Binary Additive Problems Involving Naturals with Binary Decompositions of a Special Kind

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Abstract
Let \( h \) and \( l \) be integers such that \( 0 \leq h \leq 2, \ 0 \leq l \leq 4 \). We obtain asymptotic formulas for the numbers of solutions of the equations \( n - 3m = h \), \( n - 5m = l \) in positive integers \( m \) and \( n \) of a special kind, \( m \leq X \).

Key words: binary decomposition, binary additive problems.

1. Introduction

Consider the binary decomposition of a positive integer \( n \):

\[
n = \sum_{k=0}^{\infty} \varepsilon_k 2^k,
\]

where \( \varepsilon_k = 0, 1 \) and \( k = 0, 1, \ldots \)

We split the set of positive integers into two nonintersecting classes as follows:

\[
\mathbb{N}_0 = \{ n \in \mathbb{N}, \ \sum_{k=0}^{\infty} \varepsilon_k \equiv 0 \pmod{2}\}, \quad \mathbb{N}_1 = \{ n \in \mathbb{N}, \ \sum_{k=0}^{\infty} \varepsilon_k \equiv 1 \pmod{2}\}.
\]

In 1968, A.O. Gel’fond [1] obtained the following theorem: for the number of integers \( n, n \leq X \), satisfying the conditions \( n \equiv l \pmod{m} \), \( n \in \mathbb{N}_j \ (j = 0, 1) \), the following asymptotic formula is valid:

\[
T_j(X, l, m) = \frac{X}{2m} + O(X^\lambda),
\]

where \( m, l \) are any naturals and \( \lambda = \frac{\ln 3}{\ln 4} = 0.7924818 \ldots \)

Suppose that

\[
\varepsilon(n) = \begin{cases} 
1 & \text{for } n \in \mathbb{N}_0, \\
-1 & \text{otherwise}. 
\end{cases}
\]

The proof of formula (1) is based on the estimate

\[
|S(\alpha)| \ll X^\lambda
\]
of the trigonometrical sum

\[ S(\alpha) = \sum_{n \leq X} \varepsilon(n)e^{2\pi in\alpha}, \]

which is valid for any real values of \( \alpha \).

In 1996, author [2] proved the following theorem: Let \( F_{i,k} \) be the number of solutions of the equation \( n - m = 1 \), where \( n \leq X \), \( n \in \mathbb{N}_i \), \( m \in \mathbb{N}_k \), \( i, k = 0, 1 \). Then the asymptotic formulas hold:

\[
\begin{align*}
F_{0,0}(X) &= \frac{X}{6} + O(\log X), \\
F_{1,1}(X) &= \frac{X}{6} + O(\log X), \\
F_{0,1}(X) &= \frac{X}{3} + O(\log X), \\
F_{1,0}(X) &= \frac{X}{3} + O(\log X).
\end{align*}
\]

It follows from this theorem that the orders of \( F_{i,k}(X) \) strongly depend on the values of \( i \) and \( k \).

In present paper we consider two problems in which the indicated effect vanishes. Our main results are the following theorems.

**Theorem 1.** Let \( h \) be any integer such that \( 0 \leq h \leq 2 \). Let \( I_{i,k}(X, h) \) be the number of solutions of the equation \( n - 3m = h \), where \( m \leq X \), \( m \in \mathbb{N}_i \), \( n \in \mathbb{N}_k \), \( i, k = 0, 1 \). Then the asymptotic formulas hold:

\[ I_{i,k}(X, h) = \frac{X}{4} + O(X^{\lambda}). \]

**Theorem 2.** Let \( l \) be any integer such that \( 0 \leq h \leq 4 \). Let \( J_{i,k}(X, l) \) be the number of solutions of the equation \( n - 5m = l \), where \( m \leq X \), \( m \in \mathbb{N}_i \), \( n \in \mathbb{N}_k \), \( i, k = 0, 1 \). Then the asymptotic formulas hold:

\[ J_{i,k}(X, l) = \frac{X}{4} + O(X^{\lambda}). \]

Let us introduce two sums:

\[
S_3(X, h) = \sum_{n \leq X} \varepsilon(n)\varepsilon(3n + h), \quad S_5(X, l) = \sum_{n \leq X} \varepsilon(n)\varepsilon(5n + l),
\]

where \( h \) and \( l \) are nonnegative integers.

Proofs of the theorems 1 and 2 are based on lemmas 1 and 2 consequently (see below), and also on Gel’fond’s estimate (2).

### 2. Lemmas

**Lemma 1.** Suppose that \( h \) is an integer such that \( 0 \leq h \leq 2 \). The following estimate holds:

\[ S_3(X, h) = O(\sqrt{X}). \]
Proof. Grouping summands over even and over odd \( n \) and using obvious formulae 
\[ \varepsilon(2n) = \varepsilon(n), \varepsilon(2n + 1) = -\varepsilon(n) \]
we have the following equalities
\[
\begin{align*}
S_3(X, 0) &= S_3(X^2^{-1}, 0) + S_3(X^2^{-1}, 1) + O(1), \quad (3) \\
S_3(X, 1) &= -S_3(X^2^{-1}, 0) - S_3(X^2^{-1}, 2) + O(1), \quad (4) \\
S_3(X, 2) &= S_3(X^2^{-1}, 1) + S_3(X^2^{-1}, 2) + O(1). \quad (5)
\end{align*}
\]

Consider the linear combination 
\[ \alpha_0 S_3(X, 0) + \beta_0 S_3(X, 1) + \gamma_0 S_3(X, 2), \]
where \( \alpha_0, \beta_0, \gamma_0 \) are constants. By (3)–(5) we have
\[
\alpha_0 S_3(X, 0) + \beta_0 S_3(X, 1) + \gamma_0 S_3(X, 2) = \alpha_1 S_3(X^2^{-1}, 0) + \beta_1 S_3(X^2^{-1}, 1) + \gamma_1 S_3(X^2^{-1}, 2) + O(|\alpha_0|) + O(|\beta_0|) + O(|\gamma_0|),
\]
where 
\[
\begin{align*}
\alpha_1 &= \alpha_0 - \beta_0, \\
\beta_1 &= \alpha_0 + \gamma_0, \\
\gamma_1 &= \gamma_0 - \beta_0.
\end{align*}
\]

Repeating this reasoning we arrive to the equality
\[
\alpha_0 S_3(X, 0) + \beta_0 S_3(X, 1) + \gamma_0 S_3(X, 2) = \alpha_j S_3(X_j, 0) + \beta_j S_3(X_j, 1) + \gamma_j S_3(X_j, 2) + O(|\alpha_0| + \cdots + |\alpha_{j-1}|) + O(|\beta_0| + \cdots + |\beta_{j-1}|) + O(|\gamma_0| + \cdots + |\gamma_{j-1}|),
\]
where \( j \) is any integer such that \( 0 \leq j \leq \log_2 X - 10 \), \( X_j = X 2^{-j} \), and the sequences \( \alpha_j, \beta_j, \gamma_j \) satisfy to the system of recurrent equations
\[
\begin{align*}
\alpha_{j+1} &= \alpha_j - \beta_j, \\
\beta_{j+1} &= \alpha_j + \gamma_j, \\
\gamma_{j+1} &= \gamma_j - \beta_j. \quad (6)
\end{align*}
\]

Let us write (6) in matrix form
\[
\begin{pmatrix}
\alpha_{j+1} \\
\beta_{j+1} \\
\gamma_{j+1}
\end{pmatrix}
= 
\begin{pmatrix}
1 & -1 & 0 \\
1 & 0 & 1 \\
0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_j \\
\beta_j \\
\gamma_j
\end{pmatrix}
= A
\begin{pmatrix}
\alpha_j \\
\beta_j \\
\gamma_j
\end{pmatrix}.
\]

Then we have
\[
\begin{pmatrix}
\alpha_j \\
\beta_j \\
\gamma_j
\end{pmatrix}
= A^j
\begin{pmatrix}
\alpha_0 \\
\beta_0 \\
\gamma_0
\end{pmatrix}.
\]
One can easily see that $A = CBC^{-1}$, where

$$
C = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1+i\sqrt{2} \\
-1 & 1 & 1+i\sqrt{2}
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix};
$$

where $1, \lambda_2 = \frac{1+i\sqrt{2}}{2}, \lambda_3 = \frac{1-i\sqrt{2}}{2}$ are eigenvalues of $A$.

Thus we have

$$
\begin{pmatrix}
\alpha_j \\
\beta_j \\
\gamma_j
\end{pmatrix} = C \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}^{-1} \begin{pmatrix}
\alpha_0 \\
\beta_0 \\
\gamma_0
\end{pmatrix}.
$$

(7)

Note that $|\lambda_2| = |\lambda_3| = \sqrt{2}$. It follows from (7) that if $\begin{pmatrix}
\alpha_0 \\
\beta_0 \\
\gamma_0
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}$, or

$$
\begin{pmatrix}
\alpha_0 \\
\beta_0 \\
\gamma_0
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}
\alpha_0 \\
\beta_0 \\
\gamma_0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
$$

then

$$
|\alpha_j| = O(\sqrt{2}^j), \quad |\beta_j| = O(\sqrt{2}^j), \quad |\gamma_j| = O(\sqrt{2}^j).
$$

(8)

Let $J$ be the largest natural number such that $J \leq \log_2 X - 10$, $\begin{pmatrix}
\alpha_0 \\
\beta_0 \\
\gamma_0
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}$.

It follows from (8) that

$$
|S_3(X, 0)| \leq |\alpha_J||S_3(X_J, 0)| + |\beta_J||s_3(X_J, 2)| + O(\sqrt{X}) = O(\sqrt{X}).
$$

The estimates $S_3(X, 1) = O(\sqrt{X}), S_3(X, 2) = O(\sqrt{X})$ we obtain in a similar way.

Lemma 2. Suppose that $l$ is an integer such that $0 \leq l \leq 4$. The following estimate holds:

$$
S_5(X, l) = O(X^\mu),
$$

where $\mu = 0, 60538\ldots$

Proof. Consider the linear combination

$$
\alpha_0 s_5(X, 0) + \beta_0 s_5(X, 1) + \gamma_0 s_5(X, 2) + \sigma_0 s_5(X, 3) + \tau_0 s_5(X, 4),
$$

where $\alpha_0, \beta_0, \gamma_0, \sigma_0$ and $\tau_0$ are constants.

Repeating the reasoning of the proof of lemma 1 we arrive to the equality

$$
\alpha_0 s_5(X, 0) + \beta_0 s_5(X, 1) + \gamma_0 s_5(X, 2) + \sigma_0 s_5(X, 3) + \tau_0 s_5(X, 4) =
$$

$$
\alpha_J s_5(X_J, 0) + \beta_J s_5(X_J, 1) + \gamma_J s_5(X_J, 2) + \sigma_J s_5(X_J, 3) + \tau_J s_5(X_J, 4) +
$$
+O(|α_0| + \cdots + |α_{J-1}|) + \cdots + O(|τ_0| + \cdots + τ_{J-1}|),

where \( J \) is the largest natural number such that \( J \leq \log_2 X - 10 \), \( X_J = X 2^{-J} \) and vector 
\[
\begin{pmatrix}
α_J \\
β_J \\
γ_J \\
σ_J \\
τ_J
\end{pmatrix}
\]
is defined with equation
\[
\begin{pmatrix}
α_J \\
β_J \\
γ_J \\
σ_J \\
τ_J
\end{pmatrix} = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}^J \begin{pmatrix}
α_0 \\
β_0 \\
γ_0 \\
σ_0 \\
τ_0
\end{pmatrix} = A_J^{f} \begin{pmatrix}
α_0 \\
β_0 \\
γ_0 \\
σ_0 \\
τ_0
\end{pmatrix}.
\]

Let us write out the eigenvalues of the matrix \( A_1 \):
\[
λ_1 = 1, \quad λ_2 = \frac{3^{1/3} + (9 - \sqrt{78})^{2/3}}{2^{1/3}(9 - \sqrt{78})^{1/3}}, \quad λ_3 = \frac{1 + i\sqrt{3}}{2(27 - 3\sqrt{78})^{1/3}} + \frac{(1 - i\sqrt{3})(9 - \sqrt{78})^{1/3}}{2^{3/2}}, \quad λ_4 = \frac{1 - i\sqrt{3}}{2(27 - 3\sqrt{78})^{1/3}} + \frac{(1 + i\sqrt{3})(9 - \sqrt{78})^{1/3}}{2^{3/2}}.
\]

Note that eigenvalues \( λ_2, \ λ_3, \ λ_4 \) are simple and \( λ_1 \) has multiplicity 2.

It is a well known fact of Linear Algebra that there exists a nonsingular matrix \( C_1 \) such that \( A_1 = C_1B_1C_1^{-1} \), where
\[
B_1 = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & λ_2 & 0 & 0 \\
0 & 0 & 0 & λ_3 & 0 \\
0 & 0 & 0 & 0 & λ_4
\end{pmatrix}
\]

Thus we have the equality
\[
\begin{pmatrix}
α_J \\
β_J \\
γ_J \\
σ_J \\
τ_J
\end{pmatrix} = C_1B_1^{f}C_1^{-1} \begin{pmatrix}
α_0 \\
β_0 \\
γ_0 \\
σ_0 \\
τ_0
\end{pmatrix}.
\]

Since \( |λ_2| = \max_{1 \leq j \leq 4} |λ_j| = 1, 52137 \ldots \) it follows from (9) that the inequalities
\[
|α_J| \ll |λ_2|^J, \ldots, |τ_J| \ll |λ_2|^J.
\]
hold and so for any \( l, \ 0 \leq l \leq 4 \) we have the estimate
\[
S_5(X, l) = O(X^\mu),
\]
where \( \mu = \frac{\log λ_2}{\log 2} = 0, 60538 \ldots \)

\[\square\]
3. Proofs of Theorems 1 and 2

Let us prove theorem 1. For any integer $h$, $0 \leq h \leq 2$, and for any $i, j = 0, 1$ we have

\[
I_{i,k}(X, h) = \sum_{m \leq X} \left( \frac{1 + (-1)^i \varepsilon(m)}{2} \right) \left( \frac{1 + (-1)^k \varepsilon(3m + h)}{2} \right) = \\
\frac{X}{4} + \frac{(-1)^i}{4} \sum_{m \leq X} \varepsilon(n) + \frac{(-1)^k}{4} \sum_{m \leq X} \varepsilon(3m + h) + \frac{(-1)^{i+k}}{4} S_3(X, h) + O(1) = \\
\frac{X}{4} + \frac{(-1)^i}{4} \sum_{m \leq X} \varepsilon(m) + \frac{(-1)^k}{4} \sum_{c = 1}^3 e^{-2\pi i \frac{m}{3}} \sum_{n \leq 3X + h} \varepsilon(n) e^{2\pi i \frac{2n}{3}} + \frac{(-1)^{i+k}}{4} S_3(X, h) + O(1) .
\]

Now theorem 1 follows immediately from obvious inequality $\sum_{m \leq X} \varepsilon(m) = O(1)$, Gel'fond’s estimate (2) and lemma 1.

Proof of theorem 2 essentially coincides with proof of theorem 1. The only distinction is use lemma 2 instead of lemma 1.

References

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