Tuza’s Conjecture for Graphs with Maximum Average Degree less than 7

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Abstract
Tuza’s Conjecture states that if a graph $G$ does not contain more than $k$ edge-disjoint triangles, then some set of at most $2k$ edges meets all triangles of $G$. We prove Tuza’s Conjecture for all graphs $G$ having no subgraph with average degree at least 7. As a key tool in the proof, we introduce a notion of reducible sets for Tuza’s Conjecture; these are substructures which cannot occur in a minimal counterexample to Tuza’s Conjecture. We also introduce weak König–Egerváry graphs, a generalization of the well-studied König–Egerváry graphs.

Keywords: Tuza’s Conjecture, packing and covering, triangle-free subgraphs, discharging

1. Introduction
Suppose that we wish to make a graph $G$ triangle-free by deleting a small number of edges. An obvious obstruction is the presence of a large family of edge-disjoint triangles: we must delete one edge from each such triangle. On the other hand, deleting all edges from a maximal family of edge-disjoint triangles clearly destroys all triangles in $G$. Let $\nu(G)$ denote the maximum size of a set of edge-disjoint triangles in $G$, and let $\tau(G)$ denote the minimum size of an edge set $Y$ such that $G - Y$ is triangle-free. We have just argued that $\nu(G) \leq \tau(G) \leq 3\nu(G)$. Clearly the lower bound is sharp, with equality in many instances, such as when all blocks are triangles. The desire to make the upper bound also sharp motivates the following conjecture:

Conjecture 1.1 (Tuza’s Conjecture [13, 14]). $\tau(G) \leq 2\nu(G)$ for all graphs $G$.

Any graph whose blocks are all isomorphic to $K_4$ achieves equality in the upper bound, as observed by Tuza [14].

Tuza’s Conjecture has been studied by many authors. We briefly review some results that are relevant to the results in this paper, making no pretense at a full review of all work related to the conjecture.

Tuza [14] showed that his conjecture holds for all planar graphs and for all $K_6$-free chordal graphs. Aparna Lakshmanan, Bujtás, and Tuza [9] generalized the result for planar graphs by showing that the conjecture holds for all...
“triangle-3-colorable” graphs, a class containing all 4-colorable graphs. Krivelevich [8] showed that Tuza’s Conjecture holds for all graphs having no $K_{3,3}$-minor.

Krivelevich [8] also proved that a version of Tuza’s Conjecture holds when $\tau$ or $\nu$ is replaced by its fractional relaxation. Haxell and Rödl [4] showed that if $G$ is an $n$-vertex graph and $\nu^*(G)$ is the fractional relaxation of $\nu(G)$, then $\nu^*(G) - \nu(G) = o(n^2)$. As observed by Yuster [17], these two results together imply $\tau(G) \leq 2\nu(G) + o(n^2)$; thus, Tuza’s Conjecture is asymptotically true for graphs containing a quadratic-sized family of edge-disjoint triangles. Such graphs are dense; instead, we study the conjecture on sparse graphs.

An important measure of sparseness is the maximum average degree of a graph, denoted $\text{Mad}(G)$ and defined by

$$\text{Mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|} : H \subseteq G \right\}.$$ 

In this paper, we apply the discharging method to prove the following theorem:

**Theorem 1.2.** If $\text{Mad}(G) < 7$, then $\tau(G) \leq 2\nu(G)$.

To our knowledge, this is the first application of the discharging method to Tuza’s Conjecture.

In Section 2 we introduce definitions and give the discharging argument used to prove Theorem 1.2 modulo two lemmas whose proof occupies most of the paper. The key definition in Section 2 is that of a reducible set, a particular substructure that cannot occur in a smallest counterexample to Tuza’s Conjecture. Essentially, a reducible set represents a “local solution” to the optimization problem posed by Tuza’s Conjecture.

The definition of a reducible set for Tuza’s Conjecture is perhaps the main new idea of the paper. While we use discharging to prove the existence of reducible sets, we hope that later work will be able to use these reducible sets in extremal arguments which may not involve discharging at all.

In Section 3 we discuss some consequences of Theorem 1.2. In particular, we show that Theorem 1.2 implies that Tuza’s Conjecture holds for toroidal graphs, for $K_{3,3}$-minor-free graphs, and for $K_5$-subdivision-free graphs. We also discuss how to extend the result to graphs of genus at most 2.

In Sections 4–7 we prove the two lemmas stated in Section 2. In Section 4 we introduce weak König–Egerváry graphs, which we use heavily in our removability proofs. In Section 5 we discuss the behavior of low-degree vertices in graphs with no reducible set.

The results in Sections 2–5 are sufficient to prove a weaker result than Theorem 1.2. Using these results, we can show that Tuza’s Conjecture holds for all graphs $G$ with $\text{Mad}(G) < 25/4$, a threshold which still suffices for many of the desired applications. In Section 6 we pause and sketch the proof of Tuza’s Conjecture for graphs $G$ with $\text{Mad}(G) < 25/4$.

In Section 7 we explore the relation of subsumption, which plays a prominent role in the discharging rule of Section 2 and allows us to push the maximum average degree threshold up to 7. We again explore the behavior of this relation in graphs with no reducible set.
2. Definitions and Proof Summary

When $G$ is a graph and $W \subseteq V(G)$, we write $G[W]$ for the subgraph of $G$ induced by the vertices in $W$. When $V_0 \subseteq V(G)$, we write $N(V_0)$ for $\bigcup_{v \in V_0} N(v)$, and when $U \subseteq V(G)$, we write $N_U(V_0)$ for $N(V_0) \cap U$. Similarly, $d_U(V_0)$ denotes $|N_U(V_0)|$. We write $K_n^-$ to denote the complete graph on $n$ vertices with any edge removed. When the graph $G$ is understood and $k$ is a nonnegative integer, we say that a vertex of $G$ is a $k$-vertex if its degree in $G$ is exactly $k$, a $k^+$-vertex if its degree is at least $k$, or a $k^-$-vertex if its degree is at most $k$.

While Tuza’s Conjecture involves two combinatorial optimization parameters, it can be also viewed as a single combinatorial optimization problem: in this problem, the goal is to simultaneously find a set $T$ of edge-disjoint triangles and an edge set $Y$ such that $G - Y$ is triangle-free and such that $|Y| \leq 2 |T|$. The requirement that $G - Y$ be triangle-free is a global requirement. We replace this global problem with a local problem: fixing a vertex set $V_0$, we seek a set $S$ of edge-disjoint triangles and an edge set $X$ such that $G - X$ has no triangle containing a vertex of $V_0$ and such that $|X| \leq 2 |S|$. The rough idea is to remove the vertex set $V_0$, solve the “global” problem in the resulting subgraph, and then combine the subgraph solution with the “local solution” to solve the global problem in the original graph. The main difficulty in combining solutions this way is the requirement that the final set of triangles be edge-disjoint; carelessly combining sets of triangles will violate this requirement. The definition of a reducible set is tailored to overcome this difficulty:

Definition 2.1. When $S$ is a set of triangles, an $S$-edge is an edge of some triangle in $S$. A nonempty set $V_0 \subseteq V(G)$ is reducible if there exist a set $S$ of edge-disjoint triangles in $G$ and set $X$ of edges in $G$ such that the following conditions hold:

(i) $|X| \leq 2 |S|$;

(ii) $G - X$ has no triangle containing a vertex of $V_0$; and

(iii) $X$ contains every $S$-edge whose endpoints are both outside $V_0$.

When $V_0$, $S$, and $X$ satisfy the definition above, we say that $V_0$ is reducible using $S$ and $X$.

Note that Tuza’s Conjecture holds for $G$ if and only if the entire vertex set $V(G)$ is reducible. However, if $G$ is a minimal counterexample to Tuza’s Conjecture, then $G$ has no reducible set of any size:

Lemma 2.2. Let $G$ be a graph, and let $V_0 \subseteq V(G)$ be reducible using $S$ and $X$. Let $G' = (G - X) - V_0$. If $\tau(G') \leq 2\nu(G')$, then $\tau(G) \leq 2\nu(G)$.

Proof. Let $T'$ be a largest set of edge-disjoint triangles in $G'$, and let $Y'$ be a smallest set of edges such that $G' - Y'$ is triangle-free; by hypothesis, $|Y'| \leq 2 |T'|$. Let $T = T' \cup S$ and $Y = Y' \cup X$. (The process is illustrated in Figure I.)
We show that $\mathcal{T}$ is a set of edge-disjoint triangles in $G$, that $G - Y$ is triangle-free, and that $|Y| \leq 2|\mathcal{T}|$, thus establishing the desired conclusion. The third condition is immediate from $|Y'| \leq 2|\mathcal{T}'|$ and $|X| \leq 2|S|$.

To show that the triangles in $\mathcal{T}'$ are pairwise edge-disjoint, it suffices to show that no $S$-edge is a $\mathcal{T}'$-edge. This holds because every $\mathcal{T}'$-edge is contained in $(G - X) - V_0$, while every $S$-edge is incident to $V_0$ or contained in $X$, by Condition (iii) of Definition 2.1.

Next we show that $G - Y$ is triangle-free. This holds because any triangle $T$ in $G$ satisfies one of the following three conditions:

(1) $T$ is contained in $(G - X) - V_0$; or

(2) $T$ contains a vertex of $V_0$; or

(3) $T$ contains an edge of $X$.

Triangles of the first type meet $Y'$, by hypothesis; triangles of the second type meet $X$, by Condition (ii) of Definition 2.1.

Our strategy for applying Lemma 2.2 is typical of discharging arguments: we show that various possible substructures of a graph $G$ imply the existence of a reducible set, and we show that every graph with average degree less than 7 has one of these substructures. For more background on the discharging method, see [16].

To give the list of forbidden substructures, a few new definitions are needed:

**Definition 2.3.** A graph $G$ is **robust** if for every $v \in V(G)$, every component of $G[N(v)]$ has order at least 5.

If $G$ is robust, then $\delta(G) \geq 5$. Also, $G[N(v)]$ is connected whenever $d(v) < 10$.

**Definition 2.4.** A vertex $u$ **subsumes** a vertex $v$ if $N[u] \supseteq N[v]$. 

Figure 1: Using a reducible set. Shaded triangles represent $S$ and $\mathcal{T}'$; thick edges represent $X$ and $Y'$; square white vertices represent $V_0$. 

(a) A reducible set. (b) $G' = (G - X) - V_0$.

(c) $\mathcal{T}'$ and $Y'$ in $G'$. (d) Combining $S, X$ with $\mathcal{T}', Y'$. 
Equivalently, $u$ subsumes $v$ if $u$ is a dominating vertex in $G[N(v)]$.

**Definition 2.5.** A 6-vertex $v$ is thin if $G[N(v)]$ contains a matching of size 3.

The full list of forbidden substructures is given by the following two lemmas; the proof of the second lemma will occupy most of the paper. For each part of the second lemma, we indicate which later results imply that part of the lemma.

**Lemma 2.6.** If $G$ is a minimal counterexample to Tuza’s Conjecture, then $G$ is robust.

**Lemma 2.7.** If $G$ is robust and has no reducible set, then the following conditions hold.

(a) Every $6^−$-vertex $v \in V(G)$ satisfies $\Delta(G[N(v)]) \leq 1$ and $\left|E(G[N(v)])\right| \neq 2$. \textit{(Proposition 5.1)}

(b) The $6^−$-vertices of $G$ form an independent set. \textit{(Proposition 5.2)}

(c) No 7-vertex subsumes any 6-vertex. \textit{(Lemma 7.6)}

(d) No 7-vertex is adjacent to any 6-vertex. \textit{(Lemma 7.6)}

(e) No 8^−-vertex subsumes any 5-vertex. \textit{(Lemma 7.6)}

(f) Every 8-vertex subsuming a 6-vertex has at most three $6^−$-neighbors. \textit{(Lemma 7.5)}

(g) Every 9-vertex subsumes at most three $6^−$-vertices, and a 9-vertex subsuming three $6^−$-vertices is adjacent to exactly three $6^−$-vertices. \textit{(Lemma 7.4)}

(h) Every $10^+$-vertex $v$ that subsumes some $6^−$-vertex has at most $d(v) - 6$ neighbors that are $6^−$-vertices. \textit{(Lemma 7.3)}

(i) Every vertex $v$ has at most $d(v) - 4$ neighbors that are $6^−$-vertices. \textit{(Corollary 5.3)}

Postponing the proof of Lemmas 2.6 and 2.7, we now give the proof of the main theorem.

**Lemma 2.8.** Every robust graph with average degree less than 7 has a reducible set.

**Proof.** Assuming that $G$ has no reducible set, we use the method of discharging to show that $G$ has average degree at least 7. Give every vertex $v$ initial charge $d(v)$. We apply the following discharging rule:

- Every 5-vertex takes charge $2/3$ from each vertex subsuming it;
- Every thin 6-vertex takes charge $1/6$ from each neighbor;
- Every non-thin 6-vertex takes charge $1/4$ from each vertex subsuming it.
We claim that every vertex has final charge at least 7, yielding average degree at least 7 in $G$.

First we consider the $6^-$-vertices. By part (b) of Lemma 2.7 no two such vertices are adjacent, so no $6^-$-vertex loses any charge when the discharging rule is applied. Thus we only need to check that each type of $6^-$-vertex gains enough charge to reach 7. There are no $4^-$-vertices, since $G$ is robust. By part (a) of Lemma 2.7 every $5^-$-vertex is subsumed by at least three vertices, and hence gains at least 2. Every thin $6^-$-vertex gains exactly 1, for final charge 7.

By part (a) of Lemma 2.7, the neighborhood of a non-thin $6^-$-vertex $v$ lacks at most one edge; hence $v$ is subsumed by at most four vertices, and gains at least 1.

Now we consider the higher-degree vertices. Each $7^-$-vertex starts with charge 7 and loses none, since it does not subsume any 5- or 6-vertices and is not adjacent to any thin 6-vertex, by parts (c)–(e) of Lemma 2.7.

Next, let $v$ be an $8^-$-vertex. By part (e) of Lemma 2.7, $v$ does not subsume any 5-vertices. If $v$ subsumes some $6^-$-vertex $w$, then $v$ subsumes at most three $6^-$-vertices, by part (f) of Lemma 2.7. Hence if $v$ subsumes some 6-vertex, then $v$ loses at most $3/4$ charge, yielding final charge greater than 7. On the other hand, by part (i) of Lemma 2.7 $v$ is adjacent to at most four $6^-$-vertices; hence, if $v$ subsumes no 6-vertex, then $v$ loses at most $4/6$ charge, yielding final charge greater than 7.

Now, let $v$ be a $9^-$-vertex. By part (i) of Lemma 2.7, $v$ has at most five $6^-$-neighbors in total. Hence, if $v$ subsumes at most two $6^-$-vertices, then $v$ loses at most $2(2/3) + 3(1/6)$ charge, yielding final charge greater than 7. On the other hand, if $v$ subsumes three $6^-$-vertices, then by part (f) of Lemma 2.7 we see that $v$ is adjacent to exactly those three $6^-$-vertices, so $v$ loses exactly $3(2/3)$ charge, yielding final charge 7.

Finally, let $v$ be a $k^-$-vertex with $k \geq 10$. If $v$ subsumes no $6^-$-vertex, then $v$ loses charge at most $k/6$, which yields final charge at least 7 since $k - k/6 \geq 7$. If $v$ subsumes some $6^-$-vertex, then at most $k - 6$ neighbors of $v$ are $6^-$-vertices, by part (b) of Lemma 2.7. Thus $v$ loses at most $2(k - 6)/3$, which yields final charge at least 7 since $k - 2(k - 6)/3 \geq 7$. Hence all vertices have final charge at least 7, yielding average degree at least 7.

**Theorem 1.2** If Mad($G$) < 7, then $\tau(G) \leq 2\nu(G)$.

**Proof.** If the claim fails, let $G$ be a minimal counterexample. Since Mad($G$) < 7, any proper subgraph $G'$ also satisfies Mad($G'$) < 7, so $\tau(G') \leq 2\nu(G')$ by the minimality of $G$. Thus, $G$ is a minimal counterexample to Tuza’s Conjecture among all graphs. By Lemma 2.6 $G$ is robust, so by Lemma 2.8 $G$ has a reducible set. Now Lemma 2.2 yields $\tau(G) \leq 2\nu(G)$, contradicting the choice of $G$ as a counterexample.

In the next section we explore some applications of Theorem 1.2 and its supporting lemmas. The remainder of the paper will then be devoted to proving Lemma 2.7.
3. Consequences

Several earlier results on Tuza’s Conjecture are natural consequences of Theorem 1.2. Tuza [14] proved that the conjecture holds for planar graphs. The following corollary of Euler’s Formula extends the result to toroidal graphs, which are the graphs of genus at most 1:

**Proposition 3.1.** If $G$ is an $n$-vertex graph of genus $\gamma$ with $m$ edges, then $m \leq 3(n - 2 + 2\gamma)$. In particular, $G$ has average degree at most $6 + \frac{12(\gamma - 1)}{n}$.

**Corollary 3.2.** If $G$ is toroidal, then $\tau(G) \leq 2\nu(G)$.

For higher genus, we obtain a finitization result.

**Proposition 3.3.** For any fixed $\gamma$ with $\gamma \geq 2$, if $\tau(G) \leq 2\nu(G)$ for all graphs $G$ of genus at most $\gamma$ with $|V(G)| \leq 12(\gamma - 1)$, then $\tau(G) \leq 2\nu(G)$ for all graphs $G$ of genus at most $\gamma$.

**Proof.** Suppose not; let $G$ be a minimal counterexample among the graphs of genus at most $\gamma$. All proper subgraphs $G'$ also have genus at most $\gamma$, so they satisfy $\tau(G') \leq 2\nu(G')$, by the minimality of $G$. By hypothesis, $|V(G)| > 12(\gamma - 1)$, so $G$ has average degree less than 7. By Lemma 2.6, $G$ is robust, so by Lemma 2.8 $G$ has a reducible set. Thus $\tau(G) \leq 2\nu(G)$, by Lemma 2.2, contradicting the choice of $G$ as a counterexample.

For the case $\gamma = 2$, we performed an exhaustive computer search to verify the hypothesis of Proposition 3.3. By using Lemma 2.7, we avoid explicitly checking Tuza’s Conjecture on graphs that can be shown to have reducible sets. Using the isomorph-free generation program geng [11] with a custom pruning function designed to recognize forbidden configurations (a) and (b) in Lemma 2.7, we identified a set of only 5299 graphs that contains any smallest counterexample of genus 2. (A database of these graphs, and tools for verifying the database, can be found at http://www.math.uiuc.edu/~puleo/tuzaverify.tar.gz) For higher $\gamma$, this computational approach quickly becomes intractible, even with such optimizations.

Krivelevich [8] proved Tuza’s Conjecture for graphs with no $K_{3,3}$-minor. We obtain this from Theorem 1.2. Our proof relies on a theorem of Wagner ([15], described in [12]).

**Definition 3.4.** Let $G_1$ and $G_2$ be graphs. A $k$-sum of $G_1$ and $G_2$ is any graph obtained by identifying the vertices of a $k$-clique in $G_1$ with a $k$-clique in $G_2$ and then possibly deleting some edges of the merged $k$-clique. (In particular, a 0-sum is a disjoint union.)

**Theorem 3.5** (Wagner [15]). Any graph with no $K_{3,3}$-minor can be obtained by a sequence of 0-, 1-, or 2-sums starting from planar graphs and $K_5$.

**Corollary 3.6.** If $G$ is a graph with $n$ vertices and no $K_{3,3}$-minor, then $|E(G)| \leq 3n - 5$. 7
Proof. If $G$ is planar or $G = K_5$, then the conclusion holds. It suffices to show that if $G_1$ and $G_2$ are graphs satisfying the bound, then any 0-, 1-, or 2-sum of $G_1$ and $G_2$ also satisfies the bound. This follows by straightforward algebra. In particular, for a $j$-sum of $G_1$ and $G_2$ with $j \in \{0, 1, 2\}$ and $n_i = |V(G_i)|$,

$$|E(G)| \leq |E(G_1)| + |E(G_2)| - \binom{j}{2}$$

$$\leq 3n_1 + 3n_2 - 10 - \binom{j}{2}$$

$$= 3n - 5 + \left(3j - \binom{j}{2} - 5\right) \leq 3n - 5.$$  

\[\square\]

**Theorem 3.7** (Krivelevich). If $G$ is a graph with no $K_{3,3}$-minor, then $\tau(G) \leq 2\nu(G)$.

Proof. Since all subgraphs of $G$ also have no $K_{3,3}$-minor, Corollary 3.6 implies $\text{Mad}(G) < 6$.  

Aparna Lakshmanan, Bujtás, and Tuza \cite{9} proved that if $G$ is 4-colorable, then $\tau(G) \leq 2\nu(G)$. This implies that Tuza’s Conjecture holds for all graphs with no $K_5$-minor, since (as Wagner \cite{15} showed) the Four-Color Theorem implies that all graphs with no $K_5$-minor are 4-colorable. Using a theorem of Mader \cite{10} together with Theorem 1.2, we instead obtain the result for graphs with no $K_5$-subdivision:

**Theorem 3.8.** If $G$ is a graph with no $K_5$-subdivision, then $\tau(G) \leq 2\nu(G)$.

Proof. Since $G$ has no $K_5$-subdivision, the number of edges in $G$ is at most $3|V(G)| - 6$, as proved by Mader \cite{10}. All subgraphs of $G$ are $K_5$-subdivision-free, so $\text{Mad}(G) < 6$.  

\[\square\]

4. Weak König–Egerváry Graphs

A graph $H$ is a *König–Egerváry graph* if $\alpha'(H) = \beta(H)$, where $\alpha'(H)$ is the matching number and $\beta(H)$ is the vertex cover number. The concept was introduced by Deming \cite{2}; see also Kayll \cite{6}. Let $\text{KE}$ denote the class of König–Egerváry graphs. The König–Egerváry Theorem \cite{3, 7} says that if $H$ is bipartite, then $H \in \text{KE}$. We weaken this definition, obtaining a larger class of graphs which will help streamline our removability proofs.

**Definition 4.1.** The graph $H$ is a *weak König–Egerváry graph* if $H$ has a matching $M$ and a vertex set $Q \subseteq V(H)$ such that $|Q| \leq |M|$ and $Q$ is a vertex cover in $H - M$. Let $\text{WKE}$ denote the class of weak König–Egerváry graphs. We say that a pair $(M, Q)$ as above *witnesses* $H \in \text{WKE}$.  

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Observe that $KE \subseteq WKE$: if $H \in KE$, then $(M, Q)$ witnesses $H \in WKE$, where $M$ is any maximum matching and $Q$ is any minimum vertex cover in $H$.

To relate weak König–Egerváry graphs to reducible sets, we introduce an edge version of removability:

**Definition 4.2.** A nonempty edge set $E_0 \subseteq V(G)$ is reducible if there exist a set $S$ of edge-disjoint triangles and a set $X$ of edges of $G$ such that the following conditions hold:

(i) $|X| \leq 2|S|$; and

(ii) $G - X$ has no triangle containing an edge of $E_0$; and

(iii) $X$ contains every $S$-edge that is not in $E_0$.

When $E_0$, $S$, and $X$ satisfy the definition above, we say that $E_0$ is reducible using $S$ and $X$.

An analogue of Lemma 2.2 holds for reducible edge sets. The proof is essentially the same, so we do not repeat it here:

**Lemma 4.3.** Let $G$ be a graph, and let $E_0 \subseteq E(G)$ be reducible using $S$ and $X$. Let $G' = (G - X) - E_0$. If $\tau(G') \leq 2\nu(G')$, then $\tau(G) \leq 2\nu(G)$.

**Lemma 4.4.** Let $v \in V(G)$, and let $G_0$ be a component of $G[N(v)]$. If $G_0 \in WKE$, then $G$ has a reducible set of edges. Also, if $G[N(v)] = G_0$ and $G_0 \in WKE$, then $\{v\}$ is reducible in $G$.

**Proof.** Take any pair $(M, Q)$ witnessing $G_0 \in WKE$. Define $E_0$, $S$, and $X$ as follows:

$E_0 = \{vw: w \in G_0\}$;  
$S = \{vwu: uw \in M\}$;  
$X = M \cup \{vx: x \in Q\}$.

Figure 2 illustrates the definition of $S$ and $X$; in the figure, thick edges represent $M$ and $X$, circled vertices represent $Q$, and shaded triangles represent $S$.

Since $M$ is a matching, the triangles in $S$ are pairwise edge-disjoint. We claim that $E_0$ is reducible using $S$ and $X$. Verifying each condition of Definition 4.2 in turn:
(i) Clearly \(|X| \leq 2|S|\), since \(|Q| \leq |M|\).

(ii) Any triangle of \(G\) containing an edge of \(E_0\) has the form \(vxy\), where \(xy \in E(H)\). Since \(Q\) is a vertex cover in \(G_0 - M\), either \(xy \in M\) or one of its endpoints lies in \(Q\). Thus \(G - X\) has no such triangle.

(iii) \(X\) contains every \(S\)-edge that does not contain an edge of \(E_0\), since \(M \subseteq X\).

For the second claim, we observe that when \(E_0\) is defined as above and \(G_0 = G[N(v)]\), the condition \(e \in E_0\) is equivalent to the condition \(v \in e\). Comparing Definition 2.1 to Definition 4.2 this shows that reducibility of \(E_0\) is equivalent to reducibility of \(\{v\}\).

Since all bipartite graphs are weak König–Egerváry graphs, Lemma 4.4 extends a theorem of Aparna Lakshmanan, Bujtás, and Tuza [9], who proved that if \(G\) is odd-wheel-free (i.e., locally bipartite) then Tuza’s Conjecture holds for \(G\). In the remainder of this section, we seek sufficient conditions for a graph to be a weak König–Egerváry graph. Due to Lemma 4.4, these conditions yield restrictions on the vertex neighborhoods in a minimum counterexample to Tuza’s Conjecture.

The first such result is an analogue of the König–Egerváry Theorem: if \(H\) has no odd cycle of length greater than 3, then \(H \in \text{WKE}\). The proof relies on a characterization of such graphs due to Hsu, Ikura, and Nemhauser [5].

**Theorem 4.5** (Hsu–Ikura–Nemhauser [5]). If \(H\) is 2-connected and has no odd cycle of length greater than 3, then \(H\) is either bipartite, isomorphic to \(K_4\), or isomorphic to \(K_2 \vee K_r\) for some \(r \geq 1\).

We start by proving a natural consequence of the König–Egerváry Theorem.

**Lemma 4.6.** If \(v\) is a vertex in a bipartite graph \(H\), then \(v\) lies in some minimum vertex cover of \(H\) if and only if \(v\) is covered by every maximum matching of \(H\).

**Proof.** If \(v\) lies in some minimum vertex cover \(Q\), then \(Q - v\) is a vertex cover of \(H - v\), so \(\beta(H - v) \leq \beta(H) - 1\). By the König–Egerváry Theorem, \(\alpha'(H - v) \leq \alpha'(H) - 1\). This implies that \(v\) is covered by every maximum matching of \(H\).

Now assume that \(v\) is covered in every maximum matching of \(H\). This yields \(\alpha'(H - v) = \alpha'(H) - 1\). By the König–Egerváry Theorem, \(\beta(H - v) = \alpha'(H) - 1\). Adding \(v\) to a minimum vertex cover in \(H - v\) yields the desired vertex cover. \(\square\)

Next, we consider the nonbipartite graphs in Theorem 4.5 obtaining a stronger version of the weak König–Egerváry property:

**Lemma 4.7.** If \(H\) is 2-connected and has no odd cycle of length greater than 3, then for every \(v \in V(H)\), there is a vertex set \(Q\) and a matching \(M\) such that:

1. \(|Q| \leq |M|\),
2. \(Q\) is a vertex cover in \(H - M\), and
3. Either \( v \in Q \) or \( v \) is in no edge of \( M \).

In particular, \( H \in \text{WKE} \).

Proof. If \( H \) is bipartite, then the claim follows from Lemma 4.6 together with the König–Egerváry Theorem: if \( v \) is covered by every maximum matching, then any minimum vertex cover has the desired properties. Thus, we may assume \( H \) is not bipartite. By Theorem 4.5 it suffices to consider three cases:

Case 1: \( H \cong K_3 \). Write \( V(H) = \{v, w_1, w_2\} \). Let \( Q = \{v\} \) and let \( M = \{w_1w_2\} \); the only edge of \( H \) not covered by \( v \) is \( w_1w_2 \).

Case 2: \( H \cong K_4 \) or \( H \cong K_2 \lor \overline{K}_2 \). Either way, \( H \) has a matching \( M \) of size 2. Let \( Q \) be \( v \) together with its mate in \( M \); the only edge of \( H \) not covered by \( Q \) is the other edge in \( M \).

Case 3: \( H \cong K_2 \lor \overline{K}_m \) for \( m \geq 3 \). Let \( Q \) consist of the two vertices of maximum degree. If \( v \in Q \), then let \( M \) be any matching of size 2. Otherwise, \( \alpha'(H - v) = 2 \), so we can take \( M \) to be any maximum matching in \( H - v \). \( \square \)

Proposition 4.8. If \( H \) has no odd cycle of length greater than 3, then \( H \in \text{WKE} \).

Proof. We use induction on \( |V(H)| \). If \( |V(H)| = 1 \) then clearly \( H \in \text{WKE} \). Now suppose that \( |V(H)| > 1 \) and the claim holds for all graphs with fewer vertices and no odd cycle of length greater than 3.

By the induction hypothesis, we may assume that \( H \) is connected. On the other hand, if \( H \) is 2-connected, then \( H \in \text{WKE} \), by Lemma 4.7. Thus we may assume that \( H \) is connected but not 2-connected, so \( H \) has a leaf block \( B \). Note that \( |V(B)| \geq 2 \).

Let \( v \) be the cut vertex of \( H \) contained in \( B \). By Lemma 4.7 \( B \) has a matching \( M_B \) and vertex cover \( Q_B \) such that \( |Q_B| \leq |M_B| \), \( Q_B \) is a vertex cover in \( B - M_B \), and either \( v \in Q_B \) or \( v \) is in no edge of \( M_B \). Note that \( |V(B)| \geq 2 \).

Now define a subgraph \( H' \) as follows: if \( v \in Q_B \), let \( H' = H - V(B) \); otherwise, let \( H' = H - (V(B) - v) \). By the induction hypothesis, \( H' \in \text{WKE} \); let \( M' \) and \( Q' \) witness \( H' \in \text{WKE} \). Let \( Q = Q' \cup Q_B \) and \( M = M' \cup M_B \). Whether or not \( v \in Q_B \), we see that \( M \) is a matching and that \( Q \) is a vertex cover in \( H - M \). Clearly \( |Q| \leq |M| \), so \( H \in \text{WKE} \). \( \square \)

Corollary 4.9. If \( H \) is connected and \( |V(H)| \leq 4 \), then \( H \in \text{WKE} \).

Corollary 4.10. If \( H \) is connected and \( \alpha'(H) \leq 1 \), then \( H \in \text{WKE} \). Also, if \( H \) is connected, \( |V(H)| > 5 \), and \( \alpha'(H) = 2 \), then \( H \in \text{WKE} \).

Proof. If \( H \notin \text{WKE} \), then \( H \) contains a cycle of length at least 5, which implies \( \alpha'(H) \geq 2 \). For the second claim, observe that any cycle of length at least 6 contains a matching of size 3, while if \( C \) is a 5-cycle in \( H \), then there are adjacent vertices \( v \in V(C) \) and \( w \notin V(C) \), which yields the following matching of size 3:
Recall that \( G \) is robust if for every \( v \in V(G) \), every component of \( G[N(v)] \) has order at least 5. In Section \ref{sec:main} we stated the following lemma, which now follows from the earlier results of this section:

**Lemma 2.6.** If \( G \) is a minimal counterexample to Tuza’s Conjecture, then \( G \) is robust.

**Proof.** Follows immediately from Lemma \ref{lem:Tuza} Lemma \ref{lem:Tuza2} and Corollary \ref{cor:Tuza3}.

**Corollary 4.11.** Let \( H \) be an \( n \)-vertex connected graph, where \( n \geq 6 \). If \( H \) has an independent set of size \( n - 3 \), then \( H \in \text{WKE} \).

**Proof.** If \( \alpha'(H) \leq 2 \), then \( H \in \text{WKE} \), by Lemma \ref{lem:Tuza3}. Otherwise, \( \alpha'(H) \geq 3 \), and the complement in \( V(H) \) of a maximum independent set is a vertex cover of size at most 3. Thus \( H \in \text{KE} \).

Finally, we give a sufficient condition for small graphs to be weak König–Egerváry graphs. (In fact, the condition is also necessary, but we do not need the other direction, so we omit it.)

**Proposition 4.12.** Let \( H \) be an \( n \)-vertex connected graph, where \( n \in \{5,6\} \). If \( \Delta(H) > 1 \), then \( H \in \text{WKE} \).

**Proof.** Since \( \Delta(H) > 1 \), we may take \( u, z_1, z_2 \in V(H) \) such that \( uz_1, uz_2 \notin E(H) \). By Lemma \ref{lem:Tuza3} we may assume \( \alpha'(H) = n - 3 \), since \( n \in \{5,6\} \). If \( z_1z_2 \notin E(H) \), then \( V(H) - \{u, z_1, z_2\} \) is a vertex cover in \( H \) having size \( n - 3 \), which implies \( H \in \text{KE} \). Thus we may assume \( z_1z_2 \in E(H) \). Also, if there is some maximum-size matching \( M \) containing the edge \( z_1z_2 \), then \( V(H) - \{u, z_1, z_2\} \) is a vertex cover of size \( n - 3 \) in \( H - M \), which implies \( H \in \text{WKE} \).

**Case 1:** \( n = 5 \) and \( \alpha'(H) = 2 \). Since no maximum-size matching contains \( z_1z_2 \), there are no edges in \( H - \{z_1, z_2\} \), so \( \{z_1, z_2\} \) is a vertex cover in \( H \). Hence \( H \in \text{KE} \).

**Case 2:** \( n = 6 \) and \( \alpha'(H) = 3 \). Let \( H_0 = H - \{z_1, z_2\} \); since no maximum-size matching contains \( z_1z_2 \), we have \( \alpha'(H_0) \leq 1 \). By Corollary \ref{cor:Tuza3} if \( H \) has an independent set of size 3, then \( H \in \text{WKE} \). Thus we may assume that \( \alpha(H) < 3 \), which implies that \( H_0 \) is a graph on 4 vertices such that \( \alpha'(H_0) \leq 1 \) and \( \alpha(H_0) < 3 \). This is only possible if \( H_0 \cong K_3 + K_1 \), as illustrated in Figure \ref{fig:K3K1}.

It follows that if \( M \) is any maximum-size matching in \( H \), then one edge of \( M \) must lie inside the \( K_3 \) component of \( H_0 \), one edge must join the \( K_3 \) component to \( \{z_1, z_2\} \), and one edge must join the \( K_1 \) component to \( \{z_1, z_2\} \). Fix some maximum-size matching \( M \) and label its edges \( e_1, e_2, e_3 \) respectively. Let \( y \) be the vertex in the \( K_3 \) component not contained in \( e_1 \). The set \( \{y, z_1, z_2\} \) is a vertex cover in \( H - M \), so \( H \in \text{WKE} \).
Using weak König–Egerváry graphs, we have shown that any minimum counterexample to Tuza’s Conjecture is robust (and therefore has minimum degree at least 5), and we have obtained strong restrictions on the possible neighborhoods of any 5- or 6-vertex.

5. Low-Degree Vertices

In this section, we will study the behavior of $6^-$-vertices in graphs with no reducible set. The main result of the section is Proposition 5.2 which states that the $6^-$-vertices form an independent set; this result is used heavily in Section 7.

We first obtain a stronger version of Proposition 4.12 using the observation that it is possible for $\{v\}$ to be reducible even though $G[N(v)] \not\in \text{WKE}$.

**Proposition 5.1.** Let $G$ be a robust graph. If $v \in V(G)$ with $d(v) \leq 6$, then $\{v\}$ is reducible in $G$ if $\Delta(G[N(v)]) > 1$. Also, if $d(v) \leq 6$ and $G[N(v)]$ has exactly 2 edges, then $\{v\}$ is reducible.

**Proof.** The first statement follows immediately from Lemma 4.4 and Proposition 4.12. For the second statement, we again split into cases according to $d(v)$. Let $H = G[N(v)]$, and let $w_1, \ldots, w_{d(v)}$ be the vertices of $H$, indexed so that $E(P) = \{w_1w_2, w_3w_4\}$.

**Case 1:** $d(v) = 5$. Define $S$ and $X$ by

$S = \{vw_2v_4, v_1w_3, w_1w_4w_5, w_2w_3w_5\}$,

$X = E(H)$. 

The triangles in $S$ are illustrated in Figure 1(a). We verify that $\{v\}$ is reducible using $S$ and $X$, verifying each condition of Definition 2.1:

(i) $|X| \leq 2 |S|$, since $|E(H)| = 8$.

(ii) Every triangle containing $v$ has its other two vertices in $H$, so $G - X$ has no triangle containing $v$.

(iii) $X$ contains every $S$-edge not incident to $v$, since all such edges lie in $H$.

**Case 2:** $d(v) = 6$. Define $S$ and $X$ by

$$S = \{vw_1w_4, vw_2w_3, vw_5w_6, w_1w_3w_5, w_2w_4w_5\},$$

$$X = E(H - w_6) \cup \{w_5w_6, vw_6\}.$$  

The triangles in $S$ are illustrated in Figure 1(b). We verify that $\{v\}$ is reducible using $S$ and $X$, verifying each condition of Definition 2.1:

(i) By construction, $|X| \leq 2 |S|$.

(ii) Since all edges of $H$ not incident to $w_6$ lie in $H$, any triangle of $G - X$ containing $v$ also contains $w_6$. Since $vw_6 \in X$, it follows that there is no such triangle.

(iii) By inspection, $X$ contains every $S$-edge that is not incident to $v$.  

In the rest of the paper, we will typically omit explicit verifications of Conditions (i) and (iii) of Definition 2.1 since they usually follow from a quick inspection of the definitions.

Next we show that a robust graph with no reducible set can have no edge joining “low-degree” vertices. The idea is simple: if $u$ and $v$ are adjacent low-degree vertices and neither $\{u\}$ nor $\{v\}$ is reducible, then we have a lot of information about the structure of $G[N(u)]$ and $G[N(v)]$, which will allow us to show that the set $\{u, v\}$ is reducible.

**Proposition 5.2.** Let $G$ be a robust graph. If $uv \in E(G)$ with $d(u) \leq 6$ and $d(v) \leq 6$, then one of $\{u\}$, $\{v\}$, or $\{u, v\}$ is reducible in $G$.

**Proof.** Without loss of generality, we may assume $d(u) \leq d(v)$. Assuming that neither $\{u\}$ nor $\{v\}$ is reducible in $G$, we show that $\{u, v\}$ is reducible in $G$. Since neither $\{u\}$ nor $\{v\}$ is reducible, Proposition 5.1 says that $\Delta(G[N(u)]) \leq 1$ and $\Delta(G[N(v)]) \leq 1$. Let $H = G[N(u) \cap N(v)]$. Since $u, v \notin V(H)$ and $u$ has at most one non-neighbor in $G[N(v)]$, we have $d(v) - 2 \leq |V(H)| \leq d(u) - 1$. Also, $\Delta(H) \leq 1$, since $\Delta(G[N(u)]) \leq 1$.

**Case 1:** $|V(H)| = 3$. In this case, $d(u) = d(v) = 5$ and $v$ is not a dominating vertex in $G[N(u)]$. By Proposition 5.1, $G[N(u)]$ has precisely one non-edge, and likewise for $G[N(v)]$. Let $p$ the unique vertex in $N(u) - N[v]$, and let $q$ be the unique vertex in $N(v) - N[u]$; now $pq$ is the unique non-edge in $G[N(u)]$ and $qu$ is the unique non-edge in $G[N(v)]$. Write $V(H) = \{w_1, w_2, w_3\}$. Since
Figure 5: Triangles and edges in Proposition 5.2.

(a) \(S, X\) in Case 1.

(b) Largest possible \(S, X\) in Case 2.

\(H \subseteq G[N(u)]\) and \(pw\) is the unique non-edge in \(G[N(u)]\), we have \(H \cong K_3\). Now \(\{u, v\}\) is reducible using the following sets \(S\) and \(X\), illustrated in Figure 5(a):

\[
S = \{uw_1w_2, vw_1w_3, upw_3, vqw_2\}
\]

\[
X = \{uv, vw, pv, up, pw_3, qw_2\} \cup E(H).
\]

We quickly check Condition (ii) of Definition 2.1. Let \(T\) be a triangle in \(G - X\) containing a vertex of \(V_0\), say the vertex \(u\). Since \(E(H) \subseteq X\), at most one vertex of \(H\) lies in \(T\), so \(T\) must contain a vertex in \(\{v, p, q\}\). Since \(uq \notin E(H)\) and \(\{uv, up\} \subseteq X\), no such triangle exists. If \(T\) instead contains \(v\), a similar argument holds.

**Case 2:** \(|V(H)| = 4\). Since \(\Delta(H) \leq 1\), \(H\) contains incident edges \(w_1w_2\) and \(w_1w_3\); let \(w_4\) be the remaining vertex of \(H\). We build a set \(S\) of triangles and a set \(X\) of edges step-by-step as follows: initially, \(S = \{uw_1w_2, vw_1w_3, uvw_4\}\) and \(X = E(H) \cup \{uw\}\). Note that initially, \(2|S| - |X| \geq -1\), with equality holding if and only if \(H \cong K_4\). We augment \(S\) and \(X\) according to the following rules:

- If there exists \(p \in N(u) - N[v]\), then add the triangle \(upw_3\) to \(S\) and add the edges \(pu, pw_3\) to \(X\).
- If there exists \(q \in N(v) - N[u]\), then add the triangle \(vqw_2\) to \(S\) and add the edges \(qv, qw_2\) to \(X\).
- If \(H \cong K_4\), add the triangle \(w_2w_3w_4\) to \(S\).

Figure 5(b) shows the \(S\) and \(X\) obtained when \(p\) and \(q\) both exist and \(H \cong K_4\). Note that if \(p\) exists, then \(p\) is the unique vertex of \(N(u) - N[v]\), since \(v\) has at most one non-neighbor in \(G[N(u)]\); likewise for \(q\). In all cases, we end with \(|X| \leq 2|S|\). The verification of Condition (ii) is similar to Case 1.
Case 3: $|V(H)| = 5$. In this case, $d(u) = d(v) = 6$ and $N[u] = N[v]$. Since $\Delta(H) \leq 1$, $H$ contains a subgraph $H'$ isomorphic to $C_4$, with vertices $a, b, c, d$ in order. Let $w$ be the remaining vertex of $H$. Now $\{u, v\}$ is reducible using the following sets $S$ and $X$, illustrated in Figure 6:

$$S = \{uab, ucd, vbc, vad, uvw\};$$

$$X = \{uw, vw, ua, ub, vc, vd\} \cup E(H').$$

We again check Condition (ii) of Definition 2.1. Any triangle containing the edge $uv$ is of the form $uvz$, where $z \in V(H)$. For every such $z$, either $uz \in X$ or $vz \in X$: hence $G - X$ has no triangle containing $uv$. Now suppose $T$ is a triangle containing $u$ but not $v$. Clearly $a, b, w \notin T$; hence $T = ucd$, but $cd \in X$. Hence $G - X$ has no triangle containing $u$. A similar argument holds for $v$. \hfill \Box

Corollary 5.3. Let $G$ be a robust graph. If $v \in V(G)$ has more than $d(v) - 4$ neighbors that are 6-vertices, then $\{v\}$ is reducible.

Proof. By Lemma 5.2, we may assume that $d(v) > 6$. Also by Lemma 5.2, the 6-neighborhood of $v$ form an independent set in $G[N(v)]$. By Corollary 4.11, if $G[N(v)]$ has an independent set of size $d(v) - 3$, then $G[N(v)] \in WKE$. Thus $\{v\}$ is reducible, by Lemma 4.4. \hfill \Box

6. A Weaker Result: $\text{Mad}(G) < 25/4$

We now have sufficient tools to prove the following theorem, which is weaker than Theorem 1.2 but still strong enough for many of the applications in Section 3. In particular, this theorem is strong enough to imply Corollary 3.2 on toroidal graphs, Theorem 3.7 on $K_{3,3}$-minor-free graphs, and Theorem 3.8 on $K_5$-subdivision-free graphs.

Theorem 6.1. If $\text{Mad}(G) < 25/4$, then $\tau(G) \leq 2\nu(G)$. 

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Proof Sketch. Assuming that $G$ has no reducible set, we use the method of discharging to show that $G$ has average degree at least $25/4$. Give every vertex $v$ initial charge $d(v)$. We apply the following discharging rule:

- Every $6^-$-vertex takes charge $1/4$ from every neighbor.

We claim that every vertex has final charge at least $25/4$, yielding average degree at least $25/4$ in $G$.

First we consider the $6^-$-vertices. By Lemma 2.4 $G$ is robust, so $\delta(G) \geq 5$, and by Proposition 5.2 the $6^-$-vertices form an independent set. Hence all $5$-vertices end with charge $25/4$, and all $6$-vertices end with charge $30/4$.

Next we consider the $7^+$-vertices. By Corollary 5.3 if $v$ is a $k$-vertex where $k > 6$, then $v$ has at most $k - 4$ neighbors that are $6^-$-vertices. Hence $v$ has final charge at least $3k/4 + 1$. Since $k \geq 7$, this implies that $v$ has final charge at least $25/4$, as desired. \qed

In the remaining section, we will improve the bound $\text{Mad}(G) < 25/4$ to $\text{Mad}(G) < 7$.

7. Subsumption and Related Bounds

Recall the following definitions from Section 2:

Definition 7.1. A vertex $u$ subsumes a vertex $v$ if $N[u] \supseteq N[v]$.

Definition 7.2. A $6$-vertex $v$ is thin if $G[N(v)]$ contains a matching of size $3$.

The motivation for these definitions is as follows: when $u$ subsumes a $6^-$-vertex $v$, having $d(v) - 1$ neighbors of $v$ in $G[N(u)]$ leads to better bounds on the number of $6^-$-neighbors of $u$. Thus, in the discharging rule, such a vertex $u$ can give away a lot of charge to the vertices it subsumes, since not many other $6^-$-neighbors will place demands on it. Conversely, if $u$ subsumes no $6^-$-vertex, then the bounds on the number of $6^-$-neighbors are weaker, but since $u$ does not subsume its neighbors, they need not demand much charge from $u$.

Lemma 7.3. Let $G$ be a robust graph with no reducible set. If a $10^+$-vertex $v$ subsumes a $6^-$-vertex $w$, then at most $d(v) - 6$ neighbors of $v$ are $6^-$-vertices.

Proof. Assume to the contrary that $d(v) - 5$ neighbors of $v$ are $6^-$-vertices. We obtain a contradiction by proving $G[N(v)] \in \text{WKE}$, implying that $\{v\}$ is reducible. Let $A$ be the set of $6^-$-neighbors of $v$, and let $B = N(v) - A$: note that $|A| \geq d(v) - 5 \geq 5 \geq |B|$. Since $G$ has no reducible set, Proposition 5.2 implies that $A$ is an independent set and that $N(w) - \{v\} \subseteq B$. Since $A$ is independent, $B$ is a vertex cover in $G[N(v)]$.

By Proposition 5.1 $\Delta(G[N(a)]) \leq 1$ for all $a \in A$; in particular, since $v \in N(a)$, we have $d_{G[N(a)]}(v) \geq d(a) - 2$. Since $d_{G[N(a)]}(v) = |N(a) \cap N(v)| = d_{G[N(v)]}(a)$, we have $d_{G[N(v)]}(a) \geq d(a) - 2$. Since $A$ is independent and each $d(a) \geq 5$, this implies $d_B(a) \geq 3$ for all $a \in A$. Similarly, $d_B(w) \geq 4$, since $v$ is a dominating vertex in $G[N(w)]$ and $d(w) \geq 5$.
We first argue that \( \alpha'(G[N(v)]) \geq 4 \) by greedily constructing a matching of size 4. Let \( a_1, a_2, a_3 \) be distinct elements of \( A - w \). Since each \( d_B(a_i) \geq 3 \), for each \( i \) we may choose \( b_i \in N_B(a_i) \) distinct from all earlier \( b_i \). Since \( d_B(w) \geq 4 \), we can take \( b' \in B - \{b_1, b_2, b_3\} \). Now \( \{a_1b_1, a_2b_2, a_3b_3, wb'\} \) is the desired matching of size 4. If \( |B| = 4 \), then this implies \( G[N(v)] \in \text{KE} \); thus we may assume \( |B| = 5 \). If \( d_A(b) = 0 \) for some \( b \in B \), then \( B - b \) is a vertex cover in \( G[N(v)] \), which again implies \( G[N(v)] \in \text{KE} \); thus we may also assume \( d_A(b) > 0 \) for all \( b \in B \).

**Case 1:** \( \bigcup_{z \in A - w} N_B(z) = 3 \). Let \( z_1, z_2, z_3 \) be distinct vertices in \( A - w \), and let \( b_1, b_2, b_3 \) be distinct vertices in \( \bigcup_{z \in A - w} N_B(z) \). Let \( b' \in N_B(w) - \{b_1, b_2, b_3\} \). The set \( B - b' \) is a vertex cover of size 4 in \( G[N(v)] - \{wb', z_1b_1, z_2b_2, z_3b_3\} \), so \( G[N(v)] \in \text{KE} \).

**Case 2:** \( \bigcup_{z \in A - w} N_B(z) > 3 \). We verify Hall’s Condition for \( B \). Take any \( B_0 \subseteq B \). If \( |B_0| > 2 \), then \( N_A(B_0) = A \), since each \( a \in A \) has at most 2 non-neighbors in \( B \). If \( |B_0| = 1 \), then \( |N_A(B_0)| \geq 1 \) since \( d_A(b) > 0 \) for all \( b \in B \).

Now suppose \( |B_0| = 2 \). Since \( d_B(w) \geq 4 \), we have \( w \in N_A(B_0) \). For \( z \in A - w \), if \( z \notin N_A(B_0) \), then \( N_B(z) = B - B_0 \), since \( d_B(z) \geq 3 \). Since \( \bigcup_{z \in A - w} N_B(z) > 3 \), the equality \( N_B(z) = B - B_0 \) cannot hold for all \( z \in A - w \), so \( |N_A(B_0)| \geq 2 \). By Hall’s Theorem, \( \alpha'(G[N(v)]) \geq 5 \), so \( G \in \text{KE} \). \( \square \)

When \( d(v) = 9 \) a similar statement holds, but more nuance is required, since we are no longer guaranteed that \( |A| \geq |B| \).

**Lemma 7.4.** Let \( G \) be a robust graph with no reducible set. Every 9-vertex subsumes at most three 6-vertices; furthermore, if equality holds, then it is adjacent to no other 6-vertex.

**Proof.** Let \( v \) be a 9-vertex subsuming 6-vertices \( w_1, w_2, w_3 \). Suppose to the contrary that \( v \) has another 6-neighbor \( w' \) (possibly subsuming \( w' \), possibly not). Let \( W = \{w_1, w_2, w_3, w'\} \), and let \( V_0 = W \cup \{v\} \). We show that \( V_0 \) is reducible, contradicting the hypothesis. By Proposition 5.5, we have \( \Delta(G[N(w')]) \leq 1 \), since \( G \) has no reducible set. Since \( v \in N(w') \), this implies \( |N(w') - N[v]| \leq 1 \). By the definition of subsumption, \( N(w_i) \subseteq N[v] \) for each \( i \). For convenience, let \( H_i = G[N(w_i) \cap N(v)] = G[N(w_i) - \{v\}] \), and let \( H' = G[N(w') \cap N(v)] \). By Proposition 5.2, the 6-vertices of \( G \) form an independent set, so \( V(H_i) \cap W = \emptyset \) for each \( i \). We build a set \( S \) of edge-disjoint triangles in several steps: the observation that \( V(H_i) \cap W = \emptyset \) helps guarantee that \( S \) is edge-disjoint. The algorithm begins with \( S = \emptyset \).

(Figure 7 illustrates the construction in the “worst-case” scenario where \( V(H_1) = V(H_2) = V(H_3) \), where each \( H_i \cong K_4 \), and where \( V(H') \) is a proper subset of \( V(H_1) \). In general, it is possible that the subgraphs \( H_i \) may have distinct vertex sets, but when they coincide we have less room to find edge-disjoint triangles. In the figure, dashed edges represent edges that are no longer available for use in \( S \), since they were used in earlier triangles.)
Figure 7: Constructing $S$ in Lemma 7.4

- Since $|V(H_1)| \geq 4$ and $\Delta(H_1) \leq 1$, we can find two disjoint edges $s_1s_2$ and $s_3s_4$ in $E(H_1)$. Add the triangles $w_1s_1s_2$ and $w_1s_3s_4$ to $S$.

- Since $E(H_2) \cup \{s_1s_2, s_3s_4\}$ is the union of two matchings and $|V(H_2)| \geq 4$, we see that $H_2 - \{s_1s_2, s_3s_4\}$ is a graph on at least 3 edges that is neither a star nor a triangle. Hence there are two disjoint edges $t_1t_2$ and $t_3t_4$ in $E(H_2) - \{s_1s_2, s_3s_4\}$. Add the triangles $w_2t_1t_2$ and $w_2t_3t_4$ to $S$.

- Since $\Delta(H_3) \leq 1$ and $|E(H_3)| \neq 2$, and since $|V(H_3)| \geq 4$, we have $|E(H_3)| \geq 5$. Thus $H_3 - \{s_1s_2, s_3s_4, t_1t_2, t_3t_4\}$ still contains an edge $uu'$ and a vertex $r \notin \{u, u'\}$. Add the triangles $w_3uu'$ and $vw_3r$ to $S$.

- Since $\Delta(G[N(w']]) \leq 1$, we have $|V(H')| \geq 3$. Fix any vertex $r_1 \in V(H') - \{r\}$ and add the triangle $vw'r_1$ to $S$, reaching seven triangles. Also, if $N(w') - N[w] \neq \emptyset$, let $p$ be the unique vertex in the difference. Note that $V(H') \subseteq N(p) \cap N(v)$, since otherwise $p$ would have two non-neighbors in $G[N(w')]$, contradicting $\Delta(G[N(w')]) \leq 1$. Choose $r_2 \in V(H') - \{r, r_1\}$ and add the triangle $w'r_2p$ to $S$, reaching a total of eight triangles.

Figure 7 illustrates why the triangles in $S$ are edge-disjoint. At each step, we add an edge-disjoint set of triangles, so it suffices to check that the triangles added in each step are disjoint from the earlier triangles. Since $V(H_1) \cap W = \emptyset$, edges incident to $w_i$ are used only in the step corresponding to $w_i$; similarly, edges incident to $w'$ are used only in the last step. By construction, we never use any edge in $E(H_1)$ that was previously used, so only the edges incident to $v$ and incident to neither $w_3$ nor $w'$ are liable to be reused. The only such edges are $vv$, $vr_1$, and possibly $vr_2$ and $r_2p$; since $r, r_1,$ and $r_2$ were chosen to be distinct vertices, and since $p \notin N[v]$ while all other vertices used in $S$ lie in $N[v]$, these edges are also distinct.

Let $Z = N(v) - V_0$, so that $|Z| = 5$. Define $X$ by

$$X = \begin{cases} E(G[Z]) \cup \{vw_1, vw_2, vw_3, vw\}, & \text{if } N(w') - N(v) = \emptyset; \\ E(G[Z]) \cup \{vw_1, vw_2, vw_3, vw', w'p, r_2p\}, & \text{if } N(w') - N(v) = \{p\}. \end{cases}$$

Since $|E(G[Z])| \leq 10$, we have $|X| \leq 2|S|$ in either case. By construction, $X$ contains every $S$-edge that is not incident to $V_0$. We check that $G - X$ has no
triangle containing a vertex of $V_0$. Since $E(G[Z]) \subseteq X$, any triangle in $G - X$ containing a vertex of $V_0$ must contain two vertices in $V_0 \cup (N(w') - N(v))$. Since $W$ is an independent set, the only way for a triangle to contain two vertices in $V_0$ is to contain an edge of the form $vw_i$, $vw'$, or $w'p$ if $p$ exists. All such edges also lie in $X$; thus $V_0$ is reducible using $S$ and $X$.

When $d(v) = 8$ and we are only concerned with 6-vertices, we obtain a similar result with a simple counting argument.

**Lemma 7.5.** Let $G$ be a robust graph with no reducible set. Every 8-vertex that subsumes a 6-vertex has most three 6$^{-}$-neighbors.

**Proof.** Let $v$ be an 8-vertex that subsumes a 6-vertex $w$. By Proposition 5.2, $w$ is not adjacent to any 6$^{-}$-neighbor of $v$. Since $|N(v) \cap N(w)| = 5$, this implies that $v$ has at most three 6$^{-}$-neighbors.

**Lemma 7.6.** Let $G$ be a robust graph and let $uw \in E(G)$.

If $d(u) \in \{7, 8\}$ and $d(v) = 5$ and $u$ subsumes $v$, then $G$ has a reducible set; If $d(u) = 7$ and $d(v) = 6$ and $u$ subsumes $v$, then $G$ has a reducible set; If $d(u) = 7$ and $v$ is a thin 6-vertex, then $G$ has a reducible set.

We prove each of these claims in its own proposition; the proofs are straightforward but require some case analysis.

**Proposition 7.7.** Let $G$ be a robust graph. Let $uw \in E(G)$ with $d(u) \in \{7, 8\}$ and $d(v) = 5$. If neither $\{u\}$ nor $\{v\}$ is reducible in $G$ and $u$ subsumes $v$, then $\{u, v\}$ is reducible in $G$.

**Proof.** Let $W = N(u) \cap N(v)$ and $Z = N(u) - N[v]$. By Proposition 5.1, $G[N(v)] \in \{K_5, K_4\}$. Hence $G[W] \in \{K_4, K_5\}$. Also, $|Z| = d(u) - 5$, since $u$ subsumes $v$. Hence $|Z| \in \{2, 3\}$.

If $G[W] \cong K_4$, then let $w_1w_2$ be the missing edge in $W$; otherwise, let $w_1$ and $w_2$ be distinct vertices of $W$.

**Case 1:** $G[Z]$ contains an edge $z_1z_2$. Let $Z^* = \{z \in Z : d_W(z) > 0\}$. Observe that $|Z^*| + |E(G[W])| \leq 9$, with equality holding if and only if $|Z^*| = 3$ and $|E(G[W])| = 6$.

**Case 1a:** $G[W] \cong K_4$ and $|Z^*| = 3$. Let $z_0$ be the vertex of $Z$ not in $\{z_1, z_2\}$. Choose $w \in N(z) \cap W$, relabeling if necessary so that $w, w_1, w_2$ are distinct, and let $w'$ be the unique vertex in $W - \{w, w_1, w_2\}$. Now $\{u, v\}$ is reducible using the following sets $S$ and $X$, with $S$ illustrated in Figure 8(a):

$$S = \{uw'w_1, wvw_1, wvw_2, w'w_2\} \cup \{uz_1z_2, uz_0w\},$$

$$X = E(G[W]) \cup \{uz : z \in Z\} \cup \{w, w, z_1z_2, z_0w\}.$$}

We check Condition (ii) of Definition 2.1. Let $T$ be a triangle in $G - X$ containing a vertex of $V_0$. Since $E(G[W]) \subseteq X$, we see that $T$ contains at most one vertex from $W$; hence, two vertices of $T$ must lie in $Z \cup \{u, v\}$. If $v \in T$, then $T$ cannot contain any vertex of $Z$, so $\{u, v\} \subseteq T$, which is impossible since $uv \in X$. 20
Therefore $v \notin T$, so $T$ contains $u$ and at least one vertex $z \in Z$. Since $uz \in X$, no such triangle exists.

**Case 1b:** $|Z^*| + |E(G[W])| \leq 8$. Let $w$ and $w'$ be the vertices of $W - \{w_1, w_2\}$. If $|Z^*| < 2$, then enlarge $Z^*$ to size 2 by adding arbitrary elements of $Z$. Now $\{u, v\}$ is reducible using the following sets $S$ and $X$, with $S$ illustrated in Figure 8(b):

$$S = \{uw'w_1, vww_1, www_2, wuw'w_2\} \cup \{uz_1z_2\},$$

$$X = E(G[W]) \cup \{z_1z_2, uv\} \cup \{uz: z \in Z^*\}.$$  

We again check Condition (ii) of Definition 2.1. Let $T$ be a triangle in $G - X$ containing a vertex of $V_0$. As before, $T$ contains at most one vertex of $W$, and so $v \notin T$. Thus $u \in T$, so $T$ either contains two vertices of $Z$ or a vertex of $Z$ and a vertex of $W$. Since $|Z^*| \geq 2$, if $T$ contains two vertices of $Z$ then $T$ contains some vertex of $Z^*$, which is impossible since $uz \in X$ for all $z \in Z^*$. On the other hand, if $T$ contains some $z \in Z$ and $w \in W$, then $d_T(z) > 0$, which implies $z \in Z^*$, again implying $uz \in X$.

**Case 2:** $Z$ is independent. Since $G$ is robust and $d(v) < 10$, $G[N(v)]$ is connected; thus no vertex of $Z$ can be isolated in $G[N(v)]$, so each vertex of $Z$ has a neighbor in $W$. Let $J$ be the bipartite subgraph of $G$ whose partite sets are $W$ and $Z$. Since each vertex of $Z$ has a neighbor in $W$, we have $\alpha'(J) > 0$.

**Case 2a:** $\alpha'(J) = 1$. Since every vertex of $Z$ has a neighbor in $W$, $\alpha'(J) = 1$ implies that some vertex $w \in W$ covers every edge incident to $Z$. Let $z$ be any vertex of $Z$, let $t_1t_2$ be an edge in $G[W]$ not containing $w$, and let $w'$ be the remaining vertex of $W$. Let $M = \{wz, t_1t_2, uw'\}$ and let $Q = \{v, w, w'\}$, as illustrated in Figure 8(a). All edges incident to $Z$ are covered by $W$, and the only edge of $G[W]$ not covered by $\{w, w'\}$ is $t_1t_2$, so $Q$ is a vertex cover in $G[N(u)] - M$. Hence $G[N(u)] \in \text{WKE}$, contradicting the hypothesis that $\{u\}$ is not reducible.

**Case 2b:** $\alpha'(J) = 2$. Since $\alpha'(J) = 2$, $|N_W(Z)| \geq 2$.

**Case 2b(i):** $|N_W(Z)| \geq 3$. Let $w \in N_W(Z) - \{w_1, w_2\}$, and pick $z_0 \in Z$ such that $wz \in E(G)$. Let $w'$ be the unique vertex in $W - \{w, w_1, w_2\}$, and let

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{Triangles in Proposition 7.7.}
\end{figure}
Figure 9: Matchings and vertex covers in Case 2.

{(q_1, q_2)} be a vertex cover in J. Now \{u, v\} is reducible using the following sets S and X, with S illustrated in Figure 8(c):

\[ S = \{uw, vww, uwv, wv'w_2\} \cup \{uz_0w\}, \]
\[ X = E(G[W]) \cup \{uv, z_0w\} \cup \{uq_1, uq_2\}. \]

The verification of Condition (ii) is similar to Case 1b, with the following modifications: any bad triangle T cannot contain two vertices of Z, since G[Z] has no edges; and if T contains a vertex in Z and a vertex in W, then T contains u along with an edge in J, one endpoint of which lies in \{q_1, q_2\}. The other possibilities for T are identical to Case 1b.

**Case 2b(ii):** \(|N_W(Z)| = 2\). Let \(t_1, t_2\) be the two vertices of \(W - N_W(Z)\), and let M be a maximum matching in J, as illustrated in Figure 9(b) in the case where \(t_1t_2 \in E(G)\). Clearly, M does not cover \(t_1\) or \(t_2\). Observe that \(N_W(Z) \cup \{v\}\) covers every edge in \(G[N(u)]\), except possibly the edge \(t_1t_2\) if it exists. If \(t_1t_2 \in E(G)\), then let \(M' = M \cup \{t_1t_2\}\); otherwise, let \(M' = M \cup \{vt_1\}\). In either case, \(N_W(Z) \cup \{v\}\) is a vertex cover of size 3 in \(N(v) - M'\), so \(G[N(u)] \in \text{KE}\), contradicting the hypothesis that \(\{u\}\) is not reducible.

**Case 2c:** \(\alpha'(J) = 3\). Let M be a maximum matching in J, and let w be the vertex of M not covered by M; then \(M \cup \{vw\}\) is a matching of size 4 in \(G[N(u)]\), as shown in Figure 9(c). Since W is a vertex cover of size 4 in \(G[N(u)]\), this implies \(G[N(u)] \in \text{KE}\), again contradicting the hypothesis that \(\{u\}\) is not reducible.

**Proposition 7.8.** Let G be a robust graph. Let \(uv \in E(G)\) with \(d(u) = 7\) and \(d(v) = 6\). If \(\{u\}\) is not reducible in G and u subsumes v, then \(\{u, v\}\) is reducible in G.
Figure 10: Triangles in Proposition 7.8.

Proof. By Proposition 5.1, $G[N(v)] \in \{K_6, K_6^\sim\}$ (since $G[N(v)]$ has $u$ as a dominating vertex, $G[N(v)]$ cannot be a perfect matching). Let $H = G[N(u) \cap N(v)]$ and write $V(H) = \{w_1, \ldots, w_5\}$, indexed so that $w_1w_2$ is the possible missing edge. Let $p$ be the unique vertex in $N(u) - N(v)$. Now $\{u, v\}$ is reducible using the following sets $S$ and $X$, with $S$ illustrated in Figure 10:

$$S = \{uw_2w_5, uw_3w_4\} \cup \{vw_2w_3, vw_4w_5\} \cup \{uw_1w_3w_5\},$$
$$X = E(H) \cup \{uv, up\}.$$  

We check Condition (ii) of Definition 2.1. Since $E(H) \subseteq X$, any triangle of $G - X$ containing a vertex of $V_0$ contains at most one vertex from $H$, and therefore contains two vertices from $\{u, v, p\}$. Since $uv, up \in X$ and $vp \notin E(G)$, no such triangle exists. 

Proposition 7.9. Let $G$ be a robust graph. If $G$ contains an edge $uv$ such that $d(u) = 7$ and $v$ is a thin 6-vertex, then $\{u, v\}$ is reducible in $G$.

Proof. Since $G[N(v)]$ is a matching of size 3, we know that $G[N(u) \cap N(v)] \cong C_4$; let $a, b, c, d$ be the vertices of this cycle, listed in order. Let $p_1, p_2$ be the two vertices of $N(u) - N[v]$ and let $q$ be the unique vertex in $N(v) - N[u]$. Now $\{u, v\}$ is reducible using the following sets $S$ and $X$, illustrated in Figure 11:

$$S = \{uab, ucd, vbc, vad\},$$
$$X = E(G[N(u) \cap N(v)]) \cup \{uv, up_1, up_2, vq\}.$$  

We quickly check Condition (ii) of Definition 2.1. Since $E(G[N(u) \cap N(v)]) \subseteq X$, any triangle of $G - X$ containing a vertex of $V_0$ contains at most one vertex from $N(u) \cap N(v)$, and therefore contains two vertices from $\{u, v, p_1, p_2, q\}$. Let $T$ be a triangle of $G - X$ and suppose $u \in T$. Since $uv, up_1, up_2 \in X$ and $uq \notin E(G)$, we see that $T$ cannot contain two vertices of $\{u, v, p_1, p_2, q\}$, and so $G - X$ has no triangle containing $u$. Similar logic holds for $v$. 

We have now completed the proof of Lemma 2.7, giving a list of configurations that cannot appear in a smallest counterexample to Tuza’s Conjecture. By completing the proof of this lemma, we have completed the proof of the main theorem, Theorem 1.2.
8. Acknowledgments

The author thanks his advisor Douglas B. West for many hours of careful proofreading, and for helpful suggestions regarding the design of the figures.

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