TWO KINDS OF REAL LINES ON REAL DEL PEZZO SURFACES OF DEGREE 1

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ABSTRACT. We show how the real lines on a real del Pezzo surface of degree 1 can be split into two species, elliptic and hyperbolic, via a certain distinguished, intrinsically defined, Pin\(^{-1}\)-structure on the real locus of the surface. We prove that this splitting is invariant under real automorphisms and real deformations of the surface, and that the difference between the total numbers of hyperbolic and elliptic lines is always equal to 16.

Z: How do you draw the line between algebra and topology?
L: Well, if it’s just turning the crank it’s algebra, but if it’s got an idea in it, it’s topology.

A dialog attributed to Oscar Zariski and Solomon Lefschetz.

1. Introduction

In what follows, by a real algebraic variety (real surface, real curve, etc.) we mean a complex variety equipped with an anti-holomorphic involution on its complex point set, conj : X → X. We denote by X\(_R\) the set of real points, that is the fixed point set of conj.

1.1. The setting. By definition, a compact complex surface X is a del Pezzo surface of degree 1, if X is non-singular and irreducible, its anticanonical class −K\(_X\) is ample, and K\(^2\)_\(_X\) = 1. As is known, the image of X by the bi-anticanonical map X → P\(^3\) is then a non-degenerate quadratic cone Q ⊂ P\(^3\), with X → Q being a double covering branched at the vertex of the cone and along a non-singular sextic curve C ⊂ Q (a transversal intersection of Q with a cubic surface). Thus, in particular, each del Pezzo surface of degree 1 carries a non-trivial automorphism, known as the Bertini involution, the deck transformation τ\(_X\) of the covering.

Any real structure, conj : X → X, has to commute with τ\(_X\), and this gives another real structure τ\(_X\) ∘ conj = conj ∘ τ\(_X\) called Bertini dual to conj. A pair of real structures, {conj, conj ∘ τ\(_X\)}, will be called a Bertini pair. We generally use notation conj\(^±\) for Bertini pairs of real structures and write X\(^±\) for the corresponding pairs of real del Pezzo surfaces to simplify a more formal notation (X, conj\(^±\)).

The bi-anticanonical map projects the real loci X\(_R\)\(^±\) to two complementary domains Q\(_R\)\(^±\) ⊂ Q\(_R\) on Q\(_R\), where the latter is a cone over a real non-singular conic with non-empty real locus. The branching curve C is real too, and its real locus C\(_R\) together with the vertex of the cone form the common boundary of Q\(_R\)\(^±\). Conversely, for any real non-singular curve C ⊂ Q which is a transversal intersection of Q with a real cubic surface, the surface X which is the double covering of Q branched at the vertex of Q and along C is a del Pezzo surface of degree 1 inheriting from Q a pair of Bertini dual real structures conj\(^±\).
1.2. Main results. As a starting point, we prove the following existence and uniqueness statement.

1.2.1. Theorem. There is a unique way to supply each real del Pezzo surface $X$ of degree 1 with a $\text{Pin}^-$-structure $\theta_X$ on $X_\mathbb{R}$, so that the following properties hold:

1. $\theta_X$ is invariant under real automorphisms and real deformations of $X$. In particular, the associated quadratic function $q_X : H_1(X_\mathbb{R};\mathbb{Z}/2) \to \mathbb{Z}/4$ is preserved by the Bertini involution.
2. $q_X$ vanishes on each real vanishing cycle in $H_1(X_\mathbb{R};\mathbb{Z}/2)$ and takes value 1 on the class dual to $w_1(X_\mathbb{R})$.
3. If $X^\pm$ is a Bertini pair of real del Pezzo surfaces of degree 1, then the corresponding quadratic functions $q_{X^\pm}$ take equal values on the elements represented in $H_1(X^\pm_\mathbb{R};\mathbb{Z}/2)$ by the connected components of $C_\mathbb{R}$.

By a line on $X$ we understand a $(-1)$-curve, that is a rational non-singular curve $D \subset X$ with $D^2 = -1$, and as a consequence with $D \cdot K_X = -1$. If $X$ is a real del Pezzo surface of degree 1, the quadratic function $q_X$ as in Theorem 1.2.1 splits the real lines $l \subset X$ into hyperbolic, for which $q_X(l_\mathbb{R}) = 1 \in \mathbb{Z}/4$, and elliptic, for which $q_X(l_\mathbb{R}) = -1 \in \mathbb{Z}/4$. The number of hyperbolic and elliptic real lines will be denoted by $h(X)$ and $e(X)$, respectively.

Our second goal is to prove the following invariance of a combined count of lines.

1.2.2. Theorem. For each Bertini pair $X^\pm$ of degree 1 real del Pezzo surfaces,

\[ h(X^+) - e(X^+) + h(X^-) - e(X^-) = 16. \]

As an intermediate statement, we establish the relation

\[ h(X^+) - e(X^+) = 2(\text{rk} H_2^-(X^\pm)) - 1 \]

where $H_2^-(X^\pm)$ stands for the eigenlattice $\ker(1 + \text{conj}) : H_2(X) \to H_2(X)$ (see Proposition 3.4.5). This relation allows us to get also the individual values of $h(X^\pm), e(X^\pm)$ for each of deformation classes, see Table 3 in Subsection 3.6. As can be seen from this table, the alternating sum in Theorem 1.2.2 is the only (up to a constant factor) linear combination that does not depend on the deformation class.

1.3. The context. Existing literature on del Pezzo surfaces is huge and continues to grow due to recurrent involvement of this class of surfaces in very different topics in mathematics and physics. Numerology and combinatorics of line arrangements on del Pezzo surfaces occupy there a significant place. Their properties over non-closed fields, and especially over the real field, always stood in sight. (An excellent summary of the real case is given in [Ru].

For cubic surfaces (del Pezzo surfaces of degree 3), it was B. Segre [Se] who discovered the division of real lines into elliptic and hyperbolic. The fact that the difference $h - e = 3$ (between the numbers of hyperbolic and elliptic lines) is independent of a choice of a real non-singular cubic surface, was not emphasised by Segre explicitly. It was only recently, that this remarkable rule has attracted attention and was explained in the context of a new, integer valued, real enumerative geometry. It generated also a number of generalizations, such as counting of real lines, and real projective subspaces of dimension higher, on higher dimensional varieties (and even not only over the real field), see [FK1], [FK2], [FK3], [BaWi], [KaWi], [LaVo], [OkTe]. A distinctive new feature of a treatment which we present...
here is that the varieties under consideration (del Pezzo surfaces of degree 1) have a "hidden" symmetry (Bertini involution) preserved under deformations. It is this feature that is responsible for the invariance phenomenon in Theorem 1.2.2.

For Segre, the division of real lines in species was one of the main tools in his calculation of the monodromy group action on the set of real lines for each of the deformation classes of real cubic surfaces (a mistake he made for one of the deformation classes was corrected in [ACT]). Our initial motivation came also from a study of monodromy groups arising in the general theory of real del Pezzo surfaces, the subject to which we plan to devote a separate paper. Here, instead, we indicate some other applications, as well as a few directions for generalizations.

Namely, in Section 4 we discuss a signed count of real planes tritangent to real sextics \( C \subset \mathbb{Q} \). Note that Theorem 4.2.2 and Proposition 4.3.1 presented in Section 4 provide not only "intrinsic" definitions of hyperbolicity/ellipticity for real tritangent sections, but also an alternative to our principal definition of hyperbolicity/ellipticity for lines. Next, in Section 5 we count real conics 6-tangent to a real symmetric plane sextic, and then briefly discuss extending of the line counting to the case of nodal del Pezzo surfaces together with the related wall crossing phenomena.

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For the artwork with the Hasse diagram of \( E_8 \) on Figure 2 we used Ringel’s sample in [Ri], and for the diagrams on Figure 3, McKay’s lie-hasse package [Mc].

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2. Preliminaries

2.1. Lines, roots, and vanishing cycles. Given a del Pezzo surface \( X \) of degree 1, we denote by \( L(X) \) the set of lines on \( X \), introduce the set of exceptional classes

\[
I(X) = \{ v \in H_2(X) \mid v^2 = vK_X = -1 \},
\]

and consider the mapping \( \lambda : L(X) \to I(X) \) that sends a line \( l \in L(X) \) to its fundamental class \([l]\). In the lattice \( K_X^+ = \{ x \in H_2(X) \mid x \cdot K_X = 0 \} \), which is isomorphic to \( E_8 \), we distinguish its root system

\[
R(X) = \{ e \in K_X^+ \mid e^2 = -2 \}
\]

and relate it with \( I(X) \) by a bijection

\[
\phi : I(X) \to R(X), v \mapsto -K_X - v.
\]

The following statement is a consequence of Riemann-Roch theorem, Serre duality, and the adjunction formula (see [Ma]).

2.1.1. Proposition. For any del Pezzo surface \( X \), the set \( L(X) \) is finite and the map \( \lambda : L(X) \to I(X) \) is a bijection. \( \square \)

For \( e \in R(X) \) we denote by \( l_e \in L(X) \) the unique (due to Proposition 2.1.1) line with \([l_e] = \phi^{-1}(e)\).

If \( X \) is real, we denote by \( L_{\mathbb{R}}(X) = \{ l \in L(X) \mid \text{conj}(l) = l \} \) the set of real lines, and by \( I_{\mathbb{R}}(X) = \{ v \in I(X) \mid \text{conj}_v v = -v \} \) the set of real exceptional classes. We consider also the real root system

\[
R_{\mathbb{R}}(X) = \{ x \in R(X) \mid \text{conj}_x(x) = -x \} \]
and the restriction map
\[ \phi_\mathbb{R} = \phi|_{I_\mathbb{R}(X)} : I_\mathbb{R}(X) \to R_\mathbb{R}(X). \]

2.1.2. Corollary. If a del Pezzo surface \( X \) is real, then the maps \( \lambda \) and \( \phi \) induce bijections \( \lambda_\mathbb{R} : I_\mathbb{R}(X) \to I_\mathbb{R}(X) \) and \( \phi_\mathbb{R} : I_\mathbb{R}(X) \to R_\mathbb{R}(X). \)

As is known, each element of \( R(X) \) can be represented by a vanishing cycle of a nodal degeneration of \( X \) defined over \( \mathbb{C} \). By contrary, the vanishing cycles of nodal degenerations defined over \( \mathbb{R} \) represent only a certain part, denoted \( V_\mathbb{R}(X) \), of \( R_\mathbb{R}(X) \) (see, for example, Remark 3.5.1). By this reason, we call the elements of \( R_\mathbb{R}(X) \) real roots and reserve the name real geometric vanishing cycles for the elements of \( V_\mathbb{R}(X) \).

2.2. Some elements of Smith theory. According to one of basic applications of Smith theory to topology of real algebraic surfaces (for details, see e.g. [DIK]), if a real algebraic surface \((X, \text{conj})\) has \( H_1(X; \mathbb{Z}/2) = 0 \) (which is the case if \( X \) is a del Pezzo surface), then there exists a natural homomorphism
\[ \Upsilon : H_2^-(X) \to H_1(X_\mathbb{R}; \mathbb{Z}/2), \]
where by definition \( H_2^-(X) = \ker(1 + \text{conj}_*) : H_2(X) \to H_2(X) \). Given a conj-invariant oriented smooth 2-dimensional submanifold \( \Sigma \subset X \), with conj reversing the orientation of \( \Sigma \), this homomorphism sends \( [\Sigma] \in H_2^-(X) \) to \( [\Sigma_\mathbb{R}] \in H_1(X_\mathbb{R}; \mathbb{Z}/2) \), where \( \Sigma_\mathbb{R} = \Sigma \cap X_\mathbb{R} \) is the fixed locus of \( \text{conj}|_\Sigma \) (a smooth curve in \( \Sigma \)).

The Smith theory provides also the following.

2.2.1. Proposition. If \( X \) is a real algebraic surface with \( H_1(X; \mathbb{Z}/2) = 0 \), the homomorphism \( \Upsilon \) is an epimorphism. It respects the intersection forms in the sense that
\[ \Upsilon(v_1 \cdot v_2) = v_1 \cdot v_2 \mod 2 \quad \text{for any} \quad v_1, v_2 \in H_2^-(X), \]
and has kernel
\[ \ker \Upsilon = (1 - \text{conj}_*)H_2(X). \]
Thus, \( \Upsilon \) induces an isomorphism \( H_1(X_\mathbb{R}; \mathbb{Z}/2) \cong H_2^-(X)/(1 - \text{conj}_*)H_2(X). \)

The next relation is a general property of involutions in unimodular lattices.
\[ (1 - \text{conj}_*)H_2(X) = \{ v \in H_2^-(X) | v \cdot H_2^-(X) \subset 2\mathbb{Z} \}. \]

2.3. Deformation classifications. The following real deformation classification of real degree 1 del Pezzo surfaces is well known (a proof can be found, for example, in [DIK]). From here on we use notation \( T^2 \) for a 2-torus and \( K \) for a Klein bottle.

2.3.1. Theorem. The deformation class of any real del Pezzo surface \( X \) of degree 1 is determined by the topology of \( X_\mathbb{R} \). There are 11 deformation classes. They correspond to the following topological types of \( X_\mathbb{R} : \mathbb{R}P^2 \#(4-a)T^2 \) with \( 0 \leq a \leq 4 \), \( \mathbb{R}P^2 \sqcup aS^2 \) with \( 1 \leq a \leq 4 \), \( \mathbb{R}P^2 \sqcup K \), and \( (\mathbb{R}P^2 \# T^2) \sqcup S^2 \).
and two of them, denoted \((M - 2)Ia\) and \((M - 2)Ib\), are of type I, which means that the fundamental class of \(X_\mathbb{R}\) is realizing 0 in \(H_2(X; \mathbb{Z}/2)\).

The real deformation classes of sextics \(C \subset Q\) that arise as branching locus for \(X \to Q\) are listed in Table 2 (for a proof see, for example, [DIK]). The code \(\langle \rangle \langle \rangle\) refers to \(C_\mathbb{R}\) having three “parallel” connected components \(embracing the vertex v\) of \(Q\) (see Figure 1(f)). The code \(\langle a|b\rangle\), with \(a \geq 0, b \geq 0\) means that \(C_\mathbb{R}\) contains one component which embraces the vertex and \(a + b\) components which bound disjoint discs and placed in \(Q_\mathbb{R}\) so that \(a\) of them are separated from the remaining \(b\) by the embracing component and the vertex. The components bounding a disc are called ovals.

**Table 2.** Correspondence between two deformation classifications

| Smaller type of \(X_\mathbb{R}\) | \(M\) \(I\) | \(M - 2\) \(I\) | \(M - 3\) \(I\) | \(M - 4\) \(I\) | \(M - 2\) \(Ia\) | \(M - 2\) \(Ib\) |
|----------------------------------|--------------|----------------|----------------|----------------|----------------|----------------|
| \(H_2(X) \cap K_\mathbb{R}\)  | \(0\)         | \(A_1\)       | \(2A_1\)       | \(3A_1\)       | \(A_1\)       | \(2A_1\)       |
| \(X_\mathbb{R}\)                 | \(\mathbb{RP}^2 \# 4\mathbb{S}^2\) | \(\mathbb{RP}^2 \# 3\mathbb{S}^2\) | \(\mathbb{RP}^2 \# 2\mathbb{S}^2\) | \(\mathbb{RP}^2 \# \mathbb{S}^2\) | \(\mathbb{RP}^2 \# \mathbb{R}^2\) | \(\mathbb{RP}^2 \# \mathbb{S}^2\) |
| \(X_\mathbb{R}\)                 | \(\mathbb{RP}^2 \# 4\mathbb{S}^1\) | \(\mathbb{RP}^2 \# 3\mathbb{S}^1\) | \(\mathbb{RP}^2 \# 2\mathbb{S}^1\) | \(\mathbb{RP}^2 \# \mathbb{S}^1\) | \(\mathbb{RP}^2 \# \mathbb{R}^2\) | \(\mathbb{RP}^2 \# \mathbb{S}^1\) |

2.3.2. Theorem. A real non-singular sextic \(C\) on \(Q\) not passing through the vertex is determined up to real deformations of \(C\) in \(Q\) and central symmetries of \(Q\) by the topological type of the pair \((Q_\mathbb{R}, C_\mathbb{R})\). There are 7 equivalence classes of such sextics up to deformation and central symmetry. They correspond to the following arrangements of \(C_\mathbb{R}\) on \(Q_\mathbb{R}\): \(\langle a|0\rangle\) with \(0 \leq a \leq 4\), \(\langle 1|1\rangle\), and \(\langle |||\rangle\).

By central symmetries of \(Q\) here we mean automorphisms of \(Q\) induced by real projective involutions that fix the vertex \(v \in Q\) and a hyperplane not passing through \(v\).

2.3.3. Remark. If in Theorem 2.3.2 we exclude central symmetries, then the number of classes becomes 11, since then, for each \(1 \leq a \leq 4\), we will be obliged to distinguish \(\langle a|0\rangle\) and \(\langle 0|a\rangle\).

2.3.4. Lemma. For any real non-singular sextic \(C \subset Q \setminus \{v\}\), there exists a real degeneration of \(C\) which contracts simultaneously all the ovals of \(C_\mathbb{R}\).

Proof. Theorem 2.3.2 implies that it is sufficient to prove existence of a real sextic \(C_0 \subset Q \setminus \{v\}\) which consists of a real component embracing the vertex and 2 solitary points separated by this component, as well as existence, for each \(1 \leq a \leq 4\), of a sextic consisting, besides the embracing component, of \(a\) solitary points not separated by it.
To construct such sextics, we start from a real plane quartic $B$ having either (a) one oval and 1 solitary point inside it, or (b) one oval and $a - 1$ solitary points outside it. Next, we pick a generic real support line $L$ to the oval of $B$, which intersects $B$ at a real point with multiplicity 2 and has two imaginary complex conjugate common points with $B$. We perform a double blow-up of the plane at the tangency point, $B \cap L$, so that the proper images $\hat{L}$ and $\hat{B}$ of $L$ and $B$ become disjoint. Then $\hat{L}$ is a real $(-1)$-curve, while the proper image $\hat{E}$ of the exceptional curve $E$ of the first blow-up is a real $(-2)$-curve. The contraction of $\hat{L}$ and $\hat{E}$ gives a real surface isomorphic to $Q$ and transforms $\hat{B}$ to a real sextic consisting of an embracing component and 2 solitary points separated by it in the case (a), or an embracing component and $a$ not separated solitary points in the case (b) (in both cases, the additional solitary point is given by the image of $\hat{L} \cap \hat{B}$).

As for the initial quartic $B$, it can be obtained, for instance, by a small perturbation of a suitable reducible quartic: $(x^2 + y^2)(x^2 + y^2 - z^2) + \varepsilon(x^4 + y^4) = 0$ in the case (a) and $F_1^2 + F_2^2 + \varepsilon G_1 G_2 = 0$ in the case (b), $0 < |\varepsilon| \ll 1$. Here $F_1, F_2, G_1, G_2$ are real conics chosen so that $F_1, F_2$ intersect each other at 4 real points $p_0, \ldots, p_3$, while $G_1, G_2$ pass through $p_1, \ldots, p_{a-1}$ and have $\varepsilon G_1 \cdot G_2 > 0$ at $p_a, \ldots, p_3$ and $< 0$ at $p_0$. \hfill \Box

2.4. **Pin$^-$-structures and quadratic functions.** We send a reader to [KiTa] concerning generalities on Pin$^-$-structures and recall here just a few key points.

First of all, we permanently make use of the canonical correspondence between Pin$^-$-structures $\theta$ on a smooth 2-manifold $F$ and quadratic functions $q_\theta : H^1(F; \mathbb{Z}/2) \to \mathbb{Z}/4$, that is the functions satisfying $q_\theta(x + y) = q_\theta(x) + q_\theta(y) + 2(x \cdot y) \mod 4$. The set of Pin$^-$-structures on $F$ and that of quadratic functions on $H^1(F; \mathbb{Z}/2)$ have both a natural affine structure over $H^1(F; \mathbb{Z}/2)$ (in particular, the action of $\alpha \in H^1(F; \mathbb{Z}/2)$ on quadratic functions is given by sending $q_\theta(x)$ to $q_\theta(x) + 2\alpha(x)$) and the above correspondence is an affine mapping with respect to these affine structures.

As soon as a two-sided embedding $F \subset M$ of $F$ in a smooth 3-manifold $M$ is equipped with a coorientation, the descent rule ([KiTa], Lemma 1.6) gives us a natural descent map Pin$^-$($M$) $\to$ Pin$^-$($F$) between the sets of Pin$^-$-structures on $M$ and $F$. This map is compatible with the inclusion homomorphism $H^1(M; \mathbb{Z}/2) \to H^1(F; \mathbb{Z}/2)$ and the corresponding affine group actions.

The switch of coorientation rule ([KiTa], Lemma 1.10) implies that alteration of the coorientation of $F \subset M$ results in a shift of the descent map by $w_1 \in H^1(F; \mathbb{Z}/2)$, which in terms of the function $q_\theta$ is just alternation of sign.

2.4.1. **Lemma.** Let $M$ be a smooth 3-manifold equipped with a Pin$^-$-structure and $F \subset M$ be a two-sided 2-submanifold equipped with two opposite coorientations, $\xi$ and $\xi'$. Then, the quadratic functions $q, q' : H^1(F; \mathbb{Z}/2) \to \mathbb{Z}/4$ associated with the two Pin$^-$-structures induced on $F$ from the Pin$^-$-structure on $M$ with respect to $\xi$ and $\xi'$, respectively, are opposite: $q' = -q$.

**Proof.** Due to the switch of coorientation rule, $q'(x) = q(x) + 2w_1(x)$ and thus, $q(x) + q'(x) = 2(q(x) + w_1(x)) = 2q(x) + 2(x \cdot x) = q(x + x) = q(0) = 0 \mod 4$. \hfill \Box

We will be using also the following observation.
2.4.2. Lemma. If a $\Pin^{-}$-structure $\theta$ on a real algebraic surface $X_\mathbb{R}$ is induced from a $\Pin^{-}$-structure on an ambient 3-fold $Y_\mathbb{R} \supset X_\mathbb{R}$, then $q_0(Tv) = 0$ for any geometric real vanishing cycle $v \in H_\mathbb{R}^-_2(X)$.

Proof. Due to the functoriality of $\Pin^{-}$-structures, it is sufficient to check this statement for a standard local real smoothing $x^2 + y^2 = z^2 + \varepsilon^2$, $0 < |\varepsilon| \ll 1$, of a real nodal surface singularity $x^2 + y^2 = z^2$ in $\mathbb{R}^3$, which is straightforward. □

2.4.3. Remark. Alternation of sign claimed in Lemma 2.4.1 has (and even can be deduced from) a very simple visual interpretation. It is sufficient to treat just two cases: $F$ homeomorphic to an annulus and $F$ homeomorphic to a Möbius band. In the first case, $(F, \xi)$ is isotopic to $(F, \xi')$, so that $q = q'$, which implies $q = -q'$, since in this case $q$ and $q'$ take only even values. In the second case, $(F, \xi)$ is isotopic to $(F', \xi')$ obtained from $(F, \xi')$ by performing a full twist. The latter operation results in adding 2 (the number of half-twists) to the value of $q'$ on the core-circle of the Möbius band, which implies $q = -q'$, since $q$ and $q'$ take odd values on this core-circle.

2.5. Inside a weighted projective space. Using an identification of the quadratic cone $Q$ with the weighted projective plane $\mathbb{P}(1,1,2)$, the presentation of del Pezzo surfaces $X$ of degree 1 as double coverings of $Q$ (see Section 1.2) can be enriched by embeddings of $X$ into the weighted projective space $\mathbb{P}(1,1,2,3)$ (cf. Dol).

Namely, one considers the graded anti-canonical ring $R = \sum_{m \geq 0} H^0(X; -mK_X)$ together with its graded subring $R'$ generated by $H^0(X; -K_X)$ and $H^0(X; -2K_X)$. More precisely, $R'$ is generated by two elements in $H^0(X; -K_X)$ and one element of $H^0(X; -2K_X)$, and a choice of such three elements defines a double covering $X \to \mathbb{P}(1,1,2)$ which coincides with the classical bi-anticanonical model considered above.

To generate the whole ring $R$ one adds one element of $H^0(X; -3K_X)$ and obtains a natural embedding of $X$ into the weighted projective space $\mathbb{P}(1,1,2,3)$ as a non-singular degree 6 hypersurface defined by equation

\[(2.5.1) \quad w^2 = y^3 + p_2(x_0, x_1)y^2 + p_4(x_0, x_1)y + p_6(x_0, x_1)\]

where $p_{2k}$ are binary homogeneous polynomials of degree $2k$. This embedding is unique up to automorphisms of $\mathbb{P}(1,1,2,3)$.

When $X$ is equipped with a real structure, all the ingredients in this construction can be chosen defined over the reals. However, unlike in the classical model, for the second real structure in a Bertini pair, to make it defined over the reals too, one should either switch to another real structure on $\mathbb{P}(1,1,2,3)$ that is defined by $(x_0, x_1, y, w) \mapsto (\bar{x}_0, \bar{x}_1, \bar{y}, -\bar{w})$, or pick another embedding that presents $X$ by equation $-w^2 = y^3 + p_2(x_0, x_1)y^2 + p_4(x_0, x_1)y + p_6(x_0, x_1)$.

The both models are exhaustive: every non-singular degree 6 real hypersurface in $\mathbb{P}(1,1,2,3)$ as in the second model, as well as for every real double covering of $Q$ as in the first model, is a real del Pezzo surface of degree 1. These models are also functorial: every real automorphism (or a real deformation) of a real del Pezzo surface of degree 1 extends as a real automorphism (respectively, a real deformation) to each of the models.

The topology of the real locus of $\mathbb{P}(1,1,2,3)$ is described below in a bit more general setting.
2.5.1. Lemma. The weighted projective 3-space \( V = \mathbb{P}(1,1,p,q) \), for coprime \( p,q > 1 \), has precisely two singular points, \( v_p = (0,0,1,0) \), \( v_q = (0,0,0,1) \). Furthermore:

(1) For odd \( q \), the real locus \( V^R \) is topologically non-singular at the point \( v_q \), and, thus, on \( V^R \setminus \{v_p\} \) there exists a unique up to diffeomorphism smooth structure which agrees with the natural smooth structure on \( V^R \setminus \{v_p,v_q\} \).

(2) If in addition \( p \) is even, then \( V^R \setminus \{v_p\} \) is diffeomorphic to \( \mathbb{R}P^2 \times \mathbb{R} \).

(3) If \( q \) is odd and \( p \) is even, then on \( V^R \setminus \{v_p\} \) there exist precisely two \( \mathbb{P}^n \)-structures, which differ by the class \( w_1(V^R \setminus \{v_p\}) \).

Proof. Recall that \( V = \mathbb{P}(1,1,p,q) \) is the quotient of \( \mathbb{C}^4 \setminus \{0\} \) by the action of \( \mathbb{C}^* \) sending \( (x_0:x_1:x_2:x_3) \) to \( (\lambda x_0: \lambda x_1: \lambda^p x_2: \lambda^q x_3) \) which (for coprime \( p \) and \( q \)) is free at all points of \( \mathbb{C}^4 \setminus \{0\} \) except the points of two coordinate lines, \( x_0 = x_1 = x_2 = 0 \) and \( x_0 = x_1 = x_3 = 0 \), that therefore represent the only two singular points of the quotient, \( v_p \) and \( v_q \).

On the other hand, \( \mathbb{P}(1,1,p,q) \) can be viewed as the quotient of \( \mathbb{P}^3 = \mathbb{P}(1,1,1,1) \) by the action of \( G = \mathbb{Z}/p \times \mathbb{Z}/q \) sending \( [x_0:x_1:x_2:x_3] \) to \( [x_0:x_1:ax_2:bx_3] \), where \( a,b \) are roots of unity, \( a^p = b^q = 1 \). Thus, if \( p \) and \( q \) are odd, then the real locus of \( \mathbb{P}^3 \) projects bijectively onto the real locus \( V^R \) of \( V = \mathbb{P}^3/G \), and thus the both points, \( v_p \) and \( v_q \), are topologically non-singular in \( V^R \). If \( p \) is even and \( q \) is odd, then the quotient \( \mathbb{P}^3/G \) gives only a half of the real locus \( V^R \). The other half is the quotient of a "twisted" real 3-space

\[
(1,1,c,1) \cdot \mathbb{P}^3 = \{[x_0:x_1:cx_2:x_3] | x_0,x_1,x_2,x_3 \in \mathbb{R}\} \subset \mathbb{P}^3,
\]

where \( c \) is a primitive root of 1 of degree 2p. This twisted real 3-space is indeed the fixed-point set of the twisted complex conjugation involution \( (x_0,x_1,x_2,x_3) \mapsto (\bar{x}_0,\bar{x}_1,c^2\bar{x}_2,\bar{x}_3) \). These two halves of \( V^R \) are affine cones with a common boundary \( \mathbb{P}^2 \setminus v_p \) and a common vertex at \( v_p \). This proves (1) and (2) (uniqueness of differential-topological smoothings in dimension 3 is a well known general phenomenon).

Claim (3) follows from vanishing of the class \( w_1^2 + w_2 = 0 \) for \( \mathbb{R}P^2 \times \mathbb{R} \) and \( H^1(\mathbb{R}P^2 \times \mathbb{R}; \mathbb{Z}/2) = \mathbb{Z}/2 \).

3. Proof of main theorems

3.1. Spanning root eigenlattices by vanishing cycles and geometric roots.

3.1.1. Lemma. If \( X \) is a real del Pezzo surface of degree 1 with \( X^R = \mathbb{R}P^2 \# (4-a)\mathbb{T}^2 \), then it admits a system of real geometric vanishing cycles with Coxeter-Dynkin diagram \( E_a \) for \( a = 0 \), \( E_7 \) for \( a = 1 \), \( D_6 \) for \( a = 2 \), and \( D_4 \) for \( a = 3 \).

In the case of \( X^R = (\mathbb{R}P^2 \# \mathbb{T}^2) \sqcup S^2 \), it admits a system of real geometric vanishing cycles of type \( D_4 \).

Proof. It is sufficient to check the statement for one surface in each of the five deformation classes involved.

We start with a singular sextic \( C_0 \subset Q \) on the cone \( Q \) splitting into three conic sections as on Figure 4(a) (where \( Q^R \) is sketched as a cylinder). A double covering over \( Q \) branched along \( C_0 \) is a singular (nodal) surface \( X_Q \). A perturbation of \( C_0 \) that smooths the nodes of \( C_0 \) like on Figure 4(b) leads to an M-sextic \( C \) for which the double covering \( X \to Q \) branched along \( C \) is a del Pezzo surface of degree 1 with \( X^R = \mathbb{R}P^2 \# 4\mathbb{T}^2 \). Each of the six nodes of \( C_0 \) gives a bridge in \( C \) (a purely imaginary vanishing 1-cycle) intersecting a pair of real components of \( C^R \). Each
Figure 1. Construction of real sextics by perturbation from a union of 3 conics: (b)–(f) from (a), and (h) from (g).

bridge bounds in $Q$ a disc on which the complex conjugation acts as a reflection in its diameter shown on Figure 1 as an interval joining the ovals, and it is the inverse image of such a disc that represents a real geometric vanishing cycle in $X$. On the other hand, according to Lemma 2.3.4, all the four components of $C_R$ that bound discs in $Q_R$ can be contracted simultaneously to 4 separate points, which shows that the inverse images of these 4 discs represent each a real geometric vanishing cycle in $X$. The Coxeter-Dynkin graph of the described 10 real geometric vanishing cycles is as shown at the bottom of Figure 1(b), where black vertices correspond to the ovals and white to the bridges. After dropping two of these cycles as it is shown there, we obtain the $E_8$-diagram.

To treat the other 4 cases, we use the perturbations shown in Figures 1(c,d,e,h) and apply literally the same arguments as above. □

3.1.2. Lemma. For every degree 1 real del Pezzo surface $(X, \conj)$ with $X_R \neq \mathbb{RP}^2 \sqcup K$, the lattice $H_2^+(X, K_X^\perp)$ admits a Coxeter basis of roots, $\mathcal{B}$, which has the following properties:

1. its Coxeter-Dynkin diagram is of the type shown in Table 1,
2. each element of $\mathcal{B}$ is either a real geometric vanishing cycle, or belongs to $(1 - \conj_\ast)H_2(X)$;
3. the linear span of $\Upsilon(\mathcal{B})$ is the kernel of $w_1(X_R) : H_1(X_R; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$.

Proof. As it follows from Lemma 3.1.1 and Table 1 if a component of the Coxeter-Dynkin diagram of $H_2^+(X, K_X^\perp)$ is different from $A_1$, then the corresponding summand of $H_2^+(X, K_X^\perp)$ is realized by real geometric vanishing cycles. On the other hand, the $A_1$-summands are contained in $(1 - \conj_\ast)H_2(X)$, due to (2.2.3). This shows (1) and (2).

To prove (3) note that each element of $(1 - \conj_\ast)H_2(X)$ can be represented as a union of transversely intersecting oriented surfaces $S$ and $-\conj(S)$. An equivariant smoothing of the intersection points produces an embedded closed oriented surface $F \subset X$ which is invariant under $\conj$ and has as the real locus $F \cap X_R$ a collection...
of small circles around the real intersections of $S$ with $\text{conj}(S)$. Hence, $F \cap X_\mathbb{R}$ is null-homologous, which implies (3) if $X_\mathbb{R}$ is just a union of $\mathbb{R}P^2$ and spheres.

For proving (3) in the remaining cases, note that for each real geometric vanishing cycle $v$ we have $w_1(X_\mathbb{R}) \circ \mathcal{Y} v = K_X \circ v \mod 2 = 0$, due to (2.2.1). For each pair $v_i, v_j$ of real geometric vanishing cycles we have similarly $\mathcal{Y} v_i \circ \mathcal{Y} v_j = v_i \circ v_j \mod 2$. Therefore, $\mathcal{Y}(B)$ is contained in the kernel of $w_1(X_\mathbb{R}) : H_1(X_\mathbb{R}; \mathbb{Z}/2) \to \mathbb{Z}/2$ and generate there a subspace whose dimension is equal to the rank of the non-degenerate part of the $\mathbb{Z}/2$-lattice $(H_2^+(X) \cap K_\mathbb{R}^+) \otimes \mathbb{Z}/2$. To check that the rank of the latter is equal to the rank of the former kernel, it remains to look through Table 1 and to notice that the rank of the non-degenerate part of the $\mathbb{Z}/2$-lattices $E_8 \otimes \mathbb{Z}/2$, $E_7 \otimes \mathbb{Z}/2$, $D_6 \otimes \mathbb{Z}/2$, and $D_4 \otimes \mathbb{Z}/2$ is equal to 8, 6, 4, and 2, respectively. □

3.2. Choice of Pin$^-$-structure on $X_\mathbb{R}$. Consider now an arbitrary real del Pezzo surface $X$ of degree 1 and its real graded anticanonical embedding $X \subset V = \mathbb{P}(1,1,2,3)$ as a hypersurface given by equation (2.5.1). Note that $X$ in $V$ does not contain the two singular points $v_2, v_3 \in V$ and the normal bundle to $X_\mathbb{R}$ in $V_\mathbb{R}$ is trivial, since, in terms of equation (2.5.1), $X_\mathbb{R}$ bounds in $V_\mathbb{R}$ a domain $W$ defined by inequality $w^2 \leq y^3 + p_2(x_0, x_1)y^2 + p_4(x_0, x_1)y + p_6(x_0, x_1)$. So, applying the descent map to the two Pin$^-$-structures on $V_\mathbb{R} \setminus \{v_2\}$ as in Lemma 2.5.1 with a coorientation of $X_\mathbb{R}$ given by the normal vector field of $X_\mathbb{R}$ directed outside of $W$, we get two natural induced Pin$^-$-structures on $X_\mathbb{R}$.

Among these two Pin$^-$-structures on $X_\mathbb{R}$ we choose and call monic the one whose quadratic function $g_X : H_1(X_\mathbb{R}; \mathbb{Z}/2) \to \mathbb{Z}/4$ takes value 1 on the class $w_1^* \in T$ dual to $w_1(X_\mathbb{R})$. For the other Pin$^-$-structure, due to Lemma 2.4.1 the quadratic function on $w_1^*$ will take the opposite value. This implies that the chosen Pin$^-$-structure on $X_\mathbb{R}$ does not depend on the graded anticanonical embedding considered.

3.3. Proof of Theorem 1.2.1. The monic Pin$^-$-structure on $X_\mathbb{R}$ satisfies property [1], since any real automorphism of $X$ is induced by a real automorphism of $V$ and any real automorphism or monodromy of $X$ must preserve $w_1(X_\mathbb{R})$. Our choice of the Pin$^-$-structure gives $g_\mathcal{Y}(w_1^*) = 1$, which together with Lemma 2.4.2 gives property [2].

Property [3] in the case of $X_\mathbb{R} \neq \mathbb{R}P^2 \sqcup K$ follows from [1] and [2], since the only non-oval component of the sextic $C_\mathbb{R}$ represents in $H_1(X_\mathbb{R}; \mathbb{Z}/2)$ an element dual to $w_1(X_\mathbb{R})$, while all the other components represent the image by $\mathcal{Y}$ of real geometric vanishing cycles (see Lemma 2.3.4). In the case $X_\mathbb{R} = \mathbb{R}P^2 \sqcup K$, we justify claim [3] in Lemma 3.3.1 below.

As for the uniqueness of a Pin$^-$-structure with the properties [1]–[3], the case of $X$ with $X_\mathbb{R} = \mathbb{R}P^2 \sqcup kS^2, k \geq 0$, is trivial, since $X_\mathbb{R}$ carries only two Pin$^-$-structures and only for one of them $g_X(w_1^*) = 1$. If $X_\mathbb{R} = \mathbb{R}P^2 \#(4 - a)T^2$ with $a \leq 3$, then the uniqueness follows from Lemma 3.1.2 and vanishing of $\mathcal{Y}$ on $(1 - \text{conj}^*)H_2(X)$.

In the remaining case of $X_\mathbb{R} = \mathbb{R}P^2 \sqcup K$, we note first that the real locus $C_\mathbb{R}$ of the sextic $C \subset Q$ consists of 3 connected components forming a nest surrounding the vertex of $Q_\mathbb{R}$. Let us index these connected components as $C_1, C_2, C_3$, so that $C_1, C_2$ bound an annulus in $Q_\mathbb{R}^+$ while $C_2, C_3$ bound an annulus in $Q_\mathbb{R}^-$. Note that $C_1 + C_2 + C_3$ represents both in $H_1(X_\mathbb{R}^+; \mathbb{Z}/2)$ and $H_1(X_\mathbb{R}^-; \mathbb{Z}/2)$ an element dual to $w_1(X_\mathbb{R}^\pm)$, while $C_1 + C_2$ is homologous to a real geometric vanishing cycle (a bridge between $C_1, C_2$) in $H_1(X_\mathbb{R}^+; \mathbb{Z}/2)$ and $C_2 + C_3$ to a real geometric vanishing cycle (a
bridge between $C_2, C_3$ in $H_1(X_{\mathbb{R}}; \mathbb{Z}/2)$. Therefore, the properties (1)–(3) leave only one possibility $q^\perp(C_1) = 1, q^\perp(C_2) = -1, q^\perp(C_4) = 1$. This gives the uniqueness of the Pin$^-$-structure, since the classes $[C_i], i = 1, 2, 3,$ generate $H_1(X_{\mathbb{R}}; \mathbb{Z}/2)$.

3.3.1. Lemma. If $X_{\mathbb{R}} = \mathbb{R}P^2 \sqcup \mathbb{K}$, then the property (3) of Theorem 1.2.1 is satisfied for $q_X$.

Proof. Recall that in our topological model $V_{\mathbb{R}} \setminus \{v_2\} \cong \mathbb{R}P^2 \times \mathbb{R}$, the factor $\mathbb{R}P^2$ is the real locus of the coordinate plane $\mathbb{P}(1, 1, 3) \subset \mathbb{P}(1, 1, 2, 3)$ defined by $x_2 = 0$. The real locus $\mathbb{P}_1^1$ of the coordinate line $\mathbb{P}^1 = \{x_2 = x_3 = 0\}$ lies in $\mathbb{R}P^2$ as a topological line (that is one-sided) and does not contain $v_1$. Another coordinate plane, $Q = \mathbb{P}(1, 1, 2), has punctured real locus $Q_{\mathbb{R}} \setminus \{v_3\} \cong \mathbb{P}^1 \times \mathbb{R}$. The intersection of $Q_{\mathbb{R}}$ with the domain $W = \{w^2 \leq y^3 + p_2(x_0, x_1)y^2 + p_4(x_0, x_1)y + p_6(x_0, x_1)\} \subset V_{\mathbb{R}}$ splits into the union of an annulus (between $C_1$ and $C_2$) and a region bounded by $C_3$. The Möbius bands $B_1, B_3$ lying on $X_{\mathbb{R}}$ and having $C_1, C_3$ as core and cooriented by $\nu$ are isotopic to the Möbius bands $B_2 \subset X_{\mathbb{R}}$ having $C_2$ as a core and cooriented by $-\nu$. Hence, by Lemma 2.4.1 the values of $q_X$ on $C_1, C_2, C_3$ are alternating. This implies the statement, since $w^1_1(X_{\mathbb{R}}) = C_1 + C_2 + C_3$.

3.3.2. Remark. For a graded anticanonical embedding of $X_{\mathbb{R}}$ into $V = \mathbb{P}_{\mathbb{R}}(1, 1, 2, 3)$ as a hypersurface $w^2 = y^3 + p_2(x_0, x_1)y^2 + p_4(x_0, x_1)y + p_6(x_0, x_1)$, the monic Pin$^-$-structure on $X_{\mathbb{R}}$ is obtained by the descent map from the Pin$^-$-structure on $V_{\mathbb{R}} \setminus \{v_2\} \cong \mathbb{P}_{\mathbb{R}}(1, 1, 3) \times \mathbb{R}, \mathbb{P}_{\mathbb{R}}(1, 1, 3) \cong \mathbb{R}P^2,$ which can be described “independently of $X$” as the one whose descent to $\mathbb{P}_{\mathbb{R}}(1, 1, 3) = \{y = 0\} \cap \mathbb{P}_{\mathbb{R}}(1, 1, 2, 3)$, equipped with coorientation directed outward the domain $y \geq 0$ in $V_{\mathbb{R}}$, gives value $q([\mathbb{P}_{\mathbb{R}}(1, 1)]) = 1$ on the coordinate line $P_{\mathbb{R}}(1, 1) = \{w = 0\} \cap \mathbb{P}_{\mathbb{R}}(1, 1, 3)$ (dual to $w^1_1(\mathbb{P}_{\mathbb{R}}(1, 1, 3))$). It is then straightforward to check (using Lemma 3.3.1 if $X_{\mathbb{R}} = \mathbb{R}P^2 \sqcup \mathbb{K}$ and trivial in the other cases) that the descent of this primary Pin$^-$-structure on $V_{\mathbb{R}} \setminus \{v_2\}$ to $X_{\mathbb{R}}$, equipped with the coorientation directed outward $W = \{w^2 \leq y^3 + p_2(x_0, x_1)y^2 + p_4(x_0, x_1)y + p_6(x_0, x_1)\}$, satisfies the relation $q_X(w^1_1) = 1$.

3.4. Signed count of real lines. We let $\tilde{q}(x) = q_X(Y(x)) \in \mathbb{Z}/4$ for all $x \in H_2^2(X)$ and set $s : I_{\mathbb{R}}(X) \to \{+1, -1\}, s(l_e) = i\tilde{q}(e)$ (which is well-defined due to Corollary 2.1.2). Note, that $s(l_e) = \text{Im}(\tilde{q}(l_e))$ since $l_e = -K_X - e$ and $\tilde{q}(K_X) = 1$.

3.4.1. Lemma. Any real del Pezzo surface $X$ of degree 1 with $X_{\mathbb{R}} \cong \mathbb{R}P^2 \sqcup \mathbb{K}$ admits real roots $e_i \in R_{\mathbb{R}}(X), i = 1, 2, 3, 4$ that form in the $D_4$-lattice $H_2^2(X) \cap K_X$ a Coxeter basis of roots such that $\tilde{q}(e_i) = 0$ for all $i$.

Proof. Figure 1(f) shows four disjoint real vanishing cycles $V^1, \ldots, V^4 \subset X$ whose real loci $V^i_\mathbb{R}$ are meridians of $\mathbb{K}$. Let $e_i = [V^i] \in H_2^2(X).$ By Theorem 1.2.1 $\tilde{q}([V^i]) = 0$. For any quadruple of pairwise non-intersecting roots in a $D_4$-lattice, their sum is divisible by 2, and so $e_0 = -\frac{1}{2}(e_1 + \cdots + e_4)$ belongs to the lattice and therefore, together with a triple $e_1, e_2, e_3$, forms a Coxeter basis. Another basis is formed by the same triple and $e_4' = e_0 + e_4$. Then $\tilde{q}(e_0') = \tilde{q}(e_0) + \tilde{q}(e_4') + 2 = \tilde{q}(e_0) + 2$ and, thus, one of these two bases satisfies the conditions of Lemma. □
3.4.2. Proposition. The root system $R_\mathbb{R}(X)$ of any real del Pezzo surface $X$ of degree 1 admits a Coxeter basis $B$ such that $\hat{q}(e) = 0$ for all $e \in B$.

Proof. If follows from Lemma 3.4.1 if $X_\mathbb{R} \cong \mathbb{R}P^2$ and from Lemma 3.1.2 in the other cases, since $\hat{q}(e) = 0$ for vanishing cycles by Theorem 1.2.1(2), and for $e \in (1 - \text{conj}_s)H_2(X)$ from (2.2.2) in Proposition 2.2.1 \hfill \Box

3.4.3. Lemma. Assume that $X$ is a real del Pezzo surface of degree 1 and $B \subset R_\mathbb{R}(X)$ is a Coxeter basis such that $\hat{q}(e) = 0$ for all $e \in B$. Then, for any pair of real roots $f, g \in R_\mathbb{R}(X)$ which are positive with respect to $B$ and adjacent in the corresponding Hasse diagram, the values $\hat{q}(f), \hat{q}(g) \in \{0, 2\} \subset \mathbb{Z}/4$ are distinct, and therefore, we have $s(l_f) + s(l_g) = 0$.

Proof. We may suppose that $f < g$ in the Hasse diagram, then $g = f + e$ with $e \in B$, $f \cdot e = 1$. It gives $\hat{q}(f) = \hat{q}(g + e) = \hat{q}(g) + \hat{q}(e) + 2q \cdot e = \hat{q}(g) + 2 \in \mathbb{Z}/4$. Since $\hat{q}(l_f) = \hat{q}(f + K_X) = \hat{q}(f) + q(K_X) = \hat{q}(f) + 1$ and similarly $\hat{q}(l_g) = \hat{q}(g) + 1$, we conclude that $s(l_f) + s(l_g) = 0$. \hfill \Box

3.4.4. Lemma. Let $L$ be a root lattice $E_8, E_7, D_6, D_4, A_1$, or a direct sum of such lattices, and let $L^+$ be the poset of its positive roots determined by a choice of a Coxeter basis $B \subset L$ of roots. Then $L^+ \setminus B$ can be split into pairs of adjacent roots in the Hasse diagram of $L^+$.

Proof. For $E_8$, the Hasse diagram and its splitting are shown on Fig. 2. For $E_7, D_6$, and $D_4$, whose Hasse diagrams are shown on Fig. 3 appropriate splittings are completely similar. For $A_1$, the statement is void. At last, the Hasse diagram of a direct sum is a disjoint union of Hasse diagrams of the summands. \hfill \Box

Figure 2. The Hasse diagram of $E_8$ and its splitting subdiagram

3.4.5. Proposition. For any degree 1 real del Pezzo surface $X$ the signed count of its real lines gives twice the rank of the real root system:

$$\sum_{l \in L_\mathbb{R}(X)} s(l) = 2\text{rk}(R_\mathbb{R}(X)) = 2(\text{rk}(H_2^-(X)) - 1).$$
**Figure 3.** Hasse Diagrams for $E_7$, $D_6$ and $D_4$

**Proof.** Using Corollary 2.1.2 and $s(l-e) = s(l-e)$ (since $\hat{q}(-e) = \hat{q}(e)$) we rewrite the sum

$$\sum_{l \in L_R(X)} s(l) = \sum_{e \in R_R(X)} s(l_e) = 2 \sum_{e \in R^+_R(X)} s(l_e)$$

where $R^+_R(X)$ denotes the set of positive roots with respect to a Coxeter basis $B \subset R_R(X)$. Due to Proposition 3.4.2 we may choose $B$ so that $\hat{q}(e) = 0$, and, thus, get $s(l_e) = 1$ for all $e \in B$. By Lemma 3.4.4 the roots in $R^+_R(X) \setminus B$ split into pairs of adjacent ones, and each pair contributes 0 into the above sum by Lemma 3.4.3. Thus,

$$\sum_{e \in R^+_R(X)} s(l_e) = \sum_{e \in B} s(l_e) = \text{rk}(R_R(X)).$$

\[ \square \]

3.5. **Proof of Theorem 1.2.2.** According to Proposition 3.4.5 we have

$$h(X^+) - e(X^+) + h(X^-) - e(X^-) = 2(\text{rk}(R_R(X^+)) + \text{rk}(R_R(X^-))).$$

On the other hand

$$\text{rk}(R_R(X^+)) + \text{rk}(R_R(X^-)) = \text{rk}(K^X \cap H^+_2(X)) + \text{rk}(K_X \cap H^+_2(X)) = \text{rk}(K^X) = 8.$$

3.5.1. **Remark.** On each real del Pezzo surface of degree 1 with $X_R$ containing a component of non-positive Euler characteristic, there exist real roots which are not real geometric vanishing cycles. Indeed, as it follows from Proposition 3.4.2 Lemma 3.4.3 and the Hasse diagrams shown on Figures 2-3 on any such surface there exist real roots $v$ with $q_X(\Upsilon(v)) = 2$.

3.6. **Separate count of hyperbolic and elliptic lines.** Combining Proposition 3.4.5 with the list of eigenlattices in Table 1 and the known values for the number of real lines in each deformation class (the third row in Table 3) we obtain the values of $h$ and $e$ separately (two last rows in Table 3).

In the case $X_R = \mathbb{R}\mathbb{P}^2 | \perp K$ the curve $C_R \subset Q_R$ has arrangement $\langle ||| \rangle$. Using consecutive numeration of the components $C_1, C_2, C_3$ of $C_R$ as in the proof of Lemma 3.3.1 and the alternation of values of $q_X(C_i)$ established there, it is easy to deduce...
that each of the 8 elliptic lines has odd number of common points (counting multiplicities) with $C_2$, and even number with $C_1$ and $C_3$. Among the 16 hyperbolic lines precisely 8 has odd intersection with $C_1$ and even with $C_2$, $C_3$, while the other 8 has odd intersection with $C_3$ and even with $C_1$, $C_2$. In particular, 8 hyperbolic lines are contained in the $\mathbb{RP}^2$-component of $X_R$ and the remaining 16 lines in the $\mathbb{K}$-component.

4. FROM LINES TO TRITANGENTS

4.1. Signed count of tritangents. We say that a plane $\Pi \subset \mathbb{P}^3$ transverse to $Q$, along with the plane section $A = \Pi \cap Q$, are tritangent to a sextic $C \subset Q$ if the intersection divisor $\Pi \circ C = A \circ C$ contains each point with even multiplicity.

We denote by $T(Q, C)$ the set of such tritangent sections. It contains, as is well-known, 120 elements, and the projection $L(X) \to T(Q, C)$ induced by the double covering $\pi : X \to Q$ is a two-to-one map.

Over $\mathbb{R}$, for each real $A \in T(Q, C)$ its real locus $A_{\mathbb{R}}$ lies either in $Q^+_{\mathbb{R}}$ or in $Q^-_{\mathbb{R}}$ and so lifts to a pair of real lines on $X^+_R$ or on $X^-_R$ respectively. We denote by $T_{\mathbb{R}}(Q^\pm, C)$ the corresponding sets of real tritangents and obtain induced two-to-one maps $L_{\mathbb{R}}(X^\pm) \to T_{\mathbb{R}}(Q^\pm, C)$.

By Theorem [1.2.1], for each $A \in T_{\mathbb{R}}(Q^\pm, C)$, the both real lines that form $\pi^{-1}(A)$ are of the same type. This allows to split real tritangents in two species also. Namely, we call a real tritangent hyperbolic and count it with sign plus (respectively, call elliptic and count with sign minus), if the tritangent section lifts to a pair of real hyperbolic lines (respectively, elliptic lines). Theorem [1.2.2] implies then the following result.

4.1.1. Theorem. Assume that a real sextic curve $C \subset Q$ is a transversal intersection of a real quadratic cone $Q \subset \mathbb{P}^3$ (whose base is non-singular and has non-empty real locus) with a real cubic surface. Then the difference between the numbers of hyperbolic and elliptic tritangent sections in $T_{\mathbb{R}}(Q^+, C) \cup T_{\mathbb{R}}(Q^-, C)$ is 8. $\square$

A Combination of Theorem [4.1.1] with Table 3 yields a separate count of hyperbolic and elliptic tritangents for each real deformation class of real sextics $C \subset Q$.

4.1.2. Remark. The division of tritangent sections into hyperbolic and elliptic does not extend from sextics on a quadratic cone $Q$ to those on non-singular real quadric surfaces. To see it, we can take a sextic $C_R \subset Q_R$ of type $2[0]$ (see Subsection 2.3) and pick real tritangent planes $\Pi_1, \Pi_2$ touching each of the 3 components of $C_R$ as shown on Figure 4, so that one of them is hyperbolic and another is elliptic.
Table 4. Count of real tritangent section

| Arrangement of $C_R$ in $Q_R$ | $\langle 4|0 \rangle$ | $\langle 3|0 \rangle$ | $\langle 2|0 \rangle$ | $\langle 1|0 \rangle$ | $\langle 0|0 \rangle$ | $\langle ||| \rangle$ | $\langle 1|1 \rangle$ |
|-----------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| total number                | 120            | 64             | 32             | 16             | 8              | 24             | 24             |
| hyperbolic                  | 64             | 36             | 20             | 12             | 8              | 16             | 16             |
| elliptic                    | 56             | 28             | 12             | 4              | 0              | 8              | 8              |

A small real perturbation of $Q$ into a real ellipsoid $Q'$ can be accompanied by a real deformation of $C_R$ and $\Pi_1, \Pi_2$ into a real sextic $C'_R \subset Q'_R$ and its two real tritangents $\Pi'_1, \Pi'_2$ that meet each of the 3 ovals of $C'_R$. Finally, it is not difficult to notice that these tritangents can be transformed to each other by a real deformation of $C'_R$ in $Q'_R$.

**Figure 4.** Two tritangent sections of opposite sign

4.2. **Identification of the sign via coorientation.** A coorientation of a sextic curve $C_R \subset Q_R$ is determined by choosing one of the two regions, $Q^+_R$ or $Q^-_R$, bounding $C_R$. Similarly, to coorient $A_R$ in $Q_R$ for $A \in T_R(Q^{\pm}, C)$, we choose one of the two nappes of the cone $Q_R$ bounded by $A_R$. In both cases, the direction of coorientation is outward the bounding region.

We say that chosen in such a way coorientations of $C_R$ and $A_R$ are coherent, if the corresponding region $Q^+_R$ approaches the vertex of $Q_R$ along the nappe chosen to coorient $A_R$. A tangency point of $A_R$ with $C_R$ is called positive if coherent coorientations at this point coincide and negative otherwise.

4.2.1. **Remark.** If $C_R$ has arrangement different from $\langle ||| \rangle$, then any tangency point with the component $C_0 \subset C_R$, embracing the vertex in $Q_R$, is positive. A tangency with an oval, $C_i \subset C_R$ is negative if $A_R$ separates $C_i$ and $C_0$ inside $Q^+_R$ and positive otherwise.

4.2.2. **Theorem.** A section $A \in T_R(Q^+, C) \cup T_R(Q^-, C)$ is hyperbolic (respectively, elliptic) if the number of their positive tangency points is odd (respectively, even). Accordingly, a real line $l \in L_R(X^{\pm})$ is hyperbolic (respectively, elliptic) if its projection $A = \pi(l) \in T_R(Q^{\pm}, C)$ is hyperbolic (respectively, elliptic).

**Proof.** As before, let $C$ be defined in $Q = \mathbb{P}(1, 1, 2)$ by a real equation $y^3 + p_2y^2 + p_4y + p_6 = 0$. Allowing $y$-coordinate change, we may assume that $A$ coincides with the coordinate line $\mathbb{P}(1, 1) = \{y = 0\}$ and that $A_R$ lies in the region $\{y^3 + p_2y^2 + p_4y + p_6 \geq 0\} \subset Q_R$. Then, for the covering real del Pezzo surface $V$ we may take as a model the hypersurface $\{u^2 = y^3 + p_2y^2 + p_4y + p_6\} \subset V = \mathbb{P}(1, 1, 2, 3)$.

Under such a coordinate choice, the coorientation of $A_R$ via the outward direction of $\{y \geq 0\} \subset Q_R$ is coherent with the coorientation of $C_R$ via the outward direction...
of \{y^3 + p_2y^2 + p_4y + p_6 \geq 0\} \subset Q_\mathbb{R}. Also, the both lines \(l_1^\ast, l_2^\ast \subset X_\mathbb{R}\) covering \(A_\mathbb{R}\) lie in \(\{y = 0\} \cap \mathbb{P}_\mathbb{R}(1, 1, 2, 3) = \mathbb{P}_\mathbb{R}(1, 1, 3) \cong \mathbb{R}P^2\).

Let \(\nu\) be a vector field along the smooth locus \(\mathbb{P}_\mathbb{R}(1, 1, 3) \setminus \{w_3\} \subset \mathbb{R}P^2\) directed outward the domain \(\{y \geq 0\} \subset \mathbb{P}_\mathbb{R}(1, 1, 2, 3)\). In accordance with Remark \ref{remark:3.3.2}, the primary Pin\(^+\) structure in \(\mathbb{P}_\mathbb{R}\) descends via field \(\nu\) to a Pin\(^−\) structure on \(\mathbb{R}P^2\) whose quadratic function \(q\) takes value \(q([A_\mathbb{R}]) = 1\). Then, \(q([l_2^\ast]) = 1\), since \(l_2^\ast\) are isotopic to \(A_\mathbb{R}\) in \(\mathbb{R}P^2\).

On the other hand, by definition, the monic Pin\(^−\) structure on \(X_\mathbb{R}\) is obtained by descend of the primary structure via vector field \(\nu_X\) normal to \(X_\mathbb{R} = \partial W \subset V_\mathbb{R}\) and outward-directed from \(W\). The fields \(\nu\) and \(\nu_X\) differ by some number, \(n_k\), of twists as we go along \(l_2^\ast\) and we have \(q_X(l_k^\ast) = q(l_k^\ast)\) if \(n_k\) is even and \(q_X(l_k^\ast) = -q(l_k^\ast)\) otherwise.

To find \(n_k\), note that \(\nu_X\) is collinear to \(\nu\) along \(C_\mathbb{R}\), so, collinearity on \(l_2^\ast\) happen precisely at the tangency points of \(A_\mathbb{R}\) with \(C_\mathbb{R}\). Furthermore, \(\nu_X\) is co-directed with \(\nu\) at the positive tangency points and oppositely directed for negative. Now, to complete the proof it remains to notice that the parity of \(n_k\) is the parity of the number of oppositely directed collinearities.

\[\square\]

### 4.3. The sign rule via resultants and real quadratic forms.

We can reformulate the coorienting rule of Theorem \ref{thm:4.2.2} in a more algebraic fashion, in terms of polynomials \(p_j\) from the equation \(y^3 + p_2(x_1, x_2)y^2 + p_4(x_1, x_2)y + p_6(x_1, x_2) = 0\) of a real non-singular sextic \(C \subset Q = \mathbb{P}(1, 1, 2)\), if the coordinates are chosen so that a given real tritangent section \(A \subset Q\) has equation \(y = 0\) (the convention from the proof of Theorem \ref{thm:4.2.2}). Like before, we may assume without loss of generality that \(A_\mathbb{R} \subset Q_\mathbb{R}^+\), or equivalently \(p_6 \geq 0\) (otherwise, substitute \(y \rightarrow -y\)). This means that \(p_6(x_0, x_1) = q_3^2(x_0, x_1)\) for some real cubic polynomial \(q_3\). Let \(c_1, c_2, c_3 \in \mathbb{P}^1\) be the 3 roots of \(q_3\) (they are either all real, or one is real and two are imaginary complex conjugate). Note that \(p_4(c_k) \neq 0\) for \(k = 1, 2, 3\), since otherwise the point \(c_k \in \mathbb{P}^1 = \mathbb{P}(1, 1) \subset Q = \mathbb{P}(1, 1, 2)\) would be singular for \(C\).

#### 4.3.1. Proposition. A real tritangent \(A\) is hyperbolic if \(p_4\) is positive at an odd number of real roots \(\{c_1, c_2, c_3\} \cap \mathbb{P}^1_\mathbb{R}\) (roots are counted with multiplicities) and elliptic otherwise.

**Proof.** If \(c_j\) is real, then, in appropriate local coordinates centered at \(c_j\), the curve \(C\) is defined by equation \(p_4(c_j)y + x^2 = 0\) (here we use the positivity of \(p_6\) on \(A_\mathbb{R}\)). Thus, for \(p_4(c_j) > 0\) (resp. \(p_4(c_j) < 0\)) the local branch of \(C_\mathbb{R}\) is contained in \(y \leq 0\) (resp. \(y \geq 0\)). Therefore, in the first case the coorientations introduced in Section \ref{sec:4.2} are coherent at the point \(c_j\), and they are opposite in the second case. It remains to combine this with Theorem \ref{thm:4.2.2}.

\[\square\]

#### 4.3.2. Corollary. If the resultant \(\text{Res}(p_4, q_3)\), where \(q_3^2 = p_6\), is positive (resp. negative) then the tritangent \(A = \{y = 0\}\) and the lines \(l_1, l_2\) covering this tritangent are hyperbolic (resp. elliptic).

The sign rule of Proposition \ref{prop:4.3.1} can be also reformulated in terms of a real symmetric conj-equivariant bilinear form \(\beta\) on a 3-dimensional real vector space

\[V = \{f : \{c_1, c_2, c_3\} \rightarrow \mathbb{C}\text{ such that }f \circ \text{conj} = f\},\]

\[\beta(f, g) = \sum_{j=1}^3 p_4(c_j)f(c_j)g(c_j).\]
4.3.3. Corollary. If a Gram determinant of $\beta$ is positive (resp. negative), then the tritangent $A$ and the lines $l_1, l_2$ covering $A$ are hyperbolic (resp. elliptic).

5. A few other directions

5.1. Signed count of 6-tangent conics to a symmetric plane sextic. Consider a non-singular plane sextic $S \subset \mathbb{P}^2$ which is invariant under the symmetry $s : \mathbb{P}^2 \to \mathbb{P}^2$, $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1 : -x_2]$, and transversal to the line $x_2 = 0$. Then it is easy to check that:

- $S$ does not pass through the center-point $[0 : 0 : 1]$,
- the quotient of $\mathbb{P}^2$ by $s$ is identified with $Q = \mathbb{P}(1, 1, 2)$ (making $y = x_2^2$ a coordinate for $Q$ of weight 2),
- the image of $S$ by the quotient map $\psi : \mathbb{P}^2 \to Q$ is a non-singular sextic $C \subset Q$, which does not pass through the vertex $v_2 \in Q$ and is transversal to the conic $\{y = 0\} \cap Q$.

Furthermore, every $s$-symmetric conic $B$ which is 6-tangent to $S$ is non-singular and its image $A = \psi(B)$ is a conic section tritangent to $C$. This correspondence gives a bijection between the set of tritangents to $C$ and the set of $s$-symmetric conics 6-tangent to $S$. All these properties hold over $\mathbb{R}$ as well, which, in particular, gives a bijection between the set of real tritangents to $C$ and the set of real $s$-symmetric 6-tangent conics to $S$. This bijection yields a splitting of the latter conics into hyperbolic and elliptic.

5.1.1. Theorem. If $S \subset \mathbb{P}^2$ is a generic real $s$-symmetric non-singular sextic, then

1. Every conic 6-tangent to $S$ is $s$-symmetric.
2. The number of hyperbolic 6-tangent conics is greater by 8 than the number of elliptic ones: $h - e = 8$.

Proof. Part (1) is a consequence of the surjectivity of the period map for $K3$-surfaces applied to the double covering $Y$ of the plane ramified in the sextic under consideration. Namely, $s$ lifts to a pair of involutions on $Y$ commuting with the deck transformation of the covering $Y \to \mathbb{P}^2$. One of these involutions, denoted $\tilde{s}$, fixes pointwise the pull-back of the infinity line $x_2 = 0$ (fixed by $s$) and acts as multiplication by $-1$ on $H^{2,0}(Y)$. Each 6-tangent conic $B$ lifts to a pair of $(-2)$-curves $\tilde{B}_1, \tilde{B}_2$. If $B$ is $s$-invariant, then each of $\tilde{B}_i$ is invariant under $\tilde{s}$. But if $B$ is not $s$-invariant, then the classes $[\tilde{B}_i] \in H_2(Y)$ are not $\tilde{s}$-invariant, because $\tilde{s}([\tilde{B}_i]) = [\tilde{B}_i]$ would imply a contradiction:

\[-2 = [\tilde{B}_i]^2 = [\tilde{B}_i] \cdot \tilde{s}_* [\tilde{B}_i] = [\tilde{B}_i] \cdot [\tilde{s}_* (\tilde{B}_i)] \geq 0.\]

Thus, if $B$ is not $s$-invariant it has to disappear under a generic small $s$-invariant perturbation of $S$ due to surjectivity of the period map.

Claim (2) follows immediately from claim (1) and Theorem 1.2.2. \qed

Using the same bijection in combination with a separate count of hyperbolic and elliptic tritangents, one can also perform a separate count for 6-tangent conics. For example, it is not difficult to check that if $S \subset \mathbb{P}^2$ as in Theorem 5.1.1 is an $M$- or $(M - 1)$-curve, then the corresponding $C \subset Q$ is an $M$-curve and, thus, in all these cases, in accordance with Table 4 the numbers of hyperbolic and elliptic 6-tangent to $S$ conics are 64 and 56 respectively.
For a list of deformation classes of real non-singular plane curves of degree 6 containing symmetric curves, we address an interested reader to [II]. For information available on arrangements of symmetric curves in higher degrees, one may look at [Br] and [Tr].

5.2. **Signed count of real lines on nodal surfaces and wall-crossing.** The methods of this paper can be extended, almost literally, to nodal weak del Pezzo surfaces of degree 1, that is to nodal surfaces $X$ defined in $\mathbb{P}(1, 1, 2, 3)$ by equation of the same form (2.5.1) as in non-singular case. Then, following, for example, the definition given in Remark 3.3.2 we may introduce a natural $\Pin^-$-structure on the smooth part of $X_\mathbb{R}$ with the properties like in Theorem 1.2.1 and get as a direct analog of Theorem 1.2.2 a signed count of real lines not passing through the nodes. If all the nodes of $X$ are real and $L^+_n(X)\cup L^-_n(X)$ refers to the set of real lines not passing through the nodes, we conclude that

$$
\sum_{l \in L^+_n(X)\cup L^-_n(X)} s(l) = 2 \text{rk}(H_2(X) \cap (K_X, e_1, \ldots, e_k)^\perp) = 16 - 2k
$$

where $e_1, \ldots, e_k$ stands for the roots represented in $H_2(X)$ by the $(-2)$-curves corresponding to the nodes. Respectively,

- for a real sextic $C \subset Q$ having $k$ real nodes and no other singular points, one obtains $8 - k$ as the signed count of real sections tritangent to $C$ and passing neither through its nodes nor through the vertex of $Q$.

The latter observation, combined with the idea in Subsection 5.1, shows that

- for a generic real affine $k$-nodal cubic curve $E \subset \mathbb{C}^2$, a signed count of non-singular real conics tritangent to $E$ and symmetric with respect to the origin gives $4 - k$.

In a similar vein, let us consider a generic one parametric family $X_t$, $|t| < \epsilon$, of real del Pezzo surfaces of degree 1 degenerating to a uninodal real del Pezzo surface $X^0$. Then, as $t$ is approaching 0 from the side where the Euler characteristic of $X_t$ is smaller, a certain number of real lines in $X_t$ merge pairwise to form **double lines** in $X^0$ and turn into pairs of imaginary lines as $t$ crosses 0 (cf. [IKS]). Among these double lines all except one project on $Q$ to conics not passing through the vertex of $Q$ (but passing through the node of $C_0$). They are resulted from merging of lines of opposite sign, hyperbolic and elliptic. The exceptional double line projects on $Q$ to a real line passing through the node of $C$ and the vertex of the cone. This double line results from merging of two hyperbolic lines (see Figure 5). It is the only one which, after $t$ crosses 0, splits into a pair of lines which are conjugate imaginary in $X_t$, but real with respect to the Bertini dual real structure.

**Figure 5.** Degeneration of a tritangent section into a double generating line of the cone.
This peculiar wall-crossing behavior of real lines under nodal degenerations gives an alternative explanation of the rule of combined conservation of the signed count of real lines in Bertini pairs as it is presented by Theorem 1.2.2.

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