CONORMAL VARIETIES ON THE COMINUSCULE
GRASSMANNIAN

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Abstract. Let $G$ be a simply connected, almost simple group over an algebraically closed field $k$, and $P$ a maximal parabolic subgroup corresponding to omitting a cominuscule root. We construct a compactification $\varphi : T^*G/P \to X(u)$, where $X(u)$ is a Schubert variety corresponding to the loop group $LG$. Let $N^*X(w) \subset T^*G/P$ be the conormal variety of some Schubert variety $X(w)$ in $G/P$; hence we obtain that the closure of $\varphi(N^*X(w))$ in $X(u)$ is a $B$-stable compactification of $N^*X(w)$. We further show that this compactification is a Schubert subvariety of $X(u)$ if and only if $X(uw_0) \subset G/P$ is smooth, where $w_0$ is the longest element in the Weyl group of $G$. This result is applied to compute the conormal fibre at the zero matrix in any determinantal variety.

1. Introduction

Given a quiver $Q$, let $\overline{Q} = Q \sqcup Q^{op}$ be the double of $Q$, the quiver that has the same vertex set as $Q$ and whose set of edges is a disjoint union of the sets of edges of $Q$ and of $Q^{op}$, the opposite quiver. Thus, for any edge $e \in Q$, there is also a reverse edge $e^* \in Q^{op} \subset \overline{Q}$ with the same endpoints as $e$, but in the opposite direction. For $d$ a dimension vector, the quiver variety $\text{Rep}_d\overline{Q}$ is naturally identified as the cotangent bundle of $\text{Rep}_dQ$ (cf. [Gin09]).

Orbit closures in $\text{Rep}_dA_2$ (see Table 1) are called determinantal varieties. Lakshmibai and Seshadri [LS78] have identified determinantal varieties as open subsets of certain Schubert varieties in the type $A$ Grassmannian. Further, Strickland [Str82] has identified the conormal varieties of determinantal varieties as certain nilpotent orbit closures in $T^*\text{Rep}_dA_2 = \text{Rep}_d\overline{A}_2$. On the other hand, Lusztig [Lus90] has identified nilpotent orbit closures in $\text{Rep}_d\overline{A}_2$ (and more generally in $\text{Rep}_{\tilde{A}_n}$) as open subsets of Schubert varieties in the affine Grassmannian. Inspired by these results, Lakshmibai [Lak16] has suggested an exploration of the relationship between the conormal varieties of Schubert varieties and the corresponding affine type Schubert varieties.

Let $G$ be a simply connected, almost simple algebraic group over an algebraically closed field $k$. We identify $G$ as a Kac-Moody group corresponding to some irreducible finite type Dynkin diagram $D_0$. The loop group $LG \equiv G(k[t, t^{-1}])$ is then a Kac-Moody group corresponding to the extended Dynkin diagram $\overline{D}$, obtained by attaching to $D_0$ the extra root $\alpha_0$ (see Table 1).

A simple root $\alpha_d$ is cominuscule if and only if there exists an automorphism $\iota$ of $D$ such that $\iota(\alpha_0) = \alpha_d$ (see Table 1). Let $P$ be the parabolic subgroup in $G$ corresponding to omitting a cominuscule root $\alpha_d$, and $P$ the parabolic subgroup in
Let \( \alpha_d \) and \( \alpha_0 \). For \( k = C \), Lakshmibai \[Lak10\], and Lakshmibai, Ravikumar, and Slofstra \[LRS16\] have constructed a dense embedding \( \phi \) of \( T^*G/P \) into a Schubert variety in \( LG/P \). We use the Kac-Moody functor of Tits \[Tit87\] to give a definition of \( \phi \) which works in all characteristics (see Proposition 3.25 and Theorem 3.27).

Let \( w_0 \) denote the longest element in the Weyl group of \( G \), and \( w_J \) the longest element in the Weyl group of \( P \). Our main result (Theorem 4.12) is the following:

**Theorem 1.1.** Let \( X(w) \) be a Schubert variety in \( G/P \), and let \( N^*X_J(w) \) be its conormal variety. Then the closure of \( \phi(N^*X_J(w)) \) in \( LG/P \) is a Schubert variety if and only if the Schubert variety \( X(w_0ww_J) \) in \( G/P \) is smooth.

Billey and Mitchell \[BM10\] have given a combinatorial criterion to identify smooth Schubert varieties in cominuscule Grassmannians (see Proposition 4.7). Using this, we deduce that Theorem 1.1 applies to determinantal varieties, symmetric determinantal varieties, and skew-symmetric determinantal varieties, which Lakshmibai and Seshadri \[LS78\] have identified as open subsets of certain Schubert sub-varieties of cominuscule Grassmannians of type \( A, C, \) and \( D \) respectively.

Let \( \Sigma_{r,k,n} \) denote the rank \( r \) skew-symmetric determinantal variety:

\[
\Sigma_{r,k,n} = \left\{ A \in \text{Hom}(k^n, k^n) \mid A = -A^T, \text{rank}(A) \leq r \right\}
\]

Our second result (Theorem 5.9) identifies the conormal fibre at 0 of the skew-symmetric determinantal variety.

**Theorem 1.2.** The conormal fibre of \( \Sigma_{r,k,n} \) at 0 is isomorphic to \( \Sigma_{r-k,n} \) where

\[
\pi = \begin{cases} 
  n & \text{if } n \text{ is even,} \\
  n-1 & \text{if } n \text{ is odd.}
\end{cases}
\]

Similar results are known for determinantal (see \[Str82, GR14\]) and symmetric determinantal varieties (see \[GL\]). Our proof of Theorem 5.9 can be adapted in a straightforward manner to recover these results.

The paper is arranged as follows. In Section 2, we recall the basics of (finite and affine type) root systems and the corresponding almost simple groups. We also describe how extending a finite type root system by attaching an extra root in a manner prescribed in \[Kac94\] corresponds to replacing the corresponding almost simple group with its loop group. Finally, we recall some results on Weyl groups and Schubert varieties.

In Section 3, we show that the cotangent bundle of a cominuscule Grassmannian has a compactification \( \phi \) by an affine Schubert variety. Along the way, we construct (Definition 3.8) an involution \( \iota \) of affine type Dynkin diagrams that exchanges a cominuscule root \( \alpha_d \) with the extra root \( \alpha_0 \). The involution \( \iota \) acts on the associated Weyl group by conjugation (Equation (3.13)).

In Section 4, we study the conormal variety \( N^*X_J(w) \) of a Schubert variety \( X_J(w) \) in a cominuscule Grassmannian \( G/P \). We leverage the main result of \[BM10\] to develop characterizations of smooth Schubert varieties in \( G/P \), see Proposition 4.7. We then use this to prove that \( N^*X_J(w) \) has a compactification as a Schubert
variety via the embedding \( \phi \) if and only if the Schubert variety \( X_{G}(w_{0}ww_{r}) \) is smooth, see Theorem 4.12. This yields powerful results about the geometry of \( N^{*}X_{G}(w) \) when \( X_{G}(w_{0}ww_{r}) \) is smooth, see Theorem 4.17. Further, using Littelmann’s work \[ Lit03 \] on the standard monomial theory of affine Schubert varieties, we can write down the equations defining \( N^{*}X_{G}(w) \) as a subvariety of \( T^{*}G/P \), see Theorem 4.18. In Proposition 4.19, we give a description of the fibre of \( N^{*}X_{G}(w) \) at identity as a union of Schubert varieties.

In Section 5, we apply the results of Section 4 to skew-symmetric determinantal varieties. The (usual, symmetric, skew-symmetric resp.) rank \( r \) determinantal varieties can be identified as the opposite cells of certain Schubert varieties \( X_{G}(w_{r}) \) in certain cominuscule Grassmannian (of type \( A, C, D \) resp.). Working in type D, we first verify that \( X_{G}(w_{0}ww_{r}) \) is smooth (Equation (5.10)); hence Proposition 4.19 applies. We then make explicit computations in the Weyl group (Proposition 5.14) to prove that the fibre at the zero matrix of the skew-symmetric determinantal variety is the rank \( n-r \) skew-symmetric determinantal variety, see Theorem 1.2.

Acknowledgments: We thank Terence Gaffney for fruitful discussions that pointed us towards the results in Section 5.

2. Dynkin Diagrams and Weyl Groups

In this section, we recall the basics of the theory of finite type and extended Dynkin diagrams, their root systems, and certain Kac-Moody groups associated to them. Throughout, we assume that the base field \( k \) is algebraically closed. The primary references for the combinatorial results in this section are [Bou68, Kum12, Tit87]. For the geometric results, one may refer to [Fal03, Kum12].

2.1. Finite Type Dynkin Diagrams. Let \( D_{0} \) be an irreducible finite type Dynkin diagram, and \( \Delta_{0} \) the abstract root system associated to \( D_{0} \). We shall denote by \( \Delta_{0}^{+}, \Delta_{0}^{-}, D_{0}, \theta_{0}, \mathbb{Z}D_{0}, \) and \( W_{0} \), the positive roots, negative roots, simple roots, highest root, root lattice, and the Weyl group of \( \Delta_{0} \) respectively.

2.2. Extended Dynkin Diagram. We can attach a simple root \( \alpha_{0} \) to \( D_{0} \) to get the extended Dynkin diagram \( D \) (see [Kac94, Table I]). Let \( \Delta, \Delta^{+}, \Delta^{-}, D, \mathbb{Z}D, \) and \( W \) denote the set of roots, positive roots, negative roots, simple roots, root lattice, and the Weyl group respectively of the abstract root system of \( D \).

2.3. Real and Imaginary Roots. A root \( \alpha \in \Delta \) is called a real root if there exists \( w \in W \) such that \( w(\alpha) \in D \); otherwise \( \alpha \) is called an imaginary root. The root \( \delta \overset{\text{def}}{=} \alpha_{0} + \theta_{0} \) is called the basic imaginary root. The set of real (resp. positive) roots \( \Delta_{\text{re}} \) (resp. \( \Delta^{+} \)) has the following characterization in terms of \( \delta \):

\[
\Delta_{\text{re}} = \{ \alpha + n\delta \mid \alpha \in \Delta_{0}, n \in \mathbb{Z} \}
\]

\[
\Delta^{+} = \{ \alpha + n\delta \mid \alpha \in \Delta_{0} \cup \{0\}, n > 0 \} \cup \Delta_{0}^{+}
\]
2.4. **Bruhat Order and Reduced Expressions.** The Weyl group $W$ is a Coxeter group with simple reflections $\{s_{\alpha} \mid \alpha \in \Delta\}$. The *Bruhat order* $\leq$ on $W$ is the partial order generated by the relations

\begin{align*}
ws_{\alpha} > w & \iff w(\alpha) > 0 \quad \forall \alpha \in \Delta^+ \\
s_{\alpha}w > w & \iff w^{-1}(\alpha) > 0 \quad \forall \alpha \in \Delta^+
\end{align*}

We say $w = s_1 \ldots s_l$ is a reduced expression for $w$ if each $s_i$ is a simple reflection, and any other expression $w = s'_{i_1} \ldots s'_{i_k}$ satisfies $k \geq l$. The *length* $l(w)$ of an element $w \in W$ is the number of simple reflections in a reduced expression for $w$. The length function satisfies the relation $v < w \implies l(v) < l(w)$.
2.6. The Weyl Involution. The Weyl group $W_0$ is finite, and has a unique longest element $w_0$. The element $w_0$ is an involution, i.e., $w_0^2 = 1$, and further satisfies $w_0(\Delta^+_\mathfrak{G}) = \Delta^-_{\mathfrak{G}}$. It follows that $-w_0$ induces an involution of $\mathcal{D}_0$, called the Weyl involution (see [Bou68 pg 158]).

2.7. Semi-Direct Product Decomposition. Let $\Lambda^\vee_0$ be the coroot lattice of $\Delta_0$. There exists (cf. [Kum12 §13.1.7]) a group isomorphism $W \to W_0 \ltimes \Lambda_0^\vee$ given by

\[
\begin{align*}
    s_\alpha &\mapsto (s_\alpha, 0) \quad &\text{for } \alpha \in \mathcal{D}_0 \\
    s_{\alpha_0} &\mapsto (s_{\alpha}, -\theta^\vee) 
\end{align*}
\]

where $\theta$ is the highest root in $\Delta_0$. For $q \in \Lambda_0^\vee$, we write $\tau_q \overset{\text{def}}{=} (1, q) \in W_0 \ltimes \Lambda_0^\vee$. The action of $\tau_q$ on $\Delta$ is determined by the formula $\tau_q(\theta) = \delta$, and

\[
\tau_q(\alpha) = \alpha - \alpha(q)\delta \quad \forall \alpha \in \Delta_0
\]

2.9. Support. The support of $w \in W$, denoted $\text{Supp}(w)$, is the smallest subset $\mathcal{J} \subset \mathcal{D}$ satisfying $w \in \mathcal{W}_\mathcal{J}$. For $\alpha = \sum_{\beta \in \mathcal{D}} a_\beta \beta$, we define the support of $\alpha$ to be

$$\text{Supp}(\alpha) \overset{\text{def}}{=} \{ \beta \in \mathcal{D} | a_\beta \neq 0 \}$$

If $\alpha \in \Delta^+$ and $w(\alpha) \in \Delta^-$, then $\text{Supp}(\alpha) \subset \text{Supp}(w)$. In particular, it follows from Equation (2.11) that $W_\mathcal{J} \subset W^{\mathcal{D} \setminus \mathcal{J}}$ for any $\mathcal{J} \subset \mathcal{D}$.

2.10. Minimal Representatives. Let $\mathcal{J}$ be some proper (necessarily finite type) sub-diagram of $\mathcal{D}$. We write $\Delta_{\mathcal{J}}, \Delta^+_\mathcal{J}, \Delta^-_{\mathcal{J}}, \mathcal{J}$, and $W_{\mathcal{J}}$ for the set of roots, positive roots, negative roots, simple roots, and the Weyl subgroup respectively whose support is contained in $\mathcal{J}$. Given an element $w \in W$, there exists a unique element $w_{\mathcal{J}}$, which is of minimal length in the coset $wW_{\mathcal{J}}$. The element $w_{\mathcal{J}}$ is called the minimal representative of $w$ with respect to $\mathcal{J}$. The set of minimal representatives in $W$ with respect to $\mathcal{J}$ is denoted $W^{\mathcal{J}}$. It follows from Equation (2.11) that

\[
W^\mathcal{J} = \{ w \in W | w(\alpha) > 0, \forall \alpha \in \mathcal{J} \}
\]

2.12. The Group $G$. Let $G$ be the simply connected, almost simple algebraic group over $k$ whose Dynkin diagram is $\mathcal{D}_0$. We fix a torus $T \subset G$, and a Borel subgroup $B \subset G$ satisfying $T \subset B$. We identify the root system of $(G, B, T)$ with the abstract root system $\Delta_0$, and the Weyl group $W_0$ with $N/T$, where $N$ is the normalizer of $T$ in $G$.

2.13. The Loop Group. Let $\mathcal{O} \overset{\text{def}}{=} k[t], \mathcal{O}^- \overset{\text{def}}{=} k[t^{-1}]$, and $\mathcal{K} \overset{\text{def}}{=} k[t, t^{-1}]$. The loop group $LG \overset{\text{def}}{=} G(\mathcal{K})$ is a Kac-Moody group with Dynkin diagram $\mathcal{D}$, and is ind-representable by an affine scheme over $k$. Throughout, we shall identify the Weyl group $W$ of $\mathcal{D}$ with $N(\mathcal{K})/T$.

Let $\mathfrak{g}$ be the Lie algebra of $G$. We identify the Lie algebra $L\mathfrak{g}$ of $LG$ with $\mathfrak{g} \otimes \mathcal{K}$. Let $U_{\alpha}$ denote the root subgroup corresponding to a real root $\alpha \in \Delta^\vee_{re}$ (see [Rem02 Bor12]). We can identify $G$ as the subgroup of $LG$ generated by $T$ and $\{ U_\alpha | \alpha \in \Delta_0 \}$.

Let $L^+G \overset{\text{def}}{=} G(\mathcal{O})$, $L^-G \overset{\text{def}}{=} G(\mathcal{O}^-)$, and consider the evaluation maps

\[
\begin{align*}
    \pi : L^+G &\to G, \quad t \mapsto 0 \\
    \pi^- : L^-G &\to G, \quad t^{-1} \mapsto 0
\end{align*}
\]
The subgroups $B_{\text{def}} = \pi^{-1}(B)$ and $B_{-\text{def}} = \pi^{-1}(B^-)$ are called Borel subgroups of $LG$. Suppose $B$, $B^-$ are opposite in $G$, i.e., $B \cap B^- = T$. Then $B$, $B^-$ are opposite in $LG$, i.e., $B \cap B^- = T$.

2.14. Nilpotent set of roots. (see [Tit87]) Let $\Psi$ be a finite set of real roots. We say that $\Psi$ is pre-nilpotent if there exist $w, w' \in W$ such that $w(\Psi) \subset \Delta^+$ and $w'(\Psi) \subset \Delta^-$. We say that $\Psi$ is closed if $\alpha, \beta \in \Psi$, $\alpha + \beta \in \Delta = \Rightarrow \alpha + \beta \in \Psi$. Finally, we say $\Psi$ is nilpotent if it is pre-nilpotent and closed. For $\alpha \in \Psi$, let $g_\alpha$ be the associated root space (see for example [Kac94]). If $\Psi$ is nilpotent, then so is the Lie sub-algebra

$$g_\Psi \overset{\text{def}}{=} \bigoplus_{\alpha \in \Psi} g_\alpha$$

2.16. Tits’ Functor for Kac-Moody Groups. For $\alpha$ a real root, let $U_\alpha$ be the group scheme over $k$ isomorphic to $G_\alpha$ with Lie algebra $\mathfrak{g}_\alpha$. To every nilpotent set of roots $\Psi$, Tits [Tit87] associates a group scheme $U_\Psi$ that depends only on $\mathfrak{g}_\Psi$, and is naturally a closed subgroup scheme of $LG$ (cf. [Tit87]). For any ordering of $\Psi$, the product morphism

$$\prod_{\alpha \in \Psi} U_\alpha \to U_\Psi$$

is a scheme isomorphism. Hence, we get a scheme isomorphism $\eta : g_\Psi \to U_\Psi$.

2.17. Parabolic Subgroups. For $J$ any subset of $D$, the subgroup

$$P_J \overset{\text{def}}{=} BW_J B = \{ bnb' \mid b, b' \in B, n \in N(K), n \text{mod } T \in W_J \}$$

is the parabolic subgroup of $LG$ corresponding to $J \subset D$. The parabolic subgroup $P_{D_0}$ is precisely $L^+G$. For $J \subset D_0$, the subgroup

$$P_J \overset{\text{def}}{=} BW_J B = \{ bnb' \mid b, b' \in B, n \in N, n \text{mod } T \in W_J \}$$

is the parabolic subgroup of $G$ corresponding to $J \subset D_0$.

2.18. The Bruhat decomposition. Consider some subset $J$ of $D_0$. The Bruhat decomposition of $G$ is

$$G = \bigsqcup_{w \in W_J^{\text{aff}}} BwP_J$$

where $W_J^{\text{aff}} \overset{\text{def}}{=} W_0 \cap W_J$. The partial flag variety $G/P_J$ has the decomposition

$$G/P_J = \bigsqcup_{w \in W_J^{\text{aff}}} BwP_J \mod P_J$$

Let $\preceq$ denote the Bruhat order on $W$. For $w \in W_J^{\text{aff}}$, the Schubert variety

$$X_J(w) \overset{\text{def}}{=} BwP_J \mod P_J = \bigsqcup_{w \leq w} BwP_J \mod P_J$$

is a projective variety of dimension $l(w)$. 

2.19. **Affine Schubert Varieties.** Fix a proper subset $J \subseteq D$. We have the Bruhat decomposition

$$LG = \bigsqcup_{w \in W^J} BwP_J$$

The quotient $LG/P_J$ is an ind-scheme. For $w \in W^J$, the affine Schubert variety

$$X_J(w) \overset{\text{def}}{=} BwP_J \text{mod } P_J$$

is a projective variety of dimension $l(w)$, and has the decomposition

$$X_J(w) = \bigsqcup_{v \leq w \atop v \in W^J} BvP_J \text{mod } P_J$$

Consider a proper subset $L \subseteq D$, and let $w_L$ be the longest element in $W_L$. The Schubert variety $X_J(w_J^L)$ is $P_L$-homogeneous, hence smooth. Indeed, we have

$$P_L \cap P_J = P_{\Delta^L \cap \Delta^J}$$

and further, $X_J(w_J^L) = P_L/P_L \cap P_J$.

2.20. **The Opposite Cell.** Let $J, L$ be as above. Let $e$ denote the image of the identity element in the quotient $LG/P_J$. The opposite cell $X_J^-(w)$, given by

$$X_J^-(w) \overset{\text{def}}{=} B^- e \bigcap X_J(w)$$

is an open affine subvariety of $X_J(w)$. The opposite cell of the Schubert variety $X_J(w_J^L)$ is isomorphic to the affine space $A^k$ for $k = \text{Card} (\Delta^L \setminus \Delta^J)$. Indeed, for any enumeration $\alpha_1, \ldots, \alpha_k$ of $\Delta^L \setminus \Delta^J$, the map

$$U_{\alpha_1} \times \ldots \times U_{\alpha_k} \rightarrow LG/P_J$$

$$(u_1, \ldots, u_k) \mapsto u_1 \ldots u_k \text{ mod } P_J$$

is an open immersion onto the opposite cell $X_J^-(w_J^L)$. We also mention here the $L^-G$ cell $Y_J(w) \subset X_J(w)$,

$$Y_J(w) \overset{\text{def}}{=} L^- Ge \bigcap X_J(w)$$

which is also an open affine subvariety in $X_J(w)$.

2.23. **The Demazure Product.** The Demazure product $\star$ on $W$ is the unique associative product satisfying:

$$s_\alpha \star w = \begin{cases} w & \text{if } s_\alpha w < w \\ s_\alpha w & \text{if } s_\alpha w > w \end{cases}$$

The double coset $BwB$ is called a *Bruhat cell* in $LG$. Suppose $v = s_1 \ldots s_k$ is a reduced presentation for $v \in W$. Then $v = s_1 \star \ldots \star s_k$, and

$$BvB = Bs_1B \ldots Bs_kB$$

Consider $v \in W^J, w \in W_J$. Then $v \star w = vw$. More generally, we have

$$l(vw) = l(v) + l(w) \iff v \star w = vw$$

**Remark 2.26.** The reader may refer to [KM04], Remark 3.3 for further details, including a justification of the name “Demazure product”.
3. The Cominuscule Grassmannian

Let \( D_0 \) be a finite type Dynkin diagram, and \( D \) its associated extended Dynkin diagram. In this section, we first recall the notion of cominuscule roots and cominuscule Grassmannians associated to \( D_0 \). We then fix a choice of cominuscule root \( \alpha_d \), and develop the rest of this section (and the next) for that fixed choice of \( \alpha_d \).

We introduce in Definition 3.8 a canonical Dynkin diagram involution \( \iota \) depending only on \( \alpha_d \).

Let \( G \) be the almost simple, simply connected algebraic group with Dynkin diagram \( D_0 \), and \( P \) the maximal parabolic subgroup corresponding to `omitting' the simple root \( \alpha_d \). In characteristic 0, Lakshmibai, Ravikumar, and Slofstra [LRS16] have constructed an isomorphism \( \phi \) of \( T^*G/P \) with the opposite cell of an affine Schubert variety in \( LG/P \) (see Section 3.2 for the definition of \( P \)). We give an alternate description of this Schubert variety in Equation (3.15). In Proposition 3.25, we give a characteristic free definition of \( \phi \).

**Definition 3.1.** A simple root \( \alpha_d \in D_0 \) is called cominuscule if the coefficient of \( \alpha_d \) in \( \delta \) is 1.

Observe from Table 1 that a simple root \( \alpha_d \) is cominuscule if and only if there exists an automorphism \( \iota \) of \( D \) such that \( \iota(\alpha_0) = \alpha_d \). For the remainder of this section (and the next), let \( \alpha_d \) be some fixed cominuscule root in \( D_0 \), and set

\[
D_d \overset{\text{def}}{=} D \setminus \{ \alpha_d \} \quad J \overset{\text{def}}{=} D_0 \cap D_d \quad \theta_d \overset{\text{def}}{=} \delta - \alpha_d
\]

We write \( \Delta_d, \Delta^+_d, \Delta^-_d, \mathbb{Z}D_d, \) and \( W_d \) for the set of roots, positive roots, negative roots, root lattice, and Weyl group respectively of the root system associated to \( D_d \). Observe that \( \theta_d \) is the highest root of the finite type root system \( \Delta_d \).

### 3.2. The Cominuscule Grassmannian

Let \( P_d \) be the parabolic subgroup corresponding to the set of simple roots \( D_d \). We will write \( P, \mathcal{P} \) for the parabolic subgroups \( P_d \subset G \) and \( \mathcal{P}_J \subset LG \) respectively. Observe that \( \mathcal{P} = L^+G \cap P_d \) and \( P = G \cap P_d \). The variety \( G/P \) is called a cominuscule Grassmannian of type \( D_0 \).

### 3.3. The Cotangent Space

Let \( g, p, h \) denote the Lie algebras of \( G, P, T \) respectively. We have the root space decompositions

\[
g = h \oplus \bigoplus_{\alpha \in \Delta_u} g^\alpha
\]

\[
p = h \oplus \bigoplus_{\alpha \in \Delta^+_u \cup \Delta^-_J} g^\alpha
\]

Let us identify \( g \) with its dual using the Killing form (cf. [Ser01]). In particular, the dual of a root space \( g^\alpha \) is identified with the root space \( g^{-\alpha} \). Now the tangent space at identity of \( G/P \) being \( g/p \), we can identify its dual with

\[
\bigoplus_{\alpha \in \Delta_u \setminus \Delta^-_J} g^{-\alpha} = \bigoplus_{\alpha \in \Delta^+_u \setminus \Delta^-_J} g^\alpha = u_P
\]

where \( u_P \) is the Lie algebra of the unipotent radical \( U_P \) of \( P \).
3.4. The Cotangent Bundle. The cotangent bundle $T^*G/P$ is a vector bundle over $G/P$, the fibre at any point $x \in G$ being the cotangent space to $G/P$ at $x$; the dimension of $T^*G/P$ equals $2 \dim G/P$. Also, $T^*G/P$ is the fibre bundle over $G/P$ associated to the principal $P$-bundle $G \rightarrow G/P$ for the adjoint action of $P$ on $u_P$. Thus

$$T^*G/P = G \times^P u_P = G \times u_P/\sim$$

where $\sim$ is the equivalence relation given by $(gp,u) \sim (g,p^u p^{-1})$ for $g \in G$, $u \in u_P$, $p \in P$.

Lemma 3.5. Let $w_J$ be the longest element of $W_J$. We have $w_J(\alpha_d) = \theta_0$ and $w_J(\alpha_0) = \theta_d$.

Proof. To show $w_J(\alpha_d) = \theta_0$, it is enough to show that $w_J(\alpha_d)$ is maximal in $\Delta_0^+ \setminus \Delta_J^+$. Observe first that $w_J(\Delta_0) = \Delta_0$, $w_J(\Delta_J) = \Delta_J$, and

$$\{ \alpha \in \Delta_0^+ \mid w_J(\alpha) < 0 \} = \Delta_J^+$$

Consequently, $w_J(\Delta_0^+ \setminus \Delta_J^+) \subset \Delta_0^+ \setminus \Delta_J^+$. Consider $\alpha \in \Delta_0^+ \setminus \Delta_J^+$, and let $\gamma = \alpha - \alpha_d$. Observe that $\alpha_d \leq \alpha$; hence $\gamma \geq 0$. Further, since $\alpha_d$ is cominuscule, we have $2\alpha_d \not\leq \alpha$. It follows that $\alpha_d \not\in \text{Supp}(\gamma)$, and so $\text{Supp}(\gamma) \subset J$. Hence

$$w_J(\gamma) \leq 0 \implies w_J(\alpha) = w_J(\alpha_d) + w_J(\gamma) \implies w_J(\alpha) \leq w_J(\alpha_d)$$

We see that $w_J(\alpha_d)$ is maximal in $\Delta_0^+ \setminus \Delta_J^+$, hence $w_J(\alpha_d) = \theta_0$. The formula $w_J(\alpha_0) = \theta_d$ follows similarly, by showing that $w_J(\alpha_0)$ is maximal in $\Delta_0^+ \setminus \Delta_J^+$. \qed

3.6. Bilinear Form. Let $V$ denote the real vector space with basis $\mathcal{D}$. There exists a $W$-invariant symmetric bilinear form $(\mid \ )$ on $V$ (cf. [Kac94 §3.7]) such that

$$s_{\alpha}(\beta) = \beta - 2 (\alpha \mid \beta) (\alpha \mid \alpha).$$

Definition 3.8 (The Involution $\iota$). Let $\iota$ be the linear involution of $V$ given by

$$\iota(\alpha) = \begin{cases} 
\alpha_d & \text{for } \alpha = \alpha_0 \\
\alpha_0 & \text{for } \alpha = \alpha_d \\
-w_J(\alpha) & \text{for } \alpha \in J
\end{cases}$$

Lemma 3.9. The form $(\mid \ )$ is invariant under $\iota \in GL(V)$.

Proof. Recall that $(\delta \mid \ ) = 0$ (cf. [Kac94 §5.2]). Given $\alpha, \beta \in J$, we have

$$(\iota(\alpha) \mid \iota(\beta)) = (-w_J(\alpha) \mid -w_J(\beta)) = (\alpha \mid \beta)$$

$$(\iota(\alpha_0) \mid \iota(\beta)) = (\alpha_d \mid -w_J(\beta)) = (w_J(\theta_0) \mid -w_J(\beta))$$

$$= (-\theta_0 \mid \beta) = (\alpha_0 - \delta \mid \beta) = (\alpha_0 \mid \beta)$$

$$(\iota(\alpha_d) \mid \iota(\beta)) = (\alpha_0 \mid -w_J(\beta)) = (w_J(\theta_d) \mid -w_J(\beta))$$

$$= (-\theta_d \mid \beta) = (\alpha_d - \delta \mid \beta) = (\alpha_d \mid \beta)$$

$$(\iota(\alpha_0) \mid \iota(\alpha_d)) = (\alpha_d \mid \alpha_0) = (\alpha_0 \mid \alpha_d)$$

$$\square$$

Proposition 3.10. The map $\iota$ induces an involution of the Dynkin diagram $\mathcal{D}$. \qed
Proof. It is clear from the definition that \( \iota \) is an involution. Further, since \(-w_{\mathcal{J}}\)
induces an involution of \( \mathcal{J} \) that preserves its Dynkin diagram structure (cf. [Bou68 pg 158]), it follows that \( \iota \) preserves the set of simple roots \( \mathcal{D} \). Now, it follows from \( \alpha_i^\vee (\alpha_j) = \frac{\langle \alpha_i | \alpha_j \rangle}{\langle \alpha_i | \alpha_i \rangle} \)
that the Cartan matrix \( (\alpha_i^\vee (\alpha_j))_{ij} \) is preserved under \( \iota \), and so \( \iota \) preserves the Dynkin diagram structure on \( \mathcal{D} \). \( \square \)

Corollary 3.11. We have the equality \( \iota(\delta) = \delta \).

Proof. We see from \( (\delta | _{\mathcal{J}} = 0 \) and Lemma 3.9 that \( (\iota(\delta) | _{\mathcal{J}} = 0 \). Hence \( \iota(\delta) = k\delta \) for some \( k \in \mathbb{Z} \) (cf. [Kac94 §5.6]). Further it follows from Proposition 3.10 that \( k > 0 \) and \( k^2 = 1 \). We deduce that \( k = 1 \), i.e., \( \iota(\delta) = \delta \). \( \square \)

3.12. Action on \( W \). We also define an involution \( \iota \) of \( W \) given by

\[
\iota s_{\alpha} \overset{\text{def}}{=} s_{\iota(\alpha)} \quad \text{for} \quad \alpha \in \mathcal{D}
\]

It is clear that \( \iota \) preserves the length and the Bruhat order on \( W \), and

\[
\iota W_{\mathcal{J}} = W_{\mathcal{J}} \quad \iota W^d = W^d \quad \iota w_0 = w_0 \quad \iota w_d = w_d
\]

Using Lemma 3.9 and Equation (3.7), we see that

\[
\iota(s_{\alpha}(\iota(\beta))) = \iota \left( \iota(\beta) - 2 \frac{(\alpha | \iota(\beta))}{(\alpha | \alpha)} \alpha \right) = \beta - 2 \frac{(\iota(\alpha) | \beta)}{(\iota(\alpha) | \iota(\alpha))} \iota(\alpha) = s_{\iota(\alpha)}(\beta)
\]

It follows that the action of \( \iota \) on \( w \) is the same as conjugation by \( \iota \), where both \( w \) and \( \iota \) are viewed as elements of \( GL(V) \), i.e.,

\[
\iota w = w' \iota
\]

Note also that since Schubert varieties depend only on the underlying Dynkin diagrams, there exists an isomorphism \( X_{\mathcal{J}}(w) \cong X_{\mathcal{J}}(\iota(w)) \) for any \( w \in W \).

3.14. The element \( \tau_q \). Let \( w_0, w_d \) be the maximal elements in \( W_0, W_d \) respectively, and let \( \varpi_d^\vee \) be the fundamental co-weight dual to \( \alpha_d \). Set

\[
q \overset{\text{def}}{=} w_0(\varpi_d^\vee) - \varpi_d^\vee \in \Lambda_0^\vee,
\]

and let \( \tau_q \in W \) be the element corresponding to \( q \in \Lambda_0^\vee \) (see Section 2.7).

Proposition 3.16. We have the equality \( \tau_q = w_0 w_{\mathcal{J}} w_d w_{\mathcal{J}} = w_d w_{\mathcal{J}}^d w_{\mathcal{J}} \).

Proof. The set of roots \( \Delta \) is contained in the \( \mathbb{Z} \)-span of the set \( \mathcal{D}_0 \cup \{ \delta \} \). Since the action of \( W \) on \( \Delta \) is faithful, it is enough to verify

\[
w_0 w_{\mathcal{J}} w_d w_{\mathcal{J}}(\alpha) = \tau_q(\alpha) \quad \forall \alpha \in \mathcal{D}_0 \cup \{ \delta \}
\]

Further, since \( \delta \) is fixed under the action of \( W \) (cf. Section 2.7), it is sufficient to verify Equation (3.17) for \( \alpha \in \mathcal{D}_0 \).
Recall from Section 2.6 that $-w_0$ induces an involution on $D_0$. Set $\beta = -w_0(\alpha_d)$, so that $-w_0(\varpi^\vee)^\beta = \varpi^\vee$, the fundamental co-weight dual to $\beta$. It follows from Equations (2.8) and (3.15) that

\begin{equation}
\tau_q(\alpha) = \alpha + \alpha(\varpi^\vee)^\beta - \alpha(w_0(\varpi^\vee))\delta
= \alpha + \alpha(\varpi^\vee + \varpi^\vee)^\beta
\end{equation}

Hence we can rewrite Equation (3.17) as

\begin{equation}
w_0w_\mathcal{F}w_dw_\mathcal{F}(\alpha) = \alpha + \alpha(\varpi^\vee + \varpi^\vee)^\beta \quad \forall \alpha \in D_0
\end{equation}

Next, it follows from (3.19) that

\begin{equation}
\iota(\beta) = -\iota(w_0(\alpha_d)) = -\iota(w_0(\alpha_d)) \quad \text{using Equation (3.13)}
\end{equation}

\begin{equation}
\Rightarrow \iota(\beta) = -w_d(\alpha_d) \quad \text{using Section 3.12}
\end{equation}

Further, we have

\begin{equation}
w_dw_\mathcal{F}(\alpha_d) = w_d(\theta_0) = w_d(\delta - \alpha_0) \quad \text{using Lemma 3.5}
= \delta - w_d(\alpha_0) = \delta + \iota(\beta) \quad \text{using Equation (3.20)}
\end{equation}

\begin{equation}
w_0w_\mathcal{F}(\alpha_0) = \iota(w_dw_\mathcal{F}(\iota(\alpha_d))) \quad \text{using Section 3.12}
= \iota(\delta + \iota(\beta)) = \delta + \beta \quad \text{using Corollary 3.11}
\end{equation}

We are now ready to prove that Equation (3.19) holds for $\alpha \in \{\alpha_d, \beta\}$.

**Case 1** Suppose $\beta = \alpha_d$. Then $\iota(\beta) = \alpha_0$, $\varpi^\vee_d = \varpi^\vee_{\beta}$, and $q = \tau_{\varpi^\vee_d}$. We have:

\begin{align*}
w_0w_\mathcal{F}w_dw_\mathcal{F}(\alpha_d) &= w_0w_\mathcal{F}(\delta + \iota(\beta)) \quad \text{using Equation (3.21)}
= w_0w_\mathcal{F}(\delta + \alpha_0) \quad \text{using Lemma 3.5}
= w_0(\delta + \theta_d) \quad \text{using Lemma 3.5}
= w_0(2\delta - \alpha_d) = 2\delta + \beta \quad \text{using $\beta = -w_0(\alpha_d)$}
= \alpha_d + 2\delta = \tau_q(\alpha_d) \quad \text{using $\beta = \alpha_d$ and Equation (3.18)}
\end{align*}

**Case 2** Suppose $\beta \neq \alpha_d$. Then $\beta, \iota(\beta) \in \mathcal{F}$, and $\varpi^\vee_d(\alpha_d) = \varpi^\vee(\beta) = 0$. It follows from Definition 3.8 that $\iota(\beta) = -w_\mathcal{F}(\beta)$, hence $w_\mathcal{F}(\iota(\beta)) = -\beta$. We have:

\begin{align*}
w_0w_\mathcal{F}w_dw_\mathcal{F}(\alpha_d) &= w_0w_\mathcal{F}(\delta + \iota(\beta)) \quad \text{using Equation (3.21)}
= w_0(\delta - \beta) = \delta - w_0(\beta) \quad \text{using Equation (3.18)}
= \delta + \alpha_d = \tau_q(\alpha_d) \quad \text{using Equation (3.20)}
\end{align*}

\begin{align*}
w_0w_\mathcal{F}w_dw_\mathcal{F}(\beta) &= w_0w_\mathcal{F}w_d(-\iota(\beta)) \quad \text{using Equation (3.22)}
= w_0w_\mathcal{F}w_d(\alpha_0) \quad \text{using Equation (3.20)}
= \tau_q(\beta) \quad \text{using Equation (3.18)}
\end{align*}

Finally, we prove that Equation (3.19) holds for any $\alpha \in D_0 \setminus \{\alpha_d, \beta\} = \mathcal{F} \setminus \{\beta\}$. Since $\alpha \in D_0$, we have $-w_0(\alpha) \in D_0$. Observe further that since $\alpha \neq \beta$, we have $-w_0(\alpha) \neq \alpha_d$, and so $-w_0(\alpha) \in \mathcal{F}$. Applying Definition 3.8 we get

\begin{equation}
\iota(-w_0(\alpha)) = w_\mathcal{F}w_0(\alpha)
\end{equation}

\begin{equation}
\Rightarrow w_\mathcal{F}w_0(\alpha) = -w_0(\alpha)
\end{equation}
We see from Equation (3.18) that \( \tau_\eta(\alpha) = \alpha \). We compute:

\[
\begin{align*}
    w_0 w_\eta w_d w_\eta (\alpha) &= -w_0 w_\eta^t w_0 (\alpha) & \text{using } \alpha \notin J, \text{ and Definition 3.8} \\
    &= -w_0 w_\eta t w_0 (\alpha) & \text{using Equation (3.13)} \\
    &= w_0^2 (\alpha) = \alpha = \tau_\eta (\alpha) & \text{using Equation (3.23)}
\end{align*}
\]

\( \square \)

**Lemma 3.24.** Let \( \Psi \overset{\text{def}}{=} \Delta_\eta^\perp \setminus \Delta_J \), and consider some \( \gamma \in D_0 \cup \pm J \). Then

1. All subsets of \( \Psi \) are nilpotent (see Section 2.14).
2. The set \( \Psi \cup \{ \gamma \} \) is nilpotent.

**Proof.** First observe that \( \Psi = \Delta_\eta^\perp \setminus \Delta_J = \{ \alpha \in \mathbb{Z}D_d \mid -\theta_0 \leq \alpha \leq -\alpha_0 \} \). Consider \( \alpha, \beta \in \Psi \). Then \( \text{Supp}(\alpha + \beta) \subset D_d \). Further, we have

\[
\alpha, \beta \leq -\alpha_0 \implies \alpha + \beta \leq -2\alpha_0.
\]

Now, since \( \alpha_0 \) is cominuscule in \( D_d \), it follows that \( \alpha + \beta \notin \Delta \). Consequently, every subset of \( \Psi \) is closed. Next, we prove that \( \Psi \cup \{ \gamma \} \) is closed.

1. Suppose \( \gamma = \alpha_d \). Consider \( \alpha \in \Psi \). The coefficient of \( \alpha_0 \) in \( \alpha + \gamma \) is \( -1 \), and the coefficient of \( \alpha_d \) is \( 1 \). Hence, \( \alpha + \gamma \) is not a root.
2. Suppose \( \gamma \in J \). Suppose further that \( \alpha + \gamma \in \Delta \) for some \( \alpha \in \Psi \). Since \( \text{Supp}(\alpha + \gamma) \subset D_d \), we see that \( \alpha + \gamma \in \Delta_d \). Further, since the coefficient of \( \alpha_0 \) in \( \alpha + \gamma \) is \( -1 \), we have \( \alpha + \gamma \notin \Delta_J \).
3. Suppose \( \gamma \in -J \). Suppose further that \( \alpha + \gamma \in \Delta \) for some \( \alpha \in \Psi \). Then \( \alpha + \gamma \leq -\alpha_0 \) and \( \alpha + \gamma \in \Delta_d \). It follows that \( \alpha + \gamma \in \Psi \).

Finally, consider \( u_+, u_- \in W \) given by

\[
\begin{align*}
    u_+ &= \begin{cases} 
        w_d & \text{if } \gamma = \alpha_d \\
        w_d s_\gamma & \text{if } \gamma \in J \\
        w_d & \text{if } \gamma \in -J 
    \end{cases} \\
    u_- &= \begin{cases} 
        s_{\alpha_d} & \text{if } \gamma = \alpha_d \\
        s_\gamma & \text{if } \gamma \in J \\
        1 & \text{if } \gamma \in -J
    \end{cases}
\end{align*}
\]

It is easy to verify that \( u_\pm (\Psi \cup \{ \gamma \}) \subset \Delta^\pm \). \( \square \)

Recall from Equation (2.15) the Lie sub-algebra \( g_\Psi \overset{\text{def}}{=} \sum_{\alpha \in \Psi} \mathfrak{g}^\alpha \). Recall also from Section 2.16 the isomorphism \( \eta : g_\Psi \rightarrow U_\Psi \) for some ordering on \( \Psi \).

**Proposition 3.25.** There exists a \( P \)-equivariant isomorphism \( \phi : u_P \rightarrow X_J^{-}(w_d^2) \) given by

\[
\phi(X) = \eta(t^{-1}X) \mod \mathcal{P}
\]

where \( X_J^{-}(w_d^2) \) is the opposite cell defined in Equation (2.21).

**Proof.** Observe that \( u_P = g_\Delta^\perp \setminus \Delta_J \). The map \( X \mapsto t^{-1}X \) is \( G \)-equivariant (hence also \( P \)-equivariant), and takes the root space \( \alpha \) to \( \alpha - \delta \). Now, since

\[
\begin{align*}
    \Delta_\eta^\perp \setminus \Delta_J - \delta &= \{ \alpha - \delta \mid \alpha \in \Delta_0 \setminus \Delta_J \} \\
    &= \{ \alpha - \delta \mid \alpha \in \mathbb{Z}D, \alpha_d \leq \alpha \leq \theta_0 \} \\
    &= \{ \alpha \in \mathbb{Z}D \mid -\theta_d \leq \alpha \leq -\alpha_0 \} = \Delta_d^\perp \setminus \Delta_J
\end{align*}
\]
we have a map $t^{-1} : u_P \to g^*$, where $\Psi = \Delta^- \backslash \Delta_-$. It follows from Equation (2.21) that $\eta \circ t^{-1}$ is an isomorphism from $u_P$ to $Y_{\mathcal{J}}(w_d^\mathcal{J})$. It remains to show that $\eta$ is $P$-equivariant. For $\gamma \in D_0 \cup -\mathcal{J}$, it follows from Lemma 3.24 that the group scheme $U_{\Psi \cup \{\gamma\}}$ is well-defined. The action of $U_\gamma$ on $U_{\Psi}$ (resp. $g^\Psi$) being the restriction of the adjoint action of $U_{\Psi \cup \{\gamma\}}$ on itself (resp. $g^{\Psi \cup \{\gamma\}}$), the map $\eta$ is $U_\gamma$-equivariant. It follows that $\eta$ is $P$-equivariant, since $P = (T, U_\gamma \mid \gamma \in D_0 \cup -\mathcal{J})$, and $\eta$ is $T$-equivariant by construction.  

**Theorem 3.27.** Recall that $u_P$ is the cotangent space of $G/P$ at identity. Let $q = w_0(\varpi_\mathcal{J}^\mathcal{J}) - \varpi_\mathcal{J}^\mathcal{J}$ as in Equation (3.14). The map $\phi : u_P \to X^-_{\mathcal{J}}(w_d^\mathcal{J})$ extends to a $G$-equivariant isomorphism $\phi : T^*G/P \to Y_{\mathcal{J}}(\tau_q)$ (see Equation (2.22)) given by  

$$
\phi(g, X) = g \phi(X) \mod P \quad \text{for } g \in G, \ X \in u_P
$$

Let $\theta : T^*G/P \to g$ be the map given by $(g, X) \mapsto Ad(g)X$, and $N = \Im(\theta)$. Let $\pr$ be the restriction of the quotient map $LG/P \to LG/L^+G$ to $X^-_{\mathcal{J}}(\tau_q)$. There exists an isomorphism $N \to X^-_{D_0}(\tau_q)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
T^*G/P & \xrightarrow{\phi} & Y_{\mathcal{J}}(\tau_q) \\
\downarrow{\theta} & & \downarrow{\pr} \\
N & \xrightarrow{\psi} & X^-_{D_0}(\tau_q)
\end{array}
$$

**Proof.** The proof of [LRS16, Theorem 1.3] applies to show that $\phi$ is well-defined and $G$-equivariant, and  

$$
\phi(T^*G/P) \subset X^-_{\mathcal{J}}(w_0^\mathcal{J} w_d^\mathcal{J}) = X^-_{\mathcal{J}}(\tau_q) \quad \text{using Equation (3.14)}
$$

Further, since $\phi$ is $G$-equivariant and $\phi(1, u_P) = X^-_{\mathcal{J}}(w_d^\mathcal{J})$, we have  

$$
\phi(T^*G/P) = GX^-_{\mathcal{J}}(w_d^\mathcal{J}) = GB^- X^-_{\mathcal{J}}(w_d^\mathcal{J}) = Y_{\mathcal{J}}(\tau_q).
$$

The isomorphism $\psi$ is from [AH13, AHR15] (in particular, see Proposition 6.6 of AHR15). We verify that for $X \in u_P$, we have  

$$
\pr(\phi(1, X)) = \phi(X) \mod L^+G = \psi(\theta(X)).
$$

Since $\phi, \psi$ are $G$-equivariant isomorphism (loc. cit.), the diagram commutes. \hfill \square

### 4. The Conormal Variety of a Schubert variety

Let $D_0$ be a finite type Dynkin diagram and $\mathcal{D}$ the associated extended diagram. Fix a cominuscule root $\alpha_d \in D_0$ and let $\mathcal{J}, \iota, G, LG, P, \mathcal{P}$, and $\phi : T^*G/P \hookrightarrow LG/P$ be as in the previous section. We fix $w \in W_0^- \overset{\text{def}}{=} W_0 \cap W^\mathcal{J}$ and set

(4.1)

$$
v \overset{\text{def}}{=} (w_0 w w_{\mathcal{J}}).
$$

Let $N^*X^-_{\mathcal{J}}(w)$ be the conormal variety of $X^-_{\mathcal{J}}(w)$ in $G/P$. It follows from Theorem 3.27 that the closure of $\phi(N^*X^-_{\mathcal{J}}(w))$ in $X^-_{\mathcal{J}}(\tau_q)$ is a $B$-stable compactification of $N^*X^-_{\mathcal{J}}(w)$. The primary goal of this section is Theorem 4.12. This compactification of $\phi(N^*X^-_{\mathcal{J}}(w))$ is a Schubert subvariety of $X^-_{\mathcal{J}}(\tau_q)$ if and only if $X^-_{\mathcal{J}}(w_0 w w_{\mathcal{J}})$ is smooth. This yields powerful results (Theorems 4.17 and 4.18) regarding the geometry of $N^*X^-_{\mathcal{J}}(w)$ in the case where $X^-_{\mathcal{J}}(w_0 w w_{\mathcal{J}})$ is smooth.
We first introduce the notion of the conormal variety and give a combinatorial description of \( N^*X_{\mathcal{J}}(w) \) in Proposition 4.3. We then develop a sequence of lemmas leading to the proof of Theorem 4.12. Finally, we show in Proposition 4.19 that the fibre at identity of \( N^*X_{\mathcal{J}}(w) \) can be identified as the union of opposite cells of certain Schubert varieties.

4.2. The Conormal Variety. Consider the Schubert variety \( X_{\mathcal{J}}(w) \), viewed as a subvariety of \( G/P \). For \( x \) a smooth point in \( X_{\mathcal{J}}(w) \), the conormal fibre \( N_+^*X_P(w) \) is the annihilator of \( T_xX_{\mathcal{J}}(w) \) in \( T_x^*G/P \), i.e., for \( x \) a smooth point,

\[
N_+^*X_P(w) = \{ f \in T_x^*G/P \mid f(v) = 0 \quad \forall v \in T_xX_{\mathcal{J}}(w) \}
\]

The conormal variety \( N^*X_{\mathcal{J}}(w) \) of \( X_{\mathcal{J}}(w) \) \( \hookrightarrow \) \( G/P \) is the closure in \( T^*G/P \) of the conormal bundle of the smooth locus of \( X_{\mathcal{J}}(w) \).

**Proposition 4.3.** Let \( R \overset{\text{def}}{=} \{ \alpha \in \Delta_0^+ \mid \alpha \geq \alpha_d, w(\alpha) > 0 \} \) and \( g^R = \sum_{\alpha \in R} g^\alpha \). The conormal variety \( N^*X_{\mathcal{J}}(w) \) is the closure in \( T^*G/P \) of

\[
\{(bw, X) \in G \times_P U_P \mid b \in B, X \in g^R \}
\]

**Proof.** The tangent space of \( G/P \) at identity is \( g/p \). Consider the action of \( P \) on \( g/p \) induced from the adjoint action of \( P \) on \( g \). The tangent bundle \( TG/P \) is the fibre bundle over \( G/P \) associated to the principal \( P \)-bundle \( G \rightarrow G/P \), for the aforementioned action of \( P \) on \( g/p \), i.e., \( TG/P = G \times_P g/p \). Let

\[
R' = \Delta^+ \setminus (\Delta^+_\mathcal{J} \cup R) = \{ \alpha \in \Delta^+ \mid \alpha \geq \alpha_d, w(\alpha) < 0 \}
\]

\[
U_w = (U_\alpha \mid -\alpha \in R')
\]

For any point \( b \in B \), we have (see, for example [Bor12]):

\[
BwP \pmod{P} = bBwP \pmod{P}
\]

\[
= b(wU_w w^{-1})wP \pmod{P}
\]

\[
= bwU_w P \pmod{P}
\]

It follows that the tangent subspace at \( bw \) of the big cell \( BwP \pmod{P} \) is given by

\[
T_{bw}BwP \pmod{P} = \left\{ (bw, X) \in G \times_P g/p \mid X \in \bigoplus_{-\alpha \in R'} g^\alpha/p \right\}
\]

where \( g^\alpha/p \) denotes the image of a root space \( g^\alpha \) under the map \( g \rightarrow g/p \).

Recall that the Killing form identifies the dual of a root space \( g^\alpha \) with the root space \( \bar{g}^{-\alpha} \). Consequently, a root space \( g^\alpha \subset u \) annihilates \( T_{bw}BwP \pmod{P} \) if and only if \( \alpha \in \Delta^+_0 \setminus \Delta^+_\mathcal{J} \) and \( \alpha \notin R' \), or equivalently, \( \alpha \in R \). The result now follows from the observation that \( BwP \pmod{P} \) is a dense open subset of \( X_P(w) \), and is contained in the smooth locus of \( X_P(w) \). \( \square \)

**Lemma 4.4.** Recall from Equation (4.11) that \( v = \iota(w_0wJ) \). We have:

1. \( W_0 \cap W^J = W_0^d \overset{\text{def}}{=} W_0 \cap W_d \) and \( W_d \cap W^J = W_d^0 \overset{\text{def}}{=} W_d \cap W^0 \).
2. \( v \in W_d^0 \).
3. \( l(wv) = l(w) + l(v) = \dim G/P \).
Proof. It follows from Section 2.9 that $W_0 \subset W^{\{\alpha_d\}}$, hence
\[ W_0 \cap W^J \subset W_0 \cap W^{\{\alpha_d\}} \cap W^J = W_0^d. \]
Conversely, since $W^d \subset W^J$, we have $W_0^d = W_0 \cap W^d \subset W_0 \cap W^J$. Consequently,
\[ W_0 \cap W^J = W_0^d. \]
Applying $\iota$ to this equality, we have $W_d \cap W^J = W_d^0$.

Next, we prove $v \in W^J$. It is sufficient to verify $v(\Delta^+_J) \subset \Delta^+$, see Equation (2.11).
\[
\begin{align*}
    v(\Delta^+_J) &= w_0 w w_J (\iota(\Delta^+_J)) \quad \text{using Equation (3.13)} \\
    \implies v(\Delta^+_J) &= w_0 w w_J (\Delta^+_J) \quad \text{using Section 3.12}
\end{align*}
\]
(4.5) \[ \implies v(\Delta^+_J) = w_0 w (\Delta^+_J) \quad \text{using Section 2.6} \]

Now, since $\text{Supp}(w) \subset D_0$, we see from Section 2.9 that $w(\Delta^-_J) \subset \Delta_0$. Further, since $w \in W^J$, it follows from Equation (2.11) that $w(\Delta^-_J) \subset \Delta_0$. Applying Equation (1.5), we have $v(\Delta^+_J) \subset w_0 (\Delta_0) = \iota(\Delta^+_J) \subset \Delta^+$. This proves $v \in W^J$.

Further, since $w_0, w, w_J \in W_0$, we have $w_0 w w_J \in W_0$. It follows from Section 3.12 that $v = \iota(w_0 w w_J) \in W_d$. Combining with $v \in W^J$, we get (2):
\[ v \in W^J \cap W_d = W_d^0 \]
Finally, since $v \in W^J$, we have $\iota v = w_0 w w_J \in W^J$. Consequently,
\[
\begin{align*}
    l(w_0 w) &= l(w_0 w w_J) + l(w_J) \\
    \implies \dim G - l(w) &= \dim l(v) + \dim P \\
    \implies \dim G/P &= l(w) + l(v) = l(uv)
\end{align*}
\]
where the last equality follows from the observations $w \in W_0$ and $v \in W^0$. \qed

Lemma 4.6. Let $u \in W_d^0$. Then $\text{Supp}(u)$ is a connected sub-graph of $D_d$.

Proof. Suppose $\text{Supp}(u)$ is not connected. Let $\mathcal{L}_1$ be the connected component of $\text{Supp}(u)$ containing $\alpha_d$, and let $\mathcal{L}_2 \triangleq \text{Supp}(u) \setminus \mathcal{L}_1$. Now, since $\mathcal{L}_1$ and $\mathcal{L}_2$ are disconnected, we have
\[ s_\alpha s_\beta = s_\beta s_\alpha \quad \forall s_\alpha \in \mathcal{L}_1, s_\beta \in \mathcal{L}_2. \]
Let $s_1 \ldots s_l$ be a reduced word for $u$, and let $k$ be the largest index such that $s_k \in \mathcal{L}_2$. Then $s_k s_m = s_m s_k$ for all $m > k$. It follows that
\[ u s_k = (s_1 \ldots s_l) s_k = (s_1 \ldots s_{k-1} s_{k+1} \ldots s_l) s_k = s_1 \ldots s_{k-1} s_{k+1} \ldots s_l \]
Now, since $\text{Supp}(u) \subset D_d$, we have $\alpha_0, \alpha_d \notin \mathcal{L}_2$. In particular, $s_k \in W_J$. Hence $u s_k = u (\text{mod } W_J)$ and $u s_k < u$, contradicting the assumption $u \in W_d^0 \subset W^J$. \qed

Proposition 4.7. For $u \in W_d^0$, the following are equivalent:

1. $X_J(u)$ is smooth.
2. $X_J(u)$ is $\mathcal{P}_\mathcal{L}$-homogeneous, i.e., $X_J(u) = \mathcal{P}_\mathcal{L}/(\mathcal{P}_\mathcal{L} \cap \mathcal{P})$ for some connected sub-graph $\mathcal{L} \subset D_d$.
3. $l(u^{-1} * w_J) = l(u w_J)$, where $*$ is the Demazure product, see Section 2.24.
4. $(u w_J)^{-1}(\alpha) < 0$ for all $\alpha \in \text{Supp}(u)$. 

\(\{\alpha \in \Delta^+ \mid u(\alpha) < 0\} = \Delta_{\text{Supp}(u)}^+ \setminus \Delta_J^+\).

(6) \(u = w_L w_{\mathcal{L} \cap \mathcal{J}}\), where \(\mathcal{L} = \text{Supp}(u)\), and \(w_L\), \(w_{\mathcal{L} \cap \mathcal{J}}\) denote the maximal elements in \(W_\mathcal{L}\), \(W_{\mathcal{L} \cap \mathcal{J}}\) respectively.

Proof. The claim \((1) \iff (2)\) is \([BM10\text{ ] Theorem 1.1}]. Suppose \((2)\) holds, i.e., \(X_J(u) = \mathcal{P}_L/\mathcal{P}_{\mathcal{L} \cap \mathcal{J}}\). We have the following Cartesian square:

\[
\begin{array}{ccc}
X_B(uw_J) & \longrightarrow & LG/B \\
\downarrow & & \downarrow \\
\mathcal{P}_L/\mathcal{P}_{\mathcal{L} \cap \mathcal{J}} & = & X_J(u) \longrightarrow LG/P
\end{array}
\]

Since \(X_J(u)\) is \(\mathcal{P}_L\)-stable, the same is true of its pull-back \(X_B(uw_J)\). Further, since \(u \in W_\mathcal{L}\), any lift of \(u^{-1}\) to \(N(K)\) (see Section 2.13) is in \(\mathcal{P}_L\). Consequently, \(X_B(uw_J)\) is \(u^{-1}\)-stable, and so \(u^{-1} \ast uw_J = uw_J\). Hence we obtain \((2) \implies (3)\).

It is clear from Equation (2.24) that \((3)\) is equivalent to \((4)\) holds, which holds if and only if \(s_\alpha \ast uw_J = uw_J\) for all \(\alpha \in \mathcal{L}\). This is equivalent to \((4)\) from Equation (2.5). Hence we obtain \((3) \iff (4)\).

Suppose \((4)\) holds. Then \((uw_J)^{-1}(\alpha) < 0\) for all \(\alpha \in \Delta_{\text{Supp}(u)}^+\). It follows from Section 2.9 and Equation (2.11) that

\[
\{\alpha \in \Delta^+ \mid u(\alpha) < 0\} \subset \Delta_{\text{Supp}(u)}^+ \setminus \Delta_J^+
\]

Consider \(\alpha \in \Delta_{\text{Supp}(u)}^+ \setminus \Delta_J^+\) satisfying \(u(\alpha) > 0\). Since \(u\) preserves \(\Delta_{\text{Supp}(u)}^+\), we have \(u(\alpha) \in \Delta_{\text{Supp}(u)}^+\). Applying \((4)\) to \(u(\alpha)\), we get \(w_J u^{-1}(u(\alpha)) = w_J(\alpha) < 0\). It follows that \(\alpha \in \Delta_J^+\), contradicting the assumption \(\alpha \not\in \Delta^+_J\). Hence we obtain the implication \((4) \implies (5)\).

Suppose \((5)\) holds. We verify that

\[
\{\alpha \in \Delta^+ \mid w_L w_{\mathcal{L} \cap \mathcal{J}}(\alpha) < 0\} = \Delta_L^+ \setminus \Delta_J^+
\]

Now since \(u \in W\) is uniquely determined by the set \(\{\alpha \in \Delta^+ \mid u(\alpha) < 0\}\) (cf. \([Kum12\text{ ] } \S 1.3.14]\), we get \(u = w_L w_{\mathcal{L} \cap \mathcal{J}}\). Hence we obtain \((5) \implies (6)\).

Finally, suppose \((6)\) holds. Since \(w_{\mathcal{L} \cap \mathcal{J}} \subset \mathcal{J}\), we have \(u = w_L \pmod{\mathcal{J}}\). It follows that \(X_J(u) = X_J(w_L) = \mathcal{P}_L/(\mathcal{P}_{\mathcal{L} \cap \mathcal{J}})\). Hence we obtain \((6) \implies (2)\).

Lemma 4.8. For \(\alpha \in \Delta_0^+ \setminus \Delta_J\), we have \(v(\alpha - \delta) = -uw_0 w(\alpha)\).

Proof. Since \(\alpha \in \Delta_0^+ \setminus \Delta_J\), we have \(\alpha \geq \alpha_d\). Set \(\gamma = \alpha - \alpha_d\). Since \(\alpha_d\) is cominuscule, \(\text{Supp}(\gamma) \subset \mathcal{J}\). In particular, \(\iota(\gamma) = -w_J(\gamma)\). We compute:

\[
\iota(\alpha - \delta) = \iota(\alpha_d) + \iota(\gamma) = \alpha_d - w_J(\gamma)
\]

Recall from Equation (4.1) that \(v = 'w_0 w_J\). We now compute \(v(\alpha - \delta)\):

\[
v(\alpha - \delta) = 'w_0 w_J'w_J(\alpha - \delta) = uw_0 w_J(\iota(\alpha - \delta)) \quad \text{using Section 3.12}
\]

\[
= -uw_0 w_J(\theta_0 + w_J(\gamma)) \quad \text{using Equation 1.59}
\]

\[
= -uw_0 w(\alpha_d) - uw_0 w(\gamma) \quad \text{using Lemma 5.5}
\]

\[
= -uw_0 w(\alpha_d + \gamma) = -uw_0 w(\alpha)
\]
Lemma 4.10. The map $\alpha \mapsto \alpha - \delta$ induces a bijection
\[ \{ \alpha \in \Delta_0^+ | \alpha \geq \alpha_d, w(\alpha) > 0 \} \xrightarrow{\sim} \{ \alpha \in \Delta_d^- | v(\alpha) > 0 \} \]

Proof. Observe that
\[ -w_0(\Delta_0^+) = \iota(-w_0\Delta_0^+) = \iota(\Delta_0^+) \quad \text{using Section 2.4} \]
\[ = \Delta_d^\pm \quad \text{using Section 3.12} \]

Now, since $v(\alpha - \delta) = -w_0v(\alpha)$ (see Lemma 4.8), it follows that for $\alpha \in \Delta_0^+ \setminus \Delta_d^-$, $w(\alpha) > 0$ is equivalent to $v(\alpha - \delta) > 0$. The result now follows from Equation (3.26), which states that $\alpha \mapsto \alpha - \delta$ induces a bijection from $\Delta_0^+ \setminus \Delta_d^-$ to $\Delta_d^- \setminus \Delta_d^-$. \qed

Proposition 4.11. Recall the map $\phi$ from Proposition 3.25. Let $R = \{ \alpha \in \Delta_0^+ | \alpha \geq \alpha_d, w(\alpha) > 0 \}$

Then $\phi(g^R)$ is dense in $v^{-1}X_J(v)$.

Proof. Following the proof of Proposition 3.25, we see that $\eta(t^{-1}g^R) = U_\Phi$, where $\Phi \overset{\text{def}}{=} \{ \alpha - \delta | \alpha \in R \}$. It follows from Lemma 4.10 that
\[ \Phi = \{ \alpha \in \Delta_d^- | v(\alpha) > 0 \} = v^{-1}(\Delta^+) \cap \Delta^- \]

In particular, $\# \Phi = l(v)$ (see [Kum12, §1.3.14]), and so
\[ \dim g^R = \dim U_\Phi = \# \Phi = l(v) = \dim X_J(v) = \dim v^{-1}X_J(v) \]

Further observe that
\[ \phi(g^R) = \eta(t^{-1}g^R) \pmod P \subset v^{-1}Bv \pmod P \subset v^{-1}X_J(v) \]

The result follows from the injectivity of $\phi$ and the irreducibility of $v^{-1}X_J(v)$. \qed

Theorem 4.12. The closure of $\phi(N^*X_J(w))$ in $LG/P$ is a Schubert variety if and only if $X_J(wv^{-1}w_J)$ is smooth.

Proof. For $(bw, X)$ be a generic point in $N^*X_J(w)$, we have
\[ \phi(bw, X) = bw\phi(X) \in Bwv^{-1}Bv \pmod P \]

Hence the minimal Schubert variety containing $\phi(N^*X_J(w))$ is $X_J(wv^{-1} \ast v)$. Consequently, the closure $\overline{\phi(N^*X_J(w))}$ is a Schubert variety if and only if
\[ \dim X_J(wv^{-1} \ast v) = \dim N^*X_J(w) = \dim G/P \]

Consider the following Cartesian diagram:
\[
\begin{array}{ccc}
X_B(wv^{-1} \ast v \ast w_J) & \longrightarrow & LG/B \\
\downarrow & & \downarrow \\
X_J(wv^{-1} \ast v) & \longrightarrow & LG/P
\end{array}
\]

The dimension of the generic fibre for the right projection is $\dim P/B$. Observe that $X_B(wv^{-1} \ast v \ast w_J)$ is the pullback of $X_J(wv^{-1} \ast v)$ to $LG/B$. It follows that Equation (4.13) is equivalent to
\[ \dim X_B(wv^{-1} \ast v \ast w_J) = \dim G/P + \dim P/B = \dim G/B \]
We see from Lemma 4.4 and Equation (2.25) that \( wv^{-1} = w^*v^{-1} \) and \( v^*w_{\mathcal{J}} = v_{\mathcal{J}}. \) Hence, we have:

\[
(4.15) \quad wv^{-1} \ast v \ast w_{\mathcal{J}} = w \ast v^{-1} \ast v_{\mathcal{J}}
\]

Observe that since \( v, w_{\mathcal{J}} \in W_d \), we have \( v^{-1} \ast v_{\mathcal{J}} \in W_d \). Recall also from Lemma 4.4 that \( w \in W_{\mathcal{J}} \cap W_0 \subset W_d \). It follows that

\[
(4.16) \quad l(wv^{-1} \ast v \ast w_{\mathcal{J}}) = l(w(v^{-1} \ast v_{\mathcal{J}})) \quad \text{using Equation (4.15)}
\]

Continuing Equation (4.16), we have

\[
\dim X_G(wv^{-1} \ast v \ast w_{\mathcal{J}}) \geq l(w) + l(v_{\mathcal{J}})
\]

Hence, Equation (4.14) holds if and only if \( v^{-1} \ast v_{\mathcal{J}} \in W_d \), which is equivalent to \( X_{\mathcal{J}}(v) \) being smooth, see Proposition 4.7. Finally, the Schubert varieties \( X_{\mathcal{J}}(v) \) and \( X_{\mathcal{J}}(w_0wv_{\mathcal{J}}) \) being isomorphic (since \( v = '(w_0wv_{\mathcal{J}}) \)), we deduce that the closure \( \phi(N^*X_{\mathcal{J}}(w)) \) is a Schubert variety if and only if \( X_{\mathcal{J}}(w_0wv_{\mathcal{J}}) \) is smooth. \( \square \)

**Theorem 4.17.** Let \( w \in W_0^{\mathcal{J}} \) be such that \( X_{\mathcal{J}}(w_0wv_{\mathcal{J}}) \) is smooth. Then \( N^*X_{\mathcal{J}}(w) \) is normal, Cohen-Macaulay, and has a resolution via Bott-Samelson varieties. Further, the family \( \{N^*X_{\mathcal{J}}(w) \mid X_{\mathcal{J}}(w_0wv_{\mathcal{J}}) \text{ is smooth}\} \) is compatibly Frobenius split.

**Proof.** These are standard results for Schubert varieties. One can find details in [Fal03, Lit03, MRS85]. \( \square \)

**Theorem 4.18.** Let \( w \in W_0^{\mathcal{J}} \) be such that \( X_{\mathcal{J}}(w_0wv_{\mathcal{J}}) \) is smooth. Let \( V(\lambda) \) be the simple module associated to a dominant weight \( \lambda \) corresponding to \( P \). Recall the monomial basis \( \mathbb{M}(\lambda) \) and the associated elements \( u_\pi \in V(\lambda) \) as developed in [Lit03]. The ideal sheaf of \( N^*X_{\mathcal{J}}(w) \) in \( T^*G/P \) is \( \phi^{-1}(\mathcal{L}) \), where \( \mathcal{L} = \langle u_\pi \mid \pi \leq \tau_q, \pi \not\leq wv \rangle \).

**Proof.** Recall (cf. [Lit03]) that \( \{u_\pi \mid \pi \leq \tau_q\} \) is a basis for \( H^0(X_{\mathcal{J}}(\tau_q), L(\lambda)) \), where \( L(\lambda) \) is the line bundle associated to \( \lambda \). Further, \( \mathcal{L} = \langle u_\pi \mid \pi \leq \tau_q, \pi \not\leq wv \rangle \) is the ideal sheaf of \( X_{\mathcal{J}}(wv) \) in \( X_{\mathcal{J}}(\tau_q) \). Since \( N^*X_{\mathcal{J}}(w) \) is closed in \( T^*G/P \), we have

\[
\phi(N^*X_{\mathcal{J}}(w)) = \phi(N^*X_{\mathcal{J}}(w)) \cap \phi(T^*G/P)
\]

\[
= X_{\mathcal{J}}(wv) \cap L^{-1}G \cap X_{\mathcal{J}}(\tau_q) = Y_{\mathcal{J}}(wv)
\]

It follows that the ideal sheaf of \( N^*X_{\mathcal{J}}(w) \) in \( T^*G/P \) is the pull-back (via \( \phi \)) of the restriction of \( \mathcal{L} \) to \( Y_{\mathcal{J}}(\tau_q) \), i.e., the ideal sheaf is \( \phi^{-1}(\mathcal{L}) \). \( \square \)

**Proposition 4.19.** Let \( w \in W_0^{\mathcal{J}} \) be such that \( X_{\mathcal{J}}(w_0wv_{\mathcal{J}}) \) is smooth, and let \( N_0^*X_{\mathcal{J}}(w) \) denote the fibre at identity of the conormal variety \( N^*X_{\mathcal{J}}(w) \). Then

\[
\phi(N_0^*X_{\mathcal{J}}(w)) = \bigcup_{u \in \mathcal{S}} X_{\mathcal{J}}(u)
\]
where $S = \{ u \in W_0^\sigma \mid u \leq (wv)^D \}$, and $(wv)^D$ is the minimal representative of $wv$ with respect to $D$.

Proof. Recall from Proposition 3.25 that $\phi(T^*_e G/P) = X^-_J(w_d^\sigma)$. It follows that
\[
\phi(N_d X_J(w)) = X^-_J(w_d^\sigma) \cap X_J(wv) = \bigcup X^-_J(u)
\]
where the union runs over $\{ u \in W^J \mid u \leq w_d^\sigma, u \leq wv \}$. Since $w_d^\sigma$ is maximal in $W^J_d$, the condition $\{ u \in W^J, u \leq w_d^J \}$ is equivalent to $u \in W_d \cap W^J = W_0^\sigma$, see Lemma 4.4. Hence,
\[
\{ u \in W^J \mid u \leq w_d^J, u \leq wv \} = \{ u \in W_0^\sigma \mid u \leq w_d^\sigma, u \leq wv \}
\]
Finally, consider $u \in W^0$. If $u \leq wv$, then $u \leq (wv)^D$. It follows that
\[
\{ u \in W^J \mid u \leq w_d^J, u \leq wv \} = \{ u \in W_0^\sigma \mid u \leq w_d^\sigma, u \leq wv \} = \{ u \in W_0^\sigma \mid u \leq (wv)^D \} = S
\]
\[\square\]

5. Determinantal Varieties

In this section, we use the results of Section 4 to prove the following: The conormal fibre at the zero matrix of the rank $r$ (usual, symmetric, skew-symmetric resp.) determinantal variety is the co-rank $r$ (usual, symmetric, skew-symmetric resp.) determinantal variety.

Consider a rank $r$ (usual, symmetric, skew-symmetric resp.) determinantal variety $\Sigma$. There exists a simply connected, almost simple group $G$ (of type $A$, $C$, $D$ resp.) and a cominuscule Grassmannian $G/P$ such that $\Sigma$ is naturally identified as the opposite cell of some Schubert variety $X_J(w) \subset G/P$, see [LS78]. For such $w$, we verify that the Schubert variety $X_J(w_0 w w_J)$ is smooth. This allows us to apply Proposition 4.19. Finally, we show that the union of the various Schubert varieties in Proposition 4.19 is equal to a single Schubert variety, which we further verify to be isomorphic to the co-rank $r$ (usual, symmetric, skew-symmetric resp.) determinantal variety.

We carry out the proof in detail only for the skew-symmetric determinantal varieties. The other two cases are completely analogous. For the usual determinantal varieties, this result has been proved by Strickland [Str82]. Further, the conormal fibres at the zero matrix of the usual determinantal varieties and symmetric determinantal varieties have been studied by Gaffney and Rangachev (cf. [GR14]) and Gaffney and Lira (cf. [GL]).

5.1. The Weyl Group of $D_n$. Let $D_0 = D_n$, $D = D_n$, and $W_0$ (resp. $W$) the Weyl group of $D_0$ (resp. $D$). For $1 \leq i, j \leq n - 1$, the braid relations in $W_0$ are
\[
s_i s_j = s_j s_i \quad |i - j| \geq 2
\]
\[
s_i s_j s_i = s_j s_i s_j \quad |i - j| = 1
\]
The remaining braid relations are $s_n s_i = s_i s_n$ for $i \neq n - 2$ and
\[
s_n s_{n-2} s_n = s_{n-2} s_n s_{n-2}
\]
Let $\mu$ be the involution on $\{1, \ldots, 2n\}$ given by $\mu(i) \overset{\text{def}}{=} 2n + 1 - i$. We embed the Weyl group $W_0$ into the symmetric group $S_{2n}$ as follows

\begin{equation}
W_0 = \{w \in S_{2n} \mid w\mu = \mu w, \, \text{sgn}(w) = 1\}
\end{equation}

via the homomorphism (cf. [LS78]) given by

\begin{align*}
s_i &\mapsto r_ir_{2n-i}, & i \neq n \\
s_n &\mapsto r_{n+1}r_{n+1}r_n
\end{align*}

where $r_i$ denotes the transposition $(i \ i+1)$ in $S_{2n}$. It is clear that $w \in W_0$ is uniquely determined by its value on $1, \ldots, n$. Accordingly, we represent $w$ by the string $[w(1), \ldots, w(n)]$.

5.3. The Involution $\iota$ for $D_n$. The simple root $\alpha_n$ is cominuscule in $D_0 = D_n$, and $J \overset{\text{def}}{=} D_0 \setminus \{\alpha_n\} = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is isomorphic to $A_{n-1}$. Let $\iota$ be the involution defined in Definition 3.8. Recall that the action of the Weyl involution $-w_J$ on $J \cong A_{n-1}$ is given by $-w_J(\alpha_i) = \alpha_{n-i}$ (cf. [Bou68, Ch.VI §4.7]), and so $\iota(\alpha_i) = \alpha_{n-i}$ for $1 \leq i \leq n-1$. Further, since $\iota$ interchanges $\alpha_0$ and $\alpha_n$, we have

\begin{equation}
\iota(\alpha_i) = \alpha_{n-i} \quad \forall \alpha_i \in D.
\end{equation}

5.5. Skew-Symmetric Determinantal Varieties. Let $M_n^{sk}$ be the variety of skew-symmetric $n \times n$ matrices. The rank $r$ skew-symmetric determinantal variety $\Sigma_r^{sk,n}$ is the subvariety of $M_n^{sk}$ given by:

\begin{equation}
\Sigma_r^{sk,n} = \{A \in M_n^{sk} \mid \text{rank}(A) \leq r\}
\end{equation}

Recall that the rank of a skew-symmetric matrix is necessarily even. Hence, we assume without loss of generality that $r$ is even.

Let $G$ be the simply connected, almost-simple group of type $D_n$, and let $P \subset G$ be the parabolic group corresponding to $J = \{\alpha_1, \ldots, \alpha_{n-1}\}$, see Table II. Following [LS78], we identify $M_n^{sk}$ with the opposite cell in $G/P$. Under this identification, the zero matrix corresponds to $e \in G/P$, and $\Sigma_r^{sk,n} = X^-_J(w_r)$, where

\begin{equation}
w_r \overset{\text{def}}{=} [r+1, \ldots, n, 2n-r+1, \ldots, 2n]
\end{equation}

in the sense of Equation (5.2). Observe that

\begin{equation}
w_r \in W_J \cap W_0 = W_0^J \cup \{\alpha_n\}
\end{equation}

The last equality is Lemma 4.4 (1).

Remark 5.8. In [LS78], the skew-symmetric variety is identified with a Schubert variety corresponding to the group $SO(2n)$, which is not simply connected. This is not a problem however, since Schubert varieties depend only on the underlying Dynkin diagram, and not on the group per se.

**Theorem 5.9.** The conormal fibre of $\Sigma_r^{sk,n}$ at $0$ is isomorphic to $\Sigma_r^{sk,n}$, where

\[ \Pi = \begin{cases} 
  n & \text{if } n \text{ is even}, \\
  n-1 & \text{if } n \text{ is odd}. 
\end{cases} \]
Proof. Let $D_0$ (resp. $D$, resp. $J$) be the Dynkin diagram $D_n$ (resp. $D_n \setminus \{\alpha_0\}$), and $W_0$ (resp. $W$, $W_J$) its Weyl group. Recall that $J \cong A_{n-1}$. For $L$ a sub-diagram of $D_0$, we write $w_L$ for the longest element in $W$ supported on $L$, and $w_L^J$ for its minimal representative with respect to $J$. The longest elements $w_0 \in W_0$ and $w_J \in W_J$ are given by

$$w_0 = w_{[2n, \ldots, n+2]} = [2n, \ldots, n+2, 1]$$

$$w_J = [n, \ldots, 1]$$

in the sense of Equation (5.10), see [LS78]. Let $x$ be as defined in Equation (5.6), and set $v_r = (w_0 w_r w_J)$. We have

$$w_0 w_r w_J = [1, \ldots, r, n+1, n+2, \ldots, 2n-r] = w_J$$

where $L = \{\alpha_1, \ldots, \alpha_n\}$. Hence $X_J(w_0 w_r w_J)$ is smooth, see Proposition 5.14. It now follows from Proposition 4.15 and Proposition 5.14 that

$$\phi(N^0 X_J(w_r)) = X_J(w_{[2n-r]}) \cong X_J(w_{[n-r]} \cong \Sigma_{n-r}. \hspace{1cm} \Box$$

It remains to prove Proposition 5.14. The proof is obtained as a consequence of the following two lemmas.

Lemma 5.11. Consider $x_i \in W_0$ defined inductively as

$$x_i = \begin{cases} s_n & \text{for } i = n-1 \\ s_1 s_i & \text{for } 1 \leq i < n-1 \end{cases}$$

Then $s_{i+2} x_{i+3} = x_i s_{i+1}$ for $1 \leq i \leq n-4$.

Proof. We see from the braid relations that $s_j x_i = x_i s_j$ for $j \leq i - 2$. In particular,

$$s_{i+2} x_{i+3} = x_i s_{i+1}$$

using Equation (5.12). Now

$$s_{i+2} x_{i+3} = s_{i+2} x_i s_{i+1} + s_{i+2} x_{i+1} s_{i+2} x_{i+3}$$

using Braid relations

$$= s_{i+2} x_i s_{i+1} + s_{i+2} x_{i+1} s_{i+2} x_{i+3}$$

using Braid relations

$$= s_{i+2} x_i s_{i+1} + s_{i+2} x_{i+1} s_{i+2} x_{i+3}$$

using Equation (5.12)

$$= x_i s_{i+1} \hspace{1cm} \Box$$

Lemma 5.13. Let $x_i$ be given by Equation (5.12). For $3 \leq j, k \leq n - 1$, we have

$$\tilde{t} x_{n-k} x_k = x_{k-2}^j x_{n-k+2}^j$$

and

$$\tilde{t} x_{n-k} x_k x_{k-2} x_{k-2} \cdots x_j = x_{k-2}^j x_{k-4}^j \cdots x_j^j x_{n-k}^j x_{n-k+2}^j \tilde{t} x_{n-k} x_k$$

Proof. The second equality follows from repeated applications of the first. Observe first that $x_k \in \langle s_j \ | j \geq k \rangle$, or equivalently, $\tilde{t} x_{n-k} \in \langle s_j \ | j \leq k \rangle$. Consequently, the braid relations yields $\tilde{t} x_i x_j = x_j^i x_i$ whenever $i + j \geq n + 2$. Now

$$\tilde{t} x_{n-k} x_k = \tilde{t} x_{n-k+1} x_{n-k+2} x_{n-k+2} x_{n-k+2} x_k$$

using Equation (5.12)

$$= \tilde{t} x_{n-k-1} s_{n-k-2} s_{n-k-1} s_{n-k+2} s_{n-k+1} s_{n-k+1} s_{n-k+2} x_k$$

using Equation (5.4)

$$= \tilde{t} x_{n-k-1} s_{n-k-2} s_{n-k-1} s_{n-k+2} x_{n-k+2} x_k$$

using Equation (5.12)
This proves the claim when $n$ is even. Suppose $n$ is odd, so that $\pi = n - 1$ and $k \leq n - 2$. Then

\[ \iota x_{n-k+1} x_k = \iota s_{n-k} s_{n-k-1} x_{n-k} x_k \quad \text{using Equation (5.12)} \]

\[ = s_k x_{k-2}^\iota x_{n-k+2} \quad \text{using Equation (5.4)} \]

\[ = x_k - 2 s_{k-2} s_{k-1} x_{n-k+2} \quad \text{using Lemma 5.11} \]

\[ = x_k - 2 x_{n-k+1} \quad \text{using Equation (5.12)} \]

This proves the claim when $n$ is odd.

**Proposition 5.14.** For $w_r$ given by Equation (5.6), and $v_r = \iota (w_0 w_r w_0^\iota)$, we have

\[ (w_r v_r) D_0 = w_r v_r x_{\pi - r}^{-1} = \iota w_{\pi - r} \in W_0^{\mathcal{J} \cup \{0\}} \]

Consequently, $\iota w_{\pi - r}$ is the unique maximal element in \( \{ u \in W_0^{\mathcal{J} \cup \{0\}} \mid u \leq (w_r v_r) D_0 \} \).

**Proof.** Let $x_i$ be as in Equation (5.12). We have the following formulae, which are easily verified inductively:

\[ x_i = [1, \ldots, i - 1, i + 2, \ldots, n - 2, 2n - i, 2n - i + 1] \]

\[ w_r = w_{x_{i-1} x_{i-2} \ldots x_{1}} \]

\[ w_0 w_r w_{\mathcal{J}} = x_\pi - 1 x_\pi - 2 \ldots x_{1} \]

Now

\[ \iota w_r v_r = \iota x_{i-1} x_{i-2} \ldots x_{1} x_{\pi - 1} x_{\pi - 2} \ldots x_{\pi - r+1} \]

\[ = x_{\pi - r-1} x_{\pi - r-2} \ldots x_{1} x_{\pi - 1} x_{\pi - 2} \ldots x_{\pi - r+1} \quad \text{using Lemma 5.13} \]

\[ = w_{\pi - r} \iota w_{\pi - r} \]

It follows that $\iota w_r v_r x_{\pi - r}^{-1} = w_{\pi - r}$, hence $w_r v_r x_{\pi - r}^{-1} = \iota w_{\pi - r}$.

Next, Equation (5.7) yields

\[ w_{\pi - r} \in W_0^{\mathcal{J} \cup \{0\}} \implies \iota w_{\pi - r} \in W_0^{\mathcal{J} \cup \{0\}} \subset W_{D_0} \]

Further, since $\iota w_{\pi - r} \in W_0$ (see Equation (5.10)), we have $w_r v_r = \iota w_{\pi - r} \mod W_0$. Together, we deduce $(w_r v_r) D_0 = \iota w_{\pi - r}$. \( \Box \)

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