A Biorthogonal Approach to the Infinite Dimensional Fractional Poisson Measure

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November 27, 2023

Abstract

In this paper we use a biorthogonal approach to the analysis of the infinite dimensional fractional Poisson measure \( \pi^\beta_\sigma \), \( 0 < \beta \leq 1 \), on the dual of Schwartz test function space \( D' \). The Hilbert space \( L^2(\pi^\beta_\sigma) \) of complex-valued functions is described in terms of a system of generalized Appell polynomials \( P^{\sigma,\beta,\alpha} \) associated to the measure \( \pi^\beta_\sigma \). The kernels \( C^{\sigma,\beta}_n(\cdot) \), \( n \in \mathbb{N}_0 \), of the monomials may be expressed in terms of the Stirling operators of the first and second kind as well as the falling factorials in infinite dimensions. Associated to the system \( P^{\sigma,\beta,\alpha} \), there is a generalized dual Appell system \( Q^{\sigma,\beta,\alpha} \) that is biorthogonal to \( P^{\sigma,\beta,\alpha} \). The test and generalized function spaces associated to the measure \( \pi^\beta_\sigma \) are completely characterized using an integral transform as entire functions.

Keywords: fractional Poisson measure, generalized Appell system, Wick exponential, test functions, generalized functions, Stirling operators, \( S \)-transform.

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1 Introduction

In this paper we develop a biorthogonal approach to the analysis of the infinite dimensional fractional Poisson measure (fPm) on the configuration space $\Gamma$ or over $\mathcal{D}'$ (the dual of the Schwartz test function space $\mathcal{D}$). As a special case of a non-Gaussian measure (for which this biorthogonal approach was developed in [1, 23, 21]) the fPm revealed an interesting connection with the Stirling operators and falling factorials in the context of infinite dimensional analysis introduced recently in [11].

To describe our results more precisely, let us recall that there are different ways to introduce a total set of orthogonal polynomials in the Hilbert space of square integrable functions with respect to (wrt) a probability measure. For example, applying the Gram-Schmidt method to an independent sequence of functions or using generating functions. In the case at hand, that is, the fPm $\pi_{\beta,\sigma}$ ($0 < \beta \leq 1, \sigma$ a non-degenerate and non-atomic measure in $\mathbb{R}^d$), we have chosen the generating function procedure because the Gram-Schmidt method is not practical. In addition, the generating function is picked in a way such that at $\beta = 1$, we recover the classical Charlier polynomials, that is, $\pi_1$ coincides with the standard Poisson measure $\pi_\sigma$ on $\Gamma$, see [3] for more details. In explicit, given the map

$$\alpha : \mathcal{D}_\mathbb{C} \rightarrow \mathcal{D}_\mathbb{C}, \, \varphi \mapsto \alpha(\varphi)(x) := \log(1 + \varphi(x)), \quad x \in \mathbb{R}^d,$$

we define the modified Wick exponential

$$e_{\pi_{\beta,\sigma}}(\alpha(\varphi); w) := \exp\left(\frac{\langle w, \alpha(\varphi) \rangle}{l_{\pi_{\beta,\sigma}}(\alpha(\varphi))}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_{n,\beta}^\sigma(w), \varphi^\otimes n \rangle, \quad w \in \mathcal{D}'_\mathbb{C},$$

where $\varphi$ is properly chosen from a neighborhood of zero in $\mathcal{D}_\mathbb{C}$. The monomials $\langle C_{n,\beta}^\sigma(w), \varphi^\otimes n \rangle, \, n \in \mathbb{N}_0$ generates a system of polynomials $P_{\sigma,\beta,\alpha}$ which forms a total set in the space $L^2(\pi_{\beta}^\sigma)$ of square $\pi_{\beta,\sigma}$-integrable complex functions. The kernels $C_{n,\beta}^\sigma(\cdot), \, n \in \mathbb{N}_0$, possess certain remarkable properties involving the Stirling operators of the first and second kind as well as the falling factorials $(w)_n, \, w \in \mathcal{D}'_\mathbb{C}$ introduced in [11]. We refer to Proposition 5.2 and Appendix A.2 for more details and results. Other choice of generating functions like $e_{\pi_{\beta,\sigma}}(\varphi; \cdot)$ is also possible (see the beginning of Section 5), but at $\beta = 1$, the corresponding system of polynomials do not
coincide with the classical Charlier polynomials. Thus, our natural choice goes to the modified Wick exponential generating function \( e_{\pi_\sigma^\beta}(\alpha(\varphi); \cdot) \).

On the other hand, the construction of the generalized dual Appell system \( Q^{\sigma,\beta,\alpha}_\pi \) turns out to be very appealing since it involves a differential operator of infinite order on the space of polynomials \( \mathcal{P}(\mathcal{D}') \) over \( \mathcal{D}' \) and the adjoints of the Stirling operators. This careful choice of the system \( Q^{\sigma,\beta,\alpha}_\pi \) leads us to the so-called biorthogonal property between the two systems \( \mathbb{P}^{\sigma,\beta,\alpha} \) and \( Q^{\sigma,\beta,\alpha}_\pi \), see Theorem 5.7.

The generalized Appell system \( A^{\sigma,\beta,\alpha}_\pi := (\mathbb{P}^{\sigma,\beta,\alpha}, Q^{\sigma,\beta,\alpha}_\pi) \) is used to introduce a family of test function spaces \( (\mathcal{N})^{\kappa}_{\pi_\sigma^\beta} \), \( 0 \leq \kappa \leq 1 \), which are nuclear spaces and continuously embedded in \( L^2(\pi_\sigma^\beta) \). The dual space of \( (\mathcal{N})^{\kappa}_{\pi_\sigma^\beta} \) is given by the general duality theory as \( (\mathcal{N})^{-\kappa}_{\pi_\sigma^\beta} \). In this way, we obtain the chain of continuous embeddings
\[
(\mathcal{N})^{\kappa}_{\pi_\sigma^\beta} \subset L^2(\pi_\sigma^\beta) \subset (\mathcal{N})^{-\kappa}_{\pi_\sigma^\beta}.
\]

A typical example of a test function is the modified Wick exponential \( e_{\pi_\sigma^\beta}(\alpha(\varphi); \cdot) \in (\mathcal{N})^{\kappa}_{\pi_\sigma^\beta} \) (see Example 6.1) given as a convergent series in terms of the system \( \mathbb{P}^{\sigma,\beta,\alpha} \) while a particular element in \( (\mathcal{N})^{-1}_{\pi_\sigma^\beta} \) is given by the generalized Radon-Nikodym derivative \( \rho^{\alpha}_{\pi_\sigma^\beta}(w, \cdot), w \in \mathcal{N}_\sigma^\beta \).

Moreover, the generalized function \( \rho^{\alpha}_{\pi_\sigma^\beta}(w, \cdot) \) plays the role of the generating function of the system \( Q^{\sigma,\beta,\alpha}_\pi \), that is,
\[
\rho^{\alpha}_{\pi_\sigma^\beta}(w, \cdot) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathcal{Q}^{\sigma,\alpha}_{\pi_\sigma^\beta}((\pi_\sigma^\beta)(w)_k).
\]

The spaces \( (\mathcal{N})^{\pm\kappa}_{\pi_\sigma^\beta} \) may be characterized in terms of an integral transform, called the \( S^{\sigma,\beta}_{\pi_\sigma^\beta} \)-transform. It turns out that all these spaces \( (\mathcal{N})^{\pm\kappa}_{\pi_\sigma^\beta}, 0 \leq \kappa \leq 1 \), are universal in the sense that the \( S^{\sigma,\beta}_{\pi_\sigma^\beta} \)-transform of their elements are entire functions (for \( 0 \leq \kappa < 1 \)) or holomorphic functions (\( \kappa = 1 \)) and independent of the measure \( \pi_\sigma^\beta \), see Theorem 6.6. This feature is well known in non-Gaussian analysis.

The paper is organized as follows. In Section 2, we recall some known concepts of nuclear spaces and their tensor products. As a motivation to the generalization of \( \mathbb{P}^{m} \) to infinite dimensions, we discuss its finite dimensional version in Section 3. We show that the monic polynomials \( C^\beta_n(x), n \in \mathbb{N}_0 \), obtained using Gram-Schmidt orthogonalization process to the monomials \( x^n, n \in \mathbb{N}_0 \) are orthogonal in \( L^2(\pi_\lambda^\beta) \) (\( \pi_\lambda^\beta \) is the Poisson measure in 2-dimensions) if, and only if, \( \beta = 1 \). In Section 4, we define the \( \mathbb{P}^{m} \pi_\sigma^\beta \) in infinite dimensions as a probability measure on \( (\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}')) \), where \( \mathcal{C}_\sigma(\mathcal{D}') \) is the \( \sigma \)-algebra generated by the cylinder sets. We also discuss the concept of configuration space \( \Gamma \) and then using the Kolmogorov extension theorem we define a unique measure \( \pi_\sigma^\beta \) on the configuration space \( (\Gamma, \mathcal{B}(\Gamma)) \) whose characteristic function coincides with that of \( \pi_\sigma^\beta \) on the distribution space \( \mathcal{D}' \). In Section 5, we introduce the generalized Appell system associated with the \( \mathbb{P}^{m} \pi_\sigma^\beta \). This includes the system of generalized Appell polynomials and the dual Appell system which are biorthogonal with respect to the \( \mathbb{P}^{m} \pi_\sigma^\beta \).

Finally in Section 6, we construct the test and generalized function spaces associated to the \( \mathbb{P}^{m} \pi_\sigma^\beta \) and provide some properties as well as its characterization theorems.

For completeness, in the Appendices A.1–A.3 we provide certain concepts and results already known in the literature, particularly, the Kolmogorov extension theorem on the configuration space and Stirling operators in infinite dimensions.

## 2 Tensor Powers of Nuclear Spaces

We first consider nuclear Fréchet spaces (i.e., a complete metrizable locally convex space) that may be characterized in terms of projective limits of a countable number of Hilbert spaces, see
e.g., [6], [7], [12], [16] and [30] for more details and proofs.

Let $\mathcal{H}$ be a real separable Hilbert space with inner product $(\cdot, \cdot)$ and corresponding norm $|\cdot|$. Consider a family of real separable Hilbert space $\mathcal{H}_p$, $p \in \mathbb{N}$ with Hilbert norm $|\cdot|_p$ such that the space $\bigcap_{p \in \mathbb{N}} \mathcal{H}_p$ is dense in each $\mathcal{H}_p$, and

$$\cdots \subset \mathcal{H}_p \subset \cdots \subset \mathcal{H}_1 \subset \mathcal{H}$$

with the corresponding system of norms being ordered, i.e.,

$$|\cdot| \leq |\cdot|_1 \leq \cdots \leq |\cdot|_p \leq \cdots \quad p \in \mathbb{N}.$$

Now we assume that the space $\mathcal{N} = \bigcap_{p \in \mathbb{N}} \mathcal{H}_p$ is nuclear (i.e., for each $p \in \mathbb{N}$ there is a $q > p$ such that the canonical embedding $\mathcal{H}_q \hookrightarrow \mathcal{H}_p$ is of Hilbert-Schmidt class) and on $\mathcal{N}$ we fix the projective limit topology, i.e., the coarsest topology on $\mathcal{N}$ with respect to which each canonical embedding $\mathcal{N} \hookrightarrow \mathcal{H}_p$, $p \in \mathbb{N}$, is continuous. With respect to this topology, $\mathcal{N}$ is a Fréchet space and we use the notation

$$\mathcal{N} = \text{pr lim}_{p \in \mathbb{N}} \mathcal{H}_p$$

to denote the space $\mathcal{N}$ endowed with the corresponding projective limit topology. Such a topological space is called a projective limit or a countable limit of the family $(\mathcal{H}_p)_{p \in \mathbb{N}}$.

Let us denote by $\mathcal{H}_{-p}$, $p \in \mathbb{N}$, the dual of $\mathcal{H}_p$ with respect to the space $\mathcal{H}$, with the corresponding Hilbert norm $|\cdot|_{-p}$. By the general duality theory, the dual space $\mathcal{N}'$ of $\mathcal{N}$ with respect to $\mathcal{H}$ can then be written as

$$\mathcal{N}' := \bigcup_{p \in \mathbb{N}} \mathcal{H}_{-p}$$

with the inductive limit topology, i.e., the finest topology on $\mathcal{N}'$ with respect to which all the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous. This topological space is denoted by

$$\mathcal{N}' = \text{ind lim}_{p \in \mathbb{N}} \mathcal{H}_{-p}$$

and is called an inductive limit of the family $(\mathcal{H}_{-p})_{p \in \mathbb{N}}$. In this way we have obtained the chain of spaces

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$$

called a nuclear triple or Gelfand triple. The dual pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{N}$ and $\mathcal{N}'$ is then realized as an extension of the inner product $(\cdot, \cdot)$ on $\mathcal{H}$, i.e.,

$$\langle g, \xi \rangle = (g, \xi), \quad g \in \mathcal{H}, \xi \in \mathcal{N}.$$

The tensor product of the Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{N}$, is denoted by $\mathcal{H}_p^\otimes n$. We keep the notation $|\cdot|_p$ for the Hilbert norm on this space. The subspace of $\mathcal{H}_p^\otimes n$ of symmetric elements is denoted by $\mathcal{H}_p^\hat{\otimes} n$. The $n$-th tensor power $\mathcal{N}^\otimes n$ of $\mathcal{N}$ and the $n$-th symmetric tensor power $\mathcal{N}^\hat{\otimes} n$ of $\mathcal{N}$ are the nuclear Fréchet spaces given by

$$\mathcal{N}^\otimes n := \text{pr lim}_{p \in \mathbb{N}} \mathcal{H}_p^\otimes n \quad \text{and} \quad \mathcal{N}^\hat{\otimes} n := \text{pr lim}_{p \in \mathbb{N}} \mathcal{H}_p^\hat{\otimes} n.$$

Furthermore, if $\mathcal{H}_{-p}^\otimes n$ (resp., $\mathcal{H}_{-p}^\hat{\otimes} n$) denotes the dual space of $\mathcal{H}_p^\otimes n$ (resp., $\mathcal{H}_p^\hat{\otimes} n$) with respect to $\mathcal{H}^\otimes n$, then the dual space $(\mathcal{N}^\otimes n)'$ of $\mathcal{N}^\otimes n$ with respect to $\mathcal{H}^\otimes n$ and the dual space $(\mathcal{N}^\hat{\otimes} n)'$ of $\mathcal{N}^\hat{\otimes} n$ with respect to $\mathcal{H}^\hat{\otimes} n$ can be written as

$$(\mathcal{N}^\otimes n)' = \text{ind lim}_{p \in \mathbb{N}} \mathcal{H}_{-p}^\otimes n \quad \text{and} \quad (\mathcal{N}^\hat{\otimes} n)' = \text{ind lim}_{p \in \mathbb{N}} \mathcal{H}_{-p}^\hat{\otimes} n.$$
respectively. As before we use the notation $| \cdot |_p$ for the norm on $H^\otimes n_p$, $p \in \mathbb{N}$, and $\langle \cdot , \cdot \rangle$ for the dual pairing between $(H^\otimes n)'$ and $H^\otimes n$. Thus we have defined the nuclear triples

$$\mathcal{N}^\otimes n \subset H^\otimes n \subset (\mathcal{N}^\otimes n)' \quad \text{and} \quad \mathcal{N}^\otimes n \subset H^\otimes n \subset (\mathcal{N}^\otimes n)'.$$ 

To all the real spaces in this section, we may also consider their complexifications which will be distinguished by a subscript $\mathbb{C}$, i.e., the complexification of $H$ is $H_{\mathbb{C}}$ and so on. This means that for $h \in H_{\mathbb{C}}$, we have $h = h_1 + ih_2$ where $h_1, h_2 \in H$.

Let us now introduce spaces of entire functions which will be used later in the characterization theorems in Section 6. Let $\mathcal{E}_{2^{-l}}(H_{-p,\mathbb{C}})$ denote the set of all entire functions on $H_{-p,\mathbb{C}}$ of growth $k \in [1,2]$ and type $2^{-l}$, $p, l \in \mathbb{Z}$. This is a linear space with norm

$$n_{p,l,k}(\varphi) = \sup_{w \in H_{-p,\mathbb{C}}} |\varphi(w)| \exp(-2^{-l}|z|^k_p), \quad \varphi \in \mathcal{E}_{2^{-l}}(H_{-p,\mathbb{C}}).$$

The space of entire functions on $\mathcal{N}_C^\otimes n$ of growth $k$ and minimal type is naturally introduced by

$$\mathcal{E}_{\min}^k(\mathcal{N}_C^\otimes n) := \text{pr lim}_{p,l \in \mathbb{N}} \mathcal{E}_{2^{-l}}^k(H_{-p,\mathbb{C}}),$$

see e.g., [24, 6]. We will also need the space of entire functions on $\mathcal{N}_C$ of growth $k$ and finite type given by

$$\mathcal{E}_{\max}^k(\mathcal{N}_C) := \text{ind lim}_{p,l \in \mathbb{N}} \mathcal{E}_{2^{-l}}^k(H_{p,\mathbb{C}}).$$

3 Finite Dimensional Fractional Poisson Measure

In this section we discuss the finite dimensional version of the fractional Poisson measure as a motivation to its generalization to infinite dimensions. The one dimensional version of the fractional Poisson analysis was studied in [5].

At first we introduce the Mittag-Leffler function $E_\beta$ with parameter $\beta \in (0, 1]$. The Mittag-Leffler function is an entire function defined on the complex plane by the power series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}. \quad (3.1)$$

The Mittag-Leffler function plays the same role for the fPm as the exponential function plays for Poisson measure. Note that for $\beta = 1$ we have $E_1(z) = e^z$.

For any $0 < \beta \leq 1$, the fPm $\pi_{\lambda,\beta}$ on $\mathbb{N}_0$ (or $\mathbb{R}$) with rate $\lambda > 0$ is defined for any $B \in \mathcal{P}(\mathbb{N}_0)$ by

$$\pi_{\lambda,\beta}(B) := \sum_{k \in B} \frac{\lambda^k}{k!} E^{(k)}_\beta(-\lambda),$$

where $E^{(k)}_\beta(z) := \frac{d^k}{dz^k} E_\beta(z)$ is the $k$-th derivative of the Mittag-Leffler $E_\beta$ function. In particular, if $B = \{k\} \in \mathcal{P}(\mathbb{N}_0)$, $k \in \mathbb{N}_0$, we obtain

$$\pi_{\lambda,\beta}(\{k\}) := \frac{\lambda^k}{k!} E^{(k)}_\beta(-\lambda).$$

The Laplace transform of the measure $\pi_{\lambda,\beta}$ is given for any $z \in \mathbb{C}$ by

$$l_{\pi_{\lambda,\beta}}(z) = \int_\mathbb{R} e^{zx} d\pi_{\lambda,\beta}(x) = \sum_{k=0}^{\infty} \left(\frac{e^z}{\lambda}\right)^k E^{(k)}_\beta(-\lambda) = E_\beta(\lambda(e^z - 1)) . \quad (3.2)$$
Remark 3.1. The measure $\pi_{\lambda^\beta,\beta}$ corresponds to the marginal distribution of the fractional Poisson process $N_{\lambda,\beta} = (N_{\lambda,\beta}(t))_{t \geq 0}$ with parameter $\lambda t^\beta > 0$ defined on a probability space $(\Omega, \mathcal{F}, P)$. Thus, we obtain

$$
\pi_{\lambda^\beta,\beta}(\{k\}) = P(N_{\lambda,\beta}(t) = k) = \frac{(\lambda t^\beta)^k}{k!} E_\beta^{(k)}(-\lambda t^\beta), \quad k \in \mathbb{N}_0.
$$

Remark 3.2. The fractional Poisson process $N_{\lambda,\beta}$ was proposed by O. N. Repin and A. I. Saichev [34]. Since then, it was studied by many authors see for example [25, 27, 28, 14, 37, 4, 32, 29, 8] and references therein.

A remarkable property of the fPm is that $\pi_{\lambda,\beta}$ is given as a mixture of Poisson measures with respect to a probability measure $\nu_\beta$ on $\mathbb{R}_+ := [0, \infty)$. That probability measure $\nu_\beta$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$ with a probability density $W^{-\beta,1-\beta}$, that is, the Wright function. The Laplace transform of the measure $\nu_\beta$ (or its density $W^{-\beta,1-\beta}$) is given by

$$
Z_\infty^{0} e^{-\tau z} d\nu_\beta(\tau) = Z_\infty^{0} e^{-\tau z} W^{-\beta,1-\beta}(\tau) d\tau = E_\beta(-z), \quad (3.3)
$$

for any $z \in \mathbb{C}$ such that $\text{Re}(z) \geq 0$, see [15, Cor. A.5]. Equation (3.3) is called the complete monotonicity property of the Mittag-Leffler function, see [33]. More precisely, we have the following lemma.

Lemma 3.3. For $0 < \beta \leq 1$, the fPm $\pi_{\lambda,\beta}$ is an integral (or mixture) of Poisson measure $\pi_\lambda$ with respect to the probability measure $\nu_\beta$, i.e.,

$$
\pi_{\lambda,\beta} = \int_0^\infty \pi_\lambda d\nu_\beta(\tau), \quad \forall \lambda > 0. \quad (3.4)
$$

Proof. For $\beta = 1$, we have $\nu_1 = \delta_1$, the Dirac measure at 1, and the result is clear. For $0 < \beta < 1$, we denote the right hand side of (3.4) by $\mu := \int_0^\infty \pi_\lambda W^{-\beta,1-\beta}(\tau) d\tau$. We compute the Laplace transform of $\mu$ and use Fubini’s theorem to obtain

$$
\int_0^\infty e^{\tau z} d\mu(x) = \int_0^\infty e^{\tau z} \int_0^\infty d\pi_\lambda(x) W^{-\beta,1-\beta}(\tau) d\tau
$$

$$
= \int_0^\infty \left( \int_0^\infty e^{\tau x} d\pi_\lambda(x) \right) W^{-\beta,1-\beta}(\tau) d\tau
$$

$$
= \int_0^\infty e^{\tau \lambda (e^x - 1)} W^{-\beta,1-\beta}(\tau) d\tau
$$

$$
= E_\beta(\lambda (e^x - 1)).
$$

Thus, we conclude that the Laplace transforms of $\mu$ and $\pi_{\lambda,\beta}$ (cf. (3.2)) coincide. The result follows by the uniqueness of the Laplace transform. \hfill \Box

Theorem 3.4 (Moments of $\pi_{\lambda,\beta}$, cf. [26]). The fPm $\pi_{\lambda,\beta}$ has moments of all order. More precisely, the $n$-th moment of the measure $\pi_{\lambda,\beta}$ is given by

$$
m_{\lambda,\beta}(n) := \int_\mathbb{R} x^n d\pi_{\lambda,\beta}(x) = \sum_{m=0}^{n} \frac{m!}{\Gamma(m\beta + 1)} S(n,m) \lambda^m, \quad (3.5)
$$

where $S(n,m)$ is the Stirling number of the second kind.

Here are the first few moments of the measure $\pi_{\lambda,\beta}$:
where \( s \) obtain monic polynomials

\[
m_{\lambda,\beta}(0) = 1, \\
m_{\lambda,\beta}(1) = \frac{\lambda}{\Gamma(\beta + 1)}, \\
m_{\lambda,\beta}(2) = \frac{\lambda}{\Gamma(\beta + 1)} - \frac{2\lambda^2}{\Gamma(2\beta + 1)}, \\
m_{\lambda,\beta}(3) = \frac{\lambda}{\Gamma(\beta + 1)} + \frac{6\lambda^2}{\Gamma(2\beta + 1)} + \frac{6\lambda^3}{\Gamma(3\beta + 1)}.
\]

When \( \beta = 1 \), these moments become the moments of the Poisson measure.

In addition to the Poisson measure \( \pi_{\lambda} \) and \( \mathcal{PM}_{\lambda,\beta} \) in \( \mathbb{N}_0 \) we also need the two dimensional version of both of these measures in \( \mathbb{N}_0^2 \) or \( \mathbb{R}^2 \), the reason for that is clear after Corollary 3.5. The \( d \)-dimensional Poisson measure is given by

\[
\pi^{d}_{\lambda}(\{k_1, \ldots, k_d\}) = \prod_{i=1}^{d} \frac{\lambda^{k_i}}{k_i!} e^{-\lambda_i}.
\]

The Laplace transform of \( \pi^{2}_{\lambda} \) is given by

\[
l_{\pi^{2}_{\lambda}}(z) = \int_{\mathbb{R}^2} e^{(x,s)} d\pi^{2}_{\lambda}(x) = \exp \left( \lambda_1(e^{s_1} - 1) + \lambda_2(e^{s_2} - 1) \right) \tag{3.6}
\]

where \( s = (s_1, s_2) \in \mathbb{R}^2 \). For any \( 0 < \beta \leq 1 \), \( \tilde{\lambda} \in (\mathbb{R}_+)^2 \), then a possible fractional generalization of \( \pi^{2}_{\lambda} \), denoted by \( \pi^{2}_{\lambda,\beta} \), is given, via its Laplace transform, by replacing the first exponential function on the right hand side of (3.6) by the Mittag-Leffler function. More precisely, the Laplace transform of \( \pi^{2}_{\lambda,\beta} \) is given by

\[
l_{\pi^{2}_{\lambda,\beta}}(s) = \int_{\mathbb{R}^2} e^{(x,s)} d\pi^{2}_{\lambda,\beta}(x) = E_{\beta}(\lambda_1(e^{s_1} - 1) + \lambda_2(e^{s_2} - 1)), \tag{3.7}
\]

where \( s = (s_1, s_2) \in \mathbb{R}^2 \).

The moments of the measure \( \pi^{2}_{\lambda,\beta} \), denoted by \( m^{2}_{\lambda,\beta}(n_1, n_2) \), can be obtained by applying \( \frac{d^{n_1}}{ds_1^{n_1}} \frac{d^{n_2}}{ds_2^{n_2}} \), \( n_1, n_2 \in \mathbb{N}_0 \), to Equation (3.7) and then evaluating at \( s_1 = s_2 = 0 \). As an example, here we compute the moments \( m^{2}_{\lambda,\beta}(1, 1) \) and \( m^{2}_{\lambda,\beta}(1, 2) \) of the measure \( \pi^{2}_{\lambda,\beta} \) needed later on:

\[
m^{2}_{\lambda,\beta}(1, 1) = \int_{\mathbb{R}^2} x_1 x_2 d\pi^{2}_{\lambda,\beta}(x_1, x_2) = \frac{2\lambda_1 \lambda_2}{\Gamma(2\beta + 1)}, \\
m^{2}_{\lambda,\beta}(1, 2) = \int_{\mathbb{R}^2} x_1 x_2^2 d\pi^{2}_{\lambda,\beta}(x_1, x_2) = \frac{2\lambda_1 \lambda_2}{\Gamma(2\beta + 1)} + \frac{6\lambda_1 \lambda_2^2}{\Gamma(3\beta + 1)}.
\]

We apply the Gram-Schmidt orthogonalization process to the monomials \( x^n, n \in \mathbb{N}_0 \), to obtain monic polynomials \( C_n^\beta(x) \) with \( \deg C_n^\beta(x) = n \) with respect to the inner product

\[
(p, q)_{\pi_{\lambda,\beta}} := \int_{\mathbb{R}} p(x)q(x) d\pi_{\lambda,\beta}(x).
\]

These polynomials are determined by the moments of the measure \( \pi_{\lambda,\beta} \). The first few of these polynomials are given by

\[
C_0^\beta(x) = 1, \\
C_1^\beta(x) = x - (x, C_0^\beta)_{\pi_{\lambda,\beta}} C_0^\beta(x) = x - m_{\lambda,\beta}(1), \\
C_2^\beta(x) = x^2 - (x^2, C_0^\beta)_{\pi_{\lambda,\beta}} C_0^\beta(x) - \left( x^2, \frac{C_1^\beta}{\|C_1^\beta\|^2_{\pi_{\lambda,\beta}}} \right)_{\pi_{\lambda,\beta}} C_1^\beta(x),
\]

\[
= x^2 - A(\beta, \lambda)x - m_{\lambda,\beta}(2) + A(\beta, \lambda)m_{\lambda,\beta}(1),
\]

\[
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\]
\[ F(\beta, \lambda_1=1, \lambda_2=2) \]
\[ \text{----- } F(\beta, 2, 3) \]
\[ \text{----- } F(\beta, 1, 1) \]

Figure 1: The graph of the function \( F(\cdot, \lambda_1, \lambda_2) \) with \( \lambda = (1, 1), (2, 3), (1, 2) \).

where

\[ A(\beta, \lambda) = \frac{m_{\lambda, \beta}(3) - m_{\lambda, \beta}(1)m_{\lambda, \beta}(2)}{m_{\lambda, \beta}(2) - (m_{\lambda, \beta}(1))^2}. \]

When \( \beta = 1 \), the measure \( \pi_{\lambda, 1} \) becomes the Poisson measure \( \pi_{\lambda} \) and the polynomials \( C_n^1(x) \), \( n \in \mathbb{N}_0 \), are the classical Charlier polynomials.

**Corollary 3.5.** For \( \beta \in (0, 1] \) it holds

\[ \int_{\mathbb{R}^2} C_1^\beta(x_1)C_2^\beta(x_2) \, d\pi_{\chi, \beta}^2(x_1, x_2) = 0 \]

if, and only if, \( \beta = 1 \).

**Proof.** When \( \beta = 1 \), we have the well known orthogonal property of the Charlier polynomials, that is,

\[ \int_{\mathbb{R}^2} C_1(x_1)C_2(x_2) \, d\pi_{\chi}^2(x_1, x_2) = \int_{\mathbb{R}} C_1(x_1) \, d\pi_{\chi}(x_1) \int_{\mathbb{R}} C_2(x_2) \, d\pi_{\chi}(x_2) = 0. \]

On the other hand, for \( \beta \in (0, 1) \) we have

\[ \int_{\mathbb{R}^2} C_1^\beta(x_1)C_2^\beta(x_2) \, d\pi_{\chi, \beta}^2(x_1, x_2) \]
\[ = \int_{\mathbb{R}^2} (x_1 - m_{\lambda_1, \beta}(1)) \left(x_2^2 - A(\beta, \lambda_2)x_2 - m_{\lambda_2, \beta}(2) + A(\beta, \lambda_2)m_{\lambda_2, \beta}(1)\right) \, d\pi_{\chi, \beta}^2(x_1, x_2) \]
\[ = m_{\lambda, \beta}^2(1, 2) - A(\beta, \lambda_2)m_{\lambda, \beta}^2(1, 1) - m_{\lambda_1, \beta}(1)m_{\lambda_2, \beta}(2) + A(\beta, \lambda_2)m_{\lambda_1, \beta}(1)m_{\lambda_2, \beta}(1). \quad (3.8) \]

Equation (3.3.3) defines a function \( F(\beta, \lambda_1, \lambda_2) \) which is not equal from to zero for every \( \beta \in (0, 1) \), see Figure 3.1.

Having in mind the above results, this motivate us to introduce a biorthogonal system of the fPm in higher dimension.

## 4 Infinite Dimensional Fractional Poisson Measure

After the above preparation, we are ready to define the fPm in infinite dimensions. We define the fPm in the linear space \( D' \) and then a more careful analysis shows that fPm is indeed a probability measure on the configuration space \( \Gamma \) over \( \mathbb{R}^d \).

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4.1 Fractional Poisson Measure on the Linear Space $\mathcal{D}'$

Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_d) \in (\mathbb{R}_+)^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ be given. The $d$-dimensional Poisson measure has characteristic function given by

$$C_{\vec{\lambda}}(z) = \int_{\mathbb{R}^d} e^{i(x \cdot z)} \, d\pi_\lambda^d(x) = \exp \left( \sum_{k=1}^{d} \lambda_k (e^{iz_k} - 1) \right). \tag{4.1}$$

Let us consider a Radon measure $\sigma$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, that is, $\sigma(\Lambda) < \infty$ for every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, where $\mathcal{B}_c(\mathbb{R}^d)$ the family of all $\mathcal{B}(\mathbb{R}^d)$-measurable sets with compact closure. Elements of $\mathcal{B}_c(\mathbb{R}^d)$ are called finite volumes. Here, we assume $\sigma$ to be non-degenerate (i.e., $\sigma(O) > 0$ for all non-empty open sets $O \subset \mathbb{R}^d$) and non-atomic (i.e., $\sigma(\{x\}) = 0$ for every $x \in \mathbb{R}^d$). In addition, we always assume that $\sigma(\mathbb{R}^d) = \infty$. Let $\mathcal{D} := \mathcal{D}(\mathbb{R}^d)$ be the space of $C^\infty$-functions with compact support in $\mathbb{R}^d$ and $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^d)$ be the dual of $\mathcal{D}$ with respect to the Hilbert space $L^2(\sigma) := L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \sigma)$. In this way, we obtain the triple

$$\mathcal{D} \subset L^2(\sigma) \subset \mathcal{D}'. \tag{4.2}$$

The infinite-dimensional generalization of the Poisson measure with intensity measure $\sigma$, denoted by $\pi_\sigma$, is obtained by generalizing the characteristic function (4.1) to

$$C_{\pi_\sigma}(\varphi) := \int_{\mathcal{D}'} e^{i(w \cdot \varphi)} \, d\pi_\sigma(w) = \exp \left( \int_{\mathbb{R}^d} (e^{i\varphi(x)} - 1) \, d\sigma(x) \right), \quad \varphi \in \mathcal{D}. \tag{4.3}$$

This is achieved through the Bochner-Minlos theorem (see e.g. [6]) by showing that $C_{\pi_\sigma}$ is the Fourier transform of a measure on the distribution space $\mathcal{D}'$, see [2] and references therein. Now, using the fact that the Mittag-Leffler function is a natural generalization of the exponential function, one conjectures that the characteristic functional

$$C_{\pi_\beta}(\varphi) := \int_{\mathcal{D}'} e^{i(w \cdot \varphi)} \, d\pi_\beta(w) = E_\beta \left( \int_{\mathbb{R}^d} (e^{i\varphi(x)} - 1) \, d\sigma(x) \right), \quad \varphi \in \mathcal{D}, \tag{4.4}$$

defines an infinite-dimensional version of the fPm, denoted by $\pi_\beta$. However, since the Mittag-Leffler function does not satisfy the semigroup property of the exponential, it is not obvious that this is the Fourier transform of a measure on $\mathcal{D}'$. Hence, we use the Bochner-Minlos theorem to show that $C_{\pi_\beta}$ is the Fourier transform of a probability measure $\pi_\beta$ on the distribution space $\mathcal{D}'$.

**Theorem 4.1.** For each $0 < \beta \leq 1$ fixed, the functional $C_{\pi_\beta}$ in Equation (4.4) is the characteristic functional on $\mathcal{D}$ of a probability measure $\pi_\beta$ on the distribution space $\mathcal{D}'$.

**Proof.** Using the properties of the Mittag-Leffler function, the functional $C_{\pi_\beta}$ is continuous and $C_{\pi_\beta}(0) = 1$ follow directly. To show that the functional $C_{\pi_\beta}$ is positive definite, we use the complete monotonicity property of $E_\beta$, $0 < \beta < 1$, see (3.3). For any $\varphi_i \in \mathcal{D}$, $z_i \in \mathbb{C}$, $i = 1, \ldots, n$, using Equation (4.3), we obtain

$$\sum_{k,j=1}^{n} C_{\pi_\beta}(\varphi_k - \varphi_j) z_k \bar{z}_j = \int_{0}^{\infty} \sum_{k,j=1}^{n} e^{\sigma f_{\pi_\beta}(e^{i(x_k - x_j)} - 1)} \, d\sigma(x) z_k \bar{z}_j \, d\nu_\beta(\tau)$$

$$= \int_{0}^{\infty} \sum_{k,j=1}^{n} C_{\pi_\sigma}(\varphi_k - \varphi_j) z_k \bar{z}_j \, d\nu_\beta(\tau).$$
Using the definition of $C_{\pi,\sigma}$, the integrand of the last integral may be written as

$$
\sum_{k,j=1}^{n} C_{\pi,\sigma}(\phi_k - \phi_j) z_k \bar{z}_j = \int_{D^*} \left| \sum_{k=1}^{n} e^{i(w,\phi_k)} z_k \right|^2 d\pi_{\sigma}(w) \geq 0.
$$

This implies that $C_{\nu,\sigma}$ is positive-definite. Thus by the Bochner-Minlos theorem, $C_{\nu,\sigma}$ is the characteristic functional of a probability measure $\pi_{\sigma}$ on the measurable space $(D', C_{\sigma}(D'))$. \hfill \Box

**Remark 4.2.** By the analytic property of the Mittag-Leffler function one may write (4.4) for any $\phi \in D$ such that $\text{supp } \phi \subset \Lambda \in B_c(R^d)$, as

$$
C_{\pi,\sigma}^{\beta}(\phi) = E_{\beta} \left( \int_{R^d} (e^{i\phi(x)} - 1) d\sigma(x) \right) = E_{\beta} \left( \int_{\Lambda} (e^{i\phi(x)} - 1) d\sigma(x) \right)
$$

$$
= E_{\beta} \left( \int_{\Lambda} e^{i\phi(x)} d\sigma(x) - \sigma(\Lambda) \right)
$$

$$
= \sum_{n=0}^{\infty} \frac{E_{\beta}^{(n)}(-\sigma(\Lambda))}{n!} \left( \int_{\Lambda} e^{i\phi(x)} d\sigma(x) \right)^n
$$

$$
= \sum_{n=0}^{\infty} \frac{E_{\beta}^{(n)}(-\sigma(\Lambda))}{n!} \int_{\Lambda^n} e^{i(\phi(x_1) + \cdots + \phi(x_n))} d\sigma^{\otimes n}(x_1, \ldots, x_n),
$$

where $\sigma^{\otimes n} = \sigma \otimes \cdots \otimes \sigma$ is a measure defined on the Cartesian space $(R^d)^n := R^d \times \cdots \times R^d$. In the Poisson case, we have $\exp(-\sigma(\Lambda))$ instead of $E_{\beta}^{(n)}(-\sigma(\Lambda))$, for all $n \in N_0$, while the rest of the terms are the same. Hence, the main difference between these measures ($\pi_{\sigma}^{\beta}$ and $\pi_{\sigma}$) is the different weight given in each $n$-particle space. In Subsection 4.3 we show that, indeed, the support of the measure $\pi_{\sigma}^{\beta}$ is a subset of $D'$, called the configuration space over $R^d$.

We may now generalize the result of Lemma 3.3 to the present infinite dimensional setting.

**Lemma 4.3.** For $0 < \beta < 1$, the fPm $\pi_{\sigma}^{\beta}$ is an integral (or mixture) of Poisson measure $\pi_{\sigma}$ with respect to the probability measure $\nu_{\beta}$, i.e.,

$$
\pi_{\sigma}^{\beta} = \int_{0}^{\infty} \pi_{\tau,\sigma} d\nu_{\beta}(\tau). \quad (4.5)
$$

**Proof.** For $\beta = 1$, the result is clear as in the proof of Lemma 3.3. For $0 < \beta < 1$, we use the representation (3.3) of the Mittag-Leffler function, the characteristic functional (4.4) of $\pi_{\sigma}^{\beta}$ can be rewritten as

$$
C_{\pi,\sigma}^{\beta}(\phi) = \int_{0}^{\infty} \exp\left( -\tau \int_{R^d} (1 - e^{i\phi(x)}) d\sigma(x) \right) d\nu_{\beta}(\tau)
$$

with the integrand being the characteristic function of the Poisson measure $\pi_{\tau,\sigma}$, $\tau > 0$. This implies that the characteristic functional (4.4) coincides with the characteristic functional of the measure $\int_{0}^{\infty} \pi_{\tau,\sigma} d\nu_{\beta}(\tau)$. The result follows by the uniqueness of the characteristic functional. \hfill \Box

The fPm $\pi_{\sigma}^{\beta}$ is indeed a probability measure on $(D', C_{\sigma}(D'))$. In what follows, we are going to find an appropriate support for $\pi_{\sigma}^{\beta}$. 

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4.2 Configuration Space

Recall that $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel $\sigma$-algebra on $\mathbb{R}^d$ and $\mathcal{B}_c(\mathbb{R}^d)$ the system of all sets in $\mathcal{B}(\mathbb{R}^d)$ which are bounded and have compact closure. Below we recall the configuration space over $\mathbb{R}^d$ and related concepts, see [2, 22] for more details.

**Definition 4.4.** The infinite configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$ is defined as the set of all locally finite subsets from $\mathbb{R}^d$, that is, $\Gamma := \gamma \subseteq \mathbb{R}^d : |\gamma \cap \Lambda| < \infty$ for every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, where $|B|$ denotes the cardinality of the set $B$. The elements of the space $\Gamma$ are called configurations.

Let $C_0(\mathbb{R}^d)$ denote the class of all real-valued continuous functions on $\mathbb{R}^d$ with compact support and $\mathcal{M}^+ := \mathcal{M}^+(\mathbb{R}^d)$ (resp. $\mathcal{M}^+_0 := \mathcal{M}^+_0(\mathbb{R}^d)$) denote the space of all positive (resp. positive integer-valued) Radon measures on $\mathcal{B}(\mathbb{R}^d)$.

**Definition 4.5.** Each configuration $\gamma \in \Gamma$ can be identified with a non-negative integer-valued Radon measure as follows $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} \delta_x \in \mathcal{M}^+_0 \subset \mathcal{M}^+$, where $\delta_x$ is the Dirac measure at $x \in \mathbb{R}^d$ and $\sum_{x \in \emptyset} \delta_x := 0$ (zero measure). The space $\Gamma$ can be endowed with the topology induced by the vague topology on $\mathcal{M}^+$, i.e., the weakest topology on $\Gamma$ with respect to which all mappings $\Gamma \ni \gamma \mapsto \langle \gamma, f \rangle := \langle f \rangle_{\gamma} := \int_{\mathbb{R}^d} f(x) \, d\gamma(x) = \sum_{x \in \gamma} f(x) \in \mathbb{R}$ are continuous for any $f \in C_0(\mathbb{R}^d)$.

**Definition 4.6.** Let $\mathcal{B}(\Gamma)$ be the Borel $\sigma$-algebra corresponding to the vague topology on $\Gamma$.

1. The $\sigma$-algebra $\mathcal{B}(\Gamma)$ is generated by the sets of the form $C_{\Lambda,n} = \{ \gamma \in \Gamma : |\gamma \cap \Lambda| = n \}$, \hspace{1cm} (4.6)

where $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $n \in \mathbb{N}_0$, and the set $C_{\Lambda,n}$ is a Borel set of $\Gamma$, that is, $C_{\Lambda,n} \in \mathcal{B}(\Gamma)$. Sets of the form (4.6) are called cylinder sets.

2. For any $B \subset \mathbb{R}^d$, we introduce a function $N_B : \Gamma \to \mathbb{N}_0$ such that $N_B(\gamma) := |\gamma \cap B|$, $\gamma \in \Gamma$.

Then $\mathcal{B}(\Gamma)$ is the minimal $\sigma$-algebra with which all functions $N_\Lambda$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, are measurable.

**Definition 4.7.** Let $Y \in \mathcal{B}(\mathbb{R}^d)$ be given. The space of configurations contained in $Y$ is denoted by $\Gamma_Y$, i.e., $\Gamma_Y := \{ \gamma \in \Gamma : |\gamma \cap (\mathbb{R}^d \setminus Y)| = 0 \}$.

The $\sigma$-algebra $\mathcal{B}(\Gamma_Y)$ may be introduced in a similar way $\mathcal{B}(\Gamma_Y) := \sigma \{ N_\Lambda |_{\Gamma_Y} : \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \}$, where $N_\Lambda |_{\Gamma_Y}$ denotes the restriction of the mapping $N_\Lambda$ to $\Gamma_Y$. 

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Definition 4.8. Let $Y \in \mathcal{B}(\mathbb{R}^d)$ be given. The space of $n$-point configurations $\Gamma_Y^{(n)}$ over a set $Y$ is the subset of $\Gamma_Y$ defined by

$$\Gamma_Y^{(n)} := \{ \gamma \in \Gamma_Y \mid |\gamma| = n \}, n \in \mathbb{N}, \quad \Gamma_Y^{(0)} := \{ \emptyset \}.$$  

A topological structure may be introduced on $\Gamma_Y^{(n)}$, $n \in \mathbb{N}$, through a natural surjective mapping from

$$\tilde{Y}^n = \{(x_1, \ldots, x_n) \mid x_k \in Y, x_k \neq x_j \text{ if } k \neq j \}$$

onto $\Gamma_Y^{(n)}$ defined by

$$\text{sym}_Y^n : \tilde{Y}^n \longrightarrow \Gamma_Y^{(n)}$$

$$(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}.$$  

Indeed, using the mapping $\text{sym}_Y^n$, one constructs a bijective mapping between $\Gamma_Y^{(n)}$ and the symmetrization $\tilde{Y}^n/S_n$ of $\tilde{Y}^n$, where $S_n$ is the permutation group over $\{1, \ldots, n\}$. In this way, $\text{sym}_Y^n$ induces a metric on $\Gamma_Y^{(n)}$. A set $U \subset \Gamma_Y^{(n)}$ is open in this topology if, and only if, the inverse image $(\text{sym}_Y^n)^{-1}(U)$ is open in $\tilde{Y}^n$. We denote by $\mathcal{B}(\Gamma_Y^{(n)})$ the corresponding Borel $\sigma$-algebra and $\mathcal{T}_Y^{(n)}$ the associated topology on $\Gamma_Y^{(n)}$.

For $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, each space $\Gamma_{\Lambda}$ can be described by the disjoint union

$$\Gamma_{\Lambda} = \bigcup_{n=0}^{\infty} \Gamma_{\Lambda}^{(n)}.$$  

In particular, this representation provides an equivalent description of the $\sigma$-algebra $\mathcal{B}(\Gamma_{\Lambda})$ as the $\sigma$-algebra of the disjoint union of the $\sigma$-algebras $\mathcal{B}(\Gamma_{\Lambda}^{(n)}), n \in \mathbb{N}_0$. The corresponding topology is denoted by $\mathcal{T}_{\Lambda}$ such that $(\Gamma_{\Lambda}, \mathcal{T}_{\Lambda})$ is a topological space for each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

For each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and any pair $\Lambda_1, \Lambda_2 \in \mathcal{B}_c(\mathbb{R}^d)$ such that $\Lambda_1 \subset \Lambda_2$, let us consider the natural measurable projections

$$p_{\Lambda} : \Gamma \longrightarrow \Gamma_{\Lambda} \quad \gamma \mapsto \gamma \cap \Lambda$$

$$p_{\Lambda_1, \Lambda_2} : \Gamma_{\Lambda_2} \longrightarrow \Gamma_{\Lambda_1} \quad \gamma \mapsto \gamma \cap \Lambda_1. \quad (4.7)$$  

We use now the concept of the projective limit in order to show that the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ coincides with the projective limit. More precisely, we have the following theorem.

Theorem 4.9. The family $\{(\Gamma_{\Lambda}, \mathcal{B}(\Gamma_{\Lambda})), p_{\Lambda_1, \Lambda_2}, \mathcal{B}_c(\mathbb{R}^d)\}$ is a projective system of measurable spaces with ordered index set $(\mathcal{B}_c(\mathbb{R}^d), \subset)$ and the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ is (up to an isomorphism) the projective limit together with the family of maps $p_{\Lambda} : \Gamma \longrightarrow \Gamma_{\Lambda}$ for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. In addition, we have the following commutative diagram.
Proof. It is clear from the construction above that the maps \( p_{\Lambda_1, \Lambda_2}, \Lambda_1, \Lambda_2 \in \mathcal{B}_c(\mathbb{R}^d) \) are measurable and satisfies

\[
p_{\Lambda_1, \Lambda_2} \circ p_{\Lambda_2, \Lambda_3} = p_{\Lambda_1, \Lambda_3}, \quad \Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \text{ in } \mathcal{B}_c(\mathbb{R}^d).
\]

As a result, \( \{ (\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda)), p_{\Lambda_1, \Lambda_2}, \mathcal{B}_c(\mathbb{R}^d) \} \) is a projective system. On the other hand, it is easy to see from (4.7) that the following relation

\[
p_{\Lambda_1} = p_{\Lambda_1, \Lambda_2} \circ p_{\Lambda_2}, \quad \Lambda_1 \subset \Lambda_2 \text{ in } \mathcal{B}_c(\mathbb{R}^d)
\]

holds. By definition of \( \mathcal{B}(\Gamma) \), the family of maps \( p_\Lambda, \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \) satisfy the conditions in a projective limit of measurable spaces. This concludes the proof. \( \square \)

### 4.3 Fractional Poisson Measure on \( \Gamma \)

Recall from Section 4.1 the measure \( \sigma \) on the underlying measurable space \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) and the product measure \( \sigma^{\otimes n} \) on \( ((\mathbb{R}^d)^n, \mathcal{B}((\mathbb{R}^d)^n)) \), for each \( n \in \mathbb{N} \). Then \( \sigma^{\otimes n}((\mathbb{R}^d)^n \setminus (\mathbb{R}^d)^n) = 0 \), since \( \sigma \) is non-atomic. It follows that \( \sigma^{\otimes n}(B^n \setminus \overline{B}^n) = 0 \), for every \( B \in \mathcal{B}(\mathbb{R}^d) \). For each \( \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \), let us consider the restriction of \( \sigma^{\otimes n} \) to \( (\Lambda^n, \mathcal{B}(\Lambda^n)) \), which is a finite measure, and then define the image measure \( \sigma_\Lambda^{(n)} \) on \( (\Gamma_\Lambda^{(n)}, \mathcal{B}(\Gamma_\Lambda^{(n)})) \) under the mapping \( \text{sym}^n_\Lambda \) by

\[
\sigma_\Lambda^{(n)} := \sigma^{\otimes n} \circ (\text{sym}^n_\Lambda)^{-1}.
\]

For \( n = 0 \), we set \( \sigma_\Lambda^{(0)}(\emptyset) := 1 \). Now, for each \( 0 < \beta < 1 \), one may define a probability measure \( \pi_\beta^{\sigma, \Lambda} \) on \( (\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda)) \) by

\[
\pi_\beta^{\sigma, \Lambda} := \sum_{n=0}^{\infty} \frac{E^{(n)}_{\beta}(-\sigma(\Lambda))}{n!} \sigma_\Lambda^{(n)}.
\]

(4.8)

Note that \( E^{(n)}_{\beta}(-\sigma(\Lambda)) \geq 0, n \in \mathbb{N}_0 \) due to (3.3). In addition, we have

\[
\pi_\beta^{\sigma, \Lambda}(\Gamma_\Lambda) = \pi_\beta^{\sigma, \Lambda} \left( \bigcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)} \right) = \sum_{n=0}^{\infty} \frac{E^{(n)}_{\beta}(-\sigma(\Lambda))}{n!} \sigma_\Lambda^{(n)}(\Gamma_\Lambda^{(n)})
\]

\[
= \sum_{n=0}^{\infty} \frac{E^{(n)}_{\beta}(-\sigma(\Lambda))}{n!} \sigma(\Lambda)^n = E_{\beta}(0) = 1.
\]

The family \( \{ \pi_\beta^{\sigma, \Lambda} \mid \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \} \) of probability measures yields a probability measure on \( (\Gamma, \mathcal{B}(\Gamma)) \). In fact, this family is consistent in the sense that the measure \( \pi_\beta^{\sigma, \Lambda_1} \) is the image measure of \( \pi_\beta^{\sigma, \Lambda_2} \) under \( p_{\Lambda_1, \Lambda_2} \), that is,

\[
\pi_\beta^{\sigma, \Lambda_1} = \pi_\beta^{\sigma, \Lambda_2} \circ p_{\Lambda_1, \Lambda_2}^{-1}, \quad \forall \Lambda_1, \Lambda_2 \in \mathcal{B}_c(\mathbb{R}^d), \Lambda_1 \subset \Lambda_2.
\]

By the Kolmogorov extension theorem on configuration space (see Appendix A.1) the family \( \{ \pi_\beta^{\sigma, \Lambda} \mid \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \} \) determines uniquely a measure \( \pi_\beta^{\sigma} \) on \( (\Gamma, \mathcal{B}(\Gamma)) \) such that

\[
\pi_\beta^{\sigma} = \pi_\beta^{\sigma} \circ p_{\Lambda}^{-1}, \quad \forall \Lambda \in \mathcal{B}_c(\mathbb{R}^d).
\]

Actually, we don’t need the whole family of local sets \( \mathcal{B}_c(\mathbb{R}^d) \) instead, a sub-family \( \mathcal{J}_{\mathbb{R}^d} \) with an abstract concept of local sets, see (I1)–(I3) in Appendix A.1.
Let us now compute the characteristic functional of the measure $\pi_\sigma^\beta$. Given $\varphi \in \mathcal{D}$, we have $\text{supp} \varphi \subset \Lambda$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, such that

$$\langle \gamma, \varphi \rangle = \langle p_\Lambda(\gamma), \varphi \rangle, \quad \forall \gamma \in \Gamma.$$  

Thus,

$$\int_\Gamma e^{i\langle \gamma, \varphi \rangle} d\pi_\sigma^\beta(\gamma) = \int_{\Gamma_\Lambda} e^{i\langle \gamma, \varphi \rangle} d\pi_{\sigma, \Lambda}(\gamma)$$

and the infinite divisibility (4.8) of the measure $\pi_{\sigma, \Lambda}^\beta$ yields for the right-hand side of the above equality

$$\sum_{n=0}^{\infty} \frac{E_\beta(n)}{n!} \int_{\Lambda^n} e^{i(\varphi_1 + \cdots + \varphi_n)} d\sigma^{\otimes n}(x_1, \ldots, x_n) = \sum_{n=0}^{\infty} \frac{E_\beta(n)}{n!} \left( \int_{\Lambda} e^{i\varphi(x)} d\sigma(x) \right)^n$$

which corresponds to the Taylor expansion of the function

$$E_\beta \left( \int_{\Lambda} (e^{i\varphi(x)} - 1) d\sigma(x) \right) = E_\beta \left( \int_{\mathbb{R}^d} (e^{i\varphi(x)} - 1) d\sigma(x) \right).$$

Hence, for any $\varphi \in \mathcal{D}$, we obtain

$$\int_\Gamma e^{i\langle \gamma, \varphi \rangle} d\pi_\sigma^\beta(\gamma) = E_\beta \left( \int_{\mathbb{R}^d} (e^{i\varphi(x)} - 1) d\sigma(x) \right). \tag{4.9}$$

**Remark 4.10.**
1. The characteristic functional of the measure $\pi_\sigma^\beta$ given in (4.9) coincides with the characteristic functional (4.4) of the measure $\pi_\sigma^\beta$ on the distribution space $\mathcal{D}'$. The functional (4.9) shows that the measure $\pi_\sigma^\beta$ is supported on generalized functions of the form $\sum_{\delta_x \in \Gamma, \gamma \in \Gamma}$.

2. Note that $\Gamma \subset \mathcal{D}'$ but in contrast to $\Gamma$, $\mathcal{D}'$ is a linear space. Since $\pi_\sigma^\beta(\Gamma) = 1$, the measure space $(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \pi_\sigma^\beta)$ can, in this way, be regarded as a linear extension of the fractional Poisson space $(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma^\beta)$.

## 5 Generalized Appell System

In this section we introduce the generalized Appell system associated with the fPm $\pi_\sigma^\beta$. First we consider the analytic continuation of the characteristic functional $C_{\pi_\sigma^\beta}$ to $\mathcal{D}_C := \mathcal{D} \oplus i\mathcal{D}$. By definition, an element $\varphi \in \mathcal{D}_C$ decomposes into $\varphi = \varphi_1 + i\varphi_2$, $\varphi_1, \varphi_2 \in \mathcal{D}$. Hence, computing $C_{\pi_\sigma^\beta}(-i\varphi)$, $\varphi \in \mathcal{D}$, yields the Laplace transform of the measure $\pi_\sigma^\beta$, that is,

$$l_{\pi_\sigma^\beta}(\varphi) := C_{\pi_\sigma^\beta}(-i\varphi) = E_\beta \left( \int_{\mathbb{R}^d} (e^{i\varphi(x)} - 1) d\sigma(x) \right).$$

In particular, choosing $\beta = 1$ we obtain the Laplace transform of the classical Poisson measure $\pi_\sigma := \pi_\sigma^1$ with intensity $\sigma$ on the configuration space $\Gamma$. For more details, we refer to [12, 20, 17, 18, 3] and reference therein.

The following two properties are satisfied by the fPm $\pi_\sigma^\beta$, $0 < \beta \leq 1$.

**(A1)** The measure $\pi_\sigma^\beta$ has an analytic Laplace transform in a neighborhood of zero, that is, the mapping

$$\mathcal{D}_C \ni \varphi \mapsto l_{\pi_\sigma^\beta}(\varphi) = \int_{\mathcal{D}'} e^{i\langle \varphi, \varphi \rangle} d\pi_\sigma^\beta(\varphi) = E_\beta \left( \int_{\mathbb{R}^d} (e^{i\varphi(x)} - 1) d\sigma(x) \right) \in \mathbb{C}$$

is holomorphic in a neighborhood $\mathcal{U} \subset \mathcal{D}_C$ of zero.
(A2) For any nonempty open subset $\mathcal{U} \subset \mathcal{D}'$ it should hold that $\pi_\sigma^\beta(\mathcal{U}) > 0$.

The assumption (A1) guarantees the existence of the moments of all order of the measure $\pi_\sigma^\beta$ while (A2) guarantees the embedding of the test function space on $L^2(\pi_\sigma^\beta)$, see e.g., Section 3 in [19]. In addition, the Laplace transform $l_{\pi_\sigma^\beta}(\varphi)$ of the measure $\pi_\sigma^\beta$ has the decomposition in terms of the moment kernels $M_n^\sigma,\beta$ (by the kernel theorem) given by

$$l_{\pi_\sigma^\beta}(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle M_n^\sigma,\beta, \varphi^{\otimes n} \rangle, \quad \varphi \in \mathcal{D}, \; M_n^\sigma,\beta \in (\mathcal{D}^\otimes \mathcal{N})'. \quad (5.1)$$

### 5.1 Generalized Appell Polynomials

In this subsection we follow [23] to introduce the system of Appell polynomials associated with the fPm $\pi_\sigma^\beta$. Let us consider the triple (4.2) such that

$$\mathcal{D} \subset \mathcal{N} \subset L^2(\sigma) \subset \mathcal{N'} \subset \mathcal{D}' \tag{5.2}$$

as described in Section 2. Also, the chain (5.2) holds for the tensor product of these spaces.

Then we introduce the normalized exponential $e_{\pi_\sigma^\beta}(\varphi; z)$ by

$$e_{\pi_\sigma^\beta}(\varphi; w) = e^{(w,\varphi)}, \quad w \in \mathcal{D}', \; \varphi \in \mathcal{D}. \quad (5.3)$$

Since $l_{\pi_\sigma^\beta}(0) = 1$ and $l_{\pi_\sigma^\beta}$ is holomorphic, there exists a neighborhood $\mathcal{U}_0 \subset \mathcal{D}$ of zero, such that $l_{\pi_\sigma^\beta}(\varphi) \neq 0$ for all $\varphi \in \mathcal{U}_0$. For $\varphi \in \mathcal{U}_0$, the normalized exponential $e_{\pi_\sigma^\beta}(\varphi; z)$ can be expanded in a power series and then we use the polarization identity in order to apply the kernel theorem to obtain

$$e_{\pi_\sigma^\beta}(\varphi; w) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^\sigma,\beta(w), \varphi^{\otimes n} \rangle, \quad w \in \mathcal{D}', \; \varphi \in \mathcal{U}_0, \quad (5.4)$$

for suitable $P_n^\sigma,\beta(w) \in (\mathcal{D}^\otimes \mathcal{N})'$. The family

$$\mathbb{P}^{\sigma,\beta} = \left\{ \langle P_n^\sigma,\beta(\cdot), \varphi^{(n)} \rangle \mid \varphi^{(n)} \in \mathcal{D}^\otimes \mathcal{N}, \; n \in \mathbb{N}_0 \right\} \quad (5.5)$$

is called the Appell system associated to the fPm $\pi_\sigma^\beta$. Let us now consider the transformation $\alpha : \mathcal{D} \rightarrow \mathcal{D}$ defined on a neighborhood $\mathcal{U}_a \subset \mathcal{D}$ of zero, by

$$\alpha(\varphi)(x) = \log(1 + \varphi(x)), \quad \varphi \in \mathcal{U}_a, \; x \in \mathbb{R}^d. \quad (5.6)$$

Note that for $\varphi = 0 \in \mathcal{D}$, we have $\alpha(\varphi) = 0$. Also, $\mathcal{U}_a$ is chosen in such a way that $\alpha$ is invertible and holomorphic on $\mathcal{U}_a$. Then $\alpha$ can be expanded as

$$\alpha(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n!} \overline{d^n \alpha(0)}(\varphi) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \varphi^n, \quad (5.6)$$

where

$$\overline{d^n \alpha(0)}(\varphi) = \left. \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \alpha(t_1 \varphi + \cdots + t_n \varphi) \right|_{t_1=\cdots=t_n=0}$$

for all $n \in \mathbb{N}$. For the inverse function $g_\alpha$ of $\alpha$, we have

$$(g_\alpha \varphi)(x) = e^{\varphi(x)} - 1, \quad \varphi \in \mathcal{V}_a \subset \mathcal{D}, \; x \in \mathbb{R}^d.$$
for some neighborhood $V_\alpha$ of zero in $D_C$. A similar procedure as before yields the decomposition

$$
g_\alpha(x) = \sum_{n=1}^{\infty} \frac{1}{n!} d^n g_\alpha(0) = \sum_{n=1}^{\infty} \frac{\varphi^n}{n!}. \tag{5.7}\$$

Now using the function $\alpha$, we introduce the modified normalized exponential $e_{x,\alpha}(\alpha(x); w)$ as

$$
e_{x,\alpha}(\alpha(x); w) := \frac{\exp(\langle w, \alpha(x) \rangle)}{l_{x,\alpha}(\alpha(x))} = \frac{\exp(\langle w, \log(1 + \varphi) \rangle)}{E_{\sigma} \left( \int_{\mathbb{R}^+} \varphi(x) \, d\sigma(x) \right)} = \frac{\exp(\langle w, \log(1 + \varphi) \rangle)}{E_{\sigma} (\langle \varphi \rangle)} \tag{5.8}\$$

for $x \in \mathcal{U}_\alpha \subset \mathcal{U}_0$, $w \in D_C$. Since $l_{x,\alpha}$ is holomorphic on a neighborhood of zero, for each fixed $w \in D_C$, $e_{x,\alpha}(\alpha(x); w)$ is a holomorphic function on some neighborhood $\mathcal{U}_\alpha \subset \mathcal{U}_\alpha$ of zero. Then we have the map $D_C \ni \varphi \mapsto e_{x,\alpha}(\alpha(x); w)$ which admits a power series

$$
e_{x,\alpha}(\alpha(x); w) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n^\sigma(\alpha(x)), \varphi^n \rangle, \quad \varphi \in \mathcal{U}_\alpha, \ w \in D_C, \quad \tag{5.9}\$$

where the kernels $C_n^\sigma : D_C \to (D_C^{\otimes n})'$, $n \in \mathbb{N}$, $C_0^\sigma = 1$. By Equation (5.9), it follows that for any $\varphi^{(n)} \in D_C^{\otimes n}$, $n \in \mathbb{N}_0$, the function

$$D_C \ni w \mapsto \langle C_n^\sigma(w), \varphi^{(n)} \rangle$$

is a polynomial of order $n$ on $D_C$.  

**Definition 5.1.** The family

$$\mathbb{P}^{\sigma,\beta,\alpha} = \{ \langle C_n^\sigma(\cdot), \varphi^{(n)} \rangle \mid \varphi^{(n)} \in D_C^{\otimes n}, n \in \mathbb{N}_0 \}$$

is called the generalized Appell system associated to the $\mathbb{P}^0$ or the $\mathbb{P}^{\sigma,\beta,\alpha}$ system.

In the following proposition we collect some properties of the kernels $C_n^\sigma(\cdot)$ which appeared in [21] but specific to the measure $\pi_\sigma$.

**Proposition 5.2.** For $z, w \in D_C$, $n \in \mathbb{N}_0$, the following properties hold

(P1) $C_n^\sigma(w) = \sum_{m=0}^{n} s(n, m)^* P_m^\sigma(w)$, where $s(n, m)$ is the Stirling operator of the first kind defined in (A.6) in Appendix A.2.

(P2) $w^{\otimes n} = \sum_{k=0}^{n} \sum_{m=0}^{n-k} s(k, m)^* C_m^\sigma(w) \hat{\otimes} M_{n-k}^\sigma$, where $S(n, m)$ is the Stirling operator of the second kind defined in (A.7) in Appendix A.2 and $M_n^\sigma \in (D_C^{\otimes n})'$ are the moment kernels of $\pi_\sigma$ given in (5.1).

(P3) $C_n^\sigma(z + w) = \sum_{l+t+m=n}^{\infty} \binom{n}{l, t, m} C_l^\sigma(z) \hat{\otimes} C_t^\sigma(w) \hat{\otimes} M_{m}^\sigma(\cdot)$, where $M_m^\sigma(\cdot) \in (D_C^{\otimes m})'$ is determined by

$$l_{n,\alpha}(\alpha(x)) = \sum_{m=0}^{\infty} \frac{1}{m!} \langle M_{m}^\sigma(\cdot), \varphi^{(m)} \rangle, \quad \varphi \in D_C. \tag{5.10}\$$

(P4) $C_n^\sigma(z + w) = \sum_{k=0}^{n} \binom{n}{k} C_k^\sigma(z) \hat{\otimes} (w)_{n-k}$, where $(w)_n$ is the falling factorial on $D_C$ determined by (A.1).

(P5) $C_n^\sigma(w) = \sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{n-k} C_k^\sigma(0) \hat{\otimes} (s(n - k, m)^* w^{\otimes m})$.

(P6) $E_{x,\alpha}(\langle C_n^\sigma(\cdot), \varphi^{(n)} \rangle) = \delta_{n,0}$, where $\varphi^{(n)} \in D_C^{\otimes n}$, $\delta_{n,k}$ is the Kronecker delta function and $E_{x,\alpha}(\cdot)$ is the expectation with respect to the measure $\pi_\sigma$. 

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(P7) For all $p' > p$ such that the embedding $\mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is a Hilbert-Schmidt operator and for all $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that

$$|C_n^{\sigma, \beta}(w)|_{-p'} \leq C_\varepsilon n! \varepsilon^{-n} \exp(\varepsilon |w|_{-p}), \quad w \in \mathcal{H}_{-p', \varepsilon}, \ n \in \mathbb{N}_0.$$  

Proof. (P1) In view of Equation (5.4), we have

$$e_{\pi_\sigma}(\alpha(\phi); w) = \frac{\exp(\langle w, \alpha(\phi) \rangle)}{l_{\pi_\sigma}(\alpha(\phi))} = \sum_{m=0}^{\infty} \frac{1}{m!} P_m^{\sigma, \beta}(w), \alpha(\phi)^{\otimes m}). \quad (5.11)$$

Using Equation (A.11), we obtain

$$e_{\pi_\sigma}(\alpha(\phi); w) = \sum_{m=0}^{\infty} \frac{1}{m!} \left\langle P_m^{\sigma, \beta}(w), \sum_{n=m}^{\infty} \frac{1}{n!} s(n, m) \phi^{\otimes n} \right\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{m=0}^{n} s(n, m)^* P_m^{\sigma, \beta}(w), \phi^{\otimes n} \right\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{m=0}^{n} s(n, m)^* P_m^{\sigma, \beta}(w), \phi^{\otimes n} \right\rangle.$$

On the other hand, using the equality (5.9) and comparing both series for $e_{\pi_\sigma}(\alpha(\phi), w)$ gives

$$C_n^{\sigma, \beta}(w) = \sum_{m=1}^{n} s(n, m)^* P_m^{\sigma, \beta}(w). \quad (5.12)$$

(P2) Similar as in the proof of (P1), we use Equation (5.9) and the fact that $g_\alpha$ is the inverse of $\alpha$ to obtain

$$e_{\pi_\sigma}(\phi; w) = \sum_{m=0}^{\infty} \frac{1}{m!} \left\langle C_m^{\sigma, \beta}(w), g_\alpha(\phi)^{\otimes m} \right\rangle. \quad (5.12)$$

Using Equation (A.10) we replace $g_\alpha(\phi)^{\otimes m}$ in the above Equation (5.12) and making some standard manipulations yields

$$e_{\pi_\sigma}(\phi; z) = \sum_{m=0}^{\infty} \frac{1}{m!} \left\langle C_m^{\sigma, \beta}(w), \sum_{n=m}^{\infty} \frac{1}{n!} S(n, m) \phi^{\otimes n} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{m=0}^{n} S(n, m)^* C_m^{\sigma, \beta}(w), \phi^{\otimes n} \right\rangle.$$

On the other hand, comparing the above series for $e_{\pi_\sigma}(\phi, w)$ and the Equation (5.4), we obtain

$$P_n^{\sigma, \beta}(w) = \sum_{m=0}^{n} S(n, m)^* C_m^{\sigma, \beta}(w). \quad (5.13)$$

By Equation (5.3), we have the equality

$$e^{\langle w, \phi \rangle} = e_{\pi_\sigma}(\phi; w) l_{\pi_\sigma}(\phi). \quad (5.14)$$

Now using the equations (5.4) and (5.1), we obtain the equation

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle w^{\otimes n}, \phi^{\otimes n} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{k=0}^{n} \binom{n}{k} P_n^{\sigma, \beta}(w)^{\otimes k} M^{\sigma, \beta}_{n-k}, \phi^{\otimes n} \right\rangle.$$
which implies that

$$w^\otimes n = \sum_{k=0}^{n} \binom{n}{k} P_{n-k}^{\sigma,\beta}(w) M_{n-k}^{\sigma,\beta}. \quad (5.15)$$

The claim follows by applying Equation (5.13) to Equation (5.15).

(P3) By definition of the modified normalized exponential, we have

$$e_{\pi,\sigma}^{\beta}(\alpha(\varphi); z + w) = e_{\pi,\sigma}^{\beta}(\alpha(\varphi); z) e_{\pi,\sigma}^{\beta}(\alpha(\varphi); w) l_{\pi,\sigma}^{\beta}(\alpha(\varphi)).$$

For \( l_{\pi,\sigma}^{\beta}(\alpha(\varphi)) \), we use the decomposition (5.10) such that the above equation yields

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle C_{n}^{\sigma,\beta}(z + w), \varphi \otimes \alpha \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \langle C_{k}^{\sigma,\beta}(z), \varphi \otimes k \rangle \sum_{l=0}^{\infty} \frac{1}{l!} \langle C_{l}^{\sigma,\beta}(w), \varphi \otimes l \rangle \sum_{m=0}^{\infty} \frac{1}{m!} \langle M_{m}^{\sigma,\beta,\alpha}, \varphi \otimes m \rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{k+l+m=n} \frac{n!}{k!l!m!} C_{k}^{\sigma,\beta}(z) C_{l}^{\sigma,\beta}(w) M_{m}^{\sigma,\beta,\alpha}, \varphi \otimes m \right).$$

Thus, the result follows by comparing the coefficients in both sides of the equation.

(P4) Again, by definition of the modified normalized exponential, we have

$$e_{\pi,\sigma}^{\beta}(\alpha(\varphi); z + w) = e_{\pi,\sigma}^{\beta}(\alpha(\varphi); z) \exp(\langle w, \alpha(\varphi) \rangle).$$

By Equations (A.1) and (5.9), we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle C_{n}^{\sigma,\beta}(z + w), \varphi \otimes n \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \langle C_{k}^{\sigma,\beta}(z), \varphi \otimes k \rangle \sum_{m=0}^{\infty} \frac{1}{m!} \langle (w), \varphi \otimes m \rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{k=0}^{n} \binom{n}{k} C_{k}^{\sigma,\beta}(z) \hat{\otimes} (w)_{n-k}, \varphi \otimes n \right).$$

Thus the assertion follows immediately by comparing the coefficients in both sides of the equation.

(P5) The result follows from (P4) at \( z = 0 \) and (A.9).

(P6) Note that for \( \varphi \in \mathcal{D}_C \), we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_{\pi,\sigma}^{\beta}((C_{n}^{\sigma,\beta}(w), \varphi \otimes n)) = \mathbb{E}_{\pi,\sigma}^{\beta}(e_{\pi,\sigma}^{\beta}(\alpha(\varphi); w)) = \frac{\mathbb{E}_{\pi,\sigma}^{\beta}(\exp(\langle \cdot, \alpha(\varphi) \rangle))}{l_{\pi,\sigma}^{\beta}(\alpha(\varphi))} = 1.$$

By polarization identity and comparison of coefficients, we obtain the result.

(P7) Let \( \varepsilon > 0 \) be given. Then let \( C_{\varepsilon}, \sigma_{\varepsilon} > 0 \) be chosen in such a way that \( |\alpha(\varphi)|_p \leq \varepsilon \) and \( C_{\varepsilon} \geq 1/|l_{\pi,\sigma}^{\beta}(\alpha(\varphi))| \) for \( |\varphi|_p = \sigma_{\varepsilon} \). By definition of \( C_{n}^{\sigma,\beta}(w) \) and the Cauchy formula, we have

$$|\langle C_{n}^{\sigma,\beta}(w), \varphi \otimes n \rangle| = |\langle d^n e_{\pi,\sigma}^{\beta}(0; w)(\varphi) \rangle|$$

$$\leq n! \frac{1}{\sigma_{\varepsilon}^n} \left( \sup_{|\varphi|_p = \sigma_{\varepsilon}} |\exp(|\alpha(\varphi)|_p |w|_p - p)| |\varphi|_p^n \right)$$

$$\leq n! \frac{1}{\sigma_{\varepsilon}^n} \left( \sup_{|\varphi|_p = \sigma_{\varepsilon}} \frac{1}{l_{\pi,\sigma}^{\beta}(\alpha(\varphi))} \right) \exp(\varepsilon |w|_p - p) |\varphi|_p^n$$

$$\leq C_{\varepsilon} n! \sigma_{\varepsilon}^{-n} \exp(\varepsilon |w|_p - p) |\varphi|_p^n.$$
For sufficiently small $\varepsilon$, we fix $\sigma_\varepsilon = \varepsilon \|i_{\nu', p}\|_{HS}$ so that
\[
|C_n^{\sigma,\beta}(w)|_{-\nu'} \leq n! C_\varepsilon \exp(\varepsilon |w|_{-p}) \left( \frac{1}{\sigma_\varepsilon} \|i_{\nu', p}\|_{HS} \right)^n, \quad w \in \mathcal{H}_{-p, C}.
\]
This concludes the proof. \hfill $\Box$

### 5.2 Generalized Dual Appell System

In what follows, we use again the approach in [23] of non-Gaussian analysis to introduce the generalized dual Appell system associated with the fPM $\pi_\sigma^\beta$.

**Definition 5.3.** The space of smooth polynomials $\mathcal{P}(D')$ on $D'$ is the space consisting of finite linear combinations of monomial functions, that is,
\[
\mathcal{P}(D') := \left\{ \varphi(w) = \sum_{n=0}^{N(\varphi)} \langle w^\otimes_n, \varphi^{(n)} \rangle \middle| \varphi^{(n)} \in \mathcal{D}_{C}^\otimes_n, w \in D', N(\varphi) \in \mathbb{N}_0 \right\}.
\]

The space $\mathcal{P}(D')$ shall be equipped with the natural topology, such that the mapping
\[
I : \mathcal{P}(D') \rightarrow \bigoplus_{n=0}^{\infty} \mathcal{D}_{C}^\otimes_n
\]
defined for any $\varphi(\cdot) = \sum_{n=0}^{\infty} \langle w^\otimes_n, \varphi^{(n)} \rangle \in \mathcal{P}(D')$ by
\[
I \varphi = \vec{\varphi} = (\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(n)}, \ldots)
\]
becomes a topological isomorphism from $\mathcal{P}(D')$ to the topological direct sum of symmetric tensor powers $\mathcal{D}_{C}^\otimes_n$ (see [6, 35]). Note that only a finite number of $\varphi^{(n)}$ is non-zero. With respect to this topology, a sequence $(\varphi_m)_{m \in \mathbb{N}}$ of smooth continuous polynomials, that is, $\varphi_m(w) = \sum_{n=0}^{N(\varphi_m)} \langle w^\otimes_n, \varphi_m^{(n)} \rangle$ converges to $\varphi(w) = \sum_{n=0}^{N(\varphi)} \langle w^\otimes_n, \varphi^{(n)} \rangle \in \mathcal{P}(D')$ if, and only if, the sequence $(N(\varphi_m))_{m \in \mathbb{N}}$ is bounded and $(\varphi_m^{(n)})_{m \in \mathbb{N}}$ converges to $\varphi^{(n)}$ in $\mathcal{D}_{C}^\otimes_n$ for all $n \in \mathbb{N}_0$.

Using Proposition 5.2-(P2), the space of smooth polynomials $\mathcal{P}(D')$ can also be expressed in terms of the generalized Appell polynomials associated with the measure $\pi_\sigma^\beta$ given by
\[
\mathcal{P}(D') := \left\{ \varphi(w) = \sum_{n=0}^{N(\varphi)} \langle C^{\sigma,\beta}_n(w), \varphi^{(n)} \rangle \middle| \varphi^{(n)} \in \mathcal{D}_{C}^\otimes_n, w \in D', N(\varphi) \in \mathbb{N}_0 \right\}.
\]

We denote by $\mathcal{P}'(D')$ the dual space of $\mathcal{P}(D')$ with respect to $L^2(\pi_\sigma^\beta) := L^2(D', \mathcal{C}_\sigma(D'), \pi_\sigma^\beta; \mathbb{C})$ and obtain the triple
\[
\mathcal{P}(D') \subset L^2(\pi_\sigma^\beta) \subset \mathcal{P}'(D').
\] (5.16)

The (bilinear) dual pairing $\langle \cdot, \cdot \rangle_{\pi_\sigma^\beta}$ between $\mathcal{P}(D')$ and $\mathcal{P}'(D')$ is then related to the (sesquilinear) inner product on $L^2(\pi_\sigma^\beta)$ by
\[
\langle F, \varphi \rangle_{\pi_\sigma^\beta} = \langle \langle F, \vec{\varphi} \rangle_{L^2(\pi_\sigma^\beta)} \rangle, \quad F \in L^2(\pi_\sigma^\beta), \varphi \in \mathcal{P}(D'),
\]
where \( \bar{\varphi} \) denotes the complex conjugate function of \( \varphi \). Further we introduce the constant function \( 1 \in L^2(\pi_\sigma^\beta) \subset \mathcal{P}^\beta_{\pi_\sigma^\beta}(\mathcal{D}') \) such that \( 1(w) = 1 \) for all \( w \in \mathcal{D}' \), so for any polynomial \( \varphi \in \mathcal{P}(\mathcal{D}') \),

\[
E_{\pi_\sigma^\beta}(\varphi) := \int_{\mathcal{D}'} \varphi(w) d\pi_\sigma^\beta(w) = \langle 1, \varphi \rangle_{\pi_\sigma^\beta}.
\]

Now, we will describe the distributions in \( \mathcal{P}^\beta_{\pi_\sigma^\beta}(\mathcal{D}') \) in a similar way as the smooth polynomials \( \mathcal{P}(\mathcal{D}') \), that is, for any \( \Phi \in \mathcal{P}^\beta_{\pi_\sigma^\beta}(\mathcal{D}') \), we find elements \( \Phi^{(n)} \in (\mathcal{D}^\hat{n}_C)' \) and operators \( Q_n^{\sigma,\beta,\alpha} \) on \( (\mathcal{D}^\hat{n}_C)' \), such that

\[
\Phi = \sum_{n=0}^{\infty} Q_n^{\sigma,\beta,\alpha}(\Phi^{(n)}) \in \mathcal{P}^\beta_{\pi_\sigma^\beta}(\mathcal{D}').
\]

To this end, we define first a differential operator \( D(\Phi^{(n)}) \) depending on \( \Phi^{(n)} \in (\mathcal{D}^\hat{n}_C)' \) such that when applied to the monomials \( \langle w^{\otimes m}, \varphi^{(m)} \rangle, \varphi^{(m)} \in \mathcal{D}^\hat{n}_C, m \in \mathbb{N}_0 \), gives

\[
D(\Phi^{(n)})(w^{\otimes m}, \varphi^{(m)}) := \begin{cases} m! \langle w^{\otimes (m-n)} \hat{\Phi}^{(n)}, \varphi^{(m)} \rangle, & \text{for } m \geq n \\ 0, & \text{otherwise} \end{cases}
\]

and extend by linearity from the monomials to elements in \( \mathcal{P}(\mathcal{D}') \). If we consider the space of Schwartz test function \( \mathcal{S}(\mathbb{R}) \) instead of using the space \( \mathcal{D} \) with the triple

\[
\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, dx) \subset \mathcal{S}'(\mathbb{R}),
\]

then for \( n = 1 \) and \( \Phi^{(1)} = \delta_t \in \mathcal{S}_c(\mathbb{R}) \), the differential operator \( D(\delta_t) \) coincides with the Hida derivative, see [16]. Note that \( D(\Phi^{(n)}) \) is a continuous linear operator from \( \mathcal{P}(\mathcal{D}') \) to \( \mathcal{P}(\mathcal{D}') \) (see [23, Lemma 4.13]) and this enables us to define the dual operator

\[
D(\Phi^{(n)})^* : \mathcal{P}^\beta_{\pi_\sigma^\beta}(\mathcal{D}') \rightarrow \mathcal{P}^\beta_{\pi_\sigma^\beta}(\mathcal{D}').
\]

Below we need the evaluation of the operator \( D(\Phi^{(n)}) \) on the monomials \( \langle P_m^{\sigma,\beta}(w), \varphi^{(m)} \rangle, m \in \mathbb{N}_0 \) in (5.5). We state this result in the next proposition and the proof can be found in [23, Lemma 4.14].

**Proposition 5.4.** For \( \Phi^{(n)} \in (\mathcal{D}^\hat{n}_C)' \) and \( \varphi^{(m)} \in \mathcal{D}^\hat{n}_C \) we have

\[
D(\Phi^{(n)})(P_m^{\sigma,\beta}(w), \varphi^{(m)}) = \begin{cases} \frac{m!}{(m-n)!} \langle P_{m-n}^{\sigma,\beta}(w) \hat{\Phi}^{(n)}, \varphi^{(m)} \rangle, & \text{for } m \geq n \\ 0, & \text{for } m < n. \end{cases}
\]

Now, we set \( Q_m^{\sigma,\beta}(\Phi^{(n)}) := D(\Phi^{(n)})^* 1 \) for \( \Phi^{(n)} \in (\mathcal{D}^\hat{n}_C)' \) and denote the so-called \( Q^{\sigma,\beta} \)-system in \( \mathcal{P}^\beta_{\pi_\sigma^\beta}(\mathcal{D}') \) by

\[
Q^{\sigma,\beta} := \left\{ Q_n^{\sigma,\beta}(\Phi^{(n)}) \mid \Phi^{(n)} \in (\mathcal{D}^\hat{n}_C)', n \in \mathbb{N}_0 \right\}.
\]

The pair \( \Lambda^{\sigma,\beta} = (\mathbb{P}^{\sigma,\beta}, Q^{\sigma,\beta}) \) is called the Appell system generated by the measure \( \pi_\sigma^\beta \). This system satisfies the biorthogonal property, see [23], given in the following theorem.

**Theorem 5.5.** For \( \Phi^{(m)} \in (\mathcal{D}^\hat{n}_C)' \) and \( \varphi^{(n)} \in \mathcal{D}^\hat{n}_C \) we have

\[
\langle Q_n^{\sigma,\beta}(\Phi^{(m)}), P_m^{\sigma,\beta}(\varphi^{(n)}) \rangle_{\pi_\sigma^\beta} = \delta_{m,n} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle, \quad n, m \in \mathbb{N}_0.
\] (5.17)
However, our aim is to construct the generalized dual Appell system $\mathbb{Q}^{\sigma,\beta,\alpha}$ such that $\mathbb{P}^{\sigma,\beta,\alpha}$ and $\mathbb{Q}^{\sigma,\beta,\alpha}$ are biorthogonal. The reason to do this is because when $\beta = 1$ we obtain only one system of orthogonal polynomials, so-called the Charlier polynomials, see [18].

First, recall the function $g_\alpha(\varphi), \varphi \in \mathcal{D}_C$ from Section 5.1. By Equation (A.10), we have

$$g_\alpha(\varphi)^{\otimes n} = \sum_{k=n}^{\infty} \frac{n!}{k!} S(n, k) \varphi^{\otimes k}.$$  

Then for any $\Phi^{(n)} \in (\mathcal{D}_C^\otimes)^\prime$, we have

$$\langle \Phi^{(n)}, g_\alpha(\varphi)^{\otimes n} \rangle = \sum_{k=n}^{\infty} \frac{n!}{k!} \langle \Phi^{(n)}, S(k, n) \varphi^{\otimes k} \rangle = \sum_{k=n}^{\infty} \frac{n!}{k!} \langle S(k, n)^* \Phi^{(n)}, \varphi^{\otimes k} \rangle,$$  \hspace{1cm} (5.18)

where $S(k, n)^* \Phi^{(n)} \in (\mathcal{D}_C^\otimes)^\prime$. Now, we define the operator $G(\Phi^{(n)})$ by

$$G(\Phi^{(n)}): \mathcal{P}(\mathcal{D}') \rightarrow \mathcal{P}(\mathcal{D}'), \varphi \mapsto G(\Phi^{(n)})\varphi := \sum_{k=n}^{\infty} \frac{n!}{k!} D(S(k, n)^* \Phi^{(n)})\varphi.$$  

Since $D(S(k, n)^* \Phi^{(n)})$ is continuous for any $\Phi^{(n)} \in (\mathcal{D}_C^\otimes)^\prime$, it is easy to see that $G(\Phi^{(n)})$ is also continuous and so its adjoint $G(\Phi^{(n)})^* : \mathcal{P}'_{\pi^\alpha}(\mathcal{D}') \rightarrow \mathcal{P}'_{\pi^\alpha}(\mathcal{D}')$ exists.

**Definition 5.6.** For any $\Phi^{(n)} \in (\mathcal{D}_C^\otimes)^\prime$, we define the generalized function $Q_{\sigma,\beta,\alpha}^{\sigma,\beta,\alpha}(\Phi^{(n)}) \in \mathcal{P}'_{\pi^\alpha}(\mathcal{D}')$, $n \in \mathbb{N}_0$, by

$$Q_{\sigma,\beta,\alpha}^{\sigma,\beta,\alpha}(\Phi^{(n)}):= G(\Phi^{(n)})^*1.$$  \hspace{1cm} (5.19)

The family

$$\mathbb{Q}^{\sigma,\beta,\alpha} := \left\{ Q_{\sigma,\beta,\alpha}^{\sigma,\beta,\alpha}(\Phi^{(n)}) \mid \Phi^{(n)} \in (\mathcal{D}_C^\otimes)^\prime, n \in \mathbb{N}_0 \right\}$$

is said to be the generalized dual Appell system $\mathbb{Q}^{\sigma,\beta,\alpha}$ associated with $\pi^\beta$ or the $\mathbb{Q}^{\sigma,\beta,\alpha}$-system and the pair $\mathbb{A}^{\sigma,\beta,\alpha} := (\mathbb{P}^{\sigma,\beta,\alpha}, \mathbb{Q}^{\sigma,\beta,\alpha})$ is called the generalized Appell system generated by the measure $\pi^\alpha$.

The following theorem states the biorthogonal property of the generalized Appell system $\mathbb{A}^{\sigma,\beta,\alpha}$.

**Theorem 5.7.** For $\Phi^{(n)} \in (\mathcal{D}_C^\otimes)^\prime$ and $\varphi^{(m)} \in \mathcal{D}_C^\otimes$ we have

$$\langle Q_{\sigma,\beta,\alpha}^{\sigma,\beta,\alpha}(\Phi^{(n)}), (C^{\sigma,\beta,\alpha}_m, \varphi^{(m)}) \rangle_{\pi^\alpha} = \delta_{n,m} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle, \quad n, m \in \mathbb{N}_0.$$

**Proof.** By Proposition 5.2-(P1), we have

$$\langle C^{\sigma,\beta,\alpha}_m, \varphi^{(m)} \rangle = \left\langle \sum_{i=0}^{m} s(m, i)^* P^{\sigma,\beta,\alpha}_i, \varphi^{(m)} \right\rangle = \sum_{i=0}^{m} \langle P^{\sigma,\beta,\alpha}_i, s(m, i)\varphi^{(m)} \rangle.$$  

Then it follows from Proposition 5.4 (noted below with *) and Proposition 11-(P4) in [23] (**).
that

\[
\langle\langle Q_n^{\sigma,\beta}(\Phi^{(n)}), (C_m^{\sigma,\beta}, \varphi^{(m)})\rangle\rangle_\pi = \langle\langle 1, G(\Phi^{(n)}) (C_m^{\sigma,\beta}, \varphi^{(m)})\rangle\rangle_\pi
\]

\[
= \sum_{i=0}^{m} \langle\langle 1, G(\Phi^{(n)}) (P_i^{\sigma,\beta}, s(m, i) \varphi^{(m)})\rangle\rangle_\pi
\]

\[
= \sum_{i=0}^{m} \sum_{k=0}^{\infty} \frac{n!}{k! (i-k)!} \langle\langle 1, (P_i^{\sigma,\beta} \otimes S(k,n)^* \Phi^{(n)}, s(m, i) \varphi^{(m)})\rangle\rangle_\pi
\]

\[
= \sum_{i=0}^{m} \sum_{k=0}^{\infty} \frac{n!}{k! (i-k)!} \langle\langle S(k,n)^* \Phi^{(n)}, s(m, k) \varphi^{(m)}\rangle\rangle_\pi
\]

\[
= \sum_{k=0}^{m} n! \langle\langle \Phi^{(n)}, S(k,n) s(m, k) \varphi^{(m)}\rangle\rangle_\pi
\]

\[
= \delta_{n,m} n! \langle\langle \Phi^{(n)}, \varphi^{(m)}\rangle\rangle_\pi,
\]

where the last equality is obtained using Proposition A.9 in Appendix A.2.

\[\square\]

**Remark 5.8.** In Appendix A.3, we provide an alternative proof for the biorthogonal property of the generalized Appell system \(A^\sigma,\beta,\alpha\) using the \(S_\pi\)-transform (to be introduced in Section 6) of the generalized function \(Q_n^{\sigma,\beta,\alpha} (\Phi^{(n)}) \in \mathcal{P}_\pi (\mathcal{D}')\). It is based on the fact that \(\exp(\langle\langle z, \varphi\rangle\rangle)\) is an eigenfunction of the generalized function \(G(\Phi^{(n)})\).

Using Theorem 5.7, the space \(\mathcal{P}_\pi (\mathcal{D}')\) can now be characterized in a similar way as the space \(\mathcal{P}(\mathcal{D}')\). See [21] for the proof of the following theorem.

**Theorem 5.9.** For every \(\Phi \in \mathcal{P}_\pi (\mathcal{D}')\), there exists a unique sequence \((\Phi^{(n)})_{n \in \mathbb{N}_0}\), \(\Phi^{(n)} \in (\mathcal{D}_c^{\otimes n})'\) such that

\[
\Phi = \sum_{n=0}^{\infty} Q_n^{\sigma,\beta,\alpha} (\Phi^{(n)})
\]

and vice versa, every such series generates a generalized function in \(\mathcal{P}_\pi (\mathcal{D}')\).

## 6 Test and Generalized Function Spaces

In this section, we construct the test function space and the generalized function space associated to the fPm \(\pi_\alpha\) and study some properties. Here, we consider a nuclear triple

\[
\text{pr lim}_{p \in \mathbb{N}} \mathcal{H}_p = \mathcal{N} \subset L^2(\sigma) \subset \mathcal{N}' = \text{ind lim}_{p \in \mathbb{N}} \mathcal{H}_{-p},
\]

as described in Section 2 such that

\[
\mathcal{D} \subset \mathcal{N} \subset L^2(\sigma) \subset \mathcal{N}' \subset \mathcal{D}'.
\]

Let \(\varphi = \sum_{n=0}^{N} (C_n^{\sigma,\beta}(w), \varphi^{(n)}) \in \mathcal{P}(\mathcal{D}')\) be given. Then we use the fact that \(\mathcal{D} \subset \mathcal{N}\) so that \(\varphi^{(n)} \in \mathcal{N}_{\mathcal{C}}^{\otimes n}\). Note that

\[
\mathcal{N}_{\mathcal{C}}^{\otimes n} = \text{pr lim}_{p \in \mathbb{N}} \mathcal{H}_{-p,\mathcal{C}}^{\otimes n}
\]

\[22\]
and so $\varphi^{(n)} \in \mathcal{H}_{p,q}^\alpha$ for all $p \in \mathbb{N}$. For each $p, q \in \mathbb{N}$ and $\kappa \in [0, 1]$, we introduce a norm $\| \cdot \|_{p,q,\kappa,\pi,\sigma}^\beta$ on $\mathcal{P}(\mathcal{D}')$ by

$$\| \varphi \|^2_{p,q,\kappa,\pi,\sigma}^\beta := \sum_{n=0}^{\infty} (n!)^{1+\kappa} 2^{nq} |\varphi^{(n)}|^2_p.$$ 

Let $(\mathcal{H}_p)^{\kappa}_{q,\pi,\sigma}^\beta$ be the Hilbert space obtained by completing the space $\mathcal{P}(\mathcal{D}')$ with respect to the norm $\| \cdot \|_{p,q,\kappa,\pi,\sigma}^\beta$. The Hilbert space $(\mathcal{H}_p)^{\kappa}_{q,\pi,\sigma}^\beta$ has inner product given by

$$(\varphi, \psi)^{\kappa}_{q,\pi,\sigma} := \sum_{n=0}^{\infty} (n!)^{1+\kappa} 2^{nq} \langle \varphi^{(n)}, \overline{\psi^{(n)}} \rangle_p,$$

and admits the representation

$$(\mathcal{H}_p)^{\kappa}_{q,\pi,\sigma}^\beta := \left\{ \varphi = \sum_{n=0}^{\infty} \langle C_n^{\kappa,\sigma}, \varphi^{(n)} \rangle \in L^2(\pi^\beta) \mid \| \varphi \|^2_{p,q,\kappa,\pi,\sigma}^\beta = \sum_{n=0}^{\infty} (n!)^{1+\kappa} 2^{nq} |\varphi^{(n)}|^2_p < \infty \right\}.$$

Then the test function space $(\mathcal{N})^{\kappa}_{\pi,\sigma}^\beta$ is defined by

$$(\mathcal{N})^{\kappa}_{\pi,\sigma}^\beta := \text{pr lim}_{p,q \in \mathbb{N}} (\mathcal{H}_p)^{\kappa}_{q,\pi,\sigma}^\beta.$$ 

The test function space $(\mathcal{N})^{\kappa}_{\pi,\sigma}^\beta$ is a nuclear space which is continuously embedded in $L^2(\pi^\beta)$.

**Example 6.1.** The modified normalized exponential given in (5.8) has the norm

$$\| e_{\pi,\sigma}^{\beta}(\alpha(\varphi); \cdot) \|^2_{p,q,\kappa,\pi,\sigma}^\beta = \sum_{n=0}^{\infty} (n!)^{1+\kappa} 2^{nq} \frac{|\varphi|^{2n}_p}{(n!)^{1-\kappa}}, \quad \varphi \in \mathcal{N}_C.$$ 

1. If $\kappa = 0$, we have

$$\| e_{\pi,\sigma}^{\beta}(\alpha(\varphi); \cdot) \|^2_{p,q,0,\pi,\sigma}^\beta = \exp(2q |\varphi|^2_p) < \infty, \quad \forall \varphi \in \mathcal{N}_C.$$ 

2. For $\kappa \in (0, 1)$, we use the Hölder inequality with the pair $(\frac{1}{\kappa}, 1 - \frac{1}{\kappa})$ and obtain

$$\| e_{\pi,\sigma}^{\beta}(\alpha(\varphi); \cdot) \|^2_{p,q,\kappa,\pi,\sigma}^\beta \leq \left( \sum_{n=0}^{\infty} \left( \frac{1}{2n\kappa} \right)^{\frac{1}{\kappa}} \right) \kappa \left( \sum_{n=0}^{\infty} \left( \frac{2^n 2^q |\varphi|^{2n}_p}{(n!)^{1-\kappa}} \right)^{\frac{1}{1-\kappa}} \right)^{1-\kappa} = 2^\kappa \exp \left( (1 - \kappa)2^{\frac{1}{1-\kappa}} |\varphi|^{\frac{2}{1-\kappa}}_p \right) < \infty,$$

for all $\varphi \in \mathcal{N}_C$. Thus, $e_{\pi,\sigma}^{\beta}(\alpha(\varphi); \cdot) \in (\mathcal{N})^{\kappa}_{\pi,\sigma}^\beta$, $\kappa \in [0, 1)$.

3. For $\kappa = 1$, we have

$$\| e_{\pi,\sigma}^{\beta}(\alpha(\varphi); \cdot) \|^2_{p,q,\kappa,\pi,\sigma}^\beta = \sum_{n=0}^{\infty} 2^n q |\varphi|^{2n}_p, \quad \varphi \in \mathcal{N}_C.$$ 

Hence, we have $e_{\pi,\sigma}^{\beta}(\alpha(\varphi); \cdot) \notin (\mathcal{N})^{1}_{\pi,\sigma}^\beta$ if $\varphi \neq 0$, but $e_{\pi,\sigma}^{\beta}(\alpha(\varphi); \cdot) \in (\mathcal{H}_p)^{1}_{q,\pi,\sigma}^\beta$ if $2^q |\varphi|^2_p < 1$. Moreover, the set

$$\{ e_{\pi,\sigma}^{\beta}(\alpha(\varphi); \cdot) \mid 2^q |\varphi|^2_p < 1, \varphi \in \mathcal{N}_C \}$$

is total in $(\mathcal{H}_p)^{1}_{q,\pi,\sigma}^\beta$. 

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For $\kappa = 1$, we collect below the most important properties of the space $(\mathcal{N})_{\pi,\sigma}^{1,1}$, see [21] for the proofs.

**Theorem 6.2.** (i) $(\mathcal{N})_{\pi,\sigma}^{1,1}$ is a nuclear space.

(ii) The topology in $(\mathcal{N})_{\pi,\sigma}^{1,1}$ is uniquely defined by the topology on $\mathcal{N}$, i.e., it does not depend on the choice of the family of norms $\{|·|_p\}$, $p \in \mathbb{N}$.

(iii) There exist $p', q' > 0$ such that for all $p \geq p'$, $q \geq q'$ the topological embedding $(\mathcal{H}_p)_{q,\pi,\sigma}^{1,1} \subset L^2(\pi_\sigma^β)$ holds. $(\mathcal{N})_{\pi,\sigma}^{1,1}$ is continuously and densely embedded in $L^2(\pi_\sigma^β)$.

**Proposition 6.3.** Any test function $\varphi$ in $(\mathcal{N})_{\pi,\sigma}^{1,1}$ has a uniquely defined extension to $\mathcal{N}^1_{\pi,\sigma}$ as an element of $E^{1}_{\text{min}}(\mathcal{N}_C^1)$. For all $p > p'$ such that the embedding $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}$ is of the Hilbert-Schmidt class and for all $\varepsilon > 0$, $p \in \mathbb{N}$, we obtain the following bound

$$|\varphi(w)| \leq C\|\varphi\|_{p,q,1,\pi,\sigma}E^{1}π|_{p'} \varepsilon^{-q}, \quad \varphi \in (\mathcal{N})_{\pi,\sigma}^{1,1}, \ w \in \mathcal{H}_{p,\varepsilon,\mathcal{C}},$$

where $2^q > (\varepsilon\|\varphi\|_{H^2})^{-2}$ and

$$C = C_\varepsilon(1 - 2^{-q} \varepsilon^{-2})^{-1/2}.$$

For each $p, q \in \mathbb{N}$ and $\kappa \in [0, 1]$, we denote by $(\mathcal{H}_{-p})_{-q,\pi,\sigma}^{-\kappa}$ the Hilbert space dual of the space $(\mathcal{H}_p)_{q,\pi,\sigma}^{\kappa}$ with respect to $L^2(\pi_\sigma^β)$ with the corresponding (Hilbert) norm $\|·\|_{p,q,\kappa,\pi,\sigma}$. This space admits the following representation

$$(\mathcal{H}_{-p})_{-q,\pi,\sigma}^{-\kappa} = \left\{ \Phi = \sum_{n=0}^{\infty} Q_{n}^{\pi,\sigma,\kappa}(\Phi^{(n)}) \in \mathcal{P}_\pi^{\kappa}(\mathcal{D}') : \|\Phi\|_{p,q,\mathcal{H}_{-p,\kappa,\pi,\sigma}}^2 = \sum_{n=0}^{\infty} (n!)^{1-\kappa} 2^{-nq} |\Phi^{(n)}|_{p}^2 < \infty \right\}.$$

By the general duality theory, the dual space $(\mathcal{N})_{\pi,\sigma}^{\kappa}$ of $(\mathcal{N})_{\pi,\sigma}^{-\kappa}$ with respect to $L^2(\pi_\sigma^β)$ is then given by

$$(\mathcal{N})_{\pi,\sigma}^{\kappa} = \bigcup_{p,q \in \mathbb{N}} (\mathcal{H}_{-p})_{-q,\pi,\sigma}^{-\kappa}.$$ 

Since $\mathcal{P}(\mathcal{D}') \subset (\mathcal{N})_{\pi,\sigma}^{\kappa}$, the space $(\mathcal{N})_{\pi,\sigma}^{-\kappa}$ can be viewed as a subspace of $\mathcal{P}_\pi^{\kappa}(\mathcal{D}')$ and so we extend the triple in (5.16) to the chain of spaces

$$\mathcal{P}(\mathcal{D}') \subset (\mathcal{N})_{\pi,\sigma}^{\kappa} \subset L^2(\pi_\sigma^β) \subset (\mathcal{N})_{\pi,\sigma}^{-\kappa} \subset \mathcal{P}_\pi^{\kappa}(\mathcal{D}').$$

The action of a distribution

$$\Phi = \sum_{n=0}^{\infty} Q_{n}^{\pi,\sigma,\kappa}(\Phi^{(n)}) \in (\mathcal{N})_{\pi,\sigma}^{-\kappa}$$

on a test function

$$\varphi = \sum_{n=0}^{\infty} \langle C_{n}^{\sigma,\pi}(w), \varphi^{(n)} \rangle \in (\mathcal{N})_{\pi,\sigma}^{\kappa}$$

using the biorthogonal property in Theorem 5.7 is given by

$$\langle \Phi, \varphi \rangle_{\pi,\sigma}^{\kappa} = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle.$$

Now we give two examples of the generalized functions in $(\mathcal{N})_{\pi,\sigma}^{-1}$. For a more generalized case, see [21].
Example 6.4 (Generalized Radon-Nikodym derivative). We define a generalized function \( \rho_{\pi,\sigma}^{\alpha}(w, \cdot) \in (\mathcal{N})^{-1}_{\pi,\sigma} \), \( w \in \mathcal{N}'_{\mathcal{C}} \) with the following property

\[
\langle \rho_{\pi,\sigma}^{\alpha}(w, \cdot), \varphi \rangle_{\pi,\sigma} = \int_{\mathcal{N}'} \varphi(x - w) \, d\mu(x), \quad \varphi \in (\mathcal{N})^{1}_{\pi,\sigma}.
\]

First, we have to establish the continuity of \( \rho_{\pi,\sigma}^{\alpha}(w, \cdot) \). Let \( w \in \mathcal{H}_{-p,\mathcal{C}} \) be given. Then, if \( p \geq p' \) is sufficiently large and \( \varepsilon > 0 \) is small enough, we use Proposition 6.3, that is, there exists \( q \in \mathbb{N} \) and \( C > 0 \) such that

\[
\left| \int_{\mathcal{N}'} \varphi(x - w) \, d\pi_{\sigma}^{\beta}(x) \right| \leq C \| \varphi \|_{p,q,1,\pi,\sigma} \int_{\mathcal{N}'} \exp(\varepsilon |x - w| - p') \, d\pi_{\sigma}^{\beta}(x)
\]

\[
\leq C \| \varphi \|_{p,q,1,\pi,\sigma} \exp(\varepsilon |w| - p') \int_{\mathcal{N}'} \exp(\varepsilon |x| - p') \, d\pi_{\sigma}^{\beta}(x).
\]

Since \( \varepsilon \) is sufficiently small, the last integral exists by Lemma 9 from [23]. This implies that \( \rho_{\pi,\sigma}^{\alpha}(w, \cdot) \in (\mathcal{N})^{-1}_{\pi,\sigma} \). Let us show that in \( (\mathcal{N})^{-1}_{\pi,\sigma} \) the generalized function \( \rho_{\pi,\sigma}^{\alpha}(w, \cdot) \) admits the canonical expansion

\[
\rho_{\pi,\sigma}^{\alpha}(w, \cdot) = \sum_{k=0}^{\infty} \frac{1}{k!} Q_{n}^{\sigma,\alpha,\beta}((-w)_k).
\] (6.1)

Note that the right hand side of (6.1) defines an element in \( (\mathcal{N})^{-1}_{\pi,\sigma} \). Then it is sufficient to compare the action of both sides of (6.1) on a total set from \( (\mathcal{N})^{1}_{\pi,\sigma} \). For \( \varphi^{(n)} \in \mathcal{D}_{\mathcal{C}}^{n} \), we use the biorthogonal property of \( \mathbb{P}^{\sigma,\beta,\alpha} \) and \( Q^{\sigma,\beta,\alpha} \)-systems and obtain

\[
\langle \rho_{\pi,\sigma}^{\alpha}(w, \cdot), \langle C_{n}^{\sigma,\beta}, \varphi^{(n)} \rangle \rangle_{\pi,\sigma} = \sum_{k=0}^{\infty} \frac{1}{k!} Q_{n}^{\sigma,\alpha,\beta}((-w)_k), \langle C_{n}^{\sigma,\beta}, \varphi^{(n)} \rangle_{\pi,\sigma}
\]

\[
= \langle (-w)_n, \varphi^{(n)} \rangle.
\]

On the other hand, by Proposition 5.2-(P4) and (P6),

\[
\langle \rho_{\pi,\sigma}^{\alpha}(w, \cdot), \langle C_{n}^{\sigma,\beta}, \varphi^{(n)} \rangle \rangle_{\pi,\sigma} = \int_{\mathcal{D}} \langle C_{n}^{\sigma,\beta}(x - w), \varphi^{(n)} \rangle \, d\pi_{\sigma}^{\beta}(x)
\]

\[
= \sum_{k=0}^{\infty} \binom{n}{k} \int_{\mathcal{D}} \langle C_{n}^{\sigma,\beta}(x) \hat{Q}_{n}(-w)_{n-k}, \varphi^{(n)} \rangle \, d\pi_{\sigma}^{\beta}(x)
\]

\[
= \sum_{k=0}^{\infty} \binom{n}{k} \hat{E}_{\pi,\sigma} \langle (C_{n}^{\sigma,\beta}(x) \hat{Q}_{n}(-w)_{n-k}, \varphi^{(n)} \rangle
\]

\[
= \langle (-w)_n, \varphi^{(n)} \rangle.
\]

Thus, we have shown that \( \rho_{\pi,\sigma}^{\alpha}(w, \cdot) \) is the generating function of the \( Q^{\sigma,\beta,\alpha} \)-system, i.e.,

\[
\rho_{\pi,\sigma}^{\alpha}(-w, \cdot) = \sum_{k=0}^{\infty} \frac{1}{k!} Q_{k}^{\sigma,\alpha,\beta}((-w)_k).
\]

Example 6.5 (Delta function). For \( w \in \mathcal{N}'_{\mathcal{C}} \), we define a distribution by the following \( Q^{\sigma,\beta,\alpha} \)-decomposition:

\[
\delta_w = \sum_{n=0}^{\infty} Q_{n}^{\sigma,\beta,\alpha}(C_{n}^{\sigma,\beta}(w)).
\]
If \( p \in \mathbb{N} \) is large enough and \( \varepsilon > 0 \) is sufficiently small, by Proposition 5.2-(P7), for any \( w \in \mathcal{H}_{-p, C} \) we have

\[
\|\delta_w\|_{-p, q, \pi, \alpha}^2 = \sum_{n=0}^{\infty} 2^{-nq} |\mathcal{C}_{n, \pi, \alpha}^\sigma(w)|_p^2 \leq C^2_\varepsilon \exp(2\varepsilon|w|_p) \sum_{n=0}^{\infty} \varepsilon^{-2n} 2^{-nq}
\]

which is finite for sufficiently large \( q \in \mathbb{N} \). This implies that \( \delta_w \in \mathcal{N}_{\pi, p}^{-1} \). Now, for

\[
\varphi = \sum_{n=0}^{\infty} \langle \mathcal{C}_{n, \pi, \alpha}^\sigma, \varphi(n) \rangle \in \mathcal{N}_{\pi, p}^{-1}
\]

the action of \( \delta_w \) is given by

\[
\langle \delta_w, \varphi \rangle \pi, \sigma = \sum_{n=0}^{\infty} \langle \mathcal{C}_{n, \pi, \alpha}^\sigma(w), \varphi(n) \rangle = \varphi(w)
\]

using the biorthogonal property of \( \mathbb{P}^{\sigma, \beta, \alpha} \) and \( \mathbb{Q}^{\sigma, \beta, \alpha} \)-systems. This means that \( \delta_w \) (in particular for \( w \) real) plays the role of a \( \delta \)-function (evaluation map) in the calculus we discuss.

Recall Example 6.1 where the modified normalized exponential \( e_{\pi, \beta}^\sigma(\alpha(\varphi); \cdot) \) is a test function in \( \mathcal{N}_{\pi, p}^{-1} \) only if \( 2^q|\varphi|^2_p < 1 \) for \( \varphi \in \mathcal{N}_C \). We define the \( \mathbb{S}^{\pi, \sigma} \)-transform of a distribution \( \Phi \in \mathcal{N}_{\pi, p}^{-1} \subset \mathcal{P}_{\pi, \sigma}(D') \) by

\[
\mathbb{S}^{\pi, \sigma} \Phi(\varphi) := \langle \Phi, e_{\pi, \beta}^\sigma(\varphi; \cdot) \rangle \pi, \sigma
\]

if \( \varphi \) is chosen in the above way. By the biorthogonal property of \( \mathbb{P}^{\sigma, \beta, \alpha} \) and \( \mathbb{Q}^{\sigma, \beta, \alpha} \)-systems, we have

\[
\mathbb{S}^{\pi, \sigma} \Phi(\varphi) = \sum_{n=0}^{\infty} \langle \Phi(n), g_\alpha(\varphi)^{\otimes n} \rangle.
\]

Now, we introduce the convolution of a function \( \varphi \in \mathcal{N}_{\pi, p}^{-1} \), with respect to the measure \( \pi, \sigma \) given by

\[
\mathcal{C}_{\pi, \sigma}^{\sigma, \beta}(w) = \langle \rho_{\pi, \sigma}^\sigma(-w, \cdot), \varphi \rangle \pi, \sigma
\]

where \( \rho_{\pi, \sigma}^\sigma(-w, \cdot) \in \mathcal{N}_{\pi, p}^{-1} \) is the generalized Radon-Nikodym derivative (see Example 6.4) for any \( w \in \mathcal{N}_C \). If \( \varphi \) has the representation

\[
\varphi = \sum_{n=0}^{\infty} \langle \mathcal{C}_{n, \pi, \alpha}^\sigma(w), \varphi(n) \rangle \in \mathcal{N}_{\pi, p}^{-1},
\]

then the action of \( \mathcal{C}_{\pi, \sigma}^{\sigma, \beta} \) on \( \varphi \) is given, for every \( w \in \mathcal{N}_C \), by

\[
\mathcal{C}_{\pi, \sigma}^{\sigma, \beta}(w) = \sum_{n=0}^{\infty} \langle (-w)_n, \varphi(n) \rangle \otimes k \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \langle w^{\otimes k}, s(n, k) \varphi(n) \rangle.
\]

The following is the characterization theorem of the test and generalized function spaces associated to the \( \mathbb{P}^{\pi, \sigma} \)m which is a standard result for this approach. For the proof, we refer to [23, 21].

**Theorem 6.6.** (i) The convolution \( \mathcal{C}_{\pi, \sigma}^{\sigma, \beta} \) is a topological isomorphism from \( \mathcal{N}_{\pi, p}^{-1} \) on \( \mathcal{E}_{\min}(\mathcal{N}_C) \).

(ii) The \( \mathbb{S}^{\pi, \sigma} \)-transform is a topological isomorphism from \( \mathcal{N}_{\pi, p}^{-1} \) on \( \text{Hol}_0(\mathcal{N}_C) \).

(iii) The \( \mathbb{S}^{\pi, \sigma} \)-transform is a topological isomorphism from \( \mathcal{N}_{\pi, p}^{-1} \), \( \kappa \in [0, 1] \), on \( \mathcal{E}^{2/(1-\kappa)}_{\max}(\mathcal{N}_C) \).
7 Conclusion and Outlook

In this paper, we constructed the generalized Appell system \( A_{\sigma,\beta,\alpha} = (P_{\sigma,\beta,\alpha}, Q_{\sigma,\beta,\alpha}) \) associated to the \( fPm_{\pi,\beta,\sigma} \) in infinite dimension. The Appell polynomials \( P_{\sigma,\beta,\alpha} \) (generated by the modified Wick exponential) and the dual Appell system \( Q_{\sigma,\beta,\alpha} \) are biorthogonal to each other, see Theorem 5.7. It turns out that the kernels \( C^{\sigma,\beta}_{\kappa} \) of the system \( P_{\sigma,\beta,\alpha} \) are given in terms of the Stirling operators or in terms of the falling factorials on \( D'_{\mathbb{C}} \), see Proposition 5.2. The system \( P_{\sigma,\beta,\alpha} \) is used to define the spaces of test functions \((\mathcal{N})^{\kappa}_{\pi,\beta,\sigma}\), \( 0 \leq \kappa \leq 1 \), while \( Q_{\sigma,\beta,\alpha} \) is suitable to describe the generalized functions spaces \((\mathcal{N})^{-\kappa}_{\pi,\beta,\sigma}\) arising from \( \pi,\beta,\sigma \), see Section 6. The spaces \((\mathcal{N})^{\kappa}_{\pi,\beta,\sigma}\) and \((\mathcal{N})^{-\kappa}_{\pi,\beta,\sigma}\) are universal in the sense that their characterization via \( S_{\pi,\beta,\sigma} \)-transform is independent of the measure \( \pi,\beta,\sigma \), see Theorem 6.6) as is well known from non-Gaussian analysis, [23, 21].

In a future work we plan to investigate the stochastic counterpart associated to the \( fPm \), namely the fractional Poisson process \( N_{\lambda,\beta} \) in one and infinite dimensions. In particular, their representations in terms of known processes as well as possible applications.

A Appendix

A.1 Kolmogorov extension theorem on configuration space

In this section, we discuss a version of Kolmogorov extension theorem to the configuration space \((\Gamma, \mathcal{B}(\Gamma))\). The following definitions and properties of measurable spaces can be found in [9], [13] and [31].

**Definition A.1.** Let \((X, \mathcal{A})\) and \((X', \mathcal{A}')\) be two measurable spaces.

1. The spaces \((X, \mathcal{A})\) and \((X', \mathcal{A}')\) are called isomorphic if, and only if, there exists a measurable bijective mapping \( f : X \rightarrow X' \) such that its inverse \( f^{-1} \) is also measurable.
2. \((X, \mathcal{A})\) and \((X', \mathcal{A}')\) are called \( \sigma \)-isomorphic if, and only if, there exists a bijective mapping \( F : \mathcal{A} \rightarrow \mathcal{A}' \) between the \( \sigma \)-algebras which preserves the operations in a \( \sigma \)-algebra.
3. \((X, \mathcal{A})\) is said to be countable generated if, and only if, there exists a denumerable class \( \mathcal{D} \subset \mathcal{A} \) such that \( \mathcal{D} \) generates \( \mathcal{A} \).
4. \((X, \mathcal{A})\) is said to be separable if, and only if, it is countably generated and for each \( x \in X \) the set \( \{x\} \in \mathcal{A} \).

**Definition A.2.** Let \((X, \mathcal{A})\) be a countable generated measurable space. Then \((X, \mathcal{A})\) is called the standard Borel space if, and only if, there exists a Polish space \((X', \mathcal{A}')\) (i.e., a metrizable, complete metric space which fulfills the second axiom of countability and the \( \sigma \)-algebra \( \mathcal{A}' \) coincides with the Borel \( \sigma \)-sigma) such that \((X, \mathcal{A})\) and \((X', \mathcal{B}(X'))\) are \( \sigma \)-isomorphic.

**Example A.3.** 1. Every locally compact, \( \sigma \)-compact space is a standard Borel space.

2. Polish spaces are standard Borel spaces.

**Proposition A.4.** 1. If \((X, \mathcal{A})\) is a countable generated measurable space, then there exists \( E \subset \{0, 1\}^\mathbb{N} \) such that \((X, \mathcal{A})\) is \( \sigma \)-isomorphic to \((E, \mathcal{B}(E))\). Thus \((X, \mathcal{A})\) is \( \sigma \)-isomorphic to a separable measurable space.

2. Let \((X, \mathcal{A})\) and \((X', \mathcal{A}')\) be separable measurable spaces. Then \((X, \mathcal{A})\) is \( \sigma \)-isomorphic to \((X', \mathcal{A}')\) if, and only if, they are isomorphic.
The following theorem states some operations under which separable standard Borel spaces are closed, see [31] and [9].

**Theorem A.5.**

1. Countable product, sum, and union are separable standard Borel spaces.
2. The projective limit is a separable standard Borel space.
3. Any measurable subset of a separable standard Borel space is also a separable standard Borel space.

We need also a version of Kolmogorov’s extension theorem for separable standard Borel spaces.

**Theorem A.6** (cf. [31, Chap. V, Theorem 3.2]). Let \((X_n, \mathcal{A}_n), n \in \mathbb{N}\) be separable standard Borel spaces. Let \((X, \mathcal{A})\) be the projective limit of the space \((X_n, \mathcal{A}_n)\) relative to the maps \(p_{m,n} : X_n \to X_m, m \leq n\). If \(\{\mu_n\}_{n \in \mathbb{N}}\) is a sequence of probability measures such that \(n\) is a measure on \((X_n, \mathcal{A}_n)\) and \(\mu_m = \mu_n \circ p_{m,n}^{-1}\) for \(m \leq n\). Then there exists a unique measure on \((X, \mathcal{A})\) such that \(\mu_n = \mu \circ p_n^{-1}\) for all \(n \in \mathbb{N}\) where \(p_n\) is the projection map from \(X\) on \(X_n\).

This theorem can be extended to an index set \(I\) which is a directed set with an order generating sequence, i.e., there exists a sequence \((\alpha_n)_{n \in \mathbb{N}}\) in \(I\) such that for every \(\alpha \in I\) there exists \(n \in \mathbb{N}\) with \(\alpha < \alpha_n\). We apply this general framework to our configuration space \(\Gamma\). Assume that \((X, \mathcal{X})\) is a separable standard Borel space. To use \(\mathcal{B}_c(X)\) makes this generality have no sense, hence we have to introduce an abstract concept of local sets. Let \(\mathcal{J}_X\) be a subset of \(\mathcal{X}\) with the properties:

- **(I1)** \(\Lambda_1 \cup \Lambda_2 \in \mathcal{J}_X\) for all \(\Lambda_1, \Lambda_2 \in \mathcal{J}_X\).
- **(I2)** If \(\Lambda \in \mathcal{J}_X\) and \(A \in \mathcal{X}\) with \(A \subset \Lambda\) then \(A \in \mathcal{J}_X\).
- **(I3)** There exists a sequence \(\{\Lambda_n \mid n \in \mathbb{N}\}\) from \(\mathcal{J}_X\) with \(X = \bigcup_{n \in \mathbb{N}} \Lambda_n\) such that if \(\Lambda \in \mathcal{J}_X\) then \(\Lambda \subset \Lambda_n\) for some \(n \in \mathbb{N}\).

We can then construct the configuration space as in Subsection 4.2 taking \(X = \mathbb{R}^d\) and replacing \(\mathcal{B}(\mathbb{R}^d)\) by \(\mathcal{J}_{\mathbb{R}^d}\). Our aim is to show that \((\Gamma, \mathcal{B}(\Gamma))\) is a separable standard Borel space and thus by Theorem A.6 the measure \(\tau_\mathbb{R}^d\) in Subsection 4.3 exists.

It follows from Theorem A.5 that for any \(\Lambda \in \mathcal{J}_{\mathbb{R}^d}\) and for any \(n \in \mathbb{N}\), the set \(\Lambda^n\) is a separable standard Borel space. Thus, by the same argument \(\Lambda^n / S_n\) is also a separable standard Borel space, see e.g. [36]. Now taking into account the isomorphism between \(\Lambda^n / S_n\) and \(\Gamma_{\Lambda^n}, \Gamma_{\Lambda}^{(n)}\) is also a separable standard Borel space as well as \(\Gamma_{\Lambda}\) by Theorem A.5-(1). Therefore, given \((\Gamma, \mathcal{B}(\Gamma))\) as the projective limit of the projective system \(\{(\Gamma_{\Lambda}, \mathcal{B}(\Gamma_{\Lambda})), p_{\Lambda_1, \Lambda_2}, \mathcal{J}_{\mathbb{R}^d}\}\) of separable standard Borel spaces, by Theorem A.5-(2), \((\Gamma, \mathcal{B}(\Gamma))\) is a separable standard Borel space.

### A.2 Stirling Operators

In this appendix we discuss the Stirling operators which we use in Section 5 related to the Taylor expansion of a holomorphic function typical in Poisson analysis. For more details and other applications, see [10, 11].

For \(n \in \mathbb{N}\) and \(k \in \mathbb{N}_0\), we define the *falling factorial* by

\[
(k)_n := n! \binom{k}{n} = k(k-1) \cdots (k-n+1).
\]

The latter expression allows us to define falling factorials as polynomials of a variable \(z \in \mathbb{C}\) replacing \(k\) as

\[
(z)_n := z(z-1) \cdots (z-n+1).
\]
The generating function of the falling factorials is
\[ \sum_{n=0}^{\infty} \frac{u^n}{n!} (z)_n = \exp[\log(1 + u)]. \]

The \textit{Stirling numbers of the first kind}, denoted by \( s(n,k) \), are defined as the coefficients of the expansion \((z)_n\) in \( z \), in explicit,
\[ (z)_n := \sum_{k=1}^{n} s(n,k) z^k, \]
while the \textit{Stirling numbers of the second kind}, denoted by \( S(n,k) \), are defined as the coefficients of the expansion \( z^n \) in \((z)_k\), that is,
\[ z^n = \sum_{k=1}^{n} S(n,k)(z)_k. \]

Let us consider lifting the polynomials \((z)_n\) to \( D'_C \). We call these polynomials the falling factorials on \( D'_C \), denoted by \((w)_n\), for \( w \in D'_C \) (see [10]). The generating function of the falling factorials on \( D'_C \) is given by
\[ \exp(\langle w, \log(1 + \varphi) \rangle) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle (w)_n, \varphi^{\otimes n} \rangle, \quad \varphi \in D_C, \; w \in D'_C. \] (A.1)

The falling factorial may be written recursively (see Proposition 5.4 in [10]) as follows
\[ (w)_0 = 1, \]
\[ (w)_1 = w, \]
\[ (w)_n(x_1, \ldots, x_n) = w(x_1)(w(x_2) - \delta_{x_1}(x_2)) \]
\[ \times \cdots \times (w(x_n) - \delta_{x_1}(x_n) - \delta_{x_2}(x_n) - \cdots - \delta_{x_{n-1}}(x_n)), \]
for \( n \geq 2 \) and \((x_1, \ldots, x_n) \in (\mathbb{R}^d)^n\).

Now we define the \textit{Stirling operators of the first kind} as the linear operators \( s(n,k) : D_C^{\otimes n} \rightarrow D_C^{\otimes k}, \; n \geq k \), satisfying
\[ \langle (w)_n, \varphi^{(n)} \rangle = \sum_{k=1}^{n} \langle (w)^{\otimes k}, s(n,k) \varphi^{(n)} \rangle, \quad \varphi^{(n)} \in D_C^{\otimes n}, \; w \in D'_C, \] (A.2)

and the \textit{Stirling operators of the second kind} as the linear operators \( S(n,k) : D_C^{\otimes n} \rightarrow D_C^{\otimes k}, \; n \geq k \), satisfying
\[ \langle w^{\otimes n}, \varphi^{(n)} \rangle = \sum_{k=1}^{n} \langle (w)^{\otimes k}, S(n,k) \varphi^{(n)} \rangle, \quad \varphi^{(n)} \in D_C^{\otimes n}, \; w \in D'_C. \] (A.3)

\textbf{Remark A.7.} The Stirling operators \( s(n,k) \) and \( S(n,k) \) introduced in [11] are defined on the space of measurable, bounded, compactly supported, symmetric functions, however; in this paper, we define these operators on the space \( D_C^{\otimes n} \) as a consequence of extending the falling factorials to the space of generalized functions \( D'_C \) rather than using the space of Radon measures.
Let \( n, k \in \mathbb{N}, k \leq n \) and \( i_1, \ldots, i_k \in \mathbb{N} \) such that \( i_1 + \cdots + i_k = n \). We define the operator \( \mathbb{D}_{i_1, \ldots, i_k}^{(n)} \in \mathcal{L}(\mathcal{D}_c^{\hat{n}}, \mathcal{D}_c^{\hat{k}}) \) (the space of linear operators from \( \mathcal{D}_c^{\hat{n}} \) into \( \mathcal{D}_c^{\hat{k}} \)) by

\[
(\mathbb{D}_{i_1, \ldots, i_k}^{(n)} \varphi^{(n)})(x_1, \ldots, x_k) := \frac{1}{k!} \sum_{s \in S_k} \varphi^{(n)}(x_{i(1)}, \ldots, x_{i(1)}), \ldots, x_{i(k)}, \ldots, x_{i(k)}),
\]

(A.4)

for \( \varphi^{(n)} \in \mathcal{D}_c^{\hat{n}} \) and \( (x_1, \ldots, x_k) \in (\mathbb{R}^d)^k \). In particular, we have

\[
\mathbb{D}_{i_1, \ldots, i_k}^{(n)} \varphi^{\hat{n}} = \varphi^{i_1 \cdots \hat{i}} \varphi^{i_k}, \quad \varphi \in \mathcal{D}_c.
\]

(A.5)

When \( k = 1 \), we denote \( \mathbb{D}_{n}^{(n)} := \mathbb{D}_n^{(n)} \) such that for \( \varphi^{(n)} \in \mathcal{D}_c^{\hat{n}} \),

\[
(\mathbb{D}^{(n)} \varphi^{(n)})(x) = \varphi^{(n)}(x, \ldots, x), \quad x \in \mathbb{R}^d.
\]

The operator \( \mathbb{D}_{i_1, \ldots, i_k}^{(n)} \) is continuous (see [10] and [11]), that is, its adjoint \( (\mathbb{D}_{i_1, \ldots, i_k}^{(n)})^* : (\mathcal{D}_c^{\hat{k}})' \rightarrow (\mathcal{D}_c^{\hat{n}})' \) exists and is well-defined. In fact, the operators \( s(n, k) \) and \( S(n, k) \) can be written explicitly in terms of the operator \( \mathbb{D}_{i_1, \ldots, i_k}^{(n)} \) (see Proposition 3.7 in [11]) that is, for any \( n, k \in \mathbb{N}, k \leq n \),

\[
s(n, k) = \frac{n!}{k!} \sum_{i_1 + \cdots + i_k = n} \frac{(-1)^{n-k}}{i_1 \cdots i_k} \mathbb{D}_{i_1, \ldots, i_k}^{(n)}.
\]

(A.6)

and

\[
S(n, k) = \frac{n!}{k!} \sum_{i_1 + \cdots + i_k = n} \frac{1}{i_1! \cdots i_k!} \mathbb{D}_{i_1, \ldots, i_k}^{(n)}.
\]

(A.7)

Hence, the Stirling operators are continuous (see Proposition 3.7 in [11]) and so their adjoints \( s(n, k)^* \) and \( S(n, k)^* \) are well defined, that is,

\[
s(n, k)^* : (\mathcal{D}_c^{\hat{k}})' \rightarrow (\mathcal{D}_c^{\hat{n}})' \quad \text{and} \quad S(n, k)^* : (\mathcal{D}_c^{\hat{k}})' \rightarrow (\mathcal{D}_c^{\hat{n}})'
\]

(A.8)

and satisfy

\[
\langle w^{(k)}, s(n, k) \varphi^{(n)} \rangle = \langle s(n, k)^* w^{(k)}, \varphi^{(n)} \rangle \quad \text{and} \quad \langle w^{(k)}, S(n, k) \varphi^{(n)} \rangle = \langle S(n, k)^* w^{(k)}, \varphi^{(n)} \rangle,
\]

for all \( w^{(k)} \in (\mathcal{D}_c^{\hat{k}})' \) and \( \varphi^{(n)} \in \mathcal{D}_c^{\hat{n}} \). Hence, the Equations (A.2) and (A.3) imply that

\[
(w)_n = \sum_{k=1}^{n} s(n, k)^* w^{\hat{k}}, \quad w^{\hat{n}} = \sum_{k=1}^{n} S(n, k)^* (w)_k.
\]

(A.9)

Proposition A.8 (see [11, Prop. 3.15]). For each \( k \in \mathbb{N} \) and \( \xi \in \mathcal{D}_c \),

\[
\sum_{n=k}^{\infty} \frac{1}{n!} S(n, k) \xi^{\hat{n}} = \frac{1}{k!} (e^\xi - 1)^{\hat{k}}
\]

(A.10)

and

\[
\sum_{n=k}^{\infty} \frac{1}{n!} S(n, k) \xi^{\hat{n}} = \frac{1}{k!} (\log(1 + \xi))^{\hat{k}}.
\]

(A.11)

Proposition A.9 (see [11, Prop. 3.19]). For any \( i, n \in \mathbb{N} \),

\[
\sum_{k=1}^{n} s(k, i) S(n, k) = \sum_{k=1}^{n} S(k, i) s(n, k) = \delta_{n,i} 1^{(i)},
\]

where \( 1^{(i)} \) denotes the identity operator on \( \mathcal{D}_c^{\hat{i}} \).
A.3 An Alternative Proof of the Theorem 5.7 (Biorthogonal Property)

In Section 5.2, we proved the biorthogonal property of $A_{\sigma,\beta,\alpha}$ using the definitions of $P_{\sigma,\beta,\alpha}$ and $Q_{\sigma,\beta,\alpha}$-systems. Here, we use a property of the generalized function $Q_{n,\sigma,\beta,\alpha}(\Phi(n))$ using the $S_{n,\sigma,\beta}$-transform (see Theorem A.11 below) to provide an alternative proof of the Theorem 5.7.

Lemma A.10. For every $\Phi(n) \in (D_{C}^{\hat{n}})'$, $z \in D_{C}'$ and $\varphi \in D_{C}$, we have
\[
G(\Phi(n))(\exp\langle z, \varphi \rangle) = \langle \Phi(n), g_{\alpha}(\varphi)\rangle_{n} \exp\langle z, \varphi \rangle.
\]

In other words, the function $\exp\langle z, \varphi \rangle$ is an eigenfunction of the generalized function $G(\Phi(n))$.

Proof. It follows from (A.8) that $S(k, n)\Phi(n) \in (D_{C}^{\hat{k}})'$. We use the definition of the differential operator $D(S(k, n)\Phi(n))$ to the monomial $\langle z, \varphi \rangle^{m}$, $m \geq k$, to obtain
\[
D(S(k, n)\Phi(n))\langle z, \varphi \rangle^{m} = D(S(k, n)\Phi(n))\langle z^{m}, \varphi^{m} \rangle
= \frac{m!}{(m-k)!} \langle z\langle m-k \rangle, S(k, n)\Phi(n), \varphi^{m} \rangle
= \frac{m!}{(m-k)!} \langle z, \varphi \rangle^{m-k} \langle S(k, n)\Phi(n), \varphi^{m} \rangle.
\]

Now, we apply the above result $(\ast)$ to the Taylor series of the function $\exp\langle z, \varphi \rangle$ and obtain
\[
D(S(k, n)\Phi(n))(\exp\langle z, \varphi \rangle) = D(S(k, n)\Phi(n)) \sum_{m=0}^{\infty} \frac{(z, \varphi)^{m}}{m!} \langle S(k, n)\Phi(n), \varphi^{m} \rangle
= \langle S(k, n)\Phi(n), \varphi^{m} \rangle \sum_{m=k}^{\infty} \frac{1}{(m-k)!} \langle z, \varphi \rangle^{m-k} \langle S(k, n)\Phi(n), \varphi^{m} \rangle
= \langle S(k, n)\Phi(n), \varphi^{m} \rangle \exp\langle z, \varphi \rangle.
\]

Thus, applying the operator $G(\Phi(n))$ to $\exp\langle z, \varphi \rangle$, we obtain
\[
G(\Phi(n))(\exp\langle z, \varphi \rangle) = \sum_{k=n}^{\infty} \frac{n!}{k!} D(S(k, n)\Phi(n))(\exp\langle z, \varphi \rangle)
= \langle S(k, n)\Phi(n), \varphi^{m} \rangle \exp\langle z, \varphi \rangle
= \langle \Phi(n), g_{\alpha}(\varphi)\rangle_{n} \exp\langle z, \varphi \rangle,
\]
where the last equality is a consequence of Equation (5.18).

Theorem A.11. For $\Phi(n) \in (D_{C}^{\hat{n}})'$, the generalized function $Q_{n,\sigma,\beta,\alpha}(\Phi(n))$ satisfies
\[
S_{\sigma,\beta}(Q_{n,\sigma,\beta,\alpha}(\Phi(n)))(\varphi) = \langle \Phi(n), g_{\alpha}(\varphi)\rangle_{n}, \quad \varphi \in \mathcal{V}_{\alpha} \subset D_{C}.
\]

Proof. Using Lemma A.10, the $S_{\sigma,\beta}$-transform of $Q_{n,\sigma,\beta,\alpha}(\Phi(n))$ is given by
\[
S_{\sigma,\beta}(Q_{n,\sigma,\beta,\alpha}(\Phi(n)))(\varphi) = \langle G(\Phi(n))^* \mathbf{1}, e_{\sigma,\beta}(\varphi, \cdot) \rangle_{\sigma,\beta}
= \langle \mathbf{1}, G(\Phi(n)) e_{\sigma,\beta}(\varphi, \cdot) \rangle_{\sigma,\beta}
= \frac{1}{l_{\sigma,\beta}(\varphi)} \int_{D_{\sigma}} G(\Phi(n))(\exp\langle z, \varphi \rangle) \, d\pi_{\sigma,\beta}(z)
= \frac{\langle \Phi(n), g_{\alpha}(\varphi)\rangle_{n}}{l_{\sigma,\beta}(\varphi)} \int_{D_{\sigma}} \exp\langle z, \varphi \rangle \, d\pi_{\sigma,\beta}(z)
= \langle \Phi(n), g_{\alpha}(\varphi)\rangle_{n}.
\]

\[\square\]
Now using the above result, we provide an alternative proof of Theorem 5.7.

**Proof of Theorem 5.7 (Alternative).** The $S_{\pi,\sigma}$-transform of $Q_{n}^{\sigma,\beta,\alpha}(\Phi(n))$ at $\alpha(\varphi)$ is given by

$$S_{\pi,\sigma}(Q_{n}^{\sigma,\beta,\alpha}(\Phi(n)))(\alpha(\varphi)) = \langle\langle Q_{n}^{\sigma,\beta,\alpha}(\Phi(n)), e_{\pi,\sigma}(\alpha(\varphi), \cdot) \rangle\rangle_{\pi,\beta} = \sum_{m=0}^{\infty} \frac{1}{m!} \langle\langle Q_{n}^{\sigma,\beta,\alpha}(\Phi(n)), C_{m}^{\sigma,\beta}, \varphi^{\otimes m} \rangle\rangle_{\pi,\sigma}.$$

By Theorem A.11 with $\varphi$ replaced by $\alpha(\varphi)$ we obtain

$$S_{\pi,\sigma}(Q_{n}^{\sigma,\beta,\alpha}(\Phi(n)))(\alpha(\varphi)) = \langle\Phi(m), \varphi^{\otimes m}\rangle.$$

The result follows by a comparison of coefficients and the polarization identity. \qed

**Acknowledgments**

This work was partially supported by the Center for Research in Mathematics and Applications (CIMA-UMa) related with the Statistics, Stochastic Processes and Applications (SSPA) group, through the grant UIDB/MAT/04674/2020 of FCT-Fundação para a Ciência e a Tecnologia, Portugal and by the Complex systems group of the Premièr Research Institute of Science and Mathematics (PRISM), MSU-Iligan Institute of Technology. The financial support of the Department of Science and Technology – Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP) of the Philippines under the Research Enrichment (Sandwich) Program is gratefully acknowledged. Also, special thanks to Prof. Ludwig Streit for recommending relevant reading resources.

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