Large time behavior of solutions to a diffusion approximation radiation hydrodynamics model

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Abstract: This paper concerns with the large time behavior of solutions to a diffusion approximation radiation hydrodynamics model when the initial data is a small perturbation around an equilibrium state. The global-in-time well-posedness of solutions is achieved in Sobolev spaces depending on the Littlewood-Paley decomposition technique together with certain elaborate energy estimates in frequency space. Moreover, the optimal decay rate of the solution is also yielded provided the initial data also satisfy an additional $L^1$ condition. Meanwhile, the similar results of the diffusion approximation system without the thermal conductivity could be also established.

Key Words: Diffusion approximate radiation hydrodynamics model, global well-posedness, large time behavior, Littlewood-Paley decomposition

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1 Introduction

1.1 Background and motivation

The radiation hydrodynamics system describes the coupling effect between the macroscopic description of the fluid and the statistical character of the massless photons. It finds many applications in modelling the combustion, high-temperature hydrodynamics and gaseous stars in astrophysics etc. The radiative effect is necessary to be included into the hydrodynamics equations for the high-temperature fluid, because the energy and momentum carried by the radiation field are significant in comparison with those carried by the macroscopic fluid. The readers may consult the monographs [21, 24] for more details.

Here the macroscopic fluid is described by the compressible Navier-Stokes-Fourier system which represents the conservation of mass, momentum and energy, while the motion of photons is described by the dynamics of the radiation field which is incorporated in a scalar quantity: the radiative intensity $I = I(t, x, \bar{\omega}, \nu)$. It represents the radiative intensity of the photon which moves in the direction vector $\bar{\omega} \in S^{n-1}$ ($S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$) with frequency $\nu \geq 0$ at the position $x$ and time $t$. The time evolution of $I$ is governed by a transport equation with a source term due to the absorbing and the scattering effects of the photons. That is,

$$\frac{1}{C} \partial_t I + \bar{\omega} \cdot \nabla_x I = S,$$

where $C$ is the light speed and the source term $S$ is given by

$$S = \sigma_a(B(\nu, \theta) - I) + \sigma_s \left( \frac{1}{|S^{n-1}|} \int_{S^{n-1}} I(\cdot, \bar{\omega})d\bar{\omega} - I \right).$$

The absorption coefficient $\sigma_a = \sigma_a(\nu, \rho, \theta)$ and the scattering coefficient $\sigma_s = \sigma_s(\nu, \rho, \theta)$ usually depend on the temperature $\theta$ and density $\rho$ of the macroscopic fluid. $B(\nu, \theta) \geq 0$ is the equilibrium thermal distribution of radiative intensity, $|S^{n-1}|$ is the measure of $S^{n-1}$.
The collective effect of radiation is expressed in terms of integral means (with respect to the variables \( \bar{\omega} \) and \( \nu \)) of quantities depending on \( I \).

The coupled system of the compressible Navier-Stokes-Fourier equations with the transport equation above is usually hard for both analysis and numerical simulation. So some simplified but effective models are proposed. Based on the following two assumptions, the above transport equation could be approximated for the sake of mathematical analysis and computation. One assumption is the *grey* approximation, it means that the transport coefficients \( \sigma_a, \sigma_s \) are assumed to be independent of the frequency \( \nu \); Another one is called *P1-approximation*, it is assumed that the radiative intensity \( I \) could be expanded as a linear function with respect to the angular variable \( \bar{\omega} \), that is,

\[
I = I_0 + \bar{\omega} \cdot \bar{I}_1,
\]

where \( I_0 \) and \( \bar{I}_1 \) are independent of \( \bar{\omega} \) and \( \nu \). Taking the \( \{1, \bar{\omega}\} \)-moments of equation (1.1) and integrating the resultant equation over \( \bar{\omega} \in S^{n-1} \) and \( \nu \in \mathbb{R}^+ \), it yields that

\[
\frac{1}{C} \partial_t I_0 + \frac{1}{n} \text{div} \bar{I}_1 = S_E, \quad \frac{1}{C} \partial_t \bar{I}_1 + \nabla I_0 = \bar{S}_F.
\]

Here the radiation energy and radiation flux are given by

\[
S_E = \frac{1}{C} \int_0^\infty \int_{S^{n-1}} S(\cdot, \nu, \bar{\omega}) d\bar{\omega} d\nu, \quad \text{and} \quad \bar{S}_F = \int_0^\infty \int_{S^{n-1}} \bar{\omega} S(\cdot, \nu, \bar{\omega}) d\bar{\omega} d\nu.
\]

The motion of the macroscopic fluid with the photons is achieved through additional extra source terms in the balance of momentum and energy. In the spacial dimension \( n = 3 \), the *P1-approximation* radiation hydrodynamics system takes the following form

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho \bar{u}) &= 0, \\
\partial_t (\rho \bar{u}) + \text{div}(\rho \bar{u} \otimes \bar{u}) + \nabla P &= \text{div} \mathbb{T} + \frac{1}{C} \mathcal{L}(\sigma_a + \sigma_s) \bar{I}_1, \\
\partial_t (\frac{1}{2} \rho |\bar{u}|^2 + \rho e) + \text{div}[\rho (\frac{1}{2} |\bar{u}|^2 + \rho e + P) \bar{u}] + \text{div} \bar{F} &= \text{div} (\mathbb{T} \cdot \bar{u}) - \mathcal{L} \sigma_a (b(\theta) - I_0), \\
\frac{1}{C} \partial_t I_0 + \frac{1}{3} \text{div} \bar{I}_1 &= \mathcal{L} \sigma_a (b(\theta) - I_0), \\
\frac{1}{C} \partial_t \bar{I}_1 + \nabla I_0 &= -\mathcal{L} (\sigma_a + \sigma_s) \bar{I}_1,
\end{aligned}
\]

for \((x, t) \in \mathbb{R}^3 \times [0, +\infty)\). Here \( \mathcal{L} \), \( \sigma_a \) and \( \sigma_s \) are positive dimensionless parameters related to the radiation field. The pressure of the macroscopic fluid takes the form of \( P = \frac{2}{3} \rho e \) and inner energy \( e = \frac{2}{3} R \theta \) with the density \( \rho \) and the temperature \( \theta \), \( \bar{u} \) is velocity vector. Without loss of generality, we assume the constant \( R = 1 \) for simplicity. \( \mathbb{T} \) stands for the viscous stress tensor determined by Newton’s rheological law.

\[
\mathbb{T} = \mu (\nabla u + \nabla^T u) + \lambda \text{div} u \mathbb{I}_{3 \times 3},
\]

where \( \mu \) is the shear viscosity coefficient and \( \lambda = \zeta - \frac{2}{3} \mu \) with the bulk viscosity coefficient \( \zeta \geq 0 \). \( \mathbb{I}_{3 \times 3} \) is the \( 3 \times 3 \) identity matrix. The heat flux \( \bar{F} = -\kappa \nabla \theta \) satisfies the Fourier
principle with the thermal conductivity $\kappa$. The smooth function $b(\theta)$ is the integral of $B(\nu, \theta)$ with respect to the frequency $\nu$, for example $b(\theta) = \theta^4$ for the case that $B = \frac{2h \nu^3}{k^2} (e^{\frac{k T}{h \nu}} - 1)^{-1}$ with Planck and Boltzmann constants $h$ and $k$ respectively.

In general, the first order corrector function $I_1$ changes very small with respect to time $t$ for the “almost” isotropic case. In this way, we can assume that $\partial_t I_1 = 0$ in the fifth equation in (1.2). Thus $I_0$ and $I_1$ satisfy the following Fick’s principle, that is

$$-\nabla I_0 = \mathcal{L}(\sigma_a + \sigma_s)I_1.$$ 

So, we will obtain the diffusion approximation radiation hydrodynamics system.

$$\begin{cases}
\partial_t \rho + \text{div}(\rho \bar{u}) = 0, \\
\rho \partial_t \bar{u} + (\rho \bar{u} \cdot \nabla) \bar{u} + \nabla P = \text{div} \mathcal{T} - \frac{1}{\kappa} \nabla I_0, \\
\frac{3}{2} \rho \partial_t \theta + P \text{div} \bar{u} - \kappa \Delta \theta + \mathcal{L}\sigma_a(b(\theta) - I_0) = \mathcal{T} \cdot \nabla \bar{u} + \frac{1}{\kappa} \bar{u} \cdot \nabla I_0 - \frac{3}{2} \rho \bar{u} \cdot \nabla \theta, \\
\frac{1}{\kappa} \partial_t I_0 - \frac{\mathcal{L}(\sigma_a + \sigma_s)}{\kappa} \Delta I_0 = \mathcal{L}\sigma_a(b(\theta) - I_0). 
\end{cases}$$

(1.3)

In this paper we consider the Cauchy problem for the diffusion approximation radiation hydrodynamics system of equations (1.3) with the following initial data.

$$(\rho, \bar{u}, \theta, I_0)(x, t)|_{t=0} = (\rho_0, \bar{u}_0, \theta_0, I_0^0)(x) \to (1, \bar{0}, 1, b(1)), \quad \text{as } |x| \to +\infty,$$

(1.4)

where $(1, \bar{0}, 1, b(1))$ is the equilibrium state of the system (1.3). For simplicity, we assume that the given smooth function $b(\theta)$ satisfies a natural physical assumption of $b'(1) > 0$.

Before proceeding, let us review the related known results. System (1.3) is reduced to the classical non-isentropic compressible Navier-Stokes-Fourier equations if we ignore the radiative effect. It is well-known that the strong dissipative property admits global solutions to the non-isentropic compressible Navier-Stokes-Fourier equations, see [17, 20, 35] for the global existence results with the small perturbation initial data, [20, 36] for the pointwise asymptotic behaviors of the solutions, and [21] for the global well-posedness of the equations even without the heat conductivity (i.e. the parameter $\kappa = 0$). One also refers to [17, 29] and references therein for related results of Navier-Stokes-Fourier equation in exterior domain. In addition, the solutions are proved to blow up in finite time when the initial data has compact support besides it is large in some Sobolev spaces, see [35].

When the radiation effect is taken into account, system (1.3) can be viewed as the Navier-Stokes-Fourier equations coupled with an parabolic equation with high order nonlinear term with respect to the temperature $\theta$. In the absence of both the viscosity and the heat conductivity, by denoting $q := -\nabla I_0$ and ignoring $\partial_t I_0$, system (1.3) can be formulated as the following well-known system of the radiation hydrodynamics.

$$\begin{cases}
\partial_t \rho + \text{div}(\rho \bar{u}) = 0, \\
\rho \partial_t \bar{u} + (\rho \bar{u} \cdot \nabla) \bar{u} + \nabla P = 0, \\
\frac{3}{2} \rho \partial_t \theta + P \text{div} \bar{u} + \frac{1}{\kappa} \bar{u} \cdot \nabla q = -\frac{3}{2} \rho \bar{u} \cdot \nabla \theta, \\
-\frac{1}{\kappa} \nabla \cdot \nabla \bar{u} + \mathcal{L}\sigma_a q + \mathcal{L}\sigma_a \nabla b(\theta) = 0.
\end{cases}$$

(1.5)
There are many results providing insight into the existence, uniqueness, asymptotic behavior and decay rates of solutions to the model (1.5) together with the related “baby model”.

\[
\begin{align*}
\partial_t u + \text{div} f(u) + \text{div} q &= 0, \\
-\nabla \text{div} q + q + \nabla u &= 0.
\end{align*}
\]  

(1.6)

We refer the readers to [8, 11, 10, 12, 13, 14, 15, 16, 18, 19, 25, 26, 27, 31, 32, 33, 37, 38] and the references therein. Let us come back to the radiation hydrodynamic systems (1.2) and (1.3) again. Danchin and Ducomet studied the following P1-approximation radiation hydrodynamics model in [5].

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \vec{u}) &= 0, \\
\partial_t (\rho \vec{u}) + \text{div}(\rho \vec{u} \otimes \vec{u}) + \frac{1}{(Ma)^2} \nabla P(\rho) &= \frac{1}{\kappa_c} \text{div} \mathbb{T} + \frac{1}{3} \mathcal{L}(\sigma_s + \sigma_a) \vec{I}, \\
\frac{1}{\kappa_c} \partial_t I_0 + \frac{1}{3} \text{div} \vec{I} &= \mathcal{L} \sigma_a (b(\rho) - I_0), \\
\frac{1}{\kappa_c} \partial_t \vec{I} + \nabla I_0 &= -\mathcal{L}(\sigma_a + \sigma_s) \vec{I}.
\end{align*}
\]  

(1.7)

Where the global-in-time existence of strong small perturbation solution was established in the critical Besov spaces $\dot{B}_{2,1}^{3/2}(\mathbb{R}^n)$. Moreover, the global existence of the solution in critical Besov space for radiation hydrodynamics model (1.2) also has been achieved in [6] recently. It is noticed that the large time behavior of the solutions have not been given in both of these two papers. Later, the global well-posedness and the large time behavior of the smooth solution to (1.7) in Sobolev space also have been studied by the same authors of this paper in [34]. It is shown that the interaction between fluid and the radiation effect could produce partial damping effect on the system. Together with the viscosity on the velocity, the existence and the time-decay rate of the solution could be obtained. The damping effect has also been observed in the baby model of the radiating gas (1.6) by Kawashima in [14].

As mentioned above, the energy carried by the radiation field usually dominates the total energy for the high-temperature fluids. Consequently, the energy equation should be taken into account for the precise description of the motion of the high-temperature fluids. In this way, system (1.3) is more important and more interesting in mathematical analysis from the physical point of view. In this paper, we mainly study the global in time existence and the large time behavior of the solution to the Cauchy problem (1.3)-(1.4) when the initial data is a small perturbation around the constant equilibria $(1, 0, 1, b(1))$.

\subsection{Main results}

Now, it is position to state the main results in the following theorems.

\begin{theorem}[Global existence] \label{thm:global_existence}
Assume that there is a small positive constant $\varepsilon_0$, such that the initial data satisfies

\[\|(\rho_0 - 1, \vec{u}_0, \theta_0 - 1, I_0 - b(1))\|_{H^4(\mathbb{R}^3)} \leq \varepsilon_0.\]  

(1.8)
\end{theorem}
Then the initial value problem (1.3) and (1.4) admits a global unique solution \((\rho, \bar{u}, \theta, I_0)\) in the following sense.

\[
\rho - 1 \in C^0(0, \infty; H^4(\mathbb{R}^3)) \cap C^1(0, \infty; H^3(\mathbb{R}^3)),
\]

\[
\bar{u}, \theta - 1, I_0 - b(1) \in C^0(0, \infty; H^4(\mathbb{R}^3)) \cap C^1(0, \infty; H^2(\mathbb{R}^3)).
\]

Moreover, there exists a positive constant \(C_0\) such that for any \(t \geq 0\), it holds

\[
\| (\rho - 1, \bar{u}, \theta - 1, I_0 - b(1))(t) \|_{H^4(\mathbb{R}^3)}^2
+ \int_0^t \left( \| b'(1)(\theta - 1) - (I_0 - b(1))(\tau) \|_{H^4(\mathbb{R}^3)}^2
+ \| \nabla \rho(\tau) \|_{H^4(\mathbb{R}^3)}^2 + \| \nabla (\bar{u}, \theta - 1, I_0 - b(1))(\tau) \|_{H^4(\mathbb{R}^3)}^2 \right) d\tau
\leq C_0 \| (\rho_0 - 1, \bar{u}_0, \theta_0 - 1, I_{0_0} - b(1)) \|_{H^4(\mathbb{R}^3)}^2.
\]

**Remark 1.1.** In Theorem 1.1, the appearance of the second line in (1.9) is due to the “damping” effect produced by the interaction between fluid and the radiation. This effect also provide some diffusion property in the combination of the unknown functions. In the process of proving, the main difficulty in establishing the global existence of solution for the model (1.3) comes from deriving the \(L^1_t L^1_x\)-norm estimate of \(\bar{\theta}(I_0 - b(1))\). The classical energy method doesn’t work here. We overcome this difficulty by dividing the solution \((\bar{\rho}, \bar{u}, \bar{\theta}, j_0)\) into three parts, i.e. the low frequency part, the medium frequency part and the high frequency part. By virtue of the “damping” effect and the diffusion structure of the system, the \(L^1_t L^1_x\)-norm estimate can be bounded in different frequency regions. It is worth pointing out that a suitable combination of the solution in low-frequency regimes enables us to achieve the desired a priori estimates and hence to establish the global existence of solution.

Moreover, the large time behavior of the solution is obtained in the following theorem.

**Theorem 1.2** (Large time behavior of solution). Under the assumption of Theorem 1.1, suppose further that \(\| (\rho_0 - 1, \bar{u}_0, \theta_0 - 1, I_{0_0} - b(1)) \|_{L^1(\mathbb{R}^3)}\) is bounded. Then there exists a positive constant \(\bar{C}_0\), such that

\[
\| \nabla^k (\rho - 1, \bar{u}, \theta - 1, I_0 - b(1))(t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_0 (1 + t)^{-\frac{3}{4} - \frac{k}{2}}, \quad \text{for } k = 0, 1, 2,
\]

\[
\| \nabla^k (\rho - 1, \bar{u}, \theta - 1, I_0 - b(1))(t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_0 (1 + t)^{-\frac{3}{4}}, \quad \text{for } k = 3, 4.
\]

Moreover,

\[
\| \partial_t (\rho - 1, \bar{u})(t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_0 (1 + t)^{-\frac{3}{4}},
\]

\[
\| \partial_t (\theta - 1, I_0 - b(1))(t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_0 (1 + t)^{-\frac{3}{4}},
\]

\[
\| \nabla^k [b'(1)(\theta - 1) - (I_0 - b(1))](t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_0 (1 + t)^{-\frac{3}{4} - \frac{k}{2}}, \quad \text{for } k = 0, 1, 2.
\]
Furthermore, we assume that  
\[ \| \tilde{\rho} \| \text{ is positive constant} \]  
which satisfies  
\[ (1.10) \]  
Then the initial value problem  
\begin{align*}
\frac{\partial}{\partial t} \rho &+ \text{div}(\rho \tilde{u}) = 0, \\
\rho \frac{\partial}{\partial t} \tilde{u} + (\rho \tilde{u} \cdot \nabla) \tilde{u} + \nabla P = \text{div} \mathbb{T} - \frac{1}{\kappa} \nabla I_0, \\
\frac{3}{2} \rho \frac{\partial}{\partial t} \theta + P \text{div} \tilde{u} + \mathcal{L} \sigma_a(b(\theta) - I_0) = (\mathbb{T} \cdot \nabla) \tilde{u} + \frac{1}{\kappa} \tilde{u} \cdot \nabla I_0 - \frac{3}{2} \rho \tilde{u} \cdot \nabla \theta, \\
\frac{1}{\kappa} \frac{\partial}{\partial t} I_0 - \frac{1}{3\kappa (\sigma_a + \sigma_c)} \Delta I_0 = \mathcal{L} \sigma_a(b(\theta) - I_0). 
\end{align*}
(1.10)
Precisely, we also consider the following system of equations  
\[ \begin{align*}
&\partial_t \rho + \text{div}(\rho \tilde{u}) = 0, \\
&\rho \partial_t \tilde{u} + (\rho \tilde{u} \cdot \nabla) \tilde{u} + \nabla P = \text{div} \mathbb{T} - \frac{1}{\kappa} \nabla I_0, \\
&\frac{3}{2} \rho \partial_t \theta + P \text{div} \tilde{u} + \mathcal{L} \sigma_a(b(\theta) - I_0) = (\mathbb{T} \cdot \nabla) \tilde{u} + \frac{1}{\kappa} \tilde{u} \cdot \nabla I_0 - \frac{3}{2} \rho \tilde{u} \cdot \nabla \theta, \\
&\frac{1}{\kappa} \partial_t I_0 - \frac{1}{3\kappa (\sigma_a + \sigma_c)} \Delta I_0 = \mathcal{L} \sigma_a(b(\theta) - I_0). 
\end{align*} \]

The main results are included in the following theorem.

**Theorem 1.3** (The case of $\kappa = 0$). Assume that there exists a small positive constant $\varepsilon_0$, such that the initial data satisfies  
\[ \| (\rho_0 - 1, \tilde{u}_0, \theta_0 - 1, I_0^0 - b(1)) \|_{H^4(\mathbb{R}^3)} \leq \varepsilon_0. \]  
(1.11)

Then the initial value problem (1.10) and (1.1) admits a unique global solution $(\rho, \tilde{u}, \theta, I_0)$, which satisfies  
\[ \rho - 1, \theta - 1 \in C^0(0, \infty; H^4(\mathbb{R}^3)) \cap C^1(0, \infty; H^3(\mathbb{R}^3)), \]
\[ \tilde{u}, I_0 - b(1) \in C^0(0, \infty; H^4(\mathbb{R}^3)) \cap C^1(0, \infty; H^2(\mathbb{R}^3)). \]
Moreover, there exists a positive constant $C_0'$ such that for any $t \geq 0$, it holds  
\[ \| (\rho - 1, \tilde{u}, \theta - 1, I_0 - b(1))(t) \|_{H^4(\mathbb{R}^3)}^2 \]
\[ + \int_0^t \left( \| \rho'((\theta - 1) - (I_0 - b(1)))(\tau) \|_{H^4(\mathbb{R}^3)}^2 + \| \nabla(\tilde{u}, I_0 - b(1))(\tau) \|_{H^4(\mathbb{R}^3)}^2 \right) \ d\tau \]
\[ \leq C_0'\| (\rho_0 - 1, \tilde{u}_0, \theta_0 - 1, I_0^0 - b(1)) \|_{H^4(\mathbb{R}^3)}^2. \]
Furthermore, we assume that $\| (\rho_0 - 1, \tilde{u}_0, \theta_0 - 1, I_0^0 - b(1)) \|_{L^1}$ is bound. Then there exists a positive constant $C_0'$ such that, for all $t \geq 0$  
\[ \| \nabla^k (\rho - 1, \tilde{u}, \theta - 1, I_0 - b(1))(t) \|_{L^2(\mathbb{R}^3)} \leq C_0'(1 + t)^{-\frac{3}{4} - \frac{k}{2}}, \quad \text{for } k = 0, 1, 2, \]
\[ \| \nabla^k (\rho - 1, \tilde{u}, \theta - 1, I_0 - b(1))(t) \|_{L^2(\mathbb{R}^3)} \leq C_0'(1 + t)^{-\frac{3}{4}}, \quad \text{for } k = 3, 4, \]
and
\[
\|\partial_t(\rho - 1, \bar{u})(t)\|_{L^2(\mathbb{R}^3)} \leq \tilde{C}_0(t + 1)^{-\frac{3}{4}},
\]
\[
\|\partial_t(\theta - 1, I_0 - b(1))(t)\|_{L^2(\mathbb{R}^3)} \leq \tilde{C}_0(t + 1)^{-\frac{3}{4}},
\]
\[
\|\nabla^k[b'(1)(\theta - 1) - (I_0 - b(1))](t)\|_{L^2(\mathbb{R}^3)} \leq \tilde{C}_0(t + 1)^{-\frac{3}{4} - \frac{k}{2}}, \quad \text{for } k = 0, 1, 2.
\]

**Remark 1.4.** In order to establish the global existence of solution to the system \((1.10)\), some essential observations of the structure from the system \((1.10)\) are needed. The main observation is that the dissipative property of \(\theta - 1\) comes from two aspects: the damping effect of the combination \(b'(1)(\theta - 1) - (I_0 - b(1))\) and the dissipative effect of \(I_0 - b(1)\). It is different from the work \([9]\) for Navier-Stokes-Fourier system without the heat conductivity, where the entropy doesn’t dissipate so it has no decay-in-time.

**Remark 1.5.** The results in Theorem 1.3 show that the decay properties of the solution to \((1.10)\) are the same as those of \((1.3)\). In fact, for the case of \(k = 0\), the eigenvalues of the matrix \(A(\xi)\) in \((1.32)\) with asymptotic expansion near \(0\) can be formulated as
\[
\pm i\rho \sqrt{\frac{8\gamma + 15Cb}{6\gamma + 9Cb}} + \left[ \frac{9\gamma}{(6\gamma + 9Cb)^2} + \frac{2a\gamma(\gamma + 3Cb)}{(8\gamma + 15Cb)(2\gamma + 3Cb)} + \frac{\nu}{2} \right] \varrho^2 + O(\varrho^3),
\]
and
\[
\frac{6a\gamma}{15Cb + 8\gamma} \varrho^2 + O(\varrho^3), \quad \frac{2\gamma}{3} + Cb + \frac{9Cba - \gamma}{6\gamma + 9Cb} \varrho^2 + O(\varrho^3).
\]

It enables us to follow the same procedure as in the heat conductivity case to obtain the large time behavior of solution to \((1.10)\).

### 1.3 Notations

Throughout this paper, \(C\) denotes the generic positive constant depending only on the initial data and physical coefficients but independent of time \(t\). For two quantities \(a\) and \(b\), we will employ the notation \(a \lesssim b\) to mean that \(a \leq Cb\) for a generic positive constant \(C\). And \(a \sim b\) means \(C^{-1}|b| \leq |a| \leq C|b|\). Moreover, the norms in nonhomogeneous Sobolev spaces \(H^s(\mathbb{R}^3)\) and \(W^{s,p}(\mathbb{R}^3)\) are denoted by \(\| \cdot \|_{H^s}\) and \(\| \cdot \|_{W^{s,p}}\) respectively for \(s \geq 0\) and \(p \geq 1\). \(\| \cdot \|_{H^s}\) denotes the norm in homogeneous Sobolev space \(\dot{H}(\mathbb{R}^3)\). As usual, \((\cdot)\) denotes the inner-product in \(L^2(\mathbb{R}^3)\). \(\nabla^m\) with an integer \(m \geq 0\) stands for the usual any spatial derivatives of order \(m\). In addition, we apply the Fourier transform to the variable \(x \in \mathbb{R}^3\) by \(\hat{f}(\xi, t) = \int_{\mathbb{R}^3} f(x, t)e^{-i\xi x}dx\) and the inverse Fourier transform to the variable \(\xi \in \mathbb{R}^3\) by \((\mathcal{F}^{-1}\hat{f})(x, t) = (2\pi)^{-3} \int_{\mathbb{R}^3} \hat{f}(\xi, t)e^{i\xi x}d\xi\).

The rest of this paper is organized in the following way. In Section 2, we reformulate the problem into a small perturbation frame. In Section 3, we derive a priori estimates in different frequency regimes and prove the global existence of the solution. The large time behavior of the solution is derived in Section 4. In Appendix, we give the definition of homogeneous Besov space and some useful inequalities.
2 Reformulations

Now, we linearize the system of equations (1.3) around the equilibrium \((1, \bar{\theta}, 0, 1, b(1))\). Set 
\[
\begin{align*}
\bar{\rho} &= \rho - 1, \\
\bar{u} &= \bar{u}, \\
\bar{\theta} &= \theta - 1 \\
j_0 &= I_0 - b(1),
\end{align*}
\]
we obtain
\[
\begin{align*}
\partial_t \bar{\rho} + \text{div} \bar{u} &= \hat{S}^1, \\
\partial_t \bar{u} + \nabla \bar{\rho} + \nabla \bar{\theta} + \frac{1}{3c} \nabla j_0 - \text{div} T &= \hat{S}^2, \\
\partial_t \bar{\theta} + \frac{2}{3} \text{div} \bar{u} - \frac{2}{3} \kappa \Delta \bar{\theta} + \frac{2}{3} \mathcal{L} \sigma_a (b'(1) \bar{\theta} - j_0) &= \hat{S}^3, \\
\partial_t j_0 - \frac{c}{3c(\sigma_a + \sigma_z)} \Delta j_0 - \mathcal{CL} \sigma_a (b'(1) \bar{\theta} - j_0) &= \hat{S}^4,
\end{align*}
\]
where \((\hat{S}^1, \hat{S}^2, \hat{S}^3, \hat{S}^4)\) are the nonlinear terms with
\[
\begin{align*}
\hat{S}^1 &= - \text{div}(\bar{\rho} \bar{u}), \\
\hat{S}^2 &= -(\bar{u} \cdot \nabla) \bar{u} - g(\bar{\rho}) \nabla \bar{\rho} - h(\bar{\rho}) \bar{\theta} \nabla \bar{\rho} + g(\bar{\rho}) \text{div} T - \frac{1}{3c} g(\bar{\rho}) \nabla j_0, \\
\hat{S}^3 &= - \frac{2}{3} \bar{\theta} \text{div} \bar{u} + \frac{2}{3} \kappa g(\bar{\rho}) \Delta \bar{\theta} - \frac{2}{3} \mathcal{L} \sigma_a (h(\bar{\rho}) (b(\bar{\theta} + 1) - b(1)) - b'(1) \bar{\theta}) \\
&\quad - \frac{2}{3} \mathcal{L} \sigma_a g(\bar{\rho}) (b'(1) \bar{\theta} - j_0) + \frac{2}{3} h(\bar{\rho}) (T \cdot \nabla) \cdot \bar{u} + \frac{2}{9c} h(\bar{\rho}) \bar{u} \cdot \nabla j_0 - \bar{u} \cdot \nabla \bar{\theta}, \\
\hat{S}^4 &= \mathcal{CL} \sigma_a (b(\bar{\theta} + 1) - b(1)) - b'(1) \bar{\theta}),
\end{align*}
\]
and
\[
g(\bar{\rho}) = \frac{1}{\bar{\rho} + 1} - 1, \quad h(\bar{\rho}) = \frac{1}{\bar{\rho} + 1}.
\]
The initial data is given accordingly as follows.
\[
(\bar{\rho}, \bar{u}, \bar{\theta}, j_0)(x, 0) = (\rho^0, \bar{u}^0, \bar{\theta}^0, j_0^0) \\
\quad := (\rho_0 - 1, \bar{u}_0, \theta_0 - 1, I_0 - b(1)) \rightarrow (0, \bar{\theta}, 0, 0), \quad \text{as } |x| \rightarrow +\infty. \tag{2.2}
\]

Next, we will consider the global existence of the solution \((\bar{\rho}, \bar{u}, \bar{\theta}, j_0)\) to (2.1) around the steady state \((0, \bar{\theta}, 0, 0)\). To this end, we define the function space used in this paper.

\[
X(0, T) = \left\{ (\bar{\rho}, \bar{u}, \bar{\theta}, j_0) \mid \bar{\rho} \in C^0(0, T; H^4(\mathbb{R}^3)) \cap C^1(0, T; H^3(\mathbb{R}^3)), \\
\bar{u}, \bar{\theta}, j_0 \in C^0(0, T; H^4(\mathbb{R}^3)) \cap C^1(0, T; H^2(\mathbb{R}^3)), \\
\nabla \bar{\rho} \in L^2(0, T; H^3(\mathbb{R}^3)), \quad \nabla \bar{u}, \nabla \bar{\theta}, \nabla j_0 \in L^2(0, T; H^4(\mathbb{R}^3)) \right\}.
\]

By the standard continuity argument, the global existence of solutions to the Cauchy problem (2.1) and (2.2) will be obtained by combining the local existence result with some uniform a priori estimates in Sobolev space \(H^4(\mathbb{R}^3)\).

Then, we state the local existence of smooth solutions to the Cauchy problem (2.1) and (2.2) as follows.
Proposition 2.1 (Local existence). Let \((\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0, \tilde{j}_0) \in H^4(\mathbb{R}^3)\) such that
\[
\inf_{x \in \mathbb{R}^3} \{\tilde{\rho}^0 + 1, \tilde{\theta}^0 + 1\} > 0.
\]
Then there exists a constant \(T_0 > 0\) depending on \(\| (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0, \tilde{j}_0) \|_{H^4(\mathbb{R}^3)}\), such that the initial-value problem (2.1) admits a unique solution \((\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{j}_0) \in X(0, T_0)\), which satisfies
\[
\inf_{x \in \mathbb{R}^3, 0 \leq t \leq T_0} \{\tilde{\rho} + 1, \tilde{\theta} + 1\} > 0.
\]

**Proof.** The proof can be done by using the standard iteration arguments and fixed point theorem. One also refer to [6] (see Section 2.1). We omit the details for simplicity of presentation. \(\square\)

The following proposition gives some uniform a priori estimates of smooth solutions to (2.1), which are the key parts in the proof of Theorem 1.1.

Proposition 2.2 (A priori estimates). Let \((\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0, \tilde{j}_0) \in H^4(\mathbb{R}^3)\). Suppose the initial value problem (2.1) and (2.2) has a solution \((\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{j}_0) \in X(0, T)\), where \(T\) is a positive constant. Then there exist a sufficient small positive constant \(\delta\) and a constant \(C_1 > 0\), which are independent of \(T\), such that the initial data satisfies
\[
\sup_{0 \leq t \leq T} \| (\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{j}_0)(t)\|_{H^4(\mathbb{R}^3)} \leq \delta,
\]
then for any \(t \in [0, T]\), the following estimate holds true:
\[
\begin{align*}
\| (\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{j}_0)(t)\|_{H^4(\mathbb{R}^3)}^2 &+ \int_0^t \left( \| (b'(1)\tilde{\theta} - \tilde{j}_0)(\tau)\|_{H^4(\mathbb{R}^3)}^2 
+ \| \nabla (\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{j}_0)(\tau)\|_{H^4(\mathbb{R}^3)}^2 \right) \, d\tau \\
&\leq C_1 \| (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0, \tilde{j}_0)(t)\|_{H^4(\mathbb{R}^3)}^2.
\end{align*}
\]

**Remark 2.1.** For the case that \(\kappa = 0\), the inequality (2.4) can be replaced by the following estimate.
\[
\begin{align*}
\| (\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{j}_0)(t)\|_{H^4(\mathbb{R}^3)}^2 &+ \int_0^t \left( \| (b'(1)\tilde{\theta} - \tilde{j}_0)(\tau)\|_{H^4(\mathbb{R}^3)}^2 
+ \| \nabla (\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{j}_0)(\tau)\|_{H^4(\mathbb{R}^3)}^2 \right) \, d\tau \\
&\leq C_1 \| (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0, \tilde{j}_0)(t)\|_{H^4(\mathbb{R}^3)}^2.
\end{align*}
\]

3 Global existence and uniqueness

In this section, we will show the global-in-time existence and uniqueness of solution to the Cauchy problem (2.1) and (2.2). Actually, we only need to derive the key a priori estimates stated in Proposition 2.2. Firstly, we drop the “tilde” and rewrite the unknown functions as
\((\rho, u, \theta, j_0)\) in (2.1) for simplicity of the notations. Then the system about \((\rho, u, \theta, j_0)\) can be written as follows.

\[
\begin{align*}
\partial_t \rho + \text{div} u &= S^1, \\
\partial_t u + \nabla \rho + \nabla \theta + \frac{1}{\kappa} \nabla j_0 - \text{div} T &= S^2, \\
\partial_t \theta + \frac{2}{3} \text{div} u - \frac{2}{3} \triangle \theta + \frac{2}{3} \mathcal{L} \sigma_a (b'(1) \theta - j_0) &= S^3, \\
\partial_t j_0 - \frac{C \gamma \triangle \theta}{3 \mathcal{L} (\sigma_a + \sigma_y)} \triangle j_0 - C \mathcal{L} \sigma_a (b'(1) \theta - j_0) &= S^4, \\
(\rho, u, \theta, j_0)(x,t)|_{t=0} &= (\rho^0, u^0, \theta^0, j^0_0),
\end{align*}
\]

(3.1)

where the source terms \((S^1, S^2, S^3, S^4) := (\tilde{S}^1, \tilde{S}^2, \tilde{S}^3, \tilde{S}^4)\) and the initial data \((\rho_0, u_0, \theta_0, j^0_0) := (\tilde{\rho}^0, \tilde{u}^0, \tilde{\theta}^0, j^0_0)\). Motivated by [5], we adopt the following notations.

\[
\Lambda =: (-\triangle)^{1/2}, \quad d := \Lambda^{-1} \text{div} u.
\]

(3.2)

One gets the identity \(u = -\Lambda^{-1} \nabla d - \Lambda^{-1} \text{div}(\Lambda^{-1} \text{curl} u)\) together with \(\text{div} u = \Lambda d\) and \((\text{curl} u)^j_i = \partial_j u^i - \partial_i u^j\). Setting \(\nu = \lambda + 2\mu, \gamma = \mathcal{L} \sigma_a b'(1), a = \frac{c}{3 \mathcal{L} (\sigma_a + \sigma_y)}\) and \(b = \mathcal{L} \sigma_a\), we get from (3.1) that

\[
\begin{align*}
\partial_t \rho + \Lambda d &= S^1, \\
\partial_t d - \Lambda \rho - \nu \triangle d - \Lambda \theta - \frac{1}{\kappa} \Lambda j_0 &= D, \\
\partial_t \theta + \frac{2}{3} \Lambda \triangle \theta - \frac{4}{3} \kappa \gamma \theta - \frac{2}{3} b j_0 &= S^3, \\
\partial_t j_0 - C \gamma \theta - a \triangle j_0 + C b j_0 &= S^4, \\
(\rho, d, \theta, j_0)(x,t)|_{t=0} &= (\rho^0, d^0, \theta^0, j^0_0)(x),
\end{align*}
\]

(3.3)

while \(P u = \Lambda^{-1} \text{curl} u\) where \(P\) is the projection operator on divergence-free vector fields. It satisfies

\[
\begin{align*}
\partial_t P u - \mu \triangle P u &= PS^2, \\
P u(x,t)|_{t=0} &= Pu^0(x),
\end{align*}
\]

(3.4)

with \(D := \Lambda^{-1} \text{div} S^2\) and \(d^0 := \Lambda^{-1} \text{div} u^0\). In fact, to derive the estimates of \(u\), we only need to estimate \(d\) and \(P u\). In what follows, we will give the corresponding analysis by the means of the homogeneous Littlewood-Paley decomposition \((\hat{\Lambda}_k)_{k \in \mathbb{Z}}\) (Pls. see Definition 5.1 in Appendix) and some Besov space techniques. For all \(k \in \mathbb{Z}\), applying the homogeneous frequency localized operator \(\hat{\Lambda}_k\) to (3.3) and (3.4) yields that

\[
\begin{align*}
\partial_t p_k + \Lambda d_k &= S^1_k, \\
\partial_t d_k - \Lambda p_k - \nu \triangle d_k - \Lambda \theta_k - \frac{1}{\kappa} \Lambda j_0 k &= D_k, \\
\partial_t \theta_k + \frac{2}{3} \Lambda d_k - \frac{4}{3} \kappa \triangle \theta_k + \frac{2}{3} \gamma \theta_k - \frac{2}{3} b j_0 k &= S^3_k, \\
\partial_t j_0 k - C \gamma \theta_k - a \triangle j_0 k + C b j_0 k &= S^4_k, \\
(\rho_k, d_k, \theta_k, j_0 k)(x,t)|_{t=0} &= (\rho^0_k, d^0_k, \theta^0_k, j^0_0 k)(x),
\end{align*}
\]

(3.5)
as well as
\[ \begin{align*}
\partial_t (P u)_k - \mu \Delta (P u)_k &= (P S^2)_k, \\
(P u)_k(x, t)|_{t=0} &= (P u_0)_k(x).
\end{align*} \tag{3.6} \]

Here \( \rho_k := \hat{\Delta}_k \rho \), \( d_k := \hat{\Delta}_k d \) and so on.

### 3.1 Estimates in the high-frequency regimes

In this subsection, we show the estimates of the solution to the linearized system of (3.5)–(3.6). From the system (3.5), one can get
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left( \rho_k^2 + d_k^2 + \frac{\theta_k^2}{2} + j_{0,k} \right) dx \]
\[ + \int_{\mathbb{R}^3} \left( \nu |\Delta d_k|^2 + \kappa |\Lambda \theta_k|^2 + |\theta_k|^2 + a |\Lambda j_{0,k}|^2 + C b |j_{0,k}|^2 \right) dx \]
\[ = (b + C \gamma) \int_{\mathbb{R}^3} j_{0,k} \theta_k dx + \frac{1}{3C} \int_{\mathbb{R}^3} \Lambda j_{0,k} d_k dx \]
\[ + \int_{\mathbb{R}^3} \left( \rho_k S_k^1 + d_k D_k + \frac{3}{2} \theta_k S_k^3 + j_{0,k} S_k^4 \right) dx. \tag{3.7} \]

At the same time, by the first two equations (3.5)–(3.6), we have
\[ \frac{d}{dt} \int_{\mathbb{R}^3} \left( \nu |\nabla \rho_k|^2 - \Lambda \rho_k d_k \right) dx \]
\[ = \int_{\mathbb{R}^3} \left( |\Delta d_k|^2 - |\Lambda \rho_k|^2 - \Lambda \rho_k \Lambda \theta_k - \frac{1}{3C} \Lambda \rho_k \Lambda j_{0,k} \right) dx \]
\[ + \int_{\mathbb{R}^3} \left( d_k \Lambda S_k^1 + \Lambda \rho_k D_k \right) dx + \nu \int_{\mathbb{R}^3} \nabla \rho_k \cdot \nabla S_k^1 dx. \tag{3.8} \]

Summing up (3.7) and \( \beta_1 \times (3.8) \) with a fixed constant \( \beta_1 \), one gets
\[ \frac{1}{2} \frac{d}{dt} \mathcal{L}_{h,k}(t) + \beta_1 \|\Lambda \rho_k\|^2_{L^2} + (\nu - \beta_1) \|\Delta d_k\|^2_{L^2} \]
\[ + \kappa \|\Lambda \theta_k\|^2_{L^2} + \gamma \|\theta_k\|^2_{L^2} + a \|\Lambda j_{0,k}\|^2_{L^2} + C b \|j_{0,k}\|^2_{L^2} \]
\[ = (b + C \gamma) \int_{\mathbb{R}^3} j_{0,k} \theta_k dx + \frac{1}{3C} \int_{\mathbb{R}^3} \Lambda j_{0,k} d_k dx - \beta_1 \int_{\mathbb{R}^3} \left( \Lambda \rho_k \Lambda \theta_k + \frac{1}{3C} \Lambda \rho_k \Lambda j_{0,k} \right) dx \]
\[ + \int_{\mathbb{R}^3} \left( \rho_k S_k^1 + d_k D_k + \frac{3}{2} \theta_k S_k^3 + j_{0,k} S_k^4 + \beta_1 d_k \Lambda S_k^1 + \beta_1 \Lambda \rho_k D_k \right) dx \]
\[ + \beta_1 \nu \int_{\mathbb{R}^3} \nabla \rho_k \cdot \nabla S_k^1 dx, \tag{3.9} \]

with the functional \( \mathcal{L}_{h,k}(t) \) being defined as follows.
\[ \mathcal{L}_{h,k}(t) := \int_{\mathbb{R}^3} \left( \rho_k^2 + \nu \beta_1 |\nabla \rho_k|^2 - \beta_1 \Lambda \rho_k d_k + a_k^2 + \frac{3}{2} \theta_k^2 + j_{0,k}^2 \right) dx. \tag{3.10} \]
Inserting the following inequalities

\[
(b + C\gamma) \int_{\mathbb{R}^3} j_{0,k} \theta_k dx \leq \frac{\gamma}{4} \left\| \theta_k \right\|_{L^2}^2 + \frac{(b + C\gamma)^2}{\gamma} \left\| j_{0,k} \right\|_{L^2}^2,
\]

\[
\frac{1}{3C} \int_{\mathbb{R}^3} \Lambda j_{0,k} dx \leq \frac{\alpha}{4} \left\| \Lambda j_{0,k} \right\|_{L^2}^2 + \frac{1}{9aC^2} \left\| d_k \right\|_{L^2}^2,
\]

\[
- \beta_1 \int_{\mathbb{R}^3} \Lambda \rho_k \Lambda \theta_k dx \leq \frac{\beta_1}{4} \left\| \Lambda \rho_k \right\|_{L^2}^2 + \beta_1 \left\| \Lambda \theta_k \right\|_{L^2}^2,
\]

\[
- \beta_1 \frac{1}{3C} \int_{\mathbb{R}^3} \Lambda \rho_k \Lambda j_{0,k} dx \leq \frac{\beta_1}{4} \left\| \Lambda \rho_k \right\|_{L^2}^2 + \frac{\beta_1}{9C^2} \left\| \Lambda j_{0,k} \right\|_{L^2}^2,
\]

into (3.9) leads to

\[
\frac{1}{2} d \frac{d}{dt} \mathcal{L}_{h,k}(t) + \frac{\beta_1}{2} \left\| \Lambda \rho_k \right\|_{L^2}^2 + (\nu - \beta_1) \left\| \nabla d_k \right\|_{L^2}^2 - \frac{1}{9aC^2} \left\| d_k \right\|_{L^2}^2 + (\kappa - \beta_1) \left\| \Lambda \theta_k \right\|_{L^2}^2
\]

\[
+ \frac{3\gamma}{4} \left\| \theta_k \right\|_{L^2}^2 + \frac{3\alpha}{4} \left\| \Lambda j_{0,k} \right\|_{L^2}^2 - \frac{(b + C\gamma)^2}{\gamma} \left\| j_{0,k} \right\|_{L^2}^2 + Cb \left\| j_{0,k} \right\|_{L^2}^2,
\]

\[
\leq \int_{\mathbb{R}^3} \left( \rho_k S_k^1 + d_k D_k + \frac{3}{2} \theta_k S_k^3 + j_{0,k} S_k^4 + \beta_1 d_k \Lambda S_k^1 + \beta_1 \Lambda \rho_k D_k \right) dx
\]

\[
+ \beta_1 \nu \int_{\mathbb{R}^3} \nabla \rho_k \cdot \nabla S_k^1 dx.
\]

Choosing the positive constant \( \beta_1 \) to satisfy

\[
\beta_1 \leq \min\left\{ \frac{\kappa}{2}, \frac{\nu}{4}, \frac{9aC^2}{4}, 1 \right\},
\]

and a positive integer \( k_1 \) such that

\[
2^{2k_1 - 3} > \max\left\{ \frac{1}{9a\nu C^2}, \frac{2(b + C\gamma)^2}{\alpha\gamma} \right\}.
\]

Then, we arrive at the following inequalities, for \( k > k_1 > 0 \),

\[
\nu \left\| \Lambda d_k \right\|_{L^2}^2 - \frac{1}{9aC^2} \left\| d_k \right\|_{L^2}^2 \geq \frac{\nu}{2} \left\| \Lambda d_k \right\|_{L^2}^2, \quad \frac{3\alpha}{4} \left\| \Lambda j_{0,k} \right\|_{L^2}^2 - \frac{(b + C\gamma)^2}{\gamma} \left\| j_{0,k} \right\|_{L^2}^2 \geq \frac{\alpha}{2} \left\| \Lambda j_{0,k} \right\|_{L^2}^2,
\]

where we used the Plancherel theorem and the definition of \((\Lambda_k)_{k \in \mathbb{Z}}\). So, it follows that

\[
\frac{1}{2} d \frac{d}{dt} \mathcal{L}_{h,k}(t) + \frac{\beta_1}{2} \left\| \Lambda \rho_k \right\|_{L^2}^2 + \nu \left\| \Lambda \rho_k \right\|_{L^2}^2
\]

\[
+ \frac{\kappa}{2} \left\| \Lambda \theta_k \right\|_{L^2}^2 + \frac{3\gamma}{4} \left\| \theta_k \right\|_{L^2}^2 + \frac{a}{4} \left\| \Lambda j_{0,k} \right\|_{L^2}^2 + Cb \left\| j_{0,k} \right\|_{L^2}^2
\]

\[
\leq \int_{\mathbb{R}^3} \left( \rho_k S_k^1 + d_k D_k + \frac{3}{2} \theta_k S_k^3 + j_{0,k} S_k^4 + \beta_1 d_k \Lambda S_k^1 + \beta_1 \Lambda \rho_k D_k - \beta_1 \nu \nabla \rho_k \cdot \nabla S_{12}^1 \right) dx
\]

\[
- \beta_1 \nu \int_{\mathbb{R}^3} \nabla \rho_k \cdot \nabla S_{12}^1 dx,
\]

where we split \(-S^1\) into two parts, that is, \(-S^1 = S^{11} + S^{12}\) with \(S^{11} = u \cdot \nabla \rho\) and \(S^{12} = \rho \text{div} u\). Now we estimate the terms on the r.h.s. of (3.14). For the estimates of the first term on the
r.h.s. of (3.14), by Cauchy-Schwartz inequality, one gets
\[
\int_{\mathbb{R}^3} \left( \rho_k S^1_k + d_k D_k + \frac{3}{2} \theta_k S^3_k + j_{0,k} S^4_k + \beta_1 d_k \Delta S^1_k + \beta_1 \rho_k D_k - \beta_1 \nu \nabla \rho_k \cdot \nabla S^1_k \right) dx \\
\leq \frac{\beta_1}{8} \| \Delta \rho_k \|_{L^2}^2 + \frac{\nu}{8} \| \Delta d_k \|_{L^2}^2 + \frac{\kappa}{8} \| \Delta \theta_k \|_{L^2}^2 + \frac{\alpha}{8} \| \Delta j_{0,k} \|_{L^2}^2 + C' \|(S^1_k, D_k, S^3_k, S^4_k, \nabla S^1_k)\|_{L^2}^2,
\]
where the following facts are used. For any integer \( k > k_1 > 0 \), it has
\[
\| \rho_k \|_{L^2} \leq 2^{k_1 - 1} \| \rho_k \|_{L^2} \leq \| \Delta \rho_k \|_{L^2} \quad \text{and} \quad \| \nabla \rho_k \|_{L^2} \sim \| \Delta \rho_k \|_{L^2},
\]
and so on. For the last term on the r.h.s. of (3.14), one has
\[
-\beta_1 \nu \int_{\mathbb{R}^3} \nabla \rho_k \cdot \nabla S^1_k dx \leq \beta_1 \nu \left( \left| (\nabla u)^T \cdot \nabla \rho_k \right| + \beta_1 \nu \left| (u \cdot \nabla) \nabla \rho_k \right| \right) \\
\leq \beta_1 \nu \left( \left| (\nabla u)^T \cdot \nabla \rho_k \right| + \beta_1 \nu \left| (\Delta_k u \cdot \nabla) \nabla \rho_k \right| \\
+ \beta_1 \nu \left| (u \cdot \nabla) \nabla \rho_k \right| \right) \\
\leq \frac{\beta_1}{16} \| \nabla \rho_k \|_{L^2}^2 + C \| ((\nabla u)^T \cdot \nabla \rho_k) \|_{L^2}^2 \\
+ C \left| (\Delta_k u \cdot \nabla) \nabla \rho_k \right| + C \| \div u \|_{L^\infty} \| \nabla \rho_k \|_{L^2}^2.
\]
By using Lemma 5.3 and the Young inequality, we get
\[
\left| (\Delta_k u \cdot \nabla) \nabla \rho_k \right| \lesssim \| \nabla u \|_{L^\infty} \| \nabla \rho_k \|_{L^2}^2 + \| \nabla^2 \rho \|_{L^\infty} \| \Delta_k u \|_{L^2} \| \nabla \rho_k \|_{L^2} \\
+ \| \nabla u \|_{L^\infty} \| \nabla \rho_k \|_{L^2} \sum_{l \geq k-1} 2^{k-l} \| \nabla \rho_l \|_{L^2} \leq \frac{\beta_1}{16} \| \nabla \rho_k \|_{L^2}^2 + C \| \nabla u \|_{L^\infty} \| \nabla \rho_k \|_{L^2}^2 + C \| \nabla^2 \rho \|_{L^\infty}^2 \| u_k \|_{L^2}^2 \\
+ C \| \nabla u \|_{L^\infty}^2 \left( \sum_{l \geq k-1} 2^{k-l} \| \nabla \rho_l \|_{L^2} \right)^2.
\]
Putting (3.15), (3.16) and (3.17) into (3.14) yields
\[
\frac{1}{2} \frac{d}{dt} \mathcal{L}_{h,k}(t) + \frac{3}{4} \| \Delta \rho_k \|_{L^2}^2 + \frac{\nu}{5} \| \Delta d_k \|_{L^2}^2 \\
+ \frac{\kappa}{4} \| \Delta \theta_k \|_{L^2}^2 + \frac{3}{4} \gamma \| \theta_k \|_{L^2}^2 + \frac{\alpha}{8} \| \Delta j_{0,k} \|_{L^2}^2 + Cb \| j_{0,k} \|_{L^2}^2 \\
\lesssim \|(S^1_k, D_k, S^3_k, S^4_k, \nabla S^1_k)\|_{L^2}^2 + \|(\nabla u)^T \cdot \nabla \rho_k)\|_{L^2}^2 + \| \nabla u \|_{L^\infty} \| \nabla \rho_k \|_{L^2}^2 \\
+ \| \nabla^2 \rho \|_{L^\infty}^2 \| u_k \|_{L^2}^2 + \| \nabla u \|_{L^\infty}^2 \left( \sum_{l \geq k-1} 2^{k-l} \| \nabla \rho_l \|_{L^2} \right)^2.
\]
Below, we show all of the first-order derivative estimates of \( d_k, \theta_k \) and \( j_{0,k} \). From (3.5),

Inserting the estimates above into (3.19) yields that

\[
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\| (\Delta d_k, \frac{3}{2} \Lambda \theta_k, \Lambda j_{0,k}) \|^2_{L^2} + \nu \| \Lambda^2 d_k \|^2_{L^2} \\
&+ \kappa \| \Lambda^2 \theta_k \|^2_{L^2} + \gamma \| \Lambda \theta_k \|^2_{L^2} + a \| \Lambda^2 j_{0,k} \|^2_{L^2} + C b \| \Lambda j_{0,k} \|^2_{L^2} \\
= & (b + C \gamma) (\Lambda j_{0,k} | \Lambda \theta_k) + \frac{1}{3c} (\Lambda^2 j_{0,k} | \Lambda d_k) - (\Delta \rho_k | \Lambda d_k) \\
&+ (\Lambda D_k | \Lambda d_k) + \frac{3}{2} (\Lambda S^3_k | \Lambda \theta_k) + (\Lambda S^4_k | \Lambda j_{0,k}).
\end{aligned}
\tag{3.19}
\]

The Cauchy-Schwarz inequality gives the following inequalities

\[
\begin{aligned}
(b + C \gamma) (\Lambda j_{0,k} | \Lambda \theta_k) &\leq \frac{(b + C \gamma)^2}{\gamma} \| \Lambda j_{0,k} \|^2_{L^2} + \frac{\gamma}{4} \| \Lambda \theta_k \|^2_{L^2} \\
\frac{1}{3c} (\Lambda^2 j_{0,k} | \Lambda d_k) &\leq \frac{1}{9 \alpha c^2} \| \Lambda j_{0,k} \|^2_{L^2} + \frac{\nu}{4} \| \Lambda^2 d_k \|^2_{L^2}, \\
(\Delta \rho_k | \Lambda d_k) &\leq \frac{1}{\nu} \| \Delta \rho_k \|_{L^2} + \frac{\nu}{4} \| \Lambda^2 d_k \|^2_{L^2}, \\
(\Lambda D_k | \Lambda d_k) &\leq \frac{1}{\nu} \| D_k \|^2_{L^2} + \frac{\nu}{4} \| \Lambda^2 d_k \|^2_{L^2}, \\
3 \frac{1}{2} (\Lambda S^3_k | \Lambda \theta_k) &\leq \frac{9}{4 \kappa} \| S^3_k \|^2_{L^2} + \frac{\kappa}{4} \| \Lambda^2 \theta_k \|^2_{L^2}, \\
(\Lambda S^4_k | \Lambda j_{0,k}) &\leq \frac{1}{2a} \| S^4_k \|^2_{L^2} + \frac{a}{2} \| \Lambda^2 j_{0,k} \|^2_{L^2}.
\end{aligned}
\]

Inserting the estimates above into (3.19) yields that

\[
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\| (\Delta d_k, \frac{3}{2} \Lambda \theta_k, \Lambda j_{0,k}) \|^2_{L^2} + \nu \| \Lambda^2 d_k \|^2_{L^2} \\
&+ \frac{\kappa}{2} \| \Lambda^2 \theta_k \|^2_{L^2} + \gamma \| \Lambda \theta_k \|^2_{L^2} + \frac{a}{2} \| \Lambda^2 j_{0,k} \|^2_{L^2} + C b \| \Lambda j_{0,k} \|^2_{L^2} \\
\leq & \frac{1}{\nu} \| \Delta \rho_k \|_{L^2} + \left( \frac{(b + C \gamma)^2}{\gamma} \right) \| \Lambda j_{0,k} \|^2_{L^2} + \frac{1}{\nu} \| D_k \|^2_{L^2} + \frac{9}{4 \kappa} \| S^3_k \|^2_{L^2} + \frac{1}{2a} \| S^4_k \|^2_{L^2}. 
\end{aligned}
\tag{3.20}
\]

For the estimate of \((\mathcal{P} u)_k\), it follows from (3.6) and the Young inequality that for any integer \( k > k_1 \),

\[
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\| (\mathcal{P} u)_k, \Lambda (\mathcal{P} u)_k \|_{L^2}^2 + \frac{\mu}{2} \| \Lambda^2 (\mathcal{P} u)_k \|^2_{L^2} + \mu \| \Lambda (\mathcal{P} u)_k \|^2_{L^2} - \frac{\mu}{2} \| (\mathcal{P} u)_k \|^2_{L^2} \\
\leq & \frac{1}{\mu} \| (\mathcal{P} S^2)_k \|^2_{L^2}. 
\end{aligned}
\tag{3.21}
\]

Since \( k > k_1 > 0 \), we have

\[
\| (\mathcal{P} u)_k \|^2_{L^2} \leq \| \Lambda (\mathcal{P} u)_k \|^2_{L^2},
\]

which together with (3.21) imply

\[
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\| (\mathcal{P} u)_k, \Lambda (\mathcal{P} u)_k \|_{L^2}^2 + \frac{\mu}{2} \| (\Lambda (\mathcal{P} u)_k, \Lambda^2 (\mathcal{P} u)_k) \|^2_{L^2} \\
\leq & \frac{1}{\mu} \| (\mathcal{P} S^2)_k \|^2_{L^2}. 
\end{aligned}
\tag{3.22}
\]
Then, adding $\beta_2 \times (3.20)$ and $(3.22)$ to $(3.18)$ with $\beta_2 > 0$ being a suitably small constant, we achieve the estimates of the high frequency part for $k > k_1$ as follows.

\[
\frac{1}{2} \frac{d}{dt} \left\{ \mathcal{L}_{h,k}(t) + \beta_2 \left\| \left( \Lambda d_k, \frac{3}{2} \Lambda \theta_k, \Lambda j_{0,k} \right)(t) \right\|^2_{L^2} + \left\| \left( \mathcal{P} u_k, \Lambda (\mathcal{P} u)_k \right)(t) \right\|^2_{L^2} \right\}
+ \left( \frac{\beta_1}{4} - \frac{\beta_2}{\nu} \right) \left\| \Lambda \rho_k(t) \right\|^2_{L^2} + \frac{\nu}{8} \left\| \Lambda d_k(t) \right\|^2_{L^2} + \nu \beta_2 \left\| \Lambda^2 d_k(t) \right\|^2_{L^2}
+ \frac{3}{4} \gamma \left\| \theta_k(t) \right\|^2_{L^2} + \left( \frac{\kappa}{4} + \beta_2 \gamma \right) \left\| \Lambda \theta_k(t) \right\|^2_{L^2} + \frac{\kappa}{2} \left\| \Lambda^2 \theta_k(t) \right\|^2_{L^2}
+ Cb \left\| j_{0,k}(t) \right\|^2_{L^2} + \left( \frac{a}{8} + \beta_2Cb - \frac{\beta_2(b+C_1^2)}{\gamma} - \frac{\beta_2}{9\nu C^2} \right) \left\| \Lambda j_{0,k}(t) \right\|^2_{L^2}
+ \frac{a}{2} \beta_2 \left\| \Lambda^2 j_{0,k}(t) \right\|^2_{L^2} + \frac{\mu}{2} \left\| \left( \Lambda (\mathcal{P} u)_k, \Lambda^2 (\mathcal{P} u)_k \right)(t) \right\|^2_{L^2}
\leq \left( \left\| S_1^k, S_2^k, S_3^k, S_4^k, \nabla S_{12}^k(t) \right\|^2_{L^2} d\tau + \left\| \left( \nabla u \right)^T \cdot \nabla \rho \right\|^2_{L^2} + \left\| \nabla u \right\|^2_{L^\infty} \left\| \nabla \rho \right\|^2_{L^2} + \left\| \nabla^2 \rho \right\|^2_{L^\infty} \left\| u \right\|^2_{L^2} + \left\| \nabla u \right\|^2_{L^\infty} \left( \sum_{l \geq k-1} 2 \left\| \nabla^{l+1} \rho \right\|^2_{L^2} \right) \right)^2,
\]

where we have used the facts that

\[
\left\| D_k \right\|^2_{L^2} \leq \left\| S_k^2 \right\|^2_{L^2} \quad \text{and} \quad \left\| (\mathcal{P} S_k^2) \right\|^2_{L^2} \leq \left\| S_k^2 \right\|^2_{L^2}.
\]

Setting

\[
\mathcal{H}_{h,k}(t) = \mathcal{L}_{h,k}(t) + \beta_2 \left\| \left( \Lambda d_k, \frac{3}{2} \Lambda \theta_k, \Lambda j_{0,k} \right)(t) \right\|^2_{L^2} + \left\| \left( \mathcal{P} u_k, \Lambda (\mathcal{P} u)_k \right)(t) \right\|^2_{L^2},
\]

then there exists a positive constant $C_2$ such that

\[
C_2^{-1} \mathcal{H}_{h,k}(t) \leq \left\| \left( \rho_k, u_k, \theta_k, j_{0,k} \right)(t) \right\|^2_{L^2} + 2^{2k} \left\| \left( \rho_k, u_k, \theta_k, j_{0,k} \right)(t) \right\|^2_{L^2} \leq C_2 \mathcal{H}_{h,k}(t),
\]

where we have used the identity $u_k = -\Lambda^{-1} \nabla d_k - \Lambda^{-1} \text{div}(\mathcal{P} u)_k$ and the Bernstein inequality. From $(3.23)$, by letting the constant $\beta_2$ satisfy

\[
0 < \beta_2 \leq \min \left\{ \frac{\beta_1 \nu}{8}, \frac{a}{16} \frac{(b+C_1^2)}{\gamma} + \frac{1}{9\nu C^2} - \beta_2 \right\},
\]

there is a positive constant $C_3$ depending on $\beta_1, \beta_2, \kappa, \nu, \gamma, a, C$ and $b$, such that for any $k > k_1$

\[
\frac{d}{dt} \mathcal{H}_{h,k}(t) + C_3 \left\{ \left\| \left( \theta_k, j_{0,k} \right)(t) \right\|^2_{L^2} + 2^{2k} \left\| \left( \rho_k, u_k, \theta_k, j_{0,k} \right)(t) \right\|^2_{L^2} + 2^{4k} \left\| \left( \rho_k, u_k, \theta_k, j_{0,k} \right)(t) \right\|^2_{L^2} \right\}
\leq \left( \left\| S_1^k, S_2^k, S_3^k, S_4^k, \nabla S_{12}^k(t) \right\|^2_{L^2} d\tau + \left\| \left( \nabla u \right)^T \cdot \nabla \rho \right\|^2_{L^2} + \left\| \nabla u \right\|^2_{L^\infty} \left\| \nabla \rho \right\|^2_{L^2} + \left\| \nabla^2 \rho \right\|^2_{L^\infty} \left\| u \right\|^2_{L^2} + \left\| \nabla u \right\|^2_{L^\infty} \left( \sum_{l \geq k-1} 2 \left\| \nabla^{l+1} \rho \right\|^2_{L^2} \right) \right)^2,
\]

\[
(3.27)
\]
Integrating \((3.27)\) with respect to \(t\) over \([0, t]\) and using \((3.25)\) yield for any \(k > k_1\)

\[
\|(\rho_k, u_k, \theta_k, j_{0,k})(t)\|_{L^2}^2 + 2^{2k}\|(\rho_k, u_k, \theta_k, j_{0,k})(t)\|_{L^2}^2 \\
+ \int_0^t \left\{ \|(\theta_k, j_{0,k})(\tau)\|_{L^2}^2 + 2^{2k}\|(\rho_k, u_k, \theta_k, j_{0,k})(\tau)\|_{L^2}^2 + 2^{4k}\|(u_k, \theta_k, j_{0,k})(\tau)\|_{L^2}^2 \right\} d\tau \\
\lesssim \|(\rho_k, u_k, \theta_k, j_{0,k})(0)\|_{L^2}^2 + 2^{2k}\|(\rho_k, u_k, \theta_k, j_{0,k})(0)\|_{L^2}^2 \\
+ \int_0^t \left( \|(S^1_k, S^2_k, S^3_k, S^4_k, \nabla S^{12}_k)(\tau)\|_{L^2}^2 + \|(D u^T \cdot \nabla \rho_k)(\tau)\|_{L^2}^2 \right) d\tau \\
+ \int_0^t \left( \|
abla u\|_{L^\infty} \|
abla \rho_k\|_{L^2}^2 + \|
abla^2 \rho\|_{L^\infty} \|u_k\|_{L^2}^2 + \|
abla u\|_{L^\infty}^2 \left( \sum_{l \geq k-1} 2^{k-l} \|
abla \rho_l\|_{L^2}^2 \right)^2 \right) d\tau.
\]

(3.28)

By the weighted \(l^2\) summation over \(k\) with \(k \geq k_1 + 1\) in \((3.28)\) and the definition in \((5.1)\), we have the following proposition for short wave parts.

**Proposition 3.1.** Let \(s \geq 1\) be a real number. The following inequality holds true for the solution to \((5.1)\)

\[
\|(\rho, u, \theta, j_0)^S(t)\|_{B_s^{s+1}}^2 + \|(\rho, u, \theta, j_0)^S(t)\|_{B_s^{s+1}}^2 \\
+ \int_0^t \left( \|(\rho, u, \theta, j_0)^S(\tau)\|_{B_s^{s+1}}^2 + \|(u, \theta, j_0)^S(\tau)\|_{B_s^{s+1}}^2 + \|(\theta, j_0)^S(\tau)\|_{B_s^{s+1}}^2 \right) d\tau \\
\lesssim \|(\rho, u, \theta, j_0)^S(0)\|_{B_s^{s+1}}^2 + \|(\rho, u, \theta, j_0)^S(0)\|_{B_s^{s+1}}^2 \\
+ \int_0^t \left( \|(S^{12})^S(\tau)\|_{B_s^{s+1}}^2 + \|(S^1, S^2, S^3, S^4, (\nabla u)^T \cdot \nabla \rho)^S(\tau)\|_{B_s^{s+1}}^2 \right) d\tau \\
+ \int_0^t \left( \|
abla^2 \rho\|_{L^\infty} \|u^S\|_{B_s^{s+1}}^2 + \|
abla u(\tau)\|_{L^\infty} \|
abla \rho^S(\tau)\|_{B_s^{s+1}}^2 \right) d\tau \\
+ \int_0^t \left( \|
abla u\|_{L^\infty}^2 \sum_{k > k_1} 2^{2k(s-1)} \left( \sum_{l \geq k-1} 2^{k-l} \|
abla \rho_l\|_{L^2}^2 \right)^2 \right) d\tau.
\]

(3.29)

**3.2 Estimates in the low-frequency regimes**

**3.2.1 Estimates on the compressible part**

Applying the Fourier transform on the linearized system \((3.3)\) gives that

\[
\begin{aligned}
\partial_t \hat{\rho} + |\xi| \hat{d} &= \hat{S}_1, \\
\partial_t \hat{d} - |\xi| \hat{\rho} + \nu |\xi|^2 \hat{d} - |\xi| \hat{\theta} - \frac{\gamma \rho}{\rho_0} |\xi| \hat{j}_0 &= \hat{D}, \\
\partial_t \hat{\theta} + \frac{2}{3} |\xi| \hat{d} + \frac{2}{3} \kappa |\xi|^2 \hat{\theta} + \frac{2}{3} \gamma \hat{\theta} - \frac{2}{3} \hat{b}_j &= \hat{S}_3, \\
\partial_t \hat{j}_0 - C \gamma \hat{\theta} + a |\xi|^2 \hat{j}_0 + C \hat{b}_j &= \hat{S}_4,
\end{aligned}
\]

(3.30)
In other words, it is

\[
\frac{d}{dt} \begin{pmatrix} \hat{\rho} \\ \hat{d} \\ \hat{\theta} \\ \hat{j}_0 \end{pmatrix} + A(\xi) \begin{pmatrix} \hat{\rho} \\ \hat{d} \\ \hat{\theta} \\ \hat{j}_0 \end{pmatrix} = \begin{pmatrix} \hat{S}^1 \\ \hat{D} \\ \hat{S}_3^2 \\ \hat{S}_4 \end{pmatrix},
\]

(3.31)

with

\[
A(\xi) = \begin{pmatrix} 0 & |\xi| & 0 & 0 \\ -|\xi| & \nu|\xi|^2 & -|\xi| & -\frac{|\xi|^2}{3} \\ 0 & \frac{2|\xi|}{3} & \frac{2}{3}\kappa|\xi|^2 + \frac{2\gamma}{3} & -\frac{2}{3}b \\ 0 & 0 & -C\gamma & a|\xi|^2 + Cb \end{pmatrix}.
\]

(3.32)

Denote \( \varrho = |\xi| \), the characteristic polynomial of the matrix \( A \) takes the following form.

\[
P(\lambda) = |A(\varrho) - \lambda I|
:= a_0 \lambda^4 - a_1 \lambda^3 + a_2 \lambda^2 - a_3 \lambda + a_4,
\]

where

\[
a_0 = 1, \quad a_1 = (a + 2/3 + \nu)\varrho^2 + \frac{2\gamma}{3} + Cb,
\]

\[
a_2 = \left[ (\nu a + 2/3 + \nu \kappa)\varrho^2 + \frac{2\gamma}{3} + Cb + \frac{2\gamma a}{3} + \frac{2\kappa Cb}{3} + \frac{5}{3} \right] \varrho^2,
\]

\[
a_3 = \left[ \frac{2}{3} \nu \kappa \varrho^4 + \left( \frac{2(\gamma a + \kappa Cb)\nu}{3} + \frac{5a}{3} + \frac{2\kappa}{3} \right) \varrho^2 + \frac{5}{3}Cb + \frac{8\gamma}{9} \right] \varrho^2,
\]

\[
a_4 = \frac{2}{3} \left( a\kappa \varrho^2 + a\gamma + \kappa Cb \right) \varrho^4.
\]

It is direct to check that the matrix \( A(0) \) has four eigenvalues, i.e. 0 (three multiply) and \( \frac{2\gamma}{3} + Cb \). By a direct calculation, it is clear that for small positive \( \varrho \), \( P(\lambda) \) has a pair of complex conjugated eigenvalues and two real eigenvalues with asymptotic expansion.

\[
\pm i \varrho \sqrt{\frac{8\gamma + 15Cb}{6\gamma + 9Cb}} + \left[ \frac{9\gamma}{(6\gamma + 9Cb)} + \frac{2(a\gamma + \kappa Cb)(\gamma + 3Cb)}{(8\gamma + 15Cb)(2\gamma + 3Cb)} + \frac{\nu}{2} \right] \varrho^2 + O(\varrho^3),
\]

and

\[
\frac{6(a\gamma + \kappa Cb)}{15Cb + 8\gamma} \varrho^2 + O(\varrho^3), \quad \frac{2\gamma}{3} + Cb + \frac{4\gamma \kappa + 9Cb - \gamma}{6\gamma + 9Cb} \varrho^2 + O(\varrho^3).
\]

According to these eigenvalues, we expect to have three dissipative modes (corresponding to \( \lambda_i(\varrho), \ i = 1, 2, 3 \)) and a damped mode (corresponding to \( \lambda_4(\varrho) \)). Roughly speaking, we hope that the modes corresponding to \( \lambda_i(\varrho) \) (\( i = 1, 2, 3 \)) should have decay estimates of the type \( e^{-c_0 \varrho^2 t} \) with small enough \( \varrho \) and a positive constant \( c_0 \) depended only on \( \gamma, \kappa, \nu, C, a \) and \( b \). The mode corresponding to \( \lambda_4(\varrho) \) should have better decay properties. In what follows, we use the modes:

\[
\hat{\Theta} := 3C\hat{\theta} + 2\hat{j}_0, \quad \hat{\Xi} := \gamma \hat{\theta} - b\hat{j}_0.
\]

(3.33)
Actually, to some extent, $\hat{\rho}$, $\hat{d}$ and $\hat{\Theta}$ are “dissipative modes”, and $\hat{\Xi}$ is a “damped mode”. Thus the system (3.30) could be rewritten as

$$
\begin{align*}
\partial_t \hat{\rho} + |\xi|\hat{d} &= \hat{S}^1, \\
\partial_t \hat{d} - |\xi|\hat{\rho} + \nu|\xi|^2\hat{d} - c_1|\xi|\hat{\Theta} - c_2|\xi|\hat{\Xi} &= \hat{S}^2, \\
\partial_t \hat{\Theta} + 2\mathcal{C}|\xi|\hat{d} + c_3|\xi|^2\hat{\Theta} - c_4|\xi|^2\hat{\Xi} &= \hat{S}^3, \\
\partial_t \hat{\Xi} + \frac{2\gamma}{3}|\xi|\hat{d} + c_5\hat{\Xi} - c_6|\xi|^2\hat{\Theta} &= \hat{S}^4,
\end{align*}
$$

(3.34)

where $(\hat{S}_1, \hat{S}_2, \hat{S}_3, \hat{S}_4) = (\hat{S}_1, \hat{D}, 3\mathcal{C}\hat{S}_3 + 2\hat{S}_4, \gamma \hat{S}_3 - b\hat{S}_4)$ and the coefficients $c_i$ ($i = 1, \cdots, 6$) are defined by

- $c_1 = \frac{b + \gamma}{3b\mathcal{C} + 2\gamma} > 0$, $c_2 = \frac{1}{3b\mathcal{C} + 2\gamma} > 0$,
- $c_3 = \left( \frac{2\kappa \mathcal{C}}{3b\mathcal{C} + 2\gamma} + \frac{2\alpha \gamma}{3b\mathcal{C} + 2\gamma} \right) > 0$,
- $c_4 = \left( \frac{6\alpha \mathcal{C}}{3b\mathcal{C} + 2\gamma} - \frac{4\kappa \gamma}{3b\mathcal{C} + 2\gamma} \right)$,
- $c_5 = \left[ \frac{4\kappa \gamma + 9ab\mathcal{C}}{3(3b\mathcal{C} + 2\gamma)} \right] + \left( \frac{2\gamma}{3} + b\mathcal{C} \right) > 0$,
- $c_6 = \frac{2\kappa \gamma b - 3ab\gamma}{3(3b\mathcal{C} + 2\gamma)}$.

By the energy method in the Fourier space, we can get

$$
\frac{1}{2} \frac{d}{dt} \left( |\hat{\rho}|^2 + |\hat{d}|^2 + \frac{c_1}{2\mathcal{C}}|\hat{\Theta}|^2 + \frac{3c_2}{2\gamma}|\hat{\Xi}|^2 \right) + \nu|\xi|^2|\hat{d}|^2 + \frac{c_1c_3}{2\mathcal{C}}|\xi|^2|\hat{\Theta}|^2 + \frac{3c_2c_5}{2\gamma}|\hat{\Xi}|^2
\leq \left( \frac{c_1c_4}{2\mathcal{C}} + \frac{3c_2c_6}{2\gamma} \right) |\xi|^2 \text{Re}(\hat{\Theta}\hat{\Xi})
$$

$$
+ \text{Re}(\hat{S}^1\hat{\rho}) + \text{Re}(\hat{S}^2\hat{d}) + \frac{c_1}{2\mathcal{C}} \text{Re}(\hat{S}^3\hat{\Theta}) + \frac{3c_2}{2\gamma} \text{Re}(\hat{S}^4\hat{\Xi}).
$$

(3.35)

By the similar arguments, it follows from (3.34) and (3.34) that

$$
\frac{d}{dt} \text{Re}(|\hat{\rho}d|) - |\xi||\hat{\rho}|^2 + |\xi||\hat{d}|^2 = -\nu|\xi|^2 \text{Re}(\hat{\rho}\hat{d}) + c_1|\xi| \text{Re}(\hat{\rho}\hat{\Theta}) + c_2|\xi| \text{Re}(\hat{\rho}\hat{\Xi})
\leq \text{Re}(\hat{S}^1\hat{\rho}) + \text{Re}(\hat{S}^2\hat{d}).
$$

(3.36)
And \( \beta_3 |\xi| \times (3.35) \) gives

\[
\frac{1}{2} \frac{d}{dt} \left( |\tilde{\rho}|^2 - 2 \beta_3 |\xi| \text{Re}(\tilde{\rho}d) + |\tilde{d}|^2 + \frac{c_1}{2C} |\tilde{\Theta}|^2 + \frac{3c_2}{2\gamma} |\tilde{\Xi}|^2 \right) \\
+ \beta_3 |\xi|^2 |\tilde{\rho}|^2 + (\nu - \beta_3) |\xi|^2 |\tilde{d}|^2 + \frac{c_1c_3}{2C} |\xi|^2 |\tilde{\Theta}|^2 + \frac{3c_2c_3}{2\gamma} |\tilde{\Xi}|^2 \\
= \left( \frac{c_1c_4}{2C} + \frac{3c_2c_6}{2\gamma} \right) |\xi|^2 \text{Re}(\tilde{\Theta} \tilde{\Xi}) + \beta_3 \nu |\xi|^3 \text{Re}(\tilde{\rho}d) - \beta_3 c_1 |\xi|^2 \text{Re}(\tilde{\rho} \tilde{\Theta}) \\
- \beta_3 c_2 |\xi|^2 \text{Re}(\tilde{\rho} \tilde{\Xi}) + \text{Re}(\tilde{S}^2 \tilde{\rho}) + \text{Re}(\tilde{S}^3 \tilde{d}) + \frac{c_1}{2C} \text{Re}(\tilde{S}^3 \tilde{\Theta}) \\
+ \frac{3c_2}{2\gamma} \text{Re}(\tilde{S}^3 \tilde{\Xi}) - \beta_3 \text{Re}(\tilde{S}^1 \tilde{\rho}) - \beta_3 \text{Re}(\tilde{S}^2 \tilde{d}),
\]

for some positive constant \( \beta_3 \). By the Young inequality, a direct calculation yields

\[
\text{R.H.S. terms of (3.37)} \leq \frac{c_1c_3}{4C} |\xi|^2 |\tilde{\Theta}|^2 + \left( \frac{c_1|c_4|}{4C} + \frac{3c_2|c_6|}{4\gamma} \right)^2 \frac{C}{c_1c_3} |\xi|^2 |\tilde{\Xi}|^2 \\
+ \frac{\beta_3 \nu}{2} |\xi|^3 (|\tilde{\rho}|^2 + |\tilde{d}|^2) + \frac{\beta_3}{4} |\xi|^2 |\tilde{\rho}|^2 + 2 \beta_3 c_2^2 |\xi|^2 |\tilde{\Theta}|^2 \\
+ 2 \beta_3 c_2^2 |\xi|^2 |\tilde{\Xi}|^2.
\]

Then we have

\[
\frac{d}{dt} \mathcal{L}_1(t, \xi) + \frac{3\beta_3}{4} |\xi|^2 |\tilde{\rho}|^2 + (\nu - \beta_3) |\xi|^2 |\tilde{d}|^2 + \left( \frac{c_1c_3}{4C} - 2 \beta_3 c_2^2 \right) |\xi|^2 |\tilde{\Theta}|^2 + \frac{3c_2c_3}{2\gamma} |\tilde{\Xi}|^2 \\
\leq \frac{\beta_3 \nu}{2} |\xi|^3 (|\tilde{\rho}|^2 + |\tilde{d}|^2) + \left[ \frac{c_1|c_4|}{4C} + \frac{3c_2|c_6|}{4\gamma} \right]^2 \frac{C}{c_1c_3} + 2 \beta_3 c_2 \right] |\xi|^2 |\tilde{\Xi}|^2 + \text{Re}(\tilde{S}^2 \tilde{d}) \\
+ \frac{c_1}{2C} \text{Re}(\tilde{S}^2 \tilde{\Theta}) + \frac{3c_2}{2\gamma} \text{Re}(\tilde{S}^3 \tilde{\Theta}) - \beta_3 \text{Re}(\tilde{S}^1 \tilde{\rho}) - \beta_3 \text{Re}(\tilde{S}^2 \tilde{d}),
\]

where the functional \( \mathcal{L}_1(t, \xi) \) takes the following form.

\[
\mathcal{L}_1(t, \xi) := \frac{1}{2} |\tilde{\rho}|^2 - \beta_3 |\xi| \text{Re}(\tilde{\rho}d) + \frac{1}{2} |\tilde{d}|^2 + \frac{c_1}{4C} |\tilde{\Theta}|^2 + \frac{3c_2}{4\gamma} |\tilde{\Xi}|^2.
\]

We denote that \( r_0 \) and \( R_0 \) are two fixed positive constants satisfying

\[
r_0 := \min \left\{ \frac{1}{2\nu}, \left[ \left( \frac{c_1|c_4|}{4C} + \frac{3c_2|c_6|}{4\gamma} \right)^2 \frac{C}{c_1c_3} \right]^{-\frac{1}{2}} \frac{3c_2c_3}{4\gamma}, 1 \right\},
\]

and

\[
R_0 := 2^{k_1+1},
\]

where the positive integer \( k_1 \) is defined by (3.13). Moreover, the constant \( \beta_3 \) is chosen to be a small constant which satisfies

\[
0 < \beta_3 \leq \min \left\{ \frac{2\nu}{3}, \frac{c_3}{16C_1}, \frac{1}{2R_0} \right\}.
\]
Since $\beta_3$ is small, then we get, for any $|\xi| \leq R_0$, that
\[
L(t, \xi) \sim |\hat{\rho}|^2 + |\hat{d}|^2 + |\hat{\Theta}|^2 + |\hat{\Xi}|^2,
\] (3.43)
which implies that there exists a positive constant $C_4 = C_4(\beta_3, \nu, c_1, c_2, c_3, c_5, R_0)$, such that
\[
C_4|\xi|^2 L(t, \xi) \leq \frac{3\beta_3}{4} |\xi|^2 |\hat{\rho}|^2 + (\nu - \beta_3)|\xi|^2 |\hat{d}|^2 + \left( \frac{c_1 c_3}{4C} - 2\beta_3 c_1^2 \right) |\xi|^2 |\hat{\Theta}|^2 + \frac{3c_2 c_5}{2\gamma} |\hat{\Xi}|^2. \tag{3.44}
\]
From (3.39) and (3.44), we have, for $0 \leq |\xi| \leq R_0$,
\[
\frac{d}{dt} L(t, \xi) + |\xi|^2 L(t, \xi) \leq \frac{\beta_3\nu}{2} |\xi|^3 (|\hat{\rho}|^2 + |\hat{d}|^2) + \left[ \left( \frac{c_1 |c_1|}{4C} + \frac{3c_2 |c_3|}{4\gamma} \right)^2 \frac{C}{c_1 c_3} + 2\beta_3 c_2^2 \right] |\xi|^2 |\hat{\Xi}|^2 + \Re \left( \hat{S}^2 \tilde{d} \right) + \frac{c_1}{2C} \Re \left( \hat{S}^3 \tilde{\Theta} \right) + \frac{3c_2}{2\gamma} \Re \left( \hat{S}^4 \tilde{\Xi} \right) - \beta_3 \Re \left( \hat{S}^1 \tilde{d} \right) - \beta_3 \Re \left( \hat{S}^2 \tilde{\rho} \right). \tag{3.45}
\]
Specially, for $0 \leq |\xi| \leq r_0$, from (3.39) and (3.44), we have
\[
\frac{d}{dt} L(t, \xi) + |\xi|^2 L(t, \xi) \leq \Re \left( \hat{S}^1 \tilde{\rho} \right) + \Re \left( \hat{S}^2 \tilde{d} \right) + \frac{c_1}{2C} \Re \left( \hat{S}^3 \tilde{\Theta} \right) + \frac{3c_2}{2\gamma} \Re \left( \hat{S}^4 \tilde{\Xi} \right) - \beta_3 \Re \left( \hat{S}^1 \tilde{d} \right) - \beta_3 \Re \left( \hat{S}^2 \tilde{\rho} \right). \tag{3.46}
\]
From (3.33), it is obvious that
\[
\hat{\theta} = \frac{b\Theta + 2\Xi}{3Cb + 2\gamma}, \quad \hat{j}_0 = \frac{\gamma \Theta - 3C\Xi}{3Cb + 2\gamma}. \tag{3.47}
\]
Then there exists some positive constant $C_5$, such that
\[
C_5^{-1} \left( |\hat{\theta}|^2 + |\hat{j}_0|^2 \right) \leq |\tilde{\Theta}|^2 + |\tilde{\Xi}|^2 \leq C_5 \left( |\hat{\theta}|^2 + |\hat{j}_0|^2 \right), \tag{3.48}
\]
which and (3.43) imply that
\[
L(t, \xi) \sim |\hat{\rho}|^2 + |\hat{d}|^2 + |\hat{\Theta}|^2 + |\hat{j}_0|^2. \tag{3.49}
\]
Using the inequalities that have been obtained in the different frequency regimes and the Parseval formula, we deduce the estimates for $(\rho_k, d_k, \theta_k, j_{0,k})$. More precisely, from (3.46) and (3.49), by taking $k_0 = \lfloor \log_2 r_0 \rfloor - 1$, we get for any $k \leq k_0$
\[
\| (\rho_k, d_k, \theta_k, j_{0,k}) (\tau) \|_{L^2}^2 \leq \int_0^t \left( 2^{2k} \| (\rho_k, d_k, \theta_k, j_{0,k}) (\tau) \|_{L^2}^2 + \| (\gamma \theta_k - b j_{0,k}) (\tau) \|_{L^2}^2 \right) \, d\tau \leq \int_0^t \int_{\mathbb{R}^3} \varphi_j^2 (\xi) \Re \left( \hat{S}^1 \tilde{\rho} \right) \, d\xi \, d\tau \tag{3.50}
\]
and only if the following determinants are positive:

\[ \text{det}(A(\xi)) > 0 \]

According to the analysis in Section 3.1.2, it could be checked that the eigenvalues of \( P(\lambda) \) have positive real parts for small enough \( \theta \). By Routh-Hurwitz theorem, the roots of the function \( P(\lambda) \) have positive real part if and only if the following determinants are positive:

\[ A_1 = \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix}, \quad A_2 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix}, \quad A_3 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ 0 & a_4 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{vmatrix}. \]

Below, we will derive the estimates of \( \widehat{\rho} \). By a direct calculation, it follows from (3.52) and the Bernstein inequality that for all \( \epsilon > 0 \)

\[
\int_0^t \int_{\mathbb{R}^3} \dot{\varphi}_j^2(\xi) \text{Re}\left( \widehat{S}^j \right) d\xi d\tau \leq \int_0^t \left\| \varphi_j^2(\xi) \right\| L^2 \left\| \varphi_j(\xi) \right\| L^2 d\tau \\
\leq \epsilon \int_0^t 2^{2k} \| \rho_k \|^2_{L^2} d\tau + C_\epsilon \int_0^t 2^{-2k} \| S_k \|^2_{L^2} d\tau,
\]

where \( \epsilon \) is a small positive constant. The other terms on the right hand side of (3.50) can be estimated in the same way. Therefore, for \( k \leq k_0 \), we have

\[
\| (\rho_k, d_k, \theta_k, j_{0,k})(t) \|^2_{L^2} + \int_0^t \left( 2^{2k} \| (\rho_k, d_k, \theta_k, j_{0,k})(\tau) \|^2_{L^2} + \| (\gamma_\theta - b_{j0,k})(\tau) \|^2_{L^2} \right) d\tau \\
\lesssim \| (\rho_k, d_k, \theta_k, j_{0,k})(0) \|^2_{L^2} + C_\epsilon \int_0^t 2^{-2k} \| S_k \|^2_{L^2} d\tau.
\]

### 3.2.2 Estimates on the incompressible part \((P\rho)_k\)

Below, we will derive the estimates of \( \widehat{(P\rho)}_k \). The linearized equations of (3.6) in Fourier variables take the following form.

\[
\partial_t (\widehat{(P\rho)}_k) + \mu |\xi|^2 (\widehat{(P\rho)}_k) = (PS^2)_k.
\]

By a direct calculation, it follows from (3.52) and the Bernstein inequality that for all \( |\xi| \geq 1 \)

\[
\| (P\rho)_k(t) \|^2_{L^2} + 2\mu \int_0^t 2^{2k} \| (P\rho)_k(\tau) \|^2_{L^2} d\tau \\
\lesssim \| (P\rho)_k(0) \|^2_{L^2} + \int_0^t \int_{\mathbb{R}^3} \varphi_j^2(\xi) \text{Re}(PS^2(\widehat{(P\rho)})) d\xi d\tau.
\]

Similar as the estimate (3.51), for any integer \( k \), we have

\[
\| (P\rho)_k(t) \|^2_{L^2} + \mu \int_0^t 2^{2k} \| (P\rho)_k(\tau) \|^2_{L^2} d\tau \lesssim \| (P\rho)_k(0) \|^2_{L^2} + C_\epsilon \int_0^t 2^{-2k} \| (PS^2)_k \|^2_{L^2} d\tau.
\]

### 3.3 Medium-frequency analysis

According to the analysis in Section 3.1.2, it could be checked that the eigenvalues of \( A(\xi) \) have positive real parts for small enough \( \theta \). In order to show that no condition is required for large \( \theta \), by Routh-Hurwitz theorem, the roots of the function \( P(\lambda) \) have positive real part if and only if the following determinants are positive:
It is clear that $A_1 > 0$ and $\text{sgn} A_3 = \text{sgn} A_4$. It is easy to check that

$$A_2 = a_1 a_2 - a_0 a_3$$

$$:= a_{21} \varrho^6 + a_{22} \varrho^4 + a_{23} \varrho^2 > 0,$$

where the coefficients $a_{21}$, $a_{22}$ and $a_{23}$ are defined by

$$a_{21} := (a + \frac{2}{3} \kappa + \nu)(\nu a + \frac{2}{3} \nu \kappa + \frac{2}{3} a \kappa) - \frac{2}{3} a k \nu > 0,$$

$$a_{22} := \frac{2}{3} (\gamma + C b)(\nu a + \frac{2}{3} \nu \kappa + \frac{2}{3} a \kappa) + \gamma (a + \frac{2}{3} \kappa + \nu)(\frac{2}{3} a \kappa + \frac{2}{3} \nu a + \frac{2}{3} \nu \kappa + \frac{2}{3} a \nu) + (a + \frac{2}{3} \kappa + \nu)(\frac{2}{3} a \kappa + \frac{2}{3} \nu a + \frac{2}{3} \nu \kappa + \frac{2}{3} a \nu)$$

$$- \frac{2}{3} (\gamma a + \kappa C b) \nu + 5 a + \frac{2}{3} \kappa > 0,$$

$$a_{23} := \frac{2}{3} (\gamma + C b)(\frac{2}{3} \gamma a + b \nu a + \frac{2}{3} \nu a + \frac{2}{3} \nu \kappa + \frac{2}{3} a \nu)$$

$$- (\frac{5}{3} C b + \frac{8}{9} \gamma ) > 0.$$

By a direct but tedious calculation, we get

$$A_3 = a_3 (a_1 a_2 - a_0 a_3) - a_1^2 a_4 > 0.$$

Following the arguments in Section 3.3 of [5], we have the following lemma.

Lemma 3.1 ([5]). For any given constants $r$ and $R$ with $0 < r < R$, there exists a positive constant $\iota$ (depending only on $r, R$ and the constants $\nu$, $\gamma$, $\kappa$, $C$, $a$ and $b$) such that

$$|e^{-\iota A(\xi)}| \leq C e^{-\iota t} \text{ for all } r \leq |\xi| \leq R \text{ and } t \in \mathbb{R}^+.$$

(3.55)

For the system (3.30) and the Duhamel principle, the inequality (3.55) yields for all $r \leq |\xi| \leq R$

$$|\hat{\rho}(\xi, \hat{D}, \hat{\theta}, \hat{j}_0)(t, \xi)| \leq e^{-\iota t}|\hat{\rho}(\xi, \hat{D}, \hat{\theta}, \hat{j}_0)(0, \xi)| + \int_0^t e^{-\iota (t-\tau)}|\hat{S}(\xi, \hat{D}, \hat{S}^3, \hat{S}^4)(\tau, \xi)| d\tau,$$

(3.56)

where $r$ and $R$ are any given positive constants.

Proposition 3.2. For any integer $k$ with $k_0 \leq k \leq k_1$, there exists a positive constant $C_k$ depending on $k_1$, such that

$$\int_0^t \|(\rho_k, d_k, \theta_k, j_{0,k}(\tau))\|_{L^2}^2 d\tau \leq C \|(\rho_k, d_k, \theta_k, j_{0,k}(0))\|_{L^2}^2$$

$$+ C \int_0^t \|(S_k^1, S_k^2, S_k^3, S_k^4, \rho_k, d_k, \theta_k, j_{0,k}(\tau))\|_{L^2}^2 d\tau.$$

(3.57)

Proof. Taking $r = 2^{k_0-1}$ and $R = R_0$ and using (3.57), we have for $k_0 \leq k \leq k_1$

$$\|(\rho_k, d_k, \theta_k, j_{0,k}(t))\|_{L^2} \leq C e^{-C t} \|(\rho_k, d_k, \theta_k, j_{0,k}(0))\|_{L^2}$$

$$+ C \int_0^t e^{-C (t-\tau)}\|(S_k^1, D_k, S_k^3, S_k^4, \rho_k, d_k, \theta_k, j_{0,k}(\tau))\|_{L^2} d\tau.$$
By a direct calculation, we have
\begin{equation}
\int_0^t \|(\rho_k, d_k, \theta_k, j_{0,k})(s)\|_{L^2}^2 \, ds \leq C \|(\rho_k, d_k, \theta_k, j_{0,k})(0)\|_{L^2}^2 \\
+ C \int_0^t ds \left( \int_0^s e^{-C(s-\tau)} \|(S_k^1, D_k, S_k^3, S_k^4)(\tau)\|_{L^2}^2 \right)^2.
\end{equation}
(3.59)

By using the Hölder inequality and exchanging the order of integration, the last term of (3.59) can be estimated as follows.
\begin{align*}
\int_0^t ds \left( \int_0^s e^{-C(s-\tau)} \|(S_k^1, D_k, S_k^3, S_k^4)(\tau)\|_{L^2}^2 \right)^2 \\
\leq \int_0^t ds \left( \int_0^s e^{-C(s-\tau)} \right) \left( \int_0^s e^{-C(s-\tau)} \|(S_k^1, D_k, S_k^3, S_k^4)(\tau)\|_{L^2}^2 \right) \\
\leq C \int_0^t ds \int_0^s e^{-C(s-\tau)} \|(S_k^1, D_k, S_k^3, S_k^4)(\tau)\|_{L^2}^2 \\
\leq C \int_0^t d\tau \int_0^\tau e^{-C(s-\tau)} \|(S_k^1, D_k, S_k^3, S_k^4)(\tau)\|_{L^2}^2 ds \\
\leq C \int_0^t \|(S_k^1, D_k, S_k^3, S_k^4)(\tau)\|_{L^2}^2 d\tau.
\end{align*}
(3.60)

Plugging (3.60) into (3.59) yields (3.57). \hfill \Box

Similar as the estimate (3.51), taking \( R = R_0 \), we have from (3.45) that for \( k \leq k_1 \)
\begin{align*}
\|(\rho_k, d_k, \theta_k, j_{0,k})(t)\|_{L^2}^2 + \int_0^t \left( 2^{2k} \|(\rho_k, d_k, \theta_k, j_{0,k})(\tau)\|_{L^2}^2 + \|(\gamma \theta_k - b j_{0,k})(\tau)\|_{L^2}^2 \right) d\tau \\
\leq \|(\rho_k, d_k, \theta_k, j_{0,k})(0)\|_{L^2}^2 + \int_0^t 2^{3k} \|(\rho_k, d_k)(\tau)\|_{L^2}^2 d\tau + \int_0^t 2^{2k} \|(\theta_k, j_{0,k})(\tau)\|_{L^2}^2 d\tau \\
+ C \int_0^t 2^{-2k} \|(S_k^1, S_k^2, S_k^3, S_k^4)(\tau)\|_{L^2}^2 d\tau.
\end{align*}
(3.61)

For \( k_0 \leq k \leq k_1 \), it follows from (3.57) that
\begin{align*}
\int_0^t 2^{3k} \|(\rho_k, d_k)(\tau)\|_{L^2}^2 d\tau + \int_0^t 2^{2k} \|(\theta_k, j_{0,k})(\tau)\|_{L^2}^2 d\tau \\
\leq C (2^{3k_1} + 2^{2k_1}) \|(\rho_k, d_k, \theta_k, j_{0,k})(0)\|_{L^2}^2 \\
+ C (2^{4k_1} + 2^{3k_1}) \int_0^t 2^{-2k} \|(S_k^1, S_k^2, S_k^3, S_k^4)(\tau)\|_{L^2}^2 d\tau.
\end{align*}
(3.62)

Putting (3.62) into (3.61) and recalling the definitions of \( S_k^i \) \( (i = 1, 2, 3, 4) \) yield
\begin{align*}
\|(\rho_k, d_k, \theta_k, j_{0,k})(t)\|_{L^2}^2 + \int_0^t \left( 2^{2k} \|(\rho_k, d_k, \theta_k, j_{0,k})(\tau)\|_{L^2}^2 + \|(\gamma \theta_k - b j_{0,k})(\tau)\|_{L^2}^2 \right) d\tau \\
\leq \|(\rho_k, d_k, \theta_k, j_{0,k})(0)\|_{L^2}^2 + \int_0^t 2^{-2k} \|(S_k^1, S_k^2, S_k^3, S_k^4)(\tau)\|_{L^2}^2 d\tau \\
+ C \int_0^t 2^{-2k} \|(S_k^1, S_k^2, S_k^3, S_k^4)(\tau)\|_{L^2}^2 d\tau,
\end{align*}
(3.63)
for \( k_0 \leq k \leq k_1 \).

Now, we show the estimates of long wave parts for the solution to (3.1) as follows. According to the definition in (5.1) and the definitions of \( S^i \) \((i = 1, 2, 3, 4)\), \( D_k \) and \( P S^2_k \), multiplying by \( 2^{ks'} \) and summing up over \( k \) with \( k \leq k_0 \) for (3.51), \( k_0 \leq k \leq k_1 \) for (3.63) and \( k \leq k_1 \) for (3.54), we get the following lemma.

**Proposition 3.3.** Let \( s' \) be any real number. The following inequality holds true for the solution to (3.1)

\[
\begin{align*}
\|(\rho, u, \theta, j_0)\|^2_{B_{2,2}^{s'}} + \int_0^t \left( \|\nabla \rho(\tau)\|^2_{H^3} + \|\nabla u(\tau)\|^2_{H^3} + \|\nabla \theta - b j_0(\tau)\|^2_{H^3} \right) d\tau \\
\leq \|(\rho, u, \theta, j_0)\|^2_{B_{2,2}^{s'}} + \int_0^t \left( \|(S^1, S^2, S^3, S^4)\|^2_{B_{2,2}^{s'-1}} \right) d\tau.
\end{align*}
\] (3.64)

### 3.4 Estimates of the nonlinear problem

It is ready for us to prove the global existence and uniqueness of solution stated in Theorem 1.1. Under the uniform a priori assumption (2.3), by using the Sobolev imbedding inequality, we have

\[
\frac{1}{2} \leq \rho + 1 \leq \frac{3}{2}.
\]

This will be used frequently in this section. Therefore, for some positive constant \( C \), we obtain

\[
|g(\rho)| \leq C \rho, \quad |h(\rho)| \leq C \quad \text{and} \quad |g^{(k)}(\rho)|, |h^{(k)}(\rho)| \leq C, \quad \text{for } k \geq 1.
\] (3.65)

Firstly, for \( t \in [0, T] \), we define

\[
N(t) := \sup_{0 \leq \tau \leq t} \|\rho, u, \theta, j_0(\tau)\|^2_{H^4} + \int_0^t \left( \|\nabla \rho(\tau)\|^2_{H^3} + \|\nabla u(\tau)\|^2_{H^3} + \|\nabla \theta - b j_0(\tau)\|^2_{H^3} \right) d\tau.
\] (3.66)

From Propositions 3.1 and 3.3 (with \( s = s' = 4 \) or \( s = 1, s' = 0 \)), Lemma 5.1 and Lemma 5.7 (with \( q = 2, p = \frac{8}{3} \) and \( l = 1 \)), by invoking the equivalence of the norms for \( s \geq 0 \)

\[
\|f\|_{H^s} \sim \|f\|_{L^2} + \|f\|_{H^s},
\]

we have

\[
N(t) \lesssim N(0) + \int_0^t \left( \|(S^1)^2(\tau)\|^2_{H^4} + \|(S^1, S^2, S^3, S^4, (\nabla u)^T \cdot \nabla \rho)\|^2_{H^3} \right) d\tau
\]
\[+ \int_0^t \left( \|\nabla^2 \rho\|^2_{L^\infty} \|u S\|^2_{H^3} + \|\nabla u(\tau)\|^2_{L^\infty} \|\nabla \rho S(\tau)\|^2_{H^3} \right) d\tau
\]
\[+ \int_0^t \|\nabla u\|^2_{L^\infty} \sum_{k > k_1} (1 + 2^{6k}) \left( \sum_{l \leq k-1} 2^{k-1} \|\nabla \rho_l\|^2_{L^2} \right) d\tau
\]
\[+ \int_0^t \|(S^1, S^2, S^3, S^4)\|^2_{L^{\frac{8}{3}}} d\tau,
\] (3.67)
where we used the facts that

$$
\|(S^{12})^2\|_{L^2} \lesssim \|S^{12}\|_{H^4},
$$

$$
\|(S^1, S^2, S^3, S^4)\|_{L^2} \lesssim \|(S^1, S^2, S^3, S^4)\|_{L^2} \lesssim \|(S^1, S^2, S^3, S^4)\|_{H^3},
$$

and

$$
\|(S^1, S^2, S^3, S^4)\|_{L^2} \lesssim \|\Lambda^{-1}(S^1, S^2, S^3, S^4)\|_{L^2} \lesssim \|(S^1, S^2, S^3, S^4)\|_{L^2}.
$$

For the nonlinear terms on the r.h.s. of \((3.60)\), by using the Sobolev imbedding inequality in Lemmas 5.4 and 5.5 we have

$$
\|S^{12}\|_{H^4} \lesssim \|\rho\|_{L^\infty} \|\text{div} u\|_{H^4} + \|\rho\|_{H^4} \|\text{div} u\|_{L^\infty}
$$

$$
\lesssim \|\nabla \rho\|_{H^1} \|\text{div} u\|_{H^4} + \|\rho\|_{H^4} \|\nabla \text{div} u\|_{H^1}
$$

$$
\lesssim \|\rho\|_{H^4} \|\text{div} u\|_{H^4},
$$

$$
\|S^1\|_{H^3} \lesssim \|S^{11}\|_{H^3} + \|S^{12}\|_{H^3}
$$

$$
\lesssim \|u\|_{L^\infty} \|\nabla \rho\|_{H^3} + \|u\|_{H^3} \|\nabla \rho\|_{L^\infty} + \|S^{12}\|_{H^3}
$$

$$
\lesssim \|u\|_{H^3} \|\nabla \rho\|_{H^3} + \|\rho\|_{H^3} \|\text{div} u\|_{H^3}.
$$

By using the Hölder inequality and Lemma 5.4 we have

$$
\|S^1\|_{L^2} \leq \|S^{11}\|_{L^\infty} + \|S^{12}\|_{L^\infty}
$$

$$
\lesssim \|u\|_{L^2} \|\nabla \rho\|_{L^3} + \|\rho\|_{L^2} \|\nabla u\|_{L^3}
$$

$$
\lesssim \|u\|_{L^2} \|\nabla \rho\|_{H^1} + \|\rho\|_{L^2} \|\nabla u\|_{H^1}.
$$

To handle the term of \(\|S^2\|_{H^3}\), by using \((3.65)\) and Lemmas 5.4 5.6 we have the following inequalities.

$$
\|S^2\|_{H^3} \lesssim \|u\|_{L^\infty} \|\nabla u\|_{H^3} + \|u\|_{H^3} \|\nabla u\|_{L^\infty} + \|g(\rho)\|_{L^\infty} \|\nabla \rho\|_{H^3}
$$

$$
+ \|g(\rho)\|_{H^3} \|\nabla \rho\|_{L^\infty} + \|h(\rho)\|_{L^\infty} \left(\|\theta\|_{L^\infty} \|\nabla \rho\|_{H^3} + \|\theta\|_{H^3} \|\nabla \rho\|_{L^\infty}\right)
$$

$$
+ \|h(\rho)\|_{H^3} \|\theta\|_{L^\infty} \|\nabla \rho\|_{L^\infty} + \|g(\rho)\|_{L^\infty} \|\nabla^2 u\|_{H^3}
$$

$$
+ \|g(\rho)\|_{H^3} \|\nabla^2 u\|_{L^\infty} + \|g(\rho)\|_{L^\infty} \|\nabla j_0\|_{H^3} + \|g(\rho)\|_{H^3} \|\nabla j_0\|_{L^\infty}
$$

$$
\lesssim \|u\|_{H^3} \|\nabla u\|_{H^3} + \|\theta\|_{H^3} \|\nabla \rho\|_{H^3}
$$

$$
+ \|\rho\|_{H^3} (\|\nabla \rho\|_{H^3} + \|\nabla u\|_{H^3} + \|\nabla j_0\|_{H^3} + \|\nabla j_0\|_{H^3}).
$$

Similarly,

$$
\|S^2\|_{H^3} + \|S^{11}\|_{H^3} \lesssim \|\theta\|_{H^3} \|\nabla u\|_{H^3} + \|\rho\|_{H^3} \|\nabla^2 \theta\|_{H^3} + \|\nabla \theta\|_{H^3} \|\theta\|_{H^3}
$$

$$
+ \|\nabla \rho\|_{H^3} (\|\theta\|_{H^3} + \|j_0\|_{H^3})
$$

$$
+ (\|\nabla \theta\|_{H^1} + \|\nabla j_0\|_{H^1}) \|\nabla \rho\|_{H^3} + \|\nabla u\|_{H^3}^2
$$

$$
+ \|u\|_{H^3} (\|\nabla \rho\|_{H^3} + \|\nabla j_0\|_{H^3}).
$$
We end up all of the estimates with
\[ \|(S^2, S^3, S^4)\|_{L^q_T} \lesssim \|(\rho, u, \theta)\|_{L^2} \left( \|\theta\|_{H^1} + \|\nabla(\rho, u, \theta, j_0)\|_{H^1} + \|\nabla^2(\rho, u)\|_{H^1} \right). \]
Moreover, by using Lemmas 5.3 and 5.4 and the Young inequality for series convolution, we get
\[ \|(\nabla u)^T \cdot \nabla \rho\|_{H^3} \lesssim \|\nabla \rho\|_{L^1} \|\nabla u\|_{H^3} + \|\nabla \rho\|_{H^3} \|\nabla u\|_{L^\infty} \lesssim \|\nabla \rho\|_{H^3} \|\nabla u\|_{H^3}. \]

And it follows from Lemma 5.1 that
\[ \sup_{0 \leq \tau \leq t} \|\nabla \rho\|_{H^3} \lesssim \|\nabla \rho\|_{H^3} \|\nabla u\|_{H^3}. \]

By using Lemmas 5.3 and 5.4 and the Young inequality for series convolution, we get
\[
\int_0^t \left| \sum_{k > k_1} (1 + 2^{6k}) \left( \sum_{l \geq k-1} 2^{k-l} \|\nabla \rho_l\|_{L^2}^2 \right) \right| \, d\tau \\
\lesssim \sup_{0 \leq \tau \leq t} \|\nabla u(\tau)\|_{H^2} \int_0^t \left( \sum_{k \geq k_1} \sum_{l \geq k-1} 2^{k-l} \|\nabla \rho_l\|_{L^2}^2 \right) \, d\tau \\
\lesssim \sup_{0 \leq \tau \leq t} \|\nabla u(\tau)\|_{H^2}^2 \int_0^t \left( \sum_{k \geq k_1} \sum_{l \geq k-1} 2^{k-l} \|\nabla \rho_l\|_{L^2}^2 \right) \, d\tau \\
+ \sup_{0 \leq \tau \leq t} \|\nabla u(\tau)\|_{H^2}^2 \int_0^t \sum_{k \in \mathbb{Z}} \|\nabla \rho_k\|_{L^2}^2 \, d\tau \\
\lesssim \sup_{0 \leq \tau \leq t} \|\nabla u(\tau)\|_{H^2}^2 \int_0^t \sum_{k \in \mathbb{Z}} \|\nabla \rho_k\|_{L^2}^2 \, d\tau \\
+ \sup_{0 \leq \tau \leq t} \|\nabla u(\tau)\|_{H^2}^2 \int_0^t \sum_{k \in \mathbb{Z}} 2^{6k} \|\nabla \rho_k\|_{H^3}^2 \, d\tau \\
\lesssim \sup_{0 \leq \tau \leq t} \|\nabla u(\tau)\|_{H^2}^2 \int_0^t \|\nabla \rho\|_{H^3}^2 \, d\tau.
\]

We end up all of the estimates with
\[ N(t) \leq C \left( N(0) + N^{\frac{3}{2}}(t) + N^2(t) \right). \]

This allows to close the estimates globally provided \( N(0) \) is small enough. Then, we complete the proof of Proposition 2.2. The existence and uniqueness of solutions are a direct consequence of Proposition 2.3 and Proposition 2.2 by the standard continuity argument. □
4 Decay rates

**Proposition 4.1** (Large time behavior). Under the assumptions of Proposition 2.2, if the initial data satisfies an additional condition that \( \| (\rho^0, u^0, \theta^0, j_0^0) \|_{L^1(\mathbb{R}^3)} < +\infty \), there is a constant \( \bar{C}_1 > 0 \) such that for any \( t \in [0, T] \), the global solution \((\rho, u, \theta, j_0)(x, t)\) achieved above enjoys the following decay properties.

\[
\| \nabla^k (\rho, u, \theta, j_0)(t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_1 (1 + t)^{-\frac{3}{4} - \frac{k}{2} }, \quad \text{for } k = 0, 1, 2,
\]

\[
\| \nabla^k (\rho, u, \theta, j_0)(t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_1 (1 + t)^{-\frac{3}{4}} , \quad \text{for } k = 3, 4,
\]

\[
\| \partial_t (\rho, u)(t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_1 (1 + t)^{-\frac{3}{4} },
\]

\[
\| \partial_t (\theta, j_0)(t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_1 (1 + t)^{-\frac{3}{4}},
\]

\[
\| \nabla^k (b'(1)\theta - j_0)(t) \|_{L^2(\mathbb{R}^3)} \leq \bar{C}_1 (1 + t)^{-\frac{3}{4} - \frac{k+1}{2}}, \quad \text{for } k = 0, 1, 2.
\]

4.1 Energy estimates of the short wave part

The estimate of the solution in the short wave part is stated as the following proposition.

**Proposition 4.2.** The following inequality holds true

\[
\begin{align*}
\| (\rho, u, \theta, j_0)^S(t) \|^2_{B^2_{2,2}} + \| (\rho, u, \theta, j_0)^S(t) \|^2_{B^2_{2,2}} \\
\leq & \mathcal{e}^{-C_6 t} \left( \| (\rho, u, \theta, j_0)^S(0) \|^2_{B^2_{2,2}} + \| (\rho, u, \theta, j_0)^S(0) \|^2_{B^2_{2,2}} \right) \\
+ & \delta \int_0^t \mathcal{e}^{-C_6(t-\tau)} \| (\rho, u, \theta, j_0)^L(\tau) \|^2_{B^2_{2,2}} \ d\tau \\
+ & \delta \int_0^t \mathcal{e}^{-C_6(t-\tau)} \left( \| (\rho, u, \theta, j_0)^L(\tau) \|^2_{B^2_{2,2}} + \| (u, \theta)^L(\tau) \|^2_{B^2_{2,2}} \right) \ d\tau,
\end{align*}
\]

where the positive constant \( C_6 \) is independent of \( \delta \).

**Proof.** Multiplying \((4.27)\) by \( 2^{6k} \), we get for \( k > k_1 > 0 \)

\[
\begin{align*}
\frac{d}{dt} 2^{6k} H_{h,k}(t) + C_3 2^{6k} \| (\theta_k, j_{0,k})(t) \|^2_{L^2} + \frac{C_3}{2} 2^{2k_1} 2^{6k} \| (\rho_k, u_k)(t) \|^2_{L^2} \\
+ \frac{C_3}{2} 2^{8k} \| (\rho_k, u_k)(t) \|^2_{L^2} + C_3 2^{8k} \| (\theta_k, j_{0,k})(t) \|^2_{L^2} + C_3 2^{10k} \| (u_k, \theta_k, j_{0,k})(t) \|^2_{L^2} \\
\leq 2^{6k} \| (S^1_k, S^2_k, S^3_k, S^4_k, \nabla S^1_k)(t) \|^2_{L^2} + 2^{6k} \| (\nabla u)^T \cdot \nabla \rho_k(t) \|^2_{L^2} + 2^{6k} \| \nabla u \|_{L^\infty} \| \nabla \rho_k \|_{L^2}^2 \quad (4.2)
\end{align*}
\]
The $l^2$ summation over $k$ for from $k = k_1 + 1$ to $\infty$ in (4.2) yields the following inequality

$$
\frac{d}{dt} \sum_{k > k_1} 2^{6k} \mathcal{H}_{h,k}(t) + \|(\rho, u, \theta, j_0)S(t)\|_{B^2_{2,2}}^2
+ \|(\rho, u, \theta, j_0)S(t)\|_{B^2_{2,2}}^2 + \|(\rho, u, \theta, j_0)S(t)\|_{B^2_{2,2}}^2
\lesssim \|(S^1, S^2, S^3, S^4, \nabla S^{12})(t)\|_{B^3_{2,2}}^2 + \|((\nabla u)^T \cdot \nabla \rho)S(t)\|_{B^3_{2,2}}^2 + \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{B^3_{2,2}}^2 \quad (4.3)
$$

By Lemma 5.1, Lemmas 5.4 and 5.5 and the assumption (2.3), we have

$$
\|(\nabla S^{12})(t)\|_{B^3_{2,2}}^2 \lesssim \|\nabla^4 (\rho \text{div} u)\|_{L^2}
\lesssim \|\rho\|_{L^\infty} \|\nabla^4 \text{div} u\|_{L^2} + \|\nabla^4 \rho\|_{L^2} \|\text{div} u\|_{L^\infty} \quad (4.4)
$$

$$
\|(S^1)S(t)\|_{B^3_{2,2}}^2 \lesssim \|\nabla^3 (u \cdot \nabla \rho)\|_{L^2} + \|\nabla^3 S^{12}\|_{L^2}
\lesssim \|u\|_{L^\infty} \|\nabla^4 \rho\|_{L^2} + \|\nabla^3 u\|_{L^2} \|\nabla \rho\|_{L^\infty} + \|\nabla^3 S^{12}\|_{L^2} \quad (4.5)
$$

To handle the term $\|(S^2)S\|_{B^3_{2,2}}$, by Lemma 5.1, Lemmas 5.4, 5.6 and the assumption (2.3), the following inequality holds true.

$$
\|(S^2)S(t)\|_{B^3_{2,2}}^2 \lesssim \|u\|_{L^\infty} \|\nabla^4 u\|_{L^2} + \|\nabla^3 u\|_{L^2} \|\nabla u\|_{L^\infty}
+ \|g(\rho)\|_{L^\infty} \|\nabla^4 \rho\|_{L^2} + \|\nabla^3 g(\rho)\|_{L^2} \|\nabla \rho\|_{L^\infty}
+ \|h(\rho)\|_{L^\infty} \left( \|\theta\|_{L^\infty} \|\nabla^4 \rho\|_{L^2} + \|\nabla^3 \theta\|_{L^2} \|\nabla \rho\|_{L^\infty} \right)
+ \|\nabla^3 h(\rho)\|_{L^2} \|\theta\|_{L^\infty} \|\nabla \rho\|_{L^\infty}
+ \|g(\rho)\|_{L^\infty} \|\nabla^5 u\|_{L^2} + \|\nabla^3 g(\rho)\|_{L^2} \|\nabla^2 u\|_{L^\infty}
+ \|g(\rho)\|_{L^\infty} \|\nabla^4 j_0\|_{L^2} + \|\nabla^3 g(\rho)\|_{L^2} \|\nabla j_0\|_{L^\infty}
\lesssim \delta \left( \|\rho, u, \theta\|_{B^3_{2,2}} + \|u, \theta, j_0\|_{B^4_{2,2}} + \|\rho\|_{B^2_{2,2}} \right) \quad (4.6)
$$

Similarly,

$$
\|(S^3)S(t)\|_{B^3_{2,2}}^2 \lesssim \delta \left( \|\rho, u, j_0\|_{B^3_{2,2}} + \|u, \theta, j_0\|_{B^4_{2,2}} + \|\theta\|_{B^2_{2,2}} \right) \quad (4.7)
$$

Moreover, one has

$$
\|((\nabla u)^T \cdot \nabla \rho)S(t)\|_{B^3_{2,2}} \lesssim \|\nabla \rho\|_{L^\infty} \|\nabla^4 u\|_{L^2} + \|\nabla^4 \rho\|_{L^2} \|\nabla u\|_{L^\infty}
\lesssim \delta \|\rho, u\|_{B^4_{2,2}} \quad (4.8)
$$
\[ \| \nabla u \|_{L^\infty} \| (\nabla \rho)^S (t) \|_{\dot{B}^{1,2}_2}^2 \lesssim \delta \| \rho \|_{\dot{B}^{1,2}_2}^2, \]  
(4.9)

and

\[ \| \nabla^2 \rho \|_{L^\infty}^2 \| u^S \|_{\dot{B}^{3,2}_2}^2 \leq \| \nabla^2 \rho \|_{H^2}^2 \| u \|_{\dot{B}^{3,2}_2}^2 \lesssim \delta^2 \| u \|_{\dot{B}^{3,2}_2}^2. \]

Similar to the estimate (3.68), by using Lemmas 5.3-5.4, the Young inequality for series convolution and the assumption (2.3), we get

\[ \| \nabla u \|_{L^\infty}^2 \sum_{k > k_1} 2^{6k} \left( \sum_{l \geq k-1} 2^{k-l} \| \nabla \rho_l \|_{L^2}^2 \right) \lesssim \| \nabla u \|_{H^2}^2 \| \nabla \rho \|_{\dot{B}^{3,2}_2}^2 \lesssim \delta^2 \| \nabla \rho \|_{\dot{B}^{3,2}_2}^2. \]  
(4.10)

Substituting (4.1)-(4.10) into (4.13) yields

\[ \frac{d}{dt} \sum_{k > k_1} 2^{6k} \mathcal{H}_{h,k}(t) + \| (\rho, u, \theta, j_0)^{S}(t) \|_{\dot{B}^{3,2}_2}^2 \]

\[ + \| (\rho, u, \theta, j_0)^{S}(t) \|_{\dot{B}^{3,2}_2}^2 + \| (u, \theta, j_0)^{S}(t) \|_{\dot{B}^{3,2}_2}^2 \]

\[ \lesssim \delta \left( \| (\rho, u, \theta, j_0)^{L}(t) \|_{\dot{B}^{3,2}_2}^2 + \| (\rho, u, \theta, j_0)^{L}(t) \|_{\dot{B}^{3,2}_2}^2 + \| (u, \theta)^{L}(t) \|_{\dot{B}^{3,2}_2}^2 \right). \]  
(4.11)

By using the decomposition (5.3) in Lemma 5.2 and choosing the parameter \( \delta \) is suitably small, we obtain

\[ \frac{d}{dt} \sum_{k > k_1} 2^{6k} \mathcal{H}_{h,k}(t) + \frac{1}{2} \| (\rho, u, \theta, j_0)^{S}(t) \|_{\dot{B}^{3,2}_2}^2 \]

\[ + \frac{1}{2} \| (\rho, u, \theta, j_0)^{S}(t) \|_{\dot{B}^{3,2}_2}^2 + \frac{1}{2} \| (u, \theta, j_0)^{S}(t) \|_{\dot{B}^{3,2}_2}^2 \]

\[ \lesssim \delta \left( \| (\rho, u, \theta, j_0)^{L}(t) \|_{\dot{B}^{3,2}_2}^2 + \| (\rho, u, \theta, j_0)^{L}(t) \|_{\dot{B}^{3,2}_2}^2 + \| (u, \theta)^{L}(t) \|_{\dot{B}^{3,2}_2}^2 \right). \]  
(4.12)

It follows from (3.25) that

\[ \sum_{k > k_1} 2^{6k} \mathcal{H}_{h,k}(t) \lesssim \| (\rho, u, \theta, j_0)^{S}(t) \|_{\dot{B}^{3,2}_2}^2 + \| (\rho, u, \theta, j_0)^{S}(t) \|_{\dot{B}^{3,2}_2}^2, \]  
(4.13)

for any \( 0 \leq t \leq T \). Then, from (4.12) and (4.13), there exists a positive constant \( C_6 \) independent of \( \delta \) such that

\[ \frac{d}{dt} \sum_{k > k_1} 2^{6k} \mathcal{H}_{h,k}(t) + C_6 \sum_{k > k_1} 2^{6k} \mathcal{H}_{h,k}(t) \]

\[ \lesssim \| (\rho, u, \theta, j_0)^{L}(t) \|_{\dot{B}^{3,2}_2}^2 + \| (\rho, u, \theta, j_0)^{L}(t) \|_{\dot{B}^{3,2}_2}^2 + \| (u, \theta)^{L}(t) \|_{\dot{B}^{3,2}_2}^2. \]  
(4.14)

Multiplying (4.14) with \( e^{C_6 t} \) and integrating with respect to \( t \) over \( [0, t] \), we have

\[ \sum_{k > k_1} 2^{6k} \mathcal{H}_{h,k}(t) \lesssim e^{-C_6 t} \sum_{k > k_1} 2^{6k} \mathcal{H}_{h,k}(0) + \delta \int_0^t e^{-C_6 (t-\tau)} \| (\rho, u, \theta, j_0)^{L}(\tau) \|_{\dot{B}^{3,2}_2}^2 \, d\tau \]

\[ + \delta \int_0^t e^{-C_6 (t-\tau)} \left( \| (\rho, u, \theta, j_0)^{L}(\tau) \|_{\dot{B}^{3,2}_2}^2 + \| (u, \theta)^{L}(\tau) \|_{\dot{B}^{3,2}_2}^2 \right) \, d\tau. \]  
(4.15)

Since \( \sum_{k > k_1} 2^{6k} \mathcal{H}_{h,k}(0) \sim \| (\rho_0, u_0, \theta_0, j_0)^{0} \|_{\dot{B}^{3,2}_2} + \| (\rho_0, u_0, \theta_0, j_0)^{0} \|_{\dot{B}^{3,2}_2} \), from (4.13) and (4.15), we get (4.1).
4.2 Decay rates of the long wave part

In this subsection, based on the $L^2$-norm decay estimates for Fourier analysis on the linearized system, we obtain decay estimates of the long wave parts of solutions to (3.1). Let $A$ be the following matrix of the differential operators of the form

$$
A = \begin{pmatrix}
0 & \nabla & 0 & 0 \\
\nabla & -\mu \Delta - (\mu + \lambda)\nabla \nabla & 0 & 0 \\
0 & 0 & -\frac{2}{3}k\Delta + \frac{2\gamma}{3} & -\frac{2}{3}b \\
0 & 0 & -C\gamma & -a\Delta + Cb
\end{pmatrix},
$$

and

$$
U_k(t) := (\rho_k(t), u_k(t), \theta_k(t), j_{0,k}(t))^T, \quad U_k(0) = (\rho_0^k, u_0^k, \theta_0^k, j_{0,k})^T.
$$

Applying the homogeneous frequency localized operator $\Delta_k$ to (3.1) yields for all $k \in \mathbb{Z}$ we have the following corresponding linearized problem

$$
\begin{cases}
\partial_t U_k + A U_k = 0, & \text{for } t > 0, \\
U_k|_{t=0} = U_k(0).
\end{cases}
$$

Applying the Fourier transform on (4.16) with respect to the $x$-variable and solving the ordinary differential equation with respect to $t$, we have

$$
U_k(t) = A(t)U_k(0),
$$

where $A(t) = e^{-tk}(t \geq 0)$ is the semigroup generated by the linear operator $A$ and $A(t)g := \mathcal{F}^{-1}(e^{-tk\xi})$ with

$$
A_\xi = \begin{pmatrix}
0 & i\xi^T \\
i\xi & i|\xi|^2\delta_{ij} + \xi_i\xi_j & 0 & 0 \\
0 & \frac{2}{3}i\xi^T & i|\xi|^2 + \frac{2\gamma}{3} & -\frac{2}{3}b \\
0 & 0 & -C\gamma & a|\xi|^2 + Cb
\end{pmatrix}.
$$

Lemma 4.1. Let $m \geq 0$ be an integer and $1 \leq p \leq 2$, then for any given $k \leq k_1$, it holds that

$$
\|\nabla^m (A(t)U_k(0))\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}(\frac{2}{p} - \frac{1}{2}) - \frac{m}{p}}\|U(0)\|_{L^p}.
$$

Proof. For the linearized system of (3.1), similar to the estimate (3.16), we have

$$
\frac{d}{dt}L(t, \xi) + C_4|\xi|^2L(t, \xi) \leq 0,
$$

which implies that for $|\xi| \leq r_0$

$$
L(t, \xi) \leq e^{-C_4|\xi|^2t}L(0, \xi).
$$
By using the Plancherel theorem, (3.49), (4.55) and (4.18), we have for $k \leq k_1$
\[
\|\partial_x^\alpha (\widehat{\rho_k}, \widehat{d_k}, \widehat{\theta_k}, \widehat{j_{0,k}})(t)\| = \|(i\xi)\alpha (\widehat{\rho_k}, \widehat{d_k}, \widehat{\theta_k}, \widehat{j_{0,k}})(\xi)\|_{L^2_x}
\]
\[
= \left( \int_{\mathbb{R}^3} \left| (i\xi)\alpha (\widehat{\rho_k}, \widehat{d_k}, \widehat{\theta_k}, \widehat{j_{0,k}})(\xi, t) \right|^2 d\xi \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \int_{|\xi| \leq R_0} |\xi|^{2|\alpha|} |(\widehat{\rho_k}, \widehat{d_k}, \widehat{\theta_k}, \widehat{j_{0,k}})(\xi, t)|^2 d\xi \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \int_{|\xi| \leq r_0} |\xi|^{2|\alpha|} e^{-C|\xi|^2 t} |(\widehat{\rho_k}, \widehat{d_k}, \widehat{\theta_k}, \widehat{j_{0,k}})(\xi, 0)|^2 d\xi \right)^{\frac{1}{2}}
\]
\[
+ C \left( \int_{r_0 \leq |\xi| \leq R_0} |\xi|^{2|\alpha|} e^{-|\xi|^2 t} |(\widehat{\rho_k}, \widehat{d_k}, \widehat{\theta_k}, \widehat{j_{0,k}})(\xi, 0)|^2 d\xi \right)^{\frac{1}{2}}.
\] (4.19)

From (4.19) and the Hausdorff-Young inequality, for $k \leq k_1$, we have
\[
\|\partial_x^\alpha (\widehat{\rho_k}, \widehat{d_k}, \widehat{\theta_k}, \widehat{j_{0,k}})(t)\|_{L^2} \leq C \| (\widehat{\rho_k}, \widehat{d_k}, \widehat{\theta_k}, \widehat{j_{0,k}})(0) \|_{L^{p/(p-1)}} (1 + t)^{-\frac{3}{4} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{|\alpha|}{2}}
\] (4.20)
\[
\leq C \| (\rho, u, \theta, j_0)(0) \|_{L^p} (1 + t)^{-\frac{3}{4} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{|\alpha|}{2}}.
\]

Similar as the estimates (4.20), we get for any $k \leq k_1$
\[
\|\partial_x^\alpha (\widehat{\rho_k}, \widehat{d_k}, \widehat{\theta_k}, \widehat{j_{0,k}})(t)\|_{L^2} \leq C \|u(0)\|_{L^p} (1 + t)^{-\frac{3}{4} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{|\alpha|}{2}}.
\] (4.21)

Then, from (4.20) and (4.21), we complete the proof of Lemma 4.1. \hfill\square

In what follows, based on the estimates in Lemma 4.1, we establish the time decay estimates on the long wave part of the solution to the nonlinear problem (3.1). Denote
\[
\mathbb{U}_k(t) := (\rho_k(t), u_k(t), \theta_k(t), j_{0,k}(t))^T,
\]
for any $k \leq k_1$. Then, from (3.1), we have
\[
\begin{cases}
\partial_t \mathbb{U}_k + A \mathbb{U}_k = S(\mathbb{U}_k), & \text{for } t > 0, \\
\mathbb{U}_{k|t=0} = \mathbb{U}_k(0),
\end{cases}
\] (4.22)

where
\[
S(\mathbb{U}_k) = (S^1_k, S^2_k, S^3_k, S^4_k)^T.
\]

By Duhamel’s principle, we rewrite the solution of (4.22) as follows
\[
\mathbb{U}_k(t) = \mathcal{A}(t) \mathbb{U}_k(0) + \int_0^t \mathcal{A}(t - \tau) S(\mathbb{U}_k)(\tau) d\tau.
\] (4.23)
Proposition 4.3. For any integer $m \geq 0$, there exists a positive constant $C_7$ depending on $k_1$, such that

$$
\| (\rho, u, \theta, j_0)^t(t) \|_{L^m_{t,2}} \leq C_7 (1 + t)^{-\frac{3}{4} - \frac{m}{2}} \| (\rho, u, \theta, j_0)(0) \|_{L^1} \nonumber
$$

$$
+ C_7 \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{m}{2}} \| \nabla (\rho, u, \theta, j_0)(\tau) \|_{L^2} d\tau \nonumber
$$

$$
+ C_7 \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{m}{2}} \| \nabla^2 (u, \theta)(\tau) \|_{L^2} d\tau \nonumber
$$

$$
+ C_7 \int_0^t (1 + t - \tau)^{-\frac{m}{2}} \| S(U) \|_{L^2} d\tau,
$$

where $S(U) := (S^1, S^2, S^3, S^4)$.

Proof. From (4.23), by using Lemma 4.1, we have, for $k \leq k_1$

$$
\| \nabla^m U_k(t) \|_{L^2} \leq C (1 + t)^{-\frac{3}{4} - \frac{m}{2}} \| U(0) \|_{L^1} \nonumber
$$

$$
+ C \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{m}{2}} \| S(U)(\tau) \|_{L^1} d\tau \tag{4.24}
$$

$$
+ C \int_0^t (1 + t - \tau)^{-\frac{m}{2}} \| S(U)(\tau) \|_{L^2} d\tau. \nonumber
$$

Notice that under the condition (2.3), by using the Hölder inequality, we have

$$
\| S(U)(\tau) \|_{L^1} \leq C \| (\rho, u, \theta, j_0)(\tau) \|_{H^1} \left( \| \nabla (\rho, u, \theta, j_0)(\tau) \|_{L^2} + \| \nabla^2 (u, \theta)(\tau) \|_{L^2} \right) \nonumber
$$

$$
+ \| \theta \|_{L^2} + \| \rho \|_{L^2} \left( \| \theta \|_{L^2} + \| j_0 \|_{L^2} \right) \tag{4.25}
$$

$$
\leq C \delta \left( \| \nabla (\rho, u, \theta, j_0)(\tau) \|_{L^2} + \| \nabla^2 (u, \theta)(\tau) \|_{L^2} \right) + C \| (\rho, \theta, j_0) \|_{L^2}^2.
$$

Then, putting (4.25) into (4.24) and taking a $l^2$ summation over $k$ with $k \leq k_1$, we complete the proof of Proposition 4.3.

4.3 Proof of Proposition 4.1

In this subsection, by combining Proposition 4.2 with Proposition 4.3, we get the large time behavior of solution to the nonlinear problem (3.1).

Lemma 4.2. Under the assumptions of Proposition 4.1, it holds that

$$
\| \nabla^m (\rho, u, \theta, j_0)(t) \|_{L^2} \leq C (1 + t)^{-\frac{3}{4} - \frac{m}{2}}, \quad \text{for } m = 0, 1, 2, \tag{4.26}
$$

$$
\| \nabla^m (\rho, u, \theta, j_0)(t) \|_{L^2} \leq C (1 + t)^{-\frac{m}{4}}, \quad \text{for } m = 3, 4. \tag{4.27}
$$
Proof. Denote that
\[ M(t) := \sup_{0 \leq \tau \leq t} \sum_{m=0}^{2} (1 + \tau)^{\frac{3}{4} + \frac{m}{2}} \| \nabla^m (\rho, u, \theta, j_0)(\tau) \|_{L^2}. \] (4.28)

Notice that \( M(t) \) is non-decreasing, and for \( 0 \leq m \leq 2 \)
\[ \| \nabla^m (\rho, u, \theta, j_0)(\tau) \|_{L^2}^2 \leq C_8 (1 + \tau)^{-\frac{3}{4} - \frac{m}{2}} M(t), \quad 0 \leq \tau \leq t, \] (4.29)
holds true for some positive constant \( C_8 \) independent of \( \delta \).

By using the Hölder inequality and \cite{3.65}, we have
\[ \| S(U) \|_{L^2} \lesssim \| (\rho, u, \theta) \|_{L^\infty} \| \nabla (\rho, u, \theta, j_0) \|_{L^2} + \| \rho \|_{L^\infty} \| \nabla^2 (u, \theta) \|_{L^2} + \| \nabla u \|_{H^1} \| \theta \|_{L^2} + \| \rho \|_{L^\infty} \| (\theta, j_0) \|_{L^2} \] (4.30)
Combining Proposition \cite{4.23} \cite{4.28} and \cite{4.30}, we have for \( m \geq 0 \)
\[ \| (\rho, u, \theta, j_0)^L(t) \|_{B^m_{2,2}} \leq \| (\rho, u, \theta, j_0)(0) \|_{L^1} \]
\[ + C_7 \delta \| M(t) \| \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{m}{2}} (1 + \tau)^{-\frac{7}{4}} d\tau \]
\[ + C_7 \delta \| M(t) \| \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{m}{2}} (1 + \tau)^{-\frac{7}{4}} d\tau \]
\[ + C_7 \delta \| M^T(t) \| \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{m}{2}} (1 + \tau)^{-\frac{7}{4}} d\tau \]
\[ + C_7 \delta \| M^T(t) \| \int_0^t (1 + t - \tau)^{-\frac{3}{4} - \frac{m}{2}} (1 + \tau)^{-\frac{7}{4}} d\tau. \] (4.31)

From (4.31), we arrive at
\[ \| (\rho, u, \theta, j_0)^L(t) \|_{B^m_{2,2}} \]
\[ \leq \begin{cases} 
C_9 (1 + t)^{-\frac{3}{4} - \frac{m}{2}} \| (\rho, u, \theta, j_0)(0) \|_{L^1} + \delta \| M(t) \| + \delta \| M^T(t) \|, & \text{for } 0 \leq m \leq 2, \\
C_9 (1 + t)^{-\frac{7}{4}} \| (\rho, u, \theta, j_0)(0) \|_{L^1} + \delta \| M(t) \| + \delta \| M^T(t) \|, & \text{for } m \geq 3,
\end{cases} \] (4.32)
where \( C_9 \) denotes some positive constant independent of \( \delta \). It follows from (4.1) and (4.32) that
\[ \| (\rho, u, \theta, j_0)^S(t) \|_{B^2_{3,2}} + \| (\rho, u, \theta, j_0)^S(t) \|_{B^4_{2,2}} \]
\[ \lesssim e^{-C_6 \tau} \left( \| (\rho, u, \theta, j_0)^S(0) \|_{B^2_{3,2}} + \| (\rho, u, \theta, j_0)^S(0) \|_{B^4_{2,2}} \right) \]
\[ + \delta C_9 \| (\rho, u, \theta, j_0)(0) \|_{L^1} + \delta \| M(t) \| + \delta \| M^T(t) \| \int_0^t e^{-C_6 (t-\tau)} (1 + \tau)^{-\frac{7}{4}} d\tau \] (4.33)
\[ \lesssim e^{-C_6 \tau} \left( \| (\rho, u, \theta, j_0)^S(0) \|_{B^2_{3,2}} + \| (\rho, u, \theta, j_0)^S(0) \|_{B^4_{2,2}} \right) \]
\[ + \delta C_9 (1 + t)^{-\frac{7}{4}} \| (\rho, u, \theta, j_0)(0) \|_{L^1} + \delta \| M(t) \| + \delta \| M^T(t) \|. \]
By using Lemma 5.2, we obtain for $0 \leq m \leq 2$
\[ \|\nabla^m (\rho, u, \theta, j_0)(t)\|_{L^2} \lesssim \|\nabla (\rho, u, \theta, j_0)^L(t)\|_{\dot{B}^0_{2,2}} + \|\nabla (\rho, u, \theta, j_0)^S(t)\|_{\dot{B}^0_{2,2}} \]
\[ \lesssim \|\nabla (\rho, u, \theta, j_0)^L(t)\|_{\dot{B}^0_{2,2}} + \|\nabla (\rho, u, \theta, j_0)^S(t)\|_{\dot{B}^0_{2,2}}. \tag{4.34} \]

From (4.32), (4.33) and (4.34), by noting the definition of $M(t)$ and using Lemma 5.2, there exists a positive constant $C_{10}$ independent of $\delta$, such that
\[ M(t) \leq C_{10}\{\|\nabla (\rho, u, \theta, j_0)(0)\|_{L^1 \cap H^4} + \delta^\frac{1}{8} M^\frac{1}{8}(t)\} \]
\[ \leq C_{10}\|\nabla (\rho, u, \theta, j_0)(0)\|_{L^1 \cap H^4} + \frac{1}{8} C_{10}^8 + \frac{7}{8} \delta^\frac{1}{8} M^2(t) \tag{4.35} \]
\[ := \tilde{C}_{10} + \frac{7}{8} \delta^\frac{1}{8} M^2(t). \]

Now we can claim $M(t) \leq C$. Suppose $M(t) > 2\tilde{C}_{10}$ for any $t \in [\tilde{t}, +\infty)$ with a constant $\tilde{t} > 0$. Since $M(0) = \|\nabla (\rho, u_0, \theta_0, j_0^0)\|_{H^2}$ is small (see the assumption (1.8)) and $M(t) \in C^0[0, +\infty)$, there exists $t_0 \in (0, \tilde{t})$ such that $M(t_0) = 2\tilde{C}_{10}$. From (4.35), we have
\[ M(t_0) \leq \tilde{C}_{10} + \frac{7}{8} \delta^\frac{1}{8} M^2(t_0). \]

By a directly calculation, we have
\[ M(t_0) \leq \frac{\tilde{C}_{10}}{1 - \frac{7}{8} \delta^\frac{1}{8} M(t_0)}. \tag{4.36} \]

Let $\delta$ be a small constant such that $\frac{7}{8} \delta^\frac{1}{8} < \frac{1}{4\tilde{C}_{10}}$. Then, from (4.36), we get $M(t_0) < 2\tilde{C}_{10}$. This become a contradiction with the assumption $M(t_0) = 2\tilde{C}_{10}$. So, we have $M(t) \leq 2\tilde{C}_{10}$ for any $t \in [\tilde{t}, +\infty)$. By using the continuity of $M(t)$, we have $M(t) \leq C$ for any $t \in [0, +\infty)$. By the definition of $M(t)$ in (4.23), we prove (4.26). Combining (4.32) for $m = 3, 4$ with (4.33) and using Lemma 5.2 and $M(t) \leq C$, we prove (4.27).

By using (4.26)-(4.27), from (3.1), we achieve
\[ \|\partial_t (\rho, u)(t)\|_{L^2} \lesssim \|\text{div} u(t)\|_{L^2} + \|\nabla (\rho, u, \theta, j_0)(t)\|_{L^2} + \|\text{div} T(t)\|_{L^2} \]
\[ + \|(S^1, S^2)(t)\|_{L^2} \]
\[ \lesssim \|\nabla (\rho, u)(t)\|_{L^2} + \|\nabla^2 u(t)\|_{L^2} \]
\[ \lesssim (1 + t)^{-\frac{3}{4}}, \]
and
\[ \|\partial_t (\theta, j_0)(t)\|_{L^2} \lesssim \|\text{div} u(t)\|_{L^2} + \|\Delta \theta(t)\|_{L^2} + \|(\theta, j_0)(t)\|_{L^2} \]
\[ + \|\Delta j_0(t)\|_{L^2} + \|(S^3, S^4)(t)\|_{L^2} \]
\[ \lesssim \|\nabla (\rho, u, \theta, j_0)(t)\|_{L^2} + \|(\theta, j_0)(t)\|_{L^2} + \|\nabla^2 (\theta, j_0)(t)\|_{L^2} \]
\[ \lesssim (1 + t)^{-\frac{3}{4}}. \]
Next, we show the decay estimates of the combination $L\sigma_a b'(1)\theta - L\sigma_a j_0$, i.e. $\gamma \theta - bj_0$. Set

$$\Xi = \gamma \theta - b j_0,$$

then from (3.1) and (3.1), we have

$$\partial_t \Xi + \left( \frac{2}{3} + C \right) \Xi + \frac{2}{3} \gamma \text{div } u - \frac{2\kappa}{3} \gamma \Delta \theta + ab \Delta j_0 = \gamma S^3 + b S^4. \quad (4.37)$$

Multiplying (4.37) with $\Xi$ and integrating with respect to $x$ in $\mathbb{R}^3$ and using the Young inequality, we have

$$\frac{d}{dt} \|\Xi(t)\|^2_{L^2} + \left( \frac{2}{3} + C \right) \|\Xi(t)\|^2_{L^2}$$

$$\leq \int_{\mathbb{R}^3} \left( - \frac{2}{3} \gamma \text{div } u \Xi + \frac{2\kappa}{3} \gamma \Delta \theta \Xi - ab \Delta j_0 \Xi + \gamma S^3 \Xi + b S^4 \Xi \right) dx \quad (4.38)$$

which implies

$$\frac{d}{dt} \|\Xi(t)\|^2_{L^2} + \left( \frac{1}{3} + C \right) \|\Xi(t)\|^2_{L^2}$$

$$\leq C \|\nabla u(t)\|^2_{L^2} + C \|\nabla^2(\theta, j_0)(t)\|^2_{L^2} + C \|(S^3, S^4)(t)\|^2_{L^2} \quad (4.39)$$

Multiplying (4.39) by $e^{\left(\frac{1}{3} + C\right)t}$ and integrating the resultant inequality with respect to $t$, we get

$$\|\Xi(t)\|^2_{L^2} \leq e^{-\left(\frac{1}{3} + C\right)t} \|\Xi(0)\|^2_{L^2} + \int_0^t e^{-\left(\frac{1}{3} + C\right)(t-\tau)} \|\nabla u(\tau)\|^2_{L^2} d\tau$$

$$+ \int_0^t e^{-\left(\frac{1}{3} + C\right)(t-\tau)} \left( \|\nabla^2(\theta, j_0)(\tau)\|^2_{L^2} + \|(S^3, S^4)(\tau)\|^2_{L^2} \right) d\tau. \quad (4.40)$$

From (4.40) and Lemma 4.2, we obtain

$$\|\Xi(t)\|^2_{L^2} \leq e^{-\left(\frac{1}{3} + C\right)t} \|\Xi(0)\|^2_{L^2} + C \int_0^t e^{-\left(\frac{1}{3} + C\right)(t-\tau)} (1 + \tau)^{-\frac{5}{4}} d\tau$$

$$+ C \int_0^t e^{-\left(\frac{1}{3} + C\right)(t-\tau)} (1 + \tau)^{-\frac{7}{4}} d\tau$$

$$\leq C (1 + t)^{-\frac{5}{4}}. \quad (4.41)$$

Similarly, we have

$$\|\nabla^k \Xi(t)\|^2_{L^2} \leq C (1 + t)^{-\frac{5}{4} - \frac{k+1}{2}}, \quad \text{for } k = 1, 2. \quad (4.42)$$

Thus, the proof of Proposition 4.1 is completed. □
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5 Appendix

In the Appendix, we recall some basic facts concerning Littlewood-Paley decomposition, Besov spaces and paraproduct. Let us first recall the Littlewood-Paley decomposition. For each \( j \in \mathbb{Z} \), set

\[
A_j = \{ \xi \in \mathbb{R}^3 | 2^{j-1} \leq |\xi| \leq 2^{j+1} \}.
\]

The Littlewood-Paley decomposition asserts the existence of a sequence of functions \( \{ \varphi_j \}_{j \in \mathbb{Z}} \subset S \) (\( S \) denotes the usual Schwartz class) such that

\[
\text{supp} \hat{\varphi}_j \subset A_j, \quad \hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j} \xi) \quad \text{or} \quad \varphi_j(x) = 2^{3j} \varphi_0(2^j x),
\]

and

\[
\sum_{j=-\infty}^{\infty} \hat{\varphi}_j(\xi) = \begin{cases} 
1, & \text{if } \xi \in \mathbb{R}^3 \setminus \{0\}, \\
0, & \text{if } \xi = 0.
\end{cases}
\]

Then the homogeneous Littlewood-Paley decomposition \((\hat{\Delta}_j)_{j \in \mathbb{Z}}\) over \( \mathbb{R}^3 \) is introduced by setting

\[
\hat{\Delta}_j u := \hat{\varphi}_j(D) u = 2^{3j} \int_{\mathbb{R}^3} \varphi_0(2^j y) u(x - y) dy, \quad j \in \mathbb{Z},
\]

and

\[
S_j u := \sum_{l \leq j - 1} \hat{\Delta}_l u, \quad j \in \mathbb{Z}.
\]

**Definition 5.1.** For any \( s \in \mathbb{R} \) and \((p, r) \in [1, +\infty] \times [1, +\infty] \), the homogeneous Besov space \( \dot{B}^s_{p,r}(\mathbb{R}^3) \) consists of \( f \in S'_h = S'/\mathcal{P} \) satisfying

\[
\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} := \left\| 2^{sk} \|\hat{\Delta}_k f\|_{L^p(\mathbb{R}^3)} \right\|_{l^r(\mathbb{Z})} < \infty,
\]

where \( S' \) and \( \mathcal{P} \) denote the dual of \( S \) and the space of polynomials, respectively.

Note that, for any \( f \in S'_h \), it can be rewritten as

\[
f = \sum_{k \in \mathbb{Z}} \hat{\Delta}_k f.
\]
We define that its long wave part and its short wave part are as follows

\[ f^L := \sum_{k \leq k_1} \hat{\Delta}_k f \quad \text{and} \quad f^S := \sum_{k > k_1} \hat{\Delta}_k f, \]

where the fixed positive integer \( k_1 \) is defined in (3.13). We also use the following notation

\[ \| f^L \|_{\dot{B}^s_{p,r}} := \left( \sum_{k \leq k_1} 2^{rk} \| \hat{\Delta}_k f \|_{L^p}^r \right)^{\frac{1}{r}} \quad \text{and} \quad \| f^S \|_{\dot{B}^s_{p,r}} := \left( \sum_{k > k_1} 2^{rk} \| \hat{\Delta}_k f \|_{L^p}^r \right)^{\frac{1}{r}}. \] (5.1)

In term of Definition 5.1 and the Plancherel formula, one can find that the homogeneous Sobolev space \( \dot{H}^s \) is a special case of homogeneous Besov spaces as follows, also see [1] (page 63) and [2] (Proposition A.3).

**Lemma 5.1.** For any \( s \in \mathbb{R} \),

\[ \dot{H}^s \sim \dot{B}^s_{2,2}. \]

For any \( s \in \mathbb{R} \) and \( 1 < q < \infty \),

\[ \dot{B}^s_{q,\min\{q,2\}} \hookrightarrow W^{s,q} \hookrightarrow \dot{B}^s_{q,\max\{q,2\}}, \]

and

\[ \dot{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \dot{B}^0_{q,\max\{q,2\}}. \]

In the following, we show some useful inequality in Besov space.

**Lemma 5.2.** For any \( s \geq 0 \) and \( m_1 \geq m_2 \geq 0 \), it holds that

\[ C \| \Lambda^{m_2} f^S \|_{\dot{B}^s_{2,2}} \leq \| \Lambda^{m_1} f^S \|_{\dot{B}^s_{2,2}}, \quad \| \Lambda^{m_1} f^L \|_{\dot{B}^s_{2,2}} \leq C \| \Lambda^{m_2} f^L \|_{\dot{B}^s_{2,2}}, \] (5.2)

and

\[ \| f \|_{\dot{B}^s_{2,2}} \sim \| f^L \|_{\dot{B}^s_{2,2}} + \| f^S \|_{\dot{B}^s_{2,2}}. \] (5.3)

**Proof.** The proof can be done by using the Plancherel theorem, the Bernstein inequality and the Definition 5.1 and (5.1). \[ \square \]

We recall the following estimates, cf. [2]. Here, we give the proof of Lemma 5.3 for the convenience of readers.

**Lemma 5.3.** Let \( v \) be a vector field over \( \mathbb{R}^3 \) and define the commutator

\[ [\hat{\Delta}_k, v \cdot \nabla] f = \hat{\Delta}_k (v \cdot \nabla f) - v \cdot \nabla \hat{\Delta}_k f. \]

Then the following inequality holds

\[ \int_{\mathbb{R}^3} [\hat{\Delta}_k, v \cdot \nabla] f \cdot \hat{\Delta}_k g \, dx \leq C \| \nabla v \|_{L^\infty} \| \hat{\Delta}_k f \|_{L^2} \| \hat{\Delta}_k g \|_{L^2} + C \| \nabla f \|_{L^\infty} \| \hat{\Delta}_k v \|_{L^2} \| \hat{\Delta}_k g \|_{L^2} \]

\[ + C \| \nabla v \|_{L^\infty} \| \hat{\Delta}_k g \|_{L^2} \sum_{l \geq k-1} 2^{k-l} \| \hat{\Delta}_l f \|_{L^2} . \]
Proof. By using the homogeneous Bony decomposition

\[ \| [\hat{\Delta}_k, v \cdot \nabla] f \|_{L^2} \leq \sum_{|k-l| \leq 2} \int_{\mathbb{R}^3} (\hat{\Delta}_k (S_{l-1} v \cdot \nabla \hat{\Delta}_l f) - S_{l-1} v \cdot \nabla \hat{\Delta}_k \hat{\Delta}_l f) \cdot \hat{\Delta}_k g \, dx \]

\[ + \sum_{|k-l| \leq 2} \int_{\mathbb{R}^3} (\hat{\Delta}_k (\hat{\Delta}_l v \cdot \nabla S_{l-1} f) - \hat{\Delta}_l v \cdot \nabla \hat{\Delta}_k S_{l-1} f) \cdot \hat{\Delta}_k g \, dx \]

\[ + \sum_{l \geq k-1} \int_{\mathbb{R}^3} (\hat{\Delta}_k (\hat{\Delta}_l v \cdot \nabla \hat{\Delta}_l f) - \hat{\Delta}_l v \cdot \nabla \hat{\Delta}_k \hat{\Delta}_l f) \cdot \hat{\Delta}_k g \, dx \]

:= K_1 + K_2 + K_3,

with \( \hat{\Delta}_k = \hat{\Delta}_{k-1} + \hat{\Delta}_k + \hat{\Delta}_{k+1} \). By a standard commutator estimate, the Hölder inequality and Bernstein’s inequality, one gets

\[ K_1 \leq C 2^{-k} \sum_{|k-l| \leq 2} \| \nabla S_{l-1} v \|_{L^\infty} \| \nabla \hat{\Delta}_l f \|_{L^2} \| \hat{\Delta}_k g \|_{L^2} \]

\[ \leq C \| \nabla v \|_{L^\infty} \sum_{|k-l| \leq 2} \| \hat{\Delta}_l f \|_{L^2} \| \hat{\Delta}_k g \|_{L^2}. \]

Similarly, one obtains

\[ K_2 \leq C \| \nabla f \|_{L^\infty} \sum_{|k-l| \leq 2} \| \hat{\Delta}_l v \|_{L^2} \| \hat{\Delta}_k g \|_{L^2}, \]

\[ K_3 \leq C \sum_{l \geq k-1} 2^{k-l} \| \nabla \hat{\Delta}_l v \|_{L^\infty} \| \hat{\Delta}_l f \|_{L^2} \| \hat{\Delta}_k g \|_{L^2} \]

\[ \leq C \| \nabla v \|_{L^\infty} \sum_{l \geq k-1} 2^{k-l} \| \hat{\Delta}_l f \|_{L^2} \| \hat{\Delta}_k g \|_{L^2}. \]

Since the summation over \( l \) for fixed \( k \) above consists of only a finite number of terms and the norm generated by each term is a multiple of that generated by the typical term, it suffices to keep the typical term with \( l = k \) and ignore the summation. This would help keep the presentation concise. Therefore, the proof of Lemma 5.3 is completed. \( \square \)

We list some important estimates in Sobolev space, which can be found in \cite{3, 22, 28, 30}.

**Lemma 5.4.** Let \( f \in H^2(\mathbb{R}^3) \). Then

(i) \( \| f \|_{L^\infty} \leq C \| \nabla f \|^{1/2} \| \nabla f \|^{1/2} \leq C \| \nabla f \|_{H^3}; \)

(ii) \( \| f \|_{L^6} \leq C \| \nabla f \|; \)

(iii) \( \| f \|_{L^q} \leq C \| f \|_{H^1}, \quad 2 \leq q \leq 6. \)

**Lemma 5.5.** Let \( m \geq 1 \) be an integer, then we have

\[ \| \nabla^m (fg) \|_{L^p} \leq C \| f \|_{L^{p_1}} \| \nabla^m g \|_{L^{p_2}} + C \| \nabla^m f \|_{L^{p_3}} \| g \|_{L^{p_4}}, \]  

where \( 1 \leq p_i \leq +\infty, \quad (i = 1, 2, 3, 4) \) and

\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \]
Lemma 5.6. Assume that $\|\phi\|_{L^\infty} \leq 1$. Let $f(\phi)$ be a smooth function of $\phi$ with bounded derivatives of any order, then for any integer $m \geq 1$ and $1 \leq p \leq +\infty$, we have

$$\|\nabla^m f(\phi)\|_{L^p} \leq C \|\nabla^m \phi\|_{L^p}.$$ 

Lemma 5.7. Let $0 < l < 3$, $1 < p < q < \infty$, $\frac{1}{q} + \frac{l}{3} = \frac{1}{p}$, then

$$\|\Lambda^{-l} f\|_{L^q} \lesssim \|f\|_{L^p}.$$ \hfill (5.5)

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