An integrable
time-dependent non-linear Schrödinger equation

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Abstract. The cubic non-linear Schrödinger equation (NLS), where the coefficient of the non-linear term can be a function $F(t, x)$, is shown to pass the Painlevé test of Weiss, Tabor, and Carnevale only for $F = (a + bt)^{-1}$, where $a$ and $b$ constants. This is explained by transforming the time-dependent system into the ordinary NLS (with $F = \text{const.}$) by means of a time-dependent non-linear transformation, related to the conformal properties of non-relativistic space-time.

(5/2/2020)

Physics Letter A (submitted).

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Let us consider the cubic non-linear Schrödinger equation (NLS),
\begin{equation}
iu_t + u_{xx} + F(t, x)|u|^2u = 0,
\end{equation}
where \(u = u(t, x)\) is a complex function in 1+1 space-time dimension. When the coefficient \(F(t, x)\) of the non-linearity is a constant, this is the usual NLS, which is known to be integrable. But what happens, when the coefficient \(F(t, x)\) is a function rather then just a constant? Performing the Painlevé analysis of Weiss, Tabor and Carnevale \[1\], we show

**Theorem 1**: The generalized non-linear Schrödinger equation (1) only passes the Painlevé test if the coefficient of the non-linear term is of the form
\begin{equation}
F(t, x) = \frac{1}{at + b}, \quad a, b = \text{const}.
\end{equation}

**Proof.** As it is usual in studying non-linear Schrödinger-type equations \[2\], we consider Eqn. (1) together with its complex conjugate (\(v = u^*\)),
\begin{align}
iu_t + u_{xx} + F u^2 v &= 0, \\
-iv_t + v_{xx} + F v^2 u &= 0.
\end{align}
This system will pass the Painlevé test if \(u\) et \(v\) have generalised Laurent series expansions,
\begin{equation}
\begin{aligned}
u &= \sum_{n=0}^{+\infty} u_n \xi^{n-p}, \\
v &= \sum_{n=0}^{+\infty} v_n \xi^{n-q},
\end{aligned}
\end{equation}
\((u_n \equiv u_n(x, t), v_n \equiv v_n(x, t)\) and \(\xi \equiv \xi(x, t)\)) in the neighbourhood of the singular manifold \(\xi(x, t) = 0, \xi_x \neq 0\), with a sufficient number of free coefficients. Using the results of Tabor, and of Weiss \[3\], it is enough to consider \(\xi = x + \psi(t)\); then \(u_n\) and \(v_n\) become functions de \(t\) alone, \(u_n \equiv u_n(t)\), \(v_n \equiv v_n(t)\). Checking the dominant terms, \(u \sim u_0 \xi^{-p}, v \sim v_0 \xi^{-q}\), using the above remark, we get
\begin{equation}
p = q = 1, \quad F u_0 v_0 = -2.
\end{equation}
Hence \(F\) can only depend on \(t\). Now inserting the developments (4) of \(u\) and \(v\) into (3), the terms in \(\xi^k, k \geq -3\) read
\begin{align}
i(r_{k+1, t} + (k + 2)r_{k+2, \xi}) + (k + 2)(k + 1)r_{k+3} + F \left( \sum_{i+j+l=k+3} u_i u_j v_l \right) &= 0, \\
i(r_{k+1, t} + (k + 2)r_{k+2, \xi}) + (k + 2)(k + 1)r_{k+3} + F \left( \sum_{i+j+l=k+3} v_i v_j u_l \right) &= 0.
\end{align}
(The condition (5) is recovered for \( k = -3 \)). The coefficients \( u_n, v_n \) of the series (3) are given by the system \( S_n \) \( (k = n - 3) \),

\[
\begin{align*}
[(n-1)(n-2)-4]u_n+F^3u^2v_n &= A_n, \\
Fv_0^2u_n + [(n-1)(n-2)-4]v_n &= B_n,
\end{align*}
\]

where \( A_n \) and \( B_n \) only contain those terms \( u_i, v_j \) with \( i, j < n \). The determinant of the system is

\[
\det S_n = n(n-4)(n-3)(n+1).
\]

Then (3) passes the Painlevé test if, for each \( n = 0, 3, 4 \), one of the coefficients \( u_n, v_n \) can be arbitrary. For \( n = 0 \), (5) implies that this is indeed true either for \( u_0 \) or \( v_0 \). For \( n = 1 \) and \( n = 2 \), the system (6)-(7) is readily solved, yielding

\[
\begin{align*}
u_1 &= -\frac{i}{2}u_0\xi_t, \\
v_1 &= \frac{i}{2}v_0\xi_t,
\end{align*}
\]

\[
\begin{align*}
6v_0u_2 &= iv_0,tu_0 + 2iu_0,tv_0 - \frac{1}{2}u_0v_0(\xi_t)^2, \\
6u_0v_2 &= -iu_0,tv_0 - 2iv_0,tu_0 - \frac{1}{2}u_0v_0(\xi_t)^2.
\end{align*}
\]

\( n = 3 \) has to be a resonance; using condition (5), the system (7) becomes

\[
\begin{align*}
-2v_0u_3 - 2u_0v_3 &= A_3v_0, \\
-2v_0u_3 - 2u_0v_3 &= B_3u_0,
\end{align*}
\]

which requires \( A_3v_0 = B_3u_0 \). But using the expressions of \( A_3 \) and \( B_3 \), with the help of “Mathematica” we find

\[
2FA_3 = u_0(F\xi_t - F\xi_t), \\
u_0F^2B_3 = F\xi_{tt} - F\xi_t,
\]

so that the required condition indeed holds.

\( n = 4 \) has also to be a resonance; we find, as before,

\[
\begin{align*}
2v_0u_4 - 2u_0v_4 &= A_4v_0, \\
-2v_0u_4 - 2u_0v_4 &= B_4u_0,
\end{align*}
\]

enforcing the relation \( v_0A_4 = -u_0B_4 \). Now using the expressions of \( v_0, u_1, v_1, u_2, v_2 \) as functions of \( u_0, F, u_3, v_3 \), “Mathematica” yields

\[
\begin{align*}
6u_0F^2A_4 &= \left(-F^2u_{0,t}^2 - 2iu_0^2F^2\xi_t\xi_{tt} + u_0F^2u_{0,tt} + iu_0^2F\xi_t^2F_t - u_0Fu_{0,t}F_t \right) + 2u_0F^2_t - u_0^2F\xi_{tt}, \\
3u_0^3F^3B_4 &= \left(-F^2u_{0,t}^2 - 2iu_0^2F^2\xi_t\xi_{tt} + u_0F^2u_{0,tt} + iu_0^2F\xi_t^2F_t - u_0Fu_{0,t}F_t \right) - 4u_0F^2_t + 2u_0^2F\xi_{tt}.
\end{align*}
\]

Then our constraint implies that \( 2F^2_t - FF_{tt} = 0 \), i. e. \( \frac{d^2}{dt^2}\left(\frac{1}{F}\right) = 0 \). Hence \( F^{-1}(x, t) = at + b \), as stated.
For \( a = 0 \), \( F(t, x) \) in Eqn. (1) is a constant, and we recover the usual NLS with its known solutions. For \( a \neq 0 \), the equation becomes explicitly time-dependent. Assuming, for simplicity, that \( a = 1 \) and \( b = 0 \), it reads

\[
i u_t + u_{xx} + \frac{1}{t}|u|^2 u = 0. \tag{10}
\]

This equation can also be solved. Generalizing the usual “travelling soliton”, let us seek, for example, a solution of the form

\[
u(t, x) = e^{i(x^2/4t - 1/t)} f(t, x), \tag{11}\]

where \( f(t, x) \) is some real function. Inserting the Ansatz (11) into (10), the real and imaginary parts yield

\[
\begin{align*}
&f_{xx} - \frac{1}{t^2}f + \frac{1}{t}f^3 = 0, \\
&f_t + \frac{x}{t}f_x + \frac{1}{2t}f = 0.
\end{align*} \tag{12}
\]

Time dependence can now be eliminated: setting \( f(t, x) = t^{-1/2}g(-1/t, -x/t) \) transforms (12) into

\[
\begin{align*}
g_{xx} - g + g^3 &= 0, \\
g_t &= 0. \tag{13}
\end{align*}
\]

Multiplying the first equation by \( g_x \) yields a spatial divergence; then requiring the asymptotic behaviour \( g(t, \pm \infty) = 0 = g_t(t, \pm \infty) \) and taking into account the second equation yields \( g(t, x) = \sqrt{2}/\text{sech}[x - x_0] \). In conclusion, we find the soliton

\[
u(t, x) = \frac{e^{i(x^2/4t - 1/t)}}{\sqrt{t}} \frac{\sqrt{2}}{\cosh \left[ -x/t - x_0 \right]}.
\tag{14}\]

It is worth pointing out that the eqns. (13) are essentially the same as those met when constructing travelling solitons for the ordinary NLS — and this is not a pure coincidence. We have in fact

\textbf{Theorem 2.}

\[
u(t, x) = \frac{1}{\sqrt{t}} \exp \left[ \frac{i x^2}{4t} \right] \psi \left( -1/t, -x/t \right)
\tag{15}\]

satisfies the time-dependent equation (10) if and only if \( \psi(t, x) \) solves Eqn. (1) with \( F = 1 \).
This can readily be proved by a direct calculation. Inserting (15) into (10), we find in fact

\[ t^{-5/2} \exp \left[ \frac{ix^2}{4t} \right] \left( i\psi_t + \psi_{xx} + |\psi|^2 \psi \right). \]

Our soliton (14) constructed above comes in fact from the well-known “standing soliton” solution of the NLS,

\[ \psi_0(t, x) = \frac{\sqrt{2} e^{it}}{\cosh[x - x_0]}, \]

by the transformation (15). More general solutions could be obtained starting with the “travelling soliton”

\[ \psi(t, x) = e^{i(\nu t - kx)} \frac{\sqrt{2} a}{\cosh[a(x + kt)]}, \quad a = \sqrt{k^2 + \nu}. \]

Where does the formula (15) come from? To explain it, let us remember that the non-linear space-time transformation

\[ D : \left( \begin{array}{c} t \\ x \end{array} \right) \rightarrow \left( \begin{array}{c} -1/t \\ -x/t \end{array} \right) \]

has already been met in a rather different context, namely in describing planetary motion when the gravitational “constant” changes inversely with time, as suggested by Dirac [4]. One shows in fact that \( \vec{r}(t) = t \vec{r}^* \) describes planetary motion with Newton’s “constant” varying as \( G(t) = G_0/t \), whenever \( \vec{r}^*(t) \) describes ordinary planetary motion, i.e. the one with a constant gravitational constant, \( G(t) = G_0 \) [5].

The strange-looking transformation (18) is indeed related to the conformal structure of non-relativistic space-time [6]. It has been noticed in fact almost thirty years ago, that the space-time transformations

\[ \begin{align*}
\left( \begin{array}{c} t \\ x \end{array} \right) & \rightarrow \left( \begin{array}{c} t' \\ x' \end{array} \right) = \left( \begin{array}{c} \delta^2 t \\ \delta x \end{array} \right), & 0 \neq \delta \in \mathbb{R} & \text{ dilatations} \\
\left( \begin{array}{c} t \\ x \end{array} \right) & \rightarrow \left( \begin{array}{c} t' \\ x' \end{array} \right) = \left( \begin{array}{c} t \\ \frac{1 - \kappa t}{x} \end{array} \right), & 1 \neq \kappa \in \mathbb{R} & \text{ expansions} \\
\left( \begin{array}{c} t \\ x \end{array} \right) & \rightarrow \left( \begin{array}{c} t' \\ x' \end{array} \right) = \left( \begin{array}{c} t + \epsilon \\ x \end{array} \right), & \epsilon \in \mathbb{R} & \text{ time translations}
\end{align*} \]

implemented on wave functions according to

\[ u'(t', x') = \begin{cases} \\
\frac{1}{\delta} u(t, x) & \\
(1 - \kappa t) \exp \left[ i \frac{\kappa x^2}{4(1 - \kappa t)} \right] u(t, x) & \\
u(t, x) &
\end{cases} \]
permute the solutions of the free Schrödinger equation [7]. In other words, they are symmetries for the free Schrödinger equation. (The generators in (19) span in fact an SL(2, R) group; when added to the obvious galilean symmetry, the so-called Schrödinger group is obtained. A Dirac monopole, an Aharonov-Bohm vector potential, and an inverse-square potential can also be included).

The transformation $D$ in Eqn. (18) belongs to this symmetry group: it is in fact (i) a time translation with $\epsilon = 1$, (ii) followed an expansion with $\kappa = 1$, (iii) followed by a second time-translation with $\epsilon = 1$. It is hence a symmetry for the free (linear) Schrödinger equation. Its action on $\psi$, deduced from (20), is precisely (15).

The cubic NLS with non-linearity $F = \text{const.}$ is no more $SL(2, \mathbb{R})$ invariant (1). In particular, the transformation $D$ in (18), implemented as in Eq. (15) carries the cubic term into the time-dependent term $(1/t)|u|^2u$ — just like Newton’s gravitational potential $G_0/r$ with $G_0 = \text{const.}$ is carried into the time-dependent Dirac expression $t^{-1}G_0/r$ [5].

Our results should be compared with the those of Chen et al. [8], who prove that the equation $iu_t + u_{xx} + F(|u|^2)u = 0$ can be solved by inverse scattering if and only if $F(|u|^2) = \lambda|u|^2$, where $\lambda = \text{const.}$. Note, however, that Chen et al. only study the case when the functional $F(|u|^2)$ is independent of the space-time coordinates $t$ and $x$.

In this Letter, we only studied the case of $d = 1$ space dimension. Similar results would hold for any $d \geq 1$, though.

Acknowledgements. J.-C. Y. acknowledges the Laboratoire de Mathématiques et de Physique Théorique of Tours University for hospitality, and the Gouvernement de La Côte d’Ivoire for a doctoral scholarship. We are indebted to Mokhtar Hassaïne for discussions.

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(1) Galilean symmetry can be used to produce further solutions — just like the “travelling soliton” (17) can be obtained from the “standing one” in (16) by a galilean boost. Full Schrödinger invariance yielding expanded and dilated solutions can be restored by replacing the cubic non-linear term by the fifth-order non-linearity $|\psi|^4\psi$. 

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