Hypercontractivity of a semi-Lagrangian scheme for
Hamilton-Jacobi equations

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Abstract

The equivalence between logarithmic Sobolev inequalities and hypercontractivity of
solutions of Hamilton-Jacobi equations has been proved in [5]. We consider a semi-
Lagrangian approximation scheme for the Hamilton-Jacobi equation and we prove that
the solution of the discrete problem satisfies a hypercontractivity estimate. We apply this
property to obtain an error estimate of the set where the truncation error is concentrated.

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estimate.

1 Introduction

Consider the Hamilton-Jacobi equations

\[ u_t + H(Du) = 0, \quad x \in \mathbb{R}^N, \ t > 0 \]  \hspace{1cm} (1.1)

where \( H : \mathbb{R}^N \to \mathbb{R} \) is convex and \( \lim_{|p| \to +\infty} \frac{H(p)}{|p|} = +\infty \). For \( f \) a Lipschitz continuous
function, define

\[ S_t f(x) = \inf_{y \in \mathbb{R}^N} \left\{ f(y) + t L \left( \frac{x-y}{t} \right) \right\} \hspace{1cm} (1.2)\]

where \( L \) is the Legendre transform of \( H \), i.e.

\[ L(q) = \sup_{p \in \mathbb{R}^N} \{ p \cdot q - H(p) \}. \hspace{1cm} (1.3) \]

The family \( (S_t)_{t \geq 0} \) defines a semigroup with infinitesimal generator \(-H(Df)\) and the solution
of the equation (1.1) with initial datum \( u(x, 0) = f(x) \) is given by \( u(x, t) = S_t f(x) \).

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The link between Hamilton-Jacobi equations with $H(p) = \frac{|p|^2}{2}$ and logarithmic Sobolev inequality (LSI in short) is given in [5]. We recall that the classical LSI can be written as

$$E_\mu(u^2) \leq \frac{2}{\rho} \int_{\mathbb{R}^N} |Du|^2 d\mu,$$

where

$$E_\mu(u^2) = \int_{\mathbb{R}^N} u^2 \log u^2 d\mu - \int_{\mathbb{R}^N} u^2 d\mu \log \int_{\mathbb{R}^N} u^2 d\mu,$$

$\mu$ is a probability measure and $\rho$ is a positive real number. The typical example of measure satisfying the inequality (1.4) is the canonical Gaussian measure with density $(2\pi)^{-N/2}e^{-|x|^2/2}$ with respect to the Lebesgue measure in $\mathbb{R}^N$ (in this case $\rho = 1$).

In [5] the authors prove that if $\mu$ satisfies (1.4), then for every $t > 0$ and $a \in \mathbb{R}$ the solution of (1.1) satisfies the hypercontractivity estimate

$$\|e^{S_t f}\|_{a+\rho t} \leq \|e^f\|_a$$

($\|\cdot\|_p$ is the $L^p$-norm associated to the measure $\mu$). Conversely, if (1.5) holds for all $t > 0$ and some $a \neq 0$ then (1.4) holds.

Aim of this paper is to show that a similar hypercontractivity property holds for a semi-Lagrangian approximation of (1.1). Semi-Lagrangian schemes are a well studied class of approximation schemes for Hamilton-Jacobi equation (see [2], [6]). For a fixed discretization step $h$, the semi-Lagrangian scheme generates a discrete-time semigroup $(Q_n)_{n \in \mathbb{N}}$ and the solution of the approximate equation with initial datum $u(x,0) = f(x)$ is given by $u(x,nh) = (Q_n f)(x)$. We show that if $\mu$ satisfies (1.4), then the semigroup $(Q_n)_{n \in \mathbb{N}}$ satisfies at the discrete time $nh$ the hypercontractivity estimate

$$\|e^{Q_n f}\|_{\lambda_n} \leq \|e^f\|_a \prod_{k=1}^n (1 + C\lambda_{k-1} h)^\frac{1}{\lambda_k},$$

(1.6)

where $\lambda_n = a + \rho n h$. And, as in the continuous case, also the converse is true.

It is by now classical that approximation schemes for Hamilton-Jacobi equations give an error estimate of order $h^{1/2}$ in the $L^\infty$-norm, while $L^p$-estimates are known only in some particular cases (see [3], [8], [9]). We apply (1.6) to give an estimate of the set where the truncation error is concentrated showing that its measure decays exponentially. We note that similar estimates are obtained for the approximation of stochastic differential equations via Euler schemes (see [10], [11]).

The paper is organized as follows.

In Section 2 we study the property of the discrete-time semigroup. In Section 3 we prove the equivalence between the hypercontractivity estimate and the logarithmic Sobolev inequality with respect to a gaussian measure, while in Section 4 we study a similar property for the Lebesgue measure. Finally, in Section 5 we prove the concentration estimate.

### 2 The discrete semigroup

Fixed a discretization step $h > 0$ and given a continuous function $u_0$, consider the semi-Lagrangian scheme for (1.1) (see [6])

$$\frac{u_{n+1}(x) - u_n(x)}{h} + \sup_{q \in \mathbb{R}^N} \left\{ -\frac{u_n(x - hq) - u_n(x)}{h} - L(q) \right\} = 0,$$

(2.1)
or equivalently
\[ u_{n+1}(x) = \inf_{q \in \mathbb{R}^N} \{ u_n(x - hq) + hL(q) \} \]  
(2.2)

Given a continuous function \( f \), define
\[ (Q_n f)(x) = \inf_{q_0, \ldots, q_{n-1} \in \mathbb{R}^N} \left\{ f \left( x - \sum_{k=0}^{n-1} hq_k \right) + \sum_{k=0}^{n-1} hL(q_k) \right\} \]  
(2.3)

(with the convention that \( \sum_{k=0}^{n-1} = 0 \)). In the next proposition we show that the family \( (Q_n)_{n \in \mathbb{N}} \) generates a discrete-time semigroup giving the solution of (2.4) with initial datum \( u_0(x) = f(x) \).

**Proposition 2.1.**

1) \( (Q_{n+m} f)(x) = (Q_n(Q_m f))(x) \) for any \( n, m \in \mathbb{N} \) and \( (Q_0 f)(x) = f(x) \). Moreover \( (Q_n(f + c))(x) = (Q_n f)(x) + c \) for any constant \( c \in \mathbb{R} \).

2) \( u_n(x) = (Q_n f)(x) \) is the solution of (2.1) with \( u_0(x) = f(x) \).

3) If \( f \) is bounded and Lipschitz continuous, then \( Q_n f \) is bounded and Lipschitz continuous and
\[ \|Q_{n+1} f \|_\infty \leq \|Q_n f \|_\infty + h \max \{-H(0), L(0)\} \]  
for any \( n \in \mathbb{N} \)  
(2.4)

\[ |(Q_n f)(x) - (Q_n f)(y)| \leq \|D f\|_\infty |x - y| \]  
for any \( n \in \mathbb{N}, x, y \in \mathbb{R}^N \)  
(2.5)

\[ |(Q_m f)(x) - (Q_n f)(x)| \leq C|n - m|h \]  
for any \( n \in \mathbb{N}, x \in \mathbb{R}^N \).  
(2.6)

with \( C \) independent of \( h \).

4) If \( f \) is semiconcave, then \( Q_n f \) is semiconcave in \( x \), uniformly in \( n \).

**Proof.** To prove 1), observe that
\[
(Q_n(Q_m f))(x) = \inf_{q_1, \ldots, q_n} \left\{ (Q_m f) \left( x - \sum_{k=0}^{n-1} hq_k \right) + \sum_{k=0}^{n-1} hL(q_k) \right\} \\
= \inf_{q_1, \ldots, q_n} \inf_{q_{n+1}, \ldots, q_m} \left\{ f \left( x - \sum_{k=0}^{n-1} hq_k - \sum_{k=0}^{m-1} hq_{n+1+k} \right) + \sum_{k=0}^{m-1} hL(q_{n+1+k}) \right\} + \sum_{k=0}^{n-1} hL(q_k) \\
= (Q_{n+m} f)(x).
\]

The commutativity with the constants is immediate.

Set \( u_n = Q_n f \). By 1), \( u_{n+1}(x) = (Q_{n+1} f)(x) = (Q_1(Q_n f))(x) = (Q_1 u_n)(x) \), hence \( u_n \) is the solution of (2.2) with \( u_0(x) = (Q_0 f)(x) = f(x) \).

By (2.2) for \( q = 0 \), we have
\[ u_{n+1}(x) \leq u_n(x) + hL(0). \]  
(2.7)

Moreover since \( L(q) \geq -H(0) \) for any \( q \in \mathbb{R}^N \), we have
\[ u_{n+1}(x) \geq -\|u_n\|_\infty - hH(0). \]  
(2.8)
By (2.7) and (2.8), we get (2.4) for \( u_n = Q_n f \).

Since \( u_n = Q_n f \) is continuous and \( L \) is superlinear, there exists \( R_n \) (increasing with respect to \( n \) but upper bounded uniformly in \( n \) and \( h \)) such that, defined \( M_n(x) = \arg\inf\{u_n(x - hq) + hL(q)\} \), then \( M_n(x) \in B(0, R_n) \) and the infimum in (2.2) is obtained. Given \( x, y \in \mathbb{R}^N \) and \( q^* \in M_n(x) \), by (2.2) we get

\[
    u_{n+1}(x) - u_{n+1}(y) \leq u_n(x - hq^*) - u_n(y - hq^*) \leq \|Du_n\|_{\infty}|x - y| = \|D(Q_n f)\|_{\infty}|x - y|.
\]

Iterating in the previous inequality, we get (2.5).

For \( q^* \in M_n(x) \) in (2.2), we have

\[
    u_{n+1}(x) - u_n(x) = u_n(x - hq^*) - u_n(x) + hL(q^*) \leq \|Du_n\|_{\infty}|h|q^*| + hL(q^*)
\]

which gives (2.6) since \( M_n(x) \) is uniformly bounded.

Assume that \( u_n \) is semi-concave with constant \( C_n \). If \( q^* \in M_n(x) \), then

\[
    u_{n+1}(x) = u_n(x - hq^*) + hL(q^*)
\]

\[
    u_{n+1}(x \pm y) \leq u_n(x - hq^* \pm y) + hL(q^*).
\]

Hence

\[
    u_{n+1}(x + y) + u_{n+1}(x - y) - 2u_{n+1}(x)
\]

\[
    \leq u_n(x - hq^* + y) + u_n(x - hq^* - y) - 2u_n(x - hq^*) \leq C_n |y|^2.
\]

Hence \( C_{n+1} \leq C_n \) and iterating \( C_{n+1} \leq C_0 \)

where \( C_0 \) is the semi-concavity constant of \( u_0 = f \).

\[\square\]

Remark 2.2. Note that (2.2) can be rewritten as

\[
    (Q_{n+1} f)(x) = \inf_{q \in \mathbb{R}^N} \left\{ (Q_n f)(x) + hL \left( \frac{x - y}{h} \right) \right\}.
\]

Hence the discrete semigroup \( Q_n \) is obtained by considering the continuous semigroup \( S_t \) only at the discrete time \( t = nh, n \in \mathbb{N} \).

3 Hypercontractivity of the discrete semigroup with respect to Gaussian measures

We prove the hypercontractivity of the discrete semigroup \( (Q_n)_{n \in \mathbb{N}} \) with respect to a measure \( \mu \) satisfying (1.4). In this section \( \|\cdot\|_p \) is the \( L^p \)-norm associated to the measure \( \mu \). We only consider the case \( H(p) = \frac{1}{2}|p|^2 \) and therefore \( L(q) = \frac{1}{2}|q|^2 \), but the results can be extended to more general Hamiltonians as in [3].

**Theorem 3.1.** If \( \mu \) is absolutely continuous with respect to the Lebesgue measure and satisfies the logarithmic Sobolev inequality (1.4), then for any \( f \) Lipschitz continuous and for any \( n \), any \( h > 0 \), \( a \in \mathbb{R} \)

\[
    \|e^{Q_n f}\|_{\lambda_n} \leq \|e^f\|_a \prod_{k=1}^{n} (1 + C\lambda_{k-1} h)^{\frac{1}{a}}.
\]

(3.1)
By (3.5) and (3.6), multiplying for $\lambda_n = a + \rho n h$.

Conversely if (3.1) holds for any smooth function $f$, for any $n$, any $h > 0$ and some $a \neq 0$, then the measure $\mu$ satisfies the logarithmic Sobolev inequality (1.4).

Proof. We define

$$F_n = \|e^{Q_n f}\|_{\lambda_n} = \left(\int e^{\lambda_n Q_n f(x)} d\mu\right)^{\frac{1}{\lambda_n}} \quad (3.2)$$

and we prove that

$$F_{n+1} \leq F_n (1 + C \lambda_n h)^{\frac{1}{\lambda_n+1}}. \quad (3.3)$$

Then (3.1) is obtained iterating over $n$.

We consider a measure $\mu$ which satisfies (1.4) and we start with the identity

$$F_{n+1}^{\lambda_{n+1}} - F_n^{\lambda_n} = \int e^{\lambda_{n+1} Q_{n+1} f} d\mu - \int e^{\lambda_n Q_n f} d\mu. \quad (3.4)$$

We consider first the term on the right hand side of (3.4). We have

$$F_{n+1}^{\lambda_{n+1}} - F_n^{\lambda_n} = F_{n+1}^{\lambda_{n+1}} - F_n^{\lambda_{n+1}} + F_n^{\lambda_{n+1}} - F_n^{\lambda_n} = e^{\lambda_{n+1} \ln(F_n)} \left[e^{\lambda_{n+1} (\ln(F_{n+1}) - \ln(F_n))} - 1\right] + e^{\lambda_n \ln(F_n)} \left[e^{(\lambda_{n+1} - \lambda_n) \ln(F_n)} - 1\right] = F_{n+1}^{\lambda_{n+1}} \left[\left(\frac{F_{n+1}}{F_n}\right)^{\lambda_{n+1}} - 1\right] + F_n^{\lambda_n} [e^{(\lambda_{n+1} - \lambda_n) \ln(F_n)} - 1]. \quad (3.5)$$

We now consider the term on the right hand side of (3.4)

$$\int e^{\lambda_{n+1} Q_{n+1} f} d\mu - \int e^{\lambda_n Q_n f} d\mu = \int e^{\lambda_{n+1} Q_{n+1} f} - e^{\lambda_n Q_n f} d\mu + \int e^{\lambda_n Q_n f} [e^{(\lambda_{n+1} - \lambda_n) Q_n f} - 1] d\mu. \quad (3.6)$$

By (3.5) and (3.6), multiplying for $\lambda_n$, we get

$$\lambda_n F_n^{\lambda_{n+1}} \left[\left(\frac{F_{n+1}}{F_n}\right)^{\lambda_{n+1}} - 1\right] = \lambda_n \int e^{\lambda_{n+1} Q_n f} [e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} - 1] d\mu + \lambda_n \int e^{\lambda_n Q_n f} [e^{h \rho Q_n f} - 1] d\mu - \lambda_n F_n^{\lambda_n} [e^{h \rho \ln(F_n)} - 1] = h \rho \text{Ent}(e^{\lambda_n Q_n f}) + \lambda_n \int e^{\lambda_{n+1} Q_n f} [e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} - 1] d\mu + \lambda_n \int e^{\lambda_n Q_n f} [e^{h \rho f} - 1 - h \rho Q_n f] d\mu + h \lambda_n^2 \int e^{\lambda_n Q_n f} \frac{D Q_n f}{2} d\mu + \lambda_n \int e^{\lambda_{n+1} Q_n f} [e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} - 1] d\mu \leq h \lambda_n^2 \int e^{\lambda_n Q_n f} \frac{D Q_n f}{2} d\mu + \lambda_n \int e^{\lambda_{n+1} Q_n f} [e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} - 1] d\mu + \lambda_n \int e^{\lambda_n Q_n f} [e^{h \rho f} - 1 - h \rho Q_n f] d\mu - \lambda_n F_n^{\lambda_n} [e^{h \rho \ln(F_n)} - 1 - h \rho \ln(F_n)]. \quad (3.7)$$
Observing that \( \lambda_n \leq \lambda_{n+1} \), \( |e^{h\rho Q_n f} - 1 - h\rho Q_n f| \leq C h^2 \), \( Q_{n+1} f \leq Q_n f \) and the last term on the right hand side of (3.7) is negative we get

\[
\lambda_n F^\lambda_{n+1} \left( \frac{F_{n+1}}{F_n} \right)^{\lambda_{n+1}} - 1 \leq C \lambda_n^2 h F_n^\lambda_{n+1} + C \lambda_n F_n^\lambda_{n+1} h^2 \leq C \lambda_n^2 h F_n^\lambda_{n+1}
\]

and therefore

\[
F_n^\lambda_{n+1} \leq F_n^\lambda_n (1 + C \lambda_n h) \leq F_n^\lambda_{n+1} (1 + C \lambda_n h)
\]

which gives (3.8).

To prove the converse, given \( t > 0 \), let \( t_n = nh \) converging to \( t \) for \( h \to 0 \) and \( n \to \infty \). By standard stability results in viscosity solution theory (see [2]) \( Q_n f \) converges to \( S_t f \) uniformly in \( x \), where \( S_t \) is the semigroup associated to (1.1). Moreover \( \lambda_n \to \lambda(t) = a + \rho t \) and \( (1 + C \lambda_{k-1} h)^{1/\lambda_k} \to 1 \). Hence by (3.1), for \( h \to 0 \) we get the hypercontractivity of the continuous semigroup

\[
\| e^{S_t f(x)} \|_{\lambda(t)} \leq \| e^f \|_a.
\]

Then the statement follows since it is well known that if the inequality (3.9) holds for some \( a \neq 0 \) and for any smooth function \( f \), the measure \( \mu \) satisfies (1.4) (see [5]).

![Figure 1: Behavior of \( f(h) = \prod_{k=1}^{10} (1 + C \lambda_{k-1} h)^{1/\lambda_k} \), fixed \( \rho = 1, C = 0.01 \).](image)

**4 Hypercontractivity of the discrete semigroup with respect to the Lebesgue measure**

In [12], it is proved that the semigroup \( S_t \) satisfies the following optimal hyper-contractivity inequality with respect to the Lebesgue measure

\[
\| e^{S_t f} \|_\beta \leq \left( \frac{\alpha}{\beta} \right)^{N \frac{\alpha + \beta}{2}} \| e^f \|_\alpha \left( \frac{\beta - \alpha}{t} \right)^{N \frac{\beta - \alpha}{2 \gamma \beta}} \left[ \int e^{-L(x)} dx \right]^{-\frac{\beta - \alpha}{\gamma \beta}}
\]

where \( \| \cdot \|_p \) is the \( L^p \)-norm associated to the Lebesgue measure on \( \mathbb{R}^N \). In this section we study hypercontractivity of the discrete semigroup \( Q_n \) with respect to the Lebesgue measure. We assume for simplicity that \( H(p) = \frac{1}{2} |p|^2 \) and therefore \( L(q) = \frac{1}{2} |q|^2 \).
Theorem 4.1. For any smooth function $f \in \mathbb{R}^N$, $0 < \alpha \leq \beta$, we have
\[
\|e^{Q_n f}\|_{\beta} \leq \left(\frac{\alpha}{\beta}\right)^{\frac{2N+\beta}{\alpha}} \|f\|_{\alpha} \left(\frac{\beta - \alpha}{nh}\right)^{\frac{\beta - \alpha}{\alpha} \frac{2}{\beta}} (2\pi)^{-\frac{\beta}{\alpha} \frac{2}{\beta}}.
\] (4.2)

Inequality (4.2) is optimal and equality holds if for some $0 < \alpha \leq \beta$, $x \in \mathbb{R}^N$ and $b > 0$ we have
\[
f(x) = -bL(x - \overline{x}),
\] (4.3)
and
\[
zh = \frac{\beta - \alpha(nh)^{-\frac{\alpha}{\alpha}}}{b\beta}.
\] (4.4)

Moreover, we obtain the following ultracontractive bound
\[
\|e^{Q_n f}\|_{\infty} \leq \|f\|_{1} \left(\frac{1}{nh}\right)^{\frac{\beta}{\alpha}} (2\pi)^{-\frac{\beta}{\alpha} \frac{2}{\beta}},
\] (4.5)
and the equality holds if $nh = \frac{1}{\alpha}$.

Proof. In order to prove inequality (4.2), we are going to use the following Prékopa-Leindler inequality:

Let $a, b > 0$ such that $a + b = 1$, and $u, v, w$ three non-negative functions on $\mathbb{R}^N$; suppose that for any $x, y \in \mathbb{R}^N$ we have
\[
u(x) a \leq w(ax + by).
\] (4.6)

Then the following inequality holds
\[
\left(\int_{\mathbb{R}^N} u(x)dx\right)^{a} \left(\int_{\mathbb{R}^N} v(y)dy\right)^{b} \leq \int_{\mathbb{R}^N} w(x)dx.
\]

Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq \beta$ and set $\theta = \frac{\beta - \alpha}{\alpha nh}$. For any $x \in \mathbb{R}^N$ we define
\[
u(x) = e^{\beta Q_n f(x)},
\nu(x) = e^{-\theta |y|^2},
\nu(x) = e^{af\left(\frac{x}{\alpha}\right)}.
\]

We prove that $u, v$ and $w$ verify the hypothesis of the Prékopa-Leindler inequality (4.6) with
\[
a = \frac{\alpha}{\beta}, \quad b = \frac{\beta - \alpha}{\beta}.
\] (4.7)

By (4.6) for any $\{q_1, \ldots, q_n\} \subset \mathbb{R}^N$
\[
u(x)^n e^{\beta Q_n f(x)} \leq e^{af(x-h \sum_{k=0}^{n-1} q_k)+ah \sum_{k=0}^{n-1} \frac{|q_k|^2}{2}},
\]
hence
\[
u(x)^a v(y)^b \leq e^{af(x-h \sum_{k=0}^{n-1} q_k)+ah \sum_{k=0}^{n-1} \frac{|q_k|^2}{2} - \theta \frac{\beta - \alpha}{\alpha} |y|^2}.
\]
Hence we can apply the Prékopa-Leindler inequality to obtain

\[ x - h \sum_{k=0}^{n-1} q_k = x + \frac{\beta - \alpha}{\alpha} y \]

then we get

\[ q = -\frac{\beta - \alpha}{\alpha n h} y \]

and therefore

\[
\begin{align*}
 u(x)^a v(y)^b & \leq e^{\alpha f(x - h \sum_{k=0}^{n-1} q_k)} + \alpha \frac{\beta - \alpha}{\alpha} \sum_{k=0}^{n-1} (\frac{\beta - \alpha}{\alpha n h})^2 |y|^2 - \frac{\beta - \alpha}{\alpha n h} \sum_{k=0}^{n-1} |y_k|^2 \\
& = e^{\alpha f(x + \frac{\beta - \alpha}{\alpha n h} y)} = e^{\alpha f(\frac{\beta - \alpha}{\alpha n h} y)} = e^{\alpha f(\frac{\beta - \alpha}{\alpha n h} y)} = w(ax + by).
\end{align*}
\]

Hence we can apply the Prékopa-Leindler inequality to obtain

\[
\left( \int e^{\beta Q_n f(x)} dx \right)^a \left( \int e^{-\theta |x|^2} dx \right)^b \leq \int e^{\alpha f(\frac{x}{\alpha})} dx
\]

\[ \Rightarrow \left( \int e^{\beta Q_n f(x)} dx \right)^\frac{a}{b} \leq \left( \int e^{\alpha f(\frac{x}{\alpha})} dx \right)^\frac{a}{b} \left( \int e^{-\theta |x|^2} dx \right)^{-\frac{a}{b}}. \]

It follows that

\[
\| e^{Q_n f} \|_{L^b} \leq \left( \frac{\alpha}{\beta} \right)^\frac{a}{b} \| e^f \|_{\alpha} \left( \int e^{-\theta |x|^2} dx \right)^{-\frac{a}{b}} \tag{4.8}
\]

We compute

\[
\left( \int e^{-\theta |x|^2} dx \right)^{-\frac{a}{b}} = \left( \int e^{-\frac{\theta}{2} |x|^2} dx \right)^{-\frac{a}{b}} = \left( \int e^{-\theta |y|^2} dy \right)^{-\frac{a}{b}} = \left( \frac{2\pi}{\theta} \right)^{-\frac{a}{b}} = \frac{2\pi^{\frac{\alpha}{2}}}{\left( \beta \left( \beta - \alpha \right) \right)^\frac{a}{b}}.
\]

Substituting in (4.8) we get (4.2).

In order to prove the optimality, we compute the terms \( \| e^{Q_n f} \|_{\beta} \) and \( \| e^f \|_{\alpha} \) appearing in (4.2) for \( f \) as in (4.3). We obtain

\[
\| e^{Q_n f} \|_{\beta} = \left( \int e^{\beta Q_n f(x)} dx \right)^\frac{1}{b} = \left( \int e^{-\frac{\beta}{1-n h b} L(x - \alpha') dx} \right)^\frac{1}{b} = \left( \int e^{-\frac{k}{1-n h b} L(z) dz} \right)^\frac{1}{b} = \frac{2\pi(1 - n h b)}{b \beta} \frac{\alpha}{2 \beta}
\]

and

\[
\| e^f \|_{\alpha} = \int (e^{-\alpha f(x)} dx) = \int (e^{-\alpha f(x)} dx) = \frac{2\pi^{\frac{\alpha}{2}}}{\beta \left( \beta - \alpha \right)} \frac{\alpha}{2 \beta},
\]

\[ = \int (e^{-\alpha f(x)} dx) = \frac{2\pi^{\frac{\alpha}{2}}}{\beta \left( \beta - \alpha \right)} \frac{\alpha}{2 \beta}. \]

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and we obtain an equality in (4.2) for

\[ 1 - nhb = \alpha \beta \left( \frac{1}{nh} \right)^{\frac{\alpha}{\beta}}, \]

i.e. (4.4).

The ultracontractive bound (4.5) is obtained for \( \beta \to +\infty \) and \( \alpha = 1 \) in (4.2). Furthermore, if

\[ \| e^{Q_t f} \|_{\infty} = 1, \quad \| e^f \|_1 = \left( \frac{2\pi}{b} \right)^{\frac{N}{2}}, \]

the equality in (4.5) is obtained for \( nh = \frac{1}{b} \).

\[ \Box \]

**Remark 4.2.** Consider the constant appearing in (4.2), that is

\[ C = \left( \frac{\alpha}{\beta} \right)^{\frac{N-1}{2}} \left( \frac{\beta - \alpha}{nh} \right)^{\frac{N}{2} - 1} \left( 2\pi \right)^{\frac{N}{2} - 1} \frac{\beta - \alpha}{\beta^n}. \]  

(4.9)

We observe that for fixed values of \( \alpha = \frac{\beta}{2}, nh = 1 \), we have

\[ C = \left( \frac{1}{2} \right)^{\frac{N}{2\pi}} \left( \frac{\beta}{4\pi} \right)^{\frac{N}{2\pi}}. \]

![Figure 2: Behaviour of the constant (4.9) as a function of \( \beta \), fixed \( \alpha = \frac{\beta}{2}, N = 1 \).](image)

In this case the graph of the constant coincides with the one of the constant in (4.1) for \( t = nh \) (see Fig. 4.2).

In the next proposition we give a hypercontractivity estimate with respect to the Lebesgue measure for the discrete semigroup \( Q_n \) similar to one of Theorem 3.1.

**Proposition 4.3.** For any \( f \) Lipschitz continuous and for any \( n \), any \( h > 0 \), any increasing sequence \( \{ \beta_k \}_{k \in \mathbb{N}} \) with \( \beta_0 > 0 \), we have

\[ \left\| e^{Q_n f} \right\|_{\beta_n} \leq \left\| e^f \right\|_{\beta_0} \prod_{k=1}^{n} \left( \frac{2\pi}{\beta_{k-1}} \right)^{\frac{N}{2}} \left( \frac{2\pi}{\beta_k} \right)^{-\frac{N}{2}} \frac{\beta_k - \beta_{k-1}}{\beta_k - \beta_{k-1}}. \]  

(4.10)
Proof. As in the proof of Theorem 4.1 we set
\[ u(x) = e^{\beta Q_n f(x)}, \quad v(x) = e^{-\theta |x|^2}, \quad w(x) = e^{\alpha Q_{n-1} f(\frac{x}{\alpha})} \]
and we prove that \( u, v \) and \( w \) verify the hypothesis of the Prékopa-Leindler inequality, with \( a \) and \( b \) as in (4.7). By (2.2)
\[ u(x)^a v(y)^b \leq e^{\alpha Q_{n-1} f(x-hq) + h^2 - \frac{\beta}{2} + q^2} \]
Choosing \( q = -\frac{\beta - \alpha}{\alpha h} y \) and \( \theta = -\frac{\beta - \alpha}{\alpha h} \beta, \) we obtain
\[ u(x)^a v(y)^b \leq e^{\alpha Q_{n-1} f(x + \frac{\beta - \alpha}{\alpha} y)} = e^{\alpha Q_{n-1} f(\frac{\beta}{\alpha}(ax+by))} = w(x). \]
Hence we can apply the Prékopa-Leindler inequality and arguing as for estimate (4.8), we obtain
\begin{align*}
\left\| e^{Q_n f} \right\|_\beta &\leq \left( \frac{\alpha}{\beta} \right)^{\frac{N}{2}} \left\| e^{Q_{n-1} f} \right\|_\alpha \left( \int e^{-\theta |x|^2} \right)^{-\frac{\beta - \alpha}{2 \alpha h}} \\
&= \left( \frac{\alpha}{\beta} \right)^{\frac{N}{2}} \left\| e^{Q_{n-1} f} \right\|_\alpha \left[ \frac{2\pi n h}{\beta (\beta - \alpha)} \right]^{-\frac{\beta - \alpha}{2 \alpha h}}.
\end{align*}
(4.11)
For \( \beta = \beta_n, \alpha = \beta_{n-1} \) in (4.11), we get
\[ \left\| e^{Q_n f} \right\|_\beta \leq \left( \frac{\beta_{n-1}}{\beta_n} \right)^{\frac{N}{2}} \left\| e^{Q_{n-1} f} \right\|_{\beta_{n-1}} \left( \frac{2\pi n h \beta_{n-1}}{\beta_n (\beta_n - \beta_{n-1})} \right)^{-\frac{\beta_n - \beta_{n-1}}{2 \alpha h}}. \]
Iterating the previous argument for \( n-1, n-2, \ldots, 0 \) we finally get the hypercontractivity estimate (4.10) for \( Q_n. \)

Remark 4.4. In particular, if we set \( \beta_k = \beta_0 + \rho kh, \quad \beta_{k-1} = \beta_0 + \rho (k-1)h, \) in (4.10) we have
\[ \lim_{h \to 0} \left( \frac{\beta_{k-1}}{\beta_k} \right)^{\frac{N}{2(b_k-1)}} = \lim_{h \to 0} \left( \frac{\beta_0 + \rho (k-1)h}{\beta_0 + \rho kh} \right)^{\frac{N}{2(\beta_0 + \rho (k-1)h)}} = 1. \]
and
\[ \lim_{h \to 0} \left( \frac{2\pi \beta_{k-1}}{\beta_k - \beta_{k-1}} \beta_k h \right)^{-\frac{\beta_k - \beta_{k-1}}{2 h^2}} = 1. \]
Comparing the graph in Fig.3 with the graph in Fig.1 we see that the constant in (4.10) converges to 1 by values lower than 1, whereas the constant in (3.1) by values greater than 1.
Figure 3: Behavior of \( f(h) = \prod_{k=1}^{10} \left( \frac{\beta_{k-1}}{\beta_k} \right)^{n_k-1} \left( \frac{2\pi\beta_{k-1}}{\beta_k-\beta_{k-1}} \beta_k h \right) \) for \( \rho = \beta_0 = N = 1. \)

5 A concentration estimate for the approximation error

It is well known that for \( h \to 0 \), the discrete solution computed via the scheme (2.1) converges uniformly to the solution of (1.1) with an error \( \| u - u_h \|_\infty \) of order \( h^{1/2} \) (see [4], [6]). In this section we obtain an estimate of the measure of the set where the error is concentrated.

To simplify the notation we write \( S_n f \) for \( S_{nh} f \), where \( S_t \) is the continuous semigroup associated to the equation (1.1).

**Theorem 5.1.** If \( \mu \) is absolutely continuous with respect to the Lebesgue measure and satisfies the logarithmic Sobolev inequality (1.4), then for any \( f \) semiconcave and for any \( n \in \mathbb{N} \), \( h > 0 \) and \( a \in \mathbb{R} \)

\[
\| e^{S_n f} - Q_n f \|_{\lambda_n}, \| e^{Q_n f} - S_n f \|_{\lambda_n} \leq \prod_{k=1}^{n} (1 + C \lambda_k^2 h^2)^{1/2} \lambda_n
\]  

(5.1)

where \( \lambda_n = a + \rho h \) and \( \| \cdot \|_p \) is the \( L^p \)-norm associated to the measure \( \mu \).

**Proof.** Set

\[
F_n = \| e^{Q_n f} - S_n f \|_{\lambda_n} = \left( \int e^{\lambda_n (Q_n f(x) - S_n f(x))} d\mu \right)^{\lambda_n}. 
\]

Arguing as in Theorem 3.1, we arrive, see (3.1), to the inequality

\[
\lambda_n F_n^{\lambda_{n+1}} \left[ \left( \frac{F_{n+1}}{F_n} \right)^{\lambda_{n+1}} - 1 \right] \leq h \lambda_n^2 \int e^{\lambda_n (Q_n f(x) - S_n f(x))} \frac{|DQ_n f - DS_n f|^2}{2} d\mu + \\
+ \lambda_n \int e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} \left[ e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} - (Q_{n+1} f - Q_n f) - 1 \right] d\mu + \\
\lambda_n \int e^{\lambda_n (Q_n f(x) - S_n f(x))} \left[ e^{h \rho (Q_n f(x) - S_n f(x))} - 1 - h \rho (Q_n f(x) - S_n f(x)) \right] d\mu - \\
- \lambda_n F_n^{\lambda_n} [e^{h \rho \ln(F_n)} - 1 - h \rho \ln(F_n)].
\]

Since \( \lambda_n \leq \lambda_{n+1} \), \( |e^{h \rho (Q_n f - S_n f)} - 1 - h \rho (Q_n f - S_n f)| \leq Ch^2 \) and the last term on the right
hand side of previous inequality is negative we get
\[
\lambda_n F_n^{\lambda_n} \left( \left( \frac{F_{n+1}}{F_n} \right)^{\lambda_{n+1}} - 1 \right) \leq h \lambda_n^2 \int e^{\lambda_n(Q_n f(x) - S_n f(x))} \frac{|DQ_n f - DS_n f|^2}{2} \, d\mu
\]
\[+ \lambda_n \int e^{\lambda_n(Q_n f - S_n f)} \left[ e^{\lambda_n (Q_n f(x) - S_n f(x))} - 1 \right] \, d\mu + \lambda_n F_n^{\lambda_n} Ch^2 = \]
\[
\lambda_n \int e^{\lambda_n(Q_n f - S_n f)} \left[ e^{\lambda_n (Q_n f(x) - S_n f(x))} - 1 \right] \, d\mu + \lambda_n F_n^{\lambda_n} Ch^2.
\]
We have (see (2.6) and the correspondent property for \(S_t\))
\[
\left| \frac{S_{n+1} f - S_n f}{h} \right|, \left| \frac{Q_{n+1} f - Q_n f}{h} \right| \leq C
\]
and therefore
\[
e^{\lambda_n (Q_n f(x) - S_n f(x))} - e^{\lambda_n+1(Q_n f(x) - S_n f(x))} \leq C h
\]
and by Hopf-Lax formula
\[
\frac{\partial S_n f}{\partial t}(x) - \frac{S_{n+1} f(x) - S_n f(x)}{h} = - \sup_q \{ q \cdot DS_n f(x) - L(q) \}
\]
\[+ \inf_q \left\{ \frac{S_n f(x - h q) - S_n f(x)}{h} + L(q) \right\} \leq C_2 h
\]
where \(C_2\) depends on the semiconcavity constant of \(f\). Moreover since \(|P|^2 / 2 = \sup_q \{ q \cdot P - L(q) \}\)
\[
\frac{Q_{n+1} f - Q_n f}{h} + \frac{\partial S_n f}{\partial t} + \frac{|DQ_n f - DS_n f|^2}{2} \leq
\]
\[+ \sup_q \left\{ - \frac{Q_n f(x - h q) - Q_n f(x)}{h} - L(q) \right\} + \frac{|DQ_n f|^2}{2} - \sup_q \left\{ q \cdot DQ_n f - L(q) \right\}
\]
\[+ \sup_q \left\{ q \cdot DS_n f - L(q) \right\} + \frac{|DQ_n f - DS_n f|^2}{2} \leq C_3 h.
\]
By (5.2), (5.4), (5.3) and (5.5), we get
\[
\lambda_n F_n^{\lambda_n} \left( \left( \frac{F_{n+1}}{F_n} \right)^{\lambda_{n+1}} - 1 \right) \leq \lambda_n C \lambda_{n+1} h^2 F_n^{\lambda_n}
\]
and therefore we get
\[ F_{\lambda}^{n+1} \leq F_{\lambda}^{n+1}(1 + C\lambda^{2}h^{2}). \]
Iterating over \( n \) and taking into account that \( F_{0} = \| e^{Q_{0}f-S_{0}f} \|_{\lambda_{0}} = \| e^{f-f} \|_{\lambda_{0}} = 1 \) we get the estimate
\[ \| e^{Q_{n}f-S_{n}f} \|_{\lambda_{n}} \leq \prod_{k=1}^{n}(1 + C\lambda^{2}h^{2})^{\frac{1}{\lambda_{k}}}. \]
Exchanging the role of \( S_{n}f \) and \( Q_{n}f \) we get the other estimate in (5.1).

**Corollary 5.2.** With the same notation of Theorem 5.1, if \( f \) is semi-concave, then for any \( t \in [0, T] \), \( t = nh \), we have
\[ \int (Q_{n}f - S_{n}f) d\mu, \int (Q_{n}f - S_{n}f) d\mu \leq Ch \]
with \( C \) depending on \( T \) and the semi-concavity constant of \( f \). Moreover for any \( p < 1 \)
\[ \mu\{|S_{n}f - Q_{n}f| \geq r\} \leq C e^{-1/h^{1-p}}. \]

**Proof.** We first observe that, since \( e^{t} \) is a convex function, we have \( e^{\int \lambda_{n}(Q_{n}f-S_{n}f) d\mu} \leq \int e^{\lambda_{n}(Q_{n}f-S_{n}f) d\mu} \), hence by (5.1)
\[ e^{\int \lambda_{n}(Q_{n}f-S_{n}f) d\mu} \leq \prod_{k=1}^{n}(1 + C\lambda^{2}h^{2}) \leq \prod_{k=1}^{n} e^{C\lambda^{2}h^{2}} = e^{\sum_{k=1}^{n} C\lambda^{2}h^{2}} \]
and therefore
\[ \int (Q_{n}f - S_{n}f) d\mu \leq \sum_{k=1}^{n} C\lambda^{2}h^{2} \leq C \sum_{k=1}^{n} \lambda_{k}h^{2} = C(na + \frac{1}{2}n(n+1)h^{2})h^{2} \leq Ch \]
for \( t = nh \in [0, T] \) where \( C \) depends on \( T \) and semiconcavity constant of \( f \). To prove estimate (5.7) observe that \( \mu\{|S_{n}f - Q_{n}f| \geq r\} = \mu\{S_{n}f - Q_{n}f \geq r\} + \mu\{Q_{n}f - S_{n}f \geq r\} \) and
\[ \mu\{S_{n}f - Q_{n}f \geq r\} \leq \frac{1}{e^{\lambda_{n}r}} \int e^{\lambda_{n}(S_{n}f-Q_{n}f)} d\mu \leq e^{-\lambda_{n}r} \prod_{k=1}^{n}(1 + C\lambda^{2}h^{2})^{\lambda_{n}/\lambda_{k}} \leq e^{-ar+Ch}. \]
Taking \( r = h^{p} \) and \( a = \frac{1}{h} \) in the previous estimate we get (5.7).

**Remark 5.3.** The estimate (5.7) can be interpreted as a concentration inequality of truncation error between the solution of the continuous problem and of the discrete one.

**References**

[1] Avantaggiati, A.; Loreti, P. Hypercontractivity, Hopf-Lax type formulas, Ornstein-Uhlenbeck operators. II. Discrete Contin. Dyn. Syst. Ser. S 2 (2009), no. 3, 525–545.

[2] Bardi, M., Capuzzo Dolcetta, I. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Birkhäuser, Boston, 1997.
[3] Bokanowski, O.; Forcadel, N.; Zidani, H. $L^1$-error estimates for numerical approximations of Hamilton-Jacobi-Bellman equations in dimension 1. Math. Comp. 79 (2010), no. 271, 1395–1426.

[4] Crandall, M. G.; Lions, P.-L. Two approximations of solutions of Hamilton-Jacobi equations. Math. Comp. 43 (1984), no. 167, 1–19.

[5] Bobkov, S. G.; Gentil, I.; Ledoux, M. Hypercontractivity of Hamilton-Jacobi equations. J. Math. Pures Appl. (9) 80 (2001), no. 7, 669–696.

[6] Falcone, M.; Ferretti, R. Semi-Lagrangian Approximation Schemes for Linear and Hamilton-Jacobi Equations. MOS-SIAM Series on Optimization, to appear.

[7] Gentil, I. Ultracontractive bounds on Hamilton-Jacobi equations, Bull. Sci. Math. 126 (2002), 507–524.

[8] Lepsky, O. Spectral viscosity approximations to Hamilton-Jacobi solutions. SIAM J. Numer. Anal. 38 (2000), no. 5, 1439–1453.

[9] Lin, C.-T.; Tadmor, E. $L^1$-stability and error estimates for approximate Hamilton-Jacobi solutions. Numer. Math. 87 (2001), no. 4, 701–735.

[10] Lemaire, V.; Menozzi, S. On some non asymptotic bounds for the Euler scheme, Electron. J. Probab. 15 (2010), no. 53, 1645–1681.

[11] Malrieu, F.; Talay, D. Concentration inequalities for the Euler scheme. Monte Carlo and quasi-Monte Carlo methods 2004, 355–371, Springer, Berlin, 2006.

[12] Gentil, I. The general optimal $L^p$- Euclidean logarithmic Sobolev inequality by Hamilton-Jacobi equations, J.Funct. Anal. 202 (2003), 591–599.
Hypercontractivity of semi-Lagrangian schemes for Hamilton-Jacobi equations

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Abstract

The equivalence between logarithmic Sobolev inequalities and hypercontractivity of solutions of Hamilton-Jacobi equations has been proved in [5]. We consider a semi-Lagrangian approximation scheme for the Hamilton-Jacobi equation and we prove that the solution of the discrete problem satisfies a hypercontractivity estimate. We apply this property to obtain an error estimate of the set where the truncation error is concentrated.

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1 Introduction

Consider the Hamilton-Jacobi equations

\[ u_t + H(Du) = 0, \quad x \in \mathbb{R}^N, \ t > 0 \] (1.1)

where \( H : \mathbb{R}^N \to \mathbb{R} \) is convex and \( \lim_{|p| \to +\infty} \frac{H(p)}{p} = +\infty. \) For \( f \) a Lipschitz continuous function, define

\[ S_t f(x) = \inf_{y \in \mathbb{R}^N} \left\{ f(y) + t L \left( \frac{x-y}{t} \right) \right\} \] (1.2)

where \( L \) is the Legendre transform of \( H \), i.e.

\[ L(q) = \sup_{p \in \mathbb{R}^N} \{ p \cdot q - H(p) \}. \] (1.3)

The family \( (S_t)_{t \geq 0} \) defines a semigroup with infinitesimal generator \(-H(Df)\) and the solution of the equation (1.1) with initial datum \( u(x,0) = f(x) \) is given by \( u(x,t) = S_t f(x) \).

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The link between Hamilton-Jacobi equations with $H(p) = \frac{|p|^2}{2}$ and logarithmic Sobolev inequality (LSI in short) is given in [5]. We recall that the classical LSI can be written as

$$E_\mu(u^2) \leq \frac{2}{\rho} \int_{\mathbb{R}^N} |Du|^2 d\mu, \quad (1.4)$$

where

$$E_\mu(u^2) = \int_{\mathbb{R}^n} u^2 \log u^2 d\mu - \int_{\mathbb{R}^n} u^2 d\mu \log \int_{\mathbb{R}^n} u^2 d\mu,$$

$\mu$ is a probability measure and $\rho$ is a positive real number. The typical example of measure satisfying the inequality (1.4) is the canonical Gaussian measure with density $(2\pi)^{-N/2} e^{-|x|^2/2}$ with respect to the Lebesgue measure in $\mathbb{R}^N$ (in this case $\rho = 1$).

In [5] the authors prove that if $\mu$ satisfies (1.4), then for every $t > 0$ and $a \in \mathbb{R}$ the solution of (1.1) satisfies the hypercontractivity estimate

$$\|e^{tS}f\|_{a+\rho t} \leq \|e^f\|_a \quad (1.5)$$

($\|\cdot\|_p$ is the $L^p$-norm associated to the measure $\mu$) Conversely, if (1.5) holds for all $t > 0$ and some $a \neq 0$ then (1.4) holds.

Aim of this paper is to show that a similar hypercontractivity property holds for a semi-Lagrangian approximation of (1.1). Semi-Lagrangian schemes are a well studied class of approximation schemes for Hamilton-Jacobi equation (see [2], [6]). For a fixed discretization step $h$, the semi-Lagrangian scheme generates a discrete-time semigroup $(Q_n)_{n \in \mathbb{N}}$ and the solution of the approximate equation with initial datum $u(x,0) = f(x)$ is given by $u(x, nh) = (Q_n f)(x)$. We show that if $\mu$ satisfies (1.4), then the semigroup $(Q_n)_{n \in \mathbb{N}}$ satisfies at the discrete time $nh$ the hypercontractivity estimate

$$\|e^{Q_nf}\|_{\lambda_n} \leq \|e^f\|_a \prod_{k=1}^{n} (1 + C\lambda_{k-1}h)^{\frac{1}{\lambda_k}}. \quad (1.6)$$

where $\lambda_n = a + \rho nh$. And, as in the continuous case, also the converse is true.

It is by now classical that approximation schemes for Hamilton-Jacobi equations give an error estimate of order $h^{1/2}$ in the $L^\infty$-norm, while $L^p$-estimates are known only in some particular cases (see [3], [8], [9]). We apply (1.6) to give an estimate of the set where the truncation error is concentrated showing that its measure decays exponentially. We note that similar estimates are obtained for the approximation of stochastic differential equations via Euler schemes (see [10], [11]).

The paper is organized as follows.

In Section 2 we study the property of the discrete-time semigroup. In Section 3 we prove the equivalence between the hypercontractivity estimate and the logarithmic Sobolev inequality with respect to a gaussian measure, while in Section 4 we study a similar property for the Lebesgue measure. Finally, in Section 5, we prove the concentration estimate.

### 2 The discrete semigroup

Fixed a discretization step $h > 0$ and given a continuous function $u_0$, consider the semi-Lagrangian scheme for (1.1) (see [6])

$$\frac{u_{n+1}(x) - u_n(x)}{h} + \sup_{q \in \mathbb{R}^N} \left\{-\frac{u_n(x -hq) - u_n(x)}{h} - L(q)\right\} = 0. \quad (2.1)$$
or equivalently

\[ u_{n+1}(x) = \inf_{q \in \mathbb{R}^N} \{ u_n(x - hq) + hL(q) \}. \]  \tag{2.2}

Given a continuous function \( f \), define

\[ (Q_n f)(x) = \inf_{q_0, \ldots, q_{n-1} \in \mathbb{R}^N} \left\{ f(x - \sum_{k=0}^{n-1} h q_k) + \sum_{k=0}^{n-1} hL(q_k) \right\} \]  \tag{2.3}

(with the convention that \( \sum_{k=0}^{-1} = 0 \)). In the next proposition we show that the family \((Q_n)_{n \in \mathbb{N}}\) generates a discrete-time semigroup giving the solution of (2.1) with initial datum \( u_0(x) = f(x) \).

**Proposition 2.1.**

1) \((Q_{n+m} f)(x) = (Q_n (Q_m f))(x)\) for any \(n, m \in \mathbb{N}\) and \((Q_0 f)(x) = f(x)\). Moreover \((Q_n (f + c))(x) = (Q_n f)(x) + c\) for any constant \(c \in \mathbb{R}\).

2) \(u_n(x) = (Q_n f)(x)\) is the solution of (2.1) with \( u_0(x) = f(x) \).

3) If \( f \) is bounded and Lipschitz continuous, then \( Q_n f \) is bounded and Lipschitz continuous and

\[ \|Q_{n+1} f\|_\infty \leq \|Q_n f\|_\infty + h \max \{-H(0), L(0)\} \quad \text{for any } n \in \mathbb{N} \]  \tag{2.4}

\[ |(Q_n f)(x) - (Q_n f)(y)| \leq \|Df\|_\infty |x - y| \quad \text{for any } n \in \mathbb{N}, x, y \in \mathbb{R}^N \]  \tag{2.5}

\[ |(Q_m f)(x) - (Q_n f)(x)| \leq C |n - m| h \quad \text{for any } n \in \mathbb{N}, x \in \mathbb{R}^N \]  \tag{2.6}

with \( C \) independent of \( h \).

4) If \( f \) is semiconcave, then \( Q_n f \) is semiconcave in \( x \), uniformly in \( n \).

**Proof.** To prove 1), observe that

\[
(Q_n (Q_m f))(x) = \inf_{q_1, \ldots, q_n} \left\{ (Q_m f) \left( x - \sum_{k=0}^{m-1} h q_k \right) + \sum_{k=0}^{m-1} hL(q_k) \right\}
\]

\[
= \inf_{q_1, \ldots, q_n} \left\{ \inf_{q_{n+1}, \ldots, q_m} \left\{ f \left( x - \sum_{k=0}^{n-1} h q_k - \sum_{k=0}^{m-1} h q_{n+1+k} \right) + \sum_{k=0}^{m-1} hL(q_{n+1+k}) \right\} + \sum_{k=0}^{m-1} hL(q_k) \right\}
\]

\[
= (Q_{n+m} f)(x).
\]

The commutativity with the constants it is immediate.

Set \( u_n = Q_n f \). By 1), \( u_{n+1}(x) = (Q_{n+1} f)(x) = (Q_1 (Q_n f))(x) = (Q_1 u_n)(x) \), hence \( u_n \) is the solution of (2.2) with \( u_0(x) = (Q_0 f)(x) = f(x) \).

By (2.2) for \( q = 0 \), we have

\[ u_{n+1}(x) \leq u_n(x) + hL(0) \]  \tag{2.7}

Moreover since \( L(q) \geq -H(0) \) for any \( q \in \mathbb{R}^N \), we have

\[ u_{n+1}(x) \geq -\|u_n\|_\infty - hH(0) \]  \tag{2.8}
By (2.7) and (2.8), we get (2.4) for $u_n = Q_n f$.

Since $u_n = Q_n f$ is continuous and $L$ is superlinear, there exists $R_n$ (increasing with respect to $n$ but upper bounded uniformly in $n$ and $h$) such that, defined $M_n(x) = \arg\inf\{u_n(x - hq) + hL(q)\}$, then $M_n(x) \subset B(0, R_n)$ and the infimum in (2.2) is obtained. Given $x, y \in \mathbb{R}^N$ and $q^* \in M_n(x)$, by (2.2) we get

$$u_{n+1}(x) - u_{n+1}(y) \leq u_n(x - hq^*) - u_n(y - hq^*) \leq \|Du_n\|_\infty |x - y| = \|D(Q_n f)\|_\infty |x - y|$$

Iterating in the previous inequality, we get (2.5).

For $q^* \in M_n(x)$ in (2.2), we have

$$u_{n+1}(x) - u_n(x) = u_n(x - hq^*) - u_n(x) + hL(q^*) \leq \|Du_n\|_\infty h|q^*| + hL(q^*)$$

which gives (2.6) since $M_n(x)$ is uniformly bounded.

Assume that $u_n$ is semi-concave with constant $C_n$. If $q^* \in M_n(x)$, then

$$u_{n+1}(x) = u_n(x - hq^*) + hL(q^*)$$

$$u_{n+1}(x \pm y) \leq u_n(x - hq^* \pm y) + hL(q^*).$$

Hence

$$u_{n+1}(x + y) + u_{n+1}(x - y) - 2u_{n+1}(x) \leq u_n(x - hq^* + y) + u_n(x - hq^* - y) - 2u_n(x - hq^*) \leq C_n |y|^2.$$ 

Hence $C_{n+1} \leq C_n$ and iterating

$$C_{n+1} \leq C_0$$

where $C_0$ is the semi-concavity constant of $u_0 = f$. \qed

Remark 2.2. Note that (2.2) can be rewritten as

$$(Q_{n+1} f)(x) = \inf_{q \in \mathbb{R}^N} \left\{ (Q_n f)(x) + hL \left( \frac{x - y}{h} \right) \right\}.$$ 

Hence the discrete semigroup $Q_n$ is obtained by considering the continuous semigroup $S_t$ only at the discrete time $t = nh$, $n \in \mathbb{N}$.

3 Hypercontractivity of the discrete semigroup with respect to Gaussian measures

We prove the hypercontractivity of the discrete semigroup $(Q_n)_{n \in \mathbb{N}}$ with respect to a measure $\mu$ satisfying (1.4). In this section $\| \cdot \|_p$ is the $L^p$-norm associated to the measure $\mu$. We only consider the case $H(p) = \frac{1}{2} |p|^2$ and therefore $L(q) = \frac{1}{2} |q|^2$, but the results can be extended to more general Hamiltonians as in [5].

Theorem 3.1. If $\mu$ is absolutely continuous with respect to the Lebesgue measure and satisfies the logarithmic Sobolev inequality (1.4), then for any $f$ Lipschitz continuous and for any $n$, any $h > 0$, $a \in \mathbb{R}$

$$\|e^{Q_n f}\|_{\lambda_n} \leq \|e^f\|_a \prod_{k=1}^n (1 + C\lambda_{k-1} h)^{\frac{1}{\lambda_k}}. \quad (3.1)$$
where $\lambda_n = a + \rho_n h$.

Conversely if (3.1) holds for any smooth function $f$, for any $n$, any $h > 0$ and some $a \neq 0$, then the measure $\mu$ satisfies the logarithmic Sobolev inequality (1.4).

Proof. We define

$$F_n = \| e^{Q_n f} \|_{\lambda_n} = \left( \int e^{\lambda_n Q_n f(x)} d\mu \right)^{\frac{1}{\lambda_n}}$$

and we prove that

$$F_{n+1} \leq F_n (1 + C \lambda_n h)^{\frac{1}{\lambda_n + 1}}. \quad (3.3)$$

Then (3.1) is obtained iterating over $n$.

We consider a measure $\mu$ which satisfies (1.4) and we start with the identity

$$F_{n+1}^{\lambda_{n+1}} - F_n^{\lambda_n} = \int e^{\lambda_{n+1} Q_{n+1} f} d\mu - \int e^{\lambda_n Q_n f} d\mu \quad (3.4)$$

We consider first the term on the right hand side of (3.4). We have

$$F_{n+1}^{\lambda_{n+1}} - F_n^{\lambda_n} = F_{n+1}^{\lambda_{n+1}} - F_{n+1}^{\lambda_{n+1}} + F_{n+1}^{\lambda_{n+1}} - F_n^{\lambda_n} = e^{\lambda_{n+1} \ln(F_{n+1})} \left[ e^{\lambda_{n+1} \ln(F_{n+1}) - \ln(F_n))} - 1 \right] + e^{\lambda_n \ln(F_n)} \left[ e^{(\lambda_{n+1} - \lambda_n) \ln(F_n)} - 1 \right] \quad (3.5)$$

$$F_n^{\lambda_{n+1}} \left[ \left( \frac{F_{n+1}}{F_n} \right)^{\lambda_{n+1}} - 1 \right] + F_n^{\lambda_n} \left[ e^{(\lambda_{n+1} - \lambda_n) \ln(F_n)} - 1 \right]\quad$$

We now consider the term on the right hand side of (3.4)

$$\int e^{\lambda_{n+1} Q_{n+1} f} d\mu - \int e^{\lambda_n Q_n f} d\mu = \int e^{\lambda_{n+1} Q_{n+1} f} - e^{\lambda_n Q_n f} d\mu+$$

$$\int e^{\lambda_{n+1} Q_n f} - e^{\lambda_n Q_n f} d\mu = \int e^{\lambda_{n+1} Q_n f} \left[ e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} - 1 \right] d\mu+$$

$$\int e^{\lambda_n Q_n f} \left[ e^{(\lambda_{n+1} - \lambda_n) Q_n f} - 1 \right] d\mu \quad (3.6)$$

By (3.5) and (3.6), multiplying for $\lambda_n$, we get

$$\lambda_n F_n^{\lambda_{n+1}} \left[ \left( \frac{F_{n+1}}{F_n} \right)^{\lambda_{n+1}} - 1 \right] =$$

$$\lambda_n \int e^{\lambda_{n+1} Q_n f} \left[ e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} - 1 \right] d\mu + \lambda_n \int e^{\lambda_n Q_n f} \left[ e^{h \rho Q_n f} - 1 \right] d\mu -$$

$$\lambda_n F_n^{\lambda_n} \left[ e^{h \rho \ln(F_n)} - 1 \right] = h \rho \text{Ent}(e^{\lambda_n Q_n f}) + \lambda_n \int e^{\lambda_{n+1} Q_n f} \left[ e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} - 1 \right] d\mu+$$

$$\lambda_n \int e^{\lambda_n Q_n f} \left[ e^{h \rho Q_n f} - 1 - h \rho Q_n f \right] d\mu - \lambda_n F_n^{\lambda_n} \left[ e^{h \rho \ln(F_n)} - 1 - h \rho \ln(F_n) \right] \leq$$

$$h \lambda_n^2 \int e^{\lambda_n Q_n f} \frac{D Q_n f}{2} d\mu + \lambda_n \int e^{\lambda_{n+1} Q_n f} \left[ e^{\lambda_{n+1} (Q_{n+1} f - Q_n f)} - 1 \right] d\mu+$$

$$\lambda_n \int e^{\lambda_n Q_n f} \left[ e^{h \rho Q_n f} - 1 - h \rho Q_n f \right] d\mu - \lambda_n F_n^{\lambda_n} \left[ e^{h \rho \ln(F_n)} - 1 - h \rho \ln(F_n) \right]$$

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Observing that $\lambda_n \leq \lambda_{n+1}$, $|e^{h\rho Q_n f} - 1 - h\rho Q_n f| \leq Ch^2$, $Q_{n+1} f \leq Q_n f$ and the last term on the right hand side of (3.7) is negative we get

$$\lambda_n F_{n+1}^{\lambda_{n+1}} \left[ \frac{F_{n+1}}{F_n} \right]^{\lambda_{n+1}} - 1 \leq C \lambda_n^2 h F_{n}^{\lambda_n} + C \lambda_n F_{n}^{\lambda_n} h^2 \leq C \lambda_n^2 h F_{n}^{\lambda_{n+1}}$$

and therefore

$$F_{n+1}^{\lambda_{n+1}} \leq F_n^{\lambda_n} (1 + C \lambda_n h) \leq F_n^{\lambda_{n+1}} (1 + C \lambda_n h)$$

(3.8)

which gives (3.3).

To prove the converse, given $t > 0$, let $t_n = nh$ converging to $t$ for $h \to 0$ and $n \to \infty$. By standard stability results in viscosity solution theory (see [2]) $Q_n f$ converges to $S_t f$ uniformly in $x$, where $S_t$ is the semigroup associated to (1.1). Moreover $\lambda_n \to \lambda(t) = a + \rho t$ and $(1 + C \lambda_{k-1} h)^{1/\lambda_k} \to 1$. Hence by (3.1), for $h \to 0$ we get the hypercontractivity of the continuous semigroup

$$\|e^{S_t f(x)}\|_{\lambda(t)} \leq \|e^f\|_a.$$  

(3.9)

Then the statement follows since it is well known that if the inequality (3.9) holds for some $a \neq 0$ and for any smooth function $f$, the measure $\mu$ satisfies (1.4) (see [5]).

\[\text{Figure 1: Behavior of the product } \prod_{k=1}^{n} (1 + C \lambda_k h)^{\frac{1}{\lambda_k}} \text{ as a function of } h, \text{ fixed } \rho = 1, \text{ } C = 0.01, \text{ } n = 10.\]

4 Hypercontractivity of the discrete semigroup with respect to the Lebesgue measure

We prove the hypercontractivity of the discrete semigroup $Q_n$ with respect to the Lebesgue measure. In this section $\| \cdot \|_p$ is the $L^p$-norm associated to the Lebesgue measure on $\mathbb{R}^N$ and we assume for simplicity that $H(p) = \frac{1}{2} |p|^2$ and therefore $L(q) = \frac{1}{2} |q|^2$.

We recall that for the continuous semigroup $S_t$ the following optimal inequality holds (see inequality (7) in [12])

$$\|e^{S_t f}\|_{\beta} \leq \left( \frac{\alpha}{\beta} \right)^{\frac{N}{N + 2} \frac{\beta - \alpha}{2}} \|e^f\|_\alpha \left( \frac{\beta - \alpha}{t} \right)^{\frac{N}{2} \frac{\beta - \alpha}{\alpha \beta}} \left[ \int e^{-L(x)} dx \right]^{-\frac{\beta - \alpha}{\alpha \beta}}.$$  

(4.1)
Theorem 4.1. For any smooth function \( f \in \mathbb{R}^N \), \( 0 < \alpha \leq \beta \), we have

\[
\left\| e^{Q_n f} \right\|_{\beta} \leq \left( \frac{\alpha}{\beta} \right)^{\frac{N}{2}} \left\| e^f \right\|_{\alpha} \left( \frac{\beta - \alpha}{nh} \right)^{\frac{N}{2}} \left( 2\pi \right)^{-\frac{N}{2}}
\]

(4.2)

Inequality (4.2) is optimal and equality holds if for some \( 0 < \alpha \leq \beta \), \( \bar{x} \in \mathbb{R}^N \) and \( b > 0 \) we have

\[
f(x) = -bL(x - \bar{x}),
\]

and

\[
nh = \frac{\beta - \alpha (nh) - \frac{\beta - \alpha}{\beta}}{b}. \tag{4.4}
\]

Moreover, we obtain the following ultracontractive bound

\[
\left\| e^{Q_n f} \right\|_{\infty} \leq \left\| e^f \right\|_{\frac{1}{1}} \left( \frac{1}{nh} \right)^{\frac{N}{2}} \left( 2\pi \right)^{-\frac{N}{2}}, \tag{4.5}
\]

and the equality holds if \( nh = \frac{1}{\beta} \).

Proof. In order to prove inequality (4.2), we are going to use the following Prékopa-Leindler inequality:

Let \( a, b > 0 \) such that \( a + b = 1 \), and \( u, v, w \) three non-negative functions on \( \mathbb{R}^N \). Suppose that for any \( x, y \in \mathbb{R}^N \) we have

\[
u(x) = e^{\beta Q_n f(x)},
\]

\[
v(x) = e^{-\theta |y|^2},
\]

\[
w(x) = e^{\alpha f(\frac{x}{|y|})},
\]

We prove that \( u, v \) and \( w \) verify the hypothesis of the Prékopa-Leindler inequality (4.6) with

\[
a = \frac{\alpha}{\beta}, \quad b = \frac{\beta - \alpha}{\beta}. \tag{4.7}
\]

By (2.3) for any \( \{q_1, \ldots, q_n\} \subset \mathbb{R}^N \)

\[
u(x)^a = e^{a\beta Q_n f(x)} \leq e^{\alpha f(x - h \sum_{k=0}^{n-1} q_k + ah \sum_{k=0}^{n-1} \frac{|q_k|^2}{2}) + \theta \sum_{k=0}^{n-1} \frac{|q_k|^2}{2},}
\]

hence

\[
u(x)^a v(y)^b \leq e^{\alpha f(x - h \sum_{k=0}^{n-1} q_k + ah \sum_{k=0}^{n-1} \frac{|q_k|^2}{2} - \theta \frac{\beta - \alpha}{2\beta} |y|^2}.\]
We compute

\[ x - h \sum_{k=0}^{n-1} q_k = x + \frac{\beta - \alpha}{\alpha} y \]

then we get

\[ q = -\frac{\beta - \alpha}{\alpha nh} y \]

and therefore

\[ u(x)v(y)^b \leq e^{\alpha f(x-h\sum_{k=0}^{n-1} q_k)} + \alpha^b \sum_{k=0}^{n-1} \left( -\frac{\beta - \alpha}{\alpha nh} \right)^2 |y|^2 - \frac{(\beta - \alpha)^2}{2\alpha nh} \sum |y_k|^2 \]

\[ = e^{\alpha f(x+\frac{\beta - \alpha}{\alpha} y)} = e^{\alpha f \left[ \frac{\beta}{\alpha} (\frac{\beta - \alpha}{\alpha} x + \frac{\beta - \alpha}{\alpha} y) \right]} = e^{\alpha f \left( \frac{\beta}{\alpha} (ax + by) \right)} = w(ax + by). \]

Hence we can apply the Prékopa-Leindler inequality to obtain

\[ \left( \int e^{\beta Q_n f(x)} dx \right)^{\frac{a}{b}} \leq \left( \int e^{\alpha f(x)} dx \right)^{\frac{a}{b}} \left( \int e^{-\theta |x|^2} dx \right)^{\frac{b}{2}} \]

\[ \Rightarrow \left( \int e^{\beta Q_n f(x)} dx \right)^{\frac{b}{2}} \leq \left( \int e^{\alpha f(x)} dx \right)^{\frac{a}{b}} \left( \int e^{-\theta |x|^2} dx \right)^{\frac{b}{2}} \]

It follows that

\[ \| e^{Q_n f} \|_{L^\beta} \leq \left( \frac{\alpha}{\beta} \right)^{\frac{N}{2}} \| e^f \|_{\alpha} \left( \int e^{-\theta |x|^2} dx \right)^{-\frac{\beta - \alpha}{\alpha \beta}} . \tag{4.8} \]

We compute

\[ \left( \int e^{-\theta |x|^2} dx \right)^{-\frac{\beta - \alpha}{\alpha \beta}} = \left( \int e^{-\left( \frac{\beta}{2} \right) |x|^2} dx \right)^{-\frac{\beta - \alpha}{\alpha \beta}} = \left( \int e^{-|y|^2} dy \left( \frac{\theta}{2} \right)^{-\frac{N}{2}} \right)^{-\frac{\beta - \alpha}{\alpha \beta}} \]

\[ = \left[ \left( \frac{\theta}{2} \right)^{-\frac{N}{2}} \pi^N \right]^{-\frac{\beta - \alpha}{\alpha \beta}} = \left[ \left( \frac{\theta}{2} \right)^{-\frac{N}{2}} \pi^N \right]^{-\frac{\beta - \alpha}{\alpha \beta}} = \left[ \frac{2\pi \alpha h}{\beta (\beta - \alpha)} \right]^{-\frac{N}{2}} \left[ \frac{2\pi \alpha h}{\beta (\beta - \alpha)} \right]. \]

Substituting in (4.8) we get (4.2). In order to prove the optimality, we compute the terms \( \| e^{Q_n f} \|_{\beta} \) and \( \| e^f \|_{\alpha} \) appearing in (4.2) for \( f \) as in (4.3). We obtain

\[ \| e^{Q_n f} \|_{\beta} = \left( \int e^{\beta Q_n f(x)} dx \right)^{\frac{1}{\beta}} = \left( \int e^{-\frac{b\beta}{1-nhb} L(x-y)} dx \right)^{\frac{1}{\beta}} \]

\[ = \left( \int e^{-\frac{b\beta}{1-nhb} z} dz \right)^{\frac{1}{\beta}} = \left( \int e^{-\frac{b\beta}{1-nhb} z}^2 dz \right)^{\frac{1}{\beta}} \]

\[ = \left[ 2\pi (1-nhb) \right]^{\frac{N}{2}} \]

\[ \text{and} \]

\[ \| e^f \|_{\alpha} = \int \left( e^{-\alpha b L(x-y)} dx \right)^{\frac{1}{\alpha}} \]

\[ = \int \left( e^{-\alpha b \frac{y^2}{2}} dx \right)^{\frac{1}{\alpha}} = \int \left( e^{-\alpha b \frac{z^2}{2}} dz \right)^{\frac{1}{\alpha}} = \left( \frac{2\pi}{\alpha b} \right)^{\frac{N}{2}} \]

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and we obtain an equality in (4.2) for

\[ 1 - nhb = \frac{\alpha}{\beta} \left( \frac{1}{nh} \right)^{\frac{\beta-\alpha}{\alpha}}, \]

i.e. (4.4).

The ultracontractive bound (4.5) is obtained for \( \beta \to +\infty \) and \( \alpha = 1 \) in (4.2). Furthermore, if

\[ \| e^{Q_n f} \|_\infty = 1, \quad \| e^f \|_1 = \left( \frac{2\pi}{b} \right)^{\frac{N}{2}}, \]

the equality in (4.5) is obtained for \( nh = \frac{1}{b} \).

Remark 4.2. Consider the constant appearing in (4.2), that is

\[ C = \left( \frac{\alpha}{\beta} \right)^{\frac{N}{2} \frac{\alpha+\beta}{2}} \left( \frac{\beta - \alpha}{nh} \right)^{\frac{N}{2} \frac{\beta-\alpha}{\alpha}} \left( 2\pi \right)^{\frac{N}{2} \frac{\beta-\alpha}{\alpha}} \] (4.9)

We observe that for fixed values of \( \alpha = \frac{\beta}{2}, nh = 1 \), we have

\[ C = \left( \frac{1}{2} \right)^{\frac{N}{2} \frac{\beta}{4\pi}} \left( \frac{\beta}{4\pi} \right)^{\frac{N}{2}}. \]

In this case the graph of the constant coincides with the one of the constant in (4.1) for \( t = nh \) (see Fig. 4.2).

In the next proposition we give a hyper-contractivity estimate for the discrete semigroup \( Q_n \) with respect to the Lebesgue measure similar to one of Theorem 3.1.

**Proposition 4.3.** For any \( f \) Lipschitz continuous and for any \( n, h > 0 \), \( \{\beta_k\}_{k \in \mathbb{N}} \)

\[ \| e^{Q_n f} \|_{\beta_n} \leq \| e^f \|_{\beta_0} \prod_{k=1}^{n} \left( \frac{\beta_{k-1}}{\beta_k} \right)^{\frac{N}{2} \frac{\beta_{k-1}}{\beta_k}} \left( \frac{2\pi \beta_{k-1}}{\beta_k - \beta_{k-1}} \right)^{\frac{\beta_k - \beta_{k-1}}{2\pi \beta_{k-1}}} \] (4.10)

**Proof.** As in the proof of Theorem 4.1, we set

\[ u(x) = e^{\beta Q_n f(x)}, \quad v(x) = e^{-\theta \frac{a^2}{2}}, \quad w(x) = e^{\alpha Q_{n-1} f(\frac{\beta}{x})} \]

and we prove that \( u, v \) and \( w \) verify the hypothesis of the Prékopa-Leindler inequality, with \( a \) and \( b \) as in (4.7). By (2.2)

\[ u(x)^a v(y)^b \leq e^{\alpha Q_{n-1} f(x-hq) + h\frac{a^2}{2} - \frac{b^2}{2}} \]

Choosing \( q = -\frac{\beta - \alpha}{ah} y \) and \( \theta = -\frac{\beta - \alpha}{ah} \beta \), we obtain

\[ u(x)^a v(y)^b \leq e^{\alpha Q_{n-1} f(x+\frac{\beta - \alpha}{\alpha} y)} = e^{\alpha Q_{n-1} f(\frac{\beta}{\alpha} (ax+by))} = w(x). \]
Arguing as for estimate (4.8), by the Prékopa-Leindler inequality, we obtain

$$\left\| e^{Q_n f} \right\|_\beta \leq \left( \frac{\alpha}{\beta} \right) ^ {\frac{N}{\alpha}} \left\| e^{Q_{n-1} f} \right\|_\alpha \left( \int e^{-\theta x^2 / 2} \right)^{-\frac{\beta - \alpha}{\alpha \beta}} =$$

$$\left( \frac{\alpha}{\beta} \right) ^ {\frac{N}{\alpha}} \left\| e^{Q_{n-1} f} \right\|_\alpha \left[ \frac{2\pi n h \alpha}{\beta (\beta - \alpha)} \right] ^ {\frac{N}{2} \frac{\beta - \alpha}{\alpha \beta}}$$

(4.11)

For $\beta = \beta_n, \alpha = \beta_{n-1}$ in (4.11), we get

$$\left\| e^{Q_n f} \right\|_\beta \leq \left( \frac{\beta_{n-1}}{\beta_n} \right) ^ {\frac{N}{\beta_{n-1}}} \left\| e^{Q_{n-1} f} \right\|_{\beta_{n-1}} \left[ \frac{2\pi \rho h \beta_{n-1}}{\beta_n (\beta_n - \beta_{n-1})} \right] ^ {\frac{N}{2} \frac{\beta_n - \beta_{n-1}}{\beta_{n-1} \beta_n}}.$$

Iterating the previous argument for $n - 1, n - 2, \ldots, 0$ we finally get the hyper-contractivity estimate (4.10) for $Q_n$.

**Remark 4.4.** In particular, if we set

$$\beta_k = \beta_0 + \rho kh, \quad \beta_{k-1} = \beta_0 + \rho (k-1)h,$$

in (4.10) we have

$$\lim_{h \to 0} \left( \frac{\beta_{k-1}}{\beta_k} \right) ^ {\frac{N}{\beta_{k-1}}} = \lim_{h \to 0} \left( \frac{\beta_0 + \rho (k-1)h}{\beta_0 + \rho kh} \right) ^ {\frac{N}{\beta_0 + \rho (k-1)h}} = 1.$$

and

$$\lim_{h \to 0} \left( \frac{2\pi \beta_{k-1} h}{\beta_k - \beta_{k-1}} \right) ^ {\frac{\beta_k - \beta_{k-1}}{2\beta_k \beta_{k-1}}} =$$

$$\lim_{h \to 0} \left( \frac{2\pi (\beta_0 + \rho (k-1)h)}{\rho h} (\beta_0 + \rho kh)h \right) ^ {\frac{\rho h}{2(\beta_0 + \rho (k-1)h)} (\beta_0 + \rho (k-1)h)} = 1.$$

Compare the graph in Fig.4 of the constant in (4.10) with the graph in Fig.3 of the constant in (3.1).

![Graph](image_url)

**Figure 3:** Behavior of $f(h) = \prod_{k=1}^{10} \left( \frac{\beta_{k-1}}{\beta_k} \right) ^ {\frac{N}{\beta_{k-1}}} \left( \frac{2\pi \beta_{k-1} h}{\beta_k - \beta_{k-1}} \beta_k h \right) ^ {\frac{\beta_k - \beta_{k-1}}{2\beta_k \beta_{k-1}}} \Gamma \beta_0 = N = 1.$
5  A concentration estimate for the approximation error

It is well known that for $h \to 0$, the discrete solution computed via the scheme (2.1) converges uniformly to the solution of (1.1) with an error $\|u - u_h\|_{\infty}$ of order $h^{1/2}$ (see [4], [6]). In this section we obtain an estimate of the measure of the set where the error is concentrated.

To simplify the notation we write $S_n f$ for $S_{nh} f$, where $S_t$ is the continuous semigroup associated to the equation (1.1).

**Theorem 5.1.** If $\mu$ is absolutely continuous with respect to the Lebesgue measure and satisfies the logarithmic Sobolev inequality (1.4), then for any $f$ semiconcave and for any $n \in \mathbb{N}$, $h > 0$ and $a \in \mathbb{R}$

$$\|e^{\lambda_n} f - Q_n f\|_{\lambda_n, \gamma} \leq \prod_{k=1}^{n} (1 + C \lambda_k^2 h^2)^{\frac{1}{2}}$$

where $\lambda_n = a + \rho n h$ and $\| \cdot \|_p$ is the $L^p$-norm associated to the measure $\mu$.

**Proof.** Set

$$F_n = \|e^{\lambda_n} f - S_n f\|_{\lambda_n} = \left( \int e^{\lambda_n} (Q_n f(x) - S_n f(x)) d\mu \right)^{\frac{1}{\lambda_n}}$$

Arguing as in Theorem 3.1, we arrive, see (3.7), to the inequality

$$\lambda_n F_n^{\lambda_{n+1}} \left[ \left( \frac{F_{n+1}}{F_n} \right)^{\lambda_{n+1}} - 1 \right] \leq h \lambda_n^2 \int e^{\lambda_n} (Q_n f(x) - S_n f(x)) \frac{|DQ_n f - DS_n f|^2}{2} d\mu + \lambda_n \int e^{\lambda_{n+1}} (Q_n f - S_n f) \left[ e^{\lambda_{n+1}} (Q_{n+1} f - Q_n f) - (S_{n+1} f - S_n f) \right] d\mu +$$

$$\lambda_n \int e^{\lambda_n} (Q_n f(x) - S_n f(x)) \left[ e^{\lambda_n} (Q_n f(x) - S_n f(x)) - 1 - h \rho (Q_n f(x) - S_n f(x)) \right] d\mu$$

Since $\lambda_n \leq \lambda_{n+1}$, $|e^{\lambda_{n+1}} (Q_n f - S_n f) - (S_{n+1} f - S_n f)| \leq C h^2$ and the last term on the right hand side of previous inequality is negative we get

$$\lambda_n F_n^{\lambda_{n+1}} \left[ \left( \frac{F_{n+1}}{F_n} \right)^{\lambda_{n+1}} - 1 \right] \leq h \lambda_n^2 \int e^{\lambda_n} (Q_n f(x) - S_n f(x)) \frac{|DQ_n f - DS_n f|^2}{2} d\mu +$$

$$\lambda_n \int e^{\lambda_{n+1}} (Q_n f(x) - S_n f(x)) \left[ e^{\lambda_{n+1}} (Q_{n+1} f(x) - Q_n f(x)) - (S_{n+1} f(x) - S_n f(x)) \right] d\mu +$$

$$\lambda_n \int e^{\lambda_{n+1}} (Q_n f(x) - S_n f(x)) h \lambda_{n+1} \left( \frac{\partial S_n f}{\partial t} - \frac{S_{n+1} f - S_n f}{h} \right) d\mu +$$

$$h \lambda_n^2 \int e^{\lambda_n} (Q_n f(x) - S_n f(x)) \left[ \frac{Q_{n+1} f(x) - Q_n f(x)}{h} - \frac{\partial S_n f}{\partial t} + \frac{|DQ_n f - DS_n f|^2}{2} \right] d\mu +$$

$$\lambda_n F_n^{\lambda_{n+1}} C h^2$$

We have (see (2.6) and the correspondent property for $S_t$)

$$\left| \frac{S_{n+1} f - S_n f}{h} \right|, \left| \frac{Q_{n+1} f - Q_n f}{h} \right| \leq C$$
and therefore
\[ e^{\lambda_n(\frac{Q_{n+1}f-Q_nf}{h} - \frac{S_{n+1}f-S_nf}{h})} - 1 - h\lambda_n (\frac{Q_{n+1}f-Q_nf}{h} - \frac{S_{n+1}f-S_nf}{h}) \leq C\lambda_{n+1}^2 h^2 \quad (5.2) \]
\[ e^{\lambda_n(R_nf(x)-S_nf(x))} - e^{\lambda_{n+1}(R_nf(x)-S_nf(x))} \leq Ch \quad (5.3) \]

and by Hopf-Lax formula
\[ \frac{\partial S_nf}{\partial t}(x) - \frac{S_{n+1}f(x) - S_nf(x)}{h} = -\sup_q \{ q \cdot DS_nf(x) - L(q) \} \]
\[ - \inf_q \left\{ \frac{S_nf(x-hq) - S_nf(x)}{h} + L(q) \right\} \leq C_2h \quad (5.4) \]

where \( C_2 \) depends on the semiconcavity constant of \( f \). Moreover since \( |P|^2/2 = \sup_q \{ q \cdot P - L(q) \} \)
\[ \frac{Q_{n+1}f-Q_nf}{h} + \frac{\partial S_nf}{\partial t} + \frac{|DQ_nf - DS_nf|^2}{2} \leq \]
\[ -\sup_q \left\{ -\frac{Q_nf(x-hq) - Q_nf(x)}{h} - L(q) \right\} + \frac{|DQ_nf|^2}{2} - \]
\[ \left( \sup_q \{ q \cdot DQ_nf - L(q) \} + \sup_q \{ q \cdot DS_nf - L(q) \} \right) + \frac{|DQ_nf - DS_nf|^2}{2} \leq C_3h \quad (5.5) \]

By (5.2), (5.4), (5.3) and (5.5), we get
\[ \lambda_n F_n^{\lambda_{n+1}} \left[ \left( \frac{F_{n+1}}{F_n} \right)^{\lambda_{n+1}} - 1 \right] \leq \lambda_n C\lambda_{n+1}^2 h^2 F_n^\lambda_n \]

and therefore we get
\[ F_n^{\lambda_{n+1}} \leq F_n^{\lambda_{n+1}} (1 + C\lambda_{n+1}^2 h^2) \]

Iterating over \( n \) and taking into account that \( F_0 = \|e^{Q_0f-S_0f}\|_{\lambda_0} = \|e^{f-f}\|_{\lambda_0} = 1 \) we get the estimate
\[ \|e^{\lambda_n R_nf-Q_nf}\|_{\lambda_n} \leq \prod_{k=1}^n (1 + C\lambda_k^2 h^2)^{\frac{1}{\lambda_k}} \]

Exchanging the role of \( S_nf \) and \( Q_nf \) we get the other estimate in (5.1).

**Corollary 5.2.** With the same notation of Theorem 5.1, if \( f \) is semi-concave, then for any \( t \in [0,T] \), \( t = nh \), we have
\[ \int (Q_nf-S_nf) d\mu, \int (Q_nf-S_nf) d\mu \leq Ch \quad (5.6) \]

with \( C \) depending on \( T \) and the semi-concavity constant of \( f \). Moreover for any \( p < 1 \)
\[ \mu\{|S_nf - Q_nf| \geq h^p\} \leq Ce^{-1/h^{1-p}} \quad (5.7) \]
Proof. We first observe that, since $e^t$ is a convex function, we have
\[
e^\lambda_n (Q_n f - S_n f) d\mu \leq \int e^\lambda_n (Q_n f - S_n f) d\mu,
\]
hence by (5.1)
\[
e^\lambda_n (Q_n f - S_n f) d\mu \leq \prod_{k=1}^n (1 + C \lambda_k^2 h^2) \leq \prod_{k=1}^n e^{C \lambda_k^2 h^2} = e^{\sum_{k=1}^n C \lambda_k^2 h^2}
\]
and therefore
\[
\int (Q_n f - S_n f) d\mu \leq \sum_{k=1}^n C \lambda_k^2 h^2 \leq C \sum_{k=1}^n \lambda_k h^2 = C (na + \frac{1}{2} n(n+1)h) h^2 \leq C h
\]
for $t = nh \in [0, T]$ where $C$ depends on $T$ and semiconcavity constant of $f$.

To prove estimate (5.7) observe that $\mu\{|S_n f - Q_n f| \geq r\} = \mu\{|S_n f - Q_n f| \geq r\} + \mu\{|Q_n f - S_n f| \geq r\}$ and
\[
\mu\{|S_n f - Q_n f| \geq r\} \leq \frac{1}{e^{\lambda_n r}} \int e^\lambda_n (S_n f - Q_n f) d\mu \leq e^{-\lambda_n r} \prod_{k=1}^n (1 + C \lambda_k^2 h^2)^{\lambda_n / \lambda_k} \leq e^{-ar + Ch}
\]
Taking $r = h^p$ and $a = \frac{1}{p}$ in the previous estimate we get (5.7) \[\Box\]

Remark 5.3. The estimate (5.7) can be interpreted as a concentration inequality of truncation error between the solution of the continuous problem and of the discrete one.

References

[1] Avantaggiati, A.; Loreti, P. Hypercontractivity, Hopf-Lax type formulas, Ornstein-Uhlenbeck operators. II. Discrete Contin. Dyn. Syst. Ser. S 2 (2009), no. 3, 525–545.

[2] Bardi, M., Capuzzo Dolcetta, I. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Birkäuser, Boston, 1997.

[3] Bokanowski, O.; Forcadel, N.; Zidani, H. $L^1$-error estimates for numerical approximations of Hamilton-Jacobi-Bellman equations in dimension 1. Math. Comp. 79 (2010), no. 271, 1395–1426.

[4] Crandall, M. G.; Lions, P.-L. Two approximations of solutions of Hamilton-Jacobi equations. Math. Comp. 43 (1984), no. 167, 1–19.

[5] Bobkov, S. G.; Gentil, I.; Ledoux, M. Hypercontractivity of Hamilton-Jacobi equations. J. Math. Pures Appl. (9) 80 (2001), no. 7, 669–696.

[6] Falcone, M.; Ferretti, R. Semi-Lagrangian Approximation Schemes for Linear and Hamilton-Jacobi Equations. MOS-SIAM Series on Optimization, to appear.

[7] Gentil, I. Ultracontractive bounds on Hamilton-Jacobi equations, Bull. Sci. Math. 126 (2002), 507–524.

[8] Lepsky, O. Spectral viscosity approximations to Hamilton-Jacobi solutions. SIAM J. Numer. Anal. 38 (2000), no. 5, 1439–1453.
[9] Lin, C.-T.; Tadmor, E. $L^1$-stability and error estimates for approximate Hamilton-Jacobi solutions. Numer. Math. 87 (2001), no. 4, 701–735.

[10] Lemaire, V.; Menozzi, S. On some non asymptotic bounds for the Euler scheme, Electron. J. Probab. 15 (2010), no. 53, 1645–1681.

[11] Malrieu, F.; Talay, D. Concentration inequalities for the Euler scheme. Monte Carlo and quasi-Monte Carlo methods 2004, 355–371, Springer, Berlin, 2006.

[12] Gentil, I. The general optimal $L^p$-Euclidean logarithmic Sobolev inequality by Hamilton-Jacobi equations, J.Funct. Anal. 202 (2003), 591–599.