Tensor methods for MIMO decoupling using frequency response functions

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Abstract: Decentralized control design is still commonly applied to design controllers for multivariable systems. The success of decentralized control design methodologies hinges on the quality of decoupling of the system. The aim of this paper is to develop a decoupling procedure that applies to multivariable systems and only requires a frequency response function of the system. The proposed method builds on recent tensor decomposition methods. The potential of the method is shown both in a simulation and using experimental data on an active vibration isolation system.

Keywords: identification, MIMO, tensor.

1. INTRODUCTION

Many engineering problems require a model of the system. One such example is the design of control logic, which also imposes some requirements on the model structure and simplicity. In many cases the system model can be derived based on physical knowledge about the system. However, there are quite often situations when it is not easy to obtain a precise mathematical model in such way. Even in cases when model equations are available, very often the exact parameter values for the real system are not known. System identification is often used by engineering practitioners in such real-life situations.

As in the case of control engineering design, initially linear systems were used for many practical identification problems and the results are obtained by processing them mostly in time domain, see Ljung (1999); Söderström and Stoica (1988). More recently, frequency domain methods are also becoming more widespread as found in Pintelon and Schoukens (2012); Pintelon et al. (2011) due to their advantages with respect to noise, plant operating in closed loop, but also because they permit in a very structural way to give an early estimate about the level of the system non-linearity compared to the linear system dynamics. The main concept is known as the Best Linear Approximation (BLA), which is the linear dynamics approximating best the nonlinear system output in mean square sense.

Designing models, which would be applicable for control engineers in practice imposes additional constraints on the type of useful models. Presently, many industrial controllers are still single-input single-output. Reasons include:

- they are easily understood and tuned using manual loop-shaping,
- the tuning can be based on nonparametric frequency response functions, which are often inexpensive and accurate
- they allow for manual on-site fine tuning.

Such decentralized controllers essentially require the system to be decoupled i.e., the interaction between loops is removed. If the decoupling is not perfect, then interaction may lead to performance degradation or even closed-loop instability. Several approaches to decouple systems have been developed, see, e.g., Owens (1978); Mees (1981); Maciejowski (1989). Furthermore, the success of model-based control designs also depends on successful decoupling, see, e.g., van Herpen et al. (2014). The ability to decouple a mechanical system depends on the actuator and sensor locations, the number of (dominant) modes and the alignment of mode shapes with the sensor and actuator matrices. High performance motion systems are often designed light and stiff, with the aim to move flexible mode behavior to frequencies above the intended closed loop bandwidth. Often, motion systems behave as a rigid body at low frequencies.

For several classes of systems, e.g., motion systems, frequency response functions are often used to identify sys-
tems, since these are fast, inexpensive, and accurate. Identifying a MIMO parametric model in the form of rational transfer functions, or state-space models can be the next step, which is researched in the area of system identification. It is a challenging problem especially when a large number of inputs and outputs are present. Therefore there is a motivation to use the frequency response in a non-parametric way and the most intuitive and mature approaches for this are valid for SISO case as in Vaes et al. (2004); Steinbuch et al. (2010).

Recently tensor methods have attracted attention with their inherent advantages to handle multi-dimensional data, as shown in De Lathauwer (2003, 2006); Kolda and Bader (2000); De Lathauwer et al. (2004). Some numerical tools have also been developed, which permit the application of these methods on practical data, see Andersson and Bro (2000). Although many techniques have been developed to decouple systems, the use of recent tensor-based methods seems unexplored. The main aim of this paper is to explore such methods and experimentally validate these on a mechatronic system.

To validate the approach on a practical test case we use the active vibration isolation system (AVIS) benchmark Voorhoeve et al. (2015), which provides a good publicly available data. The aim of this benchmark is to compare different black box, linear time invariant identification techniques to model complex industrial systems. The main idea is that an industrial high-tech system automatically provides practically relevant requirements for identification algorithms. The presented benchmark system is complex in the sense of high order, lightly damped flexible dynamics with a significant dimension in terms of inputs and outputs. The proposed benchmark is based on a mechanical system that has applications in motion and vibration control. In the near future the requirements for motion and vibration control will become much tighter. To meet these future demands, it is envisaged that active control of flexible dynamics is required, including the use of additional actuators and sensors and inferential control. This implies that the current trend is that the number of inputs and outputs is likely to increase, as well as the order of the relevant flexible dynamics.

2. INDUSTRIAL STATE OF THE ART

The industrial state-of-the-art control of motion systems can be summarized as follows. By appropriate system design, most systems are either decoupled or can be decoupled using static input-output transformations. Hence, most motion systems and their motion software architecture use SISO control design method and solutions. The feedback controller is typically designed using frequency domain techniques, in particular manual loop-shaping. A typical motion controller has a PID structure, with a low pass at high frequencies and one or two notch filters to compensate flexible dynamics. In addition to the feedback controller, a feedforward controller is often implemented with acceleration, velocity, and friction feedforward for the reference signal.

The first necessary step to perform any design of a controller is to obtain some knowledge regarding the system dynamics. In the case that the motion system has already been realized, system identification is an inexpensive, fast, and accurate approach to model motion systems. In particular, the first step in motion systems typically involves a frequency response function (FRF) identification using noise signals, single sine, swept sine, or multi-sine excitation.

Once an FRF is available, the controller K can be designed. First, manual loop-shaping for SISO systems is investigated. The key idea of loop-shaping is the modification of the controller such that the open-loop is made according to specifications. In MIMO systems it is much less trivial to apply loop-shaping. The stability is determined by the closed-loop polynomial, det(I + L(s)), and the characteristic loci can be used for this graphically. The characteristic loci are the eigenvalues of the FRF $L(j\omega)$ of the open-loop transfer function matrix $L(s)$. The shaping of these eigenvalue loci is not straightforward if the plant has large off-diagonal elements (interaction). In that case, a single element of the controller will affect more eigenvalue loci. This may lead to many design iterations and lost of intuition. In the work of Rosenbrock (1974) it is shown that up to some extend, when the open loop is diagonal dominant, one can allow plant interaction and still use FRF-based loop-shaping design techniques. Many classical MIMO control design methods aim at decoupling the open-loop function at some location in the feedback loop, e.g. at the plant input or plant output. The strong non-intuitive aspect of MIMO loop-shaping, and the fact that SISO loop-shaping is used often, is a major roadblock in application of modern design tools in industrial motion systems. For that reason, a step-by-step approach is used, in which design complexity is only increased if justified by the problem at hand:

1. interaction analysis,
2. decoupling transformations,
3. independent feedback control design,
4. sequential feedback control design,
5. norm based control design.

All except for the last step can be performed with a non-parametric model of the plant, i.e., an identified FRF. The norm-based control design requires a parametric model of the plant.

The goal of the interaction analysis is to identify two sided interactions in the plant dynamics. If there is no two sided interaction, then feedback design becomes a standard multi-loop SISO design problem. Two relevant measures of plant interaction include Skogestad and Postlethwaite (2005); Bristol (1966):

- Relative Gain Array per frequency, and
- structured singular value of interaction as multiplicative output uncertainty.

To reduce interaction, one may redefine the input and output of the plant using a decoupling transformation. For motion systems most transformations are found on the basis of kinematic models. Herein, combinations of the actuators are defined so that actuator variables act in independent (orthogonal) directions at the center of gravity. Similarly, combinations of the sensors are defined so that each translation and rotation of the center of gravity can be measured independently. Thus, this basically
Fig. 1. SVD and Best rank-2 approximation as in Eq. (1) amounts to the inversion of a kinematic model of the plant. In certain situations, it may be desirable to decouple the plant at other frequencies, or to use a dynamic decoupling.

3. TENSOR DECOMPOSITION (CPD)

We present some basic notions about tensors, which may help the readers to gain some intuitive understanding. However this work is in no way a rigorous presentation about tensors and their applications, such can be found in De Lathauwer (2006); Cichocki et al. (2015); Comon (2014).

We start with some ideas about matrices, which are more widely used. Using Singular Value Decomposition (SVD), we can write a matrix $M \in \mathbb{R}^{n \times m}$ as the sum of many rank one matrices:

$$
M = \sum_{i=1}^{r} s_i u_i v_i^T + E,
$$

(1)

$$
M = U \Sigma V + E.
$$

(2)

When the rank $r$ is small, the error $E$ is small and this gives a concise representation for the matrix $M$ (using $(m+n)r$ parameters instead of $mn$). Such decompositions are widely applied. An example of a rank 2 decomposition of a matrix is shown on Fig. 1.

Tensors can be regarged as high dimensional generalizations of matrices. Tensor decomposition is a generalization of low rank matrix decomposition. Although most tensor problems are NP-hard in the worst case, several natural subcases of tensor decomposition can be solved in polynomial time, as long as the tensor does not have too many components, and the components are not adversarially chosen.

In multilinear algebra, the tensor rank decomposition or canonical polyadic decomposition (CPD) may be regarded as a generalization of the matrix singular value decomposition (SVD) to tensors, which has found application in statistics, signal processing, psycho-metrics, linguistics and chemometrics. It was introduced by Hitchcock (1927) and later rediscovered several times in Carroll and Chang (1970); Harshman (1970). For this reason, the tensor rank decomposition is historically referred also as PARAFAC or CANDECOMP.

In particular a 3-dimensional tensor can be decomposed as a sum of rank-1 components, as shown on Fig. 2.

This can be written in mathematical notation as shown in (3). The computation of canonical polyadic decomposition minimizes the error $E$.

Fig. 2. CPD and Best rank-2 approximation of 3-dimensional tensor, corresponding to Eq. (3)

$$
M = \sum_{r} U^1_r \circ U^2_r \circ U^3_r + E.
$$

(3)

$$
M = U^{(1)} \circ U^{(2)} \circ U^{(3)} + E.
$$

(4)

where $U^{(n)} = [U_1^{(n)}, U_2^{(n)} \ldots U_N^{(n)}]$. Contrary to singular value decomposition (SVD) in the matrix case, no orthogonality constraints are imposed on the matrices $U^{(n)}$ to ensure uniqueness. It is clear that one can arbitrarily permute the different rank-1 terms. Also, the factors of a same rank-1 term may be arbitrarily scaled, as long as their product remains the same. CPD is considered unique when it is only subject to these trivial transformations. Several Matlab toolboxes exist to perform this computation, such as Sorber et al. (2014); Bader et al. (2015); Andersson and Bro (2000).

4. MIMO FREQUENCY RESPONSE DECOMPOSITION

The decoupling we present in this paper was inspired by the works Dreesen et al. (2015), which presents similar problem related to decoupling a multi-variable polynomials. The problem addressed in this paper is to decouple a given set of MIMO frequency response functions (FRF). Such decoupled representation, if existing, would permit the MIMO FRF to be written as a linear combination of parallel SISO FRFs. In this paper we assume that the system is square, so it has same number of inputs and outputs $N_y = N_u = N$.

A MIMO system decoupling is intuitively shown on Fig. 3. At each frequency $\omega_i$ we have a square matrix $H(\omega_i) \in \mathbb{C}^{N^2}, i = 1 \ldots N_f$ with the complex response of the system relating the inputs and the outputs. This array $H(\omega)$ of complex matrices can be ordered as a 3-dimensional tensor. The decoupled representation of this MIMO FRF is defined as

$$
H(\omega_i) = T_y S(\omega_i) T_u, \ i = 1 \ldots N_f,
$$

(5)

where $S(\omega_i) = \text{diag}(S(\omega_i)), k = 1 \ldots N$ is a diagonal matrix containing SISO FRF$s$ $S(\omega_i)$ on the main diagonal, $T_y \in \mathbb{C}^{N}$ and $T_u \in \mathbb{C}^{N}$ and we restrict the dimension of the decoupled system to the number of system inputs and outputs $N^1$.

One should note that such decomposition is not guaranteed to exist in general even in the ideal case. It may exist for some classes of systems, for example some mechanical ones. In motion systems for instance, the rigid-body dynamics can often be decoupled, i.e., for frequencies below

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1 This may be overly restrictive, and probably can be relaxed.
We are applying the diagonalising matrix transformation on the coupled data
\[ \hat{S}(\omega) = T^{-1} y H(\omega) T^{-1} u \] (13)

The above transformation effectively diagonalises the original frequency response tensor \( H \) using two transformation matrices \( T_y, T_u \). This operation is somehow similar to the result of the SVD on a single matrix, however in this case the diagonalisation occurs for a set of matrices, each describing the MIMO FRF at different frequency.

There is one practical missing part in the above method at this point. The direct application of a CPD procedure on the above complex data would result in complex solutions, including complex matrices \( T_y \in \mathbb{C}^{N \times N}, T_u \in \mathbb{C}^{N \times N} \). This is not useful for a practical decoupling of physical systems. We need to get a real solutions for the matrices \( T_y, T_u \). The simple solution for this is to take the imaginary and real part of the complex tensor \( H \in \mathbb{C}^{N \times N \times N_f} \), each of them a real tensor by itself, and stack them one behind the other in the dimension of the frequencies, thus getting an augmented real-valued tensor \( \tilde{H} \in \mathbb{R}^{N \times N \times 2 \times N_f} \). The application of the CPD procedure on the tensor \( \tilde{H} \) results in \( T_y \in \mathbb{R}^{N \times N}, T_u \in \mathbb{R}^{N \times N} \). Due to the properties of the multiplication of a complex number and real numbers, these matrices diagonalise also the original complex tensor \( H \).

5. SIMULATION AND EXPERIMENTAL VALIDATION

5.1 Simplified simulation example

Construct a \( 2 \times 2 \) coupled MIMO system as a randomly selected mixture of two SISO systems.

\[ T_y = \begin{bmatrix} 0.1298 & 0.0032 \\ 0.4304 & 0.5103 \end{bmatrix} \] (7)
\[ T_u = \begin{bmatrix} 0.3206 & 0.6716 \\ 0.9439 & 0.0064 \end{bmatrix} \] (8)
\[ S = \begin{bmatrix} 3948 & 0 \\ 0 & 3.948 \times 10^5 \end{bmatrix} \] (9)
\[ H(\omega) = T_y S(\omega) T_u, \] (10)

where we apply the coupling transformation at each frequency. The two diagonal components above are shown on Fig. 5. The resulting coupled MIMO system \( H \) is shown together with its SISO components on the diagonal on Fig. 6.

The application of the CPD procedure \( \text{cpd3} \_\text{sd} \) from Sorber et al. (2014) on the coupled FRFs on Fig. 6 results in two real matrices

\[ T_y = \begin{bmatrix} -0.0063 & -0.3180 \\ -1.0000 & -1.0544 \end{bmatrix} \] (11)
\[ T_u = \begin{bmatrix} 1.0000 & 0.4308 \\ 0.0067 & 0.9025 \end{bmatrix} \] (12)

and the diagonalised SISO components shown on Fig. 7. The comparison with Fig. 5 indicates that, as expected the CPD has recovered correctly the shape and the phase of the SISO components, but is not matching the original magnitude. This is due to the non-uniqueness properties of the transformation, which does not affect the decoupling.

To demonstrate that the procedure is useful, we apply the diagonalising matrix transformation on the coupled data
\[ \tilde{S}(\omega) = T_y^{-1} H(\omega) T_u^{-1} \] (13)
The result is shown on Fig. 8. We note that the application of the linear transformation has resulted in very big drop of the off-diagonal magnitudes, which indicates that the procedure has worked as expected. The off-diagonal elements are not exactly zeros, but they are in the range of the numerical precision of the computation. It is also possible to verify that the original coupled system can be fully restored from the available SISO components in $\hat{S}(\omega)$.

The above result is very good, but it is obtained in quite ideal conditions - in this simulation we have not introduced any errors and noise. However the idea of this procedure is to be applied to the actual FRF measurements, which are never free of noise and other deformations. In the following we show that the same approach can be applied also to the experimental data.

### 5.2 Validation on experimental data

We use the AVIS benchmark Voorhoeve et al. (2015), which provides experimental data from frequency response measurements on real-life system. The goal of the AVIS
Fig. 9. Photograph of the experimental AVIS setup

is to isolate the payload with respect to exogenous disturbances. The system consists of two main parts, i.e., a movable payload and a chassis that is connected to the floor. The payload and chassis are connected by four isolator modules. On the one hand, these isolator modules provide passive damping through a pneumatic airmount. On the other hand, the isolator modules are equipped with Lorentz motors and geophones that enable active vibration isolation. Specifically, the isolation modules are each equipped with two motors, leading to eight actuators in total. In addition, three out of four modules are equipped with two geo-phones each that construct measurements of the velocity, leading to six sensors. The system can not be exactly decoupled due to the complex dynamics it contains, but it has been transformed to a form with a dominating diagonal entries in certain frequency range i.e., the rigid-body mode, which is a -1 slope here (due to absolute velocity sensors). In the following we present results for a particular 2x2 subsystem of the original data, similar results can be obtained also for the bigger MIMO data. We see that only at certain frequency (4Hz - 100Hz) range the off-diagonal entries are suppressed. We use this data as realistic start of almost-decoupled system and couple it by applying the linear transform.

\[
H_{\text{coupled}}(\omega) = \begin{bmatrix} 1 & .05 \\ .05 & 1 \end{bmatrix} G_{\text{dec}}(\omega) \cdot \begin{bmatrix} 1 & .07 \\ .031 & .07 \end{bmatrix}
\]  

(14)

We have thus obtained a realistic data of a system, which contains noise and is not subject to full decoupling. We will show in the following that we can obtain a reasonable estimate of the original almost-decoupled data.

Due to the non-ideal properties of the data, we have to provide some guidance to the CPD procedure to concentrate on particular frequency range. The data also contains an integrator in each output, which makes low frequency magnitudes stronger and dominating the medium ones. The relevant frequency range can be targeted better by removing the integrator from the data and weighting the data. One possible weighting procedure is to apply a band-pass filter on the data in frequency domain. We use a Butterworth band-pass filter of order 5 and frequency range from 0.5 Hz to 60 Hz. We also remove the integrator by multiplying the FRF with \( j\omega \) at each frequency. The CPD procedure is applied (after converting the tensor to a real-form) on the resulting data and provides a decoupling matrices \( T_y \in \mathbb{R}^{2 \times 2}, T_u \in \mathbb{R}^{2 \times 2} \).

Fig. 11 demonstrates that the static decoupling procedure using only the two matrices \( T_y, T_u \) has almost recovered the \( G_{\text{dec}} \). This was aided by the proper selection of weighting and filtering range.
6. CONCLUSION

We present results demonstrating an application for the tensor decomposition for the design of a static decoupling of a MIMO system. The results are obtained on a non-parametric frequency domain model of the plant and indicate that the procedure is quite robust. Further work may involve improvements on the weighting procedure and handling of different structures, perhaps not only purely diagonal, extending to non-square systems, and further experimental demonstration and comparison with classical approaches.

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