one thus needs to define neighbors of a point in the point set. In one dimension the notion of neighbors is clear. In two dimensions and higher, a natural definition of neighbors uses Voronoi tiles of the modelling point set. The Voronoi tiling of the studied example is discussed at the end of our contribution.

2. Cut-and-project sets and self-similarities

A cut-and-project set is defined using a cut-and-project scheme and a bounded set $\Omega$. There are various ways of introducing a cut-and-project scheme, all of them in essence equivalent. For our purposes it is convenient to adopt the following formalism.

Let $L$ be an $s$-dimensional lattice. For an $s$-dimensional vector $x$, define projection $\pi_\parallel : \mathbb{R}^s \to \mathbb{R}^n$ by taking first $n$ coordinates of $x$ in a fixed basis, and the projection $\pi_\perp : \mathbb{R}^s \to \mathbb{R}^{s-n}$ by taking the $s-n$ remaining coordinates. A cut-and-project scheme is the lattice $L$ together with the projections $\pi_\parallel, \pi_\perp$. We denote the scheme by the pair $(L \subset \mathbb{R}^s, n)$.

Note that the projection of the lattice $L$ are $\mathbb{Z}$-modules $\pi_\parallel(L), \pi_\perp(L)$ of rank at most $s$. The cut-and-project scheme is said to be non-degenerate, if $\pi_\parallel$ restricted to $L$ is an injective map. Similarly, if $\pi_\perp$ restricted to $L$ is injective, we say that the scheme is aperiodic. If $\pi_\perp(L)$ is a set dense in $\mathbb{R}^{s-n}$, the scheme is called irreducible.

**Definition 2.1.** Let $(L \subset \mathbb{R}^s, n)$ be a non-degenerate, irreducible and aperiodic cut-and-project scheme. Let further $\Omega \subset \mathbb{R}^{s-n}$ be a bounded set with non-empty interior. A cut-and-project set $\Sigma(\Omega)$ is defined as

$$\Sigma(\Omega) = \{ \pi_\parallel(x) : x \in L, \pi_\perp(x) \in \Omega \}.$$

The conditions imposed on the cut-and-project scheme together with the requirement on $\Omega$ ensures that the set $\Sigma(\Omega)$ satisfies the Delone property and is aperiodic [16]. These two features are natural to ask for when modelling atomic structures that should represent quasicrystalline materials. A set $\Sigma \subset \mathbb{R}^n$ is Delone if there exist $0 < r, R < +\infty$ such that every ball in $\mathbb{R}^n$ of radius $r$ contains at most one point of $\Sigma$ and every ball in $\mathbb{R}^n$ of radius $R$ contains at least one point of $\Sigma$. The set $\Sigma(\Omega)$ is aperiodic if it has no translational symmetry, i.e., $y + \Sigma \subset \Sigma$ implies $y = 0$.

Unlike periodic structures, the group of geometric symmetries of quasicrystalline models is not rich; in fact, often trivial. This motivates the study of a wider range of mappings under which such a model is closed, usually organised into a semigroup structure. Most often one considers inflations, rotations, or their combination. We focus more generally on linear mappings $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$. Such a mapping $A$ is called a (linear) self-similarity of $\Sigma(\Omega)$.

It can be derived that if $A$ is a self-similarity of $\Sigma(\Omega)$, then it must be a self-similarity of the underlying $\mathbb{Z}$-module $\pi_\parallel(L)$. Consequently, the action of the mapping $A$ in the physical space naturally induces an action of a mapping on the lattice $L \subset \mathbb{R}^s$. In that case we speak about the self-similarity of the cut-and-project scheme. In [14] we provide a necessary and sufficient condition for a linear mapping given by a matrix $A$ diagonalizable over $\mathbb{C}$ to be a self-similarity of a cut-and-project scheme. The condition is formulated as the following statement.
Theorem 2.2. Let $A \in \mathbb{R}^{n \times n}$ be a matrix diagonalizable over $\mathbb{C}$. Then there exists a non-degenerated irreducible aperiodic cut-and-project scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, n)$ such that $A\pi_n(\mathcal{L}) \subset \pi_n(\mathcal{L})$ if and only if the spectrum of $A$ is composed of algebraic integers.

In the same paper [14], we also determine the minimal dimension of the lattice $\mathcal{L}$, so that the $\mathbb{Z}$-module $\pi_n(\mathcal{L})$ has the self-similarity $A$. The condition is rather technical, therefore we limit our attention to matrices whose eigenvalues are mutually conjugate algebraic integers.

Theorem 2.3. Let $A \in \mathbb{R}^{n \times n}$ be a matrix diagonalizable over $\mathbb{C}$, and let its spectrum be composed of algebraic integers. Suppose that all $\lambda \in \sigma(A)$ have the same minimal polynomial $f$ of degree $d$. Denote $l_1, \ldots, l_d$ the multiplicities of the roots $\beta_1, \ldots, \beta_d$ of $f$ in the spectrum $\sigma(A)$, and set $M := \max\{l_j : j = 1, \ldots, d\}$, $m := \min\{l_j : j = 1, \ldots, d\}$. Let $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, n)$ be a reasonable cut-and-project scheme with the dimension as small as possible. Then

$$s = \begin{cases} Md & \text{if } m < M, \\ (M + 1)d, & \text{otherwise}. \end{cases}$$

It can be shown [14] that the general case can be obtained just by direct sum of matrices considered in the above theorem. For, if $A = \oplus_{i=1}^t A_i \in \mathbb{R}^{n \times n}$ is a direct sum of matrices diagonalizable over $\mathbb{C}$ with spectrum composed of algebraic integers, then the minimal dimension of the cut-and-project scheme is the sum of the individual dimensions for matrices $A_i$, i.e.,

$$s = \sum_{i=1}^t s_{A_i}.$$  

The above description of minimal dimension of the lattice for construction of a cut-and-project scheme with given self-similarity can be regarded as a generalization of the result describing the minimal dimension only for rotations, see [5].

3. Planar quasicrystal model based on a cubic Pisot-cyclotomic number

In this section we describe a specific model of a planar cut-and-project set based on the cubic Pisot-cyclotomic number $\beta = 1 + 2 \cos \frac{2\pi}{7}$. The construction follows the lines presented in general in [14] and performed in [17] for the case of decagonal planar model. Consider the polynomial $p(x) = x^3 - 2x^2 - x + 1$ and denote by $\beta, \beta', \beta''$ its roots,

$$\beta = 1 + 2 \cos \frac{2\pi}{7} \approx 2.2470, \quad \beta' = 1 + 2 \cos \frac{4\pi}{7} \approx 0.5550, \quad \beta'' = 1 + 2 \cos \frac{6\pi}{7} \approx -0.8019. \quad (1)$$

Note that $\beta$ is a Pisot-cyclotomic number of order 7, intrinsically connected to 7-fold symmetry, as will be understood later. It is useful to realize that $\beta$ is a totally real cubic number and the extension field $\mathbb{Q}(\beta)$ is a Galois field. We have

$$\beta' = \beta^2 - 2\beta, \quad \beta'' = -\beta^2 + \beta + 2. \quad (2)$$

Let us construct C&P scheme related to $p(x)$ using its companion matrix $C_{p(x)}$, with the use of the Vandermonde matrix $Y_{3D}$ composed of eigenvectors of $C_{p(x)}$, namely

$$C_{p(x)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \quad Y_{3D} = \begin{pmatrix} 1 & 1 & 1 \\ \beta & \beta' & \beta'' \\ \beta^2 & \beta'^2 & \beta''^2 \end{pmatrix}. \quad (3)$$

For the generator vectors of the lattice we take the columns of the matrix inverse of $Y_{3D}$, in particular,

$$Y_{3D}^{-1} = \frac{1}{\det Y_{3D}} \begin{pmatrix} \beta' \beta''(\beta'' - \beta') & \beta^2 - \beta'^2 & \beta'' - \beta' \\ \beta \beta''(\beta - \beta') & \beta''^2 - \beta^2 & \beta' - \beta'' \\ \beta\beta'(\beta' - \beta) & \beta^2 - \beta'^2 & \beta' - \beta' \end{pmatrix}. \quad (4)$$
Rewritten using (2), knowing that $\det Y_{3D} = 7$, we obtain

$$L_{3D} := 7Y_{3D}^{-1} = \begin{pmatrix} 2\beta^2 - 2\beta - 5 & 3\beta^2 - 6\beta - 2 & -2\beta^2 + 3\beta + 2 \\ -2\beta - 1 & -3\beta^2 + 3\beta + 4 & \beta^2 - 2 \\ -2\beta^2 + 4\beta - 1 & 3\beta - 2 & \beta^2 - 3\beta \end{pmatrix}. \quad (5)$$

The lattice $L$ is taken to be of the form

$$L_{3D} = \mathbb{Z}\begin{pmatrix} 2\beta^2 - 2\beta - 5 \\ -2\beta - 1 \\ -2\beta^2 + 4\beta - 1 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 3\beta^2 - 6\beta - 2 \\ -3\beta^2 + 3\beta + 4 \\ 3\beta - 2 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} -2\beta^2 + 3\beta + 2 \\ \beta^2 - 2 \\ \beta^2 - 3\beta \end{pmatrix}.$$

Let us make the projection $\mathbb{R}^2 \xleftarrow{\pi_1} L_{3D} \xrightarrow{\pi_1} \mathbb{R}$. We have

$$\pi_\parallel(L_{3D}) = \mathbb{Z}\begin{pmatrix} 2\beta^2 - 2\beta - 5 \\ -2\beta - 1 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 3\beta^2 - 6\beta - 2 \\ -3\beta^2 + 3\beta + 4 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} -2\beta^2 + 3\beta + 2 \\ \beta^2 - 2 \end{pmatrix},$$

$$\pi_\perp(L_{3D}) = \mathbb{Z}\begin{pmatrix} -2\beta^2 + 4\beta - 1 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 3\beta - 2 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} \beta^2 - 3\beta \end{pmatrix}.$$

It follows from the general results in [14] that such a construction provides a cut-and-project scheme ($L_{3D} \subset \mathbb{R}^3$, 2) which is non-degenerate, aperiodic and irreducible.

By methods described in [17], we can derive which linear mappings preserve the $\mathbb{Z}$-module $\pi_\parallel(L_{3D})$ of the above defined cut-and-project scheme. It turns out that our model is not closed under any rotation $A = O \in SO(2)$ nor any pure scaling $A = \lambda I$ for $\lambda \in \mathbb{R}$, as it is seen in the following statement.

**Proposition 3.1.** Let $(L_{3D} \subset \mathbb{R}^3, 2)$ be defined as above. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$. Then $A \pi_\parallel(L_{3D}) \subset \pi_\parallel(L_{3D})$ if and only if $A$ is of the form

$$A = \begin{pmatrix} a + b\beta + c\beta^2 & 0 \\ 0 & a + b\beta' + c\beta'^2 \end{pmatrix}, \quad a, b, c \in \mathbb{Z}. \quad (6)$$

Sofar, we have only determined the self-similarities of the $\mathbb{Z}$-module $\pi_\parallel(L_{3D})$. In order to chose among all these linear mappings those that are self-similarities of a cut-and-project set, we have to proceed as follows. Any self-similarity of the $\mathbb{Z}$-module $\pi_\parallel(L)$ in the physical space naturally induces a self-similarity $C$ of the lattice $L$ in the higher-dimensional space and a self-similarity $B$ of the $\mathbb{Z}$-module $\pi_\perp(L)$ in the inner space. For explanation of this connection see for example [17]. As a cut-and-project set is defined using a bounded set $\Omega$ in the inner space, we have to control whether there exists a bounded set $\Omega$ closed under $B$, in other words, whether $B$ is a contraction. With this in mind we derive the following proposition.

**Proposition 3.2.** Let $A \in \mathbb{R}^{2 \times 2}$ be such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$ for a cut-and-project set defined using the above scheme ($L_{3D} \subset \mathbb{R}^3$, 2) and bounded $\Omega \subset \mathbb{R}$ with non-empty interior. Then $A$ is of the form (6) with $|a + b\beta'' + c\beta'^2| \leq 1$.

**Example 1.** Consider the cut-and-project set $\Sigma(\Omega) = \{\pi_\parallel(\vec{x}) : \vec{x} \in L_{3D}, \pi_\perp(\vec{x}) \in \Omega\}$ with the choice $\Omega = [0, 1]$. A section of the set $\Sigma(\Omega)$ in the neighborhood of the origin is displayed in Figure 1. This set is closed under the action of any linear mapping of the form (6) such that $\lambda = a + b\beta'' + c\beta'^2 \in [0, 1]$. This condition comes from the requirement that $\Omega$ needs to be closed under multiplication by $\lambda$. Take for example $A = \begin{pmatrix} -\beta'' & 0 \\ 0 & -\beta' \end{pmatrix}$. Then the induced map $B$ acting in the internal space is $B = -\beta''I$. Since $-\beta'' \in (0, 1)$ we have $A\Sigma(\Omega) \subset \Sigma(\Omega)$. Figure 1 illustrates this self-similarity.
Figure 1. Example of a cut-and-project set defined using the scheme \((\mathcal{L}\subset \mathbb{R}^3, 2)\) and its self-similarity. The black dots mark the elements of the set \(\Sigma(\Omega)\); the squares are in positions of elements of \(A\Sigma(\Omega)\).

4. Relation to a higher-dimensional model with seven-fold symmetry

In this section we construct four-dimensional C&P scheme with seven-fold symmetry. From the requirement of a symmetry of order 7, it follows that the minimal dimension of the lattice \(\mathcal{L}\) is 6, see the crystallographic restriction in [5]. For the construction of the lattice generators, we consider \(Y_{6D}\), the Vandermonde matrix of the roots

\[
\xi = e^{\frac{2\pi i}{7}}, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6,
\]

of the seventh cyclotomic polynomial

\[
\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1,
\]

i.e., \(Y\) is composed of eigenvectors of the companion matrix \(C_{\Phi_7}\) associated to \(\Phi_7\), namely

\[
C_{\Phi_7} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}, \quad Y_{6D} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
\xi & \xi^6 & \xi^5 & \xi^4 & \xi^3 & \xi^2 \\
\xi^2 & \xi^5 & \xi^4 & \xi^3 & \xi^6 & \xi \\
\xi^3 & \xi^4 & \xi^6 & \xi & \xi^2 & \xi^5 \\
\xi^4 & \xi^3 & \xi & \xi^6 & \xi^5 & \xi^2 \\
\xi^5 & \xi^2 & \xi^3 & \xi^4 & \xi^6 & \xi
\end{pmatrix}.
\]
The lattice generators are chosen to be the columns of the realization of the inverse $Y_{6D}^{-1}$

$$
Y_{6D}^{-1} = \frac{1}{7} \begin{pmatrix}
1 - \xi & \xi^6 - \xi & \xi^5 - \xi & \xi^4 - \xi & \xi^3 - \xi & \xi^2 - \xi \\
1 - \xi^6 & \xi - \xi^6 & \xi^2 - \xi^6 & \xi^3 - \xi^6 & \xi^4 - \xi^6 & \xi^5 - \xi^6 \\
1 - \xi^2 & \xi^5 - \xi^2 & \xi^3 - \xi^2 & \xi - \xi^2 & \xi^6 - \xi^2 & \xi^4 - \xi^2 \\
1 - \xi^5 & \xi^2 - \xi^5 & \xi^4 - \xi^5 & \xi^6 - \xi^5 & \xi - \xi^5 & \xi^3 - \xi^5 \\
1 - \xi^3 & \xi^4 - \xi^3 & \xi - \xi^3 & \xi^5 - \xi^3 & \xi^2 - \xi^3 & \xi^6 - \xi^3 \\
1 - \xi^4 & \xi^3 - \xi^4 & \xi^2 - \xi^4 & \xi^5 - \xi^4 & \xi^6 - \xi^4 & \xi - \xi^4
\end{pmatrix}.
$$

Let $P = I_3 \otimes \left( \frac{1}{2} - \frac{1}{4} \right)$, and thus $P^{-1} = \frac{1}{2} I_3 \otimes \left( \frac{1}{2} + \frac{1}{4} \right)$. Consider the matrix $L_{6D} := 7P^{-1}Y_{6D}^{-1}$, and denote its columns by $\vec{l}_1, \ldots, \vec{l}_6$. Then we have

$$
\vec{l}_1 = \frac{1}{2} \begin{pmatrix}
2 - \xi - \xi^6 \\
i(\xi^6 - \xi) \\
2 - \xi^2 - \xi^5 \\
i(\xi^5 - \xi^2) \\
2 - \xi^3 - \xi^4 \\
i(\xi^4 - \xi^3)
\end{pmatrix}, \quad \vec{l}_3 = \frac{1}{2} \begin{pmatrix}
\xi^5 + \xi^2 - (\xi^6 + \xi) \\
i(\xi^5 - \xi^2) + i(\xi^6 - \xi) \\
\xi^3 + \xi^4 - (\xi^2 + \xi^5) \\
i(\xi^3 - \xi^4) + i(\xi^2 - \xi^5) \\
\xi - \xi^6 - (\xi^3 + \xi^4) \\
i(\xi - \xi^6) + i(\xi^3 - \xi^4)
\end{pmatrix}, \quad \vec{l}_5 = \frac{1}{2} \begin{pmatrix}
\xi^3 + \xi^4 - (\xi^6 + \xi) \\
i(\xi^3 - \xi^4) + i(\xi^6 - \xi) \\
\xi^6 + \xi - (\xi^2 + \xi^5) \\
i(\xi^6 - \xi) + i(\xi^5 - \xi^2) \\
\xi^2 + \xi^5 - (\xi^3 + \xi^4) \\
i(\xi^2 - \xi^5) + i(\xi^3 - \xi^4)
\end{pmatrix},
$$

and the lattice $L_{6D}$ is given by

$$
L_{6D} = \bigoplus_{i=1}^{6} \mathbb{Z} \vec{l}_i.
$$

Again, equipped with projections onto four- and two-dimensional spaces, i.e.,

$$
\mathbb{R}^4 \xrightarrow{\pi_1} L_{6D} \xrightarrow{\pi_2} \mathbb{R}^2,
$$

by [14], the constructed cut-and-project scheme ($L_{6D} \subset \mathbb{R}^6, 4$) is non-degenerate, aperiodic and irreducible. The self-similarities of the four-dimensional $\mathbb{Z}$-module $\pi_1(L_{6D})$ can be derived using the same methods as are used in [17] for the two-dimensional $\mathbb{Z}$-module corresponding to the decagonal quasicrystalline model. We do not include the rather technical description here for the reasons of conciseness.

Let us clarify the relation of the cut-and-project scheme ($L_{3D} \subset \mathbb{R}^3, 2$) from Section 3 to the above constructed scheme ($L_{6D} \subset \mathbb{R}^6, 4$). We will show that $L_{3D}$ can be embedded into $\mathbb{R}^6$ as a sub-lattice of $L_{6D}$ in such a way that the $\mathbb{Z}$-module $\pi_1(L_{6D})$ contains the $\mathbb{Z}$-module $\pi_1(L_{3D})$, and the same holds for $\pi_2$.

Consider the realification of $Y_{6D}$ of (7) by the matrix $P = I_3 \otimes \left( \frac{1}{2} - \frac{1}{4} \right)$,

$$
Y_{6D}P = \begin{pmatrix}
2 & 0 & 2 & 0 & 2 & 0 \\
\xi + \xi^6 & i(\xi^6 - \xi) & \xi^2 + \xi^5 & i(\xi^5 - \xi^2) & \xi^3 + \xi^4 & i(\xi^4 - \xi^3) \\
\xi^2 + \xi^5 & i(\xi^5 - \xi^2) & \xi^3 + \xi^4 & i(\xi^4 - \xi^3) & \xi^6 + \xi^6 & i(\xi^6 - \xi^6) \\
\xi^3 + \xi^4 & i(\xi^4 - \xi^3) & \xi^6 + \xi^6 & i(\xi^6 - \xi^6) & \xi^2 + \xi^5 & i(\xi^5 - \xi^2) \\
\xi^3 + \xi^4 & i(\xi^4 - \xi^3) & \xi^6 + \xi^6 & i(\xi^6 - \xi^6) & \xi^2 + \xi^5 & i(\xi^5 - \xi^2) \\
\xi^2 + \xi^5 & i(\xi^5 - \xi^2) & \xi^6 + \xi^6 & i(\xi^6 - \xi^6) & \xi^2 + \xi^5 & i(\xi^5 - \xi^2)
\end{pmatrix}.
$$
The elements of this matrix belong to the algebraic field \( Q(\beta) \) where \( \beta = 2 \cos \frac{2\pi}{7} + 1 \) is the Pisot-cyclotomic number of order 7. The first, third and fifth columns of \( Y_{6D}P \) can be advantageously rewritten in terms of \( \beta \) and its algebraic conjugates \( \beta', \beta'' \), see (1). We have

\[
Y_{6D}P = \begin{pmatrix}
2 & * & 2 & * & 2 & * \\
(\beta - 1) & * & (\beta' - 1) & * & (\beta'' - 1) & * \\
(\beta') & * & (\beta'' - 1) & * & (\beta - 1) & * \\
(\beta'' - 1) & * & (\beta - 1) & * & (\beta' - 1) & * \\
(\beta' - 1) & * & (\beta'' - 1) & * & (\beta - 1) & *
\end{pmatrix},
\]

i.e., the third and fifth columns are just images of the first column under the automorphism of the field \( Q(\beta) \). Note that we put \( * \) at the place of matrix elements that are not of importance in our considerations. Consider the following rational matrix

\[
Q = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
3 & 4 & 2 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1
\end{pmatrix}.
\]

One easily verifies that it brings the vector \((2, \beta - 1, \beta' - 1, \beta'' - 1, \beta'' - 1, \beta' - 1)^T\) to \((1, \beta, \beta^2, 0, 0, 0)^T\), i.e., by application of the automorphism, we have

\[
QY_{6D}P = 2 \begin{pmatrix}
1 & * & 1 & * & 1 & * \\
\beta & * & \beta' & * & \beta'' & * \\
\beta^2 & * & \beta'^2 & * & \beta''^2 & * \\
0 & * & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & *
\end{pmatrix},
\]

i.e., we see the matrix \( Y_{3D} \) of (3) as a submatrix of \( QY_{6D}P \). Inverting this relation, we have

\[
7(QYP)^{-1} = L_{6D}Q^{-1} = \begin{pmatrix}
2\beta^2 - 2\beta - 5 & 3\beta^2 - 6\beta - 2 & -2\beta^2 + 3\beta + 2 & * & * & * \\
0 & 0 & 0 & * & * & * \\
-2\beta - 1 & -3\beta^2 + 3\beta + 4 & \beta^2 - 2 & * & * & * \\
0 & 0 & 0 & * & * & * \\
-2\beta^2 + 4\beta - 1 & 3\beta - 2 & \beta^2 - 3\beta & * & * & *
\end{pmatrix}.
\]

We see that the matrix \( L_{3D} \) of (5) is a submatrix of the above. This shows that the cut-and-project scheme \((L_{3D} \subset \mathbb{R}^3, 2)\) is contained in \((L_{6D} \subset \mathbb{R}^6, 4)\).

5. Voronoi tiling of the model

Mathematical models of quasicrystals can be viewed from two different points of view. Either we are interested in atomic positions, in that case we consider the model to be a discrete point set in the space. We can, however, study clusters of atoms around one given point. For example, when studying interatomic interactions, one needs to limit focus on a given neighbourhood of the chosen atom. Natural definition of neighbourhood in a point set which does not have a lattice structure is provided by the notion of Voronoi cell.
Figure 2. Voronoi tiling of the quasicrystal model considered in Figure 1 is displayed. Only six (up to translation and central symmetry) shapes of tiles appear in the tiling.

Given a Delone set $\Sigma \subset \mathbb{R}^d$ and a chosen point $x \in \Sigma$. The Voronoi cell of $x$ is given by

$$V(x) = \{ y \in \mathbb{R}^d \mid \|x - y\| \leq \|z - y\| \text{ for all } z \in \Sigma \}.$$ 

In other words, the Voronoi cell of $x$ is formed by the part of the space $\mathbb{R}^d$ which is closer to $x$ than to any other element of the set $\Sigma$. As $\Sigma$ has the Delone property, the Voronoi cell of any point of $\Sigma$ is a well defined convex polytope in $\mathbb{R}^d$. The cells of $\Sigma$ tile the entire space $\mathbb{R}^d$ without gaps or overlaps. Such a tiling is called a perfect tiling of the space. Voronoi tiling allows a natural definition of neighbourhood in a Delone set $\Sigma$. For neighbours of a point $x$ one chooses those elements of $\Sigma$ whose Voronoi polygons share a face of dimension $d - 1$.

Under a certain condition on the bounded set $\Omega$ in the inner space, the cut-and-project set $\Omega$ is of finite local complexity, i.e., has finitely many local configurations of any given size, up to translation. As a particular consequence, the Voronoi tiling of $\Sigma(\Omega)$ is composed of copies of only finitely many polygons. Voronoi tiling of the most prominent decagonal model dependingly on the size of the acceptance window is examined in [18].

As an example of the notion of Voronoi tiling, we consider the cut-and-project set provided in Section 3 which is defined by $\Omega \subset \mathbb{R}$ being a bounded interval. Since the inner space is one-dimensional, the condition on $\Omega$ ensuring the property of finite local complexity is simple. In fact, $\Sigma(\Omega)$ is of finite local complexity whenever the interval is semi-closed.

In Figure 2 we see the Voronoi tiling of a finite section of the cut-and-project set in Example 1. Figure 3 shows the distinct shapes which appear in the tiling.

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Figure 3. The six shapes of tiles appearing in the quasicrystal model from Figure 1. With each Voronoi cell \( V(x) \), we display the neighbours of \( x \).

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