Cobimaximal Mixing with Dirac Neutrinos

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Abstract

If neutrinos are Dirac, the conditions for cobimaximal mixing, i.e. $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm \pi/2$ in the $3 \times 3$ neutrino mixing matrix, are derived. One example with $A_4$ symmetry and radiative Dirac neutrino masses is presented.
Introduction: Neutrinos are mostly assumed to be Majorana. The associated $3 \times 3$ mass matrix has been studied in numerous papers. One particularly interesting form was discovered in 2002 [1], i.e.

$$
\mathcal{M}_\nu = \begin{pmatrix}
A & C & C^* \\
C & B & D \\
C^* & D & B^*
\end{pmatrix},
$$

(1)

where $A$ and $D$ are real, which was shown subsequently [2] to be the result of a generalized $CP$ transformation involving $\nu_{\mu,\tau}$ exchange. This form predicts the so-called cobimaximal mixing pattern [3] of neutrinos, i.e. $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm \pi/2$, which is close to what is observed [4].

To understand the cobimaximal mixing matrix $U_{CBM}$, consider its form in the PDG convention, i.e.

$$
U_{CBM} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & \mp is_{13} \\
-(1/\sqrt{2})(s_{12} \pm ic_{12}s_{13}) & (1/\sqrt{2})(c_{12} \mp is_{12}s_{13}) & -c_{13}/\sqrt{2} \\
(1/\sqrt{2})(-s_{12} \pm ic_{12}s_{13}) & (1/\sqrt{2})(c_{12} \pm is_{12}s_{13}) & c_{13}/\sqrt{2}
\end{pmatrix}.
$$

(2)

Note that the $(2i)$ and $(3i)$ entries for $i = 1, 2, 3$ are equal in magnitudes. It is easy then to obtain Eq. (1) as

$$
\mathcal{M}_\nu = U_{CBM} \mathcal{M}_{\text{diag}} U_{CBM}^T.
$$

(3)

In Eq. (1), the neutrino basis is chosen for which the charged-lepton mass matrix $M_l$ is diagonal which links the left-handed $(e, \mu, \tau)$ to their right-handed counterparts. Suppose it is not, but rather that it is digonalized on the left by the special matrix [5, 6]

$$
U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix},
$$

(4)

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$. It was discovered in 2000 [7] that

$$
U_{CBM} = U_\omega^\dagger O,
$$

(5)
where $\mathbf{O}$ is an orthogonal matrix. The proof is very simple because the product $U_\nu \mathbf{O}$ enforces the equal magnitudes of the $(2i)$ and $(3i)$ entries.

The implications of these conditions regarding Dirac (instead of Majorana) neutrinos are the subject of this paper. A specific model of radiative Dirac neutrino masses with $A_4$ symmetry [8] is also presented.

**Form of Dirac Neutrino Mass Matrix :** Consider the $3 \times 3$ mass matrix linking $\nu_L$ to $\nu_R$, i.e. $\mathcal{M}_D$. It is diagonalized by two unitary matrices, $U_L$ on the left and $U_R^\dagger$ on the right. To eliminate $U_R$ which is unobservable in the standard model (SM) of quarks and leptons, the product $\mathcal{M}_D \mathcal{M}_D^\dagger$ should be studied. It is automatically Hermitian, and is diagonalized by $U_L$ on the left and $U_L^\dagger$ on the right. Using $U_{CBM}$ of Eq. (2), it is easily seen that

$$\mathcal{M}_D \mathcal{M}_D^\dagger = \begin{pmatrix} A & C & C^* \\ C^* & B & D \\ C & D^* & B \end{pmatrix}. \quad (6)$$

This is the analog of Eq. (1) for Dirac neutrinos. An example was recently shown in Ref. [9]. However, Eq. (6) does not constrain $\mathcal{M}_D$ uniquely because of the missing arbitrary $U_R$.

Nevertheless, a possible form of $\mathcal{M}_D$ is

$$\mathcal{M}_D = \begin{pmatrix} a & c & c^* \\ d & b & e \\ d^* & e^* & b^* \end{pmatrix}, \quad (7)$$

where $a$ is real. It is then trivial to see that $\mathcal{M}_D \mathcal{M}_D^\dagger$ yields exactly Eq. (6). The origin of this $\mathcal{M}_D$ is a simple extension of the generalized $CP$ transformation of Ref. [2], i.e.

$$\nu_e \leftrightarrow \nu_e, \quad \nu_\mu \leftrightarrow \nu_\tau, \quad \nu_e^c \leftrightarrow \nu_e^c, \quad \nu_\mu^c \leftrightarrow \nu_\tau^c,$$ \quad (8)

together with complex conjugation. It is important to realize that whereas Eq. (7) guarantees Eq. (6), the former may be obtained without the latter, as shown already in Ref. [9] because of the missing arbitrary $U_R$. 

3
Scotogenic Dirac Neutrinos with Cobimaximal Mixing: The other approach to obtaining $U_{CBM}$ is through Eq. (5). Two previous models were constructed [10, 11] this way for Majorana neutrinos. Their Dirac counterpart is presented here. It is actually simpler because a technical problem is naturally avoided in this case as shown below.

Following Ref. [8], the non-Abelian discrete symmetry $A_4$ is used, under which the three families of left-handed lepton doublets transform as the $\mathbf{3}$ representation, and the three charged-lepton singlets as $\mathbf{1}, \mathbf{1}', \mathbf{1}''$. There are also three Higgs doublets $\Phi_i = (\phi_i^+, \phi_i^0)$ transforming as $\mathbf{3}$. The multiplication rules for two triplets $a_{1,2,3}$ and $b_{1,2,3}$ in this representation [8] are

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = 1, \quad a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 = 1', \quad a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 = 1''. \quad (9)$$

Assuming that $\langle \phi_i^0 \rangle$ is the same for $i = 1, 2, 3$, the $3 \times 3$ mass matrix linking $(e, \mu, \tau)_L$ to $(e, \mu, \tau)_R$ is then

$$\mathcal{M}_l = U_\omega \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad (10)$$

which is well-known since 2001.

To obtain Dirac neutrinos, three lepton singlets $\nu_R$ transforming as $\mathbf{3}$ under $A_4$ are added to the SM. Since

$$3 \times 3 = 1 + 1' + 1'' + \bar{3} + \bar{3}, \quad (11)$$

the products $(a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2)$ and $(a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1)$ are allowed, so that tree-level Dirac neutrino masses are obtained. To forbid this, a $Z_2'$ symmetry is imposed, so that $\nu_R$ are odd, and the SM fields are even as shown in Table 1. Note that all dimension-four terms in the Lagrangian are required to obey $A_4 \times Z_2'$ which is only broken together softly by the $ss'$ mass terms. Added are dark scalars and fermions which are odd under an exactly conserved $Z_2^D$ symmetry. Lepton number $L$ is conserved as shown. Dirac neutrino masses are radiatively generated by dark matter [12] as shown in Fig. 1 in analogy with the
Table 1: Fermion and scalar content of Dirac neutrino model with dark $Z_2^D$ and $A_4 \times Z_2'$ symmetries.

| fermion/scalar | $SU(2)_L$ | $U(1)_Y$ | $A_4$ | $Z_2'$ | $Z_2^D$ | $L$ |
|----------------|-----------|---------|-------|--------|---------|-----|
| $(\nu, l)_{iL}$ | 2         | $-1/2$ | 3     | $+$    | $+$     | 1   |
| $l_{iR}$       | 1         | $-1$   | $1, 1', 1''$ | $+$ | $+$     | 1   |
| $\nu_{iR}$     | 1         | 0      | 3     | $-$    | $+$     | 1   |
| $(\phi^+, \phi^0)_i$ | 2       | $1/2$  | 3     | $+$    | $+$     | 0   |
| $(\eta^+, \eta^0)$ | 2      | $1/2$  | 1     | $+$    | $+$     | 0   |
| $s_i$          | 1         | 0      | 3     | $-$    | $-$     | 0   |
| $s'_i$         | 1         | 0      | 3     | $-$    | $-$     | 0   |
| $(E^0, E^-)_{L,R}$ | 2       | $-1/2$ | 1     | $-$    | $-$     | 1   |
| $N_{L,R}$      | 1         | 0      | 1     | $+$    | $+$     | 1   |

The original scotogenic model [13]. The key for cobimaximal mixing is that the $s, s'$ scalars are real fields [10, 11].

![Figure 1: Scotogenic Dirac neutrino mass.](image)

The relevant Yukawa couplings are

$$f_{Es}E_R^0\nu_L, \quad f_{Ns'}\bar{\nu}_RN_L, \quad f_{Ne}N_LE_R^0, \quad f_{En}\bar{\eta}_0E_L^0N_R.$$ (12)

All respect $A_4 \times Z_2'$, with the latter two contributing to the $2 \times 2$ mass matrix linking $(E_L^0, N_L)$ to $(E_R^0, N_R)$, i.e.

$$M_{EN} = \begin{pmatrix} m_E & m_{EN} \\ m_{NE} & m_N \end{pmatrix} = \begin{pmatrix} \cos \theta_L & -\sin \theta_L \\ \sin \theta_L & \cos \theta_L \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \cos \theta_R & \sin \theta_R \\ -\sin \theta_R & \cos \theta_R \end{pmatrix}.$$ (13)
As for the contribution of $s$ and $s'$, the mass-squared matrix for each is proportional to the identity, whereas the $ss'$ mixing is arbitrary, breaking both $A_4$ and $Z_2'$ at the same time softly. Let it be denoted as $\mathcal{M}_{ss'}^2$ and assuming that its entries are all much smaller then the invariant masses of $s$ and $s'$, then it is clear that the Dirac neutrino mass matrix in the basis of Fig. 1 is proportional to $\mathcal{M}_{ss'}^2$ and is real up to an unobservable phase, i.e. the relative phase of $f_N$ and $f_E$. This means that it is diagonalized by an orthogonal matrix. Combined with Eq. (10), cobimaximal mixing is assured.

The explicit expression for the scotogenic Dirac neutrino mass matrix is

$$
\mathcal{M}_\nu = \frac{f^*_N f_E M_{ss'}^2}{16\pi^2} \left[ \frac{\sin \theta_L \cos \theta_R}{m_1} \left[ F(x_1) - F(y_1) \right] - \frac{\cos \theta_L \sin \theta_R}{m_2} \left[ F(x_2) - F(y_2) \right] \right],
$$

(14)

where $x_{1,2} = m^2_s/m^2_{1,2}$ and $y_{1,2} = m^2_{s'}/m^2_{1,2}$, and

$$
F(x) = \frac{x \ln x}{x - 1}.
$$

(15)

*The $A_4 \rightarrow Z_3$ Breaking:* The breaking of $A_4$ by $\langle \phi^0_i \rangle = v$ reduces this symmetry to $Z_3$ [14]. It must be maintained for $U_\omega$ to be valid. However, the addition of $s$ and $s'$ would allow the quartic terms $s_i s_j \Phi^\dagger_i \Phi_j$ and $s'_i s'_j \Phi^\dagger_i \Phi_j$. The key now is that both the quadratic mass terms $s_i s_i$ and $s'_i s'_i$ do not break $Z_2'$ and are required also not to break $A_4$. Only the $s_i s'_i$ terms break both $A_4$ and $Z_2'$ softly together. Hence the one-loop correction to $\Phi^\dagger_i \Phi_j$ is shown in Fig. 2. Two mass insertions are required, which render the diagram finite and suppressed.

![Figure 2: Finite one-loop correction to $\Phi^\dagger_i \Phi_j$.](image)
so that the residual $Z_3$ symmetry is maintained to a good approximation. In Refs. [10, 11], this option is not available for Majorana neutrinos because $s'$ is absent and $s_is_j$ breaks $A_4$, which yields only one mass insertion in Fig. 2, thus making it logarithmically divergent.

**Dark Sector** : The dark sector fermions are $(E^0, E^-)$ and $N$. The two neutral ones have masses $m_{1,2}$ and the charged one $m_E$. They are assumed greater than the masses of the scalars, $m_s$ and $m_{s'}$, with the small $M_{ss'}^2$ mixing between them. Let $m_s$ be the smaller, then the almost degenerate $s_{1,2,3}$ are dark-matter candidates. They interact with the SM Higgs boson $h$ according to

$$\mathcal{L}_{int} = -\lambda s_is_i \left( vh + \frac{h^2}{2} \right) - \lambda' s_is_j \left( vh + \frac{h^2}{2} \right),$$

where

$$h = \frac{\sqrt{2} \left[ \langle \eta^0 \rangle \text{Re}(\eta^0) + \langle \phi^0 \rangle \text{Re}(\phi^0_1 + \phi^0_2 + \phi^0_3) \right]}{\sqrt{\langle \eta^0 \rangle^2 + 3 \langle \phi^0 \rangle^2}}.$$  

(16)

This is a straightforward generation of the simplest model of dark matter, i.e. that of a real scalar. The comprehensive analysis of Ref. [15] is thus applicable.

**Conclusion** : It is shown how cobimaximal neutrino mixing, i.e. $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm \pi/2$, occurs for Dirac neutrinos. It is defined by Eq. (6) which is obtainable from Eq. (7) based on a generalized $CP$ transformation. However, because of the missing arbitrary unitary matrix $U_R$ which diagonalizes $\mathcal{M}_D$ on the right, there are certainly other solutions, one of which is discussed in Ref. [9].

Another approach is to use Eq. (5), which may be implemented with the non-Abelian discrete symmetry $A_4$ and a scotogenic Dirac neutrino mass matrix proportional to a real scalar mass-squared matrix. It is the analog of previous suggestions [10, 11] for Majorana neutrinos, but in the case of Dirac neutrinos here, it is more technically natural.

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