SEPARIING TWISTS AND THE MAGNUS REPRESENTATION
OF THE TORELLI GROUP

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Abstract. The Magnus representation of the Torelli subgroup of the mapping class group of a surface is a homomorphism \( r: \mathcal{I}_{g,1} \to \text{GL}_{2g}(\mathbb{Z}[H]) \). Here \( H \) is the first homology group of the surface. This representation is not faithful; in particular, Suzuki previously described precisely when the commutator of two Dehn twists about separating curves is in \( \ker r \). By examining the trace of the Magnus representation in greater detail, we generalize this result to determine when the commutator of two positive separating multitwists is in \( \ker r \). We also show that the images of two positive separating multitwists under the Magnus representation either commute or generate a free group.

1. Introduction

Let \( S = S_{g,1} \) be a compact orientable surface of genus \( g \) with one boundary component. The mapping class group of \( S \) is defined to be \( \text{Mod}_{g,1} = \pi_0(\text{Homeo}^+(S)) \), where \( \text{Homeo}^+(S) \) is the group of orientation-preserving self-homeomorphisms of \( S \) fixing the boundary \( \partial S \) pointwise. In other words, \( \text{Mod}_{g,1} \) is the group of isotopy classes of homeomorphisms of \( S \) relative to the boundary. The mapping class group naturally acts on \( H = H_1(S;\mathbb{Z}) \) preserving the algebraic intersection form; the Torelli group \( \mathcal{I}_{g,1} \) is defined to be the kernel of this representation, giving a short exact sequence:

\[
1 \to \mathcal{I}_{g,1} \to \text{Mod}_{g,1} \to \text{Sp}_{2g}(\mathbb{Z}) \to 1
\]

There are two basic types of elements in \( \mathcal{I}_{g,1} \): a separating twist is a Dehn twist \( T_\gamma \) about a separating simple closed curve \( \gamma \), while a bounding pair map is of the form \( T_\gamma_1 T_{\gamma_2}^{-1} \) where \( \gamma_1 \) and \( \gamma_2 \) are disjoint homologous simple closed curves. These suffice to generate \( \mathcal{I}_{g,1} \), by work of Birman [1] and Powell [9].

Let \( \Gamma = \pi_1(S) \) and let \( \mathbb{Z}[\Gamma] \) be the group ring of \( \Gamma \). The classical Magnus representation was originally defined algebraically as a crossed homomorphism \( \tau: \text{Mod}_{g,1} \to \text{GL}_{2g}(\mathbb{Z}[\Gamma]) \) using Fox calculus. This induces a homomorphism \( r: \mathcal{I}_{g,1} \to \text{GL}_{2g}(\mathbb{Z}[H]) \), which we distinguish from the classical Magnus representation \( \tau \) by simply calling \( r \) the Magnus representation. Suzuki demonstrated in [11] that there is an equivalent topological definition of the Magnus representation, which we will use throughout this paper. A homeomorphism of \( S \) can be lifted to a homeomorphism of the universal abelian cover \( \widetilde{S} \) of \( S \), which then acts on the relative first homology group \( H_1(\widetilde{S}, \pi^{-1}(\ast)) \cong \mathbb{Z}[H]^{2g} \). The resulting representation is isomorphic to \( r \); we discuss the details of this construction further in Section 2.

By exploiting the fact that the abelian cover is a surface, we obtain a \( \mathbb{Z}[H] \)-valued “higher intersection form” \( \langle \cdot, \cdot \rangle \) on \( H_1(\widetilde{S}, \pi^{-1}(\ast)) \) that is preserved by the image of the Magnus representation. A version of this form was first constructed by Papakyriakopoulos [7]. Morita verified by laborious computation that it is preserved
Theorem B. Under the Magnus representation commute:

Theorem A. proof of this result and yield the following extension. lifts

\[ [\gamma_1, \gamma_2] \]

multitwists. Then they generate a free group. (See e.g. Leininger [5]; the claim follows immediately from Proposition 5.2.) Suzuki [12, Corollary 4.4] characterized when the commutator \([T_{\gamma_1}, T_{\gamma_2}]\) of two Dehn twists around separating curves \(\gamma_1, \gamma_2\) is in \(\ker r\). Our methods allow a simpler proof of this result and yield the following extension.

Theorem A. Suppose that \(\gamma_1\) and \(\gamma_2\) are nontrivial separating curves in \(S\) with lifts \(c_1, c_2 \in H_1(S, \pi^{-1}(s))\). If \(\langle c_1, c_2 \rangle = 0\), then \(r(T_{\gamma_1}) \) and \(r(T_{\gamma_2}) \) commute. If \(\langle c_1, c_2 \rangle \neq 0\), then \(r(T_{\gamma_1}) \) and \(r(T_{\gamma_2}) \) generate a free group of rank 2 in \(\text{GL}_{2g}(\mathbb{Z}[H])\).

We characterize exactly when the images of two positive separating multitwists under the Magnus representation commute:

Theorem B. Let \(T_C = T_{\gamma_1}^{n_1} \cdots T_{\gamma_k}^{n_k}\) and \(T_D = T_{\gamma_1}'^{n_1} \cdots T_{\gamma_l}'^{n_l}\) be positive separating multitwists. Then \([T_C, T_D] \in \ker r\) if and only if \([T_{\gamma_i}, T_{\delta_j}] \in \ker r\) for all \(1 \leq i \leq k\) and \(1 \leq j \leq l\).

We also have the following result on relations between positive separating multitwists, analogous to Theorem A:

Theorem C. Let \(T_C\) and \(T_D\) be positive separating multitwists with \([T_C, T_D] \notin \ker r\). Then \(r(T_C)\) and \(r(T_D)\) generate a free group of rank 2 in \(\text{GL}_{2g}(\mathbb{Z}[H])\).

We begin by explaining Suzuki’s topological definition of the Magnus representation in Section 2. In Section 3 we give a model of the universal abelian cover and describe various topological features. In Section 4 we define the higher algebraic intersection form and develop its properties. We conclude in Section 5 by studying the trace function \(t\) and proving Theorems A, B and C. Finally, in Section 6 we outline some natural further questions generalizing these results.
Acknowledgments

We thank Masaaki Suzuki for kindly providing us with a preprint of [11], which provided the impetus for this work. This paper was begun during the REU program at Cornell University in 2005 under the supervision of Tara Brendle, funded by the National Science Foundation. We thank the REU and Cornell for their support and hospitality.

We are grateful to Khalid Bou-Rabee, Tara Brendle, Nathan Broaddus, Matthew Day, Spencer Dowdall, Benson Farb, Asaf Hadari, Ben McReynolds and Justin Malestein for reading early versions of this paper, and for their valuable comments and advice. We also thank Ken Brown, Keith Dennis, Allen Hatcher, Martin Kassabov and Karen Vogtmann for helpful discussions. The first author would additionally like to thank the Cornell Presidential Research Scholars program, whose support in part made possible the completion of this paper.

2. The Magnus representation

We fix a basepoint \( * \in \partial S \), and note that the fundamental group \( \Gamma := \pi_1(S, *) \) is free on \( 2g \) generators. We will avoid performing any computations with respect to specific generating sets or bases, but it is occasionally convenient to choose generators for \( \Gamma \). Let \( A_1, \ldots, A_g, B_1, \ldots, B_g \) be a generating set for \( \Gamma \) such that the product of commutators \([A_1, B_1] \cdots [A_g, B_g]\) is a loop around the boundary component. Note that if \( a_i, b_i \) are the homology classes of \( A_i, B_i \) respectively, then \( a_1, \ldots, a_g, b_1, \ldots, b_g \) form a symplectic basis with respect to the algebraic intersection form on \( H = H_1(S) \). (Throughout the paper, all homology groups are taken with coefficients in \( \mathbb{Z} \).)

The classical Magnus representation \( \tau \) was originally defined by interpreting mapping classes as automorphisms of \( \Gamma \) via the natural inclusion \( \text{Mod}_{g,1} \hookrightarrow \text{Aut}(\Gamma) \) and using Fox calculus (see Birman [2], for instance). We will briefly outline this definition for completeness.

Let \( z_1, \ldots, z_{2g} \) be a generating set for \( \Gamma \). The elements \( \{ z_i - 1 \} \) form a basis for the augmentation ideal in \( \mathbb{Z}[\Gamma] \) as a free left \( \mathbb{Z}[\Gamma] \)-module. The Fox derivatives

\[
\frac{\partial}{\partial z_i} : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]
\]

are determined by the identity

\[
\gamma - 1 = \sum_{i=1}^{2g} \frac{\partial}{\partial z_i} (z_i - 1).
\]

for \( \gamma \in \Gamma \). Thus the Fox derivatives are simply the coefficients with respect to the above basis. Given \( f \in \text{Mod}_{g,1} \), the matrix \( \tau(f) \in \text{GL}_{2g}(\mathbb{Z}[\Gamma]) \) has entries

\[
\tau(f)_{ij} = \frac{\partial}{\partial z_j} f(z_i).
\]

This function \( \tau \) is not a homomorphism, but it does satisfy the property

\[
\tau(fg) = \tau(f) \cdot f[\tau(g)],
\]

where \( f[\tau(g)] \) denotes the action of \( f \) on \( \text{GL}_{2g}(\mathbb{Z}[\Gamma]) \) by acting on each entry.
If $N$ is a quotient of $\Gamma$, we can compose $\tau$ with the induced map $\text{GL}_2(\mathbb{Z}[\Gamma]) \to \text{GL}_2(\mathbb{Z}[N])$; the composition becomes a homomorphism when restricted to the subgroup of $\text{Mod}_{g,1}$ that acts trivially on $N$. In particular, taking $N$ to be the abelianization $H$ gives the Magnus representation $r: \mathcal{I}_{g,1} \to \text{GL}_2(\mathbb{Z}[H])$, the primary object of our study. Note that this process actually defines a representation of $I\text{Aut}(\Gamma)$, the subgroup of $\text{Aut}(\Gamma)$ that acts trivially on $H$; for the algebraic construction, it does not matter whether an automorphism of $\Gamma$ can be realized by a homeomorphism of $S$. We will instead use a more topological definition of $r$ that was recently described by Suzuki in [11]. This forces us to restrict our attention to $\text{Mod}_{g,1}$, but lets us apply the topology of surfaces to the Magnus representation.

We now describe the topological definition of $r: \mathcal{I}_{g,1} \to \text{GL}_2(\mathbb{Z}[H])$. Let $\pi: \tilde{S} \to S$ be the regular covering space corresponding to the commutator subgroup of $\Gamma$; this is known as the universal abelian cover of $S$. The deck transformations are just $\Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}^{2g}$, and by identifying this quotient with $H$ the homology groups of $\tilde{S}$ are naturally viewed as $\mathbb{Z}[H]$–modules.

Fix a lift $\tilde{\tau} \in \pi^{-1}(\ast) \subset \partial \tilde{S}$ of the basepoint $\ast$. An element of $\text{Mod}_{g,1}$ can be represented by a homeomorphism $f: S \to S$ that fixes $\partial S$ pointwise. This lifts uniquely to a homeomorphism $\tilde{f}: \tilde{S} \to \tilde{S}$ if we require that $\tilde{f}$ fix $\tilde{\tau}$. The action of $\tilde{f}$ on the relative homology $H_1(\tilde{S}, \pi^{-1}(\ast))$ is obviously $\mathbb{Z}$–linear, but it is not necessarily $\mathbb{Z}[H]$–linear — it is twisted by the action of $f$ on $H$. If we restrict $f$ to $\mathcal{I}_{g,1}$, this ensures the lifts $\tilde{f}$ will act $\mathbb{Z}[H]$–linearly on $H_1(\tilde{S}, \pi^{-1}(\ast))$. Since $H_1(\tilde{S}, \pi^{-1}(\ast)) \cong \mathbb{Z}[H]^{2g}$ (see below), this action of $\tilde{f}$ can be identified with a representation

$$r: \mathcal{I}_{g,1} \to \text{Aut}(H_1(\tilde{S}, \pi^{-1}(\ast))) \cong \text{GL}_2(\mathbb{Z}[H]),$$

and this is exactly the Magnus representation. The classical Magnus representation $\tau$ can be defined similarly using the universal cover instead of the universal abelian cover; see [11] for more details.

We examine the details of this construction. Consider the long exact sequence of homology for the pair $(\tilde{S}, \pi^{-1}(\ast))$. The only nonzero part of this sequence is

$$0 \to H_1(\tilde{S}) \to H_1(\tilde{S}, \pi^{-1}(\ast)) \xrightarrow{\partial} H_0(\pi^{-1}(\ast)) \xrightarrow{\epsilon} H_0(\tilde{S}) \to 0,$$

where the maps $\epsilon$ and $\partial$ will be described in greater detail below. As discussed above, the action of $H$ on $\tilde{S}$ by deck transformations makes this an exact sequence of $\mathbb{Z}[H]$–modules. Elements of the Torelli group act naturally on the entire long exact sequence by $\mathbb{Z}[H]$–module automorphisms, as discussed above.

We describe the $\mathbb{Z}[H]$–module structure of each term of this sequence. We begin by noting that $H_0(\tilde{S}) = \mathbb{Z}$ because $\tilde{S}$ is connected, while $H_0(\pi^{-1}(\ast)) \cong \mathbb{Z}[H]$ because the connected components of $\pi^{-1}(\ast)$ correspond to elements of $H$. Explicitly, $h \mapsto h(\tilde{\tau})$ defines a bijection $H \to \pi^{-1}(\ast)$. The map $\epsilon$ in [11] clearly corresponds to the augmentation map $\mathbb{Z}[H] \to \mathbb{Z}$ that maps $h$ to 1 for all $h \in H$. 

We can see that $H_1(\hat{S}, \pi^{-1}(\ast)) \cong \mathbb{Z}[H]^{2g}$ by lifting a basis for $\Gamma$ to a $\mathbb{Z}[H]$-module basis for $H_1(\hat{S}, \pi^{-1}(\ast))$, as follows. For each $i = 1, \ldots, g$, define $a_i \in H_1(\hat{S}, \pi^{-1}(\ast))$ to be the unique lift of the loop $A_i$ starting at $\ast$, and similarly let $b_i$ be the lift of $B_i$. Each of these arcs must have its endpoints in $\pi^{-1}(\ast)$, so each describes an element of $H_1(\hat{S}, \pi^{-1}(\ast))$. The deck transformation $a_i$ translates the tail of the arc $a_i$ (that is, $\ast$) to its head, and similarly $b_i$ translates $\ast$ to the head of $b_i$. Cellular homology shows that $H_1(\hat{S}, \pi^{-1}(\ast))$ is a free $\mathbb{Z}[H]$-module of rank $2g$, with basis $a_1, \ldots, a_g, b_1, \ldots, b_g$. It is easily verified that the map $\partial$ in $\bigoplus$ is given by $\partial a_i = a_i - 1$ and $\partial b_i = b_i - 1$.

Finally, we have $\pi_1(\hat{S}) = [\Gamma, \Gamma]$, so $H_1(\hat{S})$ is the abelianization $[\Gamma, \Gamma]^{ab}$. The action of $\Gamma$ on $[\Gamma, \Gamma]$ by conjugation descends to a action of $H$ on $[\Gamma, \Gamma]$ by outer automorphisms. This projects to a well-defined action of $H$ on $[\Gamma, \Gamma]^{ab}$, and the resulting $\mathbb{Z}[H]$-module structure agrees with that of $H_1(\hat{S})$. The exact sequence above is thus isomorphic as an exact sequence of $\mathbb{Z}[H]$-modules to

$$0 \to [\Gamma, \Gamma]^{ab} \to \mathbb{Z}[H]^{2g} \xrightarrow{\partial} \mathbb{Z}[H] \xrightarrow{\text{c}} \mathbb{Z} \to 0.$$ 

We now briefly discuss the various types of elements in $H_1(\hat{S}, \pi^{-1}(\ast))$. First, $H_1(\hat{S}, \pi^{-1}(\ast))$ has an important subspace: the image of $H_1(\hat{S})$ in $H_1(\hat{S}, \pi^{-1}(\ast))$, which is also equal to the kernel of the boundary map $\partial$. We call elements in this subspace curves, whereas the general element of $H_1(\hat{S}, \pi^{-1}(\ast))$ is an arc. This terminology should not be confused with the usual description of curves in $S$; a separating curve in $S$ is just a null-homologous simple closed curve in $S$, not an element of some homology group.

Inside the subspace of curves in $H_1(\hat{S}, \pi^{-1}(\ast))$, there are certain special elements that we call lifting curves; these are the homology classes of simple closed curves in $\hat{S}$ that project homeomorphically under $p$ to simple closed curves in $S$. Equivalently, observe that the pre-image under $p$ of any separating curve $\gamma$ in $S$ is a disjoint union of separating curves in $\hat{S}$; we call the homology classes of these curves the lifts of $\gamma$. Distinct lifts of $\gamma$ differ by a deck transformation.

The following basic property of lifts of separating curves will be useful later:

**Lemma 2.1.** If $\gamma$ is a nontrivial separating curve, then any lift $c$ of $\gamma$ is nonzero in $H_1(\hat{S}, \pi^{-1}(\ast))$.

**Proof.** Represent the curve $\gamma$ by a based loop in $\Gamma$, which we also denote $\gamma$. Since $\gamma$ is separating, it lies in $\Gamma_2 = [\Gamma, \Gamma] \leq \Gamma$. It follows that $\gamma$ lifts to $\pi_1(\hat{S}, \ast)$, and one lift $c$ is the image of $\gamma$ under the abelianization map $\pi_1(\hat{S}, \ast) \to H_1(\hat{S})$. Since the lifts differ by deck transformations, it is enough to show that $c$ is nonzero, or equivalently that $\gamma \notin [\Gamma_2, \Gamma_2]$.

Let $k \geq 1$ be the genus of the surface cut out by $\gamma$ that does not contain $\partial S$. It is easily verified that the separating curve $\delta_k = [A_1, B_1] \cdots [A_k, B_k]$ also cuts off a surface of genus $k$, so some element of $\text{Mod}_{g,1} \leq \text{Aut}(\Gamma)$ maps $\gamma$ to $\delta_k$. Since $[\Gamma_2, \Gamma_2]$ is a characteristic subgroup of $\Gamma$, it suffices to show that $\delta_k \notin [\Gamma_2, \Gamma_2]$. Consider the quotient $\Gamma_2 \to \Gamma_2/[\Gamma, \Gamma_2] \cong \Lambda \mathbf{H}$, and note that $[A_1, B_1] \cdots [A_k, B_k]$ maps to $\sum_{i=1}^k a_i \wedge b_i$, which is nonzero. Since $[\Gamma_2, \Gamma_2] \leq [\Gamma, \Gamma_2]$, the claim follows.

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3. **The Topology of the Universal Abelian Cover**

While it is possible to use this topological definition of the Magnus representation without ever visualizing the abelian covering surface \( \tilde{S} \), a mental picture can be very useful to ground the algebraic concepts in topological intuition. In this section, we give a model for visualizing this surface, and then we discuss the different types of elements in \( H_1(\tilde{S}, \pi^{-1}(\ast)) \).

Our description will use the concept of a cyclically marked graph, which should be thought of as the spine of a surface with boundary.

**Definition 3.1.** A **cyclically marked graph** is simply a graph \( G \) endowed with a cyclic ordering \( m \) on the half-edges incident upon each vertex.

Such objects arise naturally whenever a graph is embedded in an oriented surface, since the orientation induces a cyclic ordering (e.g., order the edges by proceeding counter-clockwise around a vertex). Depending on the context, they are also known as ribbon graphs, fat graphs, cyclic graphs and dessins d’enfants.

The key feature of a cyclically marked graph \( (G, m) \) is that it can be canonically thickened into an oriented surface with boundary: expand each vertex of \( G \) to a polygon with sides corresponding to the half-edges incident on that vertex, arranged counter-clockwise with respect to the cyclic ordering \( m \), and fatten each edge of \( G \) to a ribbon connecting the sides of the vertex polygons corresponding to the two ends of the edge. (The ribbon should be attached so that its orientation agrees with the orientation of the polygons it is connected to.) Denote the resulting surface by \( S(G, m) \); observe that if \( G \) is embedded in a surface \( S \) and \( m \) is the marking induced by the orientation of \( S \), then \( S(G, m) \) can be viewed as a regular neighborhood of \( G \) in \( S \).

If \( \pi: \tilde{G} \to G \) is a covering of graphs, then a marking \( m \) on \( G \) lifts to a marking \( \tilde{m} \) on \( \tilde{G} \). This covering induces a covering \( \pi_S: S(\tilde{G}, \tilde{m}) \to S(G, m) \) extending \( \pi \). Since \( S(G, m) \) deformation retracts onto \( G \), there is a natural isomorphism \( \pi_1(G) \cong \pi_1(S(G, m)) \). It follows that the covering spaces of \( S(G, m) \) are exactly the spaces \( S(G, \tilde{m}) \to S(G, m) \) for \( (\tilde{G}, \tilde{m}) \) covering \( (G, m) \).

Note that we can describe \( S = S_{g,1} \) as the thickening of a cyclically marked graph \( (G, m) \) as follows. Let \( G \) be the graph with 1 vertex and \( 2g \) edges. Give each edge an orientation and label the edges by \( A_1, \ldots, A_g, B_1, \ldots, B_g \); for each edge \( A_i \), call the head \( A_i^+ \) and the tail \( A_i^- \) (and similarly for \( B_i \)). Then the marking \( m \) is the cyclic ordering \( A_1^+, B_1^-, A_2^+, B_2^-, \ldots, A_g^+, B_g^- \). To verify that \( S(G, m) \) is indeed homeomorphic to \( S_{g,1} \), we just need to check that \( S(G, m) \) has just one boundary component; then since \( \pi_1(S(G, m)) = \pi_1(G) \) has rank \( 2g \), \( S(G, m) \) has genus \( g \). This can be directly verified; indeed, if we treat the oriented edges \( A_i \) and \( B_i \) as a basis for \( \pi_1(G) \), the boundary is homotopic to \([A_1, B_1] \cdots [A_g, B_g] \in \pi_1(S(G, m)) \cong \pi_1(G)\). (This is the reason for our care in labeling.)
By the remark above, the abelian cover $\hat{S}$ of this $S(G, m) = S_{g, 1}$ is just $S(\hat{G}, \hat{m})$, where $\hat{G}$ is the abelian cover of $G$ and $\hat{m}$ is the lifted marking. The abelian cover $\hat{G}$ is well-known; it can be regarded as the Cayley graph of $\mathbb{Z}^2_g$ with respect to the usual generating set, or alternately as the subset of $\mathbb{R}^{2g}$ where at most one coordinate is not an integer. We can now read off a description of $\hat{S}$: first, place a $4g$-gon at each point of $\mathbb{Z}^{2g} \leq \mathbb{R}^{2g}$, and take a ribbon for every pair of adjacent points in this lattice. To each $4g$-gon we now attach the $4g$ ribbons adjacent to it, with the order as specified above. If we identify the $2g$ coordinates with the basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ of $H$, the sides of the polygon are attached, in order, to ribbons pointing in the $a_1^+, a_1^-, b_1^+, b_1^-$, $a_2^+, a_2^-, b_2^+, b_2^-$, $\ldots$, $a_g^+, a_g^-, b_g^+, b_g^-$ directions. A partial depiction of the abelian cover when $g = 2$ is given in Figure 1.

We have now completed the description of the surface $\hat{S}$, but we can describe a few more features of it. The group of deck transformations is isomorphic to $H$; its generators correspond to moving the entire lattice rigidly by one unit in each of the $2g$ directions. We can associate the shift in the $a_1$ direction with $a_1 \in H$ itself, and so on. The identification of $H_0(\pi^{-1}(\ast))$ with $\mathbb{Z}[H]$ is then immediate, since that group is generated by all the marked points making up $\pi^{-1}(\ast)$. Given a particular point $x \in \pi^{-1}(\ast)$, there is a unique deck transformation $h$ that translates $\ast$ to $x$, and we identify $x$ with the generator $h \in \mathbb{Z}[H]$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A partial depiction of the universal abelian cover of $S_{2,1}$.}
\end{figure}
The main object of our study is $H_1(\hat{S}, \pi^{-1}(\ast))$, and again we can find explicit realizations of its elements. For example, $\alpha_1$ is represented by an arc running from $\hat{a}$ along the ribbon in the $a_1$ direction, and ending at the basepoint $\alpha_1 \hat{a}$ on the neighboring polygon. The translation $h\alpha_1$ is similar, but it starts at the point $h\hat{a}$ rather than $\hat{a}$. Note that the arc that begins at $\hat{a}$, travels in the negative $a_1$ direction, and ends at the basepoint $a_1^{-1} \hat{a}$ is $-a_1^{-1} \alpha_1$, rather than $-\alpha_1$ (which is just $\alpha_1$ with the opposite orientation). The other generators of $H_1(\hat{S}, \pi^{-1}(\ast))$ are realized similarly.

4. Higher intersection forms

Papakyriakopoulos defined in [7] a biderivation on $\mathbb{Z}[\Gamma]$ which gives a $\mathbb{Z}[H]$-valued pairing for elements of $H_1(\hat{S})$. (Turaev [13] gave a geometric construction of an intersection form on $\mathbb{Z}[\Gamma]$; see Perron [8] for a modern treatment, including a proof that Turaev’s intersection form coincides with Papakyriakopoulos’ biderivation.) Hempel made frequent use of this pairing on $H_1(\hat{S})$ (see e.g. [3]), interpreting it as the “Reidemeister pairing” defined by Reidemeister in [10]. Suzuki used a version of this pairing in [12] to characterize when the commutator of two separating twists is in $\ker r$ (see Proposition 4.4). To define this higher intersection form on all of $H_1(\hat{S}, \pi^{-1}(\ast))$, we must make a choice between two pairings, which are related by an antisymmetry relation (Lemma 4.9). This pairing is analogous to the algebraic intersection number of two elements of $H_1(S)$; see Proposition 4.10 for evidence of this analogy.

We first need to define the ($\mathbb{Z}$-valued) intersection number of two elements of $H_1(\hat{S}, \pi^{-1}(\ast))$. Any element of $H_1(\hat{S}, \pi^{-1}(\ast))$ can be realized as a linear combination of closed curves in $\hat{S}$ and arcs in $\hat{S}$ with endpoints in $\pi^{-1}(\ast)$. Any pair of curves, or a curve and an arc, can be realized so that they only intersect transversely, and then the orientation of $\hat{S}$ gives a natural algebraic intersection number. This does not extend to a pair of arcs, however; $\alpha_1$ and $\beta_1$, for example, are two arcs in $H_1(\hat{S}, \pi^{-1}(\ast))$ that share one endpoint, so their intersection number is not well-defined. In order to define their algebraic intersection number, we need to move the basepoint of one arc slightly; there are two different ways of doing so, and we will keep track of the resulting differences in the algebraic intersection number. This will give us two $\mathbb{Z}$-valued bilinear forms on $H_1(\hat{S}, \pi^{-1}(\ast))$.

We now formalize the above discussion. The orientation on $S$ gives an orientation on $\partial S$. Take $\ast' \neq \ast$ to be a second basepoint in $\partial S$; there are two arcs from $\ast$ to $\ast'$ contained in $\partial S$. Call these two arcs $\gamma_+$ and $\gamma_-$, where $\gamma_+$ is the arc that is positively oriented. For $\sigma \in \{+,-\}$ let $\xi_\sigma$ be the lift of $\gamma_\sigma$ to $\hat{S}$ based at $\hat{a}$. Then we have isomorphisms $\varphi_\sigma : H_1(\hat{S}, \pi^{-1}(\ast)) \to H_1(\hat{S}, \pi^{-1}(\ast'))$ defined by $\varphi_\sigma(x) = x + \partial x \cdot \xi_\sigma$. Thus $\varphi_\sigma$, slides the basepoint along the boundary from $\pi^{-1}(\ast)$ to $\pi^{-1}(\ast')$: $\varphi_+ \in$ the positive direction, $\varphi_- \in$ the negative.

The orientation on $S$ induces an orientation of $\hat{S}$. This orientation determines a $\mathbb{Z}$-bilinear algebraic intersection form on $H_1(\hat{S}, \pi^{-1}(\ast)) \times H_1(\hat{S}, \pi^{-1}(\ast'))$, since representatives have distinct basepoints and can thus always be made transverse. We denote this form by $(\cdot, \cdot)$ and define two bilinear forms $(\cdot, \cdot)_\sigma$ on $H_1(\hat{S}, \pi^{-1}(\ast)) \times H_1(\hat{S}, \pi^{-1}(\ast))$ by $(c, d)_\sigma = (c, \varphi_\sigma(d))$, $\sigma \in \{+,-\}$. 
Definition 4.1. The two higher intersection forms $\langle \cdot, \cdot \rangle_{\sigma}$ are $\mathbb{Z}$–bilinear functions $\langle \cdot, \cdot \rangle_{\sigma} : H_1(\hat{S}, \pi^{-1}(\ast)) \times H_1(\hat{S}, \pi^{-1}(\ast)) \to \mathbb{Z}[H]$ for $\sigma \in \{+,-\}$, defined by

\begin{equation}
\langle c, d \rangle_{\sigma} = \sum_{h \in H} (c, hd)_{\sigma} h.
\end{equation}

Note that $c$ and $d$ are compactly supported, so this sum is finite. The formula (2) appears in the work of Papakyriakopoulos; his results [7, Theorem 10.13] imply that these intersection forms are equivalent to the biderivation mentioned in the introduction to this section. Thus Morita’s explicit calculations in the proof of [6, Theorem 5.3] imply that the Magnus representation preserves the forms $\langle \cdot, \cdot \rangle_{\sigma}$. In fact, it is easy to see directly that these forms are preserved by a general class of topologically defined automorphisms of $H_1(\hat{S}, \pi^{-1}(\ast))$ which includes those defining the Magnus representation.

Lemma 4.2. Suppose that $f$ is an orientation-preserving self-homeomorphism of the pair $(\hat{S}, \pi^{-1}(\ast))$ that commutes with all deck transformations $h \in H$. Then the action of $f$ on $H_1(\hat{S}, \pi^{-1}(\ast))$ preserves both forms $\langle \cdot, \cdot \rangle_{\sigma}$. In particular, $\langle \cdot, \cdot \rangle_{\sigma}$ is preserved by the action of $H$ and by the image of the Magnus representation $r$.

Proof. Using the fact that the intersection forms $\langle \cdot, \cdot \rangle_{\sigma}$ are preserved by such a homeomorphism $f$, we have that

\[
\langle f(c), f(d) \rangle_{\sigma} = \sum_{h \in H} (f(c), hf(d))_{\sigma} h
= \sum_{h \in H} (f(c), f(hd))_{\sigma} h
= \sum_{h \in H} (c, hd)_{\sigma} h = \langle c, d \rangle_{\sigma},
\]

as desired. \(\square\)

Remark. This proof of Morita’s result was first noticed by Suzuki [11].

Definition 4.3. The involution $\overline{\cdot} : \mathbb{Z}[H] \to \mathbb{Z}[H]$ is defined for $h \in H$ by $\overline{h} = h^{-1}$ and on $\mathbb{Z}[H]$ by extending $\mathbb{Z}$–linearly.

Lemma 4.4. Both forms $\langle \cdot, \cdot \rangle_{\sigma}$ are $\mathbb{Z}[H]$–sesquilinear, meaning that $\langle gc, hd \rangle_{\sigma} = g\overline{h} \langle c, d \rangle_{\sigma}$ for any $g, h \in \mathbb{Z}[H]$ and $c, d \in H_1(\hat{S}, \pi^{-1}(\ast))$.

Proof. By $\mathbb{Z}$–bilinearity, it is sufficient to prove this result when $g, h \in H$. From the definition of $\langle \cdot, \cdot \rangle_{\sigma}$, we have that

\[
\langle gc, hd \rangle_{\sigma} = \langle c, g^{-1}hd \rangle_{\sigma} = \sum_{j \in H} (c, jg^{-1}hd)_{\sigma} j
= \sum_{j \in H} (c, jd)_{\sigma} gh^{-1} j = gh^{-1} \langle c, d \rangle_{\sigma}
\]
since $H$ is abelian. The first equality holds because $\langle \cdot, \cdot \rangle_{\sigma}$ is preserved by the action of $g$ (Lemma 4.2). \(\square\)
Let \( \pi_* : H_1(\tilde{S}, \pi^{-1}(\ast)) \rightarrow H \) denote the map on first homology induced by the covering map \( \pi : \tilde{S} \rightarrow S \). Recall that \( \varepsilon : Z[H] \rightarrow Z \) is the augmentation map. Our justification for calling these forms \( \langle \cdot , \cdot \rangle_{\sigma} \) higher intersection forms is the following lifting property.

**Lemma 4.5.** For any \( c, d \in H_1(\tilde{S}, \pi^{-1}(\ast)) \) and \( \sigma \in \{+, -\} \), \( \varepsilon((c, d)_{\sigma}) = (\pi_*(c), \pi_*(d)) \).

**Proof.** We want to show that \( \sum_{h \in H} (c, hd)_{\sigma} = (\pi_*(c), \pi_*(d)) \) for \( c, d \in H_1(\tilde{S}, \pi^{-1}(\ast)) \). Since \( \pi_* (\varphi_{\sigma}(d)) = \pi_*(d) \) in \( H_1(S) \), this is equivalent to the claim that \( \sum_{h \in H} (c, hd) = (\pi_*(c), \pi_*(d)) \) for \( c \in H_1(\tilde{S}, \pi^{-1}(\ast)) \) and \( d \in H_1(\tilde{S}, \pi^{-1}(\ast)) \).

Since the projected 1-cycles \( \pi_* (c) \) and \( \pi_*(d) \) have distinct basepoints, we can realize \( c \) and \( d \) such that \( \pi_*(c) \) and \( \pi_*(d) \) intersect transversely in \( S \). Then it is easy to see that each intersection of \( \pi_*(c) \) and \( \pi_*(d) \) corresponds to an intersection of \( c \) with some translate of \( d \): if \( \pi_*(c) \) and \( \pi_*(d) \) intersect at some point \( x \in S \), then exactly one element of \( \pi^{-1}(x) \) is in the image of \( c \), and exactly one translate of \( d \) passes through that element. This yields the desired result. \( \square \)

This lifting property of the higher intersection forms allows us to deduce the nondegeneracy of \( \langle \cdot , \cdot \rangle_{\sigma} \) from the nondegeneracy of the symplectic intersection form on \( H \). In fact, we obtain the following stronger result:

**Proposition 4.6.** For any \( n \geq 0 \) and \( \sigma \in \{+, -\} \), we have that \( y \in H_1(\tilde{S}, \pi^{-1}(\ast)) \) satisfies \( (x, y)_{\sigma} \in (\ker \varepsilon)^n \) for all \( x \in H_1(\tilde{S}, \pi^{-1}(\ast)) \) if and only if \( y \in (\ker \varepsilon)^n H_1(\tilde{S}, \pi^{-1}(\ast)) \).

**Proof.** If \( y \in (\ker \varepsilon)^n H_1(\tilde{S}, \pi^{-1}(\ast)) \), it is immediate that \( (x, y)_{\sigma} \in (\ker \varepsilon)^n \) for any \( x \). We prove the other implication by induction on \( n \); this is trivial if \( n = 0 \), so assume that \( n > 0 \) and that the lemma holds for all smaller values of \( n \). Suppose for contradiction that \( y \notin (\ker \varepsilon)^n H_1(\tilde{S}, \pi^{-1}(\ast)) \) but \( (x, y)_{\sigma} \in (\ker \varepsilon)^n \) for all \( x \in H_1(\tilde{S}, \pi^{-1}(\ast)) \). Let \( \{s_1, \ldots, s_{2g}\} \) be the standard basis \( \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\} \) for \( H_1(\tilde{S}, \pi^{-1}(\ast)) \cong \mathbb{Z}[H]^{2g} \), taken in any order. By the inductive hypothesis, \( y \in (\ker \varepsilon)^{n-1} H_1(\tilde{S}, \pi^{-1}(\ast)) \), so we can write \( y = \sum_{j=1}^{2g} h_j s_j \), where each \( h_j \in (\ker \varepsilon)^{n-1} \) but \( h_{j_0} \notin (\ker \varepsilon)^n \) for some \( j_0 \).

Now, using Lemma 4.5, the nondegeneracy of the intersection form on \( H \) implies that we can find \( x \in H_1(\tilde{S}, \pi^{-1}(\ast)) \) such that \( (x, s_{j_0})_{\sigma} \in (\ker \varepsilon) \iff j \neq j_0 \). Then

\[
(x, y)_{\sigma} = \sum_{j=1}^{2g} \overline{h}_{j_0} (x, s_{j_0})_{\sigma} \in (\ker \varepsilon)^n \iff \overline{h}_{j_0} (x, s_{j_0})_{\sigma} \in (\ker \varepsilon)^n.
\]

We have

\[
\overline{h}_{j_0} (x, s_{j_0})_{\sigma} \in (\ker \varepsilon)^n \iff \overline{h}_{j_0} (\varepsilon((x, s_{j_0})_{\sigma}) \in (\ker \varepsilon)^n
\]

because the difference of the two expressions is

\[
\overline{h}_{j_0} ((x, s_{j_0})_{\sigma} - \varepsilon((x, s_{j_0})_{\sigma}) \in (\ker \varepsilon)^{n-1}(\ker \varepsilon).
\]

But \( \overline{h}_{j_0} (\varepsilon((x, s_{j_0})_{\sigma}) \not\in (\ker \varepsilon)^n \) because \( \overline{h}_{j_0} \notin (\ker \varepsilon)^n \) and \( (x, s_{j_0}) \not\in (\ker \varepsilon)^n \), so we conclude that \( (x, y)_{\sigma} \not\in (\ker \varepsilon)^n \). This contradiction completes the induction. \( \square \)

Since \( \bigcap_{n \geq 0} (\ker \varepsilon)^n H_1(\tilde{S}, \pi^{-1}(\ast)) = 0 \), we have the following corollary:

**Corollary 4.7.** Both forms \( \langle \cdot , \cdot \rangle_{\sigma} \) are nondegenerate; that is, for \( x \in H_1(\tilde{S}, \pi^{-1}(\ast)) \setminus \{0\} \) and \( \sigma \in \{+, -\} \), there exists \( y \in H_1(\tilde{S}, \pi^{-1}(\ast)) \) such that \( (x, y)_{\sigma} \neq 0 \).
The following proposition tells us the difference between the two higher intersection forms.

**Proposition 4.8.** When either \(c\) or \(d\) is a curve in \(\hat{S}\) (that is, when \(c\) or \(d\) is in \(\ker \partial\)), \(\langle c, d \rangle_+ = \langle c, d \rangle_-\). In general, \(\langle c, d \rangle_+ - \langle c, d \rangle_- = \partial \overline{c\overline{d}}\).

**Proof.** Note that

\[
(x, y)_+ - (x, y)_- = (x, \varphi_+(y)) - (x, \varphi_-(y)) = (x, (y + \partial y \cdot \xi_+)) - (y + \partial y \cdot \xi_-) = (x, \partial y(\xi_+ - \xi_-)) = (x, (\partial y)\delta)
\]

where \(\delta = \xi_+ - \xi_-\) is a curve around the boundary (one lift of \(\partial S\)). Hence we have

\[
\langle c, d \rangle_+ - \langle c, d \rangle_- = \sum_{h \in H} (c, hd)_+ h - \sum_{h \in H} (c, hd)_- h
= \sum_{h \in H} (c, \partial(hd)\delta) h
= \partial \overline{d} \sum_{h \in H} (c, h\delta) h
= \partial \overline{d} \sum_{h \in H} (h^{-1}c, \delta) h
= \partial \overline{d} \sum_{h \in H} \text{const}(\partial(h^{-1}c)) h = \partial \overline{d}\overline{d},
\]

where by analogy with power series, we use \(\text{const}(x)\) to mean the “constant term” of \(x \in \mathbb{Z}[H]\). Formally, this is the linear functional \(\text{const}: \mathbb{Z}[H] \rightarrow \mathbb{Z}\) that is the identity on \(\mathbb{Z} \leq \mathbb{Z}[H]\) and sends \(h\) to 0 for all \(h \in H \setminus \{0\}\). □

**Remark.** It follows that when \(c\) or \(d\) is a curve, we need only write \(\langle c, d \rangle\), since \(\langle c, d \rangle_+ = \langle c, d \rangle_-\).

We also have the following “antisymmetry” property of \(\langle \cdot, \cdot \rangle_\sigma\):

**Lemma 4.9.** Suppose \(\sigma \in \{+, -\}\) and \(c, d \in H_1(\hat{S}, \pi^{-1}(\ast))\). Then \(\langle d, c \rangle_\sigma = -\overline{\langle c, d \rangle}_{-\sigma}\). In particular, if \(c\) or \(d\) is a curve, then \(\langle d, c \rangle = -\overline{\langle c, d \rangle}\), so \(\langle c, d \rangle = 0 \iff \langle d, c \rangle = 0\).

**Proof.** Note that \(\langle f, c \rangle_\sigma = -\overline{\langle e, f \rangle}_{-\sigma}\) for any \(e, f \in H_1(\hat{S}, \pi^{-1}(\ast))\). Thus

\[
\langle d, c \rangle_\sigma = \sum_{h \in H} (d, hc)_\sigma h
= - \sum_{h \in H} (hc, d)_{-\sigma} h
= - \sum_{h \in H} (c, h^{-1}d)_{-\sigma} h
= - \sum_{h \in H} (c, hd)_{-\sigma} h^{-1} = -\overline{\langle c, d \rangle}_{-\sigma}.
\]

□
Proof. First, we can compute $\langle c, c \rangle = 0$ if $c$ is a curve, but only that $\langle c, c \rangle = -\langle c, c \rangle$. However, we do have the weaker statement that $\langle c, c \rangle = 0$ if $c$ is a lift of a separating curve in the base surface $S$, as the various lifts $\{hc \mid h \in H \}$ are disjoint in this case.

The following proposition is fundamental for calculations involving the higher intersection form. It tells us exactly the image of a separating twist $T_\gamma$ under the Magnus representation $r$.

**Proposition 4.10.** Let $\gamma$ be a separating curve in $S$ and let $c \in H_1(\hat{S}, \pi^{-1}(*))$ be any lift of $\gamma$. Let $T_\gamma$ denote the Dehn twist around $\gamma$. Then for any $n \in \mathbb{Z}$, the action of the power $T_\gamma^n$ on $d \in H_1(\hat{S}, \pi^{-1}(*))$ is given by

$$r(T_\gamma^n)(d) = d + \langle d, c \rangle n.$$  

**Remark.** Recall that a curve $\gamma$ in $S$ lifts to a curve rather than an arc exactly when $\gamma$ is a separating curve, so $\langle d, c \rangle = \langle d, c \rangle_+ = \langle d, c \rangle_-$ in the above formula. The result is analogous to the formula for the action of a Dehn twist on $H$ given by $T_\gamma(h) = h + (h, [\gamma][\gamma])$, where $\langle \cdot, \cdot \rangle$ is the algebraic intersection form for two elements of $H$. This may be taken as evidence that this definition of $\langle \cdot, \cdot \rangle$ is the correct one.

**Proof.** The lifted homeomorphism $\hat{T}_\gamma^n$ can be thought of as simultaneously twisting $n$ times about each lift of $\gamma$, since these lifts are nonintersecting closed curves in $\hat{S}$. For each intersection of $d$ with a lift $\hat{\gamma}$ of $\gamma$, we add or subtract $n\hat{\gamma} \in H_1(\hat{S}, \pi^{-1}(*))$, depending on the orientation of the intersection. These lifts of $\gamma$ are simply the curves $hc$ for $h \in H$. Thus

$$r(T_\gamma)(d) = d + \sum_{\hat{\gamma} \text{ lifts } \gamma} (d, \hat{\gamma}) n\hat{\gamma} = d + \sum_{h \in H} (d, hc) hc.$$  

By equation (2), this is just $d + n\langle d, c \rangle c$. \hfill $\square$

5. **Using the Trace of the Magnus Representation**

We analyze the trace of the Magnus representation $r$: this is a class function on the Torelli group. Let $t : T_{g,1} \to \mathbb{Z}[H]$ be defined by $t(f) := \text{tr}(r(f)) - 2\bar{g}$. It is easily verified that $t(T_\gamma) = t(1) = 0$ for any separating twist $T_\gamma$. Proposition 4.11 gives us a relatively easy way to compute this function in general on $K_{g,1}$. Suzuki [12, Theorem 4.3] computed the trace of the image of the product of two separating twists by laborious involving explicit matrices. We obtain the same result more simply as a demonstration of our methods:

**Proposition 5.1.** Let $\gamma_1, \gamma_2$ be separating curves in $S$ with lifts $c_1, c_2$ in $\hat{S}$. Then

$$t(T_{\gamma_1}T_{\gamma_2}) = \langle c_1, c_2 \rangle \langle c_2, c_1 \rangle.$$  

**Proof.** First, we can compute $r(T_{\gamma_1}T_{\gamma_2})$ using Proposition 4.10

$$r(T_{\gamma_1}T_{\gamma_2})(d) = r(T_{\gamma_1})(d + \langle d, c_2 \rangle c_2)$$  

$$= d + \langle d, c_2 \rangle c_2 + \langle d, c_1 \rangle c_1 + \langle d, c_2 \rangle \langle c_2, c_1 \rangle c_1$$  

for any $d \in H_1(\hat{S}, \pi^{-1}(*))$. 

□
We note that every term in the above formula is \( \mathbb{Z}[H] \)-linear in \( d \) (recall that \( \langle \cdot, \cdot \rangle \) is \( \mathbb{Z}[H] \)-linear in its first factor). Thus the trace of the whole expression is the sum of the traces of the terms (regarded as linear functions of \( d \)):

\[
\text{tr}(r(T_{\gamma_1} T_{\gamma_2})) = \text{tr}(d) + \text{tr}((d, c_1) c_1) + \text{tr}((d, c_2) c_2) + \text{tr}((d, c_2) (c_2, c_1) c_1)
\]

The first term is the trace of the identity, which is \( 2g \). For the other terms, recall that for any free module of finite rank we have \( \text{tr}(x \mapsto \lambda(x)v) = \lambda(v) \). (This is easily seen in the case when there exists a basis containing \( v \), since then the matrix of this endomorphism has only one nonzero row and the entry on the diagonal is clearly \( \lambda(v) \).) Thus \( \text{tr}((d, c_1) c_1) = \langle c_1, c_1 \rangle = 0 \) (as discussed after Lemma 4.9), and similarly \( \text{tr}((d, c_2) c_2) = \langle c_2, c_2 \rangle = 0 \) and \( \text{tr}((d, c_2) (c_2, c_1) c_1) = \langle c_1, c_2 \rangle \langle c_2, c_1 \rangle \). We conclude that

\[
\text{tr}(r(T_{\gamma_1} T_{\gamma_2})) = 2g + \langle c_1, c_2 \rangle \langle c_2, c_1 \rangle,
\]

and thus

\[
t(T_{\gamma_1} T_{\gamma_2}) = \langle c_1, c_2 \rangle \langle c_2, c_1 \rangle.
\]

It is clear how to generalize the above computation to compute the trace of an arbitrary product of separating twists.

**Proposition 5.2.** Let \( \gamma_1, \ldots, \gamma_k \) be separating curves in \( S \) with lifts \( c_1, \ldots, c_k \in H_1(S, \pi^{-1}(s)) \). Let \( n_1, n_2, \ldots, n_k \) be integers. Then

\[
t(T_{\gamma_1}^{n_1} T_{\gamma_2}^{n_2} \cdots T_{\gamma_k}^{n_k}) = \sum_{m \geq 2}^{n_1 n_2 \cdots n_m \langle c_{i_1}, c_{i_2} \rangle \cdots \langle c_{i_{m-1}}, c_{i_m} \rangle \cdots \langle c_{i_2}, c_{i_1} \rangle}.
\]

**Proof.** Use Proposition 4.10 to expand \( r(T_{\gamma_1}^{n_1} T_{\gamma_2}^{n_2} \cdots T_{\gamma_k}^{n_k})(d) \) as a sum of \( 2^k \) terms and take the trace of each one. Now exploit the fact that \( \langle c_i, c_i \rangle = 0 \) to cancel the terms corresponding to \( m = 1 \), and we obtain the formula above. \( \square \)

Recall that a separating multitwist is a product of Dehn twists \( T_C = T_{\gamma_1}^{n_1} \cdots T_{\gamma_k}^{n_k} \) such that each \( \gamma_i \) is a separating curve, and the \( \gamma_i \) are pairwise disjoint. Let \( c_i \) be a lift of \( \gamma_i \) for \( 1 \leq i \leq k \). Since the \( \gamma_i \) are disjoint, their lifts \( c_i \) are disjoint, and so we have \( \langle c_i, c_{i'} \rangle = 0 \) for all \( i, i' \). Thus we have the following corollary to Proposition 5.2.

**Corollary 5.3.** If \( T_C = T_{\gamma_1}^{n_1} \cdots T_{\gamma_k}^{n_k} \) is a separating multitwist, then \( t(T_C) = 0 \).

The following statement is equivalent to a theorem of Suzuki [12, Corollary 4.4]; we give a new proof of the reverse implication.

**Proposition 5.4.** Let \( \gamma_1, \gamma_2 \) be separating curves in \( S \) with lifts \( c_1, c_2 \) in \( \hat{S} \). Then

\[
[T_{\gamma_1}, T_{\gamma_2}] \in \ker r \iff \langle c_1, c_2 \rangle = 0.
\]

**Proof.** The implication \( \langle c_1, c_2 \rangle = 0 \implies [T_{\gamma_1}, T_{\gamma_2}] \in \ker r \) follows immediately from equation (3); in this case the last term vanishes, so the expression becomes symmetric in \( c_1 \) and \( c_2 \), and the actions of \( T_{\gamma_1} \) and \( T_{\gamma_2} \) commute, as desired.

The reverse implication follows from the formula

\[
t([T_{\gamma_1}, T_{\gamma_2}]) = \langle c_1, c_2 \rangle^2 \langle c_2, c_1 \rangle^2,
\]

which can easily be obtained using Proposition 5.2. Suppose that \( [T_{\gamma_1}, T_{\gamma_2}] \in \ker r \). Any \( f \in \ker r \) must satisfy \( t(f) = t(1) = 0 \). Thus \( \langle c_1, c_2 \rangle^2 \langle c_2, c_1 \rangle^2 = 0 \), which implies that \( \langle c_1, c_2 \rangle = \langle c_2, c_1 \rangle = 0 \) because \( \langle c_2, c_1 \rangle = 0 \implies \langle c_1, c_2 \rangle = 0 \) by Lemma 4.9.  \( \square \)
In fact, the commuting relation of Proposition [5.4] is the only relation that ever arises between the images of twists around separating curves.

**Theorem A.** Suppose that $\gamma_1$ and $\gamma_2$ are nontrivial separating curves in $S$ with lifts $c_1, c_2 \in H_1(S, \pi^{-1}(\ast))$. If $\langle c_1, c_2 \rangle = 0$, then $r(T_{\gamma_1})$ and $r(T_{\gamma_2})$ commute. If $\langle c_1, c_2 \rangle \neq 0$, then $r(T_{\gamma_1})$ and $r(T_{\gamma_2})$ generate a free group of rank 2 in $\text{GL}_{2g}(\mathbb{Z}[H])$.

**Proof.** If $\langle c_1, c_2 \rangle = 0$, then this is just Proposition [5.4]. The other case can be shown directly, but since it is a specialization of Theorem C, we defer to the proof of Theorem C below. □

**Definition 5.5.** For $a \in \mathbb{Z}[H]$, define the *pseudosquare* $||a|| \in \mathbb{Z}[H]$ to be $||a|| = a\overline{a}$.

Note that the constant term of $||a||$ is the sum of the squares of the coefficients of $a$, and thus $\text{const} ||a|| = 0 \iff a = 0$.

We now determine when the commutator of two positive separating multitwists is in $\ker r$.

**Theorem B.** Let $T_C = T_{\gamma_1}^{n_1} \cdots T_{\gamma_k}^{n_k}$ and $T_D = T_{\delta_1}^{m_1} \cdots T_{\delta_l}^{m_l}$ be positive separating multitwists. Then $[T_C, T_D] \in \ker r$ if and only if $[T_{\gamma_i}, T_{\delta_j}] \in \ker r$ for all $1 \leq i \leq k$ and $1 \leq j \leq l$.

**Proof.** The “if” direction of the implication is immediate, so suppose that $[T_C, T_D] \in \ker r$, so that $t([T_C, T_D]) = 0$. Let $c_i \in H_1(S, \pi^{-1}(\ast))$ be lifts of the $\gamma_i$ and let $d_j \in H_1(S, \pi^{-1}(\ast))$ be lifts of the $\delta_j$. We will show that $\langle c_i, d_j \rangle = 0$ for all $i$ and $d$, which will imply the desired result by Proposition [5.4].

Using Proposition [5.2] and the fact that $\langle c_i, c_i' \rangle = \langle d_j, d_j' \rangle = 0$ for any $i, i'$ or $j, j'$, we calculate that

$$t([T_C, T_D]) = \sum_{1 \leq i, i' \leq k \atop 1 \leq j, j' \leq l} n_i n_i' m_j m_j' \langle c_i, c_i' \rangle \langle d_j, c_i' \rangle \langle d_j, d_j' \rangle \langle c_i, c_i' \rangle \langle d_j, d_j' \rangle$$

Using Lemma [4.9] we can rewrite this as

$$t([T_C, T_D]) = \sum_{1 \leq i, i' \leq k \atop 1 \leq j, j' \leq l} n_i n_i' m_j m_j' \langle c_i, d_j \rangle \langle c_i, c_i' \rangle \langle d_j, c_i' \rangle \langle d_j, d_j' \rangle \langle c_i, c_i' \rangle \langle d_j, d_j' \rangle$$

$$= \sum_{1 \leq i, i' \leq k} n_i n_i' \left| \sum_{1 \leq j \leq l} m_j \langle c_i, d_j \rangle \langle c_i, c_i' \rangle \langle d_j, c_i' \rangle \langle d_j, d_j' \rangle \langle c_i, c_i' \rangle \langle d_j, d_j' \rangle \right| .$$

(Note that the pseudosquare is not linear: we have $|| \sum_i x_i || = \sum_i x_i \overline{x_i}$.) Thus we have written $0 = t([T_C, T_D])$ as a positive sum of pseudosquares. Since the constant term vanishes, we see that each pseudosquare above must vanish. In particular, taking $i = i'$ gives that

$$0 = \sum_{1 \leq j \leq l} m_j \langle c_i, d_j \rangle \langle d_j, c_i \rangle = - \sum_{1 \leq j \leq l} m_j \left| \langle c_i, d_j \rangle \right| ,$$

so taking constant terms again gives $\langle c_i, d_j \rangle = 0$ for each $i$ and $j$, as desired. □
With a little more work, we can show that the images of two separating multi-twists satisfy no relations if they do not commute.

**Theorem C.** Let $T_C$ and $T_D$ be positive separating multitwists with $[T_C, T_D] \notin \ker r$. Then $r(T_C)$ and $r(T_D)$ generate a free group of rank 2 in $GL_2\mathbb{Z}[H]$.

Proof. Proposition 4.10 together with the fact that $\langle c_i, c_j \rangle = 0$ for all $i, j$, gives that the action of $T_C$ on $H_1(\hat{S}, \pi^{-1}(\ast))$ is given by

$$r(T_C)(d) = d + \sum_{1 \leq i \leq l} \langle d, c_i \rangle c_i.$$ 

For the same reasons, if we let $A = r(T_C) - 1$ and $B = r(T_D) - 1$, regarded as $\mathbb{Z}[H]$-endomorphisms of $H_1(\hat{S}, \pi^{-1}(\ast))$, we have that $A^2 = B^2 = 0$.

Suppose for contradiction that $w$ is a nontrivial word in $T_C$ and $T_D$ such that $w \in \ker r$. By replacing $w$ with a conjugate if necessary, we can assume that $w$ is of the form

$$T_D^{m_1} T_C^{n_1} \cdots T_D^{m_k} T_C^{n_k},$$

where $m_1, \ldots, m_k$ and $n_1, \ldots, n_k$ are nonzero integers and $k \geq 1$. We then can obtain the following identity:

$$0 = A(r(w) - 1)B = A(1 + m_1 B)(1 + n_1 A) \cdots (1 + m_k B)(1 + n_k A)B - AB.$$ 

Expand this expression and recall that $A^2 = B^2 = 0$, so every term vanishes except those of the form $AB \cdots AB$. Thus this identity can be written as $0 = P(AB)$ for some polynomial $P \in \mathbb{Z}[X]$ with leading term $m_1 n_1 \cdots m_k n_k X^{k+1}$.

Now, let $K$ be the algebraic closure of the field of fractions of $\mathbb{Z}[H]$. The above identity (4) yields that every eigenvalue $\alpha \in K$ of $AB$ is a root of the integral polynomial $P$ and thus is algebraic over $\mathbb{Q}$, so the sum of these eigenvalues, $\text{tr} AB \in \mathbb{Z}[H]$, is also algebraic over $\mathbb{Q}$. But because $H$ is free abelian, the elements in $\mathbb{Z}[H]$ that are algebraic over $\mathbb{Q}$ are precisely the elements of $\mathbb{Z}$ in $\mathbb{Z}[H]$, so we must have $\text{tr} AB \in \mathbb{Z}$.

We have $AB = [r(T_C) - 1][r(T_D) - 1] = r(T_C T_D) - r(T_C) - r(T_D) + 1$. Corollary 5.3 tells us that $\text{tr}(r(T_C)) = \text{tr}(r(T_D)) = 2g$, so we conclude that $\text{tr} AB = \text{tr}(r(T_C T_D)) - 2g = t(T_C T_D)$. Now, Proposition 5.2 tells us that

$$\text{tr} AB = t(T_C T_D) = \sum_{1 \leq i \leq k, 1 \leq j \leq l} \langle c_i, d_j \rangle \langle d_j, c_i \rangle.$$ 

Lemma 4.5 implies that $\varepsilon((c_i, d_j)) = (\gamma_i, d_j) = 0$ since $\gamma_i$ is a separating curve. Thus $\text{tr} AB \in \ker \varepsilon$, so $\text{tr} AB \in \mathbb{Z} \implies \text{tr} AB = 0$. Equation (5) then gives

$$\sum_{1 \leq i \leq k, 1 \leq j \leq l} \left| \langle c_i, d_j \rangle \right| = 0,$$

so each pseudosquare must be zero. This implies that $\langle c_i, d_j \rangle = 0$, so $[T_C, T_D] \in \ker r$ by Theorem B. \qed
Remark. In the proofs of Corollary 5.3, Theorem B, and Theorem C, we use only that \( \langle c_i, c_{i'} \rangle = \langle d_j, d_{j'} \rangle = 0 \). It is not necessary that the curves \( \gamma_i \) (and \( \delta_j \)) are pairwise disjoint. Call \( T_C = T_{\gamma_1} \cdots T_{\gamma_k} \) a \textit{separating Magnus-multitwist} if each \( \gamma_i \) is a separating curve and the lifts \( c_i \) of \( \gamma_i \) satisfy \( \langle c_i, c_{i'} \rangle = 0 \) for any \( 0 \leq i, i' \leq k \).

Note that we do \textit{not} require that the curves \( \gamma_i \) be disjoint, so a Magnus-multitwist is not a multitwist in general. (For example, if \( \gamma_i \) and \( \gamma_{i'} \) have geometric intersection number 2, then we have \( \langle c_i, c_{i'} \rangle = 0 \).)

Then we have the following corollary.

Corollary 5.6. Let \( T_C \) and \( T_D \) be positive separating Magnus-multitwists. Then Theorem B and Theorem C hold verbatim.

6. Further questions

Theorem A tells us that if \( \langle c_1, c_2 \rangle = 0 \), then \( r(T_{\gamma_1}) \) and \( r(T_{\gamma_2}) \) commute, and thus generate a free abelian group of rank at most 2. This is the best result possible in the sense that there exist distinct separating curves \( \gamma_1 \) and \( \gamma_2 \) such that the lifts of \( \gamma_1 \) and \( \gamma_2 \) coincide, so that \( r(T_{\gamma_1}) = r(T_{\gamma_2}) \). For example, if \( \gamma_0 \) and \( \gamma_1 \) are separating curves with geometric intersection number 2, then \( \gamma_1 \) and \( \gamma_2 = T_{\gamma_0}(\gamma_1) \) have the same lifts to \( \hat{S} \). In general, \( r(T_{\gamma_1}) \) and \( r(T_{\gamma_2}) \) generate a cyclic group if and only if \( \gamma_1 \) and \( \gamma_2 \) have the same lifts, which occurs when \( \gamma_1\gamma_2^{-1} \) lies in \([\Gamma_2, \Gamma_2]\).

There are two natural ways in which one might try to generalize the classification described in this paper. First, what happens if we consider relations between three twists?

Question 1. \textit{What words in three separating twists} \( T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3} \) \textit{lie in the kernel of the Magnus representation} \( r \)?

Although Proposition 4.2 is still a useful tool in studying this question, it seems that a greater understanding of the higher intersection form is needed to answer it.

Second, one can study other related Magnus representations \( r_k \) of subgroups of the mapping class group, which can be obtained by using covering spaces other than the universal abelian cover (see Suzuki [11]). Let \( \Gamma_k \) be the lower central series of \( \Gamma = \Gamma_1 \), defined by \( \Gamma_{k+1} = [\Gamma, \Gamma_k] \). The Johnson filtration consists of the groups \( I_{g,1}(k) \), where \( I_{g,1}(k) \) is the subgroup of \( \text{Mod}_{g,1} \) which acts trivially on the nilpotent quotient \( N_k = \Gamma/\Gamma_k \). Note that \( I_{g,1}(1) \) is \( \text{Mod}_{g,1} \) and \( I_{g,1}(2) \) is \( I_{g,1} \). Johnson [4] proved that \( I_{g,1}(3) \) is the “Johnson kernel” \( K_{g,1} \), the subgroup of \( \text{Mod}_{g,1} \) generated by separating twists. For each \( k \), there is a representation \( r_k : I_{g,1}(k) \to \text{GL}_{2g}(\mathbb{Z}[N_k]) \). In particular, there is a representation of the Johnson kernel \( r_3 : K_{g,1} \to \text{GL}_{2g}(\mathbb{Z}[\Gamma/\Gamma_3]) \).

Question 2. \textit{When is the commutator of two separating twists} \([T_{\gamma_1}, T_{\gamma_2}]\) \textit{in ker} \( r_3 \) ?

In general, \textit{what words in two separating twists} \( T_{\gamma_1} \) and \( T_{\gamma_2} \) \textit{lie in ker} \( r_3 \) ? \textit{In ker} \( r_k \) ?

Finally, Theorems B and C suggest that we should consider multitwists as well.

Question 3. \textit{What words in three separating multitwists} \( T_C, T_D, T_E \) \textit{lie in ker} \( r \) ? \textit{In ker} \( r_k \) ?

Given the remark following Theorem C, it is natural to ask whether results in this direction also apply to Magnus-multitwists. For higher Magnus representations \( r_k \), we can define \( r_k \)–Magnus-multitwists similarly, and we extend the latter question above to \( r_k \)–Magnus-multitwists.
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