State-space Correlations and Stabilities

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Abstract

The state-space pair correlation functions and notion of stability of extremal and non-extremal black holes in string theory and M-theory are considered from the viewpoints of thermodynamic Ruppeiner geometry. From the perspective of intrinsic Riemannian geometry, the stability properties of these black branes are divulged from the positivity of principle minors of the space-state metric tensor. We have explicitly analyzed the state-space configurations for (i) the two and three charge extremal black holes, (ii) the four and six charge non-extremal black branes, which both arise from the string theory solutions. An extension is considered for the $D_6$-$D_4$-$D_2$-$D_0$ multi-centered black branes, fractional small black branes and two charge rotating fuzzy rings in the setup of Mathur’s fuzzball configurations. The state-space pair correlations and nature of stabilities have been investigated for three charged bubbling black brane foams, and thereby the M-theory solutions are brought into the present consideration. In the case of extremal black brane configurations, we have pointed out that the ratio of diagonal space-state correlations varies as inverse square of the chosen parameters, while the off diagonal components vary as inverse of the chosen parameters. We discuss the significance of this observation for the non-extremal black brane configurations, and find similar conclusion that the state-space correlations extenuate as the chosen parameters are increased. In either of the above configurations, notion of scaling property suggests that the brane-brane pair correlations, which find an asymmetric nature in comparison with the other state-space pair correlations, weaken relatively faster and they relatively swiftly come into an equilibrium statistical configuration. Novel aspects of the state-space interactions may be envisaged from the coarse graining counting entropy of underlying CFT microstates.

Keywords: Black Hole Physics, Higher-dimensional Black Branes, State-space Correlations and Statistical Configurations.

PACS numbers: 04.70.-s Physics of black holes; 04.70.Bw Classical black holes; 04.70.Dy Quantum aspects of black holes, evaporation, thermodynamics; 04.50.Gh Higher-dimensional black holes, black strings, and related objects.

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1 Introduction

State-space configurations involving thermodynamics of extremal and non-extremal black branes in string theory [1] and M-theory [2] possess rich intrinsic geometric structures [3] [4] [5]. The present article thus focus attention on thermodynamic perspectives of black branes, and thereby explicates nature of concerned state-space pair correlations and associated stability of the solutions containing large number of branes and antibranes. Besides several general notions which have earlier been analyzed in the condensed matter physics [6] [7] [8] [9] [10], we shall here consider specific string theory and M-theory configurations thus mentioned with few thermodynamic parameters and analyze possible state-space pair correlation functions and their scaling relations. Basically, the investigation which we shall follow here entails certain intriguing features of underlying statistical fluctuations which can be defined in terms of thermodynamic parameters. Given definite covariant state-space description of a consistent macroscopic black brane solution, we shall expose (i) for what conditions the considered configuration is stable?, (ii) how its state-space correlation functions scale in terms of the chosen thermodynamic parameters? In this process, we shall also enlist complete set of non-trivial relative state-space correlation functions of the configurations considered in [15]. It may further be envisaged in this direction that the similar considerations indeed remain valid over the other black holes in general relativity [11] [12] [13] [14], attractor black holes [15] [16] [17] [18] [19] [20] and Legendre transformed finite parameter chemical configurations [21] [22], quantum field theory and QCD backgrounds [23].

On other hand, the state-space configurations of four dimensional $N = 2$ black holes can be characterized by electric and magnetic charges $q, p$ which arise from usual flux integrals of the field strength tensors and their Poincaré duals. In such cases, the near horizon geometry of an extremal black hole turns out to be an $AdS_2 \times S^2$ manifold which describes the Bertotti-Robinson vacuum associated with the black hole. The area of the black hole horizon $A$ and thus the macroscopic entropy [20] [21] [22] [23] is given as $S_{\text{macro}} = \pi |Z_\infty|^2$. Such attractor solutions and their critical properties have further been explored from an effective potential defined in $N = 2, D = 4$ supergravities coupled to $n_T$ abelian vector multiplets for an asymptotically flat extremal black holes describing $(2n_T + 2)$-dyonic charges and $n_T$ complex scalar fields which parameterize $n_T$-dimensional special Kähler manifold [25] [26] [27] [28]. The statistical entropy of the supersymmetric charge black holes coming from counting the degeneracy of bound states have been examined against the macroscopic Wald entropy [29] [30] [31] which further agrees term by term with the higher derivative supergravity corrections, as well [25]. In order to study the respective cases of non-extremal black branes of such black holes, one may add some total mass or corresponding antibranes to the chosen extremal black brane configuration, and thereby possible specific computation of the black brane entropy can either be performed in the macroscopic setup, or that of associated microscopic considerations. Furthermore, the investigation of [22] shows a match between the $S_{\text{micro}}$ and $S_{\text{macro}}$ for the non-extremal configurations carrying definite brane and antibrane charges.

There has been various extremal black holes [32] [33] [34] [35] and non-extremal black brane space-times [36], multi-centered black brane configurations [37] [38], small black holes with fractional branes [39] [40] [41], fuzzy rings in Mathur’s fuzzballs as well as the subensemble theoretic set-up [42] [43] [44], for which the notion of state-space geometry has been introduced in [15]. Similar properties have further been explored for the three charge bubbling black brane solutions in M-theory as well [45]. We shall thus systematically present the status of lower dimensional black hole thermodynamic configurations from the view-points of an intrinsic Riemannian geometry. In this connection, the microscopic perspective of black branes have been analyzed in [46] [47] and their thermodynamic geometry whose basics have been motivated in [18] [21] has recently been investigated [15]. We shall show that the similar phenomena may further be explored to exhibit how state-space local correlations scales and under what conditions a chosen black brane solution correspond to stable state-space configuration. In this case, it has been possible to out-line that there is an explicit correspondence between the parameters of microscopic spectrum and macroscopic properties of a class of extremal [48] [49] and associated non-extremal black brane systems [50]. It has there after been pointed out that there should exist definite microscopic origins of underlying statistical and thermodynamic interactions among the microstates of black brane configurations which give rise to an intrinsic Riemannian geometry. In turn, we apperceive from [4] that such notions may in turn be investigated from the perspective of statistical fluctuations which arise from the coarse graining entropy of chosen configuration.

The state-space geometry thus defined introduces in particular that the thermodynamic interactions considered as the function of charges, angular momenta, and mass of a given black brane configuration may be characterized by an ensemble of equilibrium microstates of underlying microscopic configurations. Furthermore, it has been observed in all such cases that there exist a clear mechanism on the black brane side that describes the notion of interactions on the state-space which turn out to be regular intrinsic Riemannian manifold or vice-verse. In fact, one may therefore exhibit well-defined intrinsic thermodynamic geometries associated each other via conformal transformation. The physical observations thus found are consistent with existing picture of the microscopic CFT [51] [52] [53] [54] that the microscopic entropy $S_{\text{micro}}$ counts the
states of black brane configuration in a field theory description dual to the gravitational description. In fact, such conventional understanding of the entropy is based on coarse graining over a large number of microstates \[35, 37\] and thus it turns out to be a crucial ingredient in realizing equilibrium macroscopic thermodynamic geometry. It has been shown \[16, 21, 33\] that the components of state-space metric tensor defined as Hessian matrix of the entropy signify state-space pair correlation functions and the associated state-space curvature scalar implies the nature of global correlation volume of the underlying statistical system. Such intrinsic geometric local and global correlations have initially been studied for the thermodynamic configurations of general relativity black holes \[22, 35, 33, 34\], and thereafter they have been brought into focus to the string theory or M-theory black holes \[34, 37\]. Furthermore, it turns out that the state-space interactions generically remain finite and non-zero when small thermal fluctuations in the canonical ensemble are taken into account \[34\].

As mentioned above, we shall analyze the intrinsic geometric relative correlations and notion of stability for large class of extremal as well as non-extremal black hole configurations at an attractor fixed point solution. The study of chosen thermodynamic systems has rather been already intimated in \[35\], and in this paper our specific goal shall thereby be to explicate their further state-space properties for large charge rotating and non-rotating black brane configurations. A similar application has indeed been performed for the spherical horizon and non-spherical horizon M-theory configurations, see for details \[35, 37\]. Moreover, it has been demonstrated that various state-space geometric notions turn out to be well-defined, even at zero temperature \[37\]. Interestingly, such geometric studies in connection with attractor fixed point(s) are expected to give further motivations for analyzing large class of extremal and non-extremal black brane configurations under Plank length corrections \[43\] or that of the higher derivative stringy \[\alpha'\]-corrections \[35, 33, 40, 67\] being incorporated via definite Sen entropy functions \[61, 62, 63, 64, 65, 66, 67, 68\] which are obtained over an underlying supergravity effective action for a given moduli space configuration. These notions shall however be left for future investigation.

The present article is organized as follows. The first section motivates why to study state-space configurations of the string theory and M-theory black brane solutions. In either of the subsequent state-space configurations, we shall analyze scaling properties of the state-space pair correlation functions, possible-positivity of heat capacities and non-trivial state-space stabilities. In section 2, we introduce state-space properties for two charge extremal black holes or an excite strings carrying \(n_1\) number of winding modes and \(n_p\) number of momentum modes. In section 3, we demonstrate that the nature of state-space correlations find similar pattern for the three charge extremal black holes. In section 4, we consider state-space analysis for the non-extremal black branes corresponding to three charge \(D_1-D_2-P\) extremal solutions with an addition of antibrane charge. In section 5, we show explicitly that the similar conclusions hold for the six charge non-extremal black branes, as well. In section 6, we focus on the state-space correlations of multi-centered \(D_2-D_1^2-D_2^2-D_0\) configurations, and thereby expose the respective cases for the single center and double center four charge solutions. In section 7, we demonstrate state-space correlation properties for the two cluster, three cluster and then arbitrary finite cluster \(D_0\)-brane fractionations in the \(D_2-D_1\) black branes. In section 8, we explicate that the similar state-space geometric notions hold for three parameter fuzzy rings introduced in the set-up of Mathur’s fuzzball solutions. In section 9, we extend our state-space analysis to the three charge bubbling black brane solutions in M-theory. Finally, section 10 contains some concluding issues and the other implications of the state-space geometry of string theory and M-theory black brane solutions for future.

2 Two charge extremal configurations

In this section, we analyze the non-trivial state-space interactions among various microstates of given black brane configurations, and thereby explore present consideration from the perspective of relative correlations and definite state-space stabilities. To illustrate the basic idea of state-space geometry of string theory black holes, we first consider the case of simplest example describing two charge extremal configurations. It turns out that the state-space geometry of such configurations may be analyzed in terms of the winding modes and the momentum modes of an excited string carrying \(n_1\) number of winding modes and \(n_p\) number of momentum modes. To be concrete, we consider the study of state-space geometry arising from an extremal black hole whose microstates are characterized by the momentum and winding numbers, and microscopic entropy formula \[\text{13, 43, 2, 3}\] obtained from large charge degeneracy of states reduces to

\[
S_{\text{micro}} = 2\sqrt{2n_1n_p}
\]  

(1)

Macroscopically, the entropy of such two charged black holes may be computed by considering the electric magnetic charges on the \(D_2\) and \(D_0\) branes, with ascertained compactifications to obtain \(M_{3,1}\) black hole space-time. There certainly exist higher derivative corrections in string theory, like for instance the \(R^2\) corrections or \(R^4\)-corrections to the standard Einstein action, and thus these corrections make the horizon
area non-zero, as the horizon of vanishing Bekenstein-Hawking entropy black holes is being stretched by such higher derivative $\alpha'$ corrections. The computation of corresponding macroscopic entropy is usually accomplished by assuming spherically symmetric ansatz for the non-compact spatial directions \[69\]. On other hand, the microscopic entropy may whereas be counted by considering an ensemble of weakly interacting D-branes \[69\]. One indeed finds for $n_4$ number of $D_1$ branes and $n_0$ number of $D_0$ branes that the both entropy do match with

$$S_{\text{micro}} = 2\pi \sqrt{n_0 n_4} = S_{\text{macro}}$$  \hspace{1cm} (2)

First of all, an immediate goal would be to understand state-space geometric notions associated with the leading order two charge black brane solutions, which we shall thus consider via an analysis of the state-space configurations of either an excited string or that of the $D_0$, $D_4$ black holes. The analysis follows directly by computing the Hessian matrix of the entropy with respect to concerned extensive thermodynamic variables of the either configurations. It is worth to mention that the respective entropy can simply be defined as a computing the Hessian matrix of the entropy with respect to concerned extensive thermodynamic variables.

As such a configuration is uniquely related to each other and have the same expressions for their entropy. Thus, we focus our attention on the two charge $D_0$-$D_4$ configurations.

From the given expression of the entropy of two charge $D_0$-$D_4$ configuration, we observe that the statistical pair correlations may easily be accounted by simple geometric descriptions being expressed in terms of the brane numbers connoting an ensemble of microstates of the $D_0$-$D_4$ black hole solutions. Furthermore, it is not difficult to see that the components of the state-space metric tensor describing equilibrium statistical pair correlations may be computed from the negative Hessian matrix of the entropy. As an easy result, we deduce for all allowed values of the parameters of the two charge $D_0$-$D_4$ black holes that the components of underlying state-space metric tensor\[^4\] are given as

$$g_{n_0 n_4} = \frac{\pi}{2n_0} \sqrt{n_4 n_0},\quad g_{n_4 n_4} = -\frac{\pi}{2\sqrt{n_0 n_4}},\quad g_{n_4 n_0} = \frac{\pi}{2n_4} \sqrt{n_0 n_4}$$  \hspace{1cm} (3)

It is thus evident that the principle components of state-space metric tensor $\{g_{n_i n_j} | i, j = 0, 4\}$ essentially signify a set of definite heat capacities (or the related compressibilities) whose positivity in turn apprises that the $D_0$-$D_4$ black brane solutions comply an underlying equilibrium statistical configuration. In particular, it is also clear for an arbitrary number of $D_0$ and $D_4$ branes that the associated state-space metric constraints as the diagonal pair correlation functions remain positive definite, viz., we have

$$g_{n_i n_i} > 0 \; \forall \; i \in \{0, 4\} \; | n_i > 0$$  \hspace{1cm} (4)

The case of finitely many $D_0$-$D_4$ branes indeed agrees with an expectation that the non diagonal component $g_{n_0 n_4}$ of the state-space metric tensor respectively finds some non-zero negative value. Furthermore, we visualize from the definition of state-space metric tensor that the ratios of principle components of Gaussian statistical pair correlations vary as inverse square of the concerned brane charges; while that of the off-diagonal correlations modulate only inversely. Interestingly from just designated state-space pair correlations of these two charge black hole configurations, it follows for distinct $i, j \in \{0, 4\}$ that the following expressions define possible set of admissible scaling relations

$$\frac{g_{i i}}{g_{j j}} = (\frac{n_j}{n_i})^2,\quad \frac{g_{i j}}{g_{i i}} = -\frac{n_i}{n_j}$$  \hspace{1cm} (5)

In order to determine the global properties of fluctuating two charge $D_0$-$D_4$ extremal configurations, we need to determine stabilities along each intrinsic directions, each intrinsic planes, and intrinsic hyper-planes, if any, as well as on the full intrinsic state-space manifold. Nevertheless, we notice that the underlying state-space manifold in the present case is just an ordinary intrinsic surface, and thus the set of stability criteria on various possible state-space configurations could simply be determined by the two possible principle minors, viz., $p_1$ and $p_2$. For all $n_0$ and $n_4$, we find that the first minor constraint $p_1 > 0$ directly follows from the positivity of the first component of metric tensor

$$p_1 = \frac{\pi}{2n_0} \sqrt{n_4^2}$$  \hspace{1cm} (6)

Whereas, the minor constraint, $p_2 > 0$ becomes the positivity of the determinant of metric tensor which nevertheless vanishes identically for all allowed values of the $n_0$ and $n_4$. In this case, we explicitly see that the minor constraints is not fulfilled, viz., the minor $p_2 := g(n_0, n_4)$ takes the null value, and thus the leading order consideration of degeneracy of the states of large charge $D_0$-$D_4$ extremal black branes or excited strings with $n_1$ number of winding and $n_p$ number of momenta find degenerate intrinsic state-space configurations.

\[^4\] In the present and subsequent sections, we shall invariably use for given set of brane and antibrane charges and angular momenta $X \in (X_1, X_2, \cdots, X_k) \in M_k$ that the tensor notations $g_{X_i X_j}$ and $g_{i j}$ signify the same intrinsic state-space object.
3 Three charge extremal configurations

To have test of a more charged black hole state-space configuration, we may add \( n_5 \) number of \( D_5 \) branes to the above excited string configuration, and then one finds that the leading order entropy of three charge extremal black hole may be obtained from the two derivative level Einstein-Hilbert action. It is well-known that the entropy of extremal \( D_1-D_5 \) solutions arising from Einstein-Hilbert action is proportional to the area of the horizon and the corresponding microscopic entropy may as well be counted by considering an ensemble of weakly interacting D-branes. It turns out that the two entropies match and they take the following form:

\[
S_{\text{micro}} = 2\pi \sqrt{n_1 n_5 n_p} = S_{\text{macro}}
\]  

(7)

The state-space geometry describing the correlations between the equilibrium microstates of the three charged rotating extremal \( D_1-D_5 \) black holes resulting from the degeneracy of the microstates may easily be computed as earlier from the Hessian matrix of the entropy with respect to the number of \( D_1, D_5 \) branes and Kaluza-Klein momentum, viz., \( n_1, n_5 \) and \( n_p \). We thence see that the components of state-space metric tensor are given by

\[
g_{n_1 n_1} = \frac{\pi}{2n_1} \sqrt{\frac{n_5 n_p}{n_1}}, \quad g_{n_1 n_5} = -\frac{\pi}{2} \sqrt{\frac{n_p}{n_1 n_5}}, \\
g_{n_5 n_5} = \frac{\pi}{2n_5} \sqrt{\frac{n_1 n_p}{n_5}}, \\
g_{n_1 n_p} = \frac{\pi}{2n_1} \sqrt{\frac{n_5 n_p}{n_1}}, \quad g_{n_5 n_p} = \frac{\pi}{2n_5} \sqrt{\frac{n_1 n_p}{n_5}}
\]

(8)

The statistical pair correlations thus ascertained could in turn be accounted by simple microscopic descriptions being expressed in terms of the number of \( D_1-D_5 \) branes and Kaluza-Klein momentum connoting an ensemble of microstates of the extremal black hole configurations. Furthermore, it is evident that the principle components of the pair correlation functions remain positive definite for all the allowed values of concerned three parameters of the black holes. As a result, we thus easily observe that the concerned state-space metric constraints are satisfied with

\[
g_{n_i n_i} > 0 \quad \forall \ i \in \{1, 5, p\} \quad |n_i| > 0
\]

(9)

We thus see in this case that the principle components of state-space metric tensor \( \{g_{n_i n_i}, g_{n_i n_p}\} \ i = 1, 5 \) essentially signify a set of definite heat capacities (or related compressibilities) whose positivity demonstrates that the three charge \( D_1-D_5-P \) black holes comply underlying locally stable equilibrium statistical configuration. Furthermore, we inspect that an addition of Kaluza-Klein momentum charge do not alter the conclusion of excited string system that the \( D_1-D_5-P \) configuration with finitely many \( D_1-D_5 \) branes and momentum excitations agrees with our naive expectation that respective non diagonal components, viz., \( g_{ij}, g_{ij}, g_{ip} \) and \( n_p \) of the state-space metric tensor can find some non-positive values.

Interestingly, the ratios of the principle components of metric tensor describing Gaussian statistical pair correlations vary as inverse square of the brane numbers and momentum charge; while that of the off-diagonal rations of the state-space correlations modulate only inversely. It further follows from the above expressions that we may explicitly visualize for distinct \( i, j \in \{1, 5\} \) and \( p \) that the list of relative correlation functions thus described is consisting of the following scaling properties

\[
\frac{g_{ii}}{g_{jj}} = \frac{(n_i)^2}{n_i} \quad \frac{g_{ii}}{g_{pp}} = \frac{(n_p)^2}{n_i}, \quad g_{ij} = -\frac{(n_j)}{n_i}, \\
g_{ij} = -\frac{(n_p)}{n_i}, \quad g_{ij} = \frac{(n_j)}{n_i}, \quad g_{jp} = -\frac{(n_j n_p)}{n_i} \\
g_{ip} = -\frac{(n_p)}{n_i}, \quad g_{ij} = \frac{(n_j)}{n_i}, \quad g_{pp} = -\frac{(n_p)^2}{n_i n_j}
\]

(10)

Along with the positivity of principle components of state-space metric tensor, we need to demand in order to accomplish the local stability of associated system that all the principle minors should be positive definite. It is nevertheless not difficult to compute the principle minors of the Hessian matrix of the entropy of three charge \( D_1-D_5-P \) extremal black holes. In fact, after some manipulations one encounters that the local stability conditions along the principle line and that of respective two dimensional surface of concerned state-space manifold be simply measured by the following equations

\[
p_1 = \frac{\pi}{2n_1} \sqrt{\frac{n_5 n_p}{n_1}}, \quad p_2 = -\frac{\pi^2}{4n_1 n_5 n_p} \left(n_5^2 n_1 + n_5^3\right)
\]

(11)
For all physically allowed values of brane numbers and momentum charge of the $D_1$-$D_5$-$P$ extremal black holes, we thus notice that the minor constraint $p_2(n_1, n_5, n_p) > 0$ never gets satisfied for any real positive physical parameters. In particular, we may easily inspect that the nature of state-space geometry for the three charge $D_1$-$D_5$-$P$ extremal black holes is that these solutions are stable along the line on state-space, but have planer instabilities. It is easy to stipulate that our conclusion holds for arbitrary number of $D_1$-$D_5$ branes and Kaluza-Klein momentum. In the view-point of the simplest two charge extremal solutions, it turns out that the local stability on the entire equilibrium phase-space configurations of the $D_1$-$D_5$-$P$ extremal black holes may clearly be determined by computing the determinant of the underlying state-space metric tensor. As in the previous example, it is easy to observe that the state-space metric tensor is a non-degenerate and everywhere regular function of the brane charges, $n_1$ and $n_5$ and Kaluza-Klein momentum charge $n_p$. In particular, we find the under present consideration that the determinant of the metric tensor as the highest principle minor $p_3 := g$ of the Hessian matrix of the entropy obtains a simple form 

$$\|g\| = -\frac{1}{2} n_1 n_5 n_p - 3^{1/2}$$

Moreover, we observe that the determinant of the metric tensor does not take a positive definite, well-defined form, and thus there is no positive definite globally well-defined volume form on the state-space manifold $(M_1, g)$ of concerned three charge $D_1$-$D_5$-$P$ extremal system. In turn, the non-zero value of the determinant of state-space metric tensor $g(n_1, n_5, n_p)$ indicates that the extremal $D_1$-$D_5$-$P$ solution may decay into some other degenerate vacuum state configurations procuring the same corresponding entropy or microscopic degeneracy of states. Here, we further notice independent of the microscopic type-II string description or heterotic string description that the three charge $D_1$-$D_5$-$P$ black holes when considered as a bound state of the $D_1$-$D_5$-brane microstates and Kaluza-Klein excitations do not correspond to an intrinsically stable statistical configuration. It is worth to mention as introduced in [15, 57] in order to divulge phase transitions and related global state-space properties that this statistical system remains everywhere regular as long as the number of brane and Kaluza-Klein momentum charge take finite values. In the next two sections, we shall deal with the state-space geometry of non-extremal black branes in string theory with two/ three charges and two/ three anticharges of leading order entropy configurations. After defining state-space metric tensor, we shall analyze scaling properties of possible state-space pair correlation functions and stability requirements for chosen non-extremal black brane solution.

## 4 Four charge non-extremal configurations

The present section examines state-space configuration of the non-extremal black holes, and extends our intrinsic geometric assessments for the $D_1$-$D_5$ black holes having non-zero momenta along the clockwise and anticlockwise directions of Kaluza-Klein compactification circle $S^1$. For the purpose of critical ratifications, we shall focus our attention on the state-space geometry arising from the entropy of non-extremal black hole, which one can simply achieve just by adding corresponding anti-branes to the chosen extremal black brane solution. What follows in precise that we shall first consider the simplest example of such systems, viz., a string having large amount of winding and $D_5$ brane charges: $n_1, n_5$ with some extra energy, which in the microscopic description creates an equal amount of momenta running in opposite directions of the $S^1$. In this case, the entropy has been calculated from both the microscopic and macroscopic perspective [11], and matches for given total mass and brane charges. In particular, it has been shown in [11] that the either of above entropies satisfy

$$S_{\text{micro}} = 2\pi \sqrt{n_1 n_5} (\sqrt{n_p} + \sqrt{n_p}) = S_{\text{macro}}$$

We thence analyze that the state-space covariant metric tensor defined as negative Hessian matrix of entropy with respect to number of $D_1, D_5$ branes $\{n_i \mid i = 1, 5\}$ and opposite Kaluza-Klein momentum charges $\{n_p, n_p\}$ The associate components of the state-space metric tensor and stability parameters are thus easy to compute for the non-extremal $D_1$-$D_5$ black holes. In fact, a direct computation finds that the components of the metric tensor take the following expression

$$g_{n_1 n_1} = \frac{\pi}{2} \sqrt{\frac{n_5}{n_1} (\sqrt{n_p} + \sqrt{n_p})}, \quad g_{n_1 n_5} = -\frac{\pi}{2\sqrt{n_1 n_5}} (\sqrt{n_p} + \sqrt{n_p})$$

$$g_{n_1 n_p} = -\frac{\pi}{2} \sqrt{\frac{n_5}{n_1 n_p}}, \quad g_{n_1 n_p} = \frac{\pi}{2} \sqrt{\frac{n_5}{n_1 n_p}}$$

$^2$ In this section, the notations $\overline{n_p}$ and $n_p$ shall imply the same Kaluza-Klein momentum charges which are in opposite direction of the $n_p$ momentum charge, and flow along the $S^1$. 

6
\begin{align}
g_{n_{a} n_{b}} &= \frac{\pi}{2} \sqrt{\frac{n_{a}}{n_{b}} (\sqrt{n_{p}} + \sqrt{n_{p}})}, \quad g_{n_{a} n_{p}} = -\frac{\pi}{2} \sqrt{\frac{n_{a}}{n_{p}}}
g_{n_{b} n_{p}} &= -\frac{\pi}{2} \sqrt{\frac{n_{b}}{n_{p}}}, \quad g_{n_{p} n_{p}} = \frac{\pi}{2} \sqrt{\frac{n_{1} n_{2}}{n_{p}}}
g_{n_{p} n_{p}} &= 0, \quad g_{\nu_{p} \nu_{p}} = \frac{\pi}{2} \sqrt{\frac{n_{1} n_{2}}{n_{p}}}
\end{align}

It is clear that there exist an intriguing intrinsic geometric enumeration which describes possible nature of statistical pair correlations. The present framework affirms in turn that the concerned state-space pair fluctuations determined in terms of the brane and anti-brane numbers (or brane charges) of the \( D_1-D_5-\mathcal{P} \) non-extremal black holes demonstrate definite expected behavior of the underlying heat capacities. Hitherto, we see apparently that the principle components of statistical pair correlations remain positive definite quantities for all admissible values of underlying configuration parameters of the black branes. It may easily be observed that the following state-space metric constraints are satisfied

\begin{align}
g_{n_{i} n_{i}} > 0 \quad \forall \ i = 1, 5; \quad g_{n_{a} n_{a}} > 0 \quad \forall \ a = p, \mathcal{P}
\end{align}

We thus physically note that the principle components of the state-space metric tensor \( \{g_{n_{i} n_{j}} \} \) \( \forall \ i, j \in \{1, 5\} \), and \( k, l \in \{p, \mathcal{P}\} \) describing four charge non-extremal \( D_1-D_5-\mathcal{P} \) black holes that the statistical pair correlations thus proclaimed consist the following set of scaling relations

\begin{align}
g_{i j} = \left( \frac{n_{j}}{n_{i}} \right)^{2} g_{i k} = \frac{n_{k}}{n_{i}^{2}} \sqrt{n_{k} (\sqrt{n_{p}} + \sqrt{n_{p}})}, \quad \frac{g_{i j}}{g_{i k}} = -\frac{n_{j}}{n_{i}}
g_{i k} = \frac{\sqrt{n_{i}}}{n_{i}} \sqrt{n_{p} + \sqrt{n_{p}}}, \quad \frac{g_{i k}}{g_{i j}} = \frac{n_{j}}{n_{i}} \frac{n_{i}}{n_{k}} \sqrt{n_{k} (\sqrt{n_{p}} + \sqrt{n_{p}})}
g_{i k} = \frac{n_{k}}{n_{i}} \frac{g_{i j}}{g_{i k}} = \sqrt{n_{i} (\sqrt{n_{p}} + \sqrt{n_{p}})}, \quad \frac{g_{i k}}{g_{i k}} = -\frac{n_{k}}{n_{i} n_{j}} \sqrt{n_{k} (\sqrt{n_{p}} + \sqrt{n_{p}})}
\end{align}

We further see that the list of other relative correlation functions concerning the non-extremal \( D_{1}-D_{5}-\mathcal{P} \) black holes are

\begin{align}
g_{i k} = \sqrt{n_{i} n_{k}} g_{i j} = \frac{n_{j}}{n_{i}^{2}} \sqrt{n_{i} n_{k} g_{i j}}, \quad \frac{g_{i k}}{g_{i j}} = 0
\end{align}

To investigate the entire set of geometric properties of fluctuating non-extremal \( D_{1}-D_{5} \) configurations, we need to determine stability along the each intrinsic directions, each intrinsic planes, as well as on the full intrinsic state-space manifold. Here, we may adroitly compute the principle minors from the Hessian matrix of associated entropy concerning the four charge string theory non-extremal black hole solution carrying \( D_{1}, D_{5} \) charges and left and right KK momenta. In fact, a simple manipulations discovers that the set of local stability criteria on various possible surfaces and hyper-surfaces of the underlying state-space configuration are respectively determined by the following set of equations

\begin{align}
p_{0} &= 1, \quad p_{1} = \frac{\pi}{2} \sqrt{\frac{n_{5}}{n_{1}} (\sqrt{n_{p}} + \sqrt{n_{p}})}
p_{2} &= 0, \quad p_{3} = -\frac{1}{2n_{p} \sqrt{n_{1} n_{5}}} (\sqrt{n_{p}} + \sqrt{n_{p}})
\end{align}

For all physically admitted values of the brane and antibrane charges (or concerned brane numbers) of the non-extremal \( D_{1}-D_{5} \) black holes, we can thus easily ascertain that the minor constraint, viz., \( p_{2}(n_{i}, n_{p}, \mathcal{P}) = 0 \) exhibits that the two dimensional state-space configurations are not stable for any value of the brane numbers and assigned Kaluza-Klein momenta. Similarly, the positivity of \( p_{1}(n_{i}, n_{p}, \mathcal{P}) \) for arbitrary number
of branes shows that the underlying fluctuating configurations are locally stable because of the line-wise positive definiteness.

The constraint $p_3(n_i, n_p, \overline{p}_p) > 0$ respectively imposes the condition that the system may ever not attain stability on three dimensional subconfigurations for all given positive Kaluza-Klein momenta and given positive $n_i$’s. In particular, these constraints enable us to investigate potential nature of the state-space geometric stability for leading order non-extremal $D_1$-$D_5$ black branes. We thus observe that the presence of planer and hyper-planer instabilities exist for the spherical horizon non-extremal $D_1$-$D_5$ solutions. We expect altogether in the view points of subleading higher derivative contributions in the entropy that the involved systems demand some restriction on allowed value of the Kaluza-Klein momenta and number of branes and antibranes.

Moreover, it is not difficult to enquire the complete local stability of the full state-space configuration of non-extremal $D_1$-$D_5$ black branes, and in fact it may simply be acclaimed by computing the determinant of the state-space metric tensor. Nevertheless, it is possible to enumerate compact formula for the determinant of the metric tensor. For the different allowed values of brane numbers, viz., \{n_1, n_3\} and Kaluza-Klein momenta \{n_p, \overline{p}_p\}, one apparently discovers from concerned intrinsic geometric analysis that the non-extremal $D_1$-$D_5$ system admits the following expression for the determinant of the state-space metric tensor

$$g(n_1, n_5, n_p, \overline{p}_p) = -\frac{\pi^4}{4(n_p \overline{p}_p)^{3/2}}(\sqrt{n_p} + \sqrt{\overline{p}_p})^2$$  \hspace{1cm} (19)

Furthermore, we may exhibit that the nature of the statistical interactions and the other global properties of the $D_1$-$D_5$ non-extremal configurations are indeed not really perplexing to anatomize. In this regard, one computes certain global invariants of the state-space manifold $(\mathcal{M}_4, g)$ which in the present case can easily be determined in terms of the parameters of underlying brane configurations. Here, we may work in the large charge limit in which the asymptotic expansion of the entropy of non-extremal $D_1$-$D_5$ system is valid. In particular, we notice that the state-space scalar curvatures indicated in [15] generically remains non vanishes for all finite value of the brane charges and Kaluza-Klein momenta. Thus for physically acceptable parameters, the large charge non-extremal $D_1$-$D_5$ black branes having non-vanishing scalar curvature function on their state-space manifold $(\mathcal{M}_4, g)$ imply an almost everywhere weakly interacting statistical basis.

### 5 Six charge non-extremal configurations

In this section, we shall consider state-space configuration for the six parameter non-extremal string theory black holes and focus our attention to analyze concerned state-space pair correlation functions and present stability analysis in detail. In order to do so, we extrapolate the expression of the entropy of four charge non-extremal $D_1$-$D_5$ solution to a non-large charge domain, where we are no longer close to an ensemble of supersymmetric states. It is known that the leading order entropy [70] which includes all such special extremal and near-extremal cases can be written as a function of charges \{n_i\} and anticharges \{\pi m_i\} to be

$$S(n_1, n_2, n_3, n_4, m_5) := 2\pi(\sqrt{n_1} + \sqrt{m_1})(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3})$$  \hspace{1cm} (20)

Incidentally, we notice from the simple brane and antibrane description that there exist an interesting state-space interpretation which covariantly describes various statistical pair correlation formulae arising from corresponding microscopic entropy of the aforementioned (non) supersymmetric (non) extremal black brane configurations. Furthermore, we see for given charges $i, j \in A_1 := \{n_1, n_3\}$; $k, l \in A_2 := \{n_2, m_2\}$; and $m, n \in A_3 := \{n_4, m_3\}$ that the intrinsic state-space pair correlations turn out to be in precise accordance with the underlying macroscopic attractor configurations being disclosed in the special leading order limit of the non-extremal solutions.

It is again not difficult to explore the state-space geometry of equilibrium microstates of the six charge anticharge non-extremal black holes in $D = 4$ arising from the entropy expression emerging from the consideration of Einstein-Hilbert action. As stated earlier, we find that the state-space Ruppeiner metric is defined by negative Hessian matrix of the non-extremal Bekenstein-Hawking entropy with respect to the extensive variables. These variables in this case are in turn the conserved charges-anticharges carried by the non-extremal black hole. Explicitly, we obtain that the components of covariant state-space metric tensor over generic non-large charge domains are

$$g_{n_1 n_1} = \frac{\pi}{2n_1^{1/2}}(\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3}), \quad g_{n_1 m_1} = 0 \quad g_{n_1 n_2} = -\frac{\pi}{2\sqrt{n_1 n_2}}(\sqrt{n_3} + \sqrt{m_3}), \quad g_{n_1 n_3} = -\frac{\pi}{2\sqrt{n_1 n_3}}(\sqrt{n_2} + \sqrt{m_2}) \quad g_{n_1 m_2} = -\frac{\pi}{2\sqrt{n_1 m_2}}(\sqrt{n_3} + \sqrt{m_3}) \quad g_{n_1 m_3} = -\frac{\pi}{2\sqrt{n_1 m_3}}(\sqrt{n_2} + \sqrt{m_2})$$
In the entropy representation, we thus see for the non-vanishing entropy that the Hessian matrix of entropy illustrates the nature of possible Gaussian state-space correlations between the set of space-time parameters which in this case are nothing other than the charges on the brane and anti-brane, if non-extremality is violated in general. Substantially, we articulate for given non-zero value of large charges and anti charges \( \{ n_i, m_i \mid i = 1, 2, 3 \} \) that the non-vanishing principle component of underlying intrinsic state-space metric tensor are positive definite quantities. It is in fact not difficult to see for distinct \( i, j, k \in \{1, 2, 3\} \) that the component involving brane-brane state-space correlations \( g_{m_i n_i} \) and antibrane-antibrane state-space correlations \( g_{m_i m_i} \) satisfy

\[
\begin{align*}
g_{n_i n_i} &> 0 \forall \text{ finite } n_i, i = 1, 2, 3 \\
g_{m_i m_i} &> 0 \forall \text{ finite } m_i, i = 1, 2, 3
\end{align*}
\]

Furthermore, it has been observed that the ratios of diagonal components vary inversely with a multiple of well-defined factor in the underlying parameters which change under the Gaussian fluctuations, whereas the ratios involving off diagonal components in effect uniquely inversely vary in the of parameters of chosen set \( A_i \) of equilibrium black brane configurations. This suggests that the diagonal components weaken in relatively controlled fashion into an equilibrium, than the off diagonal components which vary over the domain of space metric tensor are positive definite quantities. It is in fact not difficult to see for distinct \( i, j, k \in \{1, 2, 3\} \) that the relative pair correlation functions have three type of relative correlation functions. In particular, we firstly see for \( i, j \in \{n_1, m_1\} \), and \( k, l \in \{n_2, m_2\} \) that the relative correlation functions satisfy following list of scaling relations

\[
\begin{align*}
g_{ii} &\overset{k}{=} \frac{g_{kk}}{g_{kk}} = \frac{\sqrt{n_2} + \sqrt{m_2}}{\sqrt{n_3} + \sqrt{m_3}}, \quad g_{ii} = 0 \\
g_{ik} &\overset{k}{=} \frac{g_{jk}}{g_{jk}} = -\frac{\sqrt{n_2} + \sqrt{m_2}}{\sqrt{n_3} + \sqrt{m_3}}, \quad g_{ij} = -\frac{\sqrt{n_2} + \sqrt{m_2}}{\sqrt{n_3} + \sqrt{m_3}} \\
g_{ji} &\overset{k}{=} \frac{g_{jk}}{g_{jk}} = \frac{\sqrt{n_2} + \sqrt{m_2}}{\sqrt{n_3} + \sqrt{m_3}}, \quad g_{ij} = \frac{\sqrt{n_2} + \sqrt{m_2}}{\sqrt{n_3} + \sqrt{m_3}} \\
\end{align*}
\]

The other concerned relative correlation functions are

\[
\begin{align*}
g_{lk} &\quad g_{lk} = \sqrt{\frac{l}{k}}, \quad g_{lj} = \sqrt{\frac{l}{k}}, \quad g_{kl} = n.d. \\
\end{align*}
\]

For \( k, l \in \{n_2, m_2\} \), and \( m, n \in \{n_3, m_3\} \), we have

\[
\begin{align*}
g_{km} &\overset{k}{=} \frac{\sqrt{n_2} + \sqrt{m_2}}{\sqrt{n_3} + \sqrt{m_3}}, \quad g_{km} = 0, \quad g_{kk} = \frac{\sqrt{n_3}}{m}, \quad g_{kk} = \frac{\sqrt{n_2}}{m}, \\
g_{km} &\overset{k}{=} \frac{\sqrt{n_2} + \sqrt{m_2}}{\sqrt{n_3} + \sqrt{m_3}}, \quad g_{mn} = \frac{\sqrt{n_2}}{m}, \quad g_{mm} = \frac{\sqrt{n_3}}{m}, \\
\end{align*}
\]

\[
\]

9
The other concerned relative correlation functions are

\[
\frac{g_{im}}{g_{mm}} = \sqrt{\frac{m}{i}} \frac{g_{im}}{g_{in}} = \sqrt{\frac{m}{k}} \frac{g_{im}}{g_{mn}} = n.d.
\]

\[
\frac{g_{mm}}{g_{kk}} = 0, \quad \frac{g_{mm}}{g_{nn}} = (\frac{n}{m})^{3/2}, \quad \frac{g_{nn}}{g_{mm}} = 0
\]

(26)

While for \( i, j \in \{n_1, m_1\} \), and \( m, n \in \{n_3, m_3\} \), we have

\[
\frac{g_{ij}}{g_{mm}} = (\frac{m}{i})^{3/2} (\sqrt{n_3} + \sqrt{m_3}) \frac{g_{ij}}{g_{im}} = 0, \quad \frac{g_{ij}}{g_{im}} = -\frac{m}{i} (\sqrt{n_3} + \sqrt{m_3})
\]

\[
\frac{g_{im}}{g_{jm}} = 0, \quad \frac{g_{im}}{g_{mn}} = 0, \quad \frac{g_{im}}{g_{in}} = \sqrt{\frac{m}{n}}
\]

\[
\frac{g_{mn}}{g_{mm}} = \sqrt{\frac{m}{n}}, \quad \frac{g_{mn}}{g_{nn}} = n.d., \quad \frac{g_{nn}}{g_{mm}} = 0
\]

(27)

For given \( i, j \in A_1 := \{n_1, m_1\} \); \( k, l \in A_2 := \{n_2, m_2\} \); and \( m, n \in A_3 := \{n_3, m_3\} \), we thus see by utilizing \( g_{n_1 m_1} = 0, g_{n_3 m_2} = 0 \) and \( g_{n_3 m_3} = 0 \) that there are seven non-trivial relative correlation functions for each set \( A_i \), where \( i = 1, 2, 3 \), and one non-trivial ratio in each chosen family \( A_i \). It worth to mention that the scaling relations remain similar to those obtained in the previous case, except (i) the number of relative correlation functions has been increased, and (ii) the set of cross ratios, viz., \( \{ \frac{g_{ij}}{g_{mm}}, \frac{g_{im}}{g_{mn}}, \frac{g_{mn}}{g_{mm}} \} \) being zero in the previous case become ill-defined for the six charge state-space configuration. Inspecting specific pair of distinct charge sets \( A_i \) and \( A_j \), one finds in this case that there are thus 24 type of non-trivial relative correlation functions.

Specifically, we see for three brane and three antibrane solutions that the ratios involving diagonal components in the numerator with non-diagonal components in the denominator vanishes identically \( \forall i, j, k \in \{n_1, n_2, n_3, m_2, m_1, m_3\} \). Alternatively, we thereby appraise in this case that the set of principle components denominator ratios computed from above state-space metric tensor reduce to

\[
\frac{g_{ij}}{g_{kk}} = 0 \quad \forall i, j, k \in \{n_1, m_1, n_2, m_2, n_3, m_3\}
\]

(28)

In particular for given \( i, j \in A_1 := \{n_1, m_1\} \); \( k, l \in A_2 := \{n_2, m_2\} \); and \( m, n \in A_3 := \{n_3, m_3\} \), we confirm the above fact by utilizing \( g_{n_1 m_1} = 0, g_{n_3 m_2} = 0 \) and \( g_{n_3 m_3} = 0 \) that there are total 15 type of trivial relative correlation functions. It is not difficult to see there are five such trivial ratios in each chosen family \( \{A_i \mid i = 1, 2, 3\} \). It worth to mention for each set \( A_i \) that the trivial ratio reduce the scaling relations which are nevertheless similar to those realized in the previous case, except the fact that the number of relative correlation functions has been ill-defined. Inspecting a pair of distinct charge sets \( A_i \) and \( A_j \), one finds in this case that there is an unique kind of ill-defined relative correlations, and thus there in total three type of divergent relative correlation functions.

As noticed in the previous configuration, it is not difficult to analyze the local stability for the higher charged string theory non-extremal black holes, as well. In particular, one can easily determine the principle minors associated with the state-space metric tensor and thus we argue that all the principle minors must be positive definite. In this case, we may adroitly compute the principle minors from the Hessian matrix of associated entropy concerning the three charge and three anticharged black holes. In fact, after some simple manipulations we discover that the local stability criteria on the lower dimensional hyper-surfaces and two dimensional surface of underlying state-space manifold are respectively given by the following relations

\[
p_1 = \frac{\pi}{2(n_1)^{3/2}} (\sqrt{n_2} + \sqrt{m_2})(\sqrt{n_3} + \sqrt{m_3})
\]

\[
p_2 = \frac{1}{4(n_1 m_1)^{3/2}} (\sqrt{n_2} + \sqrt{m_2})^2 (\sqrt{n_3} + \sqrt{m_3})^2
\]

\[
p_3 = \frac{1}{8(n_1 m_1 n_2)^{3/2}} \sqrt{m_2}(\sqrt{n_2} + \sqrt{m_2})^3 (\sqrt{n_2} + \sqrt{m_2}) (\sqrt{n_3} + \sqrt{m_3})
\]

\[
p_4 = 0
\]

(29)

For all physically admitted values of associated charges and anticharges of the non-extremal string theory black holes, we thus ascertain that the minor constraint, viz., \( p_2 > 0 \) inhibits the domain of assigned brane anti-brane charges that it must be a positive definite real number, while the constraint \( p_3 > 0 \) imposes that the charges must respectively satisfy desired state-space minor conditions. In particular, these constraints
enables us to investigate the nature of the state-space geometry of string theory black holes. We have further observed that the presence of planar and hyper planar instabilities exist for the non-extremal black holes. It is worth to mention that the \( p_4(n_1, m_1) = 0 \) exhibits that the four dimensional state-space configurations are not stable for any value of the brane and antibrane numbers. This altogether demand for definite restriction on the allowed value of the parameters.

Similarly we find that the principle minor \( p_5 \) remains non-vanishing for all values of charges on the constituent brane and anti branes. The generic expression of the minor \( p_5 \) may further be easily computed from the general minor formula \([55]\). An explicit calculation specifically finds that the hyper-surface minor \( p_5 \) take fairly non-trivial value in general. However, the simplest values of the brane and antibrane charges that they be identical implies that the minor \( p_5 \) reduces to the specific value of

\[
p_5(k) = -64 \pi^2 \frac{k^5}{k^{5/2}}
\]

Thus for the identical values of the brane antibrane charges, the minor \( p_5 < 0 \) respectively implies that the non-extremal black hole solutions under consideration are not stable over the possible choice of the state-space configurations. In order to obtain the highest minor \( p_5 \), we in general need to compute determinant of the metric tensor which finally reduces as the function of the charges on branes and antibranes. Moreover, it is not difficult to demonstrate the global stability on the full state-space configuration, which may in fact be carried forward by computing determinant of the state-space metric tensor. In this case, one observes that the exact expression of the determinant of the intrinsic state-space metric tensor is

\[
\|g\| = \frac{\pi^6}{16} (n_1 m_1 n_2 m_2 n_3 m_3)^{3/2} (\sqrt{m_2} + \sqrt{m_3}) (\sqrt{m_3} + \sqrt{m_1}) (\sqrt{m_1} + \sqrt{m_2}) \left( n_2 \sqrt{m_1 n_3} + n_2 \sqrt{m_1 m_3} + 2 \sqrt{n_2 m_1 n_2 n_3} + 2 \sqrt{n_2 m_1 m_2 m_3} + m_2 \sqrt{n_1 m_3} + m_2 \sqrt{n_1 n_3} + m_2 \sqrt{n_1 m_3} + m_2 \sqrt{n_1 n_3} + m_2 \sqrt{n_1 m_3} \right)
\]

which in turn never vanishes for domain of given non-zero brane antibrane charges, except for following state-space extreme values of the charges, when the brane and antibrane charges \( n_i, m_i \) belong to

\[
B := \{ (n_1, n_2, n_3, m_1, m_2, m_3) | n_2 \sqrt{m_1 n_3} + n_2 \sqrt{m_1 m_3} + 2 \sqrt{n_2 m_1 m_2 m_3} + 2 \sqrt{n_2 m_1 n_2 n_3} + 2 \sqrt{n_2 m_1 m_2 m_3} + m_2 \sqrt{n_1 n_3} + m_2 \sqrt{n_1 m_3} + m_2 \sqrt{n_1 n_3} + m_2 \sqrt{n_1 m_3} = 0 \}
\]

We may further note that the entire state-space configuration remains positive definite for potential value of the \( n_i, m_i \). We thus observe that the underlying state-space geometry of six charge non-extremal string theory configurations are in well compliance and in turn they generically correspond to a non-degenerate fluctuating statistical basis as an intrinsic Riemannian manifold \( \mathcal{M}_6 \setminus B \). Furthermore, we see that the components of the covariant Riemann tensors may become zero for definite values of the charges on branes and antibranes. In addition, the Ricci scalar curvature diverges at the same set of points on state-space manifold \( (\mathcal{M}_6, g) \), as that of the roots of the determinant of metric tensor, \( v_{i2} \), the points defined by the set \( B \).

There exists an akin single higher degree polynomial equation on which we precisely find that the Ricci scalar curvature becomes null, and exactly at these points defining the state-space configuration of the underlying (extremal or near-extremal or general) non-large charge black hole system, at which it corresponds to some non-interacting statistical system, where the state-space manifold \( (\mathcal{M}_6, g) \) is curvature free. A systematic calculation further shows that the general expression of the Ricci scalar is quite involved, and even for equal brane charges \( n_1 := n \), \( n_2 := n \), \( n_3 := n \), and equal antibrane charges \( m_1 := m \), \( m_2 := m \), \( m_3 := m \), the result do not sufficiently simplifies. Nevertheless, we find for the identical large values of brane and antibrane charges \( n := k \) and \( m := k \) [15] that there exist an attractive state-space configuration for which the expression of corresponding curvature scalar reduces to a particular small negative value of

\[
R(k) = -\frac{15}{16} \frac{1}{\pi k^{3/2}}
\]

### 6 Multi-centered \( D_6 D_4 D_2 D_0 \) Black Branes

The present section explores the state-space manifold containing the both single centered black brane solutions and multi-centered black brane configurations, viz., we shall study the state-space geometry whose co-ordinates are defined in terms of four charges of the \( D_6 D_4 D_2 D_0 \) black brane configurations. Here, we shall explicitly present the analysis arising from the entropy of stationary
single-centered black hole configurations have recently been examined by the so-called pin-sized D-brane systems [45, 46] and thus we intend to realize underlying state-space geometry arising from the counting entropy of the number of microstates of zoo of entropically dominant multi-centered black hole configurations along with usual single centered black holes.

It has been shown [45, 46] in suitable parameter regimes that the multi-centered entropy dominates the single centered entropy in the uniform large charge limit. Following [45], we shall here investigate the state-space geometric implication for the single center and two centers of the multi-centered brane systems. In this connection, we may consider a charge \( \Gamma = \sum_i \Gamma_i \), \( \Gamma_i \) obtained by wrapping the \( D_i \) branes around various cycles of a compact space \( X \), and the concerned charges are scaled as \( \Gamma \rightarrow \Lambda \Gamma \), and then there exist two centered brane solution with horizon entropy scaling as \( \Lambda^3 \); while that of the single centered entropy simply scales as \( \Lambda^2 \). More properly as analyzed in [45, 46], consider the type IIA string theory compactified on a product of three two-tori \( X = T_2^1 \times T_2^2 \times T_2^3 \). Then, the entropy as a function the charge \( \Gamma \) corresponding to \( p_0 \) \( D_0 \) branes on \( X, p \ D_4 \) branes on \( (T_2^1 \times T_2^2 + (T_2^2 \times T_2^3) + (T_2^3 \times T_2^1) \), \( q \ D_2 \) branes on \( (T_2^1 + T_2^2 + T_2^3) \) and \( q_0 \) \( D_0 \) branes is given by

\[
S(\Gamma) = \pi \sqrt{-4p^2q^2 + 3pq^2 - p^4q + 6pq - (p^2q)^2 - (p^2q)^3 - (p^2q)^4 - (p^2q)^5 - (p^2q)^6}
\]

(34)

The state-space geometry constructed out of the equilibrium state of the four charged \( D_6 \) \( D_4 \) \( D_2 \) \( D_0 \) black branes resulting from the entropy may thus be easily computed as earlier from the negative Hessian matrix of the entropy with respect to the \( D_6, D_4, D_2, D_0 \) brane charges \( \Gamma_i := (p_i^2, q_i^2) \) which in effect form the coordinates of the intrinsic state-space manifold. Explicitly, we find that the components of the covariant metric tensor are given as

\[
g_{pq} = -4\pi \frac{-3p^2q^2q^0q^0 + 3pq^2q^0 - q^6 + p^3q^3}{(-4p^2q^0 + 3p^2q^2 + 6pq^0q^0 - 4pqq^0 - (p^2q)^2)^{3/2}}
\]

\[
g_{pq} = 6\pi \frac{-p^3q^2q^0 + 2p^2q^2q^0 + p^2q^0q^0 - pq^0 - 2pq^2q^2q^2 + pq^2q^2q^2}{(-4p^2q^0 + 3p^2q^2 + 6pq^0q^0 - 4pqq^0 - (p^2q)^2)^{3/2}}
\]

\[
g_{pq} = -12\pi \frac{-12p^2q^2q^0q^0 - 3pq^2q^0q^0 + 4pq^4q^0q^0 - 6pq^2q^0q^2 + 6pq^2q^0q^2 + 6pq^2q^0q^2}{(-4p^2q^0 + 3p^2q^2 + 6pq^0q^0 - 4pqq^0 - (p^2q)^2)^{3/2}}
\]

\[
g_{pq} = -12\pi \frac{-12p^2q^2q^0q^0 - 3pq^2q^0q^0 + 4pq^4q^0q^0 - 6pq^2q^0q^2 + 6pq^2q^0q^2 + 6pq^2q^0q^2}{(-4p^2q^0 + 3p^2q^2 + 6pq^0q^0 - 4pqq^0 - (p^2q)^2)^{3/2}}
\]

\[
g_{pq} = 3\pi \frac{-12p^2q^2q^0q^0 - 3pq^2q^0q^0 + 4pq^4q^0q^0 - 6pq^2q^0q^2 + 6pq^2q^0q^2 + 6pq^2q^0q^2}{(-4p^2q^0 + 3p^2q^2 + 6pq^0q^0 - 4pqq^0 - (p^2q)^2)^{3/2}}
\]

\[
g_{pq} = -12\pi \frac{-12p^2q^2q^0q^0 + p^3q^2q^2q^0 + 3pq^2q^2q^2q^0 - 6pq^2q^2q^2q^0 - p^2q^2}{(-4p^2q^0 + 3p^2q^2 + 6pq^0q^0 - 4pqq^0 - (p^2q)^2)^{3/2}}
\]

\[
g_{pq} = 6\pi \frac{-12p^2q^2q^0q^0 - 3pq^2q^0q^0 + 4pq^4q^0q^0 - 6pq^2q^0q^2 + 6pq^2q^0q^2 + 6pq^2q^0q^2}{(-4p^2q^0 + 3p^2q^2 + 6pq^0q^0 - 4pqq^0 - (p^2q)^2)^{3/2}}
\]

\[
g_{pq} = -12\pi \frac{-12p^2q^2q^0q^0 + p^3q^2q^2q^0 + 3pq^2q^2q^2q^0 - 6pq^2q^2q^2q^0 - p^2q^2}{(-4p^2q^0 + 3p^2q^2 + 6pq^0q^0 - 4pqq^0 - (p^2q)^2)^{3/2}}
\]

(35)

In order to simplify the presentation, we shall define \( X_i = (p_0, p, q, q) \) and subsequently use the following set of notations \( 1 \leftrightarrow p_0, 2 \leftrightarrow p, 3 \leftrightarrow q, 4 \leftrightarrow q_0 \). Employing the either of above notations, we observe from the definition that the ascertained statistical pair correlations may in turn be accounted by simple microscopic descriptions which can be expressed in terms of the brane charges connoting an ensemble of microstates of the multicentered black hole configuration. Furthermore, it is in fact evident that the principle components of statistical pair correlations are positive definite for all allowed values of the concerned parameters of the \( D_6-D_4-D_2-D_0 \) black holes. As a result, we can easily see that the concerned state-space metric constraints are defined by

\[
g_{ii}(X_i) > 0 \ \forall \ i \in \{1, 2, 3, 4\} \ \text{if \ } m_{ii} < 0
\]

(36)

The principle components of state-space metric tensor \( g_{ii}(X_i) \) \( i = 1, 2, 3, 4 \) essentially signify a set of definite heat capacities (or the related compressibility) whose positivity apprises that the black brane solution complies an underlying locally equilibrium statistical configuration. It is intriguing to note that the positivity of the components \( g_{ii} \) require that the brane charges of associated multicentered \( D_6-D_4-D_2-D_0 \)
black holes should satisfy the above constraints. This is indeed admissible because of the fact that the brane configuration divulges physically stable system for all values of the brane charges satisfying Eqn. (36) with

\[
\begin{align*}
m_{11} & := -3p^2q^2q0^2 + 3pq^40q - q^6 + p^3q^3 \\
m_{22} & := p^2q^2p0^2 - p^3q^2q0 - 3p^2q0p0^3 + 4pq^30q0p \\
m_{33} & := 4p^2q0p0q - p^2q^2p0 - p^2q0^2p0^2 - 3pq^2p0^2q0 \\
m_{44} & := -p^6 + 3p^4p0q - 3p^2q^2p^2 + p^6q^3
\end{align*}
\]

(37)

From the above expressions of metric tensor, we visualize that the ratios of the principle components of statistical pair correlations vary as definite function of the asymptotic charges; while that of the off-diagonal correlations modulate slightly differently. Interestingly, it follows for the distinct \(i,j,k,l \in \{1,2,3,4\}\) that the admissible statistical pair correlations thus connoted are consisting of diverse scaling properties. The set of nontrivial relative correlations signifying possible scaling relations of the state-space correlations may nicely be depicted by

\[
C_r = \left\{ \begin{array}{cccc}
g_{11} & g_{11} & g_{11} & g_{11} \\
g_{12} & g_{13} & g_{14} & g_{23} \\
g_{21} & g_{24} & g_{34} & g_{44} \\
g_{31} & g_{32} & g_{33} & g_{34} \\
g_{41} & g_{42} & g_{43} & g_{44} \\
g_{42} & g_{43} & g_{44} & g_{44} \end{array} \right\}
\]

(38)

The local stability condition of the underlying statistical configuration under the Gaussian fluctuations requires that all the principle components of the fluctuations should be positive definite, i.e. for given set of state-space variables \(\Gamma_i := (p_i, q_{3,i})\) one must demands that \(\{g_{ii}(\Gamma_i) > 0; \forall i = 1,2\}\). In particular, it is important to note that this condition is not sufficient to insure the global stability of chosen multicentered configuration and thus one may only accomplish certain locally equilibrium statistical configuration. It is however worth to mention that the complete stability condition requires that all the principle components of the Gaussian fluctuations should be positive definite and the others components of the fluctuations should vanish. In order to ensure this condition, we can observe that all the principle components and all the principle minors of the metric tensor must be strictly positive definite. This implies that the global stability condition constrained the allowed domain of the parameters of black hole configurations, which are interestingly expressed by the following set of simultaneous equations

\[
\begin{align*}
p_1 &= -4\pi \frac{(-3p^2q^2q0^2 + 3pq^40q - q^6 + q0^3p^3)}{(-4p^4q0 + 3pq^2q^2 + 6pq0p0q0 - 4pq^2p0^2q^2)^{-3/2}} \\
p_2 &= -12\pi^2 \frac{(q0^3p^4 - 4q^2q0^2p^3 + 6q^2q0^2p^2 - 4q^2q0^2p + q^6)}{(-4p^4q0 - 3pq^2q^2 + 6pq0p0q0 + 4pq^2p0^2q^2)^{-1}} \\
p_3 &= -36\pi^3 \frac{(-3pq^2q0^2 + 3pq^40q - q^6 + q0^3p^3)}{(-4p^4q0 + 3pq^2q^2 + 6pq0p0q0 - 4pq^2p0^2q^2)^{-3/2}}
\end{align*}
\]

(39)

In addition, it is evident that the local stability of the full state-space configuration can likewise be determined by computing the determinant of the metric tensor of concerned state-space geometry. Here, we may easily compute a compact formula for the determinant of the metric tensor as the function of various possible values of brane charges, and in particular, our intrinsic geometric analysis assigns the following constant expression to the determinant of the metric tensor

\[
\|g\| = 9n^4
\]

(40)

As the determinant of basic state-space metric tensor is constant and positive quantity in the viewpoints of large charge consideration in which one acquires a non-vanishing central charge of corresponding \(D_0-D_4-D_2-D_0\) CFT configurations [25, 26]. Our analysis herewith discovers that there exists non-degenerate state-space geometry for the leading multi-centered configurations. Furthermore, it is worth to note that the determinant of the metric tensor takes positive definite form, which in turn shows that there is positive definite volume form on the concerned state-space manifold \((M_n, g)\) of the multi-centered \(D_0-D_4-D_2-D_0\) black brane configurations at the leading order contributions.

Intelligibly, it further follows from the fact that the responsible equilibrium entropy tends to its maximum value, while the same culmination may not remain valid on the chosen planes or hyper-planes of the entire state-space manifolds of the single centered and double centered configurations. It is thus envisaged for
the either single or double center descriptions or the dual CFT descriptions that the multi-centered black branes do correspond to intrinsically stable statistical configurations. Thus, it is indeed plausible that the underlying ensemble of CFT microstates upon subleading higher derivative corrections live in the same basis of $D_6\cdot D_4\cdot D_2\cdot D_0$ brane charges.

### 6.1 State-space correlations of the single center configurations

For the charges, $p^0 := 0$; $p := 6\Lambda$; $q := 0$; $q^0 := -12\Lambda$; describing the single center configurations considered in [45, 46], we see that the above state-space correlation functions reduce to the following values

$$
\begin{align*}
  g_{11} &= \pi\sqrt{2}, \quad g_{13} = \frac{3}{2}\pi\sqrt{2} = -g_{22} \\
  g_{24} &= \frac{3}{4}\pi\sqrt{2} = -g_{33}, \quad g_{44} = \frac{1}{8}\pi\sqrt{2} \\
  g_{12} &= 0 = g_{43} = g_{23} = g_{34} \\
\end{align*}
$$

Following the previously acclaimed notations, we observes that the statistical pair correlations being accounted by simple state-space characterization can be expressed in terms of the brane charges. Furthermore, an easy analysis find that all the principle components of the statistical pair correlations are positive definite for chosen value of parameters of the single center black holes. In particular, we see for $p^0 := 0$; $p := 6\Lambda$; $q := 0$; $q^0 := -12\Lambda$ that the concerned state-space metric constraints can for all $\Lambda$ be depicted as

$$
\begin{align*}
  g_{ij}(X_a) &> 0 \forall i = 1, 3 \\
  g_{ij}(X_a) &< 0 \forall j = 2, 4 \\
\end{align*}
$$

The principle components of state-space metric tensor $\{g_{ij} \mid i = 1, 2, 3, 4\}$ signifying a set of heat capacities (or the related compressibility) do not all find positive values. Here, a violation of the positivity of heat capacity apprises that the corresponding single center black brane solution corresponds to a locally unstable statistical configuration over the Gaussian fluctuations. It is thus important to mention that the positivity of principle components do not holds for the above set of brane charges associated with the single centered $D_6D_4D_2D_0$ black branes.

In analyzing the other state-space constraints, we see that the relative correlations defined as $c_{ij} := g_{ij}/g_{ii}$ reduce to the three set of constant values. Firstly following the proclaimed procedure, we find that there are only 15 non vanishing finite ratios defining the relative state-space correlation functions for the single center configuration

$\begin{align*}
  c_{1113} &= \frac{2}{3} = -c_{1122}, \quad c_{1124} = \frac{4}{3} = -c_{1133} \\
  c_{1144} &= 8, \quad c_{1222} = -1 = c_{2433} \\
  c_{1324} &= 2 = -c_{1333}, \quad c_{2233} = 2 = -c_{2224} \\
  c_{1344} &= 12 = -c_{2244}, \quad c_{2444} = 6 = -c_{3344} \\
\end{align*}$

Furthermore, an observation finds that the set of vanishing ratios of relative correlation functions is

$$
C_R^0 := \{c_{1213}, c_{1224}, c_{1222}, c_{1224}, c_{1233}, c_{1244}, c_{1422}, \\
C_{1424}, c_{1433}, c_{1444}, c_{2234}, c_{2333}, c_{2344}, c_{3344}\} = \{0\}
$$

In particular, we notice for $p^0 := 0$; $p := 6\Lambda$; $q := 0$; $q^0 := -12\Lambda$ that there exist limiting ill-defined relative correlations. In particular, the concerned ratios get numeric exception and they receive a division by zero when approaching the single centered configuration. Thus, the characterization of the relative state-space pair correlation may be accomplished by the set

$$
C_R^\infty := \{c_{1112}, c_{1114}, c_{1123}, c_{1134}, c_{1223}, c_{1234}, c_{1314}, c_{1323}, \\
C_{1334}, c_{1423}, c_{1434}, c_{2225}, c_{2234}, c_{2334}, c_{2434}, c_{3334}\} = \{\infty\}
$$

### State-space stability of single center $D_6\cdot D_4\cdot D_2\cdot D_0$ configurations

Furthermore, we see that the entropy corresponding to single center specification takes to a constant value of $S(\Gamma = \Lambda(0, 6, 0, -12)) = \pi\sqrt{10368}\Lambda^2$. Whilst, it is interesting to note that the possible stability of internal configurations under the Gaussian fluctuations reduce to the positivity of

$$
p_1 = \sqrt{2}\pi, \quad p_2 = -3\pi^2, \quad p_3 = 9\sqrt{2}\pi^3
$$
We thus find for the chosen values of brane charges which physically describe single center system that the statistical configuration has definite stability and instability character. In particular, positivity of \( p_1(0, 6Λ, 0, −12Λ) \) shows that the underlying brane configurations are locally stable on an intrinsic state-space line. Nevertheless, we observe for the chosen value of brane charges that the two dimensional surfaces of the single center state-space configurations are not stable. This has in turn been easily ascertained via the fact that the associated surface minor constraint is not satisfied. Specifically, it turns out for chosen set of charge \((0, 6Λ, 0, −12Λ)\) that the system exhibits a negative surface minor, viz., we have \( p_2(0, 6Λ, 0, −12Λ) < 0\).

Similarly, one may however notice that the system turns out to be stable on three dimensional hyper-surfaces of the single center configurations. The argument follows directly from the fact that the hypersurface minor \( p_3(0, 6Λ, 0, −12Λ) \) picks up a positive definite value.

More generally, it interesting to note that the general expression of the determinant of the metric tensor defined as \( g = 9π^2 \) remains constant for entire domain of brane parameters. In addition, we find for all non zero entropy solution that the state-space scalar curvature signifying global correlation volume of an underlying statistical system has no divergence. As expected further from [15] that the leading order entropy solutions defining the single solutions confirm for all admissible values of brane charges that their state-space correlation volume vary as an inverse function of the single center brane entropy arising from the degeneracy of equilibrium statistical configurations.

### 6.2 State-space correlations of the two center configurations

In this subsection, we shall explicitely present the state-space geometry of \( D_6-D_4-D_2-D_0 \) black holes in string theory carrying set of respective \( D \)-brane charges. We notice that these solutions carry different state-space pair correlations. For given \( Λ \), this follows from the fact that the two center black holes in general have particular correlations which are not the same for both of the centers or as that of the single center counter parts. We nevertheless find that the global state-space correlations which characterize stability of the vacuum string theory configurations of either center do not change for the choice of brane charges considered in [15] [16].

#### 6.2.1 State-space correlations at the first center

For the value of brane charges \( Λ \neq 0; \ p := 3Λ; \ q := 6Λ^2; \) and \( q0 := −6Λ \) defining first center of the two center \( D_6-D_4-D_2-D_0 \) configurations, we have the following components of the state-space metric tensor

\[
\begin{align*}
g_{11} &= 108πΛ^3 \frac{6Λ^2 + 12Λ^4 + 8Λ^6 + 1}{(3Λ^4 - 1)^{3/2}}, \quad g_{12} = -54πΛ^3 \frac{7Λ^2 + 16Λ^4 + 12Λ^6 + 1}{(3Λ^4 - 1)^{3/2}} \\
g_{13} &= 54πΛ^3 \frac{4Λ^2 + 4Λ^4 + 1}{(3Λ^4 - 1)^{3/2}}, \quad g_{14} = -π \frac{18Λ^4 + 27Λ^6 - 1}{(3Λ^4 - 1)^{3/2}} \\
g_{22} &= 18πΛ^3 \frac{13Λ^2 + 30Λ^4 + 24Λ^6 + 2}{(3Λ^4 - 1)^{3/2}}, \quad g_{23} = -3π \frac{42Λ^4 + 12Λ^2 + 45Λ^6 + 1}{(3Λ^4 - 1)^{3/2}} \\
g_{24} &= 9πΛ^3 \frac{1 + 2Λ^2}{(3Λ^4 - 1)^{3/2}}, \quad g_{33} = 3πΛ^3 \frac{2 + 9Λ^2 + 12Λ^4}{(3Λ^4 - 1)^{3/2}} \\
g_{34} &= -3πΛ^3 \frac{1 + 3Λ^2}{(3Λ^4 - 1)^{3/2}}, \quad g_{44} = \frac{1}{2} πΛ^3 \frac{1}{(3Λ^4 - 1)^{3/2}} 
\end{align*}
\]  

Simplifying subsequent notations by defining \( c_{ijkl} := g_{ij}/g_{kl} \), we then see at the first center of the two center \( D_6-D_4-D_2-D_0 \) that the relative state-space correlations describing concerned statistical system are physically sound in nature. We in this case see that it is not difficult to compute the \( c_{ijkl} \). Nevertheless, the exact expression for the set of \( c_{ijkl} \) is quite involved and thus we relegate them to the Appendix (A).

#### 6.2.2 State-space correlations at the second center

The value of charges \( Λ \neq 0; \ p := 3Λ; \ q := -6Λ^2; \) and \( q0 := -6Λ \) define the second center of the two center configurations for which we have the following limiting values of the state-space pair correlation functions

\[
\begin{align*}
g_{11} &= 108πΛ^3 \frac{6Λ^2 + 12Λ^4 + 8Λ^6 + 1}{(3Λ^4 - 1)^{3/2}}, \quad g_{12} = 54πΛ^3 \frac{7Λ^2 + 16Λ^4 + 12Λ^6 + 1}{(3Λ^4 - 1)^{3/2}} \\
g_{13} &= 54πΛ^3 \frac{4Λ^2 + 4Λ^4 + 1}{(3Λ^4 - 1)^{3/2}}, \quad g_{14} = π \frac{18Λ^4 + 27Λ^6 - 1}{(3Λ^4 - 1)^{3/2}} \\
g_{22} &= 18πΛ^3 \frac{13Λ^2 + 30Λ^4 + 24Λ^6 + 2}{(3Λ^4 - 1)^{3/2}}, \quad g_{23} = -3π \frac{42Λ^4 + 12Λ^2 + 45Λ^6 + 1}{(3Λ^4 - 1)^{3/2}} \\
g_{24} &= 9πΛ^3 \frac{1 + 2Λ^2}{(3Λ^4 - 1)^{3/2}}, \quad g_{33} = 3πΛ^3 \frac{2 + 9Λ^2 + 12Λ^4}{(3Λ^4 - 1)^{3/2}} \\
g_{34} &= -3πΛ^3 \frac{1 + 3Λ^2}{(3Λ^4 - 1)^{3/2}}, \quad g_{44} = \frac{1}{2} πΛ^3 \frac{1}{(3Λ^4 - 1)^{3/2}} 
\end{align*}
\]
\[ g_{24} = \frac{9\pi\Lambda^3}{(3\Lambda^4 - 1)^{3/2}}, \quad g_{33} = 3\pi\Lambda \frac{2 + 9\Lambda^2 + 12\Lambda^4}{(3\Lambda^4 - 1)^{3/2}} \]
\[ g_{34} = \frac{3\pi\Lambda^2}{(3\Lambda^4 - 1)^{3/2}}, \quad g_{44} = \frac{1}{2} \frac{\pi\Lambda^3}{(3\Lambda^4 - 1)^{3/2}} \quad (48) \]

Employing the previously defined notations, it has similarly seen that the relative correlations \( c_{ij;\lambda} := g_{ij}/g_{00} \) of the state-space configuration concerning second center of the \( D_6-D_4-D_2-D_0 \) system simplify to the one which have been presented in the Appendix (B).

**State-space stability of double center \( D_6-D_4-D_2-D_0 \) configurations**

We shall now consider state-space stability for the two center black brane configurations and analyzes the related positivity properties of their underlying statistical pair correlation functions and correlation volumes for the basins of \( D_6-D_4-D_2-D_0 \) configurations. At the first and second centers of the two center \( D_6-D_4-D_2-D_0 \) configurations, we find in turn that the mentioned statistical pair correlations can be simply accounted by a common factor of the charges \( \Gamma_i \). These notions further receive supports from microscopic descriptions as well that an ensemble of microstates of the multicentered black hole configurations could effectively be expressed in terms of \( \Lambda \) as such basis are simply connoted via the invariant brane charges \( D_i \).

As indicated by Denef and Moore in \([45, 46]\) that the two centered bound state configurations arise with charge centers \( \Gamma_1 = (1,3\Lambda, 6\Lambda^2, -6\Lambda) \) and \( \Gamma_2 = (-1,3\Lambda, -6\Lambda^2, -6\Lambda) \). Thus, we shall focus our attention for these charge centers and analyze their state-space quantities as the function of \( \Lambda \). It is apparent for some given \( \Lambda \) that the entropies at either of the two centers \( \Gamma_1, \Gamma_2 \) match, and in particular, we find that the double center entropy varies as

\[ S(\Gamma_1) = S(\Gamma_2) = \pi \sqrt{108\Lambda^6 - 36\Lambda^2} \sim \Lambda^3 \quad (49) \]

For given charge centers \( \Gamma_1 \) and \( \Gamma_2 \), we may apart from definite scaling in \( \Lambda \) appreciate over an equilibrim statistical basis that either of the state-space pair correlation function as defined in the Eqn. (17) and Eqn. (49) can be realized as an even function of the parameter \( \Lambda \). Similarly, one can contemplate possible nature of the pair correlation functions over the jump of one center to the other. In turn for chosen \( \Gamma_1 \) and \( \Gamma_2 \), we find that the above two center \( D_6-D_4-D_2-D_0 \) configurations form two type of state-space pair correlation functions. In particular, we see from the Eqn. (17) and Eqn. (49) that the two proclaimed set of pair correlations are

\[ C^{(1)}_{ij}(\Gamma) := \{ g_{ij}(\Gamma_1) = g_{ij}(\Gamma_2); (i, j) \in \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 3), (3, 4), (4, 4)\} \}
\[ C^{(2)}_{ij}(\Gamma) := \{ g_{ij}(\Gamma_1) = -g_{ij}(\Gamma_2); (i, j) \in \{(1, 2), (1, 4), (2, 3)\} \} \quad (50) \]

It has explicitly been seen for non-vanishing \( \Lambda \) that the state-space pair correlations belonging to \( C^{(1)} \) remain the same for both the centers, while the pair correlations belonging to the set \( C^{(2)} \) change their signature. The present analysis implies that the principle components of the metric tensor defining equilibrium statistical pair correlations are positive definite for all allowed values of the parameter \( \Lambda \). In fact, the \( \Lambda \) being the single parameter for the both first and second centers of the two center \( D_6-D_4-D_2-D_0 \) black branes describes potential stability and state-space correlation properties of the \( D_6-D_4-D_2-D_0 \) multi-center configurations. As a result, we see for all \( \Lambda \) and for either of the two centers, viz., \( \Gamma_1 \) and \( \Gamma_2 \) that the respective state-space metric constraints satisfy

\[ g_{ii}(X_a) > 0 \quad \forall \, i \in \{1, 2, 3, 4\} \quad (51) \]

Furthermore, it is intriguing to note that the double center black hole configurations arise with two different charge vectors have the same set of principle minors. In particular, we find that both the first center carrying charges, \( p_0 := 1; \quad p := 3\Lambda; \quad q := 6\Lambda^2; \quad q_0 := -6\Lambda; \) and the second center carrying charges, \( p_0 := -1; \quad p := 3\Lambda; \quad q := -6\Lambda^2; \quad q_0 := -6\Lambda \) have the same principle minors

\[ p_1 = 108\pi|\Lambda|^3 \frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}} \]
\[ p_2 = -972\pi^2|\Lambda|^4 \frac{1 + 8\Lambda^2 + 24\Lambda^4 + 32\Lambda^6 + 16\Lambda^8}{(3\Lambda^4 - 1)^{2}} \]
\[ p_3 = 972\pi^2|\Lambda|^3 \frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}} \quad (52) \]

Finally, it interesting to note that the general expression of determinant of the metric tensor implies well-defined state-space manifold \( (M_4, g) \), and in addition, we find that the state-space scalar curvature
signifying global correlation properties of underlying statistical system have no divergence for all non-zero entropy solution. As expected, this confirm for all admissible values of brane charges that the correlation volume of both the single center solution and double center solutions modulate as inverse function of the entropy associate with chosen basin of the $D_0$-$D_4$ multi-centered black brane configurations.

7 Fractionation of Branes: $D_0$-$D_4$ Black Holes

The present section as an elucidation of the state-space geometry of small black holes in string theory carries set of electric charges and a magnetic charge. We notice that the state-space geometry of these solutions are natural to analyze in the type-II string theory description. In particular, it is known that these black holes in general may carry finite number of clusters parameters, $v_{zz}$, electric charges and magnetic charge which characterize the vacuum string theory configurations made out of $D_0$ branes and $D_4$ branes [2-7]. Furthermore, the general details of [1, 74, 75, 76, 77] have been noteworthy towards some of our subsequent considerations.

In order to make contact of state-space geometry with definite microscopic perspective, let us consider the chiral primaries of $SU(1, 1 \mid 2)_Z$ and then the associated supersymmetric ground states of $\mathcal{N} = 4$ supersymmetric quantum mechanics [32] furnishes an understanding of the microscopics of small black holes and concerned electric brane fractionations. In this consideration, we may easily see that the are $24p$ bosonic chiral primaries with total $D_0$ brane charge $N$ in the background with fixed magnetic $D_4$ charge $p$. Then, the degeneracy involved with the counting of microstates arises from the combinatorics of total $N$ number of the $D_0$ brane charge splitting into $k$-small clusters with $n_i$ number of $D_0$ branes on each cluster such that the sum $\sum_{i=1}^{k} n_i = N$ corresponds to the wrapped $D_2$ branes residing on any of the $24p$ bosonic chiral primary states. Here, the counting is done with the degeneracy $d_N$ of states having level function $N$ in the $(1 + 1)$ CFT with $24p$ bosons, and thus one renders with the celebrated leading order microscopic entropy formula

$$S = \ln d_N = 4\pi \sqrt{\sum_{i=1}^{k} n_i p}$$

(53)

Below, we shall sequentially compute that the components of state-space covariant metric tensor being defined as the negative Hessian matrix with respect to given $k$-electric charges $\{n_i\}_{i=1}^{k}$ on $D_0$-branes and the magnetic charge $p$ on $D_4$-branes, and thereby divulge the state-space notion of metric positivity, relative correlations as well as planar and hyper-planer stability for the finite cluster small black brane configurations. Note that the properties of single cluster configurations are already considered in the very beginning of the present investigation.

An illustration of the basic idea of state-space geometry of these particular black holes remains the same as that of the excited string carrying $n_1$ number of winding modes and $n_p$ number of momentum modes. As we have first considered the case of simplest two charge extremal configurations, it turns out the same that the state-space geometry of single cluster configurations can be analyzed in terms of the net electric charges replacing the winding modes and net magnetic charge replacing the momentum modes of an excited string. In the next subsection, we shall first consider the two cluster configuration and analyze respective state-space scaling relations and stability properties.

7.1 Two Electric Charge Fractionation

In order to find general pattern of state-space geometric objects of brane fractionated small black holes, we shall in this subsection explain the state-space geometry for some potential values by restricting the number of electric clusters in which the total $N$ number of $D_0$ brane charge splits into specific finite partitions. In particular, we shall first explore the case for $k=2$ for which the entropy with two clusters takes the form of

$$S = 4\pi \sqrt{p(n_1 + n_2)}$$

(54)

In this case, there are two set of charges carried by the small black holes which can form coordinate charts on underlying state-space configuration. We shall take he first set of state-space variables to be the fractionated $D_0$ brane numbers $\{n_1, n_2\}$ which are simply proportional to available fraction of the electric charges present in respective clusters, while the other state-space variable is the number of $D_4$ brane which is represented by the magnetic charge $p$. We thence find that the components of state-space metric tensor arising from the Hessian of entropy of $D_0$-$D_4$ black holes are

$$g_{pp} = \frac{\pi}{p} \sqrt{\frac{n_1 + n_2}{p}}$$
For $i, j \in \{n_1, n_2\}$ and $p$, we observe that the statistical pair correlations just accounted may in turn be simply ascertained by microscopic descriptions which are being expressed in terms of large integers (or associated brane charges) of the $D_0$-$D_4$ small black brane solutions connoting an ensemble of microstates. Furthermore, it is evident for the small black brane configurations that the principle components of the statistical pair correlations are positive definite for allowed values of the concerned parameters of small black brane solution. As a result, we can easily see for all admissible set of $n_1$, $n_2$ and $p$ that the components of state-space metric tensor as given above comply

$$g_{p n_1} = g_{p n_2} = \frac{\pi}{\sqrt{p(n_1 + n_2)}}$$
$$g_{n_1 n_1} = g_{n_1 n_2} = g_{n_2 n_2} = \frac{\pi}{(n_1 + n_2)} \sqrt{\frac{p}{n_1 + n_2}}$$

(55)

The principle components of state-space metric tensor $\{g_{n_1 n_1}, g_{p p}\}$ in effect signify a set of positive definite heat capacities (or the related compressibilities) of the two cluster configurations. In fact, the positivity constraint apprises that the $D_0$-$D_4$ black branes comply an underlying locally equilibrium statistical configuration. Furthermore, it is intriguing to note that the non diagonal component $g_{n_1 n_2}$ also takes a positive value, viz., we have

$$g_{n_1 n_2} > 0 \ \forall \ (n_1, n_2, p)$$

(57)

This shows that the correlations between the associated number of $D_0$ branes in $(1 + 1)$-CFT (or the charges in dual description) remains positive in the limit of large electric charges. This is clearly perceptible because of the fact that the leading order fractionated small black brane configuration becomes unphysical for these values of the brane parameters.

Interestingly, it follows that the ratios of the principle components of statistical pair correlations involving electric charges or one electric charge in either correlations are identical, involving the both electric and magnetic charges vary as inverse square of the connoted charges; while that of the ratios involving the off-diagonal pair correlations modulate only inversely. From the above expressions, is not difficult to visualize for the distinct $i, j \in \{n_1, n_2\}$ and the magnetic charge $p$ that the admissible statistical pair correlations as described above obey the following scaling properties

$$\frac{g_{ij}}{g_{jj}} = \frac{g_{ij}}{g_{jj}} = \frac{g_{ip}}{g_{jp}} = 1$$
$$\frac{g_{ii}}{g_{ip}} = \frac{g_{ii}}{g_{ip}} = \frac{g_{ip}}{g_{pp}} = \frac{g_{ij}}{g_{ip}} = -\left(\frac{p}{n_1 + n_2}\right)$$
$$\frac{g_{ii}}{g_{pp}} = \frac{g_{ii}}{g_{pp}} = \left(\frac{p}{n_1 + n_2}\right)^2$$

(58)

Apart from the positivity of principle components of state-space metric tensor, one in order to accomplish the locally stable statistical configuration demands that all associated principle minors of the configuration should be positive definite. It is further not difficult to compute list of the principle minors, viz., $\{p_1, p_2\}$ from the Hessian matrix of associated entropy of fractional $D_0$-$D_4$ black branes. In fact, after some simple manipulations one encounters that the concerned stability conditions at a point, along one dimensional lines and the two dimensional surfaces of state-space manifold are respectively measured by

$$p_0 = 1, \ p_1 = \frac{\pi}{p} \sqrt{\frac{n_1 + n_2}{p}}, \ p_2 = 0$$

(59)

For all physically allowed values of invariant electric-magnetic charges of the $D_0$-$D_4$ black holes, one thus stipulates that the minor constraint $p_1 > 0$ obliges that the domain of ascribed magnetic charge must respectively take positive values, while the surface constraint $p_2 = 0$ implies that there are no electric-magnetic charges such that the $p_2$ remains positive real number, and thus the two dimensional subconfigurations of leading order $D_0$-$D_4$ black holes are not stable. In effect, we can further inspect complete nature of state-space configuration for the $D_0$-$D_4$ black branes with electric fractionalizations that the entire stabilities of the system do not holds for any value of electric and magnetic charges. This follows from the non existence of positive definite value of the determinant of the metric tensor. In particular, we see easily for all $i, j \in \{n_1, n_2\}$ and $p$ that the determinant of the state-space metric tensor finds vanishing value. We thence deduce that the stability of leading order $D_0$-$D_4$ system in two cluster fractionations is not very ensured over the Gaussian statistical fluctuations.
7.2 Three Electric Charge Fractionation

We focus our attention on an extension of state-space analysis for larger number of electric cluster for the $D_0$-$D_4$ black brane configurations. The exploration begins by considering three clusters of $D_0$-branes, and single cluster of $D_4$ magnetic brane for the spherical horizon four dimensional small black hole solutions. What follows here that the magnetic charge is quantified in terms of the number $D_4$ branes, while that of the electric charges render as the number of brane present in the chosen cluster of configurations. More precisely, the underlying electric and magnetic charges take large integer values in terms of the net number of constituent $D_0$ and $D_4$ branes. In turn, one arrives at the simple quantization condition that the existing charges may be inscribed as fractionated brane configuration. Such space-time solutions appear quite naturally in the string theory, see for example [4, 71, 72, 73, 74]. In this case, one finds from the general entropy expression that the three cluster entropy of $D_0$-$D_4$ black branes is given to be

$$ S = 4\pi \sqrt{p(n_1 + n_2 + n_3)} $$  \hspace{1cm} (60)

Thus, the intrinsic Riemannian geometry as the equilibrium state-space configuration may immediately be introduced as earlier from the negative Hessian matrix of the entropy of three electric charges and one magnetic charge extremal small black holes with $D_0$ brane fractionations. We find that the components of the state-space metric tensor is easily obtained with respect to the underlying electric charges $\{n_1, n_2, n_3\}$ and the magnetic charge $p$ as

$$ g_{pp} = \frac{\pi}{p} \sqrt{\frac{n_1 + n_2 + n_3}{p}} $$

$$ g_{pn_1} = g_{pn_2} = g_{pn_3} = -\frac{\pi}{\sqrt{p(n_1 + n_2 + n_3)}} $$

$$ g_{n_1n_1} = g_{n_1n_2} = g_{n_1n_3} = g_{n_2n_2} = g_{n_2n_3} = g_{n_3n_3} = \frac{\pi}{(n_1 + n_2 + n_3) \sqrt{\frac{p}{n_1 + n_2 + n_3}}} $$  \hspace{1cm} (61)

For all $i, j \in \{n_1, n_2, n_3\}$ and $p$, we notice that the similar set of positivity conditions and state-space scaling relations are followed as that of the two electric charge fractionation. Hitherto, we see apparently that the principle components of state-space pair correlations remain positive definite quantities for all admissible values of underlying electric magnetic charges of the black brane configuration. It is easy to observe for given $n_1, n_2, n_3$ and $p$ that the following state-space metric constraints are satisfied

$$ g_{pp}(n_1, n_2, n_3, p) > 0, \quad g_{n_in_i}(n_1, n_2, n_3, p) > 0 \quad \forall \ i = 1, 2, 3 $$  \hspace{1cm} (62)

Physically, one may thus note that the principle components of state-space metric tensor $\{g_{ii}, g_{pp} \mid i = n_1, n_2, n_3\}$ signify a set of heat capacities (or the associated compressibilities) whose positivity exhibits that the underlying $D_0$-$D_4$ small black brane system is in the locally equilibrium statistical configurations. Our analysis further compiles that the positivity of $g_{pp}$ obliges that the associated dual conformal field theory living on the boundary must prevail a non-vanishing value of the magnetic charge defining an associated degeneracy of large number of conformal field theory microstates. It is worth to mention for given $i, j \in \{n_1, n_2, n_3\}$ and $p$ that the inter cluster state-space correlation functions are again non-trivial in nature. In particular, we see in this case that the non diagonal components $g_{n_in_j}$ of the metric tensor take definite positive values

$$ g_{n_in_j}(n_1, n_2, n_3, p) > 0 \quad \forall \ i \neq j \in \{1, 2, 3\} $$  \hspace{1cm} (63)

We may notice further that the ratio of principle components of state-space pair correlations form three different sets of relations and specifically we find in a chosen cluster that they remain the same, vary as inverse of the involved electric magnetic charges, and vary as inverse square of the involved parameters. It is in fact not difficult to inspect for non-identical $i, j, k \in \{1, 2, 3\}$ and $p$ that the state-space pair correlations are consisting of the following type of scaling relations

$$ \frac{g_{ij}}{g_{jj}} = \frac{g_{ij}}{g_{jj}} = \frac{g_{ij}}{g_{jj}} = \frac{g_{ij}}{g_{jj}} = \frac{g_{ip}}{g_{jp}} = 1 $$

$$ \frac{g_{ii}}{g_{pp}} = \frac{g_{ij}}{g_{ij}} = \frac{g_{ik}}{g_{ik}} = \frac{g_{ip}}{g_{ip}} = -\left( \frac{p}{n_1 + n_2 + n_3} \right) $$

$$ \frac{g_{ii}}{g_{pp}} = \frac{g_{ij}}{g_{pp}} = \left( \frac{p}{n_1 + n_2 + n_3} \right)^2 $$  \hspace{1cm} (64)

An investigation of definite global properties of three electric clustered $D_0$-$D_4$ black brane configurations determines certain stability consideration along each directions, each planes and each hyper-planes, as well
as on the entire intrinsic state-space manifold. Specifically, we can determine whether the underlying $D_0$-$D_4$ configuration is locally stable on state-space planes and hyper-planes, and thus one need to compute corresponding principle minors of negative Hessian matrix of the $D_0$-$D_4$ black hole entropy. In this case, we may easily appraise for all physically likely values of magnetic charge and electric charges that the possible principle minors computed from the above state-space metric tensor are

$$p_0 = 1, \quad p_1 = \frac{\pi}{p} \sqrt{\frac{n_1 + n_2 + n_3}{p}}, \quad p_i = 0, \quad i = 2, 3$$

(65)

In the entropy representation, it could thus be seen that the principle minors defined by

$$p_2(n_1, n_2, n_3, p) := g_{11}g_{22} - g_{12}^2$$

$$p_3(n_1, n_2, n_3, p) := g_{11}g_{22}g_{33} - g_{12}^2g_{23} - g_{13}^2g_{22} - g_{11}g_{33}g_{22} + 2g_{12}g_{13}g_{23} - g_{11}g_{23}g_{33} - g_{11}g_{22}g_{33} + g_{12}g_{23}g_{31} + g_{13}g_{21}g_{32} + g_{12}g_{31}g_{23} + g_{13}g_{22}g_{31} - g_{13}g_{21}g_{32}$$

(66)

vanish identically for all admissible values of the electric charges $n_1, n_2, n_3$ and magnetic charge $p$. In turn, one can easily observe that the vanishing condition $p_{i>1}(n_1, n_2, n_3, p) = 0$ signifying the state-space configurations corresponding to the three clusters of electric $D_0$-branes indicate that the statistical system remains unstable over possible surfaces and hyper-surfaces. Furthermore, we indeed find for entire system that the positivity of final minor is just the positivity condition of the determinant of metric tensor. Thence, an easy inspection observes further that the determinant of the metric tensor vanishes as well for all three clusters of electric charges and magnetic charge $\{n_1, n_2, n_3, p\}$ which form co-ordinates on its state-space configuration.

### 7.3 Multi Electric Charge Fractionation

Now we shall consider the state-space configuration for the most general case of brane fractionation in the finite cluster of $D_0$-$D_4$ small black branes, and present our analysis from the viewpoints of associated microscopic entropy obtained for $k$-clusters. It turns out that the involved entropy can be defined via an appropriate degeneracy formula, and the concerned expression reduces to the entropy as ascribed in Eqn. (53).

The state-space geometry describing the local pair correlations between the equilibrium microstates of multi-clustered charged extremal $D_0$-$D_4$ black holes resulting from degeneracy of microstates may thence be computed as earlier from the Hessian matrix of Eqn. (53) with respect to the parameters, viz., the $D_0$ electric charges $\{n_1, n_2, \ldots, n_k\}$ and the $D_4$ magnetic charge $p$. At this juncture, we obtain that the components of underlying state-space covariant metric tensor are generically given by

$$g_{pp} = \frac{\pi}{p} \sqrt{\frac{\sum_{i=1}^{k} n_i}{p}}$$

$$g_{ii} = -\frac{\pi}{p} \sqrt{\frac{\sum_{i=1}^{k} n_i}{p}}$$

$$g_{ij} = \frac{\pi}{\sqrt{\sum_{i=1}^{k} n_i}} \sqrt{\frac{p}{\sum_{i=1}^{k} n_i}}, \quad \forall i, j = 1, 2, \ldots, k$$

(67)

For finite number of parameters of the $D_0$-$D_4$ black brane configurations, viz., the electric charges $i, j, k, l \in \{n_1, n_2, \ldots, n_k\}$ and the magnetic charge $p$, we observe that the specific inspections observed in previous subsections for the two and three cluster systems hold, as well. In effect, it is evident in general that the principle components of equilibrium statistical pair correlations are positive definite for all allowed values of concerned parameters of the $D_0$-$D_4$ small black branes in each electric clusters. As an immediate result, one finds from the present analysis that the concerned state-space metric constraints are satisfied with

$$g_{ii} > 0 \quad \forall i = 1, 2, \ldots, k$$

$$g_{pp} > 0 \quad \forall (n_1, n_2, \ldots, n_k, p)$$

(68)

Interestingly, it is worth to mention that our geometric expressions arising from the entropy of small black holes indicate that some of the brane charges are possible to be safely turn off, say $n_i = 0$, while having a well-defined state-space geometry. However, it is unfeasible to have an intrinsic state-space configuration of small black holes with no electric charge or no magnetic charge, say $n_i = 0$ or $p = 0$, since the objects inside the square-root of the statistical entropy vanishes, and thus the argued small black hole configurations with
vanishing number of either $D_0$ branes or $D_4$ branes are no more well-defined state-state configuration. The case of finitely many electric branes indeed agrees with our expectation that the non diagonal components $g_{n_i n_j}$ find their respective positive values

$$g_{n_i n_j} > 0 \forall (n_1, n_2, \ldots, n_k, p)$$

(69)

Under the present considerations, we thus observe for a given fraction of the electric charges $i \neq j \neq k \neq l \in \{n_1, n_2, \ldots, n_k\}$ and the magnetic charge $p$ that the relative state-space pair correlation functions form the same scaling qualifications as in the case of three clusters of $D_0$ electric branes delt in Eqn. (65), except the fact that now the sum in the denominator runs over $\{1, 2, \ldots, k\}$. Furthermore, one may now easily see for the $D_0$-$D_4$ configurations involving four or higher clusters of $D_0$-branes that there exist an extra identical scaling relation

$$\frac{g_{n_i}}{g_{n_j}} = 1$$

(70)

We thus see for the most general leading order brane fractionation in $D_0$-$D_4$ system that there are total 14 type of relative correlation functions at chosen state-space basis. It is worth to mention that an appraisal of exhaustive state-space stability constraints demands that all the associated principle minors must be positive definite, as the positivity of principle components of metric tensor defines the local linear stability in the neighborhood of chosen local co-ordinate chart on an underlying $(M_{k+1}, g)$ describing concerned state-space manifold of finite clustered $D_0$-$D_4$ solutions.

Specifically for $i, j \in \{n_1, n_2, \ldots, n_k\}$ and $p$, we find that the are no extra type of planer and hyper-planer stabilities as that of the relative state-space correlation functions than the linearly stable multi-clustered $D_0$-$D_4$ small black brane system. It is rather easy to divulge the physical picture of the solution set and in fact after some simplifications one discovers that the planer stability criteria on the two dimensional surfaces and planer stability criteria on the three or higher dimensional surfaces of the state-space manifold may simply be rendered from the definition of the state-space geometry.

Intriguingly, it is not difficult to compute from the consideration of the Hessian matrix of $k$-clustered $D_0$-$D_4$ black brane leading order entropy solutions that the list of non-zero principle minors remains the same as that of the two or three clustered configurations. In addition, as in the case of two and three clusters of electric charges, we observe for general $k$ electric charge configurations that the set of all possible principle minors $\{p_i(n_1, n_2, \ldots, n_k, p) \forall i > 1\}$ remain zero on the state-space manifold $(M_{k+1}, g)$ as well as on respective lower dimensional associated systems of multi-clustered $D_0$-$D_4$ black branes.

It is worth to mention in particular that the local stability of full small black brane state-space configuration is determined by computing the determinant of concerned state-space metric tensor. Herewith, we may in principle as well compute compact formula for the determinant of the metric tensor, and indispensably, our intrinsic state-space geometric analysis arising from the leading order entropy consideration demonstrates that the determinant of the metric tensor do not find a non-vanishing value for any admissible finite electric clusters of the $D_0$-$D_4$ black brane configurations.

In the next section, we shall consider implications of state-space geometry arising from the Fuzzball solutions, and explicate the nature of scaling properties of possible state-space pair correlation functions and stability requirements of the fuzzy ring solutions in the setup of Mathur’s fuzzball consideration [49].

8 The Fuzzball Solutions: Fuzzy Rings

The view-points of the Mathur’s fuzzball solutions [75] are considered in this section. To be specific, we shall analyze concerned aspects of state-space geometry for the most exhaustively studied two charge extremal black branes having electric-magnetic charges, $(Q, P)$ and an angular momentum $J$. We shall focus in particular to analyze the state-space observations in terms of concerned parameters of the fuzzball solution, and thereby shed light on the state-space quantities from Mathur recent proposal to find ensemble of microstates which form an equilibrium statistical basis over which we shall define associated thermodynamic intrinsic state-space geometry.

It is worth to mention in the fuzzball picture that one can construct classical space-time geometry with definite horizon topology when many of the quanta of underlying three parameter $D_1$-$D_2$-$P$ CFT lie in the same mode. Nevertheless, it turns out in general that the generic states will not have all the quanta placed in a few modes, so the throat of concerned black hole space-times ends in a very quantum fuzzball, see for introduction of the fuzzball solutions [49] [47] [45].

It is however interesting to note in the fuzzball picture that the actual microstates of such black branes do not have an event horizon, but it is rather the area of the boundary of fuzzy region where microstates start differing from each other satisfies Bekenstein-Hawking type relation and thereby defines an entropy inside the chosen boundary. Moreover, in turns out according to string theory picture that the different
microstates are ‘cap off’ before reaching the end of an infinite throat and thus they give rise to different near horizon space-time geometries. In particular, the average throat behaves as inverse of average radius of the fuzzballs. Thus the Bekenstein-Hawking entropy [49, 47, 48] has been obtained from the area of such a stretched horizon whose state-space interpretation may be obtained from the coarse graining statistical entropy

\[ S(Q, P, J) = C\sqrt{QP - J} \]  (71)

The associated state-space geometry of rotating two charge fuzzy ring system can hence be constructed out of the parameters which characterize the microstates of black brane. In particular, we can perform an investigation either in terms of the \( D_1 \) brane electric charge \( Q \) and \( D_5 \) brane magnetic having charge \( P \) or correspondingly \( n_1 \) number of \( D_1 \) branes and \( n_5 \) number of \( D_5 \) branes. Then, the dimension of state-space manifold is equal to the number of actual parameters which defines the fuzzy black ring solution. We shall thence study the state-space configurations whose co-ordinates deal with the charges or number of constituent branes. In particular, we shall consider the electric-magnetic charges \( (Q, P) \) and angular momentum \( J \) that they define co-ordinates on concerned state-space manifold of the two charge fuzzy black ring solution.

The state-space geometry constructed out of equilibrium state of the rotating two charged extremal black ring resulting from the entropy can now easily be computed as earlier from the negative Hessian matrix of the entropy with respect to the charges and angular momentum. Note that an understanding of the state-space correlations turns out to be in precise accordance with an associated attractor configuration being disclosed in the limiting special Bekenstein-Hawking type relation with the entropy of the fuzzball whose boundary surface becomes like a horizon only over classical time scales. We may thence see that the components of the metric tensor are explicitly given as

\[ g_{PP} = \frac{1}{4}CPQ(JQ - J)^{-3/2}, \quad g_{PQ} = \frac{1}{4}C(PQ - 2J)(PQ - J)^{-3/2} \]
\[ g_{PJ} = \frac{1}{4}C(PQ - J)^{-3/2}, \quad g_{QQ} = \frac{1}{4}CPQ(PQ - J)^{-3/2} \]
\[ g_{JQ} = \frac{1}{4}C(PQ - J)^{-3/2}, \quad g_{JJ} = \frac{1}{4}C(PQ - J)^{-3/2} \]  (72)

From the simple \( D \)-brane description, we observe that there exists an interesting brane interpretation which describe the state-space correlation formulae arising from the corresponding microscopic entropy of the aforementioned two charge rotating \( D_1-D_5 \) solutions. Furthermore, the state-space correlations turn out to be in precise accordance with an associated attractor configuration being disclosed in the limiting special Bekenstein-Hawking solution. In the entropy representation, it has thence been noticed that the Hessian matrix of the entropy illustrates basic nature of possible state-space correlations between the set of extensive variables which in this case are nothing more than the \( D_1 \) and \( D_5 \)-brane charges and angular momentum. As mentioned before, we can articulate in this case as well that for all non-zero admissible values of \( P, Q, J \), the principle components of intrinsic state-space metric tensor satisfy

\[ g_{PP} > 0, \quad g_{QQ} > 0, \quad g_{JJ} > 0 \]  (73)

Substantially, the principle components of state-space metric tensor signifies heat capacities or the associated compressibility whose positivity indicates that the underlying statistical system is in local equilibrium consisting of the \( D_1 \) and \( D_5 \)-brane configurations. Furthermore, we perceive that the ratio of possible diagonal components varies as inverse square, which weaken faster and thus relatively quickly come in to an equilibrium configuration, than those involving the off diagonal components varying inversely in the involved parameters. Incidentally, the ratios of non-diagonal components varying inversely remain comparable for a longer domain of parameters varying under the Gaussian fluctuations. We have in particular inspected \( \forall i \neq j \in \{P, Q\} \) and \( J \) that the relative pair correlation functions satisfy the following scaling relations

\[ \frac{g_{ii}}{g_{jj}} = \left( \frac{j}{i} \right)^2, \quad \frac{g_{ij}}{g_{jj}} = \frac{j}{i}, \quad \frac{g_{ii}}{g_{jj}} = \frac{1}{j^2}(PQ - 2J) \]
\[ \frac{g_{ii}}{g_{jj}} = -j, \quad \frac{g_{ij}}{g_{jj}} = \frac{j}{i}, \quad \frac{g_{ii}}{g_{jj}} = \frac{1}{j} \]
\[ \frac{g_{ij}}{g_{jj}} = -j, \quad \frac{g_{ii}}{g_{jj}} = \frac{j}{i}(PQ - 2J), \quad \frac{g_{ij}}{g_{jj}} = -(PQ - 2J) \]  (74)

An investigation of definite global properties of two charged Fuzzball configurations determines certain stability approximation along each directions, each planes, each hyper-planes and on entire intrinsic state-space manifold. In this case, as we intend to determine whether the underlying Fuzzball configuration is
locally stable on state-space planes and hyper-planes, and thus we are required to compute corresponding
principle minors of negative Hessian matrix of the entropy. Specifically, we may easily appraise for all
physically likely values of brane charge and angular momentum that the possible principle minors computed
from the above state-space metric tensor are non-zero and definite function of the electric-magnetic charges
\{P, Q\} and an angular momentum \(J\). We in effect see for all admissible parameters describing the three
parameter fuzzball solutions that the list of concerned state-space stability functions is

\[ p_1 = \frac{1}{4}C^2(Q^2 - J)^{-3/2}, \quad p_2 = \frac{1}{4}C^2J(PQ - J)^{-2} \]  

(75)

Thus, the minor constraints on \(p_1, p_2\) implies that the two charge Fuzzball solution under consideration are
stable over the lines, planes of the state-space configuration for all values of the \(D_1, D_3\) brane charges and any
positive value of the angular momentum. As we have shown in the previous examples that the determinant
of the metric tensor thus defined is non-zero for non-zero brane charges and angular momentum. In fact, it
is easy to observe that the determinant of the metric tensor reduces to

\[ \|g\| = -\frac{1}{16}C^3(PQ - J)^{-5/2} \]  

(76)

Similarly, the constraint \(p_3 := g(Q, P, J) < 0\) results in an interpretation that this configuration is
globally unstable over the full intrinsic state-space configurations. This is also intelligible from the fact that
the responsible equilibrium entropy tends to its maximum value, while the same culmination do not remain
valid over the entire state-space manifold. It may in turn be envisaged in the \(D_1, D_3\)-\(P\) description that
the fuzzball black rings do not correspond to intrinsically stable statistical basis when all the configuration
parameters fluctuate. Thus, it is very probable that the underlying ensemble of chosen CFT microstates
upon subleading higher derivative corrections may smoothly move into the more stable brane configurations.

Finally, in order to elucidate universal nature of the statistical interactions and the other properties
concerning fuzzball rotating black rings, one needs to determine definite global state-space geometric invariant
quantities on its intrinsic state-space manifold. Indeed, we notice that the indicated simplest invariant
is achieved just by computing the state-space scalar curvature, which as explained in [15] be obtained in
straightforward fashion by applying the standard method of our intrinsic geometry. In the large charge
limit in which the asymptotic expansion of the entropy of the two charge rotating ring solution is valid, we
notice in particular that the state-space scalar curvature can rather be expressed as an inverse function of the
entropy.

An exact analysis in turn finds that the constant of proportionality between the state-space scalar curva-
ture and confining entropy to be a negative constant, and thus we find fuzzy ring to be an attractive statistical
configuration, see for related interpretations [15]. Most importantly, it turns out in the limit when the fuzzy
ring is viewed in the perspective of many fuzzballs that the present analysis relies on corrected averaged
horizon configuration. Finally, it is worth to mention that the statistical systems of the \(D_1, D_3\)-\(P\) fuzzy rings
find an intriguing conclusion in the Gaussian approximations, and consequently, the present description
vindicates physically sound containments that the state-space configuration of fuzzy rings is non-degenerate,
curved and an everywhere regular intrinsic Riemannian manifold \((M_5, g)\).

9 Bubbling Black Brane Solutions: Black Brane Foams

In this section, we finally analyze the state-space geometry of an ensemble of equilibrium microstates char-
acterizing three charge foamed black brane configurations in \(M\)-theory [51]. These supergravity bubbling
solutions naturally appears in the string theory and \(M\)-theory, see for concerned details [51, 76, 77, 78]. The
study of bubbled space-time geometries and axi-symmetric merger solutions thence turn out to be interest-
ing to investigate further from the view-points of our state-space geometry. We shall here show that the
possible characterization of state-space geometry has herewith been accomplished in terms of the parameters
describing an ensemble of microstates for the three charge black foam solutions [79].

9.1 A Toy Model: Single GH-center

The state-space geometry arising from entropy of the foam configurations having single Gibbons-Hawking
center can be divulged by considering \(M\)-theory background [51] compactified on \(T^6\). In the large \(N\) limit,
one thence finds set of flux parameters which may be written in terms of brane charges. It is worth to
remark that the associated topological entropy which is independent of the number and charges on the
Gibbons-Hawking base points. The origin of such an entropy lies solely in the possible number of the choices
of positive quantized fluxes on topologically non-trivial cycles.

In effect, it turns out that these cycles satisfy definite constraints, viz one finds in particular that the
supergravity and worldvolume descriptions have the same relation between the brane parameters which
determined the entropy of the bubbled black brane foam, see for instance [51]. Note that an understanding of the state-space geometry based on the bubbling black branes require knowledge of Bekenstein-Hawking entropy which could be obtained from the area of the horizon of chosen solution. A microscopic interpretation may thence be offered as coarse graining of concerned combinatorial entropy of the foam.

To describe state-space geometry of single center Gibbons-Hawking configuration, we shall in particular consider set of flux parameters \( \{k_1^i, k_2^i, k_3^i\} \) to be positive half integers [51], and then the topological entropy coming from the leading order contributions of the fluxes \( \{k_1^i\} \) where the index \( i \) defines positions of the Gibbons-Hawking base points has been written as

\[
S(Q_1, Q_2, Q_3) := \frac{\pi}{3} \sqrt[4]{\frac{Q_3 Q_4}{Q_1}} \tag{77}
\]

As proclaimed in the previous subsections, we notice in this case as well that the state-space geometry describing the nature of equilibrium brane microstates can be constructed out of the three charges of the bubbled black brane foams. The covariant metric tensor as invoked earlier can immediately be computed from negative Hessian matrix of the foam entropy resulting from underlying statistical configuration. Thus, the brane charges, viz., \( \{Q_1, Q_2, Q_3\} \) form the coordinate charts for the state-space manifold of our interest and thus with respect to the brane parameters we may describe the typical intrinsic geometric features of the bubbled black brane foams having single GH center. In fact, we notice that the components of the covariant metric tensor can easily be presented to be

\[
g_{q_1 q_1} = \frac{5\pi \sqrt{6}}{48Q_1^2} (Q_2 Q_4)^{1/4}, \quad g_{q_2 q_2} = \frac{\pi \sqrt{6} Q_3}{48Q_1^2} (Q_2 Q_3)^{3/4}, \quad g_{q_2 q_3} = \frac{\pi \sqrt{6} Q_1}{16 Q_1} (Q_2 Q_3)^{7/4}, \quad g_{q_2 q_3} = \frac{\pi \sqrt{6} Q_3}{16 Q_1^2} (Q_2 Q_3)^{7/4} \tag{78}
\]

One thus appreciates for all \( i, j, k \in \{1, 2, 3\} \) describing the single GH center bubbling brane configuration that the possible relative state-space correlation functions for the single GH-center find

\[
g_{q_1 q_1} < 0, \quad g_{q_2 q_2} > 0, \quad g_{q_2 q_3} > 0 \tag{79}
\]

The present analysis physically proclaims that the principle components of state-space metric tensor signify heat capacities or the relevant compressibilities whose positivity connote that the underlying statistical system is in locally stable equilibrium configurations of an ensemble of dual CFT microstates. Moreover, it is rather instructive to note that the behavior of brane-brane statistical pair correlation defined as \( g_{Q_1 Q_2} \), is asymmetric in contrast to the other existing diagonal correlations. In fact, one can understand it by arguing that an increment of the \( Q_1 \) brane charge reduces the entropy and thus it correspond to locally unstable state-space interactions than the brane-brane self-interactions involving either \( Q_2 \) or \( Q_3 \) charges.

Furthermore, it has been substantiated that the ratio of diagonal components varies as inverse square of the invariant parameters which vary under the Gaussian fluctuations, whereas the ratios involving off diagonal components vary only inversely in the chosen charges. In particular, we see for \( i, j, k \in \{Q_1, Q_2, Q_3\} \) describing the single GH center configuration that the possible relative state-space correlation functions are defined as

\[
C_{BB} := \{ \frac{g_{ij}}{g_{jj}} \frac{g_{ii}}{g_{ii}} \frac{g_{ik}}{g_{jj}} \frac{g_{jk}}{g_{jj}} \frac{g_{ik}}{g_{jj}} \} \tag{80}
\]

This suggests that the diagonal components weaken faster and relatively quickly come into an equilibrium, than the off diagonal components which remain comparable for the longer domain of the parameters defining the single GH center bubbling configurations. An explicit observation shows that the relative pair correlation functions satisfy a simple set of scaling relations. In particular, we can easily observe for given distinct \( i, j, k \in \{Q_1, Q_2, Q_3\} \) that the possible relative state-space correlation functions for the single GH-center find the following values

\[
C_{BB} := \{ \frac{g_{12}}{g_{22}} \frac{g_{12}}{g_{12}} \frac{g_{12}}{g_{22}} \frac{g_{22}}{g_{22}} \} = \{ \frac{1}{3} Q_2^2 \frac{Q_1}{Q_2}, \frac{1}{3} Q_2 \frac{Q_1}{Q_2}, \frac{1}{3} Q_2 \frac{Q_1}{Q_2}, \frac{1}{3} Q_2 \frac{Q_1}{Q_2} \} \tag{81}
\]
As noticed in the previous configurations, it is not difficult to analyze the local stability for the bubbling black holes, as well. In particular, one can determine the principle minors associated with the state-space metric tensor and thereby demand that all the principle minors must be positive definite. In this case, we may adroitly compute the principle minors from the Hessian matrix of associated entropy concerning the three charge bubbling black holes. In fact, after simple manipulation, we discover that the local stability criteria on the two dimensional surfaces and the three dimensional hyper-surfaces of the underlying state-space manifold are respectively given by the following relations

\[ p_1 = -\frac{5\sqrt{6\pi}}{48}Q_1^{-3/4}Q_2^{1/4}Q_3^{1/4}, \quad p_2 = -\frac{\pi^2}{24}Q_1^{-5/2}Q_2^{-3/2}Q_3^{1/2} \]  

(82)

For all physically admitted values of brane charges of the bubbling black holes, we may thus easily ascertain that the minor constraint, viz., \( p_1(Q_i) > 0 \) inhibits the domain of assigned charges that the two of them must be positive and third be negative real number, while the constraint \( p_2(Q_i) > 0 \) imposes that the brane charges must respectively satisfy above definite brane charge conditions. In particular, these constraints enable us to investigate the nature of the state-space geometry of \( M \)-theory bubbling black holes. We thus observe that the presence of planer and hyper planer instabilities exist for the bubbling black holes, which together demand for a restriction on the allowed value of the brane charges.

As stated earlier, we find in this case that the determinant of state-space geometry describing correlations between two chosen microstates of the bubbled black brane foams may be characterized in terms of the extensive brane charges of the single GH center solution. Employing state-space consideration of negative Hessian matrix of the foam entropy with respect to the brane charges \( \{Q_1, Q_2, Q_3\} \), we find that the determinant of the metric tensor is given by

\[ (g) = -\frac{\pi^3}{384\sqrt{3}\pi^6}Q_1^{1/4}Q_2^{-1/4}Q_3^{1/4} \]  

(83)

Furthermore for equal values of charges \( Q_1 := Q_2 := Q_3 := Q \); it is easy to see that the principle minor \( p_1 := (g_{ij}) \) reduces to \( p_1 = -\frac{5\sqrt{3\pi}}{48}Q^{-1/4} \), while the surface minor \( p_2 := g_{12}g_{23} - g_{13}^2 \) shows further that the two dimensional state-space configurations of underlying single GH center solutions are unstable. In particular, we find an explicit expression for equal value of charges that the surface minor is given by \( g(Q) = -\frac{\pi^2}{24}Q^{-7/2} \).

As expected, we see for equal value of brane charges, viz., \( Q_i = Q \) that the toy model single GH bubble black brane solution remains unstable over an entire fluctuating statistical configuration. This follows from the fact that the determinant of the metric tensor as being the highest principle minor \( p_3 \) reduces to

\[ g(Q) = -\frac{\pi^3}{384\sqrt{3}\pi^6}Q^{-21/4} \]

Interestingly, it is note worthy from the general expression of the determinant of the metric tensor and in addition that of the state-space scalar curvature signifying global correlation volume of underlying statistical system that the single GH center bubbled systems are unstable and finds an attractive statistical nature for given non zero entropy solution. Finally for all admissible values of brane charges, we come up with the fact that the state-space scalar curvature signifying global correlation length of an underlying statistical system confirms no divergence, and in turn it varies as an inverse function of the entropy of chosen single center GH center bubbled configurations.

### 9.2 Black Brane Foams

In this subsection, we shall consider state-space geometry of the most general three center GH solutions, which may exhaustively be contemplated by three brane charges of the bubbled black brane configurations. The coordinate chart of underlying intrinsic state-space manifold naturally emerge from the parameters of equilibrium microstates of chosen bubbling supergravity solutions \[80, 81, 82, 83\]. It turns out from the details of brane parameters that one may easily ascribe the state-space definitions to the central charge contributions associated with the rotating black branes in Minkowski space, as well. In effect, our attention shall therefore be focused on possible U-dual configurations, and describe promising analysis in the viewpoints of \[81, 82, 83, 86, 87\].

Here, very purpose would thus be to exploit the state-space meanings of symmetric factors of brane charges arising from an elementary conformal field theory living on the boundary. As we have encountered the state-space geometry of the single GH center bubbled black branes in the previous subsection, in this subsection we shall analyze the state-space fluctuations for unrestricted 3-charge bubbled black brane foam solutions \[81\]. Thereby, we shall examine general nature of concerned state-space configurations over leading order symmetric charge contributions into topological entropy of the three charged bubbled black brane foams characterized by the charges \( Q_1, Q_2 \) and \( Q_3 \). It turns out by considering appropriate factors coming from
the partitioning of concerned flux parameters, viz., \( \{k_1, k_2, k_3\} \) that the involved topological entropy may be defined by the following formula:

\[
S(Q_1, Q_2, Q_3) := \frac{2\pi}{\sqrt{6}} \left( \frac{Q_2 Q_3}{Q_1} \right)^{1/4} + \left( \frac{Q_1 Q_2}{Q_3} \right)^{1/4} + \left( \frac{Q_1 Q_3}{Q_2} \right)^{1/4}
\]  

(84)

It is again not difficult to explore the state-space geometry of the equilibrium microstates of the three charge bubbled black brane foams arising from the entropy expression which concerns just the Einstein-Hilbert action. As stated earlier that the Ruppeiner metric on the state-space manifold is given by the negative Hessian matrix of the ring entropy with respect to the thermodynamic variables. The state-space tensor are the constituent branes. Explicitly, we find in this case that the components of covariant state-space metric tensor are

\[
\begin{align*}
g_{Q_1, Q_1} & = -\pi \left( \frac{5\sqrt{6}}{48Q_1^2} \right) \left( \frac{Q_2 Q_3}{Q_1} \right)^{1/4} - \frac{\sqrt{6}}{16Q_1} \left( \frac{Q_2}{Q_1} \right)^{1/4} - \frac{\sqrt{6}}{16Q_1} \left( \frac{Q_3}{Q_1} \right)^{1/4} \\
g_{Q_1, Q_2} & = -\pi \left( -\frac{\sqrt{6} Q_1}{48 Q_2 Q_3} \right) \left( \frac{Q_1}{Q_2 Q_3} \right)^{3/4} + \frac{\sqrt{6}}{48 Q_2} \left( \frac{Q_1}{Q_2 Q_3} \right)^{3/4} - \frac{\sqrt{6}}{48 Q_3} \left( \frac{Q_2}{Q_1 Q_3} \right)^{3/4} \\
g_{Q_1, Q_3} & = -\pi \left( -\frac{\sqrt{6} Q_1}{48 Q_2 Q_3} \right) \left( \frac{Q_1}{Q_2 Q_3} \right)^{3/4} - \frac{\sqrt{6}}{48 Q_2} \left( \frac{Q_1}{Q_2 Q_3} \right)^{3/4} + \frac{\sqrt{6}}{48 Q_3} \left( \frac{Q_2}{Q_1 Q_3} \right)^{3/4} \\
g_{Q_2, Q_2} & = -\pi \left( \frac{5\sqrt{6}}{48Q_2^2} \right) \left( \frac{Q_1 Q_3}{Q_2} \right)^{1/4} - \frac{\sqrt{6}}{16Q_2} \left( \frac{Q_1}{Q_2} \right)^{1/4} - \frac{\sqrt{6}}{16Q_2} \left( \frac{Q_3}{Q_2} \right)^{1/4} \\
g_{Q_2, Q_3} & = -\pi \left( \frac{\sqrt{6} Q_2}{48 Q_1 Q_3} \right) \left( \frac{Q_2}{Q_1 Q_3} \right)^{3/4} - \frac{\sqrt{6}}{48 Q_1} \left( \frac{Q_2}{Q_1 Q_3} \right)^{3/4} - \frac{\sqrt{6}}{48 Q_3} \left( \frac{Q_2}{Q_1 Q_3} \right)^{3/4} \\
g_{Q_3, Q_3} & = -\pi \left( \frac{5\sqrt{6}}{48Q_3^2} \right) \left( \frac{Q_1 Q_2}{Q_3} \right)^{1/4} - \frac{\sqrt{6}}{16Q_3} \left( \frac{Q_1}{Q_3} \right)^{1/4} - \frac{\sqrt{6}}{16Q_3} \left( \frac{Q_2}{Q_3} \right)^{1/4}
\end{align*}
\]  

(85)

It follows from the above expressions that the statistical pair correlations thus described can in turn be accounted by a simple geometric descriptions expressed in terms of the charge contributions counting an ensemble of fluxes for the general three GH centered bubbling black brane configurations. Furthermore, we observe that the principle components of underlying state-space configuration are positive definite for all allowed values of the bubbling parameters of the multi center GH solution. In particular, it is evident for functions \( f_{ii}(Q_1, Q_2, Q_3) \) as defined in Eqn. (86) that the state-space metric constraints defining positivity of concerned diagonal pair correlation functions are

\[
g_{Q_1, Q_i}(Q_1, Q_2, Q_3) > 0 \quad \forall \ i \in \{1, 2, 3\} \quad | f_{ii} | < 0,
\]  

(86)

Essentially, the principle components of state-space metric tensor \( \{g_{Q_i Q_i} \mid i = 1, 2, 3\} \) signify a set of definite heat capacities (or the related compressibilities) whose positivity for a range of involved charges as presented below apprises that the bubbled black holes comply an underlying locally stable equilibrium statistical configurations along each directions. It is intriguing to note that the positivity of \( g_{Q_1 Q_1} \) holds even if some of the brane charges of the associated brane charges become zero. This is clearly perceptible because of the fact that the brane configuration remains physical and locally stable for all brane charges \( (Q_1, Q_2, Q_3) \) such that the following relations defining Eqn. (86) are satisfied

\[
\begin{align*}
f_{11}(Q_1, Q_2, Q_3) & := \frac{5\sqrt{6}}{48Q_1^2} \left( \frac{Q_2 Q_3}{Q_1} \right)^{1/4} - \frac{\sqrt{6}}{16Q_1} \left( \frac{Q_2}{Q_1} \right)^{1/4} - \frac{\sqrt{6}}{16Q_1} \left( \frac{Q_3}{Q_1} \right)^{1/4} \\
f_{22}(Q_1, Q_2, Q_3) & := \frac{5\sqrt{6}}{48Q_2^2} \left( \frac{Q_1 Q_3}{Q_2} \right)^{1/4} - \frac{\sqrt{6}}{16Q_2} \left( \frac{Q_1}{Q_2} \right)^{1/4} - \frac{\sqrt{6}}{16Q_2} \left( \frac{Q_3}{Q_2} \right)^{1/4} \\
f_{33}(Q_1, Q_2, Q_3) & := \frac{5\sqrt{6}}{48Q_3^2} \left( \frac{Q_1 Q_2}{Q_3} \right)^{1/4} - \frac{\sqrt{6}}{16Q_3} \left( \frac{Q_1}{Q_3} \right)^{1/4} - \frac{\sqrt{6}}{16Q_3} \left( \frac{Q_2}{Q_3} \right)^{1/4}
\end{align*}
\]  

(87)

Interestingly, it is immediate to observe that the ratio of the associated components of statistical pair correlations vary as definite sum, which are symmetric factors of concerned brane charges; whereas we see that there is no very direct scaling relations as in the case of the single GH center bubbling brane configurations. Nevertheless, we notice for the distinct \( i, j, k \in \{1, 2, 3\} \) that the number of statistical pair correlations thus described remains same. Moreover, we find for the multiple GH center black brane foam configuration that the same type of relative correlation set is followed, except that the relative correlation functions now take realistic values over the parameters of given flux partitions. It is worth to note that the precise scaling properties is easily visualized just by considering the set \( C_{BB} \) of the possible ratios consisting of the components of the state-space metric tensor of the three charge bubbling black brane configurations.
Although there exist positivity of the principle components of state-space metric tensor, but in order to accomplish local state-space stability, one needs to further demand that all associated principle minors should be positive definite. It is rather easy to obtain the principle minors of Hessian matrix of the entropy associated with multiple GH center black brane foams. In fact, one finds after standard algebraic manipulations that the local stability conditions on the one dimensional line, two dimensional surfaces and three dimensional hyper-surfaces on the state-space manifold are respectively measured by following expressions

\[
p_l(Q_1, Q_2, Q_3) = -\frac{\sqrt{6\pi}}{48} Q_1^{15/4} Q_2^{-7/4} Q_3^{-7/4} (5Q_1^{3/2} Q_2^{3/2} Q_3^2 - 3Q_1^2 Q_2^{3/2} Q_3^2 - 3Q_1^2 Q_2^3 Q_3^2)
\]

\[
p_2(Q_1, Q_2, Q_3) = -\frac{\pi}{96} Q_1^{-7/2} Q_2^{-7/2} Q_3^{-3/2} (4Q_1^{1/2} Q_2^{3/2} Q_3 + Q_1^{1/2} Q_2^3 Q_3^2 - 8Q_1^{3/2} Q_2^{3/2} Q_3^2 - 2Q_1^2 Q_2^3 Q_3^2 - 2Q_1^2 Q_2^3 Q_3^2 + 4Q_1^2 Q_2 Q_3)
\]

An investigation of definite global properties of the general bubbled black brane foam configurations determines certain stability approximation along each directions, each planes and each hyper-planes, as well as on the entire intrinsic state-space manifold. Specifically, we need to determine whether the underlying three GH center foam configuration can be locally stable on state-space planes and hyper-planes, and thus one is required to just compute the corresponding principle minors of negative Hessian matrix of the foam entropy. Moreover, one finds that the principle minor \(p_1\) remains positive for the all \((Q_1, Q_2, Q_3)\) such that the function \(\bar{p}_1(Q_1, Q_2, Q_3)\) satisfies

\[
\bar{p}_1(Q_1, Q_2, Q_3) := 5Q_1^{3/2} Q_2^2 Q_3^2 - 3Q_1^2 Q_2^{3/2} Q_3^2 - 3Q_1^2 Q_2^3 Q_3^2 < 0
\]

It is further intriguing to mention from the view-points of our present consideration that the principle minor \(p_2 := g_{11} g_{22} - g_{12}^2\) reduces to positive values for a domain of brane charges. In particular, we see for given value of admissible brane charges that the state-space stability on two dimensional surfaces is ensured if the function

\[
\bar{p}_2(Q_1, Q_2, Q_3) := 4Q_1^{1/2} Q_2^{3/2} Q_3^2 + Q_1^{1/2} Q_2^3 Q_3^2 - 8Q_1^{3/2} Q_2^{3/2} Q_3^2 - 2Q_1^2 Q_2^3 Q_3 + Q_1^2 Q_2^3 Q_3^2 + 4Q_1^2 Q_2 Q_3^3
\]

finds definite negative value for set of given brane charges \((Q_1, Q_2, Q_3)\). Alternatively, the linear and planer stabilities require that the given foam configurations could be scarcely populated and thus the net brane charges are effectively bounded by some maximum brane charges. Moreover, it is not difficult to investigate the global stability on the full state-space configuration, which may in fact be easily carried out by computing the determinant of the state-space metric tensor. In this case, we observe that the determinant of the intrinsic state-space metric tensor is a well behaved function of brane charges. From the definition of highest principle minor, \(viz., \ p_3(Q_1, Q_2, Q_3) := \|g\|\), it is in fact not difficult to compute that the determinant of the metric tensor is

\[
\|g\| = \frac{\pi^3 \sqrt{\pi}}{384} (Q_1 Q_2 Q_3)^{-13/4} \bar{g}(Q_1, Q_2, Q_3),
\]

where the factor \(\bar{g}(Q_1, Q_2, Q_3)\) is defined by

\[
\bar{g}(Q_1, Q_2, Q_3) := -Q_1^{3/2} Q_2^3 - Q_1 Q_2^{1/2} Q_3^2 + 3Q_1^{1/2} Q_2^{3/2} Q_3^2 - Q_1^{1/2} Q_2^3 Q_3^2 - Q_1 Q_2^{3/2} Q_3^2 - Q_1 Q_2^3 Q_3^2 + Q_1^3 Q_2^2 Q_3^2 + Q_1 Q_2^3 Q_3^2 - Q_1^2 Q_2^{3/2} Q_3^2
\]

More explicitly, we see for the equal values of brane charges \(Q_1 := Q\), that the principle minors and the determinant of metric tensor being defined as the highest principle minor, \(viz., \ p_3 := g(Q)\) reduce to the following set of values

\[
p_1(Q) = \sqrt{\frac{6\pi}{48}} Q^{-7/4}, \ p_2(Q) = 0, \ g(Q) = 0
\]
10 Conclusion and Discussion

We have analyzed state-space pair correlation functions and notion of stability for the extremal and non-extremal black holes in string theory and M-theory. Our consideration is from the viewpoints of thermodynamic state-space geometry. We find from the intrinsic Riemannian geometry that the stability of these black branes have been divulged from the positivity of principle minors of the space-state metric tensor. We have explicitly analyzed the state-space configurations for (i) the two and three charge extremal black holes, (ii) the four and six charge non-extremal black branes.

The former arises from the string theory solutions containing large number of branes while the latter accounts for both the branes and antibranes. The numbers of branes and antibranes offer set of parameters to define intrinsic state-space geometry. An extension of state-space geometry is analyzed for the $D_6$-$D_2$-$D_0$ multi-centered black branes, small black holes with fractional electric branes, two charge rotating fuzzy rings in the setup of Mathur’s fuzzball configurations. The state-space pair correlations and potential nature of stabilities are thereby investigated for the three charged bubbling black brane foams. The state-space configuration finds further support from the consideration [14, 57, 15], and thus the nature of state-space geometry of rotating and non-rotating charged black branes in string theory and M-theory have respectively been examined.

In either of the black brane configurations, it has been shown that there exist an intriguing property of relative space-state correlations that the ratio of diagonal components varies as inverse square of the chosen parameters, while the off diagonal components vary as the inverse of the chosen parameters. Similarly for the corresponding non-extremal configurations, we find that the ratio of diagonal components weaken faster then the other off-diagonal components. Our analysis thus suggest that the brane-brane statistical pair correlation functions which find asymmetric nature in comparison with the other relative pair correlations weaken relatively faster and thus they swiftly come into an equilibrium statistical configuration. In both the configurations, underlying microscopic notion of the state-space interactions arise from coarse graining of the counting entropy over large number of CFT microstates of considered black branes.

It is demonstrated that the state-space configurations arising from fluctuating spherical horizon string theory and M-theory black brane solutions has as well been analyzed for finite parameter solutions. In effect, the present paper has exemplified our out-set for diverse string theory extremal and non-extremal black brane solutions, multi-centered black brane configurations, fuzzy rings and single and multi Gibbon Hawking center bubbling black brane foams. It is instructive to note in this perspective that state-space investigations of string theory and M-theory black brane configurations are based on an understanding of microscopic entropy of various black branes, in which the present consideration requires coarse graining phenomenon of large number of degenerate CFT microstates defining an equilibrium statistical basis for chosen black brane system. The present analysis thus offers a direct method to uncover statistical properties of fluctuating black brane configurations.

An exploration finds that the crucial ingredient in analyzing the state-space manifold of black brane configurations depends on the parameters carried by the space-time solution or that of an underlying microscopic conformal field theory. An illustration of state-space geometry of these black branes includes the case of extremal and non-extremal configuration which in the either of proclaimed configurations demonstrates that the stability constraints arising from the state-space pair correlation functions in effect determines potential nature of the local and global correlations. It is worth to mention that the components of a state-space metric tensor are related to the two point statistical correlation functions which in general are intertwined with the fluctuating parameters of associated boundary conformal field theory (CFT).

This is because that the required parameters of black brane configurations which describe the microstates of dual conformal field theory living on the boundary may in principle be determined via an application of AdS/CFT correspondence. Consequently, our intrinsic geometric formalism thus described deals with an ensemble of degenerate CFT ground states which at small constant positive temperature forms an equilibrium vacuum configuration over which we have defined the Gaussian statistical fluctuations. It is interesting to note that the quadratic nature of Gaussian statistical fluctuations about an equilibrium statistical configuration determines the metric tensor of associated state-space manifolds. In either case, our explicit computation shows over definite domain of black brane parameters that the principle components of state-space metric tensors are positive, while the non-identical off-diagonal ones may or may not. These notions have nonetheless been explicitly observed for the case of finite electric clusters of $D_5$-$D_4$ state-space configurations that some of ratios involving off-diagonal components of metric tensor are also positive.

Interestingly, the relative correlations weakens as the concerned parameters are increased. In particular, we find an accordance for two charge extremal black holes or an excited string with two state-space variables, e.g., brane numbers or brane charges and Kaluza-Klein momentum or three charge $D_1$-$D_2$-$P$ extremal solutions having $n_1$ number of $D_1$-branes, $n_5$ number of $D_5$-branes, $n_p$ number of Kaluza-Klein momentum. Then, we find for a pair of distinct state-space variable $\{X_i, X_j\}$ that the state-space pair correlations of
either of such extremal configurations scale as
\[ g_{ii} = \left( \frac{X_i}{X^*} \right)^2, \quad g_{jj} = -\frac{X_j}{X^*} \]

Furthermore, the particular behavior of generic statistical pair correlation functions characterizing state-space configurations of four and six charge non-extremal black holes in string theory satisfy inverse like scaling properties with integer or half-integer exponents. It may thus be envisaged that the generic state-space correlations of string theory and M-theory black holes with or without rotation decrease as an increment is made on in the parameters of concerned solution.

To appreciate definite global properties of the concerned systems, we have explained in this article that one is required to determine nature of stabilities along each directions, each planes, and each hyper-planes, as well as on the entire intrinsic state-space configurations. Our analysis has in effect demonstrated that the determinant of metric tensor are negative definite as well for the configurations having large number of branes and/or antibrane. It has however been known from the Ruppeiner geometry that only the classical fluctuations having definite thermal origin deal with the probability distribution which has a positive definite invariant intrinsic Riemannian metric tensor over an equilibrium statistical configuration. This signals that the system becomes highly quantum in nature, when all the parameters fluctuate. In fact, our state-space construction for the string theory and M-theory black holes dealing with the parameters of microscopic CFTs illustrates that the local stabilities, degeneracy and global signature of a state-space manifold can as well be indefinite and in effect these notions are sensitive to the location chosen in the moduli space geometry of the black branes.

Importantly, the sign of principle minors and determinant of the state-space metric tensor implies whether the chosen black brane solution is thermodynamically stable or not. While, the vacuum phase transitions may rather be characterized via the scalar curvature of concerned state-space configuration. The present investigation thus serves as a prelude to the state-space geometry of an arbitrary parameter black brane configuration in string theory and M-theory. Moreover, it has been explicates that the explored examples have an interesting set up of intrinsic state-space geometric characterizations which are based on general nature of the quadratic Gaussian fluctuations of chosen statistical configuration of the black branes. In this concern, we finds in general that these configurations are categorized as

1. The underlying sub-configurations turn out to be well-defined over possible domains, whenever there exist respective set of non-zero state-space principle minors.
2. The underlying full configuration turns out to be everywhere well-defined, whenever there exist a non-zero state-space determinant.
3. The underlying configuration corresponds to an interacting statistical system, whenever there exist a non-zero state-space scalar curvature.

The main line of thought which has been followed up here has first been to develop an intrinsic Riemannian geometric conception to underlying state-space geometry arising from leading order statistical interactions which exist among various CFT microstates of (rotating) black brane configurations in string theory and M-theory. The perspective notions indicates that novel scaling aspects of the state-space pair correlation and state-space stability in effect arise from negative Hessian matrix of the coarse graining entropy defined over an ensemble of large number of brane microstates characterizing considered black brane attractor configurations. Intimately, we have investigated whether the associated state-space geometries are non-degenerate and possess non-vanishing scalar curvature imply an interacting statistical basis for these configurations, like the one above for instance, the state-space configuration of the multi-centered $D_0$-$D_2$-$D_0'$ solutions have intriguingly been described, and in particular, we have presented complete list of corresponding relative state-space correlation functions. Moreover, the stabilities of underlying configurations can as well be perceived by noting the signs of principle minors. Similarly, the considerations of other string theory and M-theory configurations have been divulged in this context. Interestingly, we find that the behavior of statistical pair correlations between equilibrium microstates is governed by set of consistent parameters defining underlying CFT vacuum configurations, and thence, the same has been anticipated to remain valid for the other associated intrinsic geometric quantities on concerned state-space manifold, as well.

Finally, the higher order $\alpha'$-corrections when taken into account in the underlying effective theory are envisaged to offer diverse well-defined state-space configurations. Generically, the $\alpha'$-corrected state-space configurations are at least expected to be non-degenerate than an ill-defined intrinsic Riemannian geometry arising from the leading order entropy configuration. Such a notion has been offered for the two charge $D_0$-$D_4$ black holes or an excited string at the leading order entropy solutions. Similar motivations along these directions have been obtained in previous state-space investigations [13, 57, 15]. Herewith, we have contemplated that the state-space geometry of black branes in string theory and M-theory would ascribe
definite well-defined, non degenerate and curved intrinsic Riemannian manifold whose pair correlation functions scale as inverse functions of the parameters. There are however many caveats, many things which require further clarification and many open questions which we leave for future investigations.

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Appendix

In this appendix, we provide explicit forms of the state-space relative correlation functions at the first and second centers of the multicentered $D_6D_4D_2D_0$ black holes describing family of the four charged configurations. Our analysis illustrates that the physical properties of the specific state-space correlations may exactly be exploited in general. The definite behavior of state-space correlations as accounted in the concerned section suggests that the various intriguing single center and multi-center state-space examples of black brane solutions includes nice properties that they do have definite stability properties, except that the determinant may be non-positive definite in some cases. As mentioned in the main sections, these $D_6D_4D_2D_0$ configurations are generically well-defined and indicate an interacting statistical basis. We discover here that their state-space geometries indicate possible nature of general two center equilibrium thermodynamic configurations. Significantly, we notice from the very definition of intrinsic metric tensor that the relative state-space correlations may be analyzed as follows.

Appendix (A): State-space relative correlations at the first center

Here, we shall explicitly provide the exact expressions for the four parameter multi-centered solutions at the first center of the double centered black holes. It turns out that the functional nature of large number of branes within a small neighborhood of statistical fluctuations introduced in an equilibrium ensemble of brane configurations may precisely be divulged. Surprisingly, we can expose in this framework that the relative state-space correlations at the first center with charges $p^0 := 1$; $p := 3\Lambda$; $q := 6\Lambda^2$; and $q^0 := -6\Lambda$ take an exact and simple set of expressions

\[
\begin{align*}
\text{c}_{1112} &= -\frac{2\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{7\Lambda^2 + 16\Lambda^4 + 12\Lambda^6 + 1},
\text{c}_{1113} &= \frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{4\Lambda^2 + 4\Lambda^4 + 1},
\text{c}_{1114} &= -108\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{18\Lambda^2 + 27\Lambda^4 - 1},
\text{c}_{1122} &= 6\Lambda^2 - 13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6,
\text{c}_{1123} &= -36\Lambda\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{42\Lambda^2 + 12\Lambda^4 + 45\Lambda^6 + 1},
\text{c}_{1124} &= 12\Lambda^2 - 1 + 2\Lambda^2,
\text{c}_{1133} &= 36\Lambda\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{2 + 9\Lambda^2 + 12\Lambda^4},
\text{c}_{1134} &= -72\Lambda\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{1 + 3\Lambda^2},
\text{c}_{1144} &= 1296\Lambda^2 + 252\Lambda^4 + 1728\Lambda^6 + 216,
\text{c}_{1213} &= \frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{\Lambda(4\Lambda^2 + 1 + 4\Lambda^4)},
\text{c}_{1222} &= -3\Lambda\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6},
\text{c}_{1223} &= 18\Lambda^2 - 13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6
\end{align*}
\]

\[
\begin{align*}
\text{c}_{1224} &= -6\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{\Lambda(1 + 2\Lambda^2)},
\text{c}_{1233} &= -18\Lambda^2 - 13\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6
\end{align*}
\]

\[
\begin{align*}
\text{c}_{1244} &= 36\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{3\Lambda^2 + 1},
\text{c}_{1244} &= -108\frac{7\Lambda^2 + 16\Lambda^4 + 1 + 12\Lambda^6}{\Lambda},
\text{c}_{1314} &= -54\Lambda^2\frac{4\Lambda^2 + 1 + 4\Lambda^4}{18\Lambda^4 + 27\Lambda^6 - 1},
\text{c}_{1322} &= 3\Lambda^2\frac{4\Lambda^2 + 1 + 4\Lambda^4}{43\Lambda^2 + 30\Lambda^4 + 2 + 24\Lambda^6},
\text{c}_{1323} &= -18\Lambda^2\frac{4\Lambda^2 + 1 + 4\Lambda^4}{42\Lambda^2 + 12\Lambda^4 + 45\Lambda^6 + 1},
\text{c}_{1324} &= 6\frac{4\Lambda^2 + 1 + 4\Lambda^4}{1 + 2\Lambda^2},
\text{c}_{1333} &= 18\Lambda^2\frac{4\Lambda^2 + 1 + 4\Lambda^4}{9\Lambda^2 + 12\Lambda^4 + 2},
\text{c}_{1334} &= -36\Lambda^2\frac{4\Lambda^2 + 1 + 4\Lambda^4}{1 + 3\Lambda^2}
\end{align*}
\]
As stated earlier, the state-space metric in the multi-centered and single black brane configurations is given
Appendix (B): State-space relative correlations at the second center
respected to be extensive variables. In this case, we find that the four distinct large charges characterize the
for the second center of the
\[ D \]
at the second center are given by employing the previously defined notations. We have similarly presented
\[ c_{1112} = \frac{432\Lambda^2 + 108 + 432\Lambda^4}{1 + 4\Lambda^2 + 4\Lambda^4} \]
\[ c_{1423} = -\frac{1}{2} \frac{18\Lambda^4 + 27\Lambda^6 + 1}{3\Lambda^2 + 9\Lambda^2 + 12\Lambda^4 + 2} \]
\[ c_{1433} = \frac{1}{2} \frac{18\Lambda^4 + 27\Lambda^6 + 1}{3\Lambda^2 + 9\Lambda^2 + 12\Lambda^4 + 2} \]
\[ c_{1444} = -\frac{2}{\Lambda^2} (18\Lambda^4 + 27\Lambda^6 + 1), \]
\[ c_{2224} = \frac{2}{\Lambda^2} \frac{13\Lambda^2 + 30\Lambda^4 + 24\Lambda^6 + 2}{9\Lambda^2 + 12\Lambda^4 + 2} \]
\[ c_{2234} = -\frac{1}{\Lambda^2} \frac{42\Lambda^4 + 12\Lambda^6 + 45\Lambda^6 + 1}{9\Lambda^2 + 12\Lambda^4 + 2} \]
\[ c_{2334} = \frac{2}{\Lambda^2} \frac{42\Lambda^4 + 12\Lambda^6 + 45\Lambda^6 + 1}{9\Lambda^2 + 12\Lambda^4 + 2} \]
\[ c_{2444} = -\frac{3}{\Lambda^2} \frac{18 + 36\Lambda^2}{3\Lambda^2 + 1} \]
\[ c_{3344} = \frac{6}{\Lambda^2} (9\Lambda^2 + 12\Lambda^4 + 2), \]
\[ c_{3444} = -\frac{3}{\Lambda^2} (3\Lambda^2 + 1) \]
\[ \text{(95)} \]
\[
\begin{align*}
\text{c}_{1423} &= \frac{1}{3} \frac{18 \Lambda^4 + 27 \Lambda^6 - 1}{42 \Lambda^5 + 12 \Lambda^2 + 45 \Lambda^6 + 1} \\
\text{c}_{1433} &= \frac{1}{3 \Lambda} \frac{18 \Lambda^4 + 27 \Lambda^6 - 1}{9 \Lambda^2 + 12 \Lambda^7 + 2} \\
\text{c}_{1444} &= \frac{2}{\Lambda^2} (18 \Lambda^4 + 27 \Lambda^6 - 1), \quad c_{2223} = 6 \Lambda (1 + 3 \Lambda^2) \\
\text{c}_{2224} &= \frac{2}{\Lambda^2} \frac{13 \Lambda^4 + 30 \Lambda^4 + 24 \Lambda^6 + 2}{42 \Lambda^5 + 12 \Lambda^2 + 45 \Lambda^6 + 1} \\
\text{c}_{2234} &= -12 \frac{13 \Lambda^4 + 30 \Lambda^4 + 24 \Lambda^6 + 2}{\Lambda (1 + 3 \Lambda^2)} \\
\text{c}_{2324} &= \frac{1}{3 \Lambda^2} \frac{42 \Lambda^4 + 12 \Lambda^2 + 45 \Lambda^6 + 1}{1 + 2 \Lambda^2} \\
\text{c}_{2334} &= \frac{1}{3 \Lambda^2} \frac{42 \Lambda^4 + 12 \Lambda^2 + 45 \Lambda^6 + 1}{1 + 3 \Lambda^2} \\
\text{c}_{2433} &= \frac{3 \Lambda^2}{9 \Lambda^2 + 12 \Lambda^4 + 2} \\
\text{c}_{2444} &= \frac{18 + 36 \Lambda}{\Lambda} \\
\text{c}_{3344} &= 6 \Lambda^2 (9 \Lambda^2 + 12 \Lambda^4 + 2) \\
\end{align*}
\]

(96)

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