Exceptional Prime Number Twins, Triplets and Multiplets

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Abstract

A classification of twin primes implies special twin primes. When applied to triplets, it yields exceptional prime number triplets. These generalize yielding exceptional prime number multiplets.

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1 Introduction

Prime numbers have long fascinated people. Much is known about ordinary primes \([\mathbb{P}]\), much less about prime twins and mostly from sieve methods, and almost nothing about longer prime number multiplets. If primes and their multiplets are considered as the elements of the integers, then the aspects to be discussed in the following might be seen as part of a “periodic table” for them.

When dealing with prime number twins, triplets and/or multiplets, it is standard practice to ignore as trivial the prime pairs
(2, p) of odd distance $p - 2$ with $p$ any odd prime. In the following prime number multiplets will consist of odd primes only.

**Definition 1** A generalized twin prime consists of a pair of primes $p_i, p_f$ at distance $p_f - p_i = 2D \geq 2$. Ordinary twins are those at distance $2D = 2$.

## 2 A classification of twin primes

A triplet $p_i, p_m = p_i + 2d_1, p_f = p_m + 2d_2$ of prime numbers with $p_i < p_m < p_f$ is characterized by two distances $d_1, d_2$. Each triplet consists of three generalized twin primes $(p_i, p_m), (p_m, p_f), (p_i, p_f)$. Empirical laws governing triplets therefore are intimately tied to those of the generalized twin primes.

There is a basic parametrization of twin primes which generalizes to all prime number multiplets[2],[3].

**Theorem 2** Let $2D$ be the distance between two odd prime numbers $p_i, p_f$ of the pair, $D$ a natural number. There are three mutually exclusive classes of generalized twin primes.

- **I** : $p_i = 2a - D$, $p_f = 2a + D$, $D$ odd;
- **II** : $p_i = 3(2a - 1) - D$, $p_f = 3(2a - 1) + D$, $2|D$, $3 \nmid D$;
- **III** : $p_i = 2a + 1 - D$, $p_f = 2a + 1 + D$, $6|D$, $D \geq 6$,

where $a$ is the running integer variable. Values of $a$ for which a prime pair of distance $2D$ is reached are unpredictable (called arithmetic chaos).

Only in class II are there special twins $3, 3 + 2D$ with a median $3 + D$ not of the form $3(2a - 1)$.

Each of these three classes contains infinitely many (possibly empty) subsets of prime pairs at various even distances.

**Proof.** Let us first consider the case of odd $D$. Then $p_i = 2a - D$ for some positive integer $a$ and, therefore, $p_f = p_i + 2D = 2a + D$. The median $2a$ is the running integer variable of this class I.
For even $D$ with $D$ not divisible by 3 let $p_i = 2n + 1 - D$, so $p_f = 2n+1+D$ for an appropriate integer $n$. Let $p_i \neq 3$, excluding a possible first pair with $p_i = 3$ as special. Since one of three odd natural numbers at distance $D$ from each other is divisible by 3, the median $2n + 1$ must be so, hence $2n + 1 = 3(2a - 1)$ for an appropriate integer $a \geq 2$. Thus, the median $3(2a - 1)$ of the pair $3(2a-1)\pm D$ is again a linear function of a running integer variable $a$. These prime number pairs constitute the 2nd class $II$.

This argument is not valid for prime number pairs with $6|D$, but these can obviously be parametrized as $2a + 1 \pm 6d$, $D = 6d$. They comprise the 3rd and last class $III$ of generalized twins. Obviously these three classes are mutually exclusive and complete.

**Example 3** Ordinary twins $2a \pm 1$ for $a = 2, 3, 6, 9, 15, \ldots$ have $D = 1$ and are in class $I$. For $D = 3$, the twins $2a \pm 3$ occur for $a = 4, 5, 7, 8, 10, \ldots$. For $D = 5$, the twins $2a \pm 5$ are at $a = 4, 6, 9, 12, 18, \ldots$.

For $D = 2$, the twins $3(2a - 1) \pm 2$ for $a = 2, 3, 4, 7, 8, \ldots$ are in class $II$.

For $D = 6$, the twins $2a + 1 \pm 6$ for $a = 5, 6, 8, 11, 12, \ldots$ are in class $III$.

**Example 4** Special twins are the following prime number pairs $5 \pm 2 = (3, 7); 7 \pm 4 = (3, 11); 11 \pm 8 = (3, 19);$ etc.

How many special prime pairs are there? Since $3 + 2D$ forms an arithmetic progression with greatest common divisor $(3, 2D) = 1$, Dirichlet’s theorem [1] on arithmetic progressions says there are infinitely many of them.

Theor. 2 tells us that all triplets are in precisely one of nine classes $(I, I), (I, II), (I, III), (II, I), (II, II), (II, III), (III, I), (III, II), (III, III)$ which are labeled according to the class of their prime pairs. Quartets are distributed over $3^3$ classes, $n$–tuples over $3^{n-1}$ classes.

It is well known that, except for the first pair 3, 5, ordinary twins all have the form $(6m - 1, 6m + 1)$ for some natural number
They belong to class I. Using Theor. 2, a second classification of generalized twins involving arithmetic progressions of conductor 6 as their regular feature may be obtained [2].

Example 5.

Prime pairs at distance $2D = 4$ are in class II and of the form $6m + 1, 6(m + 1) - 1$ for $m = 1, 2, 3, \ldots$ except for the singlet $3, 7$. At distance $2D = 6$ they are in class I and have the form $6m - 1, 6(m + 1) - 1$ for $m = 1, 2, 3, \ldots$ that are intertwined with $6m + 1, 6(m + 1) + 1$ for $m = 1, 2, 5, \ldots$. At distance $2D = 12$ they are in class III and of the form $6m - 1, 6(m + 2) - 1$ for $m = 1, 3, 5, \ldots$ intertwined with $6m + 1, 6(m + 2) + 1$ for $m = 1, 3, 5, \ldots$.

Clearly, the rules governing the form $6m \pm 1, 6m + b$ depend on the arithmetic of $D$ and $a$.

3 Exceptional prime number triplets

Let us start with some prime number triplet examples.

Example 6. The triplet $3, 5, 7$ is the only one at distances $[2, 2]$; the triplet $3, 7, 11$ is the only one at distances $[4, 4]$, and $3, 11, 19$ is the only one at distances $[8, 8]$, and $3, 13, 23$ is the only at $[10, 10]$, etc. Let’s call such triplets at equal distances $[2D, 2D]$ exceptional or singlets.

There are other classes of singlets. At distances $[2, 8]$, the prime triplet $3, 5, 13$ is the only one at distances $[2d_1 = 2, 2d_2 = 8]$ and likewise is $3, 11, 13$ the only one at distances $[8, 2]$.

Rules for singlets or exceptions among generalized triplet primes are the following[2].

Theorem 7 (i) There is at most one generalized prime triplet with distances $[2D, 2D]$ for $D = 1, 2, 4, 5, \ldots$ and $3 \nmid D$.

(ii) When the distances are $[2d_1, 2d_2]$ with $3|d_2 - d_1$, and $3 \nmid d_1$, there is at most one prime triplet $p_l = 3, p_m = 3 + 2d_1, p_f = 3 + 2d_1 + 2d_2$ for appropriate integers $d_1, d_2$. 


Proof. (i) Because one of three odd numbers in a row at distances \([2D, 2D]\), with \(D\) not divisible by 3, is divisible by 3, such a triplet must start with 3. The argument fails when \(3|D\).

(ii) Of \(p_i, p_m \equiv p_i + 2d_1 \pmod{3}, p_f \equiv p_i + 4d_1 \pmod{3}\) at least one is divisible by 3, which must be \(p_i\). ∗

Naturally, the question arises: Are there infinitely many such singlets, i.e. exceptional triplets? Dirichlet’s theorem does not answer this one.

4 Exceptional prime number multiplets

It is obvious that induced special multiplets come about by adding a prime \(p_f\) to a special twin \(3, 3 + 2D = p_m\) in class \(II\) with \(p_m - D = 3 + D \neq 3(2a - 1)\). This yields an induced special triplet \(3, p_m = 3 + 2D, p_f\). Adding another prime generates an induced special quartet, etc.

Example 8 There are no prime multiplets at equal distance \(2D, 3 \not| D\) except triplets, because no other primes can be added at that distance, obviously. But there are quartets at equal distance \(2D, 3|D\). They are in class \((I, I, I)\): 5, 11, 17, 23; 41, 47, 53, 59; 61, 67, 73, 79; 251, 257, 263, 269; 641, 647, 653, 659; in class \(III, III, III\) at equal distance \(D\) with \(6|D\), etc. (Probably there is an infinite number of them.)

Thus, there are no exceptional quartets at equal distance. Likewise, there are no exceptional 6–tuples or \(q–tuples at equal distance, when \(q\) has more than one prime divisor.

Example 9 The exceptional quintet at equal distance is 5, 11, 17, 23, 29. There are no other such quintets, because one of 5 odd numbers at distance \(2D, 3|D, 5 \not| D\) is divisible by 5.

For 7 the exceptional septet is 7, 157, 307, 457, 607, 757, 907, at a surprisingly large distance.

For the primes 11, 13, 19 there are 6–tuples 11, 71, 131, 191, 251, 311; 11, 491, 971, 1451, 1931, 2411; 13, 103, 193, 283, 373, 463;
13, 223, 433, 643, 853, 1063; 13, 3673, 7333, 10993, 14653, 18313; 19, 1669, 3319, 4969, 6619, 8269 and a 7−tuple 17, 2957, 5897, 8837, 11777, 14717, 17657 for 17, but the exceptional 11−, 13−, 17−, 19−tuples are still at large, if they exist.

These are special cases of the following rule for exceptional multiplets.

**Corollary 10** For any prime \( p \geq 3 \) there is at most an exceptional \( p− \)tuple \( p, p+2D, \ldots, p+2(p−1)D \) at equal distance \( 2D, 3|D, p \not| D \).

**Proof.** A \( p− \)tuple composed of primes is exceptional, because one of any \( p \) odd numbers at equal distance \( 2D, 3|D, p \not| D \) is divisible by \( p \). \( \Diamond \)

When \( q \) has several prime divisors, there are no new exceptional \( q− \)tuples at equal distance.

Next we generalize item (ii) of Theor. 7 to quartets and subsequently higher multiplets.

Exceptional quartets with \( d_1 \equiv d_2 \pmod{3}, d_1 \not\equiv 0 \pmod{3}, d_3 \equiv 0 \pmod{3} \) are possible that are induced by triplets, etc.

**Corollary 11** New exceptional quartets at distances \([2d_1, 2d_2, 2d_3]\) when \( d_2 \equiv d_3 \pmod{3}, d_2 \not\equiv 0 \pmod{3}, d_1 \equiv 0 \pmod{3} \) and \( d_1 \equiv d_3 \pmod{3}, d_1 \not\equiv 0 \pmod{3}, d_2 \equiv 0 \pmod{3} \).

**Proof.** This is clear from the proof of (ii) in Theor. 7. \( \Diamond \) 

The generalization to higher multiplets is straightforward and left to the reader.

We conclude with an open question. Apart from exceptional multiplets, do all other multiplets repeat their pattern of given differences infinitely often?

**References**

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[3] H. J. Weber, *Generalized Twin Prime Formulas*, Global J. of Pure and Applied Math. 6(1) (2010) 101-116.