ON STABLE CMC HYPERSURFACES WITH FREE-BOUNDARY IN A EUCLIDEAN BALL

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ABSTRACT. In this note, we observe that if $B$ is a ball in a Euclidean space with dimension $n, n \geq 3$, then a stable CMC hypersurface $\Sigma$ with free boundary in $B$ satisfies

$$nA \leq L \leq nA \left( \frac{1 + \sqrt{1 + 4(n + 1)H^2}}{2} \right),$$

where $L, A$ and $H$ denote the length of $\partial \Sigma$, the area of $\Sigma$ and the mean curvature of $\Sigma$, respectively. Consequently, if the boundary $\partial \Sigma$ is embedded then $\Sigma$ must be totally geodesic or starshaped with respect to the center of the ball. This result is an improvement of a theorem proved by A. Ros and E. Vergasta [R-V]. In particular, if $n = 3$, the only stable CMC surfaces with free boundary in $B$ are the totally geodesic disks or the spherical caps. This last result was proved very recently by I. Nunes [N] using an extended stability result and a modified Hersch type balancing argument to get a better control on the genus. We don’t use that modified Hersch type argument. However, we use a Nunes type Stability Lemma and a crucial result due to A. Ros and E. Vergasta.

1. INTRODUCTION

Given a smooth compact and convex domain $B$ in $\mathbb{R}^{n+1}$, denote by $\partial B$ and $\text{int } B$ the boundary and the interior of $B$, respectively. A CMC free-boundary hypersurface in $B$ is a constant mean curvature hypersurface $\Sigma \subset B$ meeting $\partial B$ orthogonally along $\partial \Sigma$. That kind of hypersurfaces are solutions for the problem of finding critical points of the area functional among all compact hypersurfaces $\Sigma \subset B$ with $\partial \Sigma \subset \partial B$ which divides $B$ into two subsets of prescribed volumes. If a CMC free-boundary hypersurface $\Sigma \subset B$ has nonnegative second variation of area for all preserving volume variations we name it as a CMC free-boundary stable hypersurface. For more details about CMC free-boundary hypersurfaces, see the following references and references therein: [N], [R], [R-V], [So].

In [R-V], Ros and Vergasta studied stable CMC hypersurfaces with free boundary when $B$ is a ball and proved the following result. Denote by $L$ the length of the boundary $\partial \Sigma$ and by $A$ the area of $\Sigma$.

**Theorem 1.1** (Ros-Vergasta [R-V]). Let $B \subset \mathbb{R}^n, n \geq 3$, be a closed ball. Let $\Sigma \subset B$ be a CMC free-boundary stable hypersurface with embedded boundary in $B$. If $L \geq nA$ then $\Sigma$ is totally geodesic or starshaped with respect to the center of the ball.

In order to improve the result above, we use a Nunes type Stability Lemma (see Lemma 2.1) to prove that always $L \geq nA$. More precisely, we obtain the following result.
Theorem 1.2. Let $B \subset \mathbb{R}^n$, $n \geq 3$, be a closed ball. If $\Sigma \subset B$ is a CMC free-boundary stable hypersurface in $B$, then
\[
 nA \leq L \leq nA \left(1 + \sqrt{1 + 4(n+1)H^2}\right).
\]
In particular, if $\partial \Sigma$ is embedded, then $\Sigma$ is totally geodesic or starshaped with respect to the center of the ball.

As a direct consequence, we obtain the following corollary.

Corollary 1.1. Let $B \subset \mathbb{R}^n$, $n \geq 3$, be a closed ball. If $\Sigma \subset B$ is a CMC stable hypersurface with embedded free-boundary in $B$ and $0 \in \Sigma$ then $\Sigma$ is totally geodesic.

Now, noting that free-boundary surfaces must have embedded boundary (see Theorem 11 in [R-V]), we obtain a complete topological classification.

Corollary 1.2. Let $B \subset \mathbb{R}^3$ be a closed ball. If $\Sigma \subset B$ is a CMC stable surface with free-boundary then $\Sigma$ is a totally geodesic disk or a spherical cap.

It’s worthy to mention that the Corollary 1.2 was proved recently by I. Nunes [N] using a powerful stability result and a modified Hersch type balancing argument to get a better control on the genus and on the number of connected components of the boundary of the surface. In fact, I. Nunes proved a more general result which, joint with Theorem 11 in [R-V], gives us the result above as a corollary.

Theorem 1.3 (I. Nunes [N]). Let $\Omega \subset \mathbb{R}^3$ be a smooth compact convex domain. Suppose that the second fundamental form $\Pi^{\partial \Omega}$ of $\partial \Omega$ satisfies the pinching condition
\[
 k h \leq \Pi^{\partial \Omega} \leq (3/2)k h.
\]
for some constant $k > 0$, where $h$ denotes the induced metric on $\partial \Omega$. If $\Sigma \subset \Omega$ is an immersed orientable compact stable CMC surface with free boundary, then $\Sigma$ has genus zero and $\Sigma$ has at most two connected components.

In order to prove Theorem 1.2 we apply the same idea as that applied by I. Nunes in the proof of the main result for the free-boundary surfaces case in [N]. I. Nunes showed that the stability of a free-boundary CMC surface implies that the quadratic form given by the second variation of area is nonnegative for all functions $f$ such that $f = 0$ on $\partial \Sigma$ regardless of whether it satisfies $\int_{\Sigma} f \, d\nu_{\Sigma} = 0$ or not. That is what we are calling Nunes Stability Lemma. Then I. Nunes was able to apply a modified Hersch type balancing argument to obtain a better control on the genus of $\Sigma$. We use that idea for high dimension combined with some Ros-Vergasta results.

2. Nunes Type Stability Lemma

Let $B$ be a compact convex domain in $\mathbb{R}^{n+1}$. Let $\varphi : \Sigma^n \to B$ be an immersion of a smooth orientable manifold $\Sigma$ with boundary $\partial \Sigma$ such that $\varphi(\partial \Sigma) = \varphi(\Sigma) \cap \partial B$. Let’s denote the unit normal vector of the hypersurface $\Sigma$ by $N$. The immersion $\varphi$ is called free boundary if $\varphi(\Sigma)$ meets $\partial B$ orthogonally. The second fundamental form $A$ of $\Sigma$ is the endomorphism $A(X) = -\nabla_X N$, where $X \in T\Sigma$. The mean curvature of $\Sigma$ is then given by $H = \frac{1}{n} \text{Tr}A$. 
If we consider a smooth variation $\phi : \Sigma \times [0, \varepsilon) \to B$ that preserves $\partial B$ and such that $\phi(\cdot, 0) = \varphi(\cdot)$ then it natural to consider the following two functions:

$$A(t) = \int_{\Sigma} dvol_{\Sigma_t} \quad \text{and} \quad V(t) = \int_{\Sigma \times [0, t]} \phi^* dvol_{\mathbb{R}^{n+1}}.$$  

The variation $\phi$ is called volume preserving if $V(t) \equiv 0$. Let $f$ be the function defined by $f = \langle \frac{\partial}{\partial t} \phi(x, 0), N(x) \rangle$ where $x \in \Sigma$ then the first variation formula yields:

$$A'(0) = -n \int_{\Sigma} H f \, dvol_{\Sigma} + \int_{\partial \Sigma} f(N, \nu) \, ds \quad \text{and} \quad V'(0) = \int_{\Sigma} f \, dvol_{\Sigma}.$$  

It follows that CMC hypersurfaces with free boundary are critical points for the area functional $A(t)$ when restricted to volume preserving variations. The converse is also true, see [R-V]. If $A'(0) \geq 0$ is nonnegative for every volume preserving variation then the immersion $\phi$ is called Stable CMC. It can be shown that this is equivalent to have for every $f \in C^\infty(\Sigma)$ with $\int_{\Sigma} f \, dvol_{\Sigma} = 0$ that

$$I(f, f) = \int_{\Sigma} |\nabla f|^2 - |A_{\Sigma}|^2 f^2 \, dvol_{\Sigma} \geq \frac{1}{n+1} \left( \int_{\Sigma} f \, dvol_{\Sigma} \right)^2 \int_{\partial \Sigma} \Pi(N, N) \, ds.$$  

In particular, if $f \in C^\infty(\Sigma)$ is such that $f(x) = 0$ for every $x \in \partial \Sigma$ then

$$I(f, f) = \int_{\Sigma} |\nabla f|^2 - |A_{\Sigma}|^2 f^2 \, dvol_{\Sigma} \geq 0.$$  

**Lemma 2.1** (Nunes Type Stability Lemma). Let $\Sigma$ be an immersed, stable, hypersurface with constant mean curvature with free boundary in $B$. If $f \in C^\infty(\Sigma)$ is such that $f(x) = 0$ for every $x \in \partial \Sigma$ then

$$I(f, f) = \int_{\Sigma} |\nabla f|^2 - |A_{\Sigma}|^2 f^2 \, dvol_{\Sigma} \geq 0.$$  

**Proof.** Let $f_i$ be the function $f_i = \langle e_i, N \rangle$ where $\{e_i\}$ is the canonical orthonormal basis of $\mathbb{R}^{n+1}$. A simple computation yields:

$$\Delta f_i + |A_{\Sigma}|^2 f_i = 0. \quad (2.1)$$  

Plugging these functions on the quadratic form $I$ we have:

$$\sum_{i=1}^{n+1} I(f_i, f_i) = \int_{\Sigma} \sum_{i=1}^{n+1} |\nabla f_i|^2 - |A_{\Sigma}|^2 f_i^2 \, dvol_{\Sigma} - \int_{\partial \Sigma} \sum_{i=1}^{n+1} \Pi(N, N) f_i^2 \, ds$$

$$= -\sum_{i=1}^{n+1} \int_{\Sigma} f_i \Delta f_i + |A_{\Sigma}|^2 f_i \, dvol_{\Sigma} + \int_{\partial \Sigma} \sum_{i=1}^{n+1} f_i \frac{\partial f_i}{\partial \nu} - \Pi(N, N) \, ds$$

$$= \int_{\partial \Sigma} \frac{1}{2} \frac{\partial}{\partial \nu} \left( \sum_{i=1}^{n+1} f_i^2 \right) \, ds - \int_{\partial \Sigma} \Pi(N, N) \, ds = -\int_{\partial \Sigma} \Pi(N, N) \, ds.$$  

We have used that $\sum_{i=1}^{n+1} f_i^2 = |N|^2 = 1$. It follows that, given a function $f$ such that $f = 0$ on $\partial \Sigma$, at least one of the $f_i$ have the property that

$$I(f_i, f_i) \leq -\frac{1}{n+1} \int_{\partial \Sigma} \Pi(N, N) \, ds < 0 \quad \text{and} \quad f_i \neq f.$$
In fact, if for each \( f_i \) we have \( I(f_i, f_i) > -\frac{1}{n+1} \int_{\partial \Sigma} \Pi(N, N) ds \) or \( f_i = f \) then we obtain that
\[
\sum_{i=1}^{n+1} I(f_i, f_i) > -\frac{m}{n+1} \int_{\partial \Sigma} \Pi(N, N) ds
\]
for some positive integer \( m \leq n+1 \), since when \( f_i = f \) we obtain \( I(f_i, f_i) = 0 \).

Hence, let \( f_i \) be the function satisfying that condition. Note that, because of the stability of \( \Sigma \), we have that \( \int_{\partial \Sigma} f_i dvols \neq 0 \). Assume that \( \int_{\partial \Sigma} fdvols \neq 0 \). Now, consider the function \( \bar{f} = cf \), where \( c = \frac{\int_{\partial \Sigma} f_i dvols}{\int_{\partial \Sigma} f dvols} \).

We have \( \int_{\partial \Sigma} (\bar{f} - f) dvols = 0 \). Using (2.1) and that \( \bar{f} = 0 \) at \( \partial \Sigma \) we have
\[
0 \leq I(\bar{f} - f_i, \bar{f} - f_i) = I(\bar{f}, \bar{f}) - 2I(\bar{f}, f_i) \leq I(\bar{f}, \bar{f}) - \frac{1}{n+1} \int_{\partial \Sigma} \Pi(N, N) ds.
\]
This implies that
\[
I(f, f) \geq \left( \frac{\int_{\partial \Sigma} f dvols}{\int_{\partial \Sigma} f_i dvols} \right)^2 \frac{1}{n+1} \int_{\partial \Sigma} \Pi(N, N) ds.
\]
It follows from Holder’s inequality and \( \sum_{i=1}^{n+1} f_i^2 = |N|^2 = 1 \) that
\[
\left( \int_{\partial \Sigma} f_i dvols \right)^2 = \left( \int_{\partial \Sigma} f dvols \right)^2 \leq \left( \int_{\partial \Sigma} |f_i dvols| \right)^2 \leq A^2.
\]
This finishes the proof. \( \square \)

3. PROOF OF THEOREM 1.2

**Proof.** Assume that \( \Sigma \) is a stable free-boundary hypersurface in \( B \). Consider then the support function \( u = \langle \psi, N \rangle \) of \( \Sigma \), where \( \psi \) is the immersion of \( \Sigma \) in \( B \). It satisfies the following system
\[
\begin{align*}
\Delta u + |\sigma|^2 u &= -nH \quad \text{on} \quad \Sigma \\
u &= 0 \quad \text{on} \quad \partial \Sigma
\end{align*}
\]
Moreover, taking the diverge of the tangent component \( \psi - uN \) of \( \psi \) is given by
\[
\text{div}(\psi - uN) = n + nHu.
\]
It follows from the Divergence Theorem that
\[
L = n \left( A + \int_{\Sigma} H dvols \right).
\]
Since \( u = 0 \) on \( \partial \Sigma \), it follows from the stability of \( \Sigma \) and Nunes Stability Lemma that
\[
nH \int_{\Sigma} udvols = \int_{\Sigma} |\nabla \Sigma u|^2 - |A_{\Sigma}|^2 u^2 dvols \geq 0.
\]
Note that, if \( H = 0 \), then \( L = nA \). Assume that \( H \neq 0 \). First, as was done by Ros-Vergasta in [R-V], we will first prove that either \( u \geq 0 \) or \( u \leq 0 \) on \( \Sigma \). Suppose, by
contradiction, that \( u \) changes sign. Consider \( \Sigma^+ \) (resp. \( \Sigma^- \)) the subset of \( \Sigma \) where \( u \) is positive (resp. negative) and define \( u^+, u^- \in H^1(\Sigma) \) by

\[
u^+(p) = \begin{cases} u(p) & \text{if } p \in \Sigma^+ \\ 0 & \text{if } p \in \Sigma \setminus \Sigma^+ \end{cases}
\]

and

\[
u^-(p) = \begin{cases} u(p) & \text{if } p \in \Sigma^- \\ 0 & \text{if } p \in \Sigma \setminus \Sigma^- \end{cases}
\]

A direct computation gives

\[
I(u^-, u^-) = nH \int_{\Sigma} u^- \, d\text{vol}_\Sigma
\]

and

\[
I(u^+, u^+) = nH \int_{\Sigma} u^+ \, d\text{vol}_\Sigma.
\]

Now we define \( \tilde{u} = u^+ + au^- \), where \( a \) is a positive constant such that

\[
\int_{\Sigma} \tilde{u} \, d\text{vol}_\Sigma = 0.
\]

It follows that \( \tilde{u} \) is not identically null and

\[
I(\tilde{u}, \tilde{u}) = -naH \int_{\Sigma} u \, d\text{vol}_\Sigma.
\]

As in Ros-Vergasta [R-V], pag. 29, we obtain that either \( u \geq 0 \) or \( u \leq 0 \) on \( \Sigma \). We can choose the orientation on \( \Sigma \) such that \( u \geq 0 \). Since \( H \neq 0 \) and \( \int_{\Sigma} H u dA \geq 0 \), we get that \( H > 0 \). Therefore, \( u \) satisfies:

\[
u \geq 0, \quad \nu = 0 \quad \text{on } \partial \Sigma \quad \text{and} \quad \Delta \nu = |\sigma|^2 u - nH < 0.
\]

By the maximum principle for subharmonic functions we obtain that \( u \) is strictly positive on \( \text{int} \Sigma \). This gives us that \( \int_{\Sigma} u \, d\text{vol}_\Sigma \neq 0 \). It follows from the Nunes Stability Lemma that

\[
\int_{\Sigma} |\nabla \nu|^2 - |A| u^2 \, d\text{vol}_\Sigma \geq \frac{1}{n + 1} \left( \frac{\int_{\Sigma} u \, d\text{vol}_\Sigma}{A} \right)^2 \int_{\partial \Sigma} \Pi(N, N) \, ds
\]

\[
= \frac{L}{n + 1} \left( \frac{\int_{\Sigma} u \, d\text{vol}_\Sigma}{A} \right)^2.
\]

Hence, we obtain

\[
(3.3) \quad n \int_{\Sigma} H u \, d\text{vol}_\Sigma \geq \frac{L}{n + 1} \left( \frac{\int_{\Sigma} u \, d\text{vol}_\Sigma}{A} \right)^2,
\]

since

\[
\int_{\Sigma} H u \, d\text{vol}_\Sigma = \int_{\Sigma} |\nabla \nu|^2 - |A| u^2 \, d\text{vol}_\Sigma.
\]

From (3.2) we have that

\[
\frac{\int_{\Sigma} u \, d\text{vol}_\Sigma}{A} = \frac{L - nA}{nHA}.
\]

Then, from (3.2) and (3.3), we conclude that

\[
L = nA + nH \int_{\Sigma} u \, d\text{vol}_\Sigma \geq nA + \frac{L}{n + 1} \left( \frac{\int_{\Sigma} u \, d\text{vol}_\Sigma}{A} \right)^2 = nA + \frac{L}{n + 1} \left( \frac{L - nA}{nHA} \right)^2.
\]

This implies that

\[
L - nA \geq \frac{L}{n + 1} \left( \frac{L - nA}{nHA} \right)^2.
\]

Therefore,

\[
L^2 - nAL - n^2 A^2 (n + 1) H^2 \leq 0.
\]
This implies that

\[ L \leq nA \left( \frac{1 + \sqrt{1 + 4(n + 1)H^2}}{2} \right). \]

\[ \square \]

4. PROOF OF COROLLARY [L.2]

Proof. As in the proof of Theorem 11 in [R-V], we obtain that \( \partial \Sigma \) is embedded. Now, applying the Theorem [L.2] we obtain that \( \Sigma \) is totally geodesic or starshaped with respect to the center of the ball. Since starshaped surfaces must have genus 0, we obtain that \( \Sigma \) is totally geodesic or a spherical cap. \[ \square \]

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