Gauge Theory Correlators from Non-Critical String Theory

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Abstract

We suggest a means of obtaining certain Green’s functions in 3+1-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with a large number of colors via non-critical string theory. The non-critical string theory is related to critical string theory in anti-deSitter background. We introduce a boundary of the anti-deSitter space analogous to a cut-off on the Liouville coordinate of the two-dimensional string theory. Correlation functions of operators in the gauge theory are related to the dependence of the supergravity action on the boundary conditions. From the quadratic terms in supergravity we read off the anomalous dimensions. For operators that couple to massless string states it has been established through absorption calculations that the anomalous dimensions vanish, and we rederive this result. The operators that couple to massive string states at level $n$ acquire anomalous dimensions that grow as $2 \left( ng_{YM} \sqrt{2N} \right)^{1/2}$ for large ’t Hooft coupling. This is a new prediction about the strong coupling behavior of large $N$ SYM theory.
1. Introduction

Relations between gauge fields and strings present an old, fascinating and unanswered question. The full answer to this question is of great importance for theoretical physics. It will provide us with a theory of quark confinement by explaining the dynamics of color-electric fluxes. On the other hand, it will perhaps uncover the true “gauge” degrees of freedom of the fundamental string theories, and therefore of gravity.

The Wilson loops of gauge theories satisfy the loop equations which translate the Schwinger-Dyson equations into variational equations on the loop space \([1, 2]\). These equations should have a solution in the form of the sum over random surfaces bounded by the loop. These are the world surfaces of the color-electric fluxes. For the \(SU(N)\) Yang-Mills theory they are expected to carry the ‘t Hooft factor \([3]\), \(N^\chi\), where \(\chi\) is the Euler character. Hence, in the large \(N\) limit where \(g_{YM}^2 N\) is kept fixed only the simplest topologies are relevant.

Until recently, the action for the “confining string” had not been known. In \([4]\) is was suggested that it must have a rather unusual structure. Let us describe it briefly. First of all, the world surface of the electric flux propagates in at least 5 dimensions. This is because the non-critical strings are described by the fields

\[
X^\mu(\sigma), \quad g_{ij}(\sigma) = e^{n/2} \delta_{ij},
\]

where \(X^\mu\) belong to 4-dimensional (Euclidean) space and \(g_{ij}(\sigma)\) is the world sheet metric in the conformal gauge. The general form of the world sheet lagrangian compatible with the 4-dimensional symmetries is

\[
\mathcal{L} = \frac{1}{2} (\partial_i \varphi)^2 + a^2(\varphi) (\partial_i X^\mu)^2 + \Phi(\varphi) (R^{(2)} + \text{Ramond - Ramond backgrounds}),
\]

where \((R^{(2)})\) is the world sheet curvature, \(\Phi(\varphi)\) is the dilaton \([5]\), while the field

\[
\Sigma(\varphi) = a^2(\varphi)
\]

defines a variable string tension. In order to reproduce the zig-zag symmetry of the Wilson loop, the gauge fields must be located at a certain value \(\varphi = \varphi_*\) such that \(a(\varphi_*) = 0\). We will call this point “the horizon.”

The background fields \(\Phi(\varphi), a(\varphi)\) and others must be chosen to satisfy the conditions of conformal invariance on the world sheet \([6]\). After this is done, the relation between gauge fields and strings can be described as an isomorphism between the general Yang-Mills operators of the type

\[
\int d^4 x e^{ip \cdot x} \text{tr} (\nabla_{\alpha_1} \ldots F_{\mu_1 \nu_1} \ldots \nabla_{\alpha_n} \ldots F_{\mu_n \nu_n}(x))
\]
and the algebra of vertex operators of string theory, which have the form

\[ V^{\alpha_1 \ldots \alpha_n} (p) = \int d^2 \sigma \Psi^{i_1 \ldots i_n j_1 \ldots j_m} (\varphi(\sigma)) e^{ip \cdot X(\sigma)} \partial_{i_1} X^{\alpha_1} \ldots \partial_{i_n} X^{\alpha_n} \partial_{j_1} \varphi \ldots \partial_{j_m} \varphi , \]  

(4)

where the wave functions \( \Psi^{i_1 \ldots i_n j_1 \ldots j_m} (\varphi) \) are again determined by the conformal invariance on the world sheet. The isomorphism mentioned above implies the coincidence of the correlation functions of these two sets of vertex operators.

Another, seemingly unrelated, development is connected with the Dirichlet brane [7] description of black 3-branes in [8, 9, 10, 11]. The essential observation is that, on the one hand, the black branes are solitons which curve space [12] and, on the other hand, the world volume of \( N \) parallel D-branes is described by supersymmetric \( U(N) \) gauge theory with 16 supercharges [13]. A particularly interesting system is provided by the limit of a large number \( N \) of coincident D3-branes [8, 9, 10, 11], whose world volume is described by \( \mathcal{N} = 4 \) supersymmetric \( U(N) \) gauge theory in 3 + 1 dimensions. For large \( g_{YM}^2 N \) the curvature of the classical geometry becomes small compared to the string scale [9], which allows for comparison of certain correlation functions between the supergravity and the gauge theory, with perfect agreement [9, 10, 11]. Corrections in powers of \( \alpha' \) times the curvature on the string theory side correspond to corrections in powers of \( (g_{YM}^2 N)^{-1/2} \) on the gauge theory side. The string loop corrections are suppressed by powers of \( 1/N^2 \).

The vertex operators introduced in [9, 10, 11] describe the coupling of massless closed string fields to the world volume. For example, the vertex operator for the dilaton is

\[ \frac{1}{4g_{YM}^2} \int d^4 x e^{ip \cdot x} \left( \text{tr} F_{\mu \nu} F^{\mu \nu} (x) + \ldots \right) , \]  

(5)

while that for the graviton polarized along the 3-branes is

\[ \int d^4 x e^{ip \cdot x} T^{\mu \nu} (x) , \]  

(6)

where \( T^{\mu \nu} \) is the stress tensor. The low energy absorption cross-sections are related to the 2-point functions of the vertex operators, and turn out to be in complete agreement with conformal invariance and supersymmetric non-renormalization theorems [11]. An earlier calculation of the entropy as a function of temperature for \( N \) coincident D3-branes [8] exhibits a dependence expected of a field theory with \( O(N^2) \) massless fields, and turns out to be 3/4 of the free field answer. This is not a discrepancy since the free field result is valid for small \( g_{YM}^2 N \), while the result of [8] is applicable as \( g_{YM}^2 N \rightarrow \infty \). We now regard this result as a non-trivial prediction of supergravity concerning the strong coupling behavior of \( \mathcal{N} = 4 \) supersymmetric gauge theory at large \( N \) and finite temperature.

The non-critical string and the D-brane approaches to 3+1 dimensional gauge theory have been synthesized in [14] by rescaling the 3-brane metric and taking the limit in which
it has conformal symmetry, being the direct product $AdS_5 \times S^5$. This is exactly the confining string ansatz [4] with

$$a(\varphi) = e^{\varphi/R},$$

(7)

corresponding to the case of constant negative curvature of order $1/R^2$. The horizon is located at $\varphi_*= -\infty$. The Liouville field is thus related to the radial coordinate of the space transverse to the 3-brane. The extra $S^5$ part of the metric is associated with the 6 scalars and the $SU(4)$ R-symmetry present in the $\mathcal{N} = 4$ supersymmetric gauge theory.

In the present paper we make the next step and show how the masses of excited states of the “confining string” are related to the anomalous dimensions of the SYM theory. Hopefully this analysis will help future explorations of asymptotically free gauge theories needed for quark confinement.

We will suggest a potentially very rich and detailed means of analyzing the throat-brane correspondence: we propose an identification of the generating function of the Green’s functions of the superconformal world-volume theory and the supergravity action in the near horizon background geometry. We will find it necessary to introduce a boundary of the $AdS_5$ space near the place where the throat turns into the asymptotically flat space. Thus, the anti-deSitter coordinate $\varphi$ is defined on a half-line $(-\infty, 0]$, similarly to the Liouville coordinate of the 2-dimensional string theory [15, 16]. The correlation functions are specified by the dependence of the action on the boundary conditions, again in analogy with the $c=1$ case. One new prediction that we will be able to extract this way is for the anomalous dimensions of the gauge theory operators that correspond to massive string states. For a state at level $n$ we find that, for large $g_{YM}^2 N$, the anomalous dimension grows as $2\sqrt{n}(2g_{YM}^2 N)^{1/4}$.

2. Green’s functions from the supergravity action

The geometry of a large number $N$ of coincident D3-branes is

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} (-dt^2 + d\vec{x}^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2).$$

(8)

The parameter $R$, where

$$R^4 = \frac{N}{2\pi^2 T_3}, \quad T_3 = \frac{\sqrt{\pi}}{\kappa}$$

(9)

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1 The papers that put an early emphasis on the anti-deSitter nature of the near-horizon region of certain brane configurations, and its relation with string and M-theory, are [17, 18]. Other ideas on the relation between branes and AdS supergravity were recently pursued in [19, 20, 21].
is the only length scale involved in all of what we will say. \(T_3\) is the tension of a single D3-brane, and \(\kappa\) is the ten-dimensional gravitational coupling. The near-horizon geometry of \(N\) D3-branes is \(\text{AdS}_5 \times S^5\), as one can see most easily by defining the radial coordinate \(z = R^2/r\). Then

\[
\text{d}s^2 = \frac{R^2}{z^2} \left( -\text{d}t^2 + dx^2 + dz^2 \right) + R^2 d\Omega_5^2.
\]

(10)

The relation to the coordinate \(\varphi\) used in the previous section is

\[
z = Re^{-\varphi/R}.
\]

(11)

Note that the limit \(z \to 0\) is far from the brane. Of course, for \(z < \sim R\) the \(\text{AdS}\) form (10) gets modified, and for \(z \ll R\) one obtains flat ten-dimensional Minkowski space. We will freely use phrases like “far from the brane” and “near the brane” to describe regimes of small \(z\) and large \(z\), despite the fact that the geometry is geodesically complete and nonsingular.

The basic idea is to identify the generating functional of connected Green’s functions in the gauge theory with the minimum of the supergravity action, subject to some boundary conditions at \(z = R\) and \(z = \infty\):

\[
W[g_{\mu\nu}(x^\lambda)] = K[g_{\mu\nu}(x^\lambda)] = S[g_{\mu\nu}(x^\lambda, z)].
\]

(12)

\(W\) generates the connected Green’s functions of the gauge theory; \(S\) is the supergravity action on the \(\text{AdS}\) space; while \(K\) is the minimum of \(S\) subject to the boundary conditions. We have kept only the metric \(g_{\mu\nu}(x^\lambda)\) of the world-volume as an explicit argument of \(W\).

The boundary conditions subject to which the supergravity action \(S\) is minimized are

\[
\text{d}s^2 = \frac{R^2}{z^2} \left( g_{\mu\nu}dx^\mu dx^\nu + dz^2 \right) + O(1) \quad \text{as } z \to R.
\]

(13)

All fluctuations have to vanish as \(z \to \infty\).

A few refinements of the identification (12) are worth commenting on. First, it is the generator of connected Green’s functions which appears on the left hand side because the supergravity action on the right hand side is expected to follow the cluster decomposition principle. Second, because classical supergravity is reliable only for a large number \(N\) of coincident branes, (12) can only be expected to capture the leading large \(N\) behavior. Corrections in \(1/N\) should be obtained as loop effects when one replaces the classical action \(S\) with an effective action \(\Gamma\). This is sensible since the dimensionless expansion parameter \(\kappa^2/R^8 \sim 1/N^2\). We also note that, since \((\alpha')^2/R = (g_{YM}^2 N)^{-1}\), the string theoretic \(\alpha'\) corrections to the supergravity action translate into gauge theory corrections proportional to inverse powers of \(g_{YM}^2 N\). Finally, the fact that there is no covariant action for type IIB supergravity does not especially concern us: to obtain \(n\)-point Green’s functions one is actually considering the \(n^{th}\) variation of the action, which for \(n > 0\) can be regarded as the \((n - 1)^{th}\) variation of the covariant equations of motion.

In section 2 we will compute two-point functions of massless vertex operators from (12), compare them with the absorption calculations in [9, 10, 11] and find exact agreement. However it is instructive first to examine boundary conditions.
2.1. Preliminary: symmetries and boundary conditions

As a preliminary it is useful to examine the appropriate boundary conditions and how they relate to the conformal symmetry. In this discussion we follow the work of Brown and Henneaux [22]. In the consideration of geometries which are asymptotically anti-de Sitter, one would like to have a realization of the conformal group on the asymptotic form of the metric. Restrictive or less restrictive boundary conditions at small $z$ (far from the brane) corresponds, as Brown and Henneaux point out in the case of $AdS_3$, to smaller or larger asymptotic symmetry groups. On an $AdS_{d+1}$ space,

$$ds^2 = G_{mn} dx^m dx^n = \frac{R^2}{z^2} \left( -dt^2 + d\vec{x}^2 + dz^2 \right)$$  \hspace{1cm} (14)

where now $\vec{x}$ is $d - 1$ dimensional, the boundary conditions which give the conformal group as the group of asymptotic symmetries are

$$\delta G_{\mu\nu} = O(1), \quad \delta G_{z\mu} = O(z), \quad \delta G_{zz} = O(1).$$  \hspace{1cm} (15)

Our convention is to let indices $m, n$ run from 0 to $d$ (that is, over the full $AdS_{d+1}$ space) while $\mu, \nu$ run only from 0 to $d - 1$ (ie excluding $z = x^d$). Diffeomorphisms which preserve (15) are specified by a vector $\zeta^m$ which for small $z$ must have the form

$$\zeta^\mu = \xi^\mu - \frac{z^2}{d} \eta^{\mu\nu} \partial_\nu (\xi^\kappa, \kappa) + O(z^4)$$

$$\zeta^z = \frac{z}{d} \xi^\kappa, \kappa + O(z^3).$$  \hspace{1cm} (16)

Here $\xi^\mu$ is allowed to depend on $t$ and $\vec{x}$ but not $r$. (16) specifies only the asymptotic form of $\zeta^m$ at large $r$, in terms of this new vector $\xi^\mu$.

Now the condition that the variation $\delta G_{mn} = L_\zeta G_{mn} = \zeta^k \partial_k G_{mn} + G_{kn} \partial_m \zeta^k + G_{mk} \partial_n \zeta^k$ be of the allowed size specified in (15) is equivalent to

$$\xi_{\mu, \nu} + \xi_{\nu, \mu} = \frac{2}{d} \xi^\kappa, \kappa$$  \hspace{1cm} (18)

where now we are lowering indices on $\xi^\mu$ with the flat space Minkowski metric $\eta_{\mu\nu}$. Since (18) is the conformal Killing equation in $d$ dimensions ($d = 4$ for the 3-brane), we see that we indeed recover precisely the conformal group from the set of permissible $\xi^\mu$.

The spirit of [22] is to determine the central charge of an $AdS_3$ configuration by considering the commutator of deformations corresponding to Virasoro generators $L_m$ and $L_{-m}$. This method is not applicable to higher dimensional cases because the conformal group becomes finite, and there is apparently no way to read off a Schwinger term from commutators of conformal transformations. Nevertheless, the notion of central charge can be given meaning in higher dimensional conformal field theories, either via a curved space conformal anomaly (also called the gravitational anomaly) or as the normalization of the two-point function of the stress energy tensor [23]. We shall see in section 2 that a calculation reminiscent of absorption probabilities allows us to read off the two point function of stress-energy tensors in $\mathcal{N} = 4$ super-Yang-Mills, and with it the central charge.
2.2. The two point functions

First we consider the case of a minimally coupled massless scalar propagating in the anti-de Sitter near-horizon geometry (one example of such a scalar is the dilaton $\phi$ [9]). As a further simplification we assume for now that $\phi$ is in the $s$-wave (that is, there is no variation over $S^5$). Then the action becomes

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{G} \left[ \frac{1}{2} G^{MN} \partial_M \phi \partial_N \phi \right]$$

$$= \frac{\pi^3 R^8}{4\kappa^2} \int d^4x \int_0^\infty dz \frac{dz}{z^3} \left[ (\partial_z \phi)^2 + \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right].$$

Note that in (19), as well as in all the following equations, we take $\kappa$ to be the ten-dimensional gravitational constant. The equations of motion resulting from the variation of $S$ are

$$\left[ z^2 \partial_z \frac{1}{z^3} \partial_z + \eta^{\mu\nu} \partial_\mu \partial_\nu \right] \phi = 0.$$  

A complete set of normalizable solutions is

$$\phi_k(x^\ell) = \lambda_k e^{ik : x} \tilde{\phi}_k(z) \quad \text{where} \quad \tilde{\phi}_k(z) = \frac{z^2 K_2(kz)}{R^2 K_2(kR)},$$

$$k^2 = \vec{k}^2 - \omega^2.$$  

We have chosen the modified Bessel function $K_2(kz)$ rather than $I_2(kz)$ because the functions $K_\nu(kz)$ fall off exponentially for large $z$, while the functions $I_\nu(kz)$ grow exponentially. In other words, the requirement of regularity at the horizon (far down the throat) tells us which solution to keep. A connection of this choice with the absorption calculations of [9] is provided by the fact that, for time-like momenta, this is the incoming wave which corresponds to absorption from the small $z$ region. $\lambda_k$ is a coupling constant, and the normalization factor has been chosen so that $\tilde{\phi}_k = 1$ for $z = R$.

Let us consider a coupling

$$S_{\text{int}} = \int d^4x \phi(x^\lambda) O(x^\lambda)$$

in the world-volume theory. If $\phi$ is the dilaton then according to [9] one would have $O = \frac{1}{4g_Y M} \text{tr} F^2 + \ldots$. Then the analogue of (12) is the claim that

$$W[\phi(x^\lambda)] = K[\phi(x^\lambda)] = S[\phi(x^\lambda, z)]$$

where $\phi(x^\lambda, z)$ is the unique solution of the equations of motion with $\phi(x^\lambda, z) \to \phi(x^\lambda)$ as $z \to R$. Note that the existence and uniqueness of $\phi$ are guaranteed because the equation
of motion is just the laplace equation on the curved space. (One could in fact compactify \(x^\lambda\) on very large \(T^4\) and impose the boundary condition \(\phi(x^\lambda, z) = \phi(x^\lambda)\) at \(z = R\). Then the determination of \(\phi(x^\lambda, z)\) is just the Dirichlet problem for the laplacian on a compact manifold with boundary).

Analogously to the work of [15] on the \(c = 1\) matrix model, we can obtain the quadratic part of \(K[\phi(x^\lambda)]\) as a pure boundary term through integration by parts,

\[
K[\phi(x^\lambda)] = \frac{\pi^3 R^8}{4\kappa^2} \int d^4x \int_R^\infty \frac{dz}{z^3} \left[ -\phi \left( z^3 \partial_z \frac{1}{z^3} \partial_z + \eta^{\mu\nu} \partial_\mu \partial_\nu \right) \phi + z^3 \partial_z \left( \phi \frac{1}{z^3} \partial_z \phi \right) \right]
= \frac{1}{2} \int d^4k d^4q \lambda_k \lambda_q (2\pi)^4 \delta^4(k + q) \frac{N^2}{16\pi^2} F,
\]

where we have expanded

\[
\phi(x^\lambda) = \int d^4k \lambda_k e^{ik \cdot x}.
\]

The “flux factor” \(F\) (so named because of its resemblance to the particle number flux in a scattering calculation) is

\[
F = \left[ \tilde{\phi}_k \frac{1}{z^3} \partial_z \tilde{\phi}_k \right]_R^\infty.
\]

In (24) we have suppressed the boundary terms in the \(x^\lambda\) directions—again, one can consider these compactified on very large \(T^4\) so that there is no boundary. We have also used (9) to simplify the prefactor. Finally, we have cut off the integral at \(R\) as a regulator of the small \(z\) divergence. This is in fact appropriate since the D3-brane geometry is anti-de Sitter only for \(z \gg R\). Since there is exponential falloff in \(\tilde{\phi}_k\) as \(z \to \infty\), only the behavior at \(R\) matters.

To calculate the two-point function of \(O\) in the world-volume theory, we differentiate \(K\) twice with respect to the coupling constants \(\lambda^2\)

\[
\langle O(k)O(q) \rangle = \int d^4x d^4y e^{ik \cdot x + iq \cdot y} \langle O(x)O(y) \rangle
= \frac{\partial^2 K}{\partial \lambda_k \partial \lambda_q} = (2\pi)^4 \delta^4(k + q) \frac{N^2}{16\pi^2} F
= -(2\pi)^4 \delta^4(k + q) \frac{N^2}{64\pi^2} k^4 \ln(k^2 R^2) + \text{(analytic in } k^2)\]

where now the flux factor has been evaluated as

\[
F = \left[ \tilde{\phi}_k \frac{1}{z^3} \partial_z \tilde{\phi}_k \right]_R = \left[ \frac{1}{z^3} \partial_z \ln(\tilde{\phi}_k) \right]_R = -\frac{1}{4} k^4 \ln(k^2 R^2) + \text{(analytic in } k^2)\).\]

\(\text{2 The appearance of the logarithm here is analogous to the logarithmic scaling violation in the } c = 1 \text{ matrix model.}\)
Fourier transforming back to position space, we find

\[ \langle O(x)O(y) \rangle \sim \frac{N^2}{|x-y|^8}. \]  

(28)

This is consistent with the free field result for small \( g_{YM}^2 N \). Remarkably, supergravity tells us that this formula continues to hold as \( g_{YM}^2 N \to \infty \).

Another interesting application of this analysis is to the two-point function of the stress tensor, which with the normalization conventions of [11] is

\[ \langle T_{\alpha\beta}(x)T_{\gamma\delta}(0) \rangle = \frac{c}{48\pi^4} X_{\alpha\beta\gamma\delta} \left( \frac{1}{x^4} \right), \]  

(29)

where the central charge (the conformal anomaly) is \( c = N^2/4 \) and

\[
X_{\alpha\beta\gamma\delta} = 2\Box^2 \eta_{\alpha\beta}\eta_{\gamma\delta} - 3\Box^2 (\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}) - 4\partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \\
- 2\Box (\partial_\alpha \partial_\beta \eta_{\gamma\delta} + \partial_\alpha \partial_\gamma \eta_{\beta\delta} + \partial_\alpha \partial_\delta \eta_{\beta\gamma} + \partial_\beta \partial_\gamma \eta_{\alpha\delta} + \partial_\beta \partial_\delta \eta_{\alpha\gamma} + \partial_\gamma \partial_\delta \eta_{\alpha\beta}).
\]

(30)

For metric perturbations \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) around flat space, the coupling of \( h_{\mu\nu} \) at linear order is

\[ S_{\text{int}} = \int d^4x \frac{1}{2} h_{\mu\nu} T_{\mu\nu}. \]  

(31)

Furthermore, at quadratic order the supergravity action for a graviton polarized along the brane, \( h_{xy}(k) \), is exactly the minimal scalar action, provided the momentum \( k \) is orthogonal to the \( xy \) plane. We can therefore carry over the result (26) to obtain

\[ \langle T_{xy}(k)T_{xy}(q) \rangle = -(2\pi)^4 \delta^4(k+q) \frac{N^2}{64\pi^2} k^4 \ln(k^2 R^2) + \text{(analytic in } k^2), \]  

(32)

which upon Fourier transform can be compared with (29) to give \( c = N^2/4 \). In view of the conformal symmetry of both the supergravity and the gauge theory, the evaluation of this one component is a sufficient test.

The conspiracy of overall factors to give the correct normalization of (32) clearly has the same origin as the successful prediction of the minimal scalar s-wave absorption cross-section [9, 10, 11]. The absorption cross-section is, up to a constant of proportionality, the imaginary part of (32). In [9] the absorption cross-section was calculated in supergravity using propagation of a scalar field in the entire 3-brane metric, including the asymptotic region far from the brane. Here we have, in effect, replaced communication of the throat region with the asymptotic region by a boundary condition at one end of the throat. The physics of this is clear: signals coming from the asymptotic region excite the part of the throat near \( z = R \). Propagation of these excitations into the throat can then be treated just in the anti-deSitter approximation. Thus, to extract physics from anti-deSitter space
we introduce a boundary at $z = R$ and take careful account of the boundary terms that contain the dynamical information.

It now seems clear how to proceed to three-point functions: on the supergravity side one must expand to third order in the perturbing fields, including in particular the three point vertices. At higher orders the calculation is still simple in concept (the classical action is minimized subject to boundary conditions), but the complications of the $\mathcal{N} = 8$ supergravity theory seem likely to make the computation of, for instance, the four-point function, rather tedious. We leave the details of such calculations for the future. It may be very useful to compute at least the three point functions in order to have a consistency check on the normalization of fields.

One extension of the present work is to consider what fields couple to the other operators in the $N = 4$ supercurrent multiplet. The structure of the multiplet (which includes the supercurrents, the $SU(4) R$-currents, and four spin 1/2 and one scalar field) suggests a coupling to the fields of gauged $N = 4$ supergravity. The question then becomes how these fields are embedded in $N = 8$ supergravity. We leave these technical issues for the future, but with the expectation that they are “bound to work” based on supersymmetry.

The main lesson we have extracted so far is that, for certain operators that couple to the massless string states, the anomalous dimensions vanish. We expect this to hold for all vertex operators that couple to the fields of supergravity. This may be the complete set of operators that are protected by supersymmetry. As we will see in the next section, other operators acquire anomalous dimensions that grow for large ’t Hooft coupling.

3. Massive string states and anomalous dimensions

Before we proceed to the massive string states, a useful preliminary is to discuss the higher partial waves of a minimally coupled massless scalar. The action in five dimensions (with Lorentzian signature), the equations of motion, and the solutions are

$$S = \frac{\pi^3 R^3}{4\kappa^2} \int d^4x \int_R^\infty \frac{dz}{z^3} \left[ (\partial_z \phi)^2 + (\partial_\mu \phi)^2 + \frac{\ell(\ell + 4)}{z^2} \phi^2 \right]$$

$$\left[ z^3 \partial_z \frac{1}{z^3} \partial_z + \eta^{\mu\nu} \partial_\mu \partial_\nu - \frac{\ell(\ell + 4)}{z^2} \right] \phi = 0$$

$$\phi_k(x^\ell) = e^{ik\cdot x} \tilde{\phi}_k(z) \quad \text{where} \quad \tilde{\phi}_k(z) = \frac{z^{2K_{\ell+2}(kz)}}{R^2K_{\ell+2}(kR)}.$$  \hfill (35)

We have chosen the normalization such that $\tilde{\phi}_k(z) = 1$ at $z = R$. The flux factor is evaluated by expanding

$$K_{\ell+2}(kz) = 2^{\ell+1} \Gamma(\ell+2)(kz)^{-2(\ell+2)} \left( 1 + \ldots + \frac{(-1)^\ell}{2^{2\ell+3}(\ell+1)!(\ell+2)!}(kz)^{2(\ell+2)} \ln kz + \ldots \right),$$

\hfill (36)
where in parenthesis we exhibit the leading non-analytic term. We find
\[
\mathcal{F} = \left[ \frac{1}{z^3} \partial_z \ln(\tilde{\phi}_k) \right]_{z=R} = \frac{(-1)^\ell}{2^{2\ell+2}(\ell + 1)!^2} k^{4+2\ell} R^{2\ell} \ln k R .
\]
(37)

As before, we have neglected terms containing analytic powers of $k$ and focused on the leading nonanalytic term. This formula indicates that the operator that couples to $\ell$-th partial wave has dimension $4 + \ell$. In [9] it was shown that such operators with the $SO(6)$ quantum numbers of the $\ell$-th partial wave have the form
\[
\int d^4x e^{ik \cdot x} \text{ tr} \left[ \left(X^{(i_1 \ldots i_\ell)} + \ldots\right) F_{\mu\nu} F^{\mu\nu}(x) \right] ,
\]
(38)
where in parenthesis we have a traceless symmetric tensor of $SO(6)$. Thus, supergravity predicts that their non-perturbative dimensions equal their bare dimensions.

Now let us consider massive string states. Our goal is to use supergravity to calculate the anomalous dimensions of the gauge theory operators that couple to them. To simplify the discussion, let us focus on excited string states which are spacetime scalars of mass $m$. The propagation equation for such a field in the background of the 3-brane geometry is
\[
\left[ \frac{d^2}{dr^2} + \frac{5}{r} \frac{d}{dr} - k^2 \left(1 + \frac{R^4}{r^4}\right) - m^2 \left(1 + \frac{R^4}{r^4}\right)^{1/2} \right] \tilde{\phi}_k = 0 .
\]
(39)

For the state at excitation level $n$,
\[
m^2 = \frac{4n}{\alpha'} .
\]

In the throat region, $z \gg R$, (39) simplifies to
\[
\left[ \frac{d^2}{dz^2} - \frac{3}{z} \frac{d}{dz} - k^2 - \frac{m^2 R^2}{z^2} \right] \tilde{\phi}_k = 0 .
\]
(40)

Note that a massive particle with small energy $\omega \ll m$, which would be far off shell in the asymptotic region $z \ll R$, can nevertheless propagate in the throat region (i.e. it is described by an oscillatory wave function).

Equation (40) is identical to the equation encountered in the analysis of higher partial waves, except the effective angular momentum is not in general an integer: in the centrifugal barrier term $\ell(\ell + 4)$ is replaced by $(mR)^2$. Analysis of the choice of wave function goes through as before, with $\ell + 2$ replaced by $\nu$, where
\[
\nu = \sqrt{4 + (mR)^2} .
\]
(41)
In other words, the wave function falling off exponentially for large $z$ and normalized to 1 at $z = R$ is $\tilde{\phi}_k(z) = \frac{z^2 K_\nu(kz)}{R^2 K_\nu(kR)}$.

Now we recall that

$$K_\nu = \frac{\pi}{2 \sin(\pi \nu)} (I_{-\nu} - I_\nu),$$

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k! \Gamma(k + \nu + 1)}.$$

Thus,

$$K_\nu(kz) = 2^{\nu-1} \Gamma(\nu)(kz)^{-\nu}\left(1 + \ldots - \left(\frac{zk}{2}\right)^{2\nu} \frac{\Gamma(1 - \nu)}{\Gamma(1 + \nu)} + \ldots\right),$$

where in parenthesis we have exhibited the leading non-analytic term. Calculating the flux factor as before, we find that the leading non-analytic term of the 2-point function is

$$\langle O(k)O(q)\rangle = -(2\pi)^4 \delta^4(k + q) \frac{N^2 \Gamma(1 - \nu)}{8\pi^2 \Gamma(\nu)} \left(\frac{kR}{2}\right)^{2\nu} R^{-4}.$$

This implies that the dimension of the corresponding SYM operator is equal to

$$\Delta = 2 + \nu = 2 + \sqrt{4 + (mR)^2}.$$

Now, let us note that

$$R^4 = 2Ng_{YM}^2(\alpha')^2,$$

which implies

$$(mR)^2 = 4ng_{YM}\sqrt{2N}.$$

Using (41) we find that the spectrum of dimensions for operators that couple to massive string states is, for large $g_{YM}\sqrt{N}$,

$$h_n \approx 2 \left(ng_{YM}\sqrt{2N}\right)^{1/2}.$$

Equation (46) is a new non-trivial prediction of the string theoretic approach to large $N$ gauge theory.\footnote{These solutions are reminiscent of the loop correlators calculated for $c \leq 1$ matrix models in [24].}

\footnote{We do not expect this equation to be valid for arbitrarily large $n$, because application of linearized local effective actions to arbitrarily excited string states is questionable. However, we should be able to trust our approach for moderately excited states.}
We conclude that, for large 't Hooft coupling, the anomalous dimensions of the vertex operators corresponding to massive string states grow without bound. By contrast, the vertex operators that couple to the massless string states do not acquire any anomalous dimensions. This has been checked explicitly for gravitons, dilatons and RR scalars [9, 10, 11], and we believe this to be a general statement. Thus, there are several $SO(6)$ towers of operators that do not acquire anomalous dimensions, such as the dilaton tower (38). The absence of the anomalous dimensions is probably due to the fact that they are protected by SUSY. The rest of the operators are not protected and can receive arbitrarily large anomalous dimensions. While we cannot yet write down the explicit form of these operators in the gauge theory, it seems likely that they are the conventional local operators, such as (3). Indeed, the coupling of a highly excited string state to the world volume may be guessed on physical grounds. Since a string in a D3-brane is a path of electric flux, it is natural to assume that a string state couples to a Wilson loop $O = \exp \left( i \oint \gamma A \right)$. Expansion of a small loop in powers of $F$ yields the local polynomial operators.

There is one potential problem with our treatment of massive states. The minimal linear equation (39) where higher derivative terms are absent may be true only for a particular field definition (otherwise corrections in positive powers of $\alpha' \nabla^2$ will be present in the equation). Therefore, it is possible that there are energy dependent leg factors relating the operators $O$ in (44) and the gauge theory operators of the form (3). We hope that these leg factors do not change our conclusion about the anomalous dimensions. However, to completely settle this issue we need to either find an exact sigma-model which incorporates all $\alpha'$ corrections or to calculate the 3-point functions of massive vertex operators.

4. Conclusions

There are many unanswered questions that we have left for the future. So far, we have considered the limit of large 't Hooft coupling, since we used the one-loop sigma-model calculations for all operators involved. If this coupling is not large, then we have to treat the world sheet theory as an exact conformal field theory (we stress, once again, that the string loop corrections are $\sim 1/N^2$ and, therefore, vanish in the large $N$ limit). This conformal field theory is a sigma-model on a hyperboloid. It is plausible that, in addition to the global $O(2, 4)$ symmetry, this theory possesses the $O(2, 4)$ Kac-Moody algebra. If this is the case, then the sigma model is tractable with standard methods of conformal field theory.

Throughout this work we detected many formal similarities of our approach with that used in $c \leq 1$ matrix models. These models may be viewed as early examples of gauge theory – non-critical string correspondence, with the large $N$ matrix models playing the role of gauge theories. Clearly, a deeper understanding of the connection between the present work and the $c \leq 1$ matrix models is desirable.

5 A more general set of such massless fields is contained in the supermultiplet of AdS gauge fields, whose boundary couplings were recently studied in [25].
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References

[1] A. M. Polyakov, “Gauge Fields as Rings of Glue,” *Nucl. Phys.* B164 (1980) 171.

[2] Y. M. Makeenko and A. A. Migdal, “Exact Equation for the Loop Average in Multicolor QCD,” *Phys. Lett.* 88B (1979) 135.

[3] G. ’t Hooft, “A planar diagram theory for strong interactions,” *Nucl. Phys.* B72 (1974) 461.

[4] A. M. Polyakov, “String theory and quark confinement,” hep-th/9711002.

[5] E. S. Fradkin and A. A. Tseytlin, “Effective Field Theory from Quantized Strings,” *Phys. Lett.* 158B (1985) 316.

[6] C. G. Callan, E. J. Martinec, M. J. Perry, and D. Friedan, “Strings in Background Fields,” *Nucl. Phys.* B262 (1985) 593.

[7] J. Polchinski, “Dirichlet Branes and Ramond-Ramond charges,” *Phys. Rev. Lett.* 75 (1995) 4724–4727, hep-th/9510017.

[8] S. S. Gubser, I. R. Klebanov, and A. W. Peet, “Entropy and temperature of black 3-branes,” *Phys. Rev.* D54 (1996) 3915–3919, hep-th/9602135.

[9] I. R. Klebanov, “World volume approach to absorption by nondilatonic branes,” *Nucl. Phys.* B496 (1997) 231, hep-th/9702076.

[10] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, “String theory and classical absorption by three-branes,” *Nucl. Phys.* B499 (1997) 217, hep-th/9703040.

[11] S. S. Gubser and I. R. Klebanov, “Absorption by branes and Schwinger terms in the world volume theory,” *Phys. Lett.* B413 (1997) 41, hep-th/9708005.
[12] G. T. Horowitz and A. Strominger, “Black strings and P-branes,” Nucl. Phys. B360 (1991) 197–209.

[13] E. Witten, “Bound states of strings and p-branes,” Nucl. Phys. B460 (1996) 335–350, hep-th/9510135.

[14] J. Maldacena, “The Large N limit of superconformal field theories and supergravity,” hep-th/9711200.

[15] J. Polchinski, “Critical behavior of random surfaces in one dimension,” Nucl. Phys. B346 (1990) 253–263.

[16] S. R. Das and A. Jevicki, “String Field Theory and Physical Interpretation of D = 1 Strings,” Mod. Phys. Lett. A5 (1990) 1639–1650.

[17] G. W. Gibbons and P. K. Townsend, “Vacuum interpolation in supergravity via super p-branes,” Phys. Rev. Lett. 71 (1993) 3754–3757, hep-th/9307049.

[18] G. W. Gibbons, G. T. Horowitz, and P. K. Townsend, “Higher dimensional resolution of dilatonic black hole singularities,” Class. Quant. Grav. 12 (1995) 297–318, hep-th/9410073.

[19] S. Hyun, “U duality between three-dimensional and higher dimensional black holes,” hep-th/9704005.

[20] H. J. Boonstra, B. Peeters and K. Skenderis, “Duality and asymptotic geometries,” Phys. Lett. B411 (1997) 59, hep-th/9706192.

[21] K. Sfetsos and K. Skenderis, “Microscopic derivation of the Bekenstein-Hawking entropy formula for nonextremal black holes,” hep-th/9711138.

[22] J. D. Brown and M. Henneaux, “Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity,” Commun. Math. Phys. 104 (1986) 207.

[23] J. Erdmenger and H. Osborn, “Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions,” Nucl. Phys. B483 (1997) 431–474, hep-th/9605009.

[24] G. Moore, N. Seiberg, and M. Staudacher, “From loops to states in 2-D quantum gravity,” Nucl. Phys. B362 (1991) 665–709.

[25] S. Ferrara and C. Fronsdal, “Conformal Maxwell Theory As a Singleton Field Theory on AdS5, IIB Threebranes and Duality,” hep-th/9605009.