Necessary conditions for optimal control problems with sweeping systems and end point constraints

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ABSTRACT
We generalize the Maximum Principle for free end point optimal control problems involving sweeping systems derived in [de Pinho MdR, Ferreira MMA, Smirnov GV. Optimal control involving sweeping processes. Set-Valued Var Anal. 2019;27(2):523–548] to cover the case where the constraints are time dependent and the end point is constrained to a set. As in [de Pinho MdR, Ferreira MMA, Smirnov GV. Optimal control involving sweeping processes. Set-Valued Var Anal. 2019;27(2):523–548], an ingenious smooth approximating family of standard differential equations plays a crucial role.

1. Introduction
Sweeping processes are evolution differential inclusions involving the normal cone to a set. They were introduced in the seminal paper [1] by J.J. Moreau in the context of plasticity and friction theory. Since then, there has been an increasing interest in sweeping systems with its range of applications covering now problems from mechanics, engineering, economics and crowd motion control; see, for example, [2–7].

In recent years, there has been considerable research on optimal control problems involving controlled sweeping systems of the form

\[ \dot{x}(t) \in f(t, x(t), u(t)) - N_{C(t)}(x(t)), \quad u(t) \in U, \ x(0) \in C_0. \] (1)

In this respect, we refer the reader to, for example, [3–5,8–12] (see also accompanying correction [13]). A remarkable aspect of (1) is that the presence of the normal cone in this dynamics destroys the regularity of the corresponding differential inclusion under which classical results on differential inclusions and optimal control have been built.

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Assuming that the set $C$ in (1) is time independent, necessary conditions in the form of a maximum principle for optimal control problems involving such systems are derived in [4,9,14]. A special feature of [4], where $C = \{x : \psi(x) \leq 0\}$ for some function $\psi$, is that it relies on an approximating sequence of optimal control problems differing from the original problem insofar as (1) is replaced by a differential equation of the form

$$\dot{x}_{\gamma_k}(t) = f(t, x_{\gamma_k}(t), u(t)) - \gamma_k e^{\gamma_k \psi(x_{\gamma_k}(t))} \nabla \psi(x_{\gamma_k}(t))$$

for some positive sequence $\gamma_k \to +\infty$. Later, similar techniques have been applied to more general problems in [12]. For problems involving hysteresis, which can be seen as closely related to sweeping systems, we refer, for example, to [15,16], where approximation techniques are developed, and to [17,18], where optimality conditions are studied.

In this paper, we generalize the Maximum Principle proved in [4] to cover problems with additional end point constraints and time dependent set $C$. Our problem of interest is

\[
\begin{aligned}
(P) \quad \text{Minimize} & \quad \phi(x(T)) \\
\text{over processes} & \quad (x,u) \text{ such that} \\
\dot{x}(t) & \in f(t, x(t), u(t)) - N_{C(t)}(x(t)), \text{ a.e. } t \in [0, T], \\
u(t) & \in U, \text{ a.e. } t \in [0, T], \\
(x(0), x(T)) & \in C_0 \times C_T \subseteq C(0) \times C(T), \\
\end{aligned}
\]

where $T > 0$ is fixed, $\phi : \mathbb{R}^n \to \mathbb{R}$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ and

$$C(t) := \{x \in \mathbb{R}^n : \psi(t,x) \leq 0\}$$

for some function $\psi : [0, T] \times \mathbb{R}^n \to \mathbb{R}$. The paper [19] is a preliminary attempt to generalize the results of [4] to cover problems with end point constraints. Here, however, we diverge from [19]; for example, we assume the set $C$ to be time dependent whereas in [19] it is a constant set. While the present work was under review, the paper [20] was published, where the authors consider a problem that, while more general than ours in some aspects, involves a constant set $C$.

**2. Preliminaries**

In this section, we introduce a summary of the notation and state the assumptions on the data of $(P)$ enforced throughout. Furthermore, we extract information from the assumptions defining extra functions and establishing relations crucial for the forthcoming analysis.

**Notation**

For a set $S \subseteq \mathbb{R}^n$, $\partial S$, $\text{cl} S$ and $\text{int} S$ denote the **boundary**, **closure** and **interior** of $S$. 

1. 1
2. 2
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5. 5
We denote by \( g : \mathbb{R}^p \rightarrow \mathbb{R}^q \), \( \nabla g \) represents the derivative and \( \nabla^2 g \) the second derivative. If \( g : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^q \), then \( \nabla_x g \) represents the derivative w.r.t. \( x \in \mathbb{R}^p \) and \( \nabla^2_x g \) the second derivative, while \( \partial_t g(t, x) \) represents the derivative w.r.t. \( t \in \mathbb{R} \).

The Euclidean norm or the induced matrix norm on \( \mathbb{R}^{p \times q} \) is denoted by \( | \cdot | \). We denote by \( B_n \) the closed unit ball in \( \mathbb{R}^n \) centred at the origin. For some \( A \subset \mathbb{R}^n \), \( d(x, A) \) denotes the distance between \( x \) and \( A \). We denote the support function of \( A \) at \( x \) by \( S(x, A) = \sup \{ \langle x, a \rangle \mid a \in A \} \).

The spaces \( L^1([a, b]; \mathbb{R}^p) \) and \( L^\infty([a, b]; \mathbb{R}^p) \) (or simply \( L^1 \), \( L^\infty \) when the domains are clearly understood) are the Lebesgue spaces of integrable functions and of essentially bounded functions \( h : [a, b] \rightarrow \mathbb{R}^p \). The norms in these spaces are denoted by \( |h|_{L^1} \) and \( |h|_{L^\infty} \). We say that \( h \in BV([a, b]; \mathbb{R}^p) \) if \( h \) is a function of bounded variation. The space of continuous functions is denoted by \( C([a, b]; \mathbb{R}^p) \).

Standard concepts from nonsmooth analysis will also be used. Those can be found in [21–23], to name but a few. The Mordukhovich normal cone to a set \( S \) at \( s \in S \) is denoted by \( N^l_S(s) \) and \( \partial^l f(s) \) is the Mordukhovich subdifferential of \( f \) at \( s \) (also known as limiting subdifferential).

For any set \( A \subset \mathbb{R}^n \), cone \( A \) is the cone generated by the set \( A \).

We now turn to problem \((P)\). We first introduce the definition of admissible processes.

**Definition 2.1:** A pair \((x, u)\) is called an admissible process for \((P)\) when \( x \) is an absolutely continuous function and \( u \) is a measurable function satisfying the constraints of \((P)\).

**Assumptions on the data of \((P)\)**

A1: The function \( \psi \) is \( C^2 \) and the graph of the set-valued map \( C(\cdot), \text{Gra}(C(\cdot)) \), is compact. Moreover, there exist constants \( \beta > 0 \) and \( \eta > 0 \) such that

\[
\psi(t, x) \geq -\beta \implies |\nabla_x \psi(t, x)| > \eta \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^n. \tag{4}
\]

A2: The function \( f \) is continuous, \( x \rightarrow f(t, x, u) \) is continuously differentiable for all \((t, u) \in [0, T] \times \mathbb{R}^m \) and let \( M > 0 \) be such that \( |f(t, x, u)| \leq M \) and \( |\nabla_x f(t, x, u)| \leq M \) for all \((t, x, u) \in \text{Gra}(C(\cdot) + B_n) \times U \).

A3: For each \((t, x)\), the set \( f(t, x, U) \) is convex.

A4: The set \( U \) is compact.

A5: The sets \( C_0 \) and \( C_T \) are compact.

A6: There exists a constant \( L_\phi \) such that \( |\phi(x) - \phi(x')| \leq L_\phi |x - x'| \) for all \( x, x' \in \mathbb{R}^n \).

Let \( x(\cdot) \) be a solution to the differential inclusion:

\[
\dot{x}(t) \in f(t, x(t), U) - N_{C(t)}(x(t)).
\]
Under our assumptions, measurable selection theorems assert the existence of measurable functions \( u \) and \( \xi \) such that \( u(t) \in U, \xi(t) \geq 0 \ a.e. \ t \in [0, T] \) and
\[
\dot{x}(t) = f(t, x(t), u(t)) - \xi(t) \nabla_x \psi(t, x(t)) \quad a.e. \ t \in [0, T].
\]

Let \( \mu \) be such that
\[
\max \left\{ \left| (\nabla_x \psi(t, x)) \langle f(t, x, u) \rangle \right| + |\partial_t \psi(t, x)| \right\} + 1: t \in [0, T], u \in U, x \in C(t) + B_n \leq \mu.
\]

Consider the motion of the system along the boundary of \( C(\cdot) \). For any \( t \) such that \( \psi(t, x(t)) = 0 \) and \( \dot{x}(t) \) exists, we have
\[
0 = \frac{d}{dt} \psi(t, x(t)) = \langle \nabla_x \psi(t, x(t)), \dot{x}(t) \rangle + \partial_t \psi(t, x(t))
\]
\[
= \langle \nabla_x \psi(t, x(t)), f(t, x(t), u(t)) \rangle - \xi(t) |\nabla_x \psi(t, x(t))|^2 + \partial_t \psi(t, x(t)).
\]

Hence
\[
\dot{\xi}(t) = \frac{1}{|\nabla_x \psi(t, x(t))|^2} (\langle \nabla_x \psi(t, x(t)), f(t, x(t), u(t)) \rangle + \partial_t \psi(t, x(t))) \leq \frac{\mu}{\eta^2}.
\]

Define the function
\[
\mu(\gamma) = \frac{1}{\gamma} \log \frac{\mu}{\eta^2}.
\]

Consider now a sequence \( \{\sigma_k\} \) such that \( \sigma_k \downarrow 0 \). Let \( \{\gamma_k\} \) be a sequence converging to \( +\infty \) such that
\[
C(t) \subset \text{int } C^k(t) = \text{int } \{x: \psi(t, x) - \sigma_k \leq \mu_k\}
\]

where
\[
\mu_k = \mu(\gamma_k).
\]

Set
\[
\xi_k(t) = \gamma_k e^{\gamma_k (\psi(t, x_k(t)) - \sigma_k)}.
\]

Consider \( x_k \) to be a solution to the differential equation
\[
\dot{x}_k(t) = f(t, x_k(t), u_k(t)) - \xi_k(t) \nabla_x \psi(t, x_k(t))
\]
for some \( u_k(t) \in U \ a.e. \ t \in [0, T] \). Take any \( t \in [0, T] \) such that \( \dot{x}_k(t) \) exists and \( \psi(t, x_k(t)) - \sigma_k = \mu_k \). We then have
\[
\frac{d}{dt} \psi(t, x_k(t))
\]
\[
= \langle \nabla_x \psi(t, x_k(t)), f(t, x_k(t), u_k(t)) \rangle - \xi_k(t) |\nabla_x \psi(t, x_k(t))|^2 + \partial_t \psi(t, x_k(t))
\]
\[
\leq \mu - 1 - \eta^2 \gamma_k e^{\gamma_k \mu_k} = -1.
\]
Thus, if $x_k(0) \in C^k(0)$, then $x_k(t) \in C^k(t)$ for all $t \in [0, T]$, and
\[ \xi_k(t) \leq \gamma_k e^{\gamma_k \mu_k} = \frac{\mu}{\eta^2}. \] (5)

It follows that, for all $k$, we have
\[ |\dot{x}_k(t)| \leq \text{(const)}. \]

We are now in a position to state our first Theorem. This is akin to Theorem 4.1 in [12] (see also Lemma 1 in [4] when $\psi$ is independent of $t$ and convex) but we now consider a different approximating sequence of control systems. This guarantees the estimation (5), greatly simplifying the proof of Theorem 2.2.

**Theorem 2.2:** Let $\{(x_k, u_k)\}$, with $u_k(t) \in U$ a.e. be a sequence of solutions of Cauchy problems
\[ \dot{x}_k(t) = f(t, x_k(t), u_k(t)) - \xi_k(t) \nabla_x \psi(t, x_k(t)), \quad x_k(0) = b_k \in C^k(0). \] (6)

If $b_k \to x_0$, then there exists a subsequence $\{x_k\}$ (we do not relabel) converging uniformly to $x$, a unique solution to the Cauchy problem
\[ \dot{x}(t) \in f(t, x(t), u(t)) - N_{C(t)}(x(t)), \quad x(0) = x_0, \] (7)
where $u$ is a measurable function such that $u(t) \in U$ a.e. $t \in [0, T]$.

If, moreover, all the controls $u_k$ are equal, i.e. $u_k = u$, then the subsequence converges to a unique solution of (7), i.e. any solution of
\[ \dot{x}(t) \in f(t, x(t), U) - N_{C(t)}(x(t)), \quad x(0) = x_0 \in C(0) \] (8)
can be approximated by solutions of (6).

**Proof:** Consider the sequence $\{x_k\}$, where $x_k$ solves (6). Recall that $x_k(t) \in C^k(t)$ for all $t \in [0, T]$, and
\[ |\dot{x}_k(t)| \leq \text{(const)} \quad \text{and} \quad \xi_k(t) \leq \text{(const)}. \]

Then there exist subsequences (we do not relabel) weakly-$*$ converging in $L^\infty$ to some $v$ and $\xi$. Hence
\[ x_k(t) = x_0 + \int_0^t \dot{x}_k(s) \, ds \longrightarrow x(t) = x_0 + \int_0^t v(s) \, ds, \quad \forall \, t \in [0, T], \]
for an absolutely continuous function $x$. Obviously, $x(t) \in C(t)$ for all $t \in [0, T]$. 
We have
\[ \dot{x}_k(t) \in f(t, x_k(t), U) - \xi_k(t) \nabla_x \psi(t, x_k(t)). \] (9)

Inclusion (9) is equivalent to the condition
\[ \langle z, \dot{x}_k(t) \rangle \leq S(z, f(t, x_k(t), U)) - \xi_k(t) \langle z, \nabla_x \psi(t, x_k(t)) \rangle, \quad \forall z \in \mathbb{R}^n. \]

Integrating this inequality, we get
\[
\begin{align*}
\left\langle z, \frac{x_k(t + \tau) - x_k(t)}{\tau} \right\rangle & \leq \frac{1}{\tau} \int_t^{t+\tau} \left( S(z, f(s, x_k(s), U)) - \xi_k(s) \langle z, \nabla_x \psi(s, x_k(s)) \rangle \right) ds \\
& = \frac{1}{\tau} \int_t^{t+\tau} \left( S(z, f(s, x_k(s), U)) - \xi_k(s) \langle z, \nabla_x \psi(s, x(s)) \rangle \right) ds \\
& \quad + \xi_k(s) \langle z, \nabla_x \psi(s, x(s)) - \nabla_x \psi(s, x_k(s)) \rangle ds.
\end{align*}
\]

Passing to the limit as \( k \to \infty \), we obtain
\[
\left\langle z, \frac{x(t + \tau) - x(t)}{\tau} \right\rangle \leq \frac{1}{\tau} \int_t^{t+\tau} \left( S(z, f(s, x(s), U)) - \xi(s) \langle z, \nabla_x \psi(s, x(s)) \rangle \right) ds.
\]

Let \( t \in [0, T] \) be a Lebesgue point of \( x \) and \( \xi \). Passing to the limit as \( \tau \downarrow 0 \), we have
\[ \langle z, \dot{x}(t) \rangle \leq S(z, f(t, x(t), U)) - \xi(t) \langle z, \nabla_x \psi(t, x(t)) \rangle. \]

Since \( z \in \mathbb{R}^n \) is an arbitrary vector and the set \( f(t, x(t), U) \) is convex, we conclude that
\[
\dot{x}(t) \in f(t, x(t), U) - \xi(t) \nabla_x \psi(t, x(t)).
\]

By the Filippov lemma there exists a measurable control \( u(t) \in U \) such that
\[ \dot{x}(t) = f(t, x(t), u(t)) - \xi(t) \nabla_x \psi(t, x(t)). \]

Observe that \( \xi \) is zero if \( \psi(t, x(t)) < 0 \).

If \( u_k = u \) for all \( k \), then the sequence \( x_k \) converges to the solution of
\[ \dot{x}(t) = f(t, x(t), u(t)) - \xi(t) \nabla_x \psi(t, x(t)). \]

Indeed, to see this it suffices to pass to the limit as \( k \to \infty \) and then as \( \tau \downarrow 0 \) in the equality
\[
\frac{x_k(t + \tau) - x_k(t)}{\tau} = \frac{1}{\tau} \int_t^{t+\tau} \left( f(s, x_k(s), u(t)) - \xi_k(s) \nabla_x \psi(s, x_k(s)) \right) ds.
\]

We now prove the uniqueness of the solution. We follow the proof of Theorem 4.1 in [12]. Notice however that we now consider a special case and not the general case treated in [12].
Suppose that there exist two different solutions of (7): $x_1$ and $x_2$. We have

$$\frac{1}{2} \frac{d}{dt} |x_1(t) - x_2(t)|^2 = (x_1(t) - x_2(t), \dot{x}_1(t) - \dot{x}_2(t))$$

$$= (x_1(t) - x_2(t), f(t, x_1(t), u(t)) - f(t, x_2(t), u(t)))$$

$$- (x_1(t) - x_2(t), \zeta_1(t) \nabla_x \psi(t, x_1(t))$$

$$- \zeta_2(t) \nabla_x \psi(t, x_2(t))).$$

If $\psi(t, x_1(t)) < 0$ and $\psi(t, x_2(t)) < 0$, then $\zeta_1(t) = \zeta_2(t) = 0$ and we obtain

$$\frac{1}{2} \frac{d}{dt} |x_1(t) - x_2(t)|^2 \leq L_f |x_1(t) - x_2(t)|^2.$$

Suppose that $\psi(t, x_1(t)) = 0$. Then, by the Taylor formula, we get

$$\psi(t, x_2(t)) = \psi(t, x_1(t)) + \langle \nabla_x \psi(t, x_1(t)), x_2(t) - x_1(t) \rangle$$

$$+ \frac{1}{2} \langle x_2(t) - x_1(t), \nabla_x^2 \psi(t, \theta x_2(t) + (1 - \theta)x_1(t))(x_2(t) - x_1(t)) \rangle,$$

where $\theta \in [0, 1]$. Since $\psi(t, x_2(t)) \leq 0$, we have

$$\langle \nabla_x \psi(t, x_1(t)), x_2(t) - x_1(t) \rangle$$

$$\leq -\frac{1}{2} \langle x_2(t) - x_1(t), \nabla_x^2 \psi(t, \theta x_2(t) + (1 - \theta)x_1(t))(x_2(t) - x_1(t)) \rangle$$

$$\leq (\text{const}) |x_1(t) - x_2(t)|^2.$$

In the same way

$$\langle \nabla_x \psi(t, x_2(t)), x_1(t) - x_2(t) \rangle \leq (\text{const}) |x_1(t) - x_2(t)|^2.$$

Thus we have

$$\frac{1}{2} \frac{d}{dt} |x_1(t) - x_2(t)|^2 \leq (\text{const}) |x_1(t) - x_2(t)|^2.$$

Hence $|x_1(t) - x_2(t)| = 0$.

3. Approximating family of optimal control problems

In this section, we define an approximating family of optimal control problems to ($P$) and we state the corresponding necessary conditions.
Let \((\hat{x}, \hat{u})\) be a global solution to \((P)\). Consider a sequence \(\{\gamma_k\}\) as defined above. Let \(\hat{x}_k(\cdot)\) be the solution to

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), \hat{u}(t)) - \gamma_k e^{\gamma_k(y(t) - \sigma_k)} \nabla x_y(t, x(t)), \\
x(0) &= \hat{x}(0).
\end{align*}
\]

Set \(\epsilon_k = |\hat{x}_k(T) - \hat{x}(T)|.\) It follows from Theorem 2.2 that \(\epsilon_k \downarrow 0.\) Take \(\alpha > 0\) and define the problem

\[
(Q^\alpha_k)
\begin{align*}
\text{Minimize } & \phi(x(T)) + |x(0) - \hat{x}(0)|^2 + \alpha \int_0^T |u(t) - \hat{u}(t)| \, dt \\
\forall (x, u) \text{ such that } & \dot{x}(t) = f(t, x(t), u(t)) - \nabla x^\gamma_k(y(t) - \sigma_k) \text{ a.e. } t \in [0, T], \\
& u(t) \in U \text{ a.e. } t \in [0, T], \\
& x(0) \in C_0, \ x(T) \in C_T + \epsilon_k B_n.
\end{align*}
\]

Clearly, the problem \((Q^\alpha_k)\) has admissible solutions.

Consider the metric space

\[
W = \{ (c, u) \mid c \in C_0, \ u \in L^\infty \text{ with } u(t) \in U \}
\]

and the distance

\[
d_W((c_1, u_1), (c_2, u_2)) = |c_1 - c_2| + \int_0^T |u_1(t) - u_2(t)| \, dt.
\]

Endowed with \(d_W\), \(W\) is a complete metric space. Take any \((c, u) \in W\) and a solution \(y\) to the Cauchy problem

\[
\begin{align*}
\dot{y}(t) &= f(t, y(t), u(t)) - \nabla x^\gamma_k(y(t) - \sigma_k) \text{ a.e. } t \in [0, T], \\
y(0) &= c.
\end{align*}
\]

Under our conditions, the function

\[
(c, u) \to \phi(y(T)) + |c - \hat{x}(0)|^2 + \alpha \int_0^T |u - \hat{u}| \, dt
\]

is continuous on \((W, d_W)\) and bounded below.

Appealing to Ekeland’s Theorem we deduce the existence of a pair \((x_k, u_k)\) solving the following problem

\[
(AQ_k)
\begin{align*}
\text{Minimize } & \Phi(x, u) = \phi(x(T)) + |x(0) - \hat{x}(0)|^2 + \alpha \int_0^T |u(t) - \hat{u}(t)| \, dt \\
& + \epsilon_k \left( |x(0) - x_k(0)| + \int_0^T |u(t) - u_k(t)| \, dt \right), \\
\forall (x, u) \text{ such that } & \dot{x}(t) = f(t, x(t), u(t)) - \nabla x^\gamma_k(y(t) - \sigma_k) \text{ a.e. } t \in [0, T], \\
& u(t) \in U \text{ a.e. } t \in [0, T], \\
& x(0) \in C_0, \ x(T) \in C_T + \epsilon_k B_n.
\end{align*}
\]
Lemma 3.1: Take $\gamma_k \to \infty$, $\sigma_k \to 0$ and $\epsilon_k \to 0$ as defined above. For each $k$, let $(x_k, u_k)$ be the solution to $(AQ_k)$. Then there exists a subsequence (we do not relabel) such that

$$u_k(t) \to \hat{u}(t) \quad \text{a.e., } x_k \to \hat{x} \text{ uniformly in } [0, T].$$

Proof: Take any admissible process $(x, u)$ for $(P)$. Then there exists a $K > 0$ such that

$$|x(0) - x_k(0)| + \int_0^T |u(t) - u_k(t)| \, dt \leq K.$$

Let us assume that there exists a $\delta > 0$ such that

$$|x_k(0) - \hat{x}(0)|^2 + \alpha \int_0^T |u_k(t) - \hat{u}(t)| \, dt \geq \delta,$$  \hspace{1cm} (11)

for all $k$. It is an easy task to see that, for large $k$, we have $\delta > 8 \epsilon_k K$.

Now, we deduce from Theorem 2.2 that $\{x_k\}$ uniformly converges to an admissible solution $\hat{x}$ to $(P)$. Taking into account the optimality of $(\hat{x}, \hat{u})$ and (11), we conclude that

$$\Phi(x_k, u_k) \geq \phi(x_k(T)) + \delta \geq \phi(\hat{x}(T)) + \delta/2 \geq \phi(\hat{x}(T)) + \delta/2.$$  \hspace{1cm} (12)

Again from Theorem 2.2, it is also an easy task to see that $\hat{x}_k$ (see (10)) satisfies the inequality

$$\phi(\hat{x}_k(T)) \leq \phi(\hat{x}(T)) + \delta/8,$$  \hspace{1cm} (13)

for $k$ sufficiently large; in this respect, notice that each $\hat{x}_k$ corresponds to the control $\hat{u}$.

It follows from the optimality of $(x_k, u_k)$ and (12) that

$$\Phi(\hat{x}_k, \hat{u}) \geq \Phi(x_k, u_k) \geq \phi(\hat{x}(T)) + \delta/2.$$  \hspace{1cm} (14)

On the other hand, from (13), we get

$$\Phi(\hat{x}_k, \hat{u})$$

\begin{align*}
= & \phi(\hat{x}_k(T)) + |\hat{x}_k(0) - \hat{x}(0)|^2 \\
& + \epsilon_k \left( |\hat{x}_k(0) - x_k(0)| + \int_0^T |\hat{u}(t) - u_k(t)| \, dt \right) \\
\leq & \phi(\hat{x}(T)) + \delta/8 + |\hat{x}_k(0) - \hat{x}(0)|^2 + \epsilon_k K \\
< & \phi(\hat{x}(T)) + \delta/4,
\end{align*}

for sufficiently large $k$, contradicting (14). Then we conclude that

$$|x_k(0) - \hat{x}(0)|^2 + \alpha \int_0^1 |u_k(t) - \hat{u}(t)| \, dt \to 0, \quad k \to \infty$$

and the result follows.
We now finish this section with the statement of the optimality necessary conditions for the family of problems \((AQ_k)\). These can be seen as a direct consequence of Theorem 6.2.1 in [23], for example.

**Proposition 3.2:** For each \(k\), let \((x_k, u_k)\) be a solution to \((AQ_k)\). Then there exist absolutely continuous functions \(p_k\) and scalars \(\lambda_k \geq 0\) such that

(a) *(nontriviality condition)*
\[
\lambda_k + |p_k(T)| = 1, \quad (15)
\]

(b) *(adjoint equation)*
\[
\dot{p}_k = -(\nabla_x f_k)^* p_k + \gamma_k e^{\gamma_k(\psi_k - \sigma_k)} \nabla_x^2 \psi_k p_k + \gamma_k^2 e^{\gamma_k(\psi_k - \sigma_k)} \nabla_x \psi_k (\nabla_x \psi_k, p_k), \quad (16)
\]

where the superscript \(*\) stands for transpose,

(c) *(maximization condition)*
\[
\max_{u \in U} \left\{ \langle f(t, x_k, u), p_k \rangle - \alpha \lambda_k |u - \hat{u}| - \epsilon_k \lambda_k |u - u_k| \right\} \quad (17)
\]

is attained at \(u_k(t)\), for almost every \(t \in [0, T]\),

(d) *(transversality condition)*
\[
(p_k(0), -p_k(T)) \in \lambda_k \left( 2(x_k(0) - \hat{x}(0)) + \epsilon_k B_n, \partial^L \phi(x_k(T)) \right)
\]
\[
+ N_{C_0}^L (x_k(0)) \times N_{C_T + \epsilon_k B_n}^L (x_k(T)). \quad (18)
\]

To simplify the notation above, we drop the \(t\) dependence in \(p_k, \dot{p}_k, x_k, u_k, \hat{x}\) and \(\hat{u}\). Moreover, in (b), we write \(\psi_k\) instead of \(\psi(t, x_k(t))\), \(f_k\) instead of \(f(t, x_k(t), u_k(t))\). The same holds for the derivatives of \(\psi\) and \(f\).

### 4. Maximum principle for \((P)\)

In this section we establish our main result, a Maximum Principle for \((P)\). This is done by taking limits of the conclusions of Proposition 3.2, following closely the analysis done in the proof of [4, Theorem 2]. However, some changes are called for since here \(C\) is time dependent, while in [4] \(C\) is constant.

First, recall that \(\dot{\xi}_k(t) = \gamma_k e^{\gamma_k(\psi(t, x_k(t)) - \sigma_k)}\) and observe that

\[
\frac{1}{2} \frac{d}{dt} |p_k(t)|^2
\]
\[
= -\langle \nabla_x f_k p_k, p_k \rangle + \xi_k \langle \nabla_x^2 \psi_k p_k, p_k \rangle + \gamma_k \xi_k \langle \nabla_x \psi_k, p_k \rangle^2
\]
\[
\geq -\langle \nabla_x f_k p_k, p_k \rangle + \xi_k \langle \nabla_x^2 \psi_k p_k, p_k \rangle
\]
\[
\geq -M |p_k|^2 + \xi_k \langle \nabla_x^2 \psi_k p_k, p_k \rangle,
\]
where $M$ is the constant of (A2). Taking into account hypothesis (A1) and (5) we deduce the existence of a constant $K_0 > 0$ such that
\[
\frac{1}{2} \frac{d}{dt} |p_k(t)|^2 \geq -K_0 |p_k|^2.
\]
This last inequality leads to
\[
|p_k(t)|^2 \leq e^{2K_0(T-t)} |p_k(T)|^2 \leq e^{2K_0T} |p_k(T)|^2.
\]
Since, by (a) of Proposition 3.2, $|p_k(T)| \leq 1$, we deduce from the above that there exists $M_0 > 0$ such that
\[
|p_k(t)| \leq M_0. \quad (19)
\]
Next, we prove that the sequence $\{\dot{p}_k\}$ is uniformly bounded in $L^1$. We start by establishing some inequalities that differ from analogous ones in [4] due, among other things, to the dependence of $\psi$ on $t$.

We have
\[
\frac{d}{dt} |\langle \nabla_x \psi_k, p_k \rangle| = \left( \langle \nabla_x^2 \psi_k \dot{x}_k, p_k \rangle + \langle \partial_t \nabla_x \psi_k, p_k \rangle \right) \text{sign} \left( \langle \nabla_x \psi_k, p_k \rangle \right)
\]
\[
= \left( \langle p_k, \nabla_x^2 \psi_k f_k \rangle - \xi_k \langle p_k, \nabla_x^2 \psi_k \nabla_x \psi_k \rangle \right)
\]
\[
+ \langle \partial_t \nabla_x \psi_k, p_k \rangle - \langle \nabla_x \psi_k, (\nabla_x f_k)^* p_k \rangle + \xi_k \langle \nabla_x \psi_k, \nabla_x^2 \psi_k p_k \rangle
\]
\[
+ \gamma_k \xi_k |\nabla_x \psi_k|^2 \langle \nabla_x \psi_k, p_k \rangle \text{sign} \left( \langle \nabla_x \psi_k, p_k \rangle \right).
\]
Moreover, we also have
\[
\gamma_k \int_0^T \xi_k |\nabla_x \psi_k|^2 |\langle \nabla_x \psi_k, p_k \rangle| \, dt
\]
\[
= |\langle \nabla_x \psi(T, x_k(T)), p_k(T) \rangle| - |\langle \nabla_x \psi(0, x_k(0)), p_k(0) \rangle|
\]
\[
+ \int_0^T \left( \langle \nabla_x \psi_k, (\nabla_x f_k)^* p_k \rangle - \langle \partial_t \nabla_x \psi_k, p_k \rangle \right)
\]
\[
- \langle p_k, \nabla_x^2 \psi_k f_k \rangle \text{sign} \left( \langle \nabla_x \psi_k, p_k \rangle \right) \, dt \leq M_1,
\]
for some $M_1 > 0$.

We now concentrate on $\gamma_k \int_0^T \xi_k |\nabla_x \psi_k||\langle \nabla_x \psi_k, p_k \rangle| \, dt$.

Set $L_k = \gamma_k \xi_k |\nabla_x \psi_k||\langle \nabla_x \psi_k, p_k \rangle|$ and observe that
\[
\int_0^T L_k \, dt = \int_{\{t:|\nabla_x \psi_k|<\eta\}} L_k \, dt + \int_{\{t:|\nabla_x \psi_k|\geq\eta\}} L_k \, dt.
\]
Then, from (A1) we deduce that
\[
\gamma_k \int_0^T \xi_k |\nabla_x \psi_k||\langle \nabla_x \psi_k, p_k \rangle| \, dt
\]
\[
\gamma_k^2 e^{\gamma_k(-\beta-\sigma)\eta^2} \max_t |p_k(t)| + \gamma_k \int_{\{t:|\nabla_x \psi_k|\geq \eta\}} \xi_k \left| \frac{\nabla_x \psi_k}{|\nabla_x \psi_k|} \right| \langle \nabla_x \psi_k, p_k \rangle \, dt
\]
\[
\leq \gamma_k^2 e^{\gamma_k(-\beta-\sigma)\eta^2} \eta^2 M_0 + \gamma_k \int_0^T \xi_k \left| \nabla_x \psi_k \right|^2 \left| \langle \nabla_x \psi_k, p_k \rangle \right| \, dt
\]
\[
\leq \eta^2 M_0 + \frac{M_1}{\eta},
\]
for \(k\) large enough.

Summarizing, for some \(M_2 > 0\), we have
\[
\gamma_k \int_0^T \xi_k \left| \nabla_x \psi_k \right| \left| \langle \nabla_x \psi_k, p_k \rangle \right| \, dt \leq \eta^2 M_0 + \frac{M_1}{\eta} = M_2. \tag{20}
\]

Mimicking the analysis conducted in Step 1, b) and c) of the proof of Theorem 2 in [4] and taking into account (b) of Proposition 3.2 we conclude that there exist constants \(N_1 > 0\) and \(N_2 > 0\) such that
\[
\int_0^T \gamma_k \xi_k \left| \langle \nabla_x \psi_k, p_k \rangle \right| \, dt \leq N_2 \tag{21}
\]
and
\[
\int_0^T \left| \dot{p}_k(t) \right| \, dt \leq N_1, \tag{22}
\]
for \(k\) sufficiently large. With such bounds, we can then mimick the analysis of Step 2 in the proof of Theorem 2 in [4] to conclude the existence of functions \(p \in BV([0,T],\mathbb{R}^n), \xi \in L^\infty([0,T],\mathbb{R}),\xi(t) \geq 0 \ a.e.\ t, \xi(t) = 0,\ t \in I_b,\) where
\[
I_b = \{ t \in [0,T]: \psi(t,\hat{x}(t)) < 0 \},
\]
and a finite signed Radon measure \(\eta\), null in \(I_b\), such that, for any \(z \in C([0,T],\mathbb{R}^n)\)
\[
\int_0^T \left\langle z, \, dp \right\rangle = -\int_0^T \left\langle z, (\nabla_x \hat{\psi})^* p \right\rangle \, dt + \int_0^T \xi \left\langle z, \nabla_x^2 \hat{\psi} p \right\rangle \, dt + \int_0^T \left\langle z, \nabla_x \hat{\psi}(t) \right\rangle \, d\eta. \tag{23}
\]
Since the sequence \(\xi_k\) weakly-* converges in \(L^\infty\) to \(\xi \geq 0\), from (21) we have
\[
\int_0^T \left| \xi \left\langle \nabla_x \hat{\psi}, p \right\rangle \right| \, dt
\]
\[
= \lim_{k \to \infty} \int_0^T \left| \xi_k \left\langle \nabla_x \hat{\psi}, p \right\rangle \right| \, dt
\]
\[
\leq \lim_{k \to \infty} \left( \int_0^T \left| \xi_k \left\langle \nabla_x \hat{\psi}, p \right\rangle - \left\langle \nabla_x \psi_k, p_k \right\rangle \right| \, dt + \int_0^T \left| \xi_k \left\langle \nabla_x \psi_k, p_k \right\rangle \right| \, dt \right)
\]
\[
\leq \lim_{k \to \infty} \left( \left| \xi_k \right|_{L^\infty} \left| \left\langle \nabla_x \hat{\psi}, p \right\rangle - \left\langle \nabla_x \psi_k, p_k \right\rangle \right|_{L^1} + \frac{N_2}{\gamma_k} \right) = 0.
\]
Hence

$$\xi \langle \nabla_x \hat{\psi}, p \rangle = 0 \quad \text{a. e.} \quad (24)$$

It is a simple matter to see that there exists a subsequence of \( \{ \lambda_k \} \) converging to some \( \lambda \geq 0 \). This, together with convergence of \( p_k \) to \( p \) allows us to take limits in (a) and (c) of Proposition 3.2 to deduce that

$$\lambda + |p(T)| = 1$$

and

$$\langle p(t), f(t, \hat{x}(t), u) \rangle - \alpha \lambda |u - \hat{u}(t)| \leq \langle p(t), f(t, \hat{x}(t), \hat{u}(t)) \rangle \quad \text{for all } u \in U \text{ and a.e. } t \in [0, T].$$

It remains to take limits of the transversality conditions (d) in Proposition 3.2. First, observe that

$$C_T + \epsilon_k B_n = \{ x : d(x, C_T) \leq \epsilon_k \}.$$  

From the basic properties of the Mordukhovich normal cone and subdifferential (see [22], Section 1.3.3) we have

$$N_{C_T + \epsilon_k B_n}^L(x_k(T)) \subset \text{cl cone } \partial^L d(x_k(T), C_T)$$

and

$$N_{C_T}^{L^\ast} (\hat{x}(T)) = \text{cl cone } \partial^L d(\hat{x}(T), C_T).$$

Passing to the limit as \( k \to \infty \) we get

$$(p(0), -p(T)) \in N_{C_0}^L (\hat{x}(0)) \times N_{C_T}^{L^\ast} (\hat{x}(T)) + \{ 0 \} \times \lambda \partial^L \phi(\hat{x}(T)).$$

Now we mimick Step 3 in the proof of Theorem 2 in [4] to remove the dependence of the conditions on the parameter \( \alpha \) which is done, by taking further limits, this time considering a sequence of \( \alpha_i \downarrow 0 \).

We summarize our conclusions in the following Theorem.

**Theorem 4.1:** Let \( (\hat{x}, \hat{u}) \) be the optimal solution to (P). Suppose that assumptions A1–A6 are satisfied. Set \( I_b = \{ t \in [0, T] : \psi(t, \hat{x}(t)) < 0 \} \).

Then there exist a non negative scalar \( \lambda \), \( p \in BV([0, T]; \mathbb{R}^n) \), a finite signed Radon measure \( \eta \), null in \( I_b \), \( \xi \in L^\infty([0, T]; \mathbb{R}) \) with \( \xi(t) \geq 0 \text{ a.e. } t \), \( \xi(t) = 0 \text{ for } t \in I_b \) satisfying the following conditions

(a) \( \lambda + |p(T)| \neq 0 \),
(b) for any \( z \in C([0, T]; \mathbb{R}^n) \)

$$\int_0^T \langle z(t), dp(t) \rangle$$

and
\[ = - \int_0^T \langle z(t), (\nabla_x \hat{f}(t))^* p(t) \rangle \, dt + \int_0^T \xi(t) \langle z(t), \nabla_x^2 \hat{\psi}(t)p(t) \rangle \, dt \\
+ \int_0^T \langle z(t), \nabla_x \hat{\psi}(t) \rangle \, d\eta, \]

where the superscript * stands for transpose, \( \nabla_x \hat{f}(t) = \nabla_x f(t, \hat{x}(t), \hat{u}(t)), \)
\( \nabla_x \hat{\psi}(t) = \nabla_x \psi(t, \hat{x}(t)) \) and \( \nabla_x^2 \hat{\psi}(t) = \nabla_x^2 \psi(t, \hat{x}(t)) \),
\( (c) \quad \xi(t)(\nabla_x \hat{\psi}(t), p(t)) = 0 \) for a.e. \( t \in [0, T] \),
\( (d) \quad \langle p(t), f(t, \hat{x}(t), u) \rangle \leq \langle p(t), f(t, \hat{x}(t), \hat{u}(t)) \rangle \) for a.e. \( t \in [0, T] \) and all \( u \in U \),
\( (e) \quad (p(0), -p(T)) \in N_{C_0}^L (\hat{x}(0)) \times N_{\hat{C}_{T}}^L (\hat{x}(T)) + \{0\} \times \lambda \partial^L \phi(\hat{x}(T)). \)

5. Example

We consider two examples. For the first one, we construct the optimal solution and then we show that the necessary conditions given by Theorem 4.1 are satisfied by all admissible solutions. The situation changes for the second example where the necessary conditions significantly reduce the set of candidates to the solution.

Example 5.1: Set \( \rho(t) = (1 - 2t)^2 + \frac{1}{4} \) and \( C(t) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq \rho^2(t)\} \). Consider now the control system

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &\in \begin{pmatrix} u(t) \\ 0 \end{pmatrix} - N_{C(t)}(x(t), y(t)), \\

u(t) &\in [-\mu, 1].
\end{align*}
\]

Observe that

\[
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix} - \kappa(t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.
\]

In the above, \( \mu > 0 \) and \( \kappa \) is an \( L^1 \) function with \( \kappa(t) \geq 0 \).

Introducing polar coordinates \( x = r \cos \phi, y = r \sin \phi \), the system takes the form

\[
\begin{align*}
\dot{r} &= u \cos \phi - \kappa r, \\

\dot{\phi} &= -\frac{u \sin \phi}{r}.
\end{align*}
\]

Set \( t_1 = \frac{1}{4} \). Let us consider the motion of the system satisfying

\[
(r(t_1), \phi(t_1)) = \left( \rho(t_1), \frac{\pi}{3} \right).
\]

We now consider the motion of the system along the boundary of \( C(\cdot) \) for \( t \geq t_1 \).

The distance \( \hat{r}(t) \) between the origin and the vector \((\hat{x}(t), \hat{y}(t))\) is equal to \( \rho(t) \).

For large \( t \) the derivative \( \dot{\rho}(t) \) is so large that the system cannot move along the
boundary of $C(\cdot)$. To determine the moment $t_*$ after which the system is unable to move on the boundary of $C(\cdot)$, we have to solve the equation

$$
\dot{r}(t_*) = 1 \cdot \cos \phi(t_*) - 0 \cdot r(t_*) = \dot{\rho}(t_*) = 8t_* - 4. \quad (25)
$$

To find $\phi_* = \phi(t_*)$ we solve the equation

$$
\dot{\phi} = -\frac{u \sin \phi}{r}, \quad t \in [t_1, t_*],
$$

with $u = 1$ and $r(t) = \rho(t)$. It is easy to see that

$$
\phi(t) = 2 \arctan \left( \frac{1}{\sqrt{3}} \exp \left( \arctan(2(1-2t)) - \frac{\pi}{4} \right) \right), \quad t \in [t_1, t_*]. \quad (26)
$$

Set $\tau = \tan \frac{\phi_*}{2}$. From (25) and (26) we get the system of equations

$$
\frac{1 - \tau^2}{1 + \tau^2} = 8t_* - 4,
$$

$$
\tau = \frac{1}{\sqrt{3}} \exp \left( \arctan(2(1-2t_*)) - \frac{\pi}{4} \right).
$$

This system has a solution $\tau > 0$ and

$$
t_* = \frac{1}{8} \left( 4 + \frac{1 - \tau^2}{1 + \tau^2} \right) > \frac{1}{2}.
$$

Set

$$
\theta = \frac{1}{2} \left( t_* - \frac{1}{2} \right), \quad t_2 = \frac{1}{2} + \theta, \quad \text{and} \quad T = \frac{1}{2} + \frac{3}{2} \theta.
$$

In terms of $\theta$ we have

$$
\rho(T) = 9\theta^2 + \frac{1}{4}, \quad r(t_2) = \rho(t_2) = 4\theta^2 + \frac{1}{4}.
$$

Consider the function

$$
r_T^2(\mu) = y^2(t_2) + \left( x(t_2) - \frac{1}{2} \mu \theta \right)^2 = r^2(t_2) - \mu \theta r(t_2) \cos \phi(t_2) + \frac{\mu^2}{4} \theta^2.
$$

Since $r_T^2(0) = r^2(t_2)$ and $\frac{d}{dt} \rho^2(t_2) > 0$, for small $\mu > 0$ we have

$$
\rho(T) > r(t_2) > r_T(\mu) > r(t_2) \sin \phi(t_2),
$$

and

$$
\Delta = \frac{d}{dt} \rho^2(t_2) - \mu \theta \dot{x}(t_2) + 2\mu x(t_2) - \mu^2 \theta > 0.
$$

We fix such a $\mu$ satisfying $0 < \mu < \frac{1}{2\pi}$ and put $r_T = r_T(\mu)$. With such choice of the parameters, we now consider the following optimal control problem:
minimize $\Phi(x(T), y(T)) = -x(T)$ over the solutions of the above system coupled with $(x(0), y(0)) = (0, \sqrt{3}/4)$ and $(x(T), y(T)) \in C_T$, where $C_T = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r_T^2\}$.

It is now a simple matter to see that the optimal control has the form

$$\hat{u}(t) = \begin{cases} 
1, & t \in [0, t_1], \\
1, & t \in [t_1, t_2], \\
-\mu, & t \in [t_2, T],
\end{cases}$$

and that the distance $r(t)$ between the origin and the vector $(x(t), y(t))$ depends on time in the following manner:

$$\hat{r}(t) = \begin{cases} 
\sqrt{\hat{x}^2(0) + \hat{y}^2(0)}, & t \in [0, t_1], \\
\rho(t), & t \in [t_1, t_2], \\
\sqrt{\hat{y}^2(t_2) + (\hat{x}(t_2) - \mu(t - t_2))^2}, & t \in [t_2, T].
\end{cases}$$

So, the trajectory has the following structure: between $t = 0$ and $t = t_1$ the coordinate $x$ grows up, $\hat{x}(t) = t$, and $y$ is constant with $\hat{y}(t) \equiv y(0) = \sqrt{3}/4$. At the instant $t_1$ the trajectory reaches the set $C(t_1)$ and then moves along the boundary of $C(\cdot)$ for $t \in [t_1, t_2]$. Finally, at $t = t_2$ it leaves the boundary of $C(\cdot)$ and moves toward $C_T$ parallel to the axis $x$ with constant velocity $\dot{x}(t) = -\mu$. The coordinate $y$ can diminish only if the system moves along the boundary of $C(\cdot)$.

Also, the absolute minimum of $\Phi$ on $C_T$ is achieved at the point $(r_T, 0)$. Observe that $y(t)$ cannot reach zero in finite time. Thus, the minimum of $\Phi$, $-x(T) = -\sqrt{r_T^2 - y^2(T)} = -\sqrt{r_T^2 - y^2(t_2)}$, is achieved if the value of $y(t_2) > 0$ is as small as possible.

The choice of $\mu$ guarantees that $r(T) = r_T$, i.e. $(\hat{x}(T), \hat{y}(T)) \in C_T$. Moreover $\min \Phi = -\hat{x}(T)$. Indeed, consider the trajectory $(x(t), y(t))$, $t \in [t_2, t_2 + h] \cup [t_2 + h, T]$, of our system, satisfying the following conditions:

$$(x(t_2), y(t_2)) = (\hat{x}(t_2), \hat{y}(t_2)) \quad \text{and} \quad x^2(t) + y^2(t) = \rho^2(t) \text{ for } t \in [t_2, t_2 + h],$$
$$x(t, y(t)) = (x(t_2 + h) - \mu(t - t_2 - h)), y(t_2 + h)) \quad \text{for } t \in [t_2 + h, T].$$

Then we have

$$x^2(T) + y^2(T) = \left(x(t_2) + \dot{x}(t_2)h + o(h) - \mu \left(\frac{\theta}{2} - h\right)\right)^2$$
$$+ (y(t_2) + \dot{y}(t_2)h + o(h))^2$$
$$= r_T^2 + \Delta h + o(h) > r_T^2, \quad h \in [0, h_0].$$

This implies that $(x(T), y(T)) \notin C_T$.

According to the necessary conditions given by Theorem 4.1 there exist $\lambda \geq 0$, and $(p(\cdot), q(\cdot)) \in BV$ such that $\lambda + |p(T)| + |q(T)| > 0$, and

$$dp = 2\xi p \, dt + 2\dot{x} \, d\eta,$$

(27)
\[dq = 2\xi q\,dt + 2\hat{y}\,d\eta,\]  
\[\xi(\hat{x}p + \hat{y}q) = 0,\]  
\[\max_{u \in [-\mu, 1]} up = \hat{u}p,\]  
\[p(T) = \lambda - \beta\hat{x}(T), \quad q(T) = -\beta\hat{y}(T), \quad \beta \geq 0.\]  

Here \(\xi \in L^\infty, \xi \geq 0\) and \(\eta \in BV\). Observe that \(\xi(t) = 0, \eta(t) = (\text{const}), \) if \(\hat{r}(t) = (\hat{x}^2(t) + \hat{y}^2(t))^{\frac{1}{2}} < \rho(t)\).

Obviously \(p(T) \leq 0\). Otherwise, we would have \(\hat{u}(t) = 1, t \in [T - \tilde{t}, T]\) and with such a control the system could not reach the terminal set. Notice that we consider \(\mu\) small which prevents the system from reaching the boundary of \(C(\cdot)\) with control \(u(t) = -\mu\) in the initial instants of time.

It is easy to see that any admissible trajectory satisfies the necessary conditions with \(\lambda = 1, p(\cdot) = 0, q(t) = -\frac{y(T)}{x(T)}, \xi(\cdot) = 0, \) and \(\eta(\cdot) = 0\). The only useful information that we retrieve from the necessary conditions is the equality \(x^2(T) + y^2(T) = r_T^2\). Indeed, if \(x^2(T) + y^2(T) < r_T^2\), then \(\beta = 0\), and (31) implies \(q(T) = 0\) and \(0 = p(T) = \lambda\), a contradiction.

**Example 5.2:** Now, let us consider the same optimal control problem but with the dynamics governed by the following equations:

\[
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ -\sigma x(t) \end{pmatrix} - N_{C(t)}(x(t), y(t)),
\]

where \(\sigma > 0\) is a small parameter.

In this case the necessary conditions are: there exist \(\lambda \geq 0, (p(\cdot), q(\cdot)) \in BV\) such that \(\lambda + |p(T)| + |q(T)| > 0\), and

\[dp = \sigma q\,dt + 2\xi p\,dt + 2\hat{x}\,d\eta,\]  
\[dq = 2\xi q\,dt + 2\hat{y}\,d\eta,\]  
\[\xi(\hat{x}p + \hat{y}q) = 0,\]  
\[\max_{u \in [-\mu, 1]} up = \hat{u}p,\]  
\[p(T) = \lambda - \beta\hat{x}(T), \quad q(T) = -\beta\hat{y}(T), \quad \beta \geq 0.\]  

Here \(\xi \in L^\infty, \xi \geq 0, \) and \(\eta \in BV. \xi(t) = 0, \) with \(\eta(t) = (\text{const}), \) if \(\hat{r}(t) < \rho(t).\)

Now the equality \(p(t) = 0, t \in [T - \tilde{t}, T]\) is impossible. Indeed, from (32) we would get \(q(t) = 0, t \in [T - \tilde{t}, T]\). Then, (36) would imply that \(\lambda = 0\). Thus \(p(t) < 0\) and \(\hat{u}(t) = -\mu, t \in [T - \tilde{t}, T]\). In the interior of \(C(\cdot), p(\cdot)\) can change the sign only once. This means that \(\hat{x}^2(T - \tilde{t}) + \hat{y}^2(T - \tilde{t}) = \rho^2(T - \tilde{t})\). In order to reach the point \((x(T - \tilde{t}), y(T - \tilde{t}))\) and to reach the boundary of \(C(\cdot)\) from the initial point, the system must move with the control \(\hat{u} = 1\). This corresponds to \(p(t) > 0\). So, to find the optimal solution we have to analyse admissible
trajectories with the controls

\[ u(t) = \begin{cases} 
1, & t \in [0, \tilde{t}], \\
-\mu, & t \in [\tilde{t}, T], 
\end{cases} \]

and choose the optimal value of \( \tilde{t} \). Obviously, all other controls give greater values of the functional.

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No potential conflict of interest was reported by the authors.

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