Distributed Interval Observers for Bounded-Error LTI Systems

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Abstract—This paper proposes a novel distributed interval observer design for linear time-invariant (LTI) discrete-time systems subject to bounded disturbances. In the proposed observer algorithm, each agent in a networked group exchanges locally-computed framers or interval-valued state estimates with neighbors, and coordinates its update via an intersection operation. We show that the proposed framers are guaranteed to bound the true state trajectory of the system by construction, i.e., without imposing any additional assumptions or constraints. Moreover, we provide necessary and sufficient conditions for the collective stability of the distributed observer, i.e., to guarantee the uniform boundedness of the observer error sequence. In particular, we show that such conditions can be tractably satisfied through a constructive and distributed approach. Moreover, we provide an algorithm to verify some structural conditions for a given system, which guarantee the existence of the proposed observer. Finally, simulation results demonstrate the effectiveness of our proposed method compared to an existing distributed observer in the literature.

I. INTRODUCTION

Many large scale cyber-physical systems, such as electric power grids [1], intelligent transportation systems [2], and industrial infrastructures [3], are equipped with sensor networks, providing in situ and diverse measurements to monitor them. This makes possible the construction of system state estimates, which are essential to guarantee the safe and effective operation of these critical applications. Motivated by this, an intense research activity on the analysis and design of distributed estimation algorithms has ensued. In this way, each sensor, equipped with local communication and processing capabilities, interacts with neighboring nodes to compute joint estimates cooperatively.

A way to obtain such estimates is to use a centralized observer, by which a super node collects all measurements from the nodes and fuses them in an optimal way. The ubiquitous Kalman filter [4] and related approaches have been used extensively for this purpose. However, these algorithms do not scale well as the size of the network increases and are vulnerable to single-point failures. This spawned research on the design of distributed estimation filters (for systems subject to known stochastic disturbances) for sensor networks communicating only locally over a possibly time-varying network [5]. While these methods are more scalable and robust to communication failures than their centralized counterparts, they generally have comparatively worse estimation error.

When stochastic characterization of disturbances is not available, however, other techniques that leverage alternative information should be considered.

In case the disturbances are known to be bounded, interval observers are a popular method for obtaining robust, guaranteed estimates of the state, due to their simplicity and computational efficiency [6]–[10]. Hence, various approaches to design centralized interval observers for various classes of dynamical systems have been proposed [11]–[21]. The main idea in most of the aforementioned designs is to synthesize appropriate centralized observer gains to obtain a robustly stable and positive observer error system for all realizations of the existing uncertainties [11], [14]. This strategy, which usually boils down to solving centralized semi-definite programs (SDP) subject to large numbers of constraints, leads to theoretical and computational difficulties, and thus infeasible solutions, especially for large-scale systems [15]–[17]. In addition to computational issues, the communication complexity of the centralized approach does not scale well as the size of the network increases. A recent study [22] proposes a distributed interval observer for block-diagonalizable linear time-invariant (LTI) systems, which requires a certain structure on the dynamics and the output of the system. Another work [23] designs an observer for LTI systems under denial-of-service attacks. In addition, [24] proposes an internally positive representation (IPR)-based robust distributed interval observer for continuous-time LTI systems. However, the proposed design relies on similarity transformations and the satisfaction of certain algebraic constraints, which could lead to moderately-performing results. Recently, the design of distributed functional interval observers for different classes of systems have gained attention, e.g., in [25], [26]. However, all of the aforementioned works use average consensus to share estimates throughout the network, which limits the effectiveness of the proposed methods with respect to time of convergence and estimation quality.

Contributions. To overcome the aforementioned drawbacks, this work contributes to bridging the gap between interval observer design approaches and distributed estimation algorithms in the presence of distribution-free uncertainties. We introduce a novel method for synthesizing scalable distributed interval observers for discrete-time LTI systems subject to bounded additive disturbances. We provide necessary and sufficient conditions for the stability of our proposed observer. Our observer is correct by construction, i.e., the true state of the system is indeed framed by the designed framers, and we leverage this correctness to intersect interval estimates between neighboring nodes, ensuring that the tightest possible estimate among all the agents’ estimates is adopted by consensus in a finite number of iterations.

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as a consequence of intersecting. Furthermore, we introduce the intuitive notion of “collective positive detectability over neighborhoods” (CPDN) which, is sufficient to tractably compute gains that satisfy the aforementioned stability requirement in a distributed manner. This approach involves the solution to local and feasible linear programs (LP), which is potentially less conservative and computationally more efficient than SDP-based approaches. Finally, we provide an algorithm to verify if CPDN holds for a given system.

**Notation.** Let $\mathbb{R}^n$, $\mathbb{R}^{n \times p}$, $\mathbb{N}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{R}_{\geq 0}$ denote the $n$-dimensional Euclidean space, the sets of $n$ by $p$ matrices, natural numbers, nonnegative integers, and nonnegative real numbers, respectively. For $M \in \mathbb{R}^{n \times p}$, let $M_i$ and $M_{ij}$ denote the $i$th row of $M$, and the $(i, j)$th entry of $M$, respectively. Furthermore, for $M \in \mathbb{R}^{n \times p}$, we define $M^+ \in \mathbb{R}^{n \times p}$, such that $M_{ij} \triangleq \max\{M_{ij}, 0\}$, $M^- \triangleq M^+ - M$, and $|M| \triangleq M^+ + M^-$. In addition, $M \succ 0$ (or, $M \succeq 0$, resp.) denote that $M$ is positive definite (semi-definite, resp.), and $\rho(M)$ is used to denote the spectral radius of $M$. All the inequalities $\leq, \geq$, as well as $\max$ and $\min$, are considered element-wise. As usual, $e_i$ denotes the $i$th vector of the standard basis of $\mathbb{R}^n$. Finally, for $A^1, \ldots, A^N \in \mathbb{R}^{n \times n}$, $\text{diag}(A^1, \ldots, A^N) \in \mathbb{R}^{nN \times nN}$ denotes the block-diagonal matrix with block-diagonal elements being $A^i, i \in \{1, \ldots, N\}$.

**II. PROBLEM FORMULATION AND PRELIMINARIES**

This section introduces basic preliminary concepts and graph theory notions used throughout the paper.

**Graph-theoretic Notions.** Next, we recall some definitions from Graph Theory. A directed graph (digraph) $G$ is a set of nodes $V$ and a set of directed edges $E \subseteq V \times V$. The set of neighbors of node $i$, denoted $N_i$, is the set of all nodes $j$ for which there is an edge $(i, j) \in E$. We will assume that $i \in N_i$. A path from node $i$ to node $j$ is a sequence of nodes starting with $i$ and ending with $j$, such that any two consecutive nodes are joined by a directed edge. The $d$-hop neighbors of node $i$, denoted $N_i^d$, is the set of nodes connected to $i$ by a path of length no more than $d$. The diameter of a graph is the largest distance between any two nodes, i.e., $\text{diam}(G) \triangleq \max_{i,j} d(i,j)$, where $d(i, j)$ denotes the length of the shortest path between $i$ and $j$.

**Multi-dimensional Intervals.** Finally, we introduce some definitions and results regarding multi-dimensional intervals. A (multi-dimensional) interval $I \triangleq [s, \bar{s}] \subseteq \mathbb{R}^n$ is the set of all vectors $x \in \mathbb{R}^n$ that satisfy $s \leq x \leq \bar{s}$.

**Proposition 1.** [17, Lemma 1] Let $A \in \mathbb{R}^{n \times n}$ and $x \leq \bar{x} \leq \bar{x} \in \mathbb{R}^n$. Then, $A^T \bar{x} - A^T \bar{x} \leq A x \leq A^T \bar{x} - A^T \bar{x}$. As a corollary, if $A$ is non-negative, $A \bar{x} \leq A x \leq A \bar{x}$.

**III. PROBLEM FORMULATION**

**System Assumptions.** Consider a multi-agent system (MAS) consisting of $V \triangleq \{1, \ldots, N\}$ agents, which interact over a time-invariant communication graph $G = (V, E)$. The agents are able to obtain distributed measurements of a target as described by the following LTI dynamics:

$$
\begin{align*}
\mathcal{P}: \quad & f_{k+1} = A x_k + B u_k, \\
& y_k = C^T x_k + D^T v_k, \quad i \in V, \quad k \in \mathbb{Z}_{\geq 0},
\end{align*}
$$

where $x_k \in \mathbb{R}^n$ is the continuous state of the target system and $w_k \in \mathcal{I}_w \triangleq [\underline{w}, \bar{w}] \subset \mathbb{R}^{n_u}$ is bounded process disturbance. Furthermore, at time step $k$, every agent $i \in V$ takes a measurement $y_k^i \in \mathbb{R}^{n_y}$, known only to itself, which is perturbed by $v_k^i \in \mathcal{I}_v \triangleq [\underline{v}, \bar{v}] \subset \mathbb{R}^{n_v}$, a bounded sensor (measurement) noise signal. Finally, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C^i \in \mathbb{R}^{m \times n}$ and $D^i \in \mathbb{R}^{m \times n_u}$ are system matrices known to all agents. The MAS’s goal is to estimate the trajectories of (1) in a distributed manner, when they are initialized in an interval $\mathcal{I}_x \triangleq [\underline{x}, \bar{x}] \subset \mathbb{R}^n$, with $\underline{x}, \bar{x}$ known to all agents. Next, we define the notions of framer, correctness, and stability, used throughout the paper.

**Definition 1** (Framers). For an agent $i \in V$, the sequences $\{\tau_k^i\}_{k \geq 0}$ and $\{\eta_k^i\}_{k \geq 0}$ are called upper and lower individual framers for the state of $\mathcal{P}$ if $\tau_k^i \leq x_k \leq \tau_k^i$, for all $k \geq 0$. Moreover, we define the individual lower and upper framer errors as follows:

$$
\xi_k^i \triangleq x_k - \tau_k^i, \quad \eta_k^i \triangleq \tau_k^i - x_k, \quad \forall k \geq 0. \quad (2)
$$

Given an MAS with target system $\mathcal{P}$ and communication graph $G$, a distributed interval framer is a distributed algorithm over $G$ that allows each agent $i$ to cooperatively compute upper and lower individual framers for $\mathcal{P}$. Finally, $e_k \triangleq [(\xi_k^1)^T \cdots (\xi_k^N)^T \cdots (\eta_k^1)^T (\eta_k^N)^T]^T \in \mathbb{R}^{2Nn}$ is called the collective framer error, which is the vector of all individual lower and upper framer errors.

**Definition 2** (Distributed Interval Observer). A distributed interval framer is input-to-state (ISS) stable if the collective framer error is bounded as follows:

$$
\|e_k\| \leq \beta(\|e_0\|, \bar{\gamma}), \quad \forall k \in \mathbb{Z}_{\geq 0},
$$

where $\beta(\cdot)$ and $\bar{\gamma}$ are functions of classes $K\mathcal{L}$ and $K_{\infty}$, respectively. An ISS distributed interval framer is a distributed interval observer.

The observer design problem can be stated as follows:

**Problem 1.** Given a multi-agent system and the LTI system in (1), design a distributed interval observer for $\mathcal{P}$.

We finish this section by stating an assumption which characterizes the interplay between the agents’ local observations and their communication over the network and will be leveraged later as a sufficient condition to guarantee stability of the observer designed using our approach (cf. Theorem 2).

**Assumption 1** (Collective Positive Detectability over Neighborhoods (CPDN)). There is a $d^* \in \mathbb{N}$ such that for each dimension $s \in \{1, \ldots, n\}$ and every agent $i \in V$, there is an agent in the $d^*$-hop neighborhood of $i$ and denoted as $\ell(i, s) \in N_i^{d^*}$, such that there exist gains $T^{(i,s)}$, $L^{(i,s)}$, and $\Gamma^{(i,s)}$ satisfying $\|T^{(i,s)} A - L^{(i,s)} C^{(i,s)}\| < 1$.

**Remark 1.** Assumption 1 captures a broad range of conditions on the system and graph structure that can result in a stable observer. Intuitively, there is a tradeoff between the connectivity of the graph and the measurements available to individual agents. If an agent has access to measurements of

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more states, the algorithm will be stable on a less connected graph and/or with fewer network update iterations. The converse is also true, that a well connected graph (or more network update iterations) will lead to stability even if each agent has very few measurements.

Later in Section IV-C, we provide a tractable distributed procedure to verify Assumption 1 (cf. Algorithm 2).

IV. PROPOSED DISTRIBUTED INTERVAL OBSERVER

In this section, we describe our novel distributed interval observer design, a necessary and sufficient condition for stability of the proposed observer, and an LP-based distributed procedure for computing stabilizing observer gains.

A. Distributed Observer and its Framer Property

To address Problem 1, we propose a two-step distributed interval framer (cf. Definition 1) for $\mathcal{P}$. The DIO Algorithm 1 provides a pseudocode description of our observer, the details of which are further explained in this section as follows.

i) Propagation and Measurement Update: At every $k+1 \in \mathbb{Z}_{\geq 0}$, given $z^i_{k}, x^i_{k}, y^i_{k}$ and $y^i_{k+1}$, each agent $i \in \mathcal{V}$ performs a state propagation and a local measurement update step using observer gains $L^i, T^i \in \mathbb{R}^{n \times m_i}$, which will be designed to satisfy desired observer properties:

$$\begin{align*}
\dot{z}^i_{k+1} &= A^i x^i_k - \bar{A}^i \bar{x}^i_k + (T^i B^i) \bar{w}^i_k + L^i y^i_k + T^i y^i_{k+1} + ((L^i D^i)^+ + (T^i D^i)) \bar{v}^i, \\
\bar{x}^i_{k+1} &= A^i \bar{x}^i_k - \bar{A}^i \bar{x}^i_k + (T^i B^i) \bar{w}^i_k + L^i y^i_k + T^i y^i_{k+1} + ((L^i D^i)^+ + (T^i D^i)) \bar{v}^i,
\end{align*}$$

where $T^i \triangleq L^i - \Gamma^C i$ and $\bar{A}^i \triangleq T^i A - L^i C^i$. Further, $\bar{z}^i_k \triangleq z^i_k - \bar{z}^i_0$, and $\bar{x}^i_k \triangleq \bar{x}^i_k - x^i_k$ are the corresponding errors, and $\bar{e}^i_k$ is the vector of all agents' errors, as in (3). Note that similar structures to the individual state framers in (4), with additional observer gains to increase degrees of freedom, has been proposed before in a centralized setting, e.g., in [27], [28]. However, it is worth reemphasizing that the main novelty of this work is to propose a completely distributed algorithm. This decentralization leads to novel and interesting dynamics, the analysis of which is the main focus of this paper. As we will show in the numerical simulation, it is even possible that the performance of the distributed observer can exceed that of the centralized observer. This is unusual in the case of distributed approaches, e.g., in stochastic estimation, where decentralized observers typically have worse estimation error. Also note that [28] restricts the values of $T A - L C$ to be positive, whereas our method is able to relax this restriction.

ii) Network Update: After the measurement update, each agent $i$ iteratively shares its interval estimate with its neighbors in the network, and updates it by taking the tightest interval from all neighbors via intersection:

$$\begin{align*}
\bar{x}^i_{k,t} &= \max_{j \in N_i} \bar{x}^j_{k,t-1}, & \bar{x}^i_{k,t} &= \bar{x}^i_{k,t}, \\
\bar{x}^i_{k,t} &= \min_{j \in N_i} \bar{x}^j_{k,t-1}, & \bar{x}^i_{k,t} &= \bar{x}^i_{k,t},
\end{align*}$$

$$\forall t \in \{1, \ldots, d\},$$

where $d \in \mathbb{N}$ is the number of network-update iterations. Note that in case $d > 1$, this iterative procedure computes the intersection of intervals with the $d$-hop neighbors of each agent. Consequently, each agent $i$ obtains the following information:

$$\bar{x}^i_{k} = \max_{j \in \mathcal{N}_i^d} \bar{x}^j_{k} \quad \text{and} \quad \bar{x}^i_{k} = \min_{j \in \mathcal{N}_i^d} \bar{x}^j_{k},$$

used as a compact representation of the network update (5).

Lemma 1. Given the neighbors’ interval estimates $\{\bar{x}^j_{k,t} ; \bar{x}^i_{k,t}\}_{j \in \mathcal{N}_i}$, (5) results in the smallest possible interval (i.e., the one with the smallest width in all dimensions) which is guaranteed to contain the true state. Furthermore, let $e^1_k$ and $e^2_k$ be the collective errors of the DIO algorithm with the same initial conditions, but different numbers of network updates, $d_1 > d_2$. Then, $e^1_k \leq e^2_k$ for all $k \geq 0$.

Proof. The statements follow from the definition of the intersection of intervals and the fact that taking more intersection in a time step cannot worsen the interval estimate.

An important consequence of Lemma 1 is that our observer is guaranteed to perform better than one which uses a linear operation (i.e., averaging) to communicate across the network, and despite the nonlinearity of (5), we are still able to provide a necessary and sufficient condition for stability, which is a key contribution of this work. Next, we show that the proposed DIO algorithm constructs a distributed interval framer in the sense of Definition 1 for the plant $\mathcal{P}$.

Algorithm 1 Distributed Interval Observer (DIO) at node $i$

Input: $z^i_k, \bar{x}^i_0, d$; Output: $\{\bar{x}^i_k\}_{k \geq 0}, \{\bar{x}^i_k\}_{k \geq 0}$;
1: Compute $L^i, T^i$, and $\bar{e}^i_k$ using (12);
2: loop
3: Compute $\bar{x}^i_{k,0}$ and $\bar{x}^i_{k,0}$ using (4);
4: for $t = 1$ to $d$ do
5: Send $\bar{x}^i_{k,t-1}$ and $\bar{x}^i_{k,t-1}$ to $j \in \mathcal{N}_i$;
6: $\bar{x}^i_{k+1} = \min_{j \in \mathcal{N}_i} \{\bar{x}^j_{k+1} ; \bar{x}^i_{k+1}\}$, $\bar{x}^i_{k+1} = \min_{j \in \mathcal{N}_i} \{\bar{x}^j_{k+1} ; \bar{x}^i_{k+1}\}$
7: end for
8. $\bar{x}^i_{k+1} = \bar{x}^i_{k+1} ; \bar{x}^i_{k+1} = \bar{x}^i_{k+1} ; k \leftarrow k + 1$;
9: end loop
10: return $\{\bar{x}^i_k\}_{k \geq 0}, \{\bar{x}^i_k\}_{k \geq 0}$

Lemma 2 (Distributed Framer Construction). The DIO algorithm is a distributed interval framer for (1).

Proof. From (1) and the fact that $T^i = I_n - \Gamma^C i$, we have $x^i_{k+1} = (T^i C^i + T^i x^i_{k+1} = (T^i A x^i_{k+1} + B w^i_{k+1} + T^i C^i x^i_{k+1})$. (6)

Plugging $C^i x^i_{k+1} = y^i_{k+1} - D^i v^i_{k+1}$ into, as well as adding the zero term $L^i (y^i_{k+1} - C^i x^i_{k+1})$ to the right hand side of (6) results in $x^i_{k+1} = A^i x^i_{k} + T^i B w^i_{k+1} + T^i (y^i_{k+1} - D^i v^i_{k+1}) + L^i (y^i_{k+1} - D^i v^i_{k+1})$. (7)

Applying Proposition 1 to all the uncertain terms in the right hand side of (7) shows that for each $i \in \mathcal{V}$,

$$\bar{x}^i_{k+1} \leq \bar{x}^i_{k+1} \leq \bar{x}^i_{k+1} \leq \bar{x}^i_{k+1},$$

where $\bar{x}^i_{k+1}, \bar{x}^i_{k+1}$ are given in (4). This means that individual framers/interval estimates are correct. When the framer condition is satisfied for all nodes, the intersection of all the individual estimates of neighboring nodes (cf. (5)) also
results in correct interval framers, i.e.
\[ \mathbf{x}_k^{i,0} \leq x_k \leq \mathbf{x}_k^{0,i}, \quad \forall i \in \mathcal{V} \Leftrightarrow \mathbf{x}_k \leq \mathbf{x}_k^{0,i}, \quad \forall i \in \mathcal{V}. \]
Since the initial interval is known to all \( i \), by induction (4)-(5) constructs a correct distributed interval frame for (1). \hfill \blacksquare

B. Collective Input-to-State Stability

In this subsection, we investigate conditions on the observer gains \( L^i, T^i, \) and \( \Gamma^i \), as well as the communication graph \( \mathcal{G} \), that lead to an ISS distributed observer (cf. Definition 2), which equivalently results in a uniformly bounded observer error sequence \( \{e_k\}_{k \geq 0} \) (given in (2)-(3)), in the presence of bounded noise. For ease of exposition, in what follows, we ignore noise terms and focus on asymptotic stability of the noiseless error dynamics, which we will show implies collective input-to-state stability.

Switched System Perspective. We begin by stating a preliminary result that expresses the observer error dynamics in the form of a specific switched system.

**Lemma 3.** The collective error \( e_k \) has dynamics
\[ e_{k+1} = H_k \hat{A} e_k, \]
where \( \hat{A} \triangleq \text{diag}(\hat{A}^1, \ldots, \hat{A}^N) \), \( \hat{A}^i \triangleq \begin{bmatrix} ([\hat{A}^i]^+ + [\hat{A}^i]^-) - ([\hat{A}^i]^+ - [\hat{A}^i]^+) \end{bmatrix}. \)
\( H_k \in \{0, 1\}^{2Nn \times 2Nn} \) is a binary matrix which selects a single minimizer or maximizer of the framers, i.e.,
\[ (H_k)_{\text{id}(i,s), \text{id}(j,s)} = 1 \Leftrightarrow j^* = \min(\arg\max_{j \in \mathcal{N}_d^i}(z_{k,0}^i)), \]
\[ (H_k)_{\text{id}(i,s), \text{id}(j,s)} = 1 \Leftrightarrow j^* = \min(\arg\max_{j \in \mathcal{N}_d^i}(z_{k,0}^i)), \]
for \( s \in \{1, \ldots, n\} \) and \( i \in \mathcal{V} \), where \( \text{id}(i, s) = 2n(i - 1) + s - 1 \) and \( \text{id}(i, s) = 2n(i - 1) + s + n - 1 \) encode the indices associated with the upper and lower framers of system (11) at node \( i \). Furthermore, \( H_k \hat{A} \in \mathcal{F} \subseteq \mathbb{R}^{2Nn \times 2Nn} \), where
\[ \mathcal{F} \triangleq \left\{ \begin{bmatrix} a_{11}^T & \ldots & a_{1n}^T \end{bmatrix}, \ldots, \begin{bmatrix} a_{n1}^T & \ldots & a_{nn}^T \end{bmatrix} \right\} : a_{is} \in \mathcal{F}, a_{is} \in \mathcal{F}, s \in \{1, \ldots, n\}, i \in \mathcal{V} \}, \]
\[ \mathcal{F}_1 \triangleq \left\{ e_i^T \otimes ([\hat{A}^i]^+ + [\hat{A}^i]^-) \right\} \subseteq \mathbb{R}^{1 \times 2Nn} : j \in \mathcal{N}_d^i \}, \]
\[ \mathcal{F}_2 \triangleq \left\{ \begin{bmatrix} \mathbf{1} & \ldots & \mathbf{1} \end{bmatrix} \right\} \subseteq \mathbb{R}^{1 \times 2Nn} : j \in \mathcal{N}_d^i \}. \]

**Proof.** Notice that \( \hat{A} = \hat{A}^{i+} - \hat{A}^{-} \) in (7). Rewriting in terms of the error by adding (or subtracting) (7) to (4) and (5), then setting the noise to zero, we obtain \( e_{k+1} = \hat{A} e_k \) and \( e_k = H_k e_k^{0,i} \). Combining these yields (8). \hfill \blacksquare

Recall that the switching in (8) depends on the state according to (9) and always creates the smallest possible error. In order to take the advantage of this property note that the set \( \mathcal{F} \) has a specific structure known as independent row uncertainty, formally defined below.

**Definition 3 (Independent Row Uncertainty [29]).** A set of matrices \( \mathcal{M} \subseteq \mathbb{R}^{n \times n} \) has independent row uncertainties if
\[ \mathcal{M} = \left\{ \begin{bmatrix} a_1^T & \ldots & a_n^T \end{bmatrix} : a_i \in \mathcal{M}_i, i \in \{1, \ldots, n\} \right\}, \]
where all sets \( \mathcal{M}_i \subseteq \mathbb{R}^{1 \times n} \) are compact.

Next, we restate the following lemma on the spectral properties of the sets with independent row uncertainties, that will be used later in our stability analysis of system (8).

**Lemma 4.** [29, Lemma 2] Suppose \( \mathcal{M} \subseteq \mathbb{R}^{n \times n} \) has independent row uncertainties. Then, there exists \( M_\ast \in \mathcal{M} \) such that \( \rho(M_\ast) = \min_{M \in \mathcal{M}} \rho(M) = \lim_{k \to \infty} \min_{M \in \mathcal{M}, j \in \{1, \ldots, k\}} \|M_1 \cdots M_k\|^{\frac{1}{k}} \). The latter is known as the lower spectral radius of the set of matrices \( \mathcal{M} \).

We can now state our first main stability result.

**Theorem 1** (Necessary and Sufficient Conditions for Stability). The error system (8) is globally exponentially stable if and only if there exists \( H_\ast \mathcal{A} \mathcal{E}_k \) with initial condition \( e_0 = e_0 \). By the construction of \( H_k \in (9) \), which implies \( H_k \hat{A} \mathcal{E}_k \Rightarrow H_k \hat{A} \mathcal{E}_k \), \( \hat{e}_k \geq \hat{e}_k \geq 0 \) for all \( k \geq 0 \) by induction. Therefore, by comparison, (8) is globally exponentially stable. To prove necessity, note that (8) is asymptotically stable only if the lower spectral radius of \( \mathcal{F} \) is less than 1. By Lemma 4, this implies existence of a stable \( F_\ast = H_\ast A \). Finally, having studied stability of the noise-free system, we now study the ISS property of the noisy system:
\[ e_{k+1} = H_k \hat{A} e_k + H_k (W_k + V_k), \]
where
\[ W_k \triangleq \left( (\Lambda^k_1) \cdots (\Lambda^k_N) \right)^T, \Lambda^k_i \triangleq \left( (T^k_i)^T + (T^k_i)^T \right)^s_k + \left( (T^k_i)^T - (T^k_i)^T \right)^s_\mathbf{w}_k \]
with \( s_k \triangleq w_k - \mathbf{w}, \mathbf{s}_k \triangleq \mathbf{w} - w_k \) and \( V_k \) is defined similarly to \( W_k \), with \( (L^k_D)^T + (\Gamma^k_D)^T \) replacing \( (T^k_i)^T \), for \( * \in \{+, -\} \) and \( \mathbf{w}, \mathbf{w}, \) and \( e_k \) replacing \( \mathbf{s}, \mathbf{s}, \) and \( w_k \), respectively. As before, we can use the comparison system
\[ \hat{e}_{k+1} = H_\ast \hat{A} \hat{e}_k + H_\ast (W_k + V_k), \quad \hat{e}_0 = e_0 \]
It is well known that stable LTI systems are ISS [30]. Again, (9) guarantees \( \hat{e}_k \geq e_k \geq 0 \) \( \forall k \geq 0 \) by induction, regardless of the values of \( W_k \) and \( V_k \). By this comparison, the ISS property of the system (11) implies that (10) is ISS. \hfill \blacksquare

Theorem 1 is only an existence result. It does not provide a method for constructing \( H_\ast \), which could be a difficult combinatorial problem. Therefore, in the next section we provide a tractable approach that allows for the computation of stabilizing gains and the corresponding \( H_\ast \) in Theorem 1.

C. Distributed Stabilizing of the Error Dynamics

In this subsection, we show that the ISS property formalized in Theorem 1 can be tractably verified in a constructive and distributed manner. The approach is motivated by the representation (8), in which \( H_k \) exchanges rows of \( A \) to achieve the best possible estimate. The main idea is that each agent, depending on its observation of the system, contributes to stabilizing the state trajectory in some, not necessarily all, dimensions. Furthermore, the conditions in Assumption 1, can be verified using a distributed procedure.
described in Algorithm 2, summarized here. First, each node \( i \in V \) independently solves the following linear program:

\[
\min_{(E^i, T^i, \Gamma^i)} \sum_{i=1}^{n} \sum_{s=1}^{n} E_{ij}^{i} \\
\text{s.t. } -E^i \preceq T^i A - L^i C^i \preceq E^i, T^i = I_n - \Gamma^i C^i. \tag{12}
\]

Then, nodes exchange their \( \hat{A}^i \) matrices with increasingly larger neighborhoods until stabilizing agents are found for every state. The following lemma formalizes this result.

**Algorithm 2** DIO initialization at node \( i \).

**Input:** \( A, C^i, N_i; \) **Output:** \( L^i, T^i, \Gamma^i, d^* \)

1. Compute \( L^i : T^i, \Gamma^i \) by solving (12);
2. \( Q_i \leftarrow \{ T^i : A - L^i : C^i \} ; d^* \leftarrow 1 ; \)
3. while \( d^* \leq \text{diam} \mathcal{G} \) do
   4. if \( \forall s \in \{1, \ldots, n\}, \exists P \in Q_i \) s.t. \( \| (P) \|_1 < 1 \) then
      5. break
   6. end if
   7. Send \( Q_i \) to \( j \in N_i \) and receive \( Q_j \) from \( j \in N_i \);
   8. \( Q_i \leftarrow \bigcup_{j \in N_i} Q_j ; d^* \leftarrow d^* + 1 ; \)
9. end while
10. for \( t = 1 \) to \( \text{diam} \mathcal{G} \) do \( d^* \leftarrow \max_{j \in N_i} d^*_j \) end for
11. return \( L^i : T^i, \Gamma^i, d^* \)

**Lemma 5.** Assumption 1 holds if and only if Algorithm 2 returns \( d^* \leq \text{diam} \mathcal{G} \).

**Proof.** Assume Assumption 1 does not hold for some agent \( i \). Then the condition in line 4 of Algorithm 2 will never be met, resulting in \( d^* = \text{diam} \mathcal{G} + 1 \). After the max consensus on line 10, all agents will return \( d^* = \text{diam} \mathcal{G} + 1 \). On the other hand, if Assumption 1 holds, the condition in line 4 will be met after less than \( \text{diam} \mathcal{G} \) iterations for every node. \( \blacksquare \)

Next, we show that the solutions to the LP in (12) are the corresponding stabilizing observer gains.

**Theorem 2** (Distributed Interval Observer Design). Suppose Assumption 1 holds. Then the DIO algorithm is ISS with \( d = d^* \) and the corresponding observer gains \( L^*, T^*, \Gamma^* \) that are solutions to (12).

**Proof.** We will construct \( H_* \), which by Theorem 1 is sufficient for the ISS property to hold. For each node \( i \in V \) and state \( s \in \{1, \ldots, n\}, \) using \( \ell(i, s) \) from Assumption 1,

\[
(H_*)^{\text{id}(i,s)}_{\text{id}(i,s)} = 1, \quad (H_*)^{\text{id}(i,s)}_{\text{id}(\ell(i,s),s)} = 1,
\]

and all other entries are zero. Since \( \ell(i, s) \in \mathcal{N}_d^* \), then \( H_* \) can be constructed according to (9) for some \( \bar{z} \) and \( \bar{\tau} \). With \( H_* \) defined as such, rows \( \text{id}(i, s) \) and \( \text{id}(\ell(i, s)) \) of \( H_* A \) are equal to rows \( \text{id}(\ell(i, s), s) \) and \( \text{id}(\ell(i, s), s) \) of \( \bar{A} \), respectively (cf. Lemma 3). From the definition of \( \bar{A} \), it is clear that \( (\bar{A}^i)^{11} = (\bar{A}^i)^{18} = (\bar{A}^i)^{13} = 1 \). Note that the gains \( T^* \) and \( L^* \) are computed by (12), which independently minimize the 1-norm of each row of \( \bar{A} \), since the \( s^8 \) rows of \( T^* \) and \( L^* \) only affect the \( s^8 \) row of \( \bar{A} \). Moreover, Assumption 1 guarantees \( (\bar{A}(\ell(i, s), s))_1 \leq 1 \) for each \( s \). All of this implies \( (H_* A)^{\text{id}(i,s)}_{\text{id}(i,s)} = (H_* A)^{\text{id}(\ell(i,s),s)}_{\text{id}(\ell(i,s),s)} \leq 1 \). Since this holds for every row of the matrix \( H_* A \), then

\[
\rho(H_* A) \leq \| H_* A \| \leq \max_{1 \leq s \leq 2N} \sum_{s=1}^{2N} (H_* A)^{s1} \leq 1. \quad \blacksquare
\]

Since Assumption 1 is a sufficient condition in Theorem 2, even if it doesn’t hold, or if the algorithm is executed with \( d < d^* \), it is possible that solving (12) will result in stabilizing gains which can be verified by Theorem 1. In fact, since (12) is not dependent on the graph \( \mathcal{G} \), the computed observer gains will be the same for any network structure. Finally, note that the proposed algorithms admit a very low computational complexity since the design procedure is being done completely offline for each agent, i.e., all the LPs are solved only once and in a distributed manner.

**V. Numerical Examples**

This section demonstrates the effectiveness of our distributed interval observer applied to two LTI target systems. Consider the following multi-agent system (a discretized version of the example from [24], with \( \Delta_t = 0.5 \) in the form of (1) with \( N = 6, n = 12, A = I_6 \otimes [0.01 0 1 \ 0 1 0 \ 0 0 1] \), \( B = I_6 \otimes [0.01 0.0001 \ 0 0.01 \ 0 0.01 \ 0 0.01 \ 0 0.01 \ 0 0.01 \ 0 0.01] \), and \( C^i = e_{i} \otimes [1 0] \). The communication graph \( \mathcal{G} \) is a directed ring with nodes \( V = \{1, \ldots, 6\} \) and edges \( \mathcal{E} = \{(i, (i+1 \mod 6)) : i \in V\} \). Moreover, the initial state is \( x_0 = I_6 \otimes [0.7032 \ 0.0457] \) and the process noise is given by \( w_k = 1_4 \otimes [\frac{1}{2} \sin(0.01k) \ 0 \ \frac{1}{2} \cos(0.01k)]^T \), \( \bar{w} = 1_4 \otimes [\frac{1}{2} \ 0 - \frac{1}{2} \ 0 - \frac{1}{2} \ 0 - \frac{1}{2} \ 0 - \frac{1}{2} \ 0 - \frac{1}{2} \ 0 - \frac{1}{2} \ 0 - \frac{1}{2}]^T \). The measurement noise is given by \( v_k = v_k^1 = \frac{1}{2} \sin(0.01k) \), \( v_k^2 = v_k^3 = \frac{1}{2} \cos(0.02k) \), with bounds \( v_1^4 = v_1^5 = 1 \), \( v_2^4 = v_2^5 = 0 \), \( v_3^4 = v_3^5 = 0 \), \( v_4^4 = v_4^5 = 0 \), \( v_5^4 = v_5^5 = 0 \), \( v_6^4 = v_6^5 = 0 \). The DIO gains for each node \( i \in V \) are \( L^i = e_i \otimes [0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1] \). Assumption 1 is satisfied with \( d^* = 5 \), but \( d = 1 \) still results in a stable observer. Figure 1 shows the state trajectories and interval widths from the DIO algorithm with \( d = d^* \). For comparison, we also plot the results of the continuous time distributed observer proposed in the work in [24], simulated with \( \Delta_t = 0.01 \), and the centralized observer proposed in [28], also simulated with \( \Delta_t = 0.5 \). As can be seen, both our observer and the centralized [28] are able to maintain tighter intervals around the true states than [24]. In fact, while centralized and distributed framers are identical in some state dimensions (\( x_3 \) and \( x_{10} \)), the distributed framers are better than the centralized in others (\( x_9 \)). Finally, Figure 2 shows the root mean square (RMS) error of the observer trajectories compared against the number of network update iterations \( d \) from 1 to \( \text{diam} \mathcal{G} \). It is evident that the error decreases as \( d \) is increased, and eventually even surpasses the performance of the centralized observer. This performance advantage is due to the additional design variables introduced in our distributed algorithm where different gains are designed for each individual agent, rather than just a single centralized set of gains. Moreover, the network update (5) allows the agents to select the best possible estimate depending on the specific realizations of the measurement and process noise, which can further increase the performance.
In this paper, a novel distributed interval observer synthesis was introduced for linear time-invariant discrete-time systems that are subject to bounded noise. In future work, extension to nonlinear settings with time-varying networks, the effect of adversarial agents and communication and measurement failures, and effective noise suppression/interval width minimization will be considered.

VI. CONCLUSION AND FUTURE WORK

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