Boundary conditions changing operators in non conformal theories.

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Boundary conditions changing operators have played an important role in conformal field theory. Here, we study their equivalent in the case where a mass scale is introduced, in an integrable way, either in the bulk or at the boundary. More precisely, we propose an axiomatic approach to determine the general scalar products $b\langle \theta_1, \ldots, \theta_m | \theta'_1, \ldots, \theta'_n \rangle_a$ between asymptotic states in the Hilbert spaces with $a$ and $b$ boundary conditions respectively, and compute these scalar products explicitely in the case of the Ising and sinh-Gordon models with a mass and a boundary interaction. These quantities can be used to study statistical systems with inhomogeneous boundary conditions, and, more interestingly maybe, dynamical problems in quantum impurity problems. As an example, we obtain a series of new exact results for the transition probability in the double well problem of dissipative quantum mechanics.
1. Introduction

The concept of boundary conditions changing operators has been of crucial importance in the analysis of boundary conformal field theories, as well as in their applications to quantum impurity problems. In the last few years, important progress has been made in extending the solution of conformal field theories to theories perturbed either by a bulk or by a boundary operator. The latter have extremely interesting applications to the study of flows in quantum impurity problems.

As we will show here, it is possible to introduce boundary conditions changing operators even in the case where there is a mass scale, either in the bulk or at the boundary. It is not completely clear what the formal use of these objects (eg as illustrated in [1] for the conformal case) might be, but they certainly do have applications. From a 2D, statistical mechanics, point of view, one might wonder what is the effect of having different parts of the boundary with different boundary conditions [2], describing, for instance a situation with a (classical) “impurity” on the boundary. From the 1+1 point of view of quantum impurity problems, one might wish to describe situations where the coupling to the impurity is changed at some particular time, and one is interested in the subsequent time evolution of the degrees of freedom. In fact, one of the key observables in the two state problem of dissipative quantum mechanics, the well known quantity “\(P(t)\)” (see below), is essentially defined in that fashion [3].

In the conformal situation, only a discrete set of conformal boundary conditions are available, and the effect of switching from one to the other is described by the insertion on the boundary of the appropriate operator. For instance, in the Ising model, one goes from free to fixed spins by inserting the conformal operator \(\Phi_{12}\) of weight \(h = \frac{1}{16}\), and from fixed up to fixed down by inserting \(\Phi_{13}\) of weight \(h = \frac{1}{2}\) [1] (where the \(\Phi_{rs}\) are the usual degenerate conformal fields [1]). We are interested in the more general case where conformal invariance is broken at the boundary, and also maybe in the bulk. As an example, we can consider a situation where, in the Ising model, we apply a boundary magnetic field \(h_a\) for \(y < 0\), \(h_b\) for \(y > 0\) (we use coordinates \(x\) and \(y\) to describe the two dimensional space and the boundary sits at \(x = 0\) in this paper), and in addition, \(T \neq T_c\) in the bulk. In general, one does not expect this situation to be described by the insertion of a simple operator at \(y = 0\) anymore. Nevertheless, the change of boundary conditions can be fully characterized in the factorized scattering description (the Ising model with a boundary field being integrable) by scalar products of asymptotic states in the Hilbert
space with \( h_a \) and \( h_b \) (see below), and the “operator” inserted at \( y = 0 \) can thus be written, in principle, in terms of the Faddeev Zamolodchikov algebra. The problem then reduces to the determination of scalar products (we also call them transition factors), and is similar in nature to the problem of determining form-factors in integrable theories [5].

In this paper, we mostly restrict to the Ising case, where the computations are already moderately complicated, and obtain the complete solution to the problem with changing boundary magnetic fields in the massive theory. In section 2, we discuss the general problem of form-factors in the crossed channel. We determine the scalar products of interest for \( h_a h_b > 0 \) in section 3. In section 4, we discuss the limit where the bulk is massless, including the conformal invariant case. In section 5, we discuss the case \( h_a h_b < 0 \) which require a different treatment. Section 6 is devoted to extending some of these results to the sinh-Gordon model, where the non trivial bulk scattering matrix introduces further complications. In section 7, based on some reasonable conjectures, we apply and generalize our results to the determination of \( P(t) \) in dissipative quantum mechanics, for arbitrary value of the dissipation parameter \( g \).

Some of the results presented in this paper have appeared in a shorter version [6].

2. Form factors in the crossed channel

Consider the massive Ising model defined in the half plane \( x \in (-\infty, 0], y \in (-\infty, \infty) \), in the presence of a boundary magnetic field. The action reads

\[
A = \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \ a_{FF}(x, y) + \frac{1}{2} \int_{-\infty}^{\infty} dy \left[ (\psi \bar{\psi}) (x = 0) + a \dot{a} \right] + h \int_{-\infty}^{\infty} dy \sigma_B(y). \tag{2.1}
\]

Here \( a_{FF} \) is the usual massive free Majorana fermion action, \( a \) is a boundary fermion satisfying \( a^2 = 1 \), \( \sigma_B \) is the boundary spin operator, which coincides with \( \frac{1}{2} (\psi + \bar{\psi}) a \).

As discussed in [7], the problem can be studied from the point of view of the direct channel with \( x \) taken as time and \( y \) as space. In that case the Hilbert space is the usual one, and the boundary represented by a boundary state. On the other hand, in the crossed channel, one has a new Hilbert space for the theory on the half line. Setting \( z = x + iy \), the one point function of the energy is easy to compute in the direct channel. Using the boundary state formula [4]

\[
|B\rangle = \exp \left[ \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} K(\theta) Z^*(-\theta) Z^*(\theta) \right] |0\rangle, \tag{2.2}
\]
where \( K \) is related to the reflection matrix through \( K(\theta) = R\left(\frac{i\pi}{2} - \theta\right) \), and the two particle (bulk) form factor

\[
\langle 0|\epsilon|\theta_1, \theta_2 \rangle = im \sinh\left(\frac{\theta_1 - \theta_2}{2}\right),
\]

one obtains

\[
\langle \epsilon(z, \bar{z}) \rangle = -i \frac{m}{2} \int \frac{d\theta}{2\pi} K(\theta) \sinh \theta e^{2m x \cosh \theta}.
\]

An expression in the crossed channel follows by moving the contour of integration in the variable \( \theta (w = iz) \)

\[
\langle \epsilon(w, \bar{w}) \rangle = \frac{m}{2} \int \frac{d\theta}{2\pi} R(\theta) \cosh \theta e^{-2im x \sinh \theta}.
\]

There is no simple way to compute this expression directly in the crossed channel, except by solving the problem explicitly, writing mode expansions for the fermions (see appendix A). In particular, (2.5), which can be considered as a zero particle form-factor, does not, as far as we know, follow from any known nice form of the ground state of the theory with boundary.

Using the mode decomposition of fermions, it is possible to construct form factors with more particles. Because of the boundary, the states at \( \theta \) and \(-\theta\) are related: one has \( Z^*(\theta)|0\rangle_a = R(\theta)Z^*(-\theta)|0\rangle_a \) (here \( |0\rangle_a \) denotes the ground state with boundary condition of type \( a \) at \( x = 0 \), normalized such that \( a\langle 0|0\rangle_a = 1 \). We drop the label \( a \) when it is not necessary). We denote by \( ||\theta\rangle_a \) the one particle asymptotic state on the half line \( Z^*(\theta)|0\rangle_a \).

Then one has

\[
\langle 0|\epsilon(w, \bar{w})||\theta_1 \theta_2 \rangle = im e^{-my(\cosh \theta_1 + \cosh \theta_2)}
\]

\[
\times \prod_{k=1,2} \left[ 1 + R(\theta_k) t_k \right] \sinh\left(\frac{\theta_1 - \theta_2}{2}\right) e^{im(\sinh \theta_1 + \sinh \theta_2) x}
\]

where \( t_i \) is an operator acting on functions of many variables as follows

\[
t_i g(\cdots \theta_i \cdots) = g(\cdots - \theta_i \cdots), \quad t_i^2 = 1, \quad t_i t_j = t_j t_i.
\]

This expression, call it \( F_{aa}(\theta_1 \theta_2) \), obeys the relations

\[
F_{aa}(\theta_1, \theta_2) = R(\theta_1)F_{aa}(\theta_1, -\theta_2) = R(\theta_2)F_{aa}(\theta_1, -\theta_2) = -F_{aa}(\theta_2, \theta_1).
\]

1 Here, our normalization for bulk asymptotic states is \( \langle \theta_1|\theta_2 \rangle = 2\pi\delta(\theta_1 - \theta_2) \).

2 Rapidity integrals, unless specified, run from \(-\infty \) to \( \infty \).

3 The normalization is the same as in the bulk: \( \langle \theta_1||\theta_2 \rangle = 2\pi\delta(\theta_1 - \theta_2) \).
This will allow the use of integrals running from $-\infty$ to $\infty$ in the computation of correlators.

Observe that (2.6) does not have a pole at $\theta_2 = \theta_1 + i\pi$, as is only natural for the energy operator. Therefore, in the crossed channel, there is no relation between the two particle and the zero particle form-factor of the energy via kinematic poles.

In order to better understand and generalize these form factors, let us follow an axiomatic approach to try to compute them. Recently, axioms for form factors in the cross-channel in the presence of a boundary have been written for the XXZ spin chain [8]. For notations sake, we define the form factor in the cross-channel with boundary conditions of type $a$ to be

$$F_{aa}(\theta_1, \ldots, \theta_n) = \langle 0 | O Z^*(\theta_1), \ldots, Z^*(\theta_n) | 0 \rangle_a.$$  (2.9)

Following the authors of [8] and taking the continuum limit of their axioms we find, in our specific case that

$$F_{aa}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = S(\theta_i - \theta_{i+1})F_{aa}(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n)$$  (2.10)

$$F_{aa}(\theta_1, \ldots, \theta_i + 2\pi i, \ldots, \theta_n) = F_{aa}(\theta_1, \ldots, \theta_i, \ldots, \theta_n)S(\theta_{i-1} - \theta_i) \cdots S(\theta_1 - \theta_i)$$

$$\times R^*_a(\theta_i + i\pi)S(-\theta_1 - \theta_i) \cdots S(-\theta_{i-1} - \theta_i)S(-\theta_{i+1} - \theta_i) \cdots S(-\theta_n - \theta_i)$$

$$\times R^*_a(\theta_i)S(\theta_n - \theta_i) \cdots S(\theta_{i+1} - \theta_i)$$  (2.11)

The kinematic pole equation gives the residue of the form-factor as $\theta_j = \theta_1 + i\pi$, and it is given by

$$\text{Res}F_{aa}(\theta_1, \ldots, \theta_i, \ldots, \theta_n) = iF_{aa}(\hat{\theta}_1 \cdots \hat{\theta}_j \cdots \theta_n)$$

$$[S(\theta_1 - \theta_j) \cdots S(\theta_{j-1} - \theta_j) - S(\theta_j - \theta_{j+1}) \cdots S(\theta_j - \theta_n)R_a(\theta_j)$$

$$\times S(\theta_n + \theta_j) \cdots S(\theta_2 + \theta_j)R^*_a(\theta_j)].$$  (2.12)

Other residues are also generated from the bound states of the $S$ and $R$ matrices, if any.

We have written the axioms without internal indices, which are implicit in the previous equations. In the case of diagonal scattering, the $R$ matrix actually drops out from these equations due to $RR^* = 1$. The only dependence appears in the last, reflection equation

$$F_{aa}(\theta_1, \ldots, \theta_i, \ldots, \theta_n) = S(\theta_i - \theta_{i+1}) \cdots S(\theta_i - \theta_n)R_a(\theta_i)S(-\theta_n - \theta_i)$$

$$\times \cdots S(-\theta_{i+1} - \theta_i)F_{aa}(\theta_1, \cdots, -\theta_i, \cdots, \theta_n).$$  (2.13)

Up until now, solutions to these axioms have not been written and this is the task we plan to undertake here. In fact, we will first generalise these form factors slightly, and this will lead to a more general set of axioms.
3. The case of mixed boundary conditions

3.1. The two particle case

We now consider the case of inhomogeneous boundary: we suppose that the boundary conditions for \( y \in (-\infty, 0] \) are described by a boundary magnetic field \( h_a \), and for \( y \in [0, \infty) \) by a field \( h_b \). Suppose we wish to compute again \( \langle \epsilon(w, \bar{w}) \rangle \) in the cross-channel (in the direct channel, no expression for the boundary state \(|B\rangle\) is available). To do so, we have to be careful that the asymptotic states will depend on the boundary conditions in a more intricate way than through the simple relation \( ||\theta\rangle_a = R_a(\theta)|| - \theta\rangle_a \). Indeed, the scalar products

\[ b\langle \theta_n, \ldots, \theta_1||\theta_{n+1}\ldots\theta_{n+m}\rangle_a \]

have to be non zero in general, even for disjoint sets of rapidities \((m\ even)\), since the asymptotic states provide a complete set of states for any given boundary condition. As a result, the one-point function of the energy with mixed boundary conditions will read (for \( y > 0 \))

\[ \langle \epsilon(w, \bar{w})\rangle_{ba} = \int_0^\infty \frac{d\theta_1d\theta_2}{8\pi^2} \left\{ b\langle 0||\epsilon(w, \bar{w})||\theta_1\theta_2\rangle_b \ b\langle \theta_2\theta_1||0\rangle_a \\
+ b\langle 0||\epsilon(w, \bar{w})||0 \rangle_b \ b\langle 0||0\rangle_a \right\} \]

(3.1)

Let us now introduce the quantities

\[ \frac{b\langle \theta_2\theta_1||0\rangle_a}{b\langle 0||0\rangle_a} = G(\theta_1, \theta_2), \]

(3.2)

and

\[ \frac{b\langle 0||\theta_1\theta_2\rangle_a}{b\langle 0||0\rangle_a} = F(\theta_1, \theta_2). \]

(3.3)

To determine these scalar products we sometimes call transition factors, we use, as in the well known case of operators in the bulk, an axiomatic approach [5]. It is then straightforward to write the first set of axioms

\[ G(\theta_1, \theta_2) = -G(\theta_2, \theta_1) = R_b(\theta_1)G(-\theta_1, \theta_2) = R_b^*(\theta_2)G(\theta_1, -\theta_2). \]

(3.4)

To complement these, we need a residue condition. It is easily obtained as follows. Consider the one point function of the energy. It does have different expressions for \( y > 0 \), where

\[ \langle \epsilon \rangle_{ba} = \frac{m}{2} \int \frac{d\theta}{2\pi} R_b(\theta) \cosh \theta e^{-2imx\sinh \theta} - im \int \frac{d\theta_1d\theta_2}{8\pi^2} R_b(\theta_1) \sinh \frac{\theta_1 + \theta_2}{2} \]

(5)
and for $y < 0$, where

$$
\langle \epsilon \rangle_{ba} = \frac{m}{2} \int \frac{d\theta}{2\pi} R_a(\theta) \cosh \theta e^{-2imx \sinh \theta} + im \int \frac{d\theta_1 d\theta_2}{8\pi^2} R^*_a(\theta_1) \sinh \frac{\theta_1 + \theta_2}{2}
$$

$$
F(\theta_1, \theta_2) \exp[i m(\sinh \theta_1 - \sinh \theta_2) x] \exp[m(\cosh \theta_1 + \cosh \theta_2) y],
$$

where we used the symmetry relations (3.4) to write all integrals on the interval $(-\infty, \infty)$. By requiring that the second expression is the analytic continuation of the first, we find, moving the contours of integration, that $G$ must have a simple pole for $\theta_2 = \theta_1 - i\pi$, with residue as $\theta_2 \to \theta_1 - i\pi$,

$$
\text{Res} \ G(\theta, \theta - i\pi) = -i \left( 1 - \frac{R_a(\theta)}{R_b(\theta)} \right).
$$

(3.5)

This is equivalent (see below) to having a simple pole as $\theta_2 \to \theta_1 + i\pi$

$$
\text{Res} \ G(\theta, \theta + i\pi) = -i \left( 1 - \frac{R_b(\theta)}{R_a(\theta)} \right).
$$

(3.6)

Similarly, $F$ has poles for the same values $\theta_2 = \theta_1 \pm i\pi$, with residues switched.

Recall that the $R$ matrix reads [7]

$$
R_a = i \tanh \left( i \frac{\pi}{4} - \frac{\theta}{2} \right) \frac{\kappa_a - i \sinh \theta}{\kappa_a + i \sinh \theta},
$$

(3.7)

with $\kappa_a = 1 - h_a^2/(4\pi)$. It obeys the relations

$$
R(\theta + i\pi) = -\frac{1}{R(\theta)} = -R^*(\theta).
$$

(3.8)

In the following, we parametrize $\kappa_a = -\cosh \theta_a$ (and similarly for $\kappa_b$). It is useful to observe that

$$
\frac{\kappa_a - i \sinh \theta}{\kappa_a + i \sinh \theta} = i \tanh \left( \frac{\theta + \theta_a}{2} - i \frac{\pi}{4} \right) i \tanh \left( \frac{\theta - \theta_a}{2} - i \frac{\pi}{4} \right),
$$

(3.9)

Let us now try to determine the functions $F, G$. To do so, consider the equation

$$
\Phi(\theta) \Phi(\theta + i\pi) = \frac{1}{\cosh \left( \frac{\theta - \theta_a}{2} - i \frac{\pi}{4} \right) \cosh \left( \frac{\theta - \theta_a}{2} + i \frac{\pi}{4} \right)}.
$$

(3.10)

When doing so, one has to use the normalization $\langle \theta_1 | \theta_2 \rangle = 2\pi \left[ \delta(\theta_1 - \theta_2) + R^*(\theta_1) \delta(\theta_1 + \theta_2) \right]$. 

The presence of two delta terms is compensated by factors $\frac{1}{2}$ one has to introduce to get the integrations running over the whole real axis.
Its simplest solution is

\[
\Phi(\theta|\theta_b, \theta_a) \equiv \frac{1}{\cosh \left(\frac{\theta - \theta_b}{2} - \frac{i\pi}{4}\right)} \prod_{n=0}^{\infty} \frac{\Gamma \left(\frac{5}{4} + n - i \frac{\theta - \theta_a}{2\pi}\right)}{\Gamma \left(\frac{3}{4} + n - i \frac{\theta - \theta_a}{2\pi}\right)} \frac{\Gamma \left(\frac{5}{4} + n + i \frac{\theta - \theta_a}{2\pi}\right)}{\Gamma \left(\frac{3}{4} + n + i \frac{\theta - \theta_a}{2\pi}\right)},
\]

or, using an integral representation,

\[
\Phi(\theta|\theta_b, \theta_a) \equiv \frac{1}{\cosh \left(\frac{\theta - \theta_b}{2} - \frac{i\pi}{4}\right)} \exp \left[ \int_{-\infty}^{\infty} \frac{dt}{t} e^{i \frac{\theta - \theta_a}{2\pi} t} e^{-i \frac{\theta - \theta_a}{2\pi} t} \right].
\]

The function \(\Phi\) has no poles in the physical strip \(\text{Im} \, \theta \in [0, \pi]\), and a single pole \(\theta = \theta_b - \frac{i\pi}{2}\) in \(\text{Im} \, \theta \in [-\pi, 0]\). It satisfies the following identities

\[
\Phi^*(\theta|\theta_b, \theta_a) = \Phi(-\theta|\theta_b, -\theta_a) = \frac{1}{i \tanh \left(\frac{\theta - \theta_b}{2} - \frac{i\pi}{4}\right)} \Phi(\theta|\theta_b, \theta_a),
\]

together with

\[
\Phi(\theta + i\pi|\theta_b, \theta_a) = \frac{1}{i \tanh \left(\frac{\theta - \theta_b}{2} - \frac{i\pi}{4}\right)} \Phi(\theta|\theta_a, \theta_b).
\]

The asymptotics of \(\Phi\) can be worked out easily from the integral representation. If \(\theta - \theta_{a,b} \to \infty\), one has \(\Phi \approx 2\omega e^{-\theta/2} e^{(\theta_{a}+\theta_{b})/4} (\omega = e^{i\pi/4})\), while in the opposite case \(\theta - \theta_{a,b} \to -\infty\), one has \(\Phi \approx 2\omega^{-1} e^{\theta/2} e^{-(\theta_{a}+\theta_{b})/4}\).

We now introduce the quantity

\[
f(\theta|\theta_b, \theta_a) = \sqrt{-i(\kappa_a - \kappa_b)} \Phi(\theta|\theta_b, \theta_a) \Phi(\theta|\theta_b, -\theta_a).
\]

It is important to stress that, despite the notation, this function is independently an even function of \(\theta_a\) and \(\theta_b\). By convention, we always chose these variables to have positive real part in what follows. The function \(f\) obeys

\[
f(\theta)f(\theta + i\pi) = \frac{(\kappa_a - \kappa_b)}{i \cosh \left(\frac{\theta + \theta_b}{2} - \frac{i\pi}{4}\right) \cosh \left(\frac{-\theta_b}{2} - \frac{i\pi}{4}\right) \cosh \left(\frac{\theta + \theta_a}{2} + \frac{i\pi}{4}\right) \cosh \left(\frac{-\theta_a}{2} + \frac{i\pi}{4}\right)}
\]

\[
= \frac{-2}{\sinh \theta} \left(1 - \frac{R_b}{R_a}\right),
\]

together with

\[
f(\theta + i\pi|\theta_b, \theta_a) = f^*(\theta|\theta_a, \theta_b),
\]
\[ f(\theta + 2i\pi|\theta_b, \theta_a) = R_a R_b^*(\theta) f(\theta|\theta_b, \theta_a), \quad (3.18) \]

and
\[ f(\theta|\theta_b, \theta_a) = \frac{\kappa_b - i \sinh \theta}{\kappa_b + i \sinh \theta} f(-\theta|\theta_b, \theta_a) \quad (3.19) \]

Based on these identities, we can now write the minimal solution for the form factor axioms (3.4) and (3.6),
\[ G(\theta_1, \theta_2) = \frac{i}{8} \prod_{i=1,2} f(\theta_i) \sinh \theta_i \frac{\kappa_b + i \sinh \theta_i}{\kappa_b - i \sinh \theta_i} \frac{1}{\cosh(\frac{\theta_i}{2} + i \frac{\pi}{4})} \times \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) \tanh \left( \frac{\theta_1 + \theta_2}{2} \right), \quad (3.20) \]

From the expression of \( G \) together with (3.14), one easily checks that
\[ G(\theta_1 + 2i\pi, \theta_2|\theta_b, \theta_a) = -R_a(\theta_1) R_b^*(\theta_1) G(\theta_1, \theta_2|\theta_b, \theta_a), \quad (3.21) \]

so the residue axiom (3.6) is also satisfied. The other form-factor \( F \) can also be easily obtained. For this, observe that
\[ F(\theta_1, \theta_2|\theta_a, \theta_b) = [G(\theta_1, \theta_2|\theta_b, \theta_a)]^* \]

Using (3.8), (3.16), the crossing identity follows:
\[ F(\theta_1, \theta_2) = -G(\theta_1 - i\pi, \theta_2 - i\pi), \quad (3.22) \]

that is, explicitly
\[ b\langle 0|\theta_1, \theta_2\rangle_a = b\langle \theta_1 - i\pi, \theta_2 - i\pi|0\rangle_a. \]

It follows for instance that
\[ F(\theta_1 + 2i\pi, \theta_2|\theta_b, \theta_a) = -R_a^*(\theta_1) R_b(\theta_1) G(\theta_1, \theta_2|\theta_b, \theta_a). \quad (3.23) \]

Finally, from (3.22), we deduce the result
\[ \frac{b\langle \theta_2|\theta_1\rangle_a}{b\langle 0|0\rangle_a} = H(\theta_1, \theta_2) = G(\theta_1 - i\pi, \theta_2). \quad (3.24) \]

It follows that \( H \) has a pole as \( \theta_2 \to \theta_1 \), with residue \(-i \left[ 1 - R_a R_b^*(\theta_1) \right] \). It also has a pole as \( \theta_2 \to -\theta_1 \) with residue \( i \left[ R_b(\theta_1) - R_a(\theta_1) \right] \). As in the bulk case, these poles indicate that (3.24) holds only for “distinct” rapidities [5]; the term
\[ \delta(\theta_1 - \theta_2) \left[ 1 + R_a R_b^*(\theta_1) \right] + \delta(\theta_1 + \theta_2) \left[ R_a(\theta_1) + R_b(\theta_1) \right], \quad (3.25) \]
has to be added at coincident rapidities. This quantity, $H$, is probably the easiest to understand intuitively. For instance, compare our results with elementary computations for bare particles. With boundary condition $a$ at $x = 0$ (and a trivial boundary condition at $x = -L$), a state $|\theta_a\rangle$ is associated a one particle wave function

$$\psi_a(x, \theta) = \frac{1}{\sqrt{2}} \left( e^{im\sinh \theta x} + R_a(\theta)e^{-im\sinh \theta x} \right).$$

The quantization condition is $e^{2im\sinh \theta L}R_a(\theta) = 1$, and the normalization

$$\int_{-L}^{0} |\psi|^2 \, dx = L - \frac{i}{m \sinh \theta} \text{Im} \, R_a(\theta) \approx L, L >> 1.$$ 

One then finds the scalar product

$$b \langle \theta_2 | \theta_1 \rangle_a = \int_{-L}^{0} \psi_b^*(\theta_2, x)\psi_a(\theta_1, x)$$

$$= \frac{i}{2m} \left[ \frac{R_b^*(\theta_2)R_a(\theta_1) - 1}{\sinh \theta_2 - \sinh \theta_1} + \frac{R_b^*(\theta_2) - R_a(\theta_1)}{\sinh \theta_2 + \sinh \theta_1} \right],$$

where we used the quantization conditions. This expression has the same poles, with the same residues, as in (3.24). The complication for the physical theory is of course due to the fact that the ground state shifts with different boundary conditions, hence “dressing” the previous expression.

Inspection shows that $G$ has a simple pole at $\theta_i = \frac{i\pi}{2}$ and at $\theta_i = \pm \theta_b + \frac{i\pi}{2}$ (the pole of $\Phi$ at $\theta = \theta_B - \frac{i\pi}{2}$ is cancelled by the $\kappa_b - i \sinh \theta$ term in $G$). Similarly, $F$ has a simple pole at $\theta_i = -\frac{i\pi}{2}$ and at $\theta_i = \pm \theta_a - \frac{i\pi}{2}$. The poles of $G$ are thus the same as the poles of $R_b^*$, the poles of $F$ the same as those of $R_a$.

Notice that the residue condition does not prevent the quantities $F, G$ from having an added part without kinematic pole, with an expression similar to the ones written so far, but with the tanh of the sum and difference of rapidities replaced by the sinh function. However, dimensional analysis, together with the requirement of having a well defined massless limit and comparison with perturbation theory, exclude such a term.

Notice also that, like the $R$ matrices, the scalar products depend only on squares of magnetic fields. It is easy to check that this is true to the first orders in perturbation theory, see the appendix. It is important to stress that our results apply in fact to the case of fields of the same sign; when $h_ah_b < 0$, slightly different formulas have to be used, which we discuss below.
We can finally compare some of these results with perturbation theory. Consider the situation where the two boundary fields are very small; at leading order, the $R$ matrices and the functions $\Phi$ are evaluated for vanishing fields, i.e., $\theta_a = \theta_b = i\pi$. One has in particular

$$\Phi(\theta|i\pi, i\pi) = -\frac{1}{\cosh(\frac{\theta}{2} + \frac{i\pi}{4})}.$$ 

Putting these values back in the formula, we find, after some algebra

$$G(\theta_1, \theta_2) \propto \tanh\frac{\theta_1 - \theta_2}{2} \tanh\frac{\theta_1 + \theta_2}{2} \cosh\left( \frac{\theta_1}{2} + \frac{i\pi}{4} \right)$$

$$\times \cosh\left( \frac{\theta_2}{2} + \frac{i\pi}{4} \right) \frac{\sinh \theta_1 \sinh \theta_2}{\cosh^2 \theta_1 \cosh^2 \theta_2}.$$ 

On the other hand, the scalar product can be computed from straightforward perturbation theory. The fermion propagators are obtained by solving the equations of motion, using formula (4.3) of [7], which provide directly integral representations. After some lengthy algebra, one finds a result in agreement with the previous formula for $G$.

### 3.2. The general case

Following the algebraic approach of [8] we see that we could as well put different boundary conditions in the left vacuum to that of the right vacuum in the form factors

$$\mathcal{F}_{ba}(\theta_1, \ldots, \theta_n) = b \langle 0 | OZ^*(\theta_1) \cdots Z^*(\theta_n) | 0 \rangle_a$$

Then the residue axiom for the two particle form factor becomes

$$\text{Res} \mathcal{F}_{ba}(\theta_1, \theta_2) |_{\theta_2 = \theta_1 + i\pi} = i [1 - R_a(\theta_1)R_0^*(\theta_1)]_b \langle O \rangle_a$$

and can be generalised to the $n$ particles form factor in the obvious fashion. Here it is important to emphasize that the $\mathcal{F}$’s are not form factors in a theory with inhomogeneous boundary conditions, but rather “transition form factors” between two theories having different boundary conditions (and thus different Hilbert spaces). These axioms are general for operators which are local with respect to the Fadeev-Zamolodchikov operators. In particular, if we choose the case $O = 1$, i.e., the identity, then the previous axioms will yield the scalar products (or transition factors) between asymptotic states in Hilbert spaces with different boundary conditions. The other axioms have to be modified in a similar way; for instance, one has

$$\mathcal{F}_{ba}(\theta_1 + 2i\pi, \theta_2) = S(\theta_2 - \theta_1)R_a^*(\theta_1)S(-\theta_1 - \theta_2)R_b^*(\theta_1 + i\pi)\mathcal{F}_{ba}(\theta_1, \theta_2).$$
The modification of the exchange type relations is trivial.

In the special case of the Ising model where $S = -1$, we get the simpler equations (for a local operator)

$$F_{ba}(\cdots \theta_i + 2i\pi \cdots) = -R_a^*(\theta_i) R_b(\theta_i) F_{ba}(\cdots \theta_i \cdots)$$

$$\text{Res}_{\theta_j=\theta_1+i\pi} F_{ba}(\theta_1, \cdots, \theta_i, \cdots, \theta_n) = i(-1)^{j+1} [1 - R_a(\theta_1) R_b^*(\theta_1)]$$

$$\times F_{ba}(\hat{\theta}_1, \cdots, \hat{\theta}_j, \cdots, \theta_n)$$

$$F_{ba}(\cdots \theta_i \cdots) = R_a(\theta_i) F_{ba}(\cdots - \theta_i \cdots).$$

Given the function $f$ constructed before, it is now easy to write the generalisation of (3.20):

$$b \langle \theta_2, n, \cdots, \theta_1 | 0 \rangle_a = c_n \prod_i \sinh \theta_i f(\theta_i) \frac{1}{\kappa_b + i \sinh \theta_i \cosh \left( \frac{\theta_i}{2} + \frac{i\pi}{4} \right)}$$

$$\times \prod_{i<j} \tanh \frac{\theta_i - \theta_j}{2} \tanh \frac{\theta_i + \theta_j}{2}$$

with $c_n$ a proper normalisation determined by the residue condition, $c_n = \left( \frac{i}{8} \right)^n$. The various other components are obtained by crossing, in particular

$$b \langle 0 | \theta_1, \cdots, \theta_2, n \rangle_a = (-1)^n c_n \prod_i \sinh \theta_i f(\theta_i - i\pi) \frac{1}{\kappa_b - i \sinh \theta_i \cosh \left( \frac{\theta_i}{2} - \frac{i\pi}{4} \right)}$$

$$\times \prod_{i<j} \tanh \frac{\theta_i - \theta_j}{2} \tanh \frac{\theta_i + \theta_j}{2}$$

These formulas also agree with the first non trivial order in perturbation theory.

### 3.3. Compatibility of form-factors and transition factors

It must be clear that although asymptotic states with different boundary conditions have some non trivial scalar products, the form-factors of physical operators have the same general form with any boundary conditions, i.e., all they see from the asymptotic states is their property $|| \theta > = R(\theta)|| - \theta >$. This is made possible by the axioms satisfied by the transition factors, in particular the pole condition. Let us illustrate this in the case of the fermion operator. We have

$$a \langle 0 | \psi | \theta \rangle_a = \omega \left( e^{\theta/2} + R_\theta(\theta) e^{-\theta/2} \right).$$

On the other hand, we can compute this form-factor by inserting a complete set of states with $b$ boundary conditions on the left and on the right. Using the behaviour of transition
factors under reflections, this allows us to reexpress (3.32) as a sum of two types of terms. The first type is
\[
\begin{align*}
\omega \int \frac{d\theta_1}{2\pi} e^{\theta_1/2} & \left[a \langle 0 | 0 \rangle_{bb} \langle \theta_1 | \theta \rangle_a \\
+ \int & \frac{d\theta_2}{(4\pi)^2} \right] a \langle 0 | \theta_2 \rangle_{bb} \langle \theta_3 \theta_2 \theta_1 | \theta \rangle_a + \ldots \right),
\end{align*}
\]
and the second type is
\[
\begin{align*}
\bar{\omega} \int \frac{d\theta_1}{2\pi} e^{\theta_1/2} \left[ & \int \frac{d\theta_2}{4\pi} a \langle 0 | \theta_1 \theta_2 \rangle_{bb} \langle \theta_2 | \theta \rangle_a \\
+ \int & \frac{d\theta_2 d\theta_3 d\theta_4}{(4\pi)^3} a \langle 0 | \theta_1 \theta_2 \theta_3 \theta_4 \rangle_{bb} \langle \theta_4 \theta_3 \theta_2 | \theta \rangle_a + \ldots \right].
\end{align*}
\]
In all these formulas integrals are regulated by taking principal parts. Let us now consider the second sum (3.34). The \(\theta_1\) integral has no singularity on the real axis; we move the contour to \(\text{Im} (\theta_1) = \pi\); by doing so we do not encounter any singularity since the transition factors with \(\theta_1\) on the right have poles only in the lower half plane. By using crossing and the completude relation with \(b\) boundary condition, this gives rise to \(\omega \left( e^{\theta/2} + R_a(\theta) e^{-\theta/2} \right)\), up to two corrections. First, the correct crossing formula is \(a \langle 0 | \theta_1 + i\pi, \theta_2 \rangle_b = a \langle \theta_1 | \theta_2 \rangle_b^{dc}\), where on the right we mean the disconnected transition factor, without the added delta function part at coincident rapidities. On the other hand, the completude relation works with \(a \langle \theta_1 | \theta_2 \rangle_b\): this means that we have to subtract a first correction, which reads using (3.28), together with a bit of algebra, like (3.33) but with the integrand for the \(\theta_1\) variable multiplied by \(1 + \frac{1}{2} \frac{R_b}{R_a}(\theta_1) + \frac{1}{2} \frac{R_a}{R_b}(\theta_1)\). Second, after crossing, we get a \(\theta_1\) integral that runs over the whole real axis, avoiding the poles at \(\theta_2\) and \(-\theta_2\) by going under them. The completude relation on the other hand requires the principal part integrals, so we have to subtract the second correction, given by the pole contributions. This gives rise again to a formula like (3.33) but with the integrand for the \(\theta_1\) variable multiplied by \(1 - \frac{1}{2} \frac{R_a}{R_b}(\theta_1) - \frac{1}{2} \frac{R_b}{R_a}(\theta_1)\), where we used the residue conditions for \(H\). Adding up the two corrections, we see that the second type of terms (3.34) is nothing but (3.32) minus the first type of terms (3.33). Hence, adding up (3.33) and (3.34) gives (3.33), and we recover our form factor.

4. Massless limit and boundary conditions changing operators

We now consider the case where the bulk is in fact massless. This is conveniently addressed within the framework of massless scattering [9], [10], which is formally obtained
by letting the physical mass \( m \to 0 \), together with the rapidities \( \theta \to \pm \infty \). We set
\[
m = \frac{\mu}{2} e^{-\Theta}
\]
(\( \mu \) an arbitrary scale, taken equal to unity in what follows), \( \theta = \pm \Theta \pm \beta \),
to parametrize energy and momentum \( e = \pm p = \mu e^\beta \). The \( \pm \) sign corresponds to R (L) movers.

We first consider time propagation in the x-direction (direct channel). Then, L and R movers are defined by the conventions, eg for the form-factors of the fermions
\[
\langle 0| \psi_R(x,y) | \beta \rangle_R = \omega e^{\beta/2} \exp \left[ e^\beta (x + iy) \right]
\]
\[
\langle 0| \psi_L(x,y) | \beta \rangle_L = \bar{\omega} e^{\beta/2} \exp \left[ e^\beta (x - iy) \right].
\]

Consider then the one point function of the energy operator, \( \epsilon \equiv i \psi_L \psi_R \). Using the expression for the boundary state
\[
| B \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_i \frac{d\beta_i}{2\pi} K(\beta_i - \beta_B) Z_L^*(\beta_i) Z_R^*(\beta_i) | 0 \rangle,
\]
one finds (here, \( z = x + iy \))
\[
\langle \epsilon(z, \bar{z}) \rangle = \langle 0| \epsilon(z, \bar{z}) | B \rangle = -i \int \frac{d\beta}{2\pi} K(\beta - \beta_B) e^\beta e^{2e^\beta x}.
\]

In the case of fixed boundary conditions \( K = i \), and one finds \( \langle \epsilon \rangle = -\frac{1}{4\pi x} \). For free boundary conditions, \( K = -i \), and one finds the opposite result.

In contrast with the massive case, where the expression (4.3) could not be obtained directly in the cross-channel, such a computation is now possible. This is because we can represent asymptotic states as superpositions of left and right moving parts, the simplest ones being
\[
| \beta \rangle = | \beta \rangle_R + R(\beta)| \beta \rangle_L
\]
\[
| \beta_1 \beta_2 \rangle = | \beta_1, \beta_2 \rangle_{RR} + R(\beta_2)| \beta_1 \beta_2 \rangle_{RL} + R(\beta_1)| \beta_1 \beta_2 \rangle_{LR} + R(\beta_1)R(\beta_2)| \beta_1 \beta_2 \rangle_{LL},
\]
and use the LR factorized form of conformal operators. Here, the meaning of L and R is encoded in the new dependence of form-factors
\[
\langle 0| \psi_R(x,y) | \beta \rangle_R = \omega e^{\beta/2} \exp \left[ e^\beta (-y + ix) \right]
\]
\[
\langle 0| \psi_L(x,y) | \beta \rangle_L = \bar{\omega} e^{\beta/2} \exp \left[ e^\beta (-y - ix) \right].
\]

\[5 \] These states are normalized such that, for instance, \( \langle \beta_1 | \beta_2 \rangle = 2\pi \delta(\beta_1 - \beta_2) \)
Using the general crossing formula written earlier, the one point function of the energy reads then \((w = iz)\)

\[
\langle \epsilon(w, \bar{w}) \rangle = -i \int \frac{d\beta}{2\pi} \langle 0| |\psi_L(\bar{w})|\rangle \langle \psi_R(w)|0 \rangle = \int \frac{d\beta}{2\pi} R(\beta - \beta_B) e^{\beta} e^{-2ixe^\beta}.
\] (4.6)

A rotation of the contour shows that the expressions (4.3) and (4.6) are identical, with the identification \(K(\beta) = R(\frac{i\pi}{2} - \beta)\), as in the massive case. The massless reflection matrices actually follow from the massive ones by setting \(\theta_b \approx \Theta + \beta_b\): they are of the form \(R = i \tanh \left(\frac{\beta - \beta_b}{2} - \frac{i\pi}{4}\right)\).

We will define massless form-factors as form-factors on the previous asymptotic states. They follow from the massive case by taking the infinite rapidity limit \(\theta = \pm \Theta \pm \beta\) for each rapidity. For instance

\[
\langle 0| \epsilon(w, \bar{w}) |\beta_1 \beta_2 \rangle = ie^{\beta_1/2} e^{\beta_2/2} \left\{ R(\beta_2) \exp[e^{\beta_1}(-y + ix) + e^{\beta_2}(-y - ix)] - R(\beta_1) \exp[e^{\beta_1}(-y - ix) + e^{\beta_2}(-y + ix)] \right\}.
\] (4.7)

The scalar products between asymptotic states with different boundary conditions follow by the same limiting procedure. Alternatively, one could obtain them by writing a similar set of massless axioms. As an example, in the massless case, the expressions for the one point function of the energy are

\[
\langle \epsilon(x, y) \rangle_{ba} = \int \frac{d\beta}{2\pi} R_b(\beta) e^{\beta} e^{-2ixe^\beta} - i \int \frac{d\beta_1 d\beta_2}{8\pi^2} R_b(\beta_1)
\]

\[G(\beta_1, \beta_2) e^{\beta_1/2} e^{\beta_2/2} \exp \left[ e^{\beta_1}(-y - ix) + e^{\beta_2}(-y + ix) \right],
\] (4.8)

for \(y > 0\), and

\[
\langle \epsilon(x, y) \rangle_{ba} = \int \frac{d\beta}{2\pi} R_a(\beta) e^{\beta} e^{-2ixe^\beta} + i \int \frac{d\beta_1 d\beta_2}{8\pi^2} R^*_a(\beta_1)
\]

\[G(\beta_1, \beta_2) e^{\beta_1/2} e^{\beta_2/2} \exp \left[ e^{\beta_1}(y + ix) + e^{\beta_2}(y - ix) \right],
\] (4.9)

for \(y < 0\) (where we used the antisymmetry of \(F, G\) leading to the same condition as (4.4) with now the massless reflection matrices and in the variable \(\beta\). Using that \(f(\theta|\theta_b, \theta_a) \approx -2e^{-\Theta+\beta} \sqrt{\sinh \frac{\beta_b - \beta_a}{2}} \Phi(\beta|\beta_b, \beta_a)\) gives the result

\[
G(\beta_1, \beta_2) = -\frac{1}{2} \sinh \frac{\beta_b - \beta_a}{2} \left\{ \prod_{i=1,2} \Phi(\beta_i|\beta_b, \beta_a) e^{\beta_i} + iT_b \right\} \tanh \frac{\beta_1 - \beta_2}{2},
\] (4.10)
where we have set $T_{a(b)} = e^{\beta_{a(b)}}$.

We now consider the case where the mixed boundary conditions are conformal invariant: we chose free boundary conditions (F) for $y < 0$ and fixed boundary conditions (+) for $y > 0$. This is described in the previous formalism by taking the limits $\beta_a \to -\infty$ and $\beta_b \to \infty$. From previous formulas, we find

$$\frac{\langle \beta_2 \beta_1 | 0 \rangle_F + \langle 0 | 0 \rangle_F}{\langle 0 | 0 \rangle_F} = i \tanh \frac{\beta_1 - \beta_2}{2}. \quad (4.11)$$

The one point function of the energy reads then

$$\langle \epsilon(w, \bar{w}) \rangle_F = i \int \frac{d\beta_1 d\beta_2}{4\pi^2} e^{\beta_1/2} e^{\beta_2/2} \tanh(\frac{\beta_1 - \beta_2}{2}) \exp[i x(e^{\beta_1} - e^{\beta_2})]$$

$$\times \exp[-y(e^{\beta_1} + e^{\beta_2})] - \frac{1}{4\pi x}. \quad (4.12)$$

Explicit evaluation gives

$$\langle \epsilon(w, \bar{w}) \rangle_F = \frac{1}{4\pi} \left( \frac{1}{x} - \frac{y}{x \sqrt{x^2 + y^2}} \right) - \frac{1}{4\pi x}, \quad (4.13)$$

where we used the formula

$$\int_0^\infty \frac{ds_1}{\sqrt{s_1}} \frac{ds_2}{\sqrt{s_2}} e^{s_1 - s_2} e^{ix(s_1 - s_2)} e^{-y(s_1 + s_2)} = \pi \left( \frac{1}{x} - \frac{y}{x \sqrt{x^2 + y^2}} \right). \quad (4.14)$$

On the other hand, one easily gets from conformal field theory results [2] that

$$\langle \epsilon(w, \bar{w}) \rangle_F = -\frac{1}{4\pi x} \frac{y}{\sqrt{x^2 + y^2}},$$

in agreement with (4.13).

By duality, one gets similarly

$$\frac{F\langle \beta_2 \beta_1 | 0 \rangle_+ + \langle 0 | 0 \rangle_+}{F\langle 0 | 0 \rangle_+} = i \tanh \frac{\beta_1 - \beta_2}{2}. \quad (4.15)$$

The computation leading to (4.13) was done for $y > 0$. By considering $y < 0$ instead, one also gets

$$\frac{+\langle 0 | \beta_1 \beta_2 \rangle_F + \langle 0 | 0 \rangle_F}{+\langle 0 | 0 \rangle_F} = -i \tanh \frac{\beta_1 - \beta_2}{2}. \quad (4.16)$$
In a similar way, we can consider the two point function of fermions. By considering the cases where \( y_1y_2 > 0 \) and \( y_1y_2 < 0 \), one also finds the result

\[
\frac{+\langle \beta_2 | \beta_1 \rangle_F}{+\langle 0 | 0 \rangle_F} = i \coth \frac{\beta_1 - \beta_2}{2}.
\]

(4.17)

The previous formula are compatible with the standard description [1] of mixed conformal invariant boundary conditions, which involves putting a “boundary conditions changing operator” at \( y = 0 \). In the case from free to fixed spins, this operator has to be the \( \phi_{12} \) operator of conformal weight \( \frac{1}{16} \). From this identification, it follows that (4.11) generalizes to

\[
\frac{+\langle \beta_{2n} \ldots \beta_1 | 0 \rangle_F}{+\langle 0 | 0 \rangle_F} = i^n \prod_{i<j} \tanh \frac{\beta_i - \beta_j}{2},
\]

(4.18)

and that the scalar products coincide, formally, with the form factors of the (eg right moving) spin operator in the bulk. This ensures in particular that the partition function of an Ising model with a region of fixed boundary conditions of length \( L \) inserted into a wall of free boundary conditions decays as \( L^{-1/8} \).

5. The case \( h_a h_b < 0 \)

As mentioned earlier, we expect the results of section 3 to hold for the case \( h_a h_b > 0 \) only; physically, this corresponds to a situation where the boundary condition at \( x = -\infty \) is independent of \( y \). Instead, when \( h_a h_b < 0 \) and large enough magnetic fields, spins for \( y < 0 \) and \( y > 0 \) tend to be oriented in opposite directions, in particular they go to \( \pm \mathcal{M} \), where \( \mathcal{M} \) is the spontaneous magnetization, at \( x = -\infty \). There is thus a frustration line inserted in the system - in other words, following standard mappings [1], a spin operator is inserted at \( x = y = 0 \). Of course, the spin operator on the boundary acquires a dimension 1/2, so in fact we have a fermion line emitted at the transition. We thus expect that the problem of +− boundary conditions will be described by transition factors involving an odd number of particles; from that perspective, the ground state itself must be thought of as containing a particle of vanishing energy, and momentum equal to \( im \) (so the partition function of a model with a frustration line extending to \( x \) goes as \( e^{mx} \)), ie a particle at rapidity \( \theta = \frac{4\pi}{2} \), which we denote by \( I \) in what follows.

It is easy to check that this description is adequate in the massless case, and for boundary conditions \( a = +, b = - \), ie the transition from “fixed up” to “fixed down”. From conformal invariance, one expects [2],

16
\[ \langle \epsilon(w, \bar{w}) \rangle_{++} = \langle \epsilon(w, \bar{w}) \rangle_{--} = -\frac{1}{4\pi x} \left( 1 - 4 \frac{x^2}{x^2 + y^2} \right). \] (5.1)

Clearly, this cannot follow from formulas in section 3, since our scalar products \( F \) and \( G \), which are functions only of squares of fields, vanish for \( h_a = \pm h_b \). On the other hand, formula (5.1) follows from the description with an odd number of particles. To see this, observe first that in the massless case, the particle at imaginary rapidity has vanishing energy and momentum, and merely keeps the correct parity of fermion numbers. Consider then the one point function of the energy. Two processes can contribute to it: in one case, the transition simply emits the particle at imaginary rapidity, that just “goes through” the energy insertion without being affected, so the process contributes \( \langle \epsilon \rangle_{+} = -\frac{1}{4\pi x} \). The other process occurs when the transition emits a particle at real rapidity, which is then transformed into the particle at imaginary rapidity by the energy insertion. The required energy form factor is obtained by taking the usual limit:

\[ \langle 0|\epsilon(w, \bar{w})|\beta, I \rangle = \omega e^{\beta/2} \left\{ \exp \left[ e^{\beta}(-y + ix) \right] - \exp \left[ e^{\beta}(-y - ix) \right] \right\} \]

As for the scalar product \( +\langle \beta|0 \rangle_- \), it should be of the form \( \lambda e^{\beta/2} \) (\( \lambda \) an unknown constant) since the frustration line emitted by the boundary corresponds to a fermion insertion, of dimension 1/2 (in other words, the partition function of an Ising model with a region of \( - \) boundary conditions of length \( L \) inside a domain of \( + \) boundary conditions decays as \( \int d\beta e^{\beta} \exp \left[ -e^{\beta}L \right] = \frac{1}{L} \), as required from conformal invariance considerations [1]). From this, we get the second contribution to the energy one point function

\[ \lambda \omega \int \frac{d\beta}{2\pi} e^{\beta} \left\{ \exp \left[ e^{\beta}(-y + ix) \right] - \exp \left[ e^{\beta}(-y - ix) \right] \right\} = \frac{i\lambda \omega}{\pi} \frac{x}{x^2 + y^2}, \]

thus recovering (5.1) provided \( \lambda = -\omega \).

As a further check, we now compute the two point function of the energy with the \( +- \) boundary conditions. Conformal invariance gives the result

\[ \langle \epsilon(1)\epsilon(2) \rangle_{+-} = \frac{1}{4\pi^2} \left\{ \frac{1}{4x_1x_2} \left( 1 - 4 \sin^2 \alpha_1 \right) \left( 1 - 4 \sin^2 \alpha_2 \right) + \frac{1}{r^2} \left[ 1 - 4 \sin^2 (\alpha_1 - \alpha_2) \right] - \frac{1}{r^2 + 4x_1x_2} \left[ 1 - 4 \sin^2 (\alpha_1 + \alpha_2) \right] \right\}, \] (5.2)

where \( r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \), \( \sin \alpha = \frac{x}{\sqrt{x^2 + y^2}} \). Several processes are now allowed, without or with real particles emitted from the transition. In the latter case, processes
where the real particles emitted at the transition are destroyed by only one of the energy operators are identical to processes appearing in the evaluation of (5.1). The only new processes, which we call interacting processes, must add up to, using (5.2),

\[
\frac{1}{4\pi^2} \left[ \frac{4\sin^2 \alpha_1 \sin^2 \alpha_2}{x_1 x_2} - \frac{4}{r^2} \sin^2 (\alpha_1 - \alpha_2) + \frac{4}{r^2 + 4x_1 x_2} \sin^2 (\alpha_1 + \alpha_2) \right]
\]

After a bit of algebra, this reads

\[
\frac{1}{r^2 + 4x_1 x_2} \left( r_1^4 + r_2^4 \right) - 2x_1 x_2 y_1^2 y_2^2 - 2x_1^2 x_2^2 - 2x_1 x_2 \left( x_1^2 y_2^2 + x_2^2 y_1^2 \right) - \frac{4 r_2^2}{\pi^2 r_1^2 r_2^2}.
\] (5.3)

where \( r_1^2 = x_1^2 + y_1^2 \). On the other hand, there are two interaction processes. In the first one, the transition can emit a particle at rapidity \( \beta_1 \), the first insertion of the energy can destroy the particle \( \beta_1 \) and create one at rapidity \( \beta_2 \), while the second insertion of the energy destroys the latter to replace it by the imaginary one. This process has the amplitude

\[
- \int \frac{d\beta_1}{2\pi} \frac{d\beta_2}{2\pi} e^{\beta_1} e^{\beta_2} \left\{ \exp \left[ e^{\beta_1} (-y_1 - ix_1) - e^{\beta_2} (-y_1 + ix_1) \right] - \exp \left[ e^{\beta_1} (-y_1 + ix_1) - e^{\beta_2} (-y_1 - ix_1) \right] \right\} \left\{ \exp \left[ e^{\beta_2} (-y_2 - ix_2) \right] - \exp \left[ e^{\beta_2} (-y_2 + ix_2) \right] \right\}.
\]

After a bit of algebra, this reads

\[
\frac{1}{4\pi^2} \left( \frac{2}{r^2 + 4x_1 x_2} \frac{y_1 y_2 - x_1 x_2 - r_1^2}{r_1^2} - \frac{2}{r^2} \frac{y_1 y_2 + x_1 x_2 - r_1^2}{r_1^2} \right).
\] (5.4)

In the second process, the transition emits a particle at rapidity \( \beta_1 \), the first energy insertion adds up a particle at rapidity \( \beta_2 \) and the particle at imaginary rapidity (the particle at rapidity \( \beta_1 \) just going through), and the two real particles are finally destroyed by the second energy insertion. This process has the amplitude

\[
- \int \frac{d\beta_1}{2\pi} \frac{d\beta_2}{2\pi} e^{\beta_1} e^{\beta_2} \left\{ \exp \left[ e^{\beta_1} (-y_2 - ix_2) + e^{\beta_2} (-y_2 + ix_2) \right] - \exp \left[ e^{\beta_1} (-y_2 + ix_2) + e^{\beta_2} (-y_2 - ix_2) \right] \right\} \left\{ \exp \left[ -e^{\beta_2} (-y_1 - ix_1) \right] - \exp \left[ -e^{\beta_2} (-y_1 + ix_1) \right] \right\}.
\]

which is the same as the previous result, with 1 and 2 interchanged. Adding (5.4) and the similar result with 1 and 2 interchanged reproduces (5.3) indeed.
We can now complete our study of scalar products by giving results equivalent to those of section 3 when $h_a h_b < 0$. Now only odd numbers of particles are emitted. The amplitude for an even number of real particles is

$$\frac{b\langle \theta_2 \cdots \theta_1 | I \rangle_a}{b\langle I \rangle_a} = G(\theta_1, \ldots, \theta_{2n}), \quad (5.5)$$

where the expression on the right hand side coincides with (3.30), while for an odd number of real particles we have the obvious generalization of (3.30)

$$\frac{b\langle \theta_{2n+1} \cdots \theta_1 | 0 \rangle_a}{b\langle I \rangle_a} = c_n \prod_i \sinh \theta_i f(\theta_i) \kappa_b - i \sinh \theta_i \cosh \left( \frac{\theta_i}{2} + \frac{i\pi}{4} \right) \prod_{i<j} \frac{\sinh \theta_i - \sinh \theta_j}{\sinh \theta_i - \sinh \theta_j} \frac{\theta_i + \theta_j}{2} \quad (5.6)$$

The residue axioms fix $c_n$ again, except for an overall scale this time: $c_n = c \left( \frac{1}{\pi} \right)^n$.

To fix this scale, we observe that at least the one particle form-factor should not vanish when $h_a = -h_b$, and that it should vanish when $h_a h_b = 0$, since then there is no frustration left. Recall that $f$ itself goes as $\sqrt{h_b^2 - h_a^2}$; we thus chose $c_n$ to be proportional to $\sqrt{h_a h_b h_a - h_b^2}$, i.e. we expect the one particle scalar product

$$b\langle \theta | 0 \rangle_a = \frac{\omega}{2\sqrt{2}} \sqrt{h_a h_b} \frac{h_a - h_b}{\sqrt{h_b^2 - h_a^2}} \sinh \theta f(\theta) \kappa_b - i \sinh \theta \cosh \left( \frac{\theta}{2} + \frac{i\pi}{4} \right) \frac{1}{\kappa_b - i \sinh \theta \cosh \left( \frac{\theta}{2} + \frac{i\pi}{4} \right)} \quad (5.7)$$

where the field dependent prefactor and numerical factors are matched to the massless limit. Indeed, in that limit, $h \approx e^{\beta/2}$, giving rise to

$$b\langle \beta | 0 \rangle_a = -\frac{1}{2} (h_a - h_b) \Phi(\beta | \beta_b, \beta_a) \frac{e^{\beta} + i T_b}{e^{\beta} - i T_b}.$$  

Finally, in the limit of large fields $h_a, h_b \to \infty$, $h_a/h_b$ finite, this goes, using $\Phi(\beta | \beta_b, \beta_a) \approx 2 e^{i\pi/4} e^{\beta/2} e^{-\beta_a/4} e^{-\beta_b/4}$, to

$$b\langle \beta | 0 \rangle_a = \frac{\omega}{\sqrt{h_a h_b}} e^{\beta/2} \frac{h_a - h_b}{\sqrt{h_a h_b}} e^{\beta/2}$$

In the case of opposite fields we recover the previous result for $+$ $-$ boundary conditions.

Observe that, when $h_a = -h_b$, only the one particle scalar product is non zero, due to the terms $\sqrt{\kappa_a - \kappa_b}$ in the functions $f$ (since the only undetermined quantity is an overall normalization, we cannot change this without spoiling the results in the one particle case).
In the perturbative limit, when \( h_a \) and \( h_b \to 0 \), one has
\[
\langle \theta | 0 \rangle_a \propto \sqrt{h_a h_b} (h_a - h_b) \frac{\sinh \theta}{\cosh \theta \cosh \left( \frac{y}{2} - \frac{i \pi}{4} \right)}.
\] (5.8)

This can be compared with perturbative results. To do so, we have to slightly modify the action (2.1) to take into account the fermion line emitted at the origin. It is natural to represent it by a term \( \sqrt{h_a h_b} \psi_0 a(0) \), where \( a \) is the same boundary fermionic degree of freedom as in (2.1), the term \( \sqrt{h_a h_b} \) is dictated by \( a,b \) symmetry and dimensional analysis, and \( \psi_0 \) is a fermion zero mode obeying \( \langle \psi_0 \psi_{L,R} \rangle \propto e^{m \theta} \). There are now two non-trivial scalar products at second order: one does not involve insertion of this term, and thus is identical as the one we had for \( h_a h_b > 0 \), in agreement with (5.5). The second involves insertion of this term, plus a single integral along either the region \( y < 0 \) or \( y > 0 \). This goes as \( (h_a - h_b) \sqrt{h_a h_b} \int_{-\infty}^{0} dy \langle (\psi_L + \psi_R) (x = 0, y) \psi_L(x, y) \rangle \), and the integral can be shown, using the propagators in the appendix, to agree with (5.8).

6. Sinh-Gordon model.

In this section we proceed to try to construct some of the transition factors for the sinh-Gordon model. Our purpose here is to gain control of the axiomatic relations satisfied by these factors in cases where the S matrix is non trivial, hence paving the way for further study of the sine-Gordon case, which is the most important for applications. As we will see, some complications arise when we wish to identify operators creating the need for a more systematic study.

Let us first introduce some notation. The action is
\[
A = \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m_0^2}{b^2} \cosh b \phi \right] + \lambda \int_{-\infty}^{\infty} dy \cosh \frac{b}{2} (\phi - \phi_0)(x = 0, y). \] (6.1)

The two particle S matrix and the reflection matrix for this model are well known. Setting
\[
\xi(a) = \frac{\sinh(\frac{\theta}{2} + i \frac{\pi a}{4})}{\sinh(\frac{\theta}{2} - i \frac{\pi a}{4})} = \exp \left[ -4 \int_{0}^{\infty} \frac{dt \sinh \frac{\theta t}{2} \cosh(1 - \frac{a}{4})t}{\sinh t \sinh(\frac{\theta t}{2})} \right], \hspace{1cm} 0 < \text{Re} \ a < 4, \] (6.2)

they are given by
\[
S = -\frac{1}{\xi(B) \xi(2 - B)} \] \( \xi(1) \xi(2 - B/2) \xi(1 + B/2) \]
\[
R = \frac{\xi(1) \xi(2 - B/2) \xi(1 + B/2)}{\xi(1 - E(b)) \xi(1 + E(b)) \xi(1 - F(b)) \xi(1 + F(b))}, \] (6.3)
where \( B = \frac{1}{2\pi} \frac{b^2}{1+b^2/4\pi} \), and \( E, F \) are functions of \( b \), also depending on the boundary parameters: they are given by \( E = \frac{3n}{\pi} \) and \( F = i \frac{B\Theta}{\pi} \) in Ghoshal’s notation \([12]\).

In the following we will choose to work with the boundary action (6.1) having the phase \( \phi_0 = 0 \) which leads to \( \eta = E = 0 \) and simplifies the reflection matrix.

Associated with the product \( \xi(1+a)\xi(1-a) \) we define \( \Upsilon(a) \) by

\[
\Upsilon(a, \theta) = N_a \exp \left\{ \int_0^\infty \frac{dt}{t} \frac{\cosh \frac{a t}{2}}{\cosh \frac{t}{2} \sinh t} \sin^2 \left( \frac{\hat{\theta} t}{2\pi} \right) \right\}
\]

\[
N_a = \sqrt{\cos \frac{\pi a}{2}} \exp \left[ \int_0^\infty \frac{dt}{t} \frac{\cosh \frac{a t}{2}}{\cosh^2 \frac{t}{2}} \right],
\]

with \( \hat{\theta} = i\pi - \theta \). This function is constructed so that it satisfies

\[
\Upsilon(a, \theta + 2\pi i) = \Upsilon(a, -\theta) = \frac{1}{\xi(1+a)\xi(1-a)} \Upsilon(a, \theta)
\]

and has no poles nor zeroes in the physical strip. We will need this function in the next section to construct the form factors. Our choice of normalisation is such that

\[
\Upsilon(a, \theta) \Upsilon(a, \theta + i\pi) = \sinh \theta + i \cos \frac{\pi a}{2}.
\]

Boundary form-factors as well as transition factors can now be obtained by solving the set of axioms written in the previous sections. Let us start by investigating some general properties.

6.1. Two particles sector.

The two particle sector is where differences with the Ising model are emerging. Let us suppose that we have a set of boundary operators (local) denoted generically by \( k \), then the transition form factor between states in theory \( a \) and theory \( b \) can be inferred to have the form

\[
F_{ab}^{k}(\theta_1, \theta_2) = c_k^f f_{ab}(\theta_1) f_{ab}(\theta_2) \times \left\{ \frac{F_{\min}(\theta_1 - \theta_2) F_{\min}(\theta_1 + \theta_2)}{\cosh(\frac{a_1 - a_2}{2}) \cosh(\frac{a_1 + a_2}{2})} \right\} Q^k(\theta_1, \theta_2).
\]

In this formula, \( F_{\min} \) is the usual function generating the sinh-Gordon \( S \) matrix

\[
F_{\min}(\theta) = \mathcal{N}_B \exp \left[ 8 \int_0^\infty \frac{dt}{t} \frac{\sinh(\frac{tB}{4}) \sinh(\frac{t}{2} \left( 1 - \frac{B}{2} \right)) \sinh \frac{t}{2} \sin^2 \left( \frac{t\hat{\theta}}{2\pi} \right)}{\sinh^2 t} \right]
\]

\[
\mathcal{N}_B = \exp \left[ -4 \int_0^\infty \frac{dt}{t} \frac{\sinh(\frac{tB}{4}) \sinh(\frac{t}{2} \left( 1 - \frac{B}{2} \right)) \sinh \frac{t}{2}}{\sinh^2 t} \right].
\]
All the dependence on the boundary is contained in the factors $f_{ab}$. $Q^k$ is a polynomial we choose to be invariant under reflections, exchange and $2\pi i$ shifts. This leads to the following conditions on the $f_{ab}$’s

$$f_{ab}(\theta) = R_b(\theta) f_{ab}(-\theta)$$
$$f_{ab}(\theta + 2\pi i) = R_a^*(\theta + i\pi) R_b^*(\theta) f_{ab}(\theta)$$

for which a minimal solution can easily be found. Using the crossing

$$R \left( i\frac{\pi}{2} - \theta \right) = S(2\theta) R \left( i\frac{\pi}{2} + \theta \right)$$

we can write this last relation as

$$f_{ab}(\theta + 2\pi i) = \frac{R_a(\theta) R_b^*(\theta)}{S(2\theta)} f_{ab}(\theta).$$

In order to construct this function, we start with the functions $u$ and $v$ satisfying

$$u(\theta) u(\theta + i\pi) = \frac{1}{F_{\text{min}}(2\theta + i\pi)}$$
$$v(\theta + 2\pi i) = \frac{R_a(\theta)}{R_b(\theta)} v(\theta).$$

The function $u$ will generate the $S(2\theta)$ and is easily found to be:

$$u(\theta) = \exp \left[ \int_0^\infty dt \sinh \frac{Bt}{4} \sinh \left( i \frac{1 - F_a}{2} t \right) \sinh \frac{1}{2} \cos \left( \frac{2\theta}{\pi} - i \right) t \right],$$

and $v$ is given by

$$v(\theta) = \frac{\Upsilon(F_a, \theta)}{\Upsilon(F_b, \theta)}.$$

The superscript refers to boundary conditions $a, b$, corresponding to parameters $F_a, F_b$ (since we have set $\phi_0 = 0$ in the action, $a$ and $b$ correspond simply to different values of the coupling $\lambda$). Now that the monodromy relations are satisfied, we need to multiply by other terms to get the correct reflection equation. This leads to the minimal solution

$$f_{ab}(\theta) = \frac{\sinh \theta}{\sinh \left( \frac{\theta}{2} + i \frac{\pi(1-F_a)}{4} \right) \sinh \left( \frac{\theta}{2} + i \frac{\pi(1+F_a)}{4} \right)} u(\theta) \frac{\Upsilon(F_a, \theta)}{\Upsilon(F_b, \theta)}.$$
where we used the fact that \( u(\theta)/u(-\theta) = -\frac{\xi(1+B/2)\xi(2-B/2)}{\xi(1)} \). This minimal solution satisfies

\[
\frac{f_{ab}(\theta)f_{ab}(\theta + i\pi)}{\cosh(\theta\pi/2)\cosh(\theta + i\pi\pi/2)} = \frac{1}{\min(2\theta + i\pi)}
\]

This, added to the residue axiom leads to an equation for the polynomial \( Q^k \). Before embarking on the explicit form of these polynomials for specific operators, let us generalize these results to the \( N \) particle sector.

### 6.2. \( N \) particle sector.

Having understood the two particle sector the general structure emerges nicely and turns out to be a very simple generalisation of the two particle case: we have

\[
F_{ab}^k(\theta_1, \ldots, \theta_n) = c_n^k \prod_{i=1}^n f_{ab}(\theta_i) \times \prod_{i<j} \left\{ \frac{\min(\theta_i - \theta_j, \theta_i + \theta_j)}{\cosh(\theta_i - \theta_j)\cosh(\theta_i + \theta_j)} \right\} Q^k(\theta_1, \ldots, \theta_n).
\]

By construction, this satisfies all the exchange and reflection relations. Here only the residue equation needs to be fixed: it leads to an equation for the polynomial \( Q^k \) in the form

\[
Q^k(\theta + i\pi, \theta, \theta_3, \ldots, \theta_n) = P(\theta|\theta_3, \ldots \theta_n)Q^k(\theta_3, \ldots, \theta_n),
\]

with \( P \) given by

\[
P(\theta|\theta_3 \ldots \theta_n) = \frac{1}{\sinh \theta} \left\{ \left( \sinh \theta - i\cos \frac{\pi F_a}{2} \right) \left( \sinh \theta + i\cos \frac{\pi F_b}{2} \right) \times \prod_{i=3}^n \left[ \sinh(\theta - \theta_j) + \sinh \frac{i\pi B}{2} \right] \left[ \sinh(\theta + \theta_j) + \sinh \frac{i\pi B}{2} \right] 
- \left( \sinh \theta + i\cos \frac{\pi F_a}{2} \right) \left( \sinh \theta - i\cos \frac{\pi F_b}{2} \right) \times \prod_{i=3}^n \left[ \sinh(\theta - \theta_j) - \sinh \frac{i\pi B}{2} \right] \left[ \sinh(\theta + \theta_j) - \sinh \frac{i\pi B}{2} \right] \right\}
\]

and we have adjusted the constant \( c_n^k \) in order to absorb the remaining constant factors.
6.3. Operator identification.

These equations are somewhat similar to those found for operators in the bulk \[13\]. There are some essential differences though, which make the operator identification more difficult. For instance, in the bulk, an argument based on Lorentz invariance can be made to determine the degree of the polynomials \(Q^k\), depending on the spin of the operators: it is not clear how to extend this in the boundary case. In order to do the operator identification, we will use below the fact that one can analytically continue the form factors in the coupling constant \(B \rightarrow 1 + \frac{2i}{\pi} \beta_0\), leading to the so-called roaming trajectories between minimal models \[14\]. Then, as \(\beta_0 \rightarrow \infty\), at least for some simple operators, we should recover the form factors of the Ising model obtained in the previous sections. As we will see, this is not sufficient, in fact it allows to determine only part of the polynomials \(Q^k\).

Another way can be devised to identify operators by taking the massless limit: if we know the boundary dimension of the operator we can identify formally the form factor series with that of the bulk. This will allow to determine the maximal degree of the polynomials.

A specific property of the boundary case is the following: requiring the polynomials to be symmetric under exchange, and invariant under reflection, leads to the fact that they should be symmetric polynomials in the variables \(x_i = \cosh \theta_i\). Naively, one could then be tempted to construct homogeneous symmetric polynomials in these variables, but this would be wrong: since the polynomial has to be symmetric in \(e^\theta\) and \(e^{-\theta}\) we can form all invariant polynomials in these two variables. The sum is \(\cosh \theta\) but the product, which is another symmetric polynomial in these variables, is 1. This means that we should not expect the polynomials to be homogeneous.

We will not carry this study much further here, and contend ourselves by giving a few examples, mostly in the case of scalar products, ie transition form-factors for the identity operator. Since we choose our form factors to be normalised to 1 in the zero particle sector (we divide by the expectation value), we obviously have that the zero particle form factor is simply 1. For the two particle scalar product, we have the equation

\[
Q^{Id}(\theta + i\pi, \theta) = 1.
\]

(6.20)

where we adjusted the constant \(c^{Id}_2\) such that

\[
c^{Id}_2 = \frac{\cos \frac{\pi F_a}{2} - \cos \frac{\pi F_b}{2}}{2F_{min}(i\pi)}.
\]

(6.21)
Obviously there are more than one way to satisfy this equation. Choosing \( Q^{Id}(\theta_1, \theta_2) = 1 \) would seem, at first sight, to be the correct solution for the scalar products. Following [15] let us take the roaming limit \( B = 1 + \frac{2i}{\pi} \beta_0 \) with \( \beta_0 \) very large. Then we observe that

\[
F_{\text{min}}(\theta) \to -ie^{-|\beta_0|/2} \sinh \frac{\theta}{2} \\
f_{ab}(\theta)^{\text{sinh-G}} \to e^{\frac{|\beta_0|}{4}} f_{ab}(\theta)^{\text{Ising}}.
\]

Thus the scalar product becomes that of the Ising model provided we choose \( iF_a = \frac{2}{\pi} \theta_a \).

On the other hand, if we take the massless limit and choose boundary conditions \( a \) and \( b \) to be of Neumann and Dirichlet type respectively, then from conformal field theory the situation is described by the insertion of an operator of dimension \( h = 1/8 \). Then we can compare our choice for \( Q^{Id} \) with the expansion of massless operators in the bulk having \( h = 1/8 \). The conclusion, at this 2 particle level, is that there should also be a contribution proportional to \( \sigma_1 = \sum_{i=1}^2 x_i = \sum_{i=1}^2 \cosh \theta_i \). This term would not modify the residue equation. Thus, based on this analysis we expect

\[
Q^{Id}(x_1, x_2) = A\sigma_1 + 1. \tag{6.23}
\]

The main difficulty is that we have no means to fix the relative normalisation \( A \) and we are left with a free parameter.

The four particle contribution is constructed using the residue equation, we have

\[
Q^{Id}(-x, x, x_3, x_4) = P(x|x_3, x_4)Q^{Id}(x_3, x_4) \tag{6.24}
\]

which allows to construct the four particle scalar product using the property of symmetric polynomials

\[
\sigma_k^{n+2}(x, -x, x_1, \ldots, x_n) = \sigma_k^n - x^2\sigma_k^{n-2}. \tag{6.25}
\]

The part coming from the “1” can be fixed uniquely again

\[
Q^{Id,1}(x_1, \ldots, x_4) = 4\gamma(\sigma_3\sigma_2 - \gamma^2\sigma_3) + 4\gamma(\kappa_a\kappa_b - 1)(-\sigma_2\sigma_1 + \gamma^2\sigma_1) \\
- 2i(\kappa_a - \kappa_b)(\sigma_2^2 - \sigma_3\sigma_1 - 4(1 + \gamma^2)\sigma_4 - \gamma^2\sigma_4^2 - 2\gamma^2\sigma_2 + \gamma^4). \tag{6.26}
\]

Here \( x_i = \cosh \theta_i \), \( \gamma = \sinh \frac{\pi B}{2} \) and \( \kappa_{a/b} = \cos \frac{\pi F_{a/b}}{2} \). We observe that as expected, this part of the solution also collapses to that of the Ising model in the roaming limit.

\[\text{Here we use the notation } \sigma_k \text{ for the elementary symmetric polynomials defined by the generating function } \prod_k (1 + tx_k) = \sum_k t^k \sigma_k. \text{ The superscript indicates the number of indeterminates.}\]
For the other part, there is an ambiguity for the term of degree 6 coming from the kernel of the residue equation

\[
\text{Residue } (\theta, \theta + i\pi, \theta_3, \theta_4) = 0
\] (6.27)

and in general we will need physical arguments to choose the correct solution. We believe a full study of the boundary operator content of this integrable theory could lead to a complete solution but felt it is beyond the scope of this paper.

Finally, we discuss the boundary form-factor for homogeneous boundary conditions \((a = b)\), in the simple case of the trace of the stress energy tensor. The simplest guess for two particles is

\[
Q^\Theta(\theta_1, \theta_2) = \cosh \theta_1 + \cosh \theta_2
\]

\[
= 2 \cosh \left( \frac{\theta_1 - \theta_2}{2} \right) \cosh \left( \frac{\theta_1 + \theta_2}{2} \right).
\] (6.28)

Here the addition of a constant term would spoil the residue property needed. The normalisation is fixed by taking the roaming limit, where the form factor should be identical with the one of the Ising model (computed explicitly using the mode decomposition given in appendix A). Moreover, the massless limit reproduces the bulk form factor up to reflection matrices.

The four particle form factor is determined using the recurrence relation of the symmetric polynomials (6.18) and we find

\[
Q^\Theta(\theta_1, \ldots, \theta_4) = [4\gamma(\kappa_a^2 - 1)(-\sigma_2\sigma_1 + \gamma^2\sigma_4) + 4\gamma(\sigma_2\sigma_3 - \gamma^2\sigma_3)]\sigma_1.
\] (6.29)

It is clear that in the roaming limit, this form factor vanishes, as expected since in the Ising model the four particle form factor of the energy-momentum tensor is zero. Moreover, assuming that \(\sigma_1\) must be factorised, the ambiguity in this polynomial can be fixed since by taking the massless limit we know it must have at most degree 6.

We believe that these few examples are sufficient to show that the method is applicable to any integrable model, and in fact is very similar to the usual problem of finding form factors in the bulk. However, there is a need for a systematic way to identify operators at the boundary and determine the free parameters discussed above. We now conclude by some applications.
7. \( P(t) \) in dissipative quantum mechanics

The applications of our formalism to 2D statistical mechanics problems are somewhat obvious. We rather concentrate on quantum impurity problems, and discuss now a standard problem of dissipative quantum mechanics, the computation of "\( P(t) \)" in the double well problem. In the so called ohmic regime, the two state problem with dissipation can be mapped on a single channel Kondo model \[9\], with Hamiltonian

\[
H_\lambda = \int_{-\infty}^{0} dx \frac{1}{2}[\Pi^2 + (\partial_x \phi)^2] + \lambda \delta(x)(S_+ e^{i\sqrt{2\pi g}\phi} + S_- e^{-i\sqrt{2\pi g}\phi}).
\] (7.1)

Here, \( S_\pm \) are spin 1/2 operators, the values up and down corresponding to the two states of the system. The dissipation is characterized by a dimensionless number, which can be taken to coincide with the conformal weight \( h = g \) of the boundary operator.

The physical quantity of interest, \( P(t) \) is defined as follows: assuming that the spin has been fixed in the up state up to time \( t = 0 \), \( P(t) \) is its average value as a function of time after turning on the dissipation. Mathematically, introducing the ground state of the theory where the spin is not coupled to the heat bath, \( |0\rangle_{\lambda=0} \), we need to evaluate the one point function

\[
P(t) = \langle \Omega | S_z | \Omega \rangle,
\] (7.2)

where \( |\Omega\rangle = e^{-iHt}|0\rangle \), \( |0\rangle \) is the tensor product \( |0\rangle_{\lambda=0} \otimes |+\rangle \) and \( H \) is the Hamiltonian (7.1) with dissipation. Equivalently, we can write

\[
P(t) = \langle 0 | e^{iHt} S_z e^{-iHt} | 0 \rangle.
\] (7.3)

We denote the quantity inside the bracket as \( S_z(t) \) (evolution with respect to the Hamiltonian (7.1) is implied).

We adress the problem in the formalism described previously. Introducing a complete set of eigenstates of \( H_\lambda \) on the left and right hand side of (7.3), the evaluation of \( P(t) \) requires the knowledge of the scalar products of these eigenstates with \( |0\rangle_{\lambda=0} \otimes |+\rangle \). To diagonalize the Hamiltonian, we then use integrability of the anisotropic Kondo model, and the resulting massless scattering description, involving solitons and antisolitons (of charge \( \pm 1 \)) and breathers.

There is a slight ambiguity in this approach, because the scattering description is based on an infrared picture, where the spin is always screened, for \( \lambda > 0 \). For any \( \lambda > 0 \), the spin can be "extracted" from the asymptotic states: for instance, the spin correlators...
follow simply from the correlators of $\partial_x \phi$, which can be computed using the form-factors, without any reference to the spin. At $\lambda = 0$, it is not clear “where” the spin exactly is in our description. Nevertheless, one can write axioms for the scalar products of multiparticle states at $\lambda \neq 0$ with $|0\rangle_{\lambda=0} \otimes |+\rangle$, and of course they are of the same type as the ones written before for any pair of couplings $\lambda, \lambda'$, so it seems our approach should work in that case too. The effect of the spin $|+\rangle$ in $|0\rangle_{\lambda=0}$ is now simply taken into account by considering that $|0\rangle_{\lambda=0}$ has a charge equal to one, ie the only non vanishing scalar products will be of the type $\epsilon^{2n+1} \ldots \epsilon^1 \langle \beta_{2n+1} \ldots \beta_1 | 0 \rangle$, with $\sum \epsilon_i = 1$.

We discuss first the case $g = 1/2$, where the Kondo problem essentially decomposes into two decoupled Ising problems, with boundary field $h \propto \lambda$ [16]. At this value of $g$, the spectrum is made of a soliton and an antisoliton, and the $R$ matrix is

$$R_{+-} = R_{-+} = \frac{e^\beta - iT^b}{e^\beta + iT^b}, \quad R_{++} = R_{--} = 0.$$  \hfill (7.4)

and only the two particle form-factor of the current is non zero

$$\langle 0 | \partial_x \phi(x=0,t) | \beta_1 \beta_2 \rangle^{\epsilon_1 \epsilon_2} \propto \delta_{\epsilon_1 + \epsilon_2} \epsilon_1 e^{\beta_1/2} e^{\beta_2/2}$$

$$[1 + R_{+-}(\beta_1) R_{-+}(\beta_2)] \exp \left[ -it(e^{\beta_1} + e^{\beta_2}) \right].$$  \hfill (7.5)

To proceed, we have to compute the one point function of $\partial_x \phi$ by inserting complete sets of states. Many processes will contribute. In the simplest, a soliton is emitted at the transition, acted on by $\partial_x \phi$ that transforms it into another soliton, and this second soliton is destroyed at the second transition. The amplitude for this process is

$$\int \frac{d\beta_1 d\beta_2}{(2\pi)^2} \langle 0 | \beta_2 \rangle^{+} \chi^{+} \langle \beta_1 | 0 \rangle \chi^{+} \langle \beta_2 \partial_x \phi(t) | \beta_1 \rangle^{+}. \hfill (7.6)$$

This process gets “convoluted” with many others, where an additional arbitrary even number of particles is emitted at one transition, simply “goes through”, and is destroyed at the other transition. This means one has to replace $\langle 0 | \beta_2 \rangle^{+} \chi^{+} \langle \beta_1 | 0 \rangle$ in (7.6) by

$$\langle 0 | \beta_2 \rangle^{+} \chi^{+} \langle \beta_1 | 0 \rangle + \int \frac{d\beta_3 d\beta_4}{(2\pi)^2} \langle 0 | \beta_4 \beta_3 \beta_2 \rangle^{++} \chi^{+} \langle \beta_4 \beta_3 \beta_1 | 0 \rangle + \ldots.$$  \hfill (7.7)

7 In the limit $\lambda \to \infty$, the change of boundary conditions corresponds to the insertion of an operator with dimension $h = 1/8$, twice the dimension of the Ising spin (for arbitrary $g$, this dimension is $g^4$; in particular it is $1/4$ at the isotropic Kondo point [17].
The manipulation of this sort of terms follows from the general principles used in section 3.3: we simplify them by moving contours and using closure relations. Here, we move the $\beta_1, \beta_2$ integrals to $\text{Im} \beta_1 = -i\pi$ and $\text{Im} \beta_2 = +i\pi$; by doing so, we do encounter singularities of the form-factor of $\partial_x \phi$, and kinematical poles. Forgetting these singularities for a while, crossing produces a $\delta(\beta_1 - \beta_2)$, which gives a vanishing contribution because $\langle \beta | \partial_x \phi | \beta \rangle = 0$. The kinematical poles, together with the coincident terms in crossing, give rise to terms like $\delta(\beta_1 - \beta_3)$. By proceeding similarly with the new integrals, we can successively eliminate all the integrals, and we are left with the contribution of the poles of the form-factor of $\partial_x \phi$. Going back to (7.6) and (7.7), when we move the contours of $\beta_1$ and $\beta_2$ integrations, there is a pole for $\beta_1 = \beta_b - \frac{i\pi}{2}$ and $\beta_2 = \beta_b + \frac{i\pi}{2}$. For each of these poles, we are left with some integrals which we eliminate in the same fashion as before: all what remains at the end is proportional to the residue of the $\partial_x \phi$ form factor at the poles, ie a term $e^{-2T_b t}$, up to some numerical, time independent constant.

There are two other types of terms contributing to the one point function of $\partial_x \phi$. In the first type, the first transition emits two solitons and an antisoliton, a pair soliton antisoliton is destroyed by the $\partial_x \phi$ insertion, and the other soliton by the second transition. In the second type, the $\partial_x \phi$ insertion emits a pair soliton antisoliton. These processes get convoluted with many others as before. By moving the integrals, we find again a contribution proportional to $e^{-2T_b t}$.

Using now the relation between $\partial_x \phi$ and the spin [18]:

$$S_z(t) - S_z(0) = \int_0^t \partial_x \phi(x = 0, t') dt',$$

we find immediately

$$P(t) = e^{-2T_b t}, \quad g = \frac{1}{2}, \quad (7.8)$$

a well known result [19].

Remarkably, the only contribution to $P(t)$ comes from the poles of the $R$ matrices, or equivalently of the non kinematical poles of the transition factors. Since the contour manipulations did not involve the explicit form of $G$ (nor the proof that two different ways of computing the form-factors give the same result, see section (3.3)), it is very likely that this feature generalizes to the case $g \neq \frac{1}{2}$.

Assuming thus that $P(t)$ is completely determined by the poles of the transition factors, that is the poles of the $R$ matrices, we can now write a very natural set of conjectures for its general form.
First, recall that for \( g > \frac{1}{2} \), the sine-Gordon spectrum has no bound states. The \( R \) matrix for solitons antisolitons is independent of \( g \):

\[
R^\pm = i \tanh \left( \frac{\beta - \beta_b}{2} - i \frac{\pi}{4} \right), R_\pm = 0.
\]

We thus expect:

\[
P(t) = \sum_{n=1}^{\infty} a_n e^{-2nT_b t}, \quad g > \frac{1}{2}
\]

with \( \sum a_n = 1 \). There are an infinite number of terms here compared to the \( g = \frac{1}{2} \) case because now all the \( (U(1) \text{ neutral}) 2n \) particle form-factors are non zero.

On the other hand, for \( g < \frac{1}{2} \), there are bound states in the spectrum. This means that the transition factors and the \( R \) matrices now have non trivial poles. According to our conjecture, these poles contribute to \( P(t) \) by values of \( e^\beta \) having a non zero real and imaginary part, and we thus expect that \( P(t) \) will develop oscillatory components on this side of \( g \), in agreement with known numerical results, and an analytical argument in perturbation around \( g \) [20]. At large times, we expect the dominant contribution to come, as usual, from the one-breather term. Recall the one breather reflection matrix [18]:

\[
R(\beta) = -\frac{\tanh \left( \frac{\beta - \beta_b}{2} - i\frac{\pi m g}{4(1-g)} \right)}{\tanh \left( \frac{\beta - \beta_b}{2} + i\frac{\pi m g}{4(1-g)} \right)},
\]

(7.10)

together with \( \mu_1 = 2 \sin \frac{\pi g}{2(1-g)} \), the ratio of the breather mass parameter to the soliton mass.

This should give the leading behaviour of \( P(t) \) (there is no factor 2 in the exponent now because a single breather can be destroyed by \( \partial_x \phi \) )

\[
P(t) \propto \exp \left[ -2tT_b \sin^2 \frac{\pi g}{2(1-g)} \right] \cos \left[ tT_b \sin \frac{\pi g}{1-g} \right].
\]

(7.11)

In particular, setting \( g = \frac{1}{2} - \epsilon \) one gets

\[
P(t) \propto e^{-2T_b t} \cos \left( 4T_b \pi \epsilon t \right).
\]

(7.12)

The relation between \( T_b \) and the bare parameter \( \lambda \) is \( T_b \propto \lambda^{1/1-g} \) - the coefficient of proportionality can be found in [21]. Expression (7.12) agrees with the expansion near \( g = \frac{1}{2} \) in [20]. The expression (7.11) is new as far as we know: in standard notations, we predict the ratio of the period of oscillations to the damping factor to be

\[
\frac{\Omega}{\Gamma} = \cot \frac{\pi g}{2(1-g)}.
\]

(7.13)

It would be interesting to test this numerically (in fact, (7.13) agrees very well with measurements that appeared recently in a paper of K. Voelker [22]).
8. Conclusion

It is clear that more work is necessary to fully determine the transition form-factors, in particular the boundary form-factors as well as the scalar products, for general integrable theories. This seems to be a purely technical matter, and we believe that all new qualitative aspects are well described by the examples discussed here.

To conclude, we would like to make some general remarks, illustrated in the Ising case. From the knowledge of the transition factors, we have in fact obtained an expression for any state of the theory with boundary condition \( a \) in terms of the states with boundary condition \( b \). In particular, we have

\[
|0\rangle_a = b(0|0)\sum_{n=0}^{\infty} \int \prod_{i=1}^{2n} \frac{d\theta_i}{2\pi} \frac{1}{(2n)!} G(\theta_{2n}, \ldots, \theta_1) |\theta_1, \ldots, \theta_{2n}\rangle_b
\]

Using that ground states are normalized, the scalar product of the ground states follows:

\[
\frac{1}{|b(0|0)a|^2} = \sum_{n=0}^{\infty} \int \prod_{i=1}^{2n} \frac{d\theta_i}{2\pi} \frac{1}{(2n)!} |G(\theta_{2n}, \ldots, \theta_1)|^2
\]

It is not clear how useful this general expression is. For instance, in the massless case and for free and fixed boundary conditions, this is nothing but the (R part) of the spin spin correlator at coincident points in the bulk, a manifestly divergent quantity.

It is also interesting to get back to the direct channel, and express the boundary state in the case with two different boundary conditions. By matching the expressions of correlators in both channels, one finds the expression

\[
|B\rangle_{ba} = \exp \left\{ \int \frac{d\theta}{4\pi} \frac{1}{2} [K_a(\theta) + K_b(\theta)] Z^*(-\theta)Z^*(\theta) \right\}
\times \left[ \sum_{n=0}^{\infty} \int \prod_{i=1}^{2n} \frac{d\theta_i}{2\pi} \frac{1}{(2n)!} G \left( \theta_1 - \frac{i\pi}{2}, \ldots, \theta_{2n} - \frac{i\pi}{2} \right) Z^*(\theta_1) \ldots Z^*(\theta_{2n}) \right] |0\rangle,
\]

where the poles at opposite rapidities \( \theta_i = -\theta_j \) are regulated by taking the principal part.

Going to the crossed channel, the contours have first to be completed by turning around the singularities from above or below, the residues completing the term in the exponential to give either \( K_a \) or \( K_b \). The contours can then be moved by \( \pm \frac{i\pi}{2} \) to give integrals in terms of \( F \) or \( G \), reproducing the two possible expressions for correlators as discussed in details for the energy in section 3.

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Appendix A. Perturbative computations.

In this appendix we show some of the direct checks done on the form factors discussed earlier. If we use the action of the introduction, we can decompose the fermions as

\[
\begin{align*}
\psi &= \int_{-\infty}^{\infty} d\theta \left[ \omega e^{\theta/2} a(\theta) e^{imx \sinh \theta + imt \cosh \theta} + \omega e^{\theta/2} a^\dagger(\theta) e^{-imx \sinh \theta - imt \cosh \theta} \right] \\
\overline{\psi} &= \int_{-\infty}^{\infty} d\theta \left[ \omega e^{-\theta/2} a(\theta) e^{imx \sinh \theta + imt \cosh \theta} + \omega e^{-\theta/2} a^\dagger(\theta) e^{-imx \sinh \theta - imt \cosh \theta} \right]
\end{align*}
\] (A.1)

where we have used \( \omega = e^{i\pi/4} \). We evaluate the scalar products between different states in the simple case where the fields \( h_a = 0 \) and \( h_b \) are small. The states at \( t = -\infty \) are in the Hilbert space of the theory \( a \), which is at zero field, and those at \( t = +\infty \) are in the Hilbert space of theory \( b \). Then from standard perturbative arguments we obtain to second order (which is the first non trivial one)

\[
b\langle \theta_1 \theta_2 || 0 \rangle_a \propto h_a^2 \prod_{i=1,2} \sinh \theta_i \cosh \left( \theta_i^2 - i \frac{\pi}{4} \right) \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) \tanh \left( \frac{\theta_1 + \theta_2}{2} \right) 
\] (A.2)

where \( R_0 \) is the reflection matrix at zero field. The integral (A.2) are easily computed by using the mode decomposition for the fermions:

\[
b\langle \theta_1 \theta_2 || 0 \rangle_a \propto h_b^2 \prod_{i=1,2} \sinh \theta_i \cosh \left( \theta_i^2 - i \frac{\pi}{4} \right) \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) \tanh \left( \frac{\theta_1 + \theta_2}{2} \right) 
\] (A.3)

We can do a similar computation for the four particle scalar product to fourth order and we find

\[
b\langle \theta_1 \cdots \theta_4 || 0 \rangle_a \propto h_b^4 \prod_{i=1}^{4} \sinh \theta_i \cosh \left( \theta_i^2 - i \frac{\pi}{4} \right) \prod_{i<j} \tanh \left( \frac{\theta_i - \theta_j}{2} \right) \tanh \left( \frac{\theta_i + \theta_j}{2} \right)
\] (A.4)