NON-FINITE AXIOMATIZABILITY OF SOME FINITE STRUCTURES

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Dedicated to Béla Csákány on his ninetieth birthday (2022)

Abstract. We prove that finite bipartite graphs cannot be axiomatized by finitely many first-order sentences in the class of finite graphs. (The counterpart of this statement for all bipartite graphs in the class of all graphs is a well-known consequence of the compactness theorem.) Also, to exemplify that our method is applicable in various fields of mathematics, we prove that neither finite simple groups, nor the ordered sets of join-irreducible congruences of slim semimodular lattices can be described by finitely many axioms in the class of finite structures. Since a 2007 result of G. Grätzer and E. Knapp, slim semimodular lattices have constituted the most intensively studied part of lattice theory and they have already led to results even in group theory and geometry.

1. Introduction

Finite model theory is a thriving part of mathematics. This is witnessed by, say, the monograph Libkin [19] with its 250 references or by the fact that, at the time of writing, MathSciNet returns seven matches to the search “Title=(finite model theory) AND Publication Type=(Books)”. However, the following words of Fagin [11, page 4] from 1993 are still valid: “almost none of the key theorems and tools of model theory, such as the completeness theorem and the compactness theorem, apply to finite structures”. This could be the reason that, as opposed to (classical unrestricted) model theory, finite model theory has not paid to much, if any, attention to finite axiomatizability among finite structures.

This paper deals with the axiomatizability of three different classes of finite structures. We prove that none of these three classes can be defined by a finite set of first-order sentences within the class of finite structures. The first class consists of all finite bipartite graphs. While it is a trivial consequence of the compactness theorem that the class of all (not necessarily finite) bipartite graphs is not finitely axiomatizable, our result on the finite case is a bit more involved. The second class consists of all finite simple groups. Although our results on these two classes are not surprising, their proofs exemplify that, in “lucky cases”, some methods of (classical) model theory are applicable even in the finite world on condition that...
sufficiently many of the structures we deal with are not too complicated or powerful theorems apply to them.

In case of the third class, whose definition with an appropriate introduction is postponed to Section 4, our result is the opposite of what has previously been conjectured. Here we only note that Section 4, containing the main result of the paper, belongs to the most intensively studied part of lattice theory.

Prerequisities. The results of Sections 2 and 3 are easy to understand for all mathematicians and even their proofs are readable for those who have ever met the concept of ultraproducts. Section 4 is intended for lattice theorists.

2. Non-finite axiomatizability of bipartite graphs

We begin this section with recalling some known concepts and facts; they will also be needed in the subsequent sections. By a finite signature we mean a tuple

\[ \sigma = \langle p, q, \langle R_1, r_1 \rangle, \ldots, \langle R_p, r_p \rangle, \langle F_1, f_1 \rangle, \ldots, \langle F_q, f_q \rangle \]  \hspace{1cm} (2.1)

where \( p, q \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \), \( R_1, \ldots, R_p \) are relation symbols, \( F_1, \ldots, F_q \) are function symbols, and these symbols are of arities \( r_1, \ldots, r_p, f_1, \ldots, f_q \in \mathbb{N}_0 \), respectively. A structure of type \( \sigma \) or, shortly, a \( \sigma \)-structure is a \((1+p+q)\)-tuple

\[ A = \langle A, R_1^A, \ldots, R_p^A, F_1^A, \ldots, F_q^A \rangle \]  \hspace{1cm} (2.2)

where \( A \), called the underlying set, is a nonempty set, \( R_i^A \subseteq A^{r_i} \) is a relation, and \( F_j^A : A^{f_j} \to A \) is a map for all \( i \in \{1, \ldots, p\} \) and \( j \in \{1, \ldots, q\} \). Structures will be denoted by calligraphic capital letters \( \mathcal{A}, \mathcal{B}, \ldots \) while their underlying sets with the corresponding italic capitals \( A, B, \ldots \). In this paper, the first-order language with equality determined by \( \sigma \) will be denoted by \( \text{Lng}(\sigma) \).

In addition to the relation symbols and function symbols occurring in (2.1), \( \text{Lng}(\sigma) \) includes the equality symbol, which is always interpreted as the equality relation. To define the the (first-order) consequence relation modulo finiteness, denoted by \( \models_{\text{fin}} \), assume that \( \Phi \) is a set of \( \text{Lng}(\sigma) \)-sentences and \( \mu \) is an \( \text{Lng}(\sigma) \)-sentence. Then

\[ \mu \text{ is a consequence of } \Phi \text{ modulo finiteness, in notation } \Phi \models_{\text{fin}} \mu, \]  \hspace{1cm} (2.3)

if every finite \( \sigma \)-structure that satisfies all sentences belonging to \( \Phi \) also satisfies \( \mu \).

To see an example, we borrow the sentence

\[ \lambda_k : \exists x_1 \ldots \exists x_k \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j) \]  \hspace{1cm} (2.4)

for \( k \in \mathbb{N}^+ = \{1, 2, 3, \ldots \} \) from Fagin [11]. Here \( \neg \) is the negation sign. Let \( \lambda_{-1} \) be the (identically false) sentence \( \exists x(x = x \land \neg(x = x)) \). Clearly, \( \Phi \models_{\text{fin}} \lambda_{-1} \) but there is no finite subset \( \Psi \) of \( \Phi \) such that \( \Psi \models_{\text{fin}} \lambda_{-1} \). This example shows well how big the difference between \( \models_{\text{fin}} \) and the usual consequence relation (for not necessarily finite structures) is.

The signature \( \sigma_{gr} \) of graphs is the particular cases of (2.1) such that \( \langle p, q \rangle = \langle 1, 0 \rangle \) and we write \( E \) and \( x E y \) instead of \( R_1 \) and \( \langle x, y \rangle \in R_1^A \); the latter means that \( A = \langle A, E \rangle \) is a (directed) graph, \( x, y \in A \) are vertices, and there is an edge from \( x \) to \( y \). Graphs satisfying the sentence \( \forall x \forall y (x E y \Rightarrow y E x) \) are undirected. An undirected graph \( A \) is bipartite if there are disjoint nonempty subsets \( A_0 \) and \( A_1 \)
of $A$ such that $A = A_0 \cup A_1$ and $E^A \subseteq (A_0 \times A_1) \cup (A_1 \times A_0)$. Using the notation given above and in (2.2), we claim the following.

**Proposition 2.1.** The class of finite bipartite graphs is not finitely axiomatizable modulo finiteness. That is, there exists no finite set $\Sigma$ of $\text{Lng}(\sigma_{gr})$-sentences such that a finite graph $A$ is bipartite if and only if each member of $\Sigma$ holds in $A$.

For comparison, we note the following folkloric fact.

**Remark 2.2.** Let $\sigma$ be a finite signature. If $K$ is a class of finite $\sigma$-structures such that it is closed with respect to taking isomorphic copies, then there exists a set $\Phi$ of $\text{Lng}(\sigma)$-sentences such that a finite structure belongs to $K$ if and only if it satisfies every member of $\Phi$.

**Proof of Remark 2.2.** For each $k \in \mathbb{N}^+$, $K$ contains finitely many $k$-element structures (up to isomorphism). Hence there is an $\text{Lng}(\sigma)$-sentence $\nu_k$ that holds exactly in the $k$-element structures of $K$. Thus, we can let $\Phi := \{\lambda_k \Rightarrow \nu_k : k \in \mathbb{N}^+\}$. \(
\)

**Proof of Proposition 2.1.** For $2 \leq n \in \mathbb{N}^+$, the circle of length $n$ is the graph $C_n$ with base set $C_n := \{0, 1, \ldots, n-1\}$ and $E^{C_n} = \{(x, y) : |x - y| \in \{1, n-1\}\}$.

Let $J_0 := \{4, 6, 8, 10, 12, 14, \ldots\}$ and $J_1 := \{3, 5, 7, 9, 11, 13, \ldots\}$. (2.5)

Note that $C_n$ is bipartite for all $n \in J_0$ but $C_n$ is not bipartite if $n \in J_1$. The set of all subsets of $J_i$ will be denoted by $P(J_i)$. For $i \in \{0, 1\}$, let $U_i$ be a nontrivial ultrafilter over $J_i$; see, for example, Poizat [20] for this concept. What we need here is that $\emptyset \notin U_i \subseteq P(J_i)$ and $U_i$ contains all cofinite subsets of $J_i$; a subset $X \subseteq J_i$ is cofinite if $J_i \setminus X$ is finite. Let $A_i = (A_i, E)$ be the ultraproduct $\prod_{n \in J_i} C_n/U_i$. We know from Frayne, Morel, and Scott [12] or from Keisler [17] that

an ultraproduct of finite structures modulo a nontrivial ultrafilter

is either finite, or it has at least continuum many elements. (2.6)

For $i \in \{0, 1\}$, $k \in \mathbb{N}^+$, and $\lambda_k$ defined in (2.4), $\{n \in J_i : \lambda_k \text{ holds in } C_n\}$ is a cofinite set, whereby it belongs to $U_i$. Hence, $\lambda_k$ holds in $A_i$ by Loś’s Theorem; see, for example, Theorem 4.3 in Poizat [20]. Since this is true for all $k \in \mathbb{N}^+$, $A_i$ is not finite. Also, it has at most continuum many elements since the cardinality of the direct product $\prod_{n \in J_i} C_n$ is continuum. Thus (2.6) gives that, for $i \in \{0, 1\}$, the cardinality of $A_i$ is continuum; in notation, $|A_i| = 2^{\aleph_0}$. (2.7)

Note that the subsequent sections will reference (2.7) in connection with other structures defined by similar ultraproducts of finite structures.

The $\mathbb{Z}$-chain is the graph $C_\infty$ with the set $C_\infty := \mathbb{Z}$ of integer numbers as vertex set and $E^{C_\infty} := \{(x, y) : |x - y| = 1\}$.

There is an $\text{Lng}(\sigma_{gr})$-sentence expressing that for every element $x$ there are exactly two elements $y$ such that $xEy$. Apart from $n = 2$, this sentence holds in all $C_n$. Hence, by Loś’s Theorem again, this sentence holds in $A_0$ and $A_1$. This yields that $A_i$ is the disjoint union of copies of $\mathbb{Z}$-chains and circles $C_k$, $3 \leq k \in \mathbb{N}$. However, for each $3 \leq k \in \mathbb{N}^+$, there is an $\text{Lng}(\sigma_{gr})$-sentence expressing that $C_k$ is not a subgraph. This sentence holds in $C_n$ for all $n$ belonging to the cofinite set $J_i \setminus \{k\}$, whereby Loś’s Theorem gives that this sentence also holds in $A_i$. Hence, $A_i$ contains no circle. Consequently, for $i \in \{0, 1\}$, there is a cardinal number $\kappa_i$ such that $A_i$ is the disjoint union of $\kappa_i$ many copies of $\mathbb{Z}$-chains. Combining $|C_\infty| = \aleph_0$ with (2.7), it follows that $\kappa_0 = 2^{\aleph_0} = \kappa_1$. Thus, $A_0$ and $A_1$ are isomorphic graphs; in notation, $A_0 \cong A_1$. 


Next, for the sake of contradiction, suppose that Proposition 2.1 fails. Then, using that finitely many sentences can always be replaced by their conjunction, there exists a single sentence \( \varphi \) such that for every finite graph \( B \), \( \varphi \) holds in \( B \) if and only if \( B \) is a bipartite graph. In particular,

\[
\varphi \text{ holds in } C_n \text{ for all } n \in J_0 \text{ but it fails in } C_m \text{ for all } m \in J_1.
\]  

(2.8)

By Łoś’s Theorem, \( \varphi \) holds in \( \mathcal{A}_0 \). Hence, by the isomorphism \( \mathcal{A}_0 \cong \mathcal{A}_1 \), \( \varphi \) holds in \( \mathcal{A}_1 \), too. Using Łoś’s Theorem again, we obtain that the set \( \{ m \in J_1 : \varphi \text{ holds in } C_m \} \) belongs to the ultrafilter \( U_1 \). This contradicts the fact that this set is empty by (2.8), completing the proof of Proposition 2.1.

\[\square\]

3. Groups

Using the terminology of Proposition 2.1, we have the following statement.

**Proposition 3.1.** The class of finite simple groups is not finitely axiomatizable modulo finiteness.

**Proof.** Since lots of arguments used in Proposition 2.1 apply here, we give less details. According to (2.1), the signature \( \sigma_{gr} \) for groups is chosen so that \( \rho = 0 \), \( q = 1 \), \( f_1 = 2 \), and \( F_1 \) is “+”. For the sake of contradiction, suppose that there exists an \( \text{Lng}(\sigma_{gr}) \)-sentence \( \varphi \) that holds in all finite simple groups but it fails in all finite non-simple groups. For \( n \in \mathbb{N}^+ \), the cyclic group of order \( n \) will be denoted by \( C_n \). Let \( p_1 < p_2 < p_3 < \ldots \) be the list of all prime numbers, and define \( q_j := p_jp_{j+1} \) for \( j \in \mathbb{N}^+ \). Take a nontrivial ultrafilter \( U \) over \( \mathbb{N}^+ \). Let \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) be the ultraproducts \( \prod_{n \in \mathbb{N}^+} C_{n^+}/U \) and \( \prod_{n \in \mathbb{N}^+} C_{n^+}/U \), respectively. Observe that (2.7) is still valid; see the sentence right after it. For \( k \in \mathbb{N}^+ \), define the following sentence of \( \text{Lng}(\sigma_{gr}) \) with \( k \) occurrences of \( y \):

\[
\eta_k : \quad \forall x \exists y ((\ldots (y + y) + y) + \ldots) + y = x.
\]

Basic facts about linear congruences yield that the sets \( \{ n \in \mathbb{N}^+ : \eta_k \text{ holds in } C_{n^+} \} \) and \( \{ n \in \mathbb{N}^+ : \eta_k \text{ holds in } C_{q_n} \} \) are cofinite and so they belong to \( U \). Similarly, with \( k + 1 \) occurrences of \( x \) before the first equality sign, if we define

\[
\tau_k : \quad \forall x \left( (\ldots (x + x) + x) + \ldots) + x = x \Rightarrow \forall y (y + y = y) \right),
\]

then both \( \{ n \in \mathbb{N}^+ : \tau_k \text{ holds in } C_{p_n} \} \) and \( \{ n \in \mathbb{N}^+ : \tau_k \text{ holds in } C_{q_n} \} \) are cofinite and belong to \( U \). Hence, by Łoś’s Theorem, \( \eta_k \) and \( \tau_k \) hold in \( \mathcal{A}_i \) for all \( k \in \mathbb{N}^+ \) and \( i \in \{0, 1\} \). Therefore, the abelian groups \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) are torsion-free (by the sentences \( \tau_k \)) and divisible (by the \( \eta_k \)). Consequently, they are direct sums of copies of the additive group \( (\mathbb{Q}, +) \) of rational numbers; see, for example, Kurosh [18, page 165] or use the straightforward fact that a torsion-free and divisible abelian group can be considered a vector space over the field of rational numbers. (2.7) implies that each of \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) has \( 2^{|\mathbb{N}|} \)-many direct summands. Hence, \( \mathcal{A}_0 \cong \mathcal{A}_1 \).

For all \( n \in \mathbb{N}^+, C_{p_n} \) is a simple group and so it satisfies \( \varphi \). Łoś’s Theorem gives that \( \varphi \) holds in \( \mathcal{A}_0 \), whereby it holds in \( \mathcal{A}_1 \) since \( \mathcal{A}_1 \cong \mathcal{A}_0 \). Using Łoś’s Theorem again, we obtain that the set \( I := \{ n \in \mathbb{N}^+ : \varphi \text{ holds in } C_{q_n} \} \) belongs to the ultrafilter \( U \). But none of the groups \( C_{q_n} \) is simple, so none of them satisfies \( \varphi \), whence \( I = \emptyset \). The contradiction \( \emptyset \in U \) completes the proof of Proposition 3.1. \[\square\]

Note that (2.7) and the structure theorem of torsion-free divisible abelian groups in the proof above were only used to conclude that \( \mathcal{A}_0 \cong \mathcal{A}_1 \), but this isomorphism
was only needed to ensure that $\mathcal{A}_0$ and $\mathcal{A}_1$ are elementarily equivalent. There is another way to ensure this elementary equivalence that relies neither on (2.7), nor on the above-mentioned structure theorem: one can use the description of elementary equivalence of abelian groups given by Szmielew [21]. However, the use of this description would require further $\text{Lng}(\sigma_{gr})$-sentences and would make the proof more complicated.

4. The ordered sets of join-irreducible congruences of slim semimodular lattices

**Brief introduction to slim semimodular lattices.** We assume that the reader has some basic familiarity with lattices; if not then a few parts of Burris and Sankappanvar [1] or Davey and Priestley [10] or Grätzer [13] are recommended.

A lattice $\mathcal{L} = \langle L; \lor, \land \rangle$ is *semimodular* if for any $x, y, z \in L$, the covering relation $x \prec y$ implies that $x \lor z \prec y \lor z$ or $x \lor z = y \lor z$. The lattice $\mathcal{L}$ is *slim* if it is finite and the (partially) ordered set $\mathcal{J}(\mathcal{L}) = \langle \mathcal{J}(\mathcal{L}), \leq \rangle$ of its join-irreducible elements is the union of two chains. We know from Czédli and Schmidt [9, Lemma 2.3] that for finite semimodular lattices, this definition of slimmness is equivalent to the original one, which is due to Grätzer and Knapp [14] but not recalled here. We also know from Czédli and Schmidt [9, Lemma 2.2] that slim lattices are *planar*; however, the term “slim, planar, semimodular lattice” frequently occurs in the literature since the original concept of slimmness did not imply planarity. Here we write “slim semimodular lattices” and these lattices are automatically finite and planar. As usual, the set of congruence relations of a lattice $\mathcal{L}$ form a lattice, the *congruence lattice* $\text{Con} \mathcal{L}$ of $\mathcal{L}$. The study of congruence lattices of slim semimodular lattices began with Grätzer and Knapp [15]. These congruence lattices $\text{Con} \mathcal{L}$ are distributive. Hence, by the classical structure theorem of finite distributive lattices, see Grätzer [13, Theorem II.1.9] for example, these congruence lattices are economically described by simpler and smaller structures: the ordered sets $\mathcal{J}(\text{Con} \mathcal{L})$ of their join-irreducible elements.

Several properties of the ordered sets $\mathcal{J}(\text{Con} \mathcal{L})$ determined by slim semimodular lattices $\mathcal{L}$ have been discovered; they are summarized in Czédli [5] and Czédli and Grätzer [7]. In fact, the attempt to characterize these $\mathcal{J}(\text{Con} \mathcal{L})$ served as the main motive to deal with slim semimodular lattices. For surveys of these lattices, see the book chapter Czédli and Grätzer [6] and Section 2 of Czédli and Kurusa [8]. Here, as an appetizer to this section of the paper, we only mention that slim semimodular lattices were used to strengthen the Jordan–Hölder Theorem for groups from the nineteenth century, see Czédli and Schmidt [9] and Grätzer and Nation [16], and they have led to results in geometry, see Czédli [3]–[4] and Czédli and Kurusa [8] together with the survey given in it. Since 2007, when G. Grätzer and E. Knapp [14] introduced slim semimodular lattices, the study of these lattices has been the most intensive part of lattice theory. Indeed, at the time of writing, the MathSciNet search “Anywhere=(slim and semimodular)” returns 22 matches.

**The main result of the paper and its proof.** In harmony with (2.1), we assume that ordered sets are of type $\sigma_{ord} = \langle 1, 0, (\leq, 2) \rangle$. Using this notation and (2.2), we formulate our main result as follows.

**Theorem 4.1.** The class of ordered sets of join-irreducible congruences of slim semimodular lattices is not finitely axiomatizable modulo finiteness. That is, there
exists no finite set \( \Phi \) of \( \text{Lng}(\sigma_{\text{ord}}) \)-sentences such that a finite ordered set \( \mathcal{S} = \langle S, \leq \rangle \) is isomorphic to the ordered set \( \mathcal{J}(\text{Con} \mathcal{L}) = \langle J(\text{Con} \mathcal{L}), \leq \rangle \) of some slim semimodular lattice \( \mathcal{L} \) if and only if all members of \( \Phi \) hold in \( \mathcal{S} \).

**Proof.** Suppose the contrary. Then, as in the proof of Proposition 2.1, we can pick a single \( \text{Lng}(\sigma_{\text{ord}}) \)-sentence \( \varphi \) such that a finite ordered set \( \mathcal{S} \) satisfies \( \varphi \) if and only if \( \mathcal{S} \cong \mathcal{J}(\text{Con} \mathcal{L}) \) for a slim semimodular lattice \( \mathcal{L} \). For \( 2 \leq n \in \mathbb{N}^+ \), the \( n \)-crown \( \mathcal{K}_n \) is the \( 2n \)-element ordered set with maximal elements \( a_0, a_1, \ldots, a_{n−1} \) and minimal elements \( b_0, b_1, \ldots, b_{n−1} \) such that, for \( i, j \in \{0, 1, \ldots, n−1\} \), \( b_i \leq a_j \) if and only if \( i = j \) or \( i + 1 \equiv j \) (mod \( n \)). For \( n = 8 \), \( \mathcal{K}_8 \) is drawn below.

![Diagram of \( \mathcal{K}_8 \)](http://www.math.u-szeged.hu/~czedli/)

We let \( J_0 = \{2, 4, 6, \ldots\} \) and \( J_1 = \{3, 5, 7, \ldots\} \). Take a nontrivial ultrafilter \( U_i \) over \( J_i \). For \( i \in \{0, 1\} \), let \( \mathcal{A}_i \) be the ultraproduct \( \prod_{n \in J_i} \mathcal{K}_n/U_i \). We know from (2.7) and the sentence following it that \( |A_0| = |A_1| = 2^{\kappa_0} \). Although, to save space, we do not give all of them in details, we have the following \( \text{Lng}(\sigma_{\text{ord}}) \)-formulas.

\( \alpha(x) \): \( \forall y (x \leq y \Rightarrow y \leq x) \), which expresses that \( x \) is a maximal element.

\( \beta(x) \): \( \forall y (y \leq x \Rightarrow x \leq y) \), which expresses that \( x \) is a minimal element.

\( \delta_1 \): \( \forall x, \text{ exactly one of } \alpha(x) \text{ and } \beta(x) \) holds.

\( \delta_2 \): \( \forall x, \text{ if } \alpha(x) \text{, then there are exactly two elements } y \text{ such that } \beta(y) \text{ and } y \leq x \).

\( \delta_3 \): \( \forall x, \text{ if } \beta(x) \text{, then there are exactly two elements } y \text{ such that } \alpha(y) \text{ and } x \leq y \).

\( \xi_m \): there are no elements forming a subset order isomorphic to \( \mathcal{K}_m \).

The ordered set \( \mathcal{F} = \langle \{a_j : j \in \mathbb{Z}\} \cup \{b_j : j \in \mathbb{Z}\}, \leq \rangle \) such that \( \alpha(a_j), \beta(b_j), \) and \( \neg \alpha(a_j) \) for all \( j \in \mathbb{Z} \) and, in addition, \( b_j \leq a_s \) if and only if \( s \in \{j, j + 1\} \) will be called an (infinite) fence; see the Figure below.

![Diagram of \( \mathcal{F} \)](http://www.math.u-szeged.hu/~czedli/)

Recall that for ordered sets \( \mathcal{W}_h = (W_h, \leq^h), h \in H \), we obtain the cardinal sum \( \mathcal{W} = (W, \leq) \) of these ordered sets by letting \( W \) be the disjoint union of the \( \mathcal{W}_h, h \in H \), and defining \( \leq \) as the union of the \( \leq^h, h \in H \). Since \( \delta_1, \delta_2, \) and \( \delta_3 \) hold in \( \mathcal{K}_n \) for all \( n \in J_0 \cup J_1 \) and, for each \( m \geq 2 \), so does \( \xi_m \) for all \( n \in (J_0 \cup J_1) \setminus \{m\} \), Los’s Theorem yields that \( \delta_1, \delta_2, \delta_3, \) and, for all \( m \in \mathbb{N}^+ \setminus \{1\}, \xi_m \) hold in \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \). Therefore, for \( i \in \{0, 1\} \), we conclude that each element of \( \mathcal{A}_i \) belongs to a unique fence and \( \mathcal{A}_i \) is the cardinal sum of some copies, say \( \kappa_i \) copies, of fences. Using \( |A_0| = 2^{\kappa_0} = |A_1| \), we obtain that \( \kappa_0 = 2^{\kappa_0} = \kappa_1 \). Therefore, \( \mathcal{A}_0 \cong \mathcal{A}_1 \).

The rest of the proof relies heavily on Czédli [2] and mainly on [5]; these two papers\(^1\) should be near. In particular, the notation and the concepts not defined here are given there. By the “Bipartite maximal elements property”, see Corollary 3.4 in [5], \( \mathcal{K}_n \cong J(\text{Con} \mathcal{L}) \) with a slim semimodular \( \mathcal{L} \) cannot hold if \( n \in J_1 \). Hence, \( \varphi \) fails in \( \mathcal{K}_n \) for \( n \in J_1 \), and Los’s Theorem gives that \( \varphi \) does not hold in \( \mathcal{A}_1 \).

Next, we assume that \( n \in J_0 \). Let \( k := n/2 \). To construct a lattice, we begin with the direct square of the \((k + 1)-element chain\); it is a distributive lattice called a grid. For \( n = 8 \), this grid consists of the pentagon-shaped elements in Figure 1. Going downwards, we label the edges on the upper left boundary by \( a_0, a_2, \ldots \),

\(^1\)Temporary note: see [http://www.math.u-szeged.hu/~czedli/](http://www.math.u-szeged.hu/~czedli/) for their preprints.
Also, we label the edges on the upper right boundary by \(a_1, a_3, \ldots, a_{n-1}\), going downwards again. In this way, we have labeled the non-vertical thick edges of Figure 1. At this stage, the circle-shaped elements and the edges having (at least one) circle-shaped endpoints are not present. The edges of the grid determine \(k^2\) many 4-cells (that is, squares) in the plane. We obtain a slim semimodular lattice \(L_n\) from the grid in \(n\) steps in the following way. First, we insert a fork (that is, a multifork of rank 1) into \(\text{RightEnl}(a_0) \cap \text{LeftEnl}(a_1)\); this intersection is the uppermost grey-filled rectangle (which happens to be a square) in the figure. This insertion brings the \(b_0\)-labeled thick vertical edge in. (Since there would not be enough room otherwise, the label of a vertical thick edge is always below the edge in Figure 1; note that the thick edges are exactly the labeled edges.) In the second step, we insert a fork into \(\text{RightEnl}(a_0) \cap \text{LeftEnl}(a_{n-1})\), understood in the lattice obtained in the previous step, of course. In the figure, this step brings the \(b_7\)-labeled thick vertical edge in, and the intersection in question as well as the subsequent intersections are grey-filled. In the third step, we insert a fork into \(\text{LeftEnl}(a_1) \cap \text{RightEnl}(a_2)\) and we obtain the \(b_1\)-labeled thick vertical edge. And so on, inserting a fork into \(\text{RightEnl}(a_2) \cap \text{LeftEnl}(a_3)\), \(\text{LeftEnl}(a_3) \cap \text{RightEnl}(a_4)\), \(\text{RightEnl}(a_4) \cap \text{LeftEnl}(a_5)\), \ldots, \(\text{RightEnl}(a_{n-2}) \cap \text{LeftEnl}(a_{n-1})\), one by one and in this order, we obtain the thick vertical edges with labels \(b_2, b_3, b_4, \ldots, b_{n-2}\), respectively. After performing these steps, we obtain the required lattice \(L_n\). For \(n = 8\), \(L_n = L_8\) is given in Figure 1. By Theorem 3.7 of Czédli [2], \(L_n\) is a slim semimodular lattice. By (the Main) Lemma 2.11 of Czédli [5], \(\mathcal{K}_n \cong \mathcal{J}(\text{Con} L_n)\).

This isomorphism and the choice of \(\varphi\) gives that \(\varphi\) holds in \(\mathcal{K}_n\). This is true for all \(n \in J_0\), whereby Los’s Theorem implies that \(\varphi\) holds in \(\mathcal{A}_0\). But this is a contradiction since \(\mathcal{A}_0 \equiv \mathcal{A}_1\) but we have previously seen that \(\varphi\) does not hold in \(\mathcal{A}_1\). The proof of Theorem 4.1 is complete. \(\square\)
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