INITIAL DATA AND THE FINAL FATE OF INHOMOGENEOUS DUST COLLAPSE

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Abstract

We examine here the relevance of the initial state of a collapsing dust cloud towards determining it’s final fate in the course of a continuing gravitational collapse. It is shown that given any arbitrary matter distribution $M(r)$ for the cloud at the initial epoch, there is always a freedom to choose rest of the initial data, namely the initial velocities of the collapsing spherical shells, so that the collapse could result either in a black hole or a naked singularity depending on this choice. Thus, given the initial density profile, to achieve the desired end state of the gravitational collapse one has to give a suitable initial velocity to the cloud. We also characterize here a wide new family of black hole solutions resulting from inhomogeneous dust collapse. These configurations obey the usual energy conditions demanding the positivity of energy density.

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1. Introduction

One of the important questions regarding the naked singularities arising from gravitational collapse is the issue of their genericity. Even if naked singularities occurred for reasonable forms of matter satisfying energy conditions etc, if this occurrence could be shown in a suitable sense to be non-generic, arising from an initial data set of zero measure only, then such a result would open up a new path for a proper formulation and possible proof for the cosmic censorship hypothesis which prohibits the occurrence of naked singularities. To make any progress towards such a goal, it is necessary to understand the role of initial data in determining the final fate of gravitationally collapsing massive objects in terms of either a black hole or a naked singularity. The end product of such a collapse- namely the spacetime singularity- is either completely covered by an event horizon, thus producing a black hole in the spacetime, or in the case otherwise it has possible causal connection with the outside universe, which would require further investigations.

A crucial aspect of the gravitational collapse of a compact object is the choice of initial data. After the star has exhausted its nuclear fuel, it is well known that if the mass of the remnant cloud was large enough, gravitational collapse would ensue. At the time when the gravitational collapse starts, one of the sets of initial quantities are density, pressures, etc., which describe the state of matter distribution completely. Another set of initial data has to be in the form of the distribution of initial velocities of the cloud towards the center. These two sets of initial data for the collapse would actually arise as the end product of the process of the star exhausting its nuclear fuel.

Our purpose here is to analyze, from such a perspective, the role of initial data (defined in terms of physical variables such as the initial density and veloc-
ity distributions) towards determining the final fate of a gravitationally collapsing cloud. We consider here the case of pressure-free dust collapse described by the Tolman-Bondi-Lemaître models [1,2], which is a large class of collapse scenarios including the homogeneous case [3]. These models have been studied widely for the occurrence or otherwise of naked singularity [4-8] (for further details and related issues, see e.g. [9]). However, the space of initial data which would cause a naked singularity or otherwise, and it’s genericity, yet remain to be properly understood. The initial data in this case is characterized by two free functions on spacetime (obtained through the integration of Einstein equations), namely the mass function \( M(r) \) describing the matter distribution, and the velocity distribution function \( V_i(r) \) of the spherical shells of cloud at the onset of collapse. We study here the conditions causing the occurrence of a black hole or a naked singularity and it is shown quite generally that given any distribution of mass \( M(r) \) (which is part of the initial data to be specified), there always exists rest of the initial data in the form of a suitable choice of the initial velocities \( V_i(r) \) of the cloud towards the center, such that the final outcome of the collapse could be either one in the form of a black hole or a naked singularity.

Since the specification of initial state of matter and densities at the onset of collapse constitutes very important initial data physically, we investigate here in what way the choice of any particular regular distribution of matter would affect the evolution and the final fate of the system in terms of either of these outcomes. For example, in the consideration by Oppenheimer and Snyder [3], the density distribution chosen for the (pressure free) dust cloud at the initial and also at all later times is homogeneous (i.e. \( \rho = \rho(t) \)), with initial spatial curvature being constant (i.e. initial velocity distribution \( V_i(r) \propto r \)). As is well known, the collapse of such a configuration results in a black hole where the resulting singularity is
fully covered by an event horizon. A generic density profile, however, need not be homogeneous and would be typically higher at the origin and decreasing away from the center. Similarly, even if the density profile is homogeneous, the initial velocity of the spherical shells need not necessarily be proportional to \( r \). It is thus essential in realistic considerations that the effects of both the inhomogeneities present in the initial matter distribution, and the variations in the velocity profiles, be taken into account while investigating the evolution of collapse as we shall discuss below.

2. Dust Collapse

Inhomogeneous dust collapse in the comoving coordinates (i.e. \( u^i = \delta^i_t \)) is given by the Tolman-Bondi metric

\[
\begin{align*}
    ds^2 &= -dt^2 + \frac{R'^2}{1 + E} dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \\
    T^{ij} &= \epsilon \delta^i_t \delta^j_t, \quad \epsilon = \epsilon(t, r) = \frac{M'}{R^2 R'} \\
    \dot{R} &= -\sqrt{\frac{M}{R} + E} \Rightarrow t - t_0(r) = -\frac{R^{3/2} G(-ER/M)}{\sqrt{F}}
\end{align*}
\]

Here \( T^{ij} \) and \( \epsilon \) denote the energy-momentum tensor and total energy density respectively. The dot and the prime denote partial derivatives with respect to the coordinates \( t \) and \( r \) respectively, and we have taken \( \dot{R}(t, r) < 0 \) as we are considering collapse scenarios only. \( G(y) \) is a strictly real positive and bound function which has the range \( 1 \geq y \geq -\infty \), and is given by

\[
\begin{align*}
    G(y) &= \left( \frac{\sin^{-1} \sqrt{y}}{y^{3/2}} - \frac{\sqrt{1-y}}{y} \right) \quad \text{for} \quad 1 \geq y > 0 \\
    G(y) &= \frac{2}{3} \quad \text{for} \quad y = 0 \\
    G(y) &= \left( \frac{-\sinh^{-1} \sqrt{-y}}{(-y)^{3/2}} - \frac{\sqrt{1+y}}{y} \right) \quad \text{for} \quad 0 > y \geq -\infty
\end{align*}
\]
The functions $M$, $t_0(r)$ and $E$ are arbitrary functions of $r$. In the context of the cosmic censorship conjecture in a physically realistic collapse scenario, at the onset of collapse the star should have regular initial data (density, pressure, etc.), and the spacetime be singularity free at this initial epoch of time. The curvature singularity would develop later during the final stages of the gravitational collapse. We therefore require that at the onset of collapse, at time $t = t_i$, the spacetime is non-singular and initial data in terms of density etc. do not diverge. Using the remaining coordinate freedom left in the choice of coordinate $r$ we rescale the coordinate $r$ at the initial epoch of time $t = t_i$ as

$$R(t_i, r) = r$$

(5)

This then implies from equation (3) that $t_0(r)$, and the important quantity $R'$ are given by

$$t_0(r) = t_i + \frac{r^{3/2}G(-Er/M)}{\sqrt{M}}$$

(6)

$$R' = \left( (\eta - \beta)Y + (\Theta - (\eta - \frac{3}{2}\beta)Y^{2}G(pY))(-p + \frac{1}{Y})^{1} \right)^{1/2}$$

(7)

where we have put

$$Y = \frac{R}{r}, \quad \eta = \frac{r M'}{M}, \quad \beta = \frac{r E'}{E}, \quad p = p(r) = -\frac{E}{M}$$

$$\Theta = \frac{\sqrt{M}}{\sqrt{r}} t'_{0}(r) = \frac{1 + \beta - \eta}{(1 - p)^{1/2}} + (\eta - \frac{3}{2}\beta)G(p)$$

(8)

The time $t = t_0(r)$ corresponds to $R = 0$ which is the singularity of spacetime where the area of the shells of matter at a constant value of the coordinate $r$ vanishes, and corresponds to the time when the matter shells meet the physical singularity. Thus the singularity occurs at the coordinate value $r$ at the time $t = t_0(r) > t_i$ and hence the range of coordinates is given by

$$0 \leq r < \infty, -\infty < t < t_0(r)$$

(9)
From equation (3) (i.e. $\dot{R}^2 = E + M/R$), we note that $M(r)$ is interpreted as the mass function for the cloud, and is related to the density of the matter at the onset of collapse by

$$\frac{M'}{r^2} = \epsilon(t_i, r) = \rho(r)$$  \hspace{1cm} (10)

The function $E(r)$ is interpreted as the energy of the cloud, and is related to the initial radial velocity $V_i(r)$ of the particles at $t = t_i$ by the relation

$$E(r) = \dot{R}(0, r)^2 - M(r)/r = V_i^2(r) - M/r$$  \hspace{1cm} (11)

For a free fall collapse with $\dot{R}(t_i) = 0$, $V_i(r) = 0$. For a spherically symmetric star of radius $r_c$, the complete set of initial data includes the state of matter inside the star, i.e. the matter density $\rho(r)$, and the radial velocities $V_i(r)$ at $t = t_i$. Note that the Kretschmann scalar $K = R^{ijkl}R_{ijkl}$ for the Tolman-Bondi-Lemaitre spacetimes is given by

$$K = 12 \frac{M'^2}{R^4 R'^2} - 32 \frac{MM'}{R^5 R'} + 48 \frac{M^2}{R^6}$$  \hspace{1cm} (12)

As the initial data, including the density, should be regular and singularity free at $t = t_i$, this implies that throughout this initial surface (i.e. for all allowed values of $r$), we have

$$M(r) \equiv r^3 g(r)$$  \hspace{1cm} (13)

where $g(r)$ is a differentiable function. We assume that the weak energy condition ($\epsilon(t, r) \geq 0$) is satisfied, and therefore $M' \geq 0$ (note that at $t = t_i$, $\epsilon = M'/r^2$), and consequently $R$ is a monotone increasing function of $r$ i.e. $R' \geq 0$. Since $M(0) = 0$ (in the case otherwise, there will already be a singularity at $r = 0$ on the initial surface $t = t_i$ as seen from equation (12)), and due to energy conditions $M' \geq 0$, we have $M(r) \geq 0$ for $r \geq 0$. 

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As we are interested in the gravitational collapse scenario of a star, the energy density $\epsilon$ is taken to have a compact support on an initial spacelike hypersurface, and the Tolman-Bondi-Lemaitre spacetime is matched at some $r = \text{const.} = r_c$ to the exterior Schwarschild metric,

$$ds^2 = -(1 - 2M_s/r_s)dT^2 + (1 - 2M_s/r_s)^{-1}dr_s^2 + r_s^2d\Omega^2$$  \hspace{1cm} (14)

The value of the Schwarschild radial coordinate is $r_s = R(t, r_c)$ at the boundary of the star. We have $M(r_c) = 2M_s$ where $M_s$ is the total Schwarschild mass enclosed within the dust ball of coordinate radius $r = r_c$. Without going into further details of the matching conditions, we would like to say a few words regarding the apparent horizon. The apparent horizon in the interior dust ball lies at $R = M(r)$. From (3) and (6) one can see that the corresponding time $t = t_{ah}(r)$ is given by

$$t = t_{ah}(r) = t_0(r) - MG(-E)$$  \hspace{1cm} (15)

Since $t_0(r) > t_{ah}$ for all $r > 0$, and $t_0(0) = t_{ah}(0)$ at $r = 0$, it follows that, only the singularity at $r = 0$ could be naked, and rest of the singular points for the values $r > 0$ are censored.

Any radial light ray terminating at this singularity at $r = 0$ in the past could go to the future infinity if it reaches the surface of the cloud $r = r_c$ earlier than the apparent horizon at $r = r_c$. In such a case the singularity would be globally naked. The shell-focusing central singularity, which we discuss here, appears at a finite time $t = t_0(0), r = 0$; however, for $g(0) = 0$ it occurs at an infinite value of coordinate time, i.e. $t_0(0) = \infty$, unless $E(r) = r^2f_0(r)$ in such a way that $f_0(0) > 0$. One can therefore also choose suitable radial velocity $V_i(r)$ (for example, $E(r) \propto r^a$, $a < 2$) at the initial $t = t_i$, such that for a given initial density distribution the singularity does not form at $r = 0$ (i.e. it occurs at an infinite value of the coordinate time $t$).
In such a case, since any singularity forming at $r > 0$ is covered, this gives rise to a black hole as the end state of collapse.

As mentioned earlier, the main aim here is to show that for a given density distribution one can choose the initial radial velocities of the spherical shells such that gravitational collapse could result either in a black hole or a central naked singularity. We analyze here mainly the nature of the central shell-focusing singularities at $R = 0$. However, as seen from (2), there is a density singularity in space-time either when $R = 0$, or when $R' = 0$ which corresponds to a shell-crossing singularity in the spacetime. A relevant question here may be regarding the occurrence of such shell-crossing singularities, and their relevance to our conclusions. The point is, during the evolution of the collapse, if a shell-crossing singularity occurs before the formation of the physical shell-focussing singularity $R = 0$ (where the shells of matter are crushed to a vanishing area), the metric and the coordinate system cannot be readily continued beyond such a shell-cross singularity. It may be noted, however, that such shell-crosses are generally regarded to be gravitationally weak singularities, through which the spacetime may be continued. It is known that in the limit of approach to such a singularity, the spacetime curvatures do not diverge sufficiently fast in the context of the dust models being considered here [6], and at least a $C^1$ extension of the spacetime exists through such a singularity. It is also possible to discuss the extension of spacetime in a distributional sense through such a singularity (see e.g. [9] for a further discussion). In any case, for any given density distribution (i.e. for given mass function $M(r)$) we shall consider here only those energy functions $E(r)$ (i.e. the initial velocities $V_i(r)$) for which the dust collapse does not give rise to any shell-crossing singularities for $r > 0$. 

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To make this specific, we have from equations (4) to (8)

\[ R' = \frac{\sqrt{1 - pY}}{\sqrt{Y}} \left( \frac{Y^{\frac{3}{2}}G^1(pY)}{\sqrt{1 - pG^1(p)}} + \Theta(1 - \frac{Y^{\frac{3}{2}}G^1(pY)}{G^1(p)}) + \frac{\eta Y^{\frac{3}{2}}G(pY)}{2} \right) \left(1 - \frac{pYG^1(p)}{G(pY)}\right) \]

where \( G^1(f) \equiv (dG(f)/df) \). Since in a collapse scenario (i.e. \( \dot{R} < 0 \)), for \( t \geq t_i \), \( Y = R/r \leq 1 \) and because the functions \( G(y), G^1(y) \) are strictly increasing positive functions, the coefficients of \( \Theta \) and \( \eta \) are positive in the above expression. Hence the conditions for no shell-crossings basically turn out to be \( \Theta \geq 0 \) for the case when \( \eta = (rM'/M) \geq 0 \). However, for the dust cloud satisfying the weak energy condition and with the mass function \( M(r) \geq 0 \) as pointed out above, we have \( \eta \geq 0 \). Thus, to avoid shell-crossing singularities we must have \( \Theta \geq 0 \). As seen from (8), this amounts to \( t_0(r) \) being a monotone increasing function of \( r \). Thus the singularity curve must be an increasing function and successive singular points in space form at successively later times. Physically, this means that the shells of matter with increasing values of \( r \) arrive at the singularity one after the other at later and later times without crossing each other in between. In the case of density functions which are decreasing as one moves away from the center (\( g'(r) \leq 0 \)), and for the class \( E \leq 0 \), the condition above reduces to \( p' \geq 0 \) [10].

### 3. Causal Structure and Outgoing Trajectories

For a particular initial data set to develop either into a naked singularity or a black hole, one has to analyze the behavior of causal trajectories and categorize different situations as to when the spacetime permits outgoing nonspacelike geodesics from the singularity, or otherwise. We therefore consider the radial null geodesics, given by

\[ \frac{dt}{dr} = \pm \frac{R'}{\sqrt{1 + E(r)}} \]
where the positive sign represents outgoing solutions and negative sign represents ingoing ones. We consider here only outgoing geodesics, and recall for completeness some notation and terminology from [7], which will be used here. If there are no outgoing null trajectories escaping away from the singularity with their past end point at the singularity, then the resulting configuration is a black hole in any case.

Therefore, for the positive sheet of solutions, we have

\[
\frac{dt}{dr} = \frac{R'}{\sqrt{1 + E}} \Rightarrow \frac{dR}{du} = \frac{1}{u'}(R' + \frac{dt}{dr}) = (1 - \frac{\sqrt{E + \Lambda/X}}{\sqrt{1 + E}})H = U(X, u) \tag{18}
\]

where \( u = u(r) \) is a differentiable monotone increasing function of \( r \) such that \( u(r) > 0 \) for \( r > 0 \), \( u(0) = 0 \) and, from equation (7)

\[
\frac{R'}{u'} = \left( (\eta_u - \beta_u)X + (\Theta_u - (\eta_u - \frac{3}{2}\beta_u)X^2G(-PX))(P + \frac{1}{X})^{\frac{3}{2}} \right) \equiv H(X, u) \tag{19}
\]

\[
X = (R/u), \quad \eta_u = \frac{u}{ru'}\eta, \quad \beta_u = \frac{u}{ru'}\beta \quad \Theta_u = \Theta \frac{\sqrt{r}}{\sqrt{uu'}} \tag{20}
\]

The function \( \Theta_u(r) \) defined above plays an important role in deciding the end state of collapse. The singularity at \( R = 0, u = 0 \) is a singularity of the differential equation (15). For a given set of initial data (i.e. \( M(r), \) and \( V_i(r) \)), if there exists a function \( u = u(r) \) such that \( \Theta_u \sqrt{P + 1/X} \) has non-zero definite value as \( r \to 0 \) along all \( X = \text{constant} \) directions, then the null geodesics could terminate in the past at the singularity, otherwise geodesics do not terminate there. The necessary and sufficient condition [7] for the central singularity at \( R = 0, r = 0 \) (occurring at a time \( t = t_0(0) > t_i \)) to have outgoing characteristics terminating in the past at the singularity then is that there should exist a real positive value of \( X = X_0 \) such that

\[
X_0 = \lim_{R \to 0, u \to 0} R = \lim_{R \to 0, u \to 0} \frac{dR}{du} = U(X_0, 0) \tag{21}
\]
If $X_0$ is positive, this implies that geodesics are outgoing with past end point at the singularity, while for negative values of $X_0$ they are ingoing. Therefore, for the singularity to be naked, a positive real root of the following equation must exist,

$$V(X) = U(X, 0) - X = \left(1 - \frac{\sqrt{E_0 + \Lambda_0 X}}{\sqrt{1 + E_0}}\right) H(X, 0) - X = 0 \quad (22)$$

where $E_0 = E(0), \quad \Lambda_0 = \Lambda(0)$. If there are no such positive real roots existing, then the singularity must necessarily be covered in the sense that there are no outgoing null geodesics from it, and the collapse would result into a black hole. On the other hand, if such a value $X = X_0$ exists then near the singularity the characteristics have the behavior $R = X_0 u$ in the $(R, u)$ plane. In such a case, since $X_0 > \Lambda_0$ and $\Lambda_0$ describes the tangent to the apparent horizon $R = F$ at the singularity, the singularity is naked at least locally.

It follows that for a given initial density profile (i.e. for a given $M(r)$), if one could make a suitable choice of the initial velocity function, that is $V_i(r)$, such that (22) admits real positive roots, the gravitational collapse would then result in at least a locally naked singularity. The behavior of the roots equation (22) is mainly determined by the values of initial velocity $V_i(r)$, and its gradient $V'_i(r)$ at $r = 0$, for a given mass function $M(r)$. Thus, a right choice of the initial radial velocity and gradients for the spherical shells in the neighborhood of the center $r = 0$ towards the center could result in a locally naked singularity, or otherwise. In other words, for a given density distribution of matter, the velocity of the spherical shells in the neighborhood of the center $r = 0$ (or the energy function $E(r)$) at the onset of collapse determines the final state of the collapse, ending either in a naked singularity or a black hole.

However, for a faraway external observer viewing the collapse, the singularity would be visible if and only if there are causal trajectories with their past end point
at the singularity, crossing the boundary \( r = r_c \) of the star with \( dR/du > 0 \). An important question is, given a star of radius \( r = r_c \) and total mass \( M = M(r_c) \), would any singular geodesics reach the outside of the star to a faraway observer in the Schwarschild region. To be specific, we can ask for a cloud with a given value of the mass and size at the onset of collapse, when a locally naked singularity will be globally visible. Our considerations above show that for any given initial matter distribution for the cloud, it is the local behavior of rest of the initial data, namely the corresponding energy function \( E(r) \) or the initial particle velocities, which decide the final fate of collapse in terms of a black hole, or a locally naked singularity. On the other hand, the global visibility of the singularity will depend on the global behavior of the corresponding functions.

To discuss this issue of global visibility, we note that the future behavior of outgoing escaping geodesics, which terminate in the past at the singularity, is determined by the global behavior of free functions in the form of initial radial velocity \( V_i(r) \) which appear in the geodesic equations. In order that the singularity be globally visible for a given density distribution \( M(r) \), the global behavior of the initial radial velocity of the spherical shells within the cloud needs to be selected at the onset of collapse so that at least some geodesics reach the boundary of the cloud before the apparent horizon with a positive value of \( dR/du \). For this purpose, we consider the paths of these null geodesics. From equation (15),

\[
\frac{dX}{du} = \frac{1}{u} \left( \frac{dR}{du} - X \right) = \frac{U(X, u) - X}{u} \tag{23}
\]

The solution of the above gives trajectories of radial null geodesics in the form \( X = X(u) \). The necessary condition for a null geodesic to terminate at the singularity at \( R = 0, u = 0 \) is that the equation \( V(X) = 0 \) must have a real positive root. Let
$X = X_0$ be such a simple root of $V(X) = 0$. By writing the above as

$$\frac{dX}{du} = \frac{U(X, u) - X}{u} = \frac{(X - X_0)(h_0 - 1) + S}{u}$$

(24)

and integrating, the null trajectories $X = X(u)$ are given by [7],

$$X - X_0 = Du^{h_0-1} + u^{h_0-1} \int S u^{-h_0} du$$

(25)

where $V(X) \equiv (X - X_0)(h_0 - 1) + h(X)$. The function $h(X)$ contains higher order terms in $X - X_0$, i.e. $h(X_0) = (dh/dX)_{X=X_0} = 0$. Also,

$$h_0 = 1 + \left[ \frac{dV(X)}{dX} \right]_{X=X_0}$$

(26)

and $S = S(X, u) = U(X, u) - U(X, 0) + h(X)$ is a finite and continuous function in the region $t < t_0(r)$, i.e. for all values $(X, u)$, and $D$ is a constant of integration that labels different geodesics. If the singularity is the past end point of these geodesics with tangent $X = X_0$, we must have $X \to X_0$ as $u \to 0$ in (25). Note that as $X \to X_0, u \to 0$, the last term in equation (25) always vanishes near the singularity, regardless of the value of the constant $h_0$. This is due to the reason that as $u \to 0, X \to X_0$, we have $S \to 0$). The first term on the right hand side of the equation $Du^{h_0-1}$, however, vanishes only if $h_0 > 1$. Therefore, the single null geodesic described by $D = 0$ always terminates in the past at the singularity $R = 0, u = 0$, with $X = X_0$ as tangent. On the other hand, if $h_0 > 1$ a family of outgoing singular geodesics terminates at the singularity in the past, with each curve being labeled by different values of the constant $D$. Thus the necessary and sufficient condition for the singularity to be at least locally naked is that a real positive root of equation (22) exist.

In the case $h_0 > 1$, the integral curves given by equation (25) do terminate at the singularity with a positive definite value of the tangent $X = X_0$ with each null
geodesic being characterized by a value of the constant \( D \), which is determined by the boundary conditions at \( r = r_c \). For a singular geodesic reaching the boundary of the dust cloud \( u = u_c = u(r_c) \) with \( X = (R_c/u_c) = X_c \) we have,

\[
X_c - X_0 = Du_c^{h_0 - 1} + u_c^{h_0 - 1} \int_{u_c}^{u} S u^{-h_0} du \tag{27}
\]

and hence the equation of such a geodesic can be written as

\[
X - X_0 = (X_c - X_0)(\frac{u}{u_c})^{h_0 - 1} + u^{h_0 - 1} \int_{u_c}^{u} S u^{-h_0} du \tag{28}
\]

The event horizon is represented by the null geodesic for which \( X_c = \Lambda(r_c) \). For an outgoing null geodesic terminating in the past at the singularity, we have \( X = X_0 > \Lambda_0 \) as the tangent at the singularity, and the curve is ejected into the region \( R > F \) where \( dR/du \) is positive. Therefore, all the geodesics that reach the line \( r = r_c \) (where the metric (1) is matched with the Schwarschild exterior) with \( X_c > \Lambda(r_c) \) (note from equation (18) that \( dR/du \) for such geodesics at \( r = r_c \) is positive) would escape to infinity, while others would become ingoing. Such geodesics from the singularity, that reach future infinity, are then given by the equation (28) with \( X_c > \Lambda_c \).

An important subcase to note is when the root equation \( V(X) = 0 \) has only two real positive roots \( X = X_{\pm} \). In such a case, \( h_0 - 1 > 0 \) either at \( X = X_+ \) or at \( X = X_- \). If \( h_0 - 1 > 0 \) at \( X = X_- \) then \( h_0 - 1 < 0 \) at \( X = X_+ \). In this scenario the geodesics would be terminating at the singularity with tangent \( X = X_- \), i.e. \( X \to X_-, u \to 0 \). From equation (23), the behavior of these geodesics near \( X = X_+ \), assuming that \( p, \eta, \beta \) are finite as \( r \to \infty \), is given by

\[
(X - X_{+})^{\frac{1}{1-h_0}} \propto \frac{1}{u} \tag{29}
\]

and thus as \( X \to X_{+} \) the null geodesics attain arbitrarily large values of \( r \), making the singularity globally visible.
It follows from the above considerations that the occurrence of a black hole, or a locally or globally naked singularity, would generally depend on both the local as well as global behavior of the functions $M(r)$ and $V_i(r)$. In particular, the occurrence of a naked singularity (which may be locally or globally visible), for a prechosen distribution of matter, is determined completely by the initial velocity of the spherical shells towards the center at the onset of collapse. In the next section we examine the evolution of gravitational collapse from such a perspective.

4. Black Holes and Naked Singularities

Before examining the general inhomogeneous dust collapse, we first examine the special subclass of homogeneous models, where the density distribution at any given instant of time has a constant value in space. The Oppenheimer-Snyder models [3] of gravitational collapse fall within this class, where the initial density $\rho = \epsilon(t_i, r) = M'/r^2 = \rho_0 = \text{const.}$ everywhere in space, that is

$$M(r) = M_0 r^3 \quad (30)$$

In their original paper, Oppenheimer and Snyder considered the above constant density distribution, with the specific choice of the initial velocity for the cloud given by,

$$V_i(r) = -V_o r, \quad V_o = \text{const.} \quad (31)$$

It is well-known that such a set of initial data leads to a black hole.

For any given constant density profile, however, the final result of the collapse could be different for different choices of initial velocity profiles as we shall discuss here. For example, if one chooses the initial velocity differently so that $V_i(r) \propto r\sqrt{1 + ar^3}$, $a = \text{const.}$, or in terms of the function $E(r)$ as

$$E(r) = -M_0 r^2 \left( e_0 + e_1 r^3 + \gamma_0(r)r^3 \right)$$
where \( e_0, e_1 \), are constants and \( r^3 \gamma_0(r) \) is a \( C^2 \) function of \( r \) such \( \gamma_0(0) = 0 \), then such a choice of \( E(r) \) leads to a naked singularity in gravitational collapse (see also [7]). The constants \( e_0 \) and \( e_1 \) are chosen here such that

\[
\Gamma_0 = \frac{3e_1}{e_0 M_0^2 \gamma(M_0^2)} \left( \frac{1}{\sqrt{1 - e_0}} - \frac{3G(e_0)}{2} \right) = 3e_1 \frac{G^1(e_0)}{M_0^3/2} > 13 + \frac{15}{2} \sqrt{3}
\]  

(32)

With such a choice, \( u = M_0 r^3 \) and the root equation (19) becomes

\[
V(X) = 0 \Rightarrow 2x^4 + x^3 - \Gamma_0(x - 1) = 0, \quad X = x^2
\]  

(33)

For the value of \( \Gamma_0 \) chosen as above, this equation has only two real positive roots, namely \( X_\pm = x^2_\pm > 1 \), such that \( x_+ > x_- \), which implies \( X_+ > X_- \). The exact value of these roots depend on the exact value of \( \Gamma_0 \). It follows that \( h_0 > 1 \) for the root \( X = X_- \), and hence a family of outgoing null geodesics terminate at the singularity with the tangent \( X = X_- \). Also, \( h_0 < 1 \) for the other root \( X = X_+ \). The behaviour of the geodesics is given by equation (29). To give a specific example, one can take \( \Gamma_0 = 28 \), and then the roots are \( X_+ = 2.443, X_- = 1.512 \), and \( h_0 - 1 = 0.768 \) along the root \( X = X_- \) and \( h_0 - 1 = -0.433 \) along the root \( X = X_+ \). The boundary of the cloud lies at \( r = r_c \) (note that \( r_c >> 2M_s \Rightarrow 1 >> M_0 r_c^2 \)). In the neighborhood of the singularity the trajectories are described by

\[
X - X_- = (X_c - X_-) \left( \frac{u}{u_c} \right)^{h_0 - 1}
\]  

(34)

For the trajectories with \( X_c < X_- \), \( X \) decreases as \( u \) increases, and they move towards the apparent horizon which is a straight line given by \( X = 1 \) in \((R, u)\) plane. For geodesics with \( X_c > X_- \), the value of \( X \) starts increasing as \( u \) increases and they move away from the apparent horizon. The geodesic with \( X_c = 1 \) crosses the boundary of the cloud at the intersection of this boundary with the apparent
horizon. The trajectories with $X_c > 1$ reach the boundary at an earlier time than the apparent horizon and the singularity is globally naked.

We note in the above case of homogeneous dust collapse that the quantities $\Theta$ and $p$ are given by $\Theta = r^2 G^1(p)p'$, and $p = e_0 + r^3(e_1 + \gamma_0(r))$. Therefore, in order that shell-crossing singularities do not occur during the collapse, it is sufficient to require that $p' \geq 0$, or that $p$ be a monotone increasing function of $r$. This implies that while fixing the energy function $E(r)$, the choice of $\gamma_0(r)$ be such that $3r^2(e_1 + \gamma_0(r)) + r^3\gamma_0'(r) \geq 0$ for $r_c \geq r > 0$.

It follows from the considerations above that within the framework of homogeneous dust collapse itself, there is a wide variety of new collapse solutions which result either into a black hole or a naked singularity, depending on the existence or otherwise of the real positive roots of the algebraic equation (33). While it is well-known that the Oppenheimer-Snyder homogeneous dust collapse with a specific choice of velocity profile results into a black hole, we have seen here that a wide variety of different choices of particle velocity profiles would also produce the Schwarzschild black hole as the end state of gravitational collapse.

We now consider the future development of general inhomogeneous density profiles during the collapse. As pointed out earlier, for the initial data to be regular at the onset of collapse the most general mass function has to be of the form $M(r) = r^3 g(r)$, where $g(r)$ is a differentiable function. For physical reasonableness, we first consider the cases where the density is a decreasing function of $r$ away from the center (hence $g(0) > 0$), and as such $g'(r) \leq 0$. If

$$\lim_{r \rightarrow 0} \left[ \frac{r^3}{g(r) - g(0)} \right] = 0$$

that means $g(r)$ contains terms that are lower than $r^3$. In such a case, we select the energy function as that corresponding to the marginally bound case $E(r) = 0$, and
then $u$ is given by

$$u = g^{-1} - g_0^{-1}, \quad g_0 = g(0)$$  \hspace{1cm} (36)

We then have

$$V(X) \equiv X - \frac{1}{\sqrt{X}} = 0 \Rightarrow X = 1$$  \hspace{1cm} (37)

Hence the singularity is naked and the geodesics terminate at the singularity with the tangent $X = 1$ in the $(R, u)$ plane. Since $E = 0$ and $g'(r) \leq 0$ therefore there are no shell-crossing singularities within the cloud of boundary at $r = r_c$. On the other hand, if equation (35) is not satisfied that means $g(r)$ contains terms of the order $r^3$ or higher. In that case, we choose the velocity profile as given by

$$E(r) = -e_0r^2g(r)(1 + \gamma(r)r^3)$$  \hspace{1cm} (38)

which implies $p(r) = e_0(1 + r^3\gamma(r))$, where $\gamma(r)$ is a differentiable function such that $e_1 = \gamma(0) > 0$ and $3 + r\gamma'(r) \geq 0$. Then $u = r^3$, and the constants $e_0$ and $e_1$ are chosen such that

$$\Gamma_1 = \frac{3e_1}{e_0M_0^2} \left( \frac{1}{\sqrt{1 - e_0}} - \frac{3G(e_0)}{2} \right) > 13 + \frac{15}{2}\sqrt{3}$$  \hspace{1cm} (39)

where $M_0 = g(0)$. With such a choice the algebraic equation in question reduces to

$$V(X) = 0 \Rightarrow 2x^4 + x^3 - \Gamma_1(x - 1) = 0, \quad X = M_0x^2$$  \hspace{1cm} (40)

This has two real positive roots, namely $X_\pm = M_0x^2_\pm$, such that $x_+ > x_- \Rightarrow X_+ > X_-$ and therefore the singularity is naked. For this class under consideration, since $g'(r) \leq 0$, we have $p' = e_0r^2(3 + r\gamma'(r)) \geq 0$ and the shell-crossings again do not occur. Note that while the only $\gamma(0)$ and $e_0$ values determine the nakedness of the singularity, it is the global behavior of $\gamma(r)$ for the range $r_c \geq r > 0$ which determines that no shell-crossings occur.
In the above discussion of evolution of inhomogeneous density profiles, motivated by considerations of physical reasonableness, we assumed decreasing density profiles away from center with $g'(r) \leq 0$ and $g(0) \neq 0$. However, we can consider the case of a general mass function $M(r) = r^3 g(r)$ without these assumptions. We can then choose the velocity of the collapsing shells as described by the energy function $E(r)$ given by

$$E(r) = -r^2 g(r) p(r), \quad G(p) = \sqrt{g} \gamma_1(u)$$

(41)

where $u = F(r) = r^3 g(r)$, and $\gamma_1(u)$ is a monotone increasing function of $u$ with $\gamma_1(u) \geq 0$ and

$$\left[ \frac{d \gamma_1}{du} \right]_{u=0} = C_0 > 0$$

(42)

We then have

$$\Theta_u = \frac{d \gamma_1}{du} = \gamma_{1,u}(u), \quad \gamma_{1,u}(0) = C_0$$

(43)

The equation $V(x) = 0$ now becomes

$$V(X) = (1 - \sqrt{\frac{1}{X}}) \left( \frac{X + \frac{3C_0}{\sqrt{X}}}{3} \right) - X = 0$$

$$\Rightarrow 2x^4 + x^3 - 3C_0(x - 1) = 0$$

This is the same equation as (33), and would have real positive roots if $3C_0 > 13 + \frac{15}{2} \sqrt{3}$ in which case the singularity would be naked, otherwise it would be covered.

Note that for the naked singularity to occur, the above inequality is essential and it involves only $C_0$ which is the first derivative of $\gamma_1(u)$ at $u = 0$. The behavior of $\gamma_1(u)$ within the cloud $r_c \geq r > 0$ is chosen as per the model one wants and its global behavior within the cloud is taken such that there are no shell-crossing singularities. The dust models would be bounded, marginally bound or unbounded ($p > 0, = 0, < 0$) depending upon the choice of $\sqrt{g} \gamma_1 > \frac{2}{3}, = \frac{2}{3}, < \frac{2}{3}$ respectively.
For nonoccurrence of shell-crossing singularities, first note that energy conditions imply that \( \eta = rM'/M > 0 \) for \( r > 0 \) and since \( \gamma_1 \) is monotone increasing function, we have

\[
\Theta = \gamma_{1,u} M' r \sqrt{g} \geq 0
\]  

Therefore, within the cloud for \( r_c \geq r > 0 \) there are no shell-crossing singularities during the evolution of gravitational collapse.

It thus follows that given any density profile for the cloud at the initial epoch, a suitable regular choice of the velocity distribution would make the collapse of the cloud result into either a black hole or a naked singularity, depending on this choice. This characterizes a wide family of new collapse solutions for inhomogeneous dust which result into a black hole, apart from the well-known black hole solution resulting from the homogeneous dust collapse.

Although we have given above only particular initial velocity distributions at the onset of the collapse for a given density profile, it must be pointed out that many more similar velocity profiles would exist which would also ensure the fate of the collapse in terms of either a naked singularity or a black hole as desired. We have given here a treatment which includes all possible initial density profiles, while various specific density profiles have been worked out earlier by several authors (see e.g. \[9\]) for the occurrence of a naked singularity or black hole.

An interesting choice of a velocity distribution is worth pointing out in the context of the discussion here. The null geodesics in the spacetime are given by the differential equation (18). The equation involves two free functions, namely \( M(r) \) and \( E(r) \). Thus, for a given \( M(r) \) the exact behavior of the trajectories of photons, which are the null geodesics, is described by the way \( E(r) \) is chosen. Consider for example the mass function of the type given above \( M(r) = r^3 g(r) \), and let us choose
\( E(r) \) which satisfies

\[
U(a, u) = a, \quad a > 1
\]

where \( a \) is a constant. In general, \( U \) defined by equation (18) is an explicit function of \( E, M \) and \( R \) and an implicit function of \( r \); choosing \( R = au \) and \( M = u \) gives the above equation. For such an \( E(r) \) the null geodesic characterized by the equation \( R = aM \) terminates at the singularity. This geodesic never gets inside the apparent horizon and furthermore along the trajectory we have \( R' > 0 \) for \( r > 0 \), thus avoiding the shell-crossing curve also. The geodesic crosses the boundary of the cloud avoiding shell-crossings (if there are any), and makes the singularity globally naked.

We will now consider some further details on the issue of global nakedness for the naked singularity formation discussed above. In reality, for an external observer viewing the collapse, the singularity would be naked if and only if there are causal geodesics (with their past end point at the singularity) crossing the boundary of the star \( r = r_c \) with \( dR/du > 0 \), that is they cross the boundary of the star before the apparent horizon. In the above discussion, we have given several models for an arbitrary density distribution where the singularity is naked. Do every example of a locally naked singularity, under all circumstances, lead to global nakedness? The answer is certainly in the negative. As such local nakedness already implies that for a set of given \( M(r) \) and \( E(r) \), which leads to a naked singular spacetime, causal geodesics which terminate at the singularity in the past do escape into the region \( R > M \) near the singularity and as such have \( dR/du \) positive in a finite neighborhood of the singularity, however small. It follows that one can always choose the boundary at \( r = r_c > 0 \), however small, so that in this region and at the boundary we have \( dR/du > 0 \). Therefore one can always select the total mass \( M(r_c) \), and the size \( r_c \) of the cloud, such that the escaping null geodesics would
reach \( r_c \) with \( dR/du \) positive for such examples, making the singularity globally naked. Similarly one can, by choosing the boundary suitably, make it globally invisible in these cases. Thus for a given set of density and velocity distributions, which leads to a locally naked singularity, one can always choose the the set of total Schwarschild mass \( M(r_c) \), and the size of the cloud \( r_c \) so as to make the collapse either globally naked or otherwise.

For an arbitrary set of mass \( M \) and size \( r_c \), let us consider the general mass function and the example given by equation (41) where \( E(r) = -r^2 g(r)p(r) \), \( G(p) = \sqrt{g} \gamma_1(u) \), \( u = M \). For the physically reasonable case \( g(0) \neq 0 \) one can take the bound case \( p \leq 0 \). In this case, between the cloud \( r_c \geq r > 0 \) the only condition one has on the function \( \gamma_1 \) is that it be a monotone increasing function, i.e. \( \gamma'_1 = \gamma_{1,u} M' \geq 0 \) with \( \gamma_1(0) > 0 \). At the center at \( r = 0 \) one should have \( \gamma_{1,u}(0) = C_0 \geq 13 + \frac{15}{2} \sqrt{3} \), and then the family of geodesics terminate at the singularity. The root equation (22) has only two real positive roots \( X_\pm \), \( X_+ > X_- \). The geodesics terminate at the singularity with the tangent \( X = X_- > 1 \) in \((R,u)\) plane and attain arbitrarily large value of \( r \) at \( X = X_+ \). In \((R,u)\) plane the apparent horizon is a straight line \( R = u \) and \( X_\pm > 1 \) are the roots of equation (22). Geodesics with \( X_c > X_- \) will escape to the boundary with \( dR/du \) positive and the singularity would be globally naked. In cases we considered with \( g(0) = 0 \), the choice of \( E(r) \) corresponds only to the unbound class, however, the conditions on \( \gamma_1 \) remain the same for no shell-crossings as well as for the singularity to be naked.

So far we have concentrated on initial matter distributions of the star. One may also ask whether the converse is true, that is, given a regular initial data in the form of \( E(r) \), whether there always exists a regular \( M(r) \) which could produce either one of the black hole or a naked singularity. For example, the collapse initiating from rest with \( V_i(r) = 0 \) (i.e \( \dot{R} = 0 \) initially) falls under such a situation and can be considered
similarly by making a suitable choice of the mass function $M(r)$. The answer in general seems to be in affirmative, and one can develop the arguments along similar lines as above, though we will not go into the details here. Another point to note is regarding the curvature strength of the naked singularities considered here. Not all the examples considered here need be that of strong curvature type, except the ones where the root equation admits two real positive roots, which will be necessarily strong. These details will be given elsewhere [11].

5. Concluding Remarks

A relevant question here is whether results such as above could be generalized to more general equations of state. In fact, recent work [12] has indicated that the phenomena of naked singularity or black hole formation in collapse is closely related to the specification of initial data. It has been pointed out there that if a naked singular spacetime (or a black hole) exists for a particular equation of state then for all sufficiently close equations of state there would also be a naked singular solution (black hole) in spherically symmetric collapse. Hence similar conclusions as above should hold for such more general equations of state as well. Further details on this and related issues will be discussed elsewhere.

We have shown here that for gravitational collapse of dust clouds with arbitrary initial matter distributions, the final fate of collapse in terms of a black hole or a naked singularity is fully determined by the choice of rest of the initial data. It follows that given any regular matter distribution, its collapse can always lead to the formation of a black hole provided the remaining initial data is chosen suitably; and that such a choice is always possible. In other words, given the initial density distribution, we have given here a specific procedure to fine tune rest of the initial data so as to produce a black hole as the end product of gravitational collapse.
Such a conclusion no longer needs the assumption of cosmic censorship conjecture. Usually, in black hole physics, the truth of cosmic censorship is always assumed in the sense that collapse from regular initial conditions is necessarily taken to be a black hole. However, no proof or a mathematically suitable formulation of such a hypothesis is available so far. We have provided here a more specific characterization of initial data for the formation of black holes without such an assumption. Thus, these considerations bring out a wide new class of collapse solutions producing a black hole from the inhomogeneous collapse of matter. We also pointed out initial data configurations resulting into naked singularities for any given distribution of matter. In view of such a situation, it appears important to identify and determine physically the possible initial data sets responsible for either of these phenomena.
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