Multiple Magnetorotons and Spectral Sum Rules in Fractional Quantum Hall Systems

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We study numerically the charge neutral excitations (magnetorotons) in fractional quantum Hall systems, concentrating on the two Jain states near quarter filling, \( \nu = 2/7 \) and \( \nu = 2/9 \), and the \( \nu = 1/4 \) Fermi-liquid state itself. In contrast to the \( \nu = 1/3 \) states and the Jain states near half filling, on each of the two Jain states \( \nu = 2/7 \) and \( \nu = 2/9 \) the graviton spectral densities show two, instead of one, magnetoroton peaks. The magnetorotons have spin 2 and have opposite chiralities in the \( \nu = 2/7 \) state and the same chirality in the \( \nu = 2/9 \) state. We also provide a numerical verification of a sum rule relating the guiding center spin \( \hat{s} \) with the spectral densities of the stress tensor.

**Introduction.**—The exploration of topological phases of matter started with the discovery of the fractional quantum Hall (FQH) effect [1–3]. Under a strong magnetic field, electrons in two dimensions form strongly correlated quantum Hall systems. In the lowest Landau level (LLL) limit, the kinetic energy of the electrons is constant, and the two-dimensional electron system is driven to numerous exotic topological phases depending on the filling fraction and the effective interactions. The topological characteristics of the FQH ground state and its charged excitations can be understood using the wavefunction approach, pioneered by Laughlin [2] and further developed by many other authors [4–8]. The idea of the composite fermion (CF) by Jain [9, 10] provides an explanation of numerous gapped FQH states observed in experiment [11, 12] and an intuitive construction of model wavefunctions, and suggests the abelian and non-abelian braiding statistic of quasi-particles and quasi-holes [13, 14]. Inspired by Jain’s intuitive picture, a field theory description of FQH states was developed (the Halperin-Lee-Read, or HLR, theory), based on the idea of flux attachment [15], which predicts a gapless Fermi-liquid state [16] at half filling, which has been confirmed experimentally [17].

Recently, modifications to the HLR theory have been suggested to make it consistent with the symmetries of a single Landau level. Particle-hole symmetry, which has been long an issue [18, 19], is restored in the Dirac CF theory for FQH states near half filling [20]. The dipole coupling of the CF to the electric field takes care of the consistency of the theory with diffeomorphism [21, 22]. In combination, these two modifications lead to response functions consistent with all known symmetries (see, e.g., Ref. [23]).

An important feature of gapped quantum Hall states is the existence of the neutral magnetoroton mode, first suggested by Girvin, MacDonald and Platzman (GMP) [24] and later observed in experiment [25–27]. Though GMP originally introduced the magnetoroton as a charge density wave, recent works [28–30], employing the lowest Landau level symmetries, suggest that magnetoroton has spin 2 and thus can be considered as a massive “emergent graviton” in FQH systems. The magnetoroton has been studied from many different perspectives: by constructing the wavefunction [31, 32] (which conforms with the spin-2 structure), as an excitation of the CF crossing a level [33], by exact diagonalization [34–39], or within the Dirac CF theory [21, 30, 40], where it is interpreted as the shear deformation of the composite Fermi surface. In the latter studies, the chirality of the magnetoroton is determined by the direction of the residual magnetic field seen by the Dirac CFs, and the positions of the minima of the magnetoroton spectrum match the experimental results [25–27] rather well.

Very recently, the Dirac CF theory has been generalized to Jain states near \( \nu = 1/4 \) [41–43]. The situation with the magnetoroton there seems to be very different from that near \( \nu = 1/2 \); it is necessary [43] to postulate a “Haldane mode,” i.e., an extra high-energy magnetoroton (or multiple magnetorotons), in addition to the low-energy magnetoroton that emerges from the dynamics of the CFs. The additional magnetoroton(s), which contribute to the projected static structure factor (pSSF), are crucial for the Haldane bound [44]

\[
S_4 \geq \frac{|\hat{s}|}{4},
\]

where \( S_4 \) is the coefficient of the leading \( Q^4 \) [45] term in the pSSF, and \( \hat{s} \) is the guiding center spin. On the LLL \( \hat{s} = \frac{1}{2}(\mathcal{F} - 1) \) where \( \mathcal{F} \) is the Wen-Zee shift. The extra magnetoroton was heuristically suggested to arise from the microscopic structure of the CF. While the electric dipole moment of the CF is constrained by its momentum [21, 43], a higher moment deformation of its shape could in principle generate a spin-2 mode, which is the high-energy magnetoroton.

In this Letter we investigate the magnetoroton excitations, guided by the FQH spectral sum rules [43, 46, 47].
relating the chiral graviton spectral functions with $\bar{s}$ and $S_4$. These sum rules constrain the spectral densities of the $v = p/(2n p \pm 1)$ states and the Fermi-liquid-like state at $\nu = 1/4$, and suggest the chiralities of the magnetorotons there. We calculate the spectral densities numerically, from which we read out the chirality of the magnetorotons and verify the sum rules.

**Graviton spectral sum rules.**—In the LLL limit, when the interacting energy is much smaller than the cyclotron energy, one can obtain the exact sum rules involving the spectral densities of the stress tensor [46, 47]. In the complex coordinate $z = x + i y$, the two components of the traceless part of the stress tensor, $T_{zz} = \frac{i}{4}(T_{xx} - T_{yy} - 2i T_{xy})$ and $T_{z\bar{z}} = \frac{i}{4}(T_{xx} - T_{yy} + 2i T_{xy})$, can be used to define two spectral densities [46]

$$
I_-(\omega) = \frac{1}{N_e} \sum_n \langle n | \int d\mathbf{x} T_{zz}(0) | n \rangle^2 \delta(\omega - E_n),
$$

(2)

$$
I_+(\omega) = \frac{1}{N_e} \sum_n \langle n | \int d\mathbf{x} T_{z\bar{z}}(0) | n \rangle^2 \delta(\omega - E_n),
$$

(3)

where $N_e$ is the total number of electrons, $|0\rangle$ is the ground state, the sum is taken over all excited states $|n\rangle$ in the lowest Landau level, and $E_n$ is the energy of the state $|n\rangle$ relative to the ground state. Physically, (2) and (3) are the densities of spin-2 states with opposite chiralities at frequency $\omega$, and as such they depend on the microscopic details of the FQH problem. The expressions for the integrals of $T_{zz}$ and $T_{z\bar{z}}$ over space in terms of the LLL operators have been derived in Ref. [48]. We expect $I_-(\omega)$ and $I_+(\omega)$ to vanish at frequencies below the energy gap; we also expect them to rapidly go to 0 at energies much larger than the energy scale set by the Coulomb interaction. Using the $U(1)$ charge conservation and the LLL limit of momentum conservation, one can obtain the following exact sum rules [43, 46, 47, 49]

$$
\int_0^\infty \frac{d\omega}{\omega^2} [I_-(\omega) + I_+(\omega)] = S_4, \tag{4}
$$

$$
\int_0^\infty \frac{d\omega}{\omega^2} [I_-(\omega) - I_+(\omega)] = \frac{\bar{s}}{4} = \frac{\mathcal{J} - 1}{8} \text{ on LLL}. \tag{5}
$$

For derivations of the sum rules see Refs. [43, 47, 50]. Both sum rules do not rely on a microscopic details and can be applied for fractional quantum Hall states in the single Landau level limit, where Landau-level mixing is ignored. By definition $I_\pm(\omega)$ are non-negative, therefore the sum rules imply the Haldane bound (1). This bound is saturated if an only if the FQH state is chiral, i.e., when one of the spectral densities vanishes identically [i.e., $I_-(\omega) = 0$ or $I_+(\omega) = 0$].

**General Jain states**—The Wen-Zee shift $\mathcal{J}$ of the general Jain state has been found previously [43, 51, 52]

$$
v_+ = \frac{p}{2np + 1}, \quad \mathcal{J}_+ = p + 2n, \tag{6}
$$

$$
v_- = \frac{p}{2np - 1}, \quad \mathcal{J}_- = -p + 2n. \tag{7}
$$

The subscript index $+$ ($-$) corresponds to the residual magnetic field seen by CFs being in the same (opposite) direction as the applied magnetic field [21, 43]. The direction of the residual magnetic field determines the chirality of the low-energy magnetoroton, the one induced by the deformation of composite Fermi surface. If the residual magnetic field is in the same (opposite) direction of external field, the low-energy magnetoroton has negative (positive) chirality. Consequently, we expect a low energy peak in $I_-(\omega)$ ($I_+(\omega)$) of state $v_+ (v_-)$.

In Ref. [43] a Dirac CF model of the $v_\pm$ states were presented. The model is supposed to be reliable at large $p$ and its result for the spectral densities can be summarized as

$$
v_+ : \quad \frac{I_-(\omega)}{\omega^2} = \frac{p + 1}{8} \delta(\omega - \omega_L) + \frac{n - 1}{4} \delta(\omega - \omega_H), \tag{8}
$$

$$
I_+(\omega) = 0,
$$

$$
v_- : \quad \frac{I_-(\omega)}{\omega^2} = \frac{n - 1}{4} \delta(\omega - \omega_H), \tag{9}
$$

$$
I_+(\omega) = \frac{p - 1}{8} \delta(\omega - \omega_L),
$$

where $\omega_L$ and $\omega_H$ are the energies of the low- and high-energy magnetorotons, respectively. Note that the delta-function $\delta(\omega - \omega_H)$ may be broadened by the decay of the high-energy magnetoroton or splits into several peaks. One can introduce the integrated spectral densities

$$
\mathcal{J}_\pm = \int_0^\infty \frac{d\omega}{\omega} I_\pm(\omega). \tag{10}
$$

The prediction of Ref. [43] reads

$$
v_+ : \quad \mathcal{J}_- = \frac{p + 2n - 1}{8}, \quad \mathcal{J}_+ = 0, \tag{11}
$$

$$
v_- : \quad \mathcal{J}_- = \frac{n - 1}{4}, \quad \mathcal{J}_+ = \frac{p - 1}{8}. \tag{12}
$$

and from the sum rule (4) one finds the $S_4$ coefficient of the $v_\pm$ states:

$$
S_4(v_+) = \frac{p + 2n - 1}{8}, \quad S_4(v_-) = \frac{p + 2n - 3}{8}. \tag{13}
$$

Some remarks are in order. (i) For $n = 1$ (near half filling), there is no high-energy magnetoroton, and both $v_\pm$ states are chiral. (ii) For $n \neq 1$, only the $v_+$ state is chiral, with $S_4$ saturating the Haldane bound, while the $v_-$ is not chiral. (iii) Strictly speaking, the formulas presented above are obtained in the large $p$ limit, so the application of these formulas for the case of, say, $p = 2$ should be taken with a grain of salt. On the other hand, one may expect that the qualitative statements about the chirality of the magnetoroton modes are robust.

The Fermi-liquid states.—In the Fermi-liquid state with $\nu = 1/2n$ the CFs are in zero emergent magnetic field
and form a Fermi liquid, whose excitations do not contribute to the sum rule (5). The only contribution to the sum rule (5) is from the high-energy magnetoroton (the Haldane mode). Thus we find
\[ \mathcal{J}_- - \mathcal{J}_+ = \frac{n-1}{4}, \quad (14) \]
Note that since the state is ungapped, the notion of \( S_4 \) does not apply. Naively we can associate \( n-1 \) with the guiding center spin for of the Fermi-liquid state. This, in turn, can be explained if one thinks of the CF at \( \nu = 1/2n \) as a CF at \( \nu = 1/2 \) state with \( 2(n-1) \) flux quanta attached.

Looking at Eqs. (8), (9), and (14), we notice that the contribution of the Haldane mode to the guiding center spin of states near \( 1/2n \) is universal. In the Fermi-liquid state \( \nu = 1/2 \), the spectral densities \( I_-(\omega) \) and \( I_+(\omega) \) should be identical due to the particle-hole symmetry, therefore \( \mathcal{J}_- - \mathcal{J}_+ = 0 \). (The same should be valid for the PH-Pfaffian state [20].)

**Numerical results**—The graviton spectral function has been investigated on both boson and fermion FQHE states which include Moore-Read states [53] as well as Laughlin states in Refs. [54, 55]. In this Letter we present some results on the guiding center spin \( \bar{s} \), the coefficient \( S_4 \) of the \( Q^4 \) term in the pSSF, and show the FQHE graviton spectral functions.

We find good agreement with the theoretical predictions of Eqs. (4) and (5). We will present some of the results in this Letter, delegating to the Supplement Material some others not germane to the main topic of the Letter. To obtain the correct sum rules for the Coulomb interactions it is necessary to use the complete stress tensor given in the Supplement Material [56] to evaluate the spectral densities \( I_-(\omega) \) and \( I_+(\omega) \).

In Figs. 1 and 2 we present the results for the sum rules for the Jain states at fillings \( 2/5, 2/7, \) and the Fermi liquid state \( 1/4 \). We use the left vertical axis to represent \( \bar{s}/4 \), while the right vertical axis represents \( S_4 \). As predicted by Haldane [44] only chiral states saturate the \( S_4 \) bound. Generic states such as the ones obtained from the Coulomb interaction exceed this bound [43, 46]. The numerical results of \( \bar{s} \) and \( S_4 \) of Jain states converge to the theoretical predictions in Eqs. (13), (6) and (7). We also numerically verify the Wen-Zee shift of the Fermi liquid state \( \nu = 1/4 \) predicted by Eq. (14). The numerical results presented in Figs. 1 and 2 are highly non-trivial, they are the first numerical check of the exact graviton spectral sum rules in the LLL limit.

Figures 3–6 present the FQH graviton spectral functions for \( \nu = 2/5, 2/7, 1/4, \) and \( 2/9 \). For \( \nu = 2/5 \), we use just the Coulomb potential since we want to compare it to the spectrum obtained with the hard-core potential, but for the remaining three fractions we calculate the spectral functions using a modified version of Coulomb. The two forms give the same qualitative results for the distributions of the graviton weights, particularly in instances where there are two distinct energy sectors.

All the theoretical predictions on graviton’s chirality for the general Jain states from Dirac CF model of Ref. [43] are confirmed numerically. For \( n = 1 \), the chirality of gravitons are determined by the residue magnetic field seen by the CFs, therefore the graviton of \( \nu = 2/5 \) has negative chirality as showed in Fig. 3. With \( n = 2 \) the chirality of low energy graviton is also determined by the residue magnetic field, and the chirality of the high energy graviton is universal for all Jain states near \( 1/4 \). The predictions are confirmed in Figs. 4 and 6.
Interestingly, the graviton spectral functions of Fermi-liquid state $\nu = 1/4$ in Fig. 5 shows the high energy graviton with expected chirality. From both the shift sum rule and the spectral densities, we see that the Haldane mode is universal for all FQH states near 1/4: it does not care if there is a composite Fermi surface or if there is a residue magnetic field. Fig. 5 also shows the low-energy excitations of both chirality with equal weight. We expect that the spectral densities of the Fermi-liquid state $\nu = 1/2$ share the same feature with the low energy spectral densities of $\nu = 1/4$ with $I_+ (\omega)$ and $I_- (\omega)$ being similar, the only difference is the non-appearance of the Haldane mode.

Conclusion.—In this Letter, we compute numerically the graviton spectral densities for the $\nu = 2/7$, 2/9, and 1/4 states. For the first two states, we observe in the spectral densities two magnetoroton peaks, one at low energy with chirality that depends on the residual magnetic field acting on the CFs (and thus are opposite for the $\nu = 2/7$ and 2/9 states), and one at high energy with the same chirality for the two states. The higher-energy magnetoroton...
is also observed in a spectral density of the $\nu = 1/4$ state, and this magnetoroton has approximately the same energy in all three filling fractions. The result is consistent with the two-magnetoroton model of FQH states near 1/4 proposed in Ref. [43]. In addition, we have verified the FQH graviton spectral sum rules for the two Jain states and the Fermi liquid state $\nu = 1/4$.

We hope that our results will motivate the experimental exploration of the magnetoroton spectrum of the FQH states near $\nu = 1/4$, as well as more detail study of the Haldane mode which may reveal its nature.

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Note: While this manuscript is being prepared, we became aware of the work [58] that also discusses the extra graviton mode in the FQH states $\nu = 2/7$ and $\nu = 2/9$. We thank Ajit C. Balram, Zhao Liu, Andrey Gromov, and Zlatko Papić for sharing their manuscript with us before publication.

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[1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Two-Dimensional Magnetotransport in the Extreme Quantum Limit, Phys. Rev. Lett. 48, 1559 (1982).
[2] R. Laughlin, Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations, Phys. Rev. Lett. 50, 1395 (1983).
[3] X. G. Wen, Topological orders in rigid states, Int. J. Mod. Phys. B 4, 239 (1990).
[4] D. Arovas, J. R. Schrieffer, and F. Wilczek, Fractional Statistics and the Quantum Hall Effect, Phys. Rev. Lett. 53, 722 (1984).
[5] N. Read and D. Green, Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect, Phys. Rev. B 61, 10267 (2000), arXiv:cond-mat/9906453.
[6] C. Nayak and F. Wilczek, 2n-quasihole states realize $2^{n-1}$-dimensional spinor braiding statistics in paired quantum Hall states, Nucl. Phys. B 479, 529 (1996), arXiv:cond-mat/9605145.
[7] F. D. M. Haldane and E. H. Rezayi, Spin-Singlet Wave Function for the Half-Integral Quantum Hall Effect, Phys. Rev. Lett. 60, 956 (1988), [Erratum: Phys. Rev. Lett. 60, 1886 (1988)].
[8] M. Greiter, X.-G. Wen, and F. Wilczek, Paired Hall State at Half Filling, Phys. Rev. Lett. 66, 3205 (1991).
[9] J. K. Jain, Composite-Fermion Approach for the Fractional Quantum Hall Effect, Phys. Rev. Lett. 63, 199 (1989).
[10] J. K. Jain, Composite Fermions (Cambridge University Press, 2007).
[11] W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Fractional Quantum Hall Effect of Composite Fermions, Phys. Rev. Lett. 90, 016801 (2003), arXiv:cond-mat/0303429.
[12] W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Transition from an Electron Solid to the Sequence of Fractional Quantum Hall States at Very Low Landau Level Filling Factor, Phys. Rev. Lett. 88, 176802 (2002).
[13] T. H. Hansson, M. Hermans, S. H. Simon, and S. F. Viefers, Quantum Hall physics: Hierarchies and conformal field theory techniques, Rev. Mod. Phys. 89, 025005 (2017), arXiv:1601.01697.
[14] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Non-Abelian anyons and topological quantum computation, Rev. Mod. Phys. 80, 1083 (2008), arXiv:0707.1889.
[15] A. Lopez and E. Fradkin, Fractional quantum Hall effect and Chern-Simons gauge theories, Phys. Rev. B 44, 5246 (1991).
[16] B. I. Halperin, P. A. Lee, and N. Read, Theory of the half-filled Landau level, Phys. Rev. B 47, 7312 (1993).
[17] W. Kang, H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, How Real Are Composite Fermions?, Phys. Rev. Lett. 71, 3850 (1993).
[18] S. A. Kivelson, D.-H. Lee, Y. Krotov, and J. Gan, Composite-fermion Hall conductance at $\nu = 1/2$, Phys. Rev. B 55, 15552 (1997), arXiv:cond-mat/9607153.
[19] D.-H. Lee, An unsettled issue in the theory of the half-filled landau level (1999), cond-mat/9901193.
[20] D. T. Son, Is the Composite Fermion a Dirac Particle?, Phys. Rev. X 5, 031027 (2015), arXiv:1502.03446.
[21] D. X. Nguyen, S. Golkar, M. M. Roberts, and D. T. Son, Particle-hole symmetry and composite fermions in fractional quantum Hall states, Phys. Rev. B 97, 195314 (2018), arXiv:1709.07885.
[22] Y.-H. Du, U. Mehta, D. X. Nguyen, and D. T. Son, Volume-preserving diffeomorphism as nonabelian higher-rank gauge symmetry, (2021), arXiv:2103.09826.
[23] J. Hofmann, Electromagnetic response of composite Dirac fermions in the half-filled Landau level, Phys. Rev. B 104, 115401 (2021), arXiv:2102.11880.
[24] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Magneto-roton theory of collective excitations in the fractional quantum Hall effect, Phys. Rev. B 33, 2481 (1986).
[25] A. Pinczuk, B. Dennis, L. Pfeiffer, and K. West, Light scattering by collective excitations in the fractional quantum Hall regime, Physica B Condens. Matter 249-251, 40 (1998).
[26] M. Kang, A. Pinczuk, B. S. Dennis, M. A. Eriksson, L. N. Pfeiffer, and K. W. West, Inelastic Light Scattering by Gap Excitations of Fractional Quantum Hall States at $1/3 \leq \nu \leq 2/3$, Phys. Rev. Lett. 84, 546 (2000), cond-mat/9911350.
[27] I. V. Kukushkin, J. H. Smet, V. W. Scarola, V. Umansky, and K. von Klitzing, Dispersion of the Excitations of Fractional Quantum Hall States, Science 324, 1044 (2009).
[28] F. D. M. Haldane, Geometrical Description of the Fractional Quantum Hall Effect, Phys. Rev. Lett. 107, 116801 (2011), arXiv:1106.3375.
[29] F. D. M. Haldane, "Hall viscosity" and intrinsic metric of incompressible fractional Hall fluids (2009), arXiv:0906.1854.
[30] S. Golkar, D. X. Nguyen, M. M. Roberts, and D. T. Son, Higher-Spin Theory of the Magnetorotons, Phys. Rev. Lett.
D. Majumder, S. S. Mandal, and J. K. Jain, Collective excitations of composite fermions across multiple A levels, Nat. Phys. 5, 403 (2009).

J. Wang, Dirac Fermion Hierarchy of Composite Fermi Liquids, Phys. Rev. Lett. 122, 257203 (2019), arXiv:1808.07529.

H. Goldman and E. Fradkin, Dirac composite fermions and emergent reflection symmetry about even-denominator filling fractions, Phys. Rev. B 98, 165137 (2018), arXiv:1808.09314.

D. X. Nguyen and D. T. Son, Dirac composite fermion theory of general Jain sequences, Phys. Rev. Research 3, 033217 (2021), arXiv:2105.02092.

F. D. M. Haldane, Self-duality and long-wavelength behavior of the Landau-level guiding-center structure function, and the shear modulus of fractional quantum Hall fluids (2011), arXiv:1112.0899.

Here we define $Q = q \ell_B$ with $q$ being the momentum and $\ell_B$ the magnetic length.

S. Golkar, D. X. Nguyen, and D. T. Son, Spectral sum rules and magneto-roton as emergent graviton in fractional quantum Hall effect, J. High Energy Phys. 2016, 021, arXiv:1309.2638.

D. X. Nguyen, D. T. Son, and C. Wu, Lowest Landau Level Stress Tensor and Structure Factor of Triad Quantum Hall Wave Functions (2014), arXiv:1411.3316.

D. X. Nguyen and D. T. Son, Probing the spin structure of the fractional quantum hall magnetoroton with polarized raman scattering, Phys. Rev. Research 3, 023040 (2021), arXiv:2101.02213.

The upper limit $\infty$ here means a frequency which is much higher than the Coulomb gap but also smaller the the cyclotron energy.

D. T. Son, Newton-Cartan Geometry and the Quantum Hall Effect (2013), arXiv:1306.0638.

A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, Framing Anomaly in the Effective Theory of the Fractional Quantum Hall Effect, Phys. Rev. Lett. 114, 016805 (2015), [Erratum: Phys. Rev. Lett. 114, 149902 (2015)], arXiv:1410.6812.

D. X. Nguyen, T. Can, and A. Gromov, Particle-Hole Duality in the Lowest Landau Level, Phys. Rev. Lett. 118, 206602 (2017), [Erratum: Phys. Rev. Lett. 118, 269902 (2017)], arXiv:1612.07799.

G. Moore and N. Read, Nonabelions in the fractional quantum Hall effect, Nucl. Phys. B 360, 362 (1991).

S.-F. Liou, F. D. M. Haldane, K. Yang, and E. H. Rezayi, Chiral Gravitons in Fractional Quantum Hall Liquids, Phys. Rev. Lett. 123, 146801 (2019), arXiv:1904.12231.

F. D. M. Haldane, E. H. Rezayi, and K. Yang, Graviton chirality and topological order in the half-filled Landau level, Phys. Rev. B 104, L121106 (2021), arXiv:2103.11019.

See Supplemental Materials at (link) for the explicit expression of the full stress tensor operators, the numerical procedure and supplemental figures.

F. D. M. Haldane, Fractional Quantization of the Hall Effect: A Hierarchy of Incompressible Quantum Fluid States, Phys. Rev. Lett. 51, 605 (1983).

A. C. Balram, Z. Liu, A. Gromov, and Z. Papić, Very high-energy collective states of partons in fractional quantum Hall liquids (2021).

B. Yang, Z.-X. Hu, C. H. Lee, and Z. Papić, Generalized Pseudopotentials for the Anisotropic Fractional Quantum Hall Effect, Phys. Rev. Lett. 118, 146403 (2017), [Erratum: Phys. Rev. Lett. 118, 169903 (2017)], arXiv:1609.06730.

Z. Liu, A. C. Balram, Z. Papić, and A. Gromov, Quench Dynamics of Collective Modes in Fractional Quantum Hall Bilayers, Phys. Rev. Lett. 126, 076604 (2021), arXiv:2011.01955.

N. Read and E. H. Rezayi, Hall viscosities, orbital spin, and geometry: Paired superfluids and quantum Hall systems, Phys. Rev. B 84, 085316 (2011), [Erratum: Phys. Rev. B 84, 119902 (2011)], arXiv:1008.0210.
— Supplementary Material —
Multiple Magnetorotons and Spectral Sum Rules in Fractional Quantum Hall Systems
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STRESS TENSOR OPERATOR ON A GENERAL LANDAU LEVEL

In order to verify numerically the spectral sum rules, we need to use the single Landau level limit of the holomorphic components of the stress-tensor operator. The zero momentum stress tensor projected to a general Landau level \( N \) is given in Ref. [48],

\[
\int d^2x T_{ij}(x) = \frac{1}{2} \sum_q q_i q_j \frac{\partial}{\partial q} \left\{ e^{-\frac{q^2 q^2}{2}} \left[ L_N \left( \frac{q^2 q^2}{2} \right) \right] ^2 V(q) \right\} \bar{\rho}(q) \bar{\rho}(-q).
\]

(S1)

with \( L_N(x) \) being the Laguerre polynomial and \( V(q) \) the Fourier transformed of the two-body interaction,

\[
V(q) = \int d^2 x V(x) e^{i q \cdot x}.
\]

(S2)

Since we assume the interaction is rotationally invariant, we expect that \( V(q) \) only depends on \( q = p q \cdot q \).

(S3)

In the complex coordinate, we define the complex momentum as

\[
q_z = \frac{1}{2} (q_x + i q_y), \quad \bar{q}_z = \frac{1}{2} (q_x - i q_y).
\]

(S4)

Using Eq. (S1), we can write the holomorphic components of the stress tensor at zero momentum as

\[
\int d^2x T_{zz}(x) = \frac{1}{2} \sum_q q_z q_z \frac{\partial}{\partial q} \left\{ e^{-\frac{q^2 q^2}{2}} \left[ L_N \left( \frac{q^2 q^2}{2} \right) \right] ^2 V(q) \right\} \bar{\rho}(q) \bar{\rho}(-q),
\]

(S5)

\[
\int d^2x T_{\bar{z}\bar{z}}(x) = \frac{1}{2} \sum_q q_{\bar{z}} q_{\bar{z}} \frac{\partial}{\partial q} \left\{ e^{-\frac{q^2 q^2}{2}} \left[ L_N \left( \frac{q^2 q^2}{2} \right) \right] ^2 V(q) \right\} \bar{\rho}(q) \bar{\rho}(-q).
\]

(S6)

In numerical calculations, we use the explicit form of the stress tensor in Eqs. (S5), (S6) and the ground state from exact diagonalization to obtain the spectral functions \( I_{-}(\omega) \), \( I_{+}(\omega) \).

NUMERICAL PROCEDURE

Our numerical studies are based on the evaluation of two stress tensor spectral densities

\[
I_{-}(\omega) = \frac{1}{N_e} \sum_n \langle n | \int d^2 x T_{zz}(0) | n \rangle^2 \delta(\omega - E_n), \quad (S7)
\]

\[
I_{+}(\omega) = \frac{1}{N_e} \sum_n \langle n | \int d^2 x T_{\bar{z}\bar{z}}(0) | n \rangle^2 \delta(\omega - E_n), \quad (S8)
\]

where the explicit expression of the full stress tensor at zero momentum is given in Eqs. (S5) and (S6). Note that the expressions of the stress-tensor components (S5) and (S6) differs from the ones used in Refs. [54, 55, 59, 60]. As explained in Ref. [48], the expressions in Ref. [54, 55, 59, 60] includes only the “kinetic part” but omits the “potential part” of the stress tensor.

With the short-range pseudo-potential, the two expressions for the stress tensor give the same numerical result, since one can show that the contribution to zero momentum stress tensor from interaction annihilates the ground state [47]. However, to obtain the correct sum rules (4), (5) for the Coulomb interactions, it is necessary to use the complete stress tensor (S5) and (S6) to evaluate the spectral densities \( I_{-}(\omega) \) and \( I_{+}(\omega) \). The end result is that the LLL Coulomb potential...
is modified by adding to $1/\mathcal{Q}$ its cube. The first Haldane pseudo-potential is unaffected for both bosons and fermions. We’ve performed numerical calculation of both versions in this work.

The computer resources required for these types of calculations are much more demanding than those in obtaining the Hall viscosity [61] from the Berry curvature. The gain is the calculation of $S_4$ and the spectral functions of Jain (or hierarchy) states, which are new. In this Letter and in the Supplement Material, we present the numerical results of the spectral densities $I^-(\omega)$ and $I^+(\omega)$ as well as the integrated spectral densities $\mathcal{I}^-$ and $\mathcal{I}^+$.

**SUPPLEMENTAL FIGURES**

In this supplemental section, we show some more numerical results of the shift sum rule for $\nu = 1/3$ and $\nu = 2/3$ in Fig. S1. The chiralities of graviton for these states come out as expected with negative for $\nu = 1/3$ and positive for $\nu = 2/3$. We see that the sum rules are again confirmed by numerical calculations.

![Graphical representation of $\bar{s}$ and $S_4$ obtained from the sum rules at electron filling factor of $\nu = 1/3$ for both the Coulomb and the hard-core potentials. We see that $\bar{s}/4$ and $S_4$ are mostly identical, which is expected for a chiral state.](image1)

![Graphical representation of $\bar{s}$ obtained from the sum rules at electron filling factor of $\nu = 2/3$ for hard-core potentials.](image2)

We also include the graviton’s spectral function for $\nu = 2/5$, but for the hard-core ($v_1 = 1$) and the modified Coulomb potentials in Fig. S2.

![The graviton spectral functions for the filling factor $\nu = 2/5$ with the hard-core ($v_1 = 1$) and the modified Coulomb potentials. We compare these because they produce consistent values of $\bar{s}$ and $S_4$ in Fig 1. The spectrum on the left of 0.25 (in units of $e^2/(4\pi\epsilon_0\ell_B)$) is that of the modified Coulomb interactions and the spectrum to the right is for hard-core potential with $v_1 = 1$. The energies scales are different but the relative size of peaks appear to be mirrored in both cases. The modified Coulomb potential has the same scale and peak positions as the pure Coulomb potential in Fig 3. The graviton with the negative chirality dominates both spectral functions as expected.](image3)