On Variable Screening in Multiple Nonparametric Regression Model

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Abstract

In this article, we study the problem of variable screening in multiple nonparametric regression model. The proposed methodology is based on the fact that the partial derivative of the regression function with respect to the irrelevant variable should be negligible. The Statistical property of the proposed methodology is investigated under both cases:

(i) when the variance of the error term is known, and
(ii) when the variance of the error term is unknown.

Moreover, we establish the practicality of our proposed methodology for various simulated and real data related to interdisciplinary sciences such as Economics, Finance and other sciences.

Keywords: Kernel Density Estimation; Nonparametric Methods; Local Polynomial Regression; Multivariate Data; Partial Derivative; Variable Selection; Weighted Least Squares Methodology.
1 Introduction

Over the past few decades, the search for efficient methods to determine the true regression model has become central goal of a large number of research problems in Statistics. As a result, the topic of variable screening has received valuable attention from the Statistics community. It is a well known fact that, the unnecessary regressors add noise to the estimation of quantities of interest, resulting in reduction of degrees of freedom. Moreover, the collinearity may creep in the model, if there are a large number of predictors for the same response variable. In addition, the cost in terms of time and money can also be saved by not recording redundant regressors. Furthermore, the abundance of high dimensional models in recent years, has led to rapid growth in the use of model selection procedures, which are reminiscent of the variable screening procedures. For example, in labour economics, a wage equation generally has a large number of predictors, and ruling out the less important predictors is of great importance. In image processing, the number of possible image features is very high. The choice of particular features depends on the target application, and accordingly, the feature selection is implemented (see Bins and Draper (2001) for details). Feature selection also plays an important role in fault diagnosis in industrial applications, where numerous redundant sensors monitor the performance of a machine. Fault detection in these cases can be improved significantly by using feature selection (see Liu, Jiang, and Yang (2014)).

One of the noteworthy applications of feature selection is in the field of stock market analysis in finance. Technical indicators in finance refer to a collection of stock data, computed by different mathematical formulae. These indicators are used as attributes to model the stock prices, and consequently, estimate or predict certain aspects of stock market. There are a total of twelve indicators used as input factor for stock market, but to achieve certain results, all the indicators may not be of importance. Hence, the use of feature screening can provide an optimal subset of features, leading to an efficient analysis. For further details, see He, Fataliyev, and Wang (2013).

In Finance and Economics literature, many macroeconomic factors are used to predict stock market returns. There are plenty of methodologies available for modelling stock market returns. Single or multi-factor models, ARCH models, vector autoregression model, Johansen co-integration test, Granger causality test, variance decomposition method, artificial neural networks, support vector machines are some of the examples. But, which factors influence the stock market, is an important question that can be answered by feature selection methods. One example of the
implementation of feature selection in this regard can be seen in Altinbas and Biskin (2015).

Several methods for variable screening are available in the literature of linear regression model, e.g. stepwise forward selection and stepwise backward elimination, based on OLS methodology (Basu, 2019), model selection using Akaike Information Criteria (Akaike, 1974), and Bayesian Information Criteria (Schwarz, 1978), Mallows’ $C_p$ Criteria (Mallows, 1973), the Delete-1 cross-validation method (Allen, 1974 and Stone, 1974), the Delete-d cross validation method (Geisser, 1975, Burman, 1989, Shao, 1993, Zhang, 1993a). Asymptotic properties of these model selection techniques are also available, which enable us to compare the aforementioned methods when the sample size tends to infinity. The BIC is shown to be consistent (Csiszár and Shields, 2000) while AIC and Mallow’s $C_p$ Criteria are proved inconsistent (Nishii, 1984; Rao and Wu, 1989)). When the number of regressors does not increase with sample size, AIC and Mallow’s $C_p$ Criteria generally give inconsistent estimates of the true model (Shibata, 1984, Zhang, 1993b), mainly due to over-fitting. However, when the number of regressors and sample size simultaneously tend to infinity, AIC exhibits consistency (see Yanagihara, Wakaki, and Fujikoshi, 2015)).

While in conventional linear or nonlinear regression, we assume that a particular parametric model defines the relationship between the response and regressors, we make no such assumption in nonparametric regression. Instead, by estimating the regression function from observed data, we attempt to find out the true nature of relationship between the response and the predictors. When traditional parametric linear and non-linear model often fail to capture the true nature of dependence of the response variable on predictors, the need for variable screening methodologies in nonparametric set up arises. For instance, in pharmacogenomics data, which are usually high dimensional, the nature of relationship between an individual’s drug response and genetic makeup is generally quite complicated (see Fan and Liu, 2013)). As a result, variable screening in case of pharmacogenomics data is often carried out in nonparametric setup. Another application of variable screening is in biomarker discovery in genomics data. In genomics data, individual features correspond to genes, so by selecting the most relevant features, one gains important knowledge about the genes that are the most discriminative for a particular problem (see Dessi, Pascariello, and Pes, 2013 for details).

Although variable screening in multiple linear regression model has been extensively studied, it, however, has not been paid much attention in nonparametric regression. In this article, we
propose a methodology for variable screening in nonparametric regression model, based on local linear approximation (for details, see Fan and Gijbels (1996)). The central idea is that, a variable is considered to be redundant if the partial derivative of the actual regression function with respect to that particular variable is zero at every point in the covariate space. However, as the actual regression function is unknown in practice, we estimate the regression function using local linear approximation. It provides us the least squares estimates of these partial derivatives, and the assumption of normality on the distribution of error random variables gives us the approximate distributions of those partial derivatives. A regressor is retained in the model if the estimate of partial derivative of the regression function with respect to the respective variable is larger than a certain threshold everywhere, otherwise, that variable will not be included in the model as a regressor. We also prove that as the sample size tends to infinity, the methodology is capable of eliminating all redundant regressors almost surely.

The rest of the article is arranged in the following order: In Section 2.1, we fully describe the methodology, and in Section 2.2, we state the motivations in detail. Section 2.3 is dedicated to a brief discussion about advantages and disadvantages of the method, and Section 3 contains concluding remarks. All the remaining technical details and proofs along with the results provided in the tabular form are provided in the Appendix.

2 Proposed Methodology and Statistical Property

Description of the Method:

We propose a methodology for variable screening in nonparametric regression model, which is capable of eliminating all redundant regressors as the sample size grows to infinity. The method is primarily based on local linear approximation. We first estimate the unknown regression function from the data using local linear approximation, and it gives us the least squares estimates of the partial derivatives of the regression function with respect to the particular variable at any point in the sample space. Technically speaking, the partial derivative of the true regression function with respect to the particular variable will be zero everywhere in the sample space if the corresponding regressor is insignificant. Thus, we retain a regressor in the model, if the estimate obtained from local linear approximation of the partial derivative of the regression function with respect to the
corresponding component is above a certain threshold value everywhere in sample space. Otherwise, the respective variable will not be selected.

Suppose that we have the data \((x_{i1}, \ldots , x_{id}, Y_i), i = 1, \ldots , n\) on \(n\) individuals. Our regression model is

\[ Y_i = m(x_{i1}, \ldots , x_{id}) + \epsilon_i, \quad \text{where } i = 1, \ldots , n. \] (1)

Here the regression function \(m\) is unknown, and \(\epsilon_i\)'s are independent normal random variable with mean \(= 0\) and constant variance \(= \sigma^2 < \infty\). Let us now define \(\beta_0(x) = m(x), \quad \text{and } \beta_j(x) = \frac{\partial m(t)}{\partial t_j} \big|_{t=x}\) for \(j = 1, \ldots , d\). Now, using straightforward application of Taylor series expansion, one can establish that for any \(z \equiv (z_1, \ldots , z_d)^T\),

\[ m(z) = m(x) + \sum_{i=1}^{d} \beta_j(x)(z_j - x_j) + R, \] (2)

where \(R\) is the remainder term, which becomes negligible whenever \(z\) and \(x\) are close enough (see Fan and Gijbels (1996)).

Next, to formulate the decision rule, one first needs to estimate the \(\beta(x) = (\beta_0(x), \beta_1(x), \ldots , \beta_d(x))\) from the given data. In this work, the local linear estimator of \(\beta(x)\) is considered, which is described as follows:

\[
\hat{\beta}(x) = \arg \min_{\beta} \sum_{i=1}^{n} \left[ Y_i - \beta_0 - \sum_{j=1}^{d} \beta_j(x_{ij} - x_j) \right]^2 K_B(x_i - x) \\
= [X(x)^TW(x)X(x)]^{-1} X(x)^TW(x)Y, \quad \text{where}
\]

\[
X(x) = \begin{bmatrix} 1 & (x_{11} - x_1) & (x_{12} - x_2) & \cdots & (x_{1d} - x_d) \\
1 & (x_{21} - x_1) & (x_{22} - x_2) & \cdots & (x_{2d} - x_d) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (x_{n1} - x_1) & (x_{n2} - x_2) & \cdots & (x_{nd} - x_d) \end{bmatrix}, \quad \text{and } Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}. \] (4)

Here \(B\) is a bandwidth matrix, and \(K(\cdot)\) is a kernel function satisfying, \(K_B(t) = \frac{1}{|B|} K(B^{-1}t)\) vanishes outside a compact neighbourhood around \(t\), and \(W(x) = diag(K_B(x_1 - x), \ldots , K_B(x_n - x))\). Also, define \(C = [X(x)^TW(x)X(x)]^{-1} X(x)^TW(x)W(x)^TX(x) [X(x)^TW(x)X(x)]^{-1}\), and
$c_{jj}$ as the $(j + 1, j + 1)$-th element of $C$, for every $j = 1, \ldots, d$.

Finally, the decision rule is as follows:

**Case 1 : ($\sigma$ is known)**

Accept $j$-th covariate, i.e., $X_j$ if and only if

$$\inf_x \left| \frac{\beta_j(x)}{\sigma \sqrt{c_{jj}}} \right| > z_{\frac{\alpha}{2}}, \quad (5)$$

where $\beta_j(x)$ is the $j$-th ($j = 0, \ldots, d$) component of $\beta(x)$, and $z_{\frac{\alpha}{2}}$ is the $(1 - \frac{\alpha}{2})$-th quantile of standard normal distribution.

**Case 2 : ($\sigma$ is unknown)**

Accept $j$-th covariate, i.e., $X_j$ if and only if

$$\left| \frac{\beta_j(x)}{\sqrt{MS(x)c_{jj}}} \right| > t_{N(x) - d - 1, \frac{\alpha}{2}} \text{ for all } x, \quad (6)$$

where $N(x) = \sum_{i=1}^{n} I(\|x_i - x\| < \delta_n)$, $\{\delta_n\}_{n=1}^{\infty}$ is a sequence of positive numbers converging to zero, $MS(x) = \frac{\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 K_B(x_i - x)}{N(x) - d - 1}$, and $t_{N(x) - d - 1, \frac{\alpha}{2}}$ is the $(1 - \frac{\alpha}{2})$-th quantile of $t$ distribution with $(N(x) - d - 1)$ degrees of freedom.

The above decision rules are formulated based on the asymptotic distribution of $\beta_j(x)$, i.e., the estimated partial derivative of $m(x)$ with respect to the $j$-th covariate. For that reason, the threshold described in the rule, when $\sigma$ is known, is a certain quantile of the standard normal distribution (see Case 1) whereas in the case of unknown $\sigma$, the threshold value depends on a certain quantile of $t$ distribution with $(N(x) - d - 1)$ degrees of freedom. In the next subsection, we study the asymptotic performance of the proposed rule.

**Large Sample Statistical Property**

As discussed in the previous subsection, one may adopt our methodology regardless whether $\sigma$ is known or unknown. We assert that in both the cases, redundant regressors will be removed almost surely for large sample sizes. Case 1 describes the idea when $\sigma$ is known, and for unknown $\sigma$, the idea is explained in Case 2.
Case 1: \((\sigma \text{ is known})\)

We introduce the subscript \(n\) to every pre-introduced notation to emphasize its dependence on the sample size. Note that \(Y_n \sim N_n(m_n, \sigma^2 I_n)\), where \(m_n = (m(x_1), \ldots, m(x_n))^T\). Thus, \(\hat{\beta}(x)\) follows \(N_{d+1}(\mu_n(x), \sigma^2 C_n)\), where
\[
\mu_n(x) = \left[X_n(x)^T W_n(x) X_n(x)\right]^{-1} X_n(x)^T W_n(x) m_n, \quad \text{and} \\
C_n = \left[X_n(x)^T W_n(x) X_n(x)\right]^{-1} X_n(x)^T W(x) X_n(x) X_n(x)^T W_n(x) X_n(x) - 1.
\]

Here, \(K^{(n)}(\cdot)\) is a sequence of Kernel functions used in weighted least squares estimate, such that \(K_B^{(n)}(t) = 0\), whenever \(\|t\| > \delta_n\), where \(\{\delta_n\}_{n=1}^\infty\) is a sequence of positive real numbers converging to zero. To this end, we claim that
\[
P \left[ \inf_{x} \frac{\beta_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} > \frac{z_{\alpha}}{2} \left| X_j \text{ is redundant} \right| \right] \to 0,
\]
as \(n \to \infty\). The full statement is provided in Theorem 2.1 and the complete proof is given in the Appendix.

Case 2: \((\sigma \text{ is unknown})\)

If \(\sigma\) is unknown, we then define the sequence of Kernel functions as
\[
K_B^{(n)}(t) = \begin{cases} 
c_n & \text{if } \|t\| \leq \delta_n \\
0 & \text{otherwise}
\end{cases}
\]
where \(\{c_n\}_{n=1}^\infty\) is an arbitrary sequence of positive real numbers, and \(\{\delta_n\}_{n=1}^\infty\) is a sequence of positive real numbers converging to zero. Hence,
\[
\hat{\beta}(x) = \arg \min_{\{i: \|x_i - x\| \leq \delta_n\}} \left[ Y_i - \beta_0 - \sum_{j=1}^d \beta_j (x_{ij} - x_j) \right]^2.
\]

Thus, in order to find \(\hat{Y_i}\), we need to fit a linear regression model within \(\delta_n\)-neighbourhood of \(x_i\), under normal error assumptions. Consequently, imitating the formation of \(t\)-statistic in conventional multiple linear regression model, we have
\[
\frac{\beta_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \sim t_{N(x)-d-1} \quad \text{for all } x,
\]
provided
$N(x) > d + 1$, for all $x$. To this end, we claim that

$$P \left[ \inf_x \left| \frac{\hat{\beta}_{j,n}(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x) - d - 1, \frac{1}{2}} \mid X_j \text{ is redundant} \right] \to 0, \text{ as } n \to \infty.$$ 

Thus, deploying this method ensures that the redundant regressors will be eliminated almost surely as sample size grows to infinity. Both the cases are summarized in the Theorems 2.1 and 2.2 below, and the proofs are provided in the Appendix.

**Technical Assumptions**

A1. Suppose that $Y_i = m(x_{i1}, \ldots, x_{id}) + \epsilon_i$, where $\epsilon_i, i = i, \ldots, n$ are i.i.d random variables with normal distribution having zero mean and constant variance.

A2. The regression function $m$ is partially differentiable with respect to $x_j$ for all $j = 1, \ldots, d$, and the first order partial derivatives with respect to $x_j$ for all $j = 1, \ldots, n$, are bounded.

A3. Let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} \delta_n = 0$, and $\{c_n\}_{n=1}^{\infty}$ be any bounded sequence of positive real numbers.

**Theorem 2.1.** Suppose that $\sigma$ is known, and $K^{(n)}(\cdot)$ is a sequence of Kernel functions satisfying $K_B^{(n)}(t) = 0$ whenever $\|t\| > \delta_n$, then under the assumptions A1 - A3, we have

$$\beta_j(x) = 0 \text{ for all } x \iff \lim_{n \to \infty} P \left[ \inf_x \left| \frac{\hat{\beta}_{j,n}(x)}{\sqrt{\sigma c_{jj,n}}} \right| > z_{\frac{1}{2}} \right] = 0.$$ (7)

**Theorem 2.2.** Suppose that $\sigma$ is unknown, and $K^{(n)}(\cdot)$ is a sequence of Kernel functions satisfying

$$K_B^{(n)}(t) = \begin{cases} c_n & \text{if } \|t\| \leq \delta_n \\ 0 & \text{otherwise} \end{cases},$$

then under the assumptions A1 - A3, we have

$$\beta_j(x) = 0 \text{ for all } x \iff \lim_{n \to \infty} P \left[ \inf_x \left| \frac{\hat{\beta}_{j,n}(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x) - d - 1, \frac{1}{2}} \right] = 0.$$ (8)

The assertion in the above theorem indicates that, as the sample size increases, the irrelevant
regressors are eliminated almost surely. This is the reason for which one can implement this methodology in practice.

3 Simulation Study

We test the validity of our methodology using some simulated data.

Example 1(a) (σ known): For the first data set, we generate \( x \) as follows: \( x = (x_1, x_2, x_3, x_4)^T \), where

\[
\begin{align*}
x_1 &\sim \text{Beta}(1.5, 3.2), \\
x_2 &\sim \text{Beta}(2.7, 5.0), \\
x_3 &\sim \text{Beta}(4.8, 8.5), \text{ and } x_4 \sim \text{Beta}(6.2, 9.0).
\end{align*}
\]

Next, we consider the model \( y = 0.01(2x_1 + 2) + 91(2x_2^2 + 5) + 97(2x_3^3 + 8) + 98(2x_4^4 + 11) + \epsilon \), where \( \epsilon \) follows standard normal distribution. We generate five hundred instances from this model and compute the estimates of \( \hat{\beta}_{j,n}(x) \) for each \( j = 1, 2, 3, 4 \). The computed values are obtained as 0.001246265, 4.172932794, 3.245607226, and 7.142335063 for \( x_1, x_2, x_3, \) and \( x_4 \), respectively.

We consider \( \alpha = 5\% \), and accept the variables \( x_2, x_3, \) and \( x_4 \) since \( \inf_x \left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right| > z_{\alpha/2} \) for each \( j = 2, 3, 4 \), and we discard the variable \( x_1 \), since \( \inf_x \left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right| < z_{\alpha/2} \) for \( j = 1 \).

Conclusion: We repeat this analysis a hundred times to get the proportion of times a particular variable gets selected (or accepted), and the results are: \( x_1, x_2, x_3 \) and \( x_4 \) get selected 5, 100, 100 and 100 times, respectively. This result gives us strong enough evidence that \( x_2, x_3 \) and \( x_4 \) should be selected while \( x_1 \) should be rejected. The result in tabular form is provided in Table \( \Box \) in Appendix.

Example 1(b) (σ unknown): We consider the same distribution of covariates and the same model as in Example 1(a), but with error following normal distribution with zero mean and unknown standard deviation. We generate five hundred instances from the aforementioned model and compute the estimates of \( \inf_x \left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| \) for each \( j = 1, 2, 3, 4 \), and for fifty values of \( x \). The minimum with respect to \( x \) of estimates are obtained as 0.03142822, 64.09119408, 33.81495794, and 28.01598150 for \( x_1, x_2, x_3, \) and \( x_4 \), respectively. We consider \( \alpha = 5\% \), and accept the variables
\(x_2, x_3, \) and \(x_4\) since 
\[
\frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} > t_{N(x)-d-1,\frac{\alpha}{2}} \text{ for all } x \text{ and for } j = 2, 3, 4, \text{ and we discard the variable } x_1, \text{ since } \left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| < t_{N(x)-d-1,\frac{\alpha}{2}} \text{ for some } x \text{ for } j = 1.
\]

**Conclusion:** We repeat this methodology a hundred times to get the results that \(x_1, x_2, x_3 \) and \(x_4\) get selected 0, 100, 97 and 99 times, respectively. Hence the result gives us strong enough evidence that \(x_2, x_3\) and \(x_4\) should be selected while \(x_1\) should be rejected. The result in tabular form is provided in Table 1 in Appendix.

**Example 2(a) (\(\sigma\) known):** For the second data, we generate \(x\) as follows: 
\[
x = (x_1, x_2, x_3, x_4)^T,
\]
where
\[
x_1 \sim \text{Gamma}(2.0, 2.0), x_2 \sim \text{Gamma}(3.0, 2.0),
\]
\[
x_3 \sim \text{Gamma}(5.0, 1.5), \text{ and } x_4 \sim \text{Gamma}(7.0, 1.0).
\]

Next, we consider the model 
\[
y = \frac{52.0}{15}(x_1 + 5) + \frac{0.02}{15}(x_2^2 + 6) + \frac{57.0}{15}(x_3^2 + 7) + \frac{68.0}{15}\sqrt{x_4 + 8} + \epsilon,
\]
where \(\epsilon\) follows standard normal distribution. We use the factor of 1/15 to scale the data to unit interval for ease of computation. We generate five hundred instances from this model and compute the estimates of
\[
\inf_{x} \left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right| \text{ for each } j = 1, 2, 3, 4.
\]
The computed values are obtained as 4.462582, 0.000731, 4.118707, and 5.749765 for \(x_1, x_2, x_3, \) and \(x_4\), respectively. We consider \(\alpha = 5\%\), and accept the variables \(x_1, x_3, \) and \(x_4\) since 
\[
\inf_{x} \left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right| > z_{\alpha/2} \text{ for each } j = 1, 3, 4, \text{ and we discard the variable } x_2, \text{ since } \inf_{x} \left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right| < z_{\alpha/2} \text{ for } j = 2.
\]

**Conclusion:** we repeat the same study a hundred times to get the proportion of times a particular variable is selected and the results are: \(x_1, x_2, x_3 \) and \(x_4\) get selected 99, 0, 100 and 100 times, respectively. This result provides us a conclusive evidence that the variables \(x_1, x_3\) and \(x_4\) should be selected, while \(x_2\) should be rejected. The result in tabular form is provided in Table 1 in Appendix.
Example 2(b) (σ unknown): We consider the same distribution of covariates and the same model as in Example 2(a), but with error following normal distribution with zero mean and unknown standard deviation. We generate 500 instances from the aforementioned model and compute the estimates of $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right|$ for each $j = 1, 2, 3, 4$, and for fifty values of $x$. The minimum with respect to $x$ of estimates are obtained as 0.03142822, 0.09119408, 0.81495794, and 28.01598150 for $x_1, x_2, x_3$, and $x_4$, respectively. We consider $\alpha = 5\%$, and accept the variables $x_2, x_3$, and $x_4$ since $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x) - d - 1, \alpha/2}$ for all $x$ and for $j = 2, 3, 4$, and we discard the variable $x_1$, since $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| < t_{N(x) - d - 1, \alpha/2}$ for some $x$ for $j = 1$.

Conclusion: We repeat the same study a hundred times to get the proportion of times a particular variable is selected and the results are: $x_1, x_2, x_3$ and $x_4$ get selected 95, 0, 100 and 100 times, respectively. This result provides us a conclusive evidence that the variables $x_1, x_3$ and $x_4$ should be selected, while $x_2$ should be rejected. The result in tabular form is provided in Table 1 in Appendix.

Example 3(a) (σ known): For the third data, we generate $x$ as follows: $x = (x_1, x_2, x_3, x_4)^T$, where

$$
x_1 \sim Unif(0, 1), x_2 \sim Unif(1, 2),
$$
$$
x_3 \sim Unif(0, 2), \text{ and } x_4 \sim Unif(2, 4).
$$

Next, we consider the model $y = 8\sqrt{x_1} + (10.2)x_2^2 + 7\exp(x_3) + (0.03)x_4 + \epsilon$, where $\epsilon$ follows standard normal distribution. We generate five hundred instances from this model and compute the estimates of $\left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right|$ for each $j = 1, 2, 3, 4$. The computed values are obtained as 31.2972339, 49.6025809, 63.1483349, and 0.1840352 for $x_1, x_2, x_3$, and $x_4$, respectively. We consider $\alpha = 5\%$, and accept the variables $x_1, x_2$, and $x_3$ since $\left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right| > z_{\alpha/2}$ for each $j = 1, 2, 3$, and we discard the variable $x_4$, since $\left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right| < z_{\alpha/2}$ for $j = 4$. 

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Conclusion: We repeat the study a hundred time to get the proportion of times a particular variable gets selected. The results are as follows: $x_1$, $x_2$, $x_3$ and $x_4$ get selected 100, 100, 100 and 0 times, respectively. This result gives us a strong evidence that the variables $x_1$, $x_2$ and $x_3$ should be selected while the variable $x_4$ should be rejected. The result in tabular form is provided in Table 1 in Appendix.

Example 3(b) ($\sigma$ unknown): We consider the same distribution of covariates and the same model as in Example 3(a), but with error following normal distribution with mean zero (we assume that $\sigma$ is unknown). We generate five hundred instances from this model and compute the estimates of $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right|$ for each $j = 1, 2, 3, 4$, and for fifty values of $x$. The minimum with respect to $x$ of estimates are obtained as $6.7593173$, $10.0548349$, $7.1329549$, and $0.0191185$ for $x_1$, $x_2$, $x_3$, and $x_4$, respectively. We consider $\alpha = 5\%$, and accept the variables $x_1$, $x_2$, and $x_3$ since $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x)-d-1, \frac{\alpha}{2}}$ for all $x$ and for $j = 1, 2, 3$, and we discard the variable $x_4$, since $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| < t_{N(x)-d-1, \frac{\alpha}{2}}$ for some $x$ for $j = 4$.

Conclusion: We repeat the study a hundred time to get the proportion of times a particular variable gets selected. The results are as follows: $x_1$, $x_2$, $x_3$ and $x_4$ get selected 100, 100, 100 and 0 times, respectively. This result gives us a strong evidence that the variables $x_1$, $x_2$ and $x_3$ should be selected while the variable $x_4$ should be rejected. The result in tabular form is provided in Table 1 in Appendix.

Example 4(a) ($\sigma$ known): For the fourth data we generate $x$ as follows: $x = (x_1, x_2, x_3, x_4)^T$, where

$$
x_1 \sim Logistic(4, 2), x_2 \sim Logistic(5, 2), \quad x_3 \sim Logistic(6, 2), \text{ and } x_4 \sim Logistic(7, 2).
$$

Next, we consider the model $y = (5.2)x_1 + (8.5)x_2 + (0.01)x_3 + 6x_4 + \epsilon$, where $\epsilon$ follows standard normal distribution. We generate five hundred instances from this model and compute the estimates of $\inf_x \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}}$ for each $j = 1, 2, 3, 4$. The computed values are obtained as $15.7024064$, $10.2345678$, $7.1234567$, and $0.9876543$.
1.9194216, 0.1773722, and 1.8758454 for $x_1$, $x_2$, $x_3$, and $x_4$, respectively. We consider $\alpha = 5\%$, and accept the variables $x_1$, $x_2$, and $x_4$ since $\inf_x \left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right| > z_{\alpha/2}$ for each $j = 1, 2, 4$, and we discard the variable $x_3$, since $\inf_x \left| \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} \right| < z_{\alpha/2}$ for $j = 3$.

**Conclusion:** We repeat this analysis a hundred times to get the proportion of times a particular variable gets selected, and the results are: $x_1$, $x_2$, $x_3$ and $x_4$ get selected 100, 100, 0 and 97 times, respectively. This result gives us strong enough evidence that $x_1$, $x_2$ and $x_4$ should be selected while $x_3$ should be rejected. The result in tabular form is provided in Table 1 in Appendix.

**Example 4(b) ($\sigma$ unknown):** We consider the same distribution of co-variates and the same model as in Example 4(a), but with error following normal distribution with mean zero (we assume that $\sigma$ is unknown). We generate two hundred instances from this model and compute the estimates of $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{M S_n(x)c_{jj,n}}} \right|$ for each $j = 1, 2, 3, 4$, and for fifty values of $x$. The minimum with respect to $x$ of estimates are obtained as 3.3524864, 8.9557089, 0.1093094, and 4.3401730 for $x_1$, $x_2$, $x_3$, and $x_4$, respectively. We consider $\alpha = 5\%$, and accept the variables $x_1$, $x_2$, and $x_4$ since $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{M S_n(x)c_{jj,n}}} \right| > t_{N(x)-d-1, \frac{\alpha}{2}}$ for all $x$ and for $j = 1, 2, 4$, and we discard the variable $x_3$, with a similar argument.

**Conclusion:** We repeat this methodology a hundred times to get the results that $x_1$, $x_2$, $x_3$ and $x_4$ get selected 99, 100, 0 and 100 times, respectively. Hence the result gives us strong enough evidence that $x_1$, $x_2$ and $x_4$ should be selected while $x_3$ should be rejected. The result in tabular form is provided in Table 1 in Appendix.

## 4 Real Data Analysis

In this section, we implement our methodology to some real data collected from UCI Machine Learning Repository [https://archive.ics.uci.edu/ml/index.php](https://archive.ics.uci.edu/ml/index.php) and earlier research article.
Airfoil Self-Noise Data Set: This data is obtained from a series of aerodynamic and acoustic tests of two and three-dimensional airfoil blade sections conducted in an anechoic wind tunnel, and it can be accessed with the link: [https://archive.ics.uci.edu/ml/datasets/Airfoil+Self-Noise](https://archive.ics.uci.edu/ml/datasets/Airfoil+Self-Noise).

The attributes are:
1. Frequency, in Hertz,
2. Angle of attack, in degrees,
3. Chord length, in meters,
4. Free-stream velocity, in meters per second,
5. Suction side displacement thickness, in meters.

The output is scaled sound pressure level, in decibels. The attributes are labelled as $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$, respectively as regressors, and the output is labelled $y$ as response. We estimate the values of $\hat{\beta}_{j,n}(x) - \beta_j(x)$ over each $j = 1, \ldots, 5$, and for five hundred values of $x$. The minimum of the estimates with respect to the point of evaluation $x$ is obtained as 9.6986829, 3.8801889, 1.7183420, 2.7930107, and 0.5699275, for $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$, respectively. Based on these computations, we can select (or accept) $x_1$, $x_2$, $x_3$ and $x_4$, since $|\hat{\beta}_{j,n}(x) - \beta_j(x)| > t_{N(x)} - d - 1, \alpha$ for all $x$ and for $j = 1, 2, 3, 4$ (here $\alpha$ is considered as 0.05). However, for $j = 5$, $|\hat{\beta}_{j,n}(x) - \beta_j(x)| \leq t_{N(x)} - d - 1, \alpha$ for all $x$, and hence, we reject the variable $x_5$.

Conclusion: We replicate the experiment, a hundred times using bootstrap methodology and record the results as number of times a particular variable is accepted (or selected). The results are as follows: the variables $x_1, \ldots, x_5$ are accepted 100, 100, 93, 98, 7 times respectively. Our conclusion from this result is that, $x_1$, $x_2$, $x_3$ and $x_4$ can be selected, since all these variables are being selected 93% or more number of times. On the other hand, $x_5$ is only selected 7 times, and it indicates that $x_5$ may not be selected. The results in tabular form is provided in Table 2 in Appendix.

Concrete Compressive Strength Data Set: The concrete compressive strength is a highly nonlinear function of age and ingredients. There are 1030 instances and 9 attributes in this data set (see Yeh (1998) for details), and it can be accessed through the link: [https://archive.ics.uci.edu/ml/datasets/concrete+compressive+strength](https://archive.ics.uci.edu/ml/datasets/concrete+compressive+strength).

The attributes are:
- Cement (component 1) – quantitative – kg in a m3 mixture – Input Variable
Blas Furnace Slag (component 2) – quantitative – kg in a m3 mixture – Input Variable
Fly Ash (component 3) – quantitative – kg in a m3 mixture – Input Variable
Water (component 4) – quantitative – kg in a m3 mixture – Input Variable
Superplasticizer (component 5) – quantitative – kg in a m3 mixture – Input Variable
Coarse Aggregate (component 6) – quantitative – kg in a m3 mixture – Input Variable
Fine Aggregate (component 7) – quantitative – kg in a m3 mixture – Input Variable
Age – quantitative – Day (1 365) – Input Variable

Concrete compressive strength – quantitative – MPa – Output Variable.

Here, the input attributes are labelled as $x_1, \ldots, x_8$ respectively as regressors, and the output is labelled $y$ as response. We estimate the values of $\hat{\beta}_{j,n}(x) - \beta_j(x) \sqrt{M_{S_{n}}(x)c_{jj,n}}$ for each $j = 1, \ldots, 8$, and for five hundred values of $x$. The minimum of the estimates with respect to $x$ is obtained as 54.4927207, 0.2659793, 2.0637029, 3.2885670, 1.8678307, 2.9666497, 2.2933404 and 1.2256556 for $x_1, \ldots, x_8$, respectively. Based on these computations we can select (or accept) $x_1, x_3, x_4, x_5, x_6$ and $x_7$, since $|\hat{\beta}_{j,n}(x) - \beta_j(x) \sqrt{M_{S_{n}}(x)c_{jj,n}}| > t_{N(x)-d-1, \frac{\alpha}{2}}$ for all $x$ and for $j = 1, 3, 4, 5, 6, 7$ (here $\alpha$ is considered as 0.05).

However, for $j = 2$ and 8, $|\hat{\beta}_{j,n}(x) - \beta_j(x) \sqrt{M_{S_{n}}(x)c_{jj,n}}| \leq t_{N(x)-d-1, \frac{\alpha}{2}}$ for all $x$, and hence, we will not accept the rest of the variables $x_2$ and $x_8$.

**Conclusion:** We replicate the experiment, a hundred times using bootstrap methodology and record the results as number of times a particular variable is accepted (or selected). The results are as follows: the variables $x_1, \ldots, x_8$ are accepted 100, 1, 94, 96, 99, 100, 100, 2 times, respectively. This result gives us a clear conclusion that the variables $x_1, x_3, x_4, x_5, x_6$ and $x_7$ can be selected, since these variables are being selected 94% or more number of times. On the other hand, we have conclusive evidence that the variables $x_2$ and $x_8$ may not be selected, since they are being selected only 1% and 2% number of times. The results in tabular form is provided in Table 2 in Appendix.

**Istanbul Stock Exchange Data Set:** This data set includes returns of Istanbul Stock Exchange with seven other international index: SP, DAX, FTSE, NIKKEI, BOVESPA, MSCE EU, MSCI EM from June 5, 2009 to Feb 22, 2011, and it can be accessed with the link: [https://archive.ics.uci.edu/ml/datasets/ISTANBUL+STOCK+EXCHANGE](https://archive.ics.uci.edu/ml/datasets/ISTANBUL+STOCK+EXCHANGE) (see Akbilgic, Bozdogan, and Balaban (2014) for details). There are 536 instances of recorded data with 8 attributes. We consider the aforementioned international indices as the input attributes, and we label them as $x_1, \ldots, x_7$, respectively as regressors. We denote the Istanbul Stock Exchange
(ISE) by output $y$, and we label it as the response. We estimate the values of $\hat{\beta}_{j,n}(x) - \beta_j(x)$ for each $j = 1, \ldots, 7$, and for five hundred values of $x$. The minimum of the estimates with respect to $x$ is obtained as 60.529649, 20.632782, 163.734813, 355.828404, 209.169958, 4.142214, and 374.399345 for $x_1, \ldots, x_7$, respectively. Based on these computations, we may select (or accept) all the variables, i.e., $x_1, x_2, x_3, x_4, x_5, x_6,$ and $x_7$ since $|\hat{\beta}_{j,n}(x) - \beta_j(x)| \sqrt{MS_n(x)c_{jj,n}} > t_{N(x)-d-1,0.05}$ for all $x$ and for $j = 1, 2, 3, 4, 5, 6, 7$ (here $\alpha$ is considered as 0.05). Thus we are not able to reject or discard any of the independent variable.

**Conclusion:** We replicate the experiment, a hundred times using bootstrap methodology and record the results as number of times a particular variable is accepted (or selected). The results are as follows: the variables $x_1, \ldots, x_8$ are accepted 73, 100, 99, 100, 100, 100, 100 times respectively. With this result, we have enough evidence to select the variables $x_2, \ldots, x_7$, since they are being selected 99% or more number of times. But, we do not have a strong evidence to support the decision to select $x_1$, since it is only being selected 73% of the times. The results in tabular form is provided in Table 2 in Appendix.

**Profit Rates and market Structure for High-Level Advertising Firms:** This data set consists of firm profits (1963-1968) of 57 firms with several independent variables (for data set and further details, see Vernon and Nourse (1973)). There are eleven attributes in this data, out of which PT (= (Net income + interest expense)/total assets) and PE (= net income/shareholder equity) are dependent variables. The independent variables are,

- CR = Weighted Concentration Ratio of firm’s product markets (4-firm CR-1966)
- DCR = Dummy Var for CR $\geq 50$
- AS = Weighted average INDUSTRY advertising-to-sales ratio of firm’s product markets (1969 data)
- CAS = Overall 1969 advertising-to-sales ratio of the FIRM
- GR = Weighted average percent changes in industry sales in the firm’s markets
- DIV2 = Firm’s diversification in 2-digit (more broad) industries (1-sum($S^2$))
- DIV3 = Firm’s diversification in 3-digit (less broad) industries (1-sum($S^2$))
- SIZ = Firms 1968 Total assets ($1M$)
- LRSIZ = 1/ln(SIZ)

Here, the independent variables are labelled as $x_1, \ldots, x_9$, respectively as regressors, and the de-
pendent variable PT is labelled $y$ as response. We estimate the values of $\left\{ \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right\}$ for each $j = 1, \ldots, 9$, and for five hundred values of $x$. The minimum of the estimates with respect to $x$ is obtained as 90.4538868, 0.2662083, 2.2908602, 1.3122562, 3.2828394, 0.4191373, 7.5574417, 0.2233489, and 2.9207716 for $x_1, \ldots, x_9$, respectively. Based on these computations, we can select (or accept) $x_1, x_3, x_4, x_5, x_7, x_9$, since $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x)-d-1,\alpha/2}$ for all $x$ and for $j = 1, 3, 4, 5, 6, 7, 9$ (here $\alpha$ is considered as 0.05). However, for $j = 2, 6$ and $8$, $\left| \frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| \leq t_{N(x)-d-1,\alpha/2}$ for all $x$, and hence, we will not accept the rest of the variables $x_2, x_6$ and $x_8$.

**Conclusion:** We replicate the experiment, a hundred times using bootstrap methodology and record the results as number of times a particular variable is accepted (or selected). The results are as follows: the variables $x_1, \ldots, x_9$ are accepted 100, 40, 2, 38, 74, 2, 78, 87, 71 times respectively. With this result we have conclusive evidence for selection of only one variable i.e., $x_1$, since it is being selected 100% number of times. But, we do not have sufficiently strong evidence to select any other variables. The results in tabular form is provided in Table 2 in Appendix.

## 5 Concluding Remarks

In this article, we have studied a method for variable screening in nonparametric regression model. The method depends on local linear approximation to obtain estimates of partial derivative of the regression function with respect to different variables at all possible points in sample space. With this method, we include a particular regressor if and only if the estimate of partial derivative of the regression function with respect to the corresponding variable always exceeds a threshold value. It relies on the argument that if a particular regressor is useless, the partial derivative of the true regression function with respect to that component is close to zero at every point in the sample space.

Based on normality assumption of the error terms, this method ensures that when sample size grows to infinity, all redundant regressors will almost surely be eliminated from the model. However, it does not guarantee the inclusion of all the useful regressors in the model for large sample sizes. Thus, even after working with a fairly large sample, the obtained model after deploying this method, can be a subset of the true model. Moreover, this method cannot be used
in case at least one regressor is discrete or categorical.

As mentioned before, the theoretical results have been derived under the normality assumption of the error terms. However, under a few more assumptions like moment condition on the error random variable, it is possible to establish those theoretical result for a much larger class of distributions of the error random variables (see, e.g., Van der Vaart (2000), p. 21).

This method relies on local linear approximation, which involves estimating partial derivatives of the regression function with respect to different variables. As a result, this method does not readily work in case at least one regressor is categorical. Thus, one can attempt to modify this method to deal with categorical regressors. Moreover, it can also be interesting to study if this technique can be generalized for the case with non-normal error terms. Besides, such least squares estimators may not be robust against the outliers, and to overcome this problem, one may adopt various quantile based estimators or least absolute deviation based estimators (see, e.g., Dutta, Dhar, and Mitra (2019) in the context of stochastic volatility model).

6 Appendix

6.1 Proofs

Proof of Theorem 2.1. Note that, it follows from the expression of \( \hat{\beta}_{j,n}(\cdot) \) that \( \hat{\beta}_{j,n}(x) \) and \( \hat{\beta}_{j,n}(y) \) are independent for two points \( x \) and \( y \), if and only if \( S_n(x) \cap S_n(y) = \phi \), where

\[
S_n(t) = \{ v : \| v - t \| < \delta_n \}.
\]

Next, suppose that \( l_n \) is the maximum number of points in the space of \( x \), i.e., the space of points of evaluation, such that \( \hat{\beta}_{j,n}(x) \)'s are independent, then observe that

\[
\lim_{n \to \infty} \delta_n = 0 \Rightarrow \lim_{n \to \infty} l_n = \infty.
\]

Further, as \( m \) has bounded partial derivatives with respect to all regressors, the remainder term as defined in the equation (2) can be made arbitrarily small. Thus, for all \( x \),

\[
\lim_{n \to \infty} \| W_n(x) m_n - W_n(x) X_n(x) \beta(x) \| = 0 \Rightarrow \lim_{n \to \infty} \| \mu_n(x) - \beta(x) \| = 0 \text{ in probability.} \quad (9)
\]
Now, note that

\[
\frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sigma \sqrt{c_{jj,n}}} = \frac{\hat{\beta}_{j,n}(x) - \mu_n(x) + \mu_n(x) - \beta_j(x)}{\sigma \sqrt{c_{jj,n}}} = \frac{\hat{\beta}_{j,n}(x) - \mu_n(x)}{\sigma \sqrt{c_{jj,n}}} + \frac{\mu_n(x) - \beta_j(x)}{\sigma \sqrt{c_{jj,n}}} \rightarrow N(0,1),
\]

as \( n \to \infty \) for all \( j = 1, \ldots, d \), since \( \frac{\hat{\beta}_{j,n}(x) - \mu_n(x)}{\sigma \sqrt{c_{jj,n}}} \overset{d}{\to} N(0,1) \) as \( n \to \infty \) and \( \frac{\mu_n(x) - \beta_j(x)}{\sigma \sqrt{c_{jj,n}}} \overset{p}{\to} 0 \) as \( n \to \infty \), using (9). Furthermore, \( \beta_j(x) = 0 \) for all \( x \) if the variable \( X_j \) is irrelevant. Hence,

\[
P\left[ \inf_x \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} > \frac{z_{\alpha}}{2} \right] \leq \left( P\left[ \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} > \frac{z_{\alpha}}{2} \right] \right)^{l_n},
\]

because \( \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} > \frac{z_{\alpha}}{2} \) for all \( x \) implies that \( \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} > \frac{z_{\alpha}}{2} \) for \( l_n \) number of \( x \). Finally, since \( P\left[ \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} > \frac{z_{\alpha}}{2} \right] \in (0,1) \) and \( l_n \to \infty \) as \( n \to \infty \), we have \( P\left[ \inf_x \frac{\hat{\beta}_{j,n}(x)}{\sigma \sqrt{c_{jj,n}}} > \frac{z_{\alpha}}{2} \right] \to 0 \) as \( n \to \infty \), i.e., when \( l_n \to \infty \) (\( \Leftrightarrow \delta_n \to 0 \) as \( n \to \infty \)). Hence the result is proved.

\[ \square \]

**Proof of Theorem 2.2** Now, suppose that the sequence of Kernel functions is of the type,

\[
K_B^{(n)}(t) = \begin{cases} 
  c_n & \text{if } ||t|| \leq \delta_n \\
  0 & \text{otherwise}
\end{cases},
\]

then in order to find \( \hat{Y}_n \), a multiple linear regression model under normal error assumptions is to be fitted within the \( \delta_n \)-neighbourhood of \( x \). Using similar arguments as the last proof, we have for all \( x \),

\[
\lim_{n \to \infty} ||W_n(x)m_n - W_n(x)X_n(x)\beta(x)|| = 0 \implies \lim_{n \to \infty} ||\mu_n(x) - \beta(x)|| = 0 \text{ in probability.}
\]
Next, we note that

\[
\frac{\hat{\beta}_{j,n}(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} = \frac{\hat{\beta}_{j,n}(x) - \mu_n(x) + \mu_n(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} = \frac{\hat{\beta}_{j,n}(x) - \mu_n(x)}{\sqrt{MS_n(x)c_{jj,n}}} + \frac{\mu_n(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \rightarrow t_{N(x) - d - 1},
\]

as \( n \rightarrow \infty \) for all \( j = 1, \ldots, d \), provided \( N(x) > d + 1 \) for all \( x \). Note that the above convergence holds since \( \frac{\hat{\beta}_{j,n}(x) - \mu_n(x)}{\sqrt{MS_n(x)c_{jj,n}}} \rightarrow t_{N(x) - d - 1} \) as \( n \rightarrow \infty \), and \( \frac{\mu_n(x) - \beta_j(x)}{\sqrt{MS_n(x)c_{jj,n}}} \rightarrow 0 \) as \( n \rightarrow \infty \), using (9). Also recall that \( \beta_j(x) = 0 \) for all \( x \) if \( X_j \) is irrelevant. Hence,

\[
P\left[ \left| \frac{\hat{\beta}_{j,n}(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x) - d - 1, \alpha} \right] \leq \left( P\left[ \left| \frac{\hat{\beta}_{j,n}(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x) - d - 1, \frac{\alpha}{2}} \right] \right)^{l_n},
\]

because \( \left| \frac{\hat{\beta}_{j,n}(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x) - d - 1, \frac{\alpha}{2}} \) for all \( x \) implies that \( \left| \frac{\hat{\beta}_{j,n}(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x) - d - 1, \frac{\alpha}{2}} \) for \( l_n \) number of \( x \). Finally, since \( P\left[ \left| \frac{\hat{\beta}_{j,n}(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x) - d - 1, \frac{\alpha}{2}} \right] \in (0, 1) \) and \( l_n \rightarrow \infty \) as \( n \rightarrow \infty \), we have \( P\left[ \left| \frac{\hat{\beta}_{j,n}(x)}{\sqrt{MS_n(x)c_{jj,n}}} \right| > t_{N(x) - d - 1, \frac{\alpha}{2}} \right] \rightarrow 0 \) as \( n \rightarrow \infty \), and for all \( x \). This completes the proof. \( \square \)
6.2 Tables

**Simulation Study Table:** Corresponding to Section 3

Table 1: Table showing number of times out of 100, a particular variable is selected when simulation study is carried out 100 times for Examples 1, 2, 3 and 4. (see Section 3 for details on models considered)

| Example: 1          | (a) $\sigma$ Known | (b) $\sigma$ Unknown |
|---------------------|--------------------|----------------------|
| $x_1$               | 5                  | 0                    |
| $x_2$               | 100                | 100                  |
| $x_3$               | 100                | 97                   |
| $x_4$               | 100                | 99                   |
| Rejected            | Selected           | Selected             |
| Selected            | Selected           | Selected             |

| Example: 2          | (a) $\sigma$ Known | (b) $\sigma$ Unknown |
|---------------------|--------------------|----------------------|
| $x_1$               | 99                 | 95                   |
| $x_2$               | 0                  | 0                    |
| $x_3$               | 100                | 100                  |
| $x_4$               | 100                | 100                  |
| Selected            | Rejected           | Selected             |
| Selected            | Selected           | Selected             |

| Example: 3          | (a) $\sigma$ Known | (b) $\sigma$ Unknown |
|---------------------|--------------------|----------------------|
| $x_1$               | 100                | 100                  |
| $x_2$               | 100                | 100                  |
| $x_3$               | 0                  | 0                    |
| $x_4$               | 0                  | 0                    |
| Selected            | Selected           | Selected             |
| Selected            | Selected           | Selected             |

| Example: 4          | (a) $\sigma$ Known | (b) $\sigma$ Unknown |
|---------------------|--------------------|----------------------|
| $x_1$               | 100                | 99                   |
| $x_2$               | 100                | 100                  |
| $x_3$               | 0                  | 0                    |
| $x_4$               | 97                 | 100                  |
| Selected            | Selected           | Selected             |
| Selected            | Selected           | Selected             |

**Real Data Analysis Table:** Corresponding to Section 4
Table 2: Table showing number of times out of 100, a particular variable is selected when real data analysis is carried out 100 times.

| Airfoil Self-Noise Data Set |  |
|-----------------------------|--|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |  |
| 100 | 100 | 93 | 98 | 7 |  |
| Selected | Selected | Selected | Selected | Rejected |  |

| Concrete Compressive Strength Data Set |  |
|----------------------------------------|--|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |  |
| 100 | 1 | 94 | 96 | 99 | 100 | 100 | 2 |  |
| Selected | Rejected | Selected | Selected | Selected | Selected | Selected | Selected | Rejected |  |

| Istanbul Stock Exchange Data Set |  |
|----------------------------------|--|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |  |
| 73 | 100 | 99 | 100 | 100 | 100 | 100 |  |
| Undecided | Selected | Selected | Selected | Selected | Selected | Selected | Selected |  |

| Profit Rates and market Structure Data Set |  |
|---------------------------------------------|--|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ |  |
| 100 | 40 | 2 | 38 | 74 | 2 | 78 | 87 | 71 |  |
| Selected | Undecided | Rejected | Undecided | Undecided | Rejected | Undecided | Undecided | Undecided |  |
| Undecided |  |

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