Holographic Dark Energy Model: State Finder Parameters

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Abstract

In this work, we have studied interacting holographic dark energy model in the background of FRW model of the universe. The interaction is chosen either in linear combination or in product form of the matter densities for dark matter and dark energy. The IR cut off for holographic dark energy is chosen as Ricci’s length scale or radius of the future event horizon. The analysis is done using the state finder parameter and coincidence problem has been graphically presented. Finally, universal thermodynamics has been studied using state finder parameters.

Keywords : Holographic Dark energy, State finder parameter, Ricci’s length scale.

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1 Introduction

The standard cosmology is put into a great challenge by exciting observational evidences [1, 2] in the last decade. To incorporate the present accelerating phase of the universe within the frame work of Einstein gravity, it is inevitable to introduce a non-gravitating type of matter with hugely negative pressure (of the order of its energy density) called dark energy (DE). For the mysterious DE, there are only very few constraints on its form of an equation of state [2, 5]. The general and leading choice of the unknown DE is the cosmological constant ($\lambda$CDM model) which represents a vacuum energy density having constant equation of state $\omega = -1$. However, its observed value is far below than the estimation from quantum field theory (known as Cosmological constant problem). Also there is no explanation why the constant vacuum energy and matter energy densities are precisely of the same order at present epoch (known as Coincidence problem). Due to both theoretical [4, 5] and observational [6] problems related to cosmological constant other DE models (varies with time) are used in the literature. Scalar field models [7, 8, 9] (commonly known as quintessence) have attracted special attention compare to other alternatives [5].

At present DE and cold dark matter (hereafter called as DM) are dominant sources (about 70 percent of DE and 25 percent of DM) of the content of the universe. So it is natural to have a lot of interest in studying coupling in the dark sector components [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Also it is possible to have information about these components through gravitational interaction. Further, recently, it has been shown that an appropriate choice of the interaction between DE and DM can alleviate the coincidence problem [20, 21, 22, 23, 24].

For the unknown and mysterious nature of DE, it is possible to have some insight demands that DE should be compatible with holographic principle which states that "the number of relevant degrees of freedom of a system dominated by gravity must vary along with the area of the surface bounding the system" [13]. Such a DE model known as Holographic DE (HDE) model. Further the energy density of any given region should be bounded by that ascribed to a Schwarzschild black hole (BH) that fills the same volume [14]. If we denote the
DE density by $\rho_D$, $L$- size of the region (infrared cut off) and $M_p = (8\pi G)^{-\frac{1}{2}}$ the reduced planck mass, then mathematically $\rho_D \leq M_p^2 L^{-2}$. as a standard practice we write DE density $\rho_D$ as

$$\rho_D = 3M_p^2 c^2 L^2$$

where the dimensionless parameter $c^2$ takes care of the uncertainties of the theory and the factor of three has been introduced for mathematical convenience. In the present work we choose $L = R_{RC}$, the Ricci’s length and $L = R_E$, the radius of the future event horizon as the IR-cut off in two different sections.

The argument behind the choice of the Ricci’s length as [25, 26, 27, 28] as IR cut off is that it corresponds to the size of the maximal perturbatio, leading to the formation of a black hole [29]. On the other hand, radius of the future event horizon is commonly used as the IR cut off of HDE models which gives the correct equation of state and the desired accelerating universe. However, recently it has been shown [30] that future event horizon suffers from a severe circularity problem.

For the interacting DM and DE to resolve the coincidence problem (as mentioned above), the interaction term is chosen in the present work as (i) a linear combination of the energy densities of the two matter components i.e. of the form $3b^2 H(\rho_m + \rho_D)$, (ii) a natural and physically viable interaction term of the form $\gamma \rho_m \rho_D$ with $\gamma$ a dimensionally $\left(\frac{L}{m_t}\right)$ constant. I both the choices the interaction term should be positive definite (i.e. $r > 0$) so that there is transfer of energy from DE component to DM section. This choice is favorable to solve the coincidence problem. Also validity of the second law of thermodynamics and LeChâtelier’s principle [11, 31] demand positive interaction term. It should be noted that the second choice of interaction term gives the best fit to observations [11, 31] for HDE models. Lastly, we have not included baryonic matter in the interaction due to the constraints imposed by local gravity measurements [7, 31, 32].

In 2003 Sahni et al [33, 34] proposed state-finder parameters $\{r, s\}$ which are defined as

$$r = 1 \frac{1}{aH^3} \frac{d^3a}{dt^3} \quad \text{and} \quad s = \frac{r - 1}{3(q - \frac{4}{3})} \quad (1)$$

where $a$ is the scale factor for the FRW model, $H$ and $q(= -\frac{\ddot{a}}{a})$ are the Hubble parameter and the deceleration parameter respectively. In fact the parameter ‘$r$’ forms the next step in the hierarchy of geometrical cosmological parameters after $H$ and $q$. These dimensionless parameters characterize the properties of dark energy in a model-independent manner. According to Sahni et al trajectories in the $\{r, s\}$ plane plane corresponding to different cosmological models demonstrate qualitative different behavior. According to them, the state finder diagnostic together with SNAP observations may discriminate between different DE models.

In the present work, we analyze interacting holographic dark energy model in terms of state-finder parameters and possible resolution of the coincidence problem has been presented graphically. Finally, we have analyzed universal thermodynamics in terms of $\{r, s\}$—parameters. The validity of generalized second law of thermodynamics result in some summarize the whole work.

2 Interacting Holographic DE model at Ricci scale

In this section, we consider homogeneous, isotropic and spatially flat FRW model of the universe bounded by Ricci’s length $R_L = (\dot{H} + 2H^2)^{\frac{1}{2}}$. The Einstein field equations are

$$3H^2 = \rho_m + \rho_D \quad (2)$$

and

$$2\dot{H} = -(\rho_m + \rho_D + p_D) \quad (3)$$

where $\rho_m$ is the energy density of the dark energy in the form of dust while $(p_D, \rho_D)$ are the energy density and thermodynamic pressure of the dark energy in the form of a perfect fluid having equation of state $p_D = \omega_D \rho_D$ ($\omega_D$, a variable). Due to holographic nature of the DE, the energy density $\rho_D$ has the expression
where \( \Omega_D = \frac{\rho_D}{3H^2} \) is the density parameter for the HDE.

To determine the evolution equation for the density parameter we start with the energy conservation relations

\[
\dot{\rho}_m + 3H\rho_m = Q
\]

and

\[
\dot{\rho}_D + 3H\rho_D(1 + \omega_D) = -Q
\]

First of all in absence of interaction (i.e. \( Q = 0 \)) \( \Omega_D \) evolves according to

\[
\dot{\Omega}_D = H(1 - \Omega_D)(1 - \frac{2\Omega_D}{c^2})
\]

using \( x = \ln a \) we have \( \frac{dx}{dt} = \frac{1}{H} \frac{dt}{dx} \) and the above equation becomes

\[
\frac{d\Omega_D}{dx} = -(1 - \Omega_D)(1 - \frac{2\Omega_D}{c^2})
\]

which on integration gives

\[
\Omega_D = \frac{e^{\frac{2x}{\gamma} + 2c_1} - e^{2x + c^2c_1}}{e^{\frac{2x}{\gamma} + 2c_1} - 2e^{x + c^2c_1}}
\]

where \( c_1 \) is the integration constant. Now we shall introduce interaction in the following ways:

**Case I:**

\[
Q = 3b^2H(\rho_m + \rho_D)
\]

Using both the conservations, the evolution of \( \Omega_D \) is given by

\[
\dot{\Omega}_D = -3H \left[ -\frac{2\Omega_D(1 - \Omega_D)}{3c^2} + \frac{(1 - \Omega_D)}{3} + b^2 \right]
\]

or equivalently,

\[
\frac{d\Omega_D}{dx} = - \left[ (1 - \Omega_D)(1 - \frac{2\Omega_D}{c^2}) + 3b^2 \right]
\]

The solution gives

\[
\Omega_D = \frac{1}{4} \left( 2 + c^2 + K\tan \left( \frac{1}{2} \left( \frac{Kx}{c^2} + Kc_2 \right) \right) \right)
\]

where \( K = \sqrt{-4 + 4c^2 + 24b^2c_1 - c^4} \) and \( c_2 \) is the integration constant.

**Case II:**

\[
Q = \gamma \rho_m \rho_D
\]

This choice of the interaction term with the conservation equations results the evolution equation for \( \Omega_D \) as

\[
\frac{d\Omega_D}{dx} = -(1 - \Omega_D) \left[ (1 - \frac{2\Omega_D}{c^2}) + 3\gamma H\Omega_D \right]
\]
The state finder parameter as introduced by (1) can have the expressions for the present model as

\[ q = -\frac{\ddot{a}}{aH^2} = \frac{1}{2} + \frac{3}{2} \frac{p}{\rho_T} \]  \hspace{1cm} (14)

\[ r = 1 + \frac{9}{2} \left(1 + \frac{p}{\rho_T}\right) \frac{\partial p}{\partial \rho_T}, \quad s = \left(1 + \frac{\rho_T}{p}\right) \frac{\partial p}{\partial \rho_T} \]  \hspace{1cm} (15)

where \( \rho_T = \rho_m + \rho_D \) is the total energy density. Thus eliminating \( \frac{\partial p}{\partial \rho_T} \) we have the relation between \( r \) and \( s \)

\[ \frac{2(r-s)}{9s} = \frac{p}{\rho_T} = \Omega_D \omega_D \]  \hspace{1cm} (16)

i.e.

\[ 2(r - 1) = 9s \left[ \frac{1}{3} - \frac{2\Omega_D}{3c^2} \right] = 3s \left(1 - \frac{2\Omega_D}{3c^2}\right) \]

So for fixed \( \Omega_D \) we have straight line relation between the two parameters \( (r, s) \) i.e. we have a family of straight lines in the \( (r, s) \)-plane with density parameter \( \Omega_D \) as the parameter. In fact trajectories in the \( (r, s) \)-plane correspond to different cosmological models, for example the fixed point \((1, 0)\) on the horizontal axis represents \( \Lambda CDM \) model. Further as \( u = \frac{\rho_m}{\rho_D} = \frac{\Omega_D}{\Omega_D} - 1 \) so from relation (16) the coincidence problem corresponds to the straight line path \( 2(r - 1) = 3s(1 - \frac{1}{3}) \) in the \( (r, s) \) plane.

3 Interacting HDE model at future event horizon:

The commonly used IR cut off for the holographic model is chosen as the radius of the future event horizon \( (R_E) \) given by

\[ R_E = a \int_t^\infty \frac{dt}{a} \]  \hspace{1cm} (17)

Normally, \( R_E \) is chosen as IR cut off to have correct accelerating universe.

Accordingly, the energy density for the HDE can be written as

\[ \rho_D = \frac{3c^2}{R_E^2} \]  \hspace{1cm} (18)

For this choice of \( \rho_D \) we shall now find the expressions for the equation of state parameter \( \omega_D \) and the evolution of the density parameter \( \Omega_D \).

For \( Q = 3b^2H(\rho_m + \rho_D) = 3b^2H\rho_T \), we have
\[
\omega_D = -\frac{1}{3} - \frac{2\sqrt{\Omega_D}}{3c} - \frac{b^2}{\Omega_D}
\]

and
\[
\frac{d\Omega_D}{dx} = \Omega_D(1 - \Omega_D)\left[1 + \frac{\sqrt{2}\Omega_D}{c} - \frac{3b^2}{1 - \Omega_D}\right]
\]

So from (16)
\[
2(r - 1) = -3S\left[\Omega_D + \frac{2\Omega_D^2}{c} + 3b^2\right]
\]

So in this case the coincidence problem corresponds to the straight line \(2(r - 1) = -3S\left(\frac{1}{2} + \frac{1}{\sqrt{2}c} + 3b^2\right)\) in \((r, s)\)-plane. Similarly, for \(Q = \gamma \rho_m \rho_{D}\) we obtain
\[
\omega_D = -\frac{1}{3} - \frac{2\sqrt{\Omega_D}}{3c} - \gamma H(1 - \Omega_D)
\]

and
\[
\frac{d\Omega_D}{dx} = \Omega_D(1 - \Omega_D)\left[1 + \frac{\sqrt{2}\Omega_D}{c} - 3\gamma H\Omega_D\right]
\]

Thus the \((r, s)\)- relation has the form
\[
2(r - 1) = -3S\left[\Omega_D + \frac{2\Omega_D^2}{c} + 3\gamma H(1 - \Omega_D)\right]
\]

4 Thermodynamics of the HDE model

In this section, we study the thermodynamics of FRW universe filled with interacting HDE. The universe is assumed to be bounded by the horizon which we choose separately to be (i) apparent horizon or (ii) event horizon or (iii) Ricci scale. In reference [35] it has been shown that both the apparent and event horizon do not change significantly over one hubble time scale so it is reasonable to consider equilibrium thermodynamics with temperature and entropy on the horizon similar to black holes. If \(S_I\) and \(S_h\) denote the entropy of the matter distribution inside the horizon and that of the horizon respectively, then the time variation of the total entropy is given by (for details see ref [35])
\[
\frac{d}{dt}S_I + S_h = \frac{4\pi R_h^2 \rho_D}{T_h} \left\{u + (1 + \omega_D)\right\}\dot{R}_h = \frac{12\pi R_h^2 H^2}{T_h} \left[1 + \frac{2(r - 1)}{9s}\right]
\]
where \( R_h \) is the radius of the horizon and \( T_h \) is the temperature of the horizon as well as of the inside matter for equilibrium thermodynamics. To examine the validity of the generalized second law of thermodynamics we first study the evolution of the horizons.

For FRW model, the dynamical apparent horizon which is essentially the marginally trapped surface with vanishing expansion, is defined as a sphere of radius \( R = R_A \) such that

\[
h^{ab}\partial_a R \partial_b R = 0
\]

which on simplification gives

\[
R_A = \frac{1}{\sqrt{H^2 + \frac{k}{\sigma^2}}}
\]  

(26)

The event horizon on the other hand is defined as Davis (1998) and [35]

\[
\begin{align*}
-asinh\tau, & k = -1 \\
R_E = -a\tau, & k = 0 \\
-asin\tau, & k = +1
\end{align*}
\]  

(27)

where \( \tau \) is the usual conformal time defined as,

\[
\tau = -\int_{t}^{\infty} \frac{dt}{a(t)}
\]

with \(|\tau| < \infty\) for existence of event horizon. Also the horizon radius corresponding to Ricci’s length is given by

\[
R_L = (\dot{H} + 2H^2)^{1/2} = (H \sqrt{1 - q})^{-1}
\]  

(28)

where \( q = -1 \frac{\dot{H}}{H^2} \) is the deceleration parameter. As at present we are in an accelerating phase of the universe as the event horizon exists and the horizons are related by the relations

\[
R_L < R_A = R_H < R_E, \text{ for } k - 0
\]  

(29)

where \( R_H = \frac{1}{H} \) is the hubble horizon. The time variation of the horizon radii are given by

\[
\begin{align*}
\dot{R}_A &= \frac{H}{2} R_A^3 \rho_D [u + (1 + \omega_D)] \\
\dot{R}_E &= R_E \left[ \frac{1}{R_A} - \frac{1}{R_E} \right] \\
\dot{R}_L &= \frac{R_L^3 \dot{H}}{2} [1 + q] + (1 - r)]
\end{align*}
\]  

(30)

Using the expression for \( \dot{R}_A \) from (30) into (25) we obtain the standard result in the literature that generalized second law of thermodynamics (GSLT) is valid both in quintessence and in phantom era for the universe bounded by the apparent horizon.

For the event horizon, as \( R_E > R_A \) particularly in quintessence era so that GSLT is satisfied unconditionally. On the other hand, in phantom era we may have \( \dot{R}_E > 0 \) or \( s < 0 \) (for details see ref [35], so the validity of GSLT is not unconditional. In fact, as long as \( \omega_D > -(1 + u) \) i.e. \( s > 0 \) and \( \dot{R}_E > 0 \) i.e. \( R_E > R_A \) then GSLT will be satisfied but for \( \omega_D < -(1 + u) \) i.e. \( s < \frac{2(1-r)}{9} \) we must have contraction of the event horizon for the fulfillment of GSLT.

From Eq. (30) we see that at the Ricci scale \( \dot{R}_L \) increases until \( r < (2 + q) \) and GSLT will be satisfied provided \( -\frac{2}{9} < r - 1 < 1 + q \). On the other hand, if the above inequality is reserved i.e. \( 1 + q < r - 1 < -\frac{2}{9} \),
though $\dot{R}_L$ is negative and the expression within the curly bracket of Eq. $^{25}$ is negative but still GSLT will be satisfied. Further as the state finder parameter $'r'$ is always greater than unity and $'s'$ may have positive or negative values so the above inequalities are modified as

$$r < 2 + q, \ s > 0 \ \text{or} \ 2 + q < r < 1 - \frac{9s}{2}, \ s < 0$$

for the validity of GSLT. These results are presented compactly in Table 1 and the figure (III) shows the valid region in $r,s$-plane for the validity of GSLT.

### Table 1: Validity of GSLT in terms of $(r,s)$ parameters

| Universe bounded by | Condition for validity of GSLT |
|---------------------|---------------------------------|
| Apparent Horizon    | Hold both in quintessence era and in phantom era without any restriction. |
| Event Horizon       | Hold unconditionally in quintessence era. In phantom era : $R_E > R_A$ and $s < 0$, $r > 1 - \frac{9s}{2}$ or $R_E < R_A$ and $s < 0$, $r < 1 - \frac{9s}{2}$ |
| Ricci-scale length  | Either $r < 2 + q$, $s > 0$ or $2 + q < r < 1 - \frac{9s}{2}$, $s < 0$ |

### 5 Summary

The paper deals with interacting holographic dark energy model in the background of flat FRW model of the Universe. Two types of interaction are chosen for investigation of which one is the standard choice of the linear combination of the densities of the two matter system. The other one which is physically reasonable and supported by observational evidences is in the product form of the energy densities. Here dark matter is chosen in the form of dust while the HDE is chosen in the form of perfect fluid with variable equation of state. The analysis is done for two choices of the IR cut off of the HDE model namely Ricci’s length scale and the usual radius of the future event horizon. The state finder parameters are introduced and the coincidence problem has been presented graphically in the $(r,s)$-plane. Finally, universal thermodynamics has been studied for this model with universe bounded by the apparent horizon, or event horizon or by Ricci’s length scale. Validity of GSLT in all the three cases has been showed in tabular form and the valid region in $(r,s)$ plane for the validity of GSLT has been showed in figures III.
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