Magnetization in 2+1 dimensional QED at Finite Temperature and Density

Jens O. Andersen and Tor Haugset
Institute of Physics
University of Oslo
P.O. BOX 1048, Blindern
N-0316 Oslo, Norway

We consider Dirac fermions moving in a plane with a static homogeneous magnetic field orthogonal to the plane. We calculate the effective action at finite temperature and density. The magnetization is derived and it is shown that the fermion gas exhibits de Haas-van Alphen oscillations at small temperatures and weak magnetic fields. We also comment upon earlier work.

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1 Introduction

Many of the phenomena that have been discovered in condensed matter physics over the last few decades are to a very good approximation two dimensional. The most important of these are the (Fractional) Quantum Hall effect and high $T_c$ superconductivity [1].

Quantum field theories in lower dimensions have therefore become of increasing interest in recent years. Both systems mentioned above have been modelled by anyons, which are particles or excitations that obey fractional statistics. Anyons can be described in terms of Chern-Simons field theories [1-3]. The literature on 2+1d field theory in general and Chern-Simons field theory in particular is now vast (see e.g refs. [1-5]), and we shall only comment upon a few selected papers, which are relevant for the present work.

Ten years ago Redlich [6] considered fermions in a plane moving in a constant electromagnetic field. Using Schwinger’s proper time method [7] to obtain the effective action for the gauge field, he demonstrated that a Chern-Simons term was induced by radiative corrections. The Chern-Simons term is parity breaking and is gauge-invariant modulo surface terms. More recently, Lykken et al. [2] have examined two dimensional Dirac fermions coupled to a gauge field whose dynamics
is governed by a (topological) Chern-Simons term. It was found that this planar Fermi gas becomes superconducting (below a critical temperature) when the induced Chern-Simons term exactly cancels the one appearing in the classical action.

External electromagnetic fields may give rise to induced charges in the Dirac vacuum if the energy spectrum is asymmetric with respect to some arbitrarily chosen zero point. The vacuum charge comes about since the number of particles gets reduced (or increased) relative to the free case. Furthermore, induced currents may appear and are attributed to the drift of the induced charges. This only happens if the external field does not respect the translational symmetry of the system. These interesting phenomena have been examined in detail by Flekkøy and Leinaas [8] in connection with magnetic vortices and their relevance to the Hall effect has been studied by Fumita and Shizuya [9].

In the present paper we re-examine the system considered by Redlich [6]. We shall restrict ourselves to the case of a constant magnetic field, but we extend the analysis by including thermal effects and we shall mainly focus on the magnetization of the system.

In the first section we find the solutions of the Dirac equation for particles in a constant magnetic field. These solutions are then used to calculate the thermal fermion propagator, from which we derive the effective action for the gauge field. In the fourth section we compute the magnetization. It is shown that the de Haas-van Alphen oscillations [10,11] occur at low temperatures and weak magnetic fields, and that is the main result of this paper. We consider various limits of the magnetization that may be understood from a physical point of view and we explicitly demonstrate that we get the correct high temperature limit (i.e. free fermions).

Finally, we summarize and draw some conclusions in the last section.

2 Dirac Equation

In this section we shall briefly discuss the solutions of the Dirac equation in 2+1d with a constant magnetic field along the z-axis. The Dirac equation reads

\[(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0,\]  

(1)

where the gamma matrices satisfy the Clifford algebra

\[\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.\]  

(2)

In 2+1 dimensions the fundamental representation of the Clifford algebra is given by 2×2 matrices and these can be constructed from the Pauli matrices. Furthermore, in
2+1d there are two inequivalent choices of the gamma matrices, which corresponds to $\gamma^\mu \rightarrow -\gamma^\mu$. From eq. (1) we see that this extra degree of freedom may be absorbed in the sign of $m$. These choices correspond to “spin up” and “spin down”, respectively.

We would also like to point out that the angular momentum operator $\sigma$ is a pseudo vector in 2+1d, implying that the Dirac equation written in terms of these matrices does not respect parity. This is no longer the case if the Dirac equation is expressed in terms of $4 \times 4$ matrices. The latter representation is reducible and reduces to the two inequivalent fundamental representations mentioned above. Let us explicitly verify this. A natural choice is the three block diagonal $4 \times 4$ matrices

$$
\gamma^0_4 = \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix}, \quad \gamma^1_4 = \begin{pmatrix} \gamma^1 & 0 \\ 0 & -\gamma^1 \end{pmatrix}, \quad \gamma^2_4 = \begin{pmatrix} \gamma^2 & 0 \\ 0 & -\gamma^2 \end{pmatrix},
$$

where $\gamma^0$, $\gamma^1$ and $\gamma^2$ themselves are $2 \times 2$ matrices satisfying the algebra. This representation trivially reduces to the sum of the two inequivalent choices of the $2 \times 2$ gamma matrices. Furthermore, the Dirac equation written in terms of the above matrices respects parity (a parity transformation simply interchanges the coupled equations and thereby the components of the spinor).

In the following we make the choice $\gamma^0 = \sigma_3$, $\gamma^1 = -i\sigma_2$, $\gamma^2 = -i\sigma_1$ and we have chosen the asymmetric gauge $A_\mu = (0, 0, -Bx)$. The metric is diag ($-1, 1, 1$).

The Dirac equation then takes the form

$$
\left( \begin{array}{cc} i\frac{\partial}{\partial t} - m & i\frac{\partial}{\partial x} - ieBx + \frac{\partial}{\partial y} \\ -i\frac{\partial}{\partial x} - ieBx + \frac{\partial}{\partial y} & -i\frac{\partial}{\partial t} - m \end{array} \right) \psi(x, t) = 0.
$$

Here $\psi(x, t)$ is a two component spinor. We now assume that the wavefunctions may be written as

$$
\psi_\kappa(x, t) = \exp(-iEt + iky) \begin{pmatrix} f_\kappa(x) \\ g_\kappa(x) \end{pmatrix},
$$

where $\kappa$ denotes all quantum numbers necessary in order to completely characterize the solutions. Inserting this into eq. (4) one obtains

$$
\begin{pmatrix} E - m & -\xi_+ \\ \xi_- & -E - m \end{pmatrix} \begin{pmatrix} f_\kappa(x) \\ g_\kappa(x) \end{pmatrix} = 0,
$$

where

$$
\xi_\pm = -i\partial_x \mp i(k - eBx).
$$

3
The equation for $f_{\kappa}(x)$ is readily found from eq. (6):

$$
(E^2 - m^2 - \xi_+ \xi_-) f_{\kappa}(x) = 0.
$$

(8)

The eigenfunctions of $\xi_+ \xi_-$, provided that $eB > 0$, are [12]

$$
I_{n,k} = \left(\frac{eB}{\pi}\right)^{\frac{1}{4}} \exp \left[ -\frac{1}{2} (x - \frac{k}{eB})^2 eB \right] \frac{1}{\sqrt{n!}} H_n \left( \sqrt{2eB(x - \frac{k}{eB})} \right),
$$

(9)

where $H_n(x)$ is the $n$'th Hermite polynomial. Furthermore, $I_{n,k}(x)$ is normalized to unity and satisfies

$$
\xi_- I_{n,k}(x) = -i\sqrt{2eB}n I_{n-1,k}(x),
$$

$$
\xi_+ I_{n,k}(x) = i\sqrt{2eB(n+1)} I_{n+1,k}(x).
$$

Combining eqs. (8) and (9) yields

$$
f_{\kappa}(x) = I_{n,k}(x), \quad E^2 = m^2 + 2eBn.
$$

(10)

The function $g_{\kappa}(x)$ satisfies

$$
g_{\kappa}(x) = \frac{\xi_-}{E + m} f_{\kappa}(x),
$$

(11)

implying that

$$
g_{\kappa}(x) = -i\sqrt{2eB}n I_{n-1,k}(x).
$$

(12)

The normalized eigenfunctions become

$$
\psi_{n,k}^{(\pm)}(x, t) = \exp(\mp iE_n t + iky) \sqrt{\frac{E_n \pm m}{2E_n}} \left( \frac{I_{n,k}(x)}{\sqrt{\frac{2eBn}{E_n \pm m}}} I_{n-1,k}(x) \right),
$$

(13)

where $n = 0, 1, 2, ..., E_n = \sqrt{m^2 + 2eBn}$ and $\psi_{n,k}^{(\pm)}(x, t)$ are positive and negative energy solutions, respectively. Note that $\psi_{0,k}^{(-)}(x, t) = 0$ and that we have defined $I_{-1,k}(x) \equiv 0$. The spectrum is therefore asymmetric and this asymmetry is intimately related to the induced vacuum charge, as will be shown in section 4. In fig. 1 a) we have shown the spectrum for $m > 0$ and in fig. 1 b) for $m < 0$.

The field may now be expanded in the complete set of eigenmodes:

$$
\Psi(x, t) = \sum_{n=0}^{\infty} \int dk \left[ b_{n,k} \psi_{n,k}^{(+)}(x, t) + d_{n,k}^* \psi_{n,k}^{(-)}(x, t) \right].
$$

(14)

Quantization is carried out in the usual way by promoting the Fourier coefficients to operators satisfying

$$
\{b_{n,k}, b_{n',k'}^\dagger\} = \delta_{n,n'} \delta_{k,k'}, \quad \{d_{n,k}, d_{n',k'}^\dagger\} = \delta_{n,n'} \delta_{k,k'},
$$

(15)

and all other anti-commutators being zero.
3 Fermion Propagators and the Effective Action

In the previous section we solved the Dirac equation and with the wave functions at hand, we can construct the propagator. From the trace of the propagator the effective action to one-loop order is calculated.

The fermion propagator. The fermion propagator in vacuum is

\[ iS_F(x', x) = \langle 0 | T \left[ \Psi(x', t') \overline{\Psi}(x, t) \right] | 0 \rangle, \tag{16} \]

where \( T \) denotes time ordering. By use of the expansion (14) one finds

\[ iS_F(x', x) = \sum_{n=0}^{\infty} \int \frac{dk}{2\pi} \left[ \theta(t' - t) \psi^{(+)}_{n,k}(x', t') \overline{\psi}^{(+)}_{n,k}(x, t) - \theta(t - t') \psi^{(-)}_{n,k}(x', t') \overline{\psi}^{(-)}_{n,k}(x, t) \right]. \tag{17} \]

After some purely algebraic manipulations and using the integral representation of the step function, we obtain

\[ S_F(x', x)_{ab} = -\frac{1}{4\pi^2} \sum_{n=0}^{\infty} \int dk d\omega \frac{E_n + m}{2E_n} \exp \left[ -i\omega(t' - t) + ik(y' - y) \right] \frac{1}{\omega^2 - E_n^2 + i\varepsilon} S_{ab}(n, \omega, k). \tag{18} \]

Here \( S_{ab}(n, \omega, k) \) is the matrix

\[ \left( \begin{array}{cc} \frac{E_n}{E_n + m} I_{n,k}(x') I_{n,k}(x) & \frac{E_n}{E_n + m} \sqrt{2eBn} I_{n,k}(x') I_{n-1,k}(x) \\ \frac{E_n + m}{E_n} \sqrt{2eBn} I_{n-1,k}(x') I_{n,k}(x) & \frac{E_n + m}{E_n} \sqrt{2eBn} I_{n-1,k}(x') I_{n-1,k}(x) \end{array} \right). \tag{19} \]

At finite temperature and chemical potential we write the thermal propagator as (see ref. [10] for details)

\[ \langle S_F(x', x) \rangle_{\beta, \mu} = S_F(x', x) + S^\beta_{F}(x', x). \tag{20} \]

The thermal part of the propagator is

\[ iS^\beta_{F}(x', x) = -\sum_{n=0}^{\infty} \int \frac{dk}{2\pi} \left[ f_F^+(E_n) \psi^{(+)}_{n,k}(x', t') \overline{\psi}^{(+)}_{n,k}(x, t) - f_F^-(E_n) \psi^{(-)}_{n,k}(x', t') \overline{\psi}^{(-)}_{n,k}(x, t) \right], \tag{21} \]

where

\[ f_F^+(\omega) = \frac{1}{\exp \beta(\omega - \mu) + 1}, \quad f_F^-(\omega) = 1 - f_F^+(\omega) = \frac{1}{\exp \beta(\omega + \mu) + 1}. \tag{22} \]

This may be rewritten as

\[ S^\beta_{F}(x', x) = i \sum_{n=0}^{\infty} \int \frac{dk}{2\pi} d\omega \exp ik(y' - y) \exp i\omega(t' - t) f_F(\omega) \delta(\omega^2 - E_n^2 - i\varepsilon) S_{ab}(n, \omega, k). \tag{23} \]
Here $S_{ab}^\gamma(n, \omega, k)$ is the matrix
\begin{align}
\begin{pmatrix}
(\omega + m)I_{n,k}(x')I_{n,k}(x) & I_{n-1,k}(x')I_{n,k}(x) \\
I_{n,k}(x')I_{n-1,k}(x) & (\omega - m)I_{n-1,k}(x')I_{n-1,k}(x)
\end{pmatrix}.
\end{align}
(24)

As noted in ref. [10], one is not restricted to use equilibrium distributions in this approach. Single particle non-equilibrium distributions may be more appropriate if e.g. an electric field has driven the system out of equilibrium.

**The effective action.** The generating functional for fermionic Green’s functions in an external magnetic field may be written as a path integral:
\begin{align}
Z(\eta, \overline{\eta}, A_\mu) = \int D\psi D\overline{\psi} \exp \left[ i \int d^3x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\overline{\partial} - m)\psi - \eta \overline{\psi} + \overline{\psi} \eta \right) \right].
\end{align}
(25)
The functional integral describes the interaction of fermions with a classical electromagnetic field. It includes the effects of all virtual electron-positron pairs, but virtual photons are not present, which means that we are considering the weak coupling limit.

The fermion field can be integrated over since the functional integral is Gaussian:
\begin{align}
Z(\eta, \overline{\eta}, A_\mu) = \det \left( i(i\overline{\partial} - m) \right) \exp \left[ i \int d^3x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \int d^3y \overline{\eta}(x) S_F(x, y) \eta(y) \right) \right].
\end{align}
(26)

Taking the logarithm of $Z(\eta, \overline{\eta}, A_\mu)$ with vanishing sources gives the effective action
\begin{align}
S_{\text{eff}} = \int d^3x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] - i \text{Tr} \log \left[ i(i\overline{\partial} - m) \right],
\end{align}
(27)
where we have written $\log \det = \text{Tr} \log$ by the use of a complete orthogonal basis. Differentiating eq. (27) with respect to $m$ yields
\begin{align}
\frac{\partial \mathcal{L}_1}{\partial m} = i \text{tr} S_F(x, x). 
\end{align}
(28)

The trace is now over spinor indices only. By calculating the trace of the propagator and integrating this expression with respect to $m$ thus yields the one-loop contribution to the effective action. This method has been previously applied by Elmfors et al [10] in $3+1$ dimensions. The above equation may readily be generalized to finite temperature, where we separate the vacuum contribution in the effective action
\begin{align}
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}^{\beta,\mu} \equiv \mathcal{L}_0 + \mathcal{L}_{\text{eff}}
\end{align}
(29)
where $\mathcal{L}_0$ is the tree level contribution, and
\begin{align}
\frac{\partial \mathcal{L}_{\text{eff}}}{\partial m} = i \text{tr} \left[ S_F(x, x) + S_F^{\beta,\mu}(x, x) \right].
\end{align}
(30)
Using eqs. (18) and (19) a straightforward calculation gives for the vacuum contribution
\[ trS_F(x, x) = -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \int \frac{dk \, d\omega}{\omega^2 - E_n^2 + i\varepsilon} \left[ m \left( I_{n,k}^2(x) + I_{n-1,k}^2(x) \right) + \omega \left( I_{n,k}^2(x) - I_{n-1,k}^2(x) \right) \right] \]
\[ + \frac{1}{4\pi^2} \int \frac{dk \, d\omega}{\omega + m - i\varepsilon} I_0^2(x) \]
\[ = \frac{i}{2\pi} \sum_{n=1}^{\infty} \int dk \frac{m}{E_n} I_{n,k}^2(x) + \frac{ieB}{4\pi} \]
\[ = \frac{i eB}{2\pi} \sum_{n=1}^{\infty} \frac{m}{E_n} + \frac{ieB}{4\pi}. \] (31)

Integrating this expression with respect to \( m \) yields
\[ \mathcal{L}_1 = -\frac{eB}{2\pi} \sum_{n=1}^{\infty} \sqrt{m^2 + 2eBn} - \frac{eBm}{4\pi}. \] (32)

The divergence may be sidestepped by using the integral representation of the gamma function [13] and subtract a constant to make \( \mathcal{L}_1 \) vanish for \( B = 0 \),
\[ \mathcal{L}_1 = -\frac{eB}{2\pi} \sum_{n=1}^{\infty} \sqrt{m^2 + 2eBn} - \frac{eBm}{4\pi}. \] (33)

This result calls for a few comments. Firstly, one notes that the induced Chern-Simons term in the effective action vanishes. This is not in conflict with the result of Redlich; rather it is caused by our choice of gauge, \( A_0 = 0 \). Insertion of \( A_0 = 0 \) into the general expression for the Chern-Simons term obtained by Redlich, will make it vanish. Thus, our results agree. Secondly, one can show that the Chern-Simons term is gauge-invariant up to surface terms and using another gauge the Chern-Simons term needs not vanish. The choice \( A_0 \neq 0 \) gives rise to a Chern-Simons term \( A_0 Bme^2/(|m|8\pi^2) \) in this approach and the result is again in accordance with the general expression. It is always satisfactory to see that identical results can be obtained by entirely different methods.

The finite temperature part of the effective action is calculated analogously using the thermal part of the propagator (21),
\[ \mathcal{L}^{\beta, \mu} = \frac{T eB}{2\pi} \sum_{n=1}^{\infty} \left[ \log \left( 1 + \exp -\beta(E_n - \mu) \right) + \log \left( 1 + \exp -\beta(E_n + \mu) \right) \right] \]
\[ + \frac{T eB}{2\pi} \log \left( 1 + \exp -\beta(m - \mu) \right). \] (34)
Letting $B \to 0$ it can be shown that one obtains the free energy of a gas of non-interacting electrons and positrons:

$$L_{0}^{\beta,\mu} = \frac{T}{2\pi} \int_{0}^{\infty} E dE \left[ \log[1 + \exp -\beta(E - \mu)] + \log[1 + \exp -\beta(E + \mu)] \right],$$  \hspace{1cm} (35)

where $E = \sqrt{m^2 + k^2}$.

In the following we restrict ourselves to the case $\mu > 0$. Analogous results can be obtained for $\mu < 0$.

In the zero temperature limit of $L_{0}^{\beta,\mu}$ one gets

$$L_{0}^{\beta,\mu} = \frac{eB}{2\pi} \sum_{n=0}^{\prime} (\mu - E_{n}),$$  \hspace{1cm} (36)

where the prime indicates that the sum is restricted to integers less than $(\mu^2 - m^2)/2eB$. Similarly, one may derive the charge density at $T = 0$:

$$\rho = \frac{\partial L_{0}^{\beta,\mu}}{\partial \mu} = \frac{eB}{2\pi} \left[ \text{Int} \left( \frac{\mu^2 - m^2}{2eB} \right) + 1 \right], \quad \mu > m,$$  \hspace{1cm} (37)

in accordance with the result of Zeitlin [14]. Notice that at $T = 0$ the charge density is equal to the particle density, since no antiparticles are present, as can be seen by inspection of eqs. (36) and (37).

### 4 Magnetization and the de Haas-van Alphen Effect

In this section we study the physical content of the effective action which was obtained in the previous section. In particular we investigate a few limits to check the consistency of our calculations.

The magnetization is defined by

$$M = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial B}.$$  \hspace{1cm} (38)

The vacuum contribution to the magnetization is obtained from eq. (33)

$$M_1 = \frac{1}{8\pi^2} \int_{0}^{\infty} ds \frac{\exp(-m^2s)}{s^2} \left[ es \coth(es) - \frac{e^2Bs^2}{\sinh^2(es)} \right].$$  \hspace{1cm} (39)
For the thermal part of the magnetization we find

\[
M^{\beta,\mu} = \frac{T e}{2\pi} \sum_{n=1}^{\infty} \left[ \log \left( 1 + \exp -\beta (E_n - \mu) \right) + \log \left( 1 + \exp -\beta (E_n + \mu) \right) \right] \\
+ \frac{T e}{2\pi} \log \left( 1 + \exp -\beta (m - \mu) \right) \\
- \frac{e^2 B}{2\pi} \sum_{n=1}^{\infty} \frac{n}{E_n} \left[ \frac{1}{\exp \beta (E_n - \mu) + 1} + \frac{1}{\exp \beta (E_n + \mu) + 1} \right].
\]

(40)

**Magnetization at zero temperature.** In the zero temperature limit eq. (40) reduces to

\[
M^{\beta,\mu} = \frac{e}{2\pi} \sum_{n=0}^{\prime} \left[ \mu - E_n - \frac{e B n}{E_n} \right],
\]

(41)

where the sum again is restricted to integers less than \((\mu^2 - m^2)/2eB\).

In the weak \(B\)-field limit \((eB \ll \mu^2 - m^2 \ll m^2)\) the vacuum contribution becomes

\[
M_1 = \frac{e^2 B}{12\pi^{3/2}} \int_0^{\infty} ds \frac{\exp(-m^2 s)}{s^{1/2}} = \frac{e^2 B}{12\pi|m|}.
\]

(42)

This agrees with the results of refs. [14,15]. The contribution from real thermal particles is obtained by rewriting the square root in \(\mathcal{L}^{\beta,\mu}\), using the integral representation of the gamma function and treating \(Int(\mu^2 - m^2)/2eB\) as a continuous variable in the limit \(B \to 0\). The result is

\[
M^{\beta,\mu} = \frac{e}{4\pi} (\mu - m).
\]

(43)

Some comments are in order. It is perhaps somewhat surprising that the magnetization is non-zero in this limit. One should, however, bear in mind that the sign of \(m\) uniquely determines the spin of the particles (and antiparticles), implying that the system under investigation consists entirely of either spin up or spin down particles. This is not the case in 3+1d, where the representations characterized by the sign of \(m\) are equivalent. It would therefore be natural to consider a system consisting of an equal number of spin up and spin down particles, which then amounts to sum over \(\pm m\). By doing so one finds a vanishing magnetization as \(B\) goes to zero, exactly as in 3+1 dimensions. Note that this result, of course, can be obtained by using four component Dirac spinors, since the four dimensional representation reduces to the two inequivalent two dimensional ones.

\(^1\)One should use the same chemical potential irrespective of the sign of \(m\) in order to ensure the same charge density for spin up and spin down particles.
In order to get the strong field limit \( eB \gg \mu^2, m^2 \) of the vacuum contribution, we scale out \( eB \) and take \( eB \to \infty \) in the remainder. This gives

\[
L_1 \propto (eB)^{\frac{3}{2}} \Rightarrow M_1 \propto e^{\frac{3}{2}} \sqrt{B}.
\] (44)

Vacuum effects contribute to the magnetization proportional to the square root of \( B \). This should be compared with the corresponding result in 3+1d, where the magnetization goes like \( B \log(\frac{B}{m^2}) \) [10]. For real thermal particles only the lowest Landau level contributes in the strong field limit, as can be seen directly from eq. (41). The thermal part of the magnetization is then \( M^{\beta, \mu} = e(\mu - m)/2\pi \). For large \( B \)-fields all particles are in the lowest Landau level and \( \mu = m \). Hence, the contribution to the magnetization from real thermal particles vanishes. This can be understood from the following physical argument: The energy of the single particle ground state is independent of the external field \( (E_0 = m) \), so increasing \( B \) cannot lead to an increase in \( L^{\beta, \mu} \), when the charge density (and therefore the particle density) is held constant.

**High Temperature Limit** \((T^2 \gg m^2 \gg eB, \mu = 0)\). The high temperature limit is rather trivial. From a physical point of view, one expects that \( L^{\beta, \mu} \) approaches the thermodynamic potential of a gas of non-interacting particles of mass \( m \). Indeed we obtain

\[
L^{\beta, \mu} = \frac{mT^2}{2\pi} Li_2[1 + \exp(-\beta m)] + \frac{T^3}{2\pi} Li_3[1 + \exp(-\beta m)],
\] (45)

which, of course, coincides with eq. (35) when \( \mu = 0 \). Here \( Li_n(1 + x) \) is the polylogarithmic function of order \( n \):

\[
Li_n(1 + x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}.
\] (46)

Eq. (45) may be obtained by carefully replacing the sum by an integral.

**Magnetization at finite temperature.** In fig. 2 we have displayed the total magnetization as a function of the external magnetic field for different values of the temperature \((\mu/m = 1/1.5, T/m = 1/150 \text{ solid line}, 1/50 \text{ dashed line}, 1/5 \text{ dotted line})\). Fig. 3 is a magnification of fig. 2 in the oscillatory region. The de Haas-van Alphen oscillations are seen to be present for low temperatures and weak magnetic fields. Finally, we have summed the magnetizations for \( \pm m \). The resulting curve \((T/m = 1/100)\) is shown in fig. 4 and reveals that the magnetization goes to zero in the limit \( B \to 0 \).

\(^2\)This is also the case in 3+1d where the energy of the lowest Landau level is independent of \( B \). \( E_0 = \sqrt{m^2 + k_z^2} \), where \( k_z \) is the \( z \)-component of the momentum.
Induced vacuum charges and currents. Finally, we calculate the vacuum expectation value of the induced charge and current densities. Such calculations have been carried out in other contexts, e.g. in connection with magnetic flux strings (see ref. [8]). We shall employ the most commonly used definition of the current operator which can be shown to measure the spectral asymmetry relative to the spectrum of free Dirac particles.

\[ j^\mu(x) = \frac{e}{2} \left[ \bar{\Psi}_\alpha(x), (\gamma^\mu \Psi(x))_\alpha \right]. \] (47)

The induced charge density is given by

\[ \langle j^0(x) \rangle = \langle \rho(x) \rangle = -\frac{e}{2} \sum_n \int dk \text{sign}(E_n) |\psi_{n,k}(x)|^2. \] (48)

Using the complete set of eigenmodes as given by eq. (13), a straightforward calculation yields

\[ \langle \rho(x) \rangle = \frac{m}{|m|} \frac{e^2 B}{4\pi}. \] (49)

Eq. (49) is simply the Chern-Simons relation and our calculations are thus in complete agreement with the result of Fumita and Shizuya. This result has the following physical interpretation: As we turn the magnetic field on, an unpaired energy level \( E = m \) emerges (in the case \( m > 0 \)). The number of spin down negative energy electrons therefore gets reduced relative to the free case by a factor \( \frac{eB}{4\pi} \), which is the degeneracy per unit area. This can be interpreted as the appearance of spin up positrons and results in a positive charge density. For \( m < 0 \) a similar argument applies.

A similar calculation for \( \langle j(x) \rangle \) reveals that the induced current vanishes. This result should come as no surprise due to translational symmetry of the system. A non-vanishing vacuum current would arise in the presence of an external electric field and is then attributed to the drift of the induced vacuum charge.

5 Conclusions

In this paper we have calculated the effective action for fermions moving in a plane with a constant magnetic field orthogonal to the plane. We have derived the magnetization from the effective action and have shown that the system exhibits the well-known de Haas-van Alphen oscillations at small temperatures and low values of the magnetic field.
Finally, it would be of some interest to extend the present work. Firstly, one should examine improvements to our results by considering the corrections to the self-energy of the electrons in the presence of fields and a thermal heat bath. Secondly, one could treat the highly non-trivial problem of fermions in slowly varying electric and magnetic fields.

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FIGURE CAPTIONS:

Figure 1: The energy spectra of Dirac fermions in the presence of a magnetic field. a) \( m > 0 \) and b) \( m < 0 \).

Figure 2: The magnetization in units of \( em \) as a function of \( B \) in units of \( m^2/e \) for different values of temperature. \( \mu = 1.5m \).

Figure 3: A magnification of the oscillatory region in fig. 2.

Figure 4: The magnetization for a system consisting of an equal number of spin up and spin down particles.
