Remarks on curvature dimension conditions on graphs

Florentin Münch

January 26, 2015

Abstract

We show a connection between the CDE’ inequality introduced in [4] and the CDψ inequality established in [5]. In particular, we introduce a CDϕψ inequality as a slight generalization of CDψ which turns out to be equivalent to CDE’ with appropriate choices of φ and ψ. We use this to prove that the CDE’ inequality implies the classical CD inequality on graphs, and that the CDE’ inequality with curvature bound zero holds on Ricci-flat graphs.

1 Introduction

There is an immense interest in the heat equation on graphs. In this context, curvature-dimension conditions have attracted particular attention. In particular, recent works [2][4][5] have introduced a variety of such conditions. In this note, we will extend ideas of [5] to show a connection between them (Proposition 2.3 and Section 3). Moreover, we will prove that Ricci-flat graphs satisfy the CDE’ condition (Section 4).

Throughout the note, we will use notation and definitions introduced in [1][2][3][4][5] which can also be found in the appendix.

Acknowledgements

I wish to thank Matthias Keller and Daniel Lenz for their support and for sharing ideas in many fruitful discussions.

2 The connection between the CDE’ and the CDψ inequality

First, we consider the connection between Γ (cf. Definition A.6) and Γψ (cf. Definition A.11), and between ˜Γ2 (cf. Definition A.8) and Γϕψ 2 (cf. Definition A.12).

Lemma 2.1. For all f ∈ C+(V),
\[ fΓ\sqrt[f]{f} = \Gamma(\sqrt{f}), \]
\[ fΓ_2\sqrt[f]{f} = ˜Γ_2(\sqrt{f}). \]

Proof. Let f ∈ C+(V) and x ∈ V. Then for the proof of (2.1),
\[ 2 \left[ fΓ\sqrt[f]{f} \right](x) = 2f(x) \left[ \frac{\Delta f}{2f} - \sqrt{\frac{f}{f(x)}} \right](x) = [\Delta f - 2\sqrt{f}\Delta\sqrt{f}](x) = 2\Gamma(\sqrt{f})(x). \]
Next, we prove (2.2). In [2, (4.7)], it is shown that for all positive solutions $u \in C^1(V \times \mathbb{R}_0^+)$ to the heat equation, one has

$$2\tilde{\Gamma}_2 \sqrt{u} = \mathcal{L}(\sqrt{u}).$$

Now, we set $u := P_t f$ and we apply the above proven identity (2.1) and the identity $2u\Gamma_2^\psi(u) = \mathcal{L}(u \Gamma^\psi(u))$ (cf. [5, Subsection 3.2]) to obtain

$$2\tilde{\Gamma}_2(\sqrt{f}) = \left[\mathcal{L}(\Gamma(\sqrt{u}))\right]_{t=0} = \left[\mathcal{L}(u \Gamma^\psi(u))\right]_{t=0} = 2f \Gamma_2^\psi(f).$$

This finishes the proof.

The following definition extends the $CD\psi$ inequality to compare it to the $CDE'$ inequality.

**Definition 2.2 ($CD_\psi^\phi$ condition).** Let $d \in (0, \infty]$ and $K \in \mathbb{R}$. Let $\varphi, \psi \in C^1(\mathbb{R}^+)$ be concave functions. A graph $G = (V,E)$ satisfies the $CD_\psi^\phi(d,K)$ condition, if for all $f \in C^+(V)$,

$$\Gamma_2^\psi(f) \geq \frac{1}{d} \left(\Delta^\varphi f\right)^2 + K \Gamma^\psi(f).$$

Indeed, this definition is an extension of $CD\psi$ which is equivalent to $CD_\psi^\phi$.

**Proposition 2.3.** Let $G = (V,E)$ be a graph, let $d \in (0, \infty]$ and $K \in \mathbb{R}$. Then, the following statements are equivalent.

(i) $G$ satisfies the $CDE'(d,K)$ inequality.

(ii) $G$ satisfies the $CD_{\sqrt{\cdot}}(4d,K)$ inequality.

**Proof.** By definition, the $CDE'(d,K)$ inequality is equivalent to

$$\tilde{\Gamma}_2(f) \geq \frac{1}{d} f^2 (\Delta \log f)^2 + K\Gamma (f), \quad f \in C^+(V).$$

By replacing $f$ by $\sqrt{f}$ (all allowed $f \in C(V)$ are strictly positive), this is equivalent to

$$\tilde{\Gamma}_2(\sqrt{f}) \geq \frac{1}{d} f \left(\Delta \log \sqrt{f}\right)^2 + K\Gamma (\sqrt{f}), \quad f \in C^+(V).$$

By applying Lemma 2.1 and the fact that $\Delta \log = \Delta \circ \log$, this is equivalent to

$$f \Gamma_2^\psi(f) \geq \frac{1}{4d} f \left(\Delta^\varphi f\right)^2 + f K \Gamma^\psi(f), \quad f \in C^+(V).$$

By dividing by $f$ (all allowed $f \in C(V)$ are strictly positive), this is equivalent to

$$\Gamma_2^\psi(f) \geq \frac{1}{4d} \left(\Delta^\varphi f\right)^2 + K \Gamma^\psi(f), \quad f \in C^+(V).$$

By definition, this is equivalent to $CD_{\sqrt{\cdot}}^{\log}(4d,K)$. This finishes the proof.
3 The $CDE'$ inequality implies the $CD$ inequality

First, we recall a limit theorem [5, Theorem 3.18] by which it is shown that the $CD\psi$ condition implies the $CD$ condition (cf. [5, Corollary 3.20]).

**Theorem 3.1** (Limit of the $\psi$-operators). Let $G = (V, E)$ be a finite graph. Then for all $f \in C(V)$, one has the pointwise limits

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Delta \psi(1 + \varepsilon f) = \psi'(1)\Delta f \quad \text{for } \psi \in C^1(\mathbb{R}^+), \quad (3.1)
$$

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Gamma\psi(1 + \varepsilon f) = -\psi''(1)\Gamma f \quad \text{for } \psi \in C^2(\mathbb{R}^+), \quad (3.2)
$$

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Gamma_2\psi(1 + \varepsilon f) = -\psi''(1)\Gamma_2 f \quad \text{for } \psi \in C^2(\mathbb{R}^+). \quad (3.3)
$$

Since all $f \in C(V)$ are bounded, one obviously has $1 + \varepsilon f > 0$ for small enough $\varepsilon > 0$.

**Proof.** For a proof, we refer the reader to the proof of [5, Theorem 3.18].

By adapting the methods of the proof of [5, Corollary 3.20], we can show that $CD\psi$ implies $CD$ and, especially, we can handle the $CDE'$ condition.

**Theorem 3.2.** Let $\varphi, \psi \in C^2(\mathbb{R}^+)$ be concave with $\psi''(1) \neq 0 \neq \varphi'(1)$ and let $d \in \mathbb{R}^+$. Let $G = (V, E)$ be a graph satisfying the $CD\psi(d, K)$ condition. Then, $G$ also satisfies the $CD \left(\frac{-\psi''(1)}{\varphi'(1)}d, K\right)$ condition.

**Proof.** Let $f \in C(V)$. We apply [5, Theorem 3.18] in the following both equations and since $G$ satisfies the $CD\psi(d, K)$ condition,

$$
-\psi''(1)\Gamma_2 f = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Gamma_2\psi(1 + \varepsilon f) \geq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left(\frac{1}{d}[\Delta\psi(1 + \varepsilon f)]^2 + K\Gamma\psi(1 + \varepsilon f)\right) = \frac{\varphi'(1)^2}{d}(\Delta f)^2 - \psi''(1)K\Gamma(f).
$$

Since $\psi$ is concave and $\psi''(1) \neq 0$, one has $-\psi''(1) > 0$. Thus, we obtain that $G$ satisfies the $CD \left(\frac{-\psi''(1)}{\varphi'(1)}d, 0\right)$ condition.

**Corollary 3.3.** If $G = (V, E)$ satisfies the $CDE'(d, K)$, i.e., the $CD\log(4d, K)$, then $G$ also satisfies the $CD(d, K)$ condition since $-4\sqrt{\psi''(1)} = 1 = \log'(1)$.

4 The $CDE'$ inequality on Ricci-flat graphs

In [5], the $CDE'$ inequality is introduced. Examples for graphs satisfying this inequality have not been provided yet. In this section, we show that the more general $CD\psi$ condition holds on Ricci-flat graphs (cf. [5]). We will refer to the proof of the $CD\psi$ inequality on Ricci-flat graphs (cf. [5, Theorem 6.6]). Similarly to [5], we introduce a constant $C\psi$ describing the relation between the degree of the graph and the dimension parameter in the $CD\psi$ inequality.
Definition 4.1. Let $\varphi, \psi \in C^1(\mathbb{R})$. Then for all $x, y > 0$, we write

$$\tilde{\psi}(x, y) := [\psi'(x) + \psi'(y)](1 - xy) + x[\psi(y) - \psi(1/x)] + y[\psi(x) - \psi(1/y)]$$

and

$$C_{\psi}^\rho := \inf_{(x,y) \in A_\varphi} \frac{\tilde{\psi}(x, y)}{(\varphi(x) + \varphi(y) - 2\varphi(1))^2} \in [-\infty, \infty]$$

with $A_\varphi := \{(x, y) \in (\mathbb{R}^+)^2 : \varphi(x) + \varphi(y) \neq 2\varphi(1)\}$. We have $C_{\psi}^\rho = \infty$ iff $A_\varphi = \emptyset$.

Theorem 4.2 ($CD^\rho_\psi$ for Ricci-flat graphs). Let $D \in \mathbb{N}$, let $G = (V, E)$ be a $D$-Ricci-flat graph, and let $\psi, \varphi \in C^1(\mathbb{R}^+)$ be concave functions, such that $C_{\psi}^\rho > 0$. Then, $G$ satisfies the $CD^\rho_\psi(d, 0)$ inequality with $d = D/C_{\psi}^\rho$.

Proof. We can assume $\psi(1) = 0$ without loss of generality since $\Gamma_{\psi}^\rho, \Delta_{\psi}$ and $C_{\psi}$ are invariant under adding constants to $\psi$. Let $v \in V$ and $f \in C(V)$. Since $G$ is Ricci-flat, there are maps $\eta_1, \ldots, \eta_D : N(v) := \{v\} \cup \{w \sim v\} \rightarrow V$ as demanded in Definition 3.3. For all $i, j \in \{1, \ldots, D\}$, we denote $y := f(v), y_i := f(\eta_i(v)), y_{ij} := f(\eta_i(\eta_j(v)))$, $z_i := y_i/y, z_{ij} := y_{ij}/y_i$. We take the sequence of inequalities at the end of the proof of [5, Theorem 6.6]. First, we extract the inequality

$$2\Gamma_{\psi}^\rho(f)(v) \geq \frac{1}{2} \sum_i \tilde{\psi}(z_i, z_i)$$

with $\psi$ and for an adequate permutation $i \mapsto i'$.

Secondly instead of continuing this estimate as in the proof of [5, Theorem 6.6], we take the latter part applied with $\varphi$ instead of $\psi$ to see

$$\frac{1}{2} \sum_i |\varphi(z_i) + \varphi(z_i)|^2 \geq \frac{2}{D} |\Delta_{\psi} f(v)|^2.$$ 

Since $\tilde{\psi}(z_i, z_i) \geq C_{\psi}^\rho [\varphi(z_i) + \varphi(z_i)]^2$, we conclude

$$2\Gamma_{\psi}^\rho(f)(v) \geq \frac{2C_{\psi}^\rho}{D} |\Delta_{\varphi} f(v)|^2.$$ 

This finishes the proof. 

The above theorem reduces the problem, whether $CD^\rho_\psi$ holds on Ricci-flat graphs, to the question whether $C_{\psi}^\rho > 0$. By using this fact, we can give the example of the $CDE'$ condition on Ricci-flat graphs.

Example 4.3. Numerical computations indicate that $C_{\psi}^{\log \sqrt{\varphi}} > 0.1104$. Consequently by Theorem 4.2 $D$-Ricci-flat graphs satisfy the $CD^{\log \sqrt{\varphi}}(9.058d, 0)$ inequality and thus due to Proposition 2.3 also the $CDE'(2.265d, 0)$ inequality.

Now, we give an analytic estimate of $C_{\log \sqrt{\varphi}}$ by using methods similar to the proof of [5, Example 6.11] which shows $C_{\log \sqrt{\varphi}} \geq 1/2$. 

4
Lemma 4.4. $C_{\sqrt{x}}^{\log} \geq 1/16 = 0.0625$.

Proof. For $\psi = \sqrt{\cdot}$, we write

$$\tilde{\sqrt{x}}(x, y) = \tilde{\psi}(x, y) = [\psi'(x) + \psi'(y)] (1 - xy) + x[\psi(y) - \psi(1/x)] + y[\psi(x) - \psi(1/y)]$$

$$= \left[\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\right] (1 - xy) + x \left[\sqrt{y} - \frac{1}{\sqrt{x}}\right] + y \left[\sqrt{x} - \frac{1}{\sqrt{y}}\right]$$

$$= \frac{\sqrt{x} + \sqrt{y}}{2} \left(\frac{1}{\sqrt{xy}} - \sqrt{xy}\right) + \left(\sqrt{x} + \sqrt{y}\right) (\sqrt{xy} - 1)$$

$$= \frac{\sqrt{x} + \sqrt{y}}{2} \cdot \left((xy)^{1/4} - (xy)^{-1/4}\right)^2$$

$$\geq (xy)^{1/4} \cdot \left((xy)^{1/4} - (xy)^{-1/4}\right)^2.$$ 

Hence by substituting $e^{2t} := (xy)^{1/4}$,

$$\frac{\tilde{\sqrt{x}}(x, y)}{(\log x + \log y)^2} \geq (xy)^{1/4} \cdot \left((xy)^{1/4} - (xy)^{-1/4}\right)^2 = e^{2t} \cdot \left(\frac{e^{2t} - e^{-2t}}{8t}\right)^2 = \left(\frac{e^{3t} - e^{-t}}{8t}\right)^2.$$ 

We expand the fraction to

$$\frac{e^{3t} - e^{-t}}{8t} = \frac{e^{3t} - e^{-t}}{e^t - e^{-t}} \cdot \frac{e^t - e^{-t}}{8t}.$$ 

Moreover, 

$$\frac{e^{3t} - e^{-t}}{e^t - e^{-t}} = e^{2t} + 1 \geq 1$$

and, by the estimate $\frac{\sinh t}{t} \geq 1$,

$$\frac{e^t - e^{-t}}{8t} \geq 1/4.$$ 

Putting together the above estimates yields

$$C_{\sqrt{x}}^{\log} = \inf_{x, y > 0, xy \neq 1} \frac{\tilde{\sqrt{x}}(x, y)}{(\log x + \log y)^2} \geq (1/4)^2 = 1/16.$$ 

This finishes the proof. 

A Appendix

Definition A.1 (Graph). A pair $G = (V, E)$ with a finite set $V$ and a relation $E \subset V \times V$ is called a finite graph if $(v, v) \notin E$ for all $v \in V$ and if $(v, w) \in E$ implies $(w, v) \in E$ for $v, w \in V$. For $v, w \in V$, we write $v \sim w$ if $(v, w) \in E$. 

5
Definition A.2 (Laplacian $\Delta$). Let $G = (V, E)$ be a finite graph. The Laplacian $\Delta : C(V) := \mathbb{R}^V \to C(V)$ is defined for $f \in C(V)$ and $v \in V$ as $\Delta f(v) := \sum_{w \sim v} (f(w) - f(v))$.

Definition A.3. We write $\mathbb{R}^+ := (0, \infty)$ and $\mathbb{R}^+_0 := [0, \infty)$. Let $G = (V, E)$ be a finite graph. Then, we write $C^+ := \{ f : V \to \mathbb{R}^+ \}$.

Definition A.4 (Heat operator $\mathcal{L}$). Let $G = (V, E)$ be a graph. The heat operator $\mathcal{L} : C^1(V \times \mathbb{R}^+) \to C(V \times \mathbb{R}^+)$ is defined by $\mathcal{L}(u)(x, t) := \Delta u - \partial_t u$ for all $u \in C^1(V \times \mathbb{R}^+)$. We call a function $u \in C^1(V \times \mathbb{R}^+_0)$ a solution to the heat equation on $G$ if $\mathcal{L}(u) = 0$.

Definition A.5 (Ricci-flat graphs). Let $D \in \mathbb{N}$. A finite graph $G = (V, E)$ is called $D$-Ricci-flat in $v \in V$ if all $w \in N(v) := \{v\} \cup \{w \in V : w \sim v\}$ have the degree $D$, and if there are maps $\eta_1, \ldots, \eta_D : N(v) \to V$, such that for all $w \in N(v)$ and all $i, j \in \{1, \ldots, D\}$ with $i \neq j$, one has $\eta_i(w) \sim w$, $\eta_i(w) \neq \eta_j(w)$, $\bigcup_k \eta_k(\eta_k(v)) = \bigcup_k \eta_i(\eta_i(v))$. The graph $G$ is called $D$-Ricci-flat if it is $D$-Ricci-flat in all $v \in V$.

### A.1 The CD condition via $\Gamma$ calculus

We give the definition of the $\Gamma$-calculus and the CD condition following [1].

Definition A.6 (\(\Gamma\)-calculus). Let $G = (V, E)$ be a finite graph. Then, the gradient form or carré du champ operator $\Gamma : C(V) \times C(V) \to C(V)$ is defined by

$$2\Gamma(f, g) := \Delta(fg) - f\Delta g - g\Delta f.$$ 

Similarly, the second gradient form $\Gamma_2 : C(V) \times C(V) \to C(V)$ is defined by

$$2\Gamma_2(f, g) := \Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(g, \Delta f).$$

We write $\Gamma(f) := \Gamma(f, f)$ and $\Gamma_2(f) := \Gamma_2(f, f)$.

Definition A.7 (CD($d, K$) condition). Let $G = (V, E)$ be a finite graph and $d \in \mathbb{R}^+$. We say $G$ satisfies the curvature-dimension inequality $\text{CD}(d, K)$ if for all $f \in C(V)$,

$$\Gamma_2(f) \geq \frac{1}{d} (|\Delta f|^2 + K \Gamma(f)).$$

We can interpret this as meaning that the graph $G$ has a dimension (at most) $d$ and a Ricci curvature larger than $K$.

### A.2 The CDE and CDE' conditions via $\tilde{\Gamma}_2$

We give the definitions of CDE and CDE' following [2] [4].

Definition A.8 (The CDE inequality). We say that a graph $G = (V, E)$ satisfies the CDE($x, d, K$) inequality if for any $f \in C^+(V)$ such that $\Delta f < 0$, we have

$$\tilde{\Gamma}_2(f)(x) := \Gamma_2(f)(x) - \Gamma \left( f, \frac{\Gamma(f)}{f} \right) (x) \geq \frac{1}{d} (|\Delta f|^2 (x) + K \Gamma(f)(x)).$$

We say that $\text{CDE}(d, k)$ is satisfied if $\text{CDE}(x, d, K)$ is satisfied for all $x \in V$.

Definition A.9 (The CDE' inequality). We say that a graph $G = (V, E)$ satisfies the CDE'($d, K$) inequality if for any $f \in C^+(V)$, we have

$$\tilde{\Gamma}_2(f) \geq \frac{1}{d} f^2 (|\Delta \log f|^2 + K \Gamma(f)).$$
A.3 The CDψ conditions via Γψ calculus

We give the definition of the Γψ-calculus and the CDψ condition following [5].

Definition A.10 (ψ-Laplacian ∆ψ). Let ψ ∈ C^1(\mathbb{R}^+) and let G = (V, E) be a finite graph. Then, we call ∆ψ : C^+(V) → C(V), defined as

\[(\Delta \psi f)(v) := \left( \Delta \left[ \psi \left( \frac{f}{f(v)} \right) \right] \right)(v),\]

the ψ-Laplacian.

Definition A.11 (ψ-gradient Γψ). Let ψ ∈ C^1(\mathbb{R}^+) be a concave function and let G = (V, E) be a finite graph. We define

\[\overline{\psi}(x) := \psi'(1) \cdot (x - 1) - (\psi(x) - \psi(1)).\]

Moreover, we define the ψ-gradient as Γψ : C^+(V) → C(V),

\[\Gamma \psi := \Delta \overline{\psi}.\]

Definition A.12 (Second ψ-gradient Γψ^2). Let ψ ∈ C^1(\mathbb{R}^+), and let G = (V, E) be a finite graph. Then, we define Ωψ : C^+(V) → C(V) by

\[(\Omega \psi f)(v) := \left( \Delta \left[ \psi' \left( \frac{f}{f(v)} \right) \right] \cdot \frac{f}{f(v)} \left[ \frac{\Delta f}{f} - \frac{\Delta \psi f}{f(v)} \right] \right)(v).\]

Furthermore, we define the second ψ-gradient Γψ^2 : C^+(V) → C(V) by

\[2\Gamma \psi^2(f) := \Omega \psi f + \Delta f \Delta \psi f - \Delta \left( \frac{\Delta \psi f}{f} \right).\]

Definition A.13 (CDψ condition). Let G = (V, E) be a finite graph, K ∈ \mathbb{R} and d ∈ \mathbb{R}^+. We say G satisfies the CDψ(d, K) inequality if for all f ∈ C^+(V), one has

\[\Gamma \psi^2(f) \geq \frac{1}{d} (\Delta \psi f)^2 + KT \psi f.\]

References

[1] D. Bakry, M. Émery, Diffusions hypercontractives. (French) [Hypercontractive diffusions/], Séminaire de probabilité, XIX, 1983/84, 177-206, Lecture Notes in Math., 1123, Springer, Berlin, 1985.

[2] F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi, S.-T. Yau, Li-Yau inequality on graphs, to appear in J. Differential Geom., Arxiv: 1306.2561v2 (2013).

[3] F. Chung, S.-T. Yau, Logarithmic Harnack inequalities, Math. Res. Lett. 3 (1996),

[4] P. Horn, Y. Lin, S. Liu, S.-T. Yau, Volume doubling, Poincaré inequality and Gaussian heat kernel estimate for nonnegative curvature graphs, Arxiv: 1411.5087v2 (2014).

[5] F. Münch, Li-Yau inequality on finite graphs via non-linear curvature dimension conditions, Arxiv: 1412.3340v1 (2014).