Chaotic enhancement of decay. The effect of classical phase space structures

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We investigate the decay process from a time dependent potential well in the semiclassical regime. The classical dynamics is chaotic and the decay rate shows an irregular behavior as a function of the system parameters. By studying the weak-chaos regime we are able to connect the decay irregularities to the presence of nonlinear resonances in the classical phase space. A quantitative analytical prediction which accounts for the numerical results is obtained.

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I. INTRODUCTION

In the last years, many conjectures has been put forward, and tested in various system models, to answer the fundamental question of the “quantum chaos problem”: what is the signature of classical chaos in the quantum world? Among these, one of the most intriguing is the idea according to which the classical chaos can induce large scale fluctuations on a genuine quantum phenomenon such as the tunneling process. Starting from the seminal paper of Davis and Heller [1], who first noted the occurrence of coherent tunneling between regular tori separated by a chaotic region, the influence of classical chaos on quantum tunneling has been verified in many systems and is now accepted in the literature as a fingerprint of classical non integrability. It is very simple to describe this effect. Let us consider a system which is classically chaotic and invariant under a symmetry operation, like for example the space inversion. If the classical system supports a regular torus, by symmetry, there might be also a second torus which is distinct from its symmetric partner, like for instance two symmetric tori encircling the bottom of the two wells of a double-well potential. Moreover, let us suppose that the two tori are large enough to support quantum states. In this condition, the quantum system will show coherent tunneling between the states located in the two symmetric tori. If now one system parameter is changed (e.g. \(\hbar\)), contrary to the expectations of ordinary semiclassical analysis, the tunneling rate shows strong irregularities which can increase or decrease the rate by orders of magnitude. This effect does not show up in a system whose classical counterpart is integrable.

The tunneling fluctuation is usually interpreted in terms of assisted processes, or, using a widespread terminology, as “Chaos Assisted Tunneling” (CAT) [2–10].

An intuitive view of the CAT process is as follows. The presence or regular and stochastic motion in the classical phase space corresponds, from a quantum point of view, to the possibility of having two kind of states: regular ones localized inside the symmetric tori and chaotic states, which being extended through the chaotic region have a not negligible overlap with regular regions. The fluctuations in the tunneling rate are thus explained in terms of a three state tunneling process. The quantum particle first tunnels from the localized state to an extended chaotic one and then from this to the state located in the symmetric torus.

FIG. 1. Sketch of the typical behavior of the energy levels of a classically chaotic quantum system as a parameter \(p\) is changed. In a) the two solid lines describe a couple of quasi-degenerate levels of different symmetry. The dashed line describes a colliding third level. In b) we show the splitting of the two regular levels. All the units are arbitrary.

This interpretation is confirmed by the level dynamics of the tunneling system. A typical situation is sketched in Fig. 1 where we can observe the change of two quasi-degenerate levels, which correspond to the pair of tunneling regular states, as a system parameter is varied. In
almost the entire parameter range the splitting between the two states, and so the direct tunneling probability, changes smoothly. However, it may occur that, once the parameter is changed, a third level (dashed line) crosses the two quasi-degenerate levels. In the non-integrable case states belonging to the same symmetry class do not cross each other. Therefore, the appearance of a third colliding states gives rise to an avoided crossing with the state of the doublet which belongs to the same symmetry class.

The avoided crossing has a twofold consequence on the tunneling process under study. First, in the nearby of the crossing we cannot consider the tunneling as a process involving only the two quasi-degenerate states. In this condition the standard two state tunneling becomes a resonant three state process. Second, since the colliding third state modifies the energy level of only one of the doublet states, the splitting of the two levels changes. Therefore, given the small value of the energy splitting in the semiclassical regime, the avoided crossing produces a dramatic modification of the levels splitting and consequently of the tunneling rate. It is important to point out that the rate can increase by several orders of magnitude as well as vanish (see the arrow in Fig. [1]) according to the value of the parameters. Moreover, due to the fact that the energy spectrum, in the nonlinear case, does not show any regularity, the crossings with a third level do not follow a regular pattern, and the overall behavior of the tunneling rate appears to be an irregular sequence of peaks [4–7] instead of the smooth behavior expected in the regular systems.

These features motivated the widespread idea that classical chaotic trajectories can have an active role in the quantum process, helping or “assisting” the quantum particle to tunnel between the symmetric tori. Along this line, path integral techniques have been recently used to calculate the contribution to the tunneling stemming from complex orbits which connect the symmetric regular tori through the classical stochastic layer [7].

However, a real quantitative theory of the CAT process is still lacking. The main reason for this can probably be found in the chaotic nature of the third state which prevents any simple analytical treatment. Moreover, there are some aspects of the phenomenon which do not seem to fit properly with the intuitive interpretation given by the CAT picture. For example, the presence of strong decreases in the tunneling rate which, together with the enhancements, occur as a result of a parameter change, contradicts the idea of a tunnel process being “assisted” by chaos. Another controversial aspect of the effect is the nature of the third state which crosses the tunneling doublet. The key point of the CAT picture is that the perturbation in the energy splitting is relevant only for those crossings involving colliding third states which are located in the chaotic region. However, recently, it has been shown that similar strong fluctuations can be obtained also in almost integrable systems [8]. In this condition, the third state responsible for the fluctuation is by no means chaotic. It is regular and its crossing with the tunneling doublet can be directly related to the classical phase-space structure and in particular to the destruction of the regular tori which are transformed into classical non-linear resonances. In particular, a fluctuation (avoided crossing) occurs when the energy of the tunneling state corresponds classically to the energy of a nonlinear resonance. In other words, in this regime, the fluctuations (avoided crossings) are the quantum manifestations of the classical “small denominators” problem. This discovery also led to a first analytical prediction about the positions of the tunneling fluctuations which we shall review in Section [11].

In this paper we want to assess whether this picture applies also to a different tunneling process, namely the quantum escape of a particle which has been initially located inside a potential well. From a classical point of view it is clear that the particle can overcome the potential barrier of the well only if its energy is larger than the barrier height, while in the quantum framework the tunneling across the classically forbidden region is always present. Clearly this situation is going to be modified when, including the ingredient of chaos, we perturb the system by adding a forcing term, i.e., a time dependent external force. The perturbation disturbs the regular motion of the classical particle and, by increasing its energy, makes it possible to the particle to escape over the barrier. In the meanwhile, also the quantum process of tunneling changes due to the modification of the potential and both the processes contribute to the decay [11]. Our purpose is to choose a region of the system parameters where the classical and the quantum contribution to the decay can be separated, in order to study the properties of the latter process in connection with the chaotic features of the classical phase space.

By studying a decay process, we shall address a infinite system, with a continuous spectrum, and this will prevent us from using standard methods, like the diagonalization of the Floquet dynamic operator [12], to obtain directly the level splittings responsible for the tunneling. Anyway we shall be able to analyzes the system by resorting to a somewhat simpler method: we shall calculate numerically the time evolution of a quantum state initially located in the potential well and, by studying the decay of the population in the well, we shall be able to obtain the relative strength of the tunneling as a function of the system parameters. This will allow us to point out the differences between this process of chaos assisted decay and the tunneling processes in the presence of chaos. Finally, we shall be able to show that the picture which singles out the classical nonlinear resonances as the main responsible for the fluctuations of the tunneling rate, applies also in this context, and that the prediction of Ref. [3], illustrated in Section [IV] remains valid.
II. THE MODEL: CLASSICAL DYNAMICS

In order to analyze the influence of classical chaos on the quantum process of escape from a potential well, we introduce a simple one-dimensional forced system described by the following Hamiltonian:

\[ H = \frac{p^2}{2} + V(q, t) \]

(1)

\[ V(q, t) = V_0(1 - \cos(2q)) + \epsilon(1 - \cos(2q - \Omega t)) \quad q = [-\pi, \pi] \]

(2)

\[ V(q, t) = 0 \quad \text{otherwise}. \]

The particle is located initially inside a potential well which has the form of a sinusoidal function extended over two periods, as shown in Fig. 2, and then it is forced by a time periodic perturbation which is considered to be small compared to the static potential well, i.e., \( \epsilon \ll V_0 \).

We chose a perturbation term which turns system (1) into a double resonance-like Hamiltonian. This choice is dictated by the sake of simplicity. It is indeed clear that, as long as we limit our analysis to small perturbations, the particular form of the external forcing does not affect the generic features of the decay process we want to study. On the other hand, the adoption of Hamiltonian (1) presents many benefits. All the relevant information concerning the dynamic properties of our model can be derived from the dynamics of a well known system, the double resonance Hamiltonian which corresponds to eq. (1) with periodical boundary conditions [13,14].

The presence of a periodic perturbation in (1) breaks the integrability of the classical Hamiltonian. The most important features of this condition is the appearance of nonlinear resonances in the phase space together with regions characterized by extended chaotic motion (stochastic layer). The relevance of the chaotic motion depends on the strength \( \epsilon \) of the perturbation term, so that the system can be more or less chaotic. In Fig. 3 we show a stroboscopic mapping of the dynamics, namely the position in phase space at fixed intervals of time which are integer multiples of the forcing term period \( T = 2\pi/\Omega \), for a generic weak-chaos case. Some nonlinear resonances and the stochastic layer around the separatrix are clearly visible.

The nonlinear resonances are the visible consequences of the small denominators problem. These are related to the secular terms which appear in the perturbative solution of the equation of motion of non-integrable systems. In the weak-chaos condition their position in the phase space can be obtained by considering the effect of the time-dependent term as a perturbation on the dynamics expressed by the constant Hamiltonian

\[ H_0 = \frac{p^2}{2} + V_0(1 - \cos(2q))|_{-\pi<q<\pi}. \]

(4)

The KAM theorem [15] states that, as long as the perturbing term can be considered small, the main part of the phase space remains practically unperturbed and that only the tori which are resonant with the forcing term are destroyed and replaced by chains of islands like the ones shown in Fig. 3. The resonant condition can be written as

\[ \omega_0(E) = \frac{m}{n} \Omega, \]

(5)

where \( n \) and \( m \) are integer numbers and \( \omega_0(E) \) is the frequency of the unperturbed motion inside the well which depends on the energy \( E \). In our case it is possible to express \( \omega_0(E) \) in term of the elliptic function \( K(k) \) as

\[ \omega_0(E) = \pi \sqrt{V_0/K(k)} \]

\[ k = (E + V_0)/2V_0. \]

(6)

The result of Eq. (5) would actually indicate that all the tori are destroyed by the perturbation, being the rational numbers dense among the real ones. Nevertheless, the KAM theorem assures that the effects of the perturbation becomes smaller and smaller with increasing the order of the resonance, i.e., with increasing the numerator \( m \).

![FIG. 2. The unperturbed potential of Eq. (2). We also show the wave function of the two states used as initial condition in the numerical calculations.](image)

![FIG. 3. The classical dynamics inside the well. Stroboscopic Poincaré map. The values of the parameters are \( V_0 = 0.048, \epsilon = 0.005, \Omega = 2 \).](image)
This is clearly visible in Fig. 3 where we can only recognize the chains corresponding to $m = 1$, i.e., the $1/5$, $1/6$ and $1/7$ resonances. However, even for small perturbation, in the neighborhood of the separatrix of the unperturbed system the motion is always dominated by chaotic dynamics. In other words, trajectories which in the absence of perturbation are bounded inside the potential well, can now overcome the energy barrier and eventually escape from the well. In the generic weak chaos condition, the stochastic layer around the separatrix is dynamically separated from the phase space region corresponding to bounded trajectories by unbroken tori, so that the process of escape driven by chaos is limited in phase space. Therefore, in the classical case, the decay of the population of the well is possible only if some particles are initially put inside the stochastic region. This process have been extensively studied in the last years in various classical and quantum models, the most known of which is probably the Hydrogen atom in the presence of a strong radiation field \cite{1,2,7}. In that system the dynamic process described above leads to the ionization of the atom. However, in this paper we want to focus on the connection between chaos and processes which would be classically impossible, such as quantum tunneling. For that reason we shall analyze the decay of the well population for a quantum particle initially put in the phase space region which corresponds, even in the presence of chaos, to bounded motion. In connection to this it is worthwhile to remember that in quantum mechanics the situation is never as simple. Whatever the initial condition is, the wave function cannot be sharply located inside a finite region but exhibits smooth decreasing tails which extend over the stochastic layer. Therefore, to keep the classical chaotic diffusion process as small as possible, we shall consider a weak-chaos regime with a small stochastic layer like in Fig. 3, and in addition to this, we shall study the decay of quantum states well localized inside the potential well.

III. THE MODEL: QUANTUM DYNAMICS

We studied the quantum decay from the well of Fig. 2 by integrating numerically the time dependent Schrödinger equation associated with Hamiltonian (4). This can be done using a FFT splitting algorithm \cite{18} and using absorbing boundary conditions \cite{19} as described, for example, in Ref. \cite{11}.

In order to single out the effect of the chaotic perturbation on the process of quantum decay, it is necessary to choose as initial condition a state localized inside the well which has an unperturbed dynamics as simple as possible. The eigenstates of Hamiltonian (4) with $\epsilon = 0$ are not useful in this context being system (1) an open system (continuous spectrum with stationary states which do not have finite support inside the well). We thus resorted to use as initial condition metastable states which have a negligible internal dynamics and a long enough unperturbed life-time inside the well. These are the resonances of the potential well of Fig. 2 defined in the quantum theory of scattering. In addition to this, it is worthwhile to notice that, since in the parameter range we explored the unperturbed decay probability is negligible, the adoption of the eigenstates of Hamiltonian (4) supplemented by periodical boundary conditions leads essentially to the same results. Therefore, as a first approximation we can consider the initial states as stationary state of the unperturbed system. For the sake of simplicity in the following we shall refer to them as “eigenstates” of the unperturbed Hamiltonian.

FIG. 4. The time evolution of the well population. The different curves corresponds to 20 different values of $\Omega$ included between $\Omega = 1.5$ and $\Omega = 2.5$. The values of the parameters are $V_0 = 0.048$, $\epsilon = 0.005$, $\hbar = 0.025$ and the initial state is the fourth eigenstate of the unperturbed well. The horizontal dashed line correspond to the unperturbed $\epsilon = 0$ case. Note the fast oscillations in the population decay which reflect the oscillation of the forcing term.

In Fig. 4 we show the time evolution of the population inside the well $P(t) = \int_{-\pi/2}^{\pi/2} |\psi(q, t)|^2 dq$ for different values of the forcing frequency $\Omega$. This figures has been obtained by choosing as initial condition the fourth eigenstate, $|3\rangle$, of the unperturbed well. It can be easily realized that the decay probability can be strongly enhanced by the forcing term even in the small perturbation regime ($\epsilon = 0.005$). Note that in the unperturbed case the population decay is not visible in the scale of this figure, the population remaining practically unchanged in the studied interval of time. This shows that the time scale of the unperturbed process is much longer than the maximum time which we explored numerically.

To obtain a more quantitative representation of the phenomenon we could define the decay rate as the inverse of the time integral of $P(t)$, namely the inverse of the area contained under the curves of Fig. 4. This would imply a very long numerical simulation, up to a time where the population has completely leaked out. However, we are not interested in the absolute magnitude of the decay rate, but only in its relative strength as a function of
the system parameters. Thus we can simply calculate the time integral of \( P(t) \) up to a certain time \( t_{\text{max}} \) and measure the rate by studying the quantity

\[
R = 1 - \frac{1}{t_{\text{max}}} \int_0^{t_{\text{max}}} P(t) dt
\]

(7)
as a function of the frequency \( \Omega \) of the forcing term. As shown in Fig. 6, \( R \) shows a sequence of peaks, similar to a resonant dependence on the frequency of the perturbation. We repeated the calculation for two different initial conditions, namely by choosing the third and the fourth eigenstate of the unperturbed well. The decay rate for the third state is smaller, as expected, since this state is more deeply located in the potential well, but in both cases we found a similar behavior even if the position of the peaks and their intensity is not the same.

![Diagram](image_url)

**FIG. 5.** \( R \), calculated with \( t_{\text{max}} = 400 \), as function of the driving frequency \( \Omega \). The values of the parameters are \( V_0 = 0.048 \), \( \epsilon = 0.005 \), \( \hbar = 0.025 \). The two figures correspond to different initial conditions, Fig. a) by choosing as initial state the fourth eigenstate of the unperturbed well, Fig. b) by choosing the third. The numbers refer to the classical nonlinear resonances as discussed in Section [V]. The insets contain enlarged views.

Except for the presence of the peaks, the figures shows that, as a general tendency, the decay increases as the forcing frequency decreases. This can be understood using arguments based on the classical dynamics of system [I]. Indeed, as shown in Fig. 7, the chaotic features of the classical phase space (i.e., width of the stochastic layer and of the nonlinear resonances) increase by reducing \( \Omega \), thus demonstrating that the forcing term becomes more important when \( \Omega \) gets smaller. Eventually, for extremely small \( \Omega \), the stochastic layer becomes so wide that, for the chosen initial conditions, the escape from the well via the direct classical process becomes the dominant process. Since we are rather interested in the quantum mechanism of escape from the well, we shall not explore the condition of small \( \Omega \)'s corresponding to strong classical chaos.

In addition to this, there is a further reason to limit the analysis to \( \Omega \)'s not too small. Our evaluation of the decay rate, as the integral of the surviving population in the well over a finite time, is meaningful only if the time of integration is much longer than the period of the perturbation, and therefore we shall limit ourselves to study the range of large \( \Omega \) (2\( \pi/\Omega \ll t_{\text{max}} \)).

However, it is important to point out that the numerical results represented in Figs. 4 and 5 does not account for the process of classical escape from the well, even to the smallest used value of \( \Omega \). A direct calculation demonstrates indeed that the classical decay is always negligible in the considered parameter range of \( \Omega \). This can be easily assessed by numerically integrating system [I] using a classical initial particle distribution mimicking the phase-space representation of the initial quantum state (see Section [V] for details). Even for the smallest value of \( \Omega \) used to obtain the results of Figs. 4 and 5, the classical population in the well does not practically change with time.

Therefore, the peaks and the general tendency of Figs. 4 and 5 are genuine quantum effects, but, while the latter can be associated to the increased effectiveness of the forcing term in [I] with decreasing \( \Omega \), theformer effect does not have any simple interpretation. The understanding of this will be the subject of the next section.

To conclude the analysis of Fig. 5 we have to assess the role of the quantum decay process in the absence of the external perturbation (i.e., \( \epsilon = 0 \)). By direct numerical integration we observed that in the unperturbed case the decay rate is orders of magnitude smaller than that of the forced case. Indeed, for \( \epsilon = 0 \), the population inside the well remains equal to one (within the numerical precision) even at \( t = t_{\text{max}} \). This confirmed that within the observed time, the chosen initial states can indeed be regarded as “stationary states” of the unperturbed system. On the other hand, this result, as well as previous observations, suggest that the sudden increase of the decay rate is connected to the classical chaos in a way which is not directly connected to the classical process of barrier crossing via chaotic diffusion through the stochastic layer. In the next section we shall build up a theory to explain the presence of peaks.
IV. A SEMICLASSICAL ANALYSIS

The results shown if Fig. 6 resemble the typical CAT behavior: when a system parameter is changed the decay rate presents an irregular sequence of peaks on a smooth background. It is thus natural to look for a connection between CAT systems and our model. First of all, we notice that, due to the continuous nature of spectrum, the level dynamics of our system can not be simply described in terms of avoided crossings. Nevertheless, we believe that the phenomenon underlying Fig. 6 retains much of the features of the CAT processes. In particular we think that the argument introduced in Ref. 9 to explain the tunneling irregularities of a quasi-integrable system, can be effectively used also in this model. In the cited reference, a connection between the peaks in the tunneling rate and the position of the nonlinear resonances in the classical phase space was found. In this section we shall exploit the same idea.

The analysis of Ref. 9 follows the line of Refs. 21,22 and is based on a semiclassical approximation which makes use of simple arguments. We shall assume that the classical system is only weakly perturbed by the external perturbation: the size of the chaotic region is considered to be small with respect to the portion of the classical system that, by crossing, modifies the tunneling, needs the regularity of the more generic CAT one, in the weak-chaos regime. We think that the argument introduced in Ref. [9] to clarify the role of until now disregarded actors: the level dynamics of our system can not be simply described in terms of avoided crossings. Nevertheless, we believe that the phenomenon underlying Fig. 6 retains much of the features of the CAT processes. In particular we think that the argument introduced in Ref. [9] to explain the tunneling irregularities of a quasi-integrable system, can be effectively used also in this model.

In this condition, the area of the regular region, that is the phase-space region which is encircled by the last unbroken torus inside of the stochastic layer, is much larger than \( \hbar \). In this way, the regular region can accommodate several quantum states and the semiclassical approximation becomes meaningful.

As we discussed in the Introduction, all the studies about the effects of chaos on tunneling or on decay processes were concerned with the role of the stochastic layer and the classical transport therein as the main contribution to the barrier crossing. This effect is surely present, but its contribution is not always the most important, at least in a weak-chaos regime. In fact as shown in Ref. 9, by studying the tunneling process in a forced double well system in a condition on weak chaos, it has been possible to clarify the role of until now disregarded actors: the nonlinear resonances which are present inside the regular region of the well. These pre-chaotic structures cannot in fact contribute to the classical transport over the barrier, because they are embedded in a regular region of unbroken tori, but they can perturb the quantum dynamics via a indirect process similar to the one discussed in CAT. The process of avoided crossing responsible of the tunnel irregularity in CAT can in fact be present also in a weak-chaos regime, with the difference that the third state which, by crossing, modifies the tunneling, needs not to be chaotic.

This process, which we think to be a special realization of the more generic CAT one, in the weak-chaos regime can give a contribution to the tunneling rate modification even more important than the one connected to the chaotic region of the phase space. Moreover, due to the fact that the perturbing third state is not chaotic, it is possible to obtain a quantitative prediction of the avoided crossings and hence on the positions of the tunneling irregularities.

Let us note first of all that the Hamiltonian under study is time dependent and that this prevents us from adopting the Einstein-Brillouin-Keller (EBK) quantization conditions 23 in their original form. However, following the work of Breuer and Holthaus 22, who, in turn, extended the method of the canonical operator as developed by Maslov and Fedoriuk 24, it is possible to adapt the EBK prescriptions to periodically time-dependent systems. The generalization of Ref. 22 is based on the prescription of Arnold 25 and leads to semiclassical quantization rules for the Floquet quasi-energies and quasi-eigenstates 26. This is made possible by a suitable extension of the phase space, including the time \( t \) as a coordinate and adding the corresponding conjugate momentum. In the one-dimensional case the semiclassical quantization prescriptions read:

\[
J = \frac{1}{2\pi} \int_{\gamma_1} pdq = \hbar(n + \frac{\nu}{4}),
\]

\[
E_{n,m} = -\frac{1}{T} \int_{\gamma_2} (pdq - Hdt) + \hbar \Omega m.
\]

The meaning of the symbols adopted in this expression can be explained by remembering that in the one-dimensional case the extended space is the tridimensional phase space \( \{q, p, t\} \). A regular trajectory is contained on a flux tube in this tridimensional space, flux tube which repeat itself periodically along the \( t \) direction. Thus \( J \) is the action associated to the quantized trajectory; \( \gamma_1 \) is a close path winding once around the flux tube and lying in the plane at a given time \( t \) and \( \gamma_2 \) is a path stretching itself out on the surface of the flux tube such that it can be continued adopting periodic boundary conditions. In other words, the path \( \gamma_2 \) in the extended tridimensional phase space, moves from an initial point lying on the plane \( t = 0 \) to a final point lying on the plane \( t = T \). Note also that we can choose to lay the path \( \gamma_1 \) on the Poincaré section of the flux tubes. Finally, if we restrict ourselves to consider the closed orbits inside the potential well, the Maslov index assumes only the value \( \nu = 2 \).

Worth of a detailed discussion is the structure of the quantized energy \( E_{n,m} \) (for simplicity we shall use the terms energy and state as equivalents of quasi-energy and quasi-state). While the index \( n \), which values are fixed by the first of Eqs. (8), has the usual role of principal quantum number, the index \( m \), and the dependence of \( E_{n,m} \) on this one, reflects the periodicity of the time dependent term of the Hamiltonian. In fact, an important aspect of the Floquet theory is the Brillouin zone structure of the energy spectrum: for each physical solution labeled...
by $n$ we have a infinite series of representative labeled by the value of $m$. Naturally all the physical information is contained in the first Brillouin zone $0 \leq E_{n,m} < \hbar \Omega$, or equivalently, we can say that any solution of Eqs. (8) can be folded back to the first Brillouin zone by an appropriate choice of $m$.

At this point it is important to recall that the earlier formalism represents a valid quantization procedure only in closed systems, where the notion of energy levels is meaningful, and where the phase space is filled with regular tori. However, we are investigating the properties of states that lie well inside the stability region, and that present a small decay probability (see Fig. 5). In this condition, we believe that an analysis which, according to the prescriptions of CAT, connects the tunneling irregularities with the presence of avoided level crossings, can retain its validity. Let us thus proceed disregarding the continuity of the spectrum and looking for the presence of level crossings in the unperturbed Hamiltonian spectrum as a function of the forcing frequency $\Omega$.

Following Ref. [22] we can look for the level crossings by replacing $H$ with $H_0$, let us in fact recall that we are always considering a perturbative approach. The condition of crossing between states $n$ and $n'$ in the Brillouin zone yields the following equation:

$$H_0(h(n + 1/2)) + h \Omega m = H_0(h(n' + 1/2)) + h \Omega m'. \quad (9)$$

This equation can be simplified if we assume that $\hbar$ is so small as to make negligible the quantity $\hbar(n - n')$. If we now expand the RHS of this equation around this small parameter, we obtain

$$\frac{dH_0}{dn}(n - n') + h \Omega m = h \Omega m' \quad (10)$$

which can be rewritten as

$$\frac{dH_0}{dJ} \frac{dJ}{dn}(n - n') = h \Omega (m' - m) \quad (11)$$

or, by using $J = h(n + 1/2)$,

$$\omega_0 = \frac{\Delta m \Omega}{\Delta n} + \mathcal{O}(\hbar), \quad (12)$$

where $\omega_0 \equiv \omega_0(J) = \frac{dH_0}{dJ}$ is the frequency of the unperturbed motion as a function of the classical action $J$, $\Delta n \equiv n' - n$ and $\Delta m \equiv m - m'$. The condition of levels crossing, in the limit of vanishing $\hbar$ can thus be obtained by solving the classical equation which corresponds to the condition for the onset of the classical nonlinear resonances.

This is a very important results, because it means that, for sufficiently small $\hbar$’s, there is a correspondence between the presence of a nonlinear resonance in the classical phase space and a level crossing in the spectrum and therefore a correspondence between the tunneling irregularities and the nonlinear resonances. In other words, the tunneling peaks are the quantum manifestation of the non-integrable behavior of the underlying classical dynamics. The results of Eq. (12) is even more specific, in fact is says that the crossings of the two unperturbed levels $E_{n,m}^0$ and $E_{n',m'}^0$ is related to the superposition between the semiclassical quantization torus of one of the two states and the nonlinear resonance of appropriate order $\Delta m/\Delta n$.

This process admits a simple intuitive representation. Let for example consider a quantum state located deep in the well. Its quantization torus lies inside the well and the wavefunction in the semiclassical regime is mainly located around this torus. The decay rate in the unperturbed case can be obtained by the usual semiclassical calculation of the probability of barrier crossing. If we turn the perturbation on in a regime of weak chaos we have a high probability that nothing happens, due to the fact that most of the phase space inside the well remains unperturbed. But if we change an external parameter as, for example, the forcing frequency $\Omega$, we obtain that the nonlinear resonances move in the phase space and, for particular values of the parameter, one of them can intersect the quantization torus which is destroyed. This would be probably reflected in a perturbation of the quantum state and thus in a modification of its decay rate. This is exactly what is described by Eq. (12).

Notice anyway that Eq. (12) involves resonances of any order and this implies that the change of a system parameter makes a chosen level undergo a virtually infinite number of crossings. In other words, when we turn the perturbation on, the mathematical curve corresponding to a particular energy level becomes extremely complex, being fragmented by the avoided-crossing effects in an infinite number of points. However, the perturbation produced by the avoided crossings strongly depends on the order $\Delta m$ of the resonance and in practice, if we restrict ourself to the perturbative regime, the only significant crossings are those associated with the first-order resonances ($\Delta m = 1$). On the other hand the same happens in the classical dynamics, as discussed in Section (1).

The simple prediction of Eq. (12) is valid in the strong semiclassical limit and is therefore unavailable for a numerical check, being the smallest value of $\hbar$ dictated by the computer limitations. The relation between nonlinear resonances and tunneling peaks has been indeed proven in Ref. [3] by means of a further expansion of Eq. (8). To do that we proceed by evaluating the second order term in $h(n - n')$. We obtain:

$$\omega_0 + \frac{1}{2} \hbar \frac{d\omega_0}{dJ} \frac{\Delta n}{\Delta m} = \frac{\Delta m \Omega}{\Delta n} + \mathcal{O}(\hbar^2). \quad (13)$$

On the other hand,

$$\frac{d\omega_0}{dJ} \equiv \frac{d\omega_0}{dE} \frac{dE}{dJ} = \frac{d\omega_0}{dE} \omega_0. \quad (14)$$

Thus we can write (13) as follows:
\[
\omega_0(E) = \frac{\Delta m}{\Delta n} \left(1 + \frac{\Omega}{2} \frac{d\omega_0(E)}{dE} \right) + \mathcal{O}(\hbar^2). \tag{15}
\]

Where we can give an analytical expression to \( \frac{d\omega_0(E)}{dE} \), by using Eq. \( \mathcal{O} \), as

\[
\frac{d\omega_0}{dE} = \frac{\pi}{4k^2 \sqrt{\hbar V_0}} \frac{1}{K(k)} \left(1 - \frac{E(k)}{k'^2 K(k)}\right), \tag{16}
\]

where \( k'^2 = 1 - k^2 \). Eq. \( \mathcal{O} \) is the main results of Ref. \( \mathcal{O} \) and represents a generalization of the classical nonlinear resonance condition, where the frequency ratio is renormalized by means of a quantum correction proportional to \( \hbar \).

By using this prediction we are now able to verify our conjecture about the validity of this method also in the present case. From Eq. \( \mathcal{O} \) it is possible to predict the position of the peaks of decay. After fixing the values of \( \Delta m \) Eq. \( \mathcal{O} \) can be solved as a function of the energy \( E \) for several values of \( n \). The graphical solution for the first order, \( \Delta m = 1 \), crossings is shown in Fig. 5 where the thick solid horizontal lines correspond to the semiclassical energies of the third and fourth eigenstates of the unperturbed Hamiltonian \( \mathcal{O} \), i.e., of the states which we chose as initial condition in order to obtain the results of Section \( \mathcal{O} \). The dotted and dashed curves represent the quantum renormalized energies of the classical nonlinear resonances of different order \( n \) (the order is indicated by the numbers in the figure). The crossings between the horizontal lines and the decreasing curves thus indicate the solutions of Eq. \( \mathcal{O} \). Their position should also indicate the position of the peaks of the decay rate. This is in fact approximately true, as one can check by going back to Fig. 5, where we indicated the solutions showed in Fig. 5 with the numbered vertical dashed lines.

![Graph showing the solutions of Eq. (15)](image)

**FIG. 6.** The solutions of Eq. (15) represented by the crossings between the solid horizontal lines, indicating the energy of the third and fourth eigenstates, and the energy of various nonlinear resonances indicated by their order \( n \) (dashed and dotted lines). The crossings give the theoretical prediction on the position of the peaks of decay and are reported in Fig. 5 as the vertical dashed lines. The values of the parameter are \( V_0 = 0.048 \), \( \epsilon = 0.005 \), \( \Omega = 2 \), \( \hbar = 0.025 \).

We are now able to justify *a posteriori* the use of the semiclassical theory of this section in the present case. The main difference between the physical system of Ref. \( \mathcal{O} \) and Hamiltonian \( \mathcal{O} \) is the energy levels discreteness. In the present work we cannot speak about level crossings and thus the calculations above could look invalidated. On the other hand we showed that the avoided crossings are nothing more than the *trait d’union* between the nonlinear classical resonances and the peaks of the decay rate: the presence of the nonlinear resonance produces a level crossing which is reflected in a rate irregularity. The same happens for the system of Hamiltonian \( \mathcal{O} \): the classical phase space structures and the decay rate modifications are related even if the intermediate step is less clear due to the continuity of the quantum spectrum. Probably we could find a process similar to the avoided crossings, but we do not need to look for it as we showed that the quantum-classical connection works.

The role of the nonlinear resonances in the perturbation of tunneling seems thus to be established also in a system with a continuous spectrum, even if the prediction looks approximate. The not perfect agreement between theory and numerical calculation can be traced back to two major approximations. The first is the finiteness of \( \hbar \) which makes Eq. (15) slightly inaccurate, while the second and the most important would be the approximation related to consider the unperturbed states in Eq. (1). This last is in fact a double approximation, because it disregards the effect of the perturbation, but this is not so important as shown in Ref. \( \mathcal{O} \), and the effect of the unperturbed decay which actually destroys the discrete levels picture we used. Nonetheless we think that our results are quite clear and to better show the relation between the peaks in the decay and the presence of nonlinear resonances disturbing the initial state we shall now resort to a graphical picture.

**V. A PHASE-SPACE REPRESENTATION: QUANTUM-CLASSICAL COMPARISON**

As we said before, in order to connect the quantum dynamics to the classical phase-space characteristics we must extend the concept of phase space to the quantum case. This can be done by using a phase-space representation of the quantum state and among all the different possibilities we chose to use the Husimi representation, defined as:

\[
\rho(q,p) = \| x \int_{-\infty}^{\infty} dx \alpha_{q,p}(x) \psi(x) \|^2, \tag{17}
\]

where \( \alpha_{q,p} \) is a minimum indetermination state (coherent state) centered in \( (q,p) \). Using Eq. (17) we can obtain a phase representation of a quantum state in terms of the positive definite distribution \( \rho(q,p) \). In particular we can calculate the Husimi distribution of the initial
state which we chose to be an eigenstate of the unperturbed well. This choice will allow us to obtain the correspondence we are looking for in a easy way, in fact in our approximation, namely, for small decay probability, we can safely disregard the dynamics inside the well for the times we explored. This means that the phase-space representation of the quantum state practically does not change during the time interval considered in Fig. 4, and that in the comparison between quantum and classical phase space we can limit ourselves to deal with the initial quantum distribution.

In Fig. 5 we show portraits of the classical phase space for increasing values of $\Omega$. For clarity we draw only the stochastic web and the nonlinear resonances island structures, all the rest of the phase space being filled with regular tori. As explained in Section IV, due to the choice of the particular form of the time dependent perturbation, to the first order in $\epsilon$ we have only resonances of the form $\omega_0(E) = \Omega/n$ where $\omega_0(E)$ is the frequency of the motion inside the unperturbed well. The frequency $\omega_0(E)$ is a decreasing function of the energy $E$ for $0 < E < 2V_0$, being equal to $\sqrt{4V_0}$ for $E = 0$ and vanishing for $E = 2V_0$ which corresponds to the separatrix motion, see Eqs. (5). This means that as we approach the separatrix we find resonances of larger order $n$. We can also easily realize that as the value of $\Omega$ is increased the nonlinear resonances move inside the phase space getting closer to the center of the well, and eventually disappearing when the relation (5) cannot be fulfilled any more. In the meanwhile new resonances appear, moving out from the stochastic layer which is the region of the overlapping of all the infinite resonances of higher order $n$.

In this motion towards the center of the well, the various resonances cross the region of the phase space which is occupied by the Husimi distribution of the initial quantum state (shaded area in Fig. 5b) and, as discussed in Section IV, we expect this to be related to the peaks of Fig. 5. The region of parameters explored in Figs. 5 and 7 is the same and from a first inspection we actually realize that the number of resonances crossing the shaded area corresponds to the number of peaks in the decay rate. This first result is already convincing but we can be more precise by singling out the phase-space portraits which correspond to the decay peaks.

This is done in Fig. 6 where we show the classical phase-space structures and we indicate by the two continuous lines the borders of the Husimi distribution of the initial quantum state.
The expected results is confirmed, the peaks in the quantum decay rate are related to the modification of the classical phase space under the quantum initial distribution by a nonlinear resonance. The perturbation seems to be effective when a nonlinear resonance enter the external border of the initial state distribution and not when the nonlinear resonance passes over the center of the distribution where the quantization torus is located. This is qualitatively in agreement with Eq. (15) which predicts that the perturbation appears when the nonlinear resonance is close to the semiclassical quantization torus, how much close being dictated by the quantum correction to the forcing frequency $\Omega$ is positive (the term $\frac{d\omega(E)}{dE}$ is negative for energy smaller than the barrier heights and thus the renormalized frequency is always larger than $\Omega$) and this means that the decay peaks should appear for values of $\Omega$ smaller than the values for which the nonlinear resonances superpose to the quantization torus. This is precisely what happen: the peaks correspond to the approaching of the nonlinear resonances from the outside.

This result comes along with a simple intuitive explanation: when a nonlinear resonance enters the external border of the initial state part of the initial distribution is moved outward by the islands structure, this produce a higher probability of tunneling across the barrier and thus an increase of the decay. This effect becomes less important once the nonlinear resonance penetrates inside the shaded region; until it disappears when the resonance is completely embedded in the central part of the distribution.

The results of Section IV have been thus confirmed by the comparison of this section, making the picture of the role of the classical nonlinear resonances clearer and clearer.

VI. CHAOS ASSISTED DECAY VERSUS CHAOS ASSISTED TUNNELING

When we reviewed the numerical results we noticed that we found only enhancement of the decay compared to the unperturbed condition. This is not the case in the CAT, where the avoided crossings can produce both enhancement and decrease of tunneling, which can also vanish for particular values of the parameter $\epsilon$. For this reason we wrote in the Introduction that the term assisted used in CAT is not really appropriate, but it seems that it could be better used in the present context. To explain this different behavior we conjecture that this can be seen as a consequence of the continuity of the spectrum in our system. In fact the quenching of the tunneling is produced by the accidental degeneracy of the levels of the tunneling doublet, as seen in Fig. 4. This degeneracy is due to the accidental degeneracy of the levels and thus to a complete quenching of the decay. A similar situation and a graphical representation of this process can be found in a recent paper [24]. On the other hand this continuous spectrum characteristic cannot completely rule out the possibility that chaos could produce also a decrease of the unperturbed decay rate, which we think could be present in some region of the parameters.

VII. CONCLUSIONS

A numerical calculation of the decay probability due to tunneling from a potential well showed that in presence of classical chaos the decay can be strongly enhanced and that this enhancement depends on the system parameters in a resonant-like way. A qualitative inspection of the classical phase-space structure revealed a connection between the peaks in the decay probability and the presence of classical nonlinear resonances in the region of the phase space occupied by the Husimi distribution of the initial state. This correspondence has been quantitatively explained using a semiclassical result which has been shown to be valid in the case of Chaos Assisted Tunneling. We can thus conclude that the enhancement of decay from a driven well can be explained by means of a perturbation of the region of the phase space occupied...
by the initial state, perturbation produced by the emergence of nonlinear resonances. This is a direct connection between the modification of a purely quantum effect, the tunneling, and the classical phenomenon of destruction of the integrable dynamics which is at the basis of the chaotic behavior.

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