Full simulation of chiral Random Matrix Theory at non-zero chemical potential by Complex Langevin

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Abstract

It is demonstrated that the complex Langevin method can simulate chiral random matrix theory at non-zero chemical potential. The successful match with the analytic prediction for the chiral condensate is established through a shift of matrix integration variables and choosing a polar representation for the new matrix elements before complexification. Furthermore, we test the proposal to work with a Langevin-time dependent quark mass and find that it allows us to control the fluctuations of the phase of the fermion determinant throughout the Langevin trajectory.
I. INTRODUCTION

First principles non-perturbative simulations of full QCD have been limited to the region with small ratio of the quark chemical potential over temperature or heavy quark masses because of the fermion sign problem, for reviews see [1–3]. Recently, however, complex Langevin simulations of full QCD at non-zero chemical potential have been presented [4–6]. While these initially are carried out in specific parameter domains the method holds the possibility to provide first principles simulations for any value of the chemical potential, even with low temperature and light quark masses.

The introduction of chiral random matrix theory [7–9] at non-zero chemical potential [10, 11] has lead to a number of analytic insights into the non-perturbative dynamics of dense strongly interacting matter and the effect of the sign problem [10–14]. In [15] chiral random matrix theory was used to emphasize the potential problem which complex Langevin faces in simulations of QCD at low temperature and light quark masses.

In this paper we demonstrate that the complex Langevin approach can solve the sign problem in chiral random matrix theory. This is relevant for QCD at low temperature and light quark masses since chiral random matrix theory [7–11] and full QCD share a number of properties: The integral formulation of the partition function in both cases includes the determinant of a Dirac operator and the flavor symmetries and explicit breaking hereof are identical. QCD and chiral random matrix theory, therefore, have the same low energy theory in the microscopic limit, namely chiral perturbation theory at leading order in the $\epsilon$-domain [7, 16–18]. Moreover, the anti-Hermiticity of the Dirac operators are in both cases broken by the chemical potential. This is particularly relevant for the present study, since it implies that the average of the phase factor of the fermion determinant in QCD and chiral random matrix theory have the same analytic form of the exponential suppression in the limit of large volume/size of the matrix [13]. In other words the sign problem in QCD for light quarks at low temperature and in chiral random matrix theory are equally severe.

In the physical domain where chiral random matrix theory and QCD share the same low energy limit, both partition functions are independent of the chemical potential $\mu$. This is natural as the partition function is dominated by pions which have zero quark charge.
The measures in the partition functions are however strongly $\mu$-dependent. The numerical problem of realizing the $\mu$-independence in both cases becomes particular challenging once the chemical potential exceeds half of the pion mass ($\mu > m_\pi/2$), see eg. [13].

In this paper we show that complex Langevin provides a method to numerically simulate chiral random matrix theory, even in the domain of $\mu > m_\pi/2$. The success compared to the first study [15] is established through a shift of the matrix variables in the integrant as well as using a polar representation for the new matrix elements.

The advantage of simulating chiral random matrix theory compared to full QCD is that we have exact analytical solutions to test the numerics against. Tests of this kind are imperative since complex Langevin is not guaranteed to provide the correct solutions, see for example [19, 20]. One potential problem particularly relevant for QCD is that the fermion determinant renders the action non-holomorphic [15], existing proofs of correct convergence [21–23] therefore do not apply directly.

As demonstrated in [15, 24] complex Langevin may lead to wrong results if the argument of the logarithm (the fermion determinant in QCD and in chiral random matrix theory) frequently circles the origin. In [25] a practical proposal was given in order to circumvent this problem: By initially decreasing the quark mass with the Langevin time it was suggested that it might be possible to reach the desired value of the quark mass without frequent circulations of the origin by the fermion determinant. Here we test this proposal and show that it is indeed the case within chiral random matrix theory.

The results are presented as follows: In the next section the chiral random matrix theory is defined and the relevant analytic results are stated. Then in section II.A the parametrization of the matrix elements is chosen and we explicitly compute the Langevin drift. Section II.A presents the numerical results obtained and the proposal to work with an initially Langevin time dependent quark mass is tested. We draw conclusions and provide an outlook in section IV.
II. CHIRAL RANDOM MATRIX THEORY

The chiral random matrix theory we will simulate with complex Langevin has the partition function [11, 26]

\[ Z_{N_f}^{N_f}(m) = \int d\Phi_1 d\Phi_2 \det_{N_f}(D_\mu + m) e^{-2N\text{Tr}[\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2]}, \quad (1) \]

where the random matrix analogue of the Dirac operator is

\[ D_\mu + m = \begin{pmatrix} m & e^\mu \Phi_1 - e^{-\mu} \Phi_2^\dagger \\ -e^{-\mu} \Phi_1^\dagger + e^\mu \Phi_2 & m \end{pmatrix}. \quad (2) \]

The integration variables \( \Phi_1 \) and \( \Phi_2^\dagger \) are complex \( N \times (N + \nu) \) matrices, \( m \) and \( \mu \) are the quark mass and chemical potential parameters and \( N_f \) is the number of quark fields which have been integrated out. The integer \( \nu \) is the topological index, i.e. the number of exact zero eigenvalues of \( D_\mu \). In the microscopic limit where \( mN \) and \( \mu^2 N \) are fixed as \( N \to \infty \) this random matrix partition function is equivalent to that of chiral perturbation theory in the \( \epsilon \)-domain [11, 17, 18]. This limit also allow us to identify the relation between the random matrix parameters \( N, m \) and \( \mu \) and the physical four volume, quark mass and chemical potential, [11, 27, 28]

\[ 2mN \leftrightarrow m\Sigma V \quad \text{and} \quad 2\mu^2 N \leftrightarrow \mu^2 F_\pi^2 V, \quad (3) \]

where \( \Sigma \) is the chiral condensate and \( F_\pi \) is the pion decay constant. In the quenched and the phase-quenched theories a phase transition takes place at \( \mu = m_\pi / 2 \). Using the Gell-Mann - Oakes - Renner relation we can rewrite this as \( \mu^2 F_\pi^2 V = m\Sigma V / 2 \), which in the chiral random matrix variables translates to \( 2\mu^2 = m \).

For the numerical test of complex Langevin below we will naturally work with finite \( N \). It is therefore of great practical value that the partition function [11] can be computed analytically for all values of \( N_f \) and \( N \) [11, 26]

\[ Z_{N_f}^{N_f}(m) = \frac{1}{(2m)^{\nu/2N_f(N_f-1)}} \det \left[ \left( \frac{d}{dm} \right)^a L_{N+b}^{(\nu)}(-nm^2) \right]_{a=0,\ldots,N_f-1; \ b=0,\ldots,N_f-1}, \quad (4) \]

where \( L_k^{(\nu)}(x) \) is the generalized Laguerre polynomial. From this compact expression for the partition function we obtain the mass dependent chiral condensate

\[ \Sigma_{N_f}^{N_f}(m) = \frac{1}{N_f} \frac{1}{N} \frac{1}{Z_{N_f}^{N_f}(m)} \frac{d}{dm} Z_{N_f}^{N_f}(m), \quad (5) \]
Note that, as discussed in the introduction, the partition function is independent of the chemical potential even though the weight in the integral representation (1) is heavily $\mu$ dependent.

A. Complex Langevin dynamics

When the action is complex it is natural to generalize real Langevin dynamics [29] by complexifying the fields and define a complexified Langevin dynamics [30, 31] through the gradient of the action.

In order to set up the complex Langevin dynamics for the chiral random matrix theory we first express the partition function (1)

$$Z_N^N(m) = \int d\Phi_1 d\Phi_2 \exp(-S),$$

in terms of the action

$$S = 2N\text{Tr}[\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2] - N_f \text{Tr} \left[\log(m^2 - XY)\right],$$

where

$$X \equiv e^\mu \Phi_1 - e^{-\mu} \Phi_2^\dagger$$

$$Y \equiv -e^{-\mu} \Phi_2^\dagger + e^\mu \Phi_2.$$

Note the appearance of the logarithm of the fermion determinant in the action.

Next we choose to parameterize the elements of the complex $N \times (N + \nu)$ matrices $\Phi_1$ and $\Phi_2$ as

$$(\Phi_1)_{ij} = r_{1,ij} e^{i\theta_{1,ij}} \quad (\Phi_2)_{ji} = r_{2,ji} e^{i\theta_{2,ji}},$$

where $i = 1, \ldots, N$ and $j = 1, \ldots, N + \nu$. In the complex Langevin dynamics the $4N(N + \nu)$ real variables $r_{1,ij}$, $\theta_{1,ij}$, $r_{2,ij}$, $\theta_{2,ij}$ will be complexified. The motivation for the choice of parametrization [9] is that the $\mu$-independence of the partition function can be achieved if the Langevin process in effect shifts the integration contour of the $\theta_{1,ij}$ and $\theta_{2,ij}$ variables by $i\mu$ into the complex plane while the radial variables $r_{1,ij}$ and $r_{2,ij}$ are attracted to the real axis.
In the parameterization (9) the Gaussian term is simply
\[
\text{Tr} \left[ \Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2 \right] = \sum_{ij} r_{1,ij}^2 + r_{2,ji}^2,
\]
(10)
and the action is thus
\[
S = -\sum_{ij} \log(r_{1,ij}) + \log(r_{2,ji}) - \log \det(m^2 - XY) + 2N \sum_{ij} r_{1,ij}^2 + r_{2,ji}^2,
\]
(11)
with
\[
X_{ij} = e^{\mu+i\theta_1,ij} r_{1,ij} - e^{-\mu-i\theta_2,ji} r_{2,ji},
\]
(12)
\[
Y_{ij} = -e^{-\mu-i\theta_1,ji} r_{1,ji} + e^{\mu+i\theta_2,ij} r_{2,ij}.
\]
The log\((r_{1,ij})\) and log\((r_{2,ji})\) terms are from the Jacobian for the change to polar variables.

The Langevin dynamics is given by the equations
\[
egin{align*}
    r_{1, mn}^{(t+1)} &= r_{1, mn}^{(t)} - \frac{\partial S}{\partial r_{1, mn}} dt + \sqrt{dt} \eta(t), \\
    r_{2, mn}^{(t+1)} &= r_{2, mn}^{(t)} - \frac{\partial S}{\partial r_{2, mn}} dt + \sqrt{dt} \eta(t), \\
    \theta_{1, mn}^{(t+1)} &= \theta_{1, mn}^{(t)} - \frac{\partial S}{\partial \theta_{1, mn}} dt + \sqrt{dt} \eta(t), \\
    \theta_{2, mn}^{(t+1)} &= \theta_{2, mn}^{(t)} - \frac{\partial S}{\partial \theta_{2, mn}} dt + \sqrt{dt} \eta(t),
\end{align*}
\]
(13)
where the derivatives are to be evaluated at Langevin-time \(t\) and \(\eta\) is a real Gaussian white noise
\[
\langle \eta(t)\eta(t') \rangle = 2\delta(t - t').
\]
(14)

The next step is to compute the detailed form of the drift terms which enters the Langevin equations. With the notation
\[
P^{-1} \equiv (m^2 - XY)^{-1}
\]
(15)
we obtain
\[
- \frac{\partial S}{\partial \theta_{1, mn}} = -N_f \left[ \left( P^{-1} \right)_{ki} \partial_{\theta_{1, mn}} \left( X_{ij} Y_{jk} \right) \right]
\]
\[
= -N_f \left[ P^{-1}_{ki} \left( ie^{\mu+i\theta_1,ij} r_{1, mn} \delta_{mi} \delta_{nj} Y_{jk} + iX_{ij} \delta_{nj} \delta_{mk} e^{-\mu-i\theta_1, mn} r_{1, mn} \right) \right]
\]
\[
= -iN_f \left( e^{\mu+i\theta_1, mn} r_{1, mn} P^{-1}_{km} Y_{nk} + e^{-\mu-i\theta_1, mn} r_{1, mn} P^{-1}_{mi} X_{in} \right)
\]
\[
= -iN_f \left( e^{\mu+i\theta_1, mn} r_{1, mn} \left( (YP^{-1})^T \right)_{mn} + e^{-\mu-i\theta_1, mn} r_{1, mn} \left( P^{-1} X \right)_{mn} \right),
\]
(16)
while for the radial variable we have

\[- \frac{\partial S}{\partial r_{1,mm}} = -4N_{r_{1,mm}} + 1/r_{1,mm} - N_f \left[ (P^{-1})_{ki} \partial_{r_{1,mm}} (X_{ij}Y_{jk}) \right] \]

\[= -4N_{r_{1,mm}} + 1/r_{1,mm} - N_f \left[ P_{ki}^{-1} \left( e^{\mu+i\theta_{1,mm}} \delta_{mi}\delta_{nj}Y_{jk} - X_{ij}\delta_{mj}\delta_{nk}e^{-\mu-i\theta_{1,mm}} \right) \right] \]

\[= -4N_{r_{1,mm}} + 1/r_{1,mm} - N_f \left( e^{\mu+i\theta_{1,mm}} P_{km}^{-1}Y_{nk} - e^{-\mu-i\theta_{1,mm}} P_{mi}^{-1}X_{im} \right) \]

\[= -4N_{r_{1,mm}} + 1/r_{1,mm} - N_f \left[ e^{\mu+i\theta_{1,mm}} \left( (YP^{-1})^T \right)_{mn} - e^{-\mu-i\theta_{1,mm}} (P^{-1}X)^T_{mn} \right]. \]

Similarly for the angular and radial variables from $\Phi_2$ we have

\[- \frac{\partial S}{\partial \theta_{2,mm}} = -N_f \left[ (P^{-1})_{ki} \partial_{\theta_{2,mm}} (X_{ij}Y_{jk}) \right] \]

\[= -N_f \left[ P_{ki}^{-1} \left( i e^{-\mu-i\theta_{2,mm}} r_{2,mm} \delta_{ni}\delta_{mj}Y_{jk} + iX_{ij}\delta_{mj}\delta_{nk}e^{\mu+i\theta_{2,mm}} r_{2,mm} \right) \right] \]

\[= -iN_f \left( e^{-\mu-i\theta_{2,mm}} P_{kn}^{-1}Y_{mk} + e^{\mu+i\theta_{2,mm}} P_{mi}^{-1}X_{im} \right) \]

\[= -iN_f \left( e^{-\mu-i\theta_{2,mm}} r_{2,mm} (YP^{-1})_{mn} + e^{\mu+i\theta_{2,mm}} r_{2,mm} \left( (P^{-1}X)^T \right)_{mn} \right), \]

and

\[- \frac{\partial S}{\partial r_{2,mm}} = -4N_{r_{2,mm}} + 1/r_{2,mm} - N_f \left[ (P^{-1})_{ki} \partial_{r_{2,mm}} (X_{ij}Y_{jk}) \right] \]

\[= -4N_{r_{2,mm}} + 1/r_{2,mm} - N_f \left[ P_{ki}^{-1} \left( -e^{-\mu-i\theta_{2,mm}} \delta_{ni}\delta_{mj}Y_{jk} + X_{ij}\delta_{mj}\delta_{nk}e^{\mu+i\theta_{2,mm}} \right) \right] \]

\[= -4N_{r_{2,mm}} + 1/r_{2,mm} - N_f \left( -e^{-\mu-i\theta_{2,mm}} P_{kn}^{-1}Y_{mk} + e^{\mu+i\theta_{2,mm}} P_{mi}^{-1}X_{im} \right) \]

\[= -4N_{r_{2,mm}} + 1/r_{2,mm} - N_f \left[ -e^{-\mu-i\theta_{2,mm}} (YP^{-1})_{mn} + e^{\mu+i\theta_{2,mm}} \left( (P^{-1}X)^T \right)_{mn} \right]. \]

Note that we have ignored the cut of the logarithm and simply used the standard form for the derivative of the log when the argument is real and positive. This has potential consequences for the Langevin process \cite{15} in particular if the argument of the log frequently circles the origin of the complex plane. We will return to this point in section \ref{sec:III} below.

The Langevin dynamics presented above differs in two ways from that used in \cite{15}. First, the realization (1) of the chiral random matrix theory partition function is related to the partition function used in \cite{15} (see Eq. (4.1) therein) by a change of the matrices which enters as integration variables (see also the appendix of \cite{26}). Second, the parametrization of the matrix elements (9) used here is different from that used in \cite{15}. Both changes are necessary for the success of the simulations presented below.
FIG. 1: The chiral condensate as a function of the quark mass for one, two and three dynamical flavors. The full lines are the exact analytic predictions and the points are the results of the complex Langevin dynamics with adaptive step-size. The parameters chosen for the plot are $N = 20$, $\mu = 1$, $\nu = 0$ and $2T = 2000$.

III. SIMULATIONS

In order to test the complex Langevin algorithm presented above we have run a series of numerical simulations. The central observable of interest is the chiral condensate for which the analytic prediction is given in Eq. (5). This observable is not only relevant for the non-perturbative physics it is also highly sensitive to the sign problem, since the phase-quenched chiral condensate takes a drastically different form in the region of $m < 2\mu^2$, see eg. [15] for plots hereof.

To compute the chiral condensate we start the Langevin process in a random configuration from the original (not complexified) quenched ensemble. The Langevin process is then run for a period, $2T$, in time-steps of $dt$ and on the latter half of the trajectory we compute

$$
\Sigma(m) = \frac{1}{N_f} \frac{1}{N} \frac{1}{T} \sum_{t=T+dt}^{2T} \text{Re} \text{Tr} \frac{1}{D_{\mu}^{(t)}} + m \ dt , \tag{20}
$$

where $D_{\mu}^{(t)} + m$ is the Dirac operator (2) evaluated for the complexified fields generated through (16)-(19) at Langevin time $t$. The first half, $T$, of the period allows the Langevin
FIG. 2: The chiral condensate as a function of the quark mass for \( \nu = 0, 1, 2 \) with \( N_f = 2, N = 20, \mu = 1 \) and \( \nu = 0 \). The data points are obtained with complex Langevin using adaptive step-size and the full lines are the predictions [5].

process to react to the initial condition.

With increasing size of the matrices we have found it convenient to implement adaptive step-size [19, 32], since the \( 1/r \)-terms in the drift can lead to large excursions unless \( dt \) is sufficiently small. Results for the chiral condensate for \( N = 20, \mu = 1, 2T = 2000, \nu = 0 \) and adaptive step-size are shown in figure 1. Displayed are the numerical results for \( N_f = 1, 2 \) and 3 as well as the analytic predictions. We observe that the Langevin process is able to reproduce the expected mass dependence throughout the range of values for \( m \) with all three values of \( N_f \). Note that \( m < 2\mu^2 \) throughout the range displayed. The convergence is equally good for larger values of \( m \).

Next we have tested the Langevin dynamics for non-zero topological index \( \nu \). In figure 2 the numerical results for \( \nu = 0, 1 \) and 2 with adaptive step-size, \( N_f = 2, N = 20, 2T = 2000 \) are plotted against the analytic curves. The numerical data again follow the expected curves and demonstrates that the topological zero modes causes no obstacle for the complex Langevin algorithm in chiral random matrix theory.
The average of the imaginary part of the angular variable $\theta_1$ and the average of the imaginary part of the radial variable $r_1$. As a function of the Langevin time $t$ the angular variable flows towards $\mu = 1$ marked by the thin vertical line while the radial variable remains close to the real axis. The error bars are given by the square root of the variance. The parameters in the simulation are $\nu = 0$, $N_f = 2$, $N = 20$, $\mu = 1$, $m = 1$ and $\nu = 0$.

In order to gain insights in the dynamics of these successful simulations we have monitored the values of the variables throughout the Langevin process: The angular variables $\theta_{1,ij}$ and $\theta_{2,ij}$ are effectively shifted by $i\mu$ into the complex plane while the $r_{1,ij}$ and $r_{2,ij}$ are attracted toward the real axis. This cancels the $\mu$-dependence of the chiral condensate, as was indeed the motivation for the choice of parametrization (9). An example of the flow of the variable is shown in figure 3. The band is the average of the imaginary part of the elements in $\theta_1$ with the errors given by the squareroot of the variance.

The flow of the variables manifest themselves also in the distribution of the Dirac eigenvalues. In figure 4 the eigenvalues of the Dirac operator on the initial 400 configurations are plotted in black along with the eigenvalues of the final 400 configurations out of 60000 adaptive steps. The value of the quark mass, $m = 1$, is well within the initial eigenvalue distribution, however, with Langevin time the Dirac eigenvalues move inside the quark mass. As this happens the fluctuations of the phase of the fermion determinant are damped, see figure 5.

FIG. 3: The average of the imaginary part of the angular variable $\theta_1$ and the average of the imaginary part of the radial variable $r_1$. As a function of the Langevin time $t$ the angular variable flows towards $\mu = 1$ marked by the thin vertical line while the radial variable remains close to the real axis. The error bars are given by the square root of the variance. The parameters in the simulation are $\nu = 0$, $N_f = 2$, $N = 20$, $\mu = 1$, $m = 1$ and $\nu = 0$. 

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FIG. 4: The eigenvalues of the Dirac operator in the complex plane black for the first 400 time steps red for the final 400 time steps. Parameters used are \( N = 20, \ N_f = 2, \ \mu = 1, \ m = 1, \ \nu = 0 \) and 60000 adaptive steps. Note that the quark mass is initially well inside the eigenvalue distribution.

A. Decreasing the quark mass with Langevin time

In large scale simulations it becomes exceedingly hard to invert the Dirac operator if the quark mass is inside the eigenvalue distribution. Moreover, in this case the fermion determinant is likely to circulate the origin frequently (when the quark mass is inside the Dirac eigenvalues the phase of the fermion determinant is distributed according to a Lorentzian distribution [33]) and hence ignoring the cut of the logarithm is not necessarily justified [15]. In order to circumvent these issues it was proposed in [25] to allow the quark mass to decrease with the Langevin time \( t \). The proposal is to start from an initial value of the quark mass which is outside the Dirac eigenvalue distribution. With Langevin time the quark mass is then slowly decreased towards the desired value. In this way it is possible that the quark mass remains outside the distribution of the eigenvalues of the Dirac operator and that the fermion determinant does not circulate the origin at any given time throughout the Langevin process. Once the desired quark mass is reached all previous configurations are discarded from the measurement of the given observable.
FIG. 5: The argument, $\phi$, of the phase of the fermion determinant as a function of Langevin time, for $m = 1$, $\mu = 1$, $N = 20$, $N_f = 2$ and $\nu = 0$. Initially the fermion determinant frequently circles the origin but with the Langevin time the Dirac eigenvalues flow inside the quark mass and the fluctuations of the fermion determinant are damped. The fixed value of the quark mass throughout the run is indicated by the horizontal red line.

Here we test this proposal within the Langevin process for chiral random matrix theory. We start the Langevin process on a random configuration from the original quenched ensemble (not complexified) and pick a value of the quark mass parameter which is safely outside the Dirac eigenvalues, i.e. $m > 2\mu^2$. The quark mass is reduced in steps proportional to the time step, unless the angle of the determinant has been above 1.5 within the last 1000 time steps. In that case we keep the quark mass constant to allow the Langevin dynamics to dampen the fluctuations of the phase of the determinant. This procedure is repeated until $m$ reaches the desired value, here 1, see figure 6. We observe that with Langevin time it is possible to decrease the quark mass such that the fermion determinant at no point during the Langevin trajectory circulates the origin. The potential problems with the log of the fermion determinant can therefore safely be ignored.
FIG. 6: The argument of the phase of the fermion determinant along Langevin trajectory. With the Langevin time dependent quark mass (thin red curve) the fermion determinant does not circle the origin during the Langvin process. Here $\mu = 1$, $N = 20$, $N_f = 2$, $\nu = 0$ and the initial value of the quark mass 3 is outside the cloud of Dirac eigenvalues. As the quark mass reaches the desired value $m = 1$ it is kept fixed and the measurement of the chiral condensate can be performed.

IV. CONCLUSIONS

We have demonstrated that complex Langevin can simulate chiral random matrix theory at non-zero chemical potential even in the range corresponding to $\mu > m_\pi/2$. The success of the complex Langevin method in chiral random matrix theory was established by 1) a change of integration matrices and 2) a polar parametrization of these variables before complexification. This choice of variables was inspired by taking the perspective of the eigenvalues of the Dirac operator evaluated on the complexified configurations. As shown in [25] the Dirac spectrum of complex Langevin simulations must be vastly different form that with real configurations. In the application of complex Langevin to chiral random matrix theory, the natural solution is to realize an effective anti-Hermitization of the Dirac operator through the complexification of the matrix elements. The choice of matrix integration
variables and the polar parametrization of the elements hereof was handpicked to optimize the chance for complex Langevin to realize this effective anti-Hermitization. Indeed, we have checked that the success of complex Langevin in chiral random matrix theory is established through an effective shift into the complex plane of the angular part of the matrix elements. This smoothly connects the initial non-Hermitian random matrix Dirac operator to an anti-Hermitian counterpart at large Langevin time. For a discussion of the possibility to realize a similar scenario in the context of full QCD, see [25].

As the chiral random matrix Dirac operator shares the chiral symmetries of the QCD Dirac operator it allows us to address several properties directly relevant for QCD. In particular, we have tested the proposal of [25] in which the quark mass is initially Langevin time dependent: With adaptive step-size and a Langevin-time dependent quark mass we have demonstrated that it is possible to simulate the chiral random matrix theory at small mass (such that $m_\pi < 2\mu$) without the determinant frequently circulating the origin. This minimizes the potential problems due to the non-holomorfic nature of the action in the presence of a fermion determinant. Furthermore, it ensures that inversions of the Dirac operator only appear with the quark mass outside the eigenvalues of the Dirac operator.

It would be most interesting to understand if the effect of a Langevin time dependent quark mass in complex Langevin simulations of full QCD is beneficial, in particular at low temperature and light quark mass. As an intermediate step a Langevin time dependent quark mass could also be implemented for the Thirring model [34]. Another possible direction is to use the eigenvalue representation of the chiral random matrix partition function as the basis for the Langevin process. Such an angle of approach has already lead to new insights for QCD in one dimension [35].

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