COMPLEX INTERPOLATION OF NONCOMMUTATIVE HARDY SPACES ASSOCIATED SEMIFINITE VON NEUMANN ALGEBRA

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ABSTRACT. We studied complex interpolation noncommutative Hardy space associated with semi-finite von Neumann algebra and extend Pisier’s interpolation theorem for this case.

1. Introduction

In [12], Pisier give a new proof of the interpolation theorem of Peter Jones (see [10] or [5], p.414). He obtained the complex case of Peter Jones’ theorem as a consequence of the real case. The Pisier’s method does to extend to the noncommutative case and the case of Banach space valued $H^p$-spaces (see §2 and §2 in [12]). In Pisier and Xu[?] obtained noncommutative version of P. Jones’ theorem for noncommutative Hardy spaces associated with a finite subdiagonal algebra in Arveson’s sense [1] (It is stated in [13] without proof, see the remark following Lemma 8.5 there). The first named author [2], using Pisier’s method, proved that the real case of Peter Jones’ theorem for noncommutative Hardy spaces associated semifinite von Neumann algebra holds (also see [15]).

This paper is devoted to the study of complex interpolation of noncommutative Hardy spaces associated semifinite von Neumann algebra. Using the Pisier method in [12], we proved the complex case of Peter Jones’ theorem for noncommutative Hardy spaces associated semifinite von Neumann algebra.

2. Preliminaries

We use standard notation and notions from noncommutative $L_p$-spaces theory (see e.g. [13, 16]). Throughout this paper, we denote by $\mathcal{M}$ a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$ with a normal faithful semifinite trace $\tau$. A closed densely defined linear operator $x$ in $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary operators $u$ which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If $x$ is affiliated with $\mathcal{M}$, then $x$ is said to be $\tau$-measurable if for every $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that $e(\mathcal{H}) \subseteq D(x)$ and $\tau(e^+e) < \varepsilon$ (where for any projection $e$ we let $e^+ = 1 - e$). The set of all $\tau$-measurable operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closure of the algebraic
sum and product. For a positive self-adjoint operator \( x = \int_0^\infty \lambda d\nu(\lambda) \) (the spectral decomposition) affiliated with \( \mathcal{M} \), we set
\[
\tau(x) = \sup_n \tau(\int_0^n \lambda d\nu(\lambda)) = \int_0^\infty \lambda \tau(e_\lambda).
\]
For \( 0 < p < \infty \), \( L_p(\mathcal{M}) \) is defined as the set of all \( \tau \)-measurable operators \( x \) affiliated with \( \mathcal{M} \) such that
\[
\|x\|_p = \tau(|x|^p)^{1/p} < \infty.
\]
In addition, we put \( L_\infty(\mathcal{M}) = \mathcal{M} \) and denote by \( \|\cdot\|_\infty \) the usual operator norm. It is well-known that \( L_p(\mathcal{M}) \) is a Banach space under \( \|\cdot\|_p \) (\( 1 \leq p \leq \infty \)) satisfying all the expected properties such as duality.

In this paper, \( [K]_p \) denotes the closed linear span of \( K \) in \( L_p(\mathcal{M}) \) (relative to the \( w^\ast \)-topology for \( p = \infty \)) and \( J(K) \) is the family of the adjoints of the elements of \( K \).

Henceforth we will assume that \( \mathcal{D} \) is a von Neumann subalgebra of \( \mathcal{M} \) such that the restriction of \( \tau \) to \( \mathcal{D} \) is still semifinite. Let \( \mathcal{E} \) be the (unique) normal positive faithful conditional expectation of \( \mathcal{M} \) with respect to \( \mathcal{D} \) such that \( \tau \circ \mathcal{E} = \tau \).

**Definition 2.1.** A \( w^\ast \)-closed subalgebra \( \mathcal{A} \) of \( \mathcal{M} \) is called a subdiagonal algebra of \( \mathcal{M} \) with respect to \( \mathcal{E} \) (or \( \mathcal{D} \)) if
\begin{enumerate}[(i)]  
  \item \( \mathcal{A} + J(\mathcal{A}) \) is \( w^\ast \)-dense in \( \mathcal{M} \),  
  \item \( \mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y) \), \( \forall x, y \in \mathcal{A} \),  
  \item \( \mathcal{A} \cap J(\mathcal{A}) = \mathcal{D} \).  
\end{enumerate}

\( \mathcal{D} \) is then called the diagonal of \( \mathcal{A} \).

It is proved by Ji [9] that a semifinite subdiagonal algebra \( \mathcal{A} \) is automatically maximal, i.e., \( \mathcal{A} \) is not properly contained in any other subalgebra of \( \mathcal{M} \) which is subdiagonal algebra respect to \( \mathcal{E} \).

Since \( \mathcal{D} \) is semifinite, we can choose an increasing family of \( \{e_i\}_{i \in I} \) of \( \tau \)-finite projections in \( \mathcal{D} \) such that \( e_i \to 1 \) strongly, where 1 is identity of \( \mathcal{M} \) (see Theorem 2.5.6 in [14]). Throughout, the \( \{e_i\}_{i \in I} \) will be used to indicate this net.

Let \( x \in L_0(\mathcal{M}) \). We define the distribution function of \( x \) by
\[
\lambda_t(x) = \tau(e_{(t,\infty)}(|x|)), \quad t \geq 0,
\]
where \( e_{(t,\infty)}(|x|) \) is the spectral projection of \( |x| \) corresponding to the interval \( (t, \infty) \). Define
\[
\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\}, \quad t > 0.
\]
The function \( t \mapsto \mu_t(x) \) is called the generalized singular numbers of \( x \). For more details on generalized singular value function of measurable operators, see [8].

For \( 0 < p < \infty \), \( 0 < q \leq \infty \), the noncommutative Lorentz space \( L^{p,q}(\mathcal{M}) \) is defined as the space of all measurable operators \( x \) such that
\[
\|x\|_{p,q} = \left( \int_0^\infty [t^{1/p} \mu_t(x)]^q \frac{dt}{t} \right)^{1/q} < \infty.
\]
Equipped with \( \| \|_{p,q} \), \( L^{p,q}(\mathcal{M}) \) is a quasi-Banach space. Moreover, if \( p > 1 \) and \( q \geq 1 \), then \( L^{p,q}(\mathcal{M}) \) can be renormed into a Banach space. More precisely,

\[
x \mapsto \left( \int_0^\infty \left[ t^{-1+1/p} \int_0^t \mu_s(x) ds \right]^q \frac{dt}{t} \right)^{1/q}
\]
gives an equivalent norm on \( L^{p,q}(\mathcal{M}) \). It is clear that \( L^{p,p}(\mathcal{M}) = L^p(\mathcal{M}) \) (for details see [7, 18]).

The space \( L^{p,\infty}(\mathcal{M}) \) is usually called a weak \( L_p \)-space, \( 0 < p < \infty \). Its quasi-norm admits the following useful description in terms of the distribution function:

\[
\| x \|_{p,\infty} = \sup_{s > 0} s \lambda_s(x)^{1/p}
\]

(see Proposition 3.1 in [3]).

For \( 0 < p < \infty \), \( 0 < q \leq \infty \), we define the noncommutative Hardy-Lorentz space

\[
H^{p,q}(\mathcal{A}) = \text{closure of } \mathcal{A} \cap L^{p,q}(\mathcal{M}) \text{ in } L^{p,q}(\mathcal{M}).
\]

\[
H^{p,q}_0(\mathcal{A}) = \text{closure of } \mathcal{A}_0 \cap L^{p,q}(\mathcal{M}) \text{ in } L^{p,q}(\mathcal{M}).
\]

3. Complex interpolation

Let \( S \) (respectively, \( \overline{S} \)) denote the open strip \( \{ z : 0 < \Re z < 1 \} \) (respectively, the closed strip \( \{ z : 0 \leq \Re z \leq 1 \} \)) in the complex plane. Let \( (X_0, X_1) \) be a compatible couple of complex Banach spaces. Let \( \mathcal{F}(X_0, X_1) \) be the family of all functions \( f : S \to X_0 + X_1 \) satisfying the following conditions:

- \( f \) is continuous on \( S \) and analytic in \( S \);
- \( f(k + it) \in X_k \) for \( t \in \mathbb{R} \) and the function \( t \mapsto f(k + it) \) is continuous from \( \mathbb{R} \) to \( X_k \), \( k = 0, 1 \);
- \( \lim_{|t| \to \infty} \| f(k + it) \|_{X_k} = 0 \), \( k = 0, 1 \).

We equip \( \mathcal{F}(X_0, X_1) \) with the norm:

\[
\| f \|_{\mathcal{F}(X_0, X_1)} = \max \left\{ \sup_{t \in \mathbb{R}} \| f(it) \|_{X_0}, \sup_{t \in \mathbb{R}} \| f(1 + it) \|_{X_1} \right\}.
\]

Then \( \mathcal{F}(X_0, X_1) \) becomes a Banach space. Let \( 0 < \theta < 1 \). The complex interpolation space \( (X_0, X_1)_\theta \) is defined as the space of all those \( x \in X_0 + X_1 \) for which there exists \( f \in \mathcal{F}(X_0, X_1) \) such that \( f(\theta) = x \). Equipped with

\[
\| x \|_\theta = \inf \left\{ \| f \|_{\mathcal{F}(X_0, X_1)} : f(\theta) = x, \; f \in \mathcal{F}(X_0, X_1) \right\},
\]

\( (X_0, X_1)_\theta \) becomes a Banach space.

Let \( \mathcal{N} \) be the algebra of infinite diagonal matrices with entries from \( \mathcal{M} \) and \( \mathcal{B} \) be the algebra of infinite diagonal matrices with entries from \( \mathcal{A} \), i.e.

\[
\mathcal{N} = \left\{ x : x = \begin{pmatrix} x_1 & 0 & \cdots \\ 0 & x_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \; x_i \in \mathcal{M}, \; i \in \mathbb{N} \right\}
\]
and

\[ \mathcal{B} = \left\{ a : a = \begin{pmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, a_i \in \mathcal{A}, i \in \mathbb{N} \right\}. \]

For \( x \in \mathcal{N} \) with entries \( x_i \), define \( \Phi(x) \) to be the diagonal matrix with entries \( \mathcal{E}(x_i) \) and

\[ \nu(x) = \sum_{i \geq 1} \tau(x_i). \]

Then \( (\mathcal{N}, \nu) \) is a simifinite von Neumann algebra and \( \mathcal{B} \) is a simifinite subdiagonal algebra of \( (\mathcal{N}, \nu) \) respect to \( \Phi \).

Using same method as in the proof of Lemma 4.1 of [12], we obtain that

**Lemma 3.1.** Let \( 1 < p < \infty \). If \( \theta = \frac{1}{p} \), then there is a bounded nature inclusion

\[ (L^{1,\infty}(\mathcal{N})/J(H_0^{1,\infty}(\mathcal{B})), \mathcal{N}/J(\mathcal{B}_0))_{1-\theta,\infty} = H^{p,\infty}(\mathcal{B}) \quad (3.1) \]

with equivalent norms.

**Proof.** Let \( 1 < r < q < \infty \). Set \( \eta = 1 - \frac{1}{r}, \gamma = 1 - \frac{1}{q} \). It is clear that \( 0 < \eta < \gamma < 1 \).

Using Theorem 4.1 in [19] and Theorem 6.3 in [2], we know that

\[ (L^1(\mathcal{N}), \mathcal{N})_{\eta, r} = L^r(\mathcal{N}), (L^1(\mathcal{N}), \mathcal{N})_{\gamma, q} = L^q(\mathcal{N}) \]

and

\[ (H^1(\mathcal{B}), \mathcal{B})_{\eta, r} = H^r(\mathcal{B}), (H^1(\mathcal{B}), \mathcal{B})_{\gamma, q} = H^q(\mathcal{B}). \]

On the other hand, (i) and (iii) of Lemma 6.5 in [2], we have that \( (H^1(\mathcal{B}), H^s(\mathcal{B})) \) is \( K \)-closed with respect to \( (L^1(\mathcal{N}), L^s(\mathcal{N})) \) and \( (H^q(\mathcal{B}), \mathcal{B}) \) is \( K \)-closed with respect to \( (L^q(\mathcal{N}), \mathcal{N}) \). Hence by Theorem 1.2 in [11], we obtain \( (H^1(\mathcal{B}), \mathcal{B}) \) is \( K \)-closed with respect to \( (L^1(\mathcal{N}), \mathcal{N}) \). Since on can extend \( \Phi \) to a contractive projection from \( L^p(\mathcal{N}) \) onto \( L^p(\mathcal{B} \cap \mathcal{B}^*) \) for every \( 1 \leq p \leq \infty \), we deduce that \( (H^1(\mathcal{B}), \mathcal{B}_0) \) is \( K \)-closed with respect to \( (L^1(\mathcal{N}), \mathcal{N}) \). By Holmstedt’s formula (see [4], p. 52-53), we get \( (H_0^{1,\infty}(\mathcal{B}), \mathcal{B}_0) \) is \( K \)-closed with respect to \( (L^{1,\infty}(\mathcal{N}), \mathcal{N}) \). Therefore,

\[ (J(H_0^{1,\infty}(\mathcal{B})), J(\mathcal{B}_0))_{1-\eta,\infty} = J(H_0^{q,\infty}(\mathcal{B})). \]

Since

\[ (L^{1,\infty}(\mathcal{N})/J(H_0^{1,\infty}(\mathcal{B})), \mathcal{N}/J(\mathcal{B}_0)) \]

is a compatible couple (see [12], p. 351-352), the space

\[ (L^{1,\infty}(\mathcal{N})/J(H_0^{1,\infty}(\mathcal{B})), \mathcal{N}/J(\mathcal{B}_0))_{1-\theta,\infty} \]

can be identified with the space

\[ L^{p,\infty}(\mathcal{N})/J(H_0^{p,\infty}(\mathcal{B})). \]

By Theorem 4.1 in [19] and Theorem 6.3 in [2], for \( 1 < p_0 < p < p_1 < \infty \) and \( 0 < \theta < 1 \) with \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \), we have that

\[ (L^{p_0}(\mathcal{N}), L^{p_1}(\mathcal{N}))_{\theta,\infty} = L^{p,\infty}(\mathcal{N}), (H^{p_0}(\mathcal{B}), H^{p_1}(\mathcal{B}))_{\theta,\infty} = H^{p,\infty}(\mathcal{B}). \]
Applying Theorem 4.2 in [2] and interpolation, we deduce that

\[ L^{p,\infty}(\mathcal{N})/J(H^{p,\infty}_0(\mathcal{B})) \]

can be identified with \( H^{p,\infty}(\mathcal{B}) \). Hence, (3.1) holds.

For \( 1 \leq p \leq \infty \), we define \( K_p : L^p(\mathcal{M}) \to L^{p,\infty}(\mathcal{N}) \) by

\[
K_p(x) = \begin{pmatrix}
x & 0 & 0 & \ldots \\
0 & 2^{-\frac{1}{p}}x & 0 & \ldots \\
0 & 0 & 3^{-\frac{1}{p}}x & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad \forall x \in L^p(\mathcal{M}).
\]

Then

\[
\|x\|_p = \|K_p(x)\|_{p,\infty}.
\] (3.2)

Indeed, it is clear that for any \( x \in \mathcal{M} \),

\[
\|K_p(x)\|_\infty = \|x\|_\infty.
\]

If \( 1 \leq p < \infty \) and \( x \in L^p(\mathcal{M}) \), then

\[
\|x\|_p^p = \tau(|x|^p) = \int_0^{+\infty} \tau(\sum_{j\geq 1} f_{t, j+1}^p(x)) dt = \sup_{t>0} \sum_{j\geq 1} t^p \tau(\sum_{j\geq 1} f_{t,\infty}^p(x)) = \sup_{t>0} \sum_{j\geq 1} t^p \nu(e_{t,\infty}^p(K_p(x))) = \sup_{t>0} t^p \lambda_t(K_p(x)) = \|K_p(x)\|_{p,\infty}^p.
\]

Hence, \( K_p \) has norm 1. Note that \( K_p \) maps \( H^p_0(\mathcal{A}) \) into \( H^{p,\infty}_0(\mathcal{B}) \). Let

\[
\tilde{K}_p : L^p(\mathcal{M})/H^p_0(\mathcal{A}) \to L^{p,\infty}(\mathcal{N})/H^{p,\infty}_0(\mathcal{B})
\]

be the mapping canonically associated to \( K_p \). We have following result (see Lemma 4.2 in [12]).

**Lemma 3.2.** Let \( 1 < p < \infty \) and \( \theta = \frac{1}{p} \). Then \( \tilde{K}_p \) defines a contraction from \( (L^1(\mathcal{M})/J(H^1_0(\mathcal{A})), \mathcal{M}/J(\mathcal{A}_0))_{1-\theta} \) into \( H^{p,\infty}(\mathcal{B}) \).

**Proof.** Let

\[
X = \left\{ \begin{pmatrix} x & 0 & 0 & \ldots \\
0 & 2^{-\frac{1}{p}}x & 0 & \ldots \\
0 & 0 & 3^{-\frac{1}{p}}x & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} : \forall x \in L^1(\mathcal{M}) \right\}.
\]
We equip \( X \) with the norm
\[
\left\| \begin{pmatrix} x & 0 & 0 & \ldots \\ 0 & 2^{-1/p}x & 0 & \ldots \\ 0 & 0 & 3^{-1/p}x & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_X = \|x\|_1.
\]
Then \( X \) becomes a Banach space. It is clear that \( X \subset L^{1,\infty}(\mathcal{N}) \) and this inclusion has norm one. Let
\[
X_0 = \left\{ \begin{pmatrix} x & 0 & 0 & \ldots \\ 0 & 2^{-1/p}x & 0 & \ldots \\ 0 & 0 & 3^{-1/p}x & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : \forall x \in J(H_0^1(\mathcal{M})) \right\}.
\]
Then we have that
\[
X/X_0 \subset L^{1,\infty}(\mathcal{N})/J(H_0^{1,\infty}(\mathcal{B}))
\]
with norm one.

For any \( z \in S \), let
\[
K_z(x) = \begin{pmatrix} x & 0 & 0 & \ldots \\ 0 & 2^{z^{-1}}x & 0 & \ldots \\ 0 & 0 & 3^{z^{-1}}x & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \forall x \in L^1(\mathcal{M}) \cap \mathcal{M}.
\]
Then \( \{\tilde{K}_z\}_{z \in \mathbb{T}} \) is an analytic families of linear operators on
\[
L^1(\mathcal{M})/J(H_0^1(\mathcal{A})) \cap \mathcal{M}/J(\mathcal{A}_0)
\]
into
\[
X/X_0 + N/J(\mathcal{B}_0).
\]
From (3.2), it follows that if \( \text{Re}(z) = 0 \), \( \tilde{K}_z \) is a contraction from
\[
L^1(\mathcal{M})/J(H_0^1(\mathcal{A}))
\]
into \( X/X_0 \) and if \( \text{Re}(z) = 1 \), it is a contraction from \( \mathcal{M}/J(\mathcal{A}_0) \) into \( N/J(\mathcal{B}_0) \).
Hence, by Stein’s interpolation theorem for analytic families of operators (see Theorem 1 in [6]), we obtain that \( \tilde{K}_p \) is a contraction from
\[
(L^1(\mathcal{M})/J(H_0^1(\mathcal{A})), \mathcal{M}/J(\mathcal{A}_0))_{1-\frac{1}{p}}
\]
into
\[
(X/X_0, N/J(\mathcal{B}_0))_{1-\frac{1}{p}}.
\]
By Theorem 4.7.1 in [4], \( \tilde{K}_p \) is a contraction from
\[
(L^1(\mathcal{M})/J(H_0^1(\mathcal{A})), \mathcal{M}/J(\mathcal{A}_0))_{1-\frac{1}{p}}
\]
into
\[
(X/X_0, N/J(\mathcal{B}_0))_{1-\frac{1}{p}, \infty}.
\]
By (3.3), $\tilde{K}_p$ is a contraction from
\[ (L^1(\mathcal{M})/J(H_0^1(A)), \mathcal{M}/J(A_0))_{1-\frac{1}{p}} \]
into
\[ (L^{1,\infty}(\mathcal{N})/J(H_0^{1,\infty}(B)), N/J(B_0))_{1-\frac{1}{p},\infty}. \]
Using Lemma 3.1, we obtain the desired result. \hfill $\square$

**Theorem 3.3.** Let $1 < p < \infty$ and $\frac{1}{p} = 1 - \theta$. Then
\[ H^p(A) = (H^1(A), A)_\theta. \quad (3.4) \]

**Proof.** Let $q$ be the conjugate of $p$, so that $\frac{1}{p} + \frac{1}{q} = 1$. The inclusions
\[ (H^1(A), A)_\theta \subset (L^1(\mathcal{M}), \mathcal{M})_\theta = L^p(\mathcal{M}) \]
and
\[ (H^1(A), A)_\theta \subset \{ x \in L^p(\mathcal{M}) : x \perp J(H_0^1(A)) \} \]
imply $(H^1(A), A)_\theta \subset H^p(A)$. To prove the converse we dualize. Hence we have to prove that
\[ (L^1(\mathcal{M})/J(H_0^1(A)), \mathcal{M}/J(A_0))_{1-\frac{1}{q}} \subset H^q(A). \quad (3.5) \]
Let $x \in L^1(\mathcal{M}) \cap \mathcal{A}$. By (3.2) and Lemma 3.2, we have that
\[ \|x\|_q = \left\| \begin{pmatrix} x & 0 & 0 & \ldots \\ 0 & 2^{-\frac{1}{q}} x & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{q,\infty} \leq \|x\|_{(L^1(\mathcal{M})/J(H_0^1(A)), \mathcal{M}/J(A_0))_{1-\frac{1}{q}}}. \]
Hence, we get (3.5). \hfill $\square$

Let $BMO(\mathcal{M})$ be the space defined in Definition 5.1 in [2]. By Theorem 5.1 in [2], we have that $H^1(\mathcal{A})^* = BMO(\mathcal{M})$. We use the reiteration theorem (Theorem 4.6.1 in [4]) and Wolff’s theorem (Theorem 2 in [17]) to obtain that

**Corollary 3.4.** Let $1 < p < \infty$ and $\frac{1}{p} = 1 - \theta$. Then
\[ H^p(A) = (H^1(A), BMO(\mathcal{M}))_\theta. \quad (3.6) \]

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