Performance diagnosis of local filters in state estimation of nonlinear systems

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Abstract. The paper deals with state estimation of nonlinear stochastic discrete-time systems with a special focus on filters providing a point estimate of the state. The aim is concentrated on performance diagnosis of the filters, which should inform about a possible decrease of estimate quality provided by the filter. The paper proposes a diagnostics of the filter performance using the non-Gaussianity measures. More specifically, the paper focuses on a third-order moment based non-Gaussianity measure and provides relations for its recursive computation. It is shown that this measure can effectively be used for the performance diagnosis of the filters, which is illustrated using a numerical example. The example considers tracking motion of a ship using the extended Kalman filter.

Keywords: state estimation, nonlinear filters, Kalman filtering, non-Gaussianity measures, performance diagnosis

1. Introduction
The problem of state estimation of discrete-time nonlinear stochastic dynamic systems from noisy measured data plays a key role in fields such as target tracking and localization [1], navigation, signal processing [2], fault detection [3], and adaptive and optimal control problems. Hence, it has been a subject of intensive research interest for the last several decades [4, 5, 6].

A general solution to the problem is represented by the Bayesian recursive relations (BRRs) [4]. The relations compute the probability density functions (PDFs) of the state conditioned by the available measurements. These PDFs represent a complete description of the state estimate.

In many cases, the complete description of the state estimate is superfluous and only a point estimate is searched often by optimizing the mean square error (MSE) criterion [7]. The resulting minimum MSE (MMSE) estimate is given A general solution to the problem is represented by the Bayesian recursive relations (BRRs) [8]. The relations compute the probability density functions (PDFs) of the state conditioned by the available measurements. These PDFs represent a complete description of the state estimate by the conditional mean of the state conditioned by the available measurements. The estimators (for coinciding time of the state estimate and time of the available measurement also called filters) providing a point estimate of the state will henceforth be called local, owing to the locally limited estimate validity in the state space∗.

No matter which approach is chosen (either the Bayesian or the optimization), a closed-form solution to the relations is available only for a few special cases, such as the linear Gaussian system [9]. In other

∗ The algorithms of local filters may also be derived from the BRRs when an assumption on noises distributions is imposed.
cases, approximate approaches must be utilized. In the case of the optimization approach, the structure of the filter is usually fixed to be a linear function of the measurement, which results in a linear MMSE (LMMSE) estimate.

For majority of nonlinear systems, computing an LMMSE estimate requires further approximations. The performed approximations then lead to the fact, that the estimate cannot be guaranteed to converge to the true state and also the estimate error may be unbounded. In [10] it has been proven that the estimate error can be bounded with probability one for systems with mild nonlinearities, exact initial condition, and small enough disturbing noises. Nonlinearity of a system function may result in initial Gaussian PDF of the state becoming a heavy-tailed or a multi-modal distribution. Subsequently, such a distribution might have unfavourable effect on the estimate quality.

Degree of nonlinearity of the functions appearing in the system description cannot be assessed in isolation, but must be tested in connection with the location of the state. As the state has stochastic nature, uncertainty of its location may be characterized by its mean and covariance matrix. Hence, these two moments of the state should be taken into account when assessing nonlinearity of the functions.

For the assessment, several nonlinearity measures (NLMs) [11, 12, 13] have been proposed. In [11], the NLM serves for a model assessment for selection of a suitable filter, in [13] for reliable assessments of the effectiveness of nonlinear mitigation techniques and in [12] for providing information for a decision whether a filter parameter requires an adaptation. The NLMs have, however, difficulty with capturing a cumulative effect of mild nonlinearities. The effect may appear due to several subsequent transformations of a Gaussian PDF of the state that are individually only mildly nonlinear, however, their joint effect is strongly nonlinear [14].

The cumulative effect of several nonlinear transformations can be effectively captured by non-Gaussianity measures (NGMs) [14]. The NGMs measure a distance of the PDF of a transformed random variable to a Gaussian PDF. The measurement is performed based on either the complete PDFs or on higher-order moments.

Besides measuring nonlinearity of the system functions, the NGMs may also be used for assessing Gaussianity of state conditional PDFs. This is useful especially for a number of recently developed local filters (LFs) based on the assumption of Gaussian conditional PDF of the state. These include, the filters based on various deterministic (cubature or quadrature) or stochastic integration rules or on the Fourier-Hermite expansion [15, 5, 16, 17, 18, 19].

The goal of the paper is to demonstrate utilization of the NGMs for diagnosing performance of the LFs. It will be shown that besides for the LFs assuming Gaussian distribution of the conditional state PDF, other LFs such as the extended Kalman filter (EKF) may utilize the NGMs for the diagnosis. The paper will focus on the moment-based NGMs which will monitor performance of a LF indicating a potential degradation of the estimate quality. The diagnosis will be illustrated using a multi-dimensional tracking of a maneuvering ship.

The rest of the paper is organized as follows. Section 2 provides system specification and a generic LF algorithm with a focus on the EKF. Section 3 then discusses NGMs and their utilization in the LF framework for performance diagnosis. Numerical illustration and concluding remarks are given in Sections 4 and 5, respectively.

2. System Description and Local Filters

2.1. System description

Let a discrete-time nonlinear stochastic system be considered given by the following state-space model

\[ x_{k+1} = f_k(x_k) + w_k, \quad k = 0, 1, 2, \ldots, \]
\[ z_k = h_k(x_k) + v_k, \quad k = 0, 1, 2, \ldots, \]

where the vectors \( x_k \in \mathbb{R}^{n_x} \) and \( z_k \in \mathbb{R}^{n_z} \) represent the immeasurable state of the system and measurement at time instant \( k \), respectively, \( f_k : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x} \) and \( h_k : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z} \) are known vector
functions, and \(w_k \in \mathbb{R}^{n_w}\) and \(v_k \in \mathbb{R}^{n_v}\) are the state and measurement white noises. The PDFs of the noises are supposed to be Gaussian with zero means and known covariance matrices \(\Sigma_k^w\) and \(\Sigma_k^v\), i.e., \(p_w(w_k) = \mathcal{N}(w_k; 0, \Sigma_k^w)\) and \(p_v(v_k) = \mathcal{N}(v_k; 0, \Sigma_k^v)\), respectively. The PDF of the initial state is Gaussian, i.e., \(p_{x_0}(x_0) = \mathcal{N}(x_0; x_0, P_{0x})\) with known mean and covariance matrix; the initial state \(x_0\) is independent of the noises.

2.2. State estimation by local filters

The aim of a LF is to compute the first two moments of the state conditioned by the measurements, more specifically, the conditional mean \(\hat{x}_{k|k}\) and covariance matrix of the estimate error \(P_{k|k} = \text{cov}[x_k|z^k]\) in which \(z^k = \{z_0, z_1, \ldots, z_k\}\). The moments might be understood as a Gaussian approximation of the conditional PDF, i.e., \(p(x_k|z^k) \approx \mathcal{N}(x_k; \hat{x}_{k|k}, P_{k|k})\) [5, 18], depending on the type of used approximation.

The group of LFs imposing the assumption of Gaussianity of the conditional PDF of the state consists of the quadrature, cubature, and stochastic integration based filters utilizing deterministic and stochastic integration rules [20, 5, 18], and the Fourier-Hermite Kalman filter [19]. The LFs providing the point estimate of the state without the assumption are e.g., the extended Kalman filter (EKF) or the second order filter [21], the divided difference filters (DDFs) utilizing the Stirling polynomial interpolation [22] or the unscented Kalman filter (UKF) using the unscented transform [23].

All the above mentioned LF algorithms share the following steps [6, 5]:

**Algorithm 1: Generic Local Filter**

**Step 1:** (initialization) Set the time instant \(k = 0\) and define the initial condition by the predictive mean \(\hat{x}_{0|0} = \mathbb{E}[x_0] = \bar{x}_0\) and the predictive covariance matrix \(P_{0|0} = \text{cov}(x_0) = P_{0x}\).

**Step 2:** (filtering) The state filtering estimate is given by updating the state predictive estimate with respect to the last measurement \(z_k\) according to

\[
\begin{align*}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(z_k - \hat{z}_{k|k-1}), \\
P_{k|k} &= P_{k|k-1} - K_k P_{z|k-1} K_k^T,
\end{align*}
\]

(3)  (4)

where \(K_k = P_{z|k-1} (P_{z|k-1})^{-1}\) is the filter gain and

\[
\begin{align*}
\hat{z}_{k|k-1} &= \mathbb{E}[z_k|z^{k-1}] = \mathbb{E}[h_k(x_k)|z^{k-1}], \\
P_{z|k-1} &= \mathbb{E}[(z_k - \hat{z}_{k|k-1})(z_k - \hat{z}_{k|k-1})^T|z^{k-1}] = \\
&= \mathbb{E}[(h_k(x_k) - \hat{z}_{k|k-1}) \times (h_k(x_k) - \hat{z}_{k|k-1})^T|z^{k-1}] + \Sigma_k^y, \\
P_{z|k-1} &= \mathbb{E}[(x_k - \hat{x}_{k|k-1})(z_k - \hat{z}_{k|k-1})^T|z^{k-1}].
\end{align*}
\]

(5)  (6)  (7)

**Step 3:** (prediction) The state predictive estimate is given by the relations

\[
\begin{align*}
\hat{x}_{k+1|k} &= \mathbb{E}[x_{k+1}|z^k] = \mathbb{E}[f_k(x_k)|z^k], \\
P_{k+1|k} &= \mathbb{E}[(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})^T|z^k] = \mathbb{E}[(f_k(x_k) - \hat{f}_k(x_k))(f_k(x_k) - \hat{f}_k(x_k))^T|z^k] + \Sigma_k^x.
\end{align*}
\]

(8)  (9)

Let \(k = k + 1\). The algorithm then continues by **Step 2:** (filtering).
2.2.1. The first order Taylor expansion The basic approach for an approximate computation of the predictive statistics is based on an approximation of the nonlinear function by the first order Taylor expansion (TE1) [9]. The TE1 of the nonlinear function in measurement equation under assumption of the known state predictive statistics $\hat{x}_{k|k-1}$ and $P^{x}_{k|k-1}$ (defining the approximation point for the filtering step) is of the form

$$z_k = h_k(x_k) + v_k \approx h_k(\hat{x}_{k|k-1}) + H_k(\hat{x}_{k|k-1}) (x_k - \hat{x}_{k|k-1}) + v_k,$$

where $H_k(\hat{x}_{k|k-1}) = \left. \frac{\partial h(x_k)}{\partial x_k} \right|_{x_k = \hat{x}_{k|k-1}}$ is the Jacobian of $h_k(\cdot)$. The desired statistics are then computed as

$$\bar{z}_{k|k-1}^{TE1} = h_k(\hat{x}_{k|k-1}),$$

$$P^{zz}_{k|k-1}^{TE1} = H_k P^{xx}_{k|k-1} H_k^T + \Sigma^y,$$

$$P^{xz}_{k|k-1}^{TE1} = P^{x}_{k|k-1} H_k^T,$$

where $H_k = H_k(\hat{x}_{k|k-1})$.

The TE1 of the nonlinear function in the state equation under assumption of the known state filtering statistics $\hat{x}_{k|k}$ and $P^{x}_{k|k}$ (defining the approximation point for the prediction step) is of the form

$$x_{k+1} = f_k(x_k) + w_k \approx f_k(\hat{x}_{k|k}) + F_k(\hat{x}_{k|k}) (\hat{x}_k - \hat{x}_{k|k}) + w_k,$$

where $F_k(\hat{x}_{k|k}) = \left. \frac{\partial f(x_k)}{\partial x_k} \right|_{x_k = \hat{x}_{k|k}}$ is the Jacobian of $f_k(\cdot)$. The desired statistics are then computed as

$$\overline{x}_{k+1|k}^{TE1} = f_k(\hat{x}_{k|k}),$$

$$P^{xx}_{k+1|k}^{TE1} = F_k P^{xx}_{k|k} F_k^T + \Sigma^w,$$

where $F_k = F_k(\hat{x}_{k|k})$.

3. Non-Gaussianity Measures

The LFs provide reliable results if the approximation point is close to the true state. This usually happens if the nonlinearities in system description (1) and (2) are mild and if the state conditional PDFs have Gaussian-like shape (unimodal and not heavy-tailed).

Assessment of the system nonlinearity or the resulting state PDF properties is, therefore, a vital part of any filter design procedure. Hence, several NLMs have been proposed so far. A NLM can be utilized in two ways: i) as a global NLM assessing the overall nonlinearity of the given system without any reference to the usage of a specific filter [11], or ii) as a local NLM designed for an on-line monitoring of (local) filters.

While the global NLM provides an answer whether a LF is expected to perform well or a more complicated filter producing PDF estimates needs to be used, the local NLM is suitable for self-monitoring of the LFs. However, the local NLM has a very limited capability in monitoring the transformation cumululative effect (the LMs in two subsequent time instants are independent; even the NLM in the filtering step is independent of the NLM in the prediction step at the same time). As was illustrated in [14], a sequence of mildly nonlinear transformations (transformations with very low values of NLMs) with, for example, an initially Gaussian PDF, might result in a heavy-tailed or multimodal distribution. As a consequence, assumptions of some LFs are violated and the estimate quality deteriorates.

This cumulative effect of the nonlinear transformations can be monitored by NGMs, that evaluate influence of the nonlinear transformation on the PDF of a transformed state. Essentially, the NGMs are

† The statistics are generally approximate. Exact values are obtained for a few special functions $h_k(\cdot)$ only, e.g., for a linear function $h_l(\cdot)$. 

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based on a comparison of two PDFs, the estimated one and its Gaussian approximation. The comparison may consider the shape of the PDFs (e.g., based on the negentropy [14]) or be moment-based. The moment-based NGMs can, for example, compare the third and higher-order moments, skewness or kurtosis [14]. This paper will focus on the third moment, which is for a Gaussian random variable zero.

In this paper, the following parametrization of the filtering third moment of a vector state \( \mathbf{x}_k \), denoted as \( \mathbf{M}^{xxx}_{k|k} \), is adopted

\[
\mathbf{M}^{xxx}_{k|k} \triangleq \mathbb{E}[\{ \mathbf{x}_k - \hat{\mathbf{x}}_{k|k} \} \{ \mathbf{x}_k - \hat{\mathbf{x}}_{k|k} \}^T \otimes \{ \mathbf{x}_k - \hat{\mathbf{x}}_{k|k} \}^T | \mathbf{z}_k],
\]

(17)

where the symbol \( \otimes \) stands for the Kronecker product [24]. The predictive third moment \( \mathbf{M}^{xxx}_{k+1|k} \) is defined analogously. It can be seen that the 3rd moment is described by a matrix of the dimension \( n_x \times (n_x)^2 \); thus, the number of elements of the moment, which will be compared, is cubic. To reduce the costs of moments computation and comparison for higher dimensions \( n_x \), the mixed elements may be discarded and only \( n_x \) elements are compared.

As the third moment of a Gaussian distribution is zero, the NGM related to the third moment of \( n \)-th element of the state is given by

\[
J_{3,k}(\mathbf{x}(n), k) = |\mathbf{M}^{xxx}_{k|k} \{n \times (n - 1)n_x + n\}|, \quad n = 1, 2, \ldots, n_x,
\]

(18)

where \( \mathbf{M}^{xxx}_{k|k}(i, j) \) is the \((i, j)\)-th element of the matrix \( \mathbf{M}^{xxx}_{k|k} \).

When comparing the NGM (18) with a threshold \( J_{3,thr} \), the inequality \( J_{3,k}(\cdot) \geq J_{3,thr} \) indicates that the corresponding estimated PDF is far from being Gaussian. Thus, in this case the estimate quality is possibly low. The comparison is the main component of the performance diagnosis of the LF.

Note that the threshold value needs to be specified on the basis of an off-line analysis or a simulation study and might be different for each state component. Now, having the NGM specified based on the third moment, calculation of the moment will be pursued in the following subsection.

### 3.1. Recursive computation of the third moments

The filtering third moment of the state is given by (17). Substituting (3) into (17) and defining the error variables \( \mathbf{\hat{x}}_{k|k-1} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1} \) and \( \mathbf{\hat{z}}_{k|k-1} = \mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1} \), the following relation can be obtained:

\[
\mathbf{M}^{xxx}_{k|k} = \mathbb{E}[\{ \mathbf{\hat{x}}_{k|k-1} - \mathbf{K}_k \mathbf{\hat{z}}_{k|k-1} \} \{ \mathbf{\hat{x}}_{k|k-1} - \mathbf{K}_k \mathbf{\hat{z}}_{k|k-1} \}^T \otimes \{ \mathbf{\hat{x}}_{k|k-1} - \mathbf{K}_k \mathbf{\hat{z}}_{k|k-1} \}^T ]
\]

\[
\begin{align*}
&= \mathbb{E}[\{ \mathbf{\hat{x}}_{k|k-1} \otimes \mathbf{\hat{x}}_{k|k-1}^T \} - \mathbf{K}_k \mathbf{\hat{z}}_{k|k-1} \otimes \mathbf{\hat{x}}_{k|k-1}^T \} - \mathbf{\hat{x}}_{k|k-1} \otimes \mathbf{\hat{x}}_{k|k-1}^T ]
&+ \mathbf{K}_k \mathbf{\hat{z}}_{k|k-1} \otimes \mathbf{\hat{x}}_{k|k-1}^T ]
&+ \mathbf{\hat{x}}_{k|k-1} \otimes \mathbf{\hat{x}}_{k|k-1}^T] - \mathbf{K}_k \mathbf{\hat{z}}_{k|k-1} \otimes \mathbf{\hat{z}}_{k|k-1} \otimes \mathbf{\hat{z}}_{k|k-1}^T \mathbf{K}_k^T ]
&+ \mathbf{K}_k \mathbf{\hat{z}}_{k|k-1} \otimes \mathbf{\hat{z}}_{k|k-1} \otimes \mathbf{\hat{z}}_{k|k-1} \mathbf{K}_k^T ]
\end{align*}
\]

(19)

\( \dagger \) Alternative parametrizations can be found e.g., in [25].
For the scalar case (32) reduces to

\[ M_{k|k-1}^{xx} = M_{k|k-1}^{xx} - 3K_kM_{k|k-1}^{xz} + 3(K_k)^2M_{k|k-1}^{zz} - (K_k)^3M_{k|k-1}^{zzz}, \]

where the third moments are defined as

\[
\begin{align*}
E[a] &= M_{k|k-1}^{xx}, \\
E[b] &= K_kM_{k|k-1}^{xx}, \\
c &= \ddot{x}^T K^T \otimes \dddot{x}^T = \ddot{x}^T K^T \otimes \dddot{x}^T I_n = (\ddot{x}^T \otimes \dddot{x}^T)(K^T \otimes I_n), \\
E[c] &= M_{k|k-1}^{xx}K^T \otimes I_n, \\
d &= K\dddot{x}^T K^T \otimes \dddot{x}^T = K\dddot{x}^T K^T \otimes \dddot{x}^T I_n = K(\dddot{x}^T \otimes \dddot{x}^T)(K^T \otimes I_n), \\
E[d] &= KM_{k|k-1}^{zz} (K^T \otimes I_n), \\
E[e] &= M_{k|k-1}^{xz}K^T, \\
E[f] &= K_kM_{k|k-1}^{xz}, \\
g &= \ddot{x}^T K^T \otimes \dddot{x}^T K^T = (\ddot{x}^T \otimes \dddot{x}^T)(K^T \otimes K^T), \\
E[g] &= M_{k|k-1}^{xz}K^T \otimes K^T, \\
h &= K\dddot{x}^T K^T \otimes \dddot{x}^T K^T = K(\dddot{x}^T \otimes \dddot{x}^T)(K^T \otimes K^T), \\
E[h] &= K_kM_{k|k-1}^{zz} (K^T \otimes K^T), \\
\end{align*}
\]
Following the approximation technique of the EKF given by (11)–(13), where the nonlinear function $h_k(\cdot)$ in measurement equation (2) is approximated using TE1 (10), the required third moments equal to

$$M^{zzz}_{k|k-1} = (H_k)^3 (\hat{x}_{k|k-1}) M^{xxx}_{k|k-1}, \quad (35)$$

$$M^{xzz}_{k|k-1} = H_k (\hat{x}_{k|k-1}) M^{xxx}_{k|k-1}, \quad (36)$$

$$M^{zzx}_{k|k-1} = (H_k)^2 (\hat{x}_{k|k-1}) M^{xxx}_{k|k-1}. \quad (37)$$

It can be seen that the TE1-based linearization ignores the information about higher order terms caused by the nonlinear transformation. In fact, under assumption of the Gaussian initial condition (imposing $M^{xxx}_{k|k-1} = 0$), the moment $M^{xxx}_{k|k} = 0$.

Thus, to get the non-zero moment approximation, let the second order Taylor expansion (TE2)

$$z_k = h_k(x_k) + v_k \approx h_k(\hat{x}_{k|k-1}) + H_k(\hat{x}_{k|k-1}) \hat{x}_{k|k-1} + \frac{1}{2} \mathcal{H}_k(\hat{x}_{k|k-1}) \hat{x}_{k|k-1}^2 + v_k, \quad (38)$$

be used instead of the TE1 where $\mathcal{H}_k = \mathcal{H}_k(\hat{x}_{k|k-1}) = \frac{dh_k(x_k)}{dx_k} |_{x_k = \hat{x}_{k|k-1}}$ is the Hessian. The TE2 is typically used in the second order filter design [26]. Utilization of the TE2 in the higher moments computation requires conditional higher moments of the predictive state estimate, namely $M^{xxx}_{k|k-1}$, $M^{xxx}_{k|k-1}$, and $M^{zzx}_{k|k-1}$, which are not available§. The proposed solution is to approximate those values by the theoretical moments computed under assumption of the Gaussian distribution [27], i.e., as

$$M^{xxx}_{k|k-1} = 3(P^{xx}_{k|k-1})^2, \quad (39)$$

$$M^{zzx}_{k|k-1} = 0, \quad (40)$$

$$M^{zzx}_{k|k-1} = 15(P^{xx}_{k|k-1})^3. \quad (41)$$

After tedious calculations, the required third moments (35)–(37) can be expressed by:

$$M^{zzz}_{k|k-1} = (H_k)^3 M^{xxx}_{k|k-1} - \frac{3}{2} H_k(\mathcal{H}_k) M^{xxx}_{k|k-1} M^{xxx}_{k|k-1} + \mathcal{H}_k(P^{xx}_{k|k-1})^3 + 3(H_k)^2 \mathcal{H}_k(P^{xx}_{k|k-1})^2, \quad (42)$$

$$M^{xzz}_{k|k-1} = H_k M^{xxx}_{k|k-1} + \mathcal{H}_k(P^{xx}_{k|k-1})^2, \quad (43)$$

$$M^{zzx}_{k|k-1} = (H_k)^2 M^{xxx}_{k|k-1} - \frac{1}{2} \mathcal{H}_k(P^{xx}_{k|k-1})^2 P^{xx}_{k|k-1} M^{xxx}_{k|k-1} + 2H_k \mathcal{H}_k(P^{xx}_{k|k-1})^2. \quad (44)$$

The predictive moment (33) is given by

$$M^{xxx}_{k+1|k} = (F_k)^3 M^{xxx}_{k|k} - \frac{3}{2} F_k(\mathcal{F}_k) M^{xxx}_{k|k} M^{xxx}_{k|k} + \mathcal{F}_k(P^{xx}_{k|k})^3 + 3(F_k)^2 \mathcal{F}_k(P^{xx}_{k|k})^2, \quad (45)$$

where $F_k = F_k(\hat{x}_{k|k}) = \frac{df_k(x_k)}{dx_k} |_{x_k = \hat{x}_{k|k}}$ is the Jacobian of $f_k(\cdot)$ and $\mathcal{F}_k = \mathcal{F}_k(\hat{x}_{k|k}) = \frac{df_k(x_k)}{dx_k} |_{x_k = \hat{x}_{k|k}}$ is the Hessian of $f_k(\cdot)$.

### 3.3. Recursive computation of the third moment for sigma-point based filters

The relations for recursive computation of the third moments by means of the TE2 introduced in the previous section can be readily written for sigma-point based filters such as the UKF, cubature Kalman filter, quadrature Kalman filter, divided difference filter, and stochastic integration filter. For the UKF the relations can be found in [28].

§ This phenomenon is referred in the literature as the moment closure problem [4]
3.4. LF algorithm with NGMs
Having stated the relations necessary for the recursive state higher-order moment computation and the NGMs, the algorithm of the LF with steps for performance diagnosis can be summarized.

**Algorithm 2**: Local Filter with performance diagnosis

**Step 1**: (initialization) as Step 1 of Algorithm 1, supplemented with specification of an initial higher moments estimate.

**Step 2**: (filtering) as Step 2 of Algorithm 1, supplemented with computation of $M_{k|k}^{xxx}$ according to (32).

**Step 3**: (self-assessment) If any measure $J_{3,k}(x_{(n),k}), n = 1, 2, \ldots, n_x$, of the filtering estimate is greater than or equal to the respective threshold $J_{3,thr}$, then there is a possibility of a lower estimate quality.

**Step 4**: (prediction) as Step 3 of Algorithm 1, supplemented with computation of $M_{k+1|k}^{xxx}$ according to (33).

**Step 5**: (self-assessment) If any measure $J_{3,k}(x_{(n),k}), n = 1, 2, \ldots, n_x$, of the predictive estimate is greater than or equal to the respective threshold $J_{5,thr}$, then there is a possibility of a lower estimate quality.

Let $k = k + 1$. The algorithm then continues by **Step 2** (filtering).

4. Numerical Illustration
Performance diagnosis of the EKF will be illustrated using a model describing ship motion in deep water [29]. The ship motion is described by the following continuous-time model (CTM):

\[
\begin{align*}
\frac{dx}{dt} &= K_v v_0 \sin(\psi), \\
\frac{dy}{dt} &= K_v v_0 \cos(\psi), \\
\frac{d\psi}{dt} &= K_v \omega, \\
\frac{d\omega}{dt} &= \frac{v_0^2}{2\rho L^2} \left( -q_\omega \frac{L}{v_0 K_v} - s_{31} \delta \right).
\end{align*}
\]

The state vector $x = [x, y, \psi, \omega]^T$ includes the ship coordinates $x$ and $y$, heading $\psi$ and turn-rate $\omega$. The variable $\delta$ denotes the control rudder angle deviation. $L$ is the ship length, $K_v = (1 + 1.9\omega^2 L^2 v_0)^{-1}$, $v_0$ is the ship speed at zero turn-rate, $q = q_{21}r_{31} - q_{31}r_{21}$, $p = 0.5(q_{21} + r_{31})$ with $q_{21}, r_{21}, q_{31}, r_{31},$ and $s_{31}$ being hydrodynamic coefficients depending on the ship geometry and length $L$. The simulation was performed with the following realistic (for a stable ship) constants $q_{21} = 1.227, r_{21} = -0.629, q_{31} = -4.64, r_{31} = 3.88, s_{31} = -1.019, v_0 = 30$ [m/s], and $L = 99$ [m]. The control rudder angle deviation was simulated as

\[
\delta = \begin{cases} 
7.5 \sin(0.01 \cdot t) [^\circ] & \text{if } t \in (250, 450) \text{ [s]}, \\
0 & \text{if } t \notin (250, 450) \text{ [s]}.
\end{cases}
\]

The CTM (46)–(49) was discretized using the Euler method with sampling time $T = 0.5$ [s]. The turn-rate $\omega_k$ was supposed to be corrupted by a white zero-mean Gaussian noise $w_k^\omega$ with its variance depending on the turn-rate as $\Sigma_k^\omega = (\gamma_\omega + \gamma_\delta \omega_k^{-1})^2$, with $\gamma_\omega = 0.0005$ and $\gamma_\delta = 0.1$.

The state $x$ was observed as

\[
z_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \end{bmatrix} x_k + v_k.
\]
which means that the position of the ship was observed directly in Cartesian coordinates. The observation was corrupted by a white zero-mean Gaussian noise $v_k$ with covariance matrix $\Sigma_k = \text{diag}(4, 4) [m^2]$.

The motion was simulated for 900 seconds with initial conditions $x_0 = [10 \text{ [m]}, 10 \text{ [m]}, 45 \text{ [°]}, 0 \text{ [(°)/s]}]$. The EKF was initialized with $\hat{x}_0|_{-1} = \bar{x}_0$ and $P_0 = \text{diag}([10^4 \text{ [m]^2}], 10^4 \text{ [m]^2}, (1.5)^2 \text{ [(°)]^2}, 1 \text{ [(°)/s]^2}])$.

The trajectories of the state are depicted in Figure 1. The estimate error was observed using the MSE for

\[
\text{MSE}^x_k = \frac{1}{N} \sum_{i=1}^{N} (x_{(1),k}(i) - x_{(1),k}(i))^2,
\]

where $x_{(1),k}(i)$ and $x_{(1),k}(i)$ is the first state component at time $k$ and $i$-th Monte Carlo (MC) simulation and its filtering estimate calculated by the EKF, respectively. The MSE for the position in the $y$ direction $\text{MSE}^y_k$, heading $\text{MSE}^\psi_k$ and turn-rate $\text{MSE}^\omega_k$ are defined analogously. The scenario was simulated $N = 250$ times. The MSEs were compared with the third-order moments $M^{xxx}$, $M^{yyy}$, $M^{\psi\psi\psi}$, and $M^{\omega\omega\omega}$, where $M^{xxx}$ is defined as

\[
M^{xxx}_k = \frac{1}{N} \sum_{i=1}^{N} M^{xxx}_{k|i}(i)
\]

and $M^{yyy}$, $M^{\psi\psi\psi}$, and $M^{\omega\omega\omega}$ are defined analogously.

The resulting MSEs together with the corresponding third-order moments are depicted in Figures 2 and 3. To illustrate instant values, a comparison of the squared error of the state estimate and instant values of the third-order moments are given in Figures 4 and 5. From all Figures 2, 3, 4, and 5 it is clear that time instants with high values of the NGM (given by the third-order moment of the state) correspond to the time instants with deteriorated estimate quality. The deterioration of the estimate quality is caused by a significant change of the turn-rate due to control rudder angle deviation, which although known, shifts the state into a region of severe nonlinearity of the function in the dynamics. The relation between the state errors and the third-order moments confirms our hypothesis that a severe nonlinearity (leading to high values of the NGM) leads to increased estimate error. Therefore, the NGMs can effectively be used for performance diagnosis of nonlinear filters.
5. Concluding Remarks

The paper dealt with state estimation of nonlinear stochastic discrete-time systems. The aim was focused on possibilities of performance diagnosis of nonlinear filters providing point estimates of the state for such systems. Such performance diagnosis would be a convenient tool for the users as it would provide them an additional information about the estimate quality or it may be used for the filter to respond to deteriorating conditions for providing a quality estimate.
It was shown that non-Gaussianity measures may be used for this purpose as they can measure an effect of nonlinear transformations which, if they are severe, may lead to a poor estimate quality. Moreover, the NGMs are able to capture cumulating effect of a mildly nonlinear transformation in comparison with the NLMs. The paper focused on the moment-based NGMs, more specifically on the NGM based on the third-order moment. The paper provided general relations for its recursive computation and detailed relations of its computation based on the TE2. Usage of the third-order moment based NGM for performance diagnosis was demonstrated using a numerical example of a ship tracking.
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