COMPACTNESS PROPERTIES FOR MODULATION SPACES

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Abstract. We prove that if \( \omega_1 \) and \( \omega_2 \) are moderate weights and \( \mathcal{B} \) is a suitable (quasi-)Banach function space, then a necessary and sufficient condition for the embedding \( i : M(\omega_1, \mathcal{B}) \to M(\omega_2, \mathcal{B}) \) between two modulation spaces to be compact is that the quotient \( \omega_2/\omega_1 \) vanishes at infinity. Moreover we show, that the boundedness of \( \omega_2/\omega_1 \) a necessary and sufficient condition for the previous embedding to be continuous.

0. Introduction

In the paper we extend well-known compact embedding properties for classical modulation spaces to a broader family of modulation spaces. These investigations go in some sense back to [32], where M. Shubin proved that if \( t > 0 \), then the embedding \( i : Q_s \to Q_{s-t} \) is compact. In the community, the previous compactness property was not obvious since any similar fact does not hold when the Shubin spaces \( Q_s \) and \( Q_{s-t} \) are replaced by the Sobolev spaces \( H^2_s \) and \( H^2_{s-t} \) of Hilbert types. Since

\[
Q_s = M^{2,2}_{(\omega)}, \quad \omega(X) = (1 + |x| + |\xi|)^s \quad (0.1)
\]

and

\[
H^2_s = M^{2,2}_{(\omega)}, \quad \omega(X) = (1 + |\xi|)^s, \quad X = (x, \xi) \quad (0.2)
\]

the previous compact embedding properties can also be written by means of modulation spaces. In this context, a more general situation were considered by M. Dörfler, H. Feichtinger and K. Gröchenig who proved in [8, Theorem 5] that if \( p, q \in [1, \infty) \), and \( \omega_1 \) and \( \omega_2 \) are certain moderate weights of polynomial types, then

\[
i : M^{p,q}_{(\omega_1)}(\mathbb{R}^d) \to M^{p,q}_{(\omega_2)}(\mathbb{R}^d) \quad (0.3)
\]

is compact if and only if \( \omega_2/\omega_1 \) tends to zero at infinity. By choosing \( \omega_j \) in similar ways as in (0.1), the latter compactness result confirms the compactness of embedding \( i : Q_s \to Q_{s-t} \) above by Shubin, as well as confirms the lack of compactness of the embedding \( i : H^2_s \to H^2_{s-t} \) for Sobolev spaces.
In [5], the compact embedding property [8, Theorem 5] by Dörfler, Feichtinger and Gröchenig were extended in such ways that all moderate weights $\omega_j$ of polynomial type are included. That is, there are no other restrictions on $\omega_j$ than there should exists constants $N_j > 0$ such that

$$\omega_j(X + Y) \lesssim \omega_j(X)(1 + |Y|)^{N_j}, \quad j = 1, 2. \quad (0.4)$$

Moreover, in [5], the Lebesgue exponents $p$ and $q$ are allowed to attain $\infty$.

In Section 2 we extend these results to involve modulation spaces $M(\omega, B)$, which are more general in different ways. Firstly, there are no boundedness estimates of polynomial type for the involved weight $\omega$. In most of our considerations, we require that the weights are moderate, which impose boundedness estimate of exponential types

$$\omega_j(X + Y) \lesssim \omega_j(X)e^{c_j|Y|}, \quad j = 1, 2,$$

for some constants $c_1, c_2 > 0$. We notice that the latter estimate is less restrictive than the condition (0.4), which is assumed in [5,8].

Secondly, $B$ can be any general translation invariant Banach function space without restrictions that $M(\omega, B)$ should be of the form $M^{p,q}_\omega$. We may also have $M(\omega, B) = M^{p,q}_\omega$, but in contrast to [5,8], we here allow $p$ and $q$ to be smaller than 1. Here we notice that if $p < 1$ or $q < 1$, then $M^{p,q}_\omega$ fails to be a Banach space because of absence of convex topological structures.

Thirdly, we show that (0.3) is compact when $\omega_2/\omega_1$ tends to zero at infinity, and the conditions on $\omega_1$ and $\omega_2$ are relaxed into a suitable "local moderate condition" (cf. Theorem 2.9 (1)). We refer to [41] and to some extent to [43] for a detailed study of modulation spaces with such relaxed conditions assumptions on the involved weight functions.

Finally we remark that compactness properties for (0.3) can also be obtained by Gabor analysis, which transfers (0.3) into

$$i: \ell^{p,q}_{\omega_1} \to \ell^{p,q}_{\omega_2},$$

provided $\omega_1$ and $\omega_2$ are moderate weights. Since it is clear that the latter inclusion map is compact, if and only if $\omega_2/\omega_1$ tends to 0 at infinity. Hence the compactness results in [5,32] as well as some of the results in Section 2 can be deduced in such ways. We emphasise however that such technique can not be used in those situations in Section 2 when modulation spaces are of the form $M(\omega, B)$, where either $B$ is a general BF-space, or $\omega$ fails to be moderate, since the Gabor analysis seems to be insufficient in such situations.

1. Preliminaries

In this section we discuss basic properties for modulation spaces and other related spaces. The proofs are in many cases omitted since they can be found in [9,11,14,16,22,36,39].
1.1. Weight functions. A weight or weight function $\omega$ on $\mathbb{R}^d$ is a positive function such that $\omega, 1/\omega \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. Let $\omega$ and $v$ be weights on $\mathbb{R}^d$. Then $\omega$ is called $v$-moderate or moderate, if

$$\omega(x_1 + x_2) \lesssim \omega(x_1)v(x_2), \quad x_1, x_2 \in \mathbb{R}^d. \quad (1.1)$$

Here $f(\theta) \lesssim g(\theta)$ means that $f(\theta) \leq cg(\theta)$ for some constant $c > 0$ which is independent of $\theta$ in the domain of $f$ and $g$. If $v$ can be chosen as polynomial, then $\omega$ is called a weight of polynomial type.

The function $v$ is called submultiplicative, if it is even and (1.1) holds for $\omega = v$. We notice that (1.1) implies that if $v$ is submultiplicative on $\mathbb{R}^d$, then there is a constant $c > 0$ such that $v(x) \geq c$ when $x \in \mathbb{R}^d$. We let $P_E(\mathbb{R}^d)$ be the set of all moderate weights on $\mathbb{R}^d$, and $P(\mathbb{R}^d)$ be the subset of $P_E(\mathbb{R}^d)$ which consists of all polynomially moderate functions on $\mathbb{R}^d$. We also let $P_{E,s}(\mathbb{R}^d)$ ($P_{0,E,s}(\mathbb{R}^d)$) be the set of all weights $\omega$ in $\mathbb{R}^d$ such that

$$\omega(x_1 + x_2) \lesssim \omega(x_1)e^{r|x_2|^s}, \quad x_1, x_2 \in \mathbb{R}^d. \quad (1.2)$$

for some $r > 0$ (for every $r > 0$). We have

$$P \subseteq P_{E,s_1} \subseteq P_{E,s_1} \subseteq P_{E,s_2} \subseteq P_E \quad \text{when} \quad s_2 < s_1$$

and

$$P_{E,s} = P_E \quad \text{when} \quad s \leq 1,$$

where the last equality follows from the fact that if $\omega \in P_{E}(\mathbb{R}^d)$ ($\omega \in P_{0,E}(\mathbb{R}^d)$), then

$$\omega(x + y) \lesssim \omega(x)e^{r|y|^{1/s}} \quad \text{and} \quad e^{-r|x|} \leq \omega(x) \lesssim e^{r|x|}, \quad x, y \in \mathbb{R}^d \quad (1.3)$$

hold true for some $r > 0$ (for every $r > 0$) (cf. [23]).

In some situations we shall consider a more general class of weights compared to $P_E$. (Cf. [41, Definition 1.1].)

**Definition 1.1.** The set $P_Q(\mathbb{R}^d)$ consists of all weights $\omega$ on $\mathbb{R}^d$ such that

$$\omega(x)^2 \lesssim \omega(x + y)\omega(x - y) \lesssim \omega(x)^2$$

when $Rc \leq |x| \leq \frac{c}{|y|}, \quad R \geq 2, \quad (1.4)$

$$e^{-r|x|^2} \lesssim \omega(x) \lesssim e^{r|x|^2}, \quad (1.5)$$

holds for some positive constants $c$ and $r$. 


1.2. **Gelfand-Shilov spaces.** First of all let us fix $0 < h, s, t \in \mathbb{R}$ for the whole subsection. Then we denote the set of all functions $f \in C^\infty(\mathbb{R}^d)$ such that

$$
\|f\|_{S^s_{t,h}} \equiv \sup_{|\alpha|+|\beta| \leq t} \frac{|x^\alpha \partial_\beta f(x)|}{h^{\alpha+\beta}} < \infty
$$

by $S_{s,h}(\mathbb{R}^d)$. Here the supremum is taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$.

One immediately gets, that $S_{s,h}(\mathbb{R}^d)$ is a Banach space which is contained in $\mathcal{S}$. Moreover $S_{s,h}^t$ increases with $h, s$ and $t$ and we have the inclusion $S_{s,h}^t \subseteq \mathcal{S}$. We use the notation $A \rightarrow B$ for topological spaces $A$ and $B$ satisfying $A \subseteq B$ with continuous embeddings. Furthermore for sufficiently large $s, t > \frac{1}{2}$, or $s = t = \frac{1}{2}$ and $h S_{s,h}^t$ contains all finite linear combinations of the Hermite functions.

First of all let us fix $S_{s,h}(\mathbb{R}^d)$ is the strongest possible one such that $S_{s,h}(\mathbb{R}^d)$ is a Fréchet space. Additionally the inclusion map from $S_{s,h}(\mathbb{R}^d)$ to $S_{s,h}^t(\mathbb{R}^d)$ is continuous, for every choice of $h > 0$. Equipped with the seminorms $\| \cdot \|_{S_{s,h}^t}$, $h > 0$ the space $\Sigma_t(\mathbb{R}^d)$ is a Fréchet space. Additionally $\Sigma_t(\mathbb{R}^d) \neq \{0\}$, if and only if $s + t \geq 1$ and $(s, t) \neq (\frac{1}{2}, \frac{1}{2})$, and $S_{s,h}(\mathbb{R}^d) \neq \{0\}$, if and only if $s + t \geq 1$.

The **Gelfand-Shilov distribution spaces** $(S_{s,h}^t)'(\mathbb{R}^d)$ and $(\Sigma_t^s)'(\mathbb{R}^d)$ are the projective and inductive limit respectively of $(S_{s,h}^t)'(\mathbb{R}^d)$. This implies that

$$
(S_{s,h}^t)'(\mathbb{R}^d) = \bigcap_{h>0} (S_{s,h}^t)'(\mathbb{R}^d) \quad \text{and} \quad (\Sigma_t^s)'(\mathbb{R}^d) = \bigcup_{h>0} (\Sigma_t^s)'(\mathbb{R}^d). \tag{1.7}
$$

Note, that $(S_{s,h}^t)'(\mathbb{R}^d)$ is the dual of $S_{s,h}(\mathbb{R}^d)$, and $(\Sigma_t^s)'(\mathbb{R}^d)$ is the dual of $\Sigma_t(\mathbb{R}^d)$ as proved in [20]. This is also true in topological sense. In case $s = t$ we set

$$
S_s = S_s^t, \quad S'_s = (S_s^t)', \quad \Sigma_s = \Sigma_s^t \quad \text{and} \quad \Sigma'_s = (\Sigma_s^t)'.
$$

For every admissible $s, t > 0$ and $\varepsilon > 0$ the next embeddings are true:

$$
\Sigma_s(\mathbb{R}^d) \hookrightarrow S_s(\mathbb{R}^d) \hookrightarrow S_{s+t}^s(\mathbb{R}^d)
$$

and

$$
(\Sigma_{s+t}^s)'(\mathbb{R}^d) \hookrightarrow (S_{s+t}^s)'(\mathbb{R}^d) \hookrightarrow (\Sigma_s^t)'(\mathbb{R}^d). \tag{1.8}
$$
We recall that Fourier transform of \( f \in L^1(\mathbb{R}^d) \) is defined by 
\[
(\mathcal{F}f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)e^{-i(x,\xi)} \, dx,
\]
where \( \langle \cdot, \cdot \rangle \) is the usual scalar product on \( \mathbb{R}^d \). The map \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{S}'(\mathbb{R}^d) \), from \( (\mathcal{S}^\prime)^{\prime}(\mathbb{R}^d) \) to \( (\mathcal{S}^\prime)^{\prime}(\mathbb{R}^d) \) and from \( (\Sigma^\prime)^{(s)}(\mathbb{R}^d) \) to \( (\Sigma^\prime)^{(s)}(\mathbb{R}^d) \). Furthermore, \( \mathcal{F} \) restricts to homeomorphisms on \( \mathcal{S}^\prime(\mathbb{R}^d) \), from \( \Sigma^\prime(\mathbb{R}^d) \) to \( \Sigma^\prime(\mathbb{R}^d) \) and from \( \Sigma^\prime(\mathbb{R}^d) \) to \( \Sigma^\prime(\mathbb{R}^d) \), and to a unitary operator on \( L^2(\mathbb{R}^d) \). If we replace the Fourier transform by a partial Fourier transform similar results hold true for \( s = t \).

Gelfand-Shilov spaces and their distribution spaces can be characterized in a convenient way by means of estimates of the short-time Fourier transforms, see e.g. \([25, 41, 43]\). Before stating this result, we recall the definition of the short-time Fourier transform.

For a fixed \( \phi \in \mathcal{S}_s'(\mathbb{R}^d) \) the short-time Fourier transform \( V_\phi f \) of \( f \in \mathcal{S}_s'(\mathbb{R}^d) \) with respect to the window function \( \phi \) is the Gelfand-Shilov distribution on \( \mathbb{R}^{2d} \), defined by
\[
V_\phi f(x, \xi) \equiv (\mathcal{F}_2(U(f \otimes \phi)))(x, \xi) = \mathcal{F}(f \overline{\phi(\cdot - x)})(\xi),
\]
where \( (UF)(x, y) = F(y, y - x) \). Here \( \mathcal{F}_2 \) denotes the partial Fourier transform of \( F(x, y) \in \mathcal{S}_s'(\mathbb{R}^{2d}) \) with respect to the \( y \) variable. In case \( f, \phi \in \mathcal{S}_s(\mathbb{R}^d) \) the short-time Fourier transform of \( f \) can be written as
\[
V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} \int f(y)\overline{\phi(y - x)}e^{-i(y, \xi)} \, dy.
\]

The characterisation of Gelfand-Shilov functions and their distributions are formulated in the next two propositions. The proof of the characterizations can be found in e.g. \([25, 43]\) (cf. \([25, \text{Theorem 2.7}]\)) and in \([41, 43]\):

**Proposition 1.2.** Let \( s, t, s_0, t_0 > 0 \) be such that \( s_0 + t_0 \geq 1 \), \( s_0 \leq s \) and \( t_0 \leq t \). Also let \( \phi \in \mathcal{S}_{s_0}^\prime(\mathbb{R}^d) \setminus 0 \) and \( f \in (\mathcal{S}_{s_0}^\prime)^{(s)}(\mathbb{R}^d) \). Then the following is true:

1. \( f \in \mathcal{S}_{s}^{\prime}(\mathbb{R}^d) \), if and only if
\[
|V_\phi f(x, \xi)| \lesssim e^{-r(|x|^t + |\xi|^t)}, \quad (1.9)
\]
holds for some \( r > 0 \);

2. if in addition \( (s_0, t_0) \neq (\frac{1}{2}, \frac{1}{2}) \) and \( \phi \in \Sigma_{t_0}^{s_0}(\mathbb{R}^d) \), then \( f \in \Sigma_{s}^{(s)}(\mathbb{R}^d) \), if and only if \( (1.9) \) holds for every \( r > 0 \).

**Proposition 1.3.** Let \( s, t, s_0, t_0 > 0 \) be such that \( s_0 + t_0 \geq 1 \), \( s_0 \leq s \) and \( t_0 \leq t \). Also let \( \phi \in \mathcal{S}_{s}^{\prime}(\mathbb{R}^d) \setminus 0 \) and \( f \in (\mathcal{S}_{s_0}^\prime)^{(s)}(\mathbb{R}^d) \). Then the following is true:

1. \( f \in (\mathcal{S}_{s}^{\prime})^{(s)}(\mathbb{R}^d) \), if and only if
\[
|V_\phi f(x, \xi)| \lesssim \frac{1}{5} e^{r(|x|^t + |\xi|^t)}, \quad (1.10)
\]
holds for every \( r > 0 \);

(2) if in addition \((s_0, t_0) \neq (\frac{1}{2}, \frac{1}{2})\) and \( \phi \in \Sigma_{s_0}^\infty(R^d) \), then \( f \in (\Sigma_t')^s(R^d) \), if and only if \( (1.10) \) holds for some \( r > 0 \).

**Remark 1.4.** For the short-time Fourier transform the following continuity results hold: For every \( s > 0 \), the mapping \((f, \phi) \mapsto V_\phi f\) is continuous from \( S_s(R^d) \times S_s(R^d) \) to \( S_s(R^d) \) and extends uniquely to continuous mappings from \( S'_s(R^d) \times S'_s(R^d) \) to \( S'_s(R^d) \). The same is true if we replace each \( S_s \) by \( \mathcal{F} \) or by \( \Sigma_s \) (cf. e. g. \([35, 43]\)).

### 1.3. Modulation spaces.

In the whole subsection we fix some \( \phi \in \Sigma_1(R^d) \setminus 0 \), \( p, q \in (0, \infty] \) and \( \omega \in \mathcal{P}_E(R^{2d}) \). Then the modulation space \( M_{(\omega)}^{p,q}(R^d) \) is defined as the set of all \( f \in \Sigma_1(R^d) \) such that

\[
\|f\|_{M_{(\omega)}^{p,q}} \equiv \left( \int \left( \int |V_\phi f(x, \xi)\omega(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty \quad (1.11)
\]

holds. We set \( M_{(\omega)}^p = M_{(\omega)}^{p,p} \). Moreover we use the notion \( M^{p,q} = M_{(\omega)}^{p,q} \) and \( M^p = M_{(\omega)}^p \) if \( \omega = 1 \).

We summarize some well-known facts of Modulation spaces in the next Proposition. See \([11, 19, 22, 42]\) for the proof. The conjugate exponent of \( p \) is given by

\[
p' = \begin{cases} 
\infty & \text{when } p \in (0, 1], \\
\frac{p}{p-1} & \text{when } p \in (1, \infty), \\
1 & \text{when } p = \infty.
\end{cases}
\]

**Proposition 1.5.** Let \( p, q, p_j, q_j, r \in (0, \infty] \) be such that \( r \leq \min(1, p, q) \), \( j = 1, 2 \), let \( \omega, \omega_1, \omega_2, v \in \mathcal{P}_E(R^{2d}) \) be such that \( \omega \) is \( v \)-moderate, \( \phi \in M_{(\omega)}^r(R^d) \setminus 0 \), and let \( f \in \Sigma_1(R^d) \). Then the following is true:

1. \( f \in M_{(\omega)}^{p,q}(R^d) \) if and only if \( (1.11) \) holds, i.e. \( M_{(\omega)}^{p,q}(R^d) \) is independent of the choice of \( \phi \). Moreover, \( M_{(\omega)}^{p,q} \) is a quasi-Banach space under the quasi-norm in \( (1.11) \) and even a Banach space if \( p, q \geq 1 \). Different choices of \( \phi \) give rise to equivalent (quasi-)norms;

2. if \( p_1 \leq p_2, q_1 \leq q_2 \) and \( \omega_2 \leq C\omega_1 \) for some constant \( C \), then

\[
\Sigma_1(R^d) \subseteq M_{(\omega_1)}^{p_1,q_1}(R^d) \subseteq M_{(\omega_2)}^{p_2,q_2}(R^d) \subseteq \Sigma_1'(R^d).
\]

Because of Proposition \([1.5](1)\) we are allowed to be rather imprecise concerning the choice of \( \phi \in M_{(\omega)}^r \setminus 0 \) in \( (1.11) \). For instance let \( C > 0 \) be a constant and \( \Omega \) be a subset of \( \Sigma_1' \). If we then write, that \( \|a\|_{M_{(\omega)}^{p,q}} \leq C \) for every \( a \in \Omega \), we mean that the inequality holds for some choice of \( \phi \in M_{(\omega)}^r \setminus 0 \) and every \( a \in \Omega \). Additionally a similar inequality is true for any other choice of \( \phi \in M_{(\omega)}^r \setminus 0 \), although we may have to replace \( C \) by another constant.
We refer to [11, 14–16, 19, 22, 30, 42] for more facts about modulation spaces.

1.4. A broader family of modulation spaces. In this subsection we introduce a broader class of modulation spaces, by imposing certain types of translation invariant solid BF-space norms on the short-time Fourier transforms, cf. [11–15].

We recall that a quasi-norm \( \| \cdot \|_B \) of order \( r \in (0, 1] \) on the vector-space \( B \) is a nonnegative functional on \( B \) which satisfies

\[
\| f + g \|_B \leq 2^{\frac{1}{r} - 1} (\| f \|_B + \| g \|_B), \quad f, g \in B,
\]

(1.12)

\[
\| \alpha \cdot f \|_B = |\alpha| \cdot \| f \|_B, \quad \alpha \in \mathbb{C}, \quad f \in B
\]

and

\[\| f \|_B = 0 \iff f = 0.\]

The vector space \( B \) is called a quasi-Banach space if it is a complete quasi-normed space. If \( B \) is a quasi-Banach space with quasi-norm satisfying (1.12) then on account of [2, 29] there is an equivalent quasi-norm to \( \| \cdot \|_B \) which additionally satisfies

\[
\| f + g \|_B \leq \| f \|_B^r + \| g \|_B^r, \quad f, g \in B.
\]

(1.13)

From now on we always assume that the quasi-norm of the quasi-Banach space \( B \) is chosen in such way that both (1.12) and (1.13) hold.

**Definition 1.6.** Let \( B \subseteq L^r_{\text{loc}}(\mathbb{R}^d) \) be a quasi-Banach space of order \( r \in (0, 1] \) which contains \( \Sigma_1(\mathbb{R}^d) \) with continuous embedding, and let \( v_0 \in \mathcal{P}_E(\mathbb{R}^d) \). Then \( B \) is called a translation invariant Quasi-Banach Function space on \( \mathbb{R}^d \) (with respect to \( v \)), or invariant QBF space on \( \mathbb{R}^d \), if there is a constant \( C \) such that the following conditions are fulfilled:

1. if \( x \in \mathbb{R}^d \) and \( f \in B \), then \( f(\cdot - x) \in B \), and

\[
\| f(\cdot - x) \|_B \leq C v_0(x) \| f \|_B;
\]

(1.14)

2. if \( f, g \in L^r_{\text{loc}}(\mathbb{R}^d) \) satisfy \( g \in B \) and \( |f| \leq |g| \), then \( f \in B \) and

\[
\| f \|_B \leq C \| g \|_B.
\]

If the weight \( v \) even is an element of \( \mathcal{P}_{E,s}(\mathbb{R}^d) \) (\( \mathcal{P}_{E,s}^0(\mathbb{R}^d) \)), then we call \( B \) of Definition 1.6 an invariant QBF-space of Roumieu type (Beurling type) of order \( s \).

By means of (2) in Definition 1.6 we know that \( f \cdot h \in B \) if \( f \in B \) and \( h \in L^\infty \) and additionally

\[
\| f \cdot h \|_B \leq C \| f \|_B \| h \|_{L^\infty}.
\]

(1.15)

For \( r = 1 \), the invariant QBF space \( B \) of Definition 1.6 becomes a Banach space and is called an invariant BF-space (with respect to \( v \).
Because of condition (2) a translation invariant BF-space is a solid 
BF-space in the sense of (A.3) in [12]. For each invariant BF-space 
$B \subseteq L_{1\text{loc}}(R^d)$ we have Minkowski’s inequality, i.e.

$$\|f * \varphi\|_B \leq C \|f\|_B \|\varphi\|_{L_1(v)}, \quad f \in B, \ \varphi \in \Sigma_1(R^d) \quad (1.16)$$

for some $C > 0$ which is independent of $f \in B$ and $\varphi \in \Sigma_1(R^d)$. The density of $\Sigma_1$ in $L_{1\text{loc}}(v)$ provides that the definition of $f * \varphi$ extends uniquely to any $f \in B$ and $\varphi \in L_{1\text{loc}}^1(R^d)$. Hence (1.16) is also true for such $f$ and $\varphi$.

The following result shows that $v_0$ in Definition 1.6 can be replaced by a submultiplicative weight $v$ such that (1.14) is true with $v$ in place of $v_0$ and the constant $C = 1$, and such that

$$v(x + y) \leq v(x)v(y) \quad \text{and} \quad v(-x) = v(x), \quad x, y \in R^d. \quad (1.17)$$

**Proposition 1.7.** Let $B$ be an invariant BF-space on $R^d$ with respect to $v_0 \in \mathcal{P}_E(R^d)$. Then there is a $v \in \mathcal{P}_E(R^d)$ which satisfies (1.17) and such that (1.14) holds with $v$ in place of $v_0$, and $C = 1$.

**Proof.** Let

$$v_1(x) \equiv \sup_{f \in B} \left( \frac{\|f(\cdot - x)\|_B}{\|f\|_B} \right).$$

Then

$$v_1(x + y) = \sup_{f \in B} \left( \frac{\|f(\cdot - x - y)\|_B \cdot \|f(\cdot - y)\|_B}{\|f\|_B} \right)$$

$$\leq \sup_{f \in B} \left( \frac{\|f(\cdot - x)\|_B}{\|f\|_B} \right) \cdot \sup_{f \in B} \left( \frac{\|f(\cdot - y)\|_B}{\|f\|_B} \right) = v_1(x)v_1(y).$$

The result now follows by letting

$$v(x) = \max(v_1(x), v_1(-x)). \quad \square$$

From now on it is assumed that $v$ and $v_j$ are submultiplicative weights if nothing else is stated.

**Example 1.8.** For $p, q \in [1, \infty]$ the space $L_{1\text{loc}}^{p,q}(R^{2d})$ consists of all $f \in L_{1\text{loc}}^1(R^{2d})$ such that

$$\|f\|_{L_{1\text{loc}}^{p,q}} \equiv \left( \int \left( \int |f(x, \xi)|^p \, dx \right)^{q/p} \, d\xi \right)^{1/q} < \infty.$$  

Additionally $L_{2\text{loc}}^{p,q}(R^{2d})$ is the set of all $f \in L_{1\text{loc}}^1(R^{2d})$ such that

$$\|f\|_{L_{2\text{loc}}^{p,q}} \equiv \left( \int \left( \int |f(x, \xi)|^q \, dx \right)^{p/q} \, d\xi \right)^{1/p} < \infty.$$  

Then $L_{1\text{loc}}^{p,q}$ and $L_{2\text{loc}}^{p,q}$ are translation invariant BF-spaces with respect to $v = 1$.

For translation invariant BF-spaces we make the following observation.
Proposition 1.9. Assume that $v \in \mathcal{P}_E(\mathbb{R}^d)$, and that $\mathcal{B}$ is an invariant BF-space with respect to $v$ such that (1.10) holds true. Then the convolution mapping $(\varphi, f) \mapsto \varphi * f$ from $C_0^\infty(\mathbb{R}^d) \times \mathcal{B}$ to $\mathcal{B}$ extends uniquely to a continuous mapping from $L^1_v(\mathbb{R}^d) \times \mathcal{B}$ to $\mathcal{B}$, and (1.10) holds true for any $f \in \mathcal{B}$ and $\varphi \in L^1_v(\mathbb{R}^d)$.

The result is a straight-forward consequence of the fact that $C_0^\infty$ is dense in $L^1_v$.

Next we define the extended class of modulation spaces, which are of interest for us:

Definition 1.10. Assume that $\mathcal{B}$ is a translation invariant QBF-space on $\mathbb{R}^d$, $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, and that $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$. Then the set $M(\omega, \mathcal{B})$ consists of all $f \in \Sigma_1'(\mathbb{R}^d)$ such that

$$\|f\|_{M(\omega, \mathcal{B})} \equiv \|V_\phi f\omega\|_{\mathcal{B}}$$

is finite.

Obviously, we have $M_{\{\omega\}}^p,q(\mathbb{R}^d) = M(\omega, \mathcal{B})$ if $\mathcal{B} = L^p,q(\mathbb{R}^d)$, see e.g. [1.8]. We remark, that many properties of the classical modulation spaces are also true for $M(\omega, \mathcal{B})$. For instance, the definition of $M(\omega, \mathcal{B})$ is independent of the choice of $\phi$ when $\mathcal{B}$ is a Banach space. This statement is formulated in the next proposition. It can be proved by similar arguments as Proposition 11.3.2 in [22]. Hence we omit the proof.

Proposition 1.11. Let $\mathcal{B}$ be an invariant BF-space with respect to $v_0 \in \mathcal{P}_E(\mathbb{R}^d)$. Also let $\omega, v \in \mathcal{P}_E(\mathbb{R}^d)$ be such that $\omega$ is $v$-moderate, $M(\omega, \mathcal{B})$ is the same as in Definition 1.10, and let $\phi \in M_{\{v_0\}}^1(\mathbb{R}^d) \setminus 0$ and $f \in \Sigma_1'(\mathbb{R}^d)$. Then $f \in M(\omega, \mathcal{B})$ if and only if $V_\phi f\omega \in \mathcal{B}$, and different choices of $\phi$ gives rise to equivalent norms in $M(\omega, \mathcal{B})$.

In applications, the quasi-Banach space $\mathcal{B}$ is mostly a mixed quasi-normed Lebesgue space, which is defined next. Let $E = \{e_1, \ldots, e_d\}$ be an ordered basis of $\mathbb{R}^d$ and let $E' = \{e'_1, \ldots, e'_d\}$ be such that

$$\langle e_j, e'_k \rangle = 2\pi \delta_{jk}, \quad j, k = 1, \ldots, d.$$

Then $E'$ is called the dual basis of $E$. The corresponding lattice and dual lattice are

$$\Lambda_E = \{ j_1 e_1 + \cdots + j_d e_d ; (j_1, \ldots, j_d) \in \mathbb{Z}^d \},$$

and

$$\Lambda'_E = \Lambda_{E'} = \{ \iota_1 e'_1 + \cdots + \iota_d e'_d ; (\iota_1, \ldots, \iota_d) \in \mathbb{Z}^d \},$$

There is a matrix $T_E$ such that $e_1, \ldots, e_d$ and $e'_1, \ldots, e'_d$ are the images of the standard basis under $T_E$ and $T_E' = 2\pi(T_E^{-1})'$, respectively. We also let $\kappa(E)$ be the parallelepiped spanned by the basis $E$. 

\[ \text{9} \]
We define for each \( q = (q_1, \ldots, q_d) \in (0, \infty]^d \)
\[
\max q = \max(q_1, \ldots, q_d) \quad \text{and} \quad \min q = \min(q_1, \ldots, q_d).
\]

**Definition 1.12.** Let \( E = \{e_1, \ldots, e_d\} \) be an ordered basis of \( \mathbb{R}^d \), \( \omega \)
be a weight on \( \mathbb{R}^d \), \( p = (p_1, \ldots, p_d) \in (0, \infty]^d \) and \( r = \min(1, p) \). If \( f \in L^r_{\text{loc}}(\mathbb{R}^d) \), then
\[
\|f\|_{L^p_{E, \omega}}(\mathbb{R}^d) \equiv \|g_{d-1}\|_{L^p_{\mathbb{R}}},
\]
where \( g_k(z_k), z_k \in \mathbb{R}^{d-k}, k = 0, \ldots, d-1 \), are inductively defined as
\[
g_0(x_1, \ldots, x_d) \equiv |f(x_1e_1 + \cdots + x_de_d)\omega(x_1e_1 + \cdots + x_de_d)|,
\]
and
\[
g_k(z_k) \equiv \|g_{k-1}(\cdot, z_k)\|_{L^p_{\mathbb{R}}}, \quad k = 1, \ldots, d-1.
\]
The space \( L^p_{E, \omega}(\mathbb{R}^d) \) consists of all \( f \in L^r_{\text{loc}}(\mathbb{R}^d) \) such that \( \|f\|_{L^p_{E, \omega}} \) is finite, and is called \( E \)-split Lebesgue space (with respect to \( p \) and \( \omega \)).

Let \( E, p \) and \( \omega \) be the same as in Definition 1.12. Then the discrete version \( \ell^p_{E, \omega}(\Lambda_E) \) of \( L^p_{E, \omega}(\mathbb{R}^d) \) is the set of all sequences \( a = \{a(j)\}_{j \in \Lambda_E} \) such that the quasi-norm
\[
\|a\|_{\ell^p_{E, \omega}} \equiv \|f_a\|_{L^p_{E, \omega}}, \quad f_a = \sum_{j \in \Lambda_E} a(j)\chi_j,
\]
is finite. Here \( \chi_j \) is the characteristic function of \( j + \kappa(E) \). We also set \( L^p_E = L^p_{E, \omega} \) and \( \ell^p_E = \ell^p_{E, \omega} \) when \( \omega = 1 \).

**Definition 1.13.** Let \( E \) be an ordered basis of the phase space \( \mathbb{R}^{2d} \).
Then \( E \) is called phase split if there is a subset \( E_0 \subseteq E \) such that the span of \( E_0 \) equals \( \{(x, 0) \in \mathbb{R}^{2d}; x \in \mathbb{R}^d\} \), and the span of \( E \setminus E_0 \)
equals \( \{(0, \xi) \in \mathbb{R}^{2d}; \xi \in \mathbb{R}^d\} \).

1.5. Pilipović flat spaces, modulation spaces outside time-frequency analysis and the Bargmann transform. Besides the characterization by means of the short-time Fourier transform, see Proposition 1.2, Gelfand-Shilov spaces can be characterized via Hermite function expansion, too. Here the Hermite function of order \( \alpha \in \mathbb{N}^d \) is given by
\[
h_\alpha(x) = \pi^{-\frac{d}{4}}(-1)^{|\alpha|}(2^{|\alpha|}\alpha!)^{-\frac{1}{2}}e^{-\frac{|x|^2}{4}}(\partial^\alpha e^{-|x|^2}).
\]
We also can write \( h_\alpha \) via
\[
h_\alpha(x) = ((2\pi)^\frac{d}{2}\alpha!)^{-1}e^{-\frac{|x|^2}{4}}p_\alpha(x),
\]
for some polynomial \( p_\alpha \) on \( \mathbb{R}^d \). \( p_\alpha \) are called the Hermite polynomial of order \( \alpha \). It is well-known that \( \{h_\alpha\}_{\alpha \in \mathbb{N}^d} \) provides an orthonormal basis for \( L^2(\mathbb{R}^d) \).
We now can characterize the Gelfand-Shilov spaces $S_s(\mathbb{R}^d)$ ($\Sigma_s(\mathbb{R}^d)$) with $s \geq 1/2$ ($s > 1/2$) as follows: $f \in S_s(\mathbb{R}^d)$ ($f \in \Sigma_s(\mathbb{R}^d)$), if and only if the coefficients $c_\alpha(a)$ in its Hermite series expansion

$$f = \sum_{\alpha \in \mathbb{N}^d} c_\alpha(f) h_\alpha, \quad c_\alpha(f) = (f, h_\alpha)$$

fulfills

$$|c_\alpha(f)| \lesssim e^{-r|\alpha|^1_{1/2}}$$

for some $r > 0$ (for every $r > 0$). Various kinds of Fourier-invariant functions and distribution spaces can be obtained by applying suitable topologies on formal power series expansions, cf. e.g. [17, 43]. To mention one of them, which is of peculiar interest: The Pilipović flat space $H^\flat(\mathbb{R}^d)$ and its dual $H'_\flat(\mathbb{R}^d)$, are defined by all formal expansions (1.18) such that

$$|c_\alpha(f)| \lesssim r^{|\alpha|_{1/2}}$$

for some $r > 0$, respectively

$$|c_\alpha(f)| \lesssim r^{|\alpha|_{1/2}}$$

for every $r > 0$. For $f \in H'_s(\mathbb{R}^d)$ and $\phi \in H_s(\mathbb{R}^d)$, we define

$$(f, \phi)_{L^2(\mathbb{R}^d)} = \sum_{\alpha \in \mathbb{N}^d} c_\alpha(f) \overline{c_\alpha(\phi)}.$$

If $\phi, f \in L^2(\mathbb{R}^d)$, the pairing $(f, \phi)_{L^2(\mathbb{R}^d)}$ agrees with the $L^2(\mathbb{R}^d)$ scalar product of those two functions.

We remark that $H'_s(\mathbb{R}^d)$ is larger than any Fourier-invariant Gelfand-Shilov distribution space, and $H_s(\mathbb{R}^d)$ is smaller than any Fourier-invariant Gelfand-Shilov space. We already know, that any $f \in S_s(\mathbb{R}^d)$ ($f \in \Sigma_s(\mathbb{R}^d)$) with $s \geq 1/2$ ($s > 1/2$) can be expressed in a unique way by an expansion (1.18) with convergence in $S'_s(\mathbb{R}^d)$ ($\Sigma'_s(\mathbb{R}^d)$). Similarly, we have $f \in S'_s(\mathbb{R}^d)$ ($f \in \Sigma'_s(\mathbb{R}^d)$), if and only if

$$|c_\alpha(f)| \lesssim e^{r|\alpha|^1_{1/2}},$$

for every $r > 0$ (for some $r > 0$).

One reason, why the Pilipović flat space $H_s(\mathbb{R}^d)$ and its dual $H'_s(\mathbb{R}^d)$ are of particular interest, are their images under the Bargmann transform. The kernel of the Bargmann transform is given by

$$A_d(z, y) = \pi^{-d/4} \exp \left( -\frac{1}{2} (\langle z, z \rangle + |y|^2) + 2^{1/2} \langle z, y \rangle \right),$$

which is analytic in $z$. Seen as a function of $y$, $A_d$ belongs to $H'_s(\mathbb{R}^d)$.

We define the Bargmann transform $(\mathcal{U}_d f)(z)$ of a function $f \in H'_s(\mathbb{R}^d)$ by

$$\langle \mathcal{U}_d f(z), \phi \rangle = (f, A_d(z, \cdot)).$$
where \( \langle f, \phi \rangle = (f, \overline{\phi})_{L^2(\mathbb{R}^d)} \). Due to (1.19) we know that \( \mathfrak{U}_d \) is bijective between \( \mathcal{H}'(\mathbb{R}^d) \) and \( A(\mathbb{C}^d) \), the set of all entire functions on \( \mathbb{C}^d \), and restricts to a bijective map from \( \mathcal{H}_v(\mathbb{R}^d) \) and

\[
\{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R|z|}, \text{ for some } R > 0 \}.
\]

Later on we will need, that the Bargmann and the short-time Fourier transform are linked by the formula

\[
(\mathfrak{U}_d f)(x + i \xi) = (2\pi)^{\frac{d}{2}} e^{\frac{1}{4}|\xi|^2} e^{-i(x, \xi)} (V_\phi f)(2^{\frac{d}{2}} x, -2^{\frac{d}{2}} \xi), \quad \phi(x) = \pi^{-\frac{d}{2}} e^{-\frac{1}{2}|x|^2}, \quad x \in \mathbb{R}^d.
\]

This can be shown by straight-forward computations. By means of the operator

\[
(U_\mathfrak{U} F)(x, \xi) = (2\pi)^{\frac{d}{2}} e^{\frac{1}{4}|\xi|^2} e^{-i(x, \xi)} F(2^{\frac{d}{2}} x, -2^{\frac{d}{2}} \xi),
\]

when \( F \) is a function or a suitable element of \( F \in \mathcal{H}'(\mathbb{R}^d) \) we can write the Bargmann transform as

\[
(\mathfrak{U}_d f)(x + i \xi) = (U_\mathfrak{U}(V_\phi f))(x, \xi).
\]

**Definition 1.14.** Let \( \phi \) be as in (1.19), \( \omega \) be a weight on \( \mathbb{R}^{2d} \), \( \mathcal{B} \) be an invariant QBF-space with respect to \( v \in \mathcal{P}_E(\mathbb{R}^{2d}) \) on \( \mathbb{R}^{2d} \cong \mathbb{C}^d \) of order \( r \in (0, 1] \).

1. \( B(\omega, \mathcal{B}) \) consists of all \( F \in L_{\text{loc}}^r(\mathbb{R}^{2d}) = L_{\text{loc}}^r(\mathbb{C}^d) \) such that

\[
\|F\|_{B(\omega, \mathcal{B})} \equiv \|U_\mathfrak{U}^{-1} F \omega\|_{\mathcal{B}} < \infty.
\]

Here \( U_\mathfrak{U} \) is given by (1.20);

2. \( A(\omega, \mathcal{B}) \) consists of all \( F \in A(\mathbb{C}^d) \cap B(\omega, \mathcal{B}) \) with topology inherited from \( B(\omega, \mathcal{B}) \);

3. \( M(\omega, \mathcal{B}) \) consists of all \( f \in \mathcal{H}_v(\mathbb{R}^d) \) such that

\[
\|f\|_{M(\omega, \mathcal{B})} \equiv \|V_\phi f \cdot \omega\|_{\mathcal{B}}
\]

is finite.

We observe the small restrictions on \( \omega \) compared to what is the main stream, e.g. that \( \omega \) should belong to \( \mathcal{P}_E(\mathbb{R}^{2d}) \) or be moderated by functions which are bounded by polynomials. We still call the space \( M(\omega, \mathcal{B}) \) as the modulation space with respect to \( \omega \) and \( \mathcal{B} \). In contrast to earlier situations, it seems that \( M(\omega, \mathcal{B}) \) is not invariant under the choice of \( \phi \) when \( \omega \) fails to belong to \( \mathcal{P}_E \). For that reason we always assume that the weight function is given by (1.19) for such \( \omega \).

We have the following.

**Proposition 1.15.** Let \( \phi \) be as in (1.19), \( \omega \) be a weight on \( \mathbb{R}^{2d} \), and let \( \mathcal{B} \) be an invariant QBF-space with respect to \( v \in \mathcal{P}_E(\mathbb{R}^{2d}) \). Then the following is true:

1. the map \( \mathfrak{U}_d \) is an isometric bijection from \( M(\omega, \mathcal{B}) \) to \( A(\omega, \mathcal{B}) \);
(2) if in addition $\mathcal{B}$ is a mixed quasi-norm space of Lebesgue types, then $M(\omega, \mathcal{B})$ and $A(\omega, \mathcal{B})$ are quasi-Banach spaces, which are Banach spaces in the case $\mathcal{B}$ is a Banach space.

Proof. From (1.19), (1.20) and Definition 1.14 it follows that $V_d$ is an isometric injection from $M(\omega, \mathcal{B})$ to $A(\omega, \mathcal{B})$. Since any element in $A(C^d)$, and thereby any element in $A(\omega, \mathcal{B})$ is a Bargmann transform of an element in $\mathcal{H}'(\mathbb{R}^d)$, it follows that the image of $M(\omega, \mathcal{B})$ under $V_d$ contains $A(\omega, \mathcal{B})$. This gives the stated bijectivity in (1).

The completeness of $A(\omega, \mathcal{B})$, and thereby of $M(\omega, \mathcal{B})$ follows from [43]. The details are left for the reader. □

2. Compactness properties for modulation spaces

This section is devoted to the questions under which sufficient and necessary conditions the inclusion map $\iota: M(\omega_1, \mathcal{B}) \to M(\omega_2, \mathcal{B})$ is continuous or even compact for suitable invariant QBF-spaces $\mathcal{B}$.

As ingredients for the proof of our main results we need to deduce some properties for moderate weight functions. In what follows let $L^\infty_{0,\omega}(\mathbb{R}^d)$ be the set of all $f \in L^\infty_{0}(\mathbb{R}^d)$ with

$$\lim_{R \to \infty} (\text{ess sup}_{|x| \geq R} |f(x)|) = 0,$$

when $\omega$ is a weight on $\mathbb{R}^d$. We also set $L^\infty_0 = L^\infty_{0,1}(\mathbb{R}^d)$ when $\omega = 1$. If $\Lambda$ is a lattice, then the discrete Lebesgue spaces $\ell^\infty_0(\Lambda)$ and $\ell^\infty_{0,\omega}(\Lambda)$ are defined analogously.

Lemma 2.1. Let $E$ be an ordered basis of $\mathbb{R}^d$ and let $\omega \in \mathcal{P}_E(\mathbb{R}^d)$. Then the following is true:

1. $\mathcal{P}_E(\mathbb{R}^d)$ is a convex cone which is closed under multiplication, division and under compositions with power functions;
2. $\mathcal{P}_E(\mathbb{R}^d) \cap L^p_{E,\omega}(\mathbb{R}^d)$ increases with $p \in (0, \infty]^d$, and $\mathcal{P}_E(\mathbb{R}^d) \cap L^p_{E,\omega}(\mathbb{R}^d) \subseteq \mathcal{P}_E(\mathbb{R}^d) \cap L^\infty_{0,\omega}(\mathbb{R}^d)$, $p \in (0, \infty)^d$. (2.1)

Similar properties has already been shown in [5, Lemma 2.1] for the smaller weight space $\mathcal{P}$.

Proof. Claim (1) can easily be verified by means of the definition of moderate weights.

It remains to verify (2). Let $\kappa(E)$ be the (closed) parallelepiped spanned by $E$, and let $\vartheta \in \mathcal{P}_E(\mathbb{R}^d)$. By using the map $\vartheta \mapsto \vartheta \cdot \omega$, we reduce ourself to the case when $\omega = 1$.

The moderateness of $\vartheta \in \mathcal{P}_E(\mathbb{R}^d)$ implies that

$$\vartheta(x_1) \asymp \vartheta(x_1 + x_2) \quad \text{when} \quad x_2 \in \kappa(E).$$

(2.2)
Hence, if $\chi_j$ is the characteristic function of $j + \kappa(E)$, and 

$$\vartheta_0(x) = \sum_{j \in \Lambda_E} \vartheta(j) \chi_j(x),$$

then $\vartheta \asymp \vartheta_0$, giving that 

$$\|\vartheta\|_{L^p_E} \asymp \|\vartheta_0\|_{L^p_E} \asymp \|\vartheta\|_{E^p}.$$ 

The assertion now follows from the fact that $\ell^p_E$ increases with $p$ and that if in addition $p \in (0, \infty)^d$, then $\ell^p_E \subseteq \ell^\infty_E$. □

We also have the following result, which is an immediate consequence of [45, Theorem 2.5].

**Proposition 2.2.** Let $v, v_0 \in \mathcal{P}_E(\mathbb{R}^{2d})$ be submultiplicative, $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$ be $v$-moderate, and let $\mathcal{B}$ be an invariant BF-space with respect to $v_0$. Then $M(\omega, \mathcal{B})$ is a Banach space, and

$$M(\omega, \mathcal{B}) \hookrightarrow M_{(1/(v_0v))}^\infty(\mathbb{R}^d).$$

(2.3)

**Remark 2.3.** If $\mathcal{B} = \ell^p_E(\mathbb{R}^{2d})$ for some phase split basis $E$ of $\mathbb{R}^{2d}$, $p \in (0, \infty]^d$ and $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$, then $M(\omega, \mathcal{B})$ is a quasi-Banach space. Moreover $M(\omega, \ell^p_E(\mathbb{R}^{2d}))$ is increasing with $p$. In particular, (2.3) is improved into

$$M(\omega, \ell^p_E(\mathbb{R}^{2d})) \hookrightarrow M_{(\omega)}^\infty(\mathbb{R}^d).$$

We refer to [42] for the proof.

For the proof the twisted convolution $\hat{\ast}$ of two functions $F, G \in L^1(\mathbb{R}^{2d})$ defined by

$$(F \hat{\ast} G)(x, \xi) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} F(x - y, \xi - \eta) G(y, \eta) e^{-i(x-y,\eta)} \, dy \, d\eta,$$

is needed. The twisted convolution is continuous as a map between several function spaces, see e.g. [21] or Lemma 3 in [7]. For instance the map $(F, G) \mapsto F \hat{\ast} G$ is continuous from $L^1(\mathbb{R}^{2d}) \times L^1(\mathbb{R}^{2d})$ to $L^1(\mathbb{R}^{2d})$, and can be restricted to a continuous map from $\Sigma_1(\mathbb{R}^{2d}) \times \Sigma_1(\mathbb{R}^{2d})$ to $\Sigma_1(\mathbb{R}^{2d})$. The latter map can be continuously extended to a continuous map from $\Sigma'_1(\mathbb{R}^{2d}) \times \Sigma_1(\mathbb{R}^{2d})$ to $\Sigma'_1(\mathbb{R}^{2d})$.

On account of the Fourier’s inversion formula we obtain for all $f \in \Sigma'_1(\mathbb{R}^d)$ and $\phi_1, \phi_2, \phi_3 \in \Sigma_1(\mathbb{R}^d)$:

$$(\phi_3, \phi_1)_{L^2} \cdot V_{\phi_2} f = (V_{\phi_1} f) \hat{\ast} (V_{\phi_2} \phi_3).$$

(2.4)

Since $(\phi_2, \phi_3) \mapsto V_{\phi_2} \phi_3$ is continuous from $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ we get for $\phi_1 = \phi_2 = \phi_3 = \phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$ the continuity of the operator $P_\phi$, defined by

$$P_\phi F \equiv \|\phi\|_{L^2(\mathbb{R}^d)}^{-2} F \hat{\ast} (V_{\phi} \phi)$$

(2.5)

on $\Sigma'_1(\mathbb{R}^{2d})$. The operator $P_\phi$ has the following properties:

**Lemma 2.4.** Let $\phi \in \Sigma_1(\mathbb{R}^{2d})$. Then the following is true:
Proof. By (2.4) it is clear that $P_\phi$ is the identity map on $V_\phi(\Sigma'_1(\mathbb{R}^d))$ and thereby on $V_\phi(\Sigma_1(\mathbb{R}^d))$.

Let $V_\phi^*$ be the $L^2$-adjoint of $V_\phi$. That is, $V_\phi^*F$ satisfies

$$(V_\phi^*F, \psi)_{L^2(\mathbb{R}^d)} = (F, V_\phi\psi)_{L^2(\mathbb{R}^d)}, \quad F \in \Sigma'_1(\mathbb{R}^d), \ \psi \in \Sigma_1(\mathbb{R}^d).$$

By the continuity properties of $V_\phi$ on $\Sigma_1$ and $\Sigma'_1$, it follows that $V_\phi^*$ is continuous from $\Sigma'_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ and restricts to a continuous map from $\Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$.

By a straightforward application of Fourier’s inversion formula it follows that

$$P_\phi F = V_\phi f \quad \text{when} \quad f = \|\phi\|_{L^2}^{-2}V_\phi^*F,$$

which shows that the images of $\Sigma'_1(\mathbb{R}^d)$ and $\Sigma_1(\mathbb{R}^d)$ under $P_\phi$ equals $V_\phi(\Sigma'_1(\mathbb{R}^d))$ and $V_\phi(\Sigma_1(\mathbb{R}^d))$, respectively. This gives (1) and (2).

If $\mathcal{B}$ is an invariant BF-space on $\mathbb{R}^d$ and $F \in \mathcal{B}$, then it follows from the definitions that

$$|P_\phi F| \lesssim |F| \ast \Phi,$$

where $\Phi = |V_\phi \phi|$ belongs to $L^1_\omega(\mathbb{R}^d)$ for every choice of $v \in \mathcal{P}_E(\mathbb{R}^d)$. Hence, a combination of (2) in Definition 1.6 and (1.10) gives $P_\phi F \in \mathcal{B}$, and

$$\|P_\phi F\|_\mathcal{B} \lesssim \|F\|_\mathcal{B} \|\Phi\|_{L^1_\omega},$$

for some $v \in \mathcal{P}_E(\mathbb{R}^d)$, and the continuity of $P_\phi$ on $\mathcal{B}$ follows. This gives (3).

\[\square\]

**Lemma 2.5.** If $\mathcal{B}$ is an invariant BF-space of $\mathbb{R}^d$, then

$$L_{(\nu)}^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{B}$$

for some $v \in \mathcal{P}_E(\mathbb{R}^d)$. Then

Proof. Since $\Sigma_1(\mathbb{R}^d)$ is continuously embedded in $\mathcal{B}$ we have

$$\|f\|_\mathcal{B} \lesssim \sup_{\beta \in \mathbb{N}^d} \left( \frac{\|D^\beta f \cdot e^{\cdot^1/h_0}\|_{L^\infty}}{h_0^{|eta|} \beta!} \right).$$
for some $h_0 > 0$. Let $\omega = e^{-2\cdot |\cdot|/h_0}$, $\omega_0 = \omega \ast e^{-|\cdot|^2/2}$ and let $v = 1/\omega_0$. Then [1] Proposition 1.6 shows that 

$$|D^\beta \omega_0| \lesssim h^{|\beta|!} |\cdot|^{2|\beta|}/h_0$$

for every $h > 0$. By choosing $h < h_0$ we get

$$\|\omega_0\|_B \lesssim \sup_{\beta \in \mathbb{N}^d} \left( \frac{|D^\beta \omega_0 \cdot e^{-|\cdot|/h_0}|_{L^\infty}}{h^{|\beta|!}} \right) \lesssim \|e^{-2|\cdot|/h_0} \cdot e^{-|\cdot|/h_0}\|_{L^\infty} = 1 < \infty.$$ 

Hence, if $f \in L^\infty_{(v)}(\mathbb{R}^d)$, then 

$$\|f\|_B \lesssim \|\omega_0\|_B \|f \cdot v\|_{L^\infty} \asymp \|f\|_{L^\infty_{(v)}},$$

and the result follows. \hfill \Box

**Lemma 2.6.** Let $B$ be an invariant BF-space on $\mathbb{R}^d$ and $\omega \in \mathcal{P}_E(\mathbb{R}^d)$. Then

$$\mathcal{B}(\omega) \equiv \{ f \in L^1_{loc}(\mathbb{R}^d) ; f \cdot \omega \in B \}$$

is an invariant BF-space under the norm

$$f \mapsto \|f\|_{\mathcal{B}(\omega)} \equiv \|f \cdot \omega\|_B.$$

**Proof.** Let $v$ be as in . By Lemma 2.5, $L^\infty_{(\omega \cdot v)} \hookrightarrow \mathcal{B}(\omega)$. Since

$$\Sigma_1(\mathbb{R}^d) \hookrightarrow L^\infty_{(\omega \cdot v)}(\mathbb{R}^d),$$

it follows that $\Sigma_1(\mathbb{R}^d)$ is continuously embedded in $\mathcal{B}(\omega)$.

By straight-forward computations it follows that both (1) and (2) in Definition 1.6 are fulfilled with $\mathcal{B}(\omega)$ in place of $\mathcal{B}$ provided $v$ has been modified in suitable ways. \hfill \Box

**Proof of Proposition 2.2.** Let $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$ be fixed, $\mathcal{B}(\omega)$ be the Banach space which consists of all $F \in L^1_{loc}(\mathbb{R}^{2d})$ such that

$$\|F\|_{\mathcal{B}(\omega)} \equiv \|F \cdot \omega\|_B.$$

Since $\omega$ is a moderate function, it follows by the previous lemma $\mathcal{B}(\omega)$ is an invariant BF-space.

Let $\{f_j\}_{j=1}^\infty$ be a Cauchy-sequence in $M(\omega, \mathcal{B})$. Then $\{V_\phi f_j\}_{j=1}^\infty$ is a Cauchy-sequence in $\mathcal{B}(\omega)$. Since $\mathcal{B}(\omega)$ is a Banach space, there is a unique $F \in \mathcal{B}(\omega)$ such that

$$\lim_{j \to \infty} \|V_\phi f_j - F\|_{\mathcal{B}(\omega)} = 0.$$
Let \( f = \|\phi\|^2_{L^2} V_\phi^* F \). Then \( V_\phi f = P_\phi F \) belongs to \( B(\omega) \), in view of Lemma 2.4 (3). Since \( P_\phi \) is continuous on \( B(\omega) \) and satisfies the mapping properties given in Lemma 2.4, we get

\[
\lim_{j \to \infty} \| f_j - f \|_{M(\omega, B)} = \lim_{j \to \infty} \| V_\phi (f_j - f) \|_{B(\omega)} = \lim_{j \to \infty} \| P_\phi (V_\phi f_j - F) \|_{B(\omega)} \leq \lim_{j \to \infty} \| V_\phi f_j - F \|_{B(\omega)} = 0.
\]

Hence, \( f_j \to f \) in \( M(\omega, B) \), and the completeness of \( M(\omega, B) \) follows. Consequently, \( M(\omega, B) \) is a Banach space.

The embedding (2.3) is an immediate consequence of [45, Theorem 2.7] and the fact that \( M(\omega, B) \) is a Banach space.

If we assume that \( B \) is an invariant QBF-space (instead of invariant BF-space) with respect of \( v_0 \), then it seems to be an open question whether (2.3) might be violated or not.

Before studying compactness of embeddings between modulation spaces, we first consider the related continuity questions.

**Theorem 2.7.** Let \( \omega_1 \) and \( \omega_2 \) be weights on \( \mathbb{R}^{2d} \), \( B \) be an invariant BF-space on \( \mathbb{R}^{2d} \) with respect to \( v \in \mathcal{P}_E \) or a mixed quasi-normed space of Lebesgue type, and let \( i \) be the injection

\[
i : M(\omega_1, B) \to M(\omega_2, B).
\]

Then the following is true:

1. if \( \omega_2/\omega_1 \) is bounded, then the map (2.6) is continuous;
2. if in addition \( \omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( v \) is bounded, then the map (2.6) is continuous, if and only if \( \omega_2/\omega_1 \leq C \) for some \( C > 0 \).

The next lemma is related to Remark 2.3 and is needed verify the previous theorem.

**Lemma 2.8.** Let \( v \) be submultiplicative and bounded on \( \mathbb{R}^{2d} \), \( B \) be an invariant BF-space with respect \( v \) which is continuously embedded in \( \Sigma_1(\mathbb{R}^{2d}) \), and let \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \). Then \( M(\omega, B) \hookrightarrow M_1^{\infty}(\mathbb{R}^{2d}) \).

**Proof.** Let \( B' \) be the \( L^2 \)-dual of \( B \). Then it follows by straight-forward computation that both \( B \) and \( B' \) are translation invariant Banach spaces of order 1 which contain \( \Sigma_1(\mathbb{R}^{2d}) \). Let \( \phi \in \Sigma_1(\mathbb{R}^d) \) be such that \( \|\phi\|_{L^2} = 1 \), and let

\[
\Omega = \{ g \in \Sigma_1(\mathbb{R}^d) : \| g \|_{M_1(\omega)} \leq 1 \}.
\]

Since \( (M_1(\omega))(\mathbb{R}^d)' = M_1^{\infty}(\mathbb{R}^d) \) by a unique extension of the \( L^2 \)-form on \( \Sigma_1(\mathbb{R}^d) \) and that \( \Sigma_1 \) is dense in \( M_1(\omega) \), we get

\[
\| f \|_{M_1(\omega)} \approx \sup_{g \in \Omega} |(f, g)_{L^2(\mathbb{R}^d)}| = \sup_{g \in \Omega} |(V_\phi f \cdot \omega, V_\phi g/\omega)_{L^2(\mathbb{R}^d)}| \leq \sup_{g \in \Omega} \| V_\phi f \cdot \omega \|_{B'} \| V_\phi g/\omega \|_{B'} \leq \| V_\phi f \cdot \omega \|_{B'} \approx \| f \|_{M(\omega, B)}.
\]
Here we have used the fact that
\[ \|V_\phi g/\omega\|_{\mathcal{B}} \leq \|g\|_{M(1/\omega, \mathcal{B})} \leq \|g\|_{M(1/\omega)} < \infty, \]
which follows from Feichtinger’s minimality principle (cf. the extension [45, Theorem 2.4] of [21, Theorem 12.1.9]).

The previous lemma enables us to verify Theorem 2.7:

\[ \text{Claim (1) is an immediate consequence of the boundedness of } \omega_2/\omega_1 \text{ and of } \mathcal{B} \text{ being an invariant BF-space.} \]

Assume instead that the embedding \( i \) in (2.6) is continuous and all assumptions of the second claim hold. Claim (2) follows if we have proved the boundedness of \( \omega_2/\omega_1 \). We prove this boundedness by contradiction. We consider, that there is a sequence \( (x_k, \xi_k) \in \mathbb{R}^{2d} \) with \( |(x_k, \xi_k)| \to \infty \) if \( k \to \infty \) fulfilling
\[ \frac{\omega_2(x_k, \xi_k)}{\omega_1(x_k, \xi_k)} \geq k \quad \text{for all } k \in \mathbb{N}. \]

Let \( \phi \) be as in (1.19) and set
\[ f_k = \frac{1}{\omega_1(X_k)} e^{i(x \cdot \xi_k)} \phi(\cdot - x_k), \quad X_k = (x_k, \xi_k). \]

In order to show that the sequence \( f_k \) is bounded in \( M(\omega_1, \mathcal{B}) \), we choose a submultiplicative weight \( \rho \in \mathcal{P}_E(\mathbb{R}^{2d}) \) such that \( \omega_1 \) is \( \rho_0 \)-moderate and that \( \rho_0 \geq 1 \).

By
\[ V_\phi(e^{i(x \cdot \xi)} f(\cdot - x))(y, \eta) = e^{i(y \cdot \eta - \xi)} (V_\phi f)(y - x, \eta - \xi) \]
(see e.g. [22]), we get
\[ \|e^{i(x \cdot \xi)} f(\cdot - x)\|_{M(\omega_1, \mathcal{B})} \leq C \omega_1(x, \xi) \|f\|_{M(\rho_0, \mathcal{B})}, \quad f \in M(\rho_0, \mathcal{B}). \]

This gives
\[ \|f_k\|_{M(\omega_1, \mathcal{B})} = \frac{1}{\omega_1(X_k)} \|e^{i(x \cdot \xi_k)} \phi(\cdot - x_k)\|_{M(\omega_1, \mathcal{B})} \leq C \|\phi\|_{M(\rho_0, \mathcal{B})} < \infty, \]
where \( C \) is independent of \( k \in \mathbb{N} \). Then the hypothesis provides the boundedness of the sequence \( \{f_k\} \) in \( M(\omega_2, \mathcal{B}) \).

Since \( M(\omega_2, \mathcal{B}) \to M(\omega_2) \) due to Lemma 2.8 we have
\[ \sup_{X \in \mathbb{R}^{2d}} \omega_2(X)(V_\phi(f_k))(X) \leq C \|f_k\|_{M(\omega_2, \mathcal{B})} \leq C \quad \text{for all } k \in \mathbb{N} \]
for some \( C > 0 \). In particular inequality (2.7) yields if we take \( z = z_k \)
\[ \omega_2(X_k)(V_\phi(f_k))(X_k) = \frac{\omega_2(X_k)}{\omega_1(X_k)} \|V_\phi(e^{i(x \cdot \xi_k)} \phi(\cdot - x_k))(X_k)\| \]
\[ = \frac{\omega_2(X_k)}{\omega_1(X_k)} \|V_\phi\phi(0)\| = (2\pi)^{-\frac{d}{2}} \frac{\omega_2(X_k)}{\omega_1(X_k)} \leq C \quad (2.8) \]
which proves the result. □

We have now the following extension of [5, Theorem 1.2], which is our main result.

**Theorem 2.9.** Let \( \omega_1, \omega_2 \in \mathcal{P}_Q(\mathbb{R}^{2d}) \), \( v \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be submultiplicative, \( \mathcal{B} \) be an invariant BF-space on \( \mathbb{R}^{2d} \) with respect to \( v \) or a mixed quasi-normed space of Lebesgue type, and let \( i \) be the injection \( i: M(\omega_1, \mathcal{B}) \to M(\omega_2, \mathcal{B}) \).

Then the following is true:

1. If \( \omega_2/\omega_1 \in L_0^\infty(\mathbb{R}^{2d}) \), then the map \( (2.9) \) is compact;
2. If in addition \( \omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( v \) is bounded, then the map \( (2.9) \) is compact, if and only if \( \omega_2/\omega_1 \in L_0^\infty(\mathbb{R}^{2d}) \).

We need the following lemma for the proof.

**Lemma 2.10.** Let \( \mathcal{B} \) be an invariant BF space on \( \mathbb{R}^{2d} \), \( \phi(x) = \pi^{-\frac{d}{2}}e^{-\frac{1}{2}|x|^2} \), \( x \in \mathbb{R}^d \), \( \omega \in \mathcal{P}_Q(\mathbb{R}^{2d}) \) and let \( \{f_j\}_{j=1}^\infty \subseteq \Omega^i(\mathbb{R}^d) \) be a bounded set in \( M(\omega, \mathcal{B}) \). Then there is a subsequence \( \{f_{j_k}\}_{k=1}^\infty \) of \( \{f_j\}_{j=1}^\infty \) such that \( \{V_\phi f_{j_k}\}_{k=1}^\infty \) is locally uniformly convergent.

**Proof.** By the link between the Bargmann transform and Gaussian windowed short-time Fourier transforms, the result follows if we prove the assertion with \( F_j = \mathfrak{W}_df_j \) in place of \( V_\phi f_j \). For any \( R > 0 \), let \( D_R \) be the poly-disc

\[
D_R \equiv \{(x, \xi) \in \mathbb{R}^{2d}; \ x_j^2 + \xi_j^2 < R^2, \ j = 1, \ldots, d\}
\]

in \( \mathbb{R}^{2d} \) which we identify with

\[
\{x + i\xi \in \mathbb{C}^d; \ x_j^2 + \xi_j^2 < R^2, \ j = 1, \ldots, d\}
\]

in \( \mathbb{C}^d \). By [11, Theorem 3.2] and an application of Cantor’s diagonalization principle the result follows if we prove that for each \( R > 0 \), there is a subsequence \( \{f_{j_k}\}_{k=1}^\infty \) of \( \{f_j\}_{j=1}^\infty \) such that \( \{F_{j_k}\}_{k=1}^\infty \) is uniformly convergent on \( D_R \).

By [11, Theorem 3.2], it follows that \( \{f_j\}_{j=1}^\infty \) is a bounded set in \( M_{(\omega_0)}(\mathbb{R}^d) \) for some choice of \( \omega_0 \in \mathcal{P}_Q(\mathbb{R}^{2d}) \). Hence, \( \{V_\phi f_{j_k}\}_{j=1}^\infty \) and thereby \( \{F_{j_k}\}_{j=1}^\infty \) are locally uniformly bounded on \( \mathbb{R}^{2d} \). In particular,

\[
C_R \equiv \sup_{j \geq 1} \|F_j\|_{L^\infty(D_{2R})} \quad \text{and} \quad C_{R, \omega_0} \equiv \sup_{j \geq 1} \|F_j \omega_0\|_{L^\infty(D_{2R})} \quad (2.10)
\]

are finite for every weight \( \omega_0 \) on \( \mathbb{C}^d \simeq \mathbb{R}^{2d} \).

By Cauchy’s and Taylor’s formulæ we have

\[
F_j(z) = \sum_{\alpha \in \mathbb{N}^d} a_j(\alpha) z^\alpha, \quad z \in D_R, \quad (2.11)
\]
where
\[ |a_j(\alpha)| \leq C_R(2R)^{-[\alpha]}. \quad (2.12) \]

In particular, if \( \{ \beta_l \}_{l=1}^\infty \) be an enumeration of \( \mathbb{N}^d \), then for each \( l \geq 1 \), \( \{ a_j(\beta_l) \}_{j=1}^\infty \) is a bounded set in \( \mathbb{C} \). Hence, for a subsequence \( I_1 = \{ k_{1,1}, k_{1,2}, \ldots \} \) of \( \mathbb{Z}^+ = \{ 1, 2, \ldots \} \), the limit
\[
\lim_{m \to \infty} a_{k_1,m}(\beta_1)
\]
exists. By induction it follows that for some family of subsequences

\[
I_N = \{ k_{N,1}, k_{N,2}, \ldots \} \subseteq \mathbb{Z}^+,
\]
which decreases with \( N \), the limit
\[
\lim_{m \to \infty} a_{k_N,m}(\beta_n)
\]
exists for every \( n \leq N \).

By Cantor’s diagonal principle, there is a subsequence \( \{ j_k \}_{k=1}^\infty \) of \( \mathbb{Z}^+ \) and sequence \( \{ b(\alpha) \}_{\alpha \in \mathbb{N}^d} \) such that
\[
\lim_{k \to \infty} a_{j_k}(\alpha) = b(\alpha).
\]

By (2.12) we get
\[ |b(\alpha)| \leq C_R(2R)^{-[\alpha]}. \]

This in turn gives
\[
\sup_{j \geq 1} \| a_j(\alpha) z^\alpha \|_{L^\infty(D_R)} \leq C_R 2^{-[\alpha]} \quad \text{and} \quad \| b(\alpha) z^\alpha \|_{L^\infty(D_R)} \leq C_R 2^{-[\alpha]}
\]
(2.13)

Hence, (2.11) and the Taylor series
\[ F(z) \equiv \sum_{\alpha \in \mathbb{N}^d} b(\alpha) z^\alpha, \]
are uniformly convergent on \( D_R \), and by using (2.13), it follows by straight-forward computations that \( F_{j_k} \) tends to \( F \) uniformly on \( D_R \) when \( k \) tends to infinity. \( \square \)

**Proof of Theorem 2.10** In order to verify (1) we need to show, that a bounded sequence \( \{ f_j \} \) in \( M(\omega_1, \mathcal{B}) \) has a convergent subsequence in \( M(\omega_2, \mathcal{B}) \). By means of the assumptions there is a sequence of increasing balls \( B_k, k \in \mathbb{Z}_+ \), centered at the origin with radius tending to \( +\infty \) as \( k \to \infty \) such that
\[
\frac{\omega_2(x, \xi)}{\omega_1(x, \xi)} \leq \frac{1}{k}, \quad \text{when} \quad (x, \xi) \in \mathbb{R}^{2d} \setminus B_k.
\]
(2.14)

By Lemma 2.11 it follows that if \( \phi(x) = \pi^{-\frac{d}{2}} e^{-\frac{1}{2} |x|^2}, x \in \mathbb{R}^d \), then there is a subsequence \( \{ h_{j,l} \}_{j=1}^\infty \) of \( \{ f_j \}_{j=1}^\infty \) such that \( \{ V_{\phi} h_j \}_{j=1}^\infty \) converges uniformly on any \( B_k \), and converges on the whole \( \mathbb{R}^{2d} \).
We have to prove that \( \| h_{m_1} - h_{m_2} \|_{\mathcal{M}(\omega_2, \mathscr{B})} \to 0 \) as \( m_1, m_2 \to \infty \). Let \( \chi_k \) be the characteristic function of \( B_k \), \( k \geq 1 \). From the fact that \( C_R \) in (2.10) is bounded we have

\[
\| h_{m_1} - h_{m_2} \|_{\mathcal{M}(\omega_2, \mathscr{B})} = \| V_\phi h_{m_1} - V_\phi h_{m_2} \|_{\mathcal{M}(\omega_2)} \\
\leq \| (V_\phi h_{m_1} - V_\phi h_{m_2}) \chi_k \|_{\mathcal{M}(\omega_2)} + \| V_\phi h_{m_1} - V_\phi h_{m_2} \|_{\mathcal{M}(\omega_1)} / k \quad (2.15)
\]

where \( C = C_R \) is the constant in (2.10).

In order to make the right-hand side arbitrarily small, \( k \) is first chosen large enough. Then \( V_\phi h_1, V_\phi h_2, \ldots \) is a sequence of bounded continuous functions converging uniformly on the compact set \( \overline{B_k} \). Since \( \omega_2 \) is a weight and \( \mathscr{B} \) is an invariant BF-space we obtain

\[
\| (V_\phi h_{m_1} - V_\phi h_{m_2}) \chi_k \|_{\mathcal{M}(\omega_2)} = \| (V_\phi h_{m_1} - V_\phi h_{m_2}) \omega_2 \chi_k \|_{\mathscr{B}} \\
\lesssim \sup_{(x,\xi) \in B_k} |(V_\phi h_{m_1}(x,\xi) - V_\phi h_{m_2}(x,\xi)) \omega_2(x,\xi)| \| \chi_k \|_{\mathcal{M}(\omega_2)} \\
\lesssim \sup_{(x,\xi) \in B_k} |V_\phi h_{m_1}(x,\xi) - V_\phi h_{m_2}(x,\xi)|
\]

tends to zero as \( m_1 \) and \( m_2 \) tend to infinity. This proves (1).

In order to verify (2) we suppose that the embedding \( i \) in (2.11) is compact and all assumptions of the second claim hold. From the first part of the proof, the result follows if we prove that \( \omega_2 / \omega_1 \) turns to zero at infinity. We prove this claim by contradiction.

Suppose there is a sequence \( (x_k, \xi_k) \in \mathbb{R}^{2d} \) with \( |(x_k, \xi_k)| \to \infty \) if \( k \to \infty \) and a \( C > 0 \) fulfilling

\[
\frac{\omega_2(x_k, \xi_k)}{\omega_1(x_k, \xi_k)} \geq C \quad \text{for all } k \in \mathbb{N}. \quad (2.16)
\]

Let \( \phi \) be as in (1.19) and set

\[
f_k = \frac{1}{\omega_1(X_k)} e^{i(\cdot, \xi_k)} \phi(\cdot - x_k), \quad X_k = (x_k, \xi_k).
\]

By the proof of Theorem 2.7 it follows that the sequence \( \{f_k\}_{k=1}^{\infty} \) is bounded in \( M(\omega_1, \mathcal{B}) \), and by the assumptions \( \{f_k\}_{k=1}^{\infty} \) is precompact in \( M(\omega_2, \mathcal{B}) \).

Let \( \phi \in \Sigma_1(\mathbb{R}^d) \). Then \( V_\phi \phi \in \Sigma_1(\mathbb{R}^{2d}) \) by Remark 1.4. From the fact \( \omega_1 \geq e^{-|r|} \) for some \( r_0 > 0 \) we get

\[
\int \phi(x) f_k(x) \, dx = \frac{1}{\omega_1(x_k, \xi_k)} (V_\phi \phi)(x_k, \xi_k) \to 0,
\]
as \( k \to \infty \), which implies, that \( f_k \) tends to zero in \( \Sigma_1(\mathbb{R}^d) \). Hence the only possible limit point in \( M(\omega_2, \mathcal{B}) \) of \( \{f_k\}_{k=1}^{\infty} \) is zero.

As \( \{f_k\}_{k=1}^{\infty} \) is precompact in \( M(\omega_2, \mathcal{B}) \), we can then extract a subsequence \( \{f_{k_j}\}_{j=1}^{\infty} \) which converges to zero in \( M(\omega_2, \mathcal{B}) \).
Since $M(\omega_2, \mathcal{B}) \hookrightarrow M^\infty(\omega_2)$ due to Lemma 2.8 we have
\[
\sup_{X \in \mathbb{R}^{2d}} \omega_2(X)||(V_\phi(f_k))(X)|| \leq C||f_k||_{M(\omega_2, \mathcal{B})} \to 0 \quad (2.17)
\]
as $j \to \infty$. Taking $X = X_{k_j}$ in the previous inequality provides
\[
\omega_2(X_{k_j})||(V_\phi(f_k))(X_{k_j})|| = \frac{\omega_2(X_{k_j})}{\omega_1(X_{k_j})}||(V_\phi(e^{i\cdot \xi_{k_j}} \phi(\cdot - x_{k_j}))(X_{k_j}))||
\]
\[
= \frac{\omega_2(X_{k_j})}{\omega_1(X_{k_j})}||(V_\phi(\phi)(0)||/(2\pi)^{-d} \frac{\omega_2(X_{k_j})}{\omega_1(X_{k_j})} \to 0. \quad (2.18)
\]
which contradicts (2.16) and proves (2). \qed

As an immediate consequence of Lemma 2.1 and Theorem 2.9 we get:

**Corollary 2.11.** Assume that $\omega_1, \omega_2 \in \mathcal{P}(\mathbb{R}^{2n})$, and that $p, p_0, q, q_0 \in [1, \infty]$ such that $p_0, q_0 < \infty$. Assume also that $\omega_2/\omega_1 \in L^{p_0,q_0}(\mathbb{R}^{2d})$. Then the embedding (2.9) is compact.

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