New $AdS_4 \times X_7$ Geometries with $\mathcal{N} = 6$ in M Theory

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Abstract: We study supersymmetric $AdS_4 \times X_7$ solutions of 11-dim supergravity where the tri-Sasakian space $X_7$ has generically $U(1)^2 \times SU(2)_R$ isometry. The compact and regular 7-dim spaces $X_7 = S(t_1, t_2, t_3)$ is originated from 8-dim hyperkahler quotient of a 12-dim flat hyperkahler space by $U(1)$ and belongs to the class of the Eschenburg space. We calculate the volume of $X_7$ and that of the supersymmetric five cycle via localization. From this we discuss the 3-dim dual superconformal field theories with $\mathcal{N} = 3$ supersymmetry.
1. Introduction and Conclusion

The AdS-CFT correspondence predicts that the type IIB-theory on the supergravity solution $AdS_5 \times S^5$ with appropriate 5-form field strength is dual to a $\mathcal{N} = 4$ supersymmetric 4-dim $SU(N)$ gauge theory, which is a superconformal field theory[1]. When the space $S^5$ is replaced by a 5-dim Sasaki Einstein space, the dual gauge theory has less supersymmetry with more complicated group and matter structure[2]. These conformal theories can be regarded as a field theory on a stack of D3 branes sitting at the the singular tip of a Ricci flat 6-dim cone whose base is the Sasaki-Einstein space[3, 4, 5].

The AdS-CFT correspondence for the Freund-Rubin form, $AdS_4 \times X_7$, of a supersymmetric solution of 11-dim supergravity implies that the M-theory on such background is dual to a supersymmetric 3-dim superconformal field theory. Again the field theory arises as the SCFT on a stack of M2 branes at the singular apex of 8-dim Ricci-flat space with special holonomy, whose base is 7-dim $X_7$. One has to have at least one Killing spinor on $X_7$ to have a special holonomy. Recently there
have been found several countable series of Sasaki-Einstein space in 5-dim and 7-dim, and their AdS-CFT correspondence has been studied\[6, 7, 8\]. Especially the corresponding 3-dim SCFT’s have $\mathcal{N} = 1$, or 2 supersymmetries.

In this work we focus on a class of countable series of the $AdS^4 \times X_7$ spaces, which has not been studied before. The dual SCFT has $\mathcal{N} = 3$ and is the SCFT on the M2 branes on the singular tip of the 8-dim hyperkahler cone with $Sp(2)$ holonomy and with its base being a tri-Sasakian space $X_7$. The 8-dim hyperkahler cone is obtained by a hyperkahler quotient of $\mathbb{R}^{12}$ by a $U(1)$ symmetry group\[9\]. (In general, one could have started from flat $\mathbb{R}^{4n+8}$ space with $U(1)^n$ hyperkahler quotient with $n \geq 1$ but here we restrict to $n = 1$ case for simplicity.) Our tri-Sasakian space $X_7(t) = S(t_1, t_2, t_3)$ is characterized by three natural numbers $t_a$ and has $SU(2)_R \times U(1)^2$, $SU(2)_R \times SU(2) \times U(1)$ or $SU(2)_R \times SU(3)$ isometry depending on none, two, or all of $t_a$ coincide, respectively. We calculate the volume of the $X_7$ and its supersymmetric 5-cycles $\Sigma_5$ and obtain a rational expression for the ratio of the volume of $X_7$ and that of the unit 7-sphere and so on. We also discuss the dual 3-dim superconformal field theories with $\mathcal{N} = 3$. During our investigation, we found these type of space has appeared before in the mathematics literature \[10, 11, 12\] where it is known as the Eschenburg space \[13\]. However, our calculation of the volumes of the space and the super 5-cycles found in this paper seems original.

The simplest case with $X_7 = S(1, 1, 1)$ is known as $N(1, 1)$ and its cone is the relative moduli space of a single instanton in $SU(3)$ gauge group, which is hyperkahler 8-dim space with one scale and the coset space $SU(3)/U(1)$. There has been considerable work on the AdS-CFT correspondence on the $AdS^4 \times N(1, 1)$ space\[14, 15, 16\]. Especially as $N(1, 1)$ is homogeneous, one can study the Kaluza-Klein modes of the theory to compare them with the dual SCFT. More recent investigation on $AdS^4 \times X_7$ space with known $X_7$ and its marginal deformation can be found in Ref.\[17\].

Our work has been motivated in part by the effort to understand the mysterious 3-dim $\mathcal{N} = 16$ supersymmetric conformal field theory which is the low energy theory of $N$ parallel M2 branes. They can be regarded as the strong coupling limit $e^2 \rightarrow \infty$ of $\mathcal{N} = 16$ supersymmetric Yang-Mills theories, as one can easily see in the M-theory limit of D2 branes. One may deform the $\mathcal{N} = 16$ supersymmetric Yang-Mills by adding Chern-Simons terms, so that the resulting theory has a less supersymmetry $\mathcal{N} = 3$ \[18, 19\]. In the infrared limit or strong coupling limit $e^2 \rightarrow \infty$, the theory becomes purely Chern-Simons Higgs, which is superconformal. While the Chern-Simons level $k$ is quantized to be integer, the small $|k|$ limit is the strong-coupling limit. Unfortunately its physics is not well understood. One may still hope that the physics near $k = 0$ is similar to that of $N = 16$ superconformal theory. (See for a similar idea in Ref.\[20\].)
Another motivation was to try to understand further the old result on the superconformal field theory dual for the AdS geometry with tri-Sasakian space $X_7 = N(1,1)$ whose 8-dim cone is the relative moduli space of a single instanton in $SU(3)$ gauge theory. While the corresponding field theory may have some component of Chern-Simons theory, the ’t Hooft coupling from the geometry seems to be related to the parameter $N$ of the corresponding gauge group $SU(N) \times SU(N)$, instead of the generic ’t Hooft coupling $N/k$ of the Chern-Simons-Higgs theory where $k$ is the integer quantized Chern-Simons level. Our models would provide more examples along this line.

Final motivation was to try to construct new tri-Sasakian geometry similar to instanton moduli space by generalizing the moduli space of three distinct magnetic monopoles, which could be constituent of a single instanton of $SU(3)$ theory in $R^3 \times S^1$ geometry[21]. Thus one wants to generalize the moduli space of $N$ distinct magnetic monopoles in $SU(N)$ theory broken to $U(1)^{N-1}$ in $R^3 \times S^1$. There would be magnetic monopoles for each simple root of the extended Dynkin diagram of $SU(N)$ gauge group. But we generalize the interaction strength between nearest neighbor by arbitrary magnitude. Only requirement is that the geometry is smooth whenever magnetic monopoles are coming together except when all of them are coming together. In appendix this is shown to lead to the interaction strength between each link to be some natural number instead of the unity as in the $SU(N)$ case. In the limit where $N - 1$ monopoles become massless, the relative geometry has only one scale parameter which controls the overall size of the system, and so a cone-like geometry with a singularity at the apex of the cone. Our geometry $X_7$ can also obtain from this approach.

The geometry we are interested in is the $AdS_4 \times X_7$ type solutions of $D = 11$ supergravity where $X_7$ is an Einstein manifold and the four-form field strength $F_4 \sim vol_{AdS_4}$. We normalize the metric on $X_7$ so that $R_{\mu \nu}(X_7) = 6 g_{\mu \nu}(X_7)$. To preserve some of 32 supersymmetry of 11-dim supergravity, the eight dimensional cone $M_8$ over $X_7$ with the metric

$$ds^2_{M_8} = dr^2 + r^2 ds^2(X_7)$$  \hspace{1cm} (1.1)$$

should be Ricci-flat and have special holonomy. (For example see Ref. [22].) When the cone has $Sp(2) = SO(5)$ holonomy, and so hyper-Kähler, the space $X_7$ is tri-Sasakian and the dual theory is a $\mathcal{N} = 3$ SCFT.

Such a geometry arises as the near horizon limit of $M2$ branes lying at the singular apex of the Ricci-flat cone $M_8$. The dual SCFT lives on the $M2$ branes. The flux of $F_4$ on $X_7$ is proportional to the number of $M2$ branes. Baryonic states of SCFT are dual to five-branes wrapping five-cycles $\Sigma_5$ in the manifold $X_7$. For supersymmetric states the five-cycles must lift to supersymmetric 6-cycles in the cone $M_8$. A supersymmetric 6-cycle will be holomorphic with respect to one of the
three complex structures, breaking two of six supersymmetries. The dimension of the baryonic operators are given by the geometric formula \[ \Delta = \frac{\pi N \text{Vol}(\Sigma_3)}{6 \text{Vol}(X_7)}. \] (1.2)

In SCFT, they can often be predicted from their R-charges. Comparing the two predictions is then a non-trivial check for the gauge/gravity correspondence.

This Eschenburg space \( X_7 = S(t_1, t_2, t_3) \) \[10\] can be regarded as a left-quotient space of \( SU(3) \) group manifold by \( U(1) \) group whose elements are \( \text{diag}(e^{i\theta_1 \psi}, e^{i\theta_2 \psi}, e^{i\theta_3 \psi}) \). Their homological properties seem to be known. Here we provide an explicit metric and calculation of the volume of the space and supersymmetric 5-cycles. The space \( X_7 = S(t_1, t_2, t_3) \) depends only on the \( t_i \)'s up to overall common factor. It is homogeneous with \( SU(3) \times SU(2)_R \) symmetry when \( t_1 = t_2 = t_3 \). When \( t_1 = t_2 \neq t_3 \), the space has co-homogeneity one with \( SU(2) \times U(1) \times SU(2)_R \) symmetry. When all \( t_a \) are different from each other, the space has co-homogeneity two with \( U(1)^2 \times SU(2)_R \) symmetry. We find the two kind of expressions for the metric for \( X_7(t) \). The first one is explicit but not useful. The second one is more implicit but shows the symmetric and cone structures clearly.

Instead of using the metric, we use the equivariant cohomology and localization technique \[23\] to calculate the volume \( X_7(t) \). This approach is somewhat esoteric and so the detail is provided here. We find that the ratio of the volume of the tri-Sasakian space \( X_7 \) and that for any supersymmetric 5-dim cycle \( \Sigma(t) \) is independent of \( t = (t_1, t_2, t_3) \) and identical to the ratio for the volume of unit 7-sphere and that of unit 5-sphere. The explicit form for the volumes are

\[
\frac{\text{vol}(X_7(t))}{\text{vol}(S_7)} = \frac{\text{vol}(\Sigma_5(t))}{\text{vol}(S_5)} = \frac{t_1 t_2 t_3 (t_1 t_2 + t_2 t_3 + t_3 t_1)}{\text{l.c.m.}(t_1 t_2, t_2 t_3, t_3 t_1)(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)},
\] (1.3)

where \( \text{l.c.m.} \) means the least common multiple and \( \text{vol}(S_7) = \pi^4/3 \) for the unit 7 sphere and \( \text{vol}(S_5) = \pi^3 \) for the unit 5 sphere. The maximum of the above ratio for any \( t \) appears when \( t_1 = t_2 = t_3 = 1 \) for the well-known space \( N(1,1) \).

The dual superconformal field theory in three dimension is \( SU(N)_1 \times SU(N)_2 \) gauge theory with \( \mathcal{N} = 3 \) supersymmetry. The matter fields \( U_a = (u_a, -v^*_a) \) are hypermultiplets in \( \mathcal{N} = 4 \) language belonging to the symmetrized product representation \( \text{Sym}^a(N) \) of the fundamental representation of \( SU(N)_1 \) and that \( \text{Sym}^a(N) \) of the anti-fundamental representation of \( SU(N)_2 \). The internal global symmetry is again \( SU(2)_R \times U(1)^2 \) for distinct \( t_a \). The chiral primary operators and baryonic operators show that one can assign a chiral dimension \( 1/2 \) for the \( U_a \) field and its complex conjugates.

There are several directions to pursue. Our SCFT is again mysterious as the \( \mathcal{N} = 8 \) superconformal field theory in 3-dim as it is not quite the Chern-Simons theory.
They may be the strong-coupling limit of the supersymmetric Yang-Mills Chern-Simons theory with $\kappa \to 0$. One curious aspect of our AdS-CFT correspondence is that there is no obvious geometry for the dual theory when $t_1 = t_2 = t_3 \neq 1$ as the $X_7(t)$ is defined only up to common factors of $t_a$.

The plan of the paper is as follows. In Sec.2 we define the 8-dim hyperkahler space $\mathcal{M}_8(t)$ which is a singular cone and is obtained from a hyperkahler quotient of 3-dim quaternion space $\mathbb{H}^3 = \mathbb{R}^{12}$ by using a single $U(1)$ group. We find its metric explicitly and also show the space has the cone geometry. We identify its isometry. In Sec.3 we review the homology property of tri-Sasakian space $X_7(t)$ first. Then we calculate the volumes of $X_7(t)$ and supersymmetric 5-cycles $\Sigma_5$ in $X_7$ in the language of equivariant cohomology. In Sec.4, we identify the dual SCFT and study its properties. In Appendix, we generalize the caloron moduli space.

2. Hyperkahler Space $\mathcal{M}_8(t)$

Let us start from 12-dim flat hyperkahler space $\mathbb{H}^3$ defined by the three quaternions $q_1, q_2, q_3$. (See for example Ref. [24] for an introduction.) Each quaternion is defined as

$$q_a = q^\mu_a + i \sigma \cdot q_a, \quad \bar{q}_a = q^\mu_a - i \sigma \cdot q_a,$$

with four real numbers $q^\mu_a, \mu = 1, 2, 3, 4$ and three Pauli matrices $\sigma^1, \sigma^2, \sigma^3$. The Euclidean flat metric on 12-dim is

$$ds^2 = \frac{1}{2} \sum_a \text{tr}(dq_a \otimes d\bar{q}_a) = \sum_a dq_a^\mu dq_a^\mu,$$

and the three Kähler forms are

$$\omega \cdot \sigma = \frac{1}{2} dq \wedge d\bar{q}.$$

Sometimes we use complex coordinates for quaternions as

$$q_a = \begin{pmatrix} u_a & v_a \\ -\bar{v}_a & \bar{u}_a \end{pmatrix},$$

in which the metric becomes

$$ds^2 = \frac{1}{2} \sum_a (du_a \otimes d\bar{u}_a + dv_a \otimes d\bar{v}_a + \text{c.c.}).$$

Another useful coordinate for quaternions is

$$q_a = p_a e^{i \sigma_3 \psi_a}.$$
where \( p_a \) is pure imaginary, or \( \bar{p}_a = -p_a \). In terms of the 3-dim Cartesian coordinates \( \mathbf{r}_a \) such that

\[
i \mathbf{r}_a \cdot \mathbf{\sigma} = q_i \sigma_3 \bar{q} = -i p_a \sigma^3 p_a
\]  

(2.7)

and angle variable \( \psi_a \), the flat metric on 12-dim becomes

\[
ds^2 = \frac{1}{4} \sum_a \left( \frac{d\mathbf{r}_a^2}{r_a} + r_a (d\psi_a + \mathbf{w}_a \cdot d\mathbf{r}_a)^2 \right),
\]  

(2.8)

where \( r_a = |\mathbf{r}_a| \) and \( \nabla \times \mathbf{w}_a (\mathbf{r}_a) = \nabla (1/r_a) \).

For each triple natural numbers \( t_1, t_2, t_3 \), we consider a corresponding abelian symmetry \( U_t(1) \), under which

\[
q_a \rightarrow q_a e^{i \sigma_3 t_a \chi}, \quad a = 1, 2, 3.
\]  

(2.9)

The \( U_t(1) \) is unique up to a common factor on triples. The corresponding moment map \( \mu \) is

\[
\mu \cdot \mathbf{\sigma} = \sum_a t_a q_a \sigma_3 \bar{q}_a = \sum_a t_a \mathbf{r}_a
\]

\[
= \sum_a t_a \begin{pmatrix}
|u_a|^2 - |v_a|^2 & -2u_a v_a \\
-2\bar{u}_a \bar{v}_a & -|u_a|^2 + |v_a|^2
\end{pmatrix}.
\]  

(2.10)

The space which satisfies the constraint \( \mu = 0 \) becomes 9-dimensional. Once we mod out \( U(1)_t \) on this space, the resulting quotient space becomes 8-dim hyperkahler space,

\[
\mathcal{M}_8(t) = \mu^{-1}(0)/U_t(1).
\]  

(2.11)

This process of hyperkahler quotient is defined by three natural numbers \( t_1, t_2, t_3 \). As \( \mathcal{M}_8(t) \) is hyperkahler, it is Ricci-flat automatically.

Let us consider in detail the symmetry of the hyperkahler space \( \mathcal{M}_8(t) \). The first one is the \( SU(2) \) symmetry which rotates three complex structures in 12-dim space,

\[
q_a \rightarrow \exp\left(-\frac{i}{2} \epsilon \cdot \mathbf{\sigma}\right) q_a, \quad a = 1, 2, 3,
\]  

(2.12)

where \( \epsilon \) are the \( SU(2) \) parameters. Under this \( SU(2) \) transformation, \( \mathbf{r}_a \) for each \( a \) transforms as a vector. This is commuting with the hyperKähler quotient and so the resulting space has \( SU(2)_R \) symmetry. The additional symmetry arises from the transformation

\[
q_a \rightarrow q_b \left( \exp i T \sigma^3 \right)_{ba}
\]  

(2.13)

where \( T \) is the \( U(3) \) generator which commutes with the \( U(1)_t \) generator

\[
t = \text{diag}(t_1, t_2, t_3).
\]  

(2.14)
and leaves $U(1)$ invariant subspace invariant. Thus when $t_1 = t_2 = t_3$, the resulting symmetry is $SU(3)$. When $t_1 = t_2 \neq t_3$, the resulting symmetry is $SU(2) \times U(1)$. When $t_1, t_2, t_3$ are all different, the resulting symmetry would be $U(1)^2$.

For $t_1 = t_2 = t_3$, the resulting 8-dim space is the moduli space of a single SU(3) instanton in the center of mass frame. The 8 parameters denote a single scale parameter and 7 coordinates for the coset space $SU(3)/U(1)$, and so the space is cone-like. For generic $t_a$, the metric is more complicated. A simple way to write the metric is to start from the flat metric (2.8) and express the $r_3$ in terms of $r_1$ and $r_2$ by using the moment map (2.10) so that with $A = 1, 2$

$$4ds^2 = C_{AB}dr_Adr_B + C^{AB}(d\psi_A + w_{AC} \cdot dr_C)(d\psi_B + w_{BD} \cdot dr_D), \quad (2.15)$$

where

$$C_{11} = \frac{1}{r_1} + \frac{t_1^2}{t_3|t_1r_1 + t_2r_2|},$$

$$C_{22} = \frac{1}{r_2} + \frac{t_2^2}{t_3|t_1r_1 + t_2r_2|},$$

$$C_{12} = C_{21} = \frac{t_1t_2}{t_3|t_1r_1 + t_2r_2|}; \quad (2.16)$$

and the vector potential satisfies $\nabla_C \times w_{AB} = \nabla_C C_{AB}$. This metric is hyperkahler and regular unless $r_1 = r_2 = 0$ simultaneously\[25, 26\].

To express the metric so that the cone-structure is manifest needs more work. Let us focus on the generic case where all $t_a$ are different. The moment map vanishes and so $t_1r_1 + t_2r_2 + t_3r_3 = 0$, which defines a triangle whose side length are $t_1r_1, t_2r_2, t_3r_3$. Using the $SU(2)_R$ transformation, we can put this triangle on the 1-2 plane. In this case, the complex coordinates $u_a$ and $v_a$ satisfy

$$|u_a| = |v_a| = \sqrt{\frac{r_a}{2}}, \sum_{a=1}^3 t_au_av_a = 0. \quad (2.17)$$

The general configuration would be made of the rotation of the triangle in space and also the phase rotation of $u_a$ and $v_a$ in opposite way. Triangle has three independent parameters. The spatial rotation has three independent parameters. The relative phase of $u_a, v_a$ variables has three independent parameters, one of which is the global $U(1)$ which should be mode out. Thus there are eight independent parameters.

To specify the moduli parameter for the triangle, we choose the parameters to be $u_a = v_a = \sqrt{r_a/2} = |u_a|e^{i\phi_a/2}$, which implies the moduli metric of the triangle to be

$$ds^2_\Delta = \frac{1}{4} \sum_{a=1}^3 \left( \frac{1}{r_a} dr_a^2 + r_a d\phi_a^2 \right), \quad (2.18)$$
with the condition (2.17) being

\[ \sum_a t_a r_a e^{i \varphi_a} = 0. \]  

(2.19)

This is the condition on for three complex vectors \( t_a r_a e^{i \varphi_a} \) to form a triangle. This constraint depends only on the relative angles \( \theta_a \) of vectors as

\[ \theta_1 = \varphi_3 - \varphi_2, \quad \theta_2 = 2\pi + \varphi_1 - \varphi_3, \quad \theta_3 = \varphi_2 - \varphi_1, \]  

(2.20)

where \( 0 \leq \varphi_a < 2\pi \). Only two of the relative angles are independent as \( \theta_1 + \theta_2 + \theta_3 = 2\pi \). The overall orientation angle \( \varphi = \frac{\varphi_1 + \varphi_2 + \varphi_3}{3} \) of the triangle are a part of the rotational degrees from \( SU(2)_R \). The above triangle condition (2.19) implies the three following conditions on the length and relative angles as given in elementary geometry:

\[ \begin{align*}
& t_{1r_1}^2 = t_{2r_2}^2 + t_{3r_3}^2 + 2t_2 r_2 t_3 r_3 \cos \theta_1, \\
& t_{2r_2}^2 = t_{3r_3}^2 + t_{1r_1}^2 + 2t_3 r_3 t_1 r_1 \cos \theta_2, \\
& t_{3r_3}^2 = t_{1r_1}^2 + t_{2r_2}^2 + 2t_1 r_1 t_2 r_2 \cos \theta_3,
\end{align*} \]  

(2.21)

of which only two are independent. Thus these conditions reduce the independent variables to three, which we choose as one length variable, and two relative angle variables.

To solve the above constraints (2.21), let us introduce an angle variable \( A \), a length square variable \( L \), and an area variable \( S \) such that

\[ \begin{align*}
& A = -(\cot \theta_1 + \cot \theta_2 + \cot \theta_3), \\
& L = \sum_a t_a^2 r_a^2, \\
& S = t_1 r_1 t_2 r_2 \sin \theta_3 = t_3 r_3 t_1 r_1 \sin \theta_2 = t_2 r_2 t_3 r_3 \sin \theta_1.
\end{align*} \]  

(2.22)

Note that \( S \) is twice the area of the triangle, and the triangle condition implies that

\[ S = \frac{L}{2A}, \quad t_a^2 r_a^2 = L \left( 1 + \frac{\cot \theta_a}{A} \right) \text{ for each } a \]  

(2.23)

with \( A \geq 0 \). Let us now introduce the radial variable in 12-dim flat space as the length variable,

\[ r = \sqrt{|q_1|^2 + |q_2|^2 + |q_3|^2} = \sqrt{r_1 + r_2 + r_3}. \]  

(2.24)

Defining a function \( B \) of angles \( \theta_a \) as

\[ B = \sum_a \frac{1}{t_a} \sqrt{\left( 1 - \frac{\cot \theta_a}{\cot \theta_1 + \cot \theta_2 + \cot \theta_3} \right)}, \]  

(2.25)
we see the length variable $L$ is given in terms of three independent variables $r, \theta_a$ as follows,

$$L = \frac{r^4}{B^2}. \quad (2.26)$$

So the variables $r_a$ can be written in terms of scale variable $r$ and angles $\theta_a$ as follows,

$$r_a = r^2 \rho_a, \quad \rho_a \equiv \frac{1}{t_a} \sqrt{\left(1 - \frac{\cot \theta_a}{\sum_b \cot \theta_b}\right) / \sum_c \frac{1}{t_c} \sqrt{\left(1 - \frac{\cot \theta_c}{\sum_d \cot \theta_d}\right)}}. \quad (2.27)$$

Note that three functions $\rho_a$ of angles $\theta_a$ satisfies the condition $\rho_1 + \rho_2 + \rho_3 = 1$.

The moduli space of the triangle on the plane would be then

$$ds^2_\Delta = \frac{1}{4} \sum_a \left(\frac{1}{r_a} dr_a^2 + r_a d\varphi_a^2\right) = dr^2 + \frac{r^2}{4} \sum_a \left(\frac{d\rho_a^2}{\rho_a} + d\varphi_a^2\right), \quad (2.28)$$

as $\sum \rho_a = 1$. Note the $\varphi_a$ can be written as the relative angles $\theta_a$ and the overall orientation of the triangle on the plane. The metric for the $M_8$ can be now obtained by parameterizing quaternion as follows

$$Q \equiv \begin{pmatrix} u_1 & u_2 & u_3 \\ -\bar{v}_1 & -\bar{v}_2 & -\bar{v}_3 \end{pmatrix} = RQ_0 T, \quad (2.29)$$

where the $R$ is an $SU(2)$ element parameterized by Euler angle, which includes the orientation of the triangle on the plane, $T$ is a diagonal $U(3)$ element, say, $T = \text{diag}(e^{i\psi_1}, e^{i\psi_2}, e^{i\psi_3})$ and $Q_0$ is the value of $Q$ when the triangle is on 1-2 plane and so

$$Q_0 = \begin{pmatrix} \sqrt{r_1}e^{i\varphi_1/2} & \sqrt{r_2}e^{i\varphi_2/2} & \sqrt{r_3}e^{i\varphi_3/2} \\ -\sqrt{r_1}e^{-i\varphi_1/2} & -\sqrt{r_2}e^{-i\varphi_2/2} & -\sqrt{r_3}e^{-i\varphi_3/2} \end{pmatrix}. \quad (2.30)$$

The metric of the triangle is $ds^2_\Delta = dQ_0 d\bar{Q}_0$ of the metric (2.28) and so the metric on the 9-dim space is

$$ds^2_{\mu^{-1}(0)} = \frac{1}{2} (dQ \otimes d\bar{Q} + d\bar{Q} \otimes dQ) . \quad (2.31)$$

It is trivial to mode out $U_t(1)$ to get the 8-dim hyperkahler space $M_8(t)$. The 3 kahler forms are again given by

$$\omega \cdot \sigma = dQ \wedge d\bar{Q}. \quad (2.32)$$

The isometry of $M_8(t)$ can be easily read. First of all the $SU(2)$ transformation by $R$ matrix leads to $SU(2)_R$ symmetry which mixes three complex structure. In addition there are $U(1) \times U(1)$ isometries from the tranformations given by $T$ matrix modulo $U_t(1)$, which are tri-holomorphic as they leave three kahler structures invariant. When some of $t_1, t_2, t_3$ become identical, these tri-holomorphic isometries get
enhanced. If only two of three are identical, $U(1)^2$ gets enhanced to $U(1) \times SU(2)$. If all three $t_a$ are identical, $U(1)^2$ gets enhanced to $SU(3)$.

The metric (2.31) has no mixed terms for $dr$ and other angle variables including $dR, dT$ as $R$ and $T$ are unitary. Thus, the metric on the hyperKähler space $\mathcal{M}_8$ has a cone structure,

$$ds^2_{\mathcal{M}_8} = dr^2 + r^2 ds^2_{X_7}.$$  \hfill (2.33)

It is also regular everywhere except at the tip of the cone. It is Ricci-flat and the 7-dim space $X_7$ is smooth everywhere without singularity and characterized by three natural numbers $(t_1, t_2, t_3)$ without common factor. In the next section, we study properties of this 7-dim tri-Sasakian space $X_7(t)$.

### 3. Tri-Sasakian Space $X_7(t)$

We have now the 8-dimensional hyperkahler space $M_8(t)$, whose metric is cone-like and determined by three natural numbers $t_1, t_2, t_3$ modulo common factor. This space is smooth except at the tip of the cone. As the metric is written down almost explicitly, one has now the corresponding 7-dimensional tri-Sasakian space $X_7(t)$ which is smooth everywhere.

There are several properties of this space which are relevant for our consideration. The isometry of $X_7(t)$ is identical to the cone geometry $M_8(t)$. With $t_1 = t_2 = t_3$, the unique space is equivalent to $S(1, 1, 1) = N(1, 1)$. With $t_1 = t_2 \neq t_3$, there are class of geometry with $S(r, r, s)$ with coprime natural numbers $r, s$. Finally, when all $t_a$ are different, we can assume that there is no common factor in them. This space $X_7(t) = S(t_1, t_2, t_3)$ is non-singular. It is sometimes called the Eschenburg space. This toric Sasakian space is a subfamily of the more general spaces, bi-quotients of the $U(3)$ group manifold, which was studied by Eschenburg[13].

The quotient of $S(t_1, t_2, t_3)$ by $SO(3)_R$ action is a quaternionic Kähler orbifold. For any $U(1)_R$ subgroup of $SO(3)_R$, one can locally write the metric as

$$ds^2(X_7) = (d\psi + \sigma)^2 + ds^2(M_6),$$ \hfill (3.1)

where $M_6$ is locally Kahler-Einstein. If the Reeb vector $\partial/\partial\psi$ has a closed orbit, then $M_6$ is in general Kahler orbifold. A tri-Sasakian space is regular if its quotient by $SO(3)_R$ is a quaternionic Kähler manifold. Our case $S(t_1, t_2, t_3)$ would be a smooth tri-Sasakian 7-manifold which is not regular unless $t_1 = t_2 = t_3$ in which case it is homogeneous. Indeed all homogeneous tri-Sasakian spaces in $4n+3$ dimensions seem to be associated with Lie algebra and seem to be originated from the moduli space of a single instanton in a gauge theory of a given Lie group as it has only one scale.
parameter and is hyperkahler. The betti-numbers of the space $S(t_1, t_2, t_3)$ \cite{10} are

$$b_0 = b_7 = 1, b_1 = b_6 = 0, b_2 = b_5 = 1, b_3 = b_4 = 0,$$  (3.2)

which indicates one can have nontrivial wrapping of the geometry by $M2$ branes and $M5$ branes. The wrapping of 5 cycles by $M5$ branes leads to the baryonic objects in the dual SCFT.

One can obtain more general 7-dimensional toric tri-Sasakian space first proposed in Ref.\cite{11} by considering $N+2$-quaternion space $q_a, a = 1, 2, ..., N + 2$ with $N$ independent $U(1)$ groups acting on them with charge matrix

$$q_a \rightarrow q_a e^{i\sigma_3 t_A^a \chi_A},$$  (3.3)

where $A = 1, 2, ... N$ leads to $N$ abelian symmetry. There are $N$ corresponding moment map

$$\mu^A = \sum_{a=1}^{N+2} t_A^a q_a \sigma_3 q_a.$$  (3.4)

The hyperKahler quotient of the $N+2$ dimensional quaternion space by these abelian groups leads to 8-dimensional hyperkahler space. Since we propose dual SCFT only for $N = 1$ case in this work, generalization for $N \geq 2$ would be an interesting future problem.

### 3.1 The volume of tri-Sasakian space $X_7(t)$

*The Volume of hyperkahler quotient $\mathcal{M}$ and their Equivariant Deformation*

Suppose that a hyperkähler manifold $P$ with three kahler forms $\omega$ has a symmetry group $G$ generated by tri-holomorphic vector fields $V^a$, and so $L_{V^a} \omega = d(i_{V^a} \omega) + i_{V^a} d\omega = 0$. As $d\omega = 0$, there exists three moment map $\mu^a$ such that

$$i_{V^a} \omega = d\mu^a.$$  (3.5)

The hyperkahler quotient space $M = \mu^{-1}/G$ is again hyperkahler with induced three kahler forms.

Our objective in this subsection is to express the volume of the quotient $\mathcal{M}$ in terms of some integration over the ambient space $P$, which we treat as the flat hyperkahler space $P = \mathbb{H}^n \cong \mathbb{C}^{2n}$. In this work, we treat the quotient group to be a single abelian group, but the generalization to non-abelian groups is similar. Our work here is a straightforward application of the method in Ref. \cite{23}.

We can pick any kahler form out of $\omega$, say $\omega(x) \equiv \omega^3 = \omega_{\mu\nu}(x)dx^\mu \wedge dx^\nu/2$, to define the volume of a hyperkahler manifold $P$,

$$vol(P) = \int_P e^\omega = \frac{1}{(2n)!} \int_P \omega^{2n},$$  (3.6)
where \( \dim_C \mathcal{P} = 2n \). The normalization at this point is obscure, but later we will fix it to reproduce the flat volume of \( \mathbb{H}^n \cong \mathbb{C}^{2n} \). Fixing normalization for the ambient space then unambiguously determine that of the quotient space.

We introduce mutually anti-commuting Grassmann variables \( \psi^\mu \) which replace the 1-form variable \( dx^\mu \), and rewrite the Kahler form as \( \omega(x, \psi) = \omega_{\mu\nu}(x)\psi^\mu \psi^\nu / 2 \), which is a function of a bosonic coordinates \( x^\mu \) and fermionic variables \( \psi^\mu \). Any differential form \( f \) in the space of differential forms, \( \Omega^*(\mathcal{P}) \), can be regarded as a function \( f(x, \psi) \). We can consider \( (x, \psi) \) as parameterizing a supermanifold \( \mathcal{P}' \), in which usual tangent space is fermionic rather than bosonic, say \( \psi^\mu \partial / \partial x^\mu \). With this notation, the integration of a top differential form \( f \) on \( \mathcal{P} \) is written in a way that mimics supersymmetric functional integration,

\[
\int_{\mathcal{P}} f = \int_{\mathcal{P}'} dx^1 dx^2 \cdots dx^{4n} d\psi^1 d\psi^2 \cdots d\psi^{4n} f(x, \psi) = \int_{\mathcal{P}'} [dx][d\psi] f(x, \psi). \tag{3.7}
\]

It can be checked by calculating super-Jacobian that the measure \( [dx][d\psi] \) is invariant under coordinate reparametrization, and so is well-defined everywhere on \( \mathcal{P} \). The volume formula in (3.6) is then written

\[
\text{vol}(\mathcal{P}) = \int_{\mathcal{P}'} [dx][d\psi] e^\omega. \tag{3.8}
\]

Note that \( \int [dx][d\psi] \) automatically picks up the top dimensional form in the expansion of \( e^\omega \) due to the properties of the Grassmann integration.

Now let us consider the volume of the hyperkahler quotient space \( \mathcal{M} \) of the flat space \( \mathcal{P} \) by a \( U(1) \) action, which is generated by the Killing vector \( V = V^\mu(x) \partial / \partial x^\mu \) which preserves the triplet of Kahler forms, \( \mathcal{L}_V \omega = 0 \rightarrow i_V \omega = d\mu. \tag{3.9} \)

The hyperkahler quotient space \( \mathcal{M} \) is defined as \( \mathcal{M} = \mu^{-1}(0)/U(1) \), which is again hyperkahler with Kahler forms naturally inherited from \( \mathcal{P} \).

To describe the quotient procedure more explicitly, we first note that \( \mathcal{L}_V \mu = d\mu(V) = i_V \omega(V) = \omega(V, V) = 0 \), which shows that the level surface \( \mu^{-1}(0) \) is invariant under \( U(1) \) flow. Then the Killing vector \( V^\mu \) is parallel to \( \mu^{-1}(0) \). We can therefore introduce a local coordinate system \( (x^i, x^\ell) \) on \( (4n-3) \)-dim space \( \mu^{-1}(0) \) such that \( V = \partial / \partial x^\ell \), and \( x^i \) (\( i = 1, \ldots, 4n - 4 \)) are constant along \( U(1) \) trajectories. As we further quotient along the direction of \( V \) to get \( \mathcal{M} \), we naturally identify \( x^i \) as a coordinate system on \( \mathcal{M} \). In the ambient space \( \mathcal{P}, \mu^{-1}(0) \) is codimension 3, so we locally introduce \( x^\ell \) (\( \ell = 1, 2, 3 \)) around \( \mu^{-1}(0) \) as the coordinates along normal directions. In the following, we only consider points on \( \mu^{-1}(0) \) (or \( x^\ell = 0 \)) unless stated otherwise. In components, the equation \( i_V \omega = d\mu \) becomes

\[
\omega_{\nu\ell} dx^i + \omega_{i\ell} dx^n = \frac{\partial \mu}{\partial x^i} dx^i + \frac{\partial \mu}{\partial x^\ell} dx^\ell = \partial_i \mu dx^i + \partial_\ell \mu dx^\ell. \tag{3.10}
\]
Because $\mu = 0$ on $\mu^{-1}(0)$, we have $\partial_i \mu = 0$, and we get
\[ \omega_{vi} = 0, \quad \omega_{\ell i} = \partial_i \mu. \] (3.11)
From the closed-ness equation $d\omega = 0$, we have $\partial_i \omega_{ij} = \partial_i \omega_{vj} - \partial_j \omega_{vi}$, but $\omega_{vi} = 0$ on every $\mu^{-1}(0)$ and its tangent derivatives also vanish, so that $\partial_i \omega_{ij} = 0$. This means that $\omega_{ij} = \omega_{ij}(x^i) \wedge dx^j / 2$ does not vary along $V$, and so is well defined on $\mathcal{M}$, and so is identified as the induced triplet Kahler forms.

As we have identified a coordinate system and triplet Kahler forms on the quotient space $\mathcal{M}$, its symplectic volume will be
\[ \text{vol}(\mathcal{M}) = \int_{\mathcal{M}'} [dx^i][dx^j] e^{\frac{i}{2}\omega_{ij}\psi^i\psi^j}, \] (3.12)
where $\omega_{ij} = \omega_{ij}^3(x^i)$ on $\mu^{-1}(0)$ and $\mathcal{M}'$ is the corresponding supermanifold for $\mathcal{M}$. Our aim is to rewrite (3.12) as an integration over the ambient space $\mathcal{P}$, which doesn’t depend manifestly on a particular coordinate system we have chosen in the above. Firstly, because $\omega_{ij}$ is independent of $x^v$, the integral can be extended to $\mu^{-1}(0)$ as
\[ \int_{\mathcal{M}'} [dx^i][dx^j] e^{\frac{i}{2}\omega_{ij}\psi^i\psi^j} = \frac{1}{\text{vol}(U(1))} \int_{\mu^{-1}(0)} [dx^i][dx^j][dx^\ell] e^{\frac{i}{2}\omega_{ij}\psi^i\psi^j}, \] (3.13)
where $\text{vol}(U(1)) = \int dx^v$ is the range of the coordinate $x^v$. To confine the integration on $\mathcal{P}$ onto $\mu^{-1}(0) = \{ x^n = 0 \}$, we would need a $\delta$-function factor $\prod_{a=1}^3 \delta(\mu^a(x))$, and to correctly reduce $\int [dx^i][dx^v][dx^n]$ into $\int [dx^i][dx^v]$, we have to add a Jacobian factor,
\[ \int_{\mu^{-1}(0)} [dx^i][dx^v] = \int \prod_{a=1}^3 \delta(\mu^a(x)) \det (\partial_i \mu^a) \]
\[ = \frac{1}{(2\pi)^3} \int \prod_{a=1}^3 \delta(\mu^a(x)) \det (\partial_i \mu^a) \]
\[ = \frac{1}{(2\pi)^3} \int \prod_{a=1}^3 \delta(\mu^a(x)) \det (\partial_i \mu^a) \]
\[ \times e^{\frac{i}{2}\omega_{ij}\psi^i\psi^j + \omega_{i\ell}\psi^i\psi^\ell}, \] (3.14)
where $\phi_a$ are bosonic and $\chi_{a,\ell}$, $\psi^\ell$ are fermionic variables. From (3.11), we have $\partial_i \mu^3 = \omega_{i\ell} = \omega_{\ell i}$, and calling $\chi_3 \equiv \psi^v$, we have $\chi_3(\partial_i \mu^3)\psi^v = \omega_{i\ell}\psi^\ell\chi_3 = \omega_{i\ell}\psi^\ell\psi^\mu$ since $\omega_{vi} = 0$ on $\mu^{-1}(0)$. Similarly $\chi_1 \partial_i \mu^1 \psi^v + \chi_2 \partial_i \mu^2 \psi^v = \chi_1 \partial_i \mu^1 \psi^v + \chi_2 \partial_i \mu^2 \psi^v \equiv \chi_1 \partial_i \mu^1 + \chi_2 \partial_i \mu^2$ since $\partial_i \mu^a = 0$, $\partial_i \mu^a = 0$ on $\mu^{-1}(0)$. Therefore, the volume of $\mathcal{M}$ is written as
\[ \text{vol}(\mathcal{M}) = \frac{1}{(2\pi)^3} \int \prod_{a=1}^3 \delta(\mu^a(x)) \det (\partial_i \mu^a) \]
\[ \times e^{\frac{i}{2}\omega_{ij}\psi^i\psi^j + \omega_{i\ell}\psi^i\psi^\ell + i\phi_a \mu^a + \chi_1 \partial_i \mu^1 + \chi_2 \partial_i \mu^2}. \] (3.15)
where $\mathcal{P}''$ is the space a bit bigger than the supermanifold with additional coordinates $\phi_a, \chi_{1,2}$. The terms in the exponent involving $\omega$ almost comprise the Kahler form.
\( \frac{1}{2} \omega_{\mu\nu} \psi^\mu \psi^\nu \) on \( X \), except \( \omega_{\mu\nu} \psi^\mu \psi^\nu + \frac{1}{2} \omega_{mn} \psi^m \psi^n \). However, the first term can be removed by shifting \( \psi \), and also the second term by shifting \( \chi_a \). Therefore, we can replace the exponent by \( \frac{1}{2} \omega_{\mu\nu} \psi^\mu \psi^\nu + i \phi_a \mu^a + \chi_1 d\mu^1 + \chi_2 d\mu^2 \) without changing the result. Finally, the measure involves \([dx][d\psi]\) on \( X \), which is manifestly independent of a particular coordinate choice. Thus, we have

\[
\text{vol}(\mathcal{M}) = \frac{1}{(2\pi)^3 \text{vol}(U(1))} \int_{\mathcal{P}^n} [dx][d\psi][d\phi_a][d\chi_{1,2}] \ e^S, \quad (3.16)
\]

where the ‘action’ is

\[
S(x, \psi, \phi_a, \chi_1, \chi_2) = \omega + i \phi_a \mu^a + \chi_1 d\mu^1 + \chi_2 d\mu^2 \\
= \frac{1}{2} \omega_{\mu\nu}(x) \psi^\mu \psi^\nu + i \phi_a \mu^a(x) + \chi_1 \partial_\mu \mu^1(x) \psi^\mu + \chi_2 \partial_\mu \mu^2(x) \psi^\mu. \quad (3.17)
\]

The above integration looks like a path integral of a \((0+0)\)-dimensional supersymmetric system. Indeed, the action \( S \) has the following fermionic symmetry,

\[
Q x^\mu = \psi^\mu, \quad Q \psi^\mu = -i \phi_3 V^\mu(x), \quad Q \phi_a = 0, \quad Q \chi_1 = -i \phi_1, \quad Q \chi_2 = -i \phi_2, \quad (3.18)
\]

which can be easily verified using \( d\omega = 0 \) and \( i_V \omega = d\mu \). Acting \( Q \) twice, we have \( Q^2 x^\mu = -i \phi_3 V^\mu \) and \( Q^2 \psi^\mu = -i \phi_3 (\partial_\nu V^\mu) \psi^\nu \), while \( Q^2 (\phi, \chi) = 0 \). On the space of functions on \( \Omega^*(\mathcal{P}) \), that is, on the space of differential forms on \( X \), this is nothing but \( Q^2 = -i \phi_3 \mathcal{L}_V \). Therefore, if we restrict to the space of \( U(1) \)-invariant differential forms, \( Q \) is nilpotent. In fact, the right observables well-defined on \( \mathcal{M} \) are indeed \( U(1) \)-invariant differential forms on \( \mathcal{P} \), and the correlation functions of them depend only on the \( Q \)-cohomology. These correlation functions are nothing but the integrals on \( \mathcal{M} \) performed in the ambient space \( \mathcal{P} \).

Suppose that there is a \( U(1)_R \)-action on \( \mathcal{P} \) generated by \( R^\alpha(x) \) such that \( \mathcal{L}_R \omega = \mathcal{L}_R \omega^3 = 0 \), \( \mathcal{L}_R (\omega^1 - i \omega^2) = 2i(\omega^1 - i \omega^2) \), which imply that \( \mathcal{L}_R \mu^3 = R^\alpha \partial_\alpha \mu^3 = 0 \), and \( \mathcal{L}_R (\mu^1 - i \mu^2) = R^\alpha \partial_\alpha (\mu^1 - i \mu^2) = 2i(\mu^1 - i \mu^2) \). The former implies that there is a function \( H(x) \) with \( i_R \omega = dH \). We also assume that \( V \) commutes with \( R \), that is, \([V, R] = 0 \). Then we naturally assign the \( R \)-action to \( \phi_a \) and \( \chi_{1,2} \) such that the integrand \( S \) respects this symmetry;

\[
R \cdot (\phi_1 - i \phi_2) = 2i(\phi_1 - i \phi_2), \quad R \cdot (\chi_1 - i \chi_2) = 2i(\chi_1 - i \chi_2), \quad (3.19)
\]

and \( R \cdot \phi_3 = 0 \). This allows us to deform the supersymmetry \( Q \) to \( Q_\epsilon \) in the following way,

\[
Q_\epsilon x^\mu = \psi^\mu, \quad Q_\epsilon \psi^\mu = -i \phi_3 V^\mu(x) + \epsilon R^\mu(x), \quad Q_\epsilon \phi_3 = 0, \\
Q_\epsilon \phi_1 = 2i \epsilon \chi_2, \quad Q_\epsilon \phi_2 = -2i \epsilon \chi_1, \quad Q_\epsilon \chi_1 = -i \phi_1, \quad Q_\epsilon \chi_2 = -i \phi_2, \quad (3.20)
\]
with $Q^2 = -i\phi_3 \mathcal{L}_V + \epsilon R$, where $R$ acts as $\mathcal{L}_R$ on differential forms, and $\epsilon$ is a constant. Thus, $Q^2 = 0$ still holds on the space of both $U(1)$ and $U(1)_R$ invariant functions. It is straightforward to check that $S$ is both $U(1)$ and $U(1)_R$ invariant, and

$$Q_\epsilon S = \epsilon \omega_{\mu
u} R^\mu(x) \psi^\nu = \epsilon iR\omega = \epsilon dH = Q_\epsilon (\epsilon H(x)) .$$

(3.21)

Hence, $S_\epsilon \equiv S - \epsilon H$ is $Q_\epsilon$-invariant. For non-compact hyperkahler manifolds, the term $-\epsilon H$ often provides a regularization for volume [23]. In addition, the regularized volume integration would be left unchanged if we add a bosonic $Q_\epsilon$-exact term $Q_\epsilon Q$ to the deformed action $S_\epsilon$, where fermionic $\mathcal{O}$ should be $U(1)$ and $U(1)_R$-invariant, so that $Q^2_\epsilon \mathcal{O} = 0$. We then consider the regularized volume of $\mathcal{M}$,

$$\text{vol}_\epsilon(\mathcal{M}) = \frac{1}{(2\pi)^3 \text{vol}(U(1))} \int_{\mathcal{P}} [dx][d\psi][d\phi_a][d\chi_{1,2}] e^{S-\epsilon H+Q_\epsilon \mathcal{O}'} .$$

(3.22)

One useful $Q_\epsilon \mathcal{O}'$ is

$$Q_\epsilon \mathcal{O}' = Q_\epsilon \left(-it(\chi_1 \phi_1 + \chi_2 \phi_2)\right) = -t(\phi_1 \phi_1 + \phi_2 \phi_2) - 4t(\chi_1 \chi_2) ,$$

(3.23)

which will dominate the $\phi_{1,2}$ and $\chi_{1,2}$ terms in the action when we take $t \to \infty$ limit, which allows a simple integration over $\phi_{1,2}, \chi_{1,2}$. The remaining integration over $x^\mu$, $\psi^\mu$ and $\phi_3$ will then be simple Gaussian in our case and can be performed easily.

Calculations for $X_7(t)$

We now apply the previous formalism to an explicit problem of our space $X_7(t) = S(t_1, t_2, t_3)$. We started with a hyperkähler quotient of the flat 12-dim hyperkahler space $\mathcal{P} = \mathbb{H}^3 = (q_1, q_2, q_3)$. We obtained the 8-dim hyperkahler space $\mathcal{M}$ by the hyperkahler quotient of $\mathcal{P}$ by a $U(1)$ action $q_a \to q_a e^{i2\pi t a \xi}$, $a = 1, 2, 3$. Here we use the representation of the quaternions $q_a$ by the complex coordinates $u_a, v_a$ as in Eq.(2.4), which in turn be represented by the real coordiantes

$$q_a = \left(\begin{array}{cc} u_a & v_a \\ -v_a & u_a \end{array}\right) = \left(\begin{array}{cc} x_a + iy_a & \bar{x}_a + iy_a \\ -\bar{x}_a + iy_a & x_a - iy_a \end{array}\right) .$$

(3.24)

The triplet hyperkahler forms (2.3) become

$$\omega^3 = -(dx_a \wedge dy_a + d\bar{x}_a \wedge d\bar{y}_a) ,$$

$$\omega^1 - i\omega^2 = i(dx_a \wedge d\bar{x}_a - dy_a \wedge d\bar{y}_a) = -d\bar{x}_a \wedge dy_a + dy_a \wedge d\bar{x}_a .$$

(3.25)

With these canonical coordinates, the volume $\int [dx][d\psi] e^{\omega}$ of the ambient space $\mathcal{P} = \mathbb{H}^3$ is simply the flat volume $\int [dx_a][dy_a][d\bar{x}_a][d\bar{y}_a]$. This fixes the normalization of the ambient space metric to be $ds^2_\mathbb{H}^3 = dx_a dx_a + dy_a dy_a + d\bar{x}_a d\bar{x}_a + d\bar{y}_a d\bar{y}_a$. In components, the above $U(1)$ action is

$$u_a \to e^{it_a \xi} u_a , \quad v_a \to e^{-it_a \xi} v_a .$$

(3.26)
so that the generating vector field $V$ is
\[ V = \frac{\partial}{\partial \xi} = t_a \left( \frac{\partial}{\partial y_a} - y_a \frac{\partial}{\partial x_a} \right) - t_a \left( \frac{\partial}{\partial \tilde{y}_a} - \tilde{y}_a \frac{\partial}{\partial \tilde{x}_a} \right). \] (3.27)

From the definition of $i\nu \omega = d\mu$, we have
\[ \mu^3 = \frac{1}{2} t_a \left( |u_a|^2 - |v_a|^2 \right), \quad \mu^1 - i\mu^2 = -t_a u_a v_a. \] (3.28)

In addition, there is a diagonal $U(1)_R$ action of $SU(2)_R$ with the fore-mentioned properties,
\[ u_a \rightarrow e^{i\epsilon} u_a, \quad v_a \rightarrow e^{i\epsilon} v_a, \] (3.29)
which gives us $R$ as
\[ R = \left( \frac{1}{t_a} \right) \left( x_a \frac{\partial}{\partial y_a} - y_a \frac{\partial}{\partial x_a} + \tilde{x}_a \frac{\partial}{\partial \tilde{y}_a} - \tilde{y}_a \frac{\partial}{\partial \tilde{x}_a} \right), \] (3.30)
and from $i\mu \omega = dH$, we have $H = \frac{1}{2}(|u_a|^2 + |v_a|^2) = \frac{1}{2} r^2$ with $r$ being the standard radial distance in the above flat metric.

The volume of $U(1)$ is the coordinate length of $\xi$, that is, the least number $\xi$ such that $t_a \xi \in 2\pi \mathbb{Z}$ for all $a = 1, 2, 3$. It is easily seen to be
\[ vol(U(1)) = (2\pi) l.c.m \left( \frac{1}{t_a} \right) = \frac{2\pi}{t_1t_2t_3} l.c.m(t_1 t_2, t_2 t_3, t_3 t_1), \] (3.31)
where $l.c.m$ stands for least common multiple.

Let us integrate over $\phi_{1,2}$ and $\chi_{1,2}$ in the regularized volume integration (3.22) in the large $t$ limit, $\mathcal{M}_8$. The integration is dominated by $Q_x\mathcal{O}$ to give
\[ \int d\phi_1 d\phi_2 d\chi_1 d\chi_2 e^{-t(\phi_1 \phi_2 + \phi_2 \phi_3) - 4\epsilon t (\chi_1 \chi_2)} = \frac{\pi}{t}. 4\epsilon t = 4\pi \epsilon. \] (3.32)

The remaining integration is
\[ vol_x(\mathcal{M}_8(t)) = \frac{4\pi \epsilon}{(2\pi)^3 vol(U(1))} \int_{\mathbb{R}^3} [dx][d\psi] d\phi_3 e^{\frac{i}{2} \omega_{\mu\nu} \psi^\mu \psi^\nu + i\phi_3 \mu^3(x) - \epsilon H(x)}. \] (3.33)

Note that $\omega$ in (3.23) is simply constant in the flat coordinate $(x_a, y_a, \tilde{x}_a, \tilde{y}_a)$ of $\mathcal{P}$, and the $[d\psi]$ integration readily calculated to be
\[ \int [d\psi] e^{\frac{i}{2} \omega_{\mu\nu} \psi^\mu \psi^\nu} = 1. \] (3.34)

Also, $\mu^3$ and $H$ are both Gaussian functions on $x^\mu$ and a simple calculation gives
\[ \int_{\mathbb{R}^3} [dx] e^{i\phi_3 \mu^3 - \epsilon H} = \int_{\mathbb{R}^3} [dx] e^{\frac{i}{2} \phi_3 t_a (x_a^2 + (y_a)^2 - (\tilde{x}_a)^2 - (\tilde{y}_a)^2) - \frac{\epsilon}{2} ((x_a)^2 + (y_a)^2 + (\tilde{x}_a)^2 + (\tilde{y}_a)^2)} = (2\pi)^6 \prod_{a=1}^3 \frac{1}{(\epsilon - i\alpha a \phi_3)(\epsilon + i\alpha a \phi_3)}. \] (3.35)
in our cases at hand, what we would need to extract the volumes of tri-Sasakian section. This is because, dimensional hyperkahler cones, this Hamiltonian regularization happens to be exactly

$$U^M$$

volume for the hyperkahler quotient space of $$t$$ defined for any postive real number $$t$$. While we started with three distinct natural numbers $$a$$, we pick up poles at $$\phi = \frac{a}{n}$$, $$a = 1, 2, 3$$. The result is

$$\int d\phi_3 \prod_{a=1}^{3} \frac{1}{(\epsilon - it_3 \phi_3)(\epsilon + it_3 \phi_3)} = \frac{\pi}{\epsilon^5} \frac{t_1t_2 + t_2t_3 + t_3t_1}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}.$$  \hspace{1cm} (3.37)

In summary, we have the regularized volume of the 8-dim quotient space

$$\text{vol}_c(\mathcal{M}_8(t)) = \frac{16\pi^4}{\epsilon^4} \frac{t_1t_2t_3(t_1t_2 + t_2t_3 + t_3t_1)}{\text{l.c.m}(t_1t_2, t_2t_3, t_3t_1)(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}.$$  \hspace{1cm} (3.38)

While we started with three distinct natural numbers $$t$$, the above formula is well defined for any postive real number $$t$$ and so can be regarded valid even when some of $$t$$ coincide. (Note the above procedure can be easily generalized to the regularized volume for the hyperkahler quotient space $$\mathcal{M}_{4n}$$ obtained from the $$\mathbb{H}^{n+1}$$ by a single $$U(1)$$.)

As our objective is to calculate the volumes of 7-dim tri-Sasakian section of 8-dimensional hyperkahler cones, this Hamiltonian regularization happens to be exactly what we would need to extract the volumes of tri-Sasakian section. This is because, in our cases at hand, $$H$$ will turn out to be $$H = \frac{1}{2}r^2$$, when the metric is written as $$ds^2 = dr^2 + r^2 ds^2_{\Sigma_r}$$, and the regularized volume is

$$\text{vol}_c(\mathcal{M}_8) = \text{vol}(X_7) \int_0^\infty r^7 e^{-\frac{1}{2}r^2} = \frac{48}{\epsilon^4} \text{vol}(X_7).$$  \hspace{1cm} (3.39)

Finding the regularized volume would give us the volume of the tri-Sasakian section with normalization $$R_{ij} = 6g_{ij}$$.

By using (3.39), and we obtain the formula

$$\frac{\text{vol}(X_7(t))}{\text{vol}(S^7)} = \frac{t_1t_2t_3(t_1t_2 + t_2t_3 + t_3t_1)}{\text{l.c.m}(t_1t_2, t_2t_3, t_3t_1)(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}. $$  \hspace{1cm} (3.40)

The volume of the unit 7 sphere is $$\text{vol}(S^7) = \frac{\pi^4}{3}$$. The ratio of two volumes is a rational number. As a check, the well-known space $$N(1,1)$$ corresponds to $$t_1 = t_2 = t_3 = 1$$ for which we reproduce the known answer $$\text{vol}(N(1,1)) = \frac{\pi^4}{8}$$. We can show that the right hand side is less than 3/8. The reason is that for three natural numbers $$t_1,t_2,t_3$$ the following inequalities hold,

$$t_1t_2 + t_2t_3 + t_3t_1 \leq 3 \text{l.c.m}(t_1t_2, t_2t_3, t_3t_1),$$

$$t_1t_2t_3 \leq \frac{1}{8}(t_1 + t_2)(t_2 + t_3)(t_3 + t_1),$$  \hspace{1cm} (3.41)
where the last inequality comes from \(2\sqrt{t_1 t_2} \leq t_1 + t_2\) and so on. The equality holds only when \(t_1 = t_2 = t_3\).

One could study the tri-Sasakian space \(\mathcal{M}_{4n-1}\) obtained from the hyperkahler quotient of \(\mathbb{H}^{n+1}\) by a similar \(U(1)_t\) group. The volume can be calculated by the above method and is equal to

\[
\frac{\text{vol}(\mathcal{M}_{4n-1})}{\text{vol}(S_{4n-1})} = \frac{1}{\text{l.c.m.}(1/t_1, \ldots, 1/t_{n+1})} \sum_a \frac{t_a^{2n-1}}{\prod_{b \neq a} (t_a^2 - t_b^2)},
\]

where the volume of the unit sphere \(\text{vol}(S_{2n-1})\) is \((2n-1)! \pi^n/2\).

### 3.2 The volumes of supersymmetric 5-cycle \(\Sigma_5\)

The cone of the tri-Sasakian spaces \(X_7 = S(t_1, t_2, t_3)\) are 8-dimensional hyperkahler spaces, which are constructed through hyperkahler quotient. A supersymmetric 5-cycle, \(\Sigma_5\) in \(S(t_1, t_2, t_3)\) is characterized by its cone \(\Gamma\), which is a 6-dimensional subspace of the hyperkahler cone \(\mathcal{M}(t)\), defined by a single homogeneous holomorphic constraint. In fact, the two 5-cycles in \(N(1,1) = S(1,1,1)\) that were identified in Ref.[17] correspond to \(u_3 = 0\) and \(v_3 = 0\) respectively. Of course, there are others such as \(u_1 = 0\) et cetera, which are related to each other by continuous \(SU(3)\) isometry. They necessarily belong to the same homology. For generic \(S(t_1, t_2, t_3)\), the constraint \(u_a = 0\) or \(v_a = 0\) for some \(a\) again defines a supersymmetric 5-cycle, but the flavor isometry is now reduced to \(U(1)^2\) and the cycles with different \(a\)’s are separated by potential walls. The remaining \(SU(2)_R\) symmetry still relate \(u_a = 0\) and \(v_a = 0\) for the same \(a\).

In this section, we calculate the volumes of supersymmetric 5-cycles using the formalism of the previous section. Without loss of generality, we specify to a supersymmetric 5-cycle \(\Sigma_5\) obtained from the constraint, say with \(u_3 = 0\), whose 6-dim cone is \(\Gamma\). Before considering \(\Gamma\) in our hyperkahler quotient space, let us consider the 10-cycle \(\tilde{\Gamma}\) defined in the ambient space \(\mathbb{H}^3\) by the same constraint \(u_3 = 0\). Its volume, though infinite, is expressed formally as

\[
\frac{1}{5!} \int_{u_3=0} \omega^5 = \frac{1}{5!} \int_{\mathbb{H}^3} \Phi_{\tilde{\Gamma}} \wedge \omega^5,
\]

where \(\omega\) is the Kahler form, and \(\Phi_{\tilde{\Gamma}}\) is the 2-form Thom class dual to \(\tilde{\Gamma}\). In the real coordinate system introduced in the last section, \(u_3 = x_3 + iy_3\) and \(\Phi_{\Gamma} = \delta(x_3)\delta(y_3) dx_3 \wedge dy_3\). It is easily verified that \(d\Phi_{\tilde{\Gamma}} = 0\). In the formalism of the previous section where differential forms are functions on \(T_{[1]}X\), we can rewrite the above as an expectation value,

\[
\text{vol}(\tilde{\Gamma}) = \langle \Phi_{\tilde{\Gamma}} \rangle = \int_{\mathbb{H}^3} [dx][dy] \Phi_{\tilde{\Gamma}} e^\omega,
\]
where \( \Phi_{\Gamma} = \delta(x_3)\delta(y_3)\psi^x_3\psi^y_3 \) and \( \omega = \omega_{\mu\nu}\psi^\mu\psi^\nu/2 \). The ‘action’ \( S = \omega \) is invariant under a fermionic symmetry \( Qx^\mu = \psi^\mu, Q\psi^\mu = 0 \) due to \( d\omega = 0 \) (\( Q \) is in fact the de Rham d-operator on differential forms). Because \( Q\Phi_{\Gamma} = 0 \) \((d\Phi_{\Gamma} = 0)\), \( \Phi_{\Gamma} \) is a good observable of the above path integral.

For our quotient space \( \mathcal{M}_8(t) \), we have represented the regularized volume as a path integral

\[
\text{vol}_e(\mathcal{M}_8(t)) = \langle 1 \rangle_e = \frac{1}{(2\pi)^3 \text{vol}(U(1))} \int [dx][d\psi][d\phi_a][d\chi_{1,2}] \, e^{S-\epsilon H + Q.\mathcal{O}'}.
\]

(3.45)

It also has a fermionic symmetry \( Q_e \), as discussed before. Actually, there are two separate components in \( S_e \) that are \( Q_e \)-invariant; one is \( \frac{1}{2}\omega + i\phi_3\mu^3 - \epsilon H \), the other being \( i\phi_3\mu^1 + i\phi_2\mu^2 + \chi_1d\mu^1 + \chi_2d\mu^2 \). In the spirit of equivariant cohomology, a good observable \( \mathcal{O} \) satisfying \( Q_e\mathcal{O} \) represents a well-defined geometric data on the quotient space. In this respect, the first piece may be considered as the Kahler form of the quotient space, and our path integral naturally calculates the regularized volume of the quotient space.

Taking this analogy further, we expect there should exist the right observable \( \Phi_{\Gamma} \) whose expectation value calculates the regularized volume of \( \Gamma \) defined by \( u_3 = 0 \) in the quotient space. In the ambient space \( \mathbb{H}^3 \), it is \( \Phi_{\Gamma} \) and we naturally expect that for quotient space, it would be some modification of \( \Phi_{\Gamma} \) such that it is \( Q_e \)-invariant. In our case at hand, it is readily shown that \( \Phi_{\Gamma} \) itself is \( Q_e \)-invariant because of \( \delta \)-function factors. Therefore, we take \( \Phi_{\Gamma} = \Phi_{\Gamma} \), and the regularized volume of \( \Gamma \) will be

\[
\text{vol}_e(\Gamma) = \langle \Phi_{\Gamma} \rangle_e = \frac{1}{(2\pi)^3 \text{vol}(U(1))} \int [dx][d\psi][d\phi_a][d\chi_{1,2}] \, \Phi_{\Gamma} \, e^{S-\epsilon H + Q_e.\mathcal{O}'}.
\]

(3.46)

From \( \text{vol}_e(\Gamma) \), it is straightforward to extract the volume of 5-dimensional cycle \( \Sigma_5 \) that we are heading to. Writing the metric on \( \Gamma \) as \( ds_{\Gamma}^2 = dr^2 + r^2 ds_{\Sigma_5}^2 \),

\[
\text{vol}_e(\Gamma) = \text{vol}(\Sigma_5) \int_0^\infty dr \, r^5 \, e^{-\frac{4\epsilon r^2}{3}} = \text{vol}(\Sigma_5) \cdot \frac{8}{\epsilon^3}.
\]

(3.47)

The calculation of \( (3.46) \) is almost as the one in the previous section. Introducing large \( Q_e \)-exact mass term \( (3.23) \) for \( \phi^{1,2} \) and \( \chi^{1,2} \), and integrating them out results in

\[
\text{vol}_e(\Gamma) = \frac{4\pi\epsilon}{(2\pi)^3 \text{vol}(U(1))} \int_{\mathbb{H}^3} [dx][d\psi] \, \delta(x_3)\delta(y_3)\psi^x_3\psi^y_3 \, e^{\omega + i\phi^3\mu^3(x) - \epsilon H(x)}.
\]

(3.48)

By using simple integrations

\[
\int [d\psi] \psi^x_3\psi^y_3 \, e^{\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu} = 1,
\]

\[
\int_{\mathbb{H}^3} [dx] \, \delta(x_3)\delta(y_3) \, e^{i\phi^3\mu^3 - \epsilon H} = (2\pi)^5 \frac{1}{(\epsilon - it_1\phi^3)(\epsilon - it_2\phi^3)\prod_{\alpha=1}^3(\epsilon + it_\alpha\phi^3)}.
\]

(3.49)
we are left with
\[
\text{vol}_e(\Gamma) = \frac{4\pi\epsilon(2\pi)^2}{\text{vol}(U(1))} \int d\phi^3 \frac{1}{(\epsilon - it_1\phi^3)(\epsilon - it_2\phi^3)\prod_{a=1}^{3}(\epsilon + it_a\phi^3)}.
\] (3.50)

The integrand has poles at \(-\frac{\epsilon}{t_1}, -\frac{\epsilon}{t_2}\) and \(+\frac{\epsilon}{t_a}, a = 1, 2, 3\). Because it is convergent, we can close the contour in any way, and the result is
\[
\int d\phi^3 \frac{1}{(\epsilon - it_1\phi^3)(\epsilon - it_2\phi^3)\prod_{a=1}^{3}(\epsilon + it_a\phi^3)} = \frac{\pi}{\epsilon^3} \cdot \frac{t_1t_2 + t_2t_3 + t_3t_1}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)},
\] (3.51)

so that
\[
\text{vol}_e(\Gamma) = \frac{16\pi^4}{\text{vol}(U(1))} \frac{t_1t_2 + t_2t_3 + t_3t_1}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)} \cdot \frac{1}{\epsilon^3} = \text{vol}(\Sigma_5) \cdot \frac{8}{\epsilon^3},
\] (3.52)

and we finally obtain
\[
\frac{\text{vol}(\Sigma_5)}{\text{vol}(S^5)} = \frac{t_1t_2t_3(t_1t_2 + t_2t_3 + t_3t_1)}{l.c.m(t_1t_2, t_2t_3, t_3t_1)(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)},
\] (3.53)

where the volume of unit five sphere is \text{vol}(S^5) = \pi^3. The above expression is identical to \text{vol}(X_7)/\text{vol}(S^7). For \(t_1 = t_2 = t_3 = 1\), it is \frac{3\pi^3}{8} which agrees with the known value of supersymmetric 5-cycles of \(N(1, 1)\) in Ref.[17].

Interestingly, the 5-cycles \(u_a = 0\) or \(v_a = 0\) have the same volume independent of \(a\). The more striking fact is that the volume ratio between the supersymmetric 5-cycle and the total tri-Sasakian 7-manifold is independent of \(t_a\) and takes a universal value
\[
\frac{\text{vol}(\Sigma_5)}{\text{vol}(X_7(t))} = \frac{\text{vol}(S^5)}{\text{vol}(S^7)} = \frac{3}{\pi}.
\] (3.54)

This will turn out to be consistent with the SCFT expectation.

4. Dual Superconformal Field Theory

The bosonic Lagrangian for 11-dim supergravity is
\[
2\kappa^2\mathcal{L} = \sqrt{-G}R - \frac{1}{2}F_4 \wedge \ast F_4 - \frac{1}{6}C_3 \wedge F_4 \wedge F_4,
\] (4.1)
in the convention of Ref.[17]. The 11-dim Planck length \(l_{11}\) would be given by \(2\kappa^2 = (2\pi)^8 l_{11}^9\). In this convention, the \(M2\) charge and \(M5\) charges are given by the flux of \(F_4\) as
\[
N_2 = \frac{1}{(2\pi l_{11})^6} \int_{C_7} \ast F_4, \quad N_5 = \frac{1}{(2\pi l_{11})^3} \int_{C_4} F_4,
\] (4.2)
for some 7-cycle $C_7$ and 4-cycle $C_4$ surrounding branes. Our supersymmetric solution of the supergravity takes the form

$$ds^2_{11} = R_{X_7}^2 \left( \frac{1}{4} ds^2_{AdS_4} + ds^2_{X_7} \right), \quad F_4 = \frac{3}{8} R_{X_7}^3 vol_{AdS_4},$$

with $vol_{AdS_4}$ being the volume form of the AdS$_4$ space with metric $ds^2_{AdS_4}$. Here we assume that the normalization of $ds^2_{X_7}$ is such that $R_{\mu\nu}(X_7) = 6 g_{\mu\nu}(X_7)$. The radius of $R_{X_7}$ is given by the quantization condition of M2 brane charge $N_2$,

$$6 R_{X_7}^6 vol(X_7) = (2\pi l_{11})^6 N_2,$$

where $vol(X_7)$ is the volume of $X_7$, which we calculate this volume in the section ahead.

The dual field theory on M2 branes on the singular point is a $\mathcal{N} = 3$ supersymmetric conformal field theory. One naively thinks that it is the infrared limit of a gauge theory with the product gauge group

$$SU(N)_1 \times SU(N)_2,$$

where $N$ is the M2 charge $N_2$ of Eq.(4.2). In terms of $\mathcal{N} = 2$ language ($\mathcal{N} = 1$ in 4-dim), there are 6 chiral fields $u_a, v_a$, $a = 1, 2, 3$. The best way to consider the $SU(2)_R$ symmetry is to group these fields into $U^\beta_a = (u_a, -\bar{v}_a)$ and $V_a^\alpha = (v_a, \bar{u}_a)$, where $a = 1, 2, 3$ and $\beta = 1, 2$. The chiral field $U^\beta_a$ belongs to the symmetrized product representation $Sym^{\alpha_a}(N)$ of the fundamental representation of the first $SU(N)_1$ and the symmetrized product representation $Sym^{\beta_a}(\bar{N})$ of the second $SU(N)_2$. The chiral field $V_a^\beta$ transforms opposite to the chiral field $U^\beta_a$. Under the additional $U(1) \times U(1)$ global symmetries, the charges are shown in the following table.

|        | $SU(2)_R$ | $SU(N)_1$ | $SU(N)_2$ | $U(1)$            | $U(1)$            |
|--------|-----------|-----------|-----------|-------------------|-------------------|
| $U^\beta_a$ | 2         | $Sym^{\alpha_a}(N)$ | $Sym^{\beta_a}(\bar{N})$ | $(t_1, -t_2, 0)$  | $(0, t_2, -t_3)$  |
| $V_a^\beta$  | 2         | $Sym^{\alpha_a}(N)$ | $Sym^{\beta_a}(\bar{N})$ | $(-t_1, t_2, 0)$ | $(0, -t_2, t_3)$  |

As the R-symmetry is nonabelain $SU(2)$ group and these chiral fields belongs to the fundamental representation of this R-symmetry, these chiral fields would have a chiral dimension $1/2$. The chiral operators of dimension $k$ would be, for example, traces of products of $k$ U’s and $k$ V’s with totally symmetric in $SU(2)_R$ indices. There are still many kinds of such operators with different combinations of flavor indices $a = 1, 2, 3$. In the case of $N(1, 1) = S(1, 1, 1)$, KK analysis of the geometry dictates only operators with totally symmetric and traceless in flavor indices [14], and we expect a similar kind of reduction in the spectrum for generic $S(t_1, t_2, t_3)$. In SCFT, we have to assume that only these operators survive in the IR fixed point.
As was pointed out in Ref.\cite{15,16}, this contrasts to the case of \( AdS_5 \times T^{1,1} \) where a superpotential in the dual field theory selects right chiral primary operators that match with gravity analysis \cite{3}.

We can try superpotential approach as much as in \cite{16}, though it would not be sufficient to determine chiral primary operators. We introduce the complex scalar fields \( \Phi_1, \Phi_2 \) in the adjoint representation of \( SU(N)_1 \times SU(N)_2 \), respectively. They belong to the vector multiplet and in component \( \Phi^i_1, \Phi^i_2 \) with \( i = 1, 2, ..., N^2 - 1 \). We propose the superpotential to be

\[
W = \sum_{a=1}^{3} \left( g_1 \Phi^i_1 T^a U_{a\beta} V^{a\beta} + g_2 \Phi^i_2 T^a V_{a\beta} U^{a\beta} \right) + k_1 \Phi^i_1 \Phi^i_1 + k_2 \Phi^i_2 \Phi^i_2 , \tag{4.6}
\]

where \( g_1 = g_2 = g \) are the gauge coupling constants, \( T^a \) are matrix representation in \( \text{Sym}^a(N) \) and \( T^a \) is the trace operation in this representation. The Chern-Simons coefficients \( k_1 \) and \( k_2 \) may satisfy \( k_1 = -k_2 \) as in the case of \( S(1,1,1) = N(1,1,1) \) \cite{16}. Integrating out \( \Phi \)'s would produce a superpotential for \( U \)'s and \( V \)'s in IR.

The theory has a single \( U(1) \) baryon symmetry since the second and fifth betti numbers are \( b_2(S(t_1, t_2, t_3)) = b_5(S(t_1, t_2, t_3)) = 1 \). There are baryonic operators such as \( \det U_a \) or \( \det V_a \) where \( \det \) represents the product of \( N \) \( U_a \)'s or \( N \) \( V_a \)'s totally anti-symmetrized in gauge indices for both \( SU(N)_1 \) and \( SU(N)_2 \). Then \( SU(2)_R \) indices are totally symmetric. The conformal dimension would then be \( \Delta = \frac{N}{2} \) independent of \( a \) or \( t \). From the geometry, these operators correspond to M5 branes wrapping supersymmetric 5-cycles with the dimension \( \bar{1}_2 \) and the result \( \bar{3}_5 \) gives \( \Delta = \frac{N}{2} \) which agrees with the gauge theory expectation. We take this as a non-trivial evidence for our proposal of superconformal field theory.

One should caution that the detail characteristics of the corresponding \( \mathcal{N} = 3 \) SCFT is very obscure. One naively can imagine that this theory is the low energy conformal limit of the Yang-Mills-Chern-Simons theory of the gauge group \( SU(N)_1 \times SU(N)_2 \) and the matter chiral fields \( u_a, v_a \). In the low energy theory massive vector multiplet decouples, and so only the Chern-Simons kinetic term survives, with matter fields interact each other by the gauge coupling and self-coupling. The corresponding t’Hooft coupling is \( N/\kappa \) with \( \kappa \) being the Chern-Simons theory coefficient. From our geometric point of view of \( AdS_4 \times X_7 \), the Chern-Simons coefficient is not obvious at all. It would be interesting to find out further about this discrepancy.

Acknowledgments

This work is supported in part by KOSEF Grant R010-2003-000-10391-0 (K.L.,H.U.Y.), KOSEF SRC Program through CQUeST at Sogang Univ. (K.L.), KRF Grant No.
A. A Generalization of Caloron Moduli Space

We start this appendix by considering a generalization of the moduli space of distinct multi BPS magnetic monopole solutions [26]. Instead of the considering the interaction between two BPS dyonic monopoles whose interactions are fixed by the Dynkin diagram for a give Lie algebra, we introduce a somewhat more general interaction between them. Thus the generalized Lagrangian between multi-monopoles would be

\[ L = \frac{1}{2} M_{ij} (\dot{x}_i \cdot \dot{x}_j - q_i q_j) + q_i W_{ij} \cdot \dot{x}_j + q_i \dot{\xi}_i, \]  

(A.1)

where

\[ M_{ii} = m_i + \sum_{k \neq i} \frac{\lambda_{ik}}{|x_i - x_k|}, \quad M_{ij} = -\frac{\lambda_{ij}}{|x_i - x_j|} \text{ if } i \neq j, \]  

(A.2)

with non-negative mass parameters \( m_i \geq 0 \) and

\[ W_{ii} = \sum_{k \neq i} \lambda_{ik} w(x_i - x_k), \quad W_{ij} = -\lambda_{ij} w(x_i - x_j) \text{ if } i \neq j, \]  

(A.3)

with \( w \) being the value at \( x_i \) of the Dirac potential due to the \( j \)-th monopole so that

\[ \nabla \times w(x) = \frac{x}{|x|^3}. \]  

(A.4)

The range of each phase is given by

\[ 0 \leq \xi_i < 4\pi t_i, \]  

(A.5)

which implies that the quantization of charge is satisfied with

\[ q_i = n_i \frac{2}{t_i} \]  

(A.6)

with integer \( n_i \). After integration over \( q_i \), we obtain the Lagrangian

\[ L = \frac{1}{2} M_{ij} v_i \cdot v_j + \frac{1}{2} M_{ij}^{-1} (\dot{\xi}_i + W_{ik} \cdot v_k)(\dot{\xi}_j + W_{jl} \cdot v_l). \]  

(A.7)

The geometry is not necessarily regular when two points particles come together. Now we want the metric to be regular whenever any of two particles interacting each other come together. This requires that more detail analysis of two bodies. With the center of mass coordinate for any two body, say \( i = 1, 2 \),

\[ \mathbf{R} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2. \]  

(A.8)
The total charge and relative charge are defined as a linear combination
\[ q_t = \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2}, \quad q_r = \lambda_{12}(q_1 - q_2), \quad (A.9) \]
which leads to the c.m. and relative angles
\[ \chi = \xi_1 + \xi_2, \quad \psi = \frac{m_2 \xi_1 - m_1 \xi_2}{\lambda_{12}(m_1 + m_2)}. \quad (A.10) \]
In terms of new variables, the two body Lagrangian becomes a sum of \( L_{\text{cm}} \) and \( L_{\text{rel}} \),
\[ L_{\text{cm}} = \frac{1}{2}(m_1 + m_2)\ddot{R}^2 + \frac{1}{2(m_1 + m_2)}\dot{\chi}^2, \quad (A.11) \]
\[ L_{\text{rel}} = \frac{1}{2} \left( \mu + \frac{\lambda_{12}}{r} \right) \dot{r}^2 + \frac{\lambda_{12}^2}{2} \left( \mu + \frac{\lambda_{12}}{r} \right)^{-1} \left( \dot{\psi} + w(r) \cdot \dot{r} \right)^2. \quad (A.12) \]

The requirement that relative moduli space of two space being nonsingular is that the coupling constant \( \lambda_{12} \) should be positive and the relative coordinate \( \psi \) has to have a period of \( 4\pi \). Let us consider the range of the angle parameters. The shift of \( \xi_1 \) by \( 4\pi t_1 \) implies the identification
\[ (\chi, \psi) = \left( \chi + 4\pi t_1, \psi + \frac{4\pi m_2 t_1}{\lambda_{12}(m_1 + m_2)} \right), \quad (A.13) \]
and the shift of \( \xi_2 \) by \( -4\pi t_2 \) implies the identification
\[ (\chi, \psi) = \left( \chi - 4\pi t_2, \psi + \frac{4\pi m_1 t_2}{\lambda_{12}(m_1 + m_2)} \right). \quad (A.14) \]
A combination of \( \lambda_{12}/t_1 \) steps of the first shift and \( \lambda_{12}/t_2 \) steps of the second shift will lead to an identification
\[ (\chi, \psi) = (\chi, \psi + 4\pi). \quad (A.15) \]
For this operation to be minimum so that the period of \( \psi \) is \( 4\pi \), rather than a smaller number, \( \lambda_{12}/t_1 \) and \( \lambda_{12}/t_2 \) should be co-prime integers. There are several consequences from this requirement. The quantization of charge \( q_t \) leads to the relative charge as
\[ q_{\text{rel}} = \frac{\lambda_{12}}{2t_1 t_2}(t_2 n_1 - t_1 n_2), \quad (A.16) \]
with integer \( n_1, n_2 \). There are pair of integers such that \( q_{\text{rel}} = 1/2 \) as expected. The ratio of the periods \( t_1/t_2 \) should be a positive rational number. After scaling the coordinates, we can make \( t_1, t_2 \) to be integers and \( \lambda_{12} \) to be the least common multiplet of \( t_1, t_2 \).
In short distance where $x_1$ and $x_2$ particles approach each other, the corresponding metric becomes

$$G_{\text{rel}} = \frac{\lambda_{12}}{2} ds_{R^4}^2,$$

where

$$ds_{R^4}^2 = \frac{1}{r} dr^2 + r(d\psi + \cos \theta d\phi)^2$$

(A.18)

with the period of $\psi$ in $[0, 4\pi]$ is the flat metric for Euclidean four dimensional space.

Generalizing to the $N$ distinct monopoles of $SU(N)$ gauge group, the ratio of any pair of periods of monopoles in adjacent point in the root diagram should be rational for the geometry to be nonsingular when two monopoles are coming together. As all monopoles are interacting each other at least indirectly, one can scale space and time and so the periods $t_i$ are all integers, without any common factor. In our case also one can argue that the moduli space is smooth when $N - 1$ distinct monopoles are coming together by the argument similar to Ref. [26]. However, the space becomes singular when $N$ distinct monopoles coming together as they form a generalization of the moduli space of a single caloron of $SU(N)$ gauge group [27]. Our generalization of the monopole moduli space for $SU(3)$ would be exactly the three parameter $(t_1, t_2, t_3)$ generalization of the moduli space $N(1, 1)$ of single $SU(3)$ instanton. In the massless limit where two of monopole mass vanishes, the monopole moduli space becomes $\mathcal{M}_t$ after scaling of the coordinates.

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