Parametric sensitivity analysis of Aerodynamic stability for planar membrane structures

Lingling Du\textsuperscript{1} and Weiju Song\textsuperscript{2,3}

\textsuperscript{1}School of Road, Bridge and Architecture, Chongqing Vocational College of Transportation, Chongqing, China
\textsuperscript{2}School of Civil Engineering, Hebei University of Engineering, Handan, China
\textsuperscript{3}Corresponding author’s e-mail: nimrodsong@126.com

Abstract. To explore the influence of design parameters on the aerodynamic stability of planar membrane structures, a mathematical model for calculating the critical wind speed of flat panel structures was established, and the closed expression of the critical wind speed was obtained through the criterion of instability. The influence of membrane size, membrane density, pretension and other parameters on the critical wind speed was studied by monofactor analysis. The results show that, with the help of the new concept of benefit loss coefficient, the sensitivity analysis of instability critical wind speed parameters can be easily carried out, and many qualitative and quantitative laws can be obtained.

1. Introduction
In the field of construction engineering, the membrane is easy to vibrate under wind load because of its small self weight and low stiffness. If the initial pretension of the membrane material is too small, the membrane structure may appear instability, or even tear failure due to too large vibration amplitude \cite{1, 2}. The research of Shen Shizhao, Wu Yue and other scholars \cite{3} shows that when the flexible membrane structure is in a relaxed state, that is, when the pretension is insufficient, the membrane structure will appear instability and damage. In 2011, Zheng zhoulian et al. \cite{4, 5} determined the instability critical state of flat membrane structure with small sag and saddle membrane structure under wind load, and proved that the instability critical wind speed is related to the initial pretension of membrane material; In 2015, Chen Zhaoqing et al. \cite{6} conducted the wind tunnel test of aeroelastic model of unidirectional tensioned membrane structure, and discussed its instability mechanism. In 2017, Liu et al. \cite{7} used Galerkin method to study the stability of tensioned membrane structure under wind load, and the main control factors to improve the stability of the membrane structure were obtained. However, all the studies did not consider the large deformation of the membrane under wind load. That is to say, the strong nonlinearity of the membrane vibration is ignored. In 2020, song Weiju et al. \cite{8} used the perturbation method to discuss the influence of the strong nonlinear vibration characteristics of the membrane on the aerodynamic stability of the structure. However, the main impact parameters and secondary impact parameters were not analyzed in the study.

The main factors affecting the aerodynamic stability of membrane structures are the physical and geometric parameters of membrane materials, such as the density and size of membrane materials. Both of them have strong or weak uncertainty, which leads to the uncertainty of the critical wind speed. Therefore, the parameter analysis is very essential. The ratio of $V_{cr}$ after a change to that before change

\[ \frac{V_{cr}}{V_{cr}} = \text{Benefit Loss Coefficient} \]
is expressed as $R_V$, which can be called the benefit loss coefficient. When $R_V > 1$, $V_{cr}$ increased; when $R_V < 1$, $V_{cr}$ decreased.

In order to qualitatively and quantitatively grasp the influence of these factors on the critical wind speed loss coefficient $R_V$ of membrane structure instability. A unified mathematical model for calculating the critical wind speed of membrane structures is established based on the theoretical solution of the planar membrane structure in the paper. Next, the sensitivity analysis of the critical wind speed of the unstable structure is carried out with the relevant parameters of the flat membrane structure. Finally, the important properties of the critical wind speed coefficient $R_V$ are proposed and demonstrated.

2. Mathematical model

The length and width of the membrane structure are $a$ and $b$, respectively; the pretension along the length direction is $N_{0x}$, and the pretension along the width direction is $N_{0y}$; the wind blows parallel to the membrane surface. Under the condition of relaxation or insufficient pretension, single-mode divergent instability characterized by large amplitude vibration will occur [8]. According to the test results of Usenatsu and Kimoto [8], the membrane does not tear when the single-mode divergent instability occurs, only the vibration displacement amplitude jumps. Therefore, it is assumed that the membrane is still in the stage of linear elasticity when the instability occurs.

![Figure 1. Tensioned membrane model.](image)

The generalized external load includes wind load, structural damping force and inertia force[4, 5]. In the process of membrane vibration, the influence of shear stress is ignored. The calculation model as shown in Figure 1, and the dynamic governing equations of the structural membrane are:

$$
\begin{align*}
\left( N_{0x} + h \frac{\partial^2 \phi}{\partial x^2} \right) \frac{\partial^2 W}{\partial x^2} + \left( N_{0y} + h \frac{\partial^2 \phi}{\partial y^2} \right) \frac{\partial^2 W}{\partial y^2} + \rho \frac{\partial^2 W}{\partial t^2} - \rho \frac{\partial^2 W}{\partial t^2} &= 0, \\
\frac{1}{E_1} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{E_2} \frac{\partial^2 \phi}{\partial y^2} &= \left( \frac{\partial^2 W}{\partial x^2} \right)^2 - \frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} - k_{0x} \frac{\partial^2 W}{\partial x^2} - k_{0y} \frac{\partial^2 W}{\partial y^2},
\end{align*}
$$

(1)

where $h$ is the thickness of the membrane, $E_1$ and $E_2$ denote Young’s modulus in $x$ and $y$, respectively; $W$ is the normal vibration displacement of membrane, $\rho$ is the area density of membrane material; $c$ denotes structure-self’s damping coefficient, $N_{0x}$ and $N_{0y}$ denote initial tension in $x$ and $y$, respectively; $\phi = \phi(x, y, t)$ is the stress function. The initial surface function of the planar membrane structure is $z_0(x, y) = 0$. Therefore,

$$
z(x, y, t) = w(x, y, t),
$$

(2)

$$
k_{0x} = \frac{\partial^2 z_0}{\partial x^2} = 0, \quad k_x = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}, \quad k_{0y} = \frac{\partial^2 z_0}{\partial y^2} = 0, \quad k_y = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 w}{\partial y^2}
$$

(3)

The boundary conditions are

$$
w(0, y, t) = w(a, y, t) = w(x, 0, t) = w(x, b, t) = 0.
$$

(4)

Let the solution of the governing equation satisfying the boundary condition be [9]
where \( m \) and \( n \) are integer, and denotes the order of vibration.

In reference [4], the wind-induced partial differential equation of membrane structure is obtained by Galerkin method as

\[
\left\{ \begin{array}{l}
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,
\end{array} \right.
\]

where

\[
\begin{aligned}
\alpha_1 &= \int \int \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \, dx \, dy,
\end{aligned}
\]

\[
\begin{aligned}
\alpha_2 &= \int \int \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \, dx \, dy,
\end{aligned}
\]

\[
\begin{aligned}
\alpha_3 &= \int \int \left( x - \xi \right) \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \, dx \, dy,
\end{aligned}
\]

\[
\begin{aligned}
\alpha_4 &= \int \int \left( x - \xi \right) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \, dx \, dy.
\end{aligned}
\]

Observe \( \frac{\rho \pi ab + 4 \rho \alpha_1}{4\pi} \neq 0 \) in equation (6). Let \( W = W(t) \). According to the calculation principle of the improved multi-scale method [9], the perturbation parameter can be determined as

\[
\varepsilon = \frac{h \pi^2 \left( E_{a} a^{4} n^{4} + E_{b} b^{4} m^{4} \right)}{16a^{4} b^{4} \left( \rho \pi ab + 4 \rho \alpha_{1} \right)}.
\]

Then equation (6) is simplified as

\[
W^* + \alpha_0^2 W + \varepsilon \left( \mu W^* + W' \right) = 0,
\]

where

\[
\alpha_0^2 = \frac{\pi \left( m^2 \pi^2 b^2 N_{0} + n^2 \pi^2 a^2 N_{0} - 2 \rho_{\delta} \rho V \alpha_{s} \right)}{ab \left( \rho \pi ab + 4 \rho \alpha_{1} \right)}, \quad \mu = \frac{2 \left( \rho_{\delta} \pi m V \alpha_{s} - \rho \rho_{\delta} V \alpha_{s} + \rho \pi c \alpha_{s} \right)}{a \varepsilon \left( \rho \pi ab + 4 \rho \alpha_{1} \right)}.
\]

Let \( \omega \) be the frequency of membrane vibration, and expand \( \alpha_0^2 \) into a power series of \( \varepsilon \) [9, 10]. The expansion of \( \alpha_0^2 \) into a power series of \( \varepsilon \) is as follows [9, 10]

\[
\alpha_0^2 = \alpha_0^2 + \varepsilon \omega_0 \left[ 1 + \frac{1}{\alpha_0} \left( \varepsilon^2 \alpha_0 + \varepsilon^3 \alpha_3 + \ldots \right) \right] = \frac{\omega_0}{1 - \varepsilon},
\]

\[
\omega = \omega_0 \left[ 1 + \frac{1}{2} + \frac{3}{8} + \frac{5}{2} + \ldots \right],
\]

where \( \alpha = \frac{\varepsilon \omega_0}{\alpha_0} + \omega_0 \).

The solution of equation (7) can be assumed to be

\[
W(t, \alpha) = W_0(T_0, T_0) + \alpha W_1(T_0, T_0) + \alpha^2 W_2(T_0, T_0) + \ldots,
\]

where \( T_0 = t; T_1 = \alpha t \). Then, the differential operator can be obtained as follows:

\[
\frac{d}{dt} = D_0 + \alpha D_1 + \alpha^2 D_2 + \ldots,
\]

\[
\frac{d^2}{dt^2} = D_0^2 + 2 \alpha D_0 D_1 + \alpha^2 (D_1^2 + 2D_0 D_1) + \ldots
\]

Substituting equations (8) and (11) into equation (7) and sorting them out, yields
(1 - \alpha) \left[ D_0^2 + 2\alpha D_0 D_1 + \alpha^2 (D_1^2 + 2D_1D_2) \right] \left[ W_0 + \alpha W_1 + \alpha^2 W_2 + \cdots \right] + (1 - \alpha) \omega_0^2 W_0 + \alpha \omega_1^2 W_1 + \alpha^2 \omega_2^2 W_2 + \cdots = 0. \quad (12)

Expanding equation (12) and letting the coefficients of the powers of \alpha in the equation be zero, yields

\alpha^0: \quad D_0^2 W_0 + \omega_0^2 W_0 = 0, \\
\alpha^1: \quad D_0^2 W_1 + \alpha_0^2 W_1 + 2D_0 D_1 W_0 + \frac{\omega_0^2}{\omega_1^2} (D_0 W_0 + W_0^2) = 0, \\
\alpha^2: \quad D_0^2 W_2 + 2D_0 D_1 W_2 + (D_1^2 + 2D_1D_2) W_0 + \frac{\omega_0^2}{\omega_1^2} \omega_2^2 W_2 = 0.

By solving equation (13) and omitting the higher-order term, the expression of the vibration frequency can be obtained as follows

\omega = \sqrt{\omega_0^2 + \frac{3}{4} \varepsilon f^2}. \quad (14)

The research results of some scholars show that when the membrane structure reaches the critical state of instability, the frequency of the characteristic equation of the system tends to zero [2, 3]. That is

\omega_0^2 + \frac{3}{4} \varepsilon f^2 = 0, \quad (15)

where f is the vibration amplitude of the membrane.

Therefore, the critical wind speed of membranes are expressed as

V_{cr} = \sqrt{\frac{4\pi \left( m^2 a^2 b^2 N_0 + n^2 \pi^2 a^2 N_0 \right) + 3 \varepsilon f^2 (\rho \pi a^2 b^2 + 4\rho_b \alpha c)}{8\pi \rho \rho_b \alpha}} \frac{\varepsilon f^2}{8\pi \rho \rho_b \alpha}, \quad (16)

where f is the vibration amplitude corresponding to the instability critical wind speed. It shows that the critical wind speed is related to the vibration amplitude. Due to the consideration of the geometrically nonlinear effect of membrane, the stiffness of membrane will change with the amplitude change during the vibration process, which will affect the stability of the structure to a certain extent. This is consistent with the existing research results.

3. Variation of profit loss coefficient with parameters

According to equation (16), there are many parameters that affect the critical wind speed of the flat tensioned membrane structure, including the longitudinal and transverse elastic modulus of the membrane, the size of the membrane, the span ratio of the transverse wind direction, the surface density of the membrane, the pretension, and so on. The basic parameters of the example are set as follows:

\rho = 1.226 kg/m^3, \quad a = b = 20 m, \quad N_{0x} = N_{0y} = 2 kN/m.

The main design parameters of membrane structure are taken as variables and the sensitivity of parameters (\rho, a, b, N_{0x}, N_{0y}) are analyzed for the critical wind speed. The objective function is the instability critical wind speed of flat membrane structure. The initial instability critical wind speed can be calculated as

V_{cr} = 32.19 m/s

The influence of each parameter on V_{cr} was studied. The ratio of each parameter before and after change was expressed as R. Univariate changes were performed during the analysis, and the initial univariate values were as described above.
3.1. The size of the membrane a and b (when a changes, b remains unchanged, and vice versa)

$R_V$ first decreases and then increases with the increase of the membrane size $a$ along the wind direction, and decreases with the increase of the membrane size $b$ across the wind direction. The first two calculations show this rule. When $R_a = R_b$, the value of $R_V$ corresponding to the first-order vibration $R_a$ is larger, as shown in Figure 2. It also shows that the downstream dimension $a$ of the membrane has a greater influence on the critical wind velocity $V_{cr}$. With the increase of mode number, the influence of membrane size on wind speed decreases gradually, but the overall trend remains unchanged. Therefore, it is more effective to reduce the critical wind speed $V_{cr}$ by reducing the membrane size $a$ along the wind direction.

![Figure 2](image1)

**Figure 2.** relationship of $R_a$, $R_b$, and $R_V$.

3.2. Pretension of the membrane $N_{0x}$, $N_{0y}$

For each mode, the profit and loss coefficient of instability critical wind speed is the same by changing the windward pretension $N_{0x}$ and the crosswind pretension $N_{0y}$, as shown in Figure 3, which can be seen from the form of equation (16). With the increase of the mode shape, the influence of the change of the pretension on the critical instability wind speed becomes greater and greater for the higher mode shape.

![Figure 3](image2)

**Figure 3.** Relationship of $R_N$ and $R_Y$.

3.3. Density of membrane

With the increase of vibration mode, membrane density has no effect on the profit and loss coefficient of instability critical wind speed, as shown in Figure 4. For the same mode, $R_V$ decreases with the increase of membrane density, which indicates that increasing membrane density can effectively reduce the critical wind speed.

Take the first-order vibration and plot the above results as shown in Figure 5.
4. Conclusion
The divergent instability of membrane structure is a kind of divergent vibration caused by aerodynamic instability. Its aerodynamic stability is controlled by many parameters, and the change of parameters will cause the change of critical instability wind speed $V_{cr}$. Based on the formula derived from the theory, this paper quantitatively analyzes the effects of membrane size, pretension and membrane density on the critical instability wind speed benefit loss coefficient $R_\nu$ of flat membrane structure. Conclusions are as follows.

(1) $R_\nu$ decreases first and then increases with the increase of along wind dimension $a$, and decreases with the increase of across wind dimension $b$. Moreover, the influence of along wind dimension $a$ on the critical wind velocity $V_{cr}$ is greater at low-order vibration. With the increase of mode number, the influence of membrane size on wind speed decreases gradually, but the overall trend remains unchanged. Therefore, it is more effective to reduce the critical wind speed $V_{cr}$ by reducing the membrane size $a$ along the wind direction.

(2) With the increase of vibration mode, the influence of the change of pretension on the critical instability wind speed is increasing obvious. However, the membrane density has little effect on the profit and loss coefficient of the critical wind speed.

(3) For the same mode, $R_\nu$ decreases with the increase of membrane density.

Acknowledgments
This work was financially supported by Natural Science Foundation of Hebei Province of China (Grant No. E2020402061) and the Innovation Foundation of Hebei University of Engineering(Grant No. SJ010002159).
References

[1] Li Q X and Sun B N 2006 Wind-induced aerodynamic instability analysis for closed membrane roofs[J]. Journal of Vibration Engineering, 19 (3) 346-353.

[2] Zheng Z L and Song W J 2012 Study on dynamic response of rectangular orthotropic membranes under impact loading [J]. Journal of Adhesion Science and Technology, 26(2012) 1467-1479.

[3] Shen S Z and Wu Y 2006 Research progress on fluid-solid interaction effect of wind-induced vibration response of membrane structure[J]. Journal of Architecture and Civil Engineering, 23(1) 1-9.

[4] Zheng Z L, Xu Y P, Liu C J, et al. 2011 Nonlinear aerodynamic stability analysis of orthotropic membrane structures with large amplitude[J]. Structural Engineering and Mechanics, v37, n4.

[5] Xu Y P, Zheng Z L, Liu C J and Song W J. 2011 Aerodynamic stability analysis of geometrically nonlinear orthotropic membrane structure with hyperbolic paraboloid [J]. Journal of Engineering Mechanics, 137 (11) 759-768.

[6] Chen Z Q 2015 Aeroelastic instability mechanism of tensioned membrane structures [D]. Harbin: Harbin University of Technology.

[7] Liu C, Deng X and Zheng Z 2017 Nonlinear wind-induced aerodynamic stability of orthotropic saddle membrane structures[J]. Journal of Wind Engineering and Industrial Aerodynamics. 2017(164) 119-127.

[8] Song W J, Xu J, Wang X W, et al. 2020 Effect of Geometric Nonlinearity on Membrane Roof Stability in Air Flow[J]. Shock and Vibration, 2020(1) 1-13.

[9] Li Y M, Song W J and Wang X W2019 Analytical solutions to strongly nonlinear vibration of a clamped membrane structure [J]. Journal of Vibration and Shock, 38(17): 144-148, 177.

[10] Chen S H 2009 Quantitative analysis method for strong nonlinear vibration systems [M]. Beijing: Science Press.