HOMOGENIZATION OF THE DIRICHLET PROBLEM FOR ELLIPTIC SYSTEMS: TWO-PARAMETRIC ERROR ESTIMATES

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Abstract. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\Omega; C^n)$, we study a selfadjoint matrix elliptic second order differential operator $B_{D,\varepsilon}$, $0 < \varepsilon \leq 1$, with the Dirichlet boundary condition. The principal part of the operator is given in a factorized form. The operator involves lower order terms with unbounded coefficients. The coefficients of $B_{D,\varepsilon}$ are periodic and depend on $x/\varepsilon$. We study the generalized resolvent $(B_{D,\varepsilon} - \zeta Q_0(\cdot/\varepsilon))^{-1}$, where $Q_0$ is a periodic bounded and positive definite matrix-valued function, and $\zeta$ is a complex-valued parameter. We obtain approximations for the generalized resolvent in the $L_2(\Omega; C^n)$-operator norm and in the norm of operators acting from $L_2(\Omega; C^n)$ to the Sobolev space $H^1(\Omega; C^n)$, with two-parametric error estimates (depending on $\varepsilon$ and $\zeta$).

Introduction

The paper concerns homogenization theory of periodic differential operators (DO’s). A broad literature is devoted to homogenization problems. First of all, we mention the books [BeLPap, BaPa, OShaY, ZhKO]. In a series of papers [BSu1, BSu2, BSu3], M. Sh. Birman and T. A. Suslina developed an operator-theoretic (spectral) approach to homogenization problems. They studied the operator $B_{A,\varepsilon}$, and let $\Omega$ be the cell of $\Gamma$. For $\Gamma$-periodic functions in $\mathbb{R}^d$, we use the notation $\psi^\varepsilon(x) := \psi(x/\varepsilon)$ and $\overline{\psi} := |\Omega|^{-1} \int_{\Omega} \psi(x) \, dx$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\Omega; C^n)$, we consider a selfadjoint matrix strongly elliptic second order DO $B_{D,\varepsilon}$, $0 < \varepsilon \leq 1$, with the Dirichlet boundary condition. The principal part $A_{D,\varepsilon}$ of the operator $B_{D,\varepsilon}$ is given in a factorized form $A_{D,\varepsilon} = b(D)^* g^\varepsilon(x)b(D)$, where $b(D)$ is a matrix homogeneous first order DO and $g(x)$ is a bounded and positive definite $\Gamma$-periodic matrix-valued function in $\mathbb{R}^d$. (The precise assumptions on $b(D)$ and $g(x)$ are given below in Subsection 1.3.) The homogenization problem for the operator $A_{D,\varepsilon}$ was studied in [PSu, Su2, Su5]. In the present paper, we consider a more general class of selfadjoint DO’s $B_{D,\varepsilon}$ involving lower order terms:

$$B_{D,\varepsilon} = b(D)^* g^\varepsilon(x)b(D) + \sum_{j=1}^d (a_j^\varepsilon(x) D_j + D_j a_j^\varepsilon(x)^*) + Q^\varepsilon(x). \quad (0.1)$$

Here $a_j(x)$, $j = 1, \ldots, d$, and $Q(x)$ are $\Gamma$-periodic matrix-valued functions; in general, they are unbounded. (The precise assumptions on the coefficients are given below in Subsection 1.4.) The precise definition of the operator $B_{D,\varepsilon}$ is given in terms of the corresponding quadratic form defined on the Sobolev space $H^1_0(\Omega; C^n)$.

The coefficients of the operator (0.1) oscillate rapidly for small $\varepsilon$. A typical homogenization problem for the operator $B_{D,\varepsilon}$ is to approximate the resolvent $(B_{D,\varepsilon} - zI)^{-1}$ or the generalized resolvent $(B_{D,\varepsilon} - zQ_0)^{-1}$ for small $\varepsilon$. Here $Q_0(x)$ is a positive definite and bounded $\Gamma$-periodic matrix-valued function.

0.2. A survey of the results on the operator error estimates. In a series of papers [BSu1, BSu2, BSu3], M. Sh. Birman and T. A. Suslina developed an operator-theoretic (spectral) approach to homogenization problems. They studied the operator

$$A_{\varepsilon} = b(D)^* g^\varepsilon(x)b(D) \quad (0.2)$$

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acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. In [BSu1], it was shown that the resolvent $(A_\varepsilon + I)^{-1}$ converges in the $L_2(\mathbb{R}^d; \mathbb{C}^n)$-operator norm to the resolvent of the effective operator $A^0 = b(D)^*g^0b(D)$, as $\varepsilon \to 0$. Here $g^0$ is a constant positive effective matrix. It was proved that
\[
\| (A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon. \tag{0.3}
\]

In [BSu3], approximation for the resolvent $(A_\varepsilon + I)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d; \mathbb{C}^n)$ was obtained:
\[
\| (A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon) \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C\varepsilon. \tag{0.4}
\]

Here $K(\varepsilon)$ is a corrector. The operator $K(\varepsilon)$ involves rapidly oscillating factors and so depends on $\varepsilon$. Herewith, $\|\varepsilon K(\varepsilon)\|_{L_2 \to H^1} = O(1)$. Estimates (0.3) and (0.4) are order-sharp. The constants in estimates are controlled explicitly in terms of the problem data. Such inequalities are called operator error estimates in homogenization theory. The method of [BSu1] [BSu2] [BSu3] is based on the scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory.

Later the spectral method was adapted by T. A. Suslina [Su1] [Su4] to the case of the operator
\[
B_\varepsilon = A_\varepsilon + \sum_{j=1}^d (a_j^\varepsilon(x)D_j + D_j(a_j^\varepsilon(x))^*) + Q^\varepsilon(x) \tag{0.5}
\]
acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. It is convenient to fix a real-valued parameter $\lambda$ so that the operator $B_\varepsilon := B_\varepsilon + \lambda Q_0^\varepsilon$ is positive definite. In [Su1], the following analogs of estimates (0.3) and (0.4) were obtained:
\[
\| B_\varepsilon^{-1} - (B^0)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon, \tag{0.6}
\]
\[
\| B_\varepsilon^{-1} - (B^0)^{-1} - \varepsilon K(\varepsilon) \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C\varepsilon. \tag{0.7}
\]

Here $B^0$ is the corresponding effective operator and $K(\varepsilon)$ is the corresponding corrector.

A different approach to operator error estimates was suggested by V. V. Zhikov. In [Zh1] [Zh2] [ZhPas1], estimates of the form (0.3) and (0.4) were obtained for the acoustics operator and the operator of elasticity theory. The method (“modified method of first order approximation” or “shift method”) was based on analysis of the first order approximation to the solution and introduction of an additional parameter. In [Zh1] [Zh2] [ZhPas1], in addition to problems in $\mathbb{R}^d$, homogenization problems in a bounded domain $O \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary conditions were studied. Further results of V. V. Zhikov, S. E. Pastukhova, and their collaborators are discussed in the recent survey [ZhPas2].

In the presence of lower order terms, homogenization problem for the operator (0.5) in $\mathbb{R}^d$ was studied by D. I. Borisov [Bo] (this work precedes [Su1]). The effective operator was constructed and error estimates (0.6), (0.7) were obtained. Moreover, it was assumed that the coefficients depend on both fast and slow variables. However, in [Bo] the coefficients of the operator $B_\varepsilon$ were assumed to be sufficiently smooth. We also mention the very recent paper [Sc] by N. N. Senik, where the non-selfadjoint second order elliptic operator (involving lower order terms) on an infinite cylinder was studied. The coefficients oscillate along the cylinder and belong to some classes of multipliers; estimates of the form (0.6) and (0.7) were obtained.

Operator error estimates for the Dirichlet and Neumann problems for second order elliptic equations (without lower order terms) in a bounded domain with sufficiently smooth boundary were studied by many authors. Apparently, the first result is due to Shi. Moskow and M. Vogelius who proved an estimate
\[
\| A_{D,\varepsilon}^{-1} - (A_D^0)^{-1} \|_{L_2(O) \to L_2(O)} \leq C\varepsilon, \tag{0.8}
\]
see [MoV1, Corollary 2.2]. Here the operator $A_{D,\varepsilon}$ acts in $L_2(O)$, where $O \subset \mathbb{R}^2$, and is given by $-\text{div} g_\varepsilon(x)\nabla$ with the Dirichlet condition on $\partial O$. The matrix-valued function $g_\varepsilon(x)$ is assumed to be infinitely smooth. In the case of the Neumann boundary condition, a similar estimate was obtained in [MoV2, Corollary 1]. Also, in that paper the authors found approximation with corrector for the inverse operator in the norm of operators acting from $L_2(O)$ to the Sobolev space $H^1(O)$, with error estimate of order $O(\sqrt{\varepsilon})$. The order of this estimate is worse than in $\mathbb{R}^d$ because of the boundary influence.
For arbitrary dimension, homogenization problems in a bounded domain with sufficiently smooth boundary were studied in [Zh1][Zh2], and [ZhPas1]. The acoustics and elasticity operators with the Dirichlet or Neumann boundary conditions and without any smoothness assumptions on coefficients were considered. The authors obtained approximation with corrector for the inverse operator in the \((L_2 \to H^1)\)-norm with error estimate of order \(O(\sqrt{\varepsilon})\). The analog of estimate (0.8), but of order \(O(\sqrt{\varepsilon})\), was deduced. (In the case of the Dirichlet problem for the acoustics equation, the \((L_2 \to L_2)\)-estimate was improved in [ZhPas1], but the order was not sharp.) Similar results for the operator \(-\text{div} \ g'(x)\nabla\) in a smooth bounded domain \(\Omega \subset \mathbb{R}^d\) with the Dirichlet or Neumann boundary conditions were obtained by G. Griso [Gr1, Gr2] with the help of the “unfolding” method. In [Gr2], sharp-order estimate (0.8) (for the same operator) was proved. For elliptic systems similar results were independently obtained in [KeLiS] and in [PSu, Su2]. Further results and a detailed survey can be found in [Su3, Su4]. Let us only mention the forthcoming paper [SZ], where estimate of the form (0.8) and the \((L_2 \to H^1)\)-approximation were obtained for the elasticity operator with mixed (Dirichlet and Neumann) boundary conditions.

Operator error estimates for the second order matrix elliptic operator (with lower order terms) in a bounded domain \(\Omega\) with \(\varepsilon\)-parametric error estimates (with respect to \(\varepsilon\)) depending on \(\varepsilon\) and \(\xi \in \mathbb{C} \setminus \mathbb{R}_+\) were recently obtained by T. A. Suslina [Su5]. In [Su4], the operators \(A_{D,\varepsilon}\) and \(A_{N,\varepsilon}\) given by the expression (0.2) in a bounded domain with the Dirichlet or Neumann boundary conditions were also studied. Approximations for the resolvents of these operators with two-parametric error estimates (with respect to \(\varepsilon\) and \(\xi\)) were obtained. Note that investigation of the two-parametric estimates was stimulated by the study of homogenization for parabolic systems, based on the following representation of the operator exponential

\[
e^{-A_{D,\varepsilon}t} = -\frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t} (A_{D,\varepsilon} - \zeta)^{-1} d\zeta, \tag{0.9}\]

where \(\gamma \subset \mathbb{C}\) is a positively oriented contour enclosing the spectrum of \(A_{D,\varepsilon}\). (A similar representation holds for \(e^{-A_{N,\varepsilon}t}\).) Details can be found in [MSu2].

The present paper relies on the following two-parametric estimates for the operator \(B_{\varepsilon}\) obtained in [MSu1]:

\[
\| (B_{\varepsilon} - \zeta Q_0^{-1})^{-1} - (B_0 - \zeta Q_0^{-1})^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\varepsilon)|\zeta|^{-1/2}, \tag{0.10}\]

\[
\| (B_{\varepsilon} - \zeta Q_0^{-1})^{-1} - (B_0 - \zeta Q_0^{-1})^{-1} - \varepsilon K(\varepsilon; \zeta) \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C(\varepsilon). \tag{0.11}\]

Here \(\phi = \arg \zeta \in (0, 2\pi)\) and \(|\zeta| \geq 1\). The dependence of constants in estimates on \(\phi\) is traced. Estimates (0.10) and (0.11) are uniform with respect to \(\phi\) in any domain of the form

\[
\{ \zeta = |\zeta| e^{i\phi} \in \mathbb{C} : |\zeta| > 1, \phi_0 \leq \phi \leq 2\pi - \phi_0 \} \tag{0.12}\]

with arbitrarily small \(\phi_0 > 0\). (In [MSu1], error estimates in the case where \(\phi \in (0, 2\pi)\) and \(|\zeta| < 1\) were also obtained.) For details, see Section II below.

0.3. Main results. Before we formulate the results, it is convenient to turn to the positive definite operator \(B_{D,\varepsilon} = B_{D,\varepsilon} + \lambda Q_0^{-1}\) choosing an appropriate constant \(\lambda\). Let \(B_0^D\) be the corresponding effective operator. Main results are the following estimates:

\[
\| (B_{D,\varepsilon} - \zeta Q_0^{-1})^{-1} - (B_0^D - \zeta Q_0^{-1})^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\varepsilon)|\zeta|^{-1/2}, \tag{0.13}\]

\[
\| (B_{D,\varepsilon} - \zeta Q_0^{-1})^{-1} - (B_0^D - \zeta Q_0^{-1})^{-1} - \varepsilon K_D(\varepsilon; \zeta) \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C(\varepsilon)(\varepsilon^{1/2}|\zeta|^{-1/4} + \varepsilon), \tag{0.14}\]

for \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+, |\zeta| \geq 1\), and sufficiently small \(\varepsilon\). The constants \(C(\varepsilon)\) are controlled explicitly in terms of the problem data and the angle \(\phi = \arg \zeta\). Estimates (0.13) and (0.14) are uniform with respect to \(\phi\) in any domain (0.12) with arbitrarily small \(\phi_0 > 0\).

For fixed \(\xi\), estimate (0.13) has sharp order \(O(\varepsilon)\). The order of estimate (0.14) is worse than in \(\mathbb{R}^d\) (see (0.11)) because of the boundary influence. The order of the \((L_2 \to H^1)\)-estimate can
be improved up to the sharp order $O(\varepsilon)$ by passing to a strictly interior subdomain or by taking into account the boundary layer correction term. (See Theorems 2.7 and 8.1 below.)

In the general case, the corrector in (0.14) contains a smoothing operator. We distinguish the cases where a simpler corrector can be used. Besides estimates for the generalized resolvent, we also find approximation in the $(L_2 \to L_2)$-norm for the operator $g^* b(D)(B_{D, \varepsilon} - \zeta Q_0^{\varepsilon})^{-1}$ corresponding to the flux. For completeness, we find approximations for the generalized resolvent in a larger domain of the parameter $\zeta$; the corresponding error estimates have a different behavior with respect to $\zeta$. (See Section 9 below.)

When this work was finished, the authors learned about very recent paper [Xu3], where close results were obtained. Let us compare the results. On the one hand, there are some advantages of our research. First, we study the operator (0.11) which is strongly elliptic, while in [Xu3] (as well as in [KeLiS, Xu1, Xu2]) a rather restrictive condition of uniform ellipticity is imposed. Second, we admit lower order terms with unbounded coefficients (from appropriate $L_p(\Omega)$-classes), while in [Xu3] these coefficients are assumed to be bounded. Third, we obtain two-parametric error estimates (with respect to $\varepsilon$ and $\zeta$), while in [Xu3] estimates are one-parametric (with respect to $\varepsilon$). On the other hand, there are several advantages of [Xu3]: some results are obtained in the case of Lipschitz domains; the operator may be non-selfadjoint (only the principal part of the operator is assumed to be selfadjoint).

### 0.4. Method.

The proofs rely on the method developed in [PSu, Su2, Su5], it is based on consideration of the associated problem in $\mathbb{R}^d$, application of the results (0.10), (0.11) (obtained in [MSu1]), introduction of the boundary layer correction term, and a careful analysis of this term. We base our argument upon the employment of the Steklov smoothing operator (borrowed from [ZhPas1]) and estimates in the $\varepsilon$-neighborhood of the boundary. We trace the dependence of estimates on the spectral parameter carefully. Additional technical difficulties (as compared with [Su3]) are related to taking lower order terms with unbounded coefficients into account. First we prove estimate (0.14), and next we prove (0.13) using (0.14) and the duality arguments.

Approximations in a larger domain of the parameter $\zeta$ are deduced from the already proved estimates at the point $\zeta = -1$ and appropriate resolvent identities.

The results of the present paper will be applied to study homogenization for the solution $u_{\varepsilon}(x, t), x \in \Omega, t \geq 0$, of the first initial boundary value problem:

\[
\begin{cases}
Q_0^{\varepsilon}(x)\partial_t u_{\varepsilon}(x, t) = -B_{D, \varepsilon} u_{\varepsilon}(x, t), & x \in \Omega, \; t > 0; \\
u_{\varepsilon}(x, t) = 0, & x \in \partial \Omega, \; t > 0; \quad Q_0^{\varepsilon}(x)u_{\varepsilon}(x, 0) = \varphi(x),
\end{cases}
\]

where $\varphi \in L_2(\Omega; \mathbb{C}^n)$. A separate paper [MSu3] will be devoted to this subject. The method will be based on using the representation

\[
u_{\varepsilon}^{\gamma}(\cdot, t) = -\frac{1}{2\pi i} \int_\gamma e^{-\zeta t}(B_{D, \varepsilon} - \zeta Q_0^{\varepsilon})^{-1}\varphi d\zeta,
\]

where $\gamma \subset \mathbb{C}$ is a suitable contour; cf. (0.9).

### 0.5. Plan of the paper.

The paper consists of eleven sections. In Section 1 we introduce the class of operators $B_{\varepsilon}$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and formulate the results about approximations for the generalized resolvent $(B_{\varepsilon} - \zeta Q_0^{\varepsilon})^{-1}$ obtained in [MSu1]. In Section 2 the class of operators $B_{D, \varepsilon}$ is described, the effective operator $B_D^0$ is defined, and the main results are formulated. Section 3 contains auxiliary material. In Section 4 we prove the $(L_2 \to H^1)$-approximation with the boundary layer correction term. In Section 5 approximation (0.14) and approximation for the flux are obtained. In Section 6 the $(L_2 \to L_2)$-estimate (0.13) is proved. In Section 7 we distinguish the cases where the smoothing operator can be removed. In Section 8 we find approximations for the generalized resolvent in a strictly interior subdomain. Estimates in a larger domain of the spectral parameter are obtained in Section 9. Section 10 contains more results: mainly, they concern some improvements of the behavior of the right-hand sides in estimates with respect to $\phi = \arg \zeta$. Applications of the general results can be found in Section 11.

### 0.6. Notation.

Let $\mathcal{H}$ and $\mathcal{H}_s$ be complex separable Hilbert spaces. The symbols $(\cdot, \cdot)_\mathcal{H}$ and $\| \cdot \|_\mathcal{H}$ stand for the inner product and the norm in $\mathcal{H}$, respectively. The symbol $\| \cdot \|_{\mathcal{H}_s}$ denotes the norm of a linear continuous operator acting from $\mathcal{H}$ to $\mathcal{H}_s$. 
The symbols $\langle \cdot, \cdot \rangle$ and $| \cdot |$ stand for the inner product and the norm in $\mathbb{C}^n$, $1_n$ is the unit $(n \times n)$-matrix. For an $(m \times n)$-matrix $a$, the symbol $|a|$ denotes the norm of $a$ viewed as a linear operator from $\mathbb{C}^n$ to $\mathbb{C}^m$. For $z \in \mathbb{C}$, we denote by $z^n$ the complex conjugate number; this nonstandard notation is employed because we write $\mathbb{F}$ for the mean value of a periodic function $g$. Next, we use the notation $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1, \ldots, d$, $\mathbf{D} = -i \nabla = (D_1, \ldots, D_d)$. The $L_p$-classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O}; \mathbb{C}^n)$, $1 \leq p \leq \infty$. The Sobolev spaces of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $H^s(\mathcal{O}; \mathbb{C}^n)$. Next, $H^1_0(\mathcal{O}; \mathbb{C}^n)$ is the closure of $\mathbb{C}^\infty(\mathcal{O}; \mathbb{C}^n)$ in $H^1(\mathcal{O}; \mathbb{C}^n)$. If $n = 1$, we write simply $L_p(\mathcal{O})$, $H^s(\mathcal{O})$, etc., but sometimes, if this does not lead to confusion, we use this short notation also for spaces of vector-valued or matrix-valued functions.

We use the notation $\mathbb{R}_+ = [0, \infty)$. Different constants in estimates are denoted by $c, c, C, C, c, \beta, \gamma, k, \kappa$ (possibly, with indices or marks).

1. HOMOGENIZATION PROBLEM FOR ELLIPTIC OPERATOR ACTING IN $L_2(\mathbb{R}^d; \mathbb{C}^n)$

1.1. Lattices in $\mathbb{R}^d$. Let $\mathbb{G} \subset \mathbb{R}^d$ be a lattice generated by a basis $\mathbf{a}_1, \ldots, \mathbf{a}_d \in \mathbb{R}^d$, i.e., $\mathbb{G} = \{ \mathbf{a} \in \mathbb{R}^d : \mathbf{a} = \sum_{j=1}^d \nu_j \mathbf{a}_j, \nu_j \in \mathbb{Z}, \}$, and let $\Omega$ be the elementary cell of $\mathbb{G}$: $\Omega = \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \sum_{j=1}^d \tau_j \mathbf{a}_j, -\frac{1}{2} \leq \tau_j < \frac{1}{2} \}$. By $|\Omega|$ we denote the Lebesgue measure of $\Omega$: $|\Omega| = \text{meas} \, \Omega$. The basis $\mathbf{b}_1, \ldots, \mathbf{b}_d$ in $\mathbb{R}^d$ dual to the basis $\mathbf{a}_1, \ldots, \mathbf{a}_d$ is defined by the relations $\langle \mathbf{b}_i, \mathbf{a}_j \rangle = 2\pi \delta_{ij}$. This basis generates the lattice dual to $\Gamma$. Denote $2r_0 := \min_{0 \neq \mathbf{b} \in \mathbb{G}} |\mathbf{b}|$ and $2r_1 := \text{diam} \, \Omega$.

By $\mathcal{H}^1(\Omega)$ we denote the subspace of all functions in $H^1(\Omega)$ whose $\Gamma$-periodic extension to $\mathbb{R}^d$ belongs to $H^1_{\text{loc}}(\mathbb{R}^d)$. If $h(\mathbf{x})$ is a $\Gamma$-periodic measurable matrix-valued function in $\mathbb{R}^d$, we put $h^\varepsilon(\mathbf{x}) := h(\mathbf{x}/\varepsilon), \varepsilon > 0$; $\mathcal{H} := \{ |\Omega|^{-1} \int_\Omega h(\mathbf{x}) \, d\mathbf{x}, \hbar := (|\Omega|^{-1} \int_\Omega h(\mathbf{x})^{-1} d\mathbf{x})^{-1} \}. Here, in the definition of $\mathcal{H}$ it is assumed that $h \in L_{1,\text{loc}}(\mathbb{R}^d)$, and in the definition of $\mathcal{H}$ it is assumed that the matrix $h(\mathbf{x})$ is square and nondegenerate, and $h^{-1} \in L_{1,\text{loc}}(\mathbb{R}^d)$. By $[h^\varepsilon]$ we denote the operator of multiplication by the matrix-valued function $h^\varepsilon(\mathbf{x})$.

1.2. The Steklov smoothing. The Steklov smoothing operator $S^k_\varepsilon(\mathbf{x}), \varepsilon > 0$, acts in $L_2(\mathbb{R}^d; \mathbb{C}^k)$ (where $k \in \mathbb{N}$) and is given by

$$
(S^k_\varepsilon \mathbf{u})(\mathbf{x}) := |\Omega|^{-1} \int_\Omega \mathbf{u}(\mathbf{x} - \varepsilon \mathbf{z}) \, d\mathbf{z}, \quad \mathbf{u} \in L_2(\mathbb{R}^d; \mathbb{C}^k). \tag{1.1}
$$

We shall omit the index $k$ in the notation and write simply $S_\varepsilon$. Obviously, $S_\varepsilon D^\alpha \mathbf{u} = D^\alpha S_\varepsilon \mathbf{u}$ for $\mathbf{u} \in H^s(\mathbb{R}^d; \mathbb{C}^k)$ and any multindex $\alpha$ such that $|\alpha| \leq s$. Note that

$$
\| S_\varepsilon \|_{H^s(\mathbb{R}^d) \rightarrow H^s(\mathbb{R}^d)} \leq 1, \quad l \in \mathbb{Z}+. \tag{1.2}
$$

We need the following properties of the operator $S_\varepsilon$ (see [ZhPas] Lemmas 1.1 and 1.2) or [PSu] Propositions 3.1 and 3.2).

**Proposition 1.1.** For any function $\mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^k)$ we have

$$
\| S_\varepsilon \mathbf{u} - \mathbf{u} \|_{L_2(\mathbb{R}^d)} \leq \varepsilon r_1 \| D\mathbf{u} \|_{L_2(\mathbb{R}^d)}, \quad \varepsilon > 0.
$$

**Proposition 1.2.** Let $h$ be a $\Gamma$-periodic function in $\mathbb{R}^d$ such that $h \in L_2(\Omega)$. Then the operator $[h^\varepsilon]S_\varepsilon$ is continuous in $L_2(\mathbb{R}^d; \mathbb{C}^k)$, and

$$
\| [h^\varepsilon]S_\varepsilon \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq |\Omega|^{-1/2} \| h \|_{L_2(\Omega)}, \quad \varepsilon > 0.
$$

1.3. The class of operators $A_\varepsilon$. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the operator $A_\varepsilon$ given by the differential expression $A_\varepsilon = b(\mathbf{D})^\ast [g(\mathbf{x})] b(\mathbf{D})$. Here $g(\mathbf{x})$ is a $\Gamma$-periodic $(m \times m)$-matrix-valued function (in general, with complex entries). We assume that $g(\mathbf{x}) > 0$ and $g^{-1} \in L_\text{loc}(\mathbb{R}^d)$. Next, $b(\mathbf{D})$ is the DO given by $b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$, where $b_j, j = 1, \ldots, d$, are constant $(m \times n)$-matrices (in general, with complex entries). It is assumed that $m \geq n$ and that the symbol $b(\xi) = \sum_{j=1}^d b_j \xi_j$ of the operator $b(\mathbf{D})$ has maximal rank: rank $b(\xi) = n$ for $0 \neq \xi \in \mathbb{R}^d$. This condition is equivalent to the estimates

$$
a_0 1_n \leq b(\mathbf{\theta})^\ast b(\mathbf{\theta}) \leq a_1 1_n, \quad \mathbf{\theta} \in \mathbb{S}^{d-1}, \quad 0 < a_0 \leq a_1 < \infty, \tag{1.3}
$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. This condition is equivalent to the estimates
with some positive constants \( \alpha_0 \) and \( \alpha_1 \). From (1.3) it follows that
\[
|b_j| \leq \alpha_1^{1/2}, \quad j = 1, \ldots, d.
\] (1.4)

The precise definition of the operator \( A_\varepsilon \) is given in terms of the quadratic form
\[
a_\varepsilon[u, u] = \int_{\mathbb{R}^d} (g^\varepsilon(x) b(D)u, b(D)u) \, dx, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \]
Under the above assumptions, this form is closed and nonnegative. Using the Fourier transformation and condition (1.3), it is easy to check that
\[
\alpha_0 \|g^{-1}\|_{L^\infty}^2 \|Du\|_{L^2(\mathbb{R}^d)}^2 \leq a_\varepsilon[u, u] \leq \alpha_1 \|g\|_{L^\infty} \|Du\|_{L^2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \] (1.5)
Let \( c_1 := \alpha_0^{-1/2}\|g^{-1}\|_{L^\infty}^{1/2} \). Then the lower estimate (1.5) can be written as
\[
\|Du\|_{L^2(\mathbb{R}^d)}^2 \leq c_1^2 a_\varepsilon[u, u], \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \] (1.6)

1.4. **The operator** \( B_\varepsilon \). We study a self-adjoint operator \( B_\varepsilon \) whose principal part coincides with \( A_\varepsilon \). To define the lower order terms, we introduce \( \Gamma \)-periodic \((n \times n)\)-matrix-valued functions \( a_j, j = 1, \ldots, d, \) (in general, with complex entries) such that
\[
a_j \in L_\rho(\Omega), \quad \rho = 2 \text{ for } d = 1, \quad \rho > d \text{ for } d \geq 2, \quad j = 1, \ldots, d. \] (1.7)
Next, let \( Q \) and \( Q_0 \) be \( \Gamma \)-periodic Hermitian \((n \times n)\)-matrix-valued functions (with complex entries) such that
\[
Q \in L_\varepsilon(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2; \quad Q_0(x) > 0; \quad Q_0^{-1} \in L^\infty(\mathbb{R}^d). \] (1.8)
(Our assumptions on \( Q \) correspond to Example 2.4 from [MSu1].) For convenience of further references, the following set of parameters is called the “initial data”:
\[
d, m, n, \rho, s, \alpha_0, \alpha_1, \|g\|_{L^\infty}, \|g^{-1}\|_{L^\infty}, \|a_j\|_{L^\varepsilon(\Omega)}, j = 1, \ldots, d; \quad \|Q\|_{L^\varepsilon(\Omega)}; \quad \|Q_0\|_{L^\infty}, \|Q_0^{-1}\|_{L^\infty}; \quad \text{the parameters of the lattice } \Gamma. \] (1.9)

Consider the following quadratic form
\[
b_\varepsilon[u, u] = a_\varepsilon[u, u] + 2\text{Re} \sum_{j=1}^d (a_j^\varepsilon D_j u, u)_{L_2(\mathbb{R}^d)} + ((Q + \lambda Q_0^\varepsilon) u, u)_{L_2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n). \] (1.10)
We fix a constant \( \lambda \) so that the form \( b_\varepsilon \) is nonnegative (see (1.14) below). Let us check that the form \( b_\varepsilon \) is closed. By the Hölder inequality and the Sobolev embedding theorem, it is easily seen (see [Su1] (5.11)–(5.14)) that for any \( \nu > 0 \) there exist constants \( C_\nu(\nu) > 0 \) such that
\[
\|a_j^\varepsilon u\|_{L^2(\mathbb{R}^d)}^2 \leq \nu \|Du\|_{L^2(\mathbb{R}^d)}^2 + C_\nu(\nu)\|u\|_{L^2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad j = 1, \ldots, d. \]
Using the change of variable \( y := \varepsilon^{-1}x \) and denoting \( u(x) =: v(y) \), we deduce
\[
\|(a_j^\varepsilon)^* u\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |a_j(\varepsilon^{-1}x)^* u(x)|^2 \, dx = \varepsilon^d \int_{\mathbb{R}^d} |a_j(y)^* v(y)|^2 \, dy \leq \varepsilon^d \nu \int_{\mathbb{R}^d} |D_y v(y)|^2 \, dy + \varepsilon^d C_\nu(\nu) \int_{\mathbb{R}^d} |v(y)|^2 \, dy \leq \nu \|Du\|_{L^2(\mathbb{R}^d)}^2 + C_\nu(\nu)\|u\|_{L^2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.
\]
Hence, by (1.3), for any \( \nu > 0 \) there exists a constant \( C(\nu) > 0 \) such that
\[
\sum_{j=1}^d \|(a_j^\varepsilon)^* u\|_{L^2(\mathbb{R}^d)}^2 \leq \nu a_\varepsilon[u, u] + C(\nu)\|u\|_{L^2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1. \] (1.11)
If \( \nu \) is fixed, then \( C(\nu) \) depends only on \( d, \rho, \alpha_0, \) the norms \( \|g^{-1}\|_{L^\infty}, \|a_j\|_{L^\varepsilon(\Omega)}, j = 1, \ldots, d, \) and the parameters of the lattice \( \Gamma \). From (1.6) and (1.11) it follows that
\[
2 \text{Re} \sum_{j=1}^d (D_j u, (a_j^\varepsilon)^* u)_{L_2(\mathbb{R}^d)} \leq \frac{1}{4} a_\varepsilon[u, u] + c_2\|u\|_{L^2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1. \] (1.12)
where \( c_2 := 8c_1^2 C(\nu_0) \) with \( \nu_0 := 2^{-6} a_0 ||g^{-1}||_{L^1}^{-1} \).

Next, by condition (1.8) on \( Q \), for any \( \nu > 0 \) there exists a constant \( C_Q(\nu) > 0 \) such that
\[
||Q^2 u, u)||_2(\mathbb{R}^d) \leq \nu ||Du||^2_2(\mathbb{R}^d) + C_Q(\nu) ||u||^2_{L^1(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; C^n), \quad 0 < \varepsilon \leq 1. \tag{1.13}
\]

For \( \nu \) fixed, \( C_Q(\nu) \) is controlled in terms of \( d, s, ||Q||_{L^s(\Omega)} \), and the parameters of the lattice \( \Gamma \).

As in [MSu1, Subsection 2.8], we fix \( \lambda \) in (1.10) as follows:
\[
\lambda := (C_Q(\nu_s) + c_2)||Q_0^{-1}||_{L^\infty} \quad \text{for} \quad \nu_s := 2^{-1} a_0 ||g^{-1}||_{L^\infty}^{-1} \tag{1.14}
\]

Combining (1.6), (1.12), and (1.13) with \( \nu = \nu_s \), and (1.14), we deduce the following lower estimate for the form (1.10):
\[
b_\varepsilon[u, u] \geq \frac{1}{4} a_\varepsilon[u, u] \geq c_\varepsilon ||D^2 u||^2_{L^s(\mathbb{R}^d)}; \quad u \in H^1(\mathbb{R}^d; C^n); \quad c_\varepsilon := \frac{1}{4} a_0 ||g^{-1}||^{-1}_{L^\infty}. \tag{1.15}
\]

From (1.5), (1.12), and (1.13) with \( \nu = 1 \) it follows that
\[
b_\varepsilon[u, u] \leq c_\varepsilon ||u||^2_{H^1(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d; C^n), \tag{1.16}
\]

where \( c_\varepsilon := \max\{\frac{\varepsilon}{4} a_1 ||g||_{L^\infty} + 1; C_Q(1) + \lambda ||Q_0||_{L^\infty} + c_2\} \).

Thus, the form \( b_\varepsilon \) is closed and nonnegative. The selfadjoint operator in \( L_2(\mathbb{R}^d; C^n) \) generated by this form is denoted by \( B_\varepsilon \). Formally, we have
\[
B_\varepsilon = b(D)^* g^\varepsilon(x) b(D) + \sum_{j=1}^d (a_j^\varepsilon(x) D_j + D_j a_j^\varepsilon(x))^* + Q^\varepsilon(x) + \lambda Q_0^\varepsilon(x). \tag{1.17}
\]

1.5. **The effective matrix.** The effective operator for \( A_\varepsilon = b(D)^* g^\varepsilon(x) b(D) \) is given by \( A^0 = b(D)^* g^0 b(D) \), where \( g^0 \) is a constant \((m \times m)\)-matrix called the effective matrix. Suppose that a \( \Gamma \)-periodic \((m \times m)\)-matrix-valued function \( \Lambda(x) \) is the (weak) solution of the problem
\[
b(D)^* g(x)(b(D)\Lambda(x) + 1m) = 0, \quad \int_\Omega \Lambda(x) \, dx = 0. \tag{1.18}
\]

Then the effective matrix is given by
\[
g^0 = |\Omega|^{-1} \int_\Omega \tilde{g}(x) \, dx, \tag{1.19}
\]

\[
\tilde{g}(x) := g(x)(b(D)\Lambda(x) + 1m). \tag{1.20}
\]

From (1.18) it easily follows that
\[
||b(D)\Lambda||_{L^2(\Omega)} \leq ||\Omega||^{1/2} m^{1/2} ||g||_1^{1/2} ||g^{-1}||_1^{1/2}. \tag{1.21}
\]

We also need the following estimates proved in [BSu2, (6.28) and Subsection 7.3]:
\[
||\Lambda||_{L^2(\Omega)} \leq ||\Omega||^{1/2} M_1; \quad M_1 := m^{1/2}(2\alpha_0)^{-1} ||g||_1^{1/2} ||g^{-1}||_1^{1/2}, \tag{1.22}
\]
\[
||DA\Lambda||_{L^2(\Omega)} \leq ||\Omega||^{1/2} M_2; \quad M_2 := m^{1/2} \alpha_0^{-1/2} ||g||_1^{1/2} ||g^{-1}||_1^{1/2}. \tag{1.23}
\]

It can be checked that \( g^0 \) is positive definite. The effective matrix satisfies the estimates known as the Voigt–Reuss bracketing (see, e. g., [BSu1, Chapter 3, Theorem 1.5]).

**Proposition 1.3.** Let \( g^0 \) be the effective matrix (1.19). Then
\[
g \leq g^0 \leq \tilde{g}. \tag{1.24}
\]

If \( m = n \), then \( g^0 = g \).

Inequalities (1.24) imply that
\[
|g^0| \leq ||g||_{L^\infty}, \quad |(g^0)^{-1}| \leq ||g^1||_{L^\infty}. \tag{1.25}
\]

Now we distinguish the cases where one of the inequalities in (1.24) becomes an identity, see [BSu1, Chapter 3, Propositions 1.6 and 1.7].

**Proposition 1.4.** The identity \( g^0 = \tilde{g} \) is equivalent to the relations
\[
b(D)^* g_j(x) = 0, \quad j = 1, \ldots, m, \tag{1.26}
\]

where \( g_j(x), j = 1, \ldots, m, \) are the columns of the matrix \( g(x) \).
The effective operator. In order to define the effective operator for $B_ε$, consider a $Γ$-periodic $(n \times n)$-matrix-valued function $\tilde{Λ}(x)$ which is the (weak) solution of the problem
\[
b(D)^* g(x) b(D) \tilde{Λ}(x) + \sum_{j=1}^{d} D_j a_j(x) \tilde{Λ}(x) = 0, \quad \int_{\Omega} \tilde{Λ}(x) \, dx = 0.
\]
(1.28)

The following estimates for $\tilde{Λ}$ were proved in [Su6] (7.49)–(7.52):
\[
\|\tilde{Λ}\|_{L_2(Ω)} \leq (2α_0)^{-1} C_a n^{1/2} \|g^{-1}\|_{L_∞},
\]
(1.29)
\[
\|D \tilde{Λ}\|_{L_2(Ω)} \leq C_a n^{1/2} \|g^{-1}\|_{L_∞},
\]
(1.30)
\[
\|b(D) \tilde{Λ}\|_{L_2(Ω)} \leq C_a n^{1/2} \|g^{-1}\|_{L_∞},
\]
(1.31)

where $C_a := \sum_{j=1}^{d} \int_{\Omega} |a_j(x)|^2 \, dx$. Next, we define constant matrices $V$ and $W$ as follows:
\[
V := |Ω|^{-1} \int_{Ω} (b(D) \tilde{Λ}(x))^* g(x) (b(D) \tilde{Λ}(x)) \, dx,
\]
(1.32)
\[
W := |Ω|^{-1} \int_{Ω} (b(D) \tilde{Λ}(x))^* g(x) (b(D) \tilde{Λ}(x)) \, dx.
\]
(1.33)

The effective operator for the operator (1.17) is given by
\[
B^0 = b(D)^* g^0 b(D) - b(D)^* V - V^* b(D) + \sum_{j=1}^{d} (a_j + a_j^*) D_j - W + \overline{Q} + \lambda \overline{Q}.
\]
(1.34)

Lemma 1.6. The symbol (1.35) of the operator (1.34) satisfies
\[
c_a |ξ|^2 1_n \leq L(ξ) \leq C_L (|ξ|^2 + 1) 1_n, \quad ξ ∈ \mathbb{R}^d,
\]
(1.36)
where $c_a$ is defined in (1.15). The constant $C_L$ depends only on the initial data (1.9).

**Proof.** The lower estimate (1.36) is proved in [MSu1] (2.30). Let us check the upper estimate. By (1.3), (1.25), and (1.35),
\[
L(ξ) \leq α_1 ||g||_{L_∞} |ξ|^2 1_n + 2|V| |α_1|^{1/2} |ξ| 1_n + 2 \left( \sum_{j=1}^{d} |ξ|^2 j \right)^{1/2} |ξ| 1_n + (∥Q∥ + λ∥Q∥) 1_n.
\]
(1.37)

We have taken into account that, obviously, the matrix (1.33) is nonnegative. According to (1.21), (1.31), and (1.32),
\[
|V| \leq |Ω|^{-1} ||g||_{L_∞} ||b(D) Λ||_{L_2(Ω)} ||b(D) \tilde{Λ}||_{L_2(Ω)} \leq C_V,
\]
(1.38)
where $C_V := |Ω|^{-1/2} α_0^{-1/2} C_a m^{1/2} n^{1/2} ||g||_{L_∞}^{3/2} ||g^{-1}||_{L_∞}^{3/2}$. Clearly,
\[
\sum_{j=1}^{d} |ξ|^2 j \leq |Ω|^{-1} C_a^2, \quad |Q| \leq |Ω|^{-1/2} ||Q||_{L_2(Ω)}, \quad |Q| \leq ||Q||_{L_∞}.
\]
(1.39)

Now relations (1.37)–(1.39) imply the upper estimate (1.36) with the constant $C_L := \max\{α_1 ||g||_{L_∞}: |Ω|^{-1/2} ||Q||_{L_2(Ω)} + λ ||Q||_{L_∞}\} + α_1^{1/2} C_V + |Ω|^{-1/2} C_a$. \hfill □

Corollary 1.7. The quadratic form $b^0$ of the operator (1.34) satisfies
\[
c_a ||Du||^2_{L_2(\mathbb{R}^d)} \leq b^0(u,u) \leq C_L ||u||^2_{H^1(\mathbb{R}^d)}, \quad u ∈ H^1(\mathbb{R}^d; \mathbb{C}^n).
\]
(1.40)
1.7. The results about approximation of the generalized resolvent. In this subsection, we formulate the results proved in [MSn1] Theorems 5.1 and 5.2.

Theorem 1.8 ([MSn1]). Suppose that the assumptions of Subsections 1.3, 1.6 are satisfied. Let \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \), \( \zeta = |\zeta|e^{i\phi}, \phi \in (0, 2\pi) \), and \( |\zeta| \geq 1 \). Denote

\[
c(\phi) := \begin{cases} |\sin \phi|^{-1}, & \phi \in (0, \pi/2) \cup (3\pi/2, 2\pi), \\ 1, & \phi \in [\pi/2, 3\pi/2]. \end{cases}
\]

(1.41)

Then for \( 0 < \varepsilon \leq 1 \) we have

\[
\|(B_{\varepsilon} - \zeta Q_0) - (B_0 - \zeta Q_0)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1 c(\phi)^2 |\zeta|^{-1/2}.
\]

(1.42)

The constant \( C_1 \) depends only on the initial data (1.9).

Next, we introduce a corrector

\[
K(\varepsilon; \zeta) := \left( |\Lambda^{\varepsilon}| b(D) + |\tilde{\Lambda}^{\varepsilon}| \right) S_\varepsilon (B_0 - \zeta Q_0)^{-1}.
\]

(1.43)

The corrector (1.43) is a bounded operator acting from \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) to \( H^1(\mathbb{R}^d; \mathbb{C}^n) \). This can be easily checked by using Proposition 1.2 and relations \( \Lambda, \tilde{\Lambda} \in \tilde{H}^1(\Omega) \). Note that \( \|\varepsilon K(\varepsilon; \zeta)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} = O(1) \) for small \( \varepsilon \) and fixed \( \zeta \).

Theorem 1.9 ([MSn1]). Suppose that the assumptions of Theorem 1.8 are satisfied. Let \( K(\varepsilon; \zeta) \) be the operator (1.43). Then for \( 0 < \varepsilon \leq 1, \zeta \in \mathbb{C} \setminus \mathbb{R}_+, \) and \( |\zeta| \geq 1 \) we have

\[
\|(B_{\varepsilon} - \zeta Q_0) - (B_0 - \zeta Q_0)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_2 c(\phi)^2 \varepsilon |\zeta|^{-1/2},
\]

\[
\|D((B_{\varepsilon} - \zeta Q_0) - (B_0 - \zeta Q_0) - \varepsilon K(\varepsilon; \zeta))\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_3 c(\phi)^2 \varepsilon.
\]

The constants \( C_2 \) and \( C_3 \) are controlled explicitly in terms of the initial data (1.9).

We also need estimates in the case where \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \) and \( |\zeta| < 1 \). The following result is a particular case of Theorem 9.1 from [MSn1].

Theorem 1.10 ([MSn1]). Suppose that the assumptions of Subsections 1.3, 1.6 are satisfied. Let \( \zeta = |\zeta|e^{i\phi} \in \mathbb{C} \setminus \mathbb{R}_+, |\zeta| < 1, \) and \( 0 < \varepsilon \leq 1 \). Let \( K(\varepsilon; \zeta) \) be the operator (1.43). Then

\[
\|(B_{\varepsilon} - \zeta Q_0) - (B_0 - \zeta Q_0)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \hat{C}_1 c(\phi)^2 |\zeta|^{-2},
\]

(1.44)

\[
\|D((B_{\varepsilon} - \zeta Q_0) - (B_0 - \zeta Q_0) - \varepsilon K(\varepsilon; \zeta))\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq \hat{C}_2 c(\phi)^2 |\zeta|^{-2}.
\]

The constants \( \hat{C}_1 \) and \( \hat{C}_2 \) depend only on the initial data (1.9).

2. Statement of the problem. Main results

2.1. Statement of the problem. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain of class \( C^{1,1} \). In \( L_2(\Omega; \mathbb{C}^n) \), we consider the operator \( B_{D,\varepsilon} \), \( 0 < \varepsilon \leq 1 \), formally given by the differential expression (1.17) with the Dirichlet boundary condition. The precise definition of the operator \( B_{D,\varepsilon} \) is given in terms of the quadratic form

\[
b_{D,\varepsilon}[u, u] = (g^2 b(D) u, b(D) u)_{L_2(\Omega)} + 2\text{Re} \sum_{j=1}^d (D_j u, (a_j^\varepsilon)^* u)_{L_2(\Omega)}
\]

(2.1)

\[
+ (Q u, u)_{L_2(\Omega)} + \lambda (Q_0 u, u)_{L_2(\Omega)}, \quad u \in H_0^1(\Omega; \mathbb{C}^n).
\]

We extend \( u \in H_0^1(\Omega; \mathbb{C}^n) \) by zero to \( \mathbb{R}^d \setminus \Omega \). Then \( u \in H^1(\mathbb{R}^d; \mathbb{C}^n) \). By (1.10) and (1.11),

\[
c_3 \|Du\|_{L_2(\Omega)}^2 \leq b_{D,\varepsilon}[u, u] \leq c_3 \|u\|_{H^1(\Omega)}^2, \quad u \in H_0^1(\Omega; \mathbb{C}^n).
\]

(2.2)

Combining this with the Friedrichs inequality, we deduce

\[
b_{D,\varepsilon}[u, u] \geq c_4 (\text{diam } \Omega)^{-2} \|u\|_{L_2(\Omega)}^2, \quad u \in H_0^1(\Omega; \mathbb{C}^n).
\]

(2.3)

Thus, the form \( b_{D,\varepsilon} \) is closed and positive definite. It generates a selfadjoint operator in \( L_2(\Omega; \mathbb{C}^n) \), which is denoted by \( B_{D,\varepsilon} \). By (2.2) and (2.3),

\[
\|u\|_{H^1(\Omega)} \leq c_4 \|B_{D,\varepsilon}^{1/2} u\|_{L_2(\Omega)}, \quad u \in H_0^1(\Omega; \mathbb{C}^n); \quad c_4 := c_\varepsilon^{-1/2} (1 + (\text{diam } \Omega)^2)^{1/2}.
\]

(2.4)
We also need to introduce an auxiliary operator \( \tilde{B}_{D,\varepsilon} \). We factorize the matrix \( Q_0(x)^{-1} \); there exists a \( \Gamma \)-periodic matrix-valued function \( f(x) \) such that \( f, f^{-1} \in L_\infty(\mathbb{R}^d) \) and

\[
Q_0(x)^{-1} = f(x)f(x)^*.
\] (2.5)

(For instance, one can take \( f(x) = Q_0(x)^{-1/2} \).) Let \( \tilde{B}_{D,\varepsilon} \) be the selfadjoint operator in \( L_2(O; \mathbb{C}^n) \) generated by the quadratic form

\[
\tilde{b}_{D,\varepsilon}[u, u] := b_{D,\varepsilon}[f^*u, f^*u], \quad \text{Dom} \tilde{b}_{D,\varepsilon} = \{ u \in L_2(O; \mathbb{C}^n) : f^*u \in H_0^1(O; \mathbb{C}^n) \}.
\] (2.6)

Formally, \( \tilde{B}_{D,\varepsilon} = (f^*)^*B_{D,\varepsilon}f^* \). Note that

\[
(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} = f^*(\tilde{B}_{D,\varepsilon} - \zeta I)^{-1}(f^*)^*.
\] (2.7)

Our goal is to approximate the generalized resolvent \( (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} \) and to prove two-parametric error estimates (with respect to \( \varepsilon \) and \( \zeta \)). We assume that \( \zeta \in \mathbb{C}\setminus\mathbb{R}_+ \). In other words, we are interested in the behavior of the generalized solution \( u_\varepsilon \in H_0^1(O; \mathbb{C}^n) \) of the problem

\[
B_{\varepsilon}u_\varepsilon - \zeta Q_0^\varepsilon u_\varepsilon = F \quad \text{in} \quad O; \quad u_\varepsilon|_{O^c} = 0,
\] (2.8)

where \( F \in L_2(O; \mathbb{C}^n) \), for small \( \varepsilon \). We have \( u_\varepsilon = (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}F \).

**Lemma 2.1.** Let \( \zeta \in \mathbb{C}\setminus\mathbb{R}_+ \). Suppose that \( u_\varepsilon \) is the generalized solution of problem (2.8). Then for any \( 0 < \varepsilon \leq 1 \) we have

\[
\|u_\varepsilon\|_{L_2(O)} \leq c(\varepsilon)\|\zeta\|^{-1}\|Q_0^{-1}\|_{L_\infty}\|F\|_{L_2(O)},
\] (2.9)

\[
\|Du_\varepsilon\|_{L_2(O)} \leq C_1c(\varepsilon)\|\zeta\|^{-1/2}\|F\|_{L_2(O)}.
\] (2.10)

In operator terms,

\[
\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}\|_{L_2(O)\to L_2(O)} \leq c(\varepsilon)\|\zeta\|^{-1}\|Q_0^{-1}\|_{L_\infty},
\] (2.11)

\[
\|D(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}\|_{L_2(O)\to L_2(O)} \leq C_1c(\varepsilon)\|\zeta\|^{-1/2}.
\] (2.12)

Here \( c(\varepsilon) \) is given by (1.44) and \( C_1 := 2\alpha_0^{-1/2}\|g^{-1}\|_{L_\infty}^{1/2}\|Q_0^{-1}\|_{L_\infty}^{1/2} (1 + \|Q_0\|_{L_\infty}\|Q_0^{-1}\|_{L_\infty})^{1/2} \).

**Proof.** From (2.5), (2.7), and the inequality \( \tilde{B}_{D,\varepsilon} > 0 \) it follows that

\[
\|u_\varepsilon\|_{L_2(O)} \leq \|f\|_{L_\infty}\|\tilde{B}_{D,\varepsilon} - \zeta I\|_{L_2(O)\to L_2(O)}\|F\|_{L_2(O)}
\]

\[
\leq \text{dist} \{ \zeta ; \mathbb{R}_+ \}\|Q_0^{-1}\|_{L_\infty}\|F\|_{L_2(O)} = c(\varepsilon)\|\zeta\|^{-1}\|Q_0^{-1}\|_{L_\infty}\|F\|_{L_2(O)}
\]

which implies (2.9). To check (2.10), we write down the integral identity for \( u_\varepsilon \):

\[
\tilde{b}_{D,\varepsilon}[u_\varepsilon, \eta] - \zeta Q_0^\varepsilon u_\varepsilon, \eta\|_{L_2(O)} = (F, \eta)_{L_2(O)}, \quad \eta \in H_0^1(O; \mathbb{C}^n).
\]

Substituting \( \eta = u_\varepsilon \) and using the lower estimate (2.2), (2.9), and (1.15), we arrive at (2.10). □

### 2.2. The form \( b_{N,\varepsilon} \)

Apart from the form (2.1), we need the quadratic form \( b_{N,\varepsilon} \) defined by the same expression, but on the class \( H^1(O; \mathbb{C}^n) \):

\[
b_{N,\varepsilon}[u, u] := (g^*b(D)u, b(D)u)_{L_2(O)} + 2\text{Re} \sum_{j=1}^d (D_j u, (a_j^\varepsilon)^* u)_{L_2(O)}
\]

\[
+ (Q^\varepsilon u, u)_{L_2(O)} + \lambda(Q_0^\varepsilon u, u)_{L_2(O)}, \quad u \in H^1(O; \mathbb{C}^n).
\] (2.13)

This form corresponds to the Neumann problem. Let us estimate the form (2.13) from above. By (1.23), we have

\[
b_{N,\varepsilon}[u, u] \leq \alpha_1\|g\|_{L_\infty}\|Du\|^2_{L_2(O)} + \sum_{j=1}^d \int_{O} |a_j^\varepsilon(x)|^2|u(x)|^2 \, dx + \|Du\|^2_{L_2(O)}
\]

\[
+ \int_{O} |Q^\varepsilon(x)||u(x)|^2 \, dx + \lambda\|Q_0\|_{L_\infty}\|u\|^2_{L_2(O)}.
\] (2.14)

From the Hölder inequality it follows that

\[
\int_{O} |a_j^\varepsilon(x)|^2|u(x)|^2 \, dx \leq \left( \int_{O} |a_j^\varepsilon(x)|^p \, dx \right)^{2/p}\|u\|^2_{L_2(O)}.
\] (2.15)
where \( \rho \) is as in (1.47), \( q = \infty \) for \( d = 1 \), and \( q = 2\rho/(\rho - 2) \) for \( d \geq 2 \). Next, we cover the domain \( O \) by the union of cells of the lattice \( \varepsilon \Gamma \) intersecting \( O \) (here \( 0 < \varepsilon \leq 1 \)). Let \( N_\varepsilon \) be the number of cells in this covering. Clearly, this union of cells is contained in the domain \( \tilde{O} \) which is the \( 2r_1 \)-neighborhood of \( O \), where \( 2r_1 = \text{diam} \Omega \). Therefore, \( N_\varepsilon \leq c_1 \varepsilon^{-d} \), where \( c_1 \) depends only on the domain \( O \) and the parameters of the lattice \( \Gamma \). We have

\[
\int_O |a_j^*(x)|^\rho \, dx \leq c_1 \varepsilon^{-d} \int_{2\varepsilon \Omega} |a_j^*(x)|^\rho \, dx = c_1 \|a_j\|_{L_\rho(\Omega)}^\rho.
\]

(2.16)

Relations (2.15) and (2.16) imply that

\[
\int_O |a_j^*(x)|^2 |u(x)|^2 \, dx \leq c_1^2 \|a_j\|_{L_\rho(\Omega)}^2 \|u\|_{L_2(\Omega)}^2.
\]

(2.17)

By the continuous embedding \( H^1(O, \mathbb{C}^n) \hookrightarrow L_q(O, \mathbb{C}^n) \), we have

\[
\|u\|_{L_q(O)} \leq C(q, O) \|u\|_{H^1(O)},
\]

(2.18)

where \( C(q, O) \) is the corresponding embedding constant. From (2.17) and (2.18) it follows that

\[
\sum_{j=1}^d \int_O |a_j^*(x)|^2 |u(x)|^2 \, dx \leq c_1^2 \|a_j\|_{L_\rho(\Omega)}^2 \|u\|_{L_2(\Omega)}^2, \quad u \in H^1(O, \mathbb{C}^n).
\]

(2.19)

Here \( \hat{C}_a^2 := \sum_{j=1}^d \|a_j\|_{L_\rho(\Omega)}^2 \). Similarly to (2.15)–(2.19), by (1.8), we obtain

\[
\int_O \|Q^* \|u(x)\|^2 \, dx \leq c_1^{1/s} \|Q\|_{L_s(O)} C(\hat{q}, O)^{2/2} \|u\|_{H^1(O)}^2,
\]

(2.20)

where \( \hat{q} = \infty \) for \( d = 1 \) and \( \hat{q} = 2s/(s - 1) \) for \( d \geq 2 \).

Relations (2.14), (2.19), and (2.20) imply that

\[
b_{N_\varepsilon}[u, u] \leq c_2 \|u\|_{H^1(O)}^2, \quad u \in H^1(O, \mathbb{C}^n),
\]

(2.21)

where \( c_2 := 1 + d a_1 \|g\|_{L_{\infty}} + c_1^{2/s} C(q, O) \hat{C}_a^2 + c_1^{1/s} \|Q\|_{L_s(O)} C(\hat{q}, O)^2 + \lambda \|Q_0\|_{L_{\infty}} \).

2.3. The homogenized problem. In \( L_2(O, \mathbb{C}^n) \), we consider the quadratic form

\[
b_D^0[u, u] = (g^0 b(D) u, b(D) u)_{L_2(O)} + 2 \Re \sum_{j=1}^d (\overline{\tau_j} D_j u, u)_{L_2(O)} - 2 \Re (Vu, b(D) u)_{L_2(O)}.
\]

Extending \( u \in H^1_0(O, \mathbb{C}^n) \) by zero to \( \mathbb{R}^d \setminus O \), using (1.40) and the Friedrichs inequality, we obtain

\[
c_s \|Du\|_{L_2(O)}^2 \leq b_D^0[u, u] \leq C_L \|u\|_{H^1(O)}^2, \quad u \in H^1(O, \mathbb{C}^n),
\]

(2.22)

\[
b_D^0[u, u] \geq c_s (\text{diam} \Omega)^{-2} \|u\|_{L_2(O)}^2, \quad u \in H^1(O, \mathbb{C}^n).
\]

(2.23)

The selfadjoint operator in \( L_2(O, \mathbb{C}^n) \) corresponding to the form \( b_D^0 \) is denoted by \( B_D^0 \). From (2.22) and (2.23) it follows that

\[
\|u\|_{H^1(O)} \leq c_4 \left( (B_D^0)^{1/2} u \right)_{L_2(O)}, \quad u \in H^1_0(O, \mathbb{C}^n),
\]

(2.24)

where \( c_4 \) is as in (2.3). Since \( \partial O \in C^{1,1} \), the operator \( B_D^0 \) is given by the differential expression (1.34) on the domain \( H^2(O, \mathbb{C}^n) \cap H^1_0(O, \mathbb{C}^n) \). We have

\[
\|(B_D^0)^{-1}\|_{L_2(O) \rightarrow H^2(O)} \leq \tilde{c}.
\]

(2.25)

Here \( \tilde{c} \) depends only on the initial data (1.5) and the domain \( O \). This fact follows from the theorems about regularity of solutions of the strongly elliptic systems (see [McL, Chapter 4]).

Remark 2.2. Instead of the condition \( \partial O \in C^{1,1} \), one could impose the following implicit condition: a bounded domain \( O \subset \mathbb{R}^d \) with Lipschitz boundary is such that estimate (2.25) holds. The results of the paper remain valid for such domain. In the case of the scalar elliptic operators, wide sufficient conditions on \( \partial O \) ensuring (2.25) can be found in [KoP] and [MaSh, Chapter 7] (in particular, it suffices that \( \partial O \in C^\alpha, \alpha > 3/2 \)).
Let $f_0 := (Q_0)^{-1/2}$. By (2.25),
\[
|f_0| \leq \|f\|_{L_\infty} = \|Q_0^{-1}\|_{L_\infty}, \quad |f_0^{-1}| \leq \|f^{-1}\|_{L_\infty} = \|Q_0\|_{L_\infty}^{-1/2}.
\]
In what follows, we need the operator $\tilde{B}_D^0 := f_0B_D^0f_0$. Note that
\[
(B_0^0 - \xi Q_0)^{-1} = f_0(\tilde{B}_D^0 - \xi I)^{-1}f_0.
\]
The function $u_0 = (B_0^0 - \xi Q_0)^{-1}F$ is the solution of the “homogenized problem”
\[
B^0u_0 - \xi Q_0u_0 = F \text{ in } O; \quad u_0|_{\partial O} = 0.
\]

Lemma 2.3. For $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ the solution $u_0$ of problem (2.28) satisfies
\[
\|u_0\|_{L_2(O)} \leq c(\phi)\|\zeta|^{-1}\|Q_0^{-1}\|_{L_\infty}\|F\|_{L_2(O)},
\]
\[
\|Du_0\|_{L_2(O)} \leq C_1c(\phi)\|\zeta|^{-1/2}\|F\|_{L_2(O)},
\]
\[
\|u_0\|_{H^2(O)} \leq C_2c(\phi)\|F\|_{L_2(O)}.
\]
Here the constant $C_1$ is as in Lemma 2.1 and $C_2 := c(Q_0/2)^{1/2}2^{-1/2}$. In operator terms,
\[
\|(B_0^0 - \xi Q_0)^{-1}\|_{L_2(O)} \leq c(\phi)\|\zeta|^{-1}\|Q_0^{-1}\|_{L_\infty}.
\]
\[
\|D(B_0^0 - \xi Q_0)^{-1}\|_{L_2(O)} \leq C_1c(\phi)\|\zeta|^{-1/2},
\]
\[
\|B_0^0 - \xi Q_0)^{-1}\|_{L_2(O)} \leq C_2c(\phi).
\]

Proof. Estimates (2.29) and (2.30) can be checked by the same way as estimates of Lemma 2.1. Let us prove (2.31). Obviously,
\[
\|(B_0^0 - \xi Q_0)^{-1}\|_{L_2(O)} \leq \|(B_0^0)\|_{L_2(O)}\|B_0^0(B_0^0 - \xi Q_0)^{-1}\|_{L_2(O)}\|L_2(O)\|L_2(O).
\]
By (2.27), we have $B_0^0(B_0^0 - \xi Q_0)^{-1} = B_0^0f_0(\tilde{B}_D^0 - \xi I)^{-1}f_0 = f_0^{1/2}B_0^0(B_0^0 - \xi I)^{-1}f_0$. Hence,
\[
\|(B_0^0 - \xi Q_0)^{-1}\|_{L_2(O)} \leq \|f_0\|_{L_\infty} \|f_0\|_{L_\infty} \sup_{x \in \partial O} \frac{x}{x - \zeta} \leq \|Q_0\|_{L_\infty} \|Q_0\|_{L_\infty}^{1/2}c(\phi).
\]
We have taken (2.26) into account. Now, relations (2.25), (2.32), and (2.33) imply (2.31). □

2.4. Formulation of the results. We choose the numbers $\varepsilon_0, \varepsilon_1 \in (0, 1]$ according to the following condition.

Condition 2.4. Let $\varepsilon_0 \in (0, 1]$ be such that the strip $(\partial O)_\varepsilon := \{x \in \mathbb{R}_d : \text{dist } \{x ; \partial O\} < \varepsilon\}$ can be covered by a finite number of open sets admitting diffeomorphisms of class $C^{1,1}$ rectifying the boundary $\partial O$. Denote $\varepsilon_1 := \varepsilon_0(1 + r_\varepsilon)^{-1}$, where $2r_\varepsilon = \text{diam } O$.

Clearly, $\varepsilon_1$ depends only on the domain $O$ and the parameters of the lattice $\Gamma$. Note that Condition 2.4 would be provided only by the assumption that $\partial O$ is Lipschitz. We have imposed a more restrictive condition $\partial O \in C^{1,1}$ in order to ensure estimate (2.26).

Now, we formulate the main results.

Theorem 2.5. Suppose that $O \subset \mathbb{R}_d$ is a bounded domain of class $C^{1,1}$. Let $\zeta = |\zeta|e^{i\phi} \in \mathbb{C} \setminus \mathbb{R}_+$ and $|\zeta| \geq 1$. Let $u_\varepsilon$ be the solution of problem (2.8) with $F \in L_2(O; C^n)$. Let $u_0$ be the solution of problem (2.28). Suppose that $\varepsilon_1$ is subject to Condition 2.4. Then for $0 < \varepsilon \leq \varepsilon_1$ we have
\[
\|u_\varepsilon - u_0\|_{L_2(O)} \leq C_4c(\phi)^5\varepsilon|\zeta|^{-1/2}\|F\|_{L_2(O)}.
\]
Here $c(\phi)$ is given by (1.41); the constant $C_4$ depends only on the initial data (1.9) and the domain $O$. In operator terms,
\[
\|(B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_0^0 - \zeta Q_0)^{-1}\|_{L_2(O)} \leq C_4c(\phi)^5\varepsilon|\zeta|^{-1/2}.
\]

In order to approximate the solution in the Sobolev space $H^l(O; C^n)$, we introduce a corrector. For this, we fix a linear continuous extension operator
\[
P_O : H^l(O; C^n) \to H^l(\mathbb{R}_d; C^n), \quad l = 0, 1, 2.
\]
Such a “universal” extension operator exists for any bounded Lipschitz domain (see [SI] or [R]). Herewith,
\[
\|P_O\|_{H^l(O) \to H^l(\mathbb{R}_d)} \leq C(l), \quad l = 0, 1, 2,
\]
where the constant $c^{(l)}_O$ depends only on $l$ and the domain $O$. By $R_O$ we denote the operator of restriction of functions in $\mathbb{R}^d$ to the domain $O$. We put
\begin{equation}
K_D(\varepsilon; \zeta) := R_O\left( [\Lambda^\varepsilon] b(\mathbf{D}) + [\tilde{\Lambda}^\varepsilon] \right) S_P (B_D^0 - \zeta Q_0)^{-1}. \tag{2.38}
\end{equation}

The continuity of the operator $K_D(\varepsilon; \zeta): L_2(O; \mathbb{C}^n) \to H^1(O; \mathbb{C}^n)$ can be checked by analogy with the continuity of the operator $[1.13]$. Let $\tilde{u}_0 = P_O u_0$. By $v_\varepsilon$ we denote the first order approximation of the solution $u_\varepsilon$:
\begin{equation}
\tilde{v}_\varepsilon := \tilde{u}_0 + \varepsilon \tilde{\Lambda}^\varepsilon S_{\varepsilon} b(D) \tilde{u}_0 + \varepsilon \tilde{\Lambda}^\varepsilon S_{\varepsilon} \tilde{u}_0, \quad v_\varepsilon := \tilde{v}_\varepsilon|_O, \tag{2.39}
\end{equation}
\begin{equation}
v_\varepsilon = (B_D^0 - \zeta Q_0)^{-1} F + \varepsilon K_D(\varepsilon; \zeta) F, \quad \text{where } K_D(\varepsilon; \zeta) \text{ is given by (2.38)}. \tag{2.40}
\end{equation}

**Theorem 2.6.** Suppose that the assumptions of Theorem 2.5 are satisfied. Suppose that $\Lambda(x)$ and $\tilde{\Lambda}(x)$ are $\Gamma$-periodic solutions of problems (1.18) and (1.28), respectively. Let $S_{\varepsilon}$ be the smoothing operator (1.14), and let $P_O$ be the extension operator (2.36). Denote $\tilde{u}_0 = P_O u_0$. Let $v_\varepsilon$ be defined by (2.39) and (2.40). Then for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, \(|\zeta| \geq 1\), and $0 < \varepsilon \leq \varepsilon_1$ we have
\begin{equation}
\|u_\varepsilon - v_\varepsilon\|_{H^1(O)} \leq (C_5 c(\varepsilon)^2 \varepsilon^{1/2} |\zeta|^{-1/4} + C_6 c(\varepsilon)^4 \varepsilon) \|F\|_{L_2(O)}. \tag{2.41}
\end{equation}

In operator terms,
\begin{equation}
\|(B_D^0 - \zeta Q_0)^{-1} - (B_D^0 - \zeta Q_0)^{-1} - \varepsilon K_D(\varepsilon; \zeta)\|_{L_2(O)\to H^1(O)} \leq C_5 c(\varepsilon)^2 \varepsilon^{1/2} |\zeta|^{-1/4} + C_6 c(\varepsilon)^4 \varepsilon, \tag{2.42}
\end{equation}

where the operator $K_D(\varepsilon; \zeta)$ is given by (2.38). Let $g(x)$ be the matrix-valued function defined by (1.20). Let $p_{\varepsilon} := g^* b(D) u_\varepsilon$. Then for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, \(|\zeta| \geq 1\), and $0 < \varepsilon \leq \varepsilon_1$ we have
\begin{equation}
\|p_{\varepsilon} - \tilde{g}^* S_{\varepsilon} b(D) \tilde{u}_0 - \tilde{g}^* (b(D) \tilde{\Lambda}) S_{\varepsilon} \tilde{u}_0\|_{L_2(O)} \leq (\tilde{C}_5 c(\varepsilon)^2 \varepsilon^{1/2} |\zeta|^{-1/4} + \tilde{C}_6 c(\varepsilon)^4 \varepsilon) \|F\|_{L_2(O)}. \tag{2.43}
\end{equation}

In operator terms,
\begin{equation}
\|g^* b(D) (B_D^0 - \zeta Q_0)^{-1} - G_D(\varepsilon; \zeta)\|_{L_2(O)\to L_2(O)} \leq \tilde{C}_5 c(\varepsilon)^2 \varepsilon^{1/2} |\zeta|^{-1/4} + \tilde{C}_6 c(\varepsilon)^4 \varepsilon. \tag{2.44}
\end{equation}

Here $G_D(\varepsilon; \zeta) = \tilde{g}^* S_{\varepsilon} b(D) P_O (B_D^0 - \zeta Q_0)^{-1} + g^* (b(D) \tilde{\Lambda}) S_{\varepsilon} P_O (B_D^0 - \zeta Q_0)^{-1}$. The constants $C_5$, $C_6$, $\tilde{C}_5$, and $\tilde{C}_6$ depend only on the initial data (1.9) and the domain $O$.

The first order approximation $v_\varepsilon$ of the solution $u_\varepsilon$ does not satisfy the Dirichlet boundary condition. We have $v_\varepsilon|_{\partial O} = \varepsilon (\tilde{\Lambda}^\varepsilon S_{\varepsilon} b(D) \tilde{u}_0 + \tilde{\Lambda}^\varepsilon S_{\varepsilon} \tilde{u}_0)|_{\partial O}$. We consider the “discrepancy” $w_\varepsilon$, which is the solution of the problem
\begin{equation}
B_{\varepsilon} w_\varepsilon - \zeta Q_0 w_\varepsilon = 0 \quad \text{in } O; \quad w_\varepsilon|_{\partial O} = \varepsilon (\tilde{\Lambda}^\varepsilon S_{\varepsilon} b(D) \tilde{u}_0 + \tilde{\Lambda}^\varepsilon S_{\varepsilon} \tilde{u}_0)|_{\partial O}. \tag{2.45}
\end{equation}

Here the equation is understood in the weak sense, as the following identity for $w_\varepsilon \in H^1(O; \mathbb{C}^n)$:
\begin{equation}
b_{N,\varepsilon} [w_\varepsilon, \eta] - \zeta (Q_0 w_\varepsilon, \eta)_{L_2(O)} = 0, \quad \eta \in H^1_0(O; \mathbb{C}^n). \tag{2.46}
\end{equation}

The discrepancy $w_\varepsilon$ is often called the “boundary layer correction term”. Allowing some freedom, along with $w_\varepsilon$, we shall use the notation $w_\varepsilon(\cdot; \zeta)$ for the solution of problem (2.45). We introduce the operator taking $F$ to $w_\varepsilon$:
\begin{equation}
\varepsilon W_D(\varepsilon; \zeta) : L_2(O; \mathbb{C}^n) \ni F \mapsto w_\varepsilon(\cdot; \zeta) \in H^1(O; \mathbb{C}^n). \tag{2.47}
\end{equation}

Let us find more explicit expression for $W_D(\varepsilon; \zeta)$. Clearly, the function
\begin{equation}
r_\varepsilon(x; \zeta) := [w_\varepsilon(x; \zeta) - \varepsilon (K_D(\varepsilon; \zeta) F)(x)] \tag{2.48}
\end{equation}
belongs to $H^1_0(O; \mathbb{C}^n)$ and satisfies the identity
\begin{equation}
b_{D,\varepsilon} [r_\varepsilon, \eta] - \zeta (Q_0 r_\varepsilon, \eta)_{L_2(O)} = \varepsilon I(\varepsilon; \zeta)[F, \eta], \quad \eta \in H^1_0(O; \mathbb{C}^n), \tag{2.49}
\end{equation}
where
\begin{equation}
I(\varepsilon; \zeta)[F, \eta] := -b_{N,\varepsilon} [K_D(\varepsilon; \zeta) F, \eta] + \zeta (Q_0 K_D(\varepsilon; \zeta) F, \eta)_{L_2(O)} \tag{2.50}
\end{equation}

By (2.21),
\begin{equation}
|I(\varepsilon; \zeta)[F, \eta]| \leq c_2 \|K_D(\varepsilon; \zeta) F\|_{H^1(O)} \|\eta\|_{L_2(O)} + \|\eta\|_{L_2(O)} \|K_D(\varepsilon; \zeta) F\|_{L_2(O)} \|\eta\|_{L_2(O)}, \quad F \in L_2(O; \mathbb{C}^n), \quad \eta \in H^1_0(O; \mathbb{C}^n). \tag{2.51}
\end{equation}
Hence, for $F \in L_2(O; \mathbb{C}^n)$ fixed, relation (2.51) defines an antilinear continuous functional of $\eta \in H^1_0(O; \mathbb{C}^n)$, which can be identified with an element from $H^{-1}(O; \mathbb{C}^n)$. This element depends on $F$ linearly, we denote it by $T(\varepsilon; \zeta)F$. Thus,

$$I(\varepsilon; \zeta)[F, \eta] = (T(\varepsilon; \zeta)F, \eta)_{L_2(O)}, \quad F \in L_2(O; \mathbb{C}^n), \quad \eta \in H^1_0(O; \mathbb{C}^n),$$

(2.52)

where the right-hand side is understood as extension of the inner product in $L_2(O; \mathbb{C}^n)$ to pairs from $H^{-1}(O; \mathbb{C}^n) \times H^1_0(O; \mathbb{C}^n)$, and the continuity of the operator $K_D(\varepsilon; \zeta) : L_2(O; \mathbb{C}^n) \to H^1(\mathbb{C}; \mathbb{C}^n)$ it follows that the operator $T(\varepsilon; \zeta) : L_2(O; \mathbb{C}^n) \to H^{-1}(O; \mathbb{C}^n)$ is continuous.

By (2.50) and (2.53), we have

$$r_\varepsilon = \varepsilon(B_{D,\varepsilon} - \zeta Q_0)T(\varepsilon; \zeta)F,$$

(2.53)

where the generalized resolvent is extended to a continuous operator acting from $H^{-1}(O; \mathbb{C}^n)$ to $H^1_0(O; \mathbb{C}^n)$. Now, by (2.45) and (2.53),

$$w_\varepsilon(\zeta) = \varepsilon(K_D(\varepsilon; \zeta) + (B_{D,\varepsilon} - \zeta Q_0)T(\varepsilon; \zeta))F,$$

whence (see (2.47))

$$W_D(\varepsilon; \zeta) = K_D(\varepsilon; \zeta) + (B_{D,\varepsilon} - \zeta Q_0)T(\varepsilon; \zeta).$$

(2.54)

The following theorem gives approximation for the solution $u_\varepsilon$ in $H^1(O; \mathbb{C}^n)$ with error estimate of sharp order $O(\varepsilon)$; in this approximation, the discrepancy $w_\varepsilon$ is taken into account.

**Theorem 2.7.** Suppose that the assumptions of Theorem 2.6 are satisfied. Let $w_\varepsilon$ be the solution of problem (2.45). Let $W_D(\varepsilon; \zeta)$ be the operator (2.54). Then for $\zeta \in \mathbb{C} \setminus \{r_+; |\zeta| \geq 1\}$ and $0 < \varepsilon \leq 1$ we have

$$\|u_\varepsilon - v_\varepsilon + w_\varepsilon\|_{H^1(O)} \leq C\gamma(\phi)^4 \varepsilon \|F\|_{L_2(O)}.$$

(2.55)

In operator terms,

$$\|B_{D,\varepsilon} - \zeta Q_0^{\frac{1}{2}} - (B_D - \zeta Q_0^{\frac{1}{2}})^{-1} - \varepsilon K_D(\varepsilon; \zeta) + \varepsilon W_D(\varepsilon; \zeta)\|_{L_2(O) \to H^1(O)} \leq C\gamma(\phi)^4 \varepsilon.$$  

(2.56)

The constant $C\gamma$ depends only on the initial data (1.9) and the domain $O$.

3. Auxiliary statements

3.1. Estimates in the neighborhood of the boundary.

**Lemma 3.1.** Suppose that Condition 2.4 is satisfied. Then for any $u \in H^1(\mathbb{R}^d)$ we have

$$\int_{\partial(O)_\varepsilon} |u|^2 \, dx \leq \beta \varepsilon \|u\|_{H^1(\mathbb{R}^d)} \|u\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

The constant $\beta$ depends only on the domain $O$.

**Lemma 3.2.** Suppose that Condition 2.4 is satisfied. Let $h(x)$ be a $\Gamma$-periodic function in $\mathbb{R}^d$ such that $h \in L_2(\Omega)$. Let $S_\varepsilon$ be the operator (1.1). Denote $\beta_\varepsilon := \beta(1 + r_1)$, where $2r_1 = \text{diam } \Omega$. Then for $0 < \varepsilon \leq \varepsilon_1$ and any $u \in H^1(\mathbb{R}^d; \mathbb{C}^k)$ we have

$$\int_{\partial(O)_\varepsilon} |h_\varepsilon^2(x)|^2 |(S_\varepsilon u)(x)|^2 \, dx \leq \beta_\varepsilon \varepsilon \|\xi\|_{H^1(\mathbb{R}^d)} \|u\|_{H^1(\mathbb{R}^d)} \|u\|_{L_2(\mathbb{R}^d)}.$$

**Lemma 3.2** is an analogue of Lemma 2.6 from [ZhPas1]. Lemmas 3.1 and 3.2 were checked in [PSu, §5] under the condition $\partial O \in C^1$, but the proofs work also under Condition 2.4.

3.2. Properties of the matrix-valued functions $\Lambda$ and $\tilde{\Lambda}$. The following result was proved in [PSu, Corollary 2.4].

**Lemma 3.3.** Suppose that the $\Gamma$-periodic solution $\Lambda(x)$ of problem (1.18) is bounded: $\Lambda \in L_\infty$. Then for any function $u \in H^1(\mathbb{R}^d)$ and $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^d} |(D\Lambda)^{\varepsilon}(x)|^2 |u(x)|^2 \, dx \leq \beta_1 \|u\|_{L_2(\mathbb{R}^d)}^2 + \beta_2 \varepsilon \|\Lambda\|_{L_\infty}^2 \|\partial u\|_{L_2(\mathbb{R}^d)}^2.$$  

The constants $\beta_1$ and $\beta_2$ depend on $m, d, \alpha_0, \alpha_1, \|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$.

The following statement can be easily checked with the help of the Hölder inequality and the Sobolev embedding theorem; cf. [MSu] Lemma 3.5.
Lemma 3.4. Let $h(x)$ be a $\Gamma$-periodic function in $\mathbb{R}^d$ such that

$$h \in L_p(\Omega), \quad p = 2 \text{ for } d = 1, \quad p > 2 \text{ for } d = 2, \quad p \geq d \text{ for } d \geq 3.$$  \hspace{1cm} (3.1)

Then for $0 < \varepsilon \leq 1$ the operator $[h']$ is a continuous mapping of $H^1(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$, and

$$\| [h'] \|_{H^1(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \| h \|_{L_p(\Omega)} C(\hat{q}, \Omega),$$

where $C(\hat{q}, \Omega)$ is the norm of the embedding $H^1(\Omega) \hookrightarrow L_{\hat{q}}(\Omega)$. Here $\hat{q} = \infty$ for $d = 1$ and $\hat{q} = 2(p - 2)^{-1}$ for $d \geq 2$.

The following result was proved in [MSu1] Corollary 3.6.

Lemma 3.5. Suppose that the $\Gamma$-periodic solution $\tilde{\Lambda}(x)$ of problem (1.28) satisfies condition (3.1). Then for any $u \in H^2(\mathbb{R}^d)$ and $0 < \varepsilon \leq 1$ we have

$$\int_{\mathbb{R}^d} |(D\tilde{\Lambda})^c(x)|^2 |u(x)|^2 \, dx \leq \tilde{\beta}_1 \| u \|_{H^1(\mathbb{R}^d)}^2 + \tilde{\beta}_2 \varepsilon^2 \| \tilde{\Lambda} \|_{L_p(\Omega)}^2 C(\hat{q}, \Omega)^2 \| Du \|_{H^1(\mathbb{R}^d)}^2.$$  \hspace{1cm} (3.2)

Here $\tilde{q}$ is as in Lemma 3.4. The constants $\tilde{\beta}_1$ and $\tilde{\beta}_2$ depend only on $n$, $d$, $\alpha_0$, $\alpha_1$, $\rho$, $\| g \|_{L_\infty}$, $\| g^{-1} \|_{L_\infty}$, the norms $\| a_j \|_{L_p(\Omega)}$, $j = 1, \ldots, d$, and the parameters of the lattice $\Gamma$.

3.3. Lemma about $Q_0^\varepsilon - \overline{Q_0}$. The proof of the following statement is quite similar to that of Lemma 3.7 from [MSu1]. We omit the details.

Lemma 3.6. Let $Q_0(x)$ be a $\Gamma$-periodic $(n \times n)$-matrix-valued function such that $Q_0 \in L_\infty$. Then the operator $[Q_0^\varepsilon - \overline{Q_0}]$ is a continuous mapping of $H^1(\mathcal{O}; \mathbb{C}^n)$ to $H^{-1}(\mathcal{O}; \mathbb{C}^n)$, and we have

$$\| [Q_0^\varepsilon - \overline{Q_0}] \|_{H^1(\mathcal{O}) \rightarrow H^{-1}(\mathcal{O})} \leq C_{Q_0} \varepsilon.$$  \hspace{1cm} (3.3)

The constant $C_{Q_0}$ depends on $d$, $\| Q_0 \|_{L_\infty}$, and the parameters of the lattice $\Gamma$.

4. PROOF OF THEOREM 2.7. BEGINNING OF THE PROOFS OF THEOREMS 2.5 AND 2.6

In this section, we prove Theorem 2.7 and reduce the proofs of Theorems 2.5 and 2.6 to estimation of the correction term $w_\varepsilon$.

4.1. Associated problem in $\mathbb{R}^d$. By Lemma 2.3 (2.37), and the inequality $|\psi| \geq 1$, we have

$$\| \tilde{u}_0 \|_{L_2(\mathbb{R}^d)} \leq k_1 c(\phi) \| \tilde{f} \|_{L_2(\mathcal{O})}; \quad k_1 := C_{0}^{(0)} \| Q_0^{-1} \|_{L_\infty};$$  \hspace{1cm} (4.1)

$$\| \tilde{u}_0 \|_{H^1(\mathbb{R}^d)} \leq k_2 c(\phi) \| \tilde{f} \|_{L_2(\mathcal{O})}; \quad k_2 := C_{0}^{(1)} (C_1 + \| Q_0^{-1} \|_{L_\infty});$$  \hspace{1cm} (4.2)

$$\| \tilde{u}_0 \|_{H^2(\mathbb{R}^d)} \leq k_3 c(\phi) \| \tilde{f} \|_{L_2(\mathcal{O})}; \quad k_3 := C_{0}^{(2)} C_2.$$  \hspace{1cm} (4.3)

We put

$$\tilde{F} := (\tilde{B}^0 - \zeta \overline{Q_0}) \tilde{u}_0.$$  \hspace{1cm} (4.4)

Then $\tilde{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $\tilde{F}|_{\mathcal{O}} = \tilde{F}$. Relations (1.36), (4.1), and (1.13) imply that

$$\| \tilde{F} \|_{L_2(\mathbb{R}^d)} \leq C \| \tilde{u}_0 \|_{H^2(\mathbb{R}^d)} + \| \zeta \overline{Q_0} \|_{L_2(\mathbb{R}^d)} \leq C_{\tilde{F}} c(\phi) \| \tilde{f} \|_{L_2(\mathcal{O})}; \quad C_{\tilde{F}} := k_2 C_1 + k_1 \| Q_0 \|_{L_\infty},$$  \hspace{1cm} (4.5)

Let $\tilde{u}_e \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ be the solution of the following equation in $\mathbb{R}^d$:

$$B_\varepsilon \tilde{u}_e - \zeta Q_0^\varepsilon \tilde{u}_e = \tilde{F},$$  \hspace{1cm} (4.6)

i.e., $\tilde{u}_e = (B_\varepsilon - \zeta Q_0^\varepsilon)^{-1} \tilde{F}$. Combining (1.14) with (1.6) and applying Theorems 1.8 and 1.9 for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\psi| \geq 1$, and $0 < \varepsilon \leq 1$ we obtain

$$\| \tilde{u}_e - \tilde{u}_0 \|_{L_2(\mathbb{R}^d)} \leq C_1 C_{\tilde{F}} c(\phi)^3 \varepsilon |\psi|^{-1/2} \| \tilde{f} \|_{L_2(\mathcal{O})};$$  \hspace{1cm} (4.7)

$$\| \tilde{u}_e - \tilde{v}_e \|_{L_2(\mathbb{R}^d)} \leq C_2 C_{\tilde{F}} c(\phi)^3 \varepsilon |\psi|^{-1/2} \| \tilde{f} \|_{L_2(\mathcal{O})};$$  \hspace{1cm} (4.8)

$$\| D(\tilde{u}_e - \tilde{v}_e) \|_{L_2(\mathbb{R}^d)} \leq C_3 C_{\tilde{F}} c(\phi)^3 \varepsilon \| \tilde{f} \|_{L_2(\mathcal{O})}.$$  \hspace{1cm} (4.9)

Now, (1.8), (1.9), and the inequality $|\psi| \geq 1$ imply that

$$\| \tilde{u}_e - \tilde{v}_e \|_{H^1(\mathbb{R}^d)} \leq C_3 c(\phi)^3 \varepsilon \| \tilde{f} \|_{L_2(\mathcal{O})}; \quad C_3 := (C_2 + C_3) C_{\tilde{F}}.$$  \hspace{1cm} (4.10)
4.2. Proof of Theorem 2.7. Denote $V_\varepsilon := u_\varepsilon - v_\varepsilon + w_\varepsilon$. By (2.8), (2.10), and (2.14), the function $V_\varepsilon \in H^1_0(\Omega; \mathbb{C}^n)$ satisfies the identity

$$b_{D,\varepsilon}[V_\varepsilon, \eta] - \zeta(Q_0^*V_\varepsilon, \eta)_{L^2(\Omega)} = (F, \eta)_{L^2(\Omega)} - b_{N,\varepsilon}[v_\varepsilon, \eta] + \zeta(Q_0^*v_\varepsilon, \eta)_{L^2(\Omega)}, \quad \eta \in H^1_0(\Omega; \mathbb{C}^n).$$

We extend $\eta$ by zero to $\mathbb{R}^d \setminus \Omega$, keeping the same notation. Then $\eta \in H^1(\mathbb{R}^d; \mathbb{C}^n)$. Recalling that $\bar{F}$ is extension of $F$ and $\bar{v}_\varepsilon$ is extension of $v_\varepsilon$, and using (4.6), we find

$$b_{D,\varepsilon}[V_\varepsilon, \eta] - \zeta(Q_0^*V_\varepsilon, \eta)_{L^2(\Omega)} = I_\varepsilon[\eta], \quad \eta \in H^1_0(\Omega; \mathbb{C}^n),$$

where the following notation is used:

$$I_\varepsilon[\eta] := b_{\varepsilon}[ar{u}_\varepsilon - \bar{v}_\varepsilon, \eta] - \zeta(Q_0^*(\bar{u}_\varepsilon - \bar{v}_\varepsilon), \eta)_{L^2(\mathbb{R}^d)}, \quad \eta \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Next, we estimate the functional (4.12) with the help of (1.16), (4.8), and (4.10):

$$|I_\varepsilon[\eta]| \leq C_8 c(\phi)^3 \varepsilon \|\bar{F}\|_{L^2(\Omega)} \|\eta\|_{H^1(\Omega)} + C_9 c(\phi)^3 \varepsilon |\zeta|^{1/2} \|\bar{F}\|_{L^2(\Omega)} \|Q_0^*\|^{1/2} \|\eta\|_{L^2(\Omega)},$$

where $C_8 := C_3^2 C_3$ and $C_9 := \|Q_0\|_{L^\infty}^2 C_2 C_\bar{F}$.

We substitute $\eta = V_\varepsilon$ in (4.11), take the imaginary part, and apply (4.13). Then

$$\text{Im} \zeta |(Q_0^*V_\varepsilon, V_\varepsilon)_{L^2(\Omega)}| = \text{Im} I_\varepsilon[V_\varepsilon] \leq C_8 c(\phi)^3 \varepsilon \|V_\varepsilon\|_{H^1(\Omega)} \|\bar{F}\|_{L^2(\Omega)} + C_9 |\zeta|^{1/2} c(\phi)^3 \varepsilon \|\bar{F}\|_{L^2(\Omega)} \|Q_0^*\|^{1/2} \|V_\varepsilon\|_{L^2(\Omega)}.$$  

If $\text{Re} \zeta \geq 0$ (and then $\text{Im} \zeta \neq 0$), we deduce

$$(Q_0^*V_\varepsilon, V_\varepsilon)_{L^2(\Omega)} \leq 2 C_8 c(\phi)^3 \varepsilon |\zeta|^{-1} \|V_\varepsilon\|_{H^1(\Omega)} \|\bar{F}\|_{L^2(\Omega)} + C_9 |\zeta|^{1/2} c(\phi)^3 \varepsilon \|\bar{F}\|_{L^2(\Omega)}, \quad \text{Re} \zeta \geq 0.$$  

If $\text{Re} \zeta < 0$, we take the real part in identity (4.11) with $\eta = V_\varepsilon$. Note that $c(\phi) = 1$ for such $\zeta$. Using (4.13), we obtain

$$|\text{Re} \zeta| |(Q_0^*V_\varepsilon, V_\varepsilon)_{L^2(\Omega)}| \leq C_8 c(\phi)^3 \varepsilon \|V_\varepsilon\|_{H^1(\Omega)} \|\bar{F}\|_{L^2(\Omega)} + C_9 |\zeta|^{1/2} c(\phi)^3 \varepsilon \|\bar{F}\|_{L^2(\Omega)} \|Q_0^*\|^{1/2} \|V_\varepsilon\|_{L^2(\Omega)}.$$  

Summing up (4.14) and (4.16), we deduce the inequality

$$(Q_0^*V_\varepsilon, V_\varepsilon)_{L^2(\Omega)} \leq 4 C_8 c(\phi)^3 \varepsilon |\zeta|^{-1} \|V_\varepsilon\|_{H^1(\Omega)} \|\bar{F}\|_{L^2(\Omega)} + 4 C_9 c(\phi)^3 \varepsilon |\zeta|^{-1} \|\bar{F}\|_{L^2(\Omega)}, \quad \text{Re} \zeta < 0.$$  

Combining this with (4.15), for all $\zeta$ under consideration we obtain

$$(Q_0^*V_\varepsilon, V_\varepsilon)_{L^2(\Omega)} \leq 4 C_9 c(\phi)^3 \varepsilon |\zeta|^{-1} \|V_\varepsilon\|_{H^1(\Omega)} \|\bar{F}\|_{L^2(\Omega)} + 4 C_9 c(\phi)^3 \varepsilon |\zeta|^{-1} \|\bar{F}\|_{L^2(\Omega)}.$$  

Now, (4.11) with $\eta = V_\varepsilon$, (4.13), and (4.17) imply that

$$b_{D,\varepsilon}[V_\varepsilon, V_\varepsilon] \leq 7 C_9^2 c(\phi)^3 \varepsilon \|V_\varepsilon\|_{H^1(\Omega)} \|\bar{F}\|_{L^2(\Omega)} + \frac{13}{2} C_9^2 c(\phi)^3 \varepsilon \|\bar{F}\|_{L^2(\Omega)}.$$  

Taking (2.1) into account, we deduce

$$\|V_\varepsilon\|_{H^1(\Omega)}^2 \leq C_7 c(\phi)^3 \varepsilon \|F\|_{L^2(\Omega)}^2; \quad C_7^2 := 49 c_1 C_9^2 + 13 c_2^2 C_9^2,$$

which implies (2.55).

Apart from estimate (2.55), we also need to estimate the $L_2$-norm of $V_\varepsilon$.

Lemma 4.1. Under the assumptions of Theorem 2.7, for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and $0 < \varepsilon \leq 1$ we have

$$\|u_\varepsilon - v_\varepsilon + w_\varepsilon\|_{L^2(\Omega)} \leq C_{10} c(\phi)^4 \varepsilon |\zeta|^{-1/2} \|F\|_{L^2(\Omega)}.$$

The constant $C_{10}$ depends only on the initial data (1.9) and the domain $\Omega$.

Proof. By (2.55) and (4.17),

$$(Q_0^*V_\varepsilon, V_\varepsilon)_{L^2(\Omega)} \leq 4 (C_7 C_8 + C_9^2) c(\phi)^3 \varepsilon \|\bar{F}\|_{L^2(\Omega)}^2.$$

This implies (4.18) with the constant $C_{10} := 2 Q_0^{-1} 1/2 (C_7 C_8 + C_9^2)^{1/2}$.
4.3. Conclusions. 1) From (2.45) it follows that
\[ \|u_{\varepsilon} - v_{\varepsilon}\|_{W^{1}(\Omega)} \leq C_7 c(\phi) \varepsilon \|F\|_{L_2(\Omega)} + \|w_{\varepsilon}\|_{W^{1}(\Omega)}. \] (4.19)
Hence, in order to prove (2.41), it suffices to obtain an appropriate estimate for \( \|w_{\varepsilon}\|_{W^{1}(\Omega)}. \)

2) By (1.15),
\[ \|u_{\varepsilon} - u_0\|_{L_2(\Omega)} \leq C_{10} c(\phi)^{\varepsilon} \varepsilon |\varepsilon|^{-1/2} \|F\|_{L_2(\Omega)} + \|v_{\varepsilon} - u_0\|_{L_2(\Omega)} + \|w_{\varepsilon}\|_{L_2(\Omega)}. \] (4.20)
We have
\[ \|v_{\varepsilon} - u_0\|_{L_2(\Omega)} \leq \varepsilon \|\nabla^\varepsilon S_1 b(D)\tilde{u}_0\|_{L_2(\mathbb{R}^d)} + \varepsilon \|\nabla^\varepsilon S_0\tilde{u}_0\|_{L_2(\mathbb{R}^d)}. \] (4.21)
From Proposition 1.2, (1.22), and (1.29) it follows that
\[ \|\nabla^\varepsilon S_1 b(D)\tilde{u}_0\|_{L_2(\mathbb{R}^d)} \leq M_1, \] (4.22)
\[ \|\nabla^\varepsilon S_0\tilde{u}_0\|_{L_2(\mathbb{R}^d)} \leq M_1 \text{ and } M_1 := |\Omega|^{-1/2}(2r_0)^{-1} C_9 n^{1/2} \alpha_0^{-1} |g^{-1}|_{L_\infty}. \] (4.23)
Combining (1.3), (4.2), and (1.21) - (4.23), we obtain
\[ \|v_{\varepsilon} - u_0\|_{L_2(\Omega)} \leq \varepsilon (M_1^2 \alpha_1 + M_1^2)^{1/2} \|\tilde{u}_0\|_{W^{1}(\mathbb{R}^d)} \leq \varepsilon (M_1^2 \alpha_1 + M_1^2)^{1/2} k_2 c(\phi) |\varepsilon|^{-1/2} \|F\|_{L_2(\Omega)}. \] (4.24)
Now, inequalities (4.20) and (4.24) yield
\[ \|u_{\varepsilon} - u_0\|_{L_2(\Omega)} \leq C_{11} c(\phi)^{\varepsilon} \varepsilon |\varepsilon|^{-1/2} \|F\|_{L_2(\Omega)} + \|w_{\varepsilon}\|_{L_2(\Omega)}, \] (4.25)
where \( C_{11} := C_{10} + (M_1^2 \alpha_1 + M_1^2)^{1/2} k_2. \) Thus, the proof of Theorem 2.6 is reduced to appropriate estimate for \( \|w_{\varepsilon}\|_{L_2(\Omega)}. \)

5. The proof of Theorem 2.6

5.1. Localization near the boundary. Recall that \((\partial \Omega)_{\varepsilon} := \{ x \in \mathbb{R}^d : \text{dist} \{ x ; \partial \Omega \} < \varepsilon \}.\)
Fix a smooth cut-off function \( \theta_{\varepsilon}(x) \) in \( \mathbb{R}^d \) such that
\[ \theta_{\varepsilon} \in C_0^\infty(\mathbb{R}^d), \quad \text{supp} \theta_{\varepsilon} \subset (\partial \Omega)_{\varepsilon}, \quad 0 \leq \theta_{\varepsilon}(x) \leq 1, \quad \theta_{\varepsilon}(x) = 1 \text{ for } x \in \partial \Omega; \quad \varepsilon \left| \nabla \theta_{\varepsilon}(x) \right| \leq \mu = \text{Const}. \] (5.1)
The constant \( \mu \) depends only on \( d \) and the domain \( \Omega. \) Consider the following function in \( \mathbb{R}^d: \)
\[ \varphi_{\varepsilon}(x) := \varepsilon \theta_{\varepsilon}(x) \left( \nabla^\varepsilon(x)(S_0 b(D)\tilde{u}_0)(x) + \nabla^\varepsilon(x)(S_0 \tilde{u}_0)(x) \right). \] (5.2)

Lemma 5.1. Suppose that \( w_{\varepsilon} \) is the solution of problem (2.45). Suppose that \( \varphi_{\varepsilon} \) is given by (5.2). Then for \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+, |\zeta| \geq 1, \) and \( 0 < \varepsilon < \varepsilon_1 \) we have
\[ \|w_{\varepsilon}\|_{W^{1}(\Omega)} \leq c(\phi) \left( C_{12} |\zeta|^{-1/2} \|\varphi_{\varepsilon}\|_{L_2(\Omega)} + C_{13} \|\varphi_{\varepsilon}\|_{W^{1}(\Omega)} \right). \] (5.3)
The constants \( C_{12} \) and \( C_{13} \) depend only on the initial data (1.7) and the domain \( \Omega. \)

Proof. We have \( w_{\varepsilon}|_{\partial \Omega} = \varphi_{\varepsilon}|_{\partial \Omega}. \) Therefore, \( \varphi_{\varepsilon} := w_{\varepsilon} - \varphi_{\varepsilon} \in H_0^1(\Omega; \mathbb{C}^n). \) By (2.46),
\[ b_{D,\varepsilon}(\varphi_{\varepsilon}, \eta) - \zeta(Q_0^\varepsilon \varphi_{\varepsilon}, \eta)_{L_2(\Omega)} = -b_{N,\varepsilon}[\varphi_{\varepsilon}, \eta] + \zeta(Q_0^\varepsilon \varphi_{\varepsilon}, \eta)_{L_2(\Omega)}, \quad \eta \in H_0^1(\Omega; \mathbb{C}^n). \] (5.4)
We substitute \( \eta = \varphi_{\varepsilon} \) in (5.4) and take the imaginary part. Then, by (2.24),
\[ \|\text{Im} \zeta(Q_0^\varepsilon \varphi_{\varepsilon}, \varphi_{\varepsilon})_{L_2(\Omega)} \leq c_2 \|\varphi_{\varepsilon}\|_{H^1(\Omega)} \|\varphi_{\varepsilon}\|_{H^1(\Omega)} + |\zeta| \|Q_0^\varepsilon\|_{L_\infty} \|\varphi_{\varepsilon}\|_{L_4(\Omega)} \|\varphi_{\varepsilon}\|_{L_2(\Omega)} \|\varphi_{\varepsilon}\|_{L_2(\Omega)} \|\varphi_{\varepsilon}\|_{L_2(\Omega)}. \] (5.5)
If \( \Re \zeta \geq 0 \) and then \( \Im \zeta \neq 0 \), we deduce
\[ (Q_0^\varepsilon \varphi_{\varepsilon}, \varphi_{\varepsilon})_{L_2(\Omega)} \leq 2c_2 c(\phi) |\zeta|^{-1} \|\varphi_{\varepsilon}\|_{H^1(\Omega)} \|\varphi_{\varepsilon}\|_{H^1(\Omega)} + |\zeta| \|Q_0^\varepsilon\|_{L_\infty} \|\varphi_{\varepsilon}\|_{L_2(\Omega)} \|\varphi_{\varepsilon}\|_{L_2(\Omega)}, \quad \Re \zeta \geq 0. \]
If \( \Re \zeta < 0 \), we take the real part of the corresponding identity and obtain
\[ |\Re \zeta|(Q_0^\varepsilon \varphi_{\varepsilon}, \varphi_{\varepsilon})_{L_2(\Omega)} \leq c_2 \|\varphi_{\varepsilon}\|_{H^1(\Omega)} \|\varphi_{\varepsilon}\|_{H^1(\Omega)} + |\zeta| \|Q_0^\varepsilon\|_{L_\infty} \|\varphi_{\varepsilon}\|_{L_2(\Omega)} \|\varphi_{\varepsilon}\|_{L_2(\Omega)} \|\varphi_{\varepsilon}\|_{L_2(\Omega)}. \] (5.6)
Summing up (5.5) and (5.6), we deduce
\[ (Q_0^\varepsilon \varphi_{\varepsilon}, \varphi_{\varepsilon})_{L_2(\Omega)} \leq 4c_2 c(\phi) |\zeta|^{-1} \|\varphi_{\varepsilon}\|_{H^1(\Omega)} \|\varphi_{\varepsilon}\|_{H^1(\Omega)} + 4|\zeta| \|Q_0^\varepsilon\|_{L_\infty} \|\varphi_{\varepsilon}\|_{L_2(\Omega)}^2, \quad \Re \zeta < 0. \]
Thus, for all \( \zeta \) under consideration we have
\[ (Q_0^\varepsilon \varphi_{\varepsilon}, \varphi_{\varepsilon})_{L_2(\Omega)} \leq 4c_2 c(\phi) |\zeta|^{-1} \|\varphi_{\varepsilon}\|_{H^1(\Omega)} \|\varphi_{\varepsilon}\|_{H^1(\Omega)} + 4|\zeta| \|Q_0^\varepsilon\|_{L_\infty} \|\varphi_{\varepsilon}\|_{L_2(\Omega)}^2. \] (5.7)
From (5.4) with \(\eta = \varphi_\varepsilon\), (6.7), and (2.21) it follows that
\[
\|b_{D,\varepsilon}[\varphi_\varepsilon] \|_{L^2(\mathbb{R}^d)} \leq 9\varepsilon^2 c(\phi) \|\varphi_\varepsilon\|_{H^1(\mathbb{R}^d)} + 9\varepsilon(\phi)^2 \|\varphi_\varepsilon\|_{L^\infty} + 2\|\varphi_\varepsilon\|_{L^2(\mathbb{R}^d)}.
\]
Together with (2.4), this implies
\[
\|\varphi_\varepsilon\|_{H^1(\mathbb{R}^d)} \leq 9\varepsilon^2 c(\phi) \|\varphi_\varepsilon\|_{H^1(\mathbb{R}^d)} + 9\varepsilon(\phi)^2 \|\varphi_\varepsilon\|_{L^\infty} + 2\|\varphi_\varepsilon\|_{L^2(\mathbb{R}^d)}.
\]
Recalling that \(\varphi_\varepsilon = w_\varepsilon - \varphi_\varepsilon\), we obtain (5.3) with the constants \(C_{13} := 9\varepsilon^2 c_4^2 + 1\) and \(C_{12} := 3\sqrt{2}c_4\|Q_0\|_{L^\infty}^2\).

5.2. Estimates for the function \(\varphi_\varepsilon\).

**Lemma 5.2.** Let \(\varphi_\varepsilon\) be given by (5.2). Then for \(\varepsilon \in \mathbb{C} \setminus \mathbb{R}_+, \|\varphi_\varepsilon\| \geq 1\), and \(0 < \varepsilon \leq \varepsilon_1\) we have
\[
\|\varphi_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C_{14}\varepsilon^{1/2}\|F\|_{L^2(\mathbb{R}^d)},
\]
\[
\|D\varphi_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon \left(C_{15}\varepsilon^{1/2}\|\varphi_\varepsilon\|^{-1/4} + C_{16}\varepsilon\right)\|F\|_{L^2(\mathbb{R}^d)}.
\]
The constants \(C_{14}\), \(C_{15}\), and \(C_{16}\) depend only on the initial data (1.9) of (1.10) and the domain \(\mathcal{O}\).

**Proof.** First, we prove (5.8). From (1.3), (4.22), (4.23), (5.1), and (5.2) it follows that
\[
\|\varphi_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq \varepsilon M_1\alpha_1^{1/2}\|\varphi_\varepsilon\|_{H^1(\mathbb{R}^d)} + \varepsilon M_1\|u_0\|_{L^2(\mathbb{R}^d)}.
\]
Combining this with (1.11), (4.2), and the inequality \(\|\varphi_\varepsilon\| \geq 1\), we obtain (5.8) with the constant \(C_{14} := M_1\alpha_1^{1/2}k_2 + \tilde{M}_1k_1\).

To prove (5.9), consider the derivatives:
\[
\partial_j \varphi_\varepsilon = \varepsilon (\partial_j \theta_\varepsilon)(\Lambda^\varepsilon S_\varepsilon b(D)\tilde{u}_0 + \tilde{\Lambda}^\varepsilon S_\varepsilon \tilde{u}_0) + \theta_\varepsilon\left( (\partial_j \Lambda)^\varepsilon S_\varepsilon b(D)\tilde{u}_0 + (\partial_j \tilde{\Lambda})^\varepsilon S_\varepsilon \tilde{u}_0 \right)
\]
\[
+ \varepsilon \theta_\varepsilon(\Lambda^\varepsilon S_\varepsilon b(D)\partial_j \tilde{u}_0 + \tilde{\Lambda}^\varepsilon S_\varepsilon \partial_j \tilde{u}_0).
\]
Hence,
\[
\|D\varphi_\varepsilon\|_{L^2(\mathbb{R}^d)}^2 \leq 3\varepsilon^2 \|\varphi_\varepsilon\|_{H^1(\mathbb{R}^d)}^2(\Lambda^\varepsilon S_\varepsilon b(D)\tilde{u}_0 + \tilde{\Lambda}^\varepsilon S_\varepsilon \tilde{u}_0)^2 + 3\|\varphi_\varepsilon\|_{H^1(\mathbb{R}^d)}^2 b(D)\tilde{u}_0 + (\partial_j \Lambda)^\varepsilon S_\varepsilon \tilde{u}_0)\|_{L^2(\mathbb{R}^d)}^2
\]
\[
+ 3\varepsilon^2 \sum_{j=1}^d \|\varphi_\varepsilon(\Lambda^\varepsilon S_\varepsilon b(D)\partial_j \tilde{u}_0 + \tilde{\Lambda}^\varepsilon S_\varepsilon \partial_j \tilde{u}_0)\|^2_{L^2(\mathbb{R}^d)}.
\]
Denote the consecutive terms in the right-hand side of (5.10) by \(J_1(\varepsilon)\), \(J_2(\varepsilon)\), and \(J_3(\varepsilon)\). The term \(J_1(\varepsilon)\) is estimated with the help of (5.11) and Lemma 5.2
\[
J_1(\varepsilon) \leq 6\mu^2 \left( \int_{\partial\Omega} |\Lambda^\varepsilon S_\varepsilon b(D)\tilde{u}_0|^2 \ dx + \int_{\partial\Omega} |\tilde{\Lambda}^\varepsilon S_\varepsilon \tilde{u}_0|^2 \ dx \right)
\]
\[
\leq 6\mu^2 \beta_2 \varepsilon \Omega^{-1/2} \|\Lambda\|_{L^2(\mathbb{R}^d)}^2 \|b(D)\tilde{u}_0\|_{H^1(\mathbb{R}^d)} \|b(D)\tilde{u}_0\|_{L^2(\mathbb{R}^d)}
\]
\[
+ 6\mu^2 \beta_2 \varepsilon \Omega^{-1/2} \|\tilde{\Lambda}\|_{L^2(\mathbb{R}^d)}^2 \|\tilde{u}_0\|_{H^1(\mathbb{R}^d)} \|\tilde{u}_0\|_{L^2(\mathbb{R}^d)}, \quad 0 < \varepsilon \leq \varepsilon_1.
\]
According to (1.29) and (4.23), \(\|\Omega^{-1/2} \|\Lambda\|_{L^2(\mathbb{R}^d)} \leq \tilde{M}_1\). Combining this with (1.3), (1.22), and (5.11), we obtain
\[
J_1(\varepsilon) \leq 6\mu^2 \beta_2 \varepsilon \left( M_1^2 \alpha_1 \|\tilde{u}_0\|_{H^1(\mathbb{R}^d)} + \tilde{M}_1^2 \|\tilde{u}_0\|_{H^1(\mathbb{R}^d)} \|\tilde{u}_0\|_{L^2(\mathbb{R}^d)} \right), \quad 0 < \varepsilon \leq \varepsilon_1.
\]
Together with (4.1) and the inequality \(\|\varphi_\varepsilon\| \geq 1\), this implies
\[
J_1(\varepsilon) \leq \kappa_1 \varepsilon \|\varphi_\varepsilon\|^{-1/2} \|F\|_{L^2(\mathbb{R}^d)}^2, \quad 0 < \varepsilon \leq \varepsilon_1; \quad \kappa_1 := 6\mu^2 \beta_2 k_2 (M_1^2 \alpha_1 k_3 + \tilde{M}_1^2 k_1).
\]
From (1.30) it follows that \(\|\Omega^{-1/2} \|D\tilde{\Lambda}\|_{L^2(\mathbb{R}^d)} \leq \tilde{M}_2\), where
\[
\tilde{M}_2 := \|\Omega^{-1/2} C_{a1}^{1/2} \alpha_0^{-1} \|g^{-1}\|_{L^\infty},
\]
The term $J_2(\varepsilon)$ is estimated similarly to $J_1(\varepsilon)$ with the help of Lemma 3.2 and relations (1.3), (1.23), (1.41), and (1.43). We arrive at
\[ J_2(\varepsilon) \leq \kappa_2(\phi)^2 \varepsilon |\varepsilon|^{-1/2} \| F \|_{L^2(\Omega)}^2, \quad 0 < \varepsilon \leq \varepsilon_1; \quad \kappa_2 := 6\beta_2k_2(M_2^2k_3\alpha_1 + \tilde{M}_2^2k_1). \]  
(5.14)

Finally, the term $J_3(\varepsilon)$ is estimated by using (1.3), (4.22), (4.23), and (5.1):
\[ J_3(\varepsilon) \leq 6\varepsilon^2 \left( M_1^2\alpha_1 \| \bar{u}_0 \|_{H^2(\mathbb{R}^d)}^2 + \tilde{M}_1^2 \| \bar{u}_0 \|_{H^2(\mathbb{R}^d)}^2 \right). \]

Together with (4.12), (4.13), and the inequality $|\varepsilon| \geq 1$, this yields
\[ J_3(\varepsilon) \leq \kappa_3(\phi)\varepsilon^2 \| F \|_{L^2(\Omega)}^2, \quad 0 < \varepsilon \leq \varepsilon_1; \quad \kappa_3 := 6M_1^2\alpha_1k_3^2 + 6\tilde{M}_1^2k_2^2. \]  
(5.15)

Now, relations (5.10), (5.12), (5.14), and (5.15) imply (5.9) with the constants $C_{15} := (\kappa_1 + \kappa_2)^{1/2}$ and $C_{16} := \kappa_3^{1/2}$. □

5.3. Completion of the proof of Theorem 2.6 From Lemmas 5.1 and 5.2 it follows that
\[ \| \textbf{w}_\varepsilon \|_{H^1(\Omega)} \leq c(\phi)^2 \left( C_{13}C_{15}^{1/2}|\varepsilon|^{-1/4} + (C_{12}C_{14} + C_{13}C_{14} + C_{13}C_{16})\varepsilon \right) \| F \|_{L^2(\Omega)} \]
for $\varepsilon \in \mathbb{C} \setminus \mathbb{R}_+$, $|\varepsilon| \geq 1$, and $0 < \varepsilon \leq \varepsilon_1$. Together with (1.19), this implies (2.24) with the constants $C_5 := C_{12}C_{13}$ and $C_6 := C_7 + C_{12}C_{14} + C_{13}C_{14} + C_{13}C_{16}$.

It remains to check (2.23). By (1.4) and (2.24),
\[ \| \textbf{p}_\varepsilon - g^\varepsilon b(\textbf{D})\textbf{v}_\varepsilon \|_{L^2(\Omega)} \leq \| g \|_{L^\infty(d\alpha_1)^{1/2}}(C_5C(\phi)^2\varepsilon^{-1/2}|\varepsilon|^{-1/4} + C_6c(\phi)^4\varepsilon) \| F \|_{L^2(\Omega)}. \]  
(5.16)

We have
\[ g^\varepsilon b(\textbf{D})\textbf{v}_\varepsilon = g^\varepsilon b(\textbf{D})\bar{u}_0 + g^\varepsilon(b(\textbf{D})\Lambda)\varepsilon S_\varepsilon b(\textbf{D})\bar{u}_0 + g^\varepsilon(b(\textbf{D})\Lambda)\varepsilon S_\varepsilon D\bar{u}_0 + \varepsilon \sum_{l=1}^d g^\varepsilon b_l(\Lambda^\varepsilon S_\varepsilon b(\textbf{D})D_l\bar{u}_0 + \tilde{\Lambda}^\varepsilon S_\varepsilon D_l\tilde{u}_0). \]  
(5.17)

The fourth term in the right-hand side of (5.17) is estimated with the help of (1.4), (4.22), and (4.23):
\[ \left\| \varepsilon \sum_{l=1}^d g^\varepsilon b_l(\Lambda^\varepsilon S_\varepsilon b(\textbf{D})D_l\bar{u}_0 + \tilde{\Lambda}^\varepsilon S_\varepsilon D_l\tilde{u}_0) \right\|_{L^2(\Omega)} \leq \varepsilon \| g \|_{L^\infty(\Omega)}^{1/2} \left( M_1 \sum_{l=1}^d \| b(\textbf{D})D_l\bar{u}_0 \|_{L^2(\mathbb{R}^d)} + \tilde{M}_1 \sum_{l=1}^d \| D_l\tilde{u}_0 \|_{L^2(\mathbb{R}^d)} \right). \]  
(5.18)

Combining this with (1.3), (4.2), (4.3), and the condition $|\varepsilon| \geq 1$, we deduce
\[ \left\| \varepsilon \sum_{l=1}^d g^\varepsilon b_l(\Lambda^\varepsilon S_\varepsilon b(\textbf{D})D_l\bar{u}_0 + \tilde{\Lambda}^\varepsilon S_\varepsilon D_l\tilde{u}_0) \right\|_{L^2(\Omega)} \leq C_{17}c(\phi)\varepsilon \| F \|_{L^2(\Omega)}, \]  
(5.19)

where $C_{17} := \| g \|_{L^\infty(d\alpha_1)^{1/2}}(M_1\alpha_1^{1/2}k_3 + \tilde{M}_1k_2)$.

Next, by Proposition 1.11 we have
\[ \| g^\varepsilon b(\textbf{D})\bar{u}_0 - g^\varepsilon S_\varepsilon b(\textbf{D})\bar{u}_0 \|_{L^2(\mathbb{R}^d)} \leq \varepsilon r_1 \| g \|_{L^\infty} \| \textbf{D}b(\textbf{D})\bar{u}_0 \|_{L^2(\mathbb{R}^d)}. \]  
(5.20)

Together with (1.3) and (4.3), this implies
\[ \| g^\varepsilon b(\textbf{D})\bar{u}_0 - g^\varepsilon S_\varepsilon b(\textbf{D})\bar{u}_0 \|_{L^2(\mathbb{R}^d)} \leq C_{18}c(\phi)\varepsilon \| F \|_{L^2(\Omega)}, \]  
(5.21)

where $C_{18} := r_1\alpha_1^{1/2}k_3\| g \|_{L^\infty}$. Now, relations (1.20), (5.16), (5.17), (5.19), and (5.21) imply (2.23) with the constants $C_5 := (d\alpha_1)^{1/2}\| g \|_{L^\infty}C_5$ and $C_6 := (d\alpha_1)^{1/2}\| g \|_{L^\infty}C_6 + C_{17} + C_{18}$. □
6. The proof of Theorem 2.3

6.1. Estimate for the discrepancy \( w_\varepsilon \) in \( L_2 \).

**Lemma 6.1.** Suppose that \( w_\varepsilon \) is the solution of problem (2.13). Suppose that the number \( \varepsilon_1 \) is subject to Condition (2.4). Then for \( \xi \in \mathbb{C} \setminus \mathbb{R}_+ \), \( |\xi| \geq 1 \), and \( 0 < \varepsilon \leq \varepsilon_1 \) we have

\[
\|w_\varepsilon\|_{L_2(O)} \leq c(\phi)^5 (C_{19} \varepsilon |\xi|^{-1/2} + C_{20}) \|F\|_{L_2(O)}.
\]  

(6.1)

The constants \( C_{19} \) and \( C_{20} \) depend only on the initial data (1.3) and the domain \( O \).

**Proof.** Recall that \( \varphi_\varepsilon = w_\varepsilon - \varphi_\varepsilon \) satisfies (5.2). We substitute \( \eta = \eta_\varepsilon = (B_{D,\varepsilon} - \zeta^* Q_0^{-1}) \Phi \) with \( \Phi \in L_2(O; \mathbb{C}^n) \) into this identity. Then the left-hand side of (6.1) can be written as

\[
b_{D,\varepsilon} \lambda_0 \eta_\varepsilon - \zeta(Q_0 w_\varepsilon, \eta_\varepsilon)_{L_2(O)} = (\varphi_\varepsilon, \Phi)_{L_2(O)}.
\]

Hence,

\[
(w_\varepsilon - \varphi_\varepsilon, \Phi)_{L_2(O)} = -b_{N,\varepsilon}[\varphi_\varepsilon, \eta_\varepsilon] + \zeta(Q_0 \varphi_\varepsilon, \eta_\varepsilon)_{L_2(O)}.
\]  

(6.2)

To approximate \( \eta_\varepsilon \) in \( H^1(O; \mathbb{C}^n) \), we apply the already proved Theorem 2.6. Denote \( \eta_0 = (B_D^0 - \zeta^* Q_0^{-1}) \Phi \) and \( \eta_0 = P_O \eta_0 \). The first order approximation of \( \eta_\varepsilon \) is given by

\[
\eta_0 + \varepsilon \Lambda^\varepsilon S_\varepsilon(b(D) \eta_0) - \varepsilon \Lambda^\varepsilon S_\varepsilon \eta_\varepsilon.
\]

To estimate \( \eta_\varepsilon \), we apply (2.21), Theorem 2.6 and Lemma 5.2

\[
|I_4(\varepsilon)| \leq C_{21} \varepsilon \|F\|_{L_2(O)} \|\Phi\|_{L_2(O)}; \quad C_{21} := C_{14} \|Q_0\|_{L_\infty} \|Q_0^{-1}\|_{L_\infty}.
\]  

(6.4)

Next, we have

\[
I_2(\varepsilon) = -b_{N,\varepsilon}[\varphi_\varepsilon, \eta_0] = -b_{N,\varepsilon}[\varphi_\varepsilon, S_\varepsilon \eta_\varepsilon] - b_{N,\varepsilon}[\varphi_\varepsilon, S_\varepsilon \eta_0].
\]  

(6.6)

From Proposition 1.1 and estimate (4.3) for \( \eta_0 \) it follows that

\[
\|\eta_0 - S_\varepsilon \eta_\varepsilon\|_{H^1(O)} \leq \|\eta_0 - S_\varepsilon \eta_\varepsilon\|_{H^2(\mathbb{R})} = \varepsilon \eta_1 \|\eta_0\|_{H^2(\mathbb{R})} \leq c(\phi) \varepsilon \varepsilon \varepsilon_3 \|\Phi\|_{L_2(O)}.
\]

Combining this with (2.21) and Lemma 5.2, we obtain

\[
|b_{N,\varepsilon}[\varphi_\varepsilon, \eta_0] - S_\varepsilon \eta_\varepsilon| \leq c(\phi)^2 (\gamma_3 \varepsilon |\xi|^{-1/2} + \gamma_4 \varepsilon^2) \|\eta_0\|_{L_2(O)} \|\Phi\|_{L_2(O)}.
\]  

(6.7)

where \( \gamma_3 := c_2 k_3 C_{15} \) and \( \gamma_4 := c_2 k_3 C_{14} + C_{15} + C_{16} \).

Let us estimate the first term in the right-hand side of (6.6). According to (2.13),

\[
|b_{N,\varepsilon}[\varphi_\varepsilon, S_\varepsilon \eta_\varepsilon]| \leq \left| \int_O (g^\varepsilon b(D) \varphi_\varepsilon, b(D) S_\varepsilon \eta_\varepsilon) \, dx \right|
\]

\[+ \sum_{j=1}^d \int_O (|\langle a_j^\varepsilon D_j \varphi_\varepsilon, S_\varepsilon \eta_\varepsilon \rangle| + |\langle (a_j^\varepsilon)^* \varphi_\varepsilon, D_j S_\varepsilon \eta_\varepsilon \rangle|) \, dx \]

\[+ \int_O \langle Q_0 \varphi_\varepsilon, S_\varepsilon \eta_\varepsilon \rangle \, dx + \lambda \int_O \langle Q_0^* \varphi_\varepsilon, S_\varepsilon \eta_\varepsilon \rangle \, dx \]

\[= \sum_{k=1}^4 \tilde{I}_k(\varepsilon).
\]  

(6.8)
Since $\varphi_\gamma$ is supported in $(\partial \Omega)_\gamma$, all integrals in (6.8) are taken over $(\partial \Omega)_\gamma \cap \Omega$. The term $I_2^{(1)}(\varepsilon)$ is estimated with the help of Lemma 3.3, (1.2), and (1.3):

$$I_2^{(1)}(\varepsilon) \leq \|g\|_{L_\infty} \alpha_1^{1/2} \|D \varphi_\varepsilon\|_{L_2(\mathbb{R}^d)} \left(\int_{(\partial \Omega)_{\gamma}} |b(D)S_\varepsilon \tilde{n}_0|^2 \, dx\right)^{1/2}$$

$$\leq \|g\|_{L_\infty} \alpha_1^{1/2} \|D \varphi_\varepsilon\|_{L_2(\mathbb{R}^d)} (\beta \varepsilon)^{1/2} \left(\|b(D)\tilde{n}_0\|_{H^1(\mathbb{R}^d)} \|b(D)\tilde{n}_0\|_{L_2(\mathbb{R}^d)}\right)^{1/2}.$$ 

Applying (1.2), (4.2) and (1.3) for $\tilde{n}_0$, and (5.9), we see that

$$I_2^{(1)}(\varepsilon) \leq c(\phi)^2 (\gamma_5 \varepsilon |\zeta|^{-1/2} + \gamma_6 \varepsilon^2) \|F\|_{L_2(\Omega)} \|\Phi\|_{L_2(\Omega)},$$

(6.9)

where $\gamma_5 := \beta^{1/2} \|g\|_{L_\infty} \alpha_1 (k_2 k_3)^{1/2} (C_{15} + C_{16})$ and $\gamma_6 := \beta^{1/2} \|g\|_{L_\infty} \alpha_1 (k_2 k_3)^{1/2} C_{16}$.

The term $I_2^{(2)}(\varepsilon)$ satisfies

$$I_2^{(2)}(\varepsilon) \leq \sum_{j=1}^d \|D_j \varphi_\varepsilon\|_{L_2(\mathbb{R}^d)} \left(\int_{(\partial \Omega)_{\gamma}} |(a_j^\varepsilon)^* S_\varepsilon \tilde{n}_0|^2 \, dx\right)^{1/2} + \sum_{j=1}^d \|\varphi_\varepsilon\|_{L_2(\mathbb{R}^d)} |a_j^\varepsilon| \|D_j \tilde{n}_0\|_{L_2(\mathbb{R}^d)}.$$ 

By Lemma 3.2 we have

$$\int_{(\partial \Omega)_{\gamma}} |(a_j^\varepsilon)^* S_\varepsilon \tilde{n}_0|^2 \, dx \leq \beta_\varepsilon \varepsilon |\Omega|^{-1} \|a_j\|_{L_2(\Omega)}^2 \|\tilde{n}_0\|_{H^1(\mathbb{R}^d)} \|\tilde{n}_0\|_{L_2(\mathbb{R}^d)}.$$ 

(6.10)

Combining this with (4.1), (4.2) for $\tilde{n}_0$ and (5.8), we obtain the following estimate for the first summand in the right-hand side of (6.10):

$$\sum_{j=1}^d \|D_j \varphi_\varepsilon\|_{L_2(\mathbb{R}^d)} \left(\int_{(\partial \Omega)_{\gamma}} |(a_j^\varepsilon)^* S_\varepsilon \tilde{n}_0|^2 \, dx\right)^{1/2} \leq \gamma_7 c(\phi)^2 \varepsilon |\zeta|^{-1/2} \|F\|_{L_2(\Omega)} \|\Phi\|_{L_2(\Omega)},$$

(6.11)

where $\gamma_7 := C_a (\beta_\varepsilon |\Omega|^{-1} k_1 k_2)^{1/2} (C_{15} + C_{16})$. The second summand in the right-hand side of (6.10) is estimated by Proposition 1.2, (4.2) for $\tilde{n}_0$, and (5.8):

$$\sum_{j=1}^d \|\varphi_\varepsilon\|_{L_2(\mathbb{R}^d)} |a_j^\varepsilon| \|S_\varepsilon D_j \tilde{n}_0\|_{L_2(\mathbb{R}^d)} \leq |\Omega|^{-1/2} \sum_{j=1}^d \|\varphi_\varepsilon\|_{L_2(\mathbb{R}^d)} |a_j| \|D_j \tilde{n}_0\|_{L_2(\mathbb{R}^d)}$$

$$\leq \gamma_8 c(\phi)^2 \varepsilon |\zeta|^{-1/2} \|F\|_{L_2(\Omega)} \|\Phi\|_{L_2(\Omega)},$$

where $\gamma_8 := |\Omega|^{-1/2} C_a C_1 k_2$. Together with (6.10) and (6.11), this implies

$$I_2^{(2)}(\varepsilon) \leq (\gamma_7 + \gamma_8) c(\phi)^2 \varepsilon |\zeta|^{-1/2} \|F\|_{L_2(\Omega)} \|\Phi\|_{L_2(\Omega)}.$$ 

(6.12)

We proceed to estimation of the term $I_2^{(3)}(\varepsilon)$:

$$I_2^{(3)}(\varepsilon) \leq \|Q^\varepsilon\|_{L_2(\mathbb{R}^d)} \left(\int_{(\partial \Omega)_{\gamma}} |Q^\varepsilon|^2 |S_\varepsilon \tilde{n}_0|^2 \, dx\right)^{1/2}.$$ 

(6.13)

The first factor in the right-hand side of (6.13) is estimated by Lemma 3.3 and condition (1.8):

$$\|Q^\varepsilon\|_{L_2(\mathbb{R}^d)} \leq C(\hat{q}, \Omega) \|Q\|_{L_2(\Omega)} \|\varphi_\varepsilon\|_{H^1(\mathbb{R}^d)},$$

(6.14)

where $\hat{q} = \infty$ for $d = 1$, $\hat{q} = 2 s/(s - 1)$ for $d \geq 2$. The second factor in the right-hand side of (6.13) is estimated with the help of Lemma 3.2

$$\int_{(\partial \Omega)_{\gamma}} |Q^\varepsilon|^2 |S_\varepsilon \tilde{n}_0|^2 \, dx \leq \beta_\varepsilon |\Omega|^{-1} \|Q\|_{L_1(\Omega)} \|\tilde{n}_0\|_{H^1(\mathbb{R}^d)} \|\tilde{n}_0\|_{L_2(\mathbb{R}^d)}.$$ 

(6.15)

Combining (4.1) and (4.2) for $\tilde{n}_0$, (6.13)–(6.15), and using Lemma 3.2 we find

$$I_2^{(3)}(\varepsilon) \leq \gamma_9 c(\phi)^2 \varepsilon |\zeta|^{-1/2} \|F\|_{L_2(\Omega)} \|\Phi\|_{L_2(\Omega)},$$

(6.16)

where $\gamma_9 := C(\hat{q}, \Omega) \|Q\|_{L_2(\Omega)} \|Q\|_{L_1(\Omega)} (\beta_\varepsilon |\Omega|^{-1} k_1 k_2)^{1/2} (C_{14} + C_{15} + C_{16})$.

Relations (1.2), (4.1) for $\tilde{n}_0$, and (5.8) imply the following estimate for the term $I_2^{(4)}(\varepsilon)$:

$$I_2^{(4)}(\varepsilon) \leq \lambda \|Q_0\|_{L_\infty} \|\varphi_\varepsilon\|_{L_2(\mathbb{R}^d)} |S_\varepsilon \tilde{n}_0|_{L_2(\mathbb{R}^d)} \leq \gamma_{10} c(\phi)^2 \varepsilon |\zeta|^{-1/2} \|F\|_{L_2(\Omega)} \|\Phi\|_{L_2(\Omega)},$$

(6.17)
where \( \gamma_{10} := \lambda \| Q_0 \|_{L_\infty} C_{14} k_1 \).

Thus, combining (6.6), (6.9), (6.12), (6.16), and (6.17), we obtain
\[
\mathcal{I}_2(\varepsilon) \leq c(\phi)^2 \left( \tilde{\gamma} \varepsilon |\xi|^{-1/2} + \tilde{\gamma} \varepsilon^2 \right) \| F \|_{L_2(\Omega)} \| \Phi \|_{L_2(\Omega)},
\] (6.18)
where \( \tilde{\gamma} := \gamma_{13} + \gamma_{3} + \gamma_{7} + \gamma_{8} + \gamma_{9} + \gamma_{10} \) and \( \tilde{\gamma} := \gamma_{4} + \gamma_{6} \).

It remains to estimate \( \mathcal{I}_3(\varepsilon) \):
\[
\mathcal{I}_3(\varepsilon) = |b_{N, \varepsilon}[\varphi, \varepsilon \Lambda^\varepsilon S_{\varepsilon} b(D) \tilde{\eta}_0 + \varepsilon \Lambda^\varepsilon S_{\varepsilon} \tilde{\eta}_0]|
\]
\[
\leq \left| (g^2 b(D) \varphi, (b(D) \Lambda)^\varepsilon S_{\varepsilon} b(D) \tilde{\eta}_0)_{L_2(\Omega)} \right|
\]
\[
+ \left| (g^2 b(D) \varphi, (b(D) \Lambda)^\varepsilon S_{\varepsilon} \tilde{\eta}_0)_{L_2(\Omega)} \right|
\]
\[
+ \left| \left( g^2 b(D) \varphi, \varepsilon \sum_{l=1}^{d} b_{l} \Lambda^\varepsilon S_{\varepsilon} D_{l} \tilde{\eta}_0 \right)_{L_2(\Omega)} \right|
\]
\[
+ \left| \left( g^2 b(D) \varphi, \varepsilon \sum_{l=1}^{d} b_{l} \Lambda^\varepsilon S_{\varepsilon} D_{l} \tilde{\eta}_0 \right)_{L_2(\Omega)} \right|
\]
\[
+ \sum_{j=1}^{d} \left| \left( a_{j}^2 D_{j} \varphi, \varepsilon \Lambda^\varepsilon S_{\varepsilon} b(D) \tilde{\eta}_0 + \varepsilon \Lambda^\varepsilon S_{\varepsilon} \tilde{\eta}_0 \right)_{L_2(\Omega)} \right|
\]
\[
+ \sum_{j=1}^{d} \left| \left( a_{j}^2 D_{j} \varphi, \varepsilon \Lambda^\varepsilon S_{\varepsilon} b(D) \tilde{\eta}_0 + \varepsilon \Lambda^\varepsilon S_{\varepsilon} \tilde{\eta}_0 \right)_{L_2(\Omega)} \right|
\]
\[
+ \left| \left( Q^2 \varphi, \varepsilon \Lambda^\varepsilon S_{\varepsilon} b(D) \tilde{\eta}_0 + \varepsilon \Lambda^\varepsilon S_{\varepsilon} \tilde{\eta}_0 \right)_{L_2(\Omega)} \right|
\]
\[
+ \lambda \left| \left( Q_0^2 \varphi, \varepsilon \Lambda^\varepsilon S_{\varepsilon} b(D) \tilde{\eta}_0 + \varepsilon \Lambda^\varepsilon S_{\varepsilon} \tilde{\eta}_0 \right)_{L_2(\Omega)} \right|
\].

The consecutive terms in the right-hand side of (6.19) are denoted by \( \mathcal{I}_{3,j}^{(j)}(\varepsilon), j = 1, \ldots, 9 \).

Using (1.3) and Lemma 3.2 and taking into account that \( \varphi_\varepsilon \) is supported in \( (\partial \Omega)_\varepsilon \), we estimate the first term:
\[
\mathcal{I}_3^{(1)}(\varepsilon) \leq \| g \|_{L_\infty} \alpha_{1/2} \| D \varphi \|_{L_2(\mathbb{R}^d)} \left( \int_{(\partial \Omega)_\varepsilon} \| (b(D) \Lambda)^\varepsilon S_{\varepsilon} b(D) \tilde{\eta}_0 \|_{L_2(\Omega)}^2 \, dx \right)^{1/2}
\]
\[
\leq \| g \|_{L_\infty} \alpha_{1/2} \| D \varphi \|_{L_2(\mathbb{R}^d)} \| (b(D) \Lambda)^\varepsilon S_{\varepsilon} \tilde{\eta}_0 \|_{L_2(\Omega)} \| b(D) \tilde{\eta}_0 \|_{H^1(\mathbb{R}^d)} \| b(D) \tilde{\eta}_0 \|_{L_2(\mathbb{R}^d)}^{1/2}.
\]

Now we apply Lemma 5.2 and estimates (4.19), (4.23) for \( \tilde{\eta}_0 \). Taking (1.3) and (1.21) into account, we arrive at
\[
\mathcal{I}_3^{(1)}(\varepsilon) \leq c(\phi)^2 \left( \gamma_{11} |\xi|^{-1/2} + \gamma_{12} \varepsilon^2 \right) \| F \|_{L_2(\Omega)} \| \Phi \|_{L_2(\Omega)}.
\] (6.20)

Here \( \gamma_{11} := \| g \|_{L_\infty}^{3/2} g^{-1} \|_{L_\infty}^{1/2} \alpha_1 (m_{\beta, s} k_2 k_3)^{1/2} (C_{15} + C_{16}) \) and \( \gamma_{12} := \| g \|_{L_\infty}^{3/2} g^{-1} \|_{L_\infty}^{1/2} \alpha_1 (m_{\beta, s} k_2 k_3)^{1/2} C_{16} \). In a similar way, using (1.31), we obtain
\[
\mathcal{I}_3^{(2)}(\varepsilon) \leq c(\phi)^2 \varepsilon |\xi|^{-1/2} \| F \|_{L_2(\Omega)} \| \Phi \|_{L_2(\Omega)},
\] (6.21)
where \( \gamma_{13} := \| g \|_{L_\infty} \| g^{-1} \|_{L_\infty} \alpha_1 (m_{\beta, s} n k_1 k_2)^{1/2} (\Omega)^{-1/2} \alpha_0^{-1/2} C_{6} (C_{15} + C_{16}) \).

To estimate \( \mathcal{I}_3^{(3)}(\varepsilon) \), we apply (1.3), (1.4), and (1.22):
\[
\mathcal{I}_3^{(3)}(\varepsilon) \leq c \| g \|_{L_\infty} \alpha_1^{3/2} d^{1/2} M_1 \left( D \varphi \right)_{L_2(\mathbb{R}^d)} \| \tilde{\eta}_0 \|_{H^2(\mathbb{R}^d)}.
\]

Together with (1.3) for \( \tilde{\eta}_0 \) and Lemma 5.2 this implies
\[
\mathcal{I}_3^{(3)}(\varepsilon) \leq c(\phi)^2 \left( \gamma_{14} |\xi|^{-1/2} + \gamma_{15} \varepsilon^2 \right) \| F \|_{L_2(\Omega)} \| \Phi \|_{L_2(\Omega)},
\] (6.22)
where \( \gamma_{14} := \|g\|_{L\infty} \alpha_1^{3/2} d^{1/2} M_1 k_3 C_{15} \) and \( \gamma_{15} := \|g\|_{L\infty} \alpha_1^{3/2} d^{1/2} M_1 k_3 (C_{15} + C_{16}) \).

In a similar way, using (6.23), we obtain
\[
I_3^{(4)}(\varepsilon) \leq \gamma_{16} C(\phi)^2 \varepsilon |\xi|^{-1/2} \|F\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)},
\]
where \( \gamma_{16} := \|g\|_{L\infty} d^{1/2} \alpha_1 M_1 k_2 (C_{15} + C_{16}) \).

Now, we estimate the term \( I_3^{(5)}(\varepsilon) \):
\[
I_3^{(5)}(\varepsilon) \leq \varepsilon \sum_{j=1}^d \|D_j \varphi \|_{L^2(\mathbb{R}^d)} \left( \|(a_j^{*})^* \Lambda \Sigma \varphi_{b(D)} \tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} + \|(a_j^{*})^* \Lambda \Sigma \varphi_{\tilde{\eta}_0}\|_{L^2(\mathbb{R}^d)} \right).
\]

By Proposition 1.2
\[
\|(a_j^{*})^* \Lambda \Sigma \varphi_{b(D)} \tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} \leq |\Omega|^{-1/2} \|a_j^{*} \Lambda\|_{L^2(\Omega)} \|b(D) \tilde{\eta}_0\|_{L^2(\mathbb{R}^d)},
\]
From the H"older inequality and the Sobolev embedding theorem it follows that
\[
\|a_j^{*} \Lambda\|_{L^2(\Omega)} \leq C(q, \Omega) \|a_j\|_{L^\infty(\Omega)} \|\Lambda\|_{H^1(\Omega)},
\]
where \( q = \infty \) for \( d = 1 \) and \( q = 2\rho/(\rho - 2) \) for \( d \geq 2 \). Similarly,
\[
\|(a_j^{*})^* \Lambda \Sigma \varphi_{\tilde{\eta}_0}\|_{L^2(\mathbb{R}^d)} \leq |\Omega|^{-1/2} C(q, \Omega) \|a_j\|_{L^\infty(\Omega)} \|\Lambda\|_{H^1(\Omega)} \|\tilde{\eta}_0\|_{L^2(\mathbb{R}^d)}.
\]

From (6.24) and (6.27) it follows that
\[
I_3^{(5)}(\varepsilon) \leq \varepsilon \tilde{C}_a C(q, \Omega) |\Omega|^{-1/2} \|D \varphi\|_{L^2(\mathbb{R}^d)} \left( \|\Lambda\|_{H^1(\Omega)} \|b(D) \tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} + \|\Lambda\|_{H^1(\Omega)} \|\tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} \right).
\]

By (6.23),
\[
|\Omega|^{-1/2} \|\Lambda\|_{H^1(\Omega)} \leq M_1 + M_2.
\]
According to (6.20), (6.30), (6.28), and (6.34),
\[
|\Omega|^{-1/2} \|\Lambda\|_{H^1(\Omega)} \leq \tilde{M}_1 + \tilde{M}_2.
\]
Relations (6.3), (6.9), (6.28)–(6.30), and inequalities (6.18), (6.19) for \( \tilde{\eta}_0 \) imply that
\[
I_3^{(5)}(\varepsilon) \leq \gamma_{17} C(\phi)^2 \varepsilon |\xi|^{-1/2} \|F\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)}.
\]

Here \( \gamma_{17} := \tilde{C}_a C(q, \Omega) (C_{15} + C_{16}) (M_1 + M_2) \alpha_1^{1/2} k_2 + (\tilde{M}_1 + \tilde{M}_2) k_1 \).

We proceed to estimation of \( I_3^{(6)}(\varepsilon) \):
\[
I_3^{(6)}(\varepsilon) \leq \sum_{j=1}^d \|D_j \varphi\|_{L^2(\mathbb{R}^d)} \left( \int_{\partial \Omega} |(D_j \Lambda)^c \Sigma \varphi_{b(D)} \tilde{\eta}_0|^2 \, dx \right)^{1/2} + \sum_{j=1}^d \|D_j \varphi\|_{L^2(\mathbb{R}^d)} \left( \int_{\partial \Omega} |(D_j \Lambda)^c \Sigma \varphi_{\tilde{\eta}_0}|^2 \, dx \right)^{1/2}.
\]

From Lemma 3.4 it follows that
\[
\|(a_j^{*})^* \varphi\|_{L^2(\mathbb{R}^d)} \leq C(q, \Omega) \|a_j\|_{L^\infty(\Omega)} \|\varphi\|_{H^1(\mathbb{R}^d)},
\]
where \( q = \infty \) for \( d = 1 \), \( q = 2\rho/(\rho - 2) \) for \( d \geq 2 \). By (6.32), (6.33), and Lemma 5.2 we have
\[
I_3^{(6)}(\varepsilon) \leq C(q, \Omega) \tilde{C}_a (\beta_4 |\xi|^{-1} \varepsilon)^{1/2} \|\varphi\|_{H^1(\mathbb{R}^d)} 
\times \left( \|D \Lambda\|_{L^2(\Omega)} \|b(D) \tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} + \|D \Lambda\|_{L^2(\Omega)} \|\tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} \right) 
\leq C(q, \Omega) \tilde{C}_a M_2 (\alpha_1 k_3)^{1/2} + \tilde{M}_2 k_1^{1/2}.
\]

Combining this with (6.3), (6.23), (6.30), (5.13), inequalities (6.11)–(6.13) for \( \tilde{\eta}_0 \), and Lemma 5.2 we obtain
\[
I_3^{(6)}(\varepsilon) \leq C(q, \Omega) C(q, \Omega) \tilde{C}_a M_2 (\alpha_1 k_3)^{1/2} + \tilde{M}_2 k_1^{1/2},
\]
where \( \gamma_{18} := (C_{14} + C_{15} + C_{16}) C(q, \Omega) \tilde{C}_a (\beta_4 k_2)^{1/2} (M_2 (\alpha_1 k_3)^{1/2} + \tilde{M}_2 k_1^{1/2}) \) and \( \gamma_{19} := C_{16} C(q, \Omega) \tilde{C}_a M_2 (\beta_4 k_3)^{1/2} k_3^{1/2} \).
The term $I_3^{(7)}(\varepsilon)$ is estimated with the help of (4.22), (4.23), and (6.33):

$$I_3^{(7)}(\varepsilon) \leq \varepsilon C(q, \Omega) \sum_{j=1}^{d} \|a_j\|_{L^p(\Omega)} \|\varphi_j\|_{H^1(\mathbb{R}^d)} \left( M_1 \|b(D)D_j \tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} + \tilde{M}_1 \|D_j \tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} \right).$$

Now, applying Lemma 5.2, 1.3, and inequalities 4.2, 4.3 for $\tilde{\eta}_0$, we arrive at

$$I_3^{(7)}(\varepsilon) \leq \varepsilon (\phi)^2 (\gamma_{20} \varepsilon |\xi|^{-1/2} + \gamma_{21} \varepsilon^2) \|F\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)},$$

(6.35)

where $\gamma_{20} := \tilde{C}_a C(q, \Omega)(C_1 + M_1 \alpha_1^2 k_3 + (C_1 + C_1 + C_16) \tilde{M}_1 k_2)$ and $\gamma_{21} := \tilde{C}_a C(q, \Omega) M_1 \alpha_1^2 k_3 (C_1 + C_1 + C_16)$. Let us estimate the term $I_3^{(8)}(\varepsilon)$:

$$I_3^{(8)}(\varepsilon) \leq \varepsilon \|\xi^2 \Lambda^s \omega b(D)\tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} + \|\xi^2 \Lambda^s \tilde{\eta}_0\|_{L^2(\mathbb{R}^d)}.$$

(6.36)

By Proposition 1.2 and (1.3), we have

$$\|\xi^2 \Lambda^s \omega b(D)\tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} \leq \alpha_1^2 |\Omega|^{-1/2} \|Q\|_{L^2(\Omega)} \|\tilde{\eta}_0\|_{H^1(\mathbb{R}^d)}.$$

(6.37)

From the Hölder inequality and the Sobolev embedding theorem it follows that

$$\|\xi^2 \Lambda^s \omega b(D)\tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} \leq \alpha_1^2 |\Omega|^{-1/2} \|Q\|_{L^2(\Omega)} \|\tilde{\eta}_0\|_{H^1(\mathbb{R}^d)}.$$

(6.38)

where $\tilde{q} = \infty$ for $d = 1$ and $\tilde{q} = 2s/(s - 1)$ for $d \geq 2$. Similarly,

$$\|\xi^2 \Lambda^s \tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} \leq |\Omega|^{-1/2} C(\tilde{q}, \Omega) \|Q\|_{L^2(\Omega)} \|\tilde{\eta}_0\|_{H^1(\mathbb{R}^d)}.$$

(6.39)

Relations (6.14), (6.29), (6.30), and (6.36)–(6.39) imply that

$$I_3^{(8)}(\varepsilon) \leq \varepsilon C(\tilde{q}, \Omega)^2 \|Q\|_{L^2(\Omega)} \|\varphi_j\|_{H^1(\mathbb{R}^d)} \left( \alpha_1^2 (M_1 + M_2) \|\tilde{\eta}_0\|_{H^1(\mathbb{R}^d)} + (\tilde{M}_1 + \tilde{M}_2) \|\tilde{\eta}_0\|_{L^2(\mathbb{R}^d)} \right).$$

Together with estimates (4.11), (4.12) for $\tilde{\eta}_0$ and Lemma 5.2, this yields

$$I_3^{(8)}(\varepsilon) \leq \gamma_{22} C(\phi)^2 \varepsilon |\xi|^{-1/2} \|F\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)},$$

(6.40)

where $\gamma_{22} := C(\tilde{q}, \Omega)^2 \|Q\|_{L^2(\Omega)} (C_1 + C_1 + C_1) (M_1 + M_2) \alpha_1^2 k_2 + (\tilde{M}_1 + \tilde{M}_2) k_1$.

The term $I_3^{(9)}(\varepsilon)$ is estimated by using (1.3), (4.22), (4.23), inequalities (4.1), (4.2) for $\tilde{\eta}_0$, and Lemma 5.2. We arrive at

$$I_3^{(9)}(\varepsilon) \leq \gamma_{23} C(\phi)^2 \varepsilon |\xi|^{-1/2} \|F\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)},$$

(6.41)

where $\gamma_{23} := \lambda (|Q_0|_{L^infty} C_1 + M_1 \alpha_1^2 k_2 + \tilde{M}_1 k_1)$.

Finally, relations (6.19), (6.23), (6.31), (6.33), (6.35), (6.40), and (6.41) imply

$$|I_3(\varepsilon)| \leq c(\phi)^2 \left( \tilde{\gamma} \varepsilon |\xi|^{-1/2} + \tilde{\gamma} \varepsilon^2 \right) \|F\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)},$$

(6.42)

where $\tilde{\gamma} := \gamma_{11} + \gamma_{13} + \gamma_{14} + \gamma_{16} + \gamma_{17} + \gamma_{18} + \gamma_{20} + \gamma_{22} + \gamma_{23}$ and $\tilde{\gamma} := \gamma_{12} + \gamma_{15} + \gamma_{19} + \gamma_{21}$.

Thus, we have estimated all terms in the right-hand side of (6.34). From (6.34)–(6.35), (6.18), and (6.42) it follows that

$$|\langle w, \varphi, \Phi \rangle_{L^2(\Omega)}| \leq c(\phi)^5 (\gamma_* \varepsilon |\xi|^{-1/2} + \gamma_* \varepsilon^2) \|F\|_{L^2(\Omega)} \|\Phi\|_{L^2(\Omega)}, \Phi \in L^2(\Omega; C^n).$$

Here $\gamma_* := C_{21} + \gamma_{11} + \tilde{\gamma} + \tilde{\gamma}$ and $\gamma_{**} := \gamma_{12} + \tilde{\gamma} + \tilde{\gamma}$. Hence,

$$|\langle w, \varphi, \Phi \rangle_{L^2(\Omega)}| \leq c(\phi)^5 (\gamma_* \varepsilon |\xi|^{-1/2} + \gamma_{**} \varepsilon^2) \|F\|_{L^2(\Omega)}.$$

Together with (5.8), this yields (6.14) with the constants $C_{19} := \gamma_* + C_{14}$ and $C_{20} := \gamma_{**}$. □
6.2. Completion of the proof of Theorem 2.5 From (6.20) and (6.11) it follows that
\[ \| u_\varepsilon - u_0 \|_{L^2(\Omega)} \leq C_{22} c(\phi)^5 (\varepsilon |\zeta|^{-1/2} + \varepsilon^2) \| F \|_{L^2(\Omega)}, \] (6.43)
where \( C_{22} := \max \{ C_{11} + C_{19}; C_{20} \}. \) In order to deduce (7.35), we also need the following rough estimate:
\[ \|(B_{D,\varepsilon} - \zeta Q_0^{-1}) - (B_0^0 - \zeta Q_0^{-1})\|_{L^2(\Omega) \to L^2(\Omega)} \leq 2 \| Q_0^{-1} \|_{L^\infty} c(\phi) |\zeta|^{-1} \] (6.44)
for any \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \) and 0 < \( \varepsilon \leq 1 \), which follows from (2.11) and (2.29). For \( |\zeta| \leq \varepsilon^{-2} \) we use (6.33) and note that \( \varepsilon^2 \leq \varepsilon |\zeta|^{-1/2} \). For \( |\zeta| > \varepsilon^{-2} \) we apply (6.44) and take into account that \( |\zeta|^{-1} < \varepsilon |\zeta|^{-1/2} \). This implies (7.35) with \( C_4 := 2 \max \{ \| Q_0^{-1} \|_{L^\infty}; C_{22} \}. \)
}\]

7. Special cases

7.1. Removal of the smoothing operator \( S_\varepsilon \) in the corrector. It turns out that the smoothing operator \( S_\varepsilon \) in the corrector can be removed under some additional assumptions on the matrix-valued functions \( \Lambda(x) \) and \( \tilde{\Lambda}(x) \).

**Condition 7.1.** Suppose that the \( \Gamma \)-periodic solution \( \Lambda(x) \) of problem (1.18) is bounded, i. e., \( \Lambda \in L^\infty(\mathbb{R}^d) \).

**Condition 7.2.** Suppose that the \( \Gamma \)-periodic solution \( \tilde{\Lambda}(x) \) of problem (1.28) is such that \( \tilde{\Lambda} \in L_p(\Omega) \), where \( p = 2 \) for \( d = 1 \), \( p > 2 \) for \( d = 2 \), and \( p = d \) for \( d \geq 3 \).

Some cases where Conditions 7.1 and 7.2 are fulfilled were distinguished in [BSn3, Lemma 8.7] and [Su1, Proposition 8.11], respectively.

**Proposition 7.3** ([BSn3]). Suppose that at least one of the following assumptions is satisfied: 1° \( d \leq 2 \); 2° \( d \geq 1 \), and the operator \( A_\varepsilon \) is of the form \( A_\varepsilon = D^* g^\varepsilon(x) D \), where the matrix \( g(x) \) has real entries; 3° the dimension \( d \) is arbitrary, and \( g^0 = g \), i. e., relations (1.27) are valid. Then Condition 7.1 is fulfilled.

**Proposition 7.4** ([Su1]). Suppose that at least one of the following assumptions is satisfied: 1° \( d \leq 2 \); 2° the dimension \( d \) is arbitrary, and the operator \( A_\varepsilon \) is of the form \( A_\varepsilon = D^* g^{\varepsilon}(x) D \), where the matrix \( g(x) \) has real entries. Then Condition 7.2 is fulfilled.

**Remark 7.5.** If \( A_\varepsilon = D^* g^\varepsilon(x) D \), where \( g(x) \) is symmetric matrix with real entries, from [LaU, Chapter III, Theorem 13.1] it follows that \( \Lambda \in L^\infty \) and \( \tilde{\Lambda} \in L^\infty \). So, Conditions 7.1 and 7.2 are fulfilled. Moreover, the norm \( \| \Lambda \|_{L^\infty} \) does not exceed a constant depending on \( d \), \( \rho \), \( \| g \|_{L^\infty} \), \( \| g^{-1} \|_{L^\infty} \), \( |a_j|_{L^1(\Omega)} \), \( j = 1, \ldots, d \), and \( \Omega \).

In this subsection, our goal is to prove the following theorem.

**Theorem 7.6.** The assumptions of Theorem 2.6 are satisfied. Suppose also that Conditions 7.1 and 7.2 hold.

Denote
\[ K^0_{D,\varepsilon}(\varepsilon; \zeta) := (\Lambda^* b(D) + \tilde{\Lambda}^*)(B^0_D - \zeta Q_0^{-1}), \] (7.1)
\[ G^0_{D,\varepsilon}(\varepsilon; \zeta) := g^\varepsilon b(D)(B^0_D - \zeta Q_0^{-1} + g^\varepsilon b(D) \tilde{\Lambda}^*)(B^0_D - \zeta Q_0^{-1}). \] (7.2)

Then for \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \), \( |\zeta| \geq 1 \), and 0 < \( \varepsilon \leq \varepsilon_1 \) we have
\[ \|(B_{D,\varepsilon} - \zeta Q_0^{-1}) - (B^0_D - \zeta Q_0^{-1})\|_{L^2(\Omega) \to H^1(\Omega)} \leq C_5 c(\phi)^2 \varepsilon^{1/2} |\zeta|^{-1/4} + C_{23} c(\phi)^4 \varepsilon, \] (7.3)
\[ \| g^\varepsilon b(D)(B_{D,\varepsilon} - \zeta Q_0^{-1}) - G^0_{D,\varepsilon}(\varepsilon; \zeta)\|_{L^2(\Omega) \to L^2(\Omega)} \leq \tilde{C}_5 c(\phi)^2 \varepsilon^{1/2} |\zeta|^{-1/4} + \tilde{C}_{23} c(\phi)^4 \varepsilon. \] (7.4)

The constants \( C_5 \) and \( \tilde{C}_5 \) are as in Theorem 2.6. The constants \( C_{23} \) and \( \tilde{C}_{23} \) depend only on the initial data (1.9), the domain \( \Omega \), and also on \( p \) and the norms \( \| \Lambda \|_{L^\infty} \), \( \| \tilde{\Lambda} \|_{L^\infty} \).

The continuity of the operators (7.1) and (7.2) under the assumptions of Theorem 7.6 follows from Lemmas 3.3, 3.4, and 3.5.

To prove Theorem 7.6, we need the following lemmas. Their proofs are similar to the proofs of Lemmas 8.7 and 8.8 from [MSu1].
Lemma 7.7. Suppose that Condition 7.1 is satisfied. Let $S_ε$ be the Steklov smoothing operator given by (11). Then for $0 < ε ≤ 1$ we have

$$\|\{A^ε\}b(D)(S_ε - I)\|_{H^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} ≤ C_A.$$  \hfill (7.5)

The constant $C_A$ depends only on $m$, $d$, $α_0$, $α_1$, $\|g\|_{L_∞}$, $\|g^{-1}\|_{L_∞}$, the parameters of the lattice $Γ$, and the norm $\|A\|_{L_∞}$.

Proof. Let $Φ ∈ H^2(\mathbb{R}^d; \mathbb{C}^n)$. By (1.2), (1.3), and Condition 7.1,

$$\|A^εb(D)(S_ε - I)Φ\|_{L_2(\mathbb{R}^d)} ≤ 2α_1^{1/2}∥A∥_{L_∞}∥DΦ∥_{L_2(\mathbb{R}^d)}.$$  \hfill (7.6)

Clearly,

$$\|D(A^εb(D)(S_ε - I)Φ)\|^2_{L_2(\mathbb{R}^d)} ≤ 2ε^{-2}∥(DA)^ε(S_ε - I)b(D)Φ∥^2_{L_2(\mathbb{R}^d)} + 2∥A∥^2_{L_∞}∥(S_ε - I)b(D)DΦ∥^2_{L_2(\mathbb{R}^d)}.$$  \hfill (7.7)

By Lemma 3.3 this yields

$$\|D(A^εb(D)(S_ε - I)Φ)\|^2_{L_2(\mathbb{R}^d)} ≤ 2α_1^2ε^{-2}∥(S_ε - I)b(D)Φ∥^2_{L_2(\mathbb{R}^d)}.$$  \hfill (7.8)

Finally, relations (7.6) and (7.7) imply (7.8) with $C_A := α_1(2β_1r_1^2 + 8∥A∥^2_{L_∞}(β_2 + 1)).$ \hfill □

Lemma 7.8. Suppose that Condition 7.2 is satisfied. Let $S_ε$ be the Steklov smoothing operator given by (11). Then for $0 < ε ≤ 1$ we have

$$\|\{A^ε\}(S_ε - I)\|_{H^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} ≤ C_A,$$  \hfill (7.9)

The constant $C_A$ depends only on $n$, $d$, $α_0$, $α_1$, $ρ$, $∥g∥_{L_∞}$, $∥g^{-1}\|_{L_∞}$, the norms $∥a_j∥_{L_ρ(Ω)}$, $j = 1, \ldots, d$, and also on $p$, the norm $∥A∥_{L_ρ(Ω)}$, and the parameters of the lattice $Γ$.

Proof. Let $Φ ∈ H^2(\mathbb{R}^d; \mathbb{C}^n)$. From (1.2), Lemma 3.4 and Condition 7.2 it follows that

$$\|\{A^ε\}(S_ε - I)Φ\|_{L_2(\mathbb{R}^d)} ≤ 2C(\tilde{q}, Ω)∥A∥_{L_ρ(Ω)}∥Φ∥_{H^1(\mathbb{R}^d)}.$$  \hfill (7.10)

Consider the derivatives: $∂_j(\{A^ε\}(S_ε - I)Φ) = ε^{-1}(∂_j\{A^ε\}(S_ε - I)Φ + \{A^ε\}(S_ε - I)∂_jΦ)$. Together with Lemmas 3.3 and 3.5 this yields

$$\|D(\{A^ε\}(S_ε - I)Φ)\|^2_{L_2(\mathbb{R}^d)} ≤ 2β_1ε^{-2}∥(S_ε - I)Φ∥^2_{H^1(\mathbb{R}^d)} + 2∥A∥^2_{L_∞}(β_2 + 1)∥A∥^2_{L_ρ(Ω)}∥D(\{A^ε\}(S_ε - I)Φ)∥^2_{H^1(\mathbb{R}^d)}.$$  \hfill (7.11)

Combining this with (1.2) and Proposition 1.1 we obtain

$$\|D(\{A^ε\}(S_ε - I)Φ)\|^2_{L_2(\mathbb{R}^d)} ≤ 2β_1r_1^2 + (8\tilde{β}_2 + 1)∥A∥^2_{L_ρ(Ω)}∥D(\{A^ε\}(S_ε - I)Φ)∥^2_{H^1(\mathbb{R}^d)}.$$  \hfill (7.12)

Now, (7.9) and (7.10) imply (7.8) with $C_A := 2β_1r_1^2 + (8\tilde{β}_2 + 1)∥A∥^2_{L_ρ(Ω)}.$ \hfill □

7.2. Proof of Theorem 7.6. Under Condition 7.1 by (2.31), (2.37), and Lemma 7.7 we have

$$ε∥\{A^ε\}b(D)(S_ε - I)PO(B_D^0 - \zeta Q_0^{-1})∥_{L_2(\mathbb{O}) \rightarrow H^1(\mathbb{O})} ≤ C \zeta C^2_2(2)C_2(φ).ε.$$  \hfill (7.13)

Similarly, under Condition 7.2 from (2.31), (2.37), and Lemma 7.8 it follows that

$$ε∥\{A^ε\}(S_ε - I)PO(B_D^0 - \zeta Q_0^{-1})∥_{L_2(\mathbb{O}) \rightarrow H^1(\mathbb{O})} ≤ C \zeta C^2_2(2)C_2(φ).ε.$$  \hfill (7.14)

Relations (2.32), (2.41), and (2.42) imply estimate (7.8) with $C_2 := C_0 + (C_A + C_\zeta)C^2_2(2)C_2(φ).ε$

Let us check (7.8). By analogy with (5.16), from (7.3) it follows that

$$∥g^\ast b(D)(B_{D, ε} - \zeta Q_0^{-1}) - g^\ast b(D)(I + εA^εb(D) + \tilde{A}^ε)(B_D^0 - \zeta Q_0^{-1})∥_{L_2(\mathbb{O}) \rightarrow L_2(\mathbb{O})} ≤ (dα_1)^{1/2}∥g∥_{L_∞}(C_2c(φ)^{1/2}ε^{-1/4} + C_2c(φ)^4ε).$$  \hfill (7.15)
Next, by analogy with (3.17).
\[
\varepsilon g^\varepsilon b(D) (\Lambda^\varepsilon b(D) + \tilde{\Lambda}^\varepsilon) (B_0 - \zeta Q_0)^{-1}
= g^\varepsilon (b(D)\Lambda^\varepsilon b(D)(B_0 - \zeta Q_0)^{-1} + g^\varepsilon (b(D)\tilde{\Lambda}^\varepsilon (B_0 - \zeta Q_0)^{-1}
+ \varepsilon \sum_{l=1}^d g^\varepsilon b_l(\Lambda^\varepsilon b(D))D_l + \tilde{\Lambda}^\varepsilon D_l)(B_0 - \zeta Q_0)^{-1}.
\]

(7.14)

Using (1.4), (2.31), and Condition 7.1, we obtain
\[
\varepsilon \sum_{l=1}^d \|g^\varepsilon b_l(\Lambda^\varepsilon b(D))D_l (B_0 - \zeta Q_0)^{-1}\|_{L_2(\Omega)\rightarrow L_2(\Omega)}
\leq \varepsilon(\alpha_1 d\|g\|_{L_\infty} \|\Lambda\|_{L_\infty} \|D^2(B_0 - \zeta Q_0)^{-1}\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq \varepsilon(\alpha_1 d\|g\|_{L_\infty} \|\Lambda\|_{L_\infty} C_2 c(\phi).
\]

(7.15)

Next, from (1.4), (2.31), (2.37), Lemma 3.4, and Condition 7.2 it follows that
\[
\varepsilon \sum_{l=1}^d \|g^\varepsilon b_l(\Lambda^\varepsilon b(D))D_l (B_0 - \zeta Q_0)^{-1}\|_{L_2(\Omega)\rightarrow L_2(\Omega)}
\leq \varepsilon(\alpha_1 d\|g\|_{L_\infty} \|\Lambda\|_{L_\infty} \|D^2(B_0 - \zeta Q_0)^{-1}\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq \varepsilon(\alpha_1 d\|g\|_{L_\infty} \|\Lambda\|_{L_\infty} C_2 c(\phi).
\]

(7.16)

Together with (7.15), this shows that the third term in the right-hand side of (7.14) does not exceed \(\tilde{C}_{23} c(\phi) \varepsilon\), where \(\tilde{C}_{23} := (\alpha_1 d\|g\|_{L_\infty} \|\Lambda\|_{L_\infty} + C(\tilde{q},\Omega)C_2^{(1)} \|\Lambda\|_{L_\infty}).\)

Combining this with (7.13) and (7.14), we arrive at estimate (7.4) with the constant \(\tilde{C}_{23} := (\alpha_1 d\|g\|_{L_\infty} \|\Lambda\|_{L_\infty} + C(\tilde{q},\Omega)C_2^{(1)} \|\Lambda\|_{L_\infty}).\)

Remark 7.9. If only Condition 7.1 (respectively, Condition 7.2) is satisfied, then the smoothing operator \(S_\varepsilon\) can be removed only in the term of the corrector containing \(\Lambda^\varepsilon\) (respectively, \(\tilde{\Lambda}^\varepsilon\)).

7.3. The case where the corrector is equal to zero. Suppose that \(g^0 = \tilde{g}\), i.e., relations (1.26) are satisfied. Then the \(\Gamma\)-periodic solution of problem (1.18) is equal to zero: \(\Lambda(x) = 0\). In addition, suppose that
\[
\sum_{j=1}^d D_j a_j(x)^* = 0.
\]

(7.17)

Then the \(\Gamma\)-periodic solution of problem (1.28) is also equal to zero: \(\tilde{\Lambda}(x) = 0\). Hence, \(v_\varepsilon = u_0\) (see (2.39), (2.40)). The solution of problem (2.45) is equal to zero: \(w_\varepsilon = 0\). Theorem 2.7 implies the following result.

Proposition 7.10. Suppose that the assumptions of Theorem 2.3 are satisfied. Suppose that relations (1.26) and (7.17) hold. Then for \(\xi \in C \setminus \mathbb{R}_+, \|\zeta\| \geq 1\), and \(0 < \varepsilon \leq 1\) we have
\[
\|(B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_0 - \zeta Q_0)^{-1}\|_{L_2(\Omega)\rightarrow H^1(\Omega)} \leq C_{7} c(\phi)^4 \varepsilon.
\]

7.4. Special case. Suppose that \(g^0 = \tilde{g}\), i.e., relations (1.27) are satisfied. Then, by Proposition 7.3, Condition 7.4 is fulfilled. Herewith, by [BSu2, Remark 3.5], the matrix-valued function (1.29) is constant and coincides with \(g^0\), i.e., \(\tilde{g}(x) = g^0 = g\). Hence, \(\tilde{g}^\varepsilon b(D)(B_0 - \zeta Q_0)^{-1} = g^0 b(D)(B_0 - \zeta Q_0)^{-1}\). In addition, suppose that relation (7.17) holds. Then \(\tilde{\Lambda}(x) = 0\), and Theorem 7.4 implies the following result.

Proposition 7.11. Suppose that the assumptions of Theorem 2.3 are satisfied. Suppose that relations (1.27) and (7.17) hold. Then for \(\xi \in C \setminus \mathbb{R}_+, \|\zeta\| \geq 1\), and \(0 < \varepsilon \leq \varepsilon_1\) we have
\[
\|g^\varepsilon b(D)(B_{D,\varepsilon} - \zeta Q_0)^{-1} - g^0 b(D)(B_0 - \zeta Q_0)^{-1}\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq \tilde{C}_5 c(\phi)^2 \varepsilon^{1/2} \|\zeta\|^{-1/4} + \tilde{C}_{23} c(\phi)^4 \varepsilon.
\]
8. Estimates in a strictly interior subdomain

8.1. General case. Using Theorem 8.3 and the results for homogenization problem in $\mathbb{R}^d$, it is possible to improve error estimates in $H^1(\Omega')$ for any strictly interior subdomain $\Omega'$ of $\Omega$.

The following result is proved by the same method as Theorem 7.1 from [Su3].

**Theorem 8.1.** Suppose that the assumptions of Theorem 8.6 are satisfied. Let $\Omega'$ be a strictly interior subdomain of the domain $\Omega$. Denote $\delta := \text{dist} \{ \Omega'; \partial \Omega \}$. Then for $\zeta \in C \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and $0 < \varepsilon < \varepsilon_1$ we have

$$\| \mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon \|_{H^1(\Omega')} \leq c(\phi)^6 \varepsilon (C_{24}^5 |\zeta|^{-1/2} \delta^{-1} + C_{24}^6) \| \mathbf{F} \|_{L_2(\Omega)}, \quad (8.1)$$

$$\| \mathbf{p}_\varepsilon - \mathbf{p}_\varepsilon \|_{L_2(\Omega')} \leq c(\phi)^6 \varepsilon (C_{24}^5 |\zeta|^{-1/2} \delta^{-1} + C_{24}^6) \| \mathbf{F} \|_{L_2(\Omega)}. \quad (8.2)$$

The constants $C_{24}^5$, $C_{24}^6$, $C_{24}^7$, and $C_{24}^8$ depend only on the initial data (1.9) and the domain $\Omega$.

**Proof.** We fix a smooth cut-off function $\chi(x)$ such that

$$\chi \in C_0^\infty(\Omega); \quad 0 \leq \chi(x) \leq 1; \quad \chi(x) = 1 \text{ for } x \in \Omega'; \quad |\nabla \chi(x)| \leq \kappa \delta^{-1}. \quad (8.3)$$

The constant $\kappa$ depends only on the dimension $d$ and the domain $\Omega$. Let $\mathbf{u}_\varepsilon$ be the solution of problem (2.8), and let $\mathbf{u}_\varepsilon$ be the solution of equation (1.6). Then

$$b_{N,\varepsilon}[\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon, \chi]\zeta(\mathbf{Q}_\varepsilon(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon), \chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)) = 0, \quad \eta \in H^1(\Omega; \mathbb{C}^n). \quad (8.4)$$

We substitute $\eta = \chi^2(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)$ in (8.4) and denote

$$\mathcal{U}(\varepsilon) := b_{N,\varepsilon}[\chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon), \chi(\mathbf{v}_\varepsilon - \bar{\mathbf{v}}_\varepsilon)] = b_{D,\varepsilon}[\chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon), \chi(\mathbf{v}_\varepsilon - \bar{\mathbf{v}}_\varepsilon)]. \quad (8.5)$$

The corresponding identity can be written as

$$\mathcal{U}(\varepsilon) - \zeta(D \chi)(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon), (\chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon))_{L_2(\Omega)} = 2i \text{Im} \left( (g^{\varepsilon}_j z \varepsilon, b(\mathbf{D}) (\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon))_{L_2(\Omega)} + (g^{\varepsilon}_j z \varepsilon, \mathbf{z}_\varepsilon)_{L_2(\Omega)} \right) + 2i \text{Im} \sum_{j=1}^d ((D_j \chi)(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon), (a^*_j \chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon))_{L_2(\Omega)}, \quad (8.6)$$

where $z \varepsilon := \sum_{j=1}^d b_j(D_j \chi)(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)$. Denote the consecutive summands in the right-hand side of (8.6) as $\mathcal{J}_1(\varepsilon)$, $\mathcal{J}_2(\varepsilon)$, and $\mathcal{J}_3(\varepsilon)$. Let us estimate these terms. We can extend the function $\chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)$ by zero to $\mathbb{R}^d \setminus \Omega$ and apply estimates in $\mathbb{R}^d$. By (1.15) and (8.5),

$$|\mathcal{J}_1(\varepsilon)| \leq 2\|g\|_{L_\infty}^1 \|\mathbf{z}\|_{L_2(\Omega)} \|g^{\varepsilon}\|_{L_\infty} \|\mathbf{D}\chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)\|_{L_2(\Omega)} \leq 4\|g\|_{L_\infty} \|\mathbf{z}\|_{L_2(\Omega)} \|\mathcal{U}\|^{1/2}. \quad (8.7)$$

Obviously, $\mathcal{J}_2(\varepsilon) \leq \|g\|_{L_\infty} \|\mathbf{z}\|_{L_2(\Omega)}^2$. The norm of $\mathbf{z}$ is estimated with the help of (1.4) and (8.3):

$$\|\mathbf{z}\|_{L_2(\Omega)} \leq (d a_1)^{1/2} \kappa \delta^{-1} \|\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon\|_{L_2(\Omega)}. \quad (8.8)$$

Hence,

$$\mathcal{J}_2(\varepsilon) \leq \gamma_{24} \delta^{-2} \|\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon\|_{L_2(\Omega)}^2; \quad \gamma_{24} := d a_1^2 \|g\|_{L_\infty}. \quad (8.9)$$

The term $\mathcal{J}_3(\varepsilon)$ is estimated by Lemma 3.4 (2.4), (8.3), and (8.5):

$$|\mathcal{J}_3(\varepsilon)| \leq 2\|\mathbf{D}\chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)\|_{L_2(\Omega)} \left( \sum_{j=1}^d \|a^*_j \chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)\|_{L_2(\Omega)}^2 \right)^{1/2} \leq \gamma_{25} \delta^{-1} \|\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon\|_{L_2(\Omega)} \|\mathcal{U}\|^{1/2}; \quad \gamma_{25} := 2c_4 C(q, \Omega) \widehat{C}_{a_k}, \quad (8.10)$$

where $q = \infty$ for $d = 1$ and $q = 2p(\rho - 2)^{-1}$ for $d \geq 2$.

Take the imaginary part in (8.6). Then

$$\text{Im} \zeta(\|Q^{\varepsilon}_\chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)\|_{L_2(\Omega)}^2) = -\mathcal{J}_1(\varepsilon) - \mathcal{J}_3(\varepsilon).$$

Therefore, relations (8.7), (8.8), and (8.10) imply that

$$\|Q^{\varepsilon}_\chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)\|_{L_2(\Omega)} \leq \gamma_{26} \delta^{-1} \|\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon\|_{L_2(\Omega)} \|\mathcal{U}\|^{1/2}, \quad \gamma_{26} := 4\|g\|_{L_\infty} \|\mathbf{z}\|_{L_2(\Omega)} \|\mathcal{U}\|^{1/2} + \gamma_{25}. \quad (8.11)$$

where $\gamma_{26} := 4\|g\|_{L_\infty} (d a_1)^{1/2} + \gamma_{25}$. If $\Re \zeta \geq 0$ (and then $\Im \zeta \neq 0$), this yields

$$\|Q^{\varepsilon}_\chi(\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon)\|_{L_2(\Omega)} \leq \gamma_{26} C(\phi) |\zeta|^{-1} \delta^{-1} \|\mathbf{u}_\varepsilon - \bar{\mathbf{u}}_\varepsilon\|_{L_2(\Omega)} \|\mathcal{U}\|^{1/2}, \quad \Re \zeta \geq 0. \quad (8.12)$$
If $\Re \zeta < 0$, taking the real part in (8.6) and using (8.9), we have

$$|\Re \xi|\| (Q_0^{\varepsilon})^{1/2} (u_\varepsilon - \tilde{u}_\varepsilon)\|_{L^2(\Omega)}^2 \lesssim \mathcal{J}_2(\varepsilon) \lesssim \gamma_{24} \delta^{-2} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)}^2, \quad \Re \xi < 0. \tag{8.13}$$

Summing up (8.11) and (8.13), we obtain

$$|\zeta|\| (Q_0^{\varepsilon})^{1/2} (u_\varepsilon - \tilde{u}_\varepsilon)\|_{L^2(\Omega)}^2 \lesssim \gamma_{26} \delta^{-1} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)} \Omega(\varepsilon)^{1/2} + \gamma_{24} \delta^{-2} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)}^2 \tag{8.14}$$

for $\Re \zeta < 0$. As a result, (8.12) and (8.13) imply that

$$|\zeta|\| (Q_0^{\varepsilon})^{1/2} (u_\varepsilon - \tilde{u}_\varepsilon)\|_{L^2(\Omega)}^2 \lesssim \gamma_{26} \delta^{-1} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)} \Omega(\varepsilon)^{1/2} + \gamma_{24} \delta^{-2} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)}^2 \tag{8.15}$$

for all $\zeta$ under consideration. Taking the real part in (8.6) and using (8.9) and (8.15), we obtain

$$\Omega(\varepsilon) \lesssim \gamma_{27} \delta^{-1} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)} \Omega(\varepsilon)^{1/2} + 2 \gamma_{24} \delta^{-2} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)}^2,$$

Hence, $\Omega(\varepsilon) \lesssim \frac{\gamma_{27}}{\delta} \varepsilon^{1/2} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)}^2,$ where $\gamma_{27} := \gamma_{26} + 4 \gamma_{24}$. By (2.2) and (5.5), we deduce

$$\|D\chi (u_\varepsilon - \tilde{u}_\varepsilon)\|_{L^2(\Omega)} \lesssim \gamma_{27} C_{\varepsilon}^{-1} (\varepsilon) \delta^{-1} \|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)} \tag{8.16}$$

Estimates (2.34) and (4.7) imply that

$$\|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2(\Omega)} \lesssim \gamma_{28} \delta \varepsilon \|\zeta\|^{-1/2} \|\Omega(\varepsilon)^{1/2} \|_{L^2(\Omega)}, \quad 0 < \varepsilon < \varepsilon_1, \tag{8.17}$$

where $\gamma_{28} := C_s + C_F C_F$. From (8.16) and (8.17) it follows that

$$\|D\chi (u_\varepsilon - \tilde{u}_\varepsilon)\|_{L^2(\Omega)} \lesssim \gamma_{29} \delta \varepsilon \|\zeta\|^{-1/2} \|\Omega(\varepsilon)^{1/2} \|_{L^2(\Omega)}. \tag{8.18}$$

Here $\gamma_{29} := c_{\varepsilon}^{-1} \gamma_{27} \gamma_{28}$. By (8.17) and (8.18),

$$\|u_\varepsilon - \tilde{u}_\varepsilon\|_{H^1(\omega)} \lesssim \varepsilon (\delta \varepsilon \|\zeta\|^{-1/2} (\gamma_{29} \delta - 1 + \gamma_{28}) \|\Omega(\varepsilon)^{1/2} \|_{L^2(\Omega)}). \tag{8.19}$$

Combining (2.41) and (4.10), we find

$$\|\tilde{u}_\varepsilon - v_\varepsilon\|_{H^1(\Omega)} \lesssim \|\tilde{u}_\varepsilon - \varepsilon(\delta \varepsilon \|\zeta\|^{-1/2} \|\Omega(\varepsilon)^{1/2} \|_{L^2(\Omega)} \tag{8.20}$$

Now, relations (8.19) and (8.20) imply (8.1) with $C_{\varepsilon}^2 := \gamma_{29}$ and $C_{\varepsilon}^2 := \gamma_{28} + \tilde{C}_{\varepsilon}$. Let us prove (8.2). By (1.4) and (8.1),

$$\|p_\varepsilon - g^b(D)v_\varepsilon\|_{L^2(\Omega)} \lesssim (d\alpha_1)^{1/2} \|\varepsilon \|_{L^\infty} (\delta \varepsilon \|\zeta\|^{-1/2} \|\Omega(\varepsilon)^{1/2} \|_{L^2(\Omega)} + C_{\varepsilon}^2) \|\Omega(\varepsilon)^{1/2} \|_{L^2(\Omega)} \tag{8.21}$$

Combining this with (5.17), (5.19), and (5.21), we deduce estimate (8.2) with the constants $\tilde{C}_{\varepsilon} := (d\alpha_1)^{1/2} \|\varepsilon \|_{L^\infty} C_{\varepsilon}^2 + C_{\varepsilon}^2 + C_{\varepsilon}^2 + C_\varepsilon^2 + C_\varepsilon^2.$

8.2. Removal of the smoothing operator in the corrector. Provided that the matrix-valued functions $\Lambda(x)$ and $\tilde{\Lambda}(x)$ are subject to Conditions (7.1) and (7.2), respectively, the smoothing operator $S_\varepsilon$ in the corrector can be removed.

**Theorem 8.2.** Suppose that the assumptions of Theorem 8.1 are satisfied. Suppose also that Conditions (7.1) and (7.2) hold. Let $K_0^0(\varepsilon; \zeta)$ and $C_0^0(\varepsilon; \zeta)$ be given by (7.1) and (7.2), respectively. Then for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and $0 < \varepsilon \leq \varepsilon_1$, we have

$$\|\Lambda(D_\varepsilon - \zeta Q_0^{\varepsilon})^{-1} - (B_0^0 - \zeta Q_0^{\varepsilon})^{-1} - \varepsilon K_0^0(\varepsilon; \zeta)\|_{L^2(\Omega) \to L^2(\Omega)} \leq C_{\varepsilon} \delta \varepsilon (C_{\varepsilon}^2 \|\zeta\|^{-1/2} \delta^{-1} + C_{\varepsilon}^2), \tag{8.22}$$

The constants $C_{\varepsilon}^2$ and $\tilde{C}_{\varepsilon}^2$ are as in Theorem 8.1. The constants $C_{\varepsilon}^2$ and $\tilde{C}_{\varepsilon}^2$ depend only on the initial data $1.9$, the domain $\Omega$, and also on $p$ and the norms $\|\Lambda\|_{L^\infty}$, $\|\Lambda\|_{L^p(\Omega)}$.

**Proof.** Inequality (8.21) with $C_{\varepsilon}^2 := C_{\varepsilon}^2 + (\varepsilon \Lambda + \tilde{\varepsilon} \Lambda) C_{\varepsilon}^2 C_\varepsilon^2$ is a consequence of (7.11), (7.12), and (8.1).

Let us check (8.22). Similarly to (4.10), from (8.21) it follows that

$$\|g^b(D)(B_{D_\varepsilon} - \zeta Q_0^{\varepsilon})^{-1} - g^b(D)(I + \varepsilon \Lambda^b(D) + \varepsilon \tilde{\Lambda}^b(D) - \zeta Q_0^{\varepsilon})^{-1}\|_{L^2(\Omega) \to L^2(\Omega)} \leq (d\alpha_1)^{1/2} \|\varepsilon \|_{L^\infty} C_{\varepsilon} \delta \varepsilon (C_{\varepsilon}^2 \|\zeta\|^{-1/2} \delta^{-1} + C_{\varepsilon}^2).$$

Together with (7.14) - (7.15), this yields estimate (8.22) with $\tilde{C}_{\varepsilon}^2 := (d\alpha_1)^{1/2} \|\varepsilon \|_{L^\infty} C_{\varepsilon} + \tilde{C}_{\varepsilon}^2$.
9. “Another” Approximation of the Generalized Resolvent

In Theorems of Sections 2, 7, and 8 it was assumed that ζ ∈ C \ R_+ and |ζ| ≥ 1. In the present section, we obtain the results valid in a larger domain of the spectral parameter.

9.1. General case.

Condition 9.1. Let 0 < ε_0 ≤ 1. Let c_0 ≥ 0 be a common lower bound of the operators \( \tilde{B}_{D,ε} = (f^*)B_{D,ε}f^{\epsilon} \) for any 0 < ε ≤ ε_0 and \( B_D^0 = f_0B_D^0f_0 \).

Theorem 9.2. Suppose that \( O \subset R^d \) is a bounded domain of class \( C^{1,1} \). Suppose that the number \( ε_1 \) is subject to Condition 9.1. Let 0 < ε ≤ ε_1. Suppose that c_0 ≥ 0 is subject to Condition 9.1. Let ζ ∈ C \ [c_0, ∞). Denote \( ψ = \arg(ζ - c_0) \), 0 < ψ < 2π, and

\[
\varphi_0(ζ) := \begin{cases} 
(c(ψ))^2|ζ - c_0|^{-2}, & |ζ - c_0| < 1, \\
(c(ψ))^2, & |ζ - c_0| ≥ 1.
\end{cases}
\]  

(9.1)

Here c(ψ) is defined by (2.11). Let \( u_0 \) be the solution of problem (2.8), and let \( u_0 \) be the solution of problem (2.23). Let \( K_D(ε; ζ) \) be given by (2.38). Let \( v_ε \) be defined by (2.39), (2.40). Then for 0 < ε ≤ ε_0 we have

\[
\|u_ε - u_0\|_{L^2(O)} \leq C_{26}ε\varphi_0(ζ)\|F\|_{L^2(O)},
\]  

(9.2)

\[
\|u_ε - v_ε\|_{H^1(O)} \leq C_{27}(ε^{1/2}\varphi_0(ζ))^{1/2} + ε|1 + \varphi_0(ζ)|\|F\|_{L^2(O)}.
\]  

(9.3)

In operator terms,

\[
\|(B_{D,ε} - ζQ_0^{-1})^{-1} - (B_D^0 - ζQ_0^{-1})^{-1}\|_{L^2(O) → L^2(O)} \leq C_{26}\varphi_0(ζ),
\]  

(9.4)

\[
\|(B_{D,ε} - ζQ_0^{-1})^{-1} - (B_D^0 - ζQ_0^{-1})^{-1} - εK_D(ε; ζ)\|_{L^2(O) → H^1(O)} \leq C_{27}(ε^{1/2}\varphi_0(ζ))^{1/2} + ε|1 + \varphi_0(ζ)|.
\]  

(9.5)

Let \( \tilde{g}(x) \) be defined by (2.20). For 0 < ε ≤ ε_0 the flux \( p_ε = g^εb(D)u_ε \) satisfies

\[
\|p_ε - g^εS_1b(D)u_0 - g^ε(b(D)A)\tilde{u}_0\|_{L^2(O)} \leq \tilde{C}_{27}(ε^{1/2}\varphi_0(ζ))^{1/2} + ε|1 + \varphi_0(ζ)|\|F\|_{L^2(O)}.
\]  

(9.6)

The constants \( C_{26}, \tilde{C}_{27}, \) and \( \tilde{C}_{27} \) depend only on the initial data (1.9) and the domain O.

Remark 9.3. 1) Expression c(ψ)^2|ζ - c_0|^{-2} in (9.1) is inverse to the square of the distance from ζ to [c_0, ∞). 2) By (2.3), (2.5), (2.23), and (2.23), for any ε_0 ∈ (0, 1] one can take c_0 equal to \( 4^{-1}α_0g^\epsilon\|1_\infty\|_{L^\infty}Q_0^{-1}\|_{L^\infty}(\text{diam } O)^{-2} \). 3) Let \( λ_0^2 \) be the first eigenvalue of the operator \( B_D^0 \), and let ν > 0 be arbitrarily small number. By Theorem 2.3 (with \( Q_0 = 0 \)), the resolvent of \( B_{D,ε} \) converges to the resolvent of \( B_D^0 \) in the L_2-operator norm. Hence, if ε_0 is sufficiently small, the number \( λ_0^2 \) is a lower bound of the operator \( B_{D,ε} \) for any 0 < ε ≤ ε_0. Then one can take c_0 = \( |Q_0|^{-1}_\infty(λ_0^2 - ν) \). 4) It is easy to give the upper bound for c_0. By (2.2) and (2.4), we have c_0 ≤ \( \varphi_0(ζ)^{-1}_\inftyL^\mu_0 \), where \( \mu_0^1 \) is the first eigenvalue of the operator −Δ + I with the Dirichlet condition on ∂O. So, c_0 is controlled in terms of the data (1.9) and the domain O.

Remark 9.4. Estimates (9.2)–(9.6) are useful for bounded values of |ζ| and small ε\varphi_0(ζ). In this case, the value ε^{1/2}\varphi_0(ζ)^{1/2} + ε|1 + \varphi_0(ζ)|^{1/2} is majorized by C\tilde{ε}^{1/2}\varphi_0(ζ)^{1/2}. For large |ζ| (and \( \phi \) separated from 0 and 2π) application of Theorems 2.3 and 2.4 is preferable.

We start with the following two lemmas.

Lemma 9.5. Under Condition 9.1 for 0 < ε ≤ ε_0 and ζ ∈ C \ [c_0, ∞) we have

\[
\|B_{D,ε} - ζQ_0^{-1}\|_{L^2(O) → L^2(O)} \leq \|f\|_{L^\infty}^2c(ψ)|ζ - c_0|^{-1},
\]  

(9.7)

\[
\|B_{D,ε} - ζQ_0^{-1}\|_{L^2(O) → H^1(O)} \leq C_3(1 + |ζ|)^{-1/2}\varphi_0(ζ)^{1/2},
\]  

(9.8)

\[
\|B_D^0 - ζQ_0^{-1}\|_{L^2(O) → L^2(O)} \leq \|f\|_{L^\infty}^2c(ψ)|ζ - c_0|^{-1},
\]  

(9.9)

\[
\|B_D^0 - ζQ_0^{-1}\|_{L^2(O) → H^1(O)} \leq C_3(1 + |ζ|)^{-1/2}\varphi_0(ζ)^{1/2},
\]  

(9.10)

\[
\|B_D^0 - ζQ_0^{-1}\|_{L^2(O) → H^2(O)} \leq C_4\varphi_0(ζ)^{1/2}.
\]  

(9.11)

Here \( C_3 := c_4\|f\|_{L^\infty}(c_0 + 1)^{1/2}(c_0 + 2)^{1/2} \) and \( C_4 := \tilde{c}(c_0 + 2)\|f\|_{L^\infty}\|f^{-1}\|_{L^\infty} \).
Proof. Under our assumptions, the spectrum of the operator $\tilde{B}_{D,e}$ is contained in $[c_3, \infty)$. Hence, 
$(\tilde{B}_{D,e} - \zeta I)^{-1} \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq c(\zeta) |\zeta - c_3|^{-1}$. Together with (2.4), this implies (9.7).

Next, from (2.6) and (2.7) it follows that

$$
\|B_{D,e}^{1/2}(\tilde{B}_{D,e} - \zeta Q_0)^{-1/2} \|_{L_2 \rightarrow L_2} = \|\tilde{B}_{D,e}^{1/2}(\tilde{B}_{D,e} - \zeta I)^{-1} (f^*) \|_{L_2 \rightarrow L_2} \leq \|f\|_{L_\infty} \sup_{x \geq c_3} \frac{x^{1/2}}{|x - \zeta|}
$$

A calculation shows that

$$
\sup_{x \geq c_3} \frac{x}{|x - \zeta|^2} \leq \begin{cases} (c_3 + 1)(c(\zeta))^2 |\zeta - c_3|^{-2}, & |\zeta - c_3| < 1, \\
(c_3 + 1)(c(\zeta))^2 |\zeta - c_3|^{-1}, & |\zeta - c_3| \geq 1.
\end{cases}
$$

Note that $|\zeta| + 1 \leq 2 + c_3$ for $|\zeta - c_3| < 1$ and $(|\zeta| + 1)|\zeta - c_3|^{-1} \leq 2 + c_3$ for $|\zeta - c_3| \geq 1$. Therefore,

$$
(|\zeta| + 1)^{1/2} \|B_{D,e}^{1/2}(\tilde{B}_{D,e} - \zeta Q_0)^{-1/2} \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \|f\|_{L_\infty} (c_3 + 1)^{1/2}(c_3 + 2)^{1/2} \vartheta(\zeta)^{1/2}.
$$

Together with (2.3) this implies (9.8).

Estimates (9.9) and (9.10) are proved similarly to (9.7) and (9.8), respectively, with the help of (2.21), (2.26) and (2.27).

It remains to check (9.11). By (2.25) and (2.27),

$$
\|(B_0^{0} - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow H^2(\Omega)} \|_{L_2(\Omega) \rightarrow H^2(\Omega)} \|B_0^{0}(B_0^{0} - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow L_2(\Omega)}
$$

$$
\leq \tilde{c}\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \sup_{x \geq c_3} x|x - \zeta|^{-1} \leq \tilde{c}\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \sup_{x \geq c_3} (x + 1)|x - \zeta|^{-1}.
$$

A calculation shows that

$$
\sup_{x \geq c_3} \frac{(x + 1)^2}{|x - \zeta|^2} \leq (c_3 + 2)^2 \vartheta(\zeta), \quad \zeta \in \bar{C} \setminus [c_3, \infty).
$$

Relations (9.12) and (9.13) imply (9.11). \square

Lemma 9.6. Under Condition 9.1 for $0 < \varepsilon \leq \varepsilon_0$ and $\zeta \in \bar{C} \setminus [c_3, \infty)$ we have

$$
\|K_D(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_5 (1 + |\zeta|)^{-1/2} \vartheta(\zeta)^{1/2},
$$

$$
\varepsilon \|K_D(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq C_6 \left( \varepsilon + (1 + |\zeta|)^{-1/2} \right) \vartheta(\zeta)^{1/2}.
$$

The constants $C_5$ and $C_6$ depend only on the initial data 1.20.

Proof. Combining (1.3), (2.37), (2.38), (4.22), and (4.23), we obtain

$$
\|K_D(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \frac{1}{2} M_1 C^{(1)}_0 \|(B_0^{0} - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow H^1(\Omega)}
$$

$$
+ \frac{1}{2} \tilde{M}_1 C^{(0)}_0 \|(B_0^{0} - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow L_2(\Omega)}.
$$

Together with (9.10), this yields estimate (9.14) with $C_5 := C_3 (\alpha_1^{1/2} M_1 C^{(1)}_0 + \tilde{M}_1 C^{(0)}_0)$. Next,

$$
\varepsilon \|D K_D(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \|(D \Lambda)^2 b(D) + (D \tilde{\Lambda})^2\) S_{\varepsilon} P_{\varepsilon} (B_0^{0} - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow L_2(\Omega)}
$$

$$
+ \varepsilon \|(D \Lambda^2 b(D) + \tilde{\Lambda})^2\) S_{\varepsilon} P_{\varepsilon} (B_0^{0} - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow L_2(\Omega)}.
$$

By Proposition 1.2, (1.3), (1.23), (1.37), (1.37), (4.22), and (4.23), and (5.13), this implies

$$
\varepsilon \|D K_D(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq (M_2 \alpha_1^{1/2} + \tilde{M}_2 + \varepsilon \tilde{M}_1) C^{(1)}_0 \|(B_0^{0} - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow H^1(\Omega)}
$$

$$
+ \varepsilon M_1 \alpha_1^{1/2} C^{(2)}_0 \|(B_0^{0} - \zeta Q_0)^{-1}\|_{L_2(\Omega) \rightarrow H^1(\Omega)}.
$$

So, by Lemma 9.5

$$
\varepsilon \|D K_D(\varepsilon; \zeta)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_6 (1 + |\zeta|)^{-1/2} \vartheta(\zeta)^{1/2} + \tilde{C}_6 \varepsilon \vartheta(\zeta)^{1/2}.
$$

Here $\tilde{C}_6 := C_3 C^{(1)}_0 (M_2 \alpha_1^{1/2} + \tilde{M}_2 + \tilde{M}_1)$, $	ilde{C}_6 := M_1 \alpha_1^{1/2} C^{(2)}_0 C_4$.

Combining (9.14) and (9.16), we arrive at estimate (9.15) with $C_6 := \max\{C_5 + \tilde{C}_6; \tilde{C}_6\}$. \square
Similarly to (9.19), taking (2.26) into account, we obtain
\[
\|(B_{D,\varepsilon} + Q_0)^{-1} - (B_D^0 + Q_0)^{-1}\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq C_4\varepsilon.
\] (9.17)

We have
\[
(B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D^0 - \zeta Q_0)^{-1}
= (B_{D,\varepsilon} - \zeta Q_0)^{-1}(B_{D,\varepsilon} + Q_0)\left((B_{D,\varepsilon} + Q_0)^{-1} - (B_D^0 + Q_0)^{-1}\right)(B_D + Q_0)(B_D^0 - \zeta Q_0)^{-1}
+ (1 + \zeta)(B_{D,\varepsilon} - \zeta Q_0)^{-1}(Q_0^\varepsilon - \zeta Q_0)(B_D^0 - \zeta Q_0)^{-1}.
\] (9.18)

Denote the consecutive terms in the right-hand side of (9.18) by \(T_1(\varepsilon; \zeta)\) and \(T_2(\varepsilon; \zeta)\). By (2.27),
\[
\|(B_{D,\varepsilon} - \zeta Q_0)^{-1}(B_{D,\varepsilon} + Q_0)\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})}
\leq \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \|(\tilde{B}_{D,\varepsilon} - \zeta I)^{-1}(\tilde{B}_{D,\varepsilon} + I)\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \sup_{x \geq c_0} \frac{(x + 1)}{|x - \zeta|}.
\] (9.19)

Similarly to (9.19), taking (2.26) into account, we obtain
\[
\|(B_D^0 + Q_0)(B_D^0 - \zeta Q_0)^{-1}\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \sup_{x \geq c_0} \frac{(x + 1)}{|x - \zeta|}.
\] (9.20)

Now, relations (9.13), (9.17), (9.19), and (9.20) imply that
\[
\|T_1(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \gamma_{30}\varepsilon\theta_0(\varepsilon): \quad \gamma_{30} := C_4\|f\|_{L_\infty}^2 \|f^{-1}\|_{L_\infty}^2 (c_0 + 2)^2.
\] (9.21)

The second term in the right-hand side of (9.18) satisfies
\[
\|T_2(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq |1 + \zeta|\|(B_{D,\varepsilon} - \zeta Q_0)^{-1}\|_{H^{-1}(\mathcal{O}) \to L_2(\mathcal{O})}
\times \|Q_0^\varepsilon - \zeta Q_0\|_{H^1(\mathcal{O}) \to H^{-1}(\mathcal{O})} \|(B_D^0 - \zeta Q_0)^{-1}\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})}.
\] (9.22)

Note that the range of the operator \((B_{D,\varepsilon} - \zeta Q_0)^{-1}\) lies in \(H^0_1(\mathcal{O}; \mathbb{C}^n)\). Then, by duality, from (9.8) we obtain
\[
\|(B_{D,\varepsilon} - \zeta Q_0)^{-1}\|_{H^{-1}(\mathcal{O}) \to L_2(\mathcal{O})} = \|(B_{D,\varepsilon} - \zeta Q_0)^{-1}\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \leq C_3 (1 + |\zeta|)^{-1/2} \theta_0(\varepsilon) (c_0 + 2)^{1/2}.
\] (9.23)

Now, from (3.2), (9.10), (9.22), and (9.23) it follows that
\[
\|T_2(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \gamma_{31} \theta_0(\varepsilon): \quad \gamma_{31} := C_4 C_3^2.
\] (9.24)

As a result, relations (9.13), (9.21), and (9.24) yield (9.1) with the constant \(C_{26} := \gamma_{30} + \gamma_{31}\).

Let us prove (9.5). By inequality (2.42) with \(\zeta = -1\),
\[
\|(B_{D,\varepsilon} + Q_0)^{-1} - (B_D^0 + Q_0)^{-1} - \varepsilon K_D(\varepsilon; -1)\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \leq (C_5 + C_6)\varepsilon^{1/2}.
\] (9.25)

We have
\[
(B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D^0 - \zeta Q_0)^{-1} - \varepsilon K_D(\varepsilon; \zeta)
= ((B_{D,\varepsilon} + Q_0)^{-1} - (B_D^0 + Q_0)^{-1} - \varepsilon K_D(\varepsilon; -1)) (B_D^0 + Q_0)(B_D^0 - \zeta Q_0)^{-1}
+ (\zeta + 1)(B_{D,\varepsilon} - \zeta Q_0)^{-1} Q_0^\varepsilon ((B_{D,\varepsilon} + Q_0)^{-1} - (B_D^0 + Q_0)^{-1})(B_D^0 + Q_0)(B_D^0 - \zeta Q_0)^{-1}
+ (1 + \zeta)(B_{D,\varepsilon} - \zeta Q_0)^{-1}(Q_0^\varepsilon - \zeta Q_0)(B_D^0 - \zeta Q_0)^{-1}.
\] (9.26)

Denote the consecutive summands in the right-hand side of (9.26) by \(L_1(\varepsilon; \zeta)\), \(L_2(\varepsilon; \zeta)\), and \(L_3(\varepsilon; \zeta)\). (Note that \(L_3(\varepsilon; \zeta)\) coincides with \(T_2(\varepsilon; \zeta)\).) We have
\[
\|L_1(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \leq \|(B_{D,\varepsilon} + Q_0^\varepsilon)^{-1} - (B_D^0 + Q_0)^{-1} - \varepsilon K_D(\varepsilon; -1)\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})}
\times \|(B_D^0 + Q_0)(B_D^0 - \zeta Q_0)^{-1}\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})}.
\] (9.27)

Together with (9.13), (9.20), and (9.25), this yields
\[
\|L_1(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \leq \gamma_{32} \varepsilon^{1/2} \theta_0(\varepsilon)^{1/2}: \quad \gamma_{32} := (C_5 + C_6)(c_0 + 2)\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty}.
\] (9.28)
Now, consider the second term in the right-hand side of (9.20). We have
\[ \|L_2(\varepsilon; \zeta)\|_{L_2(O) \rightarrow H^1(O)} \leq |\zeta + 1|\|B_{D,\varepsilon} - \zeta Q_0^{-1}\|_{L_2(O) \rightarrow H^1(O)} \|Q_0\|_{L_\infty} \times \|B_{D,\varepsilon} + Q_0^{-1}\| - (B_0^T + Q_0^{-1})\|L_2(O) \rightarrow L_2(O)\| (B_0 + Q_0^{-1})(B_0^T - \zeta Q_0^{-1})\|L_2(O) \rightarrow L_2(O)\|. \] (9.29)

Combining this with (9.8), (9.13), (9.17), and (9.20), we obtain
\[ \|L_2(\varepsilon; \zeta)\|_{L_2(O) \rightarrow H^1(O)} \leq \gamma_33^{|1 + \zeta|^{1/2} \varphi_0(\zeta)}; \quad \gamma_33 := C_3C_4(c_0 + 2)\|f\|_{L_\infty}\|f^{-1}\|^2_{L_\infty}. \] (9.30)

It remains to estimate the third term in the right-hand side of (9.20). By (3.2),
\[ \|L_3(\varepsilon; \zeta)\|_{L_2(O) \rightarrow H^1(O)} \leq |c_1 + \zeta|^{1/2} \varphi_0(\zeta)\|B_{D,\varepsilon} - \zeta Q_0^{-1}\|_{H^{-1}(O) \rightarrow H^1(O)}\|B_0^T - \zeta Q_0^{-1}\|_{L_2(O) \rightarrow L_2(O)}. \] (9.31)

Taking (2.4) and (9.10) into account, we see that
\[ \|L_3(\varepsilon; \zeta)\|_{L_2(O) \rightarrow H^1(O)} \leq c_4C_0c_1|c_1 + \zeta|^{1/2} \varphi_0(\zeta)\|B_{D,\varepsilon} - \zeta Q_0^{-1}\|_{H^{-1}(O) \rightarrow L_2(O)}. \] (9.32)

By duality, using (2.6) and (2.7), we obtain
\[ \|B_{D,\varepsilon}^{1/2}(B_{D,\varepsilon} - \zeta Q_0^{-1})\|_{H^{-1}(O) \rightarrow L_2(O)} = \|B_{D,\varepsilon}^{1/2}(\tilde{B}_{D,\varepsilon} - \zeta I^{-1}f^*e^{f*})\|_{H^{-1}(O) \rightarrow L_2(O)} \leq \|f^*\|\|\tilde{B}_{D,\varepsilon} - \zeta I^{-1}\|_{L_2(O) \rightarrow H^1(O)}. \] (9.33)

Since the range of the operator $f^*\tilde{B}_{D,\varepsilon}(\tilde{B}_{D,\varepsilon} - \zeta I^{-1})$ lies in $H_0^1(O; \mathbb{C}^n)$, from (2.3) and (2.6) it follows that
\[ \|f^*\tilde{B}_{D,\varepsilon}^{1/2}(\tilde{B}_{D,\varepsilon} - \zeta I^{-1})\|_{L_2(O) \rightarrow H^1(O)} \leq c_4\|B_{D,\varepsilon}^{1/2}\tilde{B}_{D,\varepsilon}^{1/2}(\tilde{B}_{D,\varepsilon} - \zeta I^{-1})\|_{L_2(O) \rightarrow L_2(O)} \leq c_4\|\tilde{B}_{D,\varepsilon}(\tilde{B}_{D,\varepsilon} - \zeta I^{-1})\|_{L_2(O) \rightarrow L_2(O)}. \] (9.34)

Together with (9.13) and (9.33), this yields
\[ \|B_{D,\varepsilon}^{1/2}(B_{D,\varepsilon} - \zeta Q_0^{-1})\|_{H^{-1}(O) \rightarrow L_2(O)} \leq c_4(c_0 + 2)\varphi_0(\zeta)^{1/2}. \] (9.35)

Combining (9.32) and (9.35), we find
\[ \|L_3(\varepsilon; \zeta)\|_{L_2(O) \rightarrow H^1(O)} \leq \gamma_34|1 + \zeta|^{1/2} \varphi_0(\zeta); \quad \gamma_34 := c_3^2(c_0 + 2)C_0c_3. \] (9.36)

As a result, relations (9.20), (9.28), (9.30), and (9.36) imply that
\[ \|(B_{D,\varepsilon} - \zeta Q_0^{-1})\|_{L_2(O) \rightarrow H^1(O)} \leq \gamma_32\|\tilde{B}_{D,\varepsilon}^{1/2} \varphi_0(\zeta)^{1/2} + (\zeta_33 + \gamma_34)|c_1 + \zeta|^{1/2} \varphi_0(\zeta). \] (9.37)

This yields (9.5) with the constant $C_{27} := \max\{\gamma_32; \gamma_33 + \gamma_34\}$.

It remains to check (9.6). From (1.1) and (9.3) it follows that
\[ \|p_\varepsilon - \tilde{g}^Tb(D)v_\varepsilon\|_{L_2(O)} \leq (\alpha_1)\|g\|_{L_\infty} C_27(\varepsilon^{1/2} \varphi_0(\zeta)^{1/2} + |c_1 + \zeta|^{1/2} \varphi_0(\zeta))\|F\|_{L_2(O)}. \] (9.38)

Next, taking (1.3) into account, by analogy with (5.17), (5.18), and (5.20), we obtain
\[ \|g^Tb(D)v_\varepsilon - \tilde{g}^T S_\varepsilon b(D)\tilde{u}_0 - g^T(b(D)\tilde{\Lambda})^T S_\varepsilon \tilde{u}_0\|_{L_2(O)} \leq \gamma_35\|\tilde{u}_0\|_{H^2(\mathbb{R}^d)}. \] (9.39)

Here $\gamma_35 := \|g\|_{L_\infty}(\alpha_1d)^{1/2} + \tilde{M}_1 d^{1/2} + r_1).$ From (2.37) and (9.11) it follows that
\[ \|\tilde{u}_0\|_{H^2(\mathbb{R}^d)} \leq \gamma_36 \varphi_0(\zeta)^{1/2}\|F\|_{L_2(O)}; \quad \gamma_36 := C_0^{(2)}c_4. \] (9.40)

Combining this with (9.38) and (9.39), we arrive at estimate (9.6) with the constant $\tilde{C}_{27} := (\alpha_1)\|g\|_{L_\infty} C_{27} + \gamma_35\gamma_36$. \hfill $\square$

**Corollary 9.7.** Under the assumptions of Theorem 9.2, for $0 < \varepsilon \leq \varepsilon_\gamma$ and $\zeta \in \mathbb{C} \setminus [c_0, \infty)$ we have
\[ \|u_\varepsilon - v_\varepsilon\|_{H^1(O)} \leq C_{28}\varepsilon^{1/2} \varphi_0(\zeta)^{3/4}\|F\|_{L_2(O)}, \] (9.41)
\[ \|p_\varepsilon - \tilde{g}^T S_\varepsilon b(D)\tilde{u}_0 - g^T(b(D)\tilde{\Lambda})^T S_\varepsilon \tilde{u}_0\|_{L_2(O)} \leq \tilde{C}_{28}\varepsilon^{1/2} \varphi_0(\zeta)^{3/4}\|F\|_{L_2(O)}. \] (9.42)

The constants $C_{28}$ and $\tilde{C}_{28}$ depend only on the initial data (1.9) and the domain $O$. \hfill $\square$
Proof. Relations (9.8), (9.10), and (9.15) yield the following rough estimate:

$$\| (B_{D,e} - \zeta Q_0)^{-1} - (B_0^0 - \zeta Q_0)^{-1} - \varepsilon K_D(\varepsilon; \zeta) \|_{L_2(\Omega)\to H^1(\Omega)} \leq \gamma_{37}(\varepsilon + (1 + |\zeta|)^{-1/2})\delta_3(\zeta)^{1/2}; \quad \gamma_{37} := 2C_3 + C_6. \quad (9.43)$$

For $1 + \zeta^{1/2} \delta_3(\zeta)^{1/4} < \varepsilon^{-1/2}$ we use (9.37) and note that $\varepsilon[1 + \zeta^{1/2} \delta_3(\zeta) \leq \varepsilon^{1/2} \delta_3(\zeta)^{3/4}$.

For $1 + \zeta^{1/2} \delta_3(\zeta)^{1/4} > \varepsilon^{-1/2}$ we apply (9.43) and take the inequality $(1 + |\zeta|)^{-1/2} \delta_3(\zeta)^{1/2} < \varepsilon^{1/2} \delta_3(\zeta)^{3/4}$ into account. This yields (9.41) with $C_{28} := \max\{\gamma_{32} + \gamma_{33} + \gamma_{34}: 2\gamma_{37}\}$.

Relations (9.39), (9.40), and (9.41) imply (9.42) with $\tilde{C}_{28} := (d\alpha_1)^{1/2}\|g\|_{L_\infty}C_{28} + \gamma_{35}\gamma_{36}$.

9.3. Removal of $S_e$.

Theorem 9.8. Suppose that the assumptions of Theorem 7.2 are satisfied. Suppose also that Conditions (7.21) and (7.22) hold. Let $K^0_D(\varepsilon; \zeta)$ and $G^0_D(\varepsilon; \zeta)$ be defined by (7.21) and (7.22), respectively. Then for $0 < \varepsilon < \varepsilon_0$ and $\zeta \in \mathbb{C} \setminus [c_0, \infty)$ we have

$$\| (B_{D,e} - \zeta Q_0)^{-1} - (B_0^0 - \zeta Q_0)^{-1} - \varepsilon K^0_D(\varepsilon; \zeta) \|_{L_2(\Omega)\to H^1(\Omega)} \leq C_{29}(\varepsilon^{1/2} \delta_3(\zeta)^{1/2} + 1 + |\zeta|^{1/2} \delta_3(\zeta)), \quad (9.44)$$

$$\| g^b(D)(B_{D,e} - \zeta Q_0)^{-1} - G^0_D(\varepsilon; \zeta) \|_{L_2(\Omega)\to L_2(\Omega)} \leq \tilde{C}_{29}(\varepsilon^{1/2} \delta_3(\zeta)^{1/2} + \varepsilon[1 + |\zeta|^{1/2} \delta_3(\zeta)]). \quad (9.45)$$

The constants $C_{29}$ and $\tilde{C}_{29}$ depend only on the initial data (1.29), the domain $\Omega$, and also on $p$ and the norms $\|\Lambda\|_{L_\infty}$, $\|\tilde{\Lambda}\|_{L_\infty(\Omega)}$.

Proof. Applying Lemmas 7.7 and 7.8 together with (9.36) and (9.40), we obtain (9.44) with $C_{29} := C_{27} + (\mathcal{C}_1 + \mathcal{C}_2)\gamma_{36}$.

Let us check (9.14). By (1.4) and (9.44),

$$\| g^b(D)(B_{D,e} - \zeta Q_0)^{-1} - g^b(D)(I + \varepsilon^2 b(D) + \varepsilon^2 \tilde{\Lambda}^2)(B_0^0 - \zeta Q_0)^{-1} \|_{L_2(\Omega)\to L_2(\Omega)} \leq (d\alpha_1)^{1/2}\|g\|_{L_\infty}C_{29}(\varepsilon^{1/2} \delta_3(\zeta)^{1/2} + \varepsilon[1 + |\zeta|^{1/2} \delta_3(\zeta)]). \quad (9.46)$$

Relation (7.14) remains true. By analogy with (7.14) and (7.16), using (9.11), we obtain

$$\varepsilon \sum_{l=1}^d g^b(D)(\tilde{\Lambda}^2 b(D) D_l + \tilde{\Lambda}^2 D_l)(B_0^0 - \zeta Q_0)^{-1} \|_{L_2(\Omega)\to L_2(\Omega)} \leq \gamma_{38}\varepsilon \|\Lambda\|_{L_\infty}(\mathcal{O}1\varepsilon d\|\Lambda\|_{L_\infty} + (\alpha_1 d)^{1/2}\|\tilde{\Lambda}\|_{L_\infty(\Omega)} C_{\tilde{\Lambda}}(\gamma, \tilde{\Omega})C_O^{(1)}). \quad (9.47)$$

where $\gamma_{38} := \|\tilde{\Lambda}\|_{L_\infty} (\alpha_1 d\|\Lambda\|_{L_\infty} + (\alpha_1 d)^{1/2}\|\tilde{\Lambda}\|_{L_\infty(\Omega)} C_{\tilde{\Lambda}}(\gamma, \tilde{\Omega})C_O^{(1)})$. Now, relations (7.14), (9.46), and (9.47) imply (9.36) with $\tilde{C}_{29} := (d\alpha_1)^{1/2}\|g\|_{L_\infty}C_{29} + \gamma_{35}\gamma_{36}$.

Remark 9.9. If only Condition 7.11 (respectively, Condition 7.2) is satisfied, then the smoothing operator $S_e$ can be removed only in the term of the corrector containing $\Lambda$ (respectively, $\tilde{\Lambda}$).

9.4. Approximation with the boundary layer correction term. Now, using Theorem 7.4, we obtain “another” approximation with the boundary layer correction term.

Theorem 9.10. Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class $C^{1,1}$. Let $0 < \varepsilon_0 \leq 1$. Suppose that $c_0 \geq 0$ is subject to Condition 9.1. Let $0 < \varepsilon < \varepsilon_0$ and $\zeta \in \mathbb{C} \setminus [c_0, \infty)$. Let $u_e$ be the solution of problem (2.8), and let $v_e$ be defined by (2.29) . (2.40). Let $w_e$ be the solution of problem (2.43). Suppose that $K_D(\varepsilon; \zeta)$ and $W_D(\varepsilon; \zeta)$ are given by (2.38) and (2.54), respectively. We have

$$\| u_e - v_e + w_e \|_{H^1(\mathcal{O})} \leq (C_{30} + C_{31})|1 + \zeta^{1/2}\varepsilon \delta_3(\zeta)\|F\|_{L_2(\mathcal{O})}. \quad (9.48)$$

In operator terms,

$$\| (B_{D,e} - \zeta Q_0)^{-1} - (B_0^0 - \zeta Q_0)^{-1} - \varepsilon K_D(\varepsilon; \zeta) + \varepsilon W_D(\varepsilon; \zeta) \|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq (C_{30} + C_{31})|1 + \zeta^{1/2}\varepsilon \delta_3(\zeta)|. \quad (9.48)$$

The constants $C_{30}$ and $C_{31}$ depend only on the initial data (1.9) and the domain $\mathcal{O}$. If the matrix-valued function $Q_0(x)$ is constant, then $C_{31} = 0$.

Remark 9.11. Taking $\varepsilon_0 = 1$ and $c_0 = 0$, for $|\zeta| \geq 1$ we have $\delta_3(\zeta) = c(\phi)^2$. So, if $Q_0(x)$ is constant, then $C_{31} = 0$ and estimate (9.48) improves inequality (2.36) with respect to $\phi$.\end{proof}
The constants for any function

Theorem 9.14. Estimates in a strictly interior subdomain.

Using estimate (2.56) with

Obviously, if

Next, from the definition of \(T(\varepsilon; \zeta)\) (see (2.50), (2.52)) it is clear that

Combining this identity and (2.54), it is easy to check that

\[
(B_{D,\varepsilon} - \zeta Q_0^{-1} - (B_D - \zeta \overline{Q}_0)^{-1} - \varepsilon K_D(\varepsilon; \zeta) + \varepsilon W_D(\varepsilon; \zeta)
\]

\[
= (B_{D,\varepsilon} - \zeta Q_0^{-1}) (B_{D,\varepsilon} + (B_D + \overline{Q}_0)^{-1} + \varepsilon (B_{D,\varepsilon} + Q_0)^{-1} T(\varepsilon; -1) - (B_D + \overline{Q}_0)^{-1} T(\varepsilon; -1))
\]

\[
\times (B_D + \overline{Q}_0) (B_D - \zeta \overline{Q}_0)^{-1} + (\zeta + 1) (B_{D,\varepsilon} - \zeta Q_0^{-1}) (Q_0 - \overline{Q}_0) (B_D - \zeta \overline{Q}_0)^{-1}.
\]

Denote the first summand on the right by \(J(\varepsilon; \zeta)\). Note that the second term is \(L_3(\varepsilon; \zeta)\); cf. (9.20).

Obviously, if \(Q_0(\mathbf{x}) = \overline{Q}_0\), then \(L_3(\varepsilon; \zeta) = 0\).

From (2.2), (2.3), (2.7), and (9.13) it follows that

\[
\|B_{D,\varepsilon}^{1/2} (B_{D,\varepsilon} - \zeta Q_0^{-1} - (B_D - \zeta \overline{Q}_0)^{-1} - \varepsilon K_D(\varepsilon; \zeta) + \varepsilon W_D(\varepsilon; \zeta)) \Phi\|_{L_2(\mathcal{O})} = \|\tilde{B}_{D,\varepsilon}^{1/2} (\tilde{B}_{D,\varepsilon} - \zeta \Theta)^{-1} (\tilde{B}_{D,\varepsilon} + I) (f)^{-1} \Phi\|_{L_2(\mathcal{O})}
\]

\[
\leq \|\tilde{B}_{D,\varepsilon} - \zeta \Theta^{-1} (\tilde{B}_{D,\varepsilon} + I)\|_{L_2(\mathcal{O})} \|B_{D,\varepsilon}^{1/2} \Phi\|_{L_2(\mathcal{O})} \leq c_3^{1/2} (c_3 + 2) \theta_3 (\zeta)^{1/2} \|\Phi\|_{H^1(\mathcal{O})}
\]

for any function \(\Phi \in H^1_0(\mathcal{O}; \mathbb{C}^n)\). Hence, by (2.31) and (9.49),

\[
\|J(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) - H^1(\mathcal{O})} \leq c_3 \|B_{D,\varepsilon}^{1/2} J(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) - L_2(\mathcal{O})}
\]

\[
\leq c_4 c_3^{1/2} (c_3 + 2) \theta_3 (\zeta)^{1/2} C_7 \|B_{D,\varepsilon}^{1/2} (B_D + \overline{Q}_0)(B_D - \zeta \overline{Q}_0)^{-1}\|_{L_2(\mathcal{O}) - L_2(\mathcal{O})}.
\]

Finally, (9.36), (9.50), and (9.51) imply the required estimate (9.48) with \(C_{31} := \gamma_{34}\).

9.5. Special cases. The following statements can be checked similarly to Propositions 7.11 and 7.11.

Proposition 9.12. Suppose that \(0 < \varepsilon \leq 1\) and \(c_3\) is subject to Condition 9.1. Suppose that relations (1.26) and (7.11) hold. Then for \(0 < \varepsilon \leq \varepsilon_5\) and \(\zeta \in \mathbb{C} \setminus [c_5, \infty)\) we have

\[
\|B_{D,\varepsilon} - \zeta Q_0^{-1} - (B_D - \zeta \overline{Q}_0)^{-1}\|_{L_2(\mathcal{O}) - H^1(\mathcal{O})} \leq (C_{30} + C_{31}) (1 + \zeta^{1/2}) \varepsilon \theta_3 (\zeta).
\]

Proposition 9.13. Suppose that the assumptions of Theorem 9.2 are satisfied. Suppose that relations (1.27) and (7.11) hold. Then for \(0 < \varepsilon \leq \varepsilon_5\) and \(\zeta \in \mathbb{C} \setminus [c_5, \infty)\) we have

\[
\|g_b(D)(B_{D,\varepsilon} - \zeta Q_0^{-1}) - g_0 b(D)(B_D - \zeta \overline{Q}_0)^{-1}\|_{L_2(\mathcal{O}) - L_2(\mathcal{O})} \leq \tilde{C}_{20} (\varepsilon^{1/2} \theta_b(\zeta)^{1/2} + \varepsilon (1 + \zeta^{1/2} \theta_b(\zeta))).
\]

9.6. Estimates in a strictly interior subdomain.

Theorem 9.14. Suppose that the assumptions of Theorem 9.2 are satisfied. Let \(\mathcal{O}'\) be a strictly interior subdomain of the domain \(\mathcal{O}\). Let \(\delta := \text{dist}(\mathcal{O}', \partial \mathcal{O})\). Then for \(0 < \varepsilon \leq \varepsilon_5\) and \(\zeta \in \mathbb{C} \setminus [c_5, \infty)\) we have

\[
\|B_{D,\varepsilon} - \zeta Q_0^{-1} - (B_D - \zeta \overline{Q}_0)^{-1} - \varepsilon K_D(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) - H^1(\mathcal{O})} \leq \varepsilon (C'_{32} \delta^{-1} \theta_b(\zeta)^{1/2} + C_{32}' \|1 + \zeta^{1/2} \theta_b(\zeta)\|)
\]

\[
\|g_b(D)(B_{D,\varepsilon} - \zeta Q_0^{-1}) - G_D(\varepsilon; \zeta)\|_{L_2(\mathcal{O}) - L_2(\mathcal{O})} \leq \varepsilon (\tilde{C}'_{32} \delta^{-1} \theta_b(\zeta)^{1/2} + \tilde{C}_{32}' \|1 + \zeta^{1/2} \theta_b(\zeta)\|).
\]

The constants \(C_{32}', C_{32}''\), \(\tilde{C}_{32}', \) and \(\tilde{C}_{32}''\) depend only on the initial data (1.9) and the domain \(\mathcal{O}\).
Proof. Estimate $\| (B_{D,\varepsilon} + Q_0^\varepsilon)^{-1} - (B_D + \overline{Q_0})^{-1} - \varepsilon K_D(\varepsilon; 1) \|_{L_2(\Omega) \to H^1(\Omega')} \leq \varepsilon (C'_2\delta^{-1} + C'^2_{24}) \quad (9.54)$ for $0 < \varepsilon \leq \varepsilon_1$. We apply identity (9.26). The first term $L_1(\varepsilon; \zeta)$ satisfies

$$\| L_1(\varepsilon; \zeta) \|_{L_2(\Omega) \to H^1(\Omega')} \leq \| (B_{D,\varepsilon} + Q_0^\varepsilon)^{-1} - (B_D + \overline{Q_0})^{-1} - \varepsilon K_D(\varepsilon; 1) \|_{L_2(\Omega) \to H^1(\Omega')} \times \| (B_D + \overline{Q_0})(B_D^0 - \zeta Q_0^{-1})^{-1} \|_{L_2(\Omega) \to L_2(\Omega')}.$$ 

Combining this with (9.13), (9.20), and (9.54), we obtain

$$\| L_1(\varepsilon; \zeta) \|_{L_2(\Omega) \to H^1(\Omega')} \leq (\gamma_{38} \delta^{-1} + \gamma_{39}) \varepsilon \delta_1(\zeta)^{1/2}, \quad 0 < \varepsilon \leq \varepsilon_1, \quad (9.55)$$

where $\gamma_{38} := C'_2\delta_1 \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} (c_2 + 2) \quad (9.53)$ and $\gamma_{39} := C''_{32} \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} (c_2 + 2)$. As a result, relations (9.20), (9.30), (9.30), and (9.52) imply (9.52) with $C''_{32} = \gamma_{38}$ and $\gamma_{39} = \gamma_{33} + 3\gamma_{34} + \gamma_{39}$. Estimate (9.53) is deduced from (9.52) by analogy with (5.16)–(5.21). Instead of (4.3), we use (9.40).}

9.7. Removal of $S_\varepsilon$ in approximations in a strictly interior subdomain.

**Theorem 9.15.** Suppose that the assumptions of Theorem 9.14 hold. Suppose also that Conditions (7.1) and (7.2) are satisfied. Let $K^0_D(\varepsilon; \zeta)$ and $G^0_D(\varepsilon; \zeta)$ be given by (7.1) and (7.2), respectively. Then for $0 < \varepsilon \leq \varepsilon_1$ and $\zeta \in C \backslash \{0\}$, we have

$$\| (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta Q_0^\varepsilon)^{-1} - \varepsilon K_D^0(\varepsilon; \zeta) \|_{L_2(\Omega) \to H^1(\Omega')} \leq \varepsilon (C''_{32} \delta^{-1} \delta_1(\zeta)^{1/2} + C_{33}) \| 1 + \zeta^{1/2} \delta_1(\zeta) \|,$$

$$\| g^b(D)(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D^0(\varepsilon; \zeta) \|_{L_2(\Omega) \to H^1(\Omega')} \leq \varepsilon (\tilde{C}_{32} \delta^{-1} \delta_1(\zeta)^{1/2} + \tilde{C}_{33}) \| 1 + \zeta^{1/2} \delta_1(\zeta) \|,$$

where the constants $C''_{32}$ and $\tilde{C}_{32}$ are as in (9.52), (9.53). The constants $C_{33}$ and $\tilde{C}_{33}$ depend on the initial data (1.20), the domain $\Omega$, and also on $p$ and the norms $\| A \|_{L_\infty}$, $\| \tilde{A} \|_{L_p(\Omega)}$.

**Proof.** Combining Lemma 7.7, Lemma 7.8, and relations (2.38), (9.40), (9.52), we arrive at estimate (9.56) with $C_{33} := C''_{32} + (\zeta_0^* + \zeta_0^*) \gamma_{36}$.

Inequality (9.57) is deduced from (9.56). By analogy with (9.46), using (9.47), we obtain estimate (9.57) with $\tilde{C}_{33} := (d \delta_1)^{1/2} \| g \|_{L_\infty} C_{33} + \gamma_{38} C_{44}$.}

10. More results

In the present section, for $\text{Re} \zeta > 0$, we show that the estimates of Theorems 2.3, 2.6, and 8.1 can be "improved"; this concerns the behavior of the right-hand sides with respect to $\phi = \arg \zeta$. However, an extra "bad term" (with respect to $|\zeta|$) appears; in the case where $Q_0(\mathbf{x})$ is constant, this term is equal to zero, and we obtain the "real improvement". The method is based on the identities for generalized resolvents from Section 9. Due to these identities, we transfer the already proven estimates from the left half-plane to the symmetric point of the right one.

Also, we obtain some new versions of estimates for the fluxes.

10.1. Estimates in the $(L_2 \to L_2)$- and $(L_2 \to H^1)$-operator norms.

**Theorem 10.1.** Under the assumptions of Theorem 2.6, for $0 < \varepsilon \leq \varepsilon_1$, $\zeta \in C \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and $\text{Re} \zeta \geq 0$, we have

$$\| (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta Q_0^0)^{-1} \|_{L_2(\Omega) \to L_2(\Omega')} \leq C_{34} c(\phi)^2 c(\phi)^2 \varepsilon, \quad (10.1)$$

$$\| (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta Q_0^0)^{-1} - \varepsilon K_D^0(\varepsilon; \zeta) \|_{L_2(\Omega) \to H^1(\Omega')} \leq C_{36} \left( c(\phi)^{1/2} |\zeta|^{-1/4} + c(\phi)^2 \varepsilon \right) + C_{37} (\text{Re} \zeta)^{1/2} c(\phi)^2 \varepsilon, \quad (10.2)$$

$$\| g^b(D)(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D^0(\varepsilon; \zeta) \|_{L_2(\Omega) \to L_2(\Omega')} \leq \tilde{C}_{36} \left( c(\phi)^{1/2} |\zeta|^{-1/4} + c(\phi)^2 \varepsilon \right) + \tilde{C}_{37} (\text{Re} \zeta)^{1/2} c(\phi)^2 \varepsilon. \quad (10.3)$$

The constants $C_{34}$, $C_{35}$, $C_{36}$, $C_{37}$, $\tilde{C}_{36}$, and $\tilde{C}_{37}$ depend only on the initial data (1.9) and the domain $\Omega$. If the matrix-valued function $Q_0(\mathbf{x})$ is constant, then $C_{35} = C_{37} = \tilde{C}_{37} = 0$. 

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Proof. Let \( \zeta = \text{Re}\,\zeta + i\text{Im}\,\zeta, \text{Re}\,\zeta \geq 0, \text{Im}\,\zeta \neq 0 \). Let \( \hat{\zeta} = -\text{Re}\,\zeta + i\text{Im}\,\zeta \). Then \( |\hat{\zeta}| = |\zeta| \) and \( c(\hat{\phi}) = 1 \), where \( \hat{\phi} = \text{arg}\,\hat{\zeta} \). According to (2.35),

\[
\|(B_{D,\varepsilon} - \hat{\zeta}Q_0)^{-1} - (B_D^0 - \hat{\zeta}Q_0)^{-1}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4\varepsilon|\zeta|^{-1/2}.
\] (10.4)

Similarly to (9.18), we have

\[
(B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D^0 - \zeta Q_0)^{-1} = (B_{D,\varepsilon} - \zeta Q_0)^{-1}((B_{D,\varepsilon} - \hat{\zeta}Q_0)^{-1} - (B_D^0 - \hat{\zeta}Q_0)^{-1})(B_D^0 - \zeta Q_0)(B_D^0 - \zeta Q_0)^{-1} + (\zeta - \hat{\zeta})(B_{D,\varepsilon} - \zeta Q_0)^{-1}(Q_0 - \zeta Q_0)(B_D^0 - \zeta Q_0)^{-1}.
\] (10.5)

Denote the consecutive terms in the right-hand side of (10.5) by \( J_1(\varepsilon; \zeta) \) and \( J_2(\varepsilon; \zeta) \). By (10.4) and the analogs of (9.19) and (9.20),

\[
\|J_1(\varepsilon; \zeta)\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4\varepsilon|\zeta|^{-1/2}\|f\|_{L_\infty}^2\|f^{-1}\|_{L_\infty}^2 \sup_{x>0} \frac{|x - \hat{\zeta}|^2}{|x - x'|^2}.
\] (10.6)

The computation shows that

\[
\sup_{x>0} \frac{|x - \hat{\zeta}|}{|x - \zeta|} \leq 2c(\phi).
\] (10.7)

Estimates (10.6) and (10.7) imply the inequality

\[
\|J_1(\varepsilon; \zeta)\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4C_0(\varepsilon)^2|\zeta|^{-1/2}; \quad C_4 : = 4C_4\|f\|_{L_{\infty}}^2\|f^{-1}\|_{L_{\infty}}^2.
\] (10.8)

Since \( \zeta - \hat{\zeta} = 2\text{Re}\,\zeta \), similarly to (9.22) and (9.23), taking (3.2) into account, we obtain

\[
\|J_1(\varepsilon; \zeta)\|_{L_2(\Omega) \to L_2(\Omega)} \leq 2(\text{Re}\,\zeta)C_0(\varepsilon)^2\|B_{D,\varepsilon} - \zeta Q_0\|_{L_2(\Omega) \to L_2(\Omega)}\|B_D^0 - \zeta Q_0\|_{L_2(\Omega) \to H_\delta(\Omega)}.
\]

Together with Lemmas 2.1 and 2.3 this yields

\[
\|J_2(\varepsilon; \zeta)\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_5\varepsilon(\varepsilon)^2; \quad C_5 : = 2C_0(\varepsilon)^2|Q_0|^{-1}\|L_{\infty}\|^2.
\] (10.9)

Combining (10.5), (10.8) and (10.9), we arrive at estimate (10.1).

Now we proceed to the proof of estimate (10.2). We apply (2.22) at the point \( \hat{\zeta} \):

\[
\|(B_{D,\varepsilon} - \hat{\zeta}Q_0)^{-1} - (B_D^0 - \hat{\zeta}Q_0)^{-1} - \varepsilon K_D(\varepsilon; \hat{\zeta})\|_{L_2(\Omega) \to H_\delta(\Omega)} \leq C_5\varepsilon|\zeta|^{-1/4} + C_6\varepsilon.
\] (10.10)

Similarly to (9.26), we have

\[
(B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D^0 - \zeta Q_0)^{-1} - \varepsilon K_D(\varepsilon; \zeta) = ((B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D^0 - \zeta Q_0)^{-1} - \varepsilon K_D(\varepsilon; \hat{\zeta}))(B_D^0 - \zeta Q_0)(B_D^0 - \zeta Q_0)^{-1}
\]

\[
+ (\zeta - \hat{\zeta})(B_{D,\varepsilon} - \zeta Q_0)^{-1}(Q_0 - \zeta Q_0)(B_D^0 - \zeta Q_0)^{-1}.
\] (10.11)

Denote the consecutive summands in the right-hand side of (10.11) by \( \Sigma_1(\varepsilon; \zeta), \Sigma_2(\varepsilon; \zeta), \) and \( \Sigma_3(\varepsilon; \zeta) \). (Note that \( \Sigma_3(\varepsilon; \zeta) \) coincides with \( J_2(\varepsilon; \zeta) \).) Similarly to (9.20), by (10.7),

\[
\|(B_D^0 - \hat{\zeta}Q_0)(B_D^0 - \zeta Q_0)^{-1}\|_{L_2(\Omega) \to L_2(\Omega)} \leq 2\|f\|_{L_{\infty}}\|f^{-1}\|_{L_{\infty}}\|L_{\infty}\|c(\phi).
\] (10.12)

So, by analogy with (9.27), taking (10.10) and (10.12) into account, we have

\[
\|\Sigma_1(\varepsilon; \zeta)\|_{L_2(\Omega) \to H_\delta(\Omega)} \leq \gamma_{40}\varepsilon|\zeta|^{-1/4} + \gamma_{41}\varepsilon,
\] (10.13)

where \( \gamma_{40} : = 2C_5\|f\|_{L_{\infty}}\|f^{-1}\|_{L_{\infty}} \) and \( \gamma_{41} : = 2C_0\|f\|_{L_{\infty}}\|f^{-1}\|_{L_{\infty}} \).

Similarly to (9.29), using Lemma 2.1 and relations (10.4), (10.12), we obtain

\[
\|\Sigma_2(\varepsilon; \zeta)\|_{L_2(\Omega) \to H_\delta(\Omega)} \leq \gamma_{42}\varepsilon^2; \quad \gamma_{42} : = 4C_4(\varepsilon) + \|Q_0^{-1}\|_{L_{\infty}}\|f\|_{L_{\infty}}\|f^{-1}\|_{L_{\infty}}^3.
\] (10.14)
The term \( \mathfrak{L}_3(\varepsilon; \zeta) \) is estimated by using Lemma 2.9 and 8.2 (cf. 9.31–9.46):

\[
\|\mathfrak{L}_3(\varepsilon; \zeta)\|_{L_2 \to H^1} \lesssim 2(\text{Re}\,\zeta)C_{Q_0}(\varepsilon\|B_{D,\varepsilon} - \zeta Q_0\|_{H^{-1} \to H^1} + \|B_{D} - \zeta Q_0\|^{-1}_{L_2 \to H^1}) \\
\lesssim 2\varepsilon(\text{Re}\,\zeta)C_{Q_0}(C_1 + \|Q_0^{-1}\|_{L_\infty})c(\phi)|\zeta|^{-1/2}c_2\sup_{x \neq \zeta} \frac{x}{|x - \zeta|} \lesssim C_{37}c(\phi)^2(\text{Re}\,\zeta)^{1/2}.
\]

(10.15)

Here \( C_{37} := 2c_3^2C_{Q_0}(C_1 + \|Q_0^{-1}\|_{L_\infty}) \).

As a result, relations (10.11) and (10.13)–(10.15) imply estimate (10.2) with the constant 
\( C_{36} := \max\{\gamma_0; \gamma_4 + \gamma_2\} \). Estimate (10.3) is deduced from (5.17), (5.19), (5.21), and (10.2). \( \square \)

### 10.2. Removal of \( S_\varepsilon \)

**Theorem 10.2.** Suppose that the assumptions of Theorem 2.5 are satisfied. Suppose also that Conditions (7.3) and (7.2) hold. Let \( K^0_D(\varepsilon; \zeta) \) and \( G^0_D(\varepsilon; \zeta) \) be defined by (7.1) and (7.2), respectively. Then for \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+, \|\zeta\| \geq 1, \text{Re}\,\zeta > 0, \) we have

\[
\|g^b(D)(B_{D,\varepsilon} - \zeta Q_0) - G^0_D(\varepsilon; \zeta)\|_{L_2(\Omega) \to L_2(\Omega)} \\
\lesssim \tilde{C}_{38}\left(c(\phi)^{1/2}|\zeta|^{-1/4} + c(\phi)^2\varepsilon\right) + \tilde{C}_{37}(\text{Re}\,\zeta)^{1/2}c(\phi)^2\varepsilon.
\]

(10.17)

The constants \( C_{37} \) and \( \tilde{C}_{37} \) are the same as in Theorem 10.1. The constants \( C_{38} \) and \( \tilde{C}_{38} \) depend only on the initial data (1.9), the domain \( \Omega \), on \( p \) and the norms \( \|\Lambda\|_{L_p(\Omega)} \) and \( \|\Lambda\|_{L_p(\Omega)} \).

**Proof.** The proof is similar to that of Theorem 7.6. To obtain (10.16), we use estimates (7.11), (7.12), and (10.2). By analogy with (7.13)–(7.16), estimate (10.17) is deduced from (10.16). We omit the details. \( \square \)

### 10.3. Special case

Similarly to Proposition 7.11, the following statement can be deduced from Theorem 10.2.

**Proposition 10.3.** Suppose that the assumptions of Theorem 2.5 are satisfied. Assume that relations (1.27) and (1.17) hold. Then for \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+, \|\zeta\| \geq 1, \text{Re}\,\zeta > 0, \) we have

\[
\|g^b(D)(B_{D,\varepsilon} - \zeta Q_0) - G^0_D(\varepsilon; \zeta)\|_{L_2(\Omega) \to L_2(\Omega)} \\
\lesssim \tilde{C}_{38}\left(c(\phi)^{1/2}|\zeta|^{-1/4} + c(\phi)^2\varepsilon\right) + \tilde{C}_{37}(\text{Re}\,\zeta)^{1/2}c(\phi)^2\varepsilon.
\]

(10.17)

### 10.4. Estimates in a strictly interior subdomain

**Theorem 10.4.** Under the assumptions of Theorem 8.1 for \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+, \|\zeta\| \geq 1, \text{Re}\,\zeta > 0, \) we have

\[
\|B_{D,\varepsilon} - \zeta Q_0\|^{-1} - B_{D} - \zeta Q_0\|^{-1} - \varepsilon K^0_D(\varepsilon; \zeta)\|_{H^1(\Omega')} \\
\lesssim C_{39}\varepsilon\left(\delta^{-1}c(\phi)|\zeta|^{-1/2} + c(\phi)^2\varepsilon\right) + C_{37}(\text{Re}\,\zeta)^{1/2}c(\phi)^2\varepsilon,
\]

(10.18)

\[
\|g^b(D)(B_{D,\varepsilon} - \zeta Q_0) - G^0_D(\varepsilon; \zeta)\|_{L_2(\Omega) \to L_2(\Omega')} \\
\lesssim \tilde{C}_{39}\varepsilon\left(\delta^{-1}c(\phi)|\zeta|^{-1/2} + c(\phi)^2\varepsilon\right) + \tilde{C}_{37}(\text{Re}\,\zeta)^{1/2}c(\phi)^2\varepsilon.
\]

(10.19)

Here the constants \( C_{37} \) and \( \tilde{C}_{37} \) are the same as in Theorem 10.1. The constants \( C_{39} \) and \( \tilde{C}_{39} \) depend only on the initial data (1.9) and the domain \( \Omega \).

**Proof.** Let \( \zeta = \text{Re}\,\zeta + i\text{Im}\,\zeta, \text{Re}\,\zeta > 0, \text{Im}\,\zeta \neq 0, \) and \( \|\zeta\| \geq 1. \) Let \( \hat{\zeta} = -\text{Re}\,\zeta + i\text{Im}\,\zeta. \) Estimate (8.1) at the point \( \hat{\zeta} \) means that

\[
\|B_{D,\varepsilon} - \zeta Q_0\|^{-1} - B_{D} - \zeta Q_0\|^{-1} - \varepsilon K^0_D(\varepsilon; \hat{\zeta})\|_{L_2(\Omega) \to H^1(\Omega')} \lesssim \varepsilon (C_{24}|\zeta|^{-1/2}\delta^{-1} + C_{24}')
\]

(10.20)

for \( 0 < \varepsilon \leq \varepsilon_1 \). Next, we use identity (10.11). By (10.12) and (10.20),

\[
\|\mathfrak{L}_1(\varepsilon; \zeta)\|_{L_2(\Omega) \to H^1(\Omega')} \lesssim 2\varepsilon(C_{24}|\zeta|^{-1/2}\delta^{-1} + C_{24}')\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}c(\phi).
\]

(10.21)
Theorem 10.5. Suppose that the assumptions of Theorem 8.1 are satisfied. Suppose also that Conditions 7.1 and 7.2 hold. Let $K^0_D(\varepsilon; \zeta)$ and $G^0_D(\varepsilon; \zeta)$ be given by (7.1) and (7.2), respectively. Then for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, Re $\zeta \geq 0$, and $0 < \varepsilon \leq \varepsilon_1$ we have

$$
\| (B_{D,\varepsilon} - \zeta Q_0)^{-1} - (B_D - \zeta Q_0)^{-1} - \varepsilon K^0_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow H^1(\Omega)}
\leq C_{40c}(\phi)^2 \varepsilon(\varepsilon_1^{-1}/2 - 1) + C_{37}(\phi)^2(\text{Re} \zeta)^{1/2},
$$

and

$$
\| g^0 b(D)(B_{D,\varepsilon} - \zeta Q_0)^{-1} - G_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega)}
\leq C_{41}(\phi)^{3/2} \varepsilon^{1/2} |\zeta|^{-1/4} + C_{42}(\phi)^{3/2} \varepsilon^{1/2}.
$$

The constants $C_{37}$ and $\tilde{C}_{37}$ are the same as in Theorem 10.1. The constants $C_{40}$ and $\tilde{C}_{40}$ depend only on the initial data (1.9), the domain $\Omega$, and also on $p$ and the norms $\|A\|_{L^\infty}$, $\|A\|_{L_p(\Omega)}$.

10.5. Approximation for the flux. We obtain some new versions of estimates for the flux.

Proposition 10.6. Under the assumptions of Theorem 2.6, for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, Re $\zeta \geq 0$, and $0 < \varepsilon \leq \varepsilon_1$ we have

$$
\| g^0 b(D)(B_{D,\varepsilon} - \zeta Q_0)^{-1} - G_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_{41c}(\phi)^{3/2} \varepsilon^{1/2} |\zeta|^{-1/4} + C_{42c}(\phi)^{3/2} \varepsilon^{1/2}.
$$

(10.22)

The constants $C_{41}$ and $C_{42}$ depend only on the initial data (1.9) and the domain $\Omega$. If the matrix-valued function $Q_0(\varepsilon)$ is constant, then $C_{42} = 0$.

Proof. We start with a rough estimate for the left-hand side of (10.22). By (2.14) and (2.12),

$$
\| g^0 b(D)(B_{D,\varepsilon} - \zeta Q_0)^{-1} \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq (d\alpha_1)^{1/2} \| g \|_{L_\infty} C_1(\phi) |\zeta|^{-1/2},
$$

(10.23)

Next, using (1.3), (2.37), and Proposition 1.2 we estimate the operator $G_D(\varepsilon; \zeta)$:

$$
\| G_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq |\Omega|^{-1/2} \| g \|_{L_\infty} C_{1/2} < C_{1/2} (B_D - \zeta Q_0)^{-1} \| L_2(\Omega) \rightarrow H^1(\Omega)
$$

+ $\| g \|_{L_\infty} |\Omega|^{-1/2} \| b(D) \|_{L_2(\Omega)} C_{1/2} (B_D - \zeta Q_0)^{-1} \| L_2(\Omega) \rightarrow L_2(\Omega).
$$

(10.24)

By (2.20) and (2.21), we have $|\Omega|^{-1/2} \| g \|_{L_\infty} \leq \gamma_{43} := \| g \|_{L_\infty} \| m^1/2 \|_{L_\infty} + 1 \| g \|_{L_\infty} + 1$. Together with Lemma 2.3 (1.31), and (10.24), this implies

$$
\| G_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \gamma_{44}(\phi) \|\zeta\|^{-1/2}, \quad \zeta \in \mathbb{C} \setminus \mathbb{R}_+, \quad |\zeta| \geq 1, \quad 0 < \varepsilon \leq \varepsilon_1;
$$

$$
\gamma_{44} := \alpha_{1/2} C_{1/2} (Q_0 - c_1) + |\Omega|^{-1/2} C_{1/2} (Q_0^0 - c_1) + C_{1/2} C_{1/2} \| g \|_{L_\infty} \| g \|_{L_\infty} \| g \|_{L_\infty}.
$$

Combining this with (10.23), we obtain

$$
\| g^0 b(D)(B_{D,\varepsilon} - \zeta Q_0)^{-1} - G_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \gamma_{45}(\phi) |\zeta|^{-1/2}
$$

(10.25)

for $0 < \varepsilon \leq \varepsilon_1, \zeta \in \mathbb{C} \setminus \mathbb{R}_+, |\zeta| \geq 1$. Here $\gamma_{45} := (d\alpha_1)^{1/2} \| g \|_{L_\infty} C_1 + \gamma_{44}$.

By (10.23) and (10.25), we have

$$
\| g^0 b(D)(B_{D,\varepsilon} - \zeta Q_0)^{-1} - G_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega)}
\leq \min\{\gamma_{45}(\phi) |\zeta|^{-1/2}; C_{36}(\phi)c_{1/2}|\zeta|^{-1/4} + C_{42}(\phi)^{1/2} + \tilde{C}_{37}(\text{Re} \zeta)^{1/2} C_{1/2}^2 \}
\leq C_{41}(\phi)^{3/2} \varepsilon^{1/2} |\zeta|^{-1/4} + C_{42}(\phi)^{3/2} \varepsilon^{1/2}.
$$

where $C_{41} := \tilde{C}_{36} + (\gamma_{45} \tilde{C}_{37})^{1/2}$ and $C_{42} := (\gamma_{45} \tilde{C}_{37})^{1/2}$. By Theorem 10.1 if the matrix-valued function $Q_0(\varepsilon)$ is constant, then $\tilde{C}_{37} = 0$, whence $C_{42} = 0$.

Proposition 10.7. Under the assumptions of Theorem 2.6, for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+, |\zeta| \geq 1$, and $0 < \varepsilon \leq \varepsilon_1$ we have

$$
\| g^0 b(D)(B_{D,\varepsilon} - \zeta Q_0)^{-1} - G_D(\varepsilon; \zeta) \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_{45}(\phi)^{3/4} \varepsilon^{1/2} |\zeta|^{-1/4}.
$$

(10.26)

The constant $C_{43}$ depends only on the initial data (1.9) and the domain $\Omega$. 

References: [10.14, 10.15], and [10.24] imply [10.18] with $C_{39} := \max\{2C_{34}^2 \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty}; 2C_{34}^2 \| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} + \gamma_{42} \}$. Approximation (10.19) for the flux follows from (5.17), (5.19), (5.21), and (10.18).
Proof. Estimate (10.26) can be checked similarly to (10.22) by using (2.43) and (10.25). The constant $C_{43}$ is given by $C_{43} := \tilde{C}_5 + (\gamma_{43} \tilde{C}_6)^{1/2}$. \hfill \Box

11. Applications of the General Results

11.1. The scalar elliptic operator. Let $n = 1$, $m = d$, $b(D) = D$, and let $g(x)$ be a $\Gamma$-periodic symmetric $(d \times d)$-matrix-valued function with real entries. Suppose that $g(x) > 0$ and $g, g^{-1} \in L^\infty(\mathbb{R}^d)$. Obviously, condition (1.3) holds with $\alpha_0 = \alpha_1 = 1$. We have $b(D)^* g^\varepsilon(x) b(D) = -\text{div} g^\varepsilon(x) \nabla$. Next, let $A(x) = \text{col}\{A_1(x), \ldots, A_d(x)\}$, where $A_j(x)$, $j = 1, \ldots, d$, are $\Gamma$-periodic real-valued functions such that $A_j \in L^\rho_\varepsilon(\Omega)$, $\rho = 2$ for $d = 1$, $\rho > d$ for $d \geq 2$; $j = 1, \ldots, d$. (11.1)

Suppose that $v(x)$ and $\nabla(x)$ are real-valued $\Gamma$-periodic functions such that $v, \nabla \in L_4(\Omega)$, $s = 1$ for $d = 1$, $s > d/2$ for $d \geq 2$; $\int_\Omega v(x) dx = 0$. (11.2)

In $L^2(\mathcal{O})$, we consider the operator $\mathfrak{B}_{D,\varepsilon}$ given formally by the differential expression

$$\mathfrak{B}_{D,\varepsilon} = (D - A^\varepsilon(x))^* g^\varepsilon(x)(D - A^\varepsilon(x)) + \varepsilon^{-1} v^\varepsilon(x) + \nabla^\varepsilon(x) \tag{11.3}$$

with the Dirichlet condition on $\partial \mathcal{O}$. The precise definition of the operator $\mathfrak{B}_{D,\varepsilon}$ is given in terms of the corresponding quadratic form. The operator (11.3) can be treated as the periodic Schrödinger operator with the metric $g^\varepsilon$, the magnetic potential $A^\varepsilon$, and the electric potential $\varepsilon^{-1} v^\varepsilon + \nabla^\varepsilon$ containing the singular term $\varepsilon^{-1} v^\varepsilon$. It is easily seen (cf. [Su1 Subsection 13.1]) that the operator (11.3) can be represented as

$$\mathfrak{B}_{D,\varepsilon} = D^* g^\varepsilon(x) D + \sum_{j=1}^d \left( a_j^\varepsilon(x) D_j + D_j (a_j^\varepsilon(x))^* \right) + Q^\varepsilon(x). \tag{11.4}$$

Here the real-valued function $Q(x)$ is given by $Q(x) = \nabla(x) + (g(x) A(x), A(x))$. The complex-valued functions $a_j(x)$ are given by $a_j(x) = -\eta_j(x) + i \xi_j(x)$, $j = 1, \ldots, d$, where $\eta_j(x)$ and $\xi_j(x)$ are the components of the vector-valued function $\eta(x) = g(x) A(x)$, and $\xi_j(x)$ are defined in terms of the $\Gamma$-periodic solution $\Phi(x)$ of the problem $\Delta \Phi(x) = v(x)$, $\int_\Omega \Phi(x) dx = 0$, by the relations $\eta_j(x) = -\partial_j \Phi(x)$. We have $v(x) = -\sum_{j=1}^d \partial_j \xi_j(x)$. It is easy to check that the functions $a_j$ satisfy condition (1.7) with suitable $\rho'$ depending on $\rho$ and $s$, and the norms $\|a_j\|_{L^\rho_\varepsilon(\Omega)}$ are controlled in terms of $\|\eta\|_{L^\infty}$, $\|A\|_{L^\rho(\Omega)}$, $\|v\|_{L^s(\Omega)}$, and the parameters of the lattice $\Gamma$. (See [Su1 Subsection 13.1].) The function $Q$ satisfies condition (1.8) with suitable $s' = \min\{s; \rho'/2\}$.

Suppose that $Q_0(x)$ is a positive definite and bounded $\Gamma$-periodic function. We consider the positive definite operator $\mathfrak{B}_{D,\varepsilon} := \mathfrak{B}_{D,\varepsilon} + \lambda Q_0^\varepsilon$. Here we choose the constant $\lambda$ in accordance with condition (1.14) for the operator with the coefficients $g, a_j$, $j = 1, \ldots, d$, $Q$, and $Q_0$ defined above. We are interested in the behavior of the operator $\mathfrak{B}_{D,\varepsilon} - \zeta Q_0^\varepsilon$, where $\zeta \in \mathbb{C} \cap \mathbb{R}_+$. In the case under consideration, the initial data (1.9) reduces to the following set

$$d, \rho', s'; \|g\|_{L^\infty}, \|g^{-1}\|_{L^\infty}, \|A\|_{L^\rho(\Omega)}, \|v\|_{L^s(\Omega)}, \|\nabla\|_{L^s(\Omega)},$$

$$\|Q_0\|_{L^\infty}, \|Q_0^{-1}\|_{L^\infty}; \text{the parameters of the lattice } \Gamma. \tag{11.5}$$

Let us describe the effective operator. The $\Gamma$-periodic solution of problem (1.18) is the row $\Lambda(x) = \mathbf{r}(\Psi(x), \Psi(x) = (\psi_1(x), \ldots, \psi_d(x))$, where $\psi_j \in H^1(\Omega)$ is the solution of the problem $\text{div} g(x) \nabla \psi_j(x) + e_j = 0$, $\int_\Omega \psi_j(x) dx = 0$. Here $e_j$, $j = 1, \ldots, d$, is the standard orthonormal basis in $\mathbb{R}^d$. Clearly, the functions $\psi_j(x)$ are real-valued, while the entries of the row $\Lambda(x)$ are purely imaginary. According to (1.20), the columns of the $(d \times d)$-matrix-valued function $\tilde{g}(x)$ are given by $g(x) \nabla \psi_j(x) + e_j$, $j = 1, \ldots, d$. The effective matrix is defined by the general rule (1.19): $g^\circ = |\Omega|^{-1} \int_\Omega \tilde{g}(x) dx$. Clearly, the matrices $\tilde{g}(x)$ and $g^\circ$ have real entries.
The periodic solution of problem (11.2) can be represented as \( \tilde{\Lambda}(x) = \tilde{\Lambda}_1(x) + i\tilde{\Lambda}_2(x) \), where the real-valued \( \Gamma \)-periodic functions \( \tilde{\Lambda}_1(x) \) and \( \tilde{\Lambda}_2(x) \) are the solutions of the problems:

\[
- \text{div} \, g(x) \nabla \tilde{\Lambda}_1(x) + \nu(x) = 0, \quad \int_{\Omega} \tilde{\Lambda}_1(x) \, dx = 0;
\]

\[
- \text{div} \, g(x) \nabla \tilde{\Lambda}_2(x) + \text{div} \, g(x) A(x) = 0, \quad \int_{\Omega} \tilde{\Lambda}_2(x) \, dx = 0.
\]

The column \( V \) (see (11.34)) can be written as \( V = V_1 + iV_2 \), where \( V_1 \) and \( V_2 \) are defined by \( V_1 = \langle g\nabla \tilde{\Lambda}_2, \nabla \psi \rangle \) and \( V_2 = -\langle g\nabla \tilde{\Lambda}_1, \nabla \psi \rangle \). Clearly, \( V_1 \) and \( V_2 \) have real entries. According to (11.33), the constant \( W \) is given by \( W = \langle g\nabla \tilde{\Lambda}_1, \nabla \tilde{\Lambda}_1 \rangle + \langle g\nabla \tilde{\Lambda}_2, \nabla \tilde{\Lambda}_2 \rangle \). The effective operator for \( B_{D,\varepsilon} \) is defined by

\[
B_{D,\varepsilon}^0 u = -\text{div} \, g^0 \nabla u + 2i(\nabla u, V_1 + \Psi) + (-\nabla + \overline{\Psi} + \lambda Q_0)u, \quad u \in H^2(\Omega) \cap H^1_0(\Omega).
\]

This operator can be represented as \( B_{D,\varepsilon}^0 = (D - A_0)^* g^0(D - A_0) + \nabla^0 + \lambda Q_0 \), where \( A_0 := (g^0)^{-1}(V_1 + gA) \) and \( \nabla^0 := \overline{V} + (gA, A)^\top - (g^0 A_0, A_0)^\top - W \).

According to Remark (7.5) in the case under consideration, Conditions (7.1) and (7.2) are satisfied, and the norms \( \|\Lambda\|_{L^\infty}, \|\Lambda\|_{L^\infty} \) are controlled in terms of the initial data (11.5). Therefore, it is possible to use the simpler corrector (7.4):

\[
\mathcal{K}_D^0(\varepsilon; \zeta) := (\Lambda^\varepsilon D + \tilde{\Lambda}^\varepsilon)(B_{D,\varepsilon}^0 - \zeta Q_0)^{-1} - (\Psi^\varepsilon \nabla + \tilde{\Lambda}^\varepsilon)(B_{D,\varepsilon}^0 - \zeta Q_0)^{-1}.
\]

The operator (7.2) can be written as \( G_{D,\varepsilon}^0(\varepsilon; \zeta) := -iG_{D,\varepsilon}^0(\varepsilon; \zeta) \), where

\[
G_{D,\varepsilon}^0(\varepsilon; \zeta) := \tilde{g} \nabla (B_{D,\varepsilon}^0 - \zeta Q_0)^{-1} + g^\varepsilon (\nabla \tilde{\Lambda})^\varepsilon (B_{D,\varepsilon}^0 - \zeta Q_0)^{-1}.
\]

Applying Theorems (2.5) and (7.6) we deduce the following result.

**Proposition 11.1.** Suppose that the assumptions of Subsection 11.1 are satisfied. Let \( \zeta \in \mathbb{C} \setminus \mathbb{R}^+ \), \( \zeta = |\zeta|e^{i\phi} \), \( 0 < \phi < 2\pi \), and \( |\zeta| \geq 1 \). Suppose that \( \varepsilon_1 \) is subject to Condition (2.4). Then for \( 0 < \varepsilon \leq \varepsilon_1 \) we have

\[
\|B_{D,\varepsilon} - \zeta Q_0\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C_4 c(\phi)^2 \varepsilon |\zeta|^{-1/2},
\]

\[
\|B_{D,\varepsilon} - \zeta Q_0\|_{L^2(\Omega) \rightarrow H^1(\Omega)} \leq C_5 c(\phi)^2 \varepsilon |\zeta|^{-1/4} + C_{23} c(\phi)^4 \varepsilon,
\]

\[
\|g^\varepsilon \nabla B_{D,\varepsilon} - \zeta Q_0\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq \tilde{C}_5 c(\phi)^2 \varepsilon |\zeta|^{-1/4} + \tilde{C}_{23} c(\phi)^4 \varepsilon.
\]

Here \( c(\phi) \) is given by (1.41). The constants \( C_4, C_5, C_{23}, \tilde{C}_5, \text{and} \tilde{C}_{23} \) depend only on the initial data (11.5) and the domain \( \Omega \).

The results of Section 10 also can be applied to the operator \( B_{D,\varepsilon} \). “Another” approximation for \( (B_{D,\varepsilon} - \zeta Q_0)^{-1} \) follows from Theorems (9.2) and (9.8).

**Proposition 11.2.** Suppose that the assumptions of Subsection 11.1 are satisfied. Denote \( f(x) := Q_0(x)^{-1/2} \) and \( f_0 := (Q_0)^{-1/2} \). Let \( B_{D,\varepsilon} := f^\varepsilon B_{D,\varepsilon} f^\varepsilon \) and \( B_0 := f_0 B_0 f_0 \). Suppose that \( \varepsilon_1 \) is subject to Condition (2.4). Let \( 0 < \varepsilon \leq \varepsilon_1 \). Suppose that \( \varepsilon_0 > 0 \) is a common lower bound of the operators \( B_{D,\varepsilon} \) for any \( 0 < \varepsilon \leq \varepsilon_0 \) and \( B_0 \). Let \( \varrho_0(\zeta) \) be given by (9.1). Then for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( \zeta \in \mathbb{C} \setminus [\varepsilon_0, \infty) \) we have

\[
\|B_{D,\varepsilon} - \zeta Q_0\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C_{26} \varepsilon \varrho_0(\zeta),
\]

\[
\|B_{D,\varepsilon} - \zeta Q_0\|_{L^2(\Omega) \rightarrow H^1(\Omega)} \leq C_{29} (\varepsilon^{1/2} \varrho_0(\zeta)^{1/2} + \varepsilon |1 + \zeta|^{1/2} \varrho_0(\zeta)).
\]

The constants \( C_{26}, C_{29}, \text{and} \tilde{C}_{29} \) depend only on the initial data (11.5) and the domain \( \Omega \).
11.2. The periodic Schrödinger operator. Suppose that \( \tilde{g}(x) \) is a \( \Gamma \)-periodic symmetric \((d \times d)\)-matrix-valued function in \( \mathbb{R}^d \) with real entries and such that \( \tilde{g}(x) > 0; \tilde{g}, \tilde{g}^{-1} \in L_\infty(\mathbb{R}^d) \). Suppose that \( \tilde{v}(x) \) is a real-valued \( \Gamma \)-periodic function such that \( \tilde{v} \in L_s(\Omega), s = 1 \) for \( d = 1 \), \( s > d/2 \) for \( d \geq 2 \). By \( \mathcal{A} \) we denote the operator in \( L_2(\mathbb{R}^d) \) corresponding to the quadratic form

\[
\int_{\mathbb{R}^d} \left( \langle \tilde{g}(x)D^2 u, D^2 u \rangle + \tilde{v}(x)|u|^2 \right) dx, \quad u \in H^1(\mathbb{R}^d).
\]

Adding an appropriate constant to the potential \( \tilde{v}(x) \), we may assume that the bottom of the spectrum of \( \mathcal{A} \) is the point \( \lambda_0 = 0 \). Under this condition, the operator \( \mathcal{A} \) admits a factorization (see [BSu1] Chapter 6, Subsection 1.1).

Now, in \( L_2(\Omega) \), we consider the operator \( \mathcal{A}_D = D^*\tilde{g}(x)D + \tilde{v}(x) \) with the Dirichlet boundary condition. The precise definition of the operator \( \mathcal{A}_D \) is given in terms of the quadratic form

\[
\mathfrak{a}[u, u] = \int_\Omega \left( \langle \tilde{g}(x)D^2 u, D^2 u \rangle + \tilde{v}(x)|u|^2 \right) dx, \quad u \in H^1_0(\Omega).
\]

The operator \( \mathcal{A}_D \) inherits a factorization of \( \mathcal{A} \). To describe this factorization, we consider the equation

\[
D^*\tilde{g}(x)D\omega(x) + \tilde{v}(x)\omega(x) = 0. \tag{11.12}
\]

This equation has a \( \Gamma \)-periodic solution \( \omega \in \tilde{H}_1(\Omega) \) defined up to a constant factor. This factor can be fixed so that \( \omega(x) > 0 \) and \( \int_\Omega \omega(x) dx = |\Omega| \). Moreover, this solution is positive definite and bounded: \( 0 < \omega_0 \leq \omega(x) \leq \omega_1 < \infty \). The norms \( \|\omega\|_{L_{\infty}} \) and \( \|\omega^{-1}\|_{L_{\infty}} \) are controlled in terms of \( \|\tilde{g}\|_{L_{\infty}}, \|\tilde{g}^{-1}\|_{L_{\infty}}, \|\tilde{v}\|_{L_2(\Omega)} \). Note that \( \omega \) and \( \omega^{-1} \) are multipliers in \( H^1_0(\Omega) \).

Substituting \( u = wz, z \in H^1_0(\Omega) \), and taking (11.12) into account, we represent the form (11.11) as \( \mathfrak{a}[u, u] = \int_\Omega \omega(x)^2 \langle \tilde{g}(x)Dz, Dz \rangle dx \). Hence, the operator \( \mathcal{A}_D \) can be written in a factorized form as follows:

\[
\mathcal{A}_D = \omega^{-1}D^*\tilde{g}D\omega^{-1}, \quad g = \omega^2 \tilde{g}. \tag{11.13}
\]

Now we consider the operator

\[
\mathcal{A}_{D,\varepsilon} = (\omega^\varepsilon)^{-1}D^*g^\varepsilon D(\omega^\varepsilon)^{-1}
\]

with rapidly oscillating coefficients. In the initial terms, the operator (11.14) can be written as

\[
\mathcal{A}_{D,\varepsilon} = D^*\tilde{g}\varepsilon^2 D + \varepsilon^{-2}\tilde{v}\varepsilon. \tag{11.15}
\]

It can be interpreted as the Schrödinger operator with the rapidly oscillating metric \( \tilde{g}\varepsilon \) and the strongly singular potential \( \varepsilon^{-2}\tilde{v}\varepsilon \).

Next, let \( A = \text{col}\{A_1(x), \ldots, A_d(x)\} \), where \( A_j(x) \) are \( \Gamma \)-periodic real-valued functions satisfying (11.1). Let \( \hat{v}(x) \) and \( \hat{V}(x) \) be \( \Gamma \)-periodic real-valued functions such that

\[
\hat{v}, \hat{V} \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \geq 2; \quad \int_\Omega \hat{v}(x)\omega^2(x) dx = 0. \tag{11.16}
\]

In \( L_2(\Omega) \), consider the operator \( \mathfrak{B}_{D,\varepsilon} \) given formally by the differential expression

\[
\mathfrak{B}_{D,\varepsilon} = (D - A^\varepsilon)^*\varepsilon^2(D - A^\varepsilon) + \varepsilon^{-2}\tilde{v}\varepsilon + \varepsilon^{-1}\tilde{v}\varepsilon + \hat{V}\varepsilon
\]

with the Dirichlet boundary condition. The precise definition is given in terms of the corresponding quadratic form. The operator \( \mathfrak{B}_{D,\varepsilon} \) can be treated as the Schrödinger operator with the metric \( \tilde{g}\varepsilon \), the magnetic potential \( A^\varepsilon \), and the electric potential \( \varepsilon^{-2}\tilde{v}\varepsilon + \varepsilon^{-1}\tilde{v}\varepsilon + \varepsilon V^\varepsilon \) containing the singular summands \( \varepsilon^{-2}\tilde{v}\varepsilon \) and \( \varepsilon^{-1}\tilde{v}\varepsilon \). We put

\[
v(x) := \hat{v}(x)\omega^2(x), \quad V(x) := \hat{V}(x)\omega^2(x).
\]

Using (11.14) and (11.15), we see that \( \mathfrak{B}_{D,\varepsilon} = (\omega^\varepsilon)^{-1}\mathfrak{B}_{D,\varepsilon}(\omega^\varepsilon)^{-1} \), where the operator \( \mathfrak{B}_{D,\varepsilon} \) is given by the expression (11.13) with \( g \) defined in (11.14), and \( v, V \) defined by (11.13). Taking (11.16) into account and using the properties of the function \( \omega \), we see that the coefficients \( v \) and \( V \) satisfy conditions (11.12). Then the operator \( \mathfrak{B}_{D,\varepsilon} \) can be represented in the form (11.14), where \( a_j, j = 1, \ldots, d, \) and \( Q \) are defined in terms of \( g, A, v, \) and \( V \) as in Subsection 11.1.

Let \( Q_0(x) \) be a \( \Gamma \)-periodic positive definite and bounded real-valued function. Next, we choose the constant \( \lambda \) according to condition (11.1) for the operator whose coefficients \( g, a_j, j = 1, \ldots, d, \) and \( Q \) are the same as the coefficients of \( \mathfrak{B}_{D,\varepsilon} \), and the coefficient \( Q_0 \) is given by
Here \( c(\phi) \) is given by (11.11).

The constants \( C_1, C_5, C_{23}, C_{26}, C_{20}, \tilde{C}_5, \tilde{C}_{23}, \tilde{C}_{29}, \) and \( \|\omega\|_{L_{\infty}} \) depend only on the initial data (11.20) and the domain \( O \).

Proof. Multiplying the operators under the norm sign in (11.3) by \( \omega^\varepsilon \) from both sides and using (11.19), we arrive at (11.21).

From (11.19) it follows that \( (\omega^\varepsilon)^{-1}(\mathcal{B}_{D,\varepsilon} - \delta) = (\mathcal{B}_{D,\varepsilon} - \delta)\omega^\varepsilon \). Multiplying the operators under the norm sign in (11.3) by \( \omega^\varepsilon \) from the right, we obtain (11.22). Similarly, (11.19) implies (11.23).

The results of assertion 2° are deduced from Proposition 11.2 in a similar way. \( \square \)

Remark 11.4. Proposition 11.3 demonstrates that for the operators (11.3) and (11.17) the nature of the results is different. Because of the presence of the strongly singular potential \( \varepsilon^{-2}v^\varepsilon \), the generalized resolvent \( (\mathcal{B}_{D,\varepsilon} - \delta)\omega^\varepsilon \) has no limit in the \( L_2(O) \)-operator norm. It is approximated by the operator \( (\mathcal{B}_{D,\varepsilon} - \delta)\omega^\varepsilon \) sandwiched between the rapidly oscillating factors \( \omega^\varepsilon \).

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