Permutations that separate close elements

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Joint work with Tuvi Etzion (Technion)

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Algebraic phrasing

For $i, j \in \mathbb{Z}_n$, let $||i, j||_n$ be the distance between $i$ and $j$ when the elements of $\mathbb{Z}_n$ are written in a circle.
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**Definition (An overlapping rectangle)**

A permutation \( \pi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n \) has an \((s, k)\)-clash if there exist distinct \( i, j \in \mathbb{Z}_n \) with \(||i, j||_n < s\) and \(||\pi(i), \pi(j)||_n < k\).
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**Definition (An overlapping rectangle)**

A permutation $\pi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ has an $(s, k)$-clash if there exist distinct $i, j \in \mathbb{Z}_n$ with $||i, j||_n < s$ and $||\pi(i), \pi(j)||_n < k$.

**Definition (No overlapping rectangles)**

A permutation $\pi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ is $(s, k)$-clash-free if it has no $(s, k)$-clashes.
Related work

- Generalisations of $k = 2$ case: cyclic matching sequencability for graphs: Alspach, *Bull. ICA* 2008, Brualdi–Kiernan–Meyer, *Australas. J. Comb.* 2012; Kreher–Pastine–Tollefson, *Australas. J. Comb.* 2015.

- Non-cyclic case (cylinder or square, not torus): Mammoliti–Simpson, *Australas. J. Comb.* 2020.

- Packing diamonds rather than rectangles (large distance in the Manhattan metric): Aspvell–Liang *Stanford Tech. Report* 1980; Bevan–Homberger–Tenner *JCT-A* 2018; SRB–Homberger–Winkler *JCT-A* 2019.
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The main question

Definition (How wide can rectangles be?)

Let $n$ and $k$ be fixed. Define $\sigma(n, k)$ to be the largest $s$ such that an $(s, k)$-clash-free permutation $\pi$ of $\mathbb{Z}_n$ exists.

Theorem (Mammoliti–Simpson, Australian J. Comb. 2020)

$\sigma(n, k) \leq \left\lfloor \frac{n - 1}{k} \right\rfloor$

Proof.

$nsk \leq n^2$.

We can't have $sk = n$.

So $sk \leq n - 1$. 

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Mammoliti–Simpson conjecture

Theorem (SRB, JCT-A 2023)

\[ \left\lfloor \frac{n-1}{k} \right\rfloor - 1 \leq \sigma(n, k) \leq \left\lceil \frac{n-1}{k} \right\rceil \]

Proof. \((n, k, s) = (76, 6, 11)\).

Set \(\rho(0) = 0\), \(\rho(1) = 12\), and so on. \(\rho\) is \((k, s)\)-clash-free.

Set \(\pi = \rho - 1\). Then \(\pi\) is \((s, k)\)-clash-free.
Mammoliti–Simpson conjecture

**Theorem (SRB, JCT-A 2023)**

\[ \lfloor (n - 1)/k \rfloor - 1 \leq \sigma(n, k) \leq \lfloor (n - 1)/k \rfloor \]

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Theorem (SRB–Etzion, 2023+)

Let $n$ and $k$ be fixed positive integers, with $k < n$. Write $s = \lfloor \frac{n-1}{k} \rfloor$, so $n = sk + r$ where $1 \leq r \leq k$.

Define $d_k = \gcd(n, k)$ and $d_s = \gcd(n, s)$.

If $r \geq s$ or $k = r$, then $\sigma(n, k) = \lfloor \frac{n-1}{k} \rfloor$.

If $r < s$ and $r < k$ and $d_sd_k$ divides $n$, then $\sigma(n, k) = \lfloor \frac{n-1}{k} \rfloor$.

If $r < s$ and $r < k$ and $d_sd_k$ does not divide $n$, then $\sigma(n, k) = \lfloor \frac{n-1}{k} \rfloor - 1$. 

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A sketch proof

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Rectangles form east-west and north-south lines: *warp and weft threads.*
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Rectangles form east-west and north-south lines: *warp and weft threads*. Threads cannot change direction:
A sketch proof 2

Threads must be periodic, giving the condition that $d_s d_k$ divides $n$. 
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Can classify permutations by **jumpers**: two sequences determining sizes of gaps.
**Definition**

An \((s, k, n)\)-jumper is a pair \(((a_i), (b_i))\) of sequences of integers with the following properties:

1. \((a_i)\) has period dividing \(d_s\), and \((b_i)\) has period dividing \(d_k\).
2. We have \(1 \leq a_i < s\) and \(1 \leq b_i < k\) for \(i \geq 0\).
3. The \(d_k\) partial sums \(\sum_{i=0}^{\ell-1} b_i\) where \(0 \leq \ell < d_s\) are distinct modulo \(d_k\). Moreover, \(d_s d_k\) divides \(\sigma_b\) where \(\sigma_b = \sum_{i=0}^{d_k-1} b_i\).
4. The \(d_s\) partial sums \(\sum_{i=0}^{m-1} a_i\) where \(0 \leq m < d_s\) are distinct modulo \(d_s\). Moreover, \(d_s d_k\) divides \(\sigma_a\) where \(\sigma_a = \sum_{i=0}^{d_s-1} a_i\).
5. Defining \(\sigma_a\) and \(\sigma_b\) as above, \(\sigma_a \sigma_b = d_s d_k r\).
The classification

Theorem

Let $n$ and $k$ be fixed integers with $k < n$. Set $s = \lfloor (n - 1)/k \rfloor$, and define $r$ by $n = sk + r$ for $1 \leq r \leq k$. Define $d_s = \gcd(n, s)$ and $d_k = \gcd(n, k)$. Assume that $r < k$ and $r < s$. Furthermore, suppose that $d_s d_k$ divides $n$. 

There is a bijection between the set of clockwise $(s, k)$-clash-free permutations with $\pi(0) = 0$ and the set $J(s, k, n)$ of $(s, k, n)$-jumpers.
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Let $n$ and $k$ be fixed integers with $k < n$. Set $s = \lfloor (n - 1)/k \rfloor$, and define $r$ by $n = sk + r$ for $1 \leq r \leq k$. Define $d_s = \gcd(n, s)$ and $d_k = \gcd(n, k)$. Assume that $r < k$ and $r < s$. Furthermore, suppose that $d_s d_k$ divides $n$. There is a bijection between the set of clockwise $(s, k)$-clash-free permutations with $\pi(0) = 0$ and the set $J(s, k, n)$ of $(s, k, n)$-jumpers.
Thanks!