Pascal’s Hexagon Theorem implies a Butterfly Theorem in the Complex Projective Plane

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1 Introduction

Some time ago I attempted to prove the following for my own entertainment.

**Butterfly Theorem.** Let $ab$ be a chord of a circle with midpoint $m$. Suppose $rs$ and $uv$ are two other chords that pass through $m$, as shown below. Let $p$ and $q$ be the intersections of $rv$ and $fs$ with $ab$. Then $pm = qm$.

![Butterfly Theorem Diagram](image)

Much to my delight, the proof I came up with used Pascal’s Hexagon Theorem. A few years later, I learned of the theory of conics in the complex projective plane. I attempted to translate my proof for the circle into this more general setting, and was thrilled to see that it worked very naturally.
In what follows, $cr(abcd)$ denotes the cross ratio, and $C$ is a fixed conic in $\mathbb{CP}^2$. Here then is the generalization:

**Theorem 1** Suppose $C$ contains distinct points $a, b$. Let $m$ be any point on $ab$ which is not equal to $a$ or $b$. Let $rs$ and $fg$ be any chords of $C$ which contain $m$. Let $i = rg \cap ab$ and $j = fs \cap ab$. If $p$ is chosen on $ab$ so that $cr(pamb) = -1$, then $cr(pjmi) = -1$ as well.

The next section contains the proof of this theorem, and the final section contains a few notes on the planar case.

## 2 Proof of Theorem 1

Our initial aim is to define a natural way to reflect points around any chord of a conic. We need a lemma first.

**Lemma 1** Let $k$ be any line which intersects $C$ at two points $u$ and $v$, and let $p$ denote the pole of $k$. Let $l$ be a line through $p$ which intersects $C$ at two points $y, y'$, and let $m$ be a point on $l$. Then $cr(pymy') = -1$ if, and only if, $m = l \cap k$. 

Proof: Choose homogeneous coordinates such that \( u = (0 : 1 : 0), v = (0 : 0 : 1), y' = (1 : 0 : 0), \) and \( y = (1 : 1 : 1) \). Then the equation defining \( C \) must be of the form \( Axy + Bxz + Cyz = 0 \) with \( A + B + C = 0 \). We can see that \( l : y - z = 0 \) and \( k : x = 0 \). Let \( m = l \cap k \). Then \( m = (0 : 1 : 1) \). Using the standard formula for the tangent line to a point (see [2]), the tangent at \( u \) is \( Ax + Cz = 0 \), and the tangent at \( v \) is \( Cy + Bx = 0 \). These intersect at the point \( p = (1 : -\frac{B}{C} : \frac{A}{C}) \). But \( p \) also lies on \( l \), so that \( A = B \). The equation \( A + B = -C \) therefore forces \( p = (1 : 1/2 : 1/2) \). We can projectively map \( (p, y, m, y') \mapsto (1/2, 1, \infty, 0) \), so we see that indeed \( cr(pyym') = -1 \). The converse follows easily.

We begin by defining \( y \) and \( y' \) as referred to in the previous lemma as reflections of each other around \( k \). We see that this induces a natural mapping of \( C \) to itself. We can extend this to a map for any point \( y \) in the plane by \( y \mapsto y' \), where \( n = k \cap py \) and \( y' \) is the unique point on \( py \) such that \( cr(pyn'y') = -1 \)(this map is defined to be the identity on \( k \), and is undefined at \( m' \)). We will refer to this mapping as the reflection over \( k \).

Lemma 2 Let \( k \) be a line with pole \( p \), let \( y \) and \( y' \) be points which are reflections of each other over \( k \), and let \( u \) be a point on \( k \). Draw any other line through \( p \), and let \( t, t' \) be the intersections of this line with \( ru, r'u \). Then \( t \) and \( t' \) are reflections of each other.
Remark: In the picture it is shown that $u, y, y' \in C$, but this isn’t necessary for the lemma to hold.

Proof: Let $m = k \cap py$ and $n = k \cap pt$. Projecting $pt$ from $u$ onto $py$ we see that $cr(ptnt') = cr(pymy')$, which is $-1$ by Lemma 1. □

Lemma 3 Suppose $y, y'$ are reflections of each other over a line $k$ with pole $p$, as are $z, z'$. Then $yz \cap y'z' \in k$. 
Proof: Let \( u = yz \cap k \). Then, by Lemma 2, \( uy' \cap pz \) is the reflection of \( z \), and is therefore equal to \( z' \). Thus, \( y'z' \) passes through \( u \) as well, and \( u = yz \cap y'z' \in k \).

In this situation, we will say that the lines \( yz \) and \( y'z' \) are reflections of each other.

Lemma 4 Let \( k \) be a chord containing distinct points \( u, v \) on \( C \), and let \( p \) be the pole of \( k \). Let \( m \) be a point on \( k \), and let \( rs \) be a chord of \( C \) passing through \( m \). Let \( r' \) be the reflection of \( r \) over \( k \). Then \( x = r'v \cap su \) lies on \( pm \).

Proof: Let \( s' \) be the reflection of \( s \), and consider the hexagon \( rvr's'us \). Let \( z = rv \cap s'u \). By Lemma 4, \( r's' \cap r's \) lies on \( k \), and must therefore be equal to \( m \). By Pascal’s Theorem, \( z, m, \) and \( x \) lie on a line. The theorem will be proved if we can show that this line passes through \( p \), as is shown in this picture.
su and s'u are reflections of each other, as are rv, r'v. It follows that the reflection of x lies on both us' and vr, and is therefore equal to us' ∩ vr. Thus, z and x are reflections of each other, which implies that z, x, and p lie on a line. Hence, z, m, x, and p are collinear.

Proof of Theorem 1: In light of what has come before, we need to prove that i and j are reflections of each other. Let g' and r' be the reflections of g and r.

By Lemma 4, r'v ∩ su and fv ∩ g'u both lie on pm. Thus, by Pascal’s Theorem applied to the hexagon r'vf sug', r'g' ∩ sf = j lies on pm as well.
But $r'g'$ is the reflection of $rg$, so by Lemma 3, $rg \cap pm = i$ is the reflection of $r'g' \cap pm = j$, and we are done.

3 Remarks on the planar case

In [1], this method of proof is used to deduce the Butterfly Theorem for conics in $\mathbb{R}^2$, with one exception. The case in which the initial chord intersects a hyperbola once on each of the branches of the hyperbola could not be covered while staying entirely in $\mathbb{R}^2$, since the relevant polar in that case did not intersect the hyperbola at any point. To get around this difficulty, we consider $\mathbb{RP}^2$ as embedded in $\mathbb{CP}^2$ as the set of fixed points of the map $(z_1 : z_2 : z_3) \mapsto (\bar{z}_1 : \bar{z}_2 : \bar{z}_3)$. In this larger space, all lines intersect the conic, and we arrive at no difficulties. Therefore, the following generalization of the Butterfly Theorem in $\mathbb{R}^2$ is obtained as a corollary to the above work:

**Theorem 2** Let $C$ be a conic in the plane. Let a point $m$ be on a chord intersecting $C$ at two distinct points $a$ and $b$. Let $rs$ and $uv$ be two chords passing through $m$. Let $p$ and $q$ be the intersections of $ru$ and $sv$ with $ab$. Let $m'$ be the unique point (possibly $\infty$) on $ab$ so that $cr(m'amb) = -1$. Then $cr(m'pmq) = -1$ as well.

**Proof:** Suppose $C$ is given by $Ax^2 + By^2 + Qxy + Dx + Ey + F = 0$. Then $Ax^2 + By^2 + Qxy + Dxz + Eyz + Fz^2 = 0$ gives the extension of $C$ to $\mathbb{CP}^2$. Since $m, a, b \in \mathbb{RP}^2$, $m' \in \mathbb{RP}^2$ as well. It follows as above that $p$ and $q$ are reflections of each other over the polar of $m'$. This polar also passes through $m$, though it does not necessarily intersect $C$ in $\mathbb{RP}^2$. Whether or not the polar intersects $C$ in $\mathbb{RP}^2$, we have $cr(m'pmq) = -1$, and we are done.

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References

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