Radial boundary layers for the singular Keller-Segel model

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ABSTRACT This paper is concerned with the diffusion limit (as $\epsilon \to 0$) of radial solutions to a chemotaxis system with logarithmic singular sensitivity in a bounded interval with mixed Dirichlet and Robin boundary conditions. We use a Cole-Hopf type transformation to resolve the logarithmic singularity and prove that the solution of the transformed system has a boundary-layer profile as $\epsilon \to 0$, where the boundary layer thickness is of $O(\epsilon^{\alpha})$ with $0 < \alpha < \frac{1}{2}$. By transferring the results back to the original chemotaxis model via Cole-Hopf transformation, we find that boundary layer profile is present at the gradient of solutions and the solution itself is uniformly convergent with respect to $\epsilon > 0$.

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1 Introduction

Chemotaxis describes the oriented movement of species stimulated by uneven distribution of a chemical substance in the environment. It is a significant mechanism accounting for abundant biological process/phenomenon, such as aggregation of bacteria [34, 44], slime mould formation [15], fish pigmentation [33], tumor angiogenesis [4, 6, 7], primitive streak formation [36], blood vessel formation [12], wound healing [39]. Mathematical models of chemotaxis were first proposed by Keller and Segel in their seminal works [19–21]. In this paper, we are concerned with the following chemotaxis model:

\[
 \begin{align*}
 u_t &= \nabla \cdot [D \nabla u - \chi u \nabla \ln c], \quad (x, t) \in \Omega \times (0, \infty) \\
 c_t &= \epsilon \Delta c - \mu uc,
\end{align*}
\]

(1.1)

where $\Omega$ is a domain in $\mathbb{R}^n$ with smooth boundary. System (1.1) was first advocated in [21] to describe the traveling band propagation of bacterial chemotaxis observed in the experiment of Adler [1, 2]. It later appeared in the work by Levine et al [23] to model the initiation of tumor angiogenesis, where $u(x, t)$ represents the density of vascular endothelial cells and $c(x, t)$ denotes the concentration of signaling molecules vascular endothelial growth factor (VEGF). The parameters $D > 0$, $\epsilon \geq 0$ are diffusion coefficients of the endothelial cells and the chemical VEGF respectively, $\chi > 0$ is the chemotactic coefficient measuring the intensity of chemotaxis and $\mu \geq 0$ is the chemical consumption rate by cells. In particular it was pointed out in [23] that the chemical diffusion process is far less important comparing to its interaction with endothelial cells and thus the diffusion coefficient $\epsilon$ could be small or negligible. Despite of its biological significance, (1.1) is difficult to study mathematically due to the singularity of $\ln c$ at $c = 0$. The well-known way to overcome this singularity was applying the following Cole-Hopf transformation (cf. [22, 31]):

\[
 \tilde{v} = -\nabla \ln c = -\frac{\nabla c}{c}
\]

(1.2)
to transform (1.1) into a system of conservation laws:

\[
\begin{align*}
    u_t - \nabla \cdot (u\vec{v}) &= D\Delta u, \\
    \vec{v}_t - \nabla (u - \varepsilon |\vec{v}|^2) &= \varepsilon \Delta \vec{v}, \\
    (u, \vec{v})(x, 0) &= (u_0, \vec{v}_0)(x).
\end{align*}
\] (1.3)

The transformed system (1.3) attracts extensive attentions and numerous interesting results have been developed. We briefly recall these by the dimension of spaces. In the one dimensional case, the global well-posedness along with large time behavior of solutions was investigated when \( \Omega = \mathbb{R} \) in \([13, 25]\) with \( \varepsilon = 0 \) and in \([33, 37]\) with \( \varepsilon > 0 \). When \( \Omega = (0, 1) \), authors in \([28, 54]\) obtained the unique global solution under Neumann-Dirichlet boundary conditions for \( \varepsilon = 0 \), and the result was later extended to the case \( \varepsilon > 0 \) in \([43, 49]\). The problem with \( \varepsilon \geq 0 \) is also globally well-posed \([26]\) with Dirichlet-Dirichlet boundary conditions. Furthermore, the existence and stability of traveling wave solutions were studied in \([3, 18, 27, 29-32]\). However to the best of our knowledge, except when it is associated with radially symmetric initial data, the known well-posedness results of traveling wave solutions were studied in \([3, 18, 27, 29-32]\). However to the best of our knowledge, except when it is associated with radially symmetric initial data, the known well-posedness results of traveling wave solutions were studied in \([28, 41]\) and numerically verified in \([26]\) and rigorously proved in \([16]\).

Enlightened by these results, it is natural to expect that (1.3) in multi-dimension \( (n \geq 2) \) possesses boundary layer solutions as well when prescribing appropriate Dirichlet boundary conditions. In particular we aim to investigate this issue for its radial solutions in the present paper. To this end,
we first rewrite (1.1) in its radially symmetric form by assuming that the solutions \((u, c)\) are radially symmetric, depending only on the radial variable \(r = |x|\) and time variable \(t\). In a domain bounded by two concentric spheres, i.e., \(\Omega = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | 0 < a < |x| < b\}\), (1.1) reads as

\[
\begin{align*}
\frac{u_t}{\varepsilon} &= \frac{1}{r^n} (r^{n-1} u)_r - \frac{c}{\varepsilon} (r^{n-1} \ln c)_r, \quad (r, t) \in (a, b) \times (0, \infty) \\
\frac{c_t}{\varepsilon} &= \frac{1}{r^n} (r^{n-1} c)_r - uc, \\
u(r, 0) &= u_0(r), \quad c(r, 0) = c_0(r),
\end{align*}
\]

where \(D = \chi = \mu = 1\) have been assumed without loss of generality. Similar as deriving (1.3) from (1.1), we apply the following Cole-Hopf type transformation

\[
v = - (\ln c)_r = - \frac{c_r}{c},
\]

which turns (1.5) into

\[
\begin{align*}
\frac{u_t}{\varepsilon} &= \frac{1}{r^n} (r^{n-1} u)_r + \frac{1}{p^{n-1}} (\varepsilon^{p-1} u v)_r, \quad (r, t) \in (a, b) \times (0, \infty) \\
\frac{v_t}{\varepsilon} &= \frac{1}{r^n} (r^{n-1} v)_r - \frac{p^{n-1}}{p^{n-1} v^p - \varepsilon v^p} v - u_r + \tau, \\
u(r, 0) = (u_0, v_0)(r),
\end{align*}
\]

Similar to (1.4), the Dirichlet boundary conditions for (1.7) are prescribed as

\[
\begin{align*}
\left\{ \begin{array}{l}
u|_{r=a,b} = \tilde{u}, \\
u|_{r=a} = \tilde{v}_1, \\
u|_{r=b} = \tilde{v}_2,
\end{array} \right. \quad \text{if } \varepsilon > 0, \\
\left\{ \begin{array}{l}
u|_{r=a,b} = \tilde{u}, \\
u|_{r=a} = \tilde{v}_1, \\
u|_{r=b} = \tilde{v}_2,
\end{array} \right. \quad \text{if } \varepsilon = 0.
\]

In this paper, we shall investigate the asymptotic behavior of solutions to (1.7)-(1.8) as \(\varepsilon \to 0\) for \(n > 2\) (if \(n = 1\), it coincides with the one-dimensional model (1.3)-(1.4) which has been studied in [16] as aforementioned). In particular, the solution component \(v\) is proved to have a boundary layer due to the mismatch of its boundary values as \(\varepsilon \to 0\) (see Theorem 2.2).

### 2 Main results

To study the boundary layer effect, we first present the global well-posedness and regularity estimates for solutions of (1.7)-(1.8) with \(\varepsilon = 0\) in Theorem 2.1. By these estimates, we then state the main result on the convergence for \(u\) and boundary layer formation by \(v\) in Theorem 2.2. Finally, the result is converted to the original chemotaxis model (1.8) via (1.6). We begin with introducing some notations.

**Notations.** Without loss of generality, we assume \(0 \leq \varepsilon < 1\) since the zero diffusion limit as \(\varepsilon \to 0\) is our main concern. Throughout this paper, unless specified, we use \(C\) to denote a generic positive constant which is independent of \(\varepsilon\) and dependent on \(T\). In contrast, \(C_0\) denotes a generic constant independent of \(\varepsilon\) and \(T\). For simplicity, \(L^p\) represents \(L^p(a, b)\) with \(1 \leq p \leq \infty\), \(H^k\) denotes \(H^k(a, b)\) with \(k \in \mathbb{N}\) and \(\| \cdot \|\) stands for \(\| \cdot \|_{L^2}\). Moreover, if \(f(r, t) \in L^p(a, b)\) for fixed \(t > 0\), we use \(\| f(t) \|_{L^p}\) to denote \(\| f(\cdot, t) \|_{L^p}\).

The first result is on the global well-posedness of (1.7)-(1.8) with \(\varepsilon = 0\).

**Theorem 2.1.** Assume that \((u_0, v_0) \in H^2 \times H^2\) with \(u_0 \geq 0\) satisfy the compatible conditions \(u_0(a) = u_0(b) = \tilde{u}\). Then the initial-boundary value problem (1.7)-(1.8) with \(\varepsilon = 0\) has a unique solution \((u^0, v^0)\) in \(C((0, \infty); H^2 \times H^2)\) such that the following estimates hold true.

(i) If \(\tilde{u} > 0\), there is a constant \(C_0\) independent of \(t\) such that

\[
\| u^0(t) - \tilde{u} \|_{H^2}^2 + \| v^0(t) \|_{H^2}^2 + \int_0^t \left( \| u^0(\tau) - \tilde{u} \|_{H^2}^2 + \tilde{u} (\varepsilon^{p-1} v^0, (r^{p-1} v^0)_r) \|_{H^1}^2 \right) d\tau \leq C_0.
\]

(ii) If \(\tilde{u} = 0\), there is a constant \(C_1\) independent of \(t\) such that

\[
\| u^0(t) \|_{H^2}^2 + \| v^0(t) \|_{H^2}^2 + \int_0^t \left( \| u^0(\tau) \|_{H^2}^2 + \| v^0(\tau) \|_{H^2}^2 \right) d\tau \leq C_1.
\]
Moreover, \[ \lim_{t \to \infty} ||u^0(t) - \bar{u}||_{L^\infty} = 0. \] (2.2)

(ii) If \( \bar{u} = 0 \), for any \( 0 < T < \infty \), there exists a constant \( C \) depending on \( T \) such that

\[ ||u^0||_{L^\infty(0,T;H^2)} + ||v^0||_{L^\infty(0,T;H^2)} + ||u^0||_{L^2(0,T;H^3)} \leq C. \] (2.3)

We proceed to recall the definition of boundary layers (BLs) following the convention of \([10, 11]\).

**Definition 2.1.** Denote by \((u^\varepsilon, v^\varepsilon)\) and \((u^0, v^0)\) the solution of \((1.7) - (1.8)\) with \( \varepsilon > 0 \) and \( \varepsilon = 0 \), respectively. If there exists a non-negative function \( \delta = \delta(\varepsilon) \) satisfying \( \delta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) such that

\[ \lim_{\varepsilon \to 0} ||u^\varepsilon - u^0||_{L^\infty(0,T;C[a,b])} = 0, \]
\[ \lim_{\varepsilon \to 0} ||v^\varepsilon - v^0||_{L^\infty(0,T;C[a+\delta, b-\delta])} = 0, \]
\[ \liminf_{\varepsilon \to 0} ||v^\varepsilon - v^0||_{L^\infty(0,T;C[a,b])} > 0, \]

we say that the initial-boundary value problem \((1.7) - (1.8)\) has a boundary layer solution as \( \varepsilon \to 0 \) and \( \delta(\varepsilon) \) is called a boundary layer thickness (BL-thickness).

Our main result is as follows.

**Theorem 2.2.** Suppose that \((u_0, v_0) \in H^2 \times H^2\) with \( u_0 \geq 0 \) satisfy the compatible conditions \( u_0(a) = u_0(b) = \bar{u} \) and \( v_0(a) = \bar{v}_1, v_0(b) = \bar{v}_2 \). Let \((u^0, v^0)\) be the solution obtained in Theorem 2.1. For any \( 0 < T < \infty \), we denote

\[ \varepsilon_0 = \min \left\{ \left( 8C_0 \int_0^T F(t) \, dt \right)^{-2}, \left( 32C_0^2 TeC_0 \int_0^T F(t) \, dt \right)^{-2} \right\}, \]

where the function \( F(t) \) is defined in \((4.9)\) by \( ||u^0||_{H^2}, ||v^0||_{H^2} \) and the constant \( C_0 \) (given in \((4.19)\)) depends only on \( a,b \) and \( n \). Then \((1.7) - (1.8)\) with \( \varepsilon \in (0, \varepsilon_0) \) admits a unique solution \((u^\varepsilon, v^\varepsilon) \in C([0,T];H^2 \times H^2)\). Furthermore, any function \( \delta = \delta(\varepsilon) \) satisfying

\[ \delta(\varepsilon) \to 0 \text{ and } \varepsilon^{1/2} / \delta(\varepsilon) \to 0, \text{ as } \varepsilon \to 0 \] (2.4)

is a BL-thickness of \((1.7) - (1.8)\) such that

\[ ||u^\varepsilon - u^0||_{L^\infty(0,T;C[a,b])} \leq C\varepsilon^{1/4}, \] (2.5)
\[ ||v^\varepsilon - v^0||_{L^\infty(0,T;C[a+\delta, b-\delta])} \leq C\varepsilon^{1/4} \delta^{-1/2}. \] (2.6)

Moreover,

\[ \liminf_{\varepsilon \to 0} ||v^\varepsilon - v^0||_{L^\infty(0,T;C[a,b])} > 0 \] (2.7)

if and only if

\[ \int_0^t u^0_r(a, \tau) \, d\tau \neq 0 \quad \text{or} \quad \int_0^t u^0_r(b, \tau) \, d\tau \neq 0, \quad \text{for some } t \in [0, T]. \] (2.8)

By employing transformation \((1.6)\), we next convert the above results for \((1.7) - (1.8)\) to the pre-transformed chemotaxis model \((1.5)\). The counterpart of the original model reads as follows:

\[
\begin{cases}
    u_t = \frac{1}{r^{1-n}} \left( r^{n-1} u r_t \right)_r + \frac{1}{r^{1-n}} \left( r^{n-1} u c r_t \right)_r, \\
    c_t = \varepsilon \frac{1}{r^{1-n}} \left( r^{n-1} c r_t \right)_r - uc, \\
    u(0,r) = u_0(r), \ c(0,r) = c_0(r), \\
    u|_{r=a,b} = \bar{u}, \ [c_r + \bar{v}_1c](a,t) = 0, \ [c_r + \bar{v}_2c](b,t) = 0.
\end{cases}
\] (2.9)
Proposition 2.1. Assume $c_0 > 0$ and $(u_0, \ln c_0) \in H^2 \times H^3$. Suppose that the assumptions in Theorem 2.2 hold with $v_0 = -(\ln c_0)$. Let $0 < T < \infty$. Then (2.9) with $\varepsilon \in [0, \varepsilon_0]$ admits a unique solution $(u^\varepsilon, v^\varepsilon) \in C([0, T]; H^2 \times H^3)$ such that

\[
\begin{align*}
    \|u^\varepsilon - u^0\|_{L^\infty(0, T; C[a, b])} &\leq C\varepsilon^{1/4}, \\
    \|v^\varepsilon - v^0\|_{L^\infty(0, T; C[a, b])} &\leq C\varepsilon^{1/4}.
\end{align*}
\]

Moreover, the gradient of $c$ has a boundary layer effect as $\varepsilon \to 0$, that is

\[
\|c^\varepsilon_c - c^0_c\|_{L^\infty(0, T; C[a+\delta,b-\delta])} \leq C\varepsilon^{1/4}\delta^{-1/2},
\]

with the function $\delta(\varepsilon)$ defined (2.4) and the following estimate holds

\[
\liminf_{\varepsilon \to 0} \|c^\varepsilon_c - c^0_c\|_{L^\infty(0, T; C[a, b])} > 0,
\]

if and only if (2.8) is true.

At the end of this section, we briefly introduce the main ideas used in the paper. Although the system (1.7)-(1.8) with $n \geq 2$ is in a similar form to its counterpart with $n = 1$ for which the vanishing diffusion limit has been studied in [16] based on a $\varepsilon$-independent estimate for solutions with $\varepsilon > 0$, the methods used there cannot be applied to study the present problem since when $n \geq 2$ the system (1.7)-(1.8) with $\varepsilon > 0$ lacks an energy-like structure or a Lyapunov function to provide a preliminary estimate uniformly in $\varepsilon$. Moreover, one can not use the estimates derived in [50] for the present problem either since those estimates depend on $\varepsilon$. The difficulty in our analysis consists in deriving the $\varepsilon$-convergence estimates in (2.5) and (2.6) without any uniform-in-$\varepsilon$ priori bounds on solutions $(u^\varepsilon, v^\varepsilon)$. Inspired by the works [5, 51], this will be achieved in section 4 by regarding $(u^\varepsilon, v^\varepsilon)$ with small $\varepsilon > 0$ as a perturbation of $(u^0, v^0)$ and then estimating their difference $(u^\varepsilon - u^0, v^\varepsilon - v^0)$ by the method of energy estimates and a new Gronwall’s type inequality (see Lemma 3.1) on ODEs. The proof of Theorem 2.1 is standard and will be given in section 3.

3 Proof of Theorem 2.1

This section is to prove Theorem 2.1 based on the following lemmas where the a priori estimates on solution $(u^0, v^0)$ of (1.7)-(1.8) with $\varepsilon = 0$ are derived by the energy method. We set off by rewriting (1.7)-(1.8) with $\varepsilon = 0$ as follows:

\[
\begin{align*}
    u^0_t &= \frac{1}{r^{n-1}}(r^{n-1}u^0_r)_r + \frac{1}{\rho r^{n-1}}(r^{n-1}v^0)_r, \\
    v^0_t &= u^0, \\
    (u^0,v^0)(r,0) &= (u_0,v_0)(r), \\
    u^0(a,t) &= u^0(b,t) = \bar{u}.
\end{align*}
\]

Lemma 3.1. Suppose the assumptions in Theorem 2.1 hold and $\bar{u} > 0$. Then there exists a positive constant $C_0$ independent of $t$ such that

\[
\begin{align*}
    \int_a^b r^{n-1}[(u^0 \ln u^0 - u^0)(t) - (\bar{u} \ln \bar{u} - \bar{u}) - \ln \bar{u}(u^0(t) - \bar{u})]dr \\
    + \frac{1}{2} \int_a^b r^{n-1}(v^0)^2(t)dr + \int_0^t \int_a^b r^{n-1}(u^0)^2 u^0 drd\tau &\leq C_0
\end{align*}
\]

and

\[
\begin{align*}
    \|r^{(n-1)/2}u^0(t) - \bar{u}\|_2^2 + \int_0^t \|r^{(n-1)/2}u^0(\tau)\|^2_2 d\tau &\leq C_0.
\end{align*}
\]
Proof. Taking the $L^2$ inner products of the first and second equation of (3.1) with $r^{n-1}(\ln u^0 - \ln \bar{\nu})$ and $r^{n-1}v^0$ respectively, we then add the results and use integration by parts to get

\[
\frac{d}{dt} \int_a^b r^{n-1}[(u^0 \ln u^0 - u^0) - (\bar{u} \ln \bar{\nu} - \bar{\nu}) - \ln \bar{u}(u^0 - \bar{u})]dr
\]
\[+ \frac{1}{2} \frac{d}{dt} \int_a^b r^{n-1}(v^0)^2 dr + \int_a^b r^{n-1} \frac{(u^0)^2}{\bar{u}^2} dr = 0,
\]

which gives rise to (3.2) upon integration over $(0,t)$. To prove (3.3), we denote $\bar{u}(r,t) = u^0(r,t) - \bar{\nu}$ and find from (3.1) that $(\bar{u},v^0)(r,t)$ satisfies

\[
\begin{aligned}
\tilde{u}_r = \frac{1}{r^{n-1}}(r^{n-1}\tilde{u}_r) + \frac{1}{r^{n-1}}(r^{n-1}\tilde{\nu}_r) + \frac{\bar{u}}{r^{n-1}}(r^{n-1}\tilde{v}_r), \\
\tilde{v}_r^0 = \tilde{u}_r, \\
(\tilde{u},v^0)(r,0) = (u_0 - \bar{\nu},v_0)(r), \\
\tilde{u}(a,t) = \tilde{u}(b,t) = 0.
\end{aligned}
\]

Multiplying the first and second equation of (3.4) by $r^{n-1}\tilde{u}$ and $\bar{u}r^{n-1}v^0$, respectively. Adding the results gives

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \|r^{n-1/2}\tilde{u}\|^2 + \bar{u}\|r^{n-1/2}v^0\|^2 \right) + \|r^{n-1/2}\tilde{u}_r\|^2 \\
= - \int_a^b r^{n-1}\tilde{u}v^0 d\bar{\nu} \\
\leq \frac{1}{2} \|r^{n-1/2}\tilde{u}_r\|^2 + \frac{1}{2} \|\tilde{u}\|_L^2 \|r^{n-1/2}v^0\|^2.
\end{aligned}
\]

Note that $\|\tilde{u}\|_L^2$ can be estimated as follows

\[
|\tilde{u}(r,t)| = |u^0(r,t) - \bar{u}| = \left| \int_a^r u^0 d\bar{\nu} \right| \leq \left( \int_a^b u^0 dr \right) \left( \int_a^b \frac{(u^0)^2}{\bar{u}^2} dr \right)^{1/2}.
\]

Then substituting the above estimate into (3.5) and integrating the result over $(0,t)$ we have

\[
\frac{1}{2} \|r^{n-1/2}\tilde{u}(t)\|^2 + \frac{1}{2} \bar{u}\|r^{n-1/2}v^0(t)\|^2 + \frac{1}{2} \int_0^t \|r^{n-1/2}\tilde{u}_r\|^2 d\tau
\]
\[\leq \frac{1}{2} \left( \int_0^t \left( \int_a^b \frac{(u^0)^2}{\bar{u}^2} dr \right) d\tau \right) \|u^0\|_{L^2(0,t;L^1)} \|r^{n-1/2}v^0\|_{L^2(0,t;L^2)}^2,
\]

which, along with (3.2) and the fact

\[
\|u^0\|_{L^2(0,t;L^1)} \leq C_0 \sup_{\tau \in [0,t]} \left\{ \int_a^b r^{n-1}[(u^0 \ln u^0 - u^0)(\tau) - (\bar{u} \ln \bar{\nu} - \bar{\nu}) - \ln \bar{u}(u^0(\tau) - \bar{u})]dr \right\}
\]

implies (3.3). The proof is completed.

\[\square\]

We proceed to derive higher regularity properties for the solution $(\tilde{u},v^0)$ of (3.4).

Lemma 3.2. Suppose the assumptions in Theorem 2.1 hold and $\bar{u} > 0$. Let $(\tilde{u},v^0)(r,t)$ be the solution of (3.4). Then there is a constant $C_0$ independent of $t$ such that

\[
\|r^{n-1}v^0\|_{L^2(0,t;L^1)}^2 + \|r^{n-1/2}\tilde{u}_r\|_{L^2(0,t;L^2)}^2 + \int_0^t (\bar{u}\|r^{n-1}v^0\|_{L^2(0,t;L^1)}^2 + \|r^{n-1/2}\tilde{u}_r\|_{L^2(0,t;L^2)}^2) d\tau \leq C_0.
\]
Taking the $L^2$ inner product of (3.7) against $2(r^{n-1}v^0)_r$, to get

$$\frac{d}{dt} \| (r^{n-1}v^0)_r \|^2 + 2\bar{u}(r^{n-1}v^0)_r = 2 \int_a^b r^{n-1} \bar{u}_t (r^{n-1}v^0)_r dr - 2 \int_a^b (r^{n-1} \bar{u} v^0)_r (r^{n-1}v^0)_r dr,$$

where $M_1$ can be reorganized as

$$M_1 = \frac{d}{dt} \left( \frac{1}{2} \| (r^{n-1}v^0)_r \|^2 + 2\| r^{n-1} \bar{u} \|^2 - \| \frac{1}{\sqrt{2}} (r^{n-1}v^0)_r - \sqrt{2} r^{n-1} \bar{u} \|^2 \right),$$

and $M_2$ can be estimated by (3.7) and the Poincaré inequality as

$$M_2 = -2 \int_a^b (r^{n-1} \bar{u}) (r^{n-1} \bar{u}_r)_r dr = 2 \int_a^b (r^{n-1} \bar{u}) (r^{n-1} \bar{u}_r)_r dr \leq C_0 \| r^{(n-1)/2} \bar{u}_r \|^2.$$

Hence

$$I_1 \leq \frac{d}{dt} \left( \frac{1}{2} \| (r^{n-1}v^0)_r \|^2 + 2\| r^{n-1} \bar{u} \|^2 - \| \frac{1}{\sqrt{2}} (r^{n-1}v^0)_r - \sqrt{2} r^{n-1} \bar{u} \|^2 \right) + C_0 \| r^{(n-1)/2} \bar{u}_r \|^2.$$

To estimate $I_2$, we first note that for fixed $t > 0$ if $f(r,t) \in H^1$ satisfies $f|_{r=a,b} = 0$ it follows that $f(r,t)^2 = 2 \int_a^b f f_r dr \leq 2 \| f(t) \| \| f_r(t) \|$, which leads to

$$\| f(t) \|_{L^2} \leq \sqrt{2} \| f(t) \|^{1/2} \| f_r(t) \|^{1/2} \text{ and } \| f(t) \|_{L^2} \leq C_0 \| f_r(t) \|,$$

thanks to the Poincaré inequality $\| f(t) \| \leq C_0 \| f_r(t) \|$. Then we deduce from (3.9) and the Sobolev embedding inequality that

$$I_2 \leq \frac{\bar{u}}{2} \| (r^{n-1}v^0)_r \|^2 + \frac{4}{\bar{u}} \| \bar{u} \|_{L^2}^2 \| (r^{n-1}v^0)_r \|^2 + \frac{4}{\bar{u}} \| \bar{u}_r \|^2 \| r^{n-1}v^0 \|_{L^2}^2 \leq \frac{\bar{u}}{2} \| (r^{n-1}v^0)_r \|^2 + C_0 \| r^{(n-1)/2} \bar{u}_r \|^2 \| (r^{n-1}v^0)_r \|^2 + \| (r^{n-1}v^0)_r \|^2.$$

Substituting the above estimates for $I_1$ and $I_2$ into (3.8), one derives

$$\frac{d}{dt} \left( \frac{1}{2} \| (r^{n-1}v^0)_r \|^2 + \frac{1}{\sqrt{2}} \| (r^{n-1}v^0)_r \|^2 - \sqrt{2} r^{n-1} \bar{u} \|^2 \right) + \frac{3}{2} \bar{u} (r^{n-1}v^0)_r \|^2 \leq C_0 \| r^{(n-1)/2} \bar{u}_r \|^2 \| (r^{n-1}v^0)_r \|^2 + C_0 \| r^{(n-1)/2} \bar{u}_r \|^2 \| (r^{n-1}v^0)_r \|^2 + 1 \| r^{n-1} \bar{u} \|^2.$$
We proceed to estimate \( \| r^{(n-1)/2} \bar{u}_t(t) \| \) by multiplying the first equation of (3.4) with \( 2r^{n-1} \bar{u}_t \) in \( L^2 \) and derive

\[
\frac{d}{dt} \| r^{(n-1)/2} \bar{u}_t \|^2 + 2 \| r^{(n-1)/2} \bar{u}_t \|^2 = 2 \int_a^b (r^{n-1} \bar{u}v^0) \bar{u}_t dr + 2 \bar{u} \int_a^b (r^{n-1} v^0) \bar{u}_t dr \\
:= I_3 + I_4.
\]

By similar arguments as deriving (3.10), we estimate \( I_3 \) as

\[
I_3 \leq \frac{1}{2} \| r^{(n-1)/2} \bar{u}_t \|^2 + C_0 \| r^{(n-1)/2} \bar{u}_t \|^2 (\| r^{(n-1)/2} v^0 \|^2 + \| (r^{n-1} v^0)_r \|^2)
\]

and by the Cauchy-Schwarz inequality, \( I_4 \) is estimated as

\[
I_4 \leq \frac{1}{2} \| r^{(n-1)/2} \bar{u}_t \|^2 + C_0 \| (r^{n-1} v^0)_r \|^2.
\]

Then feeding (3.12) on the above estimates for \( I_3 \) and \( I_4 \), we have

\[
\frac{d}{dt} \| r^{(n-1)/2} \bar{u}_t \|^2 + \| r^{(n-1)/2} \bar{u}_t \|^2 \leq C_0 \| r^{(n-1)/2} \bar{u}_t \|^2 (\| r^{(n-1)/2} v^0 \|^2 + \| (r^{n-1} v^0)_r \|^2) + C_0 \| (r^{n-1} v^0)_r \|^2.
\]

Integrating (3.13) over \((0, t)\) and using (3.3) and (3.11), one arrives at

\[
\| r^{(n-1)/2} \bar{u}_t(t) \|^2 + \int_0^t \| r^{(n-1)/2} \bar{u}_t \|^2 d\tau \leq C_0,
\]

which, in conjunction with (3.11) gives (3.6). The proof is completed.

\[ \square \]

**Lemma 3.3.** Suppose that the assumptions in Theorem 2.1 hold and \( \bar{u} > 0 \). Then there exists a constant \( C_0 \) independent of \( t \) such that

\[
\| r^{(n-1)/2} \bar{u}_t(t) \|^2 + \| (r^{n-1} v^0)_{rr}(t) \|^2 + \int_0^t \left( \| r^{(n-1)/2} \bar{u}_{rr} \|^2 + \bar{u} \| (r^{n-1} v^0)_{rr} \|^2 \right) d\tau \leq C_0
\]

and

\[
\| (r^{n-1} \bar{u}_r)_r(t) \|^2 + \int_0^t \left( \| (r^{n-1} \bar{u}_r)_r \|^2 + \| (r^{n-1} \bar{u}_r)_{rr} \|^2 \right) d\tau \leq C_0.
\]

**Proof.** Differentiating the first equation of (3.4) with respect to \( t \) and multiplying the result with \( 2r^{n-1} \bar{u}_t \), we get upon integration by parts that

\[
\frac{d}{dt} \| r^{(n-1)/2} \bar{u}_t \|^2 + 2 \| r^{(n-1)/2} \bar{u}_t \|^2 \leq -2 \int_a^b (r^{n-1} \bar{u}v^0) \bar{u}_t dr - 2 \bar{u} \int_a^b (r^{n-1} v^0) \bar{u}_r \\
:= I_5 + I_6.
\]

By (3.9) and the second equation of (3.4) we have that

\[
I_5 \leq C_0 \left( \| \bar{u}_t \|_{L^2} \| r^{(n-1)/2} v^0 \| \| r^{(n-1)/2} \bar{u}_t \| + \| \bar{u} \|_{L^2} \| v^0 \| \| r^{(n-1)/2} \bar{u}_t \| \right) \\
\leq C_0 \left( \| r^{(n-1)/2} \bar{u}_t \|^2 \| r^{(n-1)/2} \bar{u}_t \| + 3/2 \| r^{n-1} \bar{u}_t \|^2 \| r^{n-1} \bar{u}_t \|^2 + \| r^{n-1} \bar{u}_t \| \| \bar{u}_t \| \| r^{n-1} \bar{u}_t \| \right) \\
\leq \frac{1}{2} \| r^{(n-1)/2} \bar{u}_t \|^2 + C_0 \left( \| r^{(n-1)/2} \bar{u}_t \|^2 \| r^{(n-1)/2} \bar{u}_t \|^2 + \| r^{n-1} \bar{u}_t \|^4 \right).
\]
We proceed to estimating the remaining part \( \|u(t)\| + \int_0^t \|u(\tau)\|^2 d\tau \leq C_0. \) (3.17)

We use again the second equation of (3.4) and Cauchy-Schwarz inequality to get

\[
\frac{d}{dt} \|u(t)\|^2 + 2\bar{u} \|u(\tau)\|^2 \leq 2C_0 \|u\| + C_0 \|u\|^2.
\]

To estimate of \( I_7, \) we note for \( g(r,t) \in L^2(a,b) \) with fixed \( t > 0, \) it follows that

\[
\frac{d}{dt} \|u(t)\|^2 + \|g(t)\|^2 \leq a^{-(n-1)}\|u(t)\|^2.
\]

Then from Cauchy-Schwarz inequality and (3.19) one derives

\[
I_7 \leq \frac{\bar{u}}{2} \|u(t)\|^2 + C_0 \|u(t)\|^2 + C_0 \|u(t)\|^2.
\]

To bound \( I_8 \) we first estimate \( \frac{d}{dt} \|u(\tau)\|^2 d\tau \) by the first equation of (3.4) as follows:

\[
\int_0^t \|u(\tau)\|^2 d\tau \leq \int_0^t \|u(\tau)\|^2 d\tau + C_0 \int_0^t \|u(\tau)\|^2 d\tau \cdot \|u(\tau)\|^2
\]

where (3.9) and Lemma 3.1 - Lemma 3.2 have been used. Then (3.20) along with (3.19) and (3.4) implies that

\[
\int_0^t \|u(\tau)\|^2 d\tau \leq C_0 \int_0^t \|u(\tau)\|^2 d\tau + C_0 \|u(\tau)\|^2 d\tau \leq C_0,
\]

where the constant \( C_0 \) depends on \( a \) and \( b. \) Noting that \( (r^{n-1}u\dot{u})_{rr} = u(r^{n-1}u)_{rr} + 2(r^{n-1}u), \) \( u(\tau) \) and the Sobolev embedding inequality that

\[
I_8 \leq \frac{\bar{u}}{2} \|u(r^{n-1}u)_{rr}\|^2 + \frac{2}{\bar{u}} \|u(r^{n-1}u)_{rr}\|^2 \leq \frac{\bar{u}}{2} \|u(r^{n-1}u)_{rr}\|^2 + C_0 \|u(\tau)\|^2 \|u(r^{n-1}u)_{rr}\|^2 \]

We feed (3.18) on the above estimates for \( I_7-I_8 \) then apply Gronwall’s inequality, Lemma 3.1 - Lemma 3.2 (3.17) and (3.21) to the result to find

\[
\|I(r^{n-1}u)_{rr}\|^2 + \bar{u} \int_0^t \|u(r^{n-1}u)_{rr}\|^2 d\tau \leq C_0,
\]

9
which, along with (3.17) yields (3.14). We next prove (3.15). By similar arguments as deriving (3.20) one gets
\[
\left\| (r^{n-1} \tilde{u}_r)_r(t) \right\|^2 \leq \left\| r^{n-1} \tilde{u}_r(t) \right\|^2 + C_0 \left\| \tilde{u}_r(t) \right\|^2 \left\| (r^{n-1} v^0)(t) \right\|^2_{H^1} \\
+ C_0 \bar{u}_r^2 \left\| (r^{n-1} v^0)_r(t) \right\|^2
\]
(3.22)
where (3.17) and Lemma 3.1 - Lemma 3.2 have been used. We differentiate (3.7) with respect to \( r \) and conclude that
\[
\int_0^t \left\| (r^{n-1} \tilde{u}_r)_rr \right\|^2 d\tau \\
\leq C_0 \left( \int_0^t \left\| r^{n-1} \tilde{u}_r \right\|^2 d\tau + \int_0^t \left\| r^{(n-1)/2} \tilde{u}_r \right\|^2 d\tau + \tilde{u}_r^2 \int_0^t \left\| (r^{n-1} v^0)_r \right\|^2 d\tau \right)
\]
(3.23)
\[
+ C_0 \int_0^t \left( \left\| \tilde{u}_r \right\|^2 + \left\| \tilde{a}_rr \right\|^2 \right) d\tau \cdot \left\| (r^{n-1} v^0) \right\|^2_{L^2(0,t;H^2)} \\
\leq C_0,
\]
where we have used (3.14), Lemma 3.1 and Lemma 3.2. Finally collecting (3.20), (3.22) and (3.23) we derive (3.15). The proof is finished.

\[
\square
\]

We are now in the position to prove Theorem 2.1 by the above Lemma 3.1 - Lemma 3.3.

**Proof of Theorem 2.1** We first prove Part (i) of Theorem 2.1. By Lemma 3.1 and (3.19), one derives
\[
\left\| v^0(t) \right\|^2 \leq C_0 \left\| r^{(n-1)/2} v^0(t) \right\|^2 \leq C_0, \quad \left\| \tilde{u}_r(t) \right\|^2 + \int_0^t \left\| \tilde{u} \right\|^2_{H^1} d\tau \leq C_0,
\]
(3.24)
where the constant \( C_0 \) depends on \( a, b \) and \( n \) and the Poincaré inequality \( \left\| \tilde{a} \right\|^2 \leq C_0 \left\| \tilde{a}_r \right\|^2 \) has been used. On the other hand, for \( f(r,t) \in H^1 \) with fixed \( t \) we have
\[
\left\| f_r \right\|^2 = \left\| r^{-(n-1)} \left( f - (n-1) r^{n-2} f \right) \right\|^2 \\
\leq a^{-2(n-2)} \left\| (r^{n-1} f) \right\|^2 + a^{-2(n-2)} (n-1) b^{2(n-2)} \left\| f \right\|^2 \\
\leq C_0 \left\| (r^{n-1} f)_r \right\|^2 + \left\| f \right\|^2
\]
(3.25)
Then it follows from Lemma 3.2, Lemma 3.1, (3.19) and (3.25) that
\[
\left\| v^0_r(t) \right\|^2 + \left\| \tilde{u}_r(t) \right\|^2 + \int_0^t \left( \tilde{u} \right\| (r^{n-1} v^0)_r \left\|^2 + \left\| \tilde{u}_r \right\|^2 \right) d\tau \leq C_0.
\]
(3.26)
Similarly, it follows from Lemma 3.3 and (3.25) that
\[
\left\| v^0_r(t) \right\|^2 + \left\| \tilde{u}_rr(t) \right\|^2 + \int_0^t \left( \tilde{u} \right\| (r^{n-1} v^0)_rr \left\|^2 + \left\| \tilde{u}_rr \right\|^2 \right) d\tau \leq C_0.
\]
(3.27)
Thus collecting (3.24), (3.26) and (3.27) we derive the desired a priori estimate (2.1), which along with the fixed point theorem implies the existence of solution \( (u^0, v^0) \) in \( C([0,\infty); H^2 \times H^2) \).

We next prove (2.2). Integrating (3.13) over \((0,\infty)\) with respect to \( t \), then using Lemma 3.1 and Lemma 3.2 we have
\[
\int_0^{\infty} \frac{d}{dt} \left\| r^{(n-1)/2} \tilde{u}_r \right\|^2 dt \\
\leq C_0 \left\| r^{(n-1)/2} \tilde{u}_r \right\|^2_{L^2(0,\infty;L^2)} \left( \left\| r^{(n-1)/2} v^0 \right\|^2_{L^2(0,\infty;L^2)} + \left\| (r^{n-1} v^0)_r \right\|^2_{L^2(0,\infty;L^2)} \right) \\
+ C_0 \left\| (r^{n-1} v^0)_r \right\|^2_{L^2(0,\infty;L^2)} \\
\leq C_0,
\]
which, along with (3.3) implies that \( \| r^{(n-1)/2} \bar{u}_t \|^2 \in W^{1,1}(0, \infty) \). Hence, it follows that

\[
\lim_{t \to \infty} \| \bar{u}_t \| \leq C_0 \lim_{t \to \infty} \| r^{(n-1)/2} \bar{u}_t \| = 0,
\]

which, along with the Gagliardo-Nirenberg inequality \( \| (u^0 - \bar{u})(t) \|_{L^2} \leq C_0 \| (u^0 - \bar{u})(t) \|_{L^2} \| (u^0 - \bar{u})(t) \|_{L^2} \) and (3.3), gives (2.2). Part (i) of Theorem 2.1 is thus proved.

We proceed to prove Part (ii). When \( \bar{u} = 0 \), for \( 0 < T < \infty \) one can easily deduce the a priori estimates (2.3) by the standard energy method that bootstraps the regularity of the solution \((u^0, v^0)\) from \(L^2\) to \(H^2\). We omit this procedure for simplicity and refer readers to [26] for details. Then the existence of solution \((u^0, v^0)\) follows from (2.3) and the fixed point theorem. The proof is finished.

4 Proof of Theorem 2.2 and Proposition 2.1

Let \((u^0, v^0)\) be the solutions of (1.7)-(1.8) corresponding to \( \varepsilon > 0 \) and \( \varepsilon = 0 \) respectively. Then the initial-boundary value problem for their differences \( h := u^\varepsilon - u^0, \ w := v^\varepsilon - v^0 \) reads:

\[
\begin{align*}
    h_t &= \frac{1}{r^{n-1}}(r^{n-1} h)_r + \frac{1}{r^{n-1}}(r^{n-1} h w)_r + \frac{1}{r^{n-1}}(r^{n-1} u^0 w)_r + \frac{1}{r^{n-1}}(r^{n-1} h^0)_r, \\
    w_t &= \varepsilon \frac{1}{r^{n-1}}(r^{n-1} v_r)_r - 2 \varepsilon w v_r + h_r + \varepsilon \frac{1}{r^{n-1}}(r^{n-1} v^0_r)_r - 2 \varepsilon (w v^0_r + v^0 w_r + v^0 v^0_r) \\
    &\quad - \frac{\varepsilon}{r^2}(w + v^0), \quad (r, t) \in (a, b) \times (0, \infty) \\
    (h, w)(r, 0) &= (0, 0), \\
    h|_{r=a,b} &= 0, \ w|_{r=a} = \bar{v}_1 - v^0(a, t), \ w|_{r=b} = \bar{v}_2 - v^0(b, t).
\end{align*}
\]

To prove Theorem 2.2 we shall invoke an elementary result (see Lemma 4.1) on an ordinary differential equation (ODE) and a series of lemmas on the a priori estimates for solutions of (4.1). In particular, the \(L^2\)-estimate for solution \((h, w)\) and higher regularity estimates for the solution component \(h\) will be established in Lemma 4.2 - Lemma 4.3 and Lemma 4.6 will give a weighted \(L^2\)-estimate for the derivative of \(w\).

We proceed to prove the following Lemma, which gives an upper bound for the solution of an ODE involving a small parameter \( \gamma \). It extends a result in [5, 51] with \( k = 2 \) to any integer \( k \geq 2 \).

**Lemma 4.1.** Let \( k \geq 2 \) be an integer and \( 0 < T < \infty \). Let \( C_0 > 1 \) be a constant independent of \( T \) and \( f_1(t) \), \( f_2(t) \geq 0 \) be two continuous functions on \([0, T]\). Consider the ODE

\[
\begin{align*}
    &\frac{dy}{dt}(t) \leq \gamma f_1(t)y(t) + f_2(t)y(t) + C_0[y^2(t) + \cdots + y^k(t)], \\
    &y(0) = 0.
\end{align*}
\]

If we set

\[
\gamma_0 = \min \left\{ [4(k-1)]^{-1} \left( \int_0^T f_1(t) \, dt \right)^{-1}, \ [8T^2 G(k-1)^2]^{-1} \left( \int_0^T f_1(t) \, dt \right)^{-1} \right\},
\]

with \( G := C_0 \left( e^{\int_0^T f_1(t) \, dt} \right)^{-k-1} \). Then for \( \gamma \in (0, \gamma_0] \), any solution \( y(t) \geq 0 \) of (4.2) satisfies

\[
y(t) \leq e^{\gamma_0 \int_0^T f_1(t) \, dt} \cdot \min \left\{ 3, \frac{3}{2T(k-1)G}, 12(k-1)\gamma \int_0^T f_1(t) \, dt \right\}, \quad t \in [0, T].
\]
Proof. Let $U(t) = y(t)e^{-\int_0^t f_2(\tau) d\tau}$. Then (4.2) can be rewritten as

$$\frac{d}{dt} U(t) \leq \gamma f_1(t) e^{-\int_0^t f_2(\tau) d\tau} + C_0 \left(e^{\int_0^t f_2(\tau) d\tau}\right) U^2 + \ldots + C_0 \left(e^{\int_0^t f_2(\tau) d\tau}\right)^k U^k. $$

Noting that $e^{\int_0^t f_2(\tau) d\tau} \geq 1$ thanks to $f_2(t) \geq 0$, we deduce that

$$\begin{cases} 
\frac{d}{dt} U(t) \leq \gamma f_1(t) + G U^2(t)(1 + U(t))^{k-2}, \\
U(0) = 0. 
\end{cases} \quad (4.5)$$

For later use, we define

$$\sigma = \min \left\{ G, \frac{1}{4T^2(k-1)^2G}, 16(k-1)^2 \gamma^2 G \left( \int_0^T f_1(t) dt \right)^2 \right\}. \quad (4.6)$$

Now dividing both sides of (4.5) by $\left(1 + \sqrt{\frac{\sigma}{\gamma}} U(t)\right)^k$, it follows that

$$\frac{d}{dt} U(t) \leq \frac{\gamma f_1(t)}{\left(1 + \sqrt{\frac{\sigma}{\gamma}} U(t)\right)^k} + \frac{G U^2(t)}{\left(1 + \sqrt{\frac{\sigma}{\gamma}} U(t)\right)^k} \cdot \frac{(1 + U(t))^{k-2}}{\left(1 + \sqrt{\frac{\sigma}{\gamma}} U(t)\right)^{k-2}}. $$

Then noting $\sigma \leq G$ due to its definition (4.6), we deduce from the above inequality that

$$\frac{d}{dt} U(t) \leq \gamma f_1(t) + \sigma,$$

which integrated over $(0, t)$ with $t \in (0, T]$, yields

$$\frac{\gamma}{k-1} \cdot \frac{1}{\left(1 + \sqrt{\frac{\sigma}{\gamma}} U(t)\right)^{k-1}} \geq \frac{\sqrt{\sigma}}{k-1} - \sigma T - \gamma \int_0^T f_1(t) dt \quad \quad (4.7)$$

$$\geq \frac{\sqrt{\sigma}}{2(k-1)} - \gamma \int_0^T f_1(t) dt,$$

where we have used the fact $\sigma T \leq \frac{\sqrt{\sigma}}{2(k-1)}$, thanks to the definition of $\sigma$. We shall prove that

$$\gamma \int_0^T f_1(t) dt \leq \frac{\sqrt{\sigma}}{4(k-1)}, \quad (4.8)$$

of which the proof is split into three cases by the value of $\sigma$.

Case 1, when $\sigma = G$, it follows from the definition of $\gamma_0$ in (4.3) that

$$\gamma \int_0^T f_1(t) dt \leq \gamma_0 \int_0^T f_1(t) dt \leq \frac{1}{4(k-1)} = \frac{\sqrt{\sigma}}{4(k-1)}.$$

Case 2, when $\sigma = \frac{1}{4T^2(k-1)^2G}$, we have by using (4.3) again that

$$\gamma \int_0^T f_1(t) dt \leq \gamma_0 \int_0^T f_1(t) dt \leq \frac{1}{8TG(k-1)^2} = \frac{\sqrt{\sigma}}{4(k-1)}.$$
Case 3, when $\sigma = 16(k - 1)^2 \gamma^2 G \left( \int_0^T f_1(t) \, dt \right)^2$, one immediately get

$$
\gamma \int_0^T f_1(t) \, dt = \frac{\sqrt{G}}{4(k - 1)}.
$$

Hence combining the above Case 1 - Case 3, we conclude that (4.8) holds true and it follows from (4.8) and (4.7) that

$$
\left( 1 + \sqrt{\frac{G}{\sigma}} U(t) \right)^{k-1} \leq 4, \quad t \in [0, T]
$$

thus

$$
U(t) \leq 3 \sqrt{\frac{\sigma}{G}}, \quad t \in [0, T]
$$

which, along with (4.6) and the definition of $U(t)$, yields the desired estimate (4.4). The proof is finished.

In the sequel, for convenience we denote

$$
\begin{align*}
E(t) & := \| r^{(n-1)/2} h(t) \|^2 + \| r^{(n-1)/2} w(t) \|^2 + \varepsilon \| r^{(n-1)/2} w_r(t) \|^2, \\
F(t) & := \| u^0(t) \|^2 + \| v^0(t) \|_{L^2}^2 + \| v^0(t) \|_{H^2}^2 + \| \tilde{v}(t) \|^2 + | \tilde{v}_2 |^2 + 1.
\end{align*}
$$

The following lemma gives the $L^2$-estimate for the solution $(h, w)$ of problem (4.1).

**Lemma 4.2.** Let $0 < t < \infty$. Then there exists a constant $C_0$ independent of $\varepsilon$ and $t$, such that

$$
\begin{align*}
\frac{d}{dt} \left( \| r^{(n-1)/2} h(t) \|^2 + \| r^{(n-1)/2} w(t) \|^2 \right) + \frac{3}{2} \| r^{(n-1)/2} h_r(t) \|^2 \\
+ 2 \varepsilon \| r^{(n-1)/2} w_r(t) \|^2 + 2(n - 1) \varepsilon \| r^{(n-3)/2} w(t) \|^2 \\
\leq C_0 \varepsilon^2 F(t) + C_0 F(t) E(t) + C_0 E^2(t) + C_0 E^3(t) + 2 \varepsilon | r^{n-1} w |_0^6.
\end{align*}
$$

**Proof.** Testing the first equation of (4.1) with $2r^{n-1} h$ in $L^2$ and using integration by parts, we get

$$
\begin{align*}
\frac{d}{dt} \| r^{(n-1)/2} h \|^2 + 2 \| r^{(n-1)/2} h_r \|^2 = & -2 \int_a^b r^{n-1} h w_h \, dr - 2 \int_a^b r^{n-1} (u^0 w + \nu v^0) h_r \, dr \\
=: & J_1 + J_2.
\end{align*}
$$

The estimate of $J_1$ follows from (3.9) and (3.19):

$$
J_1 \leq 2 \| r^{(n-1)/2} h_r \| \| h \|_{L^\infty} \| r^{(n-1)/2} w \| \\
\leq C_0 \| r^{(n-1)/2} h_r \|^2 \| r^{(n-1)/2} h \|^2 \| r^{(n-1)/2} w \| \\
\leq \frac{1}{8} \| r^{(n-1)/2} h_r \|^2 + C_0 \| r^{(n-1)/2} h \|^2 \| r^{(n-1)/2} w \|^4.
$$

On the other hand, the Sobolev embedding inequality and Cauchy-Schwarz inequality entail that

$$
J_2 \leq \frac{1}{8} \| r^{(n-1)/2} h_r \|^2 + C_0 \| u^0 \|_{H^1}^2 \| r^{(n-1)/2} w \|^2 + C_0 \| v^0 \|_{H^1}^2 \| r^{(n-1)/2} h \|^2.
$$
Collecting the above estimates for $J_1$ and $J_2$, we conclude that

$$
\frac{d}{dt} \| r^{(n-1)/2} h(t) \|^2 + \frac{7}{4} \| r^{(n-1)/2} h_r(t) \|^2 \leq C_0 F(t) E(t) + C_0 E^3(t). \tag{4.11}
$$

We proceed by taking the $L^2$ inner product of the second equation of (4.1) with $2r^{n-1}w$ and get

$$
\begin{align*}
\frac{d}{dt} \| r^{(n-1)/2} w \|^2 + 2 \varepsilon \| r^{(n-1)/2} w_r \|^2 + 2(n-1) \varepsilon \| r^{(n-3)/2} w \|^2 \\
= 2\varepsilon [r^{n-1} w, w]_a^b - 4 \varepsilon \int_a^b r^{n-1} (w w_r + w r^0) w dr \\
+ 2 \int_a^b (r^{n-1} h_r + \varepsilon (r^{n-1} v^0_r)) w dr \\
- 2 \varepsilon \int_a^b (2r^{n-1} v^0 w_r + 2r^{n-1} v^0 r + (n-1) r^{n-3} v^0) w dr \\
:= 2\varepsilon [r^{n-1} w, w]_a^b + J_3 + J_4 + J_5.
\end{align*}
$$

First Sobolev embedding inequality and (3.19) yield

$$
\begin{align*}
J_3 & \leq 4 \varepsilon \| w \|_{L^4} \| r^{(n-1)/2} w_r \| \| r^{(n-1)/2} w \| + 4 \varepsilon \| v^0 \|_{L^4} \| r^{(n-1)/2} w \| \\
& \leq C_0 \varepsilon (\| r^{(n-1)/2} w_r \| + \| r^{(n-1)/2} w \|) \| r^{(n-1)/2} w_r \| \| r^{(n-1)/2} w \| + C_0 \varepsilon \| v^0 \|_{H^2} \| r^{(n-1)/2} w \|^2.
\end{align*}
$$

It follows from Cauchy-Schwarz inequality and (3.19) that

$$
J_4 \leq \frac{1}{4} \| r^{(n-1)/2} h_r \|^2 + C_0 \| r^{(n-1)/2} w \|^2 + C_0 \varepsilon^2 \| v^0 \|_{H^2}^2.
$$

Moreover Sobolev embedding inequality, Cauchy-Schwarz inequality and (3.19) lead to

$$
J_5 \leq \varepsilon \| r^{(n-1)/2} w_r \|^2 + C_0 \varepsilon \| v^0 \|_{H^2} \| r^{(n-1)/2} w \|^2 + \| r^{(n-1)/2} w \|^2 + C_0 \varepsilon^2 \| v^0 \|_{H^2}^2 + C_0 \varepsilon^2 \| v^0 \|^2.
$$

Collecting the above estimates for $J_3-J_5$ and recalling that $0 < \varepsilon < 1$, we end up with

$$
\begin{align*}
\frac{d}{dt} \| r^{(n-1)/2} w(t) \|^2 + 2 \varepsilon \| r^{(n-1)/2} w_r(t) \|^2 + 2(n-1) \varepsilon \| r^{(n-3)/2} w(t) \|^2 \\
\leq 2\varepsilon [r^{n-1} w, w]_a^b + \frac{1}{4} \| r^{(n-1)/2} h_r(t) \|^2 + C_0 \varepsilon^2 F(t) + C_0 F(t) E(t) + C_0 E^2(t),
\end{align*}
$$

which, adding to (4.11) gives the desired estimate (4.10). The proof is completed.

We turn to estimate the derivative of $w$ and the boundary term in (4.10).

**Lemma 4.3.** Let $0 < t < \infty$. Then there exists a constant $C_0$ independent of $\varepsilon$ and $t$, such that

$$
\frac{d}{dt} (\varepsilon \| r^{(n-1)/2} w_r(t) \|^2 + \frac{1}{2} \| r^{(n-1)/2} w(t) \|^2)
\leq C_0 \varepsilon^2 F(t) + C_0 F(t) E(t) + C_0 E^2(t) + \| r^{(n-1)/2} w(t) \|^2 + 2\varepsilon [r^{n-1} w, w]_a^b \tag{4.12}
$$

and

$$
2\varepsilon [r^{n-1} w, w]_a^b + 2\varepsilon [r^{n-1} w, w]_a^b
\leq C_0 \varepsilon^1 F(t) + C_0 F(t) E(t) + C_0 E^2(t) + \frac{1}{4} \| r^{(n-1)/2} w(t) \|^2 + \frac{1}{4} \| r^{(n-1)/2} h_r(t) \|^2. \tag{4.13}
$$
Proof. Taking the $L^2$ inner product of the second equation of (4.1) with $2r^{n-1}w_t$, then using integration by parts we have
\[
\frac{d}{dt} \|r^{(n-1)/2}w_r(t)\|^2 + 2\|r^{(n-1)/2}w_t(t)\|^2
\]
\[= 2\epsilon [r^{n-1}w_r]^b_a + 4\epsilon \int_a^b r^{n-1}ww_rw_r dr - 4\epsilon \int_a^b r^{n-1}(wv_r^0 + v^0w_r + v^0v_r^0)w_r dr
\]
\[+ 2\int_a^b \left[r^{n-1}h_r + \epsilon (r^{n-1}v_r^0) - \epsilon (n-1)r^{n-3}w - \epsilon (n-1)r^{n-3}v_0\right]w_r dr
\]
\[= 2\epsilon [r^{n-1}w_rw_t]^b_a + J_6 + J_7 + J_8.
\]
We first employ Sobolev embedding inequality and $\epsilon$ (3.19) to deduce that
\[J_6 \leq C_0 \epsilon \|w\|_{H^1} \|r^{(n-1)/2}w_r\| \|r^{(n-1)/2}w_t\|\]
\[\leq \frac{1}{8}\|r^{(n-1)/2}w_r\|^2 + C_0 \epsilon^2 \left(\|r^{(n-1)/2}w_r\|^2 + \|r^{(n-1)/2}w_t\|^2\right)\|r^{(n-1)/2}w_r\|^2\]
and that
\[J_7 \leq \frac{1}{8}\|r^{(n-1)/2}w_r\|^2 + C_0 \epsilon^2 \left(\|v_r^0\|^2_{H^2} + \|r^{(n-1)/2}w_t\|^2\right).
\]
Moreover Cauchy-Schwarz inequality and (3.19) entail that
\[J_8 \leq \frac{5}{4} \|r^{(n-1)/2}w_r\|^2 + \|r^{(n-1)/2}h_r\|^2 + C_0 \epsilon^2 \left(\|v_r^0\|^2_{H^2} + \|r^{(n-1)/2}w_t\|^2\right).
\]
Then (4.12) follows from the above estimates on $J_6 - J_8$. It remains to prove (4.13). By the definition of $w$ and Gagliardo-Nirenberg interpolation inequality, one deduces that
\[2\epsilon [r^{n-1}w_rw_t]^b_a \leq C_0 \epsilon \|w_r\|_{L^\infty} \|\bar{v}_1 + |\bar{v}_2| + \|v_r^0\|_{L^\infty}
\]
\[\leq C_0 \epsilon \left(\|w_r\|^1_{H^2} + \|w_r\|^1_{H^1}\right)\|\bar{v}_1 + |\bar{v}_2| + \|v_r^0\|_{H^1}\)
\[\leq \eta \epsilon^2 \|w_{rr}\|^2 + C_0 (1 + 1/\eta) \epsilon \|w_r\|^2 + C_0 (\epsilon^{1/2} + \epsilon)(|\bar{v}_1| + |\bar{v}_2| + \|v_r^0\|_{H^2})^2,
\]
where $\eta > 0$ is a small constant to be determined. By a similar argument as deriving (4.14) and the second equation of (4.17) with $\epsilon = 0$, we further get
\[2\epsilon [r^{n-1}w_rw_t]^b_a \leq \eta \epsilon^2 \|w_{rr}\|^2 + C_0 (1 + 1/\eta) \epsilon \|w_r\|^2 + C_0 (\epsilon^{1/2} + \epsilon) \|w_r^2\|^2
\]
\[\leq \eta \epsilon^2 \|w_{rr}\|^2 + C_0 (1 + 1/\eta) \epsilon \|w_r\|^2 + C_0 (\epsilon^{1/2} + \epsilon) \|u_r^0\|^2_{H^2}.
\]
To bound the term $\|w_{rr}\|^2$ in (4.14) and (4.15), we use the second equation of (4.1), Sobolev embedding inequality and (3.19) and derive
\[\epsilon^2 \|w_{rr}\|^2 \leq C_1 \left(\|r^{(n-1)/2}w_r\|^2 + \|r^{(n-1)/2}w_{rr}\|^2 + \epsilon^2 \|r^{(n-1)/2}w_{rr}\|^2 + \epsilon^2 \|r^{(n-1)/2}w_{rr}\|^2\right)\|v_r^0\|^2_{H^2}
\]
\[+ \epsilon^2 \|r^{(n-1)/2}w_{rrr}\|^4 + \epsilon^2 \|r^{(n-1)/2}w_{rrr}\|^2 + \epsilon^2 \|r^{(n-1)/2}w_{rrr}\|^2 + \epsilon^2 \|v_r^0\|^2_{H^2}
\]
\[+ \epsilon^2 \|r^{(n-1)/2}w_{rrr}\|^4 + \epsilon^2 \|r^{(n-1)/2}w_{rrr}\|^2 + \epsilon^2 \|r^{(n-1)/2}w_{rrr}\|^2 + \epsilon^2 \|v_r^0\|^2_{H^2},
\]
where we have used the notation $C_1$ to distinguish it from the constant $C_0$ in (4.14) and (4.15). Finally feeding (4.14) and (4.15) on (4.16) then adding the results, we obtain (4.13) by taking $\eta$ small enough such that $C_1 \eta < \frac{1}{8}$ and by using $0 < \epsilon < 1$. The proof is completed.

\[\square\]
We next apply Lemma 4.4 to the combination of Lemma 4.2 and Lemma 4.3 to obtain the following result.

**Lemma 4.4.** Let $0 < T < \infty$ and $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0$ defined in Theorem 2.2. Then there exists a constant $C$ independent of $\varepsilon$, depending on $T$ such that

$$
\|h\|_{L^2(0,T;L^2)}^2 + \|w\|_{L^2(0,T;L^2)}^2 + \varepsilon \|w_r\|_{L^2(0,T;L^2)}^2 + \|h_r\|_{L^2(0,T;L^2)}^2 + \|w_t\|_{L^2(0,T;L^2)}^2 \
\leq C \varepsilon^\frac{1}{2}
$$

(4.17)

and

$$
\|w_{rr}\|_{L^2(0,T;L^2)} \leq C \varepsilon^{-3/4}.
$$

(4.18)

**Proof.** We first add (4.12) and (4.13) to (4.10) and find

$$
\frac{d}{dt} E(t) + \frac{1}{4} \left( \|r^{\frac{n-1}{2}} h_r(t)\|^2 + 2(n-1) \varepsilon \|r^{\frac{n-3}{2}} w(t)\|^2 + \frac{1}{4} \|r^{\frac{n-1}{2}} w_t(t)\|^2 \right) 
\leq C_0 \varepsilon^{\frac{1}{2}} F(t) + C_0 F(t) E(t) + C_0 E^2(t) + C_0 E^3(t),
$$

(4.19)

where $0 < \varepsilon < 1$ has been used. Then we apply Lemma 4.1 to (4.19) by taking $k = 3$, $\gamma = \varepsilon^{1/2}$ and $f_1(t) = f_2(t) = C_0 F(t)$ to conclude for $\varepsilon \in (0, \varepsilon_0]$ that

$$
\|h\|_{L^2(0,T;L^2)}^2 + \|w\|_{L^2(0,T;L^2)}^2 + \varepsilon \|w_r\|_{L^2(0,T;L^2)}^2 + \|h_r\|_{L^2(0,T;L^2)}^2 + \|w_t\|_{L^2(0,T;L^2)}^2 
\leq \left( C_0 \varepsilon^{\frac{1}{2}} \int_0^T F(t) dt \right) \varepsilon^{\frac{1}{2}},
$$

(4.20)

where (4.19) has been used. Then we integrate (4.19) over $(0,T)$ and applying (4.20) to the result to deduce that

$$
\|h\|_{L^2(0,T;L^2)}^2 + \|w\|_{L^2(0,T;L^2)}^2 + \varepsilon \|w_r\|_{L^2(0,T;L^2)}^2 + \|h_r\|_{L^2(0,T;L^2)}^2 + \|w_t\|_{L^2(0,T;L^2)}^2 
\leq C \varepsilon^{\frac{1}{2}},
$$

where the constant $C$ depending on $T$ and $\int_0^T F(t) dt$ is finite thanks to Theorem 2.1. We thus derive (4.17) and proceed to prove (4.18). Indeed, it follows from the second equation of (4.1), Sobolev embedding inequality and (4.17) that

$$
\varepsilon \|w_{rr}\|_{L^2(0,T;L^2)} \leq C \left( \|w_t\|_{L^2(0,T;L^2)}^2 + \varepsilon \|w_r\|_{L^2(0,T;L^2)}^2 + \varepsilon \|w|_{L^2(0,T;H^1)} \|w_r\|_{L^2(0,T;L^2)}^2 \right) 
+ C \left( \|h_r\|_{L^2(0,T;L^2)}^2 + \varepsilon \|w^0\|_{L^2(0,T;H^2)}^2 + \varepsilon \|w_r\|_{L^2(0,T;L^2)}^2 + \varepsilon \|w|_{L^2(0,T;H^2)}^2 \right) 
+ C \left( \varepsilon \|w^0\|_{L^2(0,T;H^2)}^2 + \varepsilon \|w_t\|_{L^2(0,T;L^2)}^2 + \varepsilon \|w^0\|_{L^2(0,T;L^2)}^2 \right)
\leq C \varepsilon^{1/4}.
$$

We thus derive (4.18) and the proof is completed.

Higher regularity estimates for the solution component $h$ is given in the following lemma.

**Lemma 4.5.** Suppose $0 < T < \infty$ and $\varepsilon \in (0, \varepsilon_0]$. Then there is a constant $C$ independent of $\varepsilon$, depending on $T$ such that

$$
\|h_r\|_{L^2(0,T;L^2)}^2 + \|h_t\|_{L^2(0,T;L^2)}^2 + \|h_{rr}\|_{L^2(0,T;L^2)}^2 \leq C \varepsilon^{1/2}.
$$

(4.21)
Proof. We first take the $L^2$ inner product of the first equation of (4.1) with $2r^{n-1}h_t$ and use integration by parts to get
\[
\frac{d}{dt}||r^{(n-1)/2}h_t||^2 + 2||r^{(n-1)/2}h_t||^2 = -2 \int_a^b r^{n-1}(hw + u^0w + hv^0)h_tdr \leq \frac{1}{2}||r^{(n-1)/2}h_t||^2 + C_0(||h_r||^2||w||^2 + ||u^0||^2 ||w||^2 + ||h_r||^2||v^0||^2). \tag{4.22}
\]
Then differentiating the first equation of (4.1) with respect to $t$ and multiplying the resulting equation with $2r^{n-1}h_t$ in $L^2$, we derive
\[
\frac{d}{dt}||r^{(n-1)/2}h_t||^2 + 2||r^{(n-1)/2}h_t||^2 = -2 \int_a^b r^{n-1}hw_rh_tdr - 2 \int_a^b r^{n-1}(hw_t + u^0w_t + u^0w_r + h^0v^0)v^0h_tdr \leq K_1 + K_2.
\]
The estimate for $K_1$ follows from (3.9) and (3.19):
\[
K_1 \leq C_0||r^{(n-1)/2}h_t|| ||h_t||_{L^2}||w|| \leq C_0||r^{(n-1)/2}h_t||^{3/2}||r^{(n-1)/2}||^{1/2}||w|| \leq \frac{1}{4}||r^{(n-1)/2}h_t||^2 + C_0||r^{(n-1)/2}h_t||^2||w||.
\]
Sobolev embedding inequality and (3.19) entail that
\[
K_2 \leq \frac{1}{4}||r^{(n-1)/2}h_t||^2 + C_0 \left( ||r^{(n-1)/2}||^2 ||w||^2 + ||u^0||^2 ||v||^2 \right) + C_0 \left( ||u^0||^2 ||w||^2 + ||v^0||^2 ||u^0||^2 ||v^0||^2 \right).
\]
Then collecting the above estimates for $K_1$ and $K_2$, one derives
\[
\frac{d}{dt}||r^{(n-1)/2}h_t||^2 + 2||r^{(n-1)/2}h_t||^2 \leq C_0 \left( ||w||^4 + ||v||^2 ||w||^2 \right) ||r^{(n-1)/2}||^2 + C_0 ||w||^2 ||r^{(n-1)/2}h_t||^2 + C_0 ||u^0||^2 ||w||^2 + C_0 ||u^0||^2 ||v||^2 + C_0 ||u^0||^2 ||h||^2,
\]where we have used inequalities $||u^0||^2 ||w||^2 \leq C_0 ||u^0||^2 ||w||^2 + ||u^0||^2 ||v||^2 ||w||^2$ and $||v||^2 \leq C_0 ||u^0||^2 ||H^2||$, thanks to the first and second equations of (3.1). Finally by adding (4.23) to (4.22) and applying Gronwall’s inequality to the result, and then using (4.17) we obtain (4.21). The proof is completed.

We turn to establish a weighted $L^2$-estimate (enlightened by (17)) on the derivative of $w$.

Lemma 4.6. For $0 < T < \infty$ and $\varepsilon \in (0, \varepsilon_0]$, there is a constant $C$ independent of $\varepsilon$, depending on $T$ such that
\[
||(r-a)(r-b)w_t||_{L^2(0,T;L^2)} + \varepsilon||(r-a)(r-b)w_{tt}||_{L^2(0,T;L^2)} \leq C\varepsilon^{1/2}.
\]
Proof. Taking the $L^2$ inner product of the second equation of (4.1) with $-2(r-a)^2(r-b)^2w_{rr}$ and using integration by parts, one gets

\[
\frac{d}{dt}\|(r-a)(r-b)w_r\|^2 + 2\varepsilon\|(r-a)(r-b)w_r\|^2
\]

\[
= -2\varepsilon\int_a^b (r-a)^2(r-b)^2w_{rr}\left[\frac{1}{r^{n-1}}(r_{r}^{n-1})_r - \frac{n-1}{r^2}(w + v_0)\right]dr
\]

\[
+ 4\varepsilon\int_a^b (r-a)^2(r-b)^2w_{rr}(wv^0 + v^0w_r + v^0v^0_r)dr
\]

\[
- 4\int_a^b (2r-a-b)(r-a)(r-b)w_rw_rdr
\]

\[
- 2\int_a^b (r-a)^2(r-b)^2w_rh_rdr
\]

\[
- 2\varepsilon\int_a^b (r-a)^2(r-b)^2w_r\left(\frac{n-1}{r}w_r - 2ww_r\right)dr
\]

\[
:= \sum_{i=3}^{7} K_i.
\]

We next estimate $K_3-K_7$. Indeed Cauchy-Schwarz inequality, Sobolev embedding inequality and (3.19) yield

\[
K_3 + K_4 \leq \frac{1}{8}\varepsilon\|(r-a)(r-b)w_r\|^2 + C_0\varepsilon(||v^0||^2_{H^2} + ||w||^2 + ||v^0||^2_{H^2} + ||w_r||^2 + ||v^0||^4_{H^1})
\]

and

\[
K_5 + K_7 \leq \frac{1}{8}\varepsilon\|(r-a)(r-b)w_r\|^2 + C_0(1 + \varepsilon + \varepsilon||w||^2 + \varepsilon||w_r||^2)\|(r-a)(r-b)w_r\|^2 + C_0||w_r||^2.
\]

For the term $K_6$, we use integration by parts and the first equation of (4.1) to get

\[
K_6 = 4\int_a^b (2r-a-b)(r-a)(r-b)w_rh_rdr + 2\int_a^b (r-a)^2(r-b)^2w_rh_rdr
\]

\[
= 4\int_a^b (2r-a-b)(r-a)(r-b)w_rh_rdr + 2\int_a^b (r-a)^2(r-b)^2w_r\left(h_r - \frac{n-1}{r}h_r\right)dr
\]

\[
- 2\int_a^b (r-a)^2(r-b)^2w_r(hw + u^0w)_rdr
\]

\[
- 2\int_a^b (r-a)^2(r-b)^2w_r \left([hv^0]_r + \frac{n-1}{r}(hw + u^0w + hv^0)\right)dr
\]

\[
:= R_1 + R_2 + R_3 + R_4.
\]

We proceed to estimate $R_1-R_4$. First it follows from Cauchy-Schwarz inequality that

\[
R_1 + R_2 \leq ||(r-a)(r-b)w_r||^2 + C_0(||h_r||^2 + ||h_r||^2).
\]

Moreover, we use Cauchy-Schwarz inequality and apply (3.9) to $h$ and $(r-a)(r-b)w$ to derive

\[
R_3 \leq 2(||h||_{L^\infty} + ||u^0||_{L^\infty})||(r-a)(r-b)w_r||^2
\]

\[
+ 2(||h_r|| + ||u^0_r||)||(r-a)(r-b)w_r||||(r-a)(r-b)w||_{L^\infty}
\]

\[
\leq C_0(||h_r|| + ||u^0||_{L^\infty})||(r-a)(r-b)w_r||^2
\]

\[
+ C_0(||h_r|| + ||u^0||_{H^2})||(r-a)(r-b)w_r||||(r-a)(r-b)w_r||
\]

\[
\leq C_0(||h_r|| + ||u^0||_{H^2} + 1)^2||(r-a)(r-b)w_r||^2 + C_0||w||^2.
\]

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The estimate for $R_4$ follows from the Sobolev embedding inequality, (3.9) and Cauchy-Schwarz inequality:

$$R_4 \leq C_0 \| (r - a)(r - b) w_r \| \left( \| hw \| + \| h v^0 \|_{H^1} + \| u^0 \| \right)$$

$$\leq \| (r - a)(r - b) w_r \|^2 + C_0 \| h v^0 \|_{H^1} \| w \| + \| h v^0 \|_{H^1} \| u^0 \| \| w \|^2 .$$

We thus conclude from the above estimates for $R_1$-$R_4$ that

$$K_6 \leq C_0 \left( \| h v^0 \|_{H^1} + \| u^0 \| \right) \| (r - a)(r - b) w_r \|^2$$

$$+ C_0 \left( \| h v^0 \|_{H^1} \| w \| + \| h v^0 \|_{H^1} \| u^0 \| \| w \|^2 + \| h v^0 \|_{H^1} \| u^0 \| \| w \|^2 \right).$$

Substituting the above estimates for $K_3$-$K_7$ into (4.24), then applying Gronwall’s inequality, Lemma 4.4 - Lemma 4.6 and Theorem 2.1 to the result, we obtain the desired estimate and the proof is finished.

We next prove Theorem 2.2 by the results derived in Lemma 4.4 - Lemma 4.6.

**Proof of Theorem 2.2** By Lemma 4.4, Lemma 4.5 and Sobolev embedding inequality, we deduce that

$$\| u^\varepsilon - u^0 \|_{L^\infty(0,T;C[a,b])} \leq C_0 \| u^\varepsilon - u^0 \|_{L^\infty(0,T;H^1)} \leq C \varepsilon^{1/4},$$

which gives (2.5). Clearly $\delta^2 \leq \frac{4}{(b - a)^2} (r - a)^2 (r - b)^2$ holds for $\delta < \frac{b - a}{2}$ and $r \in (a, b)$, thus it follows from Lemma 4.6 that

$$\delta^2 \int_{a + \delta}^{b - \delta} w_r^2(r,t) \, dr \leq \frac{4}{(b - a)^2} \int_{a + \delta}^{b - \delta} (r - a)^2 (r - b)^2 w_r^2(r,t) \, dr \leq C \varepsilon^{1/2}, \quad t \in [0,T]$$

which, along with Lemma 4.4 and Gagliardo-Nirenberg inequality entails that

$$\| v^\varepsilon - v^0 \|_{L^\infty(0,T;C[a + \delta,b - \delta])} \leq C_0 \left( \| w \|_{L^\infty(0,T;L^2(a + \delta,b - \delta))} \right)$$

$$+ C \left( \| w \|_{L^\infty(0,T;L^2(a + \delta,b - \delta))} \right)^{1/2} \| w_r \|_{L^\infty(0,T;L^2(a + \delta,b - \delta))}^{1/2}$$

$$\leq C \left( \varepsilon^{1/4} + \varepsilon^{1/8} \cdot \varepsilon^{1/8} \delta^{-1/2} \right)$$

$$\leq C \varepsilon^{1/4} \delta^{-1/2},$$

provided $\delta < 1$. Hence we derive (2.6) and we next prove the equivalence between (2.7) and (2.8). We first prove that (2.8) implies (2.7). Assume $\int_0^t u^0_r(a, \tau) \, d\tau \neq 0$ for some $t_0 \in [0,T]$. Then integrating the second equation of (1.7) with $\varepsilon = 0$ over $(0,t_0)$ along with compatible condition $v_0(a) = \bar{v}_1$ gives

$$v^0(a,t_0) = \bar{v}_1 + \int_0^{t_0} u^0_r(a, \tau) \, d\tau. \quad (4.25)$$

We thus have

$$\liminf_{\varepsilon \to 0} \| v^\varepsilon - v^0 \|_{L^\infty(0,T;C[a,b])} \geq \liminf_{\varepsilon \to 0} | \bar{v}_1 - v^0(a,t_0) | = \liminf_{\varepsilon \to 0} | \int_0^{t_0} u^0_r(a, \tau) \, d\tau | > 0.$$
which, along with the second equation of \((1.7)\) with \(\epsilon = 0\) leads to
\[
(v^\epsilon - v^0)(a,t) = -[v^0(a,t) - \bar{v}_1] = - \int_0^t u^0_r(a,\tau)d\tau = 0,
\]
\[
(v^\epsilon - v^0)(b,t) = -[v^0(b,t) - \bar{v}_2] = - \int_0^t u^0_r(b,\tau)d\tau = 0,
\]
where the compatible conditions \(v_0(a) = \bar{v}_1, v_0(b) = \bar{v}_2\) have been used. Thus \(w|_{r=a,b} = (v^\epsilon - v^0)|_{r=a,b} = 0\) and \(w_t|_{r=a,b} = [(v^\epsilon - v^0)|_{r=a,b}]_\tau = 0\) and the terms \(2\epsilon[r^{n-1}w_r]_\tau^\epsilon, 2\epsilon[r^{n-1}w_rw_r]_\tau^\epsilon\) in \((4.10)\) and \((4.12)\) would vanish. Then by similar arguments as deriving \((4.20)\), we conclude that
\[
\|h\|_{L^2(0,T;L^2)}^2 + \|w\|_{L^2(0,T;L^2)}^2 + \epsilon\|w_r\|_{L^2(0,T;L^2)}^2 \leq C\epsilon^2,
\]
which, along with Sobolev embedding inequality gives rise to
\[
\lim_{\epsilon \to 0} \|v^\epsilon - v^0\|_{L^2(0,T;C[a,b])} \leq C_0 \lim_{\epsilon \to 0} (\|w\|_{L^2(0,T;L^2)} + \|w_r\|_{L^2(0,T;L^2)}) = 0,
\]
which, contradicts with \((2.7)\), thus \((2.7)\) implies \((2.8)\). The proof is completed.

We next convert the result of Theorem \(2.2\) to the initial-boundary value problem \((2.9)\) for the original chemotaxis model to prove Proposition \(2.1\).

**Proof of Proposition 2.1**

Let
\[
c^\epsilon(r,t) = c_0(r) \exp \left\{ \int_0^t \left[ -u^\epsilon + \epsilon(v^\epsilon)^2 - \epsilon v^\epsilon_r - \frac{n-1}{r} v^\epsilon \right] (r,\tau)d\tau \right\},
\]
\[
c^0(r,t) = c_0(r) \exp \left\{ - \int_0^t u^0_r(r,\tau)d\tau \right\},
\]
(4.26)

where \((u^\epsilon, v^\epsilon)\) and \((u^0, v^0)\) are the solutions of problem \((1.7) - (1.8)\) with \(\epsilon > 0\) and \(\epsilon = 0\) respectively. It is easy to check that \((u^\epsilon, c^\epsilon)\) and \((u^0, c^0)\) with \(c^\epsilon\) and \(c^0\) defined \((4.26)\) are the unique solutions of \((2.9)\) corresponding to \(\epsilon > 0\) and \(\epsilon = 0\), respectively.

The first inequality in \((2.10)\) follows directly from the Sobolev embedding inequality, Lemma 4.4 and Lemma 4.5 as following:
\[
\|u^\epsilon - u^0\|_{L^2(0,T;C[a,b])} \leq C_0 \|u^\epsilon - u^0\|_{L^2(0,T;H^1)} \leq C\epsilon^{1/4}.
\]

To prove the second inequality in \((2.10)\), we deduce from \((4.26)\) that
\[
|c^\epsilon(r,t) - c^0(r,t)| = |c^0(r,t)| \cdot \left| \exp \left\{ \int_0^t \left[ -(u^\epsilon - u^0) + \epsilon(v^\epsilon)^2 - \epsilon v^\epsilon_r - \frac{n-1}{r} v^\epsilon \right] d\tau \right\} \right| - 1 \leq |c^0(r,t)| \cdot |e^{G^\epsilon(r,t)} - 1|,
\]
(4.27)

where we have denoted by \(G^\epsilon(r,t) = \int_0^t \left[ -(u^\epsilon - u^0) + \epsilon(v^\epsilon)^2 - \epsilon v^\epsilon_r - \frac{n-1}{r} v^\epsilon \right] d\tau\) for convenience. For
Moreover, from Theorem 2.1 we know that
\[ C_{\infty} \text{ independent of } \varepsilon < \varepsilon \]
where the assumption \( 0 < \varepsilon < 1 \) has been used in the last inequality. Then we apply Taylor expansion to \( e^{G^\varepsilon(r,t)} \) and using (4.28) to conclude that
\[ \left| e^{G^\varepsilon(r,t)} - 1 \right| \leq \sum_{k=1}^{\infty} \frac{1}{k!} \left| G^\varepsilon(x,t) \right|^k \leq \sum_{k=1}^{\infty} \frac{C_2^k}{k!} \varepsilon^{1/4} \leq C_2 \varepsilon^{1/4}, \]
for some constant \( C_1 \) independent of \( \varepsilon \) (depending on \( T \)) and the assumption \( 0 < \varepsilon < 1 \) has been used in the second inequality and the constant \( C_2 := e^{C_1} \) is independent of \( \varepsilon \). On the other hand, by the assumptions \( c_0(r) > 0 \) and \( \ln c_0 \in H^1 \) in Proposition 2.1 we derive that \( \| \ln c_0 \|_{L^\infty} \leq \| \ln c_0 \|_{H^1} \leq C_3 \) for some positive constant \( C_3 \), which along with the fact 
\[ c_0(r) = e^{\ln c_0(r)} \]
leads to
\[ e^{-C_3} \leq c_0(r) \leq e^{C_3} \quad \text{for } r \in [a,b]. \]
Moreover, from Theorem 2.1 we know that
\[ \left\| \int_0^t u^0(r, \tau) d\tau \right\|_{L^\infty(0,T;L^\infty)} \leq C_0 T \| u^0 \|_{L^\infty(0,T;H^1)} \leq C_4 \quad \text{for } t \in [0,T], \]
where the constant \( C_4 \) depending on \( T \). Thus we deduce from the second equality of (4.26), (4.30) and (4.31) that
\[ 0 < C_5^{-1} < c_0(r,t) < C_5 \quad \text{for } (r,t) \in [a,b] \times [0,T], \]
with \( C_5 = e^{(C_4+1)} \). Hence, it follows from (4.27), (4.29) and (4.32) that
\[ \| e^\varepsilon - c_0^0 \|_{L^\infty(0,T;L^\infty)} \leq C_6 \varepsilon^{1/4}, \]
where \( C_6 := C_2 C_3 \) is independent of \( \varepsilon \). We thus derive the second inequality in (2.10) and proceed to prove (2.11). It follows from transformation (1.6) that
\[ c_0(t) - c_0^0 = (v^0 - v^\varepsilon) e^\varepsilon + v^0 (c_0^0 - c^0), \]
which, in conjunction with (2.6) and (2.10) leads to
\[ \| c_0^\varepsilon - c_0^0 \|_{L^\infty(0,T;C[a,b])} \leq \| v^\varepsilon - v^0 \|_{L^\infty(0,T;C[a,b])} \left( \| e^\varepsilon \|_{L^\infty(0,T;C[a,b])} + C_6 \varepsilon^{1/4} \right) \]
\[ + C_6 \varepsilon^{1/4} \| v^0 \|_{L^\infty(0,T;C[a,b])} \]
\[ \leq C_6 \varepsilon^{1/4} \delta_{1/2}, \]
where \( \delta < 1 \) has been used thanks to \( \delta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). We thus derived (2.11). To prove the equivalence between (2.12) and (2.8), we first derives two positive constants \( C_7 \) and \( C_8 \) independent of \( \varepsilon \) such that
\[ 0 < C_7 \leq c_0^\varepsilon(r,t) \leq C_8 \quad \text{for } (r,t) \in [a,b] \times [0,T], \]
(4.35)
by choosing $\varepsilon$ small enough such that $C_6 \varepsilon^{1/4} < \frac{1}{2C_5}$ in (4.33) and using (4.32). With (4.35) in hand, we next prove the equivalence between (2.12) and (2.8). First, it follows from (4.34), (2.10) and (4.35) that

$$\liminf_{\varepsilon \to 0} \left\| c^\varepsilon - c^0 \right\|_{L^\infty(0,T;C[a,b])} \geq \liminf_{\varepsilon \to 0} \left[ \left\| c^\varepsilon \right\|_{L^\infty(0,T;L^\infty)} \right] \geq \liminf_{\varepsilon \to 0} \left[ \left\| c^\varepsilon \right\|_{L^\infty(0,T;L^\infty)} \right] - \| v^0 \|_{L^\infty(0,T;L^\infty)} \left\| c^\varepsilon - c^0 \right\|_{L^\infty(0,T;L^\infty)}$$

(4.36)

Dividing (4.34) by $c^\varepsilon$ and applying a similar argument as deriving (4.36), then using (4.35) and (2.10), one deduces that

$$\liminf_{\varepsilon \to 0} \left\| v^\varepsilon - v^0 \right\|_{L^\infty(0,T;C[a,b])} \geq \liminf_{\varepsilon \to 0} \frac{\| c^\varepsilon - c^0 \|_{L^\infty(0,T;L^\infty)}}{c^\varepsilon} \geq \frac{1}{C_8} \liminf_{\varepsilon \to 0} \left\| c^\varepsilon - c^0 \right\|_{L^\infty(0,T;L^\infty)},$$

which, in conjunction with (4.36) indicates the equivalence between (2.12) and (2.7). Then we conclude that (2.12) is equivalent to (2.8) by using Theorem 2.2. The proof is completed.

□

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References

[1] J. Adler. Chemotaxis in bacteria. Science, 153:708–716, 1966.
[2] J. Adler. Chemoreceptors in bacteria. Science, 166:1588–1597, 1969.
[3] M. Chae, K. Choi, K. Kang, and J. Lee. Stability of planar traveling waves in a Keller-Segel equation on an infinite strip domain. J. Differential Equations, 265:237–279, 2018.
[4] M.A.J. Chaplain and A.M. Stuart. A model mechanism for the chemotactic response of endothelial cells to tumor angiogenesis factor. IMA J. Math. Appl. Med., 10(3):149–168, 1993.
[5] P. Constantin. Note on loss of regularity for solutions of the 3d incompressible euler and related equations. Commun. Math. Phys., 104:311–326, 1986.
[6] L. Corrias, B. Perthame, and H. Zaag. A chemotaxis model motivated by angiogenesis. C. R. Math. Acad. Sci. Paris, 2:141–146, 2003.
[7] L. Corrias, B. Perthame, and H. Zaag. Global solutions of some chemotaxis and angiogenesis systems in high space dimensions. Milan J. Math., 72:1–29, 2004.
[8] C. Deng and T. Li. Well-posedness of a 3D parabolic-hyperbolic Keller-Segel system in the sobolev space framework. J. Differential Equations, 257:1311–1332, 2014.
[9] P.C. File. Considerations regarding the mathematical basis for Prandtl’s boundary layer theory. Arch. Ration. Mech. Anal., 28(3):184–216, 1968.

[10] H. Frid and V. Shelukhin. Boundary layers for the Navier-Stokes equations of compressible fluids. Comm. Math. Phys., 208:309–330, 1999.

[11] H. Frid and V. Shelukhin. Vanishing shear viscosity in the equations of compressible fluids for the flows with the cylinder symmetry. SIAM J. Math. Anal., 31:1144–1156, 2000.

[12] A. Gamba, D. Ambrosi, A. Coniglio, A. de Candia, S. Di Talia, E. Giraudo, G. Serini, L. Preziosi, and F. Busso. Percolation, morphogenesis, and burgers dynamics in blood vessels formation. Phys. Rev. Lett., 90:118101, 2003.

[13] J. Guo, J.X. Xiao, H.J. Zhao, and C.J. Zhu. Global solutions to a hyperbolic-parabolic coupled system with large initial data. Acta Math. Sci. Ser. B Engl. Ed, 29:629–641, 2009.

[14] C. Hao. Global well-posedness for a multidimensional chemotaxis model in critical besov spaces. Z. Angew Math. Phys., 63:825–834, 2012.

[15] H. Höfer, J.A. Sherratt, and P.K. Maini. Cellular pattern formation during Dictyostelium aggregation. Physica D., 85:425–444, 1995.

[16] Q. Hou, Z. Wang, and K. Zhao. Boundary layer problem on a hyperbolic system arising from chemotaxis. J. Differential Equations, 261:5035–5070, 2016.

[17] S. Jiang and J. Zhang. On the non-resistive limit and the magnetic boundary-layer for one-dimensional compressible magnetohydrodynamics. Nonlinearity, 30:3587–3612, 2017.

[18] H.Y. Jin, J. Li, and Z. Wang. Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity. J. Differential Equations, 255(2):193–219, 2013.

[19] E.F. Keller and L.A. Segel. Initiation of slime mold aggregation viewed as an instability. J. Theor. Biol., 26(3):399–415, 1970.

[20] E.F. Keller and L.A. Segel. Model for chemotaxis. J. Theor. Biol., 30:225–234, 1971.

[21] E.F. Keller and L.A. Segel. Traveling bands of chemotactic bacteria: A theoretical analysis. J. Theor. Biol., 30:377–380, 1971.

[22] H.A. Levine and B.D. Sleeman. A system of reaction diffusion equations arising in the theory of reinforced random walks. SIAM J. Appl. Math., 57:683–730, 1997.

[23] H.A. Levine, B.D. Sleeman, and M. Nilsen-Hamilton. A mathematical model for the roles of pericytes and macrophages in the initiation of angiogenesis. I. the role of protease inhibitors in preventing angiogenesis. Math. Biosci., 168:71–115, 2000.

[24] D. Li, T. Li, and K. Zhao. On a hyperbolic-parabolic system modeling chemotaxis. Math. Models Methods Appl. Sci., 21:1631–1650, 2011.

[25] D. Li, R. Pan, and K. Zhao. Quantitative decay of a one-dimensional hybrid chemotaxis model with large data. Nonlinearity, 7:2181–2210, 2015.

[26] H. Li and K. Zhao. Initial-boundary value problems for a system of hyperbolic balance laws arising from chemotaxis. J. Differential Equations, 258(2):302–338, 2015.

[27] J. Li, T. Li, and Z. Wang. Stability of traveling waves of the Keller-Segel system with logarithmic sensitivity. Math. Models Methods Appl. Sci., 24(14):2819–2849, 2014.

[28] T. Li, R. Pan, and K. Zhao. Global dynamics of a hyperbolic-parabolic model arising from chemotaxis. SIAM J. Appl. Math., 72(1):417–443, 2012.

[29] T. Li and Z. Wang. Nonlinear stability of travelling waves to a hyperbolic-parabolic system modeling chemotaxis. SIAM J. Appl. Math., 70(5):1522–1541, 2009.

[30] T. Li and Z. Wang. Nonlinear stability of large amplitude viscous shock waves of a hyperbolic-parabolic system arising in chemotaxis. Math. Models Methods Appl. Sci., 20(10):1967–1998,
2010.

[31] T. Li and Z. Wang. Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis. J. Differential Equations, 250(3):1310–1333, 2011.

[32] T. Li and Z. Wang. Steadily propagating waves of a chemotaxis model. Math. Biosci., 240(2):161–168, 2012.

[33] V. Martinez, Z. Wang, and K. Zhao. Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology. Indiana Univ. Math. J., 67:1383–1424, 2018.

[34] J.D. Murray. Mathematical Biology I: An Introduction. Springer, Berlin, 3rd edition, 2002.

[35] K.J. Painter, P.K. Maini, and H.G. Othmer. Stripe formation in juvenile pomacanthus explained by a generalized Turing mechanism with chemotaxis. Proc. Natl. Acad. Sci., 96:5549–5554, 1999.

[36] K.J. Painter, P.K. Maini, and H.G. Othmer. A chemotactic model for the advance and retreat of the primitive streak in avian development. Bull. Math. Biol., 62:501–525, 2000.

[37] H. Peng, L. Ruan, and C. Zhu. Convergence rates of zero diffusion limit on large amplitude solution to a conservation laws arising in chemotaxis. Kinetic and Related Models, 5:563–581, 2012.

[38] H. Peng, H. Wen, and C. Zhu. Global well-posedness and zero diffusion limit of classical solutions to 3D conservation laws arising in chemotaxis. Z. Angew Math. Phys., 65(6):1167–1188, 2014.

[39] G.J. Petter, H.M. Byrne, D.L.S. Mcelwain, and J. Norbury. A model of wound healing and angiogenesis in soft tissue. Math. Biosci., 136(1):35–63, 2003.

[40] L. Prandtl. Über Flüssigkeitsbewegungen bei sehr kleiner Reibung. In “Verh. Int. Math. Kongr., Heidelberg 1904”, Teubner, 1905.

[41] L.G. Rebholz, D. Wang, Z. Wang, C. Zerfas, and K. Zhao. Initial boundary value problems for a system of parabolic conservation laws arising from chemotaxis in multi-dimensions. Discrete Contin. Dyn. Syst., 39(7):3789–3838, 2019.

[42] H. Schlichting. Boundary layer theory. McGraw-Hill company, London-New York, 7th edition, 1987.

[43] Y.S. Tao, L.H. Wang, and Z. Wang. Large-time behavior of a parabolic-parabolic chemotaxis model with logarithmic sensitivity in one dimension. Discrete Contin. Dyn. Syst-Series B., 18:821–845, 2013.

[44] R. Tyson, S.R. Lubkin, and J. Murray. Models and analysis of chemotactic bacterial patterns in a liquid medium. J. Math. Biol., 266:299–304, 1999.

[45] D. Wang, Z. Wang, and K. Zhao. Cauchy problem of a system of parabolic conservation laws arising from the singular keller-segel model in multi-dimensions. Indiana U. Math. J., 2019.

[46] Y.G. Wang and Z. Xin. Zero-viscosity limit of the linearized compressible Navier-Stokes equations with highly oscillatory forces in the half-plane. SIAM J. Math. Anal., 37(4):1256–1298, 2005.

[47] Z. Wang. Mathematics of traveling waves in chemotaxis. Discrete Contin. Dyn. Syst-Series B, 18:601–641, 2013.

[48] Z. Wang, Z. Xiang, and P. Yu. Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis. J. Differential Equations, 260:2225–2258, 2016.

[49] Z. Wang and K. Zhao. Global dynamics and diffusion limit of a one-dimensional repulsive chemotaxis model. Comm. Pure Appl. Anal., 12:3027–3046, 2013.
[50] M. Winkler. Renormalized radial large-data solutions to the higher-dimensional keller-segel system with singular sensitivity and signal absorption. *J. Differential Equations*, 264:2310–2350, 2018.

[51] J. Wu and X. Xu. Well-posedness and inviscid limits of the boussinesq equations with fractional laplacian dissipation. *Nonlinearity*, 27:2215–2232, 2014.

[52] Z. Xin and T. Yanagisawa. Zero-viscosity limit of the linearized Navier-Stokes equations for a compressible viscous fluid in the half-plane. *Comm. Pure Appl. Math.*, 52(4):479–541, 1999.

[53] L. Yao, T. Zhang, and C. Zhu. Boundary layers for compressible Navier-Stokes equations with density-dependent viscosity and cylindrical symmetry. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(5):677–709, 2011.

[54] M. Zhang and C. Zhu. Global existence of solutions to a hyperbolic-parabolic system. *Proceedings of the American Mathematical Society*, 135:1017–1027, 2007.