Abstract

The $V$-algebras are the non-local matrix generalization of the well-known $W$-algebras. Their classical realizations are given by the second Poisson brackets associated with the matrix pseudodifferential operators. In this paper, by using the general Miura transformation, we give the decomposition theorems for the second Poisson brackets, from which we are able to construct the free field realizations for a class of $V$-algebras including $V_{(2k,2)}$-algebras that corresponds to the Lie algebra of $C_k$-type as the particular examples. The reduction of our discussion to the scalar case provides the similar result for the $W_{BKP}$-algebra.
Recently, Bilal [1-3] has proposed non-local matrix generalizations of $W$-algebras, called $V_{(m,n)}$-algebras. Their classical realizations appears naturally as the second Hamiltonian structures associated with the matrix version of the $m^{\text{th}}$-order differential operators

$$L = \partial^m + \sum_{j=0}^{m-2} U_j \partial^j$$

where $U_j$ are $n \times n$ matrix valued functions. The simplest example is the $V_{(2,2)}$-algebra. It arises in the study of the non-abelian Toda field theory as a model for strings propagating on a black hole background [1] and its Poisson bracket version can be given by the second Hamiltonian structure associated with [2]

$$L = \partial^2 - U, \quad U = \begin{pmatrix} T & -\sqrt{2} V^+ \\ -\sqrt{2} V^- & T \end{pmatrix}$$

The free field realization of the $V_{(2,2)}$-algebra was constructed first by the factorization

$$L = \partial^2 - U = (\partial + P)(\partial - P)$$

such that

$$U = P' + P^2$$

and then by expressing $P$ in terms of the vertex-like operators. In analogy to the scalar case, (4) is called the Miura transformation.

In general, the Miura transformation can be given by the similar factorization [3]

$$L = \partial^m + \sum_{j=0}^{m-2} U_j \partial^j = \prod_{j=1}^{m} (\partial + P_j)$$

and relates the second Poisson bracket of $U_j$ to much simpler ones of $P_j$, where $P_j$ are $n \times n$ matrices and satisfy the constraint $\sum P_j = 0$. However to the contrary of the scalar (i.e. $n = 1$) case, the $P_j$ are not free fields in general, and as far as we know, except $V_{(2,2)}$-algebras, it is not clear how to give the free field realization for general $V_{(m,n)}$-algebra. As pointed by Bilal [3], the reason of this difference between the $W$-algebras and their matrix generalization $V$-algebras is the existence of non-local terms in the $V$-algebras.
In this paper, we first generalize Bilal’s $V_{(m,n)}$-algebra to the $V$-algebra associated with the $m^{th}$-order matrix pseudodifferential operator (matrix $\Psi DO$)

$$L = \partial^m + \sum_{j=-\infty}^{m-1} U_j \partial^j$$

where $U_j$ are $n \times n$ matrix valued functions, then give a general decomposition theorem for the second Poisson bracket associated with (6) by using the factorization $L = L_1L_2$, where both $L_1$ and $L_2$ are matrix $\Psi$DOs with the order being $m_1$ and $m_2$ respectively and satisfying $m_1 + m_2 = m$. It is not difficult to generalize the factorization to the rational form $L = L_1L_2^{-1}$ since we may think that $L_1$ is factorized by $L_1 = LL_2$. The above discussion is nothing but matrix generalization of our previous work on the scalar case [4,5]. Thirdly, we consider the free field realization of a more general class of $V$-algebras that correspond to the $L$ of (6) with $2 \times 2$ matrix coefficients and satisfying $L = L^*$ for a proper defined adjoint action of the matrix $\Psi$DOs. Finally, as a consequence, when we restrict to the scalar case, we obtain the $W$-algebras represented by the second Poisson brackets of the BKP hierarchy and their free field realization.

Let $L$ in (6) be the $m^{th}$-order matrix $\Psi$DO. For any functional

$$\tilde{f} = \int f(U_{m-1}, U_{m-2}, \cdots) dx$$

we define

$$\frac{\delta f}{\delta L} = \sum_{j=-\infty}^{m-1} \partial^{-j-1} \frac{\delta f}{\delta U_j}$$

and

$$(\frac{\delta f}{\delta U_j})_{\alpha\beta} = \sum_{r=0}^{\infty} (-1)^r \frac{\partial^r}{\partial x^r} \frac{\partial f}{\partial (U_j^{(r)})_{\beta\alpha}}$$

is the matrix version of the Euler variation, where $(U_j^{(r)})_{\beta\alpha}$ denotes the $(\beta, \alpha)$ matrix element of $r^{th}$ derivative of $U_j$. Using (7) we find

$$d\tilde{f} = <\frac{\delta f}{\delta L}, \delta L>$$

where the product $<\cdot, \cdot>$ is defined by

$$<A, X> = \int \text{tr res} AX dx$$
for any two matrix \( \Psi DOs \) of the form
\[
A = \sum_{j=-\infty}^{m-1} A_j \partial^j \quad \text{and} \quad X = \sum_{j=-\infty}^{m-1} \partial^{-j-1} X_j.
\]
As in the scalar case, \( A \) corresponds to the “vector field” \( \partial_A \) and \( X \) is called “one form” paired by (10) with the vector field. The residues in (10) is defined to be the coefficient of \( \partial^{-1} \) term.

According to [3], the second Poisson bracket associated with (6) can be defined in analogy with the scalar case [6]
\[
\{ \tilde{f}, \tilde{g} \}_{L} = \langle H \left( \frac{\delta f}{\delta L} \right), \frac{\delta g}{\delta L} \rangle \tag{11}
\]
where
\[
H(X) = (LX)_+ L - L(XL)_- = L(XL)_- - (LX)_+ L \tag{12}
\]
mapping an one form \( X = \sum_{j=-\infty}^{m-1} \partial^{-j-1} X_j \) to the vector field \( \partial_{H(X)} \), where the subscripts “\( \pm \)” are understood as the pure differential part or the residual part of the \( \Psi DO \). If \( U_{m-1} = 0 \) is assumed the following condition
\[
\text{res} \left[ \frac{\delta f}{\delta L}, L \right] = 0 \tag{13}
\]

must be taken into account such that the leading coefficient of \( \frac{\delta f}{\delta L} \) is expressed in terms of others. The second Poisson bracket (11) constrained to \( U_{m-1} = 0 \) is called the \( V \)-algebra [1-3].

The bracket (11) is bilinear and anti-symmetric because of the apparent properties of the product (10). It will follow from the results on the Miura transformation that for a class of matrix \( \Psi DOs \) used in this paper, (11) also obeys the Jacobi identity. Nevertheless the following theorem does not depend on the property of the Jacobi identity.

**Theorem 1** By the factorization
\[
L = L_1 L_2 \tag{14}
\]
where
\[
L_i = \partial^{m_i} + \sum_{j=-\infty}^{m_i-1} U_{ij} \partial^j, \quad i = 1, 2 \tag{15}
\]
are \( m_i \text{th} \)-order matrix \( \Psi DOs \) with \( m_1 + m_2 = m \), then the Poisson bracket associated with \( L \) of (6) is decomposed to the summation of two brackets that are associated with \( L_1 \) and \( L_2 \) respectively
\[
\{ \tilde{f}, \tilde{g} \}_L = \{ \tilde{f}, \tilde{g} \}_L_1 + \{ \tilde{f}, \tilde{g} \}_L_2 \tag{16}
\]
If $U_{m-1} = U_{m_1-1} + U_{m_2-1} = 0$ is assumed, then (13) is equivalent to

$$\text{res}\left[ \frac{\delta f}{\delta L_1}, L_1 \right] + \text{res}\left[ \frac{\delta f}{\delta L_2}, \delta L_2 \right] = 0$$

(17)

The proof of this theorem is essentially the same as we shown for the scalar case in [4,5], i.e. by (14) any functional $\tilde{f}$ of $U_j$ is also a functional of $U_{1j}$ and $U_{2j}$, therefore on the one hand we have

$$d\tilde{f} = \int \text{tr} \left( \frac{\delta f}{\delta L} \delta L dx = \int \text{tr} \left( \frac{\delta f}{\delta L}(\delta L_1 L_2 + L_1 \delta L_2) dx \right)$$

and on the other hand

$$d\tilde{f} = \int \text{tr} \left( \frac{\delta f}{\delta L_1} \delta L_1 + \frac{\delta f}{\delta L_2} \delta L_2 \right) dx$$

The above two expression imply that

$$\frac{\delta f}{\delta L_1} = L_2 \frac{\delta f}{\delta L}, \quad \frac{\delta f}{\delta L_2} = \frac{\delta f}{\delta L_1} L_1$$

(18)

each of them modular an $(-m_1 - 1)^{th}$-order and $(-m_2 - 1)^{th}$-order matrix $\Psi DO$ respectively. Substitute (18) to the right hand side of (16) and by the same calculation as that in [4,5], we can prove the theorem.

It is easy to generalize Theorem 1 to the factorization $L = L_1 \cdots L_r$, in particular if $r = m$ and $L_j = \partial + P_j$, we immediately recover the result of Bilal [3]

$$\{ \tilde{f}, \tilde{g} \}_L = \sum_{j=1}^{m} \int \text{tr} \left[ \frac{\delta f}{\delta P_j}, \partial + P_j \right] \frac{\delta g}{\delta P_j} dx$$

(19)

since the second Poisson bracket associated with $L_j = \partial + P_j$ is simply

$$\{ \tilde{f}, \tilde{g} \}_{P_j} = \int \text{tr} \left[ \frac{\delta f}{\delta P_j}, \partial + P_j \right] \frac{\delta g}{\delta P_j} dx$$

(20)

The constraint $U_{m-1} = \sum P_j = 0$ is then equivalent to

$$\sum_{j=1}^{m} \left( \frac{\delta f}{\delta P_j} \right)' = 0$$

(21)
Theorem 2 If

\[ L = L_1 L_2^{-1} \]  

(22)

where for the simplicity we assume that \( L_1 \) and \( L_2 \) are \((m+k)\text{-th}\)-order and \( k\text{-th}\)-order matrix differential operators respectively, then

\[ \{ \tilde{f}, \tilde{g} \}_L = \{ \tilde{f}, \tilde{g} \}_{L_1} - \{ \tilde{f}, \tilde{g} \}_{L_2} \]  

(23)

The proof of this theorem can be completed simply by considering that \( L_1 = LL_2 \) is factorized and then by applying Theorem 1. The scalar version of the factorization \( L = L_1 L_2^{-1} \) was appeared in \([7,8]\) for the study of \( W \)-algebras.

In the following we discuss the reduction of the second Poisson bracket (11) to the subspace of matrix \( \Psi \)DOs that satisfy \( L = L^* \). For the matrix \( \Psi \)DOs \( A = \sum A_j \partial^j \), we define the matrix version of the adjoint action on \( A \) by

\[ A^* = \sum (-\partial)^j \sigma A_j^T \sigma^{-1} \]  

(24)

where “\( T \)” denotes the matrix transposition, \( \sigma \) is an \( n \times n \) constant matrix such that the adjoint action satisfies

\[
\begin{align*}
(A^*)^* &= A \\
(AB)^* &= B^* A^* \\
(A^*)_+ &= (A_+)^* \\
\int \text{tr res} A^* dx &= - \int \text{tr res} A dx
\end{align*}
\]

(25)

It is easy to see that such a matrix can be chosen freely as long as \( \sigma \) is symmetric.

Let

\[ W = L - L^* = \sum W_j \partial^j \]  

(26)

where

\[ W_j = U_j - \sum_{i=j}^{m-1} (-1)^i \binom{i}{i-j} \sigma \frac{\partial^{i-j} U_i^T}{\partial x^{i-j}} \sigma^{-1} \]  

(27)

then we can calculate that

\[ \frac{\delta (W_j)_{\alpha \beta}}{\delta L} = (E_{\beta \alpha} - (-1)^j \sigma E_{\alpha \beta} \sigma^{-1}) \partial^{-j-1} \]  

(28)
are symmetric

\[
\frac{\delta(W_j)_{\alpha\beta}}{\delta L} = (\frac{\delta(W_j)_{\alpha\beta}}{\delta L^*})^*
\]  

(29)

with respect to the matrix version of adjoint action, where \(E_{\alpha\beta}\) are the \(n \times n\) matrices only with the \((\alpha, \beta)^{th}\) matrix element being equal to one and others to zero.

If we suppose that \(m = 2k\) and \(L\) is symmetric

\[L = L^*\]

(i.e. \(W_j = 0\)), then the “vector fields” \(\partial_A\) on the submanifold \(W_j = 0\) will be parametrized by the deformations of \(L\) that remain symmetric. These \(A\) are clearly the matrix \(\Psi DOs\) of order at most \(2k - 1\) obeying the symmetric property \(A = A^*\). The “one forms” \(X = \sum \partial^{-j-1}X_j\) on the submanifold \(W_j = 0\) must be chosen to be those which are mapped via the Hamiltonian map \(H\) to the vector fields \(\partial_{H(X)}\) tangent to the submanifold \(W_j = 0\). In other words, \((H(X))^* = H(X)\). Since

\[(H(X))^* = -H(X^*),\]

(31)

\(X\) must be anti-symmetric \(X = -X^*\) modular \(a (-m - 1)^{th}\) order of matrix \(\Psi DO\) (i.e. the kernel of \(H\)). It can easily be checked that these one forms are nondegenerately paired with the vector fields \(\partial_A\), \(A = A^*\). Actually we have checked that for some simple cases for any functional \(\tilde{f} = \int fdx\) restricted on \(W_j = 0\), \(X = \frac{\delta f}{\delta L}\) really satisfies \(X = -X^*\) modular the kernel of \(H\).

Therefore the Poisson bracket of two functionals \(\tilde{f} = \int fdx\) and \(\tilde{g} = \int gdx\) on the submanifold can be given by

\[
\{\tilde{f}, \tilde{g}\}_L = \frac{1}{4} < H(\frac{\delta f}{\delta L} - (\frac{\delta f}{\delta L})^*), \frac{\delta g}{\delta L} - (\frac{\delta g}{\delta L})^*> = \frac{1}{4} < H(\frac{\delta f}{\delta L} - (\frac{\delta f}{\delta L})^*), \frac{\delta g}{\delta L} >
\]  

(32)

with \(L\) being symmetric.

The above argument is an analogue of that for the supersymmetric BKP hierarchy [9]. The following theorem will provide another argument.

**Theorem 3.** If the \(m^{th}\)-order \((m = 2k)\) symmetric matrix \(\Psi DO\) \(L\) is factorized by

\[L = L_1^*L_1\]

(33)
with
\[ L_1 = \partial^k + \sum_{j=-\infty}^{k-1} V_j \partial^j \]  
then we have
\[ \{ \tilde{f}, \tilde{g} \}_L = \frac{1}{2} \{ \tilde{f}, \tilde{g} \}_{L_1} \]  
\[ \textbf{Proof:} \] Any functional of \( U_{m-1}, U_{m-2}, \cdots \) is also a functional of \( V_{k-1}, V_{k-2}, \cdots \) via the relation of (33). Therefore
\[ d\bar{f} = \langle \frac{\delta f}{\delta L}, \delta L \rangle = \langle \frac{\delta f}{\delta L_1}, \delta L_1 \rangle \]  
so
\[ \frac{\delta f}{\delta L_1} = (\langle \frac{\delta f}{\delta L}, \delta L \rangle) - (\langle \frac{\delta f}{\delta L}, \delta L \rangle)^* L_1^* \]  
modulo a \((-k-1)^{\text{th}}\)-order matrix \( \Psi DO \).

Substitute this expression to the Poisson bracket \( \{ \cdot, \cdot \}_L \) with respect to \( L_1 \) we have the Poisson bracket (32) with respect to \( L \), which can be expressed by (35).

We may continue to factorize \( L_1 \)
\[ L_1 = \prod_{j=1}^{l}(\partial + P_j)^{-1} \prod_{j=l+1}^{k+2l}(\partial + P_j) \]  
where \( l \) is an arbitrary integer, and then apply Theorem 2 and 3, we find that the Poisson bracket (32) of \( L = L_1^* L_1 \) becomes
\[ \{ \tilde{f}, \tilde{g} \}_L = \frac{1}{2} \sum_{j=l+1}^{k+2l} \{ \tilde{f}, \tilde{g} \}_P_j - \frac{1}{2} \sum_{j=1}^{l} \{ \tilde{f}, \tilde{g} \}_P_j \]  
with each \( \{ \tilde{f}, \tilde{g} \}_P_j \) being given by (20).

Let us now calculate the coefficient of the second leading term of \( L = L_1^* L_1 \) with \( L_1 \) being in (38). It is
\[ U_{m-1} = (-1)^k \sum_{j=l+1}^{k+2l} (P_j - \sigma P_j^T \sigma^{-1}) - (-1)^l \sum_{j=1}^{l} (P_j - \sigma P_j^T \sigma^{-1}) \]  
8
We immediately find that a sufficient condition of \( U_{m-1} = 0 \) is
\[
P_j - \sigma P_j^T \sigma^{-1} = 0, \; j = 1, 2, \ldots, k + 2l
\]
(41)
namely the restriction of the Poisson bracket (32) of \( L \) to the submanifold \( U_{m-1} = 0 \) can be realized if each copy of the Poisson bracket in the form of (20) associated with \( \partial + P_j \) can be restricted to the submanifold of (41).

According to the above analysis, we choose \( n = 2 \),
\[
\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
(42)
and \( k + 2l \) copies of Bilal’s \( V_{(2,2)} \)-algebra
\[
P_j = \begin{pmatrix} T_j & -\sqrt{2}V_j^+ \\ -\sqrt{2}V_j^- & T_j \end{pmatrix}, \; 1 \leq j \leq k + 2l
\]
(43)
among them the first \( l \) copies have a sign difference with the \( V_{(2,2)} \)-algebra. It is obvious that \( P_j \) obey (41) and their elements can be expressed in terms of \( k + 2l \) independent groups of vertex-like fields. Thus we may construct the free field realization of the \( V \)-algebra that corresponds to the second Poisson bracket on the space of \( m \)-th-order (\( m = 2k \)) and \( 2 \times 2 \) matrix \( \Psi \)DOs restricted by \( L = L^\ast \). A simple case is for \( l = 0 \), i.e. if \( L_1 \) is a pure differential operator, so
\[
L = L_1^\ast L_1 = (-1)^k (\partial - P_k) \cdots (\partial - P_1)(\partial + P_1)(\partial + P_k)
\]
(44)
the \( V_{(2k,2)} \)-algebra in this case corresponds to the Lie algebra of the \( C_k \)-type [3]. Our result gives its free field realization. Note that from mathematical point of view, the \((-1)^k\) factor does not affect the structure of Poisson bracket essentially.

Finally we are going to restrict the above results to the scalar case \( n = 1 \) and connect the Poisson bracket (32) for \( n = 1 \) with the Poisson bracket for the BKP hierarchy. Let
\[
\Lambda = \partial^{2k+1} + \sum_{j=-\infty}^{2k} v_j \partial^j
\]
(45)
where \( v_j \) are scalar functions. Then we define
\[
L = \partial \Lambda = \partial^{2k+2} + \sum_{j=-\infty}^{2k+1} u_j \partial^j
\]
(46)
The relation between $v_j$ and $u_j$ can be given explicitly

$$
u_{2k+1} = v_{2k}$$
$$u_j = v'_j + v_{j-1}$$

from which we first have

$$\frac{\partial f}{\partial v_j^{(l)}} = \frac{\partial f}{\partial u_j^{(l)}} + \frac{\partial f}{\partial u_j^{(l-1)}}$$

and so

$$\frac{\delta f}{\delta v_j} = \frac{\delta f}{\delta u_j^{(l-1)}} - (\frac{\delta f}{\delta u_j})'$$

which implies that

$$\frac{\delta f}{\delta L} = \frac{\delta f}{\delta \Lambda} \partial^{-1} + (a (-2k - 3)^{th}\text{-order} \: \Psi DO)$$

If we assume that $\Lambda$ is the $\Psi DO$ associated with the BKP hierarchy [10]

$$\Lambda^* = -\partial \Lambda \partial^{-1}$$

then $L$ in (46) is symmetric $L = L^*$ and $u_{2k+1} = v_{2k} = 0$ where the adjoint action on the scalar $\Psi DO$ $A = \sum a_j \partial^j$ is defined as usual $A^* = \sum (-\partial)^j a_j$. Substitute (48) into (32) for $n = 1$ we notice that the second term of the right hand side does not contribute anything and the Poisson bracket in terms of $\Lambda$ is given by

$$\{\tilde{f}, \tilde{g}\}_\Lambda = \int \text{res}[\partial^{-1}(\partial \Lambda \frac{\delta f}{\delta \Lambda} - \Lambda (\frac{\delta f}{\delta \Lambda})_+ \partial \Lambda - \Lambda (\frac{\delta f}{\delta \Lambda})_+ \partial \Lambda]$$

We define $W_{BKP}^{(2k+1)}$-algebra corresponding to the second Poisson bracket (50) associated with the BKP hierarchy. Its free field realization is given by the following factorization

$$\Lambda = \partial^{-1} L_1^* L_1$$

with

$$L_1 = \prod_{j=1}^{l} (\partial + p_j)^{-1} \prod_{j=1+1}^{k+1+2l} (\partial + p_j)$$
where $p_j$ are independent fields and satisfy

$$
\{p_i(x), p_j(y)\} = -\delta_{ij}\delta'(x - y) \quad 1 \leq i, j \leq l
$$

$$
\{p_i(x), p_j(y)\} = \delta_{ij}\delta'(x - y) \quad l + 1 \leq i, j \leq k + 1 + 2l
$$

$$
\{p_i(x), p_j(y)\} = 0 \quad 1 \leq i \leq l, \quad l + 1 \leq j \leq k + 1 + 2l
$$

(53)

In conclusion we have discussed the properties of the second Poisson structure associated with the matrix $\Psi DO$. These properties enable us to construct the free field realizations for a more general class of $V$-algebras that correspond to the second Poisson brackets of matrix $\Psi DO$. Because of the non-locality of the $V$-algebras, the free field realizations for them become more difficult than for $W$-algebras. It would be of interest to investigate these problems for the general $V$-algebras.

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