Statistics of spatial averages and optimal averaging in the presence of missing data

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Abstract

We consider statistics of spatial averages estimated by weighting observations over an arbitrary spatial domain using identical and independent measuring devices, and derive an account of bias and variance in the presence of missing observations. We test the model relative to simulations, and the approximations for bias and variance with missing data are shown to compare well even when the probability of missing data is large. Previous authors have examined optimal averaging strategies for minimizing bias, variance and mean squared error of the spatial average, and we extend the analysis to the case of missing observations. Minimizing variance mainly requires higher weights where local variance and covariance is small, whereas minimizing bias requires higher weights where the field is closer to the true spatial average. Missing data increases variance and contributes to bias, and reducing both effects involves emphasizing locations with mean value nearer to the spatial average. The framework is applied to study spatially averaged rainfall over India. We use our model to estimate standard error in all-India rainfall as the combined effect of measurement uncertainty and bias, when weights are chosen so as to yield minimum mean squared error.

1 Introduction

Spatial averages occur frequently in climate science, for example in global or regional temperature and precipitation (Vinnikov et al. [1990]), and these are estimated by averaging across what are often point observations. The weights used in spatial averaging do not have to be uniform, even if observations themselves are uniformly spaced, but can differ according to the goal of the inquiry (Vinnikov et al. [1990], Gandin [1993], Benedetti and Palma [1995]). In applications involving statistics, not only point estimates of the spatial average but also knowledge of additional quantities such as the bias or variance becomes important (Cressie [1991], Morrissey et al. [1999], Casella and Berger [2002]). An old problem in spatial statistics is optimal averaging (OA), where investigators have examined the choice of weights minimizing bias,
random error, or mean squared error (MSE) in estimates of the spatial average of a geophysical quantity (Vinnikov et al. 1990, Gandin 1993, Shen et al. 1994, Overton and Stehman 1993, Benedetti and Palma 1995, Shen et al. 1998, Morrissey et al. 1999, Vinnikov et al. 1999).

A common feature of this previous literature (for e.g. Vinnikov et al. 1990, Gandin 1993, Shen et al. 1994, Morrissey et al. 1999) is the assumption that the relevant observations would all be reported. However the general case admits situations where some observations are missing, so that the average can only be evaluated over that part of the domain for which measurements are available. Nevertheless such situations may require prior estimates of variance or MSE, and averaging strategies must consider the goal of averaging in light of the ensemble of possible realizations over missing data. With the possibility of missing data the spatial average can be estimated as the ratio between linearly weighted observations over the domain and the fraction of the domain represented by available observations. The present paper derives estimators for bias and variance of such a spatial average in the presence of missing data.

In contrast to the case where all measurements are reported, the possibility of missing data induces variability in the denominator as well as covariance between the numerator and denominator, both of which must additionally be estimated. Statistics of a ratio between two quantities do not yield exact expressions, and one must resort to approximate methods (Hartley and Ross 1954, Oehler 1992). We estimate statistics (squared bias, variance, and MSE) based on truncation of a Taylor expansion of the ratio (Oehler 1992, van der Vaart 1998) and examine the accuracy of resulting approximations.

Missing data introduces new features, and increases the bias and variance in the spatial average. The model developed here helps understand these effects. This is applied to examine statistics of rainfall averaged over India (Mooley and Parthasarathy 1984, Gadgil 2003). The model of variance is used to describe temporal variability of all-India rainfall. In addition we estimate optimal weights that would minimize MSE in estimates of all-India rainfall. In statistical applications, for example involving variability and change (Mooley and Parthasarathy 1984), involving estimates
of spatially averaged quantities it is necessary to know the standard error associated with the estimates (Nicholls [2014]). Here we provide a rough quantification of standard error associated with estimates of all-India rainfall. The standard error measures uncertainty associated with reports of the all-India average, and arises from the the total contribution to MSE from measurement uncertainty and squared bias.

Our essential problem is as follows. Consider point observations \( r_i \) of spatially-varying field \( v \) over a domain at locations indexed by \( i \) \((1 \leq i \leq n)\) and corresponding to a fixed interval of time. Individual observations are assigned fixed weights \( \beta_i \), and relatively unbiased estimation of the spatial average requires \( \sum_i \beta_i = 1 \). Some observations might not be reported when sought, and Boolean random variable \( s_i \) represents this status: if available then \( s_i = 1 \) otherwise \( s_i = 0 \). Availability of observation at a point does not impinge on availability at another, so the \( s_i \)'s are statistically independent. If measuring and reporting instruments are nearly identical in this aspect, then each \( s_i \) has known probability \( \alpha \) of equaling one when called for.

We examine statistics of spatial average

\[
    r = \frac{R}{S} = \frac{\sum_i \beta_i s_i r_i}{\sum_i \beta_i s_i} \tag{1}
\]

which merely estimates the true spatial average \( v \). Generally, due to inherent variability, both \( v \) and \( r \) are functions of time. Point observation \( r_i \) is assumed to be related to the corresponding true value \( v_i \) through additive noise.

Section 2 derives approximate estimators for squared bias, variance and MSE of the above estimator \( r \), assuming that \( \beta_i \)'s are known. Section 3 examines how to decide weights \( \beta_i \) for minimizing the chosen statistic, given knowledge of the statistics of observations as well as probability \( \alpha \) of individual observations being available. Section 4 illustrates for the case of all-India rainfall. We illustrate major effects, including that of missing data, and consider an application: estimation of the standard error in the spatial average of rainfall.
2 Statistics of a spatial average

2.1 Approximate statistics through "Delta-method"

The spatial average is is denoted as a function \( r = f (R, S) \) of two variables, and its Taylor series about \((E_R, E_S)\), with \( E \) denoting expectation, truncated to 2\(^{nd} \) degree is

\[
f (R, S) = f (E_R, E_S) + (R - E_R) \frac{\partial f}{\partial R} + (S - E_S) \frac{\partial f}{\partial S} + \frac{1}{2} (R - E_R)^2 \frac{\partial^2 f}{\partial R^2} + \frac{1}{2} (S - E_S)^2 \frac{\partial^2 f}{\partial S^2} + (R - E_R) (S - E_S) \frac{\partial^2 f}{\partial R \partial S} \quad (2)
\]

with partial derivatives evaluated at \((E_R, E_S)\) being \( \frac{\partial f}{\partial R} = \frac{E_R}{E_S}, \frac{\partial f}{\partial S} = -\frac{E_R}{(E_S)^2}, \frac{\partial^2 f}{\partial R^2} = 0, \frac{\partial^2 f}{\partial S^2} = \frac{2E_R}{(E_S)^3}, \frac{\partial^2 f}{\partial R \partial S} = -\frac{1}{(E_S)^2} \), and taking the expectation

\[
E f (R, S) = \frac{E_R}{E_S} + \frac{E_R}{(E_S)^3} E (S - E_S)^2 - \frac{1}{(E_S)^2} E (R - E_R) (S - E_S) \quad (3)
\]

the assumption being, of course, that \( f (R, S) \) is smooth in the neighborhood of \( f (E_R, E_S) \) so that locally it can be approximated by the first few terms. This approach, sometimes called the "delta method", approximates the expectation of a function by that of its Taylor series and converges if the function is sufficiently smooth and has finite moments (Oehlert [1992], van der Vaart [1998]).

The true spatial average is \( \nu \), and the MSE in estimating it is

\[
\text{MSE} = E (f (R, S) - \nu)^2 = E (f (R, S) - Ef (R, S))^2 + E (Ef (R, S) - \nu)^2 \quad (4)
\]

being described the sum of variance \( V_r = E (f (R, S) - Ef (R, S))^2 \) and squared bias \( B_r = E (Ef (R, S) - \nu)^2 \), whose derivation is standard and therefore omitted. Henceforth we shall refer to \( B_r \) as simply bias.

In case there is no possibility of missing observations, then \( S \) is fixed and \( E (S - E_S)^2, E (R - E_R) (S - E_S) \), etc., vanish. In that case \( S = \sum_i \beta_i = 1 \), and the estimator of the spatial average is simply
\[ R = \sum_i \beta_i r_i. \] Its variance simplifies to \( V_r = \mathbf{E}R^2 - (\mathbf{E}R)^2 \) and bias \( B_r = \mathbf{E} (\mathbf{E}R - \nu)^2 \). Now, using linearity of \( \mathbf{E} \), \( \mathbf{E}R^2 = \sum_i \beta_i^2 \mathbf{E}r_i^2 + 2 \sum_{i<j} \beta_i \beta_j \mathbf{E}r_i r_j \), and \( (\mathbf{E}R)^2 = \sum_i (\mathbf{E}r_i)^2 + 2 \sum_{i<j} \beta_i \beta_j \mathbf{E}r_i \mathbf{E}r_j \) so that

\[ V_r = \sum_i \beta_i^2 \sigma^2_{r_i} + 2 \sum_{i<j} \beta_i \beta_j \text{Cov} (r_i, r_j) \tag{5} \]

where \( \text{Cov} (r_i, r_j) \) is the covariance. This formula can be depicted as positive-definite quadratic form \( V_r = \beta^T S_r \beta \) where \( S_r \) is the covariance matrix of observations and \( \beta = \left\{ \beta_1 \ldots \beta_n \right\}^T \) is the vector of weights. Similarly the squared bias reduces to \( \mathbf{E} (\sum_i \beta_i \mathbf{E}r_i - \nu)^2 \) which, given that the weights sum to unity, is estimated by average across time-series \( \frac{1}{N} \sum_t (\sum_i \beta_i (\mathbf{E}r_i (t) - \nu (t)))^2 \). Defining vector \( d_1 (t) = \left\{ \mathbf{E}r_1 (t) - \nu (t) \mathbf{E}r_2 (t) - \nu (t) \ldots \mathbf{E}r_n (t) - \nu (t) \right\}^T \), we obtain \( B_r = \frac{1}{N} \sum_t \beta^T d_1 (t) d_1 (t)^T \beta \) or equivalently

\[ B_r = \beta^T D_1 \beta \tag{6} \]

where \( D_1 = \frac{1}{N} \sum_t d_1 (t) d_1 (t)^T \) is an \( n \times n \) matrix and \( N \) is the total number of periods indexed by \( t \). These results correspond to those derived by previous authors for a situation with no missing data (Vinnikov et al. [1990], Shen et al. [1998], Vinnikov et al. [1999], Shen et al. [2007]).

We now return to the general situation where individual observations are missing with probability \( 1 - \alpha \). For estimating variance, we approximate \( \mathbf{E}f (R, S) \approx \frac{\mathbf{E}R}{\mathbf{E}S} \) because the other terms are relatively small in case \( 1 - \alpha \ll 1 \) (Appendix 1), so that variance becomes \( \sigma^2_r = \mathbf{E} \left( f (R, S) - \frac{\mathbf{E}R}{\mathbf{E}S} \right)^2 \), and using the 1st order approximation of \( f (R, S) \) the variance is

\[ V_r = \mathbf{E} \left\{ (R - \mathbf{E}R) \frac{\partial f}{\partial R} + (S - \mathbf{E}S) \frac{\partial f}{\partial S} \right\}^2 \tag{7} \]

simplifying to

\[ \sigma^2_r = \frac{1}{(\mathbf{E}S)^2} \mathbf{E} (R - \mathbf{E}R)^2 + \frac{(\mathbf{E}R)^2}{(\mathbf{E}S)^4} \mathbf{E} (S - \mathbf{E}S)^2 - 2 \frac{\mathbf{E}R}{(\mathbf{E}S)^3} \mathbf{E} (R - \mathbf{E}R) (S - \mathbf{E}S) \tag{8} \]
or equivalently
\[
\sigma^2_r = \frac{\sigma^2_R}{\mu_S^2} + \frac{\mu_R^2}{\mu_S^2} \sigma^2_S - 2 \frac{\mu_R^2}{\mu_S^2} \sigma_{RS}^2
\] (9)
where \(\mu\) and \(\sigma^2\) denote means and standard deviations (or covariance) of the subscripted variables. Likewise
\[
E f(R, S) - \nu = \left\{ \frac{\mu_R}{\mu_S} - \bar{\nu} \right\} + \left\{ \frac{\mu_R^2}{\mu_S^2} \sigma^2_S - \frac{\sigma_{RS}^2}{\mu_S^2} \right\}
\] (10)
with bias being the expectation of the square of this quantity.

### 2.2 Evaluation of the statistics

For the general situation, where individual observations are missing with probability \(1 - \alpha\), the variance of \(R\)
\[
\sigma^2_R \equiv E (R - ER)^2 = ER^2 - (ER)^2
\] (11)
is derived in Appendix 2, with result
\[
\sigma^2_R = \alpha \sum_i \beta_i^2 \left\{ Er_i^2 - \alpha (Er_i)^2 \right\} + 2\alpha^2 \sum_{i<j} \beta_i \beta_j \text{Cov} (r_i, r_j)
\] (12)
and similarly the variance of \(S\) is
\[
\sigma^2_S = \alpha (1 - \alpha) \sum_i \beta_i^2
\] (13)
arising from uncertainty about whether observations are recorded. It is largest for \(\alpha = 0.5\), increasing with uncertainty about the availability of observations. Covariance between \(R\) and \(S\) is
\[
\sigma_{RS}^2 = \alpha (1 - \alpha) \sum_i \beta_i^2 Er_i
\] (14)
Random variable \(r_i\) describing observation at the \(i^{th}\) location is modeled in relation to true value
\[ r_i = v_i + \varepsilon_i \]  

(15)

where \( \varepsilon_i \) is additive noise in the measuring and reporting instrument. We assume noise has zero mean, i.e. \( E\varepsilon_i = 0 \), and that \( \varepsilon_i \) is independent of \( v_i \). Hence

\[ Er_i^2 = E\varepsilon_i^2 + 2E\varepsilon_i \varepsilon_i = E\varepsilon_i^2 + \sigma_\varepsilon^2 \]  

(16)

where \( \sigma_\varepsilon^2 \) is the variance of \( \varepsilon_i \), independent of \( i \) because measuring instruments are assumed identical in this aspect. The last step used \( E\varepsilon_i \varepsilon_i = E\varepsilon_i E\varepsilon_i \) (from independence) and \( E\varepsilon_i = 0 \). Therefore the expectation of \( r_i \) is

\[ Er_i = E\varepsilon_i \]  

(17)

and

\[ \text{Cov} (r_i, r_j) = \text{Cov} (v_i, v_j) \]  

(18)

using Eq. (17), independence between \( v_i \) and \( \varepsilon_j \), and assuming noise terms to be mutually independent (\( E\varepsilon_i \varepsilon_j = E\varepsilon_i E\varepsilon_j = 0 \)). Substituting Eqs. (16), (17), and (18) into Eq. (12) yields the variance of \( R \)

\[ \sigma_R^2 = \alpha \sum_i \beta_i^2 \{E\varepsilon_i^2 - \alpha (E\varepsilon_i)^2\} + 2\alpha^2 \sum_{i<j} \beta_i \beta_j \text{Cov} (v_i, v_j) + \alpha \sigma_\varepsilon^2 \sum_i \beta_i^2 \]  

(19)

The first term is related to variances of true values \( v_i \). We define random variable

\[ \varsigma_i^2 = E\varepsilon_i^2 - \alpha (E\varepsilon_i)^2 \]  

(20)

which equals the variance of \( v_i \) only if \( \alpha = 1 \), with measurements available with certainty. In general \( \varsigma_i^2 > \sigma_\varepsilon^2 \) because of uncertainty about whether measurements would be available when called for. The second term in Eq. (19) arises from spatial covariance, and the last from measurement uncertainty described by noise variance \( \sigma_\varepsilon^2 \). Similarly, covariance between \( R \) and \( S \) becomes, in
terms of the field $v_i$

$$\sigma_{RS}^2 = \alpha (1 - \alpha) \sum_i \beta_i^2 \mathbf{E} v_i$$  \hspace{1cm} (21)

depending on its expected values at the sampled locations. The expectation of $R$ is $\mu_R = \alpha \sum_i \beta_i \mathbf{E} v_i$ using linearity of $\mathbf{E}$, independence between $r_i$ and $s_i$, and Eq. (17). Similarly that of $S$ is $\mu_S = \alpha$. Substituting Eqs. (13) and (14) into (10), we have

$$\mathbf{E} f(R, S) - v = \sum_i \beta_i \mathbf{E} v_i - v + \left( \frac{1 - \alpha}{\alpha} \right) \left\{ \mu_v \sum_i \beta_i^2 - \sum_i \beta_i^2 \mathbf{E} v_i \right\}$$  \hspace{1cm} (22)

with $\mu_v \equiv \sum_i \beta_i \mathbf{E} v_i$. Taking the square of the expectation of this quantity yields the squared bias $B_r$. The first contribution $\sum_i \beta_i \mathbf{E} v_i - v$ arises from finite sampling of a continuously varying field, and the second from the possibility of missing observations in case $\alpha < 1$.

As for variance, upon substituting Eqs. (13), (14) and (19) into Eq. (9) yields

$$V_r = \frac{1}{\alpha} \sum_i \beta_i^2 s_i^2 + 2 \sum_{i<j} \beta_i \beta_j \mathbf{Cov} (v_i, v_j) + \frac{1}{\alpha} \sigma_i^2 \sum_i \beta_i^2 - 2 \left( \frac{1 - \alpha}{\alpha} \right) \mu_v \sum_i \beta_i^2 \mathbf{E} v_i + \left( \frac{1 - \alpha}{\alpha} \right)^2 \mu_v^2 \sum_i \beta_i^2$$  \hspace{1cm} (23)

The first three terms in Eq. (23) are due to variance in $R$. The fourth term owes to covariance between $R$ and $S$, and the last to variance in $S$. Generally these statistics involve higher powers of $\beta_i$, and only if $\alpha = 1$ do the bias and variance reduce to quadratic forms. We also recall the use of only a 1st order approximation to the spatial average $f(R, S)$ in Eq. (7) for computing the variance. Considering 2nd and higher order terms, would have led to higher moments of $R$ and $S$ appearing in our formula.
3 Optimal averaging

Having derived approximate formulas for bias and variance, let us consider how to determine the values of $\beta_i$'s that would minimize the chosen statistic. This has been called the optimal averaging (OA) problem [Vinnikov et al. 1990, Shen et al. 1994, 1998, Vinnikov et al. 1999]. The novelty in the present work is extension to the case where observations can be missing with probability $\alpha < 1$.

3.1 Minimum Bias

We examine separately the two contributions to bias. The first contribution, from effects of finite sampling in the limit $\alpha = 1$, is $B_{r,1} = \beta^T D_1 \beta$ following Eq. (6). The weights must satisfy constraints $u^T \beta = 1$ and $\beta \geq 0$, where $u = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T$, reflecting that weights $\beta_i$ sum to 1 and are non-negative. For minimizing bias, we introduce functional $g_{b,1}(\lambda, \beta) = \beta^T D_1 \beta + 2\lambda (1 - u^T \beta) + 2\rho^T \beta$ that must be stationary at the minimum. The solution must also meet complementary slackness condition $\rho_i \beta_i = 0$, to account for inequality constraint $\beta_i \geq 0$ (Boyd and Vandenberghe 2004), and differentiating $g_{b,1}(\lambda, \beta)$ yields

$$D_1 \beta = \lambda u - \rho$$

The next section solves these equations using quadratic programming.

Here we describe an important factor influencing which observations receive higher weight. Consider initial guess $x_0 = \frac{1}{n} u$ for weight-vector $\beta$, corresponding to arithmetic averaging of the observations. Writing bias as quadratic function $B_{r,1}(x) = x^T D_1 x$, its gradient is $\text{grad} B_{r,1}(x) = 2D_1 x$. If one were to revise the vector to $x_1 = e_i$, the $i^{th}$ coordinate vector with, for e.g., $e_1 = \begin{bmatrix} 1 & \ldots & 0 \end{bmatrix}^T$, the change is $\Delta x = e_i - \frac{1}{n} u$ and the derivative of $B_{r,1}(x)$ in the direction of the change is $B'_{r,1}(x_0; \Delta x) = \{\text{grad} B_{r,1}(x)\}^T \Delta x / \|\Delta x\|$. Substituting for the gradient, the
directional derivative becomes \( B'_{r,1} (x_0; \triangle x) = \frac{2}{\| \triangle x \|} \left( \frac{1}{n} u^T D_1 e_i - \frac{1}{n^2} u^T D_1 u \right) \). This is negative if the \( i \) th column average is smaller than the average across all elements of the matrix \( D_1 \). Measurement errors have zero mean so that, for a fixed time \( t \): \( \mathbf{E} r_i (t) = v_i (t) \), and elements of \( D_1 \) are \( D_{1,ij} = \frac{1}{N} \sum_i (v_i (t) - v (t)) (v_j (t) - v (t)) \). Hence bias minimization involves generally higher weight to locations where the expectation is closer to the true spatial average. However, for observations where this holds the second directional derivative is generally positive and there is a limit to how far in the direction \( \triangle x / \| \triangle x \| \) one can go and still obtain decreasing \( B_{r,1} \). 

Turning to the second contribution to bias due to missing observations, we consider the limit where the first contribution is small so that \( \mu_v \) in Eq. (22) approximates \( \mathbf{E} v \). Then this contribution becomes \( B_{r,2} = \left( \frac{1-\alpha}{\alpha} \right)^2 \mathbf{E} \{ \mathbf{E} v \sum_i \beta_i^2 - \sum_i \beta_i^2 \mathbf{E} v_i \} \). Defining diagonal \( n \times n \) matrix \( D_2 (t) = [D_{2,ij} (t)] \) with \( D_{2,ii} (t) = \mathbf{E} (v_i - v) \) this contribution to the bias becomes \( \left( \frac{1-\alpha}{\alpha} \right)^2 \{ \sum_i \beta_i^2 (\mathbf{E} v - \mathbf{E} v_i) \} \), which is written as \( \left( \frac{1-\alpha}{\alpha} \right)^2 \beta^T D_2 \beta^T D_2 \beta \) and minimizing this requires minimum

\[
\beta^T D_2 \beta^T D_2 \beta = 0 \tag{25}
\]

Introducing functional

\[
g_{b,2} (\lambda, \beta) = \beta^T D_2 \beta^T D_2 \beta + 4 \lambda (1 - u^T \beta) \tag{26}
\]

this must be stationary at the optimal solution. Differentiating yields cubic polynomials in the \( \beta_i \)s governed by

\[
D_2 \beta^T D_2 \beta = \lambda u \tag{27}
\]

1The operation \( u^T D_1 u \) calculates the sum of all elements of \( D_1 \) whereas \( u^T D_1 e_i \) calculates the sum of elements of its \( i \) th column.

2With Hessian of \( B_{r,1} (x) \) equal to \( 2D_1 \), the second directional derivative along \( \triangle x / \| \triangle x \| \) is \( B''_{r,1} (x_0; \triangle x) = \frac{2}{\| \triangle x \|^2} \triangle x^T D_1 \triangle x \), and substituting yields \( B''_{r,1} (x_0; \triangle x) = \frac{2}{\| \triangle x \|^2} \left\{ (e_i^T D_1 e_i - \frac{1}{n} u^T D_1 u) - \| \triangle x \| B'_{r,1} (x_0; \triangle x) \right\} \). It turns out that \( \frac{1}{n} u^T D_1 u \ll e_i^T D_1 e_i \), and since \( B'_{r,1} (x_0; \triangle x) < 0 \) we have generally \( B''_{r,1} (x_0; \triangle x) > 0 \) so that the directional derivative is decreasing in magnitude.

3Although one must also introduce a term \( \rho \beta \beta^T \) in the functional along with complementary slackness condition \( \rho_i \beta_i = 0 \) [Boyd and Vandenberghe, 2004], to account for inequality constraint \( \beta_i \geq 0 \), we avoid this because, as seen here, the explicit solution to the equality-constrained problem also meets inequality constraint on weights \( \beta_i \).

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using the symmetry of $D_2$. If $n = 3$

$$
\begin{align*}
\begin{cases}
D_{2,11}\beta^3 + D_{2,11}D_{2,22}\beta_1^2 + D_{2,11}D_{2,33}\beta_1\beta_3^2 + D_{2,11}D_{2,33}\beta_1\beta_3^2 \\
D_{2,11}D_{2,22}\beta_1\beta_2 + D_{2,22}^2\beta_2^2 + D_{2,22}D_{2,33}\beta_2\beta_3^2 + D_{2,22}D_{2,33}\beta_2\beta_3^2 \\
D_{2,11}D_{2,33}\beta_1\beta_3 + D_{2,22}D_{2,33}\beta_2\beta_3^2 + D_{2,33}^2\beta_3^3 + D_{2,33}^2\beta_3^3 \\
\end{cases}
\end{align*}
\right\} = \begin{cases}
\lambda \\
\lambda \\
\lambda 
\end{cases}
(28)
$$

and subtracting each row from the previous one

$$
\begin{align*}
(D_{2,11}\beta_1^2 + D_{2,22}\beta_2^2 + D_{2,33}\beta_3^2) (D_{2,11}\beta_1 - D_{2,22}\beta_2) = 0 \\
(D_{2,11}\beta_1^2 + D_{2,22}\beta_2^2 + D_{2,33}\beta_3^2) (D_{2,22}\beta_2 - D_{2,33}\beta_3) = 0
\end{align*}
(29)\hspace{1cm}(30)
$$

Consider the case $D_{2,11} < 0$ and $D_{2,22}, D_{2,33} > 0$, so that the first location has expectation larger than the long-term spatial average, whereas the others have smaller expected values. Then $D_{2,22}\beta_2 - D_{2,33}\beta_3 = 0$ or

$$
\frac{\beta_2}{\beta_3} = \frac{E\nu_3 - E\nu}{E\nu_2 - E\nu}
(31)
$$

Higher weight is given to locations with expectation closer to the long-term spatial average, although the precise relationship is different from minimizing $B_{r,1}$. Additionally $D_{2,11}\beta_1^2 + D_{2,22}\beta_2^2 + D_{2,33}\beta_3^2 = 0$ or

$$
\frac{\beta_1}{\beta_2} = \left\{ -\frac{D_{2,22}}{D_{2,11}} \left( 1 + \frac{D_{2,22}}{D_{2,33}} \right) \right\}^{1/2}
(32)
$$

In case $D_{2,11} = -2, D_{2,22} = 1$ and $D_{2,33} = 1$, then $\beta_1 = \beta_2 = \beta_3 = \frac{1}{3}$. If $D_{2,11} = -3, D_{2,22} = 1$ and $D_{2,33} = 3$, then $\beta_1 = \frac{1}{3}, \beta_2 = \frac{1}{2}$ and $\beta_3 = \frac{1}{6}$. Generally the minimizing of $B_{r,2}$ requires higher weights for locations where the value of the field is expected to be closer to the long-term spatial average. In the extreme case with one observation having $D_{2,ii} = 0$ we obtain simply $\beta_i = 1$. If some location has the same expectation as the spatial average then only it needs to be sampled in order to minimize the bias due to missing observations.
For general $n$

$$
\begin{pmatrix}
D_{2,11}\beta_1 \sum_{k=1}^{n} D_{2,kk}\beta_k^2 \\
\vdots \\
D_{2,nn}\beta_n \sum_{k=1}^{n} D_{2,kk}\beta_k^2 \\
\end{pmatrix}
= 
\begin{pmatrix}
\lambda \\
\vdots \\
\lambda \\
\end{pmatrix}
$$

and subtracting each row from the previous one

$$
\begin{pmatrix}
(D_{2,11}\beta_1 - D_{2,22}\beta_2) \sum_{k=1}^{n} D_{2,kk}\beta_k^2 \\
\vdots \\
(D_{2,n-1,n-1}\beta_{n-1} - D_{2,nn}\beta_n) \sum_{k=1}^{n} D_{2,kk}\beta_k^2 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
\vdots \\
0 \\
\end{pmatrix}
$$

which is solved by first considering the independent relations among the two groups of locations having $D_{2,ii} > 0$ and $D_{2,ii} < 0$, for a total of $n - 2$ equations, and then solving the remaining two equations $\sum_{k=1}^{n} D_{2,kk}\beta_k^2 = 0$ and $\sum_{k=1}^{n} \beta_k = 1$. The bias increases with $(1 - \alpha)^2$.

In summary, minimizing bias due to finite sampling as well as from missing observations involves larger weights to locations where the field lies closer to the true spatial average, although the precise models are different. This is hardly surprising, because bias from finite sampling depends on a non-diagonal matrix with elements $E \left( v_i(t) - v(t) \right) \left( v_j(t) - v(t) \right)$, whereas that from missing observations depends on a diagonal matrix having elements $E \left( v_i - v \right)$.

### 3.2 Minimum Variance

As discussed in the previous section, in case $\alpha < 1$ the variance of the spatial average is generally not quadratic in the weights. However, even in this case we find it instructive to imagine the limiting case of small $B_{r,1}$ so that $\sum_i \beta_i E v_i \approx E v$, which is a constant, so that the variance of the spatial average simplifies to quadratic form

$$
\sigma_r^2 = \beta^T C \beta
$$

(35)
with
\[
C = C_1 + C_2 + C_3 + C_4 + C_5
\]  
being the sum of \(n \times n\) symmetric matrices describing respective terms in Eq. (23). The optimal weights minimize functional
\[
g_v(\lambda, \beta) = \beta^T C \beta + 2\lambda (1 - u^T \beta) + 2\rho^T \beta
\]  
along with complementary slackness condition \(\rho_i \beta_i = 0\), to account for inequality constraint \(\beta_i \geq 0\) and for stationarity
\[
C \beta = \lambda u - \rho
\]  
Let us consider some special cases to develop intuition.

3.2.1 Observations are always available

In case \(\alpha = 1\) then \(C_4 = C_5 = 0\) and \(\varsigma^2_{\zeta_i} = \sigma^2_{\zeta_i}\) so that \(C_1 + C_2 = S_v\), the covariance matrix of field \(v\), and \(C_3 = \sigma^2_{\zeta_i}I\), where \(I\) is the identity matrix. Then \(C = S_v + \sigma^2_{\zeta_i}I\). This corresponds to the formula found by previous authors who assumed that observations can be counted on being available (Gandin [1993], Vinnikov et al. [1999]).

In the limit \(\sigma^2_{\zeta_i} \to 0\) if observations are precise, \(C = S_v\) and \(\beta = P_v(\lambda u - \rho)\), where \(P_v = S_v^{-1}\) is the inverse covariance, or precision, matrix of the field. In a field with zero spatial correlation so that \(S_v\) is a diagonal matrix containing terms \(\sigma^2_{\zeta_i}\) then the precision matrix is also diagonal and weight \(\beta_i\) is proportional to \(1/\sigma^2_{\zeta_i}\), being higher for locations where the variance is smaller.

The opposite extreme where measurement uncertainty is so large that \(C \approx \sigma^2_{\zeta_i}I\) leads to uniform weights \(\beta_i = 1/n\).

Generally the weights must take into account both the precision matrix of the field and measure-
ment variance. Writing $Q_v = \frac{1}{\sigma^2_v} S_v$, $C^{-1} = \frac{1}{\sigma^2_v} (I + Q_v)^{-1}$, which is equal to $\frac{1}{\sigma^2_v} \left\{ I - (I + Q_v^{-1})^{-1} \right\}$.

In case diagonal elements of $Q_v^{-1}$ are much larger than one, corresponding to diagonal elements of the precision matrix being much larger than $1/\sigma^2_v$, one can approximate $C^{-1} \approx \frac{1}{\sigma^2_v} (I - Q_v)$. This is a small perturbation to the case of large measurement uncertainty, and the weights are proportional to $\frac{1}{\sigma^2_v} (I - Q_v) u$.

The general case needs to be considered numerically, but its interpretation is quite simple. Notice that $S_v + \sigma^2_v I$ is $S_r$, the covariance matrix of observations $r_i$, because the observation error is assumed to be independent of true value $v_i$ and errors are independent of each other.

### 3.2.2 Field is spatially uncorrelated

Let us reconsider the case where the field is spatially uncorrelated, but where any measurement goes unrecorded with probability $1 - \alpha$. Then $C$ is a diagonal matrix with

$$c_{ii} = \frac{1}{\alpha} \varsigma^2_{v_i} + \frac{1}{\alpha} \sigma^2_{\varepsilon} - \frac{2}{\alpha} (1 - \alpha) v E v_i + \frac{1 - \alpha}{\alpha} v^2$$

and, using the expression for $\varsigma_{v_i}$ in Eq. 20

$$c_{ii} = \frac{1}{\alpha} \left( \sigma^2_{v_i} + \sigma^2_{\varepsilon} \right) + \frac{1 - \alpha}{\alpha} (u - E v_i)^2$$

and $\beta_i \propto 1/c_{ii}$. Higher weights are given to locations with lower variance and those with expectation closer to the spatial mean. The second factor becomes more important if the probability of missing observations is higher. In the general case this is modified to account for effects of spatial covariance through the precision matrix.

---

4We have used identity involving matrices $U$, $V$, $W$, $Z$ $(U + WVZ)^{-1} = U^{-1} - U^{-1} W (V^{-1} + ZU^{-1} W)^{-1} ZU^{-1}$, and set $W = Z = I$ (Zhang 1999).
3.2.3 General case

From the previous development the general case can be denoted as minimizing $\sigma_r^2 = \beta^T C \beta$ where

$$C = \frac{1}{\alpha} S_r + \frac{1 - \alpha}{\alpha} F_v$$

(41)

where, as noted earlier, $S_r$ is the covariance matrix of observations and $F_v$ is a diagonal matrix with $i^{th}$ diagonal entry $(E v_i - \bar{v})^2$. The second term is similar to the contribution of missing data to bias, except for the form of dependence on $E v_i - \bar{v}$. Minimizing contributions to bias and variance from missing observations both require emphasizing in some manner observations with expected value near the spatial average. The variance is inversely proportional to probability $\alpha$ of reporting individual observations, and in the limit $\alpha \to 0$ the variance becomes infinite.

3.3 Minimum Mean squared error (MSE) through Quadratic Programming

If $\alpha < 1$ neither bias nor variance is quadratic in weights $\beta_i$. In case the first contribution to bias from finite sampling is much larger, as it can be expected to be if $\alpha$ is closer to 1, we may approximate bias as a quadratic form in $\beta$. Second, if the contribution to variance from missing data is small compared to intrinsic variability, then variance too can be approximated as a quadratic form by assuming $\sum_i \beta_i E v_i \approx E v$. As a result the MSE becomes quadratic in the weights: $\beta^T (C + D) \beta$, and the following section chooses optimal weights using quadratic programming to minimize this quantity. Of course, owing to the simplifications made, this is only an approximate minimum. However, once minimizing solutions are found, bias and variance can be computed more accurately for the corresponding averaging scheme, from Eqs. (22) and (23).
4 Computational results

We apply these developments to gridded rain-gauge data products covering the Indian mainland, released by India Meteorological Department (IMD) (Rajeevan et al. [2006], Pai et al. [2014]) at scales of $1^\circ \times 1^\circ$ (Rajeevan et al. [2005, 2006]) and subsequently at higher resolution of $0.25^\circ \times 0.25^\circ$ using a much larger network of rain-gauge stations (Pai et al. [2014]). These datasets were prepared by interpolating data from individual rain gauges onto a regular grid, with weights inversely depending on squared distance to grid’s midpoints (Rajeevan et al. [2005]), following the scheme of Shepard [1968]. The higher resolution dataset is found comparable to the previous gridded rainfall datasets in many aspects but furthermore elicits more accurately the rainfall amounts in regions exhibiting larger spatial gradients (Pai et al. [2014]). Therefore in the present paper we use the $0.25^\circ \times 0.25^\circ$ dataset for the months of April-November to estimate area-averaged rainfall, treating it as true rainfall, and analyze time-series from the $1^\circ \times 1^\circ$ dataset as if they represented a sparser sample of 357 distinct daily observations, from which the spatial average is to be estimated.

Bias and variance are modeled in Section 2 by heavily truncating a Taylor series. The number of terms in the Taylor series of $f(R, S)$ describing a ratio is infinite, and the ”delta-method” used here would converge only if all the moments of $R$ and $S$ defined in Eq. (1), appearing as they do in progressively higher terms in the series, were finite. Figure 1 considers the quality of the resulting approximation of squared bias and variance by comparing with simulations. Graphs marked ”simulations” have been computed as follows: for each period we simulate each of the 357 values of $s_i$, i.e. availability of individual observations, as independent Bernoulli random variables with $s_i = 1$ occurring with probability $\alpha$. Once the $s_i$’s are known for a given year, Indian Summer Monsoon Rainfall (ISMR) is estimated from Eq. (1) for that year, where $r_i$ corresponds to average rainfall from June through September. Repeating this process, for each of the years 1901-2011, yields a single time-series for ISMR. We simulate 5000 such realizations of the time-series of ISMR.

\footnote{We happen to choose weights $\beta_i$ to minimize MSE, but an alternative choice could well have been made for this figure, given our goal here of validating models of bias and variance.}
reflecting uncertainty in which observations are reported, and compute the ensemble mean ISMR as the mean across realizations. The simulated bias is computed from the ensemble averaged time-series of ISMR, whereas the simulated variance is the average of the temporal variance of each realization.

We compare with the squared bias and variance estimated from models in Eqs. (22) and (23). Figure 1 shows that these models, based on truncating the respective series, perform rather well, diverging from simulations only for very small $\alpha$. Even for an extreme case of 90% probability of individual point observations not being reported, the models reach within 10% of simulated bias and variance for these datasets. In realistic applications, we can expect much smaller probabilities of missing data, with $\alpha$ being close to 1, so that the models of Eqs. (22)-(23) should perform adequately in case of variables for which the series converges similarly to that of rainfall.

Another important illustration from these plots is an important effect that missing data has in terms of increasing the bias and variance in the spatial average. The effects are substantial if availability of individual observations is small. However, in the more relevant case where $\alpha$ is closer to 1, the bias and variance are dominated by intrinsic features. In particular, bias in estimates of the spatial average is dominated by effects of finite sampling of a continuous field and variance is mainly from inherent variability in the process and, to a much lesser degree, effects of measurement noise. Even so, the effects of missing data can generally not be neglected.

Figure 2 and 3 plot optimal weights for minimizing squared bias, variance, and MSE. A basic difficulty in analyzing bias and MSE for continuous fields is that the true value averaged over time and space is generally unknown. Rain gauges provide continuous measurements at what are approximately points and satellites, while providing a larger field of view, offer only brief snapshots in time (Bell and Kundu [2003]). For want of a better alternative, we treat results from IMD’s 0.25° × 0.25° gridded rainfall dataset (Pai et al. [2014]) as approximating the true ISMR upon being area-averaged. We then consider values from the 1° × 1° dataset (Rajeevan et al. [2006]) as yielding individual point-observations that must be weighted. As for availability of observations,
we only consider cases nearer to the more realistic regime, in which \( \alpha \) is close to 1, and this furthermore permits us to use quadratic programming to approximate the optimal weights for minimizing variance and MSE (Section 3.3). This is possible because, with \( \alpha \) closer to 1, the contribution of missing observations to bias and variance is relatively small, as Figure 1 shows, and optimal weights can be approximated by neglecting terms higher than quadratic in the \( \beta_i \)s. However, in case of bias, the contribution of missing data cannot be simplified through a quadratic function in the weights, and we limit analysis to \( \alpha = 1 \).

A striking feature of OA schemes (Figures 2-3) is that only a small fraction of potential observations is needed for estimating the spatial average. As described in Section 3, minimizing bias generally requires giving higher weights to locations where rainfall has expectation closer to the spatial average (Figure 2b), whereas reducing variance requires higher weights to locations with small variance and covariance. If \( \alpha \) is smaller, more locations need to be included in the OA scheme. Bias and variance both contribute significantly to MSE, and hence its minimization includes features of reducing both bias and variance.

Figure 4a shows the time-series of ISMR obtained by area-averaging the 0.25° × 0.25° dataset, which we treat as the true values. Also shown are ensemble averages for the MSE-minimizing scheme in cases of \( \alpha = 1.0 \) and \( \alpha = 0.8 \). Ensembles in this case consider different realizations of \( s_i \) for each location and year and, of course, if \( \alpha = 1.0 \) then all realizations are the same. Lower availability \( \alpha \) leads to higher temporal variance, and the graph illustrates that this is manifested through overestimating, compared to \( \alpha = 1 \), in years of above-average ISMR and underestimating in other years.

Figure 4b-d show cumulative frequency distributions of the weights for the three different OA schemes. The first bin, involving smallest \( \beta_i \)’s, has been omitted because it comprises mainly zero weights describing locations not appearing in the OA scheme. Therefore the lowest ordinates in the curves of Figure 4b-d indicate approximately what fraction of the overall domain is not involved in the corresponding OA scheme, this generally being quite large. With smaller availability of
individual observations, more of the domain participates in the OA scheme, with weights becoming slightly more evenly distributed.

We compute standard error (SE) in the spatial average. The SE measures uncertainty in reports of the all-India average, and is computed as the square root of the contribution to MSE from measurement uncertainty and squared bias as $SE = \left\{ \beta^T (D_1 + \sigma^2 \epsilon I) \beta \right\}^{1/2}$. Contribution of measurement uncertainty is $\sigma^2 \beta^T \beta$, and we assume all observations being reported, so $\alpha = 1$, in which case bias reduces to $\beta^T D_1 \beta$. Weights $\beta$ are chosen so as to minimize MSE, and are shown in Figure 3b. Figure 5 graphs the SE associated with estimates of all-India rainfall for individual months between April and November. We consider two different cases of measurement uncertainty depicted by $\sigma_e$. The small difference between the two cases shows that propagation of this uncertainty is limited by the spatial averaging process even when only a small fraction of the domain is involved in the OA scheme, so that the standard error has contribution mainly from bias. The present analysis assumes $\alpha = 1$, so bias is the result of using point measurements to estimate a spatial average.

The OA scheme used to estimate the standard error minimizes MSE in the spatial average. Instead, we might have chosen to minimize bias directly, but that would yield a time-series with much larger variance and thereby higher MSE. MSE-minimizing schemes must generally be chosen over bias-minimizing schemes if we seek to compare differences between the estimate and the true value, because bias involves only the expected value of the estimate. Such schemes that minimize the MSE lead to irreducible standard error if there is intrinsic variability in the field. Figure 5 also plots standard deviation (Stdev) of the optimal average as well as its long-term mean. The SE, while being smaller than standard deviation, is substantial.

For ISMR, averaged from June-September, after accounting for reductions in $\sigma_e$ from monthly to the 4-monthly time-scale, the SE is approximately 0.29 mm/day, mean is 7 mm/day and standard deviation is 0.39 mm/day.

---

6For uniform weights $\beta = \frac{1}{n} u$ the SE would $\frac{1}{n} \left\{ u^T (D_1 + \sigma^2 \epsilon I) u \right\}^{1/2}$
Figure 1: Verification of models of bias $B_r$ and variance $V_r$ of a spatial average (Indian Summer Monsoon Rainfall, ISMR) in Eqs. (22)–(23) for different values of availability $\alpha$, by comparison with simulations. Generally, except for small $\alpha$, corresponding to high probability of missing data, these models based on truncation of Taylor series provide good approximations.

To examine robustness of these results to assumptions about true ISMR, we repeat the analysis with true spatial average $\upsilon$ being estimated by area-weighting the $1^\circ \times 1^\circ$ (Rajeevan et al. 2005a, 2006) dataset, while continuing to use the same dataset in the OA scheme (Figure 6). Bias minimization recovers weights generally increasing with the cosine of latitude, while variance minimization yields identical results as before (Figure 3a), since variance does not depend on the true spatial average. The main result is that even MSE minimization yields a similar averaging scheme (compare Figures 3b and 6c). In the presence of variance, the OA scheme that minimizes MSE does not appear sensitive to the choice of dataset that represents true values of ISMR. Figure 6d shows that the standard error exhibits similar variation across months as in Figure 5, and numerical values are comparable to the previous analysis. The ratio of SE / mean ISMR for the two analyses is 4.1% and 4.3% respectively.
Figure 2: Optimal weights for minimizing squared bias in estimate of ISMR and graphed as a function of $(E_v - E_{v_i})^2$. Large weights occur only at locations where the deviation from the spatial average is small.

Figure 3: Optimal weights for minimizing variance and mean squared error of ISMR, in case $\alpha = 1.0$ (upper panels) and $\alpha = 0.8$ (lower panels).
Figure 4: (a) Time-series of ISMR obtained by area-averaging IMD’s gridded rainfall dataset at 0.25° × 0.25° resolution, and results of optimal averaging (OA) (for minimum-MSE) the dataset at 1° × 1° resolution in case α = 1.0 and α = 0.8; (b)-(d) Cumulative distribution of optimal weights β_i for minimizing bias, variance and MSE respectively. The first bin (i.e. smallest β_i) is not plotted, because it mostly comprises weights that are zero. Therefore the lowest ordinate of the curves indicates the fraction of area that need not be included in OA. For e.g. bias minimization requires sampling about 20% of total area. Minimizing variance or MSE requires more of the area to be sampled if α is smaller and weights are then distributed more evenly across locations.
Figure 5: Estimated standard error (SE) of the OA scheme that minimizes MSE, for individual months ranging from April through November. The formula used for calculating standard error is $SE = \{\beta^T (D_1 + \sigma^2 I) \beta\}^{1/2}$, where weights $\beta$ are chosen to minimize MSE. Also shown are standard deviation and mean rainfall. Measurement error has little effect and the standard error comes mainly from bias. For ISMR (June - September) the standard error is approximately 0.29 mm/day, mean is 7 mm/day and standard deviation 0.39 mm/day.

5 Discussion

Spatial averages appear in derived climate variables such as global mean surface temperature and regional rainfall (Mooley and Parthasarathy [1984], Vinnikov et al. [1990]). We estimate bias, variance, and mean squared error of a ratio describing a spatial average, in the presence of missing observations. The numerator is linearly weighted point observations over some spatial domain and the denominator represents the fraction of unity represented by available observations. The "delta-method" (Oehlert [1992], van der Vaart [1998]) derives estimators by taking expectations of truncated Taylor series of the ratio. The resulting estimators are non-parametric, with no assumptions being made about the distribution of the underlying variable. However, imagine a sequence of approximations to the bias and variance of a ratio, involving successive terms in the respective Taylor series. These successive terms involve progressively higher moments of the numerator and denominator, so convergence requires the moments to be finite (Oehlert [1992]).
Figure 6: Analysis repeated after treating area-averaged rainfall from $1^\circ \times 1^\circ$ rain gauge data as true values. Shown are: (a)-(c) optimal weights for minimizing bias, variance, and MSE in estimates of ISMR, for $\alpha = 1$, and (d) estimated SE of the OA scheme that minimizes MSE, for individual months ranging from April through November. Bias minimization recovers weights that are approximately increasing in cosine of latitude. Optimal weights for minimizing variance are identical to results in Figure 3a (where $0.25^\circ \times 0.25^\circ$ dataset was treated as true values). Optimal weights for minimizing MSE are similar to the results in Figure 3b. Properties of standard error in Figure 5 are similar to the present analysis in Figure 6d. Here, for ISMR the SE is approximately $0.33$ mm/day, mean is $7.7$ mm/day and standard deviation $0.43$ mm/day.
The estimators for bias and variance of a spatial average were tested on gridded rain-gauge data over India (Rajeevan et al. [2006], Pai et al. [2014]). The models, based on truncating the respective series to 1\textsuperscript{st} order for variance and 2\textsuperscript{nd} order for bias, fare very well, diverging from simulations only for very small $\alpha$, the probability of individual observations being reported. Whether or not such simplified approximations would converge for a different variable depends on behavior of the successive moments for the temporal and spatial scales of averaging.

Missing observations increase bias and variance, with effects increasing with the probability that individual observations are missing. Expressions developed here describe prior estimates of bias and variance of the sample spatial average obtained from identical measuring devices. Such prior estimates are useful when it is not known which of the potential point observations would actually be available in any particular instance. Of course, the incorporation of knowledge of which observations are actually available in any given case would affect particular estimates of bias and variance.

We also examine optimal weights that minimize bias, variance, or MSE. Previous authors (Vinnikov et al. [1990], Gandin [1993], Shen et al. [1998], Vinnikov et al. [1999]) have considered this problem and the present work is an extension to include effects of missing observations. In the present analysis both the numerator and denominator of the ratio describing a spatial average are random variables. The present estimators reduce to the previous results (Gandin [1993], Zhang [1999]) for a special case: describing that either the probability of missing data is zero or all observations are found to have been recorded, both of which are statistically indistinguishable.

Analysis of optimal sampling subsumes the problem of choosing how to locate measurements in order to minimize bias, variance, or a combination of the two as well as the problem of how to use existing observations. If a system of measurements is in place, the choice of weights can help reveal how to use potential observations in a weighting scheme whose goal is to minimize either the bias, variance or MSE. Such has been the motivation of prior discussions of optimal weighting for climate data (Vinnikov et al. [1990], Gandin [1993], Shen et al. [1998], Vinnikov et al. [1999]).
In addition, optimal weights can also reveal how to situate measuring devices.

Generally, optimal weighting procedures require to sample only a small fraction of the total area, but this fraction increases in the presence of missing observations. Minimizing variance of the spatial average requires giving higher weights to locations having smaller variance and covariance. The possibility of missing records introduces an additional factor, arising from the squared difference between the expected value at the location and the spatial mean, whose importance increases in proportion to the probability of missing data. Therefore the possibility of missing data generally affects how weights should be chosen in order to minimize variance. Bias, both due to finite sampling and that due to missing data, can be minimized by emphasizing locations where the expectation is closer the the spatial average.

We computed standard error (SE) in estimates of all-India rainfall for individual months between April and November, and for monsoon rainfall between June - September (ISMR). For this analysis, we used a $0.25^\circ \times 0.25^\circ$ gridded rain-gauge product \cite{Pai_2014} to estimate area-averaged rainfall, treating it as true rainfall, and treated time-series from a $1^\circ \times 1^\circ$ gridded rain-gauge product \cite{Rajeevan_2005} as observations in the OA procedure. The SE characterizes uncertainty in reports of the all-India average, and includes contributions to the MSE from measurement uncertainty as well as the squared bias. Because individual measurements are modeled as having noise with zero mean, the effect of measurement uncertainty at the all-India scale is reduced, and the SE mainly comes from bias due to finite sampling of a continuous field. Our analysis assumed that area-averages from gridded rain-gauge data at $0.25^\circ \times 0.25^\circ$ resolution come closest to the true values, which is unknown in practice.

Repeating the analysis with true spatial average estimated by area-weighting the $1^\circ \times 1^\circ$ dataset \cite{Rajeevan_2005} yields similar results for minimum-MSE weights and the standard error, providing a measure of confidence in this result. Due to spatiotemporal variability, the optimal averaging scheme that minimizes MSE is not sensitive to assumptions about the true value of ISMR, and hence it appears that the SE can be estimated. Its value is about 4 % of mean ISMR,
and this uncertainty in estimation should be considered in statistical inference problems involving all-India rainfall.

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**Appendix 1: Sufficient conditions for validity of variance estimator**

Our approximation of the variance of a ratio in Eq. (7) assumed \( E_f (R, S) \approx \frac{E_R}{ES} \), which from Eq. (3) requires

\[
\frac{E (S - ES)^2 \frac{E_R}{(ES)^2} - E (R - ER) (S - ES) \frac{1}{(ES)^2}}{\frac{E_R}{ES}} \tag{42}
\]

to be negligible compared to unity. Equivalently \( \frac{\sigma_S^2}{\mu_S^2} - \frac{\sigma_{RS}^2}{\mu_R \mu_S} = \left( \frac{1 - \alpha}{\alpha} \right) \left\{ \sum_i \beta_i^2 - \frac{\sum \beta_i^2 \mu_{vi}}{\sum \beta_i \mu_{vi}} \right\} \) must be small, for which it is sufficient that \( 1 - \alpha \ll 1 \). However this is not necessary, since \( \sum_i \beta_i = 1 \), and therefore generally \( \sum_i \beta_i^2 \ll 1 \).

**Appendix 2: Variance and covariance of \( R \) and \( S \)**

The variance of \( R \) is

\[
\sigma_R^2 \equiv E (R - ER)^2 = ER^2 - (ER)^2 \tag{43}
\]
with first term expanding to

$$ER^2 = E \sum_i r_i^2 a_i^2 s_i^2 + 2E \sum_{i<j} r_i r_j a_i a_j s_i s_j$$

(44)

and, using linearity of $E$, independence between observed value $r_i$ and availability $s_i$, and independence between $s_i$ and $s_j$ for $i \neq j$

$$ER^2 = \alpha \sum_i a_i^2 Er_i^2 + 2\alpha^2 \sum_{i<j} a_i a_j Er_i r_j$$

(45)

using $Es_i^2 = \alpha$ and, if $i \neq j$, $Es_i s_j = Es_i Es_j = \alpha^2$. The second term in Eq. (43), using linearity of $E$, expands to

$$(ER)^2 = \sum_i (Er_i a_i s_i)^2 + 2 \sum_{i<j} Er_i a_i s_i Er_j a_j s_j$$

(46)

and, using independence between $r_i$ and $s_i$

$$(ER)^2 = \alpha^2 \sum_i a_i^2 (Er_i)^2 + 2\alpha^2 \sum_{i<j} a_i a_j Er_i Er_j$$

(47)

using $Es_i = \alpha$. Hence

$$\sigma_R^2 = \alpha \sum_i a_i^2 \{Er_i^2 - \alpha (Er_i)^2\} + 2\alpha^2 \sum_{i<j} a_i a_j Cov (r_i, r_j)$$

(48)

where $Cov (r_i, r_j) = Er_i r_j - Er_i Er_j$.

The variance of $S$ is

$$\sigma_S^2 \equiv E(S - ES)^2 = ES^2 - (ES)^2$$

(49)

whose first term simplifies to

$$ES^2 = \alpha \sum_i a_i^2 + 2\alpha^2 \sum_{i<j} a_i a_j$$

(50)

using linearity of $E$, independence of $s_i$ and $s_j$, and $Es_i = \alpha$. Similarly the second term in Eq.
(51) simplifies to

\[(ES)^2 = \alpha^2 \sum_i a_i^2 + 2\alpha^2 \sum_{i<j} a_i a_j\]

Hence the variance of \(S\) is

\[\sigma_S^2 = \alpha (1 - \alpha) \sum_i a_i^2\]  \hspace{1cm} (52)

As for covariance between \(R\) and \(S\)

\[\sigma_{RS}^2 \equiv \text{Cov}(R, S) = ERS - ERES\]  \hspace{1cm} (53)

whose first term, using linearity of \(E\), becomes

\[ERS = \sum_i E_{r_i} a_i^2 s_i^2 + 2 \sum_{i<j} E_{r_i} a_i a_j s_i s_j\]  \hspace{1cm} (54)

simplifying to

\[\alpha \sum_i a_i^2 E_{r_i} + 2\alpha^2 \sum_{i<j} a_i a_j E_{r_i}\]  \hspace{1cm} (55)

using independence of \(r_i\) and \(s_i\) and of \(s_i\) and \(s_j\). The second term in Eq. (53) becomes

\[ERES = \sum_i E_{r_i} a_i s_i E a_i s_i + 2 \sum_{i<j} E_{r_i} a_i s_j E a_j s_j\]  \hspace{1cm} (56)

simplifying to

\[\alpha^2 \sum_i a_i^2 E_{r_i} + 2\alpha^2 \sum_{i<j} a_i a_j E_{r_i}\]  \hspace{1cm} (57)

using independence of \(r_i\) and \(s_i\).

Therefore covariance between \(R\) and \(S\) is

\[\sigma_{RS}^2 = \alpha (1 - \alpha) \sum_i a_i^2 E_{r_i}\]  \hspace{1cm} (58)
References

T L Bell and P K Kundu. Comparing satellite rainfall estimates with rain gauge data: Optimal strategies suggested by a spectral model. *Journal of Geophysical Research*, 108:1–15, 2003. doi: 10.1029/2002JD002641.

Roberto Benedetti and Daniela Palma. Optimal sampling designs for dependent spatial units. *Environmetrics*, 6:101–114, 1995. doi: 10.1002/env.3170060202.

S Boyd and L Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

G C Casella and R L Berger. *Statistical Inference*. Wadsworth, 2002.

Noel A C Cressie. *Statistics for Spatial Data*. John Wiley, 1991.

S Gadgil. The Indian Monsoon and its Variability. *Annual Review of Earth and Planetary Sciences*, 31:429–467, 2003. doi: 10.1146/annurev.earth.31.100901.141251.

Lev S Gandin. Optimal averaging of meteorological fields. US Department of Commerce Office Note 397, July 1993.

H O Hartley and A Ross. Unbiased ratio estimators. *Nature*, 174:270–271, 1954. doi: 10.1038/174270a0.

D. A. Mooley and B. Parthasarathy. Fluctuations in All-India summer monsoon rainfall during 1871-1978. *Climatic Change*, 6:287–301, 1984. doi: 10.1007/BF00142477.

Mark L. Morrissey, Jose A. Maliekal, John Scott Greene, and Jianmin Wang. The uncertainty of simple spatial averages using rain gauge networks. *Water Resources Research*, 31:2011–2017, 1999. doi: 10.1029/95WR01232.

A Nicholls. Confidence limits, error bars and method comparison in molecular modeling. part 1: The calculation of confidence intervals. *Journal of Computer-Aided Molecular Design*, 28: 887–918, 2014. doi: 10.1007/s10822-014-9753-z.
Gary W Oehlert. A note on the delta method. *The American Statistician*, 46:27–29, 1992. doi: 10.2307/2684406.

W Scott Overton and Stephen V Stehman. Properties of designs for sampling continuous spatial resources from a triangular grid. *Communications in Statistics - Theory and Methods*, 22:2641–2640, 1993. doi: 10.1080/03610928308831175.

D S Pai, L Sridhar, M Rajeevan, O P Sreejith, N S Satbhai, and B Mukhopadhyay. Development of a new high spatial resolution (0.25 deg x 0.25 deg) long period (1901-2010) daily gridded rainfall dataset over India and its comparison with existing data sets over the region. *Mausam*, 65:1–18, 2014.

M Rajeevan, J Bhave, J D Kale, and B Lal. Development of a high resolution daily gridded rainfall data for the Indian region. Met. Monograph Climatology 22/2005, National Climate Centre, India Meteorological Department, 2005.

M. Rajeevan, Jyoti Bhave, J. D. Kale, and B. Lal. High resolution daily gridded rainfall data for the Indian region: Analysis of break and active monsoon spells. *Current Science*, 91:296–306, 2006. doi: http://www.jstor.org/stable/24094135.

S S P Shen, H Y Yin, and T M Smith. An estimate of the sampling error variance of the gridded ghcn monthly surface air temperature data. *Journal of Climate*, 20:2321–2231, 2007. doi: doi.org/10.1175/JCLI4121.1.

Samuel S. Shen, Thomas M. Smith, Chester F. Ropelewski, and Robert E. Livezey. An Optimal Regional Averaging Method with Error Estimates and a Test Using Tropical Pacific SST Data. *Journal of Climate*, 11:2340–2350, 1998. doi: 10.1175/1520-0442(1998)011⟨2340:AORAMW⟩2.0.CO;2.

Samuel S. P. Shen, Gerald R. North, and Kwang-Y. Kim. Spectral approach to optimal estimation of the global average temperature. *Journal of Climate*, 7:1999–2007, 1994. doi: 10.1175/1520-0442(1994)007⟨1999:SATOEO⟩2.0.CO;2.
D Shepard. A two-dimensional interpolation function for irregularly-spaced data. In *ACM ’68: Proceedings of the 1968 23rd ACM national conference*, volume 23, pages 517–524. ACM, 1968.

A W van der Vaart. *Asymptotic Statistics*. Cambridge University Press, 1998.

K. Ya Vinnikov, P. Ya Groisman, and K. M. Lugina. Empirical Data on Contemporary Global Climate Changes (Temperature and Precipitation). *Journal of Climate*, 3:662–677, 1990. doi: 10.1175/1520-0442(1990)003⟨0662:EDOCGC⟩2.0.CO;2.

Konstantin Y. Vinnikov, Alan Robock, Shuang Qiu, and Jared K. Entin. Optimal design of surface networks for observation of soil moisture. *Journal of Geophysical Research*, 104:19743–19749, 1999. doi: 10.1029/1999JD900060.

Fuzhen Zhang. *Matrix Theory: Basic Results and Techniques*. Springer, 1999.