SHAPE-ENFORCING OPERATORS FOR POINT AND INTERVAL ESTIMATORS

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Abstract. A common problem in statistics is to estimate and make inference on functions that satisfy shape restrictions. For example, distribution functions are nondecreasing and range between zero and one, height growth charts are nondecreasing in age, and production functions are nondecreasing and quasi-concave in input quantities. We propose a method to enforce these restrictions ex post on point and interval estimates of the target function by applying functional operators. If an operator satisfies certain properties that we make precise, the shape-enforced point estimates are closer to the target function than the original point estimates and the shape-enforced interval estimates have greater coverage and shorter length than the original interval estimates. We show that these properties hold for six different operators that cover commonly used shape restrictions in practice: range, convexity, monotonicity, monotone convexity, quasi-convexity, and monotone quasi-convexity. We illustrate the results with an empirical application to the estimation of a height growth chart for infants in India.

Key words. Shape Operator, Range, Monotonicity, Convexity, Quasi-Convexity, Rearrangement, Legendre-Fenchel, Confidence Bands

1. Introduction

A common problem in statistics is to estimate and make inference on functions that satisfy shape restrictions. These restrictions might arise either from the nature of the function and variables involved or from theoretical reasons. Examples of the first case include distribution functions, which are nondecreasing and range between zero and one, and height growth charts, which are nondecreasing in age. Examples of the second case include demand functions of utility maximizing individuals, which are nonincreasing in price according to consumer demand theory; production functions of profit maximizing firms, which are nondecreasing and quasi-concave in input quantities according to production theory and can also be concave in industries that exhibit diminishing returns to scale; and bond yield curves, which are monotone and usually concave in time to maturity according to arbitrage pricing theory.

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We propose a method to enforce shape restrictions \textit{ex post} on point and interval estimates of functions by applying functional operators. If an operator satisfies certain properties that we make precise below, enforcing the shape restrictions improves the estimates. Thus, the shape-enforced point estimates are closer to the target function than the original point estimates under suitable distances, and the shape-enforced interval estimates have greater coverage and shorter length under suitable distances than the original interval estimates. We show that these properties hold for six different operators that enforce the following restrictions: range, convexity, monotonicity, convexity and monotonicity, quasi-convexity, and quasi-convexity and monotonicity. We impose the range restriction with a natural operator that censors the estimates to the desired range. The double Legendre-Fenchel transform enforces convexity by transforming the estimates into their greatest convex minorants. We rely on the monotone rearrangement to enforce monotonicity. We further develop a new operator to enforce quasi-convexity — a shape that has not been well explored in the literature, although it is common in economic theory. We also show that the compositions of the monotone rearrangement with the Fenchel-Legendre and the new quasi-convexity operators yield monotone convex and monotone quasi-convex estimates, respectively. In other words, the application of the convex and quasi-convex operators does not affect the monotonicity of the function. We also show how to modify the operators to deal with concavity, quasi-concavity, and their composition with monotonicity.

Our method is generic in that it can be applied to any point or interval estimator of the target function. For example, it works in combination with parametric, semi-parametric and nonparametric approaches to model and estimate the target function. Moreover, it applies without modification to any type of function including reduced form statistical objects such as conditional expectation, conditional probability and conditional quantile functions, or causal and structural objects such as dose response and demand curves identified and estimated using instrumental variable or other methods. The only requirement to obtain consistent point estimators or valid confidence bands is that the source point estimators be consistent or the source confidence bands be valid. There are many existing methods to construct such point estimators and confidence bands under general sampling conditions. Even if those requirements are not satisfied, the shape-enforced operators will bring improvements to the point estimators and confidence bands in a sense that we will make precise. To implement our method, we develop algorithms to compute the Legendre-Fenchel transform of multivariate functions and the new quasi-convexity enforcing operator.

We illustrate the results with an empirical applications to the estimation of a height growth chart for infants in India. In this case we impose natural monotonicity in the effect of age, together with concavity that is plausible during early childhood. We use series least square methods to flexibly estimate the conditional expectation function of interest, and construct
confidence bands using convenient bootstrap methods. We quantify the improvements that imposing shape restrictions bring to point and interval estimates in small samples through a numerical simulation calibrated to the empirical application.

Literature Review. Due to the wide range of applications of shape-restrictions, shape-restricted estimation and inference have received a lot of attention in the statistics community. Classical examples include Hildreth (1954), Ayer, Brunk, Ewing, Reid & Silverman (1955), Brunk (1955), van Eeden (1956), Grenander (1956), Groeneboom, Jongbloed & Wellner (2001), and Mammen (1991). We refer to Barlow, Bartholomew, Bremer & Brunk (1972) and Robertson, Wright & Dykstra (1988) for classical references on isotonic regression for monotonicity restrictions. In terms of estimation risk bounds, please refer to (Zhang 2002, Chatterjee, Guntuboyina & Sen 2014, Han, Wang, Chatterjee & Samworth 2017) and references therein for the recent developments in isotonic regression, and (Kuosmanen 2008, Seijo & Sen 2011, Guntuboyina & Sen 2015, Han & Wellner 2016) for convex regression. Bellec (2018) established sharp oracle inequalities for least squares estimators under various shape-restrictions. Moreover, some work (Hengartner & Stark 1995, Dümbgen 2003, Anevski & Hössjer 2006) considered the construction of confidence bands for univariate functions under monotonicity or convexity restrictions. Please refer to the book Groeneboom & Jongbloed (2014) and the survey paper Guntuboyina & Sen (2017) for more comprehensive reviews under shape constraints.

Most existing works developed constrained methods that directly produce estimated functions that satisfy the shape restrictions. This approach usually delivers estimators that are more efficient than their unconstrained counterparts at the cost of increased computational complexity. A recurrent problem with this approach is that the derivation of the statistical properties of the constrained estimators is extremely involved and specific to the estimator and shape restriction. As a consequence, there exist very few distributional results, mainly for univariate functions. The results available for the Grenander and isotonic regression estimators show that these estimators exhibit non-standard asymptotics; see Guntuboyina & Sen (2017) for a recent review. Moreover, Horowitz & Lee (2017) and Freyberger & Reeves (2017) have recently pointed out the difficulties of developing inference methods from shape-constrained estimators with good uniformity properties with respect to the data generating process. We avoid all these complications with the constrained estimators by enforcing the restrictions ex post and therefore relying on the distribution of the unrestricted estimators to construct the confidence bands. It is worthwhile noting that the idea of ex post confidence bands was mentioned in Section 4.2 in Dümbgen (2003), which only discussed two cases on the univariate monotone function and univariate convex function. Moreover, the construction of confidence bands for the convex case in Dümbgen (2003) is quite different from us (e.g., their lower bound is not necessarily a convex function).
Some of the operators that we consider have been previously analyzed in the literature. For example, Chernozhukov, Fernández-Val & Galichon (2009) followed the same approach as in this paper to produce improved point and interval estimates of monotone functions using the monotone rearrangement. Chernozhukov, Fernández-Val & Galichon (2010) applied the monotone rearrangement to deal with the quantile crossing problem and Belloni, Chernozhukov, Chetverikov & Fernández-Val (2011) to impose monotonicity in conditional quantile functions estimated using series quantile regression methods. Beare, Fang et al. (2017) used the double Legendre-Fenchel transform to construct point and interval estimates of univariate concave functions on the non-negative half-line. Other economic applications of the double Legendre-Fenchel transform include Delgado & Escanciano (2012), Beare & Moon (2015), Beare & Schmidt (2016), and Luo & Wang (2017). Chernozhukov et al. (2010) and Beare et al. (2017) used an alternative approach to make inference on shape-restricted functions. Instead of applying the shape-enforcing operator to a confidence band constructed from an unrestricted estimator, they constructed confidence bands from the estimator after applying the shape-enforcing operator. To do so, they characterized the distribution of the restricted estimator from the distribution of the unrestricted via the delta method, after showing that the shape-enforcing operator is Hadamard or Hadamard directional differentiable. This approach usually yields narrower confidence bands than ours, but it is computationally more involved and requires additional assumptions and non-standard methods. For example, Beare et al. (2017) showed that the bootstrap is inconsistent for the distribution of constrained estimators after applying the double Legendre-Fenchel transform when the target function is not strictly concave. Finally, we refer to Matzkin (1994), and Chetverikov, Santos & Shaikh (2018) for excellent up-to-date surveys on the use of shape restrictions in Econometrics.

Relative to the literature, we summarize the major contributions of this paper as follows. First, we introduce an operator to enforce quasi-convexity, which extends the notion of unimodality to multiple dimensions and generalizes convexity constraints. Despite its importance in economics, the shape-restriction of quasi-convexity has not been well studied in the literature and Guntuboyina & Sen (2017) listed quasi-convexity as an open area in shape-restricted estimation. Second, we extend the use of the Legendre-Fenchel transform to construct improved point and interval estimates of general multivariate convex functions. Third, we show that the composition of the monotone rearrangement with the Legendre-Fenchel transform can be used to construct improved point and interval estimates of monotone convex functions. Fourth, we show that the composition of the monotone rearrangement with the our quasi-convex operator can be used to construct improved point and interval estimates of monotone quasi-convex functions. Fifth, we provide a new algorithm to compute the Legendre-Fenchel transform of multivariate functions. Sixth, we develop an algorithm to compute our quasi-convex operator.
Outline. The rest of the paper is organized as follows. Section 2 introduces the functional shape-enforcing operators and their properties, together with examples of operators that enforce the shape restrictions of interest. Section 3 discusses the use of shape-enforcing operators to obtain improved point and interval estimates of functions that satisfy shape restrictions. Section 4 provides an algorithm to compute the shape-enforcing estimators. Section 5 reports the results of the empirical application and numerical simulation calibrated to the application. The proofs of the main results are gathered in the Appendix.

Notation: For any measurable function \( f : \mathcal{X} \to \mathbb{R} \) and \( p \geq 1 \), let \( \| f \|_p := \left\{ \int_X |f(x)|^p dx \right\}^{1/p} \), the \( L^p \)-norm of \( f \), with \( \| f \|_\infty := \sup_{x \in \mathcal{X}} |f(x)| \), the \( L^\infty \)-norm or sup-norm of \( f \). We drop the subscript \( p \) for the Euclidean norm, i.e. \( \| f \| := \| f \|_2 \). For \( p \geq 1 \), let \( \ell_p(\mathcal{X}) := \{ f : \mathcal{X} \to \mathbb{R} : \| f \|_p < \infty \} \), the class of all measurable functions defined on \( \mathcal{X} \) such that the \( L^p \)-norms of these functions exist. For \( x, x' \in \mathbb{R}^k \), we say \( x \geq x' \) if every entry of \( x \) is no smaller than the corresponding entry in \( x' \). We also use \( a \vee b := \max(a, b) \) and \( a \wedge b := \min(a, b) \) for any \( a, b \in \mathbb{R} \).

2. Functional Shape-Enforcing Operators

2.1. Properties of Shape-Enforcing Operators. Assume that the function of interest, \( f \), is real-valued with domain \( \mathcal{X} \subset \mathbb{R}^k \), for some positive integer \( k \). Let \( \ell_0^\infty(\mathcal{X}) \) and \( \ell_1^\infty(\mathcal{X}) \) be two subspaces of \( \ell^\infty(\mathcal{X}) \) such that \( \ell_1^\infty(\mathcal{X}) \subset \ell_0^\infty(\mathcal{X}) \) and \( \mathbf{O} : \ell_0^\infty(\mathcal{X}) \to \ell_0^\infty(\mathcal{X}) \) be a functional operator. In our case \( \ell_0^\infty(\mathcal{X}) \) will be the class of unrestricted functions and \( \ell_1^\infty(\mathcal{X}) \) will be the subclass of functions that satisfy some restriction. We first introduce three properties that an operator needs to satisfy to be considered a shape-enforcing estimator.

Definition 1 (Shape-Enforcing Operator). We say that the operator \( \mathbf{O} \) is \( \ell_1^\infty \)-enforcing with respect to \( \ell_0^\infty(\mathcal{X}) \), if it satisfies the following properties:

1. Enforcement:
   \[
   \mathbf{O}f \in \ell_1^\infty(\mathcal{X}), \text{ for any } f \in \ell_0^\infty(\mathcal{X}).
   \]

2. Neutrality:
   \[
   \mathbf{O}f = f, \text{ for any } f \in \ell_1^\infty(\mathcal{X}).
   \]

3. Order Preservation:
   \[
   \mathbf{O}f \leq \mathbf{O}g, \text{ for any } f, g \in \ell_0^\infty(\mathcal{X}) \text{ such that } f \leq g.
   \]

The first property just means that the output of the operator needs to be a function that satisfies the shape restriction. The second property implies that the operator should do nothing when the input function already satisfies the shape restriction. The third property is a global more technical condition about preserving the order of the functions. In addition to these properties, we shall also use that many operators are contractive in that the distance between
two functions is reduced after applying the operator to both functions. We call such operators as “distance contractions”. This property is defined under a certain discrepancy measure between functions that we generically denote by $\rho$.

**Definition 2 (Distance-Reducing Operator).** We say that the operator $O$ is a $\rho$-distance contraction if

$$\rho(Of, Og) \leq \rho(f, g), \text{ for any } f, g \in \ell_0^\infty(\mathcal{X}). \quad (2.4)$$

Some particularly interesting cases of restricted classes $\ell_1^\infty(\mathcal{X})$ are subsets of functions that satisfy shape restrictions. In this paper, we focus on 6 types of shape restrictions: (1) range, (2) convexity, (3) monotonicity, (4) monotone convexity, (5) quasi-convexity, and (6) monotone quasi-convexity. Our methods also apply to the restrictions of concavity and quasi-concavity by noting that if $f$ is concave (quasi-concave), then $-f$ is convex (quasi-convex). In the case of monotonicity we focus on the case of monotonically nondecreasing functions. The methods also apply to monotonically nonincreasing functions noting that if $f$ is nondecreasing then $-f$ is nonincreasing.

2.2. **Range Restrictions.** We first consider the subset of range-restricted functions $\ell_\infty^\infty(\mathcal{X}) := \{f \in \ell^\infty(\mathcal{X}) : f \leq f(x) \leq \bar{f} \text{ for all } x \in \mathcal{X}\}$ for some $\underline{f} \leq \bar{f}$. A natural range-enforcing operator is:

**Definition 3 (R-Operator).** For any set $\mathcal{X} \subset \mathbb{R}^k$, the range operator $R : \ell^\infty(\mathcal{X}) \to \ell^\infty(\mathcal{X})$ is defined by

$$Rf(x) := \underline{f} \vee f(x) \wedge \bar{f}, \text{ for any } x \in \mathcal{X}. \quad (2.5)$$

The operator $R$ censors the function $f$ to $[\underline{f}, \bar{f}]$. Let $d_p$ be the distance measure induced by the $L^p$-norm, i.e. $d_p(f, g) = \|f - g\|_p$ for any $f, g \in \ell^p(\mathcal{X})$ and $p \geq 1$. The following lemma shows that $R$ is indeed range-enforcing and distance-reducing with respect to $d_p$.

**Lemma 1 (Range Operator).** The operator $R$ is $\ell_\infty^\infty$-enforcing with respect to $\ell^\infty(\mathcal{X})$ and a $d_p$-distance contraction for any $p \geq 1$.

2.3. **Convexity.** Let $\mathcal{X}$ be a convex subset of $\mathbb{R}^k$ and $\ell_c^\infty(\mathcal{X}) := \{f \in \ell^\infty(\mathcal{X}) : f(\alpha x + (1 - \alpha)x') \leq \alpha f(x) + (1 - \alpha)f(x') \text{ for all } x, x' \in \mathcal{X}, \alpha \in [0, 1]\}$, the set of bounded convex functions on $\mathcal{X}$. We consider the Double Legendre-Fenchel (DLF) transform as a convexity-enforcing operator. To define this operator, we first recall the definition of the Legendre-Fenchel transform.

**Definition 4 (Legendre-Fenchel transform).** For any convex set $\mathcal{X} \subset \mathbb{R}^k$ and $f \in \ell^\infty(\mathcal{X})$, let $f^*(\mathcal{X}) := \{\xi \in \mathbb{R}^k : \sup_{x \in \mathcal{X}} \{\xi' x - f(x)\} < \infty\}$. The Legendre-Fenchel transform $L_\mathcal{X} : \ell^\infty(\mathcal{X}) \to \ell^\infty(f^*(\mathcal{X}))$ is defined by

$$f^*(\xi) := \sup_{x \in \mathcal{X}} \{\xi' x - f(x)\}, \text{ for any } \xi \in f^*(\mathcal{X}).$$
The function $\xi \mapsto f^*(\xi)$ is a closed convex function (see Lemma 7 in the Appendix) which is also called the convex conjugate of $f$, and the Legendre-Fenchel transform $L_X$ is also called the conjugacy operator. The Legendre-Fenchel transform is a functional operator that maps any function $f$ to a function of its family of tangent planes, which is often referred to as the dual function of $f$.

Let $\ell_\infty^S(X) := \{f \in \ell_\infty(X) : \liminf_{x' \to x} f(x') \geq f(x) \text{ for all } x \in X\}$, the set of bounded lower semi-continuous functions on $X$.

**Definition 5 (C-Operator).** For any convex set $X \subset \mathbb{R}^k$, the double Legendre-Fenchel operator $C : \ell_\infty^S(X) \to \ell_\infty^S(X)$ is defined by

$$Cf := L_{f^*(X)} \circ L_X f.$$ 

The double Legendre-Fenchel operator maps any lower semi-continuous function $f$ to its greatest convex minorant, i.e., the largest function $g \in \ell_\infty^M(X)$ such that $g \leq f$.

**Lemma 2 (Convexity Operator).** For any convex set $X$, the operator $C$ is $\ell_\infty^C$-enforcing with respect to $\ell_\infty^S(X)$ and a $d_\infty$-distance contraction.

2.4. **Monotonicity.** Let $\ell_\infty^M(X) := \{f \in \ell_\infty(X) : f(x') \leq f(x) \text{ for all } x, x' \in X \text{ such that } x' \leq x\}$, the set of bounded partially nondecreasing functions on $X$. We consider the multivariate monotone rearrangement of Chernozhukov et al. (2009) as a monotonicity-enforcing operator.

**Definition 6 (M-Operator).** For any compact set $X \subset \mathbb{R}^k$, the multivariate increasing rearrangement operator $M : \ell_\infty(X) \to \ell_\infty(X)$ is defined by

$$Mf := \frac{1}{|\Pi|} \sum_{\pi \in \Pi} M_\pi f,$$

where $\pi = (\pi_1, \ldots, \pi_k)$ is a permutation of the integers $1, \ldots, k$, $\Pi$ is the set of all possible permutations $\pi$, $M_\pi f := M_{\pi_1} \circ \cdots \circ M_{\pi_k} f$, and

$$M_j f := \inf \left\{ y \in \mathbb{R} : \int_X 1\{f(x',x_j) \leq y\} dx' \geq x_j \right\},$$

the one-dimensional increasing rearrangement applied to $x_j \mapsto f(x_j, x_{-j})$.

Proposition 2 of Chernozhukov et al. (2009) showed that the multivariate increasing rearrangement is monotonicity-enforcing and distance-reducing with respect to $d_p$ for any $p \geq 1$. We reproduce this result as a lemma for completeness.

**Lemma 3 (Monotonicity Operator).** For any compact set $X$, the operator $M$ is $\ell_\infty^M$-enforcing with respect to $\ell_\infty(X)$ and a $d_p$-distance contraction for any $p \geq 1$.

**Remark 1 (Isotonic Regression).** Chernozhukov et al. (2009) showed that isotonic regression and convex linear combinations of monotone rearrangement and isotonic regression are also $\ell_\infty^M$-enforcing operators with respect to $\ell_\infty(X)$ and $d_p$-distance contractions for any $p \geq 1$. 


2.5. Convexity and Monotonicity. Let \( \mathcal{X} \) be a convex subset of \( \mathbb{R}^k \) and \( \ell^\infty_{CM}(\mathcal{X}) := \ell^\infty_C(\mathcal{X}) \cap \ell^\infty_M(\mathcal{X}) \), the set of bounded convex and partially nondecreasing functions on \( \mathcal{X} \). We consider the composition of the \( C \) and \( M \) operators to enforce both convexity and monotonicity.

**Definition 7 (CM-Operator).** For any regular rectangular set \( \mathcal{X} \), the convex rearrangement operator \( CM : \ell^\infty_C(\mathcal{X}) \to \ell^\infty_C(\mathcal{X}) \) is defined by

\[
CMf := C \circ Mf.
\]

**Remark 2 (Rectangular Domain).** The assumption that the domain \( \mathcal{X} \) is a rectangular set is made for technical reasons. As we show in Lemma 9 in the Appendix, when \( \mathcal{X} \) is a regular rectangle the \( C \)-operator can be obtained by separate application to each face of the rectangle. Then, we show that the application of the \( C \)-operator to a face does not affect the monotonicity of the function. We do not know if the application of the \( C \)-operator to the entire domain \( \mathcal{X} \) preserves monotonicity without this convenient separability. From a practical point of view, we do not find this assumption very restrictive because the domains usually have the product form \( \mathcal{X} = [a_1, b_1] \times \cdots \times [a_k, b_k] \) in applications. If the domain of the target function is not rectangular, we can restrict the analysis to a rectangular subset of the domain.

**Lemma 4 (Convexity and Monotonicity Operator).** For any regular rectangular set \( \mathcal{X} \), the operator \( CM \) is \( \ell^\infty_{CM} \)-enforcing with respect to \( \ell^\infty_C(\mathcal{X}) \) and a \( d_\infty \)-distance contraction.

2.6. Quasi-convexity. Let \( \mathcal{X} \) be a convex subset of \( \mathbb{R}^k \) and \( \ell^\infty_Q(\mathcal{X}) := \{ f \in \ell^\infty(\mathcal{X}) : f(\alpha x + (1 - \alpha)x') \leq \min[f(x), f(x')] \text{ for all } x, x' \in \mathcal{X}, \alpha \in [0, 1] \} \), the set of bounded quasi-convex functions on \( \mathcal{X} \). We note that \( \ell^\infty_C(\mathcal{X}) \subseteq \ell^\infty_Q(\mathcal{X}) \), and that for any \( f \in \ell^\infty_Q(\mathcal{X}) \), the lower contour sets, \( \mathcal{I}_f(y) := \{ x \in \mathcal{X} : f(x) \leq y \} \), are convex for all \( y \in \mathbb{R} \). For any set \( \mathcal{Z} \subseteq \mathbb{R}^k \), let \( \text{conv}(\mathcal{Z}) \) denote the convex hull of \( \mathcal{Z} \). We consider the following new operator to impose quasi-convexity:

**Definition 8 (Q-Operator).** For any convex set \( \mathcal{X} \subseteq \mathbb{R}^k \), the \( Q \)-operator \( Q : \ell^\infty(\mathcal{X}) \to \ell^\infty(\mathcal{X}) \) is defined by

\[
Qf(x) := \min \{ y \in \mathbb{R} : x \in \text{conv}[\mathcal{I}_f(y)] \}.
\]  

(2.6)

The operator \( Q \) transform any bounded function into a quasi-convex function. To see this, recall that a function is quasi-convex if its domain and all its lower contour sets are convex. By construction, \( x \in \text{conv}[\mathcal{I}_f(y)] \) if and only if \( Qf(x) \leq y \). Therefore, the lower contour set of \( Qf \) at any level \( y \in \mathbb{R} \) is \( \mathcal{I}_{Qf}(y) = \{ x \in \mathcal{X} : Qf(x) \leq y \} = \text{conv}[\mathcal{I}_f(y)] \), which is a convex set.

For any \( f, g \in \ell^\infty(\mathcal{X}) \), define

\[
d_{H,p}(f, g) = \left( \int_{\mathbb{R}} d_H(\mathcal{I}_f(y), \mathcal{I}_g(y))^p \, dy \right)^{1/p}, \quad d_{H,\infty}(f, g) := \sup_{y \in \mathbb{R}} d_H(\mathcal{I}_f(y), \mathcal{I}_g(y)),
\]  

(2.7)

where \( d_H(\mathcal{V}, \mathcal{W}) \) is the Hausdorff distance between the sets \( \mathcal{V} \) and \( \mathcal{W} \).
Lemma 5 (Quasi-Convexity Operator). For any convex set $\mathcal{X}$, the operator $Q$ is $\ell^\infty_Q$-enforcing with respect to $\ell^\infty(Q\mathcal{X})$ and a $d_{H,p}$-distance contraction for any $p \geq 1$.

Remark 3 (Remark on $d_{H,p}$). The discrepancy function $d_{H,p}(f,g)$ measures the distance of the lower contour sets of the functions $f$ and $g$. For a sequence of functions $f_n(x)$ that converges to $f(x)$ uniformly over all $x \in \mathcal{X}$ as $n \to \infty$, the sequence of lower contour sets $I_{f_n}(y)$ converges to $I_f(y)$ uniformly over all possible values $y$ in the range of $f$. If $y$ is a regular value of $x \mapsto f(x)$, i.e., $\nabla f(x) \neq 0$ for all $x \in \{\tilde{x} \in \mathcal{X} : f(\tilde{x}) = y\}$, then $\partial I_f(y)$, the boundary of $I_f(y)$, is a $(k-1)$-dimensional manifold as described in Chernozhukov, Fernández-Val & Luo (2015), and as $d_{\infty}(f_n,f) \to 0$,

$$d_H(I_{f_n}(y), I_f(y)) \to \sup_{x \in \partial I_f(y)} \frac{|f_n(x) - f(x)|}{\|\nabla f(x)\|}.$$ 

Therefore, when $\|\nabla f(x)\| \neq 0$ for all $x \in \mathcal{X}$, as $d_{\infty}(f_n,f) \to 0$,

$$d_{H,\infty}(f_n, f) \to \sup_{x \in \mathcal{X}} \frac{|f_n(x) - f(x)|}{\|\nabla f(x)\|},$$

which is a weighted $d_{\infty}$-distance between $f$ and $f_n$.

2.7. Quasi-Convexity and Monotonicity. Let $\mathcal{X}$ be a convex subset of $\mathbb{R}^k$ and $\ell^\infty_{QM}(\mathcal{X}) := \ell^\infty_Q(\mathcal{X}) \cap \ell^\infty_M(\mathcal{X})$, the set of bounded quasi-convex and partially nondecreasing functions on $\mathcal{X}$. This case is only relevant when $k > 1$ because univariate monotone functions are quasi-convex. We consider the composition of the $Q$ and $M$ operators to impose both quasi-convexity and monotonicity.

Definition 9 (QM-Operator). For any regular rectangular set $\mathcal{X}$, the quasi-convex rearrangement operator $QM : \ell^\infty(\mathcal{X}) \mapsto \ell^\infty(\mathcal{X})$ is defined by

$$QMf := Q \circ Mf.$$ 

Lemma 6 (Quasi-Convexity and Monotonicity Operator). For any regular rectangular set $\mathcal{X}$, the operator $QM$ is $\ell^\infty_{QM}$-enforcing with respect to $\ell^\infty(\mathcal{X})$.

Remark 4 (Distance Reducing). It is not clear whether there exists a discrepancy function $\rho$ such that $QM$ is a $\rho$-distance contraction. We have found counterexamples showing that $QM$ is not distance reducing with respect to $d_{\infty}$ and $d_{H,p}$ for any $p \geq 1$.

3. IMPROVED POINT AND INTERVAL ESTIMATION

We show how to use shape-enforcing operators to improve point and interval estimators of a shape-restricted function. Let $f_0 : \mathcal{X} \to \mathbb{R}$ be the target function, which is known to satisfy a shape restriction, i.e., $f_0 \in \ell^\infty_1(\mathcal{X})$. Assume we have a point estimator $f$ of $f_0$, and an interval
estimator or uniform confidence band \([f_l, f_u]\) for \(f_0\). These estimators are unrestricted and therefore do not necessarily satisfy the shape restrictions, i.e. \(f, f_l, f_u \in \ell_0^\infty(X)\) but \(f, f_l, f_u \notin \ell_1^\infty(X)\) in general. There are many different ways to obtain these initial estimators including least squares, GMM and quantile regression approaches under general sampling conditions. A common confidence band for the function \(f_0\) is constructed as

\[ f_l(x) = f(x) - c_p s(x), \quad f_u(x) = f(x) + c_p s(x), \]

where \(s(x)\) is the standard error of \(f(x)\) and \(c_p\) is a critical values chosen such that

\[ P(f_0 \in [f_l, f_u]) := P\{f_0(x) \in [f_l(x), f_u(x)] : x \in X\} \geq p, \]

for some confidence level \(p\). Wasserman (2006) provides an excellent overview of methods for constructing the critical value. We give empirical and numerical examples in Section 5. To enforce the shape restriction, we apply a suitable shape-enforcing operator to the original point estimator and end-point functions of the confidence band. The resulting estimator, \(O_f\), and confidence band, \([O_{f_l}, O_{f_u}]\), improve over \(f\) and \([f_l, f_u]\) in a sense that we make precise in the following theorem.

**Theorem 1** (Improved Point and Interval Estimators). Suppose we have a target function \(f_0 \in \ell_1^\infty(X)\), an estimator \(f \in \ell_0^\infty(X)\) a.s., and a confidence band \([f_l, f_u]\) such that \(f_l, f_u \in \ell_0^\infty(X)\) a.s. If the operator \(O\) is \(\ell_1^\infty\)-enforcing with respect to \(\ell_0^\infty(X)\), then

1. the \(\ell_1^\infty\)-enforced confidence band \([O_{f_l}, O_{f_u}]\) has weakly greater coverage than \([f_l, f_u]\),

\[ P(f_0 \in [O_{f_l}, O_{f_u}]) \geq P(f_0 \in [f_l, f_u]). \]

If in addition \(O\) is a \(\rho\)-distance contraction, then

2. the \(\ell_1^\infty\)-enforced estimator \(O_f\) is weakly closer to \(f_0\) than \(f\) with respect to the distance \(\rho\)

\[ \rho(O_f, f_0) \leq \rho(f, f_0) \text{ a.s.}; \]

3. and the \(\ell_1^\infty\)-enforced confidence band \([O_{f_l}, O_{f_u}]\) is weakly shorter than \([f_l, f_u]\) with respect to the distance \(\rho\),

\[ \rho(O_{f_l}, O_{f_u}) \leq \rho(f_l, f_u) \text{ a.s.}. \]

Part (2) shows that the shape-enforced point estimator improves over the original estimator in terms of estimation error measured by the \(\rho\)-distance between the estimator and the target function. Parts (1) and (3) show that the shape-enforced confidence band not only has greater coverage but also is shorter with respect to the \(\rho\)-distance than the original band. These improvements apply to any sample size. In particular, they imply that enforcing the shape restriction preserves the asymptotic properties of the point and interval estimators. Thus, the shape-enforced estimator is consistent if the original estimator is consistent, and the shape-enforced confidence band has coverage at least \(p\) in large samples if the original band has
coverage $p$ in large samples. Theorem 1 can therefore be coupled with Lemmas 1–6 to yield improved inference on a function that satisfies any of the shape restrictions considered in the previous section.

Remark 5 (Model Misspecification). Model misspecification occurs when $f_\infty$, the probability limit of the estimator $f$, is different from the target function $f_0$. In this case if $f_\infty$ does not satisfy the shape-restriction, $f_\infty \notin \ell_1^- (X)$, then enforcing this restriction still improves estimation and inference. Thus, the probability limit of the shape-enforced estimator, $O_{f_\infty}$, is closer to $f_0$ in $\rho$-distance than $f_\infty$, and the shape-enforced confidence band, $[O_{f_l}, O_{f_u}]$, covers $O_{f_\infty}$ with at least the same probability as $[f_l, f_u]$ covers $f_\infty$ and $[O_{f_l}, O_{f_u}]$ is shorter than $[f_l, f_u]$ in $\rho$-distance.

4. Implementation Algorithms

We provide implementation algorithms for the different shape-enforcing operators based on a sample or grid of $n$ points $X_n = \{x_1, \ldots, x_n\}$ with corresponding values of $f$ given by the array $Y_n = \{y_1, \ldots, y_n\}$ with $y_i = f(x_i)$. The computation of the $R$-operator is trivial as it amounts to censor the elements of $Y_n$ to be between $f_\lor$ and $f_\land$, i.e.,

$$Rf(x_i) = f_\lor \lor y_i \land f_\land.$$ 

When $k = 1$, Chernozhukov et al. (2009) showed that the $M$-operator sorts the elements of $Y_n$. Thus, assume that $x_1 \leq x_2 \leq \ldots \leq x_n$ and let $y^{(1)} \leq y^{(2)} \leq \cdot \leq y^{(n)}$ denote the sorted array of $Y_n$. Then,

$$Mf(x_i) = y^{(i)}.$$ 

When $k > 1$, each $M_j$-operator in Definition 6 can be computed by applying the same sorting procedure to the dimension $j$ sequentially for each possible value of the other dimensions. We refer to Chernozhukov et al. (2009) for more details on computation. We next develop new algorithms for the $C$ and $Q$ operators.

4.1. Computation of C-Operator. When $k = 1$, we can obtain the greatest convex minorant using the standard method based on the pool adjacent violators algorithm (PAVA) described in Barlow et al. (1972). We develop a new algorithm for the case where $k > 1$. By Definitions 4 and 5, the DFL transform of $f$ is the solution to

$$Cf(x) = \sup_{\xi \in f^*(X)} \inf_{\tilde{x} \in X} \{\xi'(x - \tilde{x}) + f(\tilde{x})\}.$$ 

This is a saddle point problem that might be difficult to tackle directly. However, when $X$ is replaced by the finite grid $X_n$, the previous problem has the convenient linear programming
representation:
\[ C_f(x) = \max_{v \in \mathbb{R}, \xi \in \mathbb{R}^k} v \]
\[ \text{s.t. } v + \xi'(x_i - x) \leq f(x_i), \quad i = 1, 2, \ldots, n. \]

This program can be solved using standard linear programming methods.

The following algorithm summarizes the computation of the C-Operator.

**Algorithm 1 (C-Operator).** (1) Pick a dense enough grid of size \( n \) in \( X \), denoted as \( X_n \). One natural choice is the set of values of \( x \) observed in the data. (2) For each \( x \in X_n \), solve the linear programming problem stated in (4.1) to obtain \( C_f(x) \).

**Remark 6.** The set of points \( \{(x, C_f(x)) : x \in X_n\} \) describes the landscape of \( C_f \) on \( X_n \).

### 4.2. Computation of Q-Operator.

We propose an approximation method to compute the Q operator based on solving the problem (2.6) in a finite grid, namely
\[ Q_f(x) \approx \min \{ y \in Y : x \in \text{conv}[I_{f,n}(y)] \}, \]
where \( I_{f,n}(y) = \{ x_i : y_i = f(x_i) \leq y, i = 1, 2, \ldots, n \} \). We find the solution to the previous program using the following bi-section search algorithm:

**Algorithm 2 (Q-Operator).** For a given \( x \in X_n \): (1) Initialize \( y_L = y_{(1)} \) and \( y_U = y_{(n)} \). (2) Find the median of \( \{ y \in Y : y_L \leq y \leq y_U \} \) and assign it to \( y^* \). (3) Compute the lower contour set \( I_{f,n}(y^*) \) and its convex hull \( \text{conv}[I_{f,n}(y^*)] \). (4) If \( x \in I_{f,n}(y^*) \) (which indicates \( y^* \geq Q_f(x) \)), set \( y_U = y^* \); otherwise, set \( y_L = y^* \). (5) Repeat (2)–(4) until \( y_U = y_L \) and report \( Q_f(x) = y_U \).

## 5. Numerical Examples

We consider an empirical application to growth charts and a calibrated simulation where the target function \( f_0 \) is univariate.

### 5.1. Height Growth Charts for Indian Children.

Since their introduction by Quetelet in the 19th century, reference growth charts have become common tools to assess an individual’s health status. These charts describe the evolution of individual anthropometric measures, such as height, weight, and body mass index, across different ages. See Cole (1988) for a classical work on the subject, and Wei, Pere, Koenker & He (2006) for a recent analysis and additional references. Here we consider the estimation of height growth charts imposing monotonicity and concavity restrictions. These restrictions are plausible since an individual’s height is nondecreasing in age at a nonincreasing growth rate during early childhood.

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1The convex hull can be computed by a feasibility problem in linear programming.
We use the data from Fenske, Kneib & Hothorn (2011) and Koenker (2011) on childhood malnutrition in India. These data include a measure of height in centimeters, \( Y \), age in months, \( X \), and 22 covariates, \( Z \), for 37,623 Indian children. All the children have ages between 0-5 years, i.e., \( X \in \mathcal{X} = \{1, 2, \ldots, 60\} \). The covariates \( Z \) include the mother’s body mass index (BMI), the number of months the child was breastfed, and the mother’s age (as well as the square of the previous three covariates); the mother’s years of education and the father’s years of education; indicator variables for the child’s sex, whether the child was a single birth or multiple birth, whether or not the mother was unemployed, whether the mother’s residence is urban or rural, and whether the mother has each of: electricity, a radio, a television, a refrigerator, a bicycle, a motorcycle, and a car; and factor variables for birth order of the child, the mother’s religion and quintiles of wealth.

We assume a partially linear model for the conditional expectation of \( Y \) given \( X \) and \( Z \), namely

\[
E[Y | X = x, Z = z] = f_0(x) + Z'\gamma.
\]

The target function is the conditional average growth chart \( x \mapsto f_0(x) \), which we assume to be nondecreasing and concave. Since \( X \) is discrete, we can express \( f_0(x) = P(x)'\beta \), where \( P(x) \) is a vector of indicators for each value in \( \mathcal{X} \), i.e. \( P(X) = [1(X = 2), \ldots, 1(X = 60)]' \). We estimate \( \beta \) and \( \gamma \) by least squares of \( Y \) on \( P(X) \) and \( Z \) and construct confidence bands for \( f_0 \) on \( \mathcal{X} \) using weighted bootstrap with standard exponential weights and 200 repetitions (Præstgaard & Wellner 1993, Hahn 1995). The standard errors are estimated using bootstrap rescaled interquartile ranges (Chernozhukov, Fernández-Val & Melly 2013), and the critical value is the bootstrap 0.95-quantile of the maximal t-statistic. Weighted bootstrap is computationally convenient in this application because it is less sensitive than empirical bootstrap to singular designs, which are likely to arise in the bootstrap resampling because \( Z \) and \( P(X) \) contain many indicators.

Figures 1 and 2 report the point estimates and 95% confidence bands of \( f_0 \) for the entire sample and a random extract with 1,000 observations, respectively. We use the subsample to illustrate the deviations from the shape restrictions that are more apparent when the sample size is small. The original estimates are displayed in the left panel, and the estimates imposing monotonicity and concavity in the right panel. The original estimates in the entire sample are nondecreasing in age except at 45 months, and deviate from concavity in some areas. The \( M \) and \( C \) operators correct these deviations. The estimates in the random extract of the data clearly show deviations from both monotonicity and concavity. The \( M \) and \( C \) operators fix these deviations and produce point estimates that are closer to the estimates in the entire sample.
Figure 1. Entire Sample: Estimates and 95% confidence bands. Left: $f$ and $[f_{l}, f_{u}]$. Right: $CMf$ and $[CMf_{l}, CMf_{u}]$

Figure 2. Subsample with 1,000 observations: Estimates and 95% confidence bands. Left: $f$ and $[f_{l}, f_{u}]$. Right: $CMf$ and $[CMf_{l}, CMf_{u}]$

5.2. **Calibrated Monte Carlo Simulation.** We quantify the finite-sample improvement in the point and interval estimates of enforcing shape restrictions using simulations calibrated to
the growth chart application. The child’s height $Y$ is generated by

$$Y = \text{CM}[P(X_i)\beta] + Z_i\gamma + \sigma\epsilon_i, \quad i = 1, \ldots, n,$$

where $P(X_i)$ is the vector of indicators for all the values of $X$, $\beta$, $\gamma$ and $\sigma$ are the least squares estimates of $\beta$ and $\gamma$ and the residual standard deviation in the growth chart data, and $\epsilon_i$ are independent draws from the standard normal distribution. The application of the CM-operator guarantees that the target function $f_0(x) = \text{CM}[P(x)\beta]$ is monotone and concave. We consider 6 sample sizes, $n \in \{500, 1,000, 2,000, 4,000, 8,000, 37,623\}$, where $n = 37,623$ is the same sample size as in the empirical application. The values of $X_i$ and $Z_i$ are randomly drawn from the data without replacement. The results are based on 500 simulations. In each simulation we construct point and band estimates of $f_0$ using the same methods as in the empirical application.

Table I reports simulation averages of the $d_\infty$-distance between the estimates and target function, coverage of the target function by the confidence band and $d_\infty$-length of the confidence band for the original and shape-enforced estimators. We consider enforcing concavity with the C-operator, monotonicity with the M-operator, and both concavity and monotonicity with the CM-operator. The improvements of imposing the shape restrictions are decreasing in the sample size, but there are substantial benefits in estimation error even with the largest sample size. Enforcing monotonicity has generally stronger effects than enforcing concavity, but both help improve the estimates. For the smallest sample size, the reduction in estimation error is almost 37% and the improvement in length of the confidence band is more than 20%. The gains in coverage probability are also substantial, especially for the smaller sample sizes. Overall, the simulation results clearly showcase the benefits of enforcing shape restrictions even with large sample sizes.

**Appendix A. Proofs**

**Proof of Lemma** We first show that $R$ satisfies the three properties of Definition 1.

1. **Enforcement:** it holds because for any $f \in \ell^\infty(\mathcal{X})$, $Rf \in \ell_R(\mathcal{X})$ by construction.

2. **Neutrality:** it holds trivially because $Rf = f$ for any $f \in \ell_R(\mathcal{X})$ by definition of $R$.

3. **Order preservation:** assume that $f, g \in \ell^\infty(\mathcal{X})$ are such that $f \geq g$. For any $x \in \mathcal{X}$ there are 3 possible cases. (a) If $f(x) \geq b$, then $Rf(x) = b \geq Rg(x)$. (b) If $f(x) \leq a$, then $g(x) \leq a$, and $Rf(x) = a = Rg(x)$. (c) if $a < f(x) < b$, then $Rf(x) = f(x) \geq \max(g(x), a) = Rg(x)$ because $g(x) < b$. Thus, $Rf(x) \geq Rg(x)$ for any $x \in \mathcal{X}$.

We next show Definition 2 for $\rho = d_\rho$ for any $p \geq 1$. For any $f, g \in \ell^\infty(\mathcal{X})$ assume without loss of generality that $f(x) \geq g(x)$ for some $x \in \mathcal{X}$. We need to show that $|Rf(x) - Rg(x)| \leq f(x) - g(x)$. There are 5 possible cases. (a) If $f(x) \geq g(x) \geq b$, then $|Rf(x) - Rg(x)| = |b - b| = 0 \leq f(x) - g(x)$. (b) If $f(x) \leq g(x) \leq b$, then $Rf(x) = b \leq f(x)$, and $Rg(x) \geq g(x)$.
Table 1. Finite Sample Properties

|                | \(\|f - f_0\|_\infty\) | \(\|f_u - f_l\|_\infty\) | \(P(f_0 \in [f_l, f_u])\) | \(\|f - f_0\|_\infty\) | \(\|f_u - f_l\|_\infty\) | \(P(f_0 \in [f_l, f_u])\) |
|----------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| \(n = 500\)   |                          |                          |                          |                          |                          |                          |
| Original       | 7.28                     | 26.17                    | 0.71                     | 4.90                     | 18.56                    | 0.82                     |
| C              | 6.14                     | 21.72                    | 0.81                     | 4.28                     | 17.08                    | 0.88                     |
| M              | 5.08                     | 21.10                    | 0.91                     | 3.71                     | 16.51                    | 0.96                     |
| CM             | 4.59                     | 20.13                    | 0.92                     | 3.33                     | 15.83                    | 0.96                     |
| \(n = 1,000\)  |                          |                          |                          |                          |                          |                          |
| Original       | 3.27                     | 13.18                    | 0.92                     | 2.29                     | 9.61                     | 0.93                     |
| C              | 2.84                     | 12.83                    | 0.95                     | 1.96                     | 9.53                     | 0.97                     |
| M              | 2.73                     | 12.56                    | 0.97                     | 2.02                     | 9.45                     | 0.98                     |
| CM             | 2.44                     | 12.26                    | 0.97                     | 1.77                     | 9.38                     | 0.99                     |
| \(n = 2,000\)  |                          |                          |                          |                          |                          |                          |
| Original       | 1.59                     | 6.85                     | 0.92                     | 0.72                     | 3.21                     | 0.95                     |
| C              | 1.40                     | 6.83                     | 0.95                     | 0.63                     | 3.21                     | 0.97                     |
| M              | 1.46                     | 6.83                     | 0.97                     | 0.70                     | 3.21                     | 0.96                     |
| CM             | 1.29                     | 6.81                     | 0.98                     | 0.62                     | 3.21                     | 0.98                     |
| \(n = 4,000\)  |                          |                          |                          |                          |                          |                          |
| Original       | 1.59                     | 6.85                     | 0.92                     | 0.72                     | 3.21                     | 0.95                     |
| C              | 1.40                     | 6.83                     | 0.95                     | 0.63                     | 3.21                     | 0.97                     |
| M              | 1.46                     | 6.83                     | 0.97                     | 0.70                     | 3.21                     | 0.96                     |
| CM             | 1.29                     | 6.81                     | 0.98                     | 0.62                     | 3.21                     | 0.98                     |
| \(n = 8,000\)  |                          |                          |                          |                          |                          |                          |
| Original       | 1.59                     | 6.85                     | 0.92                     | 0.72                     | 3.21                     | 0.95                     |
| C              | 1.40                     | 6.83                     | 0.95                     | 0.63                     | 3.21                     | 0.97                     |
| M              | 1.46                     | 6.83                     | 0.97                     | 0.70                     | 3.21                     | 0.96                     |
| CM             | 1.29                     | 6.81                     | 0.98                     | 0.62                     | 3.21                     | 0.98                     |
| \(n = 37,623\) |                          |                          |                          |                          |                          |                          |
| Original       | 1.59                     | 6.85                     | 0.92                     | 0.72                     | 3.21                     | 0.95                     |
| C              | 1.40                     | 6.83                     | 0.95                     | 0.63                     | 3.21                     | 0.97                     |
| M              | 1.46                     | 6.83                     | 0.97                     | 0.70                     | 3.21                     | 0.96                     |
| CM             | 1.29                     | 6.81                     | 0.98                     | 0.62                     | 3.21                     | 0.98                     |

Notes: Based on 500 simulations. Nominal level of the confidence bands is 95%.
Confidence bands constructed by weighted bootstrap with standard exponential weights and 200 repetitions.

By the order preservation property proved in (3), \(0 \leq |Rf(x) - Rg(x)| \leq f(x) - g(x)\). (c)
If \(b > f(x) \geq g(x) \geq a\), then \(Rf(x) = f(x)\) and \(Rg(x) = g(x)\), and \(|Rf(x) - Rg(x)| = f(x) - g(x)\). (d) If \(b > f(x) > a \geq g(x)\), then \(0 \leq |Rf(x) - Rg(x)| = f(x) - a \leq f(x) - g(x)\).
(e) If \(a \geq f(x) \geq g(x)\), then \(|Rf(x) - Rg(x)| = |a - a| = 0 \leq f(x) - g(x)\).

\[\square\]

Proof of Lemma 2. Before we prove Lemma 2, we recall some useful geometric properties of the Legendre-Fenchel transform.

Lemma 7 (Properties of Legendre-Fenchel transformation). Given a convex set \(\mathcal{X} \subset \mathbb{R}^k\), suppose that \(f, g \in \ell_\infty^\mathcal{X}(\mathcal{X})\), then:

1. Lower semicontinuity: \(L_\mathcal{X} f \in \ell_\infty^\mathcal{X}(\mathbb{R}^k)\).
(2) Convexity: $L_X f$ is closed convex on $\mathbb{R}^k$.

(3) Order reversing: If $f \geq g$ for all $x \in \mathcal{X}$, then $L_X f \leq L_X g$.

(4) $d_\infty$-Distance reducing: $\|L_X f - L_X g\|_\infty \leq \|f - g\|_\infty$.

Proof of Lemma 7

(1) For any $\xi \in f^*(\mathcal{X})$ and $\epsilon > 0$, there must exist $x_0 \in \mathcal{X}$ such that $\xi' x_0 - f(x_0) \geq L_X f(\xi) - \epsilon/2$. Then, for any $\xi_1$ such that $\|\xi - \xi_1\|_2 \leq \min[1, \epsilon/(2\|x_0\|)]$, we have:

$$L_X f(\xi_1) \geq \xi_1' x_0 - f(x_0) = (\xi_1 - \xi)' x_0 + \xi' x_0 - f(x_0) \geq -\|\xi - \xi_1\|_2\|x_0\| + L_X f(\xi) - \frac{\epsilon}{2} \geq -\frac{\epsilon}{2} + L_X f(\xi) - \frac{\epsilon}{2} \geq L_X f(\xi) - \epsilon.$$

Hence, $L_X f(\xi)$ is lower semi-continuous at $\xi$. Since $\xi$ can be arbitrary, we conclude that $L_X f$ is a lower semi-continuous function.

Properties (2) and (3) are shown in Theorem 1.1.2. and Proposition 1.3.1 in Chapter E of Hiriart-Urruty & Lemaréchal (2001). For (4), it is easy to check that $\|L_X f - L_X g\|_\infty = \sup_{\xi \in \mathbb{R}^k} |L_X f(\xi) - L_X g(\xi)| \leq \sup_{\xi \in \mathbb{R}^k} \sup_{x \in \mathcal{X}} |\{\xi' x - f(x)\} - \{\xi' x - g(x)\}| \leq \|f - g\|_\infty$.

Remark 7. The Legendre-Fenchel transform $\xi \mapsto L_X f(\xi)$ is locally Lipshitz, because any convex function is locally Lipshitz. Statement (4) of Lemma 7 is known as Marshall’s Lemma (Marshall, 1970).

Next, we derive some properties for the Double Legendre-Fenchel transformation.

Lemma 8 (Properties of C-Operator). Given a convex set $\mathcal{X} \subset \mathbb{R}^k$, suppose that $f \in \ell_\infty^\infty(\mathcal{X})$, then:

(1) $C f$ is the greatest convex minorant of $f$, i.e., the largest function $g \in \ell_\infty^\infty(\mathcal{X})$ such that $g \leq f$.

(2) If $\mathcal{X}$ is compact, for any $x \in \mathcal{X}$ there exists $d \leq k$ points $x_1, x_2, \ldots, x_d$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_d$ with $\alpha_1 > 0, \alpha_2 > 0, \ldots, \alpha_d > 0$ and $\sum_{i=1}^d \alpha_i = 1$, such that

$$C f(x) = \sum_{i=1}^d \alpha_i f(x_i), \quad (A.1)$$

where $x = \sum_{i=1}^d \alpha_i x_i$ and $f(x_i) = C f(x_i), 1 \leq i \leq d$.

(3) We say that $f$ is convex at $x \in \mathcal{X}$ if there exists a supporting hyperplane with direction $\xi$ such that $f(x') \geq f(x) + \xi'(x' - x)$ for all $x' \in \mathcal{X}$. Then, $f(x) = C f(x)$ if and only if $f$ is convex at $x$. Furthermore, if $f$ is convex at every $x \in \mathcal{X}$, then $f$ is a convex function.
Proof of Lemma 8 Statement (1): it directly follows from Corollary 12.1.1 of Rockafellar (1997).

Statement (2): by Proposition 2.5.1. in Chapter B of Hiriart-Urruty & Lemaréchal (2001),

\[ C_f(x) = \inf \left\{ \sum_{j=1}^{k} \alpha_j f(x_j) : \sum_{j=1}^{k} \alpha_j x_j = x, \alpha = (\alpha_1, \ldots, \alpha_k) \in \Delta_k \right\}, \]

where \( \Delta_k = \left\{ \alpha \in \mathbb{R}^k : \alpha_j \geq 0, j = 1, 2, \ldots, k; \sum_{j=1}^{k} \alpha_j = 1 \right\} \).

By (A.2), there exists a sequence \((x^t, \alpha^t) \in \mathcal{X} \times \Delta_k\) such that \(\sum_{j=1}^{k} \alpha_j^t x_j^t = x\) and \(C_f(x) \leq \sum_{j=1}^{k} \alpha_j^t f(x_j^t) + \frac{1}{t}\). Since \(\mathcal{X} \times \Delta_k\) is compact, there must exist a limit point \((x^0, \alpha^0) \in \mathcal{X} \times \Delta_k\) of the sequence \((x^t, \alpha^t)\) such that \(\sum_{j=1}^{k} \alpha_j^0 x_j^0 = \lim_{t \to \infty} \sum_{j=1}^{k} \alpha_j^t x_j^t = x\), and by lower semi-continuity of \(f\), \(\sum_{j=1}^{k} \alpha_j^0 f(x_j^0) \leq \lim_{t \to \infty} \sum_{j=1}^{k} \alpha_j^t f(x_j^t) = C_f(x)\). Then, it follows by (A.2) that \(\sum_{j=1}^{k} \alpha_j^0 f(x_j^0) = C_f(x)\). Equivalently, \(\sum_{j=1}^{k} \alpha_j^0 x_j^0 = x\), and

\[ \sum_{j=1}^{k} \alpha_j^0 f(x_j^0) = C_f(x). \] (A.3)

Let \((\alpha_1, \ldots, \alpha_d)\) and \((x_1, \ldots, x_d)\) denote the subsets of \((\alpha_1^0, \ldots, \alpha_k^0)\) and \((x_1^0, \ldots, x_k^0)\) corresponding to the components with \(\alpha_j^0 > 0\), where \(j = 1, 2, \ldots, k\) and \(d \leq k\). Next, we show that \(f(x_j) = C_f(x_j)\), for \(1 \leq j \leq d\). By statement (1), since \(C_f\) is the convex minorant of \(f\), so \(x \mapsto C_f(x)\) is convex and \(C_f(x) \leq f(x)\) for all \(x \in \mathcal{X}\). In particular,

\[ C_f(x) \leq \sum_{j=1}^{d} \alpha_j C_f(x_j) \leq \sum_{j=1}^{d} \alpha_j f(x_j). \]

By (A.3), the two inequalities imply that

\[ \sum_{j=1}^{d} \alpha_j C_f(x_j) = \sum_{j=1}^{d} \alpha_j f(x_j). \]

Since \(C_f(x_j) \leq f(x_j)\) for all \(j = 1, 2, \ldots, d\), it follows that \(f(x_j) = C_f(x_j)\) for all \(j = 1, 2, \ldots, d\).

This completes the proof of statement (2).

Statement (3): \(C_f(x) \leq f(x)\) for all \(x \in \mathcal{X}\) by (1). If there exists a \(\xi \in \mathbb{R}^k\) such that \(f(x') \geq f(x) + \xi(x' - x)\) for any \(x' \in \mathcal{X}\), then \(g(x') = f(x) + \xi(x' - x)\) is a convex function that lies below \(f\). By (1), \(C_f(x) \geq g(x) = f(x)\). Therefore, \(C_f(x) = f(x)\). On the other hand, suppose that \(C_f(x) = f(x)\). Since \(x \mapsto C_f(x)\) is convex on \(\mathcal{X}\), there must exists \(\xi \in \mathbb{R}^k\) such that \(C_f(x') \geq C_f(x) + \xi(x' - x)\) for any \(x' \in \mathcal{X}\). By definition of greatest convex minorant, \(f(x') \geq C_f(x') \geq C_f(x) + \xi(x' - x) = f(x) + \xi(x' - x)\) for any \(x' \in \mathcal{X}\). So \(C_f(x) = f(x)\) implies that \(f\) is convex at \(x\).
If $f$ is convex at every $x \in X$, then by the results above, $f(x) = Cf(x)$ for every $x \in X$. That said, $f = Cf$, which implies that $f$ is convex on $X$ because $Cf$ is convex on $X$.

Lemma 2 follows from the properties in Lemmas 7 and 8.

Proofs for Lemma 2. The properties (1) and (3) in Definition 1 are implied by properties (2) and (3) of Lemma 7 applied to $L_X f$ and using that $L_X f \in \ell^\infty_S(\mathbb{R}^k)$ by property (1) of Lemma 7. The property (2) in Definition 1 is implied by property (3) of Lemma 8. Moreover, the $d_\infty$-contraction property is given by property (4) in Lemma 7 again applied to $L_X f$ and using that $L_X f \in \ell^\infty_S(\mathbb{R}^k)$ by property (1) of Lemma 7.

Proof of Lemma 4. We start by demonstrating the $C$-operator on a rectangle can be computed separately at a face of the rectangle.

Definition 10 (C-Operator Restricted to a Face of a Rectangle). For any regular rectangular set $X := [a_1, b_1] \times ... \times [a_k, b_k]$, a set $F_m$ is a $m$-dimensional face of $X$ if there exists a set of indexes $i(F) \in \{1, 2, ..., k\}$ with $m$ elements such that $F_m = \{x \in X : x_j \in [a_j, b_j], \text{ for any } j \in i(F), x_j \in [a_j, b_j], \text{ for any } j \notin i(F)\}$. A face of dimension 0 is called a vertex of $X$. For every $x \in F_m$, we can define the $C$-operator restricted to the face $F_m$ by applying the Legendre-Fenchel transform only to each of the coordinates of $x$ that are in $i(F_m)$. Thus, let

$$L_{X|F_m} f(\xi) := \sup_{x \in F_m} \{\xi' x_{i(F_m)} - f(x_{i(F_m)}, x_{i^c(F_m)})\},$$

where we partition $x$ into the coordinates with indexes in $i(F_m)$, $x_{i(F_m)}$, and the rest of the coordinates, $x_{i^c(F_m)}$. Then, the $C$-operator restricted to the face $F_m$ of $f \in \ell^\infty_S(X)$ is

$$C_{X|F_m} f(x) := L_{f^*(X|F_m)} \circ L_{X|F_m} f(x),$$

where $f^*(X|F_m) := \{\xi \in \mathbb{R}^m : L_{X|F_m} f(\xi) < \infty\}$. Moreover, by Proposition 2.5.1 of Hiriart-Urruty & Lemaréchal (2001), $C_{X|F_m} f(x)$ is a linear combination of the $f$-images of $m$ elements of $F_m$, that is

$$C_{X|F_m} f(x) = \inf \left\{ \sum_{j=1}^m \alpha_j f(x_j) : x_j \in F_m, \alpha_j \geq 0, \sum_{j=1}^m \alpha_j = 1 \right\}, \text{ for any } x \in F_m.$$

Lemma 9 (C-Operator on a Regular Rectangular Set). For any regular rectangular set $X$ and $f \in \ell^\infty_S(X)$, if $x \in F_m$ with $m > 0$, then

$$C f(x) = C_{X|F_m} f(x).$$
Proof of Lemma 10. Suppose that $\mathcal{X}$ is a regular rectangle in $\mathbb{R}^k$. Let $\mathcal{F}_m$ be a face of $\mathcal{X}$ with dimension $m$ such that $x \in \mathcal{F}_m$. The result follows by the following facts:

First, $x \mapsto C f(x)$ is a convex function and lies below $f(x)$ on $\mathcal{X}$, so that $x \mapsto C f(x)$ is a convex function and lies below $f(x)$ on $\mathcal{F}_m \subset \mathcal{X}$. By definition, $C_{\mathcal{X}|\mathcal{F}_m} f$ is the convex minorant of $f$ restricted on $\mathcal{F}_m$, i.e., the largest possible convex function lying below $f$ restricted on $\mathcal{F}_m$. Therefore, it must be that $C_{\mathcal{X}|\mathcal{F}_m} f \geq C f(x)$ for all $x \in \mathcal{F}_m$.

Second, by statement (2) of Lemma 8 for any $x \in \mathcal{F}_m$, there exists $d \leq k$ points $x_1, \ldots, x_d$ and $\alpha_i > 0, 1 \leq i \leq d, \sum_{i=1}^{d} \alpha_i = 1$, such that $C f(x_i) = f(x_i), \sum_{i=1}^{d} \alpha_i x_i = x$, and $\sum_{i=1}^{d} \alpha_i f(x_i) = f(x)$. It must be that, $x_i \in \mathcal{F}_m, 1 \leq i \leq d, \text{since } x \in \mathcal{F}_m$.

Third, by definition of greatest convex minorant, on the face $\mathcal{F}_m, f(x) \geq C_{\mathcal{X}|\mathcal{F}_m} f(x)$ for any $x \in \mathcal{F}_m$. Since $C_{\mathcal{X}|\mathcal{F}_m} f(x)$ is the convex minorant of $f(x)$ restricted on $\mathcal{F}_m$, and $C f$ is a convex function on $\mathcal{F}_m$, so $C_{\mathcal{X}|\mathcal{F}_m} f(x) \geq C f(x)$ for any $x \in \mathcal{F}_m$. Therefore,

$$f(x) \geq C_{\mathcal{X}|\mathcal{F}_m} f(x) \geq C f(x) \quad (A.4)$$

for all $x \in \mathcal{F}_m$.

Fourth, for each $x_i, i = 1, 2, \ldots, d$, we know that $f(x_i) = C f(x_i)$. Applying equation (A.4), it must be that $f(x_i) = C_{\mathcal{X}|\mathcal{F}_m} f(x_i) = C f(x_i)$. Therefore,

$$C f(x) = \sum_{i=1}^{d} \alpha_i f(x_i) = \sum_{i=1}^{d} \alpha_i C_{\mathcal{X}|\mathcal{F}_m} f(x_i) \geq C_{\mathcal{X}|\mathcal{F}_m} f(x), \quad (A.5)$$

where the inequality follows by convexity of $x \mapsto C_{\mathcal{X}|\mathcal{F}_m} f(x)$.

Combining inequalities (A.4) and (A.5), we conclude that $C_{\mathcal{X}|\mathcal{F}_m} f(x) = C f(x)$. \hfill \Box

Before stating the main proof of Lemma 4, we require a lemma to show that $M$ maps a function in $\ell^\infty_S(\mathcal{X})$ to $\ell^\infty_{\hat{M}}(\mathcal{X})$.

Lemma 10. Suppose $\mathcal{X} = [0, 1]^k$. The rearrangement operator maps any function $f \in \ell^\infty_S(\mathcal{X})$ to $\ell^\infty_{\hat{M}}(\mathcal{X})$.

Proof. First, it is easy to see that for any $f_1, f_2 \in \ell^\infty_S(\mathcal{X})$ and $a, b \geq 0, a f_1 + b f_2 \in \ell^\infty_S(\mathcal{X})$. Therefore, to show that $M$ maps a function $f \in \ell^\infty_S(\mathcal{X})$ to $\ell^\infty_{\hat{M}}(\mathcal{X})$, it suffices to show that $M_{\pi}$ maps a function $f \in \ell^\infty_S(\mathcal{X})$ to $\ell^\infty_S(\mathcal{X})$, since $M f = \sum_{\pi \in \Pi} M_{\pi} f/||\Pi||$. Denote $\pi = (\pi_1, \ldots, \pi_k)$, so $M_{\pi} = M_{\pi_1} \circ \ldots \circ M_{\pi_k}$. For any function $f \in \ell^\infty_S(\mathcal{X})$ and $j = 1, 2, \ldots, k$, we would like to prove that $M_{\pi_j} f \in \ell^\infty_S(\mathcal{X})$. If the statement above is true, then it follows that $M_{\pi_j} f = M_{\pi_1} \circ \ldots \circ M_{\pi_k} f \in \ell^\infty_S(\mathcal{X})$. Consequently, the conclusion of the lemma is true.
Second, we prove that for any function \( f \in \ell_S^\infty(\mathcal{X}) \) and \( j = 1, 2, ..., k, \ M_j f \in \ell^\infty(\mathcal{X}). \) Without loss of generality, we can assume \( j = 1. \) By definition,

\[
M_1 f(x_1, x_{-1}) = \inf \left\{ y \in \mathbb{R} : \int_\mathcal{X} 1\{ f(x_1', x_{-1}) \leq y \} dx_1' \geq x_1 \right\}.
\]

For any \( x_1 \in [0, 1] \) and \( x_{-1} \in [0, 1]^{k-1}, \ \int_\mathcal{X} 1\{ f(x_1', x_{-1}) \leq y_{\max} \} dx_1' = 1 \geq x_1, \) and \( \int_\mathcal{X} 1\{ f(x_1', x_{-1}) \leq (y_{\min} - \epsilon) \} dx_1' = 0 < x_1, \) where \( y_{\max} = \sup_{x \in \mathcal{X}} f(x), \ y_{\min} = \inf_{x \in \mathcal{X}} f(x) \) and \( \epsilon > 0 \) can be any arbitrarily small constant. Since \( f \in \ell_S^\infty(\mathcal{X}), \ y_{\min} \) and \( y_{\max} \) exist. Therefore, \( \inf \{ y \in \mathbb{R} : \int_\mathcal{X} 1\{ f(x_1', x_{-1}) \leq y \} dx_1' \geq x_1 \} \) must be well defined and bounded by \( y_{\max} \) from above and \( y_{\min} - \epsilon \) from below. We conclude that \( M_1 f \in \ell^\infty(\mathcal{X}). \)

Third, we show that \( M_1 f \in \ell_S^\infty(\mathcal{X}) \) if \( f \in \ell_S^\infty(\mathcal{X}). \) We prove it by contradiction: suppose that \( M_1 f \) is not lower semi-continuous at a point \( x^0 \in \mathcal{X}. \) There must exist a sequence \( x^1, ..., x^n, ... \) in \( \mathcal{X} \) and a constant \( \epsilon > 0 \) such that \( \| x^n - x^0 \| \to 0 \) as \( n \to \infty \) and \( M_1 f(x^n) \leq M_1 f(x^0) - \epsilon \) for all \( n \geq 1. \) Let \( y^n := M_1 f(x^n). \) By definition of \( M_1 f, \) it must be that \( \int_\mathcal{X} 1\{ f(x_1^n, x_{-1}^n) \leq y^n - \frac{\epsilon}{2} \} dx_1^n < x_1^n, \) and \( \int_\mathcal{X} 1\{ f(x_1^n, x_{-1}^n) \leq y^n - \epsilon \} dx_1^n \geq x_1^n \) for all \( n \geq 1. \) For any \( x_1' \in [0, 1], \) since \( f \in \ell_S^\infty(\mathcal{X}) \) and \( (x_1', x_{-1}^n) \to (x_1', x_{-1}^0) \), \( \lim_{n \to \infty} f(x_1^n, x_{-1}^n) \geq f(x_1', x_{-1}^0). \) Therefore, there exists \( N \) large enough such that \( f(x_1', x_{-1}^N) \geq f(x_1', x_{-1}^0) - \frac{\epsilon}{2} \) for all \( n \geq N. \) Consequently, \( 1\{ f(x_1', x_{-1}^N) \leq y^n - \epsilon \} \leq 1\{ f(x_1', x_{-1}^0) \leq y^n - \frac{\epsilon}{2} \} \) for all \( n \geq N. \) Then, \( \limsup_{n \to \infty} 1\{ f(x_1', x_{-1}^n) \leq y^n - \epsilon \} \leq 1\{ f(x_1', x_{-1}^0) \leq y^n - \frac{\epsilon}{2} \} \) holds for all \( x_1'. \) By reverse Fatou’s Lemma,

\[
\limsup_{n \to \infty} \int_\mathcal{X} 1\{ f(x_1', x_{-1}^n) \leq y^n - \epsilon \} dx_1^n \leq \int_\mathcal{X} \limsup_{n \to \infty} 1\{ f(x_1', x_{-1}^n) \leq y^n - \epsilon \} dx_1^n \leq \int_\mathcal{X} 1\{ f(x_1', x_{-1}^0) \leq y^n - \frac{\epsilon}{2} \} dx_1^n. \tag{A.6}
\]

However, \( \int_\mathcal{X} 1\{ f(x_1', x_{-1}^0) \leq y^n - \frac{\epsilon}{2} \} dx_1^n < x_1^n, \) while \( \limsup_{n \to \infty} \int_\mathcal{X} 1\{ f(x_1', x_{-1}^n) \leq y^n - \epsilon \} dx_1^n \geq \limsup_{n \to \infty} x_1^n = x_1^n. \) Hence,

\[
\limsup_{n \to \infty} \int_\mathcal{X} 1\{ f(x_1', x_{-1}^n) \leq y^n - \epsilon \} dx_1^n \leq \int_\mathcal{X} 1\{ f(x_1', x_{-1}^0) \leq y^n - \frac{\epsilon}{2} \} dx_1^n,
\]

which contradicts \( \text{[A.6]} \). Therefore, we conclude that \( M_1 f \in \ell_S^\infty(\mathcal{X}) \) if \( f \in \ell_S^\infty(\mathcal{X}). \) \( \square \)

We now start the proof of Lemma 4

(1) We first show that \( \text{CM} \) satisfies the enforcement property (1) of Definition 4

We know that \( \text{CM} f = \text{CM} f. \) By Lemma [10] for any \( f \in \ell_S^\infty(\mathcal{X}), \ M f \in \ell_S^\infty(\mathcal{X}). \) Consequently, \( M f \in \ell_S^\infty(\mathcal{X}) \cap \ell_M^\infty(\mathcal{X}). \)

We use induction to prove that \( C f \in \ell_M^\infty(\mathcal{X}) \) for any \( f \in \ell_S^\infty(\mathcal{X}) \cap \ell_M^\infty(\mathcal{X}), \) where \( \mathcal{X} \subset \mathbb{R}^k \) is a regular rectangular set. Without loss of generality, assume that \( \mathcal{X} = [0, 1]^k. \) Since \( C f \in \ell_S^\infty(\mathcal{X}) \) by Lemma [2], we only need to show that \( C f \in \ell_M^\infty(\mathcal{X}). \)
For dimension $k = 1$, $\mathcal{X}$ is a closed interval. We prove that $Cf$ is nondecreasing. Assume, by contradiction, that there exists a pair of points $x, x' \in \mathcal{X}$ such that $x < x'$ and $Cf(x) > Cf(x')$. Let $x_1$ be the left end point of the interval $\mathcal{X}$. By convexity, $Cf(x) \geq Cf(x) > Cf(x')$.

By Lemma 9, $Cf(x) = C_{\mathcal{X}|\mathcal{F}_0}f(x) = f(x)$. By statement (2) of Lemma 8, there exists $x_1, \ldots, x_d \in \mathcal{X}$ and $\alpha_1, \ldots, \alpha_d > 0$, $\sum_{j=1}^d \alpha_j = 1$ such that $Cf(x') = Cf(x') = \sum_{j=1}^d \alpha_j f(x_j)$. Since $f$ is nondecreasing, we have $\sum_{j=1}^d \alpha_j f(x_j) \geq \sum_{j=1}^d \alpha_j f(x_j) = f(x)$, which contradicts that $Cf(x) > Cf(x')$. Hence, for any $x < x'$, it must be that $Cf(x) \leq Cf(x')$. We conclude that $x \mapsto Cf(x)$ is nondecreasing.

Suppose that $x \mapsto Cf(x)$ is nondecreasing for $(k - 1)$-dimensional regular rectangles, $k \geq 2$. Let $\mathcal{X}$ be a $k$-dimensional rectangle. Assume, by contradiction, that there exists $x \leq x' \ (x \neq x')$ such that $Cf(x) > Cf(x')$. Consider the radial originated from $x'$ that passes through $x$, denoted as $L$. $L$ can be written as $\{z \in \mathbb{R}^k : z = \gamma x' + (1 - \gamma)x, \gamma \leq 1\}$. Therefore, there exists a $\gamma_0 \leq 0$ such that $\gamma x' + (1 - \gamma)x \in \mathcal{X} \cap L$ if and only if $1 \geq \gamma \geq \gamma_0$. Denote $l = \gamma_0 x' + (1 - \gamma_0)x$. By convexity of $x \mapsto Cf(x)$, it must be that

$$Cf(l) \geq Cf(x) > Cf(x'). \quad (A.7)$$

By statement (2) of Lemma 8, there are $d$ points $x_1, \ldots, x_d \in \mathcal{X}$ and $\alpha_1, \ldots, \alpha_d > 0$, $\sum_{j=1}^d \alpha_j = 1$, such that $\sum_{j=1}^d \alpha_j x_i = x'$ and $\sum_{j=1}^d \alpha_j f(x_j) = f(x')$. The point $l$ must be on a $k - 1$ dimensional face of $\mathcal{X}$, denoted by $\mathcal{F}_{k-1}$. Since $\mathcal{X} = [0, 1]^k$, $\mathcal{F}_{k-1}$ can be expressed as $A_1 \times A_2 \times \ldots \times A_k$, where $A_i = [0, 1]$ for $i \in i(\mathcal{F}_{k-1})$, $i(\mathcal{F}_{k-1}) \subset \{1, 2, \ldots, k\}$, $|i(\mathcal{F}_{k-1})| = k - 1$, and $A_i = \{0\}$ or $\{1\}$ if $i \notin i(\mathcal{F}_{k-1})$. Without loss of generality, we can assume that $i(\mathcal{F}_{k-1}) = \{1, 2, \ldots, k - 1\}$. Denote $A_k = \{w\}$, so $w = 0$ or 1.

Let $s$ be the projection mapping from $\mathcal{X} = [0, 1]^k$ to $\mathcal{F}_{k-1}$, so $s : (x(1), \ldots, x(k)) \mapsto (x(1), \ldots, x(k - 1), w)$ for any $x(1), \ldots, x(k) \in [0, 1]$. Since $l \in \mathcal{F}_{k-1}$, it must be that $s(l) = l$. If $w = 0$, $s(x) \leq x$ for any $x \in \mathcal{X}$. Therefore, $s(x_1) \leq x_1, \ldots, s(x_d) \leq x_d$. Then, since $x \mapsto f(x)$ is nondecreasing, $f(x_i) \geq f(s(x_i))$ for all $i = 1, 2, \ldots, d$. If $w = 1$, then any point $x \in \mathcal{F}_{k-1}$ satisfies $x(k) = 1$, including $l$. Since $x' \geq l$, the $k - th$ entry of $x'$ must equal to 1. By $\sum_{j=1}^d \alpha_j x_j = x'$, $x_j \in [0, 1]^k$, it must be that the $k - th$ entry of $x_j$ equals to 1 for all $i = 1, 2, \ldots, d$. Therefore, $s(x_j) = x_j$, and $s(x') = x'$. Therefore, regardless of the value of $w$, $x_j \geq s(x_j)$, $x' \geq s(x')$. Since $x' \geq l$, it must be that $l \leq s(x') = \sum_{j=1}^d \alpha_j s(x_j)$. By Lemma 9, $C_{\mathcal{X}|\mathcal{F}}f(l) = Cf(l)$ and by (A.7),

$$C_{\mathcal{X}|\mathcal{F}}f(l) = Cf(l) > Cf(x') = \sum_{i=1}^d \alpha_i f(x_i) \geq \sum_{i=1}^d \alpha_i f(s(x_i)) \geq \sum_{i=1}^d \alpha_i C_{\mathcal{X}|\mathcal{F}}f(s(x_i)) \geq C_{\mathcal{X}|\mathcal{F}}f(s(x')),$$
where the second inequality holds by monotonicity of \( x \mapsto f(x) \), the third inequality by \( C_{X|\mathcal{F}} \) being the convex minorant of \( f \), and the fourth by convexity of \( x \mapsto C_{X|\mathcal{F}} f(x) \). Therefore,

\[
C_{X|\mathcal{F}} f(\ell) > C_{X|\mathcal{F}} f(s(x')), \tag{A.8}
\]

By induction, \( C_{X|\mathcal{F}} f \) restricted on the \( k-1 \) dimensional regular rectangle \( \mathcal{F} \) is nondecreasing. Since \( s(x') \geq \ell \), it must be that \( C_{X|\mathcal{F}} f(s(x')) \geq C_{X|\mathcal{F}} f(\ell) \), which contradicts (A.8). Hence, the induction is complete. \( x \mapsto C f(x) \) is nondecreasing if \( x \mapsto f(x) \) is nondecreasing. Therefore, for any \( f \in \ell^\infty_\mathcal{F}(\mathcal{X}) \), \( CM f \) is monotonically increasing.

We next show that \( CM \) satisfies the rest of the properties of Definition 1 and distance reduction.

2. To show neutrality, note that if \( f \in \ell^\infty_{\mathcal{CM}}(\mathcal{X}) \), then \( Mf = f \) by Lemma 3 and therefore \( CMf = C(Mf) = Cf = f \) by definition of \( CM \) and Lemma 2.

3. Similarly, \( CM \) is order preserving because if \( f \geq g \) then \( Mf \geq Mg \) by Lemma 3 and therefore \( CMf = C(Mf) \geq C(Mg) = CMg \) by definition of \( CM \) and Lemma 2.

4. Finally, \( CM \) is a \( d_\infty \)-distance contraction because

\[
d_\infty(CMf, CMg) = d_\infty(C[Mf], C[Mg]) \leq d_\infty(Mf, Mg) \leq d_\infty(f, g),
\]

where first equality follows by definition of \( CM \), the first inequality by Lemma 2 and the second inequality by Lemma 3.

**Proof of Lemma 5.** We first show that \( Q \) satisfies the three properties of Definition 1.

1. For any \( f \in \ell^\infty(\mathcal{X}) \), the lower contour set of \( Q f \) at level \( y \) is defined as \( I_Q f(y) = \{ x \in \mathcal{X} : Q f(x) \leq y \} = \text{conv}[I_f(y)] \), where \( I_f(y) \) is the lower contour set of \( f \) at level \( y \). Since \( I_Q f(y) \) is convex for any \( y \), \( Q f \in \ell^\infty_\mathcal{Q}(\mathcal{X}) \).

2. If \( f \in \ell^\infty_\mathcal{Q}(X) \), then \( \text{conv}[I_f(y)] = I_f(y) \) for any \( y \in \mathbb{R} \). Thus, the lower contour set of \( f \) agrees with the lower contour set of \( Q f \) at any level \( y \), which implies that \( f = Q f \).

3. If \( f \geq g \), then \( I_f(y) \supseteq I_g(y) \) at any level \( y \in \mathbb{R} \). If follows that \( \text{conv}[I_f(y)] \supseteq \text{conv}[I_g(y)] \), which means that the level set of \( Q f \) contains the level set of \( Q g \) at any level \( y \), i.e., \( Q f \geq Q g \).

We next show Definition 2 for \( p = d_{H,p} \) for any \( p \geq 1 \). Recall that for any two sets \( \mathcal{V}, \mathcal{W} \subseteq \mathcal{X} \), the Hausdorff distance between \( \mathcal{V} \) and \( \mathcal{W} \) is defined as

\[
d_H(\mathcal{V}, \mathcal{W}) = \max\{ \sup_{x \in \mathcal{V}} d(x, \mathcal{W}), \sup_{y \in \mathcal{W}} d(y, \mathcal{V}) \},
\]

where \( d(c, Z) := \inf_{z \in Z} \| z - c \| \) is the distance of point \( c \in \mathbb{R}^k \) to the set \( Z \subset \mathbb{R}^k \) for any arbitrary \( c \) and \( Z \). Recall also that, for any \( Z \subset \mathbb{R}^k \), the the convex hull of \( Z \) is \( \text{conv}(Z) := \{ w \in \mathbb{R}^k : w = \alpha_1 z_1 + \ldots + \alpha_k z_k, z_1, \ldots, z_k \in Z, \alpha = (\alpha_1, \ldots, \alpha_k) \in \Delta_k \} \), where \( \Delta_k := \{ \alpha \in \mathbb{R}^k : \alpha_j \geq 0, j = 1, 2, \ldots, k, \sum_{j=1}^k \alpha_j = 1 \} \).
We first show that the application of the convex hull reduces the Hausdorff distance between two sets. For any non-empty sets $\mathcal{V}, \mathcal{W} \subset \mathbb{R}^k$, any $v_c \in \text{conv}(\mathcal{V})$ and any $\epsilon > 0$, there exists a set of points $v_1, ..., v_k \in \mathcal{V}$ and $\alpha \in \Delta_k$ such that $\|v_c - \sum_{j=1}^{k} \alpha_j v_j\| \leq \epsilon$. By definition of Hausdorff distance, there exists $w_1, ..., w_k \in \mathcal{W}$ such that $\|v_j - w_j\| \leq d(v_j, \mathcal{W}) + \epsilon \leq d_H(\mathcal{V}, \mathcal{W}) + \epsilon$. Define $w^*_c := \sum_{j=1}^{k} \alpha_j w_j$, so that $w^*_c \in \text{conv}(\mathcal{W})$. Then,

$\|v_c - w^*_c\| \leq \|v_c - \sum_{j=1}^{k} \alpha_j v_j\| + \|\sum_{j=1}^{k} \alpha_j (v_j - w_j)\| \leq \epsilon + \sum_{j=1}^{k} \alpha_j \|v_j - w_j\|$

$\leq \epsilon + \sum_{j=1}^{k} \alpha_j (d_H(\mathcal{V}, \mathcal{W}) + \epsilon) = d_H(\mathcal{V}, \mathcal{W}) + 2\epsilon.$

Therefore, $d(v_c, \text{conv}(\mathcal{W})) \leq \|v_c - w^*_c\| \leq d_H(\mathcal{V}, \mathcal{W}) + 2\epsilon$. Symmetrically, for any $w_c \in \text{conv}(\mathcal{W})$, $d(w_c, \text{conv}(\mathcal{V})) \leq d_H(\mathcal{V}, \mathcal{W}) + 2\epsilon$. The previous results imply that $d_H(\text{conv}(\mathcal{V}), \text{conv}(\mathcal{W})) \leq d_H(\mathcal{V}, \mathcal{W}) + 2\epsilon$. Since $\epsilon$ can be arbitrarily small, we conclude that for any $\mathcal{V}, \mathcal{W} \subset \mathbb{R}^k$,

$$d_H(\text{conv}(\mathcal{V}), \text{conv}(\mathcal{W})) \leq d_H(\mathcal{V}, \mathcal{W}).$$

For $\mathcal{V} = \mathcal{I}_f(y)$, and $\mathcal{W} = \mathcal{I}_g(y)$, the previous result yields

$$d_H(\text{conv}[\mathcal{I}_f(y)], \text{conv}[\mathcal{I}_g(y)]) \leq d_H(\mathcal{I}_f(y), \mathcal{I}_g(y)),$$

provided that both $\mathcal{I}_f(y)$ and $\mathcal{I}_g(y)$ are non-empty. Otherwise, the distances $d_H(\mathcal{I}_f(y), \mathcal{I}_g(y))$ and $d_H(\text{conv}[\mathcal{I}_f(y)], \text{conv}[\mathcal{I}_g(y)])$ are defined as 0. Therefore, we conclude that, for any $p > 0$,

$$d_{H,p}(f, g) = \|d_H(\mathcal{I}_f(y), \mathcal{I}_g(y))\|_p \geq \|d_H(\text{conv}[\mathcal{I}_f(y)], \text{conv}[\mathcal{I}_g(y)])\|_p = d_{H,p}(\text{QM}f, \text{QM}g).$$

\[\square\]

**Proof of Lemma 6.** Without loss of generality, we can assume that the domain $\mathcal{X} = [0, 1]^k$. For a vector $w \in \mathbb{R}^k$, denote $w(i)$ as the $i^{th}$ entry of $w$.

1. We first prove that $\text{QM}$ is shape-enforcing.

For any $f \in \ell^\infty(\mathcal{X})$, $\text{QM}f \in \ell^\infty(\mathcal{X})$ by Lemma 5. Therefore, we only need to show that $\text{QM}f \in \ell^\infty_{\mathcal{M}}(\mathcal{X})$. Let $g := \text{M}f$. By Lemma 3, $g \in \ell^\infty_{\mathcal{M}}(\mathcal{X})$, so that for any $y \in \mathbb{R}$, the lower contour set $\mathcal{I}_g(y)$ satisfies:

$$\text{For any } x \in \mathcal{I}_g(y) \text{ and } x' \in \mathcal{X} \text{ such that } x' \leq x, x' \in \mathcal{I}_g(y). \quad (A.9)$$

Therefore, we need to prove that for any $x, y$ such that $x \in \text{conv}[\mathcal{I}_g(y)], x' \in \text{conv}[\mathcal{I}_g(y)]$ for any $x' \in \mathcal{X}$ such that $x' \leq x$.

First, we show that:

If $x' = x - te_i$ for some $t \geq 0$, then $x' \in \text{conv}[\mathcal{I}_g(y)], \quad (A.10)$
where $e_i$ is defined as the $i^{th}$ standard unit vector, $i = 1, 2, ..., k$. Without loss of generality, we can simply assume that $i = 1$, so $x'$ and $x$ are the same for all entries except for the first one. By assumption that $\mathcal{X} = [0, 1]^k$, we know that the first entry of $x'$, denoted as $x'(1)$, must be non-negative. Since $x \in \text{conv}[\mathcal{I}_g(y)]$, by statement (2) of Lemma 8, there exists a finite set of points $x_1, ..., x_d$ such that $x_j \in \mathcal{I}_g(y)$, $j = 1, 2, ..., d$, and $\alpha_1, ..., \alpha_d \geq 0$, $\alpha_1 + ... + \alpha_d = 1$ such that $\sum_{j=1}^{d} \alpha_j x_j = x$. Define $\bar{x}_j = (0, x_j(2), ..., x_j(k))$ as a vector which is constructed by replacing the first entry of $x_j$ by 0. Therefore, $\bar{x} := \sum_{j=1}^{d} \alpha_j \bar{x}_j = (0, x(2), ..., x(k))$ is a vector such that $\bar{x} \leq x' \leq x$. Therefore, there must exists $x_1^*, ..., x_d^*$ such that $x_j^* = (x_j^*(1), x_j(2), ..., x_j(k))$ with $x_j^*(1) \in [0, x_j(1)]$ such that $\sum_{j=1}^{d} \alpha_j x_j^*(1) = x'(1) \in [0, x(1)]$. By construction, $x_j^* \in \mathcal{X}$. Since $x_j^* \leq x_j \in \mathcal{I}_g(y)$, (A.9) implies that $x_j^* \in \mathcal{I}_g(y)$. It follows that $\sum_{j=1}^{d} \alpha_j x_j^* = x'$, and therefore $x' \in \text{conv}[\mathcal{I}_g(y)]$.

Now, for any $x' \in \mathcal{X}$ such that $x' \leq x$, denote $v := x - x' \geq 0$. Since $x \in \text{conv}[\mathcal{I}_g(y)]$, it follows that $x - v(1)e_1 \in \text{conv}[\mathcal{I}_g(y)]$, and then that $(x - v(1)e_1) - v(2)e_2 \in \text{conv}[\mathcal{I}_g(y)]$, ... Therefore, after applying (A.10) for $k$ times, $x' = x - v(1)e_1 - v(2)e_2 - ... - v(k)e_k \in \text{conv}[\mathcal{I}_g(y)]$. By (2.6), $Qg(x) := \min\{y \in \mathbb{R} : x \in \text{conv}[\mathcal{I}_g(y)]\}$. Let $y' := Qg(x)$ so that $x \in \text{conv}[\mathcal{I}_g(y')]$. Then, for any $x' \in \mathcal{X}$ such that $x' \leq x$, $x' \in \text{conv}[\mathcal{I}_g(y'))$. That implies $Qg(x') \leq y' = Qg(x)$. Therefore, we conclude that $Qg = QMf$ is nondecreasing.

Finally, we can show that $QM$ satisfies the rest of the properties of Definition 1 using the same argument as in the proof of Lemma 4 replacing $C$ by $Q$. We omit it for brevity.

\[\square\]

**Proof of Theorem 1.** (1) We show that the event $\{f_l \leq f_0 \leq f_u\}$ implies the event $\{O f_l \leq f_0 \leq O f_u\}$ by the properties of the $O$-operator. Indeed, by order preservation $\{f_l \leq f_0 \leq f_u\}$ implies that $\{O f_l \leq O f_0 \leq O f_u\}$, which is equivalent to $\{O f_l \leq f_0 \leq O f_u\}$ because $O f_0 = f_0$ by neutrality.

(2) The result follows from $\rho(O f_0, O f) \leq \rho(f_0, f)$ by $\rho$-distance contraction and $O f_0 = f_0$ by neutrality.

(3) The result follows directly by $\rho$-distance contraction.

\[\square\]

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