A NEW GRADIENT COMPUTATIONAL FORMULA FOR OPTIMAL CONTROL PROBLEMS WITH TIME-DELAY

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Abstract. In this paper, we consider a class of time-delay optimal control problem (TDOCP) with canonical equality and inequality constraints. By applying control parameterization method together with time-scaling transformation, a TDOCP can be readily solved by gradient-based optimization methods. The partial derivative of the cost as well as the constraint functions with respect to the decision variables are obtained by variational approach, which is inefficient when the discretization for the control function is relatively dense. For general optimal control problem without time-delay, co-state approach is an effective way to compute the gradients, however, when time-delay is involved in the dynamic system, the co-state system is not known. In this paper, we derive the co-state system for TDOCP to compute the gradients of the cost and constraints. Numerical results show that the computational efficiency is much higher when compared with the traditional variational approach.

1. Introduction. An optimal control problem governed by time-delay dynamic system is called time-delay optimal control problem. It finds tremendous real-world applications in science [13, 17, 18, 24, 25] and engineering [2, 3, 4, 5, 22]. There have been decades of research on TDOCP [8, 9, 10, 15, 19, 30, 35]. Except for some academic and special examples, it is generally very difficult to obtain a closed-form analytic expression of the optimal control for TDOCP [16]. Thus, it is indispensable to solve TDOCP by numerical methods [1, 6, 20].

Traditional control parameterization method [14, 21, 26] where the control function being approximated by a piece-wise constant function and the planning time horizon being evenly partitioned [29], is an effective numerical method for solving TDOCP. However, the well-known switching time optimization technique — time-scaling transformation [12, 30, 33] was considered unable to be applied to solve TDOCP [17, 32]. In [34], a hybrid time-scaling transformation strategy is proposed to adaptively optimize the control switching time for TDOCP. This method only maps the current state and control vectors into the new time scale, while the delayed state and control vectors remain in the original time scale. Recently, Wu et al.
further improved this approach and presented a new computational method for solving TDOCP. Compared with the hybrid time-scaling transformation, the new approach provides an explicit closed-form expression for the delay time in the new time horizon. Thus, the time-delay dynamic system after transformation has fixed switching times, and the transformed problem defined on the new time scale can be solved using standard gradient-based optimization techniques [23].

For general optimal control problem without time-delay [28], there are two types of method for obtaining the gradient of the objective/constraint function: variational method (also known as the sensitivity method) [11, 18, 27] and co-state method [7, 26]. The philosophy for these two methods are quite different. The variational methods depend on the solution of the state system, the state and variational systems can be combined to form an enlarged system of differential equations. This enlarged system can then be solved numerically to yield the state and the state variation. The co-state method is normally more complicated to implement when compared with the variational method, because the state system must be solved first, after which the solution of the state system can be used to solve the co-state system. However, one advantage of the co-state method is that it generally involves less auxiliary systems of differential equations than the variational method [16].

In [31] and [34], control parametrization approach was used to discretize the control function, control switching times were optimized via time-scaling transformation technique, and the gradient formulae are obtained by variational method. To further improve the computational efficiency, in this paper, we derive the co-state method in conjunction with time-scaling transformation to convert the original TDOCP into an approximate optimization problem. In Section 3.1 and 3.2, we use the control parameterization method in conjunction with the new time scaling transformation to convert the original TDOCP into an approximate optimization problem. In Section 4, we derive the gradients of the objective and constraint functions via the co-state method. Section 5 contains some numerical results, and finally, Section 6 concludes the paper.

2. Problem statement. A general time-delay system can be modelled as follows:

\[ \dot{x}(t) = f(x(t), \dot{x}(t), u(t), \bar{u}(t)), \quad t \in [0, T], \]
\[ x(t) = \phi(t), \quad t \leq 0, \]
\[ u(t) = \varphi(t), \quad t < 0, \]

where \( t \) denotes time, \( x(t) \in \mathbb{R}^n \) is the state of the system at time \( t \), \( u(t) \in \mathbb{R}^r \) is the control signal at time \( t \), \( \dot{x}(t) = [x_1(t-h), \ldots, x_n(t-h)]^T \) and \( \bar{u}(t) = [u_1(t-h), \ldots, u_r(t-h)]^T \), in which \( h > 0 \) is a given time delay. \( f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n \) and \( \phi : (-\infty, 0] \to \mathbb{R}^n \) are given continuous differentiable functions, and \( \varphi : (-\infty, 0] \to \mathbb{R}^r \) is a given function.

Any Borel measurable function \( u(t) : (-\infty, T] \to \mathbb{R}^r \) is said to be an admissible control iff \( u(t) \in U \) for almost all \( t \in [0, T] \) and \( u(t) = \varphi(t) \) for all \( t < 0 \), where \( U \) is a compact and convex subset of \( \mathbb{R}^r \).

Any admissible control \( u \in U \) satisfying the following canonical equality and inequality constraints

\[ g_i(u) = \Phi_i(x(T|u)) + \int_0^T \mathcal{L}_i(x(t|u), \dot{x}(t|u), u(t))dt = 0, \quad i = 1, \ldots, e, \]
where $\Phi \in \mathbb{R}^n$ is called a feasible control. Let $\mathcal{F}$ denote the class of all such feasible controls.

Our goal is to find a feasible control that minimizes the following measure of system cost:

$$g_0(u) = \Phi_0(x(T|u)) + \int_0^T \mathcal{L}_i(x(t|u), \bar{x}(t|u), u(t))dt.$$  

Standard time-delay optimal control problem can now be stated as follows: Choose a feasible control $u \in \mathcal{F}$ to minimize the cost functional (6). Let's denote the problem as Problem (P).

Throughout this paper, we assume that the following conditions are satisfied.

**Assumption 1.** There exists a positive constant $C$ such that

$$\|f(\eta_1, \eta_2, \zeta_1, \zeta_2)\| \leq C(1 + \|\eta_1\| + \|\eta_2\|), \quad (\eta_1, \eta_2, \zeta_1, \zeta_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r.$$

**Assumption 2.** $\mathcal{L}_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}, i = 0 \cdots, e + m$, and $\Phi_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 0, \cdots, e + m$, are continuous differentiable with respect to each of their arguments.

3. Numerical solution process.

3.1. Control parameterization. We subdivide the planning horizon $[0, T]$ into $p \geq 1$ subintervals. Let $t_i, i = 1, \cdots, p$ denote the end points of these subintervals that satisfying

$$0 = t_0 < t_1 < t_2 < \cdots < t_p = T.$$  

Let $\Xi$ denote the set of all vectors $\sigma = [t_1, \cdots, t_p]^T$ such that (7) is satisfied. We now approximate the control by a piece-wise constant function, defined as follows:

$$u(t) \approx u_p(t) = \sum_{i=1}^{p} \delta^i \chi_{[t_{i-1}, t_i]}(t),$$  

where $\delta^i = [\delta^i_1, \cdots, \delta^i_r]^T$ is the value of the control on the $i$th subinterval, let $\Delta$ denote the set of all vectors $\delta = [(\delta^1)^T, \cdots, (\delta^p)^T]^T$. $\chi_{[t_{i-1}, t_i]}(t)$ is the characteristic function, defined by

$$\chi_{[t_{i-1}, t_i]}(t) = \begin{cases} 1, & \text{if } t \in [t_{i-1}, t_i), \\ 0, & \text{otherwise}. \end{cases}$$

Substituting (8) into (1), the time-delay system defined on the subinterval $[t_{i-1}, t_i)$ becomes

$$\begin{aligned} \dot{x}(t) &= f(x(t), \bar{x}(t), \delta^i(t)), \\
x(t) &= \phi(t), \quad t \leq 0, \\
\end{aligned}$$

where for $\delta(t)$, there are two cases: (i) if $t - h < 0$, then $\delta(t) = \varphi(t - h)$; and (ii) if $t - h \geq 0$, as reported in [31], there exists a unique time subinterval $[t_{j-1}, t_j), j = 1, \cdots, p$, such that $t - h$ belongs to it, and hence $\delta(t) = \delta^j$.

Let $x(\cdot|\sigma, \delta)$ denote the solution of (10) corresponding to $(\sigma, \delta) \in \Xi \times \Delta$. The original time-delay Problem (P) becomes

$$\min_{\sigma, \delta} g_0(\sigma, \delta) = \Phi_0(x(T|\sigma, \delta)) + \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} \mathcal{L}_i(x(t|\sigma, \delta), \bar{x}(t|\sigma, \delta), \delta^i(t))dt.$$
subject to
\[ g_k(\sigma, \delta) = \Phi_k(x(T|\sigma, \delta)) + \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} L_k(x(t|\sigma, \delta), \dot{x}(t|\sigma, \delta), \delta^i) dt = 0, \quad k = 1, \ldots, e, \] (11)
\[ g_k(\sigma, \delta) = \Phi_k(x(T|\sigma, \delta)) + \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} L_k(x(t|\sigma, \delta), \dot{x}(t|\sigma, \delta), \delta^i) dt \geq 0, \quad k = e + 1, \ldots, e + m. \] (12)

Let the approximate problem above be referred to as Problem \((P0)\).

3.2. Time-scaling transformation. Consider Problem \((P0)\) where the switching time vector \(\sigma = [t_1, \ldots, t_p]^\top\) is the decision variable. For notation convenience, the equivalent problem with fixed switching times to be introduced below is more easily expressed in terms of the durations between individuals switching times. These durations are given by
\[ \theta_i = t_i - t_{i-1}, \quad i = 1, \ldots, p, \]
where \(t_0 = 0\) and \(t_p = T\). Let \(\theta = [\theta_1, \ldots, \theta_p]^\top\). Note that (7) is equivalent to the constraints
\[ \theta_i > 0, \quad i = 1, \ldots, p, \]
and \(\theta\) must also satisfy
\[ \sum_{i=1}^{p} \theta_i = T. \]

The basic idea of the time scaling transformation is to replace the original time horizon \((-\infty, T]\) containing the variable switching time \(t_i, \quad i = 1, \ldots, p\), with a new time horizon \((-\infty, p]\) with fixed switching times at 1, 2, \ldots, \(p\).

We use \(\gamma\) to denote “time” in the new time horizon. The relationship between \(t \in (-\infty, T]\) and \(\gamma \in (-\infty, p]\) can be defined by the following differential equation
\[ \frac{dt(\gamma)}{d\gamma} = \sum_{i=1}^{p} \theta_i \chi_{[i-1,i)}(\gamma), \quad \gamma \in [0, p], \]
with the boundary condition
\[ t(\gamma) = \gamma, \quad \gamma \leq 0, \]
\[ t(p) = T, \]
where \(\chi_{[i-1,i)}(\cdot)\) is the indicator function by (9). For each \(\gamma \in (-\infty, p]\), we have
\[ t(\gamma) = \mu(\gamma|\theta) = \begin{cases} T, & \text{if } \gamma = p, \\ \sum_{i=1}^{\lceil \gamma \rceil} \theta_i + \theta_{\lceil \gamma \rceil + 1} (\gamma - \lceil \gamma \rceil), & \text{if } \gamma \in [0, p), \\ \gamma, & \text{if } \gamma < 0, \end{cases} \]
where \(\lfloor \cdot \rfloor\) denotes the floor function. It is easy to see that
\[ \mu(i|\theta) = \sum_{i=1}^{i} \theta_i = t_i, \quad i = 1, \ldots, p. \]
This shows that the time scaling function maps \( \gamma = i \) to the \( i \)th switching time \( t_i \). For simplicity, we denote \( \theta_i = 1 \) for \( i \leq 0 \).

By the nature of the time-scaling function, it is not difficult to see that a fixed time-delay \( h \) will become a variable in the new time horizon. In [31], an explicit formula for the variable time-delay in the new time horizon is given as a theorem below:

**Theorem 3.1.** [31] Let \( \zeta(\gamma) \) denote the time-delay in the new time horizon, then, let \( \theta \in \Theta \), for each \( \gamma \in (-\infty, p] \), if \( \mu(\gamma) - h < 0 \), then \( \zeta(\gamma) = \mu(\gamma) - h \). Otherwise, let \( \kappa(\gamma) \theta \) denote the unique integer such that \( \theta_{\kappa(\gamma) \theta} > 0 \) and \( \mu(\gamma) - h \in \left[ \sum_{i=0}^{\kappa(\gamma) \theta - 1} \theta_i, \sum_{i=0}^{\kappa(\gamma) \theta} \theta_i \right) \). The following equation holds:

\[
\zeta(\gamma) = \kappa(\gamma) \theta + \sum_{l=\kappa(\gamma) \theta}^{\gamma - 1} \theta_{\kappa(\gamma) \theta + 1} + \theta_{\kappa(\gamma) \theta + 1} - \theta_{\kappa(\gamma) \theta + 1}(\gamma - |\gamma|) - h\theta_{\kappa(\gamma) \theta + 1}.
\]

Applying time-scaling transformation to the time-delay dynamic system (10) gives the following new differential equation:

\[
\dot{y}(\gamma) = \frac{d}{d\gamma}(y(\gamma)) = \frac{d}{d\gamma}(x(\mu(\gamma) \theta))) = \theta_{\gamma + 1} \frac{dx(\mu(\gamma) \theta))}{dt}, \quad \gamma \in [0, p],
\]  
with the initial condition

\[
y(\gamma) = \phi(\gamma), \quad \gamma \leq 0.
\]

Substituting (13) into (10) gives the following time-delay system with fixed switching times.

\[
y(\gamma) = \theta_i f(y(\gamma | \theta, \delta), y(\gamma | \theta, \delta), \delta_i, \delta(\gamma)), \quad \gamma \in [i - 1, i], \quad i = 1, \cdots, p, \quad \gamma = \phi(\gamma), \quad \gamma \leq 0,
\]

where \( \theta \in \Theta \), \( \delta \in \Delta, \ y(\gamma | \theta, \delta) = [y_1(\gamma | \theta, \delta), \cdots, y_n(\gamma | \theta, \delta)]^T \in \mathbb{R}^n, \ y(\zeta(\gamma) | \theta, \delta) = y(y(\zeta(\gamma) | \theta, \delta) = y(\zeta(\gamma) | \theta, \delta) \in \mathbb{R}^n \). Let \( y(\cdot | \theta, \delta) \) denote the solution of (14).

For simplicity, we denote \( y(\cdot) = y(\cdot | \theta, \delta) \) and \( \dot{y}(\cdot) = \dot{y}(\cdot | \theta, \delta) \). We will use these notations throughout the paper.

Then, applying time-scaling transformation to the constraints (11) and (12) gives

\[
\tilde{g}_k(\theta, \delta) = \Phi_k(y(p)) + \sum_{i=1}^{p} \int_{i-1}^{i} \theta_i L_k(y(\gamma), \tilde{y}(\gamma), \delta^i) d\gamma = 0, \quad k = 1, \cdots, e,
\]

\[
\tilde{g}_k(\theta, \delta) = \Phi_k(y(p)) + \sum_{i=1}^{p} \int_{i-1}^{i} \theta_i L_k(y(\gamma), \tilde{y}(\gamma), \delta^i) d\gamma \geq 0, \quad k = e + 1 : \cdots, e + m.
\]

Let \( \tilde{F} \) denote the set of all pairs \( (\theta, \delta) \in \Theta \times \Delta \) satisfying above constraints. The new problem may now be stated as follows:

Choose a feasible pair \( (\theta, \delta) \in \tilde{F} \) such that the following cost function:

\[
\tilde{g}_0(\theta, \delta) = \Phi_0(y(p)) + \sum_{i=1}^{p} \int_{i-1}^{i} \theta_i L_0(y(\gamma), \tilde{y}(\gamma), \delta^i) d\gamma
\]  

is minimized over \( \tilde{F} \). Let this problem be denoted as Problem (Q). It is easy to see that Problem (P0) is equivalent to Problem (Q).
4. Gradient computation—Co-state approach. To solve Problem \((Q)\) using the gradient-based algorithms, we need the partial derivative of the cost and constraint functions with respect to each of their arguments.

If the variational method is used to obtain the gradients for Problem \((Q)\), we need to solve a sequence of auxiliary systems and the number of auxiliary systems is closely related to the number of decision variables (control heights and durations). When the discretization for the control functions is relatively dense, the number of decision variables will be increased accordingly. This means that one needs to solve more differential equations to obtain the gradients in each iteration, which is inefficient when compared with co-state approach.

For example, suppose the optimal control problem under consideration involves \(m\) constraints, the control function is discretized into \(p\) sections and \(r\) is the dimension of the control function. Then, the variational method involves \(p \times (r + 1)\) auxiliary systems (ODEs), whereas the co-state method only involves \(m + 1\) auxiliary systems of differential equations. \(p \times (r + 1)\) is normally much larger than \(m + 1\) when \(p\) is large. In the next, we shall derive the cost and constraint gradients by co-state method.

First, for \(k = 0, 1, \cdots, e + m\), we define the Hamiltonian function as follows:

\[
H_k(y(\gamma), \dot{y}(\gamma), \theta, \delta, \lambda_k(\gamma)) = \mathcal{L}_k(y(\gamma), \dot{y}(\gamma), \theta, \delta) + \lambda^T f(y(\gamma), \dot{y}(\gamma), \theta, \delta)
\]

where

\[
\mathcal{L}_k(y(\gamma), \dot{y}(\gamma), \theta, \delta) = \sum_{i=1}^{p} \theta_i \mathcal{L}_i(y(\gamma), \dot{y}(\gamma), \delta^i) \chi_{[i-1,i)}(\gamma),
\]

\[
f(y(\gamma), \dot{y}(\gamma), \theta, \delta) = \sum_{i=1}^{p} \theta_i f(y(\gamma), \dot{y}(\gamma), \delta^i) \chi_{[i-1,i)}(\gamma).
\]

Then, we consider the following co-state system

\[
\dot{\lambda}^k(\gamma) = -\left(\frac{\partial H_k(y(\gamma), \dot{y}(\gamma), \theta, \delta, \lambda_k(\gamma))}{\partial y} + \frac{\partial \mathcal{H}_k(\dot{y}(\gamma|\theta, \delta), y(\gamma), \theta, \delta, \lambda_k(\gamma))}{\partial \gamma} \right)^T
\]

with the terminal condition

\[
\lambda^k(p) = \left[\frac{\partial \Phi_k(y[p|\theta, \delta])}{\partial y}\right]^T, \quad k = 0, 1, \cdots, e + m,
\]

where

\[
\mathcal{H}_k(\dot{y}(\gamma|\theta, \delta), y(\gamma), \theta, \delta, \lambda^k(\gamma)) = H_k(\dot{y}(\gamma), y(\gamma), \theta, \delta, \lambda^k(\gamma)) \mathcal{E}(\zeta(p) - \gamma),
\]

\[
\dot{y}(\gamma|\theta, \delta) = y(\nu(\gamma|\theta)|\theta, \delta),
\]

\[
\dot{\lambda}^k(\gamma) = \lambda(\nu(\gamma|\theta)),
\]

\[
\nu(\gamma|\theta) = \mu^{-1}(\mu(\gamma|\theta) + h),
\]

and \(\mathcal{E}(\cdot)\) is the unit step function defined by

\[
e(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases}
\]
the solution of (17)-(18), denoted by $\mathbf{X}(\gamma)$, is called the co-state. Since $\theta_i > 0$, $i = 1, \cdots, p$, the time-scaling function $\mu(\gamma|\theta)$ is strictly monotonic, its inverse function is given by

$$\mu^{-1}(s|\theta) = \begin{cases} \frac{s}{\theta_1}, & s \in [0, t_1), \\ \frac{i - 1 + \frac{s - t_{i-1}}{\theta_i}}{p}, & s \in [t_{i-1}, t_i), i = 2, \cdots, p, \\ s = T, & \end{cases}$$

and hence

$$\nu(\gamma|\theta) = \begin{cases} \frac{\mu(\gamma|\theta) + h}{\theta_1}, & \mu(\gamma|\theta) + h \in [0, t_1), \\ \frac{\mu(\gamma|\theta) + h - t_{i-1}}{\theta_i}, & \mu(\gamma|\theta) + h \in [t_{i-1}, t_i), i = 2, \cdots, p, \\ \mu(\gamma|\theta) + h = T. & \end{cases}$$

For simplicity, we denote $\tilde{\gamma}(\gamma) = \tilde{y}(\gamma|\theta, \delta)$ and $\nu(\gamma) = \nu(\gamma|\theta)$. The partial derivative of the cost and constraint functions with respect to the duration vector $\theta$ is given as a theorem stated below.

**Theorem 4.1.** For each pair $(\theta, \delta) \in \Theta \times \Delta$, the partial derivative of $g_k(\theta, \delta)$, $k = 0, 1, \cdots, e + m$ with respect to $\theta$ is given by

$$\frac{\partial g_k(\theta, \delta)}{\partial \theta} = \int_{0}^{p} \frac{\partial H_k(y(\gamma), \tilde{y}(\gamma), \theta, \delta, \mathbf{X}(\gamma))}{\partial \theta} d\gamma. \quad (20)$$

**Proof.** Let $\delta$ be any but fixed vector and $\rho$ be any perturbation on $\theta$. Let $\theta^\xi = \theta + \xi \rho$, by (14), we have

$$y(\gamma) = y(0) + \int_{0}^{\gamma} \tilde{f}(y(s), \tilde{y}(s), \theta, \delta) ds,$$

and

$$y^\xi(\gamma) = y(0) + \int_{0}^{\gamma} \tilde{f}(y(s), \tilde{y}(s), \theta^\xi, \delta) ds,$$

where

$$\tilde{f}(y(\gamma), \tilde{y}(\gamma), \theta, \delta) = \sum_{i=1}^{p} \theta_i \tilde{f}(y(\gamma|\theta^\xi, \delta), \tilde{y}(\gamma|\theta^\xi, \delta), \delta, \tilde{\gamma}(\gamma))(\gamma).$$

Thus,

$$\Delta y(\gamma) = \frac{dy^\xi(\gamma)}{d\xi} \bigg|_{\xi=0} = \int_{0}^{\gamma} \frac{\partial \tilde{f}(y(s), \tilde{y}(s), \theta, \delta)}{\partial y} \Delta y(s) ds$$

$$+ \int_{0}^{\gamma} \left\{ \frac{\partial \tilde{f}(y(s), \tilde{y}(s), \theta, \delta)}{\partial y} \Delta \tilde{y}(s) + \frac{\partial \tilde{f}(y(s), \tilde{y}(s), \theta, \delta)}{\partial \theta} \Delta \theta \right\} ds$$

$$= \int_{0}^{\gamma} \left\{ \frac{\partial \tilde{f}(y(s), \tilde{y}(s), \theta, \delta)}{\partial y} \Delta y(s) + \frac{\partial \tilde{f}(y(s), \tilde{y}(s), \theta, \delta)}{\partial \theta} \Delta \tilde{y}(s) \right\} ds$$

$$+ \int_{0}^{\gamma} \frac{\partial \tilde{f}(y(s), \tilde{y}(s), \theta, \delta)}{\partial \theta} \rho ds.$$

Clearly, $\Delta y(\gamma)$ satisfies:

$$\Delta \tilde{y}(\gamma) = \frac{\partial \tilde{f}(y(\gamma), \tilde{y}(\gamma), \theta, \delta)}{\partial y} \Delta y(\gamma) + \frac{\partial \tilde{f}(y(\gamma), \tilde{y}(\gamma), \theta, \delta)}{\partial \tilde{y}} \Delta \tilde{y}(\gamma)$$
\[ + \frac{\partial f(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial \theta} \rho, \]

\[ \triangle y(\gamma) = 0, \gamma \leq 0. \]

Now, by (15), we have
\[ g_k(\theta^k, \delta) = \Phi_k(y(\theta^k, \theta^k, \delta)) + \int_0^p \tilde{L}_k(y(\gamma|\theta^k, \delta), \bar{y}(\gamma|\theta^k, \delta), \theta^k, \delta)d\gamma. \quad (21) \]

From (21), we have
\[ \triangle g_k(\theta, \delta) = \frac{dg_k(\theta^k, \delta)}{d\xi} \bigg|_{\xi = 0} = \frac{\partial g_k(\theta, \delta)}{\partial \theta} \rho \]
\[ \frac{\partial \Phi_k(y(\rho))}{\partial y} \triangle y(\rho) + \int_0^p \frac{\partial \tilde{L}_k(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial \theta} \rho d\gamma \]
\[ + \int_0^p \frac{\partial \tilde{L}_k(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial y} \triangle y(\gamma) + \frac{\partial \tilde{L}_k(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial \theta} \triangle \bar{y}(\gamma)d\gamma. \quad (22) \]

By the definition of Hamiltonian function, we have
\[ \int_0^p \frac{\partial \tilde{L}_k(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial y} \triangle y(\gamma)d\gamma \]
\[ = \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial y} \triangle y(\gamma) - \lambda^k(\gamma)^T \frac{\partial f(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial y} \triangle y(\gamma)d\gamma, \]
and
\[ \int_0^p \frac{\partial \tilde{L}_k(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial y} \triangle \bar{y}(\gamma)d\gamma \]
\[ = \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial y} \triangle \bar{y}(\gamma) - \lambda^k(\gamma)^T \frac{\partial f(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial y} \triangle \bar{y}(\gamma)d\gamma, \quad (23) \]

and
\[ \int_0^p \frac{\partial \tilde{L}_k(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial \theta} \rho d\gamma \]
\[ = \int_0^p \left\{ \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, \lambda^k(\gamma))}{\partial \theta} - \lambda^k(\gamma)^T \frac{\partial f(y(\gamma), \bar{y}(\gamma), \theta, \delta)}{\partial \theta} \right\} \rho d\gamma. \]

For the first integral term on the right-hand side of (23), let \( \zeta(\gamma) = z \), as \( \mu(\gamma) = \mu(\zeta(\gamma)) + h = \mu(z) + h \), and the inverse function of \( \mu(\gamma) \) is given in (19), we know \( \gamma = \mu^{-1}(\mu(z) + h) = \nu(z) \). Recalling the strict monotonicity of \( \mu(\gamma), \zeta(\gamma) \) is also strict monotonic.

We have
\[ \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, \lambda^k(\gamma))}{\partial y} \triangle \bar{y}(\gamma)d\gamma \]
\[ = \int_{\zeta(0)}^{\zeta(p)} \frac{\partial H_k(y(z), \bar{y}(z), \theta, \delta, \lambda^k(z))}{\partial y} \triangle y(z) \frac{\partial \nu(z)}{\partial z} dz \]
\[ = \int_{\zeta(0)}^{\zeta(p)} \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, \lambda^k(\gamma))}{\partial y} \triangle y(\gamma) \frac{\partial \nu(\gamma)}{\partial \gamma} d\gamma. \]
Clearly, $\triangle y(\gamma) = 0$ when $\gamma \leq 0$, and note that $\zeta(0) < 0$, we have,

$$
\int_0^p \partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma)) \triangle \bar{y}(\gamma) d\gamma
$$

$$
= \int_0^p \left\{ \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial y} \triangle y(\gamma) e(\zeta(p) - \gamma) \frac{\partial \nu(\gamma)}{\partial \gamma} \right\} d\gamma
$$

$$
= \int_0^p \left\{ \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial y} \triangle y(\gamma) \frac{\partial \nu(\gamma)}{\partial \gamma} \right\} d\gamma.
$$

Then, we can obtain

$\triangle g_u(\theta, \delta)$

$$
\frac{\partial \Phi_k(y(p))}{\partial y} \triangle y(p) + \int_0^p \frac{\partial \bar{L}_k(y(\gamma), y(\gamma), \theta, \delta)}{\partial \theta} d\gamma
$$

$$
+ \int_0^p \frac{\partial \bar{L}_k(y(\gamma), y(\gamma), \theta, \delta)}{\partial y} \triangle y(\gamma) + \frac{\partial \bar{L}_k(y(\gamma), y(\gamma), \theta, \delta)}{\partial y} \triangle y(\gamma) d\gamma
$$

$$
= \frac{\partial \Phi_k(y(p))}{\partial y} \triangle y(p) + \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial \theta} d\gamma
$$

$$
+ \int_0^p \left\{ \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial y} \triangle y(\gamma) + \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial y} \triangle y(\gamma) \right\} d\gamma
$$

$$
= \lambda^k(p)^T \triangle y(p) + \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial \theta} d\gamma
$$

$$
+ \int_0^p \left\{ \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial y} + \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial y} \frac{\partial \nu(\gamma)}{\partial \gamma} \right\} \triangle y(\gamma) d\gamma
$$

$$
- \int_0^p \lambda^k(\gamma)^T \triangle y(\gamma) d\gamma
$$

$$
= \lambda^k(p)^T \triangle y(p) + \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial \theta} d\gamma
$$

$$
- \int_0^p (\lambda^k(\gamma)^T \triangle y(\gamma) + \lambda^k(\gamma)^T \triangle \bar{y}(\gamma)) d\gamma
$$

$$
= \lambda^k(p)^T \triangle y(p) + \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial \theta} d\gamma
$$

$$
- \lambda^k(p)^T \triangle y(p) + \lambda^k(0)^T \triangle y(0)
$$

$$
= \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, x^k(\gamma))}{\partial \theta} d\gamma
$$
Thus, from (22), we have
\[ \Delta g_k(\theta, \delta) = \frac{\partial g_k(\theta, \delta)}{\partial \theta} \rho = \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, \lambda^k(\gamma))}{\partial \theta} \rho d\gamma. \]

Since \( \rho \) is arbitrary, the conclusion of the theorem follows readily.

Using these formulae, the partial derivatives of the cost and constraint functions with respect to \( \theta \) can be computed, and then the partial derivative of \( g_k(\theta, \delta) \), \( k = 0, 1, \cdots, e + m \), with respect to the control height \( \delta \) can be obtained in the same way. We give it as a theorem stated below:

**Theorem 4.2.** For each pair \( (\theta, \delta) \in \Theta \times \Delta \), The partial derivative of \( g_k(\theta, \delta) \), \( k = 0, 1, \cdots, e + m \) with respect to \( \delta \) is given by
\[ \frac{\partial g_k(\theta, \delta)}{\partial \delta} = \int_0^p \frac{\partial H_k(y(\gamma), \bar{y}(\gamma), \theta, \delta, \lambda^k(\gamma))}{\partial \delta} d\gamma. \]

**Proof.** The proof of the Theorem is similar to the Theorem 4.1 and hence is omitted.

5. **Numerical experiments.** To demonstrate the effectiveness and the performance of the proposed gradient computational approach, we use the built-in optimization package `fmincon` of MATLAB to solve 5 example problems. For comparison, the cost and constrain gradients are obtained by co-state as well as variational method. The numerical results for different examples are listed in Table 3. All the numerical experiments are carried out via MATLAB 2016A on a PC with CPU: 1.80GHZ and 8GB of memory.

5.1. **Problem 1: Optimal control with single time delay.** Consider the following time-delay optimal control problems [26]:
\[ \min g_0(u) = (3/2)(x(3))^2 + 1/2 \int_0^3 (u(t))^2 dt \]
subject to the time-delay system
\[ \dot{x}(t) = x(t-1) + u(t), \quad t \in [0, 3], \]
\[ x(t) = 1, \quad t \in [-1, 0], \]
where
\[ -3 \leq u(t) \leq 3, \]
the terminal inequality constraints
\[ g_1(u) = (x(3))^2 - 0.03 \geq 0, \]
\[ g_2(u) = 0.06 - (x(3))^2 \geq 0. \]

5.2. **Problem 2: Optimal control with state delay.** Consider the following multiple time-delay optimal control problem [6]:
\[ \min g_0(u) = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_0^{t_f} \{ x^T Q x + u^T R u \} dt \]
subject to the time-delay dynamic system
\[ \dot{x}(t) = A_1(t)x(t) + A_2 \bar{x}(t) + B(t)u(t), \]
\[ x(t) = [1, 0]^T, \quad t \leq 0, \]
where $\bar{x}(t) = (x_1(t-1), x_2(t-0.5))$, $A_1(t) = \begin{bmatrix} 0 & 1 \\ -4\pi^2(a + c \cos(2\pi t)) & 0 \end{bmatrix}$, $A_2(t) = \begin{bmatrix} 0 & 0 \\ -4\pi^2 b \cos(2\pi t) & 0 \end{bmatrix}$, $B(t) = [0, 1]^T$, the parameters of the problem are in Table 1, and the control constraints are

$$-3 \leq u \leq 4, \ t \in [0, t_f].$$

| a  | b  | c  | $t_f$ | Q        | R        | S        |
|----|----|----|-------|----------|----------|----------|
| 0.2| 0.5| 0.2| 1.5   | $I_{2\times 2}$ | $I_{2\times 2}$ | $10^4 I_{2\times 2}$ |

Table 1. Parameters in Problem 2

5.3. Problem 3: Optimal control problem with multiple time-delay. Consider the following time-delay optimal control problem [31], which includes different time delay in every state and control variable:

$$\min g_0 = \frac{1}{2} \bar{x}(2)^T \bar{S} \bar{x}(2) + \frac{1}{2} \int_0^2 [\bar{x}(t)^T \bar{Q} \bar{x}(t) + \bar{u}(t)^T \bar{R} \bar{u}(t)] dt$$

where

$$S = \begin{bmatrix} 1 & 2 & 0 & 0 \\
2 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 \end{bmatrix},$$

subject to the time-delay dynamic system

$$\begin{align*}
\dot{x}_1(t) &= -2x_1(t)^2 + x_1(t)x_2(t-0.2) + 2x_2(t) - u_1(t)u_2(t-0.5), \\
\dot{x}_2(t) &= -x_1(t-0.1) + 2x_3(t) + u_2(t), \\
\dot{x}_3(t) &= -x_3(t)^3 - x_1(t)x_2(t) - x_2(t-0.2)u_2(t) + u_1(t-0.4) + 2u_3(t), \\
\dot{x}_4(t) &= -x_4(t)^2 + x_2(t)x_3(t) - 2x_3(t-0.3) + 2u_4(t),
\end{align*}$$

the initial conditions

$$\begin{align*}
x_1(t-0.1) &= 1, \ t \leq 0.1; \ x_2(t-0.2) &= 1, \ t \leq 0.2; \\
x_3(t-0.3) &= 1, \ t \leq 0.3; \ x_4(t-0.4) &= 1, \ t \leq 0.4; \\
u_1(t-0.5) &= 1, \ t < 0.5; \ u_2(t-0.6) &= 1, \ t < 0.6; \\
u_3(t-0.7) &= 1, \ t < 0.7; \ u_4(t-0.8) &= 1, \ t < 0.8,
\end{align*}$$

the terminal inequality constraints

$$\begin{align*}
g_1(u) &= 4 - x_1(2)^2 - x_2(2)^2 - x_3(2)^2 - x_4(2)^2 \geq 0, \\
g_2(u) &= x_1(2)^2 + x_2(2)^2 + x_3(2)^2 + x_4(2)^2 - 0.002 \geq 0,
\end{align*}$$

and the control constraints

$$-0.9 \leq u_i(t) \leq 1, \ t \in [0, 2], \ i = 1, \cdots, 4.$$
5.4. **Problem 4: Continuous stirred tank reactor.** Consider the following time-delay optimal control problem [34], which comes from a continuous stirred tank reactor system, where the time delays in the state and control are different:

\[
\min g_0(u) = \int_0^{0.2} [\|x(t)\|_2^2 + 0.01\|u(t)\|_2^2]dt
\]

subject to the time-delay dynamic system

\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) - R(t), \\
\dot{x}_2(t) &= -x_2(t) + 0.9u_2(t - 0.02) + 0.1u_2(t) , \\
\dot{x}_3(t) &= -2x_3(t) + 0.25R(t) - 1.05u_1(t)x_3(t - 0.015),
\end{align*}
\]

where

\[
R(t) = (1 + x_1(t))(1 + x_2(t))exp\left[\frac{25x_3(t)}{1 + x_3(t)}\right],
\]

and the initial state and control are given by:

\[
\begin{align*}
x_3(t) &= -0.02, \quad -r \leq t \leq 0, \\
u_2(t) &= 1, \quad -s \leq t \leq 0, \\
x(0) &= [0.49, -0.0002, -0.02]^T.
\end{align*}
\]

5.5. **Problem 5: LQR optimal control with state delay.** Consider the following time-delay optimal control problem [6]:

\[
\min g_0(u) = \frac{1}{2}x^T(t_f)Sx(t_f) + \frac{1}{2}\int_0^{t_f} [x^TQx + u^TRu]dt
\]

subject to the time-delay dynamic system

\[
\begin{align*}
\dot{x}(t) &= A_1(t)x(t) + A_2(t)x(t-h) + B(t)u(t), \\
x(t) &= [1, 0]^T, \quad h \leq t \leq 0,
\end{align*}
\]

where

\[
A_1(t) = \begin{bmatrix} 0 & 1 \\ -4\pi^2(a + c\cos2\pi t) & 0 \end{bmatrix}, \quad A_2(t) = \begin{bmatrix} 0 & 0 \\ -4\pi^2b\cos2\pi t & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

and the parameters of the problem are in Table 2:

| a | b | c | h | tf | Q | R | S |
|---|---|---|---|----|---|---|---|
| 0.2 | 0.5 | 0.2 | 1 | 2 | $I_{2\times2}$ | $I_{2\times2}$ | $10^4I_{2\times2}$ |

**Table 2.** Parameters in Problem 5

It is clear from Table 3 that both methods can obtain the same cost for different choices of partitions for all the examples. However, the CPU time for costate method is much less than that of variational method. Furthermore, we can easily observe the trends that computational time is significantly improved when the partition number is increased for all the examples.

6. **Conclusion.** In this paper, under the computational framework of control parameterization together with time-scaling transformation, we derive the cost and constraint gradients by co-state approach. Numerical results show that the method with new gradients formulae outperform that of variational method.
### OPTIMAL CONTROL PROBLEM WITH TIME-DELAY

#### Table 3. Experimental results of co-state method and variational method

| Problem | numbers of partitions | co-state method CPU time(s) | variational method CPU time(s) | optimal cost |
|---------|-----------------------|-----------------------------|--------------------------------|--------------|
| Prob 1  | p=4                   | 9.62                        | 11.98                          | 1.7500       |
|         | p=6                   | 18.04                       | 29.49                          | 1.7407       |
|         | p=8                   | 26.14                       | 31.73                          | 1.7405       |
|         | p=12                  | 61.41                       | 117.46                         | 1.7403       |
| Prob 2  | p=5                   | 10.2                        | 147.23                         | 2.4046       |
| Prob 3  | p=10                  | 183                         | 2702                           | 2.0356       |
| Prob 4  | p=1                   | 1.180                       | 1.729                          | 0.0218       |
|         | p=3                   | 4.485                       | 22.39                          | 0.0176       |
|         | p=6                   | 12.92                       | 329.01                         | 0.0142       |
| Prob 5  | p=5                   | 8.72                        | 51.56                          | 2.1502       |

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