COMPARING DIRECT N-BODY INTEGRATION
WITH ANISOTROPIC GASEOUS MODELS OF STAR CLUSTERS

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ABSTRACT

We compare the results for the dynamical evolution of star clusters derived from anisotropic gaseous models with the data from \textit{N}-body simulations of isolated and one-component systems, each having modest number of stars. The statistical quality of \textit{N}-body data was improved by averaging results from many \textit{N}-body runs, each with the same initial parameters but with different sequences of random numbers used to initialize positions and velocities of the particles. We study the development of anisotropy, the spatial evolution and energy generation by three-body binaries and its \textit{N}-dependence. We estimate the following free parameters of anisotropic gaseous models: the time scale for collisional anisotropy decay and the coefficient in the formulae for energy generation by three-body binaries. To achieve a fair agreement between \textit{N}-body and gaseous models for the core in pre- as well as in post-collapse only the energy generation by binaries had to be varied by \textit{N}. We find that anisotropy has considerable influence on the spatial structure of the cluster particularly for the intermediate and outer regions.

Key words: celestial mechanics, stellar dynamics - globular clusters: general.

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1 INTRODUCTION

One of the grand challenges of theoretical astrophysics is to understand the dynamics of globular star clusters. Aside from the intrinsic interest of understanding the behaviour of large $N$-body systems, the importance of the problem stems from its relation with current research into the stellar content of globular star clusters. (See, for example, many of the papers in Janes 1991). Unfortunately the direct simulation of such rich stellar systems with $N$-body modelling is not yet feasible (Hut, Makino & McMillan 1988). The gap between the largest useful computer models ($N \leq 10000$) and the median globular star cluster ($N \sim 5 \times 10^5$) can only be bridged at present by use of theory. There are two main classes of theory (cf. Saslaw 1985): (i) Fokker-Planck models, which are based on the Boltzmann equation of the kinetic theory of gases, and (ii) gas models, which can be thought of as a set of moment equations of the Fokker-Planck model.

These simplified models are the only detailed models which are directly applicable to large systems such as globular star clusters. But their simplicity stems from many approximations and assumptions which are required in their formulation, and it is not clear how well these correspond to the real systems in nature. The usual assumption of spherical symmetry contradicts the asymmetry of galactic tidal fields. Almost all Fokker-Planck calculations assume isotropy of the velocity distribution, which contradicts the evidence of proper motions and of model-building. The treatment of stellar escape in Fokker-Planck models has long been problematic, and it is never considered in gas models. Three- and four-body encounters can only be modeled within our limited knowledge of the relevant scattering cross-sections.

In view of all these uncertainties, the reliance which can be placed on these simplified models is doubtful. For this reason it is of importance to test the predictions of these models against results which are not subject to this wide class of simplifying assumptions and approximations, i.e. $N$-body models. Despite the central role of the simplified models in much recent research on cluster dynamics, little has been done to test their validity. There is the already classical comparison between the fluid-dynamical model of Larson (1970), Monte-Carlo simulations, and direct $N$-body integrations of 100 and 250 particles (Aarseth, Hénon & Wielen 1974, see also Aarseth & Lecar 1975), which were at that time maximal particle numbers from the viewpoint of computational resources.

In the following years there has been much further improvement of the theoretical models as well as of computational resources. Direct numerical solutions of the Fokker-Planck equation substituted the Monte-Carlo models, and gaseous models with heat flux closure improved Larson’s original fluid-dynamical approach (Lynden-Bell & Eggleton 1980, Heggie 1984). Bettwieser & Sugimoto (1985) tried to check the main assumption of the gaseous model, the heat conductivity, by using a direct $N = 1000$ model. Their $N$-body results, although in fair agreement with expectations in pre-collapse, suffered from large statistical fluctuations especially in post-collapse due to the still too small particle number.

With the advent of several large parallel computing facilities as well as fast vector computers in general it is now possible to extend direct $N$-body calculations in two respects: first to improve the statistics for rather low $N$ (up to a few thousand) by computing many independent models simultaneously, one on each processor, improving the statistics then by averaging. On the other hand the use of one of the fastest available vector supercomputers
(CRAY YMP) yielded for the first time high accuracy $N$-body models for as much as 10,000 particles throughout most of the core collapse phase (Spurzem & Aarseth 1993).

This paper is designed to complement a larger survey of star cluster evolution (Giersz & Heggie 1993ab), which compares $N$-body models with isotropic gaseous and Fokker-Planck results, by one particular aspect; this is to check the reliability of a generalized gaseous model which includes the possible generation of anisotropy in the radial and tangential dispersions of the star cluster. On the other side there is a feedback from the $N$-body models back to some adjustments in the gaseous model code in order to reach optimal agreement of both models.

In the following section theory and numerical solutions of anisotropic gaseous models are elaborated. Sect. 3 presents some information on the $N$-body models used, to an extent that is necessary here for understanding this paper; more details will be given elsewhere (Giersz & Heggie 1993a). Sect. 4 gives an account of the results, which are finally discussed and complemented by concluding remarks in the final section.

2 ANISOTROPIC GASEOUS MODELS OF STAR CLUSTERS

2.1 The model

Observational fits of globular clusters (cf. e.g. Lupton & Gunn 1987, Lupton, Gunn & Griffin 1987) and direct $N$-body calculations show that there is a considerable amount of anisotropy ($\sigma_r^2 > \sigma_t^2$) in their halo. Including the effects of anisotropy into models of the dynamical cluster evolution has posed some difficulties in the past. The only 2-D orbit average Fokker-Planck models for anisotropic single mass clusters with and without a massive central object (Cohn & Kulsrud 1978, Cohn 1979, Cohn 1985) have not been used further and there were a variety of anisotropic gaseous models with different closures (Bettwieser 1983, Bettwieser & Spurzem 1986); however their relation to real many-body systems and the quality of the numerical solutions remained unclear. Recently, however, self-similar anisotropic models for regular pre-collapse as well as for singular post-collapse clusters were obtained (Louis & Spurzem 1991, henceforth LS). This is an occasion to revisit the quality of the numerical solutions of the anisotropic gaseous model and to utilize such a model further to compare its predictions with the results of direct $N$-body calculations, with emphasis on the results concerning the anisotropy.

For the sake of completeness and to clarify small differences in the model and notation compared with LS we will present here the full set of equations used; dependent variables are the mass $M_r$ contained in a sphere of radius $r$, the local mass density $\rho$, radial and tangential pressure $p_r, p_t$, bulk mass transport velocity $u$, and transport velocities $v_r, v_t$, of the radial and tangential energy, respectively. As auxiliary quantities we use the radial and tangential 1-D velocity dispersions $\sigma_r^2 = p_r/\rho, \sigma_t^2 = p_t/\rho$, the average velocity dispersion $\sigma^2 = (\sigma_r^2 + 2\sigma_t^2)/3$, the anisotropy $A = 2 - 2\sigma_t^2/\sigma_r^2$ (note that $A = 6a/(1 + 2a)$, where $a$ is the anisotropy measure used in LS), and the relaxation time

\[
T = \frac{9}{16\sqrt{\pi}} \frac{\sigma^3}{G^2 m \rho \log(\gamma N)}
\]
in the definition of Larson (1970), where $N$ is the total particle number of the star cluster, $m$ the individual stellar mass and $\gamma$ a numerical constant whose value will be discussed below. The equations are

$$\frac{\partial M_r}{\partial r} = 4\pi r^2 \rho \quad (2)$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(\rho u r^2) = 0 \quad (3)$$

$$\frac{\partial u}{\partial t} + \frac{u}{\partial r} \frac{\partial}{\partial r} \left( \frac{GM_r}{r^2} + \frac{1}{\rho} \frac{\partial p_r}{\partial r} + 2 \frac{p_r - p_t}{pr} \right) = 0 \quad (4)$$

$$\frac{\partial p_r}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(p_r u r^2) + 2 \frac{p_r}{r} \frac{\partial u}{\partial r} + \frac{3}{r^2} \frac{\partial}{\partial r}(p_r (v_r - u) r^2) - 4 \frac{p_t (v_t - u)}{r} = -\frac{2}{3} \frac{p_r - p_t}{\lambda A T A} + \left( \frac{\delta p_r}{\delta t} \right)_{\text{bin3}} \quad (5)$$

$$\frac{\partial p_t}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(p_t u r^2) + 2 \frac{p_t u}{r} + \frac{1}{r^2} \frac{\partial}{\partial r}(p_t (v_t - u) r^2) + 2 \frac{p_t (v_t - u)}{r} = \frac{1}{3} \frac{p_r - p_t}{\lambda A T A} + \left( \frac{\delta p_t}{\delta t} \right)_{\text{bin3}} \quad (6)$$

$$v_r - u + \frac{\lambda}{4\pi G \rho T} \frac{\partial \sigma^2}{\partial r} = 0 \quad (7)$$

$$v_r = v_t \quad (8)$$

They are nearly equivalent to model A of LS (equal velocities for the transport of radial and tangential thermal energy). Note that model A (the so-called “1 flux” model) is very similar to the gaseous model of Bettwieser & Spurzem (1986), whereas model B of LS (“2 flux” model) is related to that of Bettwieser (1983). The net transport velocities for radial and tangential energy ($v_r - u$) and ($v_t - u$) can be derived from the energy fluxes $F_r$ and $F_t$ used as variables in these papers by dividing out a convenient multiple of the relevant pressure ($2p_t$ for ($v_t - u$), $3p_r$ for ($v_r - u$)). The reader interested in more details about this and the connection of the variables to moments of the stellar velocity distribution is referred to LS.

Apart from the energy generation term due to hard binaries discussed below there is one more difference to LS in keeping the hydrodynamical terms in Eq. (4); for all applications comparing with $N$-body calculations this did not make any differences, however the stability of the code at very small timesteps and high central densities (near core bounce) is enhanced; if one follows core collapse over many orders of magnitude increase in central density these terms slightly affect the value of the anisotropy in comparison with the self-similar models of LS. Hence they were omitted only for test calculations to be compared quantitatively with the results of that paper.

The numerical constants $\lambda_A$ and $\lambda$ occurring in Eqs. (5) to (7) are related to the timescales of collisional anisotropy decay and heat transport, respectively. $\lambda$ is related to the standard $C$ constant in isotropic gaseous models (see e.g. Heggie & Stephenson 1988) by

$$\frac{\lambda}{10} = \frac{27\sqrt{\pi}}{10} C \quad (8)$$
$T_A$ is the anisotropy decay timescale for an anisotropic local velocity distribution function; in the Appendix it is outlined how one gets the final result $T_A = 10 T/9$, provided a particular velocity distribution function is assumed. Since the real distribution function is not completely known within the framework of a gaseous model a free numerical constant $\lambda_A$ is introduced in Eqs. (5) and (6); the values chosen for $\lambda_A$ and $\lambda$ will be discussed in comparison with the direct $N$-body calculations later. Note that in LS and all other previous papers the timescale $T_A$ for an isotropic stellar background distribution as derived by Larson (1970) was used, which is $5 T/6$. Thus all models of LS correspond to a choice of $\lambda_A = 3/4$ here.

For the following comparison $N$-body models consisting of equal gravitating point masses are chosen, where any effects of finite size stars or stellar evolution are neglected. In that case the dominant energy source, which finally halts core collapse is the energy generated by formation and hardening of three-body binaries, which we take in accord with the standard ansatz (Goodman 1987) as

$$\left(\frac{\delta p_r}{\delta t}\right)_{\text{bin3}} = \frac{2}{3} C_b \frac{\rho^3}{m^2} \left(\frac{Gm}{\sigma}\right)^5 \cdot \theta(t - t_{b0})$$  \hspace{1cm} (9)

$$\left(\frac{\delta p_t}{\delta t}\right)_{\text{bin3}} = \left(\frac{\delta p_r}{\delta t}\right)_{\text{bin3}}.$$  \hspace{1cm} (10)

This is an isotropic energy input. A $\theta$-function has been attached here to illustrate the possibility to start the binary energy generation not from the very beginning but at a time $t_{b0}$, which is close but not identical to the time of core collapse and will be elaborated in Sect. 4.

Although the standard ansatz chooses $C_b = 90$, we keep this value here as a free parameter for reasons which will also be discussed in Sect. 4. Such a type of energy source to model the average energy generation by three-body binary generation and hardening was first used by Bettwieser & Sugimoto (1984) and Heggie (1984) for their isotropic gaseous models. Note that our anisotropic gaseous model equations are identical up to second order to moment equations of the Boltzmann equation with a Fokker-Planck collisional term. The closure equations Eqs. (7) and (8) are related to the analogous equation of heat transfer in a gas, except that the timescale occurring in the conductivity here is the appropriate stellar dynamical relaxation time (Lynden-Bell & Eggleton 1980); thus such models are denoted as gaseous models to distinguish them from higher order fluid dynamical models (e.g. Louis 1990).

Altogether the anisotropic gaseous model itself contains three parameters, $\lambda$, $\gamma$, $\lambda_A$; including the binary energy generation adds another two parameters, namely $C_b$ and $t_{b0}$. It is shown in a separate paper comparing isotropic gaseous models with direct $N$-body calculations (Giersz & Heggie 1993a) that for all models (including also direct solutions of the Fokker-Planck equation) values close to $\gamma = 0.11$ and $C = 0.104$ (i.e. $\lambda = 0.4977$), give the best agreement; the isotropic and anisotropic gaseous models differ only marginally in the evolution of the inner mass shells during pre- and post-collapse evolution; this is shown in Fig. 1, where an isotropic model obtained with Heggie’s code is compared with
an anisotropic 1 flux model of Spurzem’s code. Since the behaviour of the inner Lagrangian radii determines the best values for $\gamma$ and $C$ as discussed by Giersz & Heggie (1993a) we use here just the same values.

It is the focus of this paper to show how a comparison with direct $N$-body calculations enables us to find unambiguously an optimal set of the remaining parameters $\lambda_A$, $C_b$, and $t_{b0}$ in order to achieve fair agreement with the $N$-body results.

Without binary energy generation the quasi-static evolution of a system governed by Eqs. (2) to (8) depends only on the product $\lambda \cdot \lambda_A$; different choices for the total particle number result merely in a rescaling of time. With binary energy generation, however, increasing $N$ means a smaller binary energy generation; thus the post-collapse evolution is qualitatively different for varying particle number, ranging from simply steady reexpansion for low $N$, through periodic oscillations at intermediate $N$ and probably chaotic large-amplitude gravothermal oscillations (originally detected by the isotropic gaseous model of Bettwieser & Sugimoto, 1984) at very large $N$ (Heggie & Ramamani 1989, Cohn, Hut & Wise 1989).

2.2 Numerical Solution of the Anisotropic Gaseous Model Equations

For the numerical solution of the model equations and comparisons with direct $N$-body results standard $N$-body units were used, where $G = 1$, the total mass of the system $M = 1$, and the total energy of Plummer’s model, which was used as initial model, is $E = -1/4$; in these units the scaling radius of Plummer’s model is $a = 3\pi/16$. The equations were discretized on an Eulerian mesh with 200 logarithmically equidistant grid points (for some test runs 400); they were distributed between a minimal and maximal radius of $2.06 \cdot 10^{-6}$ and 144 in the above units.

The equations used for discretization and numerical solution are equivalent but not identical to those of Eqs. (2) to (8). First instead of the physical quantities we used $\log M_r$, $\log \rho$, $\log p_r$ and $\log p_t$ as well as the net energy transport velocities $v_r - u$ and $v_t - u$ together with $u$ as dependent variables; in the Eulerian mesh scheme $r$ is the independent variable. In order to achieve the utmost accuracy of mass and energy conservation a partially implicit scheme was used where the equations were formulated in terms of $\zeta X + (1 - \zeta)X^*$, where $X$ is the actual value and $X^*$ the one of the previous timestep; such a scheme is numerically unstable with the optimal value of $\zeta = 0.5$, thus $\zeta = 0.55$ was used. The difference equations were discretized on a logarithmically equidistant mesh and solved iteratively with a Newton-Raphson-Henyey method. For convenient discretization in the logarithmic variables the divergence terms occurring in Eqs. (3), (5), and (6) were all split up. As a criterion for choosing the timestep we used that the positive logarithmic quantities should not change by more than $0.05$, complemented by the condition that the timestep should not exceed the central relaxation time, which ensures good accuracy at the turning points from pre- to post-collapse. As boundary conditions we imposed time-independent logarithmic gradients of density and pressures at the outer, linear variation of the velocities with respect to $r$ and of the mass with respect to $r^3$ at the inner boundary. The conservation of energy and mass in all results presented in the following section, related to the total values of 0.25 and 1, respectively, was always smaller then or equal to 0.25%; although the strict mass conservation of earlier codes was lost (due to splitting the
The divergence term in Eq. 3 and the different boundary condition) the mass error remained small as stated; therewith, however, a much better quality of energy conservation was achieved than in previously published anisotropic models of Bettwieser & Spurzem (1986) and Bettwieser (1983). Although they discretized the divergence terms in Eqs. (5) and (6) in a strictly conservative manner this did not ensure that the total energy, consisting of thermal and gravitational energy (and bulk kinetic energy, which is here not important) is strictly conserved; the energy conservation becomes worse due to the non-conservative form of terms if one has separate energy equations for radial (Eq. 5) and tangential energy (Eq. 6). On the contrary the logarithmic variables improve energy conservation since they vary linearly whenever there is a power law, which turned out to be the more important effect.

In fact it occurred that the achieved high level of accuracy was necessary here, because for the previous models the errors were larger than the remaining deviations between gaseous model and direct N-body calculation.

The numerical method was tested by comparing with the known self-similar solutions of LS; for that purpose the spatial resolution was enhanced by choosing 400 grid points with an innermost radius of $1.11 \cdot 10^{-8}$ (to be comparable with LS the two first hydrodynamic terms in Eq. (4) had been omitted for these calculations only). A pure pre-collapse solution was followed (without binary energy generation) over an increase in central density of 13 orders of magnitude; the energy error here remained below 1% until the density increased roughly 8 orders of magnitude - thereafter the resolution of the core became poorer and consequently the energy error increased further. It would be possible to increase the accuracy even after such a large growth of central density by either more meshpoints or moving them with e.g. the shrinking core radius. This is not useful for our purpose here, since we do not follow the system deep into core collapse.

Figs. 2ab show the radial profiles of the logarithmic density power law index $\alpha$ and of the anisotropy $A$ for a number of models with different evolutionary time; the onset of the self-similar collapse phase is clearly visible. In Figs. 3ab the nearly constant values of $\alpha$ and $A$ after the self-similar phase has been entered are shown as a function of $\lambda_A \lambda$ in comparison to their values in the self-similar anisotropic pre-collapse models of LS, to the isotropic gaseous models of Lynden-Bell & Eggleton (1980), and to the direct solution of the isotropic Fokker-Planck equation (Cohn 1980). (Since LS only published two different $\lambda$ parameters, additional results for different $\lambda_A \lambda$ have been obtained by P.D. Louis, priv. communication). Whereas in LS $\lambda$ was adjusted to match the asymptotic core collapse rate with the results of higher order fluid dynamical models ($\lambda = 0.186$, Louis 1990), we chose here a value of $\lambda = 0.4977$ (equivalent to $C = 0.104$), which yields fair agreement with Fokker-Planck results (Heggie & Stephenson 1988) as well as with our $N$-body results in pre- and post-collapse. Throughout most of our calculations we chose $\lambda_A = 0.1$, for reasons which will become clear in Sect. 4; a few runs were performed to show the effect of choosing $\lambda_A = 1$.

For our comparisons presented here we use only rather low particle numbers ($N = 250, 500, 1000, 2000$); the post-collapse solution in such cases is a steadily reexpanding system with a regular core, related to the stable post-collapse isotropic self-similar models of Goodman (1987). A survey of larger particle numbers with the anisotropic gaseous model presented
here where the post-collapse model is unstable to gravothermal oscillations is in progress (Spurzem & Louis 1993). Occasionally we will refer to another comparison of this model with an \( N = 10000 \) direct \( N \)-body calculation (Spurzem & Aarseth 1993).

3 \( N \)-BODY CALCULATIONS

Over the last few years a new kind of computer became available for scientific use, consisting of a set of parallel transputers or vector processors. Their use opens a new chapter for stellar dynamical study. The power of these computers can be exploited in two different ways. One direction is simply to carry out calculations more quickly. This means to study larger systems. A second way is to obtain results having better statistics. It is well known that errors of the positions and velocities of the stars grow exponentially on a time scale shorter than the crossing time (Miller 1964, Goodman, Heggie & Hut 1993). Nevertheless \( N \)-body modeling practitioners believe that statistical results obtained from \( N \)-body calculations are meaningful (cf. e.g. Aarseth & Lecar 1975). Therefore it is worth paying much more attention to improving the statistical quality of such models. This aim was achieved by running the same model several times in parallel, where the individual processes differ only with respect to the randomly generated initial positions and velocities of the stars. A full discussion of the method and the hardware used in the simulations of evolution of small \( N \)-body systems (\( N = 250, 500, 1000, 2000 \)) is described in great detail in the separate paper (Giersz & Heggie 1993a). Here we only outline them.

The computational work has been carried out on two different parallel computers installed at the Edinburgh Parallel Computing Centre. One is the Meiko transputer array consisting of 400 processors, the other is the Meiko vector processor array containing 64 processors. Models consisting of \( N = 250, 500 \) and 2000 particles were computed on the transputer array using 56, 56 and 16 processors, respectively. The model for \( N = 1000 \) bodies was computed on the vector array using 40 processors, and on the DEC Alpha superclaster machine (called “fringe”) for 20 cases.

The first sets of models (\( N = 250, 500, 1000 \)) were computed using a code developed mainly by Heggie (Heggie 1973). The \( N = 2000 \) model, not fully completed yet, and part of the \( N = 1000 \) model were computed using a version of Aarseth’s standard code NBODY5 (Aarseth 1985), as well as the \( N = 10000 \) case, which will be presented in more detail elsewhere (Spurzem & Aarseth 1993). The initial conditions for these models were drawn from Plummer’s model (isolated system), with equal masses and no primordial binaries. In intervals of one \( N \)-body time unit there was output produced for each of a parallel set of cases. This output mainly consisted of the following information: 1) - Lagrangian radii (1%, 2%, 5%, 10%, 20%, 50%, 75%) with respect to the centre of density, 2) – the mean square radial and tangential velocities for each Lagrangian shell, 3) – global information: number of bound stars, energy of the system, number and energy of escapers, 4) – information about binaries: radius, internal energy, number and energy of escapers. The output from all models (different processors) was collected in a single file and later analyzed (averaged over all models) to produce statistics for all important quantities at each time.
The improvement in the statistical quality of the data achieved in our simulations allows us to perform very detailed comparisons with other methods, as in this case with the anisotropic gaseous model.

4 RESULTS

Three different groups of complete $N$-body runs, consisting of $N = 250$, 500, and 1000 particles, have been compared with the anisotropic gaseous models; two not fully completed $N$-body models will also be partly discussed ($N = 2000$, $N = 10000$). In what follows a phrase such as “1000-body model” refers to values averaged over many models, except for $N = 10000$ where there is only one case yet available.

We compare Lagrangian radii containing certain fractions of the total mass as a function of time and the anisotropy of the velocity distributions averaged within these Lagrangian mass shells as a function of time for both the gaseous and $N$-body models. All $N$-body data had to be adjusted in their initial Lagrangian radii since the construction of an initial $N$-body model, its scaling to the required value of the initial potential energy, and the use of the density centre all result in the fact that the initial model is slightly biased with respect to Plummer’s model; the gaseous model was initially much closer to it. Fig. 4 demonstrates the amount of shift necessary for the $N = 1000$ particle number as an example.

Fig. 5 depicts the evolution of the 1% Lagrangian radius in the $N = 1000$ model in comparison to gaseous model results with a varying initial time $t_{b0}$ for the binary energy generation. The value of $t_{b0}$ controls the time at which the gaseous model curve deviates from a pure core collapse case without any energy generation. It does not alter very much the post-collapse behaviour, but we selected for the three particle numbers 250, 500, and 1000 values of $t_{b0} = 50$, 130, and 230, respectively, which appeared as best fits of the gaseous model curves to the $N$-body case. Note that the best fit is usually meant with respect to the 5%, 2% and 1% Lagrangian mass shells. We have chosen those radii because they are only slightly smaller than the core radius, within which most of the binary activity occurs. The evolution of the outer shells does not depend strongly on $t_{b0}$.

Physically $t_{b0}$ should be related to a time at which binary activity begins to play a role in the $N$-body system; the ratios of the total binary energy released in the $N$-body system until the time $t_{b0}$ to the initial total energy of the system, however, were not the same for all particle numbers (the fractions were 2.48%, 2.05%, 0.13% for $N = 250$, 500, 1000, respectively, see discussion of $x$-values below and in Sect. 5).

Another parameter is the strength of the binary energy generation $C_b$; Fig. 6a shows that the minimum of the curve for the innermost Lagrangian radius (1%) is clearly of a different shape for varying $C_b$-values; for smaller $C_b$ the minimum is less shallow, because core collapse can proceed further to higher densities, until the energy generation suffices to halt and reverse it. There is also a variation in the total energy input for different $C_b$ which can be recognized by the different post-collapse curves. Again for each particle number we could select those values of $C_b = 55$, 70, and 90, for $N = 250$, 500, 1000,
respectively, which appeared to fit best the shape of the minimum of the N-body result as well as its post-collapse slope. Since there are always some fluctuations in the N-body system we varied $C_b$ only in larger steps (45, 55, 70, 90, 135) without attempting to determine the value more exactly. Moreover, one should note that due to a loss of cases in the parallel N-body calculations (for some individual models the total energy was not preserved due to very strong interactions between binaries and field stars and between binaries themselves (Giersz & Heggie 1993a)) the statistics of the corresponding averaged model worsen towards the end of the presented post-collapse evolution. For $N = 250, 500, 1000$ at the time $t = 600$ up to half of the initial number of cases were lost.

According to Goodman (1987) (see his Eq. II.14) $C_b$ measures the product of the quantities $x = 3\phi_c/v_c^2$, where $\phi_c$ and $v_c$ are the central potential and 3-D velocity dispersion, and $f_3$, which describes the amount of energy supplied to the core by an individual binary.

Fig. 6b depicts the evolution of $x$ for different N-body models. The time for all models $N \neq 1000$ was rescaled such that the distant two-body encounter relaxation timescale has the same value as for $N = 1000$; (this leads as one should expect to very good agreement of all pre-collapse curves). The curves were obtained by smoothing the original data using the standard procedure SMOOFT from Numerical Recipes (Press et al. 1986).

Note that we find for $N = 1000$ a value of $C_b = 90$, which is expected if the average total energy available from a binary resident in the core is supplied to the core ($f_3 \approx 1$). For lower particle numbers the changes of $x$ with $N$ during the post-collapse phase cannot be only accountable for the estimated values of $C_b$. This already suggests that for these models only a fraction, $f_3$, of the energy released by 3-body binaries is deposited into the core, $f_3 \approx 0.9$ for $N = 500$ and $f_3 \approx 0.8$ for $N = 250$, although we will give some more evidence on this point later.

The best values for $t_{b0}$ and $C_b$ were used to produce the data presented in Fig. 7 for the innermost Lagrangian radius ($N = 250$: $C_b = 55$, $t_{b0} = 50$; $N = 500$: $C_b = 70$, $t_{b0} = 130$; $N = 1000$: $C_b = 90$, $t_{b0} = 230$), and in Figs. 8ab to 10ab for all three particle numbers showing the evolution of seven Lagrangian radii ranging from 1% to 75%. Looking at the innermost Lagrangian mass shells containing 1%, 2%, and 5% of the total mass we conclude that there is an excellent agreement between both models. Such a fitting procedure with the anisotropic gaseous model as described above would not be possible if the N-body results were not improved in their statistics by computing multiple cases in parallel, which suppresses the intrinsic noise of the individual N-body system.

Note, however, that for $N = 250$ the best fit to the post-collapse expansion made by $C_b = 55$ for the inner Lagrangian mass shells did not automatically produce the correct form of the minimum at core bounce in contrast to the other particle numbers $N = 500$ and $N = 1000$; for $N = 250$ the minimum of the N-body model curves is always less shallow than for the gaseous model.

For intermediate Lagrangian radii containing 10%, 20%, and 50% of the total mass at $N = 1000$ there is a tendency for the gaseous model to expand faster in post-collapse, at $N = 500$ there is fair agreement between both models here, whereas for $N = 250$ on the contrary the N-body models expand faster. The 75% radius is in all N-body models
further outside than in the gaseous model. The reasons for these remaining differences between our two models will be discussed also in Sect. 5. The larger expansion rate of the N-body system in the outer halo (75%) as compared with the gaseous model is connected both with substantial amounts of escapers (Goodman 1984) and with degradation of the statistics due to loss of cases. There is a different treatment of the outer boundary in both models; the N-body model had initially all particles within a sphere of radius 10 \(r_h\) (\(r_h\): half-mass radius) i.e. any formed in the initial Plummer model at larger radii were removed, in the course of the evolution, however, a significant fraction of very loosely bound halo particles and escapers move outwards and populate areas within a sphere of a radius much larger than the initial radius; this process starts from the very beginning with particles just moving outwards on radial orbits. The phase space is not completely covered in that region, e.g. the angular momentum of the particles there (which are approximately 5% of all particles for \(N = 10000\) in the late collapse phase) is always much less than the maximum angular momentum of a circular orbit. On the contrary the gaseous model extended much further outwards already in the initial model in order to avoid dynamical perturbations originating from the boundary, but the system was assumed to be isotropic there initially, i.e. the accessible phase space was fully covered. Any expansion of the gaseous model’s 75% radius is a dynamical expansion driven by pressure against the outer shells, which is physically different to what happens in the N-body system, where some fraction of stars is just moving outwards filling empty space. Therefore the 75% Lagrangian radius is always further outside in the N-body models. This effect is even more pronounced for the 90% radius of the \(N = 2000\) body calculation as an example (Fig. 11), where one can recognize that the \(N\)-body model started with a smaller radius than the gaseous model, but expands from the beginning faster, since particles can move freely outwards in contrast to the gaseous model. The 90% radius has therefore not been shifted, since its difference with the gaseous model was much larger than usual.

We now start to discuss the evolution of the anisotropy \(A = 2 - 2\sigma_t^2/\sigma_r^2\); it is presented as a mass-weighted average taken over the spherical shells whose inner and outer radii are given by the Lagrangian radii of the previous figures. It is now the only remaining free parameter \(\lambda_A\) which limits the anisotropy by collisional isotropization. Let us denote with \(A_5\), \(A_6\), \(A_7\) the anisotropy between Lagrangian radii of 10% to 20%, 20% and 50%, and 50% to 100%, respectively. Their time evolution is shown in comparison between \(N\)-body results and gaseous models for \(N = 250, 500, 1000, 2000,\) and \(10000\) in Figs. (12) to (16), respectively. Most gaseous models were run with \(\lambda_A = 0.1\) (solid lines in Figs.), whereas in Figs. (12)-(14) alternative results for \(\lambda_A = 1\) (dashed lines) are also presented. The anisotropy generation for the \(\lambda_A = 1\) case is clearly much too high. \(\lambda_A = 0.1\), however, yields excellent agreement of \(A_5\), \(A_6\) and \(A_7\) for \(N = 10000\) (Fig. 16), as well as for all other particle numbers until a certain time \(t_{a0}\), which turns out to be approximately equal to \(t_{b0}\) for \(N = 1000\), and a little smaller than \(t_{b0}\) for \(N = 500\) and \(N = 250\). Since for the lower particle numbers \(N < 10000\) we know that there has been already more binary activity at \(t_{b0}\) it is consistent to postulate a connection between the binary activity in the core and the larger anisotropy (more radial energy) outside.

There is further evidence that the anisotropy in the outer shells is determined (at least partially) by binary activity. Firstly, there is a strong increase in the anisotropy after \(t_{b0}\),
when the binary energy generation starts, in all $\lambda_A = 1$ gaseous models (Figs. 12-14). For the $\lambda_A = 0.1$ case it is suppressed by stronger collisional isotropization. Secondly, the total energy and number of the bound hard binaries level off during the post-collapse evolution nearly at the same time as anisotropies do (Giersz & Heggie 1993b).

To further clarify the point we performed an artificial experiment by substituting the density exponent of 3 in Eq. (9) with $3 - \beta$ for the tangential and $3 + \beta$ for the radial energy generation. We compared runs for $\beta = 1$ and $\beta = 0$, the latter being the standard case. The total energy generation was normalized such that it does not differ in both cases and such that it is still isotropic at the centre. For positive $\beta$ there is more radial binary energy generation than tangential; the physical picture is that encounters with binaries very likely produce a particle leaving the core with low angular momentum i.e. higher radial than tangential energy. It can be seen in Figs. (17) and (18) that this indeed is a good model for what happens with the anisotropy: the stronger radial energy generation does not alter $A_5$ and $A_6$ much, but there is a clear trend towards the $N$-body results for $A_7$.

As a last check we suppressed close encounter and binary activity in an $N$-body simulation using S. Aarseth’s NBODY1 with a smoothing parameter $\varepsilon = p_0$ and $\varepsilon = 2p_0$, where $p_0$ is the impact parameter for which the deflection in the relative orbit of interacting particles is equal to $\pi/2$. In $N$-body units $p_0 = 2/(NV^2)$, where for $V^2$ we use a value obtained from the virial theorem. Assuming that the maximum impact parameter is roughly equal to $r_h$ we can estimate for $N = 250$ that for $\varepsilon = 0.016$ and $\varepsilon = 0.032$ about 9% and 21% of all encounters is suppressed, respectively.

The striking result shown in Fig. (19) is the much better agreement between $N$-body and gaseous models for $A_7$ in the case $\varepsilon = 0.016$; the trend is a little too strong for $\varepsilon = 0.032$. Taking this last result into account we conclude that $\lambda_A = 0.1$ seems to be the physically realistic value for all cases, provided close encounters and binary activity play little role; this is naturally the case in large $N$ systems (for which the gaseous models are qualified). It is consistent that in all results for $\lambda_A = 0.1$ and $t < t_{bd}$ the agreement is excellent; any difference starts for low $N$ and at times after which binary activity has started.

5 CONCLUSIONS AND DISCUSSION

We have performed a set of direct $N$-body models of idealized single mass star clusters with improved statistics obtained by calculating independent models in parallel on a parallel computer for particle numbers of $N = 250$, 500, 1000, and 2000. One model of $N = 10000$ (Spurzem & Aarseth 1993) complements some of our data. The results of the parallel $N$-body project are published in more detail in Giersz & Heggie (1993ab); here we related them to time dependent evolutionary calculations with an anisotropic gaseous model, which is closely related to the models of Bettwieser & Spurzem (1986) and Louis & Spurzem (1991, LS), but complemented by an energy source appropriate to describe a (usually isotropic) energy input by formation and hardening of three-body binaries analogous to that used in the isotropic gaseous models of Bettwieser and Sugimoto (1984) and Heggie and Ramamani (1989).
Excellent agreement of the evolution of the innermost Lagrangian mass shells (less than or equal to the core radius) in both pre- and post-collapse could be found by adjusting the strength and start time of the binary activity in the gaseous model. Remarkably the numerical value of the binary strength $C_b = 90$ found in that manner agrees very well for $N \geq 1000$ with theoretical expectations for systems whose evolution is dominated by small angle gravitational encounters (Goodman 1987). For $N = 500$ and $N = 250$ such agreement could be achieved only by choosing somewhat smaller values of $C_b = 70$ and $C_b = 55$, respectively.

The generation of anisotropy during pre-collapse and its levelling off in post-collapse within the half-mass radius is in fair agreement between $N$-body and gaseous models, provided one takes as timescale for collisional decay of anisotropy $\lambda_A T_A$, where $\lambda_A = 0.1$, and $T_A$ is the local anisotropy decay time for an anisotropic Larson-type distribution function (see Eq. (5), (6), and (A15), and Larson 1970). For the outer portions of the system the agreement is satisfactory only if $N \geq 2000$ or time $t \leq t_{b0}$, the latter being the time at which binary activity becomes important.

It is important to note that in pre-collapse and without binary activity we have scaled all $N$-body and gaseous models such that their collapse phases match exactly; the scaling factor was computed theoretically and the result is that all models for all particle numbers agreed excellently for a unique value of the remaining parameters ($\lambda = 0.4977$ for heat conductivity, $\lambda_A = 0.1$ for anisotropy decay, and $\gamma = 0.11$ in the Coulomb logarithm, cf. Eqs. 2-8). The agreement shows up for the anisotropy everywhere in the system and for the evolution of the Lagrangian mass shells up to the core radius. It is a puzzle, that between the core and half-mass radius in pre- and post-collapse the evolution of the Lagrangian mass shells in the gaseous model does not agree very well with the $N$-body models. There is a tendency for the gaseous model to expand faster (this is also the case for the isotropic Fokker-Planck model, Giersz & Heggie 1993b) at these radii as can be seen for the $N = 1000$ case in Fig. (8ab); however the same result occurs in the runs available for the largest particle numbers $N = 2000$ and 10000 (no figures given in this paper). Especially for $N = 10000$ distant two-body encounters are the main force which drive the evolution. So the gaseous model based on the thermal conductivity approximation should in principle describe the evolution well. However, this is not the case, particularly for the 50% Lagrangian radius. An explanation can be connected with the fact that the gaseous model cannot cope with non-local scattering. To mimic the energy transport in the radial and tangential directions we have to take into account all interactions along a particle orbit. This could be done for example by an orbit-averaged 2-D Fokker-Planck equation or by Monte-Carlo realization of this equation. We would like to stress that the best agreement with $N$-body data for the half mass radius is achieved by Stodolikiewicz’s Monte-Carlo code (1982). Unfortunately any quantitative proof of this statement is not possible because the only obtainable data are diagrammatic.

For smaller systems ($N = 250$ and $500$) that effect is present as well but we cannot see it because it is gradually overcome by the effect of reaction products of close two- and three-body interactions, which deposit their energy by large angle encounters in the outer regions of the system.
One may argue that a variation of $\gamma$, the factor in the Coulomb logarithm, or $\lambda$, the heat conductivity coefficient, should be checked to see whether it is sufficient for full agreement of the evolution of all Lagrangian mass shells. There is some theoretical evidence about a possible dependence of $\gamma$ on position and the total number of stars (see Spitzer 1987 for detailed discussion). Spitzer (1987) suggested that $\gamma$ could be equal to $2N_c/N$. From Heggie & Stephenson (1988) one may conclude that $\lambda$ is slightly different between pre- and post-collapse. However, our results indicate that our adopted constants give very good agreement during the collapse phase. So the local effects of distant two-body encounters are correctly modeled in this phase. Discrepancies start to build up when the number of stars in the core is very small and binary activity starts to play a role. This can favour Spitzer’s definition of $\gamma$, but on the other hand non-local effects connected with the small number of stars start to be increasingly important. For example very close two- and three-body scatterings can produce high velocity stars which would disturb the velocity distribution such that our second order moment equations are not sufficient to model the system. Our anisotropic gaseous model improves previous gaseous models with respect to the inclusion of (second order) anisotropy, but it may be too simplified in the presence of high-energy scattering events, which happen in the core predominantly for small particle numbers $N \leq 1000$ in post-collapse. It would be interesting to check how an anisotropic gaseous model such as that of Louis (1990) or numerical solutions of the orbit-averaged 2-D Fokker-Planck equation could cope with this problem. So we suggest that post-collapse evolution in principle can be understood without introducing new values of $\gamma$ and $\lambda$, although we cannot totally rule out that possibility (particularly for high $N$ systems) on the basis of our results.

There is more evidence that binary and close encounter activity causes differences between gaseous and $N$-body models for low $N$; this is the much too strong anisotropy generation outside of the half-mass radius in these cases (Figs. 12-14); first the problems occur as the binary activity and the anisotropy levels off in the $N$-body systems at the same time as the binary activity begins (Giersz & Heggie 1993b). Second, a test calculation with a local, but anisotropic (stronger radial) energy generation due to binaries exhibits a trend towards the $N$-body results. Third, we may interpret our empirically found strengths of binary energy generation in terms of the parameter $f_3$ of Goodman (1987) describing the fraction of binary energy liberated in the core. Our data indicate that $f_3 < 1$ for $N \leq 500$. Since all binaries are located in the core this means a rapid non-local energy transfer to regions outside of the core radius, carried by particles moving outwards on radial orbits and creating the anisotropy in the halo. Fourth, a test $N$-body calculation with suppressed close encounters by choosing non zero smoothing parameter shows much better agreement with the gaseous model results.

Our findings are consistent with the results of earlier work of Spitzer & Mathieu (1980) and Goodman (1984) stating that the reaction products of superelastic binary-single star scatterings will be transported outwards on elongated radial orbits, sometimes even escape. Angular momentum scattering along these orbits will distribute the energy of the reaction product in the corresponding radial zones. This is the mechanism by which the above mentioned non-local energy transport is provided and we identify the reaction products as carriers of the observed high anisotropy in the $N$-body models. It is not surprising that
for systems with larger $N$ and higher $x$ (deeper central potential, see Fig. 6b) this energy distribution is confined more to the homogeneous core itself and thus more consistent with the gaseous model.

The clue to an understanding of the remaining further differences between the two models (sharp minimum of the Lagrangian radii, compare Figs. 7 and 10a, and higher binary energy generation for small $N$) lies in the nature of the binary and close encounter activity, whose character is very different for the low $N$ systems and for high $N$ systems. For the former there is a higher probability, with respect to the probability of small angle encounters, that core particles are subject to a close two-body encounter or a three-body binary will form. Therefore for $N = 250$ binaries are created earlier and they can affect the core collapse and eventually stop it earlier (in terms of $x = 3\phi_0/v_0^2$, see Fig. 6b) than in higher $N$ systems. Because of smaller $x$ the rate of collapse is higher (Cohn 1980), and binaries have to generate more energy to influence core collapse and finally reverse it. Thereby we finally have found a natural explanation why the energy generated by binaries before the time $t_{b0}$ is higher. However, due to smaller $x$ (less deep central potential) a bigger fraction of the energy can be carried by particles non-locally outwards and it is not felt as early by the core as for higher particle numbers. This explains also why the intermediate shells for $N = 250$ expand faster than in the gaseous model, because that is where much of the energy and mass is non-locally transported to.

A supplementary explanation for the sharper minimum in the time evolution of the inner Lagrangian radii in the N-body models (see Fig. 10a) is connected with the growing importance of statistical fluctuations with decreasing number of stars. For $N = 250$ the number of stars in the core at the time of core bounce is extremely small (about 14 particles). Therefore stochastic processes connected with formation of binaries and their subsequent burning cannot be properly approximated by continuous formulae (Goodman 1984, 1987). Results obtained by Giersz & Heggie (1993b) for stochastic binary formation and burning with an isotropic gaseous model suggest that the minimum is in this case much sharper than in the standard case, whereas the post-collapse expansion is nearly the same. On the basis of that result we conclude that statistical fluctuations are important and at least partially account for the sharper minimum.

It can be concluded that the core evolution for large particle numbers is excellently described by anisotropic gaseous models in pre- as well as in post-collapse with the standard phenomenological implementation of the binary energy fed into the core. Remaining problems are faced by the gaseous model in two respects: i) small particle numbers and three-body encounters: they tend to spread energy non-locally accompanied by a large generation of anisotropy, which cannot properly be taken into account by the present simple isotropic energy input formula for binaries in the gaseous model; ii) large particle numbers and two-body scatterings: in the regions outside the core of the system an appreciable difference occurs in the rate of evolution of the Lagrangian radii – the gaseous model evolves too fast. Whether this is a result of approximations inherent in the gaseous model (e.g. local approximation for all collisional effects, or neglect of higher order moments of the velocity distribution) remains a question for future work. One could for example take into account non-local scattering in the context of a numerical solution of the orbit-averaged 2-D Fokker-Planck equation or a more detailed form of the velocity distribution by a higher
order moment model.
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APPENDIX

Derivation of the collisional decay of anisotropy

According to Larson (1970) we approximate the real local (r-dependent) velocity distribution function $f$ as a function of the modulus of a star’s random velocity vector $v$ and the cosine of its angle with the radial direction $\mu$,

$$f = \sum_{j=0}^{\infty} A_j(v') P_j(\mu')$$  \hspace{1cm} (A1)

where $P_j(\mu)$ denotes the Legendre polynomials of order $j$. We are going to apply the Fokker-Planck equation in $v, \mu$ coordinates, which was computed by Rosenbluth, McDonald & Judd (1957, RMB); therefore we need to know the Rosenbluth potentials

$$h(v) = 2m \int f(v') |v - v'|^{-1} d^3v'$$  \hspace{1cm} (A2)

$$g(v) = m \int f(v') |v - v'| d^3v'$$  \hspace{1cm} (A3)

for the distribution function of Eq. (A1); they are

$$h(v, \mu) = 8\pi m \cdot \sum_{l=0}^{\infty} \frac{P_l(\mu)}{2l + 1} \int_0^{\infty} \frac{v_l^l v_{l+1}^l A_l(v')}{v_{l+1}^l} dv'$$  \hspace{1cm} (A4)

$$g(v, \mu) = 4\pi m \cdot \sum_{l=0}^{\infty} \frac{1}{2l + 1} \cdot \left\{ P_l(\mu) \int_0^{\infty} A_l(v')(v^2 + v'^2)v_{l+1}^{l+1} dv' - P_{l-1}(\mu) \frac{2l}{2l - 1} \int_0^{\infty} A_{l-1}(v') vv'^3 v_{l+1}^{l+1} dv' - P_{l+1}(\mu) \frac{2l + 2}{2l + 3} \int_0^{\infty} A_{l+1}(v') vv'^3 v_{l+1}^{l+1} dv' \right\}$$  \hspace{1cm} (A5)

with $v_\leq = \min(v, v')$, $v_\geq = \max(v, v')$.

Since we are interested mainly in distribution functions with anisotropy in the second order moments the series in Eq. (A1) shall be truncated now at that order and we use in accord
with Larson’s (1970) definitions

\[ A_0 = g(v) \]
\[ A_1 = 0 \]  \hspace{1cm} (A6)
\[ A_2 = c_2 \frac{v^2}{\sigma^2} g(v) \]

with

\[ c_2 = \frac{\sigma_t^2 - \sigma_r^2}{3\sigma_r^2} \]  \hspace{1cm} (A7)

\[ g(v) = \frac{1}{\sqrt{2\pi}^3} \sigma_r^3 \exp\left(-\frac{v^2}{2\sigma_r^2}\right) \]  \hspace{1cm} (A8)

is the isotropic Maxwell-Boltzmann distribution. These definitions just ensure that the quantities \( \sigma_r^2, \sigma_t^2 \), and \( 3\sigma_r^2(v_r - u) \) are recovered by determining the second and third order moments of \( f \); fourth order moments have been assumed to take the same value as in the case of a Maxwell-Boltzmann distribution function; in Larson’s notation this means that we have set \( \xi = 0 \) (see his Eq. 2).

Now the Rosenbluth potentials can be further evaluated in terms of the functions

\[ I_n = \int_0^v v^n g(v') dv' \]  \hspace{1cm} (A9)
\[ K_n = \int_v^\infty v^n g(v') dv' \]  \hspace{1cm} (A10)

As a final result we get

\[ h(v, \mu) = 8\pi m \left\{ \frac{1}{v^2} I_2 + K_1 + \frac{1}{3} P_2(\mu) \frac{c_2}{\sigma^2} \left[ \frac{1}{v^3} I_6 + v^2 K_1 \right] \right\} \]  \hspace{1cm} (A11)

\[ g(v, \mu) = 4\pi m \left\{ v I_2 + \frac{1}{3v} I_4 + \frac{v^2}{3} K_1 + K_3 \right\} + \frac{1}{5} \frac{c_2}{\sigma^2} P_2(\mu) \left[ -\frac{1}{3v} I_6 + \frac{1}{7v^3} I_8 + \frac{v^4}{7} K_1 - \frac{v^2}{3} K_3 \right] \]  \hspace{1cm} (A12)

With these potentials the Fokker-Planck equation of RMB is determined; let the right hand side of their Eq. (31) be denoted with FP; we then evaluate

\[ D_{pr} = \left( \frac{\delta p_r}{\delta t} \right)_c = 2\pi \rho \int_0^{\infty} v^2 dv \int_{-1}^{+1} d\mu v^2 \mu^2 \text{FP} \]  \hspace{1cm} (A13)
\[ D_{pt} = \left( \frac{\delta p_t}{\delta t} \right)_c = 2\pi \rho \int_0^{\infty} v^2 dv \int_{-1}^{+1} d\mu v^2 \mu^2 \frac{1}{2}(1 - \mu^2) \text{FP} \]
It turns finally out that

\[ Dp_r - Dp_t = -\frac{p_r - p_t}{T} \left( \frac{9}{10} - \frac{9}{140} \frac{p_r - p_t}{p} \right), \quad (A14) \]

with the average pressure \( p = (p_r + 2p_t)/3 = \rho \sigma^2 \). Since the second term in the above equation is small for reasonable values of the anisotropy we finally get

\[ T_A = -\frac{p_r - p_t}{Dp_r - Dp_t} = \frac{10}{9} T. \quad (A15) \]
FIGURE CAPTIONS

Fig. 1 Evolution of Lagrangian radii containing the stated percentage of the total mass for an isotropic (IGM) and anisotropic (AGM) gaseous model for a $N = 1000$ star cluster.

Fig. 2a Radial profile of the logarithmic density power law index $\alpha$ for a number of gaseous models with different evolutionary time. The dash-dotted line represents the self-similar anisotropic pre-collapse model of LS.

fig. 2b Radial profile of the anisotropy $A$ for a number of gaseous models with different evolutionary time. The dash-dotted line represents the self-similar anisotropic pre-collapse model of LS.

Fig. 3a Comparison between the density power law index $\alpha$ as a function of $\lambda_A\lambda$ for anisotropic gaseous models (AGM) and self-similar anisotropic pre-collapse models of LS (SIM). The values of $\alpha$ for the isotropic gaseous model of Lynden-Bell and Eggleton (LBE) and the isotropic Fokker-Planck model of Cohn (FP) are marked.

Fig. 3b Comparison between the anisotropy $A$ as a function of $\lambda_A\lambda$ for anisotropic gaseous models (AGM) and self-similar anisotropic pre-collapse models of LS (SIM).

Fig. 4 Amount of shift necessary to adjust the averaged $N = 1000$ $N$-body model to the anisotropic gaseous model for 1% Lagrangian radius.

Fig. 5 Evolution of the 1% Lagrangian radius in the averaged $N = 1000$ $N$-body model in comparison to the anisotropic gaseous model for different initial times $t_{b0}$ for switching on the binary energy generation. The cc indicates a pure core collapse gaseous model.

Fig. 6a Evolution of the 1% Lagrangian radius in the averaged $N = 1000$ $N$-body model in comparison to the anisotropic gaseous model for different strength of the binary energy generation parameter $C_b$. The cc indicates a pure core collapse gaseous model.

Fig. 6b Time variation of $x$ (Def. see text) for three particle numbers indicated at the curves, in the averaged $N$-body models; all times were normalized such that the core-collapse time is the same for all $N$ and the strongly fluctuating curves were smoothed.

Fig. 7 Evolution of the 1% Lagrangian radius in the averaged $N = 1000$, $N = 500$, $N = 250$ $N$-body models in comparison to the anisotropic gaseous models ($C_b = 90$, $t_{b0} = 230$), ($C_b = 70$, $t_{b0} = 130$), ($C_b = 55$, $t_{b0} = 50$), respectively.

Fig. 8a Evolution of the 1%, 2%, 5%, 10% Lagrangian radii in the averaged $N = 1000$ $N$-body model in comparison to the anisotropic gaseous model.

Fig. 8b Evolution of the 20%, 50%, 75% Lagrangian radii in the averaged $N = 1000$ $N$-body model in comparison to the anisotropic gaseous model.

Fig. 9a The same as in Fig. 8a but for $N = 500$.

Fig. 9b The same as in Fig. 8b but for $N = 500$.

Fig. 10a The same as in Fig. 8a but for $N = 250$.

Fig. 10b The same as in Fig. 8b but for $N = 250$. The 75% Lagrangian radius is not available for this model.

Fig. 11 Evolution of the 90% Lagrangian radius in the averaged $N = 2000$ $N$-body model in comparison to the anisotropic gaseous model.
Fig. 12 Evolution of the anisotropy between Lagrangian radii of 10% to 20% ($A_5$), 20% to 50% ($A_6$), and 50% to 100% ($A_7$) in the averaged $N = 1000$ $N$-body model in comparison to the anisotropic gaseous models for different parameters of collisional anisotropy decay, $\lambda_A$.

Fig. 13 The same as in Fig. 12 but for $N = 500$.

Fig. 14 The same as in Fig. 12 but for $N = 250$.

Fig. 15 Evolution of the anisotropy between Lagrangian radii of 40% to 50%, 50% to 75% and 75% to 90% in the averaged $N = 2000$ $N$-body model in comparison to the anisotropic gaseous model without binaries; note the different definition of Lagrangian shells here.

Fig. 16 The same as in Fig. 12 but for $N = 10000$ and $\lambda_A = 0.1$ and no binaries.

Fig. 17 Evolution of the anisotropy between Lagrangian radii of 10% to 20%, 20% to 50% and 50% to 100% in the averaged $N = 1000$ $N$-body model in comparison to the anisotropic gaseous models for different $\beta$ (see text).

Fig. 18 The same as in Fig. 17 but for $N = 500$.

Fig. 19 Evolution of the anisotropy between Lagrangian radii of 50% to 100% in the averaged $N = 1000$ $N$-body models with different smoothing parameters $\epsilon$ in comparison to the anisotropic gaseous model.