SCHWARZ REFLECTIONS AND THE TRICORN

SEUNG-YEOP LEE, MIKHAIL LYUBICH, NIKOLAI G. MAKAROV,
AND SABYASACHI MUKHERJEE

ABSTRACT. We continue our study of the family $S$ of Schwarz reflection maps with respect to a cardioid and a circle which was started in [LLMM18]. We prove that there is a natural combinatorial bijection between the geometrically finite maps of this family and those of the basilica limb of the Tricorn, which is the connectedness locus of quadratic anti-holomorphic polynomials. We also show that every geometrically finite map in $S$ arises as a conformal mating of a unique geometrically finite quadratic anti-holomorphic polynomial and a reflection map arising from the ideal triangle group. We then follow up with a combinatorial mating description for the \textit{periodically repelling} maps in $S$. Finally, we show that the locally connected topological model of the connectedness locus of $S$ is naturally homeomorphic to such a model of the basilica limb of the Tricorn.

CONTENTS

1. Introduction 2
2. Background on The Tricorn 8
2.1. Hyperbolic Components and Their Boundaries 10
2.2. Misiurewicz Parameters 19
2.3. Global Topological Structure 21
2.4. The Real Basilica Limb 24
3. Schwarz Reflection Maps, and The C&C Family 26
3.1. Quadrature Domains and Schwarz Reflections 26
3.2. The C&C Family 26
4. Tessellation of The Escape Locus of $S$ 26
5. Hyperbolic Components of $S$ 32
6. Combinatorial Rigidity of Geometrically Finite Maps 35
6.1. Combinatorics of Dynamical Rays: Orbit Portraits 35
6.2. Rigidity Theorems 40
7. Parameter Rays 41
7.1. Odd period Parabolics, and Period-doubling Bifurcations 44
7.2. Parameter Rays at Periodic Angles 48
7.3. Parameter Rays at Pre-periodic Angles 51
8. Combinatorial Straightening and Homeomorphism of Topological Models 53
8.1. Combinatorial Bijection for Hyperbolic and Parabolic Parameters 53
8.2. Combinatorial Bijection for Misiurewicz Parameters 56
8.3. Mating Description for Maps in $C(S)$ 57
9. Homeomorphism between Model Spaces 59
9. Discontinuity of Straightening 59

Date: December 5, 2018.
1. INTRODUCTION

A domain in the complex plane is called a quadrature domain if the Schwarz reflection map with respect to its boundary extends meromorphically to its interior. They first appeared in the work of Davis [Dav74], and independently in the work of Aharonov and Shapiro [AS73, AS78, AS76]. Since then, quadrature domains have played an important role in various areas of complex analysis and fluid dynamics [ASS99, CKS00, Dur83, EV92, GHMP00, Gus83, GV06, Ric72, Sak78, Sak82, Sak91, Sha92, SS00]. Moreover, topology of quadrature domains has important applications in physics, and leads to interesting classes of dynamical systems generated by Schwarz reflection maps.

It is known that except for a finite number of singular points (cusps and double points), the boundary of a quadrature domain consists of finitely many disjoint real analytic curves. Every non-singular boundary point has a neighborhood where the local reflection in $\partial \Omega$ is well-defined. The (global) Schwarz reflection $\sigma$ is an anti-holomorphic continuation of all such local reflections.

Round discs on the Riemann sphere are the simplest examples of quadrature domains. Their Schwarz reflections are just the usual circle reflections. Further examples can be constructed using univalent restrictions of polynomials or rational functions. Namely, if $\Omega$ is a simply connected domain and $\varphi : \mathbb{D} \to \Omega$ is a univalent map from the unit disc onto $\Omega$, then $\Omega$ is a quadrature domain if and only if $\varphi$ is a rational function. In this case, the Schwarz reflection $\sigma$ associated with $\Omega$ is semi-conjugate by $\varphi$ to reflection in the unit circle.

![Figure 1. The rational map $\varphi$ semi-conjugates the reflection map $1/\tau$ of $\mathbb{D}$ to the Schwarz reflection map $\sigma$ of $\Omega$.](image)

To a disjoint union of quadrature domains, one can associate a dynamical system generated by the corresponding Schwarz reflections. In [LM16], dynamical ideas were applied to the theory of quadrature domains with some physical implications. A systematic exploration of such dynamical systems was then launched in [LLMM18]. Two specific examples of Schwarz reflection maps (associated with simply connected quadrature domains) that appeared in [LLMM18] are Schwarz reflections of the interior of the cardioid curve and the exterior of the deltoid curve,

$$\left\{ \frac{z}{2} - \frac{z^2}{4} : |z| < 1 \right\} \quad \text{and} \quad \left\{ \frac{1}{z} + \frac{z^2}{2} : |z| < 1 \right\}.$$
One of the principal goals of that paper was to develop a general method of producing conformal matings between groups and anti-polynomials using Schwarz reflection maps. In particular, it was proved in [LLMM18 §5] that the Schwarz reflection map of the deltoid is a mating of the ideal triangle group and the anti-polynomial $z^2$.

Let us recall the C&C family which was introduced in [LLMM18] and is also the central object of this paper. We consider a fixed cardioid $\heartsuit$, and for each complex number $a$ (outside a slit), we consider the circle centered at $a$ circumscribing the cardioid (see Figure 2 and [LLMM18, Figure 20]). Let $r_a$ be the radius of the circumcircle, $T_a$ denote the closed disc minus the open cardioid (which we call a droplet for physical reasons, see [LLMM18 §1.2]), and $F_a$ denote the corresponding Schwarz reflection (the circle reflection $\sigma_a$ in its exterior, and the reflection $\sigma$ with respect to the cardioid in its interior). This family of Schwarz reflections maps $F_a$ is denoted by $S$ and is referred to as the C&C family.

The Schwarz reflection map $F_a$ is unicritical; indeed, the circle reflection map $\sigma_a$ is univalent, while the cardioid reflection map $\sigma$ has a unique critical point at the origin. Note that the droplet $T_a$ has two singular point on its boundary. Removing these two singular point from $T_a$, we obtain the fundamental tile. Recall that the non-escaping set of $F_a$ (denoted by $K_a$) consists of all points that do not escape to the fundamental tile under iterates of $F_a$, while the tiling set of $F_a$ (denoted by $T_a^\infty$) is the set of points that eventually escape to the fundamental tile. The connected components of $\text{int } K_a$ are called Fatou components. The boundary of the tiling set is called the limit set, and is denoted by $\Gamma_a$.

As in the case of quadratic polynomials, the non-escaping set of $F_a$ is connected if and only if it contains the unique critical point of $F_a$; i.e. the critical point does not escape to the fundamental tile. On the other hand, if the critical point escapes to the fundamental tile, the corresponding non-escaping set is totally disconnected. This leads to the notion of the connectedness locus $\mathcal{C}(S)$ as the set of parameters with connected non-escaping sets. Equivalently, $\mathcal{C}(S)$ is exactly the set of parameters for which the tiling set is unramified.

The study of the basic dynamical properties of the maps in $S$ and a closer investigation of the geometrically finite maps (in that family) led us to a combinatorial condition which guarantees a mating description for the post-critically finite (PCF for short) maps in $S$ (see [LLMM18 Proposition 8.1]). This was illustrated in [LLMM18] by a mating description for two explicit PCF maps in $S$.

The current paper has the dual objective of producing a topological model of the parameter space of the family $S$, and proving that every geometrically finite map in $S$ is a conformal mating of a unique geometrically finite quadratic anti-polynomial and the reflection map $\rho$ arising from the ideal triangle group (see [LLMM18 §3] for the definition of $\rho$).

**Theorem 1.1** (Geometrically finite Maps are Mating). *Every geometrically finite map in $S$ is a conformal mating of a unique geometrically finite quadratic anti-polynomial and the reflection map $\rho$.***

Let us mention in this respect that in the 1990s, Bullett and Penrose discovered holomorphic correspondences that are matings of quadratic holomorphic polynomials and the modular group [BP94]. More recently, Bullett and Lomonaco studied the

---

1See Proposition 8.8 for a combinatorial mating description for the periodically repelling maps in $\mathcal{C}(S)$. 

---
Figure 2. The circle \( \{ w - a = r_a \} \) is circumcircle to the cardioid. The dynamical plane of \( F_a \) is the disjoint union of the cardioid \( \heartsuit \) and the exterior disk \( \overline{B}(a, r_a) \).

The dynamical plane of \( F_a \) is the disjoint union of the cardioid \( \heartsuit \) and the exterior disk \( \overline{B}(a, r_a) \). The conclusion of Theorem 1.1 can be viewed as a similar phenomenon in the anti-holomorphic world, which produces an abundant supply of such examples.

The proof of Theorem 1.1 requires a thorough understanding of the relation between the geometrically finite maps in \( S \) and those in the basilica limb \( L \) of the Tricorn (see Subsection 2.4 for the definition of the basilica limb of the Tricorn). We establish such a relation via combinatorial models of the maps which we briefly describe below.

In usual holomorphic dynamics, the uniformization of the basin of infinity of an (anti-)polynomial extends continuously to the Julia set, provided that the Julia set is connected and locally connected. Similarly, the uniformization \( \psi_{ra} \) of the tiling set of \( F_a \) given in [LLMM18, Proposition 6.31] extends continuously to the limit set if the limit set is connected and locally connected (i.e. if \( F_a \) lies in the connectedness locus \( C(S) \)). This yields a topological model of the non-escaping set of \( F_a \) as the quotient of the closed unit disk by a geodesic lamination (analogous to the rational lamination of a polynomial).

In [LLMM18, §3], we constructed a topological conjugacy \( \mathcal{E} \) between the reflection map \( \rho \) (which models the external dynamics of the maps in \( S \)) and the anti-doubling map \( \theta \mapsto -2\theta \) on the circle (which models the external dynamics of quadratic anti-polynomials). The desired relation between the geometrically finite maps mentioned above is achieved by showing that \( \mathcal{E} \) induces a bijective correspondence between the laminations of geometrically finite maps in \( S \) and those of geometrically finite maps in \( L \).

**Theorem 1.2** (Combinatorial Bijection between Geometrically Finite Maps). There exists a natural bijection between the geometrically finite parameters in \( S \) and those
in $\mathcal{L}$ such that the laminations of the corresponding maps are related by $\mathcal{E}$ and the dynamics on the respective periodic Fatou components are conformally conjugate.

The above bijection is called “combinatorial straightening”, and is denoted by $\chi$. The existence of the map $\chi$ follows from well-known realization results in polynomial dynamics.

The fact that $\chi$ is injective depends on some crucial “rigidity” results, which state that geometrically finite maps in $\mathcal{S}$ are completely determined by their combinatorial models (or laminations) and suitable conformal invariants associated with them (see Subsection 6.2).

To prove surjectivity of $\chi$, we give a description of the combinatorial structure of the parameter space of $\mathcal{S}$. For every map $F_a$ with a disconnected limit set (i.e. when $a \notin \mathcal{C}(\mathcal{S})$), the conjugacy $\psi_a$ between $F_a$ and $\rho$ is defined on a suitable subset of the tiling set that contains the critical value $\infty$. Using the map $\psi_a$, we prove the following uniformization theorem that allows us to tessellate the exterior of the connectedness locus $\mathcal{C}(\mathcal{S})$ by dynamically meaningful tiles.

**Theorem 1.3** (Uniformization of The Escape Locus). The map

$$\Psi : \mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C}(\mathcal{S})) \to \mathbb{D}_2,$$

$$a \mapsto \psi_a(\infty)$$

is a homeomorphism.

**Figure 3.** The escape locus $\mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C}(\mathcal{S}))$ is tessellated by parameter tiles, a few of which are marked. The brown line stands for the slit $(-\infty, -1/12)$, and the connectedness locus $\mathcal{C}(\mathcal{S})$ is shown in blue.

Theorem 1.3 not only plays a key role in the proof of bijection between geometrically finite maps mentioned above, but also enables us to study the connectedness
locus $C(S)$ from outside. Indeed, we can use Theorem 1.3 to define “external tiling” and “external parameter rays” for $S$ (compare Figure 3). (Unlike in usual situations, these rays are not defined as pre-images of radial lines; see Definition 4.5 and [LLMM18, Definitions 3.2] for the precise definition.)

An analysis of the landing/accumulation properties of suitable parameter rays of $S$ combined with our knowledge of the corresponding situation for the Tricorn allows us to demonstrate that the lamination model of the connectedness locus of the C&C family precisely corresponds to that of the basilica limb of the Tricorn under the circle homeomorphism $E$ (see Figure 4 for pictures of the two parameter spaces in question). This proves that the locally connected models of the two connectedness loci are homeomorphic (see Subsection 2.4 for the definition of the abstract basilica limb $\tilde{L}$ of the Tricorn, and Subsection 8.4 for that of the abstract connectedness locus $\tilde{C}(S)$ of $S$).

Theorem 1.4 (Homeomorphism between Models). The map $E$ induces a homeomorphism between the abstract connectedness locus $\tilde{C}(S)$ of the family $S$ and the abstract basilica limb $\tilde{L}$ of the Tricorn.

Combining Theorem 1.3 and Theorem 1.4, one obtains a description of the parameter space of the C&C family as a “combinatorial mating” of the basilica limb of the tricorn with (a part of) the unit disk $D$ equipped with its tessellation arising from the ideal triangle group (compare [Dud11] for an analogous mating description of the parameter space of a certain family of quadratic rational maps).

The paper is organized as follows.

In Section 2 we give a detailed description of the dynamics of quadratic anti-polynomials and their connectedness locus, the Tricorn. Although many techniques used in the study of the Mandelbrot set can be adapted to investigate the Tricorn, lack of holomorphic parameter dependence adds complexity to the situation. Moreover, lack of quasi-conformal rigidity of parameters on the boundary of the Tricorn results in various topological subtleties. We discuss some of the essential topological differences between the Mandelbrot set and the Tricorn, and record all the results that we will need for the sequel.

Section 3 is a recapitulation of the definition of Schwarz reflection maps associated with quadrature domains and a description of the C&C family. We also recall the basic dynamically meaningful sets associated with Schwarz reflection maps in the C&C family and the corresponding objects in the parameter space. Some important
dynamical results about the maps in the C&C family (that were proved in [LLMM18]) are also collected here.

In Section 4, we begin our study of the parameter space of $S$. The main goal of this section is to prove Theorem 1.3, which states that the conformal position of the escaping critical value produces a homeomorphism between the escape locus (complement of $\mathcal{C}(S)$ in the parameter plane) and a suitable simply connected domain. The proof of Theorem 1.3 has some features in common with the proof of connectedness of the Mandelbrot set, but lack of holomorphic parameter dependence of the maps $F_a$ adds significant subtlety to the situation forcing us to adopt a more topological route. This theorem allows us to tessellate the exterior of the connectedness locus by dynamically meaningful tiles, which turns out to be a key ingredient in the rest of the paper.

In Section 5, we describe the structure of hyperbolic components of $\mathcal{C}(S)$. As in the case of the Tricorn, the hyperbolic components of odd period vastly differ from their even-period counterparts.

Section 6 concerns combinatorics of geometrically finite maps. In Subsection 6.1, we introduce an important combinatorial object called orbit portraits which, as in the polynomial case, describes the landing patterns of dynamical rays landing on a periodic orbit. Subsequently in Subsection 6.2, we prove a number of crucial rigidity statements (Theorems 6.12, 6.15) to the effect that PCF parameters in $S$ are uniquely determined by their combinatorial models (orbit portraits for centers of hyperbolic components, and laminations for Misiurewicz parameters). This subsection also contains some rigidity statements for hyperbolic and parabolic maps. The proofs are based on the “Pullback Argument”, and involve an analysis of the boundary behavior of conformal maps near cusps and double points.

In Section 7, we carry out a detailed study of the landing/accumulation properties of parameters rays of $S$ at (pre-)periodic angles (under $\rho$). This requires a complete combinatorial understanding of parabolic parameters of $\mathcal{C}(S)$. The odd period parabolic parameters of $\mathcal{C}(S)$ and the structure of bifurcations across such parameters are studied in Subsection 7.1. Subsection 7.2 contains a combinatorial realization result for parabolic parameters as landing/accumulation points of parameter rays at periodic angles. In Subsection 7.3, we investigate landing properties of parameter rays of $S$ at strictly pre-periodic angles. In particular, we characterize parameter rays landing at Misiurewicz parameters in terms of combinatorial properties of their dynamical planes. The results of this section play an important role in the proofs of our main theorems.

In Section 8, we define the combinatorial straightening map $\chi$ on all geometrically finite maps of $S$. More precisely, we send a geometrically finite map $F_a$ to the unique geometrically finite map of the Tricorn so that the homeomorphism $\mathcal{E}$ sends the lamination of the former to that of the latter, and the conformal conjugacy classes of the first return map of the characteristic Fatou components of the corresponding maps are the same. The fact that such a member of the Tricorn can be found follows from the combinatorial structure of the corresponding laminations, landing properties of external parameter rays of the Tricorn, and our understanding of the closures of the hyperbolic components.

Thanks to the combinatorial rigidity results proved in Subsection 6.2, the above combinatorial straightening map turns out to be injective.
We proceed to show that the straightening map is surjective onto all geometrically finite maps of the Tricorn. This amounts to finding geometrically finite maps in $C(S)$ with prescribed laminations and conformal data, and we achieve this “from outside”. More precisely, we use landing properties of external parameter rays of $S$ prepared in Subsection 7.3 to show that Misiurewicz maps in $S$ with prescribed lamination can be found as limit points of suitable parameter rays. To achieve the goal for hyperbolic parameters, we first realize parabolic parameters using results from Subsection 7.2. Since parabolic parameters lie on boundaries of hyperbolic components, this allows us to realize hyperbolic parameters by perturbing parabolic parameters inside hyperbolic components. This part of the argument involves a thorough understanding of odd periodic hyperbolic components of $C(S)$ and its bifurcation structure.

This yields our desired combinatorial bijection between the geometrically finite maps of $S$ and $L$, and completes the proof of Theorem 1.2. At this point, Theorem 1.1 is a straightforward consequence of Theorem 1.2 and a simple extension of [LLMM18, Proposition 8.1].

In Subsection 8.4, we construct a locally connected model for $C(S)$, and use the landing properties of parameter rays to complete the proof of Theorem 1.4.

The final Section 9 is devoted to a discussion of possible analytic improvements of the straightening map $\chi$. In fact, we show that the map $\chi$ has “built-in” discontinuities. It is worth mentioning that discontinuity of straightening maps is typical in anti-holomorphic dynamics, and is related to “non-universality” of certain conformal invariants (compare [IM16a, §9]).

To conclude, it is worth mentioning that there are several compelling reasons for adopting a combinatorial approach to describe the topology of the connectedness locus $C(S)$. The “external map” of $F_a$ is given by the map $\rho$, which is a two-to-one covering of the circle with three parabolic fixed points. On the other hand, the external map of a quadratic anti-polynomial is given by $\theta \mapsto -2\theta$, which is a two-to-one covering of the circle with three repelling fixed points. As there is no quasisymmetric conjugacy between a parabolic and a repelling fixed point, one cannot quasiconformally straighten $F_a$ to a quadratic anti-polynomial. In fact, there exists no (anti-)rational map of degree two with three parabolic fixed points (alternatively, there is no (anti-)Blaschke map with more than one parabolic fixed point). Consequently, maps in $S$ cannot be quasiconformally straightened to any family of rational maps.

In addition to the above obstacles, there are intrinsic issues with anti-holomorphic parameter spaces that make straightening maps less well-behaved (see Section 9 also compare [IM16a, Theorem 1.1]). Since the Tricorn is known to be non locally connected (with quite non-uniform wiggly features in various places), one needs to work with its locally connected topological model to develop a tractable theory. On the other hand, there are MLC-type questions of combinatorial rigidity for the Tricorn that are still open (compare [Lyu17, §38]), and go beyond the scope of the current work. Any progress in this direction would bring our topological models closer to the actual connectedness loci.

2. Background on The Tricorn

In this Section, we recall some known results on the dynamics of quadratic anti-polynomials, and their parameter space. The reason to include this fairly detailed
survey is twofold. Since the Schwarz reflection maps are anti-holomorphic and they depend only real-analytically (and not holomorphically) on the parameter, some of the purely holomorphic techniques used to study the Mandelbrot set fail to work in this setting. It is, therefore, not too surprising that the tools required to study the dynamics and parameter space of quadratic anti-polynomials find widespread applications in our study of the parameter space of Schwarz reflections. Secondly, some important topological features of the parameter space of anti-polynomials differ from their holomorphic counterpart. These differences serve as a mental guide in our analysis. Readers familiar with anti-holomorphic dynamics (or unwilling go through this lengthy exposition) may skip to Subsection 2.4 where the abstract basilica limb of the Tricorn is defined, and come back to this section whenever required.

Recall that in analogy to the holomorphic case, the set of all points which remain bounded under all iterations of \( f_c(z) = z^2 + c \) (for \( c \in \mathbb{C} \)) is called the filled Julia set \( K_c \). The boundary of the filled Julia set is defined to be the Julia set \( J_c \). This leads, as in the holomorphic case, to the notion of connectedness locus of quadratic anti-polynomials:

**Definition 2.1.** The Tricorn is defined as \( \mathcal{T} = \{ c \in \mathbb{C} : K_c \text{ is connected} \} \).

![Tricorn, the connectedness locus of quadratic anti-polynomials \( z^2 + c \).](image)

The Tricorn (see Figure 5) can be thought of as an object of intermediate complexity between one dimensional and higher dimensional parameter spaces. Combinatorially speaking, Douady’s famous ‘plough in the dynamical plane, and harvest in the parameter space’ principle continues to stand us in good stead since our maps are unicritical and our parameter space is still real two-dimensional. However, the iterates of a quadratic anti-polynomial \( f_c \) only depend real-analytically on the parameter \( c \) (unlike the iterates of \( z^2 + c \), which depend holomorphically on \( c \)). This results in significant topological differences between the Tricorn and the Mandelbrot set. Note that since the second iterate of \( f_c \) is \((z^2 + c)^2 + c\), the space of quadratic
anti-polynomials can be viewed as the real slice \( \{(a,b) \in \mathbb{C}^2 : a = b \} \) of the family of biquadratic polynomials \( \{(z^2 + a)^2 + b : a, b \in \mathbb{C}\} \). The polynomials \( (z^2 + a)^2 + b \) generically have two infinite critical orbits, much like cubic polynomials. Hence, the dynamics and parameter space of quadratic anti-polynomials also resemble in many respects the connectedness locus of cubic polynomials.

The following result was proved by Nakane [Nak93].

**Theorem 2.2 (Real-Analytic Uniformization).** The map \( \Phi : \mathbb{C} \setminus \mathcal{T} \to \mathbb{C} \setminus \overline{U} \), defined by \( c \mapsto \varphi_c(c) \) (where \( \varphi_c \) is the Böttcher coordinate near \( \infty \) for \( f_c \)) is a real-analytic diffeomorphism. In particular, the Tricorn is connected.

The previous theorem also allows us to define parameter rays of the Tricorn.

**Definition 2.3 (Parameter Ray).** The parameter ray at angle \( \theta \) of the Tricorn \( \mathcal{T} \), denoted by \( \mathcal{R}_\theta \), is defined as \( \{ \Phi^{-1}(r e^{2\pi i \theta}) : r > 1 \} \), where \( \Phi \) is the real-analytic diffeomorphism from the exterior of \( \mathcal{T} \) to the exterior of the closed unit disc in the complex plane constructed in Theorem 2.2.

### 2.1. Hyperbolic Components and Their Boundaries.

Recall that for an anti-holomorphic germ \( g \) fixing a point \( z_0 \), the quantity \( \frac{\partial g}{\partial z}|_{z_0} \) is called the *multiplier* of \( g \) at the fixed point \( z_0 \). One can use this definition to define multipliers of periodic orbits of anti-holomorphic maps (compare [Muk15a \S 1.1]). A cycle is called attracting (respectively, super-attracting or parabolic) if the associated multiplier has absolute value between 0 and 1 (respectively, is 0 or a root of unity). A map \( f_c \) is called hyperbolic if it has a (super-)attracting cycle. A hyperbolic component of \( \mathcal{T} \) is defined as a connected component of the set of all hyperbolic parameters.

#### 2.1.1. Uniformization of Hyperbolic Components.

Note that if \( c \) lies in a hyperbolic component of odd (respectively even) period of \( \mathcal{T} \), then the first return map of an attracting Fatou component of \( f_c \) is anti-holomorphic (respectively holomorphic). Due to this dichotomy, one needs to study the topology of odd and even period hyperbolic components of \( \mathcal{T} \) separately. The hyperbolic component of period 1 can be studied by explicit computation [NS03 Lemma 5.2] and is in some sense atypical. Hence we restrict our attention to higher period components, which is all we need in this paper.

Let \( H \) be a hyperbolic component of even period \( k \). For \( c \in H \), the \( k \)-periodic attracting cycle of \( f_c \) splits into two distinct attracting cycles of period \( \frac{k}{2} \) under \( f_c^{o2} \). These two attracting cycles of \( f_c^{o2} \) have complex conjugate multipliers. Let \( z_c \) be the attracting periodic point in the critical value Fatou component. We define \( \lambda_c := (f_c^{o2})'(z_c) \). The map \( c \mapsto (f_c^{o2})'(z_c) \) is called the *multiplier map* of the hyperbolic component of even period \( k \).

Now let \( H \) be a hyperbolic component of odd period \( k \) with center \( c_0 \). As before, for \( c \in H \), let \( z_c \) be the attracting periodic point of \( f_c \) contained in the critical value Fatou component \( U_c \). Let \( \text{Jac}(f_c^{o2}, z_c) \) be the Jacobian determinant of \( f_c^{o2} \) at \( z_c \). A simple computation shows that \( z_c \) is a periodic point of \( f_c^{o2} \) of period \( k \), and the associated multiplier

\[
(f_c^{o2})'(z_c) = -\text{Jac}(f_c^{o2}, z_c) = |\frac{\partial f_c^{o2}}{\partial z}(z_c)|^2
\]

is real and positive (compare [Muk15a \S 1.1]). Clearly, one has to work a bit harder to define a meaningful conformal invariant that uniformizes a hyperbolic component \( H \) of odd period. Unlike in the even period case, the natural conformal invariant
for maps with odd period attracting cycles is not a purely local quantity; it uses the conformal position of the orbit of the critical point. The following definition was introduced in [IM16a §6].

For \( c \in \mathcal{H} \setminus \{ \epsilon_0 \} \), there are two distinct critical orbits of the second iterate \( f_c^{2k} \) converging to an attracting cycle. One can choose two representatives of these two critical orbits (e.g. \( c \) and \( f_c^{-2k}(c) \)) in a fundamental domain (in the critical value Fatou component), and consider their ratio in a Koenigs linearizing coordinate. More precisely, let \( \kappa_c : U_c \to \mathbb{C} \) be a Koenigs linearizing coordinate for \( f_c^{2k} \) near \( z_c \); i.e. \( \kappa_c(f_c^{2k}(z)) = (f_c^{2k})'(z_c)\kappa_c(z) \) for all \( z \in U_c \). We define

\[
\zeta_H(c) := \frac{\kappa_c(f_c^{0k}(c))}{\kappa_c(c)}.
\]

Since a Koenigs linearizing coordinate is unique up to multiplication by a non-zero number, the above ratio is independent of the choice of \( \kappa_c \). At the center \( \epsilon_0 \), we define \( \zeta_H(\epsilon_0) = 0 \). The map \( \zeta_H \) is called the Koenigs ratio map of the hyperbolic component \( H \) of odd period \( k \).

Let \( H \) be a hyperbolic component of odd period \( k \neq 1 \) of \( \mathcal{T} \). For \( c \in \mathcal{H} \), the restriction of \( f_c^{2k} \) to the Fatou component \( U_c \) containing \( c \) is a degree 2 proper anti-holomorphic map. Moreover, \( f_c^{2k} \) has three fixed points on \( \partial U_c \). Exactly one of them is a cut point of the Julia set, this point is called the dynamical root point of \( f_c \) on \( \partial U_c \). Choosing a Riemann map of \( U_c \) that maps the attracting periodic point to 0 and the dynamical root point to 1, we obtain a conjugacy between \( f_c^{2k}|_{U_c} \) and an anti-holomorphic Blaschke product of degree 2 on \( \mathbb{D} \). By construction, such a Blaschke product must be of the form \( B_{a,\lambda}^-(z) = \lambda z^{-\frac{\pi}{2}(1-\pi)} \), with \( a \in \mathbb{D} \) and \( |\lambda| = 1 \) such that \( z = 1 \) is fixed by \( B_{a,\lambda}^- \). The unique such Blaschke product with a super-attracting fixed point is \( B_{a,\lambda}^0 \). Let \( \mathcal{B}^- \) be the space of all anti-holomorphic Blaschke products \( B_{a,\lambda}^- \) with \( a \in \mathbb{D} \) and \( |\lambda| = 1 \) such that \( z = 1 \) is fixed by \( B_{a,\lambda}^- \).

Now let \( H \) be a hyperbolic component of even period \( k \) of \( \mathcal{T} \). For \( c \in \mathcal{H} \), the restriction of \( f_c^{2k} \) to the Fatou component \( U_c \) containing \( c \) is a degree 2 proper holomorphic map. Moreover, \( f_c^{2k} \) has a unique fixed point on \( \partial U_c \). Choosing a Riemann map of \( U_c \) that maps the attracting periodic point to 0 and the unique boundary fixed point to 1, we obtain a conjugacy between \( f_c^{2k}|_{U_c} \) and a holomorphic Blaschke product of degree 2 on \( \mathbb{D} \). By construction, such a Blaschke product must be of the form \( B_{a,\lambda}^+(z) = \lambda z^{\frac{\pi}{2}(1-\pi)} \), with \( a \in \mathbb{D} \) and \( |\lambda| = 1 \) such that \( z = 1 \) is fixed by \( B_{a,\lambda}^+ \). The unique such Blaschke product with a super-attracting fixed point is \( B_{a,\lambda}^{0,1} \). Let \( \mathcal{B}^+ \) be the space of all holomorphic Blaschke products \( B_{a,\lambda}^+ \) with \( a \in \mathbb{D} \) and \( |\lambda| = 1 \) such that \( z = 1 \) is fixed by \( B_{a,\lambda}^+ \).

A direct calculation (or the Schwarz lemma) shows that 0 is necessarily an attracting fixed point for every Blaschke product in \( \mathcal{B}^\pm \). Clearly, both Blaschke product spaces \( \mathcal{B}^\pm \) are simply connected as their common parameter space is the open unit disc \( \mathbb{D} \). For both families of Blaschke products, we can define the multiplier/Koenigs ratio of the attracting fixed point. The next lemma elucidates the mapping properties of the multiplier/Koenigs ratio maps defined on \( \mathcal{B}^\pm \) [NS03 Lemma 5.4].

**Lemma 2.4.** The Blaschke product model spaces \( \mathcal{B}^\pm \) are simply connected. Moreover, the Koenigs ratio map (respectively, the multiplier map) of the attracting
fixed point defines a real-analytic 3-fold branched covering from $\mathcal{B}^-$ (respectively a real-analytic diffeomorphism from $\mathcal{B}^+$) onto $\mathbb{D}$.

The above discussion shows that we can associate a unique element of $\mathcal{B}^-$ (respectively $\mathcal{B}^+$) to every $f_c$ in an odd (respectively even) period hyperbolic component $H$. We thus have a map $\eta_H$ from $H$ to $\mathcal{B}^-$ or $\mathcal{B}^+$. The following theorem, which gives a dynamical uniformization of the hyperbolic components, was proved in [NS03, Theorem 5.6, Theorem 5.9].

**Theorem 2.5** (Uniformization of Hyperbolic Components). Let $H$ be a hyperbolic component. The map $\eta_H : H \to \mathcal{B}^-$ (respectively, $\mathcal{B}^+$) is a homeomorphism.

1. If $H$ is of odd period, then $\eta_H : H \to \mathcal{B}^-$ respects the Koenigs ratio of the attracting cycle. In particular, the Koenigs ratio map is a real-analytic 3-fold branched covering from $H$ onto the unit disk, ramified only over the origin.
2. If $H$ is of even period, then $\eta_H : H \to \mathcal{B}^+$ respects the multiplier of the attracting cycle. In particular, the multiplier map is a real-analytic diffeomorphism from $H$ onto the unit disk.

2.1.2. **Bifurcation from Even Period Hyperbolic Components.** We will now review some facts about neutral parameters and boundaries of hyperbolic components of the Tricorn. The following proposition states that every neutral (in particular, parabolic) parameter lies on the boundary of a hyperbolic component of the same period (see [MNS17, Theorem 2.1]).

**Proposition 2.6** (Neutral Parameters on Boundary). If $f_{c_0}(z) = z^2 + c_0$ has an neutral periodic point of period $k$, then every neighborhood of $c_0$ contains parameters with attracting periodic points of period $k$, so the parameter $c_0$ is on the boundary of a hyperbolic component of period $k$ of the Tricorn.

Moreover, every neighborhood of $c_0$ contains parameters for which all period $k$ orbits are repelling.

Using Theorem 2.5 one can define internal rays of hyperbolic components of $\mathcal{T}$. If $H$ is a hyperbolic component of even period, then all internal rays of $H$ land in [MI16, Lemma 2.9]. If $H$ does not bifurcate from a hyperbolic component of odd period, then the landing point of the internal ray at angle 0 is a parabolic parameter with an even-periodic parabolic cycle. This parameter is called the root of $H$.

The bifurcation structure of even period hyperbolic components of the Tricorn is analogous to that in the Mandelbrot set. The following theorem was proved in [MNS17, Theorem 1.1].

**Theorem 2.7** (Bifurcations from Even Periods). If a quadratic anti-polynomial $f_c$ has a $2k$-periodic cycle with multiplier $e^{2\pi ip/q}$ with $\gcd(p,q) = 1$, then $c$ sits on the boundary of a hyperbolic component of period $2kq$ of the Tricorn (and is the root thereof).

2.1.3. **Bifurcation from Odd Period Hyperbolic Components.** We now turn our attention to the odd period hyperbolic components of the Tricorn. One of the main features of anti-holomorphic parameter spaces is the existence of abundant parabolics. In particular, the boundaries of odd period hyperbolic components of the Tricorn consist only of parabolic parameters [MNS17, Lemma 2.5].

**Proposition 2.8** (Neutral Dynamics of Odd Period). The boundary of a hyperbolic component of odd period $k$ consists entirely of parameters having a parabolic orbit
of exact period $k$. In suitable local conformal coordinates, the $2k$-th iterate of such a map has the form $z \mapsto z + z^{q+1} + \ldots$ with $q \in \{1, 2\}$.

This leads to the following classification of odd periodic parabolic points.

**Definition 2.9 (Parabolic Cusps).** A parameter $c$ will be called a *cusp point* if it has a parabolic periodic point of odd period such that $q = 2$ in the previous proposition. Otherwise, it is called a *simple* parabolic parameter.

In holomorphic dynamics, the local dynamics in attracting petals of parabolic periodic points is well-understood: there is a local coordinate $\psi_{\text{att}}$ which conjugates the first-return dynamics to translation by $+1$ in a right half plane [Mil06, §10]. Such a coordinate $\psi_{\text{att}}$ is called a *Fatou coordinate*. Thus, the quotient of the petal by the dynamics is isomorphic to a bi-infinite cylinder, called the *Ecalle cylinder*. Note that Fatou coordinates are uniquely determined up to addition of a complex constant.

In anti-holomorphic dynamics, the situation is at the same time restricted and richer. Since the real eigenvalues of an anti-holomorphic map at a neutral fixed point are $\pm 1$, neutral dynamics of odd period is always parabolic. In particular, for a neutral periodic point of odd period $k$, the $2k$-th iterate is holomorphic with multiplier $+1$. On the other hand, additional structure is given by the anti-holomorphic intermediate iterate.

**Proposition 2.10 (Fatou Coordinates).** [HS14, Lemma 2.3] Suppose $z_0$ is a parabolic periodic point of odd period $k$ of $f_c$ with only one petal (i.e. $c$ is not a cusp), and $U$ is a periodic Fatou component with $z_0 \in \partial U$. Then there is an open subset $V \subset U$ with $z_0 \in \partial V$, and $f_c^{2k}(V) \subset V$ so that for every $z \in U$, there is an $n \in \mathbb{N}$ with $f_c^{nk}(z) \in V$. Moreover, there is a univalent map $\psi_{\text{att}}: V \to \mathbb{C}$ with $
abla(\psi_{\text{att}}(f_c^{nk}(z)) = \psi_{\text{att}}(z) + 1/2,$ and $\psi_{\text{att}}(V)$ contains a right half plane. This map $\psi_{\text{att}}$ is unique up to horizontal translation.

**Remark 1.** Note that the above proposition applies more generally to anti-holomorphic neutral periodic points such that the attracting petal(s) has (have) odd period.

The map $\psi_{\text{att}}$ will be called an *anti-holomorphic Fatou coordinate* for the petal $V$. The anti-holomorphic iterate interchanges both ends of the Ecalle cylinder, so it must fix one horizontal line around this cylinder (the *equator*). The change of coordinate has been so chosen that the equator is the projection of the real axis. We will call the vertical Fatou coordinate the *Ecalle height*. The Ecalle height vanishes precisely on the equator. Of course, the same can be done in the repelling petal as well. We will refer to the equator in the attracting (respectively repelling) petal as the attracting (respectively repelling) equator. The existence of this distinguished real line, or equivalently an intrinsic meaning to Ecalle height, is specific to anti-holomorphic maps.

The Ecalle height of the critical value plays a special role in anti-holomorphic dynamics. The next theorem, which is proved in [MNS17, Theorem 3.2], proves the existence of real-analytic arcs of simple parabolic parameters on the boundaries of odd period hyperbolic components of the Tricorn.

**Theorem 2.11 (Parabolic Arcs).** Let $\tilde{c}$ be a simple parabolic parameter of odd period. Then $\tilde{c}$ is on a parabolic arc in the following sense: there exists a real-analytic arc of simple parabolic parameters $c(h)$ (for $h \in \mathbb{R}$) with quasiconformally
equivalent but conformally distinct dynamics of which $\tilde{c}$ is an interior point, and the Ecalle height of the critical value of $f_{\tilde{c}(h)}$ is $h$.

The real-analytic arc of simple parabolic parameters constructed in the previous theorem is called a parabolic arc, and the real-analytic map $c : \mathbb{R} \to \mathcal{C}$ is called it critical Ecalle height parametrization.

Remark 2. It is worth mentioning that most of the topological differences between the Mandelbrot set and the Tricorn arise from the existence of quasiconformally conjugate parabolic parameters on the boundary of the Tricorn (while no two distinct parameters on the boundary of the Mandelbrot set are quasiconformally conjugate; compare Theorem 2.11 and [DH85, Chapter I, Proposition 7]). We do not know whether there are any non-trivial quasiconformal conjugacy classes on the boundary of the Tricorn other than odd period parabolic arcs. This question has connections with the “no invariant line fields” conjecture; in particular, non-existence of invariant line fields would imply that the parabolic arcs are the only non-trivial quasiconformal conjugacy classes on the boundary of $\mathcal{T}$.

Let $f : U \to \mathbb{C}$ be a holomorphic function on a connected open set $U \ (\subset \mathbb{C})$, and $\hat{z} \in U$ be an isolated fixed point of $f$. Then, the residue fixed point index of $f$ at $\hat{z}$ is defined to be the complex number

$$\iota(f, \hat{z}) = \frac{1}{2\pi i} \oint \frac{dz}{z - f(z)},$$

where we integrate in a small loop in the positive direction around $\hat{z}$. If the multiplier $\lambda := f'(\hat{z})$ is not equal to $+1$, then a simple computation shows that $\iota(f, \hat{z}) = 1/(1 - \lambda)$.

By the fixed point index of a periodic orbit of odd period of $f_c$, we will mean the holomorphic fixed point index of the second iterate $f_c^{\circ 2}$ at that periodic orbit.

Let $\mathcal{C}$ be a parabolic arc of odd period $k$ and $c : \mathbb{R} \to \mathcal{C}$ be its critical Ecalle height parametrization (compare Theorem 2.11). For any $h$ in $\mathbb{R}$, let us denote the residue fixed point index of the unique parabolic cycle of $f^{\circ 2}_{c(h)}$ by $\text{ind}_c(f^{\circ 2}_{c(h)})$. This defines a function

$$\text{ind}_c : \mathbb{R} \to \mathbb{C}, \ h \mapsto \text{ind}_c(f^{\circ 2}_{c(h)}).$$

Every parabolic arc limits at a parabolic cusp (of the same period) on each end. Moreover, in the dynamical plane of a parabolic cusp, the double parabolic points are formed by the merger of a simple parabolic point with a repelling point. The sum of the fixed point indices at the simple parabolic point and the repelling point converges to the fixed point index (which is necessarily a finite number) of the double parabolic point of the cusp parameter. This observation leads to the following asymptotic behavior of the parabolic fixed point index towards the ends of parabolic arcs (see [HS14, Proposition 3.7] for a proof).

**Proposition 2.12 (Fixed Point Index on Parabolic Arc).** The function $\text{ind}_c$ is real-valued and real-analytic. Moreover,

$$\lim_{h \to \pm \infty} \text{ind}_c(h) = +\infty.$$
Note that in the Mandelbrot set, bifurcation from one hyperbolic component to another occurs across a single point. The following theorem is one of the instances of the topological differences between the Mandelbrot set and the Tricorn [HS14 Theorem 3.8, Corollary 3.9], [IM16a Lemma 2.8] (see Figure 6).

**Figure 6.** Left: A hyperbolic component of even period bifurcating from another hyperbolic component of even period across a point. Right: A hyperbolic component of even period bifurcating from a hyperbolic component of odd period across arcs.

**Theorem 2.13 (Bifurcations Along Arcs).** Every parabolic arc of period $k$ intersects the boundary of a hyperbolic component of period $2k$ along an arc consisting of the set of parameters where the parabolic fixed point index is at least 1. In particular, every parabolic arc has, at both ends, an interval of positive length at which bifurcation from a hyperbolic component of odd period $k$ to a hyperbolic component of period $2k$ occurs.

Note that we have associated two important conformal invariants with odd period parabolic parameters; namely, the residue fixed point index of its parabolic cycle and the critical Ecalle height. There is no known explicit relation between these two invariants. However, some partial information is collected in the following proposition [IM16a Corollary 2.10].

We can assume without loss of generality that the set of parameters on $C$ across which bifurcation from $H$ to a hyperbolic component $H'$ (of period $2k$) occurs is precisely $c[h_0, +\infty)$; i.e. $C \cap \partial H' = c[h_0, +\infty)$.

**Proposition 2.14.** The function
\[
\text{ind}_C : [h_0, +\infty) \to [1, +\infty) \\
h \mapsto \text{ind}_C(f^{2c}(h))
\]
is strictly increasing, and hence a bijection. In particular, the bifurcating region $c[h_0, +\infty)$ can be parametrized by the fixed point index of the unique parabolic cycle.

Following [MNS17], we classify parabolic arcs into two types.

**Definition 2.15 (Root Arcs and Co-Root Arcs).** We call a parabolic arc a root arc if, in the dynamics of any parameter on this arc, the parabolic orbit disconnects the Julia set. Otherwise, we call it a co-root arc.
2.1.4. Orbit Portraits.

**Definition 2.16** (Characteristic Parabolic Point). Let \( f_c \) be a parabolic map. The **characteristic parabolic point** of \( f_c \) is the unique parabolic point on the boundary of the characteristic Fatou component of \( f_c \) (i.e. the Fatou component containing the critical value).

Orbit portraits were introduced by Goldberg and Milnor as a combinatorial tool to describe the patterns of all periodic dynamical rays landing on a periodic cycle of a complex quadratic polynomial \([\text{Go}92, \text{GM}93, \text{Mi}00]\). The usefulness of orbit portraits stems from the fact that these combinatorial objects contain substantial information on the connection between the dynamical and the parameter planes of the maps under consideration. Orbit portraits for quadratic anti-polynomials were studied in \([\text{Muk}15b]\).

**Definition 2.17** (Orbit Portraits). For a cycle \( O = \{z_1, z_2, \cdots, z_p\} \) of \( f_c \), let \( A_i \) be the set of angles of dynamical rays landing at \( z_i \). The collection \( \mathcal{P} = \{A_1, A_2, \cdots, A_p\} \) is called the **orbit portrait** associated with the orbit \( O \).

**Theorem 2.18** \([\text{Muk}15b, \text{Theorem 2.6}]\) Let \( f_c \) be a quadratic anti-polynomial, and \( O = \{z_1, z_2, \cdots, z_p\} \) be a periodic orbit such that at least one rational dynamical ray lands at some \( z_j \). Then the associated orbit portrait (which we assume to be non-trivial; i.e. \( |A_i| \geq 2 \)) \( \mathcal{P} = \{A_1, A_2, \cdots, A_p\} \) satisfies the following properties:

1. Each \( A_j \) is a finite non-empty subset of \( \mathbb{Q}/\mathbb{Z} \).
2. For each \( j \in \mathbb{Z}/p\mathbb{Z} \), the map \( m_{-2} \) maps \( A_j \) bijectively onto \( A_{j+1} \), and reverses their cyclic order.
3. For each \( i \neq j \), the sets \( A_i \) and \( A_j \) are unlinked.
4. Each \( \theta \in A_j \) is periodic under \( m_{-2} \), and there are four possibilities for their periods:
   a. If \( p \) is even, then all angles in \( \mathcal{P} \) have the same period \( rp \) for some \( r \geq 1 \).
   b. If \( p \) is odd, then one of the following three possibilities must be realized:
      i. \( |A_j| = 2 \), and both angles have period \( p \).
      ii. \( |A_j| = 2 \), and both angles have period \( 2p \).
      iii. \( |A_j| = 3 \); one angle has period \( p \), and the other two angles have period \( 2p \).

**Definition 2.19** (Formal Orbit Portraits under Anti-doubling). A finite collection \( \mathcal{P} = \{A_1, A_2, \cdots, A_p\} \) of non-empty finite subsets of \( \mathbb{Q}/\mathbb{Z} \) satisfying the conditions of Theorem 2.18 is called a **formal orbit portrait** under the anti-doubling map \( m_{-2} \) (in short, an \( m_{-2} \)-FOP).

By \([\text{Muk}15b, \text{Theorem 3.1}]\), every formal orbit portrait is realized by some \( f_c \).

**Theorem 2.20** (Realization of Orbit Portraits outside \( \mathcal{T} \)). Let \( \mathcal{P} = \{A_1, A_2, \cdots, A_p\} \) be a formal orbit portrait under the anti-doubling map \( m_{-2} \). Then there exists some \( c \in \mathbb{C} \setminus \mathcal{T} \), such that \( f_c \) has a repelling periodic orbit with associated orbit portrait \( \mathcal{P} \).

Among all the complementary arcs of the various \( A_j \), there is a unique one of minimum length. This shortest arc \( \mathcal{I}_\mathcal{P} \) is called the **characteristic arc** of the orbit portrait, and the two angles \( \{t^-, t^+\} \) at the ends of this arc are called its **characteristic angles**.
The following theorem will play an important role later in the paper.

**Theorem 2.21** (Realization of Orbit Portraits at Parabolic Parameters). Let $\mathcal{P} = \{A_1, A_2, \ldots, A_p\}$ be a formal orbit portrait under the anti-doubling map $m_{-2}$ with characteristic angles $t^-$ and $t^+.$

1) Suppose that $p$ is odd, and $t^\pm$ have period $2p.$ Then the parameter rays $R_{t^-}$ and $R_{t^+}$ accumulate on a common root parabolic arc $C$ such that for every parameter $c \in C,$ $f_c$ has a parabolic cycle of period $p$ and the orbit portrait associated with the parabolic cycle of $f_c$ is $\mathcal{P}.$

2) Suppose that $p$ is even. Then the parameter rays $R_{t^-}$ and $R_{t^+}$ land at a common parabolic parameter $c$ (whose parabolic cycle has period $p$) such that the orbit portrait associated with the parabolic cycle of $f_c$ is $\mathcal{P}.$

**Proof.** 1) By [Muk15b, Lemma 2.9], we have that $A_1 = \{t^-, t^+\},$ and hence $t^+ = (-2)^pt^-.$ It now follows from [M16b] Lemma 4.1 that the parameter rays $R_{t^-}$ and $R_{t^+}$ accumulate on a common root parabolic arc $C.$ Hence, in the dynamical plane of every $c \in C,$ the dynamical rays $R_c(t^-)$ and $R_c(t^+)$ land at the characteristic parabolic point. Finally, by [MNS17] Lemma 4.8, these are the only dynamical rays landing at the characteristic parabolic point of $f_c$ (for $c \in C$). This proves that for every parameter $c \in C,$ the map $f_c$ has a parabolic cycle with associated orbit portrait $\mathcal{P}.$

2) Arguing as in [M16b, Lemma 4.1], we can conclude that $R_{t^-}$ and $R_{t^+}$ either accumulate on a common root arc $C$ or land at a common parabolic parameter $c$ of even parabolic period.

We will first show that the former possibility cannot occur. For definiteness, we assume that $\{t^-, t^+\} \subseteq A_1.$ Let us suppose that $R_{t^-}$ and $R_{t^+}$ accumulate on a common root arc $C$ of period $k,$ and fix some $c' \in C.$ Then, the dynamical rays $R_{c'}(t^+)$ and $R_{c'}(t^-)$ land at the characteristic parabolic point of $f_{c'},$ which has odd period $k.$ It follows that $t^+ = (-2)^kt^-,$ and both these angles $t^\pm$ have period $2k.$ It is now easy to see that $p$ must divide $k$ (otherwise, $t^+$ would be contained in some $A_i$ different from $A_1$). But this is impossible as $p$ is even and $k$ is odd.

Therefore, the parameter rays $R_{t^-}$ and $R_{t^+}$ must land at a common parabolic parameter $c$ of even parabolic period. Then, the corresponding dynamical rays $R_{c}(t^+)$ and $R_{c}(t^-)$ land at the characteristic parabolic point of $f_c,$ which has even period. We denote the actual orbit portrait associated with the parabolic cycle of $f_c$ by $\mathcal{P}'.$ Since both the orbit portraits $\mathcal{P}$ and $\mathcal{P}'$ have even orbit period, it follows by [Muk15b, Lemma 3.3] that each of them is either primitive or satellite (compare [Mi100, Lemma 2.7]). The proof of [Mi100, Lemma 2.8] now applies verbatim to show that $\mathcal{P} = \mathcal{P}'.$ This completes the proof. \[\square\]

Let $H$ be a hyperbolic component of even period $k$ such that $H$ does not bifurcate from an odd period hyperbolic component. Let $A_1$ be the set of angles of the dynamical rays landing at the dynamical root of $f_c$ (where $c \in H$ or $c$ is the root point of $H$). Then, the first return map of the dynamical root either fixes every angle in $A_1$ and $|A_1| = 2,$ or permutes the angles in $A_1$ transitively. Moreover, the characteristic angles $t^-$ and $t^+$ of the orbit portrait $\mathcal{P}$ generated by $A_1$ are precisely the two adjacent angles in $A_1$ (with respect to circular order) that separate 0 from $c,$ and bound a sector of angular width less that $\frac{1}{2}$ of the root point of $H$ is the landing point of exactly two parameter rays at angles $t^-$ and $t^+.$
Let us now look at the connection between orbit portraits associated with parabolic parameters on the boundary of an odd period hyperbolic component $H$ and the angles of parameter rays accumulating on $\partial H$. Suppose that the period of $H$ is $k$ and its center is $c_0$. The first return map of the closure of the characteristic Fatou component of $c_0$ fixes exactly three points on its boundary. Only one of these fixed points disconnects the Julia set, and is the landing point of two distinct dynamical rays at $2k$-periodic angles. Let the set of the angles of these two rays be $S' = \{\alpha_1, \alpha_2\}$. Then, $\alpha_2 = (-2)^k \alpha_1$, and $S'$ is the set of characteristic angles of the corresponding orbit portrait. Each of the remaining two fixed points is the landing point of precisely one dynamical ray at a $k$-periodic angle; let the collection of the angles of these rays be $S = \{\theta_1, \theta_2\}$. We can, possibly after renumbering, assume that $0 < \alpha_1 < \theta_1 < \theta_2 < \alpha_2$ and $\alpha_2 - \alpha_1 < \frac{1}{2}$. Then, these angles satisfy the following relation (see [Muk15b, Lemma 3.5])

\[(2^k + 1)(\theta_1 - \alpha_1) = (\alpha_2 - \alpha_1) = (2^k + 1)(\alpha_2 - \theta_2).\]

Figure 7. Left: Parameter rays accumulating on the boundary of a hyperbolic component of period 5 of the Tricorn. Right: The corresponding dynamical rays landing on the boundary of the characteristic Fatou component in the dynamical plane of the center of the same hyperbolic component.

2.1.5. Boundaries Of Odd Period Hyperbolic Components. By [MNS17, Theorem 1.2], $\partial H$ is a simple closed curve consisting of three parabolic arcs, and the same number of cusp points such that every arc has two cusp points at its ends. Exactly one of these three parabolic arcs (say, $C_3$) is a root arc, and the parameter rays at angles $\alpha_1$ and $\alpha_2$ accumulate on this arc. The characteristic parabolic point in the dynamical plane of any parameter on this root arc is the landing point of precisely two dynamical rays at angles $\alpha_1$ and $\alpha_2$. The rest of the two parabolic arcs (say, $C_1$ and $C_2$) on $\partial H$ are co-root arcs. Each of these co-root arcs contains the accumulation set of exactly one parameter ray at an angle $\theta_i$, and the characteristic parabolic
point in the dynamical plane of any parameter on this co-root arc is the landing point of precisely one dynamical ray at angle \( \theta_i \) (compare Figure 7).

At the parabolic cusp on \( \partial H \) where \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) meet, the characteristic parabolic point is the landing point of exactly two dynamical rays at angles \( \theta_1 \) and \( \theta_2 \). The same is true at the center of the hyperbolic component of period 2 \( k \) that bifurcates from \( H \) across this parabolic cusp. Moreover, these angles are the characteristic angles of the corresponding orbit portrait.

On the other hand, at the parabolic cusp where \( \mathcal{C}_1 \) and \( \mathcal{C}_3 \) (respectively, \( \mathcal{C}_2 \) and \( \mathcal{C}_3 \)) meet, the characteristic parabolic point is the landing point of precisely three dynamical rays at angles \( \alpha_1, \alpha_2 \) and \( \theta_1 \) (respectively, \( \alpha_1, \alpha_2 \) and \( \theta_2 \)). As before, the same is true at the center of the hyperbolic component of period 2 \( k \) that bifurcates from \( H \) across this parabolic cusp. The characteristic angles of the corresponding orbit portrait are \( \alpha_1 \) and \( \theta_1 \) (respectively, \( \theta_2 \) and \( \alpha_2 \)).

**Theorem 2.22 (Boundaries Of Odd Period Hyperbolic Components).** The boundary of every hyperbolic component of odd period of \( T \) is a topological triangle having parabolic cusps as vertices and parabolic arcs as sides.

### 2.2. Misiurewicz Parameters

A *Misiurewicz* parameter of the Tricorn is a parameter \( c \) such that the critical point 0 is strictly pre-periodic. For a Misiurewicz parameter, the critical point eventually maps on a repelling cycle. By classification of Fatou components, the filled Julia set of such a map has empty interior. Moreover, the Julia set of a Misiurewicz parameter is locally connected [DH07, Exposé III, Proposition 4, Theorem 1], and has measure zero [DH07, Exposé V, Theorem 3].

**Theorem 2.23 (Parameter Rays Landing at Misiurewicz Parameters).** Every parameter ray of the Tricorn at a pre-periodic angle (under \( m-2 \)) lands at a Misiurewicz parameter such that in its dynamical plane, the corresponding dynamical ray lands at the critical value. Conversely, every Misiurewicz parameter \( c \) of the Tricorn is the landing point of a finite (non-zero) number of parameter rays at pre-periodic angles (under \( m-2 \)) such that the angles of these parameter rays are exactly the external angles of the dynamical rays that land at the critical value \( c \) in the dynamical plane of \( f_c \).

**Proof.** A proof of the corresponding results for the Mandelbrot set and the necessary modifications required to adapt the proof in the anti-holomorphic setting can be found in [Sch00, Theorem 1.1 (pre-periodic case)] and the remark thereafter.

Alternatively, see [GV19, Theorem 7.3] for the first part of the result (also compare [Lyu17, Theorem 37.35]). For the converse, let \( \mathcal{A} \) be the set of angles of dynamical rays landing at the critical value \( c \) of a Misiurewicz polynomial \( f_c \). Pick \( \theta \in \mathcal{A} \). If \( c' \) is the landing point of \( R_\theta \), then the dynamical ray \( R_{c'}(\theta) \) lands at the critical value \( c' \) of \( f_{c'} \). But then, the holomorphic polynomials \( f_{c'}^2 \) and \( f_c^2 \) have a common critical portrait in the sense of [PG09]. It now follows by [PG09, Theorem 1.1] that \( f_{c'}^2 = f_c^2 \); i.e. \( c = c' \). Therefore, for each \( \theta \in \mathcal{A} \), the parameter ray \( R_\theta \) lands at the Misiurewicz parameter \( c \). By the first part, no other parameter ray at a pre-periodic angle can land at \( c \).

Let \( c_0 \) be a Misiurewicz parameter, and \( \mathcal{A}' \) be the set of angles of the dynamical rays of \( f_{c_0} \) landing at the critical point 0. The set of angles of the dynamical rays that land at the critical value \( c_0 \) is then given by \( \mathcal{A} := m_{-2}(\mathcal{A}') \). Moreover, \( m_{-2} \) is two-to-one from \( \mathcal{A}' \) onto \( \mathcal{A} \). All other equivalence classes of \( \lambda(f_{c_0}) \) are mapped
bijectively onto its image class by $m_{-2}$. Note also that all angles in $A'$ are strictly pre-periodic. It is easy to see that the existence of a unique equivalence class (of $\lambda(f_{c_0})$) that maps two-to-one onto its image class under $m_{-2}$ characterizes the pre-periodic lamination of Misiurewicz maps. A formal rational lamination satisfying this condition is said to be of Misiurewicz type.

The next theorem shows that every formal rational lamination of Misiurewicz type is realized as the rational lamination of a unique Misiurewicz map $f_c$.

**Theorem 2.24 (Realization of Rational Laminations).** Every formal rational lamination of Misiurewicz type is realized as the rational lamination of a unique Misiurewicz map $f_c$ in $T$.

**Proof.** Let $\lambda$ be a formal rational lamination of Misiurewicz type. As $\lambda$ is of Misiurewicz type, there exists a unique $\lambda$-class $A'$ (consisting of strictly pre-periodic angles under $m_{-2}$) such that $m_{-2}$ maps $A'$ two-to-one onto $A := m_{-2}(A')$.

It is easy to see that $\lambda$ satisfies the properties of [Kiw01, Theorem 1.1] with $d = 4$, and hence, there exists a degree 4 holomorphic polynomial $P$ with associated rational lamination $\lambda$. Moreover, there are exactly three $\lambda$-classes on which $m_4$ (i.e. multiplication by 4 modulo one) acts in a two-to-one fashion. It follows that $P$ has three distinct simple critical points $\{\alpha_1, \alpha_2, \alpha_3\}$ such that $P(\alpha_1) = P(\alpha_2)$. By [Muk17, Lemma 3.1], $P$ is a biquadratic polynomial; i.e. $P(z) = (z^2 + a)^2 + b$, for some $a, b \in \mathbb{C}$, $a \neq 0$. Moreover, the critical points of $P$ are strictly pre-periodic.

Let $\theta \in A$, and $c \in T$ be the landing point of $R_\theta$. Then, the dynamical ray $R_c(\theta)$ lands at the critical value $c$ of $f_c$. It is now easy to see that the PCF holomorphic polynomials $P$ and $f_c^2$ have a common critical portrait in the sense of [Poi09]. Once again, [Poi09, Theorem 1.1] implies that $P = f_c^2$. Therefore, $\lambda(f_c) = \lambda$. The uniqueness statement follows by [Poi09, Theorem 1.1].

### 2.3. Global Topological Structure

There are various topological differences between the Mandelbrot set and the Tricorn. Here, we collect some of these important differences.

Note that the Mandelbrot set is conjectured to be locally connected. This is known in many cases; e.g. at most finitely renormalizable parameters with no non-repelling cycles [Hub93], parameters in embedded baby Mandelbrot sets satisfying the secondary limbs conditions [Lyu97], etc. The following theorem, which is in stark contrast to the corresponding situation for $M$, was first proved in [HS14, Theorem 6.2] and improved in [IM16a, Theorem 1.2] (see Figure 8).

**Theorem 2.25.** The Tricorn is not path connected. Moreover, no non-real hyperbolic component of odd period can be connected to the principal hyperbolic component by a path.

It should be mentioned that unlike the Mandelbrot set, not every (external) parameter ray of the Tricorn lands at a single point [IM16b] (see Figure 7).

**Theorem 2.26 (Non-Landing Parameter Rays).** The accumulation set of every parameter ray accumulating on the boundary of a hyperbolic component of odd period (except period one) of $T$ contains an arc of positive length. The fixed rays at angles 0, 1/3 and 2/3 land on the boundary of the principal hyperbolic component.

The next result, which was proved in [IM16b, §5], is also in contrast with the corresponding situation for $M$. 
Theorem 2.27 (Non-density of Misiurewicz Parameters). Misiurewicz parameters are not dense on the boundary of \( \mathcal{T} \). Indeed, there are points on the boundaries of the period 1 and period 3 hyperbolic components of \( \mathcal{T} \) that cannot be approximated by Misiurewicz parameters.

Non-density of Misiurewicz parameters on \( \partial \mathcal{T} \) can be clearly seen in Figure 9(left); for instance, the landing point of the parameter ray at angle 0 is not a limit point of Misiurewicz parameters.

Another salient difference between \( \mathcal{M} \) and \( \mathcal{T} \) is that the straightening map for “baby Tricorns” is always discontinuous [IM16a, Theorem 1.1]. The discontinuity phenomena is related to non-local connectivity and existence of quasiconformally conjugate parameters on the boundary of the Tricorn.

Theorem 2.28 (Discontinuity of Straightening in The Tricorn). Let \( c_0 \) be the center of a hyperbolic component \( H \) of odd period (other than 1) of \( \mathcal{T} \), and \( \mathcal{R}(c_0) \) be the corresponding \( c_0 \)-renormalization locus (i.e. the baby Tricorn based at \( H \)). Then the straightening map \( \chi_{c_0} : \mathcal{R}(c_0) \to \mathcal{T} \) is discontinuous at infinitely many parameters.

2.4. The Real Basilica Limb. Let us now define the real basilica limb of the Tricorn. Of course, one can give a more general definition of limbs, which can be found in [MNS17, §6]. Let us denote the hyperbolic component of period one of \( \mathcal{T} \) by \( H_0 \).

Definition 2.29 (Real Basilica Limb). The connected component of \( (\mathcal{T} \setminus \overline{H_0}) \cup \{-\frac{3}{4}\} \) intersecting the real line is called the real basilica limb of the Tricorn, and is denoted by \( \mathcal{L} \).

\( \mathcal{L} \) is precisely the set of parameters \( c \) in \( \mathcal{T} \) such that in the dynamical plane of \( f_c \), the rays \( R_c(1/3) \) and \( R_c(2/3) \) land at a common point (i.e. \( 1/3 \sim 2/3 \) in \( \lambda(f_c) \) for all \( c \in \mathcal{L} \)).

Recall that the parameter rays of the Tricorn at angles 1/3 and 2/3 are denoted by \( \mathcal{R}_{1/3} \) and \( \mathcal{R}_{2/3} \). Suppose that these two parameter rays land at the parabolic
Figure 9. Left: The real basilica limb of the Tricorn is shown. The sub-arcs of the parabolic arcs \( C_1 \) and \( C_2 \) that are contained in \( \mathcal{L} \) are marked in black. These two sub-arcs constitute \( \overline{\mathcal{L}} \setminus \mathcal{L} \). Right: A few identifications in the construction of the abstract basilica limb \( \tilde{\mathcal{L}} \).

arcs \( C_1 \) and \( C_2 \) respectively (compare [IM16] Lemma 3.1]). The closure of these two parabolic arcs intersect at \(-\frac{3}{4}\).

The real basilica limb \( \mathcal{L} \) is not compact. There are limit points of \( \mathcal{L} \) on the parabolic arcs \( C_1 \) and \( C_2 \) (of period one), but these points do not lie in \( \mathcal{L} \). Moreover, \( \overline{\mathcal{L}} \setminus \mathcal{L} \) is precisely the union of two sub-arcs of \( C_1 \) and \( C_2 \) (compare Figure 9(left)).

2.4.1. The Abstract Basilica Limb. We conclude this subsection with the construction of a locally connected model of the real basilica limb of the Tricorn. Let \( \gamma \) be the hyperbolic geodesic of \( \mathbb{D} \) connecting \( \frac{1}{3} \) and \( \frac{2}{3} \). We denote the connected component of \( \mathbb{D} \setminus \gamma \) not containing \( 0 \) by \( \mathbb{D}_2 \). The locally connected model of \( \mathcal{L} \) will be defined as the quotient of \( \mathbb{D}_2 \) under a suitable equivalence relation.

We will first construct an equivalence relation on \( \partial \mathbb{D} \cap \partial \mathbb{D}_2 \). We identify the angles of all rational parameter rays of \( \mathcal{T} \) that land at a common (parabolic or Misiurewicz) parameter or accumulate on a common root parabolic arc of \( \mathcal{L} \) (see Figure 9(right)). We also identify \( \frac{1}{3} \) and \( \frac{2}{3} \). This defines an equivalence relation on \( \mathbb{Q} / \mathbb{Z} \cap \partial \mathbb{D}_2 \). We then consider the smallest closed equivalence relation on \( \partial \mathbb{D} \cap \partial \mathbb{D}_2 \) generated by the above relation. Take the hyperbolic convex hull of each of these equivalence classes in \( \mathbb{D} \). This yields a geodesic lamination of \( \mathbb{D}_2 \) (by hyperbolic geodesics of \( \mathbb{D} \)). Finally, consider the quotient of \( \mathbb{D}_2 \) by collapsing each hyperbolic convex hull obtained above to a single point. The resulting continuum is called the abstract basilica limb \( \tilde{\mathcal{L}} \) (see [Lyu17, §9.4.2] for a general discussion on the construction of pinched disk models of planar continua).

We will now give a description of \( \tilde{\mathcal{L}} \) as a quotient space of \( \mathcal{L} \).

Definition 2.30 (Combinatorial Equivalence and Combinatorial Classes). i) Two parameters \( c \) and \( c' \) in \( \mathcal{L} \) are called combinatorially equivalent if \( f_c \) and \( f_{c'} \) have the same rational lamination.
ii) The combinatorial class $\text{Comb}(c)$ of $c \in \mathcal{L}$ is defined as the set of all parameters in $\mathcal{L}$ that are combinatorially equivalent to $c$.

iii) A combinatorial class $\text{Comb}(c)$ is called \textit{periodically repelling} if for every $c' \in \text{Comb}(c)$, each periodic orbit (excluding $\infty$) of the anti-polynomial $f_{c'}$ is repelling.

The following proposition gives a complete description of the non-repelling combinatorial classes of $\mathcal{L}$.

\textbf{Proposition 2.31 (Classification of Combinatorial Classes).} Every combinatorial class $\text{Comb}(c)$ of $\mathcal{L}$ is of one of the following four types.

1. $\text{Comb}(c)$ consists of an even period hyperbolic component (that does not bifurcate from an odd period hyperbolic component), its root point, and the irrationally neutral parameters on its boundary,
2. $\text{Comb}(c)$ consists of an even period hyperbolic component (that bifurcates from an odd period hyperbolic component), the unique parabolic cusp and the irrationally neutral parameters on its boundary,
3. $\text{Comb}(c)$ consists of an odd period hyperbolic component and the parabolic arcs on its boundary,
4. $\text{Comb}(c)$ is periodically repelling.

\textbf{Remark 3.} It is conjectured that every periodically repelling combinatorial class of $\mathcal{L}$ is a point. This is known in many cases; e.g. for all Misiurewicz parameters [Sch04], at most finitely renormalizable parameters with no non-repelling cycles [Hub93], parameters in embedded baby Mandelbrot sets satisfying the secondary limbs conditions [Lyu97], etc.

The \textit{abstract basilica limb} $\bar{\mathcal{L}}$ is obtained from $\mathcal{L}$ by

1. identifying all points in each periodically repelling combinatorial class of $\mathcal{L}$,
2. identifying all points in the non-bifurcating sub-arc of every parabolic arc of $\mathcal{L}$, and
3. identifying all points in $((\mathcal{L} \setminus \mathcal{L}) \cup \{-\frac{2}{3}\}$. 

We refer the readers to [NS03, HS14, MNS17, IM16b, IM16a] for a more comprehensive account of the combinatorics and topology of the Tricorn.

3. \textbf{Schwarz Reflection Maps, and The C&C Family}

Although we will deal with explicit quadrature domains and Schwarz reflection maps in this paper, we would like to remind the readers the general definitions of these objects. For a more detailed exposition on quadrature domains and Schwarz reflection maps, and their connection with various areas of complex analysis and statistical physics, we refer the readers to [LLMM18 §1, §4] and the references therein.

3.1. \textbf{Quadrature Domains and Schwarz Reflections.} We will denote the complex conjugation map on the Riemann sphere by $\iota$.

\textbf{Definition 3.1 (Schwarz Function).} Let $\Omega \subset \hat{\mathbb{C}}$ be a domain such that $\infty \notin \partial \Omega$ and $\text{int} \Omega = \Omega$. A \textit{Schwarz function} of $\Omega$ is a meromorphic extension of $\iota|_{\partial \Omega}$ to all of $\Omega$. More precisely, a continuous function $S : \bar{\Omega} \to \hat{\mathbb{C}}$ of $\Omega$ is called a Schwarz function of $\Omega$ if it satisfies the following two properties:
(1) \( S \) is meromorphic on \( \Omega \),
(2) \( S = \iota \) on \( \partial \Omega \).

It is easy to see from the definition that a Schwarz function of a domain (if it exists) is unique.

**Definition 3.2 (Quadrature Domains).** A domain \( \Omega \subsetneq \hat{\mathbb{C}} \) with \( \infty \notin \partial \Omega \) and \( \text{int} \overline{\Omega} = \Omega \) is called a quadrature domain if \( \Omega \) admits a Schwarz function.

Therefore, for a quadrature domain \( \Omega \), the map \( \sigma := \iota \circ S : \overline{\Omega} \to \hat{\mathbb{C}} \) is an anti-meromorphic extension of the Schwarz reflection map with respect to \( \partial \Omega \) (the reflection map fixes \( \partial \Omega \) pointwise). We will call \( \sigma \) the Schwarz reflection map of \( \Omega \).

Simply connected quadrature domains are of particular interest, and these admit a simple characterization (see AS76, Theorem 1).

**Proposition 3.3 (Simply Connected Quadrature Domains).** A simply connected domain \( \Omega \subsetneq \hat{\mathbb{C}} \) with \( \infty \notin \partial \Omega \) and \( \text{int} \overline{\Omega} = \Omega \) is a quadrature domain if and only if the Riemann map \( \varphi : \mathbb{D} \to \Omega \) is rational.

Proposition 3.3 immediately shows that the principal hyperbolic component \( \heartsuit \) of the Mandelbrot set (also called the main cardioid) is a quadrature domain. Indeed, it admits a polynomial Riemann map

\[
\varphi : \mathbb{D} \to \heartsuit
\]

\[
\varphi(\lambda) = \lambda/2 - \lambda^2/4.
\]

The Riemann map \( \varphi \) semi-conjugates the Schwarz reflection map \( \sigma \) of \( \heartsuit \) to the reflection map \( 1/z \) of the unit disk. This yields an explicit description of \( \sigma \).

\[
\sigma(\varphi(\lambda)) = \varphi(1/\lambda)
\]

(2) i.e. \( \sigma \left( \frac{\lambda}{2} - \frac{\lambda^2}{4} \right) = \left( \frac{2\lambda - 1}{4\lambda^2} \right) \)

for each \( \lambda \in \mathbb{D} \).

This allows us to study the basic mapping properties of the map \( \sigma \), see LLMM18 §6.1. In particular, \( \sigma \) has a unique critical point at 0.

3.2. The C&C Family. We are now ready to describe the main object of this paper, namely the C&C family. For any \( a \in \mathbb{C} \setminus (-\infty, -1/12) \), let \( B(a, r_a) \) be the smallest disk containing \( \heartsuit \) and centered at \( a \); i.e. \( \partial B(a, r_a) \) is the circumcircle to \( \heartsuit \) (the circle touches \( \partial \heartsuit \) at a unique point). Let \( \Omega_a := \heartsuit \cup B(a, r_a) \).

We now define our dynamical system \( F_a : \overline{\Omega_a} \to \hat{\mathbb{C}} \) as,

\[
w \mapsto \begin{cases} 
\sigma(w) & \text{if } w \in \overline{\heartsuit}, \\
\sigma_a(w) & \text{if } w \in B(a, r_a) \cap \heartsuit,
\end{cases}
\]

where \( \sigma \) is the Schwarz reflection of \( \heartsuit \), and \( \sigma_a \) is reflection with respect to the circle \( |w - a| = r_a \). It follows from our previous discussion that 0 is the only critical point of \( F_a \). We will call this family of maps \( S \); i.e.

\[
S := \left\{ F_a : \overline{\Omega_a} \to \hat{\mathbb{C}} : a \in \mathbb{C} \setminus (-\infty, -1/12) \right\}.
\]
Let \( T_a := \Omega_a^c = \overline{B(a, r_a)} \setminus \odot \) (which we call the droplet). Note that \( \partial T_a \) has two singular points; namely \( \alpha_a \) (a double point) and \( \frac{1}{4} \) (a cusp). Both of them are fixed points of \( F_a \). We define the fundamental tile (or desingularized droplet) of \( F_a \) as \( T_0^a := T_a \setminus \{\alpha_a, \frac{1}{4}\} \). Then, the restriction \( F_a : F_a^{-1}(T_0^a) \to T_0^a \) is a degree 3 covering.

**Figure 10.** Left: The tiles of rank 0 and 1 are labelled as \( T_0^a \) and \( T_1^a \) respectively. Right: Some of the initial tiles for a parameter \( a \) in the escape locus with \( n(a) = 1 \) are shown. A tile of rank 2 is ramified (i.e. it contains the critical point 0) and hence disconnects the non-escaping set \( K_a \). For such a parameter \( a \), the external conjugacy \( \psi_a \) extends conformally to all the tiles of the first rank (and all non-degenerate tiles of higher rank).

**Definition 3.4** (Tiling Set, Non-escaping set, and Limit set).

- For any \( k \geq 0 \), the connected components of \( F_a^{-k}(T_0^a) \) are called tiles (of \( F_a \)) of rank \( k \).
- The tiling set \( T_\infty^a \) of \( F_a \) is defined as the set of points in \( \Omega_a \) that eventually escape to \( T_0^a \); i.e. \( T_\infty^a = \bigcup_{k=0}^{\infty} F_a^{-k}(T_0^a) \). Equivalently, the tiling set is the union of all tiles.
- The non-escaping set \( K_a \) of \( F_a \) is the complement \( \mathcal{C} \setminus T_\infty^a \). Connected components of \( \text{int } K_a \) are called Fatou components of \( F_a \). All iterates of \( F_a \) are defined on \( K_a \).
- The boundary of \( T_\infty^a \) is called the limit set of \( F_a \), and is denoted by \( \Gamma_a \).

**Remark 4.** Note that the tiling set, non-escaping set, and limit set of \( F_a \) are the analogues of basin of infinity, filled Julia set, and Julia set (respectively) of a polynomial.

As in the case for polynomials, the tiling set \( T_\infty^a \) (respectively, the non-escaping set \( K_a \)) of \( F_a \) is open and connected [LLMM18, Proposition 6.14] (respectively, closed). By [LLMM18, Proposition 6.18], the non-escaping set of \( F_a \) is connected if and only if the critical point 0 does not escape to the fundamental tile \( T_0^a \) under iterates of \( F_a \). This leads to the definition of the connectedness locus of the family \( S \).
Definition 3.5 (Connectedness Locus and Escape Locus). The connectedness locus of the family $S$ is defined as
$$\mathcal{C}(S) = \{ a \in \mathbb{C} \setminus (-\infty, -1/12) : 0 \notin T_a^\infty \} = \{ a \in \mathbb{C} \setminus (-\infty, -1/12) : K_a is connected \}.$$ The complement of the connectedness locus in the parameter space is called the escape locus.

Note that for $a \in \mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C}(S))$, the critical value $\infty$ eventually escapes to the tile $T_a^0$. This leads to the following definition of depth for parameters in the escape locus (compare Figure 10(right)).

Definition 3.6 (Depth). For $a \in \mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C}(S))$, the smallest positive integer $n(a)$ such that $F_a^{\circ n(a)}(\infty) \in T_a^0$ is called the depth of $a$.

The dynamics of $F_a$ on the tiling set is modeled by a reflection map $\rho$ that is cooked up from the ideal triangle group (see [LLMM18, §3]). The map $\rho$ restricts to a $C^1$ double covering of $T$ (piecewise real-analytic with three parabolic fixed points), and is closely related to the anti-doubling map $m_{-2} : \theta \mapsto -2\theta$ on $\mathbb{R}/\mathbb{Z}$. In fact, there exists a homeomorphism $\mathcal{E}$ of the circle that conjugates $\rho$ to $m_{-2}$ [LLMM18, §3]. This topological conjugacy $\mathcal{E}$ will be important in more ways than one in the rest of the paper.

For $a \in \mathcal{C}(S)$, there is a dynamically defined Riemann map $\psi_a$ between the tiling set $T_a^\infty$ and the unit disk $\mathbb{D}$. The Riemann map $\psi_a$ conjugates $F_a$ to the reflection map $\rho$ [LLMM18, Proposition 6.31]. This “external” uniformization allows us to define dynamical rays of $F_a$, which in turn leads to the lamination model of the maps in the connectedness locus (see [LLMM18, Definition 6.37]).

For $a \notin \mathcal{C}(S)$, the non-escaping set is Cantor; i.e. the tiling set is infinitely connected [LLMM18, Proposition 6.40]. However, one can define a conjugacy $\psi_a$ between $F_a$ and $\rho$ on a subset of the tiling set consisting of tiles of low rank. In particular, the unique critical value $\infty$ (of $F_a$) is contained in the domain of definition of the conjugacy $\psi_a$ [LLMM18, Proposition 6.31].

Further dynamical properties of $F_a$ (including a classification of Fatou components) and some basic properties of the connectedness locus can be found in [LLMM18, §6]. We refer the readers to [LLMM18, §7, §8] for a study of some topological and analytic properties of the geometrically finite maps in $S$ (maps with an attracting/parabolic cycle or with strictly pre-periodic critical point) culminating in a mating description for two explicit PCF maps.

4. Tessellation of The Escape Locus of $S$

In this section, we will construct a homeomorphism from the escape locus of the family $S$ (see Definition 3.5) to a suitable simply connected domain. This will yield a dynamically defined tessellation of the exterior of the connectedness locus (in the spirit of a ray-equipotential structure of escape loci of polynomials). The proof will be reminiscent of the proof of connectedness of the Mandelbrot set, but lack of holomorphic parameter dependence will add some complexity to the situation.

In order to prove the main result of this subsection, we need an auxiliary lemma. Note that for $a \in (-\infty, -1/12)$, the smallest disk $B(a, r_a)$ containing $\blacktriangledown$ touches $\blacktriangledown$ at exactly two points; i.e. $\text{int}(B(a, r_a) \setminus \blacktriangledown)$ consists of two connected components [LLMM18, Proposition 6.7]. Exactly one of these two connected components is disjoint from the positive real axis, and we denote the closure of this component...
by \( K_a^- \). Note that \( K_a^- \cap \mathbb{R}_- = [q_a, -\frac{3}{4}] \), for some \( q_a < 0 \) \( \in \partial B(a, r_a) \) (compare Figure 11). As before, we denote the Schwarz reflection map with respect to \( \partial \) by \( \sigma \) and reflection with respect to \( \partial B(a, r_a) \) by \( \sigma_a \). Using these maps, we now define \( F_a \) (for \( a \in (-\infty, -1/12) \)) on \( [\infty, q_a] \cup \left[ -\frac{3}{4}, 0 \right] \) as

\[
 w \mapsto \begin{cases} 
 \sigma(w) & \text{if } w \in \left[ -\frac{3}{4}, 0 \right], \\
 \sigma_a(w) & \text{if } w \in [\infty, q_a]. 
\end{cases}
\]

**Lemma 4.1.** For all \( a \in (-\infty, -1/12) \), we have that \( F_a^{\infty}(\infty) \in \mathbb{R}_- \) whenever \( F_a^{\infty}(\infty) \) is defined.

**Proof.** This follows from the simple observation that \( F_a \left( \left[ -\frac{3}{4}, 0 \right] \right) = [\infty, -\frac{3}{4}] \subset \mathbb{R}_- \) and \( F_a \left( \left[ -\infty, q_a \right] \right) = [q_a, a] \subset \mathbb{R}_- \). \( \square \)

Note that for all \( a \in \mathbb{C} \setminus \left( (-\infty, -1/12) \cup \mathcal{C}(S) \right) \), the critical value \( \infty \) is contained in the domain of the conjugacy \( \psi_a \). We will now show that the conformal position of the critical value \( \infty \) yields the desired uniformization of the escape locus. In accordance with [LLMM18, §3], we denote the connected component of \( \mathbb{D} \setminus \Pi \) containing \( \rho_2(\Pi) \) by \( \mathbb{D}_2 \) (where \( \Pi \) is the ideal triangle in \( \mathbb{D} \) with vertices at 1, \( \omega \), and \( \omega^2 \); and \( \rho_2 \) is reflection in the side of \( \Pi \) connecting \( \omega \) and \( \omega^2 \)).

**Proof of Theorem 1.3.** First note that by [LLMM18, Proposition 6.21], \( (-\infty, -1/12) \cup \mathcal{C}(S) \) is a closed subset of the plane. So the complement of this set is open in \( \mathbb{C} \).

Since \( \infty \) lies outside \( \mathbb{D}(a, r_a) \), it follows that \( \Psi(a) := \psi_a(\infty) \in \mathbb{D}_2 \) for each \( a \in \mathbb{C} \setminus \left( (-\infty, -1/12) \cup \mathcal{C}(S) \right) \). More precisely, \( \Psi(a) \in T^{k_1, \ldots, k_n(a)} \) with \( k_1 = 2 \).
Note that as a runs over $\mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C} (S))$, the fundamental tile $T^0_a$ changes continuously, and hence so does the Riemann map $\psi_a$ restricted to $\bigcup_{k=0}^{n(a)} F^{-n}_a (T^0_a) \ni \infty$.

It follows that $\Psi (a) = \psi_a (\infty)$ depends continuously on $a$.

Our plan is to show that $\Psi$ is proper and locally invertible (compare $\text{Nak93}$ §3). From this, it will follow that $\Psi$ is a covering map onto the simply connected domain $\mathbb{D}_2$, and hence a homeomorphism.

**Lemma 4.2.** The map $\Psi$ is proper.

**Proof.** We will consider three different cases.

Let us first choose a sequence $\{a_k\} \subset \mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C} (S))$ such that $|a_k| \to \infty$. We can assume that each $a_k$ lies outside $\mathcal{C}$; i.e. $a_k \in \text{int} T^0_a$ and hence $|a_k|$ lies in some tile of first generation. Now consider the domain $U_k := \text{int} (T^0_a \cup F^{-1}_a (T^0_a))$, which is mapped biholomorphically onto $\text{int} (\Pi)$ by $\psi_a$. Since $U_k$ contains $a_k$ and $\infty$, it follows that $\psi_{a_k} (a_k)$ and $\psi_{a_k} (\infty)$ are contained in $\text{int} (\Pi)$ for each $k$. Moreover, the assumption $|a_k| \to \infty$ implies that $a_k$ is uniformly bounded away from $\partial \mathcal{C} \setminus \{a_0, a_1, \frac{1}{2}\}$ in the hyperbolic metric of $U_k$ for all $k$. Hence, $\psi_{a_k} (a_k)$ is uniformly bounded away from $\tilde{C}_1 \cup \tilde{C}_3$ in the hyperbolic metric of $\text{int} (\Pi)$ for all $k$. If $\psi_{a_k} (a_k)$ converges to $\partial \Pi \cap \partial \mathcal{D}$, then $\psi_{a_k} (\infty)$ converges to $\partial \mathcal{D}_2$ (as $\rho_2 (\psi_{a_k} (a_k)) = \psi_{a_k} (\infty)$). Otherwise, $\psi_{a_k} (a_k)$ and $\psi_{a_k} (\infty)$ are bounded away from the boundary of $\text{int} (\Pi)$. Since the spherical distance between $a_k$ and $\infty$ tends to 0 as $k$ increases, it follows that the hyperbolic distance of $\psi_{a_k} (a_k)$ and $\psi_{a_k} (\infty)$ (with respect to the hyperbolic metric of $\text{int} (\Pi)$) must also converge to 0. But this implies that both the sequences $\{\psi_{a_k} (a_k)\} \subset \mathbb{D}_2$ and $\{\psi_{a_k} (\infty)\} \subset \mathbb{D}_2$ accumulate on $\tilde{C}_2$. In either case, we conclude that $\{\Psi (a_k)\} \subset \mathbb{D}_2$.

Now let $\{a_k\} \subset \mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C} (S))$ be a sequence accumulating on $\mathcal{C} (S)$. Suppose that $\{\Psi (a_k)\} \subset \mathbb{D}_2$ converges to some $a \in \mathbb{D}_2$. Then, $\{\psi_{a_k} (\infty)\}$ is contained in a compact subset $\mathcal{C}$ of $\mathbb{D}_2$. After passing to a subsequence, we can assume that $\mathcal{C}$ is contained in a single tile of $\mathcal{D}$ (arising from $\mathcal{G}$). But this implies that each $a_k$ has a common depth $n_0$, and $\psi_{a_k} (F^n_{a_k} (\infty))$ is contained in the compact set $\rho^{n_0} (K) \subset \Pi$ for each $k$. Note that the map $F_a$, the fundamental tile $T^0_a$ as well as (the continuous extension of) the Riemann map $\psi_a : T^0_a \to \Pi$ change continuously with the parameter as $a$ runs over $\mathbb{C} \setminus (-\infty, -1/12)$. Therefore, for every accumulation point $a_0$ of $\{a_k\}$, the point $F^n_{a_0} (\infty)$ belongs to the compact set $\rho^{n_0} (K)$. In particular, the critical value of $F_{a_0}$ lies in the tiling set $T^n_{a_0}$. This contradicts the assumption that $\{a_k\}$ accumulates on $\mathcal{C} (S)$, and proves that $\{\Psi (a_k)\} \subset \mathbb{D}_2$.

Finally, let $\{a_k\}$ be a sequence of parameters converging to some parameter $a_0$ in $(-\infty, -1/12)$. Recall that $n(a_k)$ is the smallest positive integer such that $F^{n(a_k)}_{a_k} (\infty) \in T^0_{a_k}$. For $k$ sufficiently large, $F_{a_k}$ is a small perturbation of $F_{a_0}$. Hence by Lemma 4.1, we have that $\text{Re} (F^{n(a_k)}_{a_k} (\infty)) < -\frac{4}{3}$ and $\text{Im} (F^{n(a_k)}_{a_k} (\infty)) \approx 0$ (compare Figure 12). On the other hand, we can choose a point $x \in \text{hull} (\mathcal{C}) \setminus \bigcup_{k} \text{int} T^0_{a_k}$ and $\psi_{a_k} (x)$ is contained in a fixed compact subset of int $\Pi$ for all $k$. We now consider the domain $U_k := \text{int} (T^0_{a_k} \cup F^{-1}_{a_k} (T^0_{a_k}))$, which contains both $F^{n(a_k)}_{a_k} (\infty)$ and $x$. Observe that a part of the hyperbolic geodesic (in $U_k$) connecting $x$ and $F^{n(a_k)}_{a_k} (\infty)$ passes
Figure 12. For a non-real parameter $a_k$ sufficiently close to $(-\infty, -1/12)$, the fundamental tile $T_{a_k}^0$ of $F_{a_k}$ has a very narrow channel. Hence, a part of the hyperbolic geodesic of $U_k$ (the shaded domain in the figure) connecting $x$ and $F_{a_k}(\infty)$ lies close to the boundary of $U_k$. As a consequence, the hyperbolic length of this geodesic is very large.

through a very thin channel (whose thickness decreases as $k$ increases and gets pinched in the limit), and lies extremely close to the boundary of $U_k$ (compare Figure 12). Therefore, the hyperbolic distance (in $U_k$) between $x$ and $F_{a_k}(\infty)$ goes to $+\infty$ as $k$ tends to $+\infty$. Since $\psi_{a_k}$ is a conformal isomorphism between $U_k$ and $\text{int}(\Pi \cup \rho^{-1}(\Pi))$, it follows that the hyperbolic distance (in $\text{int}(\Pi \cup \rho^{-1}(\Pi))$) between $\psi_{a_k}(F_{a_k}^{\text{con}}(\infty))$ and $\psi_{a_k}(x)$ goes to $+\infty$ as $k$ tends to $+\infty$. Consequently, $\psi_{a_k}(F_{a_k}^{\text{con}}(\infty))$ converges to the boundary $\partial(\Pi \cup \rho^{-1}(\Pi))$ as $k$ tends to $+\infty$. But $\psi_{a_k}(F_{a_k}^{\text{con}}(\infty))$ is contained in $\Pi$. Therefore, $\psi_{a_k}(F_{a_k}^{\text{con}}(\infty))$ must converge to one of the (non-trivial) third roots of unity as $k$ tends to $+\infty$. In fact, Lemma 4.1 implies that for $k$ sufficiently large, each $\psi_{a_k}(F_{a_k}(\infty))$ ($1 \leq n \leq n(a_k)$) is close to one of the non-trivial third roots of unity, and hence the same is true for $\Psi(a_k)$. It follows that $\Psi(a_k)$ converges to one of the non-trivial third roots of unity (which lies on $\partial D \cap \partial D_2$) as $k$ tends to $+\infty$.

This completes the proof of the fact that $\Psi$ is a proper map. □

Lemma 4.3. The map $\Psi$ is locally invertible.

Proof. We will use a quasiconformal deformation/surgery argument to demonstrate local invertibility of $\Psi$. 
Let us first choose \( a_0 \in \mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C}(S)) \) such that \( \Psi(a_0) \) lies in the interior of some tile; i.e.

\[
u_0 := \psi_{a_0}(F_{a_0}^{\con}(a_0)(\infty)) = \rho^{\con(a_0)}(\psi_{a_0}(\infty)) \in \text{int } \Pi.
\]

Now fix an \( \varepsilon > 0 \) sufficiently small. Then, for any \( u \in B(u_0, \varepsilon) \), there exists a quasiconformal homeomorphism \( \varphi_u \) of \( \Pi \) such that \( \varphi_u(u_0) = u \), and the boundary extension of \( \varphi_u \) fixes \( 1, e^{2 \pi i/3} \) and \( e^{4 \pi i/3} \). Moreover, we can choose the maps \( \varphi_u \) such that they depend continuously on \( u \). Using the covering map \( \rho \), we can lift \( \varphi_u \)

to a quasiconformal homeomorphism of \( \bigcup_{n=0}^{n(a_0)-1} \rho^{-n}(\Pi) \) commuting with \( \rho \). Then \( \varphi_u \) defines a \( \rho \)-invariant Beltrami form on \( \bigcup_{n=0}^{n(a_0)-1} \rho^{-n}(\Pi) \).

Pulling this Beltrami form back by \( \psi_{a_0} \), we get an \( F_{a_0} \)-invariant Beltrami form \( \mu \) on \( \bigcup_{n=0}^{n(a_0)-1} F_{a_0}^{-n}(T_{a_0}^{a_n}) \). Since every point of \( T_{a_0}^{\infty} \) eventually maps to \( \bigcup_{n=0}^{n(a_0)-1} F_{a_0}^{-n}(T_{a_0}^{a_n}) \), we obtain an \( F_{a_0} \)-invariant Beltrami form on \( T_{a_0}^{\infty} \). Setting \( \mu = 0 \) on \( R_{a_0} \), we get an \( F_{a_0} \)-invariant Beltrami form on \( \hat{\mathbb{C}} \).

By [LLMM18, Lemma 6.24], there exists a quasiconformal homeomorphism \( \Phi_u \) on \( \hat{\mathbb{C}} \) that solves \( \mu \) and conjugates \( F_{a_0} \) to some map \( F_{a(u)} \) (and the maps \( \Phi_u \) depend continuously on \( u \)). By construction, the external conjugacy \( \psi_{a(u)} \) of \( F_{a(u)} \) is \( \varphi_u \circ \psi_{a_0} \circ \Phi_u^{-1} \). We therefore have \( \psi_{a(u)}(F_{a(u)}^{\con(a_0)}(\infty)) = u \), for all \( u \in B(u_0, \varepsilon) \). This proves that \( u \mapsto a(u) \) is a local continuous inverse branch of \( \Psi \) defined on \( B(u_0, \varepsilon) \). Hence, \( \Psi \) is locally invertible near \( \Psi(a_0) \).

Next let us choose \( a_0 \in \mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C}(S)) \) such that

\[
u_0 := \psi_{a_0}(F_{a_0}^{\con(a_0)}(\infty)) \in \partial \Pi \cap \mathbb{D} = \hat{C}_1 \cup \hat{C}_2 \cup \hat{C}_3.
\]

Using a quasiconformal surgery argument, we will construct parameters \( a \) near \( a_0 \) such that \( F_{a}^{\con(a_0)}(\infty) \) assumes any prescribed value near \( u_0 \). To do so, fix \( \varepsilon > 0 \) sufficiently small. By choosing a quasiconformal homeomorphism of \( \Pi \) that maps \( u_0 \) to a nearby point \( u \) on \( B(u_0, \varepsilon) \cap \partial \Pi \) (and fixes \( 1, e^{2 \pi i/3} \) and \( e^{4 \pi i/3} \)), we can easily adapt the proof of the previous case to obtain a map \( F_{a(u)} \) such that

\[
\psi_{a(u)}(F_{a(u)}^{\con(a_0)}(\infty)) = u.
\]

To construct maps \( F_{a(u)} \) such that \( \psi_{a(u)}(F_{a(u)}^{\con(a_0)}(\infty)) \) assumes values in \( B(U_0, \varepsilon) \setminus \partial \Pi \), we have to work a bit harder.

Let us choose a curve \( R \) (see Figure 13) that has double tangencies with the circular arcs \( \hat{C}_i \) at the third roots of unity. Let us denote by \( R \) the pre-image of \( R \) under the map \( \rho \). Then, \( R \) has double tangencies with the pre-images of \( \hat{C}_i \) under \( \rho \) (at the third roots of unity and their pre-images under \( \rho \)). Let \( \Pi' \) be the domain bounded by the curves \( R \). Under Möbius maps that send the third roots of unity to \( \infty \), the accesses of \( \Pi' \) to the third roots of unity map to curvilinear strips of width \( O(1) \). By [War42], the Riemann map from \( \Pi' \) onto \( \Pi \) is asymptotically linear near the cusps. Modifying the Riemann map from \( \Pi' \) to \( \Pi \), we can choose a quasiconformal homeomorphism \( \varphi_u \) from \( \Pi' \) onto \( \Pi \) such that
The inner and outer red curves \((R_i \text{ and } R_o \text{ respectively})\) form the thickening of \(\Pi\) and \(\Pi \cup \rho_1(\Pi) \cup \rho_2(\Pi) \cup \rho_3(\Pi)\) respectively. \(\rho\) maps the outer curve \(R_o\) onto the inner curve \(R_i\) as a double cover. 

\[ \varphi_u(u_0) = u \in B(u_0, \varepsilon) \cap \text{int } \Pi. \]

By construction, \(\varphi_u\) is asymptotically linear near the cusps.

Since \(\rho\) is a double cover from the outer red curves \(R_o\) onto the inner red curves \(R_i\), the map \(\varphi_u : R_i \to \partial \Pi\) lifts to \(R_o\) in an equivariant fashion. We will denote this lifted map by \(\varphi_u\) as well. Then, \(\varphi_u : R_o \to \rho^{-1}(\partial \Pi)\) is approximately linear near the cusps, and \(\varphi_u \circ \rho = \rho \circ \varphi_u\) on \(R_o\). Under Möbius maps that send the third roots of unity and their \(\rho\)-pre-images to \(\infty\), the accesses of the regions bounded by \(R_i\) and \(R_o\) to these points map to curvilinear strips of width \(O(1)\). Therefore, we can interpolate \(\varphi_u\) quasiconformally between \(R_i\) and \(R_o\). Abusing notation, let us call this interpolated map \(\varphi_u\).

Finally, we define a quasiregular map by setting it \(\psi_{a_0}^{-1} \circ \varphi_u^{-1} \circ \rho \circ \varphi_u \circ \psi_{a_0}\) in the region bounded by \(\psi_{a_0}^{-1}(R_o)\) and \(\psi_{a_0}^{-1}(R_i)\), and \(F_{a_0}\) on the complementary component of \(\psi_{a_0}^{-1}(R_o)\) not containing \(T_{a_0}^0\). Note that due to equivariance of \(\varphi_u\) on \(R_o\), these two definitions match on \(\psi_{a_0}^{-1}(R_o)\). Moreover, this newly defined map fixes \(\psi_{a_0}^{-1}(R_i)\) pointwise (intuitively, this means that for this quasiregular map, which models some map in our family, the role of the boundary of the fundamental tile is played by \(\psi_{a_0}^{-1}(R_i)\)). Now a standard argument involving the measurable Riemann mapping theorem and \([\text{LLMM18}, \text{Lemma 6.24}]\) provides us with a map \(F_{a(u)}\) with

\[ \psi_{a(u)}(F_{a(u)}^{\circ n(a_0)}(\infty)) = u \in B(u_0, \varepsilon) \cap \text{int } \Pi. \]

Finally, an opposite construction (thinning instead of thickening \(\Pi\)) yields maps \(F_{a(u)}\) with \(\psi_{a(u)}(F_{a(u)}^{\circ n(a_0)}(\infty)) = u \in B(u_0, \varepsilon) \setminus \Pi\). These constructions can now be combined to produce a continuous inverse branch \(a(u)\) of \(\Psi\) defined on \(B(u_0, \varepsilon)\). Hence, \(\Psi\) is locally invertible near \(\Psi(a_0)\). \(\square\)
The theorem now readily follows from the previous two lemmas.

As an immediate application of Theorem 1.3 we can define parameter tiles that yield a tessellation of the escape locus (see Figure 3).

**Definition 4.4 (Parameter Tiles).** For an \( M \)-admissible word \((i_1, \cdots, i_k)\) with \( i_1 = 2 \), the parameter tile \( T^{i_1, \cdots, i_k} \) is defined as
\[
T^{i_1, \cdots, i_k} := \Psi^{-1}(T^{i_1, \cdots, i_k}).
\]

Finally, we define external parameter rays of \( S \) via the map \( \Psi \) (see [LLMM18, Definitions 3.2] for the definition of \( G \)-rays).

**Definition 4.5 (Parameter Rays of \( S \)).** The pre-image of a \( G \)-ray at angle \( \theta \) (where \( \theta \in (1/3, 2/3) \)) under the map \( \Psi \) is called a \( \theta \)-parameter ray of \( S \).

**Remark 5.** It follows from the proof of properness of \( \Psi \) that every parameter ray of \( S \) accumulates on \( C(S) \); i.e. none of them accumulates on the slit \((-\infty, -1/12)\).

The set of all parameter rays of \( S \) form a binary tree (compare [LLMM18, Figure 10]). Note that if \( a \) lies on a parameter ray at angle \( \theta \), then in the dynamical plane of \( F_a \), the critical value \( \infty \) lies on a dynamical ray at angle \( \theta \). This duality will play an important role in the rest of the paper.

### 5. Hyperbolic Components of \( S \)

We now discuss the structure of hyperbolic parameters in \( C(S) \).

Since \( F_a \) depends real-analytically on \( a \), a straightforward application of the implicit function theorem shows that attracting periodic points can be locally continued as real-analytic functions of \( a \). Hence, the set of hyperbolic parameters form an open set. A connected component of the set of all hyperbolic parameters is called a hyperbolic component. It is easy to see that every hyperbolic component \( H \) has an associated positive integer \( n \) such that each parameter in \( H \) has an attracting cycle of period \( n \). We refer to such a component as a hyperbolic component of period \( n \).

A center of a hyperbolic component is a parameter \( a \) for which \( F_a \) has a super-attracting periodic cycle; i.e. the unique critical point \( 0 \) is periodic.

If \( F_a \) has an attracting periodic cycle, then the critical point \( 0 \) of \( F_a \) is attracted by the attracting cycle (see [LLMM18, Proposition 6.26]). Moreover, we can associate a dynamically defined conformal invariant to every hyperbolic map \( F_a \); namely multiplier if the attracting cycle (of \( F_a \)) has even period, and Koenigs ratio if the attracting cycle (of \( F_a \)) has odd period (see Subsection 2.1 for the corresponding definitions for anti-polynomials).

The hyperbolic components in \( C(S) \) are parametrized by the Blaschke product spaces \( B^{+} \), which model the first return map of the dynamics to the connected component of int(\( K_a \)) containing \( 0 \). The following theorem describes the topology and dynamical uniformizations of hyperbolic components in \( C(S) \).

**Theorem 5.1 (Dynamical Uniformization of Hyperbolic Components).** Let \( H \) be a hyperbolic component in \( C(S) \).

1. If \( H \) is of odd period, then there exists a homeomorphism \( \tilde{\eta}_H : H \to B^- \) that respects the Koenigs ratio of the attracting cycle. In particular, the Koenigs ratio map is a real-analytic 3-fold branched covering from \( H \) onto the open unit disk, ramified only over the origin.
(2) If $H$ is of even period, then there exists a homeomorphism $\tilde{\eta}_H : H \to B^+$ that respects the multiplier of the attracting cycle. In particular, the multiplier map is a real-analytic diffeomorphism from $H$ onto the open unit disk.

In both cases, $H$ is simply connected and has a unique center.

Proof. See [NS03, Theorem 5.6, Theorem 5.9] for a proof of the corresponding facts for quadratic anti-polynomials. It is straightforward to adapt the proof in our case. The main idea is to change the conformal dynamics of the first return map of a periodic Fatou component. More precisely, one can glue any Blaschke product belonging to the family $B^\pm$ in the connected component of $\text{int}(K_a)$ containing 0 by quasiconformal surgery. This gives the required homeomorphism between $H$ and $B^\pm$.

However, there is an important detail here. Since the original dynamics $F_a$ is modified only in a part of the connected component of $\text{int}(K_a)$ containing 0 (this is precisely where an iterate of $F_a$ is replaced by a Blaschke product), the resulting quasiregular modification $G_a$ shares some of the mapping properties of $F_a$. In particular, $G_a$ sends $\overline{B}(a, r_a)$ to $B(a, r_a)$ and maps $G_a^{-1}(\bigcirc)$ to $\bigcirc$ as a univalent map. Hence, we can adapt the proof of [LLMM18, Lemma 6.24] to show that $G_a$ is quasiconformally conjugate to some map $F_b$ in our family $S$. □

Using [LLMM18, Relation 7], it is easy to compute the centers of some low period hyperbolic components of $\mathcal{C}(S)$.

Examples. i) Since $F_a(0) = \infty$, there is no super-attracting map in $S$ with a fixed critical point. Hence, there is no hyperbolic component of period 1 in $\mathcal{C}(S)$.

![Figure 14](image)

**Figure 14.** Left: The hyperbolic component of period two (in blue) with its center 0 marked. Right: The part of the non-escaping set of $F_0$ inside the cardioid (in dark blue) with the critical point 0 marked.

ii) The unique parameter with a super-attracting 2-cycle in $\mathcal{C}(S)$ is $a = 0$. Indeed, the critical orbit of the map $F_0$ is given by $0 \leftrightarrow \infty$. It is not hard to see that the pullbacks of the leaf joining $1/3$ and $2/3$ (under $\rho$) generate the pre-periodic lamination of the corresponding non-escaping set $K_0$. Hence, its pre-periodic lamination is homeomorphic to the rational lamination of the Basilica anti-polynomial $z^2 - 1$. 
In particular, 0 is the center of the unique hyperbolic component of period two of $\mathcal{C}(S)$ (see Figure 14).

iii) The center of the unique period 3 hyperbolic component is $\frac{3}{16}$ (see Figure 15 and Figure 16). Its critical orbit is given by $0 \mapsto \infty \mapsto 3/16 \mapsto 0$.

Figure 15. Left: A part of the non-escaping set of $a = 3/16$. The corresponding map has a super-attracting 3-cycle. Right: A blow-up of the same non-escaping set around the Fatou component containing the critical point 0.

Similarly, the centers of the two period 4 hyperbolic components are $\frac{2}{5}$ (primitive) and $\frac{\sqrt{72} - 5}{36}$ (satellite).

We will conclude this subsection with a brief description of neutral parameters and boundaries of hyperbolic components of even period of $\mathcal{C}(S)$. The following proposition states that every neutral (in particular, parabolic) parameter lies on the boundary of a hyperbolic component of the same period.

**Proposition 5.2** (Neutral Parameters on Boundary). If $F_{a_0}$ has a neutral periodic point of period $k$, then every neighborhood of $a_0$ contains parameters with attracting periodic points of period $k$, so the parameter $a_0$ is on the boundary of a hyperbolic component of period $k$ of $\mathcal{C}(S)$.

*Proof.* See [MNS17, Theorem 2.1] for a proof in the Tricorn family. Since the proof given there only uses local dynamical properties of anti-holomorphic maps near neutral periodic points, it applies to the family $S$ as well. \hfill \Box

Using Theorem 5.1, one can define internal rays of hyperbolic components of $\mathcal{C}(S)$.

If $H$ is a hyperbolic component of even period, then the proof of [IM16a Lemma 2.9] can be adapted to show that all internal rays of $H$ at rational angles land (note that by Proposition 6.14, the accumulation set of such a ray is necessarily a finite set). If $H$ does not bifurcate from a hyperbolic component of odd period, then the landing point of the internal ray at angle 0 is a parabolic parameter with an even-periodic parabolic cycle. This parameter is called the *root* of $H$.
The next theorem describes the bifurcation structure of even period hyperbolic components of \( C(S) \). Once again, its proof in the Tricorn family is given in [MNS17, Theorem 1.1], which can be easily adapted for our setting.

**Theorem 5.3 (Bifurcations From Even Period Hyperbolic Components).** If \( F_a \) has a \( 2k \)-periodic cycle with multiplier \( e^{2\pi ip/q} \) with \( \gcd(p,q) = 1 \), then the parameter \( a \) sits on the boundary of a hyperbolic component of period \( 2kq \) (and is the root thereof) of \( C(S) \).

6. **Combinatorial Rigidity of Geometrically Finite Maps**

6.1. **Combinatorics of Dynamical Rays: Orbit Portraits.** We now delve into a combinatorial study of hyperbolic and parabolic maps in \( S \). The following landing property of dynamical rays for hyperbolic and parabolic maps in \( S \) follows from our topological preparation in [LLMM18, §7].

**Proposition 6.1 (Landing/bifurcation of Dynamical Rays).** 1) Let \( a \in C(S) \) and \( F_a \) be a hyperbolic or parabolic map. Then, every \( M \)-admissible sequence of tiles \( \{T_{a}^{1}, T_{a}^{1+2}, \ldots \} \) shrinks to a point on \( \Gamma_{a} \). In particular, every dynamical ray of \( F_a \) lands at some point of \( \Gamma_{a} \), and conversely.

2) If \( a \notin C(S) \), then every dynamical ray of \( F_a \) either bifurcates or lands at some point on \( K_a \).

**Proof.** 1) This follows from [LLMM18, Proposition 7.4] and Carathéodory’s theorem on boundary extension of Riemann maps.

2) This follows from [LLMM18, Proposition 6.40] and the fact that the accumulation set of a ray is connected. □

Clearly, if the dynamical ray of \( F_a \) at angle \( \theta \) lands at a point \( w \in \Gamma_{a} \), then the dynamical ray at angle \( \rho(\theta) \) lands at the point \( F_a(w) \).
6.1.1. **Hubbard Tree, Characteristic Angles, and Lamination.**

**Definition 6.2** (Characteristic Fatou Components and Parabolic Point). For a hyperbolic or parabolic map $F_a$, the Fatou component containing the critical value $\infty$ is called the characteristic Fatou component of $F_a$. If $F_a$ is parabolic, the parabolic periodic point on the boundary of the characteristic Fatou component is called the characteristic parabolic point of $F_a$.

Suppose that the period of the characteristic Fatou component $U_a$ of a hyperbolic (respectively, parabolic) map $F_a$ be $k$. Then, the sequence of iterates $\{F_a^{kn}\}$ forms a normal family on $U_a$. Hence, the $F_a^k$-orbit of every point in $U_a$ converges to the attracting (respectively, parabolic) periodic point in $U_a$ (respectively, on $\partial U_a$).

The characteristic Fatou component and the fixed points (of the first return map of the component) on its boundary will be vital in the rest of the section.

**Definition 6.3** (Roots and Co-Roots of Fatou Components). Let $F_a$ be a hyperbolic (respectively, parabolic) map, and $U_a$ be the characteristic Fatou component. Let $w$ be a boundary point of $U_a$ such that the first return map of $U_a$ fixes $w$. Then we call $w$ a dynamical root of $F_a$ if it is a cut-point of $K_a$; otherwise, we call it a dynamical co-root.

We now proceed to define Hubbard trees for super-attracting and parabolic maps.

For a super-attracting map $F_a$, the critical Fatou component $U$ (which has period $k$, say) admits a Riemann map that conjugates the first return map $F_a^k$ of $U$ to the map $z^2$ on $D$. Pre-images of radial lines in $D$ under this Riemann map are called geodesic rays in $U$. Pulling the geodesic rays in $U$ back by iterates of $F_a$, we obtain geodesic rays in all the Fatou components of $F_a$. Since the non-escaping set $K_a$ of a hyperbolic map $F_a$ is a locally connected full continuum [LLMM18, §7], it follows that there exists a unique arc in $K_a$ connecting any two points (of $K_a$) such that the intersection of the arc with every Fatou component is contained in the union of two geodesic rays (compare [DH07, Exposé II, §6, Proposition 6]). Such arcs are called allowable. The union of the allowable arcs connecting the post-critical set of $F_a$ is a tree, and we call it the Hubbard tree of the super-attracting map $F_a$.

For a parabolic map $F_a$, one can similarly define a tree in $K_a$ which connects the post-critical set and the parabolic cycle. Note that the tree obtained this way is not uniquely defined in the Fatou components, but they are all homotopic relative to the limit set. Such a tree is called a Hubbard tree of the parabolic map $F_a$ (also compare [HS14, Definition 5.4]).

In either case, the Hubbard tree is an $F_a$-invariant (up to homotopy relative to $\Gamma_a$ in the parabolic case) finite tree in $K_a$ that connects the post-critical orbit (and the parabolic cycle if $F_a$ is parabolic). We denote this tree by $H_a$. Following the arguments of [DH07, Exposé IV, §4, Proposition 4] (in the super-attracting case) or [Sch00, Lemma 3.5] (in the parabolic case), it is easy to see that the critical value $\infty$ is an endpoint of the tree, and every branch points of the tree is a pre-periodic repelling point.

One can now adapt the proof of [NS96, Lemma 3.4, Corollary 4.2] to deduce the following.

**Proposition 6.4.** Let $F_a$ be a super-attracting or parabolic map. Then, every dynamical co-root of $F_a$ has the same exact period as its Fatou component. It is the landing point of exactly one dynamic ray, and this ray has the same exact period as the component.
The characteristic Fatou component $U_a$ of $F_a$ has exactly one dynamical root. If the period of $U_a$ is even, then $U_a$ has no co-root; if the period is odd, it has exactly two co-roots. Moreover, if the period of $U_a$ is an odd integer $k$, then its dynamical root is the landing point of exactly two rays at $2k$-periodic angles.

The angles of the two adjacent rays landing at the dynamical root of $F_a$ (bounding a sector of angular width less that $\frac{1}{2}$) and separating the critical point 0 from the critical value $\infty$ are called the characteristic angles of $F_a$. The hyperbolic geodesic in $\mathbb{D}$ terminating at these two angles is referred to as the characteristic geodesic.

The characteristic angles of a hyperbolic or parabolic map $F_a$ will play a crucial role in the combinatorial study of the connectedness locus $\mathcal{C}(\mathcal{S})$. In particular, we will show that that the pre-periodic lamination of a hyperbolic or parabolic map $F_a$ (which yields a topological model of the map) can be recovered from its characteristic angles.

We start with a preliminary statement.

**Lemma 6.5.** Let $F_a$ be a super-attracting map, and $\mathcal{H}_a$ the Hubbard tree of $F_a$. Then, $\text{int } K_a \cap \mathcal{H}_a$ is dense in $\mathcal{H}_a$.

**Proof.** Let us denote the super-attracting cycle of $F_a$ by $0$.

**Claim:** The union of $\text{int } K_a \cap \mathcal{H}_a$ and the iterated pre-images of $\alpha_a$ on $\mathcal{H}_a$ is dense on $\mathcal{H}_a$.

**Proof of claim:** If it were false, then there would be an arc $\gamma_0 \in \mathcal{H}_a$ such that none of the images $F_a^n(\gamma_0)$ contains $\alpha_a$ or intersects the immediate basin of the critical point $0$.

We consider the hyperbolic metric on each of the two connected components of

$$V := \hat{\mathcal{C}} \setminus (T_a \cup 0),$$

and define the corresponding “hyperbolic metric” on each connected component $J$ of $\mathcal{H}_a \setminus \{0, \alpha_a\}$ as follows. For $x, y \in J$, a smooth path $\gamma$ connecting $x$ to $y$ in $V$ is called admissible if it can be retracted to $[x, y] \subset \mathcal{H}_a$ in $V$ relative to the end-points (clearly, $\gamma$ must be contained in a connected component of $V$). Then we let

$$d_{\text{hyp}}(x, y) := \inf_{\gamma} \ell(\gamma),$$

where $\gamma$ runs over all admissible paths connecting $x$ to $y$, and $\ell(\gamma)$ denotes hyperbolic length in $V$.

Note that if the images $x' = F_a(x)$ and $y' = F_a(y)$ also lie in the same component $J' \setminus \{0, \alpha_a\}$, then any admissible path $\gamma'$ connecting $x'$ to $y'$ lifts by $F_a$ to an admissible path $\gamma$ connecting $x$ to $y$. Moreover, $\gamma$ is contained in $\hat{\mathcal{C}} \setminus E_a \subset V$. It follows that $\ell_{\text{hyp}}(\gamma') > \ell_{\text{hyp}}(\gamma)$. Hence,

$$d_{\text{hyp}}(x', y') \geq \lambda d_{\text{hyp}}(x, y),$$

with $\lambda = \lambda(\varepsilon) > 1$ provided $x'$ and $y'$ stay $\varepsilon$-away from 0, $\frac{1}{4}$, and $\alpha_a$.

Note that by our assumption on $\gamma_0$, all iterates of $\gamma_0$ stay $\varepsilon$-away from 0. Moreover, no iterate of $\gamma_0$ contains $\alpha_a$. Since $\alpha_a$ repels nearby points in $K_a$, it follows that infinitely many iterates of $\gamma_0$ stay $\varepsilon$-away from $\alpha_a$. Finally, as $\gamma_0$ is contained in the Hubbard tree, all iterates of $\gamma_0$ are bounded $\varepsilon$-away from $\frac{1}{4}$ as well. It now follows By Relation 3 that $d_{\text{hyp}}(F_a^n(x), F_a^n(y)) \to \infty$, for $x, y \in \gamma_0$. But this is impossible as infinitely many iterates of $\gamma_0$ are uniformly bounded away from $\partial V$. This completes the proof of the claim.
To complete the proof of the lemma, it now suffices to argue that $\alpha_a$ is in the closure of $\text{int } K_a \cap \mathcal{H}_a$. We will assume the contrary, and arrive at a contradiction. To this end, let us choose a sufficiently small arc $I_0 \subset \mathcal{H}_a$ containing $\alpha_a$ and not intersecting $\text{int } K_a$, and define $I_n := F_a^{2n}(I_0)$. Since $\alpha_a$ repels nearby points in $K_a$ and $F_a^{2n}(n \geq 1)$ is injective on $I_0$, the family $\{I_n\}$ is a strictly increasing sequence of arcs in $\mathcal{H}_a$. In particular, the (right) end-points of $I_n$ form a strictly monotone sequence in the compact set $\mathcal{H}_a$, and hence must converge to an attracting fixed point of $F_a^{2}$. This is clearly impossible unless $a = 0$ (which is the only map with a super-attracting 2-cycle). Moreover, for $a = 0$, the above argument shows that the arcs $I_n$ come arbitrarily close to 0; i.e. $I_n \cap \text{int } K_a \neq \emptyset$ for $n$ large. As $\text{int } K_a$ is completely invariant, it follows that $I_0 \cap \text{int } K_a \neq \emptyset$ as well. \hfill \square

Proposition 6.6 (Characteristic Angles Determine Lamination). Let $a$ be a super-attracting parameter. Then, the iterated pre-images of the characteristic geodesic under $\rho$ are pairwise disjoint, and their closure in $\mathbb{Q}/\mathbb{Z}$ is the pre-periodic lamination $\lambda(F_a)$.

Proof. Let $\mathcal{S}_a$ be an arc in $K_a$ connecting $\frac{1}{4}$ to $\sigma_a^{-1}(\frac{1}{4})$ (it is defined up to homotopy in $\text{int } K_a$). We call $\mathcal{S}_a$ the spine of $F_a$.

Since $\rho|_S$ is expansive and $\Gamma_a$ is locally connected, the arguments of [Lyu17] Proposition 24.13 can be easily adapted for our setting to show that every cut-point of $\Gamma_a$ eventually falls on the spine of $F_a$. Moreover, every point on the spine eventually falls on the Hubbard tree $\mathcal{H}_a$ of $F_a$.

Therefore, we only need to show that every leaf of the lamination $\lambda(F_a)$ consisting of a pair of rays landing at a cut-point on the Hubbard tree can approximated by iterated pre-images of the characteristic leaf. We will proceed as in [Lyu17] Theorem 25.42.

We denote the critical Fatou component by $U_0$. Let us consider a leaf $L$ of the lamination $\lambda(F_a)$ comprising a pair of dynamical rays landing at the same cut point $w \in \mathcal{H}_a$ and bounding a minimal sector $S$ (so there are no other rays in this sector landing at $w$). By Proposition 6.5, $w$ can be approximated by components $U_{-k}$ of the basin $\text{int } K_a$ such that $F_a^{n+k}(U_{-k}) = U_0$. There are two possibilities:

1) $w \in \partial U_{-k}$ for some component $U_{-k} \subset S$. Iterating forward, we can assume that $w \in \partial U_0$. But since 0 is not a branch point of $\mathcal{H}_a$, the immediate basin $U_0$ intersects the Hubbard tree $\mathcal{H}_a$ at most at two points. Moreover, one of these two points $p$ is fixed under the first return map of $\overline{U_0}$, and the other an iterated pre-image of $p$. Hence it is sufficient to consider $w = p$. But then $L$ itself is the characteristic leaf.

2) There is a sequence of components $U_{-k} \subset S$ (such that $n_k \to \infty$) converging to $w$. Let $L_{-k}$ be the corresponding pre-image of the characteristic leaf (landing at iterated pre-image of $p$ on $\partial U_{-k}$). As $k \to \infty$, these curves converge to some curve $L_{-\infty}$ comprising two rays in $S$ landing at $w$. But $L$ is the only such curve, so $L_{-\infty} = L$.

In either case, $L$ is approximated by pre-images of the characteristic leaf. \hfill \square

6.1.2. Orbit Portraits. Following Definition 2.17, we now introduce the notion of orbit portraits which serves as a combinatorial tool to record the dynamics of these angles under $\rho$. 

Then the associated orbit portrait (which we assume to be non-trivial; i.e. $|A| \geq 2$) satisfies the following properties.

1. Each $A_j$ is a finite non-empty subset of $\text{Per}(\rho)$.
2. For each $j \in \mathbb{Z}/p\mathbb{Z}$, the map $\rho$ maps $A_j$ bijectively onto $A_{j+1}$, and reverses their cyclic order.
3. For each $i \neq j$, the sets $A_i$ and $A_j$ are unlinked.
4. Each $\theta \in A_j$ is periodic under $\rho$, and there are four possibilities for their periods:
   
   (a) If $p$ is even, then all angles in $P$ have the same period $rp$ for some $r \geq 1$.
   (b) If $p$ is odd, then one of the following three conditions holds:
      
      (i) $|A_j| = 2$, and both angles have period $p$.
      (ii) $|A_j| = 2p$, and both angles have period $2p$.
      (iii) $|A_j| = 3$; one angle has period $p$, and the other two angles have period $2p$.

Proof. Since $F_a$ is a unicritical anti-holomorphic map whose action on the angles of (landing) dynamical rays is given by $\rho : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, which is an orientation-reversing double covering with no attracting periodic point, the proof of [Muk15b Theorem 2.6] carries over verbatim to the present setting (also compare [ML10 Lemma 2.3]).

The following definition is analogous to that of formal orbit portraits under the anti-doubling map $m_{-2}$ (see Definition 2.19).

Definition 6.8 (Formal Orbit Portraits). A finite collection $P = \{A_1, A_2, \cdots, A_p\}$ of non-empty finite subsets of $\text{Per}(\rho)$ satisfying the conditions of Theorem 6.7 is called a formal orbit portrait under $\rho$ (in short, a $\rho$-FOP).

In [LLMM18 §3], we constructed a topological conjugacy $E : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ between the anti-doubling map $m_{-2}$ (which models the action of quadratic anti-polynomials on its external dynamical rays) and the reflection map $\rho$ (which models the action of $F_a$ on its external dynamical rays). Let $P = \{A_1, A_2, \cdots, A_p\}$ be an $m_{-2}$-FOP. We define the push-forward of $P$ under $E$ by

$$E_*(P) := \{E(A_1), E(A_2), \cdots, E(A_p)\}.$$ 

Similarly, we can define the pull-back $E^*(P)$ of a $\rho$-FOP $P$ under $E$.

The following proposition is a consequence of the fact that the maps $m_{-2}$ and $\rho$ are topologically conjugate via $E$. It is our first step in establishing a combinatorial bijection between the centers of $\mathcal{C}(S)$ and those of $\mathcal{L}$.

Proposition 6.9 ($E$ Preserves Orbit Portraits under $\rho$). If $P$ is an $m_{-2}$-FOP, then $E_*(P)$ is a $\rho$-FOP. Conversely, if $P$ is a $\rho$-FOP, then $E^*(P)$ is an $m_{-2}$-FOP.

Proposition 6.9 allows us to transfer combinatorial/topological results about $m_{-2}$-FOPs to corresponding results for $\rho$-FOPs. In particular, among all the
complementary arcs of the various $A_j$ of a $\rho$-FOP $\mathcal{P}$, there is a unique one of minimum length. This shortest arc $I_\mathcal{P}$ is called the characteristic arc of $\mathcal{P}$, and the two angles \{t_-, t_+\} at the ends of this arc are called its characteristic angles (compare [Muk15b, Lemma 3.2]). We can assume, without loss of generality, that \(0 < t_+ - t_- < \frac{1}{2}\).

6.2. Rigidity Theorems. This subsection will be devoted to some combinatorial rigidity results which show that PCF parameters and even-type parabolic parameters of $\mathcal{C}(S)$ are completely determined by their combinatorics (orbit portraits associated with dynamical root, or laminations).

We start with some preliminary results. Let $a_1, a_2 \in \mathcal{C}(S) \setminus \{-1/12\}$. We will denote the pre-image of $\alpha_i$ (under $\sigma$) that lies in $\heartsuit$ by $\alpha_i'$. Recall that by (see [LLMM18, Proposition 6.31]), $\psi_{a_2}^{a_1} := \psi_{a_2}^{-1} \circ \psi_{a_1} : T_{a_1}^{\infty} \to T_{a_2}^{\infty}$ is a conformal isomorphism. It restricts to a conformal isomorphism between $E_{a_1}^1$ and $E_{a_2}^1$, where $E_{a_1}$ is the union of the tiles of rank 0 and 1. It extends to a homeomorphism between $\overline{E_{a_1}^1}$ and $\overline{E_{a_2}^1}$ mapping $\frac{1}{4}$, $\alpha_{a_1}$, $\sigma_{a_1}^{-1}(\frac{1}{4})$, and $\alpha'_{a_1}$ to $\frac{1}{4}$, $\alpha_{a_2}$, $\sigma_{a_2}^{-1}(\frac{1}{4})$, and $\alpha'_{a_2}$ respectively (see Figure 17).

**Figure 17.** $E_{a_1}^1$ is the union of the tiles of rank 1 and 2; i.e. $E_{a_1}^1 = T_{a_1}^0 \cup T_{a_1}^1$. The map $\psi_{a_2}^{a_1}$ induces a homeomorphism between $\overline{E_{a_1}^1}$ and $\overline{E_{a_2}^1}$ mapping $\frac{1}{4}$, $\alpha_{a_1}$, $\sigma_{a_1}^{-1}(\frac{1}{4})$, and $\alpha'_{a_1}$ to $\frac{1}{4}$, $\alpha_{a_2}$, $\sigma_{a_2}^{-1}(\frac{1}{4})$, and $\alpha'_{a_2}$ respectively.

**Lemma 6.10.** $\psi_{a_2}^{a_1} : \overline{E_{a_1}^1} \to \overline{E_{a_2}^1}$ is asymptotically linear near $\frac{1}{4}$, $\alpha_{a_1}$, $\sigma_{a_1}^{-1}(\frac{1}{4})$, and $\alpha'_{a_1}$.

**Proof.** In [LLMMS18, Proposition 6.31], $\psi_{a_1}$ was first defined as (the homeomorphic extension of) a conformal isomorphism between $T_{a_1}$ and $\Pi$. Hence, $\psi_{a_2}^{a_1} : T_{a_1} \to T_{a_2}$ is a conformal isomorphism mapping $\frac{1}{4}$ and $\alpha_{a_1}$ to $\frac{1}{4}$ and $\alpha_{a_2}$ respectively. Moreover, on the tiles of first rank, $\psi_{a_2}^{a_1}$ is equal to $F_{a_2}^{-1} \circ (\psi_{a_2}^{a_1})|_{T_{a_1}} \circ F_{a_1}$ (choosing suitable inverse branches of $F_{a_2}$).

Since $a_1, a_2 \in \mathcal{C}(S) \setminus \{-1/12\}$, it follows from [LLMMS18, Proposition 6.10], Proposition 6.11] that the asymptotic developments of $F_{a_i}$ near $\frac{1}{4}$ and $\alpha_{a_i}$ are comparable. Moreover, $F_{a_1}$ is anti-conformal near $\sigma_{a_1}^{-1}(\frac{1}{4})$ and $\alpha_{a_2}'$. Therefore, to prove the lemma, it suffices to show that $\psi_{a_2}^{a_1} : T_{a_1} \to T_{a_2}$ is asymptotically linear near $\alpha_{a_1}$, and $\frac{1}{4}$.

Let us first prove the assertion for the access of $T_{a_1}$ to $\alpha_{a_1}$, that is mapped to the access of $\Pi$ to $\frac{1}{4}$ by $\psi_{a_1}$ (the argument for the other access to $\alpha_{a_1}$ is exactly the
same). By a Möbius map, we can send $\frac{1}{3}$ to $\infty$ such that the image of $\Pi$ (near $\frac{1}{3}$) is the horizontal strip $\{y \in \mathbb{C} : |y| < \pi/2\}$.

Let us denote the osculating circle to $\partial C_1$ at $\alpha_{a_1}$ by $C_{a_1}$. Since $a_1 \neq -1/12$, the curves $\partial B(a_1, r_{a_1})$ and $\partial C_1$ have a simple tangency at $\alpha_{a_1}$. Hence, the circle $C_{a_1}$ is different from $\partial B(a_1, r_{a_1})$. We can send $\alpha_{a_1}$ to $\infty$ by a Möbius map such that the images of $\partial B(a_1, r_{a_1})$ and $C_{a_1}$ are two horizontal straight lines $y = k_1$ and $y = k_2$ respectively ($k_1 \neq k_2$). Since $\partial C_1$ and $C_{a_1}$ have third order contact (but not fourth), the image of $\partial C_1$ (near $\alpha_{a_1}$) under the above Möbius map is a curve of the form $y = k_2 + k_3 + O(\frac{k_3}{x})$, for some $k_3 \neq 0$, and $x$ large enough. Thus, the image of the access of $T_{a_1}$ to $\alpha_{a_1}$ under consideration is mapped to a curvilinear strip bounded by $y = k_1$ and $y = k_2 + k_3 + O(\frac{1}{x})$.

$\psi_{a_1}$ induces a conformal map between the right halves of the above strips, it follows from [War42] that this map is asymptotically

$$z \mapsto k_4 + k_5 z + k_6 \ln(\text{Re}(z)) + o(1)$$

as $\text{Re}(z) \to +\infty$, where $k_5 = \pm \frac{\pi}{k_2 - k_1} \neq 0$.

Since we have the same asymptotic description for the Riemann map $\psi_{a_2} : T_{a_2} \to \Pi$ near $\alpha_{a_2}$ (with possibly different constants), it follows that $\psi_{a_2}^a : T_{a_1} \to T_{a_2}$ is asymptotically linear near $\alpha_{a_1}$.

We now proceed to prove the corresponding statement at $\frac{1}{2}$. By a Möbius map, we can send 1 to $\infty$ such that the image of $\Pi$ (near 1) is the horizontal strip $\{y \in \mathbb{C} : |y| < \pi/2\}$. Similarly, we can send $\frac{1}{2}$ to $\infty$ by a Möbius map such that the access of $T_{a_1}$ to $\frac{1}{4}$ maps to a curvilinear strip bounded by the real-analytic smooth curves $y = v_1(x) = -2(1 + x)^\frac{1}{2}$ and $y = v_2(x) = 2(1 + x)^\frac{1}{2}$. Then, $\psi_{a_1}$ induces a conformal map between the right halves of the above strips. Let us set $\theta_1(x) = v_2(x) - v_1(x) = 4(1 + x)^\frac{1}{2}$, and $\theta_2(x) = (v_2(x) + v_1(x))/2 = 0$. It now follows from [War42] that the conformal map $\psi_{a_1}$ from the curvilinear strip to the horizontal strip is

$$z = x + iy \mapsto k_7 + \pi \int_{x_0}^x \frac{1 + (\theta'_2(t))^2}{\theta_1(t)} dt + i\pi \frac{y - \theta_2(x)}{\theta_1(x)} + o(1) = O(z^2)$$

as $\text{Re} z \to +\infty$, where $k_7 \in \mathbb{R}$ and $x_0$ is a sufficiently large positive real number.

Evidently, we have the same asymptotic description for the Riemann map $\psi_{a_2} : T_{a_2} \to \Pi$ near $\frac{1}{4}$. It now follows that $\psi_{a_2}^a : T_{a_1} \to T_{a_2}$ is asymptotically linear near $\frac{1}{4}$.

We now use the above lemma to prove that the Riemann map $\psi_{a_1}^a : E_{a_1} \to E_{a_2}$ admits a quasiconformal extension to the entire Riemann sphere.

**Lemma 6.11.** Let us set $\psi$ equal to $\psi_{a_1}^a$ on $E_{a_1}$.

Proof. Let us set $\hat{\psi}$ equal to $\psi_{a_1}^a$ on $E_{a_1}$.

Note that $\hat{\psi}$ is a quasiconformal map on $E_{a_1}$.

We will demonstrate the existence of quasiconformal extensions of $\hat{\psi}$ to these three open sets separately, and then argue that the extended map is globally quasiconformal.

Let us first work with $\sigma^{-1}(B(a_1, r_{a_1}))$. Since $\partial \sigma^{-1}(B(a_1, r_{a_1}))$ is a smooth Jordan curve without any cusp, it is a quasicircle. We will show that $\hat{\psi}$ is a quasisymmetric map on $\partial \sigma^{-1}(B(a_1, r_{a_1}))$. Since $\hat{\psi}$ has a holomorphic extension
Remark with asymptotic linearity of $\hat{\psi}$.

Proof. The orbit portrait associated with the dynamical root of a super-attracting map completely determines the Hubbard tree of the maps, so the restriction of the maps on their respective Hubbard trees are topologically conjugate. Moreover,
there exist Böttcher maps conjugating \( F_{a_1} \) to \( F_{a_2} \) in a neighborhood of the super-attracting cycle.

Let us define a \( K \)-qc map \( \xi_0 \) of the sphere that agrees with \( \psi^{a_2} \) on \( E_{a_1}^1 \), and with the Böttcher conjugacies in a neighborhood \( U \) of the critical cycle. The existence of such a map is guaranteed by Lemma 6.11. Note that \( \psi_i^{a_2} \) conjugates \( F_{a_1} : \partial E_{a_1}^1 \to \partial T_{a_1}^0 \) to \( F_{a_2} : \partial E_{a_2}^1 \to \partial T_{a_2}^0 \) (both of which are double coverings). Moreover, \( F_{a_1} : \hat{C} \setminus \text{int} E_{a_1}^1 \to \hat{C} \setminus \text{int} T_{a_1}^0 \) is a two-to-one branched covering branched at 0. Since \( \xi_0 \) sends the critical value of \( F_{a_1} \) to that of \( F_{a_2} \), we can lift \( \xi_0 \) via \( F_{a_1} \) and \( F_{a_2} \) to obtain \( K \)-qc homeomorphism from \( \text{int} (\hat{C} \setminus E_{a_1}^1) \) to \( \text{int} (\hat{C} \setminus E_{a_2}^1) \) that matches continuously with \( \xi_0 \) on \( \partial E_{a_1}^1 \). By quasiconformal removability of analytic arcs, we obtain a \( K \)-qc map \( \xi_1 \) of the sphere. The lift \( \xi_1 \) becomes unique once we require \( \xi_1(\infty) = \infty \). Moreover, \( \xi_0 \) and \( \xi_1 \) agree on \( U \cup E_{a_1} \), so they are homotopic relative to the union of the super-attracting cycle and the singular points.

By iterating this lifting procedure and arguing as in [Lyu17, Lemma 38.6], we obtain a global \( K \)-qc map \( \xi \) that conjugates \( F_{a_1} \) to \( F_{a_2} \). Moreover, \( \xi \) agrees with \( \psi_i^{a_2} \) on \( T_{\infty}^1 \) and is conformal on \( \text{int} K_{a_1} \). Since \( \Gamma_{a_1} \) has measure zero (see [LLM18, Proposition 7.3]), it follows that \( \xi \) is conformal on the sphere. Note that since \( h \) fixes 0, \( \infty \), and \( \frac{1}{2} \), it must be the identity map. Therefore, \( \text{id}(F_{a_1}^{a_2}(0)) = F_{a_2}^{a_2}(0) \), i.e. \( a_1 = a_2 \).

An essentially similar argument combined with the description of hyperbolic components of \( C(S) \) given in Section 5 yields the following rigidity result for all hyperbolic maps.

**Proposition 6.13 (Rigidity of Hyperbolic Maps).** Let \( a_1 \) and \( a_2 \) be two hyperbolic parameters such that their dynamical roots have the same associated orbit portrait. Moreover, suppose that the first return map of their characteristic Fatou components are conformally conjugate. Then, \( a_1 = a_2 \).

Our next result states that parabolic parameters of even-type can also be recovered from their combinatorics.

**Proposition 6.14 (Rigidity of Parabolic Maps).** Let \( a_1 \) and \( a_2 \) be two parabolic parameters such that their characteristic Fatou components have even period. If their dynamical roots have the same associated orbit portrait, then \( a_1 = a_2 \).

**Proof.** Note that since the parabolic cycles of \( F_{a_1} \) and \( F_{a_2} \) have the same associated orbit portraits, their parabolic cycles have a common period \( k \) and a common combinatorial rotation number. Choose an attracting petal containing the critical value \( \infty \) and an attracting Fatou coordinate (in the characteristic Fatou component of \( F_{a_1} \)) that conjugates the first return map of the petal to the translation \( \zeta \mapsto \zeta + 1 \). Since Fatou coordinates are unique up to addition of a complex constant, we can arrange so that the critical values of \( F_{a_1} \) and \( F_{a_2} \) have the same image under the Fatou coordinates. Hence, the Fatou coordinates induce a conformal conjugacy between the first return maps of the petals that sends \( \infty \) to \( \infty \). Using \( F_{a_1} \) and \( F_{a_2} \), we now spread this conjugacy to suitable attracting petals in all the periodic Fatou components such that the domain of the conjugacy (which we denote by \( U \)) contains the entire post-critical set of \( F_{a_1} \).

We now construct a \( K \)-qc map \( \xi_0 \) of the sphere that agrees with \( \psi_i^{a_2} \) on \( E_{a_1}^1 \), and with the conjugacy on \( U \) constructed in the previous paragraph. Then \( \xi_0 \) lifts to a \( K \)-qc map \( \xi_1 \) of the sphere (normalized so that \( \xi_1(\infty) = \infty \)) agreeing with
\( \xi_0 \) on \( E_{a_1}^1 \cup U \). In particular, \( \xi_1 \) is homotopic to \( \xi_0 \) relative to the union of the post-critical set and the singular points. The rest of the proof is analogous to that of Proposition 6.12. \( \Box \)

**Remark 7.** We will see in Theorem 7.3 that parabolic parameters with odd-periodic characteristic Fatou components are not combinatorially rigid; i.e. they admit quasiconformal deformations that preserve combinatorics, and hence cannot be uniquely determined by their parabolic orbit portraits. However, we will prove a slightly weaker rigidity statement for such maps in Proposition 7.4.

We now prove a combinatorial rigidity principle for Misiurewicz parameters \( S \) to the effect that a Misiurewicz parameter is completely determined by its pre-periodic lamination. Since the limit set of a Misiurewicz map \( F_a \) is a dendrite [LLMM18, §7.2], there exists a unique arc (in \( \Gamma_a \)) connecting any two points of \( \Gamma_a \). The union of such arcs connecting the post-critical set of \( F_a \) is a tree, and we call it the *Hubbard tree* of a Misiurewicz map \( F_a \).

**Proposition 6.15** (Rigidity of Misiurewicz Parameters). *Let \( a_1 \) and \( a_2 \) be Misiurewicz parameters with the same pre-periodic lamination. Then, \( a_1 = a_2 \).*

**Proof.** Since \( F_{a_1} \) and \( F_{a_2} \) have the same pre-periodic lamination, the restriction of the maps on their Hubbard trees are topologically conjugate. In particular, their critical orbits have the same pre-period and period. Moreover, there exist quasiconformal maps defined on a neighborhood of the post-critical set of \( F_{a_1} \) conjugating \( F_{a_1} \) to \( F_{a_2} \). Let us now construct a K-qc map \( \psi \) of the sphere that agrees with \( \psi_{a_2} = \psi_{a_1}^{-1} \circ \psi_{a_1} \) on \( E_{a_1}^1 \), and with the quasiconformal conjugacies in a neighborhood of the post-critical set of \( F_{a_1} \). This is possible due to Lemma 6.11. Then \( \xi_0 \) lifts to a K-qc map \( \xi_1 \) of the sphere (normalized so that \( \xi_1(\infty) = \infty \)) agreeing with \( \xi_0 \) on the union of \( E_{a_1}^1 \) and some neighborhood of the post-critical set of \( F_{a_1} \). In particular, \( \xi_1 \) is homotopic to \( \xi_0 \) relative to the union of the post-critical set and the singular points.

By iterating this lifting procedure and arguing as in Proposition 6.12 we obtain a global K-qc map \( \xi \) that conjugates \( F_{a_1} \) to \( F_{a_2} \). Moreover, \( \xi \) agrees with \( \psi_{a_1} \) on \( T_{a_1}^\infty \). Since int \( K_{a_1} = \emptyset \), and \( \Gamma_{a_1} \) has measure zero [LLMM18 Propositions 7.6,7.8], it follows that \( \xi \) is conformal on the sphere. As \( h \) fixes 0, \( \infty \) and \( \frac{1}{4} \), it must be the identity map. Therefore, \( \text{id}(F_{a_1}(\infty)) = F_{a_2}(\infty) \); i.e. \( a_1 = a_2 \). \( \Box \)

**7. Parameter Rays**

In this section, we discuss landing/accumulation properties of parameter rays of \( S \) at (pre-)periodic angles.

In particular, we explore the connection between parabolic (respectively, Misiurewicz) parameters of \( C(S) \) and parameter rays at \( \rho \)-periodic (respectively, pre-periodic) angles. This, on the one hand, leads to a complete description of the boundaries of odd period hyperbolic components of \( C(S) \), and on the other hand, prepares the ground for the proofs of the main theorems of this paper.

**7.1. Odd period Parabolics, and Period-doubling Bifurcations.** We begin with a preliminary discussion of the boundaries of odd period hyperbolic components and period-doubling bifurcations associated with them. Let us first note that the boundaries of odd period hyperbolic components of \( C(S) \) consist only of parabolic parameters.
Proposition 7.1 (Neutral Dynamics of Odd Period). The boundary of a hyperbolic component of odd period \( k \) of \( \mathcal{C}(S) \) consists entirely of parameters having a parabolic orbit of exact period \( k \). In suitable local conformal coordinates, the \( 2k \)-th iterate of such a map has the form \( z \mapsto z + z^{q+1} + \ldots \) with \( q \in \{1, 2\} \).

Proof. See [MNS17, Lemma 2.5] for a proof in the case of unicritical anti-polynomials. The same proof applies to the family \( S \).

This leads to the following classification of parabolic points of odd period.

Definition 7.2 (Parabolic Cusps). A parameter \( a \) will be called a parabolic cusp if it has a parabolic periodic point of odd period such that \( q = 2 \) in the previous proposition. Otherwise, it is called a simple parabolic parameter.

Since every cycle of attracting petals of a parabolic map \( F_a \) attracts the forward orbit of 0, it follows that for a parabolic cusp with a \( k \)-periodic parabolic cycle, the period of the characteristic Fatou component is \( 2k \). Hence, Proposition 6.14 implies that there are only finitely parabolic cusps of a given period in \( \mathcal{C}(S) \).

Let us now fix a hyperbolic component \( H \) of odd period \( k \), and let \( a \in H \). Note that the first return map \( F_a^{2k} \) of a \( k \)-periodic Fatou component of \( F_a \) has precisely three fixed points (necessarily repelling) on the boundary of the component. As \( a \) tends to a simple parabolic parameter on the boundary \( \partial H \), the unique attracting periodic point of this Fatou component tends to merge with one of these three repelling periodic points. Similarly, as \( a \) tends to a parabolic cusp on the boundary \( \partial H \), the unique attracting periodic point of this Fatou component and two of the three boundary repelling periodic points merge together.

Now let \( a \) be a simple parabolic parameter of odd (parabolic) period \( k \). The holomorphic first return map \( F_a^{2k} \) of any attracting petal of \( F_a \) is conformally conjugate to translation by +1 in a right half-plane (see [Mil06, §10]). The conjugating map is called an attracting Fatou coordinate. Thus the quotient of the petal by the dynamics \( F_a^{2k} \) is isomorphic to a bi-infinite cylinder, called the attracting Ecalle cylinder. Note that Fatou coordinates are uniquely determined up to addition by a complex constant.

Since \( F_a^{2k} \) commutes with \( F_a^{2k} \), it follows that \( F_a^{2k} \) induces an anti-holomorphic involution of the attracting Ecalle cylinder \( \mathbb{C}/\mathbb{Z} \). Such a map must fix a horizontal round circle of \( \mathbb{C}/\mathbb{Z} \). By using one real additive degree of freedom of the Fatou coordinate, we can assume that this invariant circle is \( \mathbb{R}/\mathbb{Z} \). This special Fatou coordinate clearly conjugates the first anti-holomorphic return map \( F_a^{2k} \) of the attracting petal to the map \( \zeta \mapsto \zeta + \frac{1}{2} \) (compare Proposition 2.10). This coordinate is unique up to addition of a real constant. The pre-image of the real line (which is invariant under \( \zeta \mapsto \zeta + \frac{1}{2} \) under this Fatou coordinate is called the attracting equator. By construction, the attracting equator is invariant under the dynamics \( F_a^{2k} \).

The imaginary part of the image of the critical value \( \infty \) (whose forward orbit converges to the parabolic cycle, by [LLMM18, Proposition 6.26]) under this special Fatou coordinate is called the critical Ecalle height of \( F_a \) (since this Fatou coordinate is unique up to addition of a real constant, the real part of the image of \( \infty \) under this coordinate is not well-defined). It is easy to see that critical Ecalle height is a conformal conjugacy invariant of simple parabolic parameters of odd period. One can change the critical Ecalle height of simple parabolic parameters by a quasiconformal
deformation argument to obtain real-analytic arcs of parabolic parameters on the boundaries of odd period hyperbolic components.

**Theorem 7.3 (Parabolic Arcs).** Let \( \tilde{a} \) be a simple parabolic parameter of odd period. Then \( \tilde{a} \) is on a parabolic arc in the following sense: there exists a real-analytic arc of simple parabolic parameters \( a(h) \) (for \( h \in \mathbb{R} \)) with quasiconformally equivalent but conformally distinct dynamics of which \( \tilde{a} \) is an interior point, and the Ecalle height of the critical value of \( F_{a(h)} \) is \( h \). This arc is called a parabolic arc.

**Proof.** See [MNS17, Theorem 3.2] for a proof in the case of unicritical anti-polynomials. One essentially uses the same deformation in the attracting petals, and [LLMM18, Lemma 6.24] guarantees that the quasiconformal deformations of \( F_{\tilde{a}} \) also lie in the family \( S \).

The deformation of complex structure giving rise to parabolic arcs is supported on the basin of attraction of the parabolic cycle. Hence, the quasiconformal conjugacy (constructed in Theorem 7.3) between any two maps on the same parabolic arc is conformal on the tiling set. It follows that along a parabolic arc, the angles of the dynamical rays landing at the parabolic cycle remain constant. In other words, the orbit portrait associated with the parabolic cycle remains constant on a parabolic arc, so simple parabolic parameters of odd period of \( C \) orbit portrait associated with the parabolic cycle remains constant on a parabolic arc, so simple parabolic parameters of odd period of \( C \) are not combinatorially rigid. However, the next proposition shows that they can be uniquely determined by a combination of combinatorial and analytic data.

**Proposition 7.4 (Weak Rigidity of Odd Period Parabolics).** Let \( a_1 \) and \( a_2 \) be two simple parabolic parameters of odd period such that their parabolic cycles have the same orbit portrait. Then, they lie on the same parabolic arc. Moreover, if they have equal critical Ecalle height, then \( a_1 = a_2 \).

**Proof.** Suppose that the critical Ecalle heights of \( F_{a_1} \) and \( F_{a_2} \) be \( h_1 \) and \( h_2 \) respectively. Using the deformation constructed in [MNS17, Theorem 3.2], we obtain a quasiconformal conjugacy \( \phi_{a_2}^{a_1} \) between the first return maps of \( F_{a_1} \) on the attracting petals such that the conjugacy preserves the post-critical set. Let us denote the union of attracting petals (containing the post-critical set) where the conjugacy is defined by \( U \). Note that if \( h_1 \neq h_2 \), this conjugacy is not conformal.

We now construct a \( K \)-qc map \( \xi_0 \) of the sphere that agrees with \( \psi_{a_1}^{a_2} \) on \( E_{a_1}^{1} \), and with the conjugacy \( \phi_{a_1}^{a_2} \) constructed above on \( U \). Then, \( \xi_0 \) lifts to a \( K \)-qc map \( \xi_1 \) of the sphere that agrees with \( \xi_0 \) on \( E_{a_1}^{1} \cup U \). In particular, \( \xi_1 \) is homotopic to \( \xi_0 \) relative to the post-critical set and the irregular fixed points. One can now proceed as in Proposition 6.12 to conclude that iterating this lifting procedure yields a quasiconformal homeomorphism \( \xi \) of the sphere that agrees with \( \psi_{a_1}^{a_2} \) on \( T_{a_1}^{\infty} \), and conjugates \( F_{a_1} \) to \( F_{a_2} \).

Let us assume that the parameter \( a_1 \) lies on the parabolic arc \( C \). We denote the critical Ecalle height \( h_2 \) parameter on \( C \) by \( a'_2 \). By Theorem 7.3, there exists a quasiconformal conjugacy \( \xi \) between \( F_{a_1} \) and \( F_{a'_2} \) such that \( \xi \) is conformal on \( T_{a_1}^{\infty} \). It is now easy to see that \( \xi \circ \xi^{-1} \) is a quasiconformal conjugacy between \( F_{a_2} \) and \( F_{a'_2} \) such that \( \xi \circ \xi^{-1} \) agrees with \( \psi_{a_2}^{a'_2} := \psi_{a_2}^{-1} \circ \psi_{a_2} \) on the tiling set and is conformal on the parabolic basin. Since parabolic limit sets have measure zero (see [LLMM18, Corollary 7.3]), it follows that \( \xi \circ \xi^{-1} \) is a conformal conjugacy between \( F_{a_2} \) and \( F_{a'_2} \). Moreover, \( \xi \circ \xi^{-1} \) fixes \( 0, \infty, \) and \( 1/4 \). Hence, it is the identity map implying...
that \( a_2 = F_{a_2}^\infty(0) = F_{a_2}^\infty(0) = a'_2 \). Thus, \( a_1 \) and \( a_2 \) lie on the same parabolic arc \( C \), proving the first part of the proposition.

Let us now assume that \( h_1 = h_2 \). In this case, the conjugacy \( \varphi_{a_1}^{a_2} \) between the first return maps of \( F_{a_i} \) on the attracting petals (such that it preserves the post-critical set) can be chosen to be conformal. Indeed, both the maps have a simple parabolic cycle of common odd period \( k \). Choose an attracting petal containing the critical value \( \infty \) and an attracting Fatou coordinate (in the characteristic Fatou component of \( F_{a_i} \)) that conjugates the first return map of the petal to the glide reflection \( \zeta \mapsto \zeta + 1/2 \). These Fatou coordinates are unique up to addition of a real constant.

Since \( F_{a_1} \) and \( F_{a_2} \) have equal critical Ecalle height, we can arrange so that the critical values of \( F_{a_1} \) and \( F_{a_2} \) have the same image under the Fatou coordinates. Hence, the Fatou coordinates induce a conformal conjugacy \( \varphi_{a_1}^{a_2} \) between the first return maps of the petals that sends \( \infty \) to \( \infty \). Using \( F_{a_1} \) and \( F_{a_2} \), we now spread this conjugacy to suitable attracting petals in all the periodic Fatou components such that the domain of the conjugacy contains the entire post-critical set of \( F_{a_1} \).

We can now argue as in the first part of the proposition to obtain a quasiconformal conjugacy \( \xi \) between \( F_{a_1} \) to \( F_{a_2} \) that agrees with \( \psi_{a_1}^{a_2} \) on \( T_\infty \), and is conformal on the parabolic basin. But this implies that \( \xi \) is the identity map, and hence \( a_1 = a_2 \). □

**Corollary 7.5.** Two distinct parabolic arcs do not intersect.

Let us fix a parabolic arc \( C \), and its critical Ecalle height parametrization \( a : \mathbb{R} \to C \). Since \( C(S) \) is bounded, \( C \) accumulates on both ends of \( \mathbb{R} \); i.e. \( \mathcal{C} \) is a compact connected set in \( C \). It is easy to see that any accumulation point of \( C \) (as the critical Ecalle height goes to \( \pm \infty \)) is a parabolic cusp of the same period (compare [MNS17, Lemma 3.3]). Since there are only finitely many parabolic cusps of a given period, \( C \) limits at parabolic cusp points on both ends. Note also that in the dynamical plane of a parabolic cusp, the double parabolic points are formed by the merger of a simple parabolic point with a repelling point.

**Proposition 7.6** (Fixed Point Index on Parabolic Arcs). Along any parabolic arc of odd period, the holomorphic fixed point index of the parabolic cycle is a real valued real-analytic function that tends to \( +\infty \) at both ends.

**Proof.** The proof is similar to that of [HS14, Proposition 3.7]. Indeed, as we move along a parabolic arc towards one of the cusp points at its end, the simple parabolic cycle merges with a repelling cycle, and the sum of their holomorphic fixed point indices converges to the fixed point index of the double parabolic cycle of the cusp parameter (see Subsection 2.1 for the definition of holomorphic fixed point index). But it is easy to see that, as in the anti-polynomial case, the holomorphic fixed point index of the repelling cycle is real and diverges to \( -\infty \) as the parameter converges to a cusp. Since the limiting double parabolic cycle has a finite index, it follows that the holomorphic fixed point index of the simple parabolic cycle (which is also real) must tend to \( +\infty \). □

It now follows by arguments similar to those used in [HS14, Theorem 3.8, Corollary 3.9] that:

**Theorem 7.7** (Bifurcations Along Arcs). Every parabolic arc of period \( k \) intersects the boundary of a hyperbolic component of period \( 2k \) along an arc consisting of the set of parameters where the parabolic fixed point index is at least 1. In particular, every
parabolic arc has, at both ends, an interval of positive length at which bifurcation from a hyperbolic component of odd period \( k \) to a hyperbolic component of period \( 2k \) occurs.

Roughly speaking, when a parameter on such a bifurcating arc is perturbed outside the odd period hyperbolic component, the simple parabolic periodic point splits into two attracting periodic points that lie on the same orbit of \( F_a \).

The next proposition contains some partial information about the relation between critical Ecalle height and parabolic fixed point index on the bifurcating region of a parabolic arc. For any \( h \in \mathbb{R} \), we denote the residue fixed point index of the unique parabolic cycle of \( F_a \) by \( \text{ind}_C(\mathcal{F}^{\circ 2}_{a(h)}) \).

**Proposition 7.8.** Let \( H \) be a hyperbolic component of odd period \( k \) in \( \mathcal{C}(S) \), \( C \) be a parabolic arc on \( \partial H \), \( a : \mathbb{R} \to \mathcal{C} \) be the critical Ecalle height parametrization of \( C \), and let \( H' \) be a hyperbolic component of period \( 2k \) bifurcating from \( H \) across \( C \). Then there exists some \( h_0 > 0 \) such that
\[
C \cap \partial H' = a([h_0, +\infty)).
\]
Moreover, the function
\[
\text{ind}_C : [h_0, +\infty) \to [1, +\infty), \quad h \mapsto \text{ind}_C(\mathcal{F}^{\circ 2}_{a(h)}),
\]
is strictly increasing, and hence a bijection.

**Proof.** The proof of [IM16a, Lemma 2.8, Corollary 2.10] can be applied mutatis mutandis to our setting. \( \square \)

### 7.2. Parameter Rays at Periodic Angles

In this subsection, we will first look at the connection between orbit portraits associated with parabolic parameters on the boundary of an odd period hyperbolic component \( H \) and the angles of parameter rays accumulating on \( \partial H \). Subsequently, we will discuss the relation between the orbit portrait associated with the root of an even period hyperbolic component \( H \) that does not bifurcate from an odd period hyperbolic component and the angles of parameter rays landing at the root of \( H \).

We begin with a preliminary result.

**Lemma 7.9 (Orbit separation lemma).** Let \( F_a \) have a parabolic cycle. Then, the characteristic parabolic point of \( F_a \) can be separated from any other point on the parabolic orbit by two (pre-)periodic dynamical rays landing at a common repelling (pre-)periodic point.

**Proof.** The proof uses the dynamics of \( F_a \) on the parabolic Hubbard tree, and is analogous to that of [Sch00, Lemma 3.7]. \( \square \)

**Proposition 7.10 (Accumulation Points of Periodic Parameter Rays).** Let \( \theta \in (1/3, 2/3) \) be periodic under \( \rho \), and \( a_0 \in \mathcal{C}(S) \) be an accumulation point of the parameter ray at angle \( \theta \). Then, \( F_{a_0} \) has a parabolic cycle of period \( k \) dividing the period of \( \theta \) such that the corresponding dynamical ray at angle \( \theta \) lands at the characteristic parabolic point of \( F_{a_0} \).

**Proof.** The proof is similar to the classical proof of landing of rational parameter rays of the Mandelbrot set [GM93, Theorem C.7] [Sch00, Propositions 3.1, 3.2].
By [LLMM18 Proposition 6.34], the dynamical ray of $F_{a_0}$ at angle $\theta$ lands at a $k$-periodic repelling or parabolic point $w_0 \in \Gamma_{a_0}$. Clearly, $k$ divides the period of the angle $\theta$ (under $\rho$).

Let $w_0$ be a repelling periodic point. Then, by the implicit function theorem, there exists a real-analytic function $w : U(\Theta, a_0) \to \bar{\mathbb{C}}$ such that $w(a_0) = w_0$ and $F^{-k}_a(w(a)) = w(a)$ for all $a \in U$ (where, $U$ is a neighborhood of $a_0$ in the parameter space). Moreover, the dynamical ray at angle $\theta$ lands at $w(a)$ for all $a \in U$ (compare [Sch00 Lemma 2.2]). But there are parameters $a$ near $a_0$ and lying on the parameter ray at angle $\theta$. For such parameters $a$, the corresponding dynamical ray at angle $\theta$ bifurcates. This contradiction proves that $w_0$ is a parabolic periodic point.

Let us assume that $w_0$ is not the characteristic parabolic point of $F_{a_0}$. By Lemma 7.9, the characteristic parabolic point of $F_{a_0}$ can be separated from $w_0$ by two (pre-)periodic dynamical rays landing at a common repelling (pre-)periodic point. Evidently, this separation line persists under perturbation, and separates $\infty$ from the dynamical ray at angle $\theta$. However, for parameters $a$ near $a_0$ and lying on the parameter ray at angle $\theta$, the critical value $\infty$ lies on the dynamical ray at angle $\theta$. Once again, this is a contradiction, which proves that $w_0$ is the characteristic parabolic point of $F_{a_0}$.

In Definition 6.8 (respectively, in Definition 2.19), we defined $\rho$-FOP (respectively, $m_{-2}$-FOP) as a combinatorial tool to describe the patterns of all periodic dynamical rays landing on a periodic cycle of a Schwarz reflection map $F_a$ (respectively, of a quadratic anti-polynomial $f_a$). Using Proposition 6.9, one can transfer combinatorial/topological results about $m_{-2}$-FOPs to corresponding results for $\rho$-FOPs. In particular, among all the complementary arcs of the various $A_j$ of a $\rho$-FOP $\mathcal{P}$, there is a unique one of minimum length. This shortest arc $\mathcal{I}_\mathcal{P}$ is called the characteristic arc of $\mathcal{P}$, and the two angles $\{t^-, t^+\}$ at the ends of this arc are called its characteristic angles (compare [Muk15b Lemma 3.2]). We can assume, without loss of generality, that $0 < t^+ - t^- < 1/2$.

Let $t^+ \in (1/3, 2/3)$ with $t^- = S((i_1, i_2, \ldots))$, $t^+ = S((j_1, j_2, \ldots))$ (see [LLMM18 §3]). Then, there exists a complementary component of $(\mathfrak{T}^{i_1} \cup \mathfrak{T}^{i_2} \cup \cdots) \cup (\mathfrak{T}^{j_1} \cup \mathfrak{T}^{j_2} \cup \cdots) \cup \mathcal{C}(\mathcal{S})$ (see Definition 4.4 for the definition of parameter tiles) that contains the tail of any sequence of tiles determined by any $\theta \in (t^-, t^+)$. We say that $a$ lies between the parameter rays at angles $t^-$ and $t^+$ if $a$ lies in this component.

The next theorem asserts that every $\rho$-FOP with characteristic angles in $(1/3, 2/3)$ is realized by some member of $\mathcal{S}$ outside $\mathcal{C}(\mathcal{S})$.

**Theorem 7.11** (Realization of $\rho$-FOP outside $\mathcal{C}(\mathcal{S})$). Let $\mathcal{P} = \{A_1, A_2, \cdots, A_p\}$ be a formal orbit portrait under $\rho$ with its characteristic angles $t_{\pm}$ in $(1/3, 2/3)$. Then, there exists some $a \in \mathbb{C} \setminus ((-\infty, -1/12) \cup \mathcal{C}(\mathcal{S}))$ such that $F_a$ has a periodic orbit with associated orbit portrait $\mathcal{P}$.

**Proof.** Adapting the proof of [Muk15b Lemma 3.4], one can show that in the dynamical plane of every parameter $a$ lying between the parameter rays at angles $t^-$ and $t^+$, the dynamical rays at angles $t^-$ and $t^+$ land at a common point $w \in \Gamma_a$. Using Proposition 6.9 one then obtains analogues of [Muk15b Lemma 2.9, Lemma 3.3] to the effect that the characteristic angles $t^-$ and $t^+$ essentially determine the actual orbit portrait $\mathcal{P}'$ associated with the periodic point $w$. Finally, one proceeds as in [Muk15b Theorem 3.1] to conclude that for judicious choices of
that the orbit portrait associated with the parabolic cycle of $F$ associated with the parabolic cycle of angles $t^- \text{ and } t^+$, the actual orbit portrait $\mathcal{P}'$ associated with $w$ coincides with $\mathcal{P}$. 

The following proposition, which is an analogue of Theorem 2.21 for the family $S$, will play an important role in the sequel.

**Theorem 7.12 (Realization of $\rho$-FOP at Parabolic Parameters).** Let $\mathcal{P} = \{A_1, A_2, \ldots, A_p\}$ be a $\rho$-FOP with characteristic angles $t^\pm \in (1/3, 2/3)$.

1) Suppose that $p$ is odd, and $t^\pm$ have period 2. Then the parameter rays of $S$ at angles $t^-$ and $t^+$ accumulate on a common root parabolic arc $C$ such that for every parameter $a \in C$, the map $F_a$ has a parabolic cycle of period $p$ and the orbit portrait associated with the parabolic cycle of $F_a$ is $\mathcal{P}$.

2) Suppose that $p$ is even. Then the parameter rays of $S$ at angles $t^-$ and $t^+$ land at a common parabolic parameter $a$ (whose parabolic cycle has period $p$) such that the orbit portrait associated with the parabolic cycle of $F_a$ is $\mathcal{P}$.

**Proof.** 1) By Theorem 7.11, the dynamical rays at angles $t^-$ and $t^+$ land at a common point of $\Gamma_a$ for all parameters $a \notin \mathcal{C}(S)$ lying between the parameter rays at angles $t^-$ and $t^+$. One can now employ a standard wake argument, analogous to the one in [IM16] Lemma 4.1, to conclude that the parameter rays of $S$ at angles $t^-$ and $t^+$ must accumulate on a common root parabolic arc $C$ (such that the rays together with $C$ separate the plane). The details are similar to the proof of Theorem 2.21.

2) Once again, the proof is completely similar to that of Theorem 2.21. The fact that the parameter rays at angles $t^-$ and $t^+$ land follows from discreteness of (even-type) parabolic parameters with a prescribed orbit portrait (see Proposition 6.14).

Let us now fix a hyperbolic component $H$ of odd period $k$ and center $a_0$. The first return map of the closure of the characteristic Fatou component of $a_0$ fixes exactly three points on its boundary. Only one of these fixed points disconnects the non-escaping set, and is the landing point of two distinct dynamical rays at 2k-periodic angles (see Proposition 6.4). Let the set of the angles of these two rays be $S' = \{\alpha_1, \alpha_2\}$. Then, $\alpha_2 = (-2)^k \alpha_1$, and $S'$ is the set of characteristic angles of the corresponding orbit portrait. Each of the remaining two fixed points is the landing point of precisely one dynamical ray at a $k$-periodic angle; let the collection of the angles of these rays be $S = \{\theta_1, \theta_2\}$. We can, possibly after renumbering, assume that $0 < \alpha_1 < \theta_1 < \theta_2 < \alpha_2$ and $\alpha_2 - \alpha_1 < \frac{1}{k}$.

Since parabolic cusps are isolated (see Proposition 6.14) and $\partial H$ is connected, every parabolic cusp is the common limit point of two distinct parabolic arcs. By Theorem 7.14 every parabolic cusp and its nearby simple parabolic parameters are points of bifurcation from $H$ to a hyperbolic component of period $2k$. It follows that every parabolic cusp lies in the interior of $\mathcal{C}(S)$ (compare [MNS17] Lemma 5.12).

One can now argue as in the proof of [MNS17] Theorem 1.2 to give a complete description of $\partial H$. Indeed, Theorem 7.12 Proposition 7.10 and Proposition 7.14 imply that there are exactly three parabolic arcs on $\partial H$ which can be numbered in the following way.

There is a unique parabolic arc (say, $C_3$) such that the characteristic parabolic point in the dynamical plane of any parameter on $C_3$ is the landing point of precisely two dynamical rays at angles $\alpha_1$ and $\alpha_2$. Moreover, the parameter rays at angles $\alpha_1$ and $\alpha_2$ (and no other) accumulate on $C_3$. The rest of the two parabolic arcs (say,
\( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) on \( \partial H \) have the property that the characteristic parabolic point in the dynamical plane of any parameter on \( \mathcal{C}_i \) \( (i = 1, 2) \) is the landing point of precisely one dynamical ray at angle \( \theta_i \). Furthermore, \( \mathcal{C}_i \) \( (i = 1, 2) \) contains the accumulation set of exactly one parameter ray at angle \( \theta_i \).

At the parabolic cusp on \( \partial H \) where \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) meet, the characteristic parabolic point is the landing point of exactly two dynamical rays at angles \( \theta_1 \) and \( \theta_2 \). The same is true at the center of the hyperbolic component of period \( 2k \) that bifurcates from \( H \) across this parabolic cusp. Moreover, these angles are the characteristic angles of the corresponding orbit portrait.

On the other hand, at the parabolic cusp where \( \mathcal{C}_1 \) and \( \mathcal{C}_3 \) (respectively, \( \mathcal{C}_2 \) and \( \mathcal{C}_3 \)) meet, the characteristic parabolic point is the landing point of precisely three dynamical rays at angles \( \alpha_1 \), \( \alpha_2 \) and \( \theta_1 \) (respectively, \( \alpha_1 \), \( \alpha_2 \) and \( \theta_2 \)). As before, the same is true at the center of the hyperbolic component of period \( 2k \) that bifurcates from \( H \) across this parabolic cusp. The characteristic angles of the corresponding orbit portrait are \( \alpha_1 \) and \( \theta_1 \) (respectively, \( \theta_2 \) and \( \alpha_2 \)). Finally, Proposition 6.9 allows us to translate the second statement of [Muk15b, Lemma 3.5] to the current setting implying the following relation
\[
\rho^k((\alpha_1, \theta_1)) = (\theta_1, \alpha_2), \quad \rho^k(\theta_2, \alpha_2)) = (\alpha_1, \theta_2).
\]

**Theorem 7.13 (Boundaries of Odd Period Hyperbolic Components).** The boundary of every hyperbolic component of odd period of \( \mathcal{C}(S) \) is a topological triangle having parabolic cusps as vertices and parabolic arcs as sides.

Let us briefly carry out a similar analysis for even period hyperbolic components that do not bifurcate from odd period ones. Let \( H \) be a hyperbolic component of even period \( k \) such that \( H \) does not bifurcate from an odd period hyperbolic component. Let \( \mathcal{A}_1 \) be the set of angles of the dynamical rays landing at the dynamical root of \( F_a \) (where \( a \in H \) or \( a \) is the root point of \( H \)). Then, the first return map of the dynamical root either fixes every angle in \( \mathcal{A}_1 \) and \( |\mathcal{A}_1| = 2 \), or permutes the angles in \( \mathcal{A}_1 \) transitively (by Proposition 6.9 and [Muk15b, Lemma 3.3]). Moreover, the characteristic angles \( t^\pm \) of the orbit portrait \( \mathcal{P} \) generated by \( \mathcal{A}_1 \) are precisely the two adjacent angles in \( \mathcal{A}_1 \) (with respect to circular order) that separate \( 0 \) from \( \infty \), and bound a sector of angular width less that \( \frac{1}{2} \). It now follows from Theorem 7.12 and Proposition 6.14 that the parameter rays at angles \( t^\pm \) land at the root point of \( H \).

To conclude this subsection, let us state a generalization of Proposition 6.6.

**Proposition 7.14.** For any hyperbolic or parabolic map \( F_a \), the pre-periodic lamination is completely determined by the characteristic angles of \( F_a \).

**Proof.** It follows from the above discussion that the pre-periodic lamination and the characteristic angles remain unaltered throughout an odd period hyperbolic component and the parabolic arcs on its boundary (respectively, throughout an even period hyperbolic component and its root). The result now follows from Proposition 6.6. \( \square \)

### 7.3. Parameter Rays at Pre-periodic Angles

In this section, we will study the landing properties of parameter rays of \( S \) at strictly pre-periodic angles. Let \( a_0 \) be a Misiurewicz parameter, and \( \mathcal{A} \) be the set of angles of the dynamical rays of \( F_{a_0} \) landing at the critical point \( 0 \). The set of angles of the dynamical rays of \( F_{a_0} \) that land at the critical value \( \infty \) is then given by \( \mathcal{A} := \rho(\mathcal{A}) \subset (1/3, 2/3) \). Moreover,
\( \rho \) is two-to-one from \( \mathcal{A}' \) onto \( \mathcal{A} \), and is injective on all other \( \lambda(F_{a_0}) \)-classes. As for quadratic anti-polynomials, the existence of a unique \( \lambda(F_{a_0}) \)-equivalence class that maps two-to-one onto its image equivalence class (under \( \rho \)) characterizes pre-periodic laminations of Misiurewicz maps. It is called a pre-periodic lamination of Misiurewicz type.

**Proposition 7.15** (Landing of Parameter Rays at Pre-periodic Angles). Let \( \theta \in (1/3, 2/3) \) be strictly pre-periodic under \( \rho \). Then the parameter ray of \( S \) at angle \( \theta \) lands at a Misiurewicz parameter such that in the corresponding dynamical plane, the dynamical ray at angle \( \theta \) lands at the critical value \( \infty \).

**Proof.** Let \( a_0 \) be an accumulation point of the parameter ray of \( S \) at angle \( \theta \). Arguing as in the second part of [Lyu17, Theorem 37.35], we will conclude that \( a_0 \) is a Misiurewicz parameter such that in the dynamical plane of \( F_{a_0} \), the dynamical ray at angle \( \theta \) lands at the critical value \( \infty \).

By [LLMM18 Proposition 6.34], the dynamical ray of \( F_{a_0} \) at angle \( \theta \) lands at some repelling or parabolic pre-periodic point \( w \) (as \( \theta \) is strictly pre-periodic under \( \rho \), the landing point cannot be periodic). Let us suppose that \( F_{a_0} \) is a parabolic map. Note that as the landing point of the dynamical \( \theta \)-ray of \( F_{a_0} \) is not periodic, the ray does not land at the characteristic parabolic point of \( F_{a_0} \). Since the limit set of a parabolic map is locally connected and repelling periodic points are dense on the limit set [LLMM18 Theorem 1.3], it follows that there exists a cut-line through repelling periodic points on \( \Gamma_{a_0} \) separating the \( \theta \)-dynamical ray from the critical value. But such cut-lines remain stable under small perturbation. Therefore, for parameters sufficiently close to \( a_0 \), the \( \theta \)-dynamical ray stays away from the critical value. However, this is impossible as there are parameters near \( a_0 \) on the \( \theta \)-parameter ray for which the critical value lies on the \( \theta \)-dynamical ray. This contradiction shows that \( F_{a_0} \) is not parabolic: i.e. the dynamical ray of \( F_{a_0} \) at angle \( \theta \) lands at some repelling pre-periodic point \( w \).

We suppose that \( w \) is not the critical value of \( F_{a_0} \), and will arrive at a contradiction. If \( w \) is not a pre-critical point either, then for nearby parameters, the \( \theta \)-dynamical ray would land at the real-analytic continuation of the repelling pre-periodic point \( w \), and would stay away from the critical value. But there are parameters near \( a_0 \) on the \( \theta \)-parameter ray for which the critical value lies on the \( \theta \)-dynamical ray, a contradiction. Hence \( w \) must be a pre-critical point implying that the critical point of \( F_{a_0} \) is strictly pre-periodic. So \( a_0 \) is a Misiurewicz parameter. As the limit set of a Misiurewicz map is a dendrite (by [LLMM18 Proposition 7.9]) and repelling periodic points are dense on it [LLMM18 Theorem 1.3], the dynamical ray at angle \( \theta \) landing at \( w \) can be separated from the critical value by a pair of dynamical rays landing at a common repelling periodic point. Once again, this separation line remains stable under perturbation, contradicting the existence of parameters near \( a_0 \) on the \( \theta \)-parameter ray. Hence, \( w \) must be the critical value of \( F_{a_0} \).

We claim that \( a_0 \) is the unique parameter in \( \mathcal{C}(S) \) with the property that the dynamical ray at angle \( \theta \) lands at the critical value \( \infty \). Since the limit set of a ray is connected, this will prove that the parameter ray at angle \( \theta \) indeed lands at \( a_0 \).

To prove the claim, let us assume that there exists another parameter \( a_1 \) with the same property. Note that both the pre-periodic laminations \( \lambda(F_{a_0}) \) and \( \lambda(F_{a_1}) \) are of Misiurewicz type. Hence, the formal rational laminations \( \mathcal{E}_s(\lambda(F_{a_0})) \) and \( \mathcal{E}_s(\lambda(F_{a_1})) \) are also of Misiurewicz type. By Theorem [2.21], there exist Misiurewicz parameters \( c_0 \) and \( c_1 \) in \( \mathcal{L} \) realizing these rational laminations. By construction,
the dynamical ray $R_{c_0}(\mathcal{E}(\theta))$ (respectively, $R_{c_1}(\mathcal{E}(\theta))$) lands at the critical value $c_0$ (respectively, $c_1$) of $f_{c_0}$ (respectively, of $f_{c_1}$). It now follows by Theorem 2.23 that the parameter ray $R_{\mathcal{E}(\theta)}$ lands both at $c_0$ and $c_1$ implying that $c_0 = c_1$. Therefore, we have $\lambda(F_{a_0}) = \lambda(F_{a_1})$. By Proposition 6.15 we conclude that $a_0 = a_1$. This completes the proof.

Recall that for a Misiurewicz map $f_{c_0}$, the angles of the parameter rays of $\mathcal{T}$ landing at $c_0$ are exactly the external angles of the dynamical rays that land at the critical value $c_0$ in the dynamical plane of $f_{c_0}$ (see Theorem 2.23). We will now prove an analogous statement for Misiurewicz parameters of $\mathcal{C}(\mathcal{S})$.

**Proposition 7.16** (Correspondence between Dynamical and Parameter Rays). Let $a_0 \in \mathcal{C}(\mathcal{S})$ be a Misiurewicz parameter. Then, the angles of the parameter rays (at pre-periodic angles) of $\mathcal{S}$ landing at $a_0$ are exactly the external angles of the dynamical rays that land at the critical value $\infty$ in the dynamical plane of $F_{a_0}$.

**Proof.** Let $A \subset (1/3, 2/3)$ be the set of angles of the dynamical rays of $F_{a_0}$ that land at the critical value $\infty$. By Proposition 7.15 the angles of the parameter rays (at pre-periodic angles) of $\mathcal{S}$ landing at $a_0$ are contained in $A$.

Now pick $\theta \in A$, and let $a_1$ be the landing point of the parameter ray of $\mathcal{S}$ at angle $\theta$. Then, the dynamical ray of $F_{a_1}$ at angle $\theta$ lands at the critical value $\infty$. By the proof of Proposition 7.15 we know that there can be at most one parameter in $\mathcal{C}(\mathcal{S})$ whose dynamical $\theta$-ray lands at the critical value $\infty$. Therefore, $a_0 = a_1$, i.e. the parameter ray of $\mathcal{S}$ at angle $\theta$ lands at $a_0$. As $\theta$ was an arbitrary element of $A$, it follows that all parameter rays at angles in $A$ land at $a_0$. The proof is now complete. □

8. **Combinatorial Straightening and Homeomorphism of Topological Models**

In this section, we prove the main theorems of the paper.

The proof of Theorem 1.2 will be carried out in two stages. In Subsection 8.1 we will construct a lamination-preserving bijection between the centers of $\mathcal{C}(\mathcal{S})$ and the centers of $\mathcal{L}$. A lamination-preserving bijection between the Misiurewicz parameters of $\mathcal{C}(\mathcal{S})$ and $\mathcal{L}$ will be constructed in Subsection 8.2.

8.1. **Combinatorial Bijection for Hyperbolic and Parabolic Parameters.**

In this subsection, we will prove Theorem 1.2 for the hyperbolic and parabolic parameters of $\mathcal{C}(\mathcal{S})$. Recall that for every hyperbolic and parameter parameter $a$, the first return map of the characteristic Fatou component $U_a$ has a unique fixed point on $\partial U_a$ that is a cut-point of $K_a$ (which we call the dynamical root of $F_a$).

The orbit portrait (more precisely, its characteristic angles $t^\pm$) associated with the dynamical root of $F_a$ completely determines $\lambda(F_a)$ (see Propositions 6.6 and 7.14). In fact, all the iterated pullbacks of the leaf connecting $t^-$ and $t^+$ under $\rho$ are pairwise disjoint, and their closure in $\mathbb{Q}/\mathbb{Z}$ is equal to $\lambda(F_a)$.

**Lemma 8.1.** For every super-attracting map $F_{a_0} \in \mathcal{C}(\mathcal{S})$ with pre-periodic lamination $\lambda(F_{a_0})$, there exists a unique super-attracting map $f_{c_0} \in \mathcal{L}$ with associated rational lamination $\mathcal{E}_c(\lambda(F_{a_0}))$. Moreover, this correspondence is a bijection between the super-attracting maps of $\mathcal{C}(\mathcal{S})$ and $\mathcal{L}$. 
Proof. Note that super-attracting maps are precisely the centers of hyperbolic components.

Let \( a_0 \) be the center of a hyperbolic component \( H \) of even period (other than two) of \( \mathcal{C}(S) \) that does not bifurcate from a hyperbolic component of odd period. Let \( \mathcal{P} \) be the orbit portrait associated with the dynamical root of \( F_{a_0} \), the characteristic angles of \( \mathcal{P} \) be \( t^\pm \), and the pre-periodic lamination of \( F_{a_0} \) be \( \lambda(F_{a_0}) \). By Proposition 6.6, \( \mathcal{E}_s(\mathcal{P}) \) is an \( m_{-2} \)-FOP with characteristic angles \( \mathcal{E}(t^\pm) \). Since the orbit period of \( \mathcal{P} \) is even, the second statement of Theorem 2.21 implies that the parameter rays \( \mathcal{R}_{\mathcal{E}(t^-)} \) and \( \mathcal{R}_{\mathcal{E}(t^+)} \) of \( T \) land at the root point of some hyperbolic component \( H' \) of \( T \). Moreover, the orbit portrait associated with the parabolic cycle of the root of \( H' \) is given by \( \mathcal{E}_s(\mathcal{P}) \). It follows that in the dynamical plane of the center \( c_0 \) of \( H' \), the orbit portrait associated with the dynamical root is \( \mathcal{E}_s(\mathcal{P}) \). In particular, \( \mathcal{E}(t^\pm) \) are the characteristic angles of \( f_{c_0} \).

By Proposition 6.6, the two angles \( t^\pm \) (respectively, \( \mathcal{E}(t^\pm) \)) completely determine the pre-periodic lamination of \( F_{a_0} \) (respectively, the rational lamination of \( f_{c_0} \)); more precisely, all the pullbacks of the leaf connecting \( t^+ \) and \( t^- \) (respectively, \( \mathcal{E}(t^+) \) and \( \mathcal{E}(t^-) \)) under iterations of \( \rho \) (respectively, of \( m_{-2} \)) are pairwise disjoint, and their closure in \( \mathbb{Q}/\mathbb{Z} \) is equal to \( \lambda(F_{a_0}) \) (respectively, \( \lambda(f_{c_0}) \)). Therefore, \( \lambda(f_{c_0}) = \mathcal{E}_s(\lambda(F_{a_0})) \). Since \( 1/3 \sim 2/3 \) in \( \lambda(F_{a_0}) \), it follows that the dynamical rays \( R_{c_0}(1/3) \) and \( R_{c_0}(2/3) \) land at a common point of \( J(f_{c_0}) \). Hence, \( c_0 \in \mathcal{C} \). Finally, by [MNS17] Theorem 5.1 (also compare [Lyu17] Theorem 35.1), \( f_{c_0} \) is the unique PCF map in \( \mathcal{C} \) with rational lamination \( \mathcal{E}_s(\lambda(F_{a_0})) \).

Thanks to Theorem 7.12, the previous argument also goes in the opposite direction demonstrating that if \( c_0 \) is the center of a hyperbolic component of even period of \( \mathcal{C} \) that does not bifurcate from a hyperbolic component of odd period, then there exists some super-attracting map \( F_{a_0} \) with associated pre-periodic lamination \( \mathcal{E}^*(\lambda(f_{c_0})) \).

We now turn our attention to hyperbolic components of odd period and hyperbolic components of even period bifurcating from them. Let \( a_0 \) be the center of a hyperbolic component \( H \) of odd period \( k \). Let \( \mathcal{P} \) be the orbit portrait associated with the dynamical root of \( F_{a_0} \), and the characteristic angles of \( \mathcal{P} \) be \( \alpha_1 \) and \( \alpha_2 \). By Subsection 7.2, each of the two co-roots of \( F_{a_0} \) is the landing point of exactly one dynamical ray at angle \( \theta_i \) (\( i = 1, 2 \)). There are three hyperbolic components of period \( 2k \) bifurcating from \( H \), and the characteristic angles (of the orbit portraits associated with the dynamical roots) of their centers are \( \{\theta_1, \theta_2\}, \{\alpha_1, \theta_1\}, \) and \( \{\theta_2, \alpha_2\} \). Moreover, these angles satisfy Relation (4).

By Proposition 6.9 and Theorem 2.21, the parameter rays \( \mathcal{R}_{\mathcal{E}(\alpha_1)} \) and \( \mathcal{R}_{\mathcal{E}(\alpha_2)} \) of the Tricorn accumulate on a common root are \( \mathcal{C} \) of \( T \), and for every parameter \( c \in \mathcal{C} \), the parabolic orbit portrait is \( \mathcal{E}_s(\mathcal{P}) \). Let \( \mathcal{C} \subset \partial H' \) (where \( H' \) is a hyperbolic component of period \( k \) of \( T \)), and \( c_0 \) be the center of \( H' \). Then, the orbit portrait associated with the dynamical root of \( f_{c_0} \) is \( \mathcal{E}_s(\mathcal{P}) \). As in the previous case, this implies that \( \lambda(f_{c_0}) = \mathcal{E}_s(\lambda(F_{a_0})) \), and \( c_0 \in \mathcal{C} \). Moreover, \( f_{c_0} \) is the unique PCF map in \( \mathcal{C} \) with rational lamination \( \mathcal{E}_s(\lambda(F_{a_0})) \).

It also follows from the above discussion that the angles \( \mathcal{E}(\alpha_i) \) and \( \mathcal{E}(\theta_i) \) (where \( i = 1, 2 \)) satisfy Relation (4). Hence, the dynamical rays at angles \( \mathcal{E}(\theta_i) \) land at the dynamical co-roots of \( f_{c_0} \). Therefore, the characteristic angles (of the orbit portraits associated with the dynamical roots) of the centers of the hyperbolic components of period \( 2k \) bifurcating from \( H' \) are given by \( \{\mathcal{E}(\theta_1), \mathcal{E}(\theta_2)\}, \{\mathcal{E}(\alpha_1), \mathcal{E}(\theta_1)\}, \) and \( \{\mathcal{E}(\theta_2), \mathcal{E}(\alpha_2)\} \). It follows that the push-forwards of the pre-periodic laminations of
the centers of the three hyperbolic components bifurcating from $H$ are precisely the rational laminations of the centers of the three hyperbolic components bifurcating from $H'$. Moreover, this correspondence is a bijection between the parabolic component $H$ with associated rational lamination $\lambda$ and unique hyperbolic parameter $E$.

As in the even period case, one can use Theorem 7.12 and the combinatorial description of odd period hyperbolic components of $\mathcal{C}(S)$ and $\mathcal{L}$ to conclude surjectivity of the map between centers.

It remains to discuss hyperbolic components of period two. There is exactly one hyperbolic component $H_2$ (respectively, $H'_2$) of period two in $\mathcal{C}(S)$ (respectively, in $\mathcal{L}$). The center of $H_2$ (respectively, $H'_2$) is $0$ (respectively, $-1$). In the dynamical planes of the centers of these components, the dynamical rays at angles $1/3$ and $2/3$ land at the $\alpha$-fixed point (which is the dynamical root), and these are the characteristic angles of the corresponding orbit portrait. Moreover, these two angles completely determine the corresponding pre-periodic (respectively, rational) lamination (compare Proposition 6.9). Since $\mathcal{E}$ fixes $1/3$ and $2/3$, it follows that $\mathcal{E}_s(\lambda(F_0)) = \lambda(f_{-1})$.

Finally, by Proposition 6.12, two distinct super-attracting maps in $\mathcal{C}(S)$ cannot have the same pre-periodic lamination. Hence, the correspondence between the centers of $\mathcal{C}(S)$ and $\mathcal{L}$ defined above is injective.

As a corollary of the above proof (combined with our analysis of the hyperbolic components of $S$ and their boundaries, and the rigidity results of Subsection 6.2), we get a combinatorial bijection between the hyperbolic and parabolic parameters of $\mathcal{C}(S)$ and those of $\mathcal{L}$.

**Corollary 8.2** (Bijection between Hyperbolic and Parabolic Parameters of $\mathcal{C}(S)$ and $\mathcal{L}$). 1) If $a \in H \subset \mathcal{C}(S)$ is a hyperbolic parameter (contained in the hyperbolic component $H$) with associated pre-periodic lamination $\lambda(F_a)$, then there exists a unique hyperbolic parameter $c \in H' \subset \mathcal{L}$ (contained in the hyperbolic component $H'$) with associated rational lamination $\lambda(F_c) = \mathcal{E}_s(\lambda(F_a))$ satisfying $\eta_H(c) = \eta_H(a)$. Moreover, this correspondence is a bijection between the hyperbolic parameters of $\mathcal{C}(S)$ and $\mathcal{L}$.

2) If $a \in \mathcal{C}(S)$ is a parabolic parameter with associated pre-periodic lamination $\lambda(F_a)$ (and critical Ecalle height $h$, if $F_a$ has an odd-periodic simple parabolic cycle), then there exists a unique parabolic parameter $c \in \mathcal{L}$ with associated rational lamination $\mathcal{E}_s(\lambda(F_a))$ (with the same critical Ecalle height in the odd-periodic non-cusp case). Moreover, this correspondence is a bijection between the parabolic parameters of $\mathcal{C}(S)$ and $\mathcal{L}$.

**Proof.** 1) Let $a_0$ be the center of the hyperbolic component $H \ni a \in \mathcal{C}(S)$. By Lemma 8.1, there exists a super-attracting parameter $c_0 \in \mathcal{L}$ such that $\lambda(f_{c_0}) = \mathcal{E}_s(\lambda(F_{a_0}))$. We can assume that $c_0$ is the center of the hyperbolic component $H'$ of $\mathcal{L}$.

According to Theorem 5.1, there exists a homeomorphism $\tilde{\eta}_H : H \to \mathcal{B}^\pm$ (respectively, $\eta_{H'} : H' \to \mathcal{B}^\pm$) that maps the center of $H$ (respectively, $H'$) to the super-attracting Blaschke product $B_{\eta_{H}}^\pm$. We now define the parameter $c := \eta_{H'}^{-1} \circ \tilde{\eta}_H(a)$. By definition, the conformal conjugacy class of the first return map of the characteristic Fatou components of $F_a$ and $F_c$ are the same, and the laminations of the two maps are related in the desired way. In particular, the Koenigs ratio/multiplier of the corresponding attracting cycles are equal.
The fact that the above correspondence induces a bijection between the hyperbolic parameters of $C(S)$ and $L$ can now be proved mimicking the arguments of Lemma 8.1 (and invoking Proposition 6.13).

2) This is immediate from the proof of Lemma 8.1 and Propositions 6.14 and 7.4 once we define the correspondence on parabolic arcs by preserving critical Ecalle heights. □

The proof of Lemma 8.1 also implies that the map $E$ induces a bijection between the parameter rays at periodic angles landing/accumulating at the parabolic parameters of $C(S)$ and $L$.

Corollary 8.3 (Correspondence of Parameter Rays at Periodic Angles). 1) If a parabolic parameter $a \in C(S)$ of even parabolic period corresponds to the parabolic parameter $c \in L$ under the above bijection, then the angles of the two parameter rays (at periodic angles) landing at $a$ are precisely the $E$-pre-images of the angles of the two parameter rays (at periodic angles) landing at $c$.

2) If a root (respectively, co-root) parabolic arc $C \subset C(S)$ corresponds to the arc $C' \subset L$, then the angles of the two parameter rays at periodic angles (respectively, the angle of the unique parameter ray at a periodic angle) accumulating on $C$ are precisely the $E$-pre-images of the angles of the two parameter rays at periodic angles (respectively, the angle of the unique parameter ray at a periodic angle) accumulating on $C'$.

8.2. Combinatorial Bijection for Misiurewicz Parameters. We now turn our attention to the Misiurewicz parameters.

Lemma 8.4 (Bijection between Misiurewicz Parameters of $C(S)$ and $L$). For every Misiurewicz parameter $a_0 \in C(S)$ with pre-periodic lamination $\lambda(F_{a_0})$, there exists a unique Misiurewicz parameter $c_0 \in L$ with associated rational lamination $E_\ast(\lambda(F_{a_0}))$. Moreover, this correspondence is a bijection between the Misiurewicz parameters of $C(S)$ and $L$.

Proof. Pick a Misiurewicz parameter $a_0 \in C(S)$ with pre-periodic lamination $\lambda(F_{a_0})$. Then by [LLMM18, Proposition 6.39], $E_\ast(\lambda(F_{a_0}))$ is a formal rational lamination of Misiurewicz type. By Theorem 2.24 there exists a unique Misiurewicz parameter $c_0 \in L$ such that $\lambda(f_{a_0}) = E_\ast(\lambda(F_{a_0}))$. Since $1/3 \sim 2/3$ in $\lambda(F_{a_0})$, the same identification holds in $\lambda(f_{a_0})$ as well (recall that $E$ fixes $1/3$ and $2/3$). Therefore, the dynamical rays $R_{c_0}(1/3)$ and $R_{c_0}(2/3)$ land at a common point of $J_{c_0}$; hence $c_0 \in L$.

According to Proposition 6.15, two distinct Misiurewicz parameters of $C$ cannot have the same pre-periodic lamination. This shows that the map between Misiurewicz parameters of $C$ and $L$ defined above is injective.

It remains to prove surjectivity of the above map. Pick a Misiurewicz parameter $c_0 \in L$ with rational lamination $\lambda(f_{c_0})$. Suppose that the set of critical value angles of $f_{c_0}$ is $A$. Then by Theorem 2.23 the parameter $c_0$ is the landing point of the parameter rays of $T$ at angles in $A$. Moreover, as $c_0 \in L$, we have that $A \subset (1/3, 2/3)$. Pick $\theta \in E^{-1}(A)$. Clearly, $\theta \in (1/3, 2/3)$. By Proposition 7.15 the parameter ray of $S$ at angle $\theta$ lands at a Misiurewicz parameter $a_0$ such that the dynamical ray at angle $\theta$ of $F_{a_0}$ lands at the critical value $\infty$. By [LLMM18, Proposition 6.39], the push-forward $E_\ast(\lambda(F_{a_0}))$ is a formal rational lamination of Misiurewicz type, and hence is realized as the actual rational lamination of some
Misiurewicz parameter $c_1 \in \mathcal{T}$. Our construction implies that $\mathcal{E}(\theta)$ is a critical value angle for $f_{c_1}$. Once again by Theorem 2.23 the parameter ray $\mathcal{R}_{\xi(\theta)}$ (of $\mathcal{T}$) lands at $c_1$. Since $\mathcal{E}(\theta) \in \mathcal{A}$, it follows that $c_0 = c_1$. Hence, $\mathcal{E}_*(\lambda(F_{c_0})) = \lambda(f_{c_0})$. □

The following important result relates the angles of the parameter rays of $\mathcal{S}$ landing at a Misiurewicz parameter $a_0$ to those of the parameter rays of $\mathcal{T}$ landing at the corresponding parameter $c_0$.

**Corollary 8.5** (Correspondence of Parameter Rays at Pre-periodic Angles). Let $a_0$ and $c_0$ be Misiurewicz parameters in $\mathcal{C}(\mathcal{S})$ and $\mathcal{L}$ (respectively) such that $\mathcal{E}_*(\lambda(F_{a_0})) = \lambda(f_{c_0})$, and $\mathcal{A}$ be the set of angles of the parameter rays of $\mathcal{S}$ (at pre-periodic angles) landing at $a_0$. Then, $\mathcal{E}(\mathcal{A})$ is precisely the set of angles of the parameter rays of $\mathcal{T}$ (at pre-periodic angles) landing at $c_0$.

**Proof.** By Proposition 7.16 the set of external angles of the dynamical rays that land at the critical value $\infty$ in the dynamical plane of $F_{a_0}$ is precisely $\mathcal{A}$. Since $\mathcal{E}_*(\lambda(F_{a_0})) = \lambda(f_{c_0})$, the set of external angles of the dynamical rays that land at the critical value $c_0$ in the dynamical plane of $f_{c_0}$ is equal to $\mathcal{E}(\mathcal{A})$. Finally by Theorem 2.23, $\mathcal{E}(\mathcal{A})$ is the set of angles of the parameter rays of $\mathcal{T}$ (at pre-periodic angles) landing at $c_0$. □

Theorem 1.2 now readily follows.

**Proof of Theorem 1.2** This clearly follows from Corollary 8.2 and Lemma 8.4. □

Let us give a name to the bijection between the geometrically finite parameters of $\mathcal{C}(\mathcal{S})$ and $\mathcal{L}$ established in Theorem 1.2.

**Definition 8.6** (Combinatorial Straightening). For a geometrically finite parameter $a_0 \in \mathcal{C}(\mathcal{S})$, we denote by $\chi(a_0)$ the unique geometrically finite parameter $c_0 \in \mathcal{L}$ such that $\mathcal{E}_*(\lambda(F_{a_0})) = \lambda(f_{c_0})$ and the first return maps of the characteristic Fatou components (of $F_{a_0}$ and $f_{c_0}$) are conformally conjugate. The map $f_{\chi(a_0)}$ will be called the **combinatorial straightening** of $F_{a_0}$.

### 8.3. Mating Description for Maps in $\mathcal{C}(\mathcal{S})$

**Proof of Theorem 1.2** Let $a_0$ be a geometrically finite map in $\mathcal{C}(\mathcal{S})$. It follows from Theorem 1.2 and a straightforward extension of the proof of [LLMM18, Proposition 8.1] that $F_{a_0} : K_{a_0} \to K_{a_0}$ is topologically conjugate to $f_{\chi(a_0)} : K_{\chi(a_0)} \to K_{\chi(a_0)}$ such that the conjugacy is conformal in int $K_{a_0}$, and $F_{a_0} : T_{a_0}^\infty \setminus T_{a_0}^0 \to T_{a_0}^\infty$ is conformally conjugate to $\rho : \mathbb{D} \setminus \Pi \to \mathbb{D}$ (also see [LLMM18, Proposition 6.31]). The description of $F_{a_0}$ as a mating is now similar to [LLMM18, §8], we include it for completeness.

Let us consider the two conformal dynamical systems

$$f_{\chi(a_0)} : K_{\chi(a_0)} \to K_{\chi(a_0)}$$

and

$$\rho : \mathbb{D} \setminus \text{int } \Pi \to \mathbb{D}.$$ We use the mating tool $\xi := \varphi_{\chi(a_0)}^{-1} \circ \mathcal{E} : \mathbb{T} \to \mathcal{J}_{\chi(a_0)}$ (where $\varphi_{\chi(a_0)}^{-1} : \mathbb{T} \to \mathcal{J}_{\chi(a_0)}$ is the continuous boundary extension of the inverse of the Böttcher coordinate of $f_{\chi(a_0)}$) to glue $\mathbb{D}$ outside $K_{\chi(a_0)}$. Note that $\xi$ semi-conjugates $\rho$ to $f_{\chi(a_0)}$.

Denote

$$X = \mathbb{D} \cup \xi \setminus K_{\chi(a_0)}, \quad Y = X \setminus \text{int } \Pi.$$
(This is a slight abuse of notation. We have denoted the image of \( \text{int} \, \Pi \subset \mathbb{D} \) in \( X \) under the gluing by \( \text{int} \, \Pi \).)

We will argue that \( X \) is a topological sphere. Since \( K_{\chi(a_0)} \) is homeomorphic to \( \mathbb{D}/\lambda_\mathcal{R}(f_{\chi(a_0)}) \) (where \( \lambda_\mathcal{R}(f_{\chi(a_0)}) \) is the real lamination of \( f_{\chi(a_0)} \), which has a locally connected Julia set), it follows that \( X \) is topologically the quotient of the 2-sphere by a closed equivalence relation such that all equivalence classes are connected and non-separating, and not all points are equivalent. It follows by Moore’s theorem that \( X \) is a topological 2-sphere [Moo25, Theorem 25]. Moreover, \( Y \) is the union of two closed Jordan disks (with a single point of intersection) in \( X \).

The well-defined topological map

\[
\eta \equiv \rho \cap \chi f_{\chi(a_0)} : Y \rightarrow X
\]

is the topological mating between \( \rho \) and \( f_{\chi(a_0)} \).

The conjugacies obtained in [LLM18 Proposition 8.1] glue together to produce a homeomorphism

\[
\mathcal{H} : (X, Y) \rightarrow (\mathbb{C}, \mathcal{P}_{a_0})
\]

which is conformal outside \( \mathcal{H}^{-1}(\Gamma_{a_0}) \), and which conjugates \( \eta \) to \( F_{a_0} \). It endows \( X \) with a conformal structure compatible with the one on \( X \setminus \mathcal{H}^{-1}(\Gamma_{a_0}) \) that turns \( \eta \) into an anti-holomorphic map conformally conjugate to \( F_{a_0} \).

In this sense, \( F_{a_0} \) is a conformal mating of the reflection map \( \rho \) arising from the ideal triangle group and the anti-polynomial \( \pi^2 + \chi(a_0) \).

Note that the proof of Theorem [LLM18] uses local connectivity of \( \Gamma_{a_0} \) (respectively, of \( J_{\chi(a_0)} \)), rigidity of the corresponding maps, and our understanding of the dynamics of \( F_{a_0} \) (respectively, of \( f_{\chi(a_0)} \) on \( K_{a_0} \) (respectively, on \( K_{\chi(a_0)} \))) in a crucial way. This is precisely the reason why we restricted ourselves to geometrically finite maps in Theorem [LLM18].

We will now provide a weaker (and more combinatorial) mating description for the periodically repelling maps in \( C(S) \).

**Definition 8.7.**

1) A map \( F_a \in C(S) \setminus \{ -\frac{1}{12} \} \) is called **periodically repelling** if every periodic point of \( F_a \) (except the fixed points \( \frac{1}{4} \) and \( \alpha_0 \)) is repelling.

2) For a periodically repelling map \( F_a \) (respectively, \( f_c \)), we define the **real lamination** \( \lambda_\mathcal{R}(F_a) \) (respectively, \( \lambda_\mathcal{R}(f_c) \)) of \( F_a \) (respectively, of \( f_c \)) to be the smallest closed equivalence relation in \( \mathbb{R}/\mathbb{Z} \) containing \( \lambda(F_a) \) (respectively, \( \lambda(f_c) \)).

Note that by [LLM18 Proposition 6.25], a periodically repelling map \( F_a \) has no Fatou components; i.e. \( K_a = \Gamma_a \). Thus, the dynamics of a periodically repelling map \( F_a \) on its non-escaping set \( K_a \) is combinatorially modeled by the quotient of \( \rho : \mathbb{T} \rightarrow \mathbb{T} \) by the \( \rho \)-invariant lamination \( \lambda_\mathcal{R}(F_a) \). By an anti-holomorphic version of [Kiw01 Theorem 1.1] (which can be proved along the lines of Kiwi’s arguments), there exists some periodically repelling map \( f_c \in \mathcal{L} \) such that \( \lambda(f_c) = \mathcal{E}_c(\lambda(F_a)) \). Clearly, we have that \( K_c = J_c \), and the dynamics of \( f_c \) on its filled Julia set \( K_c \) is combinatorially modeled by the quotient of \( m_{-2} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} \) by the \( m_{-2} \)-invariant lamination \( \lambda_\mathcal{R}(f_c) \). Moreover, the combinatorial models of \( F_a : K_a \rightarrow K_a \) and \( f_c : K_c \rightarrow K_c \) are topologically conjugate by a factor of \( \mathcal{E} \).

Therefore, the dynamics of a periodically repelling map \( F_a \) can be decomposed into \( F_a : T^\infty_a \setminus T^0_a \rightarrow T^\infty_a \), which is conformally equivalent to \( \rho : \mathbb{D} \setminus \Pi \rightarrow \mathbb{D} \), and \( F_a : K_a \rightarrow K_a \), which is combinatorially equivalent to \( f_c : K_c \rightarrow K_c \). In this sense, every periodically repelling map \( F_a \) is a **combinatorial mating** of a (periodically
repelling) quadratic anti-polynomial $f_c$ (restricted to its filled Julia set), and the reflection map $\rho$.

**Proposition 8.8.** Every periodically repelling map $F_a$ is a combinatorial mating of a periodically repelling quadratic anti-polynomial $f_c : K_c \to K_c$ and the reflection map $\rho : \mathbb{D} \setminus \Pi \to \mathbb{D}$.

**Corollary 8.9.** If the limit set of $F_a$ and the Julia set of $f_c$ (appearing in Proposition 8.8) are locally connected, then $F_a$ is a conformal mating of $f_c$ (restricted to its Julia set) and the reflection map $\rho$.

### 8.4. Homeomorphism between Model Spaces.

In this subsection, we will construct a locally connected model $\widehat{\mathcal{C}}(S)$ of $\mathcal{C}(S)$, and show that it is homeomorphic to the combinatorial model $\widehat{L}$ of $L$.

The construction of $\widehat{\mathcal{C}}(S)$ will be similar to that of $\widehat{L}$ (compare Subsection 2.4). We first construct an equivalence relation on $\text{Per}(\rho) \cap \partial \mathbb{D}^2$ by identifying the angles of all parameter rays of $\mathcal{C}(S)$ at pre-periodic angles (under $\rho$) that land at a common (parabolic or Misiurewicz) parameter or accumulate on a common root parabolic arc of $\mathcal{C}(S)$. We also identify $1/3$ and $2/3$. We then consider the smallest closed equivalence relation on $\partial \mathbb{D} \cap \partial \mathbb{D}^2$ generated by the above relation. Taking the hyperbolic convex hull of each of these equivalence classes in $\mathbb{D}$, we obtain a geodesic lamination of $\mathbb{D}^2$ (by hyperbolic geodesics of $\mathbb{D}$). The abstract connectedness locus $\widehat{\mathcal{C}}(S)$ is defined as the quotient of $\mathbb{D}^2$ obtained by collapsing each of these hyperbolic convex hulls to a single point.

**Proof of Theorem 1.4.** By Corollaries 8.3 and 8.5, the parameter rays of $S$ at angles $\theta, \theta' \in \text{Per}(\rho)$ land at a common parabolic/Misiurewicz parameter or accumulate on a common root parabolic arc if and only if the parameter rays of $T$ at angles $E(\theta), E(\theta') \in \mathbb{Q}/\mathbb{Z}$ have the same property. Hence, $E \times E$ is a bijection between the equivalence relation on $\text{Per}(\rho) \cap \partial \mathbb{D}^2$ induced by the (co-landing or co-accumulation property of) parameter rays of $S$ at pre-periodic angles (under $\rho$) and the equivalence relation on $\mathbb{Q}/\mathbb{Z} \cap \partial \mathbb{D}^2$ induced by the rational parameter rays of $T$. Since $E$ is a homeomorphism, the smallest closed equivalent relations on $\partial \mathbb{D} \cap \partial \mathbb{D}^2$ generated by the above relations are also homeomorphic under $E \times E$. It follows that the corresponding geodesic laminations of $\mathbb{D}^2$ are topologically equivalent; i.e. there is a homeomorphism of $\mathbb{D}^2$ mapping the leaves and gaps of one lamination to those of the other. Clearly, this homeomorphism descends to a homeomorphism between the quotient spaces $\widehat{\mathcal{C}}(S)$ and $\widehat{L}$. \hfill \Box

### 9. Discontinuity of Straightening

By construction of the map $\chi$ (see Corollary 8.2), it is a homeomorphism between hyperbolic components of $\mathcal{C}(S)$ and $L$. Preserving critical Ecalle heights, we extended $\chi$ to the boundaries of odd period hyperbolic components. Thus, in light of Theorem 7.7, $\chi$ is defined in small neighborhoods of parabolic cusps. The goal of this section is to show that $\chi$ is not always continuous in neighborhoods of parabolic cusps. This is one of the reasons why we construct a homeomorphism between the models of the connectedness loci in a purely combinatorial way.

Let us now assume that $H \subset \mathcal{C}(S)$ is a hyperbolic component of odd period $k$, and $H' = \chi(H)$ is the corresponding hyperbolic component in $L$. Recall that $\partial H$ consists of three parabolic arcs and an equal number of parabolic cusps. The map
\(\chi\) sends parabolic cusps on \(\partial H\) to parabolic cusps on \(\partial H'\), and simple parabolic parameters on \(\partial H\) to simple parabolic parameters on \(\partial H'\) (preserving their parabolic orbit portraits and critical Ecalle heights).

The next proposition shows that \(\chi\) is a homeomorphism from \(\mathcal{H}\) onto \(\mathcal{H}'\).

**Proposition 9.1.** As the parameter \(a\) in \(H\) (respectively, \(c\) is \(H'\)) approaches a simple parabolic parameter with critical Ecalle height \(h\) on the boundary of \(H\) (respectively, \(H'\)), the quantity \(\frac{1}{|1-\zeta_H(a)|^2}\) (respectively, \(\frac{1}{|1-\zeta_H(c)|^2}\)) converges to \(\frac{1}{2} - 2ih\). Consequently, \(\chi\) maps the closure \(\mathcal{H}\) of the hyperbolic component \(H\) homeomorphically onto the closure \(\mathcal{H}'\) of the hyperbolic component \(H'\).

**Proof.** The proof of the first statement is similar to that of [ML16a Lemma 6.1]. Since \(\chi\) preserves Koenigs ratio (of parameters in \(H\)) and critical Ecalle height (of simple parabolic parameters on \(\partial H\)), it follows that \(\chi\) extends continuously to \(\partial H\). Since \(\chi\) is defined in an injective fashion and it is continuous on \(H\), it is a homeomorphism from \(\mathcal{H}\) onto \(\mathcal{H}'\). \(\square\)

Let \(H_1\) be a hyperbolic component of even period \(2k\) bifurcating from \(H\) across a parabolic arc \(C\). It is now easy to see from the proof of Lemma 8.1 that the hyperbolic component \(\chi(H_1)\) bifurcates from \(H' = \chi(H)\) across the parabolic arc \(\chi(C)\). We will denote the critical Ecalle height parametrization of the parabolic arc \(C\) (respectively, \(\chi(C)\)) by \(a : \mathbb{R} \to C\) (respectively, \(c : \mathbb{R} \to \chi(C)\)). Recall that for any \(h \in \mathbb{R}\), the fixed point index of the unique parabolic cycle of \(F_{a(h)}^{\circ 2}\) (respectively of \(f_{c(h)}^{\circ 2}\)) is denoted by \(\text{ind}_C(F_{a(h)}^{\circ 2})\) (respectively \(\text{ind}_{\chi(C)}(f_{c(h)}^{\circ 2})\)). This defines a pair of real-analytic functions (which we will refer to as index functions)

\[
\text{ind}_C : \mathbb{R} \to \mathbb{R}, \ h \mapsto \text{ind}_C(F_{a(h)}^{\circ 2})
\]

and

\[
\text{ind}_{\chi(C)} : \mathbb{R} \to \mathbb{R}, \ h \mapsto \text{ind}_{\chi(C)}(f_{c(h)}^{\circ 2}).
\]

We are now in a position to show that continuity of \(\chi\) on \(C\) imposes a severe restriction on the index functions \(\text{ind}_C\) and \(\text{ind}_{\chi(C)}\).

**Proposition 9.2** (Uniform Height-Index Relation). If \(\chi\) is continuous on \(C \cap \partial H_1\), then the functions \(\text{ind}_C\) and \(\text{ind}_{\chi(C)}\) are identically equal.

**Proof.** Let us pick a parameter \(a(h) \in C \cap \partial H_1\). Consider a sequence \(\{a_n\} \in H_1\) with \(a_n \to a(h)\). If \(\chi\) is continuous at \(a(h)\), then \(\chi(a_n) \to \chi(a(h)) = c(h)\). We will now show that \(\text{ind}_C(F_{a(h)}^{\circ 2}) = \text{ind}_{\chi(C)}(f_{c(h)}^{\circ 2})\).

Let, \(\text{ind}_C(F_{a(h)}^{\circ 2}) = \tau\). For any \(n\), the map \(F_{a_n}^{\circ 2}\) has two distinct \(k\)-periodic attracting cycles (which are born out of the parabolic cycle of \(F_{a(h)}^{\circ 2}\)) with multipliers \(\lambda_{a_n}\) and \(\overline{\lambda_{a_n}}\). Since \(a_n\) converges to \(a(h)\), we have that

\[
(5) \quad \frac{1}{1 - \lambda_{a_n}} + \frac{1}{1 - \overline{\lambda_{a_n}}} \to \tau
\]

as \(n \to \infty\).

Since the multipliers of attracting periodic orbits are preserved by \(\chi\), it follows that \(f_{\chi(a_n)}^{\circ 2}\) has two distinct \(k\)-periodic attracting cycles with multipliers \(\lambda_{a_n}\) and
Figure 18. The straightening map $\chi$, restricted to $H$, is a homeomorphism. On the other hand, continuity of $\chi$ on $C \cap \partial H_1$ would force parameters with equal critical Ecalle height to have the same parabolic fixed point index.

As $\{\chi(a_n)\}$ converges to the odd period parabolic parameter $c(h)$, the same limiting Relation (5) holds for the fixed point index of the parabolic cycle of $f_{c(h)}^2$ as well. In particular, the parabolic fixed point index of $f_{c(h)}^2$ is also $\tau$ (see Figure 18).

Since $h$ was arbitrarily chosen (with $|h|$ large enough so that $a(h)$ is a bifurcating parameter), we conclude that $\text{ind}_C(h) = \text{ind}_{\chi(C)}(h)$ for all $h$ in an unbounded interval. As the functions $\text{ind}_C$ and $\text{ind}_{\chi(C)}$ are real-analytic, it follows that $\text{ind}_C \equiv \text{ind}_{\chi(C)}$.

The above condition on index functions seems unlikely to hold in general. It can be numerically verified that the index function of the period 3 parabolic arc of $C(S)$ does not identically agree with the index function of the period 3 parabolic arc of $\mathcal{L}$. More precisely, it is easy to check that the parabolic cycles of the critical Ecalle height 0 parameters of these two arcs have different fixed point indices. This proves that the map $\chi$ is discontinuous on the period 3 parabolic arc of $C(S)$.

Remark 8. For an analogous discussion of discontinuity of straightening maps for the Tricorn, see [IM16a, Proposition 9.2].

References

[AS73] D. Aharonov and H. S. Shapiro. A minimal-area problem in conformal mapping - preliminary report. Research bulletin trita-mat-1973-7, Royal Institute of Technology, 1973.

[AS76] D. Aharonov and H. S. Shapiro. Domains on which analytic functions satisfy quadrature identities. J. Analyse Math., 30:39–73, 1976.

[AS78] D. Aharonov and H. S. Shapiro. A minimal-area problem in conformal mapping - preliminary report: Part ii. Research bulletin trita-mat-1978-5, Royal Institute of Technology, 1978.

[ASS99] D. Aharonov, H. S. Shapiro, and A. Solynin. A minimal area problem in conformal mapping. J. Analyse Math., 78:157–176, 1999.

[BL17a] S. Bullett and L. Lomonaco. Dynamics of modular matings. [https://arxiv.org/abs/1707.04764] 2017.

[BL17b] S. Bullett and L. Lomonaco. Mating quadratic maps with the modular group II. [https://arxiv.org/abs/1611.05257] 2017.
[BP94] S. Bullett and C. Penrose. Mating quadratic maps with the modular group. *Inventiones Mathematicae*, 115:483–511, 1994.

[CKS00] L. A. Caffarelli, L. Karp, and H. Shahgholian. Regularity of a free boundary problem with application to the pompeiu problem. *Annals of Mathematics*, 151:269–292, 2000.

[Dav74] P. J. Davis. *The Schwarz Function and its Applications*. Number 17 in Carus Math. Monographs. Math. Assoc. Amer., 1974.

[DH85] A. Douady and J. H. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. Ec. Norm. Sup.*, 18:287–343, 1985.

[DH07] A. Douady and J. H. Hubbard. Étude dynamique des polynômes complexes. Société Mathématique de France, 2007.

[Dud11] D. Dudko. Matings with laminations. [https://arxiv.org/abs/1112.4780](https://arxiv.org/abs/1112.4780), 2011.

[Dur83] P. L. Duren. *Univalent functions*, volume 259 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1983.

[EV92] P. Etingof and A. Varchenko. *Why the Boundary of a Round Drop Becomes a Curve of Order Four*, volume 3 of *University Lecture Series*. American Mathematical Society, Providence, R.I., 1992.

[GHMP00] B. Gustafsson, C. He, P. Milanfar, and M. Putinar. Reconstructing planar domains from their moments. *Inverse Problems*, 16:1053–1070, 2000.

[GM93] L. R. Goldberg and J. Milnor. Fixed points of polynomial maps II: Fixed point portraits. *Ann. Scient. École Norm. Sup.*, 4th série, 26:51–98, 1993.

[Gold92] Lisa R. Goldberg. Fixed points of polynomial maps I: Rotation subsets of the circles. *Ann. Scient. École Norm. Sup.*, 4th série, 25:679–685, 1992.

[Gus83] B. Gustafsson. Quadrature identities and the Schottky double. *Acta Appl. Math.*, 1:209–240, 1983.

[GV06] B. Gustafsson and A. Vasil’ev. *Conformal and Potential Analysis in Hele-Shaw Cell*. Birkhäuser Basel, 2006.

[GV19] T. Gauthier and G. Vigny. Distribution of postcritically finite polynomials III: combinatorial continuity. *Fundamenta Mathematicae*, 244:17–48, 2019.

[HS14] J. H. Hubbard and D. Schleicher. Multicorns are not path connected. In *Frontiers in Complex Dynamics: In Celebration of John Milnor’s 80th Birthday*, pages 73–102. Princeton University Press, 2014.

[Hub93] J. H. Hubbard. Local connectivity of Julia sets and bifurcation loci: Three theorems of J. C. Yoccoz. In *Topological methods in modern mathematics: Proceedings of the Milnor Symposium*, pages 467–511. Publish or Perish, Stony Brook, NY, 1993.

[IM16a] H. Inou and S. Mukherjee. Discontinuity of straightening in antiholomorphic dynamics. [https://arxiv.org/abs/1605.08061](https://arxiv.org/abs/1605.08061) 2016.

[IM16b] H. Inou and S. Mukherjee. Non-landing parameter rays of the multicorns. *Inventiones Mathematicae*, 204:869–893, 2016.

[Kiw01] Jan Kiwi. Rational laminations of complex polynomials. In Mikhail Lyubich, John W. Milnor, and Yair N. Minsky, editors, *Laminations and Foliations in Dynamics, Geometry and Topology*, volume 269 of *Contemporary Mathematics*, pages 111–154. American Mathematical Society, 2001.

[LLM18] S.-Y. Lee, M. Lyubich, N. G. Makarov, and S. Mukherjee. Dynamics of Schwarz reflections: the mating phenomena. [https://arxiv.org/abs/1811.04979](https://arxiv.org/abs/1811.04979) 2018.

[LM16] S. Y. Lee and N. G. Makarov. Topology of quadrature domains. *Journal of The American Mathematical Society*, 29:333–369, 2016.

[Lyu97] M. Lyubich. Dynamics of quadratic polynomials. I, II. *Acta Math.*, 178:185–247, 247–297, 1997.

[Lyu17] M. Lyubich. *Conformal Geometry and Dynamics of Quadratic Polynomials*, vol I-II. in preparation, 2017. [http://www.math.stonybrook.edu/~mlyubich/book.pdf](http://www.math.stonybrook.edu/~mlyubich/book.pdf)

[Mil00] J. Milnor. Periodic orbits, external rays and the Mandelbrot set. *Asterisque*, 261:277–333, 2000.

[Mil06] J. Milnor. *Dynamics in one complex variable*. Princeton University Press, New Jersey, 3rd edition, 2006.

[MNS17] S. Mukherjee, S. Nakane, and D. Schleicher. On Multicorns and Unicorns II: bifurcations in spaces of antiholomorphic polynomials. *Ergodic Theory and Dynamical systems*, 37:859–899, 2017.
[Moo25] R. L. Moore. Concerning upper-semicontinuous collections of continua. Transactions of the American Mathematical Society, 27(4):416–428, 1925.

[Muk15a] S. Mukherjee. Antiholomorphic Dynamics: Topology of Parameter Spaces, and Discontinuity of Straightening. PhD thesis, Jacobs University, Bremen, 2015. [http://nbn-resolving.de/urn:nbn:de:gbv:579-opus-1005222](http://nbn-resolving.de/urn:nbn:de:gbv:579-opus-1005222)

[Muk15b] S. Mukherjee. Orbit portraits of unicritical antiholomorphic polynomials. Conformal Geometry and Dynamics of the AMS, 19:35–50, 2015.

[Muk17] S. Mukherjee. Parabolic arcs of the multicorns: real-analyticity of hausdorff dimension, and singularities of Per_a(1) curves. Discrete and Continuous Dynamical Systems-A, 37:2565–2588, 2017.

[Nak93] S. Nakane. Connectedness of the tricorn. Ergodic Theory and Dynamical Systems, 13:349–356, 1993.

[Nak98] Isao Nakai. The classification of curvilinear angles in the complex plane and the groups of ± holomorphic diffeomorphisms. Annales de la faculté des sciences de Toulouse 6e série, 7:313–334, 1998.

[NS96] S. Nakane and D. Schleicher. The nonarcwise-connectedness of the tricorn. In Problems concerning complex dynamical systems (Japanese), number 959 in Sūrikaisekikenkyūshō Kōkyūroku, pages 73–83. Kyoto University, Research Institute for Mathematical Sciences, Kyoto, 1996.

[NS03] S. Nakane and D. Schleicher. On Multicorns and Unicorns I: Antiholomorphic dynamics, hyperbolic components and real cubic polynomials. International Journal of Bifurcation and Chaos, 13:2825–2844, 2003.

[Poi09] A. Fournier. Critical portraits for postcritically finite polynomials. Fundamenta Mathematicae, 203:107–163, 2009.

[Ric72] S. Richardson. Hele shaw flows with a free boundary produced by the injection of fluid into a narrow channel. J. Fluid Mech., 56:609–618, 1972.

[Sak78] M. Sakai. A moment problem in Jordan domains. Proc. Amer. Math. Soc., 70:35–38, 1978.

[Sak82] M. Sakai. Quadrature domains, volume 934 of Lect. Notes Math. Springer-Verlag, Berlin-Heidelberg, 1982.

[Sak91] M. Sakai. Regularity of a boundary having a Schwarz function. Acta Math., 166:263–297, 1991.

[Sch00] D. Schleicher. Rational parameter rays of the Mandelbrot set. Astérisque, 261:405–443, 2000.

[Sch04] D. Schleicher. On fibers and local connectivity of Mandelbrot and Multibrot sets. In Fractal geometry and applications: a jubilee of Benoît Mandelbrot, volume 72(1) of Proceedings of Symposia in Pure Mathematics, pages 477–517. American Mathematical Society, Providence, R.I., 2004.

[Sha92] H. Shahgholian. On quadrature domains and the Schwarz potential. J. Math. Anal. Appl., 171:61–78, 1992.

[SS00] T. Sheil-Small. Complex Polynomials. Cambridge University Press, 2000.

[Vor82] S. M. Voronin. Analytic classification of pairs of involutions and its applications. Funktsional. Anal. i Prilozhen, 16(2):21–29, 1982.

[War42] S. E. Warschawski. On conformal mapping of infinite strips. Transactions of the American Mathematical Society, 51(2):280–335, 1942.