Answering a question on relative countable paracompactness

M. V. Matveev
Department of Mathematics, University of California, Davis, Davis, CA, 95616, USA (address valid till June 30, 2000)
e-mail misha_matveev@hotmail.com

In [6], Yoshikazu Yasui formulates some results on relative countable paracompactness and poses some questions. Like it is the case with many other topological properties [1], countable paracompactness has several possible relativizations. Thus a subspace \( Y \subset X \) is called \textit{countably 1-paracompact} in \( X \) provided for every countable open cover \( U \) of \( X \) there is an open cover \( V \) of \( X \) which refines \( U \) and is locally finite at the points of \( Y \) (i.e. every point of \( Y \) has a neighbourhood in \( X \) that meets only finitely many elements of \( V \)). Yasui asserts that if a countably compact space \( Y \) is closed in a normal space \( X \) then \( Y \) is countably 1-paracompact in \( X \) and asks (Problem 1 in [6]) if normality can be omitted. The answer is negative as it is demonstrated by \( X = (\omega_1 + 1) \times (\omega + 1)) \setminus \{(\omega_1, \omega)\}, Y = \omega_1 \times \{\omega\} \) and \( U = \{\omega_1 \times (\omega + 1)\} \cup \{(\omega_1 + 1) \times \{n\} : n \in \omega\}. This well-known construction provides also the following general statement (recall that a space is Linearly Lindelöf iff every uncountable set of regular cardinality has a complete accumulation point see e.g. [2]).

\textbf{Theorem 1} If a Tychonoff space \( Y \) is countably 1-paracompact in every Tychonoff space \( X \) that contains \( Y \) as a closed subspace then \( Y \) is linearly Lindelöf.

\textbf{Proof:} Suppose not. Then there is an uncountable set \( Z \subset Y \) of regular cardinality and without complete accumulation points in \( Y \). Enumerate \( Z \) as \( \{z_\alpha : \alpha < \kappa\} \) where \( \kappa = |Z| \) and put \( Z^* = Z \cup \{z^*\} \) where \( z^* \notin Z \). Further, put \( X = (Z^* \times \omega) \cup (Y \times \{\omega\}) \). Topologize \( X \) as follows. The points of \( Z \times \omega \) are isolated. A basic neighbourhood of a point \( (z^*, n) \), where \( n \in \omega \), takes the form \( \{(z_\gamma, n) : \gamma > \alpha\} \cup \{(z^*, n)\} \) where \( \alpha < \kappa \). A basic neighbourhood
of a point \((y, \omega) \in Y \times \{\omega\}\) takes the form \(O_{U \cap Z} \times \{\omega\} \) where \(U\) is a neighbourhood of \(y\) in \(Y\) and \(n \in \omega\). Then \(X\) contains a closed subspace \(\tilde{Y} = Y \times \{\omega\}\) homeomorphic to \(Y\).

Now we check that \(X\) is a Tychonoff space. It is clear that the points of \(Z^* \times \omega\) have local bases consisting of clopen sets. So let \(x = (y, \omega) \in \tilde{Y}\) and let \(O\) be a neighbourhood of \(x\) in \(X\). Then there are a neighbourhood \(U\) of \(y\) in \(Y\) and \(n \in \omega\) such that \(x \in O \subset U \cap Z\). Further, there is a neighbourhood \(V\) of \(y\) in \(Y\) such that \(y \in V \subset U\) and \(|V \cap Z| < \kappa\). Since \(Y\) is Tychonoff, there is a function \(f : Y \to \mathbb{R}\) such that \(f(y) = 0\) and \(f(Y \setminus V) = \{1\}\).

Define a function \(\tilde{f} : X \to \mathbb{R}\) as

\[ \tilde{f}(u) = \begin{cases} f(v), & \text{if } u = (v, \omega) \\ f(z), & \text{if } u = (z, m) \text{ and } m > n \\ 1, & \text{otherwise.} \end{cases} \]

Then \(\tilde{f}\) is continuous; this follows from the inclusion \(\tilde{f}^{-1}(\mathbb{R} \setminus \{1\}) \subset (Z \times \omega) \cup (Y \times \{\omega\})\). Finally, \(\tilde{f}(x) = 0\) and \(\tilde{f}(X \setminus U) = \{1\}\). So \(X\) is Tychonoff.

It follows from the inequality \(\text{cf}(\kappa) > \omega\) that the open cover \(U = \{(Z \times \omega) \cup (Y \times \{\omega\})\} \cup \{Z^* \times \{n\} : n \in \omega\}\) of \(X\) does not have an open, locally finite refinement at all points of \(\tilde{Y}\). So \(\tilde{Y} \sim Y\) is not countably 1-paracompact in \(X\). ✷

**Theorem 2** A Tychonoff countably compact space \(Y\) is countably 1-paracompact in every Tychonoff \(X \supset Y\) iff \(Y\) is compact.

**Proof:** Necessity follows from the previous theorem and the fact that every countably compact (in fact, even every countably paracompact, see [3]) linearly Lindelöf space is compact.

Routinous proof of sufficiency is omitted. ✷

**Theorem 3** If a Lindelöf space \(Y\) is a closed subspace of a regular space \(X\) then \(Y\) is countably 1-paracompact in \(X\).

**Proof:** Let \(\mathcal{U} = \{U_n : n \in \omega\}\) be a countable open cover of \(X\). For every \(y \in Y\) fix \(n(y) \in \omega\) and an open set \(W_y \subset X\) so that \(y \in W_y \subset W_y \subset U_{n(y)}\). The cover \(\mathcal{W} = \{W_y : y \in Y\}\) contains a countable subcover of \(Y\), say \(\{W_{y_k} : k \in \omega\}\). Then \(\mathcal{V} = \{U_{n(y_k)} \setminus \bigcup\{W_{y_l} : l < k\} : k \in \omega\} \cup \{U_n \setminus \bigcup\{W_{y_l} : l < n\} : n \in \omega\}\) is an open refinement of \(\mathcal{U}\) and \(\mathcal{V}\) is locally finite at all points of \(Y\). ✷
Remark 1 In Theorem 1 and Theorem 2 one can replace “Tychonoff” with “regular”

Remark 2 The results above are similar to some results about normality and property (a) from [3], [4].

The paper was written when the author was visiting University of California, Davis. The author expresses his gratitude to colleagues from UC Davis for their kind hospitality.

References

[1] A. V. Arhangel’skii, *Relative topological properties and relative topological spaces*, Top. Appl. 70 (1996) 87-99.

[2] A. V. Arhangel’skii and R. Z. Buzyakova, *On linearly Lindelöf and strongly discretely Lindelöf spaces*, Proc. Amer. Math. Soc. 127 (1999) 2449-2458.

[3] A. Bella and I. V. Yaschenko, *Lindelöf property and absolute embeddings*, Proc. Amer. Math. Soc. 127 (1999) 907-913.

[4] M. V. Matveev, O. I. Pavlov and J. K. Tartir, *On relatively normal spaces, relatively regular spaces, and on relative property (a)*, Top. Appl. 93 (1999) 121-129.

[5] A. S. Mischenko, *Finally compact spaces*, Soviet Math. Dokl., 145 (1962) 1199-1202.

[6] Y. Yasui, *Results on relatively countably paracompact spaces*, Q & A in Gen. Topol. 17 (1999) 165-174.