Geometric Flow, Multiplier Ideal Sheaves and Optimal Destabilizer for a Fano Manifold

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Abstract
In (Donaldson in J Differ Geom 70(3):453–472, 2005), it was asked whether the lower bound of the Calabi functional is achieved by a sequence of the normalized Donaldson–Futaki invariants. We answer to the question for the Ricci curvature formalism, in place of the scalar curvature. Our principle is that the stability indicator is optimized by the multiplier ideal sheaves of certain weak geodesic ray asymptotic to the geometric flow. We actually obtain the results for the two cases: the inverse Monge–Ampère flow and the Kähler–Ricci flow.

Keywords  Kähler-Ricci flow · K-stability · Monge-Ampère equation

Mathematics Subject Classification  53C55 · Secondary · 53E30 · 32Q20 · 32Q26

1 Introduction
Let $X$ be a Fano manifold. Recent development of Kähler geometry, particularly represented by the cerebrated work [21], brought us the vast knowledge about the existence of the Kähler–Einstein metric and its algebraic counterpart: K-stability of $X$. On the other hand, only a few Fano manifolds admit such a standard metric. Therefore, it might be interesting to ask how general $X$ is far from and related with the Kähler–Einstein condition. In studying the curvature of each Kähler metric $\omega$ in the first Chern class $c_1(X)$ we make use of the normalized Ricci potential function $\rho$ which is characterized by

$$\text{Ric } \omega - \omega = dd^c \rho, \quad \int_X (e^\rho - 1) \omega^n = 0. \quad (1.1)$$
The volume \( V = \int_X \omega^n \) is independent of \( \omega \). The metric is Kähler–Einstein if and only if \( \rho = 0 \) and it is equivalent to say that the scalar curvature is constant. For a general polarized manifold \((X, L)\), the famous Calabi functional measures how \( \omega \) is far from constant scalar curvature and \([42]\) gives the lower bound in terms of his generalization of the Futaki invariant. For a Fano polarization \((X, -K_X)\) the above Ricci potential may work in place of the scalar curvature. In fact, in analogy with Donaldson’s lower bound, we have the inequality

\[
\inf_{\omega} \left[ \frac{1}{V} \int_X (e^\rho - 1)^2 \omega^n \right]^{\frac{1}{2}} \geq \sup_{(\mathcal{X}, \mathcal{L})} \frac{-D_{\text{NA}}(\mathcal{X}, \mathcal{L})}{\| (\mathcal{X}, \mathcal{L}) \|_2}. \tag{1.2}
\]

Here \((\mathcal{X}, \mathcal{L})\) runs through arbitrary test configurations of \((X, -K_X), \| (\mathcal{X}, \mathcal{L}) \|_2 \) is the \( L^2 \)-norm, and \( D_{\text{NA}}(\mathcal{X}, \mathcal{L}) \) is the non-Archimedean D-energy introduced in \([5], [15]\). We review the terminologies and a proof of (1.2), in the next section. From the result of \([49]\), positivity of \( D_{\text{NA}}(\mathcal{X}, \mathcal{L}) \) for every non-trivial test configuration is indeed equivalent to the K-stability condition introduced by \([41]\). The prototype of such inequalities already appears in geometric invariant theory (GIT for short), where the inequality confronts square of the moment map with Hilbert-Mumford weights. For this reason, we may call (1.2) moment-weight inequality. The precise moment map picture was explained in \([43]\), where the Kemp–Ness functional in GIT is translated into the (Archimedean) D-energy (2.2).

In the scalar curvature setting Donaldson asked whether the equality holds in the moment-weight type inequality. In our setting, \([67]\) recently proved that the equality in (1.2) actually holds for toric Fano manifolds. If there exists a test configuration accomplishes the identity, it should be the optimal destabilizer in analogy with the Harder-Narasimhan type filtration for the vector bundles. The pioneering work \([51]\) of Nadel already predicted that the certain multiplier ideal sheaf should serve as the destabilizing subsheaf of the vector bundle. See also \([55, 60]\).

In this paper, we show that the equality holds in (1.2) for general Fano manifolds, in virtue of adopting the Ricci potential formulation. The proof itself might be interesting, as it gives the asymptotic construction of the optimal destabilizer in way of the multiplier ideal sheaf. Our new ingredient is the gradient flow of the D-energy. For the definition, using the \( dd^c \)-lemma we fix the reference metric \( \omega_0 \) and represent any other metric by a function \( \varphi \) so that \( \omega = \omega_0 + dd^c \varphi \) holds. The function \( \varphi \) is determined up to a constant and we consider \( \rho = \rho_\varphi \) or other quantities as functions in \( \varphi \). In terms of \( \varphi \) we introduce the inverse Monge–Ampère flow

\[
\frac{\partial}{\partial t} \varphi = 1 - e^\rho, \tag{1.3}
\]

which imitates the Calabi flow. Although the long-time existence of the Calabi flow is still an open question, we have the solution for (1.3). This is one of the main results in our previous work \([22]\). Building on the Mabuchi geometry of the space of Kähler metrics, especially on the technique exploited by \([32]\), one can construct a weak geodesic ray \( u^t \) \((t \in [0, \infty)) \) asymptotic to the flow \( \varphi_t \). Changing variables of \( u^t \) yields the psh weight \( U \) on the product space \( X \times \mathbb{C} \). Blowing up the multiplier ideal
sheaves \( J(mU) \) for each \( m \in \mathbb{N} \), we obtain a sequence of test configurations, which canonically approximates the geodesic ray. The technology here was paved by [12] where they gave a new variational proof for existence of the Kähler–Einstein metric. In the scalar curvature setting, the variational approach has not been completed. We will show that the equality of (1.2) is then naturally achieved by the flow and these test configurations.

Note that the constructed multiplier ideal sheaves are different from the traditional construction in the literature, which was taken for the flow \( \phi_t \) at each fixed time \( t \). On the product space \( \phi_t \) is no longer psh and we need the asymptotic geodesic ray \( U \) to take the multiplier ideal.

**Theorem A** (moment-weight equality) Greatest lower bound of the Ricci–Calabi functional is given by a sequence of \( L^2 \)-normalized non-Archimedean Ding energies, that is,

\[
\inf \omega \left[ \frac{1}{V} \int_X \left( e^\rho - 1 \right)^2 \omega^n \right]^{\frac{1}{2}} = \sup_{(\mathcal{X}, \mathcal{L})} \frac{-D_{NA}(\mathcal{X}, \mathcal{L})}{\| (\mathcal{X}, \mathcal{L}) \|_2}.
\]

In fact the infimum is achieved by the inverse Monge–Ampère flow (1.3). The supremum is achieved by the test configurations which are defined as the blow-up of the associated multiplier ideal sheaves \( J(mU) \).

Conjecturally the right-hand side would be the maximum attained by some \( (\mathcal{X}, \mathcal{L}) \), provided we slightly stretches the meaning of test configurations. Indeed for toric Fano manifolds [67] constructed the optimal destabilizer as a possibly irrational but piecewise-linear convex function on the moment polytope. It implies that a single ideal sheaf can not generally optimize the stability indicator. Our proof shows that the weak geodesic ray attains the maximum in a suitable sense (see Remark 4.8). It might be challenging to clarify whether the ray, constructed transcendentally in the above, can be interpreted into certain algebraic singularities.

Replacing the Ricci–Calabi functional with the H-functional \( H(\omega) \) introduced by [46] (see Definition 5.1.), [37] established the parallel equality and showed that the optimal test configuration is equivalent to the (first one of) two-step degenerations constructed by [24]. In this formalism non-Archimedean D-energy is replaced with H-invariant (Definition 5.2) of the test configuration. The idea of the present paper as well applies to this setting. In the final section, we serve another simple proof of [37], Theorem 1.2, without using the deep result of [24, 25], nor the Cheeger-Colding theory about the non-collapsing of Riemannian manifolds. Moreover, the present argument enables us to handle with singular \( X \), thanks to the work of [63]. This point will be discussed in our future work.

**Theorem B** Greatest lower bound of the H-functional is given by a sequence of H-invariants:

\[
\inf_\omega H(\omega) = \sup_{(\mathcal{X}, \mathcal{L})} H(\mathcal{X}, \mathcal{L}).
\]
The infimum is achieved by the Kähler–Ricci flow. If one takes the weak geodesic \( u^t \) asymptotic to the flow, The supremum is achieved by the test configurations which is defined as the normalized blow-up of the associated multiplier ideal sheaves \( J(mU) \).

In the H-functional setting the maximum is attained by the test configuration constructed by [24]. Strictly speaking it is not a genuine test configuration but endowed with an irrational \( \mathbb{C}^* \)-action. In the terminology of [37] it is called \( \mathbb{R} \)-degeneration. Our argument does not construct the \( \mathbb{R} \)-degeneration in the limit, but it shows that the maximum is effectively approximated by the associated multiplier ideal sheaves.

It is known that \( H(X, L) > 0 \) for all \( (X, L) \) if and only if \( X \) is K-semistable so that the H-invariant is weaker than the non-Archimedean D-energy. For example in the toric case the optimal destabilizer for the H-functional gives a product family while the optimal destabilizer for the Ricci-Calabi functional has just two components in the central fiber. Construction and comparison of these two destabilizers in general situations should be investigated in the future work.

Just when the author was going to post the preprint on arXiv, he was informed the appearing work [66] of M. Xia. It solves the metrized version of Theorem A, and of Donaldson’s original conjecture for arbitrary compact Kähler manifolds, admitting finite energy geodesic rays in the supremum (so that the non-Archimedean D-energy is replaced with the radial D-energy of the geodesic ray) and singular \( \varphi \) in the infimum. Not a few ideas are in common and we even need [66], Lemma 5.1 critically in proving Theorem A. We focus on the Fano case but instead answer to the original version of the question and moreover clarify the relation with multiplier ideals.

2 Preliminary

2.1 Ricci Curvature Formulation

We first give the variational setting for the Kähler–Einstein problem. Throughout the paper \( X \) is an \( n \)-dimensional Fano manifold and \( \omega \) denotes a Kähler metric whose cohomology class is the first Chern class \( c_1(X) \). As in the introduction, we fix a reference metric \( \omega_0 \) to represent the metric by a function \( \varphi \). In words of the anticanonical line bundle \( -K_X \), one has a fiber metric \( h_0 \) with the Chern curvature \( \omega_0 \). Then \( h_0 e^{-\varphi} \) defines another smooth fiber metric so that \( \omega_\varphi = \omega_0 + dd^c \varphi \) gives the curvature. Here we define the real operator \( d^c = (4\pi \sqrt{-1})^{-1}(\partial - \bar{\partial}) \). We freely choose an appropriate description of the metric going back and forth between \( \omega, \varphi \) and the fiber metric \( h_0 e^{-\varphi} \). Let us denote by \( \mathcal{H} = \mathcal{H}(X, \omega_0) \), the collection of all smooth \( \varphi \) for which \( \omega = \omega_\varphi \) is strictly positive. The volume \( V = \int_X \omega_\varphi^n \) is independent of \( \varphi \). As in the introduction, one may take the Ricci potential satisfying

\[
\text{Ric } \omega - \omega = dd^c \rho
\]
with the normalization \( \int_X (e^\rho - 1) \omega^n = 0 \). In this notation, we first introduce the Ricci-Calabi functional in \( \varphi \in \mathcal{H} \), which is the curvature integration

\[
R(\varphi) := \frac{1}{V} \int_X (e^\rho - 1)^2 \omega^n. \tag{2.1}
\]

This gives the analogue of the classical functional

\[
C(\varphi) := \frac{1}{V} \int_X (S_\omega - \hat{S})^2 \omega^n
\]

introduced by E. Calabi. In the above \( \hat{S} \) denotes the mean value of the scalar curvature \( S_\omega \). Compared to the Calabi functional, it is relatively recent result [43] where the infinite-dimensional moment map picture for the Ricci-Calabi functional was demonstrated. This idea provides a natural prospect for the variational approach to the Kähler–Einstein problem. It is clever to consider for each direction \( \delta \varphi \) the pairing

\[
\delta \varphi \mapsto \frac{1}{V} \int_X \delta \varphi (e^\rho - 1) \omega^n,
\]

which indeed defines an exact 1-form on \( \mathcal{H} \). Now we introduce the potential called D-energy:

\[
D(\varphi) = L(\varphi) - E(\varphi) := -\log \frac{1}{V} \int_X e^{-\varphi + \rho_0} \omega_0^n - E(\varphi). \tag{2.2}
\]

The above second term

\[
E(\varphi) := \frac{1}{(n + 1)V} \sum_{i=0}^n \int_X \varphi \omega^i \wedge \omega_0^{n-i} \tag{2.3}
\]

is called the Monge–Ampère energy, because the differential is designed to be the Monge–Ampère measure: \( (dE)_\varphi = V^{-1} \omega^n = V^{-1} \omega_\varphi^n \). For the first term \( L(\varphi) \) we define the canonical probability measure

\[
\mu_\varphi := \frac{e^{-\varphi + \rho_0} \omega_0^n}{\int_X e^{\rho_0} \omega_0^n} = V^{-1} e^\rho \omega^n. \tag{2.4}
\]

In terms of these probability measures the D-energy is characterized by the property \( d_\varphi D = \mu_\varphi - \omega_\varphi^n \). The critical point condition \( d_\varphi D = 0 \) is well-known equivalent to the Kähler–Einstein equation. The functional first appeared in [2] and was written down to the above form by [38]. It precisely plays the role of Kemp-Ness functional in the finite-dimensional GIT. As we review in the next subsection, the D-energy is convex with respect to the natural metric structure of \( \mathcal{H}(X, \omega_0) \). Asking when the energy functional is proper we are naturally led to the definition of D-stability.
More recently in [22], we studied the gradient flow of the D-energy

\[ \frac{\partial}{\partial t} \varphi = 1 - e^\rho \]

and particularly proved that the long-time solution exists.

**Theorem 2.1** ([22], Theorem 1.3) Given an initial data, the inverse Monge–Ampère flow (1.3) has the unique solution \( \varphi = \varphi_t \) for all \( t \in [0, \infty) \). Moreover, \( E(\varphi_t) \) is constant, \( D(\varphi_t) \) and \( R(t) = \frac{d}{dt} D(\varphi_t) \) are non-increasing.

This is our key tool for speculating in which direction the D-energy worstly decays. In our notation the normalized Kähler–Ricci flow is written as

\[ \frac{\partial}{\partial t} \varphi = -\rho. \]

Both flows converge to the Kähler–Einstein metric if it exists. On the other hand, the two flows show different behaviors when \( \rho \) diverges. It is precisely the situation we are interested in.

### 2.2 Geodesic of Finite Energy Metrics

Monge–Ampère energy is affine and the D-energy is convex along any geodesic for Mabuchi’s \( L^2 \) structure. This fact strongly motivates us to exploit the general framework of convex optimization. We may also consider general \( L^p \) structure for the space of Kähler metrics. Especially \( p = 1 \) plays an important role in the proof of the main result. First from the \( dd^c \)-lemma any smooth function \( f \) can be seen as a tangent vector at \( \varphi \in \mathcal{H}(X, \omega_0) \). The \( L^p \)-norm

\[ \|f\|_p := \left[ \frac{1}{V} \int_X |f|^p \omega^n \right]^{\frac{1}{p}} \quad (2.5) \]

hence defines the distance \( d_p \) on \( \mathcal{H}(X, \omega_0) \). Since the metric space is not complete, even if the energy is proper the existence of a minimizer is not guaranteed. Therefore the completion \( \mathcal{E}^p = \mathcal{E}^p(X, \omega_0) \) comes to the forefront in the variational approach to the Kähler–Einstein problem. This is the main reason that we need to handle a singular fiber metric \( h_0 e^{-\varphi} \) for which \( \varphi \) is only assumed to be locally integrable and upper semicontinuous. Such an \( L^1 \)-function is called \( \omega_0 \)-plurisubharmonic function (psh for short) if the curvature current \( \omega = \omega_0 + dd^c \varphi \) is semipositive. One can see [10–12, 29–31] and the textbook [45] for the developments in this area.

Let us especially present the construction of \( \mathcal{E}^1 \) which is indeed closely related with the Monge–Ampère energy. It is well-known that we have the satisfactory definition of the product current \( \omega^p_0 \) and hence \( E(\varphi) \) for any bounded \( \omega_0 \)-psh function \( \varphi \), by the celebrated work of Bedford-Taylor [3]. To go further, for any \( \omega_0 \)-psh \( \varphi \) we define the
Monge–Ampère energy as

\[ E(\varphi) := \inf \left\{ E(\psi) : \psi \in L^\infty \cap \text{PSH}(X, \omega_0), \psi \geq \varphi \right\} \in \mathbb{R} \cup \{-\infty\}. \] (2.6)

The function is called of finite energy if \( E(\varphi) > -\infty \). We define the distance \( d_1(\varphi, \psi) \) approximating \( \varphi, \psi \in \mathcal{E}^1(X, \omega_0) \) by decreasing sequences of smooth \( \omega_0 \)-psh functions.

**Theorem 2.2** (Special case of [29], Theorem 2) The space of all finite energy \( \omega_0 \)-psh functions \( (\mathcal{E}^1(X, \omega_0), d_1) \) gives the completion of the metric space \( (\mathcal{H}(X, \omega_0), d_1) \). Moreover \( d_1 \) gives the coarsest refinement of the \( L^1 \)-topology for \( \omega_0 \)-psh functions so that the Monge–Ampère energy is continuous.

Note that [29] gave a similar construction for general \( (\mathcal{E}^p(X, \omega_0), d_p) \) with \( p \geq 1 \).

The \( D \)-energy is also continuous in this strong topology. This is a consequence of the uniform version of Skoda’s integrability theorem. See [10], Lemma 6.4.

We next review a certain construction of geodesics. Henceforth we distinguish the geodesic \( u^t \) from the inverse Monge–Ampère flow \( \varphi_t \). We also use superscript for the parameter of geodesics. The singularity of the metric again inevitably appears if one considers a geodesic. Indeed an \( L^2 \)-geodesic segment \( u^t (t \in [0, 1]) \) in \( \mathcal{E}^2(X, \omega_0) \) has at best \( C^{1,1} \)-regularity even if the endpoints are assumed to be smooth. \( L^1 \)-geodesics are not even unique, as it was observed in [30]. Given smooth endpoints there however exists a path \( u^t \) which is characterized as the solution of the degenerate Monge–Ampère equation and defines a geodesic for all \( d_p \). We follow [8] for the construction. Let \( \varphi, \psi \in \mathcal{H} \) and \( a, b \in \mathbb{R} \). Take the complex variable \( \tau \) of the annulus \( A := \{ \tau \in \mathbb{C} : e^{-b} < |\tau| < e^{-a} \} \), as the translation of the time parameter \( t = -\log |\tau| \). We denote the first projection by \( p_1 : X \times A \to X \). Let us consider a function \( V \in \text{PSH}(X \times A, p_1^* \omega_0) \) with the boundary condition \( V(x, e^{-a}) \leq \varphi(x), V(x, e^{-b}) \leq \psi(x) \) and define the Perron-Bremermann type upper-semicontinuous envelope as

\[ U(x, \tau) := \sup_V^* V(x, \tau). \] (2.7)

It is identified with the path \( u^t(x) := U(x, e^{-t}) \). The construction is also equivalent to the terminology psh geodesic in [12]. As a standard fact, we have \( U(x, e^{-a}) = \varphi(x), U(x, e^{-b}) = \psi(x) \). Since we assume \( \varphi, \psi \) bounded \( U \) is also bounded. A standard argument of the pluripotential theory deduces that the \( (n + 1) \)-variable Monge–Ampère measure \( (p_1^* \omega_0 + dd^c_{x, \tau} U)^{n+1} \) vanishes over \( X \times A \). In general a \( p_1^* \omega_0 \)-psh function \( U \) is called subgeodesic. From the computation of [61], for any subgeodesic \( E(u^t) \) is convex and \( (p_1^* \omega_0 + dd^c_{x, \tau} U)^{n+1} = 0 \) holds if and only if \( E(u^t) \) is affine. It follows that the above \( u^t \) is a weak geodesic for the \( L^2 \)-structure. Moreover, by [29], Theorem 4.17, \( u^t \) defines a geodesic in the \( L^p \)-Finsler metric space \( (\mathcal{E}^p, d_p) \) for an arbitrary \( p \geq 1 \).

It is rather recently proved by [27] that \( U \) has optimal \( C^{1,1} \)-regularity for the smooth boundary data. From [29], Remark 2.5, this geodesic of envelope form has a constant
speed in \( d_p \). It means that

\[
d_p(u^t, u^s) = d_p(u^a, u^b) \left| \frac{t - s}{b - a} \right| \tag{2.8}
\]

for all \( t, s \). Not all the geodesics in \((\mathcal{E}^1, d_1)\) enjoys the property. See also the discussion in [12, 30].

Convexity of the D-energy is very much related with Hörmander \( L^2\)-estimate for the \( \bar{\partial} \)-equation and was established by the seminal work of B. Berndtsson.

**Theorem 2.3** ([8], Theorem 1.1) *Let \( U \) be a (possibly non-smooth) function on \( X \times A \) such that \( p_1^*\omega_0 + dd^c_{x, \tau} U \geq 0 \) holds in the sense of current. Then for the associated subgeodesic \( u^t, L(u^t) \) is a convex function.*

### 2.3 Non-Archimedean Energies and Norms of the Test Configuration

The famous Hilbert-Mumford criterion in GIT tells that properness of the Kemp-Ness functional is examined in each direction of one-parameter subgroup. Once a given polarized manifold \((X, L)\) is embedded to the projective space each one-parameter subgroup of the linear transformation group induces a degeneration \((\mathcal{X}', \mathcal{L}')\) called a test configuration. It is then natural to ask the asymptotic behavior of the D-energy along the degeneration. In the scalar curvature setting for a general polarized manifold [41] first gave the intrinsic definition of a test configuration and introduced the Donaldson-Futaki invariant in relation to asymptotic behavior of the K-energy functional. The relationship between these invariants and energies for general test configurations was completed by [5, 15], and [16].

In this paper, we assume that a test configuration \((\mathcal{X}', \mathcal{L}')\) is a \( \mathbb{G}_m \)-equivariant family of \( \mathbb{Q} \)-polarized schemes, which is defined over the affine line \( \mathbb{A}^1 \). More generally we take account of the case when \( \mathcal{L} \) is relatively semiample. From the \( \mathbb{G}_m \)-equivariance the family is trivial outside of the origin and generic fibers are isomorphic to the anti-canonical polarization \((X, -K_X)\). In terms of the equivariant isomorphism \( \mathcal{X}'|_{\mathbb{A}^1 \setminus \{0\}} \simeq X \times (\mathbb{A}^1 \setminus \{0\}) \), it is convenient to represent a point of \( \mathcal{X}'|_{\mathbb{A}^1 \setminus \{0\}} \) as \((x, \tau)\), where \( x \in X \) and \( \tau \) is the affine coordinate centered on \( 0 \in \mathbb{A}^1 \). Moreover, \( \mathcal{X}' \) may be assumed to be a normal variety. See e.g. [15] for the detailed discussion for the singularities.

From the analytic point of view, each of the above degenerations can be regarded as a ray in the space of Kähler potentials.

**Definition 2.4** Let us endow \( \mathcal{L} \) with a semipositive curvature fiber metric, defined over the unit disk \( \mathbb{B} \subset \mathbb{A}^1 \). It then gives a function \( U \) on the punctured space such that the isomorphism \( \mathcal{X}'|_{\mathbb{B} \setminus \{0\}} \simeq X \times (\mathbb{B} \setminus \{0\}) \) translates the curvature form into \( p_1^*\omega_0 + dd^c_{x, \tau} U \). We define the associated ray \( u^t \) as

\[
u^t(x) := U(x, e^{-t}).
\]

Note that \( u^t \) is not necessarily contained in \( \mathcal{H} \). A ray of this form is called compatible with the test configuration.
Any two compatible rays $u^t$ and $v^t$ can be considered to share the same asymptotic behavior because of the bound $|U - V| \leq C$ which is uniform in $t$.

For the same reason, one may even consider a non-smooth but bounded $U$ for which $\omega^n = \omega''_{u^t}$ and $E(u^t)$ is properly defined as we already mentioned. In [5], inspired by [8], a weak geodesic ray $U$ associated with the test configuration was in fact constructed as the Perron-Bremermann envelope with the prescribed boundary value. Let us fix an initial point $u_0$ and take a function $V$ on $X \times (\mathbb{B} \setminus \{0\})$ for which the corresponding fiber metric is extended to a singular fiber metric of $\mathcal{L}$, so that the curvature is semipositive in the sense of current. The associated weak geodesic ray is defined as the upper-semicontinuous envelope of $V$ with the boundary condition $V(x, 1) \leq u_0(x)$, which we denote

$$U(x, \tau) := \sup^* V(x, \tau). \quad (2.9)$$

It should be compared with the construction of the weak geodesic segment (2.7). One can also see that it is equivalent to the previous construction of [26, 57], and [59].

Now we are interested in the asymptotic behavior of $E(u^t)$ and $D(u^t)$ for the associated rays. Gluing $(\mathcal{X}, \mathcal{L})$ with the trivial family we have the unique $\mathbb{C}_m$-equivariant family $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})$ defined over $\mathbb{P}^1$, so that this compactified family is isotrivial over $\mathbb{P}^1 \setminus \{0\}$ and the torus action as well is trivial in a neighborhood of $\infty \in \mathbb{P}^1$. As it was compactified one can take the self-intersection number $\tilde{\mathcal{L}}^{n+1}$ which in fact gives the non-Archimedean counterpart of the Monge–Ampère energy:

$$E^{\text{NA}}(\mathcal{X}, \mathcal{L}) := \frac{\tilde{\mathcal{L}}^{n+1}}{(n + 1)V}. \quad (2.10)$$

Non-Archimedean D-energy is described as the log-canonical threshold

$$D^{\text{NA}}(\mathcal{X}, \mathcal{L}) = L^{\text{NA}}(\mathcal{X}, \mathcal{L}) - E^{\text{NA}}(\mathcal{X}, \mathcal{L}) := \lct(\tilde{\mathcal{X}}, \mathcal{B})(\mathcal{X}_0) - 1 - \frac{\tilde{\mathcal{L}}^{n+1}}{(n + 1)V}. \quad (2.11)$$

Here the boundary divisor $\mathcal{B}$ is uniquely determined by the property $\mathcal{B} \sim_\mathbb{Q} - K_{\tilde{\mathcal{X}}/\mathbb{P}^1} - \tilde{\mathcal{L}}$ and $\text{supp} \mathcal{B} \subset \mathcal{X}_0$. For the substantial non-Archimedean treatment, we refer [14, 15, 18], and the survey article [9]. For our purpose it is sufficient to recall that it gives the slope of the Monge–Ampère energy. For the D-energy, definition of the invariant $D^{\text{NA}}(\mathcal{X}, \mathcal{L})$ and the slope formula was established by R. Berman. See also the milestone works [39, 64].

**Theorem 2.5** ([5], Theorem 3.11) Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of a Fano manifold and take a bounded fiber metric of $\mathcal{L}$, which is defined and has semipositive curvature over the unit disk. Then for the associated ray $u^t \in \mathcal{H}$ one has

$$E^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{E(u^t)}{t}, \quad D^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{D(u^t)}{t}. \quad (2.12)$$
It shows that the non-Archimedean $D$-energy for the Ricci curvature formulation is just in parallel with the Donaldson-Futaki invariant defined by [41] (equivalently, non-Archimedean $K$-energy defined by [15]) for the scalar curvature formulation. In terms of the positivity of $D_{NA}(X, L)$, one may define $D$-stability of $(X, -K_X)$. A Fano manifold $X$ is called $D$-semistable if $D_{NA}(X, L) \geq 0$ for any test configuration. It is called $D$-polystable if moreover the equality holds precisely when $(X, L)$ is a product family (with a possibly non-trivial $G_m$-action). As a result $D$-stability is equivalent to the $K$-stability. Indeed, from the work of [49], it is enough to consider so-called special test configuration in detecting the $K$-stability of a Fano manifold, and for these special test configurations $D_{NA}(X, L)$ is precisely equal to the Donaldson-Futaki invariant. Therefore we observe:

**Theorem 2.6** (A consequence of [21] and [49]) A Fano manifold admits a Kähler–Einstein metric if and only if it is $D$-polystable.

We refer to [12] for the variational approach to this problem.

In terms of the $G_m$-action, $E_{NA}(X, L)$ can be described as follows. Let us fix $k \in \mathbb{N}$ and write the weights $\lambda_1, \ldots, \lambda_{N_k}$ for the induced action on $H^0(X_0, kL_0)$. It is not so hard to see that

$$E_{NA}(X, L) = \lim_{k \to \infty} \frac{1}{kN_k} \sum_{j=0}^{N_k} \lambda_j.$$  \hfill (2.12)

In particular we observe that replacing $L$ with the line bundle $L + cX_0$ one has $E_{NA}(X, L + cX_0) = E_{NA}(X, L) + c$. Letting $\hat{\lambda} := N_k^{-1} \sum_{j=0}^{N_k} \lambda_j$ we may further define the $L^p$-norm

$$\|(X, L)\|_p := \lim_{k \to \infty} \left[ \frac{\sum_{j=0}^{N_k} \left| \lambda_j - \hat{\lambda} \right|^p}{k^p N_k} \right]^{\frac{1}{p}},$$

which is preserved by the above rescaling $L \mapsto L + cX_0$. The main result of [47] shows that these norms are equivalent to the $L^p$-norm of the associated weak geodesic ray. Notice that for the ray associated with the test configuration the best possible $C^{1,1}$-regularity was established by [28]. See also [58]. It follows that the time-derivative $\dot{u}^t$ in the below is well-defined.

**Theorem 2.7** ([47], Theorem 1.2) For the weak geodesic ray associated with a test configuration, we have

$$\|(X, L)\|_p = \left[ \frac{1}{V} \int_X \left| \dot{u}^t - E_{NA}(X, L) \right|^p \omega^n \right]^{\frac{1}{p}}.$$  \hfill (2.13)

In particular the right-hand side side is independent of $t \in [0, \infty)$.

Once the above results are accepted, the proof of the inequality (1.2) is immediate.
Corollary 2.8 ([47], Theorem 1.3) For any Kähler metric $\omega$ in $c_1(X)$ and test configuration $(\mathcal{X}, \mathcal{L})$, we have

$$\left[ \frac{1}{V} \int_X (e^\rho - 1)^2 \omega^n \right]^{\frac{1}{2}} \geq \frac{-D^{NA}(\mathcal{X}, \mathcal{L})}{\| (\mathcal{X}, \mathcal{L}) \|_2}.$$  

\textbf{Proof} By constant rescaling we may assume $E^{NA}(\mathcal{X}, \mathcal{L}) = 0$ and the geodesic convexity implies

$$-D^{NA}(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{-D(u^t)}{t} \leq -\frac{d}{dt} \bigg|_{t=0} D(u^t) = -\frac{1}{V} \int_X \dot{u}^0 (e^\rho - 1) \omega^n.$$

Now the statement is a simple consequence of the Cauchy-Schwarz inequality. \qed

### 3 Construction of a Weak Geodesic Ray and Test Configurations

#### 3.1 The Weak Geodesic Ray Asymptotic to the Flow

We start constructing a geodesic ray asymptotic to the inverse Monge–Ampère flow. In the sequel, we assume that $X$ admits no Kähler–Einstein metric, otherwise the identity of Theorem A is trivial. We denote the solution of the inverse Monge–Ampère flow (1.3) by $\varphi_t$ ($t \in [0, \infty)$) and fix any sequence $t_j \to \infty$. Although the solution of the flow is smooth, to consider geodesics we need the space $\mathcal{E}^1$ of finite energy metrics.

First we will take a geodesic segment $u^t_j \in \mathcal{E}^1$ ($0 \leq t \leq t_j$), which joins $\varphi_0$ to $\varphi_{t_j}$. Our normalization of the Ricci potential yields that $E(\varphi_t)$ is constant in $t$. For the supremum, we have

\textbf{Lemma 3.1} (Lemma 4.1 of [22]) \textit{The flow is linearly bounded from above:}

$$\varphi_t \leq t + A.$$  

It follows that for any fixed $T$ Aubin’s J-functional

$$J(\varphi_t) := \frac{1}{V} \int_X \varphi_t \omega^n_0 - E(\varphi_t)$$

is bounded in $t \in [0, T]$. Notice that $J(\varphi_t)$ is not bounded in $t \in [0, \infty)$, otherwise the flow converges to a weak minimizer of D-energy in $\mathcal{E}^1$, namely a Kähler–Einstein metric. It achieves equality in (1.2). In other words, we have $\sup_X \varphi_{t_j} \to +\infty$. Now by [31], Corollary 4.14, $d_1$ is explicitly described in terms of $E$ as

$$d_1(\varphi, \psi) = E(\varphi) + E(\psi) - 2E(P(\varphi, \psi)).$$
where $P(\varphi, \psi) \in \mathcal{E}^1$ is the upper envelope of $\omega_0$-functions $u$ such that $u \leq \varphi, \psi$. The formula implies that

$$\sup_X \varphi t_j - C \leq d_1(\varphi_0, \varphi t_j) \leq \sup_X \varphi t_j + C.$$ 

This is comparable with [32], Theorem 1 for the Kähler–Ricci trajectory.

Following the argument of [32] we will show that, taking a subsequence if necessary, the particular choice (2.7) of geodesics $u^j_t$ converges to a ray $u^t$ in $(\mathcal{E}^1, d_1)$.

The convergence argument focuses on the relative entropy

$$\text{Ent}(\nu | \mu) := \int_X \log \left( \frac{\nu}{\mu} \right) \nu$$

of two given probabilistic measures $\nu$ and $\mu$, due to the fundamental compactness result for the subsets with bounded entropy. We set $\text{Ent}(\nu | \mu) := \infty$ if $\nu$ is not absolutely continuous with respect to $\mu$.

**Theorem 3.2 ([11])** The subset

$$\left\{ \varphi \in \mathcal{E}^1 : \text{Ent}(V^{-1} \omega^n_\varphi | V^{-1} \omega^n_0) \leq C, \quad \sup_X \varphi = 0 \right\},$$

is compact in the $d_1$-topology.

The role of the entropy in the Kähler–Einstein problem was observed by the thermodynamical formalism of [4]. Firstly, D-energy is rather related with the entropy for the canonical probability measure $\nu = \mu_\varphi$, by the following fact.

**Proposition 3.3** The relative entropy for the probabilistic measures $\nu$ and $\mu$ can be described as the Legendre dual:

$$\text{Ent}(\nu | \mu) = \sup_{f \in C_0(X; \mathbb{R})} \left[ \int_X f \nu - \log \int_X e^f \mu \right].$$

The one-side inequality $\geq$ is obvious from Jensen’s inequality and it is actually true for arbitrary lower-semicontinuous function $f$. This point will be discussed more in section 5.

The entropy of the Monge–Ampère measure which we want to control forms the main term of Mabuchi’s K-energy functional. Indeed using the integration by parts formula of [20], we may define the K-energy by

$$M(\varphi) = \text{Ent}(V^{-1} \omega^n_\varphi | V^{-1} \omega^n_0) + \frac{1}{V} \int_X \varphi \omega^n_\varphi - E(\varphi). \quad (3.1)$$

Note that the second term is controlled by $E$, since for general non-positive $\omega_0$-psh $\varphi$ we have

$$(n + 1)E(\varphi) \leq \frac{1}{V} \int_X \varphi \omega^n_\varphi \leq E(\varphi).$$
It is immediate that the D-energy is non-increasing along the inverse Monge–Ampère flow. In addition, Monge–Ampère energy is conserved, from the normalization of $\rho$. Fortunately we have the monotonicity of the K-energy as well.

**Lemma 3.4** ([22], Lemma 4.6) *Along the the inverse Monge–Ampère flow, it holds*

$$\frac{d}{dt} M(\varphi) = -\frac{1}{V} \int_X |\nabla \rho|^2 e^{\rho} \omega^n_\varphi.$$ 

We deduce from the fact that for any fixed $T$ the entropy $\text{Ent}(V^{-1} \omega^n_\varphi \mid V^{-1} \omega^n_0)$ is bounded in $t \in [0, T]$.

From the compactness and (2.8) Ascoli’s theorem implies that (after passing to a subsequence) $u^t_j$ converges to a ray $u^t$ in $(E^1, d_1)$. Moreover, for any fixed $T$, the convergent is uniform in $t \in [0, T]$. Since $t_j \to \infty$, $u^t_j$ is defined for $t \in [0, \infty)$. In fact by [12], Theorem 1.7, $u^t$ restricted to any interval $[a, b]$ is of envelope form (2.7). Consequently the limit ray inherits the constant speed property:

$$d_1(u^t, u^s) = d_1(u^0, u^1) |t - s|.$$ \hspace{1cm} (3.2)

From the normalization we obtain

$$\sup_X u^t \leq t + A$$ \hspace{1cm} (3.3)

for any $t \in [0, \infty)$. On the other hand

$$\lim_{t \to \infty} \frac{E(u^t)}{t} = 0.$$ \hspace{1cm} (3.4)

For $p > 1$, it is not clear from the construction whether $u^t$ is asymptotic to the flow, in the sense of [32]. In general a ray $u^t$ is asymptotic to the curve $\varphi_t$, if there exists $t_j \to \infty$ and constant speed geodesic segments $u^t_j \in [0, t_j]$ connecting $\varphi_0$ and $\varphi_{t_j}$ such that for all $t$

$$\lim_{j \to \infty} d_p(u^t_j, u^t) = 0.$$ 

For the Kähler–Ricci flow [32] derived the property from the Harnack estimate which is not established for the inverse Monge–Ampère flow. At present setting, we will settle for the restrictive estimate:

**Proposition 3.5** ([66], Lemma 5.1) *For each $t$ we have $u^t \in E^2$ and*

$$d_2(u^0, u^t) \leq \liminf_{j \to \infty} d_2(u^0, u^t_j).$$

**Proof** We sketch the proof. It fully exploits the CAT(0)-property of $E^2$, which cannot be expected for the other complete length spaces $E^p$. In particular, we may generalize
the notion of weak convergence in a Hilbert space to any complete CAT(0)-space. See [1] for the general exposition.

Let us fix any \( t \). A standard argument of pluripotential theory shows that the non-increasing sequence

\[ v_j^t := \sup_{k \geq j} u_k^t \]

converges almost everywhere to \( u^t \). Since \( u_j^t \) \((j = 1, 2, \ldots)\) are bounded in \( E^2 \), one can prove that so are \( v_j^t \). The boundedness with monotonicity implies \( u^t \in E^2 \), by [29], Lemma 4.16. Now [13], Theorem 5.3 asserts that \( u_j^t \) weakly converges to \( u^t \).

The point here is that the \( d_1 \)-ball \( B_{\varepsilon}(\varphi) := \{ \psi \in E^2 : d_1(\varphi, \psi) < \varepsilon \} \)

is \( d_2 \)-closed and \( d_2 \)-convex. It follows that for any weakly convergent subsequence \( u_{jk}^t \rightarrow u^t \) \((k = 1, 2, \ldots)\) we have \( u^t = u^t \). (From the \( E^2 \)-boundedness we have at least one weakly convergent subsequence, by [1], Proposition 3.1.2.) Indeed \( u_{jk}^t \in B_{\varepsilon}(u^t) \) for any sufficiently large \( k \). Since \( B_{\varepsilon}(u^t) \) is \( d_2 \)-closed and \( d_2 \)-convex, we conclude \( u^t \in B_{\varepsilon}(u^t) \) by [1], Lemma 3.2.1. The desired inequality follows from the fact that the distance function is lower-semicontinuous with respect to the weak convergence (e.g. [1], Corollary 3.2.4).

\[ \square \]

In case \( u^t \in E^p \), from [33], Theorem 1.2, we see that \( u^t \) is a geodesic for any \((E^p, d_p)\). Moreover, each segment defines a unique geodesic ray for \( p > 1 \). In particular, it then has the constant speed for \( d_p \). Such \( u^t \) is distinguished as finite energy geodesic in [33] and studied in view of geodesic stability.

Summarizing up we obtain:

**Theorem 3.6** Let \( \varphi_t \) be the inverse Monge–Ampère flow and \( u_j^t \) \((t \in [0, t_j])\) be the weak geodesic ray of the envelope form (2.7) so as to connect \( \varphi_0 \) to \( \varphi_{t_j} \). Then there exists a ray \( u^t \) of envelope form such that \( \lim_{j \rightarrow \infty} d_1(u_j^t, u^t) = 0 \) for each \( t \). As a result \( u^t \) is a geodesic for all \((E^p(X, \omega_0), d_p)\), \( p \in [1, 2] \) and satisfies (3.2), (3.3) and (3.4).

**Remark 3.7** Provided the Harnack-type estimate for the inverse Monge–Ampère flow was established we may apply [32], Theorem 3.2 and obtain the ray directly. Such an estimate is highly non-trivial, as it implies the linear lower bound of \( \varphi_t \), or equivalently, the upper bound of the Ricci potential \( \rho \).

### 3.2 Approximative Test Configurations

Next we follow [12] to construct a canonical sequence of test configurations which approaches \( u^t \). It can be seen as the non-Archimedean analogue of Demailly’s approximation [36] for a psh function.
Changing variables as

$$U(x, e^{-t+\sqrt{-1}\theta}) := u'(x), \quad \hat{U} := U + \log |\tau|$$

we obtain an $S^1$-invariant function $U$ on $X \times (\mathbb{B} \setminus \{0\})$, which is actually $p_1^*\omega_0$-psh. From (3.3) $\hat{U}$ is uniquely extended to a $p_1^*\omega_0$-psh function on $X \times \mathbb{B}$. Since $u' \in \mathcal{E}^1$, each $u'$ has vanishing Lelong numbers. Applying the Ohsawa-Takegoshi extension theorem [52] we may conclude that $U$ has vanishing Lelong numbers in $X \times \{0\}$. Therefore, cosupport of the $S^1$-invariant multiplier ideal sheaf $\mathcal{J}(m\hat{U})$ is properly contained in $X \times \{0\}$, so that we have the normalized blow-up $\rho_m : X_m \to X \times \mathbb{C}$. It would be remarkable that the argument really requires the definition of multiplier ideal sheaves for general plurisubharmonic functions, since $U$ may have non-algebraic singularities. Let $E_m$ be the exceptional divisor of $\rho_m$. We fix some $m_0 \in \mathbb{N}$ and set the line bundle as

$$L_m := \rho_m^*p_1^*(-K_X) - \frac{1}{m+m_0}E_m + \frac{m}{m+m_0}\rho_m^*\mathcal{X}_{m,0}. \quad (3.5)$$

The number $m_0$ is chosen so that $\mathcal{O}(-(m+m_0)p_1^*K_X) \otimes \mathcal{J}(m\hat{U})$ is globally generated for all $m \geq 1$. See [12], Lemma 5.6. The term involving the central fiber $\mathcal{X}_{m,0}$ preserves the linear equivalence of $L_m$ and only adjusts the $\mathbb{G}_m$-action. The constructed semiample test configuration $(X_m, L_m)$ satisfies the following continuity property, which is crucial for the variational approach in [12].

**Theorem 3.8** ([12], Theorem 5.4, Lemmas 5.7 and 5.8) For the above constructed weak geodesic ray and test configurations the upper-semicontinuity

$$\limsup_{m \to \infty} D^{NA}(X_m, L_m) \leq \lim_{t \to \infty} \frac{D(u')}{t}$$

holds. Moreover, if $u'$ is maximal in the sense of [12], Definition 6.5, we have the continuity

$$\lim_{m \to \infty} D^{NA}(X_m, L_m) = \lim_{t \to \infty} \frac{D(u')}{t}.$$

The result tells that unlike the original version of Demailly’s approximation, for a general weak geodesic ray the multiplier ideal sheaf construction can not reach $u'$ in the limit. Since our setting looks slightly different from [12], let us repeat this part of the proof. A similar idea will appear when we compare the $L^2$-norms for several rays in the last part of the proof of Theorem A. We take an $S^1$-invariant, non-negatively curved smooth (or more generally bounded) fiber metric of the $\mathbb{Q}$-line bundle $L_m$ on $X \times \mathbb{B}$. It defines a $p_1^*\omega_0$-psh function $U_m$ endowed with the analytic singularity of $\mathcal{J}(mU)^{\frac{1}{m+m_0}}$. This is the reason why we adjusted the line bundle by $\mathcal{X}_0$, in (3.5). Using
Demailly’s approximation theorem locally, we have the estimate

\[ U_m \geq U - C_{m,r} \]

on the shrunken area \( B(0, r) \times X \). The positive constants \( C \) and \( r \) are independent of \( m \). Since the Monge–Ampère energy is non-decreasing, it follows

\[
E^{\text{NA}}(X_m, L_m) = \lim_{t \to \infty} \frac{E(u'_m t)}{t} \geq \lim_{t \to \infty} \frac{E(u' - C_{m,r})}{t} = \lim_{t \to \infty} \frac{E(u')}{t} = 0. \quad (3.6)
\]

The key point in the above is the Ohsawa-Takegoshi \( L^2 \)-extension theorem [52] used in Demailly’s approximation. Notice that such a uniform lower bound estimate of the Bergman kernel already forms a basis of the celebrated work [21] (see also [65]).

**Remark 3.9** We may ask whether the constructed weak geodesic ray asymptotic to the inverse Monge–Ampère flow is maximal in the sense of [12]. For the proof of Theorem A, however, we do not require the maximality.

### 4 Proof of the Moment-Weight Equality

#### 4.1 Test Configurations Almost Destabilize \( X \)

We devote the present section to the proof of Theorem A. Let us first check that the weak geodesic ray constructed in the previous section actually destabilizes \( X \). The inverse Monge–Ampère flow satisfies

\[
\frac{d}{dt} D(\varphi_t) = -\frac{1}{V} \int_X (e^\rho - 1)^2 \omega^n_{\varphi_t} = R(\varphi_t)
\]

and \( R(\varphi_t) \) is non-increasing, as a property of the gradient flow. In particular \( \frac{d}{dt} D(\varphi_t) \leq 0 \) and the convexity assures that \( \lim_{t \to \infty} \frac{D(\varphi_t)}{t} \in [-\infty, 0] \) exists. It then follows:

\[
\lim_{t \to \infty} \frac{D(\varphi_t)}{t} = \lim_{j \to \infty} \frac{D(\varphi_{tj})}{t_j} = \lim_{j \to \infty} \frac{D(u'_{tj})}{t_j}.
\]

Since D-energy is convex along any geodesic, for any fixed \( T \) we have

\[
\lim_{j \to \infty} \frac{D(u'_{tj})}{t_j} \geq \frac{D(u'_T)}{T}.
\]

The convergence of \( u'_{tj} \) to \( u' \) in \( (E^1, d_1) \) then yields

\[
\lim_{t \to \infty} \frac{D(\varphi_t)}{t} \geq \frac{D(u'_T)}{T}.
\]
Letting $T \to \infty$, Theorem 3.8 now implies

**Proposition 4.1** Let $\varphi_t$ be the inverse Monge–Ampère flow and $u^t$ be a weak geodesic ray asymptotic to the flow. For the test configurations which canonically approximates $u^t$ we have

$$0 \geq \lim_{t \to \infty} \frac{D(\varphi_t)}{t} \geq \limsup_{m \to \infty} D^{NA}(\mathcal{X}_m, \mathcal{L}_m).$$

The proposition already shows that $(\mathcal{X}_m, \mathcal{L}_m)$ almost destabilizes $X$. To get a more precise upper bound of $D^{NA}(\mathcal{X}_m, \mathcal{L}_m)$, we prepare to compute the differential of the energy along the flow.

**Lemma 4.2** Along the inverse Monge–Ampère flow we have

$$-\frac{d}{dt} D(\varphi_t) = -\frac{1}{V} \int_X \dot{\varphi}_t (e^{\rho_t} - 1) \omega^n_{\varphi_t} = \left[ \frac{1}{V} \int_X (\dot{\varphi}_t)^2 \omega^n_{\varphi_t} \right]^{1/2} \left[ \frac{1}{V} \int_X (e^{\rho_t} - 1)^2 \omega^n_{\varphi_t} \right]^{1/2}.$$

Proof is immediate. Indeed, from the very definition of the inverse Monge–Ampère flow, the equality holds in the Cauchy-Schwarz inequality.

**Remark 4.3** It is natural to expect the optimal destabilizer for general $L^p$-norm. See also [33], Theorem 1.6. In our argument, however, Lemma 4.2 apparently requires $L^2$-norm. In addition, the proof of Proposition 3.5 relies on the CAT(0)-property of $d_2$. For a Fano manifold without non-zero holomorphic vector fields, the existence of the Kähler-Einstein metric is equivalent to the uniform stability with respect to the $L^1$-norm, as a result of [12]. Note that the existence of an $L^p$-destabilizer does not contradict this fact.

**4.2 Comparison of the Norms**

In regard with Lemma 4.2 we thus finally should study the $L^2$-norm

$$\|\dot{\varphi}_t\|_2 := \left[ \frac{1}{V} \int_X (\dot{\varphi}_t)^2 \omega^n_{\varphi_t} \right]^{1/2}.$$

For this purpose, we need to put together the latest results of pluripotential theory. For the inverse Monge–Ampère flow we have $\dot{\varphi}_t = 1 - e^{\rho}$ so that $\|\dot{\varphi}_t\|_2 = R(\varphi_t)^{1/2}$ is non-increasing. Note that the weak geodesic ray $u^t \in \mathcal{E}^2$ is possibly apart from any test configurations and it might be not even differentiable. For this reason we make use of the choice of $u^t$ (asymptotic to the flow) and consider the “norm at infinity” as follows.
Definition 4.4 For a weak geodesic ray \( u^t \in \mathcal{E}^2(X, \omega_0) \) with constant speed, we define the \( L^2 \)-norm as
\[
\| u^t \|_2 := \lim_{t \to \infty} \frac{d_2(u^0, u^t)}{t} = \frac{d_2(u^0, u^t)}{t}.
\]
For a differentiable \( u^t \) we notice \( \| u^t \|_2 = \| \dot{u}^t \|_2 \). Observe that \( \| u^t \|_2 \) is independent of \( t \) and the initial metric \( u^0 \). Let us now take \( u^t \) as in Section 3.2.

Lemma 4.5 In the case \( u^t \) is the geodesic ray asymptotic to the flow \( \varphi_t \) we have
\[
\| \dot{\varphi}_t \|_2 \geq \| u^t \|_2.
\]

Proof If \( \| \dot{\varphi}_t \|_2 < \| u^t \|_2 \) for some \( t \), the above monotonicity implies that \( \| \dot{\varphi}_t \|_2 < \| u^t \|_2 \) holds for any sufficiently large \( t \geq T \). Proposition 3.5 implies that the right hand side is bounded from above as
\[
\| \dot{u}^t \|_2 = \lim_{j \to \infty} \frac{d_2(u^0, u^t)}{t} = \lim_{j \to \infty} \| \dot{u}^t_j \|_2.
\]
They are all independent of \( t \). Therefore we may take \( \varepsilon > 0 \) such that \( \| \dot{\varphi}_t \|_2 + \varepsilon < \| \dot{u}^t \|_2 \) for all \( j \) and \( t \geq T \). It implies \( d_2(\varphi_0, \varphi_t) < d_2(u^0_j, u^t_j) \) for a sufficiently large \( j \). On the other hand, the \( L^2 \)-geodesic connecting two metrics is unique by [31], Lemma 6.12, so that it has minimal length among all paths. It contradicts to our choice of \( u^t_j \) which is \( L^p \)-geodesic for any \( p \geq 1 \). We conclude \( \| \dot{\varphi}_t \|_2 \geq \| u^t \|_2 \). \( \square \)

Now we take a \( p^*_1 \omega_0 \)-psh function \( U_m \) as the weak geodesic ray associated with \((X_m, \mathcal{L}_m)\), and compare \( \| u^t \|_2 \) with \( \| u^t_m \|_2 \). Recall that the weak geodesic ray associated with the test configuration has \( C^{1,1} \)-regularity by [28, 58]. It implies that the norm \( \| \dot{u}^t \|_2 \) is well-defined and coincide with \( \| u^t \|_2 \) for each \( t \). Let us invoke the following Lidskii type inequality.

Theorem 4.6 ([34], Theorem 5.1) For any \( u, v, w \in \mathcal{E}^p(X, \omega_0) \) with \( u \geq v \geq w \) we have
\[
d_p(v, w)^p \leq d_p(u, w)^p - d_p(u, v)^p.
\]

Lemma 4.7 For the associated weak geodesic rays \( u^t_m \) we have
\[
\| u^t \|_2 \geq \| u^t_m \|_2 = \| \dot{u}^t_m \|_2.
\]

Proof Since \( U_m \) comes from a bounded fiber metric of \( \mathcal{L}_m \), it encodes the analytic singularity \( \mathcal{J}(mU)^{1/m+m_0} \). Again using Demailly’s approximation theorem locally, we have \( U_m \geq U - C_{m,r} \). Since \( u^t_m \) is bounded from above and \( u^0 \) is smooth there exists
a constant $B_m$ such that $u^0 + B_m \geq u^0_m$. We are ready to apply Lidskii type inequality: Theorem 4.6 to these functions and get

$$d_2(u^0 + B_m + C_{m,r}, u^0_m + C_{m,r}) \leq d_2(u^0 + B_m + C_{m,r}, u^0).$$

It follows from the triangle inequality that

$$\|u\|_2 = \lim_{t \to \infty} \frac{d_2(u^0, u^0)}{t} \geq \lim_{t \to \infty} \frac{d_2(u^0, u^0_m)}{t} = \|\dot{u}_m^0\|_2.$$  \hfill (4.1)

Combining the results all together, we obtain

$$\liminf_{m \to \infty} -D^{NA}(X_m, L_m) \geq \lim_{t \to \infty} -\frac{D(\psi_t)}{t}$$

$$= \lim_{t \to \infty} \|\dot{\psi}_t\|_2 \left[ \frac{1}{V} \int_X (e^{\rho_t} - 1)^2 \omega_{\psi_t} \right]^{1/2}$$

$$\geq \|\dot{u}_m^0\|_2 \lim_{t \to \infty} \left[ \frac{1}{V} \int_X (e^{\rho_t} - 1)^2 \omega_{\psi_t} \right]^{1/2}$$

for all $m$. For a while we denote $\varepsilon_m := E^{NA}(X_m, L_m)$ which is nonnegative from (3.6). Recall that the norm $\|(X_m, L_m)\|_2 = \|\dot{u}_m^0 - \varepsilon_m\|_2$ slightly differs from the above $\|\dot{u}_m^0\|_2$, however, we observe from $\varepsilon_m = V^{-1} \int_X \dot{u}_m^0 \omega_0^n$ that

$$\|\dot{u}_m^0 - \varepsilon_m\|_2^2 = \|\dot{u}_m^0\|_2^2 - \varepsilon_m^2 \leq \|\dot{u}_m^0\|_2^2$$

and hence conclude

$$\liminf_{m \to \infty} \frac{-D^{NA}(X_m, L_m)}{\|(X_m, L_m)\|_2} \geq \lim_{t \to \infty} \left[ \frac{1}{V} \int_X (e^{\rho_t} - 1)^2 \omega_{\psi_t} \right]^{1/2}.$$  \hfill The last inequality completes the proof of Theorem A.

Remark 4.8 The above proof of Theorem A shows that

$$\inf_{\omega} \left[ \frac{1}{V} \int_X (e^\rho - 1)^2 \omega^n \right]^{1/2} \leq \frac{1}{\|u\|_2} \lim_{t \to \infty} \frac{-D(u^0)}{t}$$

holds for a weak geodesic ray asymptotic to the inverse Monge–Ampère flow. If the geodesic $u^t$ is maximal in the sense of [12], the radial D-energy $\lim_{t \to \infty} t^{-1} D(u^t)$ equals to the non-Archimedean D-energy of the non-Archimedean potential $t^{\mathcal{NA}}$.

As a consequence of [50], the lower bound of the Calabi functional is zero if and only if $X$ is $D$-semistable (see also [12]). We may restate the result in terms of the inverse Monge–Ampère flow.
**Corollary 4.9** For any Fano manifold $X$ either there exists a Kähler metric with arbitrary small Ricci potential or a weak geodesic ray $u^t$ asymptotic to the inverse Monge–Ampère flow has negative slope:

$$\lim_{t \to \infty} \frac{D(u^t)}{t} < 0.$$ 

In particular, there exists a test configuration $(\mathcal{X}, \mathcal{L})$ with $D^{NA}(\mathcal{X}, \mathcal{L}) < 0$.

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**5 The Kähler–Ricci Flow Case**

**5.1 H-Functional and H-Invariant**

Let us recall that each metric $\varphi \in \mathcal{H}(X, \omega_0)$ defines the canonical probability measure

$$\mu_\varphi := \frac{e^{-\varphi + \rho_0} \omega_0^n}{\int_X e^{-\rho_0} \omega_0^n}. \quad (5.1)$$

In terms of the probability measure the $D$-energy is characterized by the property

$$d_\varphi D = \mu_\varphi - \omega_\varphi^n.$$

**Definition 5.1** The $H$-functional of [46] is described as the relative entropy functional:

$$H(\omega_\varphi) := \text{Ent}(\mu_\varphi | V^{-1} \omega_\varphi^n).$$

See Sect. 3 and especially Proposition 3.3 for our convention about the relative entropy. The functional first appeared in [40] and has played an important role in the study of Kähler-Ricci flow. As a consequence of Pinsker’s inequality it is at least bounded from below by the $L^1$-version of the Ricci-Calabi functional:

$$\sqrt{2H(\omega)} \geq \frac{1}{V} \int_X |e^\rho - 1| \omega^n. \quad (5.2)$$

Following [37], let us introduce the algebraic $H$-invariant of a test configuration.

**Definition 5.2** Let $(\mathcal{X}, \mathcal{L})$ be a test configuration. The H-invariant is defined as

$$H(\mathcal{X}, \mathcal{L}) = -L^{NA}(\mathcal{X}, \mathcal{L}) + F(\mathcal{X}, \mathcal{L})$$

$$:= -L^{NA}(\mathcal{X}, \mathcal{L}) + \lim_{k \to \infty} \left[ - \log \frac{1}{N_k} \sum_{j=1}^{N_k} e^{-\lambda_j} \right],$$

where $\lambda_1, \ldots, \lambda_{N_k}$ is the weights of the induced $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, k\mathcal{L}_0)$. A Fano manifold is called H-stable if $H(\mathcal{X}, \mathcal{L}) < 0$ for all non-trivial test configurations.
Comparing with weight description of the non-Archimedean Monge–Ampère energy (2.12), we observe

\[- H(\mathcal{X}, \mathcal{L}) \geq D^{\text{NA}}(\mathcal{X}, \mathcal{L}). \tag{5.3}\]

This refines [37], Lemma 2.5. It follows that D-semistability implies H-stability. Conversely, if \( X \) is H-stable Theorem B deduces \( \inf_\omega H(\omega) = 0 \) so that \( X \) is almost Kähler-Einstein and hence D-semistable. Unfortunately, the second term \( F \) is in nature more transcendental than \( E^{\text{NA}} \). It does not simply correspond to the classical energy of metrics. At least for the associated weak geodesic ray, one may observe that the limit of the “virtual slope”

\[ F(\dot{u}^t) := - \log \frac{1}{V} \int_X e^{-\dot{u}^t} \omega^n \]

gives \( F(\mathcal{X}, \mathcal{L}) \), due to the following result.

**Theorem 5.3** ([47]) Let \((\mathcal{X}, \mathcal{L})\) be a test configuration and \(u^t\) the associated \(C^{1,1}\)-weak geodesic ray. Then the pushed-forward probability measure

\[ \text{DH}(\mathcal{X}, \mathcal{L}) := \dot{u}^t_\ast (V^{-1} \omega^n_{u^t}) \]

is independent of the initial data \(u^0\) and the parameter \(t\). Moreover, we have the weak convergence of the spectral measure:

\[ \frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{x_j^t} \rightrightarrows \text{DH}(\mathcal{X}, \mathcal{L}). \]

Bearing in mind of [5], Theorem 3.11 we then obtain the slope formula

\[ H(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \left[ - \frac{L(u^t)}{t} + F(\dot{u}^t) \right]. \tag{5.4}\]

Lower bound of the H-functional is achieved by the supremum of these (unnormalized) H-invariant.

**Theorem 5.4** ([37], Theorem 1.2) For a Fano manifold we have

\[ \inf_\omega H(\omega) = \sup_{(\mathcal{X}, \mathcal{L})} H(\mathcal{X}, \mathcal{L}). \]

The one-side inequality is easier to see from (5.4). Indeed if we take \( f := -\dot{u}^0 \) in Proposition 3.3 the associated weak geodesic ray satisfies

\[ H(\omega_{u^0}) \geq - \int_X \dot{u}^0 \mu_{u^0} - \log \frac{1}{V} \int_X e^{-\dot{u}^0} \omega^n_{u^0} \]
\[
\geq - \frac{d}{dt} L(u') + F(\dot{u}')
\]
for any choice of the initial metric \( u^0 \).

The quantity \( \inf_\omega H(\omega) \) is equivalent to the supremum of Perelman’s \( \mu \)-entropy. For a smooth function \( f \) satisfying

\[
\int_X e^{-f} \omega^n = V,
\]
we define the W-functional as

\[
W(\omega, f) := \int_X (S_\omega + |\nabla f|^2 + f) e^{-f} \omega^n.
\]

Perelman’s \( \mu \)-entropy is then defined to be the infimum:

\[
\mu(\omega) := \inf_f W(\omega, f) \leq nV.
\]

**Theorem 5.5** ([37], Theorem 4.2)

\[
\sup_\omega \mu(\omega) = nV - \inf_\omega H(\omega).
\]

We also remark the relation with the greatest lower bound of the Ricci curvature

\[
R(X) := \sup \left\{ r \in [0, 1] : \text{Ric } \omega \geq r \omega \right\}.
\]

It was shown in [12] and [23] that \( R(X) = \min\{\delta_X, 1\} \), where \( \delta_X \) is the \( \delta \)-invariant of Fujita-Odaka inspired by [6]. See [44] for the definition and [18] for the non-Archimedean interpretation. In particular, it follows that \( R(X) = 1 \) iff \( X \) is D-semistable. The resent work [68] characterizes \( \delta_X \) in terms of the Moser-Trudinger type inequality for the D-energy. Using this characterization the same author greatly simplified the variational approach of [12].

**Proposition 5.6** If \( R(X) > 1/4\pi \), we have \( nVR(X) \leq \sup_\omega \mu(\omega) \leq nV \) and so that

\[
\inf_\omega H(\omega) \leq nV(1 - R(X)).
\]

**Proof** Let us take any \( r < R(X) \) close to \( R(X) \) and \( \omega \) such that \( \text{Ric } \omega \geq r \omega \). Changing variables as \( u^2 = e^{-f} \) and applying the log-Sobolev inequality, we have

\[
W(\omega, f) = \int_X (S_\omega u^2 + 4 |\nabla u|^2 - u^2 \log u^2) \omega^n
\]
\[ \geq \int_X S_\omega u^2 \omega^n + (4\pi r - 1) \int_X (u^2 \log u^2) \omega^n. \]

Since \( S_\omega \geq nr \) and \( r \geq 1/4\pi \) it yields \( \mu(\omega) \geq nVR \) and hence \( \sup_\omega \mu(\omega) \geq nV R(X) \). The last claim follows from Theorem 5.5. \( \square \)

The infimum of the H-functional seems not equivalent to \( 1 - R(X) \). One may in fact replace the Ric(\( \omega \)) with Bakry-Émery Ricci curvature so that obtain a more precise estimate in the above.

As a consequence of [48] (see also [17]), for \( n \)-dimensional Fano manifolds \( R(X) \) are uniformly bounded from below by a positive constant. The author does not know an example of Fano manifolds with \( R(X) \leq 1/4\pi \).

5.2 Weak Geodesic Ray Asymptotic to the Flow

Let us explain how ideas in the previous sections reprove Theorem 5.4. It is natural to take the normalized Kähler-Ricci flow

\[ \frac{\partial}{\partial t} \omega = - \text{Ric} \omega + \omega \]

in place of the inverse Monge–Ampère flow. We again distinguish the flow \( \varphi_t \) from the geodesic \( u^t \) using the subscript. In terms of the normalized Ricci potential this can be described as

\[ \frac{\partial}{\partial t} \varphi = -\rho. \quad (5.5) \]

In fact the equation (5.5) incorporates the slope into the H-functional in the form

\[ H(\omega_{\varphi_t}) = - \frac{d}{dt} L(\varphi_t) + F(\dot{\varphi}_t). \]

As it is shown in [53, 56], \( H(\omega_{\varphi_t}) \) is non-increasing. Notice that in [32] another normalization of the Kähler potential

\[ r_t := \varphi_t - E(\varphi_t) \]

is adopted. Our choice of \( \varphi_t \) is precisely equal to \( \tilde{r}_t \) in their notation. By Perelman’s uniform estimate for the Ricci potential we have \( \sup_X \varphi_t \leq ct + A \). See [54, 62] for the expoundation. In particular, Harnack type estimate for the Kähler-Ricci flow implies that the finite slope \( \lim_{t \to \infty} t^{-1} E(\varphi_t) \) exists. For the Monge–Ampère energy, \( E(\varphi_t) \) is non-decreasing from Jensen’s inequality. For the D-energy we obtain

\[ \frac{d}{dt} D(\varphi_t) = -\frac{1}{V} \int_X \rho(\rho e^{\rho} - 1) \omega^n_{\varphi_t} = -H(\omega_{\varphi_t}) + \frac{1}{V} \int_X \rho \omega^n_{\varphi_t}. \]
Jensen’s inequality shows
\[
\frac{1}{V} \int_X \rho \omega^n \leq \log \frac{1}{V} \int_X e^\rho \omega^n = 0
\]
so that \(D(\varphi_t)\) is non-increasing. Consequently, we may repeat the argument in Subsection 3.3 to deduce the following.

**Theorem 5.7** (Renormalization of [32], Theorem 2) Let \(\varphi_t\) be the normalized Kähler-Ricci flow and \(u^t_j\) (\(t \in [0, t_j]\)) be the weak geodesic ray of the envelope form (2.7) so as to connect \(\varphi_0\) with \(\varphi_{t_j}\). Then there exists a ray \(u^t\) such that for each \(t\) \(u^t\) is a bounded \(\omega_0\)-psh function and \(\lim_{j \to \infty} d_p(u^t_j, u^t) = 0\). As a result \(u^t\) is a constant speed geodesic for all \((E^p(X, \omega_0), d_p)\). It satisfies \(\sup_X u^t \leq ct + A\) and \(E(u^t)\) is constant.

**Proof** Theorem was originally stated for \(\tilde{\varphi}_t = \varphi_t - E(\varphi_t)\) but it is easy to modify the proof in this setting. We sketch the proof for the reader’s convenience. See [32] for the detail. Let \(\tilde{u}^t_j\) be the weak geodesic connecting \(\tilde{\varphi}_0\) with \(\tilde{\varphi}_{t_j}\). From the main theorem of [7], the K-energy functional is convex along any weak geodesic. On the other hand it is well-known that the K-energy is non-increasing along the Kähler-Ricci flow. Using Chen’s formula (3.1) we may deduce from these two facts that the entropy part is bounded in each interval \([0, t_j]\). Theorem (3.2) assures the existence of the limit \(\tilde{u}^t\) of a certain subsequence of \(\tilde{u}^t_j\), such that \(\lim_{k \to \infty} d_1(\tilde{u}^t_{jk}, \tilde{u}^t) = 0\). Moreover, we may derive \(C^0\)-bound of \(\tilde{\varphi}_t\) from \(d_p\)-bound so that obtain \(\left\| \tilde{u}^t_j \right\|_{L^\infty} \leq Ct\). It implies that \(\tilde{u}^t\) is bounded for each \(t\). Notice that the distance \(d_p(u, v)\) of the two potentials \(u, v\) is equivalent to the symmetric I-functional:

\[
d_p(u, v) \sim I_p(u, v) := \left( \int_X |u - v|^p \omega^n \right)^{\frac{1}{p}} + \left( \int_X |u - v|^p \omega^n \right)^{\frac{1}{p}}.
\]

It follows that if \(\tilde{u}^t_{jk}\) is \(d_1\)-Cauchy sequence then it is also \(d_p\)-Cauchy for any \(p \geq 1\). We conclude \(\lim_{k \to \infty} d_p(\tilde{u}^t_{jk}, \tilde{u}^t) = 0\). Let us finally consider \(u^t_j, u^t\) for \(\varphi_t\). It then easy to check that \(u^t_j = \tilde{u}^t_j + \varepsilon_j t\) holds for \(\varepsilon_j := t_j^{-1}(E(\varphi_{t_j}) - E(\varphi_0))\) which converges to \(\varepsilon := \lim_{t \to \infty} t^{-1} E(\varphi_t)\). We conclude \(u^t = \tilde{u}^t + \varepsilon t\).

Let us extend the definition of \(F(\tilde{u}^t)\) to the above (possibly not differentiable) weak geodesic ray. First we recall:

**Theorem 5.8** ([30], Theorem 1) For the \(u^t\) constructed from the envelope form (2.7) we have constants \(m\) and \(M\) such that for any \(a, b \in [0, \infty)\)

1. \(\inf_X \frac{u^a - u^b}{a - b} = m\),
2. \(\sup_X \frac{u^a - u^b}{a - b} = M\).

Solution of the Hausdorff moment problem guarantees the following definition.
Definition 5.9 Let $u'$ be the above weak geodesic ray constructed from the envelope form (2.7), which in particular has a constant speed for any $d_p$. We fix $a \in [0, \infty)$ and take a smooth approximation $u^a_j \searrow u^a$. $u^a_j$ denotes the unique $C^{1,1}$-geodesic connecting smooth $u^0$ with $u^a_j$. Define the Duistermaat-Heckman measure $DH(u^\bullet)$ as the unique measure supported on $[m, M]$ such that for any $p \geq 1$

$$\int_{\mathbb{R}} \lambda^p DH(u^\bullet) = \lim_{j \to \infty} \int_{\mathbb{R}} \lambda^p DH(u^a_j)$$

holds.

By the definition $DH(u^\bullet)$ does not depend on $t$. One has to show that the above limit exists and is independent of the approximation $u^a_j$. In fact, $u'$ is convex in $t$ so that

$$\dot{u}^0 := \inf_{t > 0} \frac{u' - u^0}{t}$$

can be defined. It is a measurable function according to Theorem 5.8. We also observe $\dot{u}^a_j \searrow \dot{u}^0$. The dominated convergence theorem then implies

$$\int_X (\dot{u}^0)^p \omega^n_{u^0} = \lim_{j \to \infty} \int_X (\dot{u}^a_j)^p \omega^n_{u^0} = \lim_{j \to \infty} \int_{\mathbb{R}} \lambda^p DH(u^a_j).$$

Therefore the above definition makes sense.

We set

$$F(u^\bullet) := -\log \int_{\mathbb{R}} e^{-\lambda} DH(u^\bullet).$$

When $u'$ is the $C^{1,1}$-weak geodesic ray associated to a test configuration, we have

$$\int_{\mathbb{R}} |\lambda|^p \ DH(X, L) = \frac{1}{V} \int_X |\dot{u}'|^p \omega^n_{u'} = \frac{d_p(u^0, u')^p}{t^p}.$$ 

It implies $DH(u^\bullet) = DH(X, L)$. For the flow $\varphi_t$ we simply set $DH(\varphi_t) := (\dot{\varphi}_t)_*(V^{-1} \omega^n)$. 

Lemma 5.10 For the above weak geodesic ray asymptotic to the normalized Kähler Ricci flow we have

$$F(u^\bullet) \geq 0 = F(\varphi_t) := -\log \int_{\mathbb{R}} e^{-\dot{\varphi}_t} \omega^n_{\varphi_t}.$$ 

Proof First we observe

$$\int_{\mathbb{R}} e^{-\lambda} DH(\varphi_t) = \frac{1}{V} \int_X e^{-\dot{\varphi}_t} \omega^n_{\varphi_t} = \frac{1}{V} \int_X e^\rho \omega^n_{\varphi_t} = 1.$$
Since DH($u'_j$) is constant in $t$, we have

$$\int_{\mathbb{R}} |\lambda|^p \text{DH}(u'_j) = \frac{1}{t_j} \int_0^{t_j} \int_{\mathbb{R}} |\lambda|^p \text{DH}(u'_j)$$

$$= \frac{d_p(u^0, u'_j)^p}{t_j} = \frac{d_p(u^0, \varphi_t)^p}{t_j}$$

$$\leq \frac{1}{t_j} \int_0^{t_j} \int_{\mathbb{R}} |\lambda|^p \text{DH}(\varphi_t)$$

for any $p > 1$. In the last inequality, we used the fact that $u'_j$ is the unique geodesic (and hence the shortest path) in $\mathcal{E}^p$ for any $p > 1$. In the same way we obtain from $\lim_{j \to \infty} d_p(u'_j, u') = 0$ that

$$\int_{\mathbb{R}} |\lambda|^p \text{DH}(u^*) = \lim_{t \to \infty} \int_{\mathbb{R}} |\lambda|^p \text{DH}(u'_j).$$

From $\varphi_t \leq ct + A$ there exists a constant $b \in \mathbb{R}$ such that the both measures $\text{DH}(u^*)$ and $\frac{1}{t_j} \int_0^{t_j} \text{DH}(\varphi_t)$ are supported in some right-bounded interval $(-\infty, b]$. In order to prove $F(u^*) \geq F(\varphi_t)$, we may assume $b = 0$. It then follows that

$$\int_{\mathbb{R}} e^{-\lambda} \text{DH}(u^*) = \int_{-\infty}^{0} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \text{DH}(u^*)$$

$$\leq \lim inf_{j \to \infty} \int_{-\infty}^{0} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{1}{t_j} \int_0^{t_j} \text{DH}(\varphi_t)$$

$$= \lim inf_{j \to \infty} \int_{\mathbb{R}} e^{-\lambda} \left[ \frac{1}{t_j} \int_0^{t_j} \text{DH}(\varphi_t) \right].$$

As we have already observed that $\int_{\mathbb{R}} e^{-\lambda} \text{DH}(\varphi_t)$ is independent of $t$, the above inequality implies $\int_{\mathbb{R}} e^{-\lambda} \text{DH}(u^*) \leq \int_{\mathbb{R}} e^{-\lambda} \text{DH}(\varphi_t)$. □

### 5.3 Multiplier Ideal Sheaves for the Asymptotic Weak Geodesic Ray

Totally in parallel with section 3.1 we may further construct approximative test configurations. Set the $S^1$-invariant $p^*_X \omega_0$-psh function $U(x, e^{-t}) := u'(x)$. The linear bound $\sup_X u' \leq ct + A$ implies that $\tilde{U} := U + c \log |\tau|$ is uniquely extended to a $p^*_X \omega_0$-psh function on $X \times \mathbb{B}$. We obtain the $S^1$-invariant multiplier ideal sheaf $\mathcal{J}(m \tilde{U})$ and the normalized blow-up $\rho_m: X_m \to X \times \mathbb{A}^1$ with exceptional divisor $E_m$. The line bundle is given by

$$\mathcal{L}_m := \rho_m^* p^*_X (-K_X) - \frac{1}{m + m_0} E_m. \quad (5.6)$$
Theorem 5.11  Let $u^t$ be the above weak geodesic ray for the normalized Kähler-Ricci flow and $(\mathcal{X}_m, L_m)$ be the canonical sequence of test configurations approximates $u^t$. Then we have

$$\liminf_{m \to \infty} H(\mathcal{X}_m, L_m) \geq \lim_{t \to \infty} \left[ \frac{-L(u^t)}{t} + F(u^*) \right].$$

Proof  For the part concerned with $L^{NA}(\mathcal{X}_m, L_m)$ it is due to [12]. The discussion for $F(\mathcal{X}_m, L_m)$ part is essentially the same as that for $E^{NA}$. Indeed, for the weak geodesic ray $U_m$ associated with $(\mathcal{X}_m, L_m)$ we obtain $U_m \geq U + c \log |\tau| - C_{m,r}$ using local Demailly approximation. We again use Theorem 4.6 to compare the $p$-moments as

$$\int_{\mathbb{R}} |\lambda|^p DH(u_m^t) = \frac{1}{t} \int_0^t \int_{\mathbb{R}} |\lambda|^p DH(u_m^s) \leq \frac{d_p(u^0, u_m^t)^p}{t} \leq \frac{d_p(u^0, u^t - ct - C_{m,r})^p}{t}.$$ 

As $t \to \infty$, just by the definition of $DH(\hat{u}^*)$, the last term converges to $\int_{\mathbb{R}} |\lambda|^p DH(\hat{u}^*)$. Notice that the supports of $DH(u_m^t)$ and $DH(\hat{u}^*)$ are both contained in $[-\infty, 0]$. The claim for $DH(u_m^t)$ is by construction and the claim for $DH(\hat{u}^*)$ follows from $U + c \log |\tau| \leq A$. In particular, we have $E^{NA}(\mathcal{X}_m, L_m) \geq \lim_{t \to \infty} t^{-1} E(u^t - ct)$ again in this situation. Moreover, using the same argument in the proof of Lemma 5.10, we obtain $F(\mathcal{X}_m, L_m) \geq F(\hat{u}^*)$. 

Combining all together, we obtain

$$\liminf_{m \to \infty} H(\mathcal{X}_m, L_m) \geq \lim_{t \to \infty} \left[ \frac{-L(u^t - ct)}{t} + F(\hat{u}^*) \right] = \lim_{t \to \infty} \left[ \frac{-L(u^t)}{t} + F(u^*) \right] \geq \lim_{t \to \infty} \left[ \frac{-L(\phi_t)}{t} + F(\phi_t) \right].$$

Finally the Kähler-Ricci flow equation translates the last term into the limit of $H(\omega_{\phi_t})$. It completes the proof of Theorem B.

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