TOEPLITZ AND HANKEL DETERMINANTS WITH SINGULARITIES:
ANNOUNCEMENT OF RESULTS

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Abstract. We obtain asymptotics for Toeplitz, Hankel, and Toeplitz+Hankel determinants whose symbols possess Fisher-Hartwig singularities. Details of the proofs will be presented in another publication.

Let $f(z)$ be a complex-valued function integrable over the unit circle with Fourier coefficients

$$f_j = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} d\theta, \quad j = 0, \pm 1, \pm 2, \ldots$$

We are interested in the $n$-dimensional Toeplitz determinant with symbol $f(z)$,

$$D_n(f(z)) = \det(f_j-k)_{j,k=0}^{n-1}.$$  

In this paper we present the asymptotics of $D_n(f(z))$ as $n \to \infty$ in the case when the symbol $f(e^{i\theta})$ has a fixed number of Fisher-Hartwig singularities [17, 25], i.e., when it has the following form on the unit circle:

$$f(z) = e^{V(e^{i\theta})} z^{\sum_{j=0}^m \beta_j} \prod_{j=0}^m |z - z_j|^{2\alpha_j} g_{z_j,\beta_j}(z) z_j^{-\beta_j}, \quad z = e^{i\theta}, \quad \theta \in [0, 2\pi),$$

for some $m = 0, 1, \ldots$, where

$$z_j = e^{i\theta_j}, \quad j = 0, \ldots, m, \quad 0 = \theta_0 < \theta_1 < \cdots < \theta_m < 2\pi;$$

$$g_{z_j,\beta_j}(z) \equiv g_{\beta_j}(z) = \begin{cases} e^{i\pi\beta_j} & 0 \leq \arg z < \theta_j \\ e^{-i\pi\beta_j} & \theta_j \leq \arg z < 2\pi \end{cases},$$

$$\Re \alpha_j > -1/2, \quad \beta_j \in \mathbb{C}, \quad j = 0, \ldots, m,$$

and $V(e^{i\theta})$ is a sufficiently smooth function on the unit circle. Here the condition on $\alpha$ insures integrability. Note that a single Fisher-Hartwig singularity at $z_j$ consists of a root-type singularity

$$|z - z_j|^{2\alpha_j} = \left| 2\sin \frac{\theta - \theta_j}{2} \right|^{2\alpha_j}$$

and a jump $g_{\beta_j}(z)$. A point $z_j$, $j = 1, \ldots, m$ is included in (3) if and only if either $\alpha_j \neq 0$ or $\beta_j \neq 0$ (or both); in contrast, the point $z_0 = 1$ is always included (note that $g_{\beta_0}(z) = e^{-i\pi\beta_0}$). Observe that for each $j$, $z_j^{\beta_j} g_{\beta_j}(z)$ is continuous at $z = 1$, and so for each $j$ each “beta” singularity produces a jump only at the point $z_j$. The factors $z_j^{-\beta_j}$ are singled out to simplify comparisons with existing literature. Indeed, (2) with the notation $b(\theta) = e^{V(e^{i\theta})}$ is exactly the symbol considered in [17, 2, 3, 4, 5, 6, 7, 10, 11, 12, 15, 16, 30]. We write the symbol,
however, in a form with \( z^{\sum_{j=0}^{m} \beta_j} \) factored out. The present way of writing is more natural for our analysis.

On the unit circle \( V(z) \) is represented by its Fourier expansion:

\[
V(z) = \sum_{k=-\infty}^{\infty} V_k z^k,
V_k = \frac{1}{2\pi} \int_{0}^{2\pi} V(e^{i\theta}) e^{-ki\theta} d\theta.
\]

The canonical Wiener-Hopf factorization of \( e^V(z) \) is

\[
e^V(z) = b_+(z)e^{V_0}b_-(z),
b_+(z) = \epsilon \sum_{k=1}^{\infty} V_k z^k,\quad b_-(z) = \epsilon \sum_{k=-\infty}^{-1} V_k z^k.
\]

First, we address the (essentially known) case when all \( \Re \beta_j \) lie in a single half-closed interval of length 1, namely \( \Re \beta_j \in (q - 1/2, q + 1/2], q \in \mathbb{R} \), reproving results of Szegő for \( \alpha_j = \beta_j = 0 \), Widom [30] for \( \beta_j = 0 \), Basor [2] for \( \Re \beta_j = 0 \), Böttcher and Silbermann [10] for \( |\Re \alpha_j| < 1/2, |\Re \beta_j| < 1/2 \), Ehrhardt [16] for \( |\Re \beta_j - \Re \beta_k| < 1 \), and other results of these authors (see [16] for a review). Note that we write the asymptotics in a form that makes it clear which branch of the roots is to be used.

**Theorem 1.** Let \( f(e^{i\theta}) \) be defined in (2) and \( \alpha_j \pm \beta_j \not= -1, -2, \ldots \) for \( j = 0, 1, \ldots \). Then as \( n \to \infty \),

\[
D_n(f) = \exp \left[ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right] \prod_{j=0}^{m} b_+(z_j)^{-\alpha_j + \beta_j} b_-(z_j)^{-\alpha_j - \beta_j}
\times \prod_{0 \leq j < k \leq m} |z_j - z_k|^{2(\beta_j \beta_k - \alpha_j \alpha_k)} \left( \frac{z_k}{z_j e^{i\pi}} \right)^{\alpha_j \beta_k - \alpha_k \beta_j}
\times \prod_{j=0}^{m} \frac{G(1 + \alpha_j + \beta_j)G(1 + \alpha_j - \beta_j)}{G(1 + 2\alpha_j)} (1 + o(1)),
\]

if \( \Re \alpha_j > -\frac{1}{2}, \quad |\Re \beta_j - \Re \beta_k| < 1, \quad j, k = 0, 1, \ldots, m, \)

where \( G(x) \) is Barnes’ \( G \)-function.

**Remark 2.** In the case of a single singularity, i.e., when \( m = 0 \) or \( m = 1 \), \( \alpha_0 = \beta_0 = 0 \), the theorem implies that the asymptotics (9) hold for

\[
\Re \alpha_m > -\frac{1}{2}, \quad \beta_m \in \mathbb{C}, \quad \alpha_m \pm \beta_m \not= -1, -2, \ldots
\]

In fact, if there is only one singularity and \( V \equiv 0 \), an explicit formula is known [10] for \( D_n(f) \) in terms of the \( G \)-functions.

**Remark 3.** Assume that the function \( V(z) \) is analytic. Then the following can be said about the remainder term. If all \( \beta_j = 0 \), the error term \( o(1) = O(n^{-1} \ln n) \). If there is only one singularity the error term is also \( O(n^{-1} \ln n) \). In the general case, the error term depends on the differences \( \beta_j - \beta_k \). For analytic \( V(z) \), our methods would allow us to calculate several asymptotic terms rather than just the main one presented in (9) (and also in (22) below).

**Remark 4.** If all \( \Re \beta_j \in (-1/2, 1/2] \) or all \( \Re \beta_j \in [-1/2, 1/2) \), the conditions \( \alpha_j \pm \beta_j \not= -1, -2, \ldots \) are satisfied automatically as \( \Re \alpha_j > -1/2 \).
Remark 5. Since $G(-k) = 0$, $k = 0, 1, \ldots$, the formula (9) no longer represents the leading asymptotics if $\alpha_j + \beta_j$ or $\alpha_j - \beta_j$ is a negative integer for some $j$. A similar situation arises in Theorem 9 below if some representations in $M$ are degenerate. We do not address this case in the paper.

As mentioned above, Theorem 1 was proved by Ehrhardt. We give an independent proof of this result using a connection of $D_n(f)$ with the system of polynomials orthogonal with weight $f(z)$ (2) on the unit circle. First, we can show that all $D_k(f) \neq 0$, $k = k_0, k_0 + 1, \ldots$, for some sufficiently large $k_0$. Then the polynomials $\phi_k(z) = \chi_k z^k + \cdots$, $\hat{\phi}_k(z) = \chi_k z^k + \cdots$ of degree $k$, $k = k_0, k_0 + 1, \ldots$, satisfying

$$
\frac{1}{2\pi} \int_0^{2\pi} \phi_k(z)z^{-j} f(z) d\theta = \chi_k^{-1} \delta_{jk}, \quad \frac{1}{2\pi} \int_0^{2\pi} \hat{\phi}_k(z^{-1}) z^j f(z) d\theta = \chi_k^{-1} \delta_{jk}, \quad z = e^{i\theta}, \quad j = 0, 1, \ldots, k,
$$

exist and are given by the following expressions:

$$
\phi_k(z) = \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix}
 f_{00} & f_{01} & \cdots & f_{0k} \\
 f_{10} & f_{11} & \cdots & f_{1k} \\
 \vdots & \vdots & \ddots & \vdots \\
 f_{k-10} & f_{k-11} & \cdots & f_{k-1k} \\
 1 & z & \cdots & z^k
\end{vmatrix},
$$

$$
\hat{\phi}_k(z^{-1}) = \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix}
 f_{00} & f_{01} & \cdots & f_{0k-1} & 1 \\
 f_{10} & f_{11} & \cdots & f_{1k-1} & z^{-1} \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 f_{k0} & f_{k1} & \cdots & f_{kk-1} & z^{-k} \\
 f_{kk} & \cdots & \cdots & \cdots & \cdots
\end{vmatrix},
$$

where

$$
f_{st} = \frac{1}{2\pi} \int_0^{2\pi} f(z)z^{-(s-t)} d\theta, \quad s, t = 0, 1, \ldots, k.
$$

We obviously have

$$
\chi_k = \sqrt{\frac{D_k}{D_{k+1}}},
$$

These polynomials satisfy a Riemann-Hilbert problem. We solve the problem asymptotically for large $n$ in case of the weight given by (2) with analytic $V(z)$, thus obtaining the large $n$ asymptotics of the orthogonal polynomials. The main new feature of the solution is a construction of the local parametrix at the points $z_j$ of Fisher-Hartwig singularities. This parametrix is given in terms of the confluent hypergeometric function. A study of the asymptotic behaviour of the polynomials orthogonal on the unit circle was initiated by Szegő [28]. Riemann-Hilbert methods developed within the last 20 years allow us to find asymptotics of orthogonal polynomials in all regions of the complex plane (see [13] and many subsequent works by many authors). Such an analysis of the polynomials with an analytic weight on the unit circle was carried out in [26], and for the case of a weight with $\alpha_j$-singularities but without jumps, in [27]. We provide, therefore, a generalization of these
results. Here we present only the following statement we will need below for the analysis of determinants.

**Theorem 6.** Let \( f(e^{i\theta}) \) be defined in (2), \( V(z) \) be analytic in a neighborhood of the unit circle, and \( \phi_k(z) = \chi_k z^k + \cdots, \hat{\phi}_k(z) = \chi_k z^k + \cdots \) be the corresponding polynomials satisfying (11). Assume that \( |\Re \beta_j - \Re \beta_k| < 1, \alpha_j \pm \beta_j \neq -1, -2, \ldots, j, k = 0, 1, \ldots, m. \) Let

\[
\delta = \max_{j,k} n^{2\Re(\beta_j - \beta_k - 1)}.
\]

Then as \( n \to \infty, \)

\[
\chi_n^2 = \exp \left[ - \int_0^{2\pi} V(e^{i\theta}) \frac{d\theta}{2\pi} \right] \left( 1 - \frac{1}{n} \sum_{k=0}^m \left( \alpha_k^2 - \beta_k^2 \right) \right) + O(\delta^2) + O(\delta/n),
\]

where

\[
\nu_j = \exp \left\{ -i\pi \left( \sum_{p=0}^{j-1} \alpha_p - \sum_{p=j+1}^m \alpha_p \right) \right\} \prod_{p \neq j} \left( \frac{z_{j}}{z_{p}} \right)^{\alpha_p} \left| z_{j} - z_{p} \right|^{2\beta_p}.
\]

Under the same conditions,

\[
\phi_n(0) = \chi_n \left( \sum_{j=0}^m n^{-2\beta_j} z_j n^{\nu_j} \frac{\Gamma(1 + \alpha_j + \beta_j) b_+(z_j)}{\Gamma(\alpha_j - \beta_j) b_-(z_j)} + O \left( \left[ \delta + 1 \right] \max_{k} n^{-2\Re \beta_k} \right) \right),
\]

\[
\hat{\phi}_n(0) = \chi_n \left( \sum_{j=0}^m n^{2\beta_j} z_j^{-n} n^{1-\nu_j} \frac{\Gamma(1 + \alpha_j - \beta_j) b_-(z_j)}{\Gamma(\alpha_j + \beta_j) b_+(z_j)} + O \left( \left[ \delta + 1 \right] \max_{k} n^{2\Re \beta_k} \right) \right).
\]

**Remark 7.** The error terms here are uniform and differentiable in all \( \alpha_j, \beta_j \) for \( \beta_j \) in compact subsets of the strip \( |\Re \beta_j - \Re \beta_k| < 1, \) for \( \alpha_j \) in compact subsets of the half-plane \( \Re \alpha_j > -1/2, \) and outside a neighborhood of the sets \( \alpha_j \pm \beta_j = -1, -2, \ldots. \) If \( \alpha_j + \beta_j = 0 \) or \( \alpha_j - \beta_j = 0 \) for some \( j, \) the corresponding terms in the above formulas vanish.

**Remark 8.** Note that the terms with \( n^{2(\beta_j - \beta_k)} \) in (16) become larger in absolute value that the \( 1/n \) term for \( \max_{j,k} \Re(\beta_j - \beta_k) > 1/2. \)

Our proof of Theorem 1 uses Theorem 6, similar results for the asymptotics of the orthogonal polynomials and their Cauchy transforms at the points \( z_j, \) and a set of differential identities for the logarithm of \( D_n, \) in the spirit of [14, 20, 23].

Our next task is to extend the result for arbitrary \( \beta_j \in \mathbb{C}, \) i.e. for the case when not all \( \Re \beta_j \)’s lie in a single interval of length less than 1. We know from examples (see, e.g. [10, 8, 16]) that in general, the formula (1) breaks down. Obviously, the general case can be reduced to \( \Re \beta_j \in (q - 1/2, q + 1/2) \) by adding integers to \( \beta_j. \) Then, apart from a constant
factor, the only change in \( f(z) \) is multiplication with \( z^\ell, \ell \in \mathbb{Z} \). However, as can be shown, the determinants \( D_n(f(z)) \) and \( D_n(z^\ell f(z)) \) are simply related. For example, for \( \ell = 1, 2, \ldots, \)

\[
D_n(z^\ell f(z)) = \frac{(-1)^\ell n}{\prod_{j=1}^{\ell-1} j!} D_n(f(z)),
\]

where

\[
F_n = \begin{vmatrix}
\Phi_n(0) & \Phi_{n+1}(0) & \cdots & \Phi_{n+\ell-1}(0) \\
\frac{d}{dz} \Phi_n(0) & \frac{d}{dz} \Phi_{n+1}(0) & \cdots & \frac{d}{dz} \Phi_{n+\ell-1}(0) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{d^{\ell-1}}{dz^{\ell-1}} \Phi_n(0) & \frac{d^{\ell-1}}{dz^{\ell-1}} \Phi_{n+1}(0) & \cdots & \frac{d^{\ell-1}}{dz^{\ell-1}} \Phi_{n+\ell-1}(0)
\end{vmatrix},
\]

and \( \Phi_k(z) = \phi_k(z)/\chi_k \). Since the \( \chi_k, \phi_k(0), \hat{\phi}_k(0) \) for large \( k \) are given by Theorem 6, and expressions for the derivatives can be found similarly, it is easy to obtain the general asymptotic formula for \( D_n \). However, this formula is implicit in the sense that one still needs to separate the main asymptotic term from the others: e.g., if the dimension \( \ell \) of \( F_{ij} \) is larger than the number of leading-order terms in (18), the obvious candidate for the leading order in \( F_n \) vanishes (this is not the case in the simplest situation given by Theorem 14). We outline below how we resolve this problem.

Following [8, 16], define a so-called representation of a symbol. Namely, for \( f(z) \) given by (2) replace \( \beta_j \) by \( \beta_j + n_j, n_j \in \mathbb{Z} \) if \( z_j \) is a singularity (i.e., if either \( \beta_j = 0 \) or \( \alpha_j \neq 0 \) or both: we set \( n_0 = 0 \) if \( z_0 = 1 \) is not a singularity). The integers \( n_j \) are arbitrary subject to the condition \( \sum_{j=0}^{m} n_j = 0 \). In a slightly different notation from [8, 16], we call the resulting function \( f(z; n_0, \ldots, n_m) \) a representation of \( f(z) \). (The original \( f(z) \) is also a representation corresponding to \( n_0 = \cdots = n_m = 0 \).) Obviously, all representations of \( f(z) \) differ only by multiplicative constants. We have

\[
f(z) = \prod_{j=0}^{m} z_j^{n_j} \times f(z; n_0, \ldots, n_m).
\]

We are interested in the representations (characterized by \( (n_j)_{j=0}^{m} \)) of \( f \) such that \( \sum_{j=0}^{m} (\Re \beta_j + n_j)^2 \) is minimal. There is a finite number of such representations and we provide an algorithm for finding them explicitly (see Remark 11). We call the set of them \( \mathcal{M} \). Furthermore, we call a representation degenerate if \( \alpha_j + (\beta_j + n_j) \) or \( \alpha_j - (\beta_j + n_j) \) is a negative integer for some \( j \). We call \( \mathcal{M} \) non-degenerate if it contains no degenerate representations. We prove

**Theorem 9.** Let \( f(z) \) be given in (2), \( \Re \alpha_j > -1/2, \beta_j \in \mathbb{C}, j = 0, 1, \ldots, m \). Let \( \mathcal{M} \) be non-degenerate. Then, as \( n \to \infty \),

\[
D_n(f) = \sum \left( \prod_{j=0}^{m} z_j^{n_j} \right)^n \Re(f(z; n_0, \ldots, n_m))(1 + o(1)),
\]

where the sum is over all representations in \( \mathcal{M} \). Each \( \Re(f(z; n_0, \ldots, n_m)) \) stands for the right-hand side of the formula (9), without the error term, corresponding to \( f(z; n_0, \ldots, n_m) \).

**Remark 10.** This theorem was conjectured by Basor and Tracy [8]. The case when the representation minimizing \( \sum_{j=0}^{m} (\Re \beta_j + n_j)^2 \) is unique, i.e. there is only one term in the sum (22), was proved by Ehrhardt [16]. Note that this case happens if and only if there
exist such \( n_j \) that \( \Re \beta_j + n_j \) belong to a half-open interval of length 1 for all \( j = 0, \ldots, m \): see next Remark. Thus, Theorem 9 in this case follows from Theorem 1 applied to this representation.

**Remark 11.** The set \( \mathcal{M} \) can be characterized as follows. Suppose that the seminorm \( \| \beta \| \equiv \max_{j,k} | \Re \beta_j - \Re \beta_k | > 1 \). Then, writing \( \beta_s^{(i)} = \beta_s + 1, \beta_t^{(i)} = \beta_t - 1, \) and \( \beta_j^{(i)} = \beta_j \) if \( j \neq s, t \), where \( \beta_s \) is one of the beta-parameters with \( \Re \beta_s = \min_j \Re \beta_j \), \( \beta_t \) is one of the beta-parameters with \( \Re \beta_t = \max_j \Re \beta_j \), we see that \( \| \beta^{(i)} \| \leq \| \beta \| \), and \( f \) corresponding to \( \beta^{(i)} \) is a representation. After a finite number, say \( r \), of such transformations we reduce an arbitrary set of \( \beta_j \) to the situation for which either \( \| \beta^{(r)} \| < 1 \) or \( \| \beta^{(r)} \| = 1 \). Note that further transformations do not change the seminorm in the second case, while in the first case the seminorm oscillates periodically taking 2 values, \( \| \beta^{(r)} \| \) and \( 2 - \| \beta^{(r)} \| \). Thus all the symbols of type (2) belong to 2 distinct classes: the first, for which \( \| \beta^{(r)} \| < 1 \), and the second, for which \( \| \beta^{(r)} \| = 1 \). For symbols of the first class, \( \mathcal{M} \) has only one member with beta-parameters \( \beta^{(r)} \). Indeed, writing \( b_j = \Re \beta_j \), if \( -1/2 < b_j^{(r)} - q \leq 1/2 \) for some \( q \in \mathbb{R} \) and all \( j \), then for any \( (k_j)_{j=0}^m \) such that \( \sum_{j=0}^m k_j = 0 \) and not all \( k_j \) are zero, we have

\[
\sum_{j=0}^m (b_j^{(r)} + k_j)^2 = \sum_{j=0}^m (b_j^{(r)})^2 + 2 \sum_{j=0}^m (b_j^{(r)} - q)k_j + \sum_{j=0}^m k_j^2 > \sum_{j=0}^m (b_j^{(r)})^2 + \sum_{j=0}^m k_j^2 - |k_j| \geq \sum_{j=0}^m (b_j^{(r)})^2,
\]

where the first inequality is strict as at least one \( k_j > 0 \). For symbols of the second class, we can find \( q \in \mathbb{R} \) such that \( -1/2 \leq b_j^{(r)} - q \leq 1/2 \) for all \( j \). Equation (23) in this case holds with “\( > \)” sign replaced by “\( \geq \)”. Clearly, there are several representations in \( \mathcal{M} \) in this case (they correspond to the equalities in (23)) and adding 1 to one of \( \beta_s^{(r)} \) with \( b_s^{(r)} = \min_j b_j^{(r)} = q - 1/2 \) while subtracting 1 from one of \( \beta_t^{(r)} \) with \( b_t^{(r)} = \max_j b_j^{(r)} = q + 1/2 \) provides the way to find all of them.

A simple explicit sufficient, but obviously not necessary, condition for \( \mathcal{M} \) to have only one member is that all \( \Re \beta_j \mod 1 \) be different.

**Remark 12.** The situation when all \( \alpha_j \pm \beta_j \) are nonnegative integers, which was considered by Böttcher and Silbermann in [11], is a particular case of the above theorem.

**Remark 13.** The case when all the representations of \( f \) are degenerate (not only those in \( \mathcal{M} \)) was considered by Ehrhardt [16] who found that in this case \( D_n(f) = O(e^{nO(n^r)}) \), where \( r \) is any real number. We can reproduce this result by our methods but do not present it here.

We prove Theorem 9 in the following way. Consider the set \( \beta_j^{(r)} \) constructed in Remark 11. We have to consider only the second class, i.e. \( \| \beta^{(r)} \| = 1 \). We then have, relabelling \( \beta_j^{(r)} \) according to increasing real part,

\[
\Re \beta_j^{(r)} = \cdots = \Re \beta_p^{(r)} < \Re \beta_{p+1}^{(r)} \leq \cdots \leq \Re \beta_{m-\ell}^{(r)} < \Re \beta_m^{(r)} = \cdots = \Re \beta_m^{(r)},
\]

for some \( p, \ell > 0 \). Here \( m' = m + 1 \) if \( \ell = 1 \) is a singularity, otherwise \( m' = m \). Now consider the symbol (not a representation of \( f \)) \( \tilde{f} \) of type (2) with beta-parameters denoted by \( \tilde{\beta} \) and given by \( \tilde{\beta}_j = \beta_j^{(r)} \) for \( j = 1, \ldots, m' - \ell \), and \( \tilde{\beta}_j = \beta_j^{(r)} - 1 \) for \( j = m' - \ell + 1, \ldots, m' \). It is easy
to see that the original symbol $f$ has $\binom{\ell}{\ell+p}$ representations in $\mathcal{M}$ obtained by shifting any $\ell$ out of $\ell+p$ parameters $\tilde{\beta}_j$ with the smallest real part to the right by 1. Thus, $f(z) = cz^\ell \tilde{f}(z)$, where $c$ is a simple constant factor. To find the asymptotics of $D_n(f)$, we now use (20) with $f$ replaced by $\tilde{f}$. The l.h.s. is then $c^{-n} D_n(f)$. In the r.h.s. we have a factor $D_n(\tilde{f})$ to which Theorem 1 is applicable since $\|\tilde{\beta}\| < 1$, and an $\ell \times \ell$ determinant $F_n$ involving the polynomials orthogonal w.r.t. $\tilde{f}$ (for simplicity, consider $V(z)$ analytic in a neighborhood of the unit circle). It is a crucial fact that the size $\ell$ of this determinant is less than the number of terms, $\ell + p$, in the expansion of $\phi_n(0)/\chi_n$ of the same largest order $O(n^{-2\beta_1-1})$ (see (18) with $\beta_j$ replaced by $\tilde{\beta}_j$). This fact enables us to extract the leading asymptotic contribution to $F_n$ (resolving the problem mentioned above). The asymptotics of $F_n$ and $D_n(\tilde{f})$ combine together and produce (22).

We will now discuss a simple particular case of Theorem 9 and present a direct independent proof in this case.

**Theorem 14** (A particular case of Theorem 9). Let the symbol $f^\pm(z)$ be obtained from $f(z)$ (2) by replacing one $\beta_{j_0}$ with $\beta_{j_0} \pm 1$ for some fixed $0 \leq j_0 \leq m$. Let $\Re \alpha_j > -\frac{1}{2}$, $\Re \beta_j \in (-1/2, 1/2]$, $j = 0, 1, \ldots, m$. Then

$$D_n(f^+(z)) = z_{j_0}^{-n} \frac{\phi_n(0)}{\chi_n} D_n(f(z)),$$

$$D_n(f^-(z)) = z_{j_0}^n \frac{\phi_n(0)}{\chi_n} D_n(f(z)).$$

These formulas together with (18,19,16,9) yield the following asymptotic description of $D_n(f^\pm)$. Let there be more than one singular points $z_j$ and all $\alpha_j \pm \beta_j \neq 0$. For $f^+(z)$, let $\beta_{p_0}$, $p = 1, \ldots, s$ be such that they have the same real part which is strictly less than the real parts of all the other $\beta_j$, i.e. $\Re \beta_{j_1} = \cdots = \Re \beta_{j_s} < \min_{j \neq j_1, \ldots, j_s} \Re \beta_j$. For $f^-(z)$ let one $\beta_{p_0}$, $p = 1, \ldots, s$ be such that $\Re \beta_{j_1} = \cdots = \Re \beta_{j_s} > \max_{j \neq j_1, \ldots, j_s} \Re \beta_j$. Then the asymptotics of $D_n(f^\pm)$ are given by:

$$D_n(f^+) = z_{j_0}^{-n} \sum_{p=1}^s z_{j_p}^{\beta_{p_0} - 1} R_{j_p, +} (1 + o(1)),$$
$$D_n(f^-) = z_{j_0}^n \sum_{p=1}^s z_{j_p}^{-\beta_{p_0} - 1} R_{j_p, -} (1 + o(1)),$$

where $R_{j_p, \pm}$ is the right-hand side of (9) (without the error term) in which $\beta_j$ is replaced by $\beta_j \pm 1$, respectively.

**Proof.** For simplicity, we present the proof only for $V(z)$ analytic in a neighborhood of the unit circle. Consider the case of $f^-(z)$. It corresponds to one of the $\beta_j$ shifted inside the interval $(-3/2, -1/2]$. Since

$$z^{\sum_{j=0}^m \beta_j - 1} = z^{-1} z^{\sum_{j=0}^m \beta_j}, \quad g_{\beta_{j_0} - 1}(z) = -g_{\beta_{j_0}}(z), \quad z_{j_0}^{-\beta_{j_0} + 1} = z_{j_0} z_{j_0}^{-\beta_{j_0}},$$

we see that

$$f^-(z) = -z_{j_0} z^{-1} f(z).$$

Therefore, using the identity (an analogue of (20) for $\ell = -1$)

$$D_n(z^{-1} f(z)) = (-1)^n \frac{\phi_n(0)}{\chi_n} D_n(f),$$

we obtain

$$D_n(f^-(z)) = z_{j_0}^n \frac{\phi_n(0)}{\chi_n} D_n(z^{-1} f(z)) = z_{j_0}^n \frac{\phi_n(0)}{\chi_n} D_n(z_{j_0} z^{-1} f(z)) = z_{j_0}^n \frac{\phi_n(0)}{\chi_n} D_n(f^-).$$

This completes the proof.
we obtain

\[ D_n(f^-(z)) = (-z_{j_0})^n D_n(z^{-1} f(z)) = z_{j_0}^{n/\chi_n} D_n(f(z)). \]

If, for some \( j_1, j_2, \ldots, j_s \), we have that \( \Re \beta_{j_1} = \cdots = \Re \beta_{j_s} > \max_{j \neq j_1, \ldots, j_s} \Re \beta_j \), then we see from (19) that only the addends with \( n^{2\beta_{j_1} - 1}, \ldots, n^{2\beta_{j_s} - 1} \) give contributions to the main asymptotic term of \( D_n(f^-(z)) \). Using the relation \( G(1 + x) = \Gamma(x)G(x) \), we obtain the formula (25) for \( D_n(f^-(z)) \). The case of \( f^+(z) \) is similar.

**Example 15.** In [8] Basor and Tracy noticed a simple example of a symbol of type (2) for which the asymptotics of the determinant can be computed directly, but are very different from (9). Up to a constant, the symbol is

\[ \tilde{f}(e^{i\theta}) = \begin{cases} -i, & 0 < \theta < \pi \\ i, & \pi < \theta < 2\pi \end{cases}. \]

We can represent \( \tilde{f} \) as a symbol with \( \beta \)-singularities \( \beta_0 = 1/2, \beta_1 = -1/2 \) at the points \( z_0 = 1 \) and \( z_1 = -1 \), respectively:

\[ \tilde{f}(z) = g_{1,1/2}(z)g_{-1,-1/2}(z)e^{i\pi/2} \]

We see that \( \tilde{f}(z) = f^-(z) \) and \( j_0 = 1 \). Therefore by the first part of Theorem 14, we have

\[ D_n(\tilde{f}(z)) = (-1)^n \frac{\phi_n(0)}{\chi_n} D_n(f(z)), \]

where \( \phi_n(z) \), \( \chi_n \), \( D_n(f(z)) \) correspond to \( f(z) \) given by (2) with \( m = 1 \), \( z_0 = 1 \), \( z_1 = e^{i\pi} \), \( \beta_0 = \beta_1 = 1/2 \), \( \alpha_0 = \alpha_1 = 0 \).

Substituting (19,9) into the above equation for \( D_n(\tilde{f}(z)) \), we obtain

\[ D_n(\tilde{f}(z)) = \frac{1 + (-1)^n}{2} \sqrt{\frac{2}{n}} G(1/2)^2 G(3/2)^2 (1 + o(1)), \]

which is the answer found in [8].

Alternatively, noting that \( s = 2, j_1 = j_0 = 1 \) and \( j_2 = 0 \) and using (25) we obtain

\[ D_n(\tilde{f}(z)) = (-1)^n ((-1)^n R_{1,-} + R_{0,-}). \]

Since \( R_{1,-} = R_{0,-} = (2n)^{-1/2} G(1/2)^2 G(3/2)^2 (1 + o(1)) \), we obtain the same result.

As noted by Basor and Tracy, \( \tilde{f}(z) \) has a different representation of type (2), namely, with \( \beta_0 = -1/2, \beta_1 = 1/2 \), and we can write

\[ \tilde{f}(z) = -g_{1,-1/2}(z)g_{-1,1/2}(z)e^{-i\pi/2}. \]

This fact was the origin of their conjecture. In the notation of Theorem 9, the symbol (28) has the two representations minimizing \( \sum_{j=0}^1 (\beta_j + n_j)^2 \), one with \( n_0 = n_1 = 0 \) and the other with \( n_0 = -1, n_1 = 1 \).

Note that in the case \( \sum_{j=0}^m \beta_j = 0 \) we can always assume that \( \Re \beta_j \in [-1/2, 1/2] \). The beta-singularities then are just piece-wise constant (step-like) functions. This case is relevant for our next result, which is on Hankel determinants.
Let \( w(x) \) be an integrable complex-valued function on the interval \([-1, 1]\). Then the Hankel determinant with symbol \( w(x) \) is

\[
D_n(w(x)) = \det \left( \int_{-1}^{1} x^{j+k} w(x) \, dx \right)_{j,k=0}^{n-1}.
\]

Define \( w(x) \) for a fixed \( r = 0, 1, \ldots \) as follows:

\[
w(x) = e^{U(x)} \prod_{j=0}^{r+1} |x - \lambda_j|^{2\alpha_j} \omega_j(x)
\]

\[1 = \lambda_0 > \lambda_1 > \cdots > \lambda_{r+1} = -1, \quad \omega_j(x) = \begin{cases} e^{i\pi \beta_j}, & \Re x < \lambda_j, \\ e^{-i\pi \beta_j}, & \Re x > \lambda_j, \end{cases}, \quad \Re \beta_j \in (-1/2, 1/2], \]

\[\beta_0 = \beta_{r+1} = 0, \quad \Re \alpha_j > -\frac{1}{2}, \quad j = 0, 1, \ldots, r+1.
\]

where \( U(x) \) is a sufficiently smooth function on the interval \([-1, 1]\). Note that we set \( \beta_0 = \beta_{r+1} = 0 \) without loss of generality as the functions \( \omega_0, \omega_{r+1} \) are just constants on \((-1, 1)\).

We prove

**Theorem 16.** Let \( w(x) \) be defined in (32). Then as \( n \to \infty \),

\[
D_n(w) = D_n(1)e^{\left[ (n+\alpha_0+\alpha_{r+1})V_0-\alpha_0V(1)-\alpha_{r+1}V(1)+\frac{i}{2} \sum_{j=1}^{r} kV_j^2 \right]}
\]

\[
\times \prod_{j=1}^{r} b_+(z_j)^{-\alpha_j-\beta_j} b_-(z_j)^{-\alpha_j+\beta_j} \times e^{\left[ 2i(n+\alpha) \sum_{j=1}^{r} \beta_j \arcsin \lambda_j + i\pi \sum_{0 \leq j < k \leq r+1} (\alpha_j \beta_j - \alpha_k \beta_j) \right]}
\]

\[
\times 4^{-\left( \alpha_0+\alpha_{r+1}+\sum_{0 \leq j < k \leq r+1} \alpha_j \alpha_k + \sum_{j=1}^{r} \beta_j^2 \right)} (2\pi)^{\alpha_0+\alpha_{r+1}} 2^{(\alpha_0^2+\alpha_{r+1}^2)} \sum_{j=1}^{r} (\alpha_j^2 - \beta_j^2)
\]

\[
\times \prod_{0 \leq j < k \leq r+1} |\lambda_j - \lambda_k|^{-2(\alpha_j \alpha_k + \beta_j \beta_k)} \left| \lambda_j \lambda_k - 1 + \sqrt{(1 - \lambda_j^2)(1 - \lambda_k^2)} \right|^{2\beta_j \beta_k}
\]

\[
\times \frac{1}{G(1+2\alpha_0)G(1+2\alpha_{r+1})} \prod_{j=1}^{r} \left( 1 - \lambda_j^2 \right)^{-(\alpha_j^2 + \beta_j^2)/2} G(1 + \alpha_j + \beta_j) G(1 + \alpha_j - \beta_j) G(1+2\alpha_j) (1+o(1)),
\]

\[A = \sum_{k=0}^{r+1} \alpha_k, \quad \Re \alpha_j > -\frac{1}{2}, \quad \Re \beta_j \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad j = 0, 1, \ldots, r+1, \quad \beta_0 = \beta_{r+1} = 0,
\]

where \( V(e^{i\theta}) = U(\cos \theta), \ z_j = e^{i\theta_j}, \ \lambda_j = \cos \theta_j, \ j = 0, \ldots, r+1, \) and the functions \( b_{\pm}(z) \) are defined in (8).

**Remark 17.** \( D_n(1) \) is an explicitly computable determinant related to the Legendre polynomials (it can also be written as a Selberg integral), c.f. [31],

\[
D_n(1) = 2^{n^2} \prod_{k=0}^{n-1} \frac{k!^3}{(n+k)!} = \frac{\pi^{n+1/2} G(1/2)^2}{2^{n(n-1)n1/4}} (1+o(1)).
\]

**Remark 18.** Since \( \beta_j \) enter the symbol only via \( e^{\pm i\pi \beta_j} \), the theorem describes the general case with the exception of the situation when some \( \Re \beta_j = 1/2 \mod 1. \)
To prove Theorem 16 we use the fact that $w(x)$ can be generated by a particular class of functions $f(z)$ given by (2). Namely, we can find an \textit{even} function $f$ of $\theta$ ($f(e^{i\theta}) = f(e^{-i\theta})$, $\theta \in [0, 2\pi]$) such that

$$w(x) = \frac{f(e^{i\theta})}{|\sin \theta|}, \quad x = \cos \theta, \quad x \in [-1, 1].$$

It turns out that we must have $m = 2r + 1$, $\theta_0 = 0$, $\theta_{r+1} = \pi$, $\theta_{m+1-j} = 2\pi - \theta_j$, $j = 1, \ldots, r$. If we denote the beta-parameters of $f(z)$ by $\tilde{\beta}_j$, we obtain $\tilde{\beta}_0 = \tilde{\beta}_{r+1} = 0$, $\tilde{\beta}_j = -\tilde{\beta}_{m+1-j} = -\beta_j$, $j = 1, \ldots, r$. In particular, $\sum_{j=0}^{m} \tilde{\beta}_j = 0$ as remarked above.

We obtain Theorem 16 from Theorem 1 and asymptotics for the orthogonal polynomials on the unit circle with weight $f(z)$ using the following connection we establish between Hankel and Toeplitz determinants:

$$D_n(w(x))^2 = \frac{\pi^{2n}}{4(n-1)!^2} \frac{(\chi_{2n} + \phi_{2n}(0))^2}{\phi_{2n}(1)\phi_{2n}(-1)} D_{2n}(f(z)), \quad n = 1, 2, \ldots,$$

where $w(x)$ and $f(z)$ are related by (35).

\textbf{Remark 19.} Asymptotics for a subset of symbols (32) which satisfy a symmetry condition and have a certain behaviour at the end-points $\pm 1$ were found by Basor and Ehrhardt in [4]. They use relations between Hankel and Toeplitz determinants which are less general than (36) but do not involve polynomials. For some other related results, see [19, 24].

\textbf{Remark 20.} Asymptotics of a Hankel determinant when some (or all) of $\beta_j$ have the real part $1/2$ can be easily obtained. For the corresponding $f(z)$ this implies that certain $\Re \tilde{\beta}_j = -1/2$ and $\Re \tilde{\beta}_{m+1-j} = 1/2$ and the rest $\Re \tilde{\beta}_k \in (-1/2, 1/2)$. Thus, Theorem 9 can be used to estimate $D_{2n}(f(z))$. For the asymptotics of $\phi_{2n}(z)$ in this case we need an additional “correction” term which is now $O(n^{-2\beta_j-1}) = O(1)$.

\textbf{Remark 21.} One can obtain the asymptotics of the polynomials orthogonal on the interval $[-1, 1]$ with weight (32) by using our results for the polynomials $\phi_k(z)$ orthogonal with the corresponding even weight on the unit circle and a Szegő relation which maps the latter polynomials to the former ones.

Our final task is to present asymptotics for the so-called Toeplitz+Hankel determinants. We consider the four most important ones appearing in the theory of classical groups and its applications to random matrices and statistical mechanics (see, e.g., [1, 18, 22]) defined in terms of the Fourier coefficients of an even $f$ (evenness implies the matrices are symmetric) as follows:

$$\det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1}, \quad \det(f_{j-k} - f_{j+k+2})_{j,k=0}^{n-1}, \quad \det(f_{j-k} \pm f_{j+k+1})_{j,k=0}^{n-1}.$$
There are simple relations [29, 21, 1] between the determinants (37) and Hankel determinants on \([-1, 1]\) with added singularities at the end-points, namely,

\[
\begin{align*}
det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1} &= \frac{2n^2 - 2n + 2}{\pi^n} D_n(f(e^{i\theta(x)})/\sqrt{1 - x^2}), \\
det(f_{j-k} - f_{j+k+2})_{j,k=0}^{n-1} &= \frac{2n^2}{\pi^n} D_n(f(e^{i\theta(x)})\sqrt{1 - x^2}), \\
det(f_{j-k} + f_{j+k+1})_{j,k=0}^{n-1} &= \frac{2n^2 - n}{\pi^n} D_n\left(f(e^{i\theta(x)})\sqrt{\frac{1 + x}{1 - x}}\right), \\
det(f_{j-k} - f_{j+k+1})_{j,k=0}^{n-1} &= \frac{2n^2 - n}{\pi^n} D_n\left(f(e^{i\theta(x)})\sqrt{\frac{1 - x}{1 + x}}\right),
\end{align*}
\]

where \(f\) is even, and \(x = \cos \theta\). It is easily seen that if \(f(z)\) is an (even) function of type (2) then the corresponding symbols of Hankel determinants belong to the class (32). Thus a combination of the above formulas and Theorem 16 gives the following:

**Theorem 22.** Let \(f(z)\) be defined in (2) with the condition \(f(e^{i\theta}) = f(e^{-i\theta})\). Let \(\theta_{r+1} = \pi\). Then as \(n \to \infty\),

\[
D_n^{T+H} = e^{nV_0 + \frac{1}{2} \left[ (a_0 + a_{r+1} + s + t)V_0 - (a_0 + s)V(1) - (a_{r+1} + t)V(-1) + \sum_{k=1}^{\infty} kV_k^2 \right]}
\times \prod_{j=1}^{r} b_+ (z_j)^{-a_j + \beta_j} b_- (z_j)^{-a_j - \beta_j} e^{-\pi \left[ \{a_0 + s + \sum_j \alpha_j \} \sum_j \beta_j + \sum_{1 \leq j < k \leq r} (a_j - k \beta_j, \beta_j) \right]}
\times \sqrt{2(1-s-t)n + p + \sum_{j=1}^{r} (a_j^2 - \beta_j^2)} \left[ -2(a_0 + a_{r+1} + s + t) + \frac{1}{2} (a_0 + a_{r+1} + s + t) n \right] \frac{\sqrt{2(1-s-t)n + p + \sum_{j=1}^{r} (a_j^2 - \beta_j^2)}}{\sqrt{2(1-s-t)n + p + \sum_{j=1}^{r} (a_j^2 - \beta_j^2)}}
\times \prod_{j=1}^{r} \frac{G(1 + \alpha_j + \beta_j)G(1 + \alpha_j - \beta_j)}{G(1 + \alpha_0 + s)G(1 + \alpha_{r+1} + t)} \prod_{j=1}^{r} G(1 + \alpha_j + \beta_j)G(1 + \alpha_j - \beta_j) \left(1 + o(1)\right),
\]

\[\tilde{A} = \frac{1}{2} (a_0 + a_{r+1} + s + t) + \sum_{j=1}^{r} \alpha_j,\]

\[\Re \alpha_j > -\frac{1}{2}, \quad \Re \beta_j \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad j = 0, 1, \ldots, r + 1, \quad \beta_0 = \beta_{r+1} = 0.\]

Here

\[
\begin{align*}
D_n^{T+H} &= \det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1}, \quad \text{with} \quad p = -2n + 2, \quad s = t = -\frac{1}{2}, \\
D_n^{T+H} &= \det(f_{j-k} - f_{j+k+2})_{j,k=0}^{n-1}, \quad \text{with} \quad p = 0, \quad s = t = \frac{1}{2}, \\
D_n^{T+H} &= \det(f_{j-k} \pm f_{j+k+1})_{j,k=0}^{n-1}, \quad \text{with} \quad p = -n, \quad s = \pm \frac{1}{2}, \quad t = \pm \frac{1}{2}.
\end{align*}
\]
Remark 23. For the case $\Re \beta_j = 1/2$ see Remark 20 above.

Remark 24. For the determinant $\det((f_{j-k} + f_{j+k+1})_{j,k=0}^{n-1})$ in the case when the symbol has no $\alpha$ singularities at $z = \pm 1$ and $|\Re \beta_j| < 1/2$, the asymptotics were obtained in [5] (see also [6] if $f$ is non-even, $\alpha_j = 0$). Note that for symbols without singularities, i.e. for $f(z) = e^{V(z)}$, the asymptotics of all the above Toeplitz+Hankel determinants (and related more general ones) were found recently in [7].

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