ON THE COMPLETE ANALYTIC STRUCTURE OF THE MASSIVE GRAVITINO PROPAGATOR IN FOUR-DIMENSIONAL DE SITTER SPACE

Giampiero Esposito\textsuperscript{1}  Raju Roychowdhury\textsuperscript{2,1}
\textsuperscript{1}Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio 6, 80126 Napoli, Italy
\textsuperscript{2}Dipartimento di Scienze Fisiche, Federico II University, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio 6, 80126 Napoli, Italy

(Dated: October 6, 2009)

Abstract

With the help of the general theory of the Heun equation, this paper completes previous work by the authors and other groups on the explicit representation of the massive gravitino propagator in four-dimensional de Sitter space. As a result of our original contribution, all weight functions which multiply the geometric invariants in the gravitino propagator are expressed through Heun functions, and the resulting plots are displayed and discussed after resorting to a suitable truncation in the series expansion of the Heun function. It turns out that there exist two ranges of values of the independent variable in which the weight functions can be divided into dominating and sub-dominating family.
I. INTRODUCTION

The investigation of Green functions has always been at the heart of important developments in quantum field theory and quantum gravity [1]. On the other hand, in recent years, developments in cosmology and string theory have led to renewed interest in supergravity theories in anti-de Sitter [2] and de Sitter space [3].

Thus, in our recent paper [4], we performed a two-component spinor analysis of geometric invariants leading to the gravitino propagator in four-dimensional de Sitter spacetime, following the two-spinor language [5] pioneered by Penrose. In that paper we also wrote down all the 10 different weight functions multiplying the invariants which occur in the massive gravitino propagator, relying upon the work by Anguelova and Langfelder [6]. It was also found there, that algebraically one can write down 8 weight functions denoted by $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega$ in terms of a pair denoted by $(\pi, \kappa)$, in case of de Sitter space. Going one step further, we also expressed $\kappa$ in terms of $\pi$ and $\pi'$ where $\pi$ was defined in this fashion: $\pi(z) = \sqrt{z} \tilde{\pi}(z)$ and $\tilde{\pi}(z)$ satisfies the Heun differential equation [7, 8], whose solutions, denoted by Heun$(a, q; b, c, d, e; z)$, with properly defined arguments, have in general four singular points, i.e. $z_0 = 0, 1, a, \infty$.

In this paper we have explicitly written down, first, the algebraic expression of all the 9 weight functions $\kappa, \alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega$ in terms of $\pi(z)$ and $\pi'(z)$, where $z$ is defined as $z = \cos^2 \frac{\mu}{2R}$, with $\mu(x, x')$ being the geodesic distance between $x$ and $x'$ as defined in [4]. Finally, we will draw a few two-dimensional plots of these weight functions and classify their parameter space with respect to $z$ in the region of our choice.

The plan of this paper is as follows. In Sec. II we set up all symbols, basically recalling all relevant definitions of use in this paper from our previous one [4]. Sec. III contains the explicit massive spin-3/2 propagator in four dimensions with all the ten invariant structures properly defined, along with the multiplicative weight functions written in terms of the $(\pi, \kappa)$ pair. In Sec. IV we give a crash course on Heun's differential equations and write down several properties of the Heun function before showing that $\tilde{\pi}(z)$ satisfies a Heun equation with properly defined arguments that we will list there. Sec. V and the appendix are devoted to build a dictionary of all the 9 weight functions $\kappa, \alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega$ written in terms of $\pi(z)$ and $\pi'(z)$ only, where prime denotes derivative with respect to $z$ instead of being the derivative with respect to $\mu$, the geodesic distance function. Then in the last Sec.
VI we display several two-dimensional plots showing the functional behavior of each of the 10 weight functions with respect to $z$. These show that there exist two ranges of values of $z$ in which the weight functions can be divided into dominating and sub-dominating family. Moreover, it appears helpful to have the result of a lengthy calculation completely worked out. Eventually, we give further details on the plots in the section devoted to concluding remarks.

In light of recent mathematical developments in \cite{9}, it might be possible to expand the Heun functions in the gravitino propagator as a combination of finitely or infinitely many hypergeometric functions, which in turn occur in the more familiar formulae for bosonic propagators in de Sitter space \cite{10}. Thus, our work might help relating fermionic and bosonic propagators in four-dimensional de Sitter space through special-function techniques, double-checking the expectations from supersymmetry.

\section*{II. A REVIEW OF A FEW USEFUL DEFINITIONS}

It has been more than two decades since Allen and co-authors used intrinsic geometric objects to calculate correlation functions in maximally symmetric spaces; their results, here exploited, were presented in two papers \cite{11,12}. In this section we would like to review, first, the elementary maximally symmetric bi-tensors which have been discussed previously by Allen and Jacobson \cite{11}. More recently, the calculation of the spinor parallel propagator has been carried out in arbitrary dimension \cite{13}.

A maximally symmetric space is a topological manifold of dimension $n$, with a metric which has the maximum number of global Killing vector fields. This type of space looks exactly the same in every direction and at every point. The simplest examples are flat space and sphere, each of which has $\frac{1}{2}n(n+1)$ independent Killing fields. For $S^n$ these generate all rotations, and for $\mathbb{R}^n$ they include both rotations and translations.

We consider a maximally symmetric space of dimension $n$ with constant scalar curvature $n(n-1)/R^2$. For the space $S^n$, the radius $R$ is real and positive, whereas for the hyperbolic space $H^n$, $R = il$ with $l$ positive, and in the flat case, $\mathbb{R}^n$, $R = \infty$. If we further consider two points $x$ and $x'$, which can be connected uniquely by a geodesic, with $\mu(x, x')$ being the geodesic distance between $x$ and $x'$, then $n^a(x, x')$ and $n'^a(x, x')$ are the tangents to the
geodesic at $x$ and $x'$, and are given in terms of the geodesic distance as follows:

$$n_a(x, x') = \nabla_a \mu(x, x') \quad \text{and} \quad n_a'(x, x') = \nabla_a' \mu(x, x'). \quad (2.1)$$

Furthermore, on denoting by $g^a_{\nu'}(x, x')$ the vector parallel propagator along the geodesic, one can then write $n^\nu = -g^\nu_{a}n^a$. Tensors that depend on two points, $x$ and $x'$, are bitensors [14]. They may carry unprimed or primed indices that live on the tangent space at $x$ or $x'$.

These geometric objects $n^a$, $n'^a$ and $g^a_{\nu'}$ satisfy the following properties [11]:

$$\nabla_a n_b = A(g_{ab} - n_a n_b), \quad \text{(2.2a)}$$

$$\nabla_a n_{\nu'} = C(g_{a\nu'} + n_a n_{\nu'}), \quad \text{(2.2b)}$$

$$\nabla_a g_{bc} = -(A + C)(g_{a b} n_{c'} + g_{a c} n_{b'}), \quad \text{(2.2c)}$$

where $A$ and $C$ are functions of the geodesic distance $\mu$ and are given by [11]

$$A = \frac{1}{R} \cot \frac{\mu}{R} \quad \text{and} \quad C = -\frac{1}{R \sin(\mu/R)}, \quad (2.3)$$

for de Sitter spacetime and thus they satisfy the relations

$$\frac{dA}{d\mu} = -C^2, \quad \frac{dC}{d\mu} = -AC \quad \text{and} \quad C^2 - A^2 = 1/R^2. \quad (2.4)$$

Last, with our convention the covariant gamma matrices satisfy the property

$$\{\Gamma^\mu, \Gamma^\nu\} = 2I g^{\mu\nu}. \quad (2.5)$$

In our previous work [4], we followed the conventions for two-component spinors, as well as all signature and curvature conventions, of Allen and Lutken [12], and hence we used dotted and undotted spinors instead of the primed and unprimed ones of Penrose and Rindler [5]. In our work a primed index indicates instead that it lives in the tangent space at $x'$, while the unprimed ones live at $x$. The fundamental object to deal with is the bispinor $D_A^{A'}(x, x')$ which parallel transports a two-component spinor $\phi^A$ at the point $x$, along the geodesic to the point $x'$, yielding a new spinor $\chi^{A'}$ at $x'$, i.e.

$$\chi^{A'} = \phi^A D_A^{A'}(x, x'). \quad (2.6)$$

Complex conjugate spinors are similarly transported by the complex conjugate of $D_A^{A'}(x, x')$, which is $\overline{D}_A^{A'}(x, x')$. A few elementary properties of $D_A^{A'}$ were listed in Sec. IV of [4]. It
is worth mentioning that the covariant derivatives of the spinor parallel propagator were defined to be

$$\nabla_{AA} D_B^{B'} = (A + C) \left[ \frac{1}{2} n_{AA} D_B^{B'} - n_{BA} D_A^{B'} \right],$$

(2.7)

where $A$ and $C$ are defined in (2.3).

The two basic massive two-point functions for spin-1/2 particle, were defined by

$$P_{AB'} \equiv \langle \phi^A(x) \overline{\phi}^{B'}(x') \rangle = f(\mu) D_A^A n^{A'B'},$$

(2.8)

$$Q_{A}^{B'} \equiv \langle \overline{\chi}_A(x) \phi^{B'}(x') \rangle = g(\mu) D_A^{A'B'},$$

(2.9)

and in de Sitter space they turned out to be [4]:

$$P_{(F)}^{AB'} = \lim_{\epsilon \to 0^+} f_{DS}(Z + i\epsilon) D_A^A n^{A'B'},$$

(2.10)

$$Q_{(F)}^{AB'} = \lim_{\epsilon \to 0^+} g_{DS}(Z + i\epsilon) D_A^{A'B'},$$

(2.11)

where $(F)$ stands for the Feynman Green functions with $f_{DS}$ and $g_{DS}$ defined in this fashion:

$$f_{DS} = N_{DS}(1 - Z)^{1/2} F(a, b; c, Z),$$

(2.12)

$$g_{DS} = -i N_{DS} 2^{-3/2} m|R| Z^{1/2} F(a, b; c + 1, Z).$$

(2.13)

Moreover, after doing some algebra one can rewrite the final answer for the constant $N_{DS}$ as

$$N_{DS} = \frac{-i|R|m(1 - m^2 R^2)}{8\sqrt{2}\pi |R|^3 \sinh \pi |R|m},$$

(2.14)

We also note that $F(a, b; c, Z)$ and $F(a, b; c + 1, Z)$ are two independent solutions of the Hypergeometric equation [15, 16]:

$$H(a, b, c; Z) w(Z) = 0,$$

(2.15)

where $H(a, b, c)$ is the hypergeometric operator

$$H(a, b, c; Z) = Z(1 - Z) \frac{d^2}{dZ^2} + [c - (a + b + 1)Z] \frac{d}{dZ} - ab.$$  

(2.16)
III. MASSIVE SPIN-3/2 PROPAGATOR

In this section we consider the propagator of the massive spin-3/2 field. Let us denote the gravitino field by $\Psi_\lambda^\alpha(x)$. In a maximally symmetric state $|s\rangle$ the propagator is

$$S_{\lambda\nu}^{\alpha\beta}(x, x') = \langle s | \Psi_\lambda^\alpha(x) \Psi_\nu^{\beta'}(x') | s \rangle. \quad (3.1)$$

The field equations imply that $S$ satisfies

$$(\Gamma^{\mu\lambda} D_\rho - m \Gamma^{\mu\lambda})^\gamma S_{\lambda\nu}^{\alpha\beta} = \frac{\delta(x - x')}{\sqrt{-g}} g^{\mu\nu} \delta^{\alpha\beta}. \quad (3.2)$$

A. The ten gravitino invariants

It is very convenient to decompose the gravitino propagator in terms of independent structures constructed out of $n_\mu, n_{\nu'}, g_{\mu\nu'}$ and $\Lambda^{\alpha\beta'}$. Thus, the propagator can be written in geometric way following Anguelova et al. [6] (see also [17]):

$$S_{\lambda\nu}^{\alpha\beta'} = \alpha(\mu) g_{\lambda\nu'} \Lambda^{\alpha\beta'} + \beta(\mu) n_\lambda n_{\nu'} \Lambda^{\alpha\beta'} + \gamma(\mu) \Lambda(\sigma) g_{\mu\nu'}(n_\sigma \Gamma^{\sigma} \Lambda)^{\alpha\beta'}$$

$$+\delta(\mu) n_\lambda n_{\nu'}(n_\sigma \Gamma^{\sigma} \Lambda)^{\alpha\beta'} + \varepsilon(\mu) n_\lambda (\Gamma_{\nu'} \Lambda)^{\alpha\beta'} + \theta(\mu) n_{\nu'} (\Gamma_{\lambda} \Lambda)^{\alpha\beta'}$$

$$+\tau(\mu) n_\lambda (n_{\sigma} \Gamma^{\sigma} \Gamma_{\nu'} \Lambda)^{\alpha\beta'} + \omega(\mu) n_{\nu'} (n_{\sigma} \Gamma^{\sigma} \Gamma_{\lambda} \Lambda)^{\alpha\beta'}$$

$$+\pi(\mu) (\Gamma_{\lambda} \Gamma_{\nu'} \Lambda)^{\alpha\beta'} + \kappa(\mu) (n_{\sigma} \Gamma^{\sigma} \Gamma_{\lambda} \Gamma_{\nu'} \Lambda)^{\alpha\beta'}. \quad (3.3)$$

B. The weight functions multiplying the invariants

A rather tedious but straightforward calculation gives a system of 10 equations for the 10 coefficient functions $\alpha, ..., \kappa$ in (3.3) as found in (see equations (3.6)-(3.15) in [6]). It was also found there that one can easily express the algebraic solutions for $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega$ in terms of the $(\pi, \kappa)$ pair in case of de Sitter space, i.e. (hereafter we set $n = 4$ in the general formulae of [6], since only in the four-dimensional case the two-component-spinor formalism
can be applied)

\[ \omega = \frac{2mC\kappa + ((A + C)^2 - m^2)\pi}{(m^2 + R^{-2})}, \]

\[ \theta = \frac{((A - C)^2 - m^2)\kappa - 2mC\pi}{(m^2 + R^{-2})}, \]

\[ \tau = \frac{2mC\kappa + ((A + C)^2 - m^2)\pi}{(m^2 + R^{-2})}, \]

\[ \varepsilon = \frac{-([A - C]^2 + 2/R^2 + m^2)\kappa + 2mC\pi}{(m^2 + R^{-2})}, \]

\[ \alpha = -\tau - 4\pi, \]

\[ \beta = 2\omega, \]

\[ \gamma = \varepsilon - 2\kappa, \]

\[ \delta = 2\varepsilon + 4(\kappa - \theta), \]

where we have used the relation \( C^2 - A^2 = 1/R^2. \)

Furthermore, from (3.4) we can immediately see that

\[ \tau = \omega \quad \text{and} \quad \varepsilon + \theta = -2\kappa. \]

(3.5)

IV. HEUN’S DIFFERENTIAL EQUATION: A PRIMER

The canonical form of the general Heun differential equation is given by (8, 18)

\[ \frac{d^2y}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) \frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0 \]

(4.1)

The four regular singular points of the equation are located at \( z = 0, 1, a, \infty. \) Here \( a \in \mathbb{C}, \) the location of the fourth singular point, is a parameter \( (a \neq 0, 1), \) and \( \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C} \) are exponent-related parameters.

The solution space of the Heun differential equation is specified uniquely by the following Riemann \( P \)-symbol:

\[ P \begin{cases} 0 & 1 & d & \infty \\ 0 & 0 & 0 & \alpha \; ; \\ 1 - \gamma & 1 - \delta & 1 - \epsilon & \beta \end{cases} \]

(4.2)

This does not uniquely specify the equation and its solutions, since it omits the accessory parameter \( q \in \mathbb{C}. \) The exponents are constrained by

\[ \alpha + \beta - \gamma - \delta - \epsilon + 1 = 0. \]

(4.3)
This is a special case of Fuchs’s relation, according to which the sum of the \(2n\) characteristic exponents of any second-order Fuchsian equation on \(\mathbb{CP}^1\) with \(n\) singular points must equal \(n - 2\) \([19]\).

There are \(2 \times 4 = 8\) local solutions of (4.1) in all: two per singular point. If \(\gamma\) is not a nonpositive integer, the solution at \(z = 0\) belonging to the exponent zero will be analytic. When normalized to unity at \(z = 0\), it is called the local Heun function, and is denoted \(Hl(a, q; \alpha, \beta, \gamma, \delta; z)\) \([8]\). It is the sum of a Heun series, which converges in a neighborhood of \(z = 0\) \([8, 20]\). In general, \(Hl(a, q; \alpha, \beta, \gamma, \delta; t)\) is not defined when \(\gamma\) is a nonpositive integer.

If \(\epsilon = 0\) and \(q = \alpha\beta d\), the Heun equation loses a singular point and becomes a hypergeometric equation. Similar losses occur if \(\delta = 0\), \(q = \alpha\beta\), or \(\gamma = 0\), \(q = 0\). This paper will exclude the case when the Heun equation has fewer than four singular points. The case, in which the solution of (4.1) can be reduced to quadratures, will also be ruled out. If \(\alpha\beta = 0\) and \(q = 0\), the Heun equation (4.1) is said to be trivial. Triviality implies that one of the exponents at \(z = \infty\) is zero (i.e., \(\alpha\beta = 0\)), and is implied by absence of the singular point at \(z = \infty\) (i.e., \(\alpha\beta = 0\), \(\alpha + \beta = 1\), \(q = 0\)).

A. Reducing Heun to hypergeometric

The transformation to Heun (\(\mathfrak{H}\)) or hypergeometric (\(\mathfrak{h}\)) of a linear second-order Fuchsian differential equation with singular points at \(z = 0, 1, d, \infty\) (resp. \(z = 0, 1, \infty\)), and with arbitrary exponents, is accomplished by certain linear changes of the dependent variable, called F-homotopies (see \([16]\) and \([8, \S\ A2 and Addendum, \S\ 1.8]\).) If an equation with singular points at \(z = 0, 1, a, \infty\) has dependent variable \(u\), carrying out the substitution \(\tilde{u}(z) = z^{-\rho}(z - 1)^{-\sigma}(z - a)^{-\tau}u(t)\) will convert the equation to a new one, with the exponents at \(z = 0, 1, d\) reduced by \(\rho, \sigma, \tau\) respectively, and those at \(z = \infty\) increased by \(\rho + \sigma + \tau\). By this technique, one exponent at each finite singular point can be shifted to zero.

In fact, the Heun equation has a group of F-homotopic automorphisms isomorphic to \((\mathbb{Z}_2)^3\), since at each of \(z = 0, 1, a\), the exponents \(\zeta\) can be shifted to \(-\zeta, 0\), i.e., to 0, \(-\zeta\). Similarly, the hypergeometric equation has a group of F-homotopic automorphisms isomorphic to \((\mathbb{Z}_2)^2\). These groups act on the 6 and 3-dimensional parameter spaces, respectively. For example, one of the latter actions is \((a, b; c) \mapsto (c - a, c - b; c)\), which is induced by an
F-homotopy at $z = 1$. From this F-homotopy follows Euler’s transformation \[21, \S\,2.2\]

\[2F_1(a, b; c; z) = (1 - z)^{c-a-b}2F_1(c-a, c-b; c; z), \tag{4.4}\]

which holds because $2F_1$ is a local solution at $z = 0$, rather than at $z = 1$. If the singular points of the differential equation are arbitrarily placed, transforming it to the Heun or hypergeometric equation will require a Möbius (i.e., projective linear or homographic) transformation, which repositions the singular points to the standard locations. A unique Möbius transformation maps any three distinct points in $\mathbb{CP}^1$ to any other three; but the same is not true of four points, which is why $(\mathfrak{H})$ has the singular point $a$ as a free parameter.

**B. The cross-ratio orbit**

The characterization of Heun equations that can be reduced to the hypergeometric equation will employ the cross-ratio orbit of $\{0, 1, d, \infty\}$, defined as follows. If $A, B, C, D \in \mathbb{CP}^1$ are distinct, their cross-ratio is

\[(A, B; C, D) \overset{\text{def}}{=} \frac{(C - A)(D - B)}{(D - A)(C - B)} \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}, \tag{4.5}\]

which is invariant under Möbius transformations. Permuting $A, B, C, D$ yields an action of the symmetric group $S_4$ on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$. The cross-ratio is invariant under interchange of $A, B$ and $C, D$, and also under simultaneous interchange of the two points in each pair. Thus, each orbit contains no more than $4!/4 = 6$ cross-ratios. The possible actions of $S_4$ on $s \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ are generated by $s \mapsto 1 - s$ and $s \mapsto 1/s$, and the orbit of $s$ comprises

\[s, \ 1 - s, \ 1/s, \ 1/(1 - s), \ s/(s - 1), \ (s - 1)/s, \tag{4.6}\]

which may not be distinct. This is called the cross-ratio orbit of $s$; or, if $s = (A, B; C, D)$, the cross-ratio orbit of the unordered set $\{A, B, C, D\} \subset \mathbb{CP}^1$. Two sets of distinct points $\{A_i, B_i, C_i, D_i\}$ ($i = 1, 2$) have the same cross-ratio orbit iff they are related by a Möbius transformation.
C. Reminder of some of the properties of Heun’s function

Our aim will be to find an integral representation of the Heun function as a Frobenius’ solution of the Heun equation, given in another form as follows [8]:

\[ z(z-1)(z-a)y''(z) + \{ \gamma(z-1)(z-a) + \delta z(z-a) + \epsilon z(z-1) \} y'(z) + (\alpha \beta z - q) y(z) = 0, \]  

(4.7)

The Frobenius’ solution, noted \( H_l(a, q; \alpha, \beta, \gamma, \delta; z) \) is the entire solution defined for the exponent zero at the point \( z = 0 \). It admits the power series expansion

\[ H_l(a, q; \alpha, \beta, \gamma, \delta; z) \equiv \sum_{n=0}^{\infty} c_n z^n, \]  

(4.8)

with \( |z| < 1 \) and \( c_0 = 1, \ c_1 = \frac{\alpha}{\gamma a} \) and \( \gamma \neq 0, -1, -2, \ldots \). The recursion relation is as follows:

\[
\begin{align*}
    a(n+2)(n+1+\gamma)c_{n+2} \\
    = \left[ q + (n+1)(\alpha + \beta - \delta + (\gamma + \delta - 1)a) + (n+1)^2(a+1) \right] c_{n+1} \\
    - (n+\alpha)(n+\beta)c_n = 0 \quad n \geq 0.
\end{align*}
\]  

(4.9)

The function \( H_l(a, q; \alpha, \beta, \gamma, \delta; z) \) is normalised with the relation

\[ H_l(a, q; \alpha, \beta, \gamma, \delta; 0) = 1. \]  

(4.10)

It admits the following important particular cases ([8], p9, formula(1.3.9)):

\[
\begin{align*}
    H_l(1, \alpha \beta; \alpha, \beta, \gamma, \delta; z) &= \, _2F_1(\alpha, \beta; \gamma; z) \quad \forall \delta \in \mathbb{C} \\
    H_l(0, 0; \alpha, \beta, \gamma, \delta; z) &= \, _2F_1(\alpha, \beta, \alpha + \beta - \delta + 1; z) \quad \forall \gamma \in \mathbb{C} \\
    H_l(a, a \alpha \beta; \alpha, \beta, \gamma, \alpha + \beta - \gamma + 1; z) &= \, _2F_1(\alpha, \beta, \gamma; z),
\end{align*}
\]  

(4.11)

where \( _2F_1(\alpha, \beta, \gamma; z) \) is the usual notation for the Gauss hypergeometric function.

D. Application of Heun’s equation to our problem

Finally we come to the punch line, why do we need these all and how does the Heun equation indeed find an application to our problem? The answer to this goes along the
On using (3.5) the differential equations for $\kappa$ and $\pi$, the equations (3.14) and (3.15) of [6], acquire the form

\[-(A + C)\theta + \kappa' + \frac{1}{2}(A - C)\kappa + m\pi = 0,
\]
\[(C - A)\omega + \pi' + \frac{1}{2}(A + C)\pi + m\kappa = 0,\]  \hspace{1cm} (4.12)

where $\theta$ and $\omega$ are given in (3.4). Clearly one can solve algebraically the second equation for $\kappa$. By differentiating the result one obtains also $\kappa'$ in terms of $\pi$, $\pi'$ and $\pi''$, and substitution of these in the first equation yields a second order ODE for $\pi(\mu)$. Now let us look at the system (4.12) in case of de Sitter spacetime. On inserting $A$ and $C$ from (5.1) below and passing to the globally defined variable $z = \cos^2 \frac{\mu}{2R}$ (see Sec. III), we obtain the following differential equation for $\pi$:

\[P_2 \frac{d^2}{dz^2} + P_1 \frac{d}{dz} + P_0 \pi = 0,\]  \hspace{1cm} (4.13)

where $P_2$ in (4.13) is a quartic polynomial in $z$, i.e.

\[P_2 = 4 \left[ m^2 R^2 + 1 \right] z^4 - 4(2m^2 R^2 + 3)z^3 + 4(m^2 R^2 + 2)z^2.\]  \hspace{1cm} (4.14)

Similarly, $P_1$ in (4.13) is a cubic polynomial in $z$,

\[P_1 = 16 \left[ m^2 R^2 + 1 \right] z^3 - 12 \left[ 2m^2 R^2 + 5 \right] z^2 + 8 \left( m^2 R^2 + 2 \right) z.\]  \hspace{1cm} (4.15)

Last, $P_0$ in (4.13) is a quadratic polynomial in $z$, i.e.

\[P_0 = \left( 4m^4 - 19m^2 + 32m^2 R^2 + 9 \right) z^2 - \left( 4m^4 - 14m^2 + 32m^2 R^2 + 21 \right) z - 3m^2 R^2 - 6.\]  \hspace{1cm} (4.16)

On making the substitution $\pi(z) = \sqrt{z} \tilde{\pi}(z)$, (4.13) becomes an equation of the type

\[z(z - 1)(z - a)y''(z) + \left\{ (b + c + 1)z^2 - [b + c + 1 + a(d + e) - e] z + ad \right\} y'(z) + (bc z - q)y(z) = 0.\]  \hspace{1cm} (4.17)

Written in canonical form it reads as follows:

\[\frac{d^2 y}{dz^2} + \left( \frac{d}{z} + \frac{e}{z - 1} + \frac{(b + c + 1) - (d + e)}{z - a} \right) \frac{dy}{dz} + \frac{bc z - q}{z(z - 1)(z - a)} y = 0,\]  \hspace{1cm} (4.18)
where the parameters in (4.18) take the values

\[ a = \frac{(m^2R^2 + 2)}{(m^2R^2 + 1)}, \]
\[ b = 2 + imR, \]
\[ c = 2 - imR, \]
\[ d = e = 3, \]
\[ q = -\frac{(m^4R^4 + 7m^2R^2 + 10)}{(m^2R^2 + 1)}. \]

The equation (4.17) is known as Heun’s differential equation [7, 8]. Its solutions, here denoted by \( H_l(a, q; b, c, d, e; z) \), have in general four singular points as we said before, i.e. \( z_0 = 0, 1, a, \infty \). Near each singularity the function behaves as a combination of two terms that are powers of \((z - z_0)\) with the following exponents: \(\{0, 1 - d\}\) for \( z_0 = 0 \), \(\{0, 1 - e\}\) for \( z_0 = 1 \), \(\{0, d + e - b - c\}\) for \( z_0 = a \), and \(\{b, c\}\) (that is, \( z^{-b} \) or \( z^{-c} \)) for \( z \rightarrow \infty \).

We now insert into the second of Eq. (4.12) the first of Eq. (3.4), finding eventually

\[ \kappa = f^{-1} \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi - (m^2 + R^{-2})\pi' \right\}, \]

where

\[ f \equiv m(m^2 + R^{-2} + 2C(C - A)), \]

and \( \pi \) and \( \pi' \) are meant to be expressed through the Heun function \( H_l(a, q; b, c, d, e; z) \). Eventually, we will show in the next section that all weight functions can be therefore expressed through such Heun function. The material covered in the present section and in the previous two is not new, and most of it is appropriate only for a physics-oriented choice of four-dimensional de Sitter space.

V. DICTIONARY OF WEIGHT FUNCTIONS FOR THE GRAVITINO PROPAGATOR

Here we will explicitly list all the weight functions as functions of \( z = \cos^2 \frac{\mu}{2R} \), in order to analyze their qualitative behavior as a function of \( z \) and de Sitter radius \( R \) in the next section. Let us recall a few definitions in de Sitter space, where \( A \) and \( C \) are functions of the geodesic distance \( \mu \) and are given by [11]

\[ A = \frac{1}{R} \cot \frac{\mu}{R} \quad \text{and} \quad C = -\frac{1}{R \sin(\mu/R)}, \]

(5.1)
Since all other weight functions $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega$ can be written in terms of the $(\pi, \kappa)$ pair, and in the last section we have seen $\kappa$ can also be expressed in a form like (4.20), it is evident that all other 9 weight functions including $\kappa$, i.e. $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega, \kappa$ can be expressed in terms of $\pi(\mu)$ and $\pi'(\mu)$ only.

We can also express $\pi$ as a function of $z$ and $R$ only as $\pi = \pi(z) = \pi(\mu = \pm 2R\cos^{-1}\sqrt{z})$. Similarly, by using a few of the familiar trigonometric identities, one can transform $\pi'(\mu)$ as

$$\pi'(\mu) = \mp \frac{1}{R} \sqrt{z(1 - z)} \pi'(z).$$ (5.2)

One can also write down the expressions of $(A + C)$ and $(A - C)$ in terms of $z$ and $R$ only as follows:

$$A + C = -\frac{1}{R} \sqrt{\frac{1 - z}{z}},$$
$$A - C = \frac{1}{R} \sqrt{\frac{z}{1 - z}}.$$ (5.3)

Another function appearing quite frequently in our evaluation of all the weight functions is $f$, which can be also expressed as a function of $z$ and $R$ only as follows:

$$f = m(m^2 + R^{-2} + R^{-2}(1 - z)^{-1}).$$ (5.4)

Now we start by listing all the weight functions in terms of $\pi(z)$ and $\pi'(z)$, bearing in mind that

$$\tilde{\pi}(z) = \text{Hl}(a, q; b, c, d, e; z),$$
$$\pi(z) = \sqrt{z} \text{ Hl}(a, q; b, c, d, e; z),$$ (5.5) (5.6)

where $\text{Hl}(a, q; b, c, d, e; z)$ is the Heun function with arguments as defined before. One has therefore the lengthy formulae for all other weight functions written down in Eqs. (A1)–(A8) of the appendix.

VI. QUALITATIVE BEHAVIORS OF THE WEIGHT FUNCTIONS

Now using the series expansion (4.8) defined before one can numerically study the behavior of each weight function, by taking the first 10 terms of the infinite series (4.8). Indeed, dealing with an infinite number of terms is impossible, and one has therefore to resort to approximations, by truncating such a series. On taking less than 10 terms, we have found
minor departures from the pattern outlined below in figures 1 to 9, whereas on taking 15 terms, the pattern in such figures is essentially confirmed.

We draw for example all these weight functions in a two-dimensional plot vs $z$, in the range (0,1). The plots, which also include $\tilde{\pi}(z)$, are as follows.

FIG. 1: Two-dimensional plot of the weight function $\alpha(z)$. The curve has two branches, depending on whether one takes the $+$ or $-$ sign in (A1). One branch of $\alpha$ cuts the horizontal $z$-axis at the points $z = 0.25, 0.75$, whereas the other branch of $\alpha$ cuts the horizontal axis at the points $z = 0.12, 0.6, 0.92$. Both branches approach the vertical axis, the first one cuts it near the value 0.5, while the other has a vertical asymptote at $z = 0.02$. The two branches intersect each other at $z = 0.15, 0.69, 1$; at these points the function $\alpha(z)$ becomes single-valued.

FIG. 2: Two-dimensional plot of the weight function $\beta(z)$. The curve has two branches, depending on whether one takes the $+$ or $-$ sign in (A2). One branch of $\beta$ cuts the horizontal $z$-axis at the points $z = 0, 0.7, 1$, whereas the other branch of $\beta$ cuts the horizontal axis at the points $z = 0.18, 0.67, 1$. The first branch never cuts the vertical axis, while the other has a vertical asymptote at $z = 0.05$. The two branches intersect each other at $z = 0.15, 0.69, 1$; at these points the function $\beta(z)$ becomes single-valued.
FIG. 3: Two-dimensional plot of the weight function \( \gamma(z) \). The curve has two branches, depending on whether one takes the + or − sign in (A3). The first branch of \( \gamma \) cuts the horizontal \( z \)-axis at the points \( z = 0.43, 0.92 \) and the vertical axis at 1, and then it has a vertical asymptote at \( z = 0.95 \). The second branch of \( \gamma \) cuts the horizontal axis at the points \( z = 0.47, 0.76 \) and the vertical axis at 0.67, and then it has a vertical asymptote at \( z = 0.87 \). The two branches intersect each other at \( z = 0.15, 0.5, 0.69 \); at these points the function \( \gamma(z) \) becomes single-valued. The first branch of \( \gamma \) has an absolute minimum, of negative sign, at \( z = 0.85 \).

FIG. 4: Two-dimensional plot of the weight function \( \delta(z) \). The first branch cuts the horizontal axis at \( z = 0, 0.47, 0.75 \), and the second branch cuts the horizontal axis at \( z = 0.45, 1 \). While the first branch never cuts the vertical axis, the second one cuts it at \(-2\). The two branches intersect each other at \( z = 0.15, 0.5, 0.69 \), where \( \delta \) becomes single-valued. The first branch has a vertical asymptote at \( z = 0.85 \), whereas the second one does have the same at \( z = 1 \).

As one can see, for values of \( z < 0.1 \), the main contribution to the gravitino propagator results from the weight functions \( \alpha(z), \beta(z), \tau(z) = \omega(z) \), whereas the other weight functions are sub-dominating. By contrast, when \( z \in ]0.8, 1[ \), the dominating contribution to the gravitino propagator results from the weight functions \( \gamma(z), \delta(z), \varepsilon(z), \theta(z), \pi(z) \), while the
FIG. 5: Two-dimensional plot of the weight function \( \varepsilon(z) \). The first branch cuts the horizontal axis at \( z = 0.2, 0.65, 0.93 \) and cuts the vertical axis at \(-1\). The curve has an absolute minimum, of negative sign, at \( z = 0.85 \), and then reaches a vertical asymptote at \( z = 0.97 \). The second branch cuts the \( z \)-axis at \( z = 0, 0.72 \). The two branches intersect each other at \( z = 0.15, 0.69 \) and at these points \( \varepsilon \) is a single-valued function. The first branch has a vertical asymptote at \( z = 0.95 \) whereas the second does the same for \( z \) in between 0.85 and 0.9.

FIG. 6: Two-dimensional plot of the weight function \( \theta(z) \). The first branch cuts the horizontal axis at \( z = 0, 0.47, 0.75 \) and the second cuts the horizontal axis at \( z = 0.42, 1 \). The first one never cuts the vertical axis, whereas the second one does it at the functional value 0.32. The second branch has a more pronounced absolute minimum, of negative sign, at \( z = 0.95 \). The two branches intersect each other at \( z = 0.15, 0.5, 0.69 \), where \( \theta \) is single-valued. The first branch has a vertical asymptote for \( z \) in between 0.85 and 0.9, whereas the second one does the same at \( z = 1 \).

others remain sub-dominating.
FIG. 7: Two-dimensional plot of the weight function $\tau(z) = \omega(z)$. The first branch cuts the horizontal $z$-axis at the points $z = 0, 0.7, 1$, and it never touches the vertical axis, whereas the second branch cuts the $z$-axis at $z = 0.18, 0.67, 1$, and it reaches a vertical asymptote at $z = 0.05$. The two branches intersect each other at $z = 0.15, 0.69, 1$, where $\tau(z)$ becomes single-valued.

FIG. 8: Two-dimensional plot of the weight functions $\pi(z)$ and $\tilde{\pi}(z)$. The $\pi(z)$ curve passes through the origin and cuts the horizontal axis at $z = 0.45, 0.83$. The $\tilde{\pi}$ curve never passes through the origin, it cuts the vertical axis at the functional value 1 and it cuts the $z$-axis at $z = 0.45, 0.83$, where it also intersects the $\pi(z)$ curve. Beyond the point $z = 0.8$ the $\pi$ and $\tilde{\pi}$ curves become virtually indistinguishable. At $z = 1$ they both have a vertical asymptote.

VII. CONCLUDING REMARKS

Our paper has obtained the complete analytic structure of massive gravitino propagators in de Sitter space. In Sec. VI we have plotted all weight functions $\alpha, \beta, \gamma, \delta, \epsilon, \theta, \tau = \omega, \pi, \kappa$ occurring in the gravitino propagator (jointly with $\tilde{\pi}$) as a function of $z$ in a two-dimensional plot where $z = \cos^2(\mu/2R)$, $\mu$ being the geodesic distance between the points $x$ and $x'$, and $R$ is the de Sitter radius. Although the series (4.8) has been truncated, it remains true that Sec. VI is the first attempt to display a supersymmetric propagator in de Sitter via
FIG. 9: Two-dimensional plot of the weight function $\kappa(z)$. The first branch cuts the $z$-axis at $z = 0.29, 0.7, 0.97$, while the second one cuts the $z$-axis at $z = 0.68, 1$. The first branch intersects the vertical axis at the functional value $1.35$, whereas the second one does the same at the functional value $0.67$. The two branches intersect each other at $z = 0.15, 0.69$, where $\kappa$ becomes single-valued.

Heun functions. As we already said in Sec. I, further interest, from the point of view of mathematical methods, arises from the possibility to expand Heun functions in terms of hypergeometric functions [9]. As we said before, direct implications of our findings on the current understanding of the propagation of gravitinos in de Sitter space are as follows: there exist two ranges of values of $z$ in which the weight functions can be divided into dominating and sub-dominating family. In other words, when $z$ is smaller than $0.1$, the weight functions $\alpha, \beta, \tau = \omega$ are dominating while the others are sub-dominating. By contrast, when $z$ is very close to 1, the weight functions $\gamma, \delta, \varepsilon, \theta, \pi$ take much larger values.

The plot range is between 0 and 1 for $z$, which is indeed the only admissible region, since the squared cos function lies always between 0 and 1. Note that the plot of $\tilde{\pi}$ is basically nothing but the plot of the Heun function with properly defined coefficients, and the plot of $\pi$ is $\sqrt{z}$ times the Heun function.

The numerical analysis of Sec. VI, as we already said therein, has been performed by taking only the first 10 terms of the infinite series representing the Heun function, by applying the Frobenius’ method. If one goes on by taking more terms, one can get even more accurate results, but roughly the qualitative features remain the same. The task of plotting Heun functions is technical but not easy, since the modern computer packages still run into difficulties. Thus, our efforts can be viewed as preparing the ground for a more systematic use of Heun functions in fundamental theoretical physics. The flat-space limit is instead a considerable simplification, since the functions $A$ and $C$ in (5.1) are then found to reduce to
\[ A = \frac{1}{\mu}, \quad C = -\frac{1}{\mu}. \] and the formulae in the appendix are therefore considerably simplified.

It also remains to be seen whether the familiarity acquired with Heun functions will prove useful in studying gravitino propagators in other backgrounds relevant for modern high energy physics.

**APPENDIX A: EXPLICIT FORM OF THE WEIGHT FUNCTIONS**

The weight functions obtained in Sec. V read, explicitly,

\[
\alpha(z) = -2mC(m^2 + R^{-2})f^{-1}(z) \times \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi(z) \pm (m^2 + R^{-2})\sqrt{\frac{z(1-z)}{R}} \pi'(z) \right\} 
- ((m^2 + R^{-2})[(A + C)^2 - m^2] - 4)\pi(z), \quad (A1)
\]

\[
\beta(z) = 4mC(m^2 + R^{-2})^{-1}f^{-1}(z) \times \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi(z) \pm (m^2 + R^{-2})\sqrt{\frac{z(1-z)}{R}} \pi'(z) \right\} 
+ 2(m^2 + R^{-2})^{-1}[(A + C)^2 - m^2] \pi(z), \quad (A2)
\]

\[
\gamma(z) = -(m^2 + R^{-2})^{-1}[(A - C)^2 - m^2] f^{-1}(z) \times \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi(z) \pm (m^2 + R^{-2})\sqrt{\frac{z(1-z)}{R}} \pi'(z) \right\} 
- 2mC(m^2 + R^{-2})^{-1}\pi(z), \quad (A3)
\]

\[
\delta(z) = -6(m^2 + R^{-2})^{-1}[(A - C)^2 - m^2] f^{-1}(z) \times \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi(z) \pm (m^2 + R^{-2})\sqrt{\frac{z(1-z)}{R}} \pi'(z) \right\} 
+ 12mC(m^2 + R^{-2})^{-1}\pi(z), \quad (A4)
\]

\[
\varepsilon(z) = -(m^2 + R^{-2})^{-1}[(A - C)^2 + \frac{2}{R^2} + m^2] f^{-1}(z) \times \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi(z) \pm (m^2 + R^{-2})\sqrt{\frac{z(1-z)}{R}} \pi'(z) \right\} 
+ 2mC(m^2 + R^{-2})^{-1}\pi(z), \quad (A5)
\]

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\[ \theta(z) = (m^2 + R^{-2})^{-1} \left[ (A - C)^2 - m^2 \right] f^{-1}(z) \times \]
\[ \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi(z) \pm (m^2 + R^{-2}) \frac{\sqrt{z(1-z)}}{R} \pi'(z) \right\} \]
\[ -2mC(m^2 + R^{-2})^{-1} \pi(z), \]
\[ \tag{A6} \]

\[ \tau(z) = 2mC(m^2 + R^{-2})^{-1} f^{-1}(z) \times \]
\[ \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi(z) \pm (m^2 + R^{-2}) \frac{\sqrt{z(1-z)}}{R} \pi'(z) \right\} \]
\[ + (m^2 + R^{-2})^{-1} \left[ (A + C)^2 - m^2 \right] \pi(z) = \omega(z), \]
\[ \tag{A7} \]

\[ \kappa(z) = f^{-1}(z) \times \]
\[ \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi(z) \right. \]
\[ \pm \left. (m^2 + R^{-2}) \frac{\sqrt{z(1-z)}}{R} \pi'(z) \right\}. \]
\[ \tag{A8} \]

These exhaust all the weight functions multiplying the invariant structure present in the gravitino propagator, written explicitly in terms of a Heun function and its derivative.

**ACKNOWLEDGMENTS**

The authors are grateful to the Dipartimento di Scienze Fisiche of Federico II University, Naples and INFN for hospitality and financial support. We also want to thank Ebrahim Karimi for his much valuable input regarding our Mathematica computations. One of us (G.E.) dedicates this work to Maria Gabriella.

[1] De Witt, B.S.: Dynamical Theory of Groups and Fields. Gordon & Breach, New York (1965)
[2] Witten, E.: Adv. Theor. Math. Phys. 2, 253 (1998)
[3] Witten, E.: hep-th/0106109
[4] Esposito, G., Roychowdhury, R.: arXiv:0902.2098 [hep-th]
[5] Penrose, R., Rindler, W.: Spinors and Space-Time. I. Cambridge University Press, Cambridge (1984).
[6] Anguelova, L., Langfelder, P.: J. High Energy Phys. JHEP 03, 057 (2003)
[7] Handbook of exact solutions for ordinary differential equations. CRC Press, Boca Raton (1995).

[8] Ronveaux, A. (eds.): Heun’s Differential Equations. Oxford University Press, Oxford (1995).

[9] Sokhoyan, R.S., Melikdzanian, D.Yu., Ishkhanyan, A.M.: [arXiv:0909.1286] [math-ph]

[10] Allen, B.: Nucl. Phys. B 292, 813 (1987)

[11] Allen, B., Jacobson, T.: Commun. Math. Phys. 103, 669 (1986)

[12] Allen, B., Lutken, C.A.: Commun. Math. Phys. 106, 201 (1986)

[13] Mück, W.: J. Phys. A 33, 3021 (2000)

[14] Synge, J.L.: Relativity: The General Theory. North–Holland, Amsterdam (1960)

[15] Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1964)

[16] Erdelyi, A.: Higher Transcendental Functions. Krieger, Malabar (1981)

[17] Basu, A., Uruchurtu, L.I.: Class. Quantum Gravit. 23, 6059 (2006)

[18] Kamke, E.: Differentialgleichungen, Lösungsmethoden und Lösungen. Vol. 1. Chelsea, New York (1974)

[19] Poole, E.G.C.: Linear Differential Equations. Oxford University Press, Oxford (1936)

[20] Snow, C. Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory, 2nd Edition, no. 19 in Applied Mathematics Series, National Bureau of Standards, Washington DC (1952)

[21] Andrews, G.E., Askey, R., Roy, R.: Special Functions, Vol. 71 of Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge (1999)