On the gravitational coupling of Hadamard states

Hadi Salehi, P. Moyassari, R. Rashidi

Department of Physics, Shahid Beheshti university
Tehran 19839, IRAN.

Abstract

We study the constraints imposed by the Hadamard condition on the two-point function of local states of a scalar quantum field conformally coupled to a gravitational background. We propose a method to assign a stress tensor to the state-dependent part of the two point function which arises as a conserved tensor with an anomalous trace. To characterize the local Hadamard states of physical interest we apply a super-selection rule relating this quantum stress tensor to the matter stress tensor of a conformal invariant gravitational model subject to a conformal symmetry breaking term. This implies that the determination of a Hadamard state may be considered as an integral part of its gravitational coupling via the back-reaction effect.

\footnote{e-mail: p-moyassari@cc.sbu.ac.ir}


1 Introduction

A central question in quantum field theory in curved space concerns the determination of the physical states of quantized matter fields propagating on a gravitational background. Usually one can assume that the two-point function of the physical states have a local structure corresponding to the Hadamard expansion [1]. A state characterized by this condition, a Hadamard state, is distinguished by the fact that the singular part of the two-point function can uniquely be determined by local geometry, but the regular part of the two-point function does not admit a geometric construction and must be considered as state-dependent. It is therefore essential for the determination of local states to apply super-selection rules characterizing the regular part of the Hadamard states. Little is known in the literature how this can be done in a physical consistent manner. The present paper deals with this issue. As a model we consider a quantum field conformally coupled to a gravitational background. Taking the local states of this field as Hadamard states we analyze the constraints imposed on the regular part (state-dependent part) of the two-point function. These constraints are then used to assign a conserved quantum stress tensor to a local state with an anomalous trace. To characterize the local Hadamard states of physical interest we apply a super-selection rule relating this quantum stress tensor to the matter stress tensor of a conformal invariant gravitational model subject to a conformal invariance breaking using a constant mass-scale. This super-selection rule arises as a statement about the nature of the gravitational coupling of a physical Hadamard state, in that it requires an indispensable change of the underlying background metric due to the back reaction of a Hadamard state. This change is taken into account through a nontrivial conformal factor. This means that a physical Hadamard state can be supported on a physical metric which is conformally related to the underlying background metric. The significance of this feature for the asymptotic particle creation is demonstrated.

2 Hadamard state condition

In this section we present the Hadamard prescription and briefly review the derivation of the local constraints on the state-dependent part of the two-
point function of local states of a linear scalar quantum field $\phi$ conformally coupled to gravity with the action functional

$$S[\phi] = -\frac{1}{2} \int d^4 x g^{1/2}(g_{\alpha\beta} \nabla^\alpha \phi \nabla^\beta \phi + \frac{1}{6} R \phi^2),$$  \hspace{1cm} (1)$$

where $R$ is the scalar curvature (In the following the semicolon and $\nabla$ indicate covariant differentiation). This gives rise to the field equation:

$$\left(\Box - \frac{1}{6} R\right) \phi(x) = 0.$$ \hspace{1cm} (2)

A state of $\phi(x)$ is characterized by a hierarchy of Wightman-functions (n-point functions)

$$<\phi(x_1), \ldots, \phi(x_n)>.$$ \hspace{1cm} (3)

We are primarily interested in those states which reflect the intuitive notion of a vacuum. For this aim, we may restrict ourselves basically to quasi-free states for which the truncated n-point functions vanish for $n > 2$. Such states may be characterized by their two-point functions. In a linear theory the anti-symmetric part of the two-point function (commutator function) is common to all states in the same representation. Therefore the individual characteristics of a state are encoded in the symmetric part of the two-point function denoted by $G^+(x, x')$. In Hadamard prescription one assumes that $G^+(x, x')$ has a singular structure represented by Hadamard expansion. This means that in a normal neighborhood of a point $x$ the function $G^+(x, x')$ can be written as [1, 2]:

$$G^+(x, x') = \frac{1}{8\pi^2} \left\{ \frac{\Delta^{1/2}(x, x')}{\sigma(x, x')} + V(x, x') \ln \sigma(x, x') + W(x, x') \right\},$$ \hspace{1cm} (4)

where $\sigma(x, x')$ is the square of the distance along the geodesic joining $x$ and $x'$ and

$$\Delta(x, x') = -g^{-1/2}(x) \text{Det} \{\sigma_{\mu\nu}\} g^{-1/2}(x').$$

$$g(x) = \text{Det} g_{\alpha\beta}$$

$$V(x, x') = \sum_{n=0}^{+\infty} V_n(x, x') \sigma^n$$

$$W(x, x') = \sum_{n=0}^{+\infty} W_n(x, x') \sigma^n.$$ \hspace{1cm} (5)
Applying (2) to $G^+(x,x')$ leads to some recursion relations in terms of $V_n$ and $W_n$. From these relations one can determine the function $V(x,x')$ uniquely in terms of local geometry, but the function $W_0(x,x')$ remains arbitrary and its specification depends significantly on the choice of a state [3]. However, there is a general constraint on $W_0(x,x')$ which can be obtained from the symmetry condition of $G^+(x,x')$ and equation (2). To get this constraint the covariant expansion of $W_0(x,x')$ is used only up to the third order in $\sigma_\alpha$ [4]. In general there are some additional constraints on the higher order expansion terms that in our analysis are neglected. It was shown in details in [4] (also see [3]) that the constraints imposed on the state-dependent part of two-point function have the form

$$\Sigma_{\alpha\beta} = 0,$$

where

$$\Sigma_{\alpha\beta} = (W_{0\alpha\beta}(x) - \frac{1}{4} g_{\alpha\beta} W_{0\gamma}(x)) - \frac{1}{8} (R_{\alpha\beta} - \frac{1}{4} R g_{\alpha\beta}) W_0(x)$$

$$- \frac{1}{3} (W_{\alpha;\beta\alpha}(x) - \frac{1}{2} g_{\alpha\beta} \Box W_0(x)) - \frac{1}{2} g_{\alpha\beta} v_1(x),$$

and

$$v_1(x) = \lim_{x' \to x} V_1(x,x') = \frac{1}{720} \{ \Box R - R_{\alpha\beta} R_{\alpha\beta} + R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \}.$$  

Functions $W_0$ and $W_{0\alpha\beta}$ are the coefficients in the covariant expansion of $W_0(x,x')$, we have namely

$$W_0(x,x') = W_0(x) - \frac{1}{2} W_{0;\alpha}(x) \sigma^{\alpha} + \frac{1}{2} W_{0\alpha\beta}(x) \sigma^{\alpha} \sigma^{\beta} + O(\sigma^{3/2}).$$

It is obvious from (7) that

$$\Sigma_\alpha = -2 v_1(x).$$

Since tensor $\Sigma_{\alpha\beta}$ arises as a conserved tensor with an anomalous trace, we take it as the quantum stress tensor induced by the two-point function [5, 6, 7]. The determination of $\Sigma_{\alpha\beta}$ is very essential for the characterization of a local Hadamard state. Any assumption about $\Sigma_{\alpha\beta}$ which respects the constraints (6) and (10) acts as a super-selection rule selecting a local Hadamard state and the corresponding Hilbert space. The application of such a super-selection rule has to respect the fundamental idea of general relativity concerning the gravitational coupling of $\Sigma_{\alpha\beta}$. In order to specify the configuration of $\Sigma_{\alpha\beta}$ we must therefore look at the equations describing the gravitational coupling of $\Sigma_{\alpha\beta}$.
3 The gravitational coupling

For the characterization of the quantum stress tensor $\Sigma_{\alpha\beta}$ we proceed to apply a super-selection rule in form of a dynamical model that is taken to describe the gravitational coupling of $\Sigma_{\alpha\beta}$ to a dynamical metric $\bar{g}_{\alpha\beta}$. This super-selection rule relates the quantum stress tensor $\Sigma_{\alpha\beta}$ to the matter stress tensor of a conformal invariant gravitational model. To arrive at the superselection rule in question we first note that the appearance of the anomalous trace in (10) suggests that the model should be defined in terms of a conformal invariance breaking. We begin with the consideration of the action functional of a scalar tensor theory, namely

$$S = -\frac{1}{2} \int d^4x \sqrt{-\bar{g}} (\bar{g}^{\alpha\beta} \bar{\nabla}_{\alpha} \psi \bar{\nabla}_{\beta} \psi + \frac{1}{6} \bar{R} \psi^2 + \mu^2 \psi^2) + S_m[\bar{g}^{\alpha\beta}].$$  \hspace{1cm} (11)

in which $S_m$ stands for a matter action, $\bar{g}_{\alpha\beta}$ is a dynamical metric and $\bar{\nabla}$ is covariant derivative with respect to $\bar{g}_{\alpha\beta}$. The parameter $\mu$ is a constant mass scale which implies the conformal symmetry breaking through the term $\mu^2 \psi^2$. Varying $S$ with respect to $\bar{g}_{\alpha\beta}$ and $\psi$ yields respectively:

$$\bar{G}_{\alpha\beta} - 3\mu^2 \bar{g}^{\alpha\beta} = -6\psi^{-2}(T_{\alpha\beta} + \tau_{\alpha\beta}),$$  \hspace{1cm} (12)

$$\Box \psi - \frac{1}{6} \bar{R} \psi - \mu^2 \psi = 0,$$  \hspace{1cm} (13)

where

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-\bar{g}}} \frac{\delta}{\delta \bar{g}^{\alpha\beta}} S_m[\bar{g}^{\alpha\beta}],$$  \hspace{1cm} (14)

and

$$\tau_{\alpha\beta} = \bar{\nabla}_{\alpha} \psi \bar{\nabla}_{\beta} \psi - \frac{1}{2} \bar{g}_{\alpha\beta} \bar{\nabla}_{\mu} \psi \bar{\nabla}^\mu \psi + \frac{1}{6} (\bar{g}_{\alpha\beta} \Box - \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta}) \psi^2,$$  \hspace{1cm} (15)

here $\Box \equiv \bar{g}^{\alpha\beta} \bar{\nabla}_{\alpha} \bar{\nabla}_{\beta}$ and $\bar{G}_{\alpha\beta}$ is the Einstein tensor of metric $\bar{g}_{\alpha\beta}$. Comparing the trace of (12) with the equation (13) yields,

$$\bar{g}^{\alpha\beta} T_{\alpha\beta} = \mu^2 \psi^2.$$  \hspace{1cm} (16)

Taking the four-divergence of (12) and using (13) leads to

$$\bar{\nabla}^{\alpha} T_{\alpha\beta} = 0.$$  \hspace{1cm} (17)
It should be noted that in the derivation of the dynamical equations no variation is performed with respect to the dynamical variables of the matter action. In theories of this type the configuration of matter is dynamically determined by the gravitational equations (12) once the metric is assumed to be fixed. We shall use this feature to apply a super-selection rule relating the (still unknown) quantum stress tensor $\Sigma_{\alpha\beta}$ to the matter stress tensor $T_{\alpha\beta}$.

In accordance with this strategy we consider the equation (12) as restricting the configuration of $T_{\alpha\beta}$, that is we fix the metric $\bar{g}_{\alpha\beta}$ and interpret (12) as constraints on $T_{\alpha\beta}$. We then apply a super-selection rule of the form

$$\Sigma_{\alpha\beta} = T_{\alpha\beta}[\bar{g}_{\alpha\beta}].$$

(18)

This super-selection rule determines the gravitational coupling of $\Sigma_{\alpha\beta}$ as

$$\Sigma_{\alpha\beta} = -(\frac{1}{6}\psi^2 \bar{G}_{\alpha\beta} - 3\mu^2 \psi^2 \bar{g}_{\alpha\beta} + \tau_{\alpha\beta}),$$

(19)

which follows from the gravitational equation (12) together with (18).

The super-selection rule (18) is complete if we know the metric $\bar{g}_{\alpha\beta}$. At this stage one may follow two different methods. Firstly one may assume the metric $\bar{g}_{\alpha\beta}$ to be equal to the background metric $g_{\alpha\beta}$. If we follow this reasoning, then the constraint (10) may not be satisfied without recourse to the higher order gravity constraints arising from the trace anomaly [4]. In the second method, according to the back reaction effect one may assume that the gravitational coupling of a Hadamard state changes the background metric. It means that a Hadamard state may not be supported on a underlying background metric. In specific terms we assume that this difference can be factorized in a conformal factor, namely

$$\bar{g}_{\alpha\beta} = e^{-2\omega} g_{\alpha\beta}.$$  

(20)

in which $\omega$ is a non-trivial conformal factor. Physically the conformal factor $\omega$ may describe a change of the background metric through a nontrivial back reaction effect of $\Sigma_{\alpha\beta}$. Therefore this approach emphasizes the indispensable role played by the back reaction in the characterization of Hadamard states. We shall follow this reasoning in order not to be confronted with higher order gravity constraints.

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\[2\] This corresponds to the g-method [8] in interpreting the gravitational equations.
The super-selection rule (18) should satisfy the constraints (6) and (10) on the metric $\bar{g}_{\alpha\beta}$. Because of (17) the constraint (6) is automatically satisfied. To satisfy the other constraint (10) we get the consistency relation

$$-2\bar{v}_1(x) = \mu^2\psi^2.$$  

(21)

In the four-dimensional case, the trace anomaly (8) for the metrics $g_{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$ are related by [4]

$$\bar{v}_1(x) = \frac{1}{720}e^{4\omega}\{720v_1(x) + 2R\Box\omega + 2R_{\alpha\omega^i\alpha} + 6\Box(\Box\omega) + 8(\Box\omega)^2$$

$$-8\omega_{\alpha\beta}\omega^{\alpha\beta} - 8R_{\alpha\beta\omega^i\omega^j\omega^k\omega^l} - 8\omega_{\alpha\beta}\omega^{\alpha\beta} - 16\omega_{\alpha\beta}\omega^{\alpha\beta}\}.$$  

(22)

The relation (21) acts as a condition on the dynamically allowed configuration of conformal factor $\omega$. It selects the physical metric as a metric in which the trace anomaly is replaced by the standard conformal symmetry breaking term $\mu^2\psi^2$. The physical characteristics of this metric depend essentially on the boundary conditions imposed on the function $\psi$ and $\omega$. In the subsequent two sections we study two examples of boundary conditions with different physical characteristics.

4 Asymptotic vacuum condition

In this section, we investigate the behavior of the conformal frame under the assumption that the background metric is approximated by an asymptotically flat metric.

We first assume that the field $\omega$ is constant at sufficiently large space-like distances, namely

$$\omega(x \rightarrow i_0) = \text{const.}$$  

(23)

where $i_0$ denotes the space-like infinity. This condition means that there is no distinction between the physical metric and the background metric at space-like infinity. In other words we ignore the back-reaction effect at space-like infinity. If a boundary condition of this type is applied we get from the asymptotic relation (23)

$$\Sigma_{\alpha\beta}(x \rightarrow i_0) = 3\mu^2\psi^2\bar{g}_{\alpha\beta} - \tau_{\alpha\beta},$$  

(24)
which follows directly from the gravitational equation (19). By choosing \( \mu = 0 \) at space-like infinity, the contribution to the quantum stress tensor comes from the variation of \( \psi \). If we assume that the conformal frame corresponds asymptotically to the Einstein frame at space-like infinity (constant configuration of \( \psi \) at \( i_0 \)) we get from (24)

\[
\Sigma_{\alpha\beta}(x \rightarrow i_0) \rightarrow 0. \tag{25}
\]

This relation shows that for an asymptotically flat background metric a Hadamard state which characterized by \( \Sigma_{\alpha\beta} \) looks like the vacuum of the Minkowski space at sufficiently large space-like distances. Therefore the super-selection rule (18) under the boundary conditions used in this section may be considered as the curved space analogue of DHR-super-selection rule [9] in flat space quantum field theory. In this case the whole quantum stress tensor vanishes at space-like infinity. This kind of behavior is desirable only in theories for which there is no asymptotic flow of vacuum energy at space-like infinity (no particle creation). In such theories the asymptotic flat background metric should reasonably taken as globally static. For problems concerning a non-vanishing vacuum energy at infinity, specifically a non-vanishing asymptotic radiation, other boundary conditions should be applied.

5 Asymptotic particle creation

In this section we consider the Schwarzschild metric as a background metric

\[
ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{26}
\]

here \( G \) is the gravitational coupling and \( M \) is the mass of the black hole. We proceed to study the physical characteristics of the physical metric \( \bar{g}_{\alpha\beta} \) in (20) with the conformal factor solutions of equation (21). We show that the corresponding conformal factor describes an outward flux of radiation. We derive this result under the assumption \( \mu = 0 \). Equations (13) and (21) in the Schwarzschild background metric lead to

\[
\Box\psi - \Box\omega\psi + \omega_{\alpha}\omega^\alpha\psi - 2\psi_{,\alpha}\omega^\alpha = 0 \tag{27}
\]
\[
\frac{M^2}{15r^6} + 6\Box(\Box \omega) + 8(\Box \omega)^2 - 8\omega_{;\alpha\beta} \omega^{;\alpha\beta} - 8\omega_{,\alpha} \omega^{,\alpha} \Box \omega - 16\omega_{;\alpha\beta} \omega^{;\alpha} \omega^{;\beta} = 0
\] (28)

We focus ourselves on those solutions having a non-vanishing flux of energy momentum at infinity. For this purpose we take the solutions satisfying these asymptotic conditions

\[
\omega(t, r) = \frac{U(t - r)}{r^2} + O\left(\frac{1}{r^3}\right), \quad \lim_{r \to \infty} \psi = \psi_0 = \text{const.}
\] (29)

here \(U\) is a smooth bounded arbitrary function of the retarded time. Clearly these solutions imply a non-vanishing energy momentum tensor at space-like infinity through the equation (19), namely

\[
\Sigma^\beta_\alpha (r \to \infty) \propto \psi_0^2 \left\{ \frac{\ddot{U}}{r^2} \begin{pmatrix}
-1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + O\left(\frac{1}{r^3}\right) \right\}
\] (30)

here the over-dot indicates differentiation with respect to \(t - r\). Thus at space-like infinity the contribution to the quantum stress tensor comes from the variation of \(\omega\). In other words, the effect of back-reaction can provide a non-vanishing asymptotic radiation.

We select a state by the condition \(\psi_0^2 \ddot{U} \propto \frac{1}{G^2 M^2}\). Under this assumption one can determine the power of this radiation (luminosity) via equation (30). At the first order the result is

\[
P \propto \frac{1}{G^2 M^2}.\] (31)

We infer that the amount of energy carried away by this radiation is the same as the energy loss by the black hole. Therefore, the power \(P\) of this radiation is the rate of loss of total energy of the black hole, that is

\[
P = -\frac{dM}{dt}.\] (32)

Equating \(P\) in equations (31) and (32) gives

\[
-\frac{dM}{dt} \propto \frac{1}{G^2 M^2}.\] (33)

which describes the black hole evaporation.
6 Summary

In this paper, we proposed a method to assign a conserved quantum stress tensor with an anomalous trace to the state-dependent part of the two-point function of a scalar quantum field conformally coupled to a gravitational background. To characterize the local Hadamard states, a super-selection rule was applied in form of a self-consistent dynamical model which describes the gravitational coupling of the quantum stress tensor to a dynamical metric. The consequence of this super-selection rule was studied under two different boundary conditions.

These considerations provided a vanishing vacuum energy at space-like infinity in the case of ignoring the back-reaction effect but a non-vanishing vacuum energy in the presence of this effect.

References

[1] Hadamard J, Lectures on Cauchy’s Problem in Linear Partial Differential Equations, Yale University Press, New Haven, (1923)
[2] Adler S, Lieberman J and Ng Y. J, Ann. Phys. 106, 279 (1977)
[3] Salehi H, Bisabr Y and Ghafarnejad H, J. Math. Phys. 41, 4582 (2000)
[4] Brown M. R, J. Math. Phys. 25, 136 (1984)
[5] Wald R. M, Phys. Rev. D. V17, N6 (1978)
[6] Wald R. M, Ann. Phys. 110, 472 (1978)
[7] Birrell N. D and Davies P. C. W, Quantum fields in Curved Space, Cambridge University Press, (1982).
[8] J.L. Synge, Relativity: The General Theory (North Holland, Amsterdam, 1966).
[9] Hagg R, Local Quantum Physics, Springer (1992)