Subspaces intersecting each element of a regulus in one point,
André-Bruck-Bose representation and clubs

Michel Lavrauw* and Corrado Zanella†

September 4, 2014

Abstract

In this paper results are proved with applications to the orbits of $(n - 1)$-dimensional subspaces disjoint from a regulus $\mathcal{R}$ of $(n - 1)$-subspaces in $\text{PG}(2n - 1, q)$, with respect to the subgroup of $\text{PGL}(2n, q)$ fixing $\mathcal{R}$. Such results have consequences on several aspects of finite geometry. First of all, a necessary condition for an $(n - 1)$-subspace $U$ and a regulus $\mathcal{R}$ of $(n - 1)$-subspaces to be extendable to a Desarguesian spread is given. The description also allows to improve results in [4] on the André-Bruck-Bose representation of a $q$-subline in $\text{PG}(2, q^n)$. Furthermore, the results in this paper are applied to the classification of linear sets, in particular clubs.

A.M.S. CLASSIFICATION: 51E20
KEY WORDS: club; linear set; subplane; André-Bruck-Bose representation; Segre variety

1 Introduction

The $(n - 1)$-dimensional projective projective space over the field $F$ is denoted by $\text{PG}(n - 1, F)$ or $\text{PG}(n - 1, q)$ if $F$ is the finite field of order $q$ (denoted by $\mathbb{F}_q$). If $L$ is an extension field $\mathbb{F}_q$, then the projective space defined by the $\mathbb{F}_q$-vector space induced by $L^d$ is denoted by $\text{PG}_q(L^d)$. For further notation and general definitions employed in this paper the reader is referred to [9, 11, 13]. For more information on Desarguesian spreads see [1].

This paper is structured as follows. In Section 2 subspaces which intersect each element of a regulus in one point are studied and a result from [6] is generalised. Section 3 contains one of the main results of this paper, determining the order of the normal rational curves obtained from $n$-dimensional subspaces on an external $(n - 1)$-dimensional subspace with respect to a regulus in $\text{PG}(2n - 1, q)$, obtained from a point and a subline after applying the field reduction map to $\text{PG}(1, q^n)$. This leads to a necessary condition on the existence of a Desarguesian spread containing a subspace and regulus (Corollary 3.4). The André-Bruck-Bose representation of...
sublines and subplanes of a finite projective plane is studied in Section 3 and improvements are obtained with respect to the known results \cite{5, 14, 15, 4}. The results from the first sections are then applied to the classification problem for clubs of rank three in PG(1, q^n) in Section 5. A study of the incidence structure of the clubs in PG(1, q^n) after field reduction yields to a partial classification, concluding that the orbits of clubs under PGL(2, q^n) are at least k − 1, where k stands for the number of divisors of n. The paper concludes with an appendix discussing a result motivated by Burau \cite{6} for the complex numbers: the result is extended to general algebraically closed fields; a new proof is provided; and counterexamples are given to some of the arguments used in the original proof.

2 Subspaces intersecting each element of a regulus in one point

Let \( \mathcal{R} \) be a regulus of subspaces in a projective space and let \( S \) be any subspace of \( \langle \mathcal{R} \rangle \). Questions about the properties of the set of intersection points, which for reasons of simplicity of notation we will denote by \( S \cap \mathcal{R} \), often turn up while investigating objects in finite geometry. If \( S \) intersects each element of the regulus \( \mathcal{R} \) in a point, then the intersection \( S \cap \mathcal{R} \) is a normal rational curve, see Lemma 2.1. This was already pointed out in \cite{6} p.173 with a proof originally intended for complex projective spaces, but actually holding in a more general setting. The notation of \cite{6} will be partly adopted.

The Segre variety representing the Cartesian product PG\((n, F) \times PG(m, F)\) in PG\(((n + 1)(m + 1) - 1, F)\) is denoted by \( S_{n,m,F} \). It is well known that \( S_{n,m,F} \) contains two families \( S^I_{n,m,F} \) and \( S^{II}_{n,m,F} \) of maximal subspaces of dimensions \( n \) and \( m \), respectively. When convenient, the notation \( S^I \) or \( S^{II} \) will be used for a subspace belonging to the first or second family. The points of \( S_{n,m,F} \) may be represented as one-dimensional subspaces spanned by rank one \((m + 1)\times(n + 1)\) matrices. This is the standard example of a regular embedding of product spaces, see \cite{16}. Note that in the finite case it is possible to embed product spaces in projective spaces of smaller dimension (see e.g. \cite{7}). A regulus \( \mathcal{R} \) of \((n − 1)\)-dimensional subspaces can also be defined as \( S^I_{n−1,1,F} \).

**Lemma 2.1.** Let \( n > 1 \) be an integer, and \( F \) a field. Let \( S_t \) be a \( t \)-subspace of PG\((2n − 1,F)\) intersecting each \( S^I \in S^I_{n−1,1,F} \) in precisely one point. Define \( \Phi = S_t \cap S_{n−1,1,F} \), and assume \( \langle \Phi \rangle = S_t \). Then \( |F| \geq t \) and the following properties hold.

1. The set \( \Phi \) is a normal rational curve of order \( t \).
2. Let \( \Xi^I \in S^I_{n−1,1,F} \). Then the set \( S(\Phi, \Xi^I) \) of the intersections of \( \Xi^I \) with all transversal lines \( l^{II} \) such that \( l^{II} \cap \Phi \neq \emptyset \) is a normal rational curve of order \( t \) or \( t − 1 \) if \( |F| = t \), and of order \( t − 1 \) if \( |F| > t \).
3. If \( \Phi \) is contained in a subvariety \( S_{t−1,1,F} \) of \( S_{n−1,1,F} \), then homogeneous coordinates can be chosen such that \( \Phi \) is represented parametrically by

\[
\begin{pmatrix}
y_0^{t−1} & y_0^{t−2}y_1 & \cdots & y_0y_1^{t−1}
y_0^{t−1} & y_0^{t−2}y_1 & \cdots & y_1^{t−1}\end{pmatrix}, \quad (y_0, y_1) \in (F^2)^*,
\]

and \( S(\Phi, \Xi^I) \), for \( z_0, z_1 \) depending only on \( \Xi^I \), by

\[
\begin{pmatrix}
y_0^{t−1}z_0 & y_0^{t−2}y_1z_0 & \cdots & y_1^{t−1}z_0
y_0^{t−1}z_1 & y_0^{t−2}y_1z_1 & \cdots & y_1^{t−1}z_1\end{pmatrix}, \quad (y_0, y_1) \in (F^2)^*.
\]
Proof. (i), (iii) The proof in [6] Sect.41 no.3], which is offered for \( F = \mathbb{C} \), works exactly the same provided that \( |F| > t \) or, more generally, that \( \Phi \) is contained in some subvariety \( S_{t-1,1,F} \) of \( S_{n-1,1,F} \). In case \( |F| \leq t \), the size of \( \Phi \) being \( |F| + 1 \) implies \( |F| = t \), so \( \Phi \) is just a set of \( t + 1 \) independent points in a subspace isomorphic to \( PG(t,t) \), hence \( \Phi \) is a normal rational curve of order \( t \).

(ii) The case \( |F| > t \) is proved in [6] immediately after the corollary at p. 175. If \( |F| \leq t \), then \( |F| = t \) and two cases are possible. If \( \Phi \) is contained in some \( S_{t-1,1,F} \subseteq S_{n-1,1,F} \), Burau’s proof is still valid as was mentioned in case (ii); so, \( S(\Phi, \Xi^I) \) is a normal rational curve of order \( t - 1 = |F| - 1 \). Otherwise \( S(\Phi, \Xi^I) \) is an independent \( (t+1) \)-set, hence a normal rational curve of order \( |F| \).

\[ \square \]

Remark 2.2. If \( |F| = t \) both cases in Lemma 2.1 (ii) can occur. The following two examples use the Segre embedding \( \sigma' = \sigma_{t-1,1,F} \) of the product space \( PG(t-1,1) \times PG(1,1) \) in \( PG(2t-1,1) \). Let \( \{ s_0, s_1, \ldots, s_t \} \) be the set of points on \( PG(1,1) \) and suppose \( \{ r_0, r_1, \ldots, r_t \} \) is a set of \( t + 1 \) points in \( PG(t-1,1) \). Put \( \Xi^I = \sigma(PG(1,1) \times s_0) \) and \( \Phi := \{ \sigma(r_i \times s_i) : i = 0, 1, \ldots, t \} \). Then \( \Phi \) consists of \( t + 1 \) points on the Segre variety \( S_{t-1,1,F} \). Depending on the set \( \{ r_0, r_1, \ldots, r_t \} \) one obtains the two cases described in Lemma 2.1 (ii).

a. If \( \{ r_0, r_1, \ldots, r_t \} \) is a frame of a hyperplane of \( PG(t-1,1) \) then \( \Phi \) generates a \( t \)-dimensional subspace of \( PG(2t-1,1) \) intersecting \( S_{t-1,1,F} \) in \( \Phi \) and \( S(\Phi, \Xi^I) \) is a normal rational curve of order \( t - 1 \).

b. If \( \{ r_0, r_1, \ldots, r_t \} \) generates \( PG(t-1,1) \) then \( \Phi \) generates a \( t \)-dimensional subspace of \( PG(2t-1,1) \) intersecting \( S_{t-1,1,F} \) in \( \Phi \) and \( S(\Phi, \Xi^I) \) is a normal rational curve of order \( t \).

Remark 2.3. By [11] and [2], the map \( \alpha : \Phi \rightarrow S(\Phi, \Xi^I) \) defined by the condition that \( X \) and \( X^\alpha \) are on a common line in \( S_{n-1,1,F}^I \) is related to a projectivity between the parametrizing projective lines. Such an \( \alpha \) is also called a projectivity.

3 The order of normal rational curves contained in \( S_{n-1,1,q} \)

Here \( n \geq 2 \) is an integer. The field reduction map \( F_{m,n,q} \) from \( PG(m-1, q^n) \) to \( PG(mn-1, q) \) will also be denoted by \( F \). If \( S \) is a set of points, in \( PG(m-1, q^n) \), then \( F(S) \) is a set of subspaces, whose union, as a set of points will be denoted by \( F(S) \). The \( F_{q^h} \)-span of a subset \( b \) of \( PG(d, q^n) \) is denoted by \( (b)_{q^h} \).

Proposition 3.1. Let \( b \) be a \( q \)-subline of \( PG(1,q^n) \), and let \( \Theta \not\in b \) be a point of \( PG(1,q^n) \). Let \( 1, \zeta \) and \( 1, \zeta' \) be homogeneous coordinates of \( \Theta \) with respect to two reference frames for \( (b)_{q^n} \), each of which consists of three points of \( b \). Then \( F_q(\zeta) = F_q(\zeta') \).

Proof. Homogeneous coordinates of a point in both reference frames, say \( (x_0, x_1) \) and \( (x'_0, x'_1) \), are related by an equation of the form \( \rho(x'_0, x'_1)^T = A(x_0, x_1)^T \), \( \rho \in F_{q^n}, A \in GL(2,q) \). Hence \( (\rho \cdot \rho')^T = A(1, \zeta)^T \) and this implies \( \zeta' \in F_q(\zeta) \). The proof of \( \zeta \in F_q(\zeta') \) is similar. \[ \square \]

By Proposition 3.1, the degree of a point over a \( q \)-subline \( b \) in a finite projective space \( PG(d, q^n) \), \([\Theta : b] = [F_q(\zeta) : F_q] \) for \( \Theta \in (b)_{q^n} \setminus b, [\Theta : b] = 1 \) for \( \Theta \in b \), is well-defined. This \( [\Theta : b] \) also equals the minimum integer \( m \) such that a subgeometry \( \Sigma \cong PG(d, q^m) \) exists containing both \( b \) and \( \Theta \).
Proposition 3.2. Any $n$-subspace of $\PG(2n - 1, q)$ containing an $(n - 1)$-subspace $S^I \in S_{n-1,1,q}$ intersects $S_{n-1,1,q}$ in the union of $S^I$ and a line in $S_{n-1,1,q}^{II}$.

Theorem 3.3. Let $b$ be a $q$-subline of $\PG(1, q^n)$, and $\Theta \not\in b$ a point of $\PG(1, q^n)$. Then in $\PG(2n - 1, q)$ any $n$-subspace $\mathcal{H}$ containing $\mathcal{F}(\Theta)$ intersects the Segre variety $S_{n-1,1,q} = \tilde{\mathcal{F}}(b)$, in a normal rational curve whose order is $\min\{q, [\Theta : b]\}$.

Proof. Set $L = \mathbb{F}_q^n$, $F = \mathbb{F}_q$. Without loss of generality, $\PG(2n - 1, q) = \PG_q(L^2)$, $\mathcal{F}(b) = \{L(x, y) \mid (x, y) \in (F^2)^*\}$ and $\Theta = L(1, \xi)$ with $[F(\xi) : F] = [\Theta : b]$. The $n$-subspace $\mathcal{H}$ intersects $L(1, 0)$ in one point $Y$ of the form $Y = F(\theta, 0)$, $\theta \in L^*$. For any $x \in F$, seeking for the intersection $\langle \mathcal{F}(\Theta), Y \rangle_q \cap L(x, 1)$, or

$$\langle L(1, \xi), F(\theta, 0) \rangle_q \cap L(x, 1)$$

gives two equations in $\alpha, \beta \in L$:

$$\alpha + \theta = \beta x, \quad \alpha \xi = \beta,$$

whence $\beta = \theta(x - \xi^{-1})^{-1}$. The intersection point is then $F(x\theta(x - \xi^{-1})^{-1}, \theta(x - \xi^{-1})^{-1})$. So, for $\Xi = L(0, 1)$, the set of the intersections of $\Xi$ with all lines in $S_{n-1,1,q}^{II}$ which meet $\mathcal{H}$ is

$$S(\mathcal{H} \cap S_{n-1,1,q}, \Xi) = \{F(0, \theta(x - \xi^{-1})^{-1}) \mid x \in F_q \} \cup \{F(0, \theta)\}.$$ 

This $S(\mathcal{H} \cap S_{n-1,1,q}, \Xi)$ is obtained by inversion from the line joining the points $F(0, \theta^{-1})$ and $F(0, \theta^{-1} \xi^{-1})$. By [10] Theorem 5, $C_\gamma$ is a normal rational curve of order $\delta' = \min\{q, [F(\xi^{-1}) : F] - 1\} = \min\{q, [\Theta : b] - 1\}$. Now apply lemma [2.1] for $\mathcal{H} = \mathcal{H} \cap S_{n-1,1,q}$ if $t \geq q$, then $t = q$ and $\delta' = q$ or $\delta' = q - 1$, so $[\Theta : b] \geq q$ and $t = \min\{q, [\Theta : b]\}$. If on the contrary $t < q$, then $t - 1 = \delta' = [\Theta : b] - 1$, so $t = [\Theta : b]$ and $t = \min\{q, [\Theta : b]\}$ again.

An important consequence of the above result answers the question of the existence of a Desarguesian spread containing a given regulus $\mathcal{R}$ and a subspace disjoint from $\mathcal{R}$.

Corollary 3.4. If a regulus $\mathcal{R} = S_{n-1,1,q}$ and an $(n - 1)$-dimensional subspace $U$, disjoint from $\mathcal{R}$, in $\PG(2n - 1, q)$ are contained in a Desarguesian spread then there is an integer $c$ such that any $n$-subspace $\mathcal{H}$ containing $U$ intersects $\mathcal{R}$ in a normal rational curve of order $c$.

The following remark illustrates that this necessary condition is not always satisfied.

Remark 3.5. For $n > 2$ by using the package FinInG [2] of GAP [3] examples can be given of $(n - 1)$-subspaces disjoint from $S_{n-1,1,q}$ contained in $n$-subspaces intersecting the Segre variety in normal rational curves of distinct orders. We include one explicit example. Let $q = 4$, $\mathbb{F}_q = \mathbb{F}_2(\omega)$, with $\omega^2 + \omega + 1 = 0$. Let $\mathcal{R}$ be the regulus of 3-dimensional subspaces of $\PG(7, 4)$ obtained from the standard subline $\PG(1, q)$ in $\PG(1, q^4)$, and put

$$S_3 := \langle (1, 0, 0, 0, \omega^2, 1, 0, 1), (0, 1, 0, 0, 1, \omega^2, 0, \omega^2), (0, 0, 1, 0, 0, \omega, 1, \omega), (0, 0, 0, 1, \omega^2, \omega^2, \omega, 1) \rangle.$$ 

Then $S_3$ is a three-dimensional subspace disjoint from the regulus $\mathcal{R}$. Moreover, the 4-dimensional subspace $\langle S_3, (1, 0, 0, 0, 0, 0, 0, 0) \rangle$ intersects the regulus $\mathcal{R}$ in a normal rational curve of degree 4, while the 4-dimensional subspace $\langle S_3, (0, 1, 0, \omega^2, 0, 0, 0, 0) \rangle$ intersects $\mathcal{R}$ in a conic.

\footnote{For $x, y \in L$, $F(x, y) = \langle (x, y) \rangle_q$, and $L(x, y) = \langle (x, y) \rangle_{q^n}$.}
4 André-Bruck-Bose representation

The André-Bruck-Bose representation of a Desarguesian affine plane of order $q^n$ is related to the image of $\text{PG}(2, q^n)$, under the field reduction map $\mathcal{F}$, by means of the following straightforward result.

**Proposition 4.1.** Let $D$ be the Desarguesian spread in $\text{PG}(3n - 1, q)$ obtained after applying the field reduction map $\mathcal{F}$ to the set of points of $\text{PG}(2, q^n)$, $l_\infty$ a line in $\text{PG}(2, q^n)$, and $K$ a $(2n)$-subspace of $\text{PG}(3n - 1, q)$, containing the spread $\mathcal{F}(l_\infty)$. Take $\text{PG}(2, q^n) \setminus l_\infty$ and $K \setminus \langle \mathcal{F}(l_\infty) \rangle_q$ as representatives of $\text{AG}(2, q^n)$ and $\text{AG}(2n, q)$, respectively. Then the map $\varphi : \text{AG}(2, q^n) \to \text{AG}(2n, q)$ defined by $\varphi(X) = \mathcal{F}(X) \cap K$ for any $X \in \text{AG}(2, q^n)$ is a bijection, mapping lines of $\text{AG}(2, q^n)$ into $n$-subspaces of $\text{AG}(2n, q)$ whose $(n-1)$-subspaces at infinity belong to the spread $\mathcal{F}(l_\infty)$.

The notation in Proposition 4.1 is assumed to hold in the whole section. The following result improves [4] Theorems 3.3 and 3.5], by determining the order of the involved normal rational curves.

**Theorem 4.2.** Let $b$ be a $q$-subline of $\text{PG}(2, q^n)$, not contained in $l_\infty$. Set $\Theta = \langle b \rangle_q \cap l_\infty$. Then the André-Bruck-Bose representation $\varphi(b \setminus l_\infty)$ is the affine part of a normal rational curve whose order is $\delta = \min\{q, [\Theta : b]\}$. More precisely, if $\delta = 1$, then $\varphi(b \setminus l_\infty)$ is an affine line; if $\delta > 1$, then $b \cap l_\infty = \emptyset$, and $\varphi(b)$ is a normal rational curve with no points at infinity.

**Proof.** The intersection $\mathcal{H} = \langle \mathcal{F}(b) \rangle_q \cap K$ is an $n$-space containing $\mathcal{F}(\Theta)$, and contained in the span of the Segre variety $S_{n-1,1,q} = \mathcal{F}(b)$. The result follows from Proposition 4.2 and Theorem 3.3.

The results in [4] Theorems 3.3 and 3.5] also characterize the normal rational curves arising from $q$-sublines in $\text{AG}(2, q^n)$.

In [5,14,15] for $n = 2$ and [4] Theorem 3.6 (a)(b)] for any $n$ the André-Bruck-Bose representation of a $q$-subplane tangent to a line at the infinity is described. Further properties are stated in the following theorem:

**Theorem 4.3.** Let $B$ be a $q$-subplane of $\text{PG}(2, q^n)$ that is tangent to $l_\infty$ at the point $T$. Let $b$ be a line of $B$ not through $T$, $\Theta = \langle b \rangle_q \cap l_\infty$, and $\delta = \min\{q, [\Theta : b]\}$. Then there are a normal rational curve $C_0$ of order $\delta$ in the $n$-subspace $\varphi(b \setminus l_\infty)$, a normal rational curve $C_1 \subset T \mathcal{F}(T)$ of order $\delta'$, with

$$
\delta' \begin{cases} 
= [\Theta : b] - 1 & \text{for } q > [\Theta : b] \\
\in \{q - 1, q\} & \text{otherwise,}
\end{cases}
$$

and a projectivity $\kappa : C_0 \to C_1$ (in the sense of Remark 2.3), such that $\varphi(B \setminus l_\infty)$ is the ruled surface union of all lines $XX^\kappa$ for $X \in C_0$.

**Proof.** By Theorem 4.2 $C_0 := \varphi(b)$ is a normal rational curve of order $\delta$ in the $n$-subspace $\varphi(b \setminus l_\infty)$, and for any $P = \varphi(X) \in C_0$, the subline $TX$ of $B$ corresponds to an affine line $P P^\kappa$ with $P^\kappa \in \mathcal{F}(T)$ at infinity. Define $C_1 = \{P^\kappa \mid P \in C_0\}$.

By the field reduction map $\mathcal{F} = \mathcal{F}_{3n,q}$, the subplane $B$ is mapped to $\mathcal{F}(B)$ which is the set of all maximal subspaces of the first family in $S_{n-1,2,q} \subset \text{PG}(3n - 1, q)$. The vector homomorphism $(\lambda, v) \in \mathbb{F}_{q^n} \times \mathbb{F}_q$ maps $\lambda \otimes_{\mathbb{F}_q} v$
corresponds to a projective embedding \( g : \text{PG}(n-1,q) \times B \rightarrow S_{n-1,2,q} \) whose image is \( S_{n-1,2,q} \), and such that \( \mathcal{F}(X) = (\text{PG}(n-1,q) \times X)^g \) for any point \( X \) in \( B \). It holds \( \varphi(B \setminus l_\infty) = S_{n-1,2,q} \cap K \setminus \mathcal{F}(T) \). For any point \( U \) in \( B \) define

\[
\kappa_U : (X,Y)^g \in S_{n-1,2,q} \mapsto (X,U)^g \in \mathcal{F}(U).
\]

Note that for any \( Y \in B \), the restriction of \( \kappa_U \) to \( \mathcal{F}(Y) \) is a projectivity. For any \( U \in B \), using the notation from Lemma 2.1, it holds \( C_0^\kappa = S(C_0, \mathcal{F}(U)) \), and as a consequence, \( C_0^\kappa \) is a normal rational curve of order \( \delta' \) as in (3). Now, since for any \( P \in C_0 \), say \( P = (X_P,Y_P)^g \), the points \( P^\kappa \) and \( P^{\kappa_T} \) are on the plane \( (X_P \times B)^g \in S_{n-1,2,q}^I \) and \( P^\kappa, P^{\kappa_T} \in \mathcal{F}(T) \), it follows that \( P^\kappa = P^{\kappa_T} \). It also follows that \( C_1 = C_0^{\kappa_U, \kappa_T} = S(C_0, \mathcal{F}(U))^{\kappa_T} \), and hence \( C_1 \) is a normal rational curve of order \( \delta' \) as in (3). Finally, \( \kappa_U : C_0 \rightarrow S(C_0, \mathcal{F}(U)) \) is a projectivity as defined in Remark 2.3 and hence so is \( \kappa \).

5 On the classification of clubs

An \( \mathbb{F}_q \)-club (or simply a club) in \( \text{PG}(1,q^n) \) is an \( \mathbb{F}_q \)-linear set of rank three, having a point of weight two, called the head of the club. An \( \mathbb{F}_q \)-club has \( q^2 + 1 \) points, and the non-head points have weight one. From now on it will be assumed that \( n > 2 \). The next proposition is a straightforward consequence of the representation of linear sets as projections of subgeometries [12, Theorem 2].

**Proposition 5.1.** Let \( L \) be an \( \mathbb{F}_q \)-club in \( \text{PG}(1,q^n) \subset \text{PG}(2,q^n) \). Then there are a \( q \)-subplane \( \Sigma \) of \( \text{PG}(2,q^n) \), a \( q \)-subline \( b \) in \( \Sigma \), and a point \( \Theta \in \langle b \rangle_{q^n} \setminus b \), such that \( L \) is the projection of \( \Sigma \) from the center \( \Theta \) onto the axis \( \text{PG}(1,q^n) \).

As before the notation \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) is used, where \( \mathcal{F} = \mathcal{F}_{2,n,q} \) denotes the field reduction map from \( \text{PG}(1,q^n) \) to \( \text{PG}(2n - 1, q) \).

**Proposition 5.2.** Let \( L \) be an \( \mathbb{F}_q \)-club of \( \text{PG}(1,q^n) \) with head \( \Upsilon \). Then \( \tilde{\mathcal{F}}(L) \) contains two collections of subspaces, say \( F_1 \) and \( F_2 \), satisfying the following properties.

(i) The subspaces in \( F_1 \) are \((n - 1)\)-dimensional, are pairwise disjoint, and any subspace in \( F_1 \) is disjoint from \( \mathcal{F}(\Upsilon) \).

(ii) Any subspace in \( F_2 \) is a plane and intersects \( \mathcal{F}(\Upsilon) \) in precisely a line.

(iii) Any point of \( \mathcal{F}(\Upsilon) \) belongs to exactly \( q + 1 \) planes in \( F_2 \).

(iv) If \( L \) is not isomorphic to \( \text{PG}(1,q^2) \), and \( l \) is any line of \( \text{PG}(2n - 1, q) \) contained in \( \tilde{\mathcal{F}}(L) \), then \( l \) is contained in \( \mathcal{F}(\Upsilon) \) or in a subspace in \( F_1 \cup F_2 \).

**Proof.** The assumptions imply the existence of \( \Sigma \) and a \( q \)-subline \( b \) in \( \Sigma \) as in Proposition 5.1. The assertions are a consequence of the fact that \( \tilde{\mathcal{F}}(\Sigma) \) is a Segre variety \( S_{n-1,2,q} \) in \( \text{PG}(3n-1,q) \). Let

\[
p_1 : \text{PG}(2,q^n) \setminus \Theta \rightarrow \text{PG}(1,q^n)
\]

be the projection with center \( \Theta \), associated with

\[
p_2 : \text{PG}(3n - 1,q) \setminus \mathcal{F}(\Theta) \rightarrow \text{PG}(2n - 1,q).
\]

6
The collections $F_1$ and $F_2$ are defined as follows:

$$F_1 = \{ \mathcal{F}(p_1(X)) \mid X \in \Sigma \setminus b \} = \mathcal{F}(L) \setminus \mathcal{F}(Y), \quad F_2 = \{ p_2(V^H) \mid V^H \in \tilde{\mathcal{F}}(\Sigma)^{II} \}.$$ 

The assertion $(i)$ is straightforward, as well as $\dim(V) = 2$ for any $V \in F_2$. For any $V^H \in \tilde{\mathcal{F}}(\Sigma)^{II}$, the intersection $V^H \cap \langle \tilde{\mathcal{F}}(b) \rangle_q$ is a line, and this with $p_2^{-1}(\mathcal{F}(Y)) = \langle \tilde{\mathcal{F}}(b) \rangle_q \setminus \mathcal{F}(\Theta)$ implies the second assertion in $(ii)$. Next, let $P$ be a point in $\mathcal{F}(Y)$. A plane $V = p_2(V^H)$ contains $P$ if, and only if, $V^H$ intersects the $n$-subspace $\langle \mathcal{F}(\Theta), P \rangle_q$, that is, $V^H$ intersects the normal rational curve $S_{n-1,2,q} \cap \langle \mathcal{F}(\Theta), P \rangle_q$; this implies $(iii)$.

Assume that a line $l \subset \tilde{\mathcal{F}}(L)$ exists which is neither contained in $\mathcal{F}(\Theta)$, nor in a $T \in F_1 \cup F_2$. Let $Q$ be a point in $l \setminus \mathcal{F}(\Theta)$, and let $V \in F_2$ such that $Q \in V$. It holds $L = B(V)$. Then $B(l)$ is a $q$-subline of $L$. Suppose that a line $l'$ in $V$ exists such that $B(l') = B(l)$. Since $B(Q) \neq B(Q')$ for any $Q' \in V$, $Q' \neq Q$, the line $l'$ contains $Q$. Then $l$, $l'$ are two distinct transversal lines in $B(l)^{II}$, a contradiction. Hence $B(l') \neq B(l)$ for any line $l'$ in $V$, that is, $B(l)$ is a so-called *irregular subline* [5]. By [5], Corollary 13, no irregular subline exists in $L$, and this contradiction implies $(iv)$. 

**Proposition 5.3.** Let $L$ be an $\mathbb{F}_q$-club with head $\Theta$. Let $\Theta$ be the point and $b$ be the subline as defined in Proposition 5.2. Then for any point $X \in \mathcal{F}(\Theta)$, the intersection lines of $\mathcal{F}(\Theta)$ with any $q$ distinct planes in $F_2$ containing $X$ span an $s$-dimensional subspace, where

$(i)$ $s = \lfloor \Theta : b \rfloor - 1$ if $q > \lfloor \Theta : b \rfloor$;

$(ii)$ $s \leq q - 1, q \}$ if $q \leq \lfloor \Theta : b \rfloor$.

**Proof.** Let $p_2$ be the projection map as defined in the proof of Proposition 5.2, $X = p_2(P)$, and $\mathcal{H} = \langle \mathcal{F}(\Theta), P \rangle_q$. For any plane $V = p_2(V^H)$, it holds $X \in V$ if, and only if $V^H \cap \mathcal{H} \neq \emptyset$. The intersection $\mathcal{H} \cap \tilde{\mathcal{F}}(b)$ is a normal rational curve of order $\min \{q, \lfloor \Theta : b \rfloor\}$ (cf. Theorem 5.3). Let $V_0 = p_2(V^H)$ be the unique plane of $F_2$ through $X$ distinct from the $q$ planes chosen in the assumptions (cf. Proposition 5.2). Let $Q = \tilde{\mathcal{F}}(b) \cap V_0^{II}$; $\mathcal{B}(Q)$ is an $(n-1)$-subspace of $\tilde{\mathcal{F}}(b)^I$. Such $\mathcal{B}(Q)$ is mapped onto $\mathcal{B}(X) = \mathcal{F}(\Theta)$ by $p_2$. Assume $V_i = p_2(V_i^{II})$, $i = 1, 2, \ldots, q$, are the $q$ planes chosen in the assumptions. Any $V_i^{II} = i = 1, 2, \ldots, q$, intersects $\mathcal{H}$, hence $V_i^{II} \cap \mathcal{B}(Q)$ is the intersection of $\mathcal{B}(Q)$ with a transversal line of $\tilde{\mathcal{F}}(b)$ intersecting the normal rational curve $\mathcal{H} \cap \tilde{\mathcal{F}}(b)$. By Lemma 2.1 $(ii)$, the set

$$S = \{ V_i^{II} \cap \mathcal{B}(Q) \mid i = 1, 2, \ldots, q \} \cup \{ Q \}$$

is a normal rational curve of order $s$ where $s$ takes the values as stated in $(i)$ and $(ii)$. Since $V_i \cap \mathcal{F}(\Theta)$ is the line through $X$ and a point of $p_2(S)$, distinct from $X$, the span of the intersection lines is the same as the span of $p_2(S)$. 

**Theorem 5.4.** Let $\mathcal{I}_{n,q}$ be the set of integers $h$ dividing $n$ and such that $1 < h < q$. For any $h \in \mathcal{I}_{n,q}$, let $L_h$ be the linear set obtained by projecting a $q$-subplane $\Sigma$ of $\text{PG}(2,q^n)$ from a point $\Theta_h$ collinear with a $q$-subline $b$ in $\Sigma$ and such that $[\Theta_h : b] = h$. Then the set $\Lambda = \{ L_h \mid h \in \mathcal{I}_{n,q} \}$ contains $\mathbb{F}_q$-clubs in $\text{PG}(1,q^n)$ all belonging to distinct orbits under $\text{PGL}(2,q^n)$.

**Proof.** If $n$ is odd, then no club is isomorphic to $\text{PG}(1,q^2)$. So, by Proposition 5.2 $(iv)$, the families $F_1$ and $F_2$ are uniquely determined. The thesis is a consequence of Proposition 5.3 taking into account that if $L$ and $L'$ are projectively equivalent, then $\tilde{\mathcal{F}}(L)$ and $\tilde{\mathcal{F}}(L')$ are projectively equivalent in $\text{PG}(2n - 1, q)$. 

7
In order to deal with the case $n$ even, it is enough to show that in $\Lambda$ at most one club is isomorphic to $\text{PG}(1,q^2)$. So assume $L_h \cong \text{PG}(1,q^2)$. Then $\tilde{\mathcal{F}}(L_h)$ has a partition $\mathcal{P}_1$ in $(n-1)$-subspaces, and a partition $\mathcal{P}_2$ in 3-subspaces. From [8 Lemma 11] it can be deduced that any line contained in $\tilde{\mathcal{F}}(L_h)$ is contained in an element of $\mathcal{P}_1$ or $\mathcal{P}_2$. The intersections of a subspace $U$ of a family $\mathcal{P}_i$ with the elements of the other family form a line spread of $U$. Hence all planes in $\mathcal{F}_2$ are contained in 3-subspaces of $\mathcal{P}_2$, and all planes of $\mathcal{F}_2$ through a point $X$ in $\mathcal{F}(\Upsilon)$ meet $\mathcal{F}(\Upsilon)$ in the same line. By Proposition 5.3 this implies $h = 2$.

Acknowledgement. The authors thank Hans Havlicek for his helpful remarks in the preparation of this paper.

References

[1] L. Bader - G. Lunardon: Desarguesian spreads. Ric. Mat. 60 (2011), 15–37.

[2] J. Bamberg - A. Betten - P. Cara - J. De Beule - M. Lavrauw - M. Law - M. Neunhoeffer - M. Pauley - S. Reichard: GAP 4 Package FinInG. cage.ugent.be/geometry/fining/manual.pdf

[3] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.5; 2014. http://www.gap-system.org

[4] S.G. Barwick - Wen-Ai Jackson: Sublines and subplanes of $\text{PG}(2,q^3)$ in the Bruck-Bose representation in $\text{PG}(6,q)$. Finite Fields App. 18 (2012), 93–107.

[5] R.C. Bose - J.W. Freeman - D.G. Glynn: On the intersection of two Baer subplanes in a finite projective plane. Utilitas Math. 17 (1980), 65–77.

[6] W. Burau: Mehrdimensionale projektive und höhere Geometrie. VEB Deutscher Verlag der Wissenschaften, Berlin, 1961.

[7] M. Lavrauw - J. Sheekey - C. Zanella: On embeddings of minimum dimension of $\text{PG}(n,q) \times \text{PG}(n,q)$. Des. Codes Cryptogr. doi: 10.1007/s10623-013-9866-8.

[8] M. Lavrauw - G. Van de Voorde: On linear sets on a projective line. Des. Codes Cryptogr. 56 (2010), 89–104.

[9] M. Lavrauw - G. Van de Voorde: Field reduction and linear sets in finite geometry. To appear in AMS Contemp. Math, American. Math Soc. arXiv:1310.8522

[10] M. Lavrauw - C. Zanella: Geometry of the inversion in a finite field and partitions of $\text{PG}(2^k - 1,q)$ in normal rational curves. J. Geom. 105 (2014), 103–110.

[11] M. Lavrauw - C. Zanella: Subgeometries and linear sets on a projective line. Preprint (2014), arXiv:1403.5754

[12] G. Lunardon - O. Polverino: Translation ovoids of orthogonal polar spaces. Forum Math. 16 (2004), 663–669.

[13] O. Polverino: Linear sets in finite projective spaces. Discrete Math. 310 (2010), 3096–3107.
claimed in the proof at page 174 that the assumption \( ⟨ \) authors the proof in [6] is obtained using an erroneous argument. As a matter of fact, it is elements. However such a generalisation would contradict Theorem 3.3. In the opinion of the

In [6] the previous result is seemingly proved using methods valid in any field with enough elements. However such a generalisation would contradict Theorem 3.3. In the opinion of the authors the proof in [6] is obtained using an erroneous argument. As a matter of fact, it is claimed in the proof at page 174 that the assumption \( ⟨ \Phi ⟩ = S_s \) is not used. However the contradiction \( S_s \subset (S_{s-2}, \mathbb{C}) \) is inferred from \( Φ \subset S_{s-2}, \mathbb{C} \).

A further counterexample, which exists whenever a hyperbolic quadric \( Q^+(3, F) \) in a three-dimensional projective space admits an external line (a condition which is not met when the field \( F \) is algebraically closed) is the following. If \( ℓ \) is the line corresponding to the two-dimensional vector space \( ⟨ e_1⟩ \otimes ⟨ e'_1, e'_2⟩ \) and \( m \) is a line external to the hyperbolic quadric obtained by the intersection of the Segre variety \( S_{2,1}, F \) with the 3-space corresponding to the vector space \( ⟨ e_2⟩ \otimes ⟨ e'_1, e'_2⟩ \), then the 3-dimensional subspace \( ⟨ ℓ, m⟩ \) intersects \( S_{2,1}, F \) in the line \( ℓ \) belonging to \( S^3_{2,1}, F \).

For the sake of completeness, a proof for corollary A.1 is given.

**Proof of corollary A.1.** Define

\[
S_t = ⟨ S_s \cap S_{s-1}, F⟩, \quad t = \dim S_t
\]

and suppose \( t < s \). It is proved in [6] p.173 (6) that \( S_t \subset (S_{t-1}, F⟩ \) for some \( S_{t-1}, F \subset S_{s-1}, F \).

Note that \( S_s \cap (S_{t-1}, F⟩ = S_t \); otherwise, comparing dimensions, \( S_s \) would intersect each \( S^I \subset S_{t-1}, F \) in more than one point. Now choose

- a subspace \( S_{s-t-1} \subset S_s \) such that \( S_{s-t-1} \cap (S_{t-1}, F⟩ = \emptyset \);
- a Segre variety \( S_{s-t-1}, F \subset S_{s-1}, F \), such that \( (S_{s-t-1}, F⟩ \cap (S_{t-1}, F⟩ = \emptyset \);
- two distinct \( A^I, B^I \subset S^I_{s-t-1}, F \).

Since \( (S_{s-t-1}, F⟩ \) and \( (S_{t-1}, F⟩ \) are complementary subspaces of \( (S_{s-1}, F⟩ \), a projection map

\[
π : (S_{s-1}, F⟩ \setminus (S_{t-1}, F⟩ \to (S_{s-t-1}, F⟩
\]

is defined by \( π(P) = (P \cup S_{t-1}, F⟩ \cap (S_{s-t-1}, F⟩ \). Now suppose \( π(S_{s-t-1}) \cap S_{s-t-1}, F \) is a point. In \( (S_{s-t-1}, F⟩ \) consider

[14] C.T. Quinn - L.R.A. Casse: Concerning a characterisation of Buekenhout-Metz units, J. Geom. 52 (1995), 159–167.

[15] R. Vincenti: Alcuni tipi di varietà \( V_2 \) di \( S_{4,q} \) e sottopiani di Baer. Boll. Un. Mat. Ital. Suppl. 1980, no. 2, 31–44.

[16] C. Zanella: Universal properties of the Corrado Segre embedding. Bull. Belg. Math. Soc. Simon Stevin 3 (1996), 65–79.

### Appendix: On a result in [6]

In [6] p.175 the following result (Korollar) is stated for \( F = \mathbb{C} \).

**Corollary A.1.** Let \( F \) be an algebraically closed field. If an \( s \)-subspace \( S_s \) of \( PG(2s-1,F) \) meets all \( S^I \subset (S_{s-1}, F⟩ \) only in points, then such points span \( S_s \).

In [6] the previous result is seemingly proved using methods valid in any field with enough elements. However such a generalisation would contradict Theorem 3.3. In the opinion of the authors the proof in [6] is obtained using an erroneous argument. As a matter of fact, it is claimed in the proof at page 174 that the assumption \( ⟨ \Phi ⟩ = S_s \) is not used. However the contradiction \( S_s \subset (S_{s-2}, \mathbb{C}) \) is inferred from \( Φ \subset (S_{s-2}, \mathbb{C} \).

A further counterexample, which exists whenever a hyperbolic quadric \( Q^+(3,F) \) in a three-dimensional projective space admits an external line (a condition which is not met when the field \( F \) is algebraically closed) is the following. If \( ℓ \) is the line corresponding to the two-dimensional vector space \( ⟨ e_1⟩ \otimes ⟨ e'_1, e'_2⟩ \) and \( m \) is a line external to the hyperbolic quadric obtained by the intersection of the Segre variety \( S_{2,1}, F \) with the 3-space corresponding to the vector space \( ⟨ e_2⟩ \otimes ⟨ e'_1, e'_2⟩ \), then the 3-dimensional subspace \( ⟨ ℓ, m⟩ \) intersects \( S_{2,1}, F \) in the line \( ℓ \) belonging to \( S^3_{2,1}, F \).

For the sake of completeness, a proof for corollary A.1 is given.
the regulus $\mathcal{R}$ corresponding to $S_{s-t-1,1,F}$, and the projectivity $\kappa : A^I \rightarrow B^I$ such that, for any $P \in A^I$, the line $\langle P, \kappa(P) \rangle$ belongs to $S_{s-t-1,1,F}^I$;

- the regulus $\mathcal{R}'$ containing $A^I$, $B^I$ and $\pi(S_{s-t-1})$, and the projectivity $\kappa' : A^I \rightarrow B^I$ such that, for any $P \in A^I$, the line $\langle P, \kappa'(P) \rangle$ is a transversal line of $\mathcal{R}'$.

Since $F$ is an algebraically closed field, $\kappa'^{-1} \circ \kappa$ has a fixed point $P$. Therefore $\kappa(P) = \kappa'(P)$, so $\mathcal{R}$ and $\mathcal{R}'$ have a common transversal. This contradicts $\pi(S_{s-t-1}) \cap S_{s-t-1,1,F} = \emptyset$. So, a point $P \in S_{s-t-1}$ exists such that $\pi(P) \in S_{s-t-1,1,F}$.

Next, let $C^I \in S_{s-1,1,F}$ be such that $\pi(P) \in C^I$, and $Q$ the point in $\langle S_{t-1,1,F} \rangle$ such that $Q$, $P$, and $\pi(P)$ are collinear. If $Q \in S_{t}$, then $\pi(P) \in S_{s}$, a contradiction; also $Q \in C^I$ leads to a contradiction (since it implies $P \in C^I$). So $Q \notin S_{t} \cup C^I$ and by a dimension argument two points $Q_1 \in C^I \setminus S_{t}$ and $Q_2 \in S_{t} \setminus C^I$ exist such that $Q$, $Q_1$ and $Q_2$ are collinear: they are on the unique line through $Q$ meeting both $C^I \cap \langle S_{t-1,1,F} \rangle$ and a $(t-1)$subspace of $S_{t}$ disjoint from $C^I$.

The plane $\langle P, Q_1, Q_2 \rangle$ contains the lines $PQ_2 \subset S_{t}$ and $\pi(P)Q_1 \subset S_{s-1,1,F}$ which meet outside $\langle S_{t-1,1,F} \rangle$. This is again a contradiction. \qed