STEADY STATE SOLUTIONS OF FERROFLUID FLOW MODELS

YOUCEF AMIRAT*
Laboratoire de Mathématiques, CNRS UMR 6620
Université Blaise Pascal (Clermont-Ferrand 2), 63177 Aubière Cedex, France

KAMEL HAMDACHE
Léonard de Vinci, Pôle Universitaire. Research Center.
92916 Paris la Défense Cedex, France

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Abstract. We study two models of differential equations for the stationary flow of an incompressible viscous magnetic fluid subjected to an external magnetic field. The first model, called Rosensweig’s model, consists of the incompressible Navier-Stokes equations, the angular momentum equation, the magnetization equation of Bloch-Torrey type, and the magnetostatic equations. The second one, called Shliomis model, is obtained by assuming that the angular momentum is given in terms of the magnetic field, the magnetization field and the vorticity. It consists of the incompressible Navier-Stokes equation, the magnetization equation and the magnetostatic equations. We prove, for each of the differential systems posed in a bounded domain of \( \mathbb{R}^3 \) and equipped with boundary conditions, existence of weak solutions by using regularization techniques, linearization and the Schauder fixed point theorem.

1. Introduction. Ferrofluids are suspensions of magnetic nanoparticles in appropriate carrier liquids. These fluids have found a variety of applications in engineering: magnetic liquid seals, cooling and resonance damping for loudspeaker coils, printing with magnetic inks, rotary shaft seals, rotating shaft seals in vacuum chambers used in the semiconductor industry, see for instance [22]. In recent years many investigations were made on the possibility of future biomedical applications of magnetic fluids, such as magnetic separation, drugs or radioisotopes targeted by magnetic guidance, magnetic resonance imaging contrast enhancement, see for instance [13].

A number of works show that ferrofluids can be treated as homogeneous monophasic fluids, see [16, 19] and the references therein. Consider the flow of an incompressible and viscous, Newtonian ferrofluid occupying a domain \( D \subset \mathbb{R}^3 \), assumed to be regular, bounded and simply connected with boundary \( \Gamma \), under the action of an external magnetic field \( H_{ext} \). The magnetic field \( H_{ext} \) induces a demagnetizing field \( H \) and a magnetic induction \( B \) given by the law \( B = \mu_0 (H + M) \), where \( M \) is the magnetization inside \( D \) and \( \mu_0 \) is the magnetic permeability constant. The

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* Corresponding author: Youcef Amirat.
flow, in the stationary case, is described by the following system, called Rosensweig’s model [14, 15, 16]:

$$\begin{align*}
\text{(P)} & \quad \begin{cases}
\text{div} U = 0,
\rho(U \cdot \nabla)U - (\eta + \zeta)\Delta U + \nabla p = \mu_0(M \cdot \nabla)H + 2\zeta\text{curl}\omega + L,
\rho\kappa I(U \cdot \nabla)\omega - \eta'\Delta\omega - (\eta' + \lambda')\nabla\text{div}\omega = \mu_0M \times H + 2\zeta(\text{curl}U - 2\omega) + G,
(U \cdot \nabla)M - \sigma\Delta M + \frac{1}{2}(M - \chi_0H) = \omega \times M,
\text{curl} H = 0, \text{ div } (\mu_0(H + M)) = F,
U = 0, \omega = 0, M \cdot n = 0, \text{curl} M \times n = 0, H \cdot n = 0 \text{ on } \Gamma.
\end{cases}
\end{align*}$$

Here $U$ is the fluid velocity, $\omega$ is the angular momentum (or spin) of the fluid particles, $I$ is the tensor of inertia, $p$ is the pressure, and $\rho, \eta, \zeta, \mu_0, \kappa, \eta', \lambda', \sigma, \chi_0, \tau$ are physical positive constants. In (P), the differential equations are posed in $D$, (P)$_1$ is the continuity equation, (P)$_2$ is the linear momentum equation, (P)$_3$ is the angular momentum equation, (P)$_4$ is the magnetization equation of Bloch-Torrey type and $\sigma$ is a magnetic diffusion coefficient which carries the spins [11, 21], (P)$_5$ are the magnetostatic equations and (P)$_6$ are the boundary conditions. The term $\mu_0(M \cdot \nabla)H$ represents the Kelvin body force due to magnetization, and $\mu_0M \times H$ is the body torque density which causes the magnetic nanoparticles and surrounding fluid to spin. The quantities $L, G$ and $F$ are given source terms; we assume that $F$ satisfies the compatibility condition $\int_D F \, dx = 0$ and is linked with the applied magnetic field $H_{\text{ext}}$ by the formula $F = -\text{div} H_{\text{ext}}$.

We also consider the following model deduced from the Rosensweig model (P) by neglecting the inertia and friction effects so that the angular momentum is given (assuming $G = 0$) by the formula [18, 19]

$$\omega = \frac{\mu_0}{4\zeta} M \times H + \frac{1}{2} \text{curl} U.$$ 

With these simplifications, problem (P) becomes

$$\begin{align*}
\text{(S)} & \quad \begin{cases}
\text{div} U = 0,
\rho(U \cdot \nabla)U - \eta\Delta U + \nabla p = \mu_0(M \cdot \nabla)H + \frac{\mu_0}{2} \text{curl} (M \times H) + L,
(U \cdot \nabla)M - \sigma\Delta M + \frac{1}{2}(M - \chi_0H) = \frac{1}{2} \text{curl} U \times M - \beta M \times (M \times H),
\text{curl} H = 0, \text{ div } (\mu_0(H + M)) = F,
U = 0, M \cdot n = 0, \text{curl} M \times n = 0, H \cdot n = 0 \text{ on } \Gamma,
\end{cases}
\end{align*}$$

where $\beta = \frac{\mu_0}{4\zeta}$. Problem (S) is called Shliomis model. For notational convenience we omit in the sequel the parameter $\mu_0$ in the magnetostatic equation by changing $F$ in $\mu_0F$ so that we will write $\text{div} (H + M) = F$.

The aim of this paper is to study the existence of weak solutions to problems (P) and (S). The corresponding time dependent problems have been discussed in [2, 3] where it was proved global existence in time of weak solutions with finite energy.

2. Main results. In the sequel we use the following functional spaces. For $1 \leq q \leq \infty$ and $s \in \mathbb{R}$, let $L^q(D)$ and $W^{s,q}(D)$ be the usual Lebesgue and Sobolev spaces of scalar functions. We denote by $\| \cdot \|_q$ the norm in $L^q(D)$. If $q = 2$, $W^{s,2}(D)$ is denoted by $H^s(D)$, and $\| \cdot \|$ and $(\cdot, \cdot)$ denote the norm and the scalar product in the Hilbert space $L^2(D)$, respectively. If $\mathcal{V}$ is a Banach space we denote by $(\cdot, \cdot)_{\mathcal{V}' \times \mathcal{V}}$ (or simply $(\cdot, \cdot)$ if no confusion arises) the duality product where $\mathcal{V}'$ is the dual space
of $V$. For vector valued functions we use the notations $L^q(D)$, $W^{s,q}(D)$, $L^2(D)$, $H^s(D)$ and the notations of norms in $L^q(D)$ and the scalar product of $L^2(D)$ are unchanged.

We introduce the functional spaces in the theory of the Navier-Stokes equations, see [9, 10, 12, 20] for example:

$$D_s(D) = \{ v \in D(D, \mathbb{R}^3) : \text{div} \, v = 0 \text{ in } D \},$$

$$\mathcal{U} = \text{closure of } D_s(D) \text{ in } H^1(D), \quad \mathcal{U}_0 = \text{closure of } D_s(D) \text{ in } L^2(D).$$

It is well known that

$$\mathcal{U} = \{ v \in H^1_0(D) : \text{div} \, v = 0 \text{ in } D \},$$

$$\mathcal{U}_0 = \{ v \in L^2(D) : \text{div} \, v = 0 \text{ in } D, \, v \cdot n = 0 \text{ on } \Gamma \},$$

$\mathcal{U} \subset \mathcal{U}_0 \subset \mathcal{U}'$ where $\mathcal{U}'$ is the dual space of $\mathcal{U}$ when $\mathcal{U}_0$ is identified with its dual. Recall that $V \cdot n$ makes sense in $H^{-\frac{1}{2}}(\Gamma)$ when $V$ belongs to the space

$$H(\text{div}, D) = \{ V \in L^2(D) : \text{div} \, V \in L^2(D) \},$$

and we have the Stokes formula

$$\forall \varphi \in H^1(D), \quad \int_D V \cdot \nabla \varphi \, dx = -\int_D \varphi \text{div} \, V \, dx + \langle V \cdot n, \varphi \rangle_{\Gamma},$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ is the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^\frac{1}{2}(\Gamma)$. Similarly, if $V$ belongs to the space

$$H(\text{curl}, D) = \{ V \in L^2(D) : \text{curl} \, V \in L^2(D) \},$$

then $V$ has a tangential component $V \times n \in H^{-\frac{1}{2}}(\Gamma)$ and the following Green’s formula holds

$$\forall W \in H^1(D), \quad \int_D \text{curl} \, V \cdot W \, dx = \int_D V \cdot \text{curl} \, W \, dx + \langle V \times n, W \rangle_{\Gamma}. $$

For solving the Bloch-Torrey equation we introduce the Hilbert space

$$H^1_1(D) = \{ M \in H^1(D) : M \cdot n = 0 \text{ on } \Gamma \},$$

equipped with the norm of $H^1(D)$. It is well known, see [8] for example, that there exists a constant $C > 0$ such that

$$\| \nabla V \| \leq C \left( \| V \|^2 + \| \text{curl} \, V \|^2 + \| \text{div} \, V \|^2 \right)^{1/2}, \quad \forall V \in H^1_1(D).$$

Hence, the norm $\| V \|_{H^1_1(D)}$ is equivalent to the norm $\left( \| V \|^2 + \| \text{curl} \, V \|^2 + \| \text{div} \, V \|^2 \right)^{1/2}$.

For solving the magnetostatic equations we introduce the Hilbert spaces

$$L^2_2(D) = \{ \varphi \in L^2(D), \, (\varphi ; 1) = 0 \}, \quad H^1_2(D) = \{ \varphi \in H^1(D), \, (\varphi ; 1) = 0 \}. $$

Due to Poincaré-Wirtinger inequality, there exists a constant $C > 0$ such that

$$\| \varphi \| \leq C \| \nabla \varphi \|, \quad \forall \varphi \in H^1_2(D). \quad (1)$$

We say that $(U, \omega, M, H)$ is a weak solution of problem $(P)$ if $(U, \omega, M, H)$ belongs to $\mathcal{U} \times H^1_0(D) \times H^1_1(D) \times H^1_2(D)$ and the conditions (i)–(v) below are satisfied:

(i) The linear momentum equation holds weakly, i.e. for all $\Phi \in \mathcal{U}$,

$$\rho \langle (U \cdot \nabla)U; \Phi \rangle + \eta \langle \nabla U; \nabla \Phi \rangle + \zeta \langle \text{curl} \, U; \text{curl} \, \Phi \rangle \noindent = \mu_0 \langle (M \cdot \nabla)H; \Phi \rangle + 2\zeta \langle \text{curl} \, \omega; \Phi \rangle + \langle L; \Phi \rangle. $$
The angular momentum equation holds weakly, i.e. for all $\Psi \in H^1_0(D)$,

$$\rho \kappa ((U \cdot \nabla) \omega; \Psi) + \eta' (\nabla \omega; \nabla \Psi) + (\eta' + \lambda') (\text{div} \omega; \text{div} \Psi)$$

$$= \mu_0 (M \times H; \Psi) + 2\zeta (\text{curl} U - 2\omega; \Psi) + (G; \Psi).$$

The magnetization equation holds weakly, i.e. for all $\Lambda \in H^1_0(D)$,

$$\langle (U \cdot \nabla) M; \Lambda \rangle + \sigma (\text{curl} M; \text{curl} \Lambda) + \sigma (\text{div} M; \text{div} \Lambda) + \frac{1}{\tau} (M - \chi_0 H; \Lambda)$$

$$= (\omega \times M; \Lambda).$$

The magnetostatic equation holds weakly, i.e. for all $v \in H^1_0(D)$,

$$\langle \nabla \varphi; \nabla v \rangle + (M; \nabla v) = - (F; v).$$

Here and in the sequel the symbol $\langle ; \rangle$ denotes the duality product $\langle ; \rangle_{L^2(D), L^2(D)}$.

For $M, H \in H^1_0(D), (M \cdot \nabla) H$ belongs to $L^2(D)$, thanks to the Sobolev embedding $H^1(D) \hookrightarrow L^6(D)$ and the Hölder inequality, then for $\Phi \in \mathcal{U}$ we have $\langle (M \cdot \nabla) H; \Phi \rangle = \int_D (M \cdot \nabla) H \cdot \Phi \, dx$. Similar arguments allow to give a sense to the terms $\langle (U \cdot \nabla) U; \Phi \rangle, \langle (U \cdot \nabla) \omega; \Psi \rangle$ and $\langle (U \cdot \nabla) M; \Lambda \rangle$. Note also that for the variational equation (iv) we used the identity $-\Delta = \text{curl}^2 - \nabla \text{div}$.

Analogously, we say that $(U, M, H)$ is a weak solution of problem $(S)$ if $(U, M, H)$ belongs to $\mathcal{U} \times H^1_0(D) \times H^1_0(D)$ and satisfies the variational equations:

$$\rho \langle (U \cdot \nabla) U; \Phi \rangle + \eta \langle (U \cdot \nabla) \omega; \Psi \rangle = \mu_0 \langle (M \cdot \nabla) \Phi \rangle + \frac{\mu_0}{2} (M \times H; \text{curl} \Phi) + (L; \Phi),$$

$$\langle (U \cdot \nabla) M; \Psi \rangle + \sigma \langle \text{curl} M; \text{curl} \Psi \rangle + \sigma \langle \text{div} M; \text{div} \Psi \rangle + \frac{1}{\tau} \langle (M - \chi_0 H); \Psi \rangle$$

$$= \frac{1}{2} \langle \text{curl} U \times M; \Psi \rangle + \beta \langle (M \times (M \times H); \Psi \rangle,$$

$$\langle \nabla \varphi; \nabla v \rangle = - (M; \nabla v) - (F; v), \quad H = \nabla \varphi,$$

for all $\Phi \in \mathcal{U}, \Psi \in H^1_0(D)$ and $v \in H^1_0(D)$.

In the sequel $C$ indicates a generic positive constant that may depend on the domain $D$ and some physical constants. When the constant $C$ depends in addition on some other parameter $m$ we will write $C(m)$. Our first main result is concerned with the existence of weak solutions to problem $(P)$.

**Theorem 2.1.** Assume that $L, G \in L^2(D)$ and $F \in L^2(D)$. Then:

(i) Problem $(P)$ has a weak solution $(U, \omega, M, H)$ satisfying the energy estimates

$$\frac{\eta}{2} \||\nabla U\|^2 + \zeta ||\text{curl} U - 2\omega\|^2 + \frac{\eta'}{2} ||\nabla \omega\|^2 + (\lambda' + \eta') ||\text{div} \omega\|^2$$

$$+ \frac{\mu_0}{\tau} \left(\frac{1}{2} + \chi_0\right) ||H||^2 + \frac{\mu_0 \sigma}{2} ||\text{div} M||^2$$

$$\leq C \left(||F||^2 + ||L||^2 + ||G||^2\right), \quad (2)$$

$$\sigma ||\text{curl} M||^2 + \sigma ||\text{div} M||^2 + \frac{1}{\tau} ||M||^2 + \frac{\chi_0}{2\tau} ||H||^2 \leq C ||F||^2, \quad (3)$$

$$||H|| \leq ||M|| + ||F||, \quad ||H||_{H^1_0(D)} \leq C(||M|| + ||\text{div} M|| + ||F||), \quad (4)$$

where $C$ is positive constant that depends only on the domain $D$ and some physical constants.

(ii) Assume in addition that $F \in W^{1, \frac{3}{2}}(D)$. Let $p$ denote the pressure and $B$ the magnetic induction given by $B = M + H$. Then the weak solution $(U, \omega, M, H)$
has the regularity: $U, \omega, B \in W^{2, \frac{2}{3}}(D)$ and $p \in W^{1, \frac{3}{2}}(D)$. Moreover, we have the estimates

$$
\|U\|_{W^{2, \frac{2}{3}}(D)} + \|p\|_{W^{1, \frac{2}{3}}(D)} + \|\omega\|_{W^{2, \frac{2}{3}}(D)} \leq C \left( e + \sqrt{c} \right),
$$

$$
\|B\|_{W^{2, \frac{2}{3}}(D)} \leq C \left( \|F\|_\sqrt{c} + \|F\|_{W^{1, \frac{2}{3}}(D)} \right),
$$

where $e = \|F\|^2 + \|L\|^2 + \|G\|^2$ and $C$ is a positive constant that depends only on the domain $D$ and some physical constants.

In a second part of this work we discuss the Shliomis model. Our second main result is the following

**Theorem 2.2.** Assume that $L \in L^2(D)$ and $F \in L^2(D)$. Then:

(i) There is a number $r_0 > 0$, depending only on the domain $D$ and some physical constants, such that if $\|F\| \leq r_0$ then problem $(S)$ has a weak solution $(U, M, H)$ satisfying the energy estimates

$$
\frac{\eta}{2} |\nabla U|^2 + \frac{\mu_0 \sigma}{2} |\text{div} M|^2 + \frac{\mu_0}{\tau} \left( \frac{1}{2} + \chi_0 \right) \|H\|^2 + \mu_0 \beta |M \times H|^2 
\leq C \left( \|F\|^2 + \|L\|^2 \right),
$$

(ii) Assume in addition that $F \in W^{1, \frac{3}{2}}(D)$. Let $p$ denote the pressure and $B$ the magnetic induction given by $B = M + H$. Then the weak solution $(U, M, H)$ has the regularity: $U, B \in W^{2, \frac{2}{3}}(D)$ and $p \in W^{1, \frac{3}{2}}(D)$. Moreover we have the estimates

$$
\|U\|_{W^{2, \frac{2}{3}}(D)} + \|p\|_{W^{1, \frac{2}{3}}(D)} \leq C \left( \|F\|^2 + \|L\|^2 + \|L\|_\frac{2}{3} \right),
$$

$$
\|B\|_{W^{2, \frac{2}{3}}(D)} \leq C \left( \|F\|^2 + \|L\|^2 + \|L\|_\frac{2}{3} \right),
$$

where $C$ is a positive constant that depends only on the domain $D$ and some physical constants.

**Remark 1.** In Theorems 2.1 and 2.2, the constant $C$ depends on the domain $D$ in the sense that $C$ depends on the constants in the Poincaré inequality, Poincaré-Wirtinger inequality, the Sobolev embedding $H^1(D) \hookrightarrow L^6(D)$, the measure of $D$ and the constants of equivalence of the natural norm of $H^1(D)$ and the norm defined by $V \mapsto \left( \|V\|^2 + \|\text{curl} V\|^2 + \|\text{div} V\|^2 \right)^{1/2}$. The same remark holds for the number $r_0$ in Theorem 2.2.

**Remark 2.** The existence result in Theorem 2.2 is obtained under a smallness assumption on $\|F\|$. The condition is imposed to ensure the continuity of the operator involved in the application of the Schauder fixed point theorem (see Section 9.3). The constant $r_0$ is proportional to the parameter $\beta$. If we neglect the trilinear term in the Bloch-Torrey equation the results of Theorem 2.2 remain true without smallness condition on $F$.

The rest of the paper is devoted to the proof of Theorems 2.1 and 2.2 and is organized as follows. Sections 3–6 deal with the proof of Theorem 2.1 and Sections 7–10
3. Energy estimates for problem \((P)\). We assume in this section that the solutions \((U, \omega, M, H)\) of problem \((P)\) are smooth enough. Multiplying the linear momentum equation by \(U\) and the angular momentum equation by \(\omega\), integrating by parts, adding the results and using the identity

\[
\zeta \|\text{curl} \, U\|^2 - 2\zeta (\text{curl} \, \omega; U) - 2\zeta (\text{curl} \, U - 2\omega; \omega) = \zeta \|\text{curl} \, U - 2\omega\|^2,
\]

we arrive at the equality

\[
\eta \|\nabla \, U\|^2 + \zeta \|\text{curl} \, U - 2\omega\|^2 + \eta' \|\nabla \, \omega\|^2 + (\lambda' + \eta') \|\text{div} \, \omega\|^2
\]

\[
= \mu_0 \left( (M \cdot \nabla H) \, U \right) + \mu_0 \left( M \times H; \omega \right) + \left( L; U \right) + \left( G; \omega \right).
\]

Using (1) and the Cauchy-Schwarz and Poincaré inequalities we deduce that

\[
\eta \|\nabla \, U\|^2 + \zeta \|\text{curl} \, U - 2\omega\|^2 + \eta' \|\nabla \, \omega\|^2 + (\lambda' + \eta') \|\text{div} \, \omega\|^2
\]

\[
+ \frac{\mu_0}{\tau} \left( 1 + \chi_0 \right) \|H\|^2 + \mu_0 \sigma \|\text{div} \, M\|^2
\]

\[
= \mu_0 \sigma \left( \text{div} \, M; F \right) + \left( L; U \right) + \left( G; \omega \right) - \frac{\mu_0}{\tau} \left( F; \varphi \right).
\]

Using (1), the Cauchy-Schwarz and Poincaré inequalities we deduce that

\[
\frac{\eta}{2} \|\nabla \, U\|^2 + \zeta \|\text{curl} \, U - 2\omega\|^2 + \frac{\eta'}{2} \|\nabla \, \omega\|^2 + (\lambda' + \eta') \|\text{div} \, \omega\|^2
\]

\[
+ \frac{\mu_0}{\tau} \left( \frac{1}{2} + \chi_0 \right) \|H\|^2 + \frac{\mu_0 \sigma}{2} \|\text{div} \, M\|^2 \leq C \left( \|F\|^2 + \|L\|^2 + \|G\|^2 \right).
\]

Multiplying the magnetostatic equation by \(M\) and integrating by parts yields

\[
\sigma \|\text{curl} \, M\|^2 + \sigma \|\text{div} \, M\|^2 + \frac{1}{\tau} \|M\|^2 - \frac{\chi_0}{\tau} \left( H; M \right) = 0.
\]

Using (1) and (13) we deduce that

\[
\sigma \|\text{curl} \, M\|^2 + \sigma \|\text{div} \, M\|^2 + \frac{1}{\tau} \|M\|^2 + \frac{\chi_0}{2\tau} \|H\|^2 \leq C \|F\|^2.
\]
Using again (1) we deduce from (13) that
\[ \|H\| \leq \|M\| + C\|F\|. \]
Writing the magnetostatic equations in the form
\[ -\Delta \varphi = \text{div}\ M - F \quad \text{in} \quad D, \quad \nabla \varphi \cdot n = 0 \quad \text{on} \quad \Gamma, \quad (14) \]
with \( \text{div}\ M \in L^2(D) \), and applying the regularity results for the second order elliptic equations with homogeneous Neumann boundary condition, we get the estimate
\[ \|\varphi\|_{H^2(D)} \leq C \left( \|\varphi\|_{H^1_0(D)} + \|\text{div}\ M\| + \|F\| \right). \]
Therefore
\[ \|H\|_{H^1_0(D)} \leq C \left( \|M\| + \|\text{div}\ M\| + \|F\| \right). \quad (15) \]

4. The regularized problem \((P^{\varepsilon})\). Let \( \varepsilon > 0 \) be a fixed small parameter. We introduce the following regularized problem
\[
\begin{aligned}
\begin{cases}
\text{div} U = 0, \\
\varepsilon^2 \Delta^2 U + \rho(U \cdot \nabla)U - (\eta + \zeta)\Delta U + \nabla p = \mu_0 (M \cdot \nabla)H + 2\zeta \text{curl}\ \omega + L, \\
\varepsilon^2 \Delta \omega + \rho \kappa (U \cdot \nabla)\omega - \eta' \Delta \omega - (\eta' + \lambda') \nabla \text{div}\ \omega = \mu_0 M \times H + 2\zeta (\text{curl}\ U - 2\omega) + G, \\
(U \cdot \nabla)M - \sigma \Delta M + \frac{1}{\tau} (M - \chi_0 H) = \omega \times M, \\
\text{curl}\ H = 0, \ \text{div} (H + M) = F, \\
U = 0, \ \nabla U \cdot n = 0, \ \omega = 0, \ \nabla \omega \cdot n = 0 \quad \text{on} \quad \Gamma, \\
M \cdot n = 0, \ \text{curl} M \times n = 0, \ \ H \cdot n = 0 \quad \text{on} \quad \Gamma.
\end{cases}
\end{aligned}
\]

The energy estimates for problem \((P^{\varepsilon})\) can be derived as that for problem \((P)\). We obtain
\[
\begin{aligned}
\varepsilon^2 \| \Delta U \|^2 + \frac{\eta}{2} \| \nabla U \|^2 + \varepsilon^2 \| \Delta \omega \|^2 + \zeta \| \text{curl} U - 2\omega \|^2 + \frac{\eta'}{2} \| \nabla \omega \|^2 \\
+ (\lambda' + \eta') \| \text{div} \omega \|^2 + \frac{\mu_0}{\tau} \left( \frac{1}{2} + \chi_0 \right) \| H \|^2 + \frac{\mu_0 \sigma}{2} \| \text{div} M \|^2 \\
\leq C \left( \| F \|^2 + \| L \|^2 + \| G \|^2 \right),
\end{aligned}
\]
\[
\sigma \| \text{curl} M \|^2 + \sigma \| \text{div} M \|^2 + \frac{1}{\tau} \| M \|^2 + \frac{\chi_0}{2\tau} \| H \|^2 \leq C \| F \|^2, \quad (16)
\]
\[
\| H \| \leq \| M \| + C \| F \|, \quad \| H \|_{H^1_0(D)} \leq C (\| M \| + \| \text{div} M \| + \| F \|). \quad (17)
\]
The regularization allows to establish some new bounds. We deduce from (16) that
\[
U \in \mathcal{U}, \ \varepsilon \Delta U \in L^2(D), \ \omega \in \mathbb{H}^1_0(D) \quad \text{and} \quad \varepsilon \Delta \omega \in L^2(D). \quad (18)
\]
Using the regularity of the Laplace equation with homogeneous Dirichlet boundary condition, we obtain that
\[
U \in \mathbb{H}^2(D) \cap \mathcal{U} \quad \text{and} \quad \omega \in \mathbb{H}^2(D) \cap \mathbb{H}^1_0(D) \quad \text{with the following estimate}
\]
\[
\begin{aligned}
\varepsilon^2 \| U \|_{\mathbb{H}^2(D)}^2 + \| U \|_{\mathbb{H}^1_0(D)}^2 + \varepsilon^2 \| \omega \|_{\mathbb{H}^2(D)}^2 + \| \omega \|_{\mathbb{H}^1_0(D)}^2 \leq C \left( \| F \|^2 + \| L \|^2 + \| G \|^2 \right) \quad (19)
\end{aligned}
\]
Note that the constant \( C \) does not depend on \( \varepsilon \).

5. Solving problem \((P^{\varepsilon})\). We establish the following result.

**Theorem 5.1.** Assume that \( L, G \in \mathbb{H}^2(D) \) and \( F \in L^2(D) \). Then problem \((P^{\varepsilon})\) has a weak solution \((U^\varepsilon, \omega^\varepsilon, M^\varepsilon, H^\varepsilon)\) with \( U^\varepsilon \in \mathbb{H}^2(D) \cap \mathcal{U}, \ \omega^\varepsilon \in \mathbb{H}^2(D) \cap \mathbb{H}^1_0(D), \ M^\varepsilon \in \mathbb{H}^1_0(D), \ H^\varepsilon = \nabla \varphi^\varepsilon \in \mathbb{H}^1_0(D) \), satisfying inequalities (16)–(19).

We will solve problem \((P^{\varepsilon})\) by linearization and use of the Schauder fixed point theorem. The proof consists in four steps.
5.1. Solvability of the magnetization equation. Let \((V, w) \in U \times H^1_0(D)\) be fixed. We denote by \((M, H)\) the pair defined to be the solution of the linear differential system

\[
(P_1) \begin{cases}
(V \cdot \nabla)M - \sigma \Delta M + \frac{1}{\tau} (M - \chi_0 H) = w \times M, \\
H = \nabla \varphi, \quad \text{div} (H + M) = F, \\
M \cdot n = 0, \quad \text{curl} M \times n = 0, \quad H \cdot n = 0 \quad \text{on } \Gamma.
\end{cases}
\]

We introduce the operator \(T_1 : U \times H^1_0(D) \to H^1_0(D) \times H^1_0(D)\) by

\[
T_1(V, w) = (M, H),
\]

where \((M, H)\) is the solution of problem \((P_1)\).

Let us introduce the bilinear form \(A\) on \((H^1_0(D) \times H^1_0(D)) \times (H^1_0(D) \times H^1_0(D))\) by

\[
A((M, \varphi), (\Phi, \psi)) = a(M, \varphi, \Phi) + \frac{\chi_0}{\tau} b(M, \varphi, \psi),
\]

with

\[
a(M, \varphi, \Phi) = \sigma (\text{curl} M; \text{curl} \Phi) + \sigma (\text{div} M; \text{div} \Phi) + \langle (V \cdot \nabla)M; \Phi \rangle \\
b(M, \varphi, \psi) = (\nabla \varphi; \nabla \psi) + (M; \nabla \psi).
\]

We denote by \(l\) the linear form defined on \(H^1_0(D) \times H^1_0(D)\) by

\[
l(\Phi, \psi) = \langle (0, -F); (\Phi, \psi) \rangle = -\langle F; \psi \rangle.
\]

We look for a solution \((M, \varphi) \in H^1_0(D) \times H^1_0(D)\) of the variational equation

\[
A((M, \varphi), (\Phi, \psi)) = l(\Phi, \psi), \quad \forall (\Phi, \psi) \in H^1_0(D) \times H^1_0(D).
\] (20)

We establish the result.

**Proposition 1.** Problem \((P_1)\) has a unique weak solution \((M, H) \in H^1_0(D) \times H^1_0(D)\). Moreover, there is a positive constant \(C\), depending only on the domain \(D\) and some physical constants, such that

\[
\sigma \|\text{curl} M\|^2 + \sigma \|\text{div} M\|^2 + \frac{1}{\tau} \|M\|^2 + \frac{\chi_0}{2\tau} \|H\|^2 \leq C \|F\|^2,
\]

\[
\|H\| \leq \|M\| + C \|F\|,
\]

\[
\|H\|_{H^1_0(D)} \leq C(\|M\| + \|\text{div} M\| + \|F\|),
\]

\[
T_1 (U \times H^1_0(D)) \subset B(0, R),
\]

where \(R = C \|F\|\) and \(B(0, R)\) is the ball of \(H^1_0(D) \times H^1_0(D)\) with center \(0\) and radius \(R\).

**Proof.** Let \((M, \Phi) \in H^1_0(D) \times H^1_0(D)\) and \((\varphi, \psi) \in H^1_0(D) \times H^1_0(D)\). Using the Hölder inequality and the Sobolev embedding \(H^1(D) \hookrightarrow L^4(D)\) we have

\[
\|w \times M\| \leq \|w\|_4 \|M\|_4 \leq C \|w\|_U \|M\|_{H^1_0(D)},
\]

\[
\|\langle (V \cdot \nabla)M; \Phi \rangle\| \leq \|V\|_4 \|\nabla M\| \|\Phi\|_4 \leq C \|V\|_U \|M\|_{H^1_0(D)} \|\Phi\|_{H^1_0(D)}.
\]
We deduce that
\[
|a(M, \varphi, \Phi)| \leq \sigma \|\text{curl} M\| \|\text{curl} \Phi\| + \sigma \|\text{div} M\| \|\text{div} \Phi\| + C\|V\|_\mathcal{U} \|M\|_{\mathcal{H}_1^1(D)} \|\Phi\|_{\mathcal{H}_1^1(D)}
+ \frac{1}{\tau} \|M\|_\mathcal{U} \|\Phi\| + \frac{\lambda_0}{\tau} \|H\|_\mathcal{U} \|\Phi\| + C\|w\|_\mathcal{U} \|M\|_{\mathcal{H}_1^1(D)} \|\Phi\|,
\]

then
\[
|a(M, \varphi, \Phi)| \leq C(V, w) \left(\|M\|_{\mathcal{H}_1^1(D)} \|\Phi\|_{\mathcal{H}_1^1(D)} + \|\varphi\|_{\mathcal{H}_1^1(D)} \|\Phi\|\right).
\]

We also have, for any \((\varphi, \psi) \in H_2^1(D) \times H_1^1(D),\)
\[
|b(M, \varphi, \psi)| \leq \|\nabla \varphi\| \|\nabla \psi\| + \|M\| \|\nabla \psi\| \leq C \left(\|\varphi\|_{\mathcal{H}_1^1(D)} + \|M\|\right) \|\nabla \psi\|.
\]

Consequently,
\[
|A((M, \varphi), (\Phi, \psi))| \leq C(V, w) \|((M, \varphi))\|_{\mathcal{H}_1^1(D) \times H_2^1(D)} \|((\Phi, \psi))\|_{\mathcal{H}_1^1(D) \times H_1^1(D)},
\]
which shows that the bilinear form \(A\) is continuous. Moreover, for all \((M, \varphi) \in \mathcal{H}_1^1(D) \times H_1^1(D)\) we have
\[
A((M, \varphi), (M, \varphi)) \geq \min(\sigma, \frac{1}{\tau}) \|M\|_{\mathcal{H}_1^1(D)}^2 + \frac{\lambda_0}{\tau} \|\nabla \varphi\|^2
\geq C \|(M, \varphi))\|_{\mathcal{H}_1^1(D) \times H_1^1(D)}^2,
\]
with \(C > 0\), then \(A\) is coercive. According to (21) and (22) one can apply the Lax-Milgram lemma to conclude that equation (20) has a unique solution, i.e. problem \((P_1)\) has a unique weak solution \((M, \varphi) \in \mathcal{H}_1^1(D) \times H_1^1(D).\) Moreover, \(H = \nabla \varphi \in \mathcal{H}_1^1(D)\) is obtained as in (14)–(15).

Writing
\[
A((M, \varphi), (M, \varphi)) = l(M, \varphi) = -(F; \varphi),
\]
using (1), (22), the Cauchy-Schwarz and Young inequalities we deduce that the solution \((M, \varphi)\) satisfies the estimate
\[
\|(M, \varphi))\|_{\mathcal{H}_1^1(D) \times H_1^1(D)}^2 \leq C\|F\|^2.
\]

The previous estimate shows that \(T_1(\mathcal{U} \times \mathcal{H}_0^1(D)) \subset B(0, R).\)

**Proposition 2.** The map \(T_1\) is Lipschitz continuous. More precisely, there is a positive constant \(C\), depending only on the domain \(D\) and some physical constants, such that
\[
\|T_1(V_1, w_1) - T_1(V_2, w_2)\|_{\mathcal{H}_1^1(D) \times \mathcal{H}_0^1(D)} \leq C\|F\| \|(V_1, w_1) - (V_2, w_2)\|_{\mathcal{U} \times \mathcal{H}_0^1(D)},
\]
\forall(V_1, w_1), (V_2, w_2) \in \mathcal{U} \times \mathcal{H}_0^1(D).

**Proof.** Let us denote \(T_1(V_i, w_i) = (M_i, H_i),\) i.e. \((M_i, H_i)\) is the solution of problem \((P_1)\) associated with \((V_i, w_i), i = 1, 2.\) We set \(V = V_1 - V_2, w = w_1 - w_2, M = M_1 - M_2\) and \(H = H_1 - H_2.\) Then \((M, H)\) satisfies the equations
\[
(V_1 + \nabla M + \sigma(\text{curl}^2 M - \nabla \text{div} M) + \frac{1}{\tau}(M - \chi_0 H) - w_2 \times M
\]
\[
= -V \cdot \nabla M_2 + w \times M_1,
\]
\[
\text{div} (H + M) = 0,
\]
(23)
and homogeneous boundary conditions. Multiplying equation (23) by \( M \) and (24) by \( H \) and integrating by parts yields, respectively,

\[
\sigma \left( \| \text{curl} \ M \|^2 + \| \text{div} \ M \|^2 \right) + \frac{1}{\tau} \| M \|^2 - \frac{\chi_0}{\tau} (H; M) = - \langle (V \cdot \nabla)M_2; M \rangle + (w \times M_1; M),
\]

and

\[
(H; M) = -\|H\|^2.
\]

Using the Young inequality and Sobolev embedding theorems we have, for any \( \alpha > 0 \),

\[
|\langle (V \cdot \nabla)M_2; M \rangle| \leq \|V\|_4 \|\nabla M_2\|_4 \|M\|_4 \leq C \|V\|_{H_1} \|M_2\|_{H_1(D)} \|M\|_{H_1(D)}
\]

\[
\leq \alpha \|M\|^2_{H_1(D)} + C(\alpha) \|M_2\|^2_{H_1(D)} \|V\|^2_{H_1(D)},
\]

\[
|\langle w \times M_1; M \rangle| \leq \|w\|_3 \|M_1\|_3 \|M\|_3
\]

\[
\leq \alpha \|M\|^2_{H_1(D)} + C(\alpha) \|M_1\|^2_{H_1(D)} \|w\|^2_{H_1(D)}.
\]

Choosing \( \alpha > 0 \) small enough and applying Proposition 1 we deduce from (25)–(28) the inequalities

\[
C \|M\|^2_{H_1} + \frac{\chi_0}{\tau} \|H\|^2 \leq C \|F\|^2 \left( \|V\|^2_{H_1} + \|w\|^2_{H_1(D)} \right),
\]

\[
\|H\|^2_{H_1(D)} \leq C \|M\|^2_{H_1(D)} \leq C \|F\|^2 \left( \|V\|^2_{H_1} + \|w\|^2_{H_1(D)} \right).
\]

Proposition 2 is proved.

5.2. The linearized system for \((U, \omega)\). Let \((M, H)\) denote the solution in \( H_1(D) \times H_1(D) \) of problem \((P_1)\) associated with \((V, w)\). We introduce the pair \((U, \omega)\) defined to be the solution of the coupled linear system

\[
\begin{align*}
\text{div} \ U &= 0, \\
\varepsilon^2 \Delta U + \rho (V \cdot \nabla)U - (\eta + \zeta) \Delta U - 2\zeta \text{curl} \omega + \nabla p &= \mu_0 (M \cdot \nabla)H + L, \\
\varepsilon^2 \Delta \omega + \rho \kappa (V \cdot \nabla)\omega - (\eta' + \Lambda') \Delta \omega - (\eta'' + \Lambda) \text{curl} \omega &= -2 \zeta (\text{curl} U - 2\omega), \\
U &= 0, \quad \nabla U \cdot \mathbf{n} = 0, \quad \omega = 0, \quad \nabla \omega \cdot \mathbf{n} = 0 \quad \text{on} \ \Gamma.
\end{align*}
\]

\[(P_2)\]

We define the map \( T_2 : H_1(D) \times H_1(D) \to (H^2(D) \cap \mathcal{U}) \times (H^2(D) \cap H_0^1(D)) \) by

\[
T_2(M, H) = (U, \omega),
\]

where \((U, \omega)\) is the solution of problem \((P_2)\) and denote by \( T \) the operator \( T = T_2 \circ T_1 \).

Let

\[
\mathcal{V} = \{ U \in \mathcal{U}, \varepsilon \Delta U \in L^2(D) \}, \quad \mathcal{W} = \{ \omega \in H_0^1(D), \varepsilon \Delta \omega \in L^2(D) \},
\]

the Hilbert embeddings equipped with their natural norms. We set \( \mathcal{E} = \mathcal{V} \times \mathcal{W} \). The continuous embeddings \( \mathcal{E} \subset \mathcal{U} \times H_0^1(D) \subset \mathcal{E}' \) hold, where we identified \( \mathcal{U} \times H_0^1(D) \) with its dual \((\mathcal{U} \times H_0^1(D))'\).

We introduce the bilinear form \( \mathcal{B} \) on \( \mathcal{E} \times \mathcal{E} \) by

\[
\mathcal{B}((U, \omega), (\Phi, \Psi)) = \varepsilon^2 (\Delta U; \Delta \Phi) + \eta (\nabla U; \nabla \Phi) + \zeta (\text{curl} U; \text{curl} \Phi) - 2\zeta (\text{curl} \omega; \Phi) + \rho ((V \cdot \nabla)U; \Phi) + \varepsilon^2 (\Delta \omega; \Delta \Psi) + \eta' (\nabla \omega; \nabla \Psi) + (\eta' + \Lambda') (\text{div} \omega; \text{div} \Psi) - 2\zeta (\text{curl} U - 2\omega; \Psi) + \rho \kappa ((V \cdot \nabla)\omega; \Psi),
\]

and
and the linear form $J$ on $\mathcal{E}$ by

$$J(\Phi, \Psi) = \mu_0 (M \cdot \nabla) H; \Phi) + (L; \Phi) + \mu_0 (M \times H; \Psi) + (G; \Psi).$$

A variational formulation of problem $(P_2^s)$ is

$$\mathcal{B}((U, \omega), (\Phi, \Psi)) = J(\Phi, \Psi), \quad \forall (\Phi, \Psi) \in \mathcal{E}. \quad (29)$$

We establish the following result.

**Proposition 3.** Assume that $L, G \in L^2(D)$ and $F \in L^2_2(D)$. Then problem $(P_2^s)$ admits a unique solution $(U, \omega) \in (H^2(D) \cap \mathcal{U}) \times (H^2(D) \cap H^1_0(D))$ satisfying the estimate

$$\epsilon^2 \|\Delta U\|^2 + \eta \|\nabla U\|^2 + \epsilon^2 \|\Delta \omega\|^2 + \eta \|\nabla \omega\|^2 + (\eta' + \lambda') \|\text{div} \, \omega\|^2 + \eta \|\text{curl} \, U - 2 \omega\|^2$$

$$\leq C (\|F\|^2 + \|L\| + \|G\|)^2, \quad (30)$$

where $C$ is a positive constant, depending only on the domain $D$ and some physical constants.

**Proof.** Using (8) we easily verify that, for all $(U, \omega) \in \mathcal{E}$,

$$\mathcal{B}((U, \omega), (U, \omega)) = \epsilon^2 \|\Delta U\|^2 + \epsilon^2 \|\Delta \omega\|^2 + \eta \|\nabla U\|^2 + (\eta' + \lambda') \|\text{div} \, \omega\|^2 + \eta \|\text{curl} \, U - 2 \omega\|^2$$

$$\geq C \|(U, \omega)\|^2_{\mathcal{E}}. \quad (31)$$

Using the Hölder inequality and the Sobolev embedding $H^1(D) \hookrightarrow L^4(D)$ we have, for all $(U, \omega), (\Phi, \Psi) \in \mathcal{E}$,

$$|\mathcal{B}((U, \omega), (\Phi, \Psi))| \leq \epsilon^2 \|\Delta U\| \|\Delta \Phi\| + \epsilon^2 \|\Delta \omega\| \|\Delta \Psi\| + \eta \|\nabla U\| \|\nabla \Phi\|$$

$$+ \eta \|\text{curl} \, U\| \|\text{curl} \, \Phi\| + 2 \eta \|\text{curl} \, \omega\| \|\Phi\| + (\eta' + \lambda') \|\text{div} \, \omega\| \|\text{div} \, \Psi\|$$

$$+ 2 \eta \|\text{curl} \, U\| \|\Psi\| + 4 \eta \|\omega\| \|\Psi\| + (\eta' + \lambda') \|\text{div} \, \omega\| \|\text{div} \, \Psi\|$$

$$+ C \rho \|V\|_{\mathcal{U}} \|\nabla U\| \|\Phi\|_{\mathcal{U}} + C \rho \|V\|_{\mathcal{U}} \|\nabla \omega\| \|\Psi\|_{H^1_0(D)},$$

which implies

$$|\mathcal{B}((U, \omega), (\Phi, \Psi))| \leq C(V) \|(U, \omega)\|_{\mathcal{E}} \|(\Phi, \Psi)\|_{\mathcal{E}}. \quad (32)$$

Similar arguments imply that

$$|J(\Phi, \Psi)| \leq C \left( \|M\|_{H^1_0(D)} \|\nabla H\| \|\Phi\|_{\mathcal{U}} + \|M\|_{H^1_0(D)} \|H\| \|\Psi\|_{H^1_0(D)} \right)$$

$$+ \|L\| \|\Phi\| + \|G\| \|\Psi\|,$$

then, applying Proposition 1 we obtain

$$|J(\Phi, \Psi)| \leq C \left( \|F\|^2 + \|L\| + \|G\| \right) \|(\Phi, \Psi)\|_{\mathcal{U} \times H^1_0(D)}, \quad (33)$$

consequently, $J \in (\mathcal{U} \times H^1_0(D))^\prime = \mathcal{U} \times H^1_0(D)$ and

$$|J|_{\mathcal{U} \times H^1_0(D)} \leq C \left( \|F\|^2 + \|L\| + \|G\| \right).$$

According to (31) and (32) one can apply the Lax-Milgram lemma to obtain that there exists a unique solution $(U, \omega) \in \mathcal{E}$ of the variational equation (29) satisfying the estimate

$$\|(U, \omega)\|_{\mathcal{E}} \leq C \left( \|F\|^2 + \|L\| + \|G\| \right), \quad (34)$$

uniformly with respect to $\epsilon$. Writing $\mathcal{B}((U, \omega), (U, \omega)) = J(U, \omega)$, we deduce from the equality in (31), using (33) and (34), inequality (30). Proposition 3 is proved. \qed
5.3. Continuity and compactness of the operator $T$. We establish the following result.

**Proposition 4.** The operator $T$ is Lipschitz continuous. There is a positive constant $C$, depending only on the domain $D$ and some physical constants, such that

$$
\|T(V_1, w_1) - T(V_2, w_2)\|_{\mathcal{U} \times \mathbb{H}_0^1(D)} \leq C \|F\|_p \|V_1\|_{\mathcal{U} \times \mathbb{H}_0^1(D)} + \|w_1\|_{\mathcal{U} \times \mathbb{H}_0^1(D)},
$$

$$
T\left(B(0,R)\right) \text{ is compactly embedded in } \mathcal{U} \times \mathbb{H}_0^1(D),
$$

where $R = C (\|F\|^2 + \|L\| + \|G\|)$ and $B(0,R)$ is the ball of $\mathcal{U} \times \mathbb{H}_0^1(D)$ with center 0 and radius $R$.

**Proof.** Let $(M_i, H_i)$ be the solution of problem $(P_1)$ associated with $(V_i, w_i) \in \mathcal{U} \times \mathbb{H}_0^1(D)$ and let $(U_i, \omega_i)$ be the solution of $(P_2^i)$ associated with $(M_i, H_i)$, for $i = 1, 2$. We set $V = V_1 - V_2$, $w = w_1 - w_2$, $M = M_1 - M_2$, $H = H_1 - H_2$, $U = U_1 - U_2$, $\omega = \omega_1 - \omega_2$, and $p = p_1 - p_2$. Then $(U, \omega)$ satisfies the differential system

$$
\begin{align*}
\text{div} U &= 0, \\
\varepsilon^2 \Delta^2 U + \rho (V \cdot \nabla) U - (\eta + \zeta) \Delta U - 2 \zeta \text{curl} \omega + \nabla p &= \mu_0 (M \cdot \nabla) H + \mu_0 (M \cdot \nabla) H - \rho (V \cdot \nabla) U, \\
\varepsilon^2 \Delta^2 \omega + \rho k (V \cdot \nabla) \omega - \eta' \Delta \omega - (\eta' + \lambda') \nabla \text{div} \omega + 2 \zeta (\text{curl} U - 2 \omega) &= \mu_0 M M \cdot \nabla H - \rho k (V \cdot \nabla) \omega_2, \\
U &= 0, \quad \nabla U \cdot \mathbf{n} = 0, \quad \omega = 0, \quad \nabla \omega \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.
\end{align*}
$$

(35)

Using the Hölder inequality, the Young inequality and Propositions 1, 2 and 3, we have for all $\alpha > 0$,

$$
\begin{align*}
\mu_0 |\langle (M_1 \cdot \nabla) H; U \rangle| &\leq \mu_0 \|M_1\|_4 \|\nabla H\|_4 \|U\|_4 \leq C(\alpha) \|M_1\|_{\mathbb{H}_0^1(D)}^2 \|H\|_{\mathbb{H}_0^1(D)}^2 + \alpha \|U\|_{\mathcal{U}^2}^2, \\
\mu_0 |\langle (M \cdot \nabla) H_2; U \rangle| &\leq \mu_0 \|\nabla H_2\|_4 \|M_2\|_4 \|U\|_4 \leq C(\alpha) \|H_2\|_{\mathbb{H}_0^1(D)}^2 \|M_2\|_{\mathbb{H}_0^1(D)}^2 + \alpha \|U\|_{\mathcal{U}^2}^2, \\
\rho|\langle (V \cdot \nabla) U_2; U \rangle| &\leq C \|U_2\|_{\mathcal{U}^2} \|V\|_4 \|U\|_4 \leq C(\alpha) \|U_2\|_{\mathcal{U}^2}^2 \|V\|_{\mathcal{U}^2}^2 + \alpha \|U\|_{\mathcal{U}^2}^2, \\
\rho|\langle (V \cdot \nabla) \omega; U \rangle| &\leq C(\alpha) \|F\|^2 + \|L\| + \|G\| \|V\|_{\mathcal{U}^2}^2 + \alpha \|U\|_{\mathcal{U}^2}^2.
\end{align*}
$$

(36)

We also have

$$
\begin{align*}
\mu_0 \|M_1 \times H; \omega\| &\leq C(\alpha) \|F\|^2 \|V, \omega\|_{\mathcal{U} \times \mathbb{H}_0^1}^2 + \alpha \|\omega\|_{\mathbb{H}_0^1}^2, \\
\mu_0 \|M_2 \times H_2; \omega\| &\leq C(\alpha) \|F\|^2 \|V, \omega\|_{\mathcal{U} \times \mathbb{H}_0^1}^2 + \alpha \|\omega\|_{\mathbb{H}_0^1}^2, \\
\rho \|V \cdot \nabla \omega; \omega\| &\leq C(\alpha) \|F\|^2 + \|L\| + \|G\| \|V\|_{\mathcal{U}^2}^2 + \alpha \|\omega\|_{\mathbb{H}_0^1}^2.
\end{align*}
$$

(37)

Multiplying the second equation of (35) by $U$, the third one by $\omega$, integrating by parts, using (8), (36), (37), and choosing $\alpha$ small enough, we get

$$
\begin{align*}
\varepsilon^2 \|\Delta U\|^2 + \frac{\eta}{2} \|\nabla U\|^2 + \varepsilon^2 \|\Delta \omega\|^2 + \frac{\eta'}{2} \|\nabla \omega\|^2 + (\eta' + \lambda') \|\text{div} \omega\|^2 + \zeta \|\text{curl} U - 2 \omega\|^2 &\leq C \|F\|^2 + \|L\| + \|G\| \|V, \omega\|_{\mathcal{U} \times \mathbb{H}_0^1}^2.
\end{align*}
$$

This estimate implies that the first two claims in Proposition 4 hold.
To prove the third claim, consider a sequence \((V_n, w_n) \subset B(0, R)\). For each \(n\), let \((M_n, H_n) = T_1(V_n, w_n)\), i.e. \((M_n, H_n)\) is the weak solution of problem \((P_1)\) associated with \((V_n, w_n)\), and let \((U_n, \omega_n) = T_2(M_n, H_n)\) i.e. \((U_n, \omega_n)\) is the weak solution of problem \((P_2)\) associated with \((M_n, H_n)\). We have \((U_n, \omega_n) = T(V_n, w_n)\).

Clearly, estimate (30) holds with \(U\) replaced by \(U_n\) and \(\omega\) by \(\omega_n\). Since \((U_n)\) is bounded in \(U\) and \((\varepsilon \Delta U_n)\) is bounded in \(L^2(D)\), by using the regularity results of the problem \(-\varepsilon \Delta U_n \in L^2(D), U_n \in U\), we have that \((U_n)\) is bounded in \(H^2(D) \cap U\).

A similar result holds for the sequence \((\omega_n)\), i.e. \((\omega_n)\) is bounded in \(H^2(D) \cap \mathbb{R}_0(D)\). Then there exists a subsequence \((m)\) of \((n)\) such that \((V_m, w_m)\) and \((U_m, \omega_m) = T(V_m, w_m)\) satisfy

\[
(V_m, w_m) \rightharpoonup (V, w) \text{ weakly in } U \times H^1_0(D),
(V_m, w_m) \to (V, w) \text{ strongly in } L^q(D) \times L^q(D) \text{ for } 1 \leq q < 6,
(U_m, \omega_m) \to (U, \omega) \text{ strongly in } U \times H^1_0(D).
\]

Let us show that \(T(V, \omega) = (U, \omega)\). To prove that we have to pass to the limit in the nonlinear terms appearing in \((P_2)\) that is on the terms \((V_m, \nabla)U_m, (V_m, \nabla)\omega_m, (M_m, \nabla)H_m, M_m \times H_m\). By using the weak-strong convergence principle we have

\[
(V_m \cdot \nabla)U_m \to (V \cdot \nabla)U, \quad (V_m \cdot \nabla)\omega_m \to (V \cdot \nabla)\omega,
\]

in \(L^\infty(D)\) strong, then in \(L^4(D)\) strong by choosing \(q = 4\). Thanks to Proposition 1, the sequence \((M_m, H_m)\) is bounded in \(H^1(D) \times H^1(D)\). Then there is a subsequence, still labeled \((M_m, H_m)\), such that \((M_m, H_m) \to (M, H)\) weakly in \(H^1(D) \times H^1(D)\) and strongly in \(L^q(D) \times L^q(D)\) for \(1 \leq q < 6\). It results that

\[
M_m \times H_m \rightharpoonup M \times H, \quad (M_m \cdot \nabla)H_m \rightharpoonup (M \cdot \nabla)H,
\]

in \(L^4(D)\) weak. As a consequence we have that

\[
(U, \omega) = T(V, w).
\]

We conclude that \(T(B(0, R))\) is compactly embedded in \(U \times H^1_0(D)\). The proof of Proposition 4 is complete.

5.4. End of the proof of Theorem 5.1. Applying the Schauder fixed point theorem to the map \(T\), we conclude that there exists \((U^\varepsilon, \omega^\varepsilon) \in U \times H^1_0(D)\) satisfying \((U^\varepsilon, \omega^\varepsilon) = T(U^\varepsilon, \omega^\varepsilon)\). Setting \((M^\varepsilon, H^\varepsilon = \nabla \varphi^\varepsilon) = T_1(U^\varepsilon, \omega^\varepsilon) \in H^1_0(D) \times H^1_0(D)\), we obtain that \((U^\varepsilon, \omega^\varepsilon, M^\varepsilon, H^\varepsilon)\) is a solution to problem \((P^\varepsilon)\), satisfying the bounds (16)–(18). Moreover, \((U^\varepsilon, \omega^\varepsilon)\) belongs to \(H^2(D) \times H^2(D)\) and satisfies inequality (19).

6. End of the proof of Theorem 2.1.

6.1. Existence of a weak solution to problem \((P)\). Let \((U^\varepsilon, \omega^\varepsilon, M^\varepsilon, H^\varepsilon)\) be the solution to problem \((P^\varepsilon)\) given by Theorem 5.1. According to the regularity of \((U^\varepsilon, \omega^\varepsilon, M^\varepsilon, H^\varepsilon)\), the uniform estimates (16)–(19) stated formally in Section 4
hold:
\[ \varepsilon^2 \| \Delta U^\varepsilon \|^2 + \frac{\eta}{2} \| \nabla U^\varepsilon \|^2 + \varepsilon^2 \| \Delta \omega^\varepsilon \|^2 + \zeta \| \text{curl} U^\varepsilon - 2\omega^\varepsilon \|^2 + \frac{\eta'}{2} \| \nabla \omega^\varepsilon \|^2 \\
+ (\lambda' + \eta') \| \text{div} \omega^\varepsilon \|^2 + \mu \left( \frac{1}{2} + \chi_0 \right) \| H^\varepsilon \|^2 + \frac{\mu_0 \sigma}{2} \| \text{div} M^\varepsilon \|^2 \]
\[ 
\leq C \left( \| F \|^2 + \| L \|^2 + \| G \|^2 \right), \quad (38) \\
\| \sigma \| \text{curl} M^\varepsilon \|^2 + \sigma \| \text{div} M^\varepsilon \|^2 + \frac{1}{\tau} \| M^\varepsilon \|^2 + \frac{\chi_0}{2\tau} \| H^\varepsilon \|^2 \leq C \| F \|^2, \quad (39) \\
\| H^\varepsilon \| \leq \| M^\varepsilon \| + C \| F \|, \quad \| H^\varepsilon \|_{H_1(D)} \leq C \left( \| M^\varepsilon \| + \| \text{div} M^\varepsilon \| + \| F \| \right). \quad (40) \]
Here C is a positive constant that depends only on the domain D and some physical constants.

We construct a weak solution of problem \( \mathcal{P} \) by passing to the limit, as \( \varepsilon \to 0 \), on the sequence \( (U^\varepsilon, \omega^\varepsilon, M^\varepsilon, H^\varepsilon) \). The weak formulation of problem \( \mathcal{P}^\varepsilon \) consists in the following variational equations.

(i) For all \( \Phi \in \mathbb{H}^2(D) \cap U \),
\[ \varepsilon^2 (\Delta U^\varepsilon; \Delta \Phi) + \rho \langle (U^\varepsilon \cdot \nabla) U^\varepsilon; \Phi \rangle + \eta \langle (\nabla U^\varepsilon \cdot \nabla \Phi) + \zeta \langle \text{curl} U^\varepsilon \cdot \text{curl} \Phi \rangle = \mu_0 \langle (M^\varepsilon \cdot \nabla) H^\varepsilon; \Phi \rangle + 2\zeta \langle \text{curl} \omega^\varepsilon \cdot \Phi \rangle + (L; \Phi). \]
(ii) For all \( \Psi \in \mathbb{H}^2(D) \cap \mathbb{H}^1_0(D) \),
\[ \varepsilon^2 (\Delta \omega^\varepsilon; \Delta \Psi) + \rho \varepsilon \langle (U^\varepsilon \cdot \nabla) \omega^\varepsilon; \Phi \rangle + \eta' \langle (\nabla \omega^\varepsilon \cdot \nabla \Phi) + \eta' + \lambda' \rangle \langle \text{div} \omega^\varepsilon \cdot \text{div} \Psi \rangle = \mu_0 \langle (M^\varepsilon \times H^\varepsilon); \Psi \rangle + 2\zeta \langle \text{curl} U^\varepsilon - 2\omega^\varepsilon; \Psi \rangle + (G; \Psi). \]
(iii) The magnetization equation holds weakly, i.e. for all \( \Lambda \in \mathbb{H}_0^1(D) \),
\[ \langle (U^\varepsilon \cdot \nabla) M^\varepsilon; \Lambda \rangle + \sigma \langle \text{curl} M^\varepsilon; \text{curl} \Lambda \rangle + \sigma \langle \text{div} M^\varepsilon; \text{div} \Lambda \rangle \\
+ \frac{1}{\tau} \langle M^\varepsilon - \chi_0 H^\varepsilon; \Lambda \rangle = (\omega^\varepsilon \times M^\varepsilon; \Lambda). \]
(iv) The magnetostatic equation holds weakly, i.e. for all \( v \in H_0^1(D) \),
\[ \langle \nabla \phi^\varepsilon; \nabla v \rangle + \langle M^\varepsilon; \nabla v \rangle = - \langle F; v \rangle. \]
Using estimates (38)–(40) we deduce the following convergence results.

**Lemma 6.1.** There exists a subsequence still denoted \( (U^\varepsilon, \omega^\varepsilon, M^\varepsilon, H^\varepsilon = \nabla \phi^\varepsilon) \) such that, as \( \varepsilon \to 0 \),
\[ \varepsilon^2 \Delta U^\varepsilon, \varepsilon^2 \Delta \omega^\varepsilon \to 0 \text{ strongly in } L^2(D), \]
\[ U^\varepsilon, \omega^\varepsilon, M^\varepsilon, H^\varepsilon \to U, \omega, M, H \text{ weakly in } \mathbb{H}^1(D), \]
\[ U^\varepsilon, \omega^\varepsilon, M^\varepsilon, H^\varepsilon \to U, \omega, M, H \text{ strongly in } L^q(D) \text{ for } 1 \leq q < 6. \]
Moreover, \( U \in U, \omega \in \mathbb{H}_0^1(D), M \in \mathbb{H}_0^1(D) \) and \( H = \nabla \phi \in \mathbb{H}_0^1(D) \).

The convergence of the nonlinear terms is given by the following

**Corollary 1.** The following convergences, as \( \varepsilon \to 0 \), hold true:
\[ \langle (M^\varepsilon \cdot \nabla) H^\varepsilon; \Phi \rangle \to \langle (M \cdot \nabla) H; \Phi \rangle, \quad \langle (M^\varepsilon \times H^\varepsilon); \Psi \rangle \to \langle M \times H; \Psi \rangle, \]
\[ \langle U^\varepsilon \otimes U^\varepsilon; \nabla \Phi \rangle \to \langle U \otimes U; \nabla \Phi \rangle, \quad \langle U^\varepsilon \otimes \omega^\varepsilon; \nabla \Psi \rangle \to \langle U \otimes \omega; \nabla \Psi \rangle, \]
\[ \langle U^\varepsilon \otimes M^\varepsilon; \nabla \Lambda \rangle \to \langle U \otimes M; \nabla \Lambda \rangle, \quad \langle \omega^\varepsilon \times M^\varepsilon; \Lambda \rangle \to \langle \omega \times M; \Lambda \rangle, \]
for all \( \Phi \in U, \Psi \in \mathbb{H}_0^1(D), \Lambda \in \mathbb{H}_0^1(D) \).
Proof. We use the weak and strong convergences stated in Lemma 6.1. Due to
Lemma 6.1, $M^\varepsilon$ converges to $M$ in $L^4(D)$ strong and $\nabla H^\varepsilon$ converges to $\nabla H$ in
$L^2(D)$ weak. It results that $M^\varepsilon \cdot \nabla H^\varepsilon \to M \cdot \nabla H$ in $L^4(D)$ weak. Let $\Phi \in U$.
According to the Sobolev embedding $H^1(D) \hookrightarrow L^4(D)$, we have that $\Phi \in L^4(D)$. Then $(\langle M^\varepsilon \cdot \nabla H^\varepsilon; \Phi \rangle) \to \langle (M \cdot \nabla)H; \Phi \rangle$. We also have $(M^\varepsilon \times H^\varepsilon; \Psi) \to (M \times H; \Psi)$, for all $\Psi \in H^1_0(D)$. By similar arguments we justify the convergence of the other nonlinear terms.

Now we can pass to the limit in the weak formulation of problem $(\mathcal{P}^\varepsilon)$ and obtain that the weak limit $(U, \omega, M, H)$ of $(U^\varepsilon, \omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ is a weak solution of problem $(\mathcal{P})$. Passing to the lower limit in $(38)$–$(40)$, we obtain that $(U, \omega, M, H)$ satisfies estimates $(2)$–(4).

6.2. Regularity of the weak solution. (i) Regularity of $U$ and $p$. The functions $U$ and $p$ satisfy
\[
\begin{cases}
\text{div } U = 0, \\
- (\eta + \zeta) \Delta U + \nabla p = F^1_r, \\
U = 0 \quad \text{on } \Gamma,
\end{cases}
\]
with
\[F^1_r = -\rho(U \cdot \nabla)U + \mu_0(M \cdot \nabla)H + 2\zeta \text{curl } \omega + L.\]
According to the Sobolev embedding $H^1(D) \hookrightarrow L^6(D)$ and the Hölder inequality, the function $F^1_r$ belongs to $L^2(\Omega)$. Applying a classical regularity result for Stokes equations with homogeneous Dirichlet boundary condition, see [6, 10], we get that $U \in W^{2, \frac{2}{3}}(D)$, $p \in W^{1, \frac{2}{3}}(D)$, and using $(2)$–(4) we deduce the estimate
\[\|U\|_{W^{2, \frac{2}{3}}(D)} + \|p\|_{W^{1, \frac{2}{3}}(D)} \leq C \|F^1_r\|^{\frac{1}{2}} \leq C (e + \sqrt{\varepsilon}),\]
with $e = \|F\|^2 + \|L\|^2 + \|G\|^2$.

(ii) Regularity of $\omega$. The functions $\omega$ satisfies the Lamé system
\[
\begin{cases}
- \eta' \Delta \omega - (\eta' + \lambda') \nabla \text{div } \omega = F^2_r, \\
\omega = 0 \quad \text{on } \Gamma,
\end{cases}
\]
with
\[F^2_r = -\rho \kappa I(U \cdot \nabla)\omega + \mu_0 M \times H + 2\zeta (\text{curl } U - 2\omega) + G.\]
Clearly, $F^2_r$ belongs to $L^2(\Omega)$. Applying a classical regularity theory for elliptic systems, see [1], and [7, 17] for domains less regular, we get that $\omega \in W^{2, \frac{2}{3}}(D)$, and using $(2)$, $(3)$ we deduce the estimate
\[\|\omega\|_{W^{2, \frac{2}{3}}(D)} \leq C \|F^2_r\|^{\frac{1}{2}} \leq C (e + \sqrt{\varepsilon}),\]
with $e = \|F\|^2 + \|L\|^2 + \|G\|^2$.

(iii) Regularity of the magnetic induction. Consider the magnetic induction $B$ defined by $B = (M + H)$. Writing
\[-\sigma \Delta B = -\sigma \Delta M - \sigma \nabla \text{div } H,
\]
we see that $B$ satisfies the Stokes system
\[
\begin{cases}
\text{div } B = F, \\
- \sigma \Delta B + \nabla \pi = G_r, \\
B \cdot n = 0, \quad \text{curl } B \times n = 0 \quad \text{on } \Gamma,
\end{cases}
\]
with \( \pi = -\sigma \text{div} H \) and
\[
\mathcal{G}_r = -(U \cdot \nabla)M - \frac{1}{\tau}(M - \chi_0 H) + \omega \times M.
\]
Clearly, \( \mathcal{G}_r \in L^2(D) \) and the compatibility condition \( (B \cdot n, 1)_\Gamma = \int_{\Gamma} F \, dx \) is satisfied since \( B \cdot n = 0 \) on \( \Gamma \) and \( F \in H^1_0(D) \). Applying a regularity result for Stokes equations with vorticity boundary conditions, see [4], we get that \( B \in W^{2, \frac{3}{2}}(D) \), and using (2), (3) we deduce the estimate
\[
\|B\|_{W^{2, \frac{3}{2}}(D)} \leq C \left( \|\mathcal{G}_r\|_{L^2} + \|F\|_{W^{1, \frac{3}{2}}(D)} \right) \leq C \left( \|F\| \sqrt{e} + \|F\|_{W^{1, \frac{3}{2}}(D)} \right),
\]
with \( e = \|F\|^2 + \|L\|^2 + \|G\|^2 \). Regularity results for Stokes equations with vorticity boundary conditions, in the hibertian setting, are given in [5]. This ends the proof of Theorem 2.1.

7. Energy estimates for problem (S). Let us now consider the Shliomis model (S). The corresponding energy estimates can be derived by mimicking the calculations done for system (P). We assume that the solutions \( (U, M, H) \) of problem (S) are smooth enough. Multiplying the linear momentum equation by \( U \) and using relation (10) we obtain
\[
\eta \|\nabla U\|^2 = -\mu_0 \langle (U \cdot \nabla)M; H \rangle + \frac{\mu_0}{2} (M \times H; \text{curl}U) + (L; U). \tag{41}
\]
Using the identity \(-\Delta = \text{curl}^2 - \nabla \text{div}\), we rewrite the magnetization equation as
\[
-(U \cdot \nabla)M = \sigma \text{curl}^2 M - \sigma \nabla \text{div} M + \frac{1}{\tau}(M - \chi_0 H)
\]
\[
= \frac{1}{2} \text{curl}U \times M + \beta M \times (M \times H).
\]
Multiplying the previous equation by \( \mu_0 H \), using the equation \( \text{curl}H = 0 \) and integrating by parts yields
\[
-\mu_0 \langle (U \cdot \nabla)M; H \rangle = \mu_0 \sigma \langle \text{div} M; \text{div} H \rangle + \frac{\mu_0}{\tau} \langle M; H \rangle - \frac{\mu_0 \chi_0}{\tau} \|H\|^2
\]
\[
- \frac{\mu_0}{2} \langle \text{curl}U \times M; H \rangle - \mu_0 \beta \|M \times H\|^2. \tag{42}
\]
Multiplying the magnetostatic equation by \( \mu_0 \sigma \text{div} M \) yields
\[
\mu_0 \sigma \langle \text{div} M; \text{div} H \rangle = -\mu_0 \sigma \|\text{div} M\|^2 + \mu_0 \sigma \langle F; \text{div} M \rangle.
\]
Multiplying the magnetostatic equation by \( \frac{\mu_0}{\tau} \varphi \) and integrating by parts yields
\[
\frac{\mu_0}{\tau} \langle M; H \rangle = -\frac{\mu_0}{\tau} \|H\|^2 - \frac{\mu_0}{\tau} \langle F; \varphi \rangle. \tag{43}
\]
Then we deduce from (42) that
\[
-\mu_0 \langle (U \cdot \nabla)M; H \rangle = -\mu_0 \sigma \|\text{div} M\|^2 + \mu_0 \sigma \langle F; \text{div} M \rangle - \frac{\mu_0}{\tau} \|H\|^2 - \frac{\mu_0}{\tau} \langle F; \varphi \rangle
\]
\[
- \frac{\mu_0 \chi_0}{\tau} \|H\|^2 - \frac{\mu_0}{2} \langle \text{curl}U \times M; H \rangle - \mu_0 \beta \|M \times H\|^2. \tag{44}
\]
Inserting (44) in (41) we obtain
\[
\eta \|\nabla U\|^2 + \mu_0 \sigma \|\text{div} M\|^2 + \frac{\mu_0}{\tau} (1 + \chi_0) \|H\|^2 + \mu_0 \beta \|M \times H\|^2
\]
\[
= \mu_0 \sigma \langle F; \text{div} M \rangle - \frac{\mu_0}{\tau} \langle F; \varphi \rangle + (L; U). \tag{45}
\]
Multiplying the magnetization equation by $M$ and integrating by parts yields

$$
\sigma \|\text{curl } M\|^2 + \sigma \|\text{div } M\|^2 + \frac{1}{2\tau} \|M\|^2 + \frac{\chi_0}{\tau} \|H\|^2 = -\frac{\chi_0}{\tau} (F; \varphi).
$$

(46)

From (45) and (46), by using inequality (1) and the Young and Cauchy-Schwarz inequalities we deduce the estimates

$$
\eta \|\nabla U\|^2 + \frac{\mu_0 \sigma}{2} \|\text{div } M\|^2 + \frac{\mu_0}{\tau} \left(\frac{1}{2} + \chi_0\right) \|H\|^2 + \mu_0 \beta \|M \times H\|^2
\leq C(\|F\|^2 + \|L\|^2),
$$

(47)

\[ \sigma \|\text{curl } M\|^2 + \sigma \|\text{div } M\|^2 + \frac{1}{2\tau} \|M\|^2 + \frac{\chi_0}{2\tau} \|H\|^2 \leq C\|F\|^2. \]

As for problem (P), we deduce from the magnetostatic equations the estimates

$$
\|H\| \leq \|M\| + C\|F\|,
$$

\[ \|H\|_{H^1(D)} \leq C(\|M\| + \|\text{div } M\| + \|F\|). \]

8. The regularized problem \((S^\varepsilon)\). The small parameter \(\varepsilon > 0\) being fixed, we consider the following regularization of problem \((S)\):

\[
\begin{aligned}
\text{div } U &= 0, \\
\varepsilon^2 \Delta^2 U + \rho(U \cdot \nabla)U - \eta \Delta U + \nabla p &= \mu_0 (M \cdot \nabla)H + \mu_0 \text{curl } (M \times H) + L, \\
U &= 0, \quad \nabla U \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \\
(M \cdot \nabla)M - \sigma \Delta M + \frac{1}{\tau} (M - \chi_0 H) &= \frac{1}{2} \text{curl } M \times M - \beta M \times (M \times H), \\
\text{curl } M \times \mathbf{n} &= 0, \quad M \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \\
\text{curl } H &= 0, \quad \text{div } (H + M) = F, \\
H \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma.
\end{aligned}
\]

\((S^\varepsilon)\)

The energy estimates associated with system \((S^\varepsilon)\) can be derived as in Section 7 for system \((S)\). Using the equality

$$
\mu_0 \langle (M \cdot \nabla)H; U \rangle = -\mu_0 \langle (U \cdot \nabla)M; H \rangle,
$$

we have that

$$
\varepsilon^2 \|\Delta U\|^2 + \eta \|\nabla U\|^2 = -\mu_0 \langle (U \cdot \nabla)M; H \rangle + \frac{\mu_0}{2} (M \times H; \text{curl } U) + (L; U).
$$

(48)

The Bloch-Torrey equation can be written in the form

$$
-(U \cdot \nabla)M = \sigma \text{curl } M - \sigma \nabla \text{div } M + \frac{1}{\tau} (M - \chi_0 H) - \frac{1}{2} \text{curl } U \times M
+ \beta M \times (M \times H).
$$

(49)

Using (43), (49) and the equation \(\text{div } (H + M) = F\), we deduce from (48) the equality

$$
\varepsilon^2 \|\Delta U\|^2 + \eta \|\nabla U\|^2 + \mu_0 \sigma \|\text{div } M\|^2 + \frac{\mu_0}{\tau} (1 + \chi_0) \|H\|^2 + \mu_0 \beta \|M \times H\|^2
= \mu_0 \sigma (\text{div } M; F) - \frac{\mu_0}{\tau} (F; \varphi) + (L; U).
$$

Using inequality (1) and the Young and Cauchy-Schwarz inequalities we deduce that

$$
\varepsilon^2 \|\Delta U\|^2 + \frac{\eta}{2} \|\nabla U\|^2 + \frac{\mu_0 \sigma}{2} \|\text{div } M\|^2 + \frac{\mu_0}{\tau} \left(\frac{1}{2} + \chi_0\right) \|H\|^2 + \mu_0 \beta \|M \times H\|^2
\leq C \left(\|F\|^2 + \|L\|^2\right).
$$

(50)
Multiplying the magnetization equation by $M$ and integrating by parts yields
\[
\sigma \|\text{curl } M\|^2 + \frac{1}{\tau} \|M\|^2 + \frac{\chi_0}{\tau} \|H\|^2 = -\frac{\chi_0}{\tau}(F; \varphi).
\]

Using again inequality (1) and the Young and Cauchy-Schwarz inequalities we deduce that
\[
\sigma \left( \|\text{curl } M\|^2 + \|\text{div } M\|^2 \right) + \frac{1}{\tau} \|M\|^2 + \frac{\chi_0}{2\tau} \|H\|^2 \leq C\|F\|^2. \tag{51}
\]

As for problem ($S_1$), the estimates (47) hold true.

9. Solving problem ($S^\varepsilon$). We will solve problem ($S^\varepsilon$) by linearization and use of the Schauder fixed point theorem. Let $V \in \mathcal{U}$ be fixed. We denote by $(M, H)$ the pair defined to be a solution of the nonlinear differential system
\[
(S_1) \left\{ \begin{array}{l}
(V \cdot \nabla)M - \sigma \Delta M + \frac{1}{2}(M - \chi_0 H) = \frac{1}{2} \text{curl } V \times M - \beta M \times (M \times H), \\
curl M \times n = 0, \quad M \cdot n = 0 \quad \text{on } \Gamma, \\
H = \nabla \varphi_1, \quad \text{div } (H + M) = F, \\
H \cdot n = 0 \quad \text{on } \Gamma.
\end{array} \right.
\]

With any $M \in H^1_t(D)$ we associate $H \in H^1_t(D)$ by $H = H(M) + H(F)$ where $H(M) = \nabla \varphi_1, \quad H(F) = \nabla \varphi_2$, and $\varphi_1, \varphi_2$ are defined as the weak solutions in $H^1_t(D)$ of the following problems
\[
\Delta \varphi_1 = -\text{div } M, \quad \nabla \varphi_1 \cdot n = 0 \quad \text{on } \Gamma, \\
\Delta \varphi_2 = F, \quad \nabla \varphi_2 \cdot n = 0 \quad \text{on } \Gamma.
\]

Clearly,
\[
\|H(M)\| \leq \|M\|, \quad \|H(F)\| \leq C\|F\|, \tag{52}
\]
where $C$ denotes the Poincaré-Wirtinger constant, see (1). We introduce the operator $S_1 : \mathcal{U} \to H^1_t(D) \times H^1_t(D)$ by
\[
S_1(V) = (M, H), \tag{53}
\]
where $(M, H)$ is the solution of problem ($S_1$). Then we introduce the function $U$ defined to be the solution of the linear differential system
\[
(S_2^\varepsilon) \left\{ \begin{array}{l}
\text{div } U = 0, \\
\varepsilon^2 \Delta^2 U + \rho(V \cdot \nabla)U - \eta \Delta U + \nabla p = \mu_0(M \cdot \nabla)H + \mu_0 \text{curl } (M \times H) + L, \\
U = 0, \quad \nabla U \cdot n = 0 \quad \text{on } \Gamma.
\end{array} \right.
\]

We define the map $S_2 : H^1_t(D) \times H^1_t(D) \to \mathcal{U}$ by
\[
S_2(M, H) = U, \tag{54}
\]
where $U$ is the solution of problem ($S_2^\varepsilon$) and denote by $S$ the map $S = S_2oS_1$. Let us recall the basic estimates satisfied by $H$ and deduced from the magnetostatic equations:
\[
\|H\| \leq \|M\| + C\|F\|, \quad \|H\|_{H^1_t(D)} \leq C(\|M\|_{H^1_t(D)} + \|F\|). \tag{55}
\]
9.1. Solving problem \((S_1)\). Let \(N \in \mathbb{H}^1(D)\) and denote \(K = H(N) + H(F)\). It results from inequality (55) that \(K \in \mathbb{H}^1(D)\). We first consider the linear differential system

\[
\begin{aligned}
(V \cdot \nabla)M - \sigma \Delta M + \frac{1}{\tau}(M - \chi_0 H) &= \frac{1}{2} \text{curl} V \times M - \beta M \times (N \times K), \\
\text{curl} M \times n &= 0, \quad M \cdot n = 0 \quad \text{on} \; \Gamma, \\
H &= \nabla \varphi, \quad \text{div} \,(H + M) = F, \\
H \cdot n &= 0 \quad \text{on} \; \Gamma.
\end{aligned}
\]

(S\(_1\),1)

Solving problem \((S_1,1)\) consists in finding \(M \in \mathbb{H}^1_\text{curl}(D)\) satisfying

\[
\begin{aligned}
(V \cdot \nabla)M - \sigma \Delta M + \frac{1}{\tau}(M - \chi_0 H(M)) - \frac{\chi_0}{\tau} H(F) \\
= \frac{1}{2} \text{curl} V \times M - \beta M \times (N \times K), \\
\text{curl} M \times n &= 0, \quad M \cdot n = 0 \quad \text{on} \; \Gamma,
\end{aligned}
\]

(56)

then we determine \(H \in \mathbb{H}^1(D)\) by

\[
H = \nabla \varphi = H(M) + H(F).
\]

Using the Hölder inequality, the Sobolev embedding and inequality (55), we deduce that, for any \(M \in \mathbb{H}^1_\text{curl}(D)\), the function \(M \times (N \times K)\) belongs to \(L^2(D)\) and we have

\[
\|M \times (N \times K)\| \leq C\|M\|_{\mathbb{H}^1(D)}\|N\|_{\mathbb{H}^1(D)} \left(\|N\|_{\mathbb{H}^1(D)} + \|F\|\right).
\]

It results that, for any \(\Phi \in \mathbb{H}^1(D)\),

\[
\left|(M \times (N \times K); \Phi)\right| \leq C\|M\|_{\mathbb{H}^1(D)}\|N\|_{\mathbb{H}^1(D)} \left(\|N\|_{\mathbb{H}^1(D)} + \|F\|\right) \|\Phi\|.
\]

We also have

\[
|(\text{curl} V \times M; \Phi)| \leq C\|M\|_{\mathbb{H}^1(D)}\|\text{curl} V\|\|\Phi\|_{\mathbb{H}^1(D)}.
\]

Let us denote by \(A\) the bilinear form defined on \(\mathbb{H}^1(D) \times \mathbb{H}^1(D)\) by

\[
A(M, \Phi) = \langle (V \cdot \nabla)M; \Phi \rangle + \sigma(\text{curl} M; \text{curl} \Phi)
\]

\[
+ \sigma(\text{div} M; \text{div} \Phi) + \frac{1}{\tau}(M - \chi_0 H(M); \Phi)
\]

\[
- \frac{1}{2} (\text{curl} V \times M; \Phi) + \beta (M \times (N \times K); \Phi).
\]

Using the Hölder inequality, Sobolev embedding theorems and (52) we obtain

\[
|A(M, \Phi)| \leq C(V, N, F)\|M\|_{\mathbb{H}^1(D)}\|\Phi\|_{\mathbb{H}^1(D)}.
\]

which implies that \(A\) is continuous on \(\mathbb{H}^1_\text{curl}(D) \times \mathbb{H}^1_\text{curl}(D)\). Moreover, for all \(M \in \mathbb{H}^1(D)\) we have

\[
A(M, M) \geq \gamma \|M\|^2_{\mathbb{H}^1(D)} + \frac{\chi_0}{\tau}\|H(M)\|^2,
\]

where \(\gamma = \min(\sigma, \frac{1}{2})\) (with \(\|M\|^2_{\mathbb{H}^1(D)} = \|M\|^2 + \|\text{curl} M\|^2 + \|\text{div} M\|^2\) then \(A\) is coercive on \(\mathbb{H}^1_\text{curl}(D) \times \mathbb{H}^1_\text{curl}(D)\). We denote by \(L\) the linear form defined on \(\mathbb{H}^1(D)\) by

\[
L(\Phi) = \frac{\chi_0}{\tau}(H(F); \Phi).
\]

Using (52) we have

\[
|L(\Phi)| \leq C\|F\|\|\Phi\|, \quad \forall \Phi \in \mathbb{H}^1(D),
\]

then \(L\) is continuous on \(\mathbb{H}^1(D)\). We establish the result.
Lemma 9.1. There exists a unique function $M \in \mathbb{H}^1(D)$ satisfying the variational equation
\[
M = \int_{\mathbb{H}^1(D)} \Phi, \quad \forall \Phi \in \mathbb{H}^1(D).
\]
Moreover, there exists a unique $\varphi \in H^2(D) \cap H^1_0(D)$ such that $\nabla \varphi = H = H(M) + H(F)$ and we have the estimates
\[
\begin{align*}
\gamma \|M\|_{\mathbb{H}^1(D)}^2 + \frac{\chi_0}{2\tau} \|H\|^2 &\leq C \|F\|^2, \\
\|H\|_{\mathbb{H}^1(D)} &\leq C(\|M\|_{\mathbb{H}^1(D)} + \|F\|),
\end{align*}
\]
where $C$ is a positive constant that depends only on the domain $D$ and some physical constants.

Proof. The proof follows from the Lax-Milgram lemma. Since $F \in L^2(D)$, there exists a unique $M \in \mathbb{H}^1(D)$, solution of the variational equation (57). Then there exists a unique $\varphi \in H^2(D) \cap H^1_0(D)$ such that $\nabla \varphi = H = H(M) + H(F)$. Writing $\Lambda(M, M) = L(M)$ and using the equality $\frac{\tau}{\chi_0}(M; H) = -\frac{\chi_0}{\tau} \|H\|^2 - \frac{\chi_0}{\tau}(F; \varphi)$, we easily deduce inequality (58). For inequality (59), see (55).

Remark 3. It results from (58) and (59) that there is a positive constant $C_1$ depending only on the domain $D$ and some physical constants such that
\[
\|M\|_{\mathbb{H}^1(D)} \leq C_1 \|F\|, \quad \|H\|_{\mathbb{H}^1(D)} \leq C_1 \|F\|.
\]

The constant $C_1$ introduced in (60) being fixed we consider the convex subset of $\mathbb{H}^1(D)$ defined by
\[
\mathcal{B}_1 = \left\{ \Phi \in \mathbb{H}^1(D) : \|\Phi\|_{\mathbb{H}^1(D)} \leq C_1 \|F\| \right\}.
\]
The set $\mathcal{B}_1$ is compact in $L^2(D)$. Indeed, being bounded in $\mathbb{H}^1(D)$, $\mathcal{B}_1$ is relatively compact in $L^2(D)$ thanks to the compact Sobolev embedding $\mathbb{H}^1(D) \hookrightarrow L^2(D)$, and $\mathcal{B}_1$ is closed in $L^2(D)$.

Let us consider the map $G : \mathcal{B}_1 \rightarrow \mathbb{H}^1(D)$ defined by
\[
G(N) = M,
\]
where $M$ is the weak solution of problem (56), i.e. $M$ is the solution of the variational equation (57). Thanks to Remark 3 we have
\[
G(\mathcal{B}_1) \subset \mathcal{B}_1.
\]
In the following lemma we are concerned with the continuity of the map $G : \mathcal{B}_1 \rightarrow \mathcal{B}_1$ when $\mathcal{B}_1$ is equipped with the topology of $L^2(D)$.

Lemma 9.2. The set $\mathcal{B}_1$ being equipped with the topology of $L^2(D)$, the map $G$ is Lipschitz continuous from $\mathcal{B}_1$ into $\mathcal{B}_1$. More precisely, there is a positive constant $C$, depending only on the domain $D$ and some physical constants, such that, for any $N_1, N_2 \in \mathcal{B}_1$,
\[
\|G(N_1) - G(N_2)\| \leq C(F, R) \|N_1 - N_2\|,
\]
with $C(F, R) = C \|F\|^2 (\|F\|^2 + R^2)$ and $R = \max \left( \|N_1\|_{\mathbb{H}^1(D)}, \|N_2\|_{\mathbb{H}^1(D)} \right)$. 

Proof. Denote \( K_i = H(N_i) + H(F) \), \( M_i = G(N_i) \), and \( H_i = H(M_i) + H(F) \), for \( i = 1, 2 \). We set \( N = N_1 - N_2 \), \( K = H(N) \), \( M = M_1 - M_2 \) and \( H = H(M) \). Using the continuous embedding

\[
\|H\|_{\mathcal{B}} \leq \|M\|_{\mathcal{B}} + \|H\|_{\mathcal{B}}
\]

it is easily seen that the functions \( M \) and \( H \) satisfy

\[
\begin{cases}
(V \cdot \nabla)M - \sigma \Delta M + \frac{1}{\tau} (M - \chi_0 H) = 0 \\
\frac{1}{2} \nabla V \times M - \beta M \times (N_1 \times K_1) - \beta M_2 \times (N \times K_1) - \beta M_2 \times (N_2 \times H(N)), \\
\text{curl } M \cdot n = 0, \quad M \cdot n = 0 \quad \text{on } \Gamma, \\
H = \nabla \varphi, \quad \text{div } (H + M) = 0, \\
H \cdot n = 0 \quad \text{on } \Gamma.
\end{cases}
\]

Multiplying the equation of \((M, H)\) by \( M \), integrating by parts, using the Hölder inequalities

\[
\left| \left( M \times (N \times H(N_1)); M \right) \right| \leq \|M\|_6 \|M_2\|_6 \|H(N_1)\|_6 \|N\|_6,
\]

and the Young inequality, we deduce that

\[
\gamma \|M\|^2_{\mathcal{H}^1(D)} + \frac{\chi_0}{\tau} \|H\|^2 \leq \alpha \|M\|^2_6 + C(\alpha) \|M_2\|^2_6 \left( \|H(N_1)\|^2_6 \|N\|^2_6 + \|N_2\|^2_6 \right),
\]

where \( \alpha > 0 \) is arbitrary and \( \gamma = \min (\sigma, \frac{1}{\tau}) \). Using the Sobolev embedding \( \mathcal{H}^1(D) \hookrightarrow L^6(D) \) and the inequalities (see (52))

\[
\|M_2\|_{\mathcal{H}^1(D)} \leq C_1 \|F\|, \quad \|H(N_1)\|_{\mathcal{H}^1(D)} \leq C \left( \|N_1\|_{\mathcal{H}^1(D)} + \|F\| \right), \quad \|H(N)\| \leq \|N\|,
\]

we deduce by choosing \( \alpha \) small enough that

\[
\frac{\gamma}{2} \|M\|^2_{\mathcal{H}^1(D)} + \frac{\chi_0}{\tau} \|H\|^2 \leq C \|F\|^2 (\|F\|^2 + R^2) \|N\|^2.
\]

Using the continuous embedding \( \mathcal{H}^1(D) \hookrightarrow L^2(D) \) we deduce (62). Lemma 9.2 is proved.

All the conditions are satisfied to apply the Schauder fixed point theorem to the map \( G : \mathcal{B} \rightarrow \mathcal{B} \) defined by (61). We conclude that there exists \( M \in \mathcal{B} \) such that \( G(M) = M \). Taking \( H = H(M) + H(F) \) we easily verify that the pair \((M, H)\) is a solution of problem \((S_1)\). Moreover, the following estimates hold:

\[
\sigma \left( \|\text{curl } M\|^2 + \|\text{div } M\|^2 \right) + \frac{1}{\tau} \|M\|^2 + \frac{\chi_0}{2\tau} \|H\|^2 \leq C \|F\|^2,
\]

\[
\|M\|_{\mathcal{H}^1(D)} \leq C_1 \|F\|, \quad \|H\|_{\mathcal{H}^1(D)} \leq C \|F\|,
\]

where \( C \) is a positive constant that depends only on the domain \( D \) and some physical constants.
9.2. Solving problem \((S_2)\). Let \((M, H)\) be the solution of problem \((S_1)\) constructed above. To solve problem \((S_2)\) we introduce the Hilbert space

\[
\mathcal{W} = \{ \Phi \in \mathcal{U}, \quad \varepsilon \Delta \Phi \in L^2(D) \},
\]

equipped with its natural norm, then define the bilinear form \(B\) on \(\mathcal{W} \times \mathcal{W}\) and the linear form \(b\) on \(\mathcal{W}\) by

\[
B(U, \Phi) = \varepsilon^2 (\Delta U; \Delta \Phi) + \rho \langle (V \cdot \nabla)U; \Phi \rangle + \eta (\nabla U; \nabla \Phi),
\]

\[
b(\Phi) = \mu_0 \langle (M \cdot \nabla)H; \Phi \rangle + \frac{\mu_0}{2} (M \times H; \text{curl} \Phi) + \langle L; \Phi \rangle.
\]

By using the Hölder inequality and the Sobolev embedding \(\mathbb{H}^1(D) \hookrightarrow L^4(D)\) we have

\[
|B(U, \Phi)| \leq \varepsilon^2 \|\Delta U\| \|\Delta \Phi\| + C \|V\|_{L^6} \|\Phi\|_{L^6} \|\nabla U\| + \eta \|\nabla U\| \|\nabla \Phi\|
\]

\[
\leq C(V) \|U\|_{\mathcal{W}} \|\Phi\|_{\mathcal{W}},
\]

and

\[
|b(\Phi)| \leq C \left( \|M\|_{\mathbb{H}^1(D)} \|\nabla H\| \|\Phi\|_{L^6} + \|M\|_{\mathbb{H}^1(D)} \|H\|_{\mathbb{H}^1(D)} \|\text{curl} \Phi\| + \|L\| \|\Phi\| \right)
\]

\[
\leq C(F) \|\Phi\|_{L^6} + \|L\| \|\Phi\|,
\]

where we used (63). Moreover

\[
B(U, U) = \varepsilon^2 \|\Delta U\|^2 + \eta \|\nabla U\|^2, \quad \forall U \in \mathcal{W}.
\]

Thus, \(B\) is continuous on \(\mathcal{W} \times \mathcal{W}\) and coercive, and \(b\) is continuous on \(\mathcal{W}\). Applying the Lax-Milgram lemma we get the following

**Lemma 9.3.** Let \(L \in L^2(D)\) and \((M, H)\) be the solution of problem \((S_1)\) constructed in Section 9.1. There exists a unique weak solution \(U \in \mathcal{W}\) of problem \((S_2)\). Moreover, there is a positive constant \(C\), depending only on the domain \(D\) and some physical constants, such that

\[
\varepsilon^2 \|\Delta U\|^2 + \frac{\eta}{2} \|\nabla U\|^2 \leq C \left( \|M\|_{\mathbb{H}^1(D)}^2 \|H\|_{\mathbb{H}^1(D)}^2 + \|L\|^2 \right).
\]

**Remark 4.** It results from (63) and (64) that there is a positive constant \(C_2\) depending only on the domain \(D\) and some physical constants such that

\[
\|U\|_{\mathcal{W}} \leq C_2 \sqrt{\|F\|^4 + \|L\|^2}.
\]

Moreover, applying a classical regularity result for the second order elliptic operator with Dirichlet boundary condition we obtain that \(U\) belongs to \(\mathbb{H}^2(D) \cap \mathcal{U}\).

9.3. Existence of a weak solution to problem \((S^c)\). We use a similar method to that in Section 9.2. The constant \(C_2\) introduced in Remark 4 being fixed we consider the convex subset of \(\mathcal{W}\) given by

\[
\mathcal{B}_2 = \left\{ V \in \mathcal{W} : \|V\|_{\mathcal{W}} \leq C_2 \sqrt{\|F\|^4 + \|L\|^2} \right\}.
\]

The set \(\mathcal{B}_2\) is compact in \(\mathcal{U}\). Indeed, being bounded in \(\mathcal{W}\), \(\mathcal{B}_2\) is relatively compact in \(\mathcal{U}\) thanks to the compact Sobolev embedding \(\mathbb{H}^2(D) \hookrightarrow \mathbb{H}^1(D)\), and \(\mathcal{B}_2\) is closed in \(\mathcal{U}\).

Let us now consider the map \(S = S_2 \circ S_1\), where \(S_1\) and \(S_2\) are defined by (53) and (54), respectively. According to Remark 4 we have

\[
S(\mathcal{B}_2) \subset \mathcal{B}_2.
\]
In the following lemma we are concerned with the continuity of the map \( S : \mathcal{B}_2 \to \mathcal{B}_2 \) when \( \mathcal{B}_2 \) is equipped with the topology of \( H^1(D) \).

**Lemma 9.4.** There is a number \( r_0 > 0 \) such that, if \( \|F\| \leq r_0 \) then the map \( S \) is Lipschitz continuous. More precisely, let \( V_1, V_2 \in \mathcal{B}_2 \) and set \( M_i = S_i(V_i), H_i = H(M_i) + H(F), U_i = S_2(M_i), \) for \( i = 1, 2 \). Then:

1. There is a positive constant \( C_0 \), depending only on the domain \( D \) and some physical constants, such that
   \[
   \beta \left( M_1 \times (M_1 \times H_1) - M_2 \times (M_2 \times H_2); M_1 - M_2 \right) \leq C_0 \beta \|F\|^2 \|M_1 - M_2\|^2_{H^1(D)}.
   \]

2. Let \( r_0 > 0 \) such that \( C_0 \beta r_0^2 < \gamma \) where \( \gamma = \min(\sigma, \frac{1}{2}) \) and assume that \( \|F\| \leq r_0 \). We have
   \[
   \|S(V_1) - S(V_2)\|_d^2 \leq C \left( \frac{\|F\|^4}{\gamma - C_0 \beta \|F\|^2} + R^2 \right) \|V_1 - V_2\|_d^2,
   \]
   where \( R = \max(\|V_1\|_d, \|V_2\|_d) \) and \( C \) is a positive constant, depending only on the domain \( D \) and some physical constants.

**Proof.** We set \( V = V_1 - V_2, M = M_1 - M_2, H = H_1 - H_2 = H(M), U = U_1 - U_2 \) and consider the equations satisfied by \( M \) and by \( U \), written in the following form, respectively:

\[
\begin{aligned}
\mathcal{S}_{1,1} & \quad \left\{ 
\begin{aligned}
(V_2 \cdot \nabla)M - \sigma \Delta M + \frac{1}{\tau} (M - \chi_0 H) &= \sigma (V_2 \cdot \nabla)M_1 + \frac{1}{2} \left( \text{curl} V_1 \times M_1 - \text{curl} V_2 \times M_2 \right) \\
- \beta (M_1 \times (M_1 \times H_1) - M_2 \times (M_2 \times H_2)) &= \text{curl} M \times n = 0, \quad M \cdot n = 0 \quad \text{on } \Gamma,
\end{aligned}
\right.

\mathcal{S}_2 & \quad \left\{ 
\begin{aligned}
\text{div} U &= 0, \\
\varepsilon^2 \Delta^2 U + \rho V_2 \cdot \nabla U - \eta \Delta U + \nabla p &= -\rho (V \cdot \nabla)U_1 + \mu_0 ((M_1 \cdot \nabla)H_1 - (M_2 \cdot \nabla \cdot H_2)) \\
+ \frac{\mu_0}{2} (\text{curl} (M_1 \times H_1) - \text{curl} (M_2 \times H_2)) &= \text{curl} (M \times H_1) - \text{curl} (M \times H_2),
\end{aligned}
\right.

\end{aligned}
\]

We have clearly:

(a) \( \beta \left( M_1 \times (M_1 \times H_1) - M_2 \times (M_2 \times H_2) \right) = \beta M \times (M_1 \times H_1) + \beta M_2 \times (M \times H_1) + \beta M_2 \times (M_2 \times H), \)

(b) \( \frac{1}{2} (\text{curl} V_1 \times M_1 - \text{curl} V_2 \times M_2) = \frac{1}{2} \text{curl} V \times M_1 + \frac{1}{2} \text{curl} V_2 \times M, \)

(c) \( (V_2 \cdot \nabla)M_1 - (V_2 \cdot \nabla)M_2 = (V \cdot \nabla)M_1 + (V_2 \cdot \nabla)M, \)

(d) \( \mu_0 \left( (M_1 \cdot \nabla)H_1 - (M_2 \cdot \nabla \cdot H_2) \right) = \mu_0 (M \cdot \nabla)H_1 - \mu_0 (M_2 \cdot \nabla)H, \)

(e) \( \mu_0 (\text{curl} (M_1 \times H_1) - \text{curl} (M_2 \times H_2)) = \mu_0 \text{curl} M \times H_1 + \mu_0 \text{curl} (M_2 \times H), \)

(f) \( \rho \left( (V_2 \cdot \nabla)U_1 - (V_2 \cdot \nabla)U_2 \right) = \rho \left( (V \cdot \nabla)U_1 + (V_2 \cdot \nabla)U \right). \)

We multiply equalities (a), (b), (c) by \( M \) and (d), (e), (f) by \( U \). Using the Hölder inequality and the Sobolev embedding theorem, we derive the inequalities
Hence item 2. Lemma 9.4 is proved.

We conclude that there exists $U \in L^2_\Omega$. We choose $(\tilde{\alpha} > 0)$ small enough and use (i), (ii) and (iii) in $(S_{1,1})$ to deduce that

\[ \gamma > C_0 \beta \| F \|_2^2 \| M \|_{B^1_2(D)}^2 + \frac{\chi_0}{\tau} \| H \|_2^2 \leq C \| F \|_2^2 \| V \|_2^2. \]

Using (iv), (v) and (vi) in $(S_2)$ we deduce that

\[ \varepsilon^2 \| \Delta U \|^2 + \frac{\mu}{2} \| V U \|^2 \leq C \left( \| F \|^2 \| M \|_{B^1_2(D)}^2 + R^2 \| V \|_2^2 \right). \]

We choose $r_0$ so that $C_0 \beta r_0^2 < \gamma$. Under the hypothesis $\| F \| \leq r_0$ we have

\[ \| M \|_{B^1_2(D)}^2 \leq \frac{C \| F \|^2}{\gamma - C_0 \beta \| F \|^2} \| V \|^2_2. \]

\[ \| H \|_{B^1_2(D)}^2 \leq C \| M \|_{B^1_2(D)}^2, \]

then

\[ \| U \|_2^2 \leq C \left( \frac{\| F \|^4}{\gamma - C_0 \beta \| F \|^2} + R^2 \right) \| V \|^2_2. \]

Hence item 2. Lemma 9.4 is proved. \( \square \)

Now one can apply the Schauder fixed point theorem to the map $S : B_2 \rightarrow B_2$. We conclude that there exists $U \in B_2$ such that $S(U) = U$. Taking $(M, H)$ as a weak solution of problem $(S_1)$ associated with $U$ we obtain that $(U, M, H)$ is a solution of problem $(S^\varepsilon)$. We can now state the following

**Proposition 5.** Let $L \in L^2(D)$ and $F \in L^2_\Omega$. There is a number $r_0 > 0$, depending only on the domain $D$ and some physical constants, such that if $\| F \| \leq r_0$ then problem $(S^\varepsilon)$ has a solution $(U^\varepsilon, M^\varepsilon, H^\varepsilon)$ with $U^\varepsilon \in H^2(D) \cap \mathcal{U}, \ M^\varepsilon \in H^1_2(D), \ H^\varepsilon = H(M^\varepsilon) + H(F) \in H^1(D)$. Moreover, $(U^\varepsilon, M^\varepsilon, H^\varepsilon)$ satisfies the estimates (50), (51) and (55).

10. **End of the proof of Theorem 2.2.**

10.1. **Existence of a weak solution to problem** $(S)$. Let $(U^\varepsilon, M^\varepsilon, H^\varepsilon)$ be the solution of problem $(S^\varepsilon)$ given by Proposition 5. It satisfies the uniform estimates stated formally in Section 8:

\[ \varepsilon^2 \| \Delta U^\varepsilon \|^2 + \frac{\mu}{2} \| \nabla U^\varepsilon \|^2 + \frac{\mu_0 \sigma}{2} \| \text{div} \ M^\varepsilon \|^2 + \frac{\mu_0}{\tau} \left( \frac{1}{2} + \chi_0 \right) \| H^\varepsilon \|^2 + \mu_0 \beta \| M^\varepsilon \times H^\varepsilon \|^2 \]

\[ \leq C \left( \| F \|^2 + \| L \|^2 \right), \quad (65) \]

\[ \sigma \left( \| \text{curl} \ M^\varepsilon \|^2 + \| \text{div} \ M^\varepsilon \|^2 \right) + \frac{1}{\tau} \| M^\varepsilon \|^2 + \frac{\chi_0}{2\tau} \| H^\varepsilon \|^2 \leq C \| F \|^2. \quad (66) \]

\[ \| H^\varepsilon \| \leq C \| M^\varepsilon \| + C \| F \|, \quad \| U^\varepsilon \|_{H^1(D)} \leq C \left( \| M^\varepsilon \|_{B^1_2(D)} + C \| F \| \right). \quad (67) \]
We construct a weak solution of problem \((P)\) by passing to the limit, as \(\varepsilon \to 0\), on the sequence \((U_\varepsilon, M_\varepsilon, H_\varepsilon)\). A weak formulation of problem \((S^\varepsilon)\) consists in the following variational equations:

\[
\varepsilon^2 \langle \Delta U_\varepsilon; \Delta \Phi \rangle + \rho \langle (U_\varepsilon \cdot \nabla) U_\varepsilon; \Phi \rangle + \eta \langle \nabla U_\varepsilon; \nabla \Phi \rangle = \mu_0 \left( \langle (M_\varepsilon \cdot \nabla) H_\varepsilon; \Phi \rangle + \frac{\mu_0}{2} \langle M_\varepsilon \times H_\varepsilon; \text{curl } \Phi \rangle \right) + (L; \Phi),
\]

\[
\langle (U_\varepsilon \cdot \nabla) M_\varepsilon; \Psi \rangle + \sigma \langle \text{curl } M_\varepsilon; \text{curl } \Psi \rangle + \sigma \langle \text{div } M_\varepsilon; \text{div } \Psi \rangle + \frac{1}{\tau} \langle M_\varepsilon - \chi_0 H_\varepsilon; \Psi \rangle = \frac{1}{2} \langle \text{curl } U \times M; \Psi \rangle - \beta \langle M \times (M \times H); \Psi \rangle,
\]

\[
\langle \nabla \varphi^\varepsilon; \nabla v \rangle + \langle 2^{\varepsilon}; \nabla v \rangle = -(F; v), \quad H^\varepsilon = \nabla \varphi^\varepsilon,
\]

for all \(\Phi \in H^2(\Omega) \cap \mathcal{U}, \Psi \in H^1_\varepsilon(\Omega)\) and \(v \in H^2_\varepsilon(\Omega)\).

From estimates (65)–(67) we deduce that, for subsequences not relabeled,

\[
\varepsilon^2 \Delta U_\varepsilon \to 0 \text{ strongly in } L^2(\Omega),
\]

\[
(U_\varepsilon^\varepsilon, M_\varepsilon) \to (U, M) \text{ weakly in } \mathcal{U} \times H^1_\varepsilon(\Omega),
\]

\[
(U_\varepsilon^\varepsilon, M_\varepsilon) \to (U, M) \text{ strongly in } L^q(\Omega) \text{ for } 1 \leq q < 6,
\]

\[
H_\varepsilon \to H = H(M) + H(F) \text{ weakly in } H^1_\varepsilon(\Omega).
\]

Given \(\Phi \in \mathcal{U}\) we claim that

\[
\langle (M_\varepsilon \cdot \nabla) H_\varepsilon; \Phi \rangle \to \langle (M \cdot \nabla) H; \Phi \rangle, \quad \text{(68)}
\]

\[
\langle \text{curl } (M_\varepsilon \times H_\varepsilon); \Phi \rangle \to \langle \text{curl } (M \times H); \Phi \rangle. \quad \text{(69)}
\]

Indeed we use the weak-strong principle. Since \(M_\varepsilon\) converges to \(M\) in \(L^4(\Omega)\) strong and \(\nabla H_\varepsilon\) converges to \(\nabla H\) in \(L^2(\Omega)\) weak, we have that \(M_\varepsilon \cdot \nabla H_\varepsilon \to M \cdot \nabla H\) in \(L^2(\Omega)\) weak. According to the Sobolev embedding \(H^1(\Omega) \to L^4(\Omega)\), we have that \(\Phi \in L^4(\Omega)\), then \(\langle (M_\varepsilon \cdot \nabla) H_\varepsilon; \Phi \rangle \to \langle (M \cdot \nabla) H; \Phi \rangle\). Hence (68).

By writing \(\langle \text{curl } (M_\varepsilon \times H_\varepsilon); \Phi \rangle \) we easily show (69). Moreover, for all \(\Psi \in H^1(\Omega)\), we have \(\langle \text{curl } U_\varepsilon \times M_\varepsilon; \Psi \rangle = \langle M_\varepsilon; \Psi \times \text{curl } U_\varepsilon \rangle\) and \(\langle M_\varepsilon \times (M_\varepsilon \times H_\varepsilon); \Psi \rangle = \langle M_\varepsilon \times H_\varepsilon; \Psi \times M_\varepsilon \rangle\). Using the strong convergence of \((M_\varepsilon)\) and the weak convergence of \((\text{curl } U_\varepsilon)\) we get

\[
\langle \text{curl } U_\varepsilon \times M_\varepsilon; \Psi \rangle \to \langle \text{curl } U \times M; \Psi \rangle.
\]

Since \((M_\varepsilon)\) and \((H_\varepsilon)\) converge strongly in \(L^4(\Omega)\) to \(M\) and \(H\), respectively, we deduce that \((M_\varepsilon \times H_\varepsilon)\) converges strongly in \(L^2(\Omega)\) to \(M \times H\). We obtain the convergence result

\[
\langle M_\varepsilon \times (M_\varepsilon \times H_\varepsilon); \Psi \rangle \to \langle M \times (M \times H); \Psi \rangle.
\]

By similar arguments we justify the convergence

\[
\langle (U_\varepsilon \cdot \nabla) U_\varepsilon; \Phi \rangle \to \langle (U \cdot \nabla) U; \Phi \rangle, \quad \forall \Phi \in \mathcal{U}.
\]

With these convergence results, we easily pass to the limit in the weak formulation of problem \((S^\varepsilon)\) and obtain that the weak limit \((U, M, H)\) of \((U_\varepsilon, M_\varepsilon, H_\varepsilon)\) is a weak solution of problem \((S)\). Passing to the lower limit in (65)–(67), we obtain that \((U, M, H)\) satisfies estimates (5)–(7).
10.2. **Regularity of the weak solution.** (i) **Regularity of** $U$ **and** $p$. The functions $U$ and $p$ satisfy

\[
\begin{aligned}
\text{div } U &= 0, \\
- \eta \Delta U + \nabla p &= \mathcal{F}_s, \\
U &= 0 \quad \text{on } \Gamma,
\end{aligned}
\]

with

\[
\mathcal{F}_s = -\rho(U \cdot \nabla)U + \mu_0 (M \cdot \nabla)H + \frac{\mu_0}{2} \text{curl} (M \times H) + L.
\]

Using the identity

\[
\text{curl} (M \times H) = (\text{div } H)M - (\text{div } M)H + (H \cdot \nabla)M - (M \cdot \nabla)H,
\]

the Sobolev embedding $\mathbb{H}^1(D) \hookrightarrow L^6(D)$ and the Hölder inequality we obtain that $\text{curl} (M \times H)$ belongs to $L^{\frac{6}{5}}(D)$. We also have that $\rho(U \cdot \nabla)U$ and $\mu_0 (M \cdot \nabla)H$ belong to $L^{\frac{6}{2}}(D)$. It results that $\mathcal{F}_s \in L^{\frac{6}{2}}(\Omega)$. Arguing as in Section 6.2 we get that $U \in W^{2,\frac{6}{5}}(D)$, $p \in W^{1,\frac{6}{2}}(D)$, and using (5)–(7) we deduce the estimate

\[
\|U\|_{W^{2,\frac{6}{5}}(D)} + \|p\|_{W^{1,\frac{6}{2}}(D)} \leq C(\|\mathcal{F}_s\|_{\frac{6}{2}} \leq C\left(\|F\|^2 + \|L\|^2 + \|\eta\|_{\frac{6}{5}}\right)).
\]

(ii) **Regularity of the magnetic induction.** Consider the magnetic induction $B$ defined by $B = (M + H)$. Writing

\[
-\sigma \Delta B = -\sigma \Delta M - \sigma \nabla \text{div } H,
\]

we see that $B$ satisfies the Stokes system

\[
\begin{aligned}
\text{div } B &= F, \\
- \sigma \Delta B + \nabla \pi &= \mathcal{G}_s, \\
B \cdot n &= 0, \quad \text{curl } B \times n = 0 \quad \text{on } \Gamma,
\end{aligned}
\]

with $\pi = -\sigma \text{div } H$ and

\[
\mathcal{G}_s = -(U \cdot \nabla)M - \frac{1}{\tau} (M - \frac{\chi_0}{\tau} H) + \frac{1}{2} \text{curl} U \times M - \beta M \times (M \times H).
\]

Clearly, $\mathcal{G}_s \in L^{\frac{6}{5}}(D)$ and the compatibility condition $(B \cdot n, 1)_{\Gamma} = \int_B F \, dx$ is satisfied since $B \cdot n = 0$ on $\Gamma$ and $F \in H^1_0(D)$. Arguing as in Section 6.2 we get that $B \in W^{2,\frac{6}{5}}(D)$, and using (5)–(7), we deduce the estimate

\[
\|B\|_{W^{2,\frac{6}{5}}(D)} \leq C\left(\|\mathcal{G}_s\|_{\frac{6}{2}} + \|F\|_{W^{1,\frac{6}{2}}(D)}\right) \leq C\left(\|F\|^2 + \|\eta\|^2 + \|\mu\|^2 + \|\sigma\|_{\frac{6}{5}}\right).
\]

This ends the proof of Theorem 2.2.

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E-mail address: amirat@math.univ-bpclermont.fr
E-mail address: kamel.hamdache@devinci.fr