On the $p$-ranks of the ideal class groups of imaginary quadratic fields

Jaitra Chattopadhyay$^1$ · Anupam Saikia$^1$

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Abstract

For a prime number $p \geq 5$, we explicitly construct a family of imaginary quadratic fields $K$ with ideal class groups $\text{Cl}_K$ having $p$-rank $\text{rk}_p(\text{Cl}_K)$ at least 2. We also quantitatively prove, under the $abc$-conjecture that for sufficiently large positive real numbers $X$ and any real number $\epsilon$ with $0 < \epsilon < \frac{1}{p-1}$, the number of imaginary quadratic fields $K$ with the absolute value of the discriminant $d_K \leq X$ and $\text{rk}_p(\text{Cl}_K) \geq 2$ is $\gg X^{\frac{1}{p-1}-\epsilon}$. This conditionally improves the previously known lower bound of $X^{\frac{1}{p-1}}$ due to Byeon and the recent bound $X^{\frac{1}{p}/(\log X)^2}$ due to Kulkarni and Levin.

Keywords Quadratic number fields · Class numbers · Ranks of class groups

Mathematics Subject Classification Primary 11R11; Secondary 11R29

1 Introduction

The question of the divisibility of class numbers of quadratic fields has been an object of much study for a long time. Due to the works of Nagell [31], and later by Ankeny and Chowla [1], it is known that there exist infinitely many imaginary quadratic fields $K$ with class numbers $h_K$ divisible by any given integer $n \geq 2$. Much later, Yamamoto [37], Weinberger [36], and many others proved the analogue for real quadratic fields.
There has also been much work done towards the quantitative study (cf. [3, 5, 7–9, 16, 17, 20, 28, 30, 34, 35, 38]). Apart from these, the topic of indivisibility of class numbers (cf. [2, 4, 11, 22, 32]) and simultaneous divisibility and indivisibility of class numbers (cf. [10, 18, 23, 24]) has received much attention in recent times.

Since the ideal class group $\text{Cl}_K$ of $K$ is a finite abelian group, it makes sense to consider the $m$-rank of $\text{Cl}_K$, for various integers $m \geq 2$. The $m$-rank of $\text{Cl}_K$, denoted by $\text{rk}_m(\text{Cl}_K)$, is defined to be the maximal integer $r \geq 0$ such that $(\mathbb{Z}/m\mathbb{Z})^r \subseteq \text{Cl}_K$. Note that if $\text{rk}_m(\text{Cl}_K) \geq 1$, then it immediately follows that $h_K$ is divisible by $m$. Thus, the study of the quantity $\text{rk}_m(\text{Cl}_K)$ sheds more light on the question of $m$-divisibility of $h_K$. In particular, the case $m = 3$ has been extensively studied (cf. [12, 13, 21, 27, 29, 39]).

In this article, we are concerned about the ranks of the ideal class groups of imaginary quadratic fields. For integers $m \geq 2$, $n \geq 0$ and a large positive real number $X$, let

$$N^m_n(X) = \{ K = \mathbb{Q}(\sqrt{-d}) : d > 0 \text{ is square-free}, |d_K| \leq X \text{ and } \text{rk}_m(\text{Cl}_K) \geq n \}.$$ 

The following question arises naturally.

**Question 1** What is the asymptotic behaviour of $N^m_n(X)$ as $X \to \infty$?

In [29], Luca and Pacelli proved that $N^2_3(X) \gg X^{\frac{1}{2}}$. Yu [39] recently improved that lower bound to $X^{\frac{1}{2} - \varepsilon}$. Also, Levin et al. [27] proved that $N^5_3(X) \gg X^{\frac{1}{2} - \varepsilon}$ is known due to Byeon [6]. Very recently, Kulkarni and Levin [26] improved this and proved that $N^2_m(X) \gg X^{\frac{1}{p-1}}/(\log X)^2$. In the same paper, Kulkari and Levin improved the lower bound of $N^5_3(X)$ to $X^{\frac{1}{15}}/(\log X)^2$. The reader is encouraged to look into [14, 15] and [25] for similar developments along this line.

In this paper, using a result of Poonen [33] on the density of square-free values of multi-variate polynomials, we slightly improve the lower bound for $N^2_p(X)$ for any prime number $p \geq 5$, by extending the method used by Yu [39]. More precisely, we prove the following theorem.

**Theorem 1** Let $p \geq 5$ be a prime number, $\varepsilon$ a real number with $0 < \varepsilon < \frac{1}{p-1}$ and $X$ a sufficiently large positive real number. Then under the validity of the abc-conjecture, we have $N^2_p(X) \gg X^{\frac{1}{p-1} - \varepsilon}$.

**Remark 1** In the course of the proof of Theorem 1, we shall construct an explicit infinite family of imaginary quadratic fields with ideal class groups having $p$-rank at least 2 and then count the number of distinct fields in the family with discriminant $\leq X$. We shall make use of the square-free values of a multi-variate polynomial of degree $p$. In order to do so, we apply a result of Poonen [33] which is valid under the assumption of the abc-conjecture.
2 A family of quadratic fields with \( \text{rk}_p(\text{Cl}_K) \geq 2 \)

In what follows, \( p \) always denotes a fixed prime number \( \geq 5 \). For two non-zero real-valued functions \( f \) and \( g \), we use the notation \( f \ll g \) or \( f = O(g) \) to mean that \( \frac{|f|}{|g|} \) is bounded and \( f \asymp g \) to mean that both \( f \ll g \) and \( g \ll f \) hold.

In this section, we prove a proposition that provides a parametric family of imaginary quadratic fields with ideal class groups having \( p \)-rank at least 2. Then by appropriately choosing the parameters, we shall count how many of these fields have the absolute values of their discriminants \( \leq X \). Our following proposition is an extension of Lemma 2.1 of Yu [39].

**Proposition 1** For a positive real number \( X \), let \( q \) be a prime number with \( q \asymp X^{\frac{1}{2p(p-1)}} \). Let \( a \) and \( b \) be integers with \( \frac{1}{2p}q^{\frac{p}{p-2}} < a, b < \left( \frac{1}{2p} + \frac{1}{2p^2} \right)q^{\frac{p}{p-2}} \) and let

\[
f_q(a, b) = 2(a^p + b^p)q^p - (a - b)^2q^{2p} - g(a, b)^2,
\]

where \( g(a, b) = \frac{a^p - b^p}{a - b} \). If \( X \) is sufficiently large and \( f_q(a, b) \) is a square-free integer \( \geq 4ab \frac{p-1}{p} \frac{p+1}{2} q^{\frac{p}{p-2}} \), then the ideal class group of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-f_q(a, b)}) \) contains a subgroup isomorphic to \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \).

**Proof** Let \( X_1 = g(a, b) + (a - b)q^p \), \( Y_1 = aq \), \( X_2 = g(a, b) - (a - b)q^p \) and \( Y_2 = bq \). Then we see that

\[
X_1^2 - 4Y_1^p = -f_q(a, b) = X_2^2 - 4Y_2^p. \tag{1}
\]

For the sake of brevity, let \( D = f_q(a, b) \). It follows that for integers \( a \) and \( b \) in the given range, we have \( D > 0 \) and \( D \asymp X \). Again, \( 4ab \frac{p-1}{p} \frac{p+1}{2} q^{\frac{p}{p-2}} \asymp X^{\frac{p}{2p-2}} \). Thus, \( D \geq 4ab \frac{p-1}{p} \frac{p+1}{2} q^{\frac{p}{p-2}} \) for sufficiently large values of \( X \).

Since \( D \) is assumed to be square-free, we get from (1) that \( X_j \) is odd and, therefore, \(-D \equiv 1 \pmod{4}\). Also, for \( j = 1, 2 \), from Eq. (1), we obtain

\[
\frac{X_j + \sqrt{-D}}{2} \cdot \frac{X_j - \sqrt{-D}}{2} = Y_j^p \text{ in } \mathcal{O}_{\mathbb{Q}(\sqrt{-D})}.
\]

Hence, \( X_j, Y_j^p \in \mathcal{I}_j := \left\langle \frac{X_j + \sqrt{-D}}{2}, \frac{X_j - \sqrt{-D}}{2} \right\rangle \). Since \( X_j^2 - 4Y_j^p = -D \) is assumed to be square-free, we conclude that \( \gcd(X_j, Y_j) = 1 \). Therefore, \( \mathcal{I}_j = \mathcal{O}_K \) and consequently, \( \left\langle \frac{X_j + \sqrt{-D}}{2} \right\rangle \) and \( \left\langle \frac{X_j - \sqrt{-D}}{2} \right\rangle \) are comaximal ideals in \( \mathcal{O}_{\mathbb{Q}(\sqrt{-D})} \). Thus, there exists an integral ideal \( a_j \subseteq \mathcal{O}_{\mathbb{Q}(\sqrt{-D})} \) such that

\[
\left\langle \frac{X_j + \sqrt{-D}}{2} \right\rangle = a_j^p \text{ and } \left\langle \frac{X_j - \sqrt{-D}}{2} \right\rangle = \overline{a}_j^p.
\]
Now, we prove that the ideal classes \([a_1]\) and \([a_2]\) of both \(a_1\) and \(a_2\) generate subgroups of order \(p\) inside \(Cl_{Q(\sqrt{-D})}\) and they intersect trivially. For that, it suffices to prove that \(a_1\), \(a_2\) and \(a_1b^t\) are all non-principal ideals in \(O_{Q(\sqrt{-D})}\), where \(b = a_2\) or \(\overline{a_2}\) and \(t = 1, \ldots, \frac{p-1}{2}\).

**Case 1** The ideal \(a_j\) is non-principal. For otherwise, \(a_j = \left\langle \frac{\alpha_j + \beta_j \sqrt{-D}}{2} \right\rangle\) for some integers \(\alpha_j\) and \(\beta_j\) of same parity. Since \(a_j^p = \left\langle \frac{X_j + \sqrt{-D}}{2} \right\rangle\), it follows that \(\beta_j \neq 0\). Therefore,

\[
Y_j^p = N(a_j^p) = \left( N\left( \frac{\alpha_j + \beta_j \sqrt{-D}}{2} \right) \right)^p = \left( \frac{\alpha_j^2 + \beta_j^2 D}{4} \right)^p \geq \left( \frac{1 + D}{4} \right)^p > (ab \frac{p-1}{2} q \frac{p+1}{2})^p > Y_j^p,
\]

which is a contradiction. Therefore, \(a_j\) is non-principal for \(j = 1, 2\).

**Case 2** For \(b = a_1^t\) or \(\overline{a_2^t}\) and \(t\) even, the ideal \(a_1b\) is non-principal. For otherwise, \(a_1b^t = \left\langle \frac{\alpha + \beta \sqrt{-D}}{2} \right\rangle\) for some integers \(\alpha\) and \(\beta\) of same parity. Then

\[
a_1^p b^{pt} = \left\langle \frac{X_1 + \sqrt{-D}}{2} \right\rangle \left\langle \frac{X_2 \pm \sqrt{-D}}{2} \right\rangle^t = \left\langle \frac{X_1 + \sqrt{-D}}{2} \right\rangle \left\langle \frac{\left( \sum_{i=0}^t \binom{t}{i} X_2^i (\pm \sqrt{-D})^{t-i} \right)}{2^t} \right\rangle.
\]

The coefficient of \(\sqrt{-D}\) is

\[
A := \pm \frac{1}{2^{t+1}} \left[ X_1 \left( \sum_{i \text{ odd}} \binom{t}{i} X_2^i (\pm \sqrt{-D})^{t-i} \right) + \sum_{i \text{ even}} \binom{t}{i} X_2^i (\pm \sqrt{-D})^i \right].
\]

Since \(t\) is even, we write \(t = 2t_1\) for an integer \(t_1\). Then \(A = 0\) implies that \(X_2\) divides \(D^{t_1}\), which in turn implies that \(\gcd(X_2, D) > 1\). This, together with the equation \(X_2^2 + D = 4Y_2^p\), implies that \(\gcd(X_2, Y_2) > 1\), which is a contradiction to the assumption that \(D\) is square-free. Consequently, we get that \(A \neq 0\) and, therefore, \(\beta \neq 0\). Hence

\[
Y_1^p Y_2^p = N(a_1^p b^{pt}) = \left( \frac{\alpha^2 + D\beta^2}{4} \right)^p \geq \left( \frac{1 + D}{4} \right)^p > (ab \frac{p-1}{2} q \frac{p+1}{2})^p,
\]

which is a contradiction.
Case 3 For $b = a_2^t$ or $\overline{a_2}^t$ and $t$ odd, the ideal $a_1 b$ is non-principal. Otherwise, $a_1 b^t = \left< \frac{\alpha + \beta \sqrt{-D}}{2} \right>$ for some integers $\alpha$ and $\beta$ of same parity. Then, similar as before, the coefficient of $\sqrt{-D}$ in $\left< \frac{X_1 + \sqrt{-D}}{2} \right> \left< \frac{X_2 + \sqrt{-D}}{2} \right>^t$ is

$$B := \pm \frac{1}{2^{t+1}} \left[ X_1 \left( \sum_{i \text{ odd}} \binom{t}{i} X_2^{i-1} (\pm \sqrt{-D})^{i-1} \right) + \sum_{i \text{ even}} \binom{t}{i} X_2^{i-1} (\pm \sqrt{-D})^i \right].$$

Since $t$ is odd, we write $t = 2t_1 + 1$. Now, if $B = 0$, then we obtain that $X_2$ divides $X_1D^t$. The assumption that $D$ is square free yields that $\gcd(X_2, D) = 1$. Thus $X_2$ divides $X_1$.

Since $X_2 \mid X_1$, we obtain that $X_2 \mid (X_1 \pm X_2)$. That is, $X_2 \mid g(a, b)$ and $X_2 \mid (a - b)q^p$. Now, we prove that $X_2$ is either $p$ or $-p$. For that, let $\ell$ be a prime divisor of $X_2$. Then $\ell$ is odd. Moreover, $\ell \mid g(a, b)$ and $\ell \mid (a - b)q^p$. We note that if $\ell = q$, then $\ell^2 \mid f_\ell(a, b) = D$, a contradiction to the assumption that $D$ is square-free. Consequently, $\ell \neq q$ and, hence, $a \equiv b \pmod{\ell}$. Again, $\ell \nmid ab$, because $D$ is square free. Therefore, $0 \equiv g(a, b) \equiv pa^{p-1} \pmod{\ell}$ implies that $\ell = \pm p$. Thus, $X_2 = \pm p^k$, for some integer $k \geq 1$. Therefore, the congruence $a \equiv b \pmod{X_2}$ translates to $a \equiv b \pmod{p^k}$. Consequently, we have $0 \equiv g(a, b) \equiv pa^{p-1} \pmod{p^k}$. Since $p \nmid a$, this forces $k = 1$. Now, we prove that the equation $X_2 = \pm p$ has no solution in integers for $a, b \in [\frac{1}{2p} q^{\frac{p-2}{2}}, (\frac{1}{2p} + \frac{1}{2^pp^p}) q^{\frac{p-2}{2}}] \cap \mathbb{Z}$.

If possible, suppose that $X_2 = a^{p-1} + ap-2b + \ldots + ab^{p-2} + b^{p-1} - (a - b)q^p = \pm p$ has an integral solution in the aforementioned range. Then we have

$$p = |(a^{p-1} + ap-2b + \ldots + ab^{p-2} + b^{p-1}) - (a - b)q^p|$$
$$\geq |a^{p-1} + ap-2b + \ldots + ab^{p-2} + b^{p-1}| - |(a - b)q^p|$$
$$\geq \left( p \left( \frac{1}{2p} \right)^{p-1} - \left( \frac{1}{2p} \right)^p \right) q^{\frac{p(p-1)}{2(2p)^p}} = \frac{2p^2 - 1}{(2p)^p} \cdot q^{\frac{p(p-1)}{2p^2}}.$$

Since $X$ is a large enough positive real number and $q \asymp X^{\frac{p-2}{2p^2}}$, we obtain that

$$p \geq \frac{2p^2 - 1}{(2p)^p} \cdot q^{\frac{p(p-1)}{2p^2}} \geq p,$$

which is a contradiction.

Therefore, $B \neq 0$ and, hence, $\beta \neq 0$. Hence,

$$Y_1^p Y_2^{p^t} = N(a_1^p b^{p^t}) = \left( \frac{\alpha^2 + D\beta^2}{4} \right)^p \geq \left( \frac{1}{4} \right)^p > \left( ab^{p-1} q^{p+1} \right)^p,$$

which is a contradiction.
Hence, both \([a_1]\) and \([a_2]\) generate a subgroup of order \(p\) inside \(Cl_{\mathbb{Q}(\sqrt{-D})}\) and they intersect trivially. Therefore, \(Cl_{\mathbb{Q}(\sqrt{-D})}\) contains a subgroup isomorphic to \(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\). This completes the proof of the proposition.

\[\square\]

### 3 Square-free values of \(f_q\)

To prove Theorem 1, we need to estimate the number of square-free values assumed by the polynomial \(f_q(a, b)\) of Proposition 1, where \(a\) and \(b\) vary in a certain range. For that, we appeal to a result of Poonen [33]. In this section, first we prove two lemmas about the square-free values of the polynomial \(f_q\). We prove that \(f_q\), considered as a polynomial in two variables over \(\mathbb{Q}\), is square-free in the polynomial ring \(\mathbb{Q}[x, y]\). We also prove that \(f_q(a, b)\) is not divisible by the square of a fixed prime number for all \((a, b) \in \mathbb{N} \times \mathbb{N}\). This is necessary when we apply Poonen’s result on the square-free values of polynomials of several variables in our context.

**Lemma 1** For a prime number \(q \geq 5\), the polynomial

\[
f_q(x, y) = 2(x^p + y^p)q^p - (x - y)^2q^{2p} - (x^{p-1} + x^{p-2}y + \ldots + y^{p-1})^2 \in \mathbb{Z}[x, y]
\]

is square-free as an element of \(\mathbb{Q}[x, y]\).

**Proof** Assume that \(f_q(x, y)\) is not square-free as an element of \(\mathbb{Q}[x, y]\). Then we can write \(f_q(x, y) = G(x, y)^2H(x, y)\) for some polynomials \(G\) and \(H\). We may also assume that \(G\) is irreducible in \(\mathbb{Q}[x, y]\). Differentiating the equation with respect to \(x\) and \(y\), we see that \(\frac{\partial f_q}{\partial x} = G^2 \frac{\partial H}{\partial x} + 2G(\frac{\partial G}{\partial x})H\) and \(\frac{\partial f_q}{\partial y} = G^2 \frac{\partial H}{\partial y} + 2G(\frac{\partial G}{\partial y})H\). That is, \(G(x, y)\) is a common divisor of \(f_q, \frac{\partial f_q}{\partial x}\) and \(\frac{\partial f_q}{\partial y}\). Therefore, \(G(x, y)\) also divides \(x\frac{\partial f_q}{\partial x} + y\frac{\partial f_q}{\partial y} = -(p - 2)((g(x, y)^2 - (x - y)^2q^{2p}) + pf_q(x, y)\) and consequently, \(G(x, y)\) divides \((g(x, y)^2 - (x - y)^2q^{2p})\). Since \(G\) is irreducible, \(G(x, y)\) divides \([g(x, y) - (x - y)q^p]\) or \([g(x, y) + (x - y)q^p]\). This implies that

\[
G(x, y)^2 \mid [g(x, y)^2 + (x - y)^2q^{2p} \pm 2(x^p - y^p)q^p].
\]

As \(G(x, y)^2\) divides \(f_q(x, y)\), we obtain

\[
G(x, y)^2 \mid [g(x, y)^2 + (x - y)^2q^{2p} - 2(x^p + y^p)q^p].
\]

From (2) and (3), it follows that \(G(x, y)^2 \mid 4x^pq^p\) or \(G(x, y)^2 \mid 4y^pq^p\). Hence, \(G\) is a power of \(x\) or a power of \(y\). This implies that \(f_q(x, 0) \equiv 0\) or \(f_q(0, y) \equiv 0\), which is clearly false. This completes the proof of the lemma.

\[\square\]

**Lemma 2** Let \(q > p\) be a prime number and let \(f_q(x, y)\) be as in Lemma 1. Then there is no prime number \(\ell\) such that \(\ell^2\) divides \(f_q(a, b)\) for all \((a, b) \in \mathbb{N} \times \mathbb{N}\).
Proof If possible, suppose that $\ell$ is a prime number number such that $\ell^2$ divides $f_q(a, b)$ for all $(a, b) \in \mathbb{N} \times \mathbb{N}$. In particular, $\ell^2 \mid f_q(a, a) = a^p(4q^p - p^2a^{p-2})$. For $a = 1$, this yields $4q^p \equiv p^2 \pmod{\ell^2}$. Since $p$ is odd, this implies that $\ell$ is also odd. Moreover, $f_q(1, 1) \equiv 0 \pmod{\ell^2}$ implies that $\ell \neq p$ and $\ell \neq q$.

Now, let $a$ be a primitive root modulo $\ell^2$. Then the congruence $f_q(a, a) \equiv 0 \pmod{\ell^2}$ boils down to $4q^p \equiv p^2a^{p-2} \pmod{\ell^2}$. Using the congruence $4q^p \equiv p^2 (\mod \ell^2)$, we obtain $p^2 \equiv p^2a^{p-2} \pmod{\ell^2}$. Since $\ell \neq p$, this gives $a^{p-2} \equiv 1 \pmod{\ell^2}$. Since $a$ is a primitive root modulo $\ell^2$, it immediately follows that $\ell(\ell - 1) \mid (p - 2)$, which is impossible because $p - 2$ is odd and $\ell(\ell - 1)$ is even. This completes the proof of the lemma. □

Our objective is to estimate the quantity $\sum_{D \in f_q(a, b)} \mu^2(D)$, where $a, b$ are integers with $\frac{1}{2p} q^p < a, b < (\frac{1}{2p} + \frac{1}{2p^p})q^p$ and $\mu$ is the Möbius function. In other words, we wish to count the number of times the bivariate polynomial $f_q$ assumes a square-free value in a certain range. In view of this, we quote a result of Poonen [33] about the density of square-free values of a multivariable polynomial with integer coefficients, under the assumption of the abc-conjecture. Before stating the result, let us recall some notation, used in [33] as follows.

For positive real numbers $B_1, \ldots, B_n$, let

$$Box := Box(B_1, \ldots, B_n) := \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : 0 < a_i \leq B_i \text{ for all } i\}.$$ 

For a set $\mathcal{S} \subseteq \mathbb{Z}^n$, define

$$\overline{\mu}_n(\mathcal{S}) := \limsup_{B_1, \ldots, B_{n-1} \to \infty} \limsup_{B_n \to \infty} \frac{|(\mathcal{S} \cap Box)|}{\#Box} \text{ and } \overline{\mu}_n(\mathcal{S}) := \liminf_{B_1, \ldots, B_{n-1} \to \infty} \liminf_{B_n \to \infty} \frac{|(\mathcal{S} \cap Box)|}{\#Box}.$$ 

If $\overline{\mu}_n(\mathcal{S}) = \mu_n(\mathcal{S})$, then $\mu_n(\mathcal{S})$ is defined to be the common value. Again,

$$\overline{\mu}_{weak}(\mathcal{S}) := \max \limsup_{B_{\sigma(1)} \to \infty} \limsup_{B_{\sigma(n)} \to \infty} \frac{|(\mathcal{S} \cap Box)|}{\#Box},$$ 

where $\sigma$ runs through all the permutations of the set $\{1, \ldots, n\}$. Similarly, we define

$$\underline{\mu}_{weak}(\mathcal{S}) := \max \liminf_{B_{\sigma(1)} \to \infty} \liminf_{B_{\sigma(n)} \to \infty} \frac{|(\mathcal{S} \cap Box)|}{\#Box}.$$ 

If $\overline{\mu}_{weak}(\mathcal{S}) = \underline{\mu}_{weak}(\mathcal{S})$, then $\mu_{weak}(\mathcal{S})$ is defined to be the common value.
We quickly recall the $\text{abc}$-conjecture, which states that if $a, b$ and $c$ are pairwise relatively prime positive integers with $a + b = c$, then for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that $c < C(\varepsilon) \left( \prod_{\ell | abc} \ell \right)^{1+\varepsilon}$.

**Proposition 2** ([33], Theorem 3.2 and Corollary 3.3) Let $F(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial that is square-free as an element of $\mathbb{Q}[x_1, \ldots, x_n]$. Suppose that $x_n$ appears in $F$. Let

$$\mathcal{S}_F := \{ x \in \mathbb{Z}^n : F(x) \text{ is square free} \}.$$  

Assume the validity of the $\text{abc}$-conjecture. Then $\mu_n(\mathcal{S}_F) = \prod_{\ell} \left( 1 - \frac{c_\ell}{\ell^{2n}} \right)$, where for each prime number $\ell$, the quantity $c_\ell$ stands for the number of $\alpha \in (\mathbb{Z}/\ell^2\mathbb{Z})^n$ satisfying $F(\alpha) = 0$ in $\mathbb{Z}/\ell^2\mathbb{Z}$. Moreover, if $x_n$ does not appear in $F$, then $\mu_{\text{weak}}(\mathcal{S}_F) = \prod_{\ell} \left( 1 - \frac{c_\ell}{\ell^{2n}} \right)$.

**Remark 2** Note that $f_q(x, y)$ satisfies the hypotheses of Proposition 2. Also, Lemma 2 asserts that the square of no fixed prime number divides $f_q(a, b)$ for all $a, b \in \mathbb{N}$. Therefore, each term in the Euler product $\prod_{\ell} \left( 1 - \frac{c_\ell}{\ell^{2n}} \right)$ is non-zero for $f_q$. Moreover, Poonen also proved in [33] that the quantity $c_\ell$ in Proposition 2 satisfies $c_\ell = O(\ell^{2n-2})$, using techniques from algebraic geometry. Therefore, each term in the Euler product $\prod_{\ell} \left( 1 - \frac{c_\ell}{\ell^{2n}} \right)$ is $1 + O \left( \frac{1}{\ell^2} \right)$ and, hence, the Euler product for $f_q$ converges to a non-zero constant.

### 4 Proof of Theorem 1

Recall that for a large positive real number $X$, let $q$ is a prime number with $q \asymp X^{\frac{p-2}{2p(p-1)}}$. That is, $c_1 X^{\frac{p-2}{2p(p-1)}} < q < c_2 X^{\frac{p-2}{2p(p-1)}}$ for two suitably chosen positive constants $c_1$ and $c_2$. For any such prime number $q$, we choose integers $a$ and $b$ with $\frac{1}{2p} q^{\frac{p}{p-2}} < a, b < (\frac{1}{2p} + \frac{1}{2p^2}) q^{\frac{p}{p-2}}$. Then we see that $f_q(a, b) > 0$ and $f_q(a, b) \asymp q^{\frac{p}{p-2}} \asymp X$. For the sake of convenience, let

$$S := \{(a, b) \in \mathbb{Z}^2 : \frac{1}{2p} q^{\frac{p}{p-2}} < a, b < \left( \frac{1}{2p} + \frac{1}{2p^2} \right) q^{\frac{p}{p-2}} \}$$

and

$$S(D) := \# \{(a, b) \in S : D = f_q(a, b) \}.$$
Also, let

\[ S_1 := \sum_{D = f_q(a,b) \leq X} \mu^2(D)S(D) \quad \text{and} \quad S_2 := \sum_{D = f_q(a,b) \leq X} \mu(D)^2S(D)^2. \]

We note that \( S_1 \) counts the number of square-free positive integral values of \( f_q(a,b) \), where \((a,b) \in \mathcal{S}\). By Lemma 1, we see that \( f_q(X,Y) \) satisfies the hypotheses of Proposition 2. Therefore, from Proposition 2 and Remark 2, we conclude that \( S_1 \gg q^{\frac{p}{p-2}} \cdot q^{\frac{p}{p-2}} = q^{2\frac{p}{p-2}} \).

Now, we proceed to find an upper bound for \( S_2 \). From the definition of \( S_2 \), we see that \( S_2 \) is the number of square-free positive integers \( D \) with \( D = f_q(a,b) \), counted with multiplicity \( S(D)^2 \). Therefore, \( S_2 \) is bounded above by the number of quadruples \((a_1, a_2, b_1, b_2)\) such that \( f_q(a_1, b_1) = f_q(a_2, b_2) \). After simplifying, this boils down to

\[ 4q^p (a_1^p - a_2^p) = (g(a_1, b_1) + (a_1 - b_1)q^p)^2 - (g(a_2, b_2) + (a_2 - b_2)q^p)^2. \]  \( (4) \)

The right-hand side of (4) can be factorized as \((u-v)(u+v)\), where \( u = g(a_1, b_1) + (a_1 - b_1)q^p \) and \( v = g(a_2, b_2) + (a_2 - b_2)q^p \). Now, if \( a_1 = a_2 \), then for a fixed value of \( b_1 \), Eq. (4) reduces to a degree 2\( (p-1) \) polynomial in the variable \( b_2 \) and thus there are at most 2\( (p-1) \) integral solutions for \( b_2 \). Since \( a_1 = a_2 \) can be chosen in \( O(q^{\frac{p}{p-2}}) \) ways and similarly \( b_1 \) can also be chosen in \( O(q^{\frac{p}{p-2}}) \) ways, the number of choices for the quadruple \((a_1, a_2, b_1, b_2)\) in this case is \( O(q^{\frac{p}{p-2}} \cdot q^{\frac{p}{p-2}}) = O(q^{\frac{2p}{p-2}}) \). If \( a_1 \neq a_2 \), we may assume without loss of any generality that \( a_1 > a_2 \). For a fixed such tuple \((a_1, a_2)\), we find by (4) that \((u-v)(u+v) \neq 0 \). Now, \( 4q^p (a_1^p - a_2^p) \) can be factorized into the product of two integers in \( \sigma_0(4q^p(a_1^p - a_2^p)) \) ways, where \( \sigma_0 \) stands for the divisor function. By using the classical result \( \sigma_0(N) = O(N^{\varepsilon}) \) for any \( \varepsilon > 0 \), we get that \( 4q^p (a_1^p - a_2^p) \) can be factorized in \( O(q^{\varepsilon}) \) ways. For each such factorization and fixed values of \( a_1, a_2, u \) and \( v \), we have

\[ a_1^{p-1} + a_1^{p-2}b_1 + \cdots + b_1^{p-1} + (a_1 - b_1)q^p = u \] \( (5) \)

and

\[ a_2^{p-1} + a_2^{p-2}b_2 + \cdots + b_2^{p-1} + (a_2 - b_2)q^p = v. \] \( (6) \)

We see that (5) and (6) are polynomials of degree \( p-1 \) in the variables \( b_1 \) and \( b_2 \), respectively. Consequently, there are at most \( p-1 \) solutions in \( b_1 \) and \( b_2 \) of (5) and (6), respectively. Thus if \( a_1 \neq a_2 \), then each \( a_1 \) and \( a_2 \) can be chosen in \( O(q^{\frac{p}{p-2}}) \) ways, and corresponding to the tuple \((a_1, a_2)\), the integers \( u \) and \( v \) can be chosen in \( O(q^{\varepsilon}) \) ways. Therefore, the number of choices for the quadruple \((a_1, a_2, b_1, b_2)\) is \( O(q^{\frac{p}{p-2}} \cdot q^{\frac{p}{p-2}} \cdot q^{\varepsilon}) = O(q^{\frac{2p}{p-2}+\varepsilon}) \) ways. Hence, \( S_2 \ll q^{\frac{2p}{p-2}+\varepsilon} \).
Now, using the Cauchy–Schwarz inequality, we obtain

\[
\left( \sum_{D \leq X \atop S(D) > 0} \mu(D)^2 \right) \left( \sum_{D \leq X \atop S(D) > 0} \mu(D)^2 S(D)^2 \right) \geq \left( \sum_{D \leq X \atop S(D) > 0} \mu(D)^2 S(D) \right)^2.
\]

From inequality (7) and the estimates of \( S_1 \) and \( S_2 \), it follows that

\[
\sum_{D \leq X \atop S(D) > 0} \mu(D)^2 \gg q^{\frac{4p}{p-1}-\frac{2p}{p-2}\varepsilon} \gg q^{\frac{2p}{p-2}-\varepsilon} \gg X^{\frac{1}{p-1}-\varepsilon}.
\]

This, together with Proposition 1, completes the proof of Theorem 1. \( \square \)

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