Scattering theory of space-time non-commutative abelian gauge field theory

Chaiho Rim$^1$ and Jae Hyung Yee$^2$

$^1$Department of Physics, Chonbuk National University
Chonju 561-756, Korea
rim@mail.chonbuk.ac.kr

$^2$Institute of Physics and Applied Physics, Yonsei University
Seoul 120-749, Korea
jhyee@phya.yonsei.ac.kr

Abstract

The unitary S-matrix for the space-time non-commutative QED is constructed using the $\ast$-time ordering which is needed in the presence of derivative interactions. Based on this S-matrix, perturbation theory is formulated and Feynman rule is presented. The gauge invariance is explicitly checked to the lowest order, using the Compton scattering process. The gauge fixing condition dependency of the classical solution of the vacuum is also discussed.
1 Introduction:

Non-commutative field theory (NCFT) \([1, 2]\) is the field theory on the non-commutative (NC) coordinates space,

\[
[x^\mu, x^{\nu}] = i\theta^{\mu \nu}.
\]

Space non-commutative theory (SSNC) involves only the space non-commuting coordinates \((\theta^{0 \nu} = 0)\), whereas space-time non-commutative theory (STNC) contains the non-commuting time \((\theta^{0 \nu} \neq 0)\). NCFT is constructed based on the Weyl’s idea \([3]\): Instead of using this non-commuting coordinates directly, one may use the \(\star\)-product of fields over commuting space-time coordinates. The \(\star\)-product encodes all the non-commuting nature of the theory and fixes the ordering ambiguity of non-commuting coordinates.

We adopt the Moyal product \([4]\) as the \(\star\)-product representations,

\[
f \star g(x) = e^{\frac{i}{2} \partial_x \wedge \partial_y} f(x)g(y) |_{y=x}
\]

(1.2)

where \(a \wedge b = \theta^{\mu \nu} a_\mu b_\nu\). \(\theta^{\mu \nu}\) is an antisymmetric c-number representing the space-time non-commutativeness. Using this idea, the commutator in (1.1) becomes the \(\star\)-commutator,

\[
[x^\mu ; x^{\nu}] \equiv x^\mu \star x^{\nu} - x^{\nu} \star x^\mu = i\theta^{\mu \nu},
\]

where the coordinates \(x\)’s are treated as the commuting ones. The merit of the Moyal product is that the \(\star\)-product maintains the ordinary form of the kinetic term of the action, and allows the conventional perturbation where the interaction terms become non-local reflecting the non-commuting nature of the interaction.

In \([5, 6]\) the unitary S-matrix was constructed using Lagrangian formalism of the second quantized operators in the Heisenberg picture. Here fixing the time-ordering ambiguity has the central role in establishing the unitary S-matrix. The solution is given as the so-called minimal realization of the time-ordering step function and \(\star\)-time ordering. The proposal by \([7]\) to avoid the unitarity problem \([8]\) in STNC is in the right direction but the time-ordering suggested in \([7]\) needs higher derivative correction. After the higher derivative correction, the right time-ordering turns out to be the \(\star\)-time ordering as given in \([5, 6]\). Based on this S-matrix, the perturbation theory of STNC is illustrated using the real scalar theory and Feynman rule is presented.

In this paper, we continue this project to investigate the STNC \(U(1)\) gauge field theory. It is well-known in the commutative gauge theory that the proper time-ordering is the covariant time-ordering due to the derivative interaction. In this sense, the time-ordering in STNC gauge theory needs to be modified from the one defined in the STNC scalar field theory, whose commutative version contains no derivative interaction.

In addition, the gauge theory possesses the gauge symmetry and its quantized version should maintain the gauge symmetry to get rid of the unphysical states from the Hilbert space. On the other hand, it was pointed out that so-called the time ordered perturbation theory (TOPT) proposed in \([9]\) does not preserve the gauge symmetry. Therefore, for the STNC gauge theory to make sense, one needs to construct the S-matrix, not only unitary but also gauge invariant.

In section 2 unitary S-matrix of NC abelian gauge theory is presented, with the proper time-ordering. Due to the presence of the derivative interaction, the covariant
time-ordering is necessary in addition to the minimal realization of the $\ast$-time-ordering in scalar theories. In section 3, Feynman rule is presented in the momentum space. In section 4, STNC quantum electrodynamics is considered and Feynman rule is presented. In section 5, the gauge invariance is explicitly checked to the lowest order using the Compton scattering amplitude. This non-trivial check provides how the $\ast$-time ordering cures the defects present in the TOPT in [9]. In section 6, classical vacuum solutions is re-analyzed for the STNC pure gauge theory using the various gauge fixing conditions, and how the gauge fixing conditions affect the vacuum solution. Section 7 is the conclusion.

2 S-matrix of NC abelian gauge theory

The NC $U(1)$ action in D-dimensional space-time is given as

$$S = -\frac{1}{4} \int d^D x F_{\mu\nu} \ast F^{\mu\nu} \tag{2.1}$$

where $F_{\mu\nu}$ is the field strength,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu \ast A_\nu]$$

with the gauge coupling constant $g$. The action is gauge invariant under the gauge transformation,

$$ie A'_\nu(x) = U(x) \ast \left( \partial_\mu - ig A_\mu(x) \right) \ast \bar{U}(x).$$

This shows that the field strength is gauge-covariant rather than gauge-invariant. For the in-coming or out-going photon, however, the field strength is gauge invariant, since according to the fundamental ansatz of the field theory, the in- or out-photon is assumed to subject to the free theory which is the commutative field theory.

As noted in the scalar theory case, the non-commutative nature of space and time does not allow the unitary transformation of the quantum field in the intermediate time into free theory. Nevertheless, S-matrix can be defined, which relates the out-going field to the in-coming field. Therefore, to construct S-matrix we do not need to transform the field strength in NCFT into the abelian commutative field strength as in SSNC gauge theory [1].

With the Lorentz gauge fixing, the action is modified as

$$S = - \int d^D x \left\{ \frac{1}{4} F_{\mu\nu} \ast F^{\mu\nu} + \frac{\lambda}{2} (\partial_\mu A^\mu) \ast (\partial_\nu A^\nu) \right\}. \tag{2.2}$$

This action can be rewritten in terms of the star-operation $\mathcal{F}$,

$$S = \int d^D x \left\{ K(x) + \mathcal{F}_x \left( \mathcal{V}(x) \right) \right\} \tag{2.3}$$

where $K$ is the usual kinetic Lagrangian density,

$$K(x) = \frac{1}{2} A^\mu \left( \partial^2 \eta_{\mu\nu} - \partial_\mu \partial_\nu (1 - \lambda) \right) A^\nu(x). \tag{2.4}$$
\( \mathcal{V}(x) \) is the interaction Lagrangian density before the star-operation, whose form should be written as the non-local form since the split space-time coordinates should be kept:

\[
\mathcal{V}(x) \equiv \mathcal{W}_3(x_1, x_2, x_3) + \mathcal{W}_4(x_1, x_2, x_3, x_4)
\]

\[
\mathcal{W}_3(x_1, x_2, x_3) = g \left( B_{\mu\nu}(x_1) C^{\mu\nu}(x_2, x_3) + C^{\mu\nu}(x_1, x_2) B_{\mu\nu}(x_3) \right)
\]

\[
\mathcal{W}_4(x_1, x_2, x_3, x_4) = g^2 C^{\mu\nu}(x_1, x_2) C^{\mu\nu}(x_3, x_4)
\]

where

\[
B_{\mu\nu}(x) = \frac{i}{2} \left( \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x) \right)
\]

\[
C_{\mu\nu}(x_1, x_2) = \frac{1}{2} \left( A_{\mu}(x_1) A_{\nu}(x_2) - A_{\nu}(x_1) A_{\mu}(x_2) \right) = -C_{\nu\mu}(x_1, x_2)
\]

Here the coordinates \( x_i \)'s are the split coordinates from \( x \) and the ordering of the split coordinates are kept in each term. This is needed to have the desired starred action after star-operation applied to \( \mathcal{V}(x) \). The star-operation \( \mathcal{F} \) is explicitly given as

\[
\mathcal{F}_x \left( \mathcal{V}(x) \right) = \left\{ e^{\frac{i}{2} \left( \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x) \right)} \mathcal{W}_3(x_1, x_2, x_3) \right\}_{x_1 = x_2 = x_3 = x}
\]

\[
+ \left\{ e^{\frac{i}{2} \left( \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x) \right)} \mathcal{W}_4(x_1, x_2, x_3, x_4) \right\}_{x_1 = x_2 = x_3 = x_4 = x}.
\] (2.5)

The Heisenberg equation of motion is given as

\[
\{ \partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu (1 - \lambda) \} A_\nu(x) = \xi_\star^\mu(x) \equiv \int d^D y \mathcal{F}_y \left( \xi^\mu(x; y) \right)
\]

(2.6)

where

\[
\xi(x; y) = \frac{\delta}{\delta A_\mu(x)} \mathcal{V}(y).
\]

Here the ordering of the gauge field operators should be taken care of. The best of this ordering can be done in term of the delta-function, whose details can be found in [6].

The solution of (2.6) is formally written in terms of an integral equation,

\[
A_\mu(x) = a_\mu(x) + D_{\text{ret}} \circ \xi_\star^\mu(x)
\]

\[
= b_\mu(x) + D_{\text{ad}} \circ \xi_{\ast\mu}(x),
\] (2.7)

where \( a_\mu(x) \) and \( b_\mu(x) \) are in- and out-fields:

\[
a_\mu(x) \equiv A_{\mu\text{in}}(x), \quad b_\mu(x) \equiv A_{\mu\text{out}}(x),
\]

and \( \circ \) denotes the convolution,

\[
G \circ \xi_\star^\nu(x) = \int d^D y G(x - y) \xi_\star^\nu(y).
\]
The retarded and advanced Green’s functions are given in terms of the free commutator function of massless scalar field $D(x)$,

$$D_{\text{ret}}(x) = -\theta(x^0)D(x), \quad D_{\text{ad}}(x) = \theta(-x^0)D(x),$$

$$D(x) = \int \frac{d^Dk}{(2\pi)^D} e^{-ikx}2\pi\delta(k^2)\epsilon(k^0).$$

Here we take the gauge parameter $\lambda = 1$ (Feynman gauge) to avoid the unnecessary complication.

It is worth to mention the effect of the derivative interaction. The commutative abelian gauge theory contains the derivative interaction terms and therefore, $\xi^\mu(x)$ contains the derivatives of delta functions in addition to the derivatives due to the direct $\star$-operation. This derivative of the delta function will result in the derivative of the retarded (or advanced) Green’s function after the convolution integral:

$$G \circ \left( A(x) \frac{\partial \delta(x)}{\partial x_\mu} \right) = \int d^Dy A(y) \frac{\partial G(x-y)}{\partial y_\mu}. \quad (2.8)$$

Then it is obvious that the derivative of the retarded (or advanced) Green’s function will bring in the contact term through the derivative of the step-function. This result will have the important effect on the S-matrix later on, especially in time-ordering.

The in-coming gauge field is assumed to satisfy the free commutation relation,

$$[ A_{\mu\text{free}}(x), A_{\nu\text{free}}(y) ] = -i\eta_{\mu\nu}D(x-y).$$

The quantized free gauge field is expressed as

$$a_\mu(x) \equiv A_{\mu\text{free}}(x) = \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{2|p|} \sum_\lambda \left( a(p,\lambda)\epsilon_\mu(p,\lambda)e^{-ip\cdot x} + a(p,\lambda)\dagger\bar{\epsilon}_\mu(p,\lambda)e^{ip\cdot x} \right)$$

where $p$ represents the $D-1$ dimensional momentum. The creation and annihilation operators satisfy the commutation relation,

$$[ a(p,\lambda), a^{\dagger}(q,\lambda') ] = -(2\pi)^{D-1}2|p|\delta(p-q)\eta^{\lambda\lambda'},$$

and the polarization vector satisfies the relation

$$\sum_\lambda \epsilon_\mu(k,\lambda)\bar{\epsilon}_\nu(k,\lambda) = \eta_{\mu\nu}, \quad \epsilon_\mu(k,\lambda)\bar{\epsilon}^{\mu}(k,\tau) = \eta^{\lambda\tau}.$$

The integral equation (2.7) is assumed to be solved iteratively in terms of the in-field $a_\mu(x)$, and, therefore, $\xi^\mu(x)$ can be written in terms of the in-fields only. As a result, the out-field is written in terms of the in-field:

$$b^\mu(x) = a^\mu(x) - D \circ \xi^\mu(x). \quad (2.9)$$

If the out-field is written in terms of an S-matrix

$$b^\mu(x) = S^{-1}a^\mu(x)\ S, \quad (2.10)$$
then, out-field also respects the free commutator relation.

One may repeat the same procedure given in the scalar case to construct the S-matrix. However, two additional points are to be considered carefully in the gauge theory; derivative interaction terms and the gauge symmetry. All of them are not new in this starred interaction but are inherent to the gauge theory. The solution to this two ingredients are already considered in the commutative gauge theory and therefore, we need to check if the solution works correctly.

The S-matrix is written as

\[
S = 1 + i \int_{-\infty}^{\infty} d^D x \mathcal{F}_x (\mathcal{V}(x)) + i^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^D x_1 d^D x_2 \mathcal{F}_{12} (\theta_{12}^* \mathcal{V}(x_1) \mathcal{V}(x_2)) \cdots + i^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^D x_1 \cdots d^D x_n \mathcal{F}_{12 \cdots n} (\theta_{12 \cdots n}^* \mathcal{V}(x_1) \cdots \mathcal{V}(x_n)) + \cdots, \tag{2.11}
\]

where \( \mathcal{F} \) is the star-operation and \( \theta_{12 \cdots n}^* \) is the covariant time-ordering, whose detailed forms will be given promptly. The time-integration has to be done outside of the star-operation. We put the space integration as well as the time integration outside the star-operation for notational convenience.

The covariant time-ordering \( \theta_{12 \cdots n}^* \) is the ordinary time-ordering when there is no derivative operators, which is defined in terms of the step function,

\[
\theta_{12}^* \phi(x_1) \phi(x_2) = \theta(t_1 - t_2) \phi(x_1) \phi(x_2).
\]

The time-ordering of derivative operators, however, need be done in the form of covariant time ordering: The covariant time ordering is defined before taking derivatives:

\[
\theta_{12}^* \left( f(\partial_1) \phi(x_1) \right) \left( g(\partial_2) \phi(x_2) \right) = f(\partial_1) g(\partial_2) \left( \theta(t_1 - t_2) \phi(x_1) \phi(x_2) \right). \tag{2.12}
\]

The composite covariant time-ordering is similarly given as

\[
\theta_{12 \cdots n}^* = \theta_{12}^* \theta_{23}^* \cdots \theta_{n-1 n}^*.
\]

The covariant time-ordering is the source of Schwinger’s contact term:

\[
\theta_{12}^* \left( \partial_\mu j_\mu(x_1) j_\nu(x_2) \right) = \delta(x_1^0 - x_2^0) \left( j_\mu^0(x_1) j_\nu^0(x_2) \right) + \theta_{12} \left( \partial_\mu j_\mu(x_1) j_\nu(x_2) \right).
\]

The composite \(*\)-operation is defined as

\[
\mathcal{F}_{12 \cdots n} \equiv \mathcal{F}_1 \mathcal{F}_2 \cdots \mathcal{F}_n,
\]

which is commutative,

\[
\mathcal{F}_{12} \equiv \mathcal{F}_1 \mathcal{F}_2 = \mathcal{F}_2 \mathcal{F}_1 = \mathcal{F}_{21},
\]

so that

\[
\mathcal{F}_{12 \cdots n} \left( \mathcal{V}(x_1) \mathcal{V}(x_2) \cdots \mathcal{V}(x_n) \right) = \mathcal{L}_f(x_1) \mathcal{L}_f(x_2) \cdots \mathcal{L}_f(x_n).
\]
The star-operation $\mathcal{F}$ needs special care in the presence of the time-ordering step function. For example, suppose one considers the star-operation on two vertices with a time step function,

$$\mathcal{F}_{xy}\left(\theta^*(x^0 - y^0)\mathcal{Z}(x)\mathcal{Z}(y)\right),$$

and the vertices are given in terms of quantum fields with no derivatives:

$$\mathcal{Z}(x) = a_{\mu_1}(x)a_{\mu_2}(x) \cdots a_{\mu_n}(x).$$

The star-operation can be done as in the scalar theory,

$$\mathcal{F}_{xy}\left(\theta^*(x^0 - y^0)\mathcal{Z}(x)\mathcal{Z}(y)\right) = \mathcal{F}_x \mathcal{F}_y \left(\theta(x^0 - y^0)\left(a_{\mu_1}(x_1)a_{\mu_2}(x_2) \cdots a_{\mu_n}(x_n)\right)\left(a_{\nu_1}(y_1)a_{\nu_2}(y_2) \cdots a_{\nu_n}(y_n)\right)\bigg|_{x_i = x, y_i = y}\right),$$

where the minimal realization is used to avoid the ambiguity of the step function. The minimal realization is to put the step function $\theta(x^0 - y^0)$ only once if the operators $a_{\mu_i}(x_i)$ and $a_{\nu_j}(x_j)$ are contracted. Even in the presence of many spectral functions the time-ordering step function should be used only once between two vertices:

$$\theta(x^0 - y^0)\prod_{i,j} \Delta(x_i - y_j) \rightarrow \theta(x_{a}^0 - y_{b}^0)\prod_{i,j} \Delta(x_i - y_j)$$

where $a$ ($b$) is just one of indices among $i$'s ($j$'s). In addition, for the connected $n$-vertices, there are only $n$-number of step functions. This minimal realization originates from the consistency condition of S-matrix \textbf{(2.10)} and Heisenberg equation of motion \textbf{(2.6)}. The details are given in \textbf{[6]}. When the vertices contain derivatives, one needs the minimal realization with the covariant time-ordering.

The proof that the S-matrix in \textbf{(2.11)} relates the in- and out-field \textbf{(2.10)} and is unitary $S^{\dagger}S = 1$ goes the same as in the scalar theory, if one takes into consideration the effect of derivative interaction: The derivative interaction necessitates the covariant time-ordering as noted in \textbf{(2.8)}.

Introducing a notation of the time-ordering $\ast$-product as,

$$T_{\ast}\left\{\mathcal{V}(t_1)\mathcal{V}(t_2)\right\} = \mathcal{F}_{12}\left(\theta_{12}^* \mathcal{V}(t_1)\mathcal{V}(t_2) + \theta_{21}^* \mathcal{V}(t_2)\mathcal{V}(t_1)\right),$$

we can put the S-matrix in a compact form as

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\infty} d^{D}x_1 \cdots \int_{-\infty}^{\infty} d^{D}x_n T_{\ast}\left\{\mathcal{V}(\phi_{\text{in}}(x_1)) \cdots \mathcal{V}(\phi_{\text{in}}(x_n))\right\}$$

$$\equiv T_{\ast} \exp \left(i \int_{-\infty}^{\infty} dx \mathcal{V}(\phi_{\text{in}}(x))\right).$$

Finally, the gauge symmetry of the system \textbf{(2.1)} requires the scattering amplitude to be gauge invariant. However, the modified action with the addition of the gauge fixing
term (2.2) allows the non-physical degree of freedom and one needs to check the non-physical degree of freedom does not affect the scattering amplitude. The is achieved by defining the physical states, \( |\text{phy} \rangle \) in a weak form,

\[
|\text{phy} \rangle \partial_\mu a^\mu |\text{phy} \rangle = 0
\]
or by introducing Faddeev-Popov ghost fields into the system. The gauge invariance of the scattering amplitude will be considered in section 5.

## 3 Feynman Rule

NC abelian gauge theory contains the derivative interaction and the 3-point and 4-point vertex function is given at the lowest order as following:

\[
\langle a_\mu(p_1)a_\nu(p_2)a_\rho(p_3) \rangle_c = -ig(2\pi)^d\delta^d(p_1 + p_2 + p_3) v_{\mu\nu\rho}(p_1, p_2, p_3)
\]

\[
\langle a_\mu(p_1)a_\nu(p_2)a_\rho(p_3)a_\sigma(p_4) \rangle_c = -ig^2(2\pi)^d\delta^d(p_1 + p_2 + p_3 + p_4) v_{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4)
\]

(3.1)

where \( \langle \cdots \rangle_c \) refers to the 1-particle irreducible vertex and

\[
v_{\mu\nu\rho}(p_1, p_2, p_3) = -2i \sin \left( \frac{p_2 \wedge p_3}{2} \right) \left( (p_1)_\nu \delta_{\mu\rho} - \nu \leftrightarrow \rho \right)
\]

\[
+ \left( (p_1, \mu) \leftrightarrow (p_2, \nu) \right) + \left( (p_1, \mu) \leftrightarrow (p_3, \rho) \right)
\]

\[
v_{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = -\sin \left( \frac{p_1 \wedge p_2}{2} \right) \sin \left( \frac{p_3 \wedge p_4}{2} \right) \delta_{\mu\rho} \delta_{\nu\sigma} - \sigma \leftrightarrow \rho
\]

\[
+ \left( (p_2, \nu) \leftrightarrow (p_3, \rho) \right) + \left( (p_2, \nu) \leftrightarrow (p_4, \sigma) \right).
\]

(3.2)

Here all the momenta are assumed to be in-coming.

The free 2-point function is given in terms of the spectral function \( D^+(\mu\nu)(x_1, x_2) \) whose Fourier transformed form is

\[
D^+(\mu\nu)(x_1, x_2) = \langle a_\mu(x_1)a_\nu(x_2) \rangle = -\eta_{\mu\nu}D^+(x_1 - x_2)
\]

(3.3)

where \( D^+(x) \) is the massless scalar spectral function,

\[
D^+(x) = \int \frac{d^Dp}{(2\pi)^D} e^{-ip \cdot x} 2\pi\delta(p^2)\theta(p^0).
\]

Therefore, the 2-point function in momentum space is given as

\[
\tilde{D}^+(\mu\nu)(p) = -\eta_{\mu\nu}2\pi\delta(p^2)\theta(p^0).
\]

(3.4)

Time-ordered spectral function is defined as

\[
D^R_{\mu\nu}(x_1, x_2) = \theta(x_1^0 - x_2^0)D^+(x_1, x_2) = \eta_{\mu\nu}D_R(x_1 - x_2)
\]

(3.5)

where \( D_R(x) \) is the massless scalar time-ordered spectral function

\[
D_R(x) = \int \frac{d^Dp}{(2\pi)^D} e^{-ip \cdot x} \frac{i}{2|\mathbf{p}|(p^0 - |\mathbf{p}| + i\epsilon)}.
\]
In momentum space the time-ordered spectral function of photon is given as

\[ \tilde{D}_R^{\mu\nu}(p) = \eta_{\mu\nu} \frac{i}{2|p| (p^0 - |p| + i\epsilon)}. \]

Note that the Feynman propagator is given in terms of the time-ordered spectral function as

\[ i\Delta F_{\mu\nu}(x) = -\eta_{\mu\nu} \left( D_R(x) + D_R(-x) \right). \quad (3.6) \]

Our Feynman rule for this theory is summarized as follows.

1. Each vertex has either 3 legs or 4 legs. 3-leg vertex is assigned as \( v_{\mu\nu\rho}(p_1, p_2, p_3) \) and 4-leg vertex as \( v_{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) \), where \( p_i \) is the in-coming momentum of each leg. The total momentum at each vertex vanishes.

![Diagram showing vertex with 3 legs and 4 legs assigned](image)

2. The legs at each vertex are either external legs or are connected to other vertex, making internal lines.

3. Each vertex is numbered so that the internal lines are assigned with arrows. The arrows point from high-numbered vertex to low-numbered one.

4. Among the arrows, one arrow between adjacent vertices can be assigned as triangled arrow, with the condition that the total number of the triangled arrows should be \( n - 1 \) for the \( n \) connected vertices \( (n > 1) \). This restriction is due to the minimal realization.

5. The diagrams with the same topology with arrows are identified and the numbering of vertices is ignored. As a result the number of diagrams are reduced from the \( n! \) diagrams. In this sense, the numbering of the vertices are considered as the intermediate step to obtain the topologically distinct diagrams with arrows.

6. The distinctive Feynman diagrams are multiplied by the symmetric factors.

7. The momentum flows along the arrows. The arrowed internal line with momentum \( k \) is assigned as \( \tilde{D}_R^{\mu\nu}(k) \) and the triangled arrowed internal line as \( \tilde{D}_R^{\mu\nu}(k) \).
4 STNC QED

When the Dirac fermions are added to the abelian gauge theory, there appears an additional action,
\[ \Delta S = \int d^Dx \left\{ \bar{\psi} \ast (i\partial - m - eA) \ast \psi \right\}. \] (4.1)

\( A \) is the abbreviation of \( A^\mu \gamma_\mu \). The gauge coupling \( g \) is identified as the charge of the electron, \( g = -e \).

The in-coming Dirac fermionic fields \( \psi \) and \( \bar{\psi} \) satisfy the free anti-commutation relation,
\[ iS_{\alpha\beta} = \{ \psi_\alpha, \bar{\psi}_\beta \} = \int \frac{d^Dk}{(2\pi)^D} e^{-ik \cdot x} (\bar{k} + m)_{\alpha\beta} 2\pi \delta(k^2 - m^2) \epsilon(k^0) \]
\[ = (i\partial + m)_{\alpha\beta} (i\Delta(x)) \] (4.2)
where \( i\Delta(x) \) is the massive scalar commutator function. The free fermionic field is written as
\[ \psi(x) = \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{m}{\omega_p} \sum_i \left( b_i(p) u^i(p) e^{-ip \cdot x} + d_i^\dagger(p) v^i(p) e^{ip \cdot x} \right) \]
\[ \bar{\psi}(x) = \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{m}{\omega_p} \sum_i \left( b_i^\dagger(p) \bar{u}^i(p) e^{ip \cdot x} + d_i(p) \bar{v}^i(p) e^{-ip \cdot x} \right) \] (4.3)
where \( b \) and \( d \) are \((b^\dagger \text{ and } d^\dagger)\) are annihilation (creation) operators,
\[ \{b_\alpha(p), b_\beta^\dagger(q)\} = (2\pi)^{D-1} \delta(p - q) \delta_{\alpha\beta} \]
\[ \{d_\alpha(p), d_\beta^\dagger(q)\} = (2\pi)^{D-1} \delta(p - q) \delta_{\alpha\beta} , \] (4.4)
and \( u \) and \( v \) are spinors and \( p \) denotes the spatial momentum of \( p \).

The retarded and advanced Green’s function are written as
\[ \left( S_{\text{ret}} \right)_{\alpha\beta}(x) = -\theta(x^0) S_{\alpha\beta}(x) \quad \left( S_{\text{adv}} \right)_{\alpha\beta}(x) = \theta(-x^0) S_{\alpha\beta}(x) \] (4.5)
since
\[ (i\partial - m) S_{\text{ret}}(x) = (i\partial - m) S_{\text{adv}}(x) = -\delta^D(x) . \]

As in the scalar and gauge theories, we solve the equation of motion in terms of the retarded Green’s function with the in-coming wave. The S-matrix which relates the in- and out- fields including fermion field are given as (2.1) if one includes the fermionic part in the interaction Lagrangian density \( \Delta V \);
\[ \Delta V = e \bar{\psi} A \psi \] (4.6)

To evaluate the S-matrix element in momentum space we need 2-point functions for the positive (particle contribution) and negative frequency part (anti-particle contribution), and their time-ordered ones. The 2-point functions are given as
\[ S_{\alpha\beta}^+(x) = \langle 0|\psi_\alpha(x) \bar{\psi}_\beta|0 \rangle = \int \frac{d^Dk}{(2\pi)^D} e^{-ik \cdot x} (\bar{k} + m)_{\alpha\beta} 2\pi \delta(k^2 - m^2) \theta(k^0) \]
\[ S_{\alpha\beta}^{-}(x) = \langle 0| \bar{\psi}_\beta \psi_\alpha(x)|0 \rangle = \int \frac{d^Dk}{(2\pi)^D} e^{ik\cdot x} (\hat{k}^\alpha - m)_{\alpha\beta} 2\pi \delta(k^2 - m^2) \theta(k^0) \]
\[ = \int \frac{d^Dk}{(2\pi)^D} e^{-ik\cdot x} (-\hat{k}^\alpha - m)_{\alpha\beta} 2\pi \delta(k^2 - m^2) \theta(-k^0), \quad (4.7) \]

and their time-ordered ones are given as
\[ S_{\alpha\beta}^{+R}(x) = \theta(x^0)S_{\alpha\beta}^{+}(x) = \int \frac{d^Dk}{(2\pi)^D} e^{-ik\cdot x} (\hat{k}^\alpha + m)_{\alpha\beta} \frac{i}{2\omega_k} k^0 - \omega_k + i\epsilon \]
\[ S_{\alpha\beta}^{-A}(x) = \theta(-x^0)S_{\alpha\beta}^{+}(x) = \int \frac{d^Dk}{(2\pi)^D} e^{-ik\cdot x} (\hat{k}^\alpha - m)_{\alpha\beta} \frac{i}{2\omega_k} k^0 + \omega_k - i\epsilon \quad (4.8) \]

where \( \omega_k = \sqrt{k^2 + m^2} \) and \( \hat{k}^\pm = (\pm \omega_k, k) \). The Feynman propagator is given in terms of the time-ordered positive and negative spectral functions,
\[ iS_{\alpha\beta}^{F} = \theta(x^0)S_{\alpha\beta}^{+}(x) - \theta(-x^0)S_{\alpha\beta}^{-}(x) = S_{\alpha\beta}^{+R}(x) - S_{\alpha\beta}^{-A}(x) = (i\partial_\mu \gamma^\mu + m)(i\Delta_F(x)) \quad (4.9) \]

where \( i\Delta_F(x) \) is the massive scalar Feynman propagator,
\[ i\Delta_F(x) = \Delta_R(x) + \Delta_R(-x). \quad (4.10) \]

Feynman rule for the fermionic theory is similarly given as in the bosonic theory. There, however, appear a few differences:

(1) The extra sign should be multiplied each time two fermionic operators interchange each other due to the fermionic statistics.

(2) There are two types of fermionic internal lines, a solid arrowed line for particle propagator and a dashed one for anti-particle one. The momentum is assumed to flow along the arrow:
\[ \begin{align*}
  & \alpha \quad k \quad \beta \quad = \quad \hat{S}_{\alpha\beta}^{+}(k) = (\hat{k}^\alpha + m)_{\alpha\beta} 2\pi \delta(k^2 - m^2) \theta(k^0) \\
  & \alpha \quad k \quad \beta \quad = \quad \hat{S}_{\alpha\beta}^{+R}(k) = \frac{(\hat{k}^\alpha + m)_{\alpha\beta}}{2\omega_k} \frac{i}{k^0 - \omega_k + i\epsilon} \\
  & \alpha \quad k \quad \beta \quad = \quad \hat{S}_{\alpha\beta}^{-}(k) = (-\hat{k}^\alpha - m)_{\alpha\beta} 2\pi \delta(k^2 - m^2) \theta(-k^0), \\
  & \alpha \quad k \quad \beta \quad = \quad \hat{S}_{\alpha\beta}^{+R}(k) = \frac{(-\hat{k}^\alpha + m)_{\alpha\beta}}{2\omega_k} \frac{i}{k^0 + \omega_k - i\epsilon}
\end{align*} \]

(3) The direction of arrows along a fermionic line should not be reversed in the Feynman graphs. If the arrow points from a high-numbered vertex to a low numbered vertex, the fermionic internal line is assigned as a solid line. Otherwise, the fermionic line is dashed.

(4) The vertex with two fermionic legs is assigned as
\[ \langle \bar{\psi}_\alpha(p)\psi_\beta(q)A_\mu(k) \rangle_c = -ie (2\pi)^D \delta(p + k - q) v_{\alpha\beta;\mu}(p, q, k) \]
where

\[-ie v_{\alpha\beta;\mu}(p, q; k) = -ie \epsilon^{\mu p q r} \gamma_{\alpha\beta} = \frac{\mu k}{p} < \frac{\alpha}{\beta} < \frac{q}{r}\]

and its total momentum vanishes \(q + k + (-p) = 0\). There are two more types of the vertex depending on the fermion legs, particle or antiparticle. However, the vertex function is independent of the particle species.

\[
\begin{align*}
\frac{\alpha}{p} &\quad \frac{\mu k}{p} \quad \frac{\beta}{q} = -ie v_{\alpha\beta;\mu}(p, q; k) \\
\frac{\alpha}{p} &\quad \frac{\beta}{q} = -ie v_{\alpha\beta;\mu}(p, q; k)
\end{align*}
\]

5 Compton Scattering and Gauge Invariance

The S-matrix should be gauge invariant. In this section, we investigate on the Compton scattering to the lowest order to prove that the scattering amplitude is gauge invariant. The Compton scattering of an electron is given as the following diagrams to the lowest order:

\[
M_{\mu\nu;\alpha\beta}(k_1, k_2; p, q) = -e^2 \left( M_{\mu\nu;\alpha\beta}^{(1)}(k_1, k_2; p, q) - M_{\mu\nu;\alpha\beta}^{(2)}(k_1, k_2; p, q) \right)
\]  \quad (5.1)

where

\[
\begin{align*}
-e^2 M_{\mu\nu;\alpha\beta}^{(1)}(k_1, k_2; p, q) &= \alpha < \frac{\beta}{q} + \alpha < \frac{\beta}{q} - \alpha < \frac{\beta}{q} - \alpha < \frac{\beta}{q} \\
+ e^2 M_{\mu\nu;\alpha\beta}^{(2)}(k_1, k_2; p, q) &= \alpha < \frac{\beta}{q} + \alpha < \frac{\beta}{q}
\end{align*}
\]
Explicitly,

\[
M^{(1)}_{\mu\nu,\alpha\beta}(k_1, k_2 ; p, q) = \sum_{\delta, \gamma} \int_{\ell} \left\{ \nu_{\alpha\delta} \mu(p, \ell; k_1) S^{+R}_{\gamma\delta}(\ell) v_{\gamma\beta,\mu}(\ell, q; k_2) \times (2\pi)^{2D} \delta(p - k_1 - \ell) \delta(\ell - q - k_2) + k_1 \leftrightarrow k_2 \right\}
\]

\[
M^{(2)}_{\mu\nu,\alpha\beta}(k_1, k_2 ; p, q) = \sum_{\delta, \gamma} \int_{\ell} \left\{ \nu_{\alpha\beta,\nu}(p, q; \ell) v_{\mu\nu}(k_1, k_2, \ell) D_{\gamma}(\ell) \times (2\pi)^{2D} \delta(p + \ell - q) \delta(k_1 + k_2 + \ell) + \ell \rightarrow -\ell \right\}.
\]  \hspace{1cm} (5.2)

The gauge invariance requires the identity

\[
k_{1}^{\mu} \epsilon^\nu(\lambda) \bar{u}_\alpha(p) M_{\mu\nu,\alpha\beta}(k_1, k_2 ; p, q) u_\beta(q) = 0.
\]  \hspace{1cm} (5.3)

For the commuting case this is trivially satisfied since \( M^{(2)}_{\mu\nu,\alpha\beta}(k_1, k_2 ; p, q) = 0 \), and due to the identity

\[\frac{1}{f - m + i\epsilon} \bar{u}_\alpha(p) k_1 \frac{1}{f - m + i\epsilon} \bar{u}_\alpha(p) \bigg|_{p = k_1 + \ell} = -\bar{u}_\alpha(p) \bigg|_{p = k_1 + \ell},\]

we have

\[
k_{1}^{\mu} \epsilon^\nu(\lambda) \bar{u}_\alpha(p) M^{(1)}_{\mu\nu,\alpha\beta}(k_1, k_2 ; p, q) u_\beta(q) = -\bar{u}_\alpha(p) \phi(\lambda) u_\beta(q) + \bar{u}_\alpha(p) \phi(\lambda) u_\beta(q) = 0.
\]  \hspace{1cm} (5.4)

For the non-commuting case, the result is not obvious at all. First, we note that

\[
M^{(1)}(k_1, k_2 ; p, q) \equiv k_{1}^{\mu} \epsilon^\nu(\lambda) \bar{u}_\alpha(p) M^{(1)}_{\mu\nu,\alpha\beta}(k_1, k_2 ; p, q) u_\beta(q) = e^{\frac{i}{2}(p - q) \wedge \ell} \bar{u}(p) \left\{ \left( \frac{\hat{q}^+ + m}{2\omega_\ell} \right) \frac{i}{\ell^0 - \omega_\ell + i\epsilon} - \left( \frac{\hat{q}^- + m}{2\omega_\ell} \right) \frac{i}{\ell^0 + \omega_\ell - i\epsilon} \right\} \phi(k_2, \lambda) u(q) \bigg|_{\ell = p - k_1} + e^{\frac{i}{2}(p - q) \wedge \ell} \bar{u}(p) \phi(k_2, \lambda) \left\{ \left( \frac{\hat{q}^+ + m}{2\omega_\ell} \right) \frac{i}{\ell^0 - \omega_\ell + i\epsilon} - \left( \frac{\hat{q}^- + m}{2\omega_\ell} \right) \frac{i}{\ell^0 + \omega_\ell - i\epsilon} \right\} k_1 u(q) \bigg|_{\ell = q + k_1}. \]  \hspace{1cm} (5.5)

Using the on-shell condition,

\[
\bar{u}(p)(\not{q} - m) = (\not{q} - m) u(p) = 0
\]

\[
(\hat{q}^+ + m)(\hat{q}^+ - m) = (\hat{q}^+ - m)(\hat{q}^+ + m) = 0
\]

\[
(\hat{q}^- + m)(\hat{q}^- - m) = (\hat{q}^- - m)(\hat{q}^- + m) = 0
\]

we may have

\[
\bar{u}(p)(\not{q} - \not{q})(\hat{q}^+ + m) = \bar{u}(p)\gamma^0(\hat{q}^+ + m)(\ell^0 - \omega_\ell)
\]

\[
\bar{u}(p)(\not{q} - \not{q})(\hat{q}^- + m) = \bar{u}(p)\gamma^0(\hat{q}^- + m)(\ell^0 + \omega_\ell)
\]

\[
(\hat{q}^+ + m)(\not{q} - \not{q}) u(q) = (\ell^0 - \omega_\ell)(\hat{q}^+ + m)\gamma^0 u(q)
\]

\[
(\hat{q}^- + m)(\not{q} - \not{q}) u(q) = (\ell^0 + \omega_\ell)(\hat{q}^- + m)\gamma^0 u(q).
\]  \hspace{1cm} (5.6)

Due to this identities, we may put \( \text{L.H.S.} \) as

\[
M^{(1)}(k_1, k_2 ; p, q) = -i \left( e^{\frac{i}{2}(p - q) \wedge (p - k_1)} - e^{\frac{i}{2}(p - q) \wedge (p - k_2)} \right) \bar{u}(p) \phi(k_2, \lambda) u(q)
\]  \hspace{1cm} (5.7)
On the other hand, $M^{(2)}$ is given as
\[
M^{(2)}(k_1, k_2; p, q) = k_1^\mu e^\nu(k_2, \lambda) \bar{u}_\alpha(p) M^{(2)}_{\mu\nu\alpha\beta}(k_1, k_2; p, q) u_\beta(q)
\]
\[
= e^{\frac{i}{2} p^{\rho} q^{\nu} \bar{u}(p) \gamma^\rho u(q)} k_1^\mu e^\nu(k_2, \lambda) v_{\mu\nu\rho}(k_1, k_2, \ell) D^R(\ell) \big|_{\ell=q-p} + e^{\frac{i}{2} p^{\rho} q^{\nu} \bar{u}(p) \gamma^\rho u(q)} k_1^\mu e^\nu(k_2, \lambda) v_{\mu\nu\rho}(k_1, k_2, -\ell) D^R(\ell) \big|_{\ell=p-q}.
\] (5.8)

Using the vertex relation,
\[
k_1^\mu v_{\mu\rho}(k_1, k_2, \ell) = -2i \left( \sin \frac{k_1^\ell + k_2^\ell}{2} (k_2 \cdot k_2 \delta_{\rho\rho} - k_2 \cdot k_1^\nu) + \sin \frac{k_2^\ell + k_1^\ell}{2} (\ell_{\nu} \cdot k_1^\rho - \ell \cdot k \delta_{\nu\rho}) \right)
\]
and on-shell conditions of the photon,
\[
2k_1 \cdot k_2 = (k_1 + k_2)^2, \quad k_2 \cdot \epsilon(k_2) = 0
\]
we have
\[
M^{(2)}(k_1, k_2; p, q) = -e^{\frac{i}{2} p^{\rho} q^{\nu} \bar{u}(p) \gamma^\rho u(q)} \frac{k_1^\ell + k_2^\ell}{2} \left( \bar{u}(p) \gamma^\rho u(q) \right) + \frac{k_1^\ell + k_2^\ell}{2} \left( \ell_{\nu} \cdot k_1^\rho - \ell \cdot k \delta_{\nu\rho} \right).
\] (5.9)

Noting that electron on shell satisfies
\[
\bar{u}(p)(\gamma - \not{p})u(q) = 0
\]
we have
\[
M^{(2)}(k_1, k_2; p, q) = -i \left( e^{\frac{i}{2} (p-q) + k_1 \wedge k_2} - e^{\frac{i}{2} (p-q) - k_1 \wedge k_2} \right) \bar{u}(p) \gamma^\rho(k_2, \lambda) u(q)
\]
\[
= -i \left( e^{\frac{i}{2} (p-q) + k_1 \wedge k_2} - e^{\frac{i}{2} (p-q) - k_1 \wedge k_2} \right) \bar{u}(p) \gamma^\rho(k_2, \lambda) u(q).
\]

Therefore, $M^{(1)}$ and $M^{(2)}$ cancel exactly each other. This proves the gauge invariance of the scattering amplitude to the lowest order.

We remark that this gauge invariance result is in contrast with the result in [10] where the gauge symmetry is seen violated to the lowest order in the time-ordered perturbation theory defined in [9]. TOPT differs from ours in the time-ordering: Our minimal realization of the time-ordering cures the ill-defined time-ordering and preserves the gauge symmetry.

6 Classical solution of the vacuum

In this section, we turn to the classical vacuum solution of STNC pure abelian gauge theory. NC abelian gauge theory is known to possess the non-abelian nature and self-sustaining magnetic flux in Euclidean NCFT [12]. One may wonder whether there is a non-trivial finite action solution in STNC gauge field theories also. Indeed, it is pointed out in [13] that non-trivial vacuum solutions exist in STNC NC gauge theory and the validity of STNC perturbative gauge theory has been questioned. We address this problem in
connection with the gauge fixing condition. For simplicity, we consider the 2 dimensional theory and use the notation, \( x^\mu = (t, \mathbf{x}) \).

The classical equation of NC abelian theory is given as
\[
[\partial_\mu - igA_\mu, F^{\mu \nu}] = 0.
\] (6.1)

Suppose we choose the temporal gauge \( A_0 = 0 \). The field strength simplifies to \( F_{01} = \partial_0 A_1 \) and satisfies the equation,
\[
\partial_0 F_{01} = \partial_0^2 A_1 = 0.
\] (6.2)

Therefore \( A_1 \) and \( F_{01} \) is of the form,
\[
A_1(x) = tc(\mathbf{x}) + d(\mathbf{x}), \quad F_{01} = c(\mathbf{x}).
\] (6.3)

The fields should satisfy the remaining field equation,
\[
0 = \partial_x c(\mathbf{x}) - ig[tc(\mathbf{x}) + d(\mathbf{x}), c(\mathbf{x})] = \partial_x c(\mathbf{x}) \left( 1 - e\theta c(\mathbf{x}) \right)
\]
and the solution is given as
\[
F_{01} = c(\mathbf{x}) = \text{constant} = 0
\] (6.4)

to have the action finite. The classical solution of the field strength with finite action is unique and vanishes. The same goes with the axial gauge, \( A_1 = 0 \).

On the other hand, suppose one chooses a radial gauge
\[
x^\mu A_\mu = 0.
\]

One may get a pure gauge form satisfying the symmetric gauge using the gauge transformation. If we choose the gauge transformation,
\[
U(x) = \frac{\theta}{2} \int \frac{d^2 k}{2\pi} e^{-ik \cdot x} = \pi \theta \delta(x),
\] (6.5)

we have the pure gauge form of the field,
\[
-igA_\mu = U\partial_\mu U^\dagger = -\frac{2}{\theta} \epsilon_{\mu \nu} x^\nu.
\] (6.6)

One may obtain a localized pure gauge field of a symmetric form
\[
A_\mu = \epsilon_{\mu \nu} x^\nu f(r)
\] (6.7)

where \( f(r) \) is a localized function with \( r^2 = t^2 + x^2 \), if one applies to \( A_\mu = 0 \), unitary gauge transformation,
\[
U_n(x) = e^{i\alpha_n g_m(x)} = 1 + (e^{i\alpha_n} - 1) g_{mn}(x).
\] (6.8)

Here \( \alpha_n \) is an arbitrary real number (conveniently put as \( \pi \)) and \( g_{mn}(x) \) is the generalized Wigner function (with \( x \to \sqrt{\theta} x \) rescaled) [11],
\[
g_{mn}(x) = 2e^{-\frac{1}{2}z} \left\lfloor \frac{m!}{n!} (-1)^n (\sqrt{2z})^{m-n} L_n^{m-n} (2r^2) = g_{mn}^* \right\rfloor \text{ for } m \geq n
\] (6.9)
where $L_n^a(t)$ is the Laguerre polynomial,
\[
L_n^a(t) = \frac{1}{n!} \frac{e^t}{t^n} \frac{d^n}{dt^n} \left( e^{-t} t^{a+n} \right)
\]
and $z = t + ix$. Note that $g_{mn}(x)$ satisfies the relation,
\[
g_{mn}(x) * g_{pq}(x) = \delta_{np} g_{mq}(x).
\]
For example, using $U_0(z) = 1 - 2g_{00}(z)$, one has the localized form of the pure gauge field,
\[
-i g(A_0 - iA_1)(x) = U_0^\dagger \left( 2\partial U_0 \right) = 2\sqrt{\frac{2}{\theta}} g_{10}(x)
\]
\[
-i g(A_0 + iA_1)(x) = U_0^\dagger \left( 2\bar{\partial} U_0 \right) = 2\sqrt{\frac{2}{\theta}} g_{10}(x)^*
\]
where $\partial = (\partial_0 - i\partial_1)/2$ and $\bar{\partial} = (\partial_0 + i\partial_1)/2$ and the following identities are used:
\[
\partial g_{nn} = -\sqrt{\frac{n+1}{2\theta}} g_{n+1} + \sqrt{\frac{n}{2\theta}} g_{n-1}, \quad \bar{\partial} g_{nn} = -\sqrt{\frac{n+1}{2\theta}} g_{n+1} + \sqrt{\frac{n}{2\theta}} g_{n-1}.
\]
Any composite operator constructed from $U_n$'s will give the localized pure gauge form.

On the other hand, using the so-called shift operator $K$, which is not unitary,
\[
K^\dagger \star K = 1 \neq K \star K^\dagger,
\]
one may construct a non-trivial finite action solution. For example,
\[
K_1 = \sum_{n=0}^{\infty} g_{n+1n}(x)
\]
is a shift operator with $K_1^\dagger \star K_1 = 1$ and $K_1 \star K_1^\dagger = 1 - g_{00}(x)$ . This results in the gauge field
\[
-i g(A_0 - iA_1)(x) = K_1^\dagger \left( 2\bar{\partial} K_1 \right) = \sum_{n=1}^{\infty} 2d_n g_{n-1n}(x)
\]
\[
-i g(A_0 + iA_1)(x) = K_1^\dagger \left( 2\partial K_1 \right) = \sum_{n=1}^{\infty} 2d_n g_{n+1n}(x).
\]
Here $d_n = \sqrt{\frac{n}{2\theta}} - \sqrt{\frac{n+1}{2\theta}}$ and the following identities are used:
\[
\partial g_{n+1n} = -\sqrt{\frac{n+1}{2\theta}} (g_{nn} - g_{n-1n+1}), \quad \bar{\partial} g_{n+1n} = -\sqrt{\frac{n+2}{2\theta}} g_{n+2n} + \sqrt{\frac{n}{2\theta}} g_{n+1n-1}.
\]
The gauge function (6.14) is localized and convergent. This gauge field gives the localized field strength, $F_{01}(x) = -g_{00}(x)/(\theta g)$ whose action is finite. If one uses a composite operation, $V(x) \equiv U(x) \star K_1(x)$ with $U(x)$ in (6.5), one then obtains
\[
-i g(A_0 - iA_1)(x) = V^\dagger \left( 2\partial V \right) = -\bar{\varepsilon} - \sum_{n=0}^{\infty} \sqrt{\frac{2n}{\theta}} g_{n+1n}(x)
\]
\[-ig\left(A_0 + iA_1\right)(x) = V^+ \star \left(2\bar{\partial}V\right) = -z - \sum_{n=0}^{\infty} \sqrt{\frac{2n}{\theta}} g_{n+1}(x) .\]

Finally, one may consider the Lorentz gauge
\[\partial^\mu A_\mu = 0 .\]

In this gauge, however, there is no such localized solution to have the finite action. The above analysis suggests that the choice of the gauge fixing may allow a different class of solutions. One may look for a singular gauge transformation which transforms the localized finite action solution in a radial gauge into a pure gauged solution in a axial gauge. So far, we could not succeed in finding this transformation.

The gauge fixing dependence of the solution is not limited to STNC but the same problem also arises in the SSNC case (Euclidean case). Thus, it remains to be seen how the gauge fixing condition affects the Hilbert space on the NCFT in general.

7 Conclusion

We construct the unitary S-matrix for the space-time non-commutative abelian gauge theory and QED. The Feynman rule is presented, based on this S-matrix.

The gauge invariance of the STNC QED perturbation is explicitly checked to the lowest order using the Compton scattering process. In this paper we choose the Feynman gauge \(\lambda = 1\) to avoid the unnecessary complication. For other choice of gauge, one may accommodate the ghost to make the S-matrix gauge invariant. To prove the gauge symmetry to all orders of perturbation theory, one may refer to the BRST symmetry as done in SSNC QED [14].

Finally, the classical vacuum solution demonstrates that the Hilbert space may differ depending on the choice of gauge. As seen in the text, the axial gauge condition and the radial gauge condition give the different class of vacuum solutions. The relation between the two different Hilbert spaces is not clearly understood yet.

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