Regularity preservation in Kolmogorov equations for non-Lipschitz coefficients under Lyapunov conditions

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Abstract

Given global Lipschitz continuity and differentiability of high enough order on the coefficients in Itô’s equation, differentiability of associated semigroups, existence of twice differentiable solutions to Kolmogorov equations and weak convergence rates of numerical approximations are known. In this work and against the counterexamples of Hairer et al. [15], the drift and diffusion coefficients having Lipschitz constants that are $o(\log V)$ and $o(\sqrt{\log V})$ respectively for a function $V$ satisfying $(\partial_t + L)V \leq CV$ is shown to be a generalizing condition in place of global Lipschitz continuity for the above.

1 Introduction

Consider for $b : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ and a standard Wiener process $W_t$, the stochastic differential equation (SDE) on $\mathbb{R}^n$ given by

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{1}$$

The underlying results of this work are moment bounds of derivatives of $X_t$ with respect to initial condition in the case that the coefficients $b$ and $\sigma$ are not globally Lipschitz continuous in space. These estimates are used to validate an Itô-Alekseev-Gröbner formula [17] and differentiability of semigroups associated with (1), which enable the existence of twice differentiable-in-space solutions to Kolmogorov equations [28], solutions to corresponding Poisson equations, weak error estimates of numerical approximations [27, 39, 40] and related derivative estimates [12]. Similar moment bounds on the first and second derivative with respect to initial value in the non-globally Lipschitz case have recently been studied in [18] under different assumptions using the stochastic Grönnwall inequality [19, 38] and related ideas had appeared earlier in [11, 21].

More specifically, it is shown that the aforementioned consequences hold for the case that the coefficients $b, \sigma$ are non-globally monotone, where higher
derivatives of $b, \sigma$ are bounded by Lyapunov functions and loosely that $b$ and $\sigma$ admit Lipschitz constants which are $o(\log V)$ and $o(\sqrt{\log V})$ respectively for a Lyapunov function $V$. The results are applicable to all of the example SDEs in [23] except in Section 4.7; in particular, for the first time, weak numerical convergence rates of order one are shown for SDEs with non-globally monotone coefficients, including the stochastic Duffing-van der Pol oscillator, stochastic Lorenz equation with additive noise, underdamped Langevin equation with variable friction, overdamped Langevin dynamics with a non-globally monotone potential gradient and the stochastic Ginzburg-Landau equation. Of note here is that a new Lyapunov function for the stochastic Duffing-van der Pol oscillator accommodating additional parameter values is given below. Order one convergence rates are obtained for the stopped increment-tamed Euler-Maruyama schemes of [23]. In particular, the result guides the choice of taming parameter in the scheme.

In contribution to regularity analysis of SDEs, the results give criteria for the positive case of regularity for associated semigroups beyond the settings of globally Lipschitz coefficients or ellipticity and against the counterexamples of [15] with globally bounded smooth coefficients. In fact, the coefficients $b$ and $\sigma$ are not directly imposed to be in any weighted $L^\infty$ spaces for spatial differentiability of semigroups, note however the coefficients are indirectly bounded by the local Lipschitz bound. More generally, the regularity with respect to initial condition demonstrated here under our assumptions has further counterexamples for SDEs with constant diffusion coefficient and drift with polynomially growing first derivatives [25]. On the other hand, in the globally Lipschitz case [28] and the globally monotone (or one-sided Lipschitz) elliptic case [8], moment bounds on derivatives with respect to initial condition are known. In addition, for infinitely differentiable $b$ and $\sigma$ satisfying Hörmander’s bracket condition [16], infinite differentiability of the associated semigroup is given by Proposition 4.18 in [15].

Our basic result about semigroup differentiability can be summarized as follows. For SDEs whose coefficients are homogeneous in time, weaker assumptions on both the coefficients of the SDE and on the observables in the expectation below for similar conclusions are given Section 5, in particular the local Lipschitz conditions on $b, \sigma$ are replaced with pointwise bounds on the first derivatives.

**Theorem 1.1.** Suppose there exists $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ twice continuously differentiable in space, continuously differentiable in time and constant $C > 0$ such that $\partial_t V(t, x) + \sum_{i=1}^n b_i(t, x) \partial_{x_i} V(t, x) + \frac{1}{2} \sum_{i,j=1}^n \sigma(t, x) \sigma(t, x)^T \partial_{x_i} \partial_{x_j} V(t, x) \leq CV(t, x)$ for all $t \in [0, T], x \in \mathbb{R}^n$ and $\lim_{|x| \to \infty} V(t, x) = \infty$. Let $f, c : [0, T] \times \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}$ be measurable functions and $p \in \mathbb{N}_0$. Suppose for any $R > 0$, it holds that $\int_0^T \sup_{x \in B_R} (|c(t, x)| + |f(t, x)|) dt < \infty, b(t, \cdot), \sigma(t, \cdot), f(t, \cdot), g, c(t, \cdot) \in C^p$, and

- there exists measurable $G : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that $G(t, \cdot) = o(\log V(t, \cdot))$
uniformly in $t$ with continuous left-hand side for all $t$ and satisfying

$$\begin{align*}
|b(t, x) - b(t, y)| &\leq (G(t, x) + G(t, y))|x - y|, \\
\|\sigma(t, x) - \sigma(t, y)\|^2 &\leq (G(t, x) + G(t, y))|x - y|^2
\end{align*}$$

(2a)

for all $t \in [0, T]$, $x \in \mathbb{R}^n$.

• for every $k > 0$, $h \in \{b, f, g, c\}$, there exists $C' > 0$ such that

$$\begin{align*}
|\partial^\alpha h(t, \lambda x + (1 - \lambda)y)| + \|\partial^\beta \sigma(t, \lambda x + (1 - \lambda)y)\|^2 &\leq C'(1 + V(t, x) + V(t, y))^{\frac{1}{k}}
\end{align*}$$

(3)

for all $t \in [0, T]$, $x, y \in \mathbb{R}^n$, $\lambda \in [0, 1]$ and multiindices $\alpha, \beta$ with $p_0 \leq |\alpha| \leq p$, $2 \leq |\beta| \leq p$, where $p_0 = 2$ if $h = b$ and $p_0 = 0$ otherwise.

For any $s \in [0, T]$, stopping time $\tau \leq T - s$, the expectation of $u(s, \tau, \cdot) : \mathbb{R}^n \to \mathbb{R}$ given by

$$
\begin{align*}
\mathbb{E}u(s, \tau, x) &= \mathbb{E}\left[ \int_0^\tau f(s + r, X_r^{s,x})e^{-\int_0^\tau c(s + w, X_r^{s,x})dw} dr + g(X_\tau^{s,x})e^{-\int_0^\tau c(s + w, X_r^{s,x})dw} \right], \tag{4}
\end{align*}
$$

is continuously differentiable in $x$ up to order $p$, where for any $s \in [0, T]$, $x \in \mathbb{R}^n$, $X_r^{s,x}$ is the solution to $X_r^{s,x} = x + \int_0^r b(s + r, X_{s+r}^{s,x})dr + \int_0^r \sigma(s + r, X_{s+r}^{s,x})dW_r$ on $t \in [0, T - s]$. Moreover, if $p \geq 2$, then the function given by $v(t, x) = \mathbb{E}u(t, T - t, x)$ is locally Lipschitz in $t$ and satisfies

$$
\partial_t v + a : D^2 v + b \cdot \nabla v - cv + f = 0 \tag{5}
$$

almost everywhere in $(0, T) \times \mathbb{R}^n$.

In the case of time homogeneous coefficients that have locally Lipschitz derivatives of order $p$, the results in Section 5 assert that (2) and (3) may be replaced respectively by

$$
\sum_i |\partial_i b(x)| + \|\partial_i \sigma(x)\|^2 \leq G(t, x)
$$

and

$$
\sum_i |\partial^\alpha h(x)| + \|\partial^\alpha \sigma(x)\|^2 \leq C'(1 + V(t, x))^{\frac{1}{k}}.
$$

Theorem 1.1 follows as corollary to Theorem 4.2 and the assertion about Kolmogorov equations can be found in Theorem 4.5. The latter appears to be the only result against the literature about almost everywhere twice differentiable-in-space solutions to Kolmogorov equations for non-hypoelliptic or non-globally monotone coefficients.

Our basic result about weak convergence rates of a numerical approximation is as follows, assuming a deterministic initial condition for (1).
Theorem 1.2. Let all of the assumptions in Theorem 1.1 hold with \( p \geq 3 \). Suppose \( b, \sigma \) are independent of \( t \) and suppose \( V \) is of the form \( V(t, x) = e^{U(x)} e^{-\rho t} \) for \( U \in C^3(\mathbb{R}^n, [0, \infty)) \), \( \rho > 0 \), such that there exist \( c \geq 1 \) satisfying

\[
|x|^\frac{1}{c} + |\partial^\alpha b(x)|^\frac{1}{c} + ||\partial^\alpha \sigma(x)||^\frac{1}{c} + |\partial^\beta U(x)| \leq c(1 + U(x))^{1-\frac{1}{c}},
\]

for all \( x \in \mathbb{R}^n \), multiindices \( \alpha, \beta \) with \( 0 \leq |\alpha| \leq 2 \) and \( 1 \leq |\beta| \leq 3 \). If \( h \in C^3(\mathbb{R}^n, \mathbb{R}) \) is such that

\[
|\partial^\alpha h(x)| \leq c(1 + |x|^c)
\]

for all \( x \in \mathbb{R}^n \) and multiindices \( \alpha \) with \( 0 \leq |\alpha| \leq 3 \), then there exists a constant \( C > 0 \) such that

\[
|\mathbb{E}[h(X_T)] - \mathbb{E}[h(Y^\delta_T)]| \leq C\delta,
\]

for all \( 0 < \delta < 1 \), where \( Y^\delta : [0, T] \rightarrow \mathbb{R}^n \) is the approximation given by \( Y^\delta_0 = X_0 \) and

\[
Y^\delta_t = Y^\delta_{k\delta} + \mathbf{1}_{\{|Y^\delta_{k\delta}| \leq \exp(|\log |\delta|)^{\frac{1}{2}}\}} \left( \frac{b(Y^\delta_{k\delta})(t - k\delta) + \sigma(Y^\delta_{k\delta})(W_t - W_{k\delta})}{1 + |b(Y^\delta_{k\delta})(t - k\delta) + \sigma(Y^\delta_{k\delta})(W_t - W_{k\delta})|^3} \right)
\]

for all \( t \in [k\delta, (k + 1)\delta] \), \( k \in \mathbb{N}_0 \cap [0, \frac{T}{\delta}) \).

Theorem 1.2 is corollary to Theorem 6.3. The numerical scheme is the stopped increment-tamed Euler-Maruyama approximation from [23] that has the key property of retaining exponential integrability properties of the continuous time SDE, which is used throughout the proof for Theorem 6.3. As is well documented [22], the classical Euler-Maruyama scheme may diverge in both the strong and weak sense for superlinearly growing, non-globally Lipschitz coefficients without this property. The power 3 appearing in (7) is chosen purposefully. Weak convergence rates of order one is only attained for exponents larger than 3, which indicates the control one should exercise over the taming parameter in [23]. The proof of Theorem 6.3 uses the recent Itô-Alekseev-Gröbner formula in order to expand the left-hand side of (6), instead of the standard method using solutions to Kolmogorov equations as in [27] that is twice continuously differentiable-in-space and once continuously differentiable-in-time. In order to apply the formula, strong completeness using a result in [11] is first shown for the derivative processes; note that a related property appeared recently in [18] using a different approach and different assumptions. Although weak convergence without rates has been established by way of convergence in probability in [23, Corollary 3.7] and [20, Corollary 3.19], weak rates of convergence analogous to the globally monotone case (of order one as above) have thus far been an open problem for non-globally monotone coefficients, see however some work [6, 33] in this direction. For weak convergence results in the former case, see references within [11] and also [40]. On the other hand, strong convergence rates of order \( \frac{1}{2} \) have been established in even the latter case, see [21].

The proofs for the moment estimates underlying both Theorems 1.1 and 1.2 use directly the results of [19], for which exponential integrability in continuous
time as in [11, 21] is an important property that is accounted for in a crucial way by our local Lipschitz condition. The core argument for these estimates, which can be thought of as a combination of the approach in [28] with ideas of [11, 21], is to consider for any \( \kappa \in \mathbb{R}^n \) processes \( X_t(\kappa) \) satisfying

\[
\sup_{t \in [0,T]} \left| \frac{X^x_{t+r\kappa} - X^x_t}{r} - X^x_{t+\kappa} \right| \to 0
\]

in probability as \( r \to 0 \), where \( X^x_t \) denotes a solution to (1) with \( X^x_0 = x \). Such processes exist [28, Theorem 4.10] for \( b, \sigma \) continuously differentiable in space satisfying some local integrability assumption and \( X^x_t(\kappa) \) satisfies the system resulting from a formal differentiation of (1) (see precisely (18)). If \( b \) and \( \sigma \) are independent of \( t \) and the derivatives of \( b \) and \( \sigma \) are locally Lipschitz, the processes \( X^x_t(\kappa) \) are almost surely continuous derivatives in the classical sense as in [36, Theorem V.39]. Higher derivatives exist for \( b \) and \( \sigma \) with higher orders of differentiability. The SDEs solved by the first order derivatives turn out to be just as considered for previous applications of the stochastic Grönwall inequality [19], whereas those for higher order derivatives have only the term involving the derivative of the highest order on the right-hand side of the dynamics requiring serious control. For the latter, the stochastic Grönwall inequality together with our Lipschitz Assumption 1 and an induction argument are sufficient to control all of the terms. The bounds for higher derivatives, as required for twice differentiability of the semigroup in the Kolmogorov equation for example, call for two-sided Lipschitz conditions as in our Assumption 1 in contrast to the one-sided conditions in [11, 21]. We use \( o(\log V) \) and \( o(\sqrt{\log V}) \) Lipschitz constants in order to control the moments for large time \( T \), but the results follow for \( O(\log V) \) and \( O(\sqrt{\log V}) \) Lipschitz constants if \( T \) is suitably small. In order to establish solutions to the Kolmogorov equation, we prove a number of intermediary results following the strategy of [28] for the case of our local Lipschitz constants. For example, it is shown by extending an argument from [35] that an Euler-type approximation converges to solutions of the SDE in probability and locally uniformly in initial time and space, that is, the SDE is regular [28, Definition 2.1].

The original motivation for this work is the Poisson equation for finding the asymptotic variance of ergodic averages associated to SDEs. In [9], a formula for the derivative of this variance with respect to a parameter in the dynamics is derived. In order to do so, the Poisson equation is interpreted as a PDE in the classical sense, which in turn made use of an appropriate solution to the Kolmogorov equation. In a setting where the coefficients are not globally Lipschitz, for example if the friction in the Langevin equation of [9] is not restricted to be constant in space, the existence of such a solution to the backward Kolmogorov equation seems unavailable in the literature. The present work fills this gap. In addition, solutions to the Poisson equation furnishes central limit theorems for additive functionals themselves by way of [7]. The results here allow some arguments there to be established rigorously for hypoelliptic diffusions, more details are given in Section 7.1.
Alternative to the related Feynman-Kac formula for making the connection between the SDE and the Kolmogorov equation satisfied by the transition semigroup is to use the theory of Dirichlet forms. In e.g. [3, 10], the order in which one works is reversed, in the sense that one begins with the partial differential operator acting on the space of smooth compactly supported functions, shows that the closure of the graph in some Banach space generates a strongly continuous semigroup and that the semigroup is associated to the transition semigroup of a Hunt process that is a weak solution to the stochastic differential equation. Note however that the Banach spaces there are $L^2(\pi)$ for an invariant measure $\pi$, so that the strongly continuous semigroup is a classical solution in the sense of an abstract Cauchy problem. There, the transition semigroup is in the domain of the infinitesimal generator, but not necessarily a twice differentiable-in-space function solving the partial differential equation almost everywhere nor in the sense of distributions without the essential $m$-dissipativity property that is a core effort in those works.

For solutions in the sense of distributions to the Kolmogorov equation, in Proposition 4.18 in [15], the authors make use of (essentially Lemma 5.12 in) [28] to obtain such solutions in the case of smooth coefficients. In fact, the proof there only makes use of local Lipschitz continuity of the coefficients and that $\sigma^T$ admits a derivative which is locally Lipschitz. The authors in [15] moreover provide existence and uniqueness results for viscosity solutions under a Lyapunov condition, which yield distributional solutions under Hölder regularity on the coefficients [24] and almost everywhere solutions given enough regularity on the solution [30, Proposition I.4]. On a similar note, the existence of a viscosity solution to the Poisson equation with possible degenerate second-order coefficients is given by [34], which appears to require the aforementioned regularity to be interpreted as a solution in the distributional sense. In the present work, Theorem 1.1 is complemented with additional sufficient conditions for the backward Kolmogorov equation to hold in the distributional sense. In particular, a situation where it suffices to assume maximal dissipativity of the associated infinitesimal generator defined on smooth compactly supported functions is shown. Moreover, to reiterate, Theorem 5.2 provides weaker conditions than those in Theorem 1.1 for the same result to hold in the case of time homogeneous coefficients.

The paper is organised as follows. In Section 2, the setting, notation and various definitions about what is meant by Lyapunov functions are given. In Section 3, moment estimates of the supremum over time on the derivative process and the difference processes in initial value are given. These results are used throughout for proving the other results in the paper. In Section 4, results on the regularity of the semigroup associated to (1) are presented, which are followed by results about twice differentiable-in-space and distributional solutions to the Kolmogorov equation. Section 6 contains the results about weak convergence rates for the stopped increment-tamed Euler-Maruyama scheme on SDEs with non-globally monotone coefficients. In Section 7, new Lyapunov functions are given for the Langevin equation with variable friction and stochastic Duffing-van der Pol equation. In the case of the former, emphasis is placed on facilitating a
solution to the associated Poisson equation.

2 Notation and preliminaries

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, \(\mathcal{F}_t, t \in [0, \infty)\), be a filtration satisfying the usual conditions and \((W_t)_{t \geq 0}\) be a standard Wiener process on \(\mathbb{R}^n\) with respect to \(\mathcal{F}_t, t \in [0, \infty)\). Let \(T \in (0, \infty)\) and \(\|M\|\) denote the Frobenius norm of a matrix \(M\). Let \(b : \Omega \times [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n\), \(\sigma : \Omega \times [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times n}\) be functions such that \(b(t, \cdot), \sigma(t, \cdot)\) are continuous for every \(t, \omega\), \(b(\cdot, x), \sigma(\cdot, x)\) are \(\mathcal{F} \otimes \mathcal{B}([0, \infty))\)-measurable for every \(x\), \(b(t, x), \sigma(t, x)\) are \(\mathcal{F}_t\)-measurable for every \(t, x\) and \(\int_0^T \sup_{|t| \leq R} (|b(t, x)| + \|\sigma(t, x)\|^2) dt < \infty\) for any \(R > 0\), \(\omega \in \Omega\). For an open set \(O \subseteq \mathbb{R}^n\) and any \(x \in O\), let \(X_t^{s, x}\) be an \(\mathcal{F}_t\)-adapted \(O\)-valued process such that \(X_t^{s, x}\) is \(\mathbb{P}\)-a.s. continuous satisfying for all \(s, t \in [0, T]\),

\[
X_t^{s, x} = x + \int_0^t b(s + r, X_r^{s, x}) dr + \int_0^t \sigma(s + r, X_r^{s, x}) dW_r. \tag{8}
\]

When the initial conditions are not important or are obvious from the context, simply \(X_t\) and similarly \(X_t^{s, x}\) is written. For \(f \in C^2(O)\) and for either \(b, \sigma\) as above or \((b^r : \Omega \times [0, T] \to \mathbb{R}^n)_{x \in O}, (\sigma^r : \Omega \times [0, T] \to \mathbb{R}^{n \times n})_{x \in O}\) that are, for each \(x\), \(\mathcal{F} \otimes \mathcal{B}([0, t])\)-measurable and \(\mathcal{F}_t\)-adapted satisfying \(\mathbb{P}\)-a.s. that \(\int_0^T (|b^r| + |\sigma^r|^2) ds < \infty\), we denote

\[
Lf = b \cdot \nabla f + a : D^2 f, \tag{9}
\]

where \(a = \frac{1}{2} \sigma \sigma^\top\), \(D^2\) denotes the Hessian and for matrices \(M, N, M : N = \sum_{i,j} M_{ij} N_{ij}\). Throughout, \(\bar{O}\) is used to denote the convex hull of \(O\), \(C_c^\infty((0, T) \times \mathbb{R}^n)\) denotes the set of compactly supported infinitely differentiable functions on \((0, T) \times \mathbb{R}^n\), \(C_b(\mathbb{R}^n)\) denotes the set of bounded continuous function on \(\mathbb{R}^n\), \(C^{1,2}((0, T) \times \mathbb{R}^n)\) denotes the set of continuous functions of the form \([0, T] \times \mathbb{R}^n \ni (t, x) \mapsto f(t, x)\) that are once continuously differentiable in \(t\) and twice so in \(x\), \(B_R(x)\) denotes the closed ball of radius \(R > 0\) around \(x \in \mathbb{R}^n\), \(B_R = B_R(0), e_i\) denotes the \(i^{th}\) Euclidean basis vector in \(\mathbb{R}^n\), and \(C > 0\) denotes a generic constant that may change from line to line. The expression \(\mathbb{1}_A\) denotes the indicator function on the set \(A\). We denote \(\Delta_T = \{(s, t) : 0 \leq s \leq t \leq T\}\). The notation \(\partial_t Z_t^{s, x} = \partial_x Z_t^{s, x}\) is used and similarly for the higher order derivatives \(\partial^\alpha Z_t^{s, x}\) for multiindices \(\alpha\). Moreover, for a multiindex \(\alpha\), we denote

\[
\kappa_\alpha = (e_1, \underbrace{\ldots, e_1}_{\alpha_1 \text{ times}}, e_2, \ldots).
\]

**Definition 2.1.** A positive random function \(V : \Omega \times [0, T] \times O \to (0, \infty)\) is referred to as a \((\bar{b}, \bar{\sigma}, \alpha, \beta, p^*, V_0)\)-Lyapunov function if \(\mathcal{F} \otimes \mathcal{B}([0, T])\)-measurable and \(\mathcal{F}_t\)-adapted processes \(\bar{b} : \Omega \times [0, T] \times O \to \mathbb{R}^n\), \(\bar{\sigma} : \Omega \times [0, T] \times O \to \mathbb{R}^{n \times n}\), \(\alpha, \beta : \Omega \times [0, T] \to [0, \infty]\), \(p^* \in [1, \infty)\) and \(V_0 \in C^{1,2}([0, T] \times O)\) are such that for all \(y \in O\) there exists an \(\mathcal{F} \otimes \mathcal{B}([0, T])\)-measurable, \(\mathcal{F}_t\)-adapted
process \( Y^y : \Omega \times [0, T] \to \mathbb{R} \) that is \( \mathbb{P} \)-a.s. continuous, \( V(t, y) = V_0(t, Y^y_t) \) and it holds \( \mathbb{P} \)-a.s. that

\[
\int_0^T (|\dot{b}^y_s| + |\dot{\sigma}^y_s|^2 + |\alpha_s|)ds < \infty,
\]

\[
Y^y_{t, \tau} = y + \int_0^t 1_{(0, \tau)}(u) \dot{b}^y_u du + \int_0^t 1_{[0, \tau)}(u) \dot{\sigma}^y_u dW_u,
\]

(10)

\[
(\partial_t + L)V_0(t, Y^y_t) + \frac{p^* - 1}{2} \left( |(\dot{\sigma}^y_t)^T \nabla V_0(t, Y^y_t)|^2 \right) \leq \alpha_t V_0(t, Y^y_t) + \beta_t
\]

(11)

for all \( t \in [0, T] \), \( y \in \Omega \) and stopping times \( \tau \), where \( L \) is given by (9) with \( \tilde{b}, \tilde{\sigma} \) replacing \( b, \sigma \).

**Definition 2.2.** A function \( V \) is referred to as a Lyapunov function if there exist \( \bar{n} \in \mathbb{N} \), \( p^* \in [1, \infty) \), open \( \bar{O} \subseteq \mathbb{R}^{\bar{n}} \), \( \bar{b} : \Omega \times [0, T] \times \bar{O} \to \mathbb{R}^\bar{n} \), \( \bar{\sigma} : \Omega \times [0, T] \times \bar{O} \to \mathbb{R}^{\bar{n} \times \bar{n}} \), \( V_0 \in C^{1,2}([0, T] \times \bar{O}) \), along with some \( \alpha \) and \( \beta \) such that \( V : \Omega \times [0, T] \times \bar{O} \to (0, \infty) \) is a \((\bar{b}, \bar{\sigma}, \alpha, \beta, p^*, V_0)\)-Lyapunov function and

\[
\left\| e^{\int_0^T |\alpha_s|ds} \right\|_{L^{p^*}([0, T] \times \bar{O})} dt + \int_0^T \left\| \frac{\beta_v}{e^{\int_0^t \alpha_s ds}} \right\|_{L^{p^*}([0, T] \times \bar{O})} dv dt < \infty.
\]

(12)

**Remark 2.1.** (i) Smooth functions \( V \) satisfying \( LV \leq CV \) for some constant \( C \) as in [26, Theorem 3.5] form Lyapunov functions with \( p^* = 1 \), \( \alpha_t = C \) and \( \beta_t = 0 \).

(ii) Lyapunov functions satisfy the stochastic Grönwall inequality as in Theorem 2.4 in [19] along with a finiteness condition on the associated processes, which are properties that will be used many times throughout the paper.

The following property allows control across families of Lyapunov functions.

**Definition 2.3.** For a family of functions \((\hat{W}_s)_{s \in [0, T]}\), we say that \((\hat{W}_s)_{s \in [0, T]}\) is \((\hat{n}, \hat{O}, V_0)\)-local in \( s \) if \( \hat{n} \in \mathbb{N} \), open \( \hat{O} \subseteq \mathbb{R}^{\hat{n}} \), \( V_0 \in C^{1,2}([0, T] \times \hat{O}) \) are such that there exists a constant \( C > 0 \) satisfying that for any \( s \in [0, T] \), \( \hat{W}_s : \Omega \times [0, T] \times \hat{O} \to (0, \infty) \) is a \((\hat{b}_s^{s, T}, \hat{\sigma}_s^{s, T}, \alpha^{s, T}, \beta^{s, T}, p^{s, T}, V_0(s + \cdot, \cdot))_{[0, T] \times \hat{O}}\)-Lyapunov function for some \( \hat{b}^{s, T}, \hat{\sigma}^{s, T} \) together with some \( \alpha^{s, T}, \beta^{s, T}, p^{s, T} \) where (12) holds uniformly with bound \( C \), that is,

\[
\left\| e^{\int_0^T |\alpha^{s, T}_u|du} \right\|_{L^{p^{s, T}}([0, T] \times \hat{O})} + \int_0^T \left\| \frac{\beta^{s, T}_v}{e^{\int_0^t \alpha^{s, T}_s ds}} \right\|_{L^{p^{s, T}}([0, T] \times \hat{O})} dv < C.
\]

(13)

We say that \((\hat{W}_s)_{s \in [0, T]}\) is local in \( s \) if there exist \( \hat{n}, \hat{O}, V_0 \) such that \((\hat{W}_s)_{s \in [0, T]}\) is \((\hat{n}, \hat{O}, V_0)\)-local in \( s \).

A family of Lyapunov functions being local in \( s \) allows terms of the form \( \mathbb{E}[\hat{W}_s(t, X^{s, T}_t)] \) to be bounded uniformly in \( s \) after applying Theorem 2.4 in [19]. This is an
important property for twice differentiable solutions to Kolmogorov equations, since such solutions and many lemmatic terms depend on a time variable via the starting times \( s \). On the other hand, such a property is in all of the examples mentioned here easily satisfied.

3 Moment estimates on derivative processes

The following assumption states our main requirement on the Lyapunov function. Alternative assumptions for the main results in the case where \( b \) and \( \sigma \) are independent of \( t \) and admit locally Lipschitz derivatives are given in Theorem 5.2.

**Assumption 1.** There exists \( G : \Omega \times [0, T] \times O \to [0, \infty) \) such that \( G \) is \( \mathcal{F} \otimes \mathcal{B}(0, T) \otimes \mathcal{B}(O) \)-measurable, \( G(t, \cdot) \) is \( \mathcal{F}_t \otimes \mathcal{B}(O) \)-measurable, continuous \( \mathbb{P} \)-a.s. for all \( t \),

\[
|b(t, x) - b(t, y)| \leq (G(t, x) + G(t, y))|x - y|, \quad (14)
\]

\[
||\sigma(t, x) - \sigma(t, y)||^2 \leq (G(t, x) + G(t, y))|x - y|^2, \quad (15)
\]

for all \( t \in [0, T] \), \( x, y \in O \) and such that for any \( s \in [0, T] \), there exist finite sets \( I_0, I'_0 \subset \mathbb{N} \), locally bounded functions \( M : (0, \infty) \to (0, \infty) \), \((\bar{x}_i)_{i \in I_0 \cup I'_0}\) and Lyapunov functions \((V_i)_{i \in I_0 \cup I'_0}\) satisfying

\[
\int_0^t G(s+r, X^{s,x}_r)dr \leq M(m) + m \left( \sum_{i \in I_0} \int_0^t \log V_i(r, \bar{x}_i(x))dr + \sum_{i' \in I'_0} \log V_{i'}(t, \bar{x}_{i'}(x)) \right) \quad (16)
\]

\( \mathbb{P} \)-a.s. for all \( m > 0 \), \( x \in O \) and stopping times \( t \leq T - s \).

Throughout the paper, the process \( Y_t \) associated with Lyapunov functions can be thought of to be equal to \( X_t \) and in the applications here, it is enough to take

\[ G \leq m \log V_0 + M \]

in place of (16); the generality is justified by a trick to increase the set of admissible Lyapunov functions, as exemplified by the inclusion of \( \bar{U} \) in [19, Corollary 3.3], see also [11, Theorem 2.24]. Assumption 1 is strictly weaker than assuming globally Lipschitz coefficients, since polynomial Lyapunov functions are easily constructed in that case. In addition, throughout, whenever continuous differentiability up to some order \( m^* \) of \( b \) and \( \sigma \) is assumed, we also assume \( \mathbb{P} \)-a.s.

\[
\sum_{\theta \in \mathbb{N}_0^m; ||\theta|| \leq m^*} \int_0^T \sup_{|s| \leq R} \left( |\partial^\theta b(t, x)| + ||\partial^\theta \sigma(t, x)|| \right) dt < \infty, \quad \forall R > 0. \quad (17)
\]

For \( x \in O \), \( s \in [0, T] \), let \( X^{s,x}_{t(e)} \) be the first \( t \)-uniform derivatives in probability of \( X^{s,x}_{t(e)} \) with respect to the initial value in any directions \( \kappa \in \mathbb{R}^{2n} \), that is, for
Lemma 3.1. Under Assumption 1, for any $p$ and $M$ consistent with Assumption 1, we omit in the notation the dependence of $\epsilon > 0$, $T > 0$, $t \leq T - s$, it holds that

$$\mathbb{P}\left( \sup_{t \in [0, T - s]} \left| \frac{X_t^{s, x + r\kappa} - X_t^{s, x}}{r} - X_t^{s, x}(\kappa) \right| > \epsilon \right) \to 0$$

as $r \to 0$ with $r \neq 0$, $x + r\kappa \in O$. If $b(t, \cdot)$ and $\sigma(t, \cdot)$ are once continuously differentiable on $O$ for all $t \in [0, \infty)$ and satisfy (17) with $m^* = 1$, then by Theorem 4.10 in [28], $X_t^{s, x}(\kappa)$ exists for any $x \in O$, $s \in [0, T]$ and satisfies the system obtained by formal differentiation of (8), that is,

$$dX_t^{s, x}(\kappa) = (X_t^{s, x}(\kappa) \cdot \nabla)b(s + t, X_t^{s, x}(\kappa))dt + (X_t^{s, x}(\kappa) \cdot \nabla)\sigma(s + t, X_t^{s, x}(\kappa))dW_t. \tag{18}$$

By induction, if $b(t, \cdot)$ and $\sigma(t, \cdot)$ are continuously differentiable on $O$ up to some order $p$ for all $t \in [0, \infty)$ and satisfy (17) with $m^* = p$, then the $p^{th}$-order $t$-uniform derivative in probability of $X_t^{s, x}$ with respect to the initial value in directions $(\kappa_i)_{1 \leq i \leq p}$, $\kappa_i \in \mathbb{R}^n$, $|\kappa_i| = 1$, $1 \leq i \leq p$ exists for any $x \in O$, $s \in [0, T]$ and satisfies the system obtained by a corresponding $p^{th}$-order formal differentiation of (8).

First we state a straightforward application of the Lyapunov property to obtain an estimate of a time integral, which will be used later and is also demonstrative for many similar derivations in the following. Throughout and consistent with Assumption 1, we omit in the notation the dependence of $V$, $\bar{x}$ and $M$ on $s$.

**Lemma 3.2.** Under Assumption 1, for any $s \in [0, T]$, $c > 0$, it holds that

$$\mathbb{E}e^{\int_t^T \frac{1}{1 - s} G(s, t, X_t^{s, x})dt} \leq e^\hat{M} \left( \frac{1}{T - s} \int_0^{T - s} \mathbb{E}[V_i(t, \bar{x}_i(x))]dt + \sum_{i \in I_0} \mathbb{E}[V_i(0, \bar{x}_i(\kappa))] \right)$$

$$\leq C \left( \sum_{i \in I_0 \cup I_0^*} \mathbb{E}[V_i(0, \bar{x}_i(\kappa))] + 1 \right) \leq \infty$$

for all $x \in O$, where $\hat{M} = M(m)$ for some $m$.

**Proof.** The first inequality is (16) together with Jensen’s inequality; the second inequality follows by Theorem 2.4 in [19] with $q_1 = 1$, $q_2 = \frac{p}{p - 1}$, $p = p^*$; the third and last inequalities follow by (12). \qed

**Lemma 3.3.** Under Assumption 1, for any $k > 0$, $s \in [0, T]$, there exists $\rho > 0$ such that

$$\mathbb{E} \sup_{0 \leq t \leq T - s} \left| X_t^{s, x}(\kappa) \right|^k \leq \rho W(x, r\kappa) \left| r \right|^k \tag{19}$$

for all $x \in O$, $r \in \mathbb{R} \setminus \{0\}$, $\kappa \in \mathbb{R}^n$, $|\kappa| = 1$, $x + r\kappa \in O$, where $X_t^{s, x}(\kappa) := X_t^{s, x + r\kappa} - X_t^{s, x}$ and $W(x, r\kappa) := 1 + \sum_{i \in I_0 \cup I_0^*} \mathbb{E}[V_i(0, \bar{x}_i(x + r\kappa))] + \mathbb{E}[V_i(0, \bar{x}_i(x))]$. 

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If in addition \( b(t, \cdot), \sigma(t, \cdot) \) are continuously differentiable for all \( t \geq 0 \) and (17) holds with \( m^* = 1 \), then

\[
E \sup_{0 \leq t \leq T-s} |X^{(r)}_{t} - t X^{(r)}_{s}|^k \leq \rho W(x, 0) \quad (20)
\]

\[
\lim_{0 \neq r \to 0} E \sup_{0 \leq t \leq T-s} |X^{(r)}_{t} - t^{-1} X^{(r)}_{s}|^k = 0 \quad (21)
\]

for all \( x \in O, \kappa \in \mathbb{R}^n \) with \( |\kappa| = 1 \). If for each \( i \in I_0, i' \in I'_0 \), the functions \( V_i, V_{i'} \) are local in \( s \), then \( \rho \) is independent of \( s \).

**Remark 3.1.** In the proof of Lemma 3.2, only a one sided Lipschitz version of (14) is necessary, see also Section 3.3 in [19] or Corollary 2.31 in [11], which gives similar estimates to (19).

**Proof.** For any \( r \),

\[
dx^{(r)}_{t} = (b(s + t, X_{s}^{(r)} x) - b(s + t, X_{s}^{(r)})) dt + (\sigma(s + t, X_{s}^{(r)}) - \sigma(s + t, X_{s}^{(r)})) dW_t. \quad (22)
\]

Since \( X_t \) is almost surely continuous in \( t \), it holds that

\[
\int_0^t (G(s + u, X_{s}^{(ru)}) + \kappa) du \leq C \int_0^t (\log V(u, x + r\kappa)) + \log V(u, x) + 1) du < \infty
\]

for any \( \omega \in \Omega \) and finite \( 0 < t < T - s \) (for everywhere continuous modifications of \( X_t \)), therefore Corollary 2.5 in [19] can be applied with

\[
a_t = b(s + t, X_{s}^{(ru)}) - b(s + t, X_{s}^{(ru)}), \quad b_t = \sigma(t, X_{s}^{(ru)}) - \sigma(t, X_{s}^{(ru)}),
\]

\[
\alpha_t = \left( \frac{1}{2} + k \lor 1 \right) (G(s + t, X_{s}^{(ru)}) + G(s + t, X_{s}^{(ru)})),
\]

\[
p = 2k \land 2, \quad \beta_t = 0, \quad q_1 = k, \quad q_2 = 3k, \quad q_3 = \frac{3k}{2},
\]

to obtain

\[
E \sup_{0 \leq t \leq T-s} |X^{(r)}_{t} - t X^{(r)}_{s}|^k \leq C \left( E\int_0^{T-s} 3k \left( \frac{1}{2} + k \lor 1 \right) (G(s + u, X_{s}^{(ru)}) + G(s + u, X_{s}^{(ru)})) du \right)^{\frac{k}{2}} |r|^k. \quad (23)
\]

By Lemma 3.1, the expectation on the right-hand side of (23) satisfies the bound

\[
E \int_0^{T-s} 3k \left( \frac{1}{2} + k \lor 1 \right) (G(s + u, X_{s}^{(ru)}) + G(s + u, X_{s}^{(ru)})) du \leq C E \left[ 1 + \sum_{i \in I_0 \cup I'_0} V_i(0, \bar{x}_i(x + r\kappa) + V_i(0, \bar{x}_i(x)) \right],
\]

which gives (19).
The statement for $X_{s(\kappa)}$ follows along the same lines, where instead $X_{s(\kappa)}$ satisfies (18) and Corollary 2.5 in [19] can be applied as above except with

$$\alpha_t = (1 + 2k \vee 2)G(s + t, X^{s,t}_t).$$  \hfill (24)

Equation (21) is a known consequence; it is immediate from the definition of $X_{u(\kappa)}$, the previous bounds and

$$E[S_k^k] \leq cP(S_1^{s,k} \leq \epsilon) + E[I_{\{S_1^{s,k} > \epsilon\}}]S_{k1}^k \leq cP(S_1^{s,k} \leq \epsilon) + E[I_{\{S_1^{s,k} > \epsilon\}}]E[S_k^k]^{\frac{k1}{k}}\quad (25)$$

with $S = \sup_{0 \leq u \leq T-s}|X_{u(\kappa)}^{s,x} - r^{-1}X^{r}_u(\kappa)|$. The final assertion follows by noting that the constants $C$ above are independent of $s$. \hfill \Box

The following Assumption 2 states our requirements on the higher derivatives of $b$ and $\sigma$ for results on the higher derivatives of solutions to (8).

**Assumption 2.** There exist $p \in \mathbb{N}_0$ such that $b(t, \cdot)|\tilde{\sigma}(t, \cdot)|_{\tilde{\sigma}^k} \in C^p$ for all $t \geq 0$, $\omega \in \Omega$ and inequality (17) holds with $m^* = p$. Moreover, for all $s \in [0, T]$ and $k \geq 2$, there exist $M' > 0$, $\hat{n}_k \in \mathbb{N}$, open $O_k \subset \mathbb{R}^\hat{n}_k$, mappings $\tilde{x}_k : O \to O_k$ and Lyapunov function $\tilde{V}_k^{s,T} : \Omega \times [0, T - s] \times O_k \to (0, \infty)$ satisfying for any $x, x' \in O$ and multiindices $\alpha$ with $2 \leq |\alpha| \leq p$ that $\mathbb{P}$-a.s.

$$\left|\partial^\alpha b(s + t, \lambda X^{s,x}_t + (1 - \lambda)X^{s,x}_t) + ||\partial^\alpha \sigma(s + t, \lambda X^{s,x}_t + (1 - \lambda)X^{s,x}_t)||^2 \right| \leq M'(1 + \tilde{V}_k^{s,T}(t, \tilde{x}_k(x)) + \tilde{V}_k^{s,T}(t, \tilde{x}_k(x')))\quad (26)$$

holds for all $t \in [0, T - s]$, $\lambda \in [0, 1]$.

In the following, for $\kappa = (\kappa_i)_{1 \leq i \leq l}$, $\kappa_i \in \mathbb{R}^n$, the $t^\text{th}$ order $t$-uniform derivatives in probability of a process $Z^s_t$ with respect to initial condition $x$ in the directions $\kappa_1, \ldots, \kappa_l \in \mathbb{R}^n$ is denoted by $\partial^{(\kappa)}Z^s_t$.

**Theorem 3.3.** Under Assumptions 1 and 2, for any $s \in [0, T]$, constants $1 \leq l \leq p - 1$, $k_l > 0$, there exist $t^* \in \mathbb{N}$, $\nu \geq k_l$, $\{\ell_i \in \{1, \ldots, t^*\} \subset (0, \infty)$ and a finite order polynomial $q_0$, the degree of which is independent of $s, T, V, \tilde{V}_k^{s,T}$, such that

$$\mathbb{E}\sup_{0 \leq t \leq T-s} |\partial^{(\kappa)}X^{s,x+r(\kappa_1)}_t - \partial^{(\kappa)}X^{s,x}_t|^ {k_1} \leq (T - s)\nu q(x, x + r(\kappa_1))|r|^{k_1}$$ \hfill (27)

$$\mathbb{E}\sup_{0 \leq t \leq T-s} |\partial^{(\kappa)}X^{s,x}_t|^ {k_1} \leq (T - s)\nu q(x, x)$$ \hfill (28)

$$\lim_{r \to 0}\mathbb{E}\sup_{0 \leq t \leq T-s} |\partial^{(\kappa)}X^{s,x}_t - r^{-1}(\partial^{(\kappa)}X^{s,x+r(\kappa_1)}_t - \partial^{(\kappa)}X^{s,x}_t)|^{k_1} = 0$$ \hfill (29)
for all initial condition \( x \in \mathbb{O}, r \in \mathbb{R} \setminus \{0\}, \kappa_i \in \mathbb{R}^n, |\kappa_i| = 1, 1 \leq i \leq l + 1, x + r\kappa_{l+1} \in \mathbb{O} \), where \( \kappa = (\kappa_i)_{1 \leq i \leq l}, \tilde{\kappa} = (\kappa_i)_{1 \leq i \leq l+1} \) and \( q : \mathbb{O} \times \mathbb{O} \rightarrow \mathbb{R} \) is given by

\[
q(y, y') = E_{q_0}((V_i(0, \bar{x}_i(y)))_{i \in I}, (\hat{V}^{s,T}_{l_i}(0, \hat{x}_{l_i}(y)))_{i \in \{1, \ldots, r\}},
(\hat{V}^{s,T}_{l_i}(0, \hat{x}_{l_i}(y')))_{i \in \{1, \ldots, r\}}).
\]

If \( V \) and \( \hat{V}^{s,T}_k \) are local in \( s \) for every \( k \), then the form of the polynomial \( q_0 \) is independent of \( s, T, V, \hat{V}^{s,T}_k \).

**Remark 3.2.** Assumption 2 can be weakened if only finite order moments of the derivatives in Theorem 3.3 are sought after, that is, if the statements in Theorem 3.3 are only required to hold for \( k_1 \) up to some finite \( k_1 \leq K_1 \). In particular, \( M', n_k, O_k, \hat{x}_k, \hat{V}_k \) in Assumption 2 need only exist for \( k \) up to some finite \( k \leq K \). In the same vein, Assumption 1 can be weakened (so that the Lipschitz constants are only required to be \( O(\log V) \) and \( O(\sqrt{\log V}) \) in this case if \( T \) is sufficiently small.

**Proof.** Fix \( k_1 > 0, s \in [0, T] \), let \( J \) be the set of strictly increasing functions from \( \mathbb{N} \) to itself and \( D^{(\kappa)}b(s + t, X_t^{s,x}) \) denote the formal derivative of \( b(s + t, X_t^{s,x}) \) with respect to \( x \) in the directions indicated by \( \kappa \). In particular,

\[
D^{(\kappa)}b(s + t, X_t^{s,x}) = \left( \partial^{(\kappa)} X_t^{s,x} : \nabla \right) b(s + t, X_t^{s,x}) + q_{\kappa, X_t^{s,x}}(\prod_{1 \leq i \leq \nu} \partial^{(\kappa_{j(i)})} X_t^{s,x}, 1 \leq l' \leq l - 1, j \in J),
\]

where the last term denotes a \( \mathbb{R}^n \)-valued polynomial taking arguments as indicated, for which exactly \( l \) of the operators \( \partial^{(\kappa_{j(i)})} \) appear in each term and coefficients are spatial derivatives between orders 2 and \( l \) of elements of \( b \) evaluated at \( (s + t, X_t^{s,x}) \). The term \( D^{(\kappa)}\sigma(s + t, X_t^{s,x}) \) is similarly defined. Denoting \( x' = x + r\kappa_{l+1} \), the difference processes of the derivatives satisfy

\[
d(\partial^{(\kappa)} X_t^{s,x'} - \partial^{(\kappa)} X_t^{s,x}) = (D^{(\kappa)}b(s + t, X_t^{s,x'}) - D^{(\kappa)}b(s + t, X_t^{s,x}))dt
+ (D^{(\kappa)}\sigma(s + t, X_t^{s,x'}) - D^{(\kappa)}\sigma(s + t, X_t^{s,x}))dW_t
\]

on \( t \in [0, T - s] \) for all \( x, x' \in \mathbb{O}, r \in \mathbb{R} \setminus \{0\}, \kappa_i \in \mathbb{R}^n, |\kappa_i| = 1, 1 \leq i \leq l + 1 \).

We proceed by strong induction in \( l \) for (27). A base case has been established in Lemma 3.2. By the fundamental theorem of calculus on derivatives
of $b$ and $\sigma$, inequalities (26), (14) and (15), it holds $\mathbb{P}$-a.s. that

$$
\left| D^{(\kappa)}b(s + t, X_t^{s,x'}) - D^{(\kappa)}b(s + t, X_t^{s,x}) \right|
\leq \sum_i \left| \left( \partial^{(\kappa)} X_t^{s,x'} - \partial^{(\kappa)} X_t^{s,x} \right)_i \| \partial b(s + t, X_t^{s,x}) \| + H(t, X_t^{s,x}, X_t^{s,x'}) \hat{q}_t \right|
\leq 2 \left| \partial^{(\kappa)} X_t^{s,x'} - \partial^{(\kappa)} X_t^{s,x} \right| G(s + t, X_t^{s,x}) + H(t, X_t^{s,x}, X_t^{s,x'}) \hat{q}_t,
$$

$$\| D^{(\kappa)}\sigma(s + t, X_t^{s,x'}) - D^{(\kappa)}\sigma(s + t, X_t^{s,x}) \|^2
\leq 2 \sum_i \left| \left( \partial^{(\kappa)} X_t^{s,x'} - \partial^{(\kappa)} X_t^{s,x} \right)_i \right|^2 \| \partial \sigma(s + t, X_t^{s,x}) \|^2 + (H(t, X_t^{s,x}, X_t^{s,x'}) \hat{q}_t)^2
\leq 4 \left| \partial^{(\kappa)} X_t^{s,x'} - \partial^{(\kappa)} X_t^{s,x} \right|^2 G(s + t, X_t^{s,x}) + (H(t, X_t^{s,x}, X_t^{s,x'}) \hat{q}_t)^2,
$$
on $t \in [0, T]$, where

$$H(t, X_t^{s,x}, X_t^{s,x'}) = M' \left( 1 + \hat{V}_{4k_1\lor 4}(t, \hat{x}_{4k_1\lor 4}(x)) + \hat{V}_{4k_1\lor 4}(t, \hat{x}_{4k_1\lor 4}(x')) \right) \frac{\mathbb{1}_{4^k}}{\mathbb{1}_{4^k}}$$

(31)

and $\hat{q}_s$ denotes a polynomial with constant coefficients taking arguments from the set $S = S_1 \cup S_2$,

$$S_1 = \left\{ \left( \prod_{1 \leq i \leq l'} \partial^{(\kappa_{\sum_{i} j_i})} \right) X_t : 1 \leq l' \leq l, j \in J, X_t \in \{ X_t^{s,x'}, X_t^{s,x} \} \right\}
$$

$$S_2 = \left\{ \left( \prod_{1 \leq i \leq l'} \partial^{(\kappa_{\sum_{i} j_i})} \right) (X_t^{s,x'} - X_t^{s,x}) : 1 \leq l' \leq l - 1, j \in J \right\}
$$

$$\cup \left\{ X_t^{s,x'} - X_t^{s,x} \right\},
$$

for which exactly $l$ of the operators $\partial^{(\kappa_{\sum_{i} j_i})}$ appear in each term of $\hat{q}_s$, and a factor from $S_2$ appears exactly once in each term. Note for $p \geq 2$ and by Lemma 3.1, it holds $\mathbb{P}$-a.s. that

$$\int_0^{T-s} \log V_i(t, \bar{x}_i(x)) dt + \log V_{i'}(t, \bar{x}_{i'}(x)) < \int_0^{T-s} V_i(t, \bar{x}_i(x)) dt + V_{i'}(t, \bar{x}_{i'}(x)) < \infty
$$
on $t \in [0, T - s]$ for all $i \in I_0$ and $i' \in I_0'$. Corollary 2.5 in [19] can then be
applied with

\[
\begin{align*}
\alpha_t &= D^{(s)}b(s + t, X_t^{s,x}) - D^{(s)}b(s + t, X_t^{s,x}), \\
b_t &= D^{(s)}\sigma(s + t, X_t^{s,x}) - D^{(s)}\sigma(s + t, X_t^{s,x}), \\
\alpha_t &= 2pG(s + t, X_t^{s,x}) + \frac{1}{2} > 0, \\
\beta_t &= \sqrt{2k_1 \vee 2H(t, X_t^{s,x}, X_t^{s,x'})\hat{q}_t}, \\
p &= 4k_1 \vee 4, \quad q_1 = k_1, \quad q_2 = \left(\frac{1}{k_1} - \frac{1}{2k_1 \vee 2}\right)^{-1}, \quad q_3 = 2k_1 \vee 2
\end{align*}
\]

(32)

to obtain

\[
\mathbb{E} \sup_{0 \leq t \leq T-s} \left| \partial^{(s)}X_t^{s,x'} - \partial^{(s)}X_t^{s,x} \right|^k \leq CA_T^{(1)}A_T^{(2)},
\]

where

\[
A_T^{(1)} := \left( \mathbb{E} \left[ \exp \left( 2q_2 p \log m + 2q_2 pm \sum_{i \in I_0} \log V_i(T - s, \bar{x}_i(x)) \right. \right. \\
&\left. \left. + \int_0^{T-s} \sum_{i \in I_0} \left( 2q_2 pm \log V_i(u, \bar{x}_i(x)) + \frac{q_2}{2} du \right) \right] \right)^{\frac{k}{k_1}}
\]

\[
A_T^{(2)} := \left( \mathbb{E} \left[ \int_0^{T-s} \left( 2k_1 \vee 2 \right) \left( H(u, X_u^{s,x}, X_u^{s,x'})\hat{q}_u \right)^2 du \right] \right)^{\frac{k}{k_3}}.
\]

By substituting our expressions for \( q_2 \), setting

\[m = \frac{1}{8(T-s)(k_1 \vee 1)(|I_0| + |I_0'|)}, \quad \left( \frac{1}{k_1} - \frac{1}{2k_1 \vee 2}\right)^{-1}
\]

with the effect that \( M(m) \) is locally bounded in \( T \) and using Lemma 3.1, the first expectation has the bound

\[
A_T^{(1)} \leq \left( \mathbb{E} \left[ \frac{C}{T-s} \int_0^{T-s} \exp \left( 2(T-s)q_2 pm \left( \sum_{i \in I_0} \log V_i(u, \bar{x}_i(x)) \right) \right. \right. \\
&\left. \left. + \sum_{i \in I_0} \log V_i(T - s, \bar{x}_i(x)) \right) \right] \right)^{\frac{k}{k_1}}
\]

\[
\leq \left( \mathbb{E} \left[ \frac{C}{T-s} \int_0^{T-s} \left( \sum_{i \in I_0} V_i(u, \bar{x}_i(x)) \right. \right. \\
&\left. \left. + \sum_{i \in I_0} V_i(T - s, \bar{x}_i(x)) \right) \right] \right)^{\frac{k}{k_1}}
\]

\[
\leq C \left( \sum_{i \in I_0'} \mathbb{E}[V_i(0, \bar{x}_i(x))] + \sum_{i \in I_0} \mathbb{E}[V_i(0, \bar{x}_i(x))] + 1 \right)^{\frac{k}{k_1}},
\]

where note \( C \) is, here and in the rest of the proof, locally bounded as a function of \( T \) and also of \( s, T \) if \( V \) is local in \( s \). On the other hand, by the inductive
argument and the form of $H$, $\hat{q}_s$ and $q_3$, it holds that

\[
A_T^{(2)} - s \leq C \left( \mathbb{E} \left( \left( \int_0^T H(u, X_u, X_u')^2 du \right)^{k_1} \sup_{0 \leq u \leq T-s} \hat{q}^{2k_1 \vee 2} \right) \right)^\frac{k_1}{2k_1 \vee 2}
\]

\[
\leq C \left( \mathbb{E} \left( \int_0^T H(u, X_u, X_u')^2 du \right)^{2k_1 \vee 2} \right)^\frac{k_1}{2k_1 \vee 2}
\]

\[
\cdot \left( \mathbb{E} \sup_{0 \leq u \leq T-s} \hat{q}^{4k_1 \vee 4} \right)^\frac{k_1}{4k_1 \vee 4}
\]

\[
\leq C \left( (T-s)^{(2k_1 \vee 2)-1} \int_0^{T-s} \mathbb{E} \left( 1 + \hat{V}_{4k_1 \vee 4}(u, \hat{x}_{4k_1 \vee 4}(x)) \right) \right)^{\frac{k_1}{2k_1 \vee 2}} \hat{q}(x, x') |r|^{k_1},
\]

where

\[
\hat{q}(x, x') = \mathbb{E} \hat{q}_0((V_i(0, \hat{x}_i(x)))_{i \in I_0 \cup I_1}, (\hat{V}_i(0, \hat{x}_i(x)))_{i \in \{1, \ldots, \hat{i}^*\}})
\]

for some $\hat{i}^* \in \mathbb{N}$, $\{i_i \}_{i \in \{1, \ldots, \hat{i}^*\}} \subset (0, \infty)$ and finite order polynomial $\hat{q}_0$ taking arguments as indicated. Therefore, by Theorem 2.4 in [19] with $q_1 = 1$, it holds that

\[
A_T^{(2)} - s \leq C \left( (T-s)^{(2k_1 \vee 2)} (1 + \mathbb{E} \hat{V}_{4k_1 \vee 4}(0, \hat{x}_{4k_1 \vee 4}(x))) \right)^{\frac{k_1}{2k_1 \vee 2}} \hat{q}(x, x') |r|^{k_1},
\]

which concludes the proof for (27). Inequality (28) follows along the same lines, therefore the argument is not repeated. Equation (29) holds by (25) with

\[
S = \sup_{0 \leq u \leq t} \left| \partial^{(\hat{k}_1)} X_u - r^{-1} (\partial^{(\hat{k}_1)} X_u X_u' - \partial^{(\hat{k}_1)} X_u X_u') \right|.
\]

\[\blacksquare\]

**Remark 3.3.** A way to prove weaker versions of Lemma 3.2 and Theorem 3.3 is instead of using the stochastic Gronwall inequality, that is, Theorem 2.4 and Corollary 2.5 in [19], to use Lemma 4.2 in [28] and Theorem 3.5 in [26]. For this, one works directly with the SDEs governing $|\partial^{(\hat{k}_1)} X_u|^k$ in the proof and inequality (11) is to be replaced by $(\partial_t + L)V_0 \leq CV_0$. 

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4 Kolmogorov’s equation

Throughout this section, we assume that \( b \) and \( \sigma \) are nonrandom functions. In Section 4.1, the moment estimates from Section 3 are used to derive \( p\)th differentiability of a Feynman-Kac semigroup and in particular of \( x \mapsto E g(X_t^{s,x}) \) for \( X_t^{s,x} \) solving (8). We allow the functions such as \( g \) to be bounded by Lyapunov functions. As such, the proof, although the approach of which is classical, demands finer attention compared to [28] for example, in which \( g \) and its derivatives are only required to be polynomially bounded. This regularity is then used to show that the semigroup solves the Kolmogorov equation in the almost everywhere sense in Section 4.2. In Section 4.3, we complement our results with a criterion for the Kolmogorov equation to be solved in the distributional sense. The criterion is based on the maximal dissipativity of the closure of the infinitesimal generator associated to (1).

4.1 Semigroup differentiability

To begin, a condition that will be imposed on functions such as \( g \) is stated.

**Definition 4.1.** For \( p \in \mathbb{N}, k \geq 1, h : \Omega \times [0,T] \times \tilde{O} \rightarrow \mathbb{R} \) with \( h(t,\cdot) \in C^p(\tilde{O}) \) for all \( (\omega,t) \in \Omega \times [0,T] \), we say that \( h \) has \((p,k)\)-Lyapunov derivatives if there exist \((V^{s,T})_{s \in [0,T]}\) local in \( s \), locally bounded \( \hat{x} \) and constant \( N > 0 \) such that for any \( s \in [0,T] \) and multiindices \( \alpha \) with \( 0 \leq |\alpha| \leq p \), it holds \( \mathbb{P}\)-a.s. that

\[
\left| \partial^\alpha h(s+t,\lambda X_t^{s,x} + (1-\lambda)X_t^{s,x'}) \right| \leq N(1 + V^{s,T}(t,\hat{x}(x)) + V^{s,T}(t,\hat{x}(x')))^{\hat{\gamma}}
\]

for all stopping times \( t \leq T-s \), \( x,x' \in O \) and \( \lambda \in [0,1] \).

We make the following mild assumptions about the Lyapunov functions \( V^{s,T} \) associated to Lyapunov derivatives in Definition 4.1 or otherwise. These are gathered with additional assumptions on the SDE (8). Assumption 3 will be referenced only when \( V^{s,T} \) has been given in the context.

**Assumption 3.** For each \( R \geq 0 \), there exists a Borel, locally integrable \( K_1(R) : [0,\infty) \rightarrow [0,\infty) \) such that

\[
2(x-y,b(t,x) - b(t,y)) + \|\sigma(t,x) - \sigma(t,y)\|^2 \leq K_1(R)|x-y|^2
\]

for all \( t \geq 0, x,y \in B_R \cap O \). For any \( s \geq 0, T > 0, x \in O \), there exists a \( \mathbb{P}\)-a.s. continuous \( O\)-valued unique solution \( X_t^{s,x} \) to (8) on \([0,T]\). Moreover, for any \( T > 0 \), there exist \( \hat{n} \in \mathbb{N}, \) open \( \tilde{O} \subseteq \mathbb{R}^n \), \( V_0 \in C^{1,2}([0,\infty) \times \tilde{O}) \), \( \hat{x} : \mathbb{R}^n \rightarrow \tilde{O}, \)

\[
G : [0,\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}, \text{ constant } C \geq 0 \text{ and } 0 < l \leq 1 \text{ such that }
\]

(i) \( (V^{s,T})_{s \in [0,T]} \) is \((\tilde{n},\tilde{O},V_0)\)-local in \( s \),

(ii) for any \( s \geq 0, V^{s,T} \) is a \((\hat{b},\hat{\sigma},\alpha,C,p^*,V_0)\)-function for some \( \tilde{b},\tilde{\sigma},\alpha,p^*,\tilde{V}_0, \)
(iii) for any $s \geq 0$, it holds $\mathbb{P}$-a.s. that
\[
V^{s+\tau,T}(0, \tilde{x}(X^s_t)) \leq C(1 + V^{s,T}(\tau, \tilde{x}(x)))
\]
for all $x \in O$ and stopping times $\tau \leq T$,

(iv) for any $s \geq 0$, it holds that $\lim_{|x|\to\infty} \inf_{t \in [0,T]} \hat{G}(t, x) = \infty$ and $\mathbb{P}$-a.s. that
\[
\hat{G}(s + t, X^s_t) \leq V^{s,T}(t, \tilde{x}(x))
\]
for all $t \in [0, T]$, $x \in O$.

Besides the first two sentences, Assumption 3 is satisfied by the Lyapunov functions considered for example in [11, Corollary 2.4]. More specifically, taking $\alpha$ and the functions $U, \hat{U}$ from there, for $n = n + 1$, one may take $V_0 = V_0(t, (x, y)) = e^{U(x)e^{-\alpha t} + y}$ and $\tilde{x} = \tilde{x}(x) = (x,0) \in \mathbb{R}^{n+1}$, then
\[
\hat{b}(t, (x, y)) = (b(t, x), \hat{U}(t, x)), \quad \hat{\sigma}(t, (x, y)) = \begin{pmatrix} \sigma(t, x) & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
\hat{G}(t, x) = e^{U(x)e^{-\alpha t}}
\]
for $t \geq 0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ and the latter statements of Assumption 3 are satisfied by the conditions on $U$ and $\hat{U}$ if $\lim_{|x|\to\infty} U(x) = \infty$ and $\hat{U} \geq C$ for some $C \in \mathbb{R}$ everywhere.

**Theorem 4.2.** Let $T > 0$, let Assumptions 1, 2 hold and let $f : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $c : \Omega \times [0,T] \times \mathbb{R}^n \to [0, \infty)$, $g : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be such that $f(\cdot, x), c(\cdot, x)$ are $\mathcal{F} \otimes \mathcal{B}([0,T])$-measurable functions for every $x \in \tilde{O}$, satisfying $\int_0^T \sup_{x \in B_R \cap \tilde{O}} (|c(t, x)| + |f(t, x)|)dt < \infty$ for every $R > 0$ and $\mathbb{P}$-a.s. $f(t, \cdot)|_{\tilde{O}}, c(t, \cdot)|_{\tilde{O}}, g|_{\tilde{O}} \in C^p(\tilde{O})$ for all $(\omega, t) \in \Omega \times [0, T]$. Suppose there exists $k_2 > 1$ such that $f$ and $g$ have $(p, k_2)$-Lyapunov derivatives. There exists $K > 1$ such that if for any $1 < k' < K$, $c$ has $(p, k')$-Lyapunov derivatives and all of the Lyapunov functions associated to Lyapunov derivatives are such that Assumption 3 is satisfied with $\tilde{I} > k_2^{-1}, K^{-1}$, then the following statements hold.

(i) For $u$ given by
\[
u(s, t, x) = \int_0^t f(s + r, X^s_r) e^{-\int_0^r c(s + w, X^s_w)dw} dr + g(X^s_t) e^{-\int_0^t c(s + w, X^s_w)dw}
\]
for $(s, x) \in [0, T] \times O$ and stopping times $t \leq T - s$, the expectation $\mathbb{E}u(s, t, x)$ is continuously differentiable in $x$ up to order $p$.

(ii) For every multiindex $\beta$ with $0 \leq |\beta| \leq p$, there exists a finite order polynomial $q^*$, the form of which is independent of $\hat{V}^{s,T}$ and $\bar{V}_k^{s,T}$, such that for all stopping times $t \leq T - s$, it holds that
\[
|\partial^\beta u(s, t, x)| \leq \mathbb{E}q^*((\hat{V}_0(0, \tilde{x}(x)))_{i \in I_0 \cup I'_0}, V^{s,T}_k(0, \tilde{x}(x)), (\bar{V}^{s,T}_k(0, \tilde{x}_i(x)))_{i \in I_1})
\]
\[
(iii) \text{for any } s \geq 0, \text{ it holds } \mathbb{P}\text{-a.s. that }
\]
\[
V^{s+\tau,T}(0, \tilde{x}(X^s_T)) \leq C(1 + V^{s,T}(\tau, \tilde{x}(x)))
\]
for all $x \in O$ and stopping times $\tau \leq T$,

(iv) for any $s \geq 0$, it holds that $\lim_{|x|\to\infty} \inf_{t \in [0,T]} \hat{G}(t, x) = \infty$ and $\mathbb{P}$-a.s. that
\[
\hat{G}(s + t, X^s_t) \leq V^{s,T}(t, \tilde{x}(x))
\]
for all $t \in [0, T]$, $x \in O$. 
on \((s, x) \in [0, T] \times O\), where \(I^* \subset \mathbb{N}\) is finite, \(l_i > 0\) and \(\tilde{x}_t, \tilde{V}_{s,t}, \tilde{V}_{s,T}^*\) is representative of any and all of the functions across \(h \in \{f, c, g\}\), multiindices \(\alpha\) with \(0 \leq |\alpha| \leq |\beta|\) and \(k \in K_0 \subset (0, K)\) for some finite \(K_0\).

(iii) If for each \(k\) and multiindex \(\alpha\) with \(0 \leq |\alpha| \leq p\), the mappings \(\tilde{x}_k\) are independent of \(s\) and \((\tilde{V}_{s,T}^*)_{s \in [0, T]}\) is local in \(s\), then \(|\partial^\alpha \tilde{x}_k|\) is locally bounded for every multiindex \(\alpha\) with \(0 \leq |\alpha| \leq p\) and if \(p \geq 2\), then for any \(R > 0\), there exists a constant \(N > 0\) such that

\[
|\mathbb{E}u(s', T - s', x) - \mathbb{E}u(s, T - s, x)| \leq N|t - s|
\]

for all \(s, s' \in (0, T)\) and \(x \in B_R\).

We prove first a lemma that will be used in the proof of Theorem 4.2. Throughout the proofs of Theorem 4.2, Lemma 4.3 and consistent with the statement of the results, we omit in the notation any dependence of \(V_{s,T}, \tilde{x}\) and \(k_2\) on \(s\) and \(h\).

**Lemma 4.3.** Let the first sentence of Theorem 4.2 hold and let \(c\) have \((p, k')\)-Lyapunov derivatives for all \(1 \leq k' < K\), for \(K\) from the same theorem. For any \(h \in \{f, c, g\}\), \(k_3 > 0\) with \(k_3 < k_2\) if \(h \in \{f, g\}\), \(s \in [0, T], x \in O, k \in \mathbb{R}^n\) with \(|\alpha| = 1, X \in [0, 1]\), multiindex \(\alpha\) with \(0 \leq |\alpha| \leq p\) and stopping time \(t \leq T - s\), it holds that

\[
\mathbb{E} \int_0^1 \left| \partial^\alpha h(s + v, \lambda x_{v,s'} + (1 - \lambda)x_{v}) - \partial^\alpha h(s + v, x_{v,s}) \right|^{k_3} dv \to 0
\]

\[
\mathbb{E} \int_0^1 \left| \partial^\alpha h(s + t, \lambda x_{v,s'}) + (1 - \lambda)x_{v,s} \right|^k \to 0
\]

\[
\mathbb{E} \int_0^1 \int_0^1 \left| \partial^\alpha h(s + v, \lambda x_{v,s'} + (1 - \lambda)x_{v,s}) - \partial^\alpha h(s + v, x_{v,s}) \right|^k dv \to 0
\]

as \(x' \to x\), where the derivatives \(\partial^\alpha\) are in the spatial argument and \(g(t, \cdot) = g\).

**Proof.** For any \(\epsilon > 0, s \in [0, T]\) and stopping time \(t \leq T - s\), note that

\[
\mathbb{P}(\sup_{0 \leq u \leq t - s}|X_{u,s'}^{x'} - X_u^x| \leq \epsilon) \leq \mathbb{P}(|X_{t,s'}^{x'} - X_t^x| \leq \epsilon),
\]

so that for any \(\lambda \in [0, 1]\), by Theorem 1.7 in [28], it holds that \(\lambda X_{t,s'}^{x'} + (1 - \lambda)X_{t,s}^{x'} = \lambda(X_{t,s'}^{x'} - X_t^x) \to 0\) in probability as \(x' \to x\) (sequentially). Therefore for any multiindex \(\alpha\), \(J := \partial^\alpha h(s + t, \lambda X_{t,s'}^{x'} + (1 - \lambda)X_{t,s}^{x'}) - \partial^\alpha h(s + t, X_{t,s}^{x'}) \to 0\) in probability by Theorem 20.5 in [4]. Moreover, if \(h \in \{f, g\}\), by (25) with \(k_1 = k_3\), \(k = k_2\) and \(S = |J|\), it holds that \(\mathbb{E}|J|^{k_3} \to 0\) as \(x' \to x\). The same holds for \(h = c\) using (25) with some \(k > k_3\) instead. By the assumption (33) and Theorem 2.4 in [19], it holds that

\[
\mathbb{E}\left| \partial^\alpha h(s + u, \lambda X_{u,s'}^{x'} + (1 - \lambda)X_{u,s}^{x'}) \right|^k \leq C\mathbb{E}(1 + V_{s,T}^* (0, \tilde{x}(x)) + V_{s,T}^* (0, \tilde{x}(x')))\]
where \( C \) here is independent of \( r, \lambda \) and \( u \), so that Jensen’s inequality, Fubini’s theorem and dominated convergence theorem concludes the proof.

**Proof of Theorem 4.2.** For \( x \in O, s \in [0, T], \) stopping time \( t \leq T - s, \kappa \in \mathbb{R}^n, r \in \mathbb{R} \setminus \{0\}, |\kappa| = 1, \) let \( x' := x + r \kappa \in O \) and for \( h \in \{f, g\}, \) let

\[
\begin{align*}
\hat{h}_t^r & := \int_0^1 \nabla h(s + t, \lambda X_t^{s,x'}) + (1 - \lambda)X_t^{s,x})d\lambda, \\
\hat{h}(t, x) & := h(s + t, X_t^{s,x}), \\
c_t^r & := \int_0^1 e^{-\lambda} f_0^c c(s + u, X_t^{s,x'})du - (1 - \lambda) f_0^c c(s + u, X_t^{s,x})du d\lambda, \\
\hat{c}(t, x) & := e^{-f_0^c c(s + u, X_t^{s,x})du},
\end{align*}
\]

where \( \nabla \) denotes the gradient in the spatial argument, \( g(s, \cdot, \cdot) = g \) and the same for its derivatives. For (i), we show first once directional differentiability. Let \( h \in \{f, g\}; \) it holds that

\[
\begin{align*}
&\left| \frac{\mathbb{E} \hat{h}(t, x') \hat{c}(t, x') - \mathbb{E} \hat{h}(t, x) \hat{c}(t, x)}{r} - \mathbb{E} \left[ \nabla h(s + t, X_t^{s,x}) \cdot X_{t(t, \kappa)}^{s,x} \hat{c}(t, x) \right] \right| \\
&\leq \left| \frac{\mathbb{E} \hat{h}(t, x') \hat{c}(t, x') - \mathbb{E} \hat{h}(t, x) \hat{c}(t, x)}{r} - \mathbb{E} h_t' \cdot X_{t(t, \kappa)}^{s,x} \hat{c}(t, x) \right| \\
&\quad + \left| \mathbb{E} \hat{h}(t, x) \hat{c}(t, x') - \mathbb{E} \hat{h}(t, x) \hat{c}(t, x) \right| + \mathbb{E} h_t' \cdot X_{t(t, \kappa)}^{s,x} \hat{c}(t, x) \\
&\quad - c(s + u, X_t^{s,x'})du \right| + \left| \mathbb{E} h_t' \cdot X_{t(t, \kappa)}^{s,x} \hat{c}(t, x) \right| \\
&\quad - \mathbb{E} \nabla h(s + t, X_t^{s,x}) \cdot X_{t(t, \kappa)}^{s,x} \hat{c}(t, x) \\
&\quad + \left| \mathbb{E} \hat{h}(t, x) c_t^r r^{-1} \left( \int_0^t (c(s + u, X_t^{s,x'}) - c(s + u, X_t^{s,x}))du \right) \\
&\quad - \mathbb{E} \hat{h}(t, x) \hat{c}(t, x) \int_0^t \nabla c(s + u, X_t^{s,x}) \cdot X_t^{s,x}du \right|.
\end{align*}
\]

The first three terms on the right-hand side of (37) converge to 0 as \( r \to 0 \) by the fundamental theorem of calculus, (33), Lemma 3.2 and Lemma 4.3. For the
bounded independently of \( r \) triangle inequality on  

For the remaining term in the second factor on the right-hand side of (38), the \( c \) on the derivatives of  

have  

where  

\( \frac{1}{k_2} + \frac{1}{k_2'} = 1. \) By (33) and Theorem 2.4 in [19], we have  

Moreover, Hölder’s inequality yields  

For the first factor on the right-hand side, note that by (36) in Lemma 4.3, we have  

in probability by the continuous mapping theorem and \( \mathbb{E}[c_0^1 - \hat{c}(t,x)]^{2k_2'} \leq \int_0^1 \mathbb{E}[\hat{S}_t]^{2k_2'} d\lambda \to 0 \) as \( r \to 0 \) by (25) with  

By setting  \( K > 2k_2' \), the second factor on the right-hand side of (39) is clearly bounded independently of \( r \) (and of \( t \)) by Hölder’s inequality, our assumption on the derivatives of \( c \) and Lemma 3.2.  

For the remaining term in the second factor on the right-hand side of (38), the triangle inequality on  

last term, Hölder’s inequality yields  

\[
\left| \mathbb{E}[\hat{t}(t,x)c_0^1 c_0^{-1} \int_0^t (c(s + u, X_u^{s,x}) - c(s + u, X_u^{s,x})) du \right| \\
- \mathbb{E}[\hat{t}(t,x)\hat{c}(t,x) \int_0^t \nabla c(s + u, X_u^{s,x}) \cdot X_u^{s,x} du] \\
\leq \left\| \hat{t}(t,x) \right\|_{L^{k_2}(\mathbb{P})} c_0^1 \left( \int_0^t \frac{(c(s + u, X_u^{s,x}) - c(s + u, X_u^{s,x}))}{r} \right. \\
- \nabla c(s + u, X_u^{s,x}) \cdot X_u^{s,x} du \\
\left. + (c_0^1 - \hat{c}(t,x)) \int_0^t \nabla c(s + u, X_u^{s,x}) \cdot X_u^{s,x} du \right\|_{L^{k_2}(\mathbb{P})},
\]

(38)
For the first term of the right-hand side of (40), by Jensen’s inequality, Theorem 2.4 in [19], setting \(K > 2k_2\) and our assumption about the derivatives of \(c\), we have

\[
\begin{align*}
\mathbb{E} \left[ \int_0^t c_u' \cdot \left( \frac{X_u^{s,x'} - X_u^{s,x}}{r} - X_u^{s,x}(\kappa) \right) \right]^{k_2'} \\
\leq T^{k_2'-1} \mathbb{E} \left[ \int_0^{T-s} c_u' \cdot \left( \frac{X_u^{s,x'} - X_u^{s,x}}{r} - X_u^{s,x}(\kappa) \right) \right]^{k_2'} du \\
\leq T^{k_2'-1} \left( \mathbb{E} \left[ \int_0^{T-s} |c_u'|^{2k_2'} du \right] \right)^{\frac{1}{2k_2'}} \left( \mathbb{E} \left[ \int_0^{T-s} \left| \frac{X_u^{s,x+r\kappa} - X_u^{s,x}}{r} - X_u^{s,x}(\kappa) \right|^{2k_2'} du \right] \right)^{\frac{1}{2k_2'}} \\
\leq C(1 + V^{s,T}(0, \hat{x}(x')) + V^{s,T}(0, \hat{x}(x)))^{\frac{1}{2}} \\
\cdot \left( \mathbb{E} \sup_{0 \leq u \leq T-s} \left| \frac{X_u^{s,x+r\kappa} - X_u^{s,x}}{r} - X_u^{s,x}(\kappa) \right|^{2k_2'} \right)^{\frac{1}{2k_2'}} \\
\leq C(1 + V^{s,T}(0, \hat{x}(x')) + V^{s,T}(0, \hat{x}(x)))^{\frac{1}{2}} \\
\cdot \left( \mathbb{E} \sup_{0 \leq u \leq T-s} \left| X_u^{s,x+r\kappa} - X_u^{s,x} \right|^{2k_2'} \right)^{\frac{1}{2k_2'}}.
\end{align*}
\]

(41)

for \(C\) independent of \(t\), which converges to 0 as \(r \to 0\) by Lemma 3.2. For the second term on the right-hand side of (40), it holds that

\[
\begin{align*}
\mathbb{E} \left[ \int_0^t (c_u' - \nabla c(s + u, X_u^{s,x})) \cdot X_u^{s,x}(\kappa) \right]^{k_2'} & \\
\leq C \left( \int_0^{T-s} \mathbb{E} |c_u' - \nabla c(s + u, X_u^{s,x})|^{2k_2'} du \right)^{\frac{1}{2k_2'}} \left( \mathbb{E} \int_0^{T-s} \left| X_u^{s,x}(\kappa) \right|^{2k_2'} du \right)^{\frac{1}{2k_2'}}. \\
\end{align*}
\]

(42)

The last factor in the right-hand side of (42) is uniformly bounded in \(r\) by Lemma 3.2 and the first factor converges to 0 as \(r \to 0\) by Lemma 4.3.

Putting together the above in (37) gives that \(\mathbb{E} Q(X_t^{s,x})c_j h \int_0^t c(s + u, X_u^{s,x}) du\) is directionally differentiable in \(x\). For the other term in (39), it suffices to check that after integrating the inequality (37) in \(t\) from 0 to \(T - s\), the same convergences hold as \(r \to 0\). This is true for the first three term on the right-hand side of (37) by the same reasoning as before. It is true for the right-hand side of (38) by dominated (in \(t\)) convergence, since the right-hand sides of (39), (41) and (42) are uniformly bounded in \(t \in [0, T - s]\) and \(r \in [0, \epsilon]\) for some \(\epsilon > 0\). By induction and largely the same arguments as above, higher order directional derivatives in \(x\) of \(\mathbb{E} h(t, x) \tilde{c}(t, x)\) exist and they are sums of expressions of the form

\[
\begin{align*}
\mathbb{E} \left[ \partial^{\beta_1} h(s + t, X_t^{s,x}) \tilde{c}(t, x) \left( \prod_{\beta_2 \in I_2} (\partial^{(\beta_2)} X_t^{s,x})_{j_{\beta_2}} \right) \right] \\
\cdot \prod_{\beta_3 \in I_3} \int_0^t \partial^{\beta_3} c(s + u, X_u^{s,x}) \prod_{\beta_4 \in I_{34}} (\partial^{(\beta_4)} X_u^{s,x})_{j_{\beta_4}} du, \\
\end{align*}
\]

(43)

\[22\]
where \( h \in \{ f, g \} \), \( \beta_1 \) is a multiindex with \( 0 \leq |\beta_1| \leq p \), \( \hat{I}_2, \hat{I}_3, \hat{I}_{\beta_3} \) are some finite sets of multiindices each with absolute value less than or equal to \( p \) and \( j_{\beta_2}, j_{\beta_4} \in \{1, \ldots, n\} \). A fully detailed argument for this is omitted.

For differentiability of the expectation of (34) in \( x \), note that Theorem 1.2 in [28] may be applied on (18) due to \( \nabla b(s + t, X_{s,x}^t) \leq C(1 + \log V(t, \bar{x}(x))) \leq C(1 + V(t, \bar{x}(x))) \) (by Assumption 1 and the same for \( \sigma \)) and Lemma 3.1, so that the derivatives in probability \( X_{t(\kappa)} \) are unique solutions to (18) for the initial condition \( \kappa \). Therefore the first directional derivatives from (37) indeed form a linear map. The same arguments apply for expressions of the form (43) that are directionally differentiable, where additionally Assumption 2, Lemma 3.2 and Theorem 3.3 are to be used to control \( K_t(1) \) from Theorem 1.2 in [28]. Next, we show continuity in \( x \) of expressions of the form (43) (for multiindices with absolute values bounded by \( p \)). Note first that \( \mathbb{P}(\sup_{0 \leq u \leq T-s} |\partial_{x} X_{s,x}^u| \leq \epsilon) \leq \mathbb{P}(|\partial_{x} X_{s,x}^u - \partial_{x} X_{s,x}^{u'}| \leq \epsilon) \), therefore \( \partial_{x} X_{s,x}^u \) is continuous in probability w.r.t. to \( x \) by Theorem 4.10 in [28]. Consequently the product w.r.t. \( \beta_2 \) in (43) and \( \partial_{x} h(s + t, X_{s,x}^t) \) are sequentially continuous in probability by Theorem 20.5 in [4]. Lemma 4.3 and continuous mapping theorem yield that \( \hat{c}(t, x) \) is continuous in probability w.r.t. \( x \). For the remaining factors in (43), for \( 1 < k < K \), we have

\[
\int_0^t \left| \partial_{x} c(s + u, X_{u}^{s,x}) \prod_{\beta_4 \in \hat{I}_{\beta_3}} (\partial_{(\beta_4)} X_{u}^{s,x})_{j_{\beta_4}} \right| \, du
\]

\[
- \partial_{x} c(s + u, X_{u}^{s,x}) \prod_{\beta_4 \in \hat{I}_{\beta_3}} (\partial_{(\beta_4)} X_{u}^{s,x})_{j_{\beta_4}} \right| \, du
\]

\[
\leq \int_0^{T-s} \left| \partial_{x} c(s + u, X_{u}^{s,x}) - c(s + u, X_{u}^{s,x}) \right| \prod_{\beta_4 \in \hat{I}_{\beta_3}} (\partial_{(\beta_4)} X_{u}^{s,x})_{j_{\beta_4}} \right| \, du
\]

\[
+ \int_0^{T-s} \left| \partial_{x} c(s + u, X_{u}^{s,x}) \prod_{\beta_4 \in \hat{I}_{\beta_3}} (\partial_{(\beta_4)} X_{u}^{s,x} - X_{u}^{s,x}) \right|_{j_{\beta_4}} \right| \, du
\]

\[
\leq C \int_0^{T-s} \left| \partial_{x} c(s + u, X_{u}^{s,x}) - c(s + u, X_{u}^{s,x}) \right| \, du \sup_{\beta_4 \in \hat{I}_{\beta_3}} \left| \partial_{(\beta_4)} X_{u}^{s,x} \right|
\]

\[
+ C \int_0^{T-s} \left| \partial_{x} c(s + u, X_{u}^{s,x}) \right| \, du \sup_{\beta_4 \in \hat{I}_{\beta_3}} \left| \partial_{(\beta_4)} X_{u}^{s,x} \right|
\]

By Hölder’s inequality, Lemma 3.2 and Theorem 3.3, the first term on the right-hand side converges to zero in mean, therefore to zero in probability, as \( x' \to x \). By Theorem 4.10 in [28] (and continuous mapping theorem), the second term on the right-hand side also converges to zero in probability. Therefore the left-hand side converges to zero in probability. By continuous mapping theorem, the term inside the square bracket in (43) is sequentially continuous in probability. Consequently, by (25) with \( k_1 = 1, k = \frac{1 + k_2}{2}, S = |J(x') - J(x)| \), where \( J(x) \)
is equal to the term inside the square brackets in (43), together with Hölder’s inequality, inequality (33), our assumption on the derivatives of $c$ with a large enough $K$, Theorem 2.4 in [19], Lemma 3.2 and Theorem 3.3, expectations of the form (43) are continuous functions w.r.t. $x$ and so are their integrals in $t$ by dominated convergence, which concludes the proof for (i).

Using the same results and denoting the expression (43) by $\hat{u}$, it holds that

$$\hat{u} \leq C(1 + V^{s,T}(0, \tilde{x}(x)))^{\frac{1}{2}} \left( \mathbb{E} \left[ \sup_{0 \leq u \leq T-s} |\partial(\beta_2) X^s,x_u|^{2k_2} \right] \right)^{\frac{1}{2k_2}} \cdot \prod_{(\beta_3, \beta_4) \in \hat{I}} \left( 1 + V^{s,T}(0, \tilde{x}(x)) \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \sup_{0 \leq u \leq T-s} |\partial(\beta_4) X^s,x_u|^{c_{\beta_4}} \right] \right)^{-\frac{1}{c_{\beta_4}}}$$

for some $c_{\beta_3}, c_{\beta_4} > 0$, $\beta_3, \beta_4 \in \hat{I}$ and in particular for some constant $C$ independent of $t$. The proof for (ii) then concludes by Theorem 3.3.

Assertion (iii) then follows by Theorem 3.5(iii) in [28], Lemma A.2 and by noting that $C$ above is independent of $s$ given that the Lyapunov functions are local in $s$.

\[ \square \]

### 4.2 Twice spatially differentiable solutions

In this section, we prove that the expectation of (34) with $t = T - s$ solves Kolmogorov’s equation by the approach in [28]. The main ingredient besides differentiability of the associated semigroups, given in Theorem 4.2, is that the SDE can be approximated in probability by an Euler-type approximation locally uniformly in initial time and space, which is given in Lemma 4.4. Throughout this section, we assume $O = \mathbb{R}^n$.

**Lemma 4.4.** Suppose for any $T > 0$, there exists a family of functions $(V^{s,T})_{s \in [0,T]}$ such that Assumption 3 holds. For $I = \{t_k\}_{k \in \mathbb{N}_0} \subset [0, \infty)$ with $t_0 = 0$, $t_{k+1} \geq t_k$, $k \in \mathbb{N}$, $t_k \to \infty$ as $k \to \infty$, $\sup_{k \geq 0} t_{k+1} - t_k < \infty$, $s \in [0, \infty)$, $x \in \mathbb{R}^n$, let $X^{s,x}_t(I)$ denote the Euler approximation given by $X^{s,x}_0(I) = x$ and

$$X^{s,x}_t(I) = X^{s,x}_t(I) + \int_{t_k}^t b(s + r, X^{s,x}_{t_k}(I))dr + \int_{t_k}^t \sigma(s + r, X^{s,x}_{t_k}(I))dW_r, \quad (44)$$

on $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$. For any $R', T' \geq 0$, $\epsilon > 0$, it holds that

$$\sup_{s \in [0,T']} \sup_{x \leq R'} \mathbb{P} \left[ \sup_{t \in [0,T']} |X^{s,x}_t - X^{s,x}_t(I)| \geq \epsilon \right] \to 0$$

as $\sup_{k \geq 0} t_{k+1} - t_k \to 0$.

**Proof.** We extend the proof of Theorem 1 in [35] to obtain convergence that is uniform with respect to $s \in [0, T]$ and $x \in B_R$. Fix the numbers $R', T' \geq 0$. For $k \in \mathbb{N}$, let $\varphi_k : \mathbb{R}^n \to [0, \infty)$ be smooth cutoff functions satisfying $\varphi_k(x) =
for all $x \in B_k$, $\varphi_k(x) = 0$ for $x \in \mathbb{R}^n \setminus B_{k+1}$ and let $b^{(k)} : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma^{(k)} : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be given by $b^{(k)} = b\varphi_k$ and $\sigma^{(k)} = \sigma\varphi_k$. Let $Y^{s,x,k}(I)$ be the unique solutions to the corresponding SDE with drift $b^{(k)}$ and diffusion coefficient $\sigma^{(k)}$. The corresponding Euler approximation is given by (44) with $Y^{s,x,k}_0 = Y^{s,x,k}_0(I) = x$. Fix w.l.o.g. $0 < \epsilon \leq 1$. In the same way as in the proof of Theorem 1 in [35], one obtains that for any $s \in [0, T']$, $x \in \mathbb{R}^n$ and $k \geq R' + 1$,

$$
\mathbb{P}\left(\sup_{0 \leq t \leq T'} |X^{s,x}_t - X^{s,x}_t(I)| > \epsilon\right) \leq \mathbb{P}\left(\sup_{0 \leq t \leq T'} |Y^{s,x,k}_t - Y^{s,x,k}_t(I)| > \epsilon\right) + \mathbb{P}(\tau_{k-1} \leq T'),
$$

where $\tau_{k-1} = \inf\{t \geq 0 : |X^{s,x}_t| > k - 1\}$. By Markov’s inequality, Theorem 2.4 in [19] and Assumption 3(iv), it holds that

$$
\mathbb{P}(\tau_{k-1} \leq T') \leq \mathbb{E}[\hat{G}(s + (\tau_{k-1} \wedge T'), X^{s,x}_{\tau_{k-1} \wedge T'}|s,s)] \leq \mathbb{E}[V^{s,T'}(\tau_{k-1} \wedge T', \tilde{x}(x))]
$$

$$
\leq \left\|e^\int_0^{(\tau_{k-1} \wedge T')} \sigma^{s,T'} du\right\|_{L^{p,T'}(\mathbb{P})} \cdot \left(\mathbb{E}[V_0(s, \tilde{x}(x)) + \int_0^{T'} \left\|\frac{1_{[0,\tau_{k-1} \wedge T']}(v)\beta_{s,T'}^{s,T'}}{e^\int_0^v \sigma^{s,T'} du}\right\|_{L^{p,T'}(\mathbb{P})} dv\right).
$$

For any $0 < \epsilon' < 1$, by the assumption that $V^{s,T'}$ is local in $s$ and continuity of $V_0$, there exists $k^s$ such that $\mathbb{P}(\tau_{k^s} \leq T') \leq \epsilon' \leq \frac{\epsilon'}{2}$ for all $s \in [0, T']$ and $x \in B_{R'}$. In addition, for any $R > 0$, it holds that

$$
2\langle x - y, b^{(k^s)}(t,x) - b^{(k^s)}(t,y)\rangle + \|\sigma^{(k^s)}(t,x) - \sigma^{(k^s)}(t,y)\|^2
$$

$$
\leq 2\langle x - y, b(t,x) - b(t,y)\rangle\varphi_{k^s}(x) + 2\|b(t,y)||x - y||\varphi_{k^s}(x) - \varphi_{k^s}(y)\|
$$

$$
+ \|\sigma(t,x) - \sigma(t,y)\|^2\varphi_{k^s}(x)^2 + \|\sigma(t,y)\|^2\varphi_{k^s}(x) - \varphi_{k^s}(y)^2
$$

$$
\leq (K_s(R) + C \sup_{y' \in B_R} (|b(t,y')| + \|\sigma(t,y')\|^2))|x - y|^2
$$

for all $x, y \in B_R$ and

$$
2\langle x, b^{(k^s)}(t,x)\rangle + \|\sigma^{(k^s)}(t,x)\|^2 \leq 2(1 + |x|) \sup_{x' \in B_{k^s+1}} (|b(t,x')| + \|\sigma(t,x')\|^2)
$$

for all $x \in \mathbb{R}^n$. Therefore Corollary 5.4 in [28] can be applied to obtain

$$
\sup_{s \in [0,T']} \sup_{x \in \mathbb{R}^n} \mathbb{P}\left(\sup_{0 \leq t \leq T'} |Y^{s,x,k'}_t - Y^{s,x,k'}_t(I)| > \epsilon\right) \to 0
$$

as $\sup_{k \geq 0} t_{k+1} - t_k \to 0$, which concludes the proof.
Theorem 4.5. Let all of the assumptions in Theorem 4.2 hold. In particular, let \( c \) have \((p,k')\)-Lyapunov derivatives for any \( 1 < k' < K \), let the mappings \( \hat{x}_k \) be independent of \( s \), \((V_k^{s,T})_{s \in [0,T]} \) be local in \( s \) for any \( k \), multiindex \( \alpha \) with \( 0 \leq |\alpha| \leq p \) and let \( p \geq 2 \). For \( v : [0,T] \times \mathbb{R}^n \to \mathbb{R} \) given by
\[
v(t,x) = u(t,T-t,x),
\]
with \( u \) as in (34), the equation
\[
\partial_t v + a : D^2 v + b \cdot \nabla v - cv + f = 0
\]
holds almost everywhere in \((0,T) \times \mathbb{R}^n\).

Proof. Theorem 4.2, Theorem 3.6 in [28] applied on the SDE (1) appended by (81) and Lemma A.2 yield \((\partial_t v + b \cdot \nabla v + a : D^2 v - cv + f) e^{-x} = 0\) almost everywhere.

Note the assumptions in Theorems 4.2 and 4.5 remain strictly weaker than those in [28, Lemma 5.10], since Lyapunov functions that are positive polynomials can easily be conjured under the global Lipschitz conditions there.

4.3 Distributional solutions under maximal dissipativity

This section complements our result about Kolmogorov equations by considering a case where maximal dissipativity of the closure of the generator acting on \( C_c^\infty \) is known. It is shown that if there exists an associated semigroup and the coefficients \( b \) and \( \sigma \) are regular enough, this is sufficient for a solution of the backward Kolmogorov equation in the distributional sense. In this section we do not assume our conditions about the Lipschitz constants or the higher derivatives of the coefficients. Below, \( \mathcal{L} \) is used to denote the differential operator
\[
\mathcal{L} = b \cdot \nabla + a : D^2
\]
defined on \( C_c^\infty \).

Proposition 4.6. Assume \( O = \mathbb{R}^n \), \( a \) and \( b \) are independent of \( \omega, t \), they admit distributional derivatives of order two and one respectively and that \( \mu \) is a probability measure on \( \mathbb{R}^n \) absolutely continuous with respect to the Lebesgue measure with density \( \rho \) satisfying, for some \( p, q \in [1,\infty) \) with \( p^{-1} + q^{-1} = 1 \),
\begin{itemize}
  \item \( \rho^{-1}, \rho^{-1} \partial^k_i \partial^l_j a_{ij}, \rho^{-1} \partial^k_i b_i \in L^p_{loc}(\mu) \), for \( i,j \in \{1,\ldots,n\} \), \( k,l \in \{0,1\} \), where \( \rho^{-1} := 0 \) whenever \( \rho = 0 \),
  \item the closure \( \bar{\mathcal{L}} \) of \( \mathcal{L} \) in \( L^q(\mu) \) generates a strongly continuous semigroup \( \{T_t\}_{t \geq 0} \) on \( L^q(\mu) \),
\end{itemize}
then for any \( g \in \mathcal{D}(\bar{\mathcal{L}}) \) and \( \eta \in C_c^\infty((0,T) \times \mathbb{R}^n) \),
\[
\int_0^T \int_{\mathbb{R}^n} T_t g \left[ \frac{\partial \eta}{\partial t} + \sum_{ij} \partial_i \partial_j (a_{ij} \eta) - \sum_i \partial_i (b_i \eta) \right] dxdt = 0.
\]
In the case when \( \det(a) > 0 \) and \( a, b \) are regular enough, Sections 3.4, 3.5, 5.2 in [5] and Section 8.1 in [31] provide results sufficient for the assumptions in Proposition 4.6. Otherwise when \( \det(a) > 0 \) does not hold everywhere, such statements are less generally available, but hold for example in the settings of [10]\(^1\) and [3]; note the assumed regularity on \( a, b \) are more than what’s required in these works but are necessary for the formulation of (47).

Stationarity of \( \mu \) is not required for the proof of Proposition 4.6, but it is the case for the assumptions to be satisfied in the above references. In addition, \( \{T_t\}_{t \geq 0} \) is not specified in terms of an expectation as in (45); for this, there must be some stochastic process \( X_t \) associated to \( \{T_t\}_{t \geq 0} \), which solves (8).

**Proof of Proposition 4.6.** By mean value theorem, there exists constant \( k_\eta > 0 \) and compact set \( K_\eta \subset (0, T) \times \mathbb{R}^n \) such that

\[
|\int T_t \eta(t) dx dt| \leq k_\eta \int T_t \eta(t) 1_{K_\eta} dt.
\]

By the inequality above, strong continuity of \( T_t \) and the assumption on \( \rho^{-1} \), the expressions above make sense and the integral under the limit has the same limit as the left-hand side of

\[
\int_0^{T+s} \left( \frac{T_{t+s}g - T_t g}{s} - \bar{L}T_t g \right) \eta(x) dx dt = \int_0^{T} \left( \frac{T_{t+s}g - T_t g}{s} - \bar{L}T_t g \right) \eta(x) dx dt,
\]

which can be dealt with by considering

\[
\int_0^{T} \left( \frac{T_{t+s}g - T_t g}{s} - \bar{L}T_t g \right) \eta(x) dx dt \leq k_\mu \int_0^{T} \left\| \frac{T_{t+s}g - T_t g}{s} - \bar{L}T_t g \right\|_{L^q(\mu)} dt \leq k_\mu \int_0^{T} \left\| \frac{T_{t+s}g - T_t g}{s} - \bar{L}T_t g \right\|_{L^q(\mu)} dt
\]

for some constant \( k_\mu > 0 \). The right hand side of (48) is converging to zero as \( s \to 0 \) by dominated convergence theorem with constant dominating function since

\[
\left\| \frac{T_{t+s}g - T_t g}{s} - \bar{L}T_t g \right\|_{L^q(\mu)} \leq \left\| \frac{T_{t+s}g - g}{s} \right\|_{L^q(\mu)} + \left\| T_t \bar{L}g \right\|_{L^q(\mu)} \leq Me^{\omega T} \left\| \frac{T}{\mu} \right\|_{L^q(\mu)}
\]

\(^1\)The state space here is slightly different to \( \mathbb{R}^n \), but the statement and proof of Proposition 4.6 can be modified accordingly.
for all \( s \in (0, S), t \in [0, T] \), some \( M, S > 0 \) and \( \omega > 0 \) by Proposition 1.1 in [14]. Therefore,
\[
\int_0^T \int_{\mathbb{R}^n} T_t g \partial_t \eta \, dx \, dt = - \int_0^T \int_{\mathbb{R}^n} \tilde{L} T_t g \eta \, dx \, dt.
\]
By assumption, for each \( t \in (0, T) \), there exist a sequence \((g_k^t)_{k \in \mathbb{N}} \subset C^\infty_c(\mathbb{R}^n)\) with \( g_k^t \to T_t g \) and \( \mathcal{L} g_k^t = \tilde{L} g_k^t \to \tilde{L} T_t g \) in \( L^q(\mu) \) as \( k \to \infty \). Since \( \rho^{-1} \in L^p_{loc}(\mu) \), we have for every \( t \),
\[
\left| \int_{\mathbb{R}^n} \left( \tilde{L} T_t g - \mathcal{L} g_k^t \right) \eta \, dx \right| \leq \| \tilde{L} T_t g - \mathcal{L} g_k^t \|_{L^q(\mu)} \| \rho^{-1} \eta \|_{L^p(\mu)}
\]
\[
\left| \int_{\mathbb{R}^n} \left( T_t g - g_k^t \right) \mathcal{L}^\top \eta \, dx \right| \leq \| T_t g - g_k^t \|_{L^q(\mu)} \| \rho^{-1} \mathcal{L}^\top \eta \|_{L^p(\mu)},
\]
where \( \mathcal{L}^\top \) denotes the \( L^2(\mathbb{R}^n) \)-adjoint of \( \mathcal{L} \), which concludes the proof.

5 Alternative assumptions for time-independent, nonrandom coefficients

In the following, we restrict to the case where \( b \) and \( \sigma \) are nonrandom and time-independent, so that we may use Theorem V.39 in [36] in order to rid the need for bounds on function values on line segments in terms of the endpoint values. In doing so, more local conditions are obtained in place of (14), (15), (26) and (33).

Lemma 5.1. Let \( p \in \mathbb{N}, b, \sigma \) be independent of \( \omega, t \) and suppose they are continuously differentiable up to order \( p \) with locally Lipschitz derivatives. For every \( s \in [0, T] \), there exists \( \Omega \times \Delta_T \times \mathbb{R}^n \ni (\omega, t, x) \mapsto \hat{X}^x_t \in \mathbb{R}^n \) that is for \( \mathbb{P}\)-a.a. \( \omega \in \Omega \) continuously differentiable in \( x \) up to order \( p \) and indistinguishable from the corresponding derivatives in probability of \( X^x_t \).

Proof. By Theorem V.38 and V.39 in [36], continuously differentiable \( \hat{X}^x_t \) up to order \( p \) exists. Moreover, it satisfies (8) and \( \hat{X}^x_t \) is indistinguishable from \( X^x_t \). The partial derivatives of \( \hat{X}^x_t \) satisfy the systems given by formal differentiation of (8). On the other hand, derivatives in probability of \( X^x_t \) as in [28, Theorem 4.10] and Theorem 3.3 above satisfy the same system. Therefore by uniqueness in the aforementioned references\(^2\), it holds that \( \partial^\alpha \hat{X}^x_t \) are the unique solutions to their respective systems for all time and are therefore indistinguishable from the corresponding derivatives in probability \( \partial^{(\alpha \alpha)} X^x_t \) for every \( s \in [0, T] \) and multiindex \( \alpha \) with \( 0 \leq |\alpha| \leq p \).

\(^2\)Alternatively, since these systems have terms on right-hand sides that are continuous functions of the partial derivatives and are in particular at most linear in the highest order derivative (see the beginning of proof for Theorem 3.3), uniqueness holds by continuity of \( X^x_t \) in \( t \), (20) in Lemma 3.2, induction in the number of derivatives and Theorem 1.2 in [28] with \( K_t(R) = K_t(1) \) constant in \( t \).
Theorem 5.2 (Alternative assumptions to Lemma 3.2 and Theorems 3.3, 4.2 and 4.5). Let $b$ and $\sigma$ be independent of $\omega, t$ and let $O = \mathbb{R}^n$. The following statements hold.

(i) Lemma 3.2 continues to hold with

$$W(x, r\kappa) = 1 + \sum_{i \in I_0 \cup I_0'} \int_0^1 EV_i(0, \bar{x}_i(x + \lambda r\kappa))d\lambda$$

if

- the coefficients $b$ and $\sigma$ admit locally Lipschitz first derivatives and
- in Assumption 1, the inequalities (14), (15) are replaced by

$$\sum_{i} |\partial_i b(x)| + \|\partial_i \sigma(x)\|^2 \leq G(t, x).$$

(ii) Theorem 3.3 continues to hold with (30) replaced by

$$q(y, y') = \mathbb{E}q_0\left(\int_0^1 V_i(0, \bar{x}_i(\lambda y + (1 - \lambda)y'))d\lambda\right)_{i \in I_0 \cup I_0'},$$

$$\int_0^1 \hat{V}_{l_i}^{s,T}(0, \hat{x}_{l_i}(\lambda y + (1 - \lambda)y'))d\lambda, (V_i(0, \bar{x}_i(y)))_{i \in I_0 \cup I_0'},$$

$$(\hat{V}_{l_i}^{s,T}(0, \hat{x}_{l_i}(y)))_{i \in \{2, \ldots, i^*\}}, (\hat{V}_{l_i}^{s,T}(0, \hat{x}_{l_i}(y')))_{i \in \{2, \ldots, i^*\}}$$

if

- the coefficients $b$ and $\sigma$ admit locally Lipschitz $p^{th}$-order derivatives,
- Assumption 1 is replaced as above and
- in Assumption 2, inequality (26) is replaced by

$$|\partial^\alpha b(X_i^x)| + \|\partial^\alpha \sigma(X_i^x')\|^2 \leq M'\left(1 + \hat{V}_{k}^{s,T}(t, \hat{x}_k(y))\right).$$

(iii) Theorems 4.2 and 4.5 continue to hold if

- the second derivatives of $b$ and $\sigma$ are locally Lipschitz,
- Assumption 1 and 2 are replaced as above and
- in Definition 4.1, inequality (33) is replaced by

$$|\partial^\alpha h(X_i^x)| \leq N(1 + V_i^{s,T}(t, \hat{x}(x))).$$

Proof. The proof strategies follow largely in the same way as in the previous proofs, the differences are specified in the following using the same notation as before. For (i), note first (20) follows unperturbed. By Lemma 5.1, classical derivatives are indistinguishable from derivatives in probability and we use the
properties of both without changing the notation in the following. In place of (22), it holds that

\[ dX^{(r)}_t = r \int_0^1 (X^{x+\lambda r}_t \cdot \nabla) b(X^{x+\lambda r}_t) d\lambda dt + r \int_0^1 (X^{x+\lambda r}_t \cdot \nabla) \sigma(X^{x+\lambda r}_t) d\lambda dW_t. \]

Note that since for every \( t \) and almost all \( \omega \), the functions \( X^{x}_t, X^{\tilde{x}}_t \) are continuous in \( x \), the integrands on the right-hand side are \( \mathcal{B}([0, T]) \otimes \mathcal{F} \otimes \mathcal{B}([0, 1]) \)-measurable by Lemma 4.51 in [1] and the integrals (in \( \lambda \)) themselves are adapted. For any \( k \geq 1 \), by (49), the drift coefficient satisfies

\[
2rX^{(r)}_t \cdot \int_0^1 (X^{x+\lambda r}_t \cdot \nabla) b(X^{x+\lambda r}_t) d\lambda \leq \sum_{ij} \int_0^1 \left( n_i^2 (X^{(r)}_t)_{ij} \partial_j (b(X^{x+\lambda r}_t)) \right) d\lambda + \sum_{ij} \int_0^1 \left( n_j^2 \max_i (X^{x+\lambda r}_t)_{ij} \right) \partial_j (b(X^{x+\lambda r}_t)) \right) d\lambda \leq \int_0^1 \left( n_i^2 \max_j (X^{x+\lambda r}_t)_{ij} \right) \partial_j (b(X^{x+\lambda r}_t)) \right) d\lambda \]

and the diffusion coefficient satisfies

\[
\left\| r \int_0^1 (X^{x+\lambda r}_t \cdot \nabla) \sigma(X^{x+\lambda r}_t) d\lambda \right\|^2 \leq r^2 \int_0^1 |X^{x+\lambda r}_t|^2 G(t, X^{x+\lambda r}_t) d\lambda.
\]

Consequently, Theorem 2.4 in [19] can be applied with

\[
a_t = r \int_0^1 (X^{x+\lambda r}_t \cdot \nabla) b(X^{x+\lambda r}_t) d\lambda, \quad b_t = r \int_0^1 (X^{x+\lambda r}_t \cdot \nabla) \sigma(X^{x+\lambda r}_t) d\lambda, \quad a_t = n^2 \int_0^1 G(t, X^{x+\lambda r}_t) d\lambda, \quad \beta_t = 4(k \vee 1) r^2 \int_0^1 |X^{x+\lambda r}_t|^2 G(t, X^{x+\lambda r}_t) d\lambda,
\]

\[
p = 2k \vee 2, \quad q_1 = \frac{k}{2}, \quad q_2 = \left( \frac{2}{k} - \frac{1}{k \vee 1} \right)^{-1}, \quad q_3 = k \vee 1, \quad V(x) = |x|^2,
\]

30
After Jensen’s inequality, the first expectation on the right-hand side can be dealt by Lemma 3.1 (the same as in the proof of Lemma 3.2). By (20), the second expectation has the bound

\[
\int_0^1 \mathbb{E} \sup_{0 \leq u \leq t} \left| X^{(r)}_{u(\kappa)} \right|^k \left( \int_0^t G(u, X^{\kappa}_u) du \right)^{k \vee 1} d\lambda \leq \int_0^1 \mathbb{E} \left\{ \sum_{i \in I_0 \cup I'_0} \mathbb{E} V_i(0, \bar{x}_i(x + \lambda \kappa)) \right\} d\lambda
\]
and, by (16) and Lemma 3.1, the third and last expectation has the bound
\[
E \int_0^1 \left( \int_0^t G(u, X_u^{x+\lambda r\kappa}) du \right)^{2k\sqrt{2}} d\lambda 
\]
\[
\leq C \int_0^1 E \left( 1 + \sum_{i \in I_0} \int_0^t \log V_i(u, \bar{x}_i(x + \lambda u\kappa)) du \right)^{2k\sqrt{2}} d\lambda 
\]
\[
+ \sum_{i' \in I'_0} \log V_{i'}(t, \bar{x}_{i'}(x + \lambda r\kappa))) d\lambda 
\]
\[
\leq C \int_0^1 \left( 1 + \sum_{i \in I_0} t^{2k\sqrt{2}-1} \int_0^t E V_i(u, \bar{x}_i(x + \lambda r\kappa)) du \right)^{2k\sqrt{2}} d\lambda 
\]
\[
+ \sum_{i' \in I'_0} E V_{i'}(t, \bar{x}_{i'}(x + \lambda r\kappa ))) d\lambda 
\]
\[
\leq C \int_0^1 \left( 1 + \sum_{i \in I_0 \cup I'_0} E V_i(0, \bar{x}_i(x + \lambda r\kappa)) \right) d\lambda. 
\]  
(52)

Gathering Lemma 3.1, (51) and (52), the bound (50) becomes
\[
E \sup_{0 \leq u \leq t} \left| X_u^{r(\kappa)} \right|^k \leq C r^k \left( \int_0^1 \left( 1 + \sum_{i \in I_0 \cup I'_0} E V_i(0, \bar{x}_i(x + \lambda r\kappa)) \right) d\lambda \right)^{\frac{k}{2k+2}} 
\]
which concludes the proof of (i) by definition of \( q_2 \).

For (ii), the conclusions of Theorem 3.3 follow with differences that have already been addressed when dealing with (i), using that expressions of the form \( h(s+u, X_u^{s,x'}) - h(s+u, X_u^{s,x}) = \int_0^1 \nabla h(s+u, \lambda X_u^{s,x'} + (1-\lambda)X_u^{s,x} - X_u^{s,x}) d\lambda \) may be replaced by \( \int_0^1 \nabla h(t, X_t^{s+x+\lambda r\kappa} \cdot rX_{t^{u(\kappa)}}^{s+\lambda r\kappa} d\lambda \) and therefore the arguments are not repeated.

For (iii), Lemma 4.3 can easily be modified using what has already been mentioned, so that Theorem 4.2(i) holds. Proofs for the other assertions of Theorem 4.2 and Theorem 4.5 follow unperturbed.

6 Weak convergence rates for approximations under Lyapunov conditions

Here, the results in Section 3 are used with the exponential integrability property of stopped increment-tamed Euler-Maruyama schemes from [23] in order to establish weak convergence rates for SDEs with non-globally monotone coefficients. Classical proofs as in [27] establishing weak rates for the Euler-Maruyama scheme approximating (8) with globally Lipschitz coefficients require bounds on derivatives of the expectation (45), the Kolmogorov equation (46) and moment bounds on the discretization. Although analogous requirements
have mostly (beside continuous differentiability of (45) in t) been shown to be met in the setting here, the Itô-Alekseev-Gröbner formula of [17] is used for a more direct proof, which uses moment estimates on derivative processes as the main prerequisites. Along the way, strong completeness (see e.g. [29] for a definition) of the derivative SDEs as in (18) (and its higher order analogues) are shown in Lemma 6.2 using a result of [11]. The same assertions as those in Lemma 6.2 up to order 2 have appeared recently in [18] under different assumptions. The approach here uses the results in [36] for continuous differentiability of (45) in $t$ value) the underlying space to be all of $\mathbb{R}^n$. Before the aforementioned strong completeness result, a local Hölder continuity in time result in the strong $L^p(\mathbb{P})$ sense for derivatives to our SDE is shown in Lemma 6.1.

We begin by stating the numerical scheme and assumptions from [23] (amongst which is a Lyapunov-type condition) used for its exponential integrability. Assumptions about the relationship between the Lyapunov(-type) functions there and those in Assumptions 1, 2 are stated alongside, as well as the mild assumptions about the relationship between the Lyapunov(-type) functions there which is a Lyapunov-type condition) used for its exponential integrability. As-

Assumption 4. (i) The filtration $\mathcal{F}_t$ satisfies $\mathcal{F}_t = \sigma(\mathcal{F}_0 \cup \sigma(W_s : s \in [0, t])) \cup \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$ and that $\mathcal{F}_0$ and $\sigma(W_s : s \in [0, T])$ are independent. It holds that $\mathcal{O} = \mathbb{R}^n$ and $b, \sigma$ are independent of $\omega, t$.

(ii) There exist $\gamma, \rho \geq 0$, $\gamma', c' > 0$, $\xi, c > 1$, $C \in \mathbb{R}$, $U \in C^2(\mathbb{R}^n, [0, \infty))$, $\hat{U} \in C(\mathbb{R}^n)$ such that $\hat{U} > C$, $U(x) \geq c'(1 + x)^{\gamma'}$ and

$$
\sup_{\kappa_1, \ldots, \kappa_j \in \mathbb{N} \setminus \{0\}, |\kappa_1| = \cdots = |\kappa_j|} \sum_{i_1, \ldots, i_j = 1}^n \partial_{i_1} \ldots \partial_{i_j} (U(x) - U(y))(\kappa_1)_{i_1} \cdots (\kappa_j)_{i_j} \\
\leq c|x - y| \left(1 + \sup_{\lambda \in [0, 1]} |U(\lambda x + (1 - \lambda)y)|\right)^{(1 - \frac{\gamma}{2})\gamma^0},
$$

$|\partial^\alpha b(x)| + ||\partial^\alpha \sigma(x)|| + |\hat{U}(x)| \leq c(1 + U(x))^{\gamma},$

$\frac{|\hat{U}(x) - \hat{U}(y)|}{|x - y|} \leq c(1 + |U(x)|^{\gamma} + |U(y)|^{\gamma}),$

$L \hat{U}(x) + \frac{1}{2} \sigma^T \nabla U(x) \| U(x) \|^2 + \hat{U}(x) \leq \rho U(x).$

for all $x, y \in \mathbb{R}^n$, $j \in \{0, 1, 2\}$ and multiindices $\alpha$ with $0 \leq |\alpha| \leq 2$.

(iii) For any $\theta \in \Theta := \{\theta = (t_0, \ldots, t_n^*) : n^* \in \mathbb{N}, t_k \in [0, T], t_k < t_{k+1}, k \in \{1, \ldots, n^* - 1\}, t_0 = 0, t_n^* = T\}$, the function $Y^\theta : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ is an $\mathcal{F}_t$-adapted, $\mathbb{P}$-a.s. continuous process satisfying $\sup_{\theta \in \Theta} \mathbb{E}[e^{U(Y^\theta_n^*)}] < \infty$.
Remark 6.1. By Theorem 3.5 in [26], the first part of Assumption 4(iv) implies assumptions in [23]. Here, of particular note is that exponential integrability results of [23]. Items (ii) and (iii) closely follow the $X_{b}$ the whole space and fix than or equal to 3 terms in the Itô-Alekseev-Gröbner expansion such that we ask convergence rate assumption is made in order to ensure well-behavedness of some higher order terms in the Lyapunov functions to have $\Theta$ of Corollary 3.3 in [19]).

\[
Y_{t}^{\theta} = Y_{t_{k}}^{\theta} + \mathbb{1}_{\{y:|y| < \exp(\log\sup_{k} t_{k+1} - t_{k}|\frac{1}{k}|)(Y_{t}^{\theta})}}(Y_{t_{k}}^{\theta})
\]

\[
\left[ \frac{b(Y_{t_{k}}^{\theta})(t - t_{k}) + \sigma(Y_{t_{k}}^{\theta})(W_{t} - W_{t_{k}})}{1 + |b(Y_{t_{k}}^{\theta})(t - t_{k}) + \sigma(Y_{t_{k}}^{\theta})(W_{t} - W_{t_{k}})|^{q'}} \right]
\]

on $t \in [t_{k}, t_{k+1})$ for each $k \in \{0, \ldots, n^{*} - 1\}$, where $q' \geq 3$.

(iv) Assumptions 1 and 2 hold and $p \geq 3$. For any $V' \in \{V, \hat{V}_{b, \sigma}^{s,T} : s \in [0, T], 2 \leq |a| \leq p - 1, k \geq 2\}$, there exist $0 < t^{*} \leq 1$, $\bar{n} \geq n$, $\bar{O} \subset \mathbb{R}^{n}$ and $\bar{b}, \bar{\sigma}$ such that $V'$ is a $(\bar{b}, \bar{\sigma}, C, 0, 1, V_{0})$-Lyapunov function for some $\hat{b}, \hat{\sigma}, V_{0}$ with $\bar{b}^{\theta} = b(t, Y_{t}^{\theta}), \bar{\sigma}^{\theta} = \tilde{\sigma}(t, Y_{t}^{\theta})$ for processes $Y_{t}^{\theta}$ satisfying (10) and $V_{0} \in C^{2}([0, T], \bar{O})$ satisfies $\mathbb{P}$-a.s. that

\[
(\partial_{t} + L)V_{0}(t, y) \leq CV_{0}(t, x), \quad (53)
\]

\[
\lim_{|x'| \to \infty} V_{0}(t, x') = \infty,
\]

\[
V'(0, \bar{x}'(X_{s,T}^{\theta}))^{\tau} \leq C(1 + V'(\tau - s, \bar{x}'(y))),
\]

\[
V_{0}(0, \bar{x}'(y))^{\tau} \leq C(1 + e^{U(y)e^{-\alpha T}})
\]

for all $s, t \in [0, T]$, stopping times $\tau = T - s$, $x \in \bar{O}$, $y \in \cup_{\theta \in \Theta}$Range$Y_{t}^{\theta}$, where $\bar{x}' = \bar{x}$ if $V' = \bar{V}$, $\bar{x}' = \tilde{x}_{k}$ otherwise, $L$ is given by (9) with $b, \sigma$ replaced by $\bar{b}, \bar{\sigma}$ and $X_{s,t}^{\theta}$ is the solution to

\[
X_{s,t}^{\theta} = y + \int_{s}^{t} b(X_{s,u}^{\theta}) du + \int_{s}^{t} \sigma(X_{s,u}^{\theta}) dW_{u} \quad (54)
\]

Remark 6.1. By Theorem 3.5 in [26], the first part of Assumption 4(iv) implies that for all $s \in [0, T], x \in \mathbb{R}^{n}$, there exists a unique up to distinguishability, $\mathcal{F}_{t}$-adapted, $\mathbb{P}$-a.s. continuous solution to (54) and for $t \in [s, T]$ it holds $\mathbb{P}$-a.s. that $X_{s,T}^{\theta} = X_{s,T}^{\theta}$. In (i), the assertions about $\mathcal{F}_{t}$ are from [17]. We set $O$ to be the whole space and fix $b$ and $\sigma$ to be time-independent and nonrandom in order to use continuous differentiability in initial value from [36] and to use the exponential integrability results of [23]. Items (ii) and (iii) closely follow the assumptions in [23]. Here, of particular note is that $q'$ is asserted to be greater than or equal to 3 rather than 1 in the denominator of the expression for $Y_{t}^{\theta}$; this assumption is made in order to ensure well-behavedness of some higher order terms in the Itô-Alekseev-Gröbner expansion such that weak convergence rate of order 1 is attained. It is worth mentioning that the Lipschitz estimate on $U$ with $j = 0$ in (ii) easily gives that $U$ is polynomially bounded, so that the set under the indicator function in (iii) indeed satisfies the assumptions in [23], as used in [21, 23]. The last assertions of item (iv) (and in general Assumption 4) are easily satisfied by all of the examples mentioned here; they collect properties of the Lyapunov-type function from (ii) required for our argument without requiring the Lyapunov functions to have $V_{0}$ be given by $e^{U(x)e^{-\alpha T} + y}$ (see the proof of Corollary 3.3 in [19]).
In the following, for any \( s \in [0,T] \), we extend the definition of any process \( Z_t \) defined on \([s,T]\) to \([0,T]\) by setting \( Z_t = Z_s \) for \( t \in [0,s) \). In the proofs, many computations are similar to those in Lemma 3.2, Theorem 3.3 and so are compressed.

**Lemma 6.1.** Under Assumption 4, for any \( k_1 > 2(n+1), R > 0 \), there exist constants \( C > 0, n+1 < \nu \leq k_1 \) such that
\[
E \sup_{u \in [s,t]} |\partial^{(\kappa)} X_{s,u}^x - \partial^{(\kappa)} X_{s,s}^x|^{k_1} < C |t - s|^{\nu}
\]
for all \((s,t) \in \Delta_T, x \in B_R, \kappa \in \{(\kappa_i)_{1 \leq i \leq p_0} : \kappa_i \in \mathbb{R}^n, |\kappa_i| = 1, 1 \leq i \leq p, p_0 \in \mathbb{N}_0 \cap [2,p]\}\). Using Assumption 4(ii), Corollary 2.5 and so are.

**Proof.** By (28) in Theorem 3.3 (with a time shifted Wiener process and filtration) and using that \( \partial^{(\kappa)} X_{s,s}^x = 0 \) (for \( \kappa \) in the following set), the existence of such constants have already been shown for \( \kappa \in \{(\kappa_i)_{1 \leq i \leq p_0} : \kappa_i \in \mathbb{R}^n, |\kappa_i| = 1, 1 \leq i \leq p, p_0 \in \mathbb{N}_0 \cap [2,p]\}\). Using Assumption 4(ii), Corollary 2.5 and Corollary 3.3 both in [19], it holds that
\[
E \sup_{u \in [s,t]} |X_{s,u}^x - x|^{k_1} \leq C e^{k_1(t-s)} \left( E \left( (t-s)^{k_1-1} \int_0^{t-s} e^{U(X_{s,s}^x+u)e^{-\rho u} - 2k_1^2 u} du \right) \right)^{\frac{1}{2}}
\]
\[
\leq C e^{k_1(t-s)} (t-s)^{k_1-1} \int_0^{t-s} e^{U(x)} du \right)^{\frac{1}{2}}
\]
\[
\leq C |t - s|^{\frac{1}{2}}
\]
for all \((s,t) \in \Delta_T, x \in B_R\). Using instead Assumption 1, it holds that
\[
E \sup_{u \in [s,t]} |\partial^{(\kappa)} X_{s,u}^x - \partial^{(\kappa)} X_{s,s}^x|^{k_1}
\leq C \left( E \left[ e^{\sum_{i \in I_0} f_i^{t-s} \log V_i(u,\bar{x}_i(x)) + \sum_{i \in I_0} \log V_i(t-s,\bar{x}_i(x))} \right]^{2k_1} \right)^{\frac{1}{2}}
\leq C \left( 1 + \sum_{i \in I_0} V_i(0,\bar{x}_i(x)) \right)^{\frac{1}{2}} \left( (t-s)^{k_1-1} \int_0^{t-s} e^{U(x)} du \right)^{\frac{1}{2}}
\leq C |t - s|^{\frac{1}{2}}
\]
for all \((s,t) \in \Delta_T, x \in B_R, \kappa_i \in \mathbb{R}^n \) with \( |\kappa_i| = 1 \). \(\Box\)

The following lemma shows that the assumptions of Theorem 3.1 in [17] hold under Assumption 4. Moreover, it is shown that the estimates therein hold uniformly with respect to the discretization \( \theta \in \Theta \).

**Lemma 6.2.** Let Assumption 4 hold. There exists a function \( \Omega \times \Delta_T \times \mathbb{R}^n \ni (\omega, (s,t), x) \mapsto \bar{X}_{s,t}^x(\omega) \in \mathbb{R}^n \) such that
\begin{itemize}
  \item it holds \(\mathbb{P}\)-a.s. that for any \((s, t) \in \Delta_T, \mathbb{R}^n \ni x \mapsto \bar{X}_{s, t}^x \in \mathbb{R}^n\) is continuously differentiable in \(x\) up to order \(p - 1\) and the derivative \(\Delta_T \times \mathbb{R}^n \ni ((s, t), x) \mapsto \partial^\alpha \bar{X}_{s, t}^x \in \mathbb{R}^n\) is continuous for all multiindices \(\alpha\) with \(0 \leq |\alpha| \leq p - 1\),
  \item for any \(s \in [0, T], \ x \in \mathbb{R}^n\), the function \(\partial^\alpha \bar{X}_{s}^x\) is indistinguishable from \(\partial^{(\kappa_0)} X_{s, t}^x\) for all multiindices \(\alpha\) with \(0 \leq |\alpha| \leq p - 1\).
\end{itemize}

Moreover, for any \(p^1 > 0\), it holds that
\[
\sup_{0 \leq |\alpha| \leq p - 1} \sup_{0 \leq r \leq t \leq T} \sup_{0 \leq \nu \leq \nu_1} \mathbb{E} \left[ |b(\bar{X}_{s, t}^x)|^{p^1} + \|\sigma(\bar{X}_{s, t}^x)\|^{p^1} + \|\partial^\alpha \bar{X}_{s, t}^x\|^{p^1} \right] < \infty.
\]

**Proof.** By Lemma 5.1 (with time-shifted Wiener process and filtration), derivatives in probability \(\partial^{(\kappa_0)} X_{s, t}^x\) are indistinguishable from classical derivatives \(\partial^\alpha \bar{X}_{s, t}^x\).

In order to use the strong completeness Corollary 3.10 in [11], we show that for each \(R > 0, \ k_1 > 2(n + 1)\), it holds that
\[
\sup_{0 \leq |\alpha| \leq p - 1} \sup_{x, x' \in B_R, s, s' \in [0, T]} \mathbb{E} \left[ \frac{\sup_{t \in [0, T]} |\partial^\alpha \bar{X}_{s, t}^x - \partial^\alpha \bar{X}_{s, t}^{x'}|^{k_1}}{(|x' - x|^2 + |s' - s|^2)^{\frac{k_1}{2}}} \right] < \infty, \quad (55)
\]

where \(\nu_1\) is the same constant from Lemma 6.1. The marginal differences in \(x\) and \(s\) in the numerator are considered separately. By Lemma 3.2 or Theorem 3.3, the difference term in \(x\) in the numerator of (55) has the bound
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |\partial^\alpha \bar{X}_{s, t}^x - \partial^\alpha \bar{X}_{s, t}^{x'}| \right] \leq C|x' - x|^{k_1}
\]

for all \(s \in (0, T), \ x, x' \in B_R\), which is the desired Hölder bound for (55). For the difference term in \(s\) in the numerator of (55), it holds that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |\partial^\alpha \bar{X}_{s', t}^x - \partial^\alpha \bar{X}_{s', t}^{x'}|^{k_1} \right] \leq \mathbb{E} \left[ \sup_{t \in [s \wedge s', s \vee s'] \wedge \nu s} |\partial^\alpha \bar{X}_{s', s, t}^x - \partial^\alpha \bar{X}_{s', s, t}^{x'}|^{k_1} \right] + \mathbb{E} \left[ \sup_{t \in [\nu s, T]} |\partial^\alpha \bar{X}_{s', t}^x - \partial^\alpha \bar{X}_{s', t}^{x'}|^{k_1} \right], \quad (56)
\]

where the first term on the right-hand side has the desired Hölder bound for (55) by Lemma 6.1. For the second term, by Assumption 4(iv) and Lemma 4.4, combined with Theorem 5.3 in [28], the joint system solved by \((\partial^\alpha \bar{X}_{s, t}^x)_{0 \leq |\alpha| \leq p - 1}\) is regular [28, Definition 2.1] and the same holds for the sum \((\partial^\alpha \bar{X}_{s', t}^x - \partial^\alpha \bar{X}_{s', t}^{x'})_{0 \leq |\alpha| \leq p - 1}\) by an easy argument; therefore the strong Markov property (Theorem 2.13...
in [28] with Proposition 4.1.5 in [14]) yields for any $R' > 0$ that

$$
E \left[ \sup_{t \in [s \wedge s', T]} |\partial^\alpha \bar{X}^x_{s', t} - \partial^\alpha \bar{X}^x_{s, t}|^{k_1} \wedge R' \right] 
$$

$$
= E \left[ \left( \sup_{t \in [s \wedge s', T]} |\partial^\alpha \bar{X}^x_{s', t} - \partial^\alpha \bar{X}^x_{s, t}|^{k_1} \wedge R' \right) \mathcal{F}_{s \vee s'} \right] 
$$

$$
= \int \int \sup_{t \in [s \wedge s', T]} \left| \partial^\alpha \bar{X}^x_{s \vee s', t}(\omega) \right|^{k_1} \wedge R' \ d\mathbb{P}(\omega') \ d\mathbb{P}(\omega), 
$$

where $\partial^\alpha \bar{X}^x_{s \vee s', t}(\omega')(\omega')$ denotes the solution to the same (joint) system as $\partial^\alpha \bar{X}^x_{s \vee s', t}(\omega')$ but with initial conditions $\partial^\beta \hat{X}^x_{s \wedge s', s \vee s'}(\omega)$ for $0 \leq |\beta| \leq p - 1$ for each respective partial derivative in place of the initial conditions $x, c_i$ or 0. Then the proofs of Lemma 3.2 and Theorem 3.3 may be slightly modified in order to obtain analogous statements for the expectation in $\omega'$ in (57); the modification is namely that the initial condition (fixed with respect to $\omega'$) as mentioned can be added with no complications\(^3\) when Corollary 2.5 in [19] is applied. Given this, it holds that

$$
E \left[ \sup_{t \in [s \wedge s', T]} |\partial^\alpha \bar{X}^x_{s', t} - \partial^\alpha \bar{X}^x_{s, t}|^{k_1} \wedge R' \right] 
$$

$$
\leq C \sum_{\beta=0}^{\alpha-1} E |\partial^\beta \hat{X}^x_{s \wedge s', s \vee s'} - \partial^\beta \hat{X}^x_{s \wedge s', s \vee s'}|^{k_1}, 
$$

$$
= C \sum_{\beta=0}^{\alpha-1} E |\partial^\beta \hat{X}^x_{s \wedge s', s \vee s'} - \partial^\beta \hat{X}^x_{s \wedge s', s \vee s'}|^{k_1}, 
$$

for all $x \in B_R$, $s, s' \in [0, T], 0 \leq |\alpha| \leq p - 1$, which, by Lemma 6.1 and dominated convergence in $R'$, implies that the last term on the right-hand side of (56) has the desired Hölder bound for (55). Gathering the above and using the triangle inequality, (57) holds. Consequently, using on the way Lemma 3.2 and Theorem 3.3, Corollary 3.10 in [11] may be applied with $\beta = \frac{\alpha}{k_1}, D = [0, T] \times \mathbb{R}^n, E = F = C([0, T], \mathbb{R}^n)$, $X = (\Omega \times [0, T] \times \mathbb{R}^n \ni (\omega, s, x) \rightarrow \partial^\alpha \hat{X}^x_{s, t}(\omega) \in C([0, T], \mathbb{R}^n))$ to obtain for $0 \leq |\alpha| \leq p - 1$ existence of an $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable $\Omega \times [0, T] \times \mathbb{R}^n \ni (\omega, s, x) \rightarrow \partial^\alpha \hat{X}^x_{s, t}(\omega) \in C([0, T], \mathbb{R}^n)$ such that for all $\omega \in \Omega$, the function $[0, T] \times \mathbb{R}^n \ni (s, x) \rightarrow \partial^\alpha \hat{X}^x_{s, t}(\omega)$ is continuous and for any $(s, x) \in [0, T] \times \mathbb{R}^n$, $\partial^\alpha \hat{X}^x_{s, t}$ is indistinguishable from $\partial^\alpha \hat{X}^x_{s, t}$.

\(^3\)Alternatively, we have uniqueness in the joint system by Theorem 3.5 in [26] and Theorem 1.2 in [28], so that Theorem IX.1.7 in [37], Itô’s rule, Theorem 4.4.2 and Proposition 4.1.5 both in [14] give together the same required Markov property.

\(^4\)actually the $T - s$ term is lost on the right-hand side of (27) but that’s not important here.
Since partial integrals of (jointly) continuous functions are still continuous, we may partially integrate $|\alpha|$ times each $\Delta_T \times \mathbb{R}^n \ni ((s,t), x) \mapsto \partial^\alpha \mathcal{X}_{s,t} \in \mathbb{R}^n$ from 0 to $x_i$ in order to obtain for each $\alpha, \omega$ a continuous function $\Delta_T \times \mathbb{R}^n \ni ((s,t), x) \mapsto \bar{\mathcal{X}}_{x,\alpha} \in \mathbb{R}^n$ (where at each integration, continuous functions of the form $((s,t), x) \mapsto \partial^\beta \mathcal{X}(x_1, \ldots, x_0, \ldots, x_n)$ and subsequently their integrals are added in line with the fundamental theorem of calculus, which have zero partial derivative). For any $(s,t) \in \Delta_T$ and $\alpha$ with $0 \leq |\alpha| \leq p - 1$, by definition of $\partial^\alpha \mathcal{X}_{s,t}$ and its continuity in $x$, it holds $\mathbb{P}$-a.s. that $\partial^\alpha \mathcal{X}_{s,t} = \partial^\alpha \hat{\mathcal{X}}_{s,t}$ for all $x \in \mathbb{R}^n$, so that their partial integrals in $x$ are also $\mathbb{P}$-a.s. equal for all $x \in \mathbb{R}^n$ and in particular it holds $\mathbb{P}$-a.s. that $\bar{\mathcal{X}}_{x,\alpha} = \hat{\mathcal{X}}_{x,\alpha}$, for all $x \in \mathbb{R}^n$. Therefore, by continuity in $(s,t), x$, these functions coincide $\mathbb{P}$-a.s. across $\alpha$, that is, it holds $\mathbb{P}$-a.s. that $\bar{\mathcal{X}}_{s,t} = \hat{\mathcal{X}}_{s,t}$ and thus $\partial^\beta \bar{\mathcal{X}}_{s,t} = \partial^\beta \hat{\mathcal{X}}_{s,t}$ for all $(s,t) \in \Delta_T$, $x \in \mathbb{R}^n$ and multiindices $\alpha, \alpha', \beta$ with $|\alpha|, |\alpha'|, |\beta| \in [0, p - 1]$. Let this $\mathbb{P}$-a.s. defined function be denoted by $\bar{\mathcal{X}}_{s,t}$, then the assertions about $\bar{\mathcal{X}}_{s,t}$ in the statement of the lemma have been shown.

For the last assertion, the Markov property (Theorem 2.13 in [28]) will be applied repeatedly without further mention. Since Assumption 4(ii) implies in particular for any $p^\dagger > 0$ that

$$|b(x)|^{p^\dagger} + ||\sigma(x)||^{p^\dagger} \leq C e^{U(x)e^{-\rho t}}$$

for all $x \in \mathbb{R}^n$, $t \in [0, T]$, by Corollary 3.3 in [19] and Assumption 4(ii), it holds that

$$\sup_{\theta \in \Theta} \sup_{0 \leq s \leq t \leq T} \mathbb{E}[|b(\bar{\mathcal{X}}_{s,t}^{Y^\theta})|^{p^\dagger} + ||\sigma(\bar{\mathcal{X}}_{s,t}^{Y^\theta})||^{p^\dagger}] \leq C \sup_{\theta \in \Theta} \sup_{0 \leq s \leq t \leq T} \mathbb{E}[e^{U(\bar{\mathcal{X}}_{s,t}^{Y^\theta})e^{-\rho(t-s)}}]$$

$$\leq C \sup_{\theta \in \Theta} \sup_{0 \leq s \leq t \leq T} \mathbb{E}[e^{f^\dagger U(\bar{\mathcal{X}}_{s,t}^{Y^\theta})e^{-\rho(t-s)}}]$$

$$\leq C \sup_{\theta \in \Theta} \sup_{0 \leq s \leq t \leq T} \mathbb{E}[e^{U(Y^\theta)}],$$

which is finite by Theorem 2.9 in [23]. For any $p^\dagger > 0$, by Assumption 4(ii),
Corollary 3.3 in [19] and that \( e^{-\rho(s-r)}, e^{-\rho r} < 1 \), it holds that

\[
\sup_{0 \leq r \leq s \leq t \leq T} \mathbb{E}\left| \tilde{X}^{Y^\theta}_{t,T} \right|^{\beta} 
\leq C \sup_{0 \leq r \leq s \leq t \leq T} \mathbb{E}\left[ \exp(U(X^{Y^\theta}_{t,T}))e^{-\rho(T-t)}e^{-\rho(s-r)}e^{-\rho r} \right. \\
\left. + \int_{t}^{T} \tilde{U}(\tilde{X}^{Y^\theta}_{t,u})e^{-\rho(u-t)}e^{-\rho(s-r)}e^{-\rho r} du \right] 
\leq C \sup_{0 \leq r \leq s \leq t \leq T} \mathbb{E}\left[ e^{U(X^{Y^\theta}_{t,T})}e^{-\rho(s-r)}e^{-\rho r} \right] 
\leq C \sup_{0 \leq r \leq s \leq t \leq T} \mathbb{E}\left[ e^{U(Y^\theta_{r,s})}e^{-\rho r} \right],
\]

for all \( \theta \in \Theta \), which is finite uniformly in \( \theta \) by Theorem 2.9 in [23].

For the higher derivatives, first note that for \( V_0 \) satisfying (53) and \( 0 < l < 1 \), (53) is also satisfied with \( V_0^l \) in place of \( V_0 \). Moreover, the respective Lyapunov functions they generate satisfy Assumptions 1 and 2. Therefore, for any \( \bar{l} \in \mathbb{N} \cap [1, p - 1] \), \( \kappa \in \{(\kappa_i)_{i=1, \ldots, \bar{l}} : \kappa_i \in \mathbb{R}^n, |\kappa_i| = 1\} \), we may choose \( l = \frac{2^\bar{l}}{\text{degree}(q_0)} \), with \( q_0 \) from Theorem 3.3, so that for \( \rho^{\bar{l}} > 0 \), by Lemma 3.2 or Theorem 3.3, Young's inequality, Assumptions 4(ii)(iv) and Theorem 2.4 in [19], it holds that

\[
\sup_{0 \leq r \leq s \leq t \leq T} \mathbb{E}\left| \partial^{(\kappa)} X^{Y^\theta}_{t,T} \right|^{\rho^{\bar{l}}} 
\leq C \sup_{0 \leq r \leq s \leq t \leq T} \mathbb{E}\left[ 1 + \sum_{i \in \mathbb{I}_0} V_i(0, \bar{x}_i(X^{Y^\theta}_{r,s}))^{2^\bar{l}} + \sum_{i = 1}^{i^*} \hat{V}_i^0(T, 0) (\bar{x}_i(X^{Y^\theta}_{r,s}))^{2^\bar{l}} \right] 
\leq C \sup_{0 \leq r \leq s \leq t \leq T} \mathbb{E}\left[ 1 + \sum_{i \in \mathbb{I}_0} V_i(s-r, \bar{x}_i(Y^\theta_{r,s}))^{i^*} + \sum_{i = 1}^{i^*} \hat{V}_i^0(T, s-r) (\bar{x}_i(Y^\theta_{r,s}))^{i^*} \right] 
\leq C \sup_{0 \leq r \leq s \leq t \leq T} \mathbb{E}\left[ 1 + \sum_{i \in \mathbb{I}_0} V_i(0, \bar{x}_i(Y^\theta_{r,s}))^{i^*} + \sum_{i = 1}^{i^*} \hat{V}_i^0(T, 0) (\bar{x}_i(Y^\theta_{r,s}))^{i^*} \right] 
\leq C \sup_{0 \leq r \leq t \leq T} \mathbb{E}\left[ 1 + e^{U(Y^\theta_{r,s})}e^{-\rho T} \right],
\]

where \( C \) is in particular independent of \( \kappa \in \{(\kappa_i)_{i=1, \ldots, \bar{l}} : \kappa_i \in \mathbb{R}^n, |\kappa_i| = 1\} \) and \( \theta \in \Theta \), so that the right-hand side is finite uniformly in \( \theta \) by Theorem 2.9 in [23] and also uniformly in \( \bar{l} \).

The main theorem of this section about weak convergence of order 1 for the stopped increment-tamed Euler-Maruyama scheme is as follows.
Theorem 6.3. Let Assumption 4 hold. For \( f \in C^3(\mathbb{R}^n, \mathbb{R}) \), if there exist constants \( q', C_f > 0 \) such that
\[
|\partial^\alpha f(x)| \leq C_f (1 + |x|^{q'})
\]
for all \( x \in \mathbb{R}^n \) and multiindices \( \alpha \) with \( 0 \leq |\alpha| \leq 3 \), then there exists a constant \( C > 0 \) such that
\[
|\mathbb{E}f(X_{0, T}^{\theta, n}) - \mathbb{E}f(Y^n_{T})| \leq C \sup_{k \in \mathbb{N}_0 \cap [0, n^*]} (t_{k+1} - t_k)
\]
for all \( \theta \in \Theta \), where \( \theta = (t_0, \ldots, t_{n^*}) \).

Proof. Throughout the proof, we write \( D_{[\theta]} = \{ y : |y| < \exp(\log\sup_t t_{k+1} - t_k|^{1/2}) \} \). To begin, we rewrite the approximation \( Y_t^\theta \) as an SDE. For every \( k \in \mathbb{N}_0 \cap [0, n^*] \), \( \theta = (t_0, \ldots, t_{n^*}) \in \Theta \), consider
\[
Z_{t, k}^{\theta, k} = \begin{cases} 
0 & \text{if } t < t_k \\
b(Y_{t_k}^\theta)(t - t_k) + \sigma(Y_{t_k}^\theta)(W_t - W_{t_k}) & \text{if } t_k \leq t < t_{k+1} \\
b(Y_{t_k}^\theta)(t_{k+1} - t_k) + \sigma(Y_{t_k}^\theta)(W_{t_{k+1}} - W_{t_k}) & \text{if } t_{k+1} \leq t 
\end{cases}
\]
\[
= \int_0^t \mathbb{1}_{[t_k, t_{k+1})}(u)b(Y_{t_k}^\theta)du + \int_0^t \mathbb{1}_{[t_k, t_{k+1})}(u)\sigma(Y_{t_k}^\theta)dW_u,
\]
defined for all \( t \in [0, T] \), then \( Y_t^\theta \) solves
\[
Y_t^\theta = Y_0^\theta + \sum_{k=0}^{n^*-1} \mathbb{1}_{D_{[\theta]}}(Y_{t_k}^\theta) \frac{Z_{t, k}^{\theta, k}}{1 + |Z_{t, k}^{\theta, k}|^{q'}}
\]
where by Itô’s rule, for \( \hat{f} : \mathbb{R}^n \to \mathbb{R}^n \) given by \( \hat{f}(z) = \frac{z}{1 + |z|^{q'}} \), it holds that
\[
\frac{Z_{t, k}^{\theta, k}}{1 + |Z_{t, k}^{\theta, k}|^{q'}} = \int_0^t \mathbb{1}_{[t_k, t_{k+1})}(u)(b(Y_{t_k}^\theta) + b^*(Y_{t_k}^\theta, Z_{u, k}^{\theta, k}))du
\]
\[
+ \int_0^t \mathbb{1}_{[t_k, t_{k+1})}(u)(\sigma(Y_{t_k}^\theta) + \sigma^*(Y_{t_k}^\theta, Z_{u, k}^{\theta, k}))dW_u
\]
and \( b^* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) and \( \sigma^* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \) are given by
\[
b^*(y, z) = -b(y)\frac{|z|^{q'}}{1 + |z|^{q'}} - q'z\left(z \cdot b(y)\frac{|z|^{q'-2}}{(1 + |z|^{q'})^2}\right)
\]
\[
+ \frac{1}{2}(\sigma\sigma^T(y)) : D^2\hat{f}(z)
\]
\[
\sigma^*(y, z) = -\sigma(y)\frac{|z|^{q'}}{1 + |z|^{q'}} - q'z\left(z^T\sigma(y)\frac{|z|^{q'-2}}{(1 + |z|^{q'})^2}\right).
\]
Note that using \( q' \geq 3 \), there exists a constant \( \nu_2 \geq 2 \) such that the second order derivatives satisfy \( |\partial_{ij}^2 \hat{f}(z)| \leq C|z|^{\nu_2} \) for all \( z \in \mathbb{R}^n, i, j \in \mathbb{N} \cap [1, n] \).
By Theorem 3.1 in [17] and Lemma 6.2, for any \( \theta \in \Theta \), it holds that

\[
\mathbb{E}[f(X_{t,T}^\theta)] - \mathbb{E}[f(Y_0^\theta)] = \sum_{k=0}^{n^* - 1} \mathbb{E} \int_{t_k}^{t_{k+1}} \left( \left( b(Y_t^\theta) - \mathbb{1}_{D_{t_k}}(Y_t^\theta)b(Y_t^\theta) \right) + b^*(Y_{t_k}, Z_{t_k}^{\theta,k}) \right) \cdot \nabla \left( X_{t,T}^\theta - X_{t_{k+1},T}^\theta \right) \cdot \nabla f(X_{t,T}^\theta) dt
\]

\[
+ \frac{1}{2} \mathbb{E} \int_{t_k}^{t_{k+1}} \sum_{i,j=1}^n \left( \sigma(Y_{t_k}^\theta) \sigma(Y_{t_k}^\theta)^\top - \mathbb{1}_{D_{t_k}}(Y_{t_k}^\theta) \sigma(Y_{t_k}^\theta) \right) \cdot \left( \sigma(Y_{t_k}^\theta) + \sigma(Y_{t_k}^\theta, Z_{t_k}^{\theta,k}) \right)^\top i_j \cdot \left( \partial_i X_{t,T}^\theta \otimes \partial_j X_{t,T}^\theta \right) \cdot \nabla f(X_{t,T}^\theta) dt
\]

(64)

For the first terms on the right-hand side of (64), denoting

\[
\hat{b}^*(y', y, z) = b(y') - \mathbb{1}_{D_{t_k}}(y)(b(y) + b^*(y, z)),
\]

it holds that

\[
\left( \left( \hat{b}^*(Y_t^\theta, Y_{t_k}^\theta, Z_{t_k}^{\theta,k}) \cdot \nabla \right) X_{t,T}^\theta \right) \cdot \nabla f(X_{t,T}^\theta)
\]

\[
= \left( \left( \hat{b}^*(Y_t^\theta, Y_{t_k}^\theta, Z_{t_k}^{\theta,k}) \cdot \nabla \right) \left( X_{t,T}^\theta - X_{t_{k+1},T}^\theta \right) \right) \cdot \nabla f(X_{t,T}^\theta)
\]

\[
+ \left( \left( \hat{b}^*(Y_t^\theta, Y_{t_k}^\theta, Z_{t_k}^{\theta,k}) \cdot \nabla \right) X_{t,T}^\theta \right) \cdot \nabla f(X_{t,T}^\theta)
\]

\[
+ \left( \left( \hat{b}^*(Y_t^\theta, Y_{t_k}^\theta, Z_{t_k}^{\theta,k}) \cdot \nabla \right) X_{t,T}^\theta \right) \cdot \nabla f(X_{t,T}^\theta).
\]

(66)

The first part of the factor involving \( b \) has the form

\[
b(Y_{t_k}^\theta) - \mathbb{1}_{D_{t_k}}(Y_{t_k}^\theta)b(Y_{t_k}^\theta)
\]

\[
= \left[ b(Y_{t_k}^\theta) - b(Y_{t_k}^\theta) \right] + \left[ b(Y_{t_k}^\theta) - \mathbb{1}_{D_{t_k}}(Y_{t_k}^\theta)b(Y_{t_k}^\theta) \right]
\]

\[
= \int_{t_k}^t \mathbb{1}_{D_y}(Y_{t_k}) \left( \left( b(Y_{t_k}) + b^*(Y_{t_k}, Z_{u}^{\theta,k}) \right) \cdot \nabla \right) b(Y_u^\theta)
\]

\[
+ \frac{1}{2} \left( \left( \sigma(Y_{t_k}^\theta) + \sigma^*(Y_{t_k}^\theta, Z_{u}^{\theta,k}) \right) \sigma(Y_{t_k}^\theta) + \sigma^*(Y_{t_k}^\theta, Z_{u}^{\theta,k}) \right)^\top \cdot D^2 \left( \hat{b}(Y_u^\theta) \right) du
\]

\[
+ \int_{t_k}^t \mathbb{1}_{D_y}(Y_{t_k}) \left( \sigma(Y_{t_k}^\theta) + \sigma^*(Y_{t_k}^\theta, Z_{u}^{\theta,k}) \right) \cdot \nabla \hat{b}(Y_u^\theta) dW_u
\]

\[
+ b(Y_{t_k}^\theta)(1 - \mathbb{1}_{D_{t_k}}(Y_{t_k}^\theta)),
\]

(67)

where the integral w.r.t. \( u \) is uniformly bounded in \( \theta \) by \( C(t - t_k) \) in \( L^2(\mathbb{P}) \) norm, the stochastic integral is uniformly bounded in \( \theta \) by \( C(t - t_k)^{1/2} \) in \( L^2(\mathbb{P}) \).
norm and the last term has the same property as the integral-in-$u$ (and in fact of arbitrary order in $t - t_k$) by the calculation of inequalities (47), (48) in [21]. Using the definition (62) for $b^*$ along with $q' \geq 3$, there exists a constant $\nu_2 \geq 2$ such that the remaining part of the factor involving $b$ from (66) has the bound
\[
|I_{D_{\theta}}(Y_{t}, Z_{t}^{\theta,k})| \leq C|b(Y_{t})||Z_{t}^{\theta,k}|^{\nu_2}
\]
for all $\theta \in \Theta$. Putting (67) and (68) into the first term on the right-hand side of (66) and using Hölder’s inequality, Assumptions 4(ii)(iv), (59), (60), (61), (62), (63), (58), Lemma 6.2, Lemma 3.2, Theorem 3.3, Markov property (Theorem 2.13 in [28]; see also justification in the proof of Lemma 6.2) and exponential integrability for $U$ as in Theorem 2.9 in [23] yield
\[
E \left| \left( \left( \tilde{b}^*(Y_{t}^{\theta}, Y_{t_k}^{\theta}, Z_{t_k}^{\theta,k}) \cdot \nabla \right) \left( \tilde{X}_{t,T}^{Y_{t}^{\theta}} - \tilde{X}_{t,T}^{Y_{t_k}^{\theta}} \right) \right) \cdot \nabla \right| f(\tilde{X}_{t,T}^{Y_{t_k}^{\theta}}) \right| \leq C(t - t_k)
\]
for all $t \in [t_k, t_{k+1}]$, $\theta \in \Theta$. The same arguments can be used for the second term on the right-hand side of (66), along with the additional estimate
\[
E[\partial_t f(\tilde{X}_{t,T}^{Y_{t}^{\theta}}) - \partial_t f(\tilde{X}_{t,T}^{Y_{t_k}^{\theta}})]^r
\]
\[
\leq E \left| \int_0^1 \nabla \partial_t f(\lambda \tilde{X}_{t,T}^{Y_{t}^{\theta}} + (1 - \lambda) \tilde{X}_{t,T}^{Y_{t_k}^{\theta}}) d\lambda \cdot (\tilde{X}_{t,T}^{Y_{t}^{\theta}} - \tilde{X}_{t,T}^{Y_{t_k}^{\theta}}) \right|^r
\]
\[
\leq C(1 + E|\tilde{X}_{t,T}^{Y_{t}^{\theta}}|^{2q} + E|\tilde{X}_{t,T}^{Y_{t_k}^{\theta}}|^{2q}) \tilde{E}(E|\tilde{X}_{t,T}^{Y_{t}^{\theta}} - \tilde{X}_{t,T}^{Y_{t_k}^{\theta}}|^{2}) \tilde{E} \leq C(\tilde{E} \tilde{E} |U(Y_{t}^{\theta})| + \tilde{E} \tilde{E} |U(Y_{t_k}^{\theta})|) E|Y_{t}^{\theta} - Y_{t_k}^{\theta}|^r
\]
\[
\leq C(t - t_k)^r
\]
where $r > 1$, in order to obtain the same right-hand bound as (69). For the last term on the right-hand side of (66), we rely more prominently on the Markov property. For any $R > 0$, it holds that
\[
E \left[ \left( \left( \tilde{b}^*(Y_{t}^{\theta}, Y_{t_k}^{\theta}, Z_{t_k}^{\theta,k}) \cdot \nabla \right) \tilde{X}_{t,T}^{Y_{t}^{\theta}} \right) \cdot \nabla \right] f(\tilde{X}_{t,T}^{Y_{t_k}^{\theta}}) \wedge R
\]
\[
= E \left[ \left( \left( \tilde{b}^*(Y_{t}^{\theta}, Y_{t_k}^{\theta}, Z_{t_k}^{\theta,k}) \cdot \nabla \right) \tilde{X}_{t,T}^{Y_{t}^{\theta}} \right) \cdot \nabla \right] f(\tilde{X}_{t,T}^{Y_{t_k}^{\theta}}) \wedge R \mid F_t
\]
\[
= \sum_{i=1}^n E \left[ \tilde{b}^*_i(Y_{t}^{\theta}, Y_{t_k}^{\theta}, Z_{t_k}^{\theta,k}) \left[ \partial_t \tilde{X}_{t,T}^{Y_{t}^{\theta}} \cdot \nabla \right] f(\tilde{X}_{t,T}^{Y_{t_k}^{\theta}}) \wedge R \right] \mid F_{t_k}
\]
so that (65), (67) and (68), where the only order $\frac{1}{2}$ term in $t - t_k$ from (67) has vanished, together with the same arguments as before and dominated convergence in $R$ yields
\[
E \left[ \left( \left( \tilde{b}^*(Y_{t}^{\theta}, Y_{t_k}^{\theta}, Z_{t_k}^{\theta,k}) \cdot \nabla \right) \tilde{X}_{t,T}^{Y_{t}^{\theta}} \right) \cdot \nabla \right] f(\tilde{X}_{t,T}^{Y_{t_k}^{\theta}}) \wedge R \leq C(t - t_k)
\]
(70)
integrals involving $b$ in (64) have been shown to be of order $t - t_k$. For the integrals involving $\sigma$ in (64), after rewriting

$$\sigma(Y_t^\theta)\sigma(Y_t^\theta) - \mathbb{I} \mathcal{D}(Y_t^\theta)\sigma(Y_t^\theta) + \sigma^*(Y_t^\theta, Z_t^{\theta,k})((\sigma(Y_t^\theta) + \sigma^*(Y_t^\theta, Z_t^{\theta,k}))^\top$$

and similarly

$$((\partial_t X_{t,T}^\theta \otimes \partial_j X_{t,T}^\theta : D^2) f(X_{t,T}^\theta) + (\partial^2_{ij} X_{t,T}^\theta \cdot \nabla) f(X_{t,T}^\theta)$$

the same bound as (70) holds for all of (64) by the same treatment as for (70).

$$\square$$

7 Examples

In this section, specific examples are provided where the results presented above are applicable. As stated in the introduction, most of the examples in [11, 23] are viable and many Lyapunov functions have already been given in these references (applicable here after a simple transformation, see Remark 6.1). A notable exception is the stochastic SIR model, where the Lipschitz constant of the diffusion coefficients grow too quickly for the Lyapunov functions given there. Here, the focus is placed on two particular examples differing in some considerable way to analogies in the aforementioned references. In Section 7.1, our results are applied to the (underdamped) Langevin dynamics with variable friction, which, by definition, does not have globally Lipschitz (nor monotone) coefficients; this is motivated by the work [9]. In Section 7.2, a Lyapunov function ($V_0$ satisfying $LV_0 \leq CV_0$) is given for the Stochastic Duffing-van der Pol equation with parameter values not accounted for in previous works mentioned above.
7.1 Langevin equation with variable friction

Here, the backward Kolmogorov equation and Poisson equation associated with the Langevin equation are shown to hold even in cases where the friction matrix depends on both position and velocity variables. The pointwise solution to the backward Kolmogorov equation may be used to obtain a distributional solution to the associated Poisson equation and in doing so, comprises a first step towards a gradient formula for the asymptotic variance as in [9]. In addition and perhaps more importantly, solutions to the Poisson equation allows one to obtain central limit theorems for additive functionals [7, Section 3]. The results here give a rigorous way to derive distributional solutions to the Poisson equation in the proof of Proposition 3.10 in [7] (in particular, it is not clear that the domain of $L^*$ includes $C_c^\infty$). In this case, hypoellipticity is required to complete the argument to obtain the central limit theorem, which means that Proposition 4.18 in [15] may also be used in the case of continuous bounded observables; the results here extend the space of observables beyond that of continuous bounded functions at the cost of stronger assumptions on the coefficients of the SDE.

Assumption 5. The function $U \in C^3(\mathbb{R}^n)$ is such that there exists $\tilde{k}, \tilde{K} > 0$ with $\nabla U(q) \cdot q \geq \tilde{k}|q|^2 - \tilde{K}$ for all $q \in \mathbb{R}^n$. The friction matrix $\Gamma \in C^\infty(\mathbb{R}^{2n}, \mathbb{R}^{n \times n}) \cap L^\infty$ is symmetric positive definite everywhere such that there exist $\beta_1 < 1$, $\tilde{m}, \tilde{M} > 0$ with $|\nabla_p \cdot \Gamma(q, p)| < \tilde{M}(1 + |q|^\beta_1) + |p|^\beta_1$ and $\Gamma(q, p) \geq \tilde{m}I$ for all $q, p \in \mathbb{R}^n$.

Note Assumption 5 implies for $R > 1$, $q \in \mathbb{R}^n$ with $|q| = 1$,

$$U(Rq) - U(q) = \int_1^R \nabla U(\lambda q) \cdot \frac{\lambda q}{\lambda} d\lambda \geq \int_1^R (\tilde{k}|\lambda q|^2 - \tilde{K})\lambda^{-1} d\lambda = \frac{\tilde{k}(R^2 - 1)}{2} - \tilde{K} \log R,$$

which yields $U(q) \geq \frac{\tilde{k}}{4}|q|^2 - C$ for all $q \in \mathbb{R}^n$ and some constant $C > 0$. Consider $\mathbb{R}^{2n}$-valued solutions $(q_t, p_t)$ to

$$dq_t = p_t dt \quad (71a)$$
$$dp_t = -\nabla U(q_t) dt + \nabla_p \cdot \Gamma(q_t, p_t) dt - \Gamma(q_t, p_t) p_t dt + \sqrt{\Gamma(q_t, p_t)} dW_t, \quad (71b)$$

where $\sqrt{\Gamma}$ denotes some matrix satisfying $\sqrt{\Gamma} \sqrt{\Gamma}^T = \Gamma$ and $(\nabla \Gamma)_i = \sum_j \nabla_{q_j} \Gamma_{i,j}$. For $b = \min(\tilde{k}^{-1}(\sup_{\mathbb{R}^{2n}} |\Gamma|)^{-1}, \tilde{m}, \tilde{K}^2)$, $a = \frac{1}{4} \min(\frac{\tilde{k}}{b}, \tilde{m})$, let

$$V_\gamma(q, p) = e^{\gamma(U(q) + a|q|^2 + b|p| + |p|^2)}.$$

In the following, $|M|$ denotes the operator norm of $M \in \mathbb{R}^{n \times n}$.

\footnote{It is possible to allow for $\beta_1 = 1$, but at the cost of more stringent bounds on the coefficients.}
Proposition 7.1. Under Assumption 5, there exists constants $c_1, c_2, c_3 > 0$ such that for all $\gamma$ satisfying

$$0 < \gamma \leq \gamma^* := \frac{1}{8} \min((\bar{k}b_{\sup|\Gamma|})^{-1}, \bar{m}(4 \sup|\Gamma|)^{-1}),$$

it holds that

$$LV_\gamma(q, p) \leq (c_1 - c_2|q|^2 - c_3|p|^2)\gamma V_\gamma(q, p)$$

for all $(q, p) \in \mathbb{R}^{2n}$, where $L$ is the generator $(9)$ associated with $(71)$. If in addition there exist $0 < \beta_2 < 1$, $M > 0$ such that

$$|\partial_i(\nabla_p \cdot \Gamma(q, p) - \nabla U(q))| \leq M(1 - \inf U + U(q) + |p|^2)^{\beta_2}$$

$$|\partial_i \Gamma(q, p)| \leq M(1 - \inf U^{-\frac{\beta_2}{2}} + U(q)^{-\frac{\beta_2}{2}} + |p|^2)^{\beta_2}$$

$$|\partial_i \partial_j(\nabla_p \cdot \Gamma(q, p) - \nabla U(q))| + |\partial_i \partial_j \Gamma(q, p)| \leq M(1 + e^{(U(q) + |p|^2)^{\beta_2}})$$

for all $q, p \in \mathbb{R}^n$, $i, j \in \{1, \ldots, 2n\}$, then Assumption 2 (with $p = 2$) is satisfied with $V_k = V_\gamma$ with any $\gamma$ satisfying $(73)$, $G(q, p) = C(1 - \inf U + U(q) + |p|^2)^{\beta_3}$ for some constants $C > 0$ and $\beta_2 < \beta_3 < 1$.

Proof. The left-hand side of $(74)$ calculates as

$$(p \cdot \nabla_q - \nabla_q U(q) \cdot \nabla_q + (\nabla_q \cdot \Gamma(q, p)) \cdot \nabla_q - \Gamma(q, p)p) \cdot \nabla_q + \Gamma(q, p) : D^2 V_\gamma(q, p)$$

$$= (2aq + b|p|^2 - b\nabla_q U(q) \cdot q + (\nabla_q \cdot \Gamma(q, p) - \Gamma(q, p)p) \cdot (bq + 2p)$$

$$+ 2\Tr \Gamma(q, p) + \Gamma(q, p) : (b^2 qq^T + 4pp^T))\gamma V_\gamma(q, p)$$

$$\leq \left(\left(a - \frac{b}{k} + \frac{1}{2}b^2 |\Gamma| + b^2 |\Gamma||\right)|q|^2 + \left(a + b + \frac{1}{2}|\Gamma| - 2\bar{m} + 4\gamma |\Gamma|\right)|p|^2$$

$$+ M(1 + |q|^{\beta_1} + |p|^{\beta_2})|bq + 2p| + bK + 2\Tr \Gamma\right)\gamma V_\gamma(q, p)$$

$$\leq \left(c - \frac{b}{16k}|q|^2 - \frac{\bar{m}}{16}|p|^2\right)\gamma V_\gamma(q, p)$$

(75)

for some constant $c > 0$. The last assertion follows by straightforward applications of Young’s inequality. □

For $U$ with locally Lipschitz third derivatives and by Theorem 5.2 (iii), the associated Poisson equation with right-hand side $\tilde{f} = f - \int_{\mathbb{R}^{2n}} f d\mu \in L^2(\mu)$ holds in the distributional sense as in Proposition 3.1 in [9] if in addition

$$|E\tilde{f}(z_t)| + \int^\infty_t E\tilde{f}(z_s) ds \to 0 \text{ in } L^2(\mu) \text{ as } t \to \infty,$$  

(76)

where for any $z \in \mathbb{R}^{2n}$, $z^T = (q_t, p_t)$ solves $(71)$, $P((q_0, p_0) = z) = 1$ and $\mu(dq, dp) = Z^{-1} e^{-U(q) - \frac{1}{2}p^2} dq dp$ is the invariant probability measure with normalizing constant $Z$. We obtain (76) in the following by using the ergodicity results of [13], see alternatively Theorem 2.4 in [41]. The proof of Proposition 1.2 in [41] can be modified for $(71)$ to obtain
Proposition 7.2. For every $x \in \mathbb{R}^{2n}$, $t > 0$, the measure $P^t(x, \cdot)$ admits a density $p_t(x, \cdot)$ satisfying $p_t(x, y) > 0$ Lebesgue almost every $y \in \mathbb{R}^{2n}$ and
\[
(x \mapsto p_t(x, \cdot)) \in C(\mathbb{R}^{2n}, L^1(\mathbb{R}^{2n})).
\]

Proof. For the Markov property, see the proof of Lemma 6.2 just before (57). The proof in the aforementioned reference follows through except in the proof of Lemma 1.1 in [41], where the Lyapunov function (72) is to be used in place of $\hat{H}(x, y) = \frac{1}{2} |y|^2 + V(x) - \inf_{\mathbb{R}^n} V + 1$ and $R^2$ in the ensuing calculations is replaced as needed.

Proposition 7.2 implies the existence of an irreducible skeleton chain, see [13]; together with Theorem 3.2 in the same reference (with $\Psi = (\Psi_1, \Psi_2)$, $\Psi_1(x) = \Psi_2(x) = (x/2)^2 + \phi(x) = x^2$, $V = V^\ast$), Theorem 3.4 in [13] (compact sets are petite by Theorem 4.1(i) in [32]) and Proposition 3.1 in the same reference (with $\phi(x) = x$, $V = V^\ast$), this yields (76) for $\hat{f}$ satisfying $\hat{f}/V^\ast \in L^\infty$. Note that the Foster-Lyapunov condition for geometric ergodicity suffice as well.

In addition, maximal dissipativity of the closure of the generator defined on $C^\infty_\mathcal{C}$ is also enough to conclude a distributional solution to the Poisson equation, which motivates the question of whether there is a relationship between this property and the Kolmogorov equation; a partial answer is given by Proposition 4.6. However, maximal dissipativity is not generally available and for example not established for (71) with $\Gamma$ dependent on both $q$ and $p$, see [3] on the other hand.

7.2 Stochastic Duffing-van der Pol equation

We show here that the Stochastic Duffing-van der Pol oscillator admits a Lyapunov function satisfying the assumptions of Theorem 3.3. Note that in doing so, the difficult parts of Assumption 4 are shown to be satisfied, so that our Theorem 6.3 about weak numerical convergence rates applies. In particular, the logarithm of the Lyapunov function described below may be used for $U$ in Assumption 4. The version of the equation considered is from [20] with $\beta_2 = 0$, which is less general than in [20] but still includes the setting of Section 13.1 in [27] and [2] for example. Moreover, it is more general than Section 4.3 in [11], which is reflected in the form of the Lyapunov function here. On the other hand, it is not more general than in [21]. Specifically, for $(W^{(1)}, W^{(3)}): [0, T] \times \Omega \to \mathbb{R}^2$ a standard $(\mathcal{F}_t)_{t \in [0, T]}$-adapted Brownian motion, $\alpha_1, \alpha_2, \beta_1, \beta_3 \in \mathbb{R}$, $\alpha_3 \geq 0$, consider $\mathbb{R}^2$-valued solutions to
\[
\begin{align*}
    dX_t^{(1)} &= X_t^{(2)} \, dt, \\
    dX_t^{(2)} &= [\alpha_1 X_t^{(1)} - \alpha_2 X_t^{(2)} - \alpha_3 X_t^{(2)} (X_t^{(1)})^2 - (X_t^{(1)})^3] \, dt \\
                   &\quad + \beta_1 X_t^{(1)} \, dW_t^{(1)} + \beta_3 dW_t^{(3)}.
\end{align*}
\]

For $\alpha_3 > 0$ (see Remark 7.1 for the case $\alpha_3 = 0$, $V$ is chosen here with regard to this case), let $a = \min(1, \frac{1}{\alpha_3})$, $b = (2 - \alpha_3) I_{\alpha_3 < 1} + \frac{a}{2} I_{\alpha_3 \geq 1}$, $c = 6 |\alpha_2|$,
\[ \gamma \leq \min(\frac{2\alpha}{\beta_1}, \frac{1}{\beta_1}) \] 
and let \( \eta : \mathbb{R} \to [0, 1] \) be a \( C^\infty \) cut off function satisfying 
\[ \eta(y) = 1 \text{ for } y^2 \leq \frac{1 + |a - 2\alpha_2b + 2\beta_3^2\gamma b^2|}{2\alpha_3b - 2\beta_3^2\gamma b^2}. \] 
Define 
\[ V(x_1, x_2) = V_1(x_1, x_2) + V_2(x_1, x_2) \]
\[ := (1 - \eta(x_1))e^{\gamma(x_1^4 + ax_1x_2 + bx_2^2)} + e^{\gamma(-cx_1x_2 + \frac{3}{2}x_2^2)}. \]

**Proposition 7.3.** If \( \alpha_3 > 0 \), then there exists a constant \( C^* > 0 \) such that \( LV \leq C^*V \), where \( L \) is the generator (9) associated with (78). Moreover, Assumptions 1 and 2 are satisfied with \( G(t, x) = (3 + 2 \sum_{i} |a_i| + \beta_1^2)(1 + |x_1|^3 + |x_2|^2) \) and \( V_k(t, x) = (|X_1^{(1)}|^4 + 2|X_1^{(2)}|^2 + 1)^k \) for \( t \geq 0 \), \( x = (x_1, x_2) \in \mathbb{R}^n \), where \( (X_1^{(1)}, X_1^{(2)}) \) solves (78) with \( (X_1^{(1)}, X_1^{(2)}) = (x_1, x_2) \).

**Proof.** The functions \( V_1 \) and \( V_2 \) satisfy 
\[ LV_1(x_1, x_2) = [(2\alpha_1b - \alpha_2a)x_1x_2 + (a - 2\alpha_2b + 2\beta_3^2\gamma b^2)x_2^2 + (\alpha_1a + 1)(\frac{1}{2}\beta_3^2\gamma a^2 + b_1^2)x_1^2 + (\frac{1}{2}\beta_3^2\gamma a^2 - a)x_1^4 + b_3^2\beta_3^2 - \frac{x_2^2\alpha_3b}{1 - \eta(x_1)}\] 
\[ + \gamma V_1(x_1, x_2) \]
\[ LV_2(x_1, x_2) = [(\frac{1}{2}\beta_3^2\gamma - c - \alpha_2)x_2^2 + (\frac{1}{2}c^2\beta_3^2\gamma - \alpha_3c + 1)(\frac{1}{2}\beta_3^2\gamma )x_1^2 + (\alpha_2c + \alpha_1)x_1x_2 + (\alpha_3c - 1)x_1^2x_2 + (c + \frac{1}{2}c^2\gamma b_1^2)x_1^4 + \frac{1}{2}\beta_3^2\gamma - \alpha_3)x_1^2x_2^2 + \frac{1}{2}\beta_3^2\gamma V_2(x_1, x_2). \]

where \( \frac{1}{1 - \eta(x_1)} := 0 \) whenever \( 1 - \eta(x_1) = 0 \). In order to see \( LV \leq CV \), consider separately the regimes \( x_1^2 \leq \frac{1 + |a - 2\alpha_2b + 2\beta_3^2\gamma b^2|}{2\alpha_3b - 2\beta_3^2\gamma b^2} \) and its complement in \( \mathbb{R}^2 \). In the former case, \( V_1(x_1, x_2) = LV_1(x_1, x_2) = 0 \) and by our choice of \( c \) and \( \gamma \), there exists a generic constant \( C > 0 \) such that \( LV_2 \leq CV_2 \), therefore \( LV \leq CV \). Otherwise in the complementary case where \( |x_1| \) is bounded below, we have \( LV_1 \leq CV_1 \) and when in addition \( x_1 \in \text{supp} \eta \cup B_1(0) \), it holds that \( LV_2 \leq CV_2 \). It remains to estimate \( LV_2 \) when \( x_1 \notin \text{supp} \eta \cup B_1(0) \), in which case we have \( |x_i|^4e^{\gamma(-cx_1x_2 + \frac{3}{2}x_2^2)} \leq Ce^{\gamma(x_1^4 + \frac{3}{2}x_2^2)} \leq CV_1(x_1, x_2) \) for \( i \leq 4 \), from which \( LV_2 \leq CV_1 \). For the second assertion, it is straightforward to see that (14), (15) hold and that the higher derivatives of the coefficients of (78) are bounded above in terms of \( V_k \) for any \( k, p \) as called-for in Assumption 2; for (16), consider separately the cases \( |x_1| \leq \sup \{ |x| : x \in \text{supp} \eta \} \) and otherwise. In the former case, it holds that 
\[ G(x_1, x_2) \leq C(1 + |x_2|^\frac{3}{2}), \]
which yields that for any \( m > 0 \), there is \( M = M(m) > 0 \) continuous in \( m \) such that 
\[ G \leq m \log(V) + M \leq m \log(V) + M. \] 
(79)
When $|x_1| > \sup \{|x| : x \in \text{supp}\eta\}$, inequalities (79) continue to hold with $V_1$ replacing $V_2$ and a corresponding continuous function $m \mapsto M(m)$.

**Remark 7.1.** When $\alpha_3 = 0$, the arguments of Proposition 7.3 fail to adapt, but one can consider (78) and its derivative dynamics appended with $d\alpha_3 = 0$ with initial condition as a positive function in $\mathbb{R}$ in order to obtain the statements of Theorem 3.3. To elaborate, for almost all $\omega \in \Omega$, the solution to the appended derivative dynamics is continuously differentiable with respect to $\alpha_3$ by Theorem V.39 in [36] and in particular is continuous, so that Fatou’s lemma can be used to obtain

$$
E \lim_{\alpha_3 \to 0} \sup_{0 \leq s \leq t} \left| \frac{\partial (\kappa)(x(t, s))}{\partial x} \right|^k_1 \leq \liminf_{\alpha_3 \to 0} E \sup_{0 \leq s \leq t} \left| \frac{\partial (\kappa)(x(t, s))}{\partial x} \right|^k_1.
$$

Therefore the same bounds in Lemma 3.2 and Theorem 3.3 apply if they are uniform with respect to small $\alpha_3$; since $\gamma$ is proportional to $\alpha_3$ (for small $\gamma$), it follows that $M$ in (16) and subsequently terms stemming from $\alpha_3$ in both the proofs of Lemma 3.2 (in Lemma 3.1 and (24)) and Theorem 3.3 (terms from equation (32)) in some sense blows up as $\alpha_3 \to 0$. This can be addressed by increasing $m$ accordingly so that $M$ is uniformly bounded.

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A Auxiliary results

Just as in the case of globally Lipschitz coefficients in [28, Lemma 5.10], the regularity of an extended system and the harmonic property of the expectation (45) are required. These properties are established for our setting in the following.

Throughout the section, we assume $O = \mathbb{R}^n$ and $b, \sigma$ are nonrandom functions. Moreover for the functions $f, c, g$, we assume nonrandomness and all of the presuppositions about them made in Theorem 4.2 regarding Assumption 3. In particular, $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, $c : [0, \infty) \times \mathbb{R}^n \to [0, \infty)$ and $g : \mathbb{R}^n \to \mathbb{R}$ are Borel functions satisfying that $f(t, \cdot), c(t, \cdot), g(\cdot)$ are continuous for every $t \in [0, T]$, $\int_0^T \sup_{x \in B_R}|c(t, x)| + |f(t, x)| \, dt < \infty$ for every $R > 0$ and such that for $h \in \{f, c, g\}$, $R > 0$, there exists $C \geq 0$, $0 < l \leq 1$, Lyapunov
functions $V^{s,T}$, locally bounded $\tilde{x}$ for which for any $s \in [0,T]$ it holds $\mathbb{P}$-a.s. that
\[
|h(s + t, X_t^s)| \leq C(1 + V^{s,T}(t, \tilde{x}(x)))^{\tilde{I}} \quad \text{(80a)}
\]
\[
|h(s + t, y) - h(s + t, y')| \leq C|y - y'| \quad \text{(80b)}
\]
\[
V^{s+\tau,T}(0, \tilde{x}(X^{s,x}_{\tau})) \leq C(1 + V^{s,T}(\tau, \tilde{x}(x))) \quad \text{(80c)}
\]
for all $t \in [0,T]$, $t \leq T - s$, stopping times $\tau \leq T$, $x \in \mathbb{R}^n$ and $y, y' \in B_R$, where continuity of the underlying $V_0$ for the relevant Lyapunov functions have been used for (80b).

For any $s \geq 0, T > 0, x \in \mathbb{R}^n$, $x', x'' \in \mathbb{R}$, consider solutions $X^{s,x}_t$ to (8) appended with the corresponding $\mathbb{R}$-valued solutions $X^{(n+1),s,x'}_t$ and $X^{(n+2),s,x'}_t$ to
\[
X^{(n+1),s,x'}_t = x' + \int_0^t c(s + r, X^{s,x}_r)dr,
\]
\[
X^{(n+2),s,x''}_t = x'' + \int_0^t f(s + r, X^{s,x}_r)e^{-X^{(n+1),s,x'}_r}dr
\]
on $[0,T]$, denoted $\tilde{X}^{s,y}_t = (X^{s,x}_t, X^{(n+1),s,x'}_t, X^{(n+2),s,x''}_t)$, $y = (x, x', x'')$. Let $\tilde{X}^{s,y}_t(I)$ be the corresponding Euler approximation analogous to (44) with $I$ as in the beginning of Lemma 4.4.

Lemma A.1. Under the assumptions of this section, for every $R, T > 0$, it holds that
\[
\sup_{s \in [0,T]} \sup_{y \in B_R} \mathbb{P}\left( \sup_{t \in [0,T]} |\tilde{X}^{s,y}_t - \tilde{X}^{s,y}_t(I)| > \epsilon \right) \to 0
\]
as $\sup_k t_{k+1} - t_k \to 0$.

Proof. For any $R' > 0$, let $R^{s,x}_X(I, R') \in \mathcal{F}$ denote the event
\[
R^{s,x}_X(I, R') = \left\{ \sup_{t \in [0,T]} |X^{s,x}_t| \leq R' \right\} \cap \left\{ \sup_{t \in [0,T]} |X^{s,x}_t(I)| \leq R' \right\}.
\]
For any $\epsilon, R' > 0$, it holds that
\[
\mathbb{P}\left( \sup_{t \in [0,T]} |\tilde{X}^{s,y}_t - \tilde{X}^{s,y}_t(I)| > \epsilon \right)
\]
\[
\leq \mathbb{P}\left( \sup_{t \in [0,T]} |X^{s,x}_t| > R' \right) + \mathbb{P}\left( \sup_{t \in [0,T]} |X^{s,x}_t(I)| > R' \right)
\]
\[
+ \mathbb{P}\left( \sup_{t \in [0,T]} |\tilde{X}^{s,y}_t - \tilde{X}^{s,y}_t(I)| > \epsilon \left| R^{s,x}_X(I, R') \right. \right).
\]
Fix $\epsilon' > 0$. For any $T, R > 0$, we may choose $R' = R''$ so that, by Lemma 2.2 in [28], the sum of the first and second term on the right-hand side is bounded

52
above by $c'/2$ uniformly in $s \in [0, T]$ and $x \in B_R$. For the last term on the right, note that by our assumptions on $c$, there exists locally bounded $\tilde{G} : \mathbb{R}^n \to [0, \infty)$ such that

$$
\sup_{t \in [0, T]} |c(s + t, X^s_x(I)) - c(s + t, X^s_x(I))| \\
\leq \sup_{t \in [0, T]} |X^s_x(I) - X^s_x(I)| (\tilde{G}(X^s_x(I)) + \tilde{G}(X^s_x(I)))
$$

(82)

and such that for $I_t := \max\{t_k : t \geq t_k\}$,

$$
\sup_{t \in [0, T]} |c(s + t, X^s_x(I)) - c(s + t, X^s_x(I))| \\
\leq \sup_{t \in [0, T]} |X^s_x(I) - X^s_x(I)| (\tilde{G}(X^s_x(I)) + \tilde{G}(X^s_x(I)))
$$

(83)

for all $s \in [0, T], y = (x, x', x'') \in \mathbb{R}^{n+2}$, where we have used that $c$ is uniformly bounded on $[0, 2T] \times B_R$ by the continuity of the underlying $V_0$ for the Lyapunov function that forms an upper bound for $c$. By (82), it holds that

$$
\left\{ \sup_{t \in [0, T]} |X^s_x(I) - X^s_x(I)| \leq \frac{\epsilon}{12\sqrt{3} T sup_{z \in B_R} \tilde{G}(z)} \cap R^e_x(I, R^*) \right\} \subset \left\{ \sup_{t \in [0, T]} |c(s + t, X^s_x(I)) - c(s + t, X^s_x(I))| \leq \frac{\epsilon}{6\sqrt{3}} \right\} \cap R^e_x(I, R^*)
$$

(84)

and by (83), there exists $\delta^* > 0$ such that $sup_{k} t_{k+1} - t_k \leq \delta^*$ implies

$$
R^e_x(I, R^*) \subset \left\{ \int_0^T |c(s + u, X^s_x(I)) - c(s + u, X^s_x(I))| du \leq \frac{\epsilon}{6\sqrt{3}} \right\}
$$

As a result, by Lemma 4.4 and our assumptions on $b$ and $\sigma$, there exists $0 < \delta \leq \delta^*$ such that for $I$ satisfying $sup_{k} t_{k+1} - t_k \leq \delta$, it holds that

$$
P \left( \sup_{t \in [0, T]} |X^{(n+1),s,x'} - X^{(n+1),s,x'}(I)| > \frac{\epsilon}{3\sqrt{3}} \right) \\
\leq P \left( \sup_{t \in [0, T]} |X^s_x - X^s_x(I)| > \frac{\epsilon}{12\sqrt{3} T sup_{z \in B_R} \tilde{G}(z)} \right) \leq \frac{\epsilon'}{6}
$$

(84)

for all $s \in [0, T]$ and $y = (x, x', x'') \in B_R \subset \mathbb{R}^{n+2}$. By a similar argument and using the above, (84) holds with $n+1$ replaced by $n+2$ and $x'$ by $x''$. Together with Lemma 4.4, the lemma is proved. 

□
Next, the harmonic property (see [28, Definition 3.1]) of (45) is shown. Let \( \bar{g} \) given by \( \bar{g}(y) = x'' + g(x)e^{-x'} \) for all \( y = (x, x', x'') \in \mathbb{R}^{n+2} \) and for \( T > 0, s \in [0, T] \), let \( \bar{v} : [0, \infty) \times \mathbb{R}^{n+2} \to \mathbb{R} \) be given by
\[
\bar{v}(s, y) = \mathbb{E}[\bar{g}(\bar{X}_{T-s}^{s,y})] = x'' + v(s, x)e^{-x'} = \mathbb{E}[X_{T-s}^{(n+2),s,x''} + g(X_{T-s}^{s,x})e^{-X_{T-s}^{(n+1),s,x'}}].
\] (85)
for \( v \) given in (45). In addition for a bounded subset \( Q \subset (0, T) \times \mathbb{R}^{n+2} \), let \( \tau \) be the stopping time
\[
\tau := \inf\{u \geq 0 : (s + u, \bar{X}_u^s) \notin Q\}. \tag{86}
\]
The next lemma establishes the equality \( \bar{v}(s, y) = \mathbb{E}[\bar{v}(s + \tau \wedge t), \bar{X}_{T}^{s,y}] \) under our setting.

**Lemma A.2.** Under the assumption of this section, for any \( T > 0 \), any bounded subset \( Q \subset (0, T) \times \mathbb{R}^{n+2}, (s, y) \in Q, t \in [0, T - s] \), it holds that
\[
\mathbb{E}[\bar{g}(\bar{X}_{T-s}^{s,y})] = \int \int \bar{g}(\bar{X}_{T-s-\tau}^{s+(\tau(\omega)\wedge t),\bar{X}_{T}^{s,y}(\omega)}, \omega')d\mathbb{P}(\omega')d\mathbb{P}(\omega),
\]
where \( \tau \) is defined by (86).

**Proof.** For any \( R, T > 0, t \in [0, T], (s, y) \in Q \) with \( y = (x, x', x'') \), by Theorem 2.13 in [28] together with Lemma A.1, it holds for \( \mathbb{P} \)-a.a. \( \omega \) that
\[
\mathbb{E}[(1_B \bar{g})(\bar{X}_{T-s}^{s,y})|\mathcal{F}_{\tau \wedge t}] = \int (1_B \bar{g})(\bar{X}_{T-s-(\tau(\omega)\wedge t)}^{s+(\tau(\omega)\wedge t),\bar{X}_{T}^{s,y}(\omega)}, \omega')d\mathbb{P}(\omega'), \tag{87}
\]
so that the right-hand side is \( \mathcal{F}_{\tau \wedge t} \)-measurable. Moreover for \( \mathbb{P} \)-a.a. \( \omega \), by (80), the absolute value of the integrand in the right-hand side is bounded independently of \( R \) as
\[
\begin{align*}
(1_B \bar{g})(\bar{X}_{T-s-(\tau(\omega)\wedge t)}^{s+(\tau(\omega)\wedge t),\bar{X}_{T}^{s,y}(\omega)}, \omega') - |X_{(\tau(\omega)\wedge t)}^{(n+2),s,x''}(
\omega)| & \\
\leq & \int_0^{T-s-(\tau(\omega)\wedge t)} |f(s + (\tau(\omega) \wedge t) + r, X_{r+(\tau(\omega)\wedge t)}^{s+(\tau(\omega)\wedge t),\bar{X}_{T}^{s,y}(\omega)}(\omega))|dr \\
& \quad + |g(X_{r+(\tau(\omega)\wedge t)}^{s+(\tau(\omega)\wedge t),\bar{X}_{T}^{s,y}(\omega)}(\omega), \omega')| \\
& \leq C \left( \int_0^{T-s-(\tau(\omega)\wedge t)} (1 + V^{s+(\tau(\omega)\wedge t),T}T(\omega', r, \tilde{x}(X_{r+(\tau(\omega)\wedge t)}^{s,y}(\omega))))dr \\
& \quad + (1 + V^{s+(\tau(\omega)\wedge t),T}T(\omega', T - s - (\tau(\omega) \wedge t), \tilde{x}(X_{r+(\tau(\omega)\wedge t)}^{s,y}(\omega)))) \right). \tag{88}
\end{align*}
\]
Since \( l^th \)-powers of Lyapunov functions are still Lyapunov functions (but with different auxiliary processes), the expectation in \( \omega' \) of the right-hand side of
which is bounded by Theorem 2.4 in [19] and (80c) as in

\[
\int \left( \int_0^{T-s-(\tau(\omega)\land t)} (1 + V^{s+(\tau(\omega)\land t),T}(\omega', r, \tilde{x}(X_{\tau(\omega)\land t}^{s,x}(\omega))))dr \\
+ (1 + V^{s+(\tau(\omega)\land t),T}(\omega', T - s - (\tau(\omega) \land t), \tilde{x}(X_{\tau(\omega)\land t}^{s,x}(\omega))))dr \right) d\mathbb{P}(\omega') \\
\leq C\|e^{\int_0^T |\alpha_r^{s+(\tau(\omega)\land t),T}(\omega')|du}\|_{L^{p\big|\frac{(\tau(\omega)\land t)}{\mathbb{P}}\big|^{-1}}} (d\mathbb{P}(\omega')) \\
\cdot \int (1 + V^{s+(\tau(\omega)\land t),T}(\omega', 0, \tilde{x}(X_{\tau(\omega)\land t}^{s,x}(\omega))))dr \right) d\mathbb{P}(\omega') \\
\leq C\left( 1 + \int V^{s,T}(\omega', \tau(\omega) \land t, \tilde{x}(x)) d\mathbb{P}(\omega') \right) \\
\leq C(1 + V^s,T(0, \tilde{x}(x))) \\
< \infty. \quad (89)
\]

Therefore by dominated convergence, the right-hand side of (87) converges to the same expression but without \(\mathbb{1}_{\bar{B}_T}\) for \(\mathbb{P}\)-a.a. \(\omega\). Moreover, by (80) and Theorem 2.4 in [19],

\[
\mathbb{E}|X^{(n+2),s,x''}_T| - |x''| \leq \mathbb{E} \int_0^{\tau \land t} |f(s + r, X_{r}^{s,x''})|dr \\
\leq C \mathbb{E} \int_0^T [1 + V^{s,T}(r, \tilde{x}(x''))]dr \\
\leq C(1 + V^s,T(0, \tilde{x}(x''))).
\]

Consequently, together with (88), (89) and dominated convergence (in \(\omega\)), it holds that

\[
\int \int (\mathbb{1}_{\bar{B}_T}\bar{g})(X_{T-s+(\tau(\omega)\land t)}^{s,x,y}(\omega'))d\mathbb{P}(\omega')d\mathbb{P}(\omega) \\
\rightarrow \int \int \bar{g}(X_{T-s+(\tau(\omega)\land t)}^{s,x,y}(\omega'))d\mathbb{P}(\omega')d\mathbb{P}(\omega)
\]
as \(R \to \infty\). On the other hand, by a similar argument as above, the expectation of the left-hand side of (87) has the limit

\[
\mathbb{E}[\mathbb{E}[\mathbb{1}_{\bar{B}_T}\bar{g})(X_{T-s}^{s,y})|\mathcal{F}_{\tau \land t}]] = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\bar{B}_T}\bar{g})(X_{T-s}^{s,y})] \rightarrow \mathbb{E}[\bar{g}(X_{T-s}^{s,y})]
\]
as \(R \to \infty\). \(\square\)