On restricted powers of complete intersections

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ABSTRACT
A restricted $d$th power of an ideal $I$ is obtained by restricting the exponent vectors allowed to appear on the “natural” generating set of $I^d$, for some integer $d$. In this paper, we study homological properties of restricted powers of complete intersections. We construct a generalization of the $L$-complex construction of Buchsbaum and Eisenbud. We use this resolution to compute an explicit basis for the Koszul homology which allows us to deduce that the quotient defined by any restricted $d$th power of a complete intersection is Golod, and construct an algebra structure on this complex.

1. Introduction
Let $I = (a_1, \ldots, a_n)$ be an ideal in some commutative ring $R$ and $w = (w_1, \ldots, w_n) \in \mathbb{Z}_{\geq 0}^n$ any vector. The $w$-restricted $d$th power of $I$ is the ideal generated by all elements of the form $a_1^{d_1} \cdot \ldots \cdot a_n^{d_n}$, where $d_i \leq w_i$ for each $1 \leq i \leq n$ and $d_1 + \ldots + d_n = d$. From a combinatorial perspective, if one imagines all exponent vectors with sum $d$ as being encoded by the dilated $(n-1)$-simplex $d \cdot \Delta^{n-1}$, then the exponent vectors for the $w$-restricted $d$th power are obtained by cutting $d \cdot \Delta^{n-1}$ by an appropriate collection of hyperplanes. This perspective is employed in, for instance, [1], where the results of Almousa, Floystad, and Lohne are extended to the case of polarizations of restricted powers of the graded maximal ideal (in a polynomial ring) with a characterization dependent on the associated $w$-restricted dilated simplex (see Figure 1).

Gasharov, Hibi, and Peeva have also considered $w$-stable monomial ideals, where the idea is similar: simply restrict the minimal generating set of a stable monomial ideal to all monomials appearing with exponent vector bounded above by $w$. As it turns out, many of the properties possessed by stable ideals are inherited by $w$-stable ideals, such as Golodness and a linear resolution (in the equigenerated case). Moreover, since the Eliahou-Kervaire resolution is a naturally multigraded minimal free resolution, the minimal free resolution of a $w$-stable ideal is obtained by restricting to all multidegrees that are also bounded above by $w$. This recovers and generalizes the squarefree Eliahou-Kervaire resolution, introduced in [2].

In this paper, we show that the idea of “restricting multidegrees” can be extended to arbitrary $w$-restricted powers of complete intersections (which need not be monomial ideals). We introduce a generalization of the well-known $L$-complexes of Buchsbaum and Eisenbud (see [4]) and prove that these complexes yield a minimal free resolution of $w$-restricted powers of complete intersections. We also give an explicit description of the basis elements of the Koszul homology of any restricted power of a complete intersection by lifting basis elements of the aforementioned minimal free resolution to the Koszul homology algebra. This allows us to prove that for $d \geq 2$, the $d$th restricted power of a complete intersection always defines a Golod ring. Finally, we employ recent results of Miller and Rahmati [10]
to show that, for large enough characteristic, the minimal free resolution of the quotient defined by the restricted power of any complete intersection admits the structure of an associative DG-algebra.

The paper is organized as follows. In Section 2, we introduce notation and conventions that will be used in the rest of the paper. This includes the definition of a Golod ring in terms of trivial Massey operations and a refresher on much of the content of [10] dealing with transferring algebra structures along special deformation retracts. In Section 3, we define the pieces that constitute the building blocks of the minimal free resolution of quotients defined by restricted powers of complete intersections. The construction generalizes the original construction of Buchsbaum and Eisenbud, and we give a brief and self-contained proof of acyclicity.

In Section 4, we use the minimal free resolution of Section 3 to obtain information about the Koszul homology algebra and Golodness. More precisely, we are able to find an explicit lift of the basis elements in the minimal free resolution to the Koszul homology algebra. With respect to this basis of the Koszul homology, we find that the trivial Massey operation with $\mu(h_1, \ldots, h_k) = 0$ for $k \geq 2$ is well-defined, allowing us to deduce Golodness of all $w$-restricted $d$th powers of complete intersections, for $d \geq 2$. This also gives an explicit minimal free resolution of the residue field due to a well-known construction of Golod.

In Section 5, we prove that the minimal free resolution constructed in Section 3 admits the structure of an associative DG-algebra. The proof of this fact relies heavily on recent techniques developed by Miller and Rahmati in [10]. In particular, we construct an explicit algebra structure on an associated total complex obtained by restricting multidegrees, and show that the scaled de Rham map satisfies the generalized Leibniz rule with respect to this product. As a consequence, the perturbation lemma will allow us to transfer the product structure on this total complex to the generalized $L$-complexes of Section 3, immediately extending the application to the Buchsbaum-Eisenbud $L$-complexes given in [10].

### 2. Restricted powers, Golod rings, and transferring algebra structures

The purpose of this section is to provide background and conventions for the material to be used for the rest of the paper. The main focal points of this section are the definition of Golod (see Definition 2.5) and the terminology introduced in the latter half, culminating in Proposition 2.11. For a general overview of DG-algebra techniques, including exposition, the reader is encouraged to consult Avramov [3]. The material on deformation retracts and the perturbation lemma comes from [10]. We begin this section by defining the $w$-restricted $d$th power of a complete intersection.

**Definition 2.1.** Let $F$ be a free $R$-module of rank $n$ with basis $f_1, \ldots, f_n$. Let $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ be any vector and let $\psi : F \to R$ be any $R$-module homomorphism. Then the $w$-restricted $d$th power of $\im(\psi)$, denoted $(\im\psi)^d_w$, is defined to be the ideal generated by all elements of the form

$$\psi(f_1)^{\alpha_1} \cdot \psi(f_2)^{\alpha_2} \cdots \psi(f_n)^{\alpha_n},$$

where $\alpha_i \leq w_i$ for all $1 \leq i \leq n$ and $\alpha_1 + \cdots + \alpha_n = d$.

The idea of restricted powers has been considered before by other authors in different contexts. Indeed:
Definition 2.4. Let $A$ be a DG-algebra over a local ring $(R, m, k)$ with $H_0(A) \cong k$. Then $A$ admits a trivial Massey operation if for some $k$-basis $B = \{h_\lambda\}_{\lambda \in \Lambda}$ of $H_{\geq 1}(A)$, there exists a function

$$
\mu : \prod_{i=1}^{\infty} B^i \to A
$$

such that

$$
\mu(h_\lambda) = z_\lambda \quad \text{with} \quad [z_\lambda] = h_\lambda, \quad \text{and}
$$

$$
d\mu(h_{\lambda_1}, \ldots, h_{\lambda_p}) = \sum_{j=1}^{p-1} \mu(h_{\lambda_1}, \ldots, h_{\lambda_j})\mu(h_{\lambda_{j+1}}, \ldots, h_{\lambda_p}).
$$

Observe that taking $p = 2$ in the above definition yields that $H_{\geq 1}(A)^2 = 0$, so the induced algebra structure on $H(A)$ is totally trivial for a DG-algebra admitting a trivial Massey operation.

Definition 2.5. Let $(R, m)$ be a local ring and let $K^R$ denote the Koszul complex on any minimal generating set of $m$. If $K^R$ admits a trivial Massey operation $\mu$, then $R$ is called a Golod ring.

Remark 2.6. It is worth mentioning that Definition 2.5 is equivalent to saying that the Poincaré series $P_k^R(t)$ of $R$ attains a coefficient-wise inequality originally established by Serre [3, Theorem 5.2.2] (this is often given as the definition of Golodness, but Definition 2.5 will be much more convenient for our
purposes). As observed above, the Koszul homology algebra of any Golod ring has trivial multiplication in positive homological degrees; it is worth noting that for rings of projective dimension \( \geq 4 \), there exist non-Golod rings with trivial Koszul homology algebras (see work by Katthän [9]).

Next, we recall some of the terminology and conventions established in [10] for transferring algebra structures. The main goal for the remainder of this section is the establishment of Proposition 2.11, which furnishes conditions for when an algebra structure can be transferred along a perturbed deformation retract.

**Definition 2.7.** Let \( F_* \) and \( G_* \) be two complexes. A deformation retract is a quasi-isomorphism of complexes

\[
F_* \xrightarrow{p} G_*
\]

satisfying:

1. \( p \circ i = 1 \)
2. \( i \circ p \simeq 1 \) via some homotopy \( h \) on \( F_* \).

A deformation retract is **special** if, furthermore, one has:

1. \( h \circ i = 0 \),
2. \( p \circ h = 0 \), and
3. \( h^2 = 0 \).

Given a deformation retract

\[
F_* \xrightarrow{p} G_*
\]

with associated homotopy \( h \), a perturbation is a map \( \delta \) such that \( d^F + \delta \) is a differential on \( F_* \); that is, \((d^F + \delta)^2 = 0\). The perturbation \( \delta \) is **small** if \( 1 - \delta h \) is invertible.

Note that if \( F_* \) is a finite length complex, then every perturbation is nilpotent and hence small.

**Setup 2.8.** Let

\[
F_* \xleftarrow{p} G_*
\]

be a special deformation retract with associated homotopy \( h \). Assume that \( \delta \) is a small perturbation on \( F_* \) and let \( A := (1 - \delta h)^{-1} \delta \). Define the following data:

1. \( i_\infty := i + hAi \),
2. \( p_\infty := p + pAh \),
3. \( d^F_\infty := d^F + \delta \),
4. \( d^G_\infty := d^G + pAi \), and
5. \( h_\infty := h + hAh \).

**Lemma 2.9** (Perturbation Lemma). Adopt notation and hypotheses as in Setup 2.8. Then the data

\[
(F_*, d^F_\infty) \xrightarrow{p_\infty} (G_*, d^G_\infty)
\]

is a special deformation retract with associated homotopy \( h_\infty \).
**Definition 2.10.** Let $F_\bullet$ be a complex of $R$-modules equipped with a product, and let $h : F_\bullet \to F_\bullet$ be any graded map. Then $h$ satisfies the **generalized Leibniz rule** if for every $f, f' \in F_\bullet$,

$$h(f \cdot f') \subseteq h(f)X + Xh(f').$$

The following Proposition is a slight variant on Proposition 3.5 of [10]; in the original statement, it was assumed that $F_\bullet$ was a DG-algebra with the unperturbed differential. It turns out that this hypothesis is unnecessary, and that the existence of an associative product for which the associated homotopy satisfies the generalized Leibniz rule is sufficient.

**Proposition 2.11.** Adopt notation and hypotheses as in Setup 2.8, and assume furthermore that:

1. $(F_\bullet, d^F_\infty)$ is an associative algebra (not necessarily satisfying the Leibniz rule),
2. $d^F_\infty = d^F + \delta$ satisfies the Leibniz rule, and
3. $h$ satisfies the generalized Leibniz rule.

Then $(G_\bullet, d^G_\infty)$ is a DG-algebra with product:

$$g \cdot g' := p_\infty (i_\infty (g) \cdot i_\infty (g')).$$

Moreover, the map $i_\infty : (G_\bullet, d^G_\infty) \to (F_\bullet, d^F_\infty)$ is a morphism of DG-algebras.

**Proof.** The proof is essentially identical to that of [10, Proposition 1.4], where one only needs to verify that $(d^F_\infty h_\infty + h_\infty d^F_\infty)(i_\infty (f) \cdot i_\infty (f')) = 0$; this is a straightforward computation. 

### 3. The minimal free resolution

In this section, we construct an explicit minimal free resolution for the quotient defined by the $w$-restricted $d$th power of a complete intersection, for any vector $w$ and power $d$. The construction is highly reminiscent of the original construction of Buchsbaum and Eisenbud; the proof of exactness here is self contained and is distinct from the proof given in [4]. Indeed, the proof of Theorem 3.10 was inspired by the proof of exactness given in [5, Theorem 2.12].

Let $F$ be a free $R$-module of rank $n$ with basis elements $f_1, \ldots, f_n$, with $\psi : F \to R$ any map such that $\psi (f_1), \ldots, \psi (f_n)$ forms a regular sequence. Throughout this section, we will assume either:

1. $R$ is a Noetherian local ring, or
2. $R$ is a $\mathbb{Z}$-graded ring, in which case the elements $\psi (f_1), \ldots, \psi (f_n)$ are assumed to be homogeneous of positive degree.

Both of the above conditions guarantee that any subsequence of the regular sequence $\psi (f_1), \ldots, \psi (f_n)$ remains regular. We begin by establishing the following notation, which will be employed tacitly for the remainder of the paper.

**Notation 3.1.** Let $F$ be a free $R$-module of rank $n$ with basis elements $f_1, \ldots, f_n$. Given integers $a$ and $b \geq 0$, the following notation will be used for conciseness:

$$\bigwedge^a := \bigwedge^a F, \quad S_b := S_b(F),$$

denoting the graded pieces of the exterior and symmetric algebras, respectively. Let $\sigma = (\sigma_1 < \cdots < \sigma_a)$ be an indexing set and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an exponent vector with $|\alpha| := \alpha_1 + \cdots + \alpha_n = b$. Then, the following notation will be used:

$$f_\sigma := f_{\sigma_1} \wedge \cdots \wedge f_{\sigma_a} \in \bigwedge^a, \quad f^\alpha := f_1^{\alpha_1} \cdots f_n^{\alpha_n} \in S_b.$$
Given any integer $1 \leq i \leq n$, the notation $\epsilon_i$ will denote the vector with a 1 in the $i$th spot and 0's elsewhere. Throughout the paper, for any two vectors $\alpha, \beta \in \mathbb{Z}^n$, write

$$\alpha \leq \beta \iff \alpha_i \leq \beta_i \text{ for all } 1 \leq i \leq n.$$  

**Definition 3.2.** Adopt notation as in Notation 3.1. Then the multidegree of an element $f_\sigma \otimes f^\alpha \in \wedge^a \otimes S_b$, denoted $mdeg$, is defined as

$$mdeg(f_\sigma \otimes f^\alpha) := \epsilon_{\sigma_1} + \cdots + \epsilon_{\sigma_a} + \alpha \in \mathbb{N}^n.$$  

The multigrading of Definition 3.2 should cause no confusion, since it is induced by the natural choice of multigrading on the symmetric and exterior algebras $S_\bullet$ and $\wedge_\bullet$, if each $f_i$ is given multidegree $\epsilon_i$.

**Setup 3.3.** Adopt notation as in Notation 3.1 and let $\psi : F \to R$ be such that grade $\text{im}(\psi) = n$. Let $w = (w_1, \ldots, w_n) \in \mathbb{Z}_{\geq 0}^n$ be any tuple and set

$$(\wedge^a \otimes S_b)_w := \{ f_\sigma \otimes f^\alpha \mid mdeg(f_\sigma \otimes f^\alpha) \leq w \}.$$  

Let $\kappa^w_{a,b} : (\wedge^a \otimes S_b)_w \to (\wedge^{a-1} \otimes S_{b+1})_w$ be the map induced by the tautological Koszul differential:

$$\begin{array}{ccc}
\wedge^a \otimes S_b & \xrightarrow{\text{comult} \otimes 1} & \wedge^{a-1} \otimes F \otimes S_b \\
& \xrightarrow{1 \otimes \text{mult}} & \wedge^{a-1} \otimes S_{b+1}.
\end{array}$$  

Observe that $\kappa^w_{a,b}$ is well defined because it preserves multidegree. With this notation, define $L^a_{b,w}(F) = L^a_{b,w} := \ker \kappa^w_{a,b} \subseteq (\wedge^a \otimes S_b)_w$.

As it turns out, the basis elements for these modules are actually quite comprehensible and can be viewed as Schur modules corresponding to hook partitions. The proof of the following proposition is a consequence of the much more general statement given in [13, Proposition 2.1.4] combined with the fact that all tableaux appearing in a given straightening relation have the same multidegree.

**Proposition 3.4.** For every $a, b \geq 0$, the module $L^a_{b,w}$ has basis represented by the set of semistandard tableaux with multidegree bounded above by $w$.

**Example 3.5.** Let $R = k[x_1, \ldots, x_3]$ and $F = Rf_1 \oplus Rf_2 \oplus Rf_3$ with $\psi : F \to R$ induced by sending $f_i \mapsto x_i$. Suppose that $w = (3, 1, 1)$; then

$$(x_1, x_2, x_3)^3_w = (x_1^3, x_1^2 x_2, x_1^2 x_3, x_1 x_2 x_3).$$  

Likewise, the module $L^3_{3,w}(F)$ has basis represented by the tableaux

$$\begin{array}{ccc}
1 & 1 & 1 \\
2 & & \\
3 & &
\end{array} \quad \begin{array}{ccc}
1 & 1 & 1 \\
3 & & \\
3 & &
\end{array} \quad \begin{array}{ccc}
1 & 1 & 2 \\
3 & & \\
2 & &
\end{array} \quad \begin{array}{ccc}
1 & 1 & 3 \\
2 & & \\
3 & &
\end{array}$$  

The following observation is a straightforward verification which shows that the horizontal and vertical differentials of Figure 2 anticommute.

**Observation 3.6.** Adopt notation and hypotheses as in Setup 3.3. Let $\text{Kos} : \wedge^a \to \wedge^{a-1}$ denote the Koszul differential induced by $\psi$, for any given $a \geq 0$. Then $\text{Kos} \otimes 1 : \wedge^\bullet \otimes S_\bullet \to \wedge^\bullet \otimes S_\bullet$ and $\kappa_{\bullet,\bullet}$ anticommute; that is,

$$(\text{Kos} \otimes 1) \circ \kappa^w_{a,b} = -\kappa^w_{a-1,b} \circ (\text{Kos} \otimes 1).$$
Figure 2. The double complex used for the proof of Theorem 3.10; recall the conventions established in Notation 3.7.

Notation 3.7. For conciseness and ease of notation, subscripts indicating homological degrees will often be omitted. Moreover, the vector \( w \) as in Setup 3.3 will often be omitted in the notation \( \kappa_w \), and the much simpler notation \( \kappa \) may be used.

Likewise, for any map \( \psi : F \to R \), there is a naturally induced map \( S_d(\psi) : S_d \to R \) defined by sending \( f^\alpha \mapsto \psi(f^1)\alpha^1 \cdots \psi(f^n)\alpha^n \). This map will also be denoted simply by \( \psi \).

Proposition 3.8. Adopt notation and hypotheses as in Setup 3.3. Let \( E \) denote the double complex of Figure 2. Then,

1. all rows of \( E \) except for the bottom-most row are exact, and
2. the complexes \( (\wedge^k S_b)_w \) for \( 0 \leq b \leq d - 1 \) are exact in homological degrees \( \geq 1 \).
**Proof.** Since the rows of $E$ are obtained from restricting the multidegrees appearing in the truncated tautological Koszul complex, they must be exact, except for the bottom-most row. At the bottom, the homology is isomorphic to $R$.

To see that the complexes $(\bigwedge^m \otimes S_b)_w$ are acyclic, observe that if $f_s \otimes f^\alpha \in (\bigwedge^1 \otimes S_b)_w$ for all $s \in S$, where $S$ is some subset of $[n] = \{1, \ldots, n\}$, then $f_s \otimes f^\alpha \in (\bigwedge^{[\sigma]} \otimes S_b)_w$ for every $\sigma \subset S$. Thus, $(\bigwedge^m \otimes S_b)_w$ decomposes as a direct sum of Koszul complexes on subsets of the set $\{\psi(f_1), \ldots, \psi(f_n)\}$, induced by writing $S_b = \bigoplus_{|\beta|=b} f^\beta$ and employing the observation of the previous sentence. \hfill\qed

**Definition 3.9.** Adopt notation and hypotheses as in Setup 3.3. Then $L^w(\psi, d)$ denotes the complex

$$0 \to L^{n-1}_{d,w}(F) \xrightarrow{\text{Kos} \otimes 1} \cdots \xrightarrow{\text{Kos} \otimes 1} L^1_{d,w}(F) \xrightarrow{\text{Kos} \otimes 1} (S_d)_w \xrightarrow{\psi} R \to 0.$$ 

We finally arrive at the main result of this section:

**Theorem 3.10.** Adopt notation and hypotheses as in Setup 3.3. The complex $L^w(\psi, d)$ of Definition 3.9 is the minimal free resolution of $R/(\text{im } \psi)_w$.

**Proof.** Throughout the proof, let $T(\cdot)$ denote the totalization of a double complex. Let $E$ denote the double complex of Figure 2, $E'$ the trivial complex arising from the bottom most $\bigwedge^0 \otimes S_0$ term, and $E''$ the rightmost nontrivial column, which is $L^w(\psi, d)$ without $R$ in homological degree 0. Index the total complex $T(E)$ such that $T(E)_0 = \left( \bigoplus_{i=0}^{d-1} (\bigwedge^0 \otimes S_i)_w \right) \oplus I^0_{d,w}$. Then the short exact sequence of total complexes

$$0 \to T(E') \to T(E) \to T(E/E') \to 0$$

combined with Proposition 3.8 (1) yields that

$$H_i(T(E)) = H_i(T(E')) = \begin{cases} R & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Likewise, the short exact sequence

$$0 \to T(E'') \to T(E) \to T(E/E'') \to 0$$

combined with Proposition 3.8 (2) yields that $H_i(T(E'')) = 0$ for $i \geq 1$ and there is an inclusion of homology

$$H_0(T(E'')) \hookrightarrow H_0(T(E)).$$

By definition, this inclusion sends the class of a cycle $f^\alpha = f_{i_1} \cdots f_{i_d} \in (S_d)_w = I^0_{d,w}$ to the class of the corresponding cycle of $T(E)$; one such lift of $f^\alpha$ to $T(E)$ is given by

$$f_{i_1} \otimes f_{i_2} \cdots f_{i_d} + \sum_{j=1}^{d-1} \psi(f_{i_1} \cdots f_{i_j})f_{i_{j+1}} \otimes f_{i_{j+2}} \cdots f_{i_d}.$$ 

Composing with the isomorphism $H_0(T(E)) \sim R$ induces the map $f^\alpha \mapsto \psi(f^\alpha)$, whence augmenting $E'$ by the map $(S_d)_w \xrightarrow{\psi} R$ remains acyclic. \hfill\qed

### 4. Koszul homology and Golodness

In this section, we study the Koszul homology of restricted powers of complete intersections in regular local rings. Since the differentials in the complexes of Definition 3.9 are induced by Koszul differentials, one can explicitly compute balancing isomorphism between the minimal free resolution and Koszul homology. The main result of this section is Corollary 4.5, but most of the work is done in the proof of
Proposition 4.4. After computing a basis for the Koszul homology, we are able to find a straightforward trivial Massey operation allowing for an easy description of the minimal free resolution of the residue field over the quotient defined by a restricted power of a complete intersection.

Throughout this section, we will assume either:

(1) $R$ is a regular local ring, or
(2) $R$ is a standard graded polynomial ring over a field $k$, in which case $\psi(f_1), \ldots, \psi(f_n)$ are assumed to be homogeneous of positive degree.

Definition 4.1. Let $M$ be a finitely generated $R$-module. The Koszul homology of $M$, denoted $H_*(M)$, is defined to be the homology of $M \otimes_R K_*$; that is

$$H_*(M) := H_*(M \otimes_R K_*),$$

where $K_*$ denotes the Koszul complex resolving the residue field, $k$.

Let $A_*$ and $B_*$ be complexes with $H_0(A) = R$ and $H_0(B) = S$, where $R$ and $S$ are quotients of some ring $Q$. Recall that there is a functorial isomorphism

$$H_*(A \otimes_Q S) \cong H_*(R \otimes_Q B)$$

induced by the natural projections

$$A \quad \leftarrow \quad A \otimes_Q B \quad \rightarrow \quad B$$

The isomorphism can be explicitly described as follows:

(1) Choose a cycle $z_1$ in $A \otimes_Q S$ representing a basis element of $H_*(A \otimes_Q S)$.
(2) Lift $z_1$ to a cycle $z_2$ in $A \otimes B$.
(3) Project $z_2$ onto a cycle $z_3$ in $R \otimes_Q B$, then descend to homology.

Definition 4.2. Adopt notation and hypotheses as in Setup 3.3. Let $K_*$ denote the Koszul complex resolving the residue field $k$ and $\bigwedge^*$ denote the exterior algebra on $F$ viewed as a complex equipped with the Koszul differential induced by $\psi$. The map $\phi^i_j : \bigwedge^i F \rightarrow K_j \otimes \bigwedge^{i-j} F$ is defined to be the composition

$$\bigwedge^i F \xrightarrow{\text{comult}} \bigwedge^i F \otimes \bigwedge^{i-j} F \xrightarrow{\text{lift} \otimes 1} K_j \otimes \bigwedge^{i-j} F,$$

where lift : $\bigwedge^j \rightarrow K_j$ denotes the projection onto $K_j$ of the cycle in $(\bigwedge^* F \otimes K_*)_j$ that represents any standard basis element $f \in \bigwedge^j$ (this map is then extended by linearity for arbitrary elements).

Example 4.3. Let $R = k[x_1, x_2, x_3]$ and let $\psi : Re_1 \oplus Re_2 \oplus Re_3 \rightarrow R$ be the map induced by sending $e_i \mapsto x_i^3$. Denote by $f_i$ the basis elements of the Koszul complex $K_*$ resolving $k$ (where the standard minimal generating set $x_1, x_2, x_3$ has been chosen).

Then the basis element $e_1 \wedge e_2 \in \bigwedge^2$ lifts to the cycle

$$e_1 \wedge e_2 + x_1^2 e_2 \otimes f_1 - x_2^2 e_1 \otimes f_2 + x_1^2 x_2 f_1 \wedge f_2 \in (\bigwedge^* \otimes K)_2.$$

Projecting onto $K_2$, it follows that

$$\text{lift}(e_1 \wedge e_2) = x_1^2 x_2^2 f_1 \wedge f_2.$$
In the notation of Definition 4.2, let \( z_\tau \) denote the lift of any \( f_\tau \in \bigwedge^i \) to the Koszul algebra. Then the map \( \phi^j_i \) of Definition 4.2 can be written explicitly:

\[
\phi^j_i(f_\tau) = \sum_{\tau \subseteq \sigma, |\tau| = j} \text{sgn}(\tau \subset \sigma)z_\tau \otimes f_{\sigma \setminus \tau},
\]

where \( \text{sgn} \) is the sign of the permutation that reorders \( (\sigma \setminus \tau) \cup \tau \) into ascending order. The next proposition tells us that the \( \phi^j_i \) maps can be used to give an explicit lift of basis elements of the complex of Definition 3.9 to the tensor product complex \( L^w(\psi, d) \otimes K_* \).

**Proposition 4.4.** Adopt notation and hypotheses as in Setup 3.3. Let \( \bigwedge^* F \) denote the Koszul complex induced by the map \( \psi : F \rightarrow R \) and let \( (\bigwedge^* F \otimes S_d)_w \) be the complex obtained by restricting to terms with multidegrees bounded above by \( w \). Given any \( f_\sigma \otimes f^\alpha \in (\bigwedge^i F \otimes S_{d-1})_w \), the element

\[
f_\sigma \otimes f^\alpha + \sum_{j=1}^{i-1} \phi^j_i(f_\sigma) \otimes f^\alpha + \psi(f^\alpha)\phi^j_i(f_\sigma)
\]

is a cycle in \( ((\bigwedge^* F \otimes S_{d-1})_w \otimes K_*)_r \).

**Proof.** Assume \( 1 \leq j \leq i-1 \) (noting that \( \phi^j_i \) is the identity). Throughout the proof, employ the shorthand notation that \( z_\tau = \text{lift}(f_\tau) \) for any subset \( \tau \subset [n] \). Then:

\[
(\text{Kos} \otimes 1 \otimes 1)(\phi^j_i(f_\sigma) \otimes f^\alpha) = (\text{Kos} \otimes 1 \otimes 1)\left( \sum_{\tau \subseteq \sigma, |\tau| = j} \text{sgn}(\tau)z_\tau \otimes f_{\sigma \setminus \tau} \otimes f^\alpha \right)
\]

\[
= \sum_{\tau \subseteq \sigma} \sum_{|\tau| = j} \text{sgn}(r \in \tau) \, \text{sgn}(\tau \subset \sigma) \, \psi(f_r)z_{\tau \setminus r} \otimes f_{\sigma \setminus \tau} \otimes f^\alpha
\]

\[
= \sum_{\tau' \subset \sigma} \sum_{|\tau'| = j-1} \text{sgn}(r \in \sigma \setminus \tau') \, \text{sgn}(\tau' \subset \sigma) \, \psi(f_r)z_{\sigma \setminus \tau'} \otimes f_{(\sigma \setminus \tau') \setminus r} \otimes f^\alpha
\]

\[
= (1 \otimes \text{Kos} \otimes 1)(\phi^j_{i-1}(f_\sigma) \otimes f^\alpha).
\]

In order to see that the above signs are equal, for any \( r \in \sigma \) and \( \tau' \subset \sigma \), let \( \tau' \) be the set \( (\tau' \cup r) \) ordered in ascending order. One can reorder the set \( (\sigma \setminus (\tau' \cup r)) \cup (\tau') \cup r \) into ascending order by either:

1. First reorder \( (\tau' \cup r) \) into ascending order, then reorder \( (\sigma \setminus (\tau' \cup r)) \cup (\tau') \cup r \) into ascending order; this permutation has sign \( \text{sgn}(r \in \tau) \, \text{sgn}(\tau \subset \sigma) \).
2. First reorder \( (\sigma \setminus (\tau' \cup r)) \cup (\tau') \cup r \) into ascending order, then reorder \( (\sigma \setminus (\tau')) \cup (\tau') \cup r \) into ascending order; this permutation has sign \( \text{sgn}(r \in \sigma \setminus \tau') \, \text{sgn}(\tau' \subset \sigma) \).

Since both of the above cases yield a permutation of the same parity, the signs are indeed equal.

In the case \( j = i \), one has:

\[
(1 \otimes \psi)(\phi^i_i(f_\sigma) \otimes f^\alpha) = \sum_{r \in \sigma} \text{sgn}(r \in \sigma) \psi(f^\alpha \cdot f_r)z_{\sigma \setminus r}
\]

\[
= (\text{Kos})(\psi(f^\alpha)z_{\sigma}).
\]

This completes the proof. \( \square \)
Corollary 4.5. Adopt notation and hypotheses as in Setup 3.3. The correspondence $f_\sigma \otimes f^\alpha \mapsto \psi(f^\alpha)\phi^i_\sigma(f_\sigma)$ induces an isomorphism of homology $L^w(\psi,d) \otimes k \to H_\bullet(R/(\im \psi)^d_w)$. Moreover, the ring $R/(\im \psi)^d_w$ is Golod whenever $d \geq 2$.

Proof. The first part of the statement is clear by construction of the isomorphism $\Tor^R_\bullet(-,k) \cong H_\bullet(- \otimes K_\bullet)$ combined with Proposition 4.4. The Golodness follows from noticing that the product of elements of the form $\psi(f^\alpha)\phi^i_\sigma(f_\sigma)$ in $R/(\im \psi)^d_w \otimes K_\bullet$ are trivial. This implies that simply choosing $\mu(h_1, \ldots, h_n) = 0$ for $i > 1$ is a well-defined trivial Massey operation on the Koszul homology. \qed

For any $\ell \geq 1$, let $V_\ell$ denote the free $R$-module with formal basis elements

$$\{v_{\sigma, \alpha} \mid \ell = |\sigma| + 1\},$$

where we are thinking of each $v_{\sigma, \alpha}$ as being a formal stand-in for the basis element $\psi(f^\alpha)z_{\sigma}$ in the Koszul homology algebra (and hence $\alpha$ ranges over all $w$-restricted exponent vectors with $|\alpha| = d$ and $\sigma$ ranges over all increasing subsets of $[n]$, where $n = \text{rank } F$). Then, combining Corollary 4.5 with Golod’s construction of the minimal free resolution of the residue field (see, for instance, [3, Theorem 5.2.2]), we have:

**Corollary 4.6.** Adopt notation and hypotheses as in Setup 3.3. Let $(T_\bullet, \partial_\bullet)$ denote the complex with

$$T_n := \bigoplus_{h+i_1+\cdots+i_p = n} K_h \otimes_R V_{i_1} \otimes_R \cdots \otimes_R V_{i_p},$$

$$\partial_n : T_n \to T_{n-1},$$

$$\partial_n(a \otimes v_{\sigma_1, \alpha_1} \otimes \cdots \otimes v_{\sigma_p, \alpha_p}) = \text{Kos}(a) \otimes v_{\sigma_1, \alpha_1} \otimes \cdots \otimes v_{\sigma_p, \alpha_p}$$

$$+ (-1)^{|a|} \alpha \psi(f^\alpha_1)z_{\sigma_1} \otimes v_{\sigma_2, \alpha_2} \otimes \cdots \otimes v_{\sigma_p, \alpha_p}.$$

Then $T_\bullet$ is the minimal free resolution of the residue field $k$ over $R/(\im \psi)^d_w$.

**Remark 4.7.** In a slightly different direction, it is easy to see that $w$-restricted powers of arbitrary monomial ideals are $d$-Golod (hence, Golod), in the terminology of [8].

## 5. Algebra structure on the minimal free resolution

In this section, we prove that the generalized $L$-complexes of Definition 3.9 admit the structure of an associative DG-algebra. The methods employed here come from techniques developed by Miller and Rahmati in [10]. The process of constructing the algebra structure consists of a few steps. First, we construct an algebra structure on an associated total complex that surjects onto the complexes of Definition 3.9. Next, we observe that the scaled de Rham map (see Lemma 5.4) satisfies the generalized Leibniz rule with respect to this algebra structure, inducing a special deformation retract. Finally, the result will follow after combining the previous two sentences with Proposition 2.11.

Let $F$ be a free $R$-module of rank $n$ with basis elements $f_1, \ldots, f_n$, with $\psi : F \to R$ any map such that $\psi(f_1), \ldots, \psi(f_n)$ forms a regular sequence. Throughout this section, we will assume either:

1. $R$ is a Noetherian local ring of characteristic $> n + 2$, or
2. $R$ is a $Z$-graded $k$-algebra of characteristic $> n + 2$, in which case the elements $\psi(f_1), \ldots, \psi(f_n)$ are assumed to be homogeneous of positive degree.

The characteristic assumptions imposed in (1) and (2) are needed because of the definition of the scaled de Rham differential given in Lemma 5.4. In the following definition, we show how to build a well-defined algebra structure after restricting multidegrees.
Definition 5.1. Let \((S_w)_w\) denote the \(R\)-submodule of \(S_w\) generated by all monomials with multidegree bounded by \(w\). Observe that \((S_w)_w\) is not a subalgebra with respect to the ordinary multiplication on \(S_w\). However, \((S_w)_w\) may be given an associative algebra structure as follows (notice: this is not necessarily graded):

\[
f^{\alpha} \cdot f^{\beta} = \psi(f^{\alpha + \beta - \min(\alpha + \beta, w)})f^{\min(\alpha + \beta, w)},
\]

where \(\alpha\) and \(\beta\) denote exponent vectors. Let \(X^w_d\) denote the total complex of the double complex obtained by deleting the rightmost nontrivial column of the double complex in Figure 2. Then \(X^w_d\) may be given the structure of an associative algebra with product defined as follows:

\[
(f_\sigma \otimes f^\alpha)(f_\tau \otimes f^\beta) = \begin{cases} 0 & \text{if } \alpha_i + \beta_i \geq w_i, \text{ and } i \in \sigma \cup \tau \text{ for some } i, \\ f_\sigma \wedge f_\tau \otimes f^\alpha \cdot f^\beta & \text{otherwise,} \end{cases}
\]

where in the above, it is understood that if \(f^\alpha \cdot f^\beta \in (S_{\geq d})_w\), then the product is 0.

Example 5.2. The product of Definition 5.1 may seem unnecessarily complicated at first sight, but it is important to notice that the “obvious” choice may not be well-defined. For instance, let \(f = k[x_1, x_2] \otimes Rf_1 \oplus Rf_2\) with \(\psi: F \to R\) induced by sending \(f_1 \mapsto x_i\). If \(w = (1, 1)\), then the standard product on \(S_w\) does not restrict to a product on \((S_w)_w\) since \(f_1 \cdot f_1 = f_1^2 \notin (S_w)_w\). The product of Definition 5.1 pulls the “overflow” out as a coefficient, so that \(f_1 \cdot f_1 = x_1f_1 \in (S_w)_w\).

Proposition 5.3. Adopt notation and hypotheses as in Setup 3.3. With product as in Definition 5.1, the total complex \(X^w_d\) is an associative DG-algebra.

Proof. Use notation as in Definition 5.1. Let \(S := \{i | \alpha_i + \beta_i \geq w_i\} \cup \delta \cup \tau\). If \(|S| = 0\), then the proof of the Leibniz rule is essentially identical to that of the Koszul complex.

If \(|S| = 1\), then let \(\ell\) be the unique integer with \(\alpha_\ell + \beta_\ell \geq w_\ell\) and \(\ell \in \sigma \cup \tau\). Assume without loss of generality that \(\ell \in \sigma\). By definition, \((f_\sigma \otimes f^\alpha) \cdot (f_\tau \otimes f^\beta) = 0\). On the other hand, one computes:

\[
d(f_\sigma \otimes f^\alpha)(f_\tau \otimes f^\beta) + (-1)^{|\sigma|}(f_\sigma \otimes f^\alpha)d(f_\tau \otimes f^\beta)
= \begin{cases} \begin{aligned} & \text{sgn}(\ell \in \sigma) \left( - \psi(f_\ell \wedge f_\tau \otimes f^\alpha \cdot f^\beta + \psi(f_\ell)(f_\sigma \wedge f_\tau \otimes f^{\alpha + \beta_\ell}) \right) \\
& = 0. \end{aligned} \end{cases}
\]

Finally, if \(|S| > 1\), then all terms appearing in the Leibniz rule still have trivial multiplication. To conclude the proof, observe that associativity follows by the associativity of the exterior algebra and the product defined in Definition 5.1.

Lemma 5.4. Adopt notation and hypotheses as in Setup 3.3. Let \(h\) denote the scaled de Rham map

\[
h_{a,b} := \frac{1}{a + b} \sum_{j=1}^n f_j \otimes \frac{\partial}{\partial f_j} : \bigwedge^a S_b \to \bigwedge^{a+1} S_{b-1}.
\]

Then,

(1) \(h^2 = 0\),

(2) \(\kappa h + h\kappa = 1\), and

(3) \(h\) restricts to a contracting homotopy \((\bigwedge^a S_b)_w \to (\bigwedge^{a+1} S_{b-1})_w\) on the \(w\)-restricted tautological Koszul complex.

Proof. The proofs of (1) and (2) may be found in [10, Lemma 4.8], and (3) follows because \(h\) preserves multidegree.
\textbf{Remark 5.5.} The explicit form for the homotopy $h$ may be given as follows:

$$h_{a,b}(f_\sigma \otimes f^\alpha) = \frac{1}{a + b} \sum_j \alpha_j f_j \wedge f_\sigma \otimes f^{\alpha - \epsilon_j}.$$ 

\textbf{Proposition 5.6.} Adopt notation and hypotheses as in Setup 3.3 and let $h$ denote the scaled de Rham map of Lemma 5.4. Then $h$ satisfies the generalized Leibniz rule with respect to the product of Definition 5.1.

Before the proof of Proposition 5.6, recall the conventions for omitting subscripts as established in Notation 3.7.

\textbf{Proof.} Let $f_\sigma \otimes f^\alpha \in \bigwedge^T \otimes S_a$ and $f_\tau \otimes f^\beta \in \bigwedge^T \otimes S_b$. If $(f_\sigma \otimes f^\alpha)(f_\tau \otimes f^\beta) = 0$, then there is nothing to prove. Assume that the product is not zero and define $T := \{ i \mid \alpha_i + \beta_i > w_i \}$. Notice that the assumption $(f_\sigma \otimes f^\alpha)(f_\tau \otimes f^\beta) \neq 0$ implies that if $i \in T$, then $i \notin \sigma \cup \tau$. For each $i \in T$, let $\alpha_i' < \alpha_i$ and $\beta_i' < \beta_i$ be such that $\alpha_i' + \beta_i = w_i$ and $\alpha_i + \beta_i' = w_i$. Observe first that by definition of $T$ there are equalities:

\[ (f_j \wedge f_\sigma \otimes f^{\alpha - \epsilon_i}) \cdot (\psi (f_\tau^{\beta_i' - \beta_i}) f_\tau \otimes f^{\beta - (\beta_i' - \beta_i)\epsilon_i}) = \begin{cases} 0 & \text{if } j \neq i, j \in T, \\ f_\tau \wedge f_\tau \wedge f_\tau \otimes f^{\alpha - \epsilon_i} \cdot f_\tau^{\beta - (\beta_i' - \beta_i)\epsilon_i} & \text{if } i = j, \text{ and } \\ f_j \wedge f_\sigma \wedge f_\tau \otimes f^{\alpha - \epsilon_i} \cdot f_\tau^{\beta - (\beta_i' - \beta_i)\epsilon_i} & \text{if } j \notin T. \end{cases} \]

\[ (\psi (f_\tau^{\alpha_i' - \alpha_i}) f_\tau \otimes f^{\alpha - (\alpha_i' - \alpha_i)\epsilon_i}) \cdot (f_\tau \wedge f_\tau \otimes f^{\beta - \epsilon_i}) = \begin{cases} 0 & \text{if } j \neq i, j \in T, \\ f_\tau \wedge f_\tau \wedge f_\tau \otimes f^{\alpha - \epsilon_i} \cdot f_\tau^{\beta - \epsilon_i} & \text{if } i = j, \text{ and } \\ f_\tau \wedge f_\tau \wedge f_\tau \otimes f^{\alpha - \epsilon_i} \cdot f_\tau^{\beta - \epsilon_i} & \text{if } j \notin T. \end{cases} \]

Combining these equalities with the explicit form of $h$ given in Remark 5.5, it follows that for all $i \in T$ there are equalities:

\[ (r + a)h(f_\sigma \otimes f^\alpha) \cdot (f_\tau \otimes f^{\beta}) = \sum_{j \notin T} f_j \wedge f_\sigma \wedge f_\tau \otimes f^\alpha \cdot \frac{\partial f^\alpha}{\partial f^\tau} \cdot f_\tau^{\beta}, \]

\[ (s + b)(\psi (f_\tau^{\alpha_i' - \alpha_i}) f_\tau \otimes f^{\alpha - (\alpha_i' - \alpha_i)\epsilon_i}) \cdot h(f_\tau \otimes f^{\beta}) = \sum_{j \notin T} f_\tau \wedge f_\tau \wedge f_\tau \otimes f^\alpha \cdot \frac{\partial f^\alpha}{\partial f^\tau} \cdot f_\tau^{\beta}, \]

\[ (r + a)h(f_\sigma \otimes f^\alpha) \cdot (f_\tau \otimes f^{\beta}) = \sum_{j \notin T} f_j \wedge f_\sigma \wedge f_\tau \otimes f^\alpha \cdot \frac{\partial f^\alpha}{\partial f^\tau} \cdot f_\tau^{\beta}, \] 

\[ (s + b)(f_\sigma \otimes f^\alpha) \cdot h(f_\tau \otimes f^{\beta}) = \sum_{j \notin T} f_\tau \wedge f_\tau \wedge f_\tau \otimes f^\alpha \cdot \frac{\partial f^\beta}{\partial f^\tau} \cdot f_\tau^{\beta}. \]

Employing the above 4 equalities, one computes:

\[ (r + a)h(f_\sigma \otimes f^\alpha)((1 - |T|)f_\tau \otimes f^{\beta}) + \sum_{i \in T} \psi (f_\tau^{\beta_i' - \beta_i}) f_\tau \otimes f^{\beta - (\beta_i' - \beta_i)\epsilon_i}) \]

\[ = (1 - |T|)(r + a)h(f_\sigma \otimes f^\alpha) \cdot (f_\tau \otimes f^{\beta}) + \sum_{i \in T} (r + a)h(f_\sigma \otimes f^\alpha) \cdot (\psi (f_\tau^{\beta_i' - \beta_i}) f_\tau \otimes f^{\beta - (\beta_i' - \beta_i)\epsilon_i}) \]
\[= (1 - |T|) \sum_{j \notin T} f_j \land f_\sigma \land f_\tau \otimes \frac{\partial f_\sigma}{\partial f_j} \cdot f_\beta + |T| \sum_{j \notin T} f_j \land f_\sigma \land f_\tau \otimes \frac{\partial f_\sigma}{\partial f_j} \cdot f_\beta + \sum_{i \in T} f_i \land f_\sigma \land f_\tau \otimes \frac{\partial f_\sigma}{\partial f_i} \cdot f_\beta \]
\[= \sum_{j=1}^n f_j \land f_\sigma \land f_\tau \otimes \frac{\partial f_\sigma}{\partial f_j} \cdot f_\beta.\]

An identical computation also shows that
\[= (s + b) \left( (1 - |T|) f_\sigma \otimes f_\alpha + \sum_{i \in T} \psi \left( f_i^{\alpha_i - \alpha_i'} f_\sigma \otimes f_\alpha - (\alpha_i - \alpha_i') \epsilon_i \right) \right) h(f_\tau \otimes f_\beta) \]
\[= \sum_{j=1}^n f_j \land f_\sigma \land f_\tau \otimes f_\alpha \cdot \frac{\partial f_\beta}{\partial f_j}.
\]

Combining the previous two equalities, one finds:
\[(r + a) h(f_\sigma \otimes f_\alpha) \left( (1 - |T|) f_\tau \otimes f_\beta + \sum_{i \in T} \psi \left( f_i^{\beta_i - \beta_i'} f_\tau \otimes f_\beta - (\beta_i - \beta_i') \epsilon_i \right) \right) h(f_\tau \otimes f_\beta) \]
\[+ (-1)^r (s + b) \left( (1 - |T|) f_\sigma \otimes f_\alpha + \sum_{i \in T} \psi \left( f_i^{\alpha_i - \alpha_i'} f_\sigma \otimes f_\alpha - (\alpha_i - \alpha_i') \epsilon_i \right) \right) h(f_\tau \otimes f_\beta) \]
\[= \sum_{j=1}^n f_j \land f_\sigma \land f_\tau \otimes \frac{\partial f_\alpha}{\partial f_j} \cdot f_\beta + (-1)^r \sum_{j=1}^n f_j \land f_\sigma \land f_\tau \otimes f_\alpha \cdot \frac{\partial f_\beta}{\partial f_j} \]
\[= \sum_{j=1}^n f_j \land f_\sigma \land f_\tau \otimes \frac{\partial (f_\alpha \cdot f_\beta)}{\partial f_j} \]
\[= (r + s + a + b) h((f_\sigma \otimes f_\alpha)(f_\tau \otimes f_\beta)).\]

Dividing the above by \(r + s + a + b\), it follows that \(h\) satisfies the generalized Leibniz rule.

5.7. Let us recall the method of transferring algebra structures used in [10] as applied to our situation. The goal of this method is to find a special deformation retract
\[(X_d^w, \kappa \otimes 1) \rightleftarrows (L^w(\psi, d), \text{Kos} \otimes 1)\]
that is a perturbation of a special deformation retract, with the homotopy \(h\) satisfying the generalized Leibniz rule. Let \(\varepsilon\) denote any choice of isomorphism \(\bigwedge^0 S_0 \cong R\). In our situation,

(1) The special deformation retract to be perturbed comes from the double complex of Figure 3, with contracting homotopy \(h\) coming from Lemma 5.4 and
\[i = \begin{cases} h & \text{on } L_{a,w}^i, \\ \varepsilon^{-1} & \text{on } R, \end{cases}\]
The unperturbed double complex.

\[ 0 \longrightarrow (\bigwedge^n \otimes S_{d-1})_w \overset{i}{\longrightarrow} L^{n-1}_{d,w} \]

\[ \vdots \quad \vdots \quad \vdots \quad \vdots \]

\[ \vdots \longrightarrow (\bigwedge^2 \otimes S_{d-1})_w \overset{i}{\longrightarrow} L^1_{d,w} \]

\[ \vdots \longrightarrow (\bigwedge^1 \otimes S_{d-1})_w \overset{i}{\longrightarrow} L^0_{d,w} \]

\[ \vdots \quad \vdots \quad \vdots \quad \vdots \longrightarrow (\bigwedge^0 \otimes S_{d-1})_w \]

\[ (\bigwedge^2 \otimes S_0)_w \overset{\kappa}{\longrightarrow} (\bigwedge^1 \otimes S_1)_w \overset{\kappa}{\longrightarrow} (\bigwedge^0 \otimes S_2)_w \]

\[ (\bigwedge^1 \otimes S_0)_w \overset{\kappa}{\longrightarrow} (\bigwedge^0 \otimes S_1)_w \overset{i}{\longrightarrow} \]

\[ (\bigwedge^0 \otimes S_0)_w \overset{\kappa}{\longrightarrow} \]

Figure 3. The unperturbed double complex.

\[
\begin{align*}
\kappa & \quad \text{on } (\bigwedge^i \otimes S_{d-1})_w, \; i > 0, \\
\varepsilon & \quad \text{on } (\bigwedge^0 \otimes S_0)_w, \\
0 & \quad \text{otherwise}.
\end{align*}
\]

\[ p = \begin{cases} 
\kappa & \text{on } (\bigwedge^i \otimes S_{d-1})_w, \; i > 0, \\
\varepsilon & \text{on } (\bigwedge^0 \otimes S_0)_w, \\
0 & \text{otherwise}.
\end{cases} \]

(2) The perturbed deformation retract comes from the double complex of Figure 4, where the perturbation is precisely the vertical Koszul differential $\text{Kos} \otimes 1$ and

\[
\begin{align*}
i_\infty &= \begin{cases} 
(1 - h(\text{Kos} \otimes 1))^{-1}h & \text{on } L^i_{d,w} \\
\varepsilon^{-1} & \text{on } R,
\end{cases} \\
p_\infty &= \begin{cases} 
\kappa & \text{on } (\bigwedge^i \otimes S_{d-1})_w, \; i > 0, \\
\varepsilon & \text{on } (\bigwedge^0 \otimes S_0)_w, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

The fact that the perturbed differential induced on the rightmost column of Figure 4 is precisely $\text{Kos} \otimes 1$ is an identical computation to that done in 4.6 of [10].

**Theorem 5.8.** Adopt notation and hypotheses as in Setup 3.3. The complex $L^w(\psi, d)$ of Definition 3.9 admits the structure of an associative DG-algebra. If $R$ is a standard graded polynomial ring, the vector
Figure 4. The perturbed double complex.

$w$ has all entries equal, and $\text{im } \psi = R_+$, then this product is invariant under the natural action of the symmetric group on the variables.

**Remark 5.9.** Assume $R = k[x_1, \ldots, x_n]$ and $\text{im } \psi = R_+$. If the entries of $w$ are not all equal, then the ideal $(R_+)_w^d$ is not $S_n$-invariant under the natural symmetric group action. There is an induced action by a product of symmetric groups that acts by permuting the variables corresponding to entries of $w$ that are equal to each other. For example, if $n = 4$ and $w = (2, 1, 2, 1)$, then $(R_+)_w^d$ is a $S_2 \times S_2$-invariant ideal with action induced by $x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4$.

In general, since this induced action by $S_{n_1} \times \cdots \times S_{n_r}$ is just a restriction of the $S_n$-action on $R$, the product of Theorem 5.8 is also $S_{n_1} \times \cdots \times S_{n_r}$-equivariant.

**Proof of Theorem 5.8.** Define the product of the classes of $f_\sigma \otimes f^\alpha$ and $f_\tau \otimes f^\beta \in L^w(\psi, d)$ via

$$(f_\sigma \otimes f^\alpha) \cdot (f_\tau \otimes f^\beta) := p_\infty(i_\infty(f_\sigma \otimes f^\alpha) \cdot i_\infty(f_\tau \otimes f^\beta)).$$

This product will yield an associative DG-algebra structure on $L^w(\psi, d)$ by Proposition 2.11 combined with Proposition 5.3, Lemma 5.4, Proposition 5.6, and the discussion of 5.7.
It is not difficult to see that in the case \( R = k[x_1, \ldots, x_n] \) and \( \text{im}(\psi) = (x_1, \ldots, x_n) \) with \( w = (1, 1, \ldots, 1) \), the complex \( L^w(\psi, d) \) is identical to the complex constructed by Galetto in [6]. Thus, Theorem 5.8 gives an explicit \( S_n \)-equivariant algebra structure on those complexes.

**Remark 5.10.** One can alternatively remove the characteristic assumption on \( k \) at the expense of losing \( S_n \)-equivariance by restricting the product of Srinivasan [12] to all standard tableaux with bounded multidegree. It is straightforward to verify that this restriction does yield a subalgebra.

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