Fermion Particle Production as a Dynamical Casimir Effect inside a Three Dimensional Sphere

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Abstract. In this paper we analyze the problem of fermion creation as a dynamical Casimir effect inside a three dimensional sphere. We present an appropriate wave function which satisfies the Dirac equation in this geometry with MIT bag model boundary condition. We consider the radius of the sphere to have dynamics and introduce the time evolution of the quantized field by expanding it over the instantaneous basis. We explain how we can obtain the average number of particles created. In this regard we find the Bogoliubov coefficients. We consider an oscillation and determine the coupling conditions between different modes that can be satisfied depending on the cavity’s spectrum. Assuming the parametric resonance case we obtain an expression for the mean number of created fermions in each mode of an oscillation.

1. Introduction
The Casimir effect is one of the most interesting manifestations of the nontrivial properties of the vacuum state in a quantum field theory (see, e.g., [1, 2, 3, 4] and references therein) and can be viewed as a polarization of the vacuum due to the boundary conditions. A new phenomenon, namely the quantum creation of particles known as the dynamical Casimir effect occurs when the geometry of the system varies in time. In two dimensional spacetime and for conformally invariant fields the problem with dynamical boundaries can be mapped to the corresponding static problem and hence allows a complete study (see [4, 5] and references therein). In higher dimensions the problem is much more complicated and only partial results are available. The vacuum stress induced by uniform acceleration of a perfectly reflecting plane has been considered in [6]. The corresponding problem for a sphere expanding in the four-dimensional spacetime with constant acceleration has been investigated by Frolov and Serebriany [7, 8] in the perfectly reflecting case and by Frolov and Singh [9] for semi-transparent boundaries. For more general cases of motion by vibrating cavities the problem of particle and energy creation has been considered on the base of various perturbation methods [10, 11, 12, 13, 14, 20, 15, 16]. It has been shown that a gradual accumulation of small changes in the quantum state of the field could result in a significant observable effect. A new application of the dynamical Casimir effect has recently appeared in connection with the suggestion by Schwinger [17] that the photon production associated with changes in the quantum electrodynamic vacuum state arising from a collapsing dielectric bubble could be relevant for sonoluminescence.

Particle creation from the quantum scalar vacuum by expanding or contracting spherical shell with Dirichlet boundary conditions have been considered in [18]. In another paper the case has
been considered when the sphere radius performs oscillation with a small amplitude and the expression are derived for the number of created particles to the first order of the perturbation theory [19].

In order to obtain the number of produced fermions, the organization of this paper is as follows, in sec. II we present a normalized eigenfuntion for a massive spinor field inside a spherical shell of finite radius. Considering that the fermionic field obeys the MIT bag condition on the spherical shell, explicit behavior for boundary dependent term is exhibited. We consider the radius of the sphere to have dynamics. Then we introduce the time evolution of the quantized field by expanding it over the instantaneous basis in sec. III. Section IV is devoted to the derivation of Bogoliubov coefficients. Following the steps given in [20] we arrive at an infinite set of coupled differential equations for the coefficients of the expansion. We explain how we can obtain the average number of particles created after the end of the motions. The mean number of created particles depends on the Bogoliubov coefficient. We consider an oscillation with small amplitudes of oscillations and determine the coupling conditions between different modes that can be satisfied depending on the cavity’s spectrum. Assuming the parametric resonance case we obtain an expression for the mean number of created fermions in each mode of an oscillation and in sec. VI. we present our conclusion.

2. Eigenfunctions for a Spinor Field

The Dirac Hamiltonian inside the sphere is

$$\hat{H}_D = \hat{\alpha} \cdot \hat{p} + \hat{\beta} m,$$

(1)

with

$$\hat{\alpha} = \begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix}.$$

(2)

Since the angular momentum operator and the parity operator commute with the Hamiltonian, states have definite energy, angular momentum and parity. Let us write the four-components spinor field $\psi$ in terms of two-components ones as

$$\psi(\vec{x}, t) = \begin{pmatrix} \zeta_{jl m} \\ \chi_{jl' m} \end{pmatrix} \exp(-i\omega t),$$

(3)

and it can be seen easily that

$$(\hat{\sigma} \cdot \hat{p}) \chi_{jl' m} = \omega \zeta_{jl m},$$

$$\omega (\hat{\hat{\sigma}} \cdot \hat{p}) \zeta_{jl m} = \omega \chi_{jl' m},$$

(4)

where $\omega$ is the frequency of the modes. The angular parts of the spinors are the standard spinor spherical harmonics $\Omega_{jl m}$ whose explicit form is given in [21]

$$\zeta_{jl m} = f(r) \Omega_{jl m},$$

$$\chi_{jl' m} = g(r) \Omega_{jl' m}$$

(5)

where $j$ specifies the value of the total angular momentum, $m$ its projection, $l = j \pm \frac{1}{2}$, and $l' = 2j - l$. By taking into account these relations, we obtain the following set of differential equations for the radial functions

$$f'(r) + \frac{1 + \kappa}{r} f(r) - (\omega + m) g(r) = 0$$

$$g'(r) + \frac{1 - \kappa}{r} g(r) + (\omega - m) f(r) = 0$$

(6)
where we use the notation
\[ \kappa = \begin{cases} -l + 1, & j = l + 1/2 \\ l, & j = l - 1/2 \end{cases}. \] (7)

These lead to second order differential equations for the separate functions
\[ f''(r) + \frac{2}{r} f'(r) + \left[ p^2 - \frac{\kappa(\kappa + 1)}{r^2} \right] f(r) = 0 \] (8)
\[ g''(r) + \frac{2}{r} g'(r) + \left[ p^2 - \frac{\kappa(\kappa - 1)}{r^2} \right] g(r) = 0 \]
with the solutions
\[ f(r) = N j_l(pr), \quad g(r) = \mp i N \sqrt{\frac{\omega - m}{\omega + m}} j_{\nu}(pr) \] (9)
where \( p = \sqrt{\omega^2 - m^2} \). In these solutions \( j_l(pr) \) represents the spherical Bessel function of order \( l \). As a result for a given \( j \) we have two types of eigenfunctions with different parities corresponding to \( j = l \pm 1/2 \). Due to the orthogonality conditions the normalization constant reads \( N = p/(\sqrt{\omega + m}). \) In order to guaranty complete confinement of the fermionic field inside the sphere it is needed to impose a boundary condition. The MIT bag model boundary condition is a proper boundary condition. Imposing the prevalent form of this boundary condition on the eigenfunction we have
\[ (1 + i\hat{r} \cdot \vec{\gamma})\psi(x) \bigg|_{r=a} = 0, \] (10)
where \( a \) is the sphere radius and the \( \hat{r} \) is the outward-pointing normal to the sphere. In terms of spinors \( \zeta_{jlm} \) and \( \chi_{j\nu m} \) this condition becomes
\[ \zeta_{jlm} + i(\vec{\sigma} \cdot \hat{r}) \chi_{j\nu m} = 0, \quad r = a \] (11)
and leads to the following quantization condition for the momentum
\[ j_l(pa) = \pm \sqrt{\frac{\omega - m}{\omega + m}} j_{l \pm 1}(pa), \quad \pm \text{for } j = l \pm 1/2. \] (12)

Let us denote the roots to Eq. (12) by \( \Lambda_{nk} = pa \) where \( n \) is the number of the root and \( \kappa \) is the quantum number introduced by Eq. (7).

3. Expansion of Dirac field over the instantaneous basis
The Fourier expansion of the Dirac field for an arbitrary moment of time, in terms of creation and annihilation operators, can be written as
\[ \psi(x, t) = \sum_n a_n^{in} u_n(x, t) + b_n^{lin} v_n(x, t), \] (13)
where \( a_n^{in} \) and \( b_n^{lin} \) are the annihilation and creation operators correspond to the particles in the ‘in’ region. By index ‘in’ we mean times in which the system is static and therefore the spherical shell has a constant radius \( a \). The mode functions \( u_n(x, t) \) form a complete orthonormal set of solutions of the wave equation with the MIT bag model boundary condition. For the static
cavity each field mode is determined by the Eq. (12) and the mode functions \( u_n(x, t) \) have the form of the static solution mentioned in Eq. (3, 5, 9).

Now consider that the radius of sphere varies as a function of time \( a(t) \), which has constant initial value \( a \) and after a finite time modulation stops and the radius takes its initial value. When the modulation begins the boundary condition varies with time. To satisfy this time-dependent boundary condition we expand the mode functions with respect to an instantaneous basis \([22]\) for particles

\[
\psi_n(x, t) = \sum \chi_n^k(t) \phi_k(x, t)
\]

and a similar instantaneous basis for anti particles where

\[
\varphi_k(x, t) = N \left( \pm i \sqrt{\frac{\omega_n}{\omega + m}} \frac{J_l(kr)\Omega_{jl}}{J_{l'}(kr)\Omega_{j'l'}} \right)
\]

and the initial conditions are

\[
Q_n^1(0) = \frac{1}{\sqrt{2\omega_n}} \delta_{n,k},
\]  
\[
\dot{Q}_n^1(0) = -i \sqrt{\frac{\omega_n}{2}} \delta_{n,k}.
\]

When the modulation occurs we can write the Fourier expansion of the Dirac field one more time, but this time with index ‘out’ to indicate this new region of time with new annihilation and creation operators

\[
\psi(x, t) = \sum_n a_n^{out} u_n(x, t) + b_n^{out} v_n(x, t).
\]

Since the expansion in Eq.(17) for the field modes must be a solution of the wave, we insert it in Dirac equation equation. Considering that the \( \phi_k \)'s form a complete and orthogonal set of solutions of the wave equation and they depend on \( t \) only through \( a(t) \), we obtain a set of coupled equations for

\[
\dot{Q}_n^1(t) \phi_k(x, t) + \omega_n k^2 Q_n^1(t) \phi_k(x, t) = -2Q_n^1(t) \dot{\phi}_k(x, t) - Q_n^1(t) \ddot{\phi}_k(x, t)
\]

where \( \omega_n^2 = m^2 + |k|^2 \).

4. Derivation of Bogoliubov coefficients

We now want to obtain the number of fermions produced after the modulation. The mean number of particles produced in the mode \( k \) is the average value of the number operator \( a_k^{out} a_k^{out} \) with respect to the initial vacuum state. Therefore we are interested in times after the modulation. In this region, the radius takes its initial value and the right hand side in Eq.(18) vanishes and it reduces to the following simple form

\[
(\partial_t^2 + \omega_p^2)Q^1_p(t) = 0,
\]

the solution reads

\[
Q^1_p(t) = A^{(n)}_p e^{i\omega_p t} + B^{(n)}_p e^{-i\omega_p t},
\]

with \( A^{(n)}_p \) and \( B^{(n)}_p \) constant coefficients to be determined. The creation and annihilation operators for particles and anti-particles in the 'in' and 'out' regions obey the usual
anticommutation relations. Using the Bogoliubov canonical transformation one can expand the ‘in’ operators in terms of the ‘out’ operators

\[ a_{in}^n = \alpha_n^* a_{out}^n - \beta_n b_{out}^n, \]
\[ b_{in}^n = \beta_n^* a_{out}^n + \alpha_n b_{out}^n. \]

(21)

Substituting Eq. (20) in Eq. (14) and then in Eq. (17), and considering Eq. (21) we expand \( \phi_k \) for out region in terms of the creation and annihilation operators in the in region. After some calculations and by means of Eq. (4) eventually we get

\[ \alpha_n^* = \frac{B_p^{(n)}}{N} \left( \frac{2m - \omega_p}{\omega_p} \omega_p^2 \frac{m + \omega_p}{\omega_p} \right). \]

(22)

Using the relation between the Bogoliubov coefficients for fermionic fields

\[ |\beta_p|^2 = 1 - |\alpha_p|^2, \]

(23)

therefore the mean number of particles produced in the mode \( \vec{p} \) through an arbitrary modulation of the single fermion mode is the average value of the number operator with respect to the initial vacuum state (defined through \( a_{in}^p |0\rangle = 0 \))

\[ \langle N_p \rangle = |\beta_p|^2 = 1 - 2\pi |B_p^{(n)}|^2. \]

(24)

Up to this point the equations are valid for an arbitrary variation of radius of the sphere. The only assumption was that \( a(0) = a \). Our purpose is to investigate the number of fermion created inside the sphere, therefore we look for harmonic oscillation of the radius which could enhance that number by means of resonance effects for some specific external frequencies. So we consider the following oscillations

\[ a(t) = a(1 + \varepsilon \sin(\Omega t)), \]

(25)

where the amplitude of oscillation is very small and we can content ourselves with its first order and Eq.(18) for a massless case takes the form [23]

\[ \ddot{Q}_p^{(n)}(t) + \omega_p^2 Q_p^{(n)}(t) = 2\varepsilon \omega_p^2 \sin(\Omega t) \dot{Q}_p^{(n)}(t) \]
\[ + \frac{2\omega_p^2}{\pi} \varepsilon \Omega \cos(\Omega t) \sum_k g_{pk} \dot{Q}_k^{(n)}(t) \]
\[ - \frac{2\omega_p^2}{\pi} \varepsilon \Omega \sin(\Omega t) \sum_k g_{pk} Q_k^{(n)}(t), \]

(26)

where

\[ g_{pk} = -a(t) \int_0^{a(t)} d^3x \, \dot{\varphi}_p^*(x, t) \frac{\partial \varphi_k(x, t)}{\partial a}. \]

(27)

Since \( \varepsilon \ll 1 \) it is natural to assume that the solution of Eq. (26) is of the form

\[ Q_p^{(n)}(t) = A_p^{(n)}(t) e^{i\omega_p t} + B_p^{(n)}(t) e^{-i\omega_p t}, \]

(28)
functions $A_p^{(n)}(t)$ and $B_p^{(n)}(t)$ varying slowly with time. We insert Eq. (28) into the Eq. (26) to obtain differential equations for $A_p^{(n)}(t)$ and $B_p^{(n)}(t)$. After neglecting their second derivatives and multiplying equation by $e^{i\omega t}$ we average over the fast oscillations, we have

\[
\frac{dA_p^{(n)}}{d\tau} = -\frac{\omega_p}{2} B_p^{(n)} \delta(2\omega_p - \Omega) + \sum_{k} (-\omega_k + \Omega/2) \delta(-\omega_p - \omega_k + \Omega) \frac{\Omega \omega_p}{2\pi} g_{pk} B_k^{(n)} \\
+ \sum_{k} \left[ (\omega_k + \Omega/2) \delta(\omega_p - \omega_k - \Omega) + (\omega_k - \Omega/2) \delta(\omega_p - \omega_k + \Omega) \right] \times \frac{\Omega \omega_p}{2\pi} g_{pk} A_k^{(n)},
\]

(29)

\[
\frac{dB_p^{(n)}}{d\tau} = -\frac{\omega_p}{2} A_p^{(n)} \delta(2\omega_p - \Omega) + \sum_{k} (-\omega_k + \Omega/2) \delta(-\omega_p - \omega_k + \Omega) \frac{\Omega \omega_p}{2\pi} g_{pk} A_k^{(n)} \\
+ \sum_{k} \left[ (\omega_k + \Omega/2) \delta(\omega_p - \omega_k - \Omega) + (\omega_k - \Omega/2) \delta(\omega_p - \omega_k + \Omega) \right] \times \frac{\Omega \omega_p}{2\pi} g_{pk} B_k^{(n)}
\]

(30)

where $\tau = \varepsilon t$ is a time scale. Now we shall solve equations (29) and (30). Depending on the radius of the static sphere and the frequency spectrum of the modes, there are different kinds of solutions. If we consider that the frequency of the boundary is twice the frequency of some unperturbed mode namely $\Omega = 2\omega_p$, in this condition if $\omega_k - \omega_p = \Omega$ the resonant mode $p$ will be coupled to some other mode $k$. Let us now suppose that the coupling conditions $|\omega_p \pm \omega_k| = \Omega$ are not fulfilled. In this case and for a massless field the equations (29) and (30) reduces to

\[
\frac{dA_p^{(n)}}{d\tau} = -\frac{\omega_p}{2} B_p^{(n)},
\]

(31)

\[
\frac{dB_p^{(n)}}{d\tau} = -\frac{\omega_p}{2} A_p^{(n)}.
\]

(32)

Satisfying the initial condition mentioned in Eq. (16), the solutions of these coupled equations are

\[
A_p^{(n)}(\tau) = -\frac{1}{\sqrt{2\omega_p}} \sinh\left(\frac{\omega_p}{2}\tau\right),
\]

(33)

\[
B_p^{(n)}(\tau) = \frac{1}{\sqrt{2\omega_p}} \cosh\left(\frac{\omega_p}{2}\tau\right).
\]

(34)

By using Eq. (24) the average number of produced fermions in the mode $p$ is

\[
\langle N_p \rangle = 1 - 2\pi \cosh^2\left(\frac{\omega_p}{2}\tau\right),
\]

(35)

where for our massless case $\omega_p = \Lambda_{nk}/a$ and $\Lambda_{1,-1} = 2.04$, $\Lambda_{2,-1} = 5.40$, etc. For each $\kappa$ there are two roots corresponding to $l' = l \pm 1$.

5. Conclusion
We have investigated the particle creation for a Dirac field in a three dimensional sphere with the MIT bag model boundary condition. We have considered the radius of the sphere to modulate during a finite time interval and when modulation stops the radius takes its initial value. In this regard we have used the Bogoliubov coefficients to obtain the number of fermions created during the motion. We have taken into account the usual parametric resonance case ($\Omega = 2\omega_p$) and find the number of fermion created in this system.
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