NEW GENERAL DECAY RESULTS IN A FINITE-MEMORY BRESSE SYSTEM

SALIM A. MESSAOUDI* AND JAMILU HASHIM HASSAN

Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
P.O. Box 546, Dhahran 31261, Saudi Arabia
(Communicated by Alain Miranville)

Abstract. This paper is concerned with the following memory-type Bresse system
\[ \begin{align*}
\rho_1 \phi_{tt} - k_1 (\phi_x + \psi + lw)_x - lk_3 (w_x - l\phi) &= 0, \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\phi_x + \psi + lw) + \int_0^t g(t-s)\psi_{xx}(\cdot,s)ds &= 0, \\
\rho_1 w_{tt} - k_3 (w_x - l\phi)_x + lk_1 (\phi_x + \psi + lw) &= 0,
\end{align*} \]
with homogeneous Dirichlet-Neumann-Neumann boundary conditions, where \((x,t) \in (0,L) \times (0,\infty)\), \(g\) is a positive strictly increasing function satisfying, for some nonnegative functions \(\xi\) and \(H\),
\[ g'(t) \leq -\xi(t)H(g(t)), \quad \forall t \geq 0. \]
Under appropriate conditions on \(\xi\) and \(H\), we prove, in cases of equal and non-equal speeds of wave propagation, some new decay results that generalize and improve the recent results in the literature.

1. Introduction. The equations of motion in the classical Bresse system are given by
\[ \begin{align*}
\rho_1 \phi_{tt} - S_x - lN &= 0 \quad \text{in} \ (0,L) \times (0,\infty), \\
\rho_2 \psi_{tt} - M_x + S &= 0 \quad \text{in} \ (0,L) \times (0,\infty), \\
\rho_1 w_{tt} - N_x + lS &= 0 \quad \text{in} \ (0,L) \times (0,\infty),
\end{align*} \]
where \(t\) represents time and \(x\) is a space variable. The unknowns \(\phi = \phi(x,t), \psi = \psi(x,t), w = w(x,t)\) denote the vertical angle, the shear angle and the longitudinal displacements, respectively. Here, the constant parameters are given by \(\rho_1 = \rho A, \rho_2 = \rho I, l = R^{-1}\), where \(\rho\) is the material density, \(A\) is its cross-sectional area, \(I\) is the second moment of the area of the cross-section and \(R\) is the radius of curvature. The axial force, the bending moment and the shear force are respectively denoted by \(N, S\) and \(M\).

In this work, we consider the viscoelastic-type Bresse system whose constitutive laws are
\[ \begin{align*}
S &= k_1 (\phi_x + \psi + lw), \\
M &= k_2 \psi_x - \int_0^t g(t-s)\psi_x(\cdot,s)ds, \\
N &= k_3 (w_x - l\phi),
\end{align*} \]

2000 Mathematics Subject Classification. Primary: 35B35, 35L05; Secondary: 35B40, 35L20.
Key words and phrases. Bresse system, viscoelastic, general and optimal decay, equal and non-equal speeds of wave propagation.
This work is funded by KFUPM under Project IN161006.
* Corresponding author.
where \( k_1 = \kappa GA, k_2 = EI, k_3 = EA, E \) is the modulus of elasticity, \( G \) is the shear modulus and \( \kappa \) is the shear factor. Many researchers have established several results dealing with the well-posedness and asymptotic stability of the Bresse system using different types of dissipation mechanisms acting on each (or some) of the equation(s) in the system. It is a well known fact that if the dissipation mechanisms are acting on only one or two of the equations, then the asymptotic behavior of the system depends completely on the speeds of wave propagation given by

\[
s_1 = \sqrt{\frac{k_1}{\rho_1}}, \quad s_2 = \sqrt{\frac{k_2}{\rho_2}}, \quad \text{and} \quad s_3 = \sqrt{\frac{k_3}{\rho_1}}.
\]

The reader is referred to [1, 2, 4, 6, 5, 8, 10, 11, 12] and the references therein for results related to stabilization of Bresse systems using different types of damping terms.

To the best of our knowledge, there is no result in the literature that dealt with the stability of Bresse system with finite memory. In this work, we study the following finite-memory Bresse system:

\[
\begin{align*}
\rho_1 \varphi_{tt} - k_1 (\varphi_x + \psi + lw)_x - l k_3 (w_x - l \varphi) &= 0, \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\varphi_x + \psi + lw) + \int_0^t g(t-s) \psi_{xx}(\cdot, s) ds &= 0, \\
\rho_1 w_{tt} - k_3 (w_x - l \varphi)_x + l k_1 (\varphi_x + \psi + lw) &= 0, \\
\varphi(0, t) &= \varphi(L, t), \quad \psi_x(0, t) = \psi_x(L, t) = w_x(0, t) = w_x(L, t) = 0, \\
\varphi(x, 0) &= \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \\
\psi(x, 0) &= \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \\
w(x, 0) &= w_0(x), \quad w_t(x, 0) = w_1(x),
\end{align*}
\]

(P)

where \((x, t) \in (0, L) \times (0, \infty), l, k_1, k_2, k_3, \rho_1, \rho_2\) are positive constants, \(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1\) are given data and \(g\) is a relaxation function satisfying some conditions to be specified in the next section. Under a smallness condition on \(l\), we prove some general decay results for the energy associated to this system in the case of equal and non-equal speeds of wave propagation. Our results will give an optimal decay rate, in the sense that the decay rate of the energy is the same as that of the relaxation function, in the case of equal speeds of wave propagation. This paper is organized as follows: in Section 2, we state some preliminary results. In Section 3, we state and prove some technical lemmas. The statement and proof of our main results are given in Sections 4 and 5. Through out this work we use \(c\) to represent a generic positive constant, independent of \(t\) but may depend on the initial data.

2. Preliminaries. In this section, we introduce our assumptions, present some useful lemmas and state the existence theorem.

**Assumptions:** We assume that the relaxation function \(g\) satisfies the following hypotheses:

\(\textbf{A.1}\) \(g : [0, +\infty) \rightarrow [0, +\infty)\) is a strictly increasing differentiable function such that

\[
g(0) > 0 \quad \text{and} \quad k_2 - \int_0^{+\infty} g(s) ds > 0.
\]

\(\textbf{A.2}\) There exist a non-increasing differentiable function \(\xi : [0, +\infty) \rightarrow (0, \infty)\) and a \(C^1\)-function \(H : [0, +\infty) \rightarrow [0, +\infty)\) which is either linear or strictly
increasing and strictly convex $C^2$-function on $(0, r]$, with $H(0) = H'(0) = 0$ such that
\[ g'(t) \leq -\xi(t)H(g(t)), \quad \forall t \geq 0. \quad (2) \]

**Remark 1.**

(1) Assumptions (A.1) and (A.2) entail that, there exists $t_0 > 0$ such that
\[ g(t_0) = r \quad \text{and} \quad g(t) \leq r, \quad \forall t \geq t_0. \]

The non-increasing property of $g$ gives
\[ 0 < g(t_0) \leq g(t) \leq g(0), \quad \forall t \in [0, t_0]. \]

A combination of this with the continuity of $H$ yields
\[ a \leq H(g(t)) \leq b, \quad \forall t \in [0, t_0], \]

for two constants $a, b > 0$. Consequently, for any $t \in [0, t_0]$, we have
\[ g'(t) \leq -\xi(t)H(g(t)) \leq -\frac{a}{g(0)}\xi(t)g(0) \leq -\frac{a}{g(0)}\xi(t)g(t) \]
and, hence,
\[ \xi(t)g(t) \leq -\frac{g(0)}{a}g'(t), \quad \forall t \in [0, t_0]. \quad (3) \]

(2) If $H$ is a strictly increasing and strictly convex $C^2$-function on $(0, r]$, with $H(0) = H'(0) = 0$, then there is a strictly convex and strictly increasing $C^2$-function $\bar{H} : [0, +\infty) \to [0, +\infty)$ which is an extension of $H$. For instance, we can define $\bar{H}$, for any $t > r$, by
\[ \bar{H}(t) := \frac{H''(r)}{2}t^2 + (H'(r) - H''(r)r)t + \left( H(r) + \frac{H''(r)}{2}r^2 - H'(r)r \right). \]

Now, integrating both sides of the second and third equations in $(P)$ over $(0, L)$ and using the boundary conditions, we get
\[ \frac{d^2}{dt^2} \int_0^L \psi(x, t)dx + \frac{k_1}{\rho_2} \int_0^L \psi(x, t)dx + \frac{lk_1}{\rho_2} \int_0^L w(x, t)dx = 0 \quad \forall t \geq 0 \]
and
\[ \frac{d^2}{dt^2} \int_0^L w(x, t)dx + \frac{l^2k_1}{\rho_1} \int_0^L w(x, t)dx + \frac{lk_1}{\rho_1} \int_0^L \psi(x, t)dx = 0 \quad \forall t \geq 0. \]

Solving these ODEs simultaneously yields
\[ \int_0^L \psi(x, t)dx = a_1 \cos(a_0 t) + a_2 \sin(a_0 t) + a_3 t + a_4 \]
and
\[ \int_0^L w(x, t)dx = \frac{a_1}{T} \left( \frac{\rho_2 a_0^2}{k_1} - 1 \right) \cos(a_0 t) + \frac{a_2}{T} \left( \frac{\rho_2 a_0^2}{k_1} - 1 \right) \sin(a_0 t) - \frac{a_3}{T} t - \frac{a_4}{T}. \]
where
\[
\begin{align*}
a_0 &= \frac{k_1}{\rho_2 + l^2 k_1}, \\
a_1 &= \frac{k_1}{\rho_2 a_0^2} \int_0^L \psi_0(x)dx + \frac{k_1}{\rho_2 a_0^2} \int_0^L w_0(x)dx, \\
a_2 &= \frac{k_1}{\rho_2 a_0^2} \int_0^L \psi_1(x)dx + \frac{k_1}{\rho_2 a_0^2} \int_0^L w_1(x)dx, \\
a_3 &= \left(1 - \frac{k_1}{\rho_2 a_0^2}\right) \int_0^L \psi_1(x)dx - \frac{k_1}{\rho_2 a_0^2} \int_0^L w_1(x)dx, \\
a_4 &= \left(1 - \frac{k_1}{\rho_2 a_0^2}\right) \int_0^L \psi_0(x)dx + \frac{k_1}{\rho_2 a_0^2} \int_0^L w_0(x)dx.
\end{align*}
\]

Therefore, we perform the following change of variables
\[
\tilde{\psi} = \psi - \frac{1}{L} (a_1 \cos(a_0 t) + a_2 \sin(a_0 t) + a_3 t + a_4), \\
\tilde{w} = w - \frac{1}{L} \left[\frac{a_1}{l} \left(\frac{a_1 a_0^2}{k_1} - 1\right) \cos(a_0 t) + \frac{a_2}{l} \left(\frac{a_2 a_0^2}{k_1} - 1\right) \sin(a_0 t) - \frac{a_3}{l} t - \frac{a_4}{l}\right]
\]
to get
\[
\int_0^L \tilde{\psi}(x,t)dx = \int_0^L \tilde{w}(x,t)dx = 0, \quad \forall t \geq 0.
\]

Furthermore, \((\varphi, \tilde{\psi}, \tilde{w})\) satisfies the equations and the boundary conditions in \((P)\) with the initial data
\[
\tilde{\psi}_0 = \psi_0 - \frac{1}{L} (a_1 + a_4), \quad \tilde{\psi}_1 = \psi_1 - \frac{1}{L} (a_0 a_2 + a_3), \\
\tilde{w}_0 = w_0 - \frac{1}{L} \left[\frac{a_1}{l} \left(\frac{a_1 a_0^2}{k_1} - 1\right) - \frac{a_4}{l}\right], \quad \tilde{w}_1 = w_1 - \frac{1}{L} \left[\frac{a_2 a_0}{l} \left(\frac{a_2 a_0^2}{k_1} - 1\right) - \frac{a_3}{l}\right].
\]

From now on, we work with \(\tilde{\psi}, \tilde{w}\) and, respectively, write \(\psi, w\) for convenience. We also introduce the following spaces,
\[
L^2_0(0,L) := \left\{ w \in L^2(0,L) : \int_0^L w(x)dx = 0 \right\}, \quad H^1_0(0,L) := H^1(0,L) \cap L^2_0(0,L),
\]
and
\[
H^2_0(0,L) := \left\{ w \in H^2(0,L) : w_x(0) = w_x(L) = 0 \right\}.
\]

Then, Poincaré’s inequality is applicable to the elements of \(H^1_0(0,L)\), that is,
\[
\exists c_0 > 0 \text{ such that } \int_0^L v^2dx \leq c_0 \int_0^L v_x^2dx \quad \forall v \in H^1_0(0,L).
\]

For completeness, we state, without proof the global existence and regularity result which can be easily established by a standard Galerkin argument.

**Theorem 2.1.** Let \((\varphi_0, \varphi_1) \in H^1_0(0,L) \times L^2(0,L)\) and \((\psi_0, \psi_1), (w_0, w_1) \in H^1_0(0,L) \times L^2_0(0,L)\) be given. Assume that \(g\) satisfies hypothesis \((A.1)\). Then, the problem \((P)\) has a unique global (weak) solution
\[
\varphi \in C(\mathbb{R}_+; H^1_0(0,L)) \cap C^1(\mathbb{R}_+; L^2(0,L)), \\
\psi, w \in C(\mathbb{R}_+; H^1_0(0,L)) \cap C^1(\mathbb{R}_+; L^2_0(0,L)).
\]

Moreover, if
\[
(\varphi_0, \varphi_1) \in (H^2(0,L) \cap H^1_0(0,L)) \times H^1_0(0,L)
\]
and
\[(\psi_0, \psi_1), (w_0, w_1) \in (H^2_0(0, L) \cap H^1_0(0, L)) \times H^1_0(0, L),\]
then
\[\varphi \in C(\mathbb{R}^+; H^2_0(0, L) \cap H^1_0(0, L)) \cap C^1(\mathbb{R}^+; H^1_0(0, L)) \cap C^2(\mathbb{R}^+; L^2(0, L)),\]
and
\[\psi, w \in C(\mathbb{R}^+; H^2_0(0, L) \cap H^1_0(0, L)) \cap C^1(\mathbb{R}^+; H^1_0(0, L)) \cap C^2(\mathbb{R}^+; L^2(0, L)).\]

Now, we introduce the energy functional
\[
E(t) = \frac{1}{2} \int_0^L \left[ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 + \left( k_2 - \int_0^t g(s)ds \right) \varphi_x^2 \right. \\
\left. + k_3 (w_x - l \varphi)^2 + k_1 (\varphi_x + \psi + lw)^2 \right] dx + \frac{1}{2} \left( g \circ \psi_x \right)(t), \quad \forall t \geq 0,
\]
where for any \(v \in L^2_{\text{loc}}([0, +\infty); L^2(0, L))\),
\[(g \circ \psi)(t) := \int_0^L \int_0^t g(t - s) (v(t) - v(s))^2 ds dx.
\]
By multiplying the equations in (P) by \(\varphi_t, \psi_t, w_t\), respectively, integrating over \((0, L)\) and exploiting the boundary conditions we have the following lemma.

**Lemma 2.2.** Let \((\varphi, \psi, w)\) be the weak solution of (P). Then,
\[
E'(t) = -\frac{1}{2} g(t) \int_0^L \psi_x^2 dx + \frac{1}{2} \left( g' \circ \psi_x \right)(t) \leq 0, \quad \forall t \geq 0.
\]

As in [9], we set, for any \(0 < \alpha < 1\),
\[C_\alpha := \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)} ds \quad \text{and} \quad h(t) := \alpha g(t) - g'(t).
\]

**Lemma 2.3 ([9]).** Assume that conditions (A.1) holds. Then for any \(v \in L^2_{\text{loc}}([0, +\infty); L^2(0, L))\), we have
\[
\int_0^L \left( \int_0^t g(t - s)(v(t) - v(s))ds \right)^2 dx \leq C_\alpha(h \circ v)(t), \quad \forall t \geq 0.
\]

**Lemma 2.4** (Jensen's inequality). Let \(G : [a, b] \rightarrow \mathbb{R}\) be a convex function. Assume that the functions \(f : \Omega \rightarrow [a, b]\) and \(h : \Omega \rightarrow \mathbb{R}\) are integrable such that \(h(x) \geq 0\), for any \(x \in \Omega\) and \(\int_\Omega h(x)dx = k > 0\). Then,
\[
G \left( \frac{1}{k} \int_\Omega f(x)h(x)dx \right) \leq \frac{1}{k} \int_\Omega G(f(x))h(x)dx.
\]

3. **Technical lemmas.** In this section, we state and prove some lemmas needed to establish our main results. All the computations are done for regular solutions but they still hold for weak and strong solutions by a density argument.

**Lemma 3.1.** Assume that conditions (A.1) and (A.2) hold. Then, for any \(0 < \delta < 1\), the functional \(I_1\) defined by
\[I_1(t) := -\rho_2 \int_0^L \psi_t \int_0^t g(t - s)(\psi(t) - \psi(s))ds dx,
\]
satisfies, along the solution of \((P)\), the estimates

\[
I_1'(t) \leq -\rho_2 \left( \int_0^t g(s) ds - \delta \right) \int_0^t \psi_t^2 dx + \rho_2 \left( \int_0^t g(s) ds \right) \int_0^t \psi_t^2 dx + \delta \int_0^t (\varphi_x + \psi + lw)^2 dx
\]

\[
+ c\delta \int_0^t \psi_x^2 dx + \frac{c}{\delta} (C_\alpha + 1)(h \circ \psi)(t).
\]

**Proof.** Differentiating \(I_1\), using equations in \((P)\) and integrating by parts, we get

\[
I_1'(t) = -\rho_2 \int_0^t \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s)) ds dx - \rho_2 \left( \int_0^t g(s) ds \right) \int_0^t \psi_t^2 dx
\]

\[
+ \left( k_2 - \int_0^t g(s) ds \right) \int_0^t \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s)) ds dx
\]

\[
+ k_1 \int_0^t (\varphi_x + \psi + lw) \int_0^t g(t-s)(\psi(t)-\psi(s)) ds dx
\]

\[
+ \int_0^t \left( \int_0^t g(t-s)(\psi_x(t)-\psi_x(s)) ds \right)^2 dx.
\]

Next, we estimate the terms on the right-hand side of the above equation.

Using Young’s inequality, Lemma 2.3 and Poicaré’s inequality, we obtain, for any \(0 < \delta < 1\),

\[
-\rho_2 \int_0^t \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s)) ds dx
\]

\[
= \rho_2 \int_0^t \psi_t \int_0^t h(t-s)(\psi(t)-\psi(s)) ds dx
\]

\[
- \rho_2 \int_0^t \psi_t \int_0^t g(t-s)(\psi(t)-\psi(s)) ds dx
\]

\[
\leq \frac{\delta}{2} \rho_2 \int_0^t \psi_t^2 dx + \frac{\rho_2}{2\delta} \int_0^t \left( \int_0^t \sqrt{h(t-s)h(t-s)} (\psi(t)-\psi(s)) ds \right)^2 dx
\]

\[
+ \frac{\delta}{2} \rho_2 \int_0^t \psi_t^2 dx + \frac{\rho_2}{2\delta} \alpha^2 \int_0^t \left( \int_0^t (\psi(t)-\psi(s)) ds \right)^2 dx
\]

\[
\leq \delta \rho_2 \int_0^t \psi_t^2 dx + \frac{\rho_2}{2\delta} \left( \int_0^t h(s) ds \right) (h \circ \psi)(t) + \frac{c}{\delta} C_\alpha (h \circ \psi)(t)
\]

\[
\leq \delta \rho_2 \int_0^t \psi_t^2 dx + \frac{C}{\delta} (C_\alpha + 1)(h \circ \psi_x)(t),
\]

\[
\left( k_2 - \int_0^t g(s) ds \right) \int_0^t \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s)) ds dx
\]

\[
\leq \delta \int_0^t \psi_x^2 dx + \frac{C}{\delta} C_\alpha (h \circ \psi_x)(t),
\]

\[
k_1 \int_0^t (\varphi_x + \psi + lw) \int_0^t g(t-s)(\psi(t)-\psi(s)) ds dx
\]

\[
\leq k_1 \delta \int_0^t (\varphi_x + \psi + lw)^2 dx + \frac{C}{\delta} C_\alpha (h \circ \psi_x)(t),
\]
Under the conditions (Lemma 3.3. and Lemma 3.2.), satisfies, along the solution of (P), the estimate

\begin{equation}
I_2(t) \leq k_1^2 \int_0^L (\varphi_x + \psi + lw)^2dx - k_3^2 \int_0^L (w_x - l\varphi)^2dx + \frac{c}{\varepsilon_0} \int_0^L \psi_i^2dx \\
+ \left(\varepsilon_0 - \rho_1 k_1 + \frac{k_1 l|k_3| - k_1|\delta_1|}{2}\right) \int_0^L \varphi_i^2dx \\
+ \rho_1 \left(k_3 + \frac{c_0 k_1|k_3 - k_1|}{2\delta_1}\right) \int_0^L w_i^2dx.
\end{equation}

Proof. Differentiation of \(I_2\), using equations in (P) and integration by parts yield

\begin{align*}
I_2' &= \rho_1 k_3 \int_0^L w_i^2dx + l\rho_1 k_3 \int_0^L \varphi_i \int_0^x w(y,t)dydx \\
&\quad - k_3^2 \int_0^L (w_x - l\varphi)^2dx + k_1^2 \int_0^L (\varphi_x + \psi + lw)^2dx \\
&\quad - \rho_1 k_1 \int_0^L \varphi_i^2dx - \rho_1 k_1 \int_0^L \varphi_i \int_0^x (\psi_i + lw_i)(y,t)dydx.
\end{align*}

Using Young's inequality, we get, for any \(\varepsilon_0, \delta_1 > 0\),

\begin{align*}
I_2' &\leq k_1^2 \int_0^L (\varphi_x + \psi + lw)^2dx - k_3^2 \int_0^L (w_x - l\varphi)^2dx + \frac{c}{\varepsilon_0} \int_0^L \psi_i^2dx \\
&\quad + \left(\varepsilon_0 - \rho_1 k_1 + \frac{k_1 l|k_3| - k_1|\delta_1|}{2}\right) \int_0^L \varphi_i^2dx + \rho_1 \left(k_3 + \frac{c_0 k_1|k_3 - k_1|}{2\delta_1}\right) \int_0^L w_i^2dx.
\end{align*}

Lemma 3.3. Under the conditions (A.1) and (A.2), the functional \(I_3\) defined by

\begin{equation}
I_3(t) := -\rho_1 \int_0^L (\varphi_x + \psi + lw) w_i dx - \frac{k_3 \rho_1}{k_1} \int_0^L (w_x - l\varphi) \varphi_i dx
\end{equation}

satisfies, along the solution of (P) and for any \(\varepsilon_0 > 0\), the estimate

\begin{align*}
I_3'(t) &\leq k_1 \int_0^L (\varphi_x + \psi + lw)^2dx - \frac{k_3^2}{k_1} \int_0^L (w_x - l\varphi)^2dx + \frac{c}{\varepsilon_0} \int_0^L \psi_i^2dx \\
&\quad + \frac{l\rho_1 k_3}{k_1} \int_0^L \varphi_i^2dx + (\varepsilon_0 - l\rho_1) \int_0^L w_i^2dx + \rho_1 \left(k_3 + \frac{k_3 - 1}{k_1}\right) \int_0^L \varphi_i w_i dx.
\end{align*}
Proof. Differentiating $I_3$, using equations in $(P)$ and integrating by parts, we have

$$I'_3 = -\rho_1 \int_0^L \psi w_t \, dx - l \rho_1 \int_0^L w_t^2 \, dx + l k_1 \int_0^L (\varphi + \psi + lw)^2 \, dx$$

$$+ \frac{l \rho_1 k_3}{k_1} \int_0^L \varphi_t^2 \, dx - \frac{l k_3}{k_1} \int_0^L (w_x - lw)^2 \, dx + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_x w_t \, dx.$$  

Use of Young’s inequality for the first term in the right-hand side gives (9).

Lemma 3.4. Assume that conditions (A.1) and (A.2) hold. Then for any $0 < \delta < 1$, the functional $I_4$ defined by

$$I_4(t) := -\int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t) \, dx$$

satisfies, along the solution of $(P)$, the estimate

$$I'_4(t) \leq -\int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2) \, dx + k_1 \int_0^L (\varphi + \psi + lw)^2 \, dx$$

$$+ k_3 \int_0^L (w_x - lw)^2 \, dx + \left( k_2 + \delta - \int_0^t g(s) \, ds \right) \int_0^L \psi_x^2 \, dx + C_\alpha (h \circ \psi_x).$$

(10)

Proof. Differentiation of $I_4$, using equations of $(P)$ gives

$$I'_4(t) = -\int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2) \, dx + k_1 \int_0^L (\varphi + \psi + lw)^2 \, dx$$

$$+ k_3 \int_0^L (w_x - lw)^2 \, dx + \left( k_2 - \int_0^t g(s) \, ds \right) \int_0^L \psi_x^2 \, dx$$

$$- \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) \, ds \, dx.$$  

Repeating the above computations yields the desired result. □

Lemma 3.5. Assume that conditions (A.1) and (A.2) hold. Then for any $0 < \delta < 1$ and $\delta_2 > 0$, the functional $I_5$ defined by

$$I_5(t) := -\rho_2 \int_0^L \psi \int_0^t \psi_t(y,t) \, dy \, dx$$

satisfies, along the solution of $(P)$, the estimate

$$I'_5(t) \leq \rho_2 \int_0^L \psi_t^2 \, dx + \left( \frac{k_1}{2 \delta_2} + \int_0^t g(s) \, ds + \delta - k_2 \right) \int_0^L \psi_x^2 \, dx$$

$$+ \frac{c_0 k_1 \delta_2}{2} \int_0^L (\varphi + lw)^2 \, dx + C_\alpha (h \circ \psi_x).$$

(11)
Lemma 3.6. Assume that the hypotheses (A.1) and (A.2) hold. Then, for any \( \varepsilon_0, \varepsilon_1, \varepsilon_2 > 0 \) and \( 0 < \delta < 1 \), the functional \( I_6 \) defined by

\[
I_6(t) := \rho_2 \int_0^L \psi_t(t) (\varphi_x + \psi + lw) \, dx + \rho_2 \int_0^L \psi_t^2 \, dx + \rho_2 \int_0^L \psi_t w_t \, dx
\]

satisfies, along the solution of (P), the estimate

\[
I_6'(t) \leq \delta \int \varphi_t^2 \, dx + \varepsilon_0 \int \varphi_t^2 \, dx + \varepsilon_0 \int \varphi_t^2 \, dx - k_1 \int (\varphi_x + \psi + lw)^2 \, dx
\]

(12)

Proof. Use of equations of (P) and integration by parts lead to

\[
I_6'(t) = -k_1 \int (\varphi_x + \psi + lw)^2 \, dx + \rho_2 \int \psi_t^2 \, dx + l \rho_2 \int \psi_t w_t \, dx
\]

Now, we estimate the terms in the right-hand side of the above equation.

Exploiting Young’s inequality, we get

\[
l \rho_2 \int \psi_t w_t \, dx \leq \varepsilon_0 \int w_t^2 \, dx + \frac{c}{\varepsilon_0} \int \psi_t^2 \, dx, \quad \forall \varepsilon_0 > 0.
\]
Using Young’s inequality and Lemma 6, we obtain, for any $\varepsilon_1, \varepsilon_2 > 0$ and $0 < \delta < 1$,
\[
\frac{lk_2k_3}{k_1} \int_0^L (w_x - l\varphi)\psi_x dx - \frac{lk_3}{k_1} \int_0^L (w_x - l\varphi) \int_0^t g(t-s)\psi_x ds ds dx
\]
\[
= \frac{lk_3}{k_1} \left( k_2 - \int_0^t g(s) ds \right) \int_0^L (w_x - l\varphi)\psi_x dx
\]
\[
+ \frac{lk_3}{k_1} \int_0^L (w_x - l\varphi) \int_0^t g(t-s)(\psi_x(t) - \psi_x(s)) ds ds dx
\]
\[
\leq \left( \frac{lk_2k_3\varepsilon_1}{2k_1} + \frac{lk_3\varepsilon_2}{2k_1} \right) \int_0^t g(s) ds + \delta \int_0^L (w_x - l\varphi)^2 dx
\]
\[
+ \left( \frac{lk_2k_3}{2k_1\varepsilon_1} + \frac{lk_3}{2k_1\varepsilon_2} \right) \int_0^L g(s) ds \int_0^L \psi_x^2 dx + \frac{c}{\delta} (g \circ \psi_x)
\]
and
\[
- \frac{\rho_1}{k_1} g(t) \int_0^L \varphi_t \psi_x dx + \frac{\rho_1}{k_1} \int_0^L \varphi_t \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s)) ds ds dx
\]
\[
\leq \delta \int_0^L \varphi_t^2 dx + \frac{c}{\delta} g(t) \int_0^L \psi_x^2 dx - \frac{c}{\delta} (g \circ \psi_x)
\]

A combination of these estimates gives the desired result. \qed

As in [9], let us define the functional
\[
J(t) := \int_0^L \int_0^t f(t-s)\psi_x^2(s) ds ds dx,
\]
where $f(t) := \int_0^t g(s) ds$ and state a lemma whose proof is similar to that in [9] taking into consideration the nature of our problem.

**Lemma 3.7 ([9]).** Assume that (A.1) and (A.2) hold. Then, the functional $J$ satisfies, along the solution of $(P)$, the estimate
\[
J'(t) \leq -\frac{1}{2} (g \circ \psi_x)(t) + 3g_0 \int_0^L \psi_x^2 dx,
\]
where $g_0 = \int_0^\infty g(s) ds$.

**Lemma 3.8.** The functional $\mathcal{L}$ defined by
\[
\mathcal{L}(t) := NE(t) + \sum_{j=1}^{6} N_j I_j(t)
\]
satisfies, for a suitable choice of $N$, $N_j \geq 0$ for $j = 1, 2, \cdots, 6$ with $N_3 = N_6 = 1$, and, for $l$ small enough,
\[
\mathcal{L}(t) \sim E(t)
\]
and the estimate
\[
\mathcal{L}'(t) \leq -mE(t) + \frac{m}{2} \left( 1 + \frac{k_2 - g_0}{12g_0} \right) (g \circ \psi_x)(t)
\]
\[
+ \left( \frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_x dx + \rho_1 \left( \frac{k_3}{k_2} - 1 \right) \int_0^L \varphi_x w_L dx, \quad \forall t \geq t_0,
\]
where $m$ is a fixed positive constant and $t_0$ has been introduced in Remark 1.
Proof. It is a routine computation to establish that $\mathcal{L}(t) \sim E(t)$. To prove (15), a combination of (5), (7)–(12), and recall that $g' = \alpha g - h$, yield, for all $t \geq t_0$,

$$
\mathcal{L}'(t) \leq \left[-\rho_1(k_1 N_2 + N_4) + \frac{l \rho_1 |k_3 - k_1| \delta_1 N_2}{2} + \frac{l \rho_1 k_3}{k_1} + \varepsilon_0 N_2 + \delta\right] \int_0^L \varphi_t^2 dx
$$

$$
+ \left[-\rho_2 \left(N_1 \int_0^t g(s) ds + N_4 - N_5\right) + \rho_2 \delta N_1 + \frac{c}{\varepsilon_0} (1 + N_2)\right] \int_0^L \psi_t^2 dx
$$

$$
+ \left[-l \rho_1 + \rho_1(k_3 N_2 - N_4) + \frac{c_0 l \rho_1 |k_3 - k_1| N_2}{2 \delta_1} + \varepsilon_0\right] \int_0^L w_t^2 dx
$$

$$
+ \left[(N_5 - N_4) \int_0^t g(s) ds + k_2 (N_4 - N_5) + \frac{k_1 N_5}{2 \delta_2} + \frac{l k_3 k_5}{2k_1}\right]
$$

$$
+ \frac{l k_2 k_3 k_5}{2k_1} \int_0^t g(s) ds + \delta \right] \int_0^L (w_x - \varphi)^2 dx
$$

$$
+ \left[-k_1 \left(1 - k_1 N_2 - l - N_4 - \frac{c_0 \delta_2 N_5}{2}\right) + \delta N_1\right] \int_0^L (\varphi_x + \psi + l w)^2 dx
$$

$$
+ \frac{c}{\delta} \left[(C_6 + 1) + N_1(C_6 + 1) + N_4 C_6 + N_5 C_6\right] (h \circ \psi_x)(t) + NE'(t)
$$

$$
+ \left(k_2 \rho_1 - \rho_2\right) \int_0^L \varphi_t \psi_t dx + \rho_1 \left(k_3 - 1\right) \int_0^L \varphi_t w_1 dx.
$$

By setting $\delta_1 = 1$, $N_4 = k_3 N_2$, $N_5 = 4k_3 N_2$, $\delta_2 = \frac{k_1}{k_2 - g_0}$, $\varepsilon_1 = \frac{k_2}{k_2}$, and $\varepsilon_2 = \frac{k_3}{g_0}$, we arrive at

$$
\mathcal{L}'(t) \leq -\rho_1 \left[(k_1 + k_3) N_2 - l \left(\frac{|k_3 - k_1|}{2} N_2 + \frac{k_3}{k_1}\right)\right] \int_0^L \varphi_t^2 dx
$$

$$
- \rho_2 \left(N_1 \int_0^t g(s) ds - 3k_3 N_2\right) \int_0^L \psi_t^2 dx
$$

$$
- l \rho_1 \left(1 - \frac{c_0 |k_3 - k_1|}{2 N_2}\right) \int_0^L w_t^2 dx
$$

$$
- \left[(k_2 - g_0) k_3 N_2 - l \left(\frac{k_2}{k_1} + g_0\right)\right] \int_0^L \psi_x^2 dx - \frac{l k_3^2}{4k_1} \int_0^L (w_x - l \varphi)^2 dx
$$

$$
- k_1 \left[1 - \left(k_1 + k_3 + \frac{c_0 k_1 k_3}{k_2 - g_0}\right) N_2 - l\right] \int_0^L (\varphi_x + \psi + l w)^2 dx
$$

$$
+ (1 + N_2) \varepsilon_0 \int_0^L (\varphi_t^2 + w_t^2) dx + \frac{c}{\varepsilon_0} (1 + N_2) \int_0^L \psi_t^2 dx + NE'(t)
$$

$$
+ \frac{c}{\delta} \left[(1 + N_1) + C_6 (1 + N_1 + 5k_3 N_2)\right] (h \circ \psi_x)(t) + \frac{c}{\delta} g(t) \int_0^L \psi_x^2 dx
$$
Choose $\varepsilon_0 = \frac{l_{\rho_1}}{2(1+N^2)}$, to get
\[
\mathcal{L}'(t) \leq -\rho_1 \left[(k_1 + k_3)N_2 - l \left(\frac{1}{2} + \frac{k_3}{k_1} + \frac{|k_3 - k_1|}{2}N_2\right)\right] \int_0^L \varphi_x^2 dx
- \rho_2 \left[N_1 \int_0^t g(s) ds - 3k_3N_2 - \frac{c(1 + N^2)}{l_{\rho_1} \rho_2}\right] \int_0^L \psi_x^2 dx
- \frac{\rho_1 k_3^2}{4k_1} \int_0^L (w_x - l\varphi)^2 dx - \frac{l_{\rho_1}}{2} (1 - c_0|k_3 - k_1|N_2) \int_0^L \psi_x^2 dx
- (k_2 - g_0)k_3N_2 - \frac{l}{k_1} \left(\frac{k_2^2}{2} + g_0^2\right) \int_0^L \psi_x^2 dx + \delta c_{N_1, N_2} E(t)
- k_1 \left[1 - \left(k_1 + k_3 + \frac{2c_0k_1k_3}{k_2 - g_0}\right)N_2 - l\right] \int_0^L (\varphi_x + \psi + lw)^2 dx
+ \left[N - \frac{c}{\delta}\right] E'(t) + \frac{c}{\delta} \left[(1 + N_1) + C_\alpha (1 + N_1 + 5k_3N_2)\right] (h \circ \psi_x)(t)
+ \left(k_2\rho_1 - \rho_2\right) \int_0^L \varphi_x \psi_t dx + \rho_1 \left(\frac{k_3}{k_1} - 1\right) \int_0^L \varphi_x w_t dx.
\]
Choose $N_2$ so small that

\[1 - c_0|k_3 - k_1|N_2 > 0 \quad \text{and} \quad 1 - \left(k_1 + k_3 + \frac{2c_0k_1k_3}{k_2 - g_0}\right)N_2 > 0.\]

Next, we select $l$ small enough so that

\[(k_1 + k_3)N_2 - l \left(\frac{1}{2} + \frac{k_3}{k_1} + \frac{|k_3 - k_1|}{2}N_2\right) > 0, \quad (k_2 - g_0)k_3N_2 - \frac{l}{k_1} \left(\frac{k_2^2}{2} + g_0^2\right) > 0,\]

and

\[1 - \left(k_1 + k_3 + \frac{2c_0k_1k_3}{k_2 - g_0}\right)N_2 - l > 0.\]

After that, we pick $N_1$ very large so that

\[N_1 \int_0^t g(s) ds - 3k_3N_2 - \frac{c(1 + N^2)}{l_{\rho_1} \rho_2} > 0.\]

Therefore, we have
\[
\mathcal{L}'(t) \leq - (\beta_0 - c\delta) E(t) + \left(N - \frac{c}{\delta}\right) E'(t)
+ \frac{c}{\delta} \left[(1 + N_1) + C_\alpha (1 + N_1 + 5k_3N_2)\right] (h \circ \psi_x)(t)
+ \left(k_2\rho_1 - \rho_2\right) \int_0^L \varphi_x \psi_t dx + \rho_1 \left(\frac{k_3}{k_1} - 1\right) \int_0^L \varphi_x w_t dx,
\]

for some $\beta_0 > 0$. At this point, we take $\delta < \frac{\beta_0}{c}$. Consequently, we obtain, for some $m > 0$,
\[
\mathcal{L}'(t) \leq - mE(t) + (N - c) E'(t) + c\left[(1 + N_1) + C_\alpha (1 + N_1 + 5k_3N_2)\right] (h \circ \psi_x)(t)
+ \left(k_2\rho_1 - \rho_2\right) \int_0^L \varphi_x \psi_t dx + \rho_1 \left(\frac{k_3}{k_1} - 1\right) \int_0^L \varphi_x w_t dx, \quad \forall t \geq t_0.
\]
Finally, we choose \( N \) so large that \( L \sim E \) and \( N^2 > c \), therefore we have, \( \forall \, t \geq t_0 \),

\[
\mathcal{L}'(t) \leq -mE(t) + \frac{\alpha}{4}N(g \circ \psi_x)(t)
- \left[ \frac{N}{4} - c(1 + N_1) - c\alpha(1 + N_1 + 5k_3N_2) \right] (h \circ \psi_x)(t)
+ \left( \frac{k_2\rho_1}{k_1} - \rho_2 \right) \int_0^L \varphi_t \psi_x dx + \rho_1 \left( \frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_x w_t dx.
\]

As \( \frac{\alpha g^2}{\alpha g - g'} < g \), it follows from (A1) and the Lebesgue Dominated Convergence Theorem that

\[
\lim_{\alpha \to 0^+} \alpha C_\alpha = \lim_{\alpha \to 0^+} \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds = 0.
\]

Consequently, there exists \( \alpha_0 \in (0, 1) \) such that

\[
\alpha C_\alpha < \frac{m}{8c(1 + N_1)} \left( 2 + \frac{k_2 - g_0}{6g_0} \right), \quad \text{whenever} \quad \alpha < \alpha_0.
\]

Then, choose \( N \) even larger (if needed) so that

\[
N > \max \left\{ 8c(1 + N_1), \frac{m}{\alpha_0} \left( 2 + \frac{k_2 - g_0}{6g_0} \right) \right\}
\]

and set

\[
\alpha = \frac{m}{N} \left( 2 + \frac{k_2 - g_0}{6g_0} \right).
\]

So

\[
\frac{1}{8}N - c(1 + N_1) > 0 \quad \text{and} \quad \alpha = \frac{m}{N} \left( 2 + \frac{k_2 - g_0}{6g_0} \right) < \alpha_0.
\]

This yields

\[
\frac{1}{4}N - c(1 + N_1) - c\alpha(1 + N_1 + 5k_3N_2) > \frac{1}{4}N - c(1 + N_1) - \frac{m}{8\alpha} \left( 2 + \frac{k_2 - g_0}{6g_0} \right)
= \frac{1}{8}N - c(1 + N_1) > 0.
\]

Hence, we arrive at the required estimate.

4. General decay rates for equal speeds of wave propagation. In this section, we state and prove a new general decay result in the case of equal speeds of wave propagation.

**Theorem 4.1.** Let \((\varphi_0, \varphi_1) \in H^1_0(0, L) \times L^2(0, L)\) and \((\psi_0, \psi_1), (w_0, w_1) \in H^1_0(0, L) \times L^2(0, L)\). Assume that (A.1) and (A.2) hold and that

\[
k_1 \frac{\rho_1}{\rho_2} = k_2 \quad \text{and} \quad k_1 = k_3.
\]

Then for \( l \) small enough, there exist two positive constants \( C \) and \( \lambda \) (independent of \( t \) but may depend on the initial data) such that the solution of \((P)\) satisfies

\[
E(t) \leq CH^{-1}_1 \left( \lambda \int_{t_0}^t \xi(s) ds \right), \quad \forall \, t > t_0,
\]

where

\[
H_1(t) = \int_t^r \frac{1}{sH(s)} ds \quad \text{and} \quad t_0 = g^{-1}(r).
\]
Proof. We start by using the non-increasing property of $\xi$ and estimates (3) and (5) to deduce, for any $t \geq t_0$,

$$
\int_0^L \int_0^t g(s)(\psi_x(t) - \psi_x(t - s))^2 ds \, dx \\
\leq \frac{1}{\xi(t_0)} \int_0^L \int_0^t \xi(s)g(s)(\psi_x(t) - \psi_x(t - s))^2 ds \, dx \\
\leq -\frac{g(0)}{a \xi(t_0)} \int_0^L \int_0^t g'(s)(\psi_x(t) - \psi_x(t - s))^2 ds \, dx \\
\leq -\frac{g(0)}{a \xi(t_0)} \int_0^L \int_0^t g'(s)(\psi_x(t) - \psi_x(t - s))^2 ds \, dx \\
\leq -\frac{2g(0)}{a \xi(t_0)} E'(t).
$$

Exploiting this last estimate and (16), the estimate (15) becomes, for any $t \geq t_0$,

$$
L'(t) \leq -mE(t) + \frac{m}{2} \left( 1 + \frac{k_2 - g_0}{12g_0} \right) (g \circ \psi_x)(t) \\
\leq -mE(t) - cE'(t) + c \int_0^t \int_0^t g(s)(\psi_x(t) - \psi_x(t - s))^2 ds \, dx.
$$

By setting $F := L + cE \sim E$, we obtain

$$
F'(t) \leq -mE(t) + c \int_0^L \int_0^t g(s)(\psi_x(t) - \psi_x(t - s))^2 ds \, dx, \quad \forall t \geq t_0. \quad (18)
$$

**Case I** $H$ is linear: Multiplying (18) by $\xi(t)$, then exploiting (A.2) and (5), we get

$$
\xi(t)F'(t) \leq -m\xi(t)E(t) + c\xi(t) \int_0^L \int_0^t \xi(s)g(s)(\psi_x(t) - \psi_x(t - s))^2 ds \, dx \\
\leq -m\xi(t)E(t) + c \int_0^L \int_0^t \xi(s)g(s)(\psi_x(t) - \psi_x(t - s))^2 ds \, dx \\
\leq -m\xi(t)E(t) - c \int_0^L \int_0^t g'(s)(\psi_x(t) - \psi_x(t - s))^2 ds \, dx \\
\leq -m\xi(t)E(t) - cE'(t), \quad \forall t \geq t_0.
$$

Using the non-increasing property of $\xi$, we have $\xi F + cE \sim E$ and

$$
(\xi F + cE)'(t) \leq -m\xi(t)E(t), \quad \forall t \geq t_0.
$$

A simple integration over $(t_0, t)$ yields, for two positive constants $C$ and $\lambda$,

$$
E(t) \leq C \exp \left( -\lambda \int_{t_0}^t \xi(s) \, ds \right), \quad \forall t > t_0.
$$

Continuity of $E$ gives

$$
E(t) \leq C \exp \left( -\lambda \int_{0}^t \xi(s) \, ds \right), \quad \forall t > 0.
$$

**Case II** $H$ is nonlinear: First, we use Lemmas 3.7 and 3.8 to conclude that, the functional $L$ defined by

$$
L(t) := L(t) + \frac{m(k_2 - g_0)}{8g_0} J(t)
$$
is nonnegative and satisfies, for some $\beta > 0$ and for any $t \geq t_0$,

$$\mathcal{L}'(t) \leq -\frac{m}{2} \int_0^L \left[ \rho_1 \varphi_1^2 + \rho_2 \psi_1^2 + \rho_1 w_1^2 + k_3 (w_x - l \varphi)^2 + k_1 (\varphi_x + \psi + lw)^2 \right] dx$$

$$- \frac{m}{2} (k_2 - g_0) \int_0^L \psi_2^2 dx - \frac{m}{2} \left[ g \circ \psi_x(t) + \frac{m}{2} \left[ 1 + \frac{k_2 - g_0}{12g_0} \right] (g \circ \psi_x(t)) \right]$$

$$- \frac{m(k_2 - g_0)}{16g_0} (g \circ \psi_x(t)) + \frac{3}{8} m(k_2 - g_0) \int_0^L \psi_2^2 dx$$

$$\leq -\frac{m}{2} \int_0^L \left[ \rho_1 \varphi_1^2 + \rho_2 \psi_1^2 + \rho_1 w_1^2 + k_3 (w_x - l \varphi)^2 + k_1 (\varphi_x + \psi + lw)^2 \right] dx$$

$$- \frac{m}{8} (k_2 - g_0) \int_0^L \psi_2^2 dx - \frac{m(k_2 - g_0)}{48g_0} (g \circ \psi_x(t))$$

$$\leq -\beta E(t).$$

From this estimate, we deduce that

$$\int_0^\infty E(s) ds < +\infty. \quad (19)$$

Now, we define a functional $\eta$ by

$$\eta(t) := \gamma \int_{t_0}^t \| \psi_x(t) - \psi_x(t - s) \|^2_2 ds,$$

where (19) allows us to choose $0 < \gamma < 1$ so that

$$\eta(t) < 1, \quad \forall t \geq t_0. \quad (20)$$

We further assume that $\eta(t) > 0$, for any $t > t_0$, otherwise we get an exponential decay from (18). Also, we define another functional $\theta$ by

$$\theta(t) := -\int_{t_0}^t g'(s) \| \psi_x(t) - \psi_x(t - s) \|^2_2 ds$$

and observe that

$$\theta(t) \leq -cE'(t), \quad \forall t \geq t_0. \quad (21)$$

In addition, it follows from the strict convexity of $H$ and the fact that $H(0) = 0$ that

$$H(s\tau) \leq sH(\tau), \quad \text{for} \quad 0 \leq s \leq 1 \quad \text{and} \quad \tau \in (0, r].$$
This fact, hypothesis (A.2), (20) and Jensen’s inequality lead to

\[ \theta(t) = -\frac{1}{\gamma \eta(t)} \int_0^t \gamma \eta(t) g'(s) \| \psi_x(t) - \psi_x(t-s) \|^2 ds \]

\[ \geq \frac{1}{\gamma \eta(t)} \int_0^t \gamma \eta(t) \xi(s) H(g(s)) \| \psi_x(t) - \psi_x(t-s) \|^2 ds \]

\[ \geq \frac{\xi(t)}{\gamma(t)} \int_0^t \gamma \eta(t) g(s) \| \psi_x(t) - \psi_x(t-s) \|^2 ds \]

\[ \geq \frac{\xi(t)}{\gamma(t)} H \left( \frac{1}{\eta(t)} \int_0^t \gamma \eta(t) g(s) \| \psi_x(t) - \psi_x(t-s) \|^2 ds \right) \]

\[ = \frac{\xi(t)}{\gamma(t)} H \left( \int_0^t g(s) \| \psi_x(t) - \psi_x(t-s) \|^2 ds \right) \]

\[ = \frac{\xi(t)}{\gamma(t)} H \left( \int_0^t g(s) \| \psi_x(t) - \psi_x(t-s) \|^2 ds \right), \quad \forall t > t_0, \]

where \( \bar{H} \) is a \( C^2 \) extension of \( H \) that is strictly increasing and strictly convex on \((0, \infty)\). This implies that

\[ \int_0^t g(s) \| \psi_x(t) - \psi_x(t-s) \|^2 ds \leq \frac{1}{\gamma} \bar{H}^{-1} \left( \frac{\gamma \theta(t)}{\xi(t)} \right), \quad \forall t > t_0 \]

and (18) becomes

\[ F'(t) \leq -mE(t) + c\bar{H}^{-1} \left( \frac{\gamma \theta(t)}{\xi(t)} \right), \quad \forall t > t_0. \]  

(22)

For \( 0 < \varepsilon_1 < r \), we define the functional \( F_1 \) by

\[ F_1(t) := \bar{H}' \left( \varepsilon_1 \frac{E(t)}{E(0)} \right) F(t) + E(t), \quad \forall t > 0. \]

Then, using estimate (22) and the fact that \( E' \leq 0 \), \( \bar{H}' > 0 \) and \( \bar{H}'' > 0 \), we deduce that \( F_1 \sim E \) and also, for any \( t > t_0 \), we have

\[ F_1'(t) = \varepsilon_1 \frac{E'(t)}{E(0)} \bar{H}'' \left( \varepsilon_1 \frac{E(t)}{E(0)} \right) F(t) + \bar{H}' \left( \varepsilon_1 \frac{E(t)}{E(0)} \right) F'(t) + E'(t) \]

\[ \leq -mE(t) \bar{H}' \left( \varepsilon_1 \frac{E(t)}{E(0)} \right) + c\bar{H}' \left( \varepsilon_1 \frac{E(t)}{E(0)} \right) \bar{H}^{-1} \left( \frac{\gamma \theta(t)}{\xi(t)} \right) + E'(t). \]

Let \( \bar{H}^* \) be the convex conjugate of \( H \) in the sense of Young (see [3, pp. 61-64]), which is given by

\[ \bar{H}^*(s) = s \left( \bar{H}' \right)^{-1}(s) - \bar{H} \left( \left( \bar{H}' \right)^{-1}(s) \right) \]  

(23)

and satisfies the following generalized Young inequality

\[ AB \leq \bar{H}^*(A) + \bar{H}(B). \]  

(24)
By taking \( A = \bar{H}'(\frac{E(t)}{E(0)}) \), \( B = \bar{H}^{-1}(\frac{\gamma(t)}{\xi(t)}) \) and combining (5), (23) and (24), we arrive at

\[
\mathcal{F}_2(t) \leq -mE(t)\bar{H}'(\frac{E(t)}{E(0)}) + c\bar{H}^*(\frac{E(t)}{E(0)}) + c\gamma(t)
\]

Multiplying this estimate by \( \xi(t) \) and using \( \xi(t) = \bar{H}'(\frac{E(t)}{E(0)}) = H'(\frac{E(t)}{E(0)}) \) and inequality (21), we get

\[
\xi(t)\mathcal{F}_2(t) \leq -mE(0) - c\bar{E}(t)\xi(t) H'(\frac{E(t)}{E(0)}) + c\gamma(t)
\]

Take \( \epsilon_2 \) smaller, if needed, to get, for some positive constant \( k \),

\[
\xi(t)\mathcal{F}_2(t) \leq -k\xi(t) H'(\frac{E(t)}{E(0)}) - cE'(t), \quad \forall t > t_0.
\]

Consequently, by setting \( \mathcal{F}_2 = \xi\mathcal{F}_2 + cE \), we obtain, for two constants \( \alpha_1, \alpha_2 > 0 \),

\[
\alpha_1\mathcal{F}_2(t) \leq E(t) \leq \alpha_2\mathcal{F}_2(t), \quad \forall t > t_0
\]

and

\[
\mathcal{F}_2(t) \leq -k\xi(t) \frac{E(t)}{E(0)} H'(\frac{E(t)}{E(0)}) = -k\xi(t)H_2(\frac{E(t)}{E(0)}), \quad \forall t > t_0,
\]

where

\[
H_2(\tau) = \tau H'(\epsilon_1 \tau).
\]

Since

\[
H_2(\tau) = H'(\epsilon_1 \tau) + \epsilon_1 \tau H''(\epsilon_1 \tau),
\]

it follows from the strict convexity of \( H \) on \( (0, r] \) that \( H_2, H_2' > 0 \) on \( (0, 1] \). Then, set

\[
R(\tau) = \frac{\alpha_1\mathcal{F}_2(\tau)}{E(0)}
\]

and exploit (25) and (26) to get

\[
R \sim E
\]

and, for some \( \lambda > 0 \),

\[
R'(t) \leq -\lambda\xi(t)H_2(R(t)), \quad \forall t > t_0.
\]

An integration over \( (t_0, t) \) gives

\[
-\int_{t_0}^{t} \frac{R'(s)}{H_2(R(s))} ds \geq \lambda \int_{t_0}^{t} \xi(s) ds
\]

or

\[
\int_{t_0}^{\epsilon_1 R(t_0)} \frac{1}{sH'(s)} ds \geq \lambda \int_{t_0}^{t} \xi(s) ds;
\]
which implies that
\[ R(t) \leq \frac{1}{\xi_1} H_1^{-1} \left( \lambda \int_{t_0}^{t} \xi(s) ds \right) \quad \forall t > t_0, \]
where \( H_1(t) := \int_{t_0}^{t} \frac{1}{\pi H(s)} ds \). A combination of this estimate with the fact that \( R \sim E \) gives
\[ E(t) \leq C H_1^{-1} \left( \lambda \int_{t_0}^{t} \xi(s) ds \right) \quad \forall t > t_0. \]
This completes the proof of Theorem 4.1.

Remark 2. Routine calculations show that the decay rate deduced from estimate (17) is optimal in the sense that it agrees with the decay rate of \( g \) deduced from (2). For the details, see [9, Remark 2.3].

Corollary 1. Assume that hypothesis (A.1) and (A.2) hold and the function \( H \) in (2) is given by
\[ H(s) = s^p, \quad \text{for} \quad s \geq 0, \quad 1 \leq p < 2. \]
Then, there exist positive constants \( C \) and \( \lambda \) such that
\[ E(t) \leq \begin{cases} 
C \exp \left( -\lambda \int_{0}^{t} \xi(s) ds \right), & \text{for} \quad p = 1, \\
C \left( 1 + \int_{0}^{t} \xi(s) ds \right)^{-\frac{1}{p-1}}, & \text{for} \quad 1 < p < 2.
\end{cases} \quad (27) \]

Example 4.1. (1) Consider the relaxation function \( g(t) = a \exp(-\alpha t) \), where \( a, \alpha \) are positive constants and \( a \) is chosen so that hypothesis (A.1) is satisfied, then
\[ g'(t) = -\alpha H(g(t)) \quad \text{with} \quad H(s) = s. \]
Therefore, it follows from (27) that
\[ E(t) \leq C e^{\lambda(1 + t)}, \quad \forall t \geq 0, \]
where \( \beta = \alpha \lambda \).

(2) Consider \( g(t) = ae^{-(1+t)\nu} \), for \( 0 < \nu < 1 \) and \( a \) is chosen so that condition (A.1) is satisfied, then
\[ g'(t) = \xi(t)H(g(t)) \quad \text{with} \quad \xi(t) = \nu(1+t)^{\nu-1} \quad \text{and} \quad H(s) = s. \]
Estimate (27) entails that
\[ E(t) \leq Ce^{\lambda(1+t)}, \quad \forall t \geq 0. \]

(3) Consider the following relaxation function, for \( \nu > 1 \),
\[ g(t) = \frac{a}{(1 + t)^\nu} \]
and \( a \) is chosen so that hypothesis (A.1) remains valid. Then
\[ g'(t) = -bH(g(t)) \quad \text{with} \quad H(s) = s^p, \]
where \( b \) is a fixed constant, \( p = \frac{1+\nu}{\nu} \) and it satisfies \( 1 < p < 2 \). Then, we deduce from (27) that
\[ E(t) \leq \frac{C}{(1 + t)^\nu}, \quad \forall t \geq 0. \]

For more examples, see [9].
5. General decay rate for different speeds of wave propagation. In this section, we state and prove a generalized decay result in the case of non-equal speeds of wave propagation. We start by differentiating both sides of the differential equations in (P) with respect to $t$ and use the fact that

$$\frac{\partial}{\partial t} \left[ \int_0^t g(t-s)\psi_{xx}(s)ds \right] = \frac{\partial}{\partial t} \left[ \int_0^t g(s)\psi_{xx}(t-s)ds \right]$$

$$= \int_0^t g(t-s)\psi_{xxx}(s)ds + g(t)\psi_{0xx},$$

to obtain the following system

$$\begin{cases}
\rho_1\phi_{tt} - k_1(\phi_{xt} + \psi_t + lw_t) - lk_3(w_{xt} - l\phi_t) = 0, \\
\rho_2\psi_{tt} - k_2\psi_{xxx} + k_1(\phi_{xt} + \psi_t + lw_t) + \int_0^t g(t-s)\psi_{xxx}(s)ds + g(t)\psi_{0xx} = 0, \\
\rho_1w_{tt} - k_3(w_{xt} - l\phi_t) + lk_1(\phi_{xt} + \psi_t + lw_t) = 0.
\end{cases}$$

(P$_*$)

The energy functional associated to (P$_*$) is given by, for any $t \geq 0$,

$$E_*(t) := \frac{1}{2} \int_0^t \left[ \rho_1\phi_{tt}^2 + \rho_2\psi_{tt}^2 + \rho_1w_{tt}^2 + \left(k_2 - \int_0^t g(s)ds\right)\psi_{xt}^2 \\
+ k_3(w_{xt} - l\phi_t)^2 + k_1(\phi_{xt} + \psi_t + lw_t)^2 \right] dx + \frac{1}{2} (g \circ \psi_{xt})(t).$$

(28)

Using similar arguments as in [7, Lemma 3.11] we have the following result.

**Lemma 5.1.** Let $(\phi, \psi, w)$ be the strong solution of (P). Then, the energy of (P$_*$) satisfies, for all $t \geq 0$,

$$E_*(t) = \frac{1}{2} g(t) \int_0^t \psi_{xx}^2 dx + \frac{1}{2} (g' \circ \psi_{xt}) - g(t) \int_0^t \psi_{tt}\psi_{0xx} dx$$

(29)

and

$$E_*(t) \leq c \left( E_*(0) + \int_0^L \psi_{0xx}^2 dx \right).$$

(30)

From estimates (29) and (30), we deduce that, for any $t \geq 0$,

$$0 \leq -(g' \circ \psi_{xt})(t) = -2E_*(t) - g(t) \int_0^L \psi_{xx}^2 dx - 2g(t) \int_0^L \psi_{tt}\psi_{0xx} dx$$

$$\leq -2E_*(t) - 2g(t) \int_0^L \psi_{tt}\psi_{0xx} dx$$

$$\leq -2E_*(t) + g(t) \int_0^L (\psi_{tt}^2 + \psi_{0xx}^2) dx$$

$$\leq -2E_*(t) + g(t) \left( \frac{2}{\rho_1} E_*(t) + \int_0^L \psi_{0xx}^2 dx \right)$$

$$\leq c \left( -E_*(t) + c_1 g(t) \right),$$

where $c_1$ is some fixed positive constant.
Lemma 5.2 ([7]). Let \((\varphi, \psi, w)\) be the strong solution of \((P)\). Then, for any \(\varepsilon > 0\), we have
\[
\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^t \varphi \psi_{xt} dx \leq \varepsilon E(t) + \frac{c}{\varepsilon} ((g \circ \psi_x)(t) - E'(t) + g(t)), \quad \forall t \geq t_0. \quad (31)
\]

Lemma 5.3. Assume that conditions \((A.1)\) and \((A.2)\) hold with \(H\) being linear. Let \((\varphi, \psi, w)\) be the strong solution of \((P)\). Then,
\[
\xi(t)(g \circ \psi_x)(t) \leq \left(-g' \circ \psi_x(t)\right) \leq c\left(-E'_*(t) + c_1 g(t)\right), \quad \forall t \geq 0, \quad (32)
\]
for some positive constant \(c_1\).

Now, we state and prove a general decay result in the case of nonequal speeds of wave propagation.

Theorem 5.4. Let
\[
(\varphi_0, \varphi_1) \in \left(H^2(0, L) \cap H^1_0(0, L)\right) \times H^1_0(0, L)
\]
and
\[
(\psi_0, \psi_1), (w_0, w_1) \in \left(H^2(0, L) \cap H^1_0(0, L)\right) \times H^1_0(0, L).
\]
Assume that conditions \((A.1)\), \((A.2)\) hold and that
\[
\frac{\rho_1}{k_1} \neq \frac{\rho_2}{k_2} \quad \text{and} \quad k_1 = k_3.
\]
Then, for \(l\) small enough, there exist some positive constants \(C\) and \(\lambda\) that depend on the initial data but independent of \(t\) and \(t_1 \geq t_0 = g^{-1}(r)\) such that the energy functional associated to problem \((P)\) satisfies the estimate
\[
E(t) \leq C(t - t_0)H^{-1}_2 \left(\frac{\lambda}{(t - t_0) \int_{t_1}^t \xi(s) ds}\right), \quad \forall t > t_1, \quad (33)
\]
where \(H_2\) is given by
\[
H_2(\tau) = \tau H'(\tau).
\]

Proof. Using Lemma 5.2 in estimate (15), we have, for some \(m > 0,\)
\[
\mathcal{L}'(t) \leq -m E(t) + c(g \circ \psi_x)(t) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^t \varphi \psi_{xt} dx
\]
\[
\leq -(m - \varepsilon) E(t) + c(g \circ \psi_x)(t) + \frac{c}{\varepsilon} (g \circ \psi_x(t) - E'(t) + g(t)), \quad \forall t \geq t_0.
\]
After fixing \(\varepsilon\) small enough, we arrive at
\[
\mathcal{L}'(t) \leq -m_1 E(t) + c(g \circ \psi_x + g \circ \psi_x)(t) - cE'(t) + c g(t), \quad \forall t \geq t_0,
\]
where \(m_1\) is a fixed positive constant. By setting \(\mathcal{F} := \mathcal{L} + cE \sim E\), we obatain, for any \(t \geq t_0,\)
\[
\mathcal{F}'(t) \leq -m_1 E(t) + c(g \circ \psi_x + g \circ \psi_x)(t) + c g(t). \quad (34)
\]

Case I. \(H\) is linear: Multiplying both sides of estimate (34) by \(\xi(t)\), then using hypothesis \((A.2)\) and Lemma 5.3 we get, for any \(t \geq t_0,\)
\[
\xi(t)\mathcal{F}'(t) \leq -m_1 \xi(t) E(t) + c\mathcal{F}(t) \left(g \circ \psi_x + g \circ \psi_x\right)(t) + c\xi(t) g(t)
\]
\[
\leq -m_1 \xi(t) E(t) - cE'(t) + c\left(-E'_*(t) + c_1 g(t)\right) + c\xi(0) g(t).
\]
From the non-increasing property of \(\xi\), we have, for some fixed positive constant \(c_2,\)
\[
(\xi \mathcal{F} + cE + cE_*)'(t) \leq -m_1 \xi(t) E(t) + c_2 g(t), \quad \forall t \geq t_0,
\]
Thus, we have, for some fixed positive constant $C$,

$$m_1 \xi(t)E(t) \leq -\left(\xi\mathcal{F} + cE + cE_s\right)'(t) + c_2g(t), \quad \forall t \geq t_0.$$  

An integration over $(t_0, t)$, exploitation of the non-increasing property of $E$ and estimate (30) yield, for any $t > t_0$,

$$m_1 E(t) \int_{t_0}^t \xi(s)ds \leq -\left(\xi\mathcal{F} + cE + cE_s\right)(t) + \left(\xi\mathcal{F} + cE + cE_s\right)(t_0) + c_2 \int_{t_0}^t g(s)ds$$

$$\leq \left(\xi\mathcal{F} + cE + cE_s\right)(0) + c \int_0^L \psi_{0xx}^2 dx + c_2(b - l).$$

Thus, we have, for some fixed positive constant $C$,

$$E(t) \leq \frac{C}{\int_{t_0}^t \xi(s)ds}, \quad \forall t > t_0.$$  

**Case II** $H$ is nonlinear: First, we use estimates (3), (5) and (32) to get, for any $t \geq t_0$,

$$\int_0^L \int_0^t g(s)(\psi_x(t) - \psi_x(t - s))^2 dsdx + \int_0^L \int_0^t g(s)(\psi_{xt}(t) - \psi_{xt}(t - s))^2 dsdx$$

$$\leq \frac{1}{\xi(t_0)} \int_0^L \int_0^t \xi(s)g(s)(\psi_x(t) - \psi_x(t - s))^2 dsdx$$

$$+ \frac{1}{\xi(t_0)} \int_0^L \int_0^t \xi(s)g(s)(\psi_{xt}(t) - \psi_{xt}(t - s))^2 dsdx$$

$$\leq - \frac{g(0)}{a\xi(t_0)} \int_0^L \int_0^t g'(s)(\psi_x(t) - \psi_x(t - s))^2 dsdx$$

$$- \frac{g(0)}{a\xi(t_0)} \int_0^L \int_0^t g'(s)(\psi_{xt}(t) - \psi_{xt}(t - s))^2 dsdx$$

$$\leq - \frac{g(0)}{a\xi(t_0)} \int_0^L \int_0^t g'(s)(\psi_x(t) - \psi_x(t - s))^2 dsdx$$

$$- \frac{g(0)}{a\xi(t_0)} \int_0^L \int_0^t g'(s)(\psi_{xt}(t) - \psi_{xt}(t - s))^2 dsdx$$

$$\leq - c(E'(t) + E_s'(t)) + c_2g(t),$$

where $c_2$ is a fixed positive constant.

Inserting this estimate in (34), we obtain, for a fixed constant $c_3 > 0$ and any $t \geq t_0$,

$$\mathcal{F}'(t) \leq - m_1 E(t) - c(E'(t) + E_s'(t)) + c_3g(t)$$

$$+ c \int_0^L \int_0^t g(s)\left[(\psi_x(t) - \psi_x(t - s))^2 + (\psi_{xt}(t) - \psi_{xt}(t - s))^2\right] dsdx$$

$$\leq - m_1 E(t) - c(E'(t) + E_s'(t)) + c_3g(t)$$

$$+ c \int_{t_0}^t g(s)(\|\psi_x(t) - \psi_x(t - s)\|^2 + \|\psi_{xt}(t) - \psi_{xt}(t - s)\|^2) ds.$$

(35)

Next, we introduce a functional $\eta$ defined by

$$\eta(t) := \frac{\gamma}{t - t_0} \int_{t_0}^t (\|\psi_x(t) - \psi_x(t - s)\|^2 + \|\psi_{xt}(t) - \psi_{xt}(t - s)\|^2) ds, \quad \forall t > t_0.$$
Then, it follows from (4), (5), (28) and (30) that
\[
\frac{1}{t-t_0} \int_{t_0}^{t} \left( \| \psi_x(t) - \psi_x(t-s) \|_2^2 + \| \psi_{xt}(t) - \psi_{xt}(t-s) \|_2^2 \right) ds \\
\leq \frac{4}{l(t-t_0)} \int_{t_0}^{t} \left( E(t) + E(t-s) + E_s(t) + E_s(t-s) \right) ds \\
\leq \frac{8}{l(t-t_0)} \int_{t_0}^{t} \left[ E(0) + c \left( E_s(0) + \int_{0}^{t} \psi_{0xx}^2 dx \right) \right] ds \\
\leq \frac{8}{l} \left[ E(0) + c \left( E_s(0) + \int_{0}^{t} \psi_{0xx}^2 dx \right) \right] < +\infty, \quad \forall \, t > t_0.
\]
This allows us to pick \( 0 < \gamma < 1 \) such that
\[
\eta(t) < 1, \quad \forall \, t > t_0. \quad (36)
\]
Furthermore, we assume that \( \eta(t) > 0 \) for any \( t > t_0 \), otherwise we obtain the following decay rate from (35)
\[
E(t) \leq \frac{c}{t-t_0}, \quad \forall \, t > t_0.
\]
We define another functional \( \theta \) by
\[
\theta(t) := \int_{t_0}^{t} g'(s) \left( \| \psi_x(t) - \psi_x(t-s) \|_2^2 + \| \psi_{xt}(t) - \psi_{xt}(t-s) \|_2^2 \right) ds
\]
and notice that
\[
\theta(t) \leq -c \left( E'(t) + E'_s(t) \right) + c_4 g(t), \quad \forall \, t > t_0, \quad (37)
\]
where \( c_4 > 0 \) is a fixed constant. Also, strict convexity of \( H \) and the fact that \( H(0) = 0 \) entail that
\[
H(s\tau) \leq sH(\tau), \quad \text{for} \quad 0 \leq s \leq 1 \quad \text{and} \quad \tau \in (0,r).
\]
Combining this with the hypothesis (A.2), Jensen’s inequality and (36), we obtain, for any \( t > t_0 \),
\[
\theta(t) = -\frac{1}{\eta(t)} \int_{t_0}^{t} \eta(t) g'(s) \left( \| \psi_x(t) - \psi_x(t-s) \|_2^2 + \| \psi_{xt}(t) - \psi_{xt}(t-s) \|_2^2 \right) ds \\
\geq \frac{1}{\eta(t)} \int_{t_0}^{t} \eta(t) \xi(s) H(g(s)) \left( \| \psi_x(t) - \psi_x(t-s) \|_2^2 + \| \psi_{xt}(t) - \psi_{xt}(t-s) \|_2^2 \right) ds \\
\geq \frac{\xi(t)}{\eta(t)} \int_{t_0}^{t} H(\eta(t)g(s)) \left( \| \psi_x(t) - \psi_x(t-s) \|_2^2 + \| \psi_{xt}(t) - \psi_{xt}(t-s) \|_2^2 \right) ds \\
\geq \frac{\xi(t)(t-t_0)}{\gamma} H \left( \frac{1}{\eta(t)} \int_{t_0}^{t} \eta(t) g(s) \cdot \frac{\gamma}{t-t_0} (\| \psi_x(t) - \psi_x(t-s) \|_2^2 \right) ds \\
\quad + \| \psi_{xt}(t) - \psi_{xt}(t-s) \|_2^2 \right) ds \\
= \frac{\xi(t)(t-t_0)}{\gamma} H \left( \frac{\gamma}{t-t_0} \int_{t_0}^{t} g(s) \left( \| \psi_x(t) - \psi_x(t-s) \|_2^2 \right. \\
\quad + \| \psi_{xt}(t) - \psi_{xt}(t-s) \|_2^2 \right) ds \right) \]
where $\bar{H}$ is a $C^2$ extension of $H$ which is strictly increasing and strictly convex on $(0, \infty)$. This yields, for any $t > t_0$,

$$\int_{t_0}^{t} g(s) \left( \| \psi_x(t) - \psi_x(t-s) \|^2_2 + \| \psi_x(t) - \psi_x(t-s) \|^2_2 \right) ds \leq \frac{t - t_0}{\gamma} \bar{H}^{-1} \left( \frac{\gamma \theta(t)}{\xi(t)(t - t_0)} \right)$$

and (35) becomes

$$F'_1(t) \leq -m_1 E(t) + \frac{c}{\gamma} (t - t_0) \bar{H}^{-1} \left( \frac{\gamma \theta(t)}{\xi(t)(t - t_0)} \right) + c_3 g(t), \quad \forall \ t > t_0, \quad (38)$$

where $F_1 := F + c E + c E_x$.

Let $0 < \varepsilon_1 < r$, then define a functional $F_2$ by

$$F_2(t) := \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) F_1(t), \quad \forall \ t > t_0.$$ 

Then, estimate (38) together with the facts that $E' \leq 0$, $H' > 0$ and $H'' > 0$ lead to

$$F_2'(t) = \left( - \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)^2} + \frac{\varepsilon_1 E'(t)}{E(0)(t - t_0)} \right) \bar{H}^\prime \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) F_1(t)$$

$$+ \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) F_1(t) \leq -m_1 E(t) \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) + c_3 \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) g(t)$$

$$+ \frac{c}{\gamma} (t - t_0) \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) \bar{H}^{-1} \left( \frac{\gamma \theta(t)}{\xi(t)(t - t_0)} \right), \quad \forall \ t > t_0.$$ 

Let $\bar{H}^*$ be the convex conjugate of $H$ as in (23), set

$$A = \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) \quad \text{and} \quad B = \bar{H}^{-1} \left( \frac{\gamma \theta(t)}{\xi(t)(t - t_0)} \right).$$

then it follows from a combination of (23), (24) and (39) that

$$F_2'(t) \leq -m_1 E(t) \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) + c_3 \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) g(t)$$

$$+ \frac{c}{\gamma} (t - t_0) \bar{H}^* \left[ \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) \right] + \frac{c}{\gamma} (t - t_0) \bar{H}' \left( \frac{\gamma \theta(t)}{\xi(t)(t - t_0)} \right)$$

$$\leq -m_1 E(t) \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) + c_3 \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) g(t)$$

$$+ c \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) g(t) + c \frac{\theta(t)}{\xi(t)}, \quad \forall \ t > t_0.$$ 

After fixing $\varepsilon_1$ much smaller (if needed), we arrive at

$$F_2'(t) \leq -m_2 E(t) \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) + c \frac{\theta(t)}{\xi(t)}$$

$$+ c_3 \bar{H}' \left( \frac{\varepsilon_1 E(t)}{E(0)(t - t_0)} \right) g(t), \quad \forall \ t > t_0,$$
where \( m_2 > 0 \).

Multiplying both sides of this estimate by \( \xi(t) \) and using \( \varepsilon_1 \frac{E(t)}{E(0)} < r \) and inequality (37), we arrive at
\[
\xi(t)\mathcal{F}_2'(t) \leq -m_2 \frac{E(t)}{E(0)} H' \left( \frac{\varepsilon_1 E(t)}{E(0)(t-t_0)} \right) \xi(t) + c\theta(t) + c_3 H' \left( \frac{\varepsilon_1 E(t)}{E(0)(t-t_0)} \right) \xi(t) g(t) \\
+ c_3 H' \left( \frac{\varepsilon_1 E(t)}{E(0)(t-t_0)} \right) \xi(t) g(t), \quad \forall t > t_0.
\]

This completes the proof of Theorem 5.4.
Example 5.1. (1) Consider the relaxation function \( g(t) = a \exp(-\alpha t) \), where \( a \) and \( a \) are positive constants and \( a \) is chosen so that hypothesis (A.1) is satisfied, then

\[
g'(t) = -\alpha H(g(t)) \quad \text{with} \quad H(s) = s.
\]

Therefore, \( H_2(t) = t \) and estimate (33) implies that there exist \( t_1 > 0 \) such that

\[
E(t) \leq \frac{c(t-t_1)}{t-t_1}, \quad \forall \, t > t_1.
\]

(2) Consider \( g(t) = ae^{-(1+t)^\nu} \), for \( 0 < \nu < 1 \) and \( a \) is chosen so that condition (A.1) is satisfied, then

\[
g'(t) = -\xi(t)H(g(t)) \quad \text{with} \quad \xi(t) = \nu(1+t)^{\nu-1} \quad \text{and} \quad H(s) = s.
\]

Therefore \( H_2(t) = t \) and estimate (33) entails that

\[
E(t) \leq \frac{c}{(1+t)^\nu}, \quad \text{for } t \text{ large enough}.
\]

(3) Consider the following relaxation function, for \( \nu > 1 \),

\[
g(t) = \frac{a}{(1+t)^\nu}
\]

and \( a \) is chosen so that hypothesis (A.1) remains valid. Then

\[
g'(t) = -bH(g(t)) \quad \text{with} \quad H(s) = s^p,
\]

where \( b \) is a fixed constant, \( p = \frac{1+\nu}{\nu} \) and it satisfies \( 1 < p < 2 \). Then, \( H_2(t) = pt^p \) and we deduce from (33) that there exist \( t_1 > 0 \) such that

\[
E(t) \leq \frac{c}{(1+t)^{(2-p)/p}} = \frac{c}{(1+t)^{(\nu-1)/(\nu+1)}}, \quad \forall \, t > t_1.
\]

Acknowledgments. The authors would like to express their gratitude to King Fahd University of Petroleum and Minerals (KFUPM) for its continuous support and to an anonymous referee for his/her careful reading.

REFERENCES

[1] M. O. Alves, L. H. Fatori, M. A. Jorge Silva and R. N. Monteriro, Stability and optimality of decay rate for a weakly dissipative Bresse system, Math. Methods Appl. Sci., 38 (2015), 898–908.
[2] M. S. Alves, O. Vera, J. Muñoz-Rivera and A. Rambaud, Exponential stability to the Bresse system with boundary dissipation conditions, (2015), arXiv:1506.01657.
[3] V. I. Arnol’d, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, 1989.
[4] F. Dell’Oro, Asymptotic stability of thermoelastic systems of Bresse type, J. Differ. Equ., 258 (2015), 3902–3927.
[5] L. H. Fatori and R. N. Monteiro, The optimal decay rate for a weak dissipative Bresse system, Appl. Math. Lett., 25 (2012), 600–604.
[6] A. Guesmia and M. Kafini, Bresse system with infinite memories, Math. Methods Appl. Sci., 38 (2015), 2389–2402.
[7] A. Guesmia and S. A. Messaoudi, On the stabilization of Timoshenko systems with memory and different speeds of wave propagation, Appl. Math. Comput., 219 (2013), 9424–9437.
[8] T. F. Ma and R. N. Monteiro, Singular limit and long-time dynamics of Bresse systems, SIAM J. Math. Anal., 49 (2017), 2468–2495.
[9] M. I. Mustafa, General decay result for nonlinear viscoelastic equations, J. Math. Anal. Appl., 457 (2018), 134–152.
[10] J. A. Soriano, J. E. Muñoz Rivera and L. H. Fatori, Bresse system with indefinite damping, J. Math. Anal. Appl., 387 (2012), 284–290.
[11] A. Soufyane and B. Said-Houari, The effect of the wave speeds and the frictional damping terms on the decay rate of the bresse system, *Evol. Equations Control Theory*, 3 (2014), 713–738.

[12] A. Wehbe and W. Youssef, Exponential and polynomial stability of an elastic Bresse system with two locally distributed feedbacks, *J. Math. Phys.*, 51 (2010), 1–17.

Received April 2018; revised July 2018.

E-mail address: messaoud0@kfupm.edu.sa
E-mail address: elhashim06@yahoo.com