NEUTRINO TRAPPING AND NEUTRINO MASS BOUNDS

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Abstract.
It has been shown recently that the exchange of virtual $\nu\bar{\nu}$ pairs leads to an unphysically large energy-density in neutron stars and white dwarfs, unless neutrinos have a minimum mass, $m_\nu \gtrsim 0.4$ eV/c$^2$. Here we consider the possibility that the presence of trapped low-energy neutrinos can suppress the exchange of virtual $\nu\bar{\nu}$ pairs, thereby avoiding a large energy-density even for massless neutrinos. We show that a) there can be subvolumes in a neutron star or white dwarf where neutrino-trapping does not take place, and which can thus have an unphysically large energy density, and b) even in those volumes where trapping does occur, the resulting suppression can be too small to alter the conclusion that neutrinos must have a minimum mass.

It is well known that the exchange of massless neutrino-antineutrino ($\nu\bar{\nu}$) pairs gives rise to a long-range interaction among electrons, protons, and neutrinos [1-7]. The 2-body potential between electrons, $V_{ee}^{(2)}(r)$, is given by [1,4,7]

$$V_{ee}^{(2)}(r) = \frac{G_F^2(2\sin^2\theta_W + \frac{1}{2})^2}{4\pi^3r^5} = 3 \times 10^{-82}eV \left(\frac{r}{1m}\right)^5. \quad (1)$$

Here $r = |\vec{r}_1 - \vec{r}_2|$ is the separation of the electrons, $G_F$ is the Fermi constant, and $\theta_W$ is the weak mixing angle. As can be seen from Eq.(1), the 2-body potential arising from neutrino-exchange is extremely weak, but its effects may nonetheless be detectable in equivalence principle experiments [6].
Neutrino-exchange can also lead to many-body interactions, and these have been studied by Feynman [2] and Hartle [3]. Among other observations, Feynman noted that higher order (in $G_F$) many-body interactions could be important because they depend on higher powers of the masses of the interacting objects. It has been shown more recently [7] that the self-energy of a compact object such as a white dwarf or neutron star arising from many-body neutrino-exchange interactions can become unphysically large, when calculated in the standard model. In the absence of some suppression mechanism, one is led to the conclusion that all neutrinos must have a minimum mass,

$$m_\nu \gtrsim 0.4 \text{ eV/c}^2. \quad (2)$$

In order to understand more clearly what any suppression mechanism would have to achieve, we briefly review the argument leading to the bound in Eq.(2). The self-energy of a spherical neutron star arising from the interaction of 4 particles, for example, must be of order $(1/R)(G_F/R^2)^4$. Since the number of 4-particle interactions that can arise among $N$ particles (where $N = \mathcal{O}(10^{57})$) is given by the binomial coefficient

$$\left( \frac{N}{4} \right) = \frac{N!}{4!(N-4)!} \approx \frac{N^4}{4!}, \quad (3)$$

the 4-body contribution $W^{(4)}$ to the total energy $W = \sum_k W^{(k)}$ is of order

$$W^{(4)} \sim \frac{1}{4!} \frac{1}{R} \left( \frac{G_F N}{R^2} \right)^4. \quad (4)$$

For a typical neutron star $(G_F N/R^2) = \mathcal{O}(10^{13})$, and hence it follows that the analogous contributions from the interactions of 6, 8,.. particles would rapidly become very large, and eventually exceed the known mass of the neutron star. For later purposes it is helpful to note that the neutrino-exchange energy in a subvolume of neutron star matter of radius $r_o \cong 1.7 \times 10^{-5}$ cm would be greater than the total mass of the neutron star.

In order to reduce the self-energy $W$ to a physically acceptable value, there must be a mechanism which suppresses the contributions from neutrino-exchange. The possibility explored in Ref.[7] is that neutrinos have a minimum mass (given in Eq.(2)), whose effect is to “saturate” the neutrino-exchange force, in analogy to nuclear forces. Smirnov and Vissani (SV) [8] have recently proposed another mechanism: Very low energy neutrinos trapped in the neutron star or white dwarf may suppress the exchange of the virtual neutrinos (via the Pauli principle) which give rise to the neutrino-exchange force. Neutrino trapping has been considered by Loeb [9], and more recently by Kiers and Weiss [10].

In qualitative terms, trapping can occur when the kinetic energy of neutrinos in a medium is less than their binding energy in the medium,
which is of order $\sqrt{2}G_F \rho \approx 50$ eV. (Since the sign of the potential energy for $\nu$ in a medium is opposite to that for $\bar{\nu}$ to order $G_F$, $\nu$ can be bound in the medium while $\bar{\nu}$ is repelled.) Under these circumstances a neutrino can have a total energy which is negative, and hence cannot escape from the medium to empty space where its total energy would be positive (see Fig. 1). Trapping can be described either in terms of a variable index of refraction in the medium [9], or as the quantum-mechanical scattering of neutrinos from a potential [10]. In either picture, however, there is no guarantee that a significant number of neutrinos will be trapped in every subvolume of the neutron star, especially one as small as $r_o$. Consider, for example, the “bubble” region shown in Fig. 1, i.e., a region of relatively low density which may form temporarily. Neutrinos from outside this region cannot enter the “bubble” for the same reason that they cannot escape into empty space. Contrariwise, neutrinos produced inside the “bubble” can escape the “bubble” region. The “bubble” can thus be viewed as a region of “anti-trapping”, i.e., where there are few trapped neutrinos but where the neutron density is nonetheless large enough to produce an unphysically large neutrino-exchange energy.

Another argument against the SV mechanism focuses on the time required for the weak interactions to produce an equilibrium distribution of trapped neutrinos. Suppose that the matter distribution in a neutron star suddenly changes, either due to a “starquake” or to the accretion of additional matter. The time $\tau_1$ required for the many-body potential to build up in a region of dimension $R$ is of order $R/c$, which for $R = r_o = 1.7 \times 10^{-5}$ cm is $\tau_1 \approx 6 \times 10^{-16}$ s. This estimate follows by noting that in the Schwinger formalism, the neutron-star “vacuum” produces a $\nu \bar{\nu}$ loop which simultaneously couples to all the neutrons located in a region of size $R \approx c\tau_1$, via many-body interactions. By contrast, the time required for the weak interactions to produce an equilibrium distribution of low-energy neutrinos via elastic neutrino scattering [11] is of order $\tau_2 = (\sigma_{\nu n} \rho c)^{-1}$, where $\sigma_{\nu n}$
is the weak $\nu n$ cross section, and $\rho$ is the number density of neutrons. For $E_\nu = 10$ eV, $\lambda_n \equiv (\sigma_{\nu n} \rho)^{-1} \cong 10^{10}$ km, and $\tau_2 \cong 4 \times 10^4$ s. Other estimates for the $\tau_2$ give similar results. We conclude from this argument that the many-body neutrino-exchange potential, and its associated energy density, comes into being on a very much shorter time scale than does the trapped neutrino distribution. Hence a distribution of trapped neutrinos will not always be present to suppress many-body neutrino exchange. This discussion is sufficient to demonstrate that one cannot rely on the presence of trapped neutrinos to suppress the large neutrino-exchange energy. In what follows it is shown that even in regions where trapped neutrinos are present they do not necessarily suppress the effect calculated in Ref. [7] sufficiently to avoid the conclusion that there is a lower bound on neutrino mass.

We begin by summarizing the argument of Smirnov and Vissani [8]. In the notation of Ref.[7] the vacuum propagator for a massless neutrino of energy $E$ is

$$S^{(0)}_{F} (\vec{r}_{ij}, E) = \gamma \cdot \eta \left[ \frac{i}{4\pi} \frac{e^{iE(r+i\epsilon)}}{r + i\epsilon} \right]$$

(5)

where $r = |\vec{r}_{ij}| = |\vec{r}_{i} - \vec{r}_{j}|$, and $\gamma \cdot \eta \equiv \vec{\gamma} \cdot \vec{\partial} - \gamma_4 E$. Eq.(5) leads directly to the 2-body potential $V^{(2)}(r)$ between neutrons arising from neutrino exchange, which is given by Eq.(1) with $(2 \sin^2 \theta_W + 1/2) \rightarrow a_n$, where $a_n = -1/2$ is neutrino-neutron coupling constant in the standard model. In the presence of a background neutrino sea at low temperature, the expression in square brackets Eq.(5) acquires an additional contribution given by

$$n_\nu \frac{\sin |E|(r+i\epsilon)}{2\pi r + i\epsilon}.$$  

(6)

The neutrino distribution function $n_\nu$ is given by

$$n_\nu = \frac{\theta(E)}{e^{(|E|-\mu)/k_BT} + 1} \frac{T^{1/2}}{\theta(E)\theta(\mu - |E|)},$$

(7)

where $k_B$ is the Boltzmann constant, $T$ is the temperature, and $\mu$ is the strength of the potential trapping the neutrinos. The effect of the additional term proportional to $n_\nu$ is to replace $V^{(2)}$ by $\tilde{V}^{(2)}$ where [5]

$$\tilde{V}^{(2)}(r) = \frac{G^2_F a_n^2}{4\pi^3 r^5} \{\cos(2\mu r) + \mu r \sin(2\mu r)\}.$$  

(8)

Smirnov and Vissani then argue that when $\tilde{V}^{(2)}(r)$ is integrated over a spherical matter distribution such as a neutron star, the arguments of the cosine and sine functions will oscillate rapidly, so that $U^{(2)} = \int d^3r \tilde{V}^{(2)}(r)$ will average to zero. Since the same considerations would apply to all the $k$-body potentials $V^{(k)} (r_{12}, r_{13}, r_{14}, ...)$, the implication is that the total energy $W = \sum_k W^{(k)}$ would be small, even for massless neutrinos.
At the heart of the above argument is the Riemann-Lebesque (RL) theorem [12]: If \( \int_a^b d\theta \psi(\theta) \) exists, and if \( \psi(\theta) \) has limited total fluctuation in the interval \( (a, b) \) then for \( \lambda \to \infty \)

\[
I(\lambda) \equiv \int_a^b d\theta \psi(\theta) \sin(\lambda\theta) = \mathcal{O}(1/\lambda). \tag{9}
\]

In the limit \( \lambda \to \infty \), \( I(\lambda) \to 0 \) which establishes the theorem in its oft-used form. [We note that the RL theorem ensures that \( I(\lambda) \) tends to zero at least as fast as \( 1/\lambda \). It may turn out that coefficient of the \( 1/\lambda \) term vanishes, so that \( I(\lambda) \) tends to zero even faster than \( 1/\lambda \), e.g. as \( 1/\lambda^2 \).] In the present context \( \lambda = \mu R \), where \( R = 10 \) km is the radius of the neutron star, and \( \mu = \sqrt{2} G_F \rho = 50 \) eV. It follows that \( 1/\lambda = 1/\mu R \cong 10^{-12} \) is indeed small, but is not zero. Moreover, \( 1/\lambda \) multiplies a factor \( (G_F \rho R) \) which is itself very large. Hence one cannot establish without a more careful analysis whether the \( 1/\mu R \) suppression is sufficient to reduce \( W \) to a physically acceptable value without having to introduce a neutrino mass.

To anticipate the results described below, we note from Eq.(8) that the term containing \( \sin(2\mu r) \) has a coefficient which is proportional to \( \mu \). Since the integration over \( r \) introduces a factor \( 1/\mu^n \) \( (n \geq 1) \) as demanded by the RL theorem, the net suppression of this term will be of order \( 1/\mu^{n-1} \). Hence if \( n = 1 \) there is no suppression at all of the term proportional to \( \sin(2\mu r) \). As we demonstrate below, the leading 2-body contribution \( W^{(2)} \) to \( W \) does in fact correspond to \( n = 1 \), so that there is no suppression at all of \( W^{(2)} \) from trapped neutrinos.

Following the discussion of Ref.[7], the average interaction energy \( \bar{U}^{(2)} \) of a single pair of neutrons having a uniform probability distribution in a spherical neutron star of radius \( R \) is given by

\[
\bar{U}^{(2)} = \int_{r_c}^{2R} dr \mathcal{P}(r) \tilde{V}^{(2)}(r). \tag{10}
\]

Here \( r_c \) is the neutron hard-core radius, and \( \mathcal{P}(r) \) is the probability density for finding two points a distance \( r \) apart in a sphere of radius \( R \) [7],

\[
\mathcal{P}(r) = \frac{3r^2}{R^3} - \frac{9}{4} \frac{r^3}{R^4} + \frac{3}{16} \frac{r^5}{R^6}. \tag{11}
\]

Combining Eqs.(8) and (11) we find
\[
\hat{U}^{(2)} = \frac{3G_F^2 a_n^2}{8\pi^3 R^3} \left[ \frac{3}{8R^2} \cos(4\mu R) + \frac{1}{r_c^2} \cos(2\mu r_c) - \frac{3}{2r_c R} \cos(2\mu r_c) + \frac{r_c}{16R^3} \cos(2\mu r_c) + \frac{3}{32\mu R^3} \sin(4\mu R) \right. \\
- \left. \frac{3}{32\mu R^3} \sin(2\mu r_c) + \frac{3\mu}{2R} \text{Si}(4\mu R) - \frac{3\mu}{2R} \text{Si}(2\mu r_c) \right].
\]

(12)

Here \( \text{Si}(z) \) is the sine integral function defined by
\[
\text{Si}(z) = \int_0^z dt \frac{\sin t}{t}.
\]

(13)

For \( \mu r_c \cong 1 \times 10^{-7} \ll 1 \) we have
\[
\hat{U}^{(2)} \approx \frac{3}{8\pi^3 r_c^2 R^3}.
\]

(14)

which is the same result found in Ref.\cite{7} for the case of no neutrino background \((\mu = 0)\). The 2-body case demonstrates that even when trapped neutrinos are present their effects may be too small to significantly modify the conclusions of Ref. \cite{7}.

The results for the many-body contributions \( W^{(k)} \) \((k \geq 4)\) can be analyzed in a similar fashion. In the absence of any effects due to trapped neutrinos, the 4-body potential \( V^{(4)} \) is given by the expression in Eq.(3.50) of Ref.\cite{7}. For present purposes it is sufficient to study the contribution from the zero-derivative term \( V^{(4)}_0 \) in \( V^{(4)} \) given by
\[
V^{(4)}_0 = \left( \frac{G_F a_n}{\sqrt{2}} \right)^4 \frac{3!}{\pi^5} \frac{1}{P_4 S_4} \left( r_{12} r_{23} r_{34} r_{41} \right) \text{ with } S_4 = r_{12} + r_{23} + r_{34} + r_{41}.
\]

(15)

The effect of \( n_\nu \) in (6) is to replace \( V^{(4)}_0 \) by
\[
\tilde{V}^{(4)}_0 = \left( \frac{G_F a_n}{2\pi^3 P_4 S_4} \right)^4 \left\{ \cos(\mu S_4) \left( \frac{3}{S_4^2} - \frac{3\mu^2}{2S_4^2} + \frac{\mu^4}{8} \right) \\
+ \frac{\mu \sin(\mu S_4)}{S_4} \left( \frac{3}{S_4^2} - \frac{\mu^2}{2} \right) \right\}.
\]

(16)

As in the 2-body case, the coefficients of the oscillatory terms \( \cos(\mu S_4) \) and \( \sin(\mu S_4) \) contain explicit powers of \( \mu \) which offset powers of \( 1/\mu \) that will
Figure 2. (a) Self-energy of neutrons arising from neutrino exchange. Lighter lines denote neutrinos, and heavier lines are neutrons. (b) Many-body contributions to $\nu\bar{\nu}$ production in a medium. (c) Many-body contributions to neutrino scattering and the MSW effect.

arise when $\tilde{V}_o^{(4)}$ is integrated over a spherical volume. This integration can be carried out in analogy to the 2-body case by writing

$$\tilde{U}_o^{(4)} = \int dr_{12} dr_{23} dr_{34} dr_{41} \mathcal{P}_4(r_{12}, r_{23}, r_{34}, r_{41}) \tilde{V}_o^{(4)}(r_{12}, r_{23}, r_{34}, r_{41})$$ (17)

where $\mathcal{P}_4(...)$ is the 4-body analog of $\mathcal{P}(r)$ in Eq.(11). Even though the functional form of $\mathcal{P}_4(r_{12}, r_{23}, r_{34}, r_{41})$ is not known, we can infer from the preceding discussion of the 2-body case that the four integrations in (17) will in general introduce a suppression factor of $(1/\mu R)^4$, via the RL theorem. However, this factor will be canceled by the explicit factor of $\mu^4$ which appears as a coefficient of $\cos(\mu S_4)$ in Eq.(16). Although it is possible that a term with a larger suppression factor, say $(1/\mu R)^5$, might be the dominant contribution, there is no reason to expect this for a general matter distribution. Moreover, even if a suppression from trapped neutrinos does arise, one cannot ensure that it would be sufficient to offset the large energy density obtained in Ref. [7].

Many-body effects may be important in other neutrino processes, as shown in Fig. 2. Fig. 2(a) exhibits the $k$-body contribution to the self-energy of a neutron star which was analyzed in Ref.[7]. By cutting the neutrino line one obtains Fig. 2(b) which represents the many-body contribution to $\nu\bar{\nu}$ production, and Fig. 2(c) which is the many-body contribution to elastic neutrino scattering or to the MSW effect [13]. If neutrinos are massless then all of these diagrams share the same enhancement factor arising from $k$-body combinatorics, and hence these may be the dominant contributions to the corresponding processes. If this were the case, then the trapping mechanism itself would have to be revisited since for $k$ even the many-body contributions could have the property of repelling both $\nu$ and $\bar{\nu}$. 

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To summarize, although low-energy neutrinos can be trapped under some conditions in a neutron star [9,10], one cannot guarantee that trapped neutrinos will be present in every subvolume of a neutron star or white dwarf, during every stage in the evolution of these objects. In an inhomogeneous medium subvolumes of lower density (“bubbles”) can be regions where “anti-trapping” occurs: $\nu\bar{\nu}$ pairs produced in this region can escape, whereas neutrinos outside the region cannot enter. Since such regions would contain few trapped neutrinos, the mechanism for suppressing neutrino-exchange forces proposed by Smirnov and Vissani [8] would not apply. More importantly, even in regions where neutrinos as trapped, their effect in suppressing neutrino-exchange forces may be too small to avoid the necessity for introducing a minimum neutrino mass.

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