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Field Theory On The World Sheet: Improvements And Generalizations

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Abstract

This article is the continuation of a project of investigating planar $\phi^3$ model in various dimensions. The idea is to reformulate them on the world sheet, and then to apply the classical (meanfield) approximation, with two goals: To show that the ground state of the model is a solitonic configuration on the world sheet, and the quantum fluctuations around the soliton lead to the formation of a transverse string. After a review of some of the earlier work, we introduce and discuss several generalizations and new results. In 1+2 dimensions, a rigorous
upper bound on the solitonic energy is established. A $\phi^4$ interaction is added to stabilize the original $\phi^3$ model. In 1+3 and 1+5 dimensions, an improved treatment of the ultraviolet divergences is given. And significantly, we show that our approximation scheme can be imbedded into a systematic strong coupling expansion. Finally, the spectrum of quantum fluctuations around the soliton confirms earlier results: In 1+2 and 1+3 dimensions, a transverse string is formed on the world sheet.
1. Introduction

The present article is in part a review and summary of an ongoing project to analyze field theory from a world sheet perspective. The original idea was to sum the planar graphs of the $\phi^3$ field theory in both $3+1$ and $5+1$ dimensions, starting with the world sheet picture developed in [1], which in turn was based on the pioneering work of 't Hooft [2]. This project went through many phases, and somewhat varied versions have appeared. In this paper, we go back to the references [3] and [4] as our starting point, and our goal will be to present a fresh approach to resolve some of the problems left open in these references.

In section 2, we review of the world sheet picture of the planar graphs of the $\phi^3$ field theory that is our starting point, and in section 3, we describe the field theory on the world sheet, developed in [3], which reproduces these graphs. This theory is formulated in terms of a complex scalar field and a two component fermionic field; a central role is played by the field $\rho$ (eq.(4)), a composite of the fermions, which roughly measures the density of graphs on the world sheet. $\rho_0$, the ground state expectation value of $\rho$, turns out to be an order parameter that distinguishes between two phases. A non-zero value for $\rho_0$ corresponds to a phase in which the world sheet densely covered with graphs, and an important question is whether this phase has lower energy than the trivial phase (empty world sheet) $\rho_0 = 0$. An old idea that a densely covered world sheet would naturally have a string description motivated some of the very early work on this subject [5, 6]. To find such a string picture has been a goal of the present, as well as of the earlier work.

To find out which phase is energetically favored, a variational calculation was carried out in [3], and $\rho_0 \neq 0$ was found to have lower energy than the trivial ground state. In [4], the same problem was tackled using a different approach: A static classical solution on the world sheet, with again $\rho_0 \neq 0$ and with energy lower than the trivial ground state, is constructed. These two approaches, although seemingly different, are actually closely related.

The computations discussed above suffer from two kinds of divergences: One of them is the standard field theoretic ultraviolet divergence which we will address later on. The second one is an infrared divergence due to the choice of the light cone coordinates. This infrared problem is temporarily circumvented by the discretizing the $\sigma$ coordinate on the world sheet in steps of $a$, which amounts to the compactification of the the light cone coordinate $x^-$. One place where the infrared divergence manifests itself is in the ground
state energy in the non-trivial phase; it is negative and proportional to $1/a^2$.

Two questions present themselves: Is this a real effect or merely a spurious infrared divergence? And if it is a real effect, is it somehow connected with classical instability of the $\phi^3$ theory?

In section 4, we answer the first question: It is a real physical effect. The model we study is $\phi^3$ in $D = 1 \,(1+2)$ dimensions. We study this lower dimensional field theory because it is free of ultraviolet divergences, and therefore we can focus on the infrared problem without being distracted by ultraviolet divergences. We use the variational method to estimate the ground state energy and find that the phase $\rho_0 \neq 0$ is energetically preferred and the corresponding ground state energy is negative and proportional to $1/a^2$. This approach has the advantage of providing a rigorous upper bound for the energy; therefore, the divergence in the $a \to 0$ limit is not a spurious effect resulting from the approximation scheme used. At the end of the section, we give an explanation of the negative sign of the energy based on statistical mechanics, and we also observe that this divergence can be cancelled by adding a counter term proportional to the area of the world sheet.

In section 5, still keeping $D = 1$, we add a $\phi^4$ term with positive coefficient to the interaction. Using the same variational method, we find the same $-1/a^2$ behaviour in the ground state energy. This answers the second question posed earlier: Since we now have at least classically stable theory, the behaviour of the ground state energy is not related to the instability of $\phi^3$.

Next, in section 6, we consider $\phi^3$ in $D = 2 \,(1+3)$ dimensions. Although we could use the variational method, in this case, it is more convenient to search for a classical static solution (soliton) on the world sheet. Such a solution was already found in [4], and the corresponding ground state energy had a logarithmic ultraviolet divergence. This the standard perturbation result, and the renormalization prescription is to eliminate it by a mass counter term. Unfortunately, the structure of the divergent term is different from the structure of the mass term, so the standard cancellation does not work. This problem can be traced back to the construction of the classical solution. In particular, $\rho$ is treated as a classical continuous variable, whereas it is in reality it takes on only the discrete values 0 and 1. Consequently, certain overlap identities (eqs.(47) and (48)), which are crucial for the correct structure of the ultraviolet divergence, are violated. We overcome this problem by using the exact overlap identities in the initial stages of the calculation, and introducing the classical approximation only after taking care of the di-
vergent term by mass renormalization. After this, things work pretty much as in the case $D = 1$: The phase $\rho_0 \neq 0$ is energetically preferred and the corresponding ground state energy is negative and proportional to $1/a^2$.

There are further complications related to renormalization in the case of $1 + 5$ dimensions ($D = 4$), which are addressed in section 7. The mass term is now quadratically divergent, and there is also a logarithmic coupling constant divergence. The classical solution is constructed using exact overlap relations (eqs.(47) and (48)) in the initial stages of the computation, same as in the previous section. Their structure is then the same as in perturbation theory, and they can readily be renormalized. The final result is, however, different from the cases $D = 1$ and $D = 2$: The ground state energy is given solely by a renormalized mass term. If the square of the mass is positive, the trivial ground state $\rho_0 = 0$ is energetically favored. The phase of interest, $\rho_0 \neq 0$, is energetically favored only if the renormalized mass is tachyonic. Whether this is physically sensible is an open question.

Section 8 deals with the quadratic quantum fluctuations around the classical solutions. This section is largely based on references [4,7] and is included here for the sake of completeness. In the cases of $D = 1$ and $D = 2$, the calculations are straightforward, and the results can be summarized as follows: The spectrum consists of two components; the heavy sector and the light sector. The energies of the heavy states go to infinity as $a \to 0$. Since the radius of compactification of the coordinate $x^-$ is $1/a$, it is natural to identify them with Kaluza-Klein states whose masses go to infinity in the decompactification limit. In contrast, the light sector, represented by a transverse string with a finite slope, remains finite. Thus, the original motivation for putting field theory on the world sheet, the formation of a string, is realized at least in lower dimensions. In contrast to the lower dimensional cases, string formation at $D = 4$ remains problematic, since it requires the renormalized mass to be tachyonic.

The existence of the light sector is the direct consequence of translation invariance in the variable $q$ (eq.(17)), which forbids a heavy mass term. Although the original action for the model (eq.(16)) was non-local in the coordinate $\sigma$, the string action is local in this coordinate. This localization is a consequence of $\rho_0$ being non-zero: Terms separated by a finite distance in $\sigma$ are suppressed [7].

The expansion around the classical configuration described above is not a systematic expansion, since we have not identified an expansion parameter. In section 9, at least in the case of $D + 1$, we remedy this defect; The
parameter
\[ e^2 = g^{-4/3}, \]
where \( g \) is the coupling constant (eq.(82)), serves as an expansion parameter.
It therefore a strong coupling expansion. We show that, by suitably scaling the parameters of the model, the mass of the soliton also scales in expected fashion (eq.(83)).

Since part of this article is a review of references [3] and [4], we would like to draw attention to the new material included here which go beyond those references. The addition of a \( \phi^4 \) term as an interaction is such a new feature. It stabilizes the original \( \phi^3 \) model, and it can serve as a first step towards more realistic models, which usually have such a term. Another new feature is the treatment of the ultraviolet divergences in mass in 1 + 3 and 1 + 5 dimensions: These divergences can now be cancelled by introducing mass counter terms in the original action. Finally, we feel that the introduction of a systematic expansion around the classical solitonic configuration is an important new development; it opens up the prospect of computing terms higher order than quadratic. It may also make it possible to investigate questions such as Lorentz invariance in a systematic fashion.

2. The World Sheet Picture

The planar graphs of \( \phi^3 \) can be represented [4] on a world sheet parameterized by the light cone coordinates \( \tau = x^+ \) and \( \sigma = p^+ \) as a collection of horizontal solid lines (Fig.1), where the \( n \)th line carries a D dimensional transverse momentum \( q_n \). Two adjacent solid lines labeled by \( n \) and \( n+1 \) correspond to the light cone propagator
\[ \Delta(p_n) = \frac{\theta(\tau)}{2p^+} \exp\left(-i\tau \frac{p_n^2 + m^2}{2p^+}\right), \quad (1) \]
where \( p_n = q_n - q_{n+1} \) is the momentum flowing through the propagator. A factor of the coupling constant \( g \) is inserted at the beginning and at the end of each line, where the interaction takes place. Ultimately, one has to integrate over all possible locations and lengths of the solid lines, as well as over the momenta they carry.

The propagator (1) is singular at \( p^+ = 0 \). It is well known that this is a spurious singularity peculiar to the light cone picture. To avoid this singularity, and as well as other technical reasons, it is convenient to temporarily discretize the \( \sigma \) coordinate in steps of length \( a \). This amounts to
compactifying the light cone coordinate $x^-$ at radius $R = 1/a$. This sort of compactification has been extensively used both in field theory [8] and in string theory [9]. A useful way of visualizing the discretized world sheet is pictured in Fig. 2. The boundaries of the propagators are marked by solid lines as before, and the bulk is filled by dotted lines spaced at a distance $a$. For the time being, we will keep $a$ finite, and later, we will discuss the limit $a \to 0$. For convenience, the $\sigma$ is compactified by imposing periodic boundary conditions at $\sigma = 0$ and $\sigma = p^-$. In contrast, the boundary conditions at $\tau = \pm \infty$ are left arbitrary.

3. The World Sheet Field Theory

It was shown in [3] that the light cone graphs described above are reproduced by a world sheet field theory, which we now briefly review. We introduce the complex scalar field $\phi(\sigma, \tau, q)$ and its conjugate $\phi^\dagger$, which at time $\tau$ annihilate (create) a solid line with coordinate $\sigma$ carrying momentum $q$. They satisfy the usual commutation relations

$$[\phi(\sigma, \tau, q), \phi^\dagger(\sigma', \tau, q')] = \delta_{\sigma, \sigma'} \delta(q - q').$$  (2)

The vacuum, annihilated by the $\phi$’s, represents the empty world sheet.

In addition, we introduce a two component fermion field $\psi_i(\sigma, \tau), i = 1, 2$, and its adjoint $\bar{\psi}_i$, which satisfy the standard anticommutation relations. The fermion with $i = 1$ is associated with the dotted lines and $i = 2$ with the solid lines. The fermions are needed to avoid unwanted configurations on
the world sheet. For example, multiple solid lines generated by the repeated
application of $\phi^\dagger$ at the same $\sigma$ would lead to overcounting of the graphs.
These redundant states can be eliminated by imposing the constraint

$$\int d\mathbf{q} \phi^\dagger(\sigma, \tau, \mathbf{q}) \phi(\sigma, \tau, \mathbf{q}) = \rho(\sigma, \tau),$$

where

$$\rho = \bar{\psi}_2 \psi_2,$$

which is equal to one on solid lines and zero on dotted lines. This constraint
ensures that there is at most one solid line at each site.

Fermions are also needed to avoid another set of unwanted configurations.
Propagators are assigned only to adjacent solid lines and not to non-adjacent
ones. To enforce this condition, it is convenient to define,

$$E(\sigma_i, \sigma_j) = \prod_{k=i+1}^{k=j-1} (1 - \rho(\sigma_k)), \quad (1)$$

for $\sigma_j > \sigma_i$, and zero for $\sigma_j < \sigma_i$. The crucial property of this function is
that it acts as a projection: It is equal to one when the two lines at $\sigma_i$ and $\sigma_j$
are separated only by the dotted lines; otherwise, it is zero. With the help
of $E$, the free Hamiltonian can be written as

$$H_0 = \frac{1}{2} \sum_{\sigma, \sigma'} \int d\mathbf{q} \int d\mathbf{q}' \frac{E(\sigma, \sigma')}{\sigma' - \sigma} \left( (\mathbf{q} - \mathbf{q}')^2 + m^2 \right)$$
\[ \times \phi^\dagger(\sigma, q)\phi(\sigma, q)\phi^\dagger(\sigma', q')\phi(\sigma', q') + \sum_\sigma \lambda(\sigma) \left( \int dq \phi^\dagger(\sigma, q)\phi(\sigma, q) - \rho(\sigma) \right), \]  

(6)

where \( \lambda \) is a lagrange multiplier enforcing the constraint (3). The evolution operator \( \exp(-i\tau H_0) \), applied to states, generates a collection of free propagators, without, however, the prefactor \( 1/(2p^+) \).

One can also think of the lagrange multiplier \( \lambda(\sigma, \tau) \) as an abelian gauge field on the world sheet. The corresponding gauge transformations are [4]

\[
\begin{align*}
\psi & \rightarrow \exp \left( -\frac{i}{2} \alpha \sigma_3 \right) \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp \left( \frac{i}{2} \alpha \sigma_3 \right), \\
\phi & \rightarrow \exp(-i\alpha) \phi, \quad \phi^\dagger \rightarrow \exp(i\alpha) \phi^\dagger, \\
\lambda & \rightarrow \lambda - \partial_\tau \alpha.
\end{align*}
\]

(7)

In what follows, we will usually gauge fix by setting

\[ \rho_+ = \rho_-; \]

where

\[ \rho_+ = \bar{\psi}_1\psi_2, \quad \rho_- = \bar{\psi}_2\psi_1. \]

(8)

Using the constraint (3), the free hamiltonian can be written in a form more convenient for later application:

\[
H_0 = \frac{1}{2} \sum_{\sigma, \sigma'} G(\sigma, \sigma') \left( \frac{1}{2} m^2 \rho(\sigma)\rho(\sigma') + \rho(\sigma') \right) \int dq \, q^2 \phi^\dagger(\sigma, q)\phi(\sigma, q)
\]

\[- \int dq \int dq' \, (q \cdot q') \phi^\dagger(\sigma, q)\phi(\sigma, q)\phi^\dagger(\sigma', q')\phi(\sigma', q') \right) \]

\[+ \sum_\sigma \lambda(\sigma) \left( \int dq \phi^\dagger(\sigma, q)\phi(\sigma, q) - \rho(\sigma) \right), \]

(9)

where we have defined

\[ G(\sigma, \sigma') = \frac{\mathcal{E}(\sigma, \sigma') + \mathcal{E}(\sigma', \sigma)}{|\sigma - \sigma'|}. \]

(10)

Next, we introduce the interaction term. Two kinds of interaction vertices, corresponding to \( \phi^\dagger \) creating a solid line or \( \phi \) destroying a solid line, are pictured in Fig.3. We also have to take care of the prefactor \( 1/(2p^+) \) in
by attaching it to the vertices. Here, as in [4], we choose a symmetric
distribution of this factor between vertices by attaching a factor of
\[
V = \frac{1}{\sqrt{8 p_{12}^+ p_{23}^+ p_{13}^+}} = \sqrt{\frac{1}{8 (\sigma_2 - \sigma_1)(\sigma_3 - \sigma_2)(\sigma_3 - \sigma_1)}}
\]
(11)
to each vertex. The interaction term in the hamiltonian can now be written as
\[
H_I = g\sqrt{a} \sum_\sigma \int d\mathbf{q} \left( V(\sigma) \rho_+(\sigma) \phi(\sigma, \mathbf{q}) + \rho_-(\sigma) V(\sigma) \phi^+(\sigma, \mathbf{q}) \right),
\]
(12)
where \( g \) is the coupling constant. \( \rho_\pm \) are given by eq.(8) and
\[
V(\sigma) = \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 < \sigma} \frac{W(\sigma_1, \sigma_2)}{\sqrt{(\sigma - \sigma_1)(\sigma_2 - \sigma_1)(\sigma_2 - \sigma)}},
\]
(13)
where,
\[
W(\sigma_1, \sigma_2) = \rho(\sigma_1) \mathcal{E}(\sigma_1, \sigma_2) \rho(\sigma_2).
\]
(14)
Here is a brief explanation of the origin of various terms in \( H_I \): The
factors of \( \rho_\pm \) are there to pair a solid line with an \( i = 2 \) fermion and a
dotted line with an \( i = 1 \) fermion. The factor of \( V \) ensures that the pair
of solid lines 12 and 23 in Fig.3 are separated by only dotted lines, without
any intervening solid lines. Apart from an overall factor, the vertex defined
above is very similar to the bosonic string interaction vertex in the light cone
picture. Taking advantage of the properties of \( \mathcal{E} \) discussed following eq.(5),
we have written an explicit representation of this overlap vertex. Finally, the factor of $\sqrt{a}$ multiplying the coupling constant comes from the replacement

$$\int d\sigma \rightarrow a \sum_{\sigma}.$$  

It is easy to verify that the factor $a$ that appears on the right is taken care of by attaching a factor of $\sqrt{a}$ to the coupling constant $g$.

The total Hamiltonian is given by

$$H = H_0 + H_I$$  

and the corresponding action by

$$S = \int d\tau \left( \sum_{\sigma} \left( i\bar{\psi}\partial_\tau \psi + i \int dq \phi^\dagger \partial_\tau \phi \right) - H(\tau) \right).$$  

For later use, we note that the theory is invariant under

$$\phi(\sigma, \tau, q) \rightarrow \phi(\sigma, \tau, q + r),$$  

where $r$ is a constant vector.

4. The Variational Treatment Of $\phi^3$ In 1+2 Dimensions

In this section, we apply the variational ansatz introduced in [3] to $\phi^3$ in 1+2 dimensions (D=1). As explained in the introduction, in this low dimension there is no ultraviolet divergence, so we can focus on the problem of removing the infrared cutoff by letting $a \rightarrow 0$ without being distracted by the ultraviolet problem. An alternative and completely equivalent approach, developed in [4], is to search for a static classical solution to the action (16). Here we prefer the variational method since it provides a rigorous upper bound to the ground state energy, which will be needed in the subsequent development. We choose the variational state

$$|s\rangle = \prod_{\sigma} \left( \int dq A(q) \phi^\dagger(\sigma, q) \rho_-(\sigma) + B \right) |0\rangle,$$  

where the product extends over all $\sigma$ and the vacuum, which satisfies

$$\rho_+(\sigma) |0\rangle = 0,$$
corresponds to the empty world sheet. By suitable gauge fixing (eq. (7) and the discussion that follows), \( A \) and \( B \) can be taken to be real, and they are \( \sigma \) independent to have a ground state that is translationally invariant in \( \sigma \). Also, assuming a rotationally invariant ground state, \( A \) can only depend on the length of the vector \( \mathbf{q} \). This trial state, introduced in [3], is just about simplest ansatz for the ground state one can think of.

Solving the constraint (3) and the normalization condition gives

\[
\int d\mathbf{q} A^2(\mathbf{q}) = \rho_0, \quad B = \sqrt{1 - \rho_0^2},
\]

where \( 0 \leq \rho_0 \leq 1 \) is the (constant) ground state expectation value of \( \rho \).

The expectation values of \( H_0 \) and \( H_I \) (eqs. (9) and (12)) are easily calculated:

\[
\langle s | H_0 | s \rangle = \sum_{\sigma' > \sigma} G(\sigma, \sigma') \left( \frac{1}{2} m^2 \rho_0^2 + \rho_0 \int d\mathbf{q} \mathbf{q}^2 A^2(\mathbf{q}) \right) + \lambda_0 \left( \int d\mathbf{q} A^2(\mathbf{q}) - \rho_0 \right),
\]

\[
\langle s | H_I | s \rangle = 2g\sqrt{a} B \sum_{\sigma} V(\sigma) \int d\mathbf{q} A(\mathbf{q}).
\]

where \( \lambda_0 \) is the (constant) expectation value of the lagrange multiplier \( \lambda \).

We now define the ground state energy by

\[
E_g = \langle s | H_0 + H_I | s \rangle,
\]

and minimize it with respect to the variational parameters \( \lambda_0 \), \( \rho_0 \) and the function \( A(\mathbf{q}) \). The variational equation

\[
\frac{\delta E_g}{\delta A(\mathbf{q})} = 0
\]

determines \( A \):

\[
A(\mathbf{q}) = -g\sqrt{a} \sqrt{1 - \rho_0} \frac{V(\sigma)}{\lambda_0 + \rho_0 \mathbf{q}^2 \sum_{\sigma' > \sigma} G(\sigma, \sigma')}.
\]

Next, we evaluate various terms in this equation, setting \( \rho(\sigma) = \rho_0 \). It is convenient to define

\[
\sum_{\sigma' > \sigma} G(\sigma, \sigma') = \frac{1}{a} F(\rho_0), \quad V(\sigma) = \frac{\rho_0^2}{a^{3/2}} Z(\rho_0),
\]

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where,

\[
F(\rho_0) = -\frac{\rho_0 \ln(\rho_0)}{1 - \rho_0},
\]

\[
Z(\rho_0) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(1 - \rho_0)^{n_1+n_2}}{\sqrt{(n_1+1)(n_2+1)(n_1+n_2+2)}}.
\]  

Eq.(23) can now be rewritten as

\[
A(q) = -g\rho_0^2 \sqrt{1 - \rho_0} \frac{Z(\rho_0)}{a\lambda_0 + F(\rho_0)} q^2.
\]  

So far, D, the dimension of the transverse space, has been arbitrary, but now, we specialize to \(D = 1\). The variational equation

\[
\frac{\partial E_g}{\partial \lambda_0} = 0,
\]  

fixes \(\lambda_0\):

\[
(a\lambda_0)^3 = \frac{\pi^2}{4} g^4 (1 - \rho_0)^2 \rho_0^6 \frac{Z^4(\rho_0)}{F(\rho_0)}.
\]  

we note that both \(A(q)\) and \(a\lambda_0\) are stay finite as \(a \to 0\).

It is convenient to separate the contribution of the mass term to \(E_g\):

\[
E_g = E_m + E_0,
\]

\[
E_m = \frac{p^+}{2a^2} m^2 \rho_0 F(\rho_0),
\]

\[
E_0 = -\frac{p^+}{3a^2} \rho_0^3 \left(4\pi g^2 (1 - \rho_0) Z^2(\rho_0)\right)^{2/3} (F(\rho_0))^{-1/3}.
\]  

It remains to minimize \(E_g\) with respect to \(\rho_0\). Fortunately, without any detailed analysis, we can deduce some qualitative general features of the two terms, which enables us to answer the important question of whether the minimum occurs at \(\rho_0 = 0\) or at a non-vanishing \(\rho_0\).

We first note that both terms are bounded; \(E_m\) is positive semi-definite, \(E_0\) is negative semi-definite. Next, we need their behaviour near the end points. At \(\rho_0 = 1\), \(E_0\) vanishes and \(E_m\) reaches a positive value, so \(E_g\) is also positive at this point. On the other hand, as \(\rho_0 \to 0\), both go to zero, but \(E_m\) vanishes faster:

\[
E_m \to -\rho_0^2 \ln(\rho_0), \quad E_0 \to \rho_0^{5/3}.
\]  

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The second limit follows from
\[
Z(\rho_0) \rightarrow \rho_0^{-1/2}, \quad F(\rho_0) \rightarrow \rho_0 \ln(\rho_0).
\] (31)

For small \(\rho_0\), since \(E_m\) vanishes faster then \(E_0\), \(E_0\) wins over \(E_m\), and \(E_g\) is therefore negative. As \(\rho_0\) increases towards one, it changes sign and becomes positive. This is sketched in Fig.4. Clearly, there is a minimum is at some \(\rho_0 \neq 0\), and \(E_g\) is negative at the minimum.

We recall that \(\rho_0\), the ground state expectation value of \(\rho\), measures the average density of graphs on the world sheet. It vanishes in any finite order of perturbation theory, so it is natural to identify the phase \(\rho_0 = 0\) with the perturbative regime. On the other hand, in the phase \(\rho_0 \neq 0\), the world sheet is densely covered with graphs, and the contribution of higher (infinite) order graphs dominate. We have seen above that it is this phase that is energetically favored.

In establishing the existence of a non-trivial ground state at a \(\rho_0 \neq 0\), the negative sign of \(E_0\) was crucial. This is a subtle entropic effect related to the counting of configurations. From the perspective of statistical mechanics, \(E_g\) is really the free energy
\[
F = E - TS
\]
which takes into account the entropy arising from the counting of the world sheet graphs. Consider the state \(|s\rangle\) of the variational ansatz (eq.(18)). It represents a superposition of spin up states (dotted lines) and spin down
states (solid lines). At $\rho_0 = 0$, the world sheet is empty, corresponding to a single state with spin up at all sites. In this case, the entropy, and also the free energy is zero. On the other hand, at $\rho_0 \neq 0$, we have a superposition of a multitude of spin up and spin down configurations, giving rise to non-zero entropy. Clearly, it is the interaction term $H_I$ that generates the entropy by causing transitions between spin up and down states. At large $g$, $H_I$ dominates, the entropy increases and finally drives $F$ to a negative value.

At this point, we face the problem of a negative ground state energy that diverges as $1/a^2$ in the continuum limit $a \to 0$ in the phase $\rho_0 \neq 0$. We emphasize that this is not an artifact of the approximation; the variational calculation which puts an upper limit on the ground state energy shows that this is a real effect. Of course, one obvious possibility is that, since we are working with an intrinsically unstable field theory, we should not be surprised to find a negative unbounded ground state energy. However, in the next section, we show that the same phenomenon persists in an at least classically stable theory, where, in addition to the original $\phi^3$ term, the new model now has a positive $\phi^4$ interaction. Using the same variational wave function, we again find a ground state at $\rho_0 \neq 0$, with an energy that goes as $-1/a^2$.

The problematic term appears to be a simple additive term to the ground state energy. One can therefore simply cancel this term by introducing a counter term in the action proportional to the area of the world sheet. This is a familiar procedure in string theory; however, a finite additive term remains undetermined. This is fixed by demanding Lorentz invariance in string theory [10]. Here, unfortunately, Lorentz invariance is still an open problem.

5. The Variational Treatment Of $\phi^3 + \phi^4$ In 1+2 Dimensions

In this section, we add a $\phi^4$ interaction to the $\phi^3$ of the previous section. The idea is to have at least a classically stable model. The total interaction hamiltonian is with this addition is given by

$$H_{I,t} = H_I + H'_I,$$  \hspace{1cm} (32)

where $H_I$ is given by eq.(12) and,

$$H'_I = g'a \sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} \int dq \int d q' \left[ f_1 (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \phi^\dagger (\sigma_2, q) \phi (\sigma_3, q') \right. \left. + f_2 (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \left( \phi (\sigma_2, q) \phi (\sigma_3, q') + \phi^\dagger (\sigma_2, q) \phi^\dagger (\sigma_3, q') \right) \right].$$  \hspace{1cm} (33)
Here, $g'$ is a positive coupling constant, scaled by $a$ in order that in the limit $a \to 0$, sums over $\sigma$ smoothly go over integrals over $\sigma$ (see section 3). The function $f_1$ is given by

$$f_1(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \left( (\sigma_2 - \sigma_1)(\sigma_4 - \sigma_2)(\sigma_4 - \sigma_3)(\sigma_3 - \sigma_1) \right)^{-1/2}$$

for

$$\sigma_1 < \sigma_2 < \sigma_4, \quad \sigma_1 < \sigma_3 < \sigma_4,$$

and it is zero otherwise. $f_2$ is given by

$$f_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \left( (\sigma_2 - \sigma_1)(\sigma_4 - \sigma_1)(\sigma_4 - \sigma_3)(\sigma_3 - \sigma_2) \right)^{-1/2}$$

for

$$\sigma_1 < \sigma_2 < \sigma_3 < \sigma_4,$$

and it is otherwise zero. $H'_I$ is the sum of three different four point vertices pictured in Fig.5.

We will now do a variational calculation for this new model, using the same form of the trial wave function given by eq.(18). The matrix element of $H'_I$ is given by

$$E' = \langle s | H'_I | s \rangle = \frac{p^+ g' \rho_0^2 (1 - \rho_0) \left( Z_1(\rho_0) + Z_2(\rho_0) \right)}{a^2} \int dq \int dq' A(q) A(q') \cdot$$

$$\cdot \int \int$$

(34)
where,

\[
Z_1(\rho_0) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \frac{(1 - \rho_0)^{n_1+n_2+n_3-3}}{(n_1 n_3 (n_2 + n_3) (n_1 + n_2))^{1/2}},
\]

\[
Z_2(\rho_0) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \frac{(1 - \rho_0)^{n_1+n_2+n_3-3}}{(n_1 n_2 n_3 (n_1 + n_2 + n_3))^{1/2}}.
\]  

(35)

Varying the ground state energy with the added term

\[
E_g = E_m + E_0 + E'
\]

with respect to \( A(q) \) (see eq.(22)), we find the solution

\[
A(q) = h \frac{(a \lambda_0 F(\rho_0))^{1/2}}{\pi a \lambda_0 + q^2 F(\rho_0)},
\]

(37)

and

\[
E' = \frac{p^+}{a^2} g' \rho_0^2 (1 - \rho_0) (Z_1(\rho_0) + Z_2(\rho_0)) h^2,
\]

(38)

where,

\[
h = \int dq \ A(q) = -\frac{g \pi \rho_0^2 (1 - \rho_0)^{1/2} Z(\rho_0)}{(a \lambda_0 F(\rho_0)) + 2\pi g' \rho_0^2 (1 - \rho_0) (Z_1(\rho_0) + Z_2(\rho_0))}.
\]

(39)

Finally, minimizing with respect to \( \lambda_0 \) (eq.(27)) gives the relation

\[
\rho_0 = \frac{h^2}{2\pi} \left( \frac{F'(\rho_0)}{a \lambda_0} \right).
\]

(40)

Combining eqs.(39) and (40), one can solve for \( \lambda_0 \) and \( h \), and express \( E' \) as a function of \( \rho_0 \) alone, and search for its minimum in this variable. Here we will be satisfied with a qualitative analysis similar to the one in the previous section. The new term in the ground state energy, \( E' \), is positive and vanishes at \( \rho_0 = 0 \). We also need its asymptotic behaviour as \( \rho_0 \to 0 \). It is not difficult to solve the above equations in this asymptotic limit. Starting with

\[
Z_{1,2}(\rho_0) \to 1/\rho_0,
\]

and (31), we have,

\[
\lambda_0 \to \rho_0 \ln(\rho_0), \quad h \to \rho_0^{1/2},
\]

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and

\[ E' \to \rho_0^2. \]  \hspace{2cm} (41)

Therefore, for small enough \( \rho_0 \), the negative term \( E_0 \) wins over the positive terms \( E_m \) and \( E' \), and \( E_g \) becomes negative. On the other hand, at \( \rho_0 = 1 \), \( E_0 \) vanishes, and \( E_g \) becomes positive. The plot of \( E_g \) against \( \rho_0 \) again looks like Fig.4, and we arrive at the same conclusion as in the last section: The minimum of \( E_g \) is at some \( \rho_0 \neq 0 \).

6. Classical Solution And Renormalization In 1+3 Dimensions

One can easily apply the variational ansatz to the \( \phi^3 \) model at \( D = 2 \) (1+3 dimensions). However, a new complication arises: There is a logarithmic mass divergence which has to be renormalized. The standard recipe is to introduce a counter term in the bare mass to cancel this divergence. The ground state energy, computed in references [3, 4], is of the form

\[ E_g = \frac{p^+}{a^2} \left( -F_1(\rho_0) \ln \left( F_2(\rho_0) \Lambda^2 \right) + \frac{1}{2} m^2 \rho_0 F(\rho_0) \right), \]  \hspace{2cm} (42)

where \( \Lambda \) is an ultraviolet cutoff, \( F_{1,2} \) and \( F \) are certain functions of \( \rho_0 \) whose explicit form will not matter. One can then cancel the cutoff dependent part of the energy against the mass counter term \( \delta m^2 \) by letting

\[ m^2 \to m_r^2 + \delta m^2. \]  \hspace{2cm} (43)

However, there is a problem with this; \( \delta m^2 \) will then be a function of \( \rho_0 \), whereas in field theory the mass counter term can only depend on the renormalized mass and the coupling constant. Furthermore, if even if we decide to allow the counter term to depend on \( \rho_0 \), this will inevitably introduce an arbitrary dependence on this variable in \( E_g \), and the position of the minimum can no longer uniquely fixed in terms of the parameters of the original field theory; namely, the mass and the coupling constant. We find this situation unsatisfactory.

Fortunately, we can modify our variational calculation to avoid this problem. First, we observe that the variational calculation is equivalent to finding solutions to the classical equations of motion [4]. For example, the classical equations motion with respect to \( \phi \) and \( \phi^\dagger \) can be solved for a static rotation invariant configuration in the form

\[ \phi_0(\sigma, q) = -g \sqrt{a} \frac{\rho_- (\sigma) V(\sigma)}{\lambda(\sigma) + \frac{1}{2} G(\sigma, \sigma') \rho(\sigma') q^2}. \]  \hspace{2cm} (44)

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Replacing $\rho(\sigma)$ and $\rho_-(\sigma)$ by their expectation values $\rho_0$ and $\sqrt{1-\rho_0}$ respectively, we see that the classical solution $\phi_0$ is identical $A(q)$ (eq.(23)). Therefore, the variational and the classical equation approaches are equivalent. We chose the variational approach in section 4 because it enabled us to put an upper bound on the ground state energy.

Being equivalent to the variational calculation, the classical approximation to the ground state energy has the same ultraviolet divergent term at $D=2$ and therefore suffers from the same problem. However, it provides a convenient starting point for a modified approach which overcomes this problem. It turns out that treating $\rho$ as a classical variable is at the root of the problem. This implies factorization of the expectation value of products. For example,

$$\langle \rho^2 \rangle \rightarrow \langle \rho \rangle \langle \rho \rangle = \rho_0^2,$$

and similarly for higher products. This is, by the way, also equivalent to the mean field approximation. On the other hand, treated exactly, $\rho$ takes on only the discrete values 0 and 1, and satisfies the identities

$$\rho^2(\sigma) = \rho(\sigma), \quad \rho_+(\sigma)\rho_-(\sigma) = 1 - \rho(\sigma), \quad \rho_-(\sigma)\rho_+(\sigma) = \rho(\sigma). \quad (46)$$

From these, one can derive two further identities

$$G(\sigma, \sigma') \rho(\sigma') \rho_-(\sigma) W(\sigma_1, \sigma_2) = \left( \delta_{\sigma', \sigma_2} \frac{1}{\sigma_2 - \sigma} + \delta_{\sigma', \sigma_1} \frac{1}{\sigma - \sigma_1} \right) \rho_-(\sigma) \ W(\sigma_1, \sigma_2), \quad (47)$$

and

$$W(\sigma_1, \sigma_2)\rho_+(\sigma)\rho_-(\sigma)W(\sigma_1', \sigma_2') = \delta_{\sigma_1, \sigma_1'} \delta_{\sigma_2, \sigma_2'} \ W(\sigma_1, \sigma_2). \quad (48)$$

One can also understand these geometrically from the overlap properties of the vertices in Fig.3. Apart from an overall factor, these are structurally the same as the corresponding string vertices, and in particular, they satisfy the same overlap relations. These overlap relations turn out to be crucial for preserving the correct structure of the ultraviolet divergent terms. The factorization ansatz (45) violates the identities (46) and consequently the above overlap relations.

This poses a dilemma: We cannot carry out an exact calculation all the way through and so we must eventually make some approximation. There is, however, a way out: We carry out an exact calculation till we get the structure of the ultraviolet divergence right, and only then we resort to the
classical (meanfield) approximation. The basic strategy is first to simplify \( \phi_0 \) and later \( E_g \) as much as possible using the relations (46),(47) and (48) before making any approximations. For example, \( \phi_0 \) can be rewritten as

\[
\phi_0(\sigma, q) = -g \sum_{\sigma_1 < \sigma < \sigma_2} \rho_- (\sigma) W(\sigma_1, \sigma_2) \left( \lambda(\sigma) + \frac{1}{2} q^2 \left( \frac{\sigma_2 - \sigma_1}{(\sigma_2 - \sigma)(\sigma - \sigma_1)} \right) \right) \sqrt{\left( \sigma - \sigma_1 \right) \left( \sigma_2 - \sigma_1 \right) \left( \sigma_2 - \sigma \right)}.
\]

This is derived by formally expanding the denominator in eq.(44) in powers of \( q^2 \) and then using the identity (47) repeatedly to simplify products of the form \( GW \). This is as far as one can go; since this equation is linear in \( W \), no further simplification is possible.

Next, we replace \( \phi \) by the above \( \phi_0 \) in the hamiltonian. The result involves products of the form \( \rho_+ \rho_- g W W \) which can be simplified with the help of (46),(47) and (48), and the integral over \( q \) can be done. We skip the algebra and give the final result:

\[
H(\phi = \phi_0) = -2\pi g^2 a \sum_{\sigma} \sum_{\sigma_1 < \sigma < \sigma_2} W(\sigma_1, \sigma_2) \ln \left( \frac{\Lambda^2}{\left( \frac{\sigma_2 - \sigma_1}{\Lambda(\sigma) \left( \sigma_2 - \sigma \right) \left( \sigma - \sigma_1 \right)} \right)} \right) + \frac{m^2}{2} \sum_{\sigma_1 < \sigma_2} \frac{W(\sigma_1, \sigma_2)}{|\sigma_1 - \sigma_2|} - \sum_{\sigma} \lambda(\sigma) \rho(\sigma),
\]

where \( \Lambda \) is an ultraviolet cutoff needed because of mass divergence. Again, since this expression is linear in \( W \), it cannot be simplified any further.

The cutoff dependent part of the above hamiltonian exactly matches the mass term in \( H_0 \) (eq.(9)). It can therefore be cancelled by letting

\[
m^2 \rightarrow m_r^2 + \delta m^2,
\]

where \( m_r \) is the renormalized mass and the counter term is

\[
\delta m^2 = -4\pi g^2 a \sum_{\sigma} \sum_{\sigma_1 < \sigma} \sum_{\sigma_2} W(\sigma_1, \sigma_2) \ln \left( \frac{\Lambda^2}{\mu^2} \right) \left( \frac{\sigma_2 - \sigma_1}{\lambda(\sigma) \left( \sigma_2 - \sigma \right) \left( \sigma - \sigma_1 \right)} \right) = -4\pi g^2 a \sum_{\sigma_1 < \sigma_2} \frac{W(\sigma_1, \sigma_2)}{|\sigma_1 - \sigma_2|} \ln \left( \frac{\mu^2}{\lambda(\sigma) \left( \sigma_2 - \sigma \right) \left( \sigma - \sigma_1 \right)} \right).
\]

The renormalized hamiltonian is then given by

\[
H_r(\phi = \phi_0) = -2\pi g^2 a \sum_{\sigma} \sum_{\sigma_1 < \sigma < \sigma_2} W(\sigma_1, \sigma_2) \ln \left( \frac{\mu^2}{\lambda(\sigma) \left( \sigma_2 - \sigma \right) \left( \sigma - \sigma_1 \right)} \right).
\]
In this result, the parameter $\mu$ is actually redundant; a change in $\mu$ can always be absorbed into the definition of $m_r$. We have therefore succeeded in renormalizing the mass divergence without introducing any extra parameters not present in the original model.

With the renormalization done, we are ready to evaluate (52) in the classical (mean field) approximation to compute the ground state energy. This amounts to replacing $\rho(\sigma)$ and $\lambda(\sigma)$ by their $\sigma$ independent ground state expectation values $\rho_0$ and $\lambda_0$. Furthermore, $\lambda_0$ can be fixed through the variational equation
\[
\frac{\delta H_r}{\delta \lambda_0} = 0,
\]
with the result
\[
\lambda_0 = \frac{2 \pi g^2}{a} F(\rho_0).
\]

Finally, putting everything together, the ground state energy is
\[
E_g = H_r (\phi = \phi_0, \rho = \rho_0, \lambda = \lambda_0)
= \frac{p^+}{a^2} \left( \left( \frac{1}{2} m_r^2 - 6 \pi g^2 \right) \rho_0 F(\rho_0) - 2 \pi g^2 \rho_0^2 \tilde{F}(\rho_0) \right) - 2 \pi g^2 \rho_0 F(\rho_0) \ln \left( \frac{\mu^2}{2 \pi g^2 F(\rho_0)} \right),
\]
where $F$ is given by (31) and
\[
\tilde{F}(\rho_0) = \sum_{n=1}^{\infty} \frac{\ln(n)}{n} (1 - \rho_0)^{n-1}.
\]

We are now ready to study the minimum of $E_g$ as a function of $\rho_0$ in the same qualitative fashion, as we have done earlier. Firstly, since $\mu$ is redundant, the above expression can be simplified without loss of generality by setting
\[
\mu^2 = 2 \pi g^2.
\]
We then note that all the terms are bounded and they vanish at $\rho_0 = 0$. Furthermore, they are all negative semi-definite with the exception of the term proportional to $m^2$, which is positive semi-definite. As before, we study
the behaviour of various terms near the two end points \( \rho_0 = 0, 1 \). For sufficiently small \( \rho_0 \), \( \tilde{F} \) dominates over the positive mass term, so \( E_g \) is negative. Near \( \rho_0 = 1 \), if
\[
m_r^2 > 12\pi g^2,
\]
\( E_g \) is positive, and we are back to Fig.4, with a minimum at \( \rho_0 \neq 0 \). If, on the other hand,
\[
m_r^2 \leq 12\pi g^2
\]
the situation is more complicated, although in this case also, the minimum is away from zero.

7. Classical Solution And Renormalization In 1+5 Dimensions

Next we consider \( D = 4 \), corresponding to \( \phi^3 \) in six dimensions. The self mass is now quadratically divergent, but this divergence can be eliminated by a mass counter term exactly as in the case \( D = 2 \). There remains, however, a residual logarithmic divergence:
\[
H_r(\phi = \phi_0) = -4\pi^2 m^2 \sum_{\sigma} \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 < \sigma} \lambda(\sigma) W(\sigma_1, \sigma_2) \frac{(\sigma_2 - \sigma)(\sigma - \sigma_1)}{(\sigma_2 - \sigma_1)^3}
\times \ln \left( \frac{\lambda(\sigma)}{\Lambda^2} \right) - \sum_{\sigma} \lambda(\sigma) \rho(\sigma) + m_r^2 \sum_{\sigma_1 < \sigma_2} W(\sigma_1, \sigma_2) \frac{|\sigma_1 - \sigma_2|}{|\sigma_1 - \sigma_2|}.
\]

(55)

This divergence can be eliminated by renormalizing the bare coupling constant \( g \) by setting
\[
g^2 = \frac{g_r^2}{\ln (\Lambda^2/\mu^2)},
\]
where \( g_r \) is the renormalized coupling constant and \( \mu \) an arbitrary mass parameter. We recall that \( \phi^3 \) is asymptotically free in 6 space-time dimensions, and the above relation between the bare and renormalized couplings is the well known lowest order result. In the limit \( \Lambda \to \infty \), the renormalized \( H_r \) is given by
\[
H_r(\phi = \phi_0) \to \sum_{\sigma} \lambda(\sigma) \left( 4\pi^2 m^2 \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 < \sigma} W(\sigma_1, \sigma_2) \frac{(\sigma_2 - \sigma)(\sigma - \sigma_1)}{(\sigma_2 - \sigma_1)^3} - \rho(\sigma) \right)
+ m_r^2 \sum_{\sigma_1 < \sigma_2} W(\sigma_1, \sigma_2) \frac{|\sigma_1 - \sigma_2|}{|\sigma_1 - \sigma_2|}.
\]

(57)
Having renormalized the ultraviolet divergences, we ready to carry out the mean field (classical) approximation. As before, we replace \( \rho \) by its constant expectation value \( \rho_0 \). But in the case of \( \lambda \), we encounter a new situation, different from the dimensions \( D = 1, 3 \). The expectation value of \( \lambda \) is undetermined; instead, \( \lambda \) acts as a Lagrange multiplier and imposes the constraint

\[
\rho_0 = 4\pi^2 g_r^2 a \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 < \sigma} \frac{W(\sigma_1, \sigma_2)}{(\sigma_2 - \sigma_1)^3} (\sigma_2 - \sigma_1) (\sigma - \sigma_1) \rho_0 = 4\pi^2 g_r^2 \rho_0^2 \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{n_1 n_2}{(n_1 + n_2)^3} (1 - \rho_0)^{n_1 + n_2 - 1}, \tag{58}
\]

and the ground state energy reduces to the mass term:

\[
E_g = H_r(\phi = \phi_0, \rho = \rho_0) = \frac{p^+}{2a^2} m_r^2 \rho_0 F(\rho_0). \tag{59}
\]

Eq.(58) is the equation that fixes \( \rho_0 \). It has two solutions: The trivial (perturbative) solution

\[ \rho_0 = 0 \]

minimizes the ground state energy at

\[ E_g = 0 \]

if \( m_r^2 > 0 \). On the other hand, for some range of \( g_r^2 \), there is also the non-perturbative solution at some

\[ \rho_0 \neq 0. \]

This the solution that minimizes the energy if \( m_r^2 < 0 \). Normally, one would tend to reject this possibility because of the tachyonic mass. However, in the case of the asymptotically free theory that we are dealing with, \( m_r \) is likely to be a mass scale associated with the running coupling constant, unrelated to the low energy physical mass spectrum. At this point, one should keep an open mind, and a deeper study of the model is needed to decide on the sign of \( m_r^2 \).

8. Quadratic Fluctuations Around The Classical Background

This section is mostly based on the earlier work on the same problem, especially reference [4]. There will, however, be some additions, corrections
and clarifications. Our goal is to determine the spectrum of quadratic fluctuations around the classical solution $\phi_0$ in the phase $\rho_0 \neq 0$. We are especially interested in the decompactification limit $a \to 0$: In this limit, the spectrum of fluctuations consists of a heavy sector and a light sector, and the masses of the heavy sector excitations go to infinity, whereas the light sector masses remain finite. It is natural to identify the heavy sector with the Kaluza-Klein modes generated by the compactification; they become infinitely heavy in the decompactification limit

$$a \to 0, \quad R = 1/a \to \infty.$$  

Interestingly, the spectrum of the light sector is that of a transverse string. This is in agreement with the results of [4].

We now sketch a brief derivation of these results. It is convenient to set

$$\phi = \phi_0 + \phi_1, \quad \phi_1 = \phi_{1,r} + i \phi_{1,i},$$  

where

$$\phi_0 = A(q)$$

for $d = 1$, and $\phi_{1,r,i}$ are hermitian fields. The contribution to the action second order in $\phi_1$ is given by the sum of kinetic and potential terms:

$$S^{(2)} = S_{k.e} - \int d\tau H^{(2)}(\tau) = S_{k.e} + S_{p.e},$$  

where,

$$S_{k.e} = 2 \sum_\sigma \int d\tau \int dq \phi_{1,i} \partial_\tau \phi_{1,r},$$  

Since the action is quadratic in both $\phi_{1,i}$ or $\phi_{1,r}$, one can carry out the functional integral over one of these fields before writing down $H^{(2)}$. We choose to integrate over $\phi_{1,i}$, with the result,

$$S_{k.e} \to \sum_\sigma \int d\tau \int dq \frac{(\partial_\tau \phi_{1,r}(\sigma, \tau, q))^2}{\lambda(\sigma) + \frac{1}{2} \sum_{\sigma'} G(\sigma, \sigma') \rho(\sigma') q^2},$$  

and, somewhat schematically,

$$H^{(2)} \to \sum_\sigma \lambda(\sigma) \int dq \phi_{1,r}^2(\sigma, q) + \sum_{\sigma, \sigma'} G(\sigma, \sigma') \left( \frac{1}{2} \rho(\sigma') \int dq q^2 \phi_{1,r}^2(\sigma, q) \right. \right.$$  

$$\left. - 2 \int dq \int dq' (q \cdot q') \left( \phi_0 \phi_{1,r}(\sigma, q) \phi_0 \phi_{1,r}(\sigma, q') \right. \right.$$  

$$\left. \left( \phi_0 \phi_{1,r}(\sigma, q) \phi_0 \phi_{1,r}(\sigma, q') \right) \right).$$  

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We first fix $\rho$ and $\lambda$ at their classical values $\rho_0$ and $\lambda_0$. Later, we will also consider their fluctuations around the classical values. The first observation is that $S_{k.e}$ reaches a finite limit as $a \to 0$. To see this, in this limit, it can be rewritten as

$$S_{k.e} \to \int d\sigma \int d\tau \int dq \frac{(\partial_\tau \phi_{1,\tau}(\sigma, \tau, q))^2}{a \lambda_0 + \frac{4}{3} \sum_{\sigma'} G(\sigma, \sigma') \rho_0 q^2},$$  \hspace{1cm} (65)$$

and since $(a \lambda_0)$ as well as

$$a \sum_{\sigma'} G(\sigma, \sigma')$$

are finite as $a \to 0$ (see eq.(24)), so is $S_{k.e}$. On the other hand, to leading order

$$H^{(2)} \to 1/a^2,$$

and therefore, in general, the spectrum becomes heavy in the decompactification limit. This argument, with a slight modification, also applies to the fluctuations of $\rho$ and $\lambda$. These fluctuations become heavy and are suppressed in the limit $a \to 0$. Consequently, in this limit, these fields become frozen at their expectation values $\rho_0$ and $\lambda_0$.

There are, however, exceptional modes which stay light. To investigate $a \to 0$ limit more carefully, following [7], we note that the term involving $G$ in $H^{(2)}$ can be written as

$$\frac{1}{2} \sum_{\sigma, \sigma'} G(\sigma, \sigma') K(\sigma) L(\sigma').$$

Next, we expand the product $KL$ in powers of $\sigma' - \sigma$:

$$K(\sigma) L(\sigma') = K(\sigma) L(\sigma) + (\sigma' - \sigma) K(\sigma) L'(\sigma) + \frac{1}{2} (\sigma' - \sigma)^2 K(\sigma) L''(\sigma) + \cdots,$$  \hspace{1cm} (66)$$

and evaluate the sums over $\sigma'$:

$$\sum_{\sigma'} G(\sigma, \sigma') = \frac{2}{a} F(\rho_0),$$

$$\sum_{\sigma'} (\sigma' - \sigma) G(\sigma, \sigma') = 0,$$

$$\sum_{\sigma'} (\sigma' - \sigma)^2 G(\sigma, \sigma') = \frac{2a}{\rho_0^2}. $$  \hspace{1cm} (67)
With the help of this expansion, we have, as \( a \to 0 \),

\[
\frac{1}{2} \sum_{\sigma,\sigma'} G(\sigma, \sigma') K(\sigma) L(\sigma') \to \int d\sigma \left( \frac{1}{a^2} F(\rho_0) K(\sigma) L(\sigma) - \frac{1}{2\rho_0^2} K'(\sigma) L'(\sigma) \right). \tag{68}
\]

If we had continued the expansion in eq.(66) to higher powers of \( \sigma' - \sigma \), we would have additional terms in the above expression involving higher derivatives of \( K \) and \( L \), but their coefficients would contain powers of \( a \) and would therefore vanish.

It is of interest to note that although we started with an action non-local in the \( \sigma \) coordinate, after taking the \( a \to 0 \) limit, we end up with a local action, with at most two derivatives with respect to \( \sigma \). This localization is a consequence of \( \rho_0 \) being non-zero. To see this, consider the term in \( H_0 \) involving the product

\[
\left( \phi^\dagger \phi \right)_\sigma \left( \phi^\dagger \phi \right)_{\sigma'}.
\]

The points \( \sigma \) and \( \sigma' \) are tied together by a factor of \( E(\sigma, \sigma') \) (eqs.(6) and (14)), which, after setting \( \rho(\sigma) = \rho_0 \), becomes

\[
E \to (1 - \rho_0)^{n-1},
\]

where,

\[
n = \frac{|\sigma - \sigma'|}{a}.
\]

If we keep \( \sigma - \sigma' \) fixed and finite as we let \( a \to 0 \), \( n \to \infty \), and since \((1 - \rho_0) < 1\),

\[
E(\sigma, \sigma') \to 0.
\]

Therefore, in the sum over \( \sigma \) and \( \sigma' \) in (68), only the terms separated by a distance of the order of \( a \) survive, and the theory localizes.

It was noted in the earlier work [7] that there is an exceptional mode for which the leading term in \( H^{(2)} \) proportional to \( 1/a^2 \) is absent, and therefore this mode survives in the decompactification limit. This is not an accident but it is the result of the invariance of the theory under

\[
q \to q + r, \tag{69}
\]
(see eq.(17)). Now consider the mode $v$ introduced by letting

$$\phi_{1,r} \to \phi_0(\sigma, q + v(\sigma, \tau)) - \phi_0(\sigma, q),$$  \hfill (70)

where $\phi_0$ is the classical solution. It is easy to see that the leading contribution of $v$ to $S_{p.e}$ would be a mass term for proportional to

$$\frac{1}{a^2} \int d\tau \int d\sigma \ v^2(\sigma, \tau).$$

But such a term is forbidden by the translation symmetry, which can be rephrased as a symmetry under

$$v(\sigma, \tau) \to v(\sigma, \tau) + r.$$  \hfill (71)

So only the finite term $K' L'$ in (68) survives. This is a familiar story: $v$ can be identified with the Goldstone mode of spontaneously broken translation invariance.

The contribution of $v$ to $S_{p.e}$ is easily calculated by making the substitution (70) in (64). The integral over $q$ can be done by simply shifting

$$q \to q + v,$$

with the result

$$S_{p.e}(v) = -\frac{1}{2} \int d\tau \int d\sigma \ (\partial_\tau v)^2.$$  \hfill (72)

This result is universal in the sense that it does not depend on the structure of $\phi_0$ or the dimension $D$. On the other hand, the kinetic energy term does depend on both $\phi_0$ and $D$. Making the substitution (70) in (63), after a short calculation, we have,

$$S_{k.e}(v) = 5\pi g^2 (4a \lambda_0)^{-7/2} \rho_0^2 (1 - \rho_0) F_1(\rho_0) \int d\tau \int d\sigma \ (\partial_\tau v)^2,$$  \hfill (73)

where $a \lambda_0$ is given by (28) and

$$F_1(\rho_0) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{(1 - \rho_0)^{n_1+n_2-2}}{n_1^{3/2} n_2^{3/2} (n_1 + n_2)^{1/2}}.$$

Combining (72) and (73),

$$S^{(2)} = \int d\tau \int_{0}^{p^+} d\sigma \left( 8\pi^2 \alpha'^2 (\partial_\tau v)^2 - \frac{1}{2} (\partial_\sigma v)^2 \right).$$  \hfill (74)
This is the action for a transverse string, where the slope is given by
\[ \alpha'^2 = \frac{5 g^2 \rho_0^2 (1 - \rho_0) F_1(\rho_0)}{8 \pi (4 a \lambda_0)^{-7/2}}. \] (75)

The detailed form of this expression is not important; what is important is that the slope is non-zero for \( \rho_0 \neq 0 \). A similar calculation for \( D = 2 \) also gives the same result: The slope is non-zero for non-zero \( \rho_0 \).

The situation in the case \( D = 4 \) is problematic. In addition to demanding that \( \rho_0 \) be non-zero, which requires a tachyonic mass term (see the discussion in section 7), we have also to specify that \((a \lambda_0)\) is finite. We recall that unlike in the lower dimensional cases, at \( D = 4 \), \( \lambda_0 \) is undetermined. So in this case, string formation remains an open problem.

9. Systematic Expansion

So far what we have done is to construct static classical solutions and expand around them. It would be very desirable to identify this expansion with a systematic expansion in powers of a free parameter. In the case of \( D = 1 \), we are able to do this. We remind the reader of the usual weak coupling expansion around a soliton. One first splits the hamiltonian into the kinetic and potential terms:
\[ H = H_V + H_K. \] (76)

After a suitable scaling of the fields, the hamiltonian is cast into the form
\[ H \to \frac{1}{e^2} \tilde{H}_V + e^\beta \tilde{H}_K \] (77)

where \( e \) is the expansion parameter, \( \beta \) is a positive number, and \( \tilde{H}_V \) and \( \tilde{H}_K \), expressed in terms of scaled fields, are independent of \( e \). In the small \( e \) limit, the saddle point equations
\[ \frac{\partial \tilde{H}_V}{\partial \phi_i} = 0 \]
determine the solitonic field configuration. One then expands in powers of \( e \) around this configuration, which becomes heavy in the weak coupling limit. The kinetic term \( \tilde{H}_K \) is also treated as a perturbation.

In the case at hand, \( D = 1 \), \( H_V \) is the total hamiltonian (eq.(15)), and \( H_K \) is
\[ H_K = \sum_\sigma \left( i \bar{\psi} \partial_\tau \psi + i \int d\mathbf{q} \phi^\dagger \partial_\tau \phi \right). \] (78)
We now define the tilde fields by
\[
\phi(\sigma, q) = g^{-1/3} \tilde{\phi}(\sigma, g^{-2/3} q), \quad \phi^\dagger(\sigma, q) = g^{-1/3} \tilde{\phi}^\dagger(\sigma, g^{-2/3} q), \\
\lambda(\sigma) = g^{4/3} \tilde{\lambda}(\sigma),
\]  
(79)
and also redefine the mass by
\[
m = g^{2/3} \tilde{m}.
\]  
(80)
The fermionic fields and $\rho$ are unchanged. Written in terms of these new fields and $\tilde{m}$, $H_V$ scales as
\[
H_V(\phi, \lambda, m) = g^{4/3} H_V(\tilde{\phi}, \tilde{\lambda}, \tilde{m}).
\]  
(81)
The tilde variables are so chosen that when $H_V$ is expressed in terms of them, it no longer depends on $g$, except for the overall factor of $g^{4/3}$. We can therefore identify the expansion parameter as
\[
e^2 = g^{-4/3}.
\]  
(82)
and the expansion is a strong coupling expansion in inverse powers of $g^{4/3}$. As expected, the soliton becomes heavy in this limit. This is a surprise, since in field theory, solitons are usually associated with the weak coupling limit.

It is easily verified that the classical solution (26) transforms as $\phi$ does, and since
\[
E_g \to g^{4/3} E_g,
\]  
(83)
$\rho_0$, the location of the minimum of $E_g(\rho_0)$, is unchanged. Also, the transformation (79) preserves the canonical commutation relations (2). Finally, $S_K$ transforms with no prefactor ($\beta = 0$ in (77)), and should therefore be treated as a perturbation. We also note that the string slope goes as
\[
\alpha'^2 \to g^{-8/3}
\]
for large $g$. The strong coupling limit is therefore the zero slope limit, when only the low lying states survive. The resulting model is then the strong coupling dual of the original model.

A similar strong coupling expansion should also be applicable $D = 2$ case once the mass renormalization is taken care of. The case $D = 4$ has additional complications due to the running coupling constant and remains a completely open problem.
10. Conclusions

The main new contribution of the present work are the items discussed at the end of the introduction. They are the introduction of a $\phi^4$ interaction in addition to the original $\phi^3$, the correct treatment of the ultraviolet divergences, and in $1+2$ dimensions, the development of a systematic strong coupling expansion around the classical configuration in section 9. Since more realistic theories such as gauge theories have $\phi^4$ type interactions, generalizations of what was done here may lead towards more realistic models. Also, a systematic expansion is always good news; it should be useful in investigating problems like Lorentz invariance in a systematic fashion.

Many open problems still remain for future research. It would very nice to extend the strong coupling expansion to dimensions $1+3$ and $1+5$. Also, generalizations to at least lower dimensional gauge theories appears to be within reach. And of course, the problem of Lorentz invariance mentioned above still remains an important goal.

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3 For some initial steps taken towards more realistic theories, see [11, 12].
4 See [13] for an investigation of renormalization and Lorentz invariance in the light cone formulation.
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