Conformism-driven phases of opinion formation on heterogeneous networks: the $q$-voter model case

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Abstract. The $q$-voter model, a variant of the classic voter model, has been analyzed by several authors. While allowing us to study opinion dynamics, this model is also believed to be one of the most representative among the many defined in the wide field of sociophysics. Here, we investigate the consequences of conformity on the consensus reaching process, by numerically simulating a $q$-voter model with agents behaving either as conformists or nonconformists, embedded on heterogeneous network topologies (as small-world and scale-free). In fact, although it is already known that conformity enhances the reaching of consensus, the related process is often studied only on fully-connected networks, thus strongly limiting our full understanding of it. This paper represents a first step in the direction of analyzing more realistic social models, showing that different opinion formation phases, driven by the conformist agents density, are observable. As a result, we identify threshold values of the density of conformist agents, varying across different topologies and separating different regimes of our system, ranging from a disordered phase, where different opinions coexist, to a gradually more ordered phase, where consensus is eventually reached.

Keywords: random graphs, networks, critical phenomena of socio-economic systems, interacting agent models
1. Introduction

Recent years have witnessed the increasing interest of scientists from different fields such as physics, mathematics and computer science, in socio-economic systems [1–5]. In particular, it has become evident that several models born within the realm of statistical physics can be successfully employed in the understanding of simplified social systems, thus gaining insight into human behavior [3].

More precisely, the influence of conformity (considered a fundamental social trait) on opinion dynamics has been extensively studied [6–10]. Although conformity is of great interest in social sciences, e.g. in social psychology [11], several authors have analyzed it by adopting the viewpoint of statistical mechanics [12, 13]. Just to cite a few, in [7, 8] the authors analyzed the role of contrarians (i.e. agents acting as nonconformists) in voting dynamics and in [6, 8] the authors analyzed how conformity affects opinion dynamics by implementing the local majority rule.

In this work, we approach the problem of understanding how conformity affects opinion dynamics by implementing the $q$-voter model [3, 14–17], i.e. a variant of the classic voter model [18], on heterogeneous networks. In fact, while it is already known that conformity enhances the reaching of consensus (i.e. an opinion shared by all agents) [6] the details of this process are still questioned [16]. Moreover, systems like the voter model and the $q$-voter model are often simulated over fully connected networks [14, 19, 20], and only to a lesser extent on more complex topologies (see for instance [21–23]). If, on the one hand, this allows us to analytically model the system under the mean-field approximation, on the other hand it strongly limits the validity of results to unrealistic scenarios as it has been proven that social systems show highly heterogeneous structures [24]. Thus, our analysis aims at exploring the behavior of the $q$-voter model by...
considering more realistic network topologies with the aim of understanding the extent to which (1) varying the amount of conformist agents and (2) varying the network structure affects the consensus reaching process. In order to do so, we rely heavily upon numerical simulations.

Results of our simulations indicate the presence of different opinion-formation regimes, driven by the density of conformist agents and varying across different network configurations. Threshold values separating different regimes vary as well. Moreover, the system seems to undergo a spontaneous symmetry-breaking, by (stochastically) choosing states with the same ‘net’ opinion but opposite signs. The remainder of the paper is organized as follows: section 2 introduces the proposed model, section 3 shows results of numerical simulations, and finally, section 4 concludes the paper.

2. The model

In order to study the role of conformity in the $q$-voter model, we defined a simple agent-based model by considering $N$ agents, provided with an opinion and a social character. Opinions are mapped to states $s_i = \pm 1$, $i = 1 \ldots N$ and are assigned to each agent of the population stochastically, i.e. according to the probability coefficients $P_+ = P_- = 1/2$; thus, our initial expected number of opinions $+1$ is $\langle N^0_+ \rangle = N/2$. Moreover, agents are provided with an individual behavior, i.e. either conformist or nonconformist. In what follows, we will adopt the definition according to which a conformist agent adopts the opinion of the majority of its neighbors, whereas a nonconformist one adopts the opposite. As for the opinion, the behavior is assigned stochastically too, according to the coefficients $P_c$ and $P_a = 1 - P_c$, i.e. the probability of behaving as a conformist or a nonconformist, respectively. As before, the initial expected number of conformist agents is $\langle N^0_c \rangle = P_c \cdot N$. The two processes of assigning opinions and behaviors are independent: so, each agent’s initial probability of being both conformist and having opinion $+1$ is $p^0_{c,+} \equiv P_c \cdot P_+ = P_+^2$. We will consider agents interacting on different configurations: while the probabilities $P_+$ and $P_-$ will remain fixed, $P_c$ and $P_a$ will vary, in order to achieve different densities of conformist (and nonconformist) agents in the population. Naturally, opinions vary as a result of the system dynamics.

The $q$-voter model extends the classic voter model, letting each agent adopt the opinion shared by a subset of neighbors of arbitrary dimension [14]. This model is described by two parameters: $q$ and $\epsilon$. The former represents the number of neighbors each agent has to consider to have its opinion defined, whereas the latter represents the probability of each agent changing its state anyway, even if not all the $q$ chosen neighbors agree. We implement the $q$-voter model setting $q = 4$ and $\epsilon = 0$ (see appendix A (ii) for different choices of $q$). Therefore, agents choose $q = 4$ neighbors at random: if they all share the same opinion, a conformist agent adopts it, whereas a nonconformist agent adopts the opposite one. Otherwise, the agent keeps its precedent opinion. In fact, setting $\epsilon = 0$ means setting to zero the probability of changing opinion stochastically, in the event the $q$ neighbors disagree.
It is worth emphasizing that the implemented updating rule has been chosen to be synchronous; this means that every agent updates its state simultaneously, on the basis of neighbors’ opinions at the previous time step. In fact, we believe asynchronous updating does not adequately capture the real dynamics of a social experiment. For instance, let us imagine many people forming groups to discuss politics: it is hard to imagine participants discussing and changing their opinion sequentially. People interact with their neighbors simultaneously, updating their opinion in real time, i.e. before being engaged in a new discussion with a different group. Another example is provided by voting scenarios, where people express their opinion at the same time. Moreover, even if asynchronous updating were applicable, it would cause the system dynamics to be dependent on the particular sequence of agents chosen.

3. Results

Numerical simulations of the proposed model have been carried out by choosing $N = 5000$ agents, embedded on different network topologies as scale-free networks, regular lattices, small world networks and completely random networks. While scale-free networks have been generated via the Barabasi–Albert model [25], the other kinds of networks have been generated via the Watts–Strogatz model [26]. The latter allows us to obtain different network configurations by varying the value of the rewiring probability $\beta$: regular lattices are achieved by setting $\beta = 0$, small-world networks by setting $0 < \beta < 1$ and completely random networks by setting $\beta = 1$. In this work, we have considered the following values: $\beta \in [0, 0.01, 0.1, 0.5, 1]$. Moreover, all the considered networks have an average degree equal to $\frac{\sum_{i=1}^{N} k_i}{N} = 8$ (i.e. agents have, on average, eight neighbors). Now, it is worth highlighting that, since these network models are stochastic, we decided to generate and to use a single network for each configuration and for each size. Each simulation has been performed with a different number of conformist agents $\rho \in [0, 0.1, 0.25, 0.5, 0.6, 0.65, 0.7, 0.75, 0.9, 1]$—notice that the expected value of $\rho$ coincides with the expected fraction of conformist agents, i.e. $\langle \rho \rangle = \langle N_c^d/N \rangle = P_c$ —and it has been run for $10^4$ time steps. For the vast majority of cases this temporal limit was long enough to reach a steady state, as only few network configurations required more time. However, in the latter scenarios (e.g. regular lattices) we performed longer simulations (see appendix A (i)). We first consider the evolution of the system magnetization over time, i.e. the absolute value of the difference between the number of agents in the two states [27], normalized to $N$:

$$M = \frac{|N_+ - N_-|}{N}. \quad (1)$$

The magnetization ranges between 0 and 1 ($0 \leq M \leq 1$), with $M = 0$ indicating the equipartition of the two opinions (i.e. the maximally disordered phase), and $M = 1$ indicating that consensus has been reached. Notice that both situations $N_+ = 0, N_- = N$ and, vice-versa, $N_- = 0, N_+ = N$ are compatible with consensus, i.e. magnetization is uninformative about the dominant opinion sign.
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Figure 1. Evolution of the magnetization for different values of $\rho$ and different configurations: (a) scale-free network; (b) regular lattice; (c) small-world network ($\beta = 0.1$); (d) small-world network ($\beta = 0.5$). Small pictorial representations are shown for each network.

Figure 1 illustrates the evolution of the magnetization upon varying the value of $\rho$ for different network configurations. Remarkably, the density of conformist agents (i.e. $\rho$) strongly affects the process of consensus reaching; in more detail, (1) values of $\rho \leq 0.5$ seem not to be sufficiently high to let the system escape the disordered phase where the two opinions coexist; (2) values of $0.5 < \rho < 1$ let the system escape the disordered phase but not reach consensus: a steady state is reached where one of the two opinions prevails on the other; (3) only the density value $\rho = 1$ allows the system to reach consensus.

Remarkably, this is valid for all the considered configurations; what changes is the number of time steps after which the steady state, or the consensus, is reached. In particular, the regular lattice (panel (b) of figure 1) is the configuration where the process is slowest (see appendix A (i) for further details). As the network is rewired more and more (panels (c) and (d) of figure 1), the process becomes faster. Interestingly, further rewiring of the network ($\beta > 0.5$) does not lead to any appreciable change. Qualitatively speaking, the scale-free configuration (panel (a) of figure 1) does not show significant differences with respect to the small-world network with $\beta = 0.5$; however, the latter reaches the steady state later, for all the values of $\rho$. It is perhaps surprising that the presence of hubs does not speed up the process. However, this apparent paradox could be explained by considering that we are implementing a q-voter model, with an update rule involving only four neighbors at a time: thus, the (potential) influence those hubs could have on large numbers of nodes is drastically
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Reduced. Notice also that, for any given configuration, rising \( \rho \) shortens the time for reaching the steady state. According to figure 1 the value \( \rho = 0.5 \) seems to play the role of a threshold, separating two phases of the system: the disordered one, characterized by \( M = 0 \), and the ordered one, with \( M \) gradually rising (as a function of \( \rho \)) until full consensus is reached. As we will show in a while, the behavior of the q-voter model on heterogeneous networks is far richer.

Figure 2 shows the value of the magnetization at the steady state (i.e. after \( 10^4 \) time steps), for two network configurations only (but the same holds true for all the others), as a function of \( \rho \). Let us focus on the scale-free configuration (left panel of figure 2). At first glance, two distinct phases are visible: the disordered one, characterized by \( M = 0 \) for all the values of \( \rho \leq 0.5 \), and the ordered one, characterized by a value \( M \neq 0 \) for \( \rho > 0.5 \). Thus, the magnetization seems to play the role of the order parameter of a continuous phase transition, while \( \rho \) plays the role of control parameter, which can be varied to change the system behavior smoothly. Actually, a closer inspection reveals three different opinion-formation regimes (indicated by different colors), with two distinct threshold values: \( \rho_1 \approx 0.45 \) separating the flat behavior (in black) from the slowly-rising linear one (in red) and \( \rho_2 \approx 0.59 \) separating the latter from the rapidly-rising linear one (in green). The insets (zooming in on the second transition) reveal that the same qualitative behavior can also be observed for networks with a lower number of agents; what changes is the trend followed by points in the third phase (linear for \( N \geq 2500 \) and quadratic for \( N < 2500 \)) with \( \rho_2 \) shifting towards lower values (\( \approx 0.56 \) for \( N = 500 \) agents). Let us now comment on our findings for the Watts–Strogatz configuration (right panel of figure 2). This time four phases are distinguishable, separated by three threshold values: \( \rho_1 \approx 0.55 \), \( \rho_2 \approx 0.65 \) and \( \rho_3 \approx 0.70 \). However, as the insets reveal, the system loses two of the phases as the number of agents is lowered, showing three linear regimes for \( N \geq 2500 \) and only one quadratic regime for \( N < 2500 \). We also emphasize that all the aforementioned critical thresholds

\[ \text{Figure 2. Phase-diagram plotting } M \text{ versus } \rho \text{ for two network configurations (left: scale-free; right: small-world with } \beta = 0.5) \text{: different phases are visible, separated by threshold values of } \rho. \text{ Insets show the same analysis for networks with (a) } N = 2500, \text{ (b) } N = 2000, \text{ (c) } N = 1000 \text{ and (d) } N = 500. \text{ Error bars represent the standard deviation over the simulations run. The average } R^2 \text{ of the fits is: scale-free—(main panel) 0.9, (a) 0.88, (b) 0.88, (c) 0.8, (d) 0.86; Watts–Strogatz—(main panel) 0.92, (a) 0.85, (b) 0.92, (c) 0.9, (d) 0.86.} \]
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$\rho_c$ represent average values computed over the simulations run for each kind of network (i.e. scale-free and small world), characterized by a standard deviation $\sigma_\rho \approx 0.02$. However, the analysis of $M$ is somehow limiting because the values of $M$ cannot be negative: this means that the situations where agents reach consensus by adopting the opinions $+1$ and $-1$ are indistinguishable. Thus, we need a quantity that is able to distinguish the sign of the system’s final state. To achieve this, we use the summation of states:

$$S = \frac{\sum s_i}{N} = \frac{N_+ - N_-}{N}$$

providing complementary information with respect to $M$. Plotting the summation $S$ versus the density of conformists $\rho$, it is possible to achieve further information on the system dynamics. As shown in figure 3, as the density of conformists rises the system chooses one of two states, characterized by the same absolute value of $S$, but with the opposite sign: remarkably, the two states revealed by crossing the thresholds are symmetrically distributed with respect to the horizontal axis. In other words, by raising the density $\rho$ the system is induced to choose one out of two possible states, a priori equally probable, thus breaking its symmetry.

Each point of the phase diagram is the result of an average over more simulations: the values obtained show very small numerical differences, amounting to a few percent in the vast majority of cases. When considering the summation of states, to not wash away the information provided by the sign of $S$, the symmetry-breaking diagram has been obtained by averaging the negative and the positive values separately, maintaining the bistable character of the system.

We have already noticed that the behavior of our system, distinguishing the disordered state with $M = 0$ from the ordered state with $M > 0$, can be interpreted, in more physical terms, as a phase transition with $\rho$ playing the role of control parameter and $M$ playing the role of order parameter. Such evidence can be better described upon
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Recalling the behavior of a system of spins as the temperature is varied. In other words, our population of agents behaves like a spin system [28], which is found to be in a paramagnetic phase for the values $\rho < \rho_c$. The latter plays the role of the critical temperature in the usual spin case. In order to describe the phase transition of our system, we move from the well known relation $\propto - \frac{M}{T - T_c}$. A similar relation can be shown to hold true for our agents.

Let us start by highlighting that the role of temperature $T$ is played, in our case, by the parameter $\rho = \rho - 1$, therefore the critical temperature $T_c$ can be mapped to $(1 - \rho_c)$. In order to find the value of the exponent $\gamma$ describing the ferromagnetic phase of our system, we have plotted the magnetization $M$ as a function of $(1 - \rho)/(1 - \rho_c)$. Figure 4 shows it for the scale-free configuration ($N = 5000$ nodes). The functional form is very well described by the analytical relation $M \propto \left(1 - \frac{(1 - \rho)}{(1 - \rho_c)}\right)^\gamma$ with $\gamma \approx 1.45$ and $\rho_c = 0.59$.

However, while for a classical spin system the analytical relation described above separates two distinct phases (the paramagnetic one, for $\frac{T}{T_c} > 1$, and the ferromagnetic one, for $\frac{T}{T_c} < 1$), in our case the ferromagnetic phase $M > 0$ can be further subdivided into different subphases which have been called ‘regimes’ by us. Nevertheless, the formalism used in physics to describe a phase transition does not allow one to take into account the possibility of observing subphases; for this reason, the value $\rho_c = 0.59$ has been chosen by us to simply distinguish the phase with $M = 0$ from the phase with $M > 0$, completely ignoring the details of the latter.

The same line of reasoning holds true for the small-world configuration.
4. Discussion and conclusions

The \( q \)-voter model shows a very rich behavior, even simply considering agents with two opinions and two characters only, as conformists and nonconformists. Notably, the density of conformist agents \( \rho \) strongly affects the consensus reaching process, defining threshold values separating different phases of opinion formation. For \( \rho = 0 \) the two original opinions equally coexist, i.e. 50\% of agents remain in the \(+1\) state and 50\% of agents remain in the \(-1\) state (with small fluctuations). Then, by progressively increasing \( \rho \), the system starts showing a non-zero magnetization, i.e. a larger number of agents start sharing the same opinion. Indeed, our control parameter \( \rho \) can be mapped to the control parameter \( T \) (i.e. the temperature) of a spin system; following the same line of reasoning we have been able to identify a critical exponent \( \gamma \) that characterizes the phase transition occurring in our agent population, as the density of conformists exceeds the threshold \( \rho_c \). More precisely, the system undergoes a sort of continuous phase transition, which can be further broken down into several regimes, suggesting different functional forms of \( M(\rho) \), separated by different values of \( \rho \).

Apart from the details of the process, the response of the \( q \)-voter model to changes in the conformist agent density is remarkably stable across different network topologies: Watts–Strogatz networks with \( \beta \geq 0.5 \) show similar phase diagrams and symmetry-breaking processes, which are in turn very similar to the ones observed for the scale-free configuration. The effect of considering heterogeneous topologies is mainly reflected in the speed of the process, which depends on the values of the parameter \( \beta \): in particular, the more random the network, the faster the process.

In conclusion, what emerges indicates that different regimes of ‘opinion growth’ are identifiable and are strongly affected by the density of conformists. Moreover, even if the percentage of conformists drives the society towards a ‘prevalent’ opinion (whose diffusion speed grows as more and more conformists are considered) in the case of agents with only two opinions (evenly distributed at \( t = 0 \)), the prevalent one cannot be predicted \textit{a priori}.

The results achieved open the way to further analyses, such as considering agents with more opinions, more and different social traits and other network configurations.

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Appendix A

This appendix is devoted to the further clarification of two important aspects of the \( q \)-voter model:

(i) how the ordered phase (i.e. consensus) is reached on ring lattices;

(ii) how the \( q \) value affects the results found in the main text.
Both issues are investigated by analyzing the magnetization evolution \( M(t) \) (averaged over more simulations), by considering a population of \( N = 5000 \) agents provided with an average degree \( \bar{k} = 8 \).

### A.1. Ordered phase on ring lattices

As shown in panel (b) of figure 1, if \( \rho = 1 \) the magnetization of the \( q \)-voter model implemented on a ring strongly increases after about 2000 time steps. However, the value of \( M = 0.25 \) is reached. In order to evaluate whether the population reaches full consensus or a different kind of steady state, we performed simulations up to \( 10^9 \) time steps. The result is shown in figure A1: a population composed of conformist agents only (i.e. \( \rho = 1 \)) is able to reach the ordered phase we looked for. It is worth highlighting that this network configuration requires the highest number of time steps to let the agents reach full consensus.

### A.2. Exploring different \( q \) values

We now explore the behavior of the \( q \)-voter model by choosing different \( q \) values. In particular, we analyze the following range \( q \in [2, 3, 6, 8] \). Notice that, if \( q = 1 \), the \( q \)-voter model reduces to the classical voter model, as each agent randomly selects one of its neighbors and then assumes the related opinion.

Figures A2 and A3 confirm that, qualitatively speaking, the behavior of the \( q \)-voter model is not affected by the particular value of \( q \), both for the scale-free and the Watts–Strogatz networks. Remarkably, the \( \rho \) threshold value above which the system escapes the disordered phase seems to stabilize around \( \rho = 0.5 \) for all the \( q \) values (fixing the number of agents at \( N = 5000 \)—see also the discussion in the main text).

As for the value \( q = 4 \) explored in the main text, scale-free networks reach consensus before the Watts–Strogatz ones: for higher values of \( q \) the difference can amount to one order of magnitude (see panels (d) of figures A2 and A3).
Figure A2. Evolution of the magnetization in scale-free networks for different values of $\rho$, for (a) $q = 2$; (b) $q = 3$; (c) $q = 6$; (d) $q = 8$. 

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Figure A3. Evolution of the magnetization in small-world networks ($\beta = 0.5$) for different values of $\rho$, for (a) $q = 2$; (b) $q = 3$; (c) $q = 6$; (d) $q = 8$. 

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Moreover, given a particular configuration, the time to reach both the steady state (for $\rho < 1$) and the ordered phase ($\rho = 1$), increases as $q$ increases. This is intuitive, considering that more time is required to find a higher number of neighbors sharing the same opinion.

Nevertheless, despite the details distinguishing the various simulations, the conclusions drawn for the case $q = 4$ can still be generalized to all the considered cases (i.e. to different values of $q$).

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