Lower Bounds for the Error of Quadrature Formulas for Hilbert Spaces

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Abstract

We prove lower bounds for the worst case error of quadrature formulas that use given sample points $X_n = \{x_1, \ldots, x_n\}$. We are mainly interested in optimal point sets $X_n$, but also prove lower bounds that hold for most randomly selected sets. As a tool, we use a recent result (and extensions thereof) of Vybiřal on the positive semi-definiteness of certain matrices related to the product theorem of Schur. The new technique also works for spaces of analytic functions where known methods based on decomposable kernels cannot be applied.

Keywords: numerical integration in high dimensions, curse of dimensionality, positive definite matrices, Schur’s product theorem.
1 Introduction

We study error bounds for quadrature formulas and assume that the integrand is from a Hilbert space \( F \) of real valued functions defined on a set \( D \). We assume that function evaluation is continuous and hence are dealing with a reproducing kernel Hilbert space (RKHS) \( F \) with a kernel \( K \). We want to compute \( S(f) \) for \( f \in F \), where \( S \) is a continuous linear functional, hence \( S(f) = \langle f, h \rangle \) for some \( h \in F \). We consider, for \( c \in \mathbb{R}^n \) and \( X_n = \{x_1, \ldots, x_n\} \subset D \), quadrature formulas \( Q_{c,X_n} : F \to \mathbb{R} \) defined by

\[
Q_{c,X_n}(f) = \sum_{j=1}^{n} c_j f(x_j).
\]

Then the worst case error (on the unit ball of \( F \)) of \( Q_{c,X_n} \) is defined by

\[
e(Q_{c,X_n}, S) := \sup_{\|f\| \leq 1} |S(f) - Q_{c,X_n}(f)|.
\]

If we fix a set \( X_n \subset D \) of sample points we may define the radius of information \( e(X_n, S) \) by

\[
e(X_n, S) = \inf_{c \in \mathbb{R}^n} e(Q_{c,X_n}, S).
\]

Our main interest is in the optimization of \( X_n \) as well as of the weights \( c \). Then we obtain the \( n \)th minimal worst case error

\[
e(n, S) = \inf_{X_n \subset D} e(X_n, S) = \inf_{c \in \mathbb{R}^n} \inf_{X_n \subset D} e(Q_{c,X_n}, S).
\]

We are mainly interested in tensor product problems. We will therefore assume that \( F_i \) is a RKHS on a domain \( D_i \) with kernel \( K_i \) for all \( i \leq d \) and that \( F_d \) is the tensor product of these spaces. That is, \( F_d \) is a RKHS on \( D_d = D_1 \times \cdots \times D_d \) with reproducing kernel

\[
K_d : D_d \times D_d \to \mathbb{R}, \quad K_d(x, y) = \prod_{i=1}^{d} K_i(x^i, y^i).
\]

If \( h_i \in F_i \) and \( S_i(f) = \langle f, h_i \rangle \) for \( f \in F_i \), we will denote by \( h_d \) the tensor product of the functions \( h_i \), i.e.,

\[
h_d(t) = (h_1 \otimes \cdots \otimes h_d)(t) = h_1(t^1) \cdots h_d(t^d), \quad t = (t^1, \ldots, t^d) \in D_d.
\]
We study the tensor product functional \( S_d = \langle \cdot, h_d \rangle \) on \( F_d \). Note that in this paper we assume that \( S_d \) is a tensor product functional, but the results can also be applied to operators, see [15].

The complexity of the tensor product problem is given by the number \( s(e(n, S_d)) \) and has been studied in many papers for a long time. Traditionally, the functional \( S_d \) and the dimension \( d \) was fixed and the interest was on large \( n \). Here we are mainly interested in the curse of dimensionality: Do we need exponentially many (in \( d \)) function values to obtain an error \( \varepsilon \) when we fix the error demand and vary the dimension?

To answer this question one has to prove upper bounds as well as lower bounds. Upper bounds for specific problems can often be proved by quasi Monte Carlo methods, see [2]. In addition there exists a general method, the analysis of the Smolyak algorithm, see [14, 20] and the recent supplement [16].

In this paper we concentrate on lower bounds, again for a fixed error demand \( \varepsilon \) and (possibly) large dimension. Such bounds were first studied in [11] for certain special problems and later in [12] with the technique of decomposable kernels. This technique is rather general as long as we consider finite smoothness. The technique does not work, however, for analytic functions.

In contrast, the approach of [19] also works for polynomials and other analytic functions. We continue this approach since it opens the door for more lower bounds under general assumptions. One result of this paper (Theorem 10) reads as follows:

**Theorem 10.** For all \( i \leq d \), let \( F_i \) be a RKHS and let \( S_i \) be a bounded linear functional on \( F_i \) with unit norm and nonnegative representer \( h_i \). Assume that there are functions \( f_i \) and \( g_i \) in \( F_i \) and a number \( \alpha_i \in (0, 1] \) such that \( (h_i, f_i, g_i) \) is orthonormal in \( F_i \) and \( \alpha_i^2 h_i^2 = f_i^2 + g_i^2 \). Then the tensor product problem \( S_d = S_1 \otimes \ldots \otimes S_d \) satisfies for all \( n \in \mathbb{N} \) that

\[
e(n, S_d)^2 \geq 1 - n \prod_{i=1}^{d} (1 + \alpha_i^2)^{-1}.
\]

In particular, we obtain the curse of dimensionality if all the \( \alpha_i \) are equal. As an application, we use this result to obtain lower bounds for the complexity of the integration problem on Korobov spaces with increasing smoothness, see Section 4.4. These lower bounds complement existing upper bounds from [14, Section 10.7.4].
The paper is organized as follows. We first provide a general connection between the worst case error of quadrature formulas and the positive semi-definiteness of certain matrices in Section 2. We then turn to tensor product problems. We start with homogeneous tensor products (i.e., all factors $F_i$ and $h_i$ are equal), see Section 3, where we also consider several examples. The non-homogeneous case is then discussed in Section 4. This section also contains the results for Korobov spaces with increasing smoothness. Section 3 and Section 4 are based on a recent generalization of Schur’s product theorem from [19]. In Section 5, we discuss further generalizations of Schur’s theorem and possible applications to numerical integration. Finally, in Section 6, we consider lower bounds for the error of quadrature formulas that use random point sets (as opposed to optimal point sets). This allows us to approach situations where we conjecture but cannot prove the curse of dimensionality for optimal point sets.

2 Lower bounds and positive definiteness

We begin with a somewhat surprising result: Lower bounds for the worst case error of quadrature formulas are equivalent to the statement that certain matrices are positive semi-definite.

**Proposition 1.** Let $F$ be a RKHS on $D$ with kernel $K$ and let $S = \langle \cdot, h \rangle$ for some $h \in F$.

(i) The following are equivalent for all $\alpha > 0$ and $X_n = \{x_1, \ldots, x_n\} \subset D$.

- The matrix $(K(x_j, x_k) - \alpha h(x_j)h(x_k))_{j,k \leq n}$ is positive semi-definite,
- $e(X_n, S)^2 \geq \|h\|^2 - \alpha^{-1}$.

(ii) The following are equivalent for all $\alpha > 0$ and $n \in \mathbb{N}$.

- The matrix $(K(x_j, x_k) - \alpha h(x_j)h(x_k))_{j,k \leq n}$ is positive semi-definite for all $x_1, \ldots, x_n \in D$,
- $e(n, S)^2 \geq \|h\|^2 - \alpha^{-1}$.

**Proof.** To prove the first part, we fix $X_n = \{x_1, \ldots, x_n\} \subset D$. For $c \in \mathbb{R}^n$ consider the quadrature rule $Q_{c,X_n} : F \to \mathbb{R}$ with

$$Q_{c,X_n}(f) = \sum_{j=1}^n c_j f(x_j).$$
Clearly, we have
\[
e(Q_c, x_n)^2 = \sup_{\|f\| \leq 1} |S(f) - Q_c, x_n(f)|^2 = \left\| h - \sum_{j=1}^n c_j K(x_j, \cdot) \right\|^2
\]
\[
= \|h\|^2 - 2 \sum_{j=1}^n c_j h(x_j) + \sum_{j,k=1}^n c_j c_k K(x_j, x_k).
\]
The function \(g : \mathbb{R} \to \mathbb{R}\) with \(g(a) = e(Q_{ac}, x_n)^2\) attains its minimum for
\[
a = \frac{\sum_{j=1}^n c_j h(x_j)}{\sum_{j,k=1}^n c_j c_k K(x_j, x_k)},
\]
where \(0/0\) is interpreted as \(0\). This yields
\[
e(X_n, S)^2 = \inf_{c \in \mathbb{R}^n} \inf_{a \in \mathbb{R}} e(Q_{ac}, x_n)^2 = \|h\|^2 - \sup_{c \in \mathbb{R}^n} \left( \frac{\sum_{j=1}^n c_j h(x_j)}{\sum_{j,k=1}^n c_j c_k K(x_j, x_k)} \right)^2.
\]
The last expression is larger or equal to \(\|h\|^2 - \alpha^{-1}\) if, and only if,
\[
\sum_{j,k=1}^n c_j c_k K(x_j, x_k) \geq \alpha \left( \sum_{j=1}^n c_j h(x_j) \right)^2
\]
holds for all \(c \in \mathbb{R}^n\), i.e. when the matrix \((K(x_j, x_k) - \alpha h(x_j) h(x_k))_{j,k \leq n}\) is positive semi-definite. This yields the statement.

The proof of the second part follows from the first part by taking the infimum over all \(X_n = \{x_1, \ldots, x_n\} \subset D\).

The idea now is to use some properties of the Schur product of matrices. We denote by \(\text{diag} M = (M_{1,1}, \ldots, M_{n,n})^T\) the diagonal entries of \(M\) whenever \(M \in \mathbb{R}^{n \times n}\). Moreover, if \(A, B \in \mathbb{R}^{n \times n}\) are two symmetric matrices, we write \(A \succeq B\) if \(A - B\) is positive semi-definite. The Schur product of \(A\) and \(B\) is the matrix \(A \circ B\) with \((A \circ B)_{i,j} = A_{i,j} B_{i,j}\) for \(i, j \leq n\). The classical Schur product theorem states that the Schur product of two positive semi-definite matrices is again positive semi-definite. However, this statement can be improved [19].

**Proposition 2.** Let \(M \in \mathbb{R}^{n \times n}\) be a positive semi-definite matrix. Then
\[
M \circ M \succeq \frac{1}{n} (\text{diag} M) (\text{diag} M)^T.
\]
A direct proof of Proposition 2 may be found in [19]. As pointed out to the authors by Dmitriy Bilyk, the result follows also from the theory of positive definite functions on the spheres as developed in the classical work of Schoenberg [17]. To sketch this approach, let \( (C_k^n(t))_{k=0}^\infty \) denote the sequence of Gegenbauer (or ultraspherical) polynomials. These are polynomials of order \( k \) on \([-1, 1]\), which are orthonormal with respect to the weight \((1 - t^2)^{\lambda - 1/2}\). Here, \( \lambda > 1/2 \) is a real parameter. By the Addition Theorem [1, Theorem 9.6.3], there is a positive constant \( C_{k,n} \), which depends only on \( k \) and \( n \), such that

\[
C_k^{(n-2)/2}(\langle x, y \rangle) = C_{k,n} \sum_{l=1}^{c_k,n} S_{k,l}(x) S_{k,l}(y), \quad x, y \in S^{n-1},
\]

where \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \) and \( S_{k,1}, \ldots, S_{k,c_k,n} \) form an orthonormal basis of the space of harmonic polynomials of degree \( k \) in \( \mathbb{R}^n \).

If now \( X = (x_{i,j})_{i,j=1}^n \in \mathbb{R}^{n \times n} \) is a positive semi-definite matrix with ones on the diagonal and \( f(t) = \sum_{k=0}^\infty a_k C_k^{(n-2)/2}(t) \) with \( a_k \geq 0 \), then \( (f(x_{i,j}))_{i,j=1}^n \) is also positive semi-definite. Indeed, we can write \( x_{i,j} = \langle x_i, x_j \rangle \) for some vectors \( x_1, \ldots, x_n \in S^{n-1} \) and use (1) to compute for every \( c \in \mathbb{R}^n \)

\[
\sum_{i,j=1}^n c_i c_j f(x_{i,j}) = \sum_{i,j=1}^n c_i c_j \sum_{k=0}^\infty a_k C_k^{(n-2)/2}(\langle x_i, x_j \rangle)
= C_{k,n} \sum_{i,j=1}^n c_i c_j \sum_{k=0}^\infty a_k \sum_{l=1}^{c_k,n} S_{k,l}(x_i) S_{k,l}(x_j)
\]

\[
\geq C_{k,n} \sum_{k=0}^\infty a_k \sum_{l=1}^{c_k,n} \left( \sum_{i=1}^n c_i S_{k,l}(x_i) \right)^2 \geq 0.
\]

For positive semi-definite matrices \( M \in \mathbb{R}^{n \times n} \) with ones on the diagonal, Proposition 2 then follows by observing that \( f(t) = t^2 - \frac{1}{n} \) is (up to a multiplicative constant) exactly the polynomial \( C_2^{(n-2)/2}(t) \). Finally, the general form of Proposition 2 is given by a simple scaling argument. \( \square \)

### 3 Homogeneous tensor products

We now use Propositions 1 and 2 in order to obtain the curse of dimensionality for certain tensor product (integration) problems. In this section,
we consider homogeneous tensor products, i.e., $F_d = F_1 \otimes \cdots \otimes F_1$ and $h_d = h_1 \otimes \cdots \otimes h_1$. Moreover, we work with normalized problems, i.e., we assume that $e(0, S_d) = \|h_d\| = 1$.

**Theorem 3.** Let $F_1$ be a RKHS on $D_1$. Assume that there are functions $e_1$ and $e_2$ on $D_1$ such that $e_1^2, e_2^2$ and $\sqrt{2}e_1e_2$ are orthonormal in $F_1$ and let $h_1 = \frac{1}{2}\sqrt{2}(e_1^2 + e_2^2)$. Then the tensor product problem $S_d = \langle \cdot, h_d \rangle$ satisfies

$$e(n, S_d)^2 \geq 1 - n 2^{-d}.$$

In particular, it suffers from the curse of dimensionality.

**Proof.** Without loss of generality, we may assume that $F_1$ is 3-dimensional, i.e., $b_1 = h_1 = \frac{1}{2}\sqrt{2}(e_1^2 + e_2^2)$, $b_2 = \frac{1}{2}\sqrt{2}(e_1^2 - e_2^2)$, and $b_3 = \sqrt{2}e_1e_2$ form an orthonormal basis. The function

$$M_1 : D_1 \times D_1 \to \mathbb{R}, \quad M_1(x, y) = \sum_{i=1}^{2} e_i(x)e_i(y),$$

is a reproducing kernel on $D_1$. The reproducing kernel $K_1$ of $F_1$ satisfies

$$K_1(x, y) = \sum_{i=1}^{3} b_i(x)b_i(y) = \left( \sum_{i=1}^{2} e_i(x)e_i(y) \right)^2 = M_1(x, y)^2$$

for all $x, y \in D_1$. Moreover, we have $h_1(x) = \frac{1}{2}\sqrt{2}M_1(x, x)$ for all $x \in D_1$. Therefore, also $K_d(x, y) = M_d(x, y)^2$ and $h_d(x) = 2^{-d/2}M_d(x, x)$ for $x, y \in D_d$, where $M_d$ is the $d$-times tensor product of $M_1$ and $h_d$ is the $d$-times tensor product of $h_1$. By Proposition 2 the matrix

$$\left( K_d(x_j, x_k) - n^{-1}2^d h_d(x_j)h_d(x_k) \right)_{j, k \leq n}$$

is positive semi-definite for all $x_1, \ldots, x_n \in D_d$. Proposition 1 yields that

$$e(n, S_d)^2 \geq 1 - n 2^{-d}.$$

The second statement is implied by the first statement; observe that the problem is normalized since $e(0, S_d) = 1$ for every $d$. \qed

Let us consider several applications of this result.
3.1 Trigonometric polynomials of degree 1

This example is already contained in Vybíral [19]; now we can see it as an application of the general Theorem 3. Take
\[ e_1(x) = 2^{1/4} \cos(\pi x) \] and \[ e_2(x) = 2^{1/4} \sin(\pi x) \] on \([0, 1]\). Then one obtains \( b_1 = h_1 = 1 \) and \( b_2(x) = \cos(2\pi x) \)
and \( b_3(x) = \sin(2\pi x) \). Hence we study, for \( d = 1 \), trigonometric polynomials
of degree 1 on the interval \([0, 1]\) with the norm
\[ \|f\|^2 = \|f\|^2_{L^2} + \frac{1}{4\pi^2} \|f'\|^2_{L^2}. \]

For \( d \in \mathbb{N} \) we take the tensor product space with the kernel
\[ K_d(x, y) = \prod_{i=1}^{d} (1 + \cos(x_i - y_i)). \]

We obtain \( \alpha_d = \sup_x K_d(x, x)^{1/2} = 2^{d/2} \) and \( \alpha_d \) is the norm of the embedding
of \( F_d \) into the space of continuous functions with the sup norm. Hence
functions in the unit ball of \( F_d \) may take large values if \( d \) is large, but the
integral is bounded by one. By applying Theorem 3 we obtain the following
result of [19] that solved an open problem of [10], see also [6].

**Corollary 4.** Let \( F_1 \) be the RKHS on \([0, 1]\) with the orthonormal system
1, \cos(2\pi x) and \sin(2\pi x). Then the integration problem \( S_d = \langle \cdot, 1 \rangle \)
on the tensor product space \( F_d \) satisfies
\[ e(n, S_d)^2 \geq 1 - n \cdot 2^{-d}. \]

In particular, it suffers from the curse of dimensionality.

**Remark 1.** The same vector space with dimension \( 3^d \) was studied earlier by Sloan and Woźniakowski [18] who proved the curse of dimensionality for a
different norm. It follows already from this work that exactly \( n = 2^d \) function
values are needed for the exact integration of trigonometric polynomials of
degree 1 (in each variable). We do not know whether this result was known
even before.

3.2 Gaussian integration for polynomials of degree 2

Let \( F_1 \) be the space of polynomials on \( \mathbb{R} \) with degree at most 2, equipped
with the scalar product
\[ \langle f, g \rangle = f(0)g(0) + \frac{1}{2} f'(0)g'(0) + \frac{1}{4} \int_{\mathbb{R}} f''(x)g''(x) \, d\mu_1(x), \]
where $\mu_1$ is the standard Gaussian measure on $\mathbb{R}$. We consider the integration problem

$$S_1: F_1 \to \mathbb{R}, \quad S_1(f) = \int_{\mathbb{R}} f(x) \, d\mu_1(x).$$

The tensor product problem for $d \in \mathbb{N}$ is given by the functional

$$S_d: F_d \to \mathbb{R}, \quad S_d(f) = \int_{\mathbb{R}^d} f(x) \, d\mu_d(x),$$

on the tensor product space $F_d$, which consists of all $d$-variate polynomials of mixed order 2 or less. Here, $\mu_d$ is the standard Gaussian measure on $\mathbb{R}^d$. By Theorem 3, this problem suffers from the curse of dimensionality. We have

$$\frac{e(n, S_d)}{e(0, S_d)} \geq \sqrt{1 - \frac{n}{2^d}}.$$

To see this, it is enough to choose $e_1(x) = 1$ and $e_2(x) = x$ and observe that the functions $\{1, x^2, \sqrt{2}x\}$ are orthonormal in $F_1$. Using the notation from the proof of Theorem 3, we obtain $b_1(x) = \sqrt{2}(1 + x^2), b_2(x) = \sqrt{2}(1 - x^2), b_3(x) = \sqrt{2}x$ and

$$S_1(f) = \sqrt{2} \cdot \langle f, b_1 \rangle \quad \text{for } f \in \{b_1, b_2, b_3\}.$$

**Corollary 5.** Take the RKHS $F_1$ on $\mathbb{R}$ which is generated by the orthonormal system $1, x^2$ and $\sqrt{2}x$. Then the problem $S_d(f) = \int_{\mathbb{R}^d} f(x) \, d\mu_d(x)$ of Gaussian integration on the tensor product space $F_d$ satisfies

$$\frac{e(n, S_d)^2}{e(0, S_d)^2} \geq 1 - n \cdot 2^{-d}.$$

In particular, we obtain the curse of dimensionality and the fact that exactly $n = 2^d$ function values are needed for exact integration.

### 3.3 Integration for polynomials of degree 2 on $[-\frac{1}{2}, \frac{1}{2}]$

Let $F_1$ be the space of polynomials on $\mathbb{R}$ with degree at most 2, defined on an interval of unit length. For convenience and symmetry we take the interval $[-1/2, 1/2]$. The univariate problem is given by $S_1(f) = \int_{-1/2}^{1/2} f(x) \, dx$ and for our construction we need $S_1(e_1^2) = \frac{1}{2} \sqrt{2}$. For $e_1 = a$ and $e_2(x) = bx$ we obtain $e_1 = 2^{-1/4}$ and $e_2(x) = 72^{1/4}x$ and hence $h_1(x) = b_1(x) = \frac{1}{2} + 6x^2$. If we apply Theorem 3 then we obtain the following.
Corollary 6. Take the RKHS $F_1$ on $I = [-1/2, 1/2]$ which is generated by the orthonormal system $\frac{1}{2}\sqrt{2}, \sqrt{72}x^2$ and $\sqrt{12}x$. Then the integration problem $S_d(f) = \int_{I} f(x) \, dx$ on $F_d$ satisfies

$$e(n, S_d)^2 \geq 1 - n 2^{-d}.$$ 

In particular, we obtain the curse of dimensionality and the fact that exactly $n = 2^d$ function values are needed for the exact integration.

The norm in $F_1$ is a weighted $\ell_2$-norm of Taylor coefficients. For $d = 1$ and $f(x) = ax^2 + bx + c$ we obtain the norm

$$\|f\|^2 = \frac{a^2}{72} + \frac{b^2}{12} + 2c^2$$

or

$$\|f\|^2 = 2f(0)^2 + \frac{1}{12}f'(0)^2 + \frac{1}{288}f''(0)^2$$

$$= 2f(0)^2 + \frac{1}{12}f'(0)^2 + \frac{1}{288} \int_{-1/2}^{1/2} f''(x)^2 \, dx.$$ 

Observe that we are “forced” by our approach to take this norm with these very specific parameters, although one can use embeddings and slightly modified norms. For the given norm we obtain

$$\alpha = \sup_{x \in I} K_1(x, x)^{1/2} = 8^{1/2}$$

and $\alpha$ (or $\alpha^d$ in the multivariate case) is the norm of the embedding of $F_d$ into the space of continuous functions with the sup norm. Hence functions in the unit ball of $F_d$ may take large values if $d$ is large, but the integral is bounded by one.

### 3.4 Integration of functions with zero boundary conditions

As another application of Theorem 3, we consider the integration of smooth functions with zero on the boundary. For that sake, let $e_1(x) = 2^{1/4} \sin(\pi x)$
and $e_2(x) = 2^{1/4} \sin(2\pi x)$ for $x \in [0, 1]$. Further, let $F_1$ be a three-dimensional space spanned by

$$
e_1^2(x) = \sqrt{2} \sin^2(\pi x), \\
e_2^2(x) = \sqrt{2} \sin^2(2\pi x), \\
\sqrt{2}e_1(x)e_2(x) = 2 \sin(\pi x)\sin(2\pi x),$$

which form an orthonormal basis of $F_1$. The functions $b_1, b_2, b_3$ are defined as in the proof of Theorem 3. We consider the integration problem on $F_1$:

$$S_1 : F_1 \to \mathbb{R}, \quad S_1(f) = \int_0^1 f(x) \, dx$$

and its tensor product version $S_d$ on $F_d$. We observe that $S_1(e_1^2) = S_1(e_2^2) = \sqrt{2}/2, S_1(e_1e_2) = 0$ and $S_1(f) = \langle f, b_1 \rangle$ for all $f \in F_1 = \text{span}\{b_1, b_2, b_3\}$.

**Corollary 7.** Let $F_1$ be the RKHS on $[0, 1]$ with the orthonormal basis defined in (2). Then the integration problem $S_d(f) = \int_{[0,1]^d} f(x) \, dx$ satisfies

$$e(n, S_d)^2 \geq 1 - n 2^{-d},$$

i.e. it suffers from the curse of dimensionality.

**Remark 2.** Let us observe that every $f \in F_1$ satisfies $f(0) = f(1) = f'(0) = f'(1) = 0$. This means that the functions from $F_d$ and all their partial derivatives of order at most one in any of the variables vanish on the boundary of the unit cube. Furthermore, the norm on $F_1$ can be given for example as

$$\|f\|^2 = \frac{1}{2} f(1/2)^2 + \frac{1}{16\pi^2} f'(1/2)^2 + \frac{1}{128\pi^4} [f''(1/4) + f''(3/4)]^2.$$  

### 3.5 Hilbert spaces with decomposable kernels

Another known method to prove lower bounds for tensor product functionals works for so called decomposable kernels and slight modifications, see [14, Chapter 11]. There is some intersection where our method and the decomposable kernel method both work.

Let $F_1$ be a RKHS on $D_1 \subset \mathbb{R}$ with reproducing kernel $K_1$. The kernel $K_1$ is called decomposable if there exists $a^* \in \mathbb{R}$ such that the sets

$$D_{(1)} = \{x \in D_1 \mid x \leq a^*\} \quad \text{and} \quad D_{(2)} = \{x \in D_1 \mid x \geq a^*\}$$

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are nonempty and $K_1(x, y) = 0$ if $(x, y) \in D_{(1)} \times D_{(2)}$ or $(x, y) \in D_{(2)} \times D_{(1)}$. If $K_1$ is decomposable, then $F_1$ is an orthonormal sum of $F_{(1)}$ and $F_{(2)}$ consisting of the functions in $F_1$ with support in $D_{(1)}$ and $D_{(2)}$, respectively.

Choosing now arbitrary suitably scaled functions $e_1$ with support in $D_{(1)}$ and $e_2$ with support in $D_{(2)}$ such that $e_1^2 \in F_{(1)}$ and $e_2^2 \in F_{(2)}$, we automatically have that $e_1^2$ and $e_2^2$ are orthonormal in $F_1$ and $e_1e_2 = 0$. The proof of Theorem 3 is easily adapted to this case and gives the next corollary.

**Corollary 8.** Let $F_1$ be a RKHS on $D_1 \subset \mathbb{R}$ with decomposable reproducing kernel. Let $e_1$ and $e_2$ be as above and let $h_1 = \frac{1}{2}\sqrt{2}(e_1^2 + e_2^2)$. Then the tensor product problem $S_d = \langle \cdot, h_d \rangle$ satisfies

$$e(n, S_d)^2 \geq 1 - n2^{-d}.$$  

In particular, it suffers from the curse of dimensionality.

One particular example, where this corollary is applicable, is the centered $L_2$-discrepancy. Here $F_1$ consists of absolutely continuous functions $f$ on $[0, 1]$ with $f(1/2) = 0$ and $f' \in L_2[0, 1]$. The norm of $f$ in $F_1$ is the $L_2$-norm of $f'$. The kernel of $F_1$ is $K_1(x, y) = (|x-1/2|+|y-1/2|-|x-y|)/2$, the normalized representer of the integration problem is $h_1(x) = (|x-1/2|-|x-1/2|^2)/2$. Then $e_1^2$ is the normalized restriction of $h_1$ to the interval $[0, 1/2]$, similarly, $e_2^2$ is the normalized restriction of $h_1$ to the interval $[1/2, 1]$. Since $h_1$ is nonnegative, such functions $e_1$ and $e_2$ exist.

Corollary 8 is a special case (for $\alpha = 1/2$) of [14, Theorem 11.8]. As such, it will not give any new results. Nevertheless, it seems appropriate to note the connection. It would be interesting to know if the full strength of [14, Theorem 11.8] can be obtained via this approach or the variants described in the next section.

### 3.6 Exact Integration

Based on the results above one may ask whether

$$e(2^d - 1, S_d) > 0$$

for all nontrivial tensor product problems. Here a problem is called trivial if $e(1, S_1) = 0$, then we have also $e(1, S_d) = 0$ for all $d$. The answer is “no”, examples with $e(d, S_d) > 0$ but $e(d + 1, S_d) = 0$ can be found in [14, Section 11.3] which is based on [11]. We obtain the following criterion.
Corollary 9. If there are functions $e_1$ and $e_2$ such that $e_1^2, e_2^2, e_1e_2 \in F_1$ are linearly independent with $S_1(e_1^2) \neq 0$ and $S_1(e_1e_2) = 0$, then

$$e(2^d - 1, S_d) > 0.$$ 

4 Non-homogeneous tensor products

We now turn to tensor products whose factors $F_i$ and $h_i$ may be different for each $i \leq d$. We start with the following generalization of Theorem 3, which involves an additional parameter $\alpha_i$.

Theorem 10. For all $i \leq d$, let $F_i$ be a RKHS and let $S_i$ be a bounded linear functional on $F_i$ with unit norm and nonnegative representer $h_i$. Assume that there are functions $f_i$ and $g_i$ in $F_i$ and a number $\alpha_i \in (0, 1]$ such that $(h_i, f_i, g_i)$ is orthonormal in $F_i$ and $\alpha_i^2 h_i^2 = f_i^2 + g_i^2$. Then the tensor product problem $S_d = S_1 \otimes \ldots \otimes S_d$ satisfies for all $n \in \mathbb{N}$ that

$$e(n, S_d)^2 \geq 1 - n \prod_{i=1}^{d}(1 + \alpha_i^2)^{-1}.$$ 

Proof. Let $D_i$ be the domain of the space $F_i$. Without loss of generality, we may assume that $(h_i, f_i, g_i)$ is an orthonormal basis of $F_i$. In this case, the reproducing kernel of $F_i$ is given by

$$K_i : D_i \times D_i \to \mathbb{R}, \quad K_i(x, y) = h_i(x)h_i(y) + f_i(x)f_i(y) + g_i(x)g_i(y).$$

Let us consider the functions

$$a_i = 2^{-1/4} \sqrt{\alpha_i h_i + f_i}, \quad b_i = 2^{-1/4} \text{sgn}(g_i) \sqrt{\alpha_i h_i - f_i}$$

on the domain $D_i$ of $F_i$. These functions are well defined since $\alpha_i h_i \geq |f_i|$ and linearly independent since $h$ and $f$ are linearly independent. The function

$$M_i : D_i \times D_i \to \mathbb{R}, \quad M_i(x, y) = a_i(x)a_i(y) + b_i(x)b_i(y)$$

is a reproducing kernel on $D_i$ and its diagonal is $\sqrt{2} \alpha_i h_i$. A simple computation shows for all $x, y \in D_i$ that

$$K_i(x, y) = M_i^2(x, y) + (1 - \alpha_i^2) h_i(x)h_i(y).$$
Let now $K_d$ be the reproducing kernel of the product space $F_d = F_1 \otimes \ldots \otimes F_d$ with domain $D_d = D_1 \times \ldots \times D_d$ and let $x_1, \ldots, x_n \in D_d$. We have

$$K_d(x_j, x_k) = \prod_{i=1}^{d} K_i(x_{j,i}, x_{k,i}) = \sum_{A \subset \{1, \ldots, d\}} K_A^d(x_j, x_k),$$

where

$$K_A^d(x_j, x_k) = \prod_{i \in A} M_i^2(x_{j,i}, x_{k,i}) \prod_{i \not\in A} (1 - \alpha_i^2) h_i(x_{j,i}) h_i(x_{k,i}).$$

The application of Proposition 2 yields

$$\left( \prod_{i \in A} M_i^2(x_{j,i}, x_{k,i}) \right)^n_{j,k=1} \geq \frac{1}{n} \left( \prod_{i \in A} 2\alpha_i^2 h_i(x_{j,i}) h_i(x_{k,i}) \right)^n_{j,k=1}$$

and hence

$$\left( K_d^A(x_j, x_k) \right)^n_{j,k=1} \geq \frac{1}{n} \prod_{i \in A} 2\alpha_i^2 \prod_{i \not\in A} (1 - \alpha_i^2) \left( h_d(x_j) h_d(x_k) \right)^n_{j,k=1},$$

where $h_d = h_1 \otimes \ldots \otimes h_d$ is the representer of the product functional $S_d$. Summing over all subsets $A$, we arrive at

$$\left( K_d(x_j, x_k) \right)^n_{j,k=1} = \sum_{A \subset \{1, \ldots, d\}} \left( K_A^d(x_j, x_k) \right)^n_{j,k=1} \geq \frac{1}{n} \sum_{A \subset \{1, \ldots, d\}} \prod_{i \in A} 2\alpha_i^2 \prod_{i \not\in A} (1 - \alpha_i^2) \left( h_d(x_j) h_d(x_k) \right)^n_{j,k=1} = \frac{1}{n} \prod_{i=1}^{d} \left( 1 + \alpha_i^2 \right) \left( h_d(x_j) h_d(x_k) \right)^n_{j,k=1}.$$

Now the statement follows by Proposition 1.

As applications of this result, we consider spaces of trigonometric polynomials, Korobov spaces with increasing smoothness and Korobov spaces with product weights.
4.1 Trigonometric polynomials

The most prominent special case of Theorem 10 is the case of trigonometric polynomials of order at most one, i.e.,

\[ h_i(x) = 1, \quad f_i(x) = \alpha_i \cos(2\pi x), \quad g_i(x) = \alpha_i \sin(2\pi x), \quad x \in [0, 1], \quad (3) \]

which leads to the following result.

**Corollary 11.** For all \( 1 \leq i \leq d \), let \( \alpha_i \in (0, 1] \) and let \( F_i \) be a RKHS on \([0, 1]\) such that \((h_i, f_i, g_i)\) defined in (3) are orthonormal in \( F_i \). Then the integration problem \( S_d(f) = \int_{[0,1]^d} f(x)dx \) satisfies on \( F_d = F_1 \otimes \cdots \otimes F_d \)

\[ e(n, S_d)^2 \geq 1 - n \prod_{i=1}^d (1 + \alpha_i^2)^{-1}. \]

Corollary 11 can be used to prove lower bounds for numerical integration on spaces with varying smoothness. Such classes were studied in [9] for the approximation problem and upper bounds for numerical integration problem were provided in [14, Section 10.7.4]. We first recall the notation.

For a non-decreasing sequence of positive integers \( r = (r_i)_{i=1}^\infty \) we consider the spaces \( H_{1,r_i} \) of 1-periodic real valued functions \( f \) defined on \([0, 1]\) such that \( f^{(r_i-1)} \) is absolutely continuous and \( f^{(r_i)} \) belongs to \( L^2([0, 1]) \). The norm on \( H_{1,r_i} \) is given by

\[ \|f\|_{H_{1,r_i}} = \left| \int_0^1 f(x)dx \right|^2 + \int_0^1 |f^{(r_i)}(x)|^2dx. \]

The Korobov space of varying smoothness is then defined by

\[ F_d = H_{1,r_1} \otimes \cdots \otimes H_{1,r_d}. \]

If we set \( \alpha_i = \sqrt{2} \cdot (2\pi)^{-r_i} \), then \((h_i, f_i, g_i)\) from (3) form an orthonormal system in \( H_{1,r_i} \) and we denote their span in \( H_{1,r_i} \) by \( \tilde{H}_{1,r_i} \). We will prove lower bounds for \( F_d \) by actually considering only the \( 3^d \)-dimensional space

\[ \tilde{F}_d = \tilde{H}_{1,r_1} \otimes \cdots \otimes \tilde{H}_{1,r_d}. \]

We consider the integration problem

\[ S_d(f) = \int_{[0,1]^d} f(x)dx, \quad f \in \tilde{F}_d. \]
The complexity of the problem is denoted by

\[ n(\varepsilon, S_d) = \min \{ n \in \mathbb{N} \mid e(n, S_d) \leq \varepsilon \} \]

We call the problem *polynomially tractable* if there are positive constants \(C, p, q > 0\) such that

\[ n(\varepsilon, S_d) \leq Cd^p \varepsilon^{-q} \]

for all \(\varepsilon > 0\) and \(d \in \mathbb{N}\). We call it *strongly polynomially tractable* if we can choose \(p = 0\) in this estimate. Moreover, the problem is called *weakly tractable* if

\[ \lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, S_d)}{\varepsilon^{-1} + d} = 0. \]

It was observed in [14, Section 10.7.4] (see also Corollary 10.5 there), that

- if \(L^{\text{sup}} := \limsup_{i \to \infty} \frac{\ln(i)}{r_i} < 2 \ln(2\pi)\), then integration on \(F_d\) is strongly polynomially tractable;
- if \(L^{\text{sup}} < +\infty\), then integration on \(F_d\) is weakly tractable.

We complement this by showing lower bounds for numerical integration on \(\tilde{F}_d\) (which of course also apply to the larger space \(F_d\)). By Corollary 11, we obtain the estimate

\[ e(n, S_d)^2 \geq 1 - n \prod_{i=1}^{d} (1 + \alpha_i^2)^{-1} = 1 - n \prod_{i=1}^{d} (1 + 2 \cdot (2\pi)^{-2r_i})^{-1}. \quad (4) \]

**Corollary 12.** For \(d \geq 2\), let \(F_d\) be the Korobov space of varying smoothness on \([0, 1]^d\) given by the sequence \(r = (r_i)_{i=1}^\infty\) and let \(\tilde{F}_d\) be its \(3^d\)-dimensional subspace of trigonometric polynomials of order at most one in each variable.

(i) If \(L^{\text{sup}} := \limsup_{i \to \infty} \frac{\ln(i)}{r_i} = \infty\), then numerical integration on \(\tilde{F}_d\) (and hence also on \(F_d\)) satisfies for any \(\varepsilon, \beta > 0\) that

\[ n(\varepsilon, S_d) \geq c_{\varepsilon, \beta} \exp \left( d^{1-\beta} \right) . \]

(ii) If \(L^{\text{inf}} := \liminf_{i \to \infty} \frac{\ln(i)}{r_i} = \infty\), then numerical integration on \(\tilde{F}_d\) (and hence also on \(F_d\)) satisfies for any \(\varepsilon, \beta > 0\) that

\[ n(\varepsilon, S_d) \geq c_{\varepsilon, \beta} \exp \left( d^{1-\beta} \right) . \]

(iii) If \(L^{\text{inf}} > 2 \ln(2\pi)\), then numerical integration on \(\tilde{F}_d\) (and hence also on \(F_d\)) is not polynomially tractable.
Proof. The proof is a direct consequence of (4). If \( r_i \leq R < \infty \) for all \( i \in \mathbb{N} \), then
\[
e(n, S_d)^2 \geq 1 - n(1 + 2 \cdot (2\pi)^{-2R})^{-d}.
\]
This implies \( n(\varepsilon, S_d) \geq (1 - \varepsilon^2)(1 + 2 \cdot (2\pi)^{-2R})^d \) and finishes the proof of (i).

To prove (ii) and (iii), we observe that there is some \( 0 < \beta < 1 \) and \( i_0 \in \mathbb{N} \) such that \( 2r_i \ln(2\pi) \leq \beta \ln(i) \) for \( i \geq i_0 \). In the case of (ii) we can even find such \( i_0 = i_0(\beta) \) for any \( 0 < \beta < 1 \). Consequently, for \( d \) large enough,
\[
n(\varepsilon, S_d) \geq (1 - \varepsilon^2) \prod_{i=1}^{d} (1 + 2(2\pi)^{-2r_i}) \geq (1 - \varepsilon^2) \prod_{i=i_0}^{d} (1 + 2i^{-\beta})
\]
\[
\geq (1 - \varepsilon^2) \prod_{i=i_0}^{d} \exp(i^{-\beta}) = (1 - \varepsilon^2) \exp\left(\sum_{i=i_0}^{d} i^{-\beta}\right)
\]
\[
\geq (1 - \varepsilon^2) \exp(c_{\beta}d^{1-\beta}),
\]
which shows both (ii) and (iii).
\( \square \)

4.2 Korobov spaces with product weights

In a quite similar manner, Corollary 11 can be used to re-prove the lower bounds for numerical integration on Korobov spaces with product weights, see [3] or [14, Section 16.8]. Again, we first recall the necessary notation, see [13, Appendix A] for details. For a real parameter \( s > 1/2 \), we define
\[
\varrho_{1,s,\gamma}(h) = \begin{cases} 1, & h = 0, \\ \frac{|2\pi h|^{2s}}{\gamma}, & h \in \mathbb{Z} \setminus \{0\}. \end{cases}
\]
The space \( H_{1,s,\gamma} \) of square-integrable functions on \([0, 1]\) is defined by the norm
\[
\|f\|_{H_{1,s,\gamma}}^2 = \sum_{h \in \mathbb{Z}} \varrho_{1,s,\gamma}(h)|\hat{f}(h)|^2,
\]
where
\[
\hat{f}(h) = \int_0^1 \exp(-2\pi i h x)f(x)dx, \quad h \in \mathbb{Z}
\]
are the Fourier coefficients of \( f \) and \( i = \sqrt{-1} \) is the imaginary unit.
If $\gamma = (\gamma_{d,j})_{d \in \mathbb{N}, 1 \leq j \leq d}$ is a sequence of positive weights, the weighted Korobov space (with product weights $\gamma$) $H_{d,s,\gamma}$ is defined as the tensor product

$$H_{d,s,\gamma} = H_{1,s,\gamma_{d,1}} \otimes \cdots \otimes H_{1,s,\gamma_{d,d}}.$$ 

If $\alpha_{d,j} = \sqrt{2\gamma_{d,j} \cdot (2\pi)^{-s}}$, the functions $(1, \alpha_{d,j} \cos(2\pi x), \alpha_{d,j} \sin(2\pi x))$ are orthonormal in $H_{1,s,\gamma_{d,j}}$. We denote their linear span in $H_{1,s,\gamma_{d,j}}$ by $\tilde{H}_{1,s,\gamma_{d,j}}$ and

$$\tilde{H}_{d,s,\gamma} = \tilde{H}_{1,s,\gamma_{d,1}} \otimes \cdots \otimes \tilde{H}_{1,s,\gamma_{d,d}}.$$

Using Corollary 11, we can re-prove (in a rather straightforward way) the lower bounds of Theorem 16.16 in [14]. Moreover, we show that the same lower bounds apply also to the much smaller subspaces $\tilde{H}_{d,s,\gamma}$.

**Proposition 13.** Let $S_d(f) = \int_{[0,1]^d} f(x)dx$ denote the multivariate integration problem defined over the sequence of Korobov spaces $H_{d,s,\gamma}$, where $s > 1/2$ and $\gamma = (\gamma_{d,j})_{d \in \mathbb{N}, 1 \leq j \leq d}$ is a bounded sequence. Let $\tilde{H}_{d,s,\gamma}$ be the $3^d$-dimensional subspaces of trigonometric polynomials of degree at most one in each variable in $H_{d,s,\gamma}$.

(i) If $(S_d)$ is strongly polynomially tractable on $\tilde{H}_{d,s,\gamma}$, then

$$\sup_{d \in \mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j} < \infty.$$ 

(ii) If $(S_d)$ is polynomially tractable on $\tilde{H}_{d,s,\gamma}$, then

$$\limsup_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}}{\ln(d+1)} < \infty.$$ 

(iii) If $(S_d)$ is weakly tractable on $\tilde{H}_{d,s,\gamma}$, then

$$\lim_{d \to \infty} \frac{1}{d} \sum_{j=1}^{d} \gamma_{d,j} = 0.$$ 

**Proof.** We put $\alpha_{d,j} = \sqrt{2\gamma_{d,j} \cdot (2\pi)^{-s}}$ and obtain by Corollary 11

$$n(\varepsilon, S_d) \geq (1 - \varepsilon^2) \prod_{j=1}^{d} (1 + \alpha_{d,j}^2) = (1 - \varepsilon^2) \prod_{j=1}^{d} \left(1 + 2\gamma_{d,j} (2\pi)^{-2s}\right).$$
If \((S_d)\) is strongly polynomially tractable, we observe from

\[
n(\varepsilon, S_d) \geq (1 - \varepsilon^2) \cdot 2 \cdot (2\pi)^{-2s} \sum_{j=1}^{d} \gamma_{d,j}
\]

that \(\sum_{j=1}^{d} \gamma_{d,j}\) must be uniformly bounded in \(d \in \mathbb{N}\).

If \((S_d)\) is polynomially tractable or weakly tractable, we use the boundedness of \(\gamma\) to estimate

\[
\ln n(\varepsilon, S_d) \geq \ln(1 - \varepsilon^2) + \sum_{j=1}^{d} \ln \left(1 + 2\gamma_{d,j}(2\pi)^{-2s}\right) \geq \ln(1 - \varepsilon^2) + C \sum_{j=1}^{d} \gamma_{d,j}.
\]

This estimate proves both (ii) and (iii).

\[\square\]

5 New variants of Schur’s Theorem

In this section we present several variants of the uniform lower bound for the Schur product obtained in [19] and several consequences for the tractability of numerical integration.

5.1 Modifications of Schur’s Theorem

The first generalization of Proposition 2 deals with matrices with reduced rank. Independently, it was also observed in [7].

**Theorem 14.** Let \(M \in \mathbb{R}^{n \times n}\) be a positive semi-definite matrix with rank \(r\). Then

\[
M \circ M \succeq \frac{1}{r} (\text{diag}(M))(\text{diag}(M))^T.
\]

**Proof.** The proof follows in the same way as in [19] but we write \(M = AA^T\), where \(A \in \mathbb{R}^{n \times r}\).

The next version deals with the Schur product of two possibly different matrices \(M \neq N\). In this sense, it addresses a problem left open in [19].
Theorem 15. Let $M, N \in \mathbb{R}^{n \times n}$ be positive semi-definite matrices with $M = AA^T$ and $N = BB^T$ with $A, B \in \mathbb{R}^{n \times D}$ and $D \geq \max(\text{rank}(M), \text{rank}(N))$. Then, for every $c \in \mathbb{R}^n$,

$$
\sum_{j,k=1}^{n} c_j c_k M_{j,k} N_{j,k} \geq \frac{1}{D} \left( \sum_{j=1}^{n} c_j \langle A^j, B^j \rangle \right)^2,
$$

(5)

where $A^j, B^j$ are the rows of $A$ and $B$, respectively.

Proof. The proof is again similar to [19]. We write

$$
\sum_{j,k=1}^{n} c_j c_k M_{j,k} N_{j,k} = \sum_{j,k=1}^{n} c_j c_k \sum_{l=1}^{D} A_{j,l} A_{k,l} \sum_{m=1}^{D} B_{j,m} B_{k,m}
$$

$$
= \sum_{l,m=1}^{D} \left( \sum_{j=1}^{n} c_j A_{j,l} B_{j,m} \right)^2 \geq \sum_{l=1}^{D} \left( \sum_{j=1}^{n} c_j A_{j,l} B_{j,l} \right)^2
$$

$$
\geq \frac{1}{D} \left( \sum_{j=1}^{n} c_j \sum_{l=1}^{D} A_{j,l} B_{j,l} \right)^2.
$$

□

Remark 3. Using $(AB^T)_{j,j} = \langle A^j, B^j \rangle$, the estimate (5) can be written as

$$
M \circ N \succeq \frac{1}{D} \text{diag}(AB^T)(\text{diag}(AB^T))^T.
$$

The last generalization of Schur’s Theorem, that, in a sense, combines Theorem 14 and Theorem 15 is the one we shall use later on.

Theorem 16. Let $M \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix with rank $r$. Let $M = AA^T = BB^T$ with $A, B \in \mathbb{R}^{n \times D}$ for some $D \geq r$. Then, for every $c \in \mathbb{R}^n$,

$$
\sum_{j,k=1}^{n} c_j c_k M_{j,k}^2 \geq \frac{1}{2^r} \left( \sum_{j=1}^{n} c_j \langle A^j, B^j \rangle \right)^2,
$$

(6)

where $A^j, B^j \in \mathbb{R}^D$ are the rows of $A$ and $B$, respectively.

Proof. We show that there exist two matrices $G, H \in \mathbb{R}^{n \times 2r}$ with rows denoted by $G^j$ and $H^j$, respectively, such that $M = GG^T = HH^T$ and
\[ \langle G^j, H^j \rangle = \langle A^j, B^j \rangle \text{ for every } j = 1, \ldots, n. \] The proof then follows by an application of Theorem 15 with \( M = N \) and \( 2r \) instead of \( r \).

Using the singular value decomposition theorem, we can write \( A = U \Sigma V^T \) and \( B = U \Sigma W^T \), where \( U \in \mathbb{R}^{n \times r} \), \( \Sigma \in \mathbb{R}^{r \times r} \) and \( V, W \in \mathbb{R}^{d \times r} \). Here, \( U, V \) and \( W \) have orthonormal columns and \( \Sigma \) is a diagonal matrix. Furthermore, \( \langle A^j, B^j \rangle = (AB^T)_{j,j} = e_j^T(U \Sigma V^T)(W \Sigma U^T)e_j = \varepsilon_j^T V^T W \varepsilon_j \), where \( \varepsilon_j = \Sigma U^T e_j \in \mathbb{R}^r \) and \( (e_j)_{j=1}^n \) is the canonical basis of \( \mathbb{R}^n \). In the same way, we are looking for \( G = U \Sigma X^T \) and \( H = U \Sigma Z^T \) with matrices \( X, Z \in \mathbb{R}^{2r \times r} \) with orthonormal columns and

\[ \langle G^j, H^j \rangle = \varepsilon_j^T X^T Z \varepsilon_j = \varepsilon_j^T V^T W \varepsilon_j = \langle A^j, B^j \rangle, \quad j = 1, \ldots, n. \quad (7) \]

The matrix \( V^T W \) is formed by the scalar products of the column vectors of \( V \) and \( W \), respectively. Using an orthogonal projection onto their common linear span (which has dimension at most \( 2r \)), we can find \( X, Z \in \mathbb{R}^{2r \times r} \) such that \( X^T Z = V^T W \), which is even stronger than (7).

5.2 Applications to numerical integration

Theorem 16 allows us to extend Theorem 3 to a larger class of tensor product problems with \( e(0, S_d) = \|h_d\| = 1. \)

**Theorem 17.** Let \( M \) be a reproducing kernel on a set \( D \) and let \( K = M^2 \). Denote by \( H(M) \) and \( H(K) \) the Hilbert spaces with reproducing kernel \( M \) and \( K \), respectively. Let \( (b_\ell)_{\ell \in I} \) and \( (\tilde{b}_\ell)_{\ell \in I} \) be two orthonormal bases of \( H(M) \) and

\[ g = \sum_{\ell \in I} b_\ell \tilde{b}_\ell \in H(K). \]

We consider the normalized problem \( S = \langle \cdot, h \rangle \) with \( h = g/\|g\| \) on \( H(K) \). Then

\[ e(n, S)^2 \geq 1 - \frac{2n}{\|g\|^2}. \]

**Proof.** For any \( x, y \in D \), we have

\[ M(x, y) = \sum_{\ell \in I} b_\ell(x) \tilde{b}_\ell(y) = \sum_{\ell \in I} b_\ell(x) \tilde{b}_\ell(y). \]
Let $x_1, \ldots, x_n \in D$ and let $M = (M(x_j, x_k))_{j,k \leq n}$. Then

$$M = BB^T = \tilde{B} \tilde{B}^T,$$

where $B = (b_\ell(x_j))_{j \leq n, \ell \in I}$ and $\tilde{B} = (\tilde{b}_\ell(x_j))_{j \leq n, \ell \in I}$. Theorem 16 yields that

$$\sum_{j,k=1}^{n} c_j c_k M_{j,k}^2 \geq \frac{1}{2n} \left( \sum_{j=1}^{n} c_j (B \tilde{B}^T)_{jj} \right)^2 = \frac{\|g\|^2}{2n} \left( \sum_{j=1}^{n} c_j h(x_j) \right)^2$$

and thus the desired lower bound follows from Proposition 1.

Let us observe that Theorem 3 is obtained by considering particular orthonormal bases of $H(M_d)$. Namely, we take

$$b_\ell(x) = \prod_{i=1}^{d} e_{\ell_i}(x_i) \quad \text{for} \quad \ell \in \{1, 2\}^d$$

and $\tilde{b}_\ell = b_\ell$, $\ell \in \{1, 2\}^d$. Then we have

$$g(x) = \prod_{i=1}^{d} \left( e_1(x_i)^2 + e_2(x_i)^2 \right).$$

and we obtain Theorem 3 (up to a factor 2).

Another interesting choice of $(b_\ell)_{\ell \in I}$ and $(\tilde{b}_\ell)_{\ell \in I}$ of $H(M_d)$ is the following. We take again $b_\ell$ defined by (8) and

$$\tilde{b}_\ell(x) = \prod_{i=1}^{d} e_{\ell_i}^{(i)}(x_i) \quad \text{for} \quad \ell \in \{1, 2\}^d$$

where

$$\begin{pmatrix} e_{1}^{(i)} \\ e_{2}^{(i)} \end{pmatrix} = U_i \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

and $U_i \in \mathbb{R}^{2 \times 2}$ is an orthogonal matrix.

If $U_i$ is the identity matrix, we obtain $e_1^{(i)} = e_1$, $e_2^{(i)} = e_2$ and $e_1^{(i)} \cdot e_1 + e_2^{(i)} \cdot e_2 = e_1^2 + e_2^2$. If, on the other hand, we choose

$$U_i = \begin{pmatrix} \cos \varphi_i & \sin \varphi_i \\ \sin \varphi_i & -\cos \varphi_i \end{pmatrix}, \quad \varphi_i \in [0, 2\pi]$$
being a reflection across a line with angle $\varphi_i/2$, we obtain
\[
\tilde{e}_1^{(i)} \cdot e_1 + \tilde{e}_2^{(i)} \cdot e_2 = \cos \varphi_i \cdot (e_1^2 - e_2^2) + 2 \sin \varphi_i \cdot e_1 e_2.
\]

Of course, we can mix these two examples by taking a different choice of $U_i$ for each dimension $i \leq d$, which leads to the following result.

**Corollary 18.** Let $F_1$ be a RKHS on $D_1$. Assume that there are functions $e_1$ and $e_2$ on $D_1$ such that $e_1^2, e_2^2$ and $\sqrt{2} e_1 e_2$ are orthonormal in $F_1$. Let
\[
h_d(x) = \prod_{i=1}^d h_i(x_i),
\]
where $h_i \in \text{span}\{e_1^2 + e_2^2\} \cup \text{span}\{e_1^2 - e_2^2, e_1 e_2\}$ has unit norm $\|h_i\| = 1$. Then the tensor product problem $S_d = \langle \cdot, h_d \rangle$ satisfies
\[
e(n, S_d)^2 \geq 1 - n 2^{-d+1}.
\]
In particular, it suffers from the curse of dimensionality.

For the next result we take again the space of trigonometric polynomials of degree 1, see Corollary 4.

**Corollary 19.** Let $F_1$ be the RKHS on $[0, 1]$ with the orthonormal system $1, \cos(2\pi x)$ and $\sin(2\pi x)$. Let $d \geq 2$ and let $\{\varphi_i\}_{i=1}^\infty \subset [0, 2\pi]$ be a bounded sequence. Let
\[
h_d(x) = \prod_{i=1}^d h_i(x_i),
\]
where
\[
h_i(x_i) = \cos \varphi_i \cdot \cos(2\pi x_i) + \sin \varphi_i \cdot \sin(2\pi x_i) = \cos(2\pi x_i - \varphi_i) \quad (10)
\]
or $h_i = 1$. Then the corresponding problem $S_d = \langle \cdot, h_d \rangle$ satisfies
\[
e(n, S_d)^2 \geq 1 - n 2^{-d+1}
\]
and the problem suffers from the curse of dimensionality.
Remark 4. Let us reformulate Corollary 19 as an integration problem. As in Section 3.1, we denote again $e_1(x) = 2^{1/4} \cos(\pi x)$ and $e_2 = 2^{1/4} \sin(\pi x)$ on $[0, 1]$. Let $\varphi \in [0, 2\pi]$ and let $h(x) = \cos(2\pi x - \varphi)$, $x \in [0, 1]$, cf. (10). Then

$$h(x) = \cos \varphi \cdot \frac{e_1^2(x) - e_2^2(x)}{\sqrt{2}} + \sin \varphi \cdot \sqrt{2} e_1(x) e_2(x).$$

Consequently, if we define $S(f) = \langle f, h \rangle$ for $f \in F_1$, it satisfies

$$S(e_1^2) = \frac{\cos \varphi}{\sqrt{2}}, \quad S(e_2^2) = -\frac{\cos \varphi}{\sqrt{2}}, \quad S(\sqrt{2} e_1 e_2) = \sin \varphi$$

and we obtain

$$S(f) = 2 \int_0^1 f(x) \cos(2\pi x - \varphi) \, dx, \quad f \in F_1.$$

Similarly, if we denote in Corollary 19 by $I \subset \{1, \ldots, d\}$ those indices, for which $h_i$ is given by (10), then

$$S_d(f) = \langle f, h_d \rangle = \int_{[0,1]^d} f(x) \prod_{i \in I} [2 \cos(2\pi x_i - \varphi_i)] \, dx.$$

We finish this section by a couple of open problems.

**Open Problem 1.** We conjecture that Theorem 16 holds true with $\frac{1}{r}$ instead of $\frac{1}{2r}$ in (6), see also [7, Theorem 1.9]. This would allow to improve the error bound in Corollary 18 and Corollary 19 to

$$e(n, S_d)^2 \geq 1 - n 2^{-d}.$$

**Open Problem 2.** Corollary 15 shows the curse for all problems $S_d = \langle \cdot, h_d \rangle$, where $h_d = \bigotimes_{i=1}^d h_i$ is a tensor product with all components $h_i$ being unit norm functions from either the span of $e_1^2 + e_2^2$ or from the span of $e_1^2 - e_2^2$ and $e_1 e_2$. Is the same true if we allow arbitrary $h_i \in F_1$?

6 Randomly chosen sample points

We continue our analysis of high dimensional integration problems. In the previous sections, we mainly studied the quality of optimal sample points.
Optimal sample points are usually hard to find. In this section, we switch our point of view and ask for the quality of random point sets \( X_n = \{x_1, \ldots, x_n\} \), where the points \( x_1, \ldots, x_n \) are independent and identically distributed in the domain of integration. Therefore, \( X_n \) is a random variable in this setting. Like for optimal points, one can ask: How many random points do we need to solve the integration problem up to the error \( \varepsilon > 0 \)? Does this number depend exponentially on \( d \), i.e., do we have the curse for random information? We can use Proposition 2 to obtain the following result for Lebesgue integration on the unit cube.

**Theorem 20.** Let \( F_1 \) be a RKHS on \([0, 1]\) with non-negative kernel \( K_1 \) such that \( 1 \in F_1 \) is the representer of the integral. If

\[
a = \inf_{x \in [0, 1]} K_1(x, x) > 1,
\]

then the integration problem on the tensor product space \( F_d \) suffers from the curse of dimensionality for random information. For all \( n < a^{d/4}/4 \), we have

\[
\mathbb{E} \left[ e(X_n, S_d)^2 \right] \geq \frac{1}{4},
\]

where \( X_n = \{x_1, \ldots, x_n\} \) with independent and uniformly distributed points \( x_1, \ldots, x_n \in [0, 1]^d \).

**Remark 5.** If the kernel is continuous, then (11) is equivalent to the claim that for any \( x \in [0, 1] \) there is some \( f \) from the unit ball of \( F_1 \) with \( f(x) > 1 \). Clearly, a statement like that of Theorem 20 also holds for any other closed interval of length 1.

**Proof.** First note that the initial error is given by

\[
e(0, S_1)^2 = \|1\|^2 = \langle 1, 1 \rangle = \int_0^1 1 \, dx = 1,
\]

so that the problem is properly normalized. Let \( n < a^{d/4}/4 \). Clearly, for all \( x \in [0, 1]^d \) and \( i \leq n \), we have

\[
\mathbb{E} [K_d(x, x_i)] = \langle K_d(x, \cdot), 1 \rangle = 1 \leq a^{-d} K_d(x, x).
\]

Using Markov’s inequality and non-negativity of \( K \), this implies for all \( j \neq i \) that

\[
K_d(x_i, x_j) \leq a^{-d/2} K_d(x_i, x_i)
\]

(12)
with probability at least $1 - a^{-d/2}$.

Thus, (12) holds simultaneously for all $j \neq i$ with probability at least $1 - n^2a^{-d/2} \geq 1/2$. In this case, we have

$$\left| K_d(x_i, x_i) - \frac{a^d}{2n} \right| - \sum_{j \neq i} \left| K_d(x_i, x_j) - \frac{a^d}{2n} \right|
\geq K_d(x_i, x_i) - \frac{a^d}{2n} - \sum_{j \neq i} \left( K_d(x_i, x_j) + \frac{a^d}{2n} \right)
\geq K_d(x_i, x_i) - \frac{n}{a^{d/2}}K_d(x_i, x_i) - \frac{a^d}{2} > \frac{K_d(x_i, x_i)}{2} - \frac{a^d}{2} \geq 0$$

for all $i \leq n$. Therefore, the matrix $(K_d(x_i, x_j) - a^d/(2n))_{i,j\leq n}$ is diagonally dominant and hence positive definite. Proposition 2 implies that

$$e(\mathcal{X}_n, S_d)^2 \geq 1 - \frac{2n}{a^d}$$

and thus

$$\mathbb{E} \left[ e(\mathcal{X}_n, S_d)^2 \right] \geq \frac{1}{2} \left( 1 - \frac{2n}{a^d} \right) \geq \frac{1}{4},$$

which proves the statement. \qed

As an example, let us consider the integration problem on the space $F_1$ of polynomials with degree at most 2 on the interval $[-1/2, 1/2]$ with scalar product

$$\langle f, g \rangle = \langle f, g \rangle_2 + \langle f', g' \rangle_2 + \langle f'', g'' \rangle_2.$$ 

The tensor product space $F_d$ consists of polynomials with mixed order 2 or less. It was raised as an open problem in [14, Open Problem 44] whether the integration problem on the tensor product space $F_d$ suffers from the curse of dimensionality. For optimal point sets, this question remains unsolved. For random point sets, Theorem 20 yields the curse of dimensionality. Indeed, the representer of the integral is 1 and the reproducing kernel on $[-1/2, 1/2]$ is

$$K_1(x, y) = 1 + \frac{xy}{1 + 1/12} + \frac{(x^2 - 1/12)(y^2 - 1/12)}{4 + 1/3 + 1/180},$$

and satisfies the assumptions of Theorem 20.

Recent results on the quality of random information suggest that random information often is almost as good as optimal information, see [4, 5, 8]. In
this sense, they indicate that integration on $F_d$ may also be intractable for optimal information. More generally, one can ask whether every integration problem from Theorem 20 also suffers from the curse of dimensionality for optimal information.

**Open Problem 3.** Let $F_d$ be a RKHS of functions on the $d$-dimensional unit cube that satisfies the conditions of Theorem 20. Prove or disprove that Lebesgue integration on $F_d$ with optimal information suffers from the curse of dimensionality.

We note that the aforementioned papers [4, 5, 8] focus on the order of convergence of the error for $n \to \infty$ and fixed dimension $d \in \mathbb{N}$. Hence, they may only serve as a weak indicator for the intractability of these problems.

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