Numerical Analysis of History-dependent Variational-hemivariational Inequalities

Shufen Wang∗, Wei Xu†, Weimin Han‡, Wenbin Chen§

Abstract In this paper, numerical analysis is carried out for a class of history-dependent variational-hemivariational inequalities arising in contact problems. Three different numerical treatments for temporal discretization are proposed to approximate the continuous model. Fixed-point iteration algorithms are employed to implement the implicit scheme and the convergence is proved with a convergence rate independent of the time step-size and mesh grid-size. A special temporal discretization is introduced for the history-dependent operator, leading to numerical schemes for which the unique solvability and error bounds for the temporally discrete systems can be proved without any restriction on the time step-size. As for spatial approximation, the finite element method is applied and an optimal order error estimate for the linear element solutions is provided under appropriate regularity assumptions. Numerical examples are presented to illustrate the theoretical results.

Keywords Variational-hemivariational inequality, history-dependent operator, fixed-point iteration, optimal order error estimate, contact mechanics

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1 Introduction

The theory of variational and hemivariational inequalities plays an important role in the study of nonlinear problems arising in Contact Mechanics, Physics, Economics and Engineering. It is generally agreed that interest in variational inequalities started with a contact problem posed by Signorini in 1930s. The mathematical theory of variational inequalities relies on the properties of monotonicity, convexity and the subdifferential of a convex function. Existence and uniqueness results can be found in [15, 3, 17]. In terms of the numerical analysis for variational inequalities, the readers are referred to, e.g., [8, 7, 15]. Hemivariational inequalities as a useful generalization of variational inequalities were introduced in early 1980s by Panagiotopoulos ([22]). For hemivariational inequalities, the notion of the subdifferential of in the sense of Clarke

∗School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
†Tongji Zhejiang College, Jiaxing, Zhejiang 314051, China.
‡Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA.
§School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, China.
(5 6), defined for locally Lipschitz function, plays an important role. Mathematical theory of hemivariational inequalities is documented in several research monographs, e.g., [23, 21, 4, 19, 27]. A comprehensive reference on the numerical solution of hemivariational inequalities is [14] where the finite element method is applied to solve hemivariational inequalities, convergence of the numerical solution is discussed, and solution algorithms are proposed and tested. More recently, there has been extensive research effort on optimal order error estimation and general convergence analysis of numerical solutions for hemivariational inequalities, e.g., [10, 2, 12, 13, 9], and the survey paper [11].

Variational-hemivariational inequalities are a particular family of hemivariational inequalities, having a special structure that include both convex and nonconvex functionals. Such inequalities arise naturally in mathematical models for many contact problems, see [27] and the references therein. A class of history-dependent variational-hemivariational inequalities with convex constraint is studied in [26]. The novel structure of the inequalities involves a history-dependent operator, unilateral constraint and two nondifferential functions, one of which is convex and the other may be non-convex. Existence, uniqueness and continuous dependence results are shown on the inequalities, and are applied to the study of a quasistatic frictionless contact problem. Numerical approximations of the history-dependent variational-hemivariational inequalities are the topic of [28], where the second order accuracy for temporal discretization is achieved by using the trapezoidal rule to approximate the history-dependent term. The spatial discretization is done using the linear finite element and an optimal order error estimate is proved. Note that for the numerical method studied in [28], a restriction on the time step-size is needed to ensure the unique solvability of the numerical solution. In this paper, we develop new numerical methods to solve the history-dependent variational-hemivariational inequalities with the property that no restriction on the time step-size is needed for the unique solvability of the numerical solution. Specifically, we use a partial trapezoidal rule to approximate the history-dependent operator, i.e., we modify the trapezoidal rule by applying the left-point rectangular rule for the sub-integral over the last time sub-interval. Consequently, the history dependent term is treated explicitly without loss of accuracy. This explicit treatment of the history dependent term eliminates the need for a restriction on the time step-size. Although the explicit treatment is given in history dependent term, other implicit terms in the numerical scheme remain. We provide a fixed-point iterative algorithm to implement the implicit scheme and prove convergence of the iterative scheme, with a convergence rate independent of the time step-size and the mesh grid-size. In addition, we propose two more schemes to solve the history-dependent variational-hemivariational inequalities. One is of first order and the other is of second order with a slightly stringent small condition compared to that of the other two schemes. For all the three schemes, optimal order error estimates with linear finite elements for spatial approximation are shown.

The rest of the paper is organized as follows. In Section 2, we review some preliminary material on functional analysis and present the history-dependent variational-hemivariational inequality problem. In Section 3, we propose three temporally semi-discrete schemes to approximate the continuous problem and error estimates are es-
tablished. The corresponding fully discrete schemes are provided in Section 4 and the error estimates are derived for the discrete problems with or without convex constraints. To implement the second order implicit scheme, in Section 5 we describe a fixed-point iterative process and prove that the iteration converges linearly with a convergence rate independent of the time step size and mesh grid size. Then in Section 6 we apply the theoretical results developed in the previous sections in the numerical solution of a viscoelastic contact problem and obtain an optimal order error estimate for the linear finite element solutions under appropriate solution regularity assumptions. In Section 7 we report results from simulation tests, focusing on the numerical evidence of the convergence orders.

2 Preliminaries

In this section we recall some notation, definitions and preliminary materials. Then we present a class of history-dependent variational-hemivariational inequalities introduced in [26].

For normed spaces $X$ and $X_j$, let $X^*$ and $X_j^*$ be their topological duals, and write $\|\cdot\|_X$, $\|\cdot\|_{X_j}$, $\|\cdot\|_{X^*}$ and $\|\cdot\|_{X_j^*}$ for their norms. The duality pairing between $X$ and $X^*$, $\langle \cdot, \cdot \rangle_{X \times X^*}$, is usually simply written as $\langle \cdot, \cdot \rangle$. Similarly, the duality pairing between $X_j^*$ and $X_j$, $\langle \cdot, \cdot \rangle_{X_j^* \times X_j}$, is usually written as $\langle \cdot, \cdot \rangle_{X_j}$.

For a convex function $\phi : X \to \mathbb{R} \cup \{+\infty\}$, the subset $\partial \phi(x) \subset X^*$ of $X^*$ is called the subdifferential ([24]) of $\phi$. If $\partial \phi(x)$ is non-empty, any element $x^* \in \partial \phi(x)$ is called a subgradient of $\phi$ at $x$. Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of $\phi$ at $x$ in the direction $v \in X$ is defined by (cf. [6])

$$\phi^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{\phi(y + \lambda v) - \phi(y)}{\lambda}.$$ 

The generalized gradient (subdifferential) of $\phi$ at $x$ is a subset of the dual space $X^*$ given by

$$\partial \phi(x) = \{x^* \in X^* | \phi^0(x; v) \geq \langle x^*, v \rangle_{X^* \times X} \forall v \in X\}.$$ 

An operator $A : X \to X^*$ is pseudomonotone ([19]) if it is bounded and $u_n \to u$ weakly in $X$ together with $\limsup_n \langle Au_n, u_n - u \rangle_{X^* \times X} \leq 0$ imply

$$\langle Au, u - v \rangle_{X^* \times X} \leq \liminf_n \langle Au_n, u_n - v \rangle_{X^* \times X} \forall v \in X.$$ 

Next we turn to some preliminary materials on function spaces and related operators. Following the standard notation, we denote by $\mathbb{N}$ the set of positive integers, $\mathbb{R}_+ = [0, +\infty)$ the set of nonnegative real numbers, $C(\mathbb{R}_+, X)$ and $C^1(\mathbb{R}_+, X)$ the spaces of continuous and continuously differentiable functions from $\mathbb{R}_+$ to $X$, respectively. It is well known that if $X$ is a Banach space, $C(\mathbb{R}_+, X)$ can be organized in a canonical way as a Fréchet space, i.e., it is a complete metric space in which the
corresponding topology is induced by a countable family of seminorms. Furthermore, 
\( x_k \to x \) in \( C(\mathbb{R}^+; X) \) as \( k \to \infty \) if and only if \( \max_{r \in [0,n]} \| x_k(r) - x(r) \|_X \to 0 \) as \( k \to \infty \) for all \( n \in \mathbb{N} \).

Let there be given two normed spaces \( X \) and \( Y \). Following \([25]\), an operator \( S : C(\mathbb{R}^+; X) \to C(\mathbb{R}^+; Y) \) is called history-dependent if for any \( n \in \mathbb{N} \), there exists an \( s_n > 0 \) such that for all \( t \in [0,n] \),

\[
\| (Su_1)(t) - (Su_2)(t) \|_Y \leq s_n \int_0^t \| u_1(s) - u_2(s) \|_X ds \quad \forall u_1, u_2 \in C(\mathbb{R}^+; X). \tag{2.1}
\]

Now we are in a position to introduce the variational-hemivariational inequalities. Let \( X, X_j, Y \) be normed spaces and \( K \subset X \). Given operators \( A : X \to X^* \), \( S : C(\mathbb{R}^+; X) \to C(\mathbb{R}^+; Y) \), \( \gamma_j : X \to X_j \) and functions \( \varphi : Y \times K \times K \to \mathbb{R} \), \( j : X_j \to \mathbb{R} \), we consider the following problem \((26, 28)\).

**PROBLEM 1** Find \( u \in C(\mathbb{R}^+; K) \) such that for all \( t \in \mathbb{R}^+ \),

\[
\langle Au(t), v - u(t) \rangle + \varphi((Su)(t), u(t), v) - \varphi((Su)(t), u(t), u(t)) + j^0(\gamma_j u(t); \gamma_j v - \gamma_j u(t)) \geq \langle f(t), v - u(t) \rangle \quad \forall v \in K. \tag{2.2}
\]

In the study of Problem 1, the following hypotheses are adopted \((26, 28)\):

- \( X \) is a reflexive Banach space, \( K \) is a closed and convex subset of \( X \) with \( 0 \in K \).
- \( X_j \) is a Banach space, \( \gamma_j \in \mathcal{L}(X; X_j) \), there exists \( c_j > 0 \) such that \( \| \gamma_j v \|_{X_j} \leq c_j \| v \|_X \quad \forall v \in X \).
- \( A : X \to X^* \) is an operator such that
  \[ (a) \text{ } A \text{ is Lipschitz continuous with a Lipschitz constant } L_A > 0. 
  \]
  \[ (b) \text{ } A \text{ is strongly monotone, i.e., there exists } m_A > 0 \text{ such that } \langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_A \| v_1 - v_2 \|_X^2 \quad \forall v_1, v_2 \in X. \] \tag{2.3}

- \( \varphi : Y \times K \times K \to \mathbb{R} \) is a function such that
  \[ (a) \text{ } \varphi(y, u, \cdot) : K \to \mathbb{R} \text{ is convex and l.s.c on } K, \forall y \in Y, \forall u \in K. 
  \]
  \[ (b) \text{ } \varphi \text{ there exists } \alpha_\varphi > 0 \text{ and } \beta_\varphi > 0 \text{ such that } \]
  \[ \langle \varphi(y_1, u_1, v_2) - \varphi(y_1, u_1, v_1) + \varphi(y_2, u_2, v_1) - \varphi(y_2, u_2, v_2) \rangle \]
  \[ \leq \alpha_\varphi \| u_1 - u_2 \|_X \| v_1 - v_2 \|_X + \beta_\varphi \| y_1 - y_2 \|_Y \| v_1 - v_2 \|_X 
  \quad \forall y_1, y_2 \in Y, \forall u_1, u_2, v_1, v_2 \in K. \] \tag{2.4}

- \( S : C(\mathbb{R}^+; X) \to C(\mathbb{R}^+; Y) \) is a history-dependent operator.
- \( j : X_j \to \mathbb{R} \) is a function such that
  \[ (a) \text{ } j \text{ is locally Lipschitz. } 
  \]
  \[ (b) \text{ } \| \partial j(z) \|_{X_j^*} \leq c_0 + c_1 \| z \|_{X_j} \quad \forall z \in X_j \text{ with } c_0, c_1 \geq 0. \] \tag{2.5}

- \( \alpha \) there exists \( \alpha_j > 0 \) such that
  \[ j^0(z_2 - z_1) \leq \alpha_j \| z_1 - z_2 \|_{X_j}^2 \quad \forall z_1, z_2 \in X_j. \] \tag{2.6}

- \( f \in C(\mathbb{R}^+; X^*) \).
- \( \alpha_\varphi + \alpha_j c_j^2 < m_A. \) \tag{2.7}
The space $X_j$ is introduced for convenience of error estimation for the discrete problems. For a specific contact problem, $X_j$ can be the space of square integrable functions over the contact boundary and $\gamma_j : X \to X_j$ is the corresponding trace operator. For a locally Lipschitz function $j$, (2.8) (c) is equivalent to the following relaxed monotonicity condition

$$\langle \partial j(z_1) - \partial j(z_2), z_1 - z_2 \rangle \geq -\alpha_j \|z_1 - z_2\|^2_{X_j} \quad \forall z_1, z_2 \in X_j.$$ 

The unique solvability of Problem 1 has been shown in [28] under the conditions (2.3)–(2.10). We will consider the following form of the operator $S$:

$$Sv(t) = R \left( \int_0^t q(t,s)v(s)ds + a_S \right) \quad \forall v \in C(\mathbb{R}_+; X), \forall t \in \mathbb{R}_+,$$  

(2.11)

where $R \in \mathcal{L}(X; Y)$, $q \in C(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))$, $a_S \in X$. It can be shown that the operator $S$ given by (2.11) is a history-dependent operator.

### 3 Temporally Semi-Discrete Approximations

In [28], a second-order numerical scheme is provided to approximate the continuous Problem 1 with a restriction on the time step-size. In this section, we handle the history-dependent term in a different manner, and propose three temporally discrete schemes for solving Problem 1 without any restriction on the time step-size. Moreover, we derive the corresponding convergence results. Below we use $C$ to represent a positive constant independent of time step-size and mesh grid-size. We use the standard notation for Sobolev spaces (cf. [1]).

For a fixed $T \in \mathbb{R}_+$, we split the time interval $I = [0, T]$ by uniform partitions. Given a positive integer $N$, let $k = T/N$ be the time step-size, and denote by $t_n = nk$, $0 \leq n \leq N$, the nodes. We comment that all the discussions below can be extended to the case with non-uniform partitions of the time interval. For a continuous function $v$ of the temporal variable $t$, we write $v_j = v(t_j)$, $0 \leq j \leq N$. For a discretization of the history dependent operator $S$ in (2.11), we employ a modified trapezoidal rule to approximate the integral $\int_0^{t_n} q(t,s)v(s)ds$ in the sense that on the last sub-interval $[t_{n-1}, t_n]$, the left-point rectangular rule is applied. Recall the trapezoidal rule

$$\int_0^{t_n} Z(s)ds \approx \frac{k}{2}Z(t_0) + k \sum_{j=1}^{n-1} Z(t_j) + \frac{k}{2}Z(t_n).$$  

(3.1)

The approximation of $S_n := S(t_n)$ can be defined as follows:

$$S_{n,L}^k v := R \left( \frac{k}{2}q(t_n, t_0)v_0 + k \sum_{j=1}^{n-1} q(t_n, t_j)v_j + \frac{k}{2}q(t_n, t_{n-1})v_{n-1} + a_S \right).$$  

(3.2)
Using arguments similar to that in [16, Section 3], for \( v \in W_{loc}^{1,\infty}(\mathbb{R}_+; X) \) and \( q \in C^1(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X)) \), we have

\[
\|S_{n,L}^k v - S_n v\| \leq C k \|v\|_{W^{1,\infty}(t; X)}, \tag{3.3}
\]

and for \( v \in W_{loc}^{2,\infty}(\mathbb{R}_+; X) \) and \( q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X)) \),

\[
\|S_{n,L}^k v - S_n v\| \leq C k^2 \|v\|_{W^{2,\infty}(t; X)}. \tag{3.4}
\]

**Remark 2** The choice of the operator \( S_{n,L}^k \) used to approximate \( S_n \) is not unique. For example, we may choose

\[
\tilde{S}_{n}^k v := R \left( k \frac{1}{2} q(t_n, t_0) v_0 + k \sum_{j=1}^{n-1} q(t_n, t_j) v_j + k \frac{1}{2} (2q(t_n, t_{n-1}) v_{n-1} - q(t_n, t_{n-2}) v_{n-2}) + a_S \right)
\]

which defines another second-order accurate approximation of \( S_n \), or choose

\[
\tilde{S}_{n}^k v := R \left( k \sum_{j=0}^{n-1} q(t_n, t_j) v_j + a_S \right)
\]

which is a first-order accurate approximation.

We note that the following weak formulation is equivalent to Problem 1

**Problem 3** Find \( u \in C(\mathbb{R}_+; K) \) such that for all \( t \in \mathbb{R}_+ \)

\[
\langle Au(t), v - u(t) \rangle + \varphi((Su)(t), u(t), v) - \varphi((Su)(t), u(t), u(t)) + j^0(\gamma_j u(t); \gamma_j v - \gamma_j u(t)) + \langle j_c(\gamma_j u(t)), \gamma_j v - \gamma_j u(t) \rangle_{X_j} \geq \langle f(t), v - u(t) \rangle + \langle j_c(\gamma_j u(t)), \gamma_j v - \gamma_j u(t) \rangle_{X_j} \quad \forall \ v \in K. \tag{3.5}
\]

In [28], \( j_c \) is chosen as the differential of a quadratic function \( \frac{a}{2} \|u\|_{X_j}^2 \). In this paper, we discuss about \( j_c \) in a more general framework. Assume

\[
\begin{cases}
  j_c : X_j \to X_j^* \text{ is a linear operator such that} \\
  (a) \|j_c(z)\|_{X_j} \leq \alpha_j \|z\|_{X_j} \quad \forall \ z \in X_j; \\
  (b) \langle j_c(z), z \rangle_{X_j} \geq \alpha_j \|z\|_{X_j}^2 \quad \forall \ z \in X_j.
\end{cases} \tag{3.6}
\]

The operator \( j_c \) can be regarded as a convexification of \( j^0 \) in the sense that

\[
\begin{align*}
  j^0(z_1; z_2 - z_1) + j^0(z_2; z_1 - z_2) &+ \langle j_c(z_1), z_2 - z_1 \rangle_{X_j} + \langle j_c(z_2), z_1 - z_2 \rangle_{X_j} \\
  &\leq \alpha_j \|z_1 - z_2\|_{X_j}^2 + \langle j_c(z_1 - z_2), z_2 - z_1 \rangle_{X_j} \tag{3.7} \\
  &\leq 0,
\end{align*}
\]

where the last equality follows from [3.6] (b).
3.1 A first-order temporally semi-discrete scheme

The first order temporally semi-discrete scheme for Problem 1 is the following.

**Problem 4** Find a discrete solution \( u^k := \{ u^k_n \}_{n=0}^N \subset K \) such that

\[
\langle Au_n^k, v - u_n^k \rangle + \varphi(S_{n,L}^k u^k_n, u^k_{n-1}, v) - \varphi(S_{n,L}^k u^k_n, u^k_{n-1}, u^k_n) \\
+ j^0(\gamma_j u^k_n, \gamma_j v - \gamma_j u^k_n) + \langle j_c(\gamma_j u^k_n), \gamma_j v - \gamma_j u^k_n \rangle_{X_j} \\
\geq \langle f_n, v - u^k_n \rangle + \langle j_c(u^k_{n-1}), \gamma_j v - \gamma_j u^k_n \rangle_{X_j} \quad \forall v \in K.
\]

**Remark 5** Note that the approximation \( S_{n,L}^k u^k \) for the history dependent operator does not involve information on the current numerical solution \( u^k_n \), and the second argument of \( \varphi \) is explicitly treated, which is important for numerical implementation. The function \( \varphi \) appeared in (3.8) is convex with respect to the unknown variable (the third argument) according to assumption (2.6). Moreover, \( j_c \) plays the role to convexify the function \( j \), i.e., \( j^0(\gamma_j u^k_n; \gamma_j v - \gamma_j u^k_n) + \langle j_c(\gamma_j u^k_n), \gamma_j v - \gamma_j u^k_n \rangle_{X_j} \) becomes the directional derivative of a convex function. Therefore, convex optimization techniques could be applied to solve the inequality (3.8) and the unique solvability of Problem 4 can be obtained without the constraint (2.10) by applying results on elliptic variational-hemivariational inequality (20). Specifically, the operator \( T_1 \) defined by \( T_1 v = Av + \partial_j(v) + j_c(v) \) is bounded, coercive and pseudomonotone, the function \( \varphi(v) \) can be extended to \( X \), denoted as \( \tilde{\varphi}(v) \) with \( \tilde{\varphi}(v) = +\infty \) for any \( v \in X \setminus K \). In this way, the operator \( T_2 \) with \( T_2 v = \partial \tilde{\varphi}(v) \) is maximal monotone. Hence, Problem 4 has a unique solution.

**Remark 6** The choice of \( j_c \) is not unique. The critical point is that \( j_c \) should be “convex” enough to have the non-convexity of \( j^0 \) under control, i.e., the inequality (3.7) is required. On the other hand, we can split \( j^0 \) in another way, e.g.,

\[
\langle j_c(\gamma_j u^k_n), \gamma_j v - \gamma_j u^k_n \rangle_{X_j} + \langle j^0(\gamma_j u^k_{n-1}; \gamma_j v - \gamma_j u^k_{n-1}) - \langle j_c(\gamma_j u^k_{n-1}), \gamma_j v - \gamma_j u^k_{n} \rangle_{X_j} \rangle
\]

could be used to approximate \( j^0(\gamma_j u^k_n; \gamma_j v - \gamma_j u^k_n) \). In this way, the inequality (3.8) becomes a convex problem with linear operators, for which efficient numerical algorithms are available.

According to the statement in Remark 5, we have the following unique solvability result for Problem 4.

**Theorem 7** Under the conditions (2.3), (2.9) and (3.6), the semi-discrete Problem 4 is uniquely solvable.

For error estimation, we first introduce some auxiliary techniques.

**Lemma 8** Let \( \{ a_n \} \) be a nonnegative sequence satisfying

\[
a_n \leq b_0 + C_1 k \sum_{j=0}^{n-1} a_j + \theta_1 a_{n-1} + \theta_2 a_{n-2} \quad \forall n \geq 2,
\]
where $a_0, a_1, b_0, \theta_1, \theta_2, C_1$ are nonnegative constants and $0 \leq \theta_1 + \theta_2 < 1$. Then

$$a_n \leq \left( \frac{b_0}{1 - \theta_1 - \theta_2} + \frac{C_1 k (a_0 + a_1)}{1 - \theta_1 - \theta_2} + \theta_1 a_1 + \theta_2 a_0 \right) \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{n-2}. \quad (3.9)$$

**Proof.** For convenience, let

$$\overline{\alpha} := \frac{b_0}{1 - \theta_1 - \theta_2} + \frac{C_1 k (a_0 + a_1)}{1 - \theta_1 - \theta_2} + \theta_1 a_1 + \theta_2 a_0.$$  

We prove the result with an induction. For $n = 2$, we have the following bound:

$$a_2 \leq b_0 + C_1 k (a_1 + a_0) + \theta_1 a_1 + \theta_2 a_0 \leq \overline{\alpha}.$$  

Thus, (3.9) holds for $n = 2$. Assume that for $n \leq m$,

$$a_n \leq \overline{\alpha} \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{n-2}.$$  

Then for $n = m + 1$,

$$a_{m+1} \leq b_0 + C_1 k \sum_{j=0}^{m} a_j + \theta_1 a_m + \theta_2 a_{m-1}$$

$$\leq b_0 + C_1 k \left[ a_0 + a_1 + \overline{\alpha} \sum_{j=2}^{m} \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{j-2} \right]$$

$$+ (\theta_1 + \theta_2) \overline{\alpha} \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{m-2}$$

$$= b_0 + C_1 k (a_0 + a_1) + \overline{\alpha} \cdot C_1 k \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{m-1} - 1 \left( 1 - \theta_1 - \theta_2 \right)$$

$$+ (\theta_1 + \theta_2) \overline{\alpha} \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{m-2}$$

$$\leq \overline{\alpha} \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{m-1} \left( 1 - \theta_1 - \theta_2 \right) + (\theta_1 + \theta_2) \overline{\alpha} \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{m-2}$$

$$\leq \overline{\alpha} \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{m-2} (1 + C_1 k) \leq \overline{\alpha} \left( 1 + \frac{C_1 k}{1 - \theta_1 - \theta_2} \right)^{m-1},$$

where we use the fact that

$$b_0 + C_1 k (a_0 + a_1) - (1 - \theta_1 - \theta_2) \overline{\alpha} \leq 0.$$  

This completes the proof. \qed

**Corollary 9** Assume that $\{a_n\}$ is a nonnegative sequence satisfying

$$a_n \leq b_0 + C_2 k \sum_{j=0}^{n-1} a_j + \theta_1 a_{n-1} \quad \forall \ n \geq 1,$$

where $a_0, b_0, \theta_1$ and $C_2$ are nonnegative constants and $\theta_1 < 1$. Then

$$a_n \leq \left( \frac{b_0}{1 - \theta_1} + \frac{C_2 k}{1 - \theta_1} a_0 + \theta_1 a_0 \right) \left( 1 + \frac{C_2 k}{1 - \theta_1} \right)^{n-1}. \quad (3.10)$$
Lemma 10 Assume $e_0, e_1$ and $e_2$ are nonnegative numbers such that

$$e_0^2 \leq e_1 e_0 + e_2^2, \quad (3.11)$$

then

$$e_0 \leq e_1 + e_2. \quad (3.12)$$

Proof. From (3.11), we have

$$\left(e_0 - \frac{e_1}{2}\right)^2 \leq \frac{e_1^2}{4} + e_2^2 \leq \left(e_1^2 + e_2^2\right). \quad (3.13)$$

Taking the square root of both sides gives (3.12). □

We now turn to an error analysis for Problem 4. For convenience, we denote $\|R\| = \|R\|_{L(X,Y)}$ and $\|q\| = \|q\|_{C(I \times I; L(X))}$. The following smallness condition is needed instead of the original one (2.10):

$$\alpha_\varphi + \alpha c_j^2 < m_A. \quad (3.14)$$

Theorem 11 Assume (2.3)–(2.9), (3.6), (3.14) and the regularity $q \in C^1(I_+ \times I_+; L(X))$, $u \in W^{1,\infty}_{\text{loc}}(I_+; X)$. Then for the semi-discrete solution of Problem 4, the following error bound holds:

$$\max_{n \leq N} \|u_n - u_n^k\|_X \leq C_3 k, \quad (3.15)$$

where $C_3 > 0$ is a constant independent of $k$.

Proof. We take $t = t_n$ in the inequality (2.2) to get

$$\langle Au_n, v - u_n \rangle + \varphi(S_n u, u_n, v) - \varphi(S_n u, u_n, u_n) + j^0(\gamma_j u_n; \gamma_j v - \gamma_j u_n) \geq \langle f_n, v - u_n \rangle \quad \forall v \in K, \quad (3.16)$$

where $S_n u = R(\int_0^{t_n} q(t_n, s) u(s) ds + a s)$. Let $v = u_n^k$ in (3.16),

$$\langle Au_n, u_n^k - u_n \rangle + \varphi(S_n u, u_n, u_n^k) - \varphi(S_n u, u_n, u_n) + j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n) \geq \langle f_n, u_n^k - u_n \rangle. \quad (3.17)$$

Taking $v = u_n$ in (3.8) yields

$$\langle Au_n^k, u_n - u_n^k \rangle + \varphi(S_n^k u_n^k, u_n^k, u_n^k) - \varphi(S_n^k u_n^k, u_n^k, u_n) + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) \geq \langle f_n, u_n - u_n^k \rangle + \langle j_c(\gamma_j u_n^k - \gamma_j u_n^k) x_j \rangle. \quad (3.18)$$
Adding (3.17) to (3.18) and employing the strong monotonicity of $A$, we obtain

$$m_A \| u_n - u_n^k \|^2_X \leq \langle Au_n - Au_n^k, u_n - u_n^k \rangle$$

$$\leq \varphi(S_n u, u_n, u_n) - \varphi(S_n u, u_n, u_n) + \varphi(S_{n,L}^k u^k, u_n^k, u_n^k)$$

$$- \varphi(S_{n,L}^k u^k, u_n^k, u_n^k) + j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n)$$

$$+ j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) + \langle j_e(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle X_j$$

$$- \langle j_e(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle X_j,$$

which is rewritten as

$$m_A \| u_n - u_n^k \|^2_X \leq E_\varphi + E_j + E_{j_e}, \quad (3.19)$$

where

$$E_\varphi = \varphi(S_{n,L}^k u^k, u_n^k, u_n^k) - \varphi(S_{n,L}^k u^k, u_n^k, u_n^k)$$

$$+ \varphi(S_n u, u_n, u_n^k) - \varphi(S_n u, u_n, u_n)$$

$$E_{j_e} = \langle j_e(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle X_j - \langle j_e(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle X_j,$$ (3.21)

$$E_j = j^0(\gamma_j u_n; \gamma_j u_n^k - \gamma_j u_n) + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k)$$

$$+ \langle j_e(\gamma_j u_n), \gamma_j u_n - \gamma_j u_n \rangle X_j + \langle j_e(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle X_j.$$ (3.22)

The term $E_j$ can be bounded by zero from above according to (3.7). Utilizing the regularity of $u$ and the properties of $j_e$ gives

$$E_{j_e} = \langle j_e(\gamma_j u_n - \gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle X_j$$

$$= \langle j_e(\gamma_j u_n - \gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle X_j$$

$$+ \langle j_e(\gamma_j u_n^k - \gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle X_j$$

$$\leq \alpha_c e^2 \| k \| u \| W^{1,\infty}(I,X) \| + \| u_n - u_n^k \| X \| u_n - u_n^k \| X.$$ (3.23)

From (2.6) we can see

$$E_\varphi \leq (\alpha_\varphi \| u_n - u_n^k \| X + \beta_\varphi \| S_n u - S_{n,L}^k u^k \| Y \| u_n - u_n^k \| X$$

$$\leq (k \alpha_\varphi \| u \| W^{1,\infty}(I,X) + \alpha_\varphi \| u_n - u_n^k \| X$$

$$+ \beta_\varphi \| S_n u - S_{n,L}^k u^k \| Y \| u_n - u_n^k \| X.$$ (3.24)

From (3.3), it holds

$$\| S_n u - S_{n,L}^k u^k \| Y \leq \| S_n u - S_{n,L}^k u \| Y + \| S_{n,L}^k u - S_{n,L}^k u^k \| Y$$

$$\leq C k \| u \| W^{1,\infty}(I,X) + \frac{3}{2} k \| R \| q \| \sum_{j=0}^{n-1} \| u_j - u_j^k \| X.$$ (3.25)

From (3.19) and (3.23)–(3.25), we obtain

$$m_A \| u_n - u_n^k \| X \leq C k \| u \| W^{1,\infty}(I,X) + \frac{3}{2} k \beta_\varphi \| R \| q \| \sum_{j=0}^{n-1} \| u_j - u_j^k \| X$$

$$+ (\alpha_\varphi + \alpha_c e^2) \| u_n - u_n^k \| X.$$ (3.26)
By applying Corollary 9

\[ \|u_n - u^n_k\|_X \leq \left( \frac{k C\|\bar{u}\|_{W^{1,\infty}(I;X)}}{m_A - \alpha \varphi - \alpha c c_j} + \left( \frac{\frac{3}{2} k \beta \varphi \|R\|\|q\|}{m_A - \alpha \varphi - \alpha c c_j^2} + \frac{\alpha \varphi + \alpha c c_j^2}{m_A} \right) \|u_0 - u^n_0\|_X \right) \cdot \left( 1 + k \frac{\frac{3}{2} \beta \varphi \|R\|\|q\|}{m_A - \alpha \varphi - \alpha c c_j^2} \right)^{n-1} \]  

(3.27)

Note that when \( t = t_0 = 0 \), the integral of history-dependent operator is zero and there is no temporally discrete error; thus \( \|u_0 - u^n_0\|_X = 0 \). Then,

\[ \|u_n - u^n_k\|_X \leq k C\|\bar{u}\|_{W^{1,\infty}(I;X)} \cdot \left( 1 + k \frac{\frac{3}{2} \beta \varphi \|R\|\|q\|}{m_A - \alpha \varphi - \alpha c c_j^2} \right)^{n-1} \leq C_3 k; \]

where

\[ C_3 = \frac{C\|\bar{u}\|_{W^{1,\infty}(I;X)}}{m_A - \alpha \varphi - \alpha c c_j^2} \cdot \exp \left\{ \frac{\frac{3}{2} \beta \varphi \|R\|\|q\|}{m_A - \alpha \varphi - \alpha c c_j^2} t_n \right\}, \]

and the error bound (3.15) follows. \( \square \)

**Remark 12** The first-order accuracy remains valid if \( \tilde{S}_n^k \) is used to approximate the history-dependent operator \( S \) in the temporally semi-discrete scheme (3.8).

### 3.2 Second-order temporally semi-discrete schemes

In this subsection, we propose and study two second-order schemes to temporally approximate Problem 1. The first scheme is the following.

**Problem 13** Find \( u^n_k := \{u^n_k\}_{n=0}^N \subset K \) such that

\[ \langle Au^n_k, v - u^n_k \rangle + \varphi(S^n_{n,L} u^n_k, u^n_k, v) - \varphi(S^n_{n,L} u^n_k, u^n_k, u^n_k) + j^0(\gamma_j u^n_k; \gamma_j v - \gamma_j u^n_k) \geq \langle f_n, v - u^n_k \rangle \quad \forall \ v \in K. \]  

(3.28)

Note that the history-dependent operator is approximated using available numerical solution values and the current unknown value \( u^n_k \) is not involved. In this way, unlike the numerical scheme studied in [28], the semi-discrete Problem 13 is ensured to have a unique solution regardless of the size of the time step-size using the same Banach fixed-point argument as in [28].

**Theorem 14** Under the conditions (2.3)–(2.10), the semi-discrete Problem 13 has a unique solution.

We turn to the error estimation of Problem 13.
Theorem 15 Assume \((2.3) - (2.10)\) and the regularity \(q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X))\), \(u \in W_{loc}^{2,\infty}(\mathbb{R}_+; X)\). Then for the semi-discrete solution of Problem 13, we have the error bound
\[
\max_{n \leq N} \|u_n - u_{n,k}\|_X \leq C_4 k^2,
\] (3.29)
where \(C_4 > 0\) is a constant independent of \(k\).

Proof. Let \(v = u_n\) in (3.28) to get
\[
\langle Au_n^k, u_n - u_{n,k}^k \rangle + \varphi(S_{n,L}^k u_n^k, u_n^k) - \varphi(S_{n,L}^k u_{n,k}^k, u_{n,k}^k)
+ j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_{n,k}^k) \geq \langle f_n, u_n - u_{n,k}^k \rangle.
\] (3.30)

Add (3.17) to (3.30) and employ the strong monotonicity of \(A\),
\[
m_A \|u_n - u_{n,k}^k\|_X^2 \leq \varphi(S_{n,L}^k u_n^k, u_n^k) - \varphi(S_{n,L}^k u_{n,k}^k, u_{n,k}^k)
+ \varphi(S_n u, u_n^k) - \varphi(S_n u, u_{n,k}^k)
+ j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_{n,k}^k) + j^0(\gamma_j u_{n,k}^k; \gamma_j u_{n,k}^k - \gamma_j u_{n,k})
\leq \alpha^2 \|u_n - u_{n,k}^k\|_X^2 + \beta^2 \|S_n u - S_{n,L}^k u_k\|_Y \|u_n - u_{n,k}^k\|_X
+ \alpha_j c_j^2 \|u_n - u_{n,k}^k\|_X^2.
\] (3.31)

Similar to (3.25) by using (3.4) instead,
\[
\|S_n u - S_{n,L}^k u_k\|_Y \leq C k^2 \|u\|_{W^2,\infty(I;X)} + \frac{3}{2} k \beta \|R\| \|q\| \sum_{j=0}^{n-1} \|u_j - u_{j,k}^k\|_X.
\] (3.32)

Apply (3.32) to (3.31),
\[
\|u_n - u_{n,k}^k\|_X \leq \frac{\beta}{m_A - \alpha^2 - \alpha_j c_j^2} \|S_n u - S_{n,L}^k u_k\|_Y
\leq \frac{C k^2 \|u\|_{W^2,\infty(I;X)} + \frac{3}{2} k \beta \|R\| \|q\| \sum_{j=0}^{n-1} \|u_j - u_{j,k}^k\|_X}{m_A - \alpha^2 - \alpha_j c_j^2} \sum_{j=0}^{n-1} \|u_j - u_{j,k}^k\|_X.
\] (3.33)

Then by Corollary 9
\[
\|u_n - u_{n,k}^k\|_X \leq \left( k^2 C \|u\|_{W^2,\infty(I;X)} + \frac{3}{2} \frac{k \beta \|R\| \|q\| \|u\|_{W^2,\infty(I;X)}}{m_A - \alpha^2 - \alpha_j c_j^2} \|u_0 - u_{0,k}^k\|_X \right) \cdot \left( 1 + k \frac{\frac{3}{2} \beta \|R\| \|q\| \|u\|_{W^2,\infty(I;X)}}{m_A - \alpha^2 - \alpha_j c_j^2} \right)^{n-1}
\leq C_4 k^2,
\] (3.34)

where
\[
C_4 = C \|u\|_{W^2,\infty(I;X)} \cdot \exp \left\{ \frac{3}{2} \frac{k \beta \|R\| \|q\| \|u\|_{W^2,\infty(I;X)}}{m_A - \alpha^2 - \alpha_j c_j^2} \right\}.
\]

Thus the second-order error estimate (3.29) is established. \(\square\)
Remark 16 For the numerical scheme in [28], the history dependent operator is implicitly treated in the sense that its approximation depends on the current unknown solution component. As a result, a restriction for the time step-size of the form \( k < (m_A - \alpha \varphi - \alpha_j c_j^2)/\beta \| R \| \| q \| \) is needed to ensure the unique solvability and for the derivation of the error bound there. In contrast, for our numerical scheme given by Problem 13, we have the unique solvability and error bound for an arbitrary time step-size.

Next we modify (3.8) and give another scheme of second-order.

Problem 17 Find a discrete solution \( u^k := \{ u_n^k \}_{n=0}^N \subset K \) such that

\[
\begin{align*}
&\langle A u_n^k, v - u_n^k \rangle + \varphi(S_n^k u_n^k, 2u_{n-1}^k - u_n^k, v) - \varphi(S_n^k u_n^k, 2u_{n-1}^k - u_n^k, u_n^k) \\
&\quad + j^0(\gamma_j v_n^k, \gamma_j v - \gamma_j u_n^k) + \langle j^c(\gamma_j u_n^k), \gamma_j v - \gamma_j u_n^k \rangle X_j \\
&\geq \langle j^c(2 \gamma_j u_{n-1}^k - \gamma_j u_{n-2}^k), \gamma_j v - \gamma_j u_n^k \rangle X_j + \langle f_n, v - u_n^k \rangle \quad \forall v \in K, n \geq 2,
\end{align*}
\]

and for \( n = 1, \)

\[
\begin{align*}
&\langle A u_1^k, v - u_1^k \rangle + \varphi(S_1^k u_1^k, u_1^k, v) - \varphi(S_1^k u_1^k, u_1^k, u_1^k) \\
&\quad + j^0(\gamma_j u_1^k, \gamma_j v - \gamma_j u_1^k) \geq \langle f_1, v - u_1^k \rangle \quad \forall v \in K.
\end{align*}
\]

The uniqueness and existence results for (3.35) are similar to that of Problem 4. As for (3.36), it can be referred to Problem 13. Then we have the following uniqueness and existence results for Problem 17.

Theorem 18 Assume (2.3)–(2.10) and (3.6). Then Problem 17 has a unique solution \( u^k = \{ u_n^k \}_{n=0}^N \subset K \).

Next we derive an error bound for semi-discrete solution of Problem 17. Meanwhile, a stronger constraint compared with (3.14) is needed, i.e.,

\[
\alpha \varphi + \alpha c_j^2 < m_A/3.
\]

Theorem 19 Assume (2.3)–(2.9), (3.6), (3.37) and the regularity \( q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathcal{L}(X)), u \in W^{2,\infty}_{\text{loc}}(\mathbb{R}_+; X) \). Then for the semi-discrete solution of Problem 17 the following error bound holds:

\[
\max_{n \leq N} \| u_n - u_n^k \|_X \leq C_5 k^2,
\]

where \( C_5 > 0 \) is a constant independent of \( k \).

Proof. For \( n = 1, \) we have a second-order accuracy result for (3.36) by Theorem 15.

\[
\| u_1 - u_1^k \|_X \leq C_4 k^2.
\]
For $n \geq 2$, taking $v = u_n$ in \((3.35)\), we have
\[
\langle Au_n^k, u_n - u_n^k \rangle + \varphi(S_{n,L}^k u_n^k, 2u_{n-1}^k - u_{n-2}^k, u_n) - \varphi(S_{n,L}^k u_n^k, 2u_{n-1}^k - u_{n-2}^k, u_n^k)
+ j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) + \langle j_c(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle_{x_j}.
\]
(3.40)
\[
\geq \langle f_n, u_n - u_n^k \rangle + \langle j_c(2\gamma_j u_{n-1}^k - \gamma_j u_{n-2}^k), \gamma_j u_n - \gamma_j u_n^k \rangle_{x_j}.
\]
Combine \((3.17)\) with \((3.40)\) and use the strong monotonicity of $A$ to obtain
\[
m_A\|u_n - u_n^k\|_X^2 \leq \varphi(S_n u, u_n, u_n^k) - \varphi(S_n u, u_n, u_n)
+ \varphi(S_{n,L}^k u_n^k, 2u_{n-1}^k - u_{n-2}^k, u_n) - \varphi(S_{n,L}^k u_n^k, 2u_{n-1}^k - u_{n-2}^k, u_n^k)
+ j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k) + j^0(\gamma_j u_n^k; \gamma_j u_n - \gamma_j u_n^k)
+ \langle j_c(\gamma_j u_n^k), \gamma_j u_n - \gamma_j u_n^k \rangle_{x_j} - \langle j_c(2\gamma_j u_{n-1}^k - \gamma_j u_{n-2}^k), \gamma_j u_n - \gamma_j u_n^k \rangle_{x_j}.
\]
(3.41)
\[
= \hat{E}_\varphi + E_j + \hat{E}_{j_c},
\]
where $E_j$ is defined in \((3.22)\) and
\[
\hat{E}_\varphi = \varphi(S_{n,L}^k u_n^k, 2u_{n-1}^k - u_{n-2}^k, u_n) - \varphi(S_{n,L}^k u_n^k, 2u_{n-1}^k - u_{n-2}^k, u_n^k)
+ \varphi(S_n u, u_n, u_n^k) - \varphi(S_n u, u_n, u_n),
\]
(3.42)
\[
\hat{E}_{j_c} = \langle j_c(\gamma_j u_n - 2\gamma_j u_{n-1} + \gamma_j u_{n-2}), \gamma_j u_n - \gamma_j u_n^k \rangle_{x_j}
+ 2\langle j_c(\gamma_j u_{n-1} - \gamma_j u_{n-2}), \gamma_j u_n - \gamma_j u_n^k \rangle_{x_j}
- \langle j_c(\gamma_j u_{n-2} - \gamma_j u_{n-1}), \gamma_j u_n - \gamma_j u_n^k \rangle_{x_j}
\leq \alpha_c c_j^2 (k^2 \|u\|_{W^{2,\infty}(I,X)} + 2\|u_{n-1} - u_{n-2}\|_X)
+ \|u_{n-2} - u_{n-2}\|_X \|u_n - u_n^k\|_X,
\]
(3.44)
\[
\hat{E}_\varphi \leq (k^2 \alpha_c \|u\|_{W^{2,\infty}(I,X)} + 2\alpha_c \|u_{n-1} - u_{n-2}\|_X)
+ \alpha_c \|u_{n-2} - u_{n-2}\|_X + \beta_c \|S_n u - S_{n,L}^k u^k\|_Y \|u_n - u_n^k\|_X.
\]
(3.45)
From \((3.41)\)–\((3.45)\) and \((3.32)\), we have
\[
m_A\|u_n - u_n^k\|_X \leq Ck^2\|u\|_{W^{2,\infty}(I,X)} + \frac{3}{2}k \beta_c \|R\|_q \sum_{j=0}^{n-1} \|u_j - u_j^k\|_X
+ (\alpha_c + \alpha_c c_j^2) (2\|u_{n-1} - u_{n-2}\|_X + \|u_{n-2} - u_{n-2}\|_X).
\]
(3.46)
Apply Lemma 8 to \((3.46)\) and combine with \((3.39)\),
\[
\|u_n - u_n^k\|_X \leq \left(\frac{\alpha_c + \alpha_c c_j^2}{m_A} (2\|u_1 - u_1^k\|_X + \|u_0 - u_0^k\|_X)
+ \frac{3}{2}k \beta_c \|R\|_q \sum_{j=0}^{n-1} \|u_j - u_j^k\|_X \right)
+ \frac{3}{2}k \beta_c \|R\|_q \sum_{j=0}^{n-1} \|u_j - u_j^k\|_X
+ k^2 \left(\frac{C \|u\|_{W^{2,\infty}(I,X)}}{m_A - 3(\alpha_c + \alpha_c c_j^2)} \right) \left(1 + \frac{3}{2}k \beta_c \|R\|_q \sum_{j=0}^{n-1} \|u_j - u_j^k\|_X \right)
\leq C_5 k^2,
\]
(3.47)
where

\[ C_5 = \left( \frac{C \|u\|_{W^{2,\infty}(I;X)}}{m_A - 3(\alpha_\varphi + \alpha c_{j}^2)} \right) + C_4 \left( \frac{3 \beta \|R\| \|q\|}{m_A - 3(\alpha_\varphi + \alpha c_{j}^2)} \right) + 2C_4 \frac{\alpha_\varphi + \alpha c_{j}^2}{m_A} \cdot \exp \left\{ \frac{3 \beta \|R\| \|q\|}{m_A - 3(\alpha_\varphi + \alpha c_{j}^2)} t_n \right\}, \]

which leads to the error bound (3.38). \( \square \)

### 4 Fully Discrete Approximation

In this section we consider fully discrete approximations of Problem 20 with or without constraints. The notation and assumptions follow from previous section, and a regular family of finite element partitions \( \{T^h\} \) with mesh grid size \( h \) is introduced for the spatial discretization. Let \( X^h \subset X \) be the conforming finite element spaces. We consider internal approximations only, i.e., \( K^h = X^h \cap K \) is nonempty, convex and closed.

Certainly, different fully discrete schemes can be constructed with different temporally semi-discrete schemes proposed in the previous section. We state these fully discrete schemes as follows.

**Problem 20** Find the discrete solution \( u^{kh} := \{u_{n}^{kh}\}_{n=0}^{N} \subset K^h \) such that

\[
\begin{align*}
\langle Au_{n}^{kh}, v^h - u_{n}^{kh} \rangle + \varphi(S_{n,L}^{k}u_{n}^{kh}, u_{n-1}^{kh}, v^h) - \varphi(S_{n,L}^{k}u_{n}^{kh}, u_{n-1}^{kh}, u_{n}^{kh}) \\
+ j^0(\gamma_{j}u_{n}^{kh}; \gamma_{j}v^h - \gamma_{j}u_{n}^{kh}) + \langle j_{c}(\gamma_{j}u_{n}^{kh}), \gamma_{j}v_{n}^{kh} - \gamma_{j}u_{n}^{kh} \rangle x_j \\
\geq \langle f_{n}, v^h - u_{n}^{kh} \rangle + \langle j_{c}(\gamma_{j}u_{n-1}^{kh}), \gamma_{j}v_{n}^{kh} - \gamma_{j}u_{n}^{kh} \rangle x_j \quad \forall v^h \in K^h.
\end{align*}
\]

(4.1)

**Problem 21** Find the discrete solution \( u^{kh} := \{u_{n}^{kh}\}_{n=0}^{N} \subset K^h \) such that

\[
\begin{align*}
\langle Au_{n}^{kh}, v^h - u_{n}^{kh} \rangle + \varphi(S_{n,L}^{k}u_{n}^{kh}, u_{n}^{kh}, v^h) - \varphi(S_{n,L}^{k}u_{n}^{kh}, u_{n}^{kh}, u_{n}^{kh}) \\
+ j^0(\gamma_{j}u_{n}^{kh}; \gamma_{j}v^h - \gamma_{j}u_{n}^{kh}) \geq \langle f_{n}, v^h - u_{n}^{kh} \rangle \quad \forall v^h \in K^h.
\end{align*}
\]

(4.2)

**Problem 22** Find the discrete solution \( u^{kh} := \{u_{n}^{kh}\}_{n=0}^{N} \subset K^h \) such that

\[
\begin{align*}
\langle Au_{n}^{kh}, v^h - u_{n}^{kh} \rangle + \varphi(S_{n,L}^{k}u_{n}^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, v^h) - \varphi(S_{n,L}^{k}u_{n}^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, u_{n}^{kh}) \\
+ j^0(\gamma_{j}u_{n}^{kh}; \gamma_{j}v^h - \gamma_{j}u_{n}^{kh}) + \langle j_{c}(\gamma_{j}u_{n}^{kh}), \gamma_{j}v_{n}^{kh} - \gamma_{j}u_{n}^{kh} \rangle x_j \\
\geq \langle f_{n}, v^h - u_{n}^{kh} \rangle + \langle j_{c}(2\gamma_{j}u_{n-1}^{kh} - \gamma_{j}u_{n-2}^{kh}), \gamma_{j}v_{n}^{kh} - \gamma_{j}u_{n}^{kh} \rangle x_j \quad \forall v^h \in K^h, n \geq 2,
\end{align*}
\]

(4.3)

for \( n = 0, 1 \) the following scheme is used

\[
\begin{align*}
\langle Au_{n}^{kh}, v^h - u_{n}^{kh} \rangle + \varphi(S_{n,L}^{k}u_{n}^{kh}, u_{n}^{kh}, v^h) - \varphi(S_{n,L}^{k}u_{n}^{kh}, u_{n}^{kh}, u_{n}^{kh}) \\
+ j^0(\gamma_{j}u_{n}^{kh}; \gamma_{j}v^h - \gamma_{j}u_{n}^{kh}) \geq \langle f_{n}, v^h - u_{n}^{kh} \rangle \quad \forall v^h \in K^h.
\end{align*}
\]

(4.4)
In the following, we will only discuss about the fully discrete Problem $22$ since the other two fully discrete schemes can be discussed similarly. Similar to the temporally semi-discrete case, we can show that under those same conditions for the temporally semi-discrete, Problem $22$ has a unique solution. An error bound for Problem $22$ is given next.

**Theorem 23** Assume $(2.3)-(2.10), (3.6)$ and $(3.37)$. Under the regularity assumptions $q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; L(\mathcal{X}))$, $u \in W^{2,\infty}_{\text{loc}}(\mathbb{R}_+; X)$, we have the error bound

$$
\max_{0 \leq n \leq N} \|u_n - u_n^{kh}\|_X \leq C_6 \max_{0 \leq n \leq N} \inf_{v^h \in K^h} \{ \|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_X^\frac{1}{2} \}
+ \|E(v^h, u_n)\|_X^\frac{1}{2} \}
+ C_6 k^2,
$$

where $C_6 > 0$ is a constant independent of $k, h$ and

$$
E(v^h, u_n) = \langle Au_n, v^h - u_n \rangle + \varphi(S_n u, u_n, v^h) - \varphi(S_n u, u_n, u_n)
+ j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n) - \langle f_n, v^h - u_n \rangle, \quad v^h \in K^h.
$$

**Proof.** First we consider the general case of $n \geq 2$. To this end, we take $t = t_n$ and $v = u_n^{kh}$ in $(2.2)$ to get

$$
\langle Au_n, u_n^{kh} - u_n \rangle + \varphi(S_n u, u_n, u_n^{kh}) - \varphi(S_n u, u_n, u_n)
+ j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) \geq \langle f_n, u_n^{kh} - u_n \rangle.
$$

On the other hand,

$$
\langle Au_n - Au_n^{kh}, u_n - u_n^{kh} \rangle = \langle Au_n, u_n - u_n^{kh} \rangle + \langle Au_n^{kh}, u_n^{kh} - v^h \rangle
+ \langle Au_n^{kh}, v^h - u_n \rangle.
$$

Combine $(2.5)(b)$ with $(4.3), (4.7)-(4.8), (4.9)$,

$$
m_A \|u_n - u_n^{kh}\|_X^2 \leq \varphi(S_n u, u_n, u_n^{kh}) - \varphi(S_n u, u_n, u_n)
+ \varphi(S_{n,L} u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, v^h) - \varphi(S_{n,L} u^{kh}, 2u_{n-1}^{kh} - u_{n-2}^{kh}, v^h)
+ \langle Au_n^{kh}, v^h - u_n \rangle = \langle f_n, v^h - u_n \rangle + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n)
+ \langle j_c(\gamma_j u_n^{kh}), \gamma_j v^h - \gamma_j u_n^{kh} \rangle x_j + j^0(\gamma_j u_n; \gamma_j v^h - \gamma_j u_n)
+ \langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}, \gamma_j v^h - \gamma_j u_n^{kh} \rangle x_j
= E_{\varphi_1} + E_{\varphi_2} + \tilde{E}_j + E_A + E(v^h, u_n).
$$
where
\[
E_{\varphi_1} = \varphi(S_{n,L}^k u_{n-1}^{kh} - u_{n-2}^{kh}) - \varphi(S_{n,L}^k u_{n-1}^{kh} - u_{n-2}^{kh}) \\
+ \varphi(S_n u, u_n, u_n^{kh}) - \varphi(S_n u, u_n, u_n)
\] (4.10)
\[
E_{\varphi_2} = \varphi(S_{n,L}^k u_{n-1}^{kh} - u_{n-2}^{kh}) - \varphi(S_{n,L}^k u_{n-1}^{kh} - u_{n-2}^{kh}), \varphi(S_n u, u_n, v_n^{kh})
\] (4.11)
\[
\dot{E}_j = j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j v_n^{kh}) + j^0(\gamma_j u_n^{kh}; \gamma_j v_n^{kh} - \gamma_j u_n^{kh}) \\
+ \langle j_c(\gamma_j u_n^{kh}), \gamma_j u_n^{kh} - \gamma_j u_n \rangle x_j + \langle j_c(\gamma_j u_n^{kh}), \gamma_j u_n - \gamma_j v_n^{kh} \rangle x_j \\
- j^0(\gamma_j u_n; \gamma_j v_n^{kh} - \gamma_j u_n)
\] (4.12)
\[
E_A = \langle A u_{n-1}^{kh}, v_n^{kh} - u_n \rangle \\
+ \langle j_c(\gamma_j u_n^{kh}), \gamma_j v_n^{kh} - \gamma_j u_n \rangle x_j - \langle j_c(\gamma_j u_n^{kh}), \gamma_j u_n^{kh} - \gamma_j u_n \rangle x_j \\
- \langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j v_n^{kh} - \gamma_j u_n \rangle x_j
\] (4.13)

Let us bound \(E_{\varphi_1}, E_{\varphi_2}, \dot{E}_j\) and \(E_A\) in turn.
\[
E_{\varphi_1} \leq \alpha \varphi \| u_n - u_n^{kh} \|_X \| u_n - 2u_{n-1}^{kh} + u_{n-2}^{kh} \|_X \\
+ \beta \varphi \| S_n u - S_{n,L}^{kh} u_{n-1} \|_Y \| u_n - u_n^{kh} \|_X
\] (4.14)
\[
E_{\varphi_2} \leq \alpha \varphi \| u_n - v_n^{kh} \|_X \| u_n - 2u_{n-1}^{kh} + u_{n-2}^{kh} \|_X \\
+ \beta \varphi \| S_n u - S_{n,L}^{kh} u_{n-1} \|_Y \| u_n - v_n^{kh} \|_X
\] (4.15)

Use the sub-additive property of generalized directional derivative,
\[
\dot{E}_j \leq j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j v_n^{kh}) + j^0(\gamma_j u_n^{kh}; \gamma_j v_n^{kh} - \gamma_j u_n^{kh}) - \alpha_j c_j^2 \| u_n - u_n^{kh} \|_X^2 \\
\leq j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v_n^{kh}) + j^0(\gamma_j u_n^{kh}; \gamma_j u_n - \gamma_j v_n^{kh}) \\
+ j^0(\gamma_j u_n^{kh}; \gamma_j v_n^{kh} - \gamma_j u_n) + j^0(\gamma_j u_n^{kh}; \gamma_j u_n - \gamma_j u_n^{kh}) - \alpha_j c_j^2 \| u_n - u_n^{kh} \|_X^2
\] (4.16)
\[
\leq (2c_0 + c_1 \| \gamma_j u_n \|_X) \| \gamma_j u_n - \gamma_j v_n^{kh} \|_X + c_1 c_j^2 \| u_n^{kh} \|_X \| u_n - v_n^{kh} \|_X.
\]

Since \(A\) is Lipschitz continuous with a Lipschitz constant \(L_A > 0\),
\[
E_A \leq L_A \| u_n - u_n^{kh} \|_X \| u_n - v_n^{kh} \|_X + \langle j_c(\gamma_j u_n - 2\gamma_j u_{n-1}^{kh} + \gamma_j u_{n-2}^{kh}), \gamma_j u_n - \gamma_j u_n^{kh} \rangle x_j \\
+ \langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j u_n - \gamma_j v_n^{kh} \rangle x_j + \langle j_c(\gamma_j u_n^{kh}), \gamma_j v_n^{kh} - \gamma_j u_n \rangle x_j.
\] (4.17)

We have
\[
\langle j_c(2\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j u_n - \gamma_j v_n^{kh} \rangle x_j + \langle j_c(\gamma_j u_n^{kh}), \gamma_j v_n^{kh} - \gamma_j u_n \rangle x_j \\
= \langle j_c(-\gamma_j u_n + 2\gamma_j u_{n-1} - \gamma_j u_{n-2}), \gamma_j u_n - \gamma_j v_n^{kh} \rangle x_j \\
+ \langle j_c(\gamma_j u_n - \gamma_j u_n^{kh}), \gamma_j u_n - \gamma_j v_n^{kh} \rangle x_j - 2 \langle j_c(\gamma_j u_{n-1}^{kh} - \gamma_j u_{n-1}^{kh}), \gamma_j u_n - \gamma_j v_n^{kh} \rangle x_j + \langle j_c(\gamma_j u_{n-2}^{kh} - \gamma_j u_{n-2}^{kh}), \gamma_j u_n - \gamma_j v_n^{kh} \rangle x_j \\
\leq \alpha c_j^2 \| u_n - v_n^{kh} \|_X (\| u_n - 2u_{n-1}^{kh} + u_{n-2}^{kh} \|_X + \| u_n - u_n^{kh} \|_X) \\
+ \alpha c_j^2 \| u_n - v_n^{kh} \|_X (2 \| u_{n-1}^{kh} - u_{n-2}^{kh} \|_X + \| u_{n-2}^{kh} - u_{n-2}^{kh} \|_X).
\] (4.18)
Together with (4.9), (4.14)–(4.18), for $\varepsilon < m_A/3 - \alpha_\varphi - \alpha_c c_j^2$, we obtain

$$m_A\| u_n - u_n^{kh} \|_X^2 \leq \left( C k^2 \| u \|_{W^{2,\infty}(I;X)} + 2(\alpha_\varphi + \alpha_c c_j^2) \| u_{n-1} - u_n^{kh} \|_X \\
+ (\alpha_\varphi + \alpha_c c_j^2) \| u_{n-2} - u_n^{kh} \|_X + \beta_\varphi \| \mathcal{S}_n u - \mathcal{S}_{n,L} u^{kh} \|_Y \right. \\
+ C \| u_n - v^h \|_X \right) \| u_n - u_n^{kh} \|_X + C k^2 \| u_n - v^h \|_X \\
+ \left. \frac{\alpha \varepsilon}{m_A} \| u_n - v^h \|_X \right) \| u_n - u_n^{kh} \|_X + C \| \gamma_j u_n - \gamma_j v^h \|_{X_j} + | E(v^h, u_n) |. \tag{4.19}$$

Apply Lemma 10 and Cauchy-Schwarz inequality,

$$\| u_n - u_n^{kh} \|_X \leq C k^2 \| u \|_{W^{2,\infty}(I;X)} + 2(\alpha_\varphi + \alpha_c c_j^2) \| u_{n-1} - u_n^{kh} \|_X \\
+ (\alpha_\varphi + \alpha_c c_j^2) \| u_{n-2} - u_n^{kh} \|_X + \frac{3}{2} k \beta_\varphi \| R \|_q \| q \| \sum_{j=0}^{n-1} \| u_j - u_j^{kh} \|_X \\
+ C \| u_n - v^h \|_X \right) \| u_n - u_n^{kh} \|_X + C \| \gamma_j u_n - \gamma_j v^h \|_{X_j} + C | E(v^h, u_n) |. \tag{4.20}$$

For $n = 0$ and $n = 1$, a slight modification based on the proof of Theorem 15 and the above arguments give

$$\| u_0 - u_0^{kh} \|_X \leq \frac{C}{m_A - \alpha_\varphi - \alpha_j c_j^2} \{ \| u_0 - v^h \|_X + \| \gamma_j u_0 - \gamma_j v^h \|_{X_j} + | E(v^h, u_0) | \}^2 \}. \tag{4.21}$$

$$\| u_1 - u_1^{kh} \|_X \leq \frac{C}{m_A - \alpha_\varphi - \alpha_j c_j^2} \{ \| u_1 - v^h \|_X + \| \gamma_j u_1 - \gamma_j v^h \|_{X_j} \}^2 \tag{4.22}$$

Apply Lemma 8 to (4.20) and combine (4.21)–(4.22) to get

$$\| u_n - u_n^{kh} \|_X \leq \left(C \{ \| u_n - v^h \|_X + \| \gamma_j u_n - \gamma_j v^h \|_{X_j} + | E(v^h, u_n) | \}^2 + k^2 \| u \|_{W^{2,\infty}(I;X)} \right) \\
+ \frac{\alpha_\varphi + \alpha_c c_j^2 + \varepsilon}{m_A} \left( 2 \| u_1 - u_1^{kh} \|_X + \| u_0 - u_0^{kh} \|_X \right) \\
+ \frac{3}{2} k \beta_\varphi \| R \|_q \| q \| \left( \| u_0 - u_0^{kh} \|_X + \| u_1 - u_1^{kh} \|_X \right) \\
\cdot \left( 1 + \frac{3}{2} k \beta_\varphi \| R \|_q \| q \| \right)^{n-2} \\
\leq C_0 \max_{0 \leq n \leq N} \left( \| u_n - v^h \|_X + \| \gamma_j u_n - \gamma_j v^h \|_{X_j} + | E(v^h, u_n) | \right)^2 + k^2, \tag{4.23}$$
where

\[ C_6 = C \| u \|_{W^{2,\infty}(I; X)} \cdot \exp \left\{ \frac{3}{2} \beta \| R \| \| q \| \left( \frac{3}{2} \alpha \| \alpha \|_{X}^{2} + \varepsilon \frac{1}{2} \right) \right\}. \]

Then we have the error bound (4.5).

Now we consider the error estimation for numerical solution of the discrete problem without constraint. We introduce the following assumption on \( \varphi \) as in [28], which allows us to simplify the error bound (4.5):

\[
\begin{align*}
\varphi : Y \times K \times K &\rightarrow \mathbb{R} \text{ is a function such that} \\
&\text{there exists a constant } c_{\varphi} > 0 \text{ satisfies} \\
&\varphi(y, u, v_1) + \varphi(y, u, v_2) - 2 \varphi(y, u, \frac{v_1 + v_2}{2}) \leq c_{\varphi} \| v_1 - v_2 \|_{X}^{2}, \\
&\forall y \in Y, \forall u, v_1, v_2 \in K. \tag{4.24}
\end{align*}
\]

**Theorem 24** Keep the assumptions stated in Theorem 23. In addition, let \( K = X \) and the function \( \varphi \) satisfy the assumption (4.24). Then the following error bound holds:

\[
\begin{align*}
\max_{0 \leq n \leq N} \| u_n - u_n^{kh} \|_{X} &\leq C \left( \max_{0 \leq n \leq N} \inf_{v^h \in K^h} \left\{ \| u_n - v^h \|_{X} + \| \gamma_j u_n - \gamma_j v^h \|_{X_j}^{\frac{3}{2}} \right\} + k^2 \right). \tag{4.25}
\end{align*}
\]

**Proof.** We start with

\[
\begin{align*}
\langle Au_n - Au_n^{kh}, u_n - u_n^{kh} \rangle &= \langle Au_n - Au_n^{kh}, u_n - v^h \rangle + \langle Au_n - Au_n^{kh}, v^h - u_n^{kh} \rangle \\
&= \langle Au_n - Au_n^{kh}, u_n - v^h \rangle + \langle Au_n, v^h - u_n \rangle \\
&+ \langle Au_n, u_n - u_n^{kh} \rangle + \langle Au_n^{kh}, u_n^{kh} - v^h \rangle. \tag{4.26}
\end{align*}
\]

Further, we replace \( v \) with \( 2u_n - v \) in (3.15) to get

\[
\begin{align*}
\langle Au_n, u_n - v \rangle + \varphi(S_n u, u_n, 2u_n - v) - \varphi(S_n u, u_n, u_n) \\
+ j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v) &\geq \langle f_n, u_n - v \rangle \quad \forall v \in X. \tag{4.27}
\end{align*}
\]

Similarly, take \( v = v^h \) in (4.27) to get

\[
\begin{align*}
\langle Au_n, u_n - v^h \rangle + \varphi(S_n u, u_n, 2u_n - v^h) - \varphi(S_n u, u_n, u_n) \\
+ j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) &\geq \langle f_n, u_n - v^h \rangle. \tag{4.28}
\end{align*}
\]

Combine (2.5), (4.3), (4.7), (4.26) and (4.28),

\[
m_A \| u_n - u_n^{kh} \|_{X}^{2} \leq \langle Au_n - Au_n^{kh}, u_n - v^h \rangle + \varphi(S_n u, u_n, 2u_n - v^h) \\
+ \varphi(S_n u, u_n, u_n) - 2 \varphi(S_n u, u_n, u_n) \\
+ \varphi(S_n^{L} u^{kh}, 2u^{kh}_{n-1} - u^{kh}_{n-2}; v^h) - \varphi(S_n^{L} u^{kh}, 2u^{kh}_{n-1} - u^{kh}_{n-2}; u_n^{kh}) \\
+ j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n) \\
+ j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) + \langle j_c(\gamma_j u_n^{kh}), \gamma_j v^h - \gamma_j u_n^{kh} \rangle_{X_j} \\
- \langle j_c(2\gamma_j u_n^{kh-1} - \gamma_j u_n^{kh-2}), \gamma_j v^h - \gamma_j u_n^{kh} \rangle_{X_j} \\
= E_{\varphi_1} + E_{\varphi_2} + E_{\varphi_3} + \tilde{E}_j + E_A. \tag{4.29}
\]
where \( E_{\varphi_1}, E_{\varphi_2}, E_A \) are the same as in (4.10), (4.11), (4.13) respectively with their bounds (4.14), (4.15), (4.17). In addition,
\[
E_{\varphi_3} = \varphi(S_n u, u_n, 2u_n - v^h) + \varphi(S_n u, u_n, v^h) - 2\varphi(S_n u, u_n, u_n),
\]
(4.30)

\[
\tilde{E}_j = j^0(\gamma_j u_n; \gamma_j u_n - \gamma_j v^h) + j^0(\gamma_j u_n; \gamma_j u_n^{kh} - \gamma_j u_n)
+ j^0(\gamma_j u_n^{kh}; \gamma_j v^h - \gamma_j u_n^{kh}) - \langle j_c(\gamma_j u_n - \gamma_j u_n^{kh}), \gamma_j u_n - \gamma_j u_n^{kh}\rangle_{X_j}.
\]
(4.31)

The assumption (4.24) shows that
\[
E_{\varphi_3} \leq C\|u_n - v^h\|_X^2.
\]
(4.32)

Using the sub-additive property again, we obtain
\[
\tilde{E}_j \leq C\|\gamma_j u_n - \gamma_j v^h\|_{X_j} + C\|u_n - u_n^{kh}\|_X\|u_n - v^h\|_X.
\]
(4.33)

Together (4.29) with (4.30)–(4.33) and analogy to (4.20),
\[
m_A\|u_n - u_n^{kh}\|_X \leq C\{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j} + C\|S_n u - S_n^{kh}\|_Y
+ (\alpha_\epsilon c_j^2 + \alpha_\varphi + \varepsilon)(\|u_{n-2} - u_{n-2}^{kh}\|_X + 2\|u_{n-1} - u_{n-1}^{kh}\|_X).
\]
(4.34)

Similar to the constrained situation, the error bounds for \( n = 0, 1 \) are
\[
\|u_0 - u_0^{kh}\|_X \leq \frac{C}{m_A - \alpha_\varphi - \alpha_\epsilon c_j^2}\{\|u_0 - v^h\|_X + \|\gamma_j u_0 - \gamma_j v^h\|_{X_j}^{1/2}\},
\]
(4.35)

\[
\|u_1 - u_1^{kh}\|_X \leq \frac{C}{m_A - \alpha_\varphi - \alpha_\epsilon c_j^2}\{\|u_1 - v^h\|_X + \|\gamma_j u_1 - \gamma_j v^h\|_{X_j}^{1/2}
+ k^2\|u\|_{W^2,\infty(I;X)}\} + Ck\|u_0 - u_0^{kh}\|_X.
\]
(4.36)

Combining (4.34)–(4.36), we find the following error bound by an application of Lemma 8
\[
\|u_n - u_n^{kh}\|_X \leq \left(C\{\|u_n - v^h\|_X + \|\gamma_j u_n - \gamma_j v^h\|_{X_j}^{1/2}\} + k^2\|u\|_{W^2,\infty(I;X)}\}
+ \frac{\alpha_\varphi + \alpha_\epsilon c_j^2 + \varepsilon}{m_A} (2\|u_1 - u_1^{kh}\|_X + \|u_0 - u_0^{kh}\|_X)
+ \frac{Ck}{m_A - 3(\alpha_\varphi + \alpha_\epsilon c_j^2 + \varepsilon)} \left(\|u_0 - u_0^{kh}\|_X + \|u_1 - u_1^{kh}\|_X\right)\right)^{n-2}
\]
(4.37)

Thus, the proof is completed. \(\square\)
5 Numerical computation using fixed-point iteration

Notice that in Problems [13 21] and in the initial steps of Problems [17 22] the implicit discretization with respect to the unknown solution component is used. Let us discuss how to implement these numerical schemes in practice. We use a fixed-point iteration approach. We first consider the fixed-point iterations for the temporally semi-discrete schemes.

**Problem 25** Let $TOL$ be a given error tolerance. For $1 \leq n \leq N$, find a sequence $\{\tilde{u}_{n,i}^k\} \subset K$ from the iterations

\[ \langle A\tilde{u}_{n,i}^k, v - \tilde{u}_{n,i}^k \rangle + \varphi(S_{n,L}^k u^k, \tilde{u}_{n,i}^k) - \varphi(S_{n,L}^k u^k, u^k) \\
+ \langle j^0(\gamma_j \tilde{u}_{n,i}^k; \gamma_j v - \gamma_j \tilde{u}_{n,i}^k) + \langle j_c(\gamma_j \tilde{u}_{n,i}^k), \gamma_j v - \gamma_j \tilde{u}_{n,i}^k \rangle x_j \rangle \\
\geq \langle f_n, v - \tilde{u}_{n,i}^k \rangle + \langle j_c(\gamma_j \tilde{u}_{n,i}^k), \gamma_j v - \gamma_j \tilde{u}_{n,i}^k \rangle x_j, \quad \forall v \in K \]  
until the relative error $\frac{\|\tilde{u}_{n,i}^k - \tilde{u}_{n,i-1}^k\|_X}{\tilde{u}_{n,i}^k} < TOL$; choose $u^n_k$ to be the last iteration $\tilde{u}_{n,i}^k$.

In Problem 25, the index $i$ refers to the $i$-th iterate at time level $t_n$. For the initialization of iteration, we may use the iterative solution from the previous step, i.e., $\tilde{u}_{n,0}^k = u_{n-1}^k$ for $n \geq 1$. Now we consider the convergence of the sequence $\{\tilde{u}_{n,i}^k\}$ generated by (5.1) to the solution of (3.28).

**Theorem 26** Assume (2.3)–(2.10). Then the iteration (5.1) converges linearly with a convergence rate $\rho = (\alpha_\varphi + \alpha_c c_j^2)/m_A$ that is independent of the time step-size $k$.

**Proof.** Take $v = \tilde{u}_{n,i}^k$ in (3.28),

\[ \langle A\tilde{u}_{n,i}^k, \tilde{u}_{n,i}^k - u_{n}^k \rangle + \varphi(S_{n,L}^k u^k, \tilde{u}_{n,i}^k) - \varphi(S_{n,L}^k u^k, u^k) \\
+ \langle j^0(\gamma_j \tilde{u}_{n,i}^k; \gamma_j \tilde{u}_{n,i}^k) \rangle \geq \langle f_n, \tilde{u}_{n,i}^k - u_{n}^k \rangle. \]  
Take $v = u_{n}^k$ in (5.1),

\[ \langle A\tilde{u}_{n,i}^k, u_{n}^k - \tilde{u}_{n,i}^k \rangle + \varphi(S_{n,L}^k u^k, \tilde{u}_{n,i}^k) - \varphi(S_{n,L}^k u^k, \tilde{u}_{n,i}^k) \\
+ \langle j^0(\gamma_j \tilde{u}_{n,i}^k; \gamma_j u_{n}^k - \gamma_j \tilde{u}_{n,i}^k) \rangle \geq \langle f_n, u_{n}^k - \tilde{u}_{n,i}^k \rangle. \]  
Combine (5.2) with (5.3),

\[ \langle A\tilde{u}_{n,i}^k, u_{n}^k - \tilde{u}_{n,i}^k \rangle + \langle A\tilde{u}_{n,i}^k, \tilde{u}_{n,i}^k - u_{n}^k \rangle \leq \alpha_\varphi \|u_{n}^k - \tilde{u}_{n,i}^k\|_X \|u_{n}^k - \tilde{u}_{n,i-1}^k\|_X \\
+ \alpha_c c_j^2 \|u_{n}^k - \tilde{u}_{n,i}^k\|_X \|u_{n}^k - \tilde{u}_{n,i-1}^k\|_X. \]  
By the strong monotonicity of $A$ and (5.4), we have the following relation:

\[ m_A \|u_{n}^k - \tilde{u}_{n,i}^k\|_X \leq (\alpha_\varphi + \alpha_c c_j^2) \|u_{n}^k - \tilde{u}_{n,i-1}^k\|_X. \]  
Therefore, the stated result is proved. □

In analogy to the temporally semi-discrete scheme, the iteration algorithm for the fully discrete scheme can be stated as follows.

\[ u_{n}^k \]
**Problem 27** Let $\text{TOL}$ be a given error tolerance. For $1 \leq n \leq N$, find a sequence $\{\tilde{u}^{kh}_{n,i}\} \subset K^h$ such that

$$
\langle A\tilde{u}^{kh}_{n,i}, v^h - \tilde{u}^{kh}_{n,i} \rangle + \varphi(S_{n,L}^{kh}\tilde{u}^{kh}_{n,i}, \tilde{u}^{kh}_{n,i-1}, v^h) - \varphi(S_{n,L}^{kh}\tilde{u}^{kh}_{n,i-1}, \tilde{u}^{kh}_{n,i}) \\
+ j^0(\gamma_j\tilde{u}^{kh}_{n,i}; \gamma_j v^h - \gamma_j \tilde{u}^{kh}_{n,i}) + \langle j_c(\gamma_j\tilde{u}^{kh}_{n,i}), \gamma_j v^h - \gamma_j \tilde{u}^{kh}_{n,i} \rangle x_j \\
\geq \langle f_n, v^h - \tilde{u}^{kh}_{n,i} \rangle + \langle j_c(\gamma_j\tilde{u}^{kh}_{n,i-1}), \gamma_j v^h - \gamma_j \tilde{u}^{kh}_{n,i} \rangle x_j,
$$

(5.6)

until the relative error $\frac{\|\tilde{u}^{kh}_{n,i} - \tilde{u}^{kh}_{n,i-1}\|_X}{\|\tilde{u}^{kh}_{n,i}\|_X} < \text{TOL}$; choose $u^{kh}_{n,i}$ to be the last iteration $\tilde{u}^{kh}_{n,i}$.

The sequence $\{\tilde{u}^{kh}_{n,i}\}$ can be similarly proved to converge to the solution of (4.2).

**Theorem 28** Keep the assumptions in Theorem 26. Then the iteration (5.6) converges linearly with a convergence rate $\rho = (\alpha_{\varphi} + \alpha_{j_c} c_j^2)/m_A$ that is independent of the time step $k$ and the mesh parameter $h$.

So far we have proposed three types of schemes and the corresponding numerical treatments to solve Problem 1. Note that the difference of the schemes lies in the way the temporal discretization is done. We list the schemes and summarize their main properties in Table 1, where CO stands for convergence order.

| semi-discrete problem | numerical method | CO      | constraint                              |
|-----------------------|------------------|---------|-----------------------------------------|
| Problem 4             | · convex optimization | first-order | $m_A > \alpha_{\varphi} + \alpha_{j_c}^2$ |
| Problem 13            | · convex optimization · fixed-point iteration (each step) | second-order | $m_A > \alpha_{\varphi} + \alpha_j c_j^2$ |
| Problem 17            | · convex optimization · extrapolation · fixed-point iteration (initial step) | second-order | $m_A/3 > \alpha_{\varphi} + \alpha_{j_c}^2$ |

We use the result of previous step to approximate the current step in Problem 4 which is easy to implement while with low accuracy. For Problem 17, the approximation for current step is performed with an extrapolation, thus an initial step is introduced and we employ a fixed-point iteration to solve it numerically. As a result, we obtain a second-order accuracy with stronger small condition constraint. Inspired by this fixed-point iterative procedure, we propose a new scheme in Problem 13 in which a fixed-point iteration is used to approximate this scheme for each step.

## 6 Application to a contact problem

In this section we apply the abstract numerical analysis results in the previous sections to a particular history-dependent variational-hemivariational inequality. A viscoelastic frictionless contact model studied in [26] will be considered. For details on the model, we refer the reader to [26, 28].
Problem 29 Find a displacement \( u : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d \) and a stress field \( \sigma : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d \) such that for all \( t \in \mathbb{R}_+ \),

\[
\sigma(t) = \mathcal{A} \varepsilon(u(t)) + \mu(\varepsilon(u(t)) - P_M(\kappa(\zeta(t)))) + \int_0^t \mathcal{B}(t-s) \varepsilon(u(s)) \, ds \quad \text{in} \ \Omega,
\]

\[
\text{Div } \sigma(t) + f_0(t) = 0 \quad \text{in } \Omega,
\]

\[
u(t) = 0 \quad \text{on } \Gamma_1,
\]

\[
\sigma(t) \nu = f_2(t) \quad \text{on } \Gamma_2,
\]

\[
\begin{aligned}
& \sigma(\nu(t)) \leq g, \ \sigma(\nu(t)) + \xi(\nu(t)) \leq 0, \\
& (\sigma(\nu(t)) + \xi(\nu(t)))(u(\nu(t)) - g) = 0, \quad \text{on } \Gamma_3, \\
& \xi(t) \in \partial j(\nu(t)) \quad \text{on } \Gamma_3.
\end{aligned}
\]

\[
(\mathcal{F}_t) = (\mathcal{F}_t) \quad \text{in } \Omega.
\]

As is standard in the literature in the area of the paper, we denote by \( \mathbb{S}^d \) the space of second order symmetric tensors on \( \mathbb{R}^d \), \( u = (u_i), \ \nu = (\nu_i), \ \sigma = (\sigma_{ij}) \), \( \varepsilon(u) = (\nabla u + (\nabla u)^T)/2 \) the displacement field, outward unit normal on the boundary, stress tensor and linearized strain tensor, respectively. In addition, \( v_\nu := v \cdot \nu \) and \( v_\tau := v - v_\nu \nu \) stand for the normal and tangential components of a vector field \( v \), \( \sigma_\nu := (\sigma \nu) \cdot \nu \) and \( \sigma_\tau := \sigma \nu - \sigma_\nu \nu \) represent the normal and tangential components of the stress field \( \sigma \), respectively. In equation (6.1) \( P_M(\kappa(\cdot)) \) denotes the projection on the Von Mises convex, \( \mathcal{A} \) and \( \mathcal{B} \) are the elastic and relaxation tensors, and \( \mu \) is a constant. In this model, time-dependent surface tractions of density \( f_2 \) and volume forces of density \( f_0 \) are considered. On \( \Gamma_3 \), the penetration is restricted by a non-negative function \( g \) and the potential function is denoted as \( j_\nu \). The function spaces \( V \) and \( \mathcal{H} \) are

\[
V = \{ v = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ a.e. on } \Gamma_1 \}.
\]

\[
\mathcal{H} = \{ \tau = (\tau_{ij}) \in L^2(\Omega; \mathbb{S}^d) \mid \tau_{ij} = \tau_{ji}, \ 1 \leq i,j \leq d \}.
\]

The inner products in the Hilbert spaces \( \mathcal{H} \) and \( V \) are

\[
(\mathbf{H}, \mathbf{v}) = \int_\Omega \mathbf{H}(x) \mathbf{v}(x) \, dx, \quad (u, v)_V = (\varepsilon(u), \varepsilon(u))_V
\]

and the associated norm are denoted by \( \| \cdot \|_\mathcal{H} \) and \( \| \cdot \|_V \). The space of fourth order tensor fields \( Q_\infty \) is given by

\[
Q_\infty = \{ E = (E_{ijkl}) \mid E_{ijkl} = E_{jikl} = E_{klij} \in L^\infty(\Omega), \ 1 \leq i,j,k,l \leq d \}.
\]

We now list the assumptions on the problem data, following [26, 28]. The elasticity tensor \( \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \) is symmetric and positive. The relaxation tensor \( \mathcal{B} \in C(\mathbb{R}_+; \mathcal{Q}_\infty) \) and the bound \( \kappa : \mathbb{R} \to \mathbb{R}_+ \) is Lipschitz continuous. The potential function \( J_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R} \) is measurable with respect to the first argument on \( \Gamma_3 \) for all \( r \in \mathbb{R} \) and is locally Lipschitz with respect to the second argument.
on $\mathbb{R}$ for a.e. $x \in \Gamma_3$; $j_{\nu}(\cdot, \tau(\cdot))$ belongs to $L^1(\Gamma_3)$ for some $\tau \in L^2(\Gamma_3)$. Besides, $|\partial j_{\nu}(x, r)| \leq \bar{c}_0 + \bar{c}_1|r|$ for a.e. $x \in \Gamma_3$, for all $r \in \mathbb{R}$ with $\bar{c}_0, \bar{c}_1 > 0$. In addition, there exists $\bar{c}_\nu \geq 0$ such that for a.e. $x \in \Gamma_3$,

$$j_{\nu}(x, r_1; r_2 - r_1) + j_{\nu}(x, r_2; r_1 - r_2) \leq \bar{c}_\nu |r_1 - r_2|^2 \quad \forall r_1, r_2 \in \mathbb{R}.$$  

For the body force and surface traction, we assume $f_0 \in C(\mathbb{R}_+; L^2(\Omega; \mathbb{R}^d))$ and $f_2 \in C(\mathbb{R}_+; L^2(\Gamma_2; \mathbb{R}^d))$. Let $U = \{v \in V \mid v_\nu \leq g \text{ a.e. on } \Gamma_3\}$ be the set of admissible displacements. Define the function $f : \mathbb{R}_+ \to V^*$ by

$$\langle f(t), v \rangle_{V^* \times V} = \langle f_0(t), v \rangle_{L^2(\Omega; \mathbb{R}^d)} + \langle f_2(t), v \rangle_{L^2(\Gamma_2; \mathbb{R}^d)} \quad \forall v \in V, \forall t \in \mathbb{R}_+.$$  

Then the weak formulation of Problem 29 can be described as following.

**Problem 30** Find a displacement $u : \mathbb{R}_+ \to U$ such that the following inequality holds:

$$(A\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)))_{\mathcal{H}} + \mu(\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)))_{\mathcal{H}}$$

$$- \mu(P_{M(c(t))}\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)))_{\mathcal{H}}$$

$$+ \left(\int_0^t \mathcal{B}(t - s)\varepsilon(u(s))ds, \varepsilon(v) - \varepsilon(u(t))\right)_{\mathcal{H}} + \int_{\Gamma_3} j_{\nu}(u_\nu(t); v_\nu - u_\nu(t))d\Gamma \geq \langle f(t), v - u(t) \rangle_{V^* \times V} \quad \forall v \in U, \forall t \in \mathbb{R}_+.$$  

(6.7)

To apply the abstract results from the previous sections to the study of this contact problem, some definitions are needed. We let $\gamma_j : V \to L^2(\Gamma_3)$ be the trace operator defined by $\gamma_j v = v_\nu$ for $v \in V$. In addition, we define the following operators (26, 28):

$$\langle Au, v \rangle_{V^* \times V} = \langle A\varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}} + \mu(\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \quad \forall u, v \in V,$$  

(6.8)

$$\|y\|_Y = |r| + \|\theta\|_{\mathcal{H}} \quad \forall y = (r, \theta) \in Y := \mathbb{R} \times \mathcal{H},$$  

(6.9)

$$\varphi(y, u, v) = -\mu(P_{M(c(r))}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (\theta, \varepsilon(v))_{\mathcal{H}} \quad \forall y = (r, \theta) \in Y, \forall u, v \in V.$$  

(6.10)

$$j_c(\gamma_j u, \gamma_j v)_{L^2(\Gamma_3)} = \alpha_j(\gamma_j u, \gamma_j v)_{L^2(\Gamma_3)} \quad \forall u, v \in V.$$  

(6.11)

$$j(\gamma_j v) = \int_{\Gamma_3} j_\nu(u_\nu)d\Gamma \quad \forall v \in V.$$  

(6.12)

$$(Su)(t) = \left(\int_0^t \|\varepsilon(u(s))\|_{\mathcal{H}}ds, \int_0^t \mathcal{B}(t - s)\varepsilon(u(s))ds\right) \quad \forall u \in C(\mathbb{R}_+; \mathcal{H}).$$  

(6.13)

Note that for $j_c$ defined in (6.11), the constants $\alpha_c$ and $\alpha_j$ in (3.6) are equal: $\alpha_c = \alpha_j$.

The unique solvability of Problem 30 has been verified in [26]. Here we consider fully discrete methods for solving Problem 30. Assume the domain $\Omega$ is polygonal/polyhedral with a regular family of partitions $\{\mathcal{T}^h\}$. The linear element space is constructed as follows:

$$V^h = \{v^h \in C(\overline{\Omega})^d \mid v^h|_{\mathcal{T}} \in P_1(\mathcal{T})^d \text{ for } \mathcal{T} \in \mathcal{T}^h, v^h = 0 \text{ on } \Gamma_1\},$$  

24
with $P_1$ being the space of polynomials of degree no greater than one. Define $U^h = \{v^h \in V^h \mid v^h \leq g\}$ at node points on $\Gamma_3$. Assume $g$ is concave; then $U^h \subseteq U$. Thus the approximation is internal and the numerical methods for Problem 30 are defined as follows:

**Problem 31** Find a discrete displacement $u^{kh} := \{u_n^{kh}\}_{n=0}^N \subset U^h$ such that

$$
(\mathcal{A}\varepsilon(u_n^{kh}), \varepsilon(v^h) - \varepsilon(u_n^{kh}))_\mathcal{H} + \mu(\varepsilon(u_n^{kh}), \varepsilon(v^h) - \varepsilon(u_n^{kh}))_\mathcal{H} \\
- \mu(P_{M(\kappa(t_n-\zeta))})\varepsilon(u_{n-1}^{kh}),\varepsilon(v^h) - \varepsilon(u_{n-1}^{kh}))_\mathcal{H} \\
+ \left(\frac{k}{2}\mathcal{B}(t_n - t_0)\varepsilon(u_0^{kh}) + k\sum_{j=1}^{n-1}\mathcal{B}(t_n - t_j)\varepsilon(u_j^{kh}) \\
+ \frac{k}{2}\mathcal{B}(t_n - t_{n-1})\varepsilon(u_{n-1}^{kh}),\varepsilon(v^h) - \varepsilon(u_{n-1}^{kh})\right)_\mathcal{H} \\
+ \int_{\Gamma_3} j_0^\varepsilon(v_n^{kh}, v^h - u_n^{kh})d\Gamma + \alpha_j(u_n^{kh}, v^h - u_n^{kh})_{L^2(\Gamma_3)} \\
\geq \alpha_j(u_{n-1}^{kh}, v_n^{kh} - u_n^{kh})_{L^2(\Gamma_3)} + \langle f, v^h - u_n^{kh} \rangle_{V^* \times V} \quad \forall v^h \in U^h,
$$

where

$$
\zeta(t_n-1) = \frac{k}{2}\|\varepsilon(u_0^{kh})\|_\mathcal{H} + k\sum_{j=1}^{n-1}\|\varepsilon(u_j^{kh})\|_\mathcal{H} + \frac{k}{2}\|\varepsilon(u_{n-1}^{kh})\|_\mathcal{H}.
$$

**Problem 32** Find a discrete displacement $u^{kh} := \{u_n^{kh}\}_{n=0}^N \subset U^h$ such that

$$
(\mathcal{A}\varepsilon(u_n^{kh}), \varepsilon(v^h) - \varepsilon(u_n^{kh}))_\mathcal{H} + \mu(\varepsilon(u_n^{kh}), \varepsilon(v^h) - \varepsilon(u_n^{kh}))_\mathcal{H} \\
- \mu(P_{M(\kappa(t_n-\zeta))})\varepsilon(u_{n-1}^{kh}),\varepsilon(v^h) - \varepsilon(u_{n-1}^{kh}))_\mathcal{H} \\
+ \left(\frac{k}{2}\mathcal{B}(t_n - t_0)\varepsilon(u_0^{kh}) + k\sum_{j=1}^{n-1}\mathcal{B}(t_n - t_j)\varepsilon(u_j^{kh}) \\
+ \frac{k}{2}\mathcal{B}(t_n - t_{n-1})\varepsilon(u_{n-1}^{kh}),\varepsilon(v^h) - \varepsilon(u_{n-1}^{kh})\right)_\mathcal{H} \\
+ \int_{\Gamma_3} j_0^\varepsilon(v_n^{kh}, v^h - u_n^{kh})d\Gamma \geq \langle f, v^h - u_n^{kh} \rangle_{V^* \times V} \quad \forall v^h \in U^h.
$$
Problem 33 Find a discrete displacement \( \mathbf{u}^{kh} := \{ \mathbf{u}_n^{kh} \}_{n=0}^N \subset U^h \) such that

\[
\begin{align*}
& (A\varepsilon(\mathbf{u}_n^{kh}), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}_n^{kh}))_H + \mu(\varepsilon(\mathbf{u}_n^{kh}), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}_n^{kh}))_H \\
& - \mu(P_{M(s(t_n-1))})\varepsilon(2\mathbf{v}_{n-1}^{kh} - \mathbf{u}_{n-2}^{kh}), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}_n^{kh}))_H \\
& + \left( k^2 \mathcal{B}(t_n - t_{n-1}) \varepsilon(\mathbf{u}_n^{kh}), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}_n^{kh}) \right)_H \\
& + \frac{k}{2} \mathcal{B}(t_n - t_{n-1}) \varepsilon(\mathbf{u}_n^{kh}), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}_n^{kh})_H \\
& + \int_{\Gamma_3} j_\nu^0(\mathbf{u}_{n,\nu}^{kh}; \mathbf{v}_n^h - \mathbf{u}_{n,\nu}^{kh}) d\Gamma + \alpha_j(u_{n,\nu}^{kh}, v_{n,\nu}^h - u_{n,\nu}^{kh}) L^2(\Gamma_3) \\
& \geq \alpha_j(2u_{n-1,\nu}^{kh} - u_{n-2,\nu}^{kh}, v_{n,\nu}^h - u_{n,\nu}^{kh}) L^2(\Gamma_3) + (f_n, \mathbf{v}^h - \mathbf{u}_n^{kh})_{V^* \times V} \quad \forall \mathbf{v}^h \in U^h, n \geq 2,
\end{align*}
\]

and for \( n = 1 \),

\[
\begin{align*}
& (A\varepsilon(\mathbf{u}_1^{kh}), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}_1^{kh}))_H + \mu(\varepsilon(\mathbf{u}_1^{kh}), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}_1^{kh}))_H \\
& - \mu(P_{M(s(t_1))})\varepsilon(\mathbf{u}_1^{kh}), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}_1^{kh})_H \\
& + \left( k^2 \mathcal{B}(t_1 - t_0) \varepsilon(\mathbf{u}_1^{kh}), \varepsilon(\mathbf{v}^h) - \varepsilon(\mathbf{u}_1^{kh}) \right)_H \\
& + \int_{\Gamma_3} j_\nu^0(\mathbf{u}_{1,\nu}^{kh}; \mathbf{v}_1^h - \mathbf{u}_{1,\nu}^{kh}) d\Gamma \geq (f_1, \mathbf{v}^h - \mathbf{u}_1^{kh})_{V^* \times V} \quad \forall \mathbf{v}^h \in U^h.
\end{align*}
\]

The numerical scheme for \( n = 0 \) is similar to (6.17) except that the approximation for the history-dependent term is omitted.

Using arguments similar to that found in [28], we can show that under the following solution regularity: \( \mathbf{u} \in W^{2,\infty}(\mathbb{R}_+; V), \sigma \in C(\mathbb{R}_+; H^1(\Omega; S^d)), \mathbf{u} \in C(\mathbb{R}_+; H^2(\Omega; \mathbb{R}^d)), \) and \( u_\nu \in C(\mathbb{R}_+; \tilde{H}^2(\Gamma_3)), \) the following optimal order error bounds hold:

\[
\max_{0 \leq n \leq N} \| \mathbf{u}_n - \mathbf{u}_n^{kh} \|_V \leq C(h + k^n),
\]

where \( \eta = 1 \) for Problem 31 and \( \eta = 2 \) for Problem 32, 33.

7 Numerical Results

In this section, we present some numerical results for the three fully discrete schemes stated in Problem 31, 33. The same physical setting as depicted in Figure 1 is employed.

Let \( \Omega = (0, L_1) \times (0, L_2) \) be a rectangle with boundary \( \Gamma \) which is divided into four parts

\( \Gamma_1 = \{0\} \times (0, L_2), \quad \Gamma_2 = \{L_1\} \times (0, L_2) \cup [0, L_1] \times \{L_2\}, \quad \Gamma_3 = [0, L_1] \times \{0\}. \)

For a given \( S > 0 \), the function \( j_\nu \) is defined as

\[
j_\nu(\xi_\nu) = S \int_0^{\xi_\nu} \mu_j(s) \, ds \quad (7.19)
\]
Figure 1: Initial configuration of the contact problem.

with

\[
\mu_j(s) = \begin{cases} 
0 & s \leq 0, \\
c_1s & 0 < s \leq s_1, \\
c_1s_1 + c_2(s - s_1) & s_1 < s \leq s_2, \\
c_1s_2 + c_2(s_2 - s_1) + c_3(s - s_2) & s > s_2,
\end{cases}
\]  
(7.20)

where \( s_1, s_2, c_1, c_2 \) and \( c_3 \) are constants. The elasticity tensor \( A \) satisfies

\[
(A \varepsilon)_{ij} = \frac{E \kappa}{1 - \kappa^2} (\varepsilon_{11} + \varepsilon_{22}) \delta_{ij} + \frac{E}{1 + \kappa} \varepsilon_{ij},
\]  
(7.21)

with \( 1 \leq i,j \leq 2 \). \( E \) is the Young modulus, \( \kappa \) the Poisson ratio of the material and \( \delta_{ij} \) denotes the Kronecker symbol. For the volume and surface forcing, we set

\[
f_0 = (0, -0.1 \sin(t)) \ N/m^2,
\]  
(7.22)

\[
f_2 = \begin{cases} 
(0, 0) \ N/m & \text{on } \{L_1\} \times (0, L_2), \\
(0, -0.2 \sin(t) \sin(\pi x/2)) \ N/m & \text{on } [0, L_1] \times \{L_2\}.
\end{cases}
\]  
(7.23)

We test the convergence behavior for the three numerical schemes. The projection on the Von Mises convex is not considered in the convergence tests; thus we let \( \mu = 0 \) in Problem 31–33. Values of the other parameters are

\[
L_1 = 2m, \ L_2 = 1m, \ E = 2N/m^2, \ \kappa = 0.3, \\
\alpha_j = 0.5, \ g = 0.15m, \ S = 1N, \ \mathcal{B}(t) = e^{-t}, \ T = 0.5, \\
s_1 = 0.1, \ s_2 = 0.15, \ c_1 = 0.1, \ c_2 = -0.1, \ c_3 = 0.4.
\]

The uniform rectangular finite element partitions are introduced to numerically solve the above problem. The numerical solution with \( h = k = 1/256 \) is used as the “reference” solution in computing numerical solution errors, and the temporal and spatial convergence orders in the \( H^1 \) norm will be shown.

**Example 1 (First order Scheme)** In Tables 2 and 3, we present the temporal and spatial convergence orders of first-order scheme respectively, and the first-order accuracy in both time and space are shown.
Example 2 (Second order scheme by fixed-point iteration) In Tables 4 and 5 we present the temporal and spatial convergence orders of second-order fixed-point iteration scheme, respectively, and the second-order accuracy in time, first-order in space are shown.

In addition, we compute the $H_1$ errors for different mesh grid sizes. Two refinement paths are taken to be $k^2 = h$ and $k = h$. The results are displayed in Table 6 and the first-order accuracy is shown for both the two refinement paths in Figure 2, which indicates the second-order convergence order in time.

| $h$  | $k$  | $\|u(\cdot,T) - u_{N}^{kh}\|_1$ | order |
|------|------|-------------------------------|-------|
| 1/256 | 1/4  | 2.30136e-3                   | -     |
| 1/256 | 1/8  | 1.01388e-2                   | 0.8433|
| 1/256 | 1/16 | 2.92633e-3                   | 0.9154|
| 1/256 | 1/32 | 4.16384e-5                   | 1.8952|

Table 4: Convergence orders with spatial step-size fixed for second-order scheme by fixed-point iteration.

Example 3 (Second order scheme with extrapolation) In Tables 7 and 8 we present the temporal and spatial convergence orders of second-order scheme with extrapolation, respectively, and the second-order accuracy in time, first-order in space are shown.

In addition, we compute the $H_1$ errors for different mesh grid sizes. Two refinement paths are taken to be $k^2 = h$ and $k = h$. The results are displayed in Table 9 and the first-order accuracy is shown for both the two refinement paths in Figure 3, which indicates the second-order convergence order in time.

In Figure 4 the normal displacement on the boundary $\Gamma_3$ at time $T = 0.5$ for the
Table 5: Convergence order of the errors with temporal step-size fixed for second-order scheme by fixed-point iteration.

| $h$   | $k$   | $\|u(\cdot, T) - u_{nk}^{kh}\|_1$ | order |
|-------|-------|----------------------------------|-------|
| 1/8   | 1/256 | 1.81822e-2                      | -     |
| 1/16  | 1/256 | 1.01334e-2                      | 0.8434|
| 1/32  | 1/256 | 5.51570e-3                      | 0.8775|
| 1/64  | 1/256 | 2.92399e-3                      | 0.9156|

Table 6: Comparison of the $H_1$ errors in the refinement path $k^2 = h$ and $k = h$ for second-order scheme by fixed-point iteration.

| mesh grid size | $\|u(\cdot, T) - u_{nk}^{kh}\|_1$ | difference between the front two |
|----------------|----------------------------------|---------------------------------|
| $h = 1/16$    | 1.03714e-2                      | 2.38e-4                         |
| $h = 1/36$    | 4.66049e-3                      | 1.09e-4                         |
| $h = 1/64$    | 2.98351e-3                      | 5.96e-5                         |
| $h = 1/100$   | 1.44761e-3                      | 4.88e-5                         |
| $h = 1/144$   | 8.58858e-4                      | 4.18e-5                         |
| $h = 1/196$   | 4.00877e-4                      | 5.12e-5                         |

Figure 2: The loglog plot of $H_1$ errors with $h = 1/16, 1/36, 1/64, 1/100, 1/144, 1/196$ for second-order fixed-point scheme.
| $h$  | $k$  | $\|u(\cdot, T) - u_N^{kh}\|_1$ | order |
|------|------|-------------------------------|-------|
| $1/256$ | $1/4$ | $1.02222e-2$                  | -     |
| $1/256$ | $1/8$ | $1.15624e-3$                  | $3.1442$ |
| $1/256$ | $1/12$ | $3.53015e-4$                  | $2.9261$ |
| $1/256$ | $1/16$ | $2.41930e-4$                  | $1.3135$ |
| $1/256$ | $1/32$ | $5.74370e-5$                  | $2.0745$ |

Table 7: Convergence orders with spatial step-size fixed for second-order scheme with extrapolation.

| $h$  | $k$  | $\|u(\cdot, T) - u_N^{kh}\|_1$ | order |
|------|------|-------------------------------|-------|
| $1/8$  | $1/256$ | $1.81823e-2$                  | -     |
| $1/16$ | $1/256$ | $1.01335e-2$                  | $0.8434$ |
| $1/32$ | $1/256$ | $5.51576e-3$                  | $0.8775$ |
| $1/64$ | $1/256$ | $2.92403e-3$                  | $0.9156$ |

Table 8: Convergence orders with temporal step-size fixed for second-order scheme with extrapolation.

| $h$  | $k$  | $\|u(\cdot, T) - u_N^{kh}\|_1$ | difference between the front two |
|------|------|-------------------------------|---------------------------------|
| $h = 1/16$ | $k = h$ | $1.42988e-2$                  | $1.01343e-2$ 4.16e-3 |
| $h = 1/36$ | $k = h$ | $6.89462e-3$                  | $4.55108e-3$ 2.34e-3 |
| $h = 1/64$ | $k = h$ | $3.13943e-3$                  | $2.92396e-3$ 2.15e-4 |
| $h = 1/100$ | $k = h$ | $1.49765e-3$                  | $1.39884e-3$ 9.88e-5 |
| $h = 1/144$ | $k = h$ | $8.90577e-4$                  | $8.17048e-4$ 7.35e-5 |
| $h = 1/196$ | $k = h$ | $3.93167e-4$                  | $3.49639e-4$ 4.35e-5 |

Table 9: Comparison of the $H_1$ errors in the refinement path $k^2 = h$ and $k = h$ for second-order scheme with extrapolation.
Error vs Mesh grid size for second-order scheme with extrapolation

Figure 3: The plot of $H_1$ errors with $h = 1/16, 1/36, 1/64, 1/100, 1/144, 1/196$ for second-order scheme with extrapolation.

three numerical schemes is shown, from which we can see, the maximum penetration is reached as the forcing increased.

Figure 4: Normal displacement on $\Gamma_3$ at time $T = 0.5$ of three numerical schemes.

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