On adelic quotient group for algebraic surface

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Abstract

We calculate explicitly an adelic quotient group for an excellent Noetherian normal integral two-dimensional separated scheme. An application to an irreducible normal projective algebraic surface over a field is given.

1 Introduction

Let $K$ be a global field or the field of rational functions of an algebraic curve over a field $k$, and $\mathbb{A}_K$ be the group of adeles of the field $K$. Then it is well-known that $\mathbb{A}_K/K$ is a compact topological space in the number theory case and a linearly compact $k$-vector space in the geometric case, see, e.g., [3] and Section 2 below. Moreover, the strong approximation theorem implies the following exact sequence, which we write in the simplest case $K = \mathbb{Q}$:

$$0 \longrightarrow \hat{\mathbb{Z}} \longrightarrow \mathbb{A}_\mathbb{Q}/\mathbb{Q} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0. \quad (1)$$

The goal of this note is to calculate explicitly, similarly to formula (1), an adelic quotient group

$$\mathbb{A}_X(\mathcal{F})/(\mathbb{A}_{X,01}(\mathcal{F}) + \mathbb{A}_{X,02}(\mathcal{F})),$$  

where $X$ is an excellent Noetherian (e.g., finite type over $k$ or over $\mathbb{Z}$) normal integral two-dimensional separated scheme $X$, $\mathbb{A}_X(\mathcal{F})$ is the group of higher adeles of a locally free sheaf $\mathcal{F}$ on $X$, the subgroup $\mathbb{A}_{X,01}(\mathcal{F}) + \mathbb{A}_{X,02}(\mathcal{F})$ is an analog of the above subgroup $K \subset \mathbb{A}_K$. Instead of fixing a valuation as, for example, the Archimedean valuation in formula (1) (or a point on a connected projective curve) we fix a reduced one-dimensional closed equidimensional subscheme $C$ of the scheme $X$ such that the open subscheme $X \setminus C$ is affine.

Higher adeles were introduced by A. N. Parshin in [13] for the case of a smooth projective surface over a field, and by A. A. Beilinson in a short note [1] (which did not contain proofs) for the case of arbitrary Noetherian schemes. Later the proofs of Beilinson’s results on higher adeles appeared in [6]. A survey of higher adeles is contained also in [9].

The main goal in the higher adeles program is the possible arithmetic applications for the study of zeta- and $L$-functions of arithmetic schemes, see the excellent survey of A. N. Parshin [15].

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Recently there were also found the interesting connections and interactions with many questions between the higher adeles theory and the Langlands program, see [16].

As an application, we deduce from our calculations of adelic quotient group (2) the corresponding quotient group when $X$ is a projective irreducible normal surface over a field $k$, and $C$ is the support of an ample divisor. As in the one-dimensional case, this quotient group will be a linearly compact $k$-vector space. We note that this quotient group when the surface $X$ is smooth and the sheaf $\mathcal{F} = \mathcal{O}_X$ was calculated in [12 § 14]. But there were the gaps in the proof of Theorem 3 from [12], see Remark 4 below.

Another application of the calculation of adelic quotient group (2) for two-dimensional schemes will be given in the subsequent paper [11]. It will be calculated explicitly an adelic quotient group on an arithmetic surface when the fibres over Archimedean points are taken into account.

This note is organized as follows. In Section 2 we recall the one-dimensional case. In Section 3.1 we recall the basic notation for adeles on two-dimensional normal excellent integral separated Noetherian schemes. In Section 3.2.1 we construct a surjective map from the adelic quotient group (2). In Section 3.2.2 we calculate the kernel of this map. We obtain in this section Theorem 1 with calculation of adelic quotient group (2). In Section 4 we apply above calculations to the case of projective normal algebraic surface, see Theorem 2.

I am grateful to A. N. Parshin, because our joint paper [12] led to this note. I am grateful to A. B. Zheglov, who provided me the reference [2].

2 One-dimensional case

To better understand the two-dimensional case we first recall the one-dimensional case.

Let $D$ be an irreducible algebraic curve over a field $k$. Let $\eta$ be the generic point of $D$, and $\mathcal{F}$ be a locally free sheaf of finite rank on $D$. We fix a (closed) point $p$ on the curve $D$.

For any point $q$ on $D$ let $\hat{\mathcal{O}}_q$ be the completion of the local ring $\mathcal{O}_q$ of the point $q$, $\hat{\mathcal{F}}_q$ be the completion of the stalk $\mathcal{F}$ at $q$, and $K_q$ be the localization of the ring $\mathcal{O}_q$ with respect to the multiplicative system $\mathcal{O}_q \setminus 0$.

Let $j : U = D \setminus p \hookrightarrow D$ be an open embedding. We consider a subgroup

$$A_U(\mathcal{F}) = H^0(U, j^* \mathcal{F}) \rightarrow \mathcal{F}_\eta,$$

and the usual adelic product

$$\mathbb{A}_D(\mathcal{F}) = \prod_{q \in D} \hat{\mathcal{F}}_q \otimes_{\mathcal{O}_q} K_q.$$

over all (closed) points of $D$.

Our goal is to construct an exact sequence:

$$0 \rightarrow \prod_{q \in D, q \neq p} \hat{\mathcal{F}}_q \lhook\joinrel\lhook\rightarrow \mathbb{A}_D(\mathcal{F})/\mathcal{F}_\eta \rightarrow \left(\hat{\mathcal{F}}_p \otimes_{\mathcal{O}_p} K_p\right)/A_U(\mathcal{F}) \rightarrow 0,$$  (3)
where the group $\mathcal{F}_\eta$ is diagonally embedded into the group $\mathbb{A}_D(\mathcal{F})$, the map $\iota$ is induced by the natural embedding $\prod_{q \in D, q \neq p} \hat{\mathcal{F}}_q \hookrightarrow \mathbb{A}_D(\mathcal{F})$, and the map $\psi$ will be defined below.

We consider the adelic complex for the sheaf $j_*j^*\mathcal{F}$ that calculates the cohomology of the sheaf $j_*j^*\mathcal{F}$ on the curve $D$:

$$
\mathcal{F}_\eta \oplus \left( \prod_{q \in D, q \neq p} \hat{\mathcal{F}}_q \right) \oplus \left( \hat{\mathcal{F}}_p \otimes \hat{\mathcal{O}}_p K_p \right) \rightarrow \mathbb{A}_D(\mathcal{F}).
$$

Since $U$ is affine, we obtain $H^1(D, j_* \mathcal{O}_U) = 0$. Hence we have

$$
\mathbb{A}_D(\mathcal{F}) = \mathcal{F}_\eta + \left( \prod_{q \in D, q \neq p} \hat{\mathcal{F}}_q \right) + \left( \hat{\mathcal{F}}_p \otimes \hat{\mathcal{O}}_p K_p \right). \quad (4)
$$

Now we construct the map $\psi$ in sequence (3) in the following way. Let $x$ be an element from the group $\mathbb{A}_D(\mathcal{F})/\mathcal{F}_\eta$, and an element $\bar{x} \in \mathbb{A}_D(\mathcal{F})$ be any lift of $x$. Then by formula (4) there is an element $f$ from the group $\mathcal{F}_\eta$ such that $f + \bar{x}$ belongs to the subgroup

$$
\left( \prod_{q \in D, q \neq p} \hat{\mathcal{F}}_q \right) \oplus \left( \hat{\mathcal{F}}_p \otimes \hat{\mathcal{O}}_p K_p \right).
$$

We define

$$
\psi(x) \overset{\text{def}}{=} s \cdot \text{pr}_p(f + \bar{x}),
$$

where the map $\text{pr}_p$ is the projection from the group $\mathbb{A}_D(\mathcal{F})$ to the group $\hat{\mathcal{F}}_p \otimes \hat{\mathcal{O}}_p K_p$, and the map $s$ is the natural map $\hat{\mathcal{F}}_p \otimes \hat{\mathcal{O}}_p K_p \rightarrow (\hat{\mathcal{F}}_p \otimes \hat{\mathcal{O}}_p K_p)/A_U(\mathcal{F})$. It is clear that the map $\psi$ is well-defined, since for any other choices $\bar{x}'$ and $f'$ we have that the element $\bar{x} - \bar{x}' + f - f'$ belongs to the subgroup

$$
\mathcal{F}_\eta \cap \left( \left( \prod_{q \in D, q \neq p} \hat{\mathcal{F}}_q \right) \oplus \left( \hat{\mathcal{F}}_p \otimes \hat{\mathcal{O}}_p K_p \right) \right) = A_U(\mathcal{F}),.
$$

By construction, it is clear that the map $\iota$ is an embedding, the map $\psi$ is a surjection, and $\psi \cdot \iota = 0$. Besides, we have $\text{Ker} \psi \subset \text{Im} \iota$, since if $\psi(x) = 0$, then there is an element $f \in \mathcal{F}_\eta$ such that

$$
f + \bar{x} \in \left( \prod_{q \in D, q \neq p} \hat{\mathcal{F}}_q \right) + \left( \mathcal{F}_\eta \cap \prod_{q \in D, q \neq p} \hat{\mathcal{F}}_q \right),
$$

and hence $\bar{x} \in \left( \prod_{q \in D, q \neq p} \hat{\mathcal{F}}_q \right) + \mathcal{F}_\eta$. Therefore sequence (3) is exact.

**Example 1.** If $\mathcal{F} = \mathcal{O}(E)$ for a divisor $E$ on the curve $D$, then sequence (3) is

$$
0 \rightarrow \prod_{q \in D, q \neq p} \hat{\mathcal{O}}_q(E) \overset{\iota}{\rightarrow} \mathbb{A}_{k(D)}/k(D) \overset{\psi}{\rightarrow} K_p/A_U(E) \rightarrow 0,
$$

where $k(D)$ is the field of rational functions on the curve $D$, and $A_U(E) = A_U(\mathcal{O}(E))$. 3
Remark 1. Clearly, exact sequence (3) is evidently generalized to the case of several points \( p_1, \ldots, p_n \) on the curve \( D \) (instead of one point \( p \)) and an affine open subscheme \( U = D \setminus \{ p_1, \ldots, p_n \} \).

3 Case of two-dimensional schemes

The goal of this and the next section is to consider the case of quotient groups of the Parshin-Beilinson adelic groups on two-dimensional schemes (on algebraic surfaces in the next section).

3.1 Adelic formalism

Let \( X \) be an excellent Noetherian (e.g., finite type over a field \( k \) or over \( \mathbb{Z} \)) normal integral two-dimensional separated scheme \( X \). For any closed point \( x \in X \) let \( \mathcal{O}_x \) be a completion of the local ring \( \mathcal{O}_x \) of the point \( x \) with respect to the maximal ideal. Then \( \mathcal{O}_x \) is again an integrally closed domain (see, e.g., [7, ch. 13, § 33, th. 79]). Let \( K_x \) be a localization of the ring \( \mathcal{O}_x \) with respect to the multiplicative system \( \mathcal{O}_x \setminus 0 \). For any one-dimensional integral closed subscheme \( D \) of \( X \) let \( K_D \) be a completion of the field of rational functions on \( X \) with respect to the discrete valuation given by \( D \).

Let \( x \in D \) be any pair such that \( x \) is a closed point on \( X \) and \( D \) is a one-dimensional integral closed subscheme of \( X \). Let \( \rho_i \), where \( 1 \leq i \leq l \), be all height one prime ideals of the ring \( \mathcal{O}_x \) which contain an ideal \( \rho_D \mathcal{O}_x \), where the prime ideal \( \rho_D \) of the ring \( \mathcal{O}_x \) defines the subscheme \( D \mid_{\text{Spec} \mathcal{O}_x} \). We define

\[
K_{x,D} = \prod_{1 \leq i \leq l} K_i,
\]

where \( K_i \) is a two-dimensional local field obtained as the completion of the field \( \text{Frac} \mathcal{O}_x \) with respect to the discrete valuation given by the prime ideal \( \rho_i \). Similarly, we define

\[
\mathcal{O}_{K_{x,D}} = \prod_{1 \leq i \leq l} \mathcal{O}_{K_i},
\]

where \( \mathcal{O}_{K_i} \) is the discrete valuation ring of the field \( K_i \).

For any quasicoherent sheaf on \( X \) there is its adelic group, see its definition, for example, in [6, 9]. We will use the following notation. For any locally free sheaf \( \mathcal{F} \) on \( X \) let \( \mathcal{F}_q \) be the completion of the stalk \( \mathcal{F}_q \) of the sheaf \( \mathcal{F} \) at a point \( q \) of \( X \). For a one-dimensional integral closed subscheme \( D \) of \( X \) by \( \mathcal{F}_D \) we mean the completion of the stalk of \( \mathcal{F} \) at the generic point of \( D \), i.e. at a non-closed point on \( X \) whose closure coincides with \( D \). We consider the adelic group of the sheaf \( \mathcal{F} \):

\[
\mathbb{A}_X(\mathcal{F}) = \prod_{x \in D} K_{x,D}(\mathcal{F}) \subset \prod_{x \in D} K_{x,D}(\mathcal{F}),
\]

where \( K_{x,D}(\mathcal{F}) = \mathcal{F}_x \otimes_{\mathcal{O}_x} K_{x,D} \), and \( x \in D \) run over all pairs as above. We define also \( \mathcal{O}_{K_{x,D}}(\mathcal{F}) = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_{K_{x,D}} \).
Similarly, there are subgroups of $\mathbb{A}_X(F)$:

$$
\mathbb{A}_X_{,01}(F) = \mathbb{A}_X(F) \cap \prod_D K_D(F), \quad \mathbb{A}_X_{,02}(F) = \mathbb{A}_X(F) \cap \prod_x K_x(F),
$$

$$
\mathbb{A}_X_{,12}(F) = \mathbb{A}_X(F) \cap \prod_{x \in D} \mathcal{O}_{K_{x,D}}(F) = \prod' \mathcal{O}_{K_{x,D}}(F),
$$

(5)

where the intersections are taken in $\prod_{x \in D} K_{x,D}(F)$, we consider the diagonal embeddings of $\prod_D$ and $\prod_x$ into $\prod_{x \in D}$, and $K_D(F) = \hat{F}_D \otimes \hat{O}_D K_D$, $K_x(F) = \hat{F}_x \otimes \hat{O}_x K_x$.

Besides, for any subset $\Delta$ of the set of all pairs $x \in D$ as above, we will use notation $\prod'_{\Delta}$ for the intersection of $\prod'_{\Delta}$ of the same factors with $\mathbb{A}_X(F)$ inside of $\prod_{x \in D} K_{x,D}(F)$ (compare with formula (5)).

### 3.2 Adelic quotient group

#### 3.2.1 Construction of a surjective map $\varphi$

We keep the notation from section 3.1.

Let $C$ be a reduced one-dimensional closed equidimensional subscheme of the scheme $X$. We consider $C = \bigcup_{1 \leq i \leq w} C_i$, where $C_i$ are integral one-dimensional closed subschemes of $X$. We suppose that $U = X \setminus C$ is an affine scheme. Let $j : U \hookrightarrow X$ be the corresponding embedding.

Let $F$ be a locally free sheaf on $X$. We note that

$$
H^i(X, j_* j^* F) = H^i(U, j^* F) = 0 \quad \text{for} \quad i \geq 1.
$$

We note also that $j_* j^* F = F \otimes_{\mathcal{O}_X} j_* \mathcal{O}_U$. Since we supposed that $X$ is an integral and normal scheme (in particular, the singular locus of $X$ is a finite number of closed points), we have

$$
\mathcal{O}_X(nC) = \lim_{\longrightarrow} \mathcal{O}_X(nC),
$$

(6)

where $\mathcal{O}_X(nC)$ is a torsion free reflexive coherent subsheaf of the constant sheaf of the field of rational functions on $X$, and this subsheaf consists of elements of $j_* \mathcal{O}_U$ which have discrete valuations given by subschemes $C_i$ (where $1 \leq i \leq w$) great or equal to $-n$. (This follows, e.g., from [2, § 3].)

From $H^2(X, j_* j^* F) = 0$ and the adelic complex for the sheaf $j_* j^* F$ we obtain

$$
\mathbb{A}_X(j_* j^* F) = \mathbb{A}_X_{,12}(j_* j^* F) + \mathbb{A}_X_{,01}(j_* j^* F) + \mathbb{A}_X_{,02}(j_* j^* F).
$$

(7)

We have

$$
\mathbb{A}_X(j_* j^* F) = \mathbb{A}_X(F), \quad \mathbb{A}_X_{,01}(j_* j^* F) = \mathbb{A}_X_{,01}(F), \quad \mathbb{A}_X_{,02}(j_* j^* F) = \mathbb{A}_X_{,02}(F).
$$
Since for any affine open subscheme \( V = \text{Spec} \ A \) of the scheme \( X \) we have (the proof is analogous to the proofs of \([8, \text{Prop. 1}(5)]\) and of \([8, \text{Prop. 4}(2)]\))

\[
A_{V,12}(j_* j^* F) \mid_V = A_{V,12}(O_V) \otimes_A H^0(V, (j_* j^* F) \mid_V),
\]

we obtain

\[
A_{X,12}(j_* j^* F) = \prod_{x \in D, D \not\subset C} O_{K_{x,D}}(F) \oplus \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(F). \tag{8}
\]

We denote

\[
A_{X,12}^U(F) = \prod_{x \in D, D \not\subset C} O_{K_{x,D}}(F). \tag{9}
\]

Hence and from formula (7) we obtain

\[
A_X(F) = A_{X,01}(F) + A_{X,02}(F) + \left( A_{X,12}^U(F) \oplus \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(F) \right). \tag{10}
\]

Now we want to construct the map \( \varphi : \):

\[
A_X(F)/ (A_{X,01}(F) + A_{X,02}(F)) \xrightarrow{\varphi} \left( \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(F) \right)/\Upsilon(F), \tag{11}
\]

where the group

\[
\Upsilon(F) = \text{pr}_C \left( (A_{X,01}(F) + A_{X,02}(F)) \cap \left( A_{X,12}^U(F) \oplus \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(F) \right) \right). \tag{12}
\]

Here the map \( \text{pr}_C \) is the projection from the group \( A_X(F) \) to the group \( \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(F) \).

We define the map \( \varphi \) as follows:

\[
\varphi(x) \overset{\text{def}}{=} s \cdot \text{pr}_C(\tilde{x} + g),
\]

where \( s \) is the natural map from the group \( \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(F) \) to the group \( A_X(F)/ (A_{X,01}(F) + A_{X,02}(F)) \), an element \( \tilde{x} \) is any lift of an element \( x \) from the group \( A_X(F) \), and an element

\[
g \in (A_{X,01}(F) + A_{X,02}(F))
\]

is chosen with the property

\[
\tilde{x} + g \in A_{X,12}^U(F) \oplus \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(F). \tag{13}
\]

(Such an element \( g \) exists by formula (10).)
The map $\varphi$ is well-defined, since if an element $\tilde{x}'$ is another lift and $g'$ is another choice of corresponding elements, then we have

$$\tilde{x}' + g' - \tilde{x} - g \in (\mathbb{A}_{X,01}(\mathcal{F}) + \mathbb{A}_{X,02}(\mathcal{F})) \cap \left( \mathbb{A}_{X,12}^U(\mathcal{F}) \oplus \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(\mathcal{F}) \right).$$

Besides, from the construction it is clear that the map $\varphi$ is surjective.

**Lemma 1.** There is an equality of subgroups of the group $\mathbb{A}_X(\mathcal{F})$:

$$\mathbb{A}_{X,12}(j_* j^* \mathcal{F}) \cap (\mathbb{A}_{X,01}(\mathcal{F}) + \mathbb{A}_{X,02}(\mathcal{F})) = (\mathbb{A}_{X,12}(j_* j^* \mathcal{F}) \cap \mathbb{A}_{X,01}(\mathcal{F})) + (\mathbb{A}_{X,12}(j_* j^* \mathcal{F}) \cap \mathbb{A}_{X,02}(\mathcal{F})).$$

**Proof.** We denote the sheaf $\mathcal{H} = j_* j^* \mathcal{F}$ on $X$ and consider a diagram with exact columns:

$$
\begin{align*}
\mathbb{A}_{X,0}(\mathcal{H}) & \longrightarrow \mathbb{A}_{X,01}(\mathcal{H}) \cap \mathbb{A}_{X,02}(\mathcal{H}) \longrightarrow 0, \\
\mathbb{A}_{X,0}(\mathcal{H}) & \oplus \mathbb{A}_{X,1}(\mathcal{H}) \oplus \mathbb{A}_{X,2}(\mathcal{H}) \longrightarrow \mathbb{A}_{X,01}(\mathcal{H}) \oplus \mathbb{A}_{X,02}(\mathcal{H}) \oplus \mathbb{A}_{X,12}(\mathcal{H}) \longrightarrow \mathbb{A}_X(\mathcal{H}), \\
\mathbb{A}_{X,1}(\mathcal{H}) \oplus \mathbb{A}_{X,2}(\mathcal{H}) & \longrightarrow (\mathbb{A}_{X,01}(\mathcal{H}) + \mathbb{A}_{X,02}(\mathcal{H})) \oplus \mathbb{A}_{X,12}(\mathcal{H}) \longrightarrow \mathbb{A}_X(\mathcal{H}),
\end{align*}
$$

where the middle row is the adelic complex for the sheaf $\mathcal{H}$ on $X$. This adelic complex calculates the cohomology of $\mathcal{H}$ on $X$. Therefore the statement of the lemma follows from the long exact sequence constructed by this diagram and from the facts: $H^1(X, \mathcal{H}) = 0$ and

$$\mathbb{A}_{X,1}(\mathcal{H}) = \mathbb{A}_{X,01}(\mathcal{H}) \cap \mathbb{A}_{X,12}(\mathcal{H}), \quad \mathbb{A}_{X,2}(\mathcal{H}) = \mathbb{A}_{X,02}(\mathcal{H}) \cap \mathbb{A}_{X,12}(\mathcal{H}). \quad (14)$$

To prove formulas (14), we note that it is enough to consider a sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nC)$ (where $n \geq 0$) instead of the sheaf $\mathcal{H}$, because $\mathcal{H} = \varprojlim_n (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nC))$, and adelic factors commute with direct limits. Now the first equality in (14) is true since it is easy to see for a locally free sheaf and the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nC)$ is invertible except for the finite number of singular points of $X$ which do not impact on the equality. The second equality is again easy to see for an open subscheme of $X$ where the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nC)$ is a locally free sheaf, because $X$ is a normal scheme. Therefore it is enough to check this equality locally for a singular (closed) point $x$ of $X$, where it follows from an equality of $\mathcal{O}_x$-submodules of $\text{Frac} \mathcal{O}_x$:

$$\mathcal{O}_x(nC) \otimes_{\mathcal{O}_x} \mathcal{O}_x = \hat{\mathcal{O}}_x(nC |_{\text{Spec} \mathcal{O}_x} ). \quad (15)$$

Equality (15) is true, because the completion of a maximal Cohen-Macaulay $\mathcal{O}_x$-module (which is the same as reflexive module in this case) is a maximal Cohen-Macaulay $\mathcal{O}_x$-module (i.e. a reflexive module).

\qed
For any pair $x \in D$ on $X$ (as in Section 3.1) we have a canonical embedding of groups

$$p_{x,D,F} : K_x(F) \hookrightarrow K_{x,D}(F).$$

For any closed point $x \in X$ we define a subgroup $B_{x,C}(F)$ of the group $K_x(F)$ in the following way:

$$B_{x,C}(F) \overset{\text{def}}{=} \bigcap_{D \ni x, D \not\subset C} p_{x,D,F}^{-1}(p_{x,D,F}(K_x(F)) \cap \mathcal{O}_{K_{x,D}(F)}).$$

**Remark 2.** From the proof of Lemma 1 we have that

$$B_{x,C}(F) = \hat{\mathcal{O}}_x \otimes \mathcal{O}_x (j_* j^* \mathcal{F})_x = \hat{\mathcal{F}}_x \otimes \mathcal{O}_x (j_* j^* \mathcal{O}_X)_x.$$

**Remark 3.** In [12, § 14] we used notation $B_x$ instead of $B_{x,C}$ (i.e., without indicating on the subscheme $C$).

Now from Lemma 1 and formula (16) we immediately obtain

$$A_{X,12}(F) \cap (A_{X,01}(F) + A_{X,02}(F)) =$$

$$= \left( \prod_{D \subset X, D \not\subset C} \hat{\mathcal{F}}_D \oplus \prod_{1 \leq i \leq w} K_{C_i}(F) \right) + \left( \prod_{x \in U} \hat{\mathcal{F}}_x \oplus \prod_{x \in C} B_{x,C}(F) \right).$$

Therefore the projection map (see formula (16)) gives

$$\text{pr}_C(A_{X,12}(F) \cap (A_{X,01}(F) + A_{X,02}(F))) = \prod_{1 \leq i \leq w} K_{C_i}(F) + \prod_{x \in C} B_{x,C}(F).$$

Thus, bearing in mind formula (16), we have that the map $\varphi$ is the following surjective map:

$$A_{X}(F)/(A_{X,01}(F) + A_{X,02}(F)) \xrightarrow{\varphi} \left( \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(F) \right) / \left( \prod_{1 \leq i \leq w} K_{C_i}(F) + \prod_{x \in C} B_{x,C}(F) \right).$$

**3.2.2 Calculation of the kernel**

We consider now a natural map

$$\prod_{x \in D, D \not\subset C} \mathcal{O}_{K_{x,D}(F)} \xrightarrow{\varphi} A_{X}(F)/(A_{X,01}(F) + A_{X,02}(F)).$$

It is clear that $\text{Im} \, \varphi \subset \text{Ker} \, \varphi$. (Indeed, for the construction of $\varphi$ we take $g = 0$ and the lift $\tilde{x}$ which comes from the image of $\phi$.)

Let us show that $\text{Ker} \, \varphi \subset \text{Im} \, \varphi$. Let $\varphi(x) = 0$ for an element $x$ from the group $A_{X}(F)/(A_{X,01}(F) + A_{X,02}(F))$. Then, by construction, there is an element $g$ from the group $A_{X,01}(F) + A_{X,02}(F)$ such that

$$\tilde{x} + g \in \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_{x,D}(F)} + (A_{X,01}(F) + A_{X,02}(F)).$$
where “∈” follows from formulas (13) and (12). Therefore we have
\[ \tilde{x} \in \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_{x,D}}(F) + (A_{X,01}(F) + A_{X,02}(F)). \]

Thus, we obtain \( \text{Im } \phi = \text{Ker } \phi \). Therefore for an explicit description of the group \( \mathbb{A}_X(F)/(A_{X,01}(F) + A_{X,02}(F)) \), we have to calculate the group \( \text{Ker } \phi \).

It is clear that
\[ \text{Ker } \phi = (A_{X,01}(F) + A_{X,02}(F)) \cap \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_{x,D}}(F). \]

Let us calculate \( \text{Ker } \phi \) more explicitly.

For any pair \( x \in D \) on \( X \) (as in Section 3.1) we have canonical embeddings of groups
\[ q_{x,D,F} : K_D(F) \hookrightarrow K_{x,D}(F). \]

Now we define a subgroup \( A_C(F) \subset \prod_{1 \leq i \leq w} K_{C_i}(F) \) as the image of the projection of the group \( \text{Ker } \Xi \) to the group \( \prod_{1 \leq i \leq w} K_{C_i}(F) \), where the map
\[ \Xi : \prod_{1 \leq i \leq w} K_{C_i}(F) \oplus \prod_{x \in C} B_{x,C}(F) \rightarrow \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C_i}(F), \quad (17) \]
and \( \Xi(z \oplus v) = \prod_{1 \leq i \leq w} \prod_{x \in C_i} q_{x,C_i,F}(z) - \prod_{1 \leq i \leq w} \prod_{x \in C_i} p_{x,C_i,F}(v) \) for elements \( z \in \prod_{1 \leq i \leq w} K_{C_i}(F) \) and \( v \in \prod_{x \in C} B_{x,C}(F) \).

We note that if \( w = 1 \), i.e. \( C = C_1 \), then
\[ A_C(F) = \bigcap_{x \in C} q_{x,C,F}^{-1}(q_{x,C,F}(K_{C_i}(F)) \cap p_{x,C,F}(B_{x,C})). \]

We have a natural embedding \( \tau : \)
\[ \tau : A_C(F) \hookrightarrow \prod_{x \in C} B_{x,C}(F) \hookrightarrow \prod_{x \in X} K_x(F) \hookrightarrow \mathbb{A}_X(F), \quad (18) \]
where the first arrow denotes the map \( z \mapsto v \) (see definition of the map \( \Xi \) in (17)).

We have also a natural embedding \( \gamma : \)
\[ \gamma : A_C(F) \hookrightarrow \prod_{1 \leq i \leq w} K_{C_i}(F) \hookrightarrow \mathbb{A}_X(F). \quad (19) \]

\[ \text{We recall that } C = \bigcup_{1 \leq i \leq w} C_i, \text{ where } C_i \text{ is an integral one-dimensional subscheme of } X. \]
We claim that
\[
\text{Ker } \phi = (A_{X,01}(\mathcal{F}) + A_{X,02}(\mathcal{F})) \cap \prod_{x \in D, D \not\subset C} O_{K_x,D}(\mathcal{F}) \supset\]
\[
\supset \prod_{D \subset X, D \not\subset C} \hat{F}_D + \prod_{x \in U} \hat{F}_x + (\tau - \gamma)(A_C(\mathcal{F})). \tag{20}
\]

Indeed, it is clear that the first two summands from the last sum belong to the group Ker $\phi$. Besides, by construction, we have that
\[
\tau(A_C(\mathcal{F})) \subset A_{X,02}(\mathcal{F}), \quad \gamma(A_C(\mathcal{F})) \subset A_{X,01}(\mathcal{F}),
\]
and
\[
(\tau - \gamma)(A_C(\mathcal{F})) \subset \prod_{x \in D, D \not\subset C} O_{K_x,D}(\mathcal{F}).
\]

Therefore, $(\tau - \gamma)(A_C(\mathcal{F})) \subset$ Ker $\phi$.

On the other hand (recall notation from formula (9)),
\[
\text{Ker } \phi = A^U_{X,12}(\mathcal{F}) \cap (A_{X,01}(\mathcal{F}) + A_{X,02}(\mathcal{F})) \subset A_{X,12}(\mathcal{F}) \cap (A_{X,01}(\mathcal{F}) + A_{X,02}(\mathcal{F})).
\]

Now, bearing in mind formula (16), and using that the projection of the group Ker $\phi$ to any group $K_{x,C_i}(\mathcal{F})$ is zero, we obtain that in the group Ker $\phi$ elements from the group \( \prod_{1 \leq i \leq w} K_{x,C_i}(\mathcal{F}) \) should be “compensated” by elements from the group \( \prod'_{x \in C} B_{x,C}(\mathcal{F}) \) to obtain zero. Hence and from formula (20) we obtain that
\[
\text{Ker } \phi = \prod_{D \subset X, D \not\subset C} \hat{F}_D + \prod_{x \in U} \hat{F}_x + (\tau - \gamma)(A_C(\mathcal{F})). \tag{21}
\]

Thus, we have proved a theorem.

**Theorem 1.** There is the following exact sequence
\[
0 \to \prod_{x \in D, D \not\subset C} O_{K_x,D}(\mathcal{F}) \xrightarrow{\phi} \prod_{D \subset X, D \not\subset C} \hat{F}_D + \prod_{x \in U} \hat{F}_x + (\tau - \gamma)(A_C(\mathcal{F})) \xrightarrow{\varphi} \prod_{1 \leq i \leq w} \prod'_{x \in C} K_{x,C_i}(\mathcal{F}) + \prod'_{x \in C} B_{x,C}(\mathcal{F}) \to 0.
\]

Now we calculate explicitly the group $A_C(\mathcal{F})$.

**Proposition 1.** There is an isomorphism
\[
A_C(\mathcal{F}) \simeq \lim_{n \to \infty} \lim_{m < n} H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X(nC)/\mathcal{O}_X(mC))), \tag{22}
\]
which comes from canonical embeddings (with the same image) of groups from the left and right hand sides of formula (22) into the group \( \prod_{1 \leq i \leq w} K_{C_i}(\mathcal{F}) \).
Proof. Let $J_C$ be the ideal sheaf of the subscheme $C$ on $X$. For integers $n > m$ we consider a 1-dimensional closed subscheme $Y_{n-m} = (C, \mathcal{O}_X/J_C^{n-m}) \subset X$ with the topological space $C$ and the structure sheaf $\mathcal{O}_X/J_C^{n-m}$. The sheaf

$$\mathcal{F}_{n,m} = \mathcal{F} \otimes \mathcal{O}_X (\mathcal{O}_X(nC)/\mathcal{O}_X(mC))$$

is a coherent sheaf on the scheme $Y_{n-m}$. Now the proof follows from the calculation of the group $H^0(X, \mathcal{F}_{n,m}) = H^0(Y_{n-m}, \mathcal{F}_{n,m})$ via the adelic complex for the sheaf $\mathcal{F}_{n,m}$ on the 1-dimensional scheme $Y_{n-m}$ (see also formula (6) and Remark 2) and the passing to injective limit on $n$ and projective limit on $m$. The final complex after the passing to injective limit on $n$ and projective limit on $m$ looks as:

$$\prod_{1 \leq i \leq w} K_{C_i}(\mathcal{F}) \oplus \prod'_{x \in C} B_{x,C}(\mathcal{F}) \longrightarrow \prod_{1 \leq i \leq w} \prod'_{x \in C_i} K_{x,C_i}(\mathcal{F}).$$

\[ \square \]

4 Case of projective algebraic surface

Now we restrict ourself to the case of a projective algebraic normal irreducible surface $X$ over a field $k$. We recall that $C = \bigcup_{1 \leq i \leq w} C_i$, where $C_i$ are irreducible closed curves on $X$.

We suppose that $\tilde{C} = \bigoplus_{1 \leq i \leq w} m_i C_i$ (where $m_i \geq 1$ are certain integers) is an ample divisor on $X$, i.e. $\mathcal{O}_X(l\tilde{C})$ is a very ample invertible sheaf on $X$ for certain integer $l \geq 0$.

Proposition 2. There is an isomorphism

$$H^0(X, j_* j^* \mathcal{F}) \simeq \lim_{n} \lim_{m < n} H^0(X, \mathcal{F} \otimes \mathcal{O}_X (\mathcal{O}_X(nC)/\mathcal{O}_X(mC))) ,$$

which comes from the composition of maps of sheaves

$$j_* j^* \mathcal{F} \simeq \lim_{n} \mathcal{F} \otimes \mathcal{O}_X \mathcal{O}_X(nC) \longrightarrow \lim_{n} \lim_{m < n} \mathcal{F} \otimes \mathcal{O}_X (\mathcal{O}_X(nC)/\mathcal{O}_X(mC)) .$$

Proof. We denote a divisor $C' = l\tilde{C}$ such that $\mathcal{O}_X(C')$ is a very ample invertible sheaf. It is clear that

$$\lim_{n} \lim_{m < n} H^0(X, \mathcal{F} \otimes \mathcal{O}_X (\mathcal{O}_X(nC)/\mathcal{O}_X(mC))) \simeq \lim_{r} \lim_{s < r} H^0(X, \mathcal{F} \otimes \mathcal{O}_X (\mathcal{O}_X(rC)/\mathcal{O}_X(sC'))) ,$$

because of isomorphism of the corresponding IndPro systems.

A normal surface is a Cohen-Macaulay scheme. Since $C'$ is a very ample divisor, by [5, ch. III, Th. 7.6] we have

$$H^i(X, \mathcal{F}(sC')) = 0 \quad \text{when} \quad i \leq 1 \quad \text{and} \quad s \ll 0 .$$
Now from application of these equalities to an exact sequence of sheaves on $X$:

$$0 \to F(sC) \to F(rC) \to F \otimes_{\mathcal{O}_X} (\mathcal{O}_X(rC)/\mathcal{O}_X(sC)) \to 0.$$ 

we obtain an isomorphism

$$\lim_{r \to s \leq r} \mathcal{H}^0(X, F \otimes_{\mathcal{O}_X} (\mathcal{O}_X(rC)/\mathcal{O}_X(sC))) \simeq \lim_{r \to s \leq r} \mathcal{H}^0(X, F \otimes_{\mathcal{O}_X} \mathcal{O}_X(rC)).$$

Finally we have

$$\lim_{r \to s \leq r} \mathcal{H}^0(X, F \otimes_{\mathcal{O}_X} \mathcal{O}_X(rC)) \simeq \mathcal{H}^0(X, j_*j^*F).$$

From Propositions 1 and 2 it follows that

$$\mathcal{A}_{C}(F) = \mathcal{H}^0(X, j_*j^*F) \subset F_{\eta},$$

where $\eta$ is the generic point of $X$. Therefore for an element $a$ from the group $\mathcal{A}_{C}(F)$ we have

$$(\tau - \gamma)(a) = I(a) - J(a),$$

where $I(a)$ is the image of the element $a$ into $\prod_{D \subset X, D \not\subset C} \hat{F}_D$ under the diagonal embedding, and $J(a)$ is the image of the element $a$ into $\prod_{x \in U} \hat{F}_x$ under the diagonal embedding.\footnote{We recall also that the maps $\tau$ and $\gamma$ were defined by formulas \ref{eq:tau} and \ref{eq:gamma} correspondingly.}

Thus we obtain that

$$(\tau - \gamma)(\mathcal{A}_{C}(F)) \subset \prod_{D \subset X, D \not\subset C} \hat{F}_D + \prod_{x \in U} \hat{F}_x.$$ 

Therefore from formula \ref{eq:ker} we obtain

$$\ker \phi = \prod_{D \subset X, D \not\subset C} \hat{F}_D + \prod_{x \in U} \hat{F}_x. \quad \text{(23)}$$

Thus, from formula \ref{eq:ker} and Theorem 1 we deduce a theorem.

**Theorem 2.** Let $\tilde{C} = \bigoplus_{1 \leq i \leq w} m_i C_i$ (with certain integers $m_i \geq 1$) be an ample divisor on a normal projective irreducible algebraic surface $X$ over a field $k$, where $C_i$ are irreducible curves. Let an open subscheme $U = X \setminus C$, where $C = \bigcup_{1 \leq i \leq w} C_i$. For any locally free sheaf $F$ on $X$ there is the following exact sequence.

$$0 \to \prod_{x \in D, D \subset C} \mathcal{O}_{K_{x,D}}(F) \to \prod_{D \subset X, D \subset C} \hat{F}_D + \prod_{x \in U} \hat{F}_x \to \mathcal{A}_X(F) \mathcal{A}_{X,01}(F) + \mathcal{A}_{X,02}(F) \to \mathcal{A}_X(B_{x,C}(F)) \to 0. \quad \text{(24)}$$
Remark 4. As we mentioned in Introduction, Theorem 2 was formulated as Theorem 3 in [12] for a smooth projective connected algebraic surface $X$, a sheaf $\mathcal{F} = \mathcal{O}_X$, and all $m_i = 1$. But the proof in [12] was incorrect: it contained gaps. In particular, an additional term $(\tau - \gamma)(A_C(\mathcal{F}))$ from Theorem 1 was not considered.

From Theorem 2 we explicitly see that $k$-vector space $\hat{\mathbb{A}}_X(\mathcal{F}) / (\hat{\mathbb{A}}_{X,01}(\mathcal{F}) + \hat{\mathbb{A}}_{X,02}(\mathcal{F}))$ is a linearly compact $k$-vector space (or a compact topological space when $k$ is a finite field.) Indeed, the first non-zero term in exact sequence (24) can be rewritten as

$$\prod_{x \in D, D \not\subset C} \mathcal{O}_{K_x,D}(\mathcal{F}) / \prod_{D \subset X, D \not\subset C} \mathcal{F}_D + \prod_{x \in U} \hat{\mathcal{F}}_x$$

and $k$-vector spaces $\hat{\mathcal{F}}_x$ for any point $x \in X$ and $\left(\prod_{x \in D} \mathcal{O}_{K_x,D}\right) / \hat{\mathcal{O}}_D$ for any irreducible curve $D \subset X$ are linearly compact $k$-vector spaces, see [12, Remark 26]. The last non-zero term in exact sequence (24) can be rewritten analogously and by the same arguments as in Theorem 4 from [12]. (On a two-dimensional local field and, more generally, on the group $\mathbb{A}_X$ there is the natural topology of inductive and projective limits, first introduced by A. N. Parshin in [14] for the case $\mathbb{F}_q(t_1) \ldots (t_n)$, see also its properties, e.g., in [4, §3.2].)

At the end we recall that using these and other calculations, it was suggested in [12, §14.3] an analogy for the first and last non-zero terms of exact sequence (24) in the case of the simplest arithmetic surface $\mathbb{P}^1_Z$. In particular, the last non-zero term should be equal to

$$\mathbb{R}((t))/ (\mathbb{Z}((t)) + \mathbb{R}[t^{-1}]) .$$

This will be justified by explicit calculations with arithmetic adeles on arithmetic surfaces in [11]. (Arithmetic adeles, i.e. adeles on an arithmetic surface when fibres over Archimedean points of the base are taken into account, were introduced in [12, Example 11]. See also applications in [10, §4].)

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