Random and quasi-coherent aspects in particle motion and their effects on transport and turbulence evolution

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Abstract
The quasi-coherent effects in two-dimensional incompressible turbulence are analyzed starting from the test particle trajectories. They can acquire coherent aspects when the stochastic potential has slow time variation and the motion is not strongly perturbed. The trajectories are, in these conditions, random sequences of large jumps and trapping or eddying events. Trapping determines quasi-coherent trajectory structures, which have a micro-confinement effect that is reflected in the transport coefficients. They determine non-Gaussian statistics and flows associated to an average velocity. Trajectory structures also influence the test modes on turbulent plasmas. Nonlinear damping and generation of zonal flow modes is found in drift turbulence in uniform magnetic field. The coupling of test particle and test mode studies permitted to evaluate the self-consistent evolution of the drift turbulence in an iterated approach. The results show an important nonlinear effect of ion diffusion, which can prevent the transition to the nonlinear regime at small drive of the instability. At larger drive, quasi-coherent trajectory structures appear and they have complex effects on turbulence.

1. Introduction
The direct numerical simulations, which have obtained important results in the last decades, largely dominate the actual research in turbulence. The analytical advance based on first principle description and mathematically justifiable approximations is comparatively very small. One cause, perhaps the principal one, is the stochastic advection process, which appears in all turbulent systems. In realistic descriptions of turbulence, the advected field are active in the sense that they influence the velocity fields. But, even the most simplified models that deal with passive fields have not been solved analytically in all cases. The cause is the basic problem of particle trajectories in stochastic velocity fields, which can be rather complex in important special cases. One example is the turbulence in incompressible media that is dominantly two-dimensional.

The two-dimensional turbulence has a self-organizing character [1–4] that consists of the generation of quasi-coherent structure (vortices), which can increase up to the size of the system in special cases as the decaying turbulence in ideal fluids. The tendency to self-organization is partly maintained in weakly perturbed systems [5–7]. This property appears at the basic level of particle (tracer) trajectories. They are random sequences of trapping or eddying events and long jumps. The trapping process [8–11] strongly modifies the diffusion coefficients and leads to non-Gaussian distribution of displacements. Trapping is essentially determined by the existence of space correlation of the velocity field. Trapping and non-Gaussian effects are also studied in a different approach based on wave-kinetic approximation and on quasiparticle representation of turbulence [12].

Theoretical methods for determining the statistics of the advected particles were developed in the last decades and used for the study of various aspects of the transport. The decorrelation trajectory method (DTM, [13]) and the nested subensemble approach (NSA, [14]) are the first semi-analytical methods that are able to go beyond the quasi-linear regime that corresponds to quasi-Gaussian transport. They were specifically developed for the case of two-dimensional incompressible turbulence that is characterized by trajectory trapping or...
eddy. The general conclusion of these studies is that trajectory trapping or eddying determines non-standard statistics of trajectories: non-Gaussian distributions, long time Lagrangian correlations (memory), strongly modified transport coefficients and an increased degree of coherence. The trapped trajectories form quasi-coherent structures similar to fluid vortices.

Very recent results [13] have shown that these methods could be the basis for the development of a theoretical approach for the study of test modes in turbulent plasmas. It is similar to the Lagrangian approach initiated by Dupree [16, 17] and developed in the 1970s. The assumption of random trajectories with Gaussian distribution limited the application of Dupree’s method to the quasilinear regime. The DTM and NSA enable the evaluation of the average propagator of the test modes in the nonlinear regime and extend the theoretical procedure to the complex processes that are generated in these conditions. The tendency of drift turbulence evolution beyond the quasilinear regime was obtained in [15].

We present a theoretical approach to the study of turbulence evolution, the iterated self-consistent method (ISC). It is based on the analysis of the test particles and test modes in turbulent plasmas. Both test particle and test mode studies start from given statistical description of the turbulence. However, the coupling of these problems can lead to the evaluation of turbulence evolution and to the understanding of the nonlinear processes that are generated. The ISC is applied here to the drift turbulence in plasmas confined in uniform magnetic fields.

The paper is organized as follows. The physical processes analyzed in this paper and the main ideas that are followed in this study are presented in section 2. Section 3 contains the test particle studies of transport, the DTM and the NSA. A discussion on the representation of the complex trajectories by these methods is also included. The two modules of the ISC (test particle and test mode studies) for the case of drift turbulence are presented in section 4. The ISC is discussed in section 5. Section 6 contains the results on the evolution of drift turbulence and the analysis of the nonlinear processes that appear in different conditions (weak and strong drive of the instability). The conclusions are summarized in section 7. We note that sections 3 and 4 contain an overview of previous work that is the basis of the present study. The new results are presented in sections 5–7.

2. Random and coherent aspects of the trajectories

Numerical simulations have shown that particle trajectories in two-dimensional incompressible stochastic velocity fields are characterized by a mixture of random and quasi-coherent aspects, which appear as a random sequence of large jumps and trapping or eddying events. They are solutions of the equation

$$\frac{dx}{dt} = v(x(t), t), \quad v_i(x(t), t) = -\frac{1}{B} \frac{\partial}{\partial y} \phi(x(t), t),$$

where $x(t)$ is the trajectory in the plane perpendicular to the uniform magnetic field $B = B_0 z$, in rectangular coordinates $x = (x_1, x_2)$, $\dot{x}_1 = 1$, $\dot{x}_2 = -1$, $\dot{z}_2 = 0$, and $\phi(x, t)$ is the potential. This is a Hamiltonian system with the conjugate variables represented by the two components of particle trajectories.

The origin of eddying is the Hamiltonian structure of equation (1). The trajectories are periodic and they wind on the contour lines of the potential when it is time independent. This defines the state of permanent trapping. The Lagrangian potential is invariant in these conditions and the transport is subdiffusive because the particles are tied on the fixed contour lines of the potential.

If the motion is weakly perturbed (by time variation of the potential or by other components of the motion that can be introduced in equation (1)), the trapping is temporary and it alternates with large jumps. The statistical importance of the trapping events depends on the strength of the perturbation that is represented by dimensionless factors, which are usually ratios of characteristic times. Two characteristic times are defined in (1): the correlation time of the potential $\tau_\phi$ that gives the scale of the time variation of $\phi(x, t)$ and the time of flight of the particles $\tau_0 = \lambda_c / V$, where $\lambda_c$ is the space scale of $\phi(x, t)$ and $V$ is the amplitude of the velocity. These parameters appear in the Eulerian correlation (EC) of the potential defined in section 3. The characteristic times define the Kubo number $K = \tau_\phi / \tau_0$, which is a measure of trapping in time dependent potentials [9]. Namely, the permanent trapping corresponds to $K \to \infty$, temporary trapping exists for $K > 1$, and the statistical relevance of trapping is a decreasing function of $K$. Any other perturbation of the basic Hamiltonian motion defines a characteristic time $\tau_d$ for the decorrelation of the trajectory from the potential, and a dimensionless parameter similar to Kubo number, the decorrelation number $K_d = \tau_d / \tau_0$. For instance, the decorrelation by an average velocity $V_\phi$ introduces the characteristic time $\tau_\phi = \lambda_c / V_\phi$. The parallel decorrelation can appear for finite correlation length along the magnetic field $\lambda_c$ and it has a characteristic time $\tau_\phi = \lambda_c / \nu_0$, where $\nu_0$ is the thermal velocity of the particles.

The trapping events appear when the trajectory arrives around the maxima or the minima of the potential. The process is quasi-coherent because it affects all the particles situated in these regions in the same way. The high degree of coherence of the trapped trajectories is evidenced in a study of the statistical properties of the
distance $\delta x$ between neighbor motion [14]. The time evolution of $\langle \delta x^2(t) \rangle$ is very slow. The neighbor particles have a coherent motion that approximately keeps the initial distance for a long time compared to $\tau_g$ and eventually $\langle \delta x^2(t) \rangle$ increases as $t^{-3}$ that is slower than the Richardson law $t^2$ [18, 19]. They are characterized by a strong clump effect. These trajectories form intermittent quasi-coherent structures similar to fluid vortices and represent eddying regions. Their average size and life-time depend on the characteristic of the turbulence.

The trapping events appear around the maxima and the minima of the potential. The other particles (that evolve on contour lines corresponding to small absolute values of the potential) have much larger displacements. The large displacements are caused by the large size of these contour lines and also by the decorrelation, which allows transitions between neighbor potential cells. These large jumps are random. The transport is essentially determined by them, while the trapping events have negligible contributions. Thus, trapping can be associated to a process of micro-confinement. At a given moment, the micro-confinement affects a fraction of the particles, and, in a large time interval, every particle is subjected to micro-confinement events during a fraction of this time. The micro-confinement determines the existence of transport reservoirs in the sense that diffusion can be strongly enhanced when these trapped particles are released by the increase of the perturbation strength.

The quasi-coherent trajectory structures have more complex effects in the presence of an average velocity $V_d$. The probability of the Lagrangian velocity has to be time invariant, as required by the zero divergence of the velocity field. In particular, the average Lagrangian velocity $V(t)$ has to be equal to the average Eulerian velocity at any time. The trapped particles do not contribute to $V(t)$, which means that the free particles have an average velocity $V_d/n_f > V_d$ in order to compensate the fraction of trapped particles $n_t$ ($n_f = 1 - n_t$ is the fraction of free trajectories). This effect leads, in the case of a potential that drifts with the velocity $V_0$, to flows in both directions: the structures move with the potential with the velocity $V_d$ and the free particles have an average motion in the opposite direction with a velocity $V_f$ such that

$$n_t V_d + n_f V_f = 0.$$  \hspace{1cm} (2)

The average velocity also determines the release of a part of the trapped particles and the decrease of the size of the trajectory structures. At large $V_d$, larger than the amplitude of the stochastic velocity, the structures are completely eliminated and $n_t = 0$, $V_f = 0$.

The aim of the paper is to analyze the effects produced by these complex trajectories on the evolution of drift turbulence and on the associated transport.

3. Test particle approach to stochastic transport

The Eulerian and the Lagrangian correlations are the main concepts in test particle transport.

The two point EC of the velocity is defined as

$$E_{ij}(x, t) = \langle v_i(x_1, t_1) v_j(x_2, t_2) \rangle,$$  \hspace{1cm} (3)

where $x = x_1 - x_2$, $t = t_2 - t_1$, $i, j = 1, 2$ account for the two components of the velocity and $\langle \rangle$ is the statistical average over the realizations of the velocity field or the space average over $x_i$. The main statistical parameters of the velocity field are evidenced by this function. The amplitudes of velocity fluctuations are $V_i^2 = E_{ii}(0, 0)$, the correlation lengths $\lambda_i$ are the characteristic decay lengths of the functions $E_{ii}(x, 0)$, and the correlation time is defined by the time decay of $E_{ij}(0, t)$. We consider here homogeneous and stationary velocity fields, which have ECs that depend only on the distance between the two points and on the difference between the two times, as (3).

The EC describes the space structure and the time variation of the stochastic velocity field

The Lagrangian velocity correlation (LVC) is defined as

$$L_{ij}(t) = \langle v_i(x_1, t_1) v_j(x(t; t_1, x_1)) \rangle,$$  \hspace{1cm} (4)

where $x(t; t_1, x_1)$ is the trajectory in a realization of the stochastic field that starts at $t = t_1$ in the point $x_1$.

The LVC is thus a time dependent function that describes statistical properties of particle motion in the stochastic velocity field. In most cases the LVC decays to zero since the Lagrangian velocity becomes statistically independent on the initial velocity (it decorrelates). The characteristic time $\tau_g$ of the LVC decay is the measure of the memory of the stochastic Lagrangian velocity. The decorrelation time $\tau_g$ is the correlation time $\tau_c$ of the EC if particle motion is determined only by the velocity field, and it can be completely different of $\tau_g$. In the presence of additional components of the motion (as average velocity, collisional diffusivity or parallel motion).

Besides this important effect of memory, the LVC determines the main quantities related to test particle transport. As shown by Taylor [20], the mean square displacement (MSD) $\langle x^2 \rangle$ and its derivative, the running diffusion coefficient $D(t)$, are integrals of the LVC

$$\langle x^2 \rangle = 2 \int_0^t d\tau L_{ii}(\tau) (t - \tau),$$  \hspace{1cm} (5)
\[
D_i(t) = \int_0^t d\tau \ L_{ii}(\tau).
\]  

(6)

The asymptotic value of \(D_i(t)\)

\[
D_i = \lim_{t \to \infty} D_i(t) = \int_0^\infty L_{ii}(\tau) d\tau
\]

(7)
is the diffusion coefficient along direction \(i\). The time dependent diffusion coefficients provide the ‘microscopic’ characteristics of the transport process. The diffusion at the transport space–time scales, which are much larger than \(\lambda_i\) and \(\tau_i\), is described by the asymptotic values \(D_i\).

The aim of the test particle studies is essentially to determine the LVC as function of the EC. The decorrelation of the Lagrangian velocity after the memory time \(\tau_i\) leads to the possibility of decomposing the statistics of the displacements at large time \(t \gg \tau_i\) in a sequence of independent small scale processes of time \(\tau_i\). The probability of the small scale displacements (the micro-probability), \(P^c(x, t)\), determines the elementary step of the transport process as the MSD

\[
\Delta^2 \equiv \langle (x_i(t_1 + \tau_i; t_0, x_i) - x_i)^2 \rangle = \int dx x_i^2 P^c(x, \tau_i).
\]

The statistics of the displacements at large times \(t \gg \tau_i\) is Gaussian, with the exception of the processes that have \(\Delta \to \infty\) or \(\tau_i \to \infty\) (see [9, 21]). The asymptotic diffusion coefficients are usually approximated by a random walk with the step \(\Delta\) performed in the time \(\tau_i\), that is \(D_{\text{rw}} \approx \Delta^2/\tau_i\), but they can be much more complicated in complex transport processes as those studied here.

In general, the diffusion coefficients \(D_{ii}(t)\) and \(D_i\) are determined by the micro-probability, which contains all statistical information on the trajectories at small time \(t \ll \tau_i\). The use of the ergodic theory is not necessary. However, long time integration of the trajectories is preferred in numerical simulations [22] because it yields more accurate results on the asymptotic diffusion coefficients.

It is important to underline that, when the space scale of the turbulence is much smaller than the characteristic lengths of the gradients, the results of the test particle transport are in very good agreement to those obtained from turbulent fluxes. This is confirmed by the numerical simulations [23, 24].

Most of the theoretical methods are essentially based on Corrsin hypothesis [25], which assume that the micro-probability \(P^c(x, t)\) is Gaussian and that the displacements are statistically independent on the velocity field. They lead to diffusive transport in the limit of static velocity fields (\(\tau_i \to \infty\)), and thus they cannot apply to the special case of zero divergence two-dimensional velocity fields. It can be shown that a diffusive transport is obtained even when the second assumption is eliminated, which suggest that the micro-probability is not Gaussian in the presence of trapping. The first theoretical approach that finds subdiffusive transport in this special case [26, 27] does not use the Gaussian assumption, but it is based on the percolation theory. It only determines the scaling of the asymptotic diffusion coefficients. The detailed statistical information that is contained in the LVC (4) was first obtained by the DTM, [13]. This semi-analytical method was developed and validated by the nNSA, [14].

A short review of the DTM and NSA and a discussion of the image they give on the micro-confinement process is presented below.

3.1. The DTM and NSA

Trajectory trapping or eddying is a consequence of the invariance of the Lagrangian potential for \(\tau_i \to \infty\), which is approximately maintained at large \(\tau_i\). The DTM and the NSA were developed having in mind the necessity of evidencing the statistical consequences of this invariance. We have also imposed the condition of performing only the approximations that do not violate this property.

The main idea is to define subensembles of the realizations of the potential that correspond to the same values of the potential, of the velocity and of its derivatives in the starting point of the trajectories \(x = 0, t = 0\). A system of nested subensembles is constructed. All the realization with the potential \(\phi(0, 0) = \phi^0\) are grouped together in a subensemble \(S_0\), then \(S_0\) is subdivided into subensembles \(S_1\) according to the values of the initial velocity \(v(0, 0) = v^0\), then each \(S_1\) is again divided into smaller subensembles \(S_2\) using the values of the derivatives of the velocity, and the procedure can continue. The trajectories in a subensemble \(S_n\) are super-determined, in the sense that they have supplementary initial condition (the potential \(\phi^0\), the initial velocity \(v^0\) and its derivatives up to the order \(n\)). This leads to a high degree of similarity of the trajectories in a subensemble \(S_n\), that increases with the increase of the nesting order \(n\).

The statistical description of the velocity in a subensemble \(S_n\) can be derived from the statistics in the whole set of realization using conditional averages. A Gaussian velocity reduced at the subensemble \(S_n\) remains Gaussian, but its average and dispersion are modified. The subensemble average \(\bar{V}^n(x, t) \equiv \langle v(x, t) \rangle_{S_n}\) exists even in the case of zero average fields. It is a function of the EC that also depends (linearly) on the set of parameters of the nested subensembles. The subensemble average velocity \(\bar{V}^n(x, t)\) is a space–time dependent
function, which has the value \( V_S(0, 0) = v_0 \), as imposed by the conditions that define \( S_1 \) in the nested subensembles. The subensemble dispersion \( \left\langle (\delta v(x, t))^2 \right\rangle_{S_n} \) (where \( \delta v(x, t) = v(x, t) - V_S \)) is much smaller than \( V_S^2 \), the amplitude of the velocity fluctuations in the whole set of realizations. It is zero for \( x = 0 \) and \( t = 0 \) (because imposing a condition on the value of the velocity eliminates any fluctuation), and it reaches the value \( V_S^2 \) at large distances and/or times \( |x_n| > \lambda_n, t > \tau_c \). The small amplitude of the velocity fluctuations in the subensemble \( S_n \) provides a second contribution to the increase of the degree of similarity of the set of trajectories that yield from the realizations that are included in \( S_n \).

The existence of the subensemble average velocity \( V_S(x, t) \) determines an average trajectory in \( S_n \) that is obtained by performing the subensemble average of the equation of motion

\[
\frac{d}{dt} \langle x(t) \rangle_{S_n} = \langle v(x(t), t) \rangle_{S_n}. \tag{8}
\]

The right-hand side of this equation is generally very difficult to be evaluated since it is the average of a stochastic function \( v(x, t) \) of a stochastic variable, the trajectory \( x(t) \) that is determined by \( v(x, t) \). However, the high degree of similarity of the trajectories in the subensemble \( S_n \) enables an important simplification that consists in neglecting the fluctuations of the trajectories. The average Lagrangian velocity is thus approximated in \( S_n \) by the subensemble Eulerian velocity calculated along the average trajectory

\[
\frac{d}{dt} X_S(t) = V_S(X_S(t), t), \tag{9}
\]

where \( X_S(t) \) is the approximate subensemble average trajectory in \( S_n \). \( X_S(t) \) is it called the decorrelation trajectory (DT), because it shows the way toward decorrelation in each subensemble. We note that the approximation (9) of equation (8) is in agreement with the invariance of the Lagrangian potential. Equation (9) has Hamiltonian structure as the equation of particle motion (1) because \( V_S(x, t) \) has the same structure as particle velocity. It can be derived from a function \( \Phi_S(x, t) \), which is the subensemble average potential

\[
V^S_i(x, t) = -\frac{\partial \Phi_S(x, t)}{\partial x_j}. \tag{10}
\]

This leads, in the case of static velocity fields, to the invariance of the Lagrangian potential \( \Phi_S(X_S(t)) \), which is equal to \( \phi^0 \) for any value of the nesting order \( n \).

The LVC, \( D_{ij}(t) \) and other Lagrangian averages are determined by summing the contribution of all subensembles. This leads to integrals over the parameters that define the subensembles.

Thus the NSM is a systematic expansion based on the nested subensembles. It is a semi-analytical method that determines the statistics of the stochastic trajectories in terms of the DTs. These trajectories and the integrals over the subensembles have to be numerically calculated. The calculations are at PC level, with run times that increase as the nesting order increases, starting from few minutes for \( n = 1 \).

The DTM [13] corresponds to the nesting order \( n = 1 \). The results obtained using the NSM of order \( n = 2 \) are presented in [14]. The DTs for the subensembles \( S_2 \) are completely different of those of the DTM. However they lead to results for the LVC that are not much different of those obtained with the DTM, which shows that the NSM converges fast. The higher order NSM provides more detailed statistical information at the expense of the increase of the number of the DTs and of the complexity of their equation. The statistics of the distance between neighbor trajectories was determined with the \( n = 2 \) NSM.

We use here the DTM that is based on the subensembles \( S_0 \) and \( S_1 \). An important simplification was recently found [28]. We have shown that the number of parameters of the subensembles can be reduced to only two without significant modifications of the diffusion coefficients. The magnitude \( \mu \) of the normalized velocity \( v_1(0, 0)/V_1 \) can be eliminated from the definition of \( S_0 \). The integral over \( \mu \) can be performed in the subensemble EC in \( S_2 \). This leads to a modified average potential, which depends only on \( \phi^0 \) and \( \theta^0 \), the orientation of the normalized velocity

\[
\Phi^S_i(x, t) = \phi^0 E(x, t) + \frac{8}{\pi} \frac{\cos(\theta^0)}{V_1} \frac{\partial E(x, t)}{V_2} = \frac{8}{\pi} \frac{\sin(\theta^0)}{V_1} \frac{\partial E(x, t)}{V_2}. \tag{11}
\]

The parameter \( \theta^0 \) defines a larger subensemble \( S'_1 \) that includes the subensembles \( S_1 \) with any value of \( \mu \).

The time dependent diffusion coefficients are obtained in this case from

\[
D_{11}(t) = \frac{V_1}{2\pi} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} d\phi^0 P(\phi^0) \int_0^{2\pi} d\theta^0 \cos(\theta^0) X^Y_1(t), \tag{12}
\]

\[
D_{22}(t) = \frac{V_1}{2\pi} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} d\phi^0 P(\phi^0) \int_0^{2\pi} d\theta^0 \sin(\theta^0) X^Y_2(t), \tag{13}
\]

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leads to the saturation of \( q \). The number of discretization points for each parameter of the subensembles \( N \) determines opposite effects in \( t \) that are either trapping events or free displacements, depending on \( E \).

At large values of \( t \), it eventually saturates the initial potential. More important, the statistical relevance of each type of motion and the in trajectory trapping is determined by the whole set of DTs.

Typical results for \( D_{ij}(t) \) obtained for isotropic turbulence with the EC of the potential \( E = \Phi^2 \exp(-x_1^2 + x_2^2)/\Lambda^2 - t/\tau_e \) and \( V_0 = 0 \) are shown in figure 1. \( D_{ii}(t) \) has different shapes at small and large amplitudes of the potential fluctuations \( \Phi \). The diffusion coefficient increases linearly at small time and it eventually saturates (for \( t > \tau_e \)) in the case of small \( \Phi \) (dashed line), while, at large \( \Phi \) (continuous line), a transitory decay appears before saturation. The decay accounts for the micro-confinement produced by trajectory trapping. This process appears when \( \tau_{\Phi, i} < \tau_e \) (in general, \( \tau_{\Phi, i} < \tau_0 \)), where \( \tau_{\Phi, i} = \lambda_i/V_i \) is the time of flight over the correlation length. One can see in figure 1 that the variation of \( \tau_e \) determines opposite effects in the two cases. The decrease of \( \tau_e \) leads to the saturation of \( D_{ii}(t) \) at smaller time. This corresponds to smaller \( D_{ij}(t) \) (smaller asymptotic \( D_i \)) in the quasilinear case (dashed line) and to larger \( D_i \) for the nonlinear transport (continuous line). In a static potential (the dotted line for \( \tau_e \to \infty \)), the decay is not limited and \( D_{ii}(t) \to 0 \).

![Figure 1. \( D_{ii}(t) \) in the quasilinear (dashed line for \( \Phi = 0.2 \)) and nonlinear (continuous line for \( \Phi = 5 \)) regimes for the correlation time \( \tau_e = 1 \). The subdiffusive transport in the static potential is represented by the dotted line.](image-url)
DTM yields subdiffusive transport for such fields where the trajectories are tied on the contour lines of the potential. The LVC (4) is negative at large time, such that its integral decays to zero. It has a negative tail that decays as a power of time. We note that similar results are obtained for more extended EC that has space decay of power low type [30].

The existence of an average velocity \( V_d \) in the equation of motion determines the decrease of the fraction of trapped trajectories and modifications of the DTs in the nonlinear regime. A new type of open DTs that turn toward the direction of \( V_d \) appears. Their number increases as \( V_d \) increases and, when \( V_d \) is much larger than the stochastic velocity, all DTs are of this type. Thus, the average velocity eliminates progressively (as \( V_d \) increases) the trapping of the trajectories and eventually leads to quasilinear transport. The open trajectories along the average velocity lead to a transitory regime characterized by the increase of \( D_{22}(t) \). It last for the interval \([\tau_0, \tau_f] \), where \( \tau_0 = \lambda_f / V_d \) is the characteristic time defined by the average velocity.

The time dependence of \( D_{22}(t) \) for the unperturbed static potential (figure 1, the dotted curve) is the representation of the micro-confined turbulence produced by the permanent trapping. The mixing of the close trajectories, which have periods that increase with the increase of their size, eliminates progressively the contributions of the trapped trajectories at the diffusion process. \( D_{22}(t) \) decreases due to the decrease of the fraction of free particles. The LVC and \( D_{22}(t) \) have long (algebraic) tails, which show that the diffusion process easily reacts at perturbations represented by the time variation of the potential or by other components of the motion. Any small perturbation (characterized by a large decorrelation time \( \tau_d \)) has significant effect that consists of releasing a part of trajectories. The transport reservoir is the maximum of \( D_{22}(t) \) (figure 1). It provides the maximum diffusion coefficient that correspond to a decorrelation process with \( \tau_d = \tau_0 \), which releases all the trapped trajectories.

Various aspects of the transport in fusion plasmas were studied using the DTM, that was developed and adapted to the study of models with increased complexity [32–40].

4. Test particles and test modes in drift turbulence

Drift instabilities are low-frequency modes generated in non-uniform magnetically confined plasmas [41]. We consider here the basic drift instability in uniform magnetic field. It belongs to the family of instabilities that have the main role in particle and energy transport in fusion research [2, 42, 43]. They have complex nonlinear evolution with generation of large correlations and zonal flow modes [5–7].

Test particle and test mode studies start from the given statistical description of turbulence. They provide answers to different questions compared to the case self-consistent studies. Namely, they evaluate the growth rates of the modes, the diffusion coefficients and the characteristics of the transport as functions of the parameters of the background turbulence. The self-consistent studies determine the characteristics of the turbulence generated in given macroscopic conditions (as density gradient, temperatures) and the associated transport. We show in section 6 that an ISC method can be developed for the drift turbulence. It is based on the coupled study of test particle and test modes. These main modules of the ISC theory are presented here.

This section is a short review of previous work [15, 28, 29] adapted and developed to the needs of the ISC method.

4.1. Test particles

We determine here the parameters required by the test mode study and present a short review of the diffusion process.

The equation for the trajectories in the drift turbulence with a zonal flow mode potential \( \phi_d \) (in the frame that moves with the potential) is

\[
\frac{dx}{dt} = -K'_x \xi \partial_x [\phi(x, t) + Z \phi_d(x, t)] + V_d, \tag{15}
\]

where dimensionless quantities are used with the units: \( \rho \) (for the distances and the correlation lengths \( \lambda_x, \lambda_d \)), \( \tau_0 = L_0 / \tau \) (for time) and \( \Phi \) (for the potential). In this normalization, the time of flight is a decreasing function of \( \Phi \), while \( \tau_0 \) and \( \tau_d \) are fixed (independent of the turbulence amplitude). The normalized amplitude of the zonal flows is \( Z = \phi_d / \Phi \) and the normalized diamagnetic velocity is \( V_d = 1 \). The dimensionless parameter \( K'_x \) is the measure of turbulence amplitude

\[
K'_x = \frac{\Phi}{B \rho V_0} = \frac{e \Phi L_0}{T_e \rho_s}, \tag{16}
\]

which naturally appears in connection with the diamagnetic velocity.

The spectrum of the drift turbulence has a special shape determined by the absence of the modes with \( k_2 = 0 \), which are stable. It has two symmetrical peaks for \( k_2 = \pm k_0 \). The ISC method shows that the evolution of
The Fourier transform gives the EC

\[
S(k) \sim k^n \exp \left( -\frac{k^2}{2\lambda^2} \right) \exp \left( -\frac{(k_0 - k_2)^2}{2\lambda^2} \right) - \exp \left( -\frac{(k_0 + k_2)^2}{2\lambda^2} \right).
\]  

(17)

The parameters of the trajectory structures are obtained using the DTM as weighted averages of the DTs seen in figure 2, the ratio of the asymptotic values in the nonlinear and quasilinear regime is much larger for

\[
\frac{D_{11}(t)}{D_{11}(t)}
\]

which is the characteristic time for the formation of the structure. The sizes \(s_i(t)\) of the trajectory structures in the two directions are

\[
s_i(t) = \int d\phi P(\phi) \int \nu_i^0 d\nu_i^0 P(\nu_i^0)P(\nu_i^0) X_i^{\text{max}}(t; \phi^0, \nu_i^0),
\]

(20)

where \(X_i^{\text{max}}(t; \phi^0, \nu_i^0)\) is the dimension of \(X^i(t; \phi^0, \nu_i^0)\) along the direction \(i\) if \(t > T(\phi^0, \nu_i^0)\) and \(X_i^{\text{max}}(t; \phi^0, \nu_i^0) = 0\) if \(t < T(\phi^0, \nu_i^0)\).

The functions \(n_i(t)\) and \(s_i(t)\) describe the growth of the trajectory structures. These functions saturate in a time \(\tau_i\), which is the characteristic time for the formation of the structure. \(\tau_i\) is an increasing function of the amplitude of the turbulence \(K_i\). The asymptotic values of \(n_i(t)\) and \(s_i(t)\) describe the parameters of the structures as functions of the characteristics of the turbulence. The dependence on the amplitude of the turbulence is
presented in figure 3. It shows that the structures continuously increase when turbulence amplitude increases and eventually they include all the trajectories \( n_{1tr} \). In the presence of a decorrelation process with characteristic time \( \tau_d < \tau_s \), the parameters of the structures are smaller because they are destroyed during the formation at the time \( \tau_s \). They can be approximated by \( n_{1tr}(\tau_d) \) and \( s_i(\tau_d) \), instead of the asymptotic values.

The micro-probability \( P^\Sigma(x, t) \) can be evaluated using the DTM as a histogram of the DTs. It can be Gaussian, or non-Gaussian, depending on turbulence parameters and on the decorrelation time \( \tau_d \). Non-Gaussian \( P^\Sigma \) are associated to the presence of trapping. The micro-confinement appears very clearly in the probability as a narrow peak in \( x = x_0 \), which remains invariant after a formation time \( \tau_s \), the same as in the time evolution of the fraction of trapped trajectories \( n_{1tr}(\tau_d) \). \( P^\Sigma \) has also a time dependent part around this maximum, which is determined by the free particles. For the drift turbulence case, the motion of the potential with the diamagnetic velocity leads to the separation of the two parts of the micro-probability: the narrow peak is displaced in the direction \( V_d e_2 \) and the free particle part concentrates in the opposite direction then moves. This result is in qualitative agreement with the process of flow generation in zero divergence velocity fields (section 2), but the velocity of the free particles obtained with the DTM is smaller than the the velocity obtained from equation (2), \( V_f = -V_d n_{1tr}/n_f \).

The analytical results for the test modes (section 4.2) are obtained using simplified approximations of \( P^\Sigma \) in agreement with equation (2) for the ion flows (see [15] for details). The trapped particle part was represented for simplicity by a Gaussian function, but with fixed dispersion given by the size \( s_i \) of the structures \( S_i = s_i^2 \). The shape of this function does not change much the estimations. The free trajectories are described by a Gaussian with dispersion that grows linearly in time: \( S_i'(\tau) = S_i + 2D_i (t - \tau) \), \( i = x, y \). The initial value \( S_i'(0) = S_i \) is an effect of trapping. It essentially means that the trajectories are spread over a surface of the order of the size of the trajectory structures when they are released by a decorrelation mechanism. Introducing the flows (2), the approximation for the micro-probability is

\[
P^\Sigma(x, y, t) = n_{1tr} G(x, y - V_d t; \mathbf{S}) + n_f G(x, y - V_f t; \mathbf{S}'),
\]

where \( G(x; \mathbf{S}) \) is the two-dimensional Gaussian distribution with dispersion \( \mathbf{S} = (S_x, S_y) \). We note that this probability is non-Gaussian.

The separation of the distribution and the existence of ion flows in drift type turbulence are confirmed by numerical simulations [24].

4.2. Test modes
We have studied in [15] drift type test modes on turbulent plasmas. The frequencies and the growth rates are obtained as functions of the characteristics of the turbulence. They show that ion diffusion (generated by the random component of the trajectories) has a stabilizing effect, while ion trapping (the quasi-coherent component) leads to strong nonlinear effects: increase of the correlation lengths, nonlinear damping of the drift modes and generation of zonal flow modes. The strength of each of these processes depends on the parameters of the turbulence.
We use the basic description of the (universal) drift turbulence provided by the drift kinetic equation in the collisionless and low density limit. We consider a plasma confined by an uniform magnetic field \( B \), taken along the \( e_3 \) axis in a rectangular system of coordinates. The density gradient with characteristic length \( L_n \) is taken along \( e_1 \). The drift wave instability that is produced by the electron kinetic effects and the ion polarization drift velocity is studied. We have chosen this simple instability for this first study of the effects produced by trajectory trapping or eddying in order to avoid interferences with other processes. The stabilizing effect produced by a sheared magnetic field on bound eigenmodes [44], or, alternatively, the instability for wavelengths below the ion Larmor scale [45] are not considered here. The effects of the magnetic shear on transport were studied using DTM in [46], which enables the extension of the present study to more complex confining configuration.

The solution of the dispersion relation for quiescent plasma is (see [41])

\[
\omega = \frac{k_z V_a}{1 + k_z^2 \rho_i^2}, \quad \gamma = \gamma_0 \omega (k_z V_a - \omega),
\]

(22)

where \( \omega \) is the frequency of the mode with wave number \( k = [k_x, k_y, k_z] \), \( k = \sqrt{k_x^2 + k_y^2} \), \( V_a = V_a e_z \), \( V_a = T_e/(\pi B L_n) = \rho_c c_s / L_n \) is the diamagnetic velocity, \( \rho_c = c_s / \Omega_i, \quad c_s = \sqrt{T_e / m_i}, \quad T_e \) is the electron temperature, \( m_i \) is the ion mass, \( \varepsilon \) is the absolute value of electron charge and \( \Omega_i = eB / m_i \) is the ion cyclotron frequency. The constant \( \gamma_0 \) that plays the role of drive of the instability is \( \gamma_0 = \sqrt{\pi/2}/(|k_z| \nu_T) \), where \( \nu_T = \sqrt{T_e / m_i} \). The drift waves are unstable (\( \gamma > 0 \)) when the phase velocity \( \omega/k_z < V_a \).

Electron and ion responses to a small perturbation \( \delta \phi \) applied on the turbulent state with potential \( \phi \) are determined. They lead to a modified propagator

\[
\Pi' = i \int_{-\infty}^t d\tau \, M(\tau) \exp[-i \omega (\tau - t)],
\]

(23)

which includes the effects of the turbulence in the function \( M(\tau) \) defined by

\[
M(\tau) \equiv \left\{ \exp \left[ i \mathbf{k} \cdot (\mathbf{x}(\tau) - \mathbf{x}) - \int_{-\infty}^\tau d\tau' \, \nabla \cdot \mathbf{u}_p(\mathbf{x}(\tau')) \right] \right\},
\]

(24)

where the average \( \langle \rangle \) is over the ion trajectories in the stochastic potential \( \phi \). The polarization drift \( \mathbf{u}_p \) determines a compressibility term due to its divergence \( \nabla \cdot \mathbf{u}_p = - \partial_t \Delta \phi / (\Omega_i B) \). The dispersion relation (quasi-neutrality condition) is the same as in the quiescent plasma in terms of the propagator

\[
-(k_z V_{ac} - \omega \rho_i^2 k^2) \Pi' = 1 + i \frac{\omega - k_z V_{ac}}{\sqrt{2 |k_z| \nu_T}}
\]

(25)

(see [15] for details).

The micro-probability of the displacements can be Gaussian or non-Gaussian, depending on the spectrum parameters and on the decorrelation time \( \tau_0 \). The simple approximation (21) includes the main features of the micro-probability. It has the advantage of enabling the derivation of simple analytical expressions for the function \( M(\tau) \), the renormalized propagator (23) and eventually to solve the dispersion relation (25) for the complex frequency of the drift modes (see [15]).

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The drift modes have modified frequency and growth rate compared to the quiescent plasmas (22), which depend on the characteristics of the trajectory structures and on the diffusion coefficients

\[
\omega_d = \frac{1}{2} \left[ \omega_0 + (1 - n) \omega_0 + \sqrt{\omega_0^2 + \frac{4n k_z^2}{1 + 4k_z^2}} \right],
\]

(26)

\[
\gamma_d = \frac{\gamma_0}{(1 - n)} \left[ (1 - n) \omega_0 - \omega_d \right] - \frac{n \omega_d^2}{(1 + 4k_z^2) + n k_z^2} \left( \omega_d - \omega_d \right),
\]

(27)

where \( \omega_0 = \omega_0 + (n - 1) \omega_0, \quad n = n_{tr} / (1 - n_{tr}), \quad \omega_0 = \mathcal{F} \omega_0 / (1 + 4k_z^2) \) is the frequency obtained for \( n \rightarrow 0 \), \( \mathcal{F} = \exp(-s^2 k_z^2/2) \) is a factor that depends on the sizes of the trajectory structures, \( s_g = \operatorname{sign}(\omega_0) \) and \( \omega_0 = \gamma_0 c_s / L_n \). Usual normalized quantities \( k_z = k_z \rho_i, \quad \omega = \omega L_n / c_s, \quad \gamma = \gamma L_n / c_s \) are used.

The expressions for \( \omega_d \) and \( \gamma_d \) show that at small amplitude, when the trajectories are random and \( n = 0 \), \( \mathcal{F} = 1 \), the background turbulence has a stabilizing effect that consists of a negative term in the growth rate \(- \omega_d^2 \mathcal{F} \) [16]. It determines the damping of the large \( k \) modes and the narrowing of the spectrum, which corresponds to the increase of the correlation lengths \( \lambda \). This effect persists at large amplitudes, but it is modified by the quasi-coherent component of the ion trajectories, which introduces the factors seen in equation (27) and changes the diffusion coefficients.

The quasi-coherent structures of the ion trajectories influence the modes through the factor \( \mathcal{F} \), while the flows generated in the moving potential (2) are represented by \( n \) in equations (26) and (27).
The effect of the structures can be understood by taking \( n = 0 \) in these equations, which become
\[
\omega_d = \omega_0 = \frac{\mathcal{F}_d^2}{1 + \mathcal{F}_d^2}, \quad \gamma_d = \gamma_0 \omega_d (\mathbf{k}_d - \omega_d) - \mathbf{k}_d^2 D_i.
\] (28)

The factor \( \mathcal{F} \) determines the decrease of \( \omega_d \) and the displacement of its maximum from \( \mathbf{k}_d \approx 1 \) to \( \mathbf{k}_d \approx 1 / \nu_i \). The maximum of \( \gamma_d \) moves toward smaller values of \( \mathbf{k}_d \) as \( \nu_i \) increase. The effect is the increase of the correlation lengths accompanied by the decrease of the diffusive damping due to the decrease of \( \mathbf{k}_d \). The physical reason for this behavior is the eddying of the ion trajectories that eliminates large \( \mathbf{k}_d \) modes by averaging.

The ion flows represented by finite values of \( n \) in equations (26) and (27) formally determine a jump of \( \gamma_d \) to negative values in the range \( \mathbf{k}_d < k_d \), where \( k_d \) is the solution of \( \omega_d = 0 \). \( k_d \) increases as \( n \) increases until \( n \geq 1 \) where \( \omega_d = 0 \) has no solution and \( \gamma_d = 1 \) in equation (26). The opposite flows (of the trapped ions that have \( V_{tr} = 1 \) and of the free ions with \( V_{tr} = -n \)) appear in equation (27). The instability condition that yields from this equation is \( V_{tr} < \omega_d / \mathbf{k}_d < V_{tr} + V_f \). The condition \( \omega_d / \mathbf{k}_d < - n \) is not fulfilled for \( n < 1 \) at small \( \mathbf{k}_d \) and for \( n > 1 \) at any \( \mathbf{k}_d \). Thus, the drift modes are damped in these conditions. The physical reason is the advection of the trapped ions with the potential that moves with the diamagnetic velocity. The effective diamagnetic velocity of this part of ions is zero, which corresponds to an effective density gradient equal to zero. The drift waves and instabilities do not exist in this case and \( \gamma_d < 0 \) in equation (27). The large \( \mathbf{k}_d \) modes have the phase velocity \( \omega_d / \mathbf{k}_d \approx - n \), which is in the instability domain. At \( n > 1 \), the trapped ions dominate leading to the damping of all modes.

The compressibility term determines unstable modes completely different from the drift modes. They have \( \mathbf{k}_d = 0 \) and very small frequencies
\[
\omega_{sf} = - k_d a \frac{1 + \mathcal{F}_d^2 n_f}{1 + \mathcal{F}_d^2},
\]
\[
\gamma_{sf} = - \frac{n_{tr} \mathcal{F}_d^2 [\gamma_0 \omega_{sf} - n_f \mathcal{F}_d^2 D_i]}{(1 + n_f \mathcal{F}_d^2)(1 + \mathcal{F}_d^2)},
\]
where \( a = a / V_{tr} \). These unstable modes that are named zonal flow modes are the consequence of trapping combined with the polarization drift. The compressibility term in the function \( M \) determines correlations with the displacements
\[
L_i(\tau) = - \frac{1}{\Omega_i B} \int_0^\tau d\tau' \langle x_i(t) \partial_{\tau'} \Delta \phi(x(t')) \rangle,
\]
which, as evaluated in [15], can be significant for \( i = 1 \) (in the direction of the density gradient) for the trapped ions. It can be approximated by
\[
L_i(\tau, t) \approx a(t - \tau), \quad a = 2 \partial_f^2 \Delta f^0(0) \frac{\tau \gamma V_k}{\Omega_i B^2}.
\] (32)

This correlation determines an average velocity for the trapped ions. One can see that when trapping is negligible \( (n_{tr} \approx 0) \), \( \omega_{sf} = - k_d a \) and \( \gamma_{sf} = 0 \), and that \( \omega_{sf}, \gamma_{sf} = 0 \) for \( a = 0 \).

5. The ISC

The idea of the ISC method is based on the difference between the description of turbulence evolution and the mode representation. One can define in the evolution equation characteristic times for each process (represented by a term in the equation). The ordering of these characteristic times shows which are the processes that have important contributions at a given stage of evolution. Neglecting the small terms leads to simplified equations that give the short time evolution, for time smaller than the characteristic times of the neglected terms. The estimation of the characteristic times provides the basis for systematic approximations that can strongly simplify the evolution equation at small time. On the other side, the mode representation of turbulence does not deal with time evolution but it determines the possible frequencies and wave numbers that are supported by the system and their tendency of amplification or damping. These quantities include effects of processes that are negligible at short time.

The combination of the study of test modes on turbulent plasma with the evaluation of the short time equilibrium of the distribution functions can provide a much simplified approach and even the possibility to develop a semi-analytical approximate method for the turbulence evolution.

More precisely, the short time solution is an approximation that is not valid at large times where the neglected small terms in the evolution equation determine the accumulation of significant effects. These effects are taken into account using the test modes. The frequency and the growth rate of a small perturbation \( \phi \) of the
There are two possible mechanisms for the attenuation of the drift turbulence. The effective calculation of these characteristics of the modes requires the statistics of the trajectories (for evaluating the average propagator). The diffusion coefficients and the probability of displacements depend on the momentary EC of the background potential. The latter is provided by the small time solutions of the evolution equations. Finally, the frequencies and the growth rates of all modes determine the evolution of the background potential.

Thus the methodology of the ISC approach consists of repeated steps that contain the following calculations:

- The evaluation of approximate (short time) equilibrium distribution functions in the presence of background turbulence and of the EC of the potential.
- The calculation of the statistical characteristics of the trajectories (diffusion coefficients, probability of displacements, characteristics of the quasi-coherent structures) as functions of the EC of the potential. This is one of the main modules, presented in section 4.2.
- The calculation of the renormalized propagator (averaged over trajectories) and evaluation of the frequencies and the growth rates of the test modes. This is the second main module of the ISC.
- The evolution of the spectrum on a small time interval is obtained using the growth rates of all modes. It is the starting point of a new step in this iterated method.

In the case of the drift instability studied here [41], the small time distributions are the adiabatic response for the electrons and \( f_i = n_0(x_i) f_{M} (1 + e \phi(x - V_d t e_z) / T_i) \) for the ions, where \( n_0(x_i) \) is plasma density, \( V_d \) is the diamagnetic velocity and \( f_{M} \) is the Maxwell distribution of ion velocities. They correspond to the stable drift waves and show that an arbitrary potential \( \phi \) in a non-uniform plasma drifts with the diamagnetic velocity. The neglected terms are taken into account in the linearized equation for the small perturbation. The kinetic effects of the electrons determine their non-adiabatic response, which is the drive of the instability. This response is the same as in quiescent plasmas due to the fast parallel motion of the electrons. The compressibility effects determined by the polarization drift of the ions make the growth rates positive. The stochastic ion trajectories in the background potential \( \phi \), which can become rather complex by acquiring coherent aspects at large amplitudes of \( \phi \), change the frequencies and the growth rate leading to strong nonlinear effects in the evolution of turbulence.

Several important simplifications can be operated in methodology of the ISC method. The most important concerns the test mode module that is practically eliminated in the present case by the analytical results available for the frequencies and the growth rates of drift and zonal flow modes in turbulent plasmas (section 4.1). Moreover, we have found that the spectra of the drift and zonal flow components obtained in the iterated evolution can be approximated by expressions of the type of equation (17). It is thus sufficient to determine the parameters in this expression at each step. This permitted to have analytical expressions for the EC in the test particle module.

A computer code was developed for the study of the drift turbulence using the ISC method. We note that this code is not performing the simulation of the turbulence, but it calculates the quantities related to test particles and test modes according the iterated procedure presented here. The run time is of few hours on a laptop.

6. Drift turbulence evolution

There are two possible mechanisms for the attenuation of the drift turbulence.

The first is the diffusive damping, which can be strongly enhanced due to the increase of \( D_z \) in the nonlinear regime. The process is produced by the diamagnetic velocity that modifies the total potential producing bundles of open contour lines with average orientation along its direction. The zonal flow modes can contribute to this process since they provide an additional component along \( V_d e_z \) [29]. This velocity is stochastic with zero average, but it determines an average effect due to the nonlinear dependence of \( n_e \) on \( V_d \) (figure 3).

The second mechanism is the nonlinear self-damping determined by the ion flows. The increase of the rate of trajectory trapping \( n = n_{tr} / (1 - n_e) \) determine the decrease of \( \gamma_d \), which becomes negative for all values of the wave numbers at \( n = 1 \), as results from equation (27).

The self-consistent evolution is expected to be much more complicated since the processes identified by a separate analysis of the two main modules can be simultaneous present or they can combine in a synergistic way. Moreover, there are other parameters that are included in both modules (the correlation lengths, the dominant wave numbers, the size of the trajectory structures) that change during the evolution and can strongly affect the quasi-coherent structures.
We have found two types of evolution depending on the strength of the drive of the instability \( \pi_0 \). The evolution at small drive (small \( \pi_0 \)) that corresponds to large \( k_2 \) is controlled by the diffusion. At high drives that appear for smaller \( k_1 \), the effects of trajectory trapping become dominant. In both cases, there is a similar initial stage.

The evolution of drift turbulence is analyzed below. We use in the figures red and blue colors for the two-dimensional quantities (red for the direction \( e_1 \) of the density gradient and blue for the direction \( e_2 \)) and black for the scalar functions or for the zonal flows.

### 6.1. The initial stage
The evolution of turbulence has an initial stage that depends on the initial condition for the spectrum. However, for all physically reasonable conditions, the shape of the spectrum changes and it always develops two symmetrical maxima in \( k_2 = \pm k_0 \) and \( k_2 = 0 \). They are produced by the diffusive damping of the modes, which acts very strongly on large \( k_2 \) and \( k_2 \lesssim 0 \) domain and makes \( \gamma < 0 \) even at small values of the diffusion coefficient \( D_1 \). The growth rate in the quasilinear limit that can be obtained from equation (27) for \( n = 0 \) and \( \mathcal{F} = 1 \) at \( k_2 = 0 \) is

\[
\tau_d = k_0^2 \left( \frac{\gamma_0 k_0^2 - D_2}{\left(1 + k_0^2\right)^2} \right).
\]

The parentheses becomes negative in the limits of both large and small \( k_0 \). The diffusive decay at large \( |k_2| \) combined with the growth rate that decay to zero at small \( |k_2| \) determine the generation of the two peaks in the spectrum. This leads to the increase of the anisotropy of the turbulence. The amplitude of the normalized velocity \( V_1 \) increases while \( V_2 \) decreases in this stage (see the small time evolution in the figures 4(a) and 5(a)).

The condition for the transition to the trapping regime \( (K_1 > \lambda_0) \) is not attained in this case since, as seen in figure 4(a), the increase rate is larger for \( \lambda_0 \) than for \( K_1 \). The structures of trajectories are practically absent, as seen in figure 4(c) where \( n_{tr} < 10^{-3} \) and \( s_i < 10^{-2} \). The normalized diffusion coefficients saturate at \( D_1 \approx 0.03 \), \( D_2 \approx 0.2 \).

### 6.2. Evolution at weak drive
Typical evolution at weak drive is presented in figure 4, where \( \pi_0 = 1 \). After the initial stage, the diffusive damping becomes stronger and it determines important changes in the evolution. The amplitude of the potential \( K_1' \) continues to increase but with a slower rate: the exponential evolution is replaced by a roughly linear increase (figure 4(a), the back curve for \( K_1' \)). Both components of the normalized velocity \( V_1' \) and \( V_2' \) decreases. They have a fast decay followed by the tendency of saturation (figure 4(a)). This is the effect of diffusive damping. The diffusion coefficients increase as \( (K_1')^2 \) and they produce the decrease of the growth rate of the drift modes.

Besides this well known effect of damping, the diffusion determines significant changes of the shape of the spectrum. The width of the two peaks decreases in both directions, which determines the increase of the correlation lengths. It also determines the displacement of the position of the peaks toward smaller wave numbers. The evolution of the spectrum parameters is shown in figure 4(b).

### 6.3. Evolution at strong drive
The increase of the drive parameter determines a strong effect on turbulence evolution, as seen in figures 5–7, where \( \pi_0 = 5 \). This large values of \( \pi_0 \) lead, since the initial stage of the evolution, to a faster increase of the amplitude of the normalized potential \( K_1' \). Ion diffusion determines the same effects as in the weak drive case, but the fast exponential increase of the potential up to large values makes \( K_1' > \lambda_0 \), and determines ion trapping.

The condition for the transition to the trapping regime \( (K_1' > \lambda_0) \) is not attained in this case since, as seen in figure 4(c), the increase rate is larger for \( \lambda_0 \) than for \( K_1' \). The structures of trajectories are practically absent, as seen in figure 4(c) where \( n_{tr} < 10^{-3} \) and \( s_i < 10^{-2} \). The normalized diffusion coefficients saturate at \( D_1 \approx 0.03 \), \( D_2 \approx 0.2 \).

Drift turbulence evolution at weak drive is essentially determined by ion diffusion, which reduces the growth rate and determines a significant change of the spectrum that prevents the development of the coherent effects produced by trapping.

Drift turbulence evolution at strong drive is analyzed below. We use in the figures red and blue colors for the two-dimensional quantities (red for the direction \( e_1 \) of the density gradient and blue for the direction \( e_2 \)) and black for the scalar functions or for the zonal flows.
Figure 4. Drift turbulence evolution for weak drive. (a) The turbulence amplitude parameter $K'_A$ (black line) compared to the nonlinear limit $\lambda_c$. The components of the normalized velocity are also plotted (red and blue curves). (b) The parameters of the EC. (c) The trajectory structure sizes $s_1$, $s_2$ and the fraction of trapped trajectories $n_{tr}$.

As seen in figure 5(a), the evolution enters in the nonlinear regime $K'_A > \lambda_c$ at the end of the initial stage determining ion trajectory trapping (figure 5(c)). The potential $K'_A$ has a fast transitory decay, followed by a regime of continuous increase of the average $K'_A$ (figure 5(a), black curve). During this stage, $K'_A > \lambda_c$ and both parameters increase with roughly the same rate. The ratio $K'_A / \lambda_c$ is approximately constant, which corresponds to small variation of the parameters of the trajectory structures (figure 5(c), the time interval [4, 6]).
The correlation lengths \( \lambda_i \) increase significantly and the dominant wave number \( k_{20} \) decreases (figure 5(b)). The evolution of the velocity is different from that of the potential as reflected in the normalized velocities \( V_1 \) and \( V_2 \), which are time dependent functions (see Figure 5).
Their evolution is actually determined by the change of the characteristics of the EC, because $V_i^2 \approx (k_i^2 n_i^2 / \lambda_i^1)^2$ and $V_2^2 \approx 1 / \lambda_i^2$.

A different evolution regime appears at $t > 6$ in the example presented in figure 5. The normalized potential drops and it saturates (figure 5(a)). The normalized velocity $V_1$ saturates and $V_2$ increases. The characteristics of the EC, $\lambda_i$ and $k_i^0$, also saturate (figure 5(b)). The condition for the existence of trajectory structures is no more fulfilled ($K_i^f$ is significantly smaller than $\lambda_i$). However, trapping becomes much stronger ($n_i$ and the size of the structures increase, as seen in figure 5(c)).

This rather non-trivial evolution of the drift turbulence can be understood by analyzing the nonlinear effects that influence the growth rate (27): the ion diffusion, the formation of ion trajectory structures and flows.

The evolution of the diffusion coefficients $D_1$ is presented in figure 6, where the amplitudes of the drift turbulence $\Phi^2$ and of the zonal flow potential $\Phi_{zf}^2$ are also shown. One can see that the zonal flow modes become unstable when trapping appears and they grow exponentially until $t \approx 6$. Their amplitude is negligible compared to that of the drift turbulence during this time interval. Thus, it is not expected an important role of the zonal flow modes at this stage.

The diffusion coefficient $D_2$ is much larger than $D_1$. It has a very large increase rate during the first transition into the nonlinear regime. The cause is ion trapping in the potential that drifts with the diamagnetic velocity, which determines a strong amplification of the diffusion along $V_i e_z$, as discusses in section 4.1 and represented in figure 2. The strong diffusion determines the transitory decay of the potential $K_i^f$ under the nonlinearity limit. Consequently, the structures of the ion trajectories are destroyed. The fraction of trapped trajectories and the size of the structures decrease when $K_i^f$ decreases (figure 3, for $t \approx 2.7$). The diffusion coefficient $D_2$ decreases due to both effects (the decrease of $\Phi$ and the reduction of structures). One can see in figures 6 and 5(c) that transitory peaks and decays appear around $t = 2.7$ for $D_2$, $\Phi$, $n_i$, and $s_i$. After that, the potential increases again and reaches a higher level in the nonlinear regime. The diffusion coefficients $D_2$ recovers large values, but without driving the decay process. The strong diffusion is supported by the potential, which increases at this stage instead of having a fast decay as in the previous event. The cause of the different behavior is the evolution of the parameters of the spectrum. The dominant wave number $k_i^0$ (maxima of the spectrum) and the width of the spectrum $\delta k_i \approx \sqrt{2 \pi} / \lambda_i$ decrease as seen in figure 5(b). As a consequence, the damping effect of the diffusion $k_i^2 D_i$ is weaker at small wave numbers and the growth rate of the modes remains positive.

The third time interval in the evolution is characterized by the presence of the zonal flow modes at amplitudes comparable to $\Phi$. The drift turbulence decays and saturates while the zonal flow amplitude increases. The EC of the drift potential also saturates ($\lambda_i$ and $k_i^0$ are roughly constant during this interval). The increase of $V_2$ shown in figure 5(a) is determined by the growing contribution of the zonal flow modes.

Zonal flow provide the explanation for the increase of trapping (figure 5(c)) that seems paradoxical in the condition of the decay of $K_i^f$ below the nonlinearity limit. The structure of the potential contour lines is modified by the zonal flow potential, which determines larger structures that capture the ions released from the micro-confinement determined by the (small scale) drift turbulence. The fraction of trapped ions and the size of the structures increase in correlation with the increase of $\Phi_{zf}$ (figures 5(c) and 6). A deformation of the trajectory
structures appears due to the stronger increase of \( s_2 \) that of \( s_1 \). They are initially elongated in the gradient direction and eventually they acquire a shape dominated by the zonal flow modes. The fraction of trapped ions strongly increases and, at \( t \approx 7 \), it overcomes the value \( n_{tr} = 1/2 \), that corresponds to negative growth rate of the drift modes \((27)\) even in the absence of diffusion.

The diffusion coefficients decrease in this stage due to the decay of the potential \( \Phi \). The increase of the diffusion \( D_2 \) determined by the zonal flow contribution discussed in \([29]\) is not observed here because the
negative growth rate leads to the transition in the quasilinear regime and the condition \( \Phi_k = \Phi > \lambda \) for this effect is not fulfilled. The decay of the drift turbulence is caused by the zonal flow modes through the amplification of trapping rather than by an increased diffusion.

The evolution of the spectrum of the drift turbulence is presented in figure 7(a), which shows the function \( S(0, k) \) at time intervals \( \Delta t = 1 \). The spectrum becomes narrower as time increases and its maxima move to smaller values of \( k \).

The EC of the potential is shown in figure 7(b). The oscillatory dependence on \( x_2 \) is determined by the narrow peaks of the spectrum. It determines the small diffusion along the density gradient \( D_1 < D_2 \) despite the amplitude of the stochastic velocity that is much larger in this direction \( \langle v_1 \rangle \gg \langle v_2 \rangle \).

The spectrum of the zonal flow modes \( S_kf (k) \) is shown in figure 7(c) at the same time moments as \( S(0, k) \). The two spectra have similar shapes, with the difference that the zonal flow modes have small tails at large \( k_1 \) while a very strong cut appears for the drift modes at large \( k_2 \). This is the consequence of the diffusive damping that is weak in the case of the zonal flow modes, as seen in equation (30). The dominant wave number of the zonal flow modes is around \( k_z \approx 1 \), and it has a weak decrease with time (as also seen in figure 5(b), black curve).

We note that the nonlinear processes found here are related to ion stochastic trapping in the structure of the potential. The growth and saturation of the zonal flows in drift turbulence were intensively studied in the last decades [7], but ion eddying was not identified as a source of zonal flow modes. It is completely different from the trapping of the drift modes represented as quasi-particles in the wave-kinetic approach (see the very recent papers [47, 48]).

7. Summary and conclusions

Particle trajectories in two-dimensional incompressible velocity fields can include both random and coherent aspects, which appear as random sequences of large jumps and trapping or eddying events. This complex behavior is determined by the Hamiltonian structure of the equation of motion, and it effectively appears when the stochastic potential is only weakly perturbed. The trapped trajectories have a high degree of coherence and they form intermittent quasi-coherent trajectory structures. Their size and life-time depend on the characteristic of the turbulence. The distribution of the displacements is non-Gaussian in the presence of these vortical structures. We have developed semi-analytical statistical methods for the study of such complex trajectories, the DTM [13] and the NSA [14]. It was shown that these methods provide very clear physical images on the nonlinear transport processes and reasonably good quantitative results.

The quasi-coherent structures, which actually produce a process of micro-confinement, have strong effects on the transport. The transport is determined by the long jumps, while the structures represent a diffusion reservoir, which leads to significant increase of the diffusion coefficients when the particles are released by a decorrelation mechanism. The transport process is completely different in the presence of structures in the sense that the dependence on the parameters is different.

The quasi-coherent structures have strong effect on the test modes on turbulent plasmas. This conclusion is drawn from an analytical study of the drift turbulence that includes the coherent aspects of the trajectories beside the random ones [15]. The frequencies and the growth rates of the test modes on strongly turbulent plasmas could be determined by the renormalized propagator procedure. A different perspective on important aspects of the physics of drift type turbulence in the strongly nonlinear regime is deduced. The main role in the nonlinear processes is shown to be played by the ion trajectory structures. The nonlinear damping of drift modes and the generation of zonal flow modes appear when trapping is strong due to the ion flows generated by the drift of the potential with the diamagnetic velocity.

These test particle and test mode studies determine the transport coefficients and the characteristics of the test modes for given statistical description of the turbulence, as functions of turbulence parameters. The conclusions on turbulence and transport evolution base on the analysis of these results are speculative, because they only suppose that the positive growth rates of the test modes eventually drive the turbulence in the nonlinear regime.

We show here that test particle and test mode studies can be connected in an ISC theoretical description of turbulence evolution. It is an approximate Lagrangian method that takes into account the quasi-coherent aspects of particle trajectories in turbulence. A computer code based on the ISC was developed for the drift turbulence in uniform magnetic field. The advantage of this approach is the clear physical image that can be deduced for rather complicated nonlinear processes. They are related to the basic problem of particle advection in turbulence.

The first results show that drift turbulence evolution can have different characteristics, which essentially depend on the drive parameter.
In the case of weak drive of the drift turbulence, the evolution is determined by the nonlinear effects of the ion diffusion. They consist not only of the diffusive damping that reduces the growth rate of the modes, but also of the modification of the spectrum. The dominant wave number decreases and the correlation length increase. In particular the increase of \( \lambda_{i} \) leads to values that are larger than those of the normalized potential \( K_{i} \) at any time in the evolution, which shows that trajectory trapping is negligible. The ion trajectories are random in this case, with negligible structures.

In the case of strong drive, both random and the quasi-coherent characteristics of the trajectories are generated during the evolution. The structures of trajectories influence the evolution of the drift turbulence by three mechanisms. All of them are connected to the quasi-coherent trajectory structures. One in the dependence of the growth rates of the test modes (27) on the parameters of the structures (the fraction of trapped trajectories \( n \) and the size of the structures \( s_{i} \)). The increase of \( n \) and of \( s_{i} \) determine the decrease of \( \tau_{g} \), which become negative for all wave numbers at \( n = 1 \) even for \( D_{2} = 0 \). The increase of the structures determines the generation of the zonal flow modes (as \( \tau_{g} \approx n_{s} \)), which represent the second possible influence on the drift turbulence. Their action is indirect and consists of the modification of the diffusion and/or of the structures. The third mechanism is provided by the diffusion, which can be strongly changed by the structures.

The results presented in the figures 5–7 show a strong action of the third mechanism at the beginning of the nonlinear stage, around \( t = 3 \). Trajectory trapping determines a strong increase of the diffusion coefficient \( D_{2} \), which drives the turbulence back in the quasilinear regime. The process last for a short time interval, and it is eliminated by the decrease of the wave numbers of the drift turbulence, which reduces strongly the diffusive damping in the growth rate (27). The decrease of the wave numbers is due to the narrowing of the spectrum as the effect of increased diffusion. Thus, a complex transient process of interaction appears at this stage, which is the result of increased diffusion which determines both the increase and the decrease of the diffusive damping. The other two mechanisms determine the evolution at later time when the zonal flow modes reach amplitudes that are comparable to \( \Phi \). The zonal flow modes determine the increase of the structures due the the modification of the potential. The fraction of trapped trajectories increases, the structures have larger size and they become elongated in the direction of the zona flow modes. This leads to the decay of the drift turbulence which acquire negative growth rates. The zonal flow effect is produced in the system studied here through the quasi-coherent structures, rather than through the diffusion.

In conclusion, the results obtained in this paper bring a new image on nonlinear processes associated to the generation of quasi-coherence in particle trajectories. They apply to the drift turbulence in a simple confinement geometry (uniform magnetic field), but we expect that similar processes can be found in more realistic models of turbulence. A semi-analytical method that could be extended to other types of turbulence was developed.

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