The Matrices of Some of the Plane Graphs and Their Dual

Rawah A. Zaben and Israa M. Tawfik

1- Salah Al-Deen's Education- The Ministry of Education, Iraq.
2- Mathematics Department, College of Education for Women, Tikrit University, Iraq.

*Corresponding Author Israa M. Tawfik israa.tawfik@tu.edu.iq

Article History: Received: xx/12/2020, Accepted: xx/xx/202x, Published: xx/xx/202x

Abstract: The importance of Graph Theory comes from being Combinatorics. It can study this subject and its relationship to the topological using plane graphs where they are embedded graphs. Also, its relationship to linear algebra which can study the using of matrices. This article describes the adjacency matrices of adjacent vertices of plane graphs and their duals by finding the formula of the adjacency matrix of some types of plane graphs and their duals.

Keywords: Adjacency matrix, Plane graphs, and Dual graphs.

1. Introduction:

A graph \( G = (V, E) \) determines an adjacency matrix denoted by \( M(G) \).

As we know, graph theory is one part of combinatorics, where matrices is a subject of linear algebra.

This research has described the relationships between graph theory, topology, and linear algebra.

The working with this research is going by taking some finite simple connected graphs and their duals. Then we try to find the adjacency matrix of each graph and their dual, by using Euler Polyhedron formula[1].

This study has deal with plan graph, which means a graph embedded on the plane. This graph called planer. A plane graph \( G \) is determined by vertices \( V \), edges \( E \), and faces \( F \). Each face bounded a region, this region is homeomorphic to a disc [2].

Every graph \( G \) embedded on the surface is called a cellularly embedded graph [3].

Euler had discovered the subject of planer, when he was working with his investigation of polyhedral. The classical results of mathematics is the Euler formula for polyhedral [1].

There are no other previous studies directly on this subject but we found some other related researches about the dual of plane graphs or about the adjacency matrices.

In [4] presented a class of graphs whose adjacency matrices were nonsingular with integral inverses, denoted by \( h \)-graph. [5] had introduced Eulerian plan graphs and studied the partial duals of plan graphs. [6] had shown that the determinate of \( M(G) \) of a plane graph \( G \).
which has the property that every face boundary is a cycle of size divisible by 4, equals $-1, 0$ or $1$, provided the inner dual graph of $G$ is a tree.

The next section of this paper contains basic results which we need to construct the third section of this paper. In this section we discover some basic result depends on Euler formula for plane graph. The third section describes the formula of matrices of plane graphs and their duals as a complete graph, bipartite graphs, tree, cobweb graph, and r-paths graph. The final section introduces the conclusions and some of the further work.

2. Basic Results

This section introduces some basic results which we need in the next section.

We suppose a graph $G = (V,E)$ is a finite connected graph, where $V = \{v_1, v_2, \ldots, v_n\}$ a set of vertices of order $n$ and $E = \{e_1, e_2, \ldots, e_m\}$ is a set of edges of size $m$. So each edge has two end-vertices. Two vertices are called adjacent vertices, if they are the end-vertices of the same edge [7].

**Theorem 1** (Euler's Formula for plan Graph) [8].

Let $G = (V,E)$ be a connected plane graph contains $n$ vertices, $m$ edges, and $f$ faces. Then

$$n - m + f = 2.$$

Now, we discuss some types of plane graphs.

**Definition 2.1** [7]

If $G = (V,E)$ is a simple graph, and the set $V$ divided into two parts $X$ and $Y$. However each vertex in $X$ adjacent to each vertex in $Y$ and there is not any other adjacent vertices. Then $G$ is called a complete bipartite graph denoted by $K_{m,n}$, where $m$ is the number of $X$'s vertices and $n$ is the number of $Y$'s vertices.

Note that, by using Kuratowski’s Theorem (Harary, 1972), we can deduce that if $K_{1,j}$ where $j = 1, 2, \ldots, n$ and $i = 1, 2$, then $K_{1,n}$ and $K_{2,n}$ are plane graphs.

**Definition 2.2** [7]

A complete graph is a graph $G = (V,E)$ which is a simple graph and each pair of vertices has an edge. It is denoted by $K_n$.

Note that, by using Kuratowski’s Theorem [7] we can conclude that $K_n$ is a planar, if $n \leq 4$.

Now, we define the type of plane graph, it had be used before as a simple graph, but we could not find a logical definition for it. So we define it in this article.

**Definition 2.3**
If $G = (V,E)$ is a simple graph, then $G$ is called a cobweb graph denoted by $C_{m,n}$, if $G$ has $n$ numbers of $C_m$ cycles. Where each vertex in $C_m$ cycle is adjacent to the symmetric vertex in the next or before cycle. See (Fig. 1).

![Fig. 1. The Cobweb Graph.](image)

Notes

1- Each cobweb graph $C_{ij}$ is a planar, $i = 1,2,3,...,m$, $j = 1,2,...,n$.

2- Each vertex in a cobweb graph has degree four except the vertices of first cycle of the graph and the end cycle of this graph has degree three.

3- Each graph has $m$ of $P_n$ paths, each path connects the symmetric vertices in the cycles.

Lemma (Hand Shaking)[9]

Let $G = (V,E)$ be a graph of $n$ vertices and $m$ edges whenever $d(v_i)$ is the degree of vertex $v_i$. Then, $\sum_{i=1}^{n} d(v_i) = 2m$.

Theorem 3

If $G = (V,E)$ is a $C_{m,n}$ graph then the following statements are true,
i- The number of the elements of $V$ is $mn$.

ii- The number of the elements of $E$ is $2mn - m$.

iii- The number of the elements of $G$ is $mn - m + 2$.

Proof

i- Since $G$ has $n$ cycles each cycle contains $m$ edges, and every path has $(n-1)$ edges. That means the number of edges of $C_{mn}$ is $m + m(n - 1) = m(2n - 1)$.

ii- From (i) a graph $C_{mn}$ has $mn$ vertices each vertex has degree 4 except $2m$ vertices has degree 3. Then the summation of the degrees of all vertices is $4(mn - 2m) + 3(2m) = 2(2mn - m)$.

Hence, by Hand Shaking Lemma the number of $E(G)$'s elements are $m(2n - 1)$.

iii- To prove this part, we need Theorem 1. $G$ has $mn$ vertices, $m(2n - 1)$ edges and $G$ is a plane graph then $G$ satisfies Euler's formula for plan graph to get.

$$f = mn - m + 2.$$ 

The following type of a graph is also a planar, so we define it by using a path $P$ of $n$ vertices and $n - 1$ edges.

Definition 2.4

Let $G = (V, E)$ be a simple graph, if $u, v \in V(G)$ and $u, v$ are joint by $r$ paths, whenever each path has $s$ edges and $r, s$ are positive integer numbers. Then $G$ is called $r$-path graph denoted by $P_{r,s}$. See Fig. (2).

![Fig. 2. The $P_{r,s}$ graph.](image)

Theorem 4

Let $G = (V, E)$ be a $P_{r,s}$ graph, then the following statements are true:

i- There are $r$ faces only.

ii- $G$ contains $rs$ edges.

iii- The number of vertices is $r(s - 1) + 2$.

Proof

i- There is a face between two sequent paths.

ii- There are $r$ paths, each path consists of $s$ edges. Hence, $G$ has $rs$ edges.

iii- Here we need to use Euler's formula. Suppose $v$ the number of vertices of $G$ then $v = rs - r + 2$. 

Now is the time to introduce the adjacency matrix.

Definition 2.5 [8]
The $n \times n$ matrix $M(G) = [a_{ij}]$ is called the adjacency matrix of a graph $G = (V, E)$, wherever $V$ has $n$ elements. $a_{ij}$ is the number of edges joining vertex $i$ and vertex $j$ of the non-diagonal entry elements of $M(G)$. So the diagonal entry $a_{ii}$ is twice the number of loops at vertex $i$.

**Notes**
1. $M(G)$ means the adjacency matrix of a graph $G$.
2. From the definition of the adjacency matrix, $M(G)$ is a square matrix. Where $a_{ij} = a_{ji}$.
3. $M(G)$ is called a binary matrix, if $a_{ij} = 0$ or $1$.
4. From 3 we can deduce that any adjacency matrix of a simple graph is a binary matrix.
5. From definition 5, $M(G)$ is a symmetric matrix.

**Theorem 5**

Let $G = (V, E)$ be a graph, $a_{ij}$ is an element of $M(G)$ then $\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} = 2m$, where $m$ is the number of $E$'s elements.

**Note** [8]
The summation of every row $i$ or column $j$ in $M(G)$ is the degree of $v_i$ or $v_j$ respectively.

3. The description of some plane graphs’ matrices and their duals

This section find some results of the adjacency matrix of some plane graphs and their duals.

The first result is got for a complete bipartite graph. As we mentioned in section 2, this type of graph leads to think in Kuratowski's Theorem. There are two types of a complete bipartite plane graph, the first is $K_{1,n}$ and the second is $K_{2,n}$. Then deals with a special bipartite graph which is a tree.

**Theorem 6**

Let $G = K_{1,n}$ where $n$ is a positive integer number, then $M(G)$ is $(n + 1) \times (n + 1)$ binary matrix.

**Proof**
Since $G$ is a simple graph then $M(G)$ is a binary matrix, and since $G$ is a complete bipartite graph has one part of vertices of order 1 and another part of order $n$ then $M(G)$ has one row and the symmetric column of summation $n$, and all other rows have summation 1.

**Theorem 7**

Let $G = K_{1,n}^*$ be a dual graph of $K_{1,n}$, where $n$ is a positive integer number, then $M(G)$ is $1 \times 1$ matrix of value $2n$.

**Proof**
Since $G = K_{1,n}^*$, and $K_{1,n}$ has one face, then $G$ has one vertex and each edge in $G$ is a loop. Hence the theorem holds.

**Theorem 8**
Let $G = K_{2,n}$ where $n$ is a positive integer number, then $M(G)$ is an $(n + 2) \times (n + 2)$ binary matrix. Every row has summation 2 except two rows have summation $n$ for each.

**Proof**

$M(G)$ is a binary matrix because $G$ is a simple graph and $M(G)$ is a symmetric matrix of $n + 2$ rows. As we know this number of rows is equal to the order of $G$. $M(G)$ has 2 rows of summation $n$ for each one and $n$ rows of summation 2 for each.

**Theorem 9**

Let $G = K_{2,n}$ be the dual graph of $K_{2,n}$, where $n$ is a positive integer number, then $M(G)$ is an $n \times n$ matrix each row of it has summation 4.

**Proof**

By using Euler formula for a plane graph we get $f = n$ of $K_{2,n}$. That means $G$ has $n$ vertices since each face of $K_{2,n}$ is bounded by four edges each pair of these edges is sharing with another face, then each vertex of $G$ adjacent to two different vertices. Each vertex of $G$ is of degree four. Hence the Theorem holds.

Let $T_n$ be a tree of order $n$ and size $n - 1$, and any tree has one face. However, any tree is a plane bipartite graph. The dual of $T_n$ is a graph of one vertex of degree $2(n - 1)$ and $M(T_n)$ is an $n \times n$ symmetric binary matrix.

**Theorem 10**

Let $G = T_n^*$ be the dual graph of $T_n$, then $M(G)$ is a $1 \times 1$ matrix where $M(G) = [2(n - 1)]$.

Also Kuratowski's Theorem leads us to determined our work with complete graph. $(n - 1)$-regular graph means a graph of degree $n - 1$ for each vertex.

**Theorem 11**

Let $G = K_n$, $n \leq 4$ $M(G)$ is an $n \times n$ binary matrix each row of it has summation $n - 1$.

**Proof**

Since $G$ is a complete graphs then it is an $(n - 1)$-regular simple graph. That means each row contains $n - 1$ column has value 1, except the $a_{ii}$ is of $0$ value. Hence, the theorem hold.

**Theorem 12**

Let $G = K_n^*$ be the dual graph of $K_n$, $n \leq 4$ then the adjacency matrix of $G$ is as in the following:

i- If $G = K_1^*$ then $M(G) = [0]$
ii- If $G = K_2^*$ then $M(G) = [2]$
iii- If $G = K_3^*$ then $M(G) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$
iv- If $G = K_4^*$ then $M(G) = M(K_4)$
Proof
i- It is obvious
ii- Since \( G \) is a loop then \( M(G) = [2] \)
iii- Since \( G \) is a graph of two vertex and three parallel edges then \( M(G) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \)
iv- Since \( G \) is a \((3)-\) regular simple graph then \( G = K_4 \).

The following two theorems discuss the adjacency matrices of the cobweb graph and their dual by using Euler formula.

**Theorem 13**

Let \( G = C_{m,n} \), then \( M(G) \) is an \( mn \times mn \) binary matrix each row has summation 4 except 2\( m \) rows have summation 3 for each.

**Proof**

Since \( C_{m,n} \) has \( n \) of \( C_m \) cycles, then \( G \) has \( mn - 2m \) vertices each of degree four and \( 2m \) vertices of degree three. Since \( C_{m,n} \) is a simple graph then each column in each row has 1 or 0 value, that means \( M(G) \) is the binary matrix.

**Theorem 14**

Let \( G = C_{m,n}^* \), the dual of \( C_{m,n} \) graph, then \( M(G) \) is an \( (m(n - 1) + 2) \times (m(n - 1) + 2) \) binary matrix each row has summation 4 except 2 rows each has summation 4\( m \).

**Proof**

Since \( C_{m,n} \) has no faces sharing in more than one edge then \( G \) is a simple graph and has \( m(n - 1) + 2 \) faces.

This mean that \( M(G) \) is a binary matrix of dimension \( (m(n - 1) + 2) \times (m(n - 1) + 2) \). Each face in \( C_{m,n} \) is bounded by four edges except two faces are surrounded by \( m \) edges. That means each row in \( M(G) \) has summation 4 except these two rows each has summation \( m \).

Now we want to discuss the matrix of \( r \)-path graph and their dual.

**Theorem 15**

Let \( G \) be a \( P_{r,s} \) graph. Then \( M(G) \) is an \( (r(s - 1) + 2) \times (r(s - 1) + 2) \) binary matrix. This matrix has two rows each has summation \( r \) and contains \( (rs - r) \) rows each has summation 2.

**Proof**

Since \( G \) is a simple graph then the adjacency matrix of it is a symmetric binary matrix.

Since \( G \) is of order \( (r(s - 1) + 2) \) then the dimension of this matrix is \( (r(s - 1) + 2) \times (r(s - 1) + 2) \). Since each vertex of any path has degree 2 except the two sharing end-vertices of all paths each has degree \( r \). This complete the proof.

The following theorem also is proved by using Euler’s Formula.

**Theorem 16**

Let \( G = P_{r,s}^* \) be the dual graph of \( P_{r,s} \). Then \( M(G) \) is an \( r \times r \) matrix. Each row of it has summation 2\( s \).
Proof

Since each face in $P_{r,s}$ is surrounded by two paths that mean each vertex in $G$ has degree $2s$.

4. Conclusions and further works

This research studied the relationships between graph theory, topology, and linear algebra by finding the adjacency matrices of some plane graphs and their duals. It described the plane graph of some ordered of complete bipartite graph, complete graph. For these two graphs, we used Kuratowski's Theorem to determine the order of plane graph. We managed to discuss the adjacency matrix of the dual tree. We got interesting matrices for two other plane graphs and their dual, we called them a cobweb graph and an r-path graph.

To get all these results, we derived some basic results which were in the main results. We plan to continue to study the adjacency matrix and study another type of matrices for other plane graphs and other types of graphs. It may be possible to develop this study through the use of polynomials of matrices or graphs.

REFERENCES

[1] Huggett, S., & Jordan, D. 2009. A Topolgical Aperitif. British Library: Springer.
[2] Huggett, S., & Tawfik, I. 2015. Embedded Graphs Whose Links have the Largest Possible number of Components. ARS MATHEMATICA CONTEMPORANCE, pp. 319-335.
[3] Robertson, N., Seymour, P., & Thomas, R. 1993. Linkless Embedding Graphs in 3-space. Bulletin of the American Mathematica Society, pp. 84-89.
[4] Tifenbach, R. M., & Kirklanad, S. J. 2009. Directed Intervals and the Dual of a Graph. Linear Algebra and its Application, pp. 792-807.
[5] Hugget, S., & Moffatt, I. 2013. Bipartite Partial Duals and Circuits in Medial Graphs. Combinatorica, pp. 231-252.
[6] Huang, L., & Yan, W. 2012. On the Determinant of the Adjacency Matrix of a Type of Plane Bipartite Graphs. MATCH Commun .Math.Comput chem, pp. 931-938.
[7] Hrarry, F.1972. Graph Theory. Menlop Park, California: Addison-Wesley.
[8] Balakrishnan, V. 1997. Schau'm's Outline of Theory and Problems of Graph Theory. United State of America: McGraw-Hill.
[9] Wilson, R.1996. Introduction to Graph Theory. Harlow: Addison Wesley Longman Limited.
[10] Cherney, D., Denton, T., Thomas, R., & Waldron, A. 2013. Linear Algebra. Davis California: Creative Commons.