The Energy for 2 + 1 Dimensional Black Hole Solutions

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ABSTRACT

The energy distributions of four 2+1 dimensional black hole solutions were obtained by using the Einstein and Møller energy-momentum complexes. While \( r \to \infty \), the energy distributions of Virbhadra’s solution for the Einstein-massless scalar equation becomes

\[
E_{\text{Ein}} \sim \frac{\pi}{\kappa} (1 - q) R^q \quad \text{and} \quad E_{\text{Møl}} \sim -\frac{2\pi}{\kappa} q(1 - q) R^q,
\]

and the energy distributions of these three solutions become

\[
E_{\text{Ein}} \sim \frac{\pi \Lambda r^2}{\kappa} \quad \text{and} \quad E_{\text{Møl}} \sim -\frac{4\pi \Lambda r^2}{\kappa}.
\]

PACS No.:04.20.-q; 11.10.Kk

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1 INTRODUCTION

One of the most interesting questions which remains unsolved in the general theory of relativity is the energy-momentum localization. Numerous attempts have been made in the past for a solution, and this question still attracts considerable attention in the literature, and remains an important issue to be settled. After the expression obtained by Einstein [1, 2, 3] for the energy-momentum complexes, many physicist, such as Landau and Lifshitz [4], Papapetrou [5], Bergmann [6], Weinberg [7] and Møller [2] had given different definitions for the energy-momentum complexes. These definitions, except that of Møller, are restricted to evaluate the energy distribution in quasi-Cartesian coordinates. In his opinion Lessner [8] sustained that the Møller energy-momentum complex is an important concept of energy and momentum in general relativity. Some interesting results recently obtained [9, 10, 11, 12] sustain the conclusion that the Møller energy-momentum complex gives reasonable results for many space-times. Also, Cooperstock [13] gave his opinion that the energy and momentum are confined to the regions of non-vanishing energy-momentum tensor of the matter and all non-gravitational fields.

2 BLACK HOLE SOLUTIONS IN 2 + 1 DIMENSIONS

In recent years the several gravity models in 2+1 dimensions have gained considerable attention [14]. Due to an expectation is that the study of 2+1 dimensional theories would provide relevant information about the corresponding theory in 3+1 dimensions. In 2+1 dimensions the number of independent components of the Riemann curvature tensor and the Einstein tensor are the same, consequently the imposition of Einstein’s equations in vacuum implies that the curvature tensor also vanishes. Therefore, the space-time described by the vacuum solutions to Einstein equations in 2+1 dimensions are no gravitational wave and no interaction between masses, and the existence of black hole would be prevented [15]. However, Bañados, Teitelboim, and Zanelli (BTZ) [16] have discovered a black hole solution to the Einstein-Maxwell equations (with a negative cosmological constant) in 2+1
dimensions, which is characterized by mass, angular momentum, and charge parameters. Because of energy-momentum is a fundamental conserved quantity associated with a symmetry of space-time geometry. Thus, there are many study about that the energy of 2+1 dimensional black hole solutions in different gravity models [17]. In this article, we will study the energy distribution of non-rotating black hole solutions in 2 + 1 dimensions with Einstein and Møller energy-momentum complexes. We use the geometrized units \((G = c = 1)\) and adopt the signature of the metric with \((- , + , +)\). It follows the convention that the Latin indices run from 1 to 2 and the Greek indices run from 0 to 2 . Now, we investigate these following 2+1 dimensional non-rotating black hole solutions with non-zero cosmological constant:

(i) Uncharged black hole solution [18]

The solution is given by

\[
ds^2 = -(\Lambda r^2 - M)dt^2 + (\Lambda r^2 - M)^{-1}dr^2 + r^2d\theta^2,
\]

where \(\Lambda\) is the cosmological constant.

(ii) Charged black hole [18]

This solution is expressed by the line element

\[
ds^2 = -(\Lambda r^2 - M - 2Q^2 \ln \left(\frac{r}{r_+}\right))dt^2 + (\Lambda r^2 - M - 2Q^2 \ln \left(\frac{r}{r_+}\right))^{-1}dr^2 + r^2d\theta^2,
\]

where \(r_+ = \sqrt{M/\Lambda}\).

(iii) Coupling to a static scalar field [18]

The solution of 2+1 dimensional gravity field coupled to a static scalar field is described by

\[
ds^2 = -\left(\frac{(r - 2B)(B + r)^2\Lambda}{r}\right)dt^2 + \left(\frac{(r - 2B)(B + r)^2\Lambda}{r}\right)^{-1}dr^2 + r^2d\theta^2
\]

(iv) The static and circularly symmetric exact solution of the Einstein-massless scalar equation [19]

The static and circularly symmetric exact solution of the Einstein-massless scalar equation is given by

\[
ds^2 = -Bdt^2 + B^{-1}dr^2 + r^2d\theta^2,
\]

where \(B = (1 - q)R^3\), \(R = r/r_0\) and \(q\) stands for the scalar charge.
3 ENERGY IN THE EINSTEIN PRESCRIPTION

The well known energy-momentum complex of Einstein [1] is defined as

$$\Theta^\nu_\mu = \frac{1}{2\kappa} \frac{\partial}{\partial x^\sigma} H^\nu_\mu,$$

with superpotential

$$H^\nu_\mu = \frac{g_{\mu\rho}}{\sqrt{-g}} \frac{\partial}{\partial x^\eta} [(-g)(g^{\nu\eta} g^{\sigma\eta} - g^{\sigma\rho} g^{\eta\rho})],$$

and the gravitational coupling constant \(\kappa\). Then we evaluate the energy and momentum by Einstein energy-momentum complex in quasi-Cartesian coordinates of 2+1 dimensional space-time,

$$P_\mu = \frac{1}{2\kappa} \int \frac{\partial H^0_\mu}{\partial x^i} d^2 x.$$ (7)

and the energy component is obtained by using the Gauss theorem

$$E_{\text{Ein}}(r) = P_0 = \frac{1}{2\kappa} \int \frac{\partial H^0_0}{\partial x^i} dx dy.$$ (8)

The metric of the non-rotating black hole solutions in 2+1 dimensions can be taken as in Cartesian coordinates

$$ds^2 = -v dt^2 + \left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right) dx^2 + \left(\frac{2xy}{r^2} w - \frac{2xy}{r^2}\right) dy dx + \left(\frac{y^2}{r^2} + \frac{x^2}{r^2}\right) dy^2.$$ (9)

Hence, the required nonvanishing components \(H^0_0\) of the Einstein energy-momentum complex are shown to be

$$H^0_0^0 = \frac{x}{r^2 \sqrt{vw}} (1 - w),$$

$$H^0_0^2 = \frac{y}{r^2 \sqrt{vw}} (1 - w).$$ (10)

Finally, we find the energy within a circle with radius \(r\) is

$$E_{\text{Ein}}(r) = \frac{1}{2\kappa} \oint \frac{v}{\sqrt{vw}} (1 - w) d\theta.$$ (12)
4 ENERGY IN THE MØLLER PRESCRIPTION

For another thing, the Møller energy-momentum complex [2] is given by

\[ \Theta^{\nu}_{\mu} = \frac{1}{\kappa} \frac{\partial}{\partial x^{\rho}} \chi^{\nu}_{\rho}, \]  

(13)

with superpotential

\[ \chi^{\nu}_{\mu} = \sqrt{-g} \left( \frac{\partial g_{\mu \beta}}{\partial x^{\alpha}} - \frac{\partial g_{\mu \alpha}}{\partial x^{\beta}} \right) g^{\nu \alpha} g^{0 \beta}. \]  

(14)

The energy and momentum using the Møller energy-momentum complex in spherical coordinates are calculated as

\[ P_{\mu} = \frac{1}{\kappa} \int \frac{\partial \chi^{\alpha}_{\mu}}{\partial x^{i}} d^{2}x. \]  

(15)

and the energy component is obtained by using the Gauss theorem

\[ E_{\text{Møl}}(r) = P_{0} = \frac{1}{\kappa} \int \frac{\partial \chi^{0}_{\mu}}{\partial x^{i}} dr d\theta. \]  

(16)

For the 2+1 dimensional non-rotating black hole solutions in spherical coordinates, the metric can be expressed in the form

\[ ds^{2} = -v(r) dt^{2} + w(r) dr^{2} + r^{2} d\theta^{2}. \]  

(17)

So, the only nonvanishing component of Møller energy-momentum complex is

\[ \chi^{0}_{0} = -\frac{r}{\sqrt{vw}} \frac{\partial v}{\partial r}. \]  

(18)

At last, the energy within a circle with the radius \( r \) is obtained

\[ E_{\text{Møl}}(r) = -\frac{1}{\kappa} \oint \frac{r}{\sqrt{vw}} \frac{\partial v}{\partial r} d\theta. \]  

(19)

5 ENERGY DISTRIBUTION OF 2 + 1 DIMENSIONAL BLACK HOLE SOLUTIONS

To put these four space-time solutions(1)-(4) into equations (6) and (14), these cases furnish:

(i) Uncharged black hole solution:

\[ E_{\text{Ein}} = \frac{\pi(\Lambda r^{2} - M - 1)}{\kappa}, \]  

(20)
and
\[ E_{\text{Møl}} = -\frac{4\pi\Lambda r^2}{\kappa}. \] (21)

(ii) Charged black hole solution:
\[ E_{\text{Ein}} = \frac{\pi[\Lambda r^2 - M - 2Q^2 \ln\left(\frac{r}{r^+}\right) - 1]}{\kappa}, \] (22)
and
\[ E_{\text{Møl}} = -\frac{4\pi\Lambda r^2 - Q^2}{\kappa}. \] (23)

(iii) Coupling to a static scalar field:
\[ E_{\text{Ein}} = \frac{\pi\Lambda (r - 2B)(B + r)^2 - r]}{\kappa r}, \] (24)
and
\[ E_{\text{Møl}} = -\frac{4\pi\Lambda (B + r)(r^2 - Br + B^2)}{\kappa r}. \] (25)

(iv) The static and circularly symmetric exact solution of the Einstein-massless scalar equation:
\[ E_{\text{Ein}} = \frac{\pi[(1 - q)R^q - 1]}{\kappa}, \] (26)
and
\[ E_{\text{Møl}} = -\frac{2\pi q(1 - q)R^q}{\kappa}. \] (27)

6 CONCLUSION

The energy distributions for some 2 + 1 dimensional black hole solutions were computed with the Einstein and Møller energy-momentum complexes. In these four solutions the energy distributions of Einstein energy-momentum complex differ from Møller, and those each solution also differ with other three with Einstein’s (or Møller’s) prescription. Specially, we found that all of the energy component of Møller energy-momentum complex are negative.
As $r$ becomes larger, the energy distribution of Virbhadra’s solution for the Einstein-massless scalar equation becomes

$$E_{\text{Ein}} \sim \frac{\pi}{\kappa} (1 - q) R^q$$

and

$$E_{\text{Møl}} \sim -\frac{2\pi}{\kappa} q(1 - q) R^q,$$

and the energy distributions of these three solutions become

$$E_{\text{Ein}} \sim \frac{\pi \Lambda r^2}{\kappa}$$

and

$$E_{\text{Møl}} \sim -\frac{4\pi \Lambda r^2}{\kappa}.$$ 

Because those solutions are not asymptotical flat, $E_{\text{Ein}}$ and $E_{\text{Møl}}$ of all solutions divergence while $r \to \infty$. Nevertheless those energy-momentum complex still could be used to study the topology of black hole solutions.

**Acknowledgements**

The authors would like to thank Prof. S. Deser, Prof. R. Jackiw, Prof. R. Mann and Dr. E.C. Vagenas for useful comments. I.-C. Yang also thanks the National Science Council of the Republic of China for financial support under the contract number NSC 92-2112-M-006-007

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