Newtonian, Post Newtonian and Parameterized Post Newtonian limits of $f(R,G)$ gravity

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We discuss in detail the weak field limit of $f(R,G)$ gravity taking into account analytic functions of the Ricci scalar $R$ and the Gauss-Bonnet invariant $G$. Specifically, we develop, in metric formalism, the Newtonian, Post Newtonian and Parameterized Post Newtonian limits starting from general $f(R,G)$ Lagrangian. The special cases of $f(R)$ and $f(G)$ gravities are considered. In the case of the Newtonian limit of $f(R,G)$ gravity, a general solution in terms of Green’s functions is achieved.

Keywords: Alternative theories of gravity, weak field limit, theory of perturbations.

I. INTRODUCTION

Any good theory of physics should satisfy three main viability criteria that are self-consistency, completeness, and agreement with experiments. These have to hold also for theories of gravity like General Relativity (GR). However, despite its successes and elegance, such a theory exhibits a number of inconsistencies and weaknesses that have led many scientists to ask, whether, it is the final theory that can explain definitively the gravitational interaction. GR disagrees with a growing number of data observed at infrared scales like cosmological scales. Furthermore, GR is not many scientists to ask, whether, it is the final theory that can explain definitively the gravitational interaction. GR disagrees with a growing number of data observed at infrared scales like cosmological scales. Furthermore, GR is not

Moreover, large amounts of dark matter and dark energy are required to address dynamics and structure of galaxies, clusters of galaxies and global accelerated expansion of the Hubble cosmic flow. If one wants to keep GR and its low energy limit, we must necessarily introduce these still unknown ingredients that, up to now, seem highly elusive. In other words, GR, from ultraviolet to infrared scales, cannot be the ultimate theory of gravity even if it addresses a wide range of phenomena.

In order to overcome the above mentioned problems and simultaneously obtain a semi-classical approach where GR and its results can be recovered, Extended Theories of Gravity (ETGs) have been recently introduced $[1-6]$. These theories are based on corrections and extensions of Einstein’s theory. The effective Hilbert-Einstein action is modified by adding higher order terms in curvature invariants as $R^2$, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $R \Box R$, $R \Box R$ or non-minimally coupled terms between scalar fields and geometry as $\phi^2 R$. The simplest generalization is the $f(R)$ gravity where the Hilbert-Einstein action, linear in the Ricci scalar $R$, is replaced by a generic function of it $[7-11]$. Such modifications of GR generate inflationary behaviors which solve a lot of shortcomings of the Standard Cosmological Model as shown by Starobinsky $[12]$, or can explain the flat rotation curves of galaxies or dynamics of galaxy clusters $[13]$. Another interesting curvature quantity that should be considered is the Gauss-Bonnet (GB) curvature invariant $G$ defined below. This term can avoid ghost contributions and contribute to the regularization of the gravitational action $[14-23]$. Furthermore, in the case in which the Lagrangian density $f$ is a function of $G$, i.e., $f(G)$ it is possible to construct viable cosmological models that are consistent with local constraints of GR $[24-31]$. In general, we can consider the most general Lovelock modification of gravity implying curvature and topological invariant, that is $\mathcal{L} = f(R,G)$ $[22]$. Beyond the motivations for considering ETGs, it is important that we understand their weak-field limit from theoretical and experimental points of view. The Parameterized Post Newtonian (PPN) formalism is the first context for considering weak-field effects to test their viability with respect to GR. Eddington, Robertson and Schiff were the first that rigorously formulated the PPN formalism and used it in interpreting Solar Systems experiments $[33-35]$. After Nordvedt and Will fixed systematically the approach $[36,37]$. Our aim, here, is to develop the weak field limit of $f(R,G)$ considering the PPN formalism. Previous results have been developed for $f(R)$ gravity in $[38,39]$ where the Post Newtonian (PN) limit is achieved by using the equivalence with scalar-tensor theories $[40]$. On the other hand, the PN limit of Gauss-Bonnet gravity has been developed in $[41]$ but not for pure $f(G)$ gravity. Here, we develop the PPN limit for $f(R,G)$, starting from the field equations. In doing so, we restrict our attention to those theories that admit a Minkowski background. Furthermore, as a special case, we develop the PPN limit even for the $f(G)$ and then make a comparison with the results obtained for $f(R)$.
The paper is organized as follows. In Sect. II, the field equations for \( f(R, \mathcal{G}) \)-gravity theories are reviewed and the Newtonian, PN and PPN limits are obtained. In Sect. III, the Newtonian limit for \( f(R, \mathcal{G}) \) gravities achieved in terms of Green’s functions. In Sect. IV, the weak field limit of the special cases of \( f(R) \) and \( f(\mathcal{G}) \) modified gravities are, respectively, studied. Finally, in Sect. V, we summarize the obtained results and give some outlooks for the approach.

II. \( f(R, \mathcal{G}) \) MODIFIED GRAVITY: THE FIELD EQUATIONS AND THE NEWTONIAN LIMIT

This section is devoted to the study of the field equations for \( f(R, \mathcal{G}) \)-gravity and their Newtonian, PN and PPN limits (for other examples of weak field limit in modified gravity theories, see [13][14]). The first remarkable characteristic of \( f(R, \mathcal{G}) \) modified gravity is that, in this case, we obtain fourth–order field equations, instead of the standard second–order ones obtained in the case of GR. This fact is due to the existence of some boundary terms that disappear in GR thanks to the divergence theorem, but they remain in other theories, as in the case of \( f(R, \mathcal{G}) \) gravity.

A. Field equations for \( f(R, \mathcal{G}) \) gravity

The starting action for \( f(R, \mathcal{G}) \) gravity is given by:

\[
S = \int d^4 x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} f(R, \mathcal{G}) + \mathcal{L}_{\text{matter}} \right\},
\]

where \( \mathcal{L}_{\text{matter}} \) is the matter Lagrangian, \( g \) is the determinant of the metric, \( \kappa^2 = 8\pi G_N \) is standard gravitational coupling, and \( \mathcal{G} \) is the Gauss-Bonnet invariant, defined as

\[
\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}.
\]

The variation of Eq.(1) with respect to the metric tensor \( g_{\mu\nu} \) gives the field equations for \( f(R, \mathcal{G}) \)-gravity that are:

\[
- \frac{1}{2} g_{\mu\nu} f(R, \mathcal{G}) + f_R(R, \mathcal{G}) R_{\mu\nu} + g_{\mu\nu} \nabla^2 (f_R(R, \mathcal{G})) - \nabla_\mu \nabla_\nu (f_R(R, \mathcal{G})) +
+ 2f_\mathcal{G}(R, \mathcal{G}) R R_{\mu\nu} - 4f_\mathcal{G}(R, \mathcal{G}) R_{\mu\rho} R_{\nu}^\rho + 2f_\mathcal{G}(R, \mathcal{G}) R_{\alpha\beta \rho\sigma} R^{\alpha\beta\rho\sigma} + 4f_\mathcal{G}(R, \mathcal{G}) R_{\mu\nu\sigma} R^{\sigma} +
+ 2g_{\mu\nu} \nabla^2 f_\mathcal{G}(R, \mathcal{G}) - 4g_{\mu\nu} R_{\rho\sigma} \nabla^\rho \nabla^\sigma f_\mathcal{G}(R, \mathcal{G}) - 2R \nabla_\mu \nabla_\nu f_\mathcal{G}(R, \mathcal{G}) - 4R_{\mu\nu} \nabla^2 f_\mathcal{G}(R, \mathcal{G}) +
+ 4R_{\nu\rho} \nabla^\rho \nabla_\mu f_\mathcal{G}(R, \mathcal{G}) + 4R_{\rho\sigma} \nabla^\rho \nabla^\sigma f_\mathcal{G}(R, \mathcal{G}) + 4R_{\mu\rho\sigma} \nabla^\rho \nabla^\sigma f_\mathcal{G}(R, \mathcal{G}) = 2\kappa^2 T_{\mu\nu}.
\]

The trace equation is

\[
- 2f(R, \mathcal{G}) + f_R(R, \mathcal{G}) R + 3 \nabla^2 f_R(R, \mathcal{G}) + 2f_\mathcal{G}(R, \mathcal{G}) \overline{G} + 2R \nabla^2 f_\mathcal{G}(R, \mathcal{G}) - 4R_{\rho\sigma} \nabla^\rho \nabla^\sigma f_\mathcal{G}(R, \mathcal{G}) = 2\kappa^2 T.
\]

In Eq.(1A) and Eq.(1), the following notation has been used: \( f_R(R, \mathcal{G}) = \frac{df(R, \mathcal{G})}{dR} \) and \( f_\mathcal{G}(R, \mathcal{G}) = \frac{df(R, \mathcal{G})}{d\mathcal{G}} \), while \( \nabla \) is the covariant derivative. The fact that the scalar curvature, \( R \), and the Gauss-Bonnet invariant, \( \mathcal{G} \), involves second derivatives of the metric tensor \( g_{\mu\nu} \) makes of Eq.(1A) and Eq.(1) fourth-order differential equations in the metric \( g_{\mu\nu} \).

B. The Newtonian, Post Newtonian and Parameterized Post Newtonian limits

All the quantities involved in the Newtonian, PN and PPN limits for \( f(R, \mathcal{G}) \) gravity, given can be expanded in powers of \( \bar{v}^2 \). As it is well known in the Solar System, the effects of the gravitational field are weak and are well represented by Newton’s theory of gravity [42], therefore, the light rays travel on straight lines at a constant speed and the test particles move with an acceleration

\[
a = \nabla U,
\]
where $U$ is the Newtonian potential produced by a rest mass density $\rho$ according to following relations
\[
\nabla^2 U = -4\pi \rho, \quad U(x, t) = G_N \int d^3x' \frac{\rho(x', t)}{|x - x'|}.
\]

The hydrodynamic equations for a perfect non-viscous fluid are the usual Eulerian equations
\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho v) = 0, \\
\rho \frac{dv}{dt} = \rho \nabla U - \nabla p, \\
\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla,
\]
where $v$ is the velocity of an element of the fluid, $\rho$ is the density and $p$ is the pressure on the element. This means
\[
U \sim \frac{p}{\rho} \sim v^2 \sim \mathcal{O}(2).
\]

Also the derivatives with respect to the time relative to spatial derivatives are:
\[
\left| \frac{\partial}{\partial t} \right| \left| \frac{\partial}{\partial x} \right| \sim \mathcal{O}(1).
\]

Here we have chosen to set $c = 1$. However, the Newtonian limit is no longer sufficient when we require that, in experiments, accuracies go beyond $10^5$ (e.g. it is not possible to explain, in the case of Mercury, the additional perihelion advance greater than $5 \times 10^{-7}$ radians per orbit). Thus we need a more accurate approximation that goes beyond the Newtonian approximation, and then the PN, PPN limits are considered. In order to build the PPN limit we need of expanding in this order of smallness. To recover the Newtonian limit we must develop $g_{00}$ up to $\mathcal{O}(2)$, while the PN limit requires
\[
\begin{align*}
g_{00} & \quad \text{up to } \mathcal{O}(4), \\
g_{0i} & \quad \text{up to } \mathcal{O}(3), \\
g_{ij} & \quad \text{up to } \mathcal{O}(2),
\end{align*}
\]
where the Latin indices denote the spatial indices. These terms must contains factors as velocity or time derivatives. Furthermore these terms could represent energy dissipation or absorption. Beyond $\mathcal{O}(4)$, modified gravity effects can take place and give different predictions. We expect that it should be possible to find a coordinate system where the metric tensor is nearly equal to the Minkowski tensor $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, the corrections being expandable in powers of $\bar{v}^2$. We will consider the following ansatz for the metric tensor:
\[
\begin{align*}
g_{00} &= g_{00}^{(0)} + g_{00}^{(2)} + g_{00}^{(4)} + \mathcal{O}(6) \quad \text{with} \quad \begin{cases} g_{00}^{(0)} = 1 \\ g_{00}^{(2)} = -2U \\ g_{00}^{(4)} = -2U^2 \\ \cdots \end{cases}, \\
g_{0i} &= g_{0i}^{(3)} + \mathcal{O}(5) \quad \text{with} \quad \begin{cases} g_{0i}^{(3)} = \cdots \\ \cdots \end{cases}, \\
g_{ij} &= g_{ij}^{(0)} + g_{ij}^{(2)} + g_{ij}^{(4)} + \mathcal{O}(6) \quad \text{with} \quad \begin{cases} g_{ij}^{(0)} = -\delta_{ij} \\ g_{ij}^{(2)} = -\delta_{ij}2V \\ \cdots \end{cases},
\end{align*}
\]
where $\delta_{ij}$ is the Kronecker delta and $U, V$ are potentials, in particular $V$, following [42], is define as
\[
V_i = G_N \int d^3x' \frac{\rho(x', t)v_i(x', t)}{|x - x'|}.
\]
The inverse metric of Eq. (5) can be calculated using the relation \( g^{\alpha\beta} g_{\rho\sigma} = \delta^\alpha_\rho \), giving the following results:

\[
\begin{align*}
g^{00} & = g^{(0)00} + g^{(2)00} + g^{(4)00} + \mathcal{O}(6) \quad \text{with} \quad \begin{cases} 
g^{(0)00} = 1 \\
g^{(2)00} = 2U \\
g^{(4)00} = -g^{(4)}_{00} + 4U^2 \end{cases} \\
g^{0i} & = g^{(3)0i} + \mathcal{O}(5) \quad \text{with} \quad g^{(3)0i} = \delta^{ij} g^{(3)}_{0j} \\
g^{ij} & = g^{(0)ij} + g^{(2)ij} + g^{(4)ij} + \mathcal{O}(6) \quad \text{with} \quad \begin{cases} 
g^{(0)ij} = -\delta^{ij} \\
g^{(2)ij} = \delta^{ij} 2V \\
g^{(4)ij} = -4V^2 \delta^{ij} - \delta^{ik} \delta^{j\ell} g^{(4)}_{k\ell} \end{cases}
\end{align*}
\]  

Given a metric tensor, the associated connection associated can be derived as \( \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu}) \), which can be expanded in powers of \( \bar{v}^2 \) introducing, in the last equation, the expressions given by Eq. (5) and Eq. (8), namely:

\[
\begin{align*}
\Gamma^0_{00} & = \Gamma^{(3)0}_{00} + \mathcal{O}(5) \quad \text{with} \quad \Gamma^{(3)0}_{00} = -\partial_0 U \\
\Gamma^0_{0i} & = \Gamma^{(2)0}_{0i} + \Gamma^{(4)0}_{0i} + \mathcal{O}(6) \quad \text{with} \quad \begin{cases} 
\Gamma^{(2)0}_{0i} = -\partial_i U \\
\Gamma^{(4)0}_{0i} = \frac{1}{2} \left( \partial_i g^{(4)}_{00} - 4U \partial_i U \right) \end{cases} \\
\Gamma^0_{ij} & = \Gamma^{(3)0}_{ij} + \mathcal{O}(5) \quad \text{with} \quad \Gamma^{(3)0}_{ij} = \frac{1}{2} \left( \partial_j g^{(3)}_{0i} + \partial_i g^{(3)}_{0j} + 2\delta_{ij} \partial_0 V \right) \\
\Gamma^i_{00} & = \Gamma^{(2)i}_{00} + \Gamma^{(4)i}_{00} + \mathcal{O}(6) \quad \text{with} \quad \begin{cases} 
\Gamma^{(2)i}_{00} = -\delta^{il} \partial_l U \\
\Gamma^{(4)i}_{00} = \frac{1}{2} \delta^{il} \left( \partial_l g^{(4)}_{00} + 4V \partial_l U - 2\partial_0 g^{(3)}_{0l} \right) \end{cases} \\
\Gamma^i_{0j} & = \Gamma^{(3)i}_{0j} + \mathcal{O}(5) \quad \text{with} \quad \Gamma^{(3)i}_{0j} = \frac{1}{2} \delta^{il} \left( \partial_l g^{(3)}_{0j} - \partial_j g^{(3)}_{0l} + 2\delta_{lj} \partial_0 V \right) \\
\Gamma^i_{jk} & = \Gamma^{(2)i}_{jk} + \Gamma^{(4)i}_{jk} + \mathcal{O}(6) \quad \text{with} \quad \begin{cases} 
\Gamma^{(2)i}_{jk} = \delta^{il} \left( -\delta_{jk} \partial_l V + \delta_{lj} \partial_k V + \delta_{lk} \partial_j V \right) \\
\Gamma^{(4)i}_{jk} = -\frac{1}{2} \delta^{il} \left[ \partial_j g^{(4)}_{kl} + \partial_k g^{(4)}_{lj} - \partial_l g^{(4)}_{jk} \right] - \\
-2V \delta^{il} \left[ \delta_{kl} \partial_j V + \delta_{lj} \partial_k V - \delta_{jk} \partial_l V \right] \end{cases}
\end{align*}
\]

Given a metric tensor, the Riemann tensor, the Ricci tensor and the scalar curvature can be immediately calculated by using the following expressions:

\[
R_{\alpha\beta\rho\sigma} = \frac{1}{2} \left( \partial_\sigma \partial_\alpha g_{\beta\rho} - \partial_\alpha \partial_\beta g_{\sigma\rho} - \partial_\rho \partial_\alpha g_{\beta\sigma} + \partial_\rho \partial_\beta g_{\alpha\sigma} \right) + g_{\mu\nu} \left( \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\beta\rho} - \Gamma^\mu_{\rho\alpha} \Gamma^\sigma_{\beta\sigma} \right),
\]

\[
R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\rho_{\rho\nu} + \Gamma^\rho_{\mu\nu} \Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\sigma\nu} \Gamma^\rho_{\rho\mu},
\]

\[
R = g^{\mu\nu} R_{\mu\nu}.
\]
The components of the Riemann tensor that we will need later can be expanded in powers of \( \delta^2 \) using Eqs. (5)-(II B): 

\[
\begin{align*}
R_{a0j0} &= R_{a0j0}^{(2)} + R_{a0j0}^{(4)} + O(6) \quad \text{with} \quad R_{a0j0}^{(2)} = \partial_i \partial_j U \\
R_{0ij0} &= R_{0ij0}^{(2)} + R_{0ij0}^{(4)} + O(6) \quad \text{with} \quad R_{0ij0}^{(2)} = -R_{0ij0}^{(2)} \\
R_{ij00} &= R_{ij00}^{(3)} + O(5) \quad \text{with} \quad R_{ij00}^{(3)} = \frac{1}{2} \left[ \partial_k \left( \partial_j g_{0i}^{(3)} - \partial_i g_{0j}^{(3)} \right) + 2 \partial_0 \left( \delta_{ik} \partial_j V - \delta_{jk} \partial_i V \right) \right]
\end{align*}
\]

By assuming the harmonic gauge, given by \( g^\mu \nabla^\nu = 0 \) (in order to simplify the expressions), and using Eqs. (5)-(II B), we can expand the Ricci tensor in powers of \( \delta^2 \):

\[
\begin{align*}
R_{00} &= R_{00}^{(2)} + R_{00}^{(4)} + O(6) \quad \text{with} \quad R_{00}^{(2)} = -\triangle U \\
R_{0i} &= R_{0i}^{(3)} + O(5) \quad \text{with} \quad R_{0i}^{(3)} = \frac{1}{2} \triangle g_{0i}^{(3)} \\
R_{ij} &= R_{ij}^{(2)} + R_{ij}^{(4)} + O(6) \quad \text{with} \quad R_{ij}^{(2)} = -\delta_{ij} \triangle V \\
R_{ij}^{(4)} &= \frac{1}{2} \left[ \partial_0 \left( \partial_j g_{0i}^{(3)} + \partial_i g_{0j}^{(3)} + 2 \partial_{ij} \partial_0 V \right) - \delta^{mn} \left( \partial_m \left( \partial_i g_{nj}^{(4)} + \partial_n g_{mj}^{(4)} \right) - \partial_i \partial_j g_{mn}^{(4)} \right) + \triangle g_{ij}^{(4)} \right] + 2 \partial_0 \partial_i \partial_j V + 2 \partial_j \partial_i U + \partial_i U \partial_j (U - V) + \partial_j V \partial_i (3V - U) + \delta_{ij} \left( \delta^{kl} \partial_i \partial_k (U + V) + 2V \triangle V \right)
\end{align*}
\]

and the scalar curvature too:

\[
\begin{align*}
R &= R^{(2)} + R^{(4)} + O(6) \quad \text{with} \quad R^{(2)} = 3 \triangle V - \triangle U \\
R^{(4)} &= \frac{1}{2} \triangle g_{00}^{(4)} + 2V (\triangle U - 3 \triangle V) + \partial_0 \partial_0 U - 2 \delta_{ij} \partial_i \partial_j U - 2U \triangle U
\end{align*}
\]

On the matter side, we start with the general definition of the energy-momentum tensor of a perfect fluid:

\[
T_{\mu \nu} = (\rho + \Pi \rho + p) u_{\mu} u_{\nu} - p g_{\mu \nu} ,
\]

where \( \Pi \) denotes the internal energy density, \( \rho \) the energy density and \( p \) the pressure. Taking into account that:

\[
\begin{align*}
u^0 &= \frac{1}{\sqrt{1 - v^2}} = 1 + \frac{1}{2} v^2 + \frac{3}{8} v^4 + O(6) , \\
\nu^i &= \frac{v^i}{\sqrt{1 - v^2}} = v^i \left( 1 + \frac{1}{2} v^2 + O(4) \right)
\end{align*}
\]
and the expressions Eq. (5) and Eq. (8), we can calculate the different components of Eq. (12): 

\[
\begin{align*}
T_{00} &= T_{00}^{(0)} + T_{00}^{(2)} + T_{00}^{(4)} + \mathcal{O}(6) \\
T_{0i} &= T_{0i}^{(1)} + T_{0i}^{(3)} + \mathcal{O}(5) \\
T_{ij} &= T_{ij}^{(2)} + \mathcal{O}(4)
\end{align*}
\]

with 

\[
\begin{align*}
T_{00}^{(0)} &= \rho \\
T_{00}^{(2)} &= \rho (\Pi + v^2 - 4U) \\
T_{00}^{(4)} &= \rho \left[ v^4 - 3Uv^2 + 4U^2 + 2g_{0i}^{(4)} + 2g_{0i}^{(3)}v^2 + \Pi(v^2 - 4U) \right] + p(v^2 - 2U) \\
T_{0i}^{(1)} &= -v^i\rho \\
T_{0i}^{(3)} &= -v^i\rho \left( \Pi + \frac{1}{2}v^2 - 2U + \frac{p}{\rho} \right) \\
T_{ij}^{(2)} &= \rho \delta_{ik}\delta_{jl}v^k v^l + p\delta_{ij} \\
T_{ij}^{(4)} &= \rho \left\{ -\delta_{ik}v^k g_{0j}^{(3)} - \delta_{jl}v^l g_{0i}^{(3)} + \delta_{ik}v^k \left( v^2 + \Pi + 4V \right) \right\} + p \left\{ \delta_{ik}v^k v^l + 2V \delta_{ij} \right\}
\end{align*}
\]

Finally, the expansion of the Gauss-Bonnet invariant \( \mathcal{G} \) in orders of \( \bar{v}^2 \) can be calculated using Eqs. (5)-13:

\[
\mathcal{G} = \mathcal{G}^{(4)} + \mathcal{O}(6),
\]

with \( \mathcal{G}^{(4)} = -3(\Delta U)^2 + (\Delta V)^2 - 6(\Delta U)(\Delta V) + 4\delta^{im}\delta^{jn} [\partial_i \partial_j (U + V)] [\partial_m \partial_n (U + V)] \). \quad (17)

We have now all the ingredients, expanded in orders of \( \bar{v}^2 \), needed to write the field equations and the trace equation in the Newtonian, PN and PPN limits. In order to do this, we assume that:

\[
f^*(R, \mathcal{G}) = f^*(0,0) + f^*_R(0,0)R + f^*_G(0,0)\mathcal{G} + \frac{1}{2} \left( f^*_{RR}(0,0)R^2 + 2f^*_R(0,0)RG + f^*_G\mathcal{G}^2 \right) + ...
\]

where \( f^*(R, \mathcal{G}) \) denotes the function \( f(R, \mathcal{G}) \) or any of its derivatives, i.e. \( f_R(R, \mathcal{G}), f_G(R, \mathcal{G}), f_{RR}(R, \mathcal{G}) \) ... Considering Eqs. (12) and (13), we can write (18) as an expansion in orders of \( \bar{v}^2 \):

\[
f^*(R, \mathcal{G}) = f^*(0,0) + f^*_R(0,0)R^{(2)} + \left( \frac{1}{2} f^*_R(0,0)R^{(2)} \right)^2 + f^*_R(0,0)R^{(4)} + f^*_G(0,0)\mathcal{G}^{(4)} + \mathcal{O}(6).
\] \quad (19)

In the following sub section we proceed to calculate the Newtonian, PN and PPN limits.

1. **The \((0,0)\)-field equation**

The \((0,0)\)-field equation is given from Eq. (12) by:

\[
-\frac{1}{2}g_{00}f(R, \mathcal{G}) + f_R(R, \mathcal{G})R_{00} + g_{00}\nabla^2 (f_R(R, \mathcal{G})) - \nabla_0 \nabla_0 (f_R(R, \mathcal{G})) + \\
+2f_G(R, \mathcal{G})RR_{00} - 4f_G(R, \mathcal{G})R_{00}R_0^\rho R_0^\rho + 2f_G(R, \mathcal{G})R_\alpha^\beta R_\beta^\alpha R_{00}^\rho R_0^\rho + 4f_G(R, \mathcal{G})R_{00}R_{0\alpha}R_{\alpha\rho \sigma}R_{\rho \sigma} + \\
+2g_{00}R\nabla^2 f_G(R, \mathcal{G}) - 4g_{00}R_{\rho \sigma}\nabla^\rho \nabla^\sigma f_G(R, \mathcal{G}) - 2R\nabla_0 \nabla_0 f_G(R, \mathcal{G}) - 4R_{00}\nabla^2 f_G(R, \mathcal{G}) + \\
+8R_{0\rho} \nabla^\rho \nabla_0 f_G(R, \mathcal{G}) + 4R_{0\rho \sigma} \nabla^\rho \nabla^\sigma f_G(R, \mathcal{G}) = 2\kappa^2 T_{00},
\] \quad (20)
At the lowest order in the velocity, we obtain: \( f(0,0) = 0 \).

In the Newtonian limit, \textit{i.e.} at \( \mathcal{O}(2) \) order in the velocity, Eq. (11B.1) reduces to:

\[
- \frac{1}{2} g_{00} f_R(0,0) R^{(2)} + f_R(0,0) R^{(2)}_{00} + \left[ g_{00} \nabla^2 (f_R(R, \mathcal{G})) - \nabla_0 \nabla_0 (f_R(R, \mathcal{G})) \right]^{(2)} = 2 \kappa^2 T^{(0)}_{00},
\]

(21)

where it has been considered that \( f(0,0) = 0 \). Introducing Eqs. (5), (12), (13) and Eq. (16) into Eq. (21), we finally obtain the following equation:

\[
f_R(0,0) (3 \Delta V + \Delta U) + 2 f_{RR}(0,0) (3 \Delta^2 V - \Delta^2 U) = -4 \kappa^2 \rho.
\]

(22)

Where the notation: \( \Delta^2 := \Delta \cdot \Delta \), has been introduced, being \( \Delta = \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \).

In the PN limit, \textit{i.e.} at \( \mathcal{O}(4) \) order in the velocity, Eq. (11B.1) reduces to:

\[
f_R(0,0) \left\{ - \frac{1}{2} g_{00}^{(2)} R^{(2)} - \frac{1}{2} g_{00} R^{(4)} + R^{(4)}_{00} \right\} + 
\]

\[
+ f_{\mathcal{G}}(0,0) \left\{ -\frac{1}{2} \mathcal{G}^{(4)} + 2 R^{(2)} R^{(2)}_{00} - 4 R^{(2)}_{00} + 2 \left( R_{\alpha \beta \rho \sigma} R^{\alpha \beta \rho \sigma} \right)^{(4)} - 4 \left( R_{\alpha \rho \sigma \alpha} R^{\rho \sigma} \right)^{(4)} \right\} + 
\]

\[
+ f_{RR}(0,0) \left\{ -\frac{1}{4} R^{(2)} + 2 R^{(2)} R^{(2)}_{00} + \left( g_{00}^{(2) ij} + g_{00}^{(2) ij} \right) \partial i \partial j R^{(2)} + g^{(0) ij} \left( \partial i \partial j R^{(4)} - \Gamma^{(2) ij} \partial k R^{(2)} \right) \right\} + 
\]

\[
+ f_{R\mathcal{G}}(0,0) \left\{ g^{(0) ij} \partial i \partial j \mathcal{G} + 2 \partial i \partial j R^{(2)} \left[ g^{(0) ij} \left( R^{(2)} - 2 R^{(2)}_{00} \right) + 2 g^{(0) ik} g^{(0) jl} \left( -R^{(2)}_{kl} + R^{(2)}_{ijkl} \right) \right] \right\} + 
\]

\[
+ f_{RR\mathcal{G}}(0,0) g^{(0) ij} \left\{ \partial i R^{(2)} \partial j R^{(2)} + R^{(2)} \partial i \partial j R^{(2)} \right\} = 2 \kappa^2 T^{(2)}_{00},
\]

(23)

where it has been considered, once more, that \( f(0,0) = 0 \). The calculations that make possible to derive Eq. (23) from Eq. (11B.1) are written in Appendix A. Introducing in Eq. (23) the results obtained in the previous section, we finally obtain:

\[
f_R(0,0) \left\{ \frac{1}{4} \Delta g_{00}^{(4)} + 3(U + V) \Delta V + V \Delta U + \frac{1}{2} \partial i \partial j U - \delta^{ij} \partial i U \partial j U \right\} + 
\]

\[
+ f_{RR}(0,0) \left\{ -\frac{1}{2} \Delta^2 g_{00}^{(4)} + \frac{15}{4} (\Delta V)^2 - \frac{7}{2} \Delta U \Delta V + \frac{11}{2} (\Delta U)^2 + 6(U + 2V) \Delta^2 V - 4V \Delta^2 U - \Delta \partial i \partial j U + \delta^{ij} \left[ 3 \partial i U \partial j (3 \Delta V - \Delta U) + 8 \partial i U \partial j \Delta U + 4 \delta^{mn} \partial m \partial n U \right] \right\} +
\]

\[
+ 2 f_{R\mathcal{G}}(0,0) \left\{ 4(\Delta U + \Delta V) \Delta^2 U - 4 \Delta V \Delta^2 V + \delta^{ij} \left[ 3 \partial i \Delta U \partial j (\Delta U + 2 \Delta V) - \partial i \Delta V \partial j \Delta V \right] + 2 \delta^{ij} \delta^{mn} \left[ \partial i \partial m (3 \Delta V - \Delta U) \partial j \partial n U - \Delta \left( \partial i \partial m (U + V) \partial j \partial n (U + V) \right) \right] \right\} -
\]

\[
- f_{RR\mathcal{G}}(0,0) \left\{ \delta^{ij} \partial i (3 \Delta V - \Delta U) \partial j (3 \Delta V - \Delta U) + (3 \Delta V - \Delta U) \Delta (3 \Delta V - \Delta U) \right\} +
\]

\[
+ f_{\mathcal{G}}(0,0) \left\{ -\frac{1}{2} (\Delta U - \Delta V)^2 + 2 \delta^{im} \delta^{jn} \partial i \partial j (U - V) \partial m \partial n (U - V) \right\} = 2 \kappa^2 \rho \left( \Pi + \nu^2 - 4U \right).
\]

(24)

In the PPN limit, \textit{i.e.} at \( \mathcal{O}(6) \) order in the velocity, using the results obtained in Appendix B, Eq. (11B.1) reduces
to:
\[
\begin{align*}
&f_R(0,0) \left\{ -\frac{1}{2} \left[ \left(g_0^{(4)} R^{(2)} + g_0^{(2)} R^{(4)} + g_0^{(0)} R^{(6)} \right) + R_0^{(6)} \right] + \\
&+ f_G(0,0) \left\{ -\frac{1}{2} \left[ g_0^{(2)} R_0^{(2)} + g_0^{(0)} G_0^{(6)} \right] + 2 \left[ R_0^{(2)} R_0^{(4)} + R_0^{(4)} R_0^{(2)} \right] - \\
&- 4 \left[ 2g_0^{(0)00} R_0^{(2)} R_0^{(4)} + g_0^{(0)00} R_0^{(2)} \right] + g_0^{(0)ij} R_0^{(3)} R_0^{(3)} + 2 \left( R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} \right)^{(6)} - 4 \left( R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} \right)^{(6)} \right\} + \\
&+ f_{RR}(0,0) \left\{ -\frac{1}{2} \left[ g_0^{(2)} R_0^{(2)} + g_0^{(0)} R_0^{(2)} R_0^{(4)} \right] + R_0^{(2)} R_0^{(4)} + R_0^{(4)} R_0^{(2)} + \\
&+ g_0^{(0)} \left[ g_0^{(2)00} \left( \partial_0 \partial_0 R^{(2)} - \Gamma_0^{(2)00} \partial_0 \partial_0 R^{(2)} \right) + \\
&+ g_0^{(0)ij} \left( \partial_0 \partial_j \partial_0 R^{(4)} - \Gamma_0^{(2)00} \partial_0 \partial_j \partial_0 R^{(4)} \right) + \\
&+ g_0^{(2)} \left( \partial_0 \partial_j \partial_0 R^{(2)} - \Gamma_0^{(2)00} \partial_0 \partial_j \partial_0 R^{(2)} \right) + g_0^{(2)ij} \partial_0 \partial_j \partial_0 R^{(2)} \right\} + \\
&+ f_{GG}(0,0) \left\{ 2 \left[ g_0^{(0)} R_0^{(2)} - 2R_0^{(2)} \right] g_0^{(0)ij} \partial_0 \partial_j G^{(4)} + 4g_0^{(0)ij} g_0^{(0)ij} \left[ R_0^{(2)} - g_0^{(0)} R_0^{(2)} \right] \partial_0 \partial_j G^{(4)} \right\} + \\
\end{align*}
\]
And introducing in Eq. (25) the results obtained in the previous section, we finally obtain:

\[
\begin{align*}
&+ f_{RRR}(0, 0) \left\{ R^{(4)} \left( -\frac{1}{12} R^{(2)} + \frac{1}{2} R_{00}^{(2)} \right) - \delta_{00} \partial_{0} R^{(2)} + \Gamma_{00}^{(2)} \partial_{0} R^{(2)} \right\} - \partial_{0} R^{(2)} \partial_{0} R^{(2)} + \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\[+2\Delta U \partial_i U \partial_j V - \frac{1}{2} \partial_i V \partial_j \Delta g_{00}^{(4)} + 2 g_{kl} \partial_k \partial_l (3 \Delta V - \Delta U) + 2 \partial_i \Delta U \left( U \partial_j (V - 8U) - 8V \partial_j U \right) - \\
-2 (V + 3U) \partial_i V \partial_j (3 \Delta V - \Delta U) \right] + \\
+ \delta^{ij} \delta^{mn} \left[ - \frac{1}{2} \left( 2 \partial_i g_{mj}^{(4)} - \partial_m g_{ij}^{(4)} \right) \partial_n (3 \Delta V - \Delta U) - g_{lm} \partial_j \partial_n (3 \Delta V - \Delta U) - \\
4 \left( 2U + V \right) \partial_i \partial_m U - \partial_i V \partial_m U \right] \partial_j \partial_n \left] + \\
+ f_{\text{GG}}(0, 0) \left\{ 12 \left( \Delta U + \Delta V \right)^2 \Delta^2 U + 4 \left( 3 \Delta V - \Delta U \right) \left( \Delta U + \Delta V \right) \Delta^2 V + \\
+ 4 \left( \Delta U + \Delta V \right) \delta^{ij} \left[ 3 \partial_i \Delta U \partial_j (\Delta U + 2 \Delta V) - \partial_i \Delta V \partial_j \Delta V \right] + \\
+ 4 \delta^{ij} \delta^{mn} \left[ \Delta \left( \partial_i \partial_m (U + V) \partial_j \partial_n (U + V) \right) + \partial_i \partial_m U \partial_j \partial_n \left( -3 \left( \Delta U \right)^2 + \left( \Delta V \right)^2 - 6 \Delta U \Delta V + \\
+ 4 \delta^{kl} \delta^{rs} \partial_k \partial_r (U + V) \partial_i \partial_s (U + V) \right] \right\} + \\
+ f_{\text{RG}}(0, 0) \left\{ -\Delta G^{(6)} + \frac{12}{7} \left( \Delta U \right)^3 - 7 \Delta V \left( \Delta U \right)^2 + \frac{21}{7} \Delta U \left( \Delta V \right)^2 + \frac{24}{7} (\Delta V)^3 - 2 \left( \Delta U + \Delta V \right) \partial_h \partial_0 \Delta U + \\
+ \Delta^2 U \left[ -\Delta g_{00}^{(4)} - 2 \partial_0 \partial_0 U - 8(U + 3V) \Delta U - 4(U + 2V) \Delta V + 4 \delta^{ij} \left( \partial_i U \partial_j U + \partial_i V \partial_j V \right) \right] + \\
+ \Delta^2 V \left[ 3 \Delta g_{00}^{(4)} + 6 \partial_0 \partial_0 U - 4 \left( 3U - 2V \right) \Delta U - 8 \left( U + 7V \right) \Delta V - 12 \delta^{ij} \left( \partial_i U \partial_j U + \partial_i V \partial_j V \right) \right] + \\
+ \Delta U \left[ -\Delta^2 g_{00}^{(4)} + 2 \delta^{ij} \left\{ 5 \partial_i V \partial_j \left( 3 \Delta V - \Delta U \right) + 8 \partial_j U \partial_j U + \delta^{mn} \left[ \partial_i \partial_m U \partial_j \partial_n (U - V) - \\
- \partial_i \partial_m V \partial_j \partial_n (U + V) \right] \right\} \right] + \\
+ \Delta V \left[ -\Delta^2 g_{00}^{(4)} + 2 \delta^{ij} \left\{ 3 \partial_i V \partial_j \left( 3 \Delta V - \Delta U \right) + 8 \partial_j U \partial_j U + \delta^{mn} \left[ \partial_i \partial_m U \partial_j \partial_n \left( 7U - 3V \right) - \\
3 \partial_i \partial_m V \partial_j \partial_n (U - V) \right] \right\} \right] - \delta^{ij} \partial_i \partial_j \left[ - \left( \Delta U \right)^2 + \left( \Delta V \right)^2 - 6 \left( \Delta U \right) \left( \Delta V \right) + \\
+ 4 \delta^{mn} \delta^{kl} \partial_m \partial_k (U + V) \partial_n \partial_l (U + V) \right] + \\
+ 4(U + V) \delta^{ij} \left[ \partial_i \Delta V \partial_j \Delta V - 3 \partial_i \Delta U \partial_j \left( \Delta U + 2 \Delta V \right) + 2 \delta^{mn} \Delta \left( \partial_i \partial_m (U + V) \partial_j \partial_n (U + V) \right) \right] + \\
+ 2 \delta^{ij} \delta^{mn} \left[ \partial_i \partial_m U \left( \partial_j \partial_n \left( \Delta g_{00}^{(4)} - 4V \left( 3 \Delta V - \Delta U \right) + 2 \partial_0 \partial_0 U - 4 \Delta \Delta U \right) - \\
- 4 \partial_j V \partial_0 (3 \Delta V - \Delta U) - 8 \partial_j \partial_0 \partial_0 (3 \Delta V - \Delta U) \right) + \partial_i \partial_m \left( 3 \Delta V - \Delta U \right) \left( - \frac{1}{2} \Delta g_{00}^{(4)} - \\
- 2 \partial_j \partial_0 \partial_0 V - 2 \partial_i \partial_j \partial_0 U - 3 \partial_0 \partial_0 \partial_0 V + \frac{1}{2} \delta^{kl} \left[ \partial_k \left( \partial_i g_{mn}^{(4)} + \partial_n g_{ij}^{(4)} \right) - \partial_j \partial_0 g_{kl}^{(4)} \right] \right) \right] - \\
- 8 \delta^{ij} \delta^{mn} \delta^{kl} \partial_i \partial_m U \partial_j \partial_n \left( \partial_k \partial_l U \partial_0 U \right) \right] + \\
+ f_{\text{RRR}}(0, 0) \left\{ \Delta^2 U \left[ \frac{1}{2} \Delta g_{00}^{(4)} + \partial_0 \partial_0 U + 2 \left( 3V - U \right) \Delta U - 18 \Delta V - 2 \delta^{ij} \partial_i \partial_j U \right] + \\
+ \Delta^2 V \left[ - \frac{1}{2} \Delta g_{00}^{(4)} - \partial_0 \partial_0 U - 2 \left( 7V + 2U \right) \Delta U + 6 \left( 7V + 3U \right) \Delta V + 2 \delta^{ij} \partial_i \partial_j U \right] - \\
- \left( \frac{1}{2} \Delta g_{00}^{(4)} + \partial_0 \partial_0 U \right) \left( 3 \Delta V - \Delta U \right) - \frac{24}{7} \left( \Delta U \right)^3 + \frac{16}{7} \Delta V \left( \Delta U \right)^2 - \frac{48}{7} \Delta U \left( \Delta V \right)^2 + \\
+ \frac{64}{7} \left( \Delta V \right)^3 + \delta^{ij} \left[ \partial_i \left( 3 \Delta V - \Delta U \right) \left( 3 \Delta V - \Delta U \right) \partial_j V + 2 \left( 3 \Delta V + \Delta U \right) \partial_j \left( 3 \Delta V - \Delta U \right) - \\
- \partial_i \left( 3 \Delta V - \Delta U \right) \left( 3 \Delta V - \Delta U \right) \partial_j V \right] \right\} + \\
+ \left( \frac{1}{2} \Delta g_{00}^{(4)} + \partial_0 \partial_0 U \right) \left( 3 \Delta V - \Delta U \right) - \frac{24}{7} \left( \Delta U \right)^3 + \frac{16}{7} \Delta V \left( \Delta U \right)^2 - \frac{48}{7} \Delta U \left( \Delta V \right)^2 + \\
+ \frac{64}{7} \left( \Delta V \right)^3 + \delta^{ij} \left[ \partial_i \left( 3 \Delta V - \Delta U \right) \left( 3 \Delta V - \Delta U \right) \partial_j V + 2 \left( 3 \Delta V + \Delta U \right) \partial_j \left( 3 \Delta V - \Delta U \right) - \\
- \partial_i \left( 3 \Delta V - \Delta U \right) \left( 3 \Delta V - \Delta U \right) \partial_j V \right] \right\} \right\} \right\} \]
Using Eq. (8), (13) and Eq. (16), Eq. (27) is given by:

\[ 2\partial_j \left( \frac{1}{2} \Delta g^{(4)}_{00} + \partial_i \partial_i U - 2U \Delta U \right) + 8 \left( 3 \Delta V - \Delta U \right) \partial_i U \partial_j \Delta U \]

\[ + 4\delta^{ij} \epsilon^{mn} \left[ \partial_i \partial_m \left( 3 \Delta V - \Delta U \right) \partial_j \partial_n \left( 3 \Delta V - \Delta U \right) + \left( 3 \Delta V - \Delta U \right) \partial_i \partial_m U \partial_j \partial_n U \right] \]

\[ + f_{RR}(0,0) \left\{ \Delta^2 U \left[ -11 \left( \Delta U \right)^2 + 25 \left( \Delta V \right)^2 + 10 \Delta U \Delta V + 4 \delta^{ij} \epsilon^{mn} \partial_i \partial_m \left( U + V \right) \partial_j \partial_n \left( U + V \right) \right] + \right. \]

\[ + \Delta^2 V \left[ 9 \left( \Delta U \right)^2 - 39 \left( \Delta V \right)^2 + 26 \Delta U \Delta V - 12 \delta^{ij} \epsilon^{mn} \partial_i \partial_m \left( U + V \right) \partial_j \partial_n \left( U + V \right) \right] + \]

\[ + 2 \delta^{ij} \left[ \left( 3 \Delta V - \Delta U \right) \left( 3 \partial_i \Delta U \partial_j \left( \Delta U + 2 \Delta V \right) - \partial_i \Delta V \partial_j \Delta V \right) - \right. \]

\[ \left. \partial_i \left( 3 \Delta V - \Delta U \right) \partial_j \left( -3 \left( \Delta U \right)^2 + \left( \Delta V \right)^2 - 6 \Delta U \Delta V + \right) \right. \]

\[ + 4 \epsilon^{mn} \delta^{ij} \partial_m \left( U + V \right) \partial_l \partial_n \left( U + V \right) \right] + \]

\[ + f_{RRR}(0,0) \left\{ - \frac{1}{2} \left( 3 \Delta V - \Delta U \right)^2 \left( 3 \Delta^2 V - \Delta^2 U \right) - \delta^{ij} \left( 3 \Delta V - \Delta U \right) \partial_i \left( 3 \Delta V - \Delta U \right) \partial_j \left( 3 \Delta V - \Delta U \right) \right\} = \]

\[ 2 \kappa^2 \left\{ \rho \left[ v^4 - 3v \Delta v + 4U^2 + 2g^{(4)}_{00} + 2g^{(3)}_{00} v^4 + \Pi (v^2 - 4U) \right] + p (v^2 - 2U) \right\} \]

Then, Eq. (22), Eq. (24) and Eq. (26) constitute the Newtonian, PN and PPN limits, respectively, for the (0,0)-field equation of \( f(R, G) \)-gravity when the metric tensor Eq. (3) is assumed.

**2. The trace equation**

We proceed to find the Newtonian, PN and PPN limits for the trace equation. From Eq. (4) at \( O(0) \) order in the velocity we obtain again: \( f(0,0) = 0 \).

In the Newtonian limit, i.e. at \( O(2) \) order in the velocity, Eq. (4) reduces to:

\[ - f_R(0,0) R^{(2)} + 3 f_{RR}(0,0) \left( \nabla^2 R \right)^{(2)} = 2 \kappa^2 T^{(0)}. \]

Using Eq. (8), (13) and Eq. (16), Eq. (27) is given by:

\[ f_R(0,0) \left( 3 \Delta V - \Delta U \right) + 3 f_{RR}(0,0) \left( 3 \Delta^2 V - \Delta^2 U \right) = -2 \kappa^2 \rho. \]

In the PN limit, i.e. at \( O(4) \) order in the velocity, using the calculations given in Appendix A, Eq. (4) reduces to:

\[ + 3 f_{RR}(0,0) \left\{ g^{(0)00} \left( \partial_i \partial_{R(2)} - 3 \partial_i \partial_{R(2)} \right) \right\} + \]

\[ + 3 f_{RRR}(0,0) \left\{ \partial_i R(2) \partial_j R(2) + R(2) \partial_i \partial_j R(2) \right\} + \]

\[ + f_{RR}(0,0) \left\{ 3 g^{(0)ij} \partial_i \partial_j \partial_{R(4)} + 2 R(2) g^{(0)ij} \partial_i \partial_j R(2) - 4 g^{(0)ijm} g^{(0)ijn} R(2) \partial_m \partial_n R(2) \right\} - \]

\[ - f_R(0,0) R^{(4)} = 2 \kappa^2 \left\{ g^{(0)00} T^{(2)}_{00} + g^{(2)00} T^{(0)}_{00} + g^{(0)ij} T^{(2)}_{ij} \right\}. \]
Using in Eq. (29) the results obtained in the previous section, the PN limit for the trace equation is given by:

\[- f_R(0, 0) \left\{ \frac{1}{2} \Delta g^{(4)}_{00} + 2V (\Delta U - 3\Delta V) + \partial_0 \partial_0 U - 2\delta^{ij} (\partial_i U) (\partial_j U) - 2U \Delta U \right\} + \]

\[+ \quad 3f_{RR}(0, 0) \left\{ - \frac{1}{2} \Delta g^{(4)}_{00} + 2 (\Delta U)^2 - 2\Delta U \Delta V + 6 (\Delta V)^2 + \partial_0 \partial_0 (3\Delta V - 2\Delta U) + 2 (U - 2V) \Delta^2 U + \right. \]

\[+ 12V \Delta^2 U + \delta^{ij} [\partial_i (3V + U) \partial_j (3\Delta V - \Delta U) + 8\partial_i U \partial_j \Delta U + 4\delta^{mn} (\partial_i \partial_m U) (\partial_j \partial_n U)] - \frac{1}{2} \Delta g^{(4)}_{00} + 2 (\Delta U)^2 - 2\Delta U \Delta V + 6 (\Delta V)^2 + \partial_0 \partial_0 (3\Delta V - 2\Delta U) + 2 (U - 2V) \Delta^2 U + \]

\[+ 12V \Delta^2 U + \delta^{ij} [\partial_i (3V + U) \partial_j (3\Delta V - \Delta U) + 8\partial_i U \partial_j \Delta U + 4\delta^{mn} (\partial_i \partial_m U) (\partial_j \partial_n U)] \left\} \right. \]

\[+ \quad 3f_{RRR}(0, 0) \left\{ \delta^{ij} \partial_i (3\Delta V - \Delta U) \partial_j (3\Delta V - \Delta U) + (3\Delta V - \Delta U) \Delta (3\Delta V - \Delta U) \right\} + \]

\[+ \quad 2f_{RG}(0, 0) \left\{ 2(4\Delta U + 5\Delta V) \Delta^2 U + 6 (2\Delta U - \Delta V) \Delta^2 V + 3\delta^{ij} [3\partial_i (\Delta U) \partial_j (\Delta U + 2\Delta V) - \right. \]

\[\left. - \partial_i (\Delta V) \partial_j (\Delta V) - 6\delta^{ij} \delta^{mn} \Delta [\partial_i \partial_m (U + V) \partial_j \partial_n (U + V)] \right\} = 2\kappa^2 \left\{ \rho (\Pi - 2U) - 3\rho \right\} \]

In the PPN limit, i.e. at \(\mathcal{O}(6)\) order in the velocity, using the calculations given in Appendix B, Eq. (11) reduces to:

\[- f_R(0, 0) R^{(6)} + \]

\[+ \quad 3f_{RR}(0, 0) \left\{ g^{(0)00} \left[ \partial_0 \partial_0 R^{(4)} - \Gamma^{(3)}_{00} \partial_0 R^{(2)} - \Gamma^{(2)} \partial_0 \partial_0 R^{(2)} - \right. \right. \]

\[\left. \left. - \Gamma^{(4)}_{00} \partial_0 R^{(2)} \right] + g^{(2)00} \left[ \partial_0 \partial_0 R^{(2)} - \Gamma^{(2)} \partial_0 \partial_0 R^{(2)} + \Gamma^{(4)} \partial_0 \partial_0 R^{(2)} \right] \right\} + \]

\[+ \quad f_{RG}(0, 0) \left\{ R^{(2)} g^{(4)} + 2 R^{(2)} g^{(0)ij} \partial_i \partial_j R^{(2)} - 4 R^{(2)} (g^{(0)00})^2 \right\} \]

\[\left\{ \partial_0 \partial_0 R^{(2)} - \Gamma^{(2)} \partial_0 \partial_0 R^{(2)} \right\} + \]

\[+ 3 \left\{ g^{(0)00} \left[ \partial_0 \partial_0 G^{(4)} - \Gamma^{(2)} \partial_0 \partial_0 G^{(4)} \right] + g^{(0)ij} \left[ \partial_0 \partial_0 G^{(4)} \right] + g^{(0)ij} \partial_i \partial_j G^{(4)} \right\} + \]

\[\left\{ 2 R^{(2)} \right\} \left\{ g^{(0)00} \left[ \partial_0 \partial_0 R^{(2)} - \Gamma^{(2)} \partial_0 \partial_0 R^{(2)} \right] + g^{(0)ij} \left[ \partial_0 \partial_0 R^{(2)} \right] + \right. \]

\[\left. g^{(0)ij} \partial_i \partial_j G^{(4)} \right\} + \]

\[+ 4 R^{(2)} \left\{ g^{(0)00} \left[ \partial_0 \partial_0 R^{(2)} - \Gamma^{(2)} \partial_0 \partial_0 R^{(2)} \right] - \right. \]

\[\left. 4 \partial_0 \partial_0 \left\{ g^{(0)00} \left[ \partial_0 \partial_0 R^{(2)} - \right. \right. \]

\[\left. \left. - \Gamma^{(2)} \partial_0 \partial_0 R^{(2)} \right] \right\} - \]

\[\left. 4 \partial_0 \partial_0 \left\{ g^{(0)00} \left[ \partial_0 \partial_0 R^{(2)} - \Gamma^{(2)} \partial_0 \partial_0 R^{(2)} \right] - \right. \right. \]

\[\left. \left. - \Gamma^{(2)} \partial_0 \partial_0 R^{(2)} \right] \right\} \right\} + \]

\[+ \quad 2 f_{RG}(0, 0) \left\{ R^{(2)} \partial_0 \partial_j G^{(4)} - 2 R^{(2)} g^{(0)00} \partial_0 \partial_j G^{(4)} \right\} + \]

\[+ \quad f_{RRR}(0, 0) \left\{ \frac{1}{8} \left( R^{(2)} \right)^3 + 3 \left\{ g^{(0)00} \left[ \partial_0 R^{(2)} \partial_0 R^{(2)} + \Gamma^{(2)} \partial_0 R^{(2)} \right] + \right. \right. \]

\[\left. \left. + g^{(0)ij} \left[ 2 \partial_0 R^{(2)} \partial_j R^{(2)} + R^{(2)} \partial_0 \partial_j R^{(2)} \right] \right\} + \right. \]

\[\left. + g^{(2)ij} \left( \partial_0 R^{(2)} \partial_j R^{(2)} + R^{(2)} \partial_0 \partial_j R^{(2)} \right) \right\} + \]

\[+ \quad f_{RRR}(0, 0) \left\{ 3 \left\{ g^{(0)ij} \left( 2 \partial_0 R^{(2)} \partial_j R^{(2)} + G^{(4)} \partial_0 \partial_j R^{(2)} + R^{(2)} \partial_0 \partial_j G^{(4)} \right) \right\} + \right. \]

\[\left. + 2 R^{(2)} g^{(0)ij} \left[ \partial_0 R^{(2)} \partial_j R^{(2)} + R^{(2)} \partial_0 \partial_j R^{(2)} \right] - \right. \]

\[\left. - 4 R^{(2)} g^{(0)ij} g^{(0)mn} \left[ \partial_0 R^{(2)} \partial_m R^{(2)} + R^{(2)} \partial_0 \partial_m R^{(2)} \right] \right\} + \]
\[
\begin{align*}
&\quad + 3f_{RRR}(0, 0) \ g^{(0) ij} R^{(2) j} \left\{ \partial_i R^{(2)} \partial_j R^{(2)} + \frac{1}{2} R^{(2)} \partial_i \partial_j R^{(2)} \right\} = \\
&= 2k^2 \left\{ g^{(0) 00} T^{(4)}_{00} + g^{(2) 00} T^{(2)}_{00} + g^{(4) 00} T^{(0)}_{00} + 2g^{(3) 0i} T^{(2)}_{ij} + g^{(0) ij} T^{(4)}_{ij} + g^{(2) ij} T^{(2)}_{ij} \right\} \quad (31)
\end{align*}
\]

Introducing in Eq. (31) the results obtained in the previous section, we finally obtain:

\[- f_R(0, 0) \ R^{(6)} +
\]

\[+ f_{RR}(0, 0) 3 \left\{ \partial_0 \partial_0 \left[ \frac{1}{2} \Delta g_{00}^{(4)} - 2V (3 \Delta V - \Delta U) + \delta_{ij} (\partial_i U) (\partial_j U) - 2U \Delta U \right] +
+ 2U \partial_0 \partial_0 (3 \Delta V - \Delta U) + \partial_0 (3V + U) \partial_0 (3 \Delta V - \Delta U) - \Delta R^{(6)} + V \Delta^2 g_{00}^{(4)} +
+ 4V (2V - U) \Delta^2 U - 24V \Delta^2 U - 4V \Delta U \Delta (U - V) -
- 12V (\Delta V)^2 + 2g^{(3) 0i} \partial_0 \partial_i (3 \Delta V - \Delta U) + \partial_{ij} \left[ \left(- \frac{1}{2} \partial_i g_{00}^{(4)} - \partial_b g_{0i}^{(3)} \right) \partial_j (3 \Delta V - \Delta U) +
+ \partial_{ij} (U - V) \left( \frac{1}{2} \partial_j g_{00}^{(4)} - 2 \partial_j V (3 \Delta V - \Delta U) + \partial_j \partial_0 \partial_0 (3 \Delta V - \Delta U) \right) \right] -
- 2U (2V - U) \partial_0 \partial_0 (V \partial_j V) \partial_j (3 \Delta V - \Delta U) - 16V \partial_0 \partial_j \partial_j U + \partial_{ij} (3 \Delta V - \Delta U) -
- \delta_{ij} \delta_{mn} \left[ 8V \partial_i \partial_m U \partial_j \partial_n U + \frac{2}{5} (2 \partial_i g_{jn}^{(4)} - \partial_n g_{ij}^{(4)}) \partial_m (3 \Delta V - \Delta U) +
+ g_{mn}^{(4)} \partial_j \partial_n (3 \Delta V - \Delta U) + 4 \partial_i (U - V) \partial_m \partial_n \partial_j \partial_n U \right] \right\} +
\]

\[+ f_{RG}(0, 0) \ \left\{ \Delta^2 U \left[ \Delta g_{00}^{(4)} - 4 (U + 6V) \Delta U - 4 (2U + 9V) \Delta V + 2 \partial_0 \partial_0 (U + 2V) +
+ 4 \delta_{ij} \left( \partial_i V \partial_j (U + V) - \partial_i U \partial_j U \right) \right] + \Delta^2 U \left[ -3 \Delta g_{00}^{(4)} + 12 (U - 6V) \Delta U + 24V \Delta V -
- 2 \partial_0 \partial_0 (3U + 2V) + 4 \delta_{ij} (3 \partial_i U \partial_j U - \partial_i V \partial_j (U + V)) \right] - 3 \Delta G^{(6)} + 3 (\Delta U)^3 +
+ 3 \partial_0 \partial_0 \left[ -3 (\Delta U)^2 + (\Delta U)^2 + 2 \partial_0 \partial_0 (3 \Delta V - \Delta U) - 2 \delta_{ij} \delta_{mn} \partial_j \partial_i \partial_m \partial_n (U + V) \partial_j \partial_n (U + V) \right] -
- 21 (\Delta V)^3 + 2 \partial_0 \partial_0 \left[ \Delta g_{00}^{(4)} + 9 \Delta V - \Delta U \right] + 6 \delta_{ij} \delta_{mn} \partial_i \partial_m (U + V) \partial_j \partial_n (U + V) \right] +
+ 4 \delta_{ij} \left( \partial_i U - (5 \Delta U + 14 \Delta V) \partial_j \Delta U + 3 (2 \Delta V - \Delta U) \partial_j \Delta V +
+ 3 \delta_{mn} \delta_{kl} \partial_j \left[ \partial_m \partial_k (U + V) \partial_n \partial_l (U + V) \right] \right) - \partial_i V \left( -2 (2 \Delta U + 5 \Delta V) \partial_j \Delta U +
+ 6 (3 \Delta U - \Delta V) \partial_j \Delta V + 3 \delta_{mn} \delta_{kl} \partial_j \left[ \partial_m \partial_k (U + V) \partial_n \partial_l (U + V) \right] \right) +
+ \Delta g_{00}^{(3)} \partial_0 \partial_0 (3 \Delta V - \Delta U) - 3 \delta_{ij} \left( \partial_i \Delta U \partial_j \Delta V + \partial_i \Delta V \partial_j \Delta U \right) +
+ 4 \delta_{ij} \delta_{mn} \left( 6 \Delta V \left( \partial_i \partial_m (U + V) \partial_j \partial_n (U + V) \right) - 4 \Delta V \partial_i \partial_m \partial_j \partial_n U -
- \partial_i \partial_m (3 \Delta V - \Delta U) \right) \left\{ \frac{1}{2} \left( \partial_0 \partial_0 g_{mn}^{(3)} + \partial_0 \partial_0 g_{0i}^{(3)} + \Delta g_{jn}^{(4)} - \partial_j \partial_n g_{00}^{(4)} -
- \delta_{kl} \left( \partial_k \partial_j g_{in}^{(4)} + \partial_k \partial_n g_{ij}^{(4)} - \partial_j \partial_n g_{kl}^{(4)} \right) \right) + 2 \partial_0 \partial_0 \partial_0 U + 2 \partial_0 \partial_0 \partial_0 V +
+ \partial_j U \partial_n (U - V) + \partial_j V \partial_n (3V - U) \right\} \right\} +
\]
By introducing the new auxiliary functions $A$, we start by considering the set of equations given by Eq. (22) and Eq. (28), this is:

$$\Delta A + \Delta^2 B = -4\kappa^2 \rho,$$

$$-4\kappa^2 \rho = \Delta A + \Delta^2 B.$$  

Then, Eq. (28), Eq. (30) and Eq. (32) constitute the Newtonian, PN and PPN limit, respectively, for the trace equation of $f(R, \mathcal{G})$-gravity when the metric tensor given by Eq. (4) is assumed.

**III. SOLVING THE FIELD EQUATIONS IN THE NEWTONIAN LIMIT**

Our aim is now to solve the system of equations for the Newtonian limit of $f(R, \mathcal{G})$-gravity, i.e. the system constituted by Eq. (22) and Eq. (28), in the most general way. In order to do this, we will search for solutions in terms of Green’s functions (see [43]).

We start by considering the set of equations given by Eq. (22) and Eq. (28), this is:

$$f_R(0, 0) (3\Delta V - \Delta U) + 3f_{RR}(0, 0) (3\Delta^2 V - \Delta^2 U) = -2\kappa^2 \rho,$$

$$f_R(0, 0) (3\Delta V + \Delta U) + 2f_{RR}(0, 0) (3\Delta^2 V + \Delta^2 U) = -4\kappa^2 \rho.$$  

By introducing the new auxiliary functions $A = f_R(0, 0)(3V + U)$ and $B = 2f_{RR}(0, 0)(3V - U)$, we can write Eq. (33) as:

$$-4\kappa^2 \rho = \Delta A + \Delta^2 B,$$

$$-4\kappa^2 \rho = \frac{f_R(0, 0)}{f_{RR}(0, 0)} A + 3\Delta^2 B.$$

$$\rho = \frac{f_R(0, 0)}{f_{RR}(0, 0)} A + 3\Delta^2 B.$$
Considering now the new function $\Phi = A + \Delta B$, Eq. (34) reduces to:

$$-4\kappa^2 \rho = \Delta \Phi,$$

$$-4\kappa^2 \rho = \frac{f_R(0,0)}{f_R(0,0)} \Delta B + 3\Delta^2 B. \tag{35}$$

It is important to remark that Eq. (35) is a set of uncoupled equations. We are interested in the solution of the second equation in Eq. (35) in terms of the Green’s function $G$ defined by:

$$B = -4\kappa^2 C \int d^3x' G(x, x'), \rho(x') \tag{36}$$

where $C$ is a constant, which is introduced for dimensional reasons. Now the set of equations given by Eq. (35) is equivalent to:

$$-4\kappa^2 \rho = \Delta \Phi,$$

$$\frac{1}{C} \delta(x - x') = \frac{f_R(0,0)}{f_R(0,0)} \Delta x G(x, x') + 3\Delta^2 G(x, x'), \tag{37}$$

where $\delta(x - x')$ is the three-dimensional Dirac $\delta$-function. The general solutions of Eqs. (33) for $U(x)$ and $V(x)$, in terms of the Green’s function $G(x, x')$ and the function $\Phi(x)$, are:

$$U(x) = \frac{1}{2f_R(0,0)} \Phi(x) + 2\kappa^2 C \left( \frac{\triangle x}{f_R(0,0)} + \frac{1}{2f_R(0,0)} \right) \int d^3x' G(x, x') \rho(x'),$$

$$V(x) = \frac{1}{6f_R(0,0)} \Phi(x) + \frac{2}{3} \kappa^2 C \left( \frac{\triangle x}{f_R(0,0)} - \frac{1}{2f_R(0,0)} \right) \int d^3x' G(x, x') \rho(x'). \tag{38}$$

In summary, the functions $U(x)$ and $V(x)$, which are related with $g^{(2)}_{00}$ and $g^{(2)}_{ij}$, respectively, by Eq. (35) have been found in terms of the Green’s function $G(x, x')$ and the function $\Phi(x)$, giving in this way a general solution to the Newtonian limit for $f(R, G)$-gravity theories.

### IV. THE WEAK FIELD LIMIT IN TWO SPECIAL CASES: $f(R)$ AND $f(G)$ GRAVITIES

The results obtained previously will be used for two special cases: $f(R)$- gravity and $f(G)$-gravity, respectively.

#### A. $f(R)$ gravity

The starting action is given by:

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} f(R) + L_m \right). \tag{39}$$

In order to obtain the Newtonian, PN and PPN limits for this theory we will use the equations of the previous section considering the change: $f(R, G) \rightarrow f(R)$. The field equations for $f(R)$- gravity are obtained from Eq. (11A):

$$-\frac{1}{2} g_{\mu\nu} f(R) + f'(R) R_{\mu\nu} + g_{\mu\nu} \nabla^2 f(R) - \nabla_\mu \nabla_\nu f'(R) = 2\kappa^2 T_{\mu\nu}, \tag{40}$$

where primes denote the derivative with respect to Ricci scalar, while the trace equation is obtained from Eq. (11):

$$-2 f(R) + f'(R) R + 3 \nabla^2 f(R) = 2\kappa^2 T. \tag{41}$$

Before analyzing the Newtonian, PN and PPN limits for this theory, it is important to remark that at the lowest order in the velocity, i.e. $O(0)$-order, from Eq. (40) and Eq. (41), we obtain: $f(0) = 0$. 
1. The Newtonian limit

The Newtonian limit of \( f(R) \)-gravity corresponds to \( \mathcal{O}(2) \)-order for Eq.\((10)\) and Eq.\((11)\). The \((0,0)\)-field equation for \( f(R) \)-gravity at Newtonian order can be obtained from Eq.\((22)\) and it is given by:

\[
f'(0) 
(3 \Delta V + \Delta U) + 2 f''(0) \left(3 \Delta^2 V - \Delta^2 U\right) = -4 \kappa^2 \rho.
\]

The trace equation for \( f(R) \) modified gravity at Newtonian order can be obtained from Eq.\((28)\) and it is given by:

\[
f'(0) \left(3 \Delta V - \Delta U\right) + 3 f''(0) \left(3 \Delta^2 V - \Delta^2 U\right) = -2 \kappa^2 \rho.
\]

2. The Post Newtonian limit

In this case, the aim is to obtain Eq.\((10)\) and Eq.\((11)\) at \( \mathcal{O}(4) \)-order in the velocity.

For the \((0,0)\)-field equation, we use Eq.\((24)\). For \( f(R) \)-gravity it reduces to:

\[
f'(0) \left\{ \frac{1}{4} \Delta g_{00}^{(4)} + 3 (U + V) \Delta V + V \Delta U + \frac{1}{2} \partial_0 \partial_0 U - \delta^{ij} \partial_i U \partial_j U \right\} +
\]

\[
f''(0) \left\{ - \frac{1}{2} \Delta^2 g_{00}^{(4)} + \frac{15}{2} (\Delta V)^2 - \frac{7}{2} \Delta U \Delta V + \frac{11}{4} (\Delta U)^2 + 6 (U + 2 V) \Delta^2 V - 4 V \Delta^2 U - \partial_0 \partial_0 \Delta U +
\]

\[
+ \delta^{ij} \left[ 3 \partial_i V \partial_j (3 \Delta V - \Delta U) + 8 \partial_i U \partial_j \Delta U + 4 \delta^{mn} \partial_i \partial_m U \partial_j \partial_n U \right] \right\} -
\]

\[
f''(0) \left\{ \delta^{ij} \partial_i (3 \Delta V - \Delta U) \partial_j (3 \Delta V - \Delta U) + \left(3 \Delta V - \Delta U\right) \Delta (3 \Delta V - \Delta U) \right\} = 2 \kappa^2 \rho \left( \Pi + \nu^2 - 4 U \right).
\]

While for the trace equation, Eq.\((30)\) gives:

\[
-f'(0) \left\{ \frac{1}{2} \Delta g_{00}^{(4)} + 2 V (\Delta U - 3 \Delta V) + \partial_0 \partial_0 U - 2 \delta^{ij} (\partial_i U) (\partial_j U) - 2 U \Delta U \right\} +
\]

\[
+ 3 f''(0) \left\{ - \frac{1}{2} \Delta^2 g_{00}^{(4)} + 2 (\Delta U)^2 - 2 \Delta U \Delta V + 6 (\Delta V)^2 - \partial_0 \partial_0 (3 \Delta V - 2 \Delta U) + 2 (U - 2 V) \Delta^2 U +
\]

\[
+ 12 \Delta V \Delta^2 U + \delta^{ij} \left[ \partial_i (3 V + U) \partial_j (3 \Delta V - \Delta U) + 8 \partial_i U \partial_j \Delta U + 4 \delta^{mn} \left( \partial_i \partial_m U \right) (\partial_j \partial_n U) \right] \right\} -
\]

\[
- 3 f''(0) \left\{ \delta^{ij} \partial_i (3 \Delta V - \Delta U) \partial_j (3 \Delta V - \Delta U) + \left(3 \Delta V - \Delta U\right) \Delta (3 \Delta V - \Delta U) \right\} = 2 \kappa^2 \{ \rho (\Pi - 2 U) - 3 p \}
\]

3. The Parameterized Post Newtonian limit

Finally, the PPN limit corresponds to \( \mathcal{O}(6) \)-order for Eq.\((10)\) and Eq.\((11)\).

For the \((0,0)\)-field equation we obtain from Eq.\((26)\) the following result:

\[
f'(0) \left\{ \frac{1}{4} \left( R_{00}^{(6)} + \delta^{ij} R_{ij}^{(6)} \right) - \delta^{ij} g_{0i}^{(3)} R_{0j}^{(3)} - \frac{1}{2} \Delta V \left[ 3 \left( g_{00}^{(4)} + 4 U V - 4 V^2 \right) + \delta^{ij} g_{ij}^{(4)} \right] -
\]

\[
- (U + V) \left[ \frac{1}{2} 6 \partial_0 \partial_0 V - \Delta g_{00}^{(4)} + 2 \delta^{ij} \left( \Delta g_{ij}^{(4)} + \partial_0 \partial_0 g_{ij}^{(3)} \right) - 2 \delta^{ij} \delta^{mn} \partial_i \partial_m g_{jn}^{(4)} \right] +
\]

\[
+ 8 \Delta V \Delta U + 2 \Delta U \Delta V + \delta^{ij} \left[ \partial_i U \partial_j (U - V) + 2 \partial_i V \partial_j (3 V + U) \right] \right\} +
\]

\[
f''(0) \left\{ - \Delta R^{(6)} + (U + V) \Delta^2 g_{00}^{(4)} + \Delta^2 U \left( g_{00}^{(4)} - 4 U^2 + 4 U V + 8 V^2 \right) - 3 \Delta^2 V \left( g_{00}^{(4)} + 4 U V + 8 V^2 \right) -
\]

\[
- (\Delta U)^2 \left( \frac{1}{2} U + 7 V \right) - 3 (\Delta V)^2 \left( \frac{1}{2} U + V \right) + 2 \Delta U \Delta V \left( 2 U + 5 V \right) +
\]

\[
+ 3 (\Delta V - \Delta U) \left( \frac{1}{4} \Delta g_{00}^{(4)} + \frac{1}{2} \partial_0 \partial_0 U - \delta^{ij} \partial_i U \partial_j U \right) + 2 V \partial_0 \partial_0 \Delta U +
\]

\[
+ \delta^{ij} \left[ \frac{1}{2} \left( 2 \partial_i g_{0j}^{(3)} - \partial_i g_{0j}^{(2)} \right) \partial_0 (3 \Delta V - \Delta U) - \partial_i V \partial_0 \partial_0 \partial_j U + 2 (3 \Delta V - \Delta U) \partial_i V \partial_j V +
\]

\[
+ 2 \Delta U \partial_0 \partial_j V - \frac{1}{2} \partial_i V \partial_j \Delta g_{00}^{(4)} + 2 \partial_0 g_{ij}^{(3)} \partial_0 \partial_j (3 \Delta V - \Delta U) + 2 \partial_i \Delta U \left( \partial_j (V - 8 U) - 8 V \partial_j U \right) -
\]

\[
\right\}.
\]
\[-2 (V + 3U) \partial_i V \partial_j (3 \Delta V - \Delta U)] + \delta^{ij} \delta^{mn} \left[ -\frac{1}{2} \left( 2 \partial_i g_{m3}^{(4)} - \partial_m g_{ij}^{(4)} \right) \partial_n (3 \Delta V - \Delta U) - \right.
\]
\[-g_{im}^{(4)} \partial_j \partial_n (3 \Delta V - \Delta U) - 4 (2(U + V) \partial_i \partial_m U - \partial_i V \partial_m U) \partial_j \partial_n U \} \right] + \\
+ f'''(0) \left\{ \Delta^2 U \left[ \frac{1}{2} \Delta g_{00}^{(4)} + \partial_0 \partial_0 U + 2(3V - U) \Delta U - 18V \Delta V - 2 \delta^{ij} \partial_i \partial_j U \right] + \\
+ \Delta^2 V \left[ -\frac{1}{2} \Delta g_{00}^{(4)} - \partial_0 \partial_0 U - 2(7V + 2U) \Delta U + 6(7V + 3U) \Delta V + 2 \delta^{ij} \partial_i \partial_j U \right] - \\
- \left( \frac{1}{2} \Delta g_{00}^{(4)} + \partial_0 \partial_0 U \right) (3 \Delta V - \Delta U) - \frac{9V}{4} \left( \Delta V \right)^3 + \frac{15}{4} \left( \Delta V \right)^3 + \frac{111}{4} \Delta V (\Delta V)^2 - \frac{17}{4} \Delta U (\Delta V)^2 + \\
+ \delta^{ij} [\partial_i (3 \Delta V - \Delta U) (3 \Delta V - \Delta U) \partial_j U + 2 (3 \Delta V + U) \partial_j (3 \Delta V - \Delta U) - \\
2 \partial_j \left( \frac{1}{2} \Delta g_{00}^{(4)} + \partial_0 \partial_0 U - 2U \Delta U \right) \right] + 8 (3 \Delta V - \Delta U) \partial_i \partial_j U \right] + \\
+ 4 \delta^{ij} \delta^{mn} \left[ \partial_i \partial_m (3 \Delta V - \Delta U) \partial_j \partial_n (3 \Delta V - \Delta U) + (3 \Delta V - \Delta U) \partial_i \partial_m U \partial_j \partial_n U \right] \} + \\
+ f''''(0) \left\{ -\frac{1}{2} (3 \Delta V - \Delta U)^2 \left( 3 \Delta^2 V - \Delta^2 U \right) - \delta^{ij} (3 \Delta V - \Delta U) \partial_i (3 \Delta V - \Delta U) \partial_j (3 \Delta V - \Delta U) \right\} = \\
= 2 \kappa^2 \left\{ \rho \left[ V^4 - 3U V^2 + 4U^2 + 2g_{00}^{(4)} + 2g_{01}^{(3)} V + \Pi (V^2 - 4U) \right] + p (V^2 - 2U) \right\}
\]
And Eq. (32) implies that the trace equation is given by:
\[- f'(0) R^{(6)} + \\
+ 3 f'''(0) \left\{ \partial_0 \partial_0 \left[ \frac{1}{2} \Delta g_{00}^{(4)} - 2 V (3 \Delta V - \Delta U) + \partial_0 \partial_0 U - 2 \delta^{ij} (\partial_i U) (\partial_j U) - 2U \Delta U \right] + \\
+ 2U \partial_0 \partial_0 (3 \Delta V - \Delta U) + \partial_0 (3V + U) \partial_0 (3 \Delta V - \Delta U) - \Delta R^{(6)} + V \Delta^2 g_{00}^{(4)} + \\
+ 4V (2V - U) \Delta^2 U - 24V \Delta^2 V - 4V \Delta U \partial (U - V) - 12V (\Delta V)^2 + \\
+ 2g_{01}^{(3)} \partial_0 \partial_1 (3 \Delta V - \Delta U) + \delta^{ij} \left[ -\left( \frac{1}{2} \partial_i g_{00}^{(4)} - \partial_0 g_{01}^{(3)} \right) \partial_j (3 \Delta V - \Delta U) + \\
+ \partial_i (U - V) \left( \frac{1}{2} \partial_j \Delta g_{00}^{(4)} - 2V \partial_j (3 \Delta V - \Delta U) + \partial_j \partial_0 \partial_0 U - 2 \partial_j U \Delta U - 2U \partial_j U \right) - \\
- 2 ((2V - U) \partial_1 U + V \partial_1 V) \partial_j (3 \Delta V - \Delta U) - 16V \partial_1 U \partial_j \Delta U + g_{01}^{(3)} \partial_0 (3 \Delta V - \Delta U) \right] - \\
- \delta^{ij} \delta^{mn} \left[ 8 \partial_0 \partial_m U \partial_j \partial_n U + \frac{1}{2} \left( 2 \partial_j g_{m3}^{(4)} - \partial_m g_{ij}^{(4)} \right) \partial_n (3 \Delta V - \Delta U) + \\
+ g_{im}^{(4)} \partial_j \partial_n (3 \Delta V - \Delta U) + 4 \partial_i (U - V) \partial_m U \partial_j \partial_n U \right] \} + \\
+ f''''(0) \left\{ -\frac{1}{2} (3 \Delta V - \Delta U) \Delta^2 g_{00}^{(4)} + 3 \Delta^2 U \left[ \frac{1}{2} \Delta g_{00}^{(4)} + \partial_0 \partial_0 U - 2(U + 3V) (3 \Delta V - \Delta U) - \\
- 2 \delta^{ij} \partial_i \partial_j U - 2U \Delta U \right] - \frac{37}{6} (\Delta U)^3 + \frac{31}{4} \left( \Delta U \right)^3 + \frac{17}{2} \Delta V (\Delta V)^2 + \frac{111}{4} (\Delta V)^3 + \\
+ 9 \Delta^2 V \left[ -\frac{1}{2} \Delta g_{00}^{(4)} - \partial_0 \partial_0 U + 6V (3 \Delta V - \Delta U) + 2 \delta^{ij} \partial_i U \partial_j U + 2U \Delta U \right] - \\
+ 3 (3 \Delta V - \Delta U) \partial_0 \partial_0 (3 \Delta V - 2 \Delta U) + 3 \partial_0 (3 \Delta V - \Delta U) \partial_0 (3 \Delta V - \Delta U) + \\
+ 3 \delta^{ij} \left[ 8 (3 \Delta V - \Delta U) \partial_i \partial_j \Delta U + \left( -\partial_i \Delta g_{00}^{(4)} + (3 \Delta V - \Delta U) \partial_i (U + 7V) + \\
\right]
\]
While the trace equation can be obtained from Eq.(28) and it reduces to:

\[ f \] is given by Eq.(44) and Eq.(45); and the PPN limit is given by Eq.(46) and Eq.(47), respectively.

For the case of the trace equation, Eq.(30) reduces to:

\[ \frac{1}{2} \frac{\partial}{\partial \rho} \left( \frac{3 \Delta V}{V} \right) + \frac{3}{2} \delta^{ij} \delta^{mn} \left[ \left( 3 \Delta V - \Delta U \right) \partial_i \partial_m U + 2 \partial_i U \partial_m \left( 3 \Delta V - \Delta U \right) \partial_j \partial_n U \right] - 3f'''(0) \left( 3 \Delta V - \Delta U \right) \left\{ \partial_i \left( 3 \Delta V - \Delta U \right) \partial_j \left( 3 \Delta V - \Delta U \right) + \frac{1}{2} \left( 3 \Delta V - \Delta U \right) \left( 3 \Delta^2 V - \Delta^2 U \right) \right\} = 2 \kappa^2 \left\{ \rho \left[ -\nabla^2 (U + 2V) + g_{00}^{(4)} + 2g_{0i}^{(3)} u^i - 2U \Pi \right] - 2U \rho \right\}. \]  

Summing up, the Newtonian limit of \( f(R) \) modified gravity theories is given by Eq.(12) and Eq.(13); the PN limit is given by Eq.(14) and Eq.(15); and the PPN limit is given by Eq.(16) and Eq.(17), respectively.

B. \( f(G) \) gravity

Let us consider now the following action:

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ R + f(G) \right] + \mathcal{L}_{\text{matter}} \right\}, \]  

where standard GR is obviously recovered for \( f(G) \to 0 \). We proceed in the same way as for \( f(R) \)-gravity, but in this case we have \( f(R,G) \to R + f(G) \).

The field equations can be directly obtained from Eq. (11a):

\[ - \frac{1}{2} g_{\mu\nu} \left[ R + f(G) \right] + R_{\mu\nu} + 2f'(G) R_{\mu\nu} - 4f'(G) R_{\alpha\beta} R^\alpha_{\mu\nu} + 2f'(G) R_{\alpha\beta\mu\nu} R^{\alpha\beta}_{\rho\lambda} - 2f'(G) R_{\mu\nu\rho\sigma} R^{\rho\sigma} + 2g_{\mu\nu} R \nabla^2 f'(G) - 4g_{\mu\nu} R_{\rho\sigma} \nabla^\rho \nabla^\sigma f'(G) - 2R \nabla_\mu \nabla_\nu f'(G) - 4R_{\mu\nu} \nabla^2 f'(G) + 4R_{\mu\nu} \nabla_\rho \nabla^\rho \nabla^\sigma f'(G) + 4R_{\mu\nu} \nabla^\rho \nabla^\sigma f'(G) + 4R_{\mu\nu} \nabla^\rho \nabla^\sigma f'(G) = 2\kappa^2 T_{\mu\nu}, \]  

while the trace equation is obtained from Eq.(43):

\[ - R - 2f(G) + 2f'(G) g_{\mu\nu} + 2R \nabla^2 f'(G) - 4R_{\mu\nu} \nabla^\rho \nabla^\sigma f'(G) = 2\kappa^2 T. \]  

From the lowest order we obtain again: \( f(0) = 0 \).

1. The Newtonian limit

In this case, the \((0,0)\)-field equation obtained from Eq.(44) is given by:

\[ \Delta U + 3 \Delta V = -4\kappa^2 \rho. \]  

While the trace equation can be obtained from Eq.(28) and it reduces to:

\[ \Delta U - 3 \Delta V = 2\kappa^2 \rho. \]

2. The Post Newtonian limit

The PN limit for the \((0,0)\)-field equation can be calculated from Eq.(44) and it reduces to:

\[ \frac{1}{2} \Delta g_{00}^{(4)} + 3(U + V) \Delta V + V \Delta U + \frac{1}{2} \partial_\rho \partial_\mu U - \delta^{ij} \partial_\rho \partial_\mu \partial_i \partial_j U + f'(0) \left\{ -\frac{1}{2} \left( \Delta U - \Delta V \right)^2 + 2 \delta^{im} \delta^{jn} \partial_i U \partial_m \partial_j (U - V) \partial_n (U - V) \right\} = 2\kappa^2 \rho \left( \Pi + \nabla^2 - 4U \right) \]  

For the case of the trace equation, Eq.(30) reduces to:

\[ - \frac{1}{2} \Delta g_{00}^{(4)} + 2V (\Delta U - 3 \Delta V) + \partial_\rho \partial_\mu U - \delta^{ij} (\partial_i U) (\partial_j U) - 2U \Delta U = 2\kappa^2 \left\{ \rho \left( \Pi - 2U \right) - 3 \rho \right\} \]
3. The Parameterized Post Newtonian limit

The PPN limit for the \((0,0)\)-field equation is obtained from Eq. (34) and can be written as:

\[
\begin{align*}
\frac{1}{2} \left( R^{(6)}_{ij} + \delta^{ij} R^{(6)}_{ij} \right) - \delta^{ij} \left( \frac{1}{2} \Delta g^{(3)}_{0j} R^{(3)}_{ij} \right) - \frac{1}{2} \Delta V \left[ 3 \left( g^{(4)}_{00} + 4 UV - 4V^2 \right) + \delta^{ij} g^{(4)}_{ij} \right] & + \\
+ \delta^{ij} \left[ \partial_i U \partial_j (U - V) + 2 \partial_i V \partial_j (3V + U) \right] + 8V \Delta V + 2U \Delta U - \\
-(U + V) \left[ \frac{1}{2} \left( 6 \partial_0 \partial_0 V - \Delta g^{(4)}_{00} + 2 \delta^{ij} \left( \Delta g^{(4)}_{ij} + \partial_i \partial_j g^{(3)}_{0j} \right) - 2 \delta^{ij} \delta^{mn} \partial_i \partial_m g^{(4)}_{jn} \right) \right] & + \\
+ f'(0) \left\{ -\frac{3}{2} g^{(6)} + (8V - 7U) \left( \Delta U \right)^2 + U \left( \Delta V \right)^2 + 2 \left( 4V - 3U \right) \Delta U \Delta V + 2U \left[ \Delta g^{(4)}_{00} + 2 \partial_0 \partial_0 (U + 2V) - \\
-4 \delta^{ij} \partial_i \partial_j (U + V) \right] + \Delta V \left[ \Delta g^{(3)}_{00} + 6 \partial_0 \partial_0 (U + 2V) + 4 \delta^{ij} \left( \partial_i \partial_j g^{(3)}_{0j} + \partial_i U \partial_j (U - 2V) \right) \right] + \\
+ \delta^{ij} \left[ \left( \Delta g^{(3)}_{0i} + 4 \partial_0 \partial_i \right) \Delta g^{(3)}_{0j} - 8 \partial_0 \partial_i V \partial_0 \partial_j V \right] + 4 \delta^{ij} \delta^{mn} \left[ U \partial_i \partial_m (U + V) \partial_j \partial_n (U + V) - \\
- \partial_i \partial_m g^{(3)}_{0j} \partial_0 \partial_n V + \frac{1}{2} \left[ \partial_0 \left( \partial_n g^{(3)}_{0j} + \partial_j g^{(3)}_{0j} - 2 \partial_j \partial_0 V \right) \right] + \delta^{ij} \left( \partial_i \partial_j g^{(4)}_{0j} + \partial_i \partial_j g^{(4)}_{0j} \right) - \\
- \Delta g^{(4)}_{ij} - \Delta g^{(4)}_{00} \right] - 2V \partial_i \partial_j V - (U + 2V) \partial_i \partial_j U + \partial_i U \partial_j (U - V) - \partial_j V \partial_n (3V + U) - \\
- \delta^{ij} \partial_i V \partial_j (U + V) + 2V \Delta V \right] \right\} + 2 \delta^{ij} \delta^{mn} \delta^{kl} \left( \partial_m g^{(3)}_{0i} + \partial_0 g^{(3)}_{0i} \right) \partial_k \left( \partial_n g^{(3)}_{0j} + \partial_j g^{(3)}_{0j} \right) + \\
+ f''(0) \left\{ 12 \left( \Delta U + \Delta V \right)^2 \Delta^2 U + 4 \left( 3 \Delta V - \Delta U \right) \left( \Delta U + \Delta V \right) \Delta^2 V + \\
+ 4 \left( \Delta U + \Delta V \right) \delta^{ij} \left[ 3 \partial_i \Delta U \partial_j \left( \Delta U + 2 \Delta V \right) - \partial_i \Delta V \partial_j \Delta V \right] + \\
+ 4 \delta^{ij} \delta^{mn} \left[ \Delta \left( \partial_i \partial_m (U + V) \partial_j \partial_n (U + V) \right) + \partial_i \partial_m U \partial_j \partial_n \left( -3 \left( \Delta U \right)^2 + \left( \Delta V \right)^2 - 6 \Delta U \Delta V + \\
+ 4 \delta^{ij} \delta^{mn} \partial_i \partial_j (U + V) \partial_i \partial_j (U + V) \right) \right] \right\} = \\
= 2 \kappa^2 \left\{ \rho \left[ v^4 - 3Uv^2 + 2U^2 + 2g^{(4)}_{00} + 2g^{(3)}_{0i} v^i + V \left( \nabla^2 - 4U \right) \right] + p \left( v^2 - 2U \right) \right\}.
\end{align*}
\]

And for the trace equation, Eq. (34) reduces to:

\[
- R^{(6)} + 2 f''(0) \left( \Delta U - \Delta V \right) \Delta \left\{ -3 \left( \Delta U \right)^2 + \left( \Delta V \right)^2 - 6 \left( \Delta U \right) \left( \Delta V \right) + 4 \delta^{im} \delta^{jn} \partial_i \partial_j (U + V) \partial_m \partial_n (U + V) \right\} = \\
= 2 \kappa^2 \left\{ \rho \left[ -v^2 \left( U + 2V \right) + g^{(4)}_{00} + 2g^{(3)}_{0i} v^i - 2U \right] \right\}.
\]

Summarizing, the Newtonian limit of \( f(G) \)- gravity theories is given by Eq. (31) and Eq. (52); the PN limit is given by Eq. (33) and Eq. (34); and the PPN limit is given by Eq. (35) and Eq. (36), respectively.

V. CONCLUSIONS

Achieving the weak field limit is the straightforward approach to compare any theory of gravity with GR. In a wide sense, this is the main consistency check in order to establish if a given theory of gravity can reproduce or not the classical experimental tests of GR and then address further phenomena and/or anomalous experimental data.

This philosophy has recently become a paradigm due to the fact that alternative theories of gravity and extended theories of gravity are aimed to reproduce GR results from one side (Solar System and laboratory scales) and address
a huge amount of phenomena starting from quantization and renormalization of gravitational interaction (ultraviolet scales) up to the large scale structure and the cosmological accelerated behavior of the Hubble flow (infrared scales).

The weak field limit approach is based on the theory of perturbations in terms of powers $c^{-2}$ where $c$ is the speed of light. Newtonian, Post Newtonian and further limits strictly depends on the accuracy of such a development and on the identification of suitable parameters that have to be confronted with the experiments.

Here we have considered a wide class of these theories, specifically the $f(R,G)$ gravity consisting of analytic models that are functions of the Ricci scalar $R$ and the Gauss-Bonnet invariant $G$. In this context, we have worked out the Newtonian, the Post Newtonian and the Parameterized Post Newtonian limits.

The main result of the approach is that new features come out with respect to GR and, due to fourth-order field equations, at least two gravitational potentials have to be considered. This fact could be extremely important in order to deal with realistic self-gravitating structures since they could results more stable and without singularities.

Specifically, we found some general solutions in the Newtonian limit and developed in details (Newtonian, Post Newtonian e Parameterized Post Newtonian limits) the cases of $f(R)$ gravity and $R + f(G)$ gravity. In particular, the presence of the topological terms seems relevant to remove singularities and give rise to stable structures. In a forthcoming paper, we will phenomenologically confront these theoretical results with experimental data.

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Appendix A: The Post Newtonian limit

In this appendix, we present the calculations needed to obtain the PN limit for the field equations and the trace equation. First, we write some calculations that will be needed after:

\[
(\nabla^2 f^*(R, G))^{(2)} = g^{(0)ij} f_R^*(0,0) \partial_i \partial_j R^{(2)}
\]

\[
(\nabla^2 f^*(R, G))^{(4)} = g^{(0)00} \left\{ f_R^*(0,0) \partial_0 \partial_0 R^{(2)} - (\nabla^2)^{(0)} \partial_0 f_R^*(0,0) \partial_0 R^{(2)} \right\} +
+ g^{(0)ij} \left[ f_{RR}^*(0,0) \partial_i \partial_j R^{(2)} + f_R^*(0,0) \partial_i \partial_j R^{(4)} + f_{RR}^*(0,0) R^{(2)} \partial_i \partial_j R^{(2)} +
+ f_R^*(0,0) \partial_i \partial_j G^{(4)} - \Gamma^{(2)k}_{ij} f_R^*(0,0) \partial_k R^{(2)} \right] +
+ g^{(2)ij} f_R^*(0,0) \partial_i \partial_j R^{(2)}
\]

\[
(\nabla^2 f^*(R, G))^{(6)} = g^{(0)00} \left\{ f_{RR}^*(0,0) \partial_0 \partial_0 R^{(2)} + f_R^*(0,0) \partial_0 \partial_0 R^{(4)} + f_{RR}^*(0,0) R^{(2)} \partial_0 \partial_0 R^{(2)} +
+ f_R^*(0,0) \partial_0 \partial_0 G^{(4)} - \Gamma^{(3)0}_{00} f_R^*(0,0) \partial_0 R^{(2)} -
- \Gamma^{(4)0}_{00} f_R^*(0,0) \partial_0 R^{(2)} -
- \Gamma^{(2)0}_{ij} f_R^*(0,0) \partial_i \partial_j R^{(2)} \right\} +
+ g^{(2)00} \left\{ f_R^*(0,0) \partial_0 \partial_0 R^{(2)} - (\nabla^2)^{(0)} \partial_0 f_R^*(0,0) \partial_0 R^{(2)} \right\} +
+ 2g^{(3)0i} f_R^*(0,0) \partial_0 \partial_i R^{(2)} +
+ g^{(0)ij} \left[ 2f_{RR}^*(0,0) \partial_i \partial_j R^{(2)} + f_{RR}^*(0,0) R^{(2)} \partial_i \partial_j R^{(2)} + 2f_{RG}^*(0,0) \partial_i \partial_0 R^{(2)} \partial_j G^{(4)} +
+ f_R^*(0,0) \partial_i \partial_j \Gamma^{(2)} - \Gamma^{(4)} \partial_i \partial_j R^{(2)} +
+ \left[ \frac{1}{2} f_{RR}^*(0,0) R^{(2)^2} + f_R^*(0,0) R^{(4)} + f_{RG}^*(0,0) G^{(4)} \right] \partial_i \partial_j R^{(2)} +
+ f_R^*(0,0) \partial_i \partial_j G^{(4)} - \Gamma^{(3)0}_{ij} f_R^*(0,0) \partial_0 R^{(2)} -
- \Gamma^{(4)0}_{ij} f_R^*(0,0) \partial_0 R^{(2)} \right] +
+ g^{(2)ij} \left[ f_R^*(0,0) \partial_i \partial_j R^{(2)} + f_R^*(0,0) \partial_i \partial_j R^{(4)} + f_{RR}^*(0,0) R^{(2)} \partial_i \partial_j R^{(2)} +
+ f_R^*(0,0) \partial_i \partial_j G^{(4)} - \Gamma^{(2)0}_{ij} f_R^*(0,0) \partial_0 R^{(2)} \right] +
+ g^{(4)ij} f_R^*(0,0) \partial_i \partial_j R^{(2)}
\]

\[
\left( R_{\alpha\beta0} R^{\alpha\beta0} \right)^{(4)} = 2 \delta^{ij} \delta^{mn} (\partial_i \partial_m U) (\partial_j \partial_n U)
\]

\[
\left( R_{\alpha\beta0} R^{\alpha\beta0} \right)^{(6)} = 4 \Delta U (\partial_0 \partial_0 V + \delta^{ij} \partial_i U \partial_j V) + 2 \delta^{ij} \delta^{mn} \partial_i \partial_m U \left[ \partial_0 \partial_0 g^{(3)}_{\alpha \beta} + \partial_0 \partial_0 g^{(3)}_{\alpha \beta} - \partial_0 g^{(4)}_{\alpha \beta} +
+ 2 \partial_0 U \partial_0 (U - V) - 2 \partial_0 U \partial_0 V + (U - 2 V) \partial_0 \partial_0 U \right] -
- \frac{1}{4} \delta^{ij} \delta^{mn} \delta^{kl} \partial_k \left( \partial_m g^{(3)}_{\alpha \beta} - \partial_i g^{(3)}_{\alpha \beta} \right) \partial_l \left( \partial_n g^{(3)}_{\alpha \beta} - \partial_j g^{(3)}_{\alpha \beta} \right)
\]
\[-2\delta^{ij}\delta^{mn}\partial_i\partial_m g_{0j}^{(3)}\partial_0 \partial_j V + 2\delta^{ij}\partial_0 \partial_i V \left(\triangle g_{0j}^{(3)} - 2\partial_0 \partial_j V\right)\]

\[(R_{0\alpha\beta\sigma} R^{\alpha\beta\sigma})^{(4)} = - (\triangle U) (\triangle V)\]

\[(R_{0\alpha\beta\sigma} R^{\alpha\beta\sigma})^{(6)} = \triangle V \left\{ \frac{1}{2} \triangle g_{00}^{(4)} - 3 \partial_0 \partial_0 V + 4V \triangle U - \delta^{ij} \left[ \frac{1}{2} \partial_0 \left( \partial_j g_{0j}^{(3)} + \partial_j g_{0j}^{(3)} \right) + 2\partial_i U \partial_j (2U - V) \right] \right\} +
\quad \delta^{ij}\delta^{mn}\partial_i\partial_m U \left\{ \frac{1}{2} \left[ \partial_0 \left( \partial_n g_{0j}^{(3)} + \partial_j g_{0n}^{(3)} + 2\delta_{jn} \partial_0 V \right) \right] -
\quad \delta^{kl} \left( \partial_k g_{in}^{(4)} + \partial_n g_{jk}^{(4)} - \partial_j \partial_n g_{ik}^{(4)} + \triangle g_{jn}^{(4)} - \partial_j \partial_n g_{00}^{(4)} \right) + 2\partial_i \partial_n V + 2U \partial_j \partial_n U +
\quad + \delta_{jn} \left( \delta^{kl} \partial_l \partial_k \left( U + V \right) + 2V \triangle V \right) \right\}

\[(\nabla^\rho \nabla^\alpha f_{GR}(R, G))^{(2)} = g^{(0)\rho\gamma} g^{(0)\sigma\gamma} f_{GR}(0, 0) \partial_\rho \partial_{\gamma} R^{(2)}\]

\[(\nabla^\rho \nabla^\alpha f_{GR}(R, G))^{(3)} = \left\{ g^{(0)\rho\gamma} g^{(0)\sigma\gamma} + g^{(0)\rho\gamma} g^{(0)\sigma\gamma} \right\} f_{GR}(0, 0) \partial_\rho \partial_{\gamma} R^{(2)}\]

\[(\nabla^\rho \nabla^\alpha f_{GR}(R, G))^{(4)} = g^{(0)\rho\gamma} g^{(0)\sigma\gamma} \left\{ f_{GR}(0, 0) \partial_\rho \partial_{\gamma} R^{(2)} - \Gamma_{i00}^{(2)} f_{GR}(0, 0) \partial_i R^{(2)} \right\} +
\quad + g^{(0)\rho\gamma} g^{(0)\sigma\gamma} \left\{ f_{GR}(0, 0) \partial_\rho \partial_{\gamma} R^{(2)} + f_{GR}(0, 0) \partial_\rho \partial_{\gamma} R^{(4)} \right\} +
\quad + f_{RR}(0, 0) R^{(2)} \partial_\rho \partial_{\gamma} R^{(2)} + f_{GR}(0, 0) \partial_\rho \partial_{\gamma} G^{(4)} - \Gamma_{i00}^{(2)} f_{GR}(0, 0) \partial_i R^{(2)} \}
\quad +
\quad + \left\{ g^{(0)\rho\gamma} g^{(0)\sigma\gamma} + g^{(0)\rho\gamma} g^{(0)\sigma\gamma} \right\} f_{GR}(0, 0) \partial_\rho \partial_{\gamma} R^{(2)}\]

For the (0, 0)-field equation the following results are necessary:

\[\left( -\frac{1}{2} g_{00} f(R, G) \right)^{(4)} = -\frac{1}{2} g_{00}^{(4)} f(0, 0) - \frac{1}{2} g_{00}^{(2)} f(R, 0) R^{(2)} -
\quad - \frac{1}{2} g_{00}^{(4)} \left[ \frac{1}{2} f_{RR}(0, 0) R^{(2)} + f_{RR}(0, 0) R^{(4)} + f_{GR}(0, 0) G^{(4)} \right] \]

\[(f_{R}(R, G) R_{00})^{(4)} = f_{R}(0, 0) R_{00}^{(4)} + f_{RR}(0, 0) R^{(2)} R_{00}^{(2)}\]

\[(g_{00} \nabla^2 f_{R}(R, G))^{(2)} = g_{00}^{(2)} g^{(0)ij} f_{RR}(0, 0) \partial_i \partial_j R^{(2)} + g_{00}^{(2)} \left[ g^{(0)ij} f_{RR}(0, 0) \left( \partial_0 \partial_j R^{(2)} - \Gamma_{i00}^{(2)} \partial_i R^{(2)} \right) \right] +
\quad + g^{(0)ij} \left( f_{RR}(0, 0) \left\{ \partial_i \partial_j R^{(2)} + R^{(2)} \partial_i \partial_j R^{(2)} \right\} + f_{GR}(0, 0) \partial_i \partial_j G^{(4)} \right) +
\quad + f_{RR}(0, 0) \left\{ \partial_i \partial_j R^{(4)} - \Gamma_{i00}^{(2)} \partial_i R^{(2)} \right\} + g^{(2)ij} f_{RR}(0, 0) \partial_i \partial_j R^{(2)}\]

\[-\nabla_0 \nabla_0 f_{R}(R, G)\]^{(4)} = - f_{RR}(0, 0) \left( \partial_0 \partial_0 R^{(2)} - \Gamma_{i00}^{(2)} \partial_i R^{(2)} \right)

\[2 f_{GR}(R, G) R_{00}^{(4)} = 2 f_{GR}(0, 0) R^{(2)} R_{00}^{(2)}\]

\[(-4 f_{GR}(R, G) g^{\alpha\beta} R_{0\alpha\beta} R_{0\alpha\beta})^{(4)} = -4 f_{GR}(0, 0) R_{00}^{(2)} R_{00}^{(2)}\]

\[2 f_{GR}(R, G) R_{\alpha\beta\rho} R^{\alpha\beta\rho}^{(4)} = 2 f_{GR}(0, 0) \left( R_{\alpha\beta\rho} R^{\alpha\beta\rho} \right)^{(4)}\]
\[-4f_G(R, \mathcal{G})R_{0\rho\sigma}R^{\rho\sigma}\]^{(4)} = -4f_G(0, 0) (R_{0\rho\sigma} R^{\rho\sigma})^{(4)}

\[(2g_{00}R \nabla^2 f_G(R, \mathcal{G}))^{(4)} = 2g_{00}^{(0)} R^{(2)} g^{(0)}_{\, \, \, ij} f_{GR}(0, 0) \partial_i \partial_j R^{(2)}\]

\[-4g_{00}R_{\rho\sigma} \nabla^\rho \nabla^\sigma f_G(R, \mathcal{G})\]^{(4)} = -4g_{00}^{(0)} R^{(2)}_{\, \, \, ij} g^{(0)}_{\, \, \, kl} f_{GR}(0, 0) \partial_i \partial_j R^{(2)}

\[-2R \nabla_0 \nabla_0 f_G(R, \mathcal{G})\]^{(4)} = 0

\[-4R_{00} \nabla^2 f_G(R, \mathcal{G})\]^{(4)} = -4R_{00}^{(2)} g^{(0)}_{\, \, \, ij} f_{GR}(0, 0) \partial_i \partial_j R^{(2)}

\[8R_{0\rho} \nabla^\rho \nabla_0 f_G(R, \mathcal{G})\]^{(4)} = 0

\[4R_{0\rho\sigma} \nabla^\rho \nabla^\sigma f_G(R, \mathcal{G})\]^{(4)} = 4R_{0\rho\sigma}^{(2)} g^{(0)}_{\, \, \, ij} f_{GR}(0, 0) \partial_i \partial_j R^{(2)}

\[(2\kappa^2 T_{00})^{(4)} = 2\kappa^2 T^{(2)}_{00}\]

In the case of the trace equation, we need:

\[(-2f(R, \mathcal{G}))^{(4)} = -2f_R(0, 0) R^{(4)} - f_{RR}(0, 0) R^{(2)} R^{(2)} - 2f_G(0, 0) \mathcal{G}^{(4)}\]

In the case of the trace equation, we need:

\[(f_R(R, \mathcal{G})R)^{(4)} = f_R(0, 0) R^{(4)} + f_{RR}(0, 0) R^{(2)} R^{(2)}\]

\[(3\nabla^2 f_R(R, \mathcal{G}))^{(4)} = 3f_{RR}(0, 0) \left\{ g^{(0)}_{\, \, \, 00} \left( \partial_0 \partial_0 R^{(2)} - \Gamma^{(2)}_{\, \, \, 00} \partial_i R^{(2)} \right) + g^{(0)}_{\, \, \, ij} \left( \partial_i \partial_j R^{(4)} - \Gamma^{(2)}_{\, \, \, ij} \partial_k R^{(2)} \right) + g^{(0)}_{\, \, \, ij} \partial_i \partial_j R^{(2)} \right\} + 3f_{RR}(0, 0) g^{(0)}_{\, \, \, ij} \left\{ \partial_i R^{(2)} \partial_j R^{(2)} + R^{(2)} \partial_i \partial_j R^{(2)} \right\} + 3f_{RG}(0, 0) g^{(0)}_{\, \, \, ij} \partial_i \partial_j \mathcal{G}^{(4)}\]

\[(2f_G(R, \mathcal{G})G)^{(4)} = 2f_G(0, 0) \mathcal{G}^{(4)}\]

\[(2R \nabla^2 f_G(R, \mathcal{G}))^{(4)} = 2R^{(2)} g^{(0)}_{\, \, \, ij} f_{RG}(0, 0) \partial_i \partial_j R^{(2)}\]

\[(-4R_{\rho\sigma} \nabla^\rho \nabla^\sigma f_G(R, \mathcal{G}))^{(4)} = -4g^{(0)}_{\, \, \, im} g^{(0)}_{\, \, \, jn} R^{(2)}_{\, \, \, ij} f_{RG}(0, 0) \partial_m \partial_n R^{(2)}\]

\[(2\kappa^2 T)^{(4)} = 2\kappa^2 T^{(2)}\]

\[= 2\kappa^2 \left\{ g^{(0)}_{\, \, \, 00} T^{(2)}_{00} + g^{(2)}_{\, \, \, 00} T^{(0)}_{00} + g^{(0)}_{\, \, \, ij} T^{(2)}_{ij} \right\}\]
Appendix B: The Parameterized Post Newtonian Limit

We present now the calculations needed to obtain the PPN limit for the field equations and the trace equation. In the case of the \((0,0)\)-field equation we need to know the following expressions:

\[
\begin{align*}
\left(-\frac{1}{2}g_{00}f(R,G)\right)^{(6)} &= -\frac{1}{2}g_{00}^{(6)}f(0,0) - \frac{1}{2}f_R(0,0) \left\{ g_{00}^{(4)}R^{(2)} + g_{00}^{(2)}R^{(4)} + g_{00}^{(0)}R^{(6)} \right\} - \\
&\quad -\frac{1}{2}f_G(0,0) \left\{ g_{00}^{(2)}G^{(4)} + g_{00}^{(0)}G^{(6)} \right\} - \frac{1}{2}f_{RR}(0,0) \left\{ \frac{1}{2}g_{00}^{(2)}R^{(2)}^2 + g_{00}^{(0)}R^{(2)}R^{(4)} \right\} - \\
&\quad -\frac{1}{2}g_{00}^{(0)}f_{RG}(0,0)R^{(2)}G^{(4)} - \frac{1}{12}f_{RRR}(0,0)R^{(2)}^3 \\
(f_R(R,G)R_{00})^{(6)} &= f_R(0,0)R_{00}^{(6)} + f_{RR}(0,0) \left\{ R^{(2)}R_{00}^{(4)} + R^{(4)}R_{00}^{(2)} \right\} + f_{RG}(0,0)G^{(4)}R_{00}^{(2)} + \\
&\quad +\frac{1}{2}f_{RRR}(0,0)R^{(2)}^2R_{00}^{(2)} \\
(g_{00}^\nabla^2 f_R(R,G))^{(6)} &= g_{00}^{(0)}(\nabla^2 f_R(R,G))^{(6)} + g_{00}^{(2)}(\nabla^2 f_R(R,G))^{(4)} + g_{00}^{(4)}(\nabla^2 f_R(R,G))^{(2)} \\
(-\nabla_0\nabla_0 f_R(R,G))^{(6)} &= -f_{RR}(0,0) \left\{ \partial_0\partial_0R^{(4)} - \Gamma^{(3)}_{\quad 00}0\partial_0R^{(2)} - \Gamma^{(2)}_{\quad 00}0\partial_0R^{(4)} - \Gamma^{(4)}_{\quad 00}0\partial_0R^{(2)} \right\} - \\
&\quad -f_{RG}(0,0) \left\{ \partial_0\partial_0G^{(4)} - \Gamma^{(2)}_{\quad 00}i\partial_iG^{(4)} \right\} - \\
&\quad -f_{RRR}(0,0) \left\{ \partial_0R^{(2)}\partial_0R^{(2)} + R^{(2)}\partial_0\partial_0R^{(2)} - \Gamma^{(2)}_{\quad 00}R^{(2)}\partial_iR^{(2)} \right\} 
\end{align*}
\]
\[(2f_{\mathcal{G}}(R, \mathcal{G}) R_{00})^{(6)} = 2f_{\mathcal{G}}(0, 0) \left\{ R^{(2)}_{00} R^{(4)}_{00} + R^{(4)}_{00} R^{(2)}_{00} \right\} + 2f_{\mathcal{RG}}(0, 0) R^{(2)}_{00} R^{(2)}_{00} \]

\[(-4f_{\mathcal{G}}(R, \mathcal{G}) g^{\rho \sigma} R_{0\rho 0\sigma} R_{\rho 0\sigma})^{(6)} = -4f_{\mathcal{G}}(0, 0) \left\{ 2g^{(0)00}_{00} R^{(4)}_{00} R^{(2)}_{00} + g^{(2)00}_{00} R^{(2)}_{00} R^{(2)}_{00} + g^{(0)ij}_{0i} R^{(3)}_{0i} R^{(3)}_{ij} \right\} - 4f_{\mathcal{RG}}(0, 0) g^{(0)00}_{00} R^{(2)}_{00} R^{(2)}_{00} \]

\[\left(2f_{\mathcal{G}}(R, \mathcal{G}) R_{\alpha \beta \rho 0} R^{\alpha \beta \rho 0}_{\rho 0} \right)^{(6)} = 2f_{\mathcal{G}}(0, 0) \left( R_{\alpha \beta \rho 0} R^{\alpha \beta \rho 0}_{\rho 0} \right)^{(6)} + 2f_{\mathcal{RG}}(0, 0) R^{(2)}_{00} \left( R_{\alpha \beta \rho 0} R^{\alpha \beta \rho 0}_{\rho 0} \right)^{(4)} \]

\[(-4f_{\mathcal{G}}(R, \mathcal{G}) R_{00\sigma} R^{\sigma} R^{(6)}_{00}) = -4f_{\mathcal{G}}(0, 0) \left( R_{00\sigma} R^{\sigma} R^{(6)}_{00} \right)^{(6)} - 4f_{\mathcal{RG}}(0, 0) R^{(2)}_{00} \left( R_{00\sigma} R^{\sigma} R^{(6)}_{00} \right)^{(4)} \]

\[(2g_{00} R \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}))^{(6)} = 2g_{00}^{(0)} R^{(2)} \left( \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(4)} + 2g_{00}^{(0)} R^{(4)} \left( \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(2)} + 2g_{00}^{(2)} R^{(2)} \left( \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(2)} \]

\[(-4g_{00} R_{\rho \sigma} \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}))^{(6)} = -4g_{00}^{(0)} R^{(2)}_{\rho \sigma} \left( \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(4)} - 4g_{00}^{(0)} R^{(4)}_{\rho \sigma} \left( \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(2)} - 4g_{00}^{(2)} R^{(2)}_{\rho \sigma} \left( \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(2)} \]

\[(-2R \nabla_0 \nabla_0 f_{\mathcal{G}}(R, \mathcal{G}))^{(6)} = -2f_{\mathcal{RG}}(0, 0) R^{(2)} \left\{ \partial_0 \partial_0 R^{(2)} - \Gamma_0^{(2)} i_{\partial_0} R^{(2)} \right\} \]

\[(-4R_{00} \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}))^{(6)} = -4R_{00}^{(2)} \left( \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(4)} - 4R_{00}^{(4)} \left( \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(2)} \]

\[(8R_{00} \nabla^\sigma \nabla_0 f_{\mathcal{G}}(R, \mathcal{G}))^{(6)} = 8f_{\mathcal{RG}}(0, 0) \left\{ R^{(2)}_{00} g^{(0)00}_{00} \left( \partial_0 \partial_0 R^{(2)} - \Gamma_0^{(2)} i_{\partial_0} R^{(2)} \right) + R^{(3)}_{0i} g^{(0)ij}_{0i} \partial_j \partial_0 R^{(2)} \right\} \]

\[(4R_{\rho 0\sigma} \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}))^{(6)} = 4R_{\rho 0\sigma}^{(2)} \left( \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(4)} + 4R_{\rho 0\sigma}^{(4)} \left( \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(2)} \]

\[(2\kappa^2 T_{00})^{(6)} = 2\kappa^2 T_{00}^{(4)} \]

Finally, in the case of the trace equation, the expressions needed are:

\[(-2 f(R, \mathcal{G}))^{(6)} = -2 \left\{ f_{R}(0, 0) R^{(6)} + f_{G}(0, 0) G^{(6)} + f_{RR}(0, 0) R^{(2)} R^{(4)} + f_{RG}(0, 0) R^{(2)} G^{(4)} + \frac{1}{6} f_{RRR}(0, 0) R^{(2)} R^{(3)} \right\} \]

\[(f_{R}(R, \mathcal{G}) R^{(6)} = f_{R}(0, 0) R^{(6)} + 2f_{RR}(0, 0) R^{(2)} R^{(4)} + f_{RG}(0, 0) R^{(2)} G^{(4)} + \frac{1}{2} f_{RRR}(0, 0) R^{(2)} R^{(3)} \]

\[(3 \nabla^2 f_{R}(R, \mathcal{G}))^{(6)} = 3 \left( \nabla^2 f_{R}(R, \mathcal{G}) \right)^{(6)} \]

\[(2 f_{\mathcal{G}}(R, \mathcal{G}) G^{(6)} = 2f_{\mathcal{G}}(0, 0) G^{(6)} + 2f_{\mathcal{RG}}(0, 0) R^{(2)} G^{(4)} \]

\[(2 R \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}))^{(6)} = 2R^{(2)} \left( \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(4)} + 2R^{(4)} \left( \nabla^2 f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(2)} \]

\[(-4 R_{\rho \sigma} \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}))^{(6)} = -4R_{\rho \sigma}^{(2)} \left( \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(4)} - 4R_{\rho \sigma}^{(4)} \left( \nabla^\rho \nabla^\sigma f_{\mathcal{G}}(R, \mathcal{G}) \right)^{(2)} \]

\[(2\kappa^2 T_{00})^{(6)} = 2\kappa^2 T_{00}^{(4)} = 2\kappa^2 \left( g^{(0)00}_{00} T_{00}^{(4)} + g^{(2)00}_{00} T_{00}^{(2)} + g^{(4)00}_{00} T_{00}^{(0)} + g^{(3)0i}_{0i} T_{0i}^{(1)} + g^{(0)ij}_{0i} T_{ij}^{(4)} + g^{(2)ij}_{0i} T_{ij}^{(2)} \right) \]

[1] S. Capozziello, M. De Laurentis, *Phys. Rept.* **509**, 167 (2011).
