Covariant stochastic products of quantum states

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Abstract. A notion of stochastic product of quantum states — a binary operation on the set of density operators preserving the convex structure — is discussed. We describe, in particular, a class of group-covariant, associative stochastic products: the twirled products. Each binary operation in this class can be constructed by means of a square integrable projective representation of a locally compact group, a probability measure on this group and a fiducial density operator in the Hilbert space of the representation. By suitably extending this operation from the convex set of density operators to the full Banach space of trace class operators, one obtains a Banach algebra, which is commutative in the case where the relevant group is abelian.

1. Introduction
The properties of operator algebras and of some related mathematical structures have shown to play a central role in modern theoretical physics. They turn out to be fundamental — in particular — in quantum mechanics, quantum field theory, quantum statistical mechanics and non-commutative geometry [1–5]. In this context, quantum states can be defined as normalized positive functionals on the C*-algebra \( B(\mathcal{H}) \) of (bounded) observables \([1–3]\) — the Banach space of bounded operators in a separable complex Hilbert space \( \mathcal{H} \), endowed with the ordinary product (composition) and with the involution \( A \mapsto A^* \) (the adjoining map). In several applications — including quantum information, quantum control and quantum measurement theory \([6, 7]\) — it is convenient to deal with a distinct class of states, the normal states \([2]\), that can be realized as normalized, positive trace class operators (the density operators). They form a convex set \( S(\mathcal{H}) \subset B_1(\mathcal{H}) \), with \( B_1(\mathcal{H}) \) denoting the complex Banach space of trace class operators in \( \mathcal{H} \).

Endowing the selfadjoint part \( B(\mathcal{H})_R \) of the C*-algebra \( B(\mathcal{H}) \) with the pair of (non-associative) products \( A \circ B := (AB + BA)/2 \) (Jordan) and \( A \diamond B := (AB - BA)/2i \) (Lie), one obtains the Jordan-Lie Banach algebra \([1]\) \( (B(\mathcal{H})_R, (\cdot) \circ (\cdot), (\cdot) \diamond (\cdot)) \) of ‘true’ observables (and, conversely, every Jordan-Lie Banach algebra gives rise, upon suitable complexification, to a C*-algebra \([1]\)); but states are only indirectly involved in this algebraic structure (as functionals).

In fact, let us observe that, for any \( \rho, \sigma \in S(\mathcal{H}) \),

- the selfadjoint trace class operator \( \rho \circ \sigma \) is a density operator if and only if \( \rho = \sigma \equiv P \), where \( P \) is a pure state (i.e., a rank-one projection) \([8]\);
- the selfadjoint trace class operator \( \rho \diamond \sigma \) cannot be a density operator because \( \text{tr}(\rho \diamond \sigma) = 0 \).

Based on these considerations, a natural question is the following \([8, 9]\): One may ask whether it is possible to endow the Banach space \( B_1(\mathcal{H}) \) with some binary operation such that

(i) one obtains an algebra structure and, moreover, the product of two states is a state too.
In order to achieve a noteworthy class of examples, such as to justify our interest in this issue, in addition to the previous condition (i) it is reasonable to require that

(ii) the aforementioned algebra be associative;

(iii) the algebra product be continuous wrt some suitable topology;

(iv) this product be — in general (i.e., apart from special cases or trivial examples) — a genuinely binary operation, namely, it should effectively depend on both its arguments.

Taking into account the remarkable parallelism of classical and quantum descriptions of physical systems [1], another natural question next arises: Does an analogous product exist in the classical setting?

The answer is in the affirmative. Consider, indeed, the convolution [10] \( \mu \ast \nu \) of two complex Radon measures \( \mu, \nu \) on a locally compact group \( G \). The Banach space \( \mathcal{M}(G) \) formed by all such measures, endowed with this binary operation, is promoted to a Banach algebra. Moreover, if \( \mu, \nu \) are classical states — i.e., probability measures — then \( \mu \ast \nu \) is a classical state too. Clearly, \( G = \mathbb{R}^n \times \mathbb{R}^m \) — the group of translations on phase space — is the standard case.

Convolution is a group-theoretical notion. Thus, regarding the classical-quantum parallelism as a good guiding principle, and considering the fundamental role of symmetry transformations in quantum physics [11–13], we will also stipulate that

(v) the product of quantum states we are going to discuss is group-theoretical and enjoys some covariance property wrt a symmetry action of a certain abstract group \( G \).

A binary operation on \( B_1(\mathcal{H}) \) satisfying (i) will be called a stochastic product, and the Banach space \( B_1(\mathcal{H}) \), endowed with a stochastic product satisfying (ii), will be called a stochastic algebra. It turns out that for such an algebra condition (iii) is automatically verified wrt the natural topology induced by the norm ((iii) being actually a consequence of condition (i) alone). We will argue that, by a suitable group-theoretical construction, one obtains a class of associative stochastic products — the twirled products [8,9] — satisfying requirements (iv) and (v), as well. These group-theoretical stochastic products give rise to the class of twirled stochastic algebras.

2. Stochastic maps and stochastic products

A map \( \Phi: \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}) \) is called stochastic if it preserves the convex structure of the space of quantum states \( \mathcal{S}(\mathcal{H}) \): namely, if it is convex-linear. Such a map can be extended to a unique (bounded) trace-preserving, positive linear map \( \Phi_{\text{ext}}: B_1(\mathcal{H}) \to B_1(\mathcal{H}) \) (equivalently, a linear map \( \Phi_{\text{ext}} \) in \( B_1(\mathcal{H}) \) mapping \( \mathcal{S}(\mathcal{H}) \) into itself) [7,9]. Analogously, we define a stochastic product on \( \mathcal{S}(\mathcal{H}) \) as a map \( (\cdot) \ast (\cdot) : \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}) \) that is convex-linear in both its arguments; i.e., for all \( \rho, \sigma, \tau, \upsilon \in \mathcal{S}(\mathcal{H}) \) and all \( \alpha, \epsilon \in [0,1] \),

\[
(\alpha \rho + (1 - \alpha)\sigma) \ast (\epsilon \tau + (1 - \epsilon)\upsilon) = \alpha \epsilon \rho \ast \tau + \alpha(1 - \epsilon)\rho \ast \upsilon + (1 - \alpha)\epsilon \sigma \ast \tau + (1 - \alpha)(1 - \epsilon)\sigma \ast \upsilon.
\]

(1)

We will also say that a binary operation \( (\cdot) \square (\cdot) \) on \( B_1(\mathcal{H}) \) is state-preserving if it is such that \( \mathcal{S}(\mathcal{H}) \square \mathcal{S}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H}) \). A stochastic product is automatically continuous wrt a natural topology and admits a bilinear extension that is analogous to the linear extension of a stochastic map [9]:

**Proposition 1.** Every stochastic product is continuous wrt the topologies on \( \mathcal{S}(\mathcal{H}) \) and on \( \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \) induced, respectively, by the metrics

\[
d_1(\rho, \sigma) := \|\rho - \sigma\|_1 \quad \text{and} \quad d_{1,1}(((\rho, \tau), (\sigma, \upsilon)) := \max\{\|\rho - \sigma\|_1, \|\tau - \upsilon\|_1\},
\]

where \( \| \cdot \|_1 \) is the norm of \( B_1(\mathcal{H}) \). For each stochastic product \( (\cdot) \ast (\cdot) : \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}) \), moreover, there is a unique stochastic map (or product) \( (\cdot) \square (\cdot) : B_1(\mathcal{H}) \times B_1(\mathcal{H}) \to B_1(\mathcal{H}) \) — namely, a state-preserving bilinear map on \( B_1(\mathcal{H}) \) — such that \( \rho \ast \sigma = \rho \square \sigma \), for all \( \rho, \sigma \in \mathcal{S}(\mathcal{H}) \).
The Banach space $B_1(\mathcal{H})$, endowed with a binary operation $(\cdot) \diamond (\cdot) : B_1(\mathcal{H}) \times B_1(\mathcal{H}) \to B_1(\mathcal{H})$ that is both stochastic (bilinear, state-preserving) and associative, is called a stochastic algebra.

By Proposition 1, every stochastic product $(\cdot) \circ (\cdot)$ on $\mathcal{S}(\mathcal{H})$ is the restriction of a unique stochastic product $(\cdot) \boxtimes (\cdot)$ on $B_1(\mathcal{H})$, which is associative if and only if $(\cdot) \circ (\cdot)$ is associative. Thus, one may equivalently define a stochastic algebra as a Banach space of trace class operators $B_1(\mathcal{H})$ together with an associative stochastic product defined on the convex subset $\mathcal{S}(\mathcal{H})$.

Let us now consider the Banach space $\mathcal{B}Z_1(\mathcal{H})$ of all bounded bilinear maps on $B_1(\mathcal{H})$, endowed with the norm $\| (\cdot) \boxtimes (\cdot) \|_{(1)} := \sup \{ \| A \boxtimes B \|_1 : \| A \|_1, \| B \|_1 \leq 1 \}$. Denoting by $B_1(\mathcal{H})_\mathbb{R}$ the real Banach space of all selfadjoint trace class operators, the following result holds [9]:

**Proposition 2.** Every stochastic map $(\cdot) \boxtimes (\cdot) : B_1(\mathcal{H}) \times B_1(\mathcal{H}) \to B_1(\mathcal{H})$ is bounded and its norm satisfies $\| (\cdot) \boxtimes (\cdot) \|_{(1)} \leq 2$; whereas, its restriction $(\cdot) \boxtimes (\cdot)$ to a bilinear map on $B_1(\mathcal{H})_\mathbb{R}$ is such that $\| (\cdot) \boxtimes (\cdot) \|_{(1)} = 1$. Thus, if the stochastic map is associative, then $(B_1(\mathcal{H})_\mathbb{R}, (\cdot) \boxtimes (\cdot))$ is a Banach algebra, because $\| A \boxtimes B \|_1 \leq \| A \|_1 \| B \|_1, \forall A, B \in B_1(\mathcal{H})_\mathbb{R}$.

**Remark 1.** The restriction of a stochastic map $(\cdot) \boxtimes (\cdot)$ on $B_1(\mathcal{H})$ to a bilinear map on $B_1(\mathcal{H})_\mathbb{R}$ is well defined, because $(\cdot) \boxtimes (\cdot)$, being bilinear and state-preserving, is also adjoint-preserving; i.e., $B_1(\mathcal{H})_\mathbb{R} \boxtimes B_1(\mathcal{H})_\mathbb{R} \subseteq B_1(\mathcal{H})_\mathbb{R}$. Moreover, the inequality $\| (\cdot) \boxtimes (\cdot) \|_{(1)} \leq 2$ may not be saturated. E.g., $(B_1(\mathcal{H}), (\cdot) \boxtimes (\cdot))$ may be a Banach algebra; see the second assertion of Theorem 1 below.

### 3. A technical detour: square integrable representations

Let $U$ be an irreducible — in general, projective — representation of a locally compact, second countable topological group (in short, a l.c.s.c. group) $G$, acting in a separable complex Hilbert space $\mathcal{H}$. We will denote by $\mu_G$ (a normalization of) the left Haar measure on $G$ and by $(\cdot, \cdot)$ the scalar product in $\mathcal{H}$. Given two vectors $\psi, \phi \in \mathcal{H}$, consider the bounded Borel coefficient function $c_{\psi\phi} : G \ni g \mapsto \langle U(g) \psi, \phi \rangle \in \mathbb{C}$. The functions of this kind allow us to define the set

$$\mathcal{A}(U) := \{ \psi \in \mathcal{H} \mid \exists \phi \in \mathcal{H} : (\cdot, \cdot) \neq 0, c_{\psi\phi} \in L^2(G, \mu_G ; \mathbb{C}) \}$$

(3)

of admissible vectors for $U$, and $U$ is called square integrable [8,14–16] if $\mathcal{A}(U) \neq \{ 0 \}$.

If $G$ is unimodular — i.e., the Haar measure $\mu_G$ on $G$ is both left and right invariant — and $U$ is square integrable, then $\mathcal{A}(U) = L^2(G, \mu_G ; \mathbb{C})$ (all vectors are admissible); moreover, all coefficient functions are square integrable wrt $\mu_G$ and satisfy the orthogonality relations; namely,

$$\int_G d\mu_G(g) \langle \eta, U(g)\phi \rangle \langle U(g)\psi, \chi \rangle = c_{\psi\phi} \langle \eta, \chi \rangle \langle \psi, \phi \rangle, \quad \forall \eta, \chi, \psi, \phi \in \mathcal{H},$$

(4)

where $c_{\psi\phi}$ is a strictly positive constant depending on $U$ (and on the normalization of $\mu_G$).

**Remark 2.** The non-unimodular case is slightly more complicated. In general, the orthogonality relations involve a positive selfadjoint operator $D_U$ in $\mathcal{H}$ (the Duflo-Moore operator [8,14–16]), whose domain coincides with $\mathcal{A}(U)$, and that is bounded if and only if $G$ is unimodular. In such case, $D_U$ is just a positive multiple of the identity $-D_U = D_U I$, $D_U \equiv c_U^{1/2}$ — and we get (4).

**Remark 3.** In the following we will consider, in particular, two remarkable classes of square integrable representations of unimodular l.c.s.c. groups (see the examples in sect. 4):

- **Every** irreducible unitary representation of a compact group $G$ is (finite-dimensional and) square integrable, because the associated Haar measure $\mu_G$ is finite. If $\mu_G$ is normalized as a probability measure, then, according to the Peter-Weyl theorem [10], $c_U = \dim(\mathcal{H})^{-1}$.

- The irreducible projective representations of the group $\mathbb{R}^n \times \mathbb{R}^n$ of phase-space translations with symplectic multiplier $\gamma_h(q, p ; \hat{q}, \hat{p}) = \exp(i(q \cdot \hat{p} - p \cdot \hat{q})/2\hbar)$, where $\hbar$ is any nonzero real number, are square integrable too and form a unitary equivalence class (two representations in the unitary equivalence classes associated, respectively, with $\hbar$ and $-\hbar$ are anti-unitarily equivalent) [16,17]. For $\hbar \equiv \hbar > 0$, this parameter can be interpreted as Planck’s constant, and an irreducible representation of $\mathbb{R}^n \times \mathbb{R}^n$ with multiplier $\gamma_h$ is called a (h-)Weyl system.
4. Covariant stochastic products from square integrable representations

We now construct the twirled products [9]. Let $G$ be a unimodular l.c.s.c. group, admitting square integrable representations; let us denote by $\mathcal{B}(G)$ the Borel $\sigma$-algebra of $G$ and by $\mathcal{PM}(G)$ the set of Borel probability measures on $G$. Then, the main ingredients of our construction are:

(i) We choose an (irreducible) square integrable projective representation $U$ of $G$ in a (separable) complex Hilbert space $\mathcal{H}$. $U$ gives rise to an isometric representation of $G$ in $B_1(\mathcal{H})$: $U \cup U(g)A := U(g)A(U(g))^*$ (a symmetry action [11–13] of $G$ on the space where the quantum states live). Notice that, although $U$ is in general projective — $U(gh) = \gamma(g, h) U(g) U(h)$ — where $\gamma(g, h) \in T$ is a group multiplier, $U \cup U$ behaves like an ordinary group representation: $U \cup U(gh) = U \cup U(g) U \cup U(h)$. We will also use the shorthand notation $\rho_\gamma \equiv U \cup U(\rho)$.

(ii) We choose a fiducial density operator $\nu \in \mathcal{S}(\mathcal{H})$ and a measure $\varpi \in \mathcal{PM}(G)$. We denote by $\varpi^g (\varpi_g)$ the left (right) $g$-translate of $\varpi$: $\varpi^g(\delta) := \varpi(g^{-1}\delta)$ ($\varpi_g(\delta) := \varpi(\delta g)$), $\delta \in \mathcal{B}(G)$.

Proposition 3. For every $\mu \in \mathcal{PM}(G)$, the linear map

$$ \mu[U]: B_1(\mathcal{H}) \ni A \mapsto \int_G d \mu(g) \ (U \cup U(g) A) \in B_1(\mathcal{H}) $$

— where a Bochner integral on $G$ is understood — is stochastic; namely: $\mu[U] S(\mathcal{H}) \subset S(\mathcal{H})$.

Proposition 4. For a suitable normalization of the Haar measure $\mu_G$ on $G$, and for every $\rho \in S(\mathcal{H})$, the map

$$ \nu_{\rho,\varpi} : \mathcal{B}(G) \ni \delta \mapsto \int_G d \mu_G(g) \ \text{tr}(\rho (U \cup U(g) \nu)) $$

is a Borel probability measure on $G$.

Remark 4. Proposition 4 relies on the square-integrability of the representation $U$ [9], whereas Proposition 3 does not [9,18]. The stochastic map $\mu[U]$, see (5), is called twirling operator [18].

Taking into account these results, we can define a binary operation on $S(\mathcal{H})$ by setting [9]

$$ \rho \oslash \sigma := \left( (\nu_{\rho,\varpi} \otimes \varpi)(U) \right) \sigma = \int_G d (\nu_{\rho,\varpi} \otimes \varpi)(g) \ (U \cup U(g) \sigma), \quad \forall \rho, \sigma \in S(\mathcal{H}). $$

Here, we have considered the fact that $\nu_{\rho,\varpi}$ is a probability measure associated with the density operators $\rho$ and $\nu$ (Proposition 4), then we have formed the convolution $\mu \equiv \nu_{\rho,\varpi} \otimes \varpi$ of this measure with the previously fixed probability measure $\varpi$ and, finally, we have constructed the stochastic map $(\nu_{\rho,\varpi} \otimes \varpi)[U]$ according to Proposition 3. More explicitly, we have:

$$ \rho \oslash \varpi = \int_G d \mu_G(g) \int_G d \varpi(h) \ \text{tr}(\rho (U \cup U(gh) \nu)) (U \cup U(gh) \sigma). $$

We stress that, in this formula, Bochner integrals of $B_1(\mathcal{H})$-valued functions are understood and the Haar measure $\mu_G$ is supposed to be normalized in such a way that $c_U = 1$ [9]; recall Remark 4 and relation (4). In particular, for $\varpi = \delta \equiv \delta_e$ (the Dirac measure at the identity $e$ of $G$), we get:

$$ \rho \oslash \delta \equiv \rho \oslash \varpi = \int_G d \mu_G(g) \ \text{tr}(\rho (U \cup U(g) \nu)) (U \cup U(g) \sigma). $$

Remark 5. Notice that we do not claim that the members of family of stochastic products

$$ \{ (\cdot) \oslash (\cdot) : v \in S(\mathcal{H}), \ \varpi \in \mathcal{PM}(G) \} $$

are square-integrable.
are all distinct one another. On the contrary, using the fact that the iterated integrals in (8) can be interchanged (by Fubini’s theorem for Bochner integrals), next performing the change of variables \( g \mapsto gh \) and, finally, inverting the integration order again and exploiting the fact that the Bochner integral wrt the measure \( \varpi \) commutes with both the trace functional and the bounded operator \( U \vee U(g) \) in \( B_1(\mathcal{H}) \), one finds out that

\[
\rho \sphericalarrow{\varpi} \sigma = \rho \sphericalarrow{\varpi} [U] \sigma,
\]

(11)

where \( \varpi \in \mathcal{P}M(G) \) is defined by \( \varpi(e) := \varpi(e^{-1}), e \in \mathcal{B}(G) \), and

\[
\varpi[U]v := \int_G d\varpi(g) (U \vee U(g)v) = \int_G d\varpi(g) (U \vee U(g^{-1})v) \in \mathcal{S}(H).\]

(12)

Moreover, certain choices of \( \upsilon \in \mathcal{S}(H) \) or \( \varpi \in \mathcal{P}M(G) \) may ‘trivialize’ the associated products; see Example 1 below.

We now formulate our main result [9].

**Theorem 1.** For every \( \upsilon \in \mathcal{S}(H) \) and \( \varpi \in \mathcal{P}M(G) \), the binary operation

\[
(\cdot) \sphericalarrow{\varpi} (\cdot) : \mathcal{S}(H) \times \mathcal{S}(H) \rightarrow \mathcal{S}(H) \quad \text{('twirled product')}
\]

(13)

is an associative stochastic product that is left-covariant wrt the representation \( U \), namely,

\[
\rho \sphericalarrow{\varpi} \upsilon \sphericalarrow{\varpi} \sigma = \rho \sphericalarrow{\varpi} \upsilon (g) \sphericalarrow{\varpi} \sigma (g), \quad \forall g \in G, \ \forall \rho, \sigma \in \mathcal{S}(H).
\]

(14)

Extending the twirled product to a stochastic product on the space \( B_1(\mathcal{H}) \) of trace class operators, one gets a Banach algebra that, in the case where the l.c.s.c. group \( G \) is abelian, is commutative.

Beside covariance, there are two properties of families of twirled products: invariance and equivariance. Let \( X \) be a \( G \)-space wrt to some group action \((\cdot)[\cdot] : G \times X \ni (g, \xi) \mapsto g[\xi] \in X\). Suppose that the points of \( X \) label a family of stochastic products:

\[
\{(\cdot) \sphericalarrow{\xi} (\cdot) : \mathcal{S}(H) \times \mathcal{S}(H) \rightarrow \mathcal{S}(H)\}_{\xi \in X}.
\]

(15)

We say that this family of products is **invariant** wrt the action \((\cdot)[\cdot]\) if

\[
\rho \sphericalarrow{\xi} \upsilon \sphericalarrow{\xi} \sigma = \rho \sphericalarrow{\xi} \upsilon (g) \sphericalarrow{\xi} \sigma (g), \quad \forall g \in G, \ \forall \xi \in X, \ \forall \rho, \sigma \in \mathcal{S}(H).
\]

(16)

We say that \( \{(\cdot) \sphericalarrow{\xi} (\cdot)\}_{\xi \in X} \) is **right inner equivariant** wrt the pair \((\cdot)[\cdot], U\) if

\[
\rho \sphericalarrow{\xi} (U \vee U(g^{-1}) \sigma) = \rho \sphericalarrow{\xi} \upsilon (g) \sphericalarrow{\xi} \sigma (g), \quad \forall g \in G, \ \forall \xi \in X, \ \forall \rho, \sigma \in \mathcal{S}(H).
\]

(17)

Then, we have the following [9]:

**Proposition 5.** The family of twirled products

\[
\{(\cdot) \sphericalarrow{\upsilon} (\cdot) : \upsilon \in \mathcal{S}(H), \ \varpi \in \mathcal{P}M(G)\}
\]

(18)

is invariant wrt the group action

\[
g((\upsilon, \varpi)) := (\upsilon_g \equiv U \vee U(g) \upsilon, \varpi_g);
\]

(19)
\[ \rho \oslash \sigma = \rho \oslash \sigma, \quad \forall g \in G, \forall \rho, \sigma \in \mathcal{S}(\mathcal{H}), \forall \varpi \in \mathcal{PM}(G). \] (20)

It is right inner equivariant wrt the pair \((\cdot|\cdot), U\), where this time \((\cdot|\cdot)\) is the group action
\[ g[(v, \varpi)] := (v, \varpi g); \] (21)

namely,
\[ \rho \oslash \sigma = \rho \oslash \sigma, \quad \forall g \in G, \forall \rho, \sigma \in \mathcal{S}(\mathcal{H}), \forall \varpi \in \mathcal{PM}(G). \] (22)

Remark 6. The invariance relation (20) can be regarded as a consequence of (11) and of the fact that \(\varpi[U]v = \varpi[U]v_g\). Moreover, in the case where \(G\) is abelian, one can derive further symmetry relations:
\[ \rho_g \oslash \sigma = \left( \rho \oslash \sigma \right)_g, \quad \forall g \in G, \forall \rho, \sigma \in \mathcal{S}(\mathcal{H}), \forall \varpi \in \mathcal{PM}(G). \] (23)

\[ \rho \oslash \sigma = \rho \oslash \sigma_g, \quad \forall \rho, \sigma, \rho_g \in \mathcal{S}(\mathcal{H}), \forall \varpi \in \mathcal{PM}(G). \] (24)

\[ \rho \oslash \sigma = \left( \rho \oslash \sigma \right)_{g^{-1}}, \quad \forall \rho, \sigma, g \in G. \] (25)

Example 1. Suppose that the topological group \(G\) is compact and \(U\) is an irreducible unitary representation of \(G\); hence, \(\gamma \equiv 1\) (the multiplier of \(U\) is trivial) and \(\text{dim}(\mathcal{H}) = n < \infty\). Because, in this case, the Haar measure \(\mu_G\) on \(G\) is finite, \(U\) is square integrable. By the Peter-Weyl theorem [10], assuming that \(\mu_G(G) = 1\) (i.e., \(\mu_G \in \mathcal{PM}(G)\)), we have: \(\mathcal{c}_U = \text{dim}(\mathcal{H})^{-1} = n^{-1}\); therefore:
\[ \rho \oslash \sigma = n \int_G \text{d}\mu_G(g) \int_G \text{d}\varpi(h) \text{tr}(\rho(Ug\varpi)(Ug\varpi h)\sigma). \] (26)

For the maximally mixed state \(\Omega := n^{-1}I\) (where \(I\) is the identity operator), we have [9]:
\[ \rho \oslash \Omega = \Omega, \quad \Omega \oslash \sigma = \Omega, \quad \rho \oslash \sigma = \Omega, \quad \forall \rho, \sigma, \in \mathcal{S}(\mathcal{H}), \forall \varpi \in \mathcal{PM}(G). \] (27)

The last of the previous relations shows that the choice of \(\Omega\) as a fiducial state ‘trivializes’ the twirled product. Similarly, putting \(\varpi = \mu_G\), one obtains a trivial stochastic product too:
\[ \rho \oslash \sigma = \Omega, \quad \forall \rho, \sigma \in \mathcal{S}(\mathcal{H}). \] (28)

Exploiting relation (11), with \(\varpi = \mu_G\), and then the fact that \(\mu_G[U]v = \Omega (\mu_G = \tilde{\mu}_G)\), for all \(v \in \mathcal{S}(\mathcal{H})\) — see formula (128) of [9] — this statement can be derived as a consequence of the last of relations (27), with \(\varpi = \delta\). Using, say, the irreducible unitary representations of the compact group \(SU(2)\), one can construct twirled products for every finite Hilbert space dimension.

Example 2. We now discuss an infinite-dimensional example. Let \(G\) be the group of translations on phase space — \(G = \mathbb{R}^n \times \mathbb{R}^n\) — let us identify \(\mathcal{H}\) with the Hilbert space \(L^2(\mathbb{R}^n)\) and, putting \(\hbar = \hbar = 1\), let \(U\) be the standard Weyl system [16, 17, 19, 20] defined by
\[ (U(q, p)f)(x) := e^{-i\eta q^2/2} e^{i\eta p} f(x - \eta), \quad (q, p) \in \mathbb{R}^n \times \mathbb{R}^n; \] (29)
i.e., \(U(q, p)\) is the ordinary (vector) position and momentum operators in \(L^2(\mathbb{R}^n)\). This is a strongly continuous, irreducible projective representation — with
symplectic multiplier $\gamma(q,p;q,\tilde{q},\tilde{p}) = \exp(i(q\cdot\tilde{p} - p\cdot\tilde{q})/2)$ — that, as previously mentioned, is square integrable. Setting $L^2(G) = L^2(\mathbb{R}^n \times \mathbb{R}^n, (2\pi)^{-n}d^nq\,d^n\hat{p}\,\mathbb{C})$, we have that $e_0 = 1$; thus:

$$\tau = \rho U^{\omega} \sigma = (2\pi)^{-n} \int d^nq\,d^n\hat{p}\,\text{tr}\left(\rho \left(e^{ip\hat{q}} e^{-iq\hat{p}} \nu \ e^{ip\hat{q}} e^{-iq\hat{p}} \right) \right) \times \int d\varpi(\tilde{q},\tilde{p}) \left(e^{i(p+\tilde{p})\hat{q}} e^{-i(q+\tilde{q})\hat{p}} \sigma \ e^{i(q+\tilde{q})\hat{p}} e^{-i(p+\tilde{p})\hat{q}} \right). \quad (30)$$

This product — the so-called phase-space stochastic product [9] — is commutative, since it stems from a representation of an abelian group. For $\varpi = \delta = \delta_e$, we get the quantum convolution [9]:

$$\tau = \rho U^{\omega} \sigma = (2\pi)^{-n} \int d^nq\,d^n\hat{p}\,\text{tr}\left(\rho \left(e^{ip\hat{q}} e^{-iq\hat{p}} \nu \ e^{ip\hat{q}} e^{-iq\hat{p}} \right) \right) \left(e^{ip\hat{q}} e^{-iq\hat{p}} \sigma \ e^{ip\hat{q}} e^{-iq\hat{p}} \right). \quad (31)$$

To justify this term, let us express this operation in terms of the Wigner distributions [8,16,19–23] $W_{\rho}$, $W_v$, $W_{\varpi}$, $W_\tau$ of the states $\rho, v, \sigma, \tau$. Setting $\hat{W}_v(x) := W_v(-x)$, $x \equiv (q,p) \in \mathbb{R}^2$, we find:

$$W_\tau(z) = \int d^2q \left( \int d^2p\, W_\rho(p) \hat{W}_v(x-y) \right) W_\sigma(z-x). \quad (32)$$

Notice that we get a double convolution of Wigner functions, where $W_\tau$ plays a pivotal role with no analogue in the classical case. The function $(q,p) \mapsto \int d^n\tilde{q} \,d^n\tilde{p} \, W_\rho(\tilde{q},\tilde{p}) \, \hat{W}_v(q-\tilde{q},p-\tilde{p})$ is a probability distribution wrt the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n$. E.g., putting $n = 1$, and choosing the pure state $\ket{\psi}\bra{\psi}$, with $\psi(q) = (2\pi)^{-1/4} e^{-q^2/4}$, as the fiducial state $v$ — whose Wigner function is $W_v(q,p) = (2\pi)^{-1} e^{-(q^2+p^2)}$ — we find the probability distribution $Q_\rho(q,p) = \pi^{-1} \int d\tilde{q} \,d\tilde{p} \, W_\rho(\tilde{q},\tilde{p}) \, e^{-(q^2+p^2)}$, the so-called Husimi-Kano function associated with $\rho$ [23]. Thus, in this case the quantum convolution, in terms of phase-space functions, is of the form

$$W_\tau(q,p) = \int d\tilde{q} \,d\tilde{p} \, Q_\rho(\tilde{q},\tilde{p}) \, W_\sigma(q-\tilde{q},p-\tilde{p}), \quad \tau = \rho U^{\omega} \sigma. \quad (33)$$

5. Final remarks and perspectives

Quantum measurements are described by POVMs and (quantum) instruments [7]. The mapping

$$\mathcal{B}(G) \ni \mathcal{E} \mapsto \int_{\mathcal{E}} d\mu_G(g) \left( U \cup U(g) \nu \right) =: E_v(\mathcal{E}) \quad (U \text{ square integrable}, \nu \in \mathcal{S}(H)) \quad (34)$$

is a covariant quantum observable; i.e., a POVM covariant wrt the irreducible representation $U$: $E_v(g\mathcal{E}) = U \cup U(g) E_v(\mathcal{E}), \forall g \in G, \forall \mathcal{E} \in \mathcal{B}(G)$. The function $p_{\nu,v}(g) = \text{tr}(\rho(U \cup U(g) \nu))$, that we used in the construction of a stochastic product, is the probability density on $G$ of the observable $E_v$ wrt the state $\rho$; namely, the Radon-Nikodym derivative of the probability measure $\nu_{\nu,v}: \mathcal{E} \mapsto \int_{\mathcal{E}} d\mu_G(g) \, \text{tr}(\rho(U \cup U(g) \nu))$ (recall Proposition 4) wrt $\mu_G$. Moreover, the mapping

$$\mathcal{B}(G) \ni \mathcal{E} \mapsto \left( B_1(\mathcal{H}) \ni A \mapsto \int_{\mathcal{E}} d\mu_G(g) \, \text{tr}(A(U \cup U(g) \nu)) \right) =: T_{\nu,\sigma}^{\mathcal{E}} A \quad (35)$$

is a covariant quantum instrument based on $G$, because $T_{\nu,\sigma}^{\mathcal{E}}(U \cup U(g)A) = U \cup U(g) \left( T_{\nu,\sigma}^{\mathcal{E}} A \right), \forall g \in G, \forall \mathcal{E} \in \mathcal{B}(G)$. Observe that the twirled product generated by the triple $(U,v,\delta)$ can be recovered by setting $\mathcal{E} = G$; i.e., $\rho \circ \delta \sigma = T_{\nu,\sigma}^{\mathcal{E}} \rho$. Also note that the associativity of the twirled stochastic product translates into the following
relation involving the pair of stochastic maps $\mathcal{I}_G^{\rho,\sigma}$ and $\mathcal{I}_G^{\sigma,\rho}$: $\mathcal{I}_G^{\rho,\sigma} \circ \mathcal{I}_G^{\nu,\rho} = \mathcal{I}_G^{\nu,\tau}$, where $\tau = \rho \circ \sigma$. If $\varpi \neq \delta$, the quantum observable $E_v$ is replaced with the smeared observable $E_{v|\varpi}$:

$$E_{v|\varpi}(\varphi) \equiv E_v \otimes \varpi(\varphi) := \int_G \mathrm{d}\varpi(h) \ E_v(\varphi h) = \int_G \mathrm{d}\varpi(h) \int_\varphi \mathrm{d}\mu_G(g) \ (U \cup U(g h^{-1}) \nu),$$

and the probability density $p_{v,\nu}(g)$ is replaced with $p_{v,\varpi}(g) = (p_{v,\nu} \otimes \varpi)(g) = \int \mathrm{d}\varpi(h) \ p_{v,\nu}(gh^{-1})$. An analogous smearing takes place, when $\varpi \neq \delta$, for the quantum instrument $\varphi \mapsto \mathcal{I}_K^{\nu,\varphi}$. It would be worth studying further aspects of stochastic products (in particular, of twirled products) [9]:

- There is a natural notion of complete positivity for a stochastic product.
- The classification of covariant stochastic products is an interesting issue.
- Twirled products generated by representations of non-unimodular groups can be considered.
- It is natural to wonder whether the twirled products are entropy-nondecreasing [24,25].

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