INCIDENCE CATEGORIES

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ABSTRACT. Given a family $\mathcal{F}$ of posets closed under disjoint unions and the operation of taking convex subposets, we construct a category $\mathcal{C}_\mathcal{F}$ called the incidence category of $\mathcal{F}$. This category is “nearly abelian” in the sense that all morphisms have kernels/cokernels, and possesses a symmetric monoidal structure akin to direct sum. The Ringel-Hall algebra of $\mathcal{C}_\mathcal{F}$ is isomorphic to the incidence Hopf algebra of the collection $\mathcal{P}(\mathcal{F})$ of order ideals of posets in $\mathcal{F}$. This construction generalizes the categories introduced by K. Kremnizer and the author in the case when $\mathcal{F}$ is the collection of posets coming from rooted forests or Feynman graphs.

1. INTRODUCTION

The notion of the incidence algebra of an interval-closed family $\mathcal{P}$ of posets was introduced by G.-C. Rota in [9]. The work of W. Schmitt demonstrated that incidence algebras frequently possess important additional structure - namely that of a Hopf algebra. The seminal papers [11, 12] established various key structural and combinatorial properties of incidence Hopf algebras.

In this paper, we show that a certain class of incidence Hopf algebras can be “categorified”. Given a family $\mathcal{F}$ of posets which are closed under disjoint unions and the operation of taking convex subposets, we construct a category $\mathcal{C}_\mathcal{F}$, whose objects are in one-to-one correspondence with the posets in $\mathcal{F}$. While $\mathcal{C}_\mathcal{F}$ is not abelian (morphisms only form a set), it is “nearly” so, in the sense that all morphisms possess kernels and cokernels, it has a null object, and symmetric monoidal structure akin to direct sum. We can therefore talk about exact sequences in $\mathcal{C}_\mathcal{F}$, its Grothendieck group, and Yoneda Ext’s.

In particular, we can define the Ringel-Hall algebra $\mathbb{H}_{\mathcal{C}_\mathcal{F}}$ of $\mathcal{C}_\mathcal{F}$. $\mathbb{H}_{\mathcal{C}_\mathcal{F}}$ is the $\mathbb{Q}$–vector space of finitely supported functions on isomorphism classes of $\mathcal{C}_\mathcal{F}$:

$$\mathbb{H}_{\mathcal{C}_\mathcal{F}} := \{ f : \text{Iso}(\mathcal{C}_\mathcal{F}) \to \mathbb{Q} ||supp(f)|| < \infty \}$$

with product given by convolution:

$$f \star g(M) = \sum_{A \subset M} f(A)g(M/A).$$

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$H_{C_F}$ possesses a co-commutative co-product given by
\begin{equation}
\Delta(f)(M, N) = f(M \oplus N)
\end{equation}

We show that $H_{C_F}$ is isomorphic to the incidence Hopf algebra of the family $\mathcal{P}(\mathcal{F})$ of order ideals of posets in $\mathcal{F}$. This is the sense in which $C_F$ is a categorification. The resulting Hopf algebra $H_{C_F}$ is graded connected and co-commutative, and so by the Milnor-Moore theorem isomorphic to $U(n_{\mathcal{F}})$, where $n_{\mathcal{F}}$ is the Lie algebra of its primitive elements.

When $\mathcal{F}$ is the family of posets coming from rooted forests or Feynman graphs, $C_F$ coincides with the categories introduced by K. Kremnizer and the author in [8].

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2. RECOLLECTIONS ON POSETS

We begin by recalling some basic notions and terminology pertaining to posets (partially ordered sets) following [12][13].

(1) An interval is a poset having unique minimal and maximal elements. For $x, y$ in a poset $P$, we denote by $[x, y]$ the interval

$$[x, y] := \{ z \in P : x \leq z \leq y \}$$

If $P$ is an interval, we will often denote by $0_P$ and $1_P$ the minimal and maximal elements.

(2) An order ideal in a poset $P$ is a subset $L \subset P$ such that whenever $y \in L$ and $x \leq y$ in $P$, then $x \in L$.

(3) A subposet $Q$ of $P$ is convex if, whenever $x \leq y$ in $Q$ and $z \in P$ satisfies $x \leq z \leq y$, then $z \in Q$. Equivalently, $Q$ is convex if $Q = L\setminus I$ for order ideals $I \subset L$ in $P$.

(4) Given two posets $P_1, P_2$, their disjoint union is naturally a poset, which we denote by $P_1 + P_2$. In $P_1 + P_2$, $x \leq y$ if both lie in either $P_1$ or $P_2$, and $x \leq y$ there.

(5) A poset which is not the union of two non-empty posets is said to be connected.

(6) The cartesian product $P_1 \times P_2$ is a poset where $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$.

(7) A distributive lattice is a poset $P$ equipped with two operations $\wedge, \vee$ that satisfy the following properties:

(a) $\wedge, \vee$ are commutative and associative
Suppose that \( (x, y) \) and \( (x, y) \) are in some additional structure (such as for instance a coloring). It follows from \( F \) as follows. Let \( P \) be a family of posets which is closed under the formation of disjoint unions and the operation of taking convex subposets, and let \( P \) be the corresponding family of distributive lattices of order ideals. For each pair \( (1, 2) \) of elements in \( P \), let \( M(1, 2) \) denote a set of maps \( 1 \rightarrow 2 \) such that the collections \( M(1, 2) \) satisfy the following properties:

1. For each \( f, g \in M(1, 2) \), \( f : 1 \rightarrow 2 \) is a poset isomorphism
2. \( M(1, 2) \) contains the identity map
3. If \( f \in M(1, 2) \) then \( f^{-1} \in M(2, 1) \).
4. If \( f, g \in M(1, 2) \) and \( g \in M(2, 3) \), then \( g \circ f \in M(1, 3) \).
5. If \( f \in M(1, 2) \) and \( g \in M(1, 2) \), then \( f \cup g \in M(1, 2) \) and \( 1 \cup 2 \in M(1, 2) \), where \( f \cup g \) denotes the map induced on the disjoint union.

Typically, the collection \( M(1, 2) \) will consist of poset isomorphisms respecting some additional structure (such as for instance a coloring). It follows from the above properties that \( M(1, 2) \) forms a group, which we denote \( \text{Aut}_M(P) \).

3. From posets to categories

Let \( \mathcal{F} \) be a family of posets which is closed under the formation of disjoint unions and the operation of taking convex subposets, and let \( \mathcal{P}(\mathcal{F}) = \{ I \in \mathcal{P} \} \) be the corresponding family of distributive lattices of order ideals. For each pair \( P_1, P_2 \in \mathcal{F} \), let \( M(P_1, P_2) \) denote a set of maps \( P_1 \rightarrow P_2 \) such that the collections \( M(P_1, P_2) \) satisfy the following properties:

1. For each \( f \in M(P_1, P_2), f : P_1 \rightarrow P_2 \) is a poset isomorphism
2. \( M(P_1, P_2) \) contains the identity map
3. If \( f \in M(P_1, P_2) \) then \( f^{-1} \in M(P_2, P_1) \).
4. If \( f \in M(P_1, P_2) \) and \( g \in M(P_2, P_3) \), then \( g \circ f \in M(P_1, P_3) \).
5. If \( f \in M(P_1, P_2) \) and \( g \in M(P_1, P_2) \), then \( f \cup g \in M(P_1, P_2) \) and \( 1 \cup 2 \in M(P_1, P_2) \), where \( f \cup g \) denotes the map induced on the disjoint union.

Typically, the collection \( M(P_1, P_2) \) will consist of poset isomorphisms respecting some additional structure (such as for instance a coloring). It follows from the above properties that \( M(P_1, P_2) \) forms a group, which we denote \( \text{Aut}_M(P) \).

3.1. The category \( \mathcal{C}_\mathcal{F} \). We proceed to define a category \( \mathcal{C}_\mathcal{F} \), called the incidence category of \( \mathcal{F} \) as follows. Let

\[
\text{Ob}(\mathcal{C}_\mathcal{F}) := \mathcal{F} = \{ X_P : P \in \mathcal{F} \}
\]

and

\[
\text{Hom}(X_P, X_Q) := \{ (I_1, I_2, f) : I_i \in P, f \in M(P_i \setminus I_i, I_2) \} \quad i = 1, 2
\]

We need to define the composition of morphisms

\[
\text{Hom}(X_P, X_Q) \times \text{Hom}(X_Q, X_R) \rightarrow \text{Hom}(X_P, X_R)
\]

Suppose that \( (I_1, I_2, f) \in \text{Hom}(X_P, X_Q) \) and \( (I_1', I_2', g) \in \text{Hom}(X_Q, X_R) \). Their composition is the morphism \( (K_1, K_3, h) \) defined as follows.

- We have \( I_2 \setminus I'_2 \subset I_2 \), and since \( f : P_1 \setminus I_1 \rightarrow I_2 \) is an isomorphism, \( f^{-1}(I_2 \setminus I'_2) \) is an order ideal of \( P_1 \setminus I_1 \). Since in \( J_{P_1} \), \( [I_1, P] \simeq J_{P_1 \setminus I_1} \), we
have that $f^{-1}(I_2 \land I_2')$ corresponds to an order ideal $K_1 \subset I_{P_1}$ such that $I_1 \subset K_1$.

- We have $I_2' \subset I_2 \lor I_2'$, and since $[I_2', P_2] \simeq I_{P_2 \setminus I_2'}$, $I_2 \lor I_2'$ corresponds to an order ideal $L_2 \subset I_{P_2 \setminus I_2'}$. Since $g : P_2 \setminus I_2' \to I_3'$ is an isomorphism, $g(L_2) \subset I_{I_3'}$, and since $I_{I_3'} \subset I_{P_3}$, $g(L_2)$ corresponds to an order ideal $K_3 \subset I_{P_3}$ contained in $I_3'$.

- The isomorphism $f : P_1 \setminus I_1 \to I_2$ restricts to an isomorphism $f : P_1 \setminus I_1 \to I_2 \setminus I_2 \land I_2' = I_2 \setminus I_2'$, and the isomorphism $g : P_2 \setminus I_2'$ restricts to an isomorphism $g : I_2 \lor I_2' \setminus I_2' = I_2 \setminus I_2' \to K_3$. Thus, $g \circ f : P_1 \setminus I_1 \to K_3$ is an isomorphism and $g \circ f \in M(P_1 \setminus I_1, K_3)$ by the property (4) above.

**Lemma 1.** Composition of morphisms is associative.

**Proof.** Suppose that $P_1, P_2, P_3, P_4 \in \mathcal{F}$, and that we have three morphisms as follows:

$$X_{P_1} \xrightarrow{(A_1, A_2, f)} X_{P_2} \xrightarrow{(B_2, B_3, g)} X_{P_3} \xrightarrow{(C_3, C_4, h)} X_{P_4}$$

Given a poset $P$ and subsets $S_1, \ldots, S_k$ of $P$, denote by $[S_1, \ldots, S_k]$ the smallest order ideal containing $\cup_{i=1}^k S_i$. We have:

$$(C_3, C_4, h) \circ ((B_2, B_3, g) \circ (A_1, A_2, f))$$

$$= (C_3, C_4, h) \circ ([A_1, f^{-1}(B_2)], g(A_2 \setminus B_2), g \circ f)$$

$$= ([A_1, f^{-1}(B_2)], (g \circ f)^{-1}(C_3)], h(g(A_2 \setminus B_2) \setminus C_3), h \circ g \circ f)$$

whereas

$$((C_3, C_4, h) \circ (B_2, B_3, g)) \circ (A_1, A_2, f)$$

$$= ([B_2, g^{-1}(C_3)], h(B_3 \setminus C_3), h \circ g) \circ (A_1, A_2, f)$$

$$= ([A_1, f^{-1}([B_2, g^{-1}(C_3)]), h \circ g(A_2 \setminus [B_2, g^{-1}(C_3)]), h \circ g \circ f)$$

We have

$$f^{-1}([B_2, g^{-1}(C_3)]) = f^{-1}(B_2 \cup g^{-1}(C_3)) = f^{-1}(B_2) \cup (g \circ f)^{-1}(C_3)$$

which implies that

$$[A_1, f^{-1}(B_2), (g \circ f)^{-1}(C_3)] = [A_1, f^{-1}([B_2, g^{-1}(C_3)])]$$

and

$$h(g(A_2 \setminus B_2) \setminus C_3) = h \circ g(A_2 \setminus (B_2 \cup g^{-1}(C_3))) = h \circ g(A_2 \setminus [B_2, g^{-1}(C_3)])$$

This proves the two compositions are equal. \hfill \Box

Finally,

- We refer to $X_{I_2}$ as the image of the morphism $(I_1, I_2, f) : X_{P_1} \to X_{P_2}$. 

We denote by $\text{Iso}(\mathcal{C}_F)$ the collection of isomorphism classes of objects in $\mathcal{C}_F$, and by $[X_P]$ the isomorphism class of $X_P \in \mathcal{C}_F$.

4. PROPERTIES OF THE CATEGORIES $\mathcal{C}_F$

We now enumerate some of the properties of the categories $\mathcal{C}_F$.

(1) The empty poset $\emptyset$ is an initial, terminal, and therefore null object. We will sometimes denote it by $X_\emptyset$.

(2) We can equip $\mathcal{C}_F$ with a symmetric monoidal structure by defining

$$X_{P_1} \oplus X_{P_2} := X_{P_1 + P_2}.$$

(3) The indecomposable objects of $\mathcal{C}_F$ are the $X_P$ with $P$ a connected poset in $\mathcal{F}$.

(4) The irreducible objects of $\mathcal{C}_F$ are the $X_P$ where $P$ is a one-element poset.

(5) Every morphism

$$(I_1, I_2, f) : X_{P_1} \to X_{P_2}$$

has a kernel

$$(\emptyset, I_1, id) : X_{I_1} \to X_{P_1}$$

(6) Similarly, every morphism $\mathcal{C}_F$ possesses a cokernel

$$(I_2, P_2 \setminus I_2, id) : X_{P_2} \to X_{P_2 \setminus I_2}$$

We will frequently use the notation $X_{P_2} / X_{P_1}$ for $\text{coker}((I_1, I_2, f))$.

Note: Properties 5 and 6 imply that the notion of exact sequence makes sense in $\mathcal{C}_F$.

(7) All monomorphisms are of the form

$$(\emptyset, I, f) : X_Q \to X_P$$

where $I \in J_P$, and $f : Q \to I \in M(Q, I)$. Monomorphisms $X_Q \to X_P$ with a fixed image $X_I$ form a torsor over $\text{Aut}_M(I)$. All epimorphisms are of the form

$$(I, \emptyset, g) : X_P \to X_Q$$

where $I \in J_P$ and $g : P \setminus I \to Q \in M(P \setminus I, Q)$. Epimorphisms with fixed kernel $X_I$ form a torsor over $\text{Aut}_M(P \setminus I)$.  

Sequences of the form

$$X_\emptyset \rightarrow X_I \rightarrow X_P \rightarrow X_{P \setminus I} \rightarrow X_\emptyset$$

with $I \in J_P$ are short exact, and all other short exact sequences with $X_P$ in the middle arise by composing with isomorphisms $X_I \rightarrow X_I'$ and $X_P \setminus I \rightarrow X_Q$ on the left and right.

Given an object $X_P$ and a subobject $X_I, I \in J_P$, the isomorphism $J_P \setminus I \simeq [I, P]$ translates into the statement that there is a bijection between sub-objects of $X_P/X_I$ and order ideals $I \in J_P$ such that $I \subset J \subset P$ via $X_I \leftrightarrow J$. The bijection is compatible with quotients, in the sense that $(X_P/X_I)/(X_J/X_I) \simeq X_J/X_I$.

Since the posets in $\mathcal{F}$ are finite, $\text{Hom}(X_{P_1}, X_{P_2})$ is a finite set.

We may define Yoneda Ext$^n(X_{P_1}, X_{P_2})$ as the equivalence class of $n$-step exact sequences with $X_{P_1}, X_{P_2}$ on the right and left respectively. Ext$^n(X_{P_1}, X_{P_2})$ is a finite set. Concatenation of exact sequences makes

$$\mathbb{Ext}^* := \bigcup_{A,B \in I(\mathcal{C}_F), n} \text{Ext}^n(A, B)$$

into a monoid.

We may define the Grothendieck group of $\mathcal{C}_F$, $K_0(\mathcal{C}_F)$, as

$$K(\mathcal{C}_F) = \bigoplus_{A \in \mathcal{C}_F} \mathbb{Z}[A]/\sim$$

where $\sim$ is generated by $A + B - C$ for short exact sequences

$$X_\emptyset \rightarrow A \rightarrow C \rightarrow B \rightarrow X_\emptyset$$

We denote by $k(A)$ the class of an object in $K_0(\mathcal{C}_F)$.

5. Ringel-Hall algebras and incidence algebras

5.1. Incidence Hopf algebras. We begin by recalling the definition of the incidence Hopf algebra of a hereditary interval-closed family of posets introduced in [12]. Incidence algebras of posets were originally introduced in [9].

A family $\mathcal{P}$ of finite intervals is said to be interval closed, if it is non-empty, and for all $P \in \mathcal{P}$ and $x \leq y \in P$, the interval $[x, y]$ belongs to $\mathcal{P}$. An order compatible relation on an interval closed family $\mathcal{P}$ is an equivalence relation $\sim$ such that whenever $P \sim Q$ in $\mathcal{P}$, there exists a bijection $\phi : P \rightarrow Q$ such that $[0_P, x] \sim [0_Q, \phi(x)]$ and $[x, 1_P] \sim [\phi(x), 1_Q]$ for all $x \in P$. Typical examples of order compatible relations are poset isomorphism, or isomorphism preserving...
some additional structure (such as a coloring). We denote by $\mathcal{P}$ the set equivalence classes of $\mathcal{P}$ under $\sim$, i.e. $\mathcal{P} = \mathcal{P} / \sim$, and by $[P]$ the equivalence class of a poset $P$ in $\mathcal{P}$. The incidence algebra of the family $(\mathcal{P}, \sim)$, denoted $H_{\mathcal{P}, \sim}$, is
\[
H_{\mathcal{P}, \sim} := \{ f : \mathcal{P} \to \mathbb{Q} : |\text{supp}(f)| < \infty \}
\]
(note that the finiteness of the support of $f$ is not standard). $H_{\mathcal{P}, \sim}$ is naturally a $\mathbb{Q}$–vector space, and becomes an associative $\mathbb{Q}$–algebra under the convolution product
\[
(5)
\]
\[
f \cdot g([P]) := \sum_{x \in P} f([0_P, x])g([x, 1_P])
\]
We would now like to equip $H_{\mathcal{P}, \sim}$ with a Hopf algebra structure. To do this, the family $\mathcal{P}$ and the relation $\sim$ must satisfy some additional properties. A hereditary family is an interval closed family $\mathcal{P}$ of posets which is closed under the formation of direct products. We will assume that $\sim$ satisfies the following two properties:
\[
\begin{align*}
\bullet & \text{ whenever } P \sim Q \text{ in } \mathcal{P}, \text{ then } P \times R \sim Q \times R \text{ and } R \times P \sim R \times Q \text{ for all } R \in \mathcal{P} \\
\bullet & \text{ if } P, Q \in \mathcal{P} \text{ and } |Q| = 1, \text{ then } P \times Q \sim Q \times P \sim P
\end{align*}
\]
An order compatible relation $\sim$ on $\mathcal{P}$ satisfying these additional properties is called a Hopf relation. In this case, $\mathcal{P}$ is a monoid under the operation $[P][Q] = [P \times Q]$ with unit the class of any one-element poset. We may now introduce a coproduct on $H_{\mathcal{P}, \sim}$
\[
\Delta(f)([M], [N]) := f([M \times N])
\]
It is shown in [12] that $\Delta$ equips $H_{\mathcal{P}, \sim}$ with the structure of a bialgebra, and furthermore, that an antipode exists, making it into a Hopf algebra, which we call the incidence Hopf algebra of $(\mathcal{P}, \sim)$.

Note: the original definition of incidence Hopf algebra given in [12] is dual to the one used here.

Suppose that $\mathcal{F}$ is a collection of finite posets which is closed under the operation of taking convex subposets and disjoint unions. The collection of order ideals
\[
\mathcal{P}(\mathcal{F}) := \{ I_P : P \in \mathcal{F} \}
\]
is then a collection of intervals which is interval-closed and hereditary. If we define $I_P \sim I_Q$ whenever there exists an $f : P \to Q \in M(P, Q)$, then $\sim$ is a Hopf relation, and so we may consider the Hopf algebra $H_{\mathcal{P}(\mathcal{F}), \sim}$. Using the
fact that for $I, L \in I_P$, $[I, L] \simeq J_{L \setminus I}$ and $J_{P+Q} \simeq J_P \times J_Q$, the product and coproduct become:

\begin{align}
(7) & \quad f \cdot g([J_P]) := \sum_{I \in J_P} f([I])g([J_P \setminus I]) \\
(8) & \quad \Delta(f)([J_P], [J_Q]) := f([J_{P+Q}])
\end{align}

5.2. Ringel-Hall algebras. For an introduction to Ringel-Hall algebras in the context of abelian categories, see [10]. We define the Ringel-Hall algebra of $C_F$, denoted $H_{C_F}$, to be the $\mathbb{Q}$–vector space of finitely supported functions on isomorphism classes of $C_F$. I.e.

$$H_{C_F} := \{ f : \text{Iso}(C_F) \to \mathbb{Q} | |\text{supp}(f)| < \infty \}$$

As a $\mathbb{Q}$–vector space it is spanned by the delta functions $\delta_A, A \in \text{Iso}(C_F)$. The algebra structure on $H_{C_F}$ is given by the convolution product:

$$f \ast g(M) = \sum_{A \subseteq M} f(A)g(M/A)$$

$H_{C_F}$ possesses a co-commutative co-product given by

$$\Delta(f)(M, N) = f(M \oplus N)$$

as well as a natural $K_0^+(C_F)$–grading in which $\delta_A$ has degree $k(A) \in K_0^+(C_F)$.

The subobjects of $X_P \in C_F$ are exactly $X_I$ for $I \in I_P$, and the product becomes

$$f \ast g([X_P]) = \sum_{I \in I_P} f([X_I])g([X_P \setminus I])$$

while the coproduct becomes

$$\Delta(f)([X_P], [X_Q]) = f([X_{P+Q}])$$

Thus, the map

$$\phi : H_{C_F} \to H_{P(F), \sim}$$

determined by

$$\phi(f)([X_P]) := f([X_P])$$

is an isomorphism of Hopf algebras. Recall that a Hopf algebra $A$ over a field $k$ is connected if it possesses a $\mathbb{Z}_{\geq 0}$–grading such that $A_0 = k$. In addition to the $K_0^+(C_F)$–grading, $H_{C_F}$ possesses a grading by the order of the poset - i.e. we may assign $\deg(\delta_{X_P}) = |P|$. This gives it the structure of graded connected Hopf algebra. The Milnor-Moore theorem implies that $H_{C_F}$ is the enveloping algebra of the Lie algebra of its primitive elements, which we denote by $n_F$ - i.e. $H_{C_F} \simeq U(n_F)$. We have thus established the following:
Theorem 1. The Ringel-Hall algebra of the category $C_{\mathcal{F}}$ is isomorphic to the Incidence Hopf algebra of the family $\mathcal{P}(\mathcal{F})$. These are graded connected Hopf algebras, graded by the order of poset, and isomorphic to $U(\mathfrak{n}_{\mathcal{F}})$, where $\mathfrak{n}_{\mathcal{F}}$ denotes the Lie algebra of its primitive elements.

6. Examples

In this section, we give some examples of families $\mathcal{F}$ of posets closed under disjoint unions and the operation of taking convex subposets.

Example 1: Let $\mathcal{F} = \text{Fin}$ be the collection of all finite posets, and let $M(P, Q)$ consist of poset isomorphisms. Fin is clearly closed under disjoint unions and taking convex subposets. I claim that $K_0(C_{\text{Fin}}) = \mathbb{Z}$. Let $P$ be a finite poset, and $m \in P$ a minimal element. Then $m$ is also an order ideal in $P$, and so we have an exact sequence
\[
\emptyset \to X_\bullet \to X_P \to X_{P \setminus m} \to \emptyset
\]
where $\bullet$ denotes the one-element poset. Repeating this procedure with $P \setminus m$, we see that in $K_0(C_{\text{Fin}})$ every element can be written as multiple of $X_\bullet$, with $X_P \sim |P|X_\bullet$, and $K_0(C_{\text{Fin}}) \simeq \mathbb{Z}$. Thus, the $K_0^+$-grading on $H_{C_{\text{Fin}}}$ coincides with that by order of poset.

$\text{Iso}(C_{\text{Fin}})$ consists of isomorphism classes of finite posets. The Lie algebra $\mathfrak{n}_{\text{Fin}}$ is spanned by $\delta_{[X_P]}$ for $P$ connected posets. For $P, Q \in \text{Fin}$, both connected, we have
\[
\delta_{[X_P]} \star \delta_{[X_Q]} = \sum_{\{R \in \text{Iso}(\text{Fin}) \mid P \in J(R), Q \simeq R \setminus P\}} N(P, Q; R) \delta_{[X_R]}
\]
where $N(P, Q; R) := |\{I \in J_R \mid I \simeq P\}|$, and
\[
[\delta_{[X_P]}, \delta_{[X_Q]}] = \delta_{[X_P]} \star \delta_{[X_Q]} - \delta_{[X_Q]} \star \delta_{[X_P]}.
\]

Example 2: Let $\mathcal{F} = \mathcal{S}$ denote the collection of all finite sets (including the empty set). A finite set can be viewed as a poset where any two distinct elements are incomparable. $\mathcal{S}$ is clearly closed under disjoint unions and taking convex subposets (which coincide with subsets). Also, the order ideals of $S \in \mathcal{S}$ are exactly the subsets of $S$. Let $M(P, Q)$ consist of set isomorphisms. By the same argument as in the previous example, we have $K_0(C_{\mathcal{S}}) \simeq \mathbb{Z}$. $\text{Iso}(C_{\mathcal{S}}) \simeq \mathbb{Z}_{\geq 0}$, and we denote by $[n]$ the isomorphism class of the set with $n$ elements. We have
\[
\delta_{[n]} \star \delta_{[m]} = \binom{m+n}{n} \delta_{[m+n]}
\]
$\mathfrak{n}_{\mathcal{S}}$ is abelian, and isomorphic the one-dimensional Lie algebra spanned by $\delta_{[1]}$. $H_{C_{\mathcal{S}}}$ is dual to the binomial Hopf algebra in [12].
Example 3:
Let $S_k$ denote the collection of all finite $k$–colored sets (including the empty set), and take $M(P,Q)$ to consist of color-preserving set isomorphism. Clearly, the previous example corresponds to $k = 1$. We have $K_0(C_{S_k}) \simeq \mathbb{Z}^k$ and $\text{Iso}(C_{S_k}) \simeq (\mathbb{Z}_{\geq 0})^k$. Denoting by $[n_1, \cdots, n_k]$ the isomorphism class of the set consisting of $n_1$ elements of color $c_1$, $\cdots$, $n_k$ elements of color $c_k$, we have
\[
\delta_{[n_1, \cdots, n_k]} \star \delta_{[m_1, \cdots, m_k]} = \left( \prod_{i} \binom{n_i + m_i}{n_i} \right) \delta_{[n_1 + m_1, \cdots, n_k + m_k]}
\]
$n_{S_k}$ is a $k$–dimensional abelian Lie algebra spanned by $\delta_{e_i}$, where $e_i$ denotes the $k$–tuple $[0, \cdots, 1, \cdots, 0]$ with 1 in the $i$th spot.

The following two examples are treated in detail in [8], and we refer the reader there for details. Please see also [12, 4]. The resulting Hopf algebras (or rather their duals) introduced in [6–2], form the algebraic backbone of the renormalization process in quantum field theory. The corresponding Lie algebras $n_\mathcal{F}$ are studied in [3].

Example 4:
Recall that a rooted tree $t$ defines a poset whose Hasse diagram is $t$. Let $\mathcal{F} = \text{RF}$ denote the family of posets defined by rooted forests (i.e. disjoint unions of rooted trees). It is obviously closed under disjoint unions. An order ideal in a rooted forest corresponds to an admissible cut in the sense of [2] – that is, a cut having the property that any path from root to leaf encounters at most one cut edge, and so $\text{RF}$ is closed under the operation of taking convex posets. We have $K_0(C_{\text{RF}}) = \mathbb{Z}$, and as shown in [8], $H_{C_{\text{RF}}}$ is isomorphic to the dual of the Connes-Kreimer Hopf algebra on forests, or equivalently, to the Grossman-Larson Hopf algebra.

This example also has a colored version, where we consider the family $\text{RF} \{k\}$ of $k$–colored rooted forests. In this case, $K_0(C_{\text{RF} \{k\}}) \simeq (\mathbb{Z}_{\geq 0})^k$.

Example 5:
A graph determines a poset of subgraphs under inclusion. We may for instance consider the poset $\mathcal{F} = \text{FG}$ of subgraphs of Feynman graphs of a quantum field theory such as $\phi^3$ theory, which is closed under disjoint unions and convex posets. The Ringel-Hall algebra $H_{C_{\text{FG}}}$ is isomorphic to the dual of the Connes-Kreimer Hopf algebra on Feynman graphs. Please see [8] for details.

In the case of $\phi^3$ theory, it is shown in [7] that $K_0(C_{\text{FG}}) \simeq \mathbb{Z}[p], p \in \mathfrak{P}$, where $\mathfrak{P}$ is the set of primitively divergent graphs.
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