A height estimate for constant mean curvature graphs and uniqueness

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Abstract

In this paper, we give a height estimate for constant mean curvature graphs. Using this result we prove two results of uniqueness for the Dirichlet problem associated to the constant mean curvature equation on unbounded domains.

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Introduction

The surfaces with constant mean curvature are the mathematical modelling of soap films. These surfaces appear as the interfaces in isoperimetric problems. There exist different points of view on constant mean curvature surfaces, one is to consider them as graphs.

Let \( \Omega \) be a domain of \( \mathbb{R}^2 \). The graph of a function \( u \) over \( \Omega \) has constant mean curvature \( H > 0 \) if it satisfies the following partial differential equation:

\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 2H \quad \text{(CMC)}
\]

The graph of such a solution is called a \( H \)-graph and has a upward pointing mean curvature vector.

The Dirichlet problem is a natural question about this point of view. For bounded domains, after the work of J. Serrin \([\text{Sc}]\), J. Spruck has given in \([\text{Sp}]\) a general answer to the existence and uniqueness questions. His results are of Jenkins-Serrin type \([\text{JS}]\) since infinite data are allowed.

On unbounded domains, there is few constructions of solutions. The examples are due to P. Collin \([\text{Co}]\) and R. López \([\text{Lo1}]\) for graphs over a strip and R. López \([\text{Lo1}, \text{Lo2}]\) for graphs with 0 boundary data.
In this paper, we investigate the uniqueness question of the Dirichlet problem. To get uniqueness, we need a control of the solutions of the Dirichlet problem, then we shall be able to bound the distance between two solutions with the same boundary data.

To get this control, we use our main result which is Theorem 2. We call this result a height estimate since it bounds the difference of height between two components of boundary of a $H$-graph. The idea of the proof is that if the difference of height is too big, we can get a sphere of radius $\frac{1}{H}$ through the $H$-graph; this is impossible because of the maximum principle.

With this height estimate, we can bound the difference between two solutions of the same dirichlet problem under certain hypotheses on the boundary value. For example, if we are on a strip, we prove that, if the boundary data is Lipschitz, we have uniqueness of possible solutions.

The first section of the paper is devoted to the statement of the height estimate and its proof.

In Section 2, we give consequences of the height estimate for solutions of the constant mean curvature equation on unbounded domains.

In the last section, we prove two uniqueness results for the Dirichlet problem on unbounded domains.

1 The height estimate

In this first section, we give a height estimate for solutions of the constant mean curvature equation (CMC). This estimate is designed for solutions on unbounded domains.

First, we need some notations and remarks. If $\Omega$ is a domain in $\mathbb{R}^2$ and $u$ is a function which is defined on $\Omega$, we note $F : \Omega \rightarrow \mathbb{R}^2$ the map with:

$$F(x, y) = (x, u(x, y))$$

Let us explain what kind of domain we shall consider in the following. For $a > \frac{1}{H}$ and $b > 0$, we note $R_{a,b} = [-a, a] \times [-b, b]$. Let $\Omega \subset R_{a,b}$ be a domain with piecewise smooth boundary. We suppose that $\Omega$ satisfies the three following hypotheses:

1. $\Omega$ is connected,
2. $\partial \Omega \cap \{-a\} \times [-b, b]$ is non empty and the same holds for $\partial \Omega \cap \{a\} \times [-b, b]$,
3. $\partial \Omega \cap [-a, a] \times \{-b\} = \emptyset$ and $\partial \Omega \cap [-a, a] \times \{-b\} = \emptyset$. 


We note \( \Lambda \) the set of the closures of connected components of \( \partial \Omega \cap R_{a,b} \) where \( R_{a,b} = (-a, a) \times (-b, b) \). Let \( \gamma \in \Lambda \) be one of these boundary components, \( \gamma \) is homeomorphic either to a circle either to \([0, 1]\). If it is homeomorphic to a segment, either it joins \( \{-a\} \times [-b, b] \) to \( \{a\} \times [-b, b] \) (by connectedness, there are exactly two such components) or the two end points are on the same edge of the rectangle \( R_{a,b} \).

We need some more notations and remarks that we shall use in the following proofs. Let \( c : [0, 1] \to R_{a,b} \) be a Lipschitz continuous path with \( c(0) \in [-a, a] \times \{-b\} \), \( c(1) \in [-a, a] \times \{b\} \) and \( c(t) \in R_{a,b} \) for \( 0 < t < 1 \). We note \( J_c \) the set of the connected components of \([0, 1] \cap c^{-1}(\overline{\Omega}) \). Let \( j \in J_c \), there exist \( e_j, o_j \in (0, 1) \) such that \( j = [e_j, o_j] \). There is a total order on \( J_c \). Let \( j, j' \in J_c \), we note \( j < j' \) if \( o_j < e_j' \); the order \( \leq \) is then a total order. We remark that \( J_c \) has a minimum \( j_{\text{min}} \) and a maximum \( j_{\text{max}} \). We then have the following lemma.

**Lemma 1.** Let \( \Omega \) and \( c \) be as above. Let \( j \in J_c \) with \( j \neq j_{\text{min}} \). We consider \( j' < j \) and note \( \gamma \) the element of \( \Lambda \) to which \( c(e_j) \) belongs. Then there exists \( j'' \) with \( j' \leq j'' < j \) such that \( c(o_{j''}) \) belongs to \( \gamma \).

**Proof.** We note \( j_0 = \sup\{i \in J_c \mid i < j\} \). There is two possibilities. First, \( j_0 < j \), in this case \( j_0 \geq j' \) and \( c(o_{j_0}, e_j) \) is a curve outside \( \overline{\Omega} \). Because of the different cases for \( \gamma \), \( c(o_{j_0}) \) is then in \( \gamma \). The second possibility is \( j_0 = j \). This implies that there exists \( i \in J_c \) with \( o_i < e_j \) and \( e_j - o_i \) as small as we want. The point \( c(e_j) \) is at a non zero distance from the complementary of \( \gamma \) in \( \partial \Omega \). Since \( c \) is Lipschitz continuous, there exists \( j' \leq i < j \) with \( c(o_i) \in \gamma \). \( \square \)

If \( c \) is injective, the \( c[e_j, o_j] \) are the connected components of \( c([0, 1]) \cap \overline{\Omega} \).

We note \( \Delta_1 \) the connected component of \( R_{a,b} \setminus \Omega \) that contains \((0, -b)\) and \( \Delta_2 \) the one that contains \((0, b)\). For \( i \in \{1, 2\} \), we note \( \gamma_i \) the element of \( \Lambda \) that is included in the boundary of \( \Delta_i \). \( \gamma_1 \) and \( \gamma_2 \) are the two elements of \( \Lambda \) that are homeomorphic to a segment and join \( \{-a\} \times [-b, b] \) to \( \{a\} \times [-b, b] \).

We are then able to give our height estimate result.

**Theorem 2.** Let \( a > \frac{1}{p} \) and \( b > 0 \) be real numbers. We consider \( \Omega \in \mathbb{R}_{a,b} \) a domain with piecewise smooth boundary that satisfies the above hypotheses 1., 2. and 3., we note \( \Lambda, \gamma_1 \) and \( \gamma_2 \) as above. Let \( \Lambda = \Lambda_1 \cup \Lambda_2 \) be a partition of \( \Lambda \) such that \( \gamma_1 \in \Lambda_1 \) and \( \gamma_2 \in \Lambda_2 \). For \( i \in \{1, 2\} \), we note \( \Gamma_i \) the part of the boundary \( \bigcup_{\gamma \in \Lambda_i} \gamma \). Let \( u \) be a solution of \( \text{CMC} \) on \( \Omega \) which is continuous.
on $\overline{\Omega}$. We then have the following upper bound:

$$d(F(\Gamma_1), F(\Gamma_2)) \leq \frac{2}{H}$$

with $d$ the distance for compact sets of $\mathbb{R}^2$ and $F$ defined as at the beginning of the section.

First we shall prove a weaker version of this result

**Theorem 2’.** Let $a > \frac{1}{H}$ and $b > 0$ be real numbers. We consider $\Omega \subseteq \mathbb{R}_{a,b}$, $\Lambda$ and $\Lambda = \Lambda_1 \cup \Lambda_2$ a partition as in Theorem 2. For $i \in \{1, 2\}$, we note $\Gamma_i = \bigcup_{\gamma \in \Lambda_i} \gamma$. Let $u$ be a solution of [CMC] on $\Omega$ which is continuous on $\overline{\Omega}$. We then have the following upper bound:

$$d(F(\Gamma_1), F(\Gamma_2)) \leq 2a$$

**Proof.** The idea of the proof is that, if the estimate on the distance does not hold, we would be able to get a sphere of radius $\frac{1}{2H}$ through the graph of $u$ and this is impossible by maximum principle. So let us assume that the distance $d(F(\Gamma_1), F(\Gamma_2))$ is greater than $2a$.

The first part of the proof consist in finding the place where the sphere will be located.

Since $\gamma_1$ and $\gamma_2$ join $\{-a\} \times [-b, b]$ to $\{a\} \times [-b, b]$ in $R_{a,b}$, $F(\gamma_1)$ and $F(\gamma_2)$ join $\{-a\} \times \mathbb{R}$ to $\{a\} \times \mathbb{R}$ in $[-a, a] \times \mathbb{R}$. Let $\gamma$ be in $\Lambda_1$ and $(x, z)$ be a point in $F(\gamma)$. Since $d(F(\Gamma_1), F(\Gamma_2)) > 2a$, no point of $F(\Gamma_2)$ has $z$ as second coordinate, then, if $\gamma' \in \Lambda_2$, $F(\gamma')$ is either above $F(\gamma)$ (i.e. $\min_{F(\gamma')} z \geq \max_{F(\gamma)} z$) or below (i.e. $\max_{F(\gamma')} z \leq \min_{F(\gamma)} z$). Then $\gamma$ defines a partition $\Lambda_2 = \Lambda_2^- (\gamma) \cup \Lambda_2^+ (\gamma)$ with $\Lambda_2^- (\gamma)$ (resp. $\Lambda_2^+ (\gamma)$) is the set of $\gamma' \in \Lambda_2$ such that $F(\gamma')$ is below (resp. above) $F(\gamma)$. In the same way, $\gamma \in \Lambda_2$ defines a partition $\Lambda_1 = \Lambda_1^- (\gamma) \cup \Lambda_1^+ (\gamma)$.

In the following, we assume that $\gamma_1 \in \Lambda_1^- (\gamma_2)$ ($F(\gamma_1)$ is below $F(\gamma_2)$). If $\gamma_1 \in \Lambda_1^+ (\gamma_2)$, the proof is the same by exchanging the labels 1 and 2.

We then define:

$$u_1 = \max \left\{ u(x, y) \mid (x, y) \in \bigcup_{\gamma \in \Lambda_1^- (\gamma_2)} \gamma, \ -\frac{1}{H} \leq x \leq \frac{1}{H} \right\}$$

We note $(x_1, y_1) \in \bigcup_{\gamma \in \Lambda_1^- (\gamma_2)} \gamma$ a point such that $u(x_1, y_1) = u_1$ and note $g_1 \in \Lambda_1^- (\gamma_2)$ the boundary component that contains $(x_1, y_1)$. We then note:

$$u_2 = \min \left\{ u(x_1, y) \mid (x_1, y) \in \bigcup_{\gamma \in \Lambda_1^+ (g_1)} \gamma \right\}$$
We remark that $u_2$ is well defined because $\gamma_2 \in \Lambda_2^+(g_1)$ and, since $\gamma_2$ join \{-a\} \times [-b, b] to \{a\} \times [-b, b]$, there is a point in $\gamma_2$ with first coordinate $x_1$. We have $u_2 > u_1$ and:

**Fact 1** For all $z \in (u_1, u_2)$, there exists $y$ such that $(x_1, y) \in \Omega$ and $u(x_1, y) = z$.

Let us prove this fact. We consider $c : [-b, b] \to R_{a,b}$ defined by $c(t) = (x_1, t)$. We consider the set $J_c$ with its order. Let $j_0 \in J_c$ be such that $c(e_{j_0})$ or $c(o_{j_0})$ is $(x_1, y_1) \in g_1$. We then note:

$$j_1 = \min\{j > j_0 \mid u(c(o_j)) \geq u_2\}$$

The segment $j_1$ exists because $u(c(o_{j_{\text{max}}})) \geq u_2$ since $c(o_{j_{\text{max}}}) \in \gamma_2$. Besides $j_0 \not\in j_1$. First let us prove that $u(c(e_{j_1})) \leq u_1$. We note $\gamma$ the element of $\Lambda$ to which $c(e_{j_1})$ belongs. By Lemma 3 there exists $i \in J_c$ with $j_0 \leq i < j_1$ such that $c(o_i) \in \gamma$. We have $u(c(o_i)) < u_2$ by definition of $j_1$. If $\gamma \in \Lambda_1$, $u(c(o_i)) < u_2$ implies that $\gamma \in \Lambda_1^-(\gamma_2)$ and then $u(c(e_{j_1})) \leq u_1$ (definition of $u_1$). If $\gamma \in \Lambda_2$, $u(c(o_i)) < u_2$ implies that $\gamma$ belongs to $\Lambda_2^-(g_1)$ (definition of $u_2$) and $u(c(e_{j_1})) \leq u_1$. Now, since $c([e_{j_1}, o_{j_1}])$ is connected and included in $\Omega$, $(u_1, u_2) \subset u \circ c(e_{j_1}, o_{j_1})$ and this prove Fact 1.

Let $t$ be in $R$ and $D_t$ be the closed disk in $[-a, a] \times R$ with center $(0, u_1+t)$ and radius $\frac{1}{2}$. $D_0$ contains the point $(x_1, u_1)$. The diameter of $D_t$ is $\frac{1}{2}$ which is less than $2a$ then we have:

$$D_t \cap F(\Gamma_1) \neq \emptyset \implies D_t \cap F(\Gamma_2) = \emptyset$$

$$D_t \cap F(\Gamma_2) \neq \emptyset \implies D_t \cap F(\Gamma_1) = \emptyset$$

We define:

$$t_0 = \inf \{t > 0 \mid D_t \cap F(\Gamma_1) = \emptyset\}$$

By compactness, $D_{t_0} \cap F(\Gamma_1) \neq \emptyset$ and then $D_t \cap F(\Gamma_2) = \emptyset$ for $0 \leq t \leq t_0$.

**Fact 2** We have $u_1 + t_0 < u_2$.

Actually, if $u_1 + t_0 \geq u_2$, $t' = u_2 - u_1$ is less than $t_0$ and $D_{t'}$ contains the point of $F(\Gamma_2)$ that realizes $u_2$. This implies that $D_{t'} \cap F(\Gamma_1) = \emptyset$ and contradicts the definition of $t_0$.

By compactness, there exists then $t_1 > t_0$ such that $u_1 + t_1 < u_2$, $D_{t_1} \cap F(\Gamma_1) = \emptyset$ and $D_t \cap F(\Gamma_2) = \emptyset$ for all $0 \leq t \leq t_1$.

**Fact 3** Let $\gamma \in \Lambda$ be a boundary component, then there are no $Z_1, Z_2 \in R$ such that there exist $X_1, X_2 \in [-\frac{1}{2}, \frac{1}{2}]$ with $(X_i, Z_i) \in F(\gamma)$ and:

$$Z_1 < u_1 + t_1 - \sqrt{\frac{1}{H^2} - X_1^2} \leq u_1 + t_1 + \sqrt{\frac{1}{H^2} - X_2^2} < Z_2$$
(\(F(\gamma)\) can not have points above and below the disk \(D_{t_1}\))

First we suppose that \(\gamma \in \Lambda_1\). Since \(Z_2 > u_1\), the definition of \(u_1\) implies that \(\gamma\) belongs to \(\Lambda_1^+ (\gamma_2)\). Since \(\gamma_2\) joins \(\{-a\} \times [-b, b]\) to \(\{a\} \times [-b, b]\), \(F(\gamma_2)\) has a point of coordinates \((x_1, z)\); by definition of \(u_2, z \geq u_2\). Then the second coordinate of every point of \(F(\gamma)\) needs to be more than \(u_2\) this contradicts \(Z_1 < u_1 + t_1\). Now if \(\gamma \in \Lambda_2, D_t\) does not intersect \(F(\Gamma_2)\) for \(0 \leq t \leq t_1\), so letting go down the disk \(D_t\) from \(t_1\) to 0, we get \(Z_1 \leq u_1\) and then \(\gamma\) belongs to \(\Lambda_2^-(g_1)\). This implies that the second coordinate of every point of \(F(\gamma)\) needs to be less than \(u_1\) this contradicts \(Z_2 > u_1 + t_1\) and proofs Fact 3.

The idea is now to get a sphere of radius \(\frac{1}{R}\) through the disk \(D_{t_1}\). We note \(S_v\) the sphere of radius \(\frac{1}{R}\) and center \((0, v, u_1 + t_1)\). When \(v\) changes, \(S_v\) moves in an horizontal cylinder with vertical section \(D_{t_1}\). For far from zero negative \(v\), \(S_v\) is out \(\Omega \times \mathbb{R}\). Since \(D_{t_1} \cap F(\Gamma_1) = \emptyset\) and \(D_{t_1} \cap F(\Gamma_2) = \emptyset\), \(S_v\) does not intersect the boundary of the graph of \(u\) for any \(v\). The graph of \(u\) splits \(\Omega \times \mathbb{R}\) into two connected components: \(G^+\), above the graph, and \(G^-\) which is below. Since \(u_1 \leq u_1 + t_1 \leq u_2\), there exists \(v\) such that \(S_v\) intersects the graph of \(u\) from Fact 1.

We start with far from zero negative \(v\) and let \(v\) increase until \(v_0\) which is the first contact between the graph and the sphere. This first contact does not occur in the boundary of the graph since the sphere never intersects it. Then, since the graph is not a piece of a sphere because of the size of \(\Omega\), the maximum principle implies that, in the neighborhood of the contact point, the sphere \(S_{v_0}\) is in \(G^-\) (we recall that the mean curvature vector of the graph points in \(G^+\) because of the equation (CMC)). But, in fact, we have: Fact 4. In the neighborhood of the contact point, the sphere \(S_{v_0}\) is in \(G^+\).

This fact is not clear since, because of the shape of the domain \(\Omega\), the sphere do not stop to get in and out \(\Omega \times \mathbb{R}\). We note \(p = (x, y, z)\) the first contact point; we know that \((x, z) \in D_{t_1}\). We define \(c : s \mapsto (x, s) \in \mathbb{R}_{a,b}\) and consider \(J_c\). We have \((x, s, z) \in G^-\) for \(s < y\) near \(y\) since the sphere \(S_{v_0}\) is in \(G^-\) in a neighborhood of \(p\). Then there exists:

\[
s_0 = \min \{s \leq -b \mid (x, s, z) \in G^-\}
\]

Since \(p\) is the first contact point, there is \(j \in J_c\) such that \(s_0 = e_j\). We know that \(D_{t_1}\) is above \(F(\gamma_1)\) then \((x, e_{j_{\text{min}}, z}) \in G^+, so j > j_{\text{min}}\). We know then that there exists \(j' < j\) such that \(c(e_j)\) and \(c(o_{j'})\) belong to the same element of \(\Lambda\); then, if \((x, e_j, z) \in G^-, (x, o_{j'}, z) \in G^-, this is due to Fact 3\). This implies that \((x, s, z) \in G^-\) for \(s < o_{j'}\) near \(o_{j'}\) and then \(s_0 < e_j\); we have our contradiction.
This end the proof of $d(F(\Gamma_1), F(\Gamma_2)) \leq 2a$. \hfill \square

We remark that Theorem 2' will be sufficient for most of the applications and Theorem 2 is just an improvement. So let us replace $2a$ by $\frac{2}{\beta}$ to get Theorem 2.

**Proof of Theorem 2.** Let us consider $a' > 0$ with $\frac{1}{\beta} < a' < a$. The idea of the proof is to apply Theorem 2' to a well chosen connected component of $\Omega \cap R_{a',b}$. We note $D^i$ the connected components of $\Omega \cap R_{a',b}$. First we remark that, among these components, there are ones that satisfy the hypotheses 1., 2. and 3.; for example, since $\gamma_1$ joins $\{-a\} \times [-b, b]$ to $\{a\} \times [-b, b]$, one connected component of $\gamma_1 \cap R_{a',b}$ joins $\{-a'\} \times [-b, b]$ to $\{a'\} \times [-b, b]$ then a $D^i$ that has this component in its boundary satisfies the three hypotheses. A component of $\Omega \cap R_{a',b}$ that satisfies the hypotheses is called a good component and the other ones are the bad ones; we rename these good components $D^1, \ldots, D^k$. There is only a finite number of such components since the length of the part of $\partial \Omega$ in $R_{a',b}$ is finite.

Let us consider a good component $D^i$. As defined at the beginning of the section, a set of boundary component $\Lambda^i$ is associated to $D^i$. In $\Lambda^i$ there is two particular elements, these are the two boundary components which are homeomorphic to a segment and joins $\{-a'\} \times [-b, b]$ to $\{a'\} \times [-b, b]$. To avoid any confusion, we note these components $\gamma^i_\alpha$ and $\gamma^i_\beta$ ($\gamma^i_\alpha$ is a part of the boundary of the connected component of $R_{a',b} \setminus D^i$ that contains $(0, -b)$ and $\gamma^i_\beta$ is the other one). Each element of $\Lambda^i$ is a part of an element of $\Lambda$, then we get a partition $\Lambda^i = \Lambda^i_1 \cup \Lambda^i_2$: an element of $\Lambda^i$ is in $\Lambda^i_1$ (resp. $\Lambda^i_2$) if it is a part of a element of $\Lambda_1$ (resp. $\Lambda_2$). Now the proof consists in applying Theorem 2 to a component $D^i$ such that $\gamma^i_\alpha \in \Lambda^i_1$ and $\gamma^i_\beta \in \Lambda^i_2$.

To each good component $D^j$, we can associate a real number which is the second coordinate of the end point of $\gamma^j_\alpha$ in $\{-a\} \times [-b, b]$. In the following, we order the good components with respect to this real number and rename the good components $D^1, \ldots, D^k$ with respect to this order. The order is the same if we consider the second coordinate of $\gamma^i_\alpha \cup \{a\} \times [-b, b]$, $\gamma^j_\beta \cup \{a\} \times [-b, b]$, $\gamma^j_\beta \cup \{-a\} \times [-b, b]$ or $\gamma^i_\beta \cup \{a\} \times [-b, b]$.

Let $D$ be a bad component of $\Omega \cap R_{a',b}$. In fact, it is a bad component only because of the hypothesis 2., then, in $D$, there is no path from $\{-a'\} \times [-b, b]$ to $\{a'\} \times [-b, b]$. This implies that, as in Figure 1, there exists a path $c: [0, 1] \rightarrow R_{a',b}$ that joins $(0, -b)$ to $(0, b)$, is outside all the bad components and such that there exist $0 < e_1 < o_1 < e_2 < \cdots < e_k < o_k < 1$ with:

$$c([0, 1]) \cap \Omega = \bigcup_i c(e_i, o_i)$$
and \( c(e_i, o_i) \subset D^i \). First we remark that \( c(e_1) \) is in \( \gamma_1 \) so it is in \( \gamma_1^1 \in \Lambda_1^1 \) and \( c(o_k) \) is in \( \gamma_2 \) so it is in \( \gamma_2^k \in \Lambda_2^k \). Then there exists:

\[
i_0 = \min \{ i \mid \gamma_\beta^i \in \Lambda_i^2 \}\]

Let us prove that the good component \( D^{i_0} \) is such that \( \gamma_{i_0}^{i_0} \in \Lambda_{i_0}^1 \). First we assume that it is not the case, \( i.e. \) \( \gamma_{i_0}^{i_0} \in \Lambda_{i_0}^2 \). Since \( \gamma_1^{i_0} \in \Lambda_1^1 \), \( i_0 > 0 \). \( \gamma_{i_0}^{i_0} \) is part of an element \( \gamma \) of \( \Lambda_2 \) then \( c(e_{i_0}) \in \gamma_{i_0}^{i_0} \subset \gamma \). Besides we know, by Lemma 1 that \( c(o_{i_0-1}) \) belongs to the same component as \( c(e_{i_0}) \) then \( c(o_{i_0-1}) \in \gamma \) and \( \gamma_{i_0-1}^{i_0-1} \subset \gamma \); this implies that \( \gamma_{i_0-1}^{i_0-1} \in \Lambda_{i_0-1}^2 \). This is a contradiction with the definition of \( i_0 \).

Now, we apply Theorem 2' to \( D^{i_0} \). We note, for \( j \in \{1, 2\} \), \( \Gamma'_j = \bigcup_{\gamma \in \Lambda_j^{i_0}} \gamma \). Then we have \( \gamma_{i_0}^{i_0} \in \Gamma'_1 \) and \( \gamma_{i_0}^{i_0} \in \Gamma'_2 \); so we can apply the theorem
and we get:
\[d(F(\Gamma_1'), F(\Gamma_2')) \leq 2a'\]

Besides, we have \(\Gamma_1' \subset \Gamma_1\) and \(\Gamma_2' \subset \Gamma_2\) then:
\[d(F(\Gamma_1), F(\Gamma_2)) \leq d(F(\Gamma_1'), F(\Gamma_2')) \leq 2a'\]

This inequality is true for every \(a' > \frac{1}{H}\), so:
\[d(F(\Gamma_1), F(\Gamma_2)) \leq \frac{2}{H}\]

\[\Box\]

### 2 Some consequences of Theorem 2

The aim of this section is to give some consequences of Theorem 2 for solutions of the Dirichlet problem associated to the constant mean curvature equation (CMC) on unbounded domains.

First we explain what kind of domains we shall consider. Let \(b_- : \mathbb{R}_+ \to \mathbb{R}\) be two continuous functions such that, for every \(x \geq 0\), \(b_-(x) < b_+(x)\). We are interested in domains of the type \(\Omega = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R} | b_-(x) < y < b_+(x)\}\). When a solution \(u\) is solution of (CMC) on \(\Omega\) and continuous on \(\Omega\), we define two continuous functions \(f_-\) and \(f_+\) by \(f_\pm(x) = u(x, b_\pm(x))\). \(f_-\) and \(f_+\) are the boundary values of \(u\).

Let us fix a last definition, if \(x \in \mathbb{R}_+\), we note \(I_x = \{x\} \times [b_-(x), b_+(x)]\).

We then have the following height estimate:

**Proposition 3.** Let \(b_-\) and \(b_+\) be continuous functions on \(\mathbb{R}_+\) with \(b_-(x) < b_+(x)\). We note \(\Omega = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R} | b_-(x) < y < b_+(x)\}\). We consider \(u\) a solution of (CMC) on \(\Omega\) continuous on \(\Omega\). We consider \(x_0 > \frac{2}{H}\) and \(M\) such that:

\[
\min_{[x_0 - \frac{2}{H}, x_0 + \frac{2}{H}]} (f_-, f_+) \geq M
\]

Then:
\[
\min_{I_{x_0}} u \geq M - \frac{3}{H}
\]

**Proof.** Let \(\varepsilon\) be a positive number. Since \(f_-\) and \(f_+\) are continuous, there exist \(\eta > 0\) such that:
\[
\min_{[x_0 - \frac{2}{H} - \eta, x_0 + \frac{2}{H} + \eta]} (f_-, f_+) \geq M - \varepsilon
\]
Let us now suppose that there is a piecewise smooth injective path \( c : [0, 1] \to \Omega \cap [x_0 - \frac{2}{H} - \eta, x_0] \times \mathbb{R} \) that joins \( I_{x_0 - \frac{2}{H} - \eta} \) to \( I_{x_0} \) such that:

\[
u \circ c(t) < M - \varepsilon - \frac{2}{H} \tag{2}\]

We consider the domain \( D \) bounded by \( c \), the curve \( y = b_-(x) \) for \( x \in [x_0 - \frac{2}{H} - \eta, x_0] \), a segment included in \( I_{x_0 - \frac{2}{H} - \eta} \) and one included in \( I_{x_0} \); \( D \) statifies the three hypotheses of Section 1. In fact, since the function \( b_- \) is only continuous the boundary of \( D \) is not piecewise smooth, but Theorem 2 can be applied because of the shape of \( D \). Then because of (1) and (2), Theorem 2 is not satisfied. Finally, this implies that there exists a path \( c_1 : [0, 1] \to \Omega \cap [x_0 - \frac{2}{H} - \eta, x_0] \times \mathbb{R} \) that joins the curve \( y = b_-(x) \) to the curve \( y = b_+(x) \) such that \( u \circ c_1(t) \geq M - \varepsilon - \frac{2}{H} \).

By the same arguments, there exists a path \( c_2 : [0, 1] \to \Omega \cap [x_0, x_0 + \frac{2}{H} + \eta] \times \mathbb{R} \) that joins the curve \( y = b_-(x) \) to the curve \( y = b_+(x) \) such that \( u \circ c_2(t) \geq M - \varepsilon - \frac{2}{H} \).

Now the domain \( D \) bounded by \( c_1, c_2 \), a piece of \( y = b_-(x) \) and a piece of \( y = b_+(x) \) contains \( I_{x_0} \) (see Figure 2). Besides on the boundary of \( D \), \( u \) is everywhere greater than \( M - \varepsilon - \frac{2}{H} \) by (1) and above. Then by a classical height estimate \( \text{Se2} \), \( u \) is greater than \( M - \varepsilon - \frac{3}{2H} \) in \( D \). This gives us:

\[
\min_{I_{x_0}} u \geq M - \varepsilon - \frac{3}{2H} \tag{3}
\]
Letting $\varepsilon$ goes to zero, we get the expected result.

We have also a simple upper-bound in this case.

**Proposition 4.** Let $b_-$ and $b_+$ be continuous functions on $\mathbb{R}^+$ with $b_-(x) < b_+(x)$. We note $\Omega = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R} | b_-(x) < y < b_+(x)\}$. We consider $u$ a solution of [CMC] on $\Omega$ continuous on $\Omega$. We consider $x_0 > \frac{1}{2H}$ and $M$ such that:

$$\max_{[x_0 - \frac{H}{2}, x_0 + \frac{H}{2}]} (f_-, f_+) \leq M$$

Then:

$$\max_{I_{x_0}} u \leq M$$

**Proof.** Let $\varepsilon$ be a positive number. Since $f_-$ and $f_+$ are continuous, there exist $\eta > 0$ such that:

$$\max_{[x_0 - \frac{H}{2} - \eta, x_0 + \frac{H}{2} + \eta]} (f_-, f_+) \leq M + \varepsilon$$

Let us consider the cylinder of radius $\frac{H}{2}$ which is centered on the axis $\{x = x_0\} \cap \{z = t\}$. For big $t$, the cylinder is above the graph and we can let $t$ decrease. Until $t = M + \frac{1}{2H}$, the cylinders can not meet the boundary of the graph of $u$. The maximum principle then says us that, for $t = M + \frac{1}{2H}$, the cylinder is still above the graph; so we get:

$$\max_{I_{x_0}} u \leq M + \varepsilon$$

Letting $\varepsilon$ goes to zero, we get the expected result.

Let $f : I \to \mathbb{R}$ be a function, we define the variation of $f$ around the point $x_0$ by:

$$V_t(x_0, f) = \sup_{[x_0 - t, x_0 + t]} f - \inf_{[x_0 - t, x_0 + t]} f$$

Let $f$ and $g$ be two continuous functions $I \to \mathbb{R}$; we define the variation of the pair $(f, g)$ around $x_0$ by:

$$V_t(x_0, f, g) = \max(V_t(x_0, f), V_t(x_0, g))$$

The two preceding propositions give us the following result:
Theorem 5. Let \( b_\pm \) be continuous functions on \( \mathbb{R}^+ \) with \( b_-(x) < b_+(x) \). We note \( \Omega = \{ (x, y) \in \mathbb{R}^+ \times \mathbb{R} | b_-(x) < y < b_+(x) \} \). We consider \( u \) a solution of CMC on \( \Omega \) continuous on \( \Omega \). We consider \( x_0 > \frac{2}{\Pi} \) and \( M \) such that:

\[
V_\frac{1}{\Pi}(x_0, f_-, f_+) \leq M
\]

Then there exists \( M' \) which depends only on \( M \) and \( H \) such that:

\[
\max_{I_{x_0}} u - \min_{I_{x_0}} u \leq M'
\]

For example, \( M' = 4M + \frac{5}{\Pi} \) works.

Proof. We have for \( x \in [x_0 - \frac{2}{\Pi}, x_0 + \frac{2}{\Pi}] \), \( |f_-(x) - f_-(x_0)| \leq M \) and \( |f_+(x) - f_+(x_0)| \leq M \). Then if we apply Theorem 2 to \( \Omega \cap [x_0 - \frac{2}{\Pi}, x_0 + \frac{2}{\Pi}] \times \mathbb{R} \), we get that the graph of \( f_- \) over this segment is at a distance less than \( \frac{2}{\Pi} \) from the one of \( f_+ \). Since, for \( \alpha \in \{ -, + \} \), the graph of \( f_\alpha \) is in the horizontal strip \( f_\alpha(x_0) - M \leq z \leq f_\alpha(x_0) + M \), we have |\( f_-(x_0) - f_+(x_0) \| \leq 2(M + \frac{1}{\Pi}) \).

This implies that, for every \( x, x' \in [x_0 - \frac{2}{\Pi}, x_0 + \frac{2}{\Pi}] \), \( |f_\alpha(x) - f_\beta(x')| \leq 4M + \frac{2}{\Pi} \) with \( \alpha, \beta \in \{ -, + \} \). Then there exist \( A \in \mathbb{R} \) such that, for every \( x \in [x_0 - \frac{2}{\Pi}, x_0 + \frac{2}{\Pi}] \), we have:

\[
|f_-(x) - A| \leq 2M + \frac{1}{\Pi}
\]

(4)

\[
|f_+(x) - A| \leq 2M + \frac{1}{\Pi}
\]

(5)

These two equations with Proposition 3 implies that:

\[
\min_{I_{x_0}} u \geq A - 2M - \frac{4}{\Pi}
\]

(6)

With Proposition 4 we get:

\[
\max_{I_{x_0}} u \leq A + 2M + \frac{1}{\Pi}
\]

(7)

Then, in bringing together (6) and (7), we obtain:

\[
\max_{I_{x_0}} u - \min_{I_{x_0}} u \leq 4M + \frac{5}{\Pi}
\]

\[\square\]

Theorem 5 has an easy corollary.
Corollary 6. Let $b_-$ and $b_+$ be continuous functions on $\mathbb{R}^+$ with $b_-(x) < b_+(x)$. We note $\Omega = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R} | b_-(x) < y < b_+(x)\}$. We consider $u$ a solution of \textbf{(CMC)} on $\Omega$ continuous on $\overline{\Omega}$. We consider $x_0 > \frac{2}{H}$ and $M$ such that:

$$V_{\frac{2}{H}}(x_0, f_-, f_+) \leq M$$

Then there exists $M'$ which depends only on $M$ and $H$ such that, for every $p \in I_{x_0}$ and $\alpha \in \{-, +\}$, we have:

$$f_\alpha(x_0) - M' \leq u(p) \leq f_\alpha(x_0) + M'$$

For example, $M' = 4M + \frac{5}{H}$ works.

Proof. It is just the fact that $(x_0, b_-(x_0))$ and $(x_0, b_+(x_0))$ are in $I_{x_0}$. $\square$

3 Two uniqueness results

In this section, we use Corollary 6 to prove uniqueness theorems for the Dirichlet problem associated to \textbf{(CMC)}.

Theorem 7. Let $b_-, b_+$ be two continuous functions on $\mathbb{R}^+$ such that $b_-(0) = b_+(0)$ and $b_-(x) < b_+(x)$ for every $x > 0$. We note $\Omega = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R} | b_-(x) < y < b_+(x)\}$. Let $f_-, f_+$ be two continuous functions on $\mathbb{R}^+$ such that $f_-(0) = f_+(0)$. We suppose that there exist an increasing positive sequence $(x_n)_{n \in \mathbb{N}}$ with $\lim x_n = +\infty$ and a sequence $(M_n)_{n \in \mathbb{N}}$ with $M_n = o(\ln x_n)$ such that, for every $n \in \mathbb{N}$, we have:

$$V_{\frac{2}{H}}(x_n, f_-, f_+) \leq M_n$$

Then, if there exists a solution $u$ of \textbf{(CMC)} on $\Omega$ with value $f_-$ and $f_+$ on the boundary, this solution is unique.

Proof. Let $u_1$ and $u_2$ be two different solutions of the Dirichlet problem with $f_-$ and $f_+$ as boundary data. We know by a result of P. Collin and R. Krust \textbf{[CK]} that:

$$\liminf_{x \to +\infty} \frac{\max_{I_{x_n}} |u_1 - u_2|}{\ln x} > 0$$

But by corollary 6 we know that:

$$\max_{I_{x_n}} |u_1 - f_-(x_n)| \leq 4M_n + \frac{5}{H}$$

$$\max_{I_{x_n}} |u_2 - f_-(x_n)| \leq 4M_n + \frac{5}{H}$$
So, we get:
\[
\max_{I_n} |u_1 - u_2| \leq 8M_n + \frac{10}{H}
\]

By the hypothesis on \(M_n\), we have:
\[
\lim_{n \to \infty} \max_{I_n} |u_1 - u_2| = 0 \quad \ln x_n = 0
\]

This gives us a contradiction since \(x_n \to +\infty\). 

We have also a second theorem.

**Theorem 8.** Let \(b_-\) be two continuous functions on \(\mathbb{R}\) such that \(b_-(x) < b_+(x)\) for every \(x \in \mathbb{R}\). We note \(\Omega = \{(x, y) \in \mathbb{R}^2 \mid b_-(x) < y < b_+(x)\}\). Let \(f_-\), \(f_+\) be two continuous functions on \(\mathbb{R}\). We suppose that there exist one increasing sequence \((x_n)_{n \in \mathbb{N}}\) with \(\lim x_n = +\infty\) and one decreasing sequence \((x'_n)_{n \in \mathbb{N}}\) with \(\lim x'_n = -\infty\) and two sequences \((M_n)_{n \in \mathbb{N}}\) and \((M'_n)_{n \in \mathbb{N}}\) such that \(M_n = o(\ln |x_n|)\), \(M'_n = o(\ln |x'_n|)\) and, for every \(n \in \mathbb{N}\), we have:
\[
V_2 H(x_n, f_-, f_+) \leq M_n \\
V_2 H(x'_n, f_-, f_+) \leq M'_n
\]
Then, if there exists a solution \(u\) of \((\text{CMC})\) on \(\Omega\) with value \(f_-\) and \(f_+\) on the boundary, this solution is unique.

**Proof.** Let \(u_1\) and \(u_2\) be two different solutions of the Dirichlet problem with \(f_-\) and \(f_+\) as boundary data. We know (\text{CK}) that:
\[
\max_{I_x \cup I_{\infty}} |u_1 - u_2| \xrightarrow{x \to +\infty} +\infty
\]

We note \(C = \max_{I_0} |u_1 - u_2|\). We then have \(u_2 - C - 1 < u_1 < u_2 + C + 1\) on \(I_0\) and because of (\text{CK}) the set \(\{|u_1 - u_2| > M + 1\}\) is non-empty. Then we can assume that there exists a subdomain \(\Omega^* \subset \Omega \cap \mathbb{R}^+ \times \mathbb{R}\) which is a connected component of \(\{u_1 > u_2 + M + 1\}\). Then we have:
\[
\liminf_{x \to -\infty} \frac{\max_{I_n \cap \Omega^*} |u_1 - u_2 - C - 1|}{\ln |x|} > 0
\]
As in the preceding proof, Corollary (\text{CK}) says us that, for every \(n\), we have:
\[
\max_{I_n} |u_1 - u_2 - C - 1| \leq 8M'_n + \frac{10}{H} + C + 1
\]
By the hypothesis on $M'_n$, we have:

$$\lim_{n \to \infty} \frac{\max_{I'_{x_n}} |u_1 - u_2 - C - 1|}{\ln |x'_n|} = 0$$

This gives us a contradiction since $x'_n \to -\infty$ and ends the proof.

This theorem can be used to study the uniqueness of the solutions which were built by P. Collin in [Co] and by R. Lopéz in [Lo1].

There are others results of uniqueness we can prove with the same arguments. For example, if we suppose that $b_+ - b_-$ is bounded in Theorems 7 and 8, we need only to assume that $M_n = o(x_n)$ and $M'_n = o(|x'_n|)$ to have the uniqueness.

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