Quantum energy teleportation across a three-spin Ising chain in a Gibbs state

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Abstract
In general, it is important to identify what is the informational resource for quantum tasks. Quantum energy teleportation (QET) is a quantum task, which attains energy transfer in an operational sense by local operations and classical communication, and is expected to play a role in future development of nano-scale smart grids. We consider QET protocols in a three-element Ising spin system with non-periodic boundary conditions coupled to a thermal bath. The open chain is the minimal model of QET between two edge spins that allows the measurement and operation steps of the QET protocol to be optimized without restriction. It is possible to analyze how two-body correlations of the system, such as mutual information, entanglement and quantum discord, can be resources of this QET at each temperature. In particular, we stress that quantum discord is not the QET resource in some cases, even if arbitrary measurements and operations are available.

Keywords: quantum energy teleportation, three spin ising chain, strong local passivity, entanglement, quantum dissonance

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum correlation has recently attracted much interest as a resource of quantum tasks including quantum parameter estimation [1–3], quantum teleportation [4] and quantum computing [5]. There exist many kinds of quantum correlation. One of the best known is quantum entanglement. Remote state transfer using quantum teleportation requires non-zero entanglement as the resource. However it is often stressed that entanglement is not the only quantum correlation. The concept of quantum discord, which was introduced in [6] and [7], can be non-zero even if entanglement exactly vanishes. Discord without entanglement is
referred to as quantum dissonance. In general, it is important to identify the correlation as the
information resource for every quantum protocol.

Quantum energy teleportation (QET) [8–11] is a quantum protocol, which attains energy
transfer in an operational sense. For quantum many-body systems, the interaction among
subsystems generates quantum correlations in the ground states. Since energy density
operators of the system do not commute with each other due to this interaction, uncertainty
relations yield zero-point fluctuation of energy density in the ground states. With a shift of the
energy such that the average energy density of the system is fixed as zero, the energy density
fluctuates around the zero value. Thus we have negative energy density in quantum theory.
This zero-point energy density fluctuations of two separate subsystems A and B are quantum
mechanically correlated. Hence, if we measure subsystem A and obtain the measurement
result \( \alpha \), this includes some information about the energy density around subsystem B. During
this measurement, a positive amount of energy \( E_A \) is injected into A, because the post-
measurement state is not the ground state but an excited state. This property is called the
passivity of the ground state. By performing an appropriate local operation on B, dependent
on \( \alpha \), the quantum fluctuation can be suppressed and a negative energy density appears
around B. Because of energy conservation, a positive amount of energy \( E_B \) is extracted by the
operation device on B. Note that B is in a state with zero energy before the measurement, thus
extraction of \( E_B \) looks like energy extraction from nothing. This is QET. \( E_A \) is regarded as the
input energy and \( E_B \) as the output energy of QET. Non-negativity of the total Hamiltonian
imposes \( E_A \geq E_B \) [12].

The QET protocol has attracted attention for future development of nano-scale smart
grids with low power consumption [13]. An experiment to verify QET using edge channel
currents in a quantum Hall system has been proposed [14]. QET is also related to information
thermodynamics. Quantum–Maxwell demons, by using the measurement results, are able to
adopt QET in order to extract more energy out of quantum systems [12]. In black hole
physics, QET plays a crucial role. By measuring the zero-point fluctuation of quantum fields
outside the horizon and performing QET, the area of the event horizon shrinks and its black
hole entropy decreases [15]. Even in low-temperature cases, QET remains effective. In Gibbs
states below a critical temperature, many-body systems which have ground states with
maximum-rank entanglement structure satisfy strong local passivity [16]. This means that a
positive amount of energy is injected during arbitrary local operations. Therefore, we are not
able to extract thermal energy from subsystem B by local operations alone. In the low-
temperature regime, thermal energy extraction requires global operations such as time evolu-
tion generated by the Hamiltonian. However, if we adopt a QET protocol, the passivity is
broken and some of the thermal energy can be extracted [17].

As with conventional quantum teleportation, it is of importance to understand what
correlation is the resource of QET. For the case of the ground state, it is known that quantum
entanglement is the resource of QET [12]. However, the finite-temperature case is non-trivial.
In a simple toy model with a two-qubit chain with non-demolition measurements of the
interaction Hamiltonian, quantum discord is seen to act as a resource for QET at high
temperatures [17]. However, so far other quantum systems have not been explored. In
addition, the two-spin model imposes a severe limitation on the QET optimization problem of
local measurements of the energy-sender qubit. Only non-demolition measurements are
available, which do not disturb the potential term between the two spins. Thus there exists no
consensus about the resource in general cases beyond the two-qubit system.

To avoid the limitation of available measurements, we consider a three-spin open chain
model and QET from one edge spin to another edge spin in this paper. The optimal mea-
surements of the energy-sender spin and optimal local unitary operations of the energy-
receiver spin, which provide the maximal amount of teleported energy, are determined in the ground-state case and the finite-temperature case. The maximum teleported energy is compared to various two-body correlations between two edge spins, including quantum mutual information, quantum discord [6, 7], concurrence [18], and negativity [19]. In contrast with the two-qubit chain, we show that quantum discord is not a perfect resource of QET in the three-qubit system. Through the variations of one of the parameters in the model, we found that the optimal teleported energy becomes zero in some cases at zero and finite temperature, where entanglement vanishes, but quantum discord is still present. This implies that quantum discord cannot become the QET resource of this system in this regime. In other words, we have found a regime where there is quantum discord between the edge spins but energy teleportation is not possible. This is an unexpected result and crucial for the problem of identification of the quantum task resource.

Because through the variation of one parameter in the model it is possible to modify the amount of teleported energy, such that it is impossible to have QET, we pose an analogy between energy transportation in the three-spin chain model and field effect transistors (FETs). This paper is structured as follows. Section 2 introduces our three-particle model and also outlines the derivation of the teleported energy when using projective measurements; the supporting calculations are given in the appendices. Section 3 explores the relationship between the QET protocol’s efficiency and the degree and type of quantum correlations in the system. Section 4 contains a detailed study, using the most general measurements that can be applied to the system, of the regime in which no teleported energy is possible, even though there are quantum correlations. Section 5 offers our conclusions.

2. Three-qubit model

Consider a system of three spin 1/2 particles, whose labels are 1, 2 and 3 respectively, with Hamiltonian

\[ H = \kappa (\sigma_{1,x} \sigma_{2,x} + \sigma_{2,z} \sigma_{3,z}) + \sigma_{2,z} + \lambda \sigma_{2,z} + \sigma_{3,z}. \] (1)

The model resembles the Ising chain model in a transverse magnetic field for only three elements in the chain, without the interaction term corresponding to the periodic boundary conditions associated with a ring topology. The \( \sigma \) operators are the Pauli operators, \( \kappa \) is a dimensionless parameter that takes real values and represents the relative strength of the coupling between spins and the intensity of the magnetic field, and \( \lambda \), also a dimensionless parameter, represents the strength of the coupling of the spin of particle 2 with the transverse magnetic field. The presence of these two parameters in the model allows us to study the two main factors necessary in order to teleport energy: the amount of correlation between subcomponents of the total system, and the ability to generate local negative energy density around subsystem B after Bob’s operation. If we restrict Bob to perform measurements on particle 3, then its local Hamiltonian is independent of \( \lambda \):

\[ H_B = \kappa \sigma_{2,z} \sigma_{3,z}. \] (2)

Therefore, though the parameter \( \lambda \) is unrelated to the generation of local negative energy at particle 3, it is related to the correlation between particles 1 and 3. In fact, for \( \lambda \to \infty \) the total Hamiltonian reduces to the \( z \) component of particle 2, meaning, in particular, that particles 1 and 3 become uncorrelated. The parameter \( \lambda \) also dictates the degeneracy of the
system: the system is completely degenerate when $\lambda = 0$ (because in this case the Hamiltonian (1) commutes with $\sigma_2$, ) and non-degenerate otherwise. On the other hand, the $\kappa$ parameter is related to the generation of local negative energy, since it is present on Bob’s local Hamiltonian (2), and to the correlations, since physically it represents the coupling between spins; in other words, in the limit $\kappa = 0$ there are no correlations. Specifically, fixing a value of $\kappa$ while changing the parameter $\lambda$ is equivalent to studying the effect of only the correlations of the system in the energy teleportation.

Since the Hamiltonian is dimensionless, so are the eigenergies (appendix A). If the three-particle system described by equation (1) is weakly coupled to a thermal bath with temperature $T$, the state of the system will be given by a Gibbs state (appendix A). Since the eigenergies are dimensionless, it is the temperature parameter $T$.

Let the three-particle system be partitioned into two subsystems, A and B, such that Alice and Bob, respectively, can make measurements on (just) A and B. The QET protocol can be applied to the system as follows.

Alice makes a projective measurement on subsystem A with output $\sigma$ ($\pm 1$):

$$M_A(\sigma) = \frac{1}{2} \left( I_A + \alpha \hat{r}_A \cdot \sigma_A \right)$$

(3)

$$\hat{r}_A = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$$

(4)

where the label A can refer to qubit 1, 2 or 3 and $\hat{r}_A$ is a unit vector to be chosen such that the maximum amount of teleported energy is achievable. Due to the strong passivity of the ground state, this measurement injects energy $E_A$ into the system. On the other hand, for a finite temperature $T$, the Gibbs states of a finite quantum system are strongly locally passive; in other words, any local operation, measurement included, will introduce further energy into the system below a critical temperature specific to the system and subsystem [16]. An additional condition for Alice’s operations is necessary in order to avoid direct energy input into Bob’s system:

$$[M_A(\sigma), V_{AB}] = 0$$

(5)

where $V_{AB}$ is the interaction term between Alice’s and Bob’s systems. Note that in the case where Alice measures only qubit 1, and Bob qubit 3, equation (5) is satisfied by the most general measurement $M_A$, since there is no interaction term $V_{13}$ in the Hamiltonian (1). This is the main difference between the three-qubit and the two-qubit system, and is the case on which the results of section 3 will be based. In section 4 we will work with more general measurements involving qubits 1 and 2.

In the second step of the QET protocol, Alice announces the measurement result $\sigma$ to Bob via a classical channel. The time evolution of the system has been neglected since it is assumed that the speed of the communication is greater than the energy diffusion of the system.

In the final step of the QET protocol, Bob performs at subsystem B a local operation $U_{B}(\alpha)$ dependent on Alice’s result $\alpha$. The operation $U_{B}(\alpha)$ is chosen such that it generates negative energy $-E_B$ on subsystem B, which is equivalent by energy conservation to the measurement yielding a positive energy $+E_B$. With the eventual cool-down of the system the input energy $E_A$ will compensate the local negative energy $-E_B$.

$$U_{B}(\alpha) = \exp \left[ -i\alpha (\mathbf{r}_B \cdot \sigma) \right]$$

(6)

$$\mathbf{r}_B = (\sin(\delta)\cos(\gamma), \sin(\delta)\sin(\gamma), \cos(\delta))$$

(7)
where \( \mathbf{r}_B \) is a vector chosen such that the maximum amount of teleported energy is achievable. In general, the label \( B \) can refer to qubit 1, 2 or 3, naturally with \( A \neq B \); however from now on Bob will be restricted to perform measurements only on qubit 3.

If the system is in equilibrium with a thermal bath at temperature \( T \), the state of the system will be given by a Gibbs state \( \rho = \rho(T) \) (appendix A); then the average energy \( E_A \) input by Alice’s measurement is given by

\[
E_A = \sum_{\alpha} \text{Tr} \left[ M^\dagger_A(\alpha)HM_A(\alpha)\rho \right] - \text{Tr} \left[ H\rho \right].
\] (8)

On the other hand, the average energy loss of the system after the QET protocol, in other words the average amount of teleported energy \( E_B \), can be calculated as the difference between the average energy after Alice’s operation and after Bob’s operation:

\[
E_B = \sum_{\alpha} \text{Tr} \left[ M^\dagger_A(\alpha)HM_A(\alpha)\rho \right] - \sum_{\alpha} \text{Tr} \left[ U^\dagger_B(\alpha)M^\dagger_A(\alpha)HM_A(\alpha)U_B(\alpha)\rho \right]
= \sum_{\alpha} \text{Tr} \left[ M^\dagger_A(\alpha)M_A(\alpha)U^\dagger_B(\alpha) \left\{ U_B(\alpha), H_B \right\} \rho \right].
\] (9)

The quantity \( E_B \) defined in the previous equation can be positive or negative; only in the case where it is positive will we have teleported energy. For the moment, let us assume that the unitary operation \( U_B(\alpha) \) is such that \( E_B \) is a positive quantity; then in order to qualify the effectiveness of the QET protocol, we introduce the efficiency \( \eta \), which allows us to compare dimensionless amount of energies, without specifying the energy scale of the system:

\[
\eta = 100 \times \frac{E_B}{E_A}.
\] (10)

3. Projective measurement results

The main result of this paper was obtained for the case in which we restrict Alice’s measurement to qubit 1, and Bob’s measurement to qubit 3. By using the results of the appendix (A), for the functions \( A, R \) and \( B \), functions of the temperature \( T \) and the parameters \( \kappa \) and \( \lambda \), the teleported energy can be written as

\[
E_B = A \sin(2r) - B(1 - \cos(2r))
\] (11)

\[
A = A \sin(\theta)\sin(\delta)\sin(\phi - \gamma)
\] (12)

\[
B = \sin^2(\delta)[2B \cos^2(\gamma) + R] - 2B.
\] (13)

The optimization of \( E_B \) in (11), such that the maximum amount of teleported energy is obtained, was done by the criteria of the second partial derivatives and the Hessian matrix, together with numerical calculations. The maximum amount of teleported energy is obtained when \( \theta = \pi/2, \phi = \pi/2 \), which implies a projective measurement of \( \sigma_3 \), by Alice, and \( \delta = \pi/2, \gamma = 0 \), which implies that the unitary operation of Bob is related to \( \sigma_3 \). Substituting these values in equation (11), and optimizing the last parameter \( r \),

\[
\tan(2r_0) = \frac{A}{R}
\] (14)
Then the maximum amount of teleported energy $E_B^{\text{max}}$ is given by

$$E_B^{\text{max}} = \sqrt{A^2(\lambda, \kappa, T)} + R^2(\lambda, \kappa, T) - R(\lambda, \kappa, T).$$  \hfill (15)

Since a separable ground state leads to zero teleported energy [12], it is thought that quantum correlations, particularly entanglement, are necessary in order to have teleported energy. Therefore the quantum correlations must be studied in detail. Since Alice and Bob are restricted to the edge spins 1 and 3 respectively, let us focus on the quantum correlations between particles 1 and 3. Considering only the edge spins, the reduced state of the system can be written as $\rho_{13} = \text{Tr}_2[\rho]$, with this density operator the concurrence [18] and the negativity [19] were calculated. The detailed calculation of the quantum correlations can be found in appendix B. The concurrence $C(\rho_{13})$ is a measure of quantum entanglement [18], that can be calculated in terms of the eigenvalues $\lambda_{j,C}$ of $\tilde{\rho}_{13} = \sigma^y \otimes \sigma^y \rho_{13} \sigma^y \otimes \sigma^y$: 

$$C(\rho_{13}) = \max \left\{ \sqrt{\lambda_{2,C}} - \sqrt{\lambda_{1,C}} - \sqrt{\lambda_{3,C}} - \sqrt{\lambda_{4,C}}, 0 \right\}. \hfill (16)$$

Another entanglement measure that is simple to calculate is the negativity $\mathcal{N}(\rho_{13})$ [19]. This is defined as the absolute sum of the negative eigenvalues $\lambda_{i,N}$ of $\rho_{13}^{(i)}$, the partial transpose of the density matrix $\rho_{13}$ with respect to qubit 1:

$$\mathcal{N}(\rho_{13}) = \sum_i \left| \lambda_{i,N} \right| - \lambda_{i,N}. \hfill (17)$$

However since both measurements only quantify the amount of entanglement, and there are other quantum correlations different from entanglement, we also calculated the discord [6, 7]. Quantum discord measures the totality of the quantum correlation, entanglement and otherwise, between two components of a quantum system. The range of the discord as a tool to study quantum systems covers physics such as the Heisenberg uncertainty principle [20], quantum key distribution protocols [21], a Heisenberg chains in a magnetic field [22], etc. In the case where there is no entanglement, the discord is called quantum dissonance. It has been proved that for some special cases of an assisted optimal state discrimination only dissonance is required [23].

The discord associated with measurement on particle 1 can be calculated analytically (appendix B). In mathematical terms the quantum discord $D_{13}(\rho_{13})$ is defined as the difference of the quantum mutual information $I(\rho_{13})$, which contains all the classical and quantum correlations between the subsystems, and the classical correlation $J(\rho_{13})$, which contains all the information that can be obtained through local measurements on one of the subsystems. If $S(\rho) = -\text{Tr}[\rho \log(\rho)]$ is the von Neumann entropy, then the quantum discord $D_{13}(\rho_{13})$ is given by

$$D_{13}(\rho_{13}) = I(\rho_{13}) - \max \rho_1 \left\{ J(\rho_{13}) \right\} \hfill (18)$$

$$I(\rho_{13}) = S(\rho_1) + S(\rho_3) - S(\rho_{13}) \hfill (19)$$

$$J(\rho_{13}) = S(\rho_1) - S(\rho_{13}|\rho_3). \hfill (20)$$

Figure 1 compares the maximum teleported energy and the quantum correlations in the ground state ($T = 0$). The expressions for the quantum correlations can be seen in appendix B. The teleported energy $E_B^{\text{max}}$, the quantum discord $D_{13}(\rho_{13})$, the concurrence $C(\rho_{13})$ and the negativity $\mathcal{N}(\rho_{13})$ show the same behavior. In the no-coupling limit $\kappa \to 0$, in which no quantum correlation exists, no energy teleportation is possible.
On the other hand, when \( \lambda = 0 \), the Hamiltonian of the system (1) commutes with the operator \( \sigma_{2x} \); therefore, all the eigenvalues are degenerate (appendix A). Taking explicitly this limit in equation (15), since the function \( \mathcal{R} \) is always positive for every value of \( \lambda, \kappa \) and \( T \), and \( A(0, \kappa, T) = 0 \) (appendix A), when \( \lambda = 0 \) it is not possible to teleport energy for the case of the ground state and also for a finite-temperature case. In particular, for the ground state, a detailed study of the negativity \( \mathcal{N}(\rho_{13}) \) and the concurrence \( C(\rho_{13}) \) shows that for \( \lambda = 0 \) there is no entanglement between qubits 1 and 3 (appendix B). In other words, even though there is dissonance, quantum correlations without entanglement, and we perform the most general projective measurements that can be applied to the system, it is not possible to teleport energy in the limit \( \lambda = 0 \), since the ground state is not entangled [12].

In figure 2 the behavior of the negativity \( \mathcal{N}(\rho_{13}) \) and the concurrence \( C(\rho_{13}) \), as a function of the \( \lambda \) parameter for \( T = 0 \) and two different values of the coupling parameter \( \kappa = \{1, 10\} \), among the teleported energy \( E_{B}^{\text{max}} \) and the quantum discord \( D_{13}(\rho_{13}) \) and mutual information \( I(\rho_{13}) \), can be found. In contrast with the negativity, concurrence and teleported energy, the discord and the mutual information are different from zero when \( \lambda = 0 \). It can also be seen that a large amount of teleported energy is associated with larger values of the coupling parameter \( \kappa \), limited to a non-large \( \lambda \), in other words a region of the phase space.

![Figure 1. For \( T = 0 \); teleported energy \( E_{B}^{\text{max}} \) (top left), quantum discord \( D_{13}(\rho_{13}) \) (top right), concurrence \( C(\rho_{13}) \) (bottom left) and negativity \( \mathcal{N}(\rho_{13}) \) (bottom right).](image-url)
in which the interaction term of the Hamiltonian (1) and the $\lambda$ term have meaningful contributions that are not overlapped between each other.

For the case of finite temperature, figure 3 shows teleported energy, quantum discord, concurrence and negativity for temperature $T = 1$. Similarly to the case of the ground state, it is not possible to teleport energy in the limit $\kappa = 0$, no correlations, and in the limit $\lambda = 0$. Furthermore, similarly to the case of $T = 0$, in the region of large $\lambda$ and small $\kappa$, in other words the region in which the Hamiltonian of the system (1) can be approximated to be only the $z$ component of particle 2, the value of teleported energy is reduced due to the decrease of the entanglement of the subsystem. As can be seen by comparing with the case of $T = 0$ (figure 1), the phase space in which it is possible to obtain the larger values of teleported energy shrinks with increasing temperature. This is a consequence of the decrease of the correlations, specifically entanglement, a natural behavior for quantum correlations.

Figure 2. Left: for $T = 0$ and $\kappa = 1$ teleported energy $E_B^{\max}$, measurements of entanglement, negativity $N(\rho_{13})$ and concurrence $C(\rho_{13})$; mutual information $I(\rho_{13})$ and quantum discord $D_{13}(\rho_{13})$. Right: for $T = 0$ and $\kappa = 10$ the same quantities as in the left-hand column.
With increasing temperature, the region of the phase space \((\kappa, \lambda)\) in which there is no entanglement expands from only \(\lambda = 0\) and \(\kappa = 0\) at \(T = 0\) to a non-trivial region whose area increases with the temperature. On the other hand, the quantum discord is different from zero in all the phase space \((\kappa, \lambda)\). Similarly, the teleported energy is different from zero but with the only exception of \(\lambda = 0\). In the right-hand column of figure 4 it can be seen that in the high-temperature limit, for \(\lambda = 10\) and \(\kappa = 2\), the quantum dissonance, and not entanglement, is the resource of the QET protocol, results similar to those found in [17] for the two-qubit system. However, in the left-hand column of figure 4 we can observe for \(T = 1\) and \(\kappa = 5\) the same behavior of the teleported energy, measures of entanglement and correlations as in the case of the ground state, meaning that only the quantum discord is different from zero in the limit of \(\lambda = 0\). This result is remarkable, since it shows that it is possible to have regimes with non-vanishing quantum dissonance in which it is impossible to teleport energy for a finite temperature.

The vanishing of the teleported energy \(E_{B}^{\text{max}}\) when \(\lambda = 0\) led us to make the analogy between the QET across the three-qubit open Ising chain and the FET. An FET is a charge carrier device with three terminals, named source (S), gate (G) and drain (D). Energy carriers enter the device on the source terminal, generating a current \(I_S\). By manipulation of the voltage of the gate \(V_{GS}\) the current \(I_D\) generated by the energy carriers leaving the device through the drain can be modified. In the case of the model, Alice’s qubit will be equivalent to the source, while Bob’s qubit will be equivalent to the drain. The energy input by Alice

![Figure 3. For \(T = 1\); teleported energy \(E_{B}^{\text{max}}\) (top left), quantum discord \(D_{13}(\rho_{13})\) (top right), concurrence \(C(\rho_{13})\) (bottom left) and negativity \(\mathcal{N}(\rho_{13})\) (bottom right).](image)
operation $E_A$ is analogous to the current generated by the energy carriers entering the source $I_S$, while the teleported energy $E_B^{\text{max}}$ is analogous to the current generated by the energy carriers leaving the drain $I_D$. The role of the voltage $V_{GS}$ will be played by the $\lambda$ parameter, representing the coupling between the spin of particle 2 and the transverse magnetic field. The limit $\lambda = 0$ corresponds to some value $V_{GS}^*$ for which the output current $I_D = 0$.

Since we are working with an energy dimensionless Hamiltonian, in order to fully appreciate the advantages of the QET protocol, it is necessary to calculate the efficiency $\eta$. In order to do so, we also needed the amount of energy that becomes the input of the protocol. Using the same optimization parameters from which the teleported energy was calculated (equation 15), the input energy $E_A$ can be calculated:

$$E_A = R(\lambda, \kappa, T) - 2B(\lambda, \kappa, T).$$ (21)
The definitions of the functions $R$ and $B$ (A25) make this quantity always positive for any value of the parameters $\lambda$ and $\kappa$ and for any temperature $T$, a necessary condition to obtain energy teleportation. The efficiency $\eta$ of the QET protocol can be seen in figure 5 for two different temperatures, $T = 0$ and $T = 1$. The maximum efficiency occurs in the case of the ground state, with a value of the order of 6%. Similarly to the teleported energy and the quantum correlations, for non-zero temperature, the maximum efficiency decreases with increasing temperature. The decrease of the correlations (figures 1 and 3) brings also a reduction of the phase space of the system in which the QET protocol can be applied efficiently. For $T = 0$, Gibbs states are strongly locally passive; in other words, no energy extraction is possible by local operations. Therefore, even with an efficiency of 6%, the QET protocol offers the possibility of energy extraction from the system.

4. General measurements for $\lambda = 0$

In this section we will study in more detail the behavior of the system when $\lambda = 0$. Let $K_A(\alpha)$ be the most general measurement that Alice can make on the system of the two qubits 1 and 2, while satisfying no energy input into qubit 3: a condition like equation (5).

$$K_A(\alpha) = [a_1 I_1 + b_1 \cdot \sigma_1] \otimes I_2 + [c_1 I_2 + d_1 \cdot \sigma_1] \otimes \sigma_2, \alpha$$

$$\sum_\alpha K_A^\dagger(\alpha)K_A(\alpha) = I_1 \otimes I_2$$

where $a$, $b$, $c$ and $d$ are complex coefficients that depend on the measurement result $\alpha$. Since these are general measurements, the result $\alpha$ is not restricted to take the values of $\pm 1$, as it was in the case of projective measurements. In fact, the $\alpha$ label in this case represents two possible results $\alpha_1$ and $\alpha_2$ that can be obtained from the measurement of qubit 1 and 2 respectively. In addition, let us consider the most general unitary operation $U_B(\alpha)$ that Bob can perform on qubit 3:

$$U_B(\alpha) = \exp[-i\alpha \cdot \sigma_3] = \cos(\alpha_3)I_3 - i\alpha_3 \cdot \sigma_3 \sin(\alpha_3)$$
where the dependence of Alice’s measurement result $\alpha$ is contained in the real vector $r_a = r_a e_a$. Then, similar to equation (9), the average energy loss of the system after the QET protocol can be written as

$$E_B = \sum_{\alpha} \text{Tr} \left[ K_\alpha^I(\alpha) K_\alpha(\alpha) U^I_B(\alpha) \left[ U_B(\alpha), H_B \right] \rho \right]$$

(24)

where $\rho$ is the Gibbs state of the system given by equation (A20) and $H_B$ (equation 2) is the local Hamiltonian around Bob. Since Bob’s operation is a unitary operation (equation 23), the energy loss can be written in terms of the amplitude of oscillation $r_a$ as

$$E_B = \sum_{\alpha} A_{\alpha} \sin(2r_a) - B_{\alpha} \sin(r_a)^2$$

(25)

$$A_{\alpha} = -\frac{i}{2} \text{Tr} \left[ K_\alpha^I(\alpha) K_\alpha(\alpha) \left[ \hat{e}_a \cdot \sigma_3, H_B \right] \rho \right]$$

(26)

$$B_{\alpha} = -\text{Tr} \left[ K_\alpha^I(\alpha) K_\alpha(\alpha) \left[ \hat{e}_a \cdot \sigma_3, H_B \right] \rho \right].$$

(27)

By explicit calculation of $A_{\alpha}$, without specifying the $\alpha$ dependence of the coefficients $a, b, c, d$ and the components of the unit vector $\hat{e}$, it can be shown that $A_{\alpha} = 0$ in the limit when $\lambda = 0$. To prove this result it is necessary to use equations (A26)–(A28) that can be found in appendix A. Therefore in this limit the average energy loss can be written as

$$E_B = -\sum_{\alpha} B_{\alpha} \sin(r_a)^2 \quad \lambda = 0.$$

(28)

As can be seen from the previous equation the sign of the energy loss on the limit $\lambda = 0$ depends only on the coefficient $B_{\alpha}$. If this is always positive, meaning that the system gains energy instead of losing it, then energy teleportation is not possible. Let us define from the coefficients of the general measurement $K_\alpha(\alpha)$ in equation (22) the complex column vector $V = (a, b_1, b_2, b_3, c, d_1, d_2, d_3)^T$, where the $T$ represents the transposition operation and the $\alpha$ dependence has been omitted for simplification of the notation. Similarly, let us write the unitary real vector $\hat{e}_a$ as a column vector $\hat{R} = (\hat{e}_a, \hat{e}_a, \hat{e}_a)^T$, where again the $\alpha$ dependence has been omitted. Then let us define the matrices of operators $M_{\beta\gamma}$ and $O_{jk}$, independent of $\alpha$, with the indexes $\beta$ and $\gamma$ from $\{1, 2, \ldots, 8\}$ and $jk$ from $\{1, 2, 3\}$ such that

$$K_\alpha^I(\alpha) K_\alpha(\alpha) = V^\dagger M V$$

(29)

$$O_{jk} = \sigma_{3,j} \left[ \sigma_{3,k}, H_B \right].$$

(30)

Then the coefficient $B_{\alpha}$ can be written as

$$B_{\alpha} = -\left( V^\dagger_j \hat{R}_j \right) \text{Tr} \left[ M_{\beta\gamma} O_{jk} \rho \right] \left( \hat{R}_k V_j^\dagger \right)$$

(31)

where the sum over repeated indexes is implied. Since the vectors $\hat{R}$ and $V$ depend on the measurement result $\alpha$, let us assume that it is possible to choose them such that $\hat{R} \otimes V$ is an eigenvalue of the $24 \times 24$ matrix $P$ with elements given by

$$P_{\beta\gamma} = -\text{Tr} \left[ M_{\beta\gamma} O_{jk} \rho \right].$$

Then to study the positivity of the coefficient $B_{\alpha}$ it is necessary to calculate the eigenvalues of $P$. 


\[
P = \begin{pmatrix}
p_0 & 0 & \kappa p_1 \\
0 & (1 + \kappa^2)p_0 & 0 \\
\kappa p_1 & 0 & \kappa^2 p_0
\end{pmatrix}
\]  
(32)

where the matrices \( p_0 \) and \( p_1 \) (A31) are 8 \times 8 complex matrices. The eigenvalues of \( P \) are each eightfold degenerate, and they are given by

\[
\text{Eig}(P) = \left\{ 0, \left(1 + \kappa^2\right) \left(C_1 \pm C_2 \sqrt{1 + \kappa^2}\right) \right\}
\]

(33)

where \( C_1 \) (A29) and \( C_2 \) (A30) are two real positive functions with \( C_1 - C_2 \sqrt{1 + \kappa^2} > 0 \). Therefore, all the eigenvalues are positive, which implies that it is impossible to teleport energy, even with general measurements. This is a remarkable result, since at the limit \( \lambda = 0 \) there are quantum correlations that could act as the resource for the QET.

5. Conclusions

We applied the QET protocol to a three-spin open chain model, from one edge spin to the other edge spin. We calculated an optimal QET protocol using projective measurements, defined such that the maximum amount of teleported energy is obtained, for the ground-state and finite-temperature case. For this optimal QET protocol, we obtained an efficiency of the order of 6% in the region in which it no energy extraction by local operations is possible due to the strong local passivity of the Gibbs states for \( T = 0 \). As opposed to local energy extraction by general operations, the QET protocol allows us to extract energy for every value of \( T, \kappa \) and \( \lambda \) with the exception of \( \lambda = 0 \).

For low-temperature regimes, negativity and concurrence are well correlated with the amount of optimal teleported energy and can be regarded as a QET resource. However, for high-temperature regimes, although the negativity and concurrence vanish at some critical temperature, energy teleportation is possible. In this case, a non-entanglement resource such as quantum dissonance yields high-temperature QET as in the two-qubit model. In addition, we found that the negativity and the concurrence between the edge spins are exactly zero in the case \( \lambda = 0 \). Even further, we proved that employing the most general operations on the system it is impossible to teleport energy in the case of \( \lambda = 0 \) and finite temperature. Even though there are quantum correlations, different from entanglement, no energy can be teleported, in contrast with the regime of high temperatures and \( \lambda \neq 0 \), in which it is possible to teleport energy due to the quantum dissonance.

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Appendix A. Eigenvalues and density matrix for the open chain

The spectrum of the Hamiltonian in equation (1) is composed by eight eigenvalues:

\[
E_0 = -E_7 = -\frac{1}{3}(\lambda + x_0)
\]  
\[
E_1 = -E_6 = -E_0 - \frac{1}{2}(x_0 + \frac{y_0}{\sqrt{3}})
\]  
\[
E_2 = -E_5 = -\lambda
\]  
\[
E_3 = -E_4 = -E_0 - \frac{1}{2}(x_0 - \frac{y_0}{\sqrt{3}})
\]

where we have defined

\[
x_0 = n_0 \cos \left(\frac{\theta}{3}\right) \quad y_0 = n_0 \sin \left(\frac{\theta}{3}\right)
\]
\[
n_0 = 4\sqrt{3 + 3\kappa^2 + \lambda^2
\]
\[
g(\lambda, \kappa) = \lambda^2(\kappa^2 + 20) + 4(3 + 3\kappa^2 + \kappa^4)
\]
\[
\tan (\theta) = \frac{27\left[4\left(\frac{\lambda^2}{\kappa^2} - 1\right)^2 + \kappa^2 g(\lambda, \kappa)\right]}{\lambda \left[16 - 9\kappa^2 - 2\left(\frac{\lambda^2}{\kappa^2} - 1\right)\right]}
\]

The eigenvalues are symmetric with the exchange of \(\kappa\) with \(-\kappa\). For \(\lambda > 0\) the lowest eigenvalue is \(E_0\); on the other hand, for \(\lambda < 0\) the lowest eigenvalue is \(E_1\). For \(\lambda = 0\) the system described by equation (1) is completely degenerated and the energies are

\[
E_0 = E_1 = -E_7 = -E_6 = -2\sqrt{1 + \kappa^2}
\]
\[
E_2 = -E_5 = E_3 = -E_4 = 0.
\]

This degeneracy of the ground state only occurs in the physical limit of \(\lambda = 0\). For any value of \(\lambda\), and the eigenvalues \(E_2\) and \(E_5\), the corresponding eigenvectors are

\[
|E_2\rangle = \frac{1}{\sqrt{2}}[-|011\rangle + |110\rangle]
\]
\[
|E_5\rangle = \frac{1}{\sqrt{2}}[-|001\rangle + |100\rangle].
\]

The eigenvectors \(|E_{AC}\rangle\) are associated with the eigenenergies \(E = \{E_1, E_3, E_7\}\).

\[
(E + \lambda)(E - 2 - \lambda)(E + 2 - \lambda) = 4\kappa^2(E - \lambda)
\]
\[
|E_{AC}\rangle = \frac{1}{N_{AC}}[A|000\rangle + |011\rangle + C|101\rangle + |110\rangle]
\]

where the normalization constant \(N_{AC}\) and the probability amplitudes \(A\) and \(C\) are functions of the associated eigenvalue \(E\).
\[ A = \frac{2\kappa}{E - 2 - \lambda} \quad C = \frac{2\kappa}{E + 2 - \lambda} \]

\[ h(\lambda, E) = 8(\lambda^2 - 1)(E - \lambda - 2)(E - \lambda + 2) \]

\[ i(\lambda, E) = 16\kappa^2 [(E - 2\lambda)(E - \lambda) + 2] \]

\[ P_{AC} = \frac{1}{h(\lambda, E) + i(\lambda, E)} \]

\[ \frac{1}{N_{AC}} = \sqrt{P_{AC}(E - \lambda + 2)^2(E - \lambda - 2)^2}. \]

On the other hand, the eigenvectors \( |E_{DF}\rangle \) are associated with the eigenenergies \( E = \{ E_0, E_4, E_6 \} \), solutions of the same equation as the eigenvalues associated with \( |E_{AC}\rangle \) with the exchange of \( \lambda \) with \( -\lambda \):

\[(E - \lambda)(E - 2 + \lambda)(E + 2 + \lambda) = 4\kappa^2(E + \lambda) \quad (A11)\]

\[ |E_{DF}\rangle = \frac{1}{N_{DF}}[|001\rangle + F|010\rangle + |100\rangle + D|111\rangle] \quad (A12)\]

\[ F(\lambda) = A(-\lambda) \quad D(\lambda) = C(-\lambda) \quad \frac{1}{N_{DF}(\lambda)} = \frac{1}{N_{AC}(-\lambda)}. \]

In particular, for \( \lambda = 0 \), due to the degeneracy of the system (A6) it is possible to rewrite the eigenstate for the ground state \(|g\rangle\) as a linear combination of \( |E_{AC}\rangle \) with energy \( E = E_1 \) and \( |E_{DF}\rangle \) with energy \( E = E_0 \):

\[ |g\rangle = a|E_{AC}\rangle + b|E_{DF}\rangle \quad (A13)\]

where one of the coefficients \( a \) and \( b \) will be determined by the normalization condition. This is possible since in this limit \( E_0 = E_1 = -2\sqrt{1 + \kappa^2} \), and a linear combination of eigenstates of a Hamiltonian with the same eigenvalue is also an eigenstate of the Hamiltonian.

Considering that the Hamiltonian (1) commutes with the Pauli operator \( \sigma_z \), it is possible to have a common base of eigenstates for both operators; then after a few calculations it is possible to prove that:

\[ |g\rangle = |\phi_1\rangle|\psi_2\rangle |\phi_3\rangle \quad (A14)\]

\[ |\phi_1\rangle = \alpha|0\rangle_1 + \beta|1\rangle_1 \quad (A15)\]

\[ |\psi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle_2 \pm |1\rangle_2) \quad (A16)\]

\[ |\phi_3\rangle = \gamma|0\rangle_3 + \delta|1\rangle_3 \quad (A17)\]

where the subscripts after the kets are related to the particle label. The probability amplitudes, in terms of only \( D \) and \( F \), evaluated for \( \lambda = 0 \) and \( E = E_0 \), are given by:

\[ |\alpha|^2 = \frac{1}{1 + D^2} \quad |\beta|^2 = \frac{D^2}{1 + D^2} \]

\[ |\gamma|^2 = \frac{1 + D^2}{D^2(F^2 + D^2 + 2)} \quad |\delta|^2 = \frac{1 + D^2}{F^2 + D^2 + 2}. \]
In other words, the ground state of the system can be written as a totally separable state in the limit of $\lambda = 0$. If the ground state is given by equation (A14), then there is no entanglement or classical correlation between qubits 1 and 3. This is possible since this ground state was constructed as a linear superposition of ground states, with the correct coefficients $a$ and $b$. However, assuming that it is possible to achieve the limit $T = 0$ and $\lambda = 0$, in the case where both parameters are gradually decreased, the ground state of the system will be given by $|E_{000}\rangle$ with $E = E_0 = -2\sqrt{1 + \kappa^2}$ for $\lambda = 0$, a non-separable state. Similar calculations are possible for the remaining six eigenstates associated with the energies $E_0, 2, 1, 2$. Therefore, since the system is degenerate, the lack or presence of correlations between spins 1 and 3, for the case $\lambda = 0$, will depend on the state in which the system is chosen to be.

If the system of the three particles is weakly coupled to a thermal bath at temperature $T$, then the probability that the system has an energy $E$ will be given as

$$p_j(T) = \frac{1}{Z} \exp\left(-\frac{E_j}{T}\right)$$

where the temperature $T$, similarly to the energies, is a dimensionless parameter, and $Z$ is the partition function in the canonical ensemble. The state of the system of the three particles in thermal equilibrium at temperature $T$, on the computational basis $\{000, 001, \ldots, 111\}$, and the partition function $Z$ are given by

$$Z = \sum_{j=0}^{3} \exp\left(-\frac{E_j}{T}\right) = 2 \left[ \cosh\left(\frac{E_0}{T}\right) + \cosh\left(\frac{E_1}{T}\right) + \cosh\left(\frac{E_2}{T}\right) + \cosh\left(\frac{E_3}{T}\right) \right]$$

$$\rho(T) = \sum_{j=0}^{3} p_j(T) |E_j\rangle \langle E_j|$$

$$\begin{pmatrix}
F_1 & 0 & 0 & F_2 & 0 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & F_{11} & F_{12} \\
0 & J_3 + p_5 & J_2 & 0 & J_3 - p_5 & 0 & 0 & J_5 \\
0 & J_2 & J_1 & 0 & J_2 & 0 & 0 & J_4 \\
F_2 & 0 & 0 & F_3 + p_5 & 0 & F_3 & F_3 - p_5 & 0 \\
0 & J_3 - p_5 & J_2 & 0 & J_3 + p_5 & 0 & 0 & J_5 \\
F_4 & 0 & 0 & F_5 & 0 & F_6 & F_5 & 0 \\
F_2 & 0 & 0 & F_3 - p_5 & 0 & F_3 & F_3 + p_5 & 0 \\
0 & J_5 & J_4 & 0 & J_5 & 0 & 0 & J_6 \\
\end{pmatrix}$$

where the $F_i = F[i]$ and $J_i = J[i]$ for $i = \{1, 2, 3, 4, 5, 6\}$ are functionals of $f_i(E)$ and $j_i(E)$ functions, that are associated with the group of eigenenergies $\{E_1, E_3, E_7\}$ and $\{E_0, E_4, E_6\}$ respectively. The functionals, which will also depend on the probabilities $p_j$ of the canonical ensemble (A18), are defined as follows:

$$F_i = F_i = 2 \sum_{k=1,3,7} f_k(E_k)p_k(T)$$

$$J_i = J_i = 2 \sum_{k=0,4,6} j_k(E_k)p_k(T).$$
The functions \( f_i \) and \( j_i \) are built with polynomials of the eigenvalue \( E \). In order to obtain the simplest representation of these functions, the characteristic equation for each set of eigenvalues \((E_1, E_3, E_7)\) and \((E_2, E_4, E_6)\) was used. The functions were found to be as follows:

\[
\begin{align*}
    f_1(E) &= P_{AC} \, 4\kappa^2(E + 2 - \lambda)^2 \\
    f_2(E) &= P_{AC} \, 2\kappa(E + 2 - \lambda)^2(E - 2 - \lambda) \\
    f_3(E) &= P_{AC} \, (E + 2 - \lambda)^2(E - 2 - \lambda)^2 \\
    f_4(E) &= P_{AC} \, 4\kappa^2(E + 2 - \lambda)(E - 2 - \lambda) \\
    f_5(E) &= P_{AC} \, 2\kappa(E - 2 - \lambda)^2(E + 2 - \lambda) \\
    f_6(E) &= P_{AC} \, 4\kappa^2(E - 2 - \lambda)^2 \\
    j_k(E) &= f_k(E) \text{ with } \lambda \rightarrow -\lambda \text{ for all } k = 1, 2, \ldots, 6.
\end{align*}
\]

Since the eigenenergies \( E_0, E_4, E_6 \) are obtained from \( E_2, E_3, E_1 \) with the change \( \lambda \rightarrow -\lambda \), and the same for the functions \( j_k(E) \) from \( f_k(E) \), any expression containing both functionals in the form \( F_i \pm J_j \) for any \( k = 1, 2, \ldots, 6 \) is symmetric under the exchange of \( \lambda \) with \( -\lambda \). On the other hand, with respect to the coupling parameter \( \kappa \), \( F_2, F_5, J_2, J_5 \) are odd functions, while the rest are even functions. For all values of \( \lambda, \kappa \) and \( T \) the functions \( F_i \) and \( J_i \) satisfy

\[
\begin{align*}
    2F_4 + \kappa(F_5 - F_2) &= 0 \\
    2J_4 + \kappa(J_5 - J_2) &= 0 \\
    \frac{1}{2}(F_1 + F_6 + J_1 + J_6) + F_3 + J_3 + p_2 + p_5 &= \sum_{j=0}^{7} p_j = 1.
\end{align*}
\]

With the definitions of the \( F \) and \( J \) functionals, the following even functions with respect to both parameters \( \kappa \) and \( \lambda \) were defined:

\[
\begin{align*}
    A(\lambda, \kappa, T) &= F_4 - F_5 - J_3 + J_4 + p_2 + p_5 \quad (A23) \\
    R(\lambda, \kappa, T) &= -\frac{1}{2}(F_1 + J_1 - F_6 - J_6) \quad (A24) \\
    B(\lambda, \kappa, T) &= \frac{\kappa}{2}(F_2 + F_5 + J_2 + J_5). \quad (A25)
\end{align*}
\]

Through numerical evaluation and analytical calculations it was verified that \( 0 \leq R(\lambda, \kappa, T) \leq 1 \) and \( B(\lambda, \kappa, T) < 0 \) for all values of the coupling parameter \( \kappa, \lambda \) and finite temperature \( T \). In particular for \( \lambda = 0 \), as can be done by analytical calculation, the function \( A(0, \kappa, T) = 0 \), since the functionals satisfy

\[
\begin{align*}
    F_4 - F_5 + p_2 &= 0 \quad \lambda = 0 \quad (A26) \\
    2(F_2 + F_5) &= \kappa(F_1 - F_6) \quad \lambda = 0 \quad (A27) \\
    2F_5 &= \kappa(2p_2 + F_4 - F_6) \quad \lambda = 0. \quad (A28)
\end{align*}
\]
In this limit of $\lambda = 0$ it is convenient also to define the functions $C_1 = F_0 - F_1$ and $C_2 = F_1 - 2F_2 + F_0 - 2\rho_2$, which can be written as

$$
\frac{4}{Z\sqrt{1 + \kappa^2}} \sinh \left( \frac{2\sqrt{1 + \kappa^2}}{T} \right) \tag{A29}
$$

$$
\frac{4}{Z(1 + \kappa^2)^{\frac{3}{2}}} \left[ \cosh \left( \frac{2\sqrt{1 + \kappa^2}}{T} \right) - 1 \right]. \tag{A30}
$$

As can be seen both functions are always positive for any value of the temperature $T$ and the coupling parameter $\kappa$. In addition, they satisfy $C_1 \geq \sqrt{1 + \kappa^2}C_2$. To end this appendix we will define the two matrices that were built with the previous functions:

$$
\rho_0 = \begin{pmatrix}
C_1 & 0 & 0 & -C_2 & 0 & -C_2k & 0 & 0 \\
0 & C_1 & -iC_2 & 0 & -C_2k & 0 & 0 & 0 \\
0 & iC_2 & C_1 & 0 & 0 & 0 & 0 & -iC_2k \\
-C_2 & 0 & 0 & C_1 & 0 & 0 & iC_2k & 0 \\
0 & -C_2k & 0 & 0 & C_1 & 0 & 0 & -C_2 \\
-C_2k & 0 & 0 & 0 & 0 & C_1 & -iC_2 & 0 \\
0 & 0 & 0 & -iC_2k & 0 & iC_2 & C_1 & 0 \\
0 & 0 & 0 & -iC_2k & 0 & -C_2 & 0 & 0 & C_1
\end{pmatrix}
$$

$$
\rho_1 = \begin{pmatrix}
C_2k & 0 & 0 & C_1 & 0 & 0 & C_2 \\
C_2k & 0 & 0 & 0 & 0 & C_1 & iC_2 & 0 \\
0 & 0 & 0 & iC_2k & 0 & -iC_2 & C_1k & 0 \\
0 & 0 & 0 & -iC_2k & 0 & C_2 & 0 & 0 & C_1 \\
C_1 & 0 & 0 & C_2 & 0 & C_2k & 0 & 0 \\
C_1 & iC_2 & 0 & C_2k & 0 & 0 & 0 & 0 \\
0 & -iC_2 & C_1 & 0 & 0 & 0 & 0 & iC_2k \\
C_2 & 0 & 0 & C_1 & 0 & 0 & -iC_2k & 0
\end{pmatrix}
$$

**Appendix B. Measures of quantum correlations of the reduced system $\rho_{13}$**

In this appendix the calculation of several quantum correlations for the reduced system $\rho_{13}$ will be explained. In particular, the concurrence, the negativity and the discord will be calculated. The state of the reduced system $\rho_{13}$ is obtained by taking the trace over qubit 2 of the state of the system $\rho$

$$
\rho_{13} = \text{Tr}_2[\rho] = \frac{1}{4} \begin{pmatrix}
1 + r + s + c_3 & 0 & 0 & c_1 - c_2 \\
0 & 1 + r - s - c_3 & c_1 + c_2 & 0 \\
0 & c_1 + c_2 & 1 - r + s - c_3 & 0 \\
c_1 - c_2 & 0 & 0 & 1 - r - s + c_3
\end{pmatrix} \tag{B1}
$$
$r = s = \frac{1}{2}(F_1 + J_1 - F_6 - J_6) = -R(\lambda, \kappa, T)$

$c_1 = F_4 + 4 + F_3 - J_4 - p_2 - p_5$
$c_2 = F_1 + J_5 - p_2 - p_5 - F_4 - J_4 = -A(\lambda, \kappa, T)$
$c_3 = \frac{1}{2}(F_1 + J_1 + F_6 + J_6) - (F_3 + J_3 + p_2 + p_5)$.

The labeling of $r, s, c_1, c_2, c_3$ was chosen in order to follow [24], where the quantum correlations for two qubits in an X state was studied. The eigenvalues of state $\rho_{13}$, to be called $\Lambda_j$, not to be confused with the $\lambda$ parameter of the three-qubit model, can be calculated in terms of the new labeling as

$A_1 = \frac{1}{4}(1 - c_3 - \sqrt{(c_1 + c_2)^2 + (r - s)^2})$
$A_2 = \frac{1}{4}(1 - c_3 + \sqrt{(c_1 + c_2)^2 + (r - s)^2})$
$A_3 = \frac{1}{4}(1 + c_3 - \sqrt{(c_1 - c_2)^2 + (r - s)^2})$
$A_4 = \frac{1}{4}(1 + c_3 + \sqrt{(c_1 - c_2)^2 + (r - s)^2})$.

The concurrence $C(\rho_{13})$, is a measure of quantum entanglement [18], that can be calculated in terms of the eigenvalues $A_C$ of $\rho_{13}^{\text{tr}}$, where $\rho_{13}^{\text{tr}} = (\sigma_1^y \otimes \sigma_3^y)\rho_{13}^* (\sigma_1^y \otimes \sigma_3^y)$. These eigenvalues $A_C$ are given by

$A_{1,C} = \frac{1}{16}(c_1 - c_2 - \sqrt{(1 + c_3)^2 - (r + s)^2})^2$
$A_{2,C} = \frac{1}{16}(c_1 - c_2 + \sqrt{(1 + c_3)^2 - (r + s)^2})^2$
$A_{3,C} = \frac{1}{16}(c_1 + c_2 - \sqrt{(1 - c_3)^2 - (r - s)^2})^2$
$A_{4,C} = \frac{1}{16}(c_1 + c_2 + \sqrt{(1 - c_3)^2 - (r - s)^2})^2$

$C(\rho_{13}) = \max \left\{ 2 \max \left\{ \sqrt{A_{1,C}}, \sqrt{A_{2,C}}, \sqrt{A_{3,C}}, \sqrt{A_{4,C}} \right\} \right.$
$- \sqrt{A_{1,C}} - \sqrt{A_{2,C}} - \sqrt{A_{3,C}} - \sqrt{A_{4,C}}, 0 \right\}$.

With the definitions of $r, s, c_1, c_2, c_3$ and through computational calculations the concurrence $C(\rho_{13})$ can be written as

$C(\rho_{13}) = \max \left\{ \sqrt{A_{2,C}} - \sqrt{A_{1,C}} - \sqrt{A_{3,C}} - \sqrt{A_{4,C}}, 0 \right\}$. (B2)

Another entanglement measure that is simple to calculate is the negativity $\mathcal{N}(\rho_{13})$ [19]. This is defined as the absolute sum of the negative eigenvalues of the partial transpose of the density matrix $\rho_{13}$ with respect to qubit 1.
\[ \mathcal{N}(\rho_{13}) = \sum_{i} \frac{|\Lambda_{i,N}| - \Lambda_{i,N}}{2} \]  

where the eigenvalues \( \Lambda_{i,N} \) are the eigenvalues of \( \rho_{13}^{(T)} \). Due to the symmetry of the X states, the partial transpose with respect to qubit 1 of state \( \rho_{13} \) is obtained by the exchange of the elements \((a)\) on the antidiagonal: \( a_{14} \) with \( a_{32} \), and \( a_{23} \) with \( a_{41} \), which is equivalent to exchanging \( c_{2} \) with \(-c_{2} \).

\[ \rho_{13}^{(T)} = \rho_{13} \quad \text{with} \quad c_{2} \rightarrow -c_{2} \]  
\[ \Lambda_{i,N} = \Lambda_{i} \quad \text{with} \quad c_{2} \rightarrow -c_{2} \quad \text{for} \ i = 1, 2, 3, 4. \]

By taking the limit \( \lambda = 0 \) it is possible to prove analytically that the eigenvalues \( \Lambda_{i,N} \) are all positive for every value of the coupling parameter \( \kappa \) and the temperature \( T \); therefore there is no entanglement between qubits 1 and 3.

The quantum discord \([6, 7]\) is a measure of the non-classical correlations between two subsystems of a quantum system. The discord includes all the quantum correlations, not only entanglement. In mathematical terms the quantum discord \( D_{13}(\rho_{13}) \) is defined as the difference of the quantum mutual information \( I(\rho_{13}) \), which contains all the classical and quantum correlations between the subsystems, and the classical correlations \( J(\rho_{13}) \), which contains all the information that can be obtained through local measurements on one of the subsystems.

\[ D_{13}(\rho_{13}) = I(\rho_{13}) - \max \left\{ J(\rho_{13}) \right\} \]  
\[ I(\rho_{13}) = S(\rho_{1}) + S(\rho_{3}) - S(\rho_{13}) \]  
\[ J(\rho_{13}) = S(\rho_{1}) - S(\rho_{13}|\hat{\rho}^{e}) \]

where \( S(\rho) = -\text{Tr}[\rho \log(\rho)] \) is the von Neumann entropy. The reduced density matrices \( \rho_{1}, \rho_{3} \) are obtained after taking the partial trace of state \( \rho_{13} \) with respect to subsystems 3 and 1 respectively.

\[ \rho_{1} = \rho_{3} = \frac{1}{2} \left[ \begin{array}{cc} 1 + r & 0 \\ 0 & 1 - r \end{array} \right] \]

With the following definition, and the eigenvalues \( \Lambda_{i} \) of \( \rho_{13} \), the mutual information can be written as

\[ h(x) = \frac{1 + x}{2} \log \left( \frac{2}{1 + x} \right) + \frac{1 - x}{2} \log \left( \frac{2}{1 - x} \right) \]

\[ I(\rho_{13}) = 2 h(r) + \sum_{i=1}^{4} \Lambda_{i} \log(\Lambda_{i}) \]

To calculate the discord \( D_{13}(\rho_{13}) \) a maximization over all possible measurements \( \hat{\Pi}_{1}^{n} \) over subsystem 1 is necessary; the maximization is introduced to eliminate any dependence of the discord on the measurements. The calculation of the classical correlations \( \max \left\{ J(\rho_{13}) \right\} \) is similar to that in \([17]\). Let \( \hat{\Pi}_{1}^{a} \) be the local projection measurements to be applied on qubit 1.

\[ \hat{\Pi}_{1}^{a} = \frac{1}{2} (I_{1} + \alpha \hat{\mathbf{r}}_{1} \cdot \mathbf{\sigma}) \quad \alpha = \pm 1 \]

\[ \hat{\mathbf{r}}_{1} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)). \]
The maximization over the measurements $\hat{H}_i$ implies by equation (B6) the calculation of the minimum:

$$\min_{\hat{H}_i} \left\{ S(\rho_{13}|\rho_{13}^a) \right\} = \min_{\hat{H}_i} \sum_a q_a S(\rho_a)$$

$$\rho_a = \frac{1}{q_a} \left( \hat{H}_i^a \otimes I_3 \right) \rho_{13} \left( \hat{H}_i^a \otimes I_3 \right)$$

$$q_a = \frac{1}{2} (1 + \alpha r \cos(\theta)).$$

The eigenvalues $\lambda_a$ of $\rho_a$ were found to be

$$\lambda_a = \left\{ 0, 0, \frac{1}{2} \pm \frac{1}{2} \sqrt{f(\theta, \phi) + \alpha \cdot 2r \cdot c_3 \cdot \cos(\theta)} \right\}$$

where

$$f(\theta, \phi) = r^2 + c_1^2 \cos(\theta)^2 + \sin(\theta)^2 \left( c_1^2 \cos(\phi)^2 - c_2^2 \sin(\phi)^2 \right).$$

The minimization was done through the calculation of the Hessian matrix and numerical calculations. The minimization is achieved when $\phi = 0$ and $\theta = \pi/2$ for all values of $\kappa$, $\lambda$ and $T$. Therefore the classical correlations $\max_{\hat{H}_i} \left\{ J(\rho_{13}) \right\}$ can be written as

$$\min_{\hat{H}_i} \left\{ S(\rho_{13}|\rho_{13}^a) \right\} = h \left( \sqrt{r^2 + c_1^2} \right)$$

(B10)

$$\max_{\hat{H}_i} \left\{ J(\rho_{13}) \right\} = h(r) - h \left( \sqrt{r^2 + c_1^2} \right).$$

(B11)

The quantum discord $D_{13}(\rho_{13})$ will be obtained by the subtraction of the mutual information (B9) and the classical correlations (B11).

$$D_{13}(\rho_{13}) = h(r) + \sum_{j=1}^{4} A_j \log(A_j) + h \left( \sqrt{r^2 + c_1^2} \right).$$

(B12)

A similar calculation can be done to obtain the quantum correlations when the measurements are made on qubit 3. It was found that the discord is symmetric for the system defined by $\rho_{13}$:

$$D_{13}(\rho_{13}) = D_{31}(\rho_{13}).$$

(B13)

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