ON THE SYZYGIES AND HODGE THEORY OF NODAL HYPERSURFACES

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Abstract. We give sharp lower bounds for the degree of the syzygies involving the partial derivatives of a homogeneous polynomial defining an even dimensional nodal hypersurface. This implies the validity of formulas due to M. Saito, L. Wotzlaw and the author for the graded pieces with respect to the Hodge filtration of the top cohomology of the hypersurface complement in many new cases.

1. Introduction

Let $S = \mathbb{C}[x_0, ..., x_n]$ be the graded ring of polynomials in $x_0, ..., x_n$ with complex coefficients and denote by $S_r$ the vector space of homogeneous polynomials in $S$ of degree $r$. For any polynomial $f \in S_r$, we define the Jacobian ideal $J_f \subset S$ as the ideal spanned by the partial derivatives $f_0, ..., f_n$ of $f$ with respect to $x_0, ..., x_n$ and the corresponding graded Milnor (or Jacobian) algebra by

(1.1) $M(f) = S/J_f$.

The Milnor algebra $M(f)$ can be seen (up to a twist in grading) as the top cohomology $H^{n+1}(K^*(f))$, where $K^*(f)$ is the Koszul complex of $f_0, ..., f_n$ with the natural grading $|x_j| = |dx_j| = 1$ defined by

(1.2) $K^*(f) : 0 \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow ... \rightarrow \Omega^{n+1} \rightarrow 0$

with all the arrows given by the wedge product by $df = f_0 dx_0 + f_1 dx_1 + ... + f_n dx_n$.

The homogeneous components of the next cohomology group, say $H^n(K^*(f))_{n+r}$, describe the syzygies

$$\sum_{j=0,n} a_j f_j = 0$$

where $a_j \in S_r$, modulo the trivial syzygies generated by

$$(f_j)f_i + (-f_i)f_j = 0.$$
Examples involving Chebyshev hypersurfaces shows that the bounds are best possible for \( n \) even, but not for \( n \) odd, see \cite{12}. The first purpose of this paper is to establish the following new bound in the case \( n \) odd, which is optimal, see Example 4.4.

**Theorem 1.2.** Let \( D : f = 0 \) be a nodal hypersurface of degree \( d > 2 \) in \( \mathbb{P}^n \).

If \( n = 2n_1 + 1 \) is odd, then \( H^m(K^*(f)) = 0 \) for any \( m \leq (n_1 + 1)d - \left\lfloor \frac{d}{2} \right\rfloor - 1 \).

The idea of proof of Theorem 1.2 is similar to that used in the proof of Theorem 1.1, namely the interplay between Hodge filtrations, pole order filtrations and some spectral sequences. In the case of Theorem 1.1 it was enough to look at the cohomology of the complement \( U = \mathbb{P}^n \setminus D \), while in the case at hand we have to consider the eigenspaces \( H^*(F)_\lambda \) of the monodromy action on the corresponding Milnor fiber \( F : f(x) - 1 = 0 \), which is technically more complicated.

The second aim of this paper is to show that the information we have obtained on the syzygies allows us to show that an algebraic description of some of the graded pieces \( Gr^p_F H^*(U, \mathbb{C}) \) given in \cite{9} holds in many cases, see Theorem 5.1. For basic facts on mixed Hodge structures we refer \cite{15}.

### 2. Spectral sequences for Milnor fibers of homogeneous polynomials

For \( k = 1, ..., d \), set \( \lambda = \exp(-2\pi ik/d) \) and let \( L_k \) be the rank one local system on \( U \) such that

\[
H^*(U, L_k) = H^*(F)_\lambda,
\]

see for details \cite{4}, p. 211 and note that here the eigenspaces are with respect to the local system monodromy \( T \) acting on \( H^*(F) \), \cite{7}. Let \( j : U \to X = \mathbb{P}^n \) be the inclusion, and let \( L_k \) be the meromorphic extension of \( L_k \otimes \mathcal{O}_U \). Then \( L_k \) is a regular holonomic \( \mathcal{D}_Y \)-module, see \cite{17}, section (4.8), and one clearly has

\[
H^*(X, DR(L_k)) = H^*(X, Rj_* L_k) = H^*(F)_\lambda.
\]

The \( \mathcal{D}_Y \)-module \( L_k \) has a natural (increasing) pole order filtration \( P_* \) such that \( P_j L_k = 0 \) for \( j < 0 \), see (3.1.3) in \cite{8}. Then the de Rham complex \( DR(L_k) \) has an induced decreasing filtration \( P^* \) and this induces the pole order filtration \( P^* \) on the eigenspaces \( H^*(F)_\lambda \) of the Milnor fiber cohomology. One has the following fundamental inclusion

\[
F^s H^j(F)_\lambda \subset P^s H^j(F)_\lambda,
\]

for any \( s \) and any \( j \), where \( F^s \) denotes the canonical Hodge filtration on the cohomology of the smooth quasi-projective variety \( F \), see (3.1.3), (4.4.7) and (4.4.8) in \cite{8}. This extends the similar result for \( H^*(U) \), see \cite{2} and \cite{16}.

On the other hand, for each \( k = 1, ..., d \) there is a spectral sequence

\[
E_1^{p,q}(f, k) = H^{p+q}(K^*(f))_{q+k} \Rightarrow H^{p+q-1}(F)_\lambda
\]

coming from the graded Gauss-Manin complex \( C^*_f \) associated with \( f \), see (4.4.4) and (4.5.3) in \cite{8}. A similar spectral sequence is obtained by using the algebraic microlocal...
Gauss-Manin complex $\tilde{C}_f^*$ and the corresponding limit is the reduced cohomology of the Milnor fiber, see the (4.5) in [3]. In fact, one has

$$H^{j+1}(C_f^*)_k = H^{j+1}(\tilde{C}_f^*)_k = H^j(F)_\lambda$$

for any $k = 1, \ldots, d$ and $j > 0$, see (4.2.3) in [3]. These spectral sequences induce a filtration $P^s$ on $H^*(F)_\lambda$ and one has $P^s = P^{s+1}$, see (4.4.7) in [3].

**Remark 2.1.** Remark 4.11 in [17] gives a very explicit description of the pole order filtration on the top cohomology group $H^n(F)$. Assume that we have a (finite) family of monomials $(g_j(x))_{j \in J}$ in $S$ such that the cohomology classes $[\omega_j]$ of the differential forms $\omega_j = g_j(x)\Delta(dx_0 \wedge \ldots \wedge dx_n)$ for $j \in J$ yield a basis of the $\mathbb{C}$-vector space $H^n(F)$. Then $Gr^{n-q}_p H^n(F)_\lambda$ is spanned by the classes $[\omega_j]$ with $\deg \omega_j = qd + k$. Note that the eigenspace $Gr^{n-q}_p H^n(F)_1$ is spanned exactly by the classes $[\omega_j]$ in $Gr^{n-q}_p H^n(F)$ having a maximal degree $\deg \omega_j = (q + 1)d$. In particular, the pole order filtrations on $H^n(F)_1$ and on $H^n(U)$, the latter constructed using a generalization of Griffiths approach in [3], Chapter 6 or in [11] are the same, as it should be since we have a natural isomorphism $H^n(F)_1 = H^n(U)$. Note that the corresponding basis for $Gr^{n-q}_p H^n(U)$ is usually written as

$$\sigma_j = \omega_j/f^{q+1}.$$ 

Similarly, the limit term $E^{n-q+1}_m(f, k)$, which is isomorphic to $Gr^{n-q}_p H^n(F)_\lambda$, has a basis given with the above notation by

$$\eta_j = g_j(x) \cdot dx_0 \wedge \ldots \wedge dx_n.$$

### 3. Hodge filtration versus pole order filtration on Milnor fibers

If $D$ is a nodal hypersurface in $\mathbb{P}^n$, then one has an equality

$$F^s H^n(U) = P^s H^n(U),$$

for any $s \geq n - m + 1$, with $m = \alpha_D = \frac{n}{2}$, see Corollary (0.12) in M. Saito [16], or the formula (1.1.3) in [9]. The purpose of this section is to prove the following similar result for the associated Milnor fibers.

**Proposition 3.1.** Let $D : f = 0$ be a nodal hypersurface of degree $d > 2$ in $\mathbb{P}^n$ with $n$ odd. Then

$$F^s H^n(F)_\lambda = P^s H^n(F)_\lambda,$$

for any $s \geq n - m + 1$ and any $\lambda \in \mu_d$, $\lambda \neq 1$, with $m = \frac{n}{2}$.

**Proof.** Consider the hypersurface $\tilde{D}$ defined in $\mathbb{P}^{n+1}$ by the equation $\tilde{f}(x, t) = 0$, with $\tilde{f}(x, t) = f(x) - t^d$. Let $\tilde{U} = \mathbb{P}^{n+1} \setminus \tilde{D}$ and let $H : t = 0$ be the hyperplane at infinity in $\mathbb{P}^{n+1}$ such that $\tilde{U} \cap H = U$. Consider the Gysin long exact sequence

$$\cdots \rightarrow H^{n-1}(U) \rightarrow H^{n+1}(\tilde{U}) \rightarrow H^{n+1}(\tilde{U} \setminus U) \rightarrow H^n(U) \rightarrow \cdots$$

The group $\mu_d$ of $d$-roots of unity acts on $\mathbb{P}^{n+1}$ via

$$\beta \cdot (x_0 : \ldots : x_n : t) = (x_0 : \ldots : x_n : \beta t).$$
This action extends the action of $\mu_d$ on $\mathbb{C}^{n+1} = \mathbb{P}^{n+1} \setminus H$, which is used to define the local system monodromy $T: F \to F$, namely

$$\beta \cdot (x_0, ..., x_n) = (\beta^{-1}x_0, ..., \beta^{-1}x_n).$$

It follows that the Gysin exact sequence (3.2) inherits a $\mu_d$-action, such that for any $\lambda \in \mu_d$, $\lambda \neq 1$, one has the following isomorphism of eigenspaces

$$i^* : H^{n+1}(\tilde{U})_\lambda \to H^{n+1}(\tilde{U} \setminus U)_\lambda.$$ 

On the other hand, one has $\tilde{U} \setminus U = \mathbb{C}^{n+1} \setminus F$, and hence a new Gysin sequence shows that one has an isomorphism

$$R : H^{n+1}(\tilde{U} \setminus U)_\lambda \to H^n(F)_\lambda,$$

induced by a residue morphism $R$ which has Hodge type $(-1, -1)$. By composing the above two isomorphisms, we get isomorphisms

$$Ri^* : F^{s+1}H^{n+1}(\tilde{U})_\lambda \to F^sH^n(F)_\lambda,$$

for any integer $s$ and any $\lambda \in \mu_d$, $\lambda \neq 1$.

Now we look at the corresponding $P^*$ filtrations and show that

$$\dim P^{s+1}H^{n+1}(\tilde{U})_\lambda = \dim P^sH^n(F)_\lambda,$$

for any integer $s$ and any $\lambda \in \mu_d$, $\lambda \neq 1$.

The cohomology of the filtered algebraic microlocal Gauss-Manin complexes $\tilde{C}_f^*$ and $\tilde{C}_f^{*'}$ are closely related, namely

$$H^{\ell+1}(\tilde{C}_f^*, P') = H^{\ell}(\tilde{C}_f^{*'}, P') \otimes H^1(\tilde{C}_d^*, P'),$$

see (4.9) in [8]. Looking at the homogeneous components corresponding to $k = d$ and taking the $\lambda$-eigenspaces yields the equality (3.6) in view of (2.6).

Finally, any singularity of the hypersurface $\tilde{D}$ has type $A_{d-1}$, and hence the corresponding $\alpha_{\tilde{D}}$ is exactly $\tilde{m} = \frac{n}{2} + \frac{1}{d}$. Using Corollary (0.12) in M. Saito [16], or the formula (1.1.3) in [9], we see that

$$F^sH^{n+1}(\tilde{U}) = P^sH^{n+1}(\tilde{U}),$$

for any $s \geq n - \tilde{m} + 2$. Using the formulas (3.5) and (3.6) and the inclusion (2.3) we complete the proof, as one may clearly replace $\tilde{m}$ by $m$ as soon as $d > 2$ and $n$ is odd.

\[\square\]
Lemma 4.1. Let $D : f = 0$ be a nodal hypersurface of degree $d > 2$ in $\mathbb{P}^{2n_1+1}$ and set $s = n_1 + 1$. Then

$$F^n H^n(F)_\lambda = P^n H^n(F)_\lambda,$$

for $\lambda = \exp(-2\pi ik/d)$ with

$$0 < k \leq k_0 = d - \left\lfloor \frac{d}{2} \right\rfloor - 1.$$

More precisely, in these conditions one has $\dim \text{Gr}_F^s H^n(F)_\lambda = \dim \text{Gr}_p^s H^n(F)_\lambda = \dim \text{Gr}_h^s H^n(F)_\lambda = \dim M(f)_{n_1d+k-n-1} = \dim M(h)_{n_1d+k-n-1}$, where $h$ is a homogeneous polynomial in $S$ of degree $d$ such that the hypersurface $D_h : h = 0$ is smooth.

Proof. In view of the inclusions in (2.3) and using Proposition 3.1 it is enough to establish the inequality

$$\dim \text{Gr}_F^s H^n(F)_\lambda \geq \dim \text{Gr}_p^s H^n(F)_\lambda.$$

We know that

$$\dim \text{Gr}_F^s H^n(F)_\lambda = \dim E_1^{n_1+2,n_1}(f, k) \leq \dim E_1^{n_1+2,n_1}(f, k) = \dim H^{n+1}(K^*(f))_{n_1d+k} = \dim M(f)_{n_1d+k-n-1} = \dim M(h)_{n_1d+k-n-1}$$

where $h \in S_d$ denotes a polynomial such that the associated hypersurface $D_h : h = 0$ is smooth. Indeed, the last equality follows from Corollary 2.2. (i) in [12], which is a direct consequence of Theorem (1.1).

On the other hand, using an argument similar to Proposition 4.1 in [5], which goes back to Lemma 3.6. in [17], it follows that for $\beta \neq \pm 1$, the mixed Hodge structure induced on

$$(4.1) H^n(F)_{\beta, \bar{\beta}} = H^n(F)_\beta \oplus H^n(F)_{\bar{\beta}}$$

is pure of weight $n$. Therefore

$$\dim \text{Gr}_F^s H^n(F)_\lambda = h^{n_1+1,n_1}(H^n(F)_\lambda) = h^{n_1+1,n_1}(H^n(\tilde{D})_\lambda)$$

in view of Corollary 1.2 in [10]. To compute the last equivariant Hodge number we use Proposition 5.2 in [10]. The first term in the sum giving $h^{n_1+1,n_1}(H^n(\tilde{D})_\lambda)$ is the corresponding number computed for the smooth hypersurface $\tilde{D}_h : \tilde{h} = h(x) - t^d = 0$. Using the standard identification going back to Griffiths [14],

$$H^{n+1,n_1}(H^n(\tilde{D}_h)) = H^{n+2}(K^*(\tilde{h}))_{(n_1+1)d},$$

and recalling that taking the $\lambda$-eigenspace means to look at forms of the form

$$g(x)t^{d-k-1}dx_0 \wedge ... \wedge dx_n \wedge dt,$$

where $g(x)$ is homogeneous of degree $n_1d + k - n - 1$, we get

$$h^{n_1+1,n_1}(H^n(\tilde{D}_h)_\lambda) = \dim M(h)_{n_1d+k-n-1}.$$

We show now that the other terms in the sum giving $h^{n_1+1,n_1}(H^n(\tilde{D})_\lambda)$ are trivial. These terms are of two types:
(a) $h^{p,q}(H^{n+1}(\tilde{D}))_\lambda$, which are zero since a $\beta$-eigenspace of the group $H^{n+1}(\tilde{D})$ under the $\mu_d$-action maybe nontrivial only for $\beta = \pm 1$, see Example 6.3.24 in [3], Theorem 1.1 in [10] and Theorem 4.1 in [12].

(b) $h^{p,q}(H^n(F(d)))_\lambda$, where $F(d)$ is the affine Milnor fiber given by

$$g(y, t) = y_1^2 + \ldots y_n^2 + t^d - 1 = 0$$

in $\mathbb{C}^{n+1}$ with the corresponding $\mu_d$-action, i.e.

$$\beta(y_1, \ldots, y_n, t) = (y_1, \ldots, y_n, \beta t)$$

replacing the monodromy action when eigenspaces are considered. Since this is the Milnor fiber of an isolated weighted homogeneous singularity whose link is a rational homology sphere, we have $h^{p,q}(H^n(F(d)))_\lambda = 0$ for $p + q \neq n$. It follows that we have just one such number to investigate, namely $h^{n_1+1,n_1}(H^n(F(d)))_\lambda$. Using the weights $wt(y_j) = 2$ and $wt(t) = d$, we get using [19]

$$h^{n_1+1,n_1}(H^n(F(d))) = \sum_{j=1,2d-1} \dim M(g(y, t))_{j-d-2}.$$  

The $\lambda$-eigenspace should come from the monomial $t^{d-k-1}$, of degree $2d - 2k - 2$. Our condition on $k$ implies that $2d - 2k - 2 > d - 3$, hence this monomial is not giving a contribution to $h^{n_1+1,n_1}(H^n(F(d)))$, i.e. $h^{n_1+1,n_1}(H^n(F(d)))_\lambda = 0$. Moreover $2k < d$ in order to avoid the case $\lambda = -1$.

This shows that $\dim Gr^*_F H^n(F)_\lambda = \dim M(h)_{n_1d+k-n-1}$, completing the proof of Lemma 4.1. \qed

Now we give the proof of Theorem 1.2. Lemma 4.1 implies that

$$E_1^{n_1+2,n_1}(f, k_0) = H^{n+1}(K^s(f))_{n_1d+k_0} = E_\infty^{n_1+2,n_1}(f, k_0).$$

Moreover, $E_1^{n_1+2+e,n_1-e}(f, k_0) = E_\infty^{n_1+1+e,n_1-e}(f, k_0)$ for all $e = 1, 2, \ldots, n_1$ by similar (and simpler) computations based on Proposition 3.1. It follows that all the differentials in the spectral sequence $E_\tau(f, k_0)$ starting from $E_\tau^{n_1+1,n_1}(f, k_0)$ are 0. Since $E_\infty^{n_1+1,n_1}(f, k_0) = 0$ as well (the only eigenvalues of the monodromy on $H^{n-1}(F)$ are $\pm 1$), we get

$$E_1^{n_1+1,n_1}(f, k_0) = H^n(K^s(f))_{n_1d+k_0} = 0$$

which completes the proof of Theorem 1.2. Indeed, recall that if the coordinates $x_0, \ldots, x_n$ are chosen such that the hyperplane at infinity $H_0 : x_0 = 0$ is transversal to $D$, then the multiplication by $x_0$ induces an injection $H^n(K^s(f))_{s-1} \to H^n(K^s(f))_s$ for any $s$ (the dual statement for the homology is part of Corollary 11 in [1]).

Before proceeding, we recall the following notions, introduced in [11].

**Definition 4.2.** For a hypersurface $D : f = 0$ of degree $d$ with isolated singularities we introduce three integers, as follows:

(i) the **coincidence threshold** $ct(D)$ defined as

$$ct(D) = \max\{q : \dim M(f)_k = \dim M(h)_k \text{ for all } k \leq q\},$$
with \( h \) a homogeneous polynomial in \( S \) of degree \( d = \deg f \) such that \( D_h : h = 0 \) is a smooth hypersurface in \( \mathbb{P}^n \).

(ii) the minimal degree of a nontrivial syzygy \( \text{mdr}(D) \) defined as

\[
\text{mdr}(D) = \min \{ q : H^n(K^*(f))_{q+n} \neq 0 \}
\]

where \( K^*(f) \) is the Koszul complex of \( f_0, \ldots, f_n \) with the natural grading defined in [11].

Moreover it is easy to see that one has

\[
\text{ct}(D) = \text{mdr}(D) + d - 2.
\]

In practice, for a given polynomial \( f \), it is easy to compute \( \text{ct}(D) \) using a number of computer algebra softwares.

**Corollary 4.3.** Let \( D : f = 0 \) be a nodal hypersurface of degree \( d > 2 \) in \( \mathbb{P}^n \). If \( n = 2n_1 + 1 \) is odd, then

\[
\text{ct}(D) \geq (n_1 + 2)d - \left\lfloor \frac{d}{2} \right\rfloor - n - 2.
\]

**Example 4.4.** Let \( \mathcal{C}(3, d) \) be the Chebyshev surface of degree \( d \) in \( \mathbb{P}^3 \) as defined in [12]. Then for \( 3 \leq d \leq 20 \), numerical computation shows that one has

\[
\text{ct}(\mathcal{C}(3, d)) = 3d - \left\lfloor \frac{d}{2} \right\rfloor - 5,
\]

i.e. we have equalities for these cases in Corollary 4.3. It follows that in any such case the bound for the vanishing in Theorem 1.2 is sharp, namely

\[
H^n(K^*(f))_{m+1} \neq 0
\]

for \( m = (n_1 + 1)d - \left\lfloor \frac{d}{2} \right\rfloor - 1 \).

**Remark 4.5.** Let \( D : f = 0 \) be a degree \( d \) hypersurface in \( \mathbb{P}^n \) having only isolated singularities. Let \( \tilde{J} \) be the saturation of the Jacobian ideal \( J \) of \( f \). Then the vector space \( \tilde{J}_d/J_d \) is naturally identified with the space of first order locally trivial deformations of \( D \) in \( \mathbb{P}^n \) modulo those arising from the above \( \text{PGL}(n+1) \)-action, see E. Sernesi [18]. The dimension of the vector space \( \tilde{J}_d/J_d \) can be determined as follows.

\[
\dim \tilde{J}_d/J_d = \dim M(f)_d - \dim M(h)_d + \dim M(f)_{T-d} - \tau(D),
\]

see G. Sticlaru [20] for this formula and a number of interesting examples of rigid and non-rigid hypersurfaces. Here \( \tau(D) \) is the total Tjurina number of the hypersurface \( D \), e.g. the number of nodes for a nodal hypersurface.

## 5. Hodge theory of nodal hypersurfaces

Let \( \mathcal{I} \subset \mathcal{O}_{\mathbb{P}^n} \) be the reduced ideal sheaf of the set of nodes \( \mathcal{N} = \text{Sing} D \subset \mathbb{P}^n \). Set \( I_k(i) = H^0(\mathbb{P}^n, \mathcal{I}(k)) \subset S_k \) and define \( I^{(i)} = \oplus_k I_k(i) \), a homogeneous ideal in the polynomial ring \( S \).

For a degree \( d \) nodal hypersurface \( D \) in \( \mathbb{P}^n \), one of the main results in [9] describe the graded pieces \( Gr_F^n H^*(U) \) of the top cohomology of the complement \( U = \mathbb{P}^n \setminus D \).
with respect to the Hodge filtration $F$ in terms of purely algebraic objects, namely

\begin{equation}
Gr_k^p H^n(U, \mathbb{C}) = (I^{(q-m+1)}/I^{(q-m)}J_f)_{(q+1)d-n-1}
\end{equation}

for $q = n - p > m := \left\lfloor \frac{n}{2} \right\rfloor$ under a certain condition (B), see Theorem 2 in [9].

Recall that for a finite set of points $N \subset \mathbb{P}^n$ we denote by

\[ \text{def } S_m(N) = |N| - \text{codim}\{ h \in S_m | h(a) = 0 \text{ for any } a \in N \} , \]

the defect (or superabundance) of the linear system of polynomials in $S_m$ vanishing at the points in $N$, see [3], p. 207. This positive integer is called the failure of $N$ to impose independent conditions on homogeneous polynomials of degree $m$ in [13]. There is a close relationship between defects $\text{def } S_m(N)$ and the syzygies described by $H^n(K^*(f))$, see [11] for nodal hypersurfaces and [6] for projective hypersurfaces with isolated singularities.

The discussion following the statement of Theorem 2 in [9] shows that in fact (B) is equivalent to the condition

\begin{equation}
(B') : \text{def } S_e(N) = 0,
\end{equation}

where $e = \left\lfloor \frac{n}{2} \right\rfloor (d - 1) - p$ and $N$ is the set of nodes of $D$.

One has the following consequence of Theorems 1.1 and 1.2.

**Theorem 5.1.** Let $D : f = 0$ be a nodal hypersurface of degree $d > 2$ in $\mathbb{P}^n$ and assume that $q = n - p > m := \left\lfloor \frac{n}{2} \right\rfloor$.

(i) If $n = 2n_1$ is even, then the isomorphism (5.1) always hold.

(ii) If $n = 2n_1 + 1$ is even, then the isomorphism (5.1) holds if

\[ p \leq n - \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{d}{2} \right\rfloor. \]

In particular, the isomorphism (5.1) holds for $d = 3$.

**Proof.** When $n = 2n_1$ is even, then $p < n_1$ and hence $e = n_1(d - 1) - p = n_1d - n_1 - p > n_1d - n$. Use now Corollary 2.2 (ii) in [12] which says that $\text{def } S_k(N) = 0$ for $k \geq n_1d - n$.

When $n = 2n_1 + 1$, we have $e = n_1(d - 1) - p$. On the other hand, one clearly has $\text{def } S_k(N) = 0$ if and only if $k \geq T - ct(D)$ and Corollary 4.3 implies

\[ T - ct(D) \leq 2(n_1 + 1)(d - 2) - \left( (n_1 + 1)d - \left\lfloor \frac{d}{2} \right\rfloor - n - 2 \right) = n_1d - n + \left\lfloor \frac{d}{2} \right\rfloor. \]

Hence $e \geq n_1d - n + \left\lfloor \frac{d}{2} \right\rfloor$ as soon as $p \leq n - \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{d}{2} \right\rfloor$. \hfill \Box

Note that Example 4.7 in [9] shows that (B) may not be satisfied by a nodal surface, where $n = 3$, $d = 4$ and $p = 1$. This shows that the condition in Theorem 5.1 (ii) is sharp, as in the case at hand we get $1 \leq 3 - 1 - 2$, which fails just by 1.
ON THE SYZYGIES AND HODGE THEORY OF NODAL HYPERSURFACES

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