On the stabilization of the elasticity system by the boundary

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Abstract
We obtain free of resonances regions for the elasticity system in the exterior of a strictly convex body in $\mathbb{R}^3$ with dissipative boundary conditions under some natural assumptions on the behaviour of the geodesics on the boundary. To do so, we use the properties of the parametrix of the Neumann operator constructed in [12]. As a consequence, we obtain time decay estimates for the local energy of the solutions of the corresponding mixed boundary value problems.

1 Introduction and statement of results
Let $\Omega \subset \mathbb{R}^3$ be a strictly convex compact set with smooth boundary $\Gamma = \partial \Omega$ and denote by $\Omega = \mathbb{R}^3 \setminus \Omega$ the exterior domain. Denote by $\Delta_e$ the elasticity operator, which is a $3 \times 3$ matrix-valued differential operator defined by

$$\Delta_e u = \mu_0 \Delta u + (\lambda_0 + \mu_0) \nabla (\nabla \cdot u),$$

$u = ^t (u_1, u_2, u_3)$. Here $\lambda_0, \mu_0$ are the Lamé constants supposed to satisfy

$$\mu_0 > 0, \quad 3\lambda_0 + 2\mu_0 > 0. \quad (1.1)$$

The Neumann boundary conditions for $\Delta_e$ are of the form

$$(Bu)_i|_{\Gamma} := \sum_{j=1}^{3} \sigma_{ij}(u)\nu_j|_{\Gamma} = 0, \quad i = 1, 2, 3, \quad (1.2)$$

where

$$\sigma_{ij}(u) = \lambda_0 \nabla \cdot u \delta_{ij} + \mu_0 \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

is the stress tensor, $\nu$ is the outer unit normal to $\Gamma$. The purpose of the present paper is to study the time decay properties of the elasticity system in $\Omega$ with dissipative boundary conditions. More precisely, we are going to study the following mixed boundary value problem

$$\begin{cases}
(\partial_t^2 - \Delta_e) u = 0 & \text{in} \ (0, +\infty) \times \Omega, \\
Bu - iAu = 0 & \text{on} \ (0, +\infty) \times \Gamma, \\
u(0) = f_1, \ \partial_t u(0) = f_2,
\end{cases} \quad (1.3)$$
where $A$ is a classical zero order $3 \times 3$ matrix-valued pseudo-differential operator on $\Gamma$, independent of $t$ and satisfying the properties $A = A^*$, $A \geq 0$. Moreover, we suppose that there exist a non-empty compact set $\Gamma_0 \subset \Gamma$ and a constant $C > 0$ so that we have

$$\langle Af, f \rangle_{L^2(\Gamma)} \geq C \|f\|_{L^2(\Gamma_0)}^2.$$  \hspace{1cm} (1.4)

The large time behaviour of the solutions to (1.3) with $A \equiv 0$ is well understood. Kawashita [6] showed that there is no uniform local energy, while Stefanov-Vodev [12], [13] proved the existence of infinitely many resonances converging polynomially fast to the real axis. The reason for this is the existence of surface waves (called Rayleigh waves), that is, a propagation of singularities of the solutions along the geodesics on $\Gamma$ with a speed $c_R > 0$ strictly less than the two other speeds in $\Omega$. Therefore, a strictly convex obstacle is trapping for the Neumann problem of the elasticity wave equation. Note that for the Dirichlet problem it is non-trapping, and in particular we have an exponential decay of the local energy similarly to the classical wave equation (see [17]). Comming back to the equation (1.3) with non-trivial $A$, note that we still have a propagation of singularities of the solutions along the geodesics on $\Gamma$ with a speed $c_R > 0$. Therefore, in order to be able to get a better decay of the local energy we need to suppose that all geodesics meet the part on $\Gamma$ where the dissipative term is non-trivial. More precisely, we suppose that there exist a non-empty open domain $\Gamma'_0 \subset \Gamma_0$ and a constant $T > 0$ so that

$$\text{for every geodesics } \gamma \text{ with } \gamma(0) \in \Gamma, \text{ there exists } 0 \leq t \leq T \text{ such that } \gamma(t) \in \Gamma'_0. \hspace{1cm} (1.5)$$

The outgoing resolvent, $R(\lambda)$, corresponding to the problem (1.3) is defined via the equation

$$\begin{cases}
(\Delta_e + \lambda^2)R(\lambda)f = f & \text{in } \Omega, \\
(B - iA)R(\lambda)f = 0 & \text{on } \Gamma, \\
R(\lambda)f - \lambda - \text{outgoing}.
\end{cases} \hspace{1cm} (1.6)$$

Recall that “$\lambda$-outgoing” means that there exist $a \gg 1$ and a compactly supported function $g$ so that

$$R(\lambda)\big|_{|x| \geq a} = R_0(\lambda)g\big|_{|x| \geq a},$$

where $R_0(\lambda)$ is the outgoing free resolvent, i.e.

$$R_0(\lambda) = (\Delta_e + \lambda^2)^{-1} \in \mathcal{L}(L^2) \text{ for } \text{Im } \lambda < 0.$$ 

Let $\chi \in C_0^\infty(\mathbb{R}^3)$, $\chi = 1$ on $\mathcal{O}$. In the same way as in the case $A \equiv 0$ we have that the cutoff resolvent

$$R_\chi(\lambda) := \chi R(\lambda)\chi$$

extends meromorphically to the whole complex plane $\mathbb{C}$ with poles in $\text{Im } \lambda > 0$ called resonances. One of our goals in the present paper is to study the distributions of the resonances near the real axis under the above assumptions. Our first result is the following

**Theorem 1.1** Under the assumptions (1.1), (1.4) and (1.5), $R_\chi(\lambda)$ extends analytically to $\{||\text{Im } \lambda|| \leq C_1||\lambda||^{-1}, |\text{Re } \lambda|| \geq C_2 > 0\}$ and satisfies there the estimate

$$\|R_\chi(\lambda)\|_{\mathcal{L}(L^2)} \leq C'.$$  \hspace{1cm} (1.7)
Moreover, under the assumption (1.1) only, there exists a constant $C > 0$ so that $R_X(\lambda)$ is analytic in the region
\[ \{ C \leq \text{Im} \lambda \leq M \log |\lambda|, \text{Re} \lambda \geq C_M \gg 1 \} \] (1.8)
for every $M \gg 1$. Furthermore, there are infinitely many resonances in $\{ 0 < \text{Im} \lambda < C \}$.

In the case $A \equiv 0$, Stefanov-Vodev [12] showed that there is a free of resonances region of the form
\[ \{ C_N |\lambda|^{-N} \leq \text{Im} \lambda \leq M \log |\lambda|, |\text{Re} \lambda| \geq C_M \gg 1 \} \]
for every $M, N \gg 1$, while in $\{ 0 < \text{Im} \lambda \leq C_N |\lambda|^{-N} \}$ there are infinitely many resonances (called Rayleigh resonances) due to the Rayleigh surface waves. Later on Sjöstrand-Vodev [10] proved that the counting function of these resonances is
\[ \tau_2 c_R^{-2} \text{Vol}(\Gamma) r^2 + O(r), \quad r \gg 1, \] (1.9)
where $c_R > 0$ is the speed of the Rayleigh waves and
\[ \tau_2 = (2\pi)^{-2} \text{Vol}(\{ x \in \mathbb{R}^3 : |x| = 1 \}). \]

We expect that the counting function of the resonances in $\{ 0 < \text{Im} \lambda < C \}$ in the general case (i.e. for non-trivial $A$) satisfies (1.9) as well with possibly an worse bound for the remainder term.

On the other hand, extending a previous result by Burq [4] to the elastic system, Bellassoued [3] obtained a free of resonances region of the form $\{ 0 < \text{Im} \lambda \leq e^{-C|\lambda|}, C > 0 \}$, so the Rayleigh resonances are concentrated in a region of the form $\{ e^{-C|\lambda|} \leq \text{Im} \lambda \leq C_N |\lambda|^{-N} \}$. Moreover, if the boundary $\Gamma$ is analytic, Vodev [14] improved this region to $\{ e^{-C|\lambda|} \leq \text{Im} \lambda \leq e^{-C|\lambda|} \}$. The presence of a non-trivial dissipative term $A$, however, changes the distribution of the resonances considerably.

As a consequence of (1.7) we get a decay rate of the local energy of the solutions to (1.3).

**Corollary 1.2** Under the assumptions (1.1), (1.4) and (1.5), for every $a \gg 1, m \geq 0$, there exists a constant $C = C(a, m) > 0$ so that we have (for $t \gg 1$)
\begin{equation}
\| \nabla_x u(t, \cdot) \|_{L^2(\Omega_a)} + \| \partial_t u(t, \cdot) \|_{L^2(\Omega_a)} \leq C \left( t^{-1} \log t \right)^m \left( \| \nabla f_1 \|_{H^m(\Omega)} + \| f_2 \|_{H^m(\Omega)} \right), \tag{1.10}
\end{equation}
where $\Omega_a := \Omega \cap \{ |x| \leq a \}$ and supp $f_j \subset \Omega_a$, $j = 1, 2$.

The fact that (1.7) implies (1.10) was proved in [9] in the case of a unitary group. In our case this can be done following the approach developed in [8] (and also in [14]). Note that in the case $A \equiv 0$, Bellassoued [3] proved (1.10) with $t^{-1} \log t$ replaced by $(\log t)^{-1}$.

It turns out that if the dissipation on the boundary is stronger, we have a uniform exponential decay of the local energy. Indeed, consider the following mixed boundary value problem
\begin{equation}
\begin{cases}
(\partial_t^2 - \Delta_x)u = 0 & \text{in } (0, +\infty) \times \Omega, \\
Bu + A\partial_t u = 0 & \text{on } (0, +\infty) \times \Gamma, \\
u(0) = f_1, \quad \partial_t u(0) = f_2,
\end{cases}
\tag{1.11}
\end{equation}
Theorem 1.3 Assume (1.1) and (1.4) fulfilled with $\Gamma_0 = \Gamma$. Then, $\tilde{R}_\lambda(\lambda)$ extends analytically to $\{\text{Im } \lambda \leq C_1, |\text{Re } \lambda| \geq C_2 > 0\}$ and satisfies there the estimate

$$\|\tilde{R}_\lambda(\lambda)\|_{L(L^2)} \leq C^\prime|\lambda|^{-1}. \tag{1.13}$$

As a consequence of (1.13) we get an exponential decay of the local energy of the solutions to (1.11).

Corollary 1.4 Under the assumptions of Theorem 1.3, for every $a \gg 1$, there exist constants $C = C(a) > 0$, $\alpha > 0$, so that we have (for $t \gg 1$)

$$\|\nabla_x u(t, \cdot)\|_{L^2(\Omega_a)} + \|\partial_t u(t, \cdot)\|_{L^2(\Omega_a)} \leq Ce^{-\alpha t} \left( \|\nabla_1 f_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)} \right), \tag{1.14}$$

provided $\text{supp } f_j \subset \Omega_a$, $j = 1, 2$.

The fact that (1.13) implies (1.14) is more or less well known in the case of unitary groups (e.g. see [15]). In the case of semi-groups the proof goes in the same way (see [7]).

It is worth noticing that an interior dissipation of the elastic wave equation with Neumann boundary conditions does not improve the decay of the local energy. Indeed, consider the following mixed boundary value problem

$$\begin{align*}
(\partial_t^2 - \Delta_e + A(x)\partial_t)u &= 0 \quad \text{in } (0, +\infty) \times \Omega, \\
Bu &= 0 \quad \text{on } (0, +\infty) \times \Gamma, \\
u(0) &= f_1, \quad \partial_t u(0) = f_2,
\end{align*} \tag{1.15}$$

where $A \in C_0^\infty(\Omega)$ is a $3 \times 3$ matrix-valued function satisfying the properties $A = A^*$, $A \geq 0$. Then, the quasi-modes constructed in [12], [13], which are due to the existence of the Rayleigh waves and hence supported in an arbitrary small neighbourhood of the boundary, are also quasi-modes for the problem with non-trivial $A$. Therefore, in the same way as in these papers one can show that there exists an infinite sequence $\{\lambda_j\}$ with $0 < \text{Im } \lambda_j \leq C_N|\lambda_j|^{-N}, \forall N \gg 1$, so that the following problem has a non-trivial solution:

$$\begin{align*}
(\Delta_e - i\lambda_j A(x) + \lambda_j^2)v_j &= 0 \quad \text{in } \Omega, \\
Bv_j &= 0 \quad \text{on } \Gamma, \\
v_j - \lambda_j - \text{outgoing}.
\end{align*} \tag{1.16}$$

Note finally that the situation is completely different for the usual scalar-valued wave equation with dissipative boundary conditions like those above. Indeed, in this case if the obstacle is non-trapping, the corresponding cut-off resolvent extends analytically through the real axis to a strip and as a consequence we have an exponential decay of the local energy without extra assumptions (e.g. see [11]). In other words, the behaviour of the cut-off resolvent and the local energy is the same as in the case of the self-adjoint problem with Neumann boundary conditions.
2 Proof of Theorem 1.1

It suffices to prove (1.7) for real $\lambda \gg 1$, only. Let $v \in L^2_{\text{comp}}(\Omega)$ and let $u$ be the solution to the equation
\[
\begin{cases}
(\Delta_e + \lambda^2)u = v & \text{in } \Omega, \\
(B - iA)u = 0 & \text{on } \Gamma, \\
u - \lambda - \text{outgoing}.
\end{cases}
\]
Clearly, (1.7) is equivalent to the estimate
\[
\|u\|_{L^2(\Omega_a)} \leq C_a \|v\|_{L^2(\Omega)}, \quad \lambda \geq \lambda_0,
\]
for every $a \gg 1$ with constants $C_a, \lambda_0 > 0$ independent of $\lambda$. To prove (2.2) we need a priori estimates of the solutions to the equation
\[
\begin{cases}
(\Delta_e + \lambda^2)u = v & \text{in } \Omega, \\
|\Gamma| = f, \quad \lambda^{-1}Bu|\Gamma = g, \\
u - \lambda - \text{outgoing}.
\end{cases}
\]
where $v \in L^2_{\text{comp}}(\Omega)$. We have the following

**Proposition 2.1** There exist constants $C, \lambda_0 > 0$ so that for $\lambda \geq \lambda_0$ we have
\[
\|u\|_{H^1(\Omega_a)} + \|g\|_{L^2(\Gamma)} \leq C\lambda^{-1}\|v\|_{L^2(\Omega)} + C\|f\|_{H^1(\Gamma)}.
\]
Hereafter the Sobolev spaces $H^1$ are equipped with the semi-classical norm (with a small parameter $\lambda^{-1}$).

*Proof.* In the case of the Euclidean Laplacian $\Delta$ the a priori estimate (2.4) is proved in [5] (see Theorem 3.1). In our case the proof goes in the same way, but we will sketch it for the sake of completeness. Observe first that the solution to (2.3) is of the form
\[
u = G(\lambda)v + K(\lambda)f,
\]
where $G(\lambda)v$ solves the problem
\[
\begin{cases}
(\Delta_e + \lambda^2)G(\lambda)v = v & \text{in } \Omega, \\
G(\lambda)v|\Gamma = 0, \\
G(\lambda)v - \lambda - \text{outgoing},
\end{cases}
\]
while $K(\lambda)f$ solves the problem
\[
\begin{cases}
(\Delta_e + \lambda^2)K(\lambda)f = 0 & \text{in } \Omega, \\
K(\lambda)f|\Gamma = f, \\
K(\lambda)f - \lambda - \text{outgoing}.
\end{cases}
\]
Since the strictly convex obstacles are non-trapping for the Dirichlet problem of the elastic wave equation (see [17]), we have the estimate

$$
\|G(\lambda)v\|_{H^1(\Omega_a)} \leq C_a \lambda^{-1}\|v\|_{L^2(\Omega)}, \quad \lambda \geq \lambda_0.
$$

Thus, to prove (2.4) we need the estimate

$$
\|K(\lambda)f\|_{H^1(\Omega_a)} \leq C_a \|f\|_{H^1(\Gamma)}, \quad \lambda \geq \lambda_0.
$$

This in turn follows from the fact that, since the obstacle is strictly convex, one can construct a parametrix of \(K(\lambda)\) near the boundary, which satisfies (2.8). More precisely, there exist a neighbourhood \(\Omega' \subset \Omega\) of \(\Gamma\) and operators

$$
\mathcal{K}(\lambda) = O(1) : H^1(\Gamma) \to H^1(\Omega'), \quad \mathcal{R}(\lambda) = O(\lambda^{-\infty}) : H^1(\Gamma) \to H^1(\Omega'),
$$

solving the equation

$$
\begin{cases}
(\Delta_e + \lambda^2)\mathcal{K}(\lambda)f = \mathcal{R}(\lambda)f & \text{in } \Omega', \\
\mathcal{K}(\lambda)f|_{\Gamma} = f.
\end{cases}
$$

(2.10)

Note that such operators are constructed in [12] (Section 2). Let \(\psi \in C^\infty(\Omega)\), \(\text{supp} \psi \subset \overline{\Omega'}\), \(\psi = 1\) near \(\Gamma\). We have

$$
\begin{cases}
(\Delta_e + \lambda^2)\psi\mathcal{K}(\lambda)f = [\Delta_e, \psi]\mathcal{K}(\lambda)f + \psi\mathcal{R}(\lambda)f & \text{in } \Omega, \\
\psi\mathcal{K}(\lambda)f|_{\Gamma} = f,
\end{cases}
$$

which leads to

$$
\begin{cases}
(\Delta_e + \lambda^2)(\mathcal{K}(\lambda)f - \psi\mathcal{K}(\lambda)f) = -[\Delta_e, \psi]\mathcal{K}(\lambda)f - \psi\mathcal{R}(\lambda)f & \text{in } \Omega, \\
(\mathcal{K}(\lambda)f - \psi\mathcal{K}(\lambda)f)|_{\Gamma} = 0.
\end{cases}
$$

Hence

$$
\mathcal{K}(\lambda)f = \psi\mathcal{K}(\lambda)f - G(\lambda) ([\Delta_e, \psi]\mathcal{K}(\lambda)f + \psi\mathcal{R}(\lambda)f).
$$

(2.11)

Thus, (2.8) follows from (2.11), (2.7) and (2.9). To complete the proof of (2.4) we need to show that

$$
\|g\|_{L^2(\Gamma)} \leq C\lambda^{-1}\|v\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega_a)} + C\|f\|_{H^1(\Gamma)}.
$$

(2.12)

To this end, we write the operator \(\Delta_e\) in normal coordinates \(y = (y_1, y') \in \mathbb{R}^+ \times \Gamma\) in a neighbourhood of the boundary. We have

$$
\partial_{x_j} = \nu_j(y')\partial_{y_1} + \beta_j(y) \cdot \nabla y', \quad j = 1, 2, 3,
$$

where \(\nu(y') = (\nu_1(y'), \nu_2(y'), \nu_3(y'))\) is the unit normal at \(y' \in \Gamma\). Hence

$$
\Delta_e = \mathcal{A}(y')\partial_{y_1}^2 + \mathcal{Q}(y, \partial_{y'}) + \mathcal{Q}_1(y, \partial_y),
$$

where \(\mathcal{Q}\) and \(\mathcal{Q}_1\) are second and first order differential operators, respectively, while \(\mathcal{A}(y')\) is a symmetric matrix-valued function defined by

$$
(\mathcal{A}(y')u)_k = \mu_0 u_k + (\lambda_0 + \mu_0)\nu_k \sum_{j=1}^3 \nu_j u_j, \quad k = 1, 2, 3.
$$
It is easy to check that
\[ \det A(y') = \mu_0^2(\lambda_0 + 2\mu_0) > 0. \]

Set
\[ E(y_1) = \left( (Q + \lambda^2)(\psi u)(y_1, \cdot), (\psi u)(y_1, \cdot) \right)_{L^2} + \left( A\partial_{y_1}(\psi u)(y_1, \cdot), \partial_{y_1}(\psi u)(y_1, \cdot) \right)_{L^2}, \]
\[ \psi \text{ being the function above. We have} \]
\[ \frac{dE(y_1)}{dy_1} = \left( [\partial_{y_1}, Q](\psi u)(y_1, \cdot), (\psi u)(y_1, \cdot) \right)_{L^2} \]
\[ + 2\text{Re} \left( (Q + \lambda^2)(\psi u)(y_1, \cdot), \partial_{y_1}(\psi u)(y_1, \cdot) \right)_{L^2} + 2\text{Re} \left( A\partial_{y_1}^2(\psi u)(y_1, \cdot), \partial_{y_1}(\psi u)(y_1, \cdot) \right)_{L^2} \]
\[ = \left( [\partial_{y_1}, Q](\psi u)(y_1, \cdot), (\psi u)(y_1, \cdot) \right)_{L^2} - 2\text{Re} \left( Q_1(\psi u)(y_1, \cdot), \partial_{y_1}(\psi u)(y_1, \cdot) \right)_{L^2} \]
\[ + 2\text{Re} \left( (\Delta_e + \lambda^2)(\psi u)(y_1, \cdot), \partial_{y_1}(\psi u)(y_1, \cdot) \right)_{L^2}. \]

Hence
\[ E(0) = -\int_0^\infty \frac{dE(y_1)}{dy_1} dy_1 \leq O(\lambda^2)\|\psi u\|^2_{H^1(\Omega)} + O(1)\|\Delta_e + \lambda^2(\psi u)\|^2_{L^2(\Omega)}. \]

On the other hand
\[ \|\partial_{y_1}(\psi u)(0, \cdot)\|^2_{L^2(\Gamma)} \leq CE(0) + O(\lambda^2)\|\psi u(0, \cdot)\|_{H^1(\Gamma)}, \]
with a constant \( C > 0 \). Combining these estimates we get
\[ \lambda^{-1}\|\partial_{y_1} u(0, \cdot)\|^2_{L^2(\Gamma)} \leq O(1)\|u(0, \cdot)\|_{H^1(\Gamma)} + O(1)\|u\|_{H^1(\Omega)} + O(1)\|\Delta_e + \lambda^2\|_{L^2(\Omega)}, \]
which clearly implies (2.12). \( \square \)

Set \( w = G(\lambda)v \), where \( v \) is as in (2.1). If \( u \) is the solution to (2.1), then the function \( u - w \) solves the equation
\[ \begin{cases} 
(\Delta_e + \lambda^2)(u - w) = 0 & \text{in } \Omega, \\
(B - iA)(u - w) = -Bw & \text{on } \Gamma, \\
(u - w) - \lambda - \text{outgoing}. 
\end{cases} \]

Set \( f = u|_\Gamma = (u - w)|_\Gamma, g = -\lambda^{-1}Bw|_\Gamma \). By (2.4),
\[ \|g\|^2_{L^2(\Gamma)} \leq C\lambda^{-1}\|v\|_{L^2(\Omega)}, \quad \lambda \geq \lambda_0. \]

Furthermore, we have
\[ \lambda^{-1}B(u - w)|_\Gamma = N(\lambda)f, \]
where \( N(\lambda) : H^1(\Gamma) \to L^2(\Gamma) \) is the outgoing Neumann operator. Thus, we get that the function \( f \) satisfies the equation
\[ \left( N(\lambda) - i\lambda^{-1}A \right) f = g \]
with \( g \) satisfying (2.14). It is easy to see that (2.2) follows from combining (2.4), (2.14) and the following
Theorem 2.2  Under the assumptions (1.1), (1.4) and (1.5), there exist constants $C, \lambda_0 > 0$ so that the solution to (2.16) satisfies the estimate
\[
\|f\|_{H^1(\Gamma)} \leq C\lambda\|g\|_{L^2(\Gamma)}, \quad \lambda \geq \lambda_0. \tag{2.17}
\]

Proof. Since the outgoing Neumann operator satisfies
\[
-\text{Im} \langle N(\lambda)f, f \rangle_{L^2(\Gamma)} \geq 0,
\]
we obtain
\[
\langle Af, f \rangle_{L^2(\Gamma)} \leq -\lambda \text{Im} \langle g, f \rangle_{L^2(\Gamma)} \leq \beta^{-2} \lambda^{2}\|g\|_{L^2(\Gamma)}^{2} + \beta^{2}\|f\|_{L^2(\Gamma)}^{2}, \tag{2.19}
\]
for every $\beta > 0$. By (1.4) and (2.19),
\[
\|f\|_{L^2(\Gamma_0)} \leq C\beta^{-1}\lambda\|g\|_{L^2(\Gamma)} + \beta\|f\|_{L^2(\Gamma)}.
\]
Now, using (1.5) together with the properties of the outgoing Neumann operator, we will prove the estimate
\[
\|f\|_{H^1(\Gamma)} \leq C\lambda\|g\|_{L^2(\Gamma)} + C\|f\|_{L^2(\Gamma)}. \tag{2.21}
\]
Clearly, (2.17) follows from (2.21) and (2.20) provided $\beta$ is taken small enough.

To prove (2.21) we will make use of the properties of the parametrix, $N(\lambda)$, of $N(\lambda)$ constructed in Section 3 of [12]. First of all, we have
\[
\|N(\lambda)f - N(\lambda)f\|_{L^2(\Gamma)} \leq O(\lambda^{-\infty})\|f\|_{L^2(\Gamma)}. \tag{2.22}
\]
Moreover, $N(\lambda)$ is a $\lambda - \Psi DO$ with a characteristic variety $\Sigma = \{\zeta \in T^*\Gamma : \|\zeta\| = c^{-1}_R\}$ belonging to the elliptic region of the corresponding boundary value problem. In the region $\{\zeta \in T^*\Gamma : \|\zeta\| > c^{-1}_R\}$ the operator $N(\lambda)$ is an elliptic $\lambda - \Psi DO$ of class $L^{0,0}_{0,0}(\Gamma)$ (hereafter we use the same notations as in the appendix of [12]), while in the region $\{\zeta \in T^*\Gamma : \|\zeta\| < c^{-1}_R\}$ it is hypoelliptic. Clearly, so is the operator $N(\lambda) - i\lambda^{-1}A$. Therefore, if $\chi \in C_0^{\infty}(T^*\Gamma)$, $\chi = 1$ on $\{\zeta \in T^*\Gamma : \|\zeta\| - c^{-1}_R\leq \epsilon\}$, $\chi = 0$ on $\{\zeta \in T^*\Gamma : \|\zeta\| - c^{-1}_R > 2\epsilon\}$, $0 < \epsilon \ll 1$, we have
\[
\|\text{Op}_\lambda(1 - \chi)f\|_{H^1(\Gamma)} \leq O(\lambda^{2/3})\bigg\|\left(N(\lambda) - i\lambda^{-1}A\right)f\bigg\|_{L^2(\Gamma)} + O(\lambda^{-\infty})\|f\|_{H^1(\Gamma)}
\]
\[
\leq O(\lambda^{2/3})\|\chi\|_{L^2(\Gamma)} + O(\lambda^{-\infty})\|f\|_{H^1(\Gamma)}. \tag{2.23}
\]
On the other hand, near $\Sigma$ the operator $N(\lambda)$ is a $\lambda - \Psi DO$ of class $L^{0,0}_{0,0}(\Gamma)$, whose principal symbol is a symmetric $3 \times 3$ matrix-valued function with eigenvalues $a_1(\zeta) = \tilde{a}_1(\zeta)(c_R\|\zeta\| - 1)$, $a_2(\zeta) > 0$, $a_2(\zeta) > 0$, $a_3(\zeta) > 0$ near $\Sigma$. It is shown in [13] (Theorem 3.1) that there exist elliptic $\lambda - \Psi DOs$, $U(\lambda)$ and $V(\lambda)$, of class $L^{0,0}_{0,0}(\Gamma)$, so that we have
\[
U(\lambda)^*N(\lambda)U(\lambda) = \begin{pmatrix}
-\lambda^{-2}c^2_R\Delta_\Gamma - \lambda^{-1}a_0 - 1 & 0 \\
0 & V(\lambda)
\end{pmatrix} \text{Op}_\lambda(\chi_1) \mod L^{0,-2}_{0,0}(\Gamma), \tag{2.24}
\]
where $-\Delta_\Gamma$ denotes the positive Laplace-Beltrami operator on $\Gamma$, $a_0$ is a classical (independent of $\lambda$) zero order $\Psi DO$ on $\Gamma$ with a real-valued principal symbol, and $\chi, \chi_1 \in C_0^{\infty}(T^*\Gamma)$, $\chi_1 = 1$ on $\{\zeta \in T^*\Gamma : \|\zeta\| - c^{-1}_R\leq 3\epsilon\}$, $\chi_1 = 0$ on $\{\zeta \in T^*\Gamma : \|\zeta\| - c^{-1}_R > 2\epsilon\}$, $\chi = 1$ on supp $\chi_1$. In
fact in [11] a better diagonalization of \( \mathcal{N}(\lambda) \) near \( \Sigma \) is carried out, but for our purposes (2.24) will suffice. Now the function \( \tilde{f} = U(\lambda)^{-1} \text{Op}_\lambda(\chi)f \) satisfies

\[
\begin{pmatrix} 0 & -\lambda - 1 \\ 0 & V(\lambda) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{f} - i \lambda^{-1} \tilde{A} \tilde{f} = \tilde{g},
\]

(2.25)

where \( \tilde{A} = U^*AU \) is a \( \lambda - \Psi DO \) of class \( L^{0,0}_{0,0}(\Gamma) \) with a principal symbol satisfying \( \sigma_p(\tilde{A}) \geq 0 \), \( \sigma_p(\tilde{A})^* = \sigma_p(\tilde{A}) \), and \( \tilde{g} \) satisfies

\[
\|\tilde{g}\|_{L^2(\Gamma)} \leq \|g\|_{L^2(\Gamma)} + O(\lambda^{-\infty})\|f\|_{H^1(\Gamma)}.
\]

(2.26)

Writing \( \tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) \), \( \tilde{g} = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3) \), we reduce (2.25) to

\[
\begin{pmatrix} 0 & -\lambda^{-2}c_R^2 \Delta \Gamma - 1 - i \lambda^{-1}b_1 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \tilde{g}_3 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} c_2 \tilde{f}_1 \\ c_3 \tilde{f}_1 \end{pmatrix},
\]

(2.27)

where \( b_j \), \( c_j \) are scalar-valued \( \lambda - \Psi DOs \) of class \( L^{0,0}_{0,0}(\Gamma) \), the principal symbol of \( b_1 \) satisfying \( \text{Re} \sigma_p(b_1) \geq 0 \), while \( \tilde{V}(\lambda) \) is a \( 2 \times 2 \) matrix-valued elliptic \( \lambda - \Psi DO \) of class \( L^{0,0}_{0,0}(\Gamma) \). Thus, the inverse \( \tilde{V}(\lambda)^{-1} \) is again a \( 2 \times 2 \) matrix-valued elliptic \( \lambda - \Psi DO \) of class \( L^{0,0}_{0,0}(\Gamma) \), so we can solve the equation (2.28). In particular, we obtain

\[
\|\tilde{f}_2\|_{L^2(\Gamma)} + \|\tilde{f}_3\|_{L^2(\Gamma)} \leq \|\tilde{g}_2\|_{L^2(\Gamma)} + \|\tilde{g}_3\|_{L^2(\Gamma)} + O(\lambda^{-1})\|\tilde{f}_1\|_{L^2(\Gamma)}.
\]

(2.29)

Furthermore, we conclude that the function \( \tilde{f}_1 \) solves an equation of the form

\[
\begin{pmatrix} 0 & -\lambda^{-2}c_R^2 \Delta \Gamma - 1 - i \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix} \tilde{f}_1 = h,
\]

(2.30)

where \( b \) is a scalar-valued \( \lambda - \Psi DO \) of class \( L^{0,0}_{0,0}(\Gamma) \) with a principal symbol satisfying \( \text{Re} \sigma_p(b) \geq 0 \), and \( h \) satisfies

\[
\|h\|_{L^2(\Gamma)} \leq \|g\|_{L^2(\Gamma)} + O(\lambda^{-\infty})\|f\|_{H^1(\Gamma)}.
\]

(2.31)

We are going to show that the assumption (1.5) leads to the estimate

\[
\|\tilde{f}_1\|_{L^2(\Gamma)} \leq C\lambda\|h\|_{L^2(\Gamma)} + C\|\tilde{f}_1\|_{L^2(\Gamma_0)}.
\]

(2.32)

Before doing so, observe that (2.32) implies (2.21). Indeed, since

\[
\|\tilde{f}_1\|_{L^2(\Gamma_0)} \leq C\|f\|_{L^2(\Gamma_0)} + O(\lambda^{-\infty})\|f\|_{L^2(\Gamma)},
\]

we deduce from (2.29), (2.31) and (2.32) that

\[
\|\text{Op}_\lambda(\chi)f\|_{L^2(\Gamma)} \leq C\lambda\|g\|_{L^2(\Gamma)} + C\|f\|_{L^2(\Gamma_0)} + O(\lambda^{-\infty})\|f\|_{H^1(\Gamma)}.
\]

(2.33)

Thus, (2.21) follows from (2.33).
Therefore, using the identity Lemma 2.3 partition of the unity in a neighbourhood of $\Sigma$, it is easy to see that (2.32) follows from (1.5) By (2.35),

\[
\frac{\partial x(t)}{\partial t} = \frac{\partial r_0(x, \xi)}{\partial t}, \quad \frac{\partial \xi(t)}{\partial t} = -\frac{\partial r_0(x, \xi)}{\partial x}, \quad x(0) = x^0, \quad \xi(0) = \xi^0.
\]

Fix a point $\zeta^0 = (x^0, \xi^0) \in \Sigma$ and choose a real-valued function $p(x, \xi) \in C^\infty_0(T^*\Gamma), 0 \leq p \leq 1$, such that $p = 1$ in a neighbourhood of $\zeta^0$ and $p = 0$ outside a bigger neighbourhood. Given a $t \in \mathbb{R}$, define the function $p_t(x, \xi) \in C^\infty_0(T^*\Gamma)$ by $p_t(x, \xi) = p(\Phi(t)(x, \xi))$. By a microlocal partition of the unity in a neighbourhood of $\Sigma$, it is easy to see that (2.32) follows from (1.5) and the following

**Lemma 2.3** For every $T > 0$ there exist positive constants $C = C(T)$ and $\lambda_0 = \lambda_0(T)$ so that the solutions to (2.30) satisfy the estimate

\[
\left\| p(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)} \leq \left\| p_t(x, \mathcal{D}_x)f_1 \right\|_{L^2(\Gamma)} + 2T\lambda \left\| h \right\|_{L^2(\Gamma)} + C\lambda^{-1} \left\| \tilde{f}_1 \right\|_{L^2(\Gamma)},
\]

for $0 \leq t \leq T$, $\lambda \geq \lambda_0$. Hereafter we denote $\mathcal{D}_x := \lambda^{-1}\mathcal{D}_x$.

**Proof.** Set $P = -\lambda^{-2}c_R^2\Delta_x - 1 - i\lambda^{-1}b$. Since

\[
\partial_t p_t + \{r_0, p_t\} = 0,
\]

the operator

\[
Q_t := \lambda \partial_t p_t(x, \mathcal{D}_x) + i\lambda^2[p, p_t(x, \mathcal{D}_x)]
\]

is a zero order $\lambda - \Psi DO$, and hence uniformly bounded on $L^2(\Gamma)$. Moreover, the fact that the principal symbol of the operator $b$ satisfies $\text{Re} \sigma_p(b) \geq 0$ implies

\[
-\text{Re} \langle bf, f \rangle_{L^2(\Gamma)} \leq O(\lambda^{-1})\|f\|_{L^2(\Gamma)}^2, \quad \forall f \in L^2(\Gamma).
\]

Therefore, using the identity

\[
\frac{1}{2}\frac{d}{dt} \left\| p_t(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)}^2 = \text{Re} \left\langle \partial_t p_t(x, \mathcal{D}_x)\tilde{f}_1, p_t(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)} + \lambda^{-1} \text{Re} \langle Q_t\tilde{f}_1, p_t(x, \mathcal{D}_x)\tilde{f}_1 \rangle_{L^2(\Gamma)}
\]

\[
= \lambda \text{Re} \left\langle [P, p_t(x, \mathcal{D}_x)]\tilde{f}_1, p_t(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)} + \lambda^{-1} \text{Re} \langle Q_t\tilde{f}_1, p_t(x, \mathcal{D}_x)\tilde{f}_1 \rangle_{L^2(\Gamma)}
\]

\[
= -\lambda \text{Re} \left\langle p_t(x, \mathcal{D}_x)P\tilde{f}_1, p_t(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)} + \lambda^{-1} \text{Re} \langle Q_t\tilde{f}_1, p_t(x, \mathcal{D}_x)\tilde{f}_1 \rangle_{L^2(\Gamma)}
\]

\[
-\text{Re} \left\langle b_p(x, \mathcal{D}_x)\tilde{f}_1, p_t(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)},
\]

we obtain

\[
\left\| \frac{d}{dt} \left\| p_t(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)} \right\| \leq 2\lambda \left\| P\tilde{f}_1 \right\|_{L^2(\Gamma)} + O(\lambda^{-1}) \left\| \tilde{f}_1 \right\|_{L^2(\Gamma)}.
\]

By (2.35),

\[
\left\| p(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)} = \left\| p_t(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)} - \int_0^t \frac{d}{d\tau} \left\| p_t(x, \mathcal{D}_x)\tilde{f}_1 \right\|_{L^2(\Gamma)} d\tau
\]
operator has a parametrix \( N_{\lambda} \) with an analogue of (2.23)
for much of the analysis still works with \( \text{Op} \)
In what follows we will keep the same notations as in the proof of Theorem 2.2 above. In fact, provided \( \text{Im} \) and hence \( \text{Im} \)
Under the assumption (1.1), for \( \text{Proposition 2.4} \) \( C > 0 \) chosen constant
By (2.36), we have
\[ \text{Im} \lambda \leq |\lambda| \|g\|_{L^2(\Gamma)}, \quad \lambda \in \Lambda_M. \]
We still have (2.29) with \( O(\lambda^{-1}) \) replaced by \( O(\lambda_1^{-1}) \) as well as (2.30) with \( h \) satisfying (2.31) with \( O(\lambda^{-\infty}) \) replaced by \( O(\lambda_1^{-\infty}) \). Thus, we get
\[ \text{Im} \lambda^2 \| \tilde{f}_1 \|_{L^2(\Gamma)}^2 = -\text{Im} \left\langle \lambda^2 h, \tilde{f}_1 \right\rangle_{L^2(\Gamma)} + \text{Re} \left\langle \lambda b \tilde{f}_1, \tilde{f}_1 \right\rangle_{L^2(\Gamma)} \]
and hence
\[ \text{Im} \lambda \| \tilde{f}_1 \|_{L^2(\Gamma)} \leq (2\text{Im} \lambda - C) \| \tilde{f}_1 \|_{L^2(\Gamma)} \leq O(|\lambda|) \| h \|_{L^2(\Gamma)}, \]
provided \( \text{Im} \lambda \geq C, \lambda \in \Lambda_M \). Combining (2.38) with (2.29) and (2.31), we obtain
\[ \left\| \text{Op}_{\Lambda_1}(\lambda)f \right\|_{L^2(\Gamma)} \leq \frac{C|\lambda|}{\text{Im} \lambda} \| g \|_{L^2(\Gamma)} + O(\lambda_1^{-\infty}) \| f \|_{H^1(\Gamma)}, \]
for \( \lambda \) belonging to the region (1.8). Now (2.36) follows from (2.37) and (2.39).
To prove the existence of infinitely many resonances (i.e. poles of \( (\lambda N(\lambda) - iA)^{-1} \)) in \( \{0 < \text{Im} \lambda < C\} \) we will proceed as in [12]. Without loss of generality we may suppose \( \text{Re} \lambda > 0 \). By (2.36), we have
\[ \left\| (\lambda N(\lambda) - iA)^{-1} \right\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C(\log |\lambda|)^{-1}, \quad \lambda \in l^\pm, \]
where \( l^\pm := \{ \lambda \in \mathbb{C} : \pm \text{Im} \lambda = \log \text{Re} \lambda, \text{Re} \lambda \geq C' \} \) with some constant \( C' \gg 1 \). If we suppose that \( (\lambda N(\lambda) - iA)^{-1} \) is analytic in \( \{ \lambda \in \mathbb{C} : 0 < \text{Im} \lambda < C, \text{Re} \lambda \geq C' \} \), so it is in \( \{ \lambda \in \mathbb{C} : |\text{Im} \lambda| \leq \log \text{Re} \lambda, \text{Re} \lambda \geq C' \} \). Then, by (2.40) together with the Frégmen-Lindelöf principle we get
\[ \left\| (\lambda N(\lambda) - iA)^{-1} \right\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C(\log |\lambda|)^{-1}, \quad \lambda \in \mathbb{R}, \lambda \geq C'. \]
On the other hand, it is shown in [12] that there exist quasi-modes \((f_j, k_j) \in L^2(\Gamma) \times \mathbb{R}\) such that \(\|f_j\|_{L^2} = 1, k_j \to +\infty\) and

\[
\|k_j N(k_j) f_j\|_{L^2} \leq \text{Const.}
\]

Hence,

\[
\|(k_j N(k_j) - iA) f_j\|_{L^2} \leq \text{Const},
\]

which combined with (2.41) lead to

\[
1 = \|f_j\|_{L^2} \leq \text{Const}(\log k_j)^{-1},
\]

which is impossible if we take \(k_j\) large enough. Therefore, the operator-valued function \((\lambda N(\lambda) - iA)^{-1}\) cannot be analytic in \(\{ \lambda \in \mathbb{C} : 0 < \text{Im} \lambda < C, \text{Re} \lambda \geq C'\}\). \(\square\)

3 Proof of Theorem 1.3

Again, it suffices to prove (1.13) for real \(\lambda \gg 1\), only. Let \(v \in L^2_{\text{comp}}(\Omega)\) and let \(u\) be the solution to the equation

\[
\begin{aligned}
(\Delta_e + \lambda^2)u &= v \quad \text{in} \quad \Omega, \\
(B - i\lambda A)u &= 0 \quad \text{on} \quad \Gamma, \\
\end{aligned}
\]

Clearly, (1.13) is equivalent to the estimate

\[
\|u\|_{L^2(\Omega_0)} \leq C\lambda^{-1} \|v\|_{L^2(\Omega)}, \quad \lambda \geq \lambda_0.
\]

The function \(f = u|_{\Gamma}\) solves the equation

\[
(N(\lambda) - iA) f = g
\]

with \(g\) satisfying

\[
\|g\|_{L^2(\Gamma)} \leq C\lambda^{-1} \|v\|_{L^2(\Omega)}, \quad \lambda \geq \lambda_0.
\]

Thus, in view of (2.4), to prove (3.2) it suffices to show that

\[
\|f\|_{H^1(\Gamma)} \leq C \|g\|_{L^2(\Gamma)}, \quad \lambda \geq \lambda_0.
\]

Using (2.18) and (1.4) with \(\Gamma_0 = \Gamma\), we get

\[
C \|f\|^2_{L^2(\Gamma)} \leq \langle Af, f \rangle_{L^2(\Gamma)} - \text{Im} \langle g, f \rangle_{L^2(\Gamma)} \leq \beta^{-2} \|g\|^2_{L^2(\Gamma)} + \beta^2 \|f\|^2_{L^2(\Gamma)},
\]

for every \(\beta > 0\). Taking \(\beta\) small enough, we deduce from (3.6),

\[
\|f\|_{L^2(\Gamma)} \leq C \|g\|_{L^2(\Gamma)}.
\]

Let \(\eta \in C^\infty_0(T^*\Gamma), \eta = 1\) on \(\{ \zeta \in T^*\Gamma : \|\zeta\| \leq c_R^{-1}\}\), \(\eta = 0\) on \(\{ \zeta \in T^*\Gamma : \|\zeta\| \geq c_R^{-1} + 2\}\). Since the parametrix \(N(\lambda)\) on \(\text{supp} (1 - \eta)\) is an elliptic \(\lambda - \Psi DO\) of class \(L^{1,0}_{0,0}(\Gamma)\), we have

\[
\|\text{Op}_\lambda (1 - \eta) f\|_{H^1(\Gamma)} \leq C \|N(\lambda) f\|_{L^2(\Gamma)} + O(\lambda^{-\infty}) \|f\|_{H^1(\Gamma)} \\
\leq C \|g\|_{L^2(\Gamma)} + C \|f\|_{L^2(\Gamma)} + O(\lambda^{-\infty}) \|f\|_{H^1(\Gamma)}.
\]

On the other hand,

\[
\|\text{Op}_\lambda (\eta) f\|_{H^1(\Gamma)} \leq C \|f\|_{L^2(\Gamma)}.
\]

Now (3.5) follows from combining (3.8) and (3.9) with (3.7).

\(\square\)
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