A new bijection on \(m\)-Dyck paths with application to random sampling

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March 29, 2016

Abstract

We present a new bijection between variants of \(m\)-Dyck paths (paths with steps in \(+1, -m\) starting and ending at height 0 and remaining at non-negative height), which generalizes a classical bijection between Dyck prefixes and pointed Łukasiewicz paths. As an application, we present a new random sampling procedure for \(m\)-Dyck paths with a linear time complexity and using a quasi-optimal number of random bits. This outperforms Devroye’s algorithm, which uses \(O(n \log n)\) random bits.

1 Introduction

Dyck paths—paths with steps in \(\{\searrow, \swarrow\}\) which start and end at height 0 and remain above the \(x\)-axis—are a cornerstone object in combinatorics. They are counted by the ubiquitous Catalan numbers and are in bijection with hundreds of other objects, among them binary plane trees (see [14] for a list).

What makes Dyck paths especially interesting is their rich combinatorics. For instance, the classical proof of the formula \(\frac{1}{n+1} \binom{2n}{n}\) for the Catalan numbers relies on the Cycle Lemma, which operates on Dyck paths. Many bijections exist between variants of Dyck paths (such as Dyck prefixes, which are not constrained to end at zero), for instance relying on the Catalan decomposition illustrated in Figure 1. Many examples can be found, among others, in [11, Chapter 9].

![Figure 1: A Dyck prefix of height \(h = 4\) with its Catalan decomposition \(q_0 u \cdots u q_h\), where \(q_0, \ldots, q_h\) are (possibly empty) Dyck paths.](image)

A natural generalization of Dyck paths are paths with steps in \(\{+1, -m\}\) for some integer \(m \geq 1\). These are called \(m\)-Dyck paths [8] and are in bijection with \(m + 1\)-ary trees. Their counting sequence is called the Fuss-Catalan numbers, equal to \(\frac{1}{m+1} \binom{(m+1)n}{n}\). The goal of this paper is to present a new bijection on \(m\)-Dyck paths, which corresponds in the case \(m = 1\) to a well-known bijection between variants of Dyck paths (Dyck prefixes and pointed Łukasiewicz paths), based on the Catalan decomposition.
As an application of our bijection, we present a random sampling procedure for $m$-Dyck paths. Random sampling—finding an algorithm that outputs an element of a given combinatorial class with a prescribed (usually uniform among the objects of a given size) random distribution, as efficiently as possible—is an important area of combinatorics with many theoretical and practical applications (for instance, it can lead to conjectures on the properties of large objects, or enable testing programs on random large inputs). The random sampling of Dyck-like paths (or equivalently, plane trees) has attracted a lot of attention. In the case of Dyck paths (or Motzkin paths, which allow → steps), efficient techniques include anticipated rejection [4, 5] and Rémy’s algorithm [13, 2]. In a more general case, including $m$-Dyck paths, we have Devroye’s algorithm [7] based on the Cycle Lemma.

The efficiency of a random sampling algorithm is measured in time, space and random bits. The random bit complexity (as opposed to, for instance, the number of calls to a uniform continuous random variable) is a realistic model of the randomness consumed by the algorithm, developed for instance in [10]. By that measure, Devroye’s algorithm uses $O(n \log n)$ random bits, since it involves drawing a uniform permutation. We show that our algorithm has a linear cost with all three respects. In fact, we show that it is asymptotically entropic in the sense that the number of random bits consumed is asymptotically equivalent to the entropy of the $m$-Dyck paths, which is an information-theoretical lower bound on the random bit complexity.

This article is organized as follows. In Section 2 we define the classes of paths between which our bijections operates. The folding bijection itself is presented in Section 3. In Section 4 we show our random sampling algorithm and prove that it has linear complexity. Finally, in Section 5 we study the limit distribution of the time complexity, which turns out to have unusual properties.

2 Definitions

Throughout the paper, let $m \geq 1$ be an integer. We consider paths with two kinds of steps, $u$ and $d$, with respective heights 1 and $-m$. The height of a path is defined as the sum of the heights of its steps. In the rest of the section, we consider a path $w$ with length $n$ and height $h$. We introduce the Euclidean divisions of $n$ and $h$ by $m + 1$:

$$n = (m + 1)n' + r; \quad h = (m + 1)h' + r.$$  

The remainders are the same since the heights of both steps $u$ and $d$ are congruent to 1 modulo $m + 1$. The height $h$ is therefore determined by the quotient $h'$, that we call the reduced height of $w$.

We say that the path $w$ is an $m$-Łukasiewicz path if every proper prefix $p$ of $w$ satisfies $h(p) \geq 0$ but the whole path satisfies $h(w) < 0$. The final height $h$ ranges between $-m$ and $-1$, as determined by $r$ (the reduced height $h'$ is $-1$). In particular, no $m$-Łukasiewicz path exists with a length divisible by $m + 1$.

Finally, we say that the path $w$ is an $m$-Dyck prefix if every prefix $p$ of $w$ satisfies $h(p) \geq 0$. If $r \neq 0$, we call decoration of $w$ a sequence $a_0, \ldots, a_{h'}$ of
integers satisfying:
\[ \begin{aligned}
1 & \leq a_i \leq m, & & i = 0, \ldots, h' - 1; \\
1 & \leq a_{h'} \leq r.
\end{aligned} \]

We call the path \( w \) thus equipped a *decorated path*. From the constraints, we see that the number of possible decorations of \( w \) is \( rm^{h'} \). In the case of Dyck prefixes (\( m = 1 \)), there is only one possible decoration for every Dyck prefix of odd length and zero for prefixes of even length.

### 3 The folding bijection

The goal of this section is to provide a bijection between the two objects defined above, decorated \( m \)-Dyck prefixes and pointed \( m \)-Łukasiewicz paths (i.e., with a distinguished step). We then provide an enumeration result as a first application. In the case \( m = 1 \), our bijection reduces to a well-known bijection between Dyck prefixes of odd length and pointed Łukasiewicz paths [11, Chapter 9].

Let \( w \) be an \( m \)-Dyck prefix equipped with a decoration \((a_0, \ldots, a_k)\). Write \( w \) in the form:
\[ w = p u q_0 \cdots u q_k, \]  
where, for \( i = 0, \ldots, k \), the path \( q_i \) is an \( m \)-Dyck prefix of height \( a_i - 1 \) (this is done by identifying first the factor \( u q_k \) as the smallest suffix of \( w \) of height \( a_k \) and working backwards to get the other factors). Let \( \phi(w) \) be the path:
\[ \phi(w) = pq_0 d \cdots q_k d, \]
pointed on the first step of \( q_0 d \). We call this operation *folding* the path \( w \).

To recover the path \( w \) from the pointed path \( \phi(w) \), let \( pq \) be the factorization of \( \phi(w) \) obtained by cutting before the pointed step. Let \( q_0 d \) be the smallest \( m \)-Łukasiewicz prefix of \( q \); repeat this process to get the factorization \( p q_0 d \cdots q_k d \). The path \( w \) is then recovered as \( p q_0 d \cdots q_k d \) and the decoration \( a_0, \ldots, a_k \) as the heights of the factors \( u q_i \). We call this operation *unfolding* the pointed path \( \phi(w) \). The folding operation is illustrated in Figure 2.

![Figure 2](image-url)

**Figure 2:** Left: a decorated 3-Dyck prefix of length 22 and height 10 (\( n' = 5 \), \( h' = 2 \) and \( r = 2 \)). Right: its image by the folding operator \( \phi \), a pointed 3-Lukasiewicz path.

**Theorem 1.** The folding operation \( \phi \) is a bijection from decorated \( m \)-Dyck prefixes to pointed \( m \)-Łukasiewicz paths.
Proof. Let \( w \) be an \( m \)-Dyck prefix of height \( h \) written as \( 1 \) and, for \( i = 0, \ldots, k \), let \( a_i \) be the height of \( u_q \) (for now, without any constraint on \( k \) or the \( a_i \)'s). The proof of the lemma hinges on the three following facts.

- For \( i = 0, \ldots, k \), the path \( q_i d \) is \( m \)-Łukasiewicz if and only if \( a_i \leq m \).
- Since \( \phi(w) \) is obtained from \( w \) by turning \( k + 1 \) up steps into down steps, we have \( h'(\phi(w)) = -1 \) if and only if \( h'(w) = k \).
- If the factors \( q_id \) are \( m \)-Łukasiewicz and if \( h'(\phi(w)) = -1 \), the folded path \( \phi(w) \) is \( m \)-Łukasiewicz if and only if the starting height of the factor \( q_k d \) is nonnegative. Since the final height is \( r - m - 1 \), this is equivalent to \( a_k \leq r \).

Together, these facts show that the folding of a decorated \( m \)-Dyck prefix is an \( m \)-Łukasiewicz path and, conversely, that unfolding a pointed \( m \)-Łukasiewicz path yields a decorated \( m \)-Dyck prefix. Moreover, the first fact shows that the folding and unfolding operations are inverse. \( \square \)

Proposition 2. The number \( L_n \) of \( m \)-Łukasiewicz paths of length \( n \) is:

\[
L_n = \frac{r}{n} \binom{n}{n'}.
\]

Let \( P_n(u) = \sum w u^{h'(w)} \) where the sum runs over all \( m \)-Dyck prefixes of length \( n \). We have:

\[
P_n(m) = \binom{n}{n'}.
\]

When \( m = 1 \), one recovers the well-known results on the enumeration of Dyck paths (the evaluation \( P_n(1) \) is simply the number of Dyck prefixes).

Proof. Let \( qd \) be an \( m \)-Łukasiewicz path of length \( n \). We transform it into the path \( u_q \), which is a path of height \( r \) staying strictly above its origin. This fits the conditions of the Cycle Lemma, which states that a proportion \( r/n \) of the paths with \( n' \) down steps have that property. This gives the formula for \( L_n \).

To enumerate \( m \)-Dyck prefixes, we use our bijection when \( r \neq 0 \). There are \( nL_n \) pointed \( m \)-Łukasiewicz paths and therefore \( nL_n \) decorated \( m \)-Dyck prefixes. Every prefix with reduced height \( h' \) has \( rm^{h'} \) possible decorations, which gives the result.

If \( r = 0 \), this breaks down because there are no \( m \)-Łukasiewicz paths. The \( m \)-Dyck prefixes of length \( n \) are then the prefixes of length \( n - 1 \) plus a single \( d \) or \( u \) step. This gives \( P_n(u) = (1 + u)P_{n-1}(u) \), which implies the formula. \( \square \)

4 Random sampling

In this section, we show how to use the unfolding bijection to build a very efficient random sampling algorithm for \( m \)-Dyck paths. In fact, our algorithm returns a uniformly distributed \( m \)-Łukasiewicz path; to draw an \( m \)-Dyck path of length \( n \), it suffices to draw an \( m \)-Łukasiewicz path of length \( n + 1 \) and delete the final \( d \) step. In the case \( m = 1 \), the algorithm already appeared in \footnote{2}, but the time complexity analysis is new.
Algorithm 1: Random $m$-Łukasiewicz path

| Line | Description |
|------|-------------|
| 1    | $w \leftarrow \epsilon$ |
| 2    | for $i = 1, \ldots, n$ do |
| 3    | $w \leftarrow (wu$ with probability $\frac{m}{m+1}$, $wd$ with probability $\frac{1}{m+1})$ |
| 4    | if $h(w) < 0$ then |
| 5    | draw uniformly a point in $w$ |
| 6    | $w \leftarrow \phi^{-1}(w)$ (forget the decoration) |
| 7    | draw uniformly a decoration of $w$ |
| 8    | $w \leftarrow \phi(w)$ (forget the point) |
| 9    | return $w$ |

Theorem 3. Algorithm 1 returns a uniformly distributed $m$-Łukasiewicz path of length $n$.

Before proving the theorem, we state our results on the complexity. We choose two models of complexity, which account for the overwhelming majority of the execution time in practice: the number $R_n$ of random bits drawn and the number $M_n$ of memory accesses. We show that both complexities are linear and derive their limit laws.

Let $\beta$ be the number of random bits necessary to draw a Bernoulli variable of parameter $\frac{1}{m+1}$. Moreover, let $S$ be an inhomogeneous Poisson process on $(0,1]$ with density $\lambda(x) = 1/2x$. Let $X$ be the random variable:

$$X = \sum_{x \in S} \text{Unif}[0,x],$$

where all uniforms are independent from each other and from $S$. Let $U \sim \text{Unif}[0,1]$ independent from $S$.

Theorem 4. The random variables $R_n$ and $M_n$ satisfy, as $n$ tends to infinity:

$$\frac{R_n}{n} \xrightarrow{d} \beta; \quad \frac{M_n}{n} \xrightarrow{d} 1 + X + U.$$  

The cost $\beta$ of drawing a Bernoulli variable is bounded from below by its entropy:

$$\beta \geq \eta, \quad \eta = -\frac{1}{m+1} \log_2 \left( \frac{1}{m+1} \right) - \frac{m}{m+1} \log_2 \left( \frac{m}{m+1} \right).$$

With Knuth and Yao’s algorithm [10], it is possible, with sufficient grouping, to reach a value of $\beta$ as close to $\eta$ as desired. Since an $m$-Dyck path of length $n$ has entropy asymptotically $\eta n$ (as can be seen from [3]), the number of random bits drawn by the algorithm can thus be made to be asymptotically optimal.

The random variable $X$ is studied in more detail in Section 5; there, we show that $E(X) = 1/4$ and $\text{Var}(X) = 1/12$. With the values $E(U) = 1/2$ and $\text{Var}(U) = 1/12$, this entails estimates for the expectation and variance of $M_n$:

$$E(M_n) \sim \frac{7n}{4}; \quad \text{Var}(M_n) \sim \frac{n^2}{6}.$$
The crucial point in the proof of both theorems is a loop invariant given in the lemma below. Consider a uniformly distributed decorated $m$-Dyck prefix; let $w$ be its underlying path. Since there are $rm^{h'(w)}$ possible decorations of $w$ and $r$ depends only on $n$, the path $w$ is distributed with a probability proportional to $m^{h'(w)}$. We denote by $\Pi_n$ that distribution on the $m$-Dyck prefixes.

**Lemma 5.** Let $0 \leq i \leq n$ and let $r = i \mod m + 1$. After $i$ iterations of the for loop, the path $w$ is distributed according to $\Pi_i$. Moreover, let $B_i$ be the event that the if branch is taken. The events $B_i$ are independent and satisfy:

$$\mathbb{P}(B_i) = \frac{r}{mi + r}.$$

**Proof.** We work by induction on $i$. First, consider the path $w$ after adding a random step. Since a $u$ step is $m$ times as likely to be drawn than a $d$ step, the probability of a given path $w$ to appear is proportional to $m^{h'(w)}$.

We now study the distribution of $w$ after the if branch, distinguishing whether or not it was taken.

- If the if branch is not taken, the path $w$ is an $m$-Dyck prefix, distributed according to $\Pi_{i+1}$.
- If the if branch is taken, the path $w$ is a uniformly distributed $m$-Łukasiewicz path since $h'(w) = -1$. After pointing and unfolding, it is therefore a uniformly distributed decorated $m$-Dyck prefix, which means that it is distributed like $\Pi_{i+1}$ after forgetting the point.

This shows that $w$ is distributed like $\Pi_{i+1}$. Moreover, the probability that the branch is taken and not taken are proportional to $L_n m^{-1}$ and $P_n(m)$, respectively (see Proposition 2), which gives the value of $\mathbb{P}(B_i)$. The independence comes from the fact that $w$ does not depend on whether the branch is taken.

**Proof of Theorem 3.** According to Lemma 5, after the execution of the for loop, the path $w$ is distributed like $\Pi_n$. Drawing a random decoration therefore yields a uniformly distributed decorated $m$-Dyck prefix. After folding, the result is thus a uniformly distributed $m$-Łukasiewicz path.

**Proof of Theorem 4.** Let us begin with the random bit cost. There are three places in the algorithm which contribute to it and we analyse them separately.

- **Drawing steps** (Line 3). This costs $n\beta$ random bits.
- **Randomly pointing the path** (Line 5). This costs $\mathcal{O}(\log i)$ if done at the $i$th iteration of the loop. According to Lemma 5, this happens with probability $\mathcal{O}(1/i)$. Summing for $i = 1, \ldots, n$, the average total cost is $\mathcal{O}(\log^2 n)$.
- **Randomly decorating the path** (Line 7). This costs $\mathcal{O}(h)$, where $h$ is the height of $w$. To estimate this, we use [3, Theorem 6]. Since the probability of a path $w$ is proportional to $m$ raised to the power of its number of up steps, that path is distributed like a random meander with drift zero (see the reference for details), which proves that the average height is $\mathcal{O}(\sqrt{n})$.

Overall, only the cost of Line 3 is significant. Let us move on to the cost in memory accesses, which also occur in three places.
• Writing steps (Line 3). This costs \( n \) memory accesses.

• Unfolding the path (Line 6). Unfolding a pointed path \( pq \) of length \( i \) only requires accessing the part \( q \). Since the point is uniformly drawn, the cost is uniformly distributed in \( \{1, \ldots, i\} \). Moreover, observe that the probability for this to occur given in Lemma 5, when averaged over \( m + 1 \) consecutive values of \( i \), is equivalent to \( 1/2i \).

Let \( S_n \) be the set of sizes \( i \) such that \( B_i \) holds. Since the \( B_i \)'s are independent, the set \( S_n/n \) converges to the Poisson point process \( S \). Therefore, the number of memory accesses divided by \( n \) tends in distribution to \( X \).

• Folding the path (Line 8). Folding a path into a pointed \( m \)-Łukasiewicz path \( pq \) only requires to access the part \( q \). Since that path is uniformly distributed, the length of \( q \) is uniformly distributed in \( \{1, \ldots, n\} \).

Summing all three contributions (which are independent) yields the result.

5 Properties of the limit distribution

This last section consists in the study of the limit law \( X \) involved in Theorem 4.

Theorem 6. The cumulants of the variable \( X \) are:

\[
\kappa_n(X) = \frac{1}{2n(n+1)}
\]

In particular, we have \( \mathbb{E}(X) = \kappa_1(X) = 1/4 \) and \( \mathbb{V}(X) = \kappa_2(X) = 1/12 \).

Moreover, the theorem can be reformulated in terms of the cumulant generating function of \( X \):

\[
K(z) = \int_0^z \frac{e^y - 1 - y}{2y^2} \, dy.
\] (5)

Proof. We compute the cumulant generating function of \( X \) from its definition, knowing that the moment generating function of \( \text{Unif}[0, x] \) is \((e^{xz} - 1)/(xz)\):

\[
K(z) = \int_0^1 \left( \frac{e^{xz} - 1}{xz} - 1 \right) \frac{dx}{2x},
\]

which is equivalent to (5) by a change of variables. The cumulants are extracted by Taylor expansion around \( z = 0 \).

Our final results concern the distribution function \( F(x) = \mathbb{P}(X \leq x) \) and tail distribution \( \bar{F}(x) = \mathbb{P}(X > x) \).

Theorem 7. The function \( F \) satisfies, for \( x > 0 \), the differential equation:

\[
F(x) + F'(x) + 2xF''(x) = F(x - 1).
\] (6)

For \( 0 \leq x \leq 1 \), its value is:

\[
F(x) = \sqrt{\frac{2e^{1-x} - 1}{\pi}} \sin \sqrt{2x},
\] (7)

where \( \gamma \) is Euler’s constant. As \( x \) tends to infinity, the tail distribution satisfies:

\[
\bar{F}(x) = x^{-x}(\log x)^{-2x}(e/2)^x + o(x).
\] (8)


Note that the equation (6), with the initial conditions (7), suffices to determine $F$. Indeed, working inductively on the intervals $[n, n+1]$, it can be seen as an inhomogeneous ordinary linear differential equation with initial conditions given by differentiability at $n$.

Moreover, we can deduce from that equation the singularity profile of $F$: since $F$ is not differentiable at 0, $F''$ is not differentiable at 1 due to the term $F(x-1)$. In the same way, $F$ has a singularity at every integer point $n$, where it is exactly $2n$ times continuously differentiable.

Finally, since $F$ is twice differentiable for $x > 0$, the distribution $X$ admits a density function $f = F'$, which shares similar properties.

\[ F_0(x) = \begin{cases} \sqrt{2e^{1-\gamma}/\pi} \sin \sqrt{2x} & \text{for } 0 < x < 1 \\ \sqrt{2e^{1-\gamma}/\pi} \sin \sqrt{2x} + 1 & \text{for } x \geq 1 \end{cases} \]

Figure 3: Left: a plot of the function $F_0(x) = \sqrt{2e^{1-\gamma}/\pi} \sin \sqrt{2x}$. Right: a plot of the distribution function $F(x)$ computed from the differential equation (6). The function $F(x)$ is equal to $F_0(x)$ until $x = 1$ and then deviates from it.

Proof. We prove these results using the Laplace transform of $F$, which is given by $L_F(z) = e^{K(-z)/z}$. From (5), we get:

\[-2z^2L_F'(z) = (e^{-z} - 3 + z)L_F(z),\]

which translates into (9) whenever $F$ is twice differentiable. We now put the Laplace transform into the form \[L_F(z) = \frac{\sqrt{e^{1-\gamma}}}{z^{3/2}} e^{-\frac{1}{z^2}} \exp \left( \int_z^\infty \frac{e^{-y} - 1}{2y^2} dy \right)\]

as $|z|$ tends to infinity in any direction. The main term transforms back into (7) (see [1, 29.3.78]). The inverse transform of the error term is supported for $x \geq 1$ (since it is $O(e^{-z})$ as $z$ tends to infinity) and is twice differentiable (since, multiplied by $z^2$, it is integrable on $i\mathbb{R}$). This proves (6) and (7).

Getting the asymptotics for large $x$ is trickier. We use a saddle point approximation, widely used in statistics to estimate tail densities and distributions [6, 12]. General saddle point asymptotics are described in detail in [9, Chapter VIII]. We compute the tail distribution using the formula, valid for $c > 0$:

\[ \tilde{F}(x) = \frac{1}{2\pi i} \int_{e^{-i\infty}}^{e^{i\infty}} \frac{e^{-xz+K(z)}}{z} \, dz. \]

\[ \gamma = \int_0^1 (e^{-y} - 1 + y)/y^2 dy + \int_1^\infty e^{-y}/y^2 dy = \gamma, \]

which can itself be derived by integrating by parts twice to get $-\int_e^\infty e^{-y} \log y dy$. That integral is also linked to the exponential integral function, in which the constant $\gamma$ famously plays a role (see for instance [4, Chapter 5]).
Write \( \xi(z) = K(z) - \log z \). We choose \( c \) to be the real point where the integrand is smallest (the saddle point), given by:

\[
\xi'(c) = \frac{e^c - 1 - 3c}{2c^2} = x.
\]

This entails:

\[
c = \log x + 2 \log \log x + \log 2 + o(1).
\]

Moreover, \( \xi(c) \) and all its derivatives are asymptotic to \( e^c/2c^2 \sim x \).

Let \( d = x^{-a} \) with \( 1/3 < a < 1/2 \) and let \( \tilde{F}_0(x) \) and \( \tilde{F}_1(x) \) be:

\[
\tilde{F}_0(x) = \frac{1}{2\pi i} \int_{c-id}^{c+id} e^{-xz + \xi(z)} \, dz;
\]

\[
\tilde{F}_1(x) = \tilde{F}(x) - \tilde{F}_0(x).
\]

We prove below that all the weight of the integral is concentrated in \( \tilde{F}_0(x) \) and that we have the saddle point approximation:

\[
\tilde{F}(x) \sim \tilde{F}_0(x) \sim \frac{e^{-xc + \xi(c)}}{\sqrt{2\pi x}}.
\]  

(9)

This evaluates to (8) (the denominator is subsumed into the error term).

To show that the approximation (9) is valid, we check that the conditions detailed in [9, Theorem VIII.3] are satisfied.

- First, we need to check that \( \tilde{F}_0(x) \) satisfies (9). To do that, we do a Taylor expansion of \( \xi(z) \) around the point \( c \):

\[
\xi(z) = \xi(c) + x(z - c) + \frac{\xi''(c)}{2} (z - c)^2 + \mathcal{O}(\xi'''(c)(z - c)^3).
\]

Since \( \xi''(c)d^3 \sim xd^3 \rightarrow 0 \), the error term tends in fact to zero, uniformly for all \( z \) such that \( |z - c| \leq d \). This entails that \( \tilde{F}_0(x) \) is approximated by the integral:

\[
\tilde{F}_0(x) \sim \frac{e^{-xc + \xi(c)}}{2\pi} \int_{-d}^{d} e^{-\frac{\xi''(c)t^2}{2}} \, dt.
\]

Since \( \xi''(c)d^2 \sim xd^2 \rightarrow \infty \), the integral can be completed to \( \mathbb{R} \), which gives a Gaussian integral evaluating to (9).

- Second, we need to check that the integral \( \tilde{F}_1(x) \) is negligible. Using the estimate (9) of \( \tilde{F}_0(x) \), we compute:

\[
\left| \frac{\tilde{F}_1(x)}{\tilde{F}_0(x)} \right| \leq \sqrt{\frac{x}{2\pi}} \int_{|t| > d} |e^{\xi(c+it) - \xi(c)}| \, dt
\]

\[
\leq \sqrt{\frac{x}{2\pi}} e^{\rho(x)} \int_{|t| > d} \left| \frac{c}{c + it} \right|^{3/2} \, dt
\]

\[
= \mathcal{O}(c\sqrt{x} e^{\rho(x)}),
\]

where:

\[
\rho(x) = \sup_{|t| > d} \Re \left( \int_{c}^{c+it} e^y - \frac{1}{2y^2} dy \right).
\]
This means that the ratio $\frac{F_1(x)}{F_0(x)}$ tends to zero as soon as $\rho(x)$ tends to $-\infty$ sufficiently fast (like a power of $x$). To show this, we set the contour of integration to $c \to 1 \to 1 + it \to c + it$, which follows the direction of steepest descent around the endpoints and avoids the singularity at zero.

The contribution of the interval $[1, 1+it]$ is bounded, as is the contribution of the term $1/y^2$. Grouping the other two intervals together, we find:

$$\rho(x) \sim \sup_{|t|>d} \int_1^c \left( \frac{e^{s+it}}{2(s+it)^2} - \frac{e^s}{2s^2} \right) ds$$

$$\leq \sup_{|t|>d} \int_1^c \left( \frac{e^s}{2(s^2+t^2)} - \frac{e^s}{2s^2} \right) ds$$

$$\sim \sup_{|t|>d} - \frac{e^c t^2}{2c^2(c^2+t^2)} = - \frac{e^c d^2}{2c^2(c^2+d^2)} \sim -x^{1-2a} \log^2 x.$$

This concludes the proof.

\[\Box\]

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