Is a system’s wave function in one-to-one correspondence with its elements of reality?

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Although quantum mechanics is one of our most successful physical theories, there has been a long-standing debate about the interpretation of the wave function—the central object of the theory. Two prominent views are that (i) it corresponds to an element of reality, i.e., an objective attribute that exists before measurement, and (ii) it is a subjective state of knowledge about some underlying reality. A recent result [Pusey et al. arXiv:1111.3328] has placed the subjective interpretation into doubt, showing that it would contradict certain physically plausible assumptions, in particular that multiple systems can be prepared such that their elements of reality are uncorrelated. Here we show, based only on the assumption that measurement settings can be chosen freely, that a system’s wave function is in one-to-one correspondence with its elements of reality. This also eliminates the possibility that it can be interpreted subjectively.

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Introduction.—Given the wave function associated with a physical system, quantum theory allows us to compute predictions for the outcomes of any measurement. Since a wave function corresponds to an extremal state and is therefore maximally informative, one possible view is that it can be considered an element of reality of the system, i.e., an objective attribute that exists before measurement. However, an alternative view, often motivated by the probabilistic nature of quantum predictions, is that the wave function represents incomplete (subjective) knowledge about some underlying reality. Which view one adopts affects how one thinks about the theory at a fundamental level.

To illuminate the difference between the above views, we give an illustrative example. Consider a meteorologist who gives a prediction about tomorrow’s weather (for example that it will be sunny with probability 33%, and cloudy with probability 67%; see left hand side of Fig. 1). We may assume that classical mechanics accurately describes the relevant processes, so that the weather depends deterministically on the initial conditions. The fact that the prediction is probabilistic then solely reflects a lack of knowledge on the part of the meteorologist on these conditions. In particular, the forecast is not an element of reality associated with the atmosphere, but rather reflects the subjective knowledge of the forecaster; a second meteorologist with different knowledge (see right hand side of Fig. 1) may issue an alternative forecast.

Moving to quantum mechanics, one may ask whether the wave function $\Psi$ that we assign to a quantum system should be seen as a subjective object (analogous to the weather forecast) representing the knowledge an experimenter has about the system, or whether $\Psi$ is an element of reality of the system (analogous to the weather being sunny). This question has been the subject of a long debate, which goes back to the early days of quantum theory [1].

The debate originated from the fact that quantum theory is inherently probabilistic: even with a full description of a system’s wave function, the theory does not allow us to predict the outcomes of future measurements with certainty. This fact is often used to motivate subjective interpretations of quantum theory, such as the Copenhagen interpretation [2–4], according to which wave functions are mere mathematical objects that allow us to calculate probabilities of future events.

Einstein, Podolsky and Rosen (EPR) advocated the view that the wave function does not provide a complete physical description of reality [5], and that a higher, complete theory is possible. In such a complete theory, any element of reality must have a counterpart in the theory. Were quantum theory not complete, it could be that the higher theory has additional parameters that complement the wave function. The wave function could then be objective, i.e., uniquely determined by the elements of reality of the higher theory. Alternatively, the wave function could take the role of a state of knowledge about the underlying parameters of the higher theory. In this case, the wave function would not be uniquely determined by these parameters and would therefore admit a subjective interpretation. To connect to some terminology in the literature (see for example Ref. [6]), in the first case the underlying model would be called $\psi$-ontic, and in the second case $\psi$-epistemic. For some recent work in support of a $\psi$-epistemic view, see for example Refs. [7–9].

In some famous works from the 1960s, several constraints were placed on higher descriptions given in terms of hidden variables [10–12], and further constraints have since been highlighted [13–15]. In addition, we have recently shown [16] that, under the assumption of free choice, if quantum theory is correct then it is non-extendible, in the sense of being maximally informative about measurement outcomes.

Very recently, Pusey, Barrett and Rudolph [17] have presented an argument showing that a subjective interpretation of the wave function would violate certain plausible assumptions. Specifically, their argument refers to
FIG. 1: Simple example illustrating the ideas. Two meteorologists attempt to predict tomorrow’s weather (whether it will be sunny or cloudy in a particular location). Both have access to historical data giving the joint distribution of the weather on successive days. However, only the meteorologist on the left has access to today’s weather, and consequently the two make different probabilistic forecasts, \( F \) and \( F' \). Assuming that the processes relevant to the weather are accurately described by classical mechanics and thus deterministic, the list of elements of reality, \( \Lambda \), may include tomorrow’s weather, \( X \). Such a list \( \Lambda \) would then necessarily satisfy \( \Gamma \leftrightarrow \Lambda \leftrightarrow X \) for any arbitrary \( \Gamma \) and therefore be complete (cf. Eq. 1). However, the analogue of Eq. 2, \( \Lambda \leftrightarrow F \leftrightarrow X \) would imply \( X \leftrightarrow F \leftrightarrow X \). This Markov chain cannot hold for the non-deterministic forecasts \( F \) and \( F' \), which are hence not complete. This is unlike the quantum-mechanical wave function, which gives a complete description for the prediction of measurement outcomes. Note that this difference explains why, in contrast to the quantum-mechanical wave function,\( F \) and \( F' \) need not be included in \( \Lambda \) and can therefore be considered subjective.

a model where each physical system possesses an individual set of (possibly hidden) elements of reality, which are the only quantities relevant for predicting the outcomes of later measurements. One of their assumptions then demands that it is possible to prepare multiple systems such that these sets are statistically independent.

Here, we present a totally different argument to show that the wave function of a quantum system is fully determined by its elements of reality. In fact, this implies that the wave function is in one-to-one correspondence with these elements of reality (see the Conclusions) and may therefore itself be considered an element of reality of the system. These claims are derived under minimal assumptions, namely that the statistical predictions of existing quantum theory are correct, and that measurement settings can (in principle) be chosen freely. In terms of the language of Ref. 6, this means that any model of reality consistent with quantum theory with free choice is \( \psi \)-complete.

General Model.—In order to state our result, we consider a general experiment where a system \( S \) is prepared in a state specified by a wave function \( \Psi \) (see Fig. 2). Then an experimenter chooses a measurement setting \( A \) (specified by an observable or a family of projectors) and records the measurement outcome, denoted \( X \). Mathematically, we model \( \Psi \) as a random variable over the set of wave functions, \( A \) as a random variable over the set of observables, and \( X \) as a random variable over the set of possible measurement outcomes. Finally, we introduce a collection of random variables, denoted \( \Gamma \), which are intended to model all information that is (in principle) available before the measurement setting, \( A \), is chosen and the measurement is carried out. Technically, we only require that \( \Gamma \) includes the wave function \( \Psi \). In the following, when we refer to a list of elements of reality, we simply mean a subset \( \Lambda \) of \( \Gamma \). Furthermore, we say that \( \Lambda \) is complete for the description of the system \( S \) if any possible prediction about the outcome \( X \) of a measurement \( A \) on \( S \) can be obtained from \( \Lambda \), i.e., we demand that the Markov condition

\[ \Gamma \leftrightarrow (\Lambda, A) \leftrightarrow X \] (1)

holds. Note that, using this definition, the aforementioned result on the non-extendibility of quantum theory [10] can be phrased as: The wave function \( \Psi \) asso-

\[ U \leftrightarrow V \leftrightarrow W \] is called a Markov chain if \( P_U|V=v, W=w = P_U|V=v \) or, equivalently, if \( P_W|V=v = P_W|V=v, U=u \) for all \( u, v, w \) with strictly positive joint probability. Here and in the following, we use upper case letters for random variables, and lower case letters for specific values they can take.

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and the system is measured, producing an outcome, \( \Psi \). The elements of reality, \( \Lambda \), may depend on this preparation. A measurement setting, \( \Lambda \), is then randomly chosen, and the system is measured, producing an outcome, \( X \). We assume that \( \Lambda \) is complete for the description of the system, in the sense that there does not exist any other parameter that provides additional information (beyond that contained in \( \Lambda \)) about the outcome of any chosen measurement. In particular, \( \Psi \) cannot provide more information than \( \Lambda \). Conversely, the non-extendibility of quantum theory \([16]\) implies that \( \Lambda \) cannot provide more information (about the outcome) than \( \Psi \). Taken together, these statements imply that \( \Psi \) and \( \Lambda \) are informationally equivalent. From this and the fact that different quantum states generally lead to different measurement statistics, we conclude that \( \Psi \) must be included in the list \( \Lambda \) and is therefore an element of reality of the system.

A system is prepared in a particular quantum state (specified by a wave function \( \Psi \)). The elements of reality, \( \Lambda \), may depend on this preparation. A measurement setting, \( \Lambda \), is then randomly chosen, and the system is measured, producing an outcome, \( X \). We assume that \( \Lambda \) is complete for the description of the system, in the sense that there does not exist any other parameter that provides additional information (beyond that contained in \( \Lambda \)) about the outcome of any chosen measurement. In particular, \( \Psi \) cannot provide more information than \( \Lambda \). Conversely, the non-extendibility of quantum theory \([16]\) implies that \( \Lambda \) cannot provide more information (about the outcome) than \( \Psi \). Taken together, these statements imply that \( \Psi \) and \( \Lambda \) are informationally equivalent. From this and the fact that different quantum states generally lead to different measurement statistics, we conclude that \( \Psi \) must be included in the list \( \Lambda \) and is therefore an element of reality of the system.

We are now ready to formulate our main technical claim.

**Theorem.**—Any list of elements of reality, \( \Lambda \), that is complete for the description of a system \( S \) includes the quantum-mechanical wave function \( \Psi \) associated with \( S \) (in the sense that \( \Psi \) is uniquely determined by \( \Lambda \)).

**Assumptions.**—The above claim is derived under the following two assumptions, which are usually implicit in the literature. (We note that very similar assumptions are also made in Ref. [17] where, as already mentioned, an additional statistical independence assumption is also used.)

- **Correctness of quantum theory:** Quantum theory gives the correct statistical predictions. For example, the distribution of \( X \) satisfies
  \[
P_X(\Psi=\psi, A=a) = \langle \psi | \Pi^a_x | \psi \rangle,
\]
  where \( \Pi^a_x \) denotes the projector corresponding to outcome \( X = x \) of the measurement specified by \( A = a \).

- **Freedom of choice:** Measurement settings can be chosen to be independent of any pre-existing value (in any frame) \(^2\). In particular, this implies that the setting \( A \) can be chosen independently of \( \Gamma \), i.e., \( P_A|_{\Gamma=\gamma} = P_A \) \(^3\).

We note that the proof of our result relies on an argument presented in Ref. [16] where these assumptions are also used (see the Supplemental Material for more details).

**Proof of the main claim.**—As shown in Ref. [16], under the above assumptions, \( \Psi \) is complete for the description of \( S \). Since \( \Lambda \) is included in \( \Gamma \), we have in particular

\[
\Lambda \leftrightarrow (\Psi, A) \leftrightarrow X.
\]

(2)

Our argument then proceeds as follows. The above condition is equivalent to the requirement that

\[
P_{X | \Lambda = \lambda, \Psi = \psi, A = a} = P_{X | \Psi = \psi, A = a}
\]

holds for all \( \lambda, \psi, a \) that have a positive joint probability, i.e., \( P_{\Lambda \Psi A}(\lambda, \psi, a) > 0 \). Furthermore, because of the assumption that \( \Lambda \) is a complete list of elements of reality, Eq. [1] and because \( \Psi \) is by definition included in \( \Gamma \), we have

\[
P_{X | \Lambda = \lambda, \Psi = \psi, A = a} = P_{X | \Lambda = \lambda, A = a}.
\]

Combining these expressions gives

\[
P_{X | \Psi = \psi, A = a} = P_{X | \Lambda = \lambda, A = a},
\]

(3)

for all values \( \lambda, \psi, a \) with \( P_{\Lambda \Psi A}(\lambda, \psi, a) > 0 \). Note that, using the free choice assumption, we have \( P_{\Lambda \Psi A} = P_{\Lambda \Psi} \times P_A \), hence this condition is equivalent to demanding \( P_{\Lambda \Psi}(\lambda, \psi) > 0 \) and \( P_A(a) > 0 \).

Now consider some fixed \( \Lambda = \lambda \) and suppose that there exist two states, \( \psi_0 \) and \( \psi_1 \), such that \( P_{\Lambda \Psi}(\lambda, \psi_0) > 0 \) and \( P_{\Lambda \Psi}(\lambda, \psi_1) > 0 \). From Eq. [3] this implies \( P_{X | \Psi = \psi_0, A = a} = P_{X | \Psi = \psi_1, A = a} \) for all \( a \) such that \( P_A(a) > 0 \). However, within quantum theory, it is easy to choose the set of measurements for which \( P_A(a) > 0 \) such that this can only be satisfied if \( \psi_0 = \psi_1 \). This holds, for example, if the set of measurements is tomographically complete. Thus, for each \( \Lambda = \lambda \), there exists only one possible value of \( \Psi = \psi \) such that \( P_{\Lambda \Psi}(\lambda, \psi) > 0 \), i.e., \( \Psi \) is uniquely determined by \( \Lambda \), which is what we set out to prove.

**Discussion and Conclusions.**—We have shown that the quantum wave function can be taken to be an element of reality of a system based on two assumptions, the correctness of quantum theory and the freedom of choice for

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\(^2\) This assumption, while often implicit, is for instance discussed (and used) in Bell’s work. In Ref. [19] he writes that “the settings of instruments are in some sense free variables . . . [which] means that the values of such variables have implications only in their future light cones.” This leads directly to the freedom of choice assumption as formulated here. We refer to the Supplemental Material for a more detailed discussion.

\(^3\) In Ref. [17] this assumption corresponds to the requirement that a quantum system can be freely prepared according to one of a number of predefined states.
measurement settings. Both of these assumptions are in principle experimentally falsifiable (see the Supplemental Material for a discussion of possible experiments).

The correctness of quantum theory is a natural assumption given that we are asking whether the quantum wave function is an element of reality of a system. Furthermore, a free choice assumption is necessary to show that the answer is yes. Without free choice, $\Lambda$ would be pre-determined and the complete list of elements of reality, $\Lambda$, could be chosen to consist of the single element $X$. In this case, Eq. 1 would be trivially satisfied. Nevertheless, since the list $\Lambda = \{X\}$ does not uniquely determine the wave function, $\Psi$, we could not consider $\Psi$ to be an element of reality of the system. This shows that the wave function would admit a subjective interpretation if the free choice assumption was dropped.

We conclude by noting that, given any complete list of elements of reality, $\Lambda$, the non-extendibility of quantum theory, Eq. 2, asserts that any information contained in $\Lambda$ that may be relevant for predicting measurement outcomes $X$ is already contained in the wave function $\Psi$. Conversely, the result shown here is that $\Psi$ is included in $\Lambda$. Since these are two seemingly opposite statements, it is somewhat intriguing that the second can be inferred from the first, as shown in this Letter. Furthermore, taken together, the two statements imply that $\Psi$ is in one-to-one correlation to $\Lambda$. This sheds new light on a principle experimentally falsifiable (see the Supplemental Material for a discussion of possible experiments).

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SUPPLEMENTAL MATERIAL

I. ADDITIONAL DISCUSSION OF THE ASSUMPTIONS

Our work relies on two assumptions, which we discuss in separate subsections. We note that these assumptions are essentially those already used in [16], upon which this work builds.

For the following exposition, it is convenient to introduce the concept of *spacetime variables (SVs)* [16]. Mathematically, these are simply random variables (which take values from an arbitrary set) together with associated coordinates \((t, r_1, r_2, r_3) \in \mathbb{R}^4\). A SV may be interpreted physically as a value that is accessible at the spacetime point specified by these coordinates (with respect to a given reference frame).

The variables described in the main text, \(A, X,\) and \(\Gamma\), can readily be modelled as SVs. In particular, the coordinates of \(A\) should specify the spacetime point where the measurement setting (for measuring the system \(S\)) is chosen. Accordingly, the coordinates of \(X\) correspond to an (arbitrary) point in the spacetime region where the measurement outcome is available. We therefore assign coordinates such that \(X\) is in the future lightcone of \(A\), whereas no SV in the set \(\Gamma\) (which models any information available before the measurement) should lie in the future lightcone of \(A\).

A. Correctness of quantum theory

This assumption refers to the statistical predictions about measurement outcomes that can be made within standard quantum theory (i.e., it does not make reference to any additional parameters of a potential higher theory\(^4\)). Following the treatment in [16], we subdivide the assumption into two parts.

- **QM\(_a\):** Consider a system whose state is described by a wave function, \(\Psi\), corresponding to an element of a Hilbert space \(H_S\). According to quantum theory, any measurement setting \(A = a\) is specified by a family of projectors, \(\{\Pi_x^a\}_x\), parameterized by the possible outcomes \(x\), such that \(\sum_x \Pi_x^a = \mathbb{1}_{H_S}\). The assumption demands that the probability of obtaining output \(X = x\) when the system is measured with setting \(A = a\) is given by

\[
P_{X|A=a}(x) = \text{tr}(|\Psi\rangle\langle\Psi|\Pi_x^a) .
\]

\(^4\) In particular, in a higher theory that has additional hidden parameters, the predictions of quantum theory should be recovered if these parameters are ignored, which corresponds mathematically to averaging over them.

- **QM\(_b\):** Consider again a measurement as in Assumption QM\(_a\) described by projectors \(\{\Pi_y^b\}_y\) on \(H_S\). According to quantum theory, for any fixed choice of the setting \(A = a_0\), the measurement can be modelled as an isometry \(E_{S \rightarrow SE}\) from states on \(H_S\) to states on a larger system \(H_S \otimes H_E\) (involving parts of the measurement apparatus and the environment) such that the restriction of \(E_{S \rightarrow SE}\) to the original system corresponds to the initial measurement, i.e., formally

\[
\text{tr}_E E_{S \rightarrow SE}(|\Psi\rangle\langle\Psi|) = \sum_x \Pi_x^{a_0} |\Psi\rangle\langle\Psi| \Pi_x^{a_0} ,
\]

where \(\text{tr}_E\) denotes the partial trace over \(H_E\) [20]. The assumption then demands that for all \(A = a\) and for any measurement (defined by a family of projectors, \(\{\Pi_y^b\}_y\), parameterized by the possible outcomes \(y\)) carried out on system \(E\), the joint statistics of the outcomes are given by

\[
P_{XY|A=a,B=b}(x,y) = \text{tr}(E_{S \rightarrow SE}(|\Psi\rangle\langle\Psi|) \Pi_x^a \otimes \Pi_y^b) .
\]

Note that both assumptions refer to the Born rule [2] for the probability distribution of measurement outcomes. In Assumption QM\(_a\), the rule is applied to a measurement on a single system, whereas Assumption QM\(_b\) demands that the rule also applies to the joint probability distribution involving the outcome of (arbitrary) additional measurements.

B. Freedom of choice

The assumption that measurement settings can be chosen freely is often left implicit in the literature. This is also true, for example, for large parts of Bell’s work, although he later mentioned the assumption explicitly [19].

The notion of freedom of choice can be expressed mathematically using the language of SVs. We say that a SV \(A\) is *free with respect to a set of SVs \(\Omega\)* if

\[
P_{A|\Omega'} = P_A \times P_{\Omega'}
\]

holds, where \(\Omega'\) is the set of all SVs from \(\Omega\) whose coordinates lie outside the future lightcone of \(A\). This captures the idea that \(A\) should be independent of any “pre-existing” values (with respect to any reference frame).

This definition is motivated by the following notion of causality. For two SVs \(A\) and \(B\), we say that \(B\) *could have been caused by \(A\)* if and only if \(B\) lies in the future lightcone of \(A\). Within a relativistic spacetime structure, this is equivalent to requiring the time coordinate of \(B\) to be larger than that of \(A\) in all reference frames. Using this notion of causality, our definition that \(A\) is free with respect to \(\Omega\) is equivalent to demanding that all SVs in \(\Omega\) that are correlated to \(A\) could have been caused by \(A\).
Connecting to the main text, we note that (by definition) all SVs in the set \( \Gamma \) defined there lie outside the future lightcone of the spacetime point where the measurement setting \( A \) is chosen. The requirement for \( A \) to be free with respect to \( \Gamma \) thus simply reads \( P_{A|\Gamma} = P_A \times P_\Gamma \).

In Ref. \( \text{[16]} \) (upon which the present result is based), the free choice assumption is used in a more general bipartite scenario. There, two measurements are carried out at spacelike separation, one of which has setting \( A \) and outcome \( X \) and the other has setting \( B \) and outcome \( Y \). In addition, as in our main argument, we consider arbitrary additional (pre-existing) information \( \Gamma \). The assumption that \( A \) and \( B \) are chosen freely (i.e., such that they are uncorrelated with any variables in their past in any frame) then corresponds mathematically to the requirements \( P_{A|BY\Gamma} = P_A \) and \( P_{B|AX\Gamma} = P_B \). We remark that these conditions are not obeyed in the de Broglie-Bohm model \( \text{[21, 22]} \) if one includes the wave function as well as the hidden particle trajectories in \( \Gamma \).

It is also worth making a few additional remarks about the connection to other work. As mentioned in the main text, in Ref. \( \text{[19]} \) Bell writes that “the settings of instruments are in some sense free variables . . . [which] means that the values of such variables have implications only in their future light cones.” When formalized, this gives the above definition. However, in spite of the motivation given in the above quote, the mathematical expression Bell writes down corresponds to a weaker notion that only requires free choices to be independent of pre-existing hidden parameters (but does not include pre-existing measurement outcomes). This weaker requirement is (as he acknowledges) a particular implication of the full freedom of choice assumption. We imagine that the reason for Bell’s reference to this weaker implication is that it is sufficient for his purpose when combined with another assumption, known as local causality. Indeed, the weaker implication of free choice together with Bell’s local causality are also sufficient to prove our result. Furthermore, in the literature the weaker notion is sometimes taken to be the definition of free choice, rather than an implication of it.

\section{Connection to Experiment}

Our main argument is based on the assumption that measurement outcomes obey the statistical predictions of quantum theory, and it is interesting to consider how closely experimental observations come to obeying these predictions. For the argument in Ref. \( \text{[16]} \) which leads to Eq. 2 in the main text, this assumption is divided into two parts, as mentioned above.

The first part of the assumption has already been subject to experimental investigation (see Refs. \( \text{[23, 24]} \), giving results compatible with quantum theory to within experimental tolerance. Note that, although no experimental result can establish the Markov chain condition of Eq. 2 precisely, the observed data can be used to bound how close (in trace distance) the Markov chain condition is to holding (see Ref. \( \text{[24]} \) for more details).

The second part of the assumption has not seen much experimental attention to date. However, were we to ever discover a measurement procedure that is demonstrably inconsistent with unitary dynamics on the microscopic scale, this would falsify the assumption and point to new physics.

The freedom of choice assumption is more difficult to probe experimentally, since it is stated in terms of \( \Gamma \), which is information in a hypothetical higher theory. Nevertheless, it would be possible to falsify the assumption in specific cases, for example using a device capable of predicting the measurement settings before they were chosen.