Characteristic cohomotopy classes for families of 4-manifolds

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Families of smooth closed oriented 4-manifolds with a complex spin structure are studied by means of a family version of the Bauer–Furuta invariants in the context of parametrised stable homotopy theory, leading to a definition of characteristic cohomotopy classes on Thom spectra associated to the classifying spaces of their complex spin diffeomorphism groups. This is illustrated with mapping tori of such diffeomorphisms and related to the equivariant invariants.

Introduction

The suggestion to extend the gauge theoretical invariants of smooth 4-manifolds to families and diffeomorphism groups of such has been around for quite a while, see [8], [9], or [4], for example. And there have already been some efforts in this direction, see [16], [17], [18], [13], and [15], mostly in the context of Seiberg-Witten invariants. This paper addresses the issue in the context of the Bauer–Furuta invariants [5], bringing affairs to a state which is conceptionally pleasing and accessible for calculations.
All 4-manifolds considered here will be closed and oriented. For simplicity it will also be assumed that the first Betti number vanishes. It is shown that there are natural characteristic cohomotopy classes for families of complex spin 4-manifolds, generalising the invariants of Bauer and Furuta [5]. As the latter are stable cohomotopy classes, the natural context for family invariants seems to be fibrewise stable homotopy theory. See [7] for an elementary approach and [14] for a more technical treatment. Section 2 explains how the monopole map of a family may be used to construct stable homotopy classes over the base of the family, following [5], see Theorem 2.1.

Section 3 discusses an important property of the family invariants: functoriality under pullbacks, see 3.1. Recall that characteristic classes for vector bundles can be described in two ways. Either by giving a universal class in the cohomology of the classifying space. Or by assigning to every vector bundle a class in the cohomology of the base such that these classes are compatible under pullbacks. These two descriptions are equivalent up to \( \lim^1 \)-terms. The former seems to be more suitable in the statements of the results, while the latter is used in the proofs. This is how we will proceed in this paper. Functoriality under pullbacks implies that for every complex spin 4-manifold \((X, \sigma_X)\) there is a universal characteristic class over classifying space of the complex spin diffeomorphism group \(\text{BSpiff}^c(X, \sigma_X)\), see 3.4. Section 1 contains a detailed description of what these spaces classify and how they relate to the ordinary diffeomorphism groups. Functoriality under pullbacks also determines the invariants of product families, see Corollary 3.3.

The family invariants are non-trivial for trivial reason, much the same as with Thom classes for vector bundles: the restriction of a family to a point in the base yields the Bauer–Furuta invariant of the fibre over that point, see 3.2. Thus, the classes constructed here behave rather like Thom classes than like Euler classes. In fact, there is an interpretation of the family invariants using the ordinary stable cohomotopy of Thom spectra for families of Fredholm operators. See Theorem 4.1 in Section 4.
Section 5 applies this to gain information about families of K3 surfaces. The example of the universal Kummer family also shows that the family invariant need not be determined by the ordinary invariant of the fibre. Section 6 illustrates how the characteristic cohomotopy classes for mapping tori may be used to define an invariant of isotopy classes of diffeomorphisms of 4-manifolds. The final Section 7 comments on the relationship of the family invariants with the equivariant invariants of [21].

1 Classifying spaces for spin diffeomorphism groups

If $X$ is a closed oriented manifold, the notation $\text{Diff}(X)$ will refer to the group of diffeomorphisms of $X$ which preserve the orientation. The space $B\text{Diff}(X)$ classifies smooth families $p : Y \to B$ with an orientation on the relative tangent bundle such that the fibres are diffeomorphic to $X$ as oriented manifolds. One may just as well assume that there is a metric on the relative tangent bundle since there is a contractible choice of these. One gets something different if one requires the fibres to be isometric to $X$ with a given metric; this defines the classifying space of the isometry group of that metric.

If $\sigma_X$ is a (real) spin structure on $X$, instead of looking at the group of diffeomorphisms $f$ which preserve the spin structure in the sense that $f^*\sigma_X$ is isomorphic to $\sigma_X$, one may consider the group $\text{Spiff}(X, \sigma_X)$ of pairs $(f, u)$, where $f$ is an orientation preserving diffeomorphism of $X$ and $u : f^*\sigma_X \to \sigma_X$ is an isomorphism of spin structures. The space $B\text{Spiff}(X)$ classifies smooth families $p : Y \to B$ with a spin structure on the relative tangent bundle such that the fibres are diffeomorphic to $X$ as spin manifolds. By definition, there is an exact sequence

$$1 \longrightarrow \text{Aut}(\sigma_X) \longrightarrow \text{Spiff}(X, \sigma_X) \longrightarrow \text{Diff}(X).$$
The 2-torus shows that the rightmost arrow need not be surjective. But since there are only finitely many spin structures on $X$, the classifying space of the image is a finite covering of $\text{BDiff}(X)$. The group $\text{Aut}(\sigma_X)$ is $\mathbb{Z}/2$. The exact sequence leads to a fibration between the classifying spaces. The 2-sphere shows that this need not split. However, this fibration shows that the map $\text{BSpiff}(X, \sigma_X) \to \text{BDiff}(X)$ induces isomorphisms between the higher homotopy groups $\pi_n$ for $n \geq 3$.

A group $\text{Spiff}^c(X, \sigma_X)$ can be defined similarly, using a complex spin structure $\sigma_X$ on $X$. Note that in the description of the families classified by $\text{BSpiff}^c(X, \sigma_X)$ one may require a unitary connection on the determinant line bundle since there is a contractible choice of these. However we do not require that the restrictions to the fibres give a fixed connection for $\sigma_X$. There is an exact sequence and a fibration as above. This time the group $\text{Aut}(\sigma_X)$ is the (gauge) group of maps from $X$ to $\mathbb{T}$. Since $b^1(X)$ is assumed to vanish, the space $\text{BAut}(\sigma_X)$ is equivalent to a copy of $\text{BT}$. Note that this differs from the description from the previous paragraph in the case the complex spin structure $\sigma_X$ comes from real spin structure. The fibration

$$
\text{BT} \longrightarrow \text{BSpiff}^c(X, \sigma_X) \longrightarrow \text{BDiff}(X)
$$

shows that the map $\text{BSpiff}^c(X, \sigma_X) \to \text{BDiff}(X)$ induces isomorphisms between the higher homotopy groups $\pi_n$ for $n \geq 4$.

## 2 Bauer–Furuta invariants for families

This section contains a framework to extend the Bauer–Furuta invariants [5] to families of smooth 4-manifolds. First an appropriate notion of a family of 4-manifolds with complex spin structure and the corresponding monopole map are explained. Then the invariants are defined.
2.1 The ingredients

A smooth family \( Y|B \) of 4-manifolds is given by a smooth submersion \( p : Y \to B \) between smooth manifolds \( Y \) and \( B \), such that the relative dimension of \( p \) is \( \dim(Y) - \dim(B) = 4 \). For any point \( b \) in \( B \), the fibre of \( p \) over \( b \) is a 4-manifold \( Y(b) \). All fibres are diffeomorphic if the base \( B \) is connected, which will be assumed throughout this section. The differential

\[
T_Y \longrightarrow p^*T_B
\]

is surjective. Its kernel \( T_{Y|B} \) is a 4-dimensional vector bundle over \( Y \), a subbundle of \( T_Y \). This is the relative tangent bundle, or vertical tangent bundle, or the tangent bundle along the fibres. The fibre in a point \( y \) is the tangent bundle in \( y \) of the fibre \( Y(p(y)) \) which passes through \( z \). Similarly, the relative cotangent bundle \( T_{Y|B}^\vee \) is the cokernel of the injection dual to (1). We will only consider oriented families, i.e. families such that the bundles \( T_{Y|B} \) and \( T_{Y|B}^\vee \) are oriented. The orientations induce orientations on every fibre. (The Klein bottle shows that families are not necessarily orientable just because the fibres and the base are.) As explained in Section 1, it can and will be assumed that there is a metric on the bundle \( T_{Y|B} \).

In order to define the monopole map for a smooth family \( Y|B \), a family of complex spin structures is required, i.e. a complex spin structure \( \sigma_{Y|B} \) on the relative tangent bundle. Note that this is a 4-dimensional vector bundle over \( Y \), so that a complex spin structure means that the structure group of that bundle is reduced to the group Spin\(^c\)(4). The spinor bundles will usually be denoted by \( W^\pm(\sigma_{Y|B}) \). As explained in Section 1, it can and will be assumed that a unitary connection \( A \) – referred to as the base connection – on the determinant line bundle \( L(\sigma_{Y|B}) \) is fixed.

One may ask the question whether or not one may always find a complex spin structure on the relative tangent bundle which restricts to the given one on the fibres. The homotopy theory from Section 1 displays a single obstruction living in \( H^3(B;\mathbb{Z}) \).
2.2 The monopole map

Given a family $p : Y \to B$ as above, let

$$\Lambda^k_{Y|B} = \Lambda^k T^*_{Y|B} \otimes L T_Y$$

be the bundle over $Y$ whose sections are the alternating $k$-linear forms on the relative tangent bundle with values in the Lie algebra $L T$ of $\mathbb{T}$. (Here and in the following, the notation $L T_Y$ is used for the trivial bundle with fibre $L T$ over $Y$.) The bundle $\Lambda^2_{Y|B}$ decomposes as $\Lambda^+_{Y|B} \oplus \Lambda^-_{Y|B}$ into the self-dual and the anti-self-dual part. Since the bundle $\Lambda^0_{Y|B}$ is the trivial line bundle, the sections thereof are just the $L T$-valued functions on $Y$. Thus, the direct image $p_* \Lambda^0_{Y|B}$ on $B$ contains a trivial line bundle isomorphic to $L T_B$ which is generated by the constants, i.e. by the functions which live on $B$.

If, in addition, a complex spin structure $\sigma_{Y|B}$ on $Y|B$ is given, consider the (infinite-dimensional) $\mathbb{T}$-vector bundles

$$\mathcal{U} = p_* W^+_{Y|B}(\sigma_{Y|B}) \oplus p_* \Lambda^1_{Y|B} \oplus L T_B$$
$$\mathcal{V} = p_* W^-_{Y|B}(\sigma_{Y|B}) \oplus p_* \Lambda^+_{Y|B} \oplus p_* \Lambda^0_{Y|B}$$

over the trivial $\mathbb{T}$-space $B$. The monopole map $\mu(Y|B, \sigma_{Y|B})$, defined as

$$(\phi, a, f) \mapsto (D_A + a)(\phi), F^+_{A+a} - \phi^2, d^*(a) - f),$$

is a $\mathbb{T}$-equivariant map from $\mathcal{U}$ to $\mathcal{V}$ over $B$.

2.3 The linearisation and its index bundles

The linearisation (at the origin) of the monopole map above is given as the $\mathbb{T}$-linear map $\lambda(Y|B, \sigma_{Y|B})$ from $\mathcal{U}$ to $\mathcal{V}$ over $B$ defined by

$$(\phi, a, f) \mapsto (D_A(\phi), d^+(a), d^*(a) - f).$$

(2)
It is fibrewise Fredholm over $B$, so that it has an index. That index is a virtual vector bundle over $B$, i.e. an element in the $K$-theory of $B$, in $KO_T(B)$ to be precise. (Since $B$ is a trivial $\mathbb{T}$-space, the group $KO_T(B)$ splits into a copy of $KO(B)$ and copies of $KU(B)$, one for each positive integer.) As one sees from (2), the index of the linearisation is the sum of the index of the family of Dirac operators and the index of the family of fundamental elliptic complexes.

The family of complex linear Dirac operators defines a class in the group $KU(B)$, which will be identified with its image in $KO_T(B)$. It seems harmless to call this class the Dirac bundle for short. Its rank is $(c_1^2 - s)/8$, where $s$ is the signature of the fibres, and $c_1$ refers to the first Chern class of the determinant line bundle restricted to the fibres.

The other summand is the index of a family of a real operators, so that the index is a class of rank $b^+(X)$ in the subgroup $KO(B)$ of $KO_T(B)$. This class is in fact the additive inverse of the plus bundle, namely of the flat vector bundle which is the self-dual subbundle $R^+ p_* \mathbb{R}Y$ of $R^2 p_* \mathbb{R}Y$. (Here, the symbol $\mathbb{R}Y$ denotes the sheaf of locally constant functions with values in $\mathbb{R}$ with the discrete topology.) See [2] and [3]. If the base space $B$ of the family is simply-connected, this flat bundle will always be trivial.

### 2.4 The invariants for families

Let $B$ be compact base manifold, and let $(Y|B, \sigma_Y|B)$ be a complex spin family over $B$ as in 2.1. The monopole map $\mu(Y|B, \sigma_Y|B)$ is a continuous map between Hilbert space bundles which is the sum of its linearisation $\lambda = \lambda(Y|B, \sigma_Y|B)$, which is fibrewise Fredholm over $B$, and another map $\kappa$. The arguments in [5] show that $\kappa$ is a fibre-preserving map which sends bounded disk bundles to subspaces which are proper over $B$. Furthermore pre-images under $\mu(Y|B, \sigma_Y|B)$ of bounded disk bundles are contained in bounded disk bundles.
Let us review the construction from [5] in this context. One chooses a finite-dimensional subbundle \( V \) inside \( \mathcal{V} \). It is required that \( V \) is sufficiently large, so that it contains the subspaces \( \text{Cok}(\lambda) \). Tautologically, \( \lambda \) maps the subbundle \( \lambda^{-1}(V) \) of \( \mathcal{U} \), which contains \( \text{Ker}(\lambda) \), into \( V \). Essentially by assumption, \( \mu \) maps \( \lambda^{-1}(V) \) near to \( V \). In particular, after possibly enlarging \( V \) further, one can achieve that the sphere bundle \( S_B(\mathcal{V} - V) \) is missed. Then the fibrewise one-point-compactification \( S_B^{\lambda^{-1}(V)} \) is mapped into \( S_B^V \setminus S_B(\mathcal{V} - V) \), which can be retracted to \( S_B^V \). This gives a map from \( S_B^{\lambda^{-1}(V)} \) to \( S_B^V \). While this is morally the map one wants, taken as it is, it does not define a class in \( [S_B^{\text{Ker}(\lambda)}; S_B^{\text{Cok}(\lambda)}]_B \) for the universe \( \mathcal{V} \). However, \( V = \text{Cok}(\lambda) \oplus (V - \text{Cok}(\lambda)) \), and the map \( \lambda \) induces an isomorphism of \( \lambda^{-1}(V) - \text{Ker}(\lambda) \) with \( V - \text{Cok}(\lambda) \). Thus, there is a unique dashed arrow such that the diagram

\[
\begin{array}{ccc}
\mathbb{Ker}(\lambda) \oplus (V - \text{Cok}(\lambda)) & \longrightarrow & S_B^{\text{Cok}(\lambda) \oplus (V - \text{Cok}(\lambda))} \\
\downarrow \cong & & \downarrow \\
S_B^{\lambda^{-1}(V)} & \longrightarrow & S_B^V \\
\end{array}
\]

commutes. It represents an element in \( [S_B^{\text{Ker}(\lambda)}; S_B^{\text{Cok}(\lambda)}]_B \) which does not depend on \( V \).

**Theorem 2.1.** For every complex spin family \( (Y|B, \sigma_Y|B) \) over \( B \), the monopole map defines a class in the group \( [S_B^{\text{Ker}(\lambda)}; S_B^{\text{Cok}(\lambda)}]_B \).

Standard arguments – using the contractibility of the choices involved – show that this class neither depends on the metric nor on the connection.

This is the Bauer–Furuta invariant of \( Y|B \) with respect to \( \sigma_Y|B \). As with any other invariant, the computability of these family invariants depends on structure the-
orems which describe how the invariants change when the families are changed. The following section contains such a structure theorem for the family invariants.

3 Functoriality

In this section, a complex spin 4-manifold \((X, \sigma_X)\) is fixed, and complex spin families with fibre \((X, \sigma_X)\) are considered for varying base \(B\). Given such a family over \(B\), a morphism \(B' \to B\) induces a pullback family over \(B'\). Pullback also induce a base change functor from the stable homotopy category over \(B\) to the stable homotopy category over \(B'\), see [7]. Inspection of the definitions immediately given the following structure result.

**Theorem 3.1.** Given a family over \(B\) and a map \(B' \to B\), the invariant of the pullback family over \(B'\) is the image of the invariant of the family over \(B\) under the homomorphism

\[
[B^\text{Ker}(\lambda), B^\text{Cok}(\lambda)]_B \to [B'^\text{Ker}(\lambda'), B'^\text{Cok}(\lambda')]_{B'},
\]

induced by the base change functor.

Let us see what this means for two distinguished classes of examples: if \(B'\) or \(B\) is a point.

3.1 Fibres

A morphism from a single point \(b\) into \(B\) as above corresponds to an identification of \((X, \sigma_X)\) with the fibre over \(b\).
Corollary 3.2. The homomorphism
\[
[S^B_{\text{Ker}(\lambda)}, S^B_{\text{Cok}(\lambda)}]^T \rightarrow [S^B_{\text{Ker}(\lambda)}, S^B_{\text{Cok}(\lambda)}]^T
\]
corresponding to the inclusion of a point in \(B\) sends the family invariant of a family over \(B\) to the ordinary invariant of its fibre \((X, \sigma_X)\) over that point.

This shows that the ordinary Bauer–Furuta invariants are contained in the family Bauer–Furuta invariants.

3.2 Products

A morphism from \(B\) to a single point corresponds to the product family \(B \times X\) with the product complex spin structure \(B \times \sigma_X\) over \(B\). All data are pullbacks from \(X\), considered as a family over a point. Note that in this case the plus bundle is the trivial bundle, and similarly for the Dirac bundle.

Corollary 3.3. The invariant of a product family \((B \times X, B \times \sigma_X)\) is the product of the ordinary invariant of \((X, \sigma_X)\) with the identity of \(B\).

Therefore the family invariants in product situations are completely understood in terms of the ordinary invariants.

3.3 A universal characteristic class

Fibrewise stable homotopy theory works well only if the base \(B\) is homotopy equivalent to a compact ENR or a finite CW-complex. In order to handle the universal base \(\text{BSpi}ff\times(X, \sigma_X)\) for the situation at hand, on better defines
\[
[?, ??]^T_{\text{BSpi}ff\times(X, \sigma_X)} = \lim_B [?, ??]^T_B,
\]
where the limit is over the subspaces $B$ of $\text{BSpiff}(X, \sigma_X)$ which satisfy the finiteness hypothesis and the corresponding restriction maps. See Section II.15 in [7]. Theorem 3.1 then immediately gives the following result.

**Theorem 3.4.** If $(X, \sigma_X)$ is a complex spin 4-manifold, the group

$$[S_B^{\text{Ker}(\lambda)}, S_B^{\text{Cok}(\lambda)}]_{\text{BSpiff}(X, \sigma_X)}$$

contains a universal characteristic class for families of complex spin 4-manifolds with fibre $(X, \sigma_X)$.

Note that this class is non-trivial for trivial reasons in general: restricted to a point it gives back the ordinary Bauer–Furuta invariant of $(X, \sigma_X)$. In this respect, this characteristic class behaves more like a Thom class than like an Euler class. In fact, an interpretation in terms of Thom spectra will be given in the next section.

## 4 Cohomotopy classes of Thom spectra

In this section, we will see how the fibrewise stable homotopy theory of the previous two sections, which is conceptionally preferable, can be translated – using one of the basic adjunctions of the game – into ordinary stable cohomotopy of Thom spectra, which might be more accessible for calculations. The motivation for the use of spectra is the same as in the ordinary case: while the sphere in a Hilbert space is contractible, it can be approximated by a sphere spectrum, which is not. Here, Thom spectra for families of Fredholm operators will be defined, such that the sphere spectrum can be recovered as the Thom spectrum of the identity over a singleton.
4.1 Thom spectra

Let $U$ and $V$ be bundles of Hilbert spaces over a compact base $B$. If $\lambda : U \to V$ is a family of Fredholm operators over $B$, we will now see how to obtain a Thom spectrum $B^\lambda$ in that situation.

Let us choose a Hilbert space $H$, which also serves as the universe for the spectrum, and a trivialisation $t : V \cong H_B$ over $B$. This exists by a theorem of Kuiper’s [11]. Consider then the composition

$$\lambda t : U \to V \cong H_B \to H$$

with the projection onto $H$. For every sufficiently large finite-dimensional subspace $V$ of $H$ the pre-image $\lambda_t^{-1}V$ is a subbundle of $U$ which is mapped to $V$ by $\lambda_t$. Let $B^{\lambda_t^{-1}V}$ be the Thom space of this bundle; this is the quotient of the fibrewise one-point-compactification $S^{\lambda_t^{-1}V}_B$ by the section at infinity. If one enlarges $V$ to $W$ the map $\lambda_t$ induces an isomorphism

$$B^{\lambda_t^{-1}W} \cong \Sigma^{W-V}B^{\lambda_t^{-1}V}$$

of Thom spaces. These define the spaces and structure maps of the Thom spectrum $B^\lambda$ of $\lambda$ with respect to the trivialisation $t$. In order to show that $B^\lambda_t$ is essentially independent of $t$, let us consider another trivialisation $B^{\lambda_t'}$. This differs from $t$ by an automorphism of $H_B$, i.e. by a map from $B$ to the unitary group of $H$, which is contractible again by Kuiper’s theorem. Using such a map one gets an essentially unique identification of the two Thom spectra $B^{\lambda_t}$ and $B^{\lambda_t'}$. This justifies the notation $B^\lambda$ to be used later.

As for functoriality, given a morphism of another space $B'$ into $B$, one may consider the pullback, say $\lambda'$, of a maps $\lambda$ as above. Then there is a commutative diagram

\[\text{Diagram}\]
The bundle \((\lambda')^{-1}V\) over \(B'\) is the pullback of the bundle \(\lambda^{-1}V\) over \(B\). Thus there is a map from \((B')^{(\lambda')^{-1}V}\) to \(B^{\lambda^{-1}V}\). These fit together to give a morphism

\[(B')^{\lambda'} \longrightarrow B^{\lambda}\]  

(3)

of spectra.

Everything works without changes in the case of a compact Lie group acting on everything in sight. See [19], for the extension of Kuiper’s theorem to this setting.

### 4.2 Cohomotopy classes

If \(T\) is a space, and \(S\) is a space over \(B\), there is a natural bijection between the set of maps from \(S\) to \(T\) and the set a fibrewise map from \(S\) to \(B \times T\) over \(B\). In the pointed context, this passes to an adjunction \([S/B, T] \cong [S, B \times T]_B\) between the direct image functor from the stable homotopy category over \(B\) to the ordinary stable homotopy category and the pullback functor in the other direction. Together with the observation that the direct image of a sphere spectrum \(S_B^V\) is the Thom spectrum \(S_B^V/B = B^V\), this yields the following result.
**Theorem 4.1.** The monopole map of a family of complex spin 4-manifolds over $B$ with fibre $(X, \sigma_X)$ defines a stable cohomotopy class in the group $\pi_0^0(B^\lambda)$. The cohomotopy class of a pullback family along a map $B' \to B$ is the composition with the morphism (3): the diagram

\[
\begin{array}{ccc}
B^\lambda & \xrightarrow{\mu} & S^0 \\
\mu' \downarrow & & \downarrow \\
(B')^{\lambda'} & \xrightarrow{\mu'} & 
\end{array}
\]

commutes. Therefore, these classes define a universal characteristic class in the group $\pi_0^0(B\text{Spiff}^c(X, \sigma_X)^\lambda)$.

The group $\pi_0^0(B\text{Spiff}^c(X, \sigma_X)^\lambda)$ should be thought of as the cohomotopy of the classifying space $B\text{Spiff}^c(X, \sigma_X)$ in degree $\lambda$.

## 5 Families of K3 surfaces

Just as many of examples of smooth 4-manifolds are given as complex surfaces, holomorphic families may be quarried for interesting examples of smooth families. However, finding non-trivial complete families of complex surfaces is esteemed to be hard. A classical example can be constructed from the tautological family of 4-tori over the moduli space $SO(4)/U(2) \cong S^2$ of complex structures on $\mathbb{R}^4$: the fibrewise Kummer construction yields a holomorphic family of K3 surfaces over the projective line, see [1], or [6] for a recent appearance. More generally, to illustrate the preceding results, let us consider families of K3 surfaces over $S^2$ with the standard (real) spin structure. The family need not come with a real spin structure on the relative tangent bundle, but there will always exits a complex spin structure which restricts to the spin structures on the fibres. The complex rank of the associated Dirac bundle is $-s(K3)/8 = 2$, and the rank of the plus bundle is $b^+(K3) = 3$. The bundle of self-dual harmonic 2-forms is clearly
trivial since the base of the family is simply-connected. However, the Dirac bundle need not be trivial. In fact, the universal Kummer family above is distinguished by the fact that its Dirac bundle has degree -2, see [1]. In order to say something in general, let us pause to review some homotopy theory of Thom spaces.

Let $V$ be a complex vector bundle of rank $r$ over a connected space $B$. If $B$ has a CW-filtration with quotients which are wedges of spheres $S^n$, the Thom space $B^V$ has a filtration with quotients which are wedges of $\mathbb{T}$-spheres $S^{r \mathbb{C} + n}$. This is non-equivariantly a CW-filtration. The bottom cell of the Thom space is given by the class of a compactified fibre. If the base space $B$ is itself an $n$-sphere, the situation is particularly easy. The CW-structure with two cells leads to a cofibre sequence

$$S^{r \mathbb{C} + (n-1)} \rightarrow S^{r \mathbb{C}} \rightarrow B^V \rightarrow S^{r \mathbb{C} + n} \rightarrow S^{r \mathbb{C} + 1}. \quad (4)$$

Therefore, the $\mathbb{T}$-equivariant stable homotopy type of the Thom space of $V$ is described by an element $J_V$ in $[S^{r \mathbb{C} + (n-1)}, S^{r \mathbb{C}}]^{\mathbb{T}} \cong \pi_{n-1}^{\mathbb{T}}$. The assignment $V \mapsto J_V$ is a version of the J-homomorphism.

Mapping the cofibre sequence (4) into a $\mathbb{T}$-trivial sphere $S^b$, one obtains a long exact sequence

$$[S^{r \mathbb{C} + (n-1)}, S^b]^{\mathbb{T}} \leftarrow [S^{r \mathbb{C}}, S^b]^{\mathbb{T}} \leftarrow [B^V, S^b]^{\mathbb{T}} \leftarrow [S^{r \mathbb{C} + n}, S^b]^{\mathbb{T}} \leftarrow [S^{r \mathbb{C} + 1}, S^b]^{\mathbb{T}}.$$ 

Note that in the particular case $n = 2$ the left most and right most groups are the same. This is relevant in the case $r = 2$, $b = 3$, and $n = 2$ encountered for the K3 families above. The structure of the sequence

$$[S^{2 \mathbb{C} + 1}, S^3]^{\mathbb{T}} \leftarrow [S^{2 \mathbb{C}}, S^3]^{\mathbb{T}} \leftarrow [B^V, S^3]^{\mathbb{T}} \leftarrow [S^{2 \mathbb{C} + 2}, S^3]^{\mathbb{T}} \leftarrow [S^{2 \mathbb{C} + 1}, S^3]^{\mathbb{T}} \quad (5)$$

is explained by the following result.
Proposition 5.1. Let $V$ be a complex vector bundle of rank 2 over a 2-sphere $B$. Consider the map

$$\mathbb{Z} \cong [S^{2C}, S^3]^T \leftarrow [B^V, S^3]^T$$

induced by the inclusion $S^{2C} \subset B^V$ of a fibre. The degree of $V$ is odd if and only if this map is injective with a cokernel of order 2. It is even if and only if this map is surjective with a kernel of order 2.

Proof. Let us consider the exact sequence (5). The two maps on the left and on the right are given by multiplication with $J_V$. To start with, let us look at the structure of the groups involved in (5). It is as follows.

$$[S^{2C}, S^3]^T \cong \mathbb{Z} \quad [S^{2C+1}, S^3]^T \cong \mathbb{Z}/2 \quad [S^{2C+2}, S^3]^T \cong \mathbb{Z}/2$$

The first two isomorphism are easy. One may proceed as in [5]. Let us turn to the third one. The group $[S^{2C+2}, S^3]^T \cong [S^{2C}, S^1]^T$ is a cokernel of the homomorphism $p^*: [S^0, S^0]^T \cong [D(2\mathbb{C})_+, S^0]^T \rightarrow [S(2\mathbb{C})_+, S^0]^T$ induced by the projection $p: S(2\mathbb{C})_+ \subset D(2\mathbb{C}) \cong_T S^0$ which sends $S(2\mathbb{C})$ to the point which is not the basepoint. While $[S^0, S^0]^T$ is a copy of the integers, generated by the identity, $[S(2\mathbb{C})_+, S^0]^T$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$, the copy of the integers being generated by $p$. Of course, $p^*(\text{id}) = p$, so that $p^*$ is a split injection, induces an isomorphism between the copies of the integers, and has cokernel $\mathbb{Z}/2$, as claimed. Up to isomorphism the exact sequence (5) thus looks like

$$\mathbb{Z}/2 \leftarrow \mathbb{Z} \leftarrow [B^V, S^3]^T \leftarrow \mathbb{Z}/2 \leftarrow \mathbb{Z}/2.$$ 

Note that the forgetful map $\Phi: [S^{2C}, S^3]^T \rightarrow [S^4, S^3] \cong \mathbb{Z}/2$ is surjective. By $\pi_*$-linearity it follows that the forgetful map $\Phi: [S^{2C+1}, S^3]^T \rightarrow [S^5, S^3] \cong \mathbb{Z}/2$ is an isomorphism and that the forgetful map $\Phi: [S^{2C+2}, S^3]^T \rightarrow [S^6, S^3] \cong \mathbb{Z}/24$ is injective. It follows that the two maps on the left and on the right of (5) are trivial if and only if $\Phi(J_V)$ is zero. They are surjective if and only if $\Phi(J_V)$ is non-zero. However, in the case of a 2-sphere, $\Phi(J_V)$ is trivial if and only the underlying real bundle of $V$ is trivial if and only the degree of $V$ is odd. \qed
In the case of a family of K3 surfaces, the map in Proposition 5.1 must be surjective: the family invariant maps to the ordinary Bauer–Furuta invariant of K3, which is a generator, as the isomorphism with $\mathbb{Z}$ sends it to the Seiberg-Witten invariant of $K3$, which is $\pm 1$. This proves the following result.

**Theorem 5.2.** Given a complex spin family of K3 surfaces over $S^2$ with the standard spin structure in each fibre, the index bundle of the Dirac operator has even degree.

Note that the family invariants are not determined by the ordinary invariants: each time there are two possible pre-images. It would be interesting to find families realizing the two different possibilities.

The universal Kummer family shows that the result is best possible. Though one is tempted to phrase it in terms of the first Chern class of the universal Dirac bundle over $\text{BSpiff}^c(K3)$, this would require knowledge of the homomorphism

$$H^2(\text{BSpiff}^c(K3); \mathbb{Z}) \longrightarrow \text{Hom}(\pi_2(\text{BSpiff}^c(K3)), \mathbb{Z}).$$

The behaviour of this map depends on the fundamental group of $\text{BSpiff}^c(K3)$, which is a problem of its own. The methods of the following section may be used to gain information about $\pi_1(\text{BSpiff}^c(X, \sigma_X))$ in general.

## 6 Mapping tori

Let $X$ be a 4-manifold as before. Every diffeomorphism of $X$ induces an automorphism of $H^2(X)$ which preserves the intersection form. As $X$ is simply-connected, the kernel of the resulting homomorphism from $\text{Diff}(X)$ to $\text{Aut}(H^2(X))$ is known to be the group of diffeomorphisms pseudo-isotopic to the identity, see [10]. However, pseudo-isotopy does not imply isotopy in dimension 4, see [12]. In this section, we show how the family invariants introduced above may be used to study these diffeomorphisms.
If \( f \) is a diffeomorphism of \( X \) which is homologically trivial, and \( \sigma_X \) is any complex spin structure on \( X \), there is an isomorphism \( u \) from \( f^*\sigma_X \) to \( \sigma_X \). The pair \((f,u)\) gives rise to a \( \mathbb{Z} \)-action on \((X,\sigma_X)\). As the discussion of Section 1 shows, the surjection from \( \text{Spi}f(X,\sigma_X) \) to the group of homologically trivial diffeomorphisms of \( X \) induces an isomorphism on components. Therefore, the choice of \( u \) does not matter, and need no longer be subject of our discussion.

Let \( Y \) be the mapping torus of \( f \), the quotient \( X \times_{\mathbb{Z}} \mathbb{R} \) of \( X \times \mathbb{R} \) by the \( \mathbb{Z} \)-action generated by \((x,t) \mapsto (f(x), t-1)\). The projection to the second component displays \( Y \) as the total space of a family over the circle \( C = \mathbb{R}/\mathbb{Z} \). In the situation given, the index bundle \( \lambda \) is trivial: the Dirac bundle is trivial as any virtual complex vector bundle over a circle is trivial, and the plus bundle is trivial by assumption on \( f \). In fact, the map induced by \( f \) on \( H^+(X) \) is the identity, so that the projection \( H^+(X) \times \mathbb{R} \to H^+(X) \times_{\mathbb{Z}} \mathbb{R} \) descends to an isomorphism \( H^+(X) \times (\mathbb{R}/\mathbb{Z}) \cong H^+(X) \times_{\mathbb{Z}} \mathbb{R} \). As for the Dirac bundle, the situation is more complicated – due to the lack of control over the action of \( f \) on the space of harmonic spinors. There does not seem to be a preferred trivialisation, but the following proposition comes for rescue.

**Proposition 6.1.** Let \( A : S^1 \to U(r) \) be any map. Then the induced self-map of the Thom space \( \Sigma^{r\mathbb{C}}S^1_+ \) of the trivial rank \( r \) bundle over \( S^1 \) is stably \( \mathbb{T} \)-homotopic to the identity.

**Proof.** The induced self-map of \( \Sigma^{r\mathbb{C}}S^1_+ \), which is the quotient of \( S^1 \times S^{r\mathbb{C}} \) by the section at infinity, is given by the formula

\[
(z,x) \mapsto (z, A(z) \cdot x).
\]  

(6)

Stably, \( S^1_+ \) splits as \( S^1 \vee S^0 \), and the self-map respects the ensuing decomposition \( \Sigma^{r\mathbb{C}}S^1_+ \cong \mathbb{T} \left( S^{r\mathbb{C}+1} \vee S^{r\mathbb{C}} \right) \) such that it is the identity on the second summand which corresponds to the base point. It remains to be shown that it is \( \mathbb{T} \)-homotopic to the identity on the first summand. But, the fixed point map

\[
[S^{r\mathbb{C}+1}, S^{r\mathbb{C}+1}]^T \longrightarrow [S^1, S^1] \cong \mathbb{Z}
\]
is an isomorphism sending the identity to the identity, and the self-map (6) induced by $A$ is clearly the identity on fixed points.

Any trivialisation of the Dirac bundle gives rise to an equivalence $C^\lambda \simeq_T \Sigma^\lambda C_+$. Given two trivialisations, the difference is a map $A$ as in the preceding proposition. As a consequence, the equivalence does not depend on the choice of the trivialisation, and we obtain from $f$ a well-defined invariant in $\pi^0_T(\Sigma^\lambda C_+)$. On balance, the element may depend on $f$, but the group does not.

If the diffeomorphism $f$ is isotopic to another diffeomorphism $g$, there is an isotopy from the identity to $gf^{-1}$, which can be used to define a diffeomorphism from the mapping torus of $f$ to the mapping torus of $g$ over the circle. Therefore, the family invariants of the two mapping tori agree.

**Theorem 6.2.** If $f$ is isotopic to $g$, then the invariants of them agree.

The invariant of the identity need not be trivial unless the ordinary invariant of $(X, \sigma_X)$ is trivial. This deficiency can easily be remedied as follows. For the identity, the family is trivial, and the characteristic class is the pullback of the ordinary Bauer–Furuta invariant. In general, the family invariant of the mapping torus of $f$ restricts to the same class in $\pi^0_T(S^\lambda)$ as does the pullback: the images are just the ordinary invariants of the fibre $(X, \sigma_X)$. Therefore, their difference is an element in the kernel of the restriction. This kernel is identified with the 0-th cohomotopy group of the sphere $\Sigma^\lambda C$ by the remarks above. This difference will be referred to as the reduced invariant of $f$.

**Corollary 6.3.** If $f$ is a diffeomorphism isotopic to the identity, then the reduced invariant of $f$ is zero.

The reduced invariants also behave well under iterations. Given a diffeomorphism $f$ and an integer $m$, it is common to consider the mapping tori $Y(m)$ of the iterate $f^m$ of $f$ as well. This is the total space of a family over a circle $C(m)$. If $m$ is a multiple of $n$, the family $Y(m)$ is the pullback of $Y(n)$ along the
map $C(m) \to C(n)$ of degree $m/n$. Thus, the reduced invariant of $f^m$ is the image of the invariant of $f^n$ under the map induced by $C(m) \to C(n)$. In particular, taking $n = 1$, this shows that the reduced invariant of $f^m$ is divisible by $m$.

### 7 Group actions

If a compact Lie group $G$ acts on $X$ preserving a complex spin structure $\sigma_X$, there is an extension $\mathbb{G}$ of $G$ by $\mathbb{T}$ such that the homomorphism $G \to \text{Diff}(X)$ lifts to a homomorphism $\mathbb{G} \to \text{Spiff}^\mathbb{R}(X, \sigma_X)$. In [21] there has been constructed an equivariant invariant which lives in $\pi^0_G(S^\mathbb{L})$ and maps to the Bauer–Furuta invariant under the forgetful map $\pi^0_G(S^\mathbb{L}) \to \pi^0_\mathbb{T}(S^\mathbb{L})$, and I will now explain how this relates to the present construction.

Given a lift as above, the universal characteristic class can be restricted from the Thom spectrum $B\text{Spiff}^\mathbb{R}(X, \sigma_X)^\mathbb{L}$ to the Thom spectrum $B\mathbb{G}^\mathbb{L}$ to give a class in the group $\pi^0_\mathbb{T}(B\mathbb{G}^\mathbb{L})$. I do not see a direct way to compare this with the equivariant class in $\pi^0_G(S^\mathbb{L})$. There is, however, a universal equivariant class in $\pi^0_G(B\text{Spiff}^\mathbb{R}(X, \sigma_X)^\mathbb{L})$ which maps to both of them with the obvious maps

$$\pi^0_\mathbb{T}(B\text{Spiff}^\mathbb{R}(X, \sigma_X)^\mathbb{L}) \leftarrow \pi^0_G(S^\mathbb{L}) \rightarrow \pi^0_\mathbb{T}(B\mathbb{G}^\mathbb{L}) \rightarrow \pi^0_G(B\text{Spiff}^\mathbb{R}(X, \sigma_X)^\mathbb{L}).$$

No further ideas are needed for that. In this sense the present paper conceptionally complements [21].
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