UNILATERAL GLOBAL INTERVAL BIFURCATION FOR KIRCHHOFF TYPE PROBLEMS AND ITS APPLICATIONS

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ABSTRACT. In this paper, we establish a unilateral global bifurcation result from interval for a class of Kirchhoff type problems with nondifferentiable nonlinearity. By applying the above result, we shall prove the existence of one-sign solutions for the following Kirchhoff type problems.

\[
\begin{align*}
- M(\int_{\Omega} |\nabla u|^2 \, dx) \Delta u &= \alpha(x) u^+ + \beta(x) u^- + ra(x) f(u), & \text{in } \Omega, \\
&= 0, & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a smooth boundary \( \partial \Omega \), \( M \) is a continuous function, \( \lambda \) is a parameter, \( \alpha(x), \beta(x) \in C(\Omega) \) is positive, \( u^+ = \max\{u, 0\}, u^- = -\min\{u, 0\} \), \( a, \beta \in C(\Omega) \); \( f \in C(\mathbb{R}, \mathbb{R}) \), \( sf(s) > 0 \) for \( s \in \mathbb{R}^+ \), and \( f_0 \in (0, \infty) \) and \( f_\infty \in [0, \infty] \) or \( f_0 = \infty \) and \( f_\infty \in [0, \infty] \), where \( f_0 = \lim_{|s|\to 0} f(s)/s, f_\infty = \lim_{|s|\to \infty} f(s)/s \). We use unilateral global bifurcation techniques and the approximation of connected components to prove our main results.

1. Introduction. In 1883, Kirchhoff [19] proposed the following problem as an extension of the classical d’Alembert’s wave equation for free vibrations of elastic strings. The model studied is

\[
u_{tt} - (a + b \int_{\Omega} |\nabla u|^2 \, dx) \Delta u = \lambda h(x) f(u),
\]

where \( u \) denotes the displacement, \( f \) is the external force, \( b \) represents the initial tension, and \( a \) is related to the intrinsic properties of the string. Kirchhoff’s model (1) takes into account the changes in length of the string produced by transverse vibrations.

Consider the following Kirchhoff type problem

\[
\begin{align*}
- M(\int_{\Omega} |\nabla u|^2 \, dx) \Delta u &= \lambda b(x) u + g(x, u, \lambda), & \text{in } \Omega, \\
&= 0, & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a smooth boundary \( \partial \Omega \), \( M \) is a continuous function, \( \lambda \) is a parameter, \( b(x) \in L^\infty(\Omega) \) is positive and the perturbation function

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$g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory condition in the first two variables and
\[
\lim_{s \to 0} \frac{g(x, s, \lambda)}{s} = 0
\] (3)
uniformly for a.e. $x \in \Omega$ and $\lambda$ on bounded sets. The problem (2) is nonlocal as the appearance of the term $\int_{\Omega} |\nabla u|^2 \, dx$ which implies that it is not a pointwise identity. This causes some mathematical difficulties which make the study for the problem (2) particularly interesting. The main difficulties when dealing with this problem lie in the presence of the nonlocal terms which arises in nonlinear vibrations and the analogous to the stationary case of equations that arise in the study of string or membrane vibrations. After the famous article by Lions [23], Eq. (2) received much attention, and some important and interesting results have been obtained, for example, see [14, 10, 15, 2, 6, 9]. In recent years, there has been considerable interest in the above problem (2) by variational method, see [28, 26, 7, 20, 14, 27].

We refer to [21, 17, 8, 4] for Kirchhoff models with critical exponents. Meanwhile, by applying the bifurcation techniques, there are few papers to study Kirchhoff-type problems, see for example [22, 18].

In 2015, Dai et al. [12] studied the Kirchhoff-type equations (2) by Rabinowitz [29, Theorem 1.3].

Dai et al. [12] also assume that $g$ and $M$ satisfies the following conditions:

(A0) There exist $c > 0$ and $p \in (1, 2^*)$ such that $|g(x, s, \lambda)| \leq c(1 + |s|^{p-1})$, for a.e. $x \in \Omega$ and $\lambda$ on bounded sets, where
\[
2^* = \begin{cases} 
\frac{2N}{N-2}, & N > 2, \\
+\infty, & N \leq 2.
\end{cases}
\] (4)

(A1) $M(t) \in C(\mathbb{R}^+) \text{ such that for some } m_0 > 0, M(t) \geq m_0, \text{ for all } t \in \mathbb{R}^+.$

(A2) there exists $m_1 > 0$, such that $\lim_{t \to +\infty} M(t) = m_1$.

By [12, Line 24-25 in Page 772], we have

**Remark 1.** The hypothesis (A1) shows that $M(0) > 0$.

However, among the above papers, the nonlinearities are differentiable at the origin. In [5], Berestycki established an important global bifurcation theorem from intervals for a class of second-order problems involving non-differentiable nonlinearity. Recently, Ma and Dai [25] improved Berestycki’s result (in [5]) to show a unilateral global bifurcation result for a class of second-order problems involving non-differentiable nonlinearity. Later, Dai and Ma [11] also considered a class of high-dimensional p-Laplacian problems involving non-differentiable nonlinearity.

Motivated by above papers, in this paper, we shall establish a Dancer-type unilateral global bifurcation result from interval for the Kirchhoff type problems

\[
\begin{aligned}
 &\left\{ -M\left(\int_{\Omega} |\nabla u|^2 \, dx\right)\Delta u = \lambda b(x)u + F(x, u, \lambda), \quad \text{in } \Omega, \\
 &u = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\] (5)

where $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Moreover, the nonlinear term $F$ has the form $F = f + g$, where $b$, $f$ and $g$ satisfy the following conditions:

(H1) $b(x) \in C(\overline{\Omega})$ is positive.

(H2) $\frac{f(x, s, \lambda)}{s} \leq M_1$, for all $x \in \overline{\Omega}$, $0 < |s| \leq 1$ and all $\lambda \in \mathbb{R}$, where $M_1$ is a positive constant.
(H3) \( g(x, s, \lambda) = o(|s|) \) near \( s = 0 \) uniformly in \( x \in \overline{\Omega} \) and \( \lambda \) on bounded sets, and \( g \) satisfies the conditions (A0).

Under the conditions (A1), (A2), and (H1)-(H3), we shall show that \([\lambda_1, M(0) - d, \lambda_1 M(0) + d]\) is a bifurcation interval of problem (5) and there are two distinct unbounded sub-continua, \( \mathcal{D}^+ \) and \( \mathcal{D}^- \), consisting of the bifurcation branch \( D \) from

\([\lambda_1, M(0) - d, \lambda_1 M(0) + d]\), where \( d = \frac{M_1 M^2(0)}{b_0 m_0} \), \( \lambda_1 \) is the principal eigenvalue of the following problem

\[
\begin{cases}
-\Delta u = \lambda b(x) u & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]

By [16], \( \lambda_1 \) is simple, isolated and is the unique principle eigenvalue of problem (6).

On the basis of the unilateral global interval bifurcation result (Theorem 3.1), we shall study the following the Kirchhoff type problem

\[
\begin{cases}
-M(\int_{\Omega} |\nabla u|^2 \, dx) \Delta u = \lambda b(x) u + \alpha(x) u^+ + \beta(x) u^- + g(x, u, \lambda), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where \( u^+ = \max\{u, 0\}, u^- = -\min\{u, 0\} \), and \( \alpha(x), \beta(x) \) satisfy

(H4) \( \alpha(x), \beta(x) \in C(\overline{\Omega}) \).

Obviously, by (H4), one may get that

Remark 2. Let \( A(x) = \alpha(x)/M_2, B(x) = \beta(x)/M_2 \) for any constant \( M_2 > 0 \). Then, it follows that \( A(x), B(x) \in C(\overline{\Omega}) \), i.e., \( A(x), B(x) \) satisfy (H4).

Furthermore, we shall investigate the existence of one-sign solutions for the following Kirchhoff-type problems

\[
\begin{cases}
-M(\int_{\Omega} |\nabla u|^2 \, dx) \Delta u = \alpha(x) u^+ + \beta(x) u^- + ra(x) f(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]

We assume that \( a, f \) satisfies the following assumptions:

(H5) \( \alpha(x) \in C(\overline{\Omega}) \) is positive.

(H6) \( s f(s) > 0 \) for \( s \neq 0 \).

(H7) \( f_0, f_\infty \in (0, +\infty) \).

(H8) \( f_0 \in (0, \infty) \) and \( f_\infty = \infty \).

(H9) \( f_0 = \infty \) and \( f_\infty \in (0, \infty) \).

(H10) \( f_0 = \infty \) and \( f_\infty = 0 \).

(H11) \( f_0 = \infty \) and \( f_\infty = \infty \).

Where

\[
f_0 = \lim_{|s| \to 0} \frac{f(s)}{s}, \quad f_\infty = \lim_{|s| \to +\infty} \frac{f(s)}{s}.
\]

In the case of \( \alpha = \beta = 0 \), Dai et al. [12] considered problem (8) by the bifurcation theory [29], in which there exist positive solutions for the problem (8) under the assumptions (H6) and (H7).

Remark 3. For the abstract unilateral global bifurcation theory, we refer the reader to [29, 5, 25, 13] and the references therein.

The rest of this paper is arranged as follows. In Section 2, we given some Preliminaries. In Section 3, we establish the unilateral global bifurcation result from the interval for the problem (5). In Section 4, on the basis of the unilateral global interval bifurcation result, we shall investigate the existence of one-sign solutions for the Kirchhoff-type problems (8).
2. Preliminaries. Let \( X := H^1_0(\Omega) \) with the norm \( \|u\| = (\int_\Omega |\nabla u|^2 \, dx)^{1/2} \). Let \( P^+ = \{ u \in X | u > 0 \text{ in } \Omega \} \) and set \( P^- = -P^+ \) and \( P = P^+ \cup P^- \). Let \( K^\pm = \mathbb{R} \times P^\pm \) under the product topology. Let \( C^\pm \) denote the closure in \( K^\pm \) of the set of nontrivial solutions of (2). We define the linear operator \( L : X \to X \) by
\[
Lu = -\Delta u, \quad u \in X.
\]
Then \( L^{-1} : X \to X \) is linear completely continuous.

From [12, p.733], it is clear that the problem (2) can be equivalently written as
\[
u = F(\lambda, u) = \lambda \cdot \frac{L^{-1}(bu)}{M(0)} + \tilde{H}(\lambda, u),
\]
where \( \tilde{H}(\lambda, u) = \frac{\lambda(M(0) - M(||u||^2))}{M(0)M(||u||^2)}L^{-1}(bu) + \frac{L^{-1}(H(\lambda, u))}{M(||u||^2)} \), where \( H(\lambda, \cdot) \) denotes the usual Nemitsky operator associated with \( g \).

From conditions (3), (A0), (A1) and noting \( 2 < 2^* \), we can see that \( \tilde{H} : \mathbb{R} \times X \to X \) is completely continuous. Furthermore, it follows that \( F : \mathbb{R} \times X \to X \) is completely continuous and \( F(\lambda, 0) = 0, \forall \lambda \in \mathbb{R} \).

Dai et al. [12] obtained that \( \lambda_1 M(0) \) is a simple characteristic value of \( L^{-1} \), where \( \lambda_1 \) be the eigenvalue of the linear problem (6). If \( \psi \) is eigenfunction corresponding to eigenvalue \( \lambda_1 M(0) \), we have
\[
\begin{cases}
-\Delta \psi = \frac{\lambda_1 b(x)}{M(0)} \psi, & \text{in } \Omega, \\
\psi = 0, & \text{on } \partial \Omega.
\end{cases}
\]
By [12, p.773-774], it follows that
\[
\lim_{\|u\| \to 0} H(\lambda, u)/\|u\| = 0, \quad \text{in } L^{p'}(\Omega)
\]
uniformly on bounded \( \lambda \) sets.

Moreover, it follows that
\[
\lim_{\|u\| \to 0} \tilde{H}(\lambda, u)/\|u\| = 0
\]
at \( u = 0 \) uniformly on bounded \( \lambda \) sets, i.e., \( \tilde{H}(\lambda, u) = o(\|u\|) \) at \( u = 0 \) uniformly on bounded \( \lambda \) sets.

Furthermore, by Rabinowitz [29, Theorem 1.3], Dai et al. [12, Theorem 1.1] obtained the following global bifurcation result.

**Lemma 2.1. (see [12, Theorem 1.1]).** Assume that (3), (A0) and (A1) hold. Then \( (\lambda_1 M(0), 0) \) is a bifurcation point of problem (2) and the associated bifurcation continuum \( C \) in \( \mathbb{R} \times H^1_0(\Omega) \), whose closure contains \( (\lambda_1 M(0), 0) \), is either unbounded or contains a pair \( (\mu M(0), 0) \), where \( \mu \) is another eigenvalue of problem (6).

By Dancer [13, Theorem 2.1], under the conditions (3), (A0) and (A1), one can obtain that the problem (2) has two distinct unbounded sub-continua \( C^+ \) and \( C^- \), consisting of the bifurcation branch \( C \) emanating from \( (\lambda_1 M(0), 0) \), which satisfy:

**Lemma 2.2. (see [13, Theorem 2.1]).** Both \( C^+ \) and \( C^- \) are unbounded and
\[
C'' \subset ((\mathbb{R} \times P^\nu) \cup \{ (\lambda_1 M(0), 0) \}),
\]
where \( \nu \in \{ +, - \} \).

Next, we give an important lemma which will be used later.
Lemma 2.3. (see [1]). Let $u, v \in C^1(\overline{\Omega}), v \neq 0$ in $\Omega$. Then we have the following identity:

$$\nabla \cdot \left[ \frac{u}{v} (v \nabla u - u \nabla v) \right] = |\nabla u|^2 + \left| \frac{u}{v} \nabla v \right|^2 - 2 \nabla u \left( \frac{u}{v} \nabla v + \left[ \frac{u}{v} (v L[u] - u L[v]) \right] \right).$$

Remark 4. (see [1]). By Young’s inequality, we get

$$|\nabla u|^2 + \left| \frac{u}{v} \nabla v \right|^2 - 2 \nabla u \left( \frac{u}{v} \nabla v \right) \geq 0$$

and the equality holds if and only if $\nabla \left( \frac{u}{v} \right) = 0$, a.e. $\Omega$, i.e., $u = kv$ for some constant $k$ in each component of $\Omega$.

By Lemma 2.3 and Remark 4, we have the following result:

Lemma 2.4. Let $u, v \in C^1(\overline{\Omega}), v \neq 0$ in $\Omega$. One can obtain

$$\int_{\Omega} \left[ \frac{u}{v} (v L[u] - u L[v]) \right] dx \leq 0.$$

Next, we summarize following Lemma from [25] which will be used later.

Lemma 2.5. ([11, Theorem 3.1 ($p = 2$)]). There exist two simple half-eigenvalues $\lambda^+$ and $\lambda^-$ for the following problem

$$\begin{cases}
-\Delta u = \lambda b(x) u + \alpha(x) u^+ + \beta(x) u^-, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega. 
\end{cases}$$

(10)

The corresponding half-linear solutions are in $\{\lambda^+\} \times \mathbb{P}^+$ and $\{\lambda^-\} \times \mathbb{P}^-$. Furthermore, aside from $\lambda^+$ and $\lambda^-$, there is no other half-eigenvalue with positive or negative eigenfunction. Where the definitions of half-eigenvalue problem (10) and principal half-eigenvalue $\lambda^\pm$ are given by [25, In Section 3].

Furthermore, by Lemma 2.5 and Remark 2, it follows that

Lemma 2.6. ([11, Theorem 3.1 ($p = 2$)]). For any constant $M_2 > 0$, there exist two simple half-eigenvalues $\lambda^+ M_2$ and $\lambda^- M_2$ for the following problem

$$\begin{cases}
-\Delta u = \frac{\lambda b(x)}{M_2} u + A(x) u^+ + B(x) u^-, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega. 
\end{cases}$$

(11)

The corresponding half-linear solutions are in $\{\lambda^+ M_2\} \times \mathbb{P}^+$ and $\{\lambda^- M_2\} \times \mathbb{P}^-$. Furthermore, aside from $\lambda^+ M_2$ and $\lambda^- M_2$, there is no other half-eigenvalue with positive or negative eigenfunction, where $A(x) = \alpha(x)/M(0)$ and $B(x) = \beta(x)/M(0)$ are given in Remark 2 by Remark 1.

Now, we also give an lemma which will be used later.

Lemma 2.7. ([1, Theorem 2.6]). Let $f_1(x) \leq f_2(x)$ be two weight functions satisfying the conditions given in Section 2.1 (see [28]) with $f_1 \neq f_2$. Let $u$ be a positive solution of

$$L[u] = f_1(x) u, x \in \Omega, u = 0, x \in \partial \Omega.$$

Then any solution $v$ of

$$L[v] = f_2(x) v, x \in \Omega$$

must change sign.

In order to treat the problems with non-asymptotic nonlinearity at 0 and $\infty$, we shall need the following definition and lemma.
Definition 2.8. (see [31]). Let $X$ be a Banach space and let $\{C_n|n = 1, 2,...\}$ be a certain infinite collection of subsets of $X$. Then the superior limit $\mathbb{D}$ of $\{C_n\}$ is defined by

$$\mathbb{D} := \limsup_{n \to \infty} C_n = \{x \in X|\exists \{u_n\} \subset N$ and $x_{n_i} \in C_{n_i},$ such that $x_{n_i} \to x\}.$$ 

Lemma 2.9. (see [24]). Let $X$ be a Banach space and let $\{C_n|n = 1, 2,...\}$ be a family of closed connected subsets of $X$. Assume that

(i) there exist $z_n \in C_n$, $n = 1, 2,...$ and $z^* \in X$, such that $z_n \to z^*$;
(ii) $r_n = \sup\{\|x\| \in C_n\} = \infty$;
(iii) for every $R > 0$, $(\cup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of $X$, where

$$B_R = \{x \in X|\|x\| \leq R\}.$$ 

Then there exists an unbounded component $C$ in $\mathbb{D}$ and $z^* \in C$.

3. Unilateral global bifurcation. Let $\mathcal{P}^{\pm}$ denote the closure in $K^{\pm}$ of the set of nontrivial solutions of (5).

The first main result for (5) is the following theorem.

Theorem 3.1. Let $(A1)$, $(A2)$, $(H1)$, $(H2)$ and $(H3)$ hold. Let $d = \frac{M_0M_2(0)}{b_0}$, where $b_0 = \min_{x \in \mathbb{R}} b(x)$, and let $I = [\lambda_1M(0) - d, \lambda_1M(0) + d]$. The component $\mathcal{C}_\nu$ of $\mathcal{P}^{\nu} \cup (I \times \{0\})$, containing $I \times \{0\}$ is unbounded and lies in $(\mathbb{R} \times \mathbb{P}^\nu) \cup (I \times \{0\})$, for $\nu = +$ and $\nu = -$.

To prove Theorem 3.1, we introduce the following auxiliary approximate problem:

$$
\begin{cases}
-M(\|u\|^2)\Delta u = \lambda b(x)u + f(x, u, |u|^\nu, \lambda) + g(x, u, \lambda), & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega.
\end{cases}
$$

(12)

To prove Theorem 3.1, the next lemma will play a key role.

Lemma 3.2. Let $\epsilon_n$, $0 < \epsilon_n < 1$, be a sequence converging to 0. If there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times \mathbb{P}^\nu$ such that $(\lambda_n, u_n)$ is a nontrivial solution of problem (12) corresponding to $\epsilon = \epsilon_n$, and $(\lambda_n, u_n)$ converges to $(\lambda M(0), 0)$ in $\mathbb{R} \times X$, then $\lambda M(0) \in I$.

Proof. Without loss of generality, we may assume that $\|u_n\| \leq 1$. Let $w_n = u_n/\|u_n\|

$$
\begin{cases}
-\Delta w_n = \frac{\lambda_n b(x)w_n}{M(0)} + \frac{f(x, u_n, |u_n|^\nu, \lambda_n)}{M(\|u_n\|^2)\|u_n\|} + \frac{H_1(\lambda_n, u_n)}{\|u_n\|}, & \text{in } \Omega, \\
\nu = 0, & \text{on } \partial\Omega,
\end{cases}
$$

where

$$H_1(\lambda_n, u_n) = \frac{\lambda_n M(0) - M(\|u_n\|^2)}{M(0)M(\|u_n\|^2)} bu + \frac{H(\lambda_n, u_n)}{M(\|u_n\|^2)},$$

where $H(\lambda_n, \cdot)$ denotes the usual Nemitsky operator associated with $g$. By $(A1)$, $(H3)$ and [12, p.773-774], it follows that

$$\lim_{\|u_n\| \to 0} \frac{H(\lambda_n, u_n)}{\|u_n\|} = 0, \quad \text{in } L^p(\Omega)$$

uniformly on bounded $\lambda$ intervals.

Furthermore, it follows that

$$\lim_{\|u_n\| \to 0} \frac{H_1(\lambda, u_n)}{\|u_n\|} = 0, \quad \text{in } L^p(\Omega)$$

(13)

uniformly on bounded $\lambda$ sets.
Since $u_n/\|u_n\|$ is bounded in $X$, so $|u_n/\|u_n\|| \leq c_0$ for some positive constant $c_0$. Furthermore, (H2) implies that
\[
\frac{f(x, u_n|\|u_n\|^n, \lambda_n)}{M(\|u\|^2)|u_n|} = \left| \frac{f(x, u_n|\|u_n\|^n, \lambda_n)}{u_n|\|u_n\|^n} \right| \leq \|u_n\| \frac{M_1c_0}{m_0} \leq \frac{M_1c_0}{m_0},
\]
for a.e. $x \in \Omega$ and $n$ large enough. Using this fact with (13) and (14), we have that $\lambda_n b(x) w_n + f(x, u_n|\|u_n\|^n, \lambda_n) + H_1(\lambda_n, u_n) |w_n|$ is bounded in $L^\infty(\Omega)$ for $n$ large enough. By the Arzela-Ascoli theorem, the completely continuous of $L^{-1}$ implies that $w_n$ is strong convergence in $C^1(\Omega)$. Without loss of generality, we may assume that $w_n \to w$ in $E$. Clearly, we have $w \in \mathcal{P}$. Now, we deduce the boundedness of $\lambda M(0)$. Let $\psi \in P'$ be an eigenfunction of problem (9) corresponding to $\lambda_1 M(0)$.

We know that $w_n$ satisfies $\Delta w_n + \lambda_n b(x) w_n + f(x, u_n|\|u_n\|^n, \lambda_n) + H_1(\lambda_n, u_n) = 0$ and $\psi$ satisfies (9).

We can assume without loss of generality that $\nu = +$. By Lemma 2.4, it follows that
\[
\int_{\Omega} \frac{w_n}{\psi} \psi L[w_n] - w_n L[\psi] dx = \int_{\Omega} \left( \lambda_1 - \lambda_n \right) \frac{b(x) w_n^2}{M(0)} - f(x, u_n|\|u_n\|^n, \lambda_n) w_n^2 - H_1(\lambda_n, u_n) w_n dx \leq 0.
\]
Similarly, we can also show that
\[
\int_{\Omega} \left( \lambda_1 - \lambda_n \right) \frac{b(x) \psi^2}{M(0)} + f(x, u_n|\|u_n\|^n, \lambda_n) \psi^2 + H_1(\lambda_n, u_n) \psi^2 dx \leq 0.
\]
If $\lambda \leq \lambda_k$, by (15), (13) and (14), we have that
\[
\int_{\Omega} \left( \lambda_1 - \lambda \right) \frac{b(x) w^2}{M(0)} \leq \lim_{n \to \infty} \int_{\Omega} \frac{f(x, u_n|\|u_n\|^n, \lambda_n)}{u_n|\|u_n\|^n} \frac{|u_n|^{\epsilon_n}}{M(\|u\|^2)} u_n^2 dx = \int_{\Omega} \frac{M_1}{m_0} w^2 dx.
\]
Hence, we get that
\[
\int_{\Omega} \left( \lambda_1 - \lambda \right) \frac{b_0 w^2}{M(0)} \leq \int_{\Omega} \frac{M_1}{m_0} w^2 dx,
\]
which implies $\lambda M(0) \geq \lambda_1 M(0) - d$.

If $\lambda \geq \lambda_k$, by (16), (13) and (14), we have that
\[
\int_{\Omega} \left( \lambda - \lambda_1 \right) \frac{b(x) \psi^2}{M(0)} \leq \lim_{n \to \infty} \int_{\Omega} \frac{-f(x, u_n|\|u_n\|^n, \lambda_n)}{u_n|\|u_n\|^n} \frac{|u_n|^{\epsilon_n}}{M(\|u\|^2)} \psi^2 dx = \int_{\Omega} \frac{M_1}{m_0} \psi^2 dx.
\]
Hence, we get that
\[
\int_{\Omega} \left( \lambda - \lambda_1 \right) \frac{b_0 \psi^2}{M(0)} \leq \int_{\Omega} \frac{M_1}{m_0} \psi^2 dx,
\]
which implies $\lambda M(0) \leq \lambda_1 M(0) + d$. Therefore, we have that $\lambda M(0) \in I$. $\square$
Proof of Theorem 3.1. We only prove the case of $\mathcal{C}^+$ since the case of $\mathcal{C}^-$ is similar. Let $\mathcal{C}^+$ be the component of $\mathcal{F}^+ \cap (I \times \{0\})$, containing $I \times \{0\}$.

We divide the rest of proofs into two steps.

Step 1. We show that $\mathcal{C}^+ \subset \mathbb{R} \times P^+ \cup (I \times \{0\})$.

For any $(\lambda M(0), u) \in \mathcal{C}^+$, there are two possibilities: ($i$) $u \geq 0$ but $u \neq 0$, or ($ii$) $u \equiv 0$. If the case ($ii$) occurs, there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times P^+$ such that $(\lambda_n, u_n)$ is a solution of problem (5), and $(\lambda_n, u_n)$ converges to $(\lambda M(0), 0)$ in $\mathbb{R} \times X$. By Lemma 3.1, we have that $\lambda M(0) \in I$, i.e., $(\lambda M(0), u) \in I \times \{0\}$. Hence, $\mathcal{C}^+ \subset \mathbb{R} \times P^+ \cup (I \times \{0\})$ in the case of ($ii$). If the case ($i$) occurs, using the similar method to prove [11, Theorem 2.1:Step 1] with obvious changes, we may prove $u > 0$ in $\Omega$. Hence, $\mathcal{C}^+ \subset \mathbb{R} \times P^+ \cup (I \times \{0\})$.

Step 2. We prove that $\mathcal{C}^+$ is unbounded.

Suppose on the contrary that $\mathcal{C}^+$ is bounded. By the similar method to prove [5, Theorem 1] with obvious changes, we can find a neighborhood $O$ of $\mathcal{C}^+$ such that $\partial O \cap \mathcal{C}^+ = \emptyset$.

In order to complete the proof of this theorem, we consider the problem (12). For $\epsilon > 0$, it is easy to show that nonlinear term $f(x, u|u|^\alpha, \lambda) + g(x, u, \lambda)$ satisfies the condition (3). Let

$$
\mathcal{F}_\epsilon = \{(\lambda, u) : (\lambda, u) \text{ satisfies } (12) \text{ and } u \neq 0\} \subset \mathbb{R} \times X.
$$

By Lemma 2.2, there exists an unbounded continuum $\mathcal{C}_\epsilon^+$ of $\mathcal{F}_\epsilon$ bifurcating from $(\lambda_1 M(0), 0)$ such that

$$
\mathcal{C}_\epsilon^+ \subset (\mathbb{R} \times P^\nu) \cup \{(\lambda_1 M(0), 0)\}, \text{ for } \nu = + \text{ and } -.
$$

So there exists $(\lambda_\epsilon, u_\epsilon) \in \mathcal{C}_\epsilon^+ \cap \partial O$ for all $\epsilon > 0$. Since $O$ is bounded in $\mathbb{R} \times P^+$, Eq. (12) shows that $(\lambda_\epsilon, u_\epsilon)$ is bounded in $\mathbb{R} \times X$ independently of $\epsilon$. By the compactness of $L^{-1}$, one can find a sequence $\epsilon_n \to 0$ such that $(\lambda_{\epsilon_n}, u_{\epsilon_n})$ converges to a solution $(\lambda M(0), u)$ of (5). So $u \in P^+$. We claim that $u \neq 0$. Suppose on the contrary that $u \equiv 0$. By Lemma 3.2, $\lambda M(0) \in I$, which contradicts the definition of $O$. Using a similar way employed in Step 1, we can show that $u > 0$ in $\Omega$. Then we have that $(\lambda M(0), u) \in \mathcal{C}^+ \cap \partial O$ which contradicts $\mathcal{C}^+ \cap \partial O = \emptyset$. \hfill $\square$

From Theorem 3.1 and its proof, we can easily get the following two corollaries.

**Corollary 1.** There exist two unbounded sub-continua $\mathcal{C}^+$ and $\mathcal{C}^-$ of solutions of (5) in $\mathbb{R} \times X$, bifurcating from $I \times \{0\}$, and $\mathcal{C}^+ \subset (\mathbb{R} \times P^+) \cup (I \times \{0\})$ for $\nu = +$ and $\nu = -$.

We relax the assumption of $b(x)$ as the following

$$(H12) \ b(x) \in C(\overline{\Omega}) \text{ is a sign-changing weight.}$$

**Corollary 2.** Let (H12) hold and $f \equiv 0$. Then there exist two unbounded sub-continua $\mathcal{C}^+$ and $\mathcal{C}^-$ of solutions of (5) in $\mathbb{R} \times X$, bifurcating from $(\lambda_1 M(0), 0)$, and $\mathcal{C}^+ \subset (\mathbb{R} \times P^+) \cup (I \times \{0\})$ for $\nu = +$ and $\nu = -$.

On the basis of Theorem 3.1 and Corollary 1, we shall obtain the following result for the Kirchhoff type problem (7).

**Theorem 3.3.** For $\nu = +, -$, $(\lambda^\nu M(0), 0)$ is a bifurcation point for problem (7). Moreover, there exists an unbounded continuum $\mathcal{C}^\nu$ of solutions of problem (7), such that $\mathcal{C}^\nu \subset ((\mathbb{R} \times P^\nu) \cup \{(\lambda^\nu M(0), 0)\})$. 

Proof. Let $\alpha^0 := \max_{x \in \Omega} |\alpha(x)|$, $\beta^0 := \max_{x \in \Omega} |\beta(x)|$ and

$$I_0 = \left[ \lambda_1 M(0) - \frac{(\alpha^0 + \beta^0)M^2(0)}{b_0 m_0}, \lambda_1 M(0) + \frac{(\alpha^0 + \beta^0)M^2(0)}{b_0 m_0} \right].$$

Corollary 1 shows that there exist two unbounded sub-continua $\mathcal{P}^+$ and $\mathcal{P}^-$ of solutions of (7) in $\mathbb{R} \times X$, bifurcating from $I_0 \times \{0\}$, and $\mathcal{P}^0 \subset (\mathbb{R} \times P^0) \cup \{I_0 \times \{0\}\}$ for $\nu = +$ and $\nu = -$. Let us show that $\mathcal{P}^0 \cap (\mathbb{R} \times \{0\}) = (\lambda^0 M(0), 0)$, i.e. $(\lambda^0 M(0), 0)$ is a bifurcation point for problem (7). Indeed, if there exists $(\lambda_n, u_n)$ be a sequence of solutions of the problem (7) converging to $(\lambda M(0), 0)$, let $v_n = \frac{u_n}{\|u_n\|}$, then $v_n$ should be a solution of problem

$$-\Delta v_n = \frac{\lambda_n b(x)v_n}{M(0)} + A(x)v_n^+ + B(x)v_n^- + \frac{H_2(\lambda_n, u_n)}{\|u_n\|},$$

where $A(x) = \alpha(x)/M(0)$ and $B(x) = \beta(x)/M(0)$ are given in Remark 2 by Remark 1.

$$H_2(\lambda_n, u_n) = \left[ \frac{\lambda_n M(0) - M(\|u_n\|^2)}{M(0) M(\|u_n\|^2)} (\lambda_n b(x)u_n + \alpha(x)u_n^+ + \beta(x)u_n^-) + \frac{H(\lambda_n, u_n)}{M(\|u_n\|^2)} \right],$$

where $H(\lambda_n, \cdot)$ denotes the usual Nemitsky operator associated with $g$. By [12, p.773-774], it follows that

$$\lim_{\|u\|\to 0} \frac{H(\lambda_n, u_n)}{\|u_n\|} = 0, \quad \text{in } L^{p'}(\Omega)$$

uniformly on bounded $\lambda_n$ sets.

Furthermore, it follows that

$$\lim_{\|u_n\|\to 0} \frac{H_2(\lambda_n, u_n)}{\|u_n\|} = 0, \quad \text{in } L^{p'}(\Omega)$$

uniformly on bounded $\lambda_n$ sets.

By (17) and the compactness of $L^{-1}$, we obtain that for some convenient subsequence $v_n \to v_0$ as $n \to +\infty$. Now $v_0$ verifies the equation

$$-\Delta v_0 = \frac{\lambda b(x)v_0}{M(0)} + A(x)v_0^+ + B(x)v_0^-$$

Moreover, by Lemma 2.6, it follows that $\lambda M(0) = \lambda^0 M(0)$ for $\nu \in \{+, -\}$. \hfill $\square$

4. One-sign solutions for Kirchhoff type problems. Let $\Sigma^\pm$ denote the closure in $K^\pm$ of the set of nontrivial solutions of (8).

To state the main result of this section, we first study the following eigenvalue problem

$$\begin{cases}
-M(\int_{\Omega} |\nabla u|^2 dx) \Delta u = \alpha(x)u^+ + \beta(x)u^- + \lambda r_\alpha(x)f(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}$$

where $\lambda > 0$ is a parameter. Let

$$f(u) = f_0 u + \zeta(u), \quad f(u) = f_\infty u + \xi(u)$$

with

$$\lim_{s \to 0^+} \frac{\zeta(s)}{s} = \lim_{s \to +\infty} \frac{\xi(s)}{s} = 0, \quad \lim_{s \to +\infty} \frac{\zeta(s)}{s} = \lim_{s \to 0^+} \frac{\xi(s)}{s} = f_\infty - f_0$$

uniformly in $\Omega$. 

\hfill $\Box$
Let
\[ H_3(\lambda, u) = \frac{\lambda(M(0) - M(\|u\|^2))}{M(0)M(\|u\|^2)}(\lambda a(x)u + \alpha(x)u^+ + \beta(x)u^-) + \frac{\lambda r a(x) \zeta(u)}{M(\|u\|^2)}. \]

By [12, p.773-774], it follows that
\[ \lim_{\|u\| \to 0} \frac{\zeta(u)}{\|u\|} = 0, \quad \text{in } L^{p'}(\Omega) \]
Moreover, we have
\[ \lim_{\|u\| \to 0} \frac{H_3(\lambda, u)}{\|u\|} = 0, \quad \text{in } L^{p'}(\Omega) \]
uniformly on bounded \( \lambda \) sets.

Let
\[ H_4(\lambda, u) = \frac{\lambda(m_1 - M(\|u\|^2))}{m_1M(\|u\|^2)}(\lambda a(x)u + \alpha(x)u^+ + \beta(x)u^-) + \frac{\lambda r a(x) \xi(u)}{M(\|u\|^2)}. \]

By (A2), (H1), (H4) and (19), it follows that for any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon \) such that
\[ |\xi(u)| \leq C_\varepsilon + \varepsilon |u_n|. \quad (20) \]
By (20) and [12, p.775-776], we can easily show that
\[ \lim_{\|u\| \to \infty} \frac{\xi(u)}{\|u\|} = 0, \quad \text{in } L^2(\Omega). \]
Furthermore, one obtain that
\[ \lim_{\|u\| \to \infty} \frac{|H_4(\lambda, u)|}{\|u\|} = 0, \quad \text{in } L^2(\Omega) \quad (21) \]
uniformly on bounded \( \lambda \) sets.

**Remark 5.** By (H1) and (H5), obviously, it follows that the results in Section 2 (Lemma 2.1, Lemma 2.2, Lemma 2.5, Lemma 2.6) and Section 3 (Theorem 3.1, Lemma 3.2, Corollary 1, Corollary 2, Theorem 3.3) are still obtained when using \( a(x) \) instead of \( b(x) \).

Let us consider
\[ \begin{cases} 
-\Delta u = \frac{\lambda r a(x) f_0}{M(0)} u + A(x)u^+ + B(x)u^- + H_3(\lambda, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega 
\end{cases} \quad (22) \]
as a bifurcation problem from the trivial solution \( u \equiv 0 \), where \( A(x) = \alpha(x)/M(0) \) and \( B(x) = \beta(x)/M(0) \) are given in Remark 2 by Remark 1, and
\[ \begin{cases} 
-\Delta u = \frac{\lambda r a(x) f_\infty}{m_1} u + A(x)u^+ + B(x)u^- + H_4(\lambda, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega 
\end{cases} \quad (23) \]
as a bifurcation problem from infinity, where \( A(x) = \alpha(x)/m_1, B(x) = \beta(x)/m_1 \) are given in Remark 2 by (A2).

We add the points \( \{(\lambda, \infty) | \lambda \in \mathbb{R}\} \) to space \( \mathbb{R} \times X \). We note that the problem (22) and the problem (23) are the same, and each of them is equivalent to the problem (18). By Theorem 3.3 and Remark 5, and the results of Rabinowitz [30], we have the following Lemma.
Lemma 4.1. Let (A1), (A2), (H1), (H4), (H5), (H6) and (H7) hold. \((\lambda_{r_f0} M(0), 0)\) and \((\lambda_{r_f\infty} m_1, \infty)\) are bifurcation points for the problem (18). Moreover, there are two distinct unbounded sub-continua \(\mathcal{D}^+\) and \(\mathcal{D}^-\) of solutions to problem (18), bifurcating from \((\lambda_{r_f0} M(0), 0)\) or \((\lambda_{r_f\infty} m_1, \infty)\). For \(\nu = +, -\), \(\mathcal{D}^\nu\) joins \((\lambda_{r_f0} M(0), 0)\) to \((\lambda_{r_f\infty} m_1, \infty)\), such that \(\mathcal{D}^\nu (\lambda_{r_f0} M(0), 0)\) and \(\mathcal{D}^\nu (\lambda_{r_f\infty} m_1, \infty)\).

Remark 6. Any solution of (18) of the form \((1, u)\) yields a solution \(u\) of (8). In order to prove our main results, one will only show that \(\mathcal{D}^\nu\) crosses the hyperplane \(\{1\} \times X\) in \(\mathbb{R} \times X\).

The main results of this section are the following theorem.

Theorem 4.2. Let (A1), (A2), (H1), (H4), (H5), (H6) and (H7) hold. For \(\nu = +, -\) and \(r \in (\min\{\frac{\lambda_{r_f0}}{\infty}, \lambda_{r_f0} M(0)\}, \max\{\frac{\lambda_{r_f\infty}}{m_1}, \lambda_{r_f0} M(0)\})\). Then the problem (8) possesses two solutions \(u^+, u^-\) such that \(nu^\nu > 0\).

Remark 7. From (H6) and (H7), we can see that there exists a positive constant \(Q\) such that \(\frac{f(s)}{s} \geq Q\) for all \(s \neq 0\).

Proof of Theorem 4.2. It is clear that any solution of (18) of the form \((1, u)\) yields solutions \(u\) of (8). We will show that \(\mathcal{D}^\nu\) crosses the hyperplane \(\{1\} \times X\). To do this, it is enough to show that \(\mathcal{D}^\nu\) joins \((\lambda_{r_f0} M(0), 0)\) to \((\lambda_{r_f\infty} m_1, \infty)\). Let \((\mu_n, u_n) \in \mathcal{D}^\nu \setminus \{\lambda_{r_f0} M(0), 0\}\) satisfy

\[|\mu_n| + \|u_n\| \to +\infty.\]

We divide the rest proof into two steps.

Step 1. We show that if there exists a constant number \(M > 0\) such that \(\mu_n \in (0, M)\) for \(n \in \mathbb{N}\) large enough, then \(\mathcal{D}^\nu\) joins \((\lambda_{r_f0} M(0), 0)\) to \((\lambda_{r_f\infty} m_1, \infty)\).

In this case, it follows that

\[\|u_n\| \to +\infty.\]

Let \(\xi \in C(\mathbb{R}, \mathbb{R})\) be such that

\[f(u) = f_\infty u + \xi(u)\]

with

\[\lim_{u \to +\infty} \frac{\xi(u)}{u} = 0, \quad \text{and} \quad \lim_{u \to 0^+} \frac{\xi(u)}{u} = f_\infty - f_0\]

uniformly in \(\Omega\), in \(\Omega\).

We divide the equation

\[\begin{cases}
-\Delta u_n = \frac{\mu_n r a(x) f_\infty}{M(\|u_n\|^2)} u_n + \frac{\alpha(x)}{M(\|u_n\|^2)} u_n^+ + \frac{\beta(x)}{M(\|u_n\|^2)} u_n^- + \frac{\mu_n r a(x) \xi(u_n)}{M(\|u_n\|^2)}, & \text{in } \Omega, \\
u_n = 0, & \text{on } \partial\Omega,
\end{cases}\]

by \(\|u_n\|\) and set \(v_n = \frac{u_n}{\|u_n\|}\). Since \(v_n\) is bounded in \(X\), choosing a subsequence and relabeling if necessary, we have that \(v_n \to v\) for some \(v \in X\) and \(v_n \to v\) in \(L^2(\Omega)\).

It follows from (24) that for any \(\varepsilon > 0\), there exists a constant \(C\) such that

\[|\xi(u_n)| \leq C + \varepsilon |u_n|.\]

By (25) and [12, p. 775-776], we can easily show that

\[\lim_{n \to \infty} \frac{\xi(u_n)}{\|u_n\|} = 0, \quad \text{in } L^2(\Omega).\]
By the compactness of $L^{-1}$, we obtain that
\[
\begin{cases}
-\Delta v = \frac{\mu r a(x) f_\infty}{m_1} v + A(x)v^+ + B(x)v^-, & \text{in } \Omega, \\
v = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where $\mu := \lim_{n \to \infty} \mu_n$, again choosing a subsequence and relabeling if necessary. Thus, by (A1), (A2), (H1) and (H4), it follows that

Theorem 4.3.

By the compactness of $L^{-1}$, we get that
\[
\mu = \frac{\lambda^\nu}{r f_\infty m_1}.
\]
Thus $\mathcal{D}^\nu$ joins $\left(\frac{\lambda^\nu}{r f_\infty} a(0), 0\right)$ to $\left(\frac{\lambda^\nu}{r f_\infty} m_1, +\infty\right)$.

Step 2. We show that there exists a constant number $M > 0$ such that $|\mu_n| \in (0, M]$, for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that $\lim_{n \to +\infty} |\mu_n| = +\infty$. Since $(\mu_n, u_n) \subset \mathcal{D}^\nu$, it follows from the compactness of $L^{-1}$ that
\[
\begin{cases}
-\Delta u_n = \frac{\mu_n r a(x) f_n(x)}{M(\|u\|^2)} u_n + \frac{\alpha(x)}{M(\|u\|^2)} u_n^+ + \frac{\beta(x)}{M(\|u\|^2)} u_n^-, & \text{in } \Omega, \\
u_n = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where
\[
\tilde{f}_n(x) = \begin{cases}
f(u_n(x)), & u_n(x) \neq 0, \\
0, & u_n(x) = 0.
\end{cases}
\]
By Remark 7, we have
\[
\lim_{n \to +\infty} \mu_n r \tilde{f}_n(x) = \pm \infty.
\]
Furthermore, by (A1), (A2), (H1) and (H4), it follows that
\[
\lim_{n \to +\infty} f_{1n}(x) = \pm \infty,
\]
where $\left[\frac{\mu_n r a(x) f_n(x)}{M(\|u\|^2)} + \frac{\alpha(x)}{M(\|u\|^2)} u_n^+ + \frac{\beta(x)}{M(\|u\|^2)} u_n^- \right] u_n = f_{1n}(x) u_n$. Let $\psi^\nu$ be an eigenfunction corresponding to $\lambda^\nu$ and
\[
\frac{\lambda^\nu r a(x)}{M(\|u\|^2)} + \frac{\alpha(x)}{M(\|\psi^\nu\|^2)} \frac{(\psi^\nu)^+}{\psi^\nu} + \frac{\beta(x)}{M(\|\psi^\nu\|^2)} \frac{(\psi^\nu)^-}{\psi^\nu}, \quad \text{where } \psi^\nu = f_2(x) \psi^\nu.
\]
But if $\lim_{n \to +\infty} f_{1n}(x) = -\infty$, applying Lemma 2.7 to $u_n$ and $\psi^\nu$ we have that $\psi^\nu$ must change sign for $n$ large enough, which is impossible. So $\lim_{n \to +\infty} f_{1n}(x) = +\infty$. By Lemma 2.7, we get that $u_n$ must change sign for $n$ large enough, and this contradicts the fact that $u_n \in P^\nu$. The result follows.

Theorem 4.3. Let (A1), (A2), (H1), (H4), (H5), (H6) and (H8) hold. For $\nu \in \{+, -, \}$, assume that one of the following conditions holds.

(i) $r \in (0, \frac{\lambda^\nu}{r_0} M(0))$ for $\lambda^\nu > 0$.

(ii) $r \in (0, \frac{\lambda^\nu}{r_0} M(0)) \cup (\frac{\lambda^\nu}{r_0} M(0), 0)$ for $\nu \lambda^\nu > 0$.

(iii) $r \in (0, \frac{\lambda^\nu}{r_0} M(0)) \cup (\frac{\lambda^\nu}{r_0} M(0), 0)$ for $\nu \lambda^\nu < 0$.

(iv) $r \in (0, \frac{\lambda^\nu}{r_0} M(0), 0)$ for $\lambda^\nu < 0$.

Then the problem (8) possesses two solutions $u^+, u^-$ such that $\nu \lambda^\nu > 0$.

Proof. We only prove the case of (i) since the proof of (ii)-(iv) can be given similarly.
Inspired by the idea of [3], we define the cut-off function of \( f \) as the following

\[
f^{[n]}(s) := \begin{cases} 
  ns, & s \in (-\infty, -2n] \cup [2n, +\infty), \\
  \frac{2n^2 + f(-n)}{n}(s + n) + f(-n), & s \in (-2n, -n), \\
  \frac{2n^2 - n}{n}(s - n) + f(n), & s \in (n, 2n), \\
  f(s), & s \in [-n, n].
\end{cases}
\]

We consider the following problem

\[
\begin{align*}
-M(\int_{\Omega} |\nabla u|^2 \, dx)\Delta u &= \alpha(x)u^+ + \beta(x)u^- + ra(x)f^{[n]}(u), & \text{in } \Omega, \\
u &= 0, & \text{on } \partial\Omega.
\end{align*}
\]

(26)

Clearly, we can see that \( \lim_{n \to +\infty} f^{[n]}(s) = f(s), \) \( (f^{[n]})_0 = f_0 \) and \( (f^{[n]})_\infty = n. \)

Similar the proof of Theorem 4.2, there exists an unbounded continuum \( \mathcal{P}^{[n]} \) of solutions of the problem (26) emanating from \( (\frac{\lambda}{f_0} M(0), 0) \), such that \( \mathcal{P}^{[n]} \subset ((\mathbb{R} \times P^\nu) \cup \{ (\frac{\lambda}{f_0} M(0), 0) \}) \), and \( \mathcal{P}^{[n]} \) joins \( (\frac{\lambda}{f_0} M(0), 0) \) to \( (\frac{\lambda}{f_0} m_1, \infty) \).

Taking \( z_n = (\frac{\lambda}{f_0} m_1, \infty) \) and \( z^* = (0, \infty) \), we have that \( z_n \to z^* \).

So condition (i) in Lemma 2.9 is satisfied with \( z^* = (0, \infty) \).

Obviously

\[ r_n = \sup \left\{ \lambda + \|u\|((\lambda, x) \in \mathcal{P}^{[n]} \right\} = \infty, \]

and accordingly, (ii) in Lemma 2.9 holds. (iii) in Lemma 2.9 can be deduced directly from the Arezela-Ascoli Theorem and the definition of \( f^{[n]} \).

Therefore, by Lemma 2.9, \( \limsup_{n \to +\infty} \mathcal{P}^{[n]} \) contains an unbounded component \( \mathcal{P}^\nu \) emanating from \( (\frac{\lambda}{f_0} M(0), 0) \) and joins to \( (0, \infty) \).

From \( \lim_{n \to +\infty} f^{[n]}(s) = f(s), \) (26) can be converted to the equivalent equation (18). Since \( \mathcal{P}^{[n]} \subset (\mathbb{R} \times P^\nu) \), we conclude \( \mathcal{P}^\nu \subset (\mathbb{R} \times P^\nu) \). Moreover, \( \mathcal{P}^\nu \subset \sum^\nu \) by (8).

Thus, there is an unbounded component \( \mathcal{P}^\nu \) of the problem (8) emanating from \( (\frac{\lambda}{f_0} M(0), 0) \) and joins to \( (0, \infty) \). \( \square \)

**Theorem 4.4.** Let (A1), (A2), (H1), (H4), (H5), (H6) and (H9) hold. For \( \nu \in \{+, -\} \), assume that one of the following conditions holds.

(i) \( r \in (0, \frac{\lambda}{f_\infty} m_1) \) for \( \lambda^\nu > 0. \)

(ii) \( r \in (\frac{\lambda}{f_\infty} m_1, 0) \cup (0, \frac{\lambda}{f_\infty} m_1) \) for \( \nu \lambda^\nu > 0. \)

(iii) \( r \in (\frac{\lambda}{f_\infty} m_1, 0) \cup (0, \frac{\lambda}{f_\infty} m_1) \) for \( \nu \lambda^\nu < 0. \)

(iv) \( r \in (\frac{\lambda}{f_\infty} m_1, 0) \) for \( \lambda^\nu < 0. \)

Then the problem (8) possesses two solutions \( u^+, u^- \) such that \( \nu \lambda^\nu > 0. \)

**Proof.** We shall only prove the case of (i) since the proofs of the cases for (ii), (iii) and (iv) are completely analogous.

If \((\lambda, u)\) is any nontrivial solution of problem (18), dividing problem (18) by \( \|u\|^2 \) and setting \( v = \frac{u}{\|u\|^2} \) yields

\[
\begin{align*}
-M(\int_{\Omega} |\nabla u|^2 \, dx)\Delta v &= \alpha(x)v^+ + \beta(x)v^- + \lambda ra(x)\frac{f(u)}{\|u\|^2}, & \text{in } \Omega, \\
v &= 0, & \text{on } \partial\Omega.
\end{align*}
\]

(27)
Define
\[ \bar{f}(v) := \begin{cases} \|v\|^2 f\left(\frac{v}{\|v\|^2}\right), & \text{if } v \neq 0, \\ 0, & \text{if } v = 0 \end{cases} \]
and
\[ \bar{M}(\|v\|^2) := \begin{cases} M\left(\frac{1}{\|v\|^2}\right), & \text{if } v \neq 0, \\ m_1, & \text{if } v = 0. \end{cases} \]

Evidently, problem (27) is equivalent to
\[ \begin{cases} -\bar{M}(\|v\|^2)\Delta v = \alpha(x)v^+ + \beta(x)v^- + \lambda ra(x)\bar{f}(v), & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \]  \tag{28}

It is obvious that \((\lambda\bar{M}(0), 0)\) is always the solution of problem (28). By simple computation, we can show that \(\bar{f}_0 = f_\infty\) and \(\bar{f}_\infty = f_0\). Now, applying Theorem 4.3, there exists an unbounded continuum \(\mathcal{C}'\) of solutions of the problem (28) emanating from \((\frac{\lambda}{r_0}\bar{M}(0)), 0)\), such that \(\mathcal{C}' \subset ((\mathbb{R} \times P\nu) \cup \{(\frac{\lambda}{r_0}\bar{M}(0), 0)\})\), and \(\mathcal{C}'\) joins \((\frac{\lambda}{r_0}\bar{M}(0), 0)\) to \((0, \infty)\).

Under the inversion \(v \to \frac{v}{\|v\|^2} = u\), we obtain \(\mathcal{C}' \to \mathcal{D}'\) being solutions of the problem (18) such that \(\mathcal{D}'\) joins \((0, 0)\) to \((\frac{\nu}{r_0}m_1, \infty)\). Moreover, \(\mathcal{D}' \subset \sum'\) by (8).

Thus, there is an unbounded component \(\mathcal{D}'\) of the problem (8) emanating from \((0, 0)\) and joins to \((\frac{\nu}{r_0}m_1, \infty)\).

**Theorem 4.5.** Let \((A1), (A2), (H1), (H4), (H5), (H6)\) and \((H10)\) hold. For \(\nu \in \{+, -\}\), assume that one of the following conditions holds.

(i) \(r \in (0, +\infty)\) for \(\lambda\nu > 0\).

(ii) \(r \in (0, +\infty) \cup (-\infty, 0)\) for \(\nu\lambda > 0\), or \(\nu\lambda < 0\).

(iii) \(r \in (-\infty, 0)\) for \(\lambda < 0\).

Then the problem (8) possesses two solutions \(u^+, u^-\) such that \(\nu x > 0\).

**Proof.** We shall only prove the case of (i) since the proofs of the cases for (ii), (iii) and (iv) are completely analogous. Define
\[ f^{[n]}(s) := \begin{cases} \frac{s}{n}, & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \frac{2}{n} - \frac{f(n)}{n} (s + n) + f(-n), & s \in (-2n, -n), \\ \frac{2}{n} - \frac{f(n)}{n} (s - n) + f(n), & s \in (n, 2n), \\ f(s), & s \in [-n, -\frac{2}{n}] \cup [\frac{2}{n}, n], \\ -\left[f\left(\frac{2}{n}\right) + 1\right] (ns + 2) + f\left(-\frac{2}{n}\right), & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ \left[f\left(\frac{2}{n}\right) - 1\right] (ns - 2) + f\left(\frac{2}{n}\right), & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ ns, & s \in \left[\frac{1}{n}, \frac{2}{n}\right]. \end{cases} \]

We consider the following problem
\[ \begin{cases} -M(\int_{\Omega} |\nabla u|^2 dx)\Delta u = \alpha(x)u^+ + \beta(x)u^- + ra(x)f^{[n]}(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \]  \tag{29}
Clearly, we can see that \( \lim_{n \to +\infty} f[n](s) = f(s) \), \( (f[n])_0 = n \) and \( (f[n])_\infty = \frac{1}{n} \).

Similar the proof of Theorem 4.2, by Lemma 4.1, there exists an unbounded continuum \( \mathcal{D}^{\nu}[n] \) of solutions of the problem (29) emanating from \( (\frac{\lambda}{n^\nu} M(0), 0) \) or \( (\frac{n\lambda}{\nu} m_1, \infty) \), such that \( \mathcal{D}^{\nu}[n] \subset (P^{\nu} \cup \{ (\frac{\lambda}{n^\nu}, 0) \}) \) and \( \mathcal{D} \subset (P^\nu \cup \{ (\frac{n\lambda}{\nu} m_1, \infty) \}) \), and \( \mathcal{D} \) joins \( (\frac{\lambda}{n^\nu} M(0), 0) \) to \( (\frac{n\lambda}{\nu} m_1, \infty) \).

Taking \( z_n = (\frac{\lambda}{n^\nu} M(0), 0) \) and \( z^* = (0, 0) \) or \( z_n = (\frac{n\lambda}{\nu} m_1, \infty) \) and \( z^* = (\infty, \infty) \), we have that \( z_n \to z^* \).

So condition (i) in Lemma 2.9 is satisfied with \( z^* = (0, 0) \) or \( z^* = (\infty, \infty) \).

Obviously
\[
r_n = \sup \left\{ \lambda + \|x\| | (\lambda, x) \in \mathcal{D}^{\nu}[n] \right\} = \infty,
\]
and accordingly, (ii) in Lemma 2.9 holds. (iii) in Lemma 2.9 can be deduced directly from the Arezela-Ascoli Theorem and the definition of \( f[n] \).

Therefore, by Lemma 2.9, \( \limsup_{n \to \infty} \mathcal{D}^{\nu}[n] \) contains an unbounded connected component \( \mathcal{D}^{\nu} \) being solutions of the problem (18) such that \( \mathcal{D}^{\nu} \) joins \( (0, 0) \) to \( (\infty, \infty) \). Moreover, \( \mathcal{D}^{\nu} \subset \sum^{\nu} \) by (8).

Thus, there is an unbounded component \( \mathcal{D}^{\nu} \) of the problem (8) emanating from \( (0, 0) \) and joins to \( (\infty, \infty) \). \( \square \)

**Theorem 4.6.** Let (A1), (A2), (H1), (H4), (H5), (H6) and (H11) hold. For \( \nu \in \{ +, - \} \), assume that one of the following conditions holds.

(i) There exists a \( \lambda^*_\nu > 0 \) for \( \lambda^* > 0 \), such that \( r \in (0, \lambda^*_\nu) \).

(ii) There exists a \( \nu \lambda^*_\nu > 0 \) for \( \nu \lambda^* > 0 \), such that \( r \in (\lambda^*_\nu, 0) \cup (0, \lambda^*_\nu) \).

(iii) There exists a \( \nu \lambda^*_\nu < 0 \) for \( \nu \lambda^* < 0 \), such that \( r \in (\lambda^*_\nu, 0) \cup (0, \lambda^*_\nu) \).

(iv) There exists a \( \lambda^*_\nu < 0 \) for \( \lambda^* < 0 \), such that \( r \in (\lambda^*_\nu, 0) \).

Then the problem (8) possesses two solutions \( u^+ \), \( u^- \) such that \( \nu u^\nu > 0 \).

**Proof.** Define

\[
f[n](s) := \begin{cases} ns, & s \in (-\infty, -2n] \cup [2n, +\infty), \\ 2n^2 + f(-n)(s + n) + f(-n), & s \in (-2n, -n), \\ 2n^2 - f(n)(s - n) + f(n), & s \in (n, 2n), \\ f(s), & s \in [-n, -\frac{2}{n}] \cup [\frac{2}{n}, n], \\ -\left[ f(-\frac{2}{n}) + 1 \right](ns + 2) + f(-\frac{2}{n}), & s \in (-\frac{2}{n}, -\frac{1}{n}), \\ \left[ f(\frac{2}{n}) + 1 \right](ns - 2) + f(\frac{2}{n}), & s \in (\frac{2}{n}, \frac{1}{n}), \\ ns, & s \in [-\frac{1}{n}, \frac{1}{n}]. \end{cases}
\]

We consider the following problem

\[
\begin{array}{ll}
-\mathcal{M} \int |\nabla u|^2 dx \Delta u = \alpha(x)u^+ + \beta(x)u^- + ra(x)f[n](u), & \text{in } \Omega, \\
u u = 0, & \text{on } \partial \Omega.
\end{array}
\]

(30)

Clearly, we can see that \( \lim_{n \to +\infty} f[n](s) = f(s) \), \( (f[n])_0 = n \) and \( (f[n])_\infty = n \).

Similar the method of the proof of Theorem 4.5, there exists an unbounded continuum \( \mathcal{D}^{\nu}[n] \) of solutions of the problem (30) emanating from \( (\frac{\lambda}{n^\nu} M(0), 0) \) or
\( \frac{\lambda}{m_1}, \infty \), such that \( \mathcal{G}^{[n]} \subset ((\mathbb{R} \times P^n) \cup \{(\frac{\lambda}{m_1}, M(0), 0)\}) \) and \( \mathcal{G}^{[n]} \subset ((\mathbb{R} \times P^n) \cup \{(\frac{\lambda}{m_1}, 0), 0\}) \), and \( \mathcal{G}^{[n]} \) joins \((\frac{\lambda}{m_1}, M(0), 0)\) to \((\frac{\lambda}{m_1}, \infty)\).

By Lemma 2.9, we obtain an unbounded component \( \mathcal{G} \subset \mathcal{P} \) with \((0, 0), (0, \infty) \in \mathcal{G}^{[n]} \).

Moreover, we can obtain the desired results.

**Remark 8.** In the case of \( \alpha = \beta = 0 \), Dai et al. [12] studied the existence of positive solutions for the problem (8) under the assumptions \((H_6)\) and \((H_7)\). In this sense, Theorems 4.2-4.6 extends and improves Theorem 1.2 of [12].

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