This paper is devoted to tetrahedron maps, which are set-theoretical solutions of the Zamolodchikov tetrahedron equation. We construct a family of tetrahedron maps on associative rings. The obtained maps are new to our knowledge. We show that matrix tetrahedron maps derived previously are a particular case of our construction. This provides an algebraic explanation of the fact that the matrix maps satisfy the tetrahedron equation. Also, Liouville integrability is established for some of the constructed maps.

Keywords: Zamolodchikov tetrahedron equation, tetrahedron map, associative ring, Liouville integrability

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1. Introduction

This paper is devoted to tetrahedron maps, which are set-theoretical solutions of the Zamolodchikov tetrahedron equation [1], [2]. The tetrahedron equation belongs to fundamental equations of mathematical physics and has applications in many diverse branches of physics and mathematics, including statistical mechanics, quantum field theories, combinatorics, low-dimensional topology, and the theory of integrable systems (see, e.g., [3]–[8] and the references therein).

Presently, the relations of tetrahedron maps to integrable systems and algebraic structures (including groups and rings) are a very active area of research (see, e.g., [6], [8]–[12]).

In this paper, we construct tetrahedron maps on associative rings and study their properties. In particular, for any associative ring $\mathcal{A}$ and any element $M \in \mathcal{A}$, we define the maps (4), (5) and prove tetrahedron equation (1) is satisfied. The obtained tetrahedron maps (4), (5) are new, to our knowledge.

Also, we study the matrix tetrahedron maps (14), (15) constructed in [10]. It is claimed in preprint [10] that one can verify by a straightforward computation that (14), (15) satisfy the tetrahedron equation (1). In Theorem 2, we present an algebraic explanation of this fact. Namely, we show in Theorem 2 that maps (14), (15) are of form (4) for $\mathcal{A} = \text{Mat}_n(K)$ and some matrices $M \in \text{Mat}_n(K)$. Here, $K$ is a field and $\text{Mat}_n(K)$ is the associative ring of $n \times n$ matrices with entries from $K$. Usually, $K$ is either $\mathbb{R}$ or $\mathbb{C}$, but arbitrary fields can also be considered.

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In Proposition 1, we recall a well-known construction that allows obtaining a new tetrahedron map from a known one. Applying Proposition 1 to maps (14), (15), we obtain new tetrahedron maps (16), (17).

In Sec. 3, we prove Liouville integrability

- for the map (16) in the case $1 \leq m \leq n/2$ in Proposition 6;
- for the map (17) in the case $3n/2 \leq \hat{m} \leq 2n - 1$ in Proposition 7.

The proofs of these results on Liouville integrability for (16), (17) are deduced from the proofs of similar results for (14), (15), taken from [10].

We use the standard notion of the Liouville integrability for maps on manifolds (see, e.g., [10], [13]–[15] and the references therein). This notion is presented in Definition 1.

2. Tetrahedron maps on sets and associative rings

Let $W$ be a set. A tetrahedron map is a map

$$T: W^3 \to W^3, \quad T(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z)), \quad x, y, z \in W,$$

satisfying the (Zamolodchikov) tetrahedron equation

$$T^{123} \circ T^{145} \circ T^{246} = T^{356} \circ T^{246} \circ T^{145} \circ T^{123}. \quad (1)$$

Here, $T^{ijk}: W^6 \to W^6$ for $i, j, k = 1, \ldots, 6$ and $i < j < k$ is the map acting as $T$ on the $i$th, $j$th, and $k$th factors of the Cartesian product $W^6$ and acting as identity on the other factors. For instance,

$$T^{246}(x, y, z, r, s, t) = (x, f(y, r, t), z, g(y, r, t), s, h(y, r, t)), \quad x, y, z, r, s, t \in W.$$ 

The statement of following Proposition is well known. The proof can be found, e.g., in [6].

**Proposition 1.** Consider the permutation map

$$P^{13}: W^3 \to W^3, \quad P^{13}(a_1, a_2, a_3) = (a_3, a_2, a_1), \quad a_i \in W.$$

If a map $T: W^3 \to W^3$ satisfies tetrahedron equation (1), then $\tilde{T} = P^{13} \circ T \circ P^{13}$ also satisfies this equation.

**Proposition 2.** For any associative ring $\mathcal{A}$, we have the tetrahedron maps

$$T: (\mathcal{A})^3 \to (\mathcal{A})^3, \quad T(X, Y, Z) = (X, Y + XZ, Z), \quad (2)$$

$$\tilde{T}: (\mathcal{A})^3 \to (\mathcal{A})^3, \quad \tilde{T}(X, Y, Z) = (X, Y + ZX, Z), \quad (3)$$

$X, Y, Z \in \mathcal{A}.$
Applying Proposition 2 to this ring, we obtain the tetrahedron maps because the initial multiplication on satisfy the tetrahedron equation.

Taking formula (6) into account, we see that maps (4), (5) coincide with tetrahedron maps (7), (8).

Therefore, map (2) satisfies tetrahedron equation (1).

Map (3) is obtained from (2) by means of Proposition 1.

**Theorem 1.** Let $\mathcal{A}$ be an associative ring. Let $M \in \mathcal{A}$. The maps

\[
T_M : (\mathcal{A})^3 \to (\mathcal{A})^3, \quad T_M(X, Y, Z) = (X, Y + XMZ, Z),
\]

\[
\tilde{T}_M : (\mathcal{A})^3 \to (\mathcal{A})^3, \quad \tilde{T}_M(X, Y, Z) = (X, Y + ZMX, Z),
\]

$satisfy the tetrahedron equation.$

**Proof.** On the associative ring $\mathcal{A}$, we have the addition and multiplication operations, which we call the initial addition and multiplication on $\mathcal{A}$. We consider a new multiplication operation on $\mathcal{A}$ defined as

\[ P \ast Q := PMQ, \quad P, Q \in \mathcal{A}. \]

Because the initial multiplication on $\mathcal{A}$ is associative, the new multiplication (6) is also associative.

Moreover, the set $\mathcal{A}$ with the initial addition and the new multiplication (6) is an associative ring. Applying Proposition 2 to this ring, we obtain the tetrahedron maps

\[
T : (\mathcal{A})^3 \to (\mathcal{A})^3, \quad T(X, Y, Z) = (X, Y + X \ast Z, Z),
\]

\[
\tilde{T} : (\mathcal{A})^3 \to (\mathcal{A})^3, \quad \tilde{T}(X, Y, Z) = (X, Y + Z \ast X, Z),
\]

$satisfy the tetrahedron equation.$

**Proof.** This follows from Theorem 1 in the case $\mathcal{A} = \text{Mat}_n(\mathbb{K})$. 

We let $\mathbb{Z}_{>0}$ denote the set of positive integers.

**Corollary 1.** Let $n \in \mathbb{Z}_{>0}$ and $M \in \text{Mat}_n(\mathbb{K})$. The maps

\[
T_M : (\text{Mat}_n(\mathbb{K}))^3 \to (\text{Mat}_n(\mathbb{K}))^3, \quad T_M(X, Y, Z) = (X, Y + XMZ, Z),
\]

\[
\tilde{T}_M : (\text{Mat}_n(\mathbb{K}))^3 \to (\text{Mat}_n(\mathbb{K}))^3, \quad \tilde{T}_M(X, Y, Z) = (X, Y + ZMX, Z),
\]

$satisfy the tetrahedron equation.$

**Proof.** This follows from Theorem 1 in the case $\mathcal{A} = \text{Mat}_n(\mathbb{K})$. 

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Remark 1. Let $\hat{A}$ be an associative ring with a unit $e \in \hat{A}$. (That is, for any $x \in \hat{A}$, we have $xe = ex = x$.) Let $K \in \hat{A}$ be an invertible element. (That is, there is $K^{-1} \in \hat{A}$ such that $KK^{-1} = K^{-1}K = e$.) In preprint [10], the following tetrahedron maps were presented:

\[
T_K: (\hat{A})^3 \to (\hat{A})^3, \quad T_K(X, Y, Z) = (X, YK + XZ, KZK^{-1}), \tag{11}
\]

\[
\tilde{T}_K: (\hat{A})^3 \to (\hat{A})^3, \quad \tilde{T}_K(X, Y, Z) = (KXK^{-1}, YK + ZX, Z), \tag{12}
\]

$X, Y, Z \in \hat{A}$.

In the case $K = e$, formulas (11), (12) coincide with (2), (3). We note that maps (2), (3) are defined for any associative ring $A$, not necessarily having a unit.

As discussed in [10], in the case $\hat{A} = \mathbb{C}$, maps (11), (12) reduce to a tetrahedron map from Sergeev’s classification [16].

Let

$$n \in \mathbb{Z}_{>0}, \quad m \in \{1, \ldots, n\}, \quad \hat{m} \in \{n + 1, \ldots, 2n - 1\}.$$  

In what follows, the elements of matrices $X, Z \in \text{Mat}_n(\mathbb{K})$ are denoted by $x_{k,l}$, $z_{k,l}$, $k, l = 1, \ldots, n$, and we use column-by-row multiplication

\[
\begin{pmatrix}
x_{1,i} \\
\vdots \\
x_{n,i}
\end{pmatrix}
\begin{pmatrix}
z_{1,j} & \ldots & z_{j,n}
\end{pmatrix}
= 
\begin{pmatrix}
x_{1,i}z_{j,1} & \ldots & x_{1,i}z_{j,n} \\
\vdots & \ddots & \vdots \\
x_{n,i}z_{j,1} & \ldots & x_{n,i}z_{j,n}
\end{pmatrix}
= XE_{i,j}Z, \tag{13}
\]

$i, j \in \{1, \ldots, n\}$.

Here, $E_{i,j} \in \text{Mat}_n(\mathbb{K})$ is the $n \times n$ matrix with $(i, j)$th entry equal to 1 and all other entries equal to zero.

In preprint [10], the following maps were presented:

\[
T_{n,m}: (\text{Mat}_n(\mathbb{K}))^3 \to (\text{Mat}_n(\mathbb{K}))^3, \quad m \in \{1, \ldots, n\},
\]

\[
T_{n,m}(X, Y, Z) = \left( X, Y + m \sum_{i=1}^{m} \begin{pmatrix} x_{1,i} \\ \vdots \\ x_{n,i} \end{pmatrix} (z_{n-m+i,1} \ldots z_{n-m+i,n}), Z \right), \tag{14}
\]

\[
T_{n,\hat{m}}: (\text{Mat}_n(\mathbb{K}))^3 \to (\text{Mat}_n(\mathbb{K}))^3, \quad \hat{m} \in \{n + 1, \ldots, 2n - 1\},
\]

\[
T_{n,\hat{m}}(X, Y, Z) = \left( X, Y + \sum_{i=1}^{2n-\hat{m}} \begin{pmatrix} x_{1,\hat{m}-n+i} \\ \vdots \\ x_{n,\hat{m}-n+i} \end{pmatrix} (z_{i,1} \ldots z_{i,n}), Z \right), \tag{15}
\]

$X, Y, Z \in \text{Mat}_n(\mathbb{K})$.

It was asserted in [10] that straightforward computation allows verifying that maps (14), (15) satisfy the tetrahedron equation. We present an algebraic explanation of this fact in Theorem 2 below.

**Theorem 2.** The map (14) is of form (9) for $M = \sum_{i=1}^{m} E_{i, n-m+i}$. The map (15) is of form (9) for $M = \sum_{i=1}^{2n-\hat{m}} E_{\hat{m}-n+i}$. Therefore, Theorem 1 and Corollary 1 explain why (14) and (15) satisfy the tetrahedron equation.

**Proof.** The statements follow from formula (13).
Proposition 3. Let \( n \in \mathbb{Z}_{>0}, \ m \in \{1, \ldots, n\}, \ m \in \{n+1, \ldots, 2n-1\} \). We have the tetrahedron maps

\[
\tilde{T}_{n,m} : (\text{Mat}_n(\mathbb{K}))^3 \to (\text{Mat}_n(\mathbb{K}))^3, \quad m \in \{1, \ldots, n\},
\]

\[
\tilde{T}_{n,m}(X, Y, Z) = \begin{pmatrix} X, Y + \sum_{i=1}^{m} \begin{pmatrix} z_{1,i} \vdots \vdots \end{pmatrix} (x_{n-m+i,1} \ldots x_{n-m+i,n}), Z \end{pmatrix}, \quad (16)
\]

\[
\tilde{T}_{n,\hat{m}} : (\text{Mat}_n(\mathbb{K}))^3 \to (\text{Mat}_n(\mathbb{K}))^3, \quad \hat{m} \in \{n+1, \ldots, 2n-1\},
\]

\[
\tilde{T}_{n,\hat{m}}(X, Y, Z) = \begin{pmatrix} X, Y + \sum_{i=1}^{2n-m} \begin{pmatrix} z_{1,m-n+i} \vdots \vdots \end{pmatrix} (x_{i,1} \ldots x_{i,n}), Z \end{pmatrix}, \quad (17)
\]

\[X, Y, Z \in \text{Mat}_n(\mathbb{K}).\]

Proof. Maps (16), (17) are obtained from (14), (15) by means of Proposition 1.

3. Liouville integrability

We recall the standard notion of the Liouville integrability for maps on manifolds (see, e.g., [10], [13]–[15] and the references therein).

Definition 1. Let \( k \in \mathbb{Z}_{>0} \). Let \( M \) be a \( k \)-dimensional manifold with (local) coordinates \( x_1, \ldots, x_k \).

A (smooth or analytic) map \( F : M \to M \) is said to be Liouville integrable (or completely integrable) if the following objects are defined on the manifold \( M \).

- A Poisson bracket \( \{ \cdot, \cdot \} \) that is invariant under the map \( F \) and is of constant rank \( 2r \) for some positive integer \( r \leq k/2 \) (i.e., the \( k \times k \) matrix with the entries \( \{x_i, x_j\} \) is of constant rank \( 2r \)). The invariance of the bracket means the following. For any functions \( g \) and \( h \) on \( M \), we have

\[
\{g, h\} \circ F = \{g \circ F, h \circ F\}. \quad (18)
\]

To prove that a bracket is invariant, it suffices to verify property (18) for \( g = x_i \) and \( h = x_j \), \( i, j = 1, \ldots, k \).

In our examples considered below, the manifold has a system of coordinates \( x_1, \ldots, x_k \) such that for any \( i, j = 1, \ldots, k \), the function \( \{x_i, x_j\} \) is constant. Then, in order to prove that the bracket is invariant under \( F \), it suffices to show that \( \{x_i \circ F, x_j \circ F\} = \{x_i, x_j\} \) for all \( i, j \).

- If \( 2r < k \), then there are \( k - 2r \) functions

\[
C_s, \quad s = 1, \ldots, k - 2r, \quad (19)
\]

that are invariant under \( F \) (i.e., \( C_s \circ F = C_s \)) and are Casimir functions (i.e., \( \{C_s, g\} = 0 \) for any function \( g \)).

- There are \( r \) functions

\[
I_l, \quad l = 1, \ldots, r, \quad (20)
\]

that are invariant under \( F \) and are in involution with respect to the Poisson bracket (i.e., \( \{I_{l_1}, I_{l_2}\} = 0 \) for all \( l_1, l_2 = 1, \ldots, r \)).

- The functions (19), (20) are functionally independent.
We assume in this section that $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then the set $(\text{Mat}_n(\mathbb{K}))^3$ is a manifold. The elements of matrices $X, Y, Z \in (\text{Mat}_n(\mathbb{K}))^3$ are denoted by $x_{i,j}, y_{i,j}, z_{i,j}, i, j = 1, \ldots, n$. Then

$$
x_{i,j}, \ y_{i,j}, \ z_{i,j}, \ i, j = 1, \ldots, n, \tag{21}
$$
can be regarded as coordinates on the $3n^2$-dimensional manifold $(\text{Mat}_n(\mathbb{K}))^3$. Clearly, the functions $x_{i,j}, z_{i,j}$ are invariant under maps (14)–(17).

Propositions 4, 5 below are proved in preprint [10]. To clarify the main ideas, we reproduce the proofs from [10].

**Proposition 4** (see [10]). Let $n, m \in \mathbb{Z}_{\geq 0}$ be such that $1 \leq m \leq n/2$. Then the map (14) is Liouville integrable.

**Proof.** On the manifold $(\text{Mat}_n(\mathbb{K}))^3$, we consider the Poisson bracket defined as follows. The bracket of two coordinates from the list in (21) is nonzero only for

$$
\{y_{p,j}, z_{p,j}\} = -\{z_{p,j}, y_{p,j}\} = 1, \quad p = 1, \ldots, n - m, \quad j = 1, \ldots, n, \\
\{y_{m,q,j}, x_{j,m+q}\} = -\{x_{j,m+q}, y_{m,q,j}\} = 1, \quad q = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{22}
$$

This Poisson bracket is of rank $2n^2$. The $n^2$ functions

$$
z_{n-m+q,j}, \ x_{j,q}, \ x_{j,2m+s}, \ q = 1, \ldots, m, \quad s = 1, \ldots, n - 2m, \quad j = 1, \ldots, n, \tag{23}
$$

are Casimir functions, because they do not appear in (22). The rank of the bracket plus the number of the Casimir functions (23) equals the dimension of the manifold.

The $n^2$ functions

$$
z_{p,j}, \ x_{j,m+q}, \ p = 1, \ldots, n - m, \quad q = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{24}
$$

are in involution with respect to the Poisson bracket, because $\{x_{i,j}, x_{i',j'}\} = \{z_{i,j}, z_{i',j'}\} = \{x_{i,j}, z_{i',j'}\} = 0$ for all $i, j, i', j' = 1, \ldots, n$. Functions (23), (24) are functionally independent and are invariant under map (14).

We show that bracket (22) is also invariant. In (14), we have

$$
T_{n,m}(X, Y, Z) = (\hat{X}, \hat{Y}, \hat{Z}),
$$

\[
\hat{X} = X, \quad \hat{Y} = Y + \sum_{i=1}^{m} \begin{pmatrix} x_{1,i} \\ \vdots \\ x_{n,i} \end{pmatrix} (z_{n-m+i,1} \ldots z_{n-m+i,n}), \quad \hat{Z} = Z. \tag{25}
\]

The elements of the matrices $\hat{X}, \hat{Y}, \hat{Z} \in (\text{Mat}_n(\mathbb{K}))^3$ are denoted by $\hat{x}_{i,j}, \hat{y}_{i,j}, \hat{z}_{i,j}$ for $i, j = 1, \ldots, n$. The functions

$$
x_{1,i}, \ x_{2,i}, \ldots, x_{n,i}, \quad z_{n-m+i,1}, \ldots, z_{n-m+i,n}, \quad i = 1, \ldots, m, \tag{26}
$$

which appear in (25), belong to the list of Casimir functions in (23). Therefore, formulas (22) and (25) yield

$$
\{\hat{x}_{i,j}, \hat{x}_{i',j'}\} = \{x_{i,j}, x_{i',j'}\}, \quad \{\hat{y}_{i,j}, \hat{y}_{i',j'}\} = \{y_{i,j}, y_{i',j'}\}, \quad \{\hat{z}_{i,j}, \hat{z}_{i',j'}\} = \{z_{i,j}, z_{i',j'}\}, \quad \{\hat{x}_{i,j}, \hat{z}_{i',j'}\} = \{x_{i,j}, z_{i',j'}\}, \quad \{\hat{y}_{i,j}, \hat{z}_{i',j'}\} = \{y_{i,j}, z_{i',j'}\},
$$

$$
i, j, i', j' = 1, \ldots, n,
$$

which implies that the Poisson bracket is invariant under map (14).

Therefore, map (14) is Liouville integrable in the case $1 \leq m \leq n/2$. 

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**Example 1.** Let \( n = 2 \) and \( m = 1 \). Then the map (14) becomes

\[
T_{2,1} : (\text{Mat}_2(\mathbb{K}))^3 \rightarrow (\text{Mat}_2(\mathbb{K}))^3,
\]

\[
T_{2,1}(X, Y, Z) = \left( X, Y + \begin{pmatrix} x_{11} \cdot z_{21} & x_{11} \cdot z_{22} \\ x_{21} \cdot z_{21} & x_{21} \cdot z_{22} \end{pmatrix}, Z \right),
\]

\[
X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, \quad Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}.
\]

We have the coordinates \( x_{ij}, y_{ij}, z_{ij}, i, j \in \{1, 2\} \), on the 12-dimensional manifold \((\text{Mat}_2(\mathbb{K}))^3\).

Poisson bracket (22) takes the form

\[
\{ y_{11}, z_{11} \} = -\{ z_{11}, y_{11} \} = 1, \quad \{ y_{12}, z_{12} \} = -\{ z_{12}, y_{12} \} = 1,
\]

\[
\{ y_{21}, x_{12} \} = -\{ x_{12}, y_{21} \} = 1, \quad \{ y_{22}, x_{12} \} = -\{ x_{12}, y_{22} \} = 1,
\]

and is of rank 8.

Casimir functions (23) are \( z_{21}, z_{22}, x_{11}, \) and \( x_{21} \). Functions (24) are \( z_{11}, z_{12}, x_{12}, \) and \( x_{22} \), and they are in involution with respect to bracket (27).

**Example 2.** We let \( n = 5 \) and \( m = 2 \). Then map (14) is

\[
T_{5,2} : (\text{Mat}_5(\mathbb{K}))^3 \rightarrow (\text{Mat}_5(\mathbb{K}))^3,
\]

\[
T_{5,2}(X, Y, Z) = \left( X, Y + \begin{pmatrix} x_{11} \cdot z_{41} + x_{12} \cdot z_{51} & \cdots & x_{11} \cdot z_{45} + x_{12} \cdot z_{55} \\ \vdots & \ddots & \vdots \\ x_{51} \cdot z_{41} + x_{52} \cdot z_{51} & \cdots & x_{51} \cdot z_{45} + x_{52} \cdot z_{55} \end{pmatrix}, Z \right),
\]

\[
X = \begin{pmatrix} x_{11} & \cdots & x_{15} \\ \vdots & \ddots & \vdots \\ x_{51} & \cdots & x_{55} \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} & \cdots & y_{15} \\ \vdots & \ddots & \vdots \\ y_{51} & \cdots & y_{55} \end{pmatrix}, \quad Z = \begin{pmatrix} z_{11} & \cdots & z_{15} \\ \vdots & \ddots & \vdots \\ z_{51} & \cdots & z_{55} \end{pmatrix}.
\]

We have the coordinates \( x_{ij}, y_{ij}, z_{ij}, i, j \in \{1, \ldots, 5\} \), on the 75-dimensional manifold \((\text{Mat}_5(\mathbb{K}))^3\).

Poisson bracket (22) becomes

\[
\{ y_{1,j}, z_{1,j} \} = -\{ z_{1,j}, y_{1,j} \} = \{ y_{2,j}, z_{2,j} \} = -\{ z_{2,j}, y_{2,j} \} = \{ y_{3,j}, z_{3,j} \} = -\{ z_{3,j}, y_{3,j} \} = 1,
\]

\[
\{ y_{4,j}, x_{j,3} \} = -\{ x_{j,3}, y_{4,j} \} = \{ y_{5,j}, x_{j,4} \} = -\{ x_{j,4}, y_{5,j} \} = 1, \quad j = 1, \ldots, 5,
\]

and is of rank 8. Casimir functions (23) are \( z_{4,1}, z_{5,1}, x_{j,1}, x_{j,2}, \) and \( x_{j,5}, j = 1, \ldots, 5 \). Functions (24) are \( z_{1,1}, z_{2,1}, z_{3,1}, x_{j,3}, \) and \( x_{j,4} \), and they are in involution with respect to bracket (28).

**Proposition 5** (see [10]). Let \( n, \hat{m} \in \mathbb{Z}_{>0} \) be such that \( 3n/2 \leq \hat{m} \leq 2n - 1 \). Then map (15) is Liouville integrable.

**Proof.** On the manifold \((\text{Mat}_n(\mathbb{K}))^3\), we consider the following Poisson bracket. The bracket of two coordinates from the list in (21) is nonzero only for

\[
\{ y_{p,j}, z_{2n-\hat{m}+p,j} \} = -\{ z_{2n-\hat{m}+p,j}, y_{p,j} \} = 1, \quad p = 1, \ldots, \hat{m} - n, \quad j = 1, \ldots, n,
\]

\[
\{ y_{\hat{m}-n+q,j}, x_{j,q} \} = -\{ x_{j,q}, y_{\hat{m}-n+q,j} \} = 1, \quad q = 1, \ldots, 2n - \hat{m}, \quad j = 1, \ldots, n.
\]
This bracket is of rank $2n^2$. The $n^2$ functions
\[ z_{q,j}, \quad x_{j,2n-\hat{m}+p}, \quad q = 1, \ldots, 2n - \hat{m}, \quad p = 1, \ldots, \hat{m} - n, \quad j = 1, \ldots, n, \quad (30) \]
are Casimir functions because they do not appear in (29). The rank of the bracket plus the number of Casimir functions (30) equals the dimension of the manifold.

The $n^2$ functions
\[ z_{2n-\hat{m}+p,j}, \quad x_{j,q}, \quad p = 1, \ldots, \hat{m} - n, \quad q = 1, \ldots, 2n - \hat{m}, \quad j = 1, \ldots, n, \quad (31) \]
are in involution with respect to the bracket because $\{x_{i,j}, x_{i',j'}\} = \{z_{i,j}, z_{i',j'}\} = \{x_{i,j}, z_{i',j'}\} = 0$ for all $i, j, i', j' = 1, \ldots, n$. Functions (30) and (31) are functionally independent and are invariant under map (15).

Similarly to the proof of Proposition 4, we can show that bracket (29) is also invariant. Hence, map (15) is Liouville integrable in the case $3n/2 \leq \hat{m} \leq 2n - 1$.

**Proposition 6.** Let $n, m \in \mathbb{Z}_{>0}$ be such that $1 \leq m \leq n/2$. Then map (16) is Liouville integrable.

**Proof.** To prove this Proposition, it suffices to interchange $x_{i,j}$ with $z_{i,j}$ in the proof of Proposition 4.

**Proposition 7.** Let $n, \hat{m} \in \mathbb{Z}_{>0}$ be such that $3n/2 \leq \hat{m} \leq 2n - 1$. Then map (17) is Liouville integrable.

**Proof.** To prove this Proposition, it suffices to interchange $x_{i,j}$ with $z_{i,j}$ in the proof of Proposition 5.

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