NON-ZERO INTEGRAL FRIEZES

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Abstract. We study non-zero integral friezes for Dynkin types $A_n$, $B_n$, $C_n$, $D_n$ and $G_2$. These differ from standard Coxeter-Conway (positive) friezes by allowing any non-zero integer to appear. In each case we show that there are either 1, 2 or 4 times as many non-zero friezes as positive friezes. This is a first step for considering friezes over general rings of integers.

1. Introduction

In 1973 Coxeter and Conway [CC] studied what they called friezes. These were grids of strictly positive (non-zero) integers on the plane satisfying a determinant condition. For instance the following is a frieze height 3:

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1 1 1 1 1 1 1
4 1 2 2 2 1 4
3 1 3 3 1 3 3
2 2 1 4 1 2 2
1 1 1 1 1 1 1
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Here the height measures the number of rows between the two rows of 1s. In the above figure, every two by two diamond $\begin{array}{cc} a & b \\ c & d \end{array}$ satisfies $ad + 1 = bc$ or $ad - bc = 1$.

Coxeter-Conway showed the following was true:

**Theorem 1.1.** [CC] There are $C(k + 1)$ friezes of height $k$ where $C(k + 1)$ is the $k + 1$-st Catalan number.

They discovered this by connecting friezes to triangulations of polygons. They observed that a frieze of height $k$ corresponds the labeling of each possible arc that can occur in a triangulation of a $k + 3$-gon with a positive integer. They required that the external arcs be labeled 1 and that for any 4 arcs forming a quadrilateral, the labels of those arcs and the two internal diagonal arcs satisfy the the Ptolemy relation: that the sum of the product of opposite sides is the product of the diagonals.

If one considers the zigzag triangulation, the labels recover a zigzag column of the frieze. Moreover, successive columns are recovered by rotating the zigzag. Finally, they show that for any such assignment of integers to polygon arcs, the set of arcs labeled 1 form a triangulation. And conversely
if one starts with a triangulation where all arcs are labeled 1, the remaining arcs are uniquely determined by the Ptolemy relations and are all positive integers.

More recently, it was noted by Caldero [ARS10], that there is a deep connection to cluster algebras. Indeed, due to the above connection to triangulations, a frieze can be interpreted as an evaluation of the cluster variables of the Dynkin type $A_k$ cluster algebra such that each cluster variable is sent to a positive integer. An alternative, but equivalent formulation is that a frieze is a ring homomorphism from the cluster algebra to $\mathbb{Z}$ such that each cluster variable is mapped to a positive integer.

Constructing a frieze is simple: pick a cluster in the cluster algebra and assign to each variable in the cluster the value 1. Then due to the positivity of the exchange relations, each other cluster variable must evaluate to a positive number. Moreover, due to the Laurent phenomenon (every cluster variable can be expressed as a Laurent polynomial with integral coefficients in terms of the cluster variables in a single cluster), every other cluster variable evaluates to an integer.

In this paper we will relax the definition by allowing negative integers to appear. Given a cluster algebra, a non-zero integral frieze is an assignment of a non-zero integer to each cluster variable satisfying the mutation rules. We will use the term positive frieze to denote a frieze which has all cluster variables evaluated to positive integers. Alternatively it is a ring homomorphism from the cluster algebra to $\mathbb{Z}$ that sends each cluster variable to a non-zero integer.

The following 3 theorems are extensions of the results in [FP]:

**Theorem 1.2.** For $A_k$, the number of non-zero integral friezes is twice the number of positive friezes when $k$ is odd and the same number when $k$ is even.

**Theorem 1.3.** For $B_k$ and $C_k$, there are twice the number of non-zero integral friezes as positive friezes.

**Theorem 1.4.** For $D_k$ there are 4 times as many non-zero friezes when $k$ is even and 2 times as many when $k$ is odd.

**Theorem 1.5.** For $G_2$ there are 9 non-zero integral friezes and they are all positive.

I would like to thank Pierre-Guy Plamondon, Allen Knutson and Richard Stanley for useful discussions and the Sage mathematics community for their software and its cluster algebra implementation.

2. **Admissible labelings of triangulations**

Consider a triangulation of an $n$-gon whose arcs (including boundary arcs) are labeled by $\pm 1$. A labeling is said to be admissible if for any 4-cycle, the product of the labels around the cycle is 1.
The boundary state of a labeled triangulation is simply the set of boundary labels considered with their cyclic order. It will be of interest to understand the structure of admissible labelings. Specifically, we want to know what the possible boundary states for an admissible labeling are and given such a boundary state, which admissible labelings have that boundary state.

**Lemma 2.1.** Given an admissible labeling of a $n$-gon where $n$ is even, the labeling formed by negating all arcs of even length (i.e. the end points of the arc have an odd number of vertices between them) gives another admissible labeling.

**Proof.** Given a 4 cycle in the triangulation, since the set of lengths of these arcs adds to $n$, which is even, it follows that there are an even number of arcs of even length in the 4 cycle. But then the negation of the signs on the arcs of even length cancels when we take the product of all the labels on the 4 cycle. Thus the product of the labels is still one for each 4 cycle. \(\square\)

Note that this operation acts as an involution on the set of admissible labelings.

**Theorem 2.2.** Given a triangulation of an $n$-gon and a boundary state consisting of $\pm 1$, it extends to an admissible labeling of the triangulation if and only if $n$ is odd or $n$ is even and the product of the boundary state is 1. Moreover, when $n$ is odd, there is a unique extension to an admissible labeling of the triangulation and when $n$ is even and the product is 1, there are exactly two extensions related by the involution of lemma 2.1

**Proof.** In the case of $n = 3$, this is true, since the triangulation is trivial, there are only the boundary labels and every state leads to an admissible labeling.
When \( n = 4 \), since the boundary is a 4-cycle, a labeling is admissible if and only if the product of the boundary labels is 1. The single internal arc of the triangulation can be labeled \( \pm 1 \) giving rise to exactly two admissible labelings for each boundary state. Since the internal arc is of length 2, these two are related by lemma 2.1.

For general \( n > 4 \) we proceed via induction. Note that any triangulation must have at least one arc of length 2. Consider cutting the triangle that the arc forms off the triangulation. The result is a triangulation of an \( n-1 \)-gon. Now we break into two cases, based on the parity of \( n \).

Let \( n \) be odd and pick an arbitrary boundary state \( a_i \). If there exists an admissible labeling for this boundary state, its restriction to the \( n-1 \)-gon will be an admissible labeling. Since \( n-1 \) is even, by induction the only way this can happen is if \( c = \prod_{i=3}^{n} b_i \), since \( c, b_3, \ldots, b_n \) is the boundary state for the \( n-1 \)-gon. By induction there are two possible admissible labelings that extend this boundary state. Note that exactly one of the two edges labeled \( e \) and \( d \) is even length and the other is odd. Since the two labelings are related by lemma 2.1 it follows that in one labeling \( de = 1 \) and in the other \( de = -1 \). Since \( b_1b_2 \) is fixed, only one of the two satisfies \( de = b_1b_2 \). I.e. there is one admissible labeling for the boundary state \( b_i \).

If \( n \) is even, once again start with a boundary state \( b_i \). Given a choice of \( c \), by induction it uniquely extends to an admissible labeling of the \( n-1 \)-gon. Thus we have at most two admissible labelings of the \( n \)-gon with boundary \( b_i \). In order for this labeling to be admissible we must have \( b_1b_2 = de \).

Let \( g = \prod_{i=2}^{k} b_i \) and \( h = \prod_{i=k+1}^{n} b_i \). Consider the edges labeled \( d \) and \( e \). They are either both of even length or odd length. If they are of odd length, consider the polygon formed from \( e \) and the boundary edges \( b_2 \) to \( b_k \) and the polygon formed from \( d \) and the boundary edges \( b_{k+1} \) to \( b_n \). These are both even, so \( eg = dh = 1 \). Otherwise if they are of even length consider the polygons formed by \( e \), \( c \) and \( b_{k+1} \) to \( b_n \) and by \( d \), \( c \) and \( b_2 \) to \( b_k \). Since they
are both even, we have \( ech = decg = 1 \). In either case this gives \( de = gh \).

Note that this is independent of our choices of \( c \).

Thus if \( \prod b_i \neq 1 \), then \( b_1b_2 \neq gh = de \) and there are no extensions.
Otherwise there are exactly 2 as determined by the sign of \( c \). \( \square \)

3. Non-zero integral \( A_{n-3} \) friezes

In \([FZ1]\), it is shown that the clusters in the type \( A_{n-3} \) cluster algebra are associated to the triangulations of \( n \)-gon with the boundary arcs labeled 1. The cluster variables are the arcs in triangulations and the relations (mutations) between them are the Ptolemy relation on quadrilaterals.

A non-zero integral frieze is an evaluation of the cluster variables such that each cluster variable evaluates to a non-zero integer. In the language of triangulations of an \( n \)-gon, this corresponds to labeling each internal arc with a non-zero integer such that the labeling satisfies the Ptolemy relation.

In order to understand non-zero integral friezes of type \( A_{n-3} \) we will need to consider friezes of a slightly enlarged cluster algebra \( FA_{n-3} \) where we add a frozen variable for each boundary arc. The value of the frozen variables corresponding to the boundary arcs will be called the boundary state and thus the non-zero integral friezes of type \( FA_{n-3} \) with boundary state 1, \ldots, 1 will be the non-zero integral friezes of type \( A_{n-1} \).

**Theorem 3.1.** There are \( 2C(n-2) \) non-zero integral \( A_{n-3} \) friezes when \( n \) is even and \( C(n-2) \) when \( n \) is odd.

Note that the the \( C(n-2) \) positive friezes of Coxeter-Conway are the portion of non-zero integral friezes that are positive. When \( n \) is even one can obtain the remaining \( C(n-2) \) non-zero integral friezes from the positive integral friezes in the following way:

**Lemma 3.2.** If \( n \) is even, there exists a non trivial involution \( \sigma \) on the set of non-zero integral \( FA_{n-3} \) friezes: Negate all labels in the frieze on arcs of even length.

**Proof.** Beginning with a non-zero integral frieze \( F \), consider any quadrilateral in the triangulation of the \( n \)-gon. Since \( n \) is even the parity of the lengths of the arcs of the quadrilateral fall into 4 general cases: all odd, all even, (odd,even,odd,even) and (odd,odd,even,even).
In each case the sign difference between $F$ and $\sigma(F)$ cancels in the Ptolemy relation, either because the summands are of total degree 2 and each term in a monomial is negated, or because one term in each of the three summands is negated. □

Lemma 3.3. Consider a non-zero integral $FA_{n-3}$ frieze. Let $\{b_i\}$ be its boundary state and $a_i$ the labels of the arcs of length 2 such that arcs labeled $a_i$, $b_i$ and $b_{i+1}$ form a triangle. Then there exists an index $i$ such that $|a_i| < |b_i| + |b_{i+1}|$.

Note that this proof is a variation of that of Coxeter-Conway in the case of positive friezes.

Proof. Fix a vertex of the polygon and let $f_i$ denote the integers labeling the $n - 1$ arcs at a fixed vertex ordered clockwise. Let $b_i$ be the boundary state, starting clockwise from the fixed vertex and $a_i$ as in the statement.

Suppose the lemma is not true, then $|a_i| \geq |b_i| + |b_{i+1}|$ for all $i$. For each $i$ the cluster mutation relations are $f_{i-1}b_{i+1} + f_{i+1}b_i = a_if_i$. Taking absolute value of both sides:

$$|f_{i-1}b_{i+1} + f_{i+1}b_i| = |a_i||f_i|.$$  

Applying triangle inequality to the left side and using $|a_i| \geq |b_i| + |b_{i+1}|$ on the right, we get

$$|f_{i-1}||b_{i+1}| + |f_{i+1}||b_i| \geq |b_i||f_i| + |b_{i+1}||f_i|.$$  

Rearranging we have

$$\frac{|f_{i+1}| - |f_i|}{|b_{i+1}|} \geq \frac{|f_i| - |f_{i-1}|}{|b_i|}.$$  

Thus the sequence $\frac{|f_i| - |f_{i-1}|}{|b_i|}$ from $i = 2$ to $n - 1$ is increasing.
When \( i = 2 \), we have
\[
\frac{|f_2| - |f_1|}{|b_2|} = \frac{|a_1| - |b_1|}{|b_2|} \geq \frac{|b_2|}{|b_2|} = 1.
\]

On the other hand when \( i = n - 1 \), we have
\[
\frac{|f_{n-1}| - |f_{n-2}|}{|b_{n-1}|} = \frac{|b_n| - |a_{n-1}|}{|b_{n-1}|} \leq -\frac{|b_{n-1}|}{|b_{n-1}|} = -1.
\]

But this is impossible, no increasing sequence can start at 1 and end at \(-1\). Thus we must have \( |a_i| < |b_i| + |b_{i+1}| \) for some \( i \).

\[\square\]

**Corollary 3.4.** Every non-zero integral \( FA_{n-3} \) frieze with boundary state composed of \( \pm 1 \) contains a triangulation whose arcs are all labeled \( \pm 1 \).

We can now prove the main result of this section:

**Proof.** By the above corollary, since a non-zero integral \( A_{n-3} \) frieze is a non-zero integral \( FA_{n-3} \) frieze there is a triangulation whose arcs are labeled \( \pm 1 \). Since no arc in the frieze is labeled 0, by the Ptolemy relation the configurations in Figure 3 are impossible. This means that the labeling is admissible. Thus the number of friezes is at most the product of the number of admissible labelings and the number of triangulations \( C(n-2) \).

But, we already know of \( C(n-2) \) non-zero integral friezes: the set of positive integral friezes. Thus, in the case that \( n \) is odd, the number of non-zero integral friezes is \( C(n-2) \). In the case of \( n \) even, for each positive integral frieze \( F \), \( \sigma(F) \) is also non-zero integral frieze, and since \( \sigma \) is an involution we have at least \( 2C(n-2) \) non-zero integral friezes.

\[\square\]

4. Non-zero integral \( D_n \) friezes

In [EST], it is shown that the cluster algebra of type \( D_n \) can be thought of in terms of (tagged) arcs and triangulations for a punctured \( n \)-gon. Thus, in parallel with the \( A_n \) case, a non-zero integral \( D_n \) frieze is equivalent to a labeling of the (possibly tagged) arcs of a punctured \( n \)-gon by non-zero integers such the boundary is labeled 1 and the relations in Figure 4 hold.
In [FP] we determined that the number of positive friezes for $D_n$ was:

$$\sum_{m=1}^{n} d(m) \binom{2n-m-1}{n-m}$$

Each positive integral $D_n$ frieze has a canonical triangulation, containing all arcs between boundary vertices labeled 1 and enough (untagged) spokes to form a triangulation. The $m$ spokes all have the same label, an integer that divides $m$.

**Theorem 4.1.** The number of non-zero integral friezes of type $D_n$ is

$$4 \sum_{m=1}^{n} d(m) \binom{2n-m-1}{n-m}$$

when $n$ is even and

$$2 \sum_{m=1}^{n} d(m) \binom{2n-m-1}{n-m}$$

when $n$ is odd.

Similarly with the non-zero integral $A_{n-3}$ friezes, we will need to consider non-zero integral friezes over an enlarged cluster algebra $FD_n$ where one adds in a frozen variable for each boundary arc of the punctured $n$-gon. In this case we have the following involutions:

**Lemma 4.2.** Given a non-zero integral $FD_n$ frieze, negating all spoke arcs (tagged and non-tagged) gives a non-zero integral frieze. Call this $\sigma_1$. 

![Figure 4. $D_n$ relations](image)
Proof. Given $F$, a non-zero integral $FD_n$ frieze, consider a tagged triangulation and an arc in that triangulation. The mutation relation for this arc in the triangulation must be one of the relations in figure 4. If the puncture is not one of the vertices, then $F$ and $\sigma_1(F)$ agree on all arcs involved in the relation and the relation is true for $\sigma_1(F)$.

On the other hand, if one of the vertices involved is the puncture, when one examines the relations in Figure 4, we see that either all terms contain one spoke and one non-spoke variable, in which case the relation for $\sigma_1(F)$ is true since it is the negation of that for $F$. Otherwise the spoke variables appear an even number of times in each multiplicand and again the $\sigma_1(F)$ relation follows from the relation in $F$. \hfill \Box

Lemma 4.3. When $n$ is even, given a non-zero integral $FD_n$ frieze, fix an untagged spoke and negate every second untagged spoke. Negate the opposite set of tagged spokes and all arcs of even length between boundary vertices. This gives a non-zero integral $FD_n$ frieze. Let $\sigma_2$ denote this involution.

Note that that there are really two involutions here, depending on which spoke one fixes, but they are related by $\sigma_1$. That is $\sigma_2 \circ \sigma_1$ is the other involution.

Proof. Let $F$ be a $FD_n$ frieze. Let $F(a)$ denote the value of $F$ on edge $a$.

For relation A and B: With out loss of generality, we may assume $F(d) = (\sigma_2(F))(d)$ otherwise we consider $\sigma_1(\sigma_2(F))$. This case then follows from the $A_{n-3}$ case: if none of the arcs are spokes it is clear. Otherwise, one considers the length a spoke to be 1 plus the distance from $d$, then $F$ and $\sigma_2(F)$ differ in sign on exactly the edges of even length. The boundary of the relation also has an even total length.

For relation C: By symmetry we can just verify the first relation. Since $n$ is even, it follows that the length of $a$ and $d$ have the same parity. If that parity is even, $\sigma_2(F)(a) = -F(a)$ and $\sigma_2(F)(d) = -F(d)$, so $\sigma_2(F)(a) + \sigma_2(F)(d) = -(F(a)+F(d))$. Considering $e$ and $b$, we have $\sigma_2(F)(e)\sigma_2(F)(b) = -F(e)F(b)$ since one is of even length and the other odd. If the parity is odd, then $\sigma_2(F)(a) + \sigma_2(F)(d) = F(a) + F(d)$ and $\sigma_2(F)(e)\sigma_2(F)(b) = F(e)F(b)$ since the lengths of $e$ and $b$ are both even or both odd. In either case we have $\sigma_2(F)(a) + \sigma_2(F)(d) = \sigma_2(F)(e)\sigma_2(F)(b)$ when $F(a) + F(b) = F(e)F(b)$.

For relation D: First note that $\sigma_2(F)(d)\sigma_2(F)(g) = -F(d)F(g)$. Suppose that $c$ and $b$ are both even or both odd. Then $\sigma_2(F)(e)\sigma_2(F)(f) = F(e)F(f)$ and $\sigma_2(F)(c)\sigma_2(F)(b) = F(c)F(b)$. Then $a$ is of even length, so $\sigma_2(F)(a) = -F(a)$. Thus $\sigma_2(F)(a)\sigma_2(F)(d)\sigma_2(F)(g) = F(a)F(d)F(g)$. On the other hand, the lengths of $c$ and $b$ have opposite parity, we have $\sigma_2(F)(e)\sigma_2(F)(f) = -F(e)F(f)$ and $\sigma_2(F)(c)\sigma_2(F)(b) = -F(c)F(b)$. Then $a$ is of odd length, so $\sigma_2(F)(a) = F(a)$. Thus in either case when the relation is satisfied under $F$ it is satisfied under $\sigma_2(F)$.

Using these two involutions, we can generate 4 non-zero integral friezes from each positive frieze when $n$ is even and 2 when $n$ is odd. All that
remains is to prove that we have no more. We will follow the method of proof from [FP], but adjusted to allow arcs labeled by $-1$. From this point onwards

**Lemma 4.4.** Given a non-zero integral $FD_n$ frieze for $n \geq 2$, with a boundary state consisting of $\pm 1$, there exists a triangulation consisting of arcs between boundary vertices labeled $\pm 1$ and $m$ untagged spokes labeled by $\pm d$ for some non-zero integer $d$.

*Proof.* When $n = 2$ we are in the situation of figure 4c. We cannot have $a = -d$, otherwise one of $e$ or $b$ will be 0. Thus $a = d$, so $|a + d| = 2$ and it follows that $|eb| = 2 = |cf|$. If $|e| = |c|$, then since $e \geq 2$ we take the arcs labeled $e$ and $c$ for the triangulation. Otherwise $|e| \neq |c|$, in which case with out loss of generality, $|e| = 2$ so $|b| = 1$, and we take the arcs labeled $c$ and $b$, and consider $b$ to be a loop rather than a tagged arc.

When $n > 2$, we fall into two cases, either there exists an internal arc of length 2 labeled $\pm 1$ or there are none. If there is one, cut the triangle subtended by the arc off, then we have a $FD_{n-1}$ frieze and by induction, it follows that the required triangulation exists.

Otherwise all arcs of length 2 have labels of norm larger than 1. Consider the triangulation by untagged spokes. Suppose the $i$-th spoke (modulo $n$) has label $f_i$ and the arc of length 2 crossing the $i$-th spoke has label $a_i$. The boundary state is given by $b_i$, where $b_i$ labels the boundary arc joining vertex $i-1$ and $i$.

Then we have:

$$a_if_i = b_{i+1}f_{i-1} + b_if_{i+1}.$$  

Taking norms and using triangle inequality, (note that $|b_i| = 1$) we have

$$|a_i||f_i| \leq |f_{i-1}| + |f_{i+1}|.$$  

And since $|a_i| \geq 2$, we have

$$2|f_i| \leq |f_{i-1}| + |f_{i+1}|,$$

or

$$|f_i| - |f_{i-1}| \leq |f_{i+1}| - |f_i|.$$  

The sequence $|f_i| - |f_{i-1}|$ is cyclic and monotone, so it is constant. Then the sequence $|f_i|$ is monotone and also cyclic, so it is also constant. $\square$

Given a non-zero integral $FD_n$ frieze and a triangulation of the punctured $n$-gon, we associate a **sign configuration**. It is a labeling of the triangulation by $\pm 1$ where an edge is labeled 1 if the corresponding label in the frieze is positive and $-1$ when the label is negative.

**Lemma 4.5.** Given a non-zero integral $FD_n$ frieze with boundary $\pm 1$ and the triangulation from lemma 4.4, the associated sign configuration is admissible in the sense of section 2 when we cut along one of the untagged spokes of the triangulation to obtain a triangulation of a $n + 2$-gon.
Proof. Note that any quadrilateral in the $n + 2$-gon either comes from a honest quadrilateral in the punctured $n$-gon or in the case where the is only one untagged spoke, from the following configuration (self folded triangle) where the cut is along $c$:

Quadrilaterals in the punctured $n$-gon come in two classes as well: those whose vertices are all boundary vertices and those which have the puncture as one boundary vertex. In the first case, all arcs of the quadrilateral are already labeled $\pm 1$ and the two diagonals are non zero. It then follows from the relation in figure 4a that there are an even number of $-1$ labels on the quadrilateral and thus this quadrilateral is admissible.

In the second case, the quadrilateral has two adjacent arcs labeled $\pm 1$ and two adjacent arcs labeled $\pm d$ (the spokes). Since the product of the diagonals of the quadrilateral is nonzero, the sum of the product of the opposite labels of the quadrilateral are either $2d$ or $-2d$, so we cannot have an odd number of the arcs of the quadrilateral be negative and thus the quadrilateral is admissible

In the self-folded case, we see that $a$ and $f$ are labeled $\pm 1$. If they have opposite sign, then $a + f = 0 = be$ so one of $b$ or $e$ would be zero. This is not the case, so $a$ and $f$ have the same sign. Cutting along $c$ revealed a quadrilateral with sides $a$, $f$, $c$ and $c$. Since the sign of $a$ and $f$ agree, the quadrilateral is admissible. \qed

In the case of a non-zero integral $D_n$ frieze, up to $\sigma_1$, we may assume that the sign configuration has at least one spoke labeled 1. Cutting along this spoke gives triangulation of the $n + 2$-gon and an admissible labeling with boundary state 1. By section 2 it follows that either all arcs are labeled 1 or $n$ is even and even length arcs are labeled $-1$.

The sign configuration falls into one of 4 categories:

1. All arcs are labeled 1.
2. All arcs are labeled 1 but spokes, which are labeled $-1$.
3. $n$ is even and all odd length arcs are labeled 1 and even length arcs are labeled $-1$. Counting from the spoke which was cut, spoke 0 is positive and the sign of spoke $i$ is given by parity of the distance along the boundary from spoke 0 with odd corresponding to $-1$ and even to 1.
4. The same as above, except the spokes have opposite labels.

But now we see that if we apply a combination of $\sigma_1$ and $\sigma_2$, we can turn a non-zero integral $D_n$ frieze a positive $D_n$ frieze. That is $\sigma_2$ moves the sign
from type 3 and 4 to 1 and 2 and $\sigma_1$ from 2 to 1. Once we have a positive
sign configuration, the entire frieze is positive since a single cluster is.

We can now prove theorem 4.1

**Proof.** In the case that $n$ is odd, applying $\sigma_1$ to the set of positive $D_n$ friezes shows that we have at least twice as many non-zero integral friezes of type $D_n$ as positive ones. On the other hand the above reasoning shows that any non-zero integral $D_n$ frieze can be brought to a positive one by applying $\sigma_1$.

So there are exactly

$$2 \sum_{m=1}^{n} d(m) \left( \frac{2n-m-1}{n-m} \right)$$

non-zero integral $D_n$ friezes when $n$ is odd.

Similarly when $n$ is even, since we have $\sigma_1$ and $\sigma_2$, we have

$$4 \sum_{m=1}^{n} d(m) \left( \frac{2n-m-1}{n-m} \right)$$

non-zero integral $D_n$ friezes when $n$ is even. \qed

5. $B_n$, $C_n$ and $G_2$

The results for $B_n$, $C_n$ and $G_2$ can be derived from the results $D_{n+1}$, $A_{2n-1}$ and $D_4$ respectively by using the folding methods of [DUP]. To summarize, if $\Delta$ is a Dynkin quiver and $G$ a group of automorphisms of $\Delta$, then $\Delta/G$ is a valued quiver and the action of $G$ lifts to the cluster algebra $A(\Delta)$. In the case of $B_n$, $G$ is generated by the automorphism of $D_{n+1}$ which swaps the short arms. For $C_n$, $G$ is generated by the automorphism of $A_{2n-1}$ which reflects the diagram through the centre node. For $G_2$, $G$ is generated by the order 3 automorphism of $D_4$ which rotates the diagram about its central node.

In this case [DUP, Corollary 5.16] shows that $A(\Delta/G)$ can be identified with a subalgebra of $A(\Delta)/G$. More over [DUP, Corollary 7.3] gives equality since $\Delta$ is Dynkin. The projection $\pi : A(\Delta) \to A(\Delta)/G$ can then be thought of as a surjective ring homomorphism from $A(\Delta)$ to $A(\Delta/G)$, which sends the cluster variables of $A(\Delta)$ to the cluster variables of $A(\Delta/G)$ via a quotient by $G$.

**Lemma 5.1.** Let $\Delta$ by a Dynkin quiver and $G$ a group of automorphisms, then each non-zero integral $\Delta/G$ frieze gives rise to a $G$ invariant non-zero integral $\Delta$ frieze. More over, each non-zero integral $\Delta$ frieze that is $G$ invariant descends to a non-zero integral $\Delta/G$ frieze.

**Proof.** We consider a non-zero integral $\Delta$ frieze to be a ring homomorphism from the cluster algebra $A(\Delta)$ to $\mathbb{Z}$ sending each cluster variable to a non-zero integer. Then a non-zero integral $\Delta/G$ frieze gives a $\Delta$ frieze by composition with $\pi$. Since $\pi$ is $G$ invariant so is the composition.
Given a non-zero integral $\Delta$ frieze, if it is $G$ invariant, it descends to a ring homomorphism from $A(\Delta)/G$ to $\mathbb{Z}$, but this gives a non-zero integral $\Delta/G$ frieze.  

**Theorem 5.2.** There are $2 \left( \frac{2n}{n} \right)$ non-zero integral $C_n$ friezes.

**Proof.** Given a non-zero integral $C_n$ frieze, lift it to a non-zero integral $A_{2n-1}$ frieze. Since $2n - 1$ is odd, if the frieze is not already positive applying $\sigma$ will make it so. Since $\sigma$ does not change the magnitude of the labels and makes all signs positive, the result will be $G$ invariant. Since there are $\left( \frac{2n}{n} \right)$ positive $G$ invariant frieze (FP), it follows that the number of non-zero integral $C_n$ friezes is at most $2 \left( \frac{2n}{n} \right)$.

On the other hand, the action of $G$ on $A_{2n-1}$ identifies exactly those arcs that are diametrically opposed. But these arcs have the same length so given a $G$ invariant $A_{2n-1}$ frieze, applying $\sigma$ to it will leave it $G$ invariant.

Thus there are twice as many non-zero integral $C_n$ friezes as there are positive $C_n$ friezes.  

**Theorem 5.3.** There are $2 \sum_{m \leq \sqrt{n+1}} \left( \frac{2n - m^2 + 1}{n} \right)$ frieze of type $B_n$.

**Proof.** Given a non-zero integral $B_n$ frieze, lift it to a $G$ invariant $D_{n+1}$ frieze. Applying some combination of $\sigma_1$ and $\sigma_2$ will give a positive frieze which will still be $G$ invariant (see above proof).

The action of $G$ on $D_{n+1}$ identifies the tagged arc with its corresponding untagged arc. The result of applying $\sigma_2$ or $\sigma_1 \circ \sigma_2$ to a $G$ invariant frieze will no longer be $G$ invariant, since the labels of a tagged/untagged pair will differ by sign. On the other hand applying $\sigma_1$ leave the frieze $G$ invariant, since it negates all spokes arcs.

Thus there are twice as many non-zero integral $B_n$ friezes as there are positive ones.

**Theorem 5.4.** Finally for the $G_2$ case, there are just 9 non-zero integral friezes and they are all positive friezes.

**Proof.** Suppose we have a non-zero integral $G_2$ frieze, we can lift it to a $G$ invariant non-zero $D_4$ frieze. More over, by applying a combination of $\sigma_1$ and $\sigma_2$ we will obtain a positive $D_4$ frieze which must still be $G$ invariant.

The $G$ action on $D_4$ leading to $G_2$ identifies the following arcs in a punctured 4-gon:
Applying $\sigma_2$ or $\sigma_1 \circ \sigma_2$ to a positive $G$ invariant $D_4$ frieze gives a non-zero frieze which is not $G$ invariant since the labels of each pair of spokes have different signs. Applying $\sigma_1$ negates all spoke labels but does not change the label of the diagonal arc in the above diagram, thus it is also not $G$ invariant.

6. Future directions

By replacing positive by non-zero, we open the doors to consider the structure of friezes over rings of integers. Interesting choices for such rings might include $\mathbb{Z}[\zeta]$ for $\zeta$ a root of unity.

The results in this direction would look slightly different than those above. For instance iterating lemma [5.3] shows that starting with a boundary state with bounded norm, one can obtain a triangulation of the $n$-gon where arcs of length $i$ have labels bounded by the sum the norms of the labels on one side of the initial polygon.

Indeed if we consider the case of $A_2$ we have the following:

**Theorem 6.1.** There are 12 nonzero $R = \mathbb{Z}[i]$ friezes of type $A_1$.

**Proof.** Since the two cluster variables are $x_1$ and $\frac{2}{x_1}$, it suffices to find all elements $x_1 \in R$ such that $2/x_1 \in R$. The original non-zero integral friezes $\pm 1$ and $\pm 2$ work. We also have $\pm 1 \pm i$, $\pm i$ and $\pm 2i$. \[\Box\]

In each case it is possible to pick a triangulation such that the norm of the label of the internal arcs is bounded (strictly) by 2.

Irrespective of this problem, we should expect the following to be true:

**Conjecture 6.2.** There are finitely many non-zero $\mathbb{Z}[\zeta]$ friezes of Dynkin type, for any $\zeta$ a root of unity.

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