On the precise value of the strong chromatic index of a planar graph with a large girth

Gerard Jennhwa Chang*1,2 and Guan-Huei Duh†1

1Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan
2National Center for Theoretical Sciences, Mathematics Division, 2F of Astronomy-Mathematics Building, National Taiwan University, Taipei 10617, Taiwan

September 24, 2015

Abstract

A strong k-edge-coloring of a graph $G$ is a mapping from $E(G)$ to $\{1, 2, \ldots, k\}$ such that every pair of distinct edges at distance at most two receive different colors. The strong chromatic index $\chi'_s(G)$ of a graph $G$ is the minimum $k$ for which $G$ has a strong $k$-edge-coloring. Denote $\sigma(G) = \max_{xy \in E(G)} \{\deg(x) + \deg(y) - 1\}$. It is easy to see that $\sigma(G) \leq \chi'_s(G)$ for any graph $G$, and the equality holds when $G$ is a tree. For a planar graph $G$ of maximum degree $\Delta$, it was proved that $\chi'_s(G) \leq 4\Delta + 4$ by using the Four Color Theorem. The upper bound was then reduced to $4\Delta, 3\Delta + 5, 3\Delta + 1, 3\Delta, 2\Delta - 1$ under different conditions for $\Delta$ and the girth. In this paper, we prove that if the girth of a planar graph $G$ is large enough and $\sigma(G) \geq \Delta(G) + 2$, then the strong chromatic index of $G$ is precisely $\sigma(G)$. This result reflects the intuition that a planar graph with a large girth locally looks like a tree.

Keywords: Strong chromatic index, planar graph, girth.

1 Introduction

A strong k-edge-coloring of a graph $G$ is a mapping from $E(G)$ to $\{1, 2, \ldots, k\}$ such that every pair of distinct edges at distance at most two receive different colors. It induces a proper vertex coloring of $L(G)^2$, the square of the line graph of $G$. The strong chromatic index $\chi'_s(G)$ of $G$ is the minimum $k$ for which $G$ has a strong $k$-edge-coloring. This concept was introduced by Fouquet and Jolivet [19, 20] to model the channel assignment in some radio networks. For more applications, see [4, 29, 32, 31, 24, 36].

*E-mail: gjchang@math.ntu.edu.tw.
†E-mail: r03221028@ntu.edu.tw.
A Vizing-type problem was asked by Erdős and Nešetřil, and further strengthened by Faudree, Schelp, Gyárfás and Tuza to give an upper bound for $\chi'_s(G)$ in terms of the maximum degree $\Delta = \Delta(G)$:

**Conjecture 1** (Erdős and Nešetřil ’88 [16] ’89 [17], Faudree et al ’90 [18]). If $G$ is a graph with maximum degree $\Delta$, then $\chi'_s(G) \leq \Delta^2 + [\frac{\Delta}{2}]^2$.

As demonstrated in [18], there are indeed some graphs reach the given upper bounds.

By a greedy algorithm, it can be easily seen that $\chi'_s(G) \leq 2\Delta(\Delta - 1) + 1$. Molloy and Reed [28] using the probabilistic method to show that $\chi'_s(G) \leq 1.998\Delta^2$ for maximum degree $\Delta$ large enough. Recently, this upper bound was improved by Bruhn and Joos [8] to $1.93\Delta^2$.

For small maximum degrees, the cases $\Delta = 3$ and 4 were studied. Andersen [1] and Horák et al [22] proved that $\chi'_s(G) \leq 10$ for $\Delta(G) \leq 3$ independently; and Cranston [13] showed that $\chi'_s(G) \leq 22$ when $\Delta(G) \leq 4$.

According to the examples in [18], the bound is tight for $\Delta = 3$, and the best we may expect for $\Delta = 4$ is 20.

The strong chromatic index of a few families of graphs are examined, such as cycles, trees, $d$-dimensional cubes, chordal graphs, Kneser graphs, $k$-degenerate graphs, chordless graphs and $C_4$-free graphs, see [9] [12] [15] [18] [27] [39] [41]. As for Halin graphs, refer to [10] [25] [26] [34] [35]. For the relation to various graph products, see [37].

Now we turn to planar graphs.

Faudree et al used the Four Color Theorem [2] [3] to prove that planar graphs with maximum degree $\Delta$ are strong $(4\Delta + 4)$-edge-colorable [18]. By the same spirit, it can be shown that $K_5$-minor free graphs are strong $(4\Delta + 4)$-edge-colorable. Moreover, every planar $G$ with girth at least 7 and $\Delta \geq 7$ is strong $3\Delta$-edge-colorable by applying a strengthened version of Vizing’s Theorem on planar graphs [33] [38] and Grötzsch’s theorem [21].

The following results are obtained by using a discharging method:

**Theorem 2** (Hudák et al ’14 [23]). If $G$ is a planar graph with girth at least 6 and maximum degree at least 4, then $\chi'_s(G) \leq 3\Delta(G) + 5$.

**Theorem 3** (Hudák et al ’14 [23]). If $G$ is a planar graph with girth at least 7, then $\chi'_s(G) \leq 3\Delta(G)$.

And the bounds are improved by Bensmail et al.

**Theorem 4** (Bensmail et al ’14 [6]). If $G$ is a planar graph with girth at least 6, then $\chi'_s(G) \leq 3\Delta(G) + 1$.

**Theorem 5** (Bensmail et al ’14 [6]). If $G$ is a planar graph with girth at least 5 or maximum degree at least 7, then $\chi'_s(G) \leq 4\Delta(G)$.
It is also interesting to see the asymptotic behavior of strong chromatic index when the girth is large enough.

**Theorem 6** (Borodin and Ivanova ’13 [7]). If $G$ is a planar graph with maximum degree $\Delta \geq 3$ and girth at least $40\lfloor \frac{\Delta}{2} \rfloor + 1$, then $\chi'_s(G) \leq 2\Delta - 1$.

**Theorem 7** (Chang et al ’13 [11]). If $G$ is a planar graph with maximum degree $\Delta \geq 4$ and girth at least $10\Delta + 46$, then $\chi'_s(G) \leq 2\Delta - 1$.

**Theorem 8** (Wang and Zhao ’15 [40]). If $G$ is a planar graph with maximum degree $\Delta \geq 4$ and girth at least $10\Delta - 4$, then $\chi'_s(G) \leq 2\Delta - 1$.

The concept of maximum average degree is also an indicator to the sparsity of a graph. Graphs with small maximum average degrees are in relation to planar graphs with large girths, as a folklore lemma that can be proved by Euler’s formula points out.

**Lemma 9.** A planar graph $G$ with girth $g$ has maximum average degree $\text{mad}(G) < 2 + \frac{4}{g-2}$.

Many results concerning planar graphs with large girths can be extended to general graphs with small maximum average degrees and large girths. Strong chromatic index is no exception.

**Theorem 10** (Wang and Zhao ’15 [40]). Let $G$ be a graph with maximum degree $\Delta \geq 4$. If the maximum average degree $\text{mad}(G) < 2 + \frac{1}{3\Delta - 2}$, the even girth is at least 6 and the odd girth is at least $2\Delta - 1$, then $\chi'_s(G) \leq 2\Delta - 1$.

In terms of maximum degree $\Delta$, the bound $2\Delta - 1$ is best possible. We seek for a better parameter as a refinement. Define

$$\sigma(G) := \max_{xy \in E(G)} \{\deg(x) + \deg(y) - 1\}.$$

An antimatching is an edge set $S \subseteq E(G)$ in which any two edges are at distance at most 2, thus any strong edge-coloring assigns distinct colors on $S$. Notice that each color set of a strong edge-coloring is an induced matching, and the intersection of an induced matching and an antimatching contains at most one edge. The fact suggests a dual problem to strong edge-coloring: finding a maximum antimatching of $G$, whose size is denoted by $\text{am}(G)$. For any edge $xy \in E(G)$, the edges incident with $xy$ form an antimatching of size $\deg(x) + \deg(y) - 1$. Together with the weak duality, this gives the inequality

$$\chi'_s(G) \geq \text{am}(G) \geq \sigma(G).$$

By induction, we see that for any nontrivial tree $T$, $\chi'_s(T) = \sigma(T)$ attains the lower bound [18]. Based on the intuition that a planar graph with large girth locally looks like a tree, in this paper, we focus on this class of graphs. More precisely, we prove the following main theorem:
Theorem 11. If $G$ is a planar graph with $\sigma = \sigma(G) \geq 5$, $\sigma \geq \Delta(G) + 2$ and girth at least $5\sigma + 16$, then $\chi'_s(G) = \sigma$.

We also make refinement on the girth constraint and gain a stronger result in Section 4.

The condition $\sigma \geq \Delta(G) + 2$ is necessary as shown in the following example. Suppose $n \geq 1$ and $d \geq 2$. Construct $G_{3n+1,d}$ from the cycle $(x_1, x_2, \ldots, x_{3n+1})$ by adding $d-2$ leaves adjacent to each $x_{3i}$ for $1 \leq i \leq n$. Then $\sigma(G_{3n+1,d}) = d + 1 < d + 2 = \Delta(G_{3n+1,d}) + 2$. See Figure 1 for $G_{3n+1,4}$.

![Figure 1: The graph $G_{3n+1,4}$.](image)

We claim that $\sigma(G_{3n+1,d}) < \chi'_s(G_{3n+1,d})$. Suppose to the contrary that $\sigma(G_{3n+1,d}) = \chi'_s(G_{3n+1,d})$. For $1 \leq i \leq n$, the $(\sigma - 1)$ edges incident to $x_{3i}$, together with $x_{3i-2}x_{3i-1}$ (or $x_{3i+1}x_{3i+2}$) use all the $\sigma$ colors, implying that $x_{3i-2}x_{3i-1}$ uses the same color as $x_{3i+1}x_{3i+2}$, where $x_{3n+2} = x_1$. Therefore, $x_1x_2, x_4x_5, \ldots, x_{3n+1}x_{3n+2}$ all use the same color, contradicting that $x_1x_2$ is adjacent to $x_{3n+1}x_1 = x_{3n+1}x_{3n+2}$.

2 The proof of the main theorem

To prove the main theorem, we need two lemmas and a key lemma (Lemma 18) to be verified in the next section.

The first lemma can be used to prove that any tree $T$ has strong chromatic index $\sigma(T)$ by induction.

Lemma 12. Suppose $x_1x_2$ is a cut edge of a graph $G$, and $G_i$ is the component of $G - x_1x_2$ containing $x_i$ joining the edge $x_1x_2$ for $i = 1, 2$. If for some integer $k$, $\deg(x_1) + \deg(x_2) - 1 \leq k$ and $\chi'_s(G_i) \leq k$ for $i = 1, 2$, then $\chi'_s(G) \leq k$.

Proof. Choose a strong $k$-edge-coloring $f_i$ of $G_i$ for $i = 1, 2$. Let $E_i$ be the set of edges incident with $x_i$ in $G_i - x_1x_2$ and $S_i = f_i(E_i)$. Since $\deg(x_1) + \deg(x_2) - 1 \leq k$, we may assume $S_1$ and $S_2$ are disjoint and $f_1(x_1x_2) = f_2(x_1x_2)$ is some element $c \in \{1, 2, \ldots, k\} \setminus (S_1 \cup S_2)$. Then

$$f(e) = \begin{cases} f_1(e), & \text{if } e \in E(G_1) - x_1x_2; \\ f_2(e), & \text{if } e \in E(G_2) - x_1x_2; \\ c, & \text{if } e = x_1x_2 \end{cases}$$

is a strong $k$-edge-coloring of $G$. \hfill \qed
The following lemma from [30] about planar graphs is also useful in the proof of the main theorem. An \(\ell\)-thread is an induced path of \(\ell+2\) vertices all of whose internal vertices are of degree 2 in the full graph.

**Lemma 13.** Any planar graph \(G\) with minimum degree at least 2 and with girth at least \(5\ell+1\) contains an \(\ell\)-thread.

**Proof.** Contract all the vertices of degree 2 to obtain \(G'\). Notice that \(G'\) is a planar graph which may have multi-edges and may be disconnected. Embed \(G' = (V,E)\) in the plane as \(P\). Then Euler’s Theorem says that \(|V| - |E| + |F| \geq 2\), where \(F\) is the set of faces of \(P\). If \(G'\) has girth larger than 5, we have \(2|E| = \sum_{f \in F} \deg(f) \geq 6|F|\). But that \(G'\) has no vertices of degree 2 implies \(2|E| = \sum_{v \in V} \deg(v) \geq 3|V|\). Combining all these produces a contradiction:

\[
2 \leq |V| - |E| + |F| \leq \frac{2}{3}|E| - |E| + \frac{1}{3}|E| = 0.
\]

Hence \(G'\) has a cycle of length at most 5. The corresponding cycle in \(G\) has length at least \(5\ell+1\). Thus one of these edges in \(G'\) is contracted from \(\ell\) vertices in \(G\), and so \(G\) has the required path. \(\square\)

These two lemmas, together with a key lemma to be verified in the next section, lead to the following proof of the main theorem:

**Proof of Theorem 11.** Since the inequality \(\chi'_s(G) \geq \sigma(G)\) is trivial, it suffices to show that \(\chi'_s(G) \leq \sigma(G)\). That is, \(G\) admits a strong \(\sigma\)-edge-coloring \(\varphi\). Suppose to the contrary that there is a counterexample \(G\) with minimum vertex number. Then there is no vertex \(x\) adjacent to \(\deg(x) - 1\) vertices of degree 1. For otherwise, there is a cut edge \(xy\), where \(y\) is not a leaf. By applying Lemma 12 to \(G\) with the cut edge \(xy\) and using the minimality of \(G\), we get a contradiction.

Consider \(H = G - \{x \in V(G) : \deg(x) = 1\}\), which clearly has the same girth as \(G\) since the deletion doesn’t break any cycle. And we have \(\delta(H) \geq 2\), otherwise \(G\) has a vertex \(x\) adjacent to \(\deg(x) - 1\) vertices of degree 1, which is impossible. Lemma 13 claims that there is a path \(x_0x_1 \ldots x_{\ell}+1\) with \(\ell = \sigma + 3\) and \(\deg_H(x_i) = 2\) for \(i = 1,2,\ldots,\ell\). Now let \(G'\) be subgraph obtained from \(G\) by deleting the leaf-neighbors of \(x_2, x_3, \ldots, x_{\ell-1}\) and the vertices \(x_3, x_4, \ldots, x_{\ell-2}\). Consider the subgraph \(T\) of \(G\) induced by \(x_1, x_2, \ldots, x_{\ell}\) and their neighbors, which is a caterpillar tree. By Lemma 18 that will be proved in the next section, \(T\) admits a strong \(\sigma\)-edge-coloring \(\varphi'\) such that \(\varphi\) and \(\varphi'\) coincides on the edges incident to \(x_1\) and \(x_{\ell}\). Gluing these two edge-colorings we construct a strong \(\sigma\)-edge-coloring of \(G\). \(\square\)

### 3 The key lemma: caterpillar with edge pre-coloring

All the graphs in this section are caterpillar trees. Let \(d_i \geq 2\) for \(i = 1,2,\ldots,\ell\). By \(T = \text{Cat}(d_1,d_2,\ldots,d_{\ell})\) we mean a caterpillar tree with spine \(x_0,x_1,\ldots,x_{\ell+1}\), whose degrees
are \(d_0, d_1, \ldots, d_{\ell+1}\), where \(d_0 = d_{\ell+1} = 1\). Call \(\ell\) the length of \(T\) and let \(E_i\) be the edges incident with \(x_i\). See Figure 2 for \(\text{Cat}(5,3,2,4,5)\).

![Figure 2: The caterpillar tree \(\text{Cat}(5,3,2,4,5)\).](image)

Collect all the tuples \((C; \alpha_0, C_1, C_\ell, \alpha_\ell)\) as \(P_\kappa(T)\), where the color sets \(C_1, C_\ell \subseteq C\) with \(|C_1| = d_1, |C_\ell| = d_\ell, |C| = \kappa\), and \(\alpha_0 \in C_1, \alpha_\ell \in C_\ell\). Fix \(\kappa \in \mathbb{N}\). For any \(P = (C; \alpha_0, C_1, C_\ell, \alpha_\ell) \in P_\kappa(T)\), the set of all strong edge-colorings \(\varphi\) using the colors in \(C\) and satisfying the following criterions is denoted by \(C_T(P)\):

\[
\varphi(E_1) = C_1, \quad \varphi(E_\ell) = C_\ell, \quad \varphi(x_0x_1) = \alpha_0 \quad \text{and} \quad \varphi(x_\ell x_{\ell+1}) = \alpha_\ell.
\]

If \(C_T(P)\) is nonempty for any \(P \in P_\kappa(T)\) with \(\kappa \geq \sigma(T)\), then \(T\) is called \(\kappa\)-two-sided strong edge-pre-colorable.

**Lemma 14.** If \(T = \text{Cat}(d_1, d_2, \ldots, d_\ell)\) is \(\kappa\)-two-sided strong edge-pre-colorable, then \(T\) is \(\kappa'\)-two-sided strong edge-pre-colorable for any \(\kappa' \geq \kappa\).

**Proof.** For any \(P' = (C'; \alpha_0', C_1', C_\ell', \alpha_\ell') \in P_\kappa'(T)\), we have to find a strong edge-coloring in \(C_T(P')\).

**Case** \(|C_1' \cup C_\ell'| \leq \kappa\). Choose a \(\kappa\)-set \(C\) so that \(C_1' \cup C_\ell' \subseteq C \subseteq C'\). By assumption, there is a strong edge-coloring in \(C_T(C; \alpha_0', C_1', C_\ell', \alpha_\ell') \subseteq C_T(P')\).

**Case** \(|C_1' \cup C_\ell'| > \kappa\). Choose a \(\kappa\)-set \(C\) so that \(C_1' \cup \{\alpha_\ell'\} \subseteq C \subseteq C_1' \cup C_\ell'\), and a \(d_\ell\)-set \(C_\ell\) so that \(C_\ell' \cap C \subseteq C_\ell \subseteq C\). By assumption, there is a strong edge-coloring \(\varphi\) in \(C_T(C; \alpha_0', C_1', C_\ell, \alpha_\ell')\). Let the edges in \(E_\ell\) with color \(C_\ell - C_\ell'\) be \(E_\ell'\). Notice \(C_\ell' - C_\ell\) and \(C\) are disjoint, so the colors in \(C_\ell' - C_\ell\) are not appeared in \(\varphi\). Hence we can change the colors of \(E_\ell'\) to \(C_\ell' - C_\ell\) and obtain a strong edge-coloring in \(C_T(P')\). \(\square\)

We now derive a series of properties regarding the two-sided strong edge-pre-colorability of a caterpillar tree and its certain subtrees.

**Lemma 15.** Suppose a caterpillar tree \(\bar{T}\) contains \(T\) as a subgraph, and both have the same length. If \(\bar{T}\) is \(\kappa\)-two-sided strong edge-pre-colorable, then \(T\) is also \(\kappa\)-two-sided strong edge-pre-colorable.

**Proof.** Suppose \((C; \alpha_0, C_1, C_\ell, \alpha_\ell) \in P_\kappa(T)\). We find \((C; \alpha_0, C_1', C_\ell', \alpha_\ell) \in P_\kappa(\bar{T})\) such that \(C_1' \supseteq C_1\) and \(C_\ell' \supseteq C_\ell\). The lemma follows that any \(\varphi' \in C_{\bar{T}}(C; \alpha_0, C_1', C_\ell', \alpha_\ell)\) has a restriction \(\varphi\) on \(T\) so that \(\varphi\) is a strong edge-coloring in \(C_T(C; \alpha_0, C_1, C_\ell, \alpha_\ell)\). \(\square\)

For \(T = \text{Cat}(d_1, d_2, \ldots, d_\ell)\), let \(T_{\ell-1}\) be the subtree \(\text{Cat}(d_1, d_2, \ldots, d_{\ell-1})\).
Lemma 16. For $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$, if $T_{-1}$ is $\kappa$-two-sided strong edge-pre-colorable, where $\kappa \geq \sigma(T)$, then so is $T$.

Proof. For any $P = (C; \alpha_0, C_1, C_\ell, \alpha_\ell) \in \mathcal{P}_\kappa(T)$, pick $\alpha_{\ell-1} \in C_\ell - \alpha_\ell$ and $C_{\ell-1}$ a $d_{\ell-1}$-subset of $C$ with $C_{\ell-1} \cap C_\ell = \{\alpha_{\ell-1}\}$. Notice that $C_{\ell-1}$ can be chosen since $d_{\ell-1} + d_\ell - 1 \leq \sigma(T) \leq \kappa$.

By the assumption, $T_{-1}$ admits a strong $\kappa$-edge-coloring $\varphi \in \mathcal{C}_{T_{-1}}(C; \alpha_0, C_1, C_\ell, \alpha_{\ell-1})$. Coloring the remaining edges with $C_\ell - \alpha_{\ell-1}$ so that $x_\ell x_{\ell+1}$ has color $\alpha_\ell$ results in a strong $\kappa$-edge-coloring in $\mathcal{C}_T(P)$. \hfill $\Box$

Hereafter, if necessary we reverse the order to view $T = \text{Cat}(d_\ell, d_{\ell-1}, \ldots, d_1)$ so that we can always assume $\sigma(T_{-1}) = \sigma(T)$. Hence the requirement $\kappa \geq \sigma(T)$ in Lemma 16 automatically holds.

For a caterpillar tree $T$, we define $T'$ and $I_T$ as follows. Call a vertex $x_i$ $\sigma$-large if $d_i \geq d^* := \lceil \frac{\sigma + 1}{2} \rceil$. The value $d^*$ is critical in the sense that

1. If $d_i + d_j \leq \sigma + 1$, then either $d_i$ or $d_j$ must be at most $d^*$.
2. If $d_i + d_j \geq \sigma + 1$, then either $d_i$ or $d_j$ must be at least $d^*$.

Let $S = \{x_i : i \in I_T\}$ be the set of all $\sigma$-large vertices, except that if there exist $i < j$ with $d_{i-1} < d^*$, $d_i = d_{i+1} = \ldots = d_j = d^*$ and $d_{j+1} < d^*$, we only take $x_i, x_{i+2}, x_{i+4}, \ldots$ till $x_j$ or $x_{j-1}$, depending on the parity. Then $S$ is a nonempty independent set. Consider a new degree sequence $d'_1, d'_2, \ldots, d'_\ell$ where

$$d'_i = \begin{cases} 
  d_i - 1, & \text{if } i \in I_T; \\
  d_i, & \text{if } i \notin I_T.
\end{cases}$$

Then $T' = \text{Cat}(d'_1, d'_2, \ldots, d'_\ell)$ is a caterpillar tree isomorphic to a subgraph of $T$, with $\sigma(T') = \sigma(T) - 1$ due to the criticalness of $d^*$ and the choice method of $S$.

It is straightforward to see that $T'_{-1} = \text{Cat}(d'_1, d'_2, \ldots, d'_{\ell-1})$ by the choice method of $S$.

Lemma 17. For $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$, suppose $\sigma(T) \geq 6$ and $T'_{-1}$ is $(\kappa - 1)$-two-sided strong edge-pre-colorable, then $T$ is $\kappa$-two-sided strong edge-pre-colorable.

Proof. For any $P = (C; \alpha_0, C_1, C_\ell, \alpha_\ell) \in \mathcal{P}_\kappa(T)$, we have to show that $\mathcal{C}_T(P)$ is nonempty.

Let $I = I_T$. Our strategy is to search for a color $\beta$ such that

$$\beta \in C_1 \text{ if and only if } 1 \in I; \text{ and } \beta \in C_\ell \text{ if and only if } \ell \in I.$$ 

Suppose such a color $\beta$ exists and $\beta \neq \alpha_\ell$. By Lemma 16, $T'$ admits a strong $(\kappa - 1)$-edge coloring in $\mathcal{C}_{T'}(C - \beta; \alpha_0, C_1 - \beta, C_\ell - \beta, \alpha_\ell)$. Coloring the remaining edges with $\beta$ then yields the required strong $\kappa$-edge-coloring in $\mathcal{C}_T(P)$. Notice that $S$ being an independent set guarantees that the edges with color $\beta$ form an induced matching. If it happens that $\beta$ coincides with $\alpha_\ell$, then we seek instead for strong-edge coloring in $\mathcal{C}_{T'}(C - \beta; C_1 - \beta, \alpha_0, C_\ell - \beta, \alpha_\ell)$. 


Similarly, there is a strong edge-coloring in $C_{t_{-1}} \subseteq C$ and $C_{t_{-1}} \cap C_t = \{\alpha_{t_{-1}}\}$, there will be a $\beta$ such that

$$\beta \in C_t \text{ if and only if } 1 \in I; \text{ and } \beta \in C_{t_{-1}} \text{ if and only if } \ell - 1 \in I.$$ 

Similarly, there is a strong edge-coloring in $C_{T_{-1}}(C; \alpha_0, C_1, \alpha_{t_{-1}})$. Color the remaining edges with $C_t - \alpha_{t_{-1}}$ so that $x_{t}x_{t_{+1}}$ has color $\alpha_t$, we gain a strong $\kappa$-edge-coloring in $C_T(P)$.

We now prove the existence of $\beta$ according to the following four cases.

**Case 1.** $1, \ell \in I$. In this case, $C_1 \cap C_\ell$ is nonempty since

$$|C_1 \cap C_\ell| = |C_1| + |C_\ell| - |C_1 \cup C_\ell| \geq 2d^* - \sigma > 0.$$ 

Pick $\beta$ to be any color in the intersection.

**Case 2.** $1 \in I$ but $\ell \notin I$. If $C_1 - C_\ell$ is nonempty, then pick $\beta$ to be any color in the difference. Otherwise, $1 \in I$ and $\ell \notin I$ imply $d_1 \geq d^* \geq d_\ell$. On the other hand, $C_1 - C_\ell = \emptyset$ implies $d_1 \leq d_\ell$. Thus the situation that $C_1 - C_\ell$ is empty occurs only when $d_1 = d_\ell = d^*$ and $C_1 = C_\ell$. We consider the subtree $T_{-1}$. Choose $\alpha_{t_{-1}}$ to be any color in $C_t - \alpha_{t_{-1}}$. Let $C_{t_{-1}}$ be $\alpha_{t_{-1}}$ together with any $(d_{t_{-1}} - 1)$-subset in $C - C_t$.

Since $d_{\ell} = d^*$ but $\ell \notin I$, it is the case that $\ell - 1 \in I$ and $d_{t_{-1}} = d^*$. Pick $\beta = \alpha_{t_{-1}}$.

**Case 3.** $\ell \in I$ but $1 \notin I$. If $C_t - C_1$ is nonempty, then let $\beta$ be any color in the difference. Otherwise, $d_1 = d_\ell = d^*$ and $C_1 = C_\ell$. But $d_1 = d^*$ implies $1 \in I$, a contradiction.

**Case 4.** $1, \ell \notin I$. If $C - (C_1 \cup C_\ell)$ is nonempty, then pick $\beta$ to be any color in the difference set. Now, suppose $C = C_1 \cup C_\ell$. We consider the subtree $T_{-1}$.

First estimate the size

$$|C_t - C_1| = |C_t \cup C_1| - |C_1| \geq \sigma - d^* \geq d^* - 2 \geq 2,$$

where $d^* \geq 4$ since $\sigma \geq 6$. Pick $\alpha_{t_{-1}}$ to be any color in $C_t - C_1 - \alpha_\ell$. Let $C_{t_{-1}}$ be a color set such that $|C_{t_{-1}}| = d_{t_{-1}}$ and $C_{t_{-1}} \cap C_t = \{\alpha_{t_{-1}}\}$.

When $\ell - 1 \in I$, pick $\beta = \alpha_{t_{-1}}$. Otherwise, let $\beta$ be chosen from $C_t - C_1 - \alpha_{t_{-1}}$. 

Now we are ready to prove the key lemma.

**Lemma 18.** Suppose $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$ is a nice caterpillar tree, i.e. it satisfies

$$\sigma = \sigma(T) \geq 5, \ \ell \geq \sigma + 3 \ \text{and} \ \sigma \geq \Delta(T) + 2.$$ 

For any $\kappa \geq \sigma(T)$, any color sets $C_1, C_\ell \subseteq C$ with $|C| = \kappa$, $|C_1| = d_1$, $|C_\ell| = d_\ell$, and any two colors $\alpha_0 \in C_1, \alpha_\ell \in C_\ell$, there is a strong $\sigma$-edge colorings $\varphi$ using the colors in $C$ such that $\varphi(E_1) = C_1$, $\varphi(E_\ell) = C_\ell$ and $\varphi(x_0x_1) = \alpha_0$, $\varphi(x_\ell x_{\ell+1}) = \alpha_\ell$. That is, $T$ is $\kappa$-two-sided strong edge-pre-colorable for any $\kappa \geq \sigma$. 

8
Proof. We prove the lemma by induction on \( \sigma = \sigma(T) \). By Lemmas 14 and 16 it suffices to consider the case \( \kappa = \sigma \) and \( \ell = \ell_\sigma \).

If \( T \) is nice and \( \sigma \geq 6 \), then \( T'_{-1} \) is also a nice caterpillar tree: The first two conditions remain since \( \sigma(T'_{-1}) = \sigma(T') = \sigma(T) - 1 \). The third one \( \sigma(T'_{-1}) + 1 \leq \Delta(T'_{-1}) + 2 \leq \Delta(T) + 2 \) and so \( \Delta(T') = \Delta(T) \). Since \( \Delta(T) \geq d^* \), in this case, \( \Delta(T') = \Delta(T) + 2 \geq \Delta(T'_{-1}) + 2 \).

By Lemma 17 we only have to discuss the base cases \( \sigma = 5 \) and \( \ell = 8 \). We may assume all degrees \( d_i = 3 \) since \( \sigma \geq \Delta + 2 \). Also assume \( C_1 = \{1, 2, 3\} \) and \( \alpha_0 = 1 \). Depending on \( C_1 \cap C_8 \) and whether \( \alpha_8 = \alpha_0 \) or not, by symmetry we color \( T \) according to \( \varphi \) shown in Table 1, where \( \alpha_i = \varphi(x_i x_{i+1}) \) and \( \hat{C}_i = \varphi(C_i) - \varphi(x_i) - \varphi(x_{i+1}) \). Or we can solve this case by the argument in [7] or the odd graph method in [11, 40].

Table 1: The 5-strong edge-colorings of \( T \) for \( \sigma = 5 \) with \( \ell = 8 \).

4 Refinement of Lemma 18

We now discuss the optimality of Lemma 18. If we take more care about the base cases, there would be a refinement:

Lemma 19. Suppose \( T \) is a caterpillar tree of length \( \ell \) satisfying

\[
\sigma = \sigma(T) \geq 5, \quad \ell \geq \ell_\sigma \quad \text{and} \quad \sigma \geq \Delta(T) + 2,
\]

where

\[
\ell_\sigma = \begin{cases} 
8, & \text{if } \sigma = 5; \\
7, & \text{if } \sigma = 6, 7; \\
\sigma, & \text{if } \sigma \geq 8.
\end{cases}
\]

Then \( T \) is \( \kappa \)-two-sided strong edge-pre-colorable for any \( \kappa \geq \sigma \).
Proof. Similar to Lemma 15, we only need to consider the base cases.

For $\sigma = 6$, we first consider the situation $\ell = 6$. By Lemma 15 and the symmetry, it suffices to discuss the caterpillar trees $\text{Cat}(4, 3, 4, 3, 4, 3)$, $\text{Cat}(4, 3, 4, 3, 4, 3, 4)$, and $\text{Cat}(3, 4, 3, 4, 3, 4, 3)$. We enumerate all the cases in Table 2 and Table 3 to show that the first two are 6-two-sided strong edge-pre-colorable.

### Table 2: The 6-strong edge-colorings for $T = \text{Cat}(4, 3, 4, 3, 4, 3)$.

| $\alpha_0$ | $\hat{C}_1$ | $\alpha_1$ | $\hat{C}_2$ | $\alpha_2$ | $\hat{C}_3$ | $\alpha_3$ | $\hat{C}_4$ | $\alpha_4$ | $\hat{C}_5$ | $\alpha_5$ | $\hat{C}_6$ | $\alpha_6$ |
|------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1          | $\{3, 4\}$ | 2           | $\{6\}$    | 5           | $\{3, 4\}$  | 1           | $\{2\}$    | 6           | $\{4, 5\}$  | 3           | $\{2\}$    | 1           |
| 1          | $\{2, 4\}$ | 3           | $\{5\}$    | 6           | $\{1, 4\}$  | 2           | $\{3\}$    | 5           | $\{4, 5\}$  | 1           | $\{3\}$    | 2           |
| 1          | $\{3, 4\}$ | 2           | $\{6\}$    | 5           | $\{3, 4\}$  | 1           | $\{2\}$    | 6           | $\{3, 4\}$  | 5           | $\{2\}$    | 1           |
| 1          | $\{2, 4\}$ | 3           | $\{5\}$    | 6           | $\{1, 4\}$  | 2           | $\{5\}$    | 3           | $\{4, 6\}$  | 1           | $\{5\}$    | 2           |
| 1          | $\{2, 4\}$ | 3           | $\{5\}$    | 6           | $\{2, 4\}$  | 1           | $\{5\}$    | 3           | $\{4, 6\}$  | 2           | $\{1\}$    | 5           |
| 1          | $\{2, 4\}$ | 3           | $\{5\}$    | 6           | $\{2, 4\}$  | 1           | $\{5\}$    | 3           | $\{2, 4\}$  | 6           | $\{5\}$    | 1           |
| 1          | $\{2, 3\}$ | 4           | $\{5\}$    | 6           | $\{2, 3\}$  | 1           | $\{5\}$    | 4           | $\{2, 3\}$  | 6           | $\{1\}$    | 5           |
| 1          | $\{3, 4\}$ | 2           | $\{6\}$    | 5           | $\{1, 4\}$  | 3           | $\{2\}$    | 6           | $\{1, 5\}$  | 4           | $\{3\}$    | 2           |
| 1          | $\{2, 3\}$ | 4           | $\{5\}$    | 6           | $\{1, 3\}$  | 2           | $\{5\}$    | 4           | $\{1, 6\}$  | 3           | $\{5\}$    | 2           |
| 1          | $\{2, 3\}$ | 4           | $\{5\}$    | 6           | $\{1, 3\}$  | 2           | $\{5\}$    | 4           | $\{1, 6\}$  | 3           | $\{2\}$    | 5           |
| 1          | $\{2, 3\}$ | 4           | $\{5\}$    | 6           | $\{1, 3\}$  | 2           | $\{5\}$    | 4           | $\{1, 3\}$  | 6           | $\{5\}$    | 2           |
| 1          | $\{2, 4\}$ | 3           | $\{5\}$    | 6           | $\{1, 4\}$  | 2           | $\{5\}$    | 3           | $\{1, 4\}$  | 6           | $\{2\}$    | 5           |

### Table 3: The 6-strong edge-colorings for $T = \text{Cat}(4, 3, 4, 3, 4, 3)$.

| $\alpha_0$ | $\hat{C}_1$ | $\alpha_1$ | $\hat{C}_2$ | $\alpha_2$ | $\hat{C}_3$ | $\alpha_3$ | $\hat{C}_4$ | $\alpha_4$ | $\hat{C}_5$ | $\alpha_5$ | $\hat{C}_6$ | $\alpha_6$ |
|------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1          | $\{2, 4\}$ | 3           | $\{5\}$    | 6           | $\{2, 4\}$  | 1           | $\{3\}$    | 5           | $\{6\}$    | 4           | $\{2, 3\}$  | 1           |
| 1          | $\{3, 4\}$ | 2           | $\{5\}$    | 6           | $\{1, 4\}$  | 3           | $\{2\}$    | 5           | $\{6\}$    | 1           | $\{3, 4\}$  | 2           |
| 1          | $\{3, 4\}$ | 2           | $\{5\}$    | 6           | $\{1, 3\}$  | 4           | $\{5\}$    | 2           | $\{6\}$    | 3           | $\{4, 5\}$  | 1           |
| 1          | $\{2, 4\}$ | 3           | $\{6\}$    | 5           | $\{1, 2\}$  | 4           | $\{3\}$    | 6           | $\{2\}$    | 1           | $\{4, 5\}$  | 3           |
| 1          | $\{2, 4\}$ | 3           | $\{6\}$    | 5           | $\{1, 2\}$  | 4           | $\{3\}$    | 6           | $\{2\}$    | 1           | $\{3, 4\}$  | 5           |
| 1          | $\{3, 4\}$ | 2           | $\{5\}$    | 6           | $\{3, 4\}$  | 1           | $\{5\}$    | 2           | $\{3\}$    | 6           | $\{4, 5\}$  | 1           |
| 1          | $\{3, 4\}$ | 2           | $\{6\}$    | 5           | $\{3, 4\}$  | 1           | $\{6\}$    | 2           | $\{3\}$    | 5           | $\{1, 6\}$  | 4           |
| 1          | $\{3, 4\}$ | 2           | $\{6\}$    | 5           | $\{1, 3\}$  | 4           | $\{6\}$    | 2           | $\{3\}$    | 1           | $\{4, 6\}$  | 5           |
| 1          | $\{3, 4\}$ | 2           | $\{6\}$    | 5           | $\{1, 4\}$  | 3           | $\{2\}$    | 6           | $\{1\}$    | 4           | $\{3, 5\}$  | 2           |
| 1          | $\{2, 4\}$ | 3           | $\{6\}$    | 5           | $\{1, 2\}$  | 4           | $\{3\}$    | 6           | $\{1\}$    | 2           | $\{3, 4\}$  | 5           |
| 1          | $\{3, 4\}$ | 2           | $\{5\}$    | 6           | $\{1, 4\}$  | 3           | $\{5\}$    | 2           | $\{1\}$    | 6           | $\{4, 5\}$  | 3           |
| 1          | $\{3, 4\}$ | 2           | $\{6\}$    | 5           | $\{1, 3\}$  | 4           | $\{6\}$    | 2           | $\{1\}$    | 3           | $\{4, 6\}$  | 5           |

If the caterpillar tree $T$ considered with $\sigma = 6$ and $\ell = 7$ has $T_{-1} = \text{Cat}(3, 4, 3, 4, 3, 4, 3)$, then $T$ is a subtree of $\text{Cat}(3, 4, 3, 4, 3, 4, 4)$. We can assume $T = \text{Cat}(3, 4, 3, 4, 3, 4, 3)$ by Lemma 15. Reverse the direction to see $T$ as $\text{Cat}(4, 3, 4, 3, 4, 3, 4)$. Then the subtree $T_{-1} = \text{Cat}(4, 3, 4, 3, 4, 3)$, which is 6-two-sided strong edge-pre-colorable. Hence all the caterpillar trees with $\sigma = 6$ and $\ell = 7$ are 6-two-sided strong edge-pre-colorable.
For $\sigma = 7$ and $\ell = 7$. It suffices to consider the caterpillar trees in Table 4.

\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|}
\hline
$T$ & $T'_{-1}$ \\
\hline
Cat(3,5,3,5,3,5,3) & Cat(3,4,3,4,3,4) \\
Cat(5,3,3,5,3,5,3) & Cat(4,3,4,3,4,3) \\
Cat(5,3,3,5,3,5,4,4) & Cat(4,3,4,3,4,3) \\
Cat(4,3,4,3,4,4,4) & Cat(3,4,3,4,3,4) \\
Cat(5,3,4,4,4,4,4) & Cat(4,3,4,3,4,3) \\
Cat(3,4,3,4,4,3,4) & Cat(3,4,3,4,3,4) \\
Cat(4,4,3,5,3,3,5) & Cat(3,4,3,4,3,4) \\
Cat(4,4,3,5,3,3,3) & Cat(3,4,3,4,3,4) \\
\hline
\end{tabular}
\end{center}
\caption{The caterpillar trees to be considered for $\sigma = 7$ and $\ell = 7$.}
\end{table}

All the trees $T$ considered except Cat(3,5,3,4,4,4,4) and Cat(3,5,3,4,4,3,5) have $T'_{-1}$ being 6-two-sided strong edge-pre-colorable, so these $T$ are 7-two-sided strong edge-pre-colorable by Lemma 17.

If we see Cat(3,5,3,4,4,4,4) as $T = Cat(4,4,4,4,3,5,3)$, then $T'_{-1} = Cat(3,4,3,4,3,4)$ is 6-two-sided strong edge-pre-colorable. Similarly, regard Cat(3,5,3,4,4,3,5) as $T = Cat(5,3,4,4,3,5,3)$, then $T'_{-1} = Cat(4,3,4,3,4,3)$ is 6-two-sided strong edge-pre-colorable. So these two trees are also 7-two-sided strong edge-pre-colorable by Lemma 17, and hence all the caterpillar trees considered with $\sigma = 7$ and $\ell = 7$ are 7-two-sided strong edge-pre-colorable.

The $\ell_\sigma$ here cannot be reduced: For $\sigma \geq 7$, consider $\ell = \sigma - 1$ and $T = Cat(d_1,d_2,\ldots,d_\ell)$, where $d_1,d_3\cdots = \lceil \frac{\sigma + 1}{2} \rceil$ and $d_2,d_4\cdots = \lceil \frac{\sigma + 1}{2} \rceil$.

If $\sigma = 2d - 1$ is an odd integer, let $P = ([1,\sigma];1,[1,d],[1,1,1]) \in P_\sigma(T)$. Suppose there is some $\varphi \in C_T(P)$. Let $C_1 = \varphi(E_1)$. Then $|C_i+C_i| = 1$ for $i = 1,2,\ldots,\ell-2$. So $|C_\ell-C_2| \leq d-2$. However, $C_1 = C_\ell$ implies $|C_\ell-C_2| = d-1$, a contradiction.

If $\sigma = 2d-2$ is an even integer, let $P = ([1,\sigma];1,[1,d-1],[2d-2],d) \in P_\sigma(T)$. Suppose there is some $\varphi \in C_T(P)$. Let $C_1 = \varphi(E_1)$. Again $|C_{i+2}-C_i| = 1$ for $i = 1,2,\ldots,\ell-2$. So $d-1 = |C_\ell-C_1| \leq d-2$, a contradiction.

For $\sigma = 6$, let $T = Cat(3,4,3,3,4,3)$ and $P = ([1,6];1,\{1,2,3\},\{4,5,6\},6) \in P_\sigma(T)$. Suppose there is some $\varphi \in C_T(P)$. Let $C_1 = \varphi(E_1)$ and $X = C_3 \cup C_4 - \varphi(x_2x_3) - \varphi(x_4x_5)$. Then $|C_1 \cap X| = |X \cap C_6| = 2$ implies $|X| \geq 4$, which is impossible since $|X| = 3$.

Exploiting Lemma 19 the main Theorem 11 can be strengthened to:
Theorem 20. If $G$ is a planar graph with $\sigma = \sigma(G) \geq 5$, $\sigma \geq \Delta(G) + 2$ and girth at least $g_\sigma$, where

$$g_\sigma = \begin{cases} 
41, & \text{if } \sigma = 5; \\
36, & \text{if } \sigma = 6, 7; \\
5\sigma + 1, & \text{if } \sigma \geq 8,
\end{cases}$$

then $\chi'_s(G) = \sigma$.

If we take off the condition $\sigma \geq \Delta + 2$ in Theorem 20, a weaker result can be obtained by using the following corollary of Lemma 19 in the proof of the main Theorem 11.

Corollary 21. Suppose $T$ is a caterpillar tree of length $\ell$ satisfying

$$\sigma = \sigma(T) \geq 4 \text{ and } \ell \geq \ell_{\sigma+1},$$

where

$$\ell_{\sigma+1} = \begin{cases} 
8, & \text{if } \sigma + 1 = 5; \\
7, & \text{if } \sigma + 1 = 6, 7; \\
\sigma + 1, & \text{if } \sigma + 1 \geq 8.
\end{cases}$$

Then $T$ is $\kappa$-two-sided strong edge-pre-colorable for any $\kappa \geq \sigma + 1$.

Proof. Add pendant edges at some vertices of $T$ with degree $\delta(T)$ such that the resulting graph $\tilde{T}$ has $\sigma(\tilde{T}) = \sigma(T) + 1$ and $\sigma(\tilde{T}) \geq \Delta(\tilde{T}) + 2$. So $\tilde{T}$ satisfies the requirements of Lemma 19 and hence it is $\kappa$-two-sided strong edge-pre-colorable for any $\kappa \geq \sigma(\tilde{T}) = \sigma(T) + 1$. The corollary then follows from Lemma 13. \qed

Theorem 22. If $G$ is a planar graph with $\sigma = \sigma(G) \geq 4$ and girth at least $g_{\sigma+1}$, where

$$g_{\sigma+1} = \begin{cases} 
41, & \text{if } \sigma + 1 = 5; \\
36, & \text{if } \sigma + 1 = 6, 7; \\
5\sigma + 6, & \text{if } \sigma + 1 \geq 8,
\end{cases}$$

then $\sigma \leq \chi'_s(G) \leq \sigma + 1$.

5 Consequences concerning the maximum average degree

The following lemma is a direct consequence of Proposition 2.2 in [14].

Lemma 23. Suppose the connected graph $G$ is not a cycle. If $G$ has minimum degree at least 2 and average degree $\frac{2|E|}{|V|} < 2 + \frac{2}{3\ell-1}$, then $G$ contains an $\ell$-thread.

A $C_n$-jellyfish is a graph by adding pendant edges at the vertices of $C_n$. In [9], it is shown that
Proposition 24. If $G$ is a $C_n$-jellyfish of $m$ edges with $\sigma(G) \geq 4$, then $\chi'_s(G) =$
\[
\begin{cases}
  m, & \text{if } n = 3; \\
  \sigma(G) + 1, & \text{if } n = 4; \\
  \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil, & \text{otherwise, if } n \text{ is odd with all } \deg(v_i) = d \text{ but } (n, d) \neq (7, 3),\
  \sigma(G) + 1, & \text{otherwise, if } (n, d) = (7, 3) \text{ with all } \deg(v_i) = d, \\
  \sigma(G), & \text{otherwise}.
\end{cases}
\]

Adopting these results leads to a strengthening of Theorem 10.

Theorem 25. If $G$ is a graph with $\sigma = \sigma(G) \geq 5$, $\sigma \geq \Delta(G) + 2$, odd girth at least $g'_\sigma$, even girth at least 6, and $\text{mad}(G) < 2 + \frac{2}{\sigma+1-1}$, where
\[
g'_\sigma = \begin{cases} 
  9, & \text{if } \sigma = 5; \\
  \sigma, & \text{if } \sigma > 5,
\end{cases}
\]

and
\[
\ell_\sigma = \begin{cases} 
  8, & \text{if } \sigma = 5; \\
  7, & \text{if } \sigma = 6; \\
  \sigma, & \text{if } \sigma \geq 8,
\end{cases}
\]

then $\chi'_s(G) = \sigma$.

Proof. In the proof of Theorem 20 alternatively use Lemma 23 to find an $\ell_\sigma$-thread in $H$. It should be noticed the girth constraints exist merely to address the problem that $H$ may be a cycle. In this case, by Proposition 24 $G$ still has strong chromatic index $\sigma$.

Indeed, suppose $H = C_n$ and $G$ is a $C_n$-jellyfish. The case $n$ is even is trivial. If $\sigma \geq \sigma(H) \geq 5$, $n$ is odd and $n \geq g'_\sigma \geq \sigma$, then
\[
\left\lfloor \frac{|E(G)|}{\frac{n}{2}} \right\rfloor \leq \left\lfloor \frac{n-1}{2} (\sigma - 1) + \frac{\sigma+1}{2} - 1 \right\rfloor \leq \sigma.
\]

Hence $\chi'_s(G) = \sigma$.

Similarly, Theorem 22 can be modified correspondingly.

Theorem 26. If $G$ is a graph with $\sigma = \sigma(G) \geq 4$, odd girth at least $\frac{\sigma+1}{2}$, and $\text{mad}(G) < 2 + \frac{2}{\sigma+1-1}$, where
\[
\ell_{\sigma+1} = \begin{cases} 
  8, & \text{if } \sigma + 1 = 5; \\
  7, & \text{if } \sigma + 1 = 6; \\
  \sigma + 1, & \text{if } \sigma + 1 \geq 8,
\end{cases}
\]

then $\sigma \leq \chi'_s(G) \leq \sigma + 1$.

Acknowledgements. This project was supported in part by the Ministry of Science and Technology (Taiwan) under grant 104-2115-M-002-006-MY2. The authors thank Tao Wang for extensive discussion and providing many useful comments.
References

[1] L. D. Andersen, The strong chromatic index of a cubic graph is at most 10, *Discrete Math.*, 108(1-3):231–252, 1992.

[2] K. Appel and W. Haken, Every planar map is four colorable. Part I: Discharging, *Illinois J. of Math.*, 21(3):429–490, 1977.

[3] K. Appel, W. Haken, and J. Koch, Every planar map is four colorable. Part II: Reducibility, *Illinois J. of Math.*, 21(3):491–567, 1977.

[4] C. Barrett, G. Istrate, A. V. S. Kumar, M. Marathe, S. Thite, and S. Thulasidasan, Strong edge coloring for channel assignment in wireless radio networks, in *PERCOMW ’06: Proceedings of the 4th Annual IEEE International Conference on Pervasive Computing and Communications Workshops*, pp. 106–110, IEEE Computer Society, Washington, DC, 2006.

[5] M. Basavaraju and M. C. Francis, Strong chromatic index of chordless graphs, *J. Graph Theory*, 80(1):58–68, 2015.

[6] J. Bensmail, A. Harutyunyan, H. Hocquard, and P. Valicov, Strong edge-colouring of sparse planar graphs, *Discrete Applied Math.*, 179:229–234, 2014.

[7] O. V. Borodin and A. O. Ivanova, Precise upper bound for the strong edge chromatic number of sparse planar graphs, *Discuss. Math. Graph Theory*, 33(4):759–770, 2013.

[8] H. Bruhn and F. Joos, A stronger bound for the strong chromatic index, preprint at *arXiv*: 1504.02583v1, 22 pages, 2015.

[9] G. J. Chang, S.-H. Chen, C.-Y. Hsu, C.-M. Hung, and H.-L. Lai, Strong edge-coloring for jellyfish graphs, *Discrete Math.*, 338(12):2348–2355, 2015.

[10] G. J. Chang and D. D.-F. Liu, Strong edge-coloring for cubic Halin graphs, *Discrete Math.*, 312(8):1468–1475, 2012.

[11] G. J. Chang, M. Montassier, A. Pêcher, and A. Raspaud, Strong chromatic index of planar graphs with large girth, *Discuss. Math. Graph Theory*, 34(4):723–733, 2014.

[12] G. J. Chang and N. Narayanan, Strong chromatic index of 2-degenerate graphs, *J. Graph Theory*, 73(2):119–126, 2013.

[13] D. W. Cranston, Strong edge-coloring of graphs with maximum degree 4 using 22 colors, *Discrete Math.*, 306(21):2772–2778, 2006.

[14] D. W. Cranston, D. B. West, A guide to the discharging method, preprint at *arXiv*: 1306.4434v1, 77 pages, 2013.
[15] M. Dębski, J. Grytczuk, and M. Śleszyńska Nowak, The strong chromatic index of sparse graphs, *Inform. Process. Lett.*, 115(2):326–330, 2015.

[16] P. Erdős, Problems and results in combinatorial analysis and graph theory, *Discrete Math.*, 72(1-3):81–92, 1988.

[17] P. Erdős and J. Nešetřil. Problems, in *Irregularities of Partitions*, pp. 162–163, Springer, Berlin, 1989.

[18] R. J. Faudree, A. Gyárfás, R. H. Schelp, and Z. Tuza, The strong chromatic index of graphs, *Ars Combin.*, 29B:205–211, 1990.

[19] J. L. Fouquet and J. L. Jolivet, Strong edge-colorings of graphs and applications to multi-k-gons, *Ars Combin.*, 16:141–150, 1983.

[20] J. L. Fouquet and J. L. Jolivet, Strong edge-coloring of cubic planar graphs, in *Progress in graph theory*, pp. 247–264, Academic Press, Toronto, 1984.

[21] H. Grötzsch, Ein dreifarbensatz fur dreikreisfreie netze auf der kugel, *Math.-Nat. Reihe*, 8:109–120, 1959.

[22] P. Horák, H. Qing, and W. T. Trotter, Induced matchings in cubic graphs, *J. Graph Theory*, 17(2):151–160, 1993.

[23] D. Hudák, B. Luzar, R. Soták, and R. Skrekovski, Strong edge-coloring of planar graphs, *Discrete Math.*, 324:41–49, 2014.

[24] J. Janssen and L. Narayanan, Approximation algorithms for channel assignment with constraints, *Theoret. Comput. Sci.*, 262(1–2):649–667, 2001.

[25] H.-H. Lai, K.-W. Lih, and P.-Y. Tsai, The strong chromatic index of Halin graphs, *Discrete Math.*, 312(9):1536–1541, 2012.

[26] K.-W. Lih and D. D.-F. Liu, On the strong chromatic index of cubic Halin graphs, *Appl. Math. Lett.*, 25(5):898–901, 2012.

[27] M. Mahdian, The strong chromatic index of $C_4$-free graphs, *Random Structures and Algorithms*, 17(3-4):357–375, 2000.

[28] M. Molloy and B. Reed, A bound on the strong chromatic index of a graph, *J. Combin. Theory Ser. B*, 69(2):103–109, 1997.

[29] T. Nandagopal, T.-E. Kim, X. Gao, and V. Bharghavan, Achieving MAC layer fairness in wireless packet networks, in *MobiCom ’00: Proceedings of the 6th annual international conference on Mobile computing and networking*, pp. 87–98, ACM, New York, 2000.
[30] J. Nešetřil, A. Raspaud, and E. Sopena, Colorings and girth of oriented planar graphs, *Discrete Math.*, 165/166:519–530, 1997.

[31] S. Ramanathan, A unified framework and algorithm for channel assignment in wireless networks, *Wireless Networks*, 5(2):81–94, 1999.

[32] S. Ramanathan and E. L. Lloyd, Scheduling algorithms for multihop radio networks, *IEEE/ACM Transactions on Networking*, 1(2):166–177, 1993.

[33] D. P. Sanders and Y. Zhao, Planar graphs of maximum degree seven are class I, *J. Combin. Theory Ser. B*, 83(2):201–212, 2001.

[34] W. C. Shiu, P. C. B. Lam, and W. K. Tam, On strong chromatic index of Halin graphs, *J. Combin. Math. Combin. Comput.*, 57:211–222, 2006.

[35] W. C. Shiu and W. K. Tam, The strong chromatic index of complete cubic Halin graphs, *Appl. Math. Lett.*, 22(5):754–758, 2009.

[36] H. Tamura, K. Watanabe, M. Sengoku, and S. Shinoda, A channel assignment problem in multihop wireless networks and graph theory, *Journal of Circuits, Systems and Computers*, 13(02):375–385, 2004.

[37] O. Togni, Strong chromatic index of products of graphs, *Discrete Math. Theor. Comput. Sci.*, 9(1):47–56, 2007.

[38] V. G. Vizing, Critical graphs with given chromatic class, *Diskretnyi Analiz*, 5:9–17, 1965.

[39] T. Wang, Strong chromatic index of $k$-degenerate graphs, *Discrete Math.*, 330:17–19, 2014.

[40] T. Wang and X. Zhao, Odd graph and its application on the strong edge coloring, preprint at arXiv: 1412.8358v3, 7 pages, 2015.

[41] G. Yu, Strong edge-colorings for $k$-degenerate graphs, *Graphs and Combin.*, 31(5):1815–1818, 2015.