MATTER SEEN AT MANY SCALES
AND
THE GEOMETRY OF AVERAGING
IN RELATIVISTIC COSMOLOGY

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Abstract

We investigate the scale–dependence of Eulerian volume averages of scalar functions on Riemannian three–manifolds. We propose a complementary view of a Lagrangian scaling of variables as opposed to their Eulerian averaging on spatial domains. This program explains rigorously the origin of the Ricci deformation flow for the metric, a flow which, on heuristic grounds, has been already suggested as a possible candidate for averaging the initial data set for cosmological spacetimes.

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1 Introduction

Averaged inhomogeneous cosmologies are in the forefront of interest, since cosmological parameters like the rate of expansion or the mass density are to be considered as volume–averaged quantities and only these can be compared with observations. For this reason the relevant parameters are intrinsically scale–dependent and one wishes to control this dependence without restricting the cosmological model by unphysical assumptions. Consider a three–dimensional manifold equipped with a Riemannian metric \((\Sigma, g)\) as a part of the initial data set for a cosmological spacetime. On such a hypersurface we may select a simply–connected spatial region and evaluate certain average properties of the physical (scalar) variables on that domain such as, e.g., the volume–averaged mass density field, or the volume–averaged scalar curvature. These (covariant) average values are functionally dependent on position and shape of the chosen domain of averaging. Let us now be more specific and relate the averages to scaling properties of the physical variables. It is then natural to identify the domain with a geodesic ball centred on a given position, and introduce the spatial scale via the geodesic radius of the ball. We may consider the variation of this radius and so explore the scaling properties of average characteristics on the whole manifold. Doing so for every position we arrive at smoothed fields that capture the effective scale–dependent dynamics and so naturally provide the theoretical values of observables that are drawn from a limited survey volume of a cosmological spacetime.

We may call this point of view Eulerian – and everybody would first think of this point of view – in the sense that the spatial manifold is explored passively by blowing up the geodesic balls covering larger and larger volumes. On the contrary, the key idea of our approach consists in demonstrating that a corresponding active averaging procedure can be devised which is Lagrangian: we hold the ball at a fixed (Lagrangian) radius, and deform the dynamical variables inside the ball such that the deformation corresponds to the smoothing of the fields. The variation with scale of, e.g., the density field or the metric is then mirrored by one–parameter families of successively deformed density fields and metrics. We shall show rigorously that this deformation corresponds to a first variation of the metric along the Ricci tensor, known as the Ricci flow. Since this flow has received great attention in the mathematical literature – major contributions are due to Hamilton – we shall so translate the averaging procedure into a well–studied deformation flow, much in the spirit of a renormalization group flow. This analysis, putting on rigorous grounds the previous, largely heuristic use of the Ricci flow in smoothing out relativistic spacetimes, is part of a larger program aimed to a full–fledged analysis of averaging and scaling in relativistic cosmology. Such a program features a mass–preserving Ricci flow and its variants in order to provide a subtle technique for modelling an effective dynamics of inhomogeneities in the Universe.
2 Averaging and Scaling put into a Geometrical Perspective

In order to characterize the correct conceptual framework for addressing averaging and scaling properties in relativistic cosmology, let us remark that in general relativity we basically have one scaling variable related to the unit of distance: we can express the unit of time in terms of the unit of distance using the speed of light. Similarly, we can express the unit of mass through the unit of distance by using Newton’s gravitational constant. This remark implies that the scaling properties of Einstein’s equations typically generate a mapping between distinct initial data sets, (in this sense we are basically dealing with a renormalization group transformation). As we shall see below in detail, a rigorous characterization of an averaging procedure in relativistic cosmology is indeed strictly connected to the scaling geometry of the initial data set for Einstein’s equations.

In order to discuss the scaling properties and the averaging procedure associated with an admissible set of initial data for a cosmological spacetime (we think of a setting within the Arnowitt–Deser–Misner formulation of general relativity), a basic issue is to characterize explicitly a scale–dependent averaging for the empirical mass distribution $\rho$. This will be the main theme of this paper.

2.1 Matter seen at many scales

In order to characterize the scale over which we are smoothing the empirical mass distribution $\rho$, we need to study the distribution $\rho$ by looking at its average behavior on (geodesic) balls on $(\Sigma, g)$ with different centers and radii. The idea is to move from the function $\rho: \Sigma \rightarrow \mathbb{R}^+$ to an associated function, defined on $\Sigma \times \mathbb{R}^+$, and which captures some aspect of the behavior of the given $\rho$ on average, at different scales and locations. The simplest function of this type is provided by the local volume average

$$\langle \rho \rangle_{B(p,r)} \doteq \frac{1}{\text{Vol}(B(p;r))} \int_{B(p;r)} \rho d\mu_g,$$

where $p \in \Sigma$ is a generic point, $d\mu_g$ is the Riemannian volume element associated with $(\Sigma, g)$, and $B(p;r)$ denotes the geodesic ball at center $p$ and radius $r$ in $(\Sigma, g)$, i.e.,

$$B(p;r) \doteq \{ q \in (\Sigma, g) : d_g(p,q) \leq r \},$$

where $d_g(p,q)$ denotes the distance, in $(\Sigma, g)$, between the point $p$ and $q$. Note that if

$$\text{diam} \doteq \sup \{ d_g(p,q) : p,q \in (\Sigma, g) \}$$
denotes the diameter of \((\Sigma, g)\), then as \(r \to \text{diam}\), we get

\[
\langle \varrho \rangle_{B(p,r)} \to \langle \varrho \rangle_{\Sigma} \equiv \frac{1}{\text{Vol}\left((\Sigma, g)\right)} \int_{\Sigma} \varrho \, d\mu_{g},
\]

at any point \(p \in \Sigma\). Conversely, if \(\varrho : \Sigma \to \mathbb{R}^+\) is locally summable then

\[
\lim_{r \to 0} \langle \varrho \rangle_{B(p,r)} = \varrho(p),
\]

for almost all points \(p \in \Sigma\). The passage from \(\varrho\) to \(\langle \varrho \rangle_{B(p,r)}\) corresponds to replacing the position–dependent empirical distribution of matter in \(B(p;r)\) by a locally uniform distribution \(\langle \varrho \rangle_{B(p,r)}\) which is constant over the typical scale \(r\). Note that for the total (material) mass contained in \(B(p;r)\), we get

\[
M(B(p;r)) \doteq \int_{B(p;r)} \varrho \, d\mu_{g} = \text{Vol}(B(p;r)) \langle \varrho \rangle_{B(p;r)},
\]

and

\[
\lim_{r \to \text{diam}} M(B(p;r)) = M(\Sigma) \doteq \int_{\Sigma} \varrho \, d\mu_{g},
\]

where \(M(\Sigma)\) is the total (material) mass contained in \((\Sigma, g)\). Our expectations in \(\langle \varrho \rangle_{B(p;r)}\) are motivated by the fact that on Euclidean 3–space \((\mathbb{R}^3, \delta_{ab})\), if \(\varrho\) is a bounded function, then the local average \(\langle \varrho \rangle_{B(p;r)}\) is a Lipschitz function on \(\mathbb{R}^3 \times \mathbb{R}^+\), i.e.,

\[
\left| \langle \varrho \rangle_{B(p;r)} - \langle \varrho \rangle_{B(q;r)} \right| \leq C_0 d_h(p,q),
\]

where \(C_0\) is a constant and \(d_h(p,q)\) is the (hyperbolic!) distance between \(p\) and \(q\). Thus, the local averages \(\langle \varrho \rangle_{B(p;r)}\) do not oscillate too wildly as \(p\) and \(r\) vary, and the replacement of \(\varrho\) by \(\{\langle \varrho \rangle_{B(p;r)}\}_{p \in \mathbb{R}^3}\) indeed provides an averaging of the original matter distribution over the length scale \(r\). It is not obvious that such a nice behavior carries over to the Riemannian manifold \((\Sigma, g)\). The point is that, even if the local averages \(\{\langle \varrho \rangle_{B(p;r)}\}_{p \in \Sigma}\) provide a controllable device of smoothing the matter distribution at the given scale \(r\), they still depend in a sensible way on the geometry of the typical ball \(B(p;r)\) as we vary the averaging radius. In this connection we need to understand how, as we rescale the domain \(B(p;r)\), the local average \(\langle \varrho \rangle_{B(p;r)}\) depends on the underlying geometry of \((\Sigma, g)\). The reasoning here is slightly delicate, so we go into a few details that require some geometric preliminaries.

Let us denote by

\[
\exp_p : T_p \Sigma \to \Sigma \quad (\vec{v}, r) \mapsto \exp_p(r\vec{v})
\]
the exponential mapping at \( p \in (\Sigma, g) \), i.e., the map which to the vector \( r \vec{v} \in T_p \Sigma \cong \mathbb{R}^3 \) associates the point \( \exp_p(r \vec{v}) \in \Sigma \) reached at “time” \( r \in \mathbb{R}_+ \) by the unique geodesic issued at \( p \in \Sigma \) with unit speed \( \vec{v} \in \mathbb{S}^2(1) \). Let \( r > 0 \) be such that \( \exp_p \) is defined on the Euclidean ball

\[
B_E(0, r) = \{ y \in T_p \Sigma \cong \mathbb{R}^3 : |y| \leq r \}
\]

and \( \exp_p : B_E(0, r) \to B(p, r) \subset \Sigma \) is a diffeomorphism onto its image. The largest radius \( r \) for which this happens, as \( p \) varies in \( \Sigma \), is the injectivity radius \( inj_M \) of \((\Sigma, g)\). Recall that a curve \( c_{p, q} : \mathbb{R} \supset I \to (\Sigma, g) \) connecting the base point \( p \) with the point \( q \) is called a segment, if its length \( L(c_{p, q}) \) is such that \( L(c_{p, q}) = d_g(p, q) \), and if it is parametrized proportional to arclength. The domain in \( T_p \Sigma \) over which geodesics issued from \( p \) are segments (i.e., are distance–realizing) is called the segment domain and is defined according to

\[
\overline{\text{Seg}}(p) = \{ \vec{v} \in T_p \Sigma; \exp_p(r \vec{v}) : [0, 1] \to \Sigma \text{ is a segment} \}.
\]

Note that \( \overline{\text{Seg}}(p) \) is a closed star–shaped subset of \( T_p \Sigma \), and \( \Sigma = \exp_p(\overline{\text{Seg}}(p)) \).

The exponential map acts diffeomorphically over the interior \( \text{Seg}(p) \) of \( \overline{\text{Seg}}(p) \), and \( \text{Seg}(p) - \overline{\text{Seg}}(p) \), the cut locus of \( p \) in \( T_p \Sigma \), is a set whose image under \( \exp_p \) has zero measure in \( \Sigma \). In particular, it follows that

\[
M(\Sigma) \cong \int_\Sigma g(y) d\mu_g = \int_{\overline{\text{Seg}}(p)} g(\exp_p) \exp_p^*(d\mu_g),
\]

where \( \exp_p^*(d\mu_g) \) denotes the pull–back of the Riemannian measure \( d\mu_g \) over \( \text{Seg}(p) \subset T_p \Sigma \) under the exponential mapping.

Let \( B(p; r_0) \) denote a given geodesic ball of radius \( r_0 < inj_M \), and for vector fields \( X, Y, \) and \( Z \), in \((B(p; r), g)\) let \( R(X, Y)Z = R^a_{bcd} X^c Y^d Z^b \) be the corresponding curvature operator. Since \( r : (B(p; r), g) \to \mathbb{R} \) is a distance function (i.e., \( |\nabla r| = 1 \)), the geometry of \( B(p; r) \) can be described by the following set of equations \cite{12}:

\[
(\nabla_{\partial_r} S)(X) + S^2(X) = -R(X, \partial_r)\partial_r, \tag{13}
\]

\[
(L_{\partial_r} g)(X, Y) = 2g(S(X), Y), \tag{14}
\]

\[
\nabla_{\partial_r} S = L_{\partial_r} S, \tag{15}
\]

where \( \partial_r = \nabla r \) is the gradient of \( r \). \( L_{\partial_r} g \) is the Lie derivative of the metric in the radial direction \( \partial_r \), and the shape tensor \( S = \nabla^2 r \) is the Hessian of \( r \).

Such equations follow from the Gauss–Weingarten relations applied to study the \( r \)-constant slices \( U_r = \{ \exp_p(r \vec{v}) \in \Sigma : r = \text{const.} \} \) which are the images in \((B(p; r), g)\) of the standard Euclidean 2–spheres \( \mathbb{S}^2(r) \subset T_p \Sigma \cong \mathbb{R}^3 \).

\footnote{The Hessian \( S = \nabla^2 r \) is basically the second fundamental form of the immersion \( U_r \to (B(p; r), g) \). We use the equivalent characterization of \textit{shape tensor} in order to avoid confusion with the standard second fundamental form of use in relativity.}
shape tensor $S$ measures how the bidimensional metric $g^{(2)}$ induced on $U_r$ by the embedding in $(B(p; r), g)$ rescales as the radial distance $r$ varies. Further properties of the equations $(13)$ and $(14)$ that we need are best seen by using polar geodesic coordinates and harmonic coordinates. Recall that normal exponential coordinates at $p$ are geometrically defined by

$$\exp_p^{-1} : B(p; r) \to T_p \Sigma \simeq \mathbb{R}^3$$

$$q \mapsto \exp_p^{-1}(q) = (y^i),$$

where $(y^i)$ are the Cartesian components of the velocity vector $\vec{v} \in T_p \Sigma$ characterizing the geodesic segment from $p$ to $q$. Such coordinates are unique up to the chosen identification of $T_p \Sigma$ with $\mathbb{R}^3$. Since we are dealing with a radial rescaling, for our purposes a suitable identification is the one associated with the use of polar coordinates in $T_p \Sigma \simeq \mathbb{R}^3$. We therefore introduce an orthonormal frame $\{e_1, e_2, e_3\}$ in $T_p \Sigma$ such that $e_1 = \partial_r$ and with $\{e_2, e_3\}$ an orthonormal frame on the unit 2–sphere $S^2(1) \subset T_p \Sigma$. We can extend such vector fields radially to the whole $T_p \Sigma$; we consider also the dual coframe $\{\theta^2, \theta^3\}$ associated with $\{e_2, e_3\}$. The introduction of such a polar coordinate system in $T_p \Sigma$ is independent of the metric $g$ and thus is ideally suited for discussing geometrically the $r$–scaling properties of $\langle g \rangle_{B(p; r)}$. If we pull–back to $T_p \Sigma$ the metric $g$ of $B(p; r) \subset \Sigma$, we get

$$g = dr^2 + g(e_A, e_B)\theta^A\theta^B, \quad A, B = 1, 2,$$

where the components $g_{AB} = g(e_A, e_B) = g^{(2)}(r, \vartheta, \varphi)$ are functions of the polar coordinates $(\vartheta, \varphi)$ in $T_p \Sigma \simeq \mathbb{R}^3$, associated with the coframe $\{\theta^2, \theta^3\}$. Note that such a local representation of the metric holds throughout the local chart $(B(p; r), \exp_p^{-1})$ and not just at $p$. In Cartesian coordinates in $T_p \Sigma \simeq \mathbb{R}^3$ one recovers the familiar expression

$$g_{ab} = \delta_{ab} - \frac{1}{3}R_{kbl}^{\ ab}(p)y^k y^l + O(|y|^3).$$

In polar geodesic coordinates $(r, \vartheta, \varphi)$ the equations $(13)$ and $(14)$ take the explicit form

$$\partial_r S^i_j = -S^i_k S^k_j - R^i_{\ rjr},$$

$$\partial_r g_{ij} = 2S^i_j g_{kj},$$

where $R^i_{\ rjr}$ denotes the radial components of the curvature tensor. By taking the trace of both equations we get

$$\partial_r S = -S^i_k S^k_i - Ric(\partial_r, \partial_r),$$

$$g^{ij} \partial_r g_{ij} = 2S,$$

$^4$Latin indices run through 1, 2, 3; we adopt the summation convention.
where $\text{Ric}(\partial_r, \partial_r) = R^i_{r_ir}$ denotes the $\partial_r$ component of the Ricci tensor, and $S = S^i_k \delta^k_i$ is the rate of volume expansion of $g^{(2)}(r, \vartheta, \phi)$, (the first equation in (20) is nothing but the Jacobi operator coming from the second variation of the area for $g^{(2)}(r, \vartheta, \phi)$). If we assume that the curvature $R(X,Y)Z$ shall then be given by fixing the $(\vartheta, \phi)$–dependence in the factorized metric (13), we can consider (19) as a system of decoupled ordinary differential equations describing the rescaling of the geometry of $B(p; r)$ in terms of the one–parameter flow of immersions $S^2(r) \hookrightarrow (U_r, g^{(2)}(r, \vartheta, \phi))$. Note in particular that the shape tensor matrix $(S^i_j)$ is characterized, from the first set of equations (19), as a functional of the given curvature tensor $(R^i_{rjr})$. In such a setting the equation $\partial_r g_{ij} = 2S^i_k g_{kj}$ can be interpreted by saying that the metric rescales radially along the (curvature–dependent) shape tensor $S^i_j$. In a sufficiently small neighborhood of the pole $p$, from the expression (18) and from $\partial_r = 1/r \partial_i$, we get that in Cartesian coordinates we can write

$$\partial_r g_{ab} = -\frac{2}{9} r R_{ab}(p) - \frac{2}{3} \left[ \frac{y^i y^k}{r} - \frac{1}{3} \delta^{ik} r \right] R_{abk}(p) + O(r^2),$$

where we have evidentiated the trace–free part $\left[ y^i y^k - \frac{1}{3} \delta^{ik} r^2 \right]$ of the Cartesian tensor $y^i y^k$. Roughly speaking, the term $\frac{2}{3} \left[ \frac{y^i y^k}{r} - \frac{1}{3} \delta^{ik} r \right] R_{abk}(p)$ represents the shear part of the rescaling of the geometry of the spherical surfaces $U_r \equiv \{ \exp_p(r\vec{v}) \in \Sigma : r = \text{const.} \}$ as $r$ varies in the neighborhood of the pole $p$. The term $-\frac{2}{3} r R_{ab}(p)$ is instead responsible for the radial rescaling in the geometry of $U_r$. Thus, one may say that in a sufficiently small neighborhood of the point $p$ the metric rescales radially in the direction of its Ricci tensor. This latter remark will turn out quite useful in understanding the geometric rationale behind the choice of a proper averaging procedure for the geometry of $(\Sigma, g)$.

Even if polar geodesic coordinates suggest themselves as the most natural labels for the points of $B(p; r)$, they suffer from the basic drawback that their domain of definition cannot be a priori estimated and strongly depends on the local geometry of $(\Sigma, g)$. In this connection, a much better control on the geometry of the balls $B(p; r)$, and hence on $(\vec{q})_{B(p; r)}$, can be achieved by labelling the points $\exp_p^{-1}(q) \in B_E(0, r)$ with harmonic coordinates, i.e., a coordinate system $\{X^i\}$ such that the coordinate functions $X^i$ are harmonic functions with respect to the Laplacian on $(\Sigma, g)$. We can do this by starting from the given (Cartesian) normal coordinates $\{y^k\}$, and look for a diffeomorphism on a sufficiently small Euclidean ball $B_E(0, r) \subset \mathbb{R}^3$,

$$\Phi : B_E(0, r) \to B_E(0, r)$$

$$y^k = \exp_p^{-1}(q) \mapsto \Phi(y^k) = X^i$$

such that

$$\left\{ \begin{array}{lcl}
\Delta \Phi^k &=& \frac{1}{\sqrt{\det(g_{ab})}} \partial_i \left( \sqrt{\det(g_{ab})} g^{ij} \partial_j \Phi^k \right) = 0 \\
\Phi^k|_{\partial B_E(0, r)} &=& Id.
\end{array} \right.$$
The standard theory of elliptic partial differential equations implies that the harmonic functions so characterized do form a coordinate system in \( B^E(0, r) \). The important observation is that such harmonic coordinates can be introduced on balls of an a priori size as soon as the manifold \((\Sigma, g)\) has bounded sectional curvature and its injectivity radius is bounded below. Note that from \( \exp_p^{-1} : B(p, r) \to T_p \Sigma \) and a generic (i.e., not necessarily harmonic) diffeomorphism \( \Phi : B_E(0, r) \to B_E(0, r) \) we can define the map

\[
F = \Phi \circ \exp_p^{-1} : (B(p, r), g) \xrightarrow{\exp_p^{-1}} (B_E(0, r), \delta) \xrightarrow{\Phi} (B_E(0, r), \delta),
\]

where \( \delta \) denotes the Euclidean metric on \( B_E(0, r) \). To any such map we can associate the scalar “energy density” \( e(x) \).

\[
e(x) = \|dF\|^2 = g^{ij} \delta_{lm} \frac{\partial F^l}{\partial x^i} \frac{\partial F^m}{\partial x^j},
\]

where the \( \{x^i\} \) denote given generic coordinates of the points \( q \) of \((B(p, r), g)\) and where

\[
dF = \frac{\partial F^l}{\partial x^i} dx^i \otimes \frac{\partial}{\partial F^l}
\]

is the differential of the map \( F \), thought of as a section of the bundle

\[
T^*\Sigma \otimes F^{-1}T\mathbb{R}^3\big|_{B(p, r)}.
\]

Note that the pull–back bundle \( F^{-1}T\mathbb{R}^3 \) over \( B(p, r) \) inherits the flat metric \( \delta_{lm}(F(x)) \). Thus \( T^*\Sigma \otimes F^{-1}T\mathbb{R}^3\big|_{B(p, r)} \) is naturally endowed with the metric \( g^{ij} \) on \( T^*\Sigma \) and \( \delta_{lm} \) on \( F^{-1}T\mathbb{R}^3 \), a metric which characterizes the norm \( \|dF\|^2 \).

Alternatively, \( e(x) \) is the trace with respect to the metric of \( T^*\Sigma \) of the pull–back by \( F \) of the flat metric tensor \((B_E(0, r), \delta)\). Such remarks emphasize the fact that, even if the value of \( e(x) \) seems to depend on the choice of local coordinates, it is indeed a well–known geometrical quantity: the harmonic map energy density associated with the map \( F : (B(p, r), g) \to (B_E(0, r), \delta) \). As to its meaning in our setting, note that we can always choose the given generic coordinates \( \{x^i\} \) to be the original geodesic coordinates \( \{y^i\} \). By exploiting such a remark, from the asymptotic expansion \( \chi \), we easily get that in a sufficiently small neighborhood of the point \( p \), the scalar \( e(x) \) can be written as

\[
e(y) = \left( \delta^{ab} + \frac{1}{3} R_{kli}(p)y^k y^l + O(|y|^2) \right) \delta_{rs} \frac{\partial F^r}{\partial y^a} \frac{\partial F^s}{\partial y^b}.
\]

If we further assume that the diffeomorphism \( \Phi : B_E(0, r) \to B_E(0, r) \) is the identity, (i.e., \( F \) reduces to \( \exp_p^{-1} \)), then \( \frac{\partial F^r}{\partial y^a} = \delta^r_a \), and we get

\[
e(y)|_{F=\exp_p^{-1}} = 3 + \frac{1}{3} R_{kli}(p)y^k y^l + O(|y|^2)
\]
showing that \( e(x) \big|_{F=\exp_p^{-1}} \) is, roughly speaking, a measure of the curvature associated with \( (B(p; r), g; \exp_p^{-1}) \). As we shall see, the quantity \( e(x) \) will play a basic role in understanding the geometric rationale for a proper averaging strategy of the geometry of \( (\Sigma, g) \).

Guided by such geometrical features of geodesic balls, let us go back to the study of the scaling properties of \( \langle \rho \rangle_{B(p;r)} \). To this end, for any \( r \) and \( s \) such that \( r + s < inj \Sigma \), let us consider the one–parameter family of diffeomorphisms (geodesic ball dilatations)

\[
H_s : (\Sigma, p) \rightarrow (\Sigma, p)
\]

defined by flowing each point \( q \in B(p; r) \) a distance \( s \) along the unique radial geodesic segment issued at \( p \in \Sigma \) and passing through \( q \). Let us remark that, for any \( r \) such that \( r_0 \leq r < inj_M \), we can formally write

\[
B(p; r) = H(r - r_0)B(p; r_0).
\]

Thus, in a neighborhood of \( B(p; r_0) \) and for sufficiently small \( r \), we get

\[
M(B(p; r)) \doteq \int_{B(p; r)} \rho d\mu_g = \int_{B(p; r_0)} H^*_{(r-r_0)}(\rho d\mu_g),
\]

where \( H^*_{(r-r_0)}d(\rho d\mu_g) \) is the Riemannian measure obtained by pulling back \( \rho d\mu_g \) under the action of \( H_{(r-r_0)} \). By differentiating (32) with respect to \( r \), we have

\[
\frac{d}{dr} M(B(p; r)) = \lim_{h \to 0} \frac{[M(B(p; r + h)) - M(B(p; r))]}{h}
\]

\[
= \lim_{h \to 0} \left[ \int_{B(p; r_0)} H^*_{(r-r_0)+h}(\rho d\mu_g) - \int_{B(p; r_0)} H^*_{(r-r_0)}(\rho d\mu_g) \right]/h.
\]

Since \( H_{(r-r_0)+h} = H_{(r-r_0)} \circ H_h \), we can write the above expression as

\[
\lim_{h \to 0} \left[ \int_{B(p; r_0)} \frac{H^*_{(r-r_0)}[H^*_{h}(\rho d\mu_g) - (\rho d\mu_g)]}{h} \right]
\]

\[
= \lim_{h \to 0} \left[ \int_{B(p; r)} \frac{H^*_{h}(\rho d\mu_g) - (\rho d\mu_g)}{h} \right]
\]

\[
= \int_{B(p; r)} \lim_{h \to 0} \left[ \frac{H^*_{h}(\rho d\mu_g) - (\rho d\mu_g)}{h} \right],
\]
from which it follows that

$$\frac{d}{dr} M(B(p;r)) = \int_{B(p;r)} L_{\partial_r} (\varrho d\mu_g)$$

(35)

$$= \int_{B(p;r)} \left( \frac{\partial}{\partial r} \varrho + g \text{div}(\partial_r) \right) d\mu_g$$

$$= \int_{B(p;r)} \left( \frac{\partial}{\partial r} \varrho + \frac{1}{2} g^{ab} \frac{\partial}{\partial r} g_{ab} \right) d\mu_g,$$

where $L_{\partial_r}$ and $\text{div}(\partial_r)$ denote the Lie derivative along the vector field $\partial_r$ and its divergence, respectively, and where we have exploited the well-known expression for the derivative of a volume density,

$$\frac{\partial}{\partial r} \sqrt{g} = \frac{1}{2} \sqrt{g} g^{ab} \frac{\partial}{\partial r} g_{ab}.$$  

(36)

With these preliminary remarks along the way, it is straightforward to compute the rate of variation with $r$ of the local average $\langle \varrho \rangle_{B(p;r)}$ since (35) implies

$$\frac{d}{dr} \langle \varrho \rangle_{B(p;r)} = \left\langle \frac{\partial}{\partial r} \varrho \right\rangle_{B(p;r)}$$

(37)

$$+ \frac{1}{2} \left\langle gg^{ab} \frac{\partial}{\partial r} g_{ab} \right\rangle_{B(p;r)} - \frac{1}{2} \langle \varrho \rangle_{B(p;r)} \left\langle g^{ab} \frac{\partial}{\partial r} g_{ab} \right\rangle_{B(p;r)},$$

where $\langle f \rangle_{B(p;r)}$ denotes the volume average of $f$ over the ball $B(p;r)$. Explicitly, by exploiting (36), we get

$$\frac{d}{dr} \langle \varrho \rangle_{B(p;r)} = \left\langle \frac{\partial}{\partial r} \varrho \right\rangle_{B(p;r)} + \langle \varrho S \rangle_{B(p;r)} - \langle \varrho \rangle_{B(p;r)} \langle S \rangle_{B(p;r)}.$$  

(38)

Thus, the local average $\langle \varrho \rangle_{B(p;r)}$ feels the fluctuations in the geometry as we vary the scale, fluctuations represented by the shape tensor terms

$$\langle \varrho S \rangle_{B(p;r)} - \langle \varrho \rangle_{B_E(0;r)} \langle S \rangle_{B(p;r)}$$  

(39)

governed by the curvature in $B(p;r)$ according to (20), and expressing a geometric non-commutativity between averaging over the ball $B(p;r)$ and rescaling its size. Since the curvature varies both in the given $B(p;r)$ and when we consider distinct base points $p$, the above remarks indicate that the local averages $\langle \varrho \rangle_{B(p;r)}$ are subjected to the accidents of the fluctuating geometry of $(\Sigma, g)$. In other words, there is no way of obtaining a proper smoothing of $g$ without smoothing out at the same time the geometry of $(\Sigma, g)$.

### 2.2 Eulerian averaging and Lagrangian scaling

The use of the exponential mapping in discussing the geometry behind the local averages $\langle \varrho \rangle_{B(p;r)}$ makes it clear that we are trying to measure how different
the $\langle g \rangle_{B(p,r)}$ are from the standard average over Euclidean balls. In so doing we think of $\exp_p : T_p\Sigma \to \Sigma$ as maps from the fixed space $B_E(0, r)$ into the manifold $(\Sigma, g)$. In this way we are implicitly trying to transfer information from the manifold $(\Sigma, g)$ into domains of $\mathbb{R}^3$ which we would like to be, as far as possible, to be independent of the accidental geometry of $(\Sigma, g)$ itself. Indeed, any averaging would be quite difficult to implement if the reference model varies with the geometry to be averaged. As already emphasized, this latter task is only partially accomplished by the exponential mapping, since the domain over which $\exp_p : T_p\Sigma \to \Sigma$ is a diffeomorphism depends on $p$ and on the actual geometry of $(\Sigma, g)$. We can go a step further in this direction by not just the given $(\Sigma, g)$ but rather a whole family of Riemannian manifolds. To start with let us remark that, if we fix the radius $r_0$ of the Euclidean ball $B_E(0; r_0) \subset T_p\Sigma$ and consider the family of exponential mappings, $\exp_{(p,\beta)} : T_p\Sigma \to (\Sigma, g(\beta))$, associated with a corresponding one–parameter family of Riemannian metrics $g_{ab}(\beta)$, $0 \leq \beta < +\infty$, with $g_{ab}(\beta = 0) = g_{ab}$, then $B(p; r_0)$ becomes a functional of the set of Riemannian structures associated with $g(\beta)_{ab}$, $0 \leq \beta < +\infty$, i.e.,

$$B(p; r_0) \mapsto B_{\beta}(p; r_0) = \exp_{(p,\beta)} [B_E(0; r_0)].$$

In this way, instead of considering just a given geodesic ball $B(p; r_0)$, we can consider, as $\beta$ varies, a family of geodesic balls $B_{\beta}(p; r_0)$, all with the same radius $r_0$ but with distinct inner geometries $g_{ab}(\beta)$. Since $B_{\beta=0}(p; r_0) = B(p; r_0)$, such balls can be thought of as obtained from the given one $B(p; r_0)$ by a smooth continuous deformation of its original geometry. Under such deformation also $\langle g \rangle_{B_{(p,\beta)}}$ becomes a functional, $\langle g \rangle_{B_{\beta}(p; r_0)}$, of the one–parameter family of Riemannian structures associated with $g_{ab}(\beta)$, $0 \leq \beta < +\infty$. The elementary but basic observation in order to take properly care of the geometrical fluctuations in $\langle g \rangle_{B_{(p,\beta)}}$ is that the right member of (57) has precisely the formal structure of the linearization (i.e., of the variation) of the functional $\langle g \rangle_{B_{(p,\beta)}}$ in the direction of the (infinitesimally) deformed Riemannian metric $\partial/\partial\beta[g_{ab}(\beta)]$, viz.,

$$\frac{d}{d\beta} \langle g \rangle_{B_{\beta}(p; r_0)} = \left. \frac{\partial}{\partial\beta} \langle g \rangle \right|_{B_{\beta}(p; r_0)} + \left. \langle g \right|_{B_{\beta}(p; r_0)} g_{ab}(\beta) \frac{\partial}{\partial\beta} g_{ab}(\beta) \right|_{B_{\beta}(p; r_0)},$$

where the ball $B_E(0; r_0)$ is kept fixed while its image $B(p; r_0)$ is deformed according to the flow of metrics $g_{ab}(\beta)$, $0 \leq \beta \leq \infty$.

In a rather obvious sense, (43) represents the active interpretation corresponding to the Eulerian passive view associated with the ball variation $B(p; r_0) \to B(p; r)$. In other words, we are here dealing with the Lagrangian point of view of following a fluid domain in its deformation, where the fluid particles here are the points of $B(p; r_0)$ suitably labelled. This latter remark suggests that in order to optimize the local averaging procedure associated with the local average $\langle g \rangle_{B_{(p,\beta)}}$, instead of studying its scaling behavior as $r$ increases,
and consequently be subjected to the accidents of the fluctuating geometry of \((B(p; r), g_{ab})\), we may keep fixed the domain \(B(p; r_0)\) and rescale the geometry inside \(B(p; r_0)\) according to a suitable flow of metrics \(g_{ab} \to g_{ab}(\beta)\), \(0 \leq \beta \leq \infty\). Correspondingly, also the matter density \(\varrho\) will be forced to rescale \(\varrho \to \varrho(\beta)\), and if we are able to choose the flow \(g_{ab} \to g_{ab}(\beta)\) in such a way that the local inhomogeneities of the original geometry of \((\Sigma, g)\) are smoothly eliminated, then the local average \(<\varrho>_{B(p; r)}\) comes closer and closer to represent a matter averaging over a homogeneous geometry.

In order to put such Lagrangian picture on a firmer ground we need to consider the important issue of how to label the points of \((B(p; r_0), g(\beta))\) during the deformation process. For \(\beta = 0\) the natural choice are (polar) geodesic coordinates associated with \(\exp_p\). However, as the metric changes along the flow \((B(p; r_0), g(\beta))\), the remarks of the previous section suggest that polar geodesic coordinates may not be the optimal choice as \(\beta\) varies, since we do not have an a priori control on the domain of validity of such a coordinate system. However, if we select the flow \(g_{ab} \to g_{ab}(\beta)\), \(0 \leq \beta \leq \infty\), in such a way that the sectional curvatures of \(g(\beta)\) stay bounded and the injectivity radius is bounded below as \(\beta\) varies, then such an a priori control can be achieved by labelling the points of \(\exp_{(p,\beta)}^{-1}\) by harmonic coordinates. Thus, in order to properly label the points of \((B(p; r_0), g(\beta))\), instead of considering the one–parameter family of maps \(\exp_{(p,\beta)}^{-1} : (B(p; r_0), g(\beta)) \to (B_E(0; r_0), \delta)\), we shall consider the family of maps

\[
F(\beta) \equiv \Phi(\beta) \circ \exp_{(p,\beta)}^{-1} : (B(p; r_0), g(\beta)) \xrightarrow{\exp_{(p,\beta)}^{-1}} (B_E(0; r_0), \delta) \xrightarrow{\Phi(\beta)} (B_E(0; r_0), \delta),
\]

(42)

where \(\Phi(\beta) : (B_E(0; r_0), \delta) \to (B_E(0; r_0), \delta)\) is a \(\beta\)-dependent diffeomorphism on \((B_E(0; r_0), \delta)\), (inducing a coordinate change in \((B_E(0; r_0), \delta)\)). By mimicking (23), we associate the harmonic map energy density with the family of maps \(F(\beta)\) providing the Lagrangian coordinates that allow to follow the points of \((B(p; r_0), g(\beta))\) during the deformation, i.e.,

\[
e(\beta; x) \equiv \|dF(\beta)\|^2 = g^{ij}(\beta)\delta_{lm}\frac{\partial F_l(\beta)}{\partial x^i}\frac{\partial F_m(\beta)}{\partial x^j},
\]

(43)

where the \(\{x^i\}\) denote the fixed generic (Eulerian) coordinates of the points of \((\Sigma_p, g(\beta))\). If we compute the rate of variation with \(\beta\) of \(e(\beta; x)\) we get

\[
\frac{\partial}{\partial \beta} e(\beta; x) = \delta_{lm}\frac{\partial F_l(\beta)}{\partial x^i}\frac{\partial F_m(\beta)}{\partial x^j}\frac{\partial}{\partial \beta} g^{ij}(\beta) + 2g^{ij}(\beta)\delta_{lm}\frac{\partial F_l(\beta)}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial \beta} F^m(\beta)\right).
\]

(44)

Since we eventually want to label the points of \((B(p; r_0), g(\beta))\) with harmonic coordinates, the most natural thing to do is to deform \(F(\beta)\) according to the
harmonic map flow, i.e.,

\[
\begin{align*}
\frac{\partial}{\partial \beta} F^m(\beta) &= \Delta_\beta F^m(\beta) \\
F^m(\beta = 0) &= \exp_p^{-1},
\end{align*}
\]

(45)

where

\[
\Delta_\beta F^m(\beta) = g^{ij}(\beta) \left[ \frac{\partial^2}{\partial x^i \partial x^j} F^m(\beta) - \Gamma^k_{ij}(g(\beta)) \frac{\partial}{\partial x^k} F^m(\beta) \right]
\]

(46)

is the harmonic map Laplacian associated with the Riemannian connection \(\Gamma^k_{ij}(g(\beta))\) of \((B(p, r_0), g(\beta))\). In this way \(F^m(\beta)\) evolves towards harmonic maps (i.e., the Lagrangian coordinates labelling the points of \((B(p, r_0), g(\beta))\) will be harmonic coordinates as \(\beta \to \infty\)). Note that (45) is a strictly parabolic initial value problem admitting a solution for sufficiently small \(\beta > 0\), (in our case, standard theorems in harmonic map theory imply that the solution actually exists for all \(0 \leq \beta < \infty\). If we insert (45) into (44) we get

\[
\frac{\partial}{\partial \beta} e(\beta; x) = \delta_{lm} \frac{\partial F^l(\beta)}{\partial x^i} \frac{\partial F^m(\beta)}{\partial x^j} g^{ij}(\beta)
\]

(47)

\[
+ 2g^{ij}(\beta) \delta_{lm} \frac{\partial F^l(\beta)}{\partial x^i} \frac{\partial}{\partial x^j} (\Delta_\beta F^m(\beta)).
\]

(48)

Now, note that a Bochner type formula allows to compute the Laplacian of \(e(\beta; x)\) according to

\[
\Delta_\beta (e(\beta; x)) = \Delta_\beta \left( |\nabla F(\beta)|^2 \right)
\]

= 2 \(|\nabla^2 F(\beta)|^2 + 2 \text{Ric}(\nabla F(\beta), \nabla F(\beta)) + 2 g(\nabla F(\beta), \nabla \Delta_\beta F(\beta))\),

where the notation is as follows:

\[
|\nabla^2 F(\beta)|^2 = g^{ik}(\beta) g^{il}(\beta) \delta_{rs} \nabla^2_{ij} F^r(\beta) \nabla^2_{kl} F^s(\beta),
\]

(49)

\[
\nabla^2_{ij} F^r(\beta) = \frac{\partial^2}{\partial x^i \partial x^j} F^r(\beta) - \Gamma^k_{ij}(g(\beta)) \frac{\partial}{\partial x^k} F^r(\beta),
\]

(50)

\[
g(\nabla F(\beta), \nabla \Delta_\beta F(\beta)) = g^{ik}(\beta) \delta_{rs} \nabla(\Delta_\beta F(\beta)) \nabla_k F^s(\beta),
\]

(51)

\[
\text{Ric}(\nabla F(\beta), \nabla F(\beta)) = \delta_{lm} \frac{\partial F^l(\beta)}{\partial x^i} \frac{\partial F^m(\beta)}{\partial x^j} R^{ij}(\beta).
\]

(52)

Equation (48) implies that we can write (47) as

\[
\frac{\partial}{\partial \beta} e(\beta; x) = \delta_{lm} \frac{\partial F^l(\beta)}{\partial x^i} \frac{\partial F^m(\beta)}{\partial x^j} g^{ij}(\beta) + \Delta_\beta (e(\beta; x))
\]

(53)

\[
- 2 |\nabla^2 F(\beta)|^2 - 2 \delta_{lm} \frac{\partial F^l(\beta)}{\partial x^i} \frac{\partial F^m(\beta)}{\partial x^j} R^{ij}(\beta).
\]

(54)

This latter expression shows that, as expected, the \(\beta\)-evolution of \(e(\beta; x)\) is strongly influenced by the chosen evolution of \(g^{ij}(\beta)\). However, such an evolution becomes particularly simple if we choose the flow of metrics \(g^{ij}(\beta)\) in a
way that is strongly reminiscent of the radial scaling properties of the metric (21), viz.,
\[
\frac{\partial}{\partial \beta} g^{ij}(\beta) = 2 R^{ij}(\beta),
\]
(54)
or in terms of the covariant components \(g_{ij}(\beta)\)
\[
\frac{\partial}{\partial \beta} g_{ij}(\beta) = -2 R_{ij}(\beta).
\]
(55)
Under such a choice (53) reduces to
\[
\frac{\partial}{\partial \beta} e(\beta; x) = \Delta_{\beta} (e(\beta; x)) - 2 \left| \nabla^2 F(\beta) \right|^2,
\]
(56)
which implies that, as \(\beta\) increases, the maximum of \(e(\beta; x)\) is (weakly) decreasing. According to the geometrical meaning of \(e(\beta; x)\), (see (29)), curvature inhomogeneities tend to decrease along the flow (55), at least in a sufficiently small neighborhood (in the \(C^k\)-topology) of \((B(p; r_0), g(\beta))\).

Obviously, these remarks are strongly reminiscent of the properties of the Ricci flow on a Riemannian manifold \((\Sigma, g)\), i.e.,
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial \eta} g_{ab}(\eta) = -2 R_{ab}(\eta) \\
g_{ab}(\eta = 0) = g_{ab},
\end{array} \right.
\]
(57)
studied by Richard Hamilton and his co–workers in connection with an analytic attempt to proving Thurston’s geometrization conjecture. As is well–known, the flow (57) is weakly–parabolic, and it is always solvable for sufficiently small \(\eta\). Moreover, it is such that any symmetries of \(g_{ab}(\eta = 0) = g_{ab}\) are preserved along the flow. The flow (57) may be reparametrized \(\eta \to \lambda\) by an \(\eta\)–dependent rescaling and by an \(\eta\)–dependent homothety \(g_{ab}(\eta) \to \tilde{g}_{ab}(\lambda)\) so as to preserve the original volume of \((\Sigma, g)\). In this latter case in place of (57) we get the associated flow
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial \lambda} \tilde{g}_{ab}(\lambda) = -2 \tilde{R}_{ab}(\lambda) + \frac{2}{3} \left( \frac{\partial}{\partial \lambda} \right) \tilde{R}(\lambda) \left( \Sigma(\lambda) \right) \tilde{g}_{ab}(\lambda) \\
\tilde{g}_{ab}(\lambda = 0) = g_{ab},
\end{array} \right.
\]
(58)
whose global solutions (if attained)
\[
\overline{g}_{ab} = \lim_{\lambda \to +\infty} \tilde{g}_{ab}(\lambda)
\]
are constant curvature metrics.

Thus, a Lagrangian picture shows in a very direct way that the natural flow \(g_{ab} \to g_{ab}(\beta)\) providing the simultaneous rescaling of the matter distribution and the geometry is a (suitably normalized) Ricci flow. Clearly, its normalization
cannot be selected simply on the basis of geometric convenience (e.g., fixed volume, as in the standard Ricci–Hamilton flow\cite{58}), and we can realize our matter averaging program only if there exists a global solution $g_{ab} \rightarrow g_{ab}(\beta)$, $0 \leq \beta \leq \infty$ of the Ricci flow which preserves the total matter content $M(\Sigma)$ of $(\Sigma, g)$. Such a problem together with the set up of a renormalized effective dynamics of smoothed out cosmological spacetimes is the headline of our forthcoming work.

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