ALGEBRAIC ANABELIAN FUNCTORS

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Abstract. In this paper we will prove that there exists a covariant functor, called algebraic anabelian functor, from the category of algebraic schemes over a given field to the category of outer homomorphism sets of groups. The algebraic anabelian functor, given in a canonical manner, is full and faithful. It reformulates the anabelian geometry over a field. As an application of the anabelian functor, we will also give a proof of the section conjecture of Grothendieck for the case of algebraic schemes.

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Introduction

In this paper we will prove that there exists an algebraic anabelian functor, a covariant functor, given in a canonical manner, from the category of algebraic schemes over a given field to the category of outer homomorphism sets of groups.

Fortunately, the algebraic anabelian functor is full and faithful. In deed, it reformulates the anabelian geometry over a field in the sense

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of Grothendieck. For detail, see Theorem 1.3, the main theorem of the paper.

As an application of the algebraic anabelian functor, we will give a proof of the section conjecture of Grothendieck for the case of algebraic schemes. See §2 for detail.

We will prove the Main Theorem of the paper in §12 after we make several preparations in §§3-11.

In particular, in §9 we will give the proofs that the arithmetic unramified extension in [6] and the formally unramified extension in [7] are both well-defined. These unramified extensions are used to give the computations of étale fundamental groups for arithmetic schemes and algebraic schemes, respectively.

Note that there exists another anabelian functor, the **arithmetic anabelian functor**, which is a covariant functor defined canonically on the category of arithmetic schemes surjectively over the ring $\mathcal{O}_K$ of algebraic integers of a number field $K$. Such a functor is also full and faithful.

However, the arithmetic anabelian functors, which are related to class field theory, are very different from the algebraic ones. This is due to the fact that their étale fundamental groups are very different. For example, it is clear that

$$\pi_1^{et}(\text{Spec}(\mathbb{Q})) \cong \text{Gal}(\mathbb{Q}/\mathbb{Q})$$

holds (or see Theorem 10.2 for a generalized result). On the other hand, by [5] [9] we have

$$\pi_1^{et}(\text{Spec}(\mathbb{Z})) \cong \text{Gal}(\mathbb{Q}^{\text{un}}/\mathbb{Q}) = \{0\}.$$  

Here, $\mathbb{Q}^{\text{un}}$ (=$\mathbb{Q}$) denotes the (nonabelian) maximal unramified extension of the rational field $\mathbb{Q}$.

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**1. Statement of the Main Theorem**

1.1. **Notation.** Let $K$ be a field. By an **algebraic $K$-variety** we will understand an integral scheme $X$ over $K$ of finite type.

For an integral scheme $Z$, put

- $k(Z) \triangleq \mathcal{O}_{X,\xi}$, the function field of an integral scheme $Z$ with generic point $\xi$;
- $\pi_1^{et}(Z) \triangleq$ the étale fundamental group of $Z$ for a geometric point of $Z$ over a separable closure of the function field $k(Z)$. 
In particular, for a field $L$, set

$$\pi^\text{et}_1(L) \triangleq \pi^\text{et}_1(\text{Spec}(L)).$$

1.2. **Outer homomorphism set.** Let $G, H, \pi_1, \pi_2$ be four groups with homomorphisms $p : G \to \pi_1$ and $q : H \to \pi_2$, respectively.

**Definition 1.1.** The outer homomorphism set from $G$ into $H$ over $\pi_1$ and $\pi_2$ respectively, denoted by $\text{Hom}^{\text{out}}_{\pi_1,\pi_2}(G,H)$, is the set of the maps $\sigma$ from the quotient $\pi_1 \frac{p}{p(G)}$ into the quotient $\pi_2 \frac{q}{q(H)}$, given by a group homomorphism $f : G \to H$ in such a manner:

$$\sigma : \pi_1 \frac{p}{p(G)} \to \pi_2 \frac{q}{q(H)}, x \cdot p(G) \mapsto f(x) \cdot q(H)$$

for any $x \in \pi_1$.

In particular, such a $\sigma$ is said to be bijective if the $f$ above is an isomorphism such that $q(H) = f \circ p(G)$.

**Remark 1.2.** Suppose that $G$ and $H$ are normal subgroups of $\pi_1$ and $\pi_2$, respectively. Then $\text{Hom}^{\text{out}}_{\pi_1,\pi_2}(G,H)$ can be regarded as a subset of $\text{Hom}(\text{Out}(G),\text{Out}(H))$.

Let $\mathbb{P}G$ be the category of group pairs $(G, \pi)$ as objects, with outer homomorphisms between group pairs as morphisms. Here, By a group pair $(G, \pi)$ we understand two groups $G$ and $\pi$ together with a given homomorphism $p : G \to \pi$ of groups. By an outer homomorphism from $(G, \pi_1)$ into $(H, \pi_2)$ we understand a map

$$\sigma : \pi_1 \frac{p}{p(G)} \to \pi_2 \frac{q}{q(H)}$$

given in Definition 1.1.

$\mathbb{P}G$ will be called the category of outer homomorphism sets of groups in the paper.

1.3. **Statement of the main theorem.** Fixed a field $K$. Here $K$ is not necessarily of characteristic zero. For any algebraic $K$-variety $X$, there canonically exists a group pair $(\pi^\text{et}_1(X), \pi^\text{et}_1(k(X)))$.

Let $\mathcal{S}\mathcal{C}\mathcal{H}(K)$ denote the category of algebraic $K$-varieties as objects, with scheme morphisms as morphisms satisfying the condition:

For any $X, Y \in \mathcal{S}\mathcal{C}\mathcal{H}(K)$, a scheme morphism $f : X \to Y$ is said to be contained in the category $\mathcal{S}\mathcal{C}\mathcal{H}(K)$ if $k(X)$ is separably generated over $k(Y)$ canonically.

Here is the main theorem of the present paper.
Theorem 1.3. (Main Theorem) For any field $K$, there is a covariant functor $\tau$ from category $\mathcal{S}/\text{CW}(K)$ to category $\mathcal{P}G$ given in a canonical manner:

- An algebraic $K$-variety $X \in \mathcal{S}/\text{CW}(K)$ is mapped into a group pair $(\pi_1^{\text{et}}(X), \pi_1^{\text{et}}(k(X))) \in \mathcal{P}G$;
- A scheme morphism $f \in \text{Hom}(X, Y)$ is mapped into an outer homomorphism $\tau(f) \in \text{Hom}^{\text{out}}_{\pi_1^{\text{et}}(k(X)), \pi_1^{\text{et}}(k(Y))}(\pi_1^{\text{et}}(X), \pi_1^{\text{et}}(Y))$ given by $f$.

Furthermore, $\tau$ is full and faithful.

In particular, an $f \in \text{Hom}(X, Y)$ is an isomorphism if and only if $X$ and $Y$ have a common sp-completion and $\tau(f)$ is a bijective outer homomorphism.

Here, for sp-completion, see [9] or see below §8.2 in the present paper. Roughly speaking, an sp-completion of an integral scheme is such a one that contains all the separably closed points.

We will prove Theorem 1.3 in §12 after we make preparations in §§3-11.

Remark 1.4. The functor $\tau(K)$ is said to be the anabelian functor over a field $K$, or algebraic anabelian functor. The Main Theorem above says that the anabelian functor over a field reformulates the anabelian geometry in the sense of Grothendieck. In deed, it will also give an answer to the section conjecture of Grothendieck for the case of algebraic schemes (see §2).

Remark 1.5. There exists an arithmetic anabelian functor, which is the anabelian functor over the ring $\mathcal{O}_K$ of algebraic integers of a number field $K$. Such a functor is also full and faithful. However, the arithmetic anabelian functors, which are related to class field theory, are very different from the algebraic ones. This is due to the fact that their étale fundamental groups are very different.

Remark 1.6. Let $X$ and $Y$ be two integral $K$-varieties. It is seen that the group pairs $(\pi_1^{\text{et}}(X), \pi_1^{\text{et}}(k(X)))$ and $(\pi_1^{\text{et}}(Y), \pi_1^{\text{et}}(k(Y)))$ are indeed the ramified groups $\pi_1^{\text{br}}(X)$ and $\pi_1^{\text{et}}(Y)$, respectively. Hence, the outer homomorphism set

$$\text{Hom}^{\text{out}}_{\pi_1^{\text{et}}(k(X)), \pi_1^{\text{et}}(k(Y))}(\pi_1^{\text{et}}(X), \pi_1^{\text{et}}(Y))$$

is exactly equal to the set

$$\text{Hom}(\pi_1^{\text{br}}(X), \pi_1^{\text{br}}(Y))$$

of homomorphisms between the ramified groups. For ramified groups, see §10-11 below in the paper.
2. Application to the Section Conjecture

2.1. Algebraic anabelian functor: Special case. Let $\text{Sch}(K)_0$ be the category of algebraic $K$-varieties as objects, together with scheme morphisms as morphisms satisfying the condition:

For any $X, Y \in \text{Sch}(K)_0$, a morphism $f : X \to Y$ of schemes is said to be contained in the category $\text{Sch}(K)_0$ if $X$ and $Y$ have a common sp-completion and $k(X)$ is separable over $k(Y)$ canonically.

Here is a result on algebraic anabelian functor on the category $\text{Sch}(K)_0$.

**Theorem 2.1.** For any field $K$, there exists a covariant functor $\tau_0$ from category $\text{Sch}(K)_0$ to category $\text{PG}$ given in a canonical manner:

- An algebraic $K$-variety $X \in \text{Sch}(K)_0$ is mapped into a group pair $(\pi^\text{et}_1(X), \pi^\text{et}_1(k(X))) \in \text{PG}$;
- A scheme morphism $f \in \text{Hom}(X, Y)$ is mapped into an outer homomorphism $\tau_0(f) \in \text{Hom}^\text{out}_{\pi^\text{et}_1(k(X)), \pi^\text{et}_1(k(Y))}((\pi^\text{et}_1(X), \pi^\text{et}_1(Y))$ given by $f$.

Furthermore, $\tau_0$ is full and faithful.

In particular, an $f \in \text{Hom}(X, Y)$ is an isomorphism if and only if the outer homomorphism $\tau_0(f)$ is bijective.

**Proof.** It is immediate from Theorem 1.3 and Lemma 12.2. \qed

2.2. Section conjecture for algebraic schemes. Fixed a field $K$. There are the following results on anabelian geometry of algebraic schemes over $K$.

**Theorem 2.2.** Let $X$ and $Y$ be two algebraic $K$-varieties. Suppose that $k(X)$ is canonically separable over $k(Y)$ and $X, Y$ have a common sp-completion. Then there is a bijection

$$\text{Hom}(X, Y) \cong \text{Hom}^\text{out}_{\pi^\text{et}_1(k(X)), \pi^\text{et}_1(k(Y))}((\pi^\text{et}_1(X), \pi^\text{et}_1(Y))$$

between sets.

**Proof.** It is immediate from Theorem 2.1. \qed

**Theorem 2.3.** Let $X$ be an algebraic $K$-variety. Suppose that $k(X)$ is separably generated over $K$. Then there is a bijection

$$\Gamma(X/K) \cong \text{Hom}^\text{out}_{\pi^\text{et}_1(K), \pi^\text{et}_1(k(X))}((\pi^\text{et}_1(K), \pi^\text{et}_1(X))$$

between sets.

**Proof.** It is immediate from Theorem 1.3 and Lemma 12.4. \qed
2.3. An interpretation given by Galois groups. For a field \( L \), set the following symbols

- \( G(L) \triangleq \text{the absolute Galois group } \text{Gal}(L^{\text{sep}}/L) \);
- \( G(L)^{\text{au}} \triangleq \text{the Galois group } \text{Gal}(L^{\text{au}}/L) \) of the maximal formally unramified extension \( L^{\text{au}} \) of \( L \) (see Definition 9.7 below).

For Theorems 2.2-3, there is the following version of Galois groups of fields.

Theorem 2.4. Let \( X \) and \( Y \) be two algebraic \( K \)-varieties. Suppose that \( k(X) \) is canonically separable over \( k(Y) \) and \( X, Y \) have a common sp-completion. Then there is a bijection

\[
\text{Hom}(X, Y) \cong \text{Hom}_{G(k(X)), G(k(Y))}^{\text{out}}(G(k(X))^{\text{au}}, G(k(Y))^{\text{au}})
\]

between sets.

Theorem 2.5. Let \( X \) be an algebraic \( K \)-variety. Suppose that \( k(X) \) is separably generated over \( K \). Then there is a bijection

\[
\Gamma(X/K) \cong \text{Hom}_{G(k(X)), G(k(X))}^{\text{out}}(G(k(X))^{\text{au}}, G(k(X))^{\text{au}})
\]

between sets.

It is seen that Theorems 2.4-5 hold from Theorems 2.2-3 above.

3. Basic Definitions

Let’s fix notation and terminology in the present paper. They will be used in the following sections.

3.1. Convention. For an integral domain \( D \), let \( Fr(D) \) denote the field of fractions of \( D \).

In particular, let \( D \) be contained in a field \( \Omega \). In the paper \( Fr(D) \) will always be assumed to be contained in \( \Omega \).

For a field \( L \), set

- \( L^{\text{sep}} \triangleq \text{the separable closure of } L \);
- \( L^{\text{al}} \) (or \( \overline{L} \)) \( \triangleq \) an algebraic closure of \( L \).

By an integral \( K \)-variety \( X \) in the paper we will understand an integral scheme over a field \( K \) (not necessarily of finite type).

3.2. Quasi-galois extension of a function field. Assume that \( L \) is an extension over a field \( K \). Let \( \text{Gal}(L/K) \) be the Galois group of \( L \) over \( K \). Note that here \( L \) is not necessarily algebraic over \( K \).

Definition 3.1. The field \( L \) is said to be \textbf{Galois} over \( K \) if \( K \) is the invariant subfield of \( \text{Gal}(L/K) \).
For example, $\mathbb{Q}(t)$ is Galois over $\mathbb{Q}$. Here, $t$ is a variable over $\mathbb{Q}$.

Now we extend the notion of quasi-galois from algebraic extensions to function fields.

**Definition 3.2.** The field $L$ is said to be **quasi-galois** over $K$ if each irreducible polynomial $f(X) \in F[X]$ that has a root in $L$ factors completely in $L[X]$ into linear factors for any subfield $F$ with $K \subseteq F \subseteq L$.

Let $D \subseteq D_1 \cap D_2$ be three integral domains. Then $D_1$ is said to be **quasi-galois** over $D$ if the fraction field $Fr(D_1)$ is quasi-galois over $Fr(D)$.

**Definition 3.3.** The ring $D_1$ is said to be a **conjugation** of $D_2$ over $D$ if there is an $F$–isomorphism $\tau : Fr(D_1) \rightarrow Fr(D_2)$ such that $\tau(D_1) = D_2$, where $F \triangleq k(\Delta)$, $k \triangleq Fr(D)$, $\Delta$ is a transcendental basis of the field $Fr(D_1)$ over $k$, and $F$ is contained in $Fr(D_1) \cap Fr(D_2)$.

In such a case, $D_1$ is also said to be a $D$–**conjugation** of $D_2$.

Replacing rings by fields, we have a definition that a field $L_1$ is said to be a **conjugation** of a field $L_2$ over a field $K$, where $K$ is assumed to be contained in the intersection $L_1 \cap L_2$. Note that in such a case, we must have $L_1 = L_2$.

### 3.3. Essentially affine scheme.

Let $X$ be a scheme. As usual, an **affine covering** of $X$ is a family $\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ such that for each $\alpha \in \Delta$, $\phi_\alpha$ is an isomorphism from scheme $(U_\alpha, O_X|_{U_\alpha})$ onto scheme $(Spec(A_\alpha), O_{Spec(A_\alpha)})$, where $A_\alpha$ is a commutative ring with identity.

Each element $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ is called a **local chart**. For the sake of brevity, a local chart $(U_\alpha, \phi_\alpha; A_\alpha)$ will be denoted by $U_\alpha$ or $(U_\alpha, \phi_\alpha)$.

An affine covering $\mathcal{C}_X$ of $(X, O_X)$ is said to be **reduced** if $U_\alpha \neq U_\beta$ holds for any $\alpha \neq \beta$ in $\Delta$.

**Definition 3.4.** An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of $X$ is said to be an **affine patching** of $X$ if $U_\alpha = Spec(A_\alpha)$ and the map $\phi_\alpha$ is the identity map on the underlying space $U_\alpha$ for each $\alpha \in \Delta$.

Let $\mathcal{Comm}$ be the category of commutative rings with identity. For a given field $\Omega$, let $\mathcal{Comm}(\Omega)$ be the category consisting of the subrings of $\Omega$ and their isomorphisms.

**Definition 3.5.** Let $\mathcal{Comm}_0$ be a subcategory of $\mathcal{Comm}$. An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of $X$ is said to be **with values** in $\mathcal{Comm}_0$ if for each $\alpha \in \Delta$ there are $O_X(U_\alpha) = A_\alpha$ and $U_\alpha = Spec(A_\alpha)$, where $A_\alpha$ is a ring contained in $\mathcal{Comm}_0$.

In particular, an affine covering $\mathcal{C}_X$ of $X$ with values in $\mathcal{Comm}(\Omega)$ is said to be **with values** in the field $\Omega$. 
Let $C_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ be a reduced affine covering of $X$ with values in a field $\Omega$. For any $(U_\alpha, \phi_\alpha; A_\alpha), (U_\beta, \phi_\beta; A_\beta) \in C_X$, we say

$$(U_\alpha, \phi_\alpha; A_\alpha) = (U_\beta, \phi_\beta; A_\beta)$$

if and only if

$$U_\alpha = U_\beta, \phi_\alpha = \phi_\beta.$$

That is, we will always neglect the map $\phi_\alpha$ for a local chart $(U_\alpha, \phi_\alpha; A_\alpha)$ in such a $C_X$.

For brevity, a scheme is said to be essentially affine in $\Omega$ if it has a reduced affine covering with values in $\Omega$.

It will be seen that essentially affine schemes have many properties like affine schemes.

3.4. Essentially equal scheme. By affine covering with values in a field, it is seen that affine open sets in a scheme is measurable and the non-affine open sets are unmeasurable. So we can neglect the non-affine open sets in an evident manner, where almost every property of the scheme is preserved.

Now suppose that there are two structure sheaves $\mathcal{O}_X$ and $\mathcal{O}'_X$ on the underlying space of an integral scheme $X$.

**Definition 3.6.** The two integral schemes $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}'_X)$ are said to be essentially equal provided that for any open set $U$ in $X$, there is an equivalence relation

$$U \text{ is affine open in } (X, \mathcal{O}_X) \iff \text{ so is } U \text{ in } (X, \mathcal{O}'_X)$$

and in such a case, either $D_1 = D_2$ holds or the two conditions below are both satisfied$^1$.

- $Fr(D_1) = Fr(D_2)$.
- For any nonzero $x \in Fr(D_1)$, there is a relation

  $$x \in D_1 \bigcap D_2$$

  or there is an equivalence relation

  $$x \in D_1 \setminus D_2 \iff x^{-1} \in D_2 \setminus D_1.$$

Here, $D_1 = \mathcal{O}_X(U)$ and $D_2 = \mathcal{O}'_X(U)$.

For example, consider the discrete valuation ring

$$\mathbb{Z}(p) = \left\{ \frac{r}{s} : r, s \in \mathbb{Z}, s \neq 0, (p, s) = 1 \right\}$$

$^1$By such additional conditions, we have a sufficiently large number of integral schemes so that we can give a computation of étale fundamental groups.
of $\text{Spec}(\mathbb{Q})$ for a prime $p$. It is clear that $\text{Spec}(\mathbb{Z}_{(3)})$ and $\text{Spec}(\mathbb{Z}_{(5)})$ are not essentially equal.

In deed, let $\dim X = 1$. Suppose that the integral schemes $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}_X')$ are essentially equal. Then $\mathcal{O}_X(U)$ and $\mathcal{O}_X'(U)$ have the same discrete valuation for an affine open set $U$ in $X$.

**Definition 3.7.** Any two schemes $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ are said to be **essentially equal** if the underlying spaces of $X$ and $Z$ coincide with each other and the schemes $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}_Z)$ are essentially equal.

It is seen that scheme that are essentially equal must be isomorphic.

### 3.5. Quasi-galois closed affine covering

Assume that $f : X \to Y$ is a surjective morphism between integral schemes. Fixed an algebraic closure $\Omega$ of the function field $k(X)$.

**Definition 3.8.** A reduced affine covering $\mathcal{C}_X$ of $X$ with values in $\Omega$ is said to be **quasi-galois closed** over $Y$ by $f$ if there exists a local chart $(U'_\alpha, \phi'_\alpha; A'_\alpha) \in \mathcal{C}_X$ such that $U'_\alpha \subseteq \varphi^{-1}(V_\alpha)$ holds

- for any affine open set $V_\alpha$ in $Y$;
- for any $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ with $U_\alpha \subseteq f^{-1}(V_\alpha)$;
- for any conjugate $A'_\alpha$ of $A_\alpha$ over $B_\alpha$,

where $B_\alpha$ is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in $k(X)$ via $f$.

### 3.6. Quasi-galois closed scheme

Let $X$ and $Y$ be integral schemes. Suppose that $f : X \to Y$ is a surjective morphism. Denote by $\text{Aut}(X/Y)$ the group of automorphisms of $X$ over $Y$.

An integral scheme $Z$ is said to be a **conjugate** of $X$ over $Y$ if there is an isomorphism $\sigma : X \to Z$ over $Y$.

**Definition 3.9.** The scheme $X$ is said to be **quasi-galois closed** (or $\text{qc}$ for short) over $Y$ by $f$ if there is an algebraically closed field $\Omega$ and a reduced affine covering $\mathcal{C}_X$ of $X$ with values in $\Omega$ such that for any conjugate $Z$ of $X$ over $Y$ the two conditions are both satisfied:

- $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ are essentially equal if $Z$ is essentially affine in $\Omega$.
- $\mathcal{C}_Z \subseteq \mathcal{C}_X$ holds if $\mathcal{C}_Z$ is a reduced affine covering of $Z$ with values in $\Omega$.

**Remark 3.10.** In the above definition, the field $\Omega$ enables the affine open subschemes of the integral scheme $X$ to be **measurable** while the other open subschemes of $X$ are still **unmeasurable**. In particular, each scheme $X$ that is $\text{qc}$ over $Y$ must be essentially affine.
Remark 3.11. Let $\Omega$ be the algebraically closed field in Definition 3.9.

(i) By $\Omega$, all the rings of affine open sets in $X$ are taken to be as subrings of the same ring $\Omega$ so that they can be compared with each other.

(ii) By $\Omega$, we can restrict ourselves only to consider the function fields which have the same variables over a given field.

Remark 3.12. It is seen that in Definition 3.9, the affine covering $C_X$ of $X$ is maximal by set inclusion. In fact, $C_X$ is the natural affine structure of $X$ with values in $\Omega$ (see [2] for definition). Conversely, it can be proved that a quasi-galois closed scheme has a unique natural affine structure with values in $\Omega$ (see [2, 8]).

In other words, $\Omega$ can be chosen to be an algebraic closure of the function field $k(X); C_X$ is the unique maximal affine covering of $X$ with values in $\Omega$ (see Remark 5.2 in §5 below).

4. Galois Extensions of Function Fields

4.1. Galois extension of a function field. In this section we will prove the following result about a Galois extension of a function field.

**Theorem 4.1.** Let $L$ be a finitely generated extension of a field $K$. Then $L$ is Galois over $K$ if and only if $L$ is quasi-galois and separably generated over $K$.

Note that here $L$ is not necessarily algebraic over $K$. After several lemmas, we will prove Theorem 4.1 at the end of the present section.

4.2. Recalling preliminary facts on quasi-galois extensions. Let $L$ be a finitely generated extension of a field $K$.

The elements $w_1, w_2, \cdots, w_n \in L$ are said to be a $(r, n)$—nice basis of $L$ over $K$ if the conditions below are satisfied:

- $L = K(w_1, w_2, \cdots, w_n)$;
- $w_1, w_2, \cdots, w_r$ are a transcendental basis of $L$ over $K$;
- $w_{r+1}, w_{r+2}, \cdots, w_n$ are a linear basis of $L$ over $K(w_1, w_2, \cdots, w_r)$.

Here $0 \leq r \leq n$.

Let’s recall preliminary facts on quasi-galois extensions of function fields.

**Lemma 4.2.** ([2, 3]) Fixed an intermediate field $K \subseteq F \subseteq L$ and an $x \in L$ that is algebraic over $F$. Let $z$ be a conjugate of $x$ over $F$. Then there exists a $(s, m)$—nice basis $v_1, v_2, \cdots, v_m$ of $L$ over $F(x)$ and an $F$—isomorphism $\tau$ from the field

$$L = F(x, v_1, v_2, \cdots, v_s, v_{s+1}, \cdots, v_m)$$
onto a field of the form

\[ F(z, v_1, v_2, \cdots, v_s, w_{s+1}, \cdots, w_m) \]

such that

\[ \tau(x) = z, \tau(v_1) = v_1, \cdots, \tau(v_s) = v_s. \]

Here, \( w_{s+1}, w_{s+2}, \cdots, w_m \) are elements contained in an extension of \( F \).

In particular, we have

\[ w_{s+1} = v_{s+1}, w_{s+2} = v_{s+2}, \cdots, w_m = v_m \]

if \( z \) is not contained in \( F(v_1, v_2, \cdots, v_m) \).

**Proof.** It is immediate from preliminary facts on fields. \( \square \)

**Lemma 4.3.** ([2, 3]) The following statements are equivalent.

(i) \( L \) is quasi-galois over \( K \).

(ii) Any conjugation of \( L \) over \( K \) is contained in \( L \).

(iii) There exists one and only one conjugation of \( L \) over \( K \).

(iv) Take any \( x \in L \) and any subfield \( K \subseteq F \subseteq L \). Then \( L \) contains all conjugations of \( F(x) \) over \( F \).

**Proof.** Prove \((i) \implies (iv)\). Fixed an \( x \in L \) and a subfield \( K \subseteq F \subseteq L \). If \( x \) is a variable over \( F \), the field \( F(x) \) must be contained in \( L \) since \( F(x) \) is the unique conjugation of \( F(x) \) over \( F \) by \((i)\).

Let \( x \) be algebraic over \( F \). Then an \( F \)-conjugation of \( F(x) \), which is exactly an \( F \)-conjugate of \( F(x) \), must be contained in \( L \) by \((i)\).

Prove \((iv) \implies (i)\). Fixed any subfield \( K \subseteq F \subseteq L \). Let \( f(X) \) be an irreducible polynomial over \( F \). Take any \( x \in L \) such that \( f(x) = 0 \). It is seen that an \( F \)-conjugation of \( F(x) \) is nothing other than an \( F \)-conjugate. Then every \( F \)-conjugate of \( F(x) \) is contained in \( L \) by \((iv)\). Hence, \( L \) is quasi-galois over \( K \).

Prove \((iv) \implies (ii)\). Let \( H \) be a conjugation of \( L \) over \( K \). Fixed any \( x_0 \in H \). Take a \((r,n)\)-nice basis \( w_1, w_2, \cdots, w_n \) of \( L \) over \( K \) and an isomorphism \( \sigma : H \to L \) over

\[ K_0 \triangleq K(w_1, w_2, \cdots, w_r) \]

such that \( H \) is a \( K \)-conjugation of \( L \) by \( \sigma \).

It is clear that \( w_1, w_2, \cdots, w_r \) are all contained in the intersection of \( H \) and \( L \). It follows that \( x_0 \) must be algebraic over \( K_0 \).

Evidently, the field \( K_0[x_0] \) is a conjugate of the field \( K_0[\sigma(x_0)] \) over \( K_0 \) and then is a conjugation of \( K_0[\sigma(x_0)] \) over \( K \). From \((iv)\) we have \( x_0 \in K_0[x_0] \subseteq L \). Hence, \( H \subseteq L \).

Prove \((ii) \implies (iv)\). Take any \( x \in L \) and any subfield \( K \subseteq F \subseteq L \).

If \( x \) is a variable over \( F \), the field \( F(x) \) that is the unique conjugation of \( F(x) \) itself over \( F \) must be contained in \( L \) by \((ii)\).
Suppose that $x$ is algebraic over $F$. Let $z$ be an $F$–conjugate of $x$. If $F = L$, we have $z = x \in L$ by (ii).

Now let $F \neq L$. From Lemma 4.2 we have a field of the form

$$F(z, v_1, v_2, \ldots, v_s, w_{s+1}, \ldots, w_m),$$

which is an $F$–conjugation of $L$. As $K \subseteq F$, we must have

$$z \in F(z, v_1, v_2, \ldots, v_s, w_{s+1}, \ldots, w_m) \subseteq L.$$

by (ii) again. Hence, $z \in L$.

Prove (iii) $\implies$ (ii). Trivial.

Prove (i) $\implies$ (iii). Let $L$ be quasi-galois over $K$ and let $H$ be a conjugation of $L$ over $K$. In the following we will prove $H = L$.

In fact, choose a $(s, m)$–nice basis $v_1, v_2, \ldots, v_m$ of $L$ over $K$ and an $F$–isomorphism $\tau$ of $H$ onto $L$ such that via $\tau$ the field $H$ is a conjugate of $L$ over $F$, where

$$F \triangleq k(v_1, v_2, \ldots, v_s).$$

It is seen that $F \subseteq H \subseteq L$ hold by Definition 3.2.

Hypothesize $H \nsubseteq L$. Take any $x_0 \in L \setminus H$. There are two cases.

Case (i). Let $x_0$ be a variable over $H$. We have

$$\dim_K H = \dim_K L = s < \infty$$

since $H$ and $L$ are conjugations over $K$. On the other hand, we have

$$1 + \dim_K H = \dim_K H(x_0) \leq \dim_K L$$

from $x_0 \in L \setminus H$, which will be in contradiction.

Case (ii). Let $x_0$ be algebraic over $H$. It is seen that $x_0$ is algebraic over $F$. We have

$$[H : F] = [L : F] < \infty$$

since $H$ is a conjugate of $L$ over $F$. On the other hand, we have

$$2 + [H : F] \leq [H[x_0] : F] \leq [L : F]$$

from $x_0 \in L \setminus H$, which will be in contradiction.

Then $L \setminus H$ must be empty. Hence, $L = H$.

This completes the proof. \(\square\)

4.3. Proof of Theorem 4.1. Now we give the proof of Theorem 4.1:

Proof. Prove $\iff$. Let $L$ be quasi-galois and separably generated over $K$. If $L$ is algebraic over $K$, it is clear that $L$ is Galois over $K$.

Now suppose that $L$ is a transcendental extension over $K$. It suffices to prove that there exists an automorphism $\sigma_0 \in Gal(L/K)$ such that $K$ is the invariant subfield of $\sigma_0$. 
In fact, fixed any \((r,n)\)–nice basis \(v_1, v_2, \cdots, v_n\) of \(L\) over \(K\). Put
\[
F_0 \triangleq K(v_1, v_2, \cdots, v_r).
\]

Then \(L\) is algebraic over \(F_0\). Here, we have \(r \geq 1\). By Lemma 4.3 it is seen that every conjugation of \(L\) over \(K\) is exactly \(L\) itself. It follows that there is one and only conjugate of \(L\) over \(F_0\). Then \(L\) is a quasi-galois algebraic extension of \(F_0\).

Hence, \(L\) is Galois over \(F_0\) since \(L\) is separable over \(F_0\) from the assumption. Fixed any \(\tau_0 \in \text{Gal}(L/F_0)\) with \(\tau_0 \neq id_{L}\).

Let \(\tau_1\) be an automorphism of \(F_0\) over \(K\) given by
\[
v_1 \mapsto \frac{1}{v_1}, v_2 \mapsto \frac{1}{v_2}, \cdots, v_r \mapsto \frac{1}{v_r}.
\]

Then we have an automorphism \(\sigma_0 \in \text{Gal}(L/K)\) defined by \(\tau_0\) and \(\tau_1\) in such a manner
\[
\frac{f(v_1, v_2, \cdots, v_n)}{g(v_1, v_2, \cdots, v_n)} \in L \\
\mapsto \frac{\tau_0(v_1), \tau_1(v_2), \cdots, \tau_1(v_r), \tau_0(v_{r+1}), \cdots, \tau_0(v_n)}{\tau_0(v_1), \tau_1(v_2), \cdots, \tau_1(v_r), \tau_0(v_{r+1}), \cdots, \tau_0(v_n)} \in L
\]
for any polynomials \(f(X_1, X_2, \cdots, X_n)\) and \(g(X_1, X_2, \cdots, X_n) \neq 0\) over the field \(K\) with \(g(v_1, v_2, \cdots, v_n) \neq 0\).

It is easily seen that we have
\[
g(v_1, v_2, \cdots, v_n) = 0
\]
if and only if
\[
g(v_1, v_2, \cdots, v_r, \tau_0(v_{r+1}), \cdots, \tau_0(v_n)) = 0
\]
if and only if
\[
g(\tau_1(v_1), \tau_1(v_2), \cdots, \tau_1(v_r), \tau_0(v_{r+1}), \cdots, \tau_0(v_n)) = 0.
\]

Hence, \(\sigma_0\) is well-defined.

It is seen that \(K\) is the invariant subfield of the automorphism \(\sigma_0\) of \(L\) over \(K\). Hence, \(K\) is the invariant subfield of the Galois group \(\text{Gal}(L/K)\). This proves that \(L\) is Galois over \(K\).

Prove \(\implies\). Let \(L\) is Galois over \(K\). Fixed any \((r,n)\)–nice basis of \(L\) over \(K\), namely \(v_1, v_2, \cdots, v_n\). Set
\[
F_0 \triangleq K(v_1, v_2, \cdots, v_r).
\]

By Lemma 3.1 it is seen that the field \(F_0\) must be invariant under the Galois group \(\text{Gal}(L/F_0)\). It follows that \(L\) is a Galois algebraic extension over \(F_0\). Hence, \(L\) is separably generated over \(K\).

Let \(H\) be a conjugate of \(L\) over \(F_0\). Then \(H\) is a conjugation of \(L\) over \(K\). As \(L\) is Galois over \(F_0\), we have \(H = L\); hence, there exists
one and only one conjugation of $L$ over $K$ under a $(r, n)$−nice basis of $L$ over $K$.

Now let

$$v_1, v_2, \ldots, v_n$$

run through all possible $(r, n)$−nice bases of $L$ over $K$. It is seen that there exists one and only one conjugation of $L$ over $K$. From Lemma 4.3 it is immediate that $L$ is quasi-galois over $K$. □

5. Key Property of qc Scheme

In this section we will prove a key property of a qc scheme and then give a criterion for such a scheme. By these results, we will obtain an essential and sufficient condition for a qc scheme.

5.1. Key property of a qc scheme. Let $X$ and $Y$ be integral schemes and let $f : X \to Y$ be a surjective morphism. We have the following key property for qc schemes.

**Lemma 5.1.** Let $\Omega$ be an algebraic closure of the function field $k(X)$. Suppose that $X$ is qc over $Y$ by $f$. Then there is a unique maximal affine covering $C_X$ of $X$ with values in $\Omega$; moreover, $C_X$ is quasi-galois closed over $Y$ by $f$.

**Proof.** From Definition 3.9 we have an algebraically closed field $\Omega'$ and a reduced affine covering $C'_X$ of $X$ with values in $\Omega'$ such that for any conjugate $Z$ of $X$ over $Y$ the two conditions are satisfied:

- $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ are essentially equal if $Z$ has a reduced affine covering with values in $\Omega'$.
- $C_Z \subseteq C'_X$ holds if $C_Z$ is a reduced affine covering of $Z$ with values in $\Omega'$.

It is clear that $\Omega'$ contains the function field $k(X)$ and $C'_X$ is maximal by set inclusion.

Prove the uniqueness of $C'_X$. In deed, let $C''_X$ be another reduced affine covering of $X$ with values in $\Omega'$ satisfying the two conditions above. Then we must have $C''_X = C'_X$ according to the second condition.

Prove that $C'_X$ is quasi-galois closed over $Y$ by $f$. In fact, take

- an affine open set $V_\alpha$ in $Y$;
- a $(U_\alpha, \phi_\alpha; A_\alpha) \in C'_X$ with $U_\alpha \subseteq f^{-1}(V_\alpha)$;
- a conjugate $A'_\alpha$ of $A_\alpha$ over $B_\alpha$,

where $B_\alpha$ is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in $k(X)$ via $f$.

Then there must exist a local chart $(U'_\alpha, \phi'_\alpha; A'_\alpha) \in C'_X$ such that $U'_\alpha \subseteq \varphi^{-1}(V_\alpha)$ holds; otherwise, if there is some $(U'_\alpha, \phi'_\alpha; A'_\alpha) \notin C'_X$, we
will obtain a new reduced affine covering
\[
\{(U'_\alpha, \phi'_\alpha; A'_\alpha)\} \bigcup C'_X
\]
of \(X\), which is in contradiction to the uniqueness of \(C'_X\).

It follows that \(C'_X\) is with values in an algebraic closure \(\Omega\) of the function field \(k(X)\) from Definition 3.3. \(\square\)

By Lemma 5.1 we have the following remarks.

**Remark 5.2.** Let \(C_X\) and \(\Omega\) be assumed as in Definition 3.9. There are the following statements.

- The field \(\Omega\) can be chosen to be an algebraic closure of the function field \(k(X)\).
- The affine covering \(C_X\) is quasi-galois closed over \(Y\) by \(f\) and is the unique maximal affine covering of \(X\) with values in \(\Omega\).

**Remark 5.3.** The affine covering \(C_X\) above is the unique maximal affine structure of \(X\) with values in the algebraic closure \(\Omega\) of \(k(X)\). In \([2]\), we use affine structures to get the key property as in Lemma 5.1.

### 5.2. Criterion for qc schemes

Fixed integral schemes \(X\) and \(Y\). Let \(\Omega\) be an algebraically closed closure of the function field \(k(X)\). In \([2]\), we use affine structures to get the key property as in Lemma 5.1.

**Lemma 5.4.** Let \(f : X \rightarrow Y\) be a surjective morphism of schemes. Then \(X\) is qc over \(Y\) by \(f\) if there is a unique maximal reduced affine covering \(C_X\) of \(X\) with values in \(\Omega\) such that \(C_X\) is quasi-galois closed over \(Y\) by \(f\).

**Proof.** Assume that \(X\) has a unique maximal reduced affine covering \(C_X\) with values in \(\Omega\) that is quasi-galois closed over \(Y\) by \(f\).

Fixed any conjugate \(Z\) of \(X\) over \(Y\) and any isomorphism \(\sigma : Z \rightarrow X\) over \(Y\). Suppose that \(Z\) has a reduced affine covering \(C_Z\) with values in \(\Omega\).

Take a local chart \((W, \delta, C) \in C_Z\). Put
\[
U = \sigma(W); \quad A = O_X(U); \quad C = O_Z(W).
\]

We have
\[
U = \text{Spec}(A); \quad W = \text{Spec}(C); \quad A \subseteq \Omega; \quad C \subseteq \Omega.
\]

It is seen that there exists an affine open subset \(U'\) in \(X\) such that
\[
C' = O_X(U')
\]
by the assumption that \(C_X\) is quasi-galois closed over \(Y\). As
\[
U' = \text{Spec}(C) = W,
\]
we have

\[ W = \sigma^{-1}(U) = U' \subseteq X. \]

This proves \( Z \subseteq X. \)

On the other hand, all such local charts \((W, \delta, C)\) with \( \text{Spec}(C) = W \) constitute a reduced affine covering \( C'_Z \) of \( Z \) with values in \( \Omega \) that is unique, maximal, and quasi-galois closed over \( Y \) by \( f \circ \delta \).

In a similar manner, we have \( X \subseteq Z \).

Hence, \( Z = X \). It is seen that \((X, \mathcal{O}_X)\) and \((Z, \mathcal{O}_Z)\) are essentially equal. \( \square \)

**Lemma 5.5.** (c.f. [9]) Let \( f : X \rightarrow Y \) be a surjective morphism. Then \( X \) is qc over \( Y \) if there is a unique maximal affine patching \( C_X \) of \( X \) with values in \( \Omega \) satisfying the condition:

\[ A_{\alpha} \] has one and only one conjugate over \( B_{\alpha} \) for any \( (U_{\alpha}, \phi_{\alpha}; A_{\alpha}) \in C_X \) and for any affine open set \( V_{\alpha} \) in \( Y \) with \( U_{\alpha} \subseteq f^{-1}(V_{\alpha}) \), where \( B_{\alpha} \) is the canonical image of \( \mathcal{O}_Y(V_{\alpha}) \) in \( k(X) \).

**Proof.** It is immediate from Lemma 5.4. \( \square \)

### 5.3. Equivalent condition.

Now we give an essential and sufficient condition for qc schemes.

**Theorem 5.6.** Let \( f : X \rightarrow Y \) be a surjective morphism of schemes. The following statements are equivalent:

- The scheme \( X \) is qc over \( Y \) by \( f \).
- There is a unique maximal affine covering \( C_X \) of \( X \) with values in \( \Omega \) such that \( C_X \) is quasi-galois closed over \( Y \) by \( f \).
- There is a unique maximal affine patching \( C_X \) of \( X \) with values in \( \Omega \) such that \( C_X \) is quasi-galois closed over \( Y \) by \( f \).

**Proof.** It is immediate from Lemmas 5.1, 5.4-5 above and Lemma 5.7 below. \( \square \)

We also need the following lemma to prove the third statement in Theorem 5.6.

**Lemma 5.7.** Let \( Z \) be an integral scheme and let \( C_X = \{(U_{\alpha}, \phi_{\alpha}; A_{\alpha})\} \) be an affine covering of \( X \) with values in \( \Omega \). Then \( C_X \) induces an affine patching \( D_X = \{(U_{\alpha}, id_{\alpha}; \sigma_{\alpha}(A_{\alpha}))\} \) of \( X \) with values in \( \Omega \) given in a natural manner:

\[(U_{\alpha}, \phi_{\alpha}; A_{\alpha}) \in C_X \mapsto (U_{\alpha}, id_{\alpha}; \sigma_{\alpha}(A_{\alpha})) \in D_X\]

where \( \sigma_{\alpha} : A_{\alpha} \rightarrow A_{\alpha} \)
is a ring isomorphism induced from the homeomorphism
\[ \phi_{\alpha}^{-1} : \text{Spec}(A_{\alpha}) \to U_{\alpha}. \]

Proof. It is immediate from Definitions 3.4-5. \qed

6. Universal Construction for qc Schemes

In this section we will give a universal construction for a qc scheme over a given integral scheme. That is, we will prove the existence of a qc cover of an integral scheme.

6.1. A preliminary lemma. Let \( X \) be an integral scheme that is not essentially affine in \( k(X) \). By the lemma below we can change \( X \) into a scheme \( Z \) that is essentially affine in \( k(Z) \). That is, any integral scheme is isomorphic to an essentially affine scheme.

Lemma 6.1. For any integral scheme \( X \), there is an integral scheme \( Z \) satisfying the properties:

\begin{itemize}
  \item \( k(X) = k(Z) \);
  \item \( X \cong Z \) are isomorphic schemes;
  \item \( Z \) is essentially affine in the field \( k(Z)^{\text{al}} \).
\end{itemize}

Proof. Let \( \Omega = k(X)^{\text{al}} \) and let \( \xi \) be the generic point of \( X \). We have \( k(X) \triangleq O_{X,\xi} \). For any open set \( U \) of \( X \), there is the following canonical embedding
\[ i_{U} : O_{X}(U) \to k(X). \]

Now take an affine covering \( C_{X} = \{ (U_{\alpha}, \phi_{\alpha}; A_{\alpha}) \}_{\alpha \in \Delta} \) of \( X \). Fixed an \( \alpha \in \Delta \). We have ring isomorphisms
\[ \phi^{\sharp}_{\alpha} : A_{\alpha} \to O_{X}(U_{\alpha}); \]
\[ i_{U_{\alpha}} : O_{X}(U_{\alpha}) \to B_{\alpha} \subseteq k(X). \]

The isomorphism
\[ t_{\alpha} \triangleq i_{U_{\alpha}} \circ \phi^{\sharp}_{\alpha} : A_{\alpha} \to B_{\alpha} \]
between rings induces an isomorphism
\[ \tau_{\alpha} : (\text{Spec}(A_{\alpha}), O_{\text{Spec}(A_{\alpha})}) \to (\text{Spec}(B_{\alpha}), O_{\text{Spec}(B_{\alpha})}) \]
between schemes.

Then we obtain a new scheme \((X, O'_{X})\), namely \( Z \), by gluing these schemes \((U_{\alpha}, O_{X\mid U_{\alpha}})\) along
\[ \psi_{\alpha} \triangleq \tau_{\alpha} \circ \phi_{\alpha} \]
for \( \alpha \in \Delta \).

It is seen that \( Z \) has the desired properties. \qed
6.2. **A universal construction for qc covers.** Fixed a field \( K \) (not necessarily of characteristic zero). Let \( Y \) be an integral \( K \)-variety. Suppose that \( Y \) is essentially affine in an algebraic closure of \( M \triangleq k(Y) \).

Fixed an extension \( L \) of \( M \) such that \( L \) is Galois over \( M \). Note that here \( L \) is not necessarily finitely generated over \( M \).

**Lemma 6.2.** There exists an integral \( K \)-variety \( X \) and a surjective morphism \( f : X \to Y \) such that
- \( L = k(X) \);
- \( f \) is affine;
- \( X \) is qc over \( Y \) by \( f \);
- \( X \) is essentially affine in \( L^{al} \).

**Proof.** (Universal Construction for qc Covers) Here, we repeat this construction developed in [4]. We will proceed in several steps.

**Step 1.** Take algebraic closures \( \Omega_M \) of \( M \) and \( \Omega_L \) of \( L \), respectively. Suppose \( \Omega_M \subseteq \Omega_L \).

Let \( \Delta_1 \) be a transcendental basis of \( L \) over \( M \) and let \( \Delta_2 \) be a linear basis of \( L \) as a vector space over \( M \). Put

\[
\Delta = \Delta_1 \cup \Delta_2.
\]

Fixed a reduced affine coverings \( \mathcal{C}_Y \) of \( Y \) with values in \( \Omega_M \) from the assumption that \( Y \) is essentially affine in \( \Omega_M \). Suppose that \( \mathcal{C}_Y \) is maximal (by set inclusion).

**Step 2.** Take a local chart \((V, \psi_V, B_V) \in \mathcal{C}_Y \). It is seen that \( V \) is an affine open subset of \( Y \) and we have

\[
Fr(B_V) = M; \quad O_Y(V) = B_V \subseteq \Omega_M.
\]

Define

\[
A_V \triangleq B_V [\Delta_V],
\]

i.e., the subring of \( L \) generated over \( B_V \) by the set

\[
\Delta_V \triangleq \{ \sigma(w) \in L : \sigma \in Gal(L/M), w \in \Delta \}.
\]

Then \( Fr(A_V) = L \) holds. It is seen that \( B_V \) is exactly the invariant subring of the natural action of the Galois group \( Gal(L/M) \) on \( A_V \).

Set

\[
i_V : B_V \to A_V
\]

to be the inclusion.

**Step 3.** Define the disjoint union

\[
\Sigma = \coprod_{(V, \psi_V, B_V) \in \mathcal{C}_Y} Spec(A_V).
\]
Then $\Sigma$ is a topological space, where the topology $\tau_\Sigma$ on $\Sigma$ is naturally determined by the Zariski topologies on all $\text{Spec}(A_V)$.

Let

$$\pi_Y : \Sigma \to Y$$

be the projection.

**Step 4.** Given an equivalence relation $R_\Sigma$ in $\Sigma$ in such a manner:

For any $x_1, x_2 \in \Sigma$, we say

$$x_1 \sim x_2$$

if and only if

$$j_{x_1} = j_{x_2}$$

holds in $L$.

Here, $j_x$ denotes the corresponding prime ideal of $A_V$ to a point $x \in \text{Spec}(A_V)$ (see [12]).

Let

$$X = \Sigma / \sim$$

and let

$$\pi_X : \Sigma \to X$$

be the projection.

It is seen that $X$ is a topological space as a quotient of $\Sigma$.

**Step 5.** Define a map

$$f : X \to Y$$

by

$$\pi_X(z) \mapsto \pi_Y(z)$$

for each $z \in \Sigma$.

**Step 6.** Define

$$\mathcal{C}_X = \{(U_V, \varphi_V, A_V) \}_{(V, \psi_V, B_V) \in \mathcal{C}_Y}$$

where $U_V \triangleq \pi_Y^{-1}(V)$ is an open set in $X$ and $\varphi_V : U_V \to \text{Spec}(A_V)$ is the identity map on $U_V$ for each $(V, \psi_V, B_V) \in \mathcal{C}_Y$.

Then we have an integral scheme $(X, \mathcal{O}_X)$ by gluing the affine schemes $\text{Spec}(A_V)$ for all local charts $(V, \psi_V, B_V) \in \mathcal{C}_Y$ with respect to the equivalence relation $R_\Sigma$ (see [12, 15]).

It is seen that $\mathcal{C}_X$ is an affine patching on the scheme $X$ with values in $\Omega_L$.

In particular, $\mathcal{C}_X$ is maximal and quasi-galois closed over $Y$ by $f$.

By Theorem 5.6 it is seen that $X$ and $f$ have the desired property.

This completes the proof. $\square$
6.3. Existence of $qc$ covers. Now we give the existence of $qc$ covers.

**Theorem 6.3.** Fixed an integral $K$-variety $Y$ and a Galois extension $L$ over $k(Y)$. Then there exists an integral $K$-variety $X$ and a surjective morphism $f : X \to Y$ such that

- $L = k(X)$;
- $f$ is affine;
- $X$ is a $qc$ over $Y$ by $f$;
- $X$ is essentially affine in $L^{al}$.

Such an integral $K$-variety $X$ with a morphism $f$, denoted by $(X, f)$, is said to be a $qc$ cover of $Y$.

**Proof.** It is immediate from Lemmas 6.1-2. \hfill $\square$

**Lemma 6.4.** Let $X$, $Y$ and $Z$ be integral $K$-varieties such that $X$ and $Z$ are $qc$ over $Y$. Then $X$ and $Z$ are essentially equal if $k(X) = k(Z)$ and $X$ and $Z$ are isomorphic.

**Proof.** It is immediate from Definition 3.9. \hfill $\square$

**Remark 6.5.** Let $X$ and $Z$ both be $qc$ over an integral $K$-variety $Y$. Suppose $k(X) = k(Z)$. In general, it is not true that $X$ and $Z$ are essentially equal.

7. Main Property of $qc$ Schemes

Let $L/K$ be a field extension. The integral schemes $X/Y$ are said to be a geometric model for the extension $L/K$ if there is a group isomorphism $\text{Aut}(X/Y) \cong \text{Gal}(L/K)$ (e.g., see [13, 18, 19]).

In this section we will prove that $qc$ schemes afford such a geometric model for an extension of a function field.

7.1. Function fields of $qc$ schemes. The function fields of $qc$ schemes are quasi-galois.

**Lemma 7.1.** Let $X$ and $Y$ be two integral schemes such that $X$ is $qc$ over $Y$ by a surjective morphism $f$ of finite type. Then the function field $k(X)$ is canonically quasi-galois over the function field $f(Y)$.

**Proof.** For brevity, assume that $H \triangleq k(Y)$ is contained in $L \triangleq k(X)$. Let $M$ be a conjugation of $L$ over $H$.

Fixed an element $w \in M \setminus H$. There is an element $u \in L \setminus H$ and an $H$–isomorphism $\sigma : L \to M$ such that $w = \sigma(u)$.

As $X$ is essentially affine in an algebraic closure $\Omega$ of $L$, we must have some affine open set $U$ in $X$ such that $u$ is contained in the ring

$$A \triangleq \mathcal{O}_X(U) \subseteq \Omega$$
where $U$ is contained in $f^{-1}(V)$ for some affine open set $V$ in $Y$.

Put $B = \sigma(A)$. The ring $B$ is a conjugation of $A$ (canonically) over the ring $O_Y(V)$. By Theorem 5.6 it is seen that $B$ must be contained in $L$ and then the element $w \in B$ is contained in $L$. Hence, we have $M \subseteq L$.

From Lemma 4.3 it is seen that $L$ is quasi-galois over $H$. □

7.2. qc schemes as geometric models. Let $X$ and $Y$ be two integral $K$-varieties and let $f : X \to Y$ be a surjective morphism.

Lemma 7.2. Suppose that $X$ is qc over $Y$ by $f$ and $k(X)$ is canonically Galois over $k(Y)$. Then there is a group isomorphism

$$\text{Aut}(X/Y) \cong \text{Gal}(k(X)/k(Y)) .$$

Proof. In the following we will use the trick developed in [2, 3] to prove the second statement above.

In fact, define a mapping

$$t : \text{Aut}(X/Y) \to \text{Gal}(k(X)/k(Y))$$

by

$$\sigma = (\sigma, \sigma^\sharp) \mapsto t(\sigma) = \langle \sigma, \sigma^{\sharp^{-1}} \rangle$$

where $\langle \sigma, \sigma^{\sharp^{-1}} \rangle$ is the map of $k(X)$ into $k(X)$ given by

$$(U, f) \in O_X(U) \subseteq k(X) \mapsto (\sigma(U), \sigma^{\sharp^{-1}}(f)) \in O_X(\sigma(U)) \subseteq k(X)$$

for any open set $U$ in $X$ and any element $f \in O_X(U)$. Here the function field $k(X)$ is taken canonically as the set of elements of the form $(U, f)$.

It is easily seen that $t$ is well-defined. We will proceed in several steps to prove that $t$ is a group isomorphism.

Step 1. Prove that $t$ is injective. Fixed any $\sigma, \sigma' \in \text{Aut}(X/Y)$ such that $t(\sigma) = t(\sigma')$. We have

$$(\sigma(U), \sigma^{\sharp^{-1}}(f)) = (\sigma'(U), \sigma'^{\sharp^{-1}}(f))$$

for any $(U, f) \in k(X)$. In particular, for any $f \in O_X(U_0)$ we have

$$(\sigma(U_0), \sigma^{\sharp^{-1}}(f)) = (\sigma'(U_0), \sigma'^{\sharp^{-1}}(f))$$

where $U_0$ is an affine open subset of $X$ such that $\sigma(U_0)$ and $\sigma'(U_0)$ are both contained in $\sigma(U) \cap \sigma'(U)$.

It is seen that

$$\sigma|_{U_0} = \sigma'|_{U_0}$$

holds as isomorphisms of schemes. As $U_0$ is dense in $X$, we have

$$\sigma = \sigma|_{U_0} = \sigma'|_{U_0} = \sigma'$$
on the whole of \( X \); then
\[
\sigma(U) = \sigma'(U);
\]
hence,
\[
\sigma = \sigma'.
\]
This proves that \( t \) is an injection.

**Step 2.** Prove that \( t \) is surjective. Fixed any element \( \rho \) of the group \( \text{Gal}(k(X)/k(Y)) \).

As \( k(X) = \{(U_f, f) : f \in \mathcal{O}_X(U_f) \text{ and } U_f \subseteq X \text{ is open}\} \), we have
\[
\rho : (U_f, f) \in k(X) \mapsto (U_{\rho(f)}, \rho(f)) \in k(X),
\]
where \( U_f \) and \( U_{\rho(f)} \) are open sets in \( X \), \( f \) is contained in \( \mathcal{O}_X(U_f) \), and \( \rho(f) \) is contained in \( \mathcal{O}_X(U_{\rho(f)}) \).

It is seen that each element of \( \text{Gal}(k(X)/k(Y)) \) gives a unique element of \( \text{Aut}(X/Y) \).

In fact, fixed any affine open set \( V \) of \( Y \). It is easily seen that for each affine open set \( U \subseteq \phi^{-1}(V) \) there is an affine open set \( U_\rho \) in \( X \) such that \( \rho \) determines an isomorphism \( \lambda_U \) between affine schemes \( (U, \mathcal{O}_X|_U) \) and \( (U_\rho, \mathcal{O}_X|_{U_\rho}) \). Then
\[
\lambda_U|_{U \cap U'} = \lambda_{U'}|_{U \cap U'}
\]
holds as morphisms of schemes for any affine open sets \( U, U' \subseteq \phi^{-1}(V) \).

Glue \( \lambda_U \) along all such affine open subsets \( U \subseteq \phi^{-1}(V) \), where \( V \) runs through all affine open sets in \( Y \). Then we have an automorphism \( \lambda \) of the scheme \( X \) such that \( \lambda|_U = \lambda_U \) for any affine open set \( U \) in \( X \).

It is clear that \( t(\lambda) = \rho \). Hence, \( t \) is a surjection.

This completes the proof. \( \square \)

**Lemma 7.3.** Suppose that \( X \) is qc over \( Y \) by \( f \) and \( k(X) \) is canonically Galois over \( k(Y) \). Then \( f \) is an affine morphism and there is a natural isomorphism
\[
\mathcal{O}_Y \cong f_*(\mathcal{O}_X)^{\text{Aut}(X/Y)}
\]
where \( (\mathcal{O}_X)^{\text{Aut}(X/Y)}(U) \) denotes the invariant subring of \( \mathcal{O}_X(U) \) under the natural action of \( \text{Aut}(X/Y) \) for any open subset \( U \) of \( X \).

**Proof.** Let \( \Omega \) be an algebraic closure of the function field \( k(X) \). By Lemma 6.1 assume that \( Y \) is essentially affine in \( \Omega \) without loss of generality. In particular, suppose \( k(Y) \subseteq k(X) \) for brevity.

It is clear that \( X \) is essentially affine in \( \Omega \). Put
\[
G = \text{Aut}(X/Y).
\]

Fixed a point \( x \in X \) and an affine open set \( U \) in \( X \) with \( x \in U \). Then there must be
\[
\mathcal{O}_X(U)^G = \mathcal{O}_Y(V)
\]
for an affine open set $V$ in $Y$ such that $f(x) \in V$ and $U \subseteq f^{-1}(V)$.

Otherwise, if there is some

$$w \in \mathcal{O}_X(U)^G \setminus \mathcal{O}_Y(V),$$

we have

$$w \not\in k(Y),
\quad w^{-1} \in k(Y)$$

or

$$w \in k(Y),
\quad w^{-1} \not\in k(Y).$$

If $w \in k(Y)$ and $w^{-1} \not\in k(Y)$, it is seen that $w \not\in k(Y)$ holds since we have

$$\sigma(w \cdot w^{-1}) = \sigma(w) \cdot \sigma(w^{-1}) = 1$$

for any $\sigma \in G$.

Hence, for both cases we will have $w \not\in k(Y)$, which will be in contradiction to the fact that

$$k(X)^G = k(Y)$$

holds by Lemma 7.2.

Now consider any open set $U$ in $X$. We have

$$\mathcal{O}_X(U)^G = \mathcal{O}_Y(V)$$

for an open set $V$ in $Y$ such that $U \subseteq f^{-1}(V)$ since $\mathcal{O}_X(U)$ can be regarded as a subring of $\mathcal{O}_X(U_0)$ for an affine open set $U_0 \subseteq U$. This prove that

$$\mathcal{O}_Y = (\mathcal{O}_X)^G$$

holds.

Conversely, take any affine open set $V$ of $Y$. As $f$ is surjective, it is seen that there is an affine open $U \subseteq f^{-1}(V)$ of $X$ such that

$$\mathcal{O}_X(U)^G \supseteq \mathcal{O}_Y(V).$$

Repeating the same procedure above, we can choose $U$ to be such that

$$\mathcal{O}_X(U)^G = \mathcal{O}_Y(V).$$

It follows that

$$U = f^{-1}(V)$$

holds. This proves that $f$ is affine. \qed
7.3. Main property of qc schemes. The qc schemes behave like quasi-galois extensions of fields.

**Theorem 7.4.** Let $X$ and $Y$ be two algebraic $K$-varieties such that $k(X)$ is separably generated over $k(Y)$ canonically. Suppose that $X$ is qc over $Y$ by a surjective morphism $f$ of finite type. Then there are the following statements:

- $f$ is affine.
- $k(X)$ is Galois over $k(Y)$ canonically.
- There is a group isomorphism
  \[ \text{Aut}(X/Y) \cong \text{Gal}(k(X)/k(Y)). \]

In particular, let $\dim X = \dim Y$. Then $X$ is a pseudo-galois cover of $Y$ in the sense of Suslin-Voevodsky.

**Proof.** It is immediate from Theorem 4.1 and Lemmas 7.1-3. \qed

Here for pseudo-galois cover, see [18, 19] for definition and property.

8. sp-Completion

By the graph functor, there is an sp-completion of a given integral scheme, which is an integral scheme of the same length but have the maximal combinatorial graph.

The sp-completion can give a type of completions of rational maps between schemes.

In the present paper, the sp-completion will be applied to definitions of formally unramified extension of fields in §9 and of monodromy actions of automorphism groups in §12.

8.1. The graph functor $\Gamma$ from schemes to graphs. For convenience, in this subsection we will review the graph functor developed in [1, 9]. See [1, 9] for proofs of the results listed below.

Let $X$ be a scheme. Take any points $x, y$ in $X$.

If $y$ is in the (topological) closure $\{x\}$, $y$ is a specialization of $x$ (or, $x$ is a generalization of $y$) in $X$, denoted by $x \rightarrow y$.

Put $Sp(x) = \{y \in X \mid x \rightarrow y\}$. Then $Sp(x) = \{x\}$ is an irreducible closed subset in $X$.

If $x \rightarrow y$ and $y \rightarrow x$ both hold in $X$, $y$ is a generic specialization of $x$ in $X$, denoted by $x \leftrightarrow y$.

$x$ is said to be generic (or initial) in $X$ if we must have $x \leftrightarrow z$ for any $z \in X$ such that $z \rightarrow x$.

$x$ is said to be closed (or final) if we must have $x \leftrightarrow z$ for any $z \in X$ such that $x \rightarrow z$.

We have the following preliminary facts for specializations.
Lemma 8.1. (1, 9) For any points \(x, y \in X\), we have \(x \leftrightarrow y\) in \(X\) if and only if \(x = y\).

Lemma 8.2. (1, 9) Fixed any specialization \(x \to y\) in \(X\). Then there is an affine open subset \(U\) of \(X\) such that the two points \(x\) and \(y\) are both contained in \(U\). In particular, any affine open set in \(X\) containing (the specialization) \(y\) must contain (the generalization) \(x\).

Lemma 8.3. (1, 9) Any morphism between schemes is specialization-preserving. That is, fixed any morphism \(f : X \to Y\) between schemes. Then there is a specialization \(f(x) \to f(y)\) in \(Y\) for any specialization \(x \to y\) in \(X\).

8.2. \(sp\)-completion. Let \(G\) and \(H\) be combinatorial graphs. Recall that an isomorphism \(t\) from \(G\) onto \(H\) is an ordered pair \((t_V, t_E)\) satisfying the conditions:

- \(t_V\) is a bijection from \(V(G)\) onto \(V(H)\);
- \(t_E\) is a bijection from \(E(G)\) onto \(E(H)\);
- Let \(x \in V(G)\) and \(L \in E(G)\). Then \(x\) is incident with \(L\) if and only if \(t_V(x) \in V(H)\) is incident with \(t_E(L) \in E(H)\).
Definition 8.5. An integral scheme $X$ is said to be $sp$-complete if $X$ must be essentially equal to $Y$ for any integral scheme $Y$ such that

- $\Gamma(X)$ is isomorphic to a subgraph of $\Gamma(Y)$;
- $k(Y)$ is contained in a separable closure of $k(X)$.

Remark 8.6. The function field of an $sp$-complete integral scheme must be a separable closure. In other words, there is no other separably closed points that can be added to an $sp$-complete integral scheme.

Now we give the existence of the $sp$-completion of an integral scheme.

Theorem 8.7. For any integral scheme $X$, there is an integral scheme $X_{sp}$ and a surjective morphism $\lambda_X : X_{sp} \to X$ satisfying the following properties:

- $\lambda_X$ is affine;
- $X_{sp}$ is $sp$-complete;
- $X_{sp}$ is essentially affine in $k(X)_{al}$;
- $k(X_{sp})$ is a separable closure of $k(X)$;
- $X_{sp}$ is quasi-galois closed over $X$ by $\lambda_X$.

Such a scheme $X_{sp}$ with a morphism $\lambda_X$, denoted by $(X_{sp}, \lambda_X)$, is said to be an $sp$-completion of $X$. We will denote by $Sp[X]$ the set of all $sp$-completions of an integral scheme $X$.

Proof. (Universal Construction for $sp$-Completion) Here repeat the construction developed in [9].

Let $K = k(X)$ and $L = K^{sep}$. Fixed a transcendental basis $\Delta_1$ of $L$ over $K$ and a linear basis $\Delta_2$ of $L$ as a vector space over $K(\Delta_1)$. Put

$$G = Gal(L/K); \Delta = \Delta_1 \bigcup \Delta_2.$$ 

By Lemma 6.1, without loss of generality, assume that $X$ has a reduced affine covering $\mathcal{C}_X$ with values in $L$. We choose $\mathcal{C}_X$ to be maximal (by set inclusion).

We proceed in several steps such as the following to give the construction:

- Fixed a local chart $(V, \psi_V, B_V) \in \mathcal{C}_X$. Define $A_V = B_V[\Delta_V]$, where $\Delta_V = \left\{ \sigma(x) \in L : \sigma \in G, x \in \Delta \right\}$. Set $i_V : B_V \to A_V$ to be the inclusion.
- Let
  $$\Sigma = \coprod_{(V, \psi_V, B_V) \in \mathcal{C}_X} Spec(A_V)$$ 
  be the disjoint union. Denote by $\pi_X : \Sigma \to X$ the projection induced by the inclusions $i_V$.
- Given an equivalence relation $R_\Sigma$ in $\Sigma$ in such a manner:
For any $x_1, x_2 \in \Sigma$, we say $x_1 \sim x_2$ if and only if $j_{x_1} = j_{x_2}$ holds in $L$. Here $j_x$ denotes the corresponding prime ideal of $A_V$ to a point $x \in \text{Spec}(A_V)$.

Let $X_{sp}$ be the quotient space $\Sigma/\sim$ and let $\pi_{sp} : \Sigma \to X_{sp}$ be the projection of spaces.

- Set a map $\lambda_X : X_{sp} \to X$ of topological spaces by $\pi_{sp}(z) \mapsto \pi_X(z)$ for each $z \in \Sigma$.
- Suppose $C_{X_{sp}} = \{(U_V, \varphi_V, A_V) \} (V, \psi_V, B_V) \in C_X$.

Here $U_V = \pi_{X_{sp}}^{-1}(V)$ and $\varphi_V : U_V \to \text{Spec}(A_V)$ is the identity map for each $(V, \psi_V, B_V) \in C_X$.

- There is a scheme, namely $X_{sp}$, by gluing the affine schemes $\text{Spec}(A_V)$ for all $(U_V, \varphi_V, A_V) \in C_X$ with respect to the equivalence relation $R_{\Sigma}$. Naturally, $\lambda_X$ becomes a morphism of schemes.

It is seen that $X_{sp}$ and $\lambda_X$ are the desired scheme and morphism, respectively.

There is the uniqueness of $sp$-completions such as the following.

**Lemma 8.8.** Fixed any integral $K$-varieties $X$. Then all $sp$-completions of $X$ are essentially equal.

*Proof.* It is seen from Definition 8.5, Remark 8.6, and Theorem 8.7. \qed

**Lemma 8.9.** Fixed any two integral $K$-varieties $X$ and $Y$. Suppose that $k(X)$ and $k(Y)$ have the same separable closure. Then either

$$Sp[X] = Sp[Y]$$

or

$$Sp[X] \cap Sp[Y] = \emptyset$$

holds.

*Proof.* It is immediate from Lemma 8.8. \qed

**Remark 8.10.** An $sp$-completion of an integral scheme is $sp$-complete. By $sp$-completion we can give a completion of rational maps between integral schemes.

**Remark 8.11.** An integral scheme $X$ and its $sp$-completion $X_{sp}$ have the same dimension. However, the $sp$-completion is very complicated and exotic. In general, it is not true that $X_{sp}$ is of finite type over $X$. For example, let $t$ be a variable over $\mathbb{Q}$. It is seen that $\text{Spec}(\mathbb{Q})$ and
$\text{Spec}(\overline{\mathbb{Q}(t)})$ are $sp$-completions of $\text{Spec}(\mathbb{Q})$ and $\text{Spec}(\overline{\mathbb{Q}(t)})$, respectively. Their underlying spaces are very different.

**Remark 8.12.** By Theorem 8.7 it is seen that an $sp$-completion of an integral scheme behaves like a separable closure of a field.

### 9. Unramified Extensions of Function Fields

In this section we will use $sp$-complete schemes to introduce a notion of formally unramified extensions over function fields and then give several preliminary properties. The formally unramified extensions will be applied to the computation of étale fundamental groups.

#### 9.1. Basic lemma

**Fixed a field $K$.** We have the following basic result.

**Lemma 9.1.** Let $X$ and $Y$ be two integral $K$-varieties satisfying the two conditions:

- $\text{Sp}[X] = \text{Sp}[Y]$ are equal sets.
- $\text{Aut}(X_{sp}/X) \cong \text{Aut}(Y_{sp}/Y)$ are isomorphic groups.

Then $X$ and $Y$ are isomorphic schemes.

**Proof.** Fixed any $sp$-completions $(X_{sp}, \lambda_X)$ of $X$ and $(Y_{sp}, \lambda_Y)$ of $Y$, respectively. As $\text{Sp}[X] = \text{Sp}[Y]$, we have an isomorphism $t : X_{sp} \to Y_{sp}$.

Let

$$\sigma : \text{Aut}(X_{sp}/X) \to \text{Aut}(Y_{sp}/Y)$$

be an isomorphism between groups.

Take any point $x_0 \in X$. From Lemmas 7.2-3 we have

$$\lambda_X^{-1}(x_0) = \{ g(x_0) \in X_{sp} : g \in \text{Aut}(X_{sp}/X) \};$$

$$\lambda_Y^{-1}(t(x_0)) = \{ h(t(x_0)) \in Y_{sp} : h \in \text{Aut}(Y_{sp}/Y) \}.$$

It follows that there exists a morphism $f_{sp} : X_{sp} \to Y_{sp}$ given by

$$g(x_0) \mapsto \sigma(g)(t(x_0)).$$

From $f_{sp}$ we obtain a morphism $f : X \to Y$ given by

$$x_0 = \lambda_X(g(x_0)) \mapsto \lambda_Y(\sigma(g)(t(x_0)))$$

satisfying the property

$$f \circ \lambda_X = \lambda_Y \circ f_{sp}.$$

It is easily seen that $f_{sp} : X_{sp} \to Y_{sp}$ is an isomorphism. Hence, $f : X \to Y$ is an isomorphism. \qed
9.2. Formally unramified extensions. Fixed an integral $K$-variety $X$ over a field $K$. Let $L_1$ and $L_2$ be two algebraic extensions over the function field $k(X)$, respectively.

**Definition 9.2.** $L_2$ is said to be a finite $X$-formally unramified Galois extension over $L_1$ if there are two integral $K$-varieties $X_1$ and $X_2$ and a surjective morphism $f : X_2 \to X_1$ such that

- $\text{Sp}[X] = \text{Sp}[X_1] = \text{Sp}[X_2]$;
- $k(X_1) = L_1$, $k(X_2) = L_2$;
- $X_2$ is a finite étale Galois cover of $X_1$ by $f$.

In such a case, $X_2/X_1$ are said to be a $X$-geometric model of the field extension $L_2/L_1$.

**Remark 9.3.** It is seen that such geometric models are unique up to isomorphisms. It follows that the formally unramified extension above is well-defined. In fact, fixed any two geometric models $X_2/X_1$ and $Y_2/Y_1$ for the extension $L_2/L_1$, respectively. By Lemma 9.1 we must have isomorphisms $X_2 \cong Y_2$ and $X_1 \cong Y_1$, respectively. This is due to the preliminary fact that we have

$$\text{Aut}(X_{sp}/X_i) \cong \text{Gal}(k(X)^{sep}/L_i) \cong \text{Aut}(X_{sp}/Y_i)$$

for $i = 1, 2$.

**Remark 9.4.** Suppose that $L_3/L_2$ and $L_2/L_1$ both are $X$-formally unramified extensions. Then $L_3/L_1$ must be $X$-formally unramified.

**Remark 9.5.** Note that even for the case that $L_1$ and $L_2$ are both algebraic extensions of $K$, in general, the formally unramified defined in Definition 9.2 does not coincide with unramified that is defined in algebraic number theory.

**Remark 9.6.** Note that we define another unramified extensions in [5, 6, 9] for arithmetic schemes, which is a generalization of unramified extensions in algebraic number theory and hence is different from the above one defined in Definition 9.2.

**Definition 9.7.** Let $X$ be an integral $K$-variety over a field $K$. Set

$$k(X)^{au} \triangleq \text{the smallest field containing all finite } X\text{-formally unramified subextensions over } L \text{ contained in } L^{al}.$$  

The field $k(X)^{au}$ is said to be the maximal formally unramified extension of the function field $k(X)$.

**Lemma 9.8.** Let $X$ be an integral $K$-variety and let $L \subseteq k(K)^{au}$ be a finite Galois extension of $k(X)$. Then there are the following statements.
(i) $k(X)^{au}$ is an algebraic Galois extension of $k(X)$. In particular, $k(X)^{au}$ is a subfield of $k(X)^{sep}$.

(ii) There is a finite $X$-formally unramified Galois extension $M$ of $k(X)$ such that $M \supseteq L$.

(iii) Let $M$ be a finite $X$-formally unramified Galois extension of $k(X)$ such that $M \supseteq L$. Then so is $M$ over $L$.

(iv) $L$ is a finite $X$-formally unramified Galois extension of $k(X)$.

Proof. (i) It is immediate from preliminary facts on field theory.

(ii) It is clear from the assumption that $L \subseteq k(K)^{au}$ holds.

(iii) Take a geometric model $X_M/X$ for the extension $M/k(X)$. It reduces to the case that $X_M$ and $X$ are both affine schemes.

Suppose

$$X = \text{Spec}(K_0), \ K_0 = K[t_1, t_2, \cdots, t_n];$$

$$L = \text{Fr}(L_0), \ L_0 = K[t_1, t_2, \cdots, t_n, s_1, s_2, \cdots, s_l];$$

$$X_M = \text{Spec}(M_0), \ M_0 = K[t_1, t_2, \cdots, t_n, s_1, s_2, \cdots, s_l, s_{l+1}, \cdots, s_m].$$

Here, the elements

$$t_1, t_2, \cdots, t_n, s_1, s_2, \cdots, s_l, s_{l+1}, \cdots, s_m$$

are all contained in $k(X)^{au}$, and

$$s_1, s_2, \cdots, s_l$$

and

$$s_1, s_2, \cdots, s_l, s_{l+1}, \cdots, s_m$$

are both supposed to contain all conjugates over the field $k(X)$.

It is easily seen that $X_M$ is a finite étale Galois cover of $\text{Spec}(L_0)$ from base change of étale morphisms.

(iv) It reduces to consider affine schemes. Take $K_0, L_0, M_0$ as in (iii) above.

Let $\mathfrak{P}$ be a maximal ideal of $M_0$. Just check what one has done in algebraic number theory. Put

$$\mathfrak{p} = \mathfrak{P} \cap K_0.$$

Then $\mathfrak{p}$ is a maximal ideal of $K_0$.

Conversely, let $\mathfrak{p}$ be a maximal ideal of $K_0$. From definition for étale morphisms, we have one and only one maximal ideal $\mathfrak{P}$ of $M_0$ that is over the maximal ideal $\mathfrak{p}$.

Likewise, there is one and only one maximal ideal $\mathfrak{P}_0$ of $L_0$ such that $\mathfrak{P} | \mathfrak{P}_0$ and $\mathfrak{P}_0 | \mathfrak{p}$ hold.

It follows that $\text{Spec}(L_0)$ must be unramified over $X$ and hence étale over $X$. This completes the proof. □
9.3. **Arithmetic unramified extension.** There is another type of unramified extensions, the arithmetic unramified extensions over the ring $\mathcal{O}_K$ of algebraic integers of a number field $K$.

**Convention.** In this subsection, an integral $\mathbb{Z}$-variety is defined to be an integral scheme surjectively over $\text{Spec}(\mathbb{Z})$; an arithmetic variety is an integral scheme surjectively over $\text{Spec}(\mathbb{Z})$ of finite type.

Likewise, we have the following basic lemma.

**Lemma 9.9.** Let $X$ and $Y$ be two integral $\mathbb{Z}$-varieties satisfying the two conditions:

- $\text{Sp}[X] = \text{Sp}[Y]$ are equal sets.
- $\text{Aut}(X_{sp}/X) \cong \text{Aut}(Y_{sp}/Y)$ are isomorphic groups.

Then $X$ and $Y$ are isomorphic schemes.

Fixed an integral $\mathbb{Z}$-variety $X$ over a field $K$. Let $L_1$ and $L_2$ be two algebraic extensions over the function field $k(X)$, respectively.

**Definition 9.10.** The field $L_2$ is said to be a finite $X$-unramified Galois extension over $L_1$ if there are two integral $\mathbb{Z}$-varieties $X_1$ and $X_2$ and a surjective morphism $f : X_2 \to X_1$ such that

- $\text{Sp}[X] = \text{Sp}[X_1] = \text{Sp}[X_2]$;
- $k(X_1) = L_1$, $k(X_2) = L_2$;
- $X_2$ is a finite étale Galois cover of $X_1$ by $f$.

In such a case, $X_2/X_1$ are said to be a $X$-geometric model of the field extension $L_2/L_1$.

**Remark 9.11.** Let $L_1 \subseteq L_2 \subseteq L_3$ be function fields over a number field $K$. Suppose that $L_2/L_1$ and $L_3/L_2$ are $X$-unramified extensions. Then $L_3$ is $X$-unramified over $L_1$.

**Definition 9.12.** Let $X$ be an integral $\mathbb{Z}$-variety. Set

$$k(X)^{\text{un}} \triangleq \text{the smallest field containing all finite } X\text{-unramified subextensions over } L \text{ contained in } L^{\text{al}}.$$  

The field $k(X)^{\text{un}}$ is said to be the maximal unramified extension of the function field $k(X)$.

**Lemma 9.13.** Let $X$ be an integral $\mathbb{Z}$-variety and let $L \subseteq k(K)^{\text{un}}$ be a finite Galois extension of $k(X)$. Then there are the following statements.

(i) $k(X)^{\text{un}}$ is an algebraic Galois extension of $k(X)$.

(ii) There is a finite $X$-unramified Galois extension $M$ of $k(X)$ such that $M \supseteq L$.

(iii) Let $M$ be a finite $X$-unramified Galois extension of $k(X)$ such that $M \supseteq L$. Then so is $M$ over $L$. 


(iv) \( L \) is a finite \( X \)-unramified Galois extension of \( k(X) \).

Proof. Repeat what we have done in proving Lemma 9.8. \( \square \)

Remark 9.14. It is seen that for the case of an algebraic extension, the unramified extension defined in Definition 9.12 coincides exactly with that in algebraic number theory.

It appears that unramified extensions for an arithmetic variety and for an algebraic \( K \)-variety have some common properties. However, they are very different. For example, we have

\[
k(\text{Spec}(\mathbb{Z})) = \mathbb{Q}^{\text{un}}; \quad k(\text{Spec}(\mathbb{Q})) = \mathbb{Q}^{\text{sep}} = \overline{\mathbb{Q}}.
\]

In particular, for arithmetic varieties, we have a stronger result such as the following.

Theorem 9.15. Let \( X \) and \( Y \) be two arithmetic varieties such that \( k(X) = k(Y) \). Then \( Sp[X] = Sp[Y] \) holds, i.e., \( X \) and \( Y \) have the same sp-completions; moreover, \( X \) and \( Y \) are isomorphic.

Proof. By Lemma 9.9 it suffices to prove that \( X \) and \( Y \) have a common sp-completion. Fixed any sp-completions \( (X_{sp}, \lambda_X) \) of \( X \) and \( (Y_{sp}, \lambda_Y) \) of \( Y \), respectively. It reduces to prove that \( X_{sp} \) and \( Y_{sp} \) are isomorphic. In the following we will proceed in several steps to prove that there exists an isomorphism

\[
f_{sp} : X_{sp} \rightarrow Y_{sp}.
\]

Step 1. We have \( \Omega \triangleq k(X)^{al} = k(Y)^{al} \). As \( X_{sp} \) and \( Y_{sp} \) are both qc over \( \text{Spec}(\mathbb{Z}) \), it is seen that for any affine open set \( U \) in \( X \) there must be an affine open set \( V \) in \( Y \) such that

\[
f_U : U \rightarrow V
\]

is an isomorphism of schemes which is induced from an isomorphism

\[
\sigma_U : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U)
\]

between subrings of \( \Omega \).

By the universal construction in §8.2, we have a morphism

\[
f_{sp} : X_{sp} \rightarrow Y_{sp}
\]

such that there is the restriction

\[
f_{sp}|_U = f_U
\]

to each affine open set \( U \) in \( X \).

Step 2. By Theorems 5.6,8.7, assume that \( \sigma_U \) is an identity map without loss of generality. It is seen that \( f_{sp} \) is an injective morphism.
Step 3. Take any closed point $y_0$ in $Y_{sp}$. Let $B(y_0) \subseteq \Omega$ be a subring such that the affine open set $V(y_0) = \text{Spec}(B(y_0))$ in $Y$ containing the point $y_0$. Denote by $j_{y_0}$ the prime ideal in $B(y_0)$ corresponding to $y_0$.

By Theorem 5.6 it is seen that $B(y_0)$ is a ring over $\mathbb{Z}$ generated by the set $$\{t_1, \ldots, t_n\} \cup \Delta$$ where $t_1, \ldots, t_n$ are variables over $\mathbb{Q}$, $\text{dim } X = n$, and $$\Delta = \overline{\mathbb{Q}(t_1, \ldots, t_n)} \setminus \mathbb{Z}.$$ This is due to the fact that $Y_{sp}$ is qc over $\text{Spec}(\mathbb{Z})$.

It is seen that such a prime ideal $j_{y_0}$ contains a unique prime $\mathfrak{p} \in \mathcal{O}_K$ over a prime $p \in \mathbb{N}$, where $\mathcal{O}_K$ is the ring of the algebraic integers of a number field $K$ and $j_{y_0}$ is a maximal ideal of the ring $B(y_0)$ generated by a set $\Delta_{\mathfrak{p}}$ containing the subset $$\{\mathfrak{p}\} \cup \{t_1, \ldots, t_n\}$$ of $\Delta$.

As $X_{sp}$ is qc over $\text{Spec}(\mathbb{Z})$, we have a point $x_0 \in X$ such that $$j_{x_0} = j_{y_0}.$$ Then we have $f_{sp}(x_0) = y_0$.

This completes the proof. \[\square\]

Remark 9.16. By Theorem 9.15 it is seen that unramified extension, the notion for arithmetic varieties given in [5, 6, 9], as in Definition 9.12, are well-defined.

10. Algebraic Fundamental Groups

In this section we will give the computation of algebraic fundamental groups.

10.1. A universal cover for an étale fundamental group. For an integral $K$-variety $X$, let $k(X)^{au}$ denote the maximal formally unramified extension of the function field $k(X)$.

Lemma 10.1. For any integral $K$-variety $X$, there exists an integral $K$-variety $X_{et}$ and a surjective morphism $p_X : X_{et} \to X$ satisfying the properties:

- $p_X$ is affine;
- $k(X_{et}) = k(X)^{au}$;
- $X_{et}$ is qc over $X$ by $p_X$;
- $k(X_{et})$ is Galois over $k(X)$;
- $X_{et}$ is essentially affine in $k(X)^{au}$. 

Such an integral $K$-variety $X_{et}$ with a morphism $p_X$, denoted by $(X_{et}, p_X)$, is called a **universal cover** over $X$ for the étale fundamental group $\pi^1_{et}(X)$.

**Proof. (Universal Construction for the Cover)** By Lemma 6.1, without loss of generality, assume that $X$ has a reduced affine covering $C_X$ with values in $k(X)^{ad}$. Let $C_X$ be maximal by set inclusion.

We will proceed in several steps:

- **Fixed a set $\Delta$ of generators of the field $k(X)^{au}$ over $k(X)$.**
  - For any local chart $(V, \psi_V, B_V) \in C_X$, define $A_V = B_V[\Delta_V]$, that is, $A_V$ over $B_V$ generated by the set $\Delta_V = \{ \sigma (x) \in k(X)^{au} : \sigma \in \text{Gal}(k(X)^{au}/k(X)), x \in \Delta \}$. Let $i_V : B_V \rightarrow A_V$ be the inclusion.
  - Assume that $\Sigma = \coprod_{(V, \psi_V, B_V) \in C_X} \text{Spec}(A_V)$ is the disjoint union. Let $\pi_X : \Sigma \rightarrow X$ be the projection induced by the inclusions $i_V$.
  - Define an equivalence relation $R_\Sigma$ in $\Sigma$ in such a manner:
    - For any $x_1, x_2 \in \Sigma$, we say $x_1 \sim x_2$ if and only if $j_{x_1} = j_{x_2}$ holds in $L$, where $j_x$ denotes the corresponding prime ideal of $A_V$ to a point $x$ in $\text{Spec}(A_V)$.
    - Let $X_{et}$ be the quotient space $\Sigma/\sim$ and let $\pi_{et} : \Sigma \rightarrow X_{et}$ be the projection of spaces.
  - Set a map $p_X : X_{et} \rightarrow X$ of spaces by $\pi_{et}(z) \mapsto \pi_X(z)$ for each $z \in \Sigma$.
  - Suppose $C_{X_{et}} = \{(U_V, \varphi_V, A_V) \}_{(V, \psi_V, B_V) \in C_X}$. Here $U_V = \pi_X^{-1}(V)$ and $\varphi_V : U_V \rightarrow \text{Spec}(A_V)$ is the identity map for each $(V, \psi_V, B_V) \in C_X$.
  - There is a scheme, namely $X_{et}$, obtained by gluing the affine schemes $\text{Spec}(A_V)$ for all $(U_V, \varphi_V, A_V) \in C_X$ with respect to the equivalence relation $R_\Sigma$. Naturally, $p_X$ becomes a morphism of schemes.

It is seen that $X_{et}$ and $p_X$ satisfy the properties. This completes the proof. \qed
10.2. **A computation of étale fundamental groups.** By Lemma 10.1 we have the following result.

**Theorem 10.2.** For any integral $K$-variety $X$, there exists a group isomorphism

$$\pi_1^{et}(X) \cong \text{Gal}(k(X)_{au}/k(X)).$$

**Proof.** Assume that $X$ has a reduced a reduced affine covering $C_X$ with values in $k(X)_{au}$ without loss of generality.

Let $\Delta \subseteq k(X)_{au} \setminus k(X)$ be a set of generators of the field $k(X)_{au}$ over $k(X)$. Put

$$I = \{\text{finite subsets of } \Delta\}.$$ 

We will proceed in several steps to give the proof.

**Step 1.** Fixed any $\alpha$ in $I$. Repeating the universal construction in §10.1 for $\alpha$, i.e., replacing $\Delta$ by $\alpha$, we have an integral $K$-variety $X_\alpha$ and a surjective morphism $f_\alpha : X_\alpha \to X$ satisfying the properties:

- $f_\alpha$ is affine;
- $k(X_\alpha) \subseteq k(X)_{au}$;
- $X_\alpha$ is qc over $X$ by $f_\alpha$;
- $k(X_\alpha)$ is Galois over $k(X)$;
- $X_\alpha$ is essentially affine in $k(X)_{au}$.

**Step 2.** Let $\alpha \subseteq \beta$ be in $I$. By Step 1 we have integral $K$-varieties $X_\alpha$ and $X_\beta$ which are qc over $X$, respectively. There is a surjective morphism $f_\alpha^\beta : X_\beta \to X_\alpha$ satisfying the properties:

- $f_\alpha^\beta$ is affine;
- $f_\beta = f_\alpha \circ f_\alpha^\beta$;
- $X_\beta$ is qc over $X_\alpha$ by $f_\alpha^\beta$;
- $k(X_\beta)$ is Galois over $k(X_\alpha)$.

Here $f_\alpha^\beta$ is obtained in a canonical manner similar to $f_\alpha$.

It is clear that there is a $\gamma$ in $I$ such that $\gamma \supseteq \alpha$ and $\gamma \supseteq \beta$. Hence, we have an integral $K$-variety $X_\gamma$ that is qc over $X_\beta$ and over $X_\alpha$, respectively.

**Step 3.** For any $\alpha, \beta$ in $I$, we say $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$. Then $I$ is a partially ordered set.

Hence, $\{k(X_\alpha) ; i_\alpha^\beta \} \alpha \in I$ is a direct system of groups, where each

$$i_\alpha^\beta : k(X_\alpha) \to k(X_\beta)$$

is a homomorphism of fields canonically induced by $f_\alpha^\beta$.

Let $(X_{et}, p_X)$ be a universal cover for $\pi_1^{et}(X)$. For the fields, we have

$$k(X_{et}) = k(X)_{au} = \lim_{\alpha \in I} k(X_\alpha).$$
For the Galois groups, we have
\[
Gal \left( k \left( X_{et} \right) / k \left( X \right) \right) \cong \lim_{\leftarrow \alpha \in I} Gal \left( k \left( X_\alpha \right) / k \left( X \right) \right).
\]

Step 4. Let
\[
\left[ X \right]_{au} = \{ X_\alpha : \alpha \in I \}.
\]
Then \([X]_{au}\) is a directed set. Here for any \(X_\alpha, X_\beta \in \left[ X \right]_{au}\), we say
\[
X_\alpha \leq X_\beta
\]
if and only if \(X_\beta\) is qc over \(X_\alpha\).

Fixed a geometric point \(s\) of \(X\) over \(k\left( X \right)\). Put
\[
\left[ X \right]_{et} = \{ \text{finite étale Galois covers of } X \text{ over } s \}.
\]
Then \([X]_{et}\) is a directed set. Here for any \(X_1, X_2 \in \left[ X \right]_{et}\), we say
\[
X_1 \leq X_2
\]
if and only if \(X_2\) is a finite étale Galois cover over \(X_1\).

Step 5. Fixed any \(X_\alpha, X_\beta \in \left[ X \right]_{au}\).

It is seen that \(X_\alpha\) and \(X_\beta\) both are finite étale Galois covers of \(X\) by Lemma 9.8.

Let \(X_\beta\) be \(qc\) over \(X_\alpha\). Then \(X_\beta\) is a étale finite Galois cover of \(X_\alpha\) from Lemma 9.8 again.

Hence, \(\left[ X \right]_{au}\) is a directed subset of \(\left[ X \right]_{et}\).

Step 6. Let \(Z \in \left[ X \right]_{et}\). We have \(k\left( Z \right) \subseteq k\left( X \right)_{au}\). It is seen that \(k\left( Z \right)\) is a finite unramified Galois extension of \(k\left( X \right)\).

Let \(\alpha \subseteq k\left( Z \right) \setminus k\left( X \right)\) be a set of generators of the field \(k\left( Z \right)\) over \(k\left( X \right)\). As \(\alpha \in I\) is finite and \(\Delta\) is infinite, there is a finite set \(\beta \in I\) such that
\[
\alpha \subseteq \beta \subseteq \Delta.
\]
We have
\[
X_\beta \in \left[ X \right]_{au}
\]
such that \(X_\beta\) is \(qc\) over \(Z\).

Hence, \(\left[ X \right]_{au}\) is a co-final directed subset in \(X_{et}\).

Step 7. Now by Steps 1-6 above we have
\[
\pi_1^{et} \left( X \right) \\
= \lim_{\leftarrow Z \in \left[ X \right]_{et}} Aut \left( Z/X \right) \\
\cong \lim_{\leftarrow Z \in \left[ X \right]_{au}} Aut \left( Z/X \right) \\
\cong \lim_{\leftarrow Z \in \left[ X \right]_{au}} Gal \left( k \left( Z \right) / k \left( X \right) \right) \\
= \lim_{\leftarrow \alpha \in I} Gal \left( k \left( X_\alpha \right) / k \left( X \right) \right) \\
\cong Gal \left( k \left( X_{et} \right) / k \left( X \right) \right) \\
= Gal \left( k \left( X \right)_{au} / k \left( X \right) \right).
\]

This completes the proof. □
Remark 10.3. Let $X$ be an integral $K$-variety. From Theorem 10.2 we have
\[ \pi^\text{et}_1(X_\text{et}) \cong \{0\}. \]
For example, let $X = \text{Spec}(\mathbb{Q})$. We have $X_\text{et} = \text{Spec}(\overline{\mathbb{Q}})$ and hence
\[ \pi^\text{et}_1\left(\text{Spec}(\overline{\mathbb{Q}})\right) \cong \{0\}. \]

10.3. A prior estimate of the \'{e}tale fundamental group. In this subsection we will introduce the qc fundamental group of an algebraic variety. We will prove that the \'{e}tale fundamental group is a normal subgroup of the qc fundamental group.

Fixed an algebraic $K$-variety $X$. Let $\Omega$ be the separable closure of a separably generated extension of the function field $k(X)$.

Define $\left[ X; \Omega\right]_{\text{qc}}$ to be the set of algebraic $K$-varieties $Z$ satisfying the conditions:

- $k(Z)$ is contained in $\Omega$;
- There is a surjective morphism $f : Z \to X$ of finite type such that $Z$ is $qc$ over $X$.

In [7], we require such an additional condition that $Z$ has a reduced affine covering with values in $\Omega$.

However, from Lemma 6.1 it is seen that there is no essential difference between the two conditions.

There are preliminary facts on the set $\left[ X; \Omega\right]_{\text{qc}}$ such as the following.

Lemma 10.4. For any $Z_1, Z_2 \in \left[ X; \Omega\right]_{\text{qc}}$, there is a third $Z_3 \in \left[ X; \Omega\right]_{\text{qc}}$ such that $Z_2$ is qc over $Z_1$ and $Z_2$, respectively.

Lemma 10.5. Let $Z_1, Z_2, Z_3 \in \left[ X; \Omega\right]_{\text{qc}}$. Suppose that $Z_2$ is qc over $Z_1$ and $Z_3$ is qc over $Z_2$. Then $Z_3$ is qc over $Z_1$.

Here, Lemmas 10.4-5 above can be proved in a manner similar to what we have done for the proof of Theorem 10.2.

Set a partial order $\leq$ in the set $\left[ X; \Omega\right]_{\text{qc}}$ in such a manner:

For any $Z_1, Z_2 \in \left[ X; \Omega\right]_{\text{qc}}$, we say
\[ Z_1 \leq Z_2 \]
if and only if there is a surjective morphism $\varphi : Z_2 \to Z_1$ of finite type such that $Z_2$ is qc over $Z_1$.

By Lemmas 10.4-5 it is seen that $\left[ X; \Omega\right]_{\text{qc}}$ is a directed set and
\[ \{\text{Aut}(Z/X) : Z \in \left[ X; \Omega\right]_{\text{qc}}\} \]
is an inverse system of groups.

Now we introduce the following definition.
**Definition 10.6.** Let \( X \) be an algebraic \( K \)-variety. Suppose that \( \Omega \) is the separable closure of a separably generated extension of \( k(X) \). The inverse limit

\[
\pi_{1}^{qc}(X; \Omega) \triangleq \lim_{\longrightarrow} \text{Aut}(Z/X)
\]

of the inverse system \( \{ \text{Aut}(Z/X) : Z \in [X; \Omega]_{qc} \} \) of groups is said to be the **qc fundamental group** of \( X \) with coefficients in \( \Omega \).

We have the following result on the \(qc\) fundamental group, which will be prove in §10.5.

**Theorem 10.7.** Let \( X \) be an algebraic \( K \)-variety. Suppose that \( \Omega \) is the separable closure of a separably generated extension of \( k(X) \). There are the following statements.

(i) There is a group isomorphism

\[
\pi_{1}^{qc}(X; \Omega) \cong \text{Gal}(\Omega/k(X)).
\]

(ii) There is a group isomorphism

\[
\pi_{1}^{et}(X; s) \cong \pi_{1}^{qc}(X; \Omega)_{et}
\]

for a geometric point \( s \) of \( X \) over \( \Omega \), where \( \pi_{1}^{qc}(X; \Omega)_{et} \) is a normal subgroup of \( \pi_{1}^{qc}(X; \Omega) \).

**Remark 10.8.** Let \( X \) be an algebraic \( K \)-variety. Define

\[
\pi_{1}^{qc}(X) \triangleq \pi_{1}^{qc}(X; k(X)^{sep}).
\]

It is seen that there is a group isomorphism

\[
\pi_{1}^{qc}(X) \cong \text{Gal}(k(X)^{sep}/k(X)).
\]

**Remark 10.9.** Let \( X \) be an algebraic \( K \)-variety. The quotient group

\[
\pi_{1}^{br}(X) = \frac{\pi_{1}^{qc}(X; k(X)^{sep})}{\pi_{1}^{qc}(X; k(X)^{sep})_{et}}
\]

is said to be the **ramified group** (or **branched group**) of \( X \). By Lemma 9.8 it is seen that \( k(X)^{sep} \) is a Galois extension of \( k(X)^{au} \); from Theorems 10.2,10.7 we have

\[
\pi_{1}^{br}(X) = \frac{\text{Gal}(k(X)^{sep}/k(X))}{\text{Gal}(k(X)^{au}/k(X))}.
\]

The ramified group \( \pi_{1}^{br}(X) \) reflects the topological properties of the scheme \( X \) such as the branched covers. In deed, such ramified groups will play an important role in giving the anabelian functors in the present paper.
10.4. A universal cover for the qc fundamental group. Let $X$ be an algebraic $K$-variety. Suppose that $\Omega$ is the separable closure of a separably generated extension of the function field $k(X)$.

Assume that $X$ has a reduced affine covering with values in $\Omega^{ad}$ without loss of generality. Let

$$G = Gal (\Omega/k(X))$$

and let

$$\Delta \subseteq \Omega \setminus k(X)$$

be a set of generators of $\Omega$ over $k(X)$. By Theorem 4.1 it is seen that $\Omega$ is Galois over $k(X)$.

Repeating the universal construction for étale fundamental group in §10.1, we have an integral variety $X_\Omega$ and a morphism $f_\Omega$ such as in the following lemma.

Lemma 10.10. There is an integral $K$-variety $X_\Omega$ and a surjective morphism $f_\Omega : X_\Omega \to X$ satisfying the conditions:

- $k(X_\Omega) = \Omega$;
- $f_\Omega$ is affine;
- $X_\Omega$ is qc over $X$ by $f_\Omega$;
- $k(X_\Omega)$ is Galois over $k(X)$;
- $X_\Omega$ is essentially affine in $\Omega$.

Such an integral $K$-variety $X_\Omega$ with a morphism $f_\Omega$, denoted by $(X_\Omega, f_\Omega)$, is said to be a universal cover over $X$ for the qc fundamental group group $\pi_{1}^{qc}(X; \Omega)$.

10.5. Proof of Theorem 10.7. Now we can prove the main result above on the qc fundamental group in §10.4.

Proof. (Proof of Theorem 10.7) We will proceed in several steps.

Step 1. By Theorem 7.4 we have

$$Gal (\Omega/k(X)) \cong \lim_{\leftarrow Z \in [X; \Omega]_{qc}} Gal (k(Z)/k(X)) \cong \lim_{\leftarrow Z \in [X; \Omega]_{qc}} Aut (Z/X) = \pi_{1}^{qc}(X; \Omega)$$

according to preliminary facts on Galois groups.

Step 2. Fixed a geometric point $s$ of $X$ over $\Omega$. Let $[X; s]_{et}$ be the set of finite étale Galois covers of $X$ over the geometric point $s$. For any $Z_1, Z_2 \in [X; s]_{et}$ we say

$$Z_1 \leq Z_2$$
if and only if $Z_2$ is a finite étale Galois cover of $Z_1$. Then $[X; s]_{et}$ is a partially ordered set. Put

$$[X; s]_{qc} \triangleq [X; \Omega]_{qc} \cap [X; s]_{et}.$$ 

Let $Z_1, Z_2 \in [X; s]_{qc}$. It is easily seen that $Z_2$ is a finite étale Galois cover of $Z_1$ if and only if $Z_2$ is $qc$ over $Z_1$.

It follows that $[X; s]_{qc}$ is a co-final directed subset in $[X; s]_{et}$.

Step 3. Now consider the universal covers $X_\Omega$ and $X_{et}$ of $X$ for the groups $\pi^qc_1(X; \Omega)$ and $\pi^et_1(X; s)$, respectively. From Step 7 in §10.2 we have

$$Gal(k(X_{et})/k(X)) \cong \lim_{\leftarrow Z \in [X; s]_{et}} Gal(k(Z)/k(X))$$

$$\cong \lim_{\leftarrow Z \in [X; s]_{et}} Aut(Z/X)$$

$$\cong \pi^et_1(X; s).$$

By Step 2 we have

$$Gal(k(X_{et})/k(X))$$

$$\cong \lim_{\leftarrow Z \in [X; s]_{et}} Gal(k(Z)/k(X))$$

$$\cong \lim_{\leftarrow Z \in [X; s]_{qc}} Gal(k(Z)/k(X))$$

since $[X; s]_{qc}$ is co-final in $[X; s]_{et}$.

On the other hand, we have

$$k(X_\Omega) = \lim_{\leftarrow Z \in [X; \Omega]_{qc}} k(Z)$$

and

$$k(X_{et}) = \lim_{\leftarrow Z \in [X; s]_{et}} k(Z) = \lim_{\leftarrow Z \in [X; s]_{qc}} k(Z)$$

as direct limits of direct systems of groups for the function fields.

It is seen that

$$\lim_{\leftarrow Z \in [X; \Omega]_{qc}} k(Z)$$

is an extension of the field

$$\lim_{\leftarrow Z \in [X; s]_{qc}} k(Z).$$

It follows that

$$k(X)^{au} = k(X_{et})$$

is a subfield of

$$\Omega = k(X_\Omega).$$

Then we have a tower of Galois extensions of function fields

$$k(X) \subseteq k(X)^{au} \subseteq \Omega$$

from Lemma 10.1.
It is seen that \( \pi_1^{et}(X; s) \) is isomorphic to the normal subgroup 
\[ \text{Gal}(k(X)^{au}/k(X)) \]
of the group \( \text{Gal}(\Omega/k(X)) \). Hence, \( \pi_1^{et}(X; s) \) is isomorphic to a normal subgroup of \( \pi_1^{qc}(X; \Omega) \) since by Step 1 we have 
\[ \text{Gal}(\Omega/k(X)) \cong \pi_1^{qc}(X; \Omega). \]
This completes the proof. \( \square \)

11. Monodromy Actions

Naturally there exist three types of monodromy actions for a given integral \( K \)-variety, as we have done for arithmetic varieties in \( [9] \):

- Monodromy action of étale fundamental group on the universal cover;
- Monodromy action of absolute Galois group on the \( sp \)-completion;
- Monodromy action of ramified group on the \( sp \)-completion.

11.1. Monodromy actions of étale fundamental groups. From Lemma 10.1 and Theorem 10.2 it is seen there is the below preliminary fact on the étale fundamental group of an integral variety.

**Lemma 11.1.** For an integral \( K \)-variety \( X \), there is an isomorphism 
\[ \text{Aut}(X_{et}/X) \cong \pi_1^{et}(X) \]
between groups, where \( (X_{et}, p_X) \) is a universal cover of \( X \) for the étale fundamental group \( \pi_1^{et}(X) \).

Now let \( X \) and \( Y \) be two integral \( K \)-varieties. Suppose that \( (X_{et}, p_X) \) and \( (Y_{et}, p_Y) \) are universal covers of \( X \) and \( Y \) for the étale fundamental groups \( \pi_1^{et}(X) \) and \( \pi_1^{et}(Y) \), respectively.

By Lemma 11.1 it is seen that each group homomorphism 
\[ \sigma : \pi_1^{et}(X) \to \pi_1^{et}(Y). \]
gives a group homomorphism, namely 
\[ \sigma : \text{Aut}(X_{et}/X) \to \text{Aut}(Y_{et}/Y). \]
The converse is true.

Here is the monodromy action of étale fundamental groups on the universal covers.

**Lemma 11.2.** Assume that there is a group homomorphism 
\[ \sigma : \text{Aut}(X_{et}/X) \to \text{Aut}(Y_{et}/Y). \]

Then there is a bijection 
\[ \tau : \text{Hom}(X, Y) \to \text{Hom}(X_{et}, Y_{et}), f \mapsto f_{et} \]
between sets given in a canonical manner:

- Let \( f \in \text{Hom}(X,Y) \). Then the map
  \[
  g(x_0) \mapsto \sigma(g)(f(x_0))
  \]
defines a morphism
  \[
  f_{\text{et}} : X_{\text{et}} \to Y_{\text{et}}
  \]
  for any \( x_0 \in X \) and any \( g \in \text{Aut}(X_{\text{et}}/X) \).
- Let \( f_{\text{et}} \in \text{Hom}(X_{\text{et}},Y_{\text{et}}) \). Then the map
  \[
  p_X(x) \mapsto p_Y(f_{\text{et}}(x))
  \]
defines a morphism
  \[
  f : X \to Y
  \]
  for any \( x \in X_{\text{et}} \).

In particular, we have
\[
f \circ p_X = p_Y \circ f_{\text{et}}.
\]

**Proof.** It is immediate from Lemma 10.1 and Theorem 10.2. \( \square \)

### 11.2. Monodromy actions of absolute Galois groups.

Let \( X \) and \( Y \) be two integral \( K \)-varieties. Consider the \( \text{sp} \)-completions \( (X_{\text{sp}}, \lambda_X) \) and \( (Y_{\text{sp}}, \lambda_Y) \) and the universal covers \( (X_{\text{et}}, p_X) \) and \( (Y_{\text{et}}, p_Y) \) for \( \pi_1^{\text{et}}(X) \) and \( \pi_1^{\text{et}}(Y) \), respectively.

It is easily seen that there are isomorphisms
\[
\text{Gal}(k(X)^{\text{sep}}/k(X)) \cong \text{Aut}(X_{\text{sp}}/X);
\]
\[
\text{Gal}(k(Y)^{\text{sep}}/k(Y)) \cong \text{Aut}(Y_{\text{sp}}/Y).
\]
between groups from Lemma 7.2 and Theorem 8.7.

Here is the monodromy action of absolute Galois groups on the \( \text{sp} \)-completions.

**Lemma 11.3.** Suppose that there is a group homomorphism
\[
\sigma : \text{Aut}(X_{\text{sp}}/X) \to \text{Aut}(Y_{\text{sp}}/Y).
\]
Then there is a bijection
\[
\tau : \text{Hom}(X,Y) \to \text{Hom}(X_{\text{sp}},Y_{\text{sp}}), f \mapsto f_{\text{sp}}
\]
between sets given in a canonical manner:
• Let \( f \in \text{Hom}(X,Y) \). Then the map
\[
g(x_0) \mapsto \sigma(g)(f(x_0))
\]
defines a morphism
\[
f_{sp} : X_{sp} \rightarrow Y_{sp}
\]
for any \( x_0 \in X \) and any \( g \in \text{Aut}(X_{sp}/X) \).

• Let \( f_{sp} \in \text{Hom}(X_{sp}, Y_{sp}) \). Then the map
\[
\lambda_X(x) \mapsto \lambda_Y(f_{sp}(x))
\]
defines a morphism
\[
f : X \rightarrow Y
\]
for any \( x \in X_{sp} \).
In particular, we have
\[
f \circ \lambda_X = \lambda_Y \circ f_{sp}.
\]

Proof. It is immediate from Lemma 7.2 and Theorem 8.7.

11.3. Monodromy actions of ramified groups. To start with, let’s prove a preparatory lemma.

Lemma 11.4. Let \( X \) be an integral \( K \)-variety. Suppose that \((X_{sp}, \lambda_X)\) is an \( sp \)-completion of \( X \) and \((X_{et}, p_X)\) is a universal cover of \( X \) for the étale fundamental group \( \pi^\text{et}_1(X) \). Then there exists canonically a surjective morphism \( q_X : X_{sp} \rightarrow X_{et} \) satisfying the below properties:

• \( q_X \) is affine;
• \( \lambda_X = p_X \circ q_X \);
• \( X_{sp} \) is qc over \( X_{et} \) by \( q_X \);
• \((X_{sp}, q_X)\) is an \( sp \)-completion of \( X_{et} \).
In particular, we have \( q_X = \lambda_{X_{et}} \).

Proof. Repeat the universal construction for an \( sp \)-completion of the integral scheme \( X_{et} \) in §8. Then we have a morphism
\[
q_X = \lambda_{X_{et}} : X_{sp} \rightarrow X_{et}
\]
such that \((X_{sp}, \lambda_{X_{et}})\) is an \( sp \)-completion of \( X_{et} \).

Let \( X \) and \( Y \) be two integral \( K \)-varieties. Consider the \( sp \)-completions \((X_{sp}, \lambda_X)\) and \((Y_{sp}, \lambda_Y)\) and the universal covers \((X_{et}, p_X)\) and \((Y_{et}, p_Y)\) for \( \pi^\text{et}_1(X) \) and \( \pi^\text{et}_1(Y) \), respectively.

From Remark 10.9 we have the ramified groups
\[
\pi^\text{br}_1(X) = \frac{\text{Gal}(k(X)^{sp}/k(X))}{\text{Gal}(k(X)^{au}/k(X))};
\]
\[
\pi^\text{br}_1(Y) = \frac{\text{Gal}(k(Y)_{\text{sep}}/k(Y))}{\text{Gal}(k(Y)_{\text{au}}/k(Y))}.
\]

It follows that we have the following result.

**Lemma 11.5.** For any integral \(K\)-varieties \(X\) and \(Y\), there are group isomorphisms

\[
\pi^\text{br}_1(X) \cong \text{Gal}(k(X)_{\text{sep}}/k(X)_{\text{au}}) \cong \text{Aut}(X_{\text{sp}}/X_{\text{et}});
\]

\[
\pi^\text{br}_1(Y) \cong \text{Gal}(k(Y)_{\text{sep}}/k(Y)_{\text{au}}) \cong \text{Aut}(Y_{\text{sp}}/Y_{\text{et}}).
\]

Here is the monodromy action of ramified groups on the \(sp\)-completions. It will play an important role in the anabelian geometry.

**Lemma 11.6.** Suppose that there is a group homomorphism

\[
\sigma : \text{Aut}(X_{\text{sp}}/X_{\text{et}}) \rightarrow \text{Aut}(Y_{\text{sp}}/Y_{\text{et}}).
\]

Then there is a bijection

\[
\tau : \text{Hom}(X_{\text{et}}, Y_{\text{et}}) \rightarrow \text{Hom}(X_{\text{sp}}, Y_{\text{sp}}), f \mapsto f_{\text{sp}}
\]

between sets given in a canonical manner:

- Let \(f \in \text{Hom}(X_{\text{et}}, Y_{\text{et}})\). Then the map
  \[
g(x_0) \mapsto \sigma(g)(f(x_0))
  \]
  defines a morphism
  \[
f_{\text{sp}} : X_{\text{sp}} \rightarrow Y_{\text{sp}}
  \]
  for any \(x_0 \in X_{\text{et}}\) and any \(g \in \text{Aut}(X_{\text{sp}}/X_{\text{et}})\).

- Let \(f_{\text{sp}} \in \text{Hom}(X_{\text{sp}}, Y_{\text{sp}})\). Then the map
  \[
  \lambda_{X_{\text{et}}} (x) \mapsto \lambda_{Y_{\text{et}}} (f_{\text{sp}}(x))
  \]
  defines a morphism
  \[
f : X_{\text{et}} \rightarrow Y_{\text{et}}
  \]
  for any \(x \in X_{\text{sp}}\).

In particular, we have

\[
f \circ \lambda_{X_{\text{et}}} = \lambda_{Y_{\text{et}}} \circ f_{\text{sp}}.
\]

**Proof.** It is immediate from Lemmas 11.3-4. \qed
12. Proof of the Main Theorem

12.1. Preliminary lemmas. Let $X$ and $Y$ be two integral $K$-varieties. Fixed any $sp$-completions $(X_{sp}, \lambda_X)$ and $(Y_{sp}, \lambda_Y)$ of $X$ and $Y$, and any universal covers $(X_{et}, p_X)$ and $(Y_{et}, p_Y)$ of $X$ and $Y$ for the groups $\pi^{et}_1(X)$ and $\pi^{et}_1(Y)$, respectively.

There are several results on the $sp$-completions and the universal covers of $X$ and $Y$, respectively (c.f. [9]).

Remark 12.1. From a viewpoint of $sp$-completion, it is seen that arithmetic varieties and integral $K$-varieties are very different. In fact, for arithmetic varieties $Z_1, Z_2$, by Theorem 9.15 we have

$$k(Z_1) = k(Z_2) \implies Sp[Z_1] = Sp[Z_2].$$

However, for integral $K$-varieties $Z_1, Z_2$, from Lemma 8.9 it is seen that

$$k(Z_1) = k(Z_2) \implies Sp[Z_1] = Sp[Z_2]$$

does not hold in general.

Lemma 12.2. Suppose $k(X) = k(Y)$ and $Sp[X] = Sp[Y]$. Then there exists a bijection $\tau$ from $Hom(X,Y)$ onto $Hom(X_{sp}, Y_{sp})$ given in a canonical manner. In particular, $Hom(X,Y)$ must be a non-void set.

Proof. Fixed any $sp$-completions $X_{sp}$ and $Y_{sp}$ of $X$ and $Y$, respectively. By Lemma 8.8 there is an isomorphism

$$f_{sp} : X_{sp} \to Y_{sp}$$

according to the assumption that $Sp[X] = Sp[Y]$ holds.

By Theorem 8.7 we have

$$Aut(X_{sp}/X) \cong Gal(k(X)_{sep}/k(X)) \cong Aut(Y_{sp}/Y).$$

From Lemma 11.3 we immediately obtain the desired properties. □

Remark 12.3. From a viewpoint of graph functor $\Gamma$, it is seen that arithmetic varieties and integral $K$-varieties are also very different. In fact, for arithmetic varieties $Z_1, Z_2$, we have

$$k(Z_1) \subseteq k(Z_2) \implies \Gamma(Z_1) \subseteq \Gamma(Z_2).$$

However, for integral $K$-varieties $Z_1, Z_2$, in general, it is not true that

$$k(Z_1) \subseteq k(Z_2) \implies \Gamma(Z_1) \subseteq \Gamma(Z_2)$$

holds.

Lemma 12.4. Let $k(X)$ be separably generated over $k(Y)$. Then there are the following statements:
• There is a homomorphism
  \[ \sigma_{sp} : \text{Gal}(k(X)^{\text{sep}}/k(X)) \to \text{Gal}(k(Y)^{\text{sep}}/k(Y)). \]

• There is a bijection \( \tau \) from \( \text{Hom}(X, Y) \) onto \( \text{Hom}(X_{sp}, Y_{sp}) \) given in a canonical manner. In particular, \( \text{Hom}(X, Y) \) is empty if and only if so is \( \text{Hom}(X_{sp}, Y_{sp}) \).

• Let \( \Gamma(X_{sp}) \supseteq \Gamma(Y_{sp}) \). Then \( \text{Hom}(X, Y) \) must be a non-void set.

**Proof.** It suffices to prove the third statement. Suppose that \( \Gamma(Y_{sp}) \) is a subgraph of \( \Gamma(X_{sp}) \). Take the \( sp \)-completions \( (X_{sp}, \lambda_X) \) and \( (Y_{sp}, \lambda_Y) \) of \( X \) and \( Y \), respectively.

It is seen that the ring \( B \) of an affine open set \( V \) in \( Y_{sp} \) must be embedded into the ring \( A \) of some certain affine open set \( U \) in \( X_{sp} \) as a subring. In deed, choose \( A \) to be the ring over \( B \) generated by the set

\[ \Delta_B = \{ \sigma(w) : w \in \Delta, \sigma \in \text{Gal}(k(X)^{\text{sep}}/k(Y)^{\text{sep}}) \} \]

where \( \Delta \subseteq k(X)^{\text{sep}} \setminus k(Y)^{\text{sep}} \) is a set of generators of \( k(X)^{\text{sep}} \) over \( k(Y)^{\text{sep}} \). It is easily seen that \( U = \text{Spec}(A) \) is an affine open set in an \( sp \)-completion of \( X \) that is essentially equal to the given \( X_{sp} \) from the universal construction for \( X_{sp} \) in §8.2.

Conversely, each \( A \) must contain some \( B \). In fact, let \( \xi \) and \( \eta \) be the generic points of \( X \) and \( Y \), respectively. Take any point \( y_0 \) in \( V \). We have the specializations

\[ \eta \to y_0 \text{ in } Y_{sp}; \]

\[ \xi \to y_0 \text{ in } X_{sp}. \]

From Lemma 8.2 we have an affine open set \( U = \text{Spec}(A) \) in \( X_{sp} \) containing \( \xi \) and \( y_0 \); then, \( U \) also contains \( \eta \). Hence, \( A \) contains some \( B \) such that \( y_0 \in V = \text{Spec}(B) \).

It follows that there is a homomorphism

\[ f_U : U = \text{Spec}(A) \to V = \text{Spec}(B) \]

defined by the inclusion. This gives us a scheme homomorphism

\[ f_{sp} : X_{sp} \to Y_{sp}. \]

By the projections \( \lambda_X : X_{sp} \to X \) and \( \lambda_Y : Y_{sp} \to Y \) we have a unique homomorphism \( f : X \to Y \) satisfying the condition

\[ \lambda_{sp} \circ f_{sp} = f \circ \lambda_{sp}. \]

This completes the proof. \( \Box \)

**Lemma 12.5.** Let \( k(X) \) be separably generated over \( k(Y) \). Then there are the following statements:
There is a homomorphism
\[ \sigma_{br}: \text{Gal}(k(X)^{sep}/k(X)^{au}) \to \text{Gal}(k(Y)^{sep}/k(Y)^{au}). \]

There is a bijection \( \tau \) from \( \text{Hom}(X_{et},Y_{et}) \) onto \( \text{Hom}(X_{sp},Y_{sp}) \) given in a canonical manner. In particular, \( \text{Hom}(X_{et},Y_{et}) \) is empty if and only if so is \( \text{Hom}(X_{sp},Y_{sp}) \).

Let \( \Gamma(X_{sp}) \supseteq \Gamma(Y_{sp}) \). Then \( \text{Hom}(X_{et},Y_{et}) \) must be a non-void set.

**Proof.** It is immediate from Lemmas 11.4-6,12.4.

**Lemma 12.6.** Let \( k(X) \) be separably generated over \( k(Y) \). Then there are the following statements:

- There is a homomorphism
  \[ \sigma_{et}: \text{Gal}(k(X)^{au}/k(X)) \to \text{Gal}(k(Y)^{au}/k(Y)). \]

- There is a bijection \( \tau \) from \( \text{Hom}(X,Y) \) onto \( \text{Hom}(X_{au},Y_{au}) \) given in a canonical manner. In particular, \( \text{Hom}(X,Y) \) is empty if and only if so is \( \text{Hom}(X_{au},Y_{au}) \).

- Let \( \Gamma(X_{sp}) \supseteq \Gamma(Y_{sp}) \). Then \( \text{Hom}(X_{et},Y_{et}) \) must be a non-void set.

**Proof.** It is immediate from Lemmas 11.2,12.4.

12.2. **Proof of the main theorem.** Now we can give the proof of the main theorem in the paper.

**Proof. (Proof of Theorem 1.3)** Noted that from Lemma 11.5 we have the ramified groups

\[ \pi_1^{br}(X) = \frac{\text{Gal}(k(X)^{sep}/k(X))}{\text{Gal}(k(X)^{au}/k(X))} \cong \text{Aut}(X_{sp}/X_{et}); \]

\[ \pi_1^{br}(Y) = \frac{\text{Gal}(k(Y)^{sep}/k(Y))}{\text{Gal}(k(Y)^{au}/k(Y))} \cong \text{Aut}(Y_{sp}/Y_{et}). \]

Then we have

\[ \text{Hom}(X,Y) \cong \text{Hom}(\pi_1^{br}(X),\pi_1^{br}(Y)) \]

\[ \cong \text{Hom}(\frac{\text{Aut}(X_{sp}/X)}{\text{Aut}(X_{et}/X)},\frac{\text{Aut}(Y_{sp}/Y)}{\text{Aut}(Y_{et}/Y)}) \]

from Lemmas 11.1-4,11.6,12.2,12.4-6.

This completes the proof.
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