AN INDEX THEOREM FOR FAMILIES INVARIANT WITH RESPECT TO A BUNDLE OF LIE GROUPS

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Abstract. We define the equivariant family index of a family of elliptic operators invariant with respect to the free action of a bundle $G$ of Lie groups. If the fibers of $G \to B$ are simply-connected solvable, we then compute the Chern character of the (equivariant family) index, the result being given by an Atiyah-Singer type formula. We also study traces on the corresponding algebras of pseudodifferential operators and obtain a local index formula for such families of invariant operators, using the Fedosov product. For topologically non-trivial bundles we have to use methods of non-commutative geometry. We discuss then as an application the construction of “higher-eta invariants,” which are morphisms $K_n(\Psi_{\text{inv}}^{\infty}(Y)) \to \mathbb{C}$. The algebras of invariant pseudodifferential operators that we study, $\Psi_{\text{inv}}^{\infty}(Y)$ and $\Psi_{\text{inv}}^{\infty}(Y)$, are generalizations of “parameter dependent” algebras of pseudodifferential operators (with parameter in $\mathbb{R}^n$), so our results provide also an index theorem for elliptic, parameter dependent pseudodifferential operators.

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Introduction

Families of Dirac operators invariant with respect to a bundle of Lie groups appear in the analysis of the Dirac operator on certain non-compact manifolds. They arise, for example, in the analysis of the Dirac operator on an $S^1$-manifold $M$, if we desingularize the action of $S^1$ by replacing the original metric $g$ with $\phi^{-2}g$, where $\phi$ is the length of the infinitesimal generator $X$ of the $S^1$-action. In this way, $X$ becomes of length one in the new metric. The main result of [34] states that

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the kernel of the new Dirac operator on the open manifold $M \setminus M^{S^3}$ is naturally isomorphic to the kernel of the original Dirac operator.

It turns out that the Fredholm property of the resulting Dirac operator (obtained by the above procedure on the non-compact manifold $M \setminus M^{S^3}$) is controlled by the invertibility of a family of operators invariant with respect to the action of a bundle of non-abelian, solvable Lie groups. This follows from the results of [22] and it will be discussed in greater detail in a future paper. In general, neither the bundle $Y \rightarrow B$ on which these operators act, nor the bundle of Lie groups $\mathcal{G} \rightarrow B$ acting on $Y$, are trivial. A natural problem then is to study the invertibility of these invariant families of operators, and more generally, their index.

We define the (equivariant family) index of a family of invariant, elliptic operators using $K$-theory. For operators acting between sections of the same vector bundle, we can define the index using the boundary map in algebraic $K$-Theory. For operators acting between sections of different bundles, one has to use Kasparov’s bivariant $K$-Theory [12] or a smooth variant of it [12]. The algebraic $K$-theory definition of the index gives a little bit more than the one using bivariant $K$-theory, but it applies only to elliptic operators acting between isomorphic vector bundles, which is however almost always the case in applications, and hence only a minor drawback. In any case, it turns out that the (equivariant family) index of such an invariant elliptic family is the obstruction to finding an invertible perturbation of the original family by families of invariant, regularizing operators, if we exclude the degenerate case $\dim Y = \dim \mathcal{G}$. This shows the relevance of computing the index to the problem of determining the invertibility of a given family.

In this paper, we study the index and certain non-local invariants of families of elliptic operators invariant with respect to a bundle of simply-connected, solvable Lie groups $\mathcal{G}$. One of the main results is a formula for the Chern character of the index bundle that is similar to the Atiyah-Singer index formula for families. We analyse the local behavior of these families, when $\mathcal{G}$ is a vector bundle. This leads us then to the construction of several traces on the algebras $\Psi^*_{inv}(Y)$. We use these traces to obtain local index theorems. For more general bundles $\mathcal{G}$, the local analysis is likely to be much harder because there are no good candidates for the construction of convolution algebras closed under functional analytic calculus on non-commutative solvable Lie groups.

In [10], Bismut and Cheeger have generalized the Atiyah-Patodi-Singer index theorem [4] to families of Dirac operators on manifolds with boundary (see also [30]). Their results apply to operators whose “indicial parts” are invertible. These indicial parts are actually families of Dirac operators invariant with respect to a one-parameter group, so they fit into the framework of this paper (with $\mathcal{G} = B \times \mathbb{R}$). In addition to the usual ingredients of an index theorem – curvature and characteristic classes – their result was stated in terms of a new invariant, called the “eta-form,” in analogy to the additional invariant appearing in the Atiyah-Patodi-Singer index formula for operators on manifolds with boundary. Thus, the results of this paper are relevant also to the problem of determining an explicit formula for the index of a family of pseudodifferential elliptic operators on a bundle of manifolds with boundary. (See also [44].)

With an eye towards this problem, we also give a new proof of the regularity of the eta function at the origin and discuss some possible generalizations the eta invariant. Actually, we discuss two possible generalizations, one which is a direct
generalization of a result of [28] and one using higher algebraic $K$-theory. The first possible generalization is to associate to a Dirac operator invariant with respect to $R^q$ the quantity defined by the formula $\text{Tr}_q\left((D^{-1}dD)^{2k-1}\right)$. This possible generalization was considered before by Lesch and Pflaum [24], who proved that this formula does not lead to new invariants for Dirac operators and also that this formula is not additive for a product of two invertible operators, except for $k=1$, when one recovers the usual eta invariant [28]. The second possible generalization, which has the advantage of being additive, is to define the higher eta invariant as a morphism on higher algebraic $K$-theory. This definition can be found in Section 6.

We now describe the contents of each section of this paper. In Section 1, we discuss the action of a bundle of Lie groups $G$ on a fiber bundle $Y$ and we introduce the algebras $\psi^\infty_{\text{inv}}(Y)$ and $\Psi^\infty_{\text{inv}}(Y)$, which will be our main object of study. (Both these algebras consist of families of invariant pseudodifferential operators.) Families of operators invariant with respect to a bundle of Lie groups were probably considered for the first time in [26]. (See also [25].) When $B$ is reduced to a point and $G = R^q$, the algebras $\Psi^\infty_{\text{inv}}(Y)$ were studied in [27]. Torsion invariants for families of Dirac operators invariant with respect to a vector bundle were defined and studied by Bismut, see [9]. We prove that, when $G$ is a vector bundle, the group of gauge transformations of $G$ acts on $\psi^\infty_{\text{inv}}(Y)$ and $\Psi^\infty_{\text{inv}}(Y)$. In Section 2, we define the index of a family of elliptic, invariant pseudodifferential operators $A$. We shall sometimes use the term “the equivariant family index” of an elliptic $A \in \psi^\infty_{\text{inv}}(Y)$, for the index of such a family. We prove that the index of $A$ is the obstruction to finding a regularizing $R$ such that $A + R$, acting between suitable Sobolev spaces, is invertible in each fiber (excluding the degenerate case $\dim Y = \dim G$). This generalizes the usual property of the Fredholm index of a Fredholm operator. In Section 3, we prove that when $G$ consists of simply-connected solvable Lie groups $K^* (C^*(G)) \cong K^*(g)$, (1) if $g$ is the vector bundle of the Lie algebras defined by $G$. Then we obtain a formula for the Chern character of the index of an elliptic operator $A \in \psi^\infty_{\text{inv}}(Y)$. (For simplicity, we called $A$ “an operator,” although it is really a family of operators. We shall do this repeatedly.)

Beginning with Section 4, we assume that $G$ is a vector bundle, because then we can construct natural algebras that are spectrally invariant (even closed under functional analytic calculus), and hence we obtain more refined results. We then develop the necessary facts about the asymptotics of the trace of the operators in $\Psi^\infty_{\text{inv}}(Y)$. This allows us to define various regularized and residue traces. Using these traces, we obtain in Section 5 two local formulae for the (equivariant family) index of an elliptic operator $A \in \psi^\infty_{\text{inv}}(Y)$. In Section 6, we discuss two possible generalizations of the eta invariant, which is suggested by the formula

$$\eta(D_0, s) = \text{Tr}_1(D^{-1}D'),$$

proved in [28] using the local index theorem (see also [24]). Operators invariant with respect to $R^q$, a particular case of our operators when the base is reduced to a point, appear in the formulation of elliptic (or Fredholm) boundary conditions for pseudodifferential operators on manifolds with corners. The equivariant index can be used to study this problem, which is relevant to the question of extending the Atiyah-Patodi-Singer boundary conditions to manifolds with corners.
The algebras of invariant pseudodifferential operators that we study, $\psi^\infty_{\text{inv}}(Y)$ and $\Psi^\infty_{\text{inv}}(Y)$, are generalizations of “parameter dependent” algebras of pseudodifferential operators considered by Agmon [1], Grubb and Seeley [18], Lesch and Pflaum [24], Melrose [28], Shubin [43], and others. Our index theorem, Theorem 5, hence gives a solution to the problem of determining the index of elliptic, parameter dependent families of pseudodifferential operators, parameterized by $\lambda \in \mathbb{R}^q$. We note that the concept of index of such a family requires a proper definition, and that the Fredholm index, “dimension of the kernel” - “dimension of the cokernel,” is not appropriate for $q \geq 1$. Our definition of the (equivariant, family) index is somewhat closer to the definition of the $L^2$-index for covering spaces given in [3] and [41] than to the definition of the Fredholm index. However, an essential difference between their definition and ours is that no trace is involved in our definition.

For the proof of our local index theorem, we use ideas of non-commutative geometry [14], more precisely, the general approach to index theorems using cyclic cohomology as developed in [36]. These computations are also an example of a computation of a bivariant Chern-Connes character [33]. We plan to use the results of this paper for some problems in $M$-theory [40]. We also expect our results to have applications to adiabatic limits of eta invariants [11, 45].

There exist several potential extensions of our results, although none of them seems to be straightforward. These extensions will presumably involve more general conditions on $G$ and its action on $Y$. Probably the most general conditions under which one can reasonably expect to obtain definite results are those when the fibers of $G$ are connected and the action is proper. We assume that the fibers of $G$ are connected to be able to use general results on connected Lie groups. Therefore, let us consider a general family of connected Lie groups $G$ and successively weaken our assumptions on $G$ and its action on $Y$ and then try to see to what extend our results extend to this general setting. If we continue to assume that the action of $G$ is free, then the results of Sections 1 and 2 remain true, because we do not use any conditions on the fibers of $G$ in these two section. However, the results of the subsequent sections are no longer valid. If we drop the assumption that the action of $G$ be free, then the definition of the algebras $\psi^\infty_{\text{inv}}(Y)$ still makes sense in this more general situation, provided that we assume the action of $G$ to be proper, or otherwise our algebras might be too small.

Let us then assume henceforth that the action of $G$ is proper. For simply-connected solvable families, this implies that the action is free, which justifies our choices and moreover gives that $Y \cong Z \times_B G$, as $G$-spaces. For general families though, the topology on $Y$ and the action of $G$ will be more complicated if the action of $G$ is not free. Moreover, except the definition of the algebras $\psi^\infty_{\text{inv}}(Y)$, few of our results extend to the general case of connected fibers and proper actions. For example, the results of Section 3 are not true for non-free actions, in general. This is the case, in particular, of Lemma 3, so we cannot define the equivariant family index as an element of $K_0(C^*_r(G))$, as we do in this paper, but we get it as an element of $K_0(C^*_r(Y; G))$. Further work is necessary to show that the natural map $K_0(C^*_r(Y; G)) \rightarrow K_0(C^*_r(G))$ is injective in the cases of interest.

Let us now take a closer look at the case when the fibers of $G \rightarrow B$ are compact. Then we do get that $K_0(C^*_r(Y; G)) \rightarrow K_0(C^*_r(G))$ is injective, and hence we can still define the index as an element of $K_0(C^*_r(G))$. It is not difficult to show that all the fibers $G_b$ are isomorphic as groups to a fixed Lie group, say $G$. The fundamental group $\pi$ of $B$ then acts by holonomy on $R(G)$, the representation ring of $G$, and if
If $G$ is connected and semisimple, we have

\[(2) \quad K_\ast(C^*(\mathcal{G})) \otimes \mathbb{Q} \cong K_\ast(B) \otimes R(G)^\pi \otimes \mathbb{Q},\]

In general, however, there will be no such isomorphism if we do not tensor by $\mathbb{Q}$. Thus, when the fibers of $\mathcal{G}$ are compact, the groups $K_\ast(C^r(B))$ are completely different from the corresponding groups when the fibers of $\mathcal{G}$ are solvable (see also Equation (1)).

Nevertheless, the methods of this paper will probably be useful to treat the general case, allowing us to reduce the general case of connected fibers and proper actions to the case of a bundle of semisimple Lie groups. The “Dirac” and “dual Dirac” bivariant $K$-theory elements will probably then allow us to further reduce the semisimple case to the compact case, in the spirit of the Connes-Kasparov conjecture, using Kasparov’s bivariant $K$-theory. The case of compact fibers can then be treated directly, although this is not straightforward and may require fields of matrices with non-trivial Diximier-Douady invariants. Moreover, some conceptual difficulties in determining the equivariant index arise when the bundle $\mathcal{G}$ is non-trivial.

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All pseudodifferential operators considered in this paper are “classical,” that is, one-step polyhomogeneous.

1. INVARIANT PSEUDODIFFERENTIAL OPERATORS

We now describe the settings in which we shall work. Let $B$ be a smooth compact manifold and

\[d : \mathcal{G} \to B \quad \text{and} \quad \pi : Y \to B\]

be two smooth fiber bundles with fibers $\mathcal{G}_b := d^{-1}(b)$ and $Y_b := \pi^{-1}(b)$. We shall assume that $\mathcal{G}$ is a bundle of Lie groups acting smoothly on $Y$, and then we shall consider families of operators along the fibers of $Y$ and invariant under the action of $\mathcal{G}$. We can restrict in our discussion to a connected component of $B$, so for simplicity, we shall assume that $B$ is connected. The index and local invariant of these operators will form our main object of study. We now make all these assumptions and concepts more precise.

Throughout this paper, $\mathcal{G}$ will denote a bundle of Lie groups on a manifold usually denoted $B$. By this we mean that $\mathcal{G} \to B$ is a smooth fiber bundle, that each $\mathcal{G}_b$ is a Lie group, and that the multiplication and inverse depend differentiably on $b$. Hence the map

\[(3) \quad \mathcal{G} \times_B \mathcal{G} := \{(g', g) \in \mathcal{G} \times \mathcal{G}, d(g') = d(g)\} \ni (g', g) \mapsto g'g^{-1} \in \mathcal{G}\]

is differentiable. This implies, by standard arguments, that the map sending a point $b \in \mathcal{G}_b$ to $e_b$, the identity element of $\mathcal{G}_b$, is a diffeomorphism onto a smooth submanifold of $\mathcal{G}$. It also implies that the map $\mathcal{G} \to g \mapsto g^{-1} \in \mathcal{G}$ is differentiable.

We also assume that $\mathcal{G}$ acts smoothly on $Y$. This means that there are given actions $\mathcal{G}_b \times Y_b \to Y_b$ of $\mathcal{G}_b$ on $Y_b$, for each $b$, such that the induced map,

\[\mathcal{G} \times_B Y := \{(g, y) \in \mathcal{G} \times Y, d(g) = \pi(y)\} \ni (g, y) \mapsto gy \in Y,\]

is differentiable. We shall also assume that the action of $\mathcal{G}$ on $Y$ is free, that is, that the action of $\mathcal{G}_b$ on $Y_b$ is free for each $b$. 
We do not assume, however, that the groups \( \mathcal{G}_b \) are isomorphic, although this is true in most applications. For us, two important particular cases of Lie groups bundles are when the fibers of \( \mathcal{G} \) are compact and the case when the fibers of \( \mathcal{G} \) are simply-connected, solvable Lie groups. Let \( \mathfrak{g} \rightarrow \mathcal{B} \) be the bundle of Lie algebras of associated to \( \mathcal{G} \), that is, the bundle whose fiber above \( b \), \( \mathfrak{g}_b \), is \( \text{Lie}(\mathcal{G}_b) \). We shall also treat in detail the case when \( \mathcal{G} \) consists of simply-connected abelian groups, that is, it is a vector bundle \( \mathcal{G} \cong \mathfrak{g} \). This case is very important in applications and can be treated more completely because it does not involve any complications related to the harmonic analysis on non-compact, non-abelian Lie groups.

On \( Y \), we consider smooth families \( A = (A_b), b \in B, \) of classical pseudodifferential operators acting on the fibers of \( Y \rightarrow \mathcal{B} \) such that each \( A_b \) is invariant with respect to the action of the group \( \mathcal{G}_b \). Unless mentioned otherwise, we assume that these operators act on half densities along each fiber. The algebra that we are interested in consists of such invariant operators satisfying also a support condition. To state this support condition, first notice that a family \( A = (A_b) \) defines a continuous map \( \mathcal{C}_c^\infty (Y) \rightarrow \mathcal{C}^\infty (Y) \), and, as such, it has a distribution (or Schwartz) kernel, which is a distribution \( K_A \) on \( Y \times_B Y \subset Y \times Y \). (We ignore the vector bundles in which this distribution takes its values.) Because the family \( A = (A_b) \) is invariant, the distribution \( K_A \) is also invariant with respect to the action of \( \mathcal{G} \). Consequently, \( K_A \) is the pull back of a distribution \( k_A \) on \( (Y \times_B Y)/\mathcal{G} \). We will require that \( k_A \) have compact support. We shall sometimes call \( k_A \) the convolution kernel of \( A \). This condition on the support of \( k_A \) ensures that each \( A_b \) is a properly supported pseudodifferential operator, and hence it maps compactly supported functions (or sections of a vector bundle, if we consider operators acting on sections of a smooth vector bundle) to compactly supported functions (or sections). This support condition is automatically satisfied if \( Y/\mathcal{G} \) is compact and each \( A_b \) is a differential operator. The space of smooth, invariant families \( A \) of order \( m \) pseudodifferential operators acting on the fibers of \( Y \rightarrow \mathcal{B} \) such that \( k_A \) has compact support will be denoted by \( \Psi^m_{\text{inv}} (Y) \). Then

\[
\Psi^\infty_{\text{inv}} (Y) := \bigcup_{m \in \mathbb{Z}} \Psi^m_{\text{inv}} (Y)
\]

is an algebra, by classical results \([19]\). Note also that \( \Psi^m_{\text{inv}} (Y) \) makes sense also for \( m \) not an integer.

We now discuss the principal symbols of the invariant operators that we study. Let

\[
T_{\text{vert}} Y := \ker (TY \rightarrow TB)
\]

be the bundle of vertical tangent vectors to \( Y \), and let \( T^*_{\text{vert}} Y \) be its dual. We fix compatible metrics on \( T_{\text{vert}} Y \) and \( T^*_{\text{vert}} Y \), and define \( S^*_{\text{vert}} Y \), the cosphere bundle of the vertical tangent bundle to \( Y \), to be the set of vectors of length one of \( T^*_{\text{vert}} Y \). Also, let

\[
\sigma_m : \Psi^m (Y_b) \rightarrow C^\infty (S^*_{\text{vert}} Y_b \cap T^* Y_b)
\]

be the usual principal symbol map, defined on the space of pseudodifferential operators of order \( m \) on \( Y_b \). The definition of \( \sigma_m \) depends on the choice of a trivialization of the bundle of homogeneous functions of order \( m \) on \( T^*_{\text{vert}} Y_b \), regarded as a bundle over \( S^*_{\text{vert}} Y \). The principal symbols \( \sigma_m (A_b) \) of an element (or family) \( A = (A_b) \in \Psi^m_{\text{inv}} (Y) \) then gives rise to a smooth function on \( C^\infty (S^*_{\text{vert}} Y) \), which is invariant with respect to \( \mathcal{G} \), and hence descends to a smooth function on \( S^*_{\text{vert}} Y \), which has compact support because of the support condition on the kernel of \( A \).
The resulting function,

\[ \sigma_m(A) \in \mathcal{C}_c^\infty((S_{vert}^*Y)/\mathcal{G}), \]

will be referred to as the \textit{principal symbol} of an element (or operator) in \( \psi_{inv}^m(Y) \).

In the particular case \( Y = \mathcal{G} \), \( \psi_{inv}^\infty(\mathcal{G}) \) identifies with families of convolution operators on the fibers \( \mathcal{G}_b \) with kernels contained in a compact subset of \( \mathcal{G} \), smooth outside the identity, and only with conormal singularities at the identity. In particular, \( \psi_{inv}^{-\infty}(\mathcal{G}) = \mathcal{C}_c^\infty(\mathcal{G}) \), with the fiberwise convolution product.

Suppose now that the quotient \( Y/\mathcal{G} \) is compact, which implies that \( (S_{vert}^*Y)/\mathcal{G} \) is also compact. As it is customary, an operator \( A \in \psi_{inv}^m(Y) \) is called \textit{elliptic} if, and only if, its principal symbol is everywhere invertible. The same definition applies to \( A = [A_{ij}] \in M_N(\psi_{inv}^m(Y)) : \) the operator \( A \), regarded as acting on sections of the trivial vector bundle \( \mathbb{C}^N \), is elliptic if, and only if, its principal symbol

\[ \sigma_m(A) := [\sigma_m(A_{ij})] \in M_n(\mathcal{C}_c^\infty(S_{vert}^*Y/\mathcal{G})) \]

is invertible.

Assume that there is given a \( \mathcal{G} \)-invariant metric on \( T_{vert}Y \), the bundle of vertical tangent vectors, and a \( \mathcal{G} \)-equivariant bundle \( W \) of modules over the Clifford algebras of \( T_{vert}Y \). Then a typical example of a family \( D = (D_b) \in \psi_{inv}^\infty(Y) \) is that of the family of Dirac operators \( D_b \) acting on the fibers \( Y_b \) of \( Y \rightarrow B \). (Each \( D_b \) acts on sections of \( W|_{Y_b} \), the restriction of the given Clifford module \( W \) to that fiber.)

Before proceeding, in the next section, to define the equivariant family index of an elliptic family invariant with respect to a bundle of Lie groups, let us take a closer look at a particular case of the previous construction.

Take \( Y = B \times Z \times \mathbb{R}^q \), with \( Z \) a compact manifold and \( \mathcal{G} = B \times \mathbb{R}^q \), \( \pi \) and \( d \) being the projections onto the first components of each product. The action of \( \mathcal{G} \) on \( Y \) is given by translation on the last component of \( Y \). Then the \( \mathcal{G} \)-invariance condition becomes simply \( \mathbb{R}^q \) invariance with respect to the resulting \( \mathbb{R}^q \) action. If \( Y \) and \( \mathcal{G} \) are as described here, then we call \( Y \) a \textit{flat} \( \mathcal{G} \)-space.

One disadvantage of the algebras \( \psi_{inv}^\infty(Y) \) is the following. It is possible to find families \( A = (A_b) \in \psi_{inv}^0(Y) \) such that each \( A_b \) is invertible as a bounded operator, but the family \( (A_b^{-1}) \) is not in \( \psi_{inv}^0(Y) \), although it consists of invariant, pseudodifferential operators. This pathology is due to the support condition. Nevertheless, for \( \mathcal{G} \) consisting of abelian groups, it is easy to remedy this pathology by enlarging the algebra \( \psi_{inv}^{-\infty}(Y) \), as follows.

Since the enlargement of the algebra \( \psi_{inv}^{-\infty}(Y) \) is done locally, we may assume that \( Y \) is a flat \( \mathcal{G} \)-space. The residual ideal of the algebra \( \psi_{inv}^\infty(Y) \) is \( \psi_{inv}^{-\infty}(Y) \) and consists of operators that are regularizing along each fiber. More precisely, it consists of those families of smoothing operators on \( Y = B \times Z \times \mathbb{R}^q \) that are translation-invariant under the action of \( \mathbb{R}^q \) and have \textit{compactly supported} convolution kernels. Thus

\[ \psi_{inv}^{-\infty}(Y) \cong \mathcal{C}_c^\infty(B \times \mathbb{R}^q; \Psi^{-\infty}(Z)) \subset \mathcal{S}(B \times \mathbb{R}^q; \Psi^{-\infty}(Z)) \cong \mathcal{S}(B \times Z \times \mathbb{R}^q). \]

(Here \( \mathcal{S} \) is the generic notation for the space of Schwartz functions on a suitable space, in this case on \( B \times \mathbb{R}^q \), with values regularizing operators.) The second isomorphism above is obtained from the isomorphism

\[ \Psi^{-\infty}(Z) \cong \mathcal{C}_c^\infty(Z \times Z), \]

defined by the choice of a nowhere vanishing density on \( Z \). If we also endow \( \Psi^{-\infty}(Z) \) with the locally convex topology induced by this isomorphism, then it becomes a
Proof. The vector representation of $GL_{q}$ Assume Lemma 1. Variables of $S$ in invariant, regularizing operators whose kernels are in $B$ operators. This property extends right away to the action of a vector bundle, $\psi_{\text{inv}}^{\infty}(Y)$ is isomorphically to itself. From the isomorphism (6), we see that $C_{\text{inv}}^{\infty}(B, GL_{q}(\mathbb{R}))$ maps $\psi_{\text{inv}}^{\infty}(Y)$ to itself. This gives an action by automorphisms of $C_{\text{inv}}^{\infty}(B, GL_{q}(\mathbb{R}))$ on $\Psi_{\text{inv}}^{\infty}(Y)$ and on $\Psi_{\text{inv}}^{\infty}(Y)$.

Lemma 1. Assume $Y = B \times Z \times \mathbb{R}^{q}$ is a flat $\mathcal{G}$-space. Then the action of the group $GL_{q}(\mathbb{R})$ on the last factor of $Y = B \times Z \times \mathbb{R}^{q}$ extends to an action by automorphisms of $C_{\text{inv}}^{\infty}(B, GL_{q}(\mathbb{R}))$ on $\Psi_{\text{inv}}^{\infty}(Y)$ and on $\Psi_{\text{inv}}^{\infty}(Y)$.

Proof. The vector representation of $GL_{q}(\mathbb{R})$ on the second component of $Z \times \mathbb{R}^{q}$ defines an action of $GL_{q}(\mathbb{R})$ on $\Psi_{\text{inv}}^{\infty}(Z \times \mathbb{R}^{q})$ that preserves the class of properly supported operators and the products of such operators. It also normalizes the group $\mathbb{R}^{q}$ of translations, and hence it maps $\mathbb{R}^{q}$-invariant operators to $\mathbb{R}^{q}$-invariant operators. This property extends right away to the action of $C_{\text{inv}}^{\infty}(B, GL_{q}(\mathbb{R}))$ on families of operators on $B \times Z \times \mathbb{R}^{q}$, and hence $C_{\text{inv}}^{\infty}(B, GL_{q}(\mathbb{R}))$ maps $\psi_{\text{inv}}^{\infty}(Y)$ isomorphically to itself. From the isomorphism (6), we see that $C_{\text{inv}}^{\infty}(B, GL_{q}(\mathbb{R}))$ also maps $\Psi_{\text{inv}}^{\infty}(Y)$ to itself. This gives an action by automorphisms of $C_{\text{inv}}^{\infty}(B, GL_{q}(\mathbb{R}))$ on $\Psi_{\text{inv}}^{\infty}(Y)$, which is the sum of $\Psi_{\text{inv}}^{\infty}(Y)$ and $\psi_{\text{inv}}^{\infty}(Y)$. 

Suppose the family of Lie groups $\mathcal{G}$ consists of abelian Lie groups, so that $\mathcal{G}$ is a vector bundle, $\mathcal{G} \cong g$. By choosing a lift of $Y/\mathcal{G} \to Y$, which is possible because the fibers are contractable, we obtain that locally the bundle $Y$ is isomorphic to a flat $\mathcal{G}$ space. Then the above lemma allows us to extend the previous definitions, including those of the algebras $\Psi_{\text{inv}}^{\infty}(Y)$ and of the indicial family from the flat case to the case $\mathcal{G}$ abelian. The indicial family $\tilde{A}$ of an operator $A \in \Psi_{\text{inv}}^{\infty}(Y)$, will then
be a family of pseudodifferential operators acting on the fibers of $Y \times B \to \mathfrak{g}^*$ (here $\mathfrak{g}^*$ is the dual of the vector bundle $\mathfrak{g}$):

\begin{equation}
\hat{A}(\tau) \in \Psi^\infty(Y_b/\mathcal{G}_b), \quad \text{if } \tau \in \mathfrak{g}_b^*.
\end{equation}

The action of $GL_q(\mathbb{R})$ in the above lemma will have to be replaced with the group of gauge-transformations of $\mathfrak{g}$.

We can look at the general abelian $\mathcal{G}$ from the point an other, related point of view also. Let $Z := Y/\mathcal{G}$, which is again a fiber bundle $Z \to B$. The algebra $\psi^\infty_{\text{inv}}(Z)$ of smooth families of pseudodifferential operators along the fibers of $Z \to B$ can be regarded as the algebra of sections of a vector bundle on $Z$ (with infinite dimensional fibers). We can lift this vector bundle to $\mathfrak{g}^*$ and then $\hat{A}$ is a section of this lifted vector bundle over $\mathfrak{g}^*$, and we write this as follows:

\begin{equation}
\hat{A} \in \Gamma(\mathfrak{g}^*, \psi^\infty_{\text{inv}}(Z)).
\end{equation}

The considerations of this section extend immediately to operators acting between sections of a $\mathcal{G}$-equivariant vector bundles. If $E_0$ and $E_1$ are $\mathcal{G}$-equivariant vector bundles, we denote by $\psi^m_m(Y; E_0, E_1)$ the space of $\mathcal{G}$-invariant families of order $m$ pseudodifferential operators acting on sections of $E_0$, with values sections of $E_1$, whose convolution kernels have compact support.

2. The equivariant family index: Definition

We now define study invariants of elliptic operators in $M_N(\psi^\infty_{\text{inv}}(Y))$, the main invariant being the (equivariant family) index of such an invariant, elliptic family. For $\mathcal{G}$ consisting of simply connected, solvable Lie groups and $\dim Y > \dim \mathcal{G}$, we then show that the (equivariant family) index gives the obstruction for family $A = (A_b) \in M_N(\psi^m_{\text{inv}}(Y))$ to have a perturbation by a family $R = (R_b) \in M_N(\psi^{-\infty}_{\text{inv}}(Y))$, such that $A + R = (A_b + R_b)$ be invertible, for all $b \in B$, between suitable Sobolev spaces, see Theorem 1. For families of abelian Lie groups $\mathcal{G}$, we give an interpretation of the index of an elliptic operator in terms of its indicial family. This leads to an Atiyah-Singer index type formula for the Chern character of the index of a family of invariant, elliptic operators. If $\mathcal{G}$ is abelian (that is, if its fibers are abelian Lie groups), then we can consider the algebra $\Psi^\infty_{\text{inv}}(Y)$ instead of $\psi^\infty_{\text{inv}}(Y)$.

We now proceed to define the index of an elliptic family $A \in \psi^m_{\text{inv}}(Y)$. This will be done using the $K$-theory of Banach algebras. Let $C^*_r(Y, \mathcal{G})$ be the closure of $\psi^{-\infty}_{\text{inv}}(Y)$ with respect to the norm

\[ \|A\| = \sup_{b \in B} \|A_b\| \]

each operator $A_b$ acting on the Hilbert space of square integrable densities on the fiber $Y_b$. If $Y = \mathcal{G}$, then we also write $C^*_r(\mathcal{G}, \mathcal{G}) = C^*_r(\mathcal{G})$, $C^*_r(\mathcal{G})$ is the reduced $C^*$-algebra associated to $\mathcal{G}$, regarded as a groupoid. For each locally compact space $X$, we denote by $C_0(X)$ the space of continuous functions vanishing at infinity on $X$. Then, if $\mathcal{G}$ is abelian, we have $C^*_r(\mathcal{G}) \simeq C_0(\mathcal{G}^*)$.

We shall use below $\otimes$, the minimal tensor product of $C^*$-algebras. This minimal tensor product is defined to be (isomorphic to) the completion of the image of $\pi_1 \otimes \pi_2$, the tensor product of two injective representations $\pi_1$ and $\pi_2$. For the cases we are interested in, the minimal and the maximal tensor product coincide.\[ \text{\underline{\text{\footnotesize{30}}}} \]
Lemma 2. Assume \( \dim Y > \dim G \). Also, let \( K = K(Y_b/G_b) \) denote the algebra of compact operators on one of the fibers \( Y_b/G_b \), for some fixed but arbitrary \( b \in B \). Then

\[
C^*_r(Y, G) \cong C^*_r(G) \otimes K.
\]

Consequently, \( K_i(C^*_r(Y, G)) \cong K_i(C^*_r(G)) \).

Proof. Let \( \mathfrak{A} \) be the space of sections of the bundle of algebras \( K(Y_b/G_b) \). If \( Y \) is a flat \( G \)-space, then the isomorphism \( C^*_r(Y, G) \cong C^*_r(G) \otimes K \) follows, for example, from the results of [22]. In general, this local isomorphism gives \( C^*_r(Y, G) \cong C^*_r(G) \otimes C_0(B) \otimes K \).

Our assumptions imply that \( K \) is infinite dimensional, and hence its group of automorphisms is contractible, see [4]. Consequently, there is no obstruction to trivializing the bundle of algebras \( K(Y_b/G_b) \), and hence \( \mathfrak{A} \cong C_0(B) \otimes K \). We obtain

\[
C^*_r(Y, G) \cong C^*_r(G) \otimes C_0(B) \otimes \mathfrak{A} \cong C^*_r(G) \otimes C_0(B) \otimes K \cong C^*_r(G) \otimes K.
\]

The last part of lemma follows from the above results and from the natural isomorphism \( K_i(A \otimes K) \cong K_i(A) \), valid for any C*-algebra \( A \). \( \square \)

We proceed now to define the index of an elliptic, invariant family of operators

\[
A = (A_b) \in M_N(\psi_{\text{inv}}^m(Y)) = \psi_{\text{inv}}^m(Y; \mathbb{C}^N).
\]

We assume first that \( Y/G \) is compact, for simplicity; otherwise, we need to use algebras with adjoint units. We observe that there exists an exact sequence

\[
(10) \quad 0 \to C^*_r(Y, G) \to E \to C^\infty(S^*_{\text{vert}} Y) \to 0, \quad E := \psi_{\text{inv}}^0(Y) + C^*_r(Y, G),
\]

obtained using the results of [22]. The operator \( A \) (or, rather, the family of operators \( A = (A_b) \)) has an invertible principal symbol, and hence the family \( T = (T_b) \)

\[
T_b := (1 + A_b^* A_b)^{-1/2} A_b,
\]

consists of elliptic, invariant operators. Moreover, \( T \) is an element of \( E : = \psi_{\text{inv}}^0(Y) + C^*_r(Y, G) \), its principal symbol is still invertible, and hence defines a class \( [T] \in K_1(C^\infty(S^*_{\text{vert}} Y)) \cong K^1(S^*_{\text{vert}} Y) \).

Let

\[
\delta : K_1^\text{alg}(S^*_{\text{vert}} Y) \to K_0^\text{alg}(C^*_r(Y, G)) \cong K_0(C^*_r(Y, G))
\]

be the boundary map in the K-theory exact sequence

\[
K_1^\text{alg}(C^*_r(Y, G)) \to K_1^\text{alg}(E) \to K_1^\text{alg}(S^*_{\text{vert}} Y) \xrightarrow{\delta} K_0^\text{alg}(C^*_r(Y, G)) \to K_0^\text{alg}(E) \to K_0^\text{alg}(S^*_{\text{vert}} Y)
\]

associated to the exact sequence (10). Because \( K_0(C^*_r(Y, G)) \cong K_0(C^*_r(G)) \), by Lemma 4, we get a group morphism

\[
(11) \quad \text{ind}_a : K_1^\text{alg}(C^\infty(S^*_{\text{vert}} Y)) \to K_0(C^*_r(G)),
\]

which we shall call the analytic index morphism. The image of \( A \) under the composition of the above maps is \( \text{ind}_a([T]) \), will be denoted \( \text{ind}_a(A) \), and will be called the analytic index of \( A \). A more direct but longer definition is contained in the proof of Theorem 1 below (see Equation (13)). The analytic index morphism descends in this case to a group morphism \( K_1^\text{alg}(C^\infty(S^*_{\text{vert}} Y)) \to K_0(C^*_r(G)) \) denoted in the same way. For \( G \) abelian, we can replace \( E \) with \( \Psi_{\text{inv}}^0(Y) \) and \( C^*_r(G) \) with \( \Psi_{\text{inv}}^{-1}(Y) \).

We denote by \( I_N \) the unit of the matrix algebra \( M_N(E) \). If \( A \in \psi_{\text{inv}}^0(Y; E) \), we can find \( N \) large such that \( \psi_{\text{inv}}^0(Y; E) \subset M_N(\psi_{\text{inv}}^0(Y)) \) with 1 mapping to the
projector $e$ under this isomorphism. Then $A + (I_N - e)$ is invertible in this matrix algebra, and we define then $\text{ind}_a(A) = \text{ind}_a(A + I_N - e)$. If $Y/G$ is not compact, this definition of the index applies only to elliptic operators in $I_N + M_N(\psi^0_\text{inv}(Y; E))$. All results below extend to these operators, after obvious changes.

For differential operators acting between sections of different bundles, we can define the analytic index using the adiabatic groupoid of $G$ as in [22]. For arbitrary elliptic pseudodifferential operators $A \in \psi^m_\text{inv}(Y; E_0, E_1)$, acting between sections of possibly different vector bundles, we can define the index using Kasparov’s bivariant $K$-Theory. Since for $Y/G$ non-compact, no element in $\psi^m_\text{inv}(Y; E_0, E_1)$ is invertible modulo regularizing operators, we must allow more general operators in this case. For example, we can take $A \in \Gamma(Y, \text{Hom}(E_0, E_1))^{\sigma} + \psi^0_\text{inv}(Y; E_0, E_1)$, provided that it is bounded. If $A \in \psi^m_\text{inv}(Y; E_0, E_1)$, $m > 0$, we replace $A$ by $(1 + AA^*)^{-1/2}A$, which will be an operator in the closure of $\psi^0_\text{inv}(Y; E_0, E_1)$, by the results of [23]. If $A$ is elliptic, then the resulting family defines by definition an element of $KK(\mathbb{C}, C^*(G))$. To obtain the index, we use the isomorphism $KK(\mathbb{C}, C^*(G)) \cong K_0(C^*(G))$. This definition has the disadvantage that we must always work with $C^*(G)$, the closure of $\psi^0_\text{inv}(Y)$, and we cannot use $\Psi^\infty_\text{inv}(G)$.

The main property of the analytic index of an operator $A$ is that it gives the obstruction to the existence of invertible perturbations of $A$ by lower order operators. We denote by $H^s(Y_b)$ the $s$th Sobolev space of $1/2$-densities on $Y_b$, which is uniquely defined because of the bounded geometry of $Y_b$, for $Y/G$ compact. More precisely, $H^s(Y_b)$ is, by definition, the domain of $(1 + D^*D)^{s/2m}$, if $D \in \psi^m_\text{inv}(Y)$ is elliptic and $s \geq 0$. For $s \leq 0$, $H^s(Y_b)$ is, by definition, the dual of $H^{-s}(Y_b)$.

**Theorem 1.** Let $G \to B$ be a bundle of Lie groups acting on the fiber bundle $Y \to B$, as above, and assume that $Y/G$ is compact, of positive dimension. Let $A \in \psi^m_\text{inv}(Y, \mathbb{C}^N)$ be an elliptic operator. Then we can find $R \in \psi^{m-1}_\text{inv}(Y, \mathbb{C}^N)$ such that

$$A_b + R_b : H^s(Y_b)^N \to H^{s-m}(Y_b)^N$$

is invertible for all $b \in B$ if, and only if, $\text{ind}_a(A) = 0$. Moreover, if $\text{ind}_a(A) = 0$, then we can choose $R \in \psi^\infty_\text{inv}(Y)$. The same result holds if $Y/G$ is non-compact and $A \in \Gamma(Y, \text{Hom}(E_0, E_1))^{\sigma} + \psi^0_\text{inv}(Y; E_0, E_1)$ is elliptic and bounded.

**Proof.** It is clear from definition that if we can find $R$ with the desired properties, then $\text{ind}_a(A) = 0 \in K^0(\mathfrak{g}^*)$. Suppose now that $A \in \psi^m_\text{inv}(Y, \mathbb{C}^N)$ is elliptic and has vanishing analytic index. Using the notation $T = (1 + AA^*)^{-1/2}A$, we see that $A_b$ is invertible between the indicated Sobolev spaces if, and only if, $T_b$ is invertible as a bounded operator on $L^2(Y_b)^N$. We can hence assume that $m = 0$ and $T = A$.

Because $C^*_r(Y, G)$ satisfies $C^*_r(Y, G) \simeq C^*_r(G) \otimes K$, by Lemma 2, we can use some general techniques to prove that the vanishing of $\text{ind}_a(A)$ implies that $A$ has a perturbation by invariant, regularizing operators in $\psi^{-\infty}_\text{inv}(Y, \mathbb{C}^N)$ that is invertible on each fiber. We fix an isomorphism $M_N(C^*_r(Y, G)) \cong C^*_r(G) \otimes K$. We now review this general technique using a generalization of an argument from [22]. Let $\mathcal{E}$ be the algebra introduced in Equation (10). We denote by $I_N$ the unit of the matrix algebra $M_N(\mathcal{E})$. Also, denote by $\overline{\mathcal{E}}$ the closure of $\mathcal{E}$ in norm.

Choose a sequence of projections $p_n \in K$, $\dim p_n = n$, such that $p_n \to 1$ in the strong topology. Because $A_b$ is invertible modulo $C^*_r(G_b) \otimes K$, we can find a large $n$ and $R \in M_N(\psi^{-\infty}_\text{inv}(Y))$ such that

$$A'_b := A_b \oplus R_b : L^2(Y_b)^N \oplus L^2(Y_b)^N \to L^2(Y_b)^N.$$
is surjective and \((1 \otimes p_n)R_b(1 \otimes p_n) = R_b\), for all \(b \in B\). Then \(\{0\}\) is not in the spectrum of \(A'A'^*\), and we can consider \(V := (A'A'^*)^{-1/2}A' \in M_N(\overline{E})\), which by construction satisfies \(VV^* = I_N \in M_N(\overline{E})\). Consequently, \(V^*V\) is a projection in \(M_{2N}(\overline{E})\). Because \(A\) is invertible modulo \(M_N(C^*_r(\mathcal{G}) \otimes K)\),
\[
V^*V - VV^* \in M_{2N}(C^*_r(\mathcal{G}) \otimes K).
\]
Let \(e = I_N \oplus (1 \otimes p_n) - V^*V\), which is also a projection, by construction. Moreover,
\[
e - (1 \otimes p_n) \in M_{2N}(C^*_r(\mathcal{G}) \otimes K),
\]
and hence both \(e\) and \(1 \otimes p_n\) are projection in \(M_{2N}(C^*_r(\mathcal{G})^+ \otimes K)\) (for any algebra \(B\), we denote by \(B^+\) the algebra with adjoint unit). Equation (12) gives that, by definition, \([e] - [1 \otimes p_n]\) defines a \(K\)-theory class in \(K_0(C^*_r(\mathcal{G}))\). From definition, we get then
\[
\text{ind}_a(A) = [e] - [1 \otimes p_n].
\]

Now, if \(\text{ind}_a(A) = 0\), then we can find \(k\) such that \(e \oplus 1 \otimes p_k\) is (Murray-von Neumann) equivalent to \(1 \otimes p_{n+k}\). By replacing our original choice for \(n\) with \(n + k\), we may assume that \(e\) and \(1 \otimes p_n\) are equivalent, and hence that we can find \(u \in r \mathbb{C}I_{2N} + M_{2N}(C^*_r(\mathcal{G}) \otimes K)\) with the following properties: there exists a large \(l\) and \(x \in M_{2N}(C^*_r(\mathcal{G}))\) such that, if we denote \(e_0 = 1 \otimes p_l \oplus 1 \otimes p_n\), then \(u = I_{2N} + e_0x e_0\) and \(e = u(1 \otimes p_n)u^{-1}\). Then \(Vu\) is in \(M_n(\overline{E})\) (more precisely \(I_NVI_N = V\)) and is invertible. Consequently \(B := (A'A'^*)^{1/2}\) is also invertible. But \(B\) is a perturbation of \(A'\), and hence also of \(A\), by an element in \(M_{2N}(C^*_r(\mathcal{G}) \otimes K)\). Since \(\psi_{\text{inv}}^{-\infty}(Y)\) is dense in \(C^*_r(\mathcal{G})\), this gives the result.

\section{The Index for Bundles of Solvable Lie Groups}

We now treat in more detail the case of bundles of solvable Lie groups, when more precise results can be obtained. Other classes of groups will lead to completely different problems and results, so we leave their study for the future. The class of simply-connected solvable fibers is rich enough and has many specific features, so we content ourselves from now on with this case only.

\textbf{Assumption.} From now on and throughout this paper, we shall assume that the family \(\mathcal{G}\) consists of simply-connected solvable Lie groups. By “simply-connected” we mean, as usual, “connected with trivial fundamental group.”

We shall denote by \(\mathfrak{g} \to B\) the bundle of Lie algebras of the groups \(\mathcal{G}_b\), \(\mathfrak{g}_b \simeq \text{Lie}(\mathcal{G}_b)\) and by
\[
\exp : \mathfrak{g} \to \mathcal{G}.
\]
the exponential map.

Because all the groups \(\mathcal{G}_b\) are solvable, we have that the enveloping \(C^*\)-algebra of \(\mathcal{G}\), that is, \(C^*(\mathcal{G})\) is isomorphic to the reduced \(C^*-\)algebra of \(\mathcal{G}\): \(C^*(\mathcal{G}) \simeq C^*_r(\mathcal{G})\), so we can drop the index "\(r\)" from the notation.

In order to study the algebra \(C^*_r(\mathcal{G}) = C^*(\mathcal{G})\) and its \(K\)-groups, we shall deform it to a commutative algebra. This deformation is obtained as follows. Let \(\mathcal{G}_{ad} = \{0\} \times \mathfrak{g} \cup \mathcal{G}, B_1 = [0, 1] \times B\), and \(d : \mathcal{G}_{ad} \to B_1\) be the natural projection. On \(\mathcal{G}_{ad}\) we put the smooth structure induced by
\[
\phi : B_1 \times \mathfrak{g} \to \mathcal{G}_{ad}
\]
Corollary 1. Let \( \mathcal{G} \) be a bundle of simply connected, solvable Lie groups. Then

\[
K_i(C^*(\mathcal{G})) \simeq K_i(C_0(\mathfrak{g}^*)) \simeq K^i(\mathfrak{g}^*).
\]
We now give an interpretation of $\text{ind}_a(A)$, for $\mathcal{G}$ abelian, using the properties of the indicial family $\tilde{A}(\tau)$ of $A$. We assume that $Y/\mathcal{G}$ is compact.

We shall also use the following construction. Let $X$ be a compact manifold with boundary. Let $T(x)$ be a family of elliptic pseudodifferential operators acting between sections of two smooth vector bundles, $E_0$ and $E_1$, on the fibers of a fiber bundle $M \to X$ whose fibers are compact manifolds without corners. Then we can realize the index of $T$ as an element in the relative group $K^0(X, \partial X)$. This can be done directly using Kasparov’s theory or by the “Atiyah-Singer trick” as follows. We and proceed as in [7], Proposition (2.2), to define a smooth family of maps $R(x) : C^N \to C^\infty(Y)$, such that the induced map

$$V := T \oplus R : C^\infty(X)^N \oplus C^\infty(X, L^2(Y; E_0)) \to C^\infty(X, L^2(Y; E_1))$$

is onto for each $x$. Since $T(x)$ is invertible for $x \in \partial X$, we can choose $R(x) = 0$ for $x \in \partial X$. Then $\ker(V)$ is a vector bundle on $X$, which is canonically trivial on the boundary $\partial X$. The general definition of the index of the family $T$ in [7] is that of the difference of the kernel bundle $\ker(V)$ and the trivial bundle $X \times C^\infty$. Since the bundle $\ker(V)$ is canonically trivial on the boundary of $X$, we obtain an element

$$\text{deg}(T) \in K^0(X, \partial X).$$

The degree is invariant with respect to homotopies $T_t$ of families of operators on $X$ that are invertible on $\partial X$ throughout the homotopy. We shall use the degree in Theorem 3 for $X = B_R$, a large closed ball in $\mathcal{G}^* \simeq \mathfrak{g}^*$, or for $X$ being the radial compactification of $\mathfrak{g}^*$. If the boundary of $X$ is empty, this construction goes back to Atiyah and Singer and then $\text{deg}(T)$ is simply the family index of $T$. If $\partial X$ is not empty, this definition of $\text{deg}(T)$ is due to Melrose. We note that when $\partial X \neq \emptyset$, the degree is not a local quantity in $T$, in the sense that it depends on more than just the principal symbol.

Assume now that the family $T$ above consists of order zero operators and $T(x)$ is a multiplication operator for each $x \in \partial X$. We want to compute the Chern character of $\text{deg}(T)$ using the Atiyah-Singer family index formula [4, 5]. To introduce the main ingredients of the index formula, denote by $S^*_{\text{vert}}M$ the set of unit vectors in the dual of the vertical tangent bundle $T^*_{\text{vert}}M$ to the fibers of $M \to X$. Because the family $T$ is elliptic, the principal symbols define an invertible matrix of functions

$$a = \sigma_0(T) \in C^\infty(S^*_{\text{vert}}M; \text{Hom}(E_0, E_1)).$$

Since the operators $T(x)$ are multiplication operators, we can then extend $a$ to an invertible endomorphism on $S_M := S^*_{\text{vert}}M \cup B^*$, with $B^*$ denoting the set of vertical cotangent vectors of length $\leq 1$ above $\partial X$, as in [5]. The constructions of [4, 5] are in terms of $[a^*] \in K^0(T_{\text{vert}}^*M, T_{\text{vert}}^*M|_{\partial M})$ obtained by applying the clutching (or difference) construction to $a$. Explicitly, $[a^*]$ is represented by $(E_0, E_1, a_1)$ (where $a_1$ is a smooth function that coincides with $a$ outside a neighborhood of the zero section). It defines an element in

$$K^0(T_{\text{vert}}^*M, T_{\text{vert}}^*M|_{\partial M}) = K^0(T_{\text{vert}}^*M \times T_{\text{vert}}^*M|_{\partial M}),$$

because $a$ defines an endomorphism of the trivial bundle $C^N$ which is invertible outside a compact set (see [4]).

When $E_0 \cong E_1$, we can assume that $E_0 = E_1 = M \times C^N$ are trivial of rank $N$, and we have that $a$ is an invertible matrix valued function, which hence defines an element $[a] \in K^1(S_M)$. Let $B_1$ be the set of vectors of norm at most 1 in $T^*M$. 
After the identification (up to homeomorphism) of $B_1 \setminus S_M$, the interior of $S_M$, with the difference $T^*_{vert} M \setminus T^*_{vert} M|_{\partial M}$, we have

$$[a'] = \partial[a].$$

We denote by $\pi_* : H^*(S_M) \to H^{*+2n+1}(X, \partial X)$ the integration along the fibers, where $n$ is the dimension of the fibers of $M \to X$. Integration along the fibers in this case is the composition of

$$\partial : H^*(S_M) \to H^{*+1}(B_1, S_M) \cong H^{*+1}(T^*_{vert} M \setminus T^*_{vert} M|_{\partial M})$$

and

$$\tilde{\pi}_* : H^*(T^*_{vert} M \setminus T^*_{vert} M|_{\partial M}) \to H^{*-2n}(X, \partial X)$$

obtained by integration along the fibers of the bundle $T^*_{vert} M \setminus T^*_{vert} M|_{\partial M} \to X \setminus \partial X$:

$$\pi_* = \tilde{\pi}_* \circ \partial.$$

To state the following result, we also need $Ch : K_1(S_M) \to H^{odd}(S_M)$, the Chern character in $K$-Theory and $T$, the Todd class of $(T^*_{vert} M) \otimes \mathbb{C}$, the complexified vertical tangent bundle of the fibration $M \to X$, as in \[.] Use the notation introduced above, we have:

**Theorem 2.** Let $M \to X$ be a smooth fiber bundle whose fibers are smooth manifolds (without corners), and let $T$ be a family of order zero elliptic pseudodifferential operators acting along the fibers of $M \to X$. Assume $X$ is a manifold with boundary $\partial X$ such that the operators $T(x)$ are multiplication operators on $\partial X$, also let $[a']$ and $[a]$ be the classes defined above. Then

$$Ch(\deg(T)) = (-1)^n \tilde{\pi}_*(Ch[a']|T) \in H^*(X, \partial X),$$

If $E_0 \cong E_1$, then we also have $Ch(\deg(T)) = (-1)^n \pi_*(Ch[a]|T)$.

**Proof.** For continuous families $T(x)$ that are multiplication operators on the boundary $\partial X$, the degree is a local quantity – it depends only on the principal symbol – so we can follow word for word \[.] to prove that

$$Ch(\deg(T)) = (-1)^n \tilde{\pi}_*(Ch[a']|T) \in H^*(X, \partial X).$$

When $E_0 \cong E_1$, using $Ch[a'] = Ch(\partial[a]) = \partial Ch[a]$, we get

$$Ch(\deg(T)) = (-1)^n \tilde{\pi}_*(\partial Ch[a]|T) = (-1)^n \tilde{\pi}_* \circ \partial(Ch[a]|T)$$

$$= (-1)^n \pi_*(Ch[a]|T) \in H^*(X, \partial X).$$

This completes the proof.

It is interesting to note that it is not possible in general to give a formula for $Ch(\deg(T))$ only in terms of its principal symbol, without further assumptions on $T$. A consequence is that, in general, the formula for $Ch(\deg(T))$ will involve some non-local invariants. It would be nevertheless useful to find such a formula.

Returning to our considerations, we continue to assume that $Z := Y/G$ is compact, and we fix a metric on $G$ (which, we recall, is a vector bundle in these considerations). If $A \in \Psi^\infty(Y; E_0, E_1)$ is elliptic (in the sense that its principal symbol is invertible outside the zero section), then the indicial operators $\hat{A}(\tau)$ are invertible for $|\tau| \geq R, \tau \in G^*$, and some large $R$. In particular, by restricting the family $A$ to the ball

$$B_R := \{|\tau| \leq R\},$$
we obtain a family of elliptic operators that are invertible on the boundary of $B_R$, and hence $\hat{A}$ defines an element
\begin{equation}
\deg_G(\hat{A}) := \deg(\hat{A}) \in K^0(B_R, \partial B_R) \simeq K^0(G^*)
\end{equation}
in the $K$-group of the ball of radius $R$, relative to its boundary, as explained above, called also the degree of $A$.

We want a formula for the Chern character of the degree of $A \in \Psi_{\text{inv}}^\infty(Y; E_0, E_1)$. Because $A$ is elliptic, its principal symbol defines a class $[a'] \in K^0((T^*_\text{vert}Y)/G)$. If $E_0$ and $E_1$ are isomorphic, then it also defines a class $[a] \in K^0((S^*_\text{vert}Y)/G)$. Denote by $n$ the dimension of the quotient $Z_b = Y_b/G_b$ (which is independent of $b$ because we assumed $B$ connected), and let $\pi : (S^*_\text{vert}Y)/G \to B$ be the projection and
\begin{equation}
\begin{aligned}
\tilde{\pi}_* : H^*((T^*_\text{vert}Y)/G) &\to H_c^{*-2n}(g^*) \\
\pi_* : H^*((S^*_\text{vert}Y)/G) &\to H_c^{*-2n+1}(g^*)
\end{aligned}
\end{equation}
be the integration along the fibers in cohomology. Then
\[
\text{Ch}(\deg_G(\hat{A})) \in H^*_c(g^*) \simeq H^{**n}(B, O),
\]
and the following theorem gives a formula for this cohomology class in terms of the classes $[a']$ or $[a]$ defined above.

We denote by $T$ the Todd class of the vector bundle $(T^*_\text{vert}Y)/G \otimes \mathbb{C} \to Y/G$. We assume $B$ to be compact.

**Theorem 3.** If $A \in \Psi_{\text{inv}}^\infty(Y; E_0, E_1)$ is elliptic, then the Chern character of $\deg_G(\hat{A})$ is given by
\[
\text{Ch}(\deg_G(\hat{A})) = (-1)^n \tilde{\pi}_* (\text{Ch}([a']T)) \in H^*_c(g^*),
\]
Moreover, $\text{Ch}(\deg_G(\hat{A})) = (-1)^n \pi_* (\text{Ch}([a]T))$ if $E_0 \cong E_1$.

**Observations.** It is almost always the case that $E_0 \cong E_1$. For example, it is easy to see that this must happen if the Euler characteristic of $(T^*_\text{vert}Y)/G$ vanishes. Since $G = B \times \mathbb{R}^q$ in many applications, this assumption is satisfied if $q > 0$.

Another observation is that if the set of elliptic elements in $\Psi_{\text{inv}}^\infty(Y; E_0, E_1)$ is not empty, $Z := Y/G$ is compact.

**Proof.** We cannot use Theorem 4 directly because our family $\hat{A}$ does not consist of multiplication operators on the boundary. Nevertheless, we can deform $\hat{A}$ to a family of operators that are multiplication operators at $\infty$, for suitable $A$. We now construct this deformation.

Let $E$ be a vector bundle over $V$. We consider classical symbols $S^m_\text{cl}(E)$ whose support projects onto a compact subset of $V$. Let
\[
A_V := (T^*_\text{vert}Y)/G \cong T^*_\text{vert}Z \times_B G.
\]
First we need to define a nice quantization map $q : S^m(A_V^*) \to \psi^m_{\text{inv}}(Y)$. To this end, we proceed as usual, using local coordinates, local quantization maps, and partitions of unity, but being careful to keep into account the extra structure afforded by our settings: the fibration over $B$ and the action of $G$. Here are the details of how this is done.

Fix a cross section for $Y \to Z := Y/G$ and, using it, identify $Y$ with $Z \times_B G$ as $G$-spaces. Denote by $p_0 : Z \to B$ the natural projection. We cover $B$ with open sets $U_{\alpha'}$ that are diffeomorphic to open balls in a Euclidean space such that
Denote by \( q_\alpha : S^m(A_V^*|W_\alpha) \to \psi^m_{\text{inv}}(W_\alpha \times_B \mathcal{G}), \quad q_\alpha(a) = a(b, x, D_x, D_t), \)
where we identify
\[
S^m(A_V^*|W_\alpha) = S^m(T^*_{\text{vert}}(W_\alpha) \times_B \mathcal{G}) = S^m(U_{\alpha'} \times T^*V_{\alpha''} \times \mathbb{R}^q).
\]
Denote by \( b \in U_{\alpha'}, (x, y) \in T^*V_{\alpha''} \cong V_{\alpha''} \times \mathbb{R}^n, \) and \( \tau \in \mathbb{R}^q \) the corresponding coordinate maps. Then \( q_\alpha(a) = a(b, x, D_x, D_t) \) acts on \( C_c^\infty(U_{\alpha'} \times T^*V_{\alpha''} \times \mathbb{R}^q) \) as
\[
a(b, x, D_x, D_t)u(b, x, t) = (2\pi)^{-n-q} \int_{\mathbb{R}^{n+q}} \left( \int_{\mathbb{R}^{n+q}} e^{i(x-z)\cdot y + i(t-s)\cdot \tau} a(b, x, y, \tau)u(b, z, s)dsdz \right)dydr.
\]
Choose now a partition of unity \( \phi_\alpha^2 \) subordinated to \( W_\alpha \) and let
\[
q(a) = \sum_\alpha q_\alpha(\phi_\alpha a)\phi_\alpha.
\]
The main properties of \( q \) are the following:
1. if \( a \) has order \( m \), then \( \sigma_m(q(a)) = a, \) modulo symbols of lower order;
2. there exist maps \( q_b : S^m(T^*Z_b) \to \Psi^m(Z_b) \) such that
\[
\hat{q}(a)(\tau) = q_b(a(\cdot, \tau)) = a(b, x, \tau + D_x), \quad \text{if } \tau \in g_b^* \text{ and } x \in Z_b;
\]
3. the maps \( q_b \) define a quantization map
\[
\hat{q} : \Gamma(g^*, S^m(T^*_{\text{vert}}Z)) \to \Gamma(g^*, \Psi^m(Z)),
\]
where we regard \( \Psi^m(Z_b) \) as defining a bundle of algebras, \( \Psi^m(Z) \), on \( B \), first, and then on \( g^* \), by pull-back. Similarly, we regard \( S^m(T^*_{\text{vert}}Z) \) as defining a bundle over \( Z \), which we then pull back to a bundle on \( g^* \). (See also the discussion related to Equations (3) and (4).)

The deformation of our family is obtained as follows. Let \( |\tau| \) and \( |y| \) be the norms \( \tau \in g^* \) and \( y \in T^*_{\text{vert}}Z \). Define then
\[
\phi_\lambda^2(y, \tau) = 1 + \lambda|y|^2 + \lambda(1 + \lambda|\tau|^2)^{-1}|y|^2 \quad \text{and} \quad \psi_\lambda^{-2}(y, \tau) = 1 + \lambda|\tau|^2,
\]
which are chosen to satisfy \( 1 + \phi_\lambda^2|\tau|^2 + \psi_\lambda^2|y|^2 = 1 + |\tau|^2 + |y|^2 + \lambda|\tau|^2|y|^2 \). For any symbol \( a \in S^0(A_V^*), A_V^* = T^*_{\text{vert}}Z \times_B g^* \), we let
\[
a_{\lambda, \tau}(y) = a(\psi_\lambda y, \phi_\lambda \tau), \quad \lambda \in [0, 1], \quad \tau \in g_b^*, \quad \text{and} \quad y \in T^*Z_b.
\]
We can define then \( A_\lambda(\tau) := q_b(a_{\lambda, \tau}) \), which is the same as saying that \( A_\lambda = \hat{q}(a_{\lambda, \tau}) \), and these operators will satisfy the following properties:

1. For each fixed \( \lambda \), the operators \( A_\lambda(\tau) \) define a section of \( \Psi^m(Z) \) over \( g^* \) and these sections depend smoothly on \( \lambda \) (in other words, \( A_\lambda(\tau) \) depends smoothly on both \( \lambda \) and \( \tau \), in any trivialization);
2. \( A_0(\tau) = q(a(\tau)) \), for all \( \tau \);
3. For each nonzero \( \tau' \in g^* \) and \( \lambda > 0 \), the limit \( \lim_{t \to \infty} A_\lambda(t\tau') \) exists and is a multiplication operator;
4. If \( a, b \in S^0(A^*_\tau) \) are such that \( ab = 1 \), \( a \) is homogeneous of order zero outside the unit ball, and if we define \( A_\lambda := \tilde{q}(a_{\lambda, \tau}) \) and \( B_{\lambda, \tau} := \tilde{q}(b_{\lambda, \tau}) \), then there exists a constant \( C > 0 \) such that
\[
\|A_\lambda(\tau)B_\lambda(\tau) - 1\| \leq C(1 + |\tau|)^{-1}
\]
and similarly
\[
\|B_{\lambda, \tau} - 1\| \leq C(1 + |\tau|)^{-1}
\]
For all \( \tau \) and \( \lambda \).

5. All these estimates extend in an obvious way to matrix valued symbols.

These properties are proved as follows. We first recall that, for any vector bundle \( E \), we can identify the space of classical symbols \( S^0(E) \) with \( C^\infty_c(E_1) \), the space of compactly supported functions on \( E_1 \), the unit ball of \( E \), by \( E_1 \setminus \partial E_1 \ni \xi \to (1 - \|\xi\|^2)^{-1}\xi \in E \). For any quantization map, the norm of the resulting operator will depend on finitely many derivatives. Because we can extend \( a_{\lambda, \tau} \) to a smooth function on the radial compactification of \( A^*_\tau \), the first property follows. The second property is obvious. The third property is obtained using the same argument and observing that, for \( \lambda > 0 \), we can further extend our function to the radial compactification in \( \lambda \) also. By investigating what this limit is along various rays, we obtain the third property.

The fourth property is obtained using the following observation: there exist a constant \( C > 0 \) and seminorms \( \|\cdot\|_0 \) on \( S^0_c(T^*\mathbb{R}^n) \) and \( \|\cdot\|_{-1} \) on \( S^{-1}(T^*\mathbb{R}^n) \) such that, for any symbols \( a, b \in S^0_c(T^*\mathbb{R}^n) \),
\[
\|(ab)(x, D_x) - a(x, D_x)b(x, D_x)\| \leq \sum_j C(\|a\|_0\|\partial_{y_j} b\|_{-1} + \|\partial_{y_j} a\|_{-1}\|b\|_0),
\]
\( \partial_{y_j} \) being all derivatives in the symbolic directions (whose coordinates are denoted by \( y \)). Finally, the fifth property is obvious.

We now turn to the proof of the formula for the degree of \( A \) stated in our theorem. We prove it by a sequence of successive reductions, using the facts established above. First, it is easy to see that \( \text{deg}_q(A) \) depends only on its principal symbol, and hence we can assume that \( A \) has order zero and \( A = q(a) \), where \( a = \sigma_0(A) \).

The above deformation can be used to prove our theorem as follows. Fix \( R \) large enough, and restrict the families \( A_\lambda \) to the closed ball of radius \( R \) in \( \mathfrak{g}^* \). For \( |\tau| = R \) large enough, all operators \( A_\lambda(\tau) \), \( \lambda \in [0, 1] \) are invertible, so the degree \( \text{deg}(A_\lambda) \) of these families is defined and does not depend on \( \lambda \) or \( R \), provided that \( R \) is large enough. Since \( \text{deg}_q(A) = \text{deg}(A_0) \), by definition, it is enough to compute \( \text{deg}(A_\lambda) \), for any given \( \lambda \). Choose then \( \lambda > 0 \) arbitrary, and let \( R \to \infty \). Then the family \( A_\lambda \) extends to a continuous family on the radial compactification of \( \mathfrak{g}^* \), which consists of multiplication operators on the boundary. Moreover, the symbol class of \( A_\lambda \) is nothing but the extension of \( a_{\lambda, \tau}(\tau) \) to the radial compactification in \( \tau \) and \( \lambda \) (which is a manifold with corners of codimension two).

We can use then Theorem 2 to conclude that
\[
\text{deg}(A_\lambda) = (-1)^n \tilde{\pi}_* (\text{Ch}[a_{\lambda}]T) \in H^*_c(\mathfrak{g}^*).
\]
But \( a_\lambda \) is homotopic to \( a \) through symbols that are invertible outside a fixed compact set, so \( [a_{\lambda}] = [a] \). We get
\[
\text{deg}(A) = (-1)^n \tilde{\pi}_* (\text{Ch}[a]T) \in H^*_c(\mathfrak{g}^*).
\]
To obtain the second form of our formula for $E_0 \cong E_1$, and thus finish the proof, we proceed as at the end of the proof of Theorem 3 using $Ch[\varphi] = \partial Ch[\varphi].$  

To prove the following result, we shall use terminology from algebraic topology: if $I_k \subset A_k$ are two-sided ideal of some algebras $A_0$ and $A_1$ and $\phi : A_0 \to A_1$ is an algebra morphism, we say that $\phi$ induces a morphism of pairs $\phi : (A_0, I_0) \to (A_1, I_1)$ if, by definition, $\phi(I_0) \subset I_1$.

**Theorem 4.** Let $G$ be a bundle of abelian Lie groups and $A \in \Psi^m_{\text{inv}}(Y; \mathbb{C}^N)$ be an elliptic element. Then

$$\text{ind}_a(A) = \deg_g(A) \in K^0(\mathfrak{g}^*).$$

**Proof.** Let $B_R = \{|r| \leq R\} \subset \mathfrak{g}^*$ be as above. The algebra

$$\mathfrak{A}_R := C^\infty(B_R, \Psi^\infty(Y_b))$$

of $C^\infty$-families of pseudodifferential operators on $B_R$ acting on fibers of $Y \times_B B_R \to B_R$, contains as an ideal $\mathcal{I}_R = C^\infty_0(B_R, \Psi^{-\infty}(Y_b)),$ the space of families of smoothing operators that vanish to infinite order at the boundary of $B_R$. If $A$ is an elliptic family, as in the statement of the lemma, and if $R$ is large enough, then $A$, the indicial family of $A$, defines by restriction an element of $M_N(\mathfrak{A}_R)$ that is invertible modulo $M_N(\mathcal{I}_R)$.

Recall that the boundary map $\partial$ in algebraic $K$-theory associated to the ideal $\mathcal{I}_R$ of the algebra $\mathfrak{A}_R$ gives $\partial_1[A] = \deg_g(A)$, by definition. Also, the boundary map $\partial_0$ in algebraic $K$-theory associated to the ideal $\Psi^{-\infty}(Y)$ of the algebra $\Psi^{-\infty}(Y)$ gives $\partial_0[A] = \text{ind}_a(A)$. We want to prove that $\partial_1[A] = \partial_0[A]$. The desired equality will follow by a deformation argument, which involves constructing an algebra smoothly connecting the ideals $\mathcal{I}_R$ and $\Psi^{-\infty}(Y)$.

Consider inside $C^\infty([-1, 1], \Psi^{-\infty}(Y))$ the subalgebra of families $T = (T_x)$ such that $T_x(\tau) = 0$ for $|\tau| \geq x^{-1}$. (In other words, $T_x \in \mathcal{I}_{x^{-1}}$, if $x \neq 0$, and $T_0$ is arbitrary.) Denote this subalgebra by $\mathcal{I}_{R, x}$. Also, let $\mathfrak{A}_{R, x}$ be the set of families $A = (A_x), x \in [-1, 1], A_x \in \mathfrak{A}_{x^{-1}}$, if $x \neq 0, A_0 \in \Psi^{-\infty}(Y)$ arbitrary such that the families $AT := (A_x T_x)$ and $TA := (T_x A_x)$ are in $\mathcal{I}_{R, x}$, for all families $T = (T_x) \in \mathcal{I}_{R, x}$.

It follows that $\mathcal{I}_{R, x}$ is a two-sided ideal in $\mathfrak{A}_{R, x}$ and that the natural restrictions of operators to $x = R^{-1}$ and, respectively, to $x = 0$, give rise to morphisms of pairs

$$e_1 : (\mathfrak{A}_{R, x}, \mathcal{I}_{R, x}) \to (\mathfrak{A}_R, \mathcal{I}_R),$$

and

$$e_0 : (\mathfrak{A}_{R, x}, \mathcal{I}_{R, x}) \to (\Psi^{-\infty}(Y), \Psi^{-\infty}(Y)).$$

Moreover, the indicial family of the operator $A$ gives rise, by restriction to larger and larger balls $B_r$, to an invertible element in $\mathfrak{A}_{R, x}$, also denoted by $A$. Let $\partial$ be the boundary map in algebraic $K$-theory associated to the pair $(\mathfrak{A}_{R, x}, \mathcal{I}_{R, x})$. Then $(e_0)_*, \partial[A] = \partial_0[A]$ and $(e_1)_*, \partial[A] = \partial_1[A]$. Since $(e_0)_* : K_0(\mathcal{I}_{R, x}) \to K_0(\Psi^{-\infty}(Y))$ and $(e_1)_* : K_0(\mathcal{I}_{R, x}) \to K_0(\mathcal{I}_R)$ are natural isomorphisms, our result follows.

We now drop the assumption above that $G$ consist of abelian Lie groups, assuming instead that $G$ consists of simply-connected solvable Lie groups, and want to compute the Chern character of the analytic index $\text{ind}_a(A)$, for an elliptic family $A \in \psi^m_{\text{inv}}(Y)$. One difficulty that we encounter is that the space on which the principal symbols are defined, that is $(S^*_\text{vert} Y)/G$, is not orientable in general. (Recall that $S^*_\text{vert} Y$ is the space of vectors of length one of $T^*_\text{vert} Y$, the dual of the vertical tangent bundle $T_{\text{vert}} Y$ to the fibers of $Y \to B$.)
We denote by $T$ the Todd class of the vector bundle $(T_{\text{vert}}Y)/\mathcal{G} \otimes \mathbb{C} \to Y/\mathcal{G}$ and by $\pi_*$ the integration along the fibers of $(S'_{\text{vert}}Y)/\mathcal{G} \to B$, as above. We assume $B$ to be compact.

**Theorem 5.** Let $\mathcal{G}$ be a bundle of Lie groups whose fibers are simply-connected, solvable Lie groups. Let $A \in \psi^m_{\text{inv}}(Y, \mathbb{C}^N)$ be an elliptic, invariant family, and let $[\sigma_m(A)] \in K^1((S'_{\text{vert}}Y)/\mathcal{G})$ be the class defined by the principal symbol $\sigma_m(A)$ of $A$. Then the Chern character of the analytic index of $A$ is given by

$$Ch(\text{ind}_a(A)) = (-1)^n \pi_*(Ch[\sigma_m(A)]T) \in H^*_c(\mathfrak{g}^*),$$

where $n$ is the dimension of the fibers of $(S'_{\text{vert}}Y)/\mathcal{G} \to B$.

**Proof.** Note first that we can deform the bundle of Lie groups $\mathcal{G}$ to the bundle of commutative Lie groups $\mathfrak{g}$ as before, using $\mathcal{G}_{\text{ad}}$. Moreover, we can keep the principal symbol of $A$ constant along this deformation. This shows that we may assume $\mathcal{G}$ to consist of commutative Lie groups, i.e. that $\mathcal{G}$ is a vector bundle. The result then follows from Theorems 3 and 4. \hfill \Box

**Observations.** We can extend the above theorems in several ways. First, we can drop the assumption that $Z = Y/\mathcal{G}$ be compact, but then we need to consider bounded, elliptic elements $A \in \text{Hom}(E_0, E_1) + \psi^o_{\text{inv}}(Y; E_0, E_1)$ (or, if $\mathcal{G}$ consists of abelian Lie groups, then $A \in \text{Hom}(E_0, E_1) + \psi^o_{\text{inv}}(Y; E_0, E_1)$). Also, in the last two theorems, we can allow operators acting between sections of different vector bundles. This will require to slightly modify the proof of Theorem 4, either by using a smooth version of bivariant $K$-theory [32], or by using the usual bivariant $K$-theory after we have taken the norm closures of the various ideals $I$ decorated with various indices. We can also further integrate along the fibers of $\mathfrak{g}^* \to B$ to obtain a cohomological formula with values in $H^*_c(B; \mathcal{O})$, the cohomology with local coefficients in the orientation sheaf of $\mathfrak{g}^* \to B$. This will be useful in Section 4.

4. Regularized traces

Having in mind future applications, we also want to give a local formula for the equivariant family index of an invariant, elliptic family of operators, as considered in the previous section. This will be done in terms of various residue type traces. In this section, we develop the analytic tools required to define these regularized traces.

**Assumption.** Throughout the rest of this paper, we shall assume that $\mathcal{G}$ consist of simply-connected, non-trivial abelian Lie groups, and hence that it is a non-zero vector bundle.

Recall that $Y$ is a flat $\mathcal{G}$-bundle if $Y = B \times Z \times \mathbb{R}^q$ and $\mathcal{G} = B \times \mathbb{R}^q$ ($q > 0$). The results we will establish are local in $B$, and hence we can reduce the general case to the flat case. Actually, it is easier to assume first that $B$ is reduced to a point. We thus carry the analysis first in this case, and then we extend the results to the general case. The Lemma 8 and Proposition 7 are probably not new. We nevertheless include their proofs for completeness and to fix notation.

There is an action of $\mathbb{R}^q$ on $\Psi^\infty_{\text{inv}}(Z \times \mathbb{R}^q)$, the action of $\xi \in \mathbb{R}^q$ is obtained by multiplying the convolution kernel of an operator $A \in \Psi^0_{\text{inv}}(Z \times \mathbb{R}^q)$ by $\exp(it \cdot \xi)$, where $t \in \mathbb{R}^q$ are coordinates for the second component in $Z \times \mathbb{R}^q$. In terms of the Fourier transform representation of these operators, the action of $\xi$ becomes translation by $\xi \in \mathbb{R}^q$. 


We shall denote by $Tr$ the usual (Fredholm) trace on the space of trace class operators on a given Hilbert space.

**Lemma 4.** The space of $\mathbb{R}^q$-invariant traces on $\Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q)$ is one-dimensional.

**Proof.** Consider the map

$$\overline{\text{Tr}}(A) = \int \text{Tr}(\hat{A}(\tau)) \, d\tau.$$  

We need to show that this is the only invariant trace functional. In terms of indicial families, the infinitesimal generators of the $\mathbb{R}^q$-action correspond to the functions $t_k$. Let

$$\text{HH}_0(\Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q)) := \Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q)/[\Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q), \Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q)]$$

be the first homology group of $\Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q)$. It remains to show that the subspace of $\text{HH}_0(\Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q)) \simeq \mathcal{S}(\mathbb{R}^q)$ spanned by $t_k \text{HH}_0(\Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q))$ has codimension 1. Indeed, the kernel $\text{HH}_0(\Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q)) \to \mathbb{C}$ of the evaluation at 0 is the span of $t_k \text{HH}_0(\Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q))$. This proves the lemma.

Let $x$ be the identity function on $[0, \infty)$ and $l_s$ be a smooth function on $[0, \infty)$ such that $l'_s(x) = x^{s-1}$ for $x \geq 1$. (So that, in particular, $l_0(x) = \ln x$, for $x$ large.) We define then the spaces of functions

$$\mathcal{M}_k = \mathcal{S}^\infty([0, \infty)) + \mathbb{C}[x]l_0, \quad \text{for } k \in \mathbb{Z},$$

and

$$\mathcal{M}_s = \mathcal{S}^\infty([0, \infty))l_s, \quad \text{for } s \in \mathbb{C} \setminus \mathbb{Z}.$$  

Thus, $\mathcal{M}_0$ consists of smooth functions on $[0, \infty)$, that can be written, for any $M \in \mathbb{Z}_+$, as

$$f(x) = h_M(x) + \sum_{k=-M}^{-1} a_k x^k + \sum_{k=0}^{N} (b_k + c_k \log x)x^k, \quad \forall x \geq 1,$$

where $a_k, b_k, c_k$ are complex parameters, $N \in \mathbb{Z}_+$, and $h_M \in \mathcal{S}^{-M-1}([0, \infty))$. Similarly, the space $\mathcal{M}_s$, $s \not\in \mathbb{Z}$, consists of smooth functions $f \in \mathcal{C}^\infty([0, \infty))$ that can be written, for any $M \in \mathbb{Z}_+$, as

$$f(x) = h_M(x)x^s + \sum_{k=-M}^{N} \alpha_k x^{k+s}, \quad \forall x \geq 1,$$

for some constants $\alpha_k \in \mathbb{C}$, $N \in \mathbb{Z}$, and $h_M \in \mathcal{S}^{-M-1}([0, \infty))$. Fix $M > \max\{s, 0\}$ and $R \geq 1$, and define

$$I(f)(x) = \int_0^x f(x) \, dx,$$

$$\text{pv-} \int f = \int_0^R f(x) \, dx + \int_R^\infty h_M(x) \, dx - \sum_{k=-M}^{-2} \frac{a_k R^{k+1}}{k+1} - a_{-1} \log R,$$

$$- \sum_{k=0}^{N} \frac{R^{k+1}}{k+1} (b_k - \frac{c_k}{(k+1)} + c_k \log R), \quad \text{if } f \in \mathcal{M}_l, \; l \in \mathbb{Z}$$

Thus, we have defined a natural family of $\mathbb{R}^q$-invariant traces on $\mathcal{M}_k$. By Lemma 4, these traces are all the $\mathbb{R}^q$-invariant traces on $\Psi_{\text{inv}}^{-\infty}(Z \times \mathbb{R}^q)$.
Lemma 5. Let \( f(x) \in S^m((0, \infty)) \) be holomorphic in \( z \) and \( g_z(x) := f_z(x)(1+x)^{-\alpha} \). Then \( \pv \int_0^\infty g_z(x)dx \) is holomorphic on \( \mathbb{C} \setminus (m+\mathbb{Z}) \), with at most simple poles at \( m+\mathbb{Z} \), and \( \pv \int_0^\infty g_z(x)dx = \int_0^\infty g_z(x)dx \) for \( \Re(z) > m+1 \), where the second integral is absolutely convergent.

Proof. This follows from the fact that the right hand side of Equation (25) is holomorphic in \( f \) and \( s \) for \( s \not\in \mathbb{Z} \). The same formula guarantees at most simple poles at \( s \in \mathbb{Z} \). Since \( s = z - m \), in our case, the result follows.

If \( f \in C^\infty(\mathbb{R}) \) is such that \( f_+, f_- \in M_0 + \mathbb{C}[X] \), for some \( s \), where \( f_+(\tau) = f(\tau) \) and \( f_-(\tau) = f(-\tau) \), \( \tau \geq 0 \), we define

\[
\pv \int f = \pv \int f_+ + \pv \int f_- .
\]

We fix from now on a positive, invertible operator \( D \in \Psi^1_\text{inv}(Z \times \mathbb{R}^q) \). Let \( \mathbb{C} \ni z \to A(z) \in \Psi^m_\text{inv}(Z \times \mathbb{R}^q) \) be an entire function. Then

\[
f_z(\tau) = \text{Tr} [\partial^{\alpha}_{\tau}(|\tau|^k D(\tau)^{-\alpha} A(z, \tau))] , \quad k \in \mathbb{Z}_+ ,
\]

is defined and holomorphic for any multi-index \( \alpha \), for any \( \Re(z-m) > \dim Z - |\alpha| = d - |\alpha| \), and for any fixed \( \tau \in \mathbb{R}^q \). Moreover, by classical results (see \cite{17}, for example), the function \( z \to f_z(\tau) \) has a meromorphic extension to \( \mathbb{C} \), for each fixed \( \tau \), with at most simple poles at integers. Let

\[
\Omega = (\mathbb{C} \setminus \mathbb{Z}) \cup \{z \in \mathbb{C}, \Re(z-m) > d - |\alpha| \} .
\]
(Recall that $d = \dim Z$). In the following proposition $m$ does not have to be an integer.

**Proposition 1.** Let $A(z) \in \Psi^m_{\text{inv}}(Z \times \mathbb{R}^q)$ be an entire function and $D \in \Psi_{\text{inv}}^1(Z \times \mathbb{R}^q)$ be an invertible, positive operator. Also, let $f_z(\tau)$, defined for $z \in \Omega$, be as in Equation (28) above.

(i) The function $f_z(\tau)$ is in $C^\infty(\Omega \times \mathbb{R}^q)$ and the map $z \to f_z(\tau)$ is holomorphic on $\Omega$, for each fixed $\tau \in \mathbb{R}^q$.

(ii) There is a holomorphic $g : \Omega \to S^{m+d-|\alpha|}(|\mathbb{R}^q|)$ such that $f_z(\tau) = g_z(\tau)|\tau|^{k-z}$, for all $\tau$ such that $|\tau| \geq 1$, and hence $f_z(\tau) \in \mathcal{M}_{m+d+k-1}$ (as a function of $x$), for each fixed $\tau \neq 0$.

(iii) The function $z \to \frac{1}{2\pi} \int_0^\infty f_z(\tau)dx$ is holomorphic on $\mathbb{C} \setminus (m + \mathbb{Z})$, with at most simple poles at $m + \mathbb{Z}$.

**Proof.** The proof for $k \neq 0$ or $\alpha \neq (0, \ldots, 0)$ is the same as that for $k = 0$ and $\alpha = (0, \ldots, 0)$, so we shall assume that we are in the latter situation. Also, by replacing $z$ by $z - m$, we can assume that $m \in \mathbb{Z}$.

We first prove the lemma for $m = -\infty$, that is for $A(z) \in \Psi^{-\infty}_\text{inv}(Z \times \mathbb{R}^q)$. Denote by $\mathcal{K}$ the algebra of compact operators acting on $L^2(\mathbb{Z})$ and by $\mathcal{C}_1 \subset \mathcal{K}$ the normed ideal of trace class operators. For any $M \in \mathbb{Z}_+$, the product $\hat{D}^M(\tau)A(z, \tau)$ is in $S(\mathbb{R}, \mathcal{C}_1) = S(\mathbb{R}) \hat{\otimes} \mathcal{C}_1$ (here $\hat{\otimes}$ denotes the completed projective tensor product). Also, because $D$ is invertible and positive, the function $(z, \tau) \to \hat{D}(\tau)^{z} \in \mathcal{K}$ is differentiable, with bounded derivatives, and holomorphic in $z$, for $\text{Re}(z) \geq 1$.

Since $\text{Tr} : K \hat{\otimes} \mathcal{C}_1 \to \mathbb{C}$ is continuous, it follows that the function

$$(z, \tau) \to \text{Tr}(\hat{D}(\tau)^{-z-M-1}\hat{D}(\tau)^{M+1}A(z, \tau)) \in \mathbb{C}$$

is differentiable, with bounded derivatives, and holomorphic in $z$ for $\text{Re}(z) \geq -M$.

Since $M$ is arbitrary, this proves (i) and (ii) for $A(z) \in \Psi^{-\infty}_\text{inv}(Z \times \mathbb{R}^q)$. The last statement is an immediate consequence of (ii), because $z \to g_z \in S(|\mathbb{R}^q|)$ is holomorphic.

Using now the fact that the lemma is true for $A(z)$ in the residual ideal, we may assume, using a partition of unity, that $Z = \mathbb{R}^d$ and that the Schwartz convolution kernels of $A(z, \tau)$ are contained in a fixed compact set.

Let $\Delta_0, \Delta_1 \geq 0$ be the constant coefficient Laplacians on $Z = \mathbb{R}^d$ and $\mathbb{R}^q$, respectively. We define $D_0 = (1 + \Delta_0 + \Delta_1)^{1/2} \in \Psi^1_{\text{inv}}(Z \times \mathbb{R}^q)$. To prove the lemma for $A(z) \in \Psi^m_{\text{inv}}(Z \times \mathbb{R}^q)$, $m > \infty$, we shall first assume that $D = D_0$. Clearly, $D_0 \in \Psi^1_{\text{inv}}(Z \times \mathbb{R}^q)$. If $A(z) = a(z, x, D_x, D_z)$, for a symbol $a(z, x, D_x, D_z) \in S^m(T^*Z \times \mathbb{R}^q) = S^m(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^q)$, then

$$\hat{A}(z, \tau) = a(z, x, D_x, \tau) \quad \text{and} \quad \hat{D}_0(\tau)^{z} = (1 + \Delta_0 + |\tau|^2)^{-z/2}.$$  

This gives, by the standard calculus, that $\hat{A}(z, \tau)\hat{D}_0(\tau)^{-z} = a_1(z, x, D_x, \tau)$ for

$$a_1(z, x, \xi, \tau) = a(z, x, \xi, \tau)(1 + |\tau|^2 + |\xi|^2)^{-z/2}. \tag{29}$$

From the above relation, we obtain

$$f_z(\tau) := \text{Tr}(\hat{D}(\tau)^{-z}\hat{A}(\tau)) = \text{Tr}(\hat{A}(\tau)\hat{D}(\tau)^{-z})$$

$$= (2\pi)^{-d} \int_{T^*Z} a(z, x, \xi, \tau)(1 + |\tau|^2 + |\xi|^2)^{-z/2} d\xi dx, \quad \text{for} \quad \tau \geq 1.$$
Using the asymptotic expansion of $a$ in homogeneous functions in $(\xi, \tau)$ and the substitution $\xi \to (1 + |\tau|^2)^{1/2} \xi$, and the asymptotic expansion of $(1 + |\tau|^2)^{1/2}$ in powers of $|\tau|$ at $\infty$, we obtain (i) and (ii) for this particular choice of $D = D_0$.

We obtain (iii) directly from (ii) using Lemma 5.

The case $D$ arbitrary follows by writing $D^{-z} = D_0^{-z} (D_0^z D^{-z})$ and observing that $\mathbb{C} \ni z \to D_0^z D^{-z} \in \Psi^0_{\text{inv}}(Z \times \mathbb{R}^q)$ is an entire function.

See also [21].

We shall also need the following consequence of Proposition 1, above.

**Corollary 2.** Using the notation of the above proposition, we have that the function

$$F(s) := \int_{S^{q-1}} \left( \text{pv-}\int_0^\infty I^l \circ \partial_x^l [x^{q-1} f_z(x\tau)] dx \right) d\tau$$

is holomorphic on \{ $z \in \mathbb{C}, \text{Re}(z - m) > q + d$ \} $\cup \mathbb{C} \smallsetminus (m + \mathbb{Z})$, with at most simple poles at $m + \mathbb{Z}$, and extends the function $\int_{\mathbb{R}^q} f_z(x\tau) dx$, which is defined for Re$(z - m) > q + d$.

**Proof.** Assume $l = 0$. The proof for arbitrary $l$ is completely similar. The function $\text{pv-}\int_0^\infty x^{q-1} f_z(x\tau) dx$ is a holomorphic extension of the function $\int_0^\infty x^{q-1} f_z(x\tau) dx$, which is convergent for Re$(z)$ large. The result is obtained then by integration in polar coordinates and by combining Lemma 6 with (ii) of the above proposition.

Assume for the moment that $q = 1$ (and hence that $Y = Z \times \mathbb{R}$). Using the above lemma and the functionals $\text{pv-}$ and $I$, we obtain, as in [22], a functional $\mathcal{T}_1$ on $\Psi^{\infty}_{\text{inv}}(Z \times \mathbb{R})[|\tau|]$, by the formula

$$\mathcal{T}_1([\tau]A) = \text{pv-}\int I^k (f_+ + f_-),$$

where $f_+(\tau) = \text{Tr}[\partial_\tau^k([\tau] A(\tau))]$, $f_-(\tau) = \text{Tr}[\partial_\tau^k([\tau] A(-\tau))]$, if $\tau \leq 0$, and $k \in \mathbb{Z}_+, \ k > m + \dim Z + 1$. From equation (28), we see that this definition is independent on $k$. The tracial property of $\mathcal{T}_1$ follows, as in [22], from

$$\partial_\tau [A(\tau), B(\tau)] = [\partial_\tau A(\tau), B(\tau)] + [A(\tau), \partial_\tau B(\tau)].$$

Let now $q$ be arbitrary, but we continue to assume that $B$ is reduced to a point. The following lemma will allow us to generalize the definition of $\mathcal{T}_1$.

**Lemma 6.** Restriction of the indicial family $\hat{A}$ to $\mathbb{R}x$, $x \in S^{q-1}$, defines an $O(q)$-equivariant family of algebra morphism $r_x : \Psi^{\infty}_{\text{inv}}(Z \times \mathbb{R}^q) \to \Psi^{\infty}_{\text{inv}}(Z \times \mathbb{R})$. Each $r_x$ restricts to a degree preserving isomorphism $\Psi^{\infty}_{\text{inv}}(Z \times \mathbb{R})^{O(q)} \simeq \Psi^{\infty}_{\text{inv}}(Z \times \mathbb{R})^Z$, which is independent of $x$.

**Proof.** It is clear that the restriction of $\hat{A}$ to a line $\mathbb{R}x$ is in $S(\mathbb{R}x, \Psi^{-\infty}(Z))$, whenever $A$ is in $\Psi^{-\infty}_{\text{inv}}(Z \times \mathbb{R}^q)$. Moreover, we obtain isomorphisms

$$\Psi^{\infty}_{\text{inv}}(Z \times \mathbb{R}^q)^{O(q)} \simeq S(\mathbb{R}^q, \Psi^{-\infty}(Z))^{O(q)} \simeq S(\mathbb{R}, \Psi^{-\infty}(Z))^Z \simeq \Psi^{-\infty}(Z \times \mathbb{R})^Z.$$

These isomorphisms allow us to assume, using a partition of unity argument, that $Z = \mathbb{R}^l$. Using the fact that a symbol $a \in S^m(T^*Z \times \mathbb{R}^q)$ restricts to a symbol in $S^m(T^*Z \times \mathbb{R}x)$, $x \in S^{q-1}$, and the relation

$$\hat{A}(\tau) = a(x, D_x, \tau),$$

we obtain the desired conclusion.

The case $D$ arbitrary follows by writing $D^{-z} = D_0^{-z} (D_0^z D^{-z})$ and observing that $\mathbb{C} \ni z \to D_0^z D^{-z} \in \Psi^0_{\text{inv}}(Z \times \mathbb{R}^q)$ is an entire function.
if $A = o(x, D_z, D_\tau)$, we see that the restriction of $\hat{A}$ to $\mathbb{R}_x$ is the indicial family of an operator in $\Psi^m_{\text{inv}}(Z \times \mathbb{R})$, denoted $r_x(A)$. Since 
$$S^m(T^*Z \times \mathbb{R}^q)O(q) \simeq S^m(T^*Z \times \mathbb{R})^{\mathbb{Z}_2},$$
the isomorphism $\Psi^\infty_{\text{inv}}(Z \times \mathbb{R})^O(q) \simeq \Psi^\infty_{\text{inv}}(Z \times \mathbb{R})^{\mathbb{Z}_2}$ follows.

Let $A \in \Psi^\infty_{\text{inv}}(Z \times \mathbb{R}^q)$ and denote by $A_1 = \int_{O(q)} v(A)dv$ its average over $O(q)$ with respect to its normalized Haar measure, which we identify with an element of $\Psi^\infty_{\text{inv}}(Z \times \mathbb{R})^{\mathbb{Z}_2}$, thanks to Lemma 3. Define

$$(31) \quad \mathbb{T}_{\tau^q}(A) = \text{vol}(S^{q-1})\mathbb{T}_1(|\tau|^{q-1}A_1).$$

**Lemma 7.** The functional $\mathbb{T}_{\tau^q}$ is an $O(q)$-invariant trace on $\Psi^\infty_{\text{inv}}(Z \times \mathbb{R}^q)$, which extends the trace $\mathbb{T}_1$ defined on $\Psi^\infty_{\text{inv}}(Z \times \mathbb{R}^q)$, $\text{Re}(s) > d + q$ ($d = \dim Z$), by Equation (31).

**Proof.** The map $\mathbb{T}_{\tau^q}$ is obviously well defined and $O(q)$-invariant, in view of the above lemma. In order to check the tracial property, we use the definition. Fix $x \in \mathbb{R}^q$ of length one, arbitrarily, then

$$\mathbb{T}_{\tau^q}(A) = \text{vol}(S^{q-1})\int_{O(q)} \mathbb{T}_1(|\tau|^{q-1}r_{x\tau}(A))dv$$

Since $r_x$ is a morphism, $|\tau|$ is central, and $\mathbb{T}_1$ is a trace [28], the tracial property of $\mathbb{T}_{\tau^q}$ follows.

To complete the proof, we need only prove that $\mathbb{T}_{\tau^q}$ extends $\mathbb{T}_1$, and this follows by integration in polar coordinates. \hfill \Box

It is not essential in the above statements that $A$ have integral order. Both the formula for $r_x(A)$ and the definition of $\mathbb{T}_{\tau^q}(A)$ make sense for $A \in \Psi^s_{\text{inv}}(Y)$, with $s$ not necessarily integral. We shall use this for operators of the form $D^{-z}A$ in the next proposition. Actually, more is true of the $\mathbb{T}_{\tau^q}$-traces of elements of non-integral order than for elements of integral order: let $s \notin \mathbb{Z}$, then the action of $GL_q(\mathbb{R})$ by automorphisms on $\Psi^s_{\text{inv}}(Z \times \mathbb{R}^q)$ has the property

$$(32) \quad \mathbb{T}_{\tau^q}(T(A)) = |\det(T)|^{-1}\mathbb{T}_{\tau^q}(A).$$

This follows by using one of the several equivalent definitions of $\mathbb{T}_{\tau^q}(A)$, for $A$ of non-integral order, provided in the following Lemma.

**Lemma 8.** Let $A \in \Psi^s_{\text{inv}}(Z \times \mathbb{R}^q)$, $s \notin \mathbb{Z}$. Then the functions $\text{Tr}(\hat{D}(\tau)^{-z}\hat{A}(\tau))$, and $\int_{\mathbb{R}_x} \text{Tr}(\hat{D}(\tau)^{-z}\hat{A}(\tau))d\tau$ are holomorphic for $\text{Re}(z - s) > d + q$ and extend to holomorphic functions on $\mathbb{C} \setminus (\mathbb{Z} + s)$. At $z_0 \notin s + \mathbb{Z}$, these holomorphic extensions satisfy

$$\mathbb{T}_{\tau^q}(D^{-z_0}A) = \left(\int_{\mathbb{R}_x} \text{Tr}(\hat{D}(\tau)^{-z}\hat{A}(\tau))d\tau\right)_{z=z_0}$$

$$= \int_{S^{q-1}} \left(\text{pv}\int_0^\infty \left[x^{q-1}\text{Tr}(\hat{D}(\tau x)^{-z}\hat{A}(\tau x))\right]_{z=z_0}d\tau\right) dx.$$

**Proof.** The function $\mathbb{T}_{\tau^q}(D^{-z}A)$ is defined for all $z$, and it is seen to be holomorphic on $\mathbb{C} \setminus (\mathbb{Z} + s)$ using the definition of $\mathbb{T}_{\tau^q}$ and Corollary 3, which also gives the existence of the desired holomorphic extensions.
Finally, using again Corollary 2 and Lemma 2, we see that $\overline{\text{Tr}}_q(D^{-z}A)$ and all the other functions in the stated equation coincide for $\text{Re}(z-s) > d + q$. Because they are holomorphic on a connected open set containing both $z_0 \in \mathbb{C} \smallsetminus (s + Z)$ and $\text{Re}(z-s) > d + q$ ($d = \dim Z$), they must coincide at $z_0$ also. □

**Proposition 2.** For any invertible, positive element $D \in \Psi_1^{\text{inv}}(Z \times \mathbb{R}^q)$ and any holomorphic function $A : \mathbb{C} \to \Psi_{\text{inv}}^m(Z \times \mathbb{R}^q)$, the function 

$$F_D(A; z) = \overline{\text{Tr}}_q(D^{-z}A(z)), \quad \text{Re}(z) > m + q + \dim Z,$$

is holomorphic on $(\mathbb{C} \smallsetminus Z) \cup \{ \text{Re} z > m + q + \dim Z \}$ with at most simple poles at the integers. The residue of this holomorphic function at 0 depends only on $A(0)$ and will be denoted by $\text{Tr}_R(A(0))$. Moreover, $\text{Tr}_R(A(0))$ vanishes on regularizing elements, is independent of $D$, and defines a trace on $\Psi_{\text{inv}}^\infty(Z \times \mathbb{R}^q)$.

**Proof.** The function $F_D$ is holomorphic on the indicated domain by 8. Its poles are simple by Corollary 2 and by the definition of $\overline{\text{Tr}}_q$ in terms of $\overline{\text{Tr}}_1$, see Equation (B). For the rest of the proof, it is enough to assume that $A(z)$ is independent of $z$. We set then $A = A(z) = A(0)$. For example, the proof of the fact that $\text{Tr}_R$ is a trace and that it is independent of the choice of $D$ is obtained from a standard reasoning, as follows. We first write 

$$\overline{\text{Tr}}_q(D^{-z}[A, B]) = \overline{\text{Tr}}_q(D^{-z}, D^2A[B])$$

and observe that $[D^{-z}, D^2A]B$ is a holomorphic function vanishing at 0. This shows that $\overline{\text{Tr}}_q$ vanishes on commutators. The independence of $\overline{\text{Tr}}_q$ of $D$ is a consequence of 

$$\overline{\text{Tr}}_q(D^{-z}A) - \overline{\text{Tr}}_q(D_1^{-z}A) = \overline{\text{Tr}}_q(D^{-z}(\text{Id} - D^2D_1^{-z})A),$$

using that $(\text{Id} - D^2D_1^{-z})A$ is a holomorphic function vanishing at 0. □

**Proposition 3.** Let $D \in \Psi_1^{\text{inv}}(Z \times \mathbb{R}^q)$, $D \geq 0$, invertible as above, and let $A \in \Psi_{\text{inv}}^m(Z \times \mathbb{R}^q)$. We denote $B_z = D^{-z}A \in \Psi_{\text{inv}}^{m^{-z}}(Z \times \mathbb{R}^q)$, and write the asymptotic expansion

$$(33) \quad \text{Tr}(\hat{B}_z(\tau)) \sim \sum_{k \leq m+d} \beta_k(z, \tau/|\tau|)|\tau|^{k-z}, \quad |\tau| \to \infty,$$

defined for $\text{Re}(z)$ large or for $z$ not an integer. Then the coefficient $\beta_{-1}$ is holomorphic in a neighborhood of 0. $\text{Tr}_R(A) = \int_{|\tau|=1} \beta_{-1}(0, \tau)$, and

$$\lim_{z \to 0} (\overline{\text{Tr}}_q(D^{-z}A) - z^{-1} \text{Tr}_R(A)) = \overline{\text{Tr}}_q(A) + \int_{|\tau|=1} \partial_\tau \beta_{-1}(z, \tau)|_{z=0}.$$

**Proof.** By the definition of $\overline{\text{Tr}}_q$ using integration with respect to the orthogonal group, it is enough to prove the result for $q = 1$. By definition, $\overline{\text{Tr}}_1(A)$ is completely determined by $\text{Tr}(\partial^2_x A(x))$. This reduces our analysis to a lemma about integrals of functions in $\mathcal{M}_s$.

Fix $\epsilon > 0$, and define $C_\epsilon$ to be the space of functions $f(s, x), -\epsilon < \text{Re}(s) < \epsilon$, $x \geq 0$, with the following properties:

1. $f(s, x)$ is smooth in $(s, x)$ and holomorphic in $s$, for each fixed $x$;
2. For any $M \in \mathbb{N}$ there exist $R > 0$, $c_k \in \mathbb{C}$, and complex valued functions $\alpha_k(s), b_k(s),$ and $h_M(s, x)$ satisfying

$$\begin{align*}
f(0, x) &= h_M(0, x) + \sum_{k=-M}^{-1} \alpha_k(0)x^k + \sum_{k=0}^{N} (\alpha_k(0) + b_k(0) + c_k \log x)x^k, \\
f(s, x) &= h_M(s, x)x^{-s} + \sum_{k=-M}^{-1} \alpha_k(s)x^{k-s} + \sum_{k=0}^{N} (b_k(s) + c_k s^{-1}(1-x^{-s}))x^k,
\end{align*}$$

for $x \geq R$, and

$$\begin{align*}
f(s, x) &= h_M(s, x)x^{-s} + \sum_{k=-M}^{-1} \alpha_k(s)x^{k-s} + \sum_{k=0}^{N} (b_k(s) + c_k s^{-1}(1-x^{-s}))x^k,
\end{align*}$$

for $s \neq 0$, $x \geq R$, and $-\epsilon < \text{Re}(s) < \epsilon$.

3. $\alpha_k(s), b_k(s)$ are holomorphic in $s$, $h_M(s, x)$ is holomorphic in $s$, for each fixed $x$, and $h_M(s, \cdot) \in S^{-M-1}([0, \infty))$, for each fixed $s$ in the strip $-\epsilon < \text{Re}(s) < \epsilon$.

Of course, the choice of $R > 0$ is not important in the above definition. A simple but crucial observation is that

$$\begin{align*}
\text{lim}_{s \to 0} \text{pv-} \int f(s, x)dx - \alpha_{-1}(s)s^{-1}R^{-s} &= \text{pv-} \int f(0, x)dx + \alpha_{-1}(0) \log R,
\end{align*}$$

which gives

$$\begin{align*}
\text{lim}_{s \to 0} \text{pv-} \int_{0}^{\infty} f(s, x)dx - \alpha_{-1}(0)s^{-1} &= \text{pv-} \int_{0}^{\infty} f(0, x)dx + \alpha'_{-1}(0).
\end{align*}$$

The main idea is to prove that the familiar function

$$f_l(s, x) := \text{Tr}[(\varphi_x(D^{-s}(x))\hat{A}(x) + \hat{D}^{-s}(-x)\hat{A}(-x))]$$

is in $\mathcal{C}_c$, for $l$ large. Then $I^l(f_l) \in \mathcal{C}_c$. This is, of course, a refinement of Proposition 3.

Now, by definition, $\text{Tr}_1(D^{-s}A) = \text{pv-} \int_{0}^{\infty} I^l(f_l)(s, x)dx$, for any $l > m + d + 1$. For any such $l$, $f_l(s, x)$ is holomorphic in $s$, for $\text{Re}(s) > -1$, and hence $f_l \in \mathcal{C}_c$ (and $c_k = 0$, but this will play no role in our reasoning), by Proposition 3. Consequently, $I^l(f_l)(s, x) \in \mathcal{C}_c$. Let $\alpha_k$ be the corresponding coefficients in the canonical asymptotic expansion of $I^l(f_l)$.

We know, by classical results, that the functions $I^l(f_l)(s, x)$ and $f_0(s, x)$ have holomorphic extensions in $s \notin \mathbb{Z}$. Moreover, for $s \notin \mathbb{Z}$, the difference $I^l(f_l)(s, x) - f_0(s, x)$ is a polynomial in $x$, which shows that the coefficients of $x^{-1}$ in the asymptotic expansions of these two functions are the same. Consequently, $\beta_{-1}(s) = \alpha_{-1}(s)$, for $s \neq 0$. But by the definition of $\mathcal{C}_c$, $\alpha_{-1}$ is holomorphic in a neighborhood of $0$, and hence $\beta_{-1}$ has a holomorphic extension to a neighborhood of $0$, as claimed.

Finally, using $\text{Tr}_1(D^{-s}A) = \text{pv-} \int_{0}^{\infty} I^l(f_l)(s, x)dx$, for $z$ in a small neighborhood of $0$, and Equation (34), for $f = I^l(f_l)$, we obtain

$$\begin{align*}
\text{lim}_{s \to 0} \text{pv-} \int_{0}^{\infty} (\text{Tr}_1(D^{-s}A) - \beta_{-1}(0)s^{-1}) &= \text{lim}_{s \to 0} \text{pv-} \int_{0}^{\infty} I^l(f_l)(s, x)dx - \alpha_{-1}(0)s^{-1} \\
&= \text{pv-} \int_{0}^{\infty} I^l(f_l)(0, x)dx + \alpha'_{-1}(0) = \text{Tr}_1(A) + \beta'_{-1}(0).
\end{align*}$$

This completes the proof. $\square$
We now drop the assumption that \( B \) be reduced to a point. To extend the above results to the general case, we proceed to a large extent as we did when \( B \) was reduced to a point.

Fix an invertible positive operator \( D \in \Psi^1_{\text{inv}}(Y) \), and let \( C \ni z \to A(z) \in \Psi^m_{\text{inv}}(Y) \) be an entire function. Then

\[
(35) \quad f_z(\tau) = Tr(\hat{D}(\tau)^{-1} |\tau|^k \hat{A}(\tau)), \quad k \in \mathbb{Z}_+,
\]

is defined and holomorphic for any \( \text{Re}(z) > m + d = m + \dim Z \) and any fixed \( \tau \in \mathcal{G}^* \); moreover, the function \( z \to f_z(\tau) \) has a meromorphic extension to \( \mathbb{C} \), for each fixed \( \tau \), with at most simple poles at integers.

Let \( \Omega = (\mathbb{C} \setminus \mathbb{Z}) \cup \{z, \text{Re}(z) > m + \dim Z\} \).

**Lemma 9.** Let \( A(z) \in \Psi^m_{\text{inv}}(Y) \) be an entire function. Also, let \( f_z(\tau) \) be as above, with \( z \in \Omega \) and \( \tau \in \mathcal{G}^* \).

(i) The function \( f_z(\tau) \) is in \( C^\infty(\Omega \times \mathcal{G}^*) \) and the map \( z \to f_z(\tau) \) is holomorphic on \( \Omega \), for each fixed \( \tau \in \mathcal{G}^* \).

(ii) There is \( g \in C^\infty(\Omega \times \mathcal{G}^*) \) such that \( g(z, \cdot) \in S^{m+d}(\mathcal{G}^*) \), for each fixed \( z \), such that \( g(z, \tau) \) is holomorphic in \( z \), for each fixed \( \tau \), and such that \( f_z(\tau) = g(z, \tau)|\tau|^{k-d} \), for all \( \tau \geq 1 \). Consequently, \( f_z(x\tau) \in \mathcal{M}_{m+d+k-d} \), for all \( \tau \neq 0 \) \((d = \dim Y - \dim \mathcal{G})\).

(iii) The function \( z \to \text{pv-} f_z \) is holomorphic on \( \mathbb{C} \setminus \mathbb{Z} \), with at most simple poles at integers.

**Proof.** Since all the statements of the above theorem are statements about the local behavior in \( b \in B \) of certain functions, we may assume that \( Y \) is a flat \( \mathcal{G} \)-space. This means, we recall, that \( Y = B \times Z \times \mathbb{R}^q \) and \( \mathcal{G} = B \times \mathbb{R}^q \). Then we just repeat the proof of Proposition 6 including an extra parameter \( b \in B \), with respect to which all functions involved are smooth. \( \square \)

We now extend the definition of the various traces and functionals we considered above when \( B \) was reduced to a point. This is not completely canonical, because we need to fix a metric on \( \mathcal{G}^* \) in order to obtain a volume form on the fibers of \( \mathcal{G}^* \to B \). The choice of the metric defines the group \( O(\mathcal{G}) \) of fiberwise orthogonal isomorphisms of \( \mathcal{G} \). We also fix a lifting \( Y/\mathcal{G} \to Y \), which gives an isomorphism

\[
Y \simeq Y/\mathcal{G} \times_B \mathcal{G}.
\]

This isomorphism and the metric on \( \mathcal{G} \) give an action of \( O(\mathcal{G}) \) on \( Y \), which normalizes the structural action of \( \mathcal{G} \) by translations on \( Y \). Consequently, the group \( O(\mathcal{G}) \) acts by isomorphisms on the algebra \( \psi_{\text{inv}}^\infty(Y) \). By Lemma 4, the group \( O(\mathcal{G}) \) also acts by isomorphisms on \( \psi_{\text{inv}}^\infty(Y) \).

**Lemma 10.** Fix a metric on \( \mathcal{G} \) and choose a lifting \( Y/\mathcal{G} \to Y \), which gives rise then to an isomorphism \( Y \simeq Y/\mathcal{G} \times_B \mathcal{G} \) and an action by automorphisms of the group \( O(\mathcal{G}) \) on \( \psi_{\text{inv}}^\infty(Y) \), as above. Then there exists an \( O(\mathcal{G}) \)-linear map

\[
E_Y : \psi_{\text{inv}}^\infty(Y) \to \psi_{\text{inv}}^\infty(Y)^{O(\mathcal{G})}
\]

such that \( E_Y(AB) = AE_Y(B) \) and \( E_Y(BA) = E_Y(B)A \), for all \( A \in \psi_{\text{inv}}^\infty(Y)^{O(\mathcal{G})} \) and \( B \in \psi_{\text{inv}}^\infty(Y) \). Moreover, \( \psi_{\text{inv}}^\infty(Y)^{O(\mathcal{G})} \simeq \psi_{\text{inv}}^\infty(Y/\mathcal{G} \times \mathbb{R}^q)^{\mathbb{Z}/2\mathbb{Z}} \), and hence the isomorphism class of the algebra \( \psi_{\text{inv}}^\infty(Y)^{O(\mathcal{G})} \) depends only on \( Y/\mathcal{G} \).
Proof. If $Y$ is a flat $G$ space, then this result follows right away from Lemma 1. In general, we can choose trivializations of $Y$ such that the transition functions preserve the metric on $G$, and hence the transition functions are in $O(G)$. Because the isomorphisms of $G$ commute with the action of the orthogonal group, the result follows.

We now consider $C^\infty(B)$–linear traces on $\Psi_{\text{inv}}^m(Y)$, for a general $G$ space $Y$. That is, we consider $C^\infty(B)$–linear maps $T : \Psi_{\text{inv}}^m(Y) \to C^\infty(B)$ such that

$$T(fA) = fT(A), \quad \text{for } f \in C^\infty(B) \text{ and } A \in \Psi_{\text{inv}}^m(Y),$$

and

$$T([A, B]) = 0, \quad \text{for } A, B \in \Psi_{\text{inv}}^m(Y).$$

If we fix a metric on $G$, then we obtain a $C^\infty(B)$–linear trace $\mathcal{T}_Y$ that generalizes the $\mathcal{T}_q$–traces as follows. Suppose $Y = B \times Z \times \mathbb{R}^q$ and $A = (A_b) \in \Psi_{\text{inv}}^m(Y)$, then we set

$$\mathcal{T}_Y(A)(b) = \mathcal{T}_q(A_b).$$

Because trace $\mathcal{T}_Y$, for $Y = B \times Z \times \mathbb{R}^q$, is invariant with respect to the action of the orthogonal group, the choice of an isomorphism $Y \cong Y/G \times_B G$ of $G$–spaces and of a metric on $G$ allow us to extend the definition of $\mathcal{T}_Y$ to arbitrary $Y$. We stress that this trace depends on the choices we have made. This new trace satisfies

$$\mathcal{T}_Y(A) = \mathcal{T}_{Y/G \times \mathbb{R}}(E_Y(A)).$$

We then have the following immediate generalization of Proposition 3 above:

**Proposition 4.** Fix a lifting $Y/G \to Y$, which is used to define $\mathcal{T}_Y(\cdot)$, as above. For any self-adjoint, invertible, positive element $D \in \Psi_{\text{inv}}^1(Y)$ and any holomorphic function $A : \mathbb{C} \to \Psi_{\text{inv}}^m(Y)$, the function

$$F_D(A; z) = \mathcal{T}_Y(D^{-z}A(z))$$

is holomorphic in $(\mathbb{C} \smallsetminus \mathbb{Z}) \cup \{\Re z > m + q + \dim Y - \dim G\}$ and has at most simple poles at the integers. The residue of this holomorphic function depends only on $A(0)$ and will be denoted by $\mathcal{T}_{R_Y}(A(0))$. Moreover, $\mathcal{T}_{R_Y}(A(0))$ vanishes on regularizing elements, is independent of $D$, and defines a $C^\infty(B)$–linear trace on $\Psi_{\text{inv}}^m(Z \times \mathbb{R}^q)$. This trace is independent of the choice of the isomorphism $Y \cong Y/G \times_B G$.

Proof. Everything in this proposition follows from the case when $B$ is reduced to a point, except the independence of isomorphism $Y \cong Y/G \times_B G$. For this we also use Equation (32).

We also note that Proposition 3 extends virtually without change to families (that is, to the case $B$–nontrivial).

The traces $\mathcal{T}_q$ and $\mathcal{T}_Y$ extend to matrix algebras by taking the sum of the traces of the entries on the main diagonal.

### 5. Local index formulae

We now return to the the study of the index of a family of invariant, elliptic operators. More precisely, we want local formulae for $Ch(\text{ind}_q(A))$ when the fibers $\mathcal{G}_b$ of $\mathcal{G} \to B$ are simply-connected abelian Lie groups, that is, when $\mathcal{G}$ is a vector bundle. To this end, we shall use regularized traces and their properties developed in the previous section.
If \( A \) is a family of Dirac operators and \( \mathcal{G} \) is trivial, local formulae for \( Ch(\text{ind}_a(A)) \) were obtained using heat kernels by Bismut in a remarkable paper, \(^8\). Our results are a step towards a similar result for arbitrary families of pseudodifferential operators invariant with respect to a bundle of Lie groups. The choice of the case when \( \mathcal{G} \) is a vector bundle may seem, at first sight, to be a long way from the general case. It is clear however that each type of Lie group bundle will have its specific features and hence have to be treated separately. From the point of view of \( K \)-theory, vector bundles and bundles of simply-connected solvable Lie groups behave quite similarly. Moreover, by homotopy, the general case of simply-connected solvable Lie groups can be reduced to the particular case of a vector bundle, as we have shown in Section \(^2\).

No immediate generalizations of the results of this section to other classes of Lie group bundles seem possible. The case when the fibers of \( \mathcal{G} \to B \) are not simply-connected solvable has a quite different flavor, and even the case where \( B \) is reduced to a point and \( \mathcal{G} \) has compact fibers is not well enough understood from the local perspective adopted in this section. The case when \( \mathcal{G} \to B \) has simply-connected, non-commutative solvable fibers might seem more manageable at first sight, however, then we have some very difficult issues related to the choice of algebras closed under holomorphic functional calculus, not to mention that these groups may be not type I, so that the Fourier transform approach has no meaning.

This should fully justify the choice of treating the case of vector bundles in such a great detail.

Fix now a group morphism \( \chi : H^m_\mathbb{C}(g^*) \to \mathbb{C} \). We want to obtain local formulae for \( \chi(Ch(\text{ind}_a(A))) \). For simplicity, we shall that \( B \) and \( Y/\mathcal{G} \) are compact. Since the general case requires some additional ideas, we shall assume first that \( Y \) is a flat \( \mathcal{G} \)-space, that is, that \( \mathcal{G} = B \times \mathbb{R}^q \) and \( Y = B \times Z \times \mathbb{R}^q \).

Our approach is based on an interpretation of \( \chi(Ch(\text{ind}_a(A))) \) using the Fedosov (or \( \star \)) product. We begin by recalling the definition of the Fedosov product and by making some general remarks on traces and their pairing with the \( K \)-theory of the algebras we consider.

Let \( \mathfrak{A} = \bigoplus_{k=0}^N \mathfrak{A}_k, \; N < \infty \) be a graded algebra endowed with a graded derivation \( d : \mathfrak{A}_k \to \mathfrak{A}_{k+1} \), the Fedosov product is defined by

\[
a \star b = ab + (-1)^{\deg a}(da)(db).
\]

(The name is due to Cuntz and Quillen who have thoroughly studied the Fedosov product in connection to their approach to Non-commutative de Rham cohomology, see \(^{15}\).) We shall denote by \( Q\mathfrak{A} \) the algebra \( \mathfrak{A} \) with the Fedosov (or \( \star \)) product and by \( Q_{ev}\mathfrak{A} \subset Q\mathfrak{A} \) the subalgebra of even elements.

Since we shall work with non-unital algebras also, it is sometimes necessary to adjoin a unit "1" to \( Q\mathfrak{A} \). The resulting algebra will be simply denoted by \( Q^+\mathfrak{A} \simeq Q\mathfrak{A} \oplus \mathbb{C} \). Similarly, \( Q_{ev}^+\mathfrak{A} := Q_{ev}\mathfrak{A} \oplus \mathbb{C} \).

A graded trace \( \tau \) on \( Q^+\mathfrak{A} \) restricts to an ordinary trace on \( Q_{ev}^+\mathfrak{A} \), and hence it gives rise to a morphism

\[
\tau_* : K_0^{alg}(Q_{ev}\mathfrak{A}) \to \mathbb{C}, \quad \tau_*(e) = \sum_j \tau(e_{jj}),
\]

for any idempotent \( e = [e_{jj}] \in M_k(Q_{ev}\mathfrak{A}) \). If \( \pi_* : K_0^{alg}(Q\mathfrak{A}) \to K_0^{alg}(\mathfrak{A}_0) \) is the natural morphism induced by \( \pi : Q_{ev}\mathfrak{A} \to \mathfrak{A}_0 \oplus \mathbb{C} \), then \( \pi_* \) is an isomorphism, by
standard algebra results. Consequently, the trace \( \tau \) also gives rise to a morphism

\[
(38) \quad \tilde{\tau} := \tau_* \circ \pi_*^{-1} : K^a_k(\mathfrak{A}_0 \oplus \mathbb{C}) \longrightarrow \mathbb{C},
\]

see (54, 58).

The explicit form of the morphism \( \tilde{\tau} \) is not difficult to determine. Let \( e \in \mathfrak{A}_0 \oplus \mathbb{C} \) be an idempotent, then

\[
(39) \quad \bar{e} = \frac{1}{2} + \sum_{k \geq 0} (-1)^k \frac{(2k)!}{k!} (e - \frac{1}{2} + dede)(de)^2k
\]
is an idempotent in \( Q^+_{ev} \mathfrak{A} \) lifting \( e \). (Note that the sum defining \( \bar{e} \) is actually finite.)

Assume the trace \( \tau \) is concentrated on \( \mathfrak{A}_{2k}, k \in \mathbb{N} \) and \( \tau(d\mathfrak{A}_{2k-1}) = 0 \). Then the explicit formula for \( \tilde{\tau}([e]) \) is

\[
(40) \quad \tilde{\tau}([e]) = (-1)^k \frac{(2k)!}{k!} \tau((eedede)^k).
\]

(We used \( e(de)^k = (eedede)^k \), valid for all \( e \) satisfying \( e^2 = e \).)

Traces on \( Q^+_{ev} \mathfrak{A} \) are easy to obtain. Indeed, if \( \tau \) is an even graded trace on \( \mathfrak{A} \) satisfying \( \tau(\mathfrak{A}_k) = 0 \), if \( k \neq p \), and \( \tau(d\mathfrak{A}) = 0 \), then

\[
\tau(a \star b - (-1)^j b \star a) = 0,
\]

for any \( a \in \mathfrak{A}_i \) and \( b \in \mathfrak{A}_j \), and hence \( \tau \) defines a graded trace on \( QA \). The trace \( \tilde{\tau} \) defined above then extends to a trace on \( Q^+_{ev} \mathfrak{A} \) by setting \( \tau(1) = 0 \).

We now define the algebras to which we shall apply the above considerations, if \( Y = B \times Z \times \mathbb{R}^q \) is a flat \( \mathcal{G} \) space (\( \mathcal{G} = B \times \mathbb{R}^q \)). Let \( \Omega^*(B) \) be the space of smooth forms on \( B \), and consider the differential graded algebra

\[
\mathfrak{A} := \Psi^\infty_{inv}(Y; C^N) \otimes C^\infty(B) \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q \simeq M_N(\Psi^\infty_{inv}(Y)) \otimes C^\infty(B) \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q
\]

\[
\simeq M_N(\Psi^\infty(Y)) \otimes \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q \quad \text{for } Y \text{ a flat } \mathcal{G} - \text{space}.
\]

Let \( d_B \) the de Rham differential in the \( B \) variables (here we use the assumption that \( Y \) is a flat \( \mathcal{G} \)-space). The differential of \( \mathfrak{A} \) is then the usual de Rham differential \( d_{DR} \):

\[
d_{DR}(A) = \sum [t_i, A] d\tau_i + d_B(A), \quad \text{and } d(A \xi) = d(A) \xi,
\]

if \( A \in \Psi^\infty_{inv}(Y; C^N) \otimes C^\infty(B) \Omega^*(B) \simeq M_N(\Psi^\infty_{inv}(Y)) \otimes C^\infty(B) \Omega^*(B) \) and \( \xi \) is a product of some of the “constant” forms \( d\tau_1, \ldots, d\tau_q \). Thus the differential \( d_{DR} \) is with respect to the \( B \times \mathbb{R}^q \) variables.

Let \( \pi^* \Lambda^* T^* B \) be the pull back to \( Y \) of the exterior algebra of the cotangent bundle of \( B \) and \( F = \pi^* \Lambda^* T^* B \otimes \Lambda^* \mathbb{R}^q \). Then

\[
\mathfrak{A} \simeq \Psi^{-\infty}_{inv}(Y; F \otimes C^N).
\]

Inside \( \mathfrak{A} \) we have the ideal of regularizing operators

\[
\mathcal{J} := \Psi^{-\infty}_{inv}(Y; C^N) \otimes C^\infty(B) \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q \simeq \Psi^{-\infty}_{inv}(Y; F \otimes C^N),
\]

with quotient algebra

\[
\mathfrak{B} := \mathfrak{A}/\mathcal{J} = M_N(\Psi^{-\infty}_{inv}(Y)/\Psi^{-\infty}_{inv}(Y)) \otimes C^\infty(B) \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q
\]

\[
\simeq \Psi^{-\infty}_{inv}(Y; F \otimes C^N)/\Psi^{-\infty}_{inv}(Y; F \otimes C^N).
\]
We consequently obtain the exact sequence of algebras
\begin{equation}
0 \rightarrow Q_{ev}\mathcal{J} \rightarrow Q_{ev}\mathcal{A} \rightarrow Q_{ev}\mathcal{B} \rightarrow 0,
\end{equation}
which gives rise to the boundary map $\partial_Q$,
\[ \partial_Q : K_1^{al}(Q_{ev}\mathcal{B}) \rightarrow K_0^{al}(Q_{ev}\mathcal{J}) = K_0^{al}(Q_{ev}\mathcal{J}_0), \]
on algebraic $K$-theory.

We now define the traces we shall consider on the algebras $\mathcal{A}, \mathcal{B}, \ldots, \mathcal{Q}$. Let $\omega : \Omega^k(B) \rightarrow \mathbb{C}$ be a closed current, that is, a continuous map such that $\omega (d\eta) = 0$ for any form $\eta$. Then $\omega$ defines a morphism $\chi_\omega : H^*_\mathbb{C} (\mathfrak{g}^*) \rightarrow \mathbb{C}$. Similarly, let
\[ A \otimes \zeta \otimes \xi \in \mathcal{A} := \Psi_{inv}^{\infty}(Y; \mathbb{C}^N) \otimes_{C^\infty(B)} \Omega^*(B) \otimes \Lambda^*p \mathbb{R}^q, \]
and define
\begin{equation}
\tau_\omega(A \otimes \zeta \otimes \xi) = \overline{\text{Tr}}_q(A) \omega(\zeta) \int_{\mathbb{R}^q} \xi.
\end{equation}
Then $\tau_\omega$ is a trace on $\mathcal{A}$ satisfying $\tau_\omega(d(a)) = 0$ if $a \in \mathcal{J}$.

If $e = (e_0, \lambda) \in M_N(\Psi_{inv}^{\infty}(Y) + \mathbb{C}) = M_N(\Psi_{inv}^{\infty}(Y)) \oplus M_N(\mathbb{C})$ is an idempotent, then $e$ defines a class $x_e \in K_0^0(\mathfrak{g}^*) \cong K_0^0(B \times \mathbb{R}^q)$. It is useful to review the definition of this class. First, we can replace $e = (e_0, \lambda)$ by an equivalent projection such that there exists another projection $p = (p_0, \lambda)$ satisfying $dp = 0$ and $pe = ep = p$. More precisely, the projection $p$ is such that its indicial family, $\tilde{p} := p_0 + \lambda$ consists of a smooth family of projections on $\mathfrak{g}^*$, acting on the fibers of $\mathfrak{g}^* \times_B Y \rightarrow \mathfrak{g}^*$, with values in $M_N(\Psi_{inv}^{\infty}(Y_\theta) + \mathbb{C})$, and constant along the fibers of $\mathfrak{g}^* \rightarrow B$. This projection $p$ has the property that $p\mathcal{J}_0 \subset \mathcal{J}_0$. Then we define the vector bundle $V_e$ on $\mathfrak{g}^*$ such that its fiber at $\tau$ is the range of the indicial operator $\tilde{e}(\tau) - \tilde{p}$. Finally, the class $x_e$ defined by $e$, which we are looking for, is $[V_e] - r[1]$, where $r$ is the rank of $V_e$.

It is interesting to compare the Chern character of the bundle $V_e$ to the pairing $\tilde{\tau}_\omega[e]$. First, the vector bundle $V_e$ is trivial at infinity. The curvature of the Grassmannian connection $\nabla^e = e \circ d$ is $R^e := (e \circ d)^2 = ede d e$, by a standard computation. If $k + q = 2p > 0$ is even, then $\tilde{\tau}_\omega$ is an even, graded trace, and hence it pairs with $[e]$. Using the explicit formula from Equation (40), we obtain that
\begin{equation}
\tilde{\tau}_\omega[e] = \frac{(2p)!}{p!} \chi_\omega(\frac{(-R^e)^p}{p!})
\end{equation}
recovers (up to a constant) the pairing of $Ch(V_e)$, the Chern character of $V_e$, with the cohomology class of $\chi_\omega$. In the notation introduced above, we have
\begin{equation}
\tilde{\tau}_\omega[e] = \frac{(2p)!}{p!} \chi_\omega(Ch(V_e)) = \frac{(2p)!}{p!} \chi_\omega(Ch(x_e)).
\end{equation}
See [14]. It also follows that all morphisms $K_0^{al}(\Psi_{inv}^{\infty}(Y)) \simeq K_0(\Psi_{inv}^{\infty}(Y)) \rightarrow \mathbb{C}$ are of the form $\tilde{\tau}_\omega$, for a suitable $\omega$. (Recall that $\tilde{\tau}_\omega$ is defined by Equations (43) and (44).)

Recall now that we defined the analytic index $\text{ind}_a$ to be the composite map
\[ \text{ind}_a : K_1^{al}(\mathcal{B}_0) \xrightarrow{\partial} K_0^{al}(\mathcal{J}_0) \simeq K^0(\mathfrak{g}^*), \]
where $\mathcal{J}_0 = M_N(\Psi_{inv}^{\infty}(Y))$ and $\partial$ is the boundary map in algebraic $K$-theory associated to the exact sequence $0 \rightarrow \mathcal{J}_0 \rightarrow \mathcal{A}_0 \rightarrow \mathcal{B}_0 \rightarrow 0$. Moreover, for the case we are currently discussing, that where $Y$ is a flat $G$-space, $\mathfrak{g}^* \cong B \times \mathbb{R}^q$. Let
\( \chi : H^*_c(\mathfrak{g}^*) \to \mathbb{C} \) be an arbitrary group morphism. As we explained at the beginning of the section, we are interested in understanding the morphism

\[
\chi \circ Ch \circ \text{ind}_a : K^\text{alg}_1(\Psi^\infty_{\text{inv}}(Y; \mathbb{C}^N)/\Psi^-\infty(\mathcal{N})) = K^\text{alg}_1(\mathcal{B}_0) \to \mathbb{C}.
\]

By the above discussion and linearity, we may assume that \( \chi = \chi_\omega, \) for some closed current \( \omega : \Omega^k(B) \to \mathbb{C}. \) Then a preliminary formula for the composition \( \chi_\omega \circ Ch \circ \text{ind}_a \) is given in the following lemma.

**Lemma 11.** Let \( \omega : \Omega^k(B) \to \mathbb{C} \) be a closed current such that \( k + q = 2p > 0 \) is even. Denote by \( \chi_\omega : H^*_c(\mathfrak{g}^*) \to \mathbb{C} \) the morphism defined by \( \omega. \) Then

\[
\chi_\omega \circ Ch \circ \text{ind}_a = \tilde{\tau}_\omega \circ \partial : K^\text{alg}_1(\mathcal{B}_0) \to \mathbb{C}.
\]

**Proof.** This follows by applying the above constructions to \( \text{ind}_a(A), \) \( A \) elliptic, using Equation (44).

We now turn to the computation of \( \tilde{\tau}_\omega \circ \partial : K^\text{alg}_1(\mathcal{B}_0) \to \mathbb{C}. \) We shall use the generic notation \( \pi \) for all the quotient morphisms \( Q\mathfrak{A} \to \mathfrak{A}_0 \) and \( Q\mathfrak{B} \to \mathfrak{B}_0, \) and \( Q\mathcal{J} \to \mathcal{J}_0. \) Also, we shall denote by

\[
[A, B]_\ast = A \ast B - (-1)^{ij} B \ast A,
\]

for \( A \in \mathfrak{A}_i, B \in \mathfrak{A}_j, \) the graded commutator in \( Q\mathfrak{A} \) with respect to the \( \ast \)-product. (Although in most cases \( i \) and \( j \) are even, as it is the case in the lemma below.)

**Lemma 12.** Let \( u \in \mathfrak{B}_0 \) be an invertible element with inverse \( v. \) Choose liftings \( A \) and \( B \) of \( u \) and, respectively, \( v. \) Also, let \( B' := \sum_{k=0}^{\infty} (-1)^k B(dAdB)^k \) and let \( \tau \) be a closed graded trace on \( \mathcal{J} \) satisfying \( \tau([\mathfrak{A}, \mathfrak{J}]) = 0. \) Then

\[
\tilde{\tau} \circ \partial[u] = \tau([A, B']_\ast).
\]

**Proof.** The map

\[
\pi_* : K^\text{alg}_1(Q\mathfrak{B}) \to K^\text{alg}_1(\mathfrak{B}_0)
\]

is onto because if \( u \in \mathfrak{B}_0 = \Psi^\infty_{\text{inv}}(Y; \mathbb{C}^N)/\Psi^-\infty(\mathcal{N}) \) is invertible in \( \mathfrak{B}_0 \) with inverse \( v, \) then its image \( u' \) in \( Q\mathfrak{B} \) is also invertible with inverse

\[
v' := \sum_{k=0}^{\infty} (-1)^k v(dudv)^k.
\]

(The sum is actually finite for our algebras.) The relations

\[
u' \ast v' = 1 = v' \ast u'
\]

are easily checked. From the naturality of the boundary map in algebraic \( K \)-theory, we obtain that

\[
\pi_* \circ \partial_Q = \partial \circ \pi_* : K^\text{alg}_1(Q\mathfrak{B}) \to K^\text{alg}_0(\mathcal{J}_0),
\]

and hence

\[
(45) \quad \tilde{\tau} \circ \partial[u] = \tau_* \circ \partial_Q[u'].
\]

This simple relation will play an important role in what follows because it reduces the computation of \( \tilde{\tau} \circ \partial \) to the computation of \( \tau_* \circ \partial_Q. \)

Lift \( u \in M_N(\mathfrak{B}_0) \) to an element \( A \in M_N(\mathfrak{A}_0) = M_N(\Psi^\infty_{\text{inv}}(Y)) \) and its inverse \( v \) to an element \( B \in M_N(\Psi^-\infty(\mathcal{J})) \), as in the statement of the lemma. This gives
for \( u' \) (the inverse of the image \( u' \) of \( u \) in \( Q^2A \) with respect to the \( \ast \) product) the explicit lift

\[
B' := \sum_{k=0}^{\infty} (-1)^kB(dAdB)^k.
\]

We now proceed by direct computation (as in [36], for example), using the explicit formula \( \partial_Q([u']) = [\epsilon_1] - [\epsilon_0] \) with \( \epsilon_0 = 1 \oplus 0 \) and

\[
(46) \quad \epsilon_1 = \begin{bmatrix}
2A \ast B' - (A \ast B')^2 & A \ast (2 - B' \ast A) \ast (1 - B' \ast A) \\
(1 - B' \ast A) \ast B' & (1 - B' \ast A)^2
\end{bmatrix}.
\]

(all products and powers are with respect to the \( \ast \)-product). Then,

\[
\tau_\ast \circ \partial_Q([u']) = \tau([\epsilon_1] - [\epsilon_0]) = \\
\tau(2A \ast B' - (A \ast B')^2 + (1 - B' \ast A)^2 - 1) = \tau((1 - B' \ast A)^2 - (1 - A \ast B')^2).
\]

Then we notice that \( \tau([A, B' - B' \ast A \ast B'_1]) = 0 \), because \( B' - B' \ast A \ast B' \in J \). This relation and its analogue obtained by switching \( A \) with \( B' \) then give that \( \tau_\ast \circ \partial_Q([u']) = \tau(A \ast B' - B' \ast A) \), and the lemma follows.

Let \( D \in \Psi^1(Y) \) be the operator used to define \( \overline{\text{TR}}_q \) and \( \text{TR}_R \) in the previous section. Also, let \( \iota : A^q \rightarrow \mathbb{C} \) be the isomorphism given by contraction with the (dual) of the top form on \( \mathbb{R}^q \). This gives rise to maps

\[
(47) \quad \overline{\text{TR}}_Y \otimes \iota, \text{TR}_R \otimes \iota : Q\Psi^\infty(Y) \rightarrow \Omega^*(B),
\]

which vanish on forms of degree less than \( q \) in \( d\tau_1, \ldots, d\tau_q \). Because \( \overline{\text{TR}}_Y \) and \( \text{TR}_R \) are \( C^\infty(B) \)-linear graded traces on \( A_0 \), \( \overline{\text{TR}}_Y \otimes \iota, \text{TR}_R \otimes \iota \) are \( \Omega^*(B) \)-linear (graded) traces. If \( \omega : \Omega^*(B) \rightarrow \mathbb{C} \) is a closed current, we denote

\[
(48) \quad \rho_\omega(A) = \langle \omega, \overline{\text{TR}}_Y \otimes \iota(A) \rangle.
\]

and note that \( \rho_\omega \) is a closed graded trace. Also, note that \( \tau_\omega(A) = \langle \omega, \overline{\text{TR}}_Y \otimes \iota(A) \rangle \), which we shall use to extend \( \tau_\omega \) to operators of non-integral orders, still preserving the tracial property. Moreover, \( \tau_\omega(dA) = 0 \), if \( A \) has non-integer order. Consequently, \( \tau_\omega([A, B'_1]) = 0 \) if \( \text{ord}A + \text{ord}B \) is not an integer. This is seen by noticing that \( \tau_\omega(d(D^{-2}A)) = 0 \) for \( \text{Re}(z) \) large first, and then for \( z \) such that \( z \text{ord}D + \text{ord}A \) not an integer, by analytic continuation.

We then have

**Lemma 13.** If \( A(z) \in Q^2A \) is holomorphic in a neighborhood of \( 0 \in \mathbb{C} \setminus \mathbb{Z}^* \), then the function \( \tau_\omega(D^{-z} \ast A(z)) \) holomorphic and at \( 0 \) it has a simple pole with residue \( \rho_\omega(A(0)) \).

**Proof.** We have that

\[
D^{-z} \ast A(z) = D^{-z}A(z) + dD^{-z}dA(z).
\]

Now we observe that

\[
\lim_{z \to 0} z^{-1}dD^{-z} = \lim_{z \to 0} z^{-1} \sum [t_j, D^{-z}]d\tau_j = -\sum [t_j, \log D]d\tau_j = -d\log D
\]

Consequently, \( D^{-z} \ast A(z) = D^{-z}B(z) \) for some holomorphic function \( B \) such that \( B(0) = A(0) \). The result then is an immediate consequence of Proposition [3].
Let \( \partial : K_1^{alg}(\mathcal{B}_0) = K_1^{alg}(\Psi_{inv}^\infty(Y) / \Psi_{inv}^\infty(Y)) \to K_0^{alg}(\Psi_{inv}^\infty(Y)) = K_0^{alg}(\mathcal{J}_0) \) be the boundary map in algebraic K-theory, as above. Recall that the map \( \tau_\omega : K_0^{alg}(\mathcal{J}_0) \to \mathbb{C} \) is given by the Equations (38) and (43) and that \( \rho_\omega \) is given by the Equation (13).

We continue to assume that \( Y = B \times Z \times \mathbb{R}^q \) is a flat \( \mathcal{G} = B \times \mathbb{R}^q \)-space, that \( q > 0 \), and that \( B \) and \( Z \) are compact.

**Theorem 6.** Let \( u \in M_N(\Psi_{inv}^\infty(Y) / \Psi_{inv}^\infty(Y)) \) be an invertible element, and choose \( A, B \in M_N(\Psi_{inv}^\infty(Y)) \) such that \( A \) maps to \( u \) and \( B \) maps to \( u^{-1} \). If \( \omega : \Omega^k(B) \to \mathbb{C} \) is a closed current such the \( k + q = 2p > 0 \) is even, then

\[
\tilde{\tau}_\omega \circ \partial \left[ u \right] = -2(-1)^p \tau_\omega((dAdB)^p) = 2(-1)^p \tau_\omega((dBdA)^p)
\]

\[
= -\rho_\omega(u^{-1}[\log D, u](u^{-1}du)^2p) - \rho_\omega(u^{-1}[\log D, du](u^{-1}du)^{2p-2}).
\]

**Proof.** We shall denote

\[
\mathfrak{A} = M_N(\Psi_{inv}^\infty(Y) \otimes \mathcal{C}^{\infty}(B) \Omega^* (B) \otimes \Lambda^* \mathbb{R}^q),
\]

as before. Moreover, \( \mathfrak{B} \) and \( \mathfrak{J} \) will have the meaning they had before.

We shall use Lemma [12]. Let \( B' = \sum_{k=0}^\infty (-1)^k B(dAdB)^k \) be as in that lemma and evaluate the commutator \([A, B']_* \) (with respect to the * product). We obtain

\[
[A, B']_* = \sum_{l=0}^\infty (-1)^l AB(dAdB)^l - \sum_{l=0}^\infty (-1)^l B(dAdB)^l A
\]

\[
+ \sum_{l=0}^\infty (-1)^l (dAdB)^{l+1} - \sum_{l=0}^\infty (-1)^l (dBdA)^{l+1},
\]

the sums being of course finite.

We next observe that \( \tau_\omega((ABdAdB)^l) = \tau_\omega(B(dAdB)^l A) \) and \( \tau_\omega((dAdB)^l) = -\tau_\omega((dBdA)^l) \) because \( \tau_\omega \) is a graded trace on \( \mathfrak{A} \) (with the usual product). Using this we obtain from Lemma [12] that

\[
\tilde{\tau}_\omega \circ \partial [u] = \tau_\omega(A \star B' - B' \star A) = 2(-1)^p \tau_\omega((dAdB)^p).
\]

This proves the first part of our formula.

We now prove the second part of our formula. The commutator \([A, B']_* = A \star B' - B' \star A \) maps to \( u \star v - v \star u = 0 \) in \( Q\mathfrak{B} \), and hence \([A, B']_* \) is in \( Q\mathfrak{J} \). Consequently,

\[
\tau_\omega([A, B']_*) = \lim_{z \to 0} \tau_\omega(D^{-z} \star [A, B']_*).
\]

Next we observe that \( \lim_{z \to 0} z^{-1} dD^{-z} = -d \log D \) and hence \( d \log D \) (defined in the canonical representation) is actually in \( \Psi_{inv}^\infty(Y) \), in spite of the fact that \( \log D \) is not in the algebra \( \Psi_{inv}^\infty(Y) \). Moreover, \( z^{-1} dD^{-z} \) is holomorphic at 0.

Using that \( \tau_\omega([A, D^{-z} B']_*) = 0 \) for all \( z \) such that \( -z + \text{ord } A + \text{ord } B \) is not an integer, we finally obtain

\[
\tau_\omega([A, B']_*) = \lim_{z \to 0} \tau_\omega(D^{-z} \star [A, B']_*)
\]

\[
= \lim_{z \to 0} \tau_\omega([D^{-z} A]_* \star B') = \lim_{z \to 0} z \tau_\omega(D^{-z} \star F(z)),
\]

where \( F(z) = z^{-1}[D^{-z}, D^z \star A]_* \star B' \). Since \( F \) is a holomorphic function in a neighborhood of 0 with

\[
F(0) = -[\log D, A]_* B' - d[\log D, A]_* dB',
\]
it further follows from Lemma 13 that
\[
\lim_{z \to 0} z \tau_\omega(D^{-z} \ast F(z)) = \rho_\omega(F(0)) = -\sum_k \rho_\omega(u^{-1}[\log D, u] \ast (u^{-1}du)^{2k}),
\]
because \(\rho_\omega\) is a closed graded trace.

Putting together the above formulae, we obtain the desired result.

The method we used in the previous theorem works even when \(Y\) is not a flat \(\mathcal{G}\) bundle, with only minor changes. Actually, using a trick of Connes, we can formally treat the general case to a large extent as if it was a flat bundle. For example, the definition of \(\mathfrak{A}, \mathfrak{J}, \) and \(\mathfrak{B}\) extend to this case without change. However, \(\omega\) and the differential \(d\) cannot be defined as before to enjoy all the previous properties. Here is what changes. First, we need to take care of the fact that \(\mathfrak{g}\) may not be orientable. This is easy dealt with by considering linear functionals on forms twisted with the orientation sheaf instead currents (which are linear functional on ordinary forms). We say then that the current \(\omega\) must be twisted with the orientation sheaf of \(\mathfrak{g}\).

Note that it still makes sense to talk about a closed twisted current, because the de Rham differential extends to the space \(\Omega^j \otimes \mathcal{O}\) of forms twisted with the orientation sheaf:
\[
d : \Omega^j \otimes \mathcal{O} \to \Omega^{j+1} \otimes \mathcal{O},
\]
and \(d^2 = 0\). Once we understand the nature of \(\omega\), the definitions of \(\tau_\omega\) and \(\rho_\omega\) carry through without any change.

The problem is to define the differential structure on \(\mathfrak{A}\) in the non-flat case. Choose a trivializing covering of \(B\) and a partition of unity \(\phi_\alpha\), subordinated to that covering. Using the construction from the flat case, on each of the trivializing open sets \(U_\alpha\) of the covering we have an operator \(d_\alpha : \mathfrak{A}|_{U_\alpha} \to \mathfrak{A}|_{U_\alpha}\) (which satisfies \(d_\alpha^2 = 0\), but this is irrelevant to us). Let \(\nabla := \sum \phi_\alpha d_\alpha\). Then \(\nabla : \mathfrak{A} \to \mathfrak{A}\) is a degree one derivation and \(\nabla^2(A) = [\Theta, A]\), for some \(\Theta \in \mathfrak{A}\). Moreover, \(\tau_\omega(\nabla(a)) = 0\) if \(a \in \mathfrak{J}\), \(\rho_\omega(\nabla(a)) = 0\) if \(a \in \mathfrak{J}\), and both \(\tau_\omega\) and \(\rho_\omega\) are traces on \(\mathfrak{A}\).

The fact that \(\nabla^2\) is not zero in general means that we cannot use it to define the Fedosov product using \(\nabla\). This is not a big issue, however, because we can proceed as in [14] (see also [35]). The idea, due to Connes, is to enlarge the algebra \(\mathfrak{A}\) and to perturb \(\nabla\) such that it becomes a differential. We now review this construction following [35].

We first introduce a formal variable \(X\) such that
\[
aXb = 0 \quad \text{and} \quad (aX)(bX) = a\Theta b \quad \text{if} \ a, b \in \mathfrak{A}
\]
(we never consider \(X\) alone, only in formulae like \(aX, Xb, \) or \(XaX\)). Define \(\overline{\mathfrak{A}} = \mathfrak{A} + X\mathfrak{A} + \mathfrak{A}X + X\mathfrak{A}X\), with the induced grading such that \(\deg X = 1\). We define then \(\overline{\nabla} = \nabla + (-1)^{\deg a}aX\). We can then extend \(\overline{\nabla}\) to a derivation of \(\overline{\mathfrak{A}}\) such that \(\overline{\nabla}(X) = 0\) and we can also extend \(\tau_\omega\) to a graded trace, call it \(\overline{\tau}_\omega\), on \(\overline{\mathfrak{A}}\).

The explicit formula for this trace is
\[
\tau_\omega(a_{00} + a_{01}X + Xa_{10} + Xa_{11}X) = \tau_\omega(a_{00}) - (-1)^{\deg a_{11}}\tau_\omega(\Theta a_{11}).
\]
The preceding theorem will then extend to this new setting where \(\overline{\mathfrak{A}}\) replaces \(\mathfrak{A}\), \(\overline{\nabla}\) replaces \(d\), and \(\overline{\tau}_\omega\) replaces \(\tau_\omega\). The place of \(\rho_\omega\) will be taken by \(\overline{\rho}_\omega\), defined by a formula similar for that for \(\overline{\tau}_\omega\) above:
\[
\overline{\tau}_\omega(a_{00} + a_{01}X + Xa_{10} + Xa_{11}X) = \overline{\tau}_\omega(a_{00}) - (-1)^{\deg a_{11}}\rho_\omega(\Theta a_{11}).
\]
This shows that \(\overline{\rho}_\omega\), unlike \(\tau_\omega\), is a closed graded trace (\(\overline{\tau}_\omega\) is not closed).
Note, in the following theorem, that the meaning of \( \tilde{\tau}_\omega \) does not change in the case of non-flat bundles.

**Theorem 7.** Let \( u \in M_N(\Psi^\infty_{\text{inv}}(Y)/\Psi^{-\infty}_{\text{inv}}(Y)) \) and \( A, B \in M_N(\Psi^\infty_{\text{inv}}(Y)) \) be as in Theorem 3. Also, let \( \omega : \Omega^k(B) \to \mathbb{C} \) is a closed twisted current (i.e., with values in the orientation sheaf of \( g \)) such the \( k + q = 2p \) is even. If \( \overline{d}, \overline{\tau}_\omega \), and \( \overline{\tau}_\omega \) are as above, then

\[
\tilde{\tau}_\omega \circ \partial[u] = -2(-1)^p \tau'(u^{-1}|\log D, u|(u^{-1}du)^2p) = 2(-1)^p \tau'(u^{-1}|\log D, u|u^{-1}du)^2p - \overline{\tau}_\omega(u^{-1}|d\log D, du|u^{-1}du)^{2p-2}).
\]

**Proof.** The proof is word for word the same, if we replace \( \tau \) with \( \tau' \), \( \rho \) with \( \tilde{\rho} \), \( \mathfrak{A} \) with \( \overline{\mathfrak{A}} \), and so on.

After the above discussion, the matter of extending Theorem 6 to the non-flat case seems a triviality. It is however far from being so, because the multiplication in the algebra \( \overline{\mathfrak{A}} \) is more complicated. In particular, it introduces the curvature \( \Theta \) of the vertical bundle \( T_{\text{vert}}(Y/G) \). The appearance of \( \Theta \) in formulae is actually a good thing, because we know from Theorem 7 above. Obtaining Theorem 5 from the Theorem 7 is a question that remains to be solved.

**6. Higher eta invariants in algebraic \( K \)-theory**

We consider the same setting as in the previous section. More precisely, \( Z \to B \) is a fiber bundle, \( G \to B \) is a vector bundle and \( Y = Z \times_B G \), with the induced action of \( G \). We state our results below for the case when both \( Z \) and \( G \) are trivial bundles. When dealing with non-trivial bundles, we replace \( \tau \) with \( \tau' \), \( d \) with \( \partial \), and \( \mathfrak{A} \) with \( \overline{\mathfrak{A}} \), as we did at the end of that section.

Consider then a closed current \( \omega : \Omega^k(B) \to \mathbb{C} \) and let \( \tau_\omega \) be the associated trace on \( \mathfrak{A} \). Then a direct computation gives that

\[
\phi_\omega(a_0, a_1, \ldots, a_l) = \tau_\omega(a_0a_1 \ldots da_l)/l!, \quad l = q + k
\]

is a \( l \)-Hochschild cocycle on \( \Psi^\infty_{\text{inv}}(Y) \). The Dennis trace map (24)

\[
K_i^{\text{alg}}(\Psi^\infty_{\text{inv}}(Y)) \to \text{HH}_i(\Psi^\infty_{\text{inv}}(Y))
\]

and the morphism \( \text{HH}_i(\Psi^\infty_{\text{inv}}(Y)) \to \mathbb{C} \) defined by \( \phi_\omega \) give rise by composition to a morphism

(50)

\[
\eta_\omega : K_i^{\text{alg}}(\Psi^\infty_{\text{inv}}(Y)) \to \mathbb{C}.
\]

Because the restriction of \( \phi_\omega \) to \( \Psi^{-\infty}_{\text{inv}}(Y) \) is cyclic (so it defines a cyclic cocycle), the composition

\[
K_i^{\text{alg}}(\Psi^{-\infty}_{\text{inv}}(Y)) \to K_i^{\text{alg}}(\Psi^\infty_{\text{inv}}(Y)) \xrightarrow{\eta_\omega} \mathbb{C}
\]

factors as

\[
K_i^{\text{alg}}(\Psi^{-\infty}_{\text{inv}}(Y)) \to K_i^{\text{top}}(\Psi^{-\infty}_{\text{inv}}(Y)) \xrightarrow{(\phi_\omega)_\ast} \mathbb{C},
\]

where \( (\phi_\omega)_\ast \) is the pairing of cyclic homology with topological \( K \)-theory. In particular, \( \eta_\omega \) is non-zero if \( \omega \) is not exact.

The morphism \( \eta_\omega \) does not factor through topological \( K \)-theory though. This is seen by noticing that

\[
K_i^{\text{top}}(\Psi^{-\infty}_{\text{inv}}(Y)) \to K_i^{\text{top}}(\Psi^0_{\text{inv}}(Y))
\]
vanishes for any \( p \), as proved in [20]. Moreover, for \( B \) reduced to a point, \( G = \mathbb{R} \), \( k = 1 \), \( \omega(f) = \int_{\mathbb{R}} f(t)dt \), and
\[
D = D_0 + \partial t \in M_N(\Psi_{\text{inv}}^\infty(Y)),
\]
the indicial map of an admissible (chiral) Dirac operator on \( Y \times \mathbb{R} \), the main result of [28] states that \( \eta_k(D) = \eta(D_0)/2 \), where \( \eta(D_0) \) is the “eta”–invariant introduced by Atiyah, Patodi, and Singer in [5]. As proved in [25], this gives that the eta invariant of \( D_0 \) is the value at \( D \) of a group morphism \( K_1(\Psi_{\text{inv}}^\infty(M \times \mathbb{R})) \to \mathbb{C} \). This group morphism coincides with \( \eta \), \( \omega(f) = \int_{\mathbb{R}} f \), in the above notation.

It is tempting then to try to define a higher eta invariant on \( \Psi_{\text{inv}}^\infty(M \times \mathbb{R}^q) \), \( q = 2k - 1 \), by the formula \( \eta_k(D_0) = \prod_q((D^{-1}dD)(2k-1)), \) where \( D = D_0 + c(\tau) \); however, as it was proved by Lesch and Pflaum, this is not multiplicative, and besides, it coincides with usual eta invariant of \( D_0 \) (up to a multiple depending only on \( k \)).

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