Existence and uniqueness of limit cycles in a class of second order ODE’s

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Abstract

We prove a uniqueness result for limit cycles of a class of second order ODE’s. As a special case, we prove limit cycle’s uniqueness for an ODE studied in [1].

1 Introduction

Let us consider a first order differential system in the real plane,

\begin{align}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y).
\end{align}

(1)

The study of the dynamics of (1) strongly depends on the existence and stability properties of special solutions such as equilibrium points and non-constant periodic solutions. In particular, if an attracting non-constant periodic solution exists, then it dominates the dynamics of (1) in an open, connected subset of the plane, its region of attraction. In some cases such a region of attraction can even extend to cover the whole plane, with the unique exception of an equilibrium point. Uniqueness theorems for non-constant periodic solutions, i. e. limit cycles, have been extensively studied, see [2] and [4] for recent results and extensive bibliographies. Most of the results known are concerned with the classical Liénard system,

\begin{align}
\dot{x} &= y - F(x), \\
\dot{y} &= -g(x).
\end{align}

(2)

and its generalizations, such as

\begin{align}
\dot{x} &= \beta(x) [\varphi(y) - F(x)], \\
\dot{y} &= -\alpha(y)g(x).
\end{align}

(3)

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Such a class of systems also contain Lotka-Volterra systems and systems equivalent to Rayleigh equation

\[ \ddot{x} + f(\dot{x}) + g(x) = 0, \]  

(4)
as special cases. A very recent result [2] is concerned with systems equivalent to

\[ \ddot{x} + \sum_{k=0}^{N} f_{2k+1}(x) \dot{x}^{2k+1} + x = 0, \]  

(5)
with \( f_{2k+1}(x) \geq 0 \), increasing for \( x > 0 \), decreasing for \( x < 0 \), \( k = 0, \ldots, N \). On the other hand, there exist classes of second order ODE’s which are not covered by the above cases. This is the case of a model developed in [1], which led to the equation

\[ \ddot{x} + \epsilon \dot{x}(x^2 + \dot{x} + \dot{x}^2 - 1) + x = 0, \quad \epsilon > 0. \]  

(6)
In this paper we prove a uniqueness result for systems equivalent to

\[ \ddot{x} + \dot{x} \phi(x, \dot{x}) + x = 0, \]  

(7)
under the assumption that \( \phi(x, y) \) be a function with star-shaped level sets. As a consequence, we are able to prove existence and uniqueness of the limit cycle for the equation (6).

2 Risultati preliminari

Let \( \Omega \subset \mathbb{R}^2 \) be a star-shaped set. We say that a function \( \phi \in C^1(\Omega, \mathbb{R}) \) is **star-shaped** if \( (x, y) \cdot \nabla \phi = x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \) does not change sign. We say that \( \phi \) is **strictly star-shaped** if \( (x, y) \cdot \nabla \phi \neq 0 \). We call ray a half-line having origin at the point \((0, 0)\).

Let us consider a system equivalent to the equation (7)

\[ \dot{x} = y, \quad \dot{y} = -x - y \phi(x, y). \]  

(8)

We denote by \( \gamma(t, x^*, y^*) \) the unique solution to the system (8) such that \( \gamma(0, x^*, y^*) = (x^*, y^*) \). We first consider a sufficient condition for limit cycles’ uniqueness.

**Theorem 1.** Let \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a strictly star-shaped function. Then (8) has at most one limit cycle.

**Proof.** Let us assume that, for \( (x, y) \neq (0, 0) \),

\[ x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} > 0. \]
The proof can be performed analogously for the opposite inequality.

Applying Corollary 6 in [3] requires to compute the expression
\[ \nu = P \left( x \frac{\partial Q}{\partial x} + y \frac{\partial Q}{\partial y} \right) - Q \left( x \frac{\partial P}{\partial x} + y \frac{\partial P}{\partial y} \right), \]
where \( P \) and \( Q \) are the components of the considered vector field. For system (8), one has
\[ \nu = y \left( -x - xy \frac{\partial \phi}{\partial x} - y \frac{\partial \phi}{\partial y} + y \phi \right) - \left( -x - \phi(x, y) \right) y = -y^2 \left( x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right) \leq 0. \]
The function \( \nu \) vanishes only for \( y = 0 \). Let us assume, by absurd, that two distinct limit cycles exist, \( \gamma_1 \) and \( \gamma_2 \). Since the system (8) has only one critical point, the two cycles have to be concentric. Let us assume that \( \gamma_2 \) encloses \( \gamma_1 \). For both cycles one has:
\[ \int_0^{T_i} \nu(\gamma_i(t)) \, dt < 0, \quad i = 1, 2, \]
where \( T_i \) is the period of \( \gamma_i, \ i = 1, 2 \). Hence both cycles, by theorem 1 in [3], are attractive. Let \( A_1 \) be the region of attraction of \( \gamma_1 \). \( A_1 \) is bounded, because it is enclosed by \( \gamma_2 \), which is not attracted to \( \gamma_1 \). The external component of \( A_1 \)'s boundary is itself a cycle \( \gamma_3 \), because (8) has just one critical point at the origin. Again,
\[ \int_0^{T_3} \nu(\gamma_3(t)) \, dt < 0, \]
hence \( \gamma_3 \) is attractive, too. This contradicts the fact that the solutions of (8) starting from its inner side are attracted to \( \gamma_1 \). Hence the system (8) can have at most a single limit cycle.

In particular, the equation (6) considered in [1] has at most one limit cycle. In fact, in this case one has \( \phi(x, y) = \epsilon(x^2 + xy + y^2 - 1) \), so that one has
\[ \nu = x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} = 2xy^2(x^2 + xy + y^2) > 0 \quad \text{for} \quad (x, y) \neq (0, 0). \]
It should be noted that even if the proof is essentially based on a stability argument, the divergence cannot be used in order to replace the function \( \nu \). In fact, the divergence of system (8) is
\[ \text{div}(y, -x - y\phi(x, y)) = -\phi \frac{\partial \phi}{\partial y}. \]
Lasalle's invariance principle, since $\dot{V}$ has just one critical point at the origin, hence by Poincaré-Bendixon theorem there exist a limit cycle. By theorem 1, such a limit cycle is unique.

Now we care about the existence of limit cycles. We say that $\gamma(t)$ is **positively bounded** if the semi-orbit $\gamma^+ = \{\gamma(t), \ t \geq 0\}$ is contained in a bounded set. Let us denote by $D_r$ the disk $\{(x,y): \text{dist}((x,y),O) \leq r\}$, and by $B_r$ its boundary $\{(x,y): \text{dist}((x,y),O) = r\}$. In the following, we use the function $V(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$ as a Liapunov function. Its derivative along the solutions of $\dot{\gamma}$ is

$$\dot{V}(x,y) = -y^2 \phi(x,y).$$

**Lemma 1.** Let $U$ be a bounded set, with $\sigma := \sup\{\text{dist}((x,y),O), (x,y) \in U\}$. If $\phi(x,y) \geq 0$ out of $U$, and $\phi(x,y)$ does not vanish identically on any $B_r$, for $r > \sigma$, then every $\gamma(t)$ definitely enters the disk $D_\sigma$ and does not leave it.

**Proof.** The level curves of $V(x,y)$ are circumferences. For every $r \geq \sigma$, the disk $D_r$ contains $U$. Since $\dot{V}(x,y) = -y^2 \phi(x,y) \leq 0$ on its boundary, such a disk is positively invariant. Let $\gamma$ be an orbit with a point $\gamma(t^*)$ such that $d^* = \text{dist}(\gamma(t^*),O) > \sigma$. Then $\gamma$ does not leave the disk $D_{d^*}$, hence it is positively bounded. Moreover $\gamma(t)$ cannot be definitely contained in $B_r$, for any $r > \sigma$, since $\dot{V}(x,y)$ does not vanish identically on any $B_r$, for $r > \sigma$. Now, assume by absurd that $\gamma(t)$ does not intersect $B_r$. Then its positive limit set is a cycle $\gamma(t)$, having no points in $D_\sigma$. The cycle $\gamma(t)$ cannot cross outwards any $B_r$, hence it has to be contained in $B_r$, for some $r > \sigma$, contradicting the fact that $\dot{V}(x,y)$ does not vanish identically on any $B_r$, for $r > \sigma$. Hence there exists $t^+ > t^*$ such that $\gamma(t^+) \in D_\sigma$. Then, for every $t > t^+$, one has $\gamma(t) \in D_\sigma$, because $\dot{V}(x,y) \leq 0$ on $B_\sigma$.

Collecting the results of the above statements, we may state a theorem of existence and uniqueness for limit cycles of a class of second order equations. We say that an equilibrium point $O$ is **negatively asymptotically stable** if it is asymptotically stable for the system obtained by reversing the time direction.

**Theorem 2.** If the hypotheses of theorem [1] and lemma[1] hold, and $\phi(0,0) < 0$, then the system [3] has exactly one limit cycle, which attracts every non-constant solution.

**Proof.** By the above lemma, all the solutions are definitely contained in $D_\sigma$. The condition $\phi(0,0) < 0$ implies by continuity $\phi(x,y) < 0$ in a neighbourhood $N_O$ of the origin. This gives the negative asymptotic stability of the origin by Lasalle’s invariance principle, since $\dot{V}(x,y) \geq 0$ in $N_O$, and the set $\{\dot{V}(x,y) = 0\} \cap N_O = \{y = 0\} \cap N_O$ does not contain any positive semi-orbit. The system has just one critical point at the origin, hence by Poincaré-Bendixson theorem there exist a limit cycle. By theorem[1] such a limit cycle is unique.
This proves that every non-constant solution to the equation \(6\) studied in [1] is attracted to the unique limit cycle.

We can produce more complex systems with such a property. Let us set

\[
\phi(x, y) = -M + \sum_{k=1}^{n} H_{2k}(x, y),
\]

with \(H_{2k}(x, y)\) is a homogeneous function of degree \(2k\), positive except at the origin, \(M\) is a positive constant. Then, by Euler’s identity, one has

\[
\nu = \sum_{k=1}^{n} \left( x \frac{\partial H_{2k}}{\partial x} + y \frac{\partial H_{2k}}{\partial y} \right) = \sum_{k=0}^{n} 2kH_{2k}(x, y) > 0 \quad \text{for} \quad (x, y) \neq (0, 0).
\]

If \(\phi(x, y)\) does not vanish identically on any \(B_{r}\), for instance if \(H_{2k}(x, y) = (x^2 + xy + y^2)^k\), then the corresponding system \([5]\) has a unique limit cycle. In general, it is not necessary to assume the positiveness of all of the homogeneous functions \(H_{2k}(x, y)\), as the following example shows. Let us set \(Q(x, y) = x^2 + xy + y^2\). Then take

\[
\phi(x, y) = -1 + Q - Q^2 + Q^3.
\]

One has

\[
\nu = x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} = 2Q - 4Q^2 + 6Q^3 = Q(2 - 4Q + 6Q^2).
\]

The discriminant of the quadratic polynomial \(2 - 4Q + 6Q^2\) is \(\Delta = -32 < 0\) hence \(\nu > 0\) everywhere but at the origin. Moreover, \(\phi(x, y)\) does not vanish identically on any circumference, hence the corresponding system \([5]\) has a unique limit cycle.

References

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