UNIFORM BOUNDS FOR EXPRESSIONS INVOLVING MODIFIED BESSEL FUNCTIONS

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Abstract. In this paper we obtain uniform bounds for a number of expressions that involve derivatives and integrals of modified Bessel functions. These uniform bounds are motivated by the need to bound such expressions in the study of Variance-Gamma approximations via Stein’s method.

1. Introduction

In developing Stein’s method for the class of Variance-Gamma distributions, Gaunt [4] required uniform bounds on the first four derivatives of the function

\[ f(x) = -e^{-\beta x} \frac{K_\nu(|x|)}{|x|^\nu} \int_0^x e^{\beta y} y^\nu I_\nu(|y|) \tilde{h}(y) dy - e^{-\beta x} \frac{I_\nu(|x|)}{|x|^\nu} \int_x^\infty e^{\beta y} y^\nu K_\nu(|y|) \tilde{h}(y) dy, \quad x \in \mathbb{R}, \]

where

\[ \tilde{h}(y) = h(y) - \int_{-\infty}^{\infty} (1 - \beta^2)^{\nu+1/2} \frac{1}{\pi^{1/2} (\nu + 1/2)} e^{\beta t} t^\nu K_\nu(|t|) \tilde{h}(t) dt, \]

and \( \nu > -1/2 \), \( -1 < \beta < 1 \), and \( h \) is a real-valued three times differentiable function with bounded derivatives. To achieve uniform bounds on these derivatives we require bounds for a number of terms involving derivatives and integrals involving the modified Bessel functions \( I_\nu(x) \) and \( K_\nu(x) \). In this paper we establish uniform bounds for these terms. Before presenting the expressions that we obtain bounds for, we introduce some notation.

Notation. For a function \( g : \mathbb{R} \to \mathbb{R} \), we write \( \|g\| = \sup_{x \geq 0} |g(x)| \). We also have the following notation for the the repeated integral of the function \( e^{\beta x} x^\nu I_\nu(x) \):

\[ I_{(\nu,\beta,0)}(x) = e^{\beta x} x^\nu I_\nu(x), \quad I_{(\nu,\beta,n+1)}(x) = \int_0^x I_{(\nu,\beta,n)}(y) dy, \quad n = 0, 1, 2, 3, \ldots. \]

In this paper we shall firstly consider bounding the expressions

\[ \left\| \left[ \frac{d^n}{dx^n} \left( e^{-\beta x} I_\nu(x) \right) \right] \int_x^\infty e^{\beta t} t^\nu K_\nu(t) dt \right\| \]

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and

$$\left\| I_{(\nu,\beta,n)}(x) \frac{d^n}{dx^n} \left( \frac{e^{-\beta x} K_\nu(x)}{x^\nu} \right) \right\|,$$

where in both cases \( n \geq 0 \). We shall bound (1.2) for all \( n \geq 0 \), \( \nu > -1/2 \) and

\(-1 < \beta < 1\) (see Theorem 3.2), but will only bound (1.3) for the case \( n = 1 \) (see Proposition 3.1). We shall then move on to consider the case \( \beta = 0 \). For this case, we obtain improved bounds for (1.2) (see Theorems 3.4 and 3.5) and are able to bound (1.3) for all \( n \geq 1 \) (see Theorem 3.3).

In Theorem 3.6 obtain uniform bounds for

$$K_{\nu+n+1}(x) \frac{d}{dx} \int_0^x t^{\nu-k} I_{\nu+n}(t) dt$$

and

$$I_{\nu+n}(x) \int_x^\infty t^{\nu-k} K_{\nu+n}(t) dt,$$

where \( 0 \leq k \leq n \) and \( n \geq 1 \), in the region \( x \geq 1 \). It is worth noting that the expressions in Theorem 3.6 are undefined in the limit \( x \downarrow 0 \). This is easily verified using the asymptotic formulas (A.7) and (A.8). They are, however, defined for all \( x > 0 \). The approach we use to obtain these bounds the region \( x \geq 1 \) could easily be extended to obtain bounds that hold for all \( x > 0 \).

Finally, in Theorems 3.13–3.16 we obtain uniform bounds for the expressions

$$\left| \frac{1}{x} - (-1)^n I_{(\nu,0,n-1)}(x) \frac{d^n}{dx^n} \left( \frac{K_\nu(x)}{x^\nu} \right) \right|, \quad n \geq 2,$$

$$\left| \frac{2\nu + 2}{x^2} + I_{(\nu,0,1)}(x) \frac{d^3}{dx^3} \left( \frac{K_\nu(x)}{x^\nu} \right) \right|,$$

$$\left| \frac{2\nu + 3}{x^2} - I_{(\nu,0,2)}(x) \frac{d^4}{dx^4} \left( \frac{K_\nu(x)}{x^\nu} \right) \right|,$$

$$\left| \frac{(2\nu + 2)(2\nu + 3)}{x^3} + \frac{1}{x} - I_{(\nu,0,1)}(x) \frac{d^4}{dx^4} \left( \frac{K_\nu(x)}{x^\nu} \right) \right|$$

in the region \( 0 \leq x \leq 1 \). We use a series expansion approach to bound these expressions, which would lead to poor bounds for large \( x \). Moreover, we use the series expansion (A.2) to arrive at our bounds for (1.7), (1.8) and (1.9), and an artefact of this approach is that a term involving \( 1/\sin(\pi \nu) \) appears in our bounds for the case \( \nu \notin \mathbb{Z} \) (see Remark 3.17 for further details). As a result, our bounds perform poorly when \( 2\nu \) is close to but not equal to an integer. We end this paper by considering the special case \( \nu = 0 \) (see Theorem 3.18), for which we are able to obtain uniform bounds for all \( x \geq 0 \). An open problem is to establish uniform bounds for these expressions for all \( \nu > -1/2 \) in the region \( x \geq 0 \), with these bounds not involving the rogue \( 1/\sin(\pi \nu) \) term.

This paper is organised as follows. In Section 2, we state a number of results from the literature that we will make repeated use of. We present formulas and inequalities for modified Bessel functions and their derivatives and integrals that we make use in this paper. In Section 3, we obtain uniform bounds for the expressions involving modified Bessel functions that have been presented in this section. We separately consider bounds in the regions \( 0 \leq x < \infty \), \( 1 \leq x < \infty \) and \( 0 \leq x \leq 1 \).
Proposition 2.1. Suppose \( x \geq 0 \) and \( \nu > -1/2 \). Then for \( n \in \mathbb{N} \),

\[
(2.1) \quad \frac{d^{2n}}{dx^{2n}} \left( \frac{I_{\nu}(x)}{x^\nu} \right) = \sum_{k=0}^{n} A_k^n(\nu) \frac{I_{\nu+2k}(x)}{x^\nu},
\]

\[
(2.2) \quad \frac{d^{2n}}{dx^{2n}} \left( \frac{K_{\nu}(x)}{x^\nu} \right) = \sum_{k=0}^{n} A_k^n(\nu) \frac{K_{\nu+2k}(x)}{x^\nu},
\]

\[
(2.3) \quad \frac{d^{2n+1}}{dx^{2n+1}} \left( \frac{I_{\nu}(x)}{x^\nu} \right) = -\sum_{k=0}^{n} B_k^n(\nu) \frac{I_{\nu+2k+1}(x)}{x^\nu},
\]

where

\[
A_k^n(\nu) = \frac{(2n)!\nu(\nu+2k)\prod_{j=0}^{k-1}(2\nu+2j+1)}{2^{2n-k}(2k)!(n-k)!\prod_{j=0}^{n}(\nu+k+j)}, \quad k = 0, 1, \ldots, n,
\]

\[
B_k^n(\nu) = \frac{(2n+1)!\nu(2\nu+2k+1)\prod_{j=0}^{k-1}(2\nu+2j+1)}{2^{2n-k}(2k+1)!(n-k)!\prod_{j=0}^{n}(\nu+k+j+1)}, \quad k = 0, 1, \ldots, n,
\]

and we set \( \prod_{j=0}^{n}(2\nu+2j+1) = 1 \). The coefficients \( A_k^n(\nu) \) and \( B_k^n(\nu) \) satisfy the inequalities \( 0 < A_k^n(\nu), B_k^n(\nu) \leq 1 \) and the identity

\[
\sum_{k=0}^{n} A_k^n(\nu) = \sum_{k=0}^{n} B_k^n(\nu) = 1.
\]

In this paper we use the following special cases of the formulas of Proposition 2.1.

\[
(2.4) \quad \frac{d^2}{dx^2}(K_0(x)) = \frac{1}{2}(K_2(x) + K_0(x)),
\]

\[
(2.5) \quad \frac{d^3}{dx^3} \left( \frac{K_{\nu}(x)}{x^\nu} \right) = \frac{2\nu+1}{2(\nu+2)} \frac{K_{\nu+3}(x)}{x^\nu} - \frac{3}{2(\nu+2)} \frac{K_{\nu+1}(x)}{x^\nu},
\]

\[
\frac{d^4}{dx^4} \left( \frac{K_{\nu}(x)}{x^\nu} \right) = \frac{(2\nu+1)(2\nu+3)}{4(\nu+2)(\nu+3)} \frac{K_{\nu+4}(x)}{x^\nu} + \frac{3(2\nu+1)}{2(\nu+1)(\nu+3)} \frac{K_{\nu+2}(x)}{x^\nu}
\]

\[
+ \frac{3}{4(\nu+1)(\nu+2)} \frac{K_{\nu}(x)}{x^\nu},
\]

We also have the following simple corollary to Proposition 2.1.
Lemma 2.3. Let \( n \in \mathbb{N} \) and suppose that \( \nu \) and \( x \) are real-valued, with \( \nu \geq -1/2 \) and \( x > 0 \), then the following inequalities hold
\[
0 < \frac{d^{2n}}{dx^{2n}} \left( \frac{K_{\nu}(x)}{x^\nu} \right) \leq \frac{K_{\nu+2n}(x)}{x^\nu}, \quad -\frac{K_{\nu+2n+1}(x)}{x^\nu} \leq \frac{d^{2n+1}}{dx^{2n+1}} \left( \frac{K_{\nu}(x)}{x^\nu} \right) < 0,
\]
and
\[
0 < \frac{d^n}{dx^n} \left( \frac{I_{\nu}(x)}{x^\nu} \right) \leq \begin{cases} x^{-\nu}I_{\nu+1}(x), & \text{odd } n, \\ x^{-\nu}I_{\nu}(x), & \text{even } n. \end{cases}
\]

Corollary 2.2. Let \( n \in \mathbb{N} \) and suppose that \( \nu \) and \( x \) are real-valued, with \( \nu \geq -1/2 \) and \( x > 0 \), then the following inequalities hold
\[
\int_0^x t^\nu I_{\nu+n}(t)dt < \frac{2(\nu + n + 1)}{2\nu + n + 1} x^\nu I_{\nu+n+1}(x), \quad \nu > -1/2, \ n \geq 0,
\]
\[
I_{(\nu,0,n)}(x) < \left\{ \prod_{k=1}^{\infty} \frac{2\nu + 2k}{2\nu + k} \right\} x^\nu I_{\nu+n}(x), \quad \nu \geq 0, \ n = 1, 2, 3, \ldots,
\]
where \( I_{(\nu,0,n)}(x) \) is defined as in \([1,7]\).

Lemma 2.4. Let \( -1 \leq \beta < 1 \), then for all \( x > 0 \) the following inequalities hold
\[
\int_x^\infty t^\nu K_{\nu}(t)dt < x^\nu K_{\nu+1}(x), \quad \nu \in \mathbb{R},
\]
\[
\int_x^\infty e^{\beta t} t^\nu K_{\nu}(t)dt < \frac{1}{1 - |\beta|} e^{\beta x} x^\nu K_{\nu}(x), \quad \nu < 1/2,
\]
\[
\int_x^\infty t^\nu K_{\nu}(t)dt < \frac{\sqrt{\pi} \Gamma(\nu + 1/2)}{\Gamma(\nu)} x^\nu K_{\nu}(x), \quad \nu \geq 1/2,
\]
\[
\int_x^\infty e^{\beta t} t^\nu K_{\nu}(t)dt < \frac{2\sqrt{\pi} \Gamma(\nu + 1/2)}{(1 - \beta^2)^{\nu+1/2} \Gamma(\nu)} e^{\beta x} x^\nu K_{\nu}(x), \quad \nu \geq 1/2.
\]

Lemma 2.5. Let \( \mu > 1 \), then
\[
0 < \frac{1}{x^2} - \frac{x^{\mu-2}K_{\mu}(x)}{2^{\mu-1}\Gamma(\mu)} \leq \frac{1}{4(\mu-1)}, \quad \text{for all } x \geq 0.
\]

The following proposition is proved in Baricz \([2]\). The result has a simple corollary (stated below), which we will make repeated use of.

Proposition 2.6. For positive real argument and \( \nu > -1/2 \) the product \( K_{\nu}(x)I_{\nu}(x) \) is a strictly monotone decreasing function of \( x \).

Corollary 2.7. Let \( \nu > 0 \), then for all \( x \geq 0 \), then the following inequality holds
\[
0 < K_{\nu}(x)I_{\nu}(x) \leq \frac{1}{2\nu}.
\]

Proof. The proof follows from Proposition 2.6 and the asymptotic expansions \([A.7]\), \([A.8]\), \([A.9]\) and \([A.10]\) for modified Bessel functions in the limit \( x \) tends to 0 and \( \infty \).
3. Uniform bounds for some expressions involving modified Bessel functions

With our preliminary results stated, we are now able to present our bounds for the expressions involving modified Bessel functions that were presented in the introduction. We consider bounds in the regions $0 \leq x < \infty$, $1 \leq x < \infty$ and $0 \leq x < 1$ separately.

3.1. Bounds for $0 \leq x < \infty$. Here we bound $L_2$ and $L_3$. We begin by considering the general case $-1 < \beta < 1$ and $\nu > -1/2$ and then specialise to the case $\beta = 0$.

**Proposition 3.1.** Let $-1 < \beta < 1$ and $\nu > -1/2$, then for all $x \geq 0$ the following inequalities hold

\[
e^{-\beta x} K_{\nu+1}(x) \int_0^\infty e^{\beta t^\nu} I_\nu(x) dt \leq \frac{2}{2\nu + 1} x K_{\nu+1}(x) I_\nu(x) < \infty,
\]

\[
e^{-\beta x} K_\nu(x) \int_0^x e^{\beta t^\nu} I_\nu(x) dt \leq \frac{2}{2\nu + 1} x K_{\nu+1}(x) I_\nu(x) < \infty,
\]

\[
\left| \frac{d}{dx} \left( e^{-\beta x} K_{\nu+1}(x) \right) \right| \int_0^x e^{\beta t^\nu} I_\nu(x) dt \leq \frac{2(\beta + 1)}{2\nu + 1} x K_{\nu+1}(x) I_\nu(x) < \infty.
\]

**Proof.** (i) Note that

\[
\frac{d}{d\beta} \left( e^{-\beta x} K_{\nu+1}(x) \right) \int_0^x e^{\beta t^\nu} I_\nu(x) dt = e^{-\beta x} K_{\nu+1}(x) \int_0^x (t - x) e^{\beta t^\nu} I_\nu(x) dt \leq 0.
\]

Therefore, for $-1 < \beta < 1$, we have

\[
e^{-\beta x} K_{\nu+1}(x) \int_0^x e^{\beta t^\nu} I_\nu(x) dt \leq e^{2\nu+1} K_{\nu+1}(x) \int_0^x e^{-t^\nu} I_\nu(x) dt
\]

\[= \frac{1}{2\nu + 1} x K_\nu(x) [I_\nu(x) + I_{\nu+1}(x)]
\]

\[\leq \frac{2}{2\nu + 1} x K_{\nu+1}(x) I_\nu(x),
\]

where we used the integral formula (A.19) to evaluate the integral and inequality (A.12) to obtain the final inequality.

Use of the asymptotic formulas (A.7), (A.8), (A.9) and (A.10) for $K_\nu(x)$ and $I_\nu(x)$ verify that the function $x K_{\nu+1}(x) I_\nu(x)$ is bounded in the limits as $x$ tends to $0$ and $\infty$, and its clearly bounded for all other $x$, and hence is bounded for all $x \geq 0$. This completes the proof of part (i).

(ii) This follows immediately from (i), since, by inequality (A.14), $K_\nu(x) \leq K_{\nu+1}(x)$ for $\nu > -1/2$.

(iii) By the differentiation formula (A.17), we have

\[
\frac{d}{dx} \left( e^{-\beta x} K_\nu(x) \right) = -e^{-\beta x} \left( \beta K_\nu(x) \frac{x}{\nu} + \frac{K_{\nu+1}(x)}{x^\nu} \right).
\]

We may then obtain the desired inequality by applying parts (i) and (ii). \qed

**Theorem 3.2.** Suppose $-1 < \beta < 1$ and $n = 0, 1, 2, \ldots$, then the following inequalities hold

\[
\left| \frac{d^n}{dx^n} \left( e^{-\beta x} I_\nu(x) \right) \right| \int_0^\infty e^{\beta t^\nu} K_\nu(t) dt < \frac{2^n \sqrt{x} \Gamma(\nu + 1/2)}{(1 - \beta^2)^{n+1/2} \Gamma(n + 1)}, \quad \nu \geq 1/2,
\]
and
\[ \left\| \frac{d^n}{dx^n} \left( e^{-\beta x} I_{\nu}(x) \right) \right\| \int_x^\infty e^{\beta t} K_{\nu}(t) dt < \frac{(e + 1)^{2n+1/2} \Gamma(\nu + 1/2)}{1 - |\beta|}, \quad |\nu| < 1/2. \]

Proof. We begin by bounding \( \frac{d^n}{dx^n} \left( e^{-\beta x} I_{\nu}(x) \right) \). By the Leibniz rule for differentiating products we have
\[ \frac{d^n}{dx^n} \left( e^{-\beta x} I_{\nu}(x) \right) = \sum_{k=0}^{n} \binom{n}{k} (-\beta)^{n-k} e^{-\beta x} \frac{d^k}{dx^k} \left( I_{\nu}(x) \right). \]

Using that \(-1 < \beta < 1\) and inequality (2.8) gives
\[ (3.1) \quad \left\| \frac{d^n}{dx^n} \left( e^{-\beta x} I_{\nu}(x) \right) \right\| < \sum_{k=0}^{n} \binom{n}{k} \frac{e^{-\beta x}}{x^\nu} = \frac{2^n e^{-\beta x}}{x^\nu}. \]

(i) Suppose that \( \nu \geq 1/2 \). By inequalities (3.1) and (2.14), and that \( I_{\nu+1}(x) \leq I_{\nu}(x) \) for \( \nu \geq 1/2 \), we have
\[ \left\| \frac{d^n}{dx^n} \left( e^{-\beta x} I_{\nu}(x) \right) \right\| \int_x^\infty e^{\beta t} K_{\nu}(t) dt < \frac{2^{n+1} \sqrt{\pi} \Gamma(\nu + 1/2)}{(1 - \beta^2)^{\nu+1/2} \Gamma(\nu)} I_{\nu}(x)K_{\nu}(x). \]

Using inequality (2.16) gives
\[ \left\| \frac{d^n}{dx^n} \left( e^{-\beta x} I_{\nu}(x) \right) \right\| \int_x^\infty e^{\beta t} K_{\nu}(t) dt < \frac{2^{n+1} \sqrt{\pi} \Gamma(\nu + 1/2)}{(1 - \beta^2)^{\nu+1/2} \Gamma(\nu)} \cdot \frac{1}{2^\nu} \]
\[ = \frac{2^n \sqrt{\pi} \Gamma(\nu + 1/2)}{(1 - \beta^2)^{\nu+1/2} \Gamma(\nu + 1)}, \]

as required.

(ii) Suppose now that \(-1/2 < \nu < 1/2\). We begin by proving that the bound holds in the region \( x \geq 1/2 \). By inequalities (3.1) and (2.12), and \( I_{\nu+1}(x) \leq I_{\nu}(x) \) for \( \nu > -1/2 \), we have
\[ \left\| \frac{d^n}{dx^n} \left( e^{-\beta x} I_{\nu}(x) \right) \right\| \int_x^\infty e^{\beta t} K_{\nu}(t) dt < \frac{2^n e^{-\beta x}}{1 - |\beta|} I_{\nu}(x)K_{\nu}(x). \]

By Proposition 2.6, \( I_{\nu}(x)K_{\nu}(x) \) is a monotone decreasing function of \( x \) for \( x > 0 \), and therefore we may bound this product for \( x \geq 1/2 \) by \( I_{\nu}(1/2)K_{\nu}(1/2) \). In fact, we may produce a bound for all \( 0 \leq \nu < 1/2 \) using (A.11) and (A.13), which gives \( I_{\nu}(1/2)K_{\nu}(1/2) < I_{-1/2}(1/2)K_{1/2}(1/2) = 1 + e^{-1} \), where we used the formulas (A.3) and (A.5) for \( I_{-1/2}(x) \) and \( K_{1/2}(x) \), respectively, to obtain the equality. Putting this together we have
\[ e^{-\beta x} \left\| \frac{d^n}{dx^n} \left( I_{\nu}(x) \right) \right\| \int_x^\infty e^{\beta t} K_{\nu}(t) dt < \frac{(1 + e^{-1})2^n}{1 - |\beta|}, \quad x \geq 1/2, |\nu| < 1/2. \]

For \(-1/2 < \nu < 1/2, \Gamma(\nu+1/2) > \Gamma(1) = 1 \). Therefore \( 1 + e^{-1} < (e+1)^{2\nu} \Gamma(\nu+1/2) \) for \(-1/2 < \nu < 1/2 \), and thus the bound holds in the region \( x \geq 1/2 \).

We now verify that the bound holds in the region \( 0 \leq x \leq 1/2 \). By inequality (3.1), we have
\[ \left\| \frac{d^n}{dx^n} \left( e^{-\beta x} I_{\nu}(x) \right) \right\| \int_x^\infty e^{\beta t} K_{\nu}(t) dt < \frac{2^n e^{-\beta x} I_{\nu}(x)}{x^\nu} \int_x^\infty e^{\beta t} K_{\nu}(t) dt. \]
From the series expansion (A.1) for \( I_\nu(x) \) we can easily deduce that \( x^{-\nu} I_\nu(x) \) is an increasing function of \( x \). The integral \( \int_{-\infty}^{\infty} e^{\beta t} K_\nu(|t|) dt \) is a decreasing function of \( x \), and so we may bound the right-hand side of the previous display by

\[
(3.2) \quad \frac{2^n e^{-\beta/2} I_\nu(1/2)}{(1/2)^\nu} \int_{-\infty}^{\infty} e^{\beta t} |t|^\nu K_\nu(|t|) dt = \frac{\sqrt{\pi} e^{-\beta/2} I_\nu(1/2) 2^{n+\nu} \Gamma(\nu + 1/2)}{(1 - \beta^2)^{\nu+1/2}},
\]

where the integral was evaluated using formula (A.18). For \(-1 < \beta < 1 \) and \(-1/2 < \nu < 1/2 \) the following inequalities hold: \( e^{-\beta/2} < e^{1/2} \), \( I_\nu(1/2) < I_{-1/2}(1/2) = \pi^{-1/2}(e^{1/2} + e^{-1/2}) \) and \((1 - \beta^2)^{\nu+1/2} > 1 - |\beta| \). With these bounds we may bound the right-hand side of (3.2), and thus obtain, for \( 0 \leq x \leq 1/2 \) and \(-1/2 < \nu < 1/2 \),

\[
\frac{2^n e^{-\beta/2} I_\nu(1/2)}{(1/2)^\nu} \int_{-\infty}^{\infty} e^{\beta t} |t|^\nu K_\nu(|t|) dt < \frac{(e + 1) 2^{n+1/2} \Gamma(\nu + 1/2)}{1 - |\beta|}.
\]

Combining this bound with the bound for \( x \geq 1/2 \), and using that \( \Gamma(\nu + 1/2) > 1 \) for \(-1/2 < \nu < 1/2 \), completes the proof of part (ii).

We now specialise to the case \( \beta = 0 \). The bounds in the following lemmas are of order \( \nu^{-1} \) as \( \nu \to \infty \), except for the bound given in Theorem 3.3, which is of order \( \nu^{-1/2} \) as \( \nu \to \infty \) (see Olver et al. [12], formula 5.11.12.).

**Theorem 3.3.** Let \( I_{(\nu,0,n)}(x) \) be defined as per equation (1.7). Suppose \( \nu > -1/2 \), then

\[
\left\| I_{(\nu,0,n)}(x) \frac{d^n}{dx^n} \left( \frac{K_\nu(x)}{x^\nu} \right) \right\| \leq \frac{2^{n-1}}{2\nu + 1}, \quad n \geq 1,
\]

and

\[
\left\| I_{(\nu,0,1)}(x) \frac{K_\nu(x)}{x^\nu} \right\| \leq \frac{1}{2\nu + 1}.
\]

**Proof.** (i) Applying inequalities (2.7) and (2.10) gives

\[
\left| I_{(\nu,0,n)}(x) \frac{d^n}{dx^n} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| \leq \left\{ \prod_{k=1}^{n} \frac{2\nu + 2k}{2\nu + k} \right\} K_{\nu+n}(x) I_{\nu+n+1}(x).
\]

We now use that \( K_{\nu+n}(x) < K_{\nu+n+1}(x) \) and inequality (2.16) to obtain

\[
\left| I_{(\nu,0,n)}(x) \frac{d^n}{dx^n} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| < \left\{ \prod_{k=1}^{n} \frac{2\nu + 2k}{2\nu + k} \right\} K_{\nu+n+1}(x) I_{\nu+n+1}(x)
\]

\[
\leq \frac{1}{2(\nu + n + 1)} \prod_{k=1}^{n} \frac{2\nu + 2k}{2\nu + k}.
\]

We can simplify the bound given in the above display by noting

\[
\frac{1}{2(\nu + n + 1)} \prod_{k=1}^{n} \frac{2\nu + 2k}{2\nu + k} = \frac{1}{2\nu + 1} \cdot \frac{2(\nu + n)}{2(\nu + n + 1)} \prod_{k=1}^{n} \frac{2\nu + 2k}{2\nu + k + 1} < \frac{2^{n-1}}{2\nu + 1},
\]

as \( \frac{2\nu + 2k}{2\nu + k + 1} \leq \frac{2k - 1}{k} < 2 \), for \( \nu > -1/2 \) and \( k \geq 2 \). This completes the proof of part (i).

(ii) We note that, by inequality (A.14) and the differentiation formula (A.16),

\[
x^{-\nu} K_\nu(x) < x^{-\nu} K_{\nu+1}(x) = -\frac{d}{dx} (K_\nu(x)) \text{ and apply part (i).} \]

\( \square \)
Theorem 3.4. Let \( \nu > -1/2 \), then
\[
\left\| \frac{I_\nu(x)}{x^\nu} \int_x^\infty t^\nu K_\nu(t) dt \right\| = \frac{\sqrt{\pi} \Gamma(\nu + 1/2)}{2 \Gamma(\nu + 1)}.
\]

Proof. We begin by proving that \( \frac{I_\nu(x)}{x^\nu} \int_x^\infty t^\nu K_\nu(t) dt \) is a decreasing function. By the differentiation formula (A.16) and inequality (2.11), we have, for \( x \geq 0 \),
\[
\frac{d}{dx} \left( \frac{I_\nu(x)}{x^\nu} \int_x^\infty t^\nu K_\nu(t) dt \right) = \frac{I_{\nu+1}(x)}{x^\nu} \int_x^\infty t^\nu K_\nu(t) dt - I_\nu(x) K_\nu(x)
\]
\[
\leq I_{\nu+1}(x) K_{\nu+1}(x) - I_\nu(x) K_\nu(x).
\]

Theorem 1 of Segura [14] states that for \( x > 0 \) and \( \mu > -1/2 \) the inequality \( I_{\nu+1}(x) K_{\nu+1}(x) - I_\nu(x) K_\nu(x) < 0 \) holds. Hence, \( \frac{I_\nu(x)}{x^\nu} \int_x^\infty t^\nu K_\nu(t) dt \) is a decreasing function of \( x \).

Using the asymptotic property (A.7) of modified Bessel functions \( I_\nu(x) \) and that \( \int_0^\infty t^\nu K_\nu(t) dt = \sqrt{\pi} \Gamma(\nu + 1/2) 2^{\nu-1} \) (see (A.18)), we have
\[
\lim_{x \to 0^+} \left( \frac{I_\nu(x)}{x^\nu} \int_x^\infty t^\nu K_\nu(t) dt \right) = \frac{\sqrt{\pi} \Gamma(\nu + 1/2)}{2 \Gamma(\nu + 1)},
\]
proving the result. \( \square \)

Theorem 3.5. Let \( \nu > -1/2 \) and \( n = 0, 1, 2, \ldots \), then
\[
\left\| \left[ \frac{d^{2n}}{dx^{2n}} \left( \frac{I_\nu(x)}{x^\nu} \right) \right] \int_x^\infty t^\nu K_\nu(t) dt \right\| < \frac{1}{2(\nu + 2)} + \frac{\sqrt{\pi}}{2(2\nu + 1)},
\]
and
\[
\left\| \left[ \frac{d^{2n+1}}{dx^{2n+1}} \left( \frac{I_\nu(x)}{x^\nu} \right) \right] \int_x^\infty t^\nu K_\nu(y) dt \right\| < \frac{1}{2(\nu + 1)}.
\]

Proof. (i) We begin by obtaining the following inequality:
\[
\frac{d^{2n}}{dx^{2n}} \left( \frac{I_\nu(x)}{x^\nu} \right) = \sum_{k=0}^n A_k^n(\nu) \frac{I_{\nu+2k}(x)}{x^\nu}
\leq \sum_{k=1}^n A_k^n(\nu) \frac{I_{\nu+2k}(x)}{x^\nu} + A_0^n(\nu) \frac{I_\nu(x)}{x^\nu}
\leq \left( 1 - A_0^n(\nu) \right) \frac{I_{\nu+2}(x)}{x^\nu} + A_0^n(\nu) \frac{I_\nu(x)}{x^\nu}
\]
as \( \sum_{k=0}^n A_k^n(\nu) = 1 \)
\[
\leq I_{\nu+2}(x) + \frac{I_\nu(x)}{2(\nu + 1)} x^\nu
\]
for \( n \geq 1 \). Therefore
\[
\left[ \frac{d^{2n}}{dx^{2n}} \left( \frac{I_\nu(x)}{x^\nu} \right) \right] \int_x^\infty y^\nu K_\nu(y) dy \leq \frac{I_{\nu+2}(x)}{x^\nu} \int_x^\infty y^\nu K_\nu(y) dy
\]
\[
+ \frac{I_\nu(x)}{2(\nu + 1)x^\nu} \int_x^\infty y^\nu K_\nu(y) dy
\]
\[
< I_{\nu+2}(x) K_{\nu+1}(x) + \frac{1}{2(\nu + 1)} \frac{\sqrt{\pi} \Gamma(\nu + 1/2)}{2 \Gamma(\nu + 1)}.
\]
where we used inequalities (2.11) and (3.3) to obtain the second inequality. We now use that $K_{\nu+1} < K_{\nu+2}(x)$ for $\nu > -1/2$ and inequality (2.1) to obtain
\[
\left[ \frac{d^{2n}}{dx^{2n}} \left( \frac{I_\nu(x)}{x^\nu} \right) \right] \int_x^\infty y^\nu K_\nu(y) dy < I_{\nu+2}(x)K_{\nu+2}(x) + \frac{1}{2(\nu + 1)} \cdot \frac{\sqrt{\pi} \Gamma(\nu + 1/2)}{2\Gamma(\nu + 1)} \leq \frac{1}{2(\nu + 2)} + \frac{\sqrt{\pi} \Gamma(\nu + 3/2)}{2(2\nu + 1)\Gamma(\nu + 2)} \leq \frac{1}{2(\nu + 2)} + \frac{\sqrt{\pi}}{2(2\nu + 1)}.
\]

(ii) Applying inequalities (2.8) and (2.11) and then inequality (2.16) gives
\[
\left[ \frac{d^{2n+1}}{dx^{2n+1}} \left( \frac{I_\nu(x)}{x^\nu} \right) \right] \int_x^\infty y^\nu K_\nu(y) dy < I_{\nu+1}(x)K_{\nu+1}(x) \leq \frac{1}{2(\nu + 1)},
\]
as required. \(\square\)

3.2. **Bounds for** $x \geq 1$. We now obtain uniform bounds for (1.4) and (1.5) in the region $x \geq 1$. As was mentioned in Section 1, the approach used to obtain bounds in the region $x \geq 1$ could be applied to establish uniform bounds in the region $x \geq c$, where $c > 0$.

**Theorem 3.6.** Let $\nu > -1/2$ and $0 \leq k \leq n$, where $n \geq 1$, then for $x \geq 1$,
\[
\frac{K_{\nu+n+1}(x)}{x^\nu} \int_0^x t^{\nu-k} I_{\nu+n}(t) dt \leq \frac{1}{2\nu + n - k + 1},
\]
and
\[
\frac{I_{\nu+n}(x)}{x^\nu} \int_x^\infty t^{\nu-k} K_{\nu+n}(t) dt < \frac{\sqrt{\pi}}{\sqrt{4(\nu + n) + 1}}.
\]

**Proof.** (i) Suppose $x \geq 1$. Then,
\[
\frac{K_{\nu+n+1}(x)}{x^\nu} \int_0^x t^{\nu-k} I_{\nu+n}(t) dt \leq \frac{2(\nu + n + 1)}{2\nu + n - k + 1} \cdot \frac{x^{\nu-k} I_{\nu+n+1}(x)}{x^\nu} \cdot \frac{K_{\nu+n+1}(x)}{x^\nu} \text{ by (2.9)}
\]
\[
\leq \frac{2(\nu + n + 1)}{2\nu + n - k + 1} K_{\nu+n+1}(x) I_{\nu+n+1}(x) \text{ as } x \geq 1
\]
\[
\leq \frac{1}{2\nu + n - k + 1} \text{ by (2.16)}
\]
as required.

(ii) Again, we suppose $x \geq 1$. We have
\[
\frac{I_{\nu+n}(x)}{x^\nu} \int_x^\infty t^{\nu-k} K_{\nu+n}(t) dt \leq \frac{I_{\nu+n}(x)}{x^{\nu+n+k}} \int_x^\infty t^{\nu+n} K_{\nu+n}(t) dt \leq \frac{\sqrt{\pi} \Gamma(\nu + n + 1/2)}{\Gamma(\nu + n)} x^{-k} I_{\nu+n}(x) K_{\nu+n}(x) \text{ by (2.16)}
\]
\[
\leq \frac{\sqrt{\pi} \Gamma(\nu + n + 1/2)}{\Gamma(\nu + n)} I_{\nu+n}(x) K_{\nu+n}(x) \text{ as } x \geq 1
\]
\[
\leq \frac{\sqrt{\pi} \Gamma(\nu + n + 1/2)}{2\Gamma(\nu + n + 1)} \text{ by (2.13)}
\]
where the final inequality was obtained by an application of the following inequality, which can be found in Elezović et al. [3]:

$$\frac{\Gamma(u + 1/2)}{\Gamma(u + 1)} < \frac{1}{\sqrt{u + 1/4}}, \quad u > -1/4.$$ 

The proof is complete. \(\square\)

3.3. Bounds for \(0 \leq x \leq 1\). Here we consider the problem of bounding the terms \((1.6) - (1.9)\). We begin by considering the general case \(\nu > -1/2\) and obtain uniform bounds in the region \(0 \leq x \leq 1\). In Lemma 3.18 we specialise to the case \(\nu = 0\) and obtain uniform bounds in the region \(0 \leq x < \infty\).

We begin by stating some preliminary lemmas, which are required in the proofs of Lemmas 3.14, 3.15 and 3.16 (given below). These preliminary lemmas bound the following expressions:

\[\alpha_{p,q,r}(x) = \frac{2^{p-r-1}\Gamma(\nu + r)}{x^{\nu+r}} \sum_{k=p}^{\infty} \frac{1}{k!} \left\{ \prod_{i=1}^{q} \frac{1}{2\nu + 2k + i} \right\} \left( \frac{x}{2} \right)^{2\nu + 2k},\]

\[\beta_{p,q,r}(x) = \frac{I_{\nu,0,q}(x)}{x^\nu} \left( \sum_{k=p}^{\infty} \frac{(-1)^k \Gamma(\nu + r - k)}{k!} \left( \frac{x}{2} \right)^{2k - \nu - r} \right),\]

\[\gamma_{r}(x) = \frac{I_{\nu,0,q}(x)}{x^\nu} \left( \log \left( \frac{x}{2} \right) + \gamma \right) I_{\nu+r}(x),\]

\[\delta_{r}(x) = \frac{I_{\nu,0,q}(x)}{x^\nu} \cdot \frac{1}{2} \sum_{j=0}^{\infty} \frac{\psi(j) + \psi(\nu + j + r)}{j!(\nu + j + r)!} \left( \frac{x}{2} \right)^{\nu + 2j + r},\]

\[\epsilon_{r}(x) = \frac{I_{\nu,0,q}(x)}{x^\nu} \cdot \frac{\pi}{2|\sin(\nu + r)\pi|} \sum_{l=\nu+r}^{\infty} \frac{1}{l!} \left( \frac{x}{2} \right)^{2l - \nu - r},\]

\[\zeta_{r}(x) = \frac{I_{\nu,0,q}(x)}{x^\nu} \cdot \frac{\pi}{2|\sin(\nu + r)\pi|} I_{\nu+r}(x),\]

where \(p, q\) and \(r\) are positive integers, \(\gamma\) is the Euler-Mascheroni constant and \(\psi(k) = \sum_{j=1}^{k} \frac{1}{j} \).

**Lemma 3.7.** Suppose that positive integers \(p, q\) and \(r\) satisfy \(2p + q \geq r\), then for \(\nu > -1/2\) and \(0 \leq x \leq 1\),

\[\alpha_{p,q,r}(x) < \frac{2^{r-2}\Gamma(\nu + r)}{p!\Gamma(\nu + p + 1)} \prod_{i=1}^{q} \frac{1}{2\nu + 2p + i}.\]

**Proof.** Setting \(k = j + p\) gives

\[\alpha_{p,q,r}(x) = \frac{2^{r-2p-1}\Gamma(\nu + r)x^{2p+q-r}}{p!\Gamma(\nu + p + 1)} \left\{ \prod_{i=1}^{q} \frac{1}{2\nu + 2j + 2p + i} \right\} \left( \frac{x}{2} \right)^{2j} \]

\[\leq \frac{2^{r-2p-1}\Gamma(\nu + r)x^{2p+q-r}}{p!\Gamma(\nu + p + 1)} \sum_{j=0}^{\infty} \frac{1}{\prod_{i=1}^{q} \frac{1}{2\nu + 2p + i}} \left( \frac{x}{2} \right)^{2j} \]

\[\leq \frac{2^{r-2p-1}\Gamma(\nu + r)x^{2p+q-r}}{p!\Gamma(\nu + p + 1)} \sum_{j=0}^{\infty} \frac{1}{\prod_{i=1}^{q} \frac{1}{2\nu + 2p + i}} \left( \frac{x}{2} \right)^{2j} \]

\[\leq \frac{2^{r-2p-1}\Gamma(\nu + r)x^{2p+q-r}}{p!\Gamma(\nu + p + 1)} \sum_{j=0}^{\infty} \frac{1}{\prod_{i=1}^{q} \frac{1}{2\nu + 2p + i}} \left( \frac{x}{2} \right)^{2j} \]

\[\leq \frac{2^{r-2p-1}\Gamma(\nu + r)x^{2p+q-r}}{p!\Gamma(\nu + p + 1)} \sum_{j=0}^{\infty} \frac{1}{\prod_{i=1}^{q} \frac{1}{2\nu + 2p + i}} \left( \frac{x}{2} \right)^{2j} \]

\[\leq \frac{2^{r-2p-1}\Gamma(\nu + r)x^{2p+q-r}}{p!\Gamma(\nu + p + 1)} \sum_{j=0}^{\infty} \frac{1}{\prod_{i=1}^{q} \frac{1}{2\nu + 2p + i}} \left( \frac{x}{2} \right)^{2j} \]
Lemma 3.8. Suppose that positive integers $p, q$ and $r$ satisfy $2p + q \geq r$, then for \( \nu > -1/2 \) and $0 \leq x \leq 1$, 
\[
\beta_{p,q,r}(x) < \frac{\Gamma(\nu + r - p)}{2^{2p-r}p!\Gamma(\nu + 1)} \prod_{k=1}^{q} \frac{1}{2\nu + k}.
\]

Proof. We bound $\beta_{p,q,r}(x)$ as follows 
\[
\beta_{p,q,r}(x) \leq \frac{I_{(\nu,q,r)}(x)}{2x^\nu} \cdot \frac{\Gamma(\nu + r - p)}{p!} \left( \frac{x}{2} \right)^{2p-\nu-r} 
\leq \left\{ \prod_{k=1}^{q} \frac{2\nu + 2k}{2\nu + k} \right\} I_{\nu+q}(x) \cdot \frac{\Gamma(\nu + r - p)}{p!} \left( \frac{x}{2} \right)^{2p-\nu-r} \quad \text{by (2.10)} 
\leq \frac{\Gamma(\nu + r - p)}{2^{2p+q-r+1}p!\Gamma(\nu + q + 1)} \left\{ \prod_{k=1}^{q} \frac{2\nu + 2k}{2\nu + k} \right\} 2^{2p+q-r} \cosh(x) \quad \text{by (A.15)} 
\leq \frac{\Gamma(\nu + r - p)}{2^{2p+q-r}p!(\nu + q + 1)} \prod_{k=1}^{q} \frac{2\nu + 2k}{2\nu + k} 
= \frac{\Gamma(\nu + r - p)}{2^{2p+q-r}p!} \sum_{j=1}^{q} (\nu + j) \prod_{k=1}^{q} \frac{1}{2\nu + k} 
= \frac{\Gamma(\nu + r - p)}{2^{2p+q-r}p!\Gamma(\nu + 1)} \prod_{k=1}^{q} \frac{1}{2\nu + k}.
\]
where to obtain the third inequality we used that for $2p + q \geq r$ and $0 \leq x \leq 1$ we have $x^{2p+q-r} \cosh(x) \leq \cosh(1) = 1.54 \ldots < 2$. 

Lemma 3.9. Suppose that $q$ and $r$ are positive integers, then for $\nu \in \mathbb{N}$ and $0 \leq x \leq 1$, 
\[
\gamma_{q,r}(x) < \frac{1}{2^{2\nu+r-1}r!(\nu + r)!} \prod_{k=1}^{q} \frac{1}{2\nu + k}.
\]

Proof. Using inequalities (2.10) and (A.15) and that $|\log(\frac{x}{2}) + \gamma| \leq -\log(\frac{x}{2})$ for $0 \leq x \leq 1$, we have 
\[
\gamma_{q,r}(x) \leq \left\{ \prod_{k=1}^{q} \frac{2\nu + 2k}{2\nu + k} \right\} I_{\nu+q}(x) \cdot \log \left( \frac{x}{2} \right) + \gamma \left| I_{\nu+r}(x) \right| 
\leq -\left\{ \prod_{k=1}^{q} \frac{2\nu + 2k}{2\nu + k} \right\} \log \left( \frac{x}{2} \right) \frac{x^{2\nu+q+r}(\cosh(x))^2}{2^{2\nu+q+r}(\nu + q)!}.
\]
Since $2\nu + q + r \geq 1$ and $0 \leq x \leq 1$ we have that $-x^{2\nu+q+r} \log(x/2) \leq -\frac{r}{2} \log(x/2)$, and by elementary calculus $-\frac{r}{2} \log(x/2) \leq 2e^{-1}$, and therefore

$$\gamma_{q,r}(x) \leq \left\{ \prod_{k=1}^{q} \frac{2\nu + 2k}{2\nu + k} \right\} \frac{2e^{-1}(\cosh(1))^2}{2^{2\nu+q+r}(\nu + q)!} \left( \frac{1}{\nu - 1} \right) \left( \frac{x}{2} \right)^{\nu + r} \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{x}{2} \right)^{2j}$$

where we used that $2e^{-1}(\cosh(1))^2 = 1.75 \ldots < 2$ to obtain the final inequality.

**Lemma 3.10.** Suppose that $q$ and $r$ are positive integers, then for $\nu \in \mathbb{N}$ and $0 \leq x \leq 1$,

$$\delta_{q,r}(x) < \frac{1}{2^{2\nu+q+r-1} \nu!(\nu + r - 1)!} \prod_{k=1}^{q} \frac{1}{2\nu + k}.$$ 

**Proof.** We begin by noting that

$$\frac{\psi(j) + \psi(j + r)}{(\nu + j + r)!} \leq \frac{2\psi(\nu + j + r)}{(\nu + j + r)!} \leq \frac{2(\nu + j + r)}{(\nu + j + r)!} = \frac{2}{(\nu + j + r)!}.$$ 

Hence,

$$\delta_{q,r}(x) \leq \left\{ \prod_{k=1}^{q} \frac{2\nu + 2k}{2\nu + k} \right\} I_{\nu+q}(x) \cdot \frac{1}{2^{\nu + r}(\nu + r - 1)!} \sum_{j=0}^{q} \frac{1}{j!} \left( \frac{x}{2} \right)^{2j} \leq \frac{1}{2^{\nu + r + q - 1}(\nu + q)!} \prod_{k=1}^{q} \frac{2\nu + 2k}{2\nu + k} \frac{1}{2^{\nu + r - 1} \nu!(\nu + r - 1)!} \prod_{k=1}^{q} \frac{1}{2\nu + k},$$

where we used to obtain the final inequality we used for that, for $2\nu + q + r \geq 0$ and $0 \leq x \leq 1$, we have $x^{2\nu+q+r} e^{x^2/4} \cosh(x) \leq e^{1/4} \cosh(1) = 1.98 \ldots < 2$.

**Lemma 3.11.** Suppose that $q$ and $r$ are positive integers, then for $0 \leq x \leq 1$,

$$\epsilon_{q,r}(x) < \frac{1}{\sin(\nu \pi) \cdot \Gamma(\nu + 1) \cdot \Gamma(\nu + r + 1)} \prod_{k=1}^{q} \frac{1}{2\nu + k}.$$ 

**Proof.** Setting $l = \lfloor \nu + r \rfloor + j$ gives

$$\frac{\pi}{2 |\sin(\nu \pi)|} \sum_{l=\lfloor \nu + r \rfloor}^\infty \frac{1}{\Gamma(l - \nu - r + 1)!} \left( \frac{x}{2} \right)^{2l - \nu - r}$$

$$= \frac{\pi}{2 |\sin(\nu \pi)|} \sum_{j=0}^\infty \frac{1}{\Gamma(k + \lfloor \nu + r \rfloor + j - \nu - r + 1)!} \left( \frac{x}{2} \right)^{2j + 2\lfloor \nu + r \rfloor - \nu - r}$$

$$\leq \frac{\pi}{2 |\sin(\nu \pi)| \cdot \Gamma(\lfloor \nu + r \rfloor + j - \nu - r + 1)!} \left( \frac{x}{2} \right)^{2\lfloor \nu + r \rfloor - \nu - r} \sum_{j=0}^\infty \frac{1}{j!} \left( \frac{x}{2} \right)^{2j}$$

$$= \frac{\pi}{2 |\sin(\nu \pi)| \cdot \Gamma(\nu + r + 1)!} \left( \frac{x}{2} \right)^{2\nu + q + r - \nu - r} \sum_{j=0}^\infty \frac{1}{j!} \left( \frac{x}{2} \right)^{2j}$$
where the final inequality follows from noting that $(\frac{x}{2})^{2[\nu+r]-\nu-r}e^{x^2/4} \leq \frac{\pi e^{x^2/4}}{2^{r+1} \cdot 0.88 |\sin(\nu\pi)| \Gamma(\nu + r + 1)}$

Proof. The proof is similar to the proof of Lemma 3.11, with the only difference

Lemma 3.12. Suppose that $q$ and $r$ are positive integers, then for $0 \leq x \leq 1$,

$$
\zeta_{q,r}(x) < \frac{1}{\sin(\nu\pi)|22^r \cdot 2\nu + r - 2\Gamma(\nu + 1)|\Gamma(\nu + r + 1)} \prod_{k=1}^{q} \frac{1}{2\nu + k}.
$$

Proof. The proof is similar to the proof of Lemma 3.11 with the only difference being that we don’t have to deal with the term $|\log(\frac{x}{2}) + \gamma|$, and that we now use that $\frac{x}{2}(cosh(1))^2 = 3.74 \ldots < 4$.}

With the preliminary lemmas now stated, we are now in a position to obtain bounds for $0 \leq x \leq 1$ for the expressions of type (iv) for the case $\beta = 0$.

Theorem 3.13. Suppose $\nu > -1/2$ and $n \geq 2$, then for $0 \leq x \leq 1$,

$$
\left| \frac{1}{x} - (-1)^n I_{\nu,0,n-1}(x) \frac{d^n}{dx^n} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| < \frac{1 + 2^{n-1}}{2\nu + 1}.
$$

Proof. We begin by proving the result for the case of even $n$. Let $n = 2m$. Using the differentiation formula (2.2) and the triangle inequality gives

$$
\left| \frac{1}{x} - I_{\nu,0,2m-1}(x) \frac{d^{2m}}{dx^{2m}} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| \leq R_1 + R_2,
$$

where

\[ R_1 = I_{\nu,0,2m-1}(x) \sum_{k=0}^{m-1} A_k^{m}(\nu) \frac{K_{\nu+2k}(x)}{x^\nu}, \]

\[ R_2 = \left| \frac{1}{x} - A_m^{m}(\nu) \frac{K_{\nu+2m}(x)I_{\nu,0,2m-1}(x)}{x^\nu} \right|. \]
We begin by bounding $R_1$. Using the inequality $K_\nu(x) < K_{\nu+1}(x)$ for $\nu > -1/2$, and the identity $\sum_{k=0}^m A_k^m(\nu) = 1$, we have

$$R_1 < x^{-\nu} K_{\nu+2m-2}(x) I_{(\nu,0,2m-1)}(x) \sum_{k=0}^{m-1} A_k^m(\nu)$$

$$= (1 - A_m^m(\nu)) x^{-\nu} K_{\nu+2m-2}(x) I_{(\nu,0,2m-1)}(x).$$

We therefore have

$$R_1 < (1 - A_m^m(\nu)) \left\{ \prod_{k=1}^{2m-1} \frac{2\nu + 2k}{2\nu + k} \right\} K_{\nu+2m-2}(x) I_{\nu+2m-1}(x) \quad \text{by (2.10)}$$

$$< \left\{ \prod_{k=1}^{2m-1} \frac{2\nu + 2k}{2\nu + k} \right\} K_{\nu+2m-1}(x) I_{\nu+2m-1}(x) \quad \text{by (A.14)}$$

$$\leq \frac{1}{2(\nu + 2m - 1)} \prod_{k=1}^{2m-1} \frac{2\nu + 2k}{2\nu + k} \quad \text{by (2.10)}$$

$$= \frac{1}{2\nu + 1} \prod_{k=1}^{2m-2} \frac{2\nu + 2k}{2\nu + k + 1} \quad \text{recalling } n = 2m,$$

where to obtain the final inequality we used that $\frac{2\nu + 2k}{2\nu + k + 1} \leq \frac{2\nu + 1}{2\nu} < 2$, for $\nu > -1/2$ and $k \geq 2$. We now bound $R_2$. By the Mean Value Theorem we have

$$I_\nu(x) - \frac{x^{2\nu}}{\Gamma(\nu + 1)2\nu} \leq \frac{d}{dx} \left( \frac{I_\nu(x)}{x^{2\nu}} \right) \quad \text{at } x = \eta,$$

where $0 < \eta < x$, and we used that, by the series expansion (A.1) of $I_\nu(x)$, we have

$$\lim_{t \to 0^+} x^{-\nu} I_\nu(x) = \frac{1}{\Gamma(\nu + 1)2\nu}.$$

As $\frac{d}{dx} (x^{-\nu} I_\nu(x)) = x^{-\nu} I_{\nu+1}(x)$ is a positive monotone increasing function of $x$, it follows that

$$\frac{x^{2\nu}}{\Gamma(\nu + 1)2\nu} \leq x^{-\nu} I_\nu(x) \leq \frac{x^{2\nu}}{\Gamma(\nu + 1)2\nu} + x^{\nu+1} I_{\nu+1}(x).$$

Integrating $n - 1$ times with respect to $x$ over the interval $[0,t]$ gives

$$\frac{t^{2\nu+n-1}}{\Gamma(\nu + 1)2\nu \prod_{j=0}^{n-1}(2\nu + j + 1)} \leq I_{(\nu,0,n-1)}(t)$$

$$\leq \frac{t^{2\nu+n-1}}{\Gamma(\nu + 1)2\nu \prod_{j=0}^{n-1}(2\nu + j + 1)} + I_{(\nu+1,0,n-1)}(t).$$

We may therefore bound $R_2$ as follows

$$R_2 \leq \frac{1}{x^2} \left\{ \frac{A_m^m(\nu)x^{\nu+2m-1} K_{\nu+2m}(x)}{\Gamma(\nu + 1)2\nu \prod_{j=0}^{2m-1}(2\nu + j + 1)} + A_m^m(\nu)x^{-\nu} K_{\nu+2m}(x) I_{(\nu+1,0,2m-1)}(x) \right\}$$

$$= \frac{1}{x^2} \left\{ \frac{x^{\nu+2m-2} K_{\nu+2m}(x)}{2\nu+2m-1 \Gamma(\nu + 2m)} + A_m^m(\nu)x^{-\nu} K_{\nu+2m}(x) I_{(\nu+1,0,2m-1)}(x), \right\}$$
where we used inequality (A.14) for $K_{\mu}(x)$ to obtain the first inequality and to obtain the equality we used that

$$
A_m^m(\nu) = \frac{\prod_{j=0}^{m-1} (2\nu + 2j + 1)}{2^m \prod_{j=0}^{m-1} (\nu + m + j)} = \frac{\prod_{k=0}^{2m-1} (2\nu + k + 1)}{2^{2m-1} \prod_{k=0}^{2m-1}(\nu + k)}
$$

$$
= \frac{\Gamma(\nu + 1) \prod_{k=0}^{2m-1}(2\nu + k + 1)}{2^{2m-1} \Gamma(\nu + 2m)}.
$$

Using Lemma 2.5 gives

$$
x \left| 1 - \frac{x^{\nu+2m-2} K_{\nu+2m}(x)}{2^{\nu+2m-1} \Gamma(\nu + 2m)} \right| \leq \frac{x}{4(\nu + 2m)} \leq \frac{1}{4(\nu + 2m)}, \quad \text{for } 0 \leq x \leq 1.
$$

We also have

$$
A_m^m(\nu)x^{-\nu} K_{\nu+2m}(x) I_{\nu(1, 0, 2m-1)}(x) 
\leq x^{-\nu} K_{\nu+2m}(x) I_{\nu(1, 1, 0, 2m-1)}(x) \quad \text{since } A_m^m(\nu) \leq 1
$$

$$
\leq K_{\nu+2m}(x) I_{\nu+2m}(x) \prod_{k=1}^{2m-1} \frac{2\nu + 2k}{2\nu + k}
$$

by (2.10)

$$
\leq \frac{1}{2(\nu + 2m)} \prod_{k=1}^{2m-1} \frac{2\nu + 2k}{2\nu + k}
$$

$$
\leq \frac{1}{2(\nu + 2m - 1)} \prod_{k=1}^{2m-1} \frac{2\nu + 2k}{2\nu + k}
$$

$$
\leq \frac{2^{n-2}}{2\nu + 1},
$$

where the final inequality follows from (3.4) and (3.5). Therefore, for $0 \leq x \leq 1$, we have

$$
R_2 \leq \frac{1}{4(\nu + 2m)} + \frac{2^{n-2}}{2\nu + 1} < \frac{1}{2\nu + 1} + \frac{2^{n-2}}{2\nu + 1} = \frac{1 + 2^{n-2}}{2\nu + 1}.
$$

Summing up the bounds for $R_1$ and $R_2$ gives the result for the case of even $n$. The proof for odd $n$ is similar, the only difference being that we make use of the derivative formula (2.3) for $x^{-\nu} K_{\mu}(x)$ rather than formula (2.2).\hfill \Box

We now present some series expansions for $K_{\mu}(x)$ and $I_{(\mu, 0, n)}(x)$ which we shall use in our proofs of Lemmas 3.14, 3.15 and 3.16 (below). The series expansion for $K_{\mu}(x)$ can be written as follows:

$$
K_{\mu}(x) = \frac{1}{2} \sum_{k=0}^{[\mu]-1} \frac{\Gamma(\mu - k)(-1)^k}{k!} \left( \frac{x}{2} \right)^{2k-\mu} + u_{\mu}(x),
$$

where

$$
u(\mu)(x) = \frac{\pi}{2 \sin(\pi \mu)} \left[ \sum_{k=\lceil \mu \rceil}^{\infty} \frac{1}{\Gamma(k - \mu + 1)k!} \left( \frac{x}{2} \right)^{2k-\mu} \right. - \left. \sum_{k=0}^{\infty} \frac{1}{\Gamma(\mu + k + 1)k!} \left( \frac{x}{2} \right)^{\mu + 2k} \right], \quad \mu \notin \mathbb{Z},
$$
and
\[ u_\mu(x) = (-1)^{\mu-1} \left\{ \log \left( \frac{x}{2} \right) + \gamma \right\} I_\mu(x) + \frac{(-1)^\mu}{2} \sum_{k=0}^{\infty} \frac{\psi(k) + \psi(\mu + k)}{k!(\mu + k)!} \left( \frac{x}{2} \right)^{\mu + 2k}, \quad \mu \in \mathbb{N}, \]

where \( \psi(k) = \sum_{j=1}^{k} \frac{1}{j} \) and the ceiling function \( \lceil x \rceil \) is the smallest integer that is not less than \( x \). We can see that (3.7) is true for \( \mu \notin \mathbb{Z} \). To see that it holds for \( \mu \notin \mathbb{N} \), we note that the formula \( \Gamma(\mu)\Gamma(1 - \mu) = \frac{\pi}{\sin(\pi \mu)} \) (see, for example, Olver et al. [12]) yields
\[ \frac{\pi}{\sin(\pi \mu)\Gamma(k - \mu + 1)} = \frac{\sin(\pi (k - \mu + 1))\Gamma(\mu - k)}{\sin(\pi \mu)} = (-1)^k \Gamma(\mu - k), \]

and substituting this formula in (A.2) shows that (3.7) holds for \( \mu \notin \mathbb{Z} \).

From the series expansion (A.1) for \( I_\mu(x) \) we may deduce the following series expansion for \( I_{(\mu,0,n)}(x) \):
\[
I_{(\mu,0,n)}(x) = \int_0^x \int_0^{t_{n-1}} \cdots \int_0^{t_1} t_1^{\mu} I_\mu(t_1) dt_1 \cdots dt_{n-1} dx \]
\[
= \int_0^x \int_0^{t_{n-1}} \cdots \int_0^{t_1} \frac{1}{\Gamma(\mu + k + 1)} \left( \frac{t_1}{2} \right)^{\mu + 2k} dt_1 \cdots dt_{n-1} dx \]
\[
= \sum_{k=0}^{\infty} \int_0^x \int_0^{t_{n-1}} \cdots \int_0^{t_1} \frac{t_1^{2\mu + 2k}}{\Gamma(\mu + k + 1)\prod_{j=1}^{n}(2\mu + 2k + j)} dt_1 \cdots dt_{n-1} dx \]
\[
= \sum_{k=0}^{\infty} x^{\mu + 2k + n} \frac{2\mu + 2k + 1}{\Gamma(\mu + k + 1)\prod_{j=1}^{n}(2\mu + 2k + j)}, \quad n \geq 1,
\]

where the interchange of integration and summation is easily justified by using a corollary of the monotone convergence theorem (see Theorem 17.2 of Priestley [13]).

**Theorem 3.14.** Let \( \nu > -1/2 \), then for \( 0 \leq x \leq 1 \) we have
\[
\left| \frac{2\nu + 2}{x^2} + I_{(\nu,0,1)}(x) \frac{d^3}{dx^3} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| < \frac{25}{12(2\nu + 1)} + \frac{1}{v_1(\nu)},
\]

where
\[ v_1(\nu) = \begin{cases} 
2^{\nu+1} \nu (\nu + 2)! (2\nu + 1), & \nu \in \mathbb{N}, \\
|\sin(\pi \nu)| 2^{2\nu} \Gamma(\nu + 1) \Gamma(\nu + 4) (2\nu + 1), & \nu > -1/2 \text{ and } \nu \notin \mathbb{N}.
\end{cases} \]

When \( \nu \in \{0, 1/2, 1, 3/2, \ldots\} \) we have, for \( 0 \leq x \leq 1 \),
\[
\left| \frac{2\nu + 2}{x^2} + I_{(\nu,0,1)}(x) \frac{d^3}{dx^3} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| < \frac{3}{2\nu + 1}.
\]

**Proof.** Using the differentiation formula (2.9) and the triangle inequality gives
\[
\left| \frac{2\nu + 2}{x^2} + I_{(\nu,0,1)}(x) \frac{d^3}{dx^3} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| \leq R_1 + R_2,
\]

where
\[
R_1 = \left| \frac{2\nu + 2}{x^2} - \frac{2\nu + 1}{2(\nu + 2)} K_{\nu+3}(x) I_{(\nu,0,1)}(x) \right|,
\]

and
\[
R_2 = \left| \frac{2\nu + 2}{x^2} - \frac{2\nu + 1}{2(\nu + 2)} K_{\nu+3}(x) I_{(\nu,0,1)}(x) \right|.
\]
\[ R_2 = \frac{3K_{\nu+1}(x)I_{(\nu,0,1)}(x)}{2(\nu+2)x^\nu} \]
\[ \leq \frac{3}{2(\nu+2)} \frac{2(\nu+1)}{2\nu+1} K_{\nu+1}(x)I_{\nu+1}(x) \quad \text{by (2.9)} \]
\[ \leq \frac{3}{2(\nu+2)} \frac{2(\nu+1)}{2\nu+1} \frac{1}{3} \quad \text{by (2.10)} \]
\[ = \frac{1}{3(2\nu+1)(2\nu+4)}. \]

We now bound \( R_1 \). Using the series expansion (3.7) of \( K_{\nu+3}(x) \) and the series expansion (3.8) of \( I_{(\nu,0,1)}(x) \) gives

\[ R_1 = \frac{2\nu + 2}{x^2} - \frac{2\nu + 1}{2(\nu + 2)} \left[ \frac{2^{\nu+2} \Gamma(\nu + 3)}{x^{\nu+3}} - \frac{2^\nu \Gamma(\nu + 2)}{x^{\nu+1}} \right] + R_3 \]
\[ \times \left[ \frac{x^{\nu+1}}{\Gamma(\nu + 1)(2\nu + 1)^{2\nu}} + \frac{x^{\nu+3}}{\Gamma(\nu + 2)(2\nu + 3)^{2\nu+2}} + R_4 \right], \]

where

\[ R_3 = \frac{1}{2} \sum_{k=2}^{[\nu+3]-1} \frac{\Gamma(\nu + 3 - k)(-1)^k}{k!} \left( \frac{x}{2} \right)^{2k-3} + u_{\nu+3}(x), \]
\[ R_4 = \sum_{k=2}^{\infty} \frac{x^{\nu+2k+1}}{\Gamma(\nu + k + 1)k!(2\nu + 2k + 1)^{2\nu+2k}}. \]

Combining terms and simplifying gives

\[ R_1 = \left| \frac{2\nu + 2}{x^2} - \frac{2\nu + 1}{2(\nu + 2)} \left[ \frac{2^{\nu+2} \Gamma(\nu + 3)}{x^{\nu+3}} - \frac{2^\nu \Gamma(\nu + 2)}{x^{\nu+1}} \right] \frac{1}{x^2} \right| \]
\[ - \frac{2\nu + 1}{2(\nu + 2)} \left( \frac{2^{\nu+2} \Gamma(\nu + 3)}{\Gamma(\nu + 1)(2\nu + 1)^{2\nu}} - \frac{2^\nu \Gamma(\nu + 2)}{\Gamma(\nu + 1)(2\nu + 1)^{2\nu}} \right) + R_5 \]
\[ = \frac{1}{(2\nu + 3)(2\nu + 4)} + R_5 \]
\[ \leq \frac{1}{(2\nu + 3)(2\nu + 4)} + |R_5|, \]

where

\[ R_5 = -\frac{2\nu + 1}{2(\nu + 2)} \left[ R_3 \frac{I_{(\nu,0,1)}(x)}{x^\nu} + \left( \frac{2^{\nu+2} \Gamma(\nu + 3)}{x^{\nu+3}} - \frac{2^\nu \Gamma(\nu + 2)}{x^{\nu+1}} \right) R_4 + \frac{x^2}{4(2\nu + 3)} \right]. \]

Applying the triangle inequality, and using that \( \frac{2\nu + 1}{2(\nu + 2)} < 1 \), we have

\[ |R_5| < R_6 + R_7 + R_8, \]

where

\[ R_6 = \frac{|R_3| I_{(\nu,0,1)}(x)}{x^\nu}, \]
\[ R_7 = \left( \frac{2^{\nu+2} \Gamma(\nu + 3)}{x^{\nu+3}} - \frac{2^\nu \Gamma(\nu + 2)}{x^{\nu+1}} \right) |R_4| < \frac{2^{\nu+2} \Gamma(\nu + 3)}{x^{\nu+3}} |R_4|, \quad \text{for } 0 \leq x \leq 1, \]
and
\[
R_8 = \frac{x^2}{4(2\nu + 3)} \leq \frac{1}{4(2\nu + 3)}, \quad \text{for } 0 \leq x \leq 1.
\]

By the triangle inequality we have
\[
R_6 \leq \frac{I_{(\nu,0,1)}(x)}{x^\nu} \left\{ \frac{1}{2} \sum_{k=2}^{[\nu+3]-1} \frac{\Gamma(\nu + 3 - k)(-1)^k}{k!} \left( \frac{x}{2} \right)^{2k-\nu-3} + |u_{\nu+3}(x)| \right\}
\]

Suppose \( \nu \in \mathbb{N} \). Recalling the notation of Lemmas 3.8, 3.9 and 3.10, we have
\[
R_6 \leq \beta_{2,1,3}(x) + \gamma_{1,3}(x) + \delta_{1,3}(x).
\]

Using the bounds that are given in Lemmas 3.8, 3.9 and 3.10 gives, for \( 0 \leq x \leq 1 \),
\[
R_6 \leq \frac{1}{4(2\nu + 1)} + \frac{1}{2^\nu + 2\nu!(\nu + 3)!(2\nu + 1)} + \frac{1}{2^\nu + 2\nu!(\nu + 2)!(2\nu + 1)}.
\]

Suppose now that \( \nu \notin \mathbb{Z} \). Then by, Lemmas 3.8, 3.11 and 3.12 we have that for \( \nu \notin \mathbb{Z} \) and \( 0 \leq x \leq 1 \),
\[
R_6 \leq \beta_{2,1,3}(x) + \epsilon_{1,3}(x) + \zeta_{1,3}(x)
\]

\[
\leq \frac{1}{4(2\nu + 1)} + \frac{1}{|\sin(\pi\nu)|2^\nu \Gamma(\nu + 1)\Gamma(\nu + 4)(2\nu + 1)}.
\]

Let \( \alpha_{2,1,3}(x) \) be defined as per Lemma 3.7 then by Lemma 3.7 we have
\[
R_7 < \alpha_{2,1,3}(x) < \frac{1}{4(2\nu + 5)}, \quad 0 \leq x \leq 1.
\]

Summing up the remainder terms gives
\[
\left| \frac{2\nu + 2}{x^\nu} + I_{(\nu,0,1)}(x) \frac{d^3}{dx^3} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| \leq \frac{3}{(2\nu + 1)(2\nu + 4)} + \frac{1}{(2\nu + 3)(2\nu + 4)}
\]

\[
+ \frac{1}{4(2\nu + 5)} + \frac{1}{4(2\nu + 3)} + \frac{1}{4(2\nu + 1)}
\]

\[
+ \frac{1}{v_1(\nu)}
\]

\[
< \left( 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) \frac{1}{2\nu + 1} + \frac{1}{v_1(\nu)}
\]

\[
= \frac{25}{12(2\nu + 1)} + \frac{1}{v_1(\nu)},
\]

where
\[
v_1(\nu) = \begin{cases} 
2^\nu + 1\nu!(\nu + 2)!(2\nu + 1), & \nu \in \mathbb{N}, \\
|\sin(\pi\nu)|2^\nu \Gamma(\nu + 1)\Gamma(\nu + 4)(2\nu + 1), & \nu > -1/2 \text{ and } \nu \notin \mathbb{N},
\end{cases}
\]

and to obtain the second inequality we used that \( \nu > -1/2 \) and thus, for example, that \( 2\nu + 4 > 3 \). This completes the proof of inequality (3.9).

We now prove that inequality (3.10) holds. Suppose that \( \nu \in \mathbb{N} \), then
\[
\frac{1}{v_1(\nu)} \leq \frac{1}{2^{1/2} \cdot 2!(2\nu + 1)} = \frac{1}{4(2\nu + 1)}.
\]
Suppose now that \( \nu = 1/2, 1, 3/2, 5/2, \ldots \), then
\[
\frac{1}{v_1(\nu)} \leq \frac{1}{2^1 \Gamma(3/2) \Gamma(9/2)(2\nu + 1)} = \frac{16}{105\pi(2\nu + 1)} < \frac{1}{4(2\nu + 1)},
\]
where we used that \( \Gamma(3/2) = \frac{\sqrt{\pi}}{2} \) and \( \Gamma(9/2) = \frac{105\sqrt{\pi}}{16} \). Therefore, for \( \nu = 0, 1/2, 1, 3/2, \ldots \) we have
\[
\left| \frac{2\nu + 2}{x^2} + I_{(\nu,0,1)}(x) \frac{d^3}{dx^3} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| < \left[ \frac{25}{12} + \frac{1}{4} \right] \frac{1}{2\nu + 1} \leq \frac{7}{3(2\nu + 1)} \leq \frac{3}{2\nu + 1},
\]
as required. \( \square \)

**Theorem 3.15.** Let \( \nu > -1/2 \), then for \( 0 \leq x \leq 1 \) we have
\[
(3.11) \quad \left| \frac{2\nu + 3}{x^2} - I_{(\nu,0,2)}(x) \frac{d^4}{dx^4} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| < \frac{77}{20(2\nu + 1)} + \frac{1}{v_2(\nu)},
\]
where
\[
v_2(\nu) = \begin{cases} 2^{2\nu+3}(\nu + 1)!(\nu + 3)!(2\nu + 1), & \nu \in \mathbb{N}, \\ |\sin(\pi\nu)|2^{2\nu+2}\Gamma(\nu + 2)\Gamma(\nu + 5)(2\nu + 1), & \nu > -1/2 \text{ and } \nu \notin \mathbb{N}. \end{cases}
\]
When \( \nu \in \{0, 1/2, 1, 3/2, \ldots\} \) we have, for \( 0 \leq x \leq 1 \),
\[
(3.12) \quad \left| \frac{2\nu + 3}{x^2} - I_{(\nu,0,2)}(x) \frac{d^4}{dx^4} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| < \frac{4}{2\nu + 1}.
\]

**Proof.** Using the differentiation formula (2.6) and the triangle inequality gives
\[
\left| \frac{2\nu + 3}{x^2} - I_{(\nu,0,2)}(x) \frac{d^4}{dx^4} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| \leq R_1 + R_2
\]
where
\[
R_1 = (2\nu + 3) \left| \frac{1}{x^2} - \frac{2\nu + 1}{4(2\nu + 1)} \right| \frac{K_{\nu+4}(x)I_{(\nu,0,2)}(x)}{x^\nu}
\]
and
\[
R_2 = \left( \frac{3(2\nu + 1)K_{\nu+2}(x)}{2(\nu + 1)(\nu + 3)x^\nu} + \frac{3K_\nu(x)}{4(\nu + 1)(\nu + 2)x^\nu} \right) I_{(\nu,0,2)}(x)
\]
\[
< \left\{ \prod_{k=1}^{3} \frac{2\nu + 2k}{2\nu + k} \right\} K_{\nu+2}(x)I_{\nu+2}(x) \quad \text{by (2.9) and (A.14)}
\]
\[
\leq \frac{(2\nu + 2)(2\nu + 4)(2\nu + 6)}{(2\nu + 1)(2\nu + 2)(2\nu + 3)} \cdot \frac{1}{2(\nu + 2)} \quad \text{by (2.16)}
\]
\[
= \frac{2\nu + 6}{(2\nu + 1)(2\nu + 3)}
\]
\[
< \frac{5}{2(\nu + 1)},
\]
where the final inequality holds because for \( \nu > -1/2 \) we have \( \frac{2\nu + 6}{2\nu + 3} < \frac{5}{2} \). Using the series expansion (3.7) for \( K_{\nu+4}(x) \) and the series expansion (3.8) for \( I_{(\nu,0,2)}(x) \)
gives

\[
R_1 = (2\nu + 3) \left[ \frac{1}{x^2} - \frac{2\nu + 1}{4(\nu + 2)(\nu + 3)} \left[ \frac{2^\nu 3^\nu (\nu + 4)}{x^{\nu + 4}} - \frac{2^\nu 4^\nu (\nu + 3)}{x^{\nu + 2}} + R_3 \right] \times \left[ \frac{x^\nu + 2}{\Gamma(\nu + 1)(2\nu + 1)(2\nu + 2)2^\nu} + \frac{x^\nu + 4}{\Gamma(\nu + 2)(2\nu + 3)(2\nu + 4)2^\nu} + R_4 \right] \right]
\]

where

\[
R_3 = \frac{1}{2} \sum_{k=2}^{[\nu+4]-1} \frac{\Gamma(\nu + 4 - k)(-1)^k (x/2)^{2k-\nu-4}}{k!} + u_{\nu+4}(x),
\]

\[
R_4 = \sum_{k=2}^{\infty} \frac{x^{\nu+2k+2}}{\Gamma(\nu + k + 1)(2\nu + 2k + 1)(2\nu + 2k + 2)^2}.
\]

Combining terms and simplifying gives

\[
R_1 = (2\nu + 3) \left[ \left(1 - \frac{2\nu + 1}{4(\nu + 2)(\nu + 3)} \cdot \frac{2^\nu 3^\nu (\nu + 4)}{\Gamma(\nu + 1)(2\nu + 1)(2\nu + 2)2^\nu}\right) \frac{1}{x^2} \right.
\]

\[
- \frac{2\nu + 1}{4(\nu + 2)(\nu + 3)} \left( \frac{\Gamma(\nu + 2)2^\nu 3^\nu (\nu + 4)}{\Gamma(\nu + 1)(2\nu + 1)(2\nu + 2)2^\nu} \right) \frac{2^\nu 3^\nu (\nu + 4)}{\Gamma(\nu + 1)(2\nu + 1)(2\nu + 2)2^\nu} + R_5 \right]
\]

\[
= \left| \left( \frac{3}{(2\nu + 3)(2\nu + 4)} + R_5 \right) \right|
\]

\[
\leq \frac{3}{(2\nu + 3)(2\nu + 4)} + |R_5|,
\]

where

\[
R_5 = \frac{2\nu + 1}{4(\nu + 2)(\nu + 3)} \left[ R_3 \left( I_{(\nu,0,2)}(x) \right) + \left( \frac{2^\nu 3^\nu (\nu + 4)}{x^{\nu + 4}} - \frac{2^\nu 4^\nu (\nu + 3)}{x^{\nu + 2}} \right) \frac{2^\nu 3^\nu (\nu + 4)}{x^{\nu + 4}} \right] R_4
\]

\[
+ \frac{x^2}{4(2\nu + 3)} \right].
\]

Applying the triangle inequality, and using that \( \frac{(2\nu + 1)(2\nu + 3)}{4(\nu + 2)(\nu + 3)} < 1 \), we have

\[
|R_5| < R_6 + R_7 + R_8,
\]

where

\[
R_6 = \frac{|R_3| I_{(\nu,0,2)}(x)}{x^\nu},
\]

\[
R_7 = \left( \frac{2^\nu 3^\nu (\nu + 4)}{x^{\nu + 4}} - \frac{2^\nu 4^\nu (\nu + 3)}{x^{\nu + 2}} \right) |R_4| < \frac{2^\nu 3^\nu (\nu + 4)}{x^{\nu + 4}} |R_4|, \quad \text{for } 0 \leq x \leq 1,
\]

\[
R_8 = \frac{x^2}{4(2\nu + 3)} \leq \frac{1}{4(2\nu + 3)}, \quad \text{for } 0 \leq x \leq 1.
\]
We bound $R_6$ and $R_7$ by using the same approach as for Lemma 3.14. By Lemmas 3.8, 3.9 and 3.10, we have that for $\nu \in \mathbb{N}$ and $0 \leq x \leq 1$,

$$R_6 \leq \beta_{2,2,4}(x) + \gamma_{2,4}(x) + \delta_{2,4}(x)$$

$$< \frac{1}{4(2\nu + 1)} + \frac{1}{2^{2\nu + 4}(\nu + 1)!(\nu + 4)!(2\nu + 1)} + \frac{1}{2^{2\nu + 4}(\nu + 1)!(\nu + 3)!(2\nu + 1)}$$

$$< \frac{1}{4(2\nu + 1)} + \frac{1}{2^{2\nu + 3}(\nu + 1)!(\nu + 3)!(2\nu + 1)}.$$  

By Lemmas 3.8, 3.11 and 3.12, we have that for $\nu \notin \mathbb{Z}$ and $0 \leq x \leq 1$,

$$R_6 \leq \beta_{2,2,4}(x) + \epsilon_{2,4}(x) + \zeta_{2,4}(x)$$

$$< \frac{1}{4(2\nu + 1)} + \frac{1}{\sin(\pi\nu)2^{2\nu + 2}\Gamma(\nu + 2)\Gamma(\nu + 5)(2\nu + 1)}.$$  

We use Lemma 3.7 to bound $R_7$:

$$R_7 \leq \zeta_{2,2,4}(x) < \frac{1}{4(2\nu + 5)}, \quad 0 \leq x \leq 1.$$  

Summing up the remainder terms gives

$$\left| \frac{2\nu + 3}{x^2} - I_{\nu,0,2}(x) \frac{d^4}{dx^4} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| \leq \frac{5}{2(2\nu + 1)} + \frac{3}{(2\nu + 3)(2\nu + 4)} + \frac{1}{4(2\nu + 5)}$$

$$+ \frac{1}{4(2\nu + 3)} + \frac{1}{4(2\nu + 1)} + \frac{1}{v_2(\nu)}$$

$$< \left( \frac{5}{2} + \frac{3}{5} + \frac{1}{4} + \frac{1}{4} + 1 \right) \frac{1}{2\nu + 1} + \frac{1}{v_2(\nu)}$$

$$= \frac{77}{20(2\nu + 1)} + \frac{1}{v_2(\nu)},$$

where

$$v_2(\nu) = \begin{cases} 
2^{2\nu + 3}(\nu + 1)!(\nu + 3)!(2\nu + 1), & \nu \in \mathbb{N}, \\
\sin(\pi\nu)2^{2\nu + 2}\Gamma(\nu + 2)\Gamma(\nu + 5)(2\nu + 1), & \nu > -1/2 \text{ and } \nu \notin \mathbb{N},
\end{cases}$$

and to obtain the second inequality we used that $\nu > -1/2$ and thus, for example, that $2\nu + 4 > 3$. This completes the proof of inequality (3.11).

We now prove that inequality (3.12) holds. Suppose that $\nu \in \mathbb{N}$, then

$$\frac{1}{v_2(\nu)} \leq \frac{1}{2^3 \cdot 3!(2\nu + 1)} = \frac{1}{48(2\nu + 1)}.$$  

Suppose now that $\nu = 1/2, 1, 3/2, 5/2, \ldots$, then

$$\frac{1}{v_2(\nu)} \leq \frac{1}{2^3 \Gamma(5/2)\Gamma(11/2)(2\nu + 1)} = \frac{16}{2835\pi(2\nu + 1)} < \frac{1}{48(2\nu + 1)}.$$  

Therefore, for $\nu = 0, 1/2, 1, 3/2, \ldots$ we have

$$\left| \frac{2\nu + 3}{x^2} - I_{\nu,0,2}(x) \frac{d^4}{dx^4} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| < \left( \frac{77}{20} + \frac{1}{48} \right) \frac{1}{2\nu + 1} = \frac{929}{240(2\nu + 1)} < \frac{4}{2\nu + 1},$$

as required. \qed
Theorem 3.16. Let $\nu > -1/2$, then for $0 \leq x \leq 1$ we have

\begin{equation}
(3.13) \quad \left| \frac{(2\nu + 2)(2\nu + 3)}{x^3} + \frac{1}{x} - I_{(\nu, 0, 1)}(x) \frac{d^4}{dx^4} K_{\nu}(x) \frac{x^{\nu}}{x^{\nu}} \right| < \frac{2779}{768(2\nu + 1)} + \frac{1}{v_3(\nu)}.
\end{equation}

where

\begin{equation}
v_3(\nu) = \begin{cases} 2^{2\nu+2}\nu!(\nu + 3)!(2\nu + 1), & \nu \in \mathbb{N}, \\ |\sin(\pi\nu)|2^{2\nu+1}\Gamma(\nu + 1)\Gamma(\nu + 5)(2\nu + 1), & \nu > -1/2 \text{ and } \nu \notin \mathbb{N}. \end{cases}
\end{equation}

When $\nu \in \{0, 1/2, 1, 3/2, \ldots\}$ we have, for $0 \leq x \leq 1$,

\begin{equation}
(3.14) \quad \left| \frac{(2\nu + 2)(2\nu + 3)}{x^3} + \frac{1}{x} - I_{(\nu, 0, 1)}(x) \frac{d^4}{dx^4} K_{\nu}(x) \frac{x^{\nu}}{x^{\nu}} \right| < \frac{4}{2\nu + 1}.
\end{equation}

Proof. Using the differentiation formula (2.6) and the triangle inequality gives

\begin{equation}
\left| \frac{(2\nu + 2)(2\nu + 3)}{x^3} + \frac{1}{x} - I_{(\nu, 0, 1)}(x) \frac{d^4}{dx^4} K_{\nu}(x) \frac{x^{\nu}}{x^{\nu}} \right| \leq R_1 + R_2
\end{equation}

where

\begin{equation}
R_1 = \left| \frac{(2\nu + 2)(2\nu + 3)}{x^3} + \frac{1}{x} - \frac{(2\nu + 1)(2\nu + 3)}{4(\nu + 2)(\nu + 3)} K_{\nu+4}(x) I_{(\nu, 0, 1)}(x) \frac{x^{\nu}}{x^{\nu}} \right|
\end{equation}

and

\begin{equation}
R_2 = \frac{3K_{\nu}(x) I_{(\nu, 0, 1)}(x)}{4(\nu + 1)(\nu + 2)x^{\nu}}
\end{equation}

\begin{align*}
&\leq \frac{3}{4(\nu + 1)(\nu + 2)} \cdot \frac{2\nu + 2}{2\nu + 1} K_{\nu}(x) I_{\nu+1}(x) \quad \text{by (2.9)} \\
&\leq \frac{3}{4(\nu + 1)(\nu + 2)} \cdot \frac{2\nu + 2}{2\nu + 1} \cdot \frac{1}{2(\nu + 1)} \quad \text{by (A.13) and (2.10)} \\
&= \frac{3}{(2\nu + 1)(2\nu + 2)(2\nu + 4)}.
\end{align*}

We now bound $R_1$. By the triangle inequality

\begin{equation}
R_1 \leq R_3 + R_4,
\end{equation}

where

\begin{align*}
R_3 &= \left| \frac{1}{x} - \frac{2\nu + 1}{2(\nu + 1)} K_{\nu+2}(x) I_{(\nu, 0, 1)}(x) \frac{x^{\nu}}{x^{\nu}} \right|, \\
R_4 &= \left| \frac{(2\nu + 2)(2\nu + 3)}{x^3} + \frac{\nu}{(\nu + 3)x} - \frac{(2\nu + 1)(2\nu + 3)}{4(\nu + 2)(\nu + 3)} K_{\nu+4}(x) I_{(\nu, 0, 1)}(x) \frac{x^{\nu}}{x^{\nu}} \right|.
\end{align*}

In the proof of Lemma 3.13 (see inequality 3.8) we showed that:

\begin{equation}
\left| \frac{1}{x} - A_m^m(\nu) K_{\nu+2m}(x) I_{(\nu, 0, 2m-1)}(x) \frac{x^{\nu}}{x^{\nu}} \right| < \frac{1 + 2^{2m-2}}{2\nu + 1}, \quad \text{for } 0 \leq x \leq 1.
\end{equation}

Since $A_1^1(\nu) = \frac{2\nu + 1}{2(\nu + 1)}$, we have the following bound on $R_3$:

\begin{equation}
R_3 \leq \frac{3}{\nu + 3} \cdot \frac{2}{2\nu + 1} = \frac{6}{(2\nu + 1)(\nu + 3)}, \quad \text{for } 0 \leq x \leq 1.
\end{equation}
Using the series expansion (3.7) for $K_{\nu+4}(x)$ and the series expansion (3.8) for $I_{(\nu,0,1)}(x)$ gives

\[
R_4 = \left| \frac{(2\nu+2)(2\nu+3)}{x^3} + \frac{\nu}{(\nu+3)x} - \frac{(2\nu+1)(2\nu+3)}{4(\nu+2)(\nu+3)} \cdot \frac{2^{\nu+3}\Gamma(\nu+4)}{x^{\nu+4}} \right| \frac{1}{x^3}
\]

\[
+ \left( \frac{\nu}{\nu+3} - \frac{(2\nu+1)(2\nu+3)}{4(\nu+2)(\nu+3)} \cdot \frac{2^{\nu+3}\Gamma(\nu+4)}{\Gamma(\nu+1)(2\nu+1)2^{\nu+2}} - \frac{2^{\nu+1}\Gamma(\nu+3)}{\Gamma(\nu+1)(2\nu+1)2^{\nu+2}} \right) \frac{1}{x} + R_5
\]

\[
R_5 = \frac{1}{2} \sum_{k=2}^{[\nu+4]-1} \frac{\Gamma(\nu+4-k)(-1)^k}{k!(2\nu+1)(2\nu+3)(2\nu+5)2^{\nu+2k}} + u_{\nu+4}(x),
\]

\[
R_6 = \sum_{k=3}^{\infty} \frac{\Gamma(\nu+k+1)k!}{\Gamma(\nu+1)(\nu+2)(\nu+3)(\nu+5)2^{\nu+2k}},
\]

Combining terms and simplifying gives

\[
R_4 = \left| \frac{(2\nu+2)(2\nu+3)}{x^3} + \frac{\nu}{(\nu+3)x} - \frac{(2\nu+1)(2\nu+3)}{4(\nu+2)(\nu+3)} \cdot \frac{2^{\nu+3}\Gamma(\nu+4)}{x^{\nu+4}} \right| \frac{1}{x^3}
\]

\[
+ \left( \frac{\nu}{\nu+3} - \frac{(2\nu+1)(2\nu+3)}{4(\nu+2)(\nu+3)} \cdot \frac{2^{\nu+3}\Gamma(\nu+4)}{\Gamma(\nu+1)(2\nu+1)2^{\nu+2}} - \frac{2^{\nu+1}\Gamma(\nu+3)}{\Gamma(\nu+1)(2\nu+1)2^{\nu+2}} \right) \frac{1}{x} + R_5
\]

\[
= \left| \frac{x}{(2\nu+1)(2\nu+3)(2\nu+5)} + R_7 \right|
\]

\[
\leq \frac{1}{(2\nu+1)(2\nu+3)(2\nu+5)} + |R_7|,
\]

where

\[
R_7 = -\frac{(2\nu+1)(2\nu+3)}{4(\nu+2)(\nu+3)} \left[ R_5 \left( \frac{I_{(\nu,0,1)}(x)}{x^\nu} + \frac{2^{\nu+3}\Gamma(\nu+4)}{x^{\nu+4}} - \frac{2^{\nu+1}\Gamma(\nu+3)}{x^{\nu+2}} \right) \frac{1}{x^3}
\]

\[
+ \frac{2^{\nu-2}\Gamma(\nu+2)}{x^{\nu-2}} \right) R_6 + \left( \frac{2^{\nu-2}\Gamma(\nu+2)}{\Gamma(\nu+3)(2\nu+3)(2\nu+5)} x^{\nu+5} \Gamma(\nu+3)(2\nu+5)2^{\nu+5} \right)x^3
\]

\[
+ \frac{2^{\nu-2}(\nu+1)x^5}{\Gamma(\nu+3)(2\nu+5)2^{\nu+5}},
\]

and applying the triangle inequality, and using that $\frac{(2\nu+1)(2\nu+3)}{4(\nu+2)(\nu+3)} < 1$, we have

\[
|R_7| < R_8 + R_9 + R_{10},
\]

where

\[
R_8 = \left| \frac{R_5}{x^\nu} I_{(\nu,0,1)}(x) \right|
\]

\[
R_9 = \left( \frac{2^{\nu+3}\Gamma(\nu+4)}{x^{\nu+4}} - \frac{2^{\nu+1}\Gamma(\nu+3)}{x^{\nu+2}} + \frac{2^{\nu-2}\Gamma(\nu+2)}{x^{\nu}} \right) |R_6|,
\]

\[
R_{10} = \left| \frac{x}{(2\nu+1)(2\nu+3)(2\nu+5)} + R_7 \right|
\]

\[
\leq \frac{1}{(2\nu+1)(2\nu+3)(2\nu+5)} + |R_7|.
\]
and
\[ R_{10} = \frac{x^3}{8(2\nu + 3)(2\nu + 5)} + \frac{2^{\nu-2}\Gamma(\nu + 2)x^5}{\Gamma(\nu + 3)(2\nu + 5)2^{\nu+5}}. \]

We can bound \( R_9 \) and \( R_{10} \), for \( 0 \leq x \leq 1 \), as follows
\[
R_9 < \left( \frac{2^{\nu+1}\Gamma(\nu + 4)}{x^{\nu+4}} - \frac{2^{\nu+1}\Gamma(\nu + 3)}{x^{\nu+2}} + \frac{2^{\nu-1}\Gamma(\nu + 2)}{x^{\nu}} \right) |R_6| < \frac{2^{\nu+3}\Gamma(\nu + 4)}{x^{\nu+4}} |R_6|,
\]
\[
R_{10} < \left( \frac{1}{8} + \frac{1}{64} \right) \frac{1}{(2\nu + 3)(2\nu + 5)} = \frac{9}{64(2\nu + 3)(2\nu + 5)}.
\]

We bound \( R_6 \) and \( R_7 \) by using the same approach as for Lemmas 3.14 and 3.15.

By Lemmas 3.8, 3.9 and 3.10, we have that for \( \nu \in \mathbb{N} \) and \( 0 \leq x \leq 1 \),
\[
R_9 \leq \beta_{3,1,4}(x) + \gamma_{1,4}(x) + \delta_{1,4}(x)
\]
\[
< \frac{1}{24(2\nu + 1)} + \frac{2^{\nu+2}\nu!(\nu + 4)!}{(2\nu + 3)!} \frac{1}{2^{\nu+1} + 1} + \frac{2^{\nu+3}\nu!(\nu + 3)!}{(2\nu + 1)!} \frac{1}{2^{\nu+2} + 1},
\]
and by Lemmas 3.8 [3.11] and 3.12 we have that, for \( \nu \notin \mathbb{Z} \) and \( 0 \leq x \leq 1 \),
\[
R_8 \leq \beta_{3,1,4}(x) + \epsilon_{1,4}(x) + \zeta_{1,4}(x)
\]
\[
< \frac{1}{24(2\nu + 1)} + \frac{1}{|\sin(\pi\nu)|2^{\nu+1}\Gamma(\nu + 1)\Gamma(\nu + 5)(2\nu + 1)}.\]

We use Lemma 3.7 to bound \( R_7 \):
\[
R_7 < \alpha_{3,1,4}(x) < \frac{1}{24(2\nu + 7)}, \quad 0 \leq x \leq 1.
\]

Summing up the remainder terms gives
\[
\left| \frac{(2\nu + 2)(2\nu + 3)}{x^3} + \frac{1}{x} - I_{(\nu,0,1)}(x) \right| \frac{d^4}{dx^4} \left( \frac{K_\nu(x)}{x^\nu} \right)
\]
\[
\leq \frac{3}{2(2\nu + 1)(2\nu + 2)(2\nu + 4)} + \frac{9}{24(2\nu + 1)} + \frac{6}{24(2\nu + 7)} + \frac{1}{24(2\nu + 7)}
\]
\[
+ \frac{64(2\nu + 3)(2\nu + 4)}{24(2\nu + 1)} + \frac{1}{v_3(\nu)}
\]
\[
< \left( \frac{1}{2} + \frac{1}{24} + \frac{9}{256} + \frac{1}{24} \right) \frac{1}{2\nu + 1} + \frac{1}{v_3(\nu)}
\]
\[
= \frac{2779}{768(2\nu + 1)} + \frac{1}{v_3(\nu)},
\]
where
\[
v_3(\nu) = \begin{cases} 2^{\nu+2}\nu!(\nu + 3)!/(2\nu + 1), & \nu \in \mathbb{N}, \\ |\sin(\pi\nu)|2^{\nu+1}\Gamma(\nu + 1)\Gamma(\nu + 5)(2\nu + 1), & \nu > -1/2 \text{ and } \nu \notin \mathbb{N}, \end{cases}
\]
and to obtain the second inequality we used that \( \nu > -1/2 \) and thus, for example, that \( 2\nu + 4 > 3 \). This completes the proof of inequality (3.13).

We now prove that inequality (3.14) holds. Suppose that \( \nu \in \mathbb{N} \), then
\[
\frac{1}{v_3(\nu)} \leq \frac{1}{2^2 \cdot 3!(2\nu + 1)} = \frac{1}{24(2\nu + 1)}.
\]
Suppose now that $\nu = 1/2, 1, 3/2, 5/2, \ldots$, then
\[
\frac{1}{\nu_3(\nu)} \leq \frac{1}{2^2 \Gamma(3/2) \Gamma(11/2)(2\nu + 1)} = \frac{16}{945 \pi (2\nu + 1)} < \frac{1}{24(2\nu + 1)}.
\]
Therefore, for $\nu = 0, 1/2, 1, 3/2, \ldots$ we have
\[
\left| \frac{2\nu + 3}{x^2} - \frac{d^4}{dx^4} \left( \frac{K_\nu(x)}{x^\nu} \right) \right| < \left[ \frac{2779}{768} + \frac{1}{24} \right] \frac{1}{2\nu + 1} = \frac{937}{256(2\nu + 1)} < \frac{4}{2\nu + 1},
\]
as required.

**Remark 3.17.** The bounds of Lemmas 3.13–3.15 and 3.16 perform poorly when $2\nu$ is very close but not equal to an integer, due to the presence of a term involving $1/\sin(\pi\nu)$. This term is an artefact of our series expansion method and we consider that an alternative approach should yield bounds of the form $C(2\nu + 1)^{-1}$ for all $\nu > -1/2$, where $C$ is a constant not involving $\nu$.

It would also be of interest for future research to establish $O(\nu^{-1})$ bounds for all $x \in [0, \infty)$ for each of the expressions in Lemmas 3.13–3.15 and 3.16. Whilst the series expansion method that we used in our proofs leads to good bounds in the region $0 \leq x \leq 1$, it would give poor bounds for large $x$, and so a different approach would be needed to obtain $O(\nu^{-1})$ bounds for large $x$.

For the case $\nu = 0$ we can actually use a relatively straightforward approach to achieve good bounds for all $x \in [0, \infty)$, as we shall see in the following lemma. The approach could be easily extended to general $\nu > -1/2$, but would lead to poor bounds for large $\nu$.

**Theorem 3.18.** For $x \geq 0$ we have
\[
(3.15) \quad \left| \frac{1}{x} - I_{(0,0,1)}(x)K_0''(x) \right| < 3,
\]
\[
(3.16) \quad \left| \frac{1}{x} + I_{(0,0,2)}(x)K_0^{(3)}(x) \right| < 5,
\]
\[
(3.17) \quad \left| \frac{1}{x} - I_{(0,0,3)}(x)K_0^{(4)}(x) \right| < 9,
\]
\[
(3.18) \quad \left| \frac{2}{x^2} + I_{(0,0,1)}(x)K_0^{(3)}(x) \right| < 4.39,
\]
\[
(3.19) \quad \left| \frac{3}{x^2} - I_{(0,0,2)}(x)K_0^{(4)}(x) \right| < 6.81,
\]
\[
(3.20) \quad \left| \frac{6}{x^3} + \frac{1}{x} - I_{(0,0,1)}(x)K_0^{(4)}(x) \right| < 14.61.
\]

**Proof.** In Lemmas 3.13–3.16 we established bounds for these expressions in the region $0 \leq x \leq 1$. We now obtain bounds in the region $x \geq 1$, and combining these bounds will give us bounds in the region $0 \leq x \leq \infty$.

(i) For $x \geq 1$ we have
\[
\left| \frac{1}{x} - I_{(0,0,1)}(x)K_0''(x) \right| \leq 1 + \left| I_{(0,0,1)}(x)K_0''(x) \right|
\]
\[
= 1 + \frac{1}{2} I_{(0,0,1)}(x)\{K_2(x) + K_0(x)\} \quad \text{by (2.4)}
\]
\[
< 1 + I_1(x)\{K_2(x) + K_0(x)\} \quad \text{by (2.10)}
\]
$< 1 + I_1(x)[K_2(x) + K_1(x)].$ by (A.14)

Now, Theorem 1.1 of Laforgia and Natalini \[9\] states that, for $x > 0$ and $\mu \geq 0$,

\[
\frac{I_{\mu-1}(x)}{I_{\mu}(x)} < \frac{x}{-\mu + \sqrt{\mu^2 + x^2}} = \frac{\mu + \sqrt{\mu^2 + x^2}}{x}.
\]

Straightforward calculus shows that for $\mu \geq 0$ the function $\frac{\mu + \sqrt{\mu^2 + x^2}}{x}$ is strictly monotone decreasing in the region $(0, \infty)$. Hence, in the region $x \geq 1$ the following inequality holds

\[
I_{\mu-1}(x) < (\mu + \sqrt{\mu^2 + 1})I_{\mu}(x), \quad \mu \geq 0.
\]

Applying inequality (3.22) to (3.21) gives, for $x \geq 1$,

\[
\frac{1}{x} - I_{(0,0,1)}(x)K_0''(x) < 1 + (2 + \sqrt{5})I_2(x)K_2(x) + I_1(x)K_1(x)
\]

\[
< 1 + (2 + \sqrt{5})I_2(1)K_2(1) + I_1(1)K_1(1)
\]

\[
= 2.274\ldots,
\]

where we used that $I_\nu(x)K_\nu(x)$ is a monotone decreasing function of $x$ in $(0, \infty)
(\text{see Proposition \[2.6\] to obtain the second inequality, and the values of } I_1(1), I_2(1), K_1(1) \text{ and } K_2(1) \text{ were calculated using Table 9.8 of Abramowitz and Stegun \[1\].})

Combining inequality (3.10) with $\nu = 0$ and inequality and (3.23) yields (3.15).

(ii) The proof is similar to that of inequality (3.15). For $x \geq 1$ we have

\[
\frac{1}{x^2} + I_{(0,0,2)}(x)K_0^{(3)}(x) < 1 + (3 + \sqrt{10})I_3(1)K_3(1) + I_2(1)K_2(1) = 2.631\ldots.
\]

Combining inequalities (3.10) and (3.24) yields (3.10).

(iii) For $x \geq 1$ we can show that

\[
\frac{1}{x^4} - I_{(0,0,3)}(x)K_0^{(4)}(x) < 1 + (4 + \sqrt{17})I_4(1)K_4(1) + 7I_3(1)K_3(1) = 3.085\ldots.
\]

Combining inequalities (3.10) and (3.25) yields (3.17).

(iv) The proof is similar to that of inequality (3.15), but slightly longer. For $x \geq 1$ we have

\[
\frac{2}{x^4} + I_{(0,0,1)}(x)K_0^{(3)}(x)
\]

\[
\leq 2 + \left| I_{(0,0,1)}(x)K_0^{(3)}(x) \right|
\]

\[
= 2 + \frac{1}{4}I_{(0,0,1)}(x)[K_3(x) + 3K_1(x)] \quad \text{by (2.5)}
\]

\[
< 2 + \frac{1}{2}I_1(x)[K_3(x) + 3K_1(x)] \quad \text{by (2.10)}
\]

\[
< 2 + \frac{1}{2}(2 + \sqrt{5})I_2(x)K_2(x) + \frac{3}{2}I_1(x)K_1(x) \quad \text{by (3.22)}
\]

\[
< 2 + \frac{1}{2}(2 + \sqrt{5})(3 + \sqrt{10})I_3(x)K_3(x) + \frac{3}{2}I_1(x)K_1(x) \quad \text{by (3.22)}
\]

\[
< 2 + \frac{1}{2}(2 + \sqrt{5})(3 + \sqrt{10})I_3(1)K_3(1) + \frac{3}{2}I_1(1)K_1(1)
\]

\[
(3.26) \quad = 4.385\ldots
\]
Combining inequalities (3.12) and (3.26) yields (3.18).

(v) For $x \geq 1$ we can show that
\[
\left| \frac{3}{x^2} - I_{(0,0,2)}(x)K_{0}^{(4)}(x) \right| < 3 + \frac{1}{2}(3 + \sqrt{10})(4 + \sqrt{17})I_{4}(1)K_{4}(1) + \frac{7}{2}I_{2}(1)K_{2}(1)
\]
(3.27)
\[
= 6.802 \ldots
\]

Combining inequalities (3.12) and (3.27) yields (3.19).

(vi) For $x \geq 1$ we can show that
\[
\left| \frac{6}{x^3} + \frac{1}{x} - I_{(0,0,1)}(x)K_{0}^{(4)}(x) \right| < 7 + \frac{1}{4}(2 + \sqrt{5})(3 + \sqrt{10})(4 + \sqrt{17})I_{4}(1)K_{4}(1)
\]
\[
+ (2 + \sqrt{5})I_{2}(1)K_{2}(1) + \frac{3}{4}I_{1}(1)K_{1}(1)
\]
(3.28)
\[
= 14.607 \ldots
\]

Combining inequalities (3.14) and (3.28) yields (3.20). □

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Appendix A. Elementary properties of modified Bessel functions

Here we list standard properties of modified Bessel functions that are used throughout this paper. All these formulas can be found in Olver et al. [12], unless otherwise stated. Inequalities (A.11) and (A.12) can be found in Jones [8] and Näsell [11]; inequalities (A.13) and (A.14) can easily be deduced from (A.6);
inequality (A.15) is given in Luke [10]. The integration formulas can be found in Gradshetyn and Ryzhik [7].

A.1. Basic properties. The modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ are both regular functions of $x$. For positive values of $x$ the functions $I_\nu(x)$ and $K_\nu(x)$ are positive for $\nu > -1$ and all $\nu \in \mathbb{R}$, respectively.

A.2. Series expansions.

(A.1) \[ I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu + k + 1)k!} \left( \frac{x}{2} \right)^{\nu + 2k}, \]
\[ K_\nu(x) = \frac{\pi}{2 \sin(\pi \nu)} \left[ \sum_{k=0}^{\infty} \frac{1}{\Gamma(k - \nu + 1)k!} \left( \frac{x}{2} \right)^{2k - \nu} \right], \quad \nu \not\in \mathbb{Z}, \]
\[ K_\nu(x) = (-1)^{\nu-1} \left\{ \log \left( \frac{x}{2} \right) + \gamma \right\} I_\nu(x) + \frac{1}{2} \sum_{k=0}^{\nu-1} \frac{(-1)^k (\nu - k - 1)!}{k!} \left( \frac{x}{2} \right)^{2k - \nu} \]
\[ + \frac{(-1)^\nu}{2} \sum_{k=0}^{\infty} \frac{\psi(k) + \psi(\nu + k)}{k!(\nu + k)!} \left( \frac{x}{2} \right)^{\nu + 2k}, \quad \nu \in \mathbb{N}, \]

where $\psi(k) = \sum_{j=1}^{k} \frac{1}{j}$.

A.3. Spherical Bessel functions.

(A.4) \[ I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh(x), \]
\[ K_{1/2}(x) = K_{-1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}. \]

A.4. Integral representations.

(A.6) \[ K_\nu(x) = \int_{0}^{\infty} e^{-x \cosh(t)} \cosh(\nu t) dt, \quad x > 0. \]

A.5. Asymptotic expansions.

(A.7) \[ I_\nu(x) \sim \frac{1}{\Gamma(\nu + 1)} \left( \frac{x}{2} \right)^\nu, \quad x \downarrow 0, \]
\[ K_\nu(x) \sim \begin{cases} 2^{\nu-1} \Gamma(|\nu|) x^{-|\nu|}, & x \downarrow 0, \nu \neq 0, \\ -\log x, & x \downarrow 0, \nu = 0, \end{cases} \]
\[ I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \quad x \to \infty, \]
\[ K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \to \infty. \]
A.6. **Inequalities.** Let $x > 0$, then following inequalities hold

(A.11) \[ I_{\mu}(x) < I_{\nu}(x), \quad 0 \leq \nu < \mu, \]

(A.12) \[ I_{\nu}(x) < I_{\nu-1}(x), \quad \nu \geq 1/2, \]

(A.13) \[ K_{\mu}(x) > K_{\nu}(x), \quad 0 \leq \nu < \mu, \]

(A.14) \[ K_{\nu}(x) \geq K_{\nu-1}(x), \quad \nu \geq 1/2. \]

We have equality in (A.14) if and only if $\nu = 1/2$.

(A.15) \[ \Gamma(\nu + 1) \left(\frac{2}{x}\right)^\nu I_{\nu}(x) < \cosh(x), \quad x > 0, \quad \nu > -\frac{1}{2}. \]

A.7. **Differentiation.**

(A.16) \[ \frac{d}{dx} \left( \frac{I_{\nu}(x)}{x^\nu} \right) = \frac{I_{\nu+1}(x)}{x^\nu}, \]

(A.17) \[ \frac{d}{dx} \left( \frac{K_{\nu}(x)}{x^\nu} \right) = -\frac{K_{\nu+1}(x)}{x^\nu}. \]

A.8. **Integration.**

(A.18) \[ \int_{-\infty}^{\infty} e^{\beta t} |t|^\nu K_{\nu}(|t|) dt = \frac{\sqrt{\pi} \Gamma(\nu + 1/2) 2^\nu}{(1 - \beta^2)^{\nu+1/2}}, \quad \nu > -1/2, \quad -1 < \beta < 1, \]

(A.19) \[ \int_{0}^{x} e^{-t^\nu} I_{\nu}(t) dt = \frac{e^{-x} x^{\nu+1}}{2\nu + 1} [I_{\nu}(x) + I_{\nu+1}(x)], \quad \nu > -1/2. \]