UNIQUENESS OF STEADY 1-D SHOCK SOLUTIONS IN A FINITE NOZZLE VIA VANISHING VISCOSITY ARGUMENTS

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Dedicated to Professor Shuxing Chen on the occasion of his 80th birthday

Abstract. This paper studies the uniqueness of steady 1-D shock solutions in a finite flat nozzle via vanishing viscosity arguments. It is proved that, for both barotropic gases and non-isentropic gases, the steady viscous shock solutions converge under the $L^1$ norm. Hence only one shock solution of the inviscid Euler system could be the limit as the viscosity coefficient goes to 0, which shows the uniqueness of the steady 1-D shock solution in a finite flat nozzle. Moreover, the position of the shock front for the limit shock solution is also obtained.

1. Introduction. This paper studies the uniqueness of steady 1-D shock solutions in a finite flat nozzle via vanishing viscosity arguments. In a flat nozzle with parallel walls, the steady inviscid flow can be described by the following steady 1-D Euler system under the assumption that the flow parameters depend only on the space variable $x$:

$$
\partial_x (\rho u) = 0, \quad (1.1)
$$
$$
\partial_x (\rho u^2 + p) = 0, \quad (1.2)
$$
$$
\partial_x (\rho u \Phi) = 0, \quad (1.3)
$$

where $u$ is the velocity, $(p, \rho)$ represents the pressure and the density, and $\Phi := \frac{1}{2}u^2 + i$ with $i := e + p/\rho$ the enthalpy and $e$ the internal energy. Only two of the thermal parameters $(p, \rho, e)$ are independent. In this paper, the fluid is supposed to be a polytropic gas satisfying the state equation

$$
e = \frac{1}{\gamma - 1} \cdot \frac{p}{\rho}, \quad (1.4)
$$

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with $\gamma > 1$ the adiabatic exponent, or

$$p = e^{S/c_v} \rho^\gamma,$$  \hspace{1cm} (1.5)

with $S$ the entropy, and $c_v$ the specific heat at constant volume. Let $c := \sqrt{\partial_p \rho} = \sqrt{\gamma p/\rho}$ be the sonic speed and $M := u/c$ be the Mach number. Then the flow is supersonic in case $u > c$, or equivalently, $M > 1$, and it is subsonic in case $u < c$, or equivalently, $M < 1$.

Let $U = (u, p, \rho)^\top$ represents the state of the flow. Then for a supersonic state $U_0 := (q_0, p_0, \rho_0)^\top$, that is,

$$q_0 > \sqrt{\gamma p_0/\rho_0}, \quad \text{or equivalently,} \quad M_0^2 = \frac{\rho_0 q_0^2}{\gamma p_0} > 1,$$  \hspace{1cm} (1.6)

there exists a unique subsonic state $U_1 := (q_1, p_1, \rho_1)^\top$ such that the following Rankine-Hugoniot conditions hold, which should be satisfied across a normal shock front:

$$\rho_0 q_0 = \rho_1 q_1,$$  \hspace{1cm} (1.7)

$$\rho_0 q_0^2 + p_0 = \rho_1 q_1^2 + p_1,$$  \hspace{1cm} (1.8)

$$\frac{1}{2} q_0^2 + \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} = \frac{1}{2} q_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1}.$$  \hspace{1cm} (1.9)

Direct computation yields that, with $\mu^2 := \frac{\gamma - 1}{\gamma + 1}$, and $c_+^2 = \mu^2(q_0^2 + \frac{2\gamma}{\gamma - 1} \cdot \frac{p_0}{\rho_0})$,

$$q_1 = c_+^2/q_0,$$  

$$p_1 = ((1 + \mu^2)M_0^2 - \mu^2) \cdot p_0,$$  

$$\rho_1 = \frac{q_0^2}{c_+^2} \cdot \rho_0.$$  

And it is easy to check that $q_1 < q_0$, $p_1 > p_0$, and $\rho_1 > \rho_0$. Without loss of generality, we may assume that

$$\rho_0 q_0 = 1.$$  \hspace{1cm} (1.10)

Let the nozzle be bounded in the interval

$$\mathcal{N} := \{x : 0 < x < 1\}.$$
Then for any $0 < x_s < 1$, with $x = x_s$ being the position of the shock front,

$$U^0(x; x_s) := \begin{cases} U_0, & 0 \leq x < x_s, \\ U_1, & x_s < x \leq 1, \end{cases}$$

(1.11) gives a normal shock solution, which also satisfies the entropy condition $p_1 > p_0$, to the steady 1-D Euler system (1.1) - (1.3) (see Figure 1). That is, for a given supersonic state $U_0$ at the entrance of the nozzle, the state $U_1$ at the exit is uniquely determined, while the position of the shock front can be arbitrary. This non-uniqueness of normal shock solutions in a flat nozzle has been a well-known fact from the viewpoint of the inviscous flows. In this paper, we are going to investigate the uniqueness of the inviscid shock solution by regarding it as a vanishing viscosity limit of smooth viscous solutions $U = U^\varepsilon(x)$, with $\varepsilon > 0$ the viscosity coefficient, to the steady 1-D Navier-Stokes system below

$$\partial_x (\rho u) = 0, \quad \partial_x (\rho u^2 + p) = \varepsilon \partial_{xx} u, \quad \partial_x (\rho u \Phi) = \varepsilon \partial_x (u \partial_x u),$$

(1.12) - (1.14)

with the boundary conditions

$$U(0) = U_0, \quad u(1) = q_1.$$  

(1.15) - (1.16)

It is interesting to find that, among all the possible shock solutions in (1.11) with different positions of the shock front, only one could be the limit of the viscous solutions $U^\varepsilon(x)$ to the boundary value problem of the equations (1.12) - (1.14) with the boundary conditions (1.15) - (1.16) as the viscosity coefficient $\varepsilon \to 0+$. This fact yields the uniqueness of the steady 1-D normal shock solution to the Euler system (1.1) - (1.3) for given $U_0$ from the viewpoint of vanishing viscosity limits. This paper will present vanishing viscosity arguments for the uniqueness of the steady 1-D shock solutions for barotropic gases and polytropic gases.

**Uniqueness of shock solutions for barotropic gases.**

Under the assumption that the entropy $S$ remains constant in the whole flow fields, the fluid can be further supposed to be barotropic with the state equation

$$p = \rho^\gamma.$$  

(1.17)

Then the steady Euler system (1.1)-(1.3) could be simplified as the following isentropic Euler system:

$$\partial_x (\rho u) = 0, \quad \partial_x (\rho u^2 + p) = 0,$$

(1.18) - (1.19)

and the steady Navier-Stokes system (1.12) - (1.14) is simplified as

$$\partial_x (\rho u) = 0, \quad \partial_x (\rho u^2 + p) = \varepsilon \partial_{xx} u.$$

(1.20) - (1.21)

This paper is going to prove the following theorem, which shows the uniqueness of the steady normal shock solutions for barotropic gases.

**Theorem 1.1.** For any $\varepsilon > 0$, there exists a unique solution $U = U_0^\varepsilon(x) \in C^2[0, 1]$ to the equations (1.20) - (1.21) with the boundary conditions (1.15) and (1.16).
Furthermore, let
\[ U_0^b(x) := \begin{cases} U_0, & 0 \leq x < X_s, \\ U_1, & X_s < x \leq 1, \end{cases} \] (1.22)
where, with \( M_1 := \frac{q_1}{\sqrt{\gamma p_1/\rho_1}} \) being the Mach number at the exit,
\[ X_s := \frac{q_0^{\gamma+1}}{\gamma} \cdot \frac{\gamma - q_0^{\gamma+1}}{q_0^{\gamma+1} - q_1^{\gamma+1}} = M_0^2 \cdot \frac{1 - M_1^2}{M_0^2 - M_1^2}. \] (1.23)
Then \( U_0^b(x) \) is a shock solution to the equations (1.18) - (1.19) with the boundary condition (1.15), and \( U_\varepsilon^b \) converges to \( U_0^b \) in \( L^1(0,1) \) as \( \varepsilon \to 0^+ \), namely,
\[ \lim_{\varepsilon \to 0^+} \int_0^1 |U_\varepsilon^b(x) - U_0^b(x)| \, dx = 0. \] (1.24)

**Remark 1.1.** The convergence of the viscous solutions \( U_\varepsilon^b \) as the viscosity coefficient \( \varepsilon \to 0^+ \) yields the uniqueness of shock solutions to the isentropic Euler system (1.18) - (1.19). That is, from the viewpoint of vanishing viscosity limit, \( U_0^b \) is the only admissible shock solution and the position of the shock front is \( x = X_s \). See Figure 2 for the graphs for the velocity functions \( u^\varepsilon \) for different viscosity \( \varepsilon \) and the limit function \( u^0 \) as \( \varepsilon \to 0^+ \).

![Figure 2](image_url)

**Figure 2.** The velocity functions \( u^\varepsilon(x) \) for different viscosity \( \varepsilon > 0 \) and their limit as \( \varepsilon \to 0^+ \).

**Uniqueness of shock solutions for polytropic gases.**
For the non-isentropic polytropic gases with the state equation (1.4) or (1.5), this paper will establish a similar result as Theorem 1.1 under an additional assumption on the Mach number of the supersonic state \( U_0 \). The following theorem will be proved.

**Theorem 1.2.** Suppose that
\[ 1 < M_0^2 < \frac{2\gamma}{\gamma - 1}. \] (1.25)
Then for any \( \varepsilon > 0 \), there exists a unique solution \( U = U_\varepsilon^p(x) \in C^2[0,1] \) to the equations (1.12)-(1.14) with the boundary conditions (1.15) and (1.16).
Furthermore, let

\[ U_0^0(p)(x) := \begin{cases} U_0, & 0 \leq x < X_s, \\ U_1, & X_s < x \leq 1, \end{cases} \]  

(1.26)

where

\[ X_s := M_0^2 \frac{1 - M_s^2}{M_s^2 - M_1^2}. \]  

(1.27)

Then \( U_0^0(p)(x) \) is a shock solution to the equations (1.1)-(1.4) with the boundary condition (1.15), and \( U_\varepsilon^p \) converges to \( U_0^0(p) \) in \( L^1(0,1) \) as \( \varepsilon \to 0+ \), namely,

\[ \lim_{\varepsilon \to 0+} \int_0^1 |U_\varepsilon^p(x) - U_0^0(p)(x)| \, dx = 0. \]  

(1.28)

Remark 1.2. The condition (1.25) is technically required in the analysis. Whether it can be removed or not is still open by now.

Remark 1.3. For barotropic gases, by (1.20) and the state equation (1.17), the velocity condition (2.6) at the exit of the nozzle is equivalent to the pressure condition, which is proposed by Courant and Friedrichs in [7]. However, for non-isentropic gases, this is not true, namely, the velocity condition (1.16) at the exit is not equivalent to the pressure condition. It is also open whether or not Theorem 1.2 still holds if (1.16) is replaced by a pressure condition.

The key difficulty in proving Theorem 1.1 and 1.2 is that one has no information on the limit of viscous solutions as the viscosity coefficient goes to 0, even whether the limit exists or not is not clear in advance. Therefore, one has to directly prove that the viscous solutions converge in some sense. The key observation in this paper is that the velocity functions of the viscous solutions are strictly decreasing such that the inverse functions are available, and the inverse functions converge pointwisely a.e. (see Figure 2). Then the pointwise convergence yields the convergence in \( L^1 \) space.

The mathematical analysis on gas flows with shocks in a finite nozzle plays a fundamental role in the theory for fluid dynamics. In [7], Courant and Friedrichs first gave a systematic analysis for various type of inviscous gas flows in a nozzle from viewpoint of nonlinear partial differential equations and special shock solutions had been established. Important progresses on the well-posedness of these shock solutions within the framework of inviscous flows have been made since then. In particular, in [19, 20], for gas flows governed by the quasi-one-dimensional model, Liu showed that the shock occurs in the expanding portion of the nozzle is dynamically stable, while the shock in the contracting portion is not. For steady 2-D flows in an expanding nozzle, with appropriate pressure condition at the exit, one can obtain a shock solution by assuming that the flow depends only on the radius parameter and the position of the shock front can be uniquely determined. The well-posedness of the shock solution has been established, for instance, in [5, 6, 8, 16, 17, 18]. While for steady 2-D flows in a finite flat nozzle, there exist infinite shock solutions and the position of the shock front can be arbitrary. By presuming that the shock front goes through a fixed point, given in advance artificially, the stability of the shock solution can also be established, for instance, see [4, 3, 24] and references therein. Without the assumption, it is showed recently in [10] that, for a generic perturbation of the flat nozzle with curved boundaries, there may exist multiple shock solutions with the same given pressure at the exit, similar with the phenomena observed in [9] for steady flows governed by quasi-one-dimensional model. Hence, it seems impossible
to determine the position of the shock front for steady flows in a flat nozzle within the framework of inviscid flows. Since it is generally physically accepted that the inviscid flows can be regarded as limits of the viscous flows as the viscosity parameter goes to 0, it is natural to ask whether it is true or not that all shock solutions in the flat nozzle could be the limits of the viscous flows with the same boundary data. For one-dimensional flows in the whole space, it has been shown that the viscous shock solutions converge to inviscid shock solutions, see, for instance, [2, 11, 12, 15, 23] and references therein. See also, for instance, [13, 14] and references therein for analysis on multi-dimensional shocks. For one-dimensional flows in a finite interval, it has been established in [1, 21, 22] the existence, uniqueness and stability of steady viscous shock solutions with certain inflow-outflow boundary data. However, it is not clear whether or not these viscous shock solutions converge as the viscous parameter goes to 0. In this paper, we are going to give a positive answer to this question by verifying Theorem 1.1 and 1.2.

The paper is organized as follows. In Section 2, we are going to prove Theorem 1.1 for barotropic gases by careful analysis on the viscous shock solutions and their limit behavior as the viscosity coefficient goes to 0. The problem is reformulated as a singular limit problem of an ordinary differential equation for the velocity \( u^\varepsilon \) with an unknown parameter. The velocity \( u^\varepsilon \) of the viscous shock solutions are proved to be strictly decreasing such that their inverse functions exist. The convergence of the solutions \( u^\varepsilon \) can be established by verifying that their inverse functions converge point-wisely, which implies that Theorem 1.1 holds. In Section 3, Theorem 1.2 for non-isentropic gases will be proved by employing the similar ideas as in Section 2.

2. Uniqueness of shock solutions for barotropic gases. In this section, we are going to show the uniqueness of shock solutions for barotropic gases via vanishing viscosity limit arguments and prove Theorem 1.1. Suppose that the fluid is a barotropic gas satisfying the state equation (1.17). Let \( U := (u, \rho)^\top \) represents the state of the flow. Then if the flow is inviscid, \( U \) satisfies the steady isentropic Euler equations (1.18) - (1.19); and if the viscosity is taken into account with the viscosity coefficient \( \varepsilon > 0 \), \( U \) satisfies the steady isentropic Navier-Stokes equations (1.20) - (1.21).

Let \( U_0 := (q_0, \rho_0)^\top \) be a supersonic state such that
\[
q_0 > \sqrt{\gamma \rho_0^{\gamma-1}}, \quad \text{or equivalently,} \quad M_0 = \frac{q_0}{\sqrt{\gamma \rho_0^{\gamma-1}}} > 1. \tag{2.1}
\]
Without loss of generality, we further assume that \( \rho_0 q_0 = 1 \), that is, (1.10) holds. Then there exists a unique subsonic state \( U_1 := (q_1, \rho_1)^\top \) satisfying the Rankine-Hugoniot conditions below for steady isentropic Euler system:
\[
\rho_0 q_0 = \rho_1 q_1, \tag{2.2}
\]
\[
\rho_0 q_0^2 + \rho_0^\gamma = \rho_1 q_1^2 + \rho_1^\gamma, \tag{2.3}
\]
and for any \( 0 < x_s < 1 \), with \( x = x_s \) being the position of the shock front,
\[
U^0(x; x_s) := \begin{cases} 
U_0, & 0 \leq x < x_s, \\
U_1, & x_s < x \leq 1,
\end{cases} \tag{2.4}
\]
is a normal shock solution to the steady isentropic Euler equations (1.18)-(1.19). It can be easily verified that \( 0 < q_1 < q_0 \) and \( \rho_1 > \rho_0 \).
Let $U = U^\varepsilon(x) := (u^\varepsilon(x), \rho^\varepsilon(x))^\top$ be the viscous shock solution to the steady isentropic Navier-Stokes equations (1.20)-(1.21) under the following boundary conditions

\begin{align}
U^\varepsilon(0) &= U_0, \\
u^\varepsilon(1) &= q_1.
\end{align}

We are going to show that $U^\varepsilon$ exists for any $\varepsilon > 0$ and they are convergent as $\varepsilon \to 0^+$, which yields that only one shock solution among $\{U^0(x; x_s); 0 < x_s < 1\}$ could be the limit of viscous solutions $U^\varepsilon(x)$.

2.1. Reformulation of the problem for the viscous shock solutions. By (1.20), it holds that

\[ \rho^\varepsilon u^\varepsilon \equiv \rho_0 q_0 = 1, \]

which implies

\[ \rho^\varepsilon = \frac{1}{u^\varepsilon}. \quad (2.7) \]

Then substituting (2.7) into (1.21), one obtains that

\[ \partial_x P(u^\varepsilon) = \varepsilon \partial_{xx} u^\varepsilon, \quad (2.8) \]

where

\[ P(u) := u + \frac{1}{u^\gamma}. \quad (2.9) \]

Then the boundary value problem of the equations (1.20) - (1.21) with the boundary conditions (2.5) - (2.6) is equivalent to the boundary value problem of the equation (2.8) with the following boundary conditions:

\[ u^\varepsilon(0) = q_0, \quad u^\varepsilon(1) = q_1. \quad (2.10) \]

Note that, by the Rankine-Hugoniot conditions (2.2) - (2.3), it holds that

\[ P(q_0) = P(q_1). \quad (2.11) \]

Lemma 2.1. Let $\varepsilon > 0$ and $u^\varepsilon \in C^2(0,1) \cap C^1[0,1]$ be a solution to the boundary value problem of the equation (2.8) with the boundary conditions (2.10). Then it holds that

\[ q_1 < u^\varepsilon(x) < q_0, \quad \text{for any } x \in (0,1). \quad (2.12) \]

Moreover, it also holds that

\[ \partial_{x} u^\varepsilon(0) < 0, \quad \text{and } \partial_{x} u^\varepsilon(1) < 0. \quad (2.13) \]

Proof. By employing the weak maximum principle for elliptic equations of second order, it is obvious that

\[ \min_{x \in [0,1]} u^\varepsilon(x) = q_1, \quad \text{and} \quad \max_{x \in [0,1]} u^\varepsilon(x) = q_0. \quad (2.14) \]

Furthermore, since $u^\varepsilon$ is not a constant function on $[0,1]$, by applying the strong maximum principle, one obtains that $u^\varepsilon$ cannot attain its maximum and its minimum in the interior of the interval $[0,1]$. That is, the estimate (2.12) holds. Then (2.13) follows immediately by applying the Hopf lemma.

Integrating (2.8) over the interval $(0,x)$, one obtains

\[ \varepsilon \partial_x u^\varepsilon(x) = F(u^\varepsilon(x); \alpha^\varepsilon), \quad (2.15) \]

where, with $f(u) := P(u) - P(q_0)$,

\[ F(u; \alpha) := f(u) + \alpha, \quad (2.16) \]
and $\alpha^\varepsilon := \varepsilon \partial_x u^\varepsilon (0)$ is an unknown constant which should be determined together with $u^\varepsilon$ by the boundary condition (2.10). Obviously, $f$ is a smooth function defined in $(0, +\infty)$ and it is strictly convex, namely, $f''(u) > 0$ (see Figure 3). Moreover,

$$f(q_0) = f(q_1) = 0,$$

(2.17)

which implies that

$$f(u) < 0, \quad \text{for any } u \in (q_1, q_0),$$

(2.18)

and there exists a unique $q_* := \gamma \frac{1}{\sqrt{1+\varepsilon}} \in (q_1, q_0)$, such that $f'(q_*) = 0$.

By Lemma 2.1, one obtains that, for any $\varepsilon > 0$,

$$\alpha^\varepsilon < 0,$$

(2.19)

and if $u^\varepsilon$ is the solution to the equation (2.8) with the boundary condition (2.10),

$$F(u^\varepsilon; \alpha^\varepsilon) < 0.$$  

(2.20)

Thus, the boundary value problem of the equation (2.8) with the boundary conditions (2.10) is further reduced to the following problem.

The reduced viscous problem $\text{[RVP-B]}$.

Let $\varepsilon > 0$. Try to determine $(u^\varepsilon; \alpha^\varepsilon)$ satisfying the equation (2.15) and the boundary condition (2.10) under the restrictions of (2.12) and (2.19).

Obviously, as $\varepsilon \to 0^+$, the reduced viscous problem $\text{[RVP-B]}$ formally tends to the following problem.

The reduced viscous limit problem $\text{[RVLP-B]}$.

Try to determine $(u^0; \alpha^0)$ such that the following equation holds

$$F(u^0(x); \alpha^0) = 0,$$

(2.21)

in the weak sense that, for any test functions $\phi(x) \in C^\infty [0, 1]$,

$$\int_0^1 F(u^0(x); \alpha^0) \cdot \phi(x) \, dx = 0,$$

and the boundary condition (2.10) holds.

Obviously, there exists a solution to the problem $\text{[RVLP-B]}$ if and only if $\alpha^0 = 0$, and the function $u^0$ only takes values in the set $\{q_0, q_1\}$, namely, with $A \subset [0, 1]$ and $A^c = [0, 1] \setminus A$,

$$u^0(x) = \begin{cases} 
q_0, & x \in A, \\
q_1, & x \in A^c.
\end{cases}$$
Then the aim of this section is to solve the reduced viscous problem \([\text{RVP-B}]\) and to show that the solutions \(u^\varepsilon\) are convergent as \(\varepsilon\) goes to 0 with the limit function being a solution to the problem \([\text{RVLP-B}]\).

2.2. **Unique existence of the solution to the problem \([\text{RVP-B}]\).** By (2.20), one immediately has that the solution \(u^\varepsilon\) is strictly decreasing in \([0, 1]\), which implies that its inverse function exists, which will be denoted by \(x = X_\varepsilon(u)\) for \(u \in [q_1, q_0]\). Then \(X_\varepsilon\) can be determined by solving the boundary value problem below

\[
\begin{aligned}
\frac{dX_\varepsilon}{du} &= \frac{\varepsilon}{F(u; \alpha^\varepsilon)}, \quad u \in (q_1, q_0), \\
X_\varepsilon(q_1) &= 1, \quad X_\varepsilon(q_0) = 0.
\end{aligned}
\] (2.22)

Obviously, to determine the solution \((u^\varepsilon; \alpha^\varepsilon)\) of the problem \([\text{RVP-B}]\) is equivalent to determine \((X_\varepsilon; \alpha^\varepsilon)\) by solving the problem (2.22).

By (2.22), one immediately has that

\[
X_\varepsilon(u) = \int_{u}^{q_0} \frac{\varepsilon}{F(v, \alpha^\varepsilon)} dv,
\] (2.23)

where \(\alpha^\varepsilon < 0\) is determined by

\[
-\int_{q_1}^{q_0} \frac{\varepsilon}{F(v, \alpha^\varepsilon)} dv = 1.
\] (2.24)

Therefore, it suffices to show the unique existence of the constant \(\alpha^\varepsilon < 0\).

**Lemma 2.2.** For any \(\varepsilon > 0\), there exists a unique constant \(\alpha^\varepsilon < 0\) such that

\[
H_\varepsilon(\alpha^\varepsilon) = 1,
\] (2.25)

where

\[
H_\varepsilon(\alpha) := -\int_{q_1}^{q_0} \frac{\varepsilon}{F(v, \alpha)} dv, \quad \alpha < 0.
\]

**Proof.** For any \(\varepsilon > 0\), it is easy to see that \(H_\varepsilon(\alpha)\) is a continuous strictly increasing function with respect to \(\alpha \in (-\infty, 0)\). Moreover,

\[
\lim_{\alpha \to -\infty} H_\varepsilon(\alpha) = 0.
\]

We claim that

\[
\lim_{\alpha \to 0^-} H_\varepsilon(\alpha) = +\infty,
\] (2.26)

then there exists a unique constant \(\alpha^\varepsilon < 0\) such that (2.25) holds. Hence, it suffices to prove (2.26).

Since \(f\) is strictly convex in \([q_1, q_0]\) and \(f(q_1) = 0\), it holds that (see Figure 4), \(f'(q_1) < 0\) and for any \(u \in [q_1, q_0]\),

\[
0 > f(u) > f'(q_1) \cdot (u - q_1).
\]

Then

\[
H_\varepsilon(\alpha) = \int_{q_1}^{q_0} \frac{\varepsilon}{-f(v) - \alpha} dv > \int_{q_1}^{q_0} \frac{\varepsilon}{-f'(q_1) \cdot (v - q_1) - \alpha} dv
\]

\[
= -\frac{\varepsilon}{f'(q_1)} \ln \frac{-f'(q_1) \cdot (q_0 - q_1) - \alpha}{-\alpha}.
\]

Hence, (2.26) holds, which concludes the proof of the lemma. \(\square\)

Concluding the above argument, one has the following lemma.
Lemma 2.3. For any $\varepsilon > 0$, there exists a unique solution $(u^{\varepsilon}; \alpha^{\varepsilon})$ to the reduced viscous problem $[RVP\text{-}B]$.  

2.3. Convergence of $u^{\varepsilon}$ as $\varepsilon \to 0^+$. Now we are going to investigate the limit behavior of the viscous solutions $u^{\varepsilon}$ as $\varepsilon \to 0^+$. Since $u^{\varepsilon}$ is strictly decreasing, one has that

$$T.V. u^{\varepsilon} = q_0 - q_1,$$  (2.27)

where $T.V. u^{\varepsilon}$ represents the total variation of $u^{\varepsilon}$. That is, the total variation of $u^{\varepsilon}$ is uniformly bounded with respect to the viscous parameter $\varepsilon > 0$. Then by Helly theorem, there exists a subsequence $\{\varepsilon_n\}$ and a function $u^0$ of bounded variation, such that $\varepsilon_n \to 0$ as $n \to \infty$, and for a.e. $x \in [0,1]$,

$$u^{\varepsilon_n}(x) \to u^0(x), \quad \text{as } n \to \infty. \quad (2.28)$$

Then it is interesting to ask:

(Q1) whether or not the limit function $u^0$ is a weak solution of the limit problem $[RVLP\text{-}B]$.

(Q2) whether or not there exists another function $\tilde{u}^0$ with another subsequence $\{\tilde{\varepsilon}_n\}$, such that $\tilde{\varepsilon}_n \to 0$ as $n \to \infty$, and for a.e. $x \in [0,1]$,

$$u^{\tilde{\varepsilon}_n}(x) \to \tilde{u}^0(x), \quad \text{as } n \to \infty.$$

To answer the above questions, the key step is to investigate the limit behavior of $\alpha^{\varepsilon}$ as $\varepsilon$ goes to 0. One has the following lemma.

Lemma 2.4. It holds that

$$\lim_{\varepsilon \to 0^+} \alpha^{\varepsilon} = 0. \quad (2.29)$$

Proof. Let $\hat{u} := \frac{q_0 + q_1}{2}$, and

\[
\hat{L}(u) := \begin{cases} 
-k(u - q_1), & q_1 \leq u \leq \hat{u}, \\
 k(u - q_0), & \hat{u} \leq u \leq q_0,
\end{cases}
\]

where

\[
k := \frac{f(\hat{u})}{\hat{u} - q_0} = -\frac{f(\hat{u})}{\hat{u} - q_1} > 0.
\]

Since $f$ is strictly convex and (2.17) holds, one has that (see Figure 4),

$$f(u) \leq \hat{L}(u) \leq 0, \quad \text{for any } u \in [q_1, q_0],$$

Figure 4. The auxiliary functions for $f(u)$. 

\[
\hat{u}
\]

\[
-q_1
\]

\[
q_0
\]

\[
\hat{L}(u)
\]

\[
f(u)
\]

\[
f'(q_1)(u - q_1)
\]
and \( f(u) = \tilde{L}(u) \) if and only if \( u \) takes the values \( q_0 \), \( q_1 \) and \( \hat{u} \). Therefore, it holds that, for any \( \alpha < 0 \),
\[
H_\varepsilon(\alpha) = \int_{q_1}^{q_0} \frac{\varepsilon}{f(v) - \alpha} dv < \int_{q_1}^{q_0} \frac{\varepsilon}{\tilde{L}(v) - \alpha} dv
\]
\[
= \int_{q_1}^{\hat{u}} \frac{\varepsilon}{k(u-q_1) - \alpha} dv + \int_{\hat{u}}^{q_0} \frac{\varepsilon}{-k(u-q_0) - \alpha} dv
\]
\[
= \frac{\varepsilon}{k} \ln \frac{k(u-q_1) - \alpha}{-\alpha} + \frac{\varepsilon}{k} \ln \frac{-k(u-q_0) - \alpha}{-\alpha}
\]
\[
= \frac{2\varepsilon}{k} \ln \left( 1 - k(q_0-q_1) \cdot \frac{1}{2\alpha} \right),
\]
that is,
\[
H_\varepsilon(\alpha) < \tilde{H}_\varepsilon(\alpha) := \frac{2\varepsilon}{k} \ln \left( 1 - k(q_0-q_1) \cdot \frac{1}{2\alpha} \right), \quad \text{for} \ \alpha \in (-\infty, 0).
\]

Let, with sufficiently small \( \varepsilon > 0 \),
\[
\hat{\alpha}_\varepsilon := \frac{k(q_0-q_1)}{2(1-e^{k/(2\varepsilon)})} < 0,
\]
such that
\[
\tilde{H}_\varepsilon(\hat{\alpha}_\varepsilon) = 1.
\]
Then one obtains that
\[
H_\varepsilon(\hat{\alpha}_\varepsilon) < \tilde{H}_\varepsilon(\hat{\alpha}_\varepsilon) = 1 = H_\varepsilon(\alpha^\varepsilon).
\]
Since \( H_\varepsilon \) is strictly increasing with respect to \( \alpha \in (-\infty, 0) \), it holds that
\[
\hat{\alpha}_\varepsilon < \alpha^\varepsilon < 0.
\]

Then (2.29) holds since
\[
\lim_{\varepsilon \to 0^+} \hat{\alpha}_\varepsilon = 0,
\]
which concludes the proof of the lemma.

With Lemma 2.4, one can give a positive answer to the question (Q1) by verifying the following lemma.

\textbf{Lemma 2.5.} The limit function \( u^0 \) is a weak solution to the limit problem [\textit{RVLP-B}].

\textit{Proof.} For any function \( \phi(x) \in C^\infty[0,1] \), it holds that
\[
\varepsilon_n \int_0^1 \partial_x u^{\varepsilon_n} \phi dx = \varepsilon_n u^{\varepsilon_n} \phi |_{x=1} - \varepsilon_n \int_0^1 u^{\varepsilon_n} \partial_x \phi dx \to 0, \quad \text{as} \ n \to \infty,
\]
since \( \varepsilon_n \) tends to 0 as \( n \to \infty \). Moreover,
\[
\lim_{n \to \infty} \int_0^1 F(u^{\varepsilon_n}; \alpha^{\varepsilon_n}) \cdot \phi dx = \int_0^1 F(u^0; 0) \cdot \phi dx,
\]
which implies the lemma holds. \qed
To answer the question (Q2), we are going to show that the viscous solutions \( u^\varepsilon \) are convergent in \( L^1(0, 1) \), which yields that \( u^0 \) is the only possible limit function. To this aim, we are going to investigate the limit behavior of the inverse functions \( X_\varepsilon(u) \) as \( \varepsilon \to 0^+ \).

Consider the following function for \( u \in (q_1, q_0) \):

\[
I_\varepsilon(u) := \frac{X_\varepsilon(u)}{1 - X_\varepsilon(u)} = \frac{\int_{q_0}^u \frac{1}{F(v, \alpha^\varepsilon)} \, dv}{\int_{q_1}^u \frac{1}{F(v, \alpha^\varepsilon)} \, dv}.
\]

Then the following lemma holds.

**Lemma 2.6.** For any \( u \in (q_1, q_0) \), it holds that

\[
\lim_{\varepsilon \to 0^+} I_\varepsilon(u) = -\frac{f'(q_1)}{f'(q_0)}.
\]

**Proof.** Fixed \( u \in (q_1, q_0) \), for any sufficiently small \( \delta > 0 \) such that, with \( q_* \) satisfying \( f'(q_*) = 0 \),

\[
q_1 + \delta < \min \{u, q_*\} \leq \max \{u, q_*\} < q_0 - \delta,
\]

it holds that (see Figure 5)

\[
\int_u^{q_0} \frac{1}{F(v, \alpha^\varepsilon)} \, dv = \int_u^{q_0} \frac{1}{f(v) + \alpha^\varepsilon} \, dv + \int_u^{q_0-\delta} \frac{1}{f(v) + \alpha^\varepsilon} \, dv,
\]

\[
\int_{q_1}^{u} \frac{1}{F(v, \alpha^\varepsilon)} \, dv = \int_{q_1}^{u} \frac{1}{f(v) + \alpha^\varepsilon} \, dv + \int_{q_1+\delta}^{u} \frac{1}{f(v) + \alpha^\varepsilon} \, dv.
\]

Since \( f \) is strictly convex, one has that, for any \( \xi \in (q_1, q_1 + \delta) \),

\[
f'(q_1) < f'(\xi) < f'(q_1 + \delta).
\]

Thus, as \( v \in (q_1, q_1 + \delta) \), since

\[
f(v) = \int_0^1 f'(q_1 + t(v - q_1)) \, dt \cdot (v - q_1),
\]

it holds that

\[
f'(q_1) \cdot (v - q_1) < f(v) < f'(q_1 + \delta) \cdot (v - q_1) < 0.
\]

Therefore,

\[
\int_{q_1}^{q_1+\delta} \frac{1}{f(v) + \alpha^\varepsilon} \, dv \leq \int_{q_1}^{q_1+\delta} \frac{1}{f'(q_1) \cdot (v - q_1) + \alpha^\varepsilon} \, dv
\]

\[\text{Figure 5.} \ \text{Auxilliary points for } f(u).\]
Hence, one obtains that
\[ \int_{q_1}^{q_1 + \delta} \frac{1}{f'(v) + \alpha^\varepsilon} \, dv \geq \int_{q_1}^{q_1 + \delta} \frac{1}{f'(q_1 + \delta) \cdot (v - q_1) + \alpha^\varepsilon} \, dv = \frac{1}{f'(q_1 + \delta)} \cdot \ln \frac{\delta \cdot f'(q_1 + \delta) + \alpha^\varepsilon}{\alpha^\varepsilon}, \]
that is,
\[ \frac{1}{f'(q_1 + \delta)} \cdot \ln \frac{\delta \cdot f'(q_1 + \delta) + \alpha^\varepsilon}{\alpha^\varepsilon} \leq \int_{q_1}^{q_1 + \delta} \frac{1}{f'(v) + \alpha^\varepsilon} \, dv \leq \frac{1}{f'(q_1 + \delta)} \cdot \ln \frac{\delta \cdot f'(q_1) + \alpha^\varepsilon}{\alpha^\varepsilon}. \]
Moreover, as \( v \in (q_1 + \delta, u) \), since
\[ f(q_1) \leq f(v) \leq f(q_1 + \delta), \]
it holds that
\[ \int_{q_1 + \delta}^{u} \frac{1}{f'(v) + \alpha^\varepsilon} \, dv \leq \int_{q_1 + \delta}^{u} \frac{1}{f(q_1) + \alpha^\varepsilon} \, dv = \frac{u - q_1 - \delta}{f(q_1) + \alpha^\varepsilon}, \]
and
\[ \int_{q_1 + \delta}^{u} \frac{1}{f(q_1 + \delta) + \alpha^\varepsilon} \, dv \leq \int_{q_1 + \delta}^{u} \frac{1}{f(q_1) + \alpha^\varepsilon} \, dv = \frac{u - q_1 - \delta}{f(q_1 + \delta) + \alpha^\varepsilon}. \]
that is,
\[ \frac{u - q_1 - \delta}{f(q_1 + \delta) + \alpha^\varepsilon} \leq \int_{q_1 + \delta}^{u} \frac{1}{f'(v) + \alpha^\varepsilon} \, dv \leq \frac{u - q_1 - \delta}{f(q_1) + \alpha^\varepsilon}. \]
Hence, one obtains that
\[
\frac{1}{f'(q_1 + \delta)} \cdot \ln \frac{\delta \cdot f'(q_1 + \delta) + \alpha^\varepsilon}{\alpha^\varepsilon} + \frac{u - q_1 - \delta}{f(q_1 + \delta) + \alpha^\varepsilon} \leq \int_{q_1}^{u} \frac{1}{F(v, \alpha^\varepsilon)} \, dv = \int_{q_1}^{q_1 + \delta} \frac{1}{f'(v) + \alpha^\varepsilon} \, dv + \int_{q_1 + \delta}^{u} \frac{1}{f'(v) + \alpha^\varepsilon} \, dv \leq \frac{1}{f'(q_1)} \cdot \ln \frac{\delta \cdot f'(q_1) + \alpha^\varepsilon}{\alpha^\varepsilon} + \frac{u - q_1 - \delta}{f(q_1) + \alpha^\varepsilon} < 0. \tag{2.31}
\]
Analogously, one obtains that
\[
\frac{1}{f'(q_0 - \delta)} \cdot \ln \frac{-\alpha^\varepsilon}{\delta \cdot f'(q_0 - \delta) - \alpha^\varepsilon} + \frac{q_0 - u - \delta}{f(q_0 - \delta) + \alpha^\varepsilon} \leq \int_{q_0}^{u} \frac{1}{F(v, \alpha^\varepsilon)} \, dv = \int_{q_0 - \delta}^{q_0} \frac{1}{f'(v) + \alpha^\varepsilon} \, dv + \int_{q_0 - \delta}^{u} \frac{1}{f'(v) + \alpha^\varepsilon} \, dv \leq \frac{1}{f'(q_0)} \cdot \ln \frac{-\alpha^\varepsilon}{\delta \cdot f'(q_0) - \alpha^\varepsilon} + \frac{q_0 - u - \delta}{f(q_0) + \alpha^\varepsilon} < 0. \tag{2.32}
\]
Hence, applying Lemma 2.4, one obtains that, for given \( u \in (q_1, q_0) \) and for any sufficiently small \( \delta > 0 \),
\[
\limsup_{\varepsilon \to 0+} I_\varepsilon (u) = \limsup_{\varepsilon \to 0+} \frac{\int_{q_1}^{q_0} \frac{1}{F(v, \alpha^\varepsilon)} \, dv}{\int_{q_1}^{q_0} \frac{1}{F(v, \alpha^\varepsilon)} \, dv}.
\]
\[
\begin{align*}
\leq \limsup_{\varepsilon \to 0^+} \frac{1}{f'(q_0 - \delta)} \cdot \ln \frac{-\alpha \varepsilon}{\delta \cdot f'(q_0 - \delta) - \alpha \varepsilon + \frac{q_0 - u - \delta}{f(q_0 - \delta) + \alpha \varepsilon}} + \frac{u - q_1}{f(q_*) + \alpha \varepsilon} \\
= - \frac{f'(q_1)}{f'(q_0 - \delta)},
\end{align*}
\]
and
\[
\begin{align*}
\liminf_{\varepsilon \to 0^+} I_\varepsilon (u) &= \liminf_{\varepsilon \to 0^+} \frac{\int_{u}^{q_0} 1}{F'(v, \alpha \varepsilon)} dv \\
&\geq \liminf_{\varepsilon \to 0^+} \frac{1}{f'(q_0)} \cdot \ln \frac{-\alpha \varepsilon}{\delta \cdot f'(q_0) - \alpha \varepsilon + \frac{q_0 - u}{f(q_0) + \alpha \varepsilon}} + \frac{u - q_1 - \delta}{f(q_0 + \delta) + \alpha \varepsilon} \\
&= - \frac{f'(q_1 + \delta)}{f'(q_0)},
\end{align*}
\]
that is,
\[
- \frac{f'(q_1 + \delta)}{f'(q_0)} \leq \liminf_{\varepsilon \to 0^+} I_\varepsilon (u) \leq \limsup_{\varepsilon \to 0^+} I_\varepsilon (u) \leq - \frac{f'(q_1)}{f'(q_0 - \delta)}. \tag{2.33}
\]
Then (2.30) follows by letting \( \delta \to 0^+ \).

It follows immediately from Lemma 2.6 that, for any \( u \in (q_1, q_0) \),
\[
\lim_{\varepsilon \to 0^+} X_\varepsilon (u) = - \frac{f'(q_1)}{f'(q_0) - f'(q_1)} := X_0 (u), \tag{2.34}
\]
which yields that
\[
\lim_{\varepsilon \to 0^+} \int_{q_1}^{q_0} |X_\varepsilon (u) - X_0 (u)| du = 0. \tag{2.35}
\]
That is, \( \{ X_\varepsilon (u); \varepsilon > 0 \} \) is Cauchy in \( L^1 (q_1, q_0) \) as \( \varepsilon \to 0^+ \). Since, for any \( \varepsilon_1, \varepsilon_2 > 0 \),
\[
\int_{0}^{1} |u^{\varepsilon_1} (x) - u^{\varepsilon_2} (x)| dx = \int_{q_1}^{q_0} \int_{0}^{1} |X_{\varepsilon_1} (u) - X_{\varepsilon_2} (u)| du,
\]
it follows that \( \{ u^{\varepsilon} (x); \varepsilon > 0 \} \) is Cauchy in \( L^1 (0, 1) \) as \( \varepsilon \to 0^+ \), namely, \( \{ u^{\varepsilon} (x); \varepsilon > 0 \} \) converges to some function \( u^0_\varepsilon (x) \in L^1 (0, 1) \) as \( \varepsilon \to 0^+ \). By (2.34), it is easy to check that
\[
u^0_\varepsilon (x) = \begin{cases}
q_0, & 0 \leq x < X_s, \\
q_1, & X_s < x \leq 1,
\end{cases}
\tag{2.36}
\]
where
\[
X_s := - \frac{f'(q_1)}{f'(q_0) - f'(q_1)} = \frac{q_0^{\gamma + 1}}{\gamma}, \quad \frac{\gamma - q_1^{\gamma + 1}}{q_0^{\gamma + 1} - q_1^{\gamma + 1}} = M_0^2 \cdot \frac{1 - M_1^2}{M_2^2 - M_1^2},
\]
and
\[
\lim_{\varepsilon \to 0^+} \int_{0}^{1} |u^{\varepsilon} (x) - u^0_\varepsilon (x)| dx = 0. \tag{2.37}
\]
Moreover, for any pointwisely convergent subsequence \( \{ u^\varepsilon_n; n \geq 1 \} \) with the limit function \( u^0 \) satisfying (2.28), it holds that
\[
u^0 (x) = u^0_s (x), \quad \text{a.e. } x \in [0,1].
\]

Concluding the above argument, one obtains the following lemma, which yields the negative answer to the question (Q2).

**Lemma 2.7.** The viscous solutions \( \{ u^\varepsilon (x); \varepsilon > 0 \} \) to the problems \([RVP-B]\) converge to \( u^0 (x) \), defined by (2.36), in \( L^1 (0,1) \) as \( \varepsilon \to 0+ \), namely, (2.37) holds.

Finally, it is obvious that Theorem 1.1 is an immediate consequence of Lemma 2.7.

3. **Uniqueness of shock solutions for non-isentropic gases.** In this section, we are going to prove Theorem 1.2 which shows the uniqueness of steady normal shock solutions for non-isentropic gases via vanishing viscosity arguments. The idea is similar as in Section 2 for barotropic gases and similar notations will be employed as there is no confusion taking place.

Let \( U_0 \) be a supersonic state satisfying (1.6), and \( U_1 \) be the unique subsonic state satisfying the Rankine-Hugoniot conditions (1.7)-(1.9). Then, for any \( 0 < x_s < 1 \), \( U^0 (x; x_s) \) defined by (1.11) gives a normal shock solution to the steady Euler system (1.1)-(1.3).

Let \( U = U^\varepsilon (x) := (u^\varepsilon (x), p^\varepsilon (x), \rho^\varepsilon (x))^T \) be the viscous shock solution to the steady non-isentropic Navier-Stokes equations (1.12) - (1.14) with the boundary conditions (1.15) and (1.16). The aim of this section is to show the existence of the viscous shock solutions \( U^\varepsilon \) for any \( \varepsilon > 0 \) and to verify their convergence in \( L^1 (0,1) \) as \( \varepsilon \to 0+ \). Hence, among all possible shock solutions \( \{ U^0 (x; x_s) \} \), only one could be the limit of the viscous solutions \( U^\varepsilon \).

3.1. **Reformulation of the problem for the viscous shock solutions.** By (1.12), it holds that
\[
\rho^\varepsilon u^\varepsilon \equiv \rho_0 q_0 = 1,
\]
which implies
\[
\rho^\varepsilon = \frac{1}{u^\varepsilon}.
\]

Then (1.13) and (1.14) become
\[
\partial_x (u^\varepsilon + p^\varepsilon) = \varepsilon \partial_{xx} u^\varepsilon,
\]
\[
\partial_x \left( \frac{1}{2} (u^\varepsilon)^2 + \frac{\gamma}{\gamma - 1} p^\varepsilon \right) = \varepsilon \left( u \partial_{xx} u^\varepsilon + (\partial_x u^\varepsilon)^2 \right).
\]

By eliminating the terms \( \partial_{xx} u \) in (3.2) and (3.3), one obtains
\[
\frac{1}{\gamma - 1} u^\varepsilon \partial_{xx} p^\varepsilon = - \frac{\gamma}{\gamma - 1} p^\varepsilon \partial_x u^\varepsilon + \varepsilon (\partial_x u^\varepsilon)^2.
\]

Substituting the above equation into (3.2), one has
\[
\partial_x u^\varepsilon - \gamma p^\varepsilon u^\varepsilon \partial_x u^\varepsilon + \frac{\varepsilon (\gamma - 1)}{u^\varepsilon} (\partial_x u^\varepsilon)^2 = \varepsilon \partial_{xx} u^\varepsilon.
\]

Then by employing the maximum principles for elliptic equations of second order, one obtains
\[
q_1 < u^\varepsilon (x) < q_0, \quad \text{for any } x \in (0,1),
\]
since, with $0 < q_1 < q_0$,
\[
    u^\varepsilon (0) = q_0, \quad u^\varepsilon (1) = q_1.
\]  
Moreover, by applying the Hopf lemma, it also holds that
\[
    \partial_x u^\varepsilon (0) < 0, \quad \text{and} \quad \partial_x u^\varepsilon (1) < 0.
\]  
Concluding the above argument, one has the following lemma.

**Lemma 3.1.** Let $\varepsilon > 0$ and $U^\varepsilon \in C^2(0,1) \cap C^1[0,1]$ be a solution to the boundary value problem of the equations (1.12)-(1.14) with the boundary conditions (1.15) and (1.16). Then the a priori estimates (3.5) and (3.7) for the velocity $u^\varepsilon$ hold.

With the estimates (3.5) and (3.7) for the velocity $u^\varepsilon$, we are going to reformulate the boundary value problem of the equations (1.12)-(1.14) with the boundary conditions (1.15) and (1.16) into a boundary value problem of an ordinary differential equation with an unknown parameter similar as the problem $[\textbf{RVP-B}]$ in Section 2.

Let
\[
    P_0 := \rho_0 q_0^2 + p_0, \quad \Phi_0 := \frac{1}{2} q_0^2 + \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0},
\]
and, by Lemma 3.1,
\[
    \alpha^\varepsilon := \varepsilon \partial_x u^\varepsilon (0) < 0.\tag{3.8}
\]
Integrating the equations (1.13) and (1.14) over the interval $(0, x)$, one obtains
\[
    (u^\varepsilon (x) + p^\varepsilon (x)) - P_0 = \varepsilon \partial_x u^\varepsilon (x) - \alpha^\varepsilon, \tag{3.9}
\]
\[
    \left( \frac{1}{2} (u^\varepsilon (x))^2 + \frac{\gamma}{\gamma - 1} p^\varepsilon (x) u^\varepsilon (x) \right) - \Phi_0 = \varepsilon u^\varepsilon (x) \partial_x u^\varepsilon (x) - q_0 \alpha^\varepsilon. \tag{3.10}
\]
Then it holds that
\[
    \left( \frac{1}{2} (u^\varepsilon (x))^2 - \frac{1}{\gamma - 1} p^\varepsilon (x) u^\varepsilon (x) \right) - P_0 u^\varepsilon (x) + \Phi_0 = (q_0 - u^\varepsilon (x)) \cdot \alpha^\varepsilon,
\]
which yields that
\[
    p^\varepsilon (x) = \frac{\gamma - 1}{2} \left\{ u (x) + \frac{2\Phi_0}{u (x)} - 2P_0 \right\} - (\gamma - 1) \left( \frac{q_0}{u^\varepsilon (x)} - 1 \right) \cdot \alpha^\varepsilon. \tag{3.11}
\]
Then substituting (3.11) into (3.9), one obtains an ordinary differential equation of first order for $u^\varepsilon$ with an unknown parameter $\alpha^\varepsilon < 0$ similar as (2.15):
\[
    \varepsilon \partial_x u^\varepsilon (x) = F(u^\varepsilon (x); \alpha^\varepsilon), \tag{3.12}
\]
with the right hand side being defined by
\[
    F (u; \alpha) := f (u) + \alpha \cdot g (u), \tag{3.13}
\]
where
\[
    f (u) := \frac{\gamma + 1}{2} u + (\gamma - 1) \frac{\Phi_0}{u} - \gamma P_0, \quad g (u) := \frac{\gamma}{\gamma - 1} \frac{q_0}{u}.
\]
Obviously, $f$ and $g$ are smooth functions defined in $(0, +\infty)$. Moreover, $f$ is strictly convex, namely, $f'' (u) > 0$ for any $u \in \mathbb{R}_+$, and by the Rankine-Hugoniot conditions (1.7)-(1.9), it holds that
\[
    f (q_1) = f (q_0) = 0, \tag{3.14}
\]
which implies that
\[
    f (u) < 0, \quad \text{for any} \quad u \in (q_1, q_0), \tag{3.15}
\]
and there exists a unique $q_* \in (q_1, q_0)$, such that $f'(q_*) = 0$ (see Figure 3).
Thus, the boundary value problem of the equations (1.12)-(1.14) with the boundary

conditions (1.15) and (1.16) could be reformulated as the following problem.

**The reduced viscous problem [RVP-P].**

Let \( \varepsilon > 0 \). Try to determine \((u^\varepsilon; \alpha^\varepsilon)\) satisfying the equation (3.12) and
the boundary condition (3.6) under the restrictions of (3.5) and (3.7).

As \( \varepsilon \to 0^+ \), the reduced viscous problem [RVP-P] formally tends to the following
problem.

**The reduced viscous limit problem [RVLP-P].**

Try to determine \((u^0; \alpha^0)\) such that the following equation holds

\[
F (u^0(x); \alpha^0) = 0, \tag{3.16}
\]

in the weak sense that, for any test functions \( \phi(x) \in C^\infty [0, 1] \),

\[
\int_0^1 F (u^0(x); \alpha^0) \cdot \phi(x) \, dx = 0,
\]

and the boundary condition (3.6) holds.

Obviously, there exists a solution to the problem [RVLP-P] if and only if \( \alpha^0 = 0 \),
and the function \( u^0 \) only takes values in the set \( \{q_0, q_1\} \); namely, with \( A \subset [0, 1] \) and \( A^c = [0, 1] \setminus A \),

\[
u^0(x) = \begin{cases}
q_0, & x \in A, \\
q_1, & x \in A^c.
\end{cases}
\]

### 3.2. Unique existence and convergence of the viscous solutions \((u^\varepsilon; \alpha^\varepsilon)\).

The aim of this section is to solve the reduced viscous problem [RVP-P] and to show
that the solutions \( u^\varepsilon \) are convergent as \( \varepsilon \) goes to 0 with the limit function being
a weak solution to the problem [RVLP-P]. In order to use the ideas in
Section 2, one needs that \( u^\varepsilon \) is strictly monotone such that it is invertible. The
following lemma presents a sufficient condition for the monotonicity.

**Lemma 3.2.** Suppose that (1.25) holds. Then for any \( u \in [q_1, q_0] \), and \( \alpha \in (-\infty, 0) \),
it holds that

\[
g(u) \geq \left(\gamma - (\gamma - 1) \frac{q_0}{q_1}\right) > 0, \tag{3.17}
\]

\[
F (u; \alpha) < 0. \tag{3.18}
\]

**Proof.** It is obvious that \( g(u) \) is strictly decreasing in \([q_1, q_0] \). Thus, for any \( u \in [q_1, q_0] \), one has

\[
g(u) \geq g(q_1) = \left(\gamma - (\gamma - 1) \frac{q_0}{q_1}\right).
\]

By applying the Rankine-Hugoniot conditions (1.7) - (1.9), it can be easily verified
that (3.17) holds under the condition (1.25).

Moreover, direct computation yields that

\[
\partial_u F (u; \alpha) = \frac{\gamma + 1}{2} - (\gamma - 1) (\Phi_0 - q_0\alpha) \cdot \frac{1}{u^2}.
\]

For any \( \alpha \in (-\infty, 0) \), let \( u_*(\alpha) > 0 \) be the constant satisfy \( \partial_u F (u_*(\alpha); \alpha) = 0 \), namely,

\[
\ \ 
\]

Therefore, \( u_*(\alpha) > q_1 \). Moreover, for any \( \alpha \in (-\infty, 0) \), if \( u_*(\alpha) \geq q_0 \), then \( F(u; \alpha) \)
is strictly decreasing in \([q_1, q_0] \); and if \( q_1 < u_*(\alpha) < q_0 \), then \( F(u; \alpha) \) is strictly
decreasing in \([q_1, u_*(\alpha)]\), while strictly increasing in \([u_*(\alpha), q_0]\). In both cases, in the interval \(u \in [q_1, q_0]\), the function \(F(u; \alpha)\) attains its maximum at either \(u = q_1\) or \(u = q_0\), that is,

\[
\max_{u \in [q_1, q_0]} F(u; \alpha) = \max \{F(q_1; \alpha), F(q_0; \alpha)\}.
\]

Since, under the condition (1.25),

\[
F(q_1; \alpha) = (\gamma - (\gamma - 1) \frac{q_0}{q_1})^{\alpha} < 0, \quad F(q_0; \alpha) = \alpha < 0,
\]

it follows that (3.18) holds, which complete the proof of the lemma. \(\square\)

By (3.12) and Lemma 3.2, one has that, under the condition (1.25), the solution \(u^\varepsilon\) is strictly decreasing. Therefore, one can follow the arguments in Section 2, with slight modifications, to conclude the following lemma.

**Lemma 3.3.** Suppose that (1.25) holds. Then for any \(\varepsilon > 0\), there exists a unique solution \((u^\varepsilon; \alpha^\varepsilon)\) to the reduced viscous problem [RVP-P]

Moreover, it holds that

\[
\lim_{\varepsilon \to 0^+} \alpha^\varepsilon = 0. \tag{3.19}
\]

Finally, as \(\varepsilon\) goes to 0, \(u^\varepsilon\) converges in \(L^1(0,1)\) to a weak solution to the problem [RVLP-P]:

\[
u_0^\varepsilon(x) = \begin{cases} q_0, & \text{for } 0 \leq x < X_s, \\ q_1, & \text{for } X_s < x \leq 1, \end{cases} \tag{3.20}
\]

where

\[
X_s := -\frac{f'(q_1)}{f'(q_0) - f'(q_1)} = \frac{1}{M_0^2 - M_1^2},
\]

that is,

\[
\lim_{\varepsilon \to 0^+} \int_0^1 |u^\varepsilon(x) - u_0^\varepsilon(x)| \, dx = 0. \tag{3.21}
\]

It is obvious that Theorem 1.2 is an immediate consequence of Lemma 3.3.

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