Asymptotic formulas for determinants of a sum of finite Toeplitz and Hankel matrices

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Abstract

The purpose of this paper is to describe asymptotic formulas for determinants of a sum of finite Toeplitz and Hankel matrices with singular generating functions. The formulas are similar to those of the analogous problem for finite Toeplitz matrices for a certain class of symbols. However, the appearance of the Hankel matrices changes the nature of the asymptotics in some instances depending on the location of the singularities. Several concrete examples are also described in the paper.

1 Introduction

In the theory of random matrices [10], one is led naturally to consider the asymptotics of determinants of Fredholm operators of the form $I + W + H$ where $W$ is a finite Wiener–Hopf operator and $H$ is a finite Hankel operator. This problem arises when investigating the probability distribution function of a random variable thought of as a function of the eigenvalues of a positive random Hermitian matrix. The random matrix connections show that the constant term in the asymptotic expansion of determinants is fundamentally connected to the mean and variance of the distribution function. We will not describe the random

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matrix connections any further, but simply refer the reader to [10] for general information and also [1, 4, 6, 8] for more specific tie-ins to the random variable problem.

The focus of this paper is to study the discrete analogue of this problem. This is not exactly the desired situation for those interested in random matrix theory. However, it is a natural starting place for cases where the random variable is discontinuous, since then the discrete nature of the computations make things a bit more accessible and the mathematical questions that arise are quite interesting in themselves.

The discrete analogue of the Fredholm determinant problem is precisely to find an asymptotic expansion of the determinants of the following Toeplitz + Hankel matrices

\[ M_n(\phi) = T_n(\phi) + H_n(\phi). \]

Here the \( n \times n \) Toeplitz and Hankel matrices are defined as usual by

\[ T_n(\phi) = (\phi_{j-k})^{n-1}_{j,k=0}, \quad H_n(\phi) = (\phi_{j+k+1})^{n-1}_{j,k=0}. \]

The entries \( \phi_k \) are given by the \( k \)-th Fourier coefficient of \( \phi \) where \( \phi \) is a sufficiently well-behaved function on the circle. If \( \phi \) is continuous, even, and sufficiently smooth, then the continuous analogue of the problem (i.e. the Toeplitz + Hankel matrices are replaced by finite Wiener–Hopf + Hankel operators) is solved in [4]. There it is shown that the asymptotics are very similar to the ones given in the Szegö–Kac–Widom Strong Limit Theorem. Indeed, it is only in the constant, or third order term that the answers differ. This is no surprise since if \( \phi \) is continuous, then the Toeplitz operator is perturbed by a compact Hankel operator. However, if the symbol \( \phi \) is singular, then the problem, as in the Toeplitz case, is not easy to solve.

The purpose of this paper is to compute the asymptotics of \( \det M_n(\phi) \) as \( n \to \infty \) for certain piecewise continuous functions \( \phi \). The main general result that we will obtain is as follows. We consider piecewise continuous functions of the form

\[ \phi(e^{i\theta}) = b(e^{i\theta})t_{\beta_+}(e^{i\theta})t_{\beta_-}(e^{i(\theta - \pi)}) \prod_{r=1}^{R} t_{\beta_r}(e^{i(\theta - \theta_r)}), \]

where

\[ t_{\beta}(e^{i\theta}) = \exp(i\beta(\theta - \pi)), \quad 0 < \theta < 2\pi, \]

and \( b \) is a smooth nonvanishing function defined on the circle with winding number zero. We also need conditions on the parameters \( \beta_+, \beta_-, \beta_1, \ldots, \beta_R \). These conditions on the “size” of the jumps and the precise smoothness conditions on \( b \) will be described later on. In addition, we have to assume that \( \theta_1, \ldots, \theta_R \in (-\pi, 0) \cup (0, \pi) \) are certain distinct numbers satisfying \( \theta_r + \theta_s \neq 0 \) for each \( r \) and \( s \). The latter condition excludes piecewise continuous functions
with jumps at both a point on the unit circle and its complex conjugate. However, the function \( \phi \) may have jumps at the points 1 and \(-1\). Our conditions on \( b \) guarantee that the following functions

\[
b_+(t) = \exp \left( \sum_{k=1}^{\infty} t^k [\log b]_k \right), \quad t \in \mathbb{T},
\]

\[
b_-(t) = \exp \left( \sum_{k=1}^{\infty} t^{-k} [\log b]_{-k} \right), \quad t \in \mathbb{T},
\]

are well defined and smooth. Here \([\log b]_k\) denotes the \(k\)-th Fourier coefficient of the logarithm of \( b \). Then the asymptotic formula reads

\[
\det M_n(\phi) \sim G[b]^n n^{\Omega_M} E_M
\]

as \( n \to \infty \), where

\[
G[b] = \exp[\log b]_0,
\]

\[
\Omega_M = -\frac{3\beta^2_+}{2} - \frac{\beta_+}{2} - \frac{3\beta^2_-}{2} + \frac{\beta_-}{2} - \sum_{r=1}^{R} \beta_r^2,
\]

\[
E_M = E[b] F[b]
\]

\[
\times G(1 + \beta_+) G(1 - \beta_+) G(1/2 - \beta_+) G(1/2)^{-1}(2\pi)^{\beta_+}/2 2^{3\beta_+^2}/2
\]

\[
\times G(1 + \beta_-) G(1 - \beta_-) G(3/2 - \beta_-) G(3/2)^{-1}(2\pi)^{\beta_-}/2 2^{3\beta_-^2}/2
\]

\[
\times \prod_{r=1}^{R} G(1 + \beta_r) G(1 - \beta_r) \left( 1 - e^{-i\theta_r} \right)^{\beta_r^2/2 + \beta_r/2} \left( 1 + e^{-i\theta_r} \right)^{\beta_r^2/2 - \beta_r/2}
\]

\[
\times b_+(1)^{2\beta_+} b_-(-1)^{-\beta_+} b_+(-1)^{2\beta_-} b_-(-1)^{-\beta_-} 2^{3\beta_+ \beta_-}
\]

\[
\times \prod_{r=1}^{R} b_+(e^{i\theta_r})^{\beta_r} b_-(-e^{i\theta_r})^{-\beta_r} b_+(-e^{-i\theta_r})^{\beta_r}
\]

\[
\times \prod_{r=1}^{R} (1 - e^{-i\theta_r})^{2\beta_+ \beta_r} (1 - e^{i\theta_r})^{\beta_+ \beta_r} (1 + e^{-i\theta_r})^{2\beta_- \beta_r} (1 + e^{i\theta_r})^{\beta_- \beta_r}
\]

\[
\times \prod_{1 \leq s < r \leq R} (1 - e^{i(\theta_s - \theta_r)})^{\beta_r \beta_s} (1 - e^{i(\theta_s - \theta_r)})^{\beta_r \beta_s} (1 - e^{-i(\theta_s + \theta_r)})^{\beta_r \beta_s}.
\]

The constants \( E[b] \) and \( F[b] \) are defined by

\[
E[b] = \exp \left( \sum_{k=1}^{\infty} k[\log b]_k [\log b]_{-k} \right),
\]

\[
F[b] = \left( \frac{b_+(1)}{b_+(-1)} \right)^{1/2} \exp \left( -\frac{1}{2} \sum_{k=1}^{\infty} [\log b]^2_k \right),
\]

\[
_3\prod_{r=1}^{R}
\]
and $G(*)$ is the Barnes G–function \[2, 18\] defined by
\[
G(1 + z) = (2\pi)^{z/2} e^{-\gamma E z^2/2} \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k} \right)^{k} e^{-z^2/2k}
\] (13)
with $\gamma E$ being Euler’s constant.

It is interesting to compare this asymptotic formula with the corresponding formula for Toeplitz determinants. The asymptotic expansion of Toeplitz determinants for a certain class of singular generating functions is described by the Fisher–Hartwig conjecture. For instance, it is well known \[3, 9\] that if $\phi$ is of the form
\[
\phi(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} t_{\beta_r}(e^{i(\theta - \theta_r)}),
\] (14)
where $b$ is a sufficiently smooth and nonvanishing on the unit circle, $\theta_1, \ldots, \theta_R \in (-\pi, \pi]$ are distinct numbers, and if $|\text{Re} \beta_r| < 1/2$ holds for each $1 \leq r \leq R$, then the asymptotic behavior of the Toeplitz determinants is given by
\[
det T_n(\phi) \sim G[b]^n n^{\Omega_T} E_T
\] (15)
as $n \to \infty$, where
\[
\Omega_T = - \sum_{r=1}^{R} \beta_r^2,
\] (16)
\[
E_T = E[b] \prod_{r=1}^{R} G(1 + \beta_r) G(1 - \beta_r)
\] \[
\times \prod_{r=1}^{R} b_+(e^{i\theta_r})^{\beta_r} b_-(e^{i\theta_r})^{-\beta_r} \prod_{1 \leq r \neq s \leq R} \left( 1 - e^{i(\theta_s - \theta_r)} \right)^{\beta_r \beta_s}.
\] (17)
A general account of the Fisher–Hartwig conjecture can be found in \[10\] and more recent work in \[3, 12, 11\]. It has been proved in many cases and reformulated in others.

The paper is organized as follows. In Section 2 we compute the asymptotics of $\det M_n(\phi)$ in the case of smooth nonvanishing functions $\phi$ with winding number zero. As in the continuous analogue, the asymptotics are very similar to the Toeplitz case. In fact, we prove that the quotient $\det M_n(\phi) / \det T_n(\phi)$ converges to a nonzero constant.

In Section 3 we recall several operator theoretic results, in particular those related to Toeplitz operators and Toeplitz + Hankel operators.

In Section 4 we prove the asymptotic formula (7) in the special case of piecewise continuous functions \[3\] without jumps at 1 and $-1$ (i.e. with $\beta_+ = \beta_- = 0$), without jumps at both
a point on the unit circle and its complex conjugate (i.e. \(\theta_r + \theta_s \neq 0\)) and under the assumption \(|\Re \beta_r| < 1/2\), \(1 \leq r \leq R\). As before we show that \(\det M_n(\phi)/\det T_n(\phi)\) converges to a nonzero constant (although the corresponding Hankel operator is not compact). In other words, if the symbol has jumps in the “proper” locations, then the asymptotic behavior is again like in the Toeplitz case. Note that the condition on the location of the jumps is extremely important in the Toeplitz + Hankel case, as contrasted to the Toeplitz case.

In Section 5 we show that the quotient \(\det M_n(\phi \psi)/(\det M_n(\phi) \det M_n(\psi))\) converges to a nonzero constant for certain functions \(\phi\) and \(\psi\). This result allows us to localize at certain points on the unit circle, in particular at 1 or \(-1\). However, it is not possible to localize at a point on the unit circle and its complex conjugate. Thus the localization result reduces the asymptotics for general piecewise continuous functions to those for functions \(t_\beta(e^{i\theta})\) and \(t_\beta(e^{i(\theta-\pi)})\) with a single pure jump at 1 or \(-1\) and to those for functions \(t_{\beta^+}(e^{i(\theta-\theta_r)})t_{\beta^-}(e^{i(\theta+\theta_s)})\) with two pure jumps. Note that it is exactly this last class of functions for which we are not able to determine the asymptotics in general.

In Section 6 we then consider the case of functions \(t_\beta(e^{i\theta})\) and \(t_\beta(e^{i(\theta-\pi)})\). In this case, the jump at 1 or \(-1\) on the circle changes the nature of the asymptotics in the second order term. The computations are based on the fact that the corresponding matrices are Cauchy matrices times certain diagonal matrices. Finally, this result in conjunction with the localization and the results of Section 4 gives the above mentioned main general result (7).

In the last section, we illustrate with additional concrete examples. The first class of examples is for piecewise continuous functions with two jumps either at \(\pm 1\) or at \(\pm i\). These functions are special cases where the parameters describing the jumps are connected with each other in some way. The significance is that they show that the one jump results, for functions with jumps at \(-i\) and \(-i\) cannot be pieced together to obtain the asymptotics for a symbol that has jumps at both these points. It should be pointed out that this does work for the points 1 and \(-1\) and this is confirmed by the examples. Note that one special case of these examples is a piecewise constant even function with jumps at \(\pm i\).

The second class of examples in the last section is for the even functions

\[
u_\alpha(e^{i\theta}) = (2 - 2 \cos \theta)^\alpha \quad \text{and} \quad \nu_\alpha(e^{i(\theta-\pi)}) = (2 + 2 \cos \theta)^\alpha.
\]

These functions are singular or zero at 1 or \(-1\), respectively, and are particularly interesting since they represent a more general class of examples of even functions.

For random matrix theory even functions are most important. It would be helpful in the future to extend these results to that case and also to the continuous analogue of Wiener–Hopf + Hankel operators. But we believe the present paper is a good start and leave the other questions to some other time.
2 Operator theoretic results and Toeplitz + Hankel determinants in the case of smooth functions

In this section, we compute the asymptotic behavior of determinants of Toeplitz + Hankel matrices $M_n(\phi)$ in the case of smooth nonvanishing functions defined on the unit circle with winding number zero. What we exactly mean by smoothness will be explained shortly. In the first part of this section, however, we will recall certain operator theoretic results that will be needed later on.

We first introduce the following linear bounded operators acting on the Hilbert space $\ell_2 = \ell_2(\mathbb{Z}_+)$ of one-sided square-summable sequences. Given $\phi \in L^\infty(\mathbb{T})$, define

$$M(\phi) = T(\phi) + H(\phi)$$

(19)

where the Toeplitz and Hankel operators are given by the infinite matrices

$$T(\phi) = (\phi_{j-k})_{j,k=0}^\infty, \quad H(\phi) = (\phi_{j+k+1})_{j,k=0}^\infty.$$  

(20)

Note that the Hardy spaces $H^\infty$ and $\overline{H^\infty}$ consist of those functions $\phi \in L^\infty(\mathbb{T})$ for which the Fourier coefficients $\phi_n$ vanish for each $n < 0$ or $n > 0$, respectively. We also write

$$\tilde{\phi}(e^{i\theta}) = \phi(e^{-i\theta}),$$  

(21)

and call $\phi$ even if $\tilde{\phi} = \phi$. Finally, we introduce the projection $P_n$ acting on $\ell_2$ by

$$P_n(f_0, f_1, \ldots) = (f_0, f_1, \ldots, f_{n-2}, f_{n-1}, 0, 0, \ldots).$$

(22)

Note that $T_n(\phi) = P_nT(\phi)P_n$, $H_n(\phi) = P_nH(\phi)P_n$ and $M_n(\phi) = P_nM(\phi)P_n$.

It is well known that Toeplitz and Hankel operators are related to each other by the formulas

$$T(\phi\psi) = T(\phi)T(\psi) + H(\phi)H(\tilde{\psi}),$$  

(23)

$$H(\phi\psi) = T(\phi)H(\psi) + H(\phi)T(\tilde{\psi}).$$  

(24)

If $\psi_+ \in H^\infty$ and $\psi_- \in \overline{H^\infty}$, then we have

$$T(\psi_-\phi\psi_+) = T(\psi_-)T(\phi)T(\psi_+),$$  

(25)

$$H(\psi_-\phi\tilde{\psi}_+) = T(\psi_-)H(\phi)T(\psi_+).$$  

(26)

Combining equations (23) and (24), it follows that

$$M(\phi\psi) = T(\phi)M(\psi) + H(\phi)M(\tilde{\psi}).$$

(27)
This implies
\[ M(\phi \psi) = M(\phi)M(\psi) + H(\phi)M(\tilde{\psi} - \psi). \quad (28) \]

If \( \psi \) is even, then the latter equation simplifies to
\[ M(\phi \psi) = M(\phi)M(\psi). \quad (29) \]

These and other results concerning \( M(\phi) \) are discussed and proved in [7].

Two important notions are stability and strong convergence. Let \( A_n \) be a sequence of operators. This sequence is said to be stable if there exists an \( n_0 \) such that the operators \( A_n \) are invertible for each \( n \geq n_0 \) and \( \sup_{n \geq n_0} \| A_n^{-1} \| < \infty \). Moreover, we say that \( A_n \) converges strongly on \( \ell_2 \) to an operator \( A \) as \( n \to \infty \) if \( A_n x \to Ax \) in the norm of \( \ell_2 \) for each \( x \in \ell_2 \). When dealing with finite matrices \( A_n \), we identify the matrices and their inverses with operators acting on \( \ell_2 \). It is interesting to note that stability is related to strong convergence of the inverses (and their adjoints) in the following sense.

Lemma 2.1 Suppose that \( A_n \) is a stable sequence such that \( A_n \to A \) and \( A_n^* \to A^* \) strongly. Then \( A \) is invertible, and \( A_n^{-1} \to A^{-1} \) and \( (A_n^{-1})^* \to (A^{-1})^* \) strongly.

Another important set of operators is the ideal of trace class operators (see e.g. [15]). For such operators, the trace “\( \text{tr} A \)” and the operator determinant “\( \det(I + A) \)” are well defined and continuous with respect to \( A \) in the trace class norm. The following result shows the connection with strong convergence.

Lemma 2.2 Let \( B \) be a trace class operator and suppose that \( A_n \) and \( C_n \) are sequences such that \( A_n \to A \) and \( C_n^* \to C^* \) strongly. Then \( A_n B C_n \to ABC \) in the trace class norm.

We proceed with describing the smoothness conditions. We therefore introduce certain function spaces. Let \( F^\beta_{1/2} \) stand for the Banach space of all functions \( b \in L^1(\mathbb{T}) \) for which
\[ \| b \|_{F^\beta_{1/2}} := \left( \sum_{n=-\infty}^{\infty} (1 + |n|^{2\beta}) |b_n|^2 \right)^{1/2} < \infty, \quad (30) \]
and let \( W \) denote the Wiener algebra. It is well known that \( F^\beta_{1/2} \cap W \) is a Banach algebra of continuous functions on the unit circle. The Besov class \( B^1_1 \) is the Banach algebra of all functions \( b \in L^1(\mathbb{T}) \) for which
\[ \| b \|_{B^1_1} := \int_{-\pi}^{\pi} \frac{1}{y^2} \int_{-\pi}^{\pi} \left| b(e^{ix+iy}) + b(e^{ix-iy}) - 2b(e^{ix}) \right| \, dx \, dy < \infty. \quad (31) \]

Using results of Peller [17] one can show that \( b \in B^1_1 \) if and only if both \( H(b) \) and \( H(\tilde{b}) \) are trace class operators. Moreover, the Riesz projection acts boundedly on \( B^1_1 \). An equivalent
norm in $B^1_1$ is given by $|b_0| + \|H(b)\|_1 + \|H(\tilde{b})\|_1$. Finally, one has the following continuous and dense embeddings
\[ F^\beta_2 \subset B^1_1 \subset \left( F^{1/2}_2 \cap W \right) \] if $\beta > 1$. (32)

For more information on these and related classes of smooth functions we refer the reader to [10] and the literature cited there.

The Besov class $B^1_1$ exactly fits our purposes in the sense that the function $b$ appearing in (3) will be assumed to be in $B^1_1$. In order to simplify notation we denote by $G_0 B^1_1$ the group of all nonvanishing functions in $B^1_1$ with winding number zero. Remark that the asymptotic formula for Toeplitz determinants as given in (15) has been proved for functions of the form (14) with $b \in G_0 B^1_1$ (see e.g. [10]). The following proposition shows that all definitions involving the function $b$ which were made in the introduction make sense.

**Proposition 2.3** Let $b \in G_0 B^1_1$. Then $b$ possesses a logarithm $\log b \in B^1_1$, and the constants $G[b]$, $E[b]$ and $F[b]$ as well as the functions $b_+$ and $b_-$ make sense. Moreover,
\[ b(e^{i\theta}) = b_-(e^{i\theta})G[b]b_+(e^{i\theta}), \quad 0 \leq \theta < 2\pi, \]
is the (normalized) Wiener–Hopf factorization with $b_+ \in G(B^1_1 \cap H^\infty)$ and $b_- \in G(B^1_1 \cap H^\infty)$.

**Proof.** Approximating the function $b$ by polynomials and using that $B^1_1$ is a Banach algebra, one can show that $b$ possesses a logarithm $\log b \in B^1_1$. Now one need only use the boundedness of the Riesz projection and the fact that $B^1_1$ is contained in $F^{1/2}_2 \cap W$. \[\Box\]

We now establish a limit relation for the quotient of two determinants with smooth generating functions. In Section 4 this relation will be generalized to the case of certain piecewise continuous functions.

**Proposition 2.4** Let $b \in G_0 B^1_1$. Then $\det M_n(b)/\det T_n(b) \to F(b)$ as $n \to \infty$ where $F(b)$ is the operator determinant defined by
\[ F(b) := \det \left( I + T^{-1}(\phi)H(\phi) \right). \] (33)

**Proof.** It is well known [14] that under the above assumptions on $b$, the sequence $T_n(b)$ is stable and the inverses converge strongly on $\ell_2$ to the inverse of $T(b)$. Since $H(b)$ is trace class, we obtain that
\[ \frac{\det M_n(b)}{\det T_n(b)} = \det T_n^{-1}(b)M_n(b) = \det \left( P_n + T_n^{-1}(b)P_nH(b)P_n \right) \]
converges to the operator determinant $F(b)$. Note that $P_n^* = P_n \to I$ strongly. \[\Box\]

Next we establish the one main result of this section, the evaluation of the operator determinant $F(b)$. The computation is especially remarkable since it relies on a “differentiation” trick, which was recently used in a modified form in [1].

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Theorem 2.5 Let $b \in G_0 B_1$. Then $F(b) = F[b]$.

Proof. Using the Wiener–Hopf factorization of $b$ and formula (23) it is easily seen that $T(b) = G[b]T(b_-)T(b_+)$, and hence the inverse equals

$$T^{-1}(b) = G[b]^{-1}T(b_+^{-1})T(b_-^{-1}).$$

From equation (26) it follows that

$$T^{-1}(b)H(b) = T(b_+^{-1})T(b_-^{-1})H(b_-b_+) = T^{-1}(b_+)H(b_+).$$

Hence $F(b) = F(b_+)$. Now let $c = \tilde{b}_+b_+$. One can conclude analogously that $F(c) = F(b_+)$. Thus we are left with the evaluation of $F(c)$. Obviously, $[\log c]_n = [\log b]_n$ for $n \in \mathbb{Z}\{0\}$, and hence $\log \tilde{c} = \log c$. Since the Riesz projection is bounded on $B_1^1$, we obtain that $\log c \in B_1^1$. Next we define the $B_1^1$–valued analytic (in $\lambda$) function

$$c_\lambda = \exp(\lambda \log c), \quad \lambda \in \mathbb{C}.$$ 

Note that the derivative of $c_\lambda$ with respect to $\lambda$ equals $c_\lambda \log c$. Let $Y(\lambda)$ be the analytic operator–valued function

$$Y(\lambda) = I + T^{-1}(c_\lambda)H(c_\lambda) = T^{-1}(c_\lambda)M(c_\lambda).$$

It is easy to compute the inverse and the derivative of $Y(\lambda)$. We obtain

$$Y'(\lambda)Y^{-1}(\lambda) = -T^{-1}(c_\lambda)T(c_\lambda \log c)T^{-1}(c_\lambda)M(c_\lambda)M^{-1}(c_\lambda)T(c_\lambda)$$

$$+ T^{-1}(c_\lambda)M(c_\lambda \log c)M^{-1}(c_\lambda)T(c_\lambda)$$

$$= -T^{-1}(c_\lambda)T(c_\lambda \log c) + T^{-1}(c_\lambda)M(\log c)T(c_\lambda).$$

Note that $\tilde{c}_\lambda = c_\lambda$, and thus (29) implies $M(c_\lambda \log c) = M(\log c)M(c_\lambda)$. Because $det Y(\lambda) \neq 0$ for all $\lambda$, the scalar function $y(\lambda) = \log det Y(\lambda)$ is an entire analytic function. We conclude

$$\frac{dy}{d\lambda} = \frac{(det Y)'}{det Y} = tr Y'Y^{-1}$$

$$= tr \left( -T(c_\lambda \log c)T^{-1}(c_\lambda) + M(\log c) \right).$$

Differentiating again yields

$$\frac{d^2y}{d\lambda^2} = tr \left( -T(c_\lambda \log^2 c)T^{-1}(c_\lambda) + T(c_\lambda \log c)T^{-1}(c_\lambda)T(c_\lambda \log c)T^{-1}(c_\lambda) \right)$$

$$= tr \left( -T(\log^2 c) + T(\log c)T(\log c) \right) = -tr H(\log c)H(\log \tilde{c}).$$
The last equality follows from $T^{-1}(c_\lambda) = T(c_\lambda^{-1})$ where $c_\lambda = c_{\lambda^-}c_{\lambda^+}$ is the Wiener–Hopf factorization of $c_\lambda$. By repeated application of (25), all occurring functions $c_{\lambda^\pm}$ cancel each other. Hence the second derivative of $y$ does not depend on $\lambda$. Note that

\[
\begin{align*}
y(0) &= \log \det I = 0, \\
y'(0) &= \operatorname{tr} \left( -T(\log c) + M(\log c) \right) = \operatorname{tr} H(\log c)
\end{align*}
\]

since $c_\lambda|_{\lambda=0} \equiv 1$. It follows that

\[
y(\lambda) = -\frac{\lambda^2}{2} \operatorname{tr} H(\log c) H(\log \tilde{c}) + \lambda \operatorname{tr} H(\log c).
\]

Because $F(c) = \det Y(1) = \exp y(1)$, we obtain

\[
F(c) = \exp \left( -\frac{1}{2} \operatorname{tr} H(\log c) H(\log \tilde{c}) + \operatorname{tr} H(\log c) \right)
\]

\[
= \exp \left( -\frac{1}{2} \sum_{n=1}^{\infty} n[\log c]_n[\log c]_{-n} + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ [\log c]_n - (-1)^n[\log c]_n \right\} \right).
\]

Writing $\log c$ in terms of $\log b$, we arrive at

\[
F(b) = \exp \left( -\frac{1}{2} \sum_{n=1}^{\infty} n[\log b]_n^2 + \frac{1}{2} \log b_+(1) - \frac{1}{2} \log b_+(-1) \right).
\]

This immediately implies the assertion.

Finally, we can combine the previous results with the well known Szegö–Widom Limit Theorem [19]. This limit theorem says that $\det T_n(b) \sim G[b]^n E[b]$ as $n \to \infty$ for nonvanishing functions $b \in F_{1/2}^1 \cap W$ with winding number zero.

**Corollary 2.6** Let $b \in G_0 B_1^1$. Then

\[
\det M_n(b) \sim G[b]^n E[b] F[b] \quad \text{as } n \to \infty.
\]

### 3 Further operator theoretic results

The proofs of the results that will presented in the following two sections require further operator theoretic preliminaries. In particular, we need some results about Toeplitz operators and Hankel operators as well as about Toeplitz + Hankel operators $M(\phi)$ (see [10] and [7] for the general theory). First of all, in addition to the projection $P_n$, we define $Q_n = I - P_n$ and

\[
\begin{align*}
W_n(f_0, f_1, \ldots) &= (f_{n-1}, f_{n-2}, \ldots, f_1, f_0, 0, 0, \ldots), \\
V_n(f_0, f_1, \ldots) &= (0, 0, \ldots, 0, 0, f_0, f_1, f_2, \ldots), \\
V_{-n}(f_0, f_1, \ldots) &= (f_n, f_{n+1}, f_{n+2}, \ldots).
\end{align*}
\]
It is easily seen that $W_n^2 = P_n$, $W_n = W_nP_n = P_nW_n$, $V_n V_n = Q_n$ and $V_n V_n = I$. Note also that
\begin{equation}
P_n T(\phi) V_n = W_n H(\bar{\phi}), \quad V_n T(\phi) P_n = H(\phi) W_n.
\end{equation}

Moreover, we have
\begin{align}
V_n H(\phi) &= H(\phi) V_n, \quad (36) \\
W_n T_n(\phi) W_n &= T_n(\bar{\phi}). \quad (37)
\end{align}

Using equations (35) we can write
\begin{equation}
P_n T(\phi) Q_n T(\psi) P_n = P_n T(\phi) V_n V_n T(\psi) P_n = W_n H(\bar{\phi}) H(\psi) W_n. \quad (38)
\end{equation}

Taking (23) into account we arrive at the fundamental identity due to Widom [19]
\begin{equation}
T_n(\phi \psi) = T_n(\phi) T_n(\psi) + P_n H(\phi) H(\psi) P_n + W_n H(\bar{\phi}) H(\psi) W_n. \quad (39)
\end{equation}

The following result deals with strong convergence. For brevity of notation, we will henceforth write $A = s - \lim A_n$ if both $A_n \to A$ and $A^*_n \to A^*$ strongly.

**Proposition 3.1** Let $\phi \in L^\infty(\mathbb{T})$. Then
\begin{align}
T(\phi) &= s - \lim T_n(\phi), \quad T(\bar{\phi}) = s - \lim W_n T_n(\bar{\phi}) W_n, \\
M(\phi) &= s - \lim M_n(\phi), \quad T(\bar{\phi}) = s - \lim W_n M_n(\bar{\phi}) W_n.
\end{align}

**Proof.** The relations without the $W_n$’s are trivial because $P_n^* = P_n \to I$ strongly. Also the second assertion is easy to show by taking account of (37). Finally, using (35) we obtain
\begin{equation}
W_n M_n(\phi) W_n = T_n(\bar{\phi}) + W_n V_n T(\phi) P_n = T_n(\bar{\phi}) + P_n T(\bar{\phi}) V_n W_n.
\end{equation}

Because $V_n^* = V_n \to 0$ strongly, the second term and its adjoint tend strongly to zero. This settles the last assertion. \hfill \square

We can now combine the previous result with Lemma 2.1 and obtain information about the strong convergence of the inverses. Note that $T(\bar{\phi})$ is the transpose of $T(\phi)$.

**Corollary 3.2** Let $\phi \in L^\infty(\mathbb{T})$. If $T_n(\phi)$ is stable, then $T(\phi)$ is invertible and
\begin{align}
T^{-1}(\phi) &= s - \lim T_n^{-1}(\phi), \quad T^{-1}(\bar{\phi}) = s - \lim W_n T_n^{-1}(\bar{\phi}) W_n. \\
If M_n(\phi) is stable, then M(\phi) and T(\phi) are invertible and
M^{-1}(\phi) &= s - \lim M_n^{-1}(\phi), \quad T^{-1}(\bar{\phi}) = s - \lim W_n M_n^{-1}(\bar{\phi}) W_n.
\end{align}
The next step is the description of invertibility and stability in the case of functions \( \phi \) contained in the set \( PC \) of all piecewise continuous functions on the unit circle. For this we need more notation. In what follows, let \( A \) stand for the set \( C \) of continuous functions, the set \( B^1 \), or the set \( C^\infty \) of infinitely differentiable functions. We denote by \( PC[A; K] \) the set of all functions \( \phi \) that can be written in the form

\[
\phi(e^{i\theta}) = b(e^{i\theta}) t_{\beta_+}(e^{i\theta}) t_{\beta_-}(e^{i(\theta-\pi)}) \prod_{r=1}^{R} t_{\beta^+_r}(e^{i(\theta-\theta_r)}) t_{\beta^-_r}(e^{i(\theta+\theta_r)}),
\]

where \( b \in A \) is a nonvanishing function with winding number zero, \( \theta_1, \ldots, \theta_R \in (0, \pi) \) are distinct points, the set \( K \subseteq \{1, -1, e^{i\theta_1}, \ldots, e^{i\theta_R}, e^{-i\theta_1}, \ldots, e^{-i\theta_R} \} \) is the set of jump discontinuities of \( \phi \), and \( \beta_\pm, \beta^\pm_1, \ldots, \beta^\pm_R \) are certain complex parameters. If some these parameters are zero, then no jumps occur at the corresponding points.

The representation (40) is essentially the same as (14), however it displays the special role of jumps at 1 and \(-1\) and a connection between jumps at a point on the unit circle and its complex conjugate. As will be seen below, this distinction is not necessary for the pure Toeplitz case, however, it is for the Toeplitz + Hankel case. Note that \( PC[C; K] \) is the set of all invertible functions in \( PC \) with finitely many jumps at \( K \).

Let \( PC^1[A; K] \), \( PC^II[A; K] \) and \( PC^III[A; K] \) be the sets of functions of the above form such that in addition the following conditions (I), (II) and (III), respectively, are satisfied:

1. \( |\text{Re } \beta_\pm| < 1/2 \) and \( |\text{Re } \beta^\pm_r| < 1/2 \) for each \( r \);
2. \(-3/4 < \text{Re } \beta_+ < 1/4 \) and \(-1/4 < \text{Re } \beta_- < 3/4 \) and \( |\text{Re } (\beta^+_r + \beta^-_r)| < 1/2 \) for each \( r \);
3. \(-1/2 < \text{Re } \beta_+ < 1/4 \) and \(-1/4 < \text{Re } \beta_- < 1/2 \) and
   \[|\text{Re } \beta^+_r| < 1/2 \) and \( |\text{Re } \beta^-_r| < 1/2 \) and \( |\text{Re } (\beta^+_r + \beta^-_r)| < 1/2 \) for each \( r \).

We want to emphasize that \( PC^III[A; K] \subseteq PC^1[A; K] \cap PC^II[A; K] \) and that this inclusion is proper. This seems strange at first glance, but it finds its solution in the fact that the representation (40) and in particular the \( \beta \)'s need not be unique. Examples of functions showing this are given in [7, Sect. 4]. Finally, note that \( \phi \in PC^1[A; K] \) if and only if \( \tilde{\phi} \in PC^1[A; \tilde{K}] \) where

\[
\tilde{K} = \{ 1/t : t \in K \}.
\]

Invertibility and stability criteria for Toeplitz and Toeplitz + Hankel operators with piecewise continuous generating functions can now be stated as follows. For proofs we refer to \([10]\) and \([7]\), respectively.

**Proposition 3.3** Let \( \phi \in PC \) be a function with jumps in a finite set \( K \subset \mathbb{T} \). Then
(a) $T(\phi)$ is invertible if and only if $\phi \in PC_1[C; K]$;
(b) $T_n(\phi)$ is stable if and only if $\phi \in PC_1[C; K]$;
(c) $M(\phi)$ is invertible if and only if $\phi \in PC_1[C; K]$;
(b) $M_n(\phi)$ is stable if and only if $\phi \in PC_1[C; K]$.

We also need to introduce “approximate” functions $\phi_\mu$, $0 \leq \mu < 1$, associated to a piecewise continuous function $\phi \in PC[A; K]$ of the form (33):

$$\phi_\mu(e^{i\theta}) = b(e^{i\theta})t_{\beta+\mu}(e^{i\theta})t_{\beta-\mu}(e^{i(\theta-\pi)}) \prod_{r=1}^R t_{\beta-r,\mu}(e^{i(\theta-\theta_r)})t_{\beta+r,\mu}(e^{i(\theta+\theta_r)})$$ (42)

Here $t_{\beta,\mu}$ is the smooth nonvanishing function with winding number zero defined by

$$t_{\beta,\mu}(e^{i\theta}) = \left(1 - e^{i\theta}\right)^\beta \left(1 - e^{-i\theta}\right)^{-\beta}, \quad 0 \leq \theta < 2\pi.$$ (43)

Note that also $\phi_\mu \in A$ is a nonvanishing function with winding number zero.

For a (generalized) sequence $A_\mu$ of operators acting on $\ell_2$ depending on a parameter $\mu \in [0, 1)$, one can define the concepts of strong convergence (as $\mu \to 1$) and stability in the same way as for (discrete) sequences $A_n$. The analogues of Lemma 2.1 and Lemma 2.2 also remain true. We will write $A = s - \lim A_\mu$ if both $A_\mu \to A$ and $A_\mu^* \to A^*$ strongly.

Proposition 3.4 Let $K$ be a finite subset of $\mathbb{T}$, let $\phi \in PC[C; K]$, and let $\phi_\mu$, $0 \leq \mu < 1$, be the associated approximate functions. Then

$$H(\phi) = s - \lim H(\phi_\mu), \quad T(\phi) = s - \lim T(\phi_\mu), \quad M(\phi) = s - \lim M(\phi_\mu).$$

Moreover, if $\phi \in PC[B_1; K]$ and $f \in C^\infty$ vanishes on an open neighborhood of $K$, then $H(f\phi_\mu) \to H(f\phi)$ and $H(f/\phi_\mu) \to H(f/\phi)$ in the trace class norm.

Proof. First of all note that $t_\beta(e^{i\theta}) = (1 - e^{i\theta})^\beta (1 - e^{-i\theta})^{-\beta}$, and thus

$$t_\beta(e^{i\theta}) = \exp\left(2i\beta \arg(1 - e^{i\theta})\right), \quad t_{\beta,\mu}(e^{i\theta}) = \exp\left(2i\beta \arg(1 - \mu e^{i\theta})\right),$$

where the argument is taken in $(-\pi/2, \pi/2)$. It is easy to see $t_{\beta,\mu} \to t_\beta$ locally uniformly on $\mathbb{T} \setminus \{1\}$ as $\mu \to 1$. The same holds for the derivatives of arbitrary order. We obtain that $\phi_\mu \to \phi$ in measure. Because the sequence $\phi_\mu$ is uniformly bounded in the norm of $L^\infty(\mathbb{T})$, the Laurent operators generated by $\phi_\mu$ (which are unitarily equivalent to multiplication operators on $L^2(\mathbb{T})$) converge strongly to the Laurent operator generated by $\phi$. This settles the first part of the proposition.
Now write $\phi_\mu = b\psi_\mu$ and $\phi = b\psi$. From the above statements, it follows that $f_\psi_\mu \to f\psi$ in the sense of $C^\infty$. Multiplying with $b \in B_1^1$ we obtain that $f_\phi_\mu \to f\phi$ in the norm of $B_1^1$ and hence the desired convergence of the Hankel operators. The last assertion can be shown analogously.

Next we state the necessary and sufficient conditions for the stability of $T(\phi_\mu)$ and $M(\phi_\mu)$. Proofs are given in [13] and [7]. These results can be combined with Proposition 3.4 and the analogue of Lemma 2.1 in order to obtain a result about the strong convergence of the inverses and the adjoints of the inverses. We leave the details to the reader.

**Proposition 3.5** Let $K$ be a finite subset of $\mathbb{T}$, and let $\phi_\mu$, $0 \leq \mu < 1$, be the approximate functions of the form (42) associated to a function $\phi \in PC[I;K]$ of the form (40). Then

(a) $T(\phi_\mu)$ is stable if and only if the parameters satisfy condition (I).

(b) $M(\phi_\mu)$ is stable if and only if the parameters satisfy condition (II).

Note that it does not suffice to require $\phi \in PC_I[C;K]$ or $\phi \in PC_{II}[C;K]$, respectively, because the representation of $\phi$ is not unique and the parameters of $\phi_\mu$ must be chosen properly.

Finally, we need the following basic results.

**Lemma 3.6** Let $A : H_1 \to H_2$ and $B : H_2 \to H_1$ be linear bounded operators acting on Hilbert spaces $H_1$ and $H_2$. Then the following assertions hold.

(a) The operator $I + AB$ is invertible if and only if so is $I + BA$.

(b) The operator $I + AB$ is a Fredholm operator if and only if so is $I + BA$. If this is true, then $\text{ind}(I + AB) = \text{ind}(I + BA)$.

(c) If $A$ or $B$ is a trace class operator, then $\det(I + AB) = \det(I + BA)$.

**Proof.** Part (a) can be proved by using the formula $(I + BA)^{-1} = I - B(I + AB)^{-1}A$. Assertion (b) can be proved in the same way, by thinking of the inverses as Fredholm regularizers. Also, we use the fact that the kernels (resp. cokernels) of $I + AB$ and $I + BA$ have the same dimension. Finally, $AB$ and $BA$ have the same nonzero eigenvalues (taking multiplicities into account).

**Lemma 3.7** Let $A_n = P_n + P_n K P_n + W_n L W_n + C_n$ be a sequence of $n \times n$ matrices where $K$ and $L$ are trace class operators, and $C_n$ tends to zero in the trace class norm. Then $\lim_{n \to \infty} \det A_n = \det(I + K) \det(I + L)$. 

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Proof. Because $W_n \to 0$ weakly on $\ell_2$, the sequence $KW_nL$ tends to zero in the trace class norm. Hence

$$\det A_n = \det \left( I + P_nKP_n + W_nLW_n + C_n \right)$$

$$= \det \left( (I + P_nKP_n)(I + W_nLW_n) + C'_n \right)$$

with $C'_n \to 0$ in the trace class norm. Noting that $\det(I + W_nLW_n) = \det(I + P_nLP_n)$ completes the proof.

\[\square\]

4 The limit theorem for piecewise continuous functions

In Section 3, we have proved that for functions $\varphi \in G_0B^1$, the quotient $\det M_n(\varphi) / \det T_n(\varphi)$ converges to a nonzero constant. Surprisingly, the same turns out to be true for certain piecewise continuous functions satisfying a particular condition on the location of the jumps. This condition excludes functions with jumps at 1 or $-1$ or at both a point on the unit circle and its complex conjugate. The fact that the Hankel operator $H(\varphi)$ need not be compact (and hence $F(\varphi)$ need not be defined) makes the proof of the limit relation for piecewise continuous functions more complicated than the proof of Proposition 2.4.

In order to overcome this obstacle we introduce another operator determinant which is related to $F(\varphi)$. In this connection, we use the notion of a smooth partition of unity. By this we here mean two smooth functions (in $C^\infty$) whose sums equals the constant function with value one on the unit circle.

**Theorem 4.1** Let $\varphi \in L^\infty(T)$. Suppose that $T(\varphi)$ is invertible and that there exists a smooth partition of unity, $f + \tilde{f} = 1$, such that $H(f\varphi)$ and $H(f/\varphi)$ are trace class. Moreover, introduce the following operators:

$$A_{11} = T(f)T^{-1}(\varphi)H(\varphi),$$

$$A_{12} = T(f)T^{-1}(\varphi)H(\varphi)T(\tilde{f}),$$

$$A_{21} = T^{-1}(\varphi)H(\varphi),$$

$$A_{22} = T^{-1}(\varphi)H(\varphi)T(\tilde{f}).$$

Then $A_{11}, A_{12}$ and $A_{22}$ are trace class, and the operator determinant

$$F(\varphi; f) := \det \left( \begin{array}{cc} I + A_{11} & A_{12} \\ -A_{21}A_{11} & I + A_{22} - A_{21}A_{12} \end{array} \right) \quad (44)$$

is well defined and nonzero. If in addition $\varphi \in B^1$, then $F(\varphi; f) = F(\varphi)$. 15
Proof. First of all note that if $T(\phi)$ is invertible, then the function $\phi$ is invertible in $L^\infty(\mathbb{T})$. Using the identities (23) and (24) one can show that $A_{11}$ and $A_{22}$ are trace class:

\begin{align*}
A_{22} &= T^{-1}(\phi)H(\phi)T(\tilde{f}) \\
&= T^{-1}(\phi) \left( H(\phi f) - T(\phi)H(f) \right) \\
&= T^{-1}(\phi)H(\phi f) - H(f),
\end{align*}

(45)

\begin{align*}
A_{11} &= T(f)T^{-1}(\phi)H(\phi) \\
&= \left( T(f/\phi)T(\phi) + H(f/\phi)H(\tilde{\phi}) \right) T^{-1}(\phi)H(\phi) \\
&= T(f/\phi)H(\phi) + H(f/\phi)H(\tilde{\phi})T^{-1}(\phi)H(\phi) \\
&= H(f) - H(f/\phi)T(\tilde{\phi}) + H(f/\phi)H(\tilde{\phi})T^{-1}(\phi)H(\phi).
\end{align*}

(46)

Indeed, in each of these terms there appears a Hankel operator which is trace class. Hence also $A_{12}$ is trace class. Notice however that $A_{21}$ need not be trace class. In any case, it follows that $F(\phi; f)$ is well defined. Next, we have

\begin{align*}
\begin{pmatrix}
I + A_{11} \\
-A_{21}A_{11} \\
-A_{21}A_{11} + A_{22} - A_{21}A_{12}
\end{pmatrix}
&= \begin{pmatrix}
I & A_{12} \\
-A_{21} & I + A_{22} - A_{21}A_{12}
\end{pmatrix}
= \begin{pmatrix}
I + A_{11} & A_{12} \\
-A_{21} & I + A_{22}
\end{pmatrix},
\end{align*}

(47)

and

\begin{align*}
\begin{pmatrix}
I + A_{11} & A_{12} \\
A_{21} & I + A_{22}
\end{pmatrix}
&= \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix} + \begin{pmatrix}
T(f) \\
I
\end{pmatrix} T^{-1}(\phi)H(\phi) \begin{pmatrix}
I & T(\tilde{f})
\end{pmatrix}.
\end{align*}

(48)

Note that $F(\phi; f) \neq 0$ if and only if the operator appearing in (47) is invertible, or, equivalently, if the one in (48) is invertible. Using Lemma 3.6(a) and the fact that

\begin{align*}
\begin{pmatrix}
I, & T(\tilde{f})
\end{pmatrix}
\begin{pmatrix}
T(f) \\
I
\end{pmatrix}
= I,
\end{align*}

it follows that (48) is invertible if and only if $I + T^{-1}(\phi)H(\phi)$ is invertible. Hence, we arrive at the conclusion that $F(\phi; f) \neq 0$ if and only if $M(\phi)$ is invertible. On the other hand, the operator in (47) equals identity plus a trace class operator, hence it is a Fredholm operator with index zero. It follows that so is (48). Using Lemma 3.6(b) we obtain that also $I + T^{-1}(\phi)H(\phi)$ and thus $M(\phi)$ is Fredholm with index zero. Now we need only use the fact that $M(\phi)$ is invertible if and only if $M(\phi)$ is Fredholm with index zero (see [6, Corollary 2.7]).

Finally, suppose that $\phi \in B_1^1$. Then the operators $A_{11}, A_{12}, A_{21}, A_{22}$ are trace class, and one can take the determinant of (47) and (48). Noting that the determinant of the first matrix on the right hand side of (47) is equal to one and employing Lemma 3.6(c), it follows that $F(\phi; f) = F(\phi)$.

\[ \square \]
Although we have not a proof in the general setting, it seems reasonable that $F(\phi; f)$ does not depend on the particular choice of $f$ (see also the remark below).

The next result is a quite general version of the limit theorem, which will be applied afterwards to certain piecewise continuous functions.

**Theorem 4.2** Let $\phi \in L^\infty(\mathbb{T})$. Suppose that the sequence $T_n(\phi)$ is stable and that there exists a smooth partition of unity, $f + \tilde{f} = 1$, such that $H(f\phi)$, $H(f/\phi)$ and $H(\tilde{f}/\phi)$ are trace class. Then $\lim_{n \to \infty} \det M_n(\phi)/\det T_n(\phi) = F(\phi; f)$.

**Proof.** First of all remark that the stability of $T_n(\phi)$ implies the invertibility of $T(\phi)$. Hence the assumptions of Theorem [4.1] are fulfilled and $F(\phi; f)$ is well defined. Let $A_{11}, \ldots, A_{22}$ be the operators introduced there, and define the following sequences of matrices:

\[
A_{11}^{(n)} = T_n(f)T_n^{-1}(\phi)H_n(\phi),
A_{12}^{(n)} = T_n(f)T_n^{-1}(\phi)H_n(\phi)T_n(\tilde{f}),
A_{21}^{(n)} = T_n^{-1}(\phi)H_n(\phi),
A_{22}^{(n)} = T_n^{-1}(\phi)H_n(\phi)T_n(\tilde{f}).
\]

We first claim that $A_{11}^{(n)} \to A_{11}$ and $A_{22}^{(n)} \to A_{22}$ in the trace class norm as $n \to \infty$. Indeed, using (24), (33) and (36), we can write

\[
A_{22}^{(n)} = T_n^{-1}(\phi)H_n(\phi)T_n(\tilde{f})
= T_n^{-1}(\phi)P_nH(\phi)T(\tilde{f})P_n - T_n^{-1}(\phi)P_nH(\phi)Q_nT(\tilde{f})P_n
= T_n^{-1}(\phi)P_n\left(H(\phi f) - T(\phi)H(f)\right)P_n - T_n^{-1}(\phi)P_nV_nH(\phi)H(\tilde{f})W_n.
\]

The last term in the sum tends to zero in the trace class norm because $H(\phi)H(\tilde{f})$ is trace class and $V_n \to 0$ strongly. The first term converges to $A_{22}$ in the trace class norm since $T_n^{-1}(\phi) \to T^{-1}(\phi)$ and $P^* = P_n \to I$ strongly and the expression in the middle is a trace class operator. Now we employ identity (39) to rewrite $T_n(f)$, and we obtain

\[
A_{11}^{(n)} = T_n(f)T_n^{-1}(\phi)H_n(\phi)
= T_n(f/\phi)H_n(\phi) + P_nH(f/\phi)H(f/\phi)P_nT_n^{-1}(\phi)H_n(\phi)
+ W_nH(\tilde{f}/\phi)H(\phi)W_nT_n^{-1}(\phi)H_n(\phi).
\]

Analyzing the first summand yields (see again (24), (33) and (36))

\[
T_n(f/\phi)H_n(\phi) = P_nT(f/\phi)H(\phi)P_n - P_nT(f/\phi)Q_nH(\phi)P_n
= P_n\left(H(f) - H(f/\phi)T(\phi)\right)P_n - W_nH(\tilde{f}/\phi)H(\phi)V_nP_n.
\]
Because $H(\tilde{f}/\tilde{\phi})H(\phi)$ is trace class and $V_n^* = V_{-n} \to 0$ strongly, we obtain that

$$T_n(f/\phi)H_n(\phi) \to H(f) - H(f/\phi)T(\tilde{\phi})$$

in the trace class norm as $n \to \infty$. Note that the adjoint of $T_n^{-1}(\tilde{\phi})$ converges strongly to the adjoint of $T^{-1}(\tilde{\phi})$. Hence it is easily seen that

$$P_n H(f/\phi)H(\tilde{\phi})P_n T_n^{-1}(\phi)H_n(\phi) \to H(f/\phi)H(\tilde{\phi})T^{-1}(\phi)H(\phi)$$

in the trace class norm as $n \to \infty$. Finally, (35) and (37) imply that

$$W_n H(\tilde{f}/\tilde{\phi})H(\phi)W_n T_n^{-1}(\phi)H_n(\phi) = W_n H(\tilde{f}/\tilde{\phi})H(\phi)T_n^{-1}(\tilde{\phi})W_n H_n(\phi) = W_n H(\tilde{f}/\tilde{\phi})H(\phi)T_n^{-1}(\tilde{\phi})T(\tilde{\phi})V_n P_n.$$

Because $H(\tilde{f}/\tilde{\phi})H(\phi)$ is trace class, the adjoint of $T_n^{-1}(\tilde{\phi})$ converges strongly to the adjoint of $T^{-1}(\tilde{\phi})$ and $V_n^* = V_{-n} \to 0$ strongly, it follows that the latter term converges to zero in the trace class norm. Thus we have proved that $A_{11}^{(n)} \to A_{11}$ and $A_{22}^{(n)} \to A_{22}$ in the trace class norm. Now one can immediately conclude that also $A_{12}^{(n)} \to A_{12}$ in the trace class norm. Moreover, it is obvious that $A_{21}^{(n)} \to A_{21}$ strongly.

The desired limit relation can now be proved as follows. Let

$$S_n = \det M_n(\phi) / \det T_n(\phi) = \det \left( P_n + T_n^{-1}(\phi)H_n(\phi) \right).$$

Since $f + \tilde{f} = 1$, we have

$$P_n = \begin{pmatrix} P_n & T_n(\tilde{f}) \\ T_n(f) & P_n \end{pmatrix}.$$

This in conjunction with Lemma 3.6(c) (for matrices) implies that

$$S_n = \det \left\{ P_n + \begin{pmatrix} P_n & T_n(\tilde{f}) \\ T_n(f) & P_n \end{pmatrix} \begin{pmatrix} T_n(f) \\ P_n \end{pmatrix} T_n^{-1}(\phi)H_n(\phi) \right\} = \det \left\{ \begin{pmatrix} P_n & A_{11}^{(n)} \\ A_{21}^{(n)} & P_n + A_{22}^{(n)} \end{pmatrix} \begin{pmatrix} T_n(f) \\ P_n \end{pmatrix} T_n^{-1}(\phi)H_n(\phi) \begin{pmatrix} P_n & T_n(\tilde{f}) \end{pmatrix} \right\} = \det \begin{pmatrix} P_n + A_{11}^{(n)} & A_{12}^{(n)} \\ A_{21}^{(n)} & P_n + A_{22}^{(n)} \end{pmatrix} \begin{pmatrix} T_n(f) \\ P_n \end{pmatrix} T_n^{-1}(\phi)H_n(\phi) \begin{pmatrix} P_n & T_n(\tilde{f}) \end{pmatrix}.$$
Note that the last identity follows from

\[
\begin{pmatrix}
  P_n & 0 \\
  -A_{21}^{(n)} & P_n
\end{pmatrix}
\begin{pmatrix}
  P_n + A_{11}^{(n)} & A_{12}^{(n)} \\
  A_{21}^{(n)} & P_n + A_{22}^{(n)}
\end{pmatrix}
= \begin{pmatrix}
  P_n + A_{11}^{(n)} & A_{12}^{(n)} \\
  -A_{21}^{(n)}A_{11}^{(n)} & P_n + A_{22}^{(n)} - A_{21}^{(n)}A_{12}^{(n)}
\end{pmatrix},
\]

where the determinant of the first matrix on the left is equal to one. Taking the limit \( n \to \infty \), we obtain that \( S_n \to F(\phi; f) \).

Note that if the assumptions of Theorem 4.2 are satisfied (which are slightly stronger than those of Theorem 4.1), then the value of \( F(\phi; f) \) does not depend on the particular choice of \( f \). For it is the limit of a sequence independent of \( f \).

Now we specialize to the case of piecewise continuous functions with jumps on a finite set \( K \). We show how the constant \( F(\phi, f) \) can be evaluated from the constant \( F(\phi) \). The proof will reveal that the partition of unity condition is in some sense responsible for the afore-mentioned condition on the location of the jumps of \( \phi \). Note that this condition can be expressed as \( K \cap \tilde{K} = \emptyset \) where \( \tilde{K} \) is defined as in (11).

**Lemma 4.3** Let \( \phi \in PC_1[B_1^1; K] \) with \( K \cap \tilde{K} = \emptyset \). Moreover, let \( \phi_\mu \), \( 0 \leq \mu < 1 \), be the approximate functions associated to \( \phi \). Then there exists a function \( f \) such that the assumptions of Theorem 4.1 and Theorem 4.2 are fulfilled, and \( F(\phi; f) = \lim_{\mu \to 1} F(\phi_\mu) \).

**Proof.** The invertibility of \( T(\phi) \) and stability of \( T_n(\phi) \) follows from Proposition 3.3(ab). Because \( K \cap \tilde{K} = \emptyset \), there exists an \( f \in C^\infty \) with \( f + \tilde{f} = 1 \) such that \( f \) vanishes on an open neighborhood of \( K \). It is easy to see that \( f\phi \in B_1^1 \) and \( f/\phi \in B_1^1 \). Thus \( H(f\phi) \), \( H(f/\phi) \) and \( H(\tilde{f}/\tilde{\phi}) \) are trace class. In addition to \( A_{11}, \ldots, A_{22} \) defined in Theorem 4.1, let

\[
\begin{align*}
A_{11}^{(\mu)} &= T(f)T^{-1}(\phi_\mu)H(\phi_\mu), \\
A_{12}^{(\mu)} &= T(f)T^{-1}(\phi_\mu)H(\phi_\mu)T(\tilde{f}), \\
A_{21}^{(\mu)} &= T^{-1}(\phi_\mu)H(\phi_\mu), \\
A_{22}^{(\mu)} &= T^{-1}(\phi_\mu)H(\phi_\mu)T(\tilde{f}).
\end{align*}
\]

Recall that \( A_{11} \) and \( A_{22} \) can be written in the form (15) and (16), and analogously so can \( A_{11}^{(\mu)} \) and \( A_{22}^{(\mu)} \) with \( \phi \) replaced by \( \phi_\mu \). Applying Proposition 3.4 and Proposition 3.5 to these modified expressions, we conclude that \( A_{11}^{(\mu)} \to A_{11} \) and \( A_{22}^{(\mu)} \to A_{22} \) in the trace class norm as \( \mu \to 1 \). Moreover, also \( A_{12}^{(\mu)} \to A_{12} \) in the trace class norm and \( A_{21}^{(\mu)} \to A_{21} \) strongly. The convergence of these operators implies that \( F(\phi; f) = \lim_{\mu \to 1} F(\phi_\mu; f) \). Finally, observe that \( \phi_\mu \in G_0B_1^1 \). Hence \( F(\phi_\mu; f) = F(\phi_\mu) \) by Theorem 4.1.

The evaluation of the constant \( F(\phi; f) \) is now straightforward. We consider functions \( \phi \in PC_1[B_1^1; K] \) with \( K \cap \tilde{K} = \emptyset \). Such a function can be written in the form (3). In terms of this representation, the assumptions on the parameters of \( \phi \) can be expressed as follows:
(i) \( \beta_+ = \beta_- = 0 \) and \( |\text{Re} \beta_r| < 1/2 \) for each \( r \);

(ii) \( \theta_r \in (-\pi, 0) \cup (0, \pi) \) for each \( r \) and \( \theta_r + \theta_s \neq 0 \) for each \( r \) and \( s \).

Hence the final result of this section will nearly treat the situation which was promised in the introduction. However, we still must exclude jumps at 1 and -1.

**Corollary 4.4** Let \( \phi \in PC_1[B^1; K] \) with \( K \cap \tilde{K} = \emptyset \) be a function of the form (3). Then

\[
F(\phi; f) = F[b]\prod_{r=1}^{R} \left(1 - e^{-i\theta_r}\right)^{\beta_r^2/2} \left(1 + e^{-i\theta_r}\right)^{\beta_r^2/2 - \beta_r/2} 
\times \prod_{r=1}^{R} b_+(e^{-i\theta_r})^{\beta_r} \prod_{1 \leq s < r \leq R} \left(1 - e^{-i(\theta_s + \theta_r)}\right)^{\beta_r \beta_s}.
\]

**Proof.** Using Theorem 2.5 and Lemma 4.3 we can evaluate \( F(\phi_\mu) \) and then pass to the limit \( \mu \to 1 \). It is easy to see that

\[
\log \phi_\mu(e^{i\theta}) = \log b(e^{i\theta}) + \sum_{r=1}^{R} \beta_r \left( \log(1 - \mu e^{i(\theta - \theta_r)}) - \log(1 - \mu e^{-i(\theta - \theta_r)}) \right).
\]

Employing the Taylor series expansion of \( \log(1 - z) \) at \( z = 0 \) we obtain

\[
[\log \phi_\mu]_n = [\log b]_n - \sum_{r=1}^{R} \beta_r \mu^n e^{-in\theta_r}/n, \quad n > 0,
\]

\[
[\phi_\mu]^+(e^{i\theta}) = b_+(e^{i\theta}) \prod_{r=1}^{R} \left(1 - \mu e^{i(\theta - \theta_r)}\right)^{\beta_r}.
\]

It follows that

\[
-\frac{1}{2} \sum_{n=1}^{\infty} n[\log \phi_\mu]^2_n = -\frac{1}{2} \sum_{n=1}^{\infty} n[\log b]^2_n + \sum_{r=1}^{R} \sum_{n=1}^{\infty} \beta_r [\log b]_n \mu^n e^{-in\theta_r} - \frac{1}{2} \sum_{1 \leq r, s \leq R} \sum_{n=1}^{\infty} \beta_r \beta_s \mu^{2n} e^{-in(\theta_r + \theta_s)}/n
\]

\[
= -\frac{1}{2} \sum_{n=1}^{\infty} n[\log b]^2_n + \sum_{r=1}^{R} \beta_r \log b_+(\mu e^{-i\theta_r}) + \frac{1}{2} \sum_{1 \leq r, s \leq R} \beta_r \beta_s \log(1 - \mu^2 e^{-i(\theta_r + \theta_s)})
\]

\[
\left(\frac{\phi_{\mu,+}(1)}{\phi_{\mu,+}(-1)}\right)^{1/2} = \left(\frac{b_+(1)}{b_+(-1)}\right)^{1/2} \prod_{r=1}^{R} \left(1 - \mu e^{-i\theta_r}/1 + \mu e^{-i\theta_r}\right)^{\beta_r/2}.
\]

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Now it is easy to complete the proof. □

**Theorem 4.5** Let \( \phi \in PC_{1}[B_1; K] \) with \( K \cap \tilde{K} = \emptyset \) be a function of the form (3). Then
\[
\det M_n(\phi) \sim G[b]^n n^{\Omega M} E_M \quad \text{as } n \to \infty.
\]

**Proof.** The previous results say that \( \det M_n(\phi)/\det T_n(\phi) \to F(\phi; f) \) where \( F(\phi; f) \) is as given in Corollary 4.4. Now we need only apply the asymptotic formula for Toeplitz determinants given in (15). □

## 5 The localization theorem for piecewise continuous functions

In this section, we establish a localization theorem for determinants of Toeplitz + Hankel matrices. We will show that the quotient \( \det M_n(\phi\psi)/\det M_n(\phi) \det M_n(\psi) \) converges to some nonzero constant for certain functions \( \phi \) and \( \psi \). Using this localization, we can reduce the asymptotics of \( \det M_n(\phi) \) for certain piecewise continuous functions to the asymptotics for functions of a particular form with only one or two “pure” jumps.

We first define certain constants in terms of operator determinants.

**Theorem 5.1** Let \( \phi, \psi \in L^\infty(\mathbb{T}) \).

(a) Suppose that \( T(\phi) \) and \( T(\psi) \) are invertible and \( H(\phi)H(\tilde{\psi}) \) is trace class. Then
\[
E(\phi, \psi) = \det \left( I + T^{-1}(\phi)H(\phi)H(\tilde{\psi})T^{-1}(\psi) \right) = \det T^{-1}(\phi)T(\phi\psi)T^{-1}(\psi)
\]

is well defined and nonzero. Moreover, \( E(\tilde{\psi}, \tilde{\phi}) \) is well defined and \( E(\tilde{\psi}, \tilde{\phi}) = E(\phi, \psi) \).

(b) Suppose that \( M(\phi) \) and \( M(\psi) \) are invertible and \( H(\phi)M(\tilde{\psi} - \psi) \) is trace class. Then
\[
G(\phi, \psi) = \det \left( I + M^{-1}(\phi)H(\phi)M(\tilde{\psi} - \psi)M^{-1}(\psi) \right) = \det M^{-1}(\phi)M(\phi\psi)M^{-1}(\psi)
\]

is well defined and nonzero.

(c) Suppose that \( T(\phi) \) and \( T(\psi) \) are invertible and \( H(\phi)H(\tilde{\psi}), H(\tilde{\phi})H(\psi) \) and \( H(\phi)H(\psi) \) are trace class. Then
\[
H(\phi, \psi) = E(\phi, \psi)E(\tilde{\phi}, \tilde{\psi})/E(\phi, \tilde{\psi})
\]

is well defined and nonzero. Moreover, \( H(\psi, \phi) \) is well defined and \( H(\phi, \psi) = H(\psi, \phi) \).
Proof. The equality of the expressions in (49) and (50) follow from (23) and (28). Note that the operator
\[ I + T^{-1}(\phi)H(\phi)H(\tilde{\psi})T^{-1}(\psi) = T^{-1}(\phi)T(\phi\psi)T^{-1}(\psi) \]  
(52)
is Fredholm with index zero. Hence so is \( T(\phi\psi) \). This implies that \( T(\phi\psi) \) is invertible. It follows that also (52) is invertible and thus \( E(\phi, \psi) \neq 0 \). We can argue similarly in the case of \( G(\phi, \psi) \neq 0 \). The point is that the fact that \( M(\phi\psi) \) is Fredholm with index zero implies the invertibility of \( M(\phi\psi) \) (see [7, Corollary 2.7]). The equality \( E(\phi, \psi) = E(\tilde{\phi}, \tilde{\psi}) \) is obtained by passing to the transposed operators in (49). Recall that the transposed operators of \( T(\phi) \) and \( H(\phi) \) are \( T(\tilde{\phi}) \) and \( H(\tilde{\phi}) \), respectively. Finally, part (c) follows directly from (a).

Obviously, if \( \phi \in B_1^1 \), then the above trace class conditions are fulfilled. However, the trace class conditions are also fulfilled under weaker assumptions.

Lemma 5.2 Let \( \phi, \psi \in L^\infty(\mathbb{T}) \).

(a) If there exists a smooth partition of unity, \( f + g = 1 \), such that \( f\phi \in B_1^1 \) and \( g\psi \in B_1^1 \), then \( H(\phi)H(\tilde{\psi}) \) is trace class.

(b) If there exists a smooth partition of unity, \( f + g = 1 \), such that \( f\phi \in B_1^1 \) and \( g\tilde{\psi} = g\psi \), then \( H(\phi)M(\tilde{\psi} - \psi) \) is trace class.

Proof. Using equation (24) we can write
\[ H(\phi)H(\tilde{\psi}) = H(\phi)T(f)H(\tilde{\psi}) + H(\phi)T(g)H(\tilde{\psi}) = \left( H(\phi f) - T(\phi)H(f) \right)H(\tilde{\psi}) + H(\phi) \left( H(g\tilde{\psi}) - H(g)T(\psi) \right) \]
(53)

Analogously, from equation (24), we can conclude
\[ H(\phi)M(\tilde{\psi} - \psi) = H(\phi)T(f)M(\tilde{\psi} - \psi) + H(\phi)T(g)M(\tilde{\psi} - \psi) = \left( H(\phi f) - T(\phi)H(f) \right)M(\tilde{\psi} - \psi) + H(\phi) \left( M(g\tilde{\psi} - g\psi) - H(g)M(\psi - \tilde{\psi}) \right) \]
(54)

Hence the operators under consideration are trace class.

The next result is a first (general) version of a localization theorem for Toeplitz + Hankel determinants. All further work is based on it.

Theorem 5.3 Let \( \phi, \psi \in L^\infty(\mathbb{T}) \) such that the sequences \( M_n(\phi) \) and \( M_n(\psi) \) are stable. Suppose in addition that \( H(\phi)M(\tilde{\psi} - \psi) \), \( H(\phi)H(\psi) \) and \( H(\tilde{\phi})H(\psi) \) are trace class. Then
\[ \lim_{n \to \infty} \frac{\det M_n(\phi\psi)}{\det M_n(\phi) \det M_n(\psi)} = G(\phi, \psi)E(\tilde{\phi}, \tilde{\psi}) \]

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Proof. We obtain from equation (28) that

\[ M_n(\phi \psi) = M_n(\phi)M_n(\psi) + P_nH(\phi)M(\tilde{\psi} - \psi)P_n + P_nM(\phi)Q_nM(\psi)P_n, \]

and from (35) and (36) it follows that

The trace class norm. Hence

Because \( V_n^* = V_n \to 0 \) strongly, it is easy to see that the last three terms tend to zero in the trace class norm. Hence

\[ M_n(\phi \psi) = M_n(\phi)M_n(\psi) + P_nH(\phi)M(\tilde{\psi} - \psi)P_n + W_nH(\tilde{\phi})H(\psi)W_n + C_n, \]

where \( C_n \to 0 \) in the trace class norm. Multiplying with the inverses of \( M_n(\phi) \) and \( M_n(\psi) \) and applying Proposition 3.3(c) and Corollary 3.2, it follows that

\[ M_n^{-1}(\phi)M_n(\phi \psi)M_n^{-1}(\psi) = P_n + P_nM_n^{-1}(\phi)P_nH(\phi)M(\tilde{\psi} - \psi)P_n + W_n\left(W_nM_n^{-1}(\phi)W_n\right)H(\tilde{\phi})H(\psi)\left(W_nM_n^{-1}(\psi)W_n\right)W_n + C'_n \]

\[ = P_n + P_nM_n^{-1}(\phi)H(\tilde{\phi})H(\psi)M_n^{-1}(\psi)P_n + W_nT^{-1}(\tilde{\phi})H(\tilde{\phi})H(\psi)T^{-1}(\tilde{\psi})W_n + C''_n \]

with \( C'_n \to 0 \) and \( C''_n \to 0 \) in the trace class norm. Lemma 3.7 completes the proof.

Note that the trace class assumptions required in the theorem can be replaced by the conditions given in Lemma 5.2.

Now we proceed with establishing basic properties of the operator determinants \( E(\ast, \ast) \) and \( G(\ast, \ast) \) in the case of smooth functions. These properties allow their computation. Note that the constant \( E(\ast, \ast) \) is already known for a long time [3].

**Theorem 5.4** Let \( b, c \in G_0B_1^1 \). Then

\[ E(b, c) = F(b)F(c)/F(bc), \quad (55) \]

\[ G(b, c) = E(b, c)/E(b, \tilde{c}). \quad (56) \]

In particular, we have

\[ E(b, c) = \exp \left( \sum_{n=1}^{\infty} n[\log b]_n[\log c]_{-n} \right). \quad (57) \]
Proof. We start with proving (55). Suppose first that \( b = \tilde{b} \) and \( c = \tilde{c} \). Because of (29) we have \( M(bc) = M(b)M(c) \). Using the definition (33) of \( F(\ast) \), we can write

\[
F(b)F(c) = \det T^{-1}(b)M(b)\det T^{-1}(c)M(c) = \det T^{-1}(b)M(b)\det M(c)T^{-1}(c) = \det T^{-1}(b)M(b)M(c)T^{-1}(c) = \det T^{-1}(c)T^{-1}(b)T(bc)\det T^{-1}(bc)M(bc) = E(b,c)F(bc).
\]

Hence \( F(b)F(c) = E(b,c)F(bc) \). Now consider arbitrary \( b \) and \( c \). Let \( b = b_{+}G[b]b_{+} \) and \( c = c_{-}G[c]c_{-} \) be the Wiener–Hopf factorization. In the proof of Theorem 2.5, we have shown that \( F(b) = F(\tilde{b}_{+}b_{+}) \). Analogously, \( F(\tilde{c}) = F(c_{-}\tilde{c}_{-}) \) and \( F(b\tilde{c}) = F(\tilde{b}_{+}b_{+}c_{-}\tilde{c}_{-}) \). Doing a similar computation as in (34), we obtain

\[
T^{-1}(b)H(b)H(\tilde{c})T^{-1}(c) = T^{-1}(b_{+})H(b_{+})H(\tilde{c}_{-})T^{-1}(c_{-}).
\]

Hence \( E(b,c) = E(b_{+},c_{-}) = E(\tilde{b}_{+}b_{+},c_{-}\tilde{c}_{-}) \). Because \( \tilde{b}_{+}b_{+} \) and \( c_{-}\tilde{c}_{-} \) are even, we can apply the above results. It follows \( E(b,c) = E(\tilde{b}_{+}b_{+},c_{-}\tilde{c}_{-}) = F(\tilde{b}_{+}b_{+})F(c_{-}\tilde{c}_{-})/F(\tilde{b}_{+}b_{+}c_{-}\tilde{c}_{-}) = F(b)F(\tilde{c})/F(b\tilde{c}) \). Note that (53) implies (57) by using Theorem 2.3.

We are now going to prove (56). We write

\[
G(b,c) = \det M^{-1}(b)M(bc)M^{-1}(c) = \det M^{-1}(b)T(b)\det T^{-1}(b)M(bc)T^{-1}(c)\det T(c)M^{-1}(c) = \det T^{-1}(c)T^{-1}(b)M(bc)/(F(b)F(c)) = \det T^{-1}(c)T^{-1}(b)T(bc)\det T^{-1}(bc)M(bc)/(F(b)F(c)) = E(b,c)F(bc)/(F(b)F(c)) = E(b,c)/E(b,\tilde{c}).
\]

Here we have used the equations (33), (49), (50) and (55). \( \square \)

Next we address the question under which conditions \( E(\ast,\ast) \) and \( G(\ast,\ast) \) are well defined for certain piecewise continuous functions and how to evaluate them.

**Lemma 5.5** Let \( K \) and \( L \) be finite subsets of \( \mathbb{T} \).

(a) Let \( \phi \in PC_{1}[B_{1};K] \) and \( \psi \in PC_{1}[B_{1};L] \) with \( K \cap L = \emptyset \), and let \( \phi_{\mu} \) and \( \psi_{\mu} \), \( 0 \leq \mu < 1 \), be the approximate functions associated to \( \phi \) and \( \psi \). Then \( E(\phi,\psi) \) is well defined, and

\[
E(\phi,\psi) = \lim_{\mu_{1} \to 1} \lim_{\mu_{2} \to 1} E(\phi_{\mu_{1}},\psi_{\mu_{2}}) = \lim_{\mu_{2} \to 1} \lim_{\mu_{1} \to 1} E(\phi_{\mu_{1}},\psi_{\mu_{2}}).
\]

(b) Let \( \phi \in PC_{II}[B_{1};K] \) and \( \psi \in PC_{II}[B_{1};L] \), assume that there exists an open neighborhood \( U \) of \( K \) such that \( \psi|_{U} \equiv \psi|_{U} \), and let \( \phi_{\mu} \) and \( \psi_{\mu} \), \( 0 \leq \mu < 1 \), be the approximate functions associated to \( \phi \) and \( \psi \). Then \( G(\phi,\psi) \) is well defined, and

\[
G(\phi,\psi) = \lim_{\mu_{1} \to 1} \lim_{\mu_{2} \to 1} G(\phi_{\mu_{1}},\psi_{\mu_{2}}).
\]
Proof. (a) Because $K$ and $L$ are disjoint, there exists a smooth partition of unity, $f + g = 1$, such that $f$ vanishes identically on an open neighborhood of $K$ and $g$ vanishes on an open neighborhood of $L$. With this partition the assumptions of Lemma 5.2(a) are fulfilled, and hence $H(\phi)H(\bar{\psi})$ is trace class. We write this operator and $H(\phi_{\mu_1})H(\bar{\psi}_{\mu_2})$ in the form (23), and conclude from Proposition 3.4 that $H(\phi_{\mu_1})H(\bar{\psi}_{\mu_2}) \rightarrow H(\phi)H(\bar{\psi})$ in the trace class norm as $\mu_1 \rightarrow 1$ and $\mu_2 \rightarrow 1$. Finally, we apply Proposition 3.5(a). Note that the order of $\mu_1$ and $\mu_2$ in the limit does not play a role.

(b) We choose a smooth partition of unity, $f + g = 1$, such that $f$ vanishes identically on an open neighborhood of $K$ and $g$ vanishes on $T \setminus U$. Because $f\phi \in B_1^1$ and $g\bar{\psi} = g\psi$, Lemma 5.2(b) can be applied, and thus the constant $G(\phi, \psi)$ is well defined. As $H(\phi_{\mu_1})$ is trace class, we conclude from Proposition 3.4 and Proposition 3.5(b) that for fixed $\mu_1$

$$M^{-1}(\phi_{\mu_1})H(\phi_{\mu_1})M(\bar{\psi}_{\mu_2} - \psi_{\mu_2})M^{-1}(\psi_{\mu_2}) \rightarrow M^{-1}(\phi_{\mu_1})H(\phi_{\mu_1})M(\bar{\psi} - \psi)M^{-1}(\psi)$$

in the trace class norm as $\mu_2 \rightarrow 1$. Writing $H(\phi_{\mu_1})M(\bar{\psi} - \psi)$ in the form (24), we obtain from Proposition 3.4 and Proposition 3.5(b) that

$$M^{-1}(\phi_{\mu_1})H(\phi_{\mu_1})M(\bar{\psi} - \psi)M^{-1}(\psi) \rightarrow M^{-1}(\phi)H(\phi)M(\bar{\psi} - \psi)M^{-1}(\psi)$$

in the trace class norm as $\mu_1 \rightarrow 1$. This completes the proof. \square

The previous result reduces the evaluation of the constants for piecewise continuous functions to the case of smooth functions. However, we must warn the reader to be careful with the choice of $\phi_\mu$ and $\psi_\mu$. (see the remark after Proposition 3.3). It may happen, for instance, that given a function $\phi \in PC_1[B_1^1; K] \cap PC_1[B_1^1; L]$, the associated approximate functions which satisfy (I) or (II), respectively, are not the same. Later on, we will restrict to condition (III), and then this nonuniqueness does not occur.

Now we establish further relations between these constants.

Lemma 5.6 Let $K$, $L$ and $M$ be finite subsets of $T$.

(a) Let $\phi \in PC_1[B_1^1; K]$, $\psi_1 \in PC_1[B_1^1; L]$, $\psi_2 \in PC_1[B_1^1; M]$, and suppose $K \cap (L \cup M) = L \cap M = \emptyset$. Then $E(\phi, \psi_1 \psi_2) = E(\phi, \psi_1)E(\phi, \psi_2)$. If in addition, $\bar{K} \cap (L \cup M) = \emptyset$, then $H(\phi, \psi_1 \psi_2) = H(\phi, \psi_1)H(\phi, \psi_2)$.

(b) Let $\phi \in PC_3[B_1^1; K]$ and $\psi \in PC_3[B_1^1; L]$ with $K \cap L = K \cap \bar{L} = \emptyset$, and suppose that there exists an open neighborhood $U$ of $K$ such that $\psi|_U \equiv \bar{\psi}|_U$. Then $G(\phi, \psi) = E(\phi, \psi)/E(\phi, \bar{\psi})$.

Proof. (a) It follows from (27) that the first relation holds for smooth $\phi$, $\psi_1$ and $\psi_2$. The general case can is obtained from Lemma 5.3(a) by approximation. Note that $\psi_1 \psi_2 \in PC_1[B_1^1; L \cup M]$ and $(\psi_1 \psi_2)_\mu = \psi_{1,\mu} \psi_{2,\mu}$ because of $L \cap M = \emptyset$. Finally, the relation for $H(*,*)$ follows from the definition by using the relation for $E(*,*)$.
(b) The equality for smooth functions is stated in (56). In the general case, we approximate by smooth functions as indicated in Lemma 5.5(ab). Note that the approximate functions φµ and ψµ, respectively, are the same for G(*,*) and E(*,*).

Now we establish a first version of the localization theorem for the piecewise continuous functions.

**Corollary 5.7** Let φ ∈ PCIII[B1; K] and ψ ∈ PCIII[B1; L] with K and L being finite subsets of T such that K ∩ L = K ∩ L = ∅. Suppose also that there exists an open neighborhood U of K such that ψ|U = ˜ψ|U. Then

\[
\lim_{n \to \infty} \frac{\det M_n(\phi \psi)}{\det M_n(\phi) \det M_n(\psi)} = H(\phi, \psi).
\]

**Proof.** Similar as in the proof of Lemma 5.6 (see also Lemma 5.2) one can show that the conditions on K, L and U imply that the operators H(φ)H(ψ), H(φ)H(ψ) and H(φ)M(˜ψ − ψ) are trace class. Moreover, because of Proposition 3.3, the sequences M_n(φ) and M_n(ψ) are stable. Hence all the assumptions required in Theorem 5.3 are fulfilled. Note that H(φ, ψ) = G(φ, ψ)E(˜ψ, ˜φ) by Lemma 5.6(b).

The previous result requires a “strange” assumption, namely, ψ|U = ˜ψ|U. Note that in case K = ∅ (i.e. φ ∈ G0B1), one can choose U = ∅, and hence this assumption is redundant. In fact, it turns out that this requirement can be dropped also in general, as shown in the following lemma. For technical reasons we sharpen the smoothness condition.

**Lemma 5.8** Let K and L be finite subsets of T with K ∩ L = K ∩ L = ∅, and suppose that ψ ∈ PC[CI∞; L]. Then we can factor ψ = ψ1ψ2 such that ψ1 ∈ CI∞(T) is nonvanishing and has winding number zero, and that there exists an open neighborhood U of K such that ψ2|U ≡ ˜ψ2|U.

**Proof.** Let U and V be open and disjoint neighborhoods of K and L, respectively, which are both non-empty, consist of a finite union of open subarcs, and satisfy ˜U = U and ˜V = V. We put ψ1(t) = ˜ψ(t) for t ∈ U, and continue ψ1 on T \ U such that ψ1 ∈ CI∞(T) is nonvanishing and has winding number zero. This is possible because ψ is infinitely differentiable and nonzero on T \ L ⊇ T \ V ⊇ closU. The winding number condition can be fulfilled by choosing the values of ψ appropriately on some subarc of V. Finally put ψ2 = ψ/ψ1. Note that ψ2|U ≡ ˜ψ2|U ≡ 1.

**Corollary 5.9** Let φ ∈ PCIII[CI∞; K] and ψ ∈ PCIII[CI∞; L] with K and L being finite subsets of T satisfying K ∩ L = K ∩ L = ∅. Then

\[
\lim_{n \to \infty} \frac{\det M_n(\phi \psi)}{\det M_n(\phi) \det M_n(\psi)} = H(\phi, \psi).
\]
Proof. We factor \( \psi = \psi_1 \psi_2 \) as stated in the Lemma 5.8, and write
\[
\frac{\det M_n(\phi \psi)}{\det M_n(\phi) \det M_n(\psi)} = \frac{\det M_n(\phi \psi_1 \psi_2)}{\det M_n(\phi_1) \det M_n(\psi_2)} \times \frac{\det M_n(\phi \psi_2)}{\det M_n(\phi) \det M_n(\psi_2)}.
\]
Because of the conditions on \( K \) and \( L \) and on the function \( \psi_1 \), Proposition 3.3 shows that the stability of \( M_n(\phi) \) and \( M_n(\psi) \) implies the stability of all other sequences \( M_n(*) \) which occur above. Employing Corollary 5.7 one can take the limit of the above expression, which equals
\[
\frac{H(\psi_1, \psi_2)H(\phi, \psi_2)}{H(\psi_1, \psi_2)} = H(\psi_1, \phi)H(\phi, \psi_2) = H(\phi, \psi).
\]
Note that Lemma 5.6(a) and Theorem 5.1(c) has been used here. \( \square \)

Corollary 5.10 Let \( b \in G_0 B_1^1, \phi(e^{i\theta}) = t_{\beta}(e^{i(\theta - \theta_1)} \) and \( \psi(e^{i\theta}) = t_{\beta_2}(e^{i(\theta - \theta_2)} \). Assume that \( |\text{Re} \beta_1| < 1/2, |\text{Re} \beta_2| < 1/2, \theta_1, \theta_2 \in (-\pi, \pi], \theta_1 \neq \theta_2 \) and \( \theta_1 + \theta_2 \neq 0 \). Then
\[
H(b, \psi) = b_+(e^{i\theta_2})^\beta_1 b_-(e^{i\theta_2})^{-\beta_2} b_+(e^{-i\theta_2})^\beta_2,
\]
\[
H(\phi, \psi) = \left(1 - e^{i(\theta_1 - \theta_2)}\right)^{\beta_1 \beta_2} \left(1 - e^{i(\theta_2 - \theta_1)}\right)^{\beta_2} \left(1 - e^{-i(\theta_1 + \theta_2)}\right)^{\beta_2}.
\]
Here \( b_\pm \) are the functions defined as in (31) and (32).

Proof. The calculation is similar to the one given in the proof of Corollary 4.4. We are using (31) and Lemma 5.3(a). The functions \( \phi \) and \( \psi \) are approximated by \( \phi_\mu \) and \( \psi_\mu \). We obtain
\[
E(b, \psi) = b_+(e^{i\theta_2})^\beta_1,
\]
\[
E(\phi, \psi) = \left(1 - e^{i(\theta_1 - \theta_2)}\right)^{\beta_1 \beta_2}.
\]
Because of \( t_\beta(e^{-i\theta}) = t_{-\beta}(e^{i\theta}) \), we have \( \tilde{\phi}(e^{i\theta}) = t_{-\beta}(e^{i(\theta + \theta_1)} \) and \( \tilde{\psi}(e^{i\theta}) = t_{-\beta}(e^{i(\theta_1 + \theta_2)}) \). The values of the constant \( H(*) \) follow now immediately. \( \square \)

Now we establish the main result of this section. We consider functions \( \phi \in \text{PC}_{III}[B_1^1; K] \) of the form (31) and localize as much as possible. This allows us to eliminate functions with pure jumps at 1 or at -1. However, using this localization technique one cannot separate singularities at both a point on the unit circle and its complex conjugate.

Theorem 5.11 Let \( \phi \) be a function of the form
\[
\phi(e^{i\theta}) = b(e^{i\theta})\phi_1(e^{i\theta})\phi_2(e^{i\theta}) \prod_{r=1}^{R} \phi_r(e^{i\theta}), \quad (58)
\]
where \( b \in G_0 B^1_1 \) and
\[
\phi^+(e^{i\theta}) = t_{\beta_+}(e^{i\theta}), \\
\phi^-(e^{i\theta}) = t_{\beta_-}(e^{i(\theta - \pi)}), \\
\phi_r(e^{i\theta}) = t_{\beta_+}^r(e^{i(\theta - \theta_r)})t_{\beta_-}^r(e^{i(\theta + \theta_r)}), \quad 1 \leq r \leq R.
\]

Suppose that \( \theta_1, \ldots, \theta_R \in (0, \pi) \) are distinct numbers and that
(a) \(-1/2 < \text{Re} \beta_+ < 1/4 \) and \(-1/4 < \text{Re} \beta_- < 1/2; \)
(b) \( |\text{Re} \beta_r^+| < 1/2 \) and \( |\text{Re} \beta_r^-| < 1/2 \) and \( |\text{Re} (\beta_r^+ + \beta_r^-)| < 1/2 \) for each \( 1 \leq r \leq R. \)

Then
\[
\lim_{n \to \infty} \frac{\det M_n(\phi)}{\det M_n(b) \det M_n(\phi^+) \det M_n(\phi^-) \prod_{r=1}^R \det M_n(\phi_r)} = H,
\]
where
\[
H = b_+(1)^{2\beta_+} b_-(1)^{-\beta_+} b_-(1)^{-2\beta_-} b_+(1)^{-\beta_-} 2^{3\beta_+ \beta_-}
\times \prod_{r=1}^R b_+(e^{i\theta_r})^{\beta_+^r + \beta_-^r} b_-(e^{-i\theta_r})^{-\beta_+^r} b_+(e^{-i\theta_r})^{\beta_+^r + \beta_-^r} b_-(e^{i\theta_r})^{\beta_-^r}
\times \prod_{r=1}^R (1 - e^{i\theta_r})^{\beta_+(\beta_+^r + 2\beta_-^r)} (1 - e^{-i\theta_r})^{\beta_+(2\beta_+^r + \beta_-^r)}
\times \prod_{r=1}^R (1 + e^{i\theta_r})^{\beta_-(\beta_+^r + 2\beta_-^r)} (1 + e^{-i\theta_r})^{\beta_-(2\beta_+^r + \beta_-^r)}
\times \prod_{1 \leq r < s \leq R} (1 - e^{i(\theta_r + \theta_s)})^{\beta_r^+ \beta_s^+ + \beta_r^- \beta_s^- + \beta_r^+ \beta_s^- + \beta_r^- \beta_s^+}
\times \prod_{1 \leq r < s \leq R} (1 - e^{-i(\theta_r + \theta_s)})^{\beta_r^+ \beta_s^+ + \beta_r^- \beta_s^- + \beta_r^+ \beta_s^- + \beta_r^- \beta_s^+}.
\]

Proof. Note that \( \phi \) belongs to \( PC_{\text{III}}[B^1_1; K] \) with \( K \subseteq \{1, -1, e^{i\theta_1}, \ldots, e^{i\theta_R}, e^{-i\theta_1}, \ldots, e^{-i\theta_R}\} \)
being the set of jump discontinuities of \( \phi \), and consequently, so does each product which involves only some of the factors appearing in (58). We first apply Corollary 5.7 and eliminate the factor \( b \). This yields a constant term
\[
H(b, \phi^+)H(b, \phi^-) \prod_{r=1}^R H(b, \phi_r)
\]
in the asymptotics (see also Lemma 5.4(a)). The remaining function (i.e. \( \phi \) with \( b \) being dropped) is contained even in \( PC_{\text{III}}[C^\infty; K] \), and we may
6 Functions with one jump and the main theorem

As noted in previous sections, using the localization technique and the limit theorem, we have reduced the computation of the asymptotics of the determinants for all piecewise continuous functions (satisfying appropriate conditions on the size of the jumps) to those with pure jumps at 1 or $-1$, and to those with two jumps at a point on the unit circle and its complex conjugate.

In this section we compute the asymptotics of the corresponding determinants for the functions $t_\beta(e^{i\theta})$ and $t_\beta(e^{i(\theta-\pi)})$ and thus with these examples the promised asymptotic formula given in the introduction is proved.

It is interesting to note that we are able to describe the asymptotic behavior of the determinants for the above pure function with arbitrary complex parameters $\beta$. Notice that we may exclude the cases $\beta \in \mathbb{Z}$ as they lead to trivial results.

In computing these examples, and also in the next section where other interesting examples are computed, several Cauchy type determinants arise. The next lemma shows how to evaluate several of the products that occur in the Cauchy determinants in terms of the Barnes G-function and how to then evaluate the asymptotics that arise from the Barnes G-function. It will be used several times in this section and the next.

Lemma 6.1 (a) For each nonnegative integer $n$ and each $z \notin \mathbb{Z}$ we have
\[ G(1+z-n) G(1+z) = (-1)^{n(n-1)/2} \left( \frac{\sin \pi z}{\pi} \right)^n \frac{G(1-z+n)}{G(1-z)}. \] (59)

(b) If $x_1 + \ldots + x_R = y_1 + \ldots + y_R$ and $\omega := x_1^2 + \ldots + x_R^2 - y_1^2 - \ldots - y_R^2$, then
\[ \prod_{r=1}^R \frac{G(1+x_r+n)}{G(1+y_r+n)} \sim n^{\omega/2}, \quad \text{as } n \to \infty. \] (60)

(c) For each nonnegative integers $n_1$, $n_2$ and $n$, the following identities hold:
\[ \prod_{0 \leq j < k \leq n-1} (k-j) = G(1+n), \] (61)
\[ \prod_{0 \leq j < k \leq n-1} (k+j+z) = \frac{G(2n-1+z)G(1+z)}{G(n+z)G(n-1+z)} \frac{G(1+z)n!}{2^{n-1}}, \] (62)
\[ \prod_{0 \leq k_1 \leq n_1-1} G(z+n_1+k_2) = \frac{G(z+n_1+n_2)G(z)}{G(z+n_1)G(z+n_2)}, \] (63)
\[ \prod_{0 \leq k_2 \leq n_2-1} (z+k_1-k_2) = \frac{G(1+z+n_1)G(1-z+n_2)}{G(1+z+n_1-n_2)G(1-z)}(-1)^{n_2(n_2-1)/2} \left( \frac{\sin \pi z}{\pi} \right)^{n_2}. \] (64)
Here we assume $z \notin \{0, -1, -2, \ldots\}$ in (52) and (53), and $z \notin \mathbb{Z}$ in (54).

**Proof.** (a) Using the recurrence relation $G(1 + z) = \Gamma(z)G(z)$ and the well known formula $\Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z$, we can write

$$
\frac{G(1 + z - n)}{G(1 + z)} = \prod_{k=1}^{n} \frac{1}{\Gamma(1 + z - k)} = \prod_{k=1}^{n} \frac{\sin \pi(k - z)}{\pi} \Gamma(k - z)
$$

$$
= \prod_{k=1}^{n} (-1)^{k-1} \frac{\sin \pi z}{\pi} \Gamma(k - z)
$$

$$
= (-1)^{n(n-1)/2} \left( \frac{\sin \pi z}{\pi} \right)^n \frac{G(1 + z + n)}{G(1 - z)}.
$$

(b) In [12, Proof of Corollary 3.2] it is shown that

$$
G(1 + z + n) \sim a_n b^n z^{n+1}, \quad \text{as } n \to \infty,
$$

where $a_n$ and $b_n$ are sequences of positive numbers not depending on $z$.

(c) Noting that $G(1) = 1$, formula (61) can be proved as follows:

$$
\prod_{j=0}^{n-2} \prod_{k=j+1}^{n-1} (k - j) = \prod_{j=0}^{n-2} \Gamma(n - j) = G(1 + n).
$$

Formula (52) can be proved in the same sort of way, but requires a little more work. First write

$$
\prod_{0 \leq j < k \leq n-1} (k + j + z) = \prod_{j=0}^{n-2} \prod_{k=j+1}^{n-1} (j + k + z) = \prod_{j=0}^{n-2} \frac{\Gamma(j + n + z)}{\Gamma(2j + 1 + z)}.
$$

Now we can write this last product as two products, and then apply the duplication formula for the Gamma function. We have

$$
\prod_{j=0}^{n-2} \frac{1}{\Gamma(2j + 1 + z)} = \frac{G(2n - 1 + z)}{G(n + z)} \prod_{j=0}^{n-2} \frac{\pi^{1/2}}{\Gamma(j + 1 + \frac{z}{2}) \Gamma(j + 1 + \frac{z}{2})^{2^{2j+z}}},
$$

and using the basic properties of the Barnes function this is equal to

$$
\frac{G(2n - 1 + z)G(\frac{1}{2} + \frac{z}{2})G(1 + \frac{z}{2})\pi^{\frac{z-1}{2}}}{G(n + z)G(n - \frac{1}{2} + \frac{z}{2})G(n + \frac{z}{2})^{2(n-1)(n-2+z)}}.
$$
Formula (63) can be shown by writing
\[ \prod_{k_1=0}^{n_1-1} \prod_{k_2=0}^{n_2-1} (z + k_1 + k_2) = \prod_{k_1=0}^{n_1-1} \frac{\Gamma(z + n_2 + k_1)}{\Gamma(z + k_1)} = \frac{G(z + n_1 + n_2)G(z)}{G(z + n_1)G(z + n_2)}. \]

In order to obtain (64), we make an index substitution, and then use (63):
\[ \prod_{0 \leq k_1 \leq n_1-1} \prod_{0 \leq k_2 \leq n_2-1} (z + k_1 - k_2) = \prod_{0 \leq k_1 \leq n_1-1} \prod_{0 \leq k_2 \leq n_2-1} (z + k_1 + k_2 - n_2 + 1) = \frac{G(1 + z + n_1)G(1 + z - n_2)}{G(1 + z + n_1 - n_2)G(1 + z)}. \]

Finally, we apply (59). □

We now prove the asymptotic formula for the special function \( t_\beta(e^{i\theta}) \). We first note that if \( A \) is a matrix of Cauchy type, that is, if \( \{a_j\}_{j=0}^{n-1} \) and \( \{b_k\}_{k=0}^{n-1} \) are sequences of complex numbers such that the following matrix is well defined
\[ A = \begin{bmatrix} (a_j + b_k)^{-1} \end{bmatrix}_{j,k=0}^{n-1}, \]
then \( \det A = p/q \) where
\[ p = \prod_{0 \leq j < k \leq n-1} (a_k - a_j)(b_k - b_j), \]
\[ q = \prod_{0 \leq j,k \leq n-1} (a_j + b_k). \]

We will use this identity in the next theorem and also for many of the examples that follow.

**Theorem 6.2** Let \( \phi(e^{i\theta}) = t_\beta(e^{i\theta}) \) and assume \( \beta \notin \mathbb{Z} \). Then
\[ \det M_n(\phi) \sim n^{-3\beta^2/2-\beta/2}(2\pi)^{\beta/2}2^{3\beta^2/2}G(1/2)^{-1}G(1/2 - \beta)G(1 - \beta)G(1 + \beta). \]
Moreover, \( \det M_n(\phi) = 0 \) if and only if \( \beta \in \{1/2, 3/2, \ldots, n - 1/2\} \).

**Proof.** The Fourier coefficients of \( \phi = t_\beta \) are given by
\[ [t_\beta]_n = \frac{\sin \pi \beta}{\pi (\beta - n)}. \] (65)

Thus the matrices \( T_n(t_\beta) + H_n(t_\beta) \) have \( j, k \) entry
\[ \frac{\sin \pi \beta}{\pi} \left( \frac{1}{\beta - j + k} + \frac{1}{\beta - 1 - j - k} \right) = \frac{\sin \pi \beta}{\pi} \cdot \frac{2\beta - 2j - 1}{\beta^2 - \beta - 2\beta j + j + j^2 - k - k^2}, \] (66)
and except for terms that can be factored out of rows, the corresponding determinant is a Cauchy determinant of the above form with \( a_j = \beta^2 - \beta - 2\beta j + j^2 \) and \( b_k = -k - k^2 \).

Our remarks above concerning Cauchy determinants show that

\[
\det M_n(t_\beta) = \frac{p}{q} \left( \frac{\sin \pi \beta}{\pi} \right)^n \prod_{j=0}^{n-1} (2\beta - 2j - 1),
\]

where

\[
p = \prod_{0 \leq j < k \leq n-1} (k^2 + k - 2\beta k - j^2 - j + 2\beta j)(-k^2 - k + j^2 + j)
\]

and

\[
q = \prod_{0 \leq j, k \leq n-1} (-\beta + j - k)(1 - \beta + j + k).
\]

We can evaluate the \( p \) term by first writing

\[
p = \prod_{0 \leq j < k \leq n-1} (k-j)(k+j+1)(k+j+1-2\beta)(-1).
\]

Then we apply (61) and (62) to find that \( p \) is equal to

\[
(-1)^{n(n-1)/2} \frac{\pi^{n-1}G(1+n)^2G(2n)G(1)G(\frac{3}{2})G(2n-2\beta)G(1-\beta)G(\frac{1}{2} - \beta)}{2^{2(n-1)(1-\beta)}G(n+1)G(n)G(n+\frac{1}{2})G(n+1-2\beta)G(n-\beta)G(n+\frac{1}{2} - \beta)}.
\]

To evaluate the \( q \) term use (63) and (64) to see that

\[
q = \frac{G(1-\beta+2n)G(1-\beta)G(1-\beta+n)G(1+\beta+n)}{G(1-\beta+n)^2G(1-\beta)G(1+\beta)}(-1)^{n(n+1)/2} \left( \frac{\sin \pi \beta}{\pi} \right)^n.
\]

We write the product in (67) as

\[
\prod_{j=0}^{n-1} (2\beta - 2j - 1) = (-1)^n 2^n \frac{\Gamma(n+\frac{1}{2} - \beta)}{\Gamma(\frac{1}{2} - \beta)},
\]

and then simplify and collect terms to obtain

\[
\det M_n(t_\beta) = \pi^{n-1} 2^{n-2(n-1)(n-1-\beta)} \frac{\Gamma(n+\frac{1}{2} - \beta)}{\Gamma(\frac{1}{2} - \beta)} \times \frac{G(1+n)G(2n)G(\frac{3}{2})G(2n-2\beta)G(1-\beta)G(\frac{3}{2} - \beta)}{G(n)G(n+\frac{1}{2})G(n+1-2\beta)G(n-\beta)G(n+\frac{1}{2} - \beta)} \times \frac{G(1-\beta+n)G(1+\beta)}{G(1-\beta+2n)G(1+\beta+n)}.
\]
In the above expression group together the terms with 2n factors, that is consider
\[
\frac{G(2n)G(2n - 2\beta)}{G(1 - \beta + 2n)}.
\]

We wish to apply (60) so we multiply and divide by \(G(2n - \beta - 1)\) to find that the terms involving 2n are asymptotic to
\[
(2n)^{3\beta - 1} G(2n - \beta - 1).
\]

Before we evaluate the rest asymptotically it is convenient to use the duplication formula for the Barnes G-function \(G\) which reads
\[
G(z)G(z + \frac{1}{2})^2 = G(\frac{1}{2})^2 \pi z^{2(-2z^2 + 3z - 1)} G(2z).
\]

We let \(z = n - \beta/2 - 1/2\) in this formula to obtain
\[
G(2n - \beta - 1) = \pi^{-n + \beta/2 + 1/2} 2^{2n^2 - 2n\beta - 5n + \beta^2/2 + 5\beta/2 + 3} \times G(n - \frac{\beta}{2} - \frac{1}{2})G(n - \frac{\beta}{2}) G(n - \frac{\beta}{2} + \frac{1}{2}) G(\frac{1}{2})^{-2}
\]

and then use this substitution in our formula for \(\det M_n(t_\beta)\).

So at this point we have, gathering all terms, that \(\det M_n(t_\beta)\) is asymptotically
\[
n^{-3\beta^2/2 - \beta/2} (2\pi)^{\beta/2} 2^{3\beta^2/2} G(\frac{1}{2})^{-1} G(1 - \beta) G(\frac{1}{2} - \beta) G(1 + \beta).
\]

Write \(\Gamma(n + \frac{1}{2} - \beta) = G(n + \frac{\beta}{2} - \beta)/G(n + \frac{1}{2} - \beta)\) and apply (68) to the above expression in a straightforward way to finally arrive at
\[
\det M_n(t_\beta) \sim n^{-3\beta^2/2 - \beta/2} (2\pi)^{\beta/2} 2^{3\beta^2/2} G(\frac{1}{2})^{-1} G(1 - \beta) G(\frac{1}{2} - \beta) G(1 + \beta).
\]

This completes the proof of the asymptotic formula. Finally note that the determinant vanishes if and only if the \(p\) term or the product (68) vanish.

For functions that have jumps discontinuities at the point \(-1\), the analogous results of above theorem are contained in the following.

**Theorem 6.3** Let \(\phi(e^{i\theta}) = t_\beta(e^{i(\theta - \pi)})\) and assume \(\beta \notin \mathbb{Z}\). Then
\[
\det M_n(\phi) \sim n^{-3\beta^2/2 + \beta/2} (2\pi)^{\beta/2} 2^{3\beta^2/2} G(3/2)^{-1} G(3/2 - \beta) G(1 - \beta) G(1 + \beta).
\]

Moreover, \(\det M_n(\phi) = 0\) if and only if \(\beta \in \{3/2, 5/2, \ldots, n - 1/2\}\).
Proof. The only effect of the move of the discontinuity is that we need to now evaluate the determinant of $T_n(t_\beta) - H_n(t_\beta)$ since the Fourier coefficients change by a factor of $(-1)^n$. In the computation in the previous theorem this replaces the factor $(2\beta - 2j - 1)$ which appears in the numerator of the matrices (66) with $(-2k - 1)$. A simple check of the computation shows that this only changes the first product in the computation, i.e. the term $\Gamma(n+1/2-\beta)/\Gamma(1/2-\beta)$ appearing in (67) and in the formulas afterwards has to be replaced by the factor $\Gamma(n+1/2)/\Gamma(1/2)$. We leave the details to the reader. 

With these two theorems we have completed all the parts of pieces that go together to prove formula (7). For completeness sake we now state the main result with all the necessary restrictions on the $\beta$ parameters. The theorem follows directly from the two theorems of this section combined with the localization result of the last section and finally the limit theorem of Section 4.

Theorem 6.4 (Main theorem) Let $\phi$ be a function of the form

$$\phi(e^{i\theta}) = b(e^{i\theta})t_{\beta_+}(e^{i\theta})t_{\beta_-}(e^{i(\theta-\pi)}),$$

where $b \in G_0B_1^1$ and where $\theta_1, \ldots, \theta_R \in (-\pi, 0) \cup (0, \pi)$ are distinct numbers satisfying $\theta_r + \theta_s \neq 0$ for each $r$ and $s$. Assume also that

(a) $-1/2 < \text{Re} \beta_+ < 1/4$ and $-1/4 < \text{Re} \beta_- < 1/2$;

(b) $|\text{Re} \beta_r| < 1/2$ for each $1 \leq r \leq R$.

Then as $n \to \infty$,

$$\det M_n(\phi) \sim G[b] n^{\Omega_M} E_M,$$

where the constants $G[b], \Omega_M$ and $E_M$ are defined as in (8), (9) and (10).

7 Other interesting examples

In this section we evaluate the asymptotics of the determinant $\det M_n(\phi)$ for some classes of generating functions with two jumps and for a class of functions that have a singularity of a different type. The functions we consider here are special cases where certain assumptions on the location and the size of the singularities are supposed. Only one class of functions considered here and then only for certain values of the parameters is covered by the previous theorems. We begin by considering the following two functions $\phi^{(1,\beta)}$ and $\phi^{(2,\beta)}$:

$$\phi^{(1,\beta)}(e^{i\theta}) = t_{\beta-1/2}(e^{i\theta})t_{\beta+1/2}(e^{i(\theta-\pi)}),$$

$$\phi^{(2,\beta)}(e^{i\theta}) = t_{\beta}(e^{i\theta})t_{\beta}(e^{i(\theta-\pi)}).$$
These functions have two jumps at 1 and $-1$. Note that the functions $\phi^{(1,\beta)}$ with $\beta \in \mathbb{Z} + 1/2$ and $\phi^{(2,\beta)}$ with $\beta \in \mathbb{Z}$ are up to a constant equal to the functions $\phi(e^{i\theta}) = e^{in\theta}$, $n \in \mathbb{Z}$. These trivial cases may be excluded without loss of generality. We remark also that $\phi^{(1,\beta)}$ admits another representation of a similar form:

$$\phi^{(1,\beta)}(e^{i\theta}) = -t_{\beta+1/2}(e^{i\theta})t_{\beta-1/2}(e^{i(\theta-\pi)}).$$

(73)

In the next proposition we evaluate the Fourier coefficients of $\phi^{(1,\beta)}$ and $\phi^{(2,\beta)}$ explicitly.

**Proposition 7.1** Let $\phi^{(1,\beta)}$ and $\phi^{(2,\beta)}$ be as above. If $\beta \notin \mathbb{Z} + 1/2$, then

$$\phi^{(1,\beta)}_n = \begin{cases} -\frac{\cos \pi \beta}{\pi (\beta - n/2)} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

(74)

If $\beta \notin \mathbb{Z}$, then

$$\phi^{(2,\beta)}_n = \begin{cases} \frac{\sin \pi \beta}{\pi (\beta - n/2)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

(75)

**Proof.** Using the definition (4) of the functions $t_\beta$, it can be easily verified that

$$\phi^{(1,\beta)}(e^{i\theta}) = e^{i\theta}t_{\beta-1/2}(e^{2i\theta}),$$

$$\phi^{(2,\beta)}(e^{i\theta}) = t_\beta(e^{2i\theta}).$$

The Fourier series expansion of $\phi^{(1,\beta)}$ and $\phi^{(2,\beta)}$ can be obtained from that of $t_{\beta-1/2}$ and $t_\beta$, respectively. Recall that the Fourier coefficients of $t_\beta$ has been given in (65). 

Because of the special form of the Fourier coefficients of $\phi^{(1,\beta)}$ and $\phi^{(2,\beta)}$, the corresponding matrices $T_n(\phi) + H_n(\phi)$ also have a particular structure. In fact, it turns out that they can be transformed into matrices of Cauchy form. Thus, using similar computations from the previous sections the determinants can be computed.

**Proposition 7.2** Let $\{a_j\}_{j=0}^{m_1-1}$, $\{\tilde{a}_j\}_{j=0}^{m_2-1}$, $\{b_k\}_{k=0}^{m_1-1}$ and $\{\tilde{b}_k\}_{k=0}^{m_2-1}$ such that the block matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is well defined, where

$$A_{11} = \begin{pmatrix} (a_j + b_k)^{-1} \end{pmatrix}_{j,k} 0 \leq j \leq m_1 - 1; \quad 0 \leq k \leq m_1 - 1,$n

$$A_{12} = \begin{pmatrix} (a_j + \tilde{b}_k)^{-1} \end{pmatrix}_{j,k} 0 \leq j \leq m_1 - 1; \quad 0 \leq k \leq m_2 - 1,$n

$$A_{21} = \begin{pmatrix} (\tilde{a}_j + b_k)^{-1} \end{pmatrix}_{j,k} 0 \leq j \leq m_2 - 1; \quad 0 \leq k \leq m_1 - 1,$n

$$A_{22} = \begin{pmatrix} (\tilde{a}_j + \tilde{b}_k)^{-1} \end{pmatrix}_{j,k} 0 \leq j \leq m_2 - 1; \quad 0 \leq k \leq m_2 - 1.$$
Then \( \det A = p/q \), where

\[
p = \prod_{0 \leq j < k \leq m_1 - 1} (a_k - a_j)(b_k - b_j) \prod_{0 \leq j < k \leq m_2 - 1} (\tilde{a}_k - \tilde{a}_j)(\tilde{b}_k - \tilde{b}_j) \prod_{0 \leq j \leq m_1 - 1} (\tilde{a}_k - a_j) (\tilde{b}_k - b_j),
\]

\[
q = \prod_{0 \leq j, k \leq m_1 - 1} (a_j + b_k) \prod_{0 \leq j, k \leq m_2 - 1} (\tilde{a}_j + \tilde{b}_k) \prod_{0 \leq j \leq m_1 - 1} (a_j + \tilde{b}_k) \prod_{0 \leq j \leq m_1 - 1} (\tilde{a}_j + b_k).
\]

The matrix \( A \) considered in (76) is also of Cauchy form, and therefore the above follows from the standard products that arise in the Cauchy determinants. The reason for writing \( A \) in block form is only for convenience in regard to what follows shortly.

Now we are able to establish the first main results of this section. In the following theorem note that although the values of the parameters \( \beta - 1/2 \) and \( \beta + 1/2 \) never fit the requirements of the main theorem (Theorem 6.4) for any value of \( \beta \), the answer agrees with the results from that theorem. This can be seen by a straightforward computation taking into account the duplication formula for the Barnes \( G \)-function (70) with \( z = 1/2 - \beta \). Notice, however, that if we take the parameters corresponding to representation (73) instead of (71), then the asymptotic formula given in Theorem 6.4 does not coincide with the following result.

**Theorem 7.3** Let \( \phi^{(1, \beta)} \) be as defined in (71) and assume \( \beta \notin \mathbb{Z} + 1/2 \). Then

\[
\det M_n(\phi^{(1, \beta)}) \sim n^{-1/4 - 3\beta^2} 2^{4\beta} G(1 - 2\beta) G(1/2 + \beta) G(3/2 + \beta).
\]

Moreover, \( \det M_n(\phi^{(1, \beta)}) = 0 \) if and only if \( \beta \in \{1, 2, 3, \ldots\} \) and \( n \geq 2\beta + 1 \).

**Proof.** For \( n \) fixed, put \( m_1 = m_2 = n/2 \) if \( n \) is even, and put \( m_1 = (n + 1)/2 \) and \( m_2 = (n - 1)/2 \) if \( n \) is odd. Let \( \sigma = m_1 - m_2 \). Denote by \( \varphi_{jk} \), \( 0 \leq j, k \leq n - 1 \), the \((j, k)\)-entry of the matrix \( T_n(\phi^{(1, \beta)}) + H_n(\phi^{(1, \beta)}) \). We permute the rows and columns of this matrix in such a way that we take first the even and then the odd rows and columns. This rearrangement results in a matrix \( B \) with the same determinant. This matrix is a \( 2 \times 2 \) block matrix with the same structure as (76) and with a size determined by \( m_1 \) and \( m_2 \):

\[
B = \begin{pmatrix}
\varphi_{2j, 2k} & \varphi_{2j, 2k+1} \\
\varphi_{2j+1, 2k} & \varphi_{2j+1, 2k+1}
\end{pmatrix}
\]  

(77)

Because the even Fourier coefficients of \( \phi^{(1, \beta)} \) vanish, it is not hard to see that

\[
B = \begin{pmatrix}
\phi^{(1, \beta)}_{2j+2k+1} & \phi^{(1, \beta)}_{2j+2k-1} \\
\phi^{(1, \beta)}_{2j+2k+3} & \phi^{(1, \beta)}_{2j+2k-1}
\end{pmatrix}
\]  

(78)

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It follows that

$$B = -\frac{\cos \pi \beta}{\pi} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} A,$$

(79)

where \( A \) is of the form \((76)\) with \(a_j = \beta - j - 1/2, \bar{a}_j = -\beta + j + 1/2, b_k = -k\) and \(\bar{b}_k = k + 1\).

We obtain that

$$\det B = \left(-\frac{\cos \pi \beta}{\pi}\right)^n (-1)^{m_2} \det A,$$

(80)

where \( \det A = p/q \) with

$$p = \prod_{0 \leq j < k \leq m_1 - 1} (k - j)^2 \prod_{0 \leq j < k \leq m_2 - 1} (k - j)^2 \prod_{0 \leq j \leq m_1 - 1} (1 - 2\beta + k + j)(1 + k + j)$$

$$= G(1 + m_1)^2 G(1 + m_2)^2 \frac{G(1 - 2\beta + n)G(1 - 2\beta)}{G(1 - 2\beta + m_1)G(1 - 2\beta + m_2)} \cdot \frac{G(1 + n)}{G(1 + m_1)G(1 + m_2)}$$

$$= \frac{G(1 + m_1)G(1 + m_2)G(1 - 2\beta + n)G(1 + n)G(1 - 2\beta)}{G(1 - 2\beta + m_1)G(1 - 2\beta + m_2)},$$

(81)

$$q = \prod_{0 \leq j, k \leq m_1 - 1} (\beta - 1/2 - j - k) \prod_{0 \leq j, k \leq m_2 - 1} (-\beta + 3/2 + j + k)$$

$$\times \prod_{0 \leq j \leq m_1 - 1} (\beta + 1/2 - j + k) \prod_{0 \leq j \leq m_2 - 1} (-\beta + 1/2 + j - k)$$

$$= (-1)^{m_2} \frac{G(1/2 - \beta + n + \sigma)G(1/2 - \beta)}{G(1/2 - \beta + m_1)^2} \cdot \frac{G(3/2 - \beta + n - \sigma)G(3/2 - \beta)}{G(3/2 - \beta + m_2)^2}$$

$$\times (-1)^{m_1 m_2} \frac{G(1/2 - \beta + m_1)G(3/2 + \beta + m_2)}{G(1/2 - \beta + \sigma)G(3/2 + \beta)} (-1)^{m_2(m_2 - 1)/2} \left(-\frac{\cos \pi \beta}{\pi}\right)^{m_2}$$

$$\times \frac{G(3/2 - \beta + m_2)G(1/2 + \beta + m_1)}{G(3/2 - \beta - \sigma)G(1/2 + \beta)} (-1)^{m_1(m_1 - 1)/2} \left(\frac{\cos \pi \beta}{\pi}\right)^{m_1}$$

$$= (-1)^{n(n - 1)/2} \left(-\frac{\cos \pi \beta}{\pi}\right)^n \frac{G(1/2 - \beta + n)G(3/2 - \beta + n)}{G(1/2 + \beta)G(3/2 + \beta)}$$

$$\times \frac{G(1/2 + \beta + m_1)G(3/2 + \beta + m_2)}{G(1/2 - \beta + m_1)G(3/2 - \beta + m_2)}.$$
products by means of (61), (63), and (14). Recall that \(m_1 + m_2 = n\) and \(m_1 - m_2 = \sigma\), and observe that \(2m_1 = n + \sigma\) and \(2m_2 = n - \sigma\). Note also that \(\sigma = 0\) or \(\sigma = 1\) depending on whether \(n\) is even or odd. Now we can combine (60), (81) and (82), and it follows that

\[
\det B = G(1 - 2\beta)G(1/2 + \beta)G(3/2 + \beta) \cdot \frac{G(1 - 2\beta + n)G(1 + n)}{G(1/2 - \beta + n)G(3/2 - \beta + n)} \\
\times \frac{G(1 + m_1)G(1/2 - \beta + m_1)}{G(1 - 2\beta + m_1)G(1/2 + \beta + m_1)} \cdot \frac{G(1 + m_2)G(3/2 - \beta + m_2)}{G(1 - 2\beta + m_2)G(3/2 + \beta + m_2)}.
\]

Here we have used the fact that \(n(n - 1)/2 + m_2\) is always even. We apply (84), and we can conclude that the first fraction in the last expression behaves asymptotically as \(n^{3\beta^2 - 1/4}\), the second fraction as \((n/2)^{\beta - 2\beta^2}\) and the third one as \((n/2)^{-\beta - 2\beta^2}\). This yields the desired limit behavior of \(\det M_n(\phi^{(1, \beta)})\). It can be read off from the third product in the \(p\)-term that the determinant vanishes if and only if \(2\beta \in \{1, \ldots, n - 1\}\).

The results from the next theorem just as in the previous one agree with our main theorem (Theorem 7.4) and this example is partially covered by this theorem. However, we point out that the allowed values of \(\beta_+\) and \(\beta_-\) while of a special form may not satisfy the conditions of the main theorem. As before, the duplication formula for the Barnes \(G\)-function shows the equality of both asymptotic formulas.

**Theorem 7.4** Let \(\phi^{(2, \beta)}\) be as defined in (72) and assume \(\beta \notin \mathbb{Z}\). Then

\[
\det M_n(\phi^{(2, \beta)}) \sim n^{-3\beta^2} 2^{4\beta^2} G(1 - 2\beta)G(1 + \beta)^2.
\]

Moreover, \(\det M_n(\phi^{(2, \beta)}) = 0\) if and only if \(\beta \in \{1/2, 3/2, 5/2, \ldots\\}\) and \(n \geq 2\beta + 1\).

**Proof.** The proof is similar to the one of the previous theorem. We introduce also the numbers \(m_1\) and \(m_2\) and rearrange the rows and columns of the matrix \(T_n(\phi^{(2, \beta)}) + H_n(\phi^{(2, \beta)})\) in the same way. We obtain a matrix \(B\) that can be written in the form (74). However, because now the odd Fourier coefficients vanish, formula (78) must be modified as follows:

\[
B = \left(\begin{array}{c}
\phi^{(2, \beta)}_{2j - 2k} & \phi^{(2, \beta)}_{2j + 2k + 2} \\
\phi^{(2, \beta)}_{2j + 2k + 2} & \phi^{(2, \beta)}_{2j - 2k}
\end{array}\right)_{j,k} \cdot (83)
\]

It follows that

\[
B = \frac{\sin \pi \beta}{\pi} \left(\begin{array}{cc}
I & 0 \\
0 & -I
\end{array}\right) A,
\]

where \(A\) is of the form (76) with \(a_j = \beta - j\), \(\tilde{a}_j = -\beta + j + 1\), \(b_k = k\), \(\tilde{b}_k = -k - 1\). Hence

\[
\det B = \left(\frac{\sin \pi \beta}{\pi}\right)^n (-1)^{m_2} \det A,
\]

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where \( \det A = p/q \) with

\[
p = \prod_{0 \leq j < k \leq m_1-1} -(k-j)^2 \prod_{0 \leq j < k \leq m_2-1} -(k-j)^2 \prod_{0 \leq j \leq m_1-1} \prod_{0 \leq k \leq m_2-1} (1-2\beta+k+j)(-1-k-j)
\]

\[
= (-1)^{n(n-1)/2} G(1+m_1)^2 G(1+m_2)^2
\times \frac{G(1-2\beta+n)G(1-2\beta)}{G(1-2\beta+m_1)G(1-2\beta+m_2)} \cdot \frac{G(1+n)}{G(1+m_1)G(1+m_2)}
\]

\[
= (-1)^{n(n-1)/2} \frac{G(1+m_1)G(1+m_2)G(1-2\beta+n)G(1+n)G(1-2\beta)}{G(1-2\beta+m_1)G(1-2\beta+m_2)}, \tag{86}
\]

\[
q = \prod_{0 \leq j \leq m_1-1} (\beta-j+k) \prod_{0 \leq j \leq m_2-1} (-\beta+j-k)
\times \prod_{0 \leq j \leq m_1-1} (\beta-1-j-k) \prod_{0 \leq j \leq m_2-1} (-\beta+1+j+k)
\]

\[
= \frac{G(1+\beta+m_1)G(1-\beta+m_1)}{G(1+\beta)G(1-\beta)} \cdot (-1)^{m_1(m_1-1)/2} \left( \frac{\sin \pi \beta}{\pi} \right)^{m_1}
\times \frac{G(1+\beta+m_2)G(1-\beta+m_2)}{G(1+\beta)G(1-\beta)} \cdot (-1)^{m_2(m_2-2)/2} \left( \frac{-\sin \pi \beta}{\pi} \right)^{m_2}
\times (-1)^{m_1m_2} \frac{G(1-\beta+n)^2G(1-\beta)^2}{G(1-\beta+m_1)^2G(1-\beta+m_2)^2}
\]

\[
= (-1)^{n(n-1)/2} (-1)^{m_2} \left( \frac{\sin \pi \beta}{\pi} \right)^n \frac{G(1-\beta+n)^2}{G(1+\beta)^2} \cdot \frac{G(1+\beta+m_1)G(1+\beta+m_2)}{G(1-\beta+m_1)G(1-\beta+m_2)}. \tag{87}
\]

Here we have again employed Proposition \[72\] pulled out a factor \(-1\) in each of the products of the \(p\)-term, and finally evaluated the products by formula \[39\]. Combining \[85\], \[86\] and \[87\] it follows that

\[
\det B = \frac{G(1-2\beta)G(1+\beta)^2}{G(1-\beta+n)^2} \cdot \frac{G(1-2\beta+n)G(1+n)}{G(1-\beta+n)^2}
\times \frac{G(1+m_1)G(1-\beta+m_1)}{G(1-2\beta+m_1)G(1+\beta+m_1)} \cdot \frac{G(1+m_2)G(1-\beta+m_2)}{G(1-2\beta+m_2)G(1+\beta+m_2)}.
\]

By \[84\], the first fraction in this expression behaves as \(n^{\beta^2}\) and the second and third fraction as \((n/2)^{-2\beta^2}\). This yields the limit behavior of \(\det M_n(\phi(2,\beta))\). Finally, the third product in the \(p\)-term shows that the determinant vanishes if and only if \(2\beta \in \{1, \ldots, n-1\} \). \(\Box\)
Next we consider two further classes of functions \( \phi^{(3,\beta)} \) and \( \phi^{(4,\beta)} \):

\[
\phi^{(3,\beta)}(e^{i\theta}) = t_\beta \frac{1}{2}(e^{i(\theta+\pi/2)})t_{\beta+1/2}(e^{i(\theta-\pi/2)}),
\]

\[
\phi^{(4,\beta)}(e^{i\theta}) = t_\beta(e^{i(\theta+\pi/2)})t_\beta(e^{i(\theta-\pi/2)}).
\]

These functions can be directly obtained from \( \phi^{(1,\beta)} \) and \( \phi^{(2,\beta)} \) if one rotates them by \(-\pi/2\) on the unit circle. The functions \( \phi^{(3,\beta)} \) and \( \phi^{(4,\beta)} \) have two jumps at \( i \) and \(-i\).

The following theorem relates the determinants generated by \( \phi^{(3,\beta)} \) and \( \phi^{(4,\beta)} \) to those computed in the previous two theorems. Note that the corresponding asymptotic formulas, which can easily be established, cannot be obtained by piecing together the asymptotics for two functions with a single jump at \( i \) and \(-i\). This fact clearly indicates the limitations of the localization idea.

**Theorem 7.5** Let \( \phi^{(1,\beta)}, \ldots, \phi^{(4,\beta)} \) be the functions defined above. Then

\[
\det M_n(\phi^{(3,\beta)}) = i^\sigma \det M_n(\phi^{(1,\beta)}),
\]

\[
\det M_n(\phi^{(4,\beta)}) = \det M_n(\phi^{(2,\beta)}),
\]

where \( \sigma = 0 \) if \( n \) is even and \( \sigma = 1 \) if \( n \) is odd.

**Proof.** First note that the Fourier coefficients are related by \( \phi_n^{(3,\beta)} = i^n \phi_n^{(1,\beta)} \). We make the same rearrangement of the rows and columns as in the proof of Theorem 7.3 and arrive at a matrix \( B \) for \( \phi^{(1,\beta)} \) and a matrix \( \tilde{B} \) for \( \phi^{(3,\beta)} \) both being of the form (78). We have

\[
\tilde{B} = \begin{pmatrix}
\text{diag } (i^{2j})_{j=0}^{m_1-1} & 0 \\
0 & \text{diag } (i^{2j})_{j=0}^{m_2-1}
\end{pmatrix}
B
\begin{pmatrix}
\text{diag } (i^{2k+1})_{k=0}^{m_1-1} & 0 \\
0 & \text{diag } (i^{2k-1})_{k=0}^{m_2-1}
\end{pmatrix}.
\]

Now we take the determinant. Analogously, \( \phi_n^{(4,\beta)} = i^n \phi_n^{(2,\beta)} \). After the same modification, we obtain a matrix \( B \) for \( \phi^{(2,\beta)} \) and a matrix \( \tilde{B} \) for \( \phi^{(4,\beta)} \), which are related by

\[
\tilde{B} = \begin{pmatrix}
\text{diag } (i^{2j})_{j=0}^{m_1-1} & 0 \\
0 & \text{diag } (i^{2j+2})_{j=0}^{m_2-1}
\end{pmatrix}
B
\begin{pmatrix}
\text{diag } (i^{2k})_{k=0}^{m_1-1} & 0 \\
0 & \text{diag } (i^{2k+2})_{k=0}^{m_2-1}
\end{pmatrix}.
\]

Taking the determinant completes the proof. \( \square \)

Among all the functions we have considered so far in this section, there is only one non-trivial function which is even, i.e. which satisfies \( \phi(t) = \phi(1/t), \ t \in \mathbb{T} \). This is the function \( \phi^{(3,\beta)} \) with \( \beta = 0 \). Note that

\[
\phi^{(3,0)}(e^{i\theta}) = \begin{cases} 
  i & -\pi/2 < \theta < \pi/2 \\
  -i & \pi/2 < \theta < 3\pi/2.
\end{cases}
\]
However, there are two more interesting examples that can be done which are not piecewise continuous functions, but which are even and have singularities of Fisher–Hartwig type.

We begin by introducing the function \( u_\alpha \), defined by \( u_\alpha(e^{i\theta}) = (2 - 2 \cos \theta)^\alpha \). In what follows we assume that \( \text{Re} \alpha > -1/2 \). We first note that if we define the functions

\[
\eta_\gamma(t) = (1 - t)\gamma, \quad \xi_\delta(t) = (1 - 1/t)^\delta, \quad t \in \mathbb{T},
\]

where the branches of \( \xi \) and \( \eta \) are chosen so that \( \eta_\gamma(0) = \xi_\delta(\infty) = 1 \) for their analytic continuations, then \( u_\alpha = \xi_\alpha \eta_\alpha \) and \( t_\beta = \xi_{-\beta} \eta_{-\beta} \). In the following proposition we list some facts already proven in [12, Sect. 3] which will eventually be used to show how the determinants generated by \( u_\alpha \) can be obtained in terms of our previous computations.

**Proposition 7.6** Let \( D_{\alpha,n} \) be the \( n \times n \) diagonal matrix defined by

\[
D_{\alpha,n} = \text{diag}(\mu_0^{(\alpha)}, \mu_1^{(\alpha)}, \ldots, \mu_{n-1}^{(\alpha)})
\]

where

\[
\mu_j^{(\alpha)} = \frac{\Gamma(1 + \alpha + j)}{j! \Gamma(1 + \alpha)}, \quad \Gamma_{\gamma,\delta} = \frac{\Gamma(1 + \gamma) \Gamma(1 + \delta)}{\Gamma(1 + \delta + \gamma)}.
\]

Then

\[
T_n(\xi_\delta \eta_\gamma) = \Gamma_{\delta,\gamma,n}^{-1} T_n(\eta_\gamma) D_{\delta+\gamma,n} \Gamma_{\delta+\gamma,n}^{-1}
\]

\[
H_n(\xi_\delta \eta_\gamma) = \Gamma_{\delta,\gamma,n}^{-1} T_n(\eta_\gamma) D_{\delta+\gamma,n} H_n(\tau_\delta) D_{-\gamma,n}
\]

where the finite Hankel matrix \( H_n(\tau_\delta) \) is defined by

\[
H_n(\tau_\delta) = \left( \frac{-(i + j) \Gamma(1 + \delta)}{\Gamma(2 + i + j + \delta)} \right)_{i,j=0}^{n-1}.
\]

The above identities allow us to reduce the computations of the asymptotics for the function \( u_\alpha \) to those for a function that we have already done.

**Theorem 7.7** Let \( \phi(e^{i\theta}) = u_\alpha(e^{i\theta}) \) with \( \alpha \notin \mathbb{Z} \) and \( \text{Re} \alpha > -1/2 \). Then

\[
\det M_n(\phi) \sim n^{(\alpha^2 - \alpha)/2} (2\pi)^{-\alpha}/2^\alpha 3^\alpha G(3/2 + \alpha) G(1 + \alpha)^2 / G(3/2) G(1 + 2\alpha).
\]

**Proof.** From the above identities we may write

\[
T_n(t_\beta) - H_n(t_\beta) = \Gamma_{\beta,\gamma,n}^{-1} D_{\beta,\gamma,n}^{-1} T_n(\eta_\beta) (T_n(\xi_\beta) D_{\beta,n}^{-1} - \beta H_n(\tau_\beta) D_{-\beta,n})
\]

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and
\[ T_n(u_\alpha) + H_n(u_\alpha) = \Gamma_{\alpha,\alpha}^{-1} D_{\alpha,n}^{-1} T_n(\xi_\alpha) D_{\alpha,n}^{-1} + \alpha H_n(\tau_\alpha) D_{\alpha,n}. \]

Pulling out to the right \( D_{\beta,n}^{-1} \) and \( D_{\alpha,n}^{-1} \) and taking the determinant, this yields with \( \beta = -\alpha \),
\[
\frac{\det(T_n(u_\alpha) + H_n(u_\alpha))}{\det(T_n(t_{-\alpha}) - H_n(t_{-\alpha}))} = \frac{\det(\Gamma_{\alpha,\alpha}^{-1} D_{\alpha,n}^{-1} T_n(\eta_\alpha) D_{\alpha,n})}{\det(\Gamma_{\alpha,\alpha}^{-1} D_{\alpha,n}^{-1} T_n(\eta_{-\alpha}))} \cdot \frac{\det D_{\alpha,n}}{\det D_{\alpha,n}}.
\]

Each of these terms can be computed. The determinants of \( T_n(\eta_\alpha) \) and \( T_n(\eta_{-\alpha}) \) are equal to one since they are triangular matrices. The other matrices on the right hand side are diagonal matrices and thus the last term equals
\[
\prod_{j=0}^{n-1} \frac{\Gamma(1 - \alpha + j) \Gamma(1 + 2\alpha + j)}{j! \Gamma(1 + \alpha + j)}.
\]

This product can be simplified using the basic recurrence relation of the Barnes G-function and an application of (60). The end result is that asymptotically
\[
\frac{\det(T_n(u_\alpha) + H_n(u_\alpha))}{\det(T_n(t_{-\alpha}) - H_n(t_{-\alpha}))} \sim n^{2\alpha^2} \frac{G(1 + \alpha) G(1 - \alpha) G(1 + 2\alpha)}{G(1/2) G(1 + 2\alpha)}.
\]

Finally note that the asymptotics of
\[
\det(T_n(t_{-\alpha}) - H_n(t_{-\alpha})) = \det M_n(t_{-\alpha}(e^{i(\theta - \pi)})],
\]
has been computed in Theorem 6.3. Collecting all terms gives the desired formula. \( \square \)

It is easy to modify this last theorem to find the asymptotic formula for one last example. We consider \( \phi(\epsilon^{i\theta}) = u_\alpha(e^{i(\theta - \pi)}) \). This is a change in the location of the singularity/zero to the point \(-1\). As in the examples of the previous section this change in the location of the singularity only requires a small modification in the proof.

**Theorem 7.8** Let \( \phi(\epsilon^{i\theta}) = u_\alpha(e^{i(\theta - \pi)}) \) with \( \alpha \notin \mathbb{Z} \) and \( \text{Re}\alpha > -1/2 \). Then
\[
\det M_n(\phi) \sim n^{(\alpha^2 + \alpha)/2(2\pi)^{-\alpha/2}} \frac{G(1/2 + \alpha) G(1 + \alpha)^2}{G(1/2) G(1 + 2\alpha)}.
\]

**Proof.** As before the only effect of the move of the discontinuity is that we need to evaluate the determinant of \( T_n(u_\alpha) - H_n(u_\alpha) \). This means the only change in the above computation is that the term \( M_n(t_{-\alpha}(e^{i(\theta - \pi)}) \)) is replaced by \( M_n(t_{-\alpha}(\epsilon^{i\theta})) \). Thus using the results of Theorem 6.3 the asymptotic formula is established. \( \square \)

Let us make some final remarks concerning the last two theorems. The assumptions \( \alpha \notin \mathbb{Z} \) have been imposed because of corresponding assumptions in Theorem 6.2 and Theorem 6.3.
However, in the last two theorems these assumptions are redundant. To see this, one has to elaborate a bit more on the proofs given in the previous section. In fact, one can establish an explicit expression for the determinants. In this expression (using analyticity) the term \( G(1 - \alpha) \) appearing in (94) cancels with the term \( G(1 + \beta) \) appearing in (69).

Note that the condition \( \text{Re}\alpha > -1/2 \) is exactly the condition for the integrability of the function \( u_\alpha \). Also this condition can be weakened. One can replace it with the assumption \( 2\alpha \notin \{-1, -2, \ldots\} \). Notice that in this case one has to understand \( u_\alpha \) as a distribution with well defined Fourier coefficients. For more details we refer to [12, 11].

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