Towards the trace formula for convex-cocompact groups

Ulrich Bunke* and Martin Olbrich†

March 28, 2022

Contents

1 Introduction 1

2 The distribution Ψ 3

2.1 Invariant distributions 3

2.2 An invariant distribution associated to Γ 4

2.3 The Fourier inversion formula 4

2.4 The Fourier transform of Ψ 5

2.5 The distribution Ψ as a regularized trace 5

3 The Plancherel theorem and integral kernels 7

3.1 The Plancherel theorems for \( L^2(G) \) and \( L^2(\Gamma \backslash G) \). Support of Plancherel measures 7

3.2 Extension of \( R \) and \( R_\Gamma \) 8

3.3 The absolute continuous part of \( L^2(\Gamma \backslash G) \). Integral kernels for \( R_\Gamma(h) \) 10

4 Poisson transforms and asymptotic computations 12

4.1 Motivation 12

4.2 Poisson transformation, \( \phi \)-functions, and asymptotics 12

4.3 An estimate 14

4.4 A computation 16

*Mathematisches Institut, Universität Göttingen, Bunsenstr. 3-5, 37073 Göttingen, GERMANY, E-mail:bunke@uni-math.gwdg.de

†Mathematisches Institut, Universität Göttingen, Bunsenstr. 3-5, 37073 Göttingen, GERMANY, E-mail:olbrich@uni-math.gwdg.de
1 Introduction

In this paper we develop a part of the harmonic analysis associated with a convex cocompact subgroup $\Gamma$ of a semisimple Lie group $G$ of real rank one that could play the same role as the trace formula in the case of cocompact groups or groups of finite covolume. In these classical situations a smooth, compactly supported, and $K$-finite function $f$ on $G$ acts by right convolution $R_\Gamma(f)$ on the Hilbert space $L^2(\Gamma \backslash G)$. The trace formula is an expression of the trace of the restriction of this operator to the discrete subspace in terms of $f$ and its Fourier transform $\hat{f}$. The part involving $f$ is called the geometric side and usually written as a sum of orbital integrals. The Fourier transform enters the trace formula in the case of non-cocompact subgroups, where it is combined with the scattering matrix. We will call this part the contribution of the scattering matrix.

In the present paper we assume that $\Gamma$ is a convex cocompact subgroup of $G$. Let $X$ be the symmetric space of $G$ and $\partial X$ be its geodesic boundary. If $\Gamma \subset G$ is a discrete torsion-free subgroup, then there is a $\Gamma$-equivariant decomposition $\partial X = \Omega \cup \Lambda$, where $\Lambda$ is the limit set of $\Gamma$. We call $\Gamma$ convex-cocompact if $\Gamma \backslash X \cup \Omega$ is a compact manifold with boundary.

Since the discrete spectrum of $L^2(\Gamma \backslash G)$ is rather sparse we take the point of view that the contribution of the scattering matrix is essentially (up to the contribution of the discrete spectrum) the Fourier transform of the geometric side of the trace formula.

Thus our starting point is the geometric side. It is a distribution $\Psi$ on $G$ given as a sum of suitably normalized orbital integrals associated to the hyperbolic conjugacy classes of $\Gamma$ (see Subsection 2.2). The objective of the trace formula in the case of convex cocompact $\Gamma$ is an explicit expression for the Fourier transform of $\Psi$. We are looking for a "measure" $\Phi$ on the
unitary dual $\hat{G}$ such that

$$\Psi(f) = \int_{\hat{G}} \theta_\pi(f) \Phi(d\pi) ,$$

(1)

where $\theta_\pi(f) := \text{Tr} \hat{f}(\pi)$ is the character of $\pi$. In the present paper we will formulate a precise Conjecture 5.2 about $\Phi$, but we are not able to prove the formula (1) in the general case.

The unitary dual $\hat{G}$ has a natural topology. Now observe that the intersection of the support of $\hat{f}$ and the support of the Plancherel measure of $L^2(\Gamma \backslash G)$ is the spectrum of $R_\Gamma(f)$. The Fourier transform of a compactly supported function $f$ is never compactly supported. In order to do our computations we have to approximate $R_\Gamma(f)$ by operators which have compact spectrum. The missing piece is some estimate which eventually allows for dropping the cut-off. Conjecture 5.2 can easily be verified in the case of a negative critical exponent.

In the present paper we will prove a formula which is similar to (1), but where $\Psi$ has a different interpretation. Let $R(f)$ denote the right-convolution operator on $L^2(G)$ induced by $f$. Then both, $R(f)$ and $R_\Gamma(f)$, have smooth integral kernels $K_R(f)$, $K_{R_\Gamma}(f)$, and, by Lemma 2.2, the value $\Psi(f)$ is nothing else than the integral

$$\Psi'(f) := \int_{\Gamma \backslash G} (K_{R_\Gamma}(f)(g,g) - K_R(f)(g,g)) \mu_G(dg) .$$

We will show that $\Psi'$ can be applied to functions with compactly-supported Fourier transform, and our main Theorem 5.1 is a formula

$$\Psi'(f) = \int_{\hat{G}} \theta_\pi(f) \Phi(d\pi) .$$

(2)

together with an explicit expression for $\Phi$.

We will describe $\Phi$ in terms of multiplicities $N_\Gamma(\pi)$ and a density $L_\Gamma(\pi)$. If $\pi$ is a representation of the complementary series (a non-tempered unitary representation of $G$), then the integer $N_\Gamma(\pi) := \Phi(\{\pi\})$ is the multiplicity of $\pi$ in $L^2(\Gamma \backslash G)$. If $\pi$ is a discrete series representation, then by Proposition 6.14 $N_\Gamma(\pi) := \Phi(\{\pi\})$ is an integer. It is an interesting problem to study this number in detail.

There are interesting operators with non-compact spectrum to which $\Psi'$ can be applied. Let $K \subset G$ be a maximal compact subgroup and $\Omega$ be the Casimir operator of $G$. We fix a $K$-type $\gamma$ and consider $(z - \Omega)^{-1}$ on $L^2(\Gamma \backslash G)(\gamma)$ and $L^2(G)(\gamma)$ if $z$ is not in the spectrum. Let $K_\Gamma(z)$ and $K(z)$ denote the corresponding integral kernels. The difference $K_\Gamma(z) - K(z)$ is smooth on the diagonal and goes into $\Psi'$ if $\text{Re}(z) \ll 0$. The consideration of these operators provides the link between the continuous part of $\Phi$ and the Selberg zeta functions. In fact, using the analysis of the present paper we can show the meromorphic continuation of the logarithmic derivative and the functional equation (Theorem 6.12) of the Selberg zeta functions (By other methods we apriori know that the Selberg zeta functions are meromorphic, see below.). Our work extends
previous results \cite{8} in the two-dimensional case and \cite{11} in the spherical case of $G = SO(1, n)$, $n \geq 2$.

Using symbolic dynamics of the geodesic flow and the thermodynamic formalism one can show that the Selberg zeta functions themselves are meromorphic functions of finite order \cite{11}. This gives information about the growth of $\Phi$ and on the counting function of resonances. It in particular shows that $\Phi$ can be applied to Schwartz functions like the Fourier transform $\hat{f}$ of a $K$-finite smooth function $f$ of compact support on $G$.

\section{The distribution $\Psi$}

\subsection{Invariant distributions}

Let $G$ be a semisimple Lie group. We fix once and for all a Haar measure $\mu_G$ on $G$. In this subsection we describe two sorts of conjugation invariant distributions on $G$, namely orbital integrals and characters of irreducible representations.

Let $\gamma \in G$ be a semisimple element. The orbit $O_\gamma := \{ g\gamma g^{-1} \mid g \in G \}$ of $\gamma$ under conjugation by $G$ is a submanifold of $G$ which can be identified with $G_\gamma \backslash G$, where $G_\gamma$ denotes the centralizer of $\gamma$. The inclusion $i_\gamma : G_\gamma \backslash G \cong O_\gamma \hookrightarrow G$ is a proper map. Therefore the pull-back by $i_\gamma$ is a continuous map

$$i_\gamma^* : C_c^\infty(G) \to C_c^\infty(G_\gamma \backslash G).$$

If we choose a Haar measure $\mu_{G_\gamma}$ on $G_\gamma$, then we obtain an induced measure $\mu_{G_\gamma \backslash G}$ on $G_\gamma \backslash G$ such that

$$\int_G f(g)\mu_G(dg) = \int_{G_\gamma \backslash G} \int_{G_\gamma} f(hg)\mu_{G_\gamma}(dh)\mu_{G_\gamma \backslash G}(dg).$$

The orbital integral $\theta_\gamma$ associated to $\gamma$ and the choice of the Haar measure $\mu_{G_\gamma}$ is, by definition, the composition of $i_\gamma^*$ and the measure $\mu_{G_\gamma \backslash G}$, i.e.

$$\theta_\gamma(f) := \mu_{G_\gamma \backslash G} \circ i_\gamma^*(f) = \int_{G_\gamma \backslash G} f(g\gamma g^{-1})\mu_{G_\gamma \backslash G}(dg).$$

We now introduce the character $\theta_\pi$ associated to an irreducible admissible representation $\pi$ of $G$ on a Hilbert space $V_\pi$. If $f \in C_c^\infty(G)$, then

$$\pi(f) := \int_G f(g)\pi(g)\mu_G(dg)$$

is a trace class operator on $V_\pi$. The character $\theta_\pi$ is the distribution on $G$ given by

$$\theta_\pi(f) := \text{Tr} \pi(f).$$
2.2 An invariant distribution asociated to $\Gamma$

Let $G$ be a semisimple linear connected Lie group of real rank one. We consider a torsion-free discrete convex-cocompact, non-cocompact subgroup $\Gamma \subset G$ (see [3], Sec. 2). Let $\mathcal{O}_\Gamma$ denote the disjoint union of manifolds $G_\gamma \backslash G$, where $\gamma$ runs over a set $C\mathcal{T}$ of representatives of the set $C\Gamma \setminus \{[1]\}$ of non-trivial conjugacy classes of $\Gamma$:

$$\mathcal{O}_\Gamma = \bigcup_{\gamma \in C\mathcal{T}} G_\gamma \backslash G.$$ 

The natural map $i_\Gamma : \mathcal{O}_\Gamma \to G$ is proper, and we obtain a continuous map

$$i_\Gamma^* : C_\infty^c (G) \to C_\infty^c (\mathcal{O}_\Gamma).$$

For each $\gamma \in C\mathcal{T}$ we fix a Haar measure $\mu_{G_\gamma}$. Then we define a measure $\mu_\Gamma$ on $\mathcal{O}_\Gamma$ such that its restriction to $G_\gamma \backslash G$ is $\text{vol}(\Gamma_\gamma \backslash G_\gamma)\mu_{G_\gamma \backslash G}$. Note that this measure only depends on the Haar measure $\mu_G$ and not on the choices of $\mu_{G_\gamma}$.

**Definition 2.1** The geometric side of the trace formula is the distribution $\Psi$ on $G$ given by

$$\Psi := \mu_\Gamma \circ i_\Gamma^*.$$ 

In terms of orbital integrals we can write

$$\Psi(f) = \sum_{\gamma \in C\mathcal{T}} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \theta_\gamma(f).$$

Note that this distribution is in fact a measure, invariant under conjugation, and it only depends on $\Gamma$ and the Haar measure $\mu_G$.

2.3 The Fourier inversion formula

Let $\hat{G}$ denote the unitary dual of $G$. This is the set of equivalence classes of irreducible unitary representations of $G$ equipped with a natural structure of a measurable space. For $\pi \in \hat{G}$ the operator $\pi(f)$ is, by definition, the value of the Fourier transform of $f$ at $\pi$, which we will also denote by $\hat{f} (\pi)$.

It is a consequence of the Plancherel theorem for $G$ that there is a measure $p$ on $\hat{G}$ such that for any $f \in C^\infty_c (G)$ and $g \in G$ we have

$$f(g) = \int_{\hat{G}} \text{Tr} \pi(g)^{-1} \hat{f}(\pi) \, p(d\pi).$$

Note that $p(d\pi)$ depends on the choice of the Haar measure $\mu_G$. Later in the present paper we will state a more precise version of the Plancherel theorem.
2.4 The Fourier Transform of $\Psi$

If $h$ is a function on $\hat{G}$ such that $h(\pi)$ is a trace class operator on $V_\pi$ for almost all $\pi$ (mod. $p$), then we form the function

$$\bar{h}(g) := \int_{\hat{G}} \text{Tr} \pi(g)^{-1} h(\pi) p(d\pi)$$

if the integral exists for all $g \in G$.

2.4 The Fourier Transform of $\Psi$

The contents of a trace formula for convex-cocompact groups $\Gamma$ would be an expression of $\Psi(f)$ in terms of the Fourier transform $\hat{f}$. In other words, we are interested in the Fourier transform of the distribution $\Psi$. Since $\Psi$ is invariant this expression should only involve the characters $\theta_\pi(f) = \text{Tr} \hat{f}(\pi)$. Thus there should exist a certain measure $\Phi$ on $\hat{G}$ such that the following equality holds true for all $f \in C_c^\infty(G)$:

$$\Psi(f) = \int_{\hat{G}} \theta_\pi(f) \Phi(d\pi).$$

Note that there is a Paley-Wiener theorem for $G$ which characterizes the range of the Fourier transform as a certain Paley-Wiener space. Apriori, $\Phi$ is a functional on this Paley-Wiener space, and it would be a non-trivial statement that this functional is in fact induced by a measure on $\hat{G}$.

2.5 The distribution $\Psi$ as a regularized trace

In the present paper we will not compute the Fourier transform $\Phi$ of $\Psi$ in the sense of Subsection 2.4. Rather we will compute the candidate for $\Phi$ using a different interpretation of $\Psi$.

Let $R$ denote the right-regular representation of $G$ on $L^2(G)$. It extends to the convolution algebra $L^1(G)$ by the formula

$$R(f) = \int_G f(g) R(g) \mu_G(dg).$$

If $f \in C_c^\infty(G)$, then $R(f)$ is an integral operator with smooth integral kernel $K_{R(f)}(g,h) = f(g^{-1}h)$. In a similar manner we have an unitary right-regular representation $R_\Gamma$ of $G$ on the Hilbert space $L^2(\Gamma \backslash G)$ which can be extended to $L^1(G)$ using the formula

$$R_\Gamma(f) = \int_G f(g) R_\Gamma(g) \mu_G(dg).$$

If $f \in C_c^\infty(G)$, then $R_\Gamma(f)$ is an integral operator with smooth kernel

$$K_{R_\Gamma(f)}(g,h) = \sum_{\gamma \in G} f(g^{-1}\gamma h).$$
Indeed, for $\phi \in L^2(\Gamma \backslash G)$ we have

$$R(f)\phi(g) = \int_G \phi(gh)f(h)\mu_G(\text{d}h) = \int_G \phi(h)f(g^{-1}h)\mu_G(\text{d}h) = \int_{\Gamma \backslash G} \phi(h)f(g^{-1}\gamma h)\mu_G(\text{d}h).$$

**Lemma 2.2** For $f \in C_c^\infty(G)$ we have

$$\Psi(f) = \int_{\Gamma \backslash G} [K_{R(f)}(\Gamma g, \Gamma g) - K_{R(\gamma)}(g, g)]\mu_G(\text{d}g).$$

**Proof.** We compute

$$\Psi(f) = \sum_{\gamma \in \tilde{C}^\Gamma} \text{vol}(\Gamma \gamma \backslash G) \theta_\gamma(f) = \sum_{\gamma \in \tilde{C}^\Gamma} \text{vol}(\Gamma \gamma \backslash G) \int_{G \gamma \backslash G} f(g^{-1}g)\mu_{G \gamma \backslash G}(\text{d}g) = \sum_{\gamma \in \tilde{C}^\Gamma} \int_{\Gamma \gamma \backslash G} f(g^{-1}g)\mu_G(\text{d}g) = \int_{\Gamma \backslash G} \sum_{\gamma \in \tilde{C}^\Gamma} \sum_{h \in \Gamma \gamma \backslash \Gamma} f(g^{-1}h^{-1}g)\mu_G(\text{d}g) = \int_{\Gamma \backslash G} \sum_{\gamma \in \tilde{C}^\Gamma} f(g^{-1}g)\mu_G(\text{d}g) = \int_{\Gamma \backslash G} [K_{R(f)}(g, g) - K_{R(\gamma)}(g, g)]\mu_G(\text{d}g).$$

The expression of $\Psi(f)$ in terms of the integral kernels of $R(f)$ and $R_\Gamma(f)$ can be used to define $\Psi$ on other classes of functions or even on certain distributions.

Using the Plancherel theorems for $L^2(G)$ and $L^2(\Gamma \backslash G)$ the right-regular representations $R$ and $R_\Gamma$ can be extended. If $f$ is $K$-finite and $\hat{f}$ is smooth and has compact support, then we will see that $g \mapsto [K_{R(f)}(g, g) - K_{R(f)}(g, g)]$ belongs to $L^1(\Gamma \backslash G)$, and thus

$$\Psi'(f) := \int_{\Gamma \backslash G} [K_{R(f)}(g, g) - K_{R(f)}(g, g)]\mu_G(\text{d}g)$$

is well-defined. The main result of the present paper is an expression of $\Psi'(f)$ in terms of $\hat{f}$ for those functions.
As mentioned in the introduction we are going to apply $\Psi'$ to the difference of distribution kernels of the resolvents $(z - \Omega)^{-1}$ of the Casimir operator restricted to a $K$-type of $L^2(\Gamma\backslash G)$ and $L^2(G)$, respectively. In this example the single kernels are not smooth, but their difference is so on the diagonal of $\Gamma\backslash G$. Strictly speaking, the integral defining $\Psi'$ exists for $\text{Re}(z) \ll 0$. For other values of $z$ we introduce a truncated version $\Psi'_R$, $R > 0$, and we define the value of $\Psi'$ as the constant term of the asymptotic expansion of $\Psi'_R$ as $R \to \infty$. It seems to be an interesting problem to characterize the functions of $\Omega$ (restricted to a $K$-type) with the property that $\Psi'_R$ (applied to the corresponding distribution kernels) admits such an asymptotic expansion.

Given a discrete series representation $\pi$ of $G$ we can consider the corresponding isotypic components of $L^2(G)$ and $L^2(\Gamma\backslash G)$. If we further consider a $K$-type of $\pi$, then the projections onto these components have smooth integral kernels. As a byproduct of the investigation of the resolvents we can show that $\Psi'$ can be applied to these integral kernels and that its values are integers.

3 The Plancherel theorem and integral kernels

3.1 The Plancherel theorems for $L^2(G)$ and $L^2(\Gamma\backslash G)$. Support of Plancherel measures

We start with describing the rough structure of the unitary dual $\hat{G}$. First there is a countable family of square integrable unitary representations, the discrete series $\hat{G}_d$. The discretely decomposable subspace $L^2(G)_d \subset L^2(G)$ is composed out of these representations each occurring with infinite multiplicity.

The orthogonal complement $L^2(G)_{ac}$ of $L^2(G)_d$ is given by a countable direct sum of direct integrals of unitary principal series representations. We are going to describe their parametrization. Let $G = KAN$ be an Iwasawa decomposition of $G$. The abelian group $A$ is isomorphic to the multiplicative group $\mathbb{R}^+$. Let $a$ and $n$ denote the Lie algebras of $A$ and $N$. Then $\text{dim}_{\mathbb{R}}(a) = 1$ and the roots of $(a, n)$, fix an order on $a$. Let $M = Z_K(A)$ denote the centralizer of $A$ in $K$. The unitary principal series representations $\pi^{\sigma,\lambda}$ of $G$ are parametrized by the set $\sigma, \lambda \in \hat{M} \times \text{ia}^*$. Let $W$ denote the Weyl group $N_K(A)/M$, where $N_K(A)$ denotes the normalizer of $A$ in $K$. It is isomorphic to $\mathbb{Z}_2$, and we can choose a representative of the non-trivial element $w \in N_K(A)$ such that $w^{-1} = w$. One knows that $\pi^{\sigma,\lambda}$ is equivalent to $\pi^{\sigma^w, -\lambda}$, where $\sigma^w$ denotes the Weyl conjugate representation of $\sigma$ given by $\sigma^w(m) := \sigma(m^w)$. For $\lambda \neq 0$ the representation $\pi^{\sigma,\lambda}$ is irreducible. If $\sigma$ is equivalent to $\sigma^w$, i.e. $\sigma$ is Weyl invariant, then it may happen that $\pi^{\sigma,0}$ is reducible. In this case it decomposes into a sum $\pi^{\sigma^+,0} \oplus \pi^{\sigma^-,0}$ of limits of discrete series representations.
The set of equivalence classes of unitary representations of $G$ which we have listed above is the set of tempered representations. We refer to Sec. 8 of [3] for a discussion of the notion of temperedness for $L^2(G)$ and $L^2(\Gamma \backslash G)$.

The Plancherel theorem for $L^2(G)$ is, of course, explicitly known for a long time [7]. The Plancherel measure $p$ is supported on the set of tempered representations (compare [1]). In particular, it is absolutely continuous with respect to the Lebesgue measure on $i\mathfrak{a}$. Thus we can neglect the point $\lambda = 0$. Then $L^2(G)_{ac}$ decomposes as a direct integral of unitary principal series representations over $\hat{M} \times i\mathfrak{a}^*_+$ with infinite multiplicity, and the Plancherel measure has full support. Note that the multiplicity space of the representation $\pi$ can be realized as $V^*_{\tilde{\pi}}$.

By $\hat{G}_{ac}$ we denote the set of irreducible unitary principal series representations $\pi^{\sigma,\lambda}$, $\lambda \neq 0$. The remaining unitary representations $\hat{G}_c = \hat{G} \setminus (\hat{G}_d \cup \hat{G}_{ac})$ can be realized as subspaces of principal series representations $\pi^{\sigma,\lambda}$ with $\lambda \in \mathfrak{a}^*_+ \cup \{0\}$. The case of limits of discrete series $\hat{G}_{ld}$ (in this case $\lambda = 0$) was mentioned above. The representations with parameter $\lambda > 0$ are not tempered and belong to the complementary series $\hat{G}_{cs}$.

In [3] we studied the Plancherel theorem for $L^2(\Gamma \backslash G)$. Let us recall its rough structure. The support of the corresponding Plancherel measure $p_{\Gamma}$ is the union of $\hat{G}_d$, $\hat{G}_{ac}$, and a countable subset of $\hat{G}_c$. $L^2(\Gamma \backslash G)$ decomposes into sum of subspaces $L^2(\Gamma \backslash G)_{cusp}$, $L^2(\Gamma \backslash G)_{ac}$, and $L^2(\Gamma \backslash G)_c$. Here $L^2(\Gamma \backslash G)_{cusp}$ is discretely decomposable into representations of the discrete series, each occurring with infinite multiplicity, $L^2(\Gamma \backslash G)_c$ is discretely decomposable into representations belonging to $\hat{G}_c$, each occurring with finite multiplicity, and $L^2(\Gamma \backslash G)_{ac}$ is a direct integral of unitary principal series representations with infinite multiplicity over the parameter set $\hat{M} \times i\mathfrak{a}^*_+$. On this set the Plancherel measure $p_{\Gamma}$ is absolutely continuous to the Lebesgue measure and has full support. The multiplicity space $M_{\pi}$ can be realized as a subspace of the $\Gamma$-invariant distribution vectors of $V_{\tilde{\pi}}$, i.e., $M_{\pi} \subset \Gamma V_{\tilde{\pi}},_{-\infty}$, where $\tilde{\pi}$ denotes the dual representation of $\pi$. For $\pi \in \hat{G}_{ac}$ we are going to describe $M_{\pi}$ explicitly in Subsection 3.3.

### 3.2 Extension of $R$ and $R_{\Gamma}$

The Plancherel theorem for $G$ provides a $G$-equivariant unitary equivalence

$$U : L^2(G) \xrightarrow{\sim} \int_{\hat{G}} V^*_\pi \hat{\otimes} V_\pi \ p(d\pi) \ ,$$

where $G$ acts on $L^2(G)$ by the right-regular representation $R$, and the action on the direct integral is given by $g \mapsto \{ \pi \mapsto \text{id}_{V^*_\pi} \otimes \pi(g) \}$. We can identify $V^*_\pi \hat{\otimes} V_\pi$ with the space of Hilbert-Schmidt operators on $V_\pi$. For $\phi \in C^\infty_c(G)$ we set

$$U(\phi) := \{ \pi \mapsto \hat{\phi}(\pi) \} \ ,$$
where $\tilde{\phi}(g) := \phi(g^{-1})$. This fixes the normalization of the Plancherel measure $p$.

The inverse transformation maps the family $\pi \mapsto h(\pi)$ to the function

$$U^{-1}(h)(g) = \int_G \text{Tr} \, \pi(g)h(\pi)p(d\pi).$$

If $f \in L^1(G)$, then $R(f)$ is given by $g \mapsto \{ \pi \mapsto \text{id}_{V^\pi} \otimes \hat{f}(\pi)\}$.

The Plancherel theorem for $\Gamma \backslash G$ provides a $G$-equivariant unitary equivalence

$$U_\Gamma : L^2(\Gamma \backslash G) \overset{\sim}{\to} \int_G M_\pi \hat{\otimes} V_\pi \, p_\Gamma(d\pi),$$

(4)

where $G$ acts on $L^2(\Gamma \backslash G)$ by the right-regular representation $R_G$, and the representation of $G$ on the direct integral is given by $g \mapsto \{ \pi \mapsto \text{id}_{M_\pi} \otimes \pi(g)\}$. Again, if $f \in L^1(G)$, than $R_G(f)$ is given by $g \mapsto \{ \pi \mapsto \text{id}_{M_\pi} \otimes \hat{f}(\pi)\}$. In order to write down an explicit formula for $U_\Gamma$ we first identify $M_\pi^*$ with $M_\pi$ and embed $M_\pi \hat{\otimes} V_\pi$ into $\text{Hom}(M_\pi, V_\pi)$. For $\phi \in C_c^\infty(\Gamma \backslash G)$ we define

$$U_\Gamma(\phi)(\pi) := \{ M_\pi \ni v \mapsto \int_{\Gamma \backslash G} \phi(g)\pi(g^{-1})v \, \mu_G(dg) \in V_\pi \}.$$

This fixes the normalization of $p_\Gamma$.

Let now $h$ be a function on $\text{supp}(p)$ such that $h(\pi)$ is a bounded operator on $V_\pi$. If $h$ is essentially bounded (and measurable in the appropriate sense), then it acts on the direct integral (3) by $\pi \mapsto \{ \text{id}_{V^\pi} \otimes h(\pi)\}$ and thus defines a bounded operator $\hat{R}(h)$ on $L^2(G)$ commuting with the left-regular action of $G$.

In a similar manner, if $h$ is a function on $\text{supp}(p_\Gamma)$ such that $h(\pi)$ is a bounded operator on $V_\pi$, and $h$ is essentially bounded, then it acts on the direct integral (4) by $\pi \mapsto \{ \text{id}_{M_\pi} \otimes h(\pi)\}$ and thus defines a bounded operator $\hat{R}_\Gamma(h)$ on $L^2(\Gamma \backslash G)$.

Let us now assume that the $h(\pi)$ are of trace-class, and that $\int_G \|h(\pi)\|_1 p(d\pi)$ is finite, where $\|\cdot\|_1$ denotes the trace norm $\|A\|_1 = \text{Tr} |A|$ for a trace class operator $A$ on $V_\pi$. Then $\hat{R}(h)$ is an integral operator with integral kernel $K_{\hat{R}(h)}(g,k) = \hat{h}(g^{-1}k)$. Indeed, for $\phi \in C_c^\infty(G)$ we have

$$\begin{align*}
(\hat{R}(h)\phi)(g) &= \int_G \text{Tr} \, \pi(g)h(\pi)\tilde{\phi}(\pi)p(d\pi) \\
&= \int_G \text{Tr} \int_G \pi(g)h(\pi)\pi(k^{-1})\phi(k)\mu_G(dk)p(d\pi) \\
&= \int_G \int_G \text{Tr} \pi(k^{-1}g)h(\pi)p(d\pi)\phi(k)\mu_G(dk) \\
&= \int_G \hat{h}(g^{-1}k)\phi(k)\mu_G(dk).
\end{align*}$$

We are looking for a similar formula for the integral kernel of $\hat{R}_\Gamma(h)$ in Subsection 3.3.
3.3 The absolute continuous part of $L^2(\Gamma \backslash G)$. Integral kernels for $\check{R}_\Gamma(h)$.

In this subsection we describe in detail the Plancherel decomposition of $L^2(\Gamma \backslash G)_{\text{ac}}$. The goal is to exhibit a class of functions $\pi \mapsto h(\pi)$ with the property that $\check{R}_\Gamma(h)$ is an integral operator.

Let $P = MAN$ be a fixed parabolic subgroup. If $(\sigma, \lambda) \in \hat{M} \times a_C^*$, then we define the representation $\sigma_\lambda$ of $P$ by $\sigma_\lambda(\sigma(m)a) := \sigma(\sigma(m)) a^{\rho - \lambda}$, where $\rho \in a^*$ is given by $2\rho(H) = \text{tr} \text{ad}(H)|_n$, $H \in a$, and for $\lambda \in a_C^*$ and $a = \exp(H) \in A$ we put $a^\lambda = e^{\lambda(H)}$. We realize the principal series representation $\pi^\sigma,\lambda := \sigma \sigma_\lambda$ as the subspace of $C^{-\infty}(G \times_P V_{\sigma_\lambda})$ of functions such that $f|_K \in L^2(K \times_M V_\sigma)$. Then $H^{\sigma,\lambda}_{-\infty} = C_{-\infty}^{-\infty}(G \times_P V_{\sigma_\lambda})$ are the spaces of smooth (resp. distribution) vectors of $\pi^\sigma,\lambda$. By restriction to $K$ we obtain canonical isomorphisms $H^{\sigma,\lambda}_{-\infty} \cong L^2(K \times_M V_\sigma)$. It therefore makes sense to speak of smooth functions $f$ on $ia^*$ such that $f(\lambda) \in H^{\sigma,\lambda}_{-\infty}$.

Note that $\partial X$ can be identified with $G/P$. Let $G/P = \Omega \cup \Lambda$ be the $\Gamma$-equivariant decomposition of the space $G/P$ into the (open) domain of discontinuity $\Omega$ and the (closed) limit set $\Lambda$. As a convex-cocompact subgroup $\Gamma$ acts freely and cocompactly on $\Omega$. We put $B := \Gamma \setminus \Omega$. Furthermore, we define the bundle $V_B(\sigma_\lambda) \to B$ by $V_B(\sigma_\lambda) := \Gamma \setminus (G \times_P V_{\sigma_\lambda})|_\Omega$. If $\lambda \in ia^*$, then we have a natural Hilbert space $L^2(B, V_B(\sigma_\lambda))$. Again, fixing a volume form on $B$ we obtain identifications of the spaces $L^2(B, V_B(\sigma_\lambda))$ with the fixed space $L^2(B, V_B(\sigma_0))$ so that it makes sense to speak of smooth functions $f$ on $ia^*$ such that $f(\lambda) \in L^2(B, V_B(\sigma_\lambda))$. We refer to [3], Sec. 3, for more details.

In [3] we defined a family of extension maps $\text{ext} : L^2(B, V_B(\sigma_\lambda)) \to \Gamma H^{\sigma,\lambda}_{-\infty}$. For $\lambda \in ia^*$, $\lambda \neq 0$, the extension map provides an explicit identification of the space of multiplicities $M_{\pi^\sigma,\lambda} \subset \Gamma H^{\sigma,\lambda}_{-\infty}$ with $L^2(B, V_B(\tilde{\sigma}_{-\lambda}))$.

The Plancherel measures $p$ and $p_\Gamma$ on $\{\sigma\} \times ia^*_+$ are given by $\frac{\dim(V_\sigma)}{2\pi \omega_X} p_\sigma(\lambda) d\lambda$, where $p_\sigma$ is a smooth symmetric function on $ia^*$ of polynomial growth (see [3], Lemma 5.5. (3)), and $\omega_X := \lim_{a \to 0} a^{-2p} \text{vol}_{G/K} (KaK)$ (see [3], Sec. 11). Note that $d\lambda$ is the real Lebesgue measure on $ia^*$.

We now describe the embedding

$$U^{-1}_\Gamma : \frac{\dim(V_\sigma)}{2\pi \omega_X} \int_{\{\sigma\} \times ia^*_+} L^2(B, V_B(\tilde{\sigma}_{-\lambda})) \otimes H^{\sigma,\lambda}_{-\infty} p_\sigma(\lambda) d\lambda \to L^2(\Gamma \setminus G)_{\text{ac}}.$$

If $v \otimes w \in L^2(B, V_B(\tilde{\sigma}_{-\lambda})) \otimes H^{\sigma,\lambda}_{-\infty}$, then we define $\langle v \otimes w \rangle := \langle \text{ext}(v), w \rangle$. Let $\phi$ be a smooth function of compact support on $ia^*_+ \cup \{0\}$ such that $\phi(\lambda) \in L^2(B, V_B(\tilde{\sigma}_{-\lambda})) \otimes H^{\sigma,\lambda}_{-\infty}$, then we have

$$U^{-1}_\Gamma(\phi)(g) = \frac{\dim(V_\sigma)}{2\pi \omega_X} \int_{ia^*_+} (1 \otimes \pi^\sigma(\lambda)) \phi(\lambda) > p_\sigma(\lambda) d\lambda.$$

Note that $\text{ext}$ may be singular at $\lambda = 0$. In this case it has a first-order pole and $p_\sigma(0) = 0$ (see [3], Prop. 7.4) such that the integral is still well-defined.
3.3 The absolute continuous part of \( L^2(\Gamma \backslash G) \). Integral kernels for \( \check{R}_\Gamma(h) \).

We now fix a \( K \)-type \( \gamma \in \check{K} \). Let \( H^{\sigma,\lambda}(\gamma) \) denote the \( \gamma \)-isotypic component of \( H^{\sigma,\lambda} \). By Frobenius reciprocity we have a canonical identification

\[
H^{\sigma,\lambda}(\gamma) \cong V_\gamma \otimes \text{Hom}_K(V_\gamma, H^{\sigma,\lambda}) \overset{\alpha^{-1}}{\cong} V_\gamma \otimes \text{Hom}_M(V_\gamma, V_\sigma) .
\]

Here \( \alpha^{-1}(v \otimes U) := U(v) \) and \( \beta(U)(v) := U(v)(1) \). Any operator \( A \in \text{End}(V_\gamma \otimes \text{Hom}_M(V_\gamma, V_\sigma)) \) gives rise to a finite-dimensional operator \( F(A) \in \text{End}(H^{\sigma,\lambda}) \) which is trivial on the orthogonal complement of \( H^{\sigma,\lambda}(\gamma) \).

Let \( q \) be a smooth function of compact support on \( \check{M} \times i\mathfrak{a}^* \) such that \( q(\sigma,\lambda) \in \text{End}(V_\gamma \otimes \text{Hom}_M(V_\gamma, V_\sigma)) \). We call \( q \) symmetric if it is compatible with the equivalences \( J^{\sigma,\lambda}_{w,\gamma} : H^{\sigma,\lambda} \to H^{\sigma,\lambda}_{\gamma,w} \), i.e. if \( F(q(\sigma^w, -\lambda)) = J^{\sigma,\lambda}_{w,\gamma} \circ F(q(\sigma, \lambda)) \circ (J^{\sigma,\lambda}_{\gamma,w})^{-1} \). If \( q \) is symmetric, then we can define the function \( h_q \) on \( \check{G} \) such that \( h_q(\pi) \in \text{End}(V_\pi) \) by \( h_q(\pi) := F(q(\sigma, \lambda)) \) for \((\sigma, \lambda) \in \check{M} \times i\mathfrak{a}^*, \lambda \neq 0, \) and by \( h_q(\pi) = 0 \) for all other representations.

Let \( \pi_* : H^{\sigma,\lambda}_\infty \to L^2(B, V_B(\sigma_\lambda)) \) denote the push-down map which can be considered here as the adjoint of the extension \( \text{ext} : L^2(B, V_B(\check{\sigma}_\lambda)) \to H^{\check{\sigma}_\lambda, -\lambda}_\infty \). The composition

\[
\pi^{\sigma,\lambda}(g)h_q(\pi^{\sigma,\lambda})\pi^{\sigma,\lambda}(k^{-1})\text{ext} \circ \pi_*
\]

is a finite-dimensional map from \( H^{\sigma,\lambda}_\infty \) to \( H^{\sigma,\lambda}_\infty \). It is therefore nuclear and has a well-defined trace.

**Lemma 3.1** The operator \( \check{R}_\Gamma(h_q) \) has a smooth integral kernel given by

\[
K_{\check{R}_\Gamma(h_q)}(g, k) = \sum_{\sigma \in M} \frac{\text{dim}(V_\sigma)}{4\pi \omega_X} \int_{i\mathfrak{a}^*} \text{Tr} \pi^{\sigma,\lambda}(g)h_q(\pi^{\sigma,\lambda})\pi^{\sigma,\lambda}(k^{-1})\text{ext} \circ \pi_*p_\sigma(\lambda)d\lambda .
\]

**Proof.** First of all note that the integral is well defined at \( \lambda = 0 \). If \( \text{ext} \circ \pi_* \) is singular at this point, then it has a pole of at most second order. But then the Plancherel density vanishes at least of second order, too.

Let \( \phi \in C_c^\infty(\Gamma \backslash G) \). In the Plancherel decomposition it is represented by the function \( \pi \mapsto U_\Gamma(\phi)(\pi) \in M_\pi \hat{\otimes} V_\pi \). We fix \( \lambda \) for a moment and choose orthonormal bases \( \{v_i\} \) of \( L^2(B, V_B(\check{\sigma}_-\lambda)) \) and \( \{w_i\} \) of \( H^{\sigma,\lambda} \) consisting of smooth sections. Furthermore, let \( \{v^i\} \) and \( \{w^j\} \) be dual bases of \( L^2(B, V_B(\sigma_\lambda)) \) and \( H^{\check{\sigma}_-, -\lambda} \), respectively. Then we have

\[
U_\Gamma(\phi)(\pi^{\sigma,\lambda}) = \sum_{i,j} \int_{\Gamma \backslash G} \langle v^i \otimes \pi^{\check{\sigma},-\lambda}(k)w^j, \phi(k)\mu_G(dk) \rangle v_i \otimes w_j .
\]

We have

\[
\check{R}_\Gamma(h_q)(\phi)(g) = \frac{\text{dim}(V_\gamma)}{4\pi \omega_X} \sum_{\sigma \in M} \int_{i\mathfrak{a}^*} < (1 \otimes \pi^{\sigma,\lambda}(g)h_q(\pi^{\sigma,\lambda}))U_\Gamma(\phi)(\pi^{\sigma,\lambda}) > p_\sigma(\lambda)d\lambda .\]
We now compute

\[
< (1 \otimes \pi^\sigma,\lambda)(g)h_q(\pi^\sigma,\lambda))U_T(\phi)(\pi^\sigma,\lambda) > \\
= \sum_{i,j} \int_G < v^i \otimes \pi^\sigma,\lambda(k)w^j > \phi(k)\mu_G(dk) < (1 \otimes \pi^\sigma,\lambda)(g)h_q(\pi^\sigma,\lambda))v_i \otimes w_j > \\
= \int_G \sum_{i,j} < v^i \otimes w^j > \phi(k) < (1 \otimes \pi^\sigma,\lambda)(g)h_q(\pi^\sigma,\lambda))\pi^\sigma,\lambda(k^{-1})v_i \otimes w_j > \mu_G(dk) \\
= \int_G \sum_{i,j} \phi(k)\langle v^i, \pi_*(w^j) \rangle_{ext}(v_i), \pi^\sigma,\lambda(g)h_q(\pi^\sigma,\lambda)\pi^\sigma,\lambda(k^{-1})w_j)\mu_G(dk) \\
= \int_G \sum_{j,l} \phi(k)\langle e_{ext} \circ \pi_*(w^j), \pi^\sigma,\lambda(g)h_q(\pi^\sigma,\lambda)\pi^\sigma,\lambda(k^{-1})w_j)\mu_G(dk) \\
= \int_G \sum_{i,j} \phi(k)\langle w^i, \pi^\sigma,\lambda(g)h_q(\pi^\sigma,\lambda)\pi^\sigma,\lambda(k^{-1})w_j\rangle_{ext} \circ \pi_*(w_j)\mu_G(dk) \\
= \int_G \phi(k)\text{Tr} \pi^\sigma,\lambda(g)h_q(\pi^\sigma,\lambda)\pi^\sigma,\lambda(k^{-1})w_{ext} \circ \pi_*(\mu_G(dk)).
\]

Inserting this computation into (3), the we obtain the desired formula for the integral kernel of \( R_T(h_q) \).

\[\blacksquare\]

4 Poisson transforms and asymptotic computations

4.1 Motivation

Let \( q \) be symmetric and define and \( h_q \) as in subsection 3.3. We want to show that the function \( g \mapsto [K_{R_T(h_q)}(g, g) - K_{\hat{R}(h_q)}(g, g)] \) belongs to \( L^1(\Gamma \backslash G) \). It follows that

\[
\Psi'(\tilde{h}_q) = \int_{\Gamma \backslash G} [K_{R_T(h_q)}(\Gamma g, \Gamma g) - K_{\hat{R}(h_q)}(g, g)]\mu_G(dg)
\]

is well-defined, and we are asking for an expression of \( \Psi'(\tilde{h}_q) \) in terms of \( q \), respectively \( h_q \).

In the present section we show related results using the language of Poisson transformations. In Subsection 5.1 we will provide the translation of these results and solve the problems above.

4.2 Poisson transformation, \( \epsilon \)-functions, and asymptotics

We fix a \( K \)-type \( \gamma \) and a \( M \)-type \( \sigma \). Let \( T \in \text{Hom}_M(V_\sigma, V_\gamma) \) and \( \lambda \in a_+^* \). If \( w \in V_\gamma \), then by Frobenius reciprocity we consider \( w \otimes T^* \) as an element of \( H^{\bar{\sigma}, -\lambda}(\tilde{\gamma}) \) which is given by the
4.2 Poisson transformation, c-functions, and asymptotics

function $k \mapsto T^*(\tilde{\gamma}(k^{-1})w)$ under the canonical identification $\phi_{-\lambda} : H_{\infty}^\sigma \to C^\infty(K \times_M V_{\tilde{\gamma}})$. We further put $\Phi_{\lambda,\mu} := \phi_{-\lambda} \circ \phi_{\mu}$. We will also use the notation $\Phi_{0,\lambda}$ for $\phi_{\lambda}$.

The Poisson transformation

$$P^T_\lambda : H_{-\infty}^\sigma \to C^\infty(G \times_K V_{\tilde{\gamma}})$$

is a $G$-equivariant map which is defined by the relation

$$\langle w, P^T_\lambda(\psi)(g) \rangle = \langle w \otimes T^*, \pi^\sigma \gamma^{-1}(g) \psi \rangle,$$

for all $\psi \in H_{-\infty}^\sigma$, $w \in V_{\tilde{\gamma}}$.

For the definition of the function $c_{\sigma}$ we refer to [3], Sec. 5. We have the relation

$$c_{\sigma}(\lambda)c_{\bar{\sigma}}(-\lambda) = p_{\sigma}(\lambda)^{-1}.$$ 

It turns out to be useful to introduce the normalized Poisson transformation $0P^T_\lambda := c_{\sigma}(-\lambda)^{-1}P^T_\lambda$.

We introduce the family of operators

$$P^T_{\lambda,a} : H_{-\infty}^\sigma \to C^\infty(K \times_M V_{\gamma}), \quad a \in A_+$$

by

$$P^T_{\lambda,a}(f)(k) := 0P^T_\lambda(f)(ka).$$

In order to discuss the asymptotic behaviour of $P^T_{\lambda,a}$ as $a \to \infty$ we need the normalized Knapp-Stein intertwining operators

$$J^w_{\sigma,\lambda} : H_{-\infty}^\sigma \to H_{-\infty}^{\sigma_{-w,-\lambda}}.$$

Note that $J^w_{\sigma,\lambda} \circ J^w_{\sigma_{-w,-\lambda}} = \text{id}$. We again refer to [3], Sec. 5, for more details. The following is a consequence of [3], Lemma 6.2. Let $\alpha \in a^*$ denote the short root of $(a,n)$. For $\lambda \in i\mathfrak{a}^*$ we have

$$P^T_{\lambda,a} = a^{\lambda - \rho} \frac{c_{\gamma}(\lambda)}{c_{\sigma}(-\lambda)} T \circ \Phi_{0,\lambda} + a^{-\lambda - \rho} \gamma(w) T \circ \Phi_{0,-\lambda} \circ J^w_{\sigma,\lambda} + a^{-\rho - \alpha} \mathcal{R}_{-\infty}(\lambda, a), \quad (7)$$

where $\mathcal{R}_{-\infty}(\lambda, a) \circ \Phi_{\lambda,0}$ remains bounded in $C^\infty(i\mathfrak{a}^*, \text{Hom}(H_{-\infty}^{\sigma_{0}}, C^{-\infty}(K \times_M V_{\gamma})))$ as $a \to \infty$. Multiplication by $T$ (resp. $\gamma(w)T$) is here considered as a map from $H_{-\infty}^{\sigma_{0}}$ (resp. $H_{-\infty}^{\sigma_{-w,0}}$) to $C^{-\infty}(K \times_M V_{\gamma})$ in the natural way. If $\chi, \tilde{\chi}$ are smooth functions on $K/M$ with disjoint support, then $\chi \mathcal{R}_{-\infty}(\lambda, a)\tilde{\chi} \circ \Phi_{\lambda,0}$ remains bounded in $C^\infty(i\mathfrak{a}^*, \text{Hom}(H_{-\infty}^{\sigma_{0}}, C^\infty(K \times_M V_{\gamma})))$ as $a \to \infty$. 

4.3 An estimate

In order to formulate the result appropriately we introduce the following space $C_{\Gamma}(G)$ of functions on $G$. For each compact $V \subset \Omega$ and integer $N$ we consider the seminorm

$$|\phi|_{V,N} := \sup_{kah \in V_{\lambda,M}} (1 + |\log(a)|)^N a^{2\rho} |\phi(kah)|, \phi \in C(G).$$

Here we consider $V$ as a subset of $K$ using the identification $G/P = K/M$. We define the Fréchet space $C_{\Gamma}(G)$ as the space of all continuous functions $\phi$ on $G$ such that $|\phi|_{V,N} < \infty$ for all compact $V \subset \Omega$ and $N \in \mathbb{N}$. If $\phi$ is $\Gamma$-invariant and belongs to $C_{\Gamma}(G)$, then clearly $\phi \in L^1(\Gamma \backslash G)$.

Now let $T \in \text{Hom}_{\mathbb{C}}(V_\gamma, V_\gamma)$, $R \in \text{Hom}_{\mathbb{C}}(V_\gamma, V_\gamma)$, and $q \in C_c^\infty(ia^*)$. Then we can define the operator

$$A_q = A_q(T, R) := \int_{\text{ia}^*} 0P_T^\gamma \circ (\text{ext} \circ \pi_\ast - 1) \circ (0P_R^\gamma)^* q(\lambda) d\lambda \in \text{Hom}(C_c^{-\infty}(G \times_K V_\gamma), C_c^\infty(G \times_K V_\gamma)).$$

This operator has a smooth integral kernel $(g, h) \mapsto A_q(g, h) \in \text{End}(V_\gamma)$. The main result of the present subsection is the following estimate.

**Proposition 4.1**

$$|A_q(g, g)| \in C_{\Gamma}(G).$$

**Proof.** Note, that we only have to show finiteness of the norms $|.|_{V,N}$, where $V \subset \Omega$ is compact and has the additional property that $V$ is contained in the interior of a compact subset $V_1 \subset \Omega$ satisfying $\gamma V_1 \cap V_1 = \emptyset$ for all $1 \neq \gamma \in \Gamma$. Indeed, any seminorm of $C_{\Gamma}(G)$ can be majorized by the maximum of finite number of these special ones.

We choose a smooth cut-off function $\tilde{\chi}$ on $\Omega$ such that $\text{supp}(\tilde{\chi}) \subset V_1$ and $\text{supp}(1 - \tilde{\chi}) \cap V = \emptyset$. We further choose a compact $V_2$ containing $V$ in its interior and being contained in the interior of $V_1$, and a cut-off function $\chi$ on $\Omega$ such that $\text{supp}(\chi) \subset V_2$ and $\text{supp}(1 - \chi) \cap V = \emptyset$.

Then we can write for $k \in V$

$$0P_{\lambda}^T \circ (\text{ext} \circ \pi_\ast - 1) \circ (0P_{\lambda}^R)^*(kah, kah) \quad = \quad \gamma(h)^{-1} \chi(k) \circ [P_{\lambda,a}^T (\text{ext} \circ \pi_\ast - 1) \circ (P_{-\lambda,a}^R)^*] (k, k) \circ \chi(k) \gamma(h). \quad (8)$$

In order to employ the off-diagonal localization of the Poisson transformation we write

$$\chi \circ P_{\lambda,a}^T (\text{ext} \circ \pi_\ast - 1) \circ (P_{-\lambda,a}^R)^* \circ \chi \quad = \quad \chi \circ P_{\lambda,a}^T (1 - \tilde{\chi}) \circ (\text{ext} \circ \pi_\ast - 1) \circ \tilde{\chi} \circ (P_{-\lambda,a}^R)^* \circ \chi \quad + \chi \circ P_{\lambda,a}^T (\text{ext} \circ \pi_\ast - 1) \circ (1 - \tilde{\chi}) \circ (P_{-\lambda,a}^R)^* \circ \chi. \quad (9)$$
4.3 An estimate

In (11) we could insert the factor \((1 - \tilde{\chi})\) since \(\tilde{\chi} \circ (ext \circ \pi_\ast - 1) \circ \tilde{\chi} = 0\). Using that \(\text{supp}(\chi) \cap \text{supp}(1 - \tilde{\chi}) = \emptyset\) we have

\[
\chi \circ \mathcal{P}_{\lambda,a}^T \circ (1 - \tilde{\chi}) = a^{-\lambda - \rho} \chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ J_{\sigma, \lambda}^w \circ (1 - \tilde{\chi}) + a^{-\rho - \sigma} \mathcal{R}_\infty(\lambda,a),
\]

where \(\mathcal{R}_\infty(\lambda,a) \circ \Phi_{\lambda,0}\) remains bounded in \(C^\infty(ia^*, \text{Hom}(H_{-\infty}^\sigma, C^\infty(K \times_M V_\gamma)))\) as \(a \to \infty\).

We obtain

\[
\chi \circ \mathcal{P}_{\lambda,a}^T \circ (ext \circ \pi_\ast - 1) \circ (\mathcal{P}_{-\lambda,a}^R)^* \circ \chi =
\]

\[
\begin{align*}
&\quad a^{-2\rho} a^{-2\lambda} \chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ J_{\sigma, \lambda}^w \circ (1 - \tilde{\chi}) \circ (ext \circ \pi_\ast - 1) \circ \tilde{\chi} \circ \Phi_{\lambda,0} \circ R^* \frac{c_\gamma(-\lambda)^*}{c_\sigma(\lambda)} \circ \chi \quad (10) \\
&+ a^{-2\rho} \chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ J_{\sigma, \lambda}^w \circ (1 - \tilde{\chi}) \circ (ext \circ \pi_\ast - 1) \circ \tilde{\chi} \circ (J_{\sigma,- \lambda}^w)^* \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^*(\Phi_0) \\
&+ a^{-2\rho} a^{-2\lambda} \chi \circ \frac{c_\sigma(\lambda)}{c_\sigma(-\lambda)} T \circ \Phi_{0, \lambda} \circ (ext \circ \pi_\ast - 1) \circ (1 - \tilde{\chi}) \circ (J_{\sigma,- \lambda}^w)^* \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^* \circ \chi \\
&+ a^{-2\rho} \chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ J_{\sigma, \lambda}^w \circ (ext \circ \pi_\ast - 1) \circ (1 - \tilde{\chi}) \circ (J_{\sigma,- \lambda}^w)^* \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^* \circ \chi \\
&+ Q(\lambda,a),
\end{align*}
\]

where \(a^{2\rho + \sigma} Q(\lambda,a)\) remains bounded in \(C^\infty(ia^*, \text{Hom}(C^{-\infty}(K \times_M V_\gamma), C^\infty(K \times_M V_\gamma)))\) as \(a \to \infty\).

We further compute using that the intertwining operators commute with \(ext \circ \pi_\ast\) (compare the proof of Lemma 12 for a similar argument), the functional equation of the intertwining operators, and

\[
\begin{align*}
\chi \circ (ext \circ \pi_\ast - 1) \circ \chi &= 0 \\
(11) + (13)
\end{align*}
\]

\[
\begin{align*}
&= a^{-2\rho} \chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ J_{\sigma, \lambda}^w \circ (ext \circ \pi_\ast - 1) \circ \tilde{\chi} \circ (J_{\sigma,- \lambda}^w)^* \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^* \circ \chi \\
&+ a^{-2\rho} \chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ J_{\sigma, \lambda}^w \circ (ext \circ \pi_\ast - 1) \circ (1 - \tilde{\chi}) \circ (J_{\sigma,- \lambda}^w)^* \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^* \circ \chi \\
&= a^{-2\rho} \chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ (ext \circ \pi_\ast - 1) \circ J_{\sigma, \lambda}^w \circ (J_{\sigma,- \lambda}^w)^* \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^* \circ \chi \\
&= a^{-2\rho} \chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ (ext \circ \pi_\ast - 1) \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^* \circ \chi \\
&= a^{-2\rho} \chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ \chi \circ (ext \circ \pi_\ast - 1) \circ \chi \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^* \\
&= 0.
\end{align*}
\]

Note that in (10) and (12) one of the intertwining operators is localized off-diagonally. We conclude that the following families of operators

\[
\begin{align*}
&\chi \circ \gamma(w)T \circ \Phi_{0,- \lambda} \circ J_{\sigma, \lambda}^w \circ (1 - \tilde{\chi}) \circ (ext \circ \pi_\ast - 1) \circ \tilde{\chi} \circ \Phi_{\lambda,0} \circ R^* \frac{c_\gamma(-\lambda)}{c_\sigma(\lambda)} \circ \chi \\
&\chi \circ \frac{c_\sigma(\lambda)}{c_\sigma(-\lambda)} T \circ \Phi_{0, \lambda} \circ (ext \circ \pi_\ast - 1) \circ (1 - \tilde{\chi}) \circ (J_{\sigma,- \lambda}^w)^* \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^* \circ \chi
\end{align*}
\]
We have shown in fact that 

\[ C^\infty(\mathbb{C}^*, \text{Hom}(C^{-\infty}(K \times_M V_\gamma), C^\infty(K \times_M V_\gamma))) \]. 

Restricting the smooth distribution kernel to the diagonal, multiplying by the smooth compactly supported function \( q \), and integrating over \( \mathbb{C}^* \) we obtain the following estimates using the standard theory of the Euclidean Fourier transform. For any \( N \in \mathbb{N} \)

\[
\sup_{k \in K} \sup_{a \in A_+} \left| \int_{ia^*} Q(\lambda, a)(k, k)q(\lambda)d\lambda \right| a^{2N}(1 + |\log(a)|)^N < \infty
\]

\[
\left| \int_{ia^*} \chi(k)^2[\gamma(w) T \circ \Phi_{0,-\lambda} \circ J^w_{\sigma,\lambda} \circ (1 - \bar{\chi}) \circ (\text{ext} \circ \pi_+ - 1) \circ \Phi_{\lambda,0} \circ R^* c_{\gamma}(-\lambda)](k, k)q(\lambda)a^{-2\lambda}d\lambda \right|
\]

\[
(1 + |\log(a)|)^N < \infty
\]

\[
\sup_{k \in K} \sup_{a \in A_+} \left| \int_{ia^*} \chi(k)^2[\frac{c_\gamma(\lambda)}{c_\sigma(-\lambda)} T \circ \Phi_{0,\lambda} \circ (\text{ext} \circ \pi_+ - 1) \circ (1 - \bar{\chi}) \circ (J^w_{\delta,-\lambda})^* \circ \Phi_{-\lambda,0} \circ R^* \tilde{\gamma}(w)^*[k, k)q(\lambda)a^{2\lambda}d\lambda \right|
\]

\[
(1 + |\log(a)|)^N < \infty
\].

This implies the proposition. \( \square \)

Remark: We have shown in fact that \( q \mapsto |(A_q)_{diag}| \) is a continuous map from \( C_c^\infty(\mathbb{C}^*) \) to \( L^1(\Gamma \backslash G) \). It would be desirable to extend this map from \( C_c^\infty(\mathbb{C}^*) \) to the Schwartz space \( \mathcal{S}(\mathbb{C}^*) \).

It is this technical problem that prevents us to prove that the Fourier transform \( \Phi \) of \( \Psi \) restricted to the unitary principal series representations is a tempered distribution. If this would true then it is in fact a measure and given by our computations below.

If we would like to show that the map \( q \mapsto |(A_q)_{diag}| \) extends to a map from the Schwartz space to \( L^1(\Gamma \backslash G) \) along the lines above we need estimates on the growth of \( \text{ext} \) as the parameter \( \lambda \) tends to infinity along the imaginary axis. If the imaginary axis is in the domain of convergence of \( \text{ext} \), i.e. the critical exponent \( \delta_\Gamma \) of \( \Gamma \) is negative, then such an estimate is easy to obtain. In the general case one has to estimate the meromorphic continuation of \( \text{ext} \), and this is an open problem.

### 4.4 A computation

In this subsection we want to express \( \int_{\Gamma \backslash G} \text{tr} A_q(g, g)\mu_G(dg) \) in terms of \( q \).

Recall that the symmetric space \( X = G/K \) can be compactified by adjoining the boundary \( \partial X = G/P \). As a convex-cocompact group \( \Gamma \) acts freely and properly on \( X \cup \Omega \) with compact quotient. Therefore, we can choose a smooth function \( \chi^\Gamma \in C_c^\infty(X \cup \Omega) \) such that \( \sum_{\gamma \in \Gamma} \gamma^* \chi^\Gamma \equiv 1 \) on \( X \cup \Omega \). The restriction of \( \chi^\Gamma \) to \( X \) can be lifted to \( G \) as a right-\( K \)-invariant function which
we still denote by $\chi^\Gamma$. We denote by $\chi^\Gamma_\infty$ the right-$M$-invariant lift to $K$ of the restriction of $\chi^\Gamma$ to $\partial X = K/M$. We write
\[
\int_{\Gamma \backslash G} \text{tr} \ A_q(g,g)\mu_G(dg) = \int_G \chi^\Gamma(g)\text{tr} \ A_q(g,g)\mu_G(dg).
\]
Let $\chi_U$ be the characteristic function of the ball $B_U$ in $X$ of radius $U$ centered at the origin $[K]$. Again, we denote its right-$K$-invariant lift to $G$ by the same symbol. Then we can write
\[
\int_G \chi^\Gamma(g)\text{tr} \ A_q(g,g)\mu_G(dg) = \lim_{U \to \infty} \int_G \chi^\Gamma(g)\chi_U(g)\text{tr} \ A_q(g,g)\mu_G(dg).
\]
Given $U$ we fix a function $\chi_1 \in C^\infty_c(G/K)$ such that
\[
\chi_1\chi_U \chi^\Gamma = \chi_U \chi^\Gamma.
\] (14)

The operator $\chi_U \chi^\Gamma_0 \mathcal{P}^\sigma_{\mathcal{A}} \circ (\text{ext} \circ \pi_\ast - 1) \circ (0\mathcal{P}^R_{-\lambda})^\ast \chi_1$ has an integral kernel of compact support. Since the kernel is smooth in the interior of the support it is of trace class. We can write
\[
\int_G \chi^\Gamma(g)\chi_U(g)\mu_G(dg) = \int_{i\mathbb{R}^+} \text{Tr} \ [\chi_U \chi^\Gamma_0 \mathcal{P}^\sigma_{\mathcal{A}} \circ (\text{ext} \circ \pi_\ast - 1) \circ (0\mathcal{P}^R_{-\lambda})^\ast \chi_1]g(\lambda)d\lambda.
\]
Note that
\[
i\lambda^* \ni \lambda \mapsto \text{Tr} \ [\chi_U \chi^\Gamma_0 \mathcal{P}^\sigma_{\mathcal{A}} \circ (\text{ext} \circ \pi_\ast - 1) \circ (0\mathcal{P}^R_{-\lambda})^\ast \chi_1]
\]
is a smooth function. We want to compute its limit in the sense of distributions as $U \to \infty$ using Green’s formula.

Note that $V_B(1_{\rho+\alpha})$ is a complex bundle with a real structure which is trivial together with this structure. Indeed, $B$ is orientable, and $V_B(1_{\rho+\alpha})$ is a real power of $\Lambda^\max T^*B$. We choose any non-vanishing positive section $\phi \in C^\infty(B, V_B(1_{\rho+\alpha}))$. For any $z \in \mathbb{C}$ we can form $\phi^z \in C^\infty(B, V_B(1_{\rho+z\alpha}))$. In particular, if we choose $z$ such that $z\alpha = \lambda - \mu$, then multiplication by $\phi^z$ gives an isomorphism $\Phi_{\lambda,\mu} : C^\infty(B, V_B(\sigma_\mu)) \to C^\infty(B, V_B(\sigma_\lambda))$, and similar isomorphisms of the spaces of $L^2$- and distribution sections. If $\text{Re}(z) = 0$, then $\text{ext} : C^\infty(B, V_B(1_{\rho+z\alpha})) \to \Gamma H^1_{-\infty} H^\sigma_{\mu}$ is regular (indeed $\rho + z\alpha$ belongs to the domain of convergence). Multiplication by $\text{ext}(\phi^z)$ gives a continuous map
\[
\Phi_{\lambda,\mu} : H^\sigma_{\infty} \to H^\sigma_{-\infty}.
\]
This map is $\Gamma$-equivariant and extends in fact to larger subspaces of $H^\sigma_{-\infty}$ of distributions which are smooth on neighbourhoods of the limit set $\Lambda$.

The usual trick to bring in Green’s formula is to write
\[
\text{Tr} \ [\chi_U \chi^\Gamma_0 \mathcal{P}^\sigma_{\mathcal{A}} \circ (\text{ext} \circ \pi_\ast - 1) \circ (0\mathcal{P}^R_{-\lambda})^\ast \chi_1] = \lim_{\mu \to \lambda} \text{Tr} \ [\chi_U \chi^\Gamma_0 \mathcal{P}^\sigma_{\mathcal{A}} \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0\mathcal{P}^R_{-\mu})^\ast \chi_1].
\]
Let $\nabla^\gamma$ denote the invariant connection of the bundle $V(\gamma) = G \times_K V_\gamma$ over $X$ and $\Delta_\gamma := -(\nabla^\gamma)^* \nabla^\gamma$ be the Laplace operator. Then there exists a constant $c \in \mathbb{R}$ such that $(\Delta_\gamma + c + \lambda^2) \circ P_\lambda^T = 0$. Let $n$ denote the outer unit-normal vector field at $B_U$.

By Green’s formula we have for $\lambda \neq \pm \mu$

\[
\text{Tr} \left[ \chi_{K} \Gamma_0 P_\lambda^T \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (0 P_{-\mu}^R)^* \chi_1 \right] = \frac{1}{\lambda^2 - \mu^2} \text{Tr} \left[ \Delta_\gamma \chi_{K} \Gamma_0 P_\lambda^T \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (0 P_{-\mu}^R)^* \chi_1 \right] - \frac{1}{\lambda^2 - \mu^2} \text{Tr} \left[ \chi_{\partial B_U} \circ (0 P_{-\mu}^R)^* \chi_{\partial B_U} \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (\nabla^\gamma_0 P_{-\mu}^R)^* \right] + \frac{1}{\lambda^2 - \mu^2} \text{Tr} \left[ \chi_{\partial B_U} \circ (\nabla^\gamma_0 P_{-\mu}^R)^* \chi_{\partial B_U} \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (0 P_{-\mu}^R)^* \right].
\]

Note that the derivatives of $\chi_1$ drop out because of (14). Moreover, $(0 P_{-\mu}^R)^* : H^\sigma_{-\infty} \to C^\infty(\partial B_U, V(\gamma)_{\partial B_U})$ denotes the composition of the Poisson transform and restriction to the boundary of $B_U$, and this operator can be expressed in terms of $\mathcal{P}_{\lambda,a}^T$.

We introduce the following notation. Let $a_U \in A_+$ be such that $\text{dist}_X(a_U; K) = U$. We define $\omega(U) := a_U^{-\rho} \text{vol}(\partial B_U)$. Note that $\omega_X := \lim_{U \to \infty} \omega(U)$ exists. Let $\chi_a^\Gamma \in C^\infty(K)$ denote the function $k \mapsto \chi^\Gamma(ka)$. Note that $\lim_{a \to \infty} \chi_a = \chi^\infty$. Then we can write

\[(16) + (17) \quad - \frac{\omega_{a_U}^{2\rho}}{\lambda^2 - \mu^2} \text{Tr} \chi_{a_U} \circ \mathcal{P}_{\lambda,a_U}^T \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (\partial \mathcal{P}_{\lambda,a,U}^R)^* \quad + \quad \frac{\omega_{a_U}^{2\rho}}{\lambda^2 - \mu^2} \text{Tr} \chi_{a_U} \circ \partial \mathcal{P}_{\lambda,a_U}^T \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (\mathcal{P}_{\lambda,a,U}^R)^* \]

Here $\partial \mathcal{P}_{\lambda,a}^T$ stands for the derivative of the function $a \mapsto \mathcal{P}_{\lambda,a}^T$ with respect to the positive fundamental unit vector field on $A$.

Let $\chi$ be a smooth cut-off function on $X \cup \Omega$ of compact support such that

\[
\gamma \text{supp}(\chi) \cap \text{supp}(\chi) = \emptyset, \quad \forall \gamma \in \Gamma, \gamma \neq 1.
\]

Note that $\chi^\Gamma$ can be decomposed into a finite sum $\chi^\Gamma = \sum_i \chi^i$ such that each $\chi^i$ satisfies (18).

We fix a cut-off function $\tilde{\chi}$ on $\Omega$ satisfying (15) and $\tilde{\chi} \equiv 1$ on a neighbourhood of $\text{supp}(\chi) \cap \Omega$. We further define $\chi_a(k) = \chi(kaK)$, $\chi_\infty(k) = \chi_{\partial X}(kM)$ and observe that $|\chi_a - \chi_\infty| = O(a^{-\alpha})$ for any seminorm $|\cdot|$ of $C^\infty(K)$. Using that $\tilde{\chi}(\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \tilde{\chi} = 0$ we can write

\[(16) + (17) \quad (\chi^\Gamma \text{ replaced by } \chi) \quad - \frac{\omega_{a_U}^{2\rho}}{\lambda^2 - \mu^2} \text{Tr} \chi_{a_U} \circ \mathcal{P}_{\lambda,a_U}^T \circ (1 - \tilde{\chi}) \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (\partial \mathcal{P}_{\lambda,a,U}^R)^* \quad - \quad \frac{\omega_{a_U}^{2\rho}}{\lambda^2 - \mu^2} \text{Tr} \chi_{a_U} \circ \partial \mathcal{P}_{\lambda,a_U}^T \circ \tilde{\chi} \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (1 - \tilde{\chi}) \circ (\partial \mathcal{P}_{\lambda,a,U}^R)^* \]

Note that $\chi^\Gamma$ can be decomposed into a finite sum $\chi^\Gamma = \sum_i \chi^i$ such that each $\chi^i$ satisfies (18).
\[ + \frac{\omega U_{aU}^2}{\lambda^2 - \mu^2} \text{Tr} \chi^*_U \circ \partial \mathcal{P}^T_{\lambda, aU} \circ (1 - \tilde{\chi}) \circ (\text{ext} \circ \tilde{\Phi}_{\lambda, \mu} \circ \pi_* - \tilde{\Phi}_{\lambda, \mu}) \circ (\mathcal{P}^R_{\mu, aU})^* \]

\[ + \frac{\omega U_{aU}^2}{\lambda^2 - \mu^2} \text{Tr} \chi^*_U \circ \partial \mathcal{P}^T_{\lambda, aU} \circ \tilde{\chi} \circ (\text{ext} \circ \tilde{\Phi}_{\lambda, \mu} \circ \pi_* - \tilde{\Phi}_{\lambda, \mu}) \circ (1 - \tilde{\chi}) \circ (\mathcal{P}^R_{\mu, aU})^*. \] (19)

We now insert the asymptotic decomposition (7) of the operators \( \mathcal{P}^T_{\lambda, a} \) as \( a \to \infty \) noting that in each line one of these operators is localized off-diagonally. In order to stay in trace class operators we choose a function \( \chi_1 \in C^\infty(K) \) such that \( \text{supp}(1 - \chi_1) \cap \text{supp}(\chi) = \emptyset \) and \( \text{supp}(1 - \tilde{\chi}) \cap \text{supp}(\chi_1) = \emptyset. \) We obtain

\[ (16) + \chi^T \text{ replaced by } \chi \]

\[ = - \frac{\omega U_{aU}^2}{\lambda^2 - \mu^2} \text{Tr} R^* \frac{c_\gamma(-\mu)}{c_\sigma(\mu)} \gamma(w) T \chi^*_U \Phi_{0, \lambda} J^w_{w, \lambda} (1 - \tilde{\chi}) \circ (\text{ext} \circ \tilde{\Phi}_{\lambda, \mu} \circ \pi_* - \tilde{\Phi}_{\lambda, \mu}) \Phi_{\mu, 0} \chi_1 \]

\[ - \frac{\omega U_{aU}^2}{\lambda^2 - \mu^2} \text{Tr} R^* \frac{c_\gamma(-\mu)}{c_\sigma(\mu)} \gamma(w) T \chi^*_U \Phi_{0, \lambda} J^w_{w, \lambda} (1 - \tilde{\chi}) \circ (\text{ext} \circ \tilde{\Phi}_{\lambda, \mu} \circ \pi_* - \tilde{\Phi}_{\lambda, \mu}) \Phi_{\mu, 0} \chi_1 \]

\[ - \frac{\omega U_{aU}^2}{\lambda^2 - \mu^2} \text{Tr} R^* \frac{c_\gamma(-\mu)}{c_\sigma(\mu)} \gamma(w) T \chi^*_U \Phi_{0, \lambda} J^w_{w, \lambda} (1 - \tilde{\chi}) \circ (\text{ext} \circ \tilde{\Phi}_{\lambda, \mu} \circ \pi_* - \tilde{\Phi}_{\lambda, \mu}) \Phi_{\mu, 0} \chi_1 \]

\[ + a_{U}^{-\alpha} \frac{1}{\lambda^2 - \mu^2} R_{\chi}(\lambda, \mu, a_U) \]

The remainder \( R_{\chi}(\lambda, \mu, a) \) is holomorphic and remains bounded in \( C^\infty(\text{i}a^* \times \text{i}a^*) \) as \( a \to \infty. \)
We define \( \langle R, T \rangle \in \mathbb{C} \) such that \( R^* \circ T = \langle R, T \rangle \text{id}_{V_\sigma} \). If \( \sigma \) is not Weyl-invariant, i.e. \( \sigma^w \not\equiv \sigma \), then the compositions \( R^* c_\gamma(\mu)^* \gamma(w)T, R^* \gamma(w)^{-1} c_\gamma(\lambda)T \) vanish. If the representation \( \sigma \) is Weyl-invariant, then it can be extended to the normalizer \( N_K(M) \) of \( M \). In particular, we can define \( \sigma(w) \). In this case we define \( \langle R, T \rangle(\lambda) \in \mathbb{C} \) such that

\[
\sigma(w)R^* \gamma(w)^{-1} \frac{c_\gamma(\lambda)}{c_\gamma(-\lambda)} T = \langle R, T \rangle(\lambda) \text{id}_{V_\sigma}.
\]

Note that \( R^* \frac{c_\gamma(-\mu)^*}{c_\gamma(\mu)} \gamma(w)T \sigma(w)^{-1} = \langle R, T \rangle(-\mu) \text{id}_{V_\sigma} \). Further we put \( J_{\sigma,\lambda} = \sigma(w)J_{\sigma,\lambda}^w : H_{-\infty}^{\sigma,\lambda} \to H_{-\infty}^{-\lambda} \). Then we can write

\[
\text{Tr}[\chi U \circ 0 P_T^* \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P_{-\mu}^R)^* \chi_1] = -\frac{\omega_U a_{\mu,\lambda}^* (R, T)(-\mu)}{\lambda + \mu} \text{Tr} \Phi_{\mu,-\lambda} \circ \chi \circ \sigma, (1 - \bar{\chi}) \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ \chi_1(20)
\]

\[
-\frac{\omega_U a_{\mu,\lambda}^* (R, T)}{\lambda^2 - \mu^2} \text{Tr} \chi \circ J_{\sigma,\lambda}^w \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (J_{\sigma,-\mu}^w)^* \circ \Phi_{-\mu,-\lambda} \circ \chi_1(21)
\]

\[
+ \frac{1}{\lambda^2 - \mu^2} \text{Tr}[\Delta, \chi] \circ 0 P_T^* \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P_{-\mu}^R)^*
\]

\[
+ \frac{a_U^\sigma}{\lambda^2 - \mu^2} Q_\chi(\lambda, \mu, a_U)
\]

\[
Q_\chi(\lambda, \mu, a_U) = R_\chi(\lambda, \mu, a_U)
\]

\[
-\frac{a_U^\sigma}{\lambda - \mu} \text{Tr}[\Delta, \chi] \circ (1 - \chi_\sigma) \circ 0 P_T^* \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P_{-\mu}^R)^*
\]

We defer the justification of the terms (22), (23) to Lemma 4.3 below. The functional \( \text{Tr}' \) here is applied to operators with distribution kernels which are continuous on the diagonal, and it takes the integral of its local trace over the diagonal. Note that the remainder \( Q_\chi \) is independent of the choice of \( \chi_1 \).

The left-hand side of this formula is holomorphic on \( a_\sigma^* \times a_\sigma^* \). The terms on the right-hand side may have poles. To aim of the following discussion is to understand these singularities properly.

**Lemma 4.2**

\[
\frac{1}{\lambda - \mu} \text{Tr} \chi \circ J_{\sigma,\lambda}^w \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (J_{\sigma,-\mu}^w)^* \circ \Phi_{-\mu,-\lambda} \circ \chi_1
\]

is regular for \( \mu = \lambda \).
Proof. We must show that
\[ \text{Tr} \chi_\infty \circ J_{\sigma,\lambda}^w \circ (ext \circ \pi_* - 1) \circ (J_{\sigma,-\lambda}^w)^* \circ \chi_1 = 0 . \]
Recall the definition of the scattering matrix \( S_{\sigma,\lambda}^w \) from [3], Def. 5.6. We are going to employ the relations
\[ ext \circ S_{\sigma,\lambda}^w = J_{\sigma,\lambda}^w \circ ext , \pi_* \circ (J_{\sigma,-\lambda}^w)^* = (S_{\sigma,-\lambda}^w)^* \circ \pi_* , \]
\[ S_{\sigma,\lambda}^w \circ (S_{\sigma,-\lambda}^w)^* = \text{id} . \]
We now compute
\[
\begin{align*}
\text{Tr} \chi_\infty \circ J_{\sigma,\lambda}^w \circ (ext \circ \pi_* - 1) \circ (J_{\sigma,-\lambda}^w)^* \circ \chi_1 &= \text{Tr} \chi_\infty \circ ext \circ J_{\sigma,\lambda}^w \circ (S_{\sigma,-\lambda}^w)^* \circ \pi_* \circ \chi_1 - \chi_\infty \circ J_{\sigma,\lambda}^w \circ (J_{\sigma,-\lambda}^w)^* \circ \chi_1 \\
&= \text{Tr} \chi_\infty \circ (ext \circ \pi_* - 1) \circ \chi_1 \\
&= 0 .
\end{align*}
\]
In particular we have
\[
\lim_{U \to \infty} \lim_{\mu \to \lambda} \omega(U) d_U^w - \frac{\lambda}{\lambda - \mu} \text{Tr} \chi_\infty \circ J_{\sigma,\lambda}^w \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (J_{\sigma,-\lambda}^w)^* \circ \Phi_{-\mu,\lambda} \circ \chi_1 = \omega \chi \langle R, T \rangle \frac{d}{d\mu} \bigg|_{\mu=\lambda} \text{Tr} \chi \circ J_{\sigma,\lambda}^w \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (J_{\sigma,-\lambda}^w)^* \circ \Phi_{-\mu,\lambda} \circ \chi_1 .
\]
(25)

If the distribution kernel of an operator admits a continuous restriction to the diagonal, then let \( DA \) denote this restriction.

Lemma 4.3 1. For any compact subset \( Q \subset i\mathfrak{a}^* \) there is a constant \( C \) such that for all \( k \in K \) and \( \mu, \lambda \in Q \)
\[
|D[\Delta, \chi] \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (0 P^R_{-\mu})^* (ka)| < Ca^{-2\rho-\alpha} .
\]

2. We have
\[
\text{Tr} \Gamma [\Delta, \chi] \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (0 P^R_{-\mu})^* = 0
\]
(note that we consider the cut-off function \( \chi^\Gamma \) here).

Proof. The reason that 1. holds true is that \( |d\chi(ka)| \leq Ca^{-\alpha} \) and \( |\Delta \chi(ka)| \leq Ca^{-\alpha} \) uniformly in \( k \in K \) and \( a \in A \). We use the decomposition
\[
D[\Delta, \chi] \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (0 P^R_{-\mu})^* (ka)
\]
\[
= D[\Delta, \chi] \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (1 - \chi) \circ (0 P^R_{-\mu})^* (ka)
\]
\[
+ D[\Delta, \chi] \circ 0 P^T_\lambda \circ (1 - \chi) \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (0 P^R_{-\mu})^* (ka) .
\]
(26)
The asymptotic behaviour (6) of the operators $P^T_{\lambda,a}$ is uniform for $\lambda$ in compact subsets of $i\mathbb{R}^*$ and can be differentiated with respect to $a$. We conclude that for any compact subset $Q \subset i\mathbb{R}^*$ there is a constant $C \in \mathbb{R}$ such for all $\lambda, \mu \in Q$ we have

$$\sup_k |D[\Delta, \chi]_a \circ 0 P^T_\lambda \circ (1 - \chi) \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P^*_\mu)^*(ka)| \leq Ca^{-2\rho - \delta}$$

We can write $\chi^R$ as a finite sum $\chi^R = \sum_i \chi^i$, where the cut-off functions $\chi^i$ obeying (18). For each index $i$ we choose an appropriate cut-off function $\chi_1^i$ as above. It follows from 1. that $\text{tr} D[\Delta, \chi^R] \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P^*_\mu)^*$ is integrable over $G$. We compute

$$\begin{align*}
\text{Tr}^\prime [\Delta, \chi^R] & \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P^*_\mu)^* \\
& = \sum_{\gamma \in \Gamma} \text{Tr}^\prime \pi(\gamma)^{-1} \chi^R \circ (\Delta, \chi^R) \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P^*_\mu)^* \\
& = \sum_{\gamma \in \Gamma} \text{Tr}^\prime \chi^R \circ [\Delta, (\gamma^{-1})^* \chi^R] \circ \pi(\gamma) \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P^*_\mu)^* \\
& = \sum_{\gamma \in \Gamma} \text{Tr}^\prime \chi^R \circ [\Delta, (\gamma^{-1})^* \chi^R] \circ 0 P^T_\lambda \circ (\pi(\gamma) \circ ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \pi(\gamma) \circ \Phi_{\lambda,\mu}) \circ (0 P^*_\mu)^* \\
& = \sum_{\gamma \in \Gamma} \text{Tr}^\prime \chi^R \circ [\Delta, (\gamma^{-1})^* \chi^R] \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P^*_\mu)^* \\
& = 0
\end{align*}$$

since $\sum_{\gamma \in \Gamma} (\gamma^{-1})^* \chi^R \equiv 1$.

Note that the second assertion of the lemma implies

$$\sum_i \text{Tr} [\Delta, \chi^i] \circ 0 P^T_\lambda \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (0 P^*_\mu)^* \circ \chi^i = 0$$

We now combine (20) and (21) and write

$$(20) + (21)$$

$$\begin{align*}
& = \frac{\omega(U)(a^{\mu + \lambda}_{i\xi} - a^{-\mu - \lambda}_{i\xi})}{\lambda + \mu} (R, T)(-\mu) \\
& \quad \text{Tr} \Phi_{\mu,-\lambda} \circ \chi_{\infty} \circ J_{\sigma,\lambda} \circ (1 - \tilde{\chi}) \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ \chi_1 \\
& + \omega(U) a^{\mu + \lambda}_{i\xi} \text{Tr} \left[ (R, T)(\lambda) \chi_{\infty} \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (1 - \tilde{\chi}) \circ (J_{\delta,-\lambda})^* \circ \Phi_{\delta,-\lambda} \circ \chi_1 \\
& \quad - (R, T)(-\mu) \Phi_{\mu,-\lambda} \circ \chi_{\infty} \circ J_{\sigma,\lambda} \circ (1 - \tilde{\chi}) \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ \chi_1 \right]
\end{align*}$$
Note that (28) is regular at $\lambda = \mu = 0$. In fact, if $\text{ext}$ has a pole at $\lambda = 0$, then it is of first order and $J_{\sigma,0} = \text{id}$ (see [3], Prop. 7.4). Hence, the composition $\chi_\infty \circ J_{\sigma,0} \circ (1 - \tilde{\chi})$ vanishes.

In order to see that (29) is regular at $\lambda = \mu = 0$, too, observe in addition that

$$\text{Tr} [\chi \circ (\text{ext} \circ \pi_* - 1) \circ (1 - \tilde{\chi}) \circ (J_{\tilde{\sigma},0})^* \circ \chi_1 - \chi \circ J_{\sigma,0} \circ (1 - \tilde{\chi}) \circ (\text{ext} \circ \pi_* - 1) \circ \chi_1] = 0 \ . \quad (30)$$

Combining Lemma 4.2 and 4.3., and (30) we conclude that $\frac{1}{\lambda^2 - \mu^2} Q_{\chi^r}(\lambda, \mu, a_U)$ is regular at $\mu = \lambda$, where we set $Q_{\chi^r} := \sum_i Q_{\chi^r}$. Furthermore, locally uniformly in $\lambda$

$$\lim_{U \to \infty} \frac{a_{\chi^r}^{-\alpha}}{\lambda^2 - \mu^2} Q_{\chi^r}(\lambda, \mu, a_U) = 0 \ .$$

By the Lemma of Riemann-Lebesgue we have

$$\lim_{U \to \infty} \frac{\omega(U) a_{\chi^r}^{2\lambda}}{2\lambda} \text{Tr} [(R, T)(\lambda) \chi_\infty \circ (\text{ext} \circ \pi_* - 1) \circ (1 - \tilde{\chi}) \circ (J_{\tilde{\sigma},-\lambda})^* \circ \Phi_{\lambda,-\lambda} \circ \chi_1$$

$$- (R, T)(-\lambda) \Phi_{\lambda,-\lambda} \circ \chi_\infty \circ J_{\sigma,\lambda} \circ (1 - \tilde{\chi}) \circ (\text{ext} \circ \pi_* - 1) \circ \chi_1] = 0$$

as distributions on $i\mathfrak{a}^*$. Moreover,

$$\lim_{U \to \infty} \frac{\omega(U)(a_{\chi^r}^{2\lambda} - a_{\chi^r}^{2\lambda})}{2\lambda} \langle R, T \rangle(-\lambda)$$

$$\text{Tr} \Phi_{\lambda,-\lambda} \circ \chi_\infty \circ J_{\sigma,\lambda} \circ (1 - \tilde{\chi}) \circ (\text{ext} \circ \pi_* - 1) \circ \chi_1 $$

$$= \pi \omega_X \langle R, T \rangle(0) \delta_0(\lambda) \lim_{\lambda \to 0} \left[ \text{Tr} \Phi_{\lambda,-\lambda} \circ \chi_\infty \circ J_{\sigma,\lambda} \circ (1 - \tilde{\chi}) \circ (\text{ext} \circ \pi_* - 1) \circ \chi_1 \right] \quad (32)$$

in the sense of distributions on $i\mathfrak{a}^*$. Combining (25) and (32) we now have shown the following proposition.

**Proposition 4.4**

$$\int_{\Gamma \backslash G} \text{tr} A_q(\sigma, g) \mu_G(dg) \quad (33)$$

$$= \omega_X \langle R, T \rangle \int_{i\mathfrak{a}^*} \frac{d}{d\mu|_{\mu=\lambda}} \text{Tr} \left[ \chi_\infty^i \circ J_{\sigma,\lambda}^i \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \tilde{\Phi}_{\lambda,\mu}) \circ (J_{\tilde{\theta},-\mu}^w)^* \circ \Phi_{-\mu,-\lambda} \circ \chi_1^i \right] q(\lambda) d\lambda$$

$$+ \pi \omega_X \langle R, T \rangle(0) \sum_{i} \lim_{\lambda \to 0} \text{Tr} \left[ \Phi_{\lambda,-\lambda} \circ \chi_\infty^i \circ J_{\sigma,\lambda} \circ (\text{ext} \circ \pi_* - 1) \circ \chi_1^i \right] q(0) .$$

Observe that we can rewrite this in the more invariant form

$$\omega_X \langle R, T \rangle \int_{i\mathfrak{a}^*} \frac{d}{d\mu|_{\mu=\lambda}} \text{Tr} \left[ \chi_\infty^i \circ J_{\sigma,\lambda}^i \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \tilde{\Phi}_{\lambda,\mu}) \circ (J_{\tilde{\theta},-\mu}^w)^* \circ \Phi_{-\mu,-\lambda} \right] q(\lambda) d\lambda$$

$$+ \pi \omega_X \langle R, T \rangle(0) \lim_{\lambda \to 0} \text{Tr} \left[ \Phi_{\lambda,-\lambda} \circ \chi_\infty^i \circ J_{\sigma,\lambda} \circ (\text{ext} \circ \pi_* - 1) \right] q(0) .$$
5 The Fourier transform of $\Psi$

5.1 The contribution of the scattering matrix

We consider a symmetric function $q \in C^\infty_c(\mathfrak{a}^*, \text{End}(V_{\gamma} \otimes \text{Hom}_M(V_{\gamma}, V_{\sigma})))$. For $\pi \in \hat{G}$ we form $h_q(\pi) \in \text{End}(V_{\pi})$ as in Subsection 3.3. There we have seen that $\hat{R}(h_q)$ and $\hat{\Gamma}(h_q)$ have smooth integral kernels.

We choose a basis \{\(v_i\)\}_{i=1,\ldots,\dim(\gamma)} of \(V_\gamma\) and a basis \{\(T_j\)\}_{j=1,\ldots,\dim(\text{Hom}_M(V_\gamma, V_{\sigma}))} of \(\text{Hom}_M(V_{\gamma}, V_{\sigma})\). Let \{\(v^i\)\} be a dual basis of \(V_\gamma\) and \{\(T^j\)\} be a dual basis of \(\text{Hom}_M(V_{\sigma}, V_\gamma) = \text{Hom}_M(V_\gamma, V_{\sigma})^*\) (with respect to the pairing \(\langle T', T \rangle = \text{tr}_{V_{\gamma}} T' \circ T, T \in \text{Hom}_M(V_{\gamma}, V_{\sigma}), T' \in \text{Hom}_M(V_{\sigma}, V_\gamma)\)). Then \{\(\phi_{ij} := v_i \otimes T_j\)\} can be considered as a basis of \(H^{\sigma, \lambda}(\gamma)\). Furthermore, \{\(\phi^{ij} := v^i \otimes (T^j)^*\)\} can be considered as a basis of \(H^{\sigma, -\lambda}(\gamma)\). We can write

$$\langle \phi_{ij}, \phi^{kl} \rangle = \int_K \langle T_j \circ \gamma(k)^{-1}(v_i), (T^k)^* \circ \gamma(k)^{-1}(v^l) \rangle dk \quad \text{for } k \in K,$$

where \(\gamma(k)^{-1}(v_i) \rightarrow v^i\) as \(k \rightarrow 0\) and \(\gamma(k)^{-1}(v^i) \rightarrow v_i\) as \(k \rightarrow 0\).

For \(g \in G\) and \(w \in V_\gamma\) we define \(p^T_{\lambda,w}(g) \in H^{2\sigma, -\lambda}_{\infty}\) such that \(\langle p^T_{\lambda,w}(\phi)(g), w \rangle = \langle p^T_{\lambda,w}(g), \phi \rangle\) for all \(\phi \in H^{\sigma, \lambda}_{\infty}\). Using (34) we can write \(p^T_{\lambda,w}(g) = \pi^{\sigma, -\lambda}(g)(w \otimes T^*)\). We can write

$$q = \sum_{i,j,k,l} q_{ijkl} v_i \otimes T_j \otimes v^k \otimes T^l,$$

where the functions \(q_{ijkl} := \langle v^i \otimes T^j, q(v_k \otimes T_l) \rangle\) belong to \(C^\infty_c(\mathfrak{a}^*)\). Now we can compute

$$\text{Tr } \pi^{\sigma, \lambda}(g) h_q(\pi^{\sigma, \lambda}) = \dim(V_\gamma) \sum_{i,j} \langle \phi_{ij}^{ij}, \pi^{\sigma, \lambda}(g) h_q(\pi^{\sigma, \lambda}) \phi_{ij} \rangle = \dim(V_\gamma) \sum_{i,j} \langle (p^T_{\lambda}(h_q(\pi^{\sigma, \lambda}) \phi_{ij}))(g^{-1}), v^i \rangle \quad \text{for } \phi_{ij} \in H^{\sigma, \lambda}_{\infty},$$

$$= \dim(V_\gamma) \sum_{i,j} \langle h_q(\pi^{\sigma, \lambda}) \phi_{ij}, p^T_{\lambda,w}(g^{-1}) \rangle$$

$$= \dim(V_\gamma) \sum_{i,j} \langle q_{ijkl}(\lambda)(v_k \otimes T_l, p^T_{\lambda,w}(g^{-1})) \rangle$$

$$= \dim(V_\gamma) \sum_{i,j,k,l} q_{ijkl}(\lambda)(v^i, p^T_{\lambda,w}(g^{-1}))(1, v_k)$$

$$= \dim(V_\gamma) \sum_{i,j,k,l} q_{ijkl}(\lambda)(v^i, p^T_{\lambda}(p^T_{\lambda,w}(g^{-1}))(1, v_k)).$$
In the last line of this computation $P_{\Lambda}^{T_{j}} \circ (P_{\Lambda}^{T_{j}})^{*} (g, g')$ is the integral kernel of the $G$-equivariant operator

$$P_{\Lambda}^{T_{j}} \circ (P_{\Lambda}^{T_{j}})^{*} : C_{c}^{-\infty}(G \times_{K} V_{\gamma}) \to C^{\infty}(G \times_{K} V_{\gamma}).$$

We can express the integral kernel of $R(h_{q})$ via Poisson transforms as follows:

$$K_{R(h_{q})}(g, g_{1}) = \int_{G} \text{Tr} \pi(g)h_{q}((\pi(g_{1}^{-1})p(d\pi)
= \frac{\text{dim}(V_{\gamma})}{4\pi \omega_{X}} \int_{\text{ia}^{*}} \text{Tr} \pi(g_{1}^{-1})h_{q}(\pi) \pi_{\sigma}(\lambda) d\lambda
= \sum_{i,j,k,l} \frac{\text{dim}(V_{\gamma}) \text{dim}(V_{\gamma})}{4\pi \omega_{X}} \int_{\text{ia}^{*}} q_{kl}^{ij} \langle \phi_{kl}, \pi_{\sigma}(\lambda) \pi_{\sigma}(g_{1}^{-1}) \phi_{j}^{} \rangle d\lambda
= \sum_{i,j,k,l} \frac{\text{dim}(V_{\gamma}) \text{dim}(V_{\gamma})}{4\pi \omega_{X}} \int_{\text{ia}^{*}} q_{kl}^{ij} \langle \pi_{\sigma}(g_{1}^{-1}) \phi_{j}^{}, \pi_{\sigma}(\lambda) \rangle d\lambda
= \frac{\text{dim}(V_{\gamma})}{4\pi \omega_{X}} \int_{\text{ia}^{*}} \text{Tr} \pi_{\sigma}(\lambda) h_{q}(\pi_{\sigma}(g_{1}^{-1})) \pi_{\sigma}(\lambda) d\lambda$$

In a similar manner we obtain

$$\text{Tr} \pi_{\sigma}(\lambda) h_{q}(\pi_{\sigma}(\lambda)) \pi_{\sigma}(\lambda) \pi_{\sigma}(g_{1}^{-1}) \pi_{\sigma}(\lambda)$$

and thus

$$K_{R(h_{q})}(g, g_{1}) = \frac{\text{dim}(V_{\sigma})}{4\pi \omega_{X}} \int_{\text{ia}^{*}} \text{Tr} \pi_{\sigma}(\lambda) h_{q}(\pi_{\sigma}(\lambda)) \pi_{\sigma}(\lambda) \pi_{\sigma}(g_{1}^{-1}) \pi_{\sigma}(\lambda) d\lambda$$
By Proposition 4.1 the difference

\[
K_{\tilde{R}^T(h_\lambda)}(g, g_1) - K_{\tilde{R}(h_\lambda)}(g, g_1)
\]

is integrable over \( \Gamma \setminus G \). Using the fact that \( \gamma \) is irreducible we compute

\[
\sum_{i,k} \int_{\Gamma \setminus G} \langle v^i, A_{q_{kij}}(T^j, T^*_l)(g, g)v_k \rangle \mu_G(dg)
\]

\[
= \sum_{i,k} \int_{\Gamma \setminus G} \int_K \langle v^i, A_{q_{kij}}(T^j, T^*_l)(gh, gh)v_k \rangle \mu_K(dh) \mu_G(dg)
\]

\[
= \sum_{i,k} \int_{\Gamma \setminus G} \int_K \langle v^i, \gamma(h)^{-1} A_{q_{kij}}(T^j, T^*_l)(g, g) \gamma(h)v_k \rangle \mu_K(dh) \mu_G(dg)
\]

\[
= \sum_k \int_{\Gamma \setminus G} \text{tr} A_{q_{kikj}}(T^j, T^*_l)(g, g) \mu_G(dg)
\]

The following formula is now an immediate consequence of Proposition 4.4,

\[
\int_{\Gamma \setminus G} \left( K_{\tilde{R}^T(h_\lambda)}(g, g) - K_{\tilde{R}(h_\lambda)}(g, g) \right) \mu_G(dg)
\]

\[
= \frac{\dim(V_\sigma) \dim(V_\gamma)}{4\pi} \int_{i,\ast} \sum_{k,j,l} \langle T^*_l, T^j \rangle
\]

\[
\frac{d}{d\mu_{\lambda_\ast} = \lambda} \text{Tr} \left[ \chi_{\infty} \circ J_{\sigma,\lambda} \circ (ext \circ \Phi_{\lambda,\mu} \circ \pi_\ast - \Phi_{\lambda,\mu}) \circ (J_{\sigma,\mu}^w)^\ast \circ \Phi_{-\mu, -\lambda} \circ \chi_1^1 \right] q_{kikj}(\lambda) d\lambda
\]

\[
+ \frac{\dim(V_\sigma) \dim(V_\gamma)}{4} \sum_{i, k, j, l} \langle T^*_l, T^j \rangle(0) \lim_{\lambda \to 0} \left( \text{Tr} \left[ \Phi_{\lambda, -\lambda} \circ \chi_{\infty}^i \circ J_{\sigma,\lambda} \circ (ext \circ \pi_\ast - 1) \circ \chi_1^1 \right] \right) q_{kikj}(0).
\]

Now we will rewrite this formula in a more invariant fashion. Using [34], we first compute

\[
\sum_{k, j, l} \langle T^*_l, T^j \rangle q_{kikj}(\lambda)
\]
5.1 The contribution of the scattering matrix

\[ \frac{1}{\dim(V_\sigma)} \sum_{j,k,l} T_j^i(T^j)^i_\gamma(v^k \otimes T^l, q(\lambda)v_k \otimes T_j) \]
\[ = \frac{1}{\dim(V_\sigma)} \sum_{l,k} (v^k \otimes T^l, q(\lambda)v_k \otimes T_l) \]
\[ = \frac{1}{\dim(V_\gamma) \dim(V_\sigma)} \Tr h_q(\pi_{\sigma,\lambda}) \]

(39)

We assume for a moment that \( \sigma \) is Weyl-invariant. For \( T \in \Hom_M(V_\sigma, V_\gamma) \) let \( T^u \in \Hom(H^\sigma_{-\infty}, V_\gamma) \) be given by \( T^u(f) = P_T^u(f)(1) \). Recall the relation (3, Lemma 5.5, 1.)

\[ T^u \circ \sigma(w) \circ J^w_{\sigma,\lambda} = c_\sigma(-\lambda)[\gamma(w) \circ c_\gamma(\lambda) \circ T \circ \sigma(w)^{-1}]^u \]

We compute

\[ \langle R, T \rangle(0) = \frac{1}{\dim(V_\sigma)c_\sigma(0)} \Tr \sigma(w) \circ R^* \circ \gamma(w)^{-1} \circ c_\gamma(0) \circ T \]
\[ = \frac{1}{\dim(V_\sigma)c_\sigma(0)} \Tr \gamma(w) \circ c_\gamma(0) \circ T \circ \sigma(w)^{-1} \circ R^* \]
\[ = \frac{1}{\dim(V_\sigma)c_\sigma(0)} \int_K \Tr \gamma(k)^{-1} \circ \gamma(w) \circ c_\gamma(0) \circ T \circ \sigma(w)^{-1} \circ R^* \circ \gamma(k)\mu_K(dk) \]
\[ = \frac{1}{\dim(V_\sigma)c_\sigma(0)} \sum_i \int_K \langle \gamma(k)^{-1} \circ \gamma(w) \circ c_\gamma(0) \circ T \circ \sigma(w)^{-1} \circ R^* \circ \gamma(k)(v_i), \nu^i \rangle \mu_K(dk) \]
\[ = \frac{1}{\dim(V_\sigma)c_\sigma(0)} \sum_i \langle [\gamma(w) \circ c_\gamma(0) \circ T \circ \sigma(w)^{-1}]^u(v_i \otimes R^*), \nu^i \rangle \]
\[ = \frac{1}{\dim(V_\sigma)c_\sigma(0)} \sum_i \langle T^u \circ \sigma(w) \circ J^u_{\sigma,0}(v_i \otimes R^*), \nu^i \rangle \]
\[ = \frac{1}{\dim(V_\sigma)} \sum_i p_0^T \circ J_{\sigma,0}(v_i \otimes R)(1), \nu^i \]
\[ = \frac{1}{\dim(V_\sigma)} \sum_i \langle J_{\sigma,0}(v_i \otimes R^*), \nu^i \otimes T^u \rangle \]

Since \( J_{\sigma,0} \) is \( K \)-equivariant, we can write

\[ J_{\sigma,0}(v_i \otimes R) = v_i \otimes j_{\sigma,0}(R) \]

for some \( j_{\sigma,0} \in \End(Hom_M(V_\sigma, V_\gamma)) \). We now have

\[ \sum_{k,j,l} (T^*_{k,l}, T^j_{l,k})(0)q_{klj}(\lambda) = \frac{1}{\dim(V_\sigma)} \sum_{k,j,l} \sum_i \langle v_i \otimes j_{\sigma,0}(T^*_{k,l}), v^i \otimes T^j \rangle \langle v^k \otimes T^l, q(\lambda)v_k \otimes T_j \rangle \]
\[ = \frac{1}{\dim(V_\sigma)} \sum_{k,l} \langle v^k \otimes T^l, q(\lambda)(v_k \otimes j_{\sigma,0}(T^*_{k,l})) \rangle \]

(40)
\[ \frac{1}{\dim(V_\sigma)} \sum_{k,l} \langle v_k \otimes T^l, q(\lambda) \circ J_{\sigma,0}(v_k \otimes T^l) \rangle \]

\[ = \frac{1}{\dim(V_\sigma) \dim(V_\gamma)} \text{Tr} h_q(\pi^{\sigma,0}) \circ J_{\sigma,0} \]

Inserting (39) and (41) into (37) we obtain the following theorem.

**Theorem 5.1** If \( q \) is a smooth compactly supported symmetric function on \( \hat{M} \times \mathfrak{a}^* \) such that \( q(\sigma,\lambda) \in \text{End}(V_\gamma \otimes \text{Hom}_{\text{M}}(V_\gamma, V_\sigma)) \), then the difference

\[ g \mapsto K_{\hat{R}(h_q)}(g,g) - K_{\hat{R}(h_q)}(g,g) \]

is integrable over \( \Gamma \setminus \mathbb{G} \), and we have

\[ \int_{\Gamma \setminus \mathbb{G}} \left( K_{\hat{R}(h_q)}(g,g) - K_{\hat{R}(h_q)}(g,g) \right) \mu_G(dg) \]

\[ = \sum_{\sigma \in \hat{M}} \frac{1}{4\pi} \sum_i \int_{\mathfrak{a}^*} \frac{d}{d\mu}_{\mu=\lambda} \text{Tr} \left[ \chi_\infty^i \circ J^{\gamma}_{\sigma,\lambda} \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (J^{\gamma}_{\sigma,-\lambda})^* \circ \Phi_{-\mu,-\lambda} \circ \chi_1^i \right] \text{Tr} h_q(\pi^{\sigma,\lambda}) d\lambda \]

\[ + \frac{1}{4} \sum_i \lim_{\lambda \to 0} \left( \text{Tr} \left[ \Phi_{\lambda,-\lambda} \circ \chi_\infty^i \circ J_{\sigma,\lambda} \circ (\text{ext} \circ \pi_* - 1) \circ \chi_1^i \right] \right) \text{Tr} h_q(\pi^{\sigma,0}) \circ J_{\sigma,0} . \]

We can again rewrite this formula as follows

\[ \int_{\Gamma \setminus \mathbb{G}} \left( K_{\hat{R}(h_q)}(g,g) - K_{\hat{R}(h_q)}(g,g) \right) \mu_G(dg) \]

\[ = \sum_{\sigma \in \hat{M}} \frac{1}{4\pi} \int_{\mathfrak{a}^*} \frac{d}{d\mu}_{\mu=\lambda} \text{Tr}^\gamma \left[ \chi_\infty^\gamma \circ J^{\gamma}_{\sigma,\lambda} \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_* - \Phi_{\lambda,\mu}) \circ (J^{\gamma}_{\sigma,-\lambda})^* \circ \Phi_{-\mu,-\lambda} \right] \text{Tr} h_q(\pi^{\sigma,\lambda}) d\lambda \]

\[ + \frac{1}{4} \lim_{\lambda \to 0} \left( \text{Tr}^\gamma \left[ \Phi_{\lambda,-\lambda} \circ \chi_\infty^\gamma \circ J_{\sigma,0} \circ (\text{ext} \circ \pi_* - 1) \right] \right) \text{Tr} h_q(\pi^{\sigma,0}) \circ J_{\sigma,0} . \]

### 5.2 The Fourier transform of \( \Psi \). A conjecture.

Observe that Theorem 5.1 does not solve our initial problem of computing the Fourier transform of the distribution \( \Psi \). The point is that there is no function \( f \in C^\infty_c(\mathbb{G}) \) such that its Fourier transform \( \hat{f} \) has compact support. In order to extend Theorem 5.1 to \( \hat{f} \) we must extend Proposition 4.1 to Schwartz functions. As explained in the remark at the end of Subsection 4.3 the main obstacle to do this is an estimate of the growth of the extension map \( \text{ext} = \text{ext}_\lambda : C^\infty(B, V_B(\sigma, \lambda)) \to H^{\sigma,\lambda}_\infty \) as \( \lambda \) tends to infinity along the imaginary axis.
The goal of the present subsection is to rewrite the result of the computation of $\Psi'$ in terms of characters thus obtaining the candidate of the measure $\Phi$. We will also take the discrete spectrum of $L^2(\Gamma\backslash G)$ into account.

Recall that

$$\operatorname{Tr} \hat{f}(\pi^{\sigma,\lambda}) = \theta_{\pi^{\sigma,\lambda}}(f) .$$

If $\pi^{\sigma,0}$ is reducible, then it decomposes into a sum of $\pi^{\sigma,+} \oplus \pi^{\sigma,-}$ of limits of discrete series representations which are just the $\pm 1$ eigenspaces of $J_{\sigma,0}$. In this case $\operatorname{ext}$ is regular at $\lambda = 0$ (13, Prop. 7.4.). We can write

$$\operatorname{Tr} \hat{f}(\pi^{\sigma,0}) \circ J_{\sigma,0} = \theta_{\pi^{\sigma,+}}(f) - \theta_{\pi^{\sigma,-}}(f) .$$

If we replace $h_q$ by $\hat{f}$ then formulas (23) and (30) just give the contributions of the continuous spectrum $K_{R^0}(f)$ and $K_{R^c}(f)$ to the integral kernels $K_{R}(f)$ and $K_{R^c}(f)$. We have $K_{R}(f) = K_{R^0}(f) + K_{R^c}(f)$, where

$$K_{R^c}(f)(g, g_1) = \sum_{\pi \in \hat{G}_c} \operatorname{Tr}(\pi) \hat{f}(\pi) \pi(g_1^{-1}) .$$

Since we assume that $f$ is $K$-finite, and there are only finitely many discrete series representations containing a given $K$-type, this sum is finite.

Furthermore, $K_{R^c}(f) = K_{R^0}(f) + K_{R^c}(f) + K_{R^c}(f)$. Here $K_{R^c}(f) = \sum_{\pi \in \hat{G}_d} K_{R^c}(f)$ is the contribution of discrete series representations (again a finite sum),

$$K_{R^c}(f)(g, g_1) = \sum_{i,j} \langle \psi_i, \phi_j \rangle \langle \psi_i, \pi(g_1^{-1}) \phi_j \rangle ,$$

where $\{\phi_j\}$ and $\{\psi_i\}$ are orthonormal bases of the infinite-dimensional Hilbert spaces $V_\pi$ and $M_\pi$, respectively. The finite sum $K_{R^c}(f) = \sum_{\pi \in \hat{G}_c} K_{R^c}(f)$ is the discrete contribution of representations belonging to $\hat{G}_c$. If we choose orthogonal bases $\{\phi_j\}$ and $\{\psi_i\}$ of the Hilbert spaces $V_\pi$ and $M_\pi$ (note that $\dim(M_\pi) < \infty$), then we can write

$$K_{R^c}(f)(g, g_1) = \sum_{i,j} \langle \psi_i, \phi_j \rangle \langle \psi_i, \pi(g_1^{-1}) \phi_j \rangle .$$

We define the multiplicity of $\pi$ by $N_G(\pi) := \dim(M_\pi)$. It is clear that

$$\int_{\Gamma\backslash G} K_{R^c}(f)(g, g) \mu_G(dg) = \sum_{\pi \in \hat{G}_c} N_G(\pi) \theta_{\pi}(f) .$$

In Lemma 6.13 we will show that if $f$ is $K$-finite and invariant under conjugation by $K$, then for each $\pi \in \hat{G}_d$ we have

$$K_{R^d}(f)(g, g) - K_{R^d}(f)(g, g) \in L^1(\Gamma\backslash G) .$$
Given any \( A \in V_{\hat{\pi}, K} \otimes V_{\pi, K} \) we define the function \( \hat{G} \ni \pi' \mapsto h_A(\pi') \) to be zero for all \( \pi' \neq \pi \) and \( h_A(\pi) := \bar{A} \), where \( \bar{A} := \int_K \pi(k) \otimes \pi(k) Adk \). The map

\[
V_{\hat{\pi}, K} \otimes V_{\pi, K} \ni A \mapsto T(A) := \int_{\Gamma \setminus G} [K_{R^c}(h_A)(g, g) - K_{R}(h_A)(g, g)] \mu_G(dg)
\]

is well-defined and a \((g, K)\)-invariant functional on \( V_{\hat{\pi}, K} \otimes V_{\pi, K} \). Since \( V_{\pi, K} \) is irreducible it follows that

\[
T(A) = N_{\Gamma}(\pi) \theta_\pi(A)
\]

for some number \( N_{\Gamma}(\pi) \in \mathbb{C} \) which plays the role of the multiplicity of \( \pi \). Here we consider \( A \) as a finite-dimensional operator on \( V_{\pi} \).

It follows from Lemma 6.13 that

\[
K_{R^c}(f)(g, g) - K_{R}(f)(g, g)
\]

belongs to \( L^1(\Gamma \setminus G) \) not only if \( \hat{f} \) is smooth of compact support, \( K\)-finite and \( K\)-invariant (Proposition 4.1), but also in the case that \( f \in C_c^\infty(G) \) is \( K\)-finite and \( K\)-conjugation invariant.

The following conjecture provides the candidate for the measure \( \Phi \). Its discrete part is expressed in terms of multiplicities \( N_{\Gamma}(\pi) \). If \( \pi \in \hat{G}_c \), then \( N_{\Gamma}(\pi) \) is just the dimension of the space of multiplicities \( M_\pi \) and thus a non-negative integer. If \( \pi \in \hat{G}_d \), then \( N_{\Gamma}(\pi) \) is a sort of regularized dimension of \( M_\pi \). We will show in Proposition 6.14 that \( N_{\Gamma}(\pi) \) is an integer in this case, too. The continuous part of the spectrum will contribute a point measure supported on the irreducible constituents of the representations \( \pi^{\sigma,0} \), and the corresponding weight will be denoted by \( \tilde{N}_{\Gamma}(\pi) \). The remaining contribution of the continuous spectrum is absolute continuous to the Lebesgue measure \( d\lambda \) on \( \hat{G}_{ac} = \hat{M} \times i\alpha^*_{+} \) and will be described by the density \( L_{\Gamma}(\pi^{\sigma,\lambda}) \). By Theorem 6.12 this density appears in the functional equation of the Selberg zeta function. In particular, as it can be already seen from the definition below, \( L_{\Gamma}(\pi^{\sigma,\lambda}) \) admits a meromorphic continuation to all of \( \alpha^*_{+} \) as a function of \( \lambda \). Its residues are closely related to the multiplicities of resonances.

**Conjecture 5.2** If \( f \in C_c^\infty(G) \) is bi-\( K\)-finite, then we have

\[
\Psi(f) = \sum_{\sigma \in \hat{M}} \frac{1}{4\pi} \int_{ia^*} L_{\Gamma}(\pi^{\sigma,\lambda}) \theta_{\pi^{\sigma,\lambda}}(f) d\lambda
\]

\[
+ \sum_{\sigma \in \hat{M}, \sigma \geq \sigma^{w, \pi^{\sigma,0}_{\text{red.}}}} \sum_{\epsilon \in \{+, -\}} \tilde{N}_{\Gamma}(\pi^{\sigma,\epsilon}) \theta_{\pi^{\sigma,\epsilon}}(f)
\]

\[
+ \sum_{\sigma \in \hat{M}, \sigma \geq \sigma^{w, \pi^{\sigma,0}_{\text{irred.}}}} \tilde{N}_{\Gamma}(\pi^{\sigma,0}) \theta_{\pi^{\sigma,0}}(f)
\]

\[
+ \sum_{\pi \in \hat{G}_c \cup \hat{G}_d} N_{\Gamma}(\pi) \theta_{\pi}(f),
\]
where

\[
L_\Gamma(\pi^{\sigma,\lambda}) := \frac{d}{d\mu|_{\mu=\lambda}} \text{Tr}' \left[ \chi_\infty \circ J^w_{\sigma,\lambda} \circ (\text{ext} \circ \Phi_{\lambda,\mu} \circ \pi_{\ast} - \Phi_{\lambda,\mu}) \circ (J^w_{\sigma,-\mu}^* \circ \Phi_{-\mu,-\lambda}) \right]
\]

and

\[
\tilde{N}_\Gamma(\pi^{\sigma,\pm}) := \pm \frac{1}{4} \text{Tr}' \lim_{\lambda \to 0} \left( [\Phi_{-\lambda,-} \circ \chi_\infty \circ J_{\sigma,-} \circ (\text{ext} \circ \pi_{\ast} - 1)] \right)
\]

\[
\tilde{N}_\Gamma(\pi^{\sigma,0}) := \frac{1}{4} \text{Tr}' \lim_{\lambda \to 0} \left( [\Phi_{-\lambda,-} \circ \chi_\infty \circ J_{\sigma,-} \circ (\text{ext} \circ \pi_{\ast} - 1)] \right).
\]

Note that the discussion above does not prove this conjecture. What it does prove is the following theorem.

**Theorem 5.3** If \( \hat{G} \ni \pi \mapsto h(\pi) \in \text{End}_K(V_\pi) \) is smooth, of compact support, and factorizes over finitely many \( K \)-types, then we have

\[
\Psi'(\tilde{h}) = \sum_{\sigma \in \hat{M}} \frac{1}{4\pi} \int_{i\alpha^*} L_\Gamma(\pi^{\sigma,\lambda}) \text{Tr} h(\pi^{\sigma,\lambda}) d\lambda + \sum_{\sigma \in \hat{M}^{'}, \sigma \equiv \sigma^{\mu,\pi,\sigma,0}_{\text{red.}}} \sum_{\epsilon \in \{+,-\}} \tilde{N}_\Gamma(\pi^{\sigma,\epsilon}) \text{Tr} h(\pi^{\sigma,\epsilon}) + \sum_{\sigma \in \hat{M}, \sigma \equiv \sigma^{\mu,\pi,\sigma,0}_{\text{irred.}}} \tilde{N}_\Gamma(\pi^{\sigma,0}) \text{Tr} h(\pi^{\sigma,0}) + \sum_{\pi \in \hat{G}_c \cup \hat{G}_d} N_\Gamma(\pi) \text{Tr} h(\pi).
\]

6 Resolvent kernels and Selberg zeta functions

6.1 Meromorphic continuation of resolvent kernels

We fix any \( K \)-type \( \gamma \). The Casimir operator \( \Omega \) of \( G \) gives rise to an unbounded selfadjoint operator on the Hilbert space of sections \( L^2(X, V(\gamma)) \). To be precise it is the unique selfadjoint extension of the restriction of \( \Omega \) to the space of smooth sections with compact support, where we normalize \( \Omega \) such that it is bounded from below. For each complex number \( z \) which is not contained in the spectrum \( \sigma(\Omega) \) we let \( R(z) \) be the operator \( (z - \Omega)^{-1} \). By \( r(z) \) we denote its distribution kernel.

The Casimir operator descends to an operator acting on sections of \( V_Y(\gamma) \) and induces an unique unbounded selfadjoint operator \( \Omega_Y \) on \( L^2(Y, V_Y(\gamma)) \). For \( z \not\in \sigma(\Omega_Y) \) we define the resolvent \( R_Y(z) \) of the operator \( (z - \Omega_Y)^{-1} \) and denote by \( r_Y(z) \) its distribution kernel.

We consider both, \( r(z) \) and \( r_Y(z) \), as distribution sections of the bundle \( V(\gamma) \otimes V(\tilde{\gamma}) \) over \( X \times X \). We will in particular be interested in the difference \( d(z) := r_Y(z) - r(z) \).
Let $\hat{M}(\gamma) \subset \hat{M}$ denote the set of irreducible representations of $M$ which appear in the restriction of $\gamma$ to $M$. For each $\sigma \in \hat{M}$ and $\lambda \in \mathfrak{a}_c^*$ let $z_\sigma(\lambda)$ denote the value of $\Omega$ on the principal series representation $H^\sigma$. Note that $z_\sigma(\lambda)$ is of the form $z_\sigma(\lambda) = c_\sigma - \lambda^2$ for some $c_\sigma \in \mathbb{R}$. We define $\mathbb{C}_\gamma$ to be the branched cover of $\mathbb{C}$ to which the inverse functions

$$\lambda_\sigma(z) = \sqrt{c_\sigma - z}$$

extend holomorphically for all $\sigma \in \hat{M}(\gamma)$. We fix one sheet $\mathbb{C}_\gamma^{\text{phys}}$ of $\mathbb{C}_\gamma$ over the set $\mathbb{C} \setminus [b, \infty)$, $b := \min_{\sigma \in \hat{M}} c_\sigma$, which we call physical. We will often consider $\mathbb{C}_\gamma^{\text{phys}}$ as a subset of $\mathbb{C}$. It follows from the Plancherel theorem for $L^2(X, V(\gamma))$ and $L^2(Y, V_\gamma(\gamma))$ that $[b, \infty)$ is the continuous spectrum of $\Omega$ and $\Omega_Y$. Thus $d(z)$ is defined on the complement of finitely many points of $\mathbb{C}_\gamma^{\text{phys}}$ which belong to the discrete spectrum of $\Omega$ and $\Omega_Y$.

Let $\Delta$ denote the diagonal in $X \times X$ and define $S := \bigcup_{1 \neq \gamma \in \Gamma} (1 \times \gamma) \Delta \subset X \times X$. Let $\Omega_i$, $i = 1, 2$, denote the Casimir operators of $G$ acting on the first and the second variable of the product $X \times X$. The distribution $d(z)$ satisfies the elliptic differential equation

$$(2z - \Omega_1 - \Omega_2)d(z) = 0 \quad (42)$$

on $X \times X \setminus S$ and is therefore smooth on this set.

**Lemma 6.1** $d(z)$ extends to $\mathbb{C}_\gamma$ as a meromorphic family of smooth sections of $V(\gamma) \otimes V(\tilde{\gamma})$ on $X \times X \setminus S$.

**Proof.** We first show

**Lemma 6.2** $r(z)$ and $r_Y(z)$ extend to $\mathbb{C}_\gamma$ as meromorphic families of distributions.

**Proof.** We give the argument for $r_Y(z)$ since $r(z)$ can be considered as a special case where $\Gamma$ is trivial. Let $V_i \subset X$, $i = 1, 2$ be open subsets such that the restriction to $V_i$ of the projection $X \rightarrow Y$ is a diffeomorphism. We consider $\phi \in C_c^\infty(V_1, V(\tilde{\gamma})), \psi \in C_c^\infty(V_2, V(\gamma))$ as compactly supported sections over $Y$.

We now employ the Plancherel theorem for $L^2(Y, V_\gamma(\gamma))$ in order to show that

$$r_{\phi, \psi}(z) := r_Y(z)(\phi \otimes \psi) = \langle \phi, (z - \Omega_Y)^{-1}\psi \rangle$$

extends meromorphically to $\mathbb{C}_\gamma$. We decompose $\phi = \sum_{s \in \sigma_p(\Omega_Y)} \phi_s + \phi_{\text{ac}}, \psi = \sum_{s \in \sigma_p(\Omega_Y)} \psi_s + \psi_{\text{ac}}$ according to the discrete and continuous spectrum of $\Omega_Y$. We have $r_{\phi, \psi}(z) = \sum_{s \in \sigma_p(\Omega_Y)} (z - s)^{-1}\langle \phi_s, \psi_s \rangle + \langle \phi_{\text{ac}}, (z - \Omega)^{-1}\psi_{\text{ac}} \rangle$. We now employ the Eisenstein Fourier transformation in order to rewrite the last term of this equation.
6.1 Meromorphic continuation of resolvent kernels

For each $\sigma \in \tilde{M}(\gamma)$, we consider the normalized Eisenstein series as a meromorphic family of maps

$$C^{-\infty}(B, V_B(\sigma_\lambda)) \otimes \text{Hom}_M(V_\sigma, V_\gamma) \ni f \otimes T \mapsto 0^E_T(f) := 0^P_T \circ \text{ext}(f) \in C^\infty(Y, V_Y(\gamma)).$$

For each $\sigma \in \{T_i(\sigma)\}_{i}, T_i(\sigma) \in \text{Hom}(V_\sigma, V_\gamma)$, be a base, and let $T^j(\sigma) \in \text{Hom}(V_\sigma, V_\gamma)$ be the dual base such that $T_i(\sigma)^* T^j(\sigma) = \delta_i^j$. If $\phi \in C^\infty_c(Y, V_Y(\tilde{\gamma}))$, then its Eisenstein Fourier transform $EFT_\sigma(\phi)(\lambda) \in C^\infty(B, V_B(\tilde{\sigma}_\lambda)) \otimes \text{Hom}_M(V_{\tilde{\sigma}}, V_\tilde{\gamma})$ is given by

$$\langle EFT_\sigma(\phi)(\lambda), f \otimes T_i(\sigma) \rangle := \langle \phi, 0^E_{-\lambda}(T_i(\sigma)) \rangle = \langle (0^E_{-\lambda})^*(\phi), f \rangle.$$ 

As a consequence of the Plancherel theorem we obtain for $\phi \in C^\infty_c(V_1, V(\tilde{\gamma})), \psi \in C^\infty_c(V_2, V(\gamma))$ that

$$\langle \phi_{ac}, \psi_{ac} \rangle = \sum_{\sigma \in \tilde{M}(\gamma)} \frac{1}{4 \pi \omega_X} \int_{i\alpha^*} \langle EFT_\sigma(\phi)(-\lambda), EFT_\sigma(\psi)(\lambda) \rangle d\lambda$$

$$= \sum_{\sigma \in \tilde{M}(\gamma)} \frac{1}{4 \pi \omega_X} \sum_j \int_{i\alpha^*} \langle (0^E_{-\lambda})(T_j(\sigma))^*(\phi), (0^E_{-\lambda})(T_j(\sigma))^*(\psi) \rangle d\lambda$$

$$= \sum_{\sigma \in \tilde{M}(\gamma)} \frac{1}{4 \pi \omega_X} \sum_j \int_{i\alpha^*} \langle \phi, 0^E_{\lambda}(T_j(\sigma))^* \circ (0^E_{-\lambda})(T_j(\sigma))^*(\psi) \rangle d\lambda.$$ 

In a similar manner we obtain

$$\langle \phi_{ac}, (z - \Omega)^{-1} \psi_{ac} \rangle = \sum_{\sigma \in \tilde{M}(\gamma)} \frac{1}{4 \pi \omega_X} \sum_j \int_{i\alpha^*} \langle z - z_\sigma(\lambda) \rangle^{-1} \langle \phi, 0^E_{\lambda}(T_j(\sigma))^* \circ (0^E_{-\lambda})(T_j(\sigma))^*(\psi) \rangle d\lambda. \quad (43)$$

We further investigate the summands in (43) for each $\sigma$ separately. So for $z \in \mathbb{C}^{\text{phys}}$ we put

$$u(z) := \int_{i\alpha^*} \langle z - z_\sigma(\lambda) \rangle^{-1} \sum_j \langle \phi, 0^E_{\lambda}(T_j(\sigma))^* \circ (0^E_{-\lambda})(T_j(\sigma))^*(\psi) \rangle d\lambda. \quad (44)$$

If we define $U(\lambda) := u(z_\sigma(\lambda))$, then it is defined for $\text{Re}(\lambda) > 0$. We claim that

$$U(-\lambda) := U(\lambda) - \frac{2\pi}{\lambda} \sum_j \langle \phi, 0^E_{\lambda}(T_j(\sigma))^* \circ (0^E_{-\lambda})(T_j(\sigma))^*(\psi) \rangle$$

provides a meromorphic continuation of $U$ to all of $\mathbb{C}^{\text{phys}}$. Indeed, $U(\lambda)$ is meromorphic for $\text{Re}(\lambda) < 0$, too. We let $\lambda = \epsilon + i\mu$ and show that the jump $U(\epsilon + i\mu) - U(-\epsilon + i\mu)$ vanishes as $\epsilon \to 0$.

Using the functional equation of the Eisenstein series and unitarity of the scattering matrix we see that the integrand in (44) is a symmetric function. We obtain

$$\lim_{\epsilon \to 0} \left[ u(z_\sigma(\epsilon + i\mu)) - u(z_\sigma(-\epsilon + i\mu)) \right]$$
Given $\epsilon > 0$ and a compact subset $W \subset \{ \Re(z) < b \}$ there is a constant $C > 0$ such that $|d(z)(x,y)| < Ce^{-(d_0(x,y) - \epsilon)\sqrt{b - \Re(z)}}$ for all $(x,y) \notin S_{2\epsilon}$ and $z \in W$. \hfill \Box
Proof. The proof is based on the finite propagation speed of the wave operators \( \cos(tA), \cos(tAY) \) where \( A := \sqrt{\Omega - b}, \ A_Y := \sqrt{\Omega_Y - b} \). We write

\[
R(z) = \int_0^\infty e^{-t\sqrt{b-z}} \cos(tA) dt, \quad R_Y(z) = \int_0^\infty e^{-t\sqrt{b-z}} \cos(tAY) dt.
\]

Finite propagation speed gives

\[
d(z)(x,y) = \int_{d_0(x,y)-\epsilon}^\infty e^{-t\sqrt{b-z}} [\cos(tAY) - \cos(tA)] dt \quad (x,y)
\]
on the level of distribution kernels. By partial integration

\[
(\Omega_1 - b)^N(\Omega_2 - b)^N d(z)(x,y) = (b-z)^{2N} \int_{d_0(x,y)-\epsilon}^\infty e^{-t\sqrt{b-z}} [\cos(tAY) - \cos(tA)] dt (x,y).
\]

We now employ the fact that \( \int_{d_0(x,y)-\epsilon}^\infty e^{-t\sqrt{b-z}} \cos(tAY) dt \) is a bounded operator on \( L^2(Y,V_Y(\gamma)) \) with norm bounded by \( C_1 e^{-(d_0(x,y)-\epsilon)\Re(\sqrt{b-z})} \). A similar estimate holds for the other term. If we choose \( N > \text{dim}(X)/4 \), then we can conclude that

\[
|d(z)(x,y)| < Ce^{-(d_0(x,y)-\epsilon)\Re(\sqrt{b-z})},
\]

where \( C \) depends on \( W, C_1, \) and a uniform estimate of norms of delta distributions as functionals on the Sobolev spaces \( W^{2N,2}(X,V(\gamma)) \) and \( W^{2N,2}(Y,V_Y(\gamma)) \) which hold because \( X, Y \) as well as the bundles \( V(\gamma), V_Y(\gamma) \) have bounded geometry. We further have employed the fact (which is again a consequence of bounded geometry) that we can use powers of the operator \( \Omega, \Omega_Y \) in order to define the norm of the Sobolev spaces. The assertion of the lemma now follows from \( \Re(\sqrt{b-z}) \geq \sqrt{b-\Re(z)} \).

The distribution \( r(z) \) is smooth outside the diagonal \( \Delta \) because it satisfies a differential equation similar to (12). For \( \epsilon > 0 \) we define the neighbourhood \( \Delta_\epsilon := \{ d \leq \epsilon \} \) of \( \Delta \).

Lemma 6.4 For \( \epsilon > 0 \) and a compact subset \( W \subset \{ \Re(z) < b \} \) there is a constant and \( C > 0 \) such that \( |r(z)(x,y)| < Ce^{-(d(x,y)-\epsilon)\sqrt{b-\Re(z)}} \) for all \( (x,y) \notin \Delta_2 \epsilon \) and \( z \in W \).

Proof. The proof is similar to that of Lemma 6.3. Using finite propagation speed we can write

\[
r(z)(x,y) = \int_{d(x,y)-\epsilon}^\infty e^{-t\sqrt{b-z}} \cos(tA) dt (x,y),
\]

\[
(\Omega_1 - b)^N(\Omega_2 - b)^N r(z)(x,y) = (b-z)^{2N} \int_{d(x,y)-\epsilon}^\infty e^{-t\sqrt{b-z}} \cos(tA) dt (x,y).
\]

We now argue as in the proof of Lemma 6.3 in order to conclude the estimate.

Let \( L_\gamma \) denote the action of \( \gamma \in \Gamma \) on sections of \( V(\gamma) \).
Lemma 6.5 If \( \sqrt{b - \text{Re}(z)} > \delta_{\Gamma} + \rho \), then on \( X \times X \setminus S \) we have \( d(z) = \sum_{1 \neq \gamma \in \Gamma} (1 \otimes L_{\gamma})r(z) \).

Proof. It follows from Lemma 3.4 that \( |(1 \otimes L_{\gamma})r(z)(x, y)| < Ce^{-d(x, \gamma^{-1}y)}\sqrt{b - \text{Re}(z)} \). Therefore the sum converges locally uniformly on \( X \times X \setminus S \). The distribution \( u(z) := r_{\gamma}(z) - \sum_{\gamma \in \Gamma} (1 \otimes L_{\gamma})r(z) \) satisfies the differential equations

\[
(z - \Omega_{1})u(z) = 0, (z - \Omega_{2})u(z) = 0,
\]

and is therefore a smooth section depending meromorphically on \( z \in \mathbb{C}^{\text{phys}} \). We further have the estimate \( \sum_{1 \neq \gamma \in \Gamma} |(1 \otimes L_{\gamma})r(z)(x, y)| < Ce^{-d_{0}(x, y)} \), where \( r < \sqrt{b - \text{Re}(z)} - \delta_{\Gamma} - \rho \). For \( \text{Re}(z) \ll 0 \) we see that \( u(z) \) defines a bounded operator on \( L^{2}(Y, V_{\gamma}(\gamma)) \) and therefore vanishes. Since \( u \) is meromorphic in \( z \) it vanishes for all \( z \) with \( \sqrt{b - \text{Re}(z)} > \delta_{\Gamma} + \rho \). This proves the lemma.

\[\Box\]

6.3 Boundary values

The meromorphic family of eigenfunctions \( d(z) \) on \( X \times X \setminus S \) has meromorphic families of hyperfunction boundary values. Since we consider a product of rank one spaces it is easy to determine the leading exponents of a joint eigenfunction of \( \Omega_{1}, \Omega_{2} \) with eigenvalue \( z \). These exponents are pairs of elements of \( \mathfrak{a}_{\mathbb{C}}^{*} \).

Lemma 6.6 The set of leading exponents of a joint eigenfunction of \( \Omega_{1}, \Omega_{2} \) in the bundle \( V(\gamma) \otimes V(\tilde{\gamma}) \) with generic eigenvalue \( z \) is

\[
\{ \mu(\sigma, \epsilon), (\sigma', \epsilon') \mid \sigma \in \tilde{M}(\gamma), \sigma' \in \tilde{M}(\tilde{\gamma}), \epsilon, \epsilon' \in \{+,-\} \},
\]

where \( \mu(\sigma, \epsilon), (\sigma', \epsilon') \) is \( (-\rho + \epsilon \lambda_{\sigma}(z), -\rho + \epsilon' \lambda_{\sigma'}(z)) \). The corresponding boundary value is a section of the bundle \( V(\gamma(\sigma)_{\epsilon \lambda_{\sigma}(z)}) \otimes V(\tilde{\gamma}(\sigma')_{\epsilon' \lambda_{\sigma'}(z)}) \rightarrow \partial X \times \partial X \), where \( \gamma(\sigma), \tilde{\gamma}(\sigma') \) denote the isotypic components.

Proof. An eigensection of \( \Omega \) in \( V(\gamma) \rightarrow X \) has leading exponents \( -\rho + \epsilon \lambda_{\sigma}(z), \epsilon \in \{+,-\} \), and the corresponding boundary value is a section of \( V(\gamma(\sigma)_{\epsilon \lambda_{\sigma}(z)}) \). This implies the lemma.

\[\Box\]

Let \( \partial \Delta \subset \partial X \times \partial X \) be the diagonal in the boundary and define \( \partial S := \bigcup_{1 \neq \gamma \in \Gamma}(\gamma \times 1)(\partial \Delta \cap \Omega \times \Omega) \). Note that \( d(z) \) is a joint eigenfunction in a neighbourhood in \( X \times X \) of \( \Omega \times \Omega \setminus \partial S \). Therefore, for generic \( z \) it has boundary values along this set \( S \). We denote the boundary value associated to the leading exponent \( \tau := \mu(\sigma, \epsilon), (\sigma', \epsilon') \) by \( \beta_{\tau}(f) \).
Lemma 6.7 We have $\beta_\tau(d(z)) = 0$ (the meromorphic family of hyperfunctions vanishes) except for $\tau = \mu(\sigma, -), (\tilde{\sigma}^w, -)(z)$, $\sigma \in \tilde{M}(\gamma)$, in which case $\beta_\tau(d(z))$ is a meromorphic family of real analytic sections.

Proof. We employ the fact that $\beta_\tau(d(z))$ depends meromorphically on $z$. Let $U \subset \Omega$ be such that the restriction of the projection $\Omega \to B$ is a diffeomorphism. There is a constant $c > 0$ such that for all $(k_1, k_2) \in U \times U$ we have $e^{d_0(k_1a_1, k_2a_2)} > c|\max(a_1, a_2)|$ (see \textbf{[B, Cor. 2.4]}). Using Lemma \textbf{3.3} we see that for $\text{Re}(z) < b$ we have $|d(z)(k_1a_1, k_2a_2)| < C|\max(a_1, a_2)|^{-\sqrt{b-\text{Re}(z)}}$, where $C$ depends on $z$. If one of the signs $\epsilon, \epsilon'$ is positive, for $z \ll 0$ we have $\sqrt{b-z} - 2\rho + \epsilon\lambda_\sigma(z) + \epsilon'\lambda_{\sigma'}(z) > 0$. For those $z$ we have $\lim_{\min(a_1, a_2) \to \infty} d(z)(k_1a_1, k_2a_2)(a_1, a_2)^{-\mu(\sigma, \nu), (\sigma', \nu')(z)}(z) = 0$ uniformly in $(k_1, k_2)$, where $(a, b)^{(\mu, \nu)} := a^\mu b^\nu$. This shows that $\beta_\tau(d(z)) = 0$ if one of $\epsilon, \epsilon'$ is positive.

We now consider the kernel $r(z)$ on $X \times X \setminus \Delta$. It is a joint eigenfunction of $\Omega_1, \Omega_2$ to the eigenvalue $z$ on a neighbourhood of $\partial X \times \partial X \setminus \partial \Delta$ and therefore has hyperfunction boundary values along this set. A similar argument as above but using Lemma \textbf{3.4} instead of \textbf{3.3} shows that $\beta_\tau(r(z)) = 0$ except for $\epsilon = \epsilon' = -$.

Note that $r(z)$ is $G$-invariant in the sense that for $g \in G$ we have $L_g \otimes L_g r(z) = r(z)$. If $\tau = \mu(\sigma, -), (\tilde{\sigma}^w, -)(z)$, then $\beta_\tau(r(z))$ is a $G$-invariant hyperfunction section of $V(\gamma(\sigma)_{-} \lambda_\sigma(z)) \otimes V(\tilde{\gamma}(\sigma')_{-} \lambda_{\sigma'}(z))$ over $\partial X \times \partial X \setminus \partial \Delta$. Since this set is an orbit of $G$, an invariant hyperfunction on this set is smooth, and the evaluation at the point $(w, 1) \in \partial X \times \partial X$ provides an injection of the space of invariant sections into $V_{\gamma}(\sigma) \otimes V_{\tilde{\gamma}}(\sigma')$. If $b$ is such a $G$-invariant section, then we have for $ma \in MA$

$$b(w, 1) = b(maw, ma) = b(wm^w a^{-1}, ma) = \gamma(wm^w) \otimes \tilde{\gamma}(m^{-1}a^{-1} \lambda_{\sigma'}(z) - \lambda_\sigma(z)) b(w, 1).$$

Thus $b(w, 1) \in [V_{\gamma}(\sigma^w) \otimes V_{\tilde{\gamma}}(\sigma')]^M$. We conclude that $\sigma' \cong \tilde{\sigma}^w$, and in this case $\lambda_{\sigma'}(z) = \lambda_\sigma(z)$ holds automatically. Thus $\beta_\tau(r(z)) = 0$ if $\sigma' \not\cong \tilde{\sigma}^w$.

We write

$$V(\gamma(\sigma)_{-}) \otimes V(\tilde{\gamma}(\tilde{\sigma}^w)_{-}) = V(\sigma_{-}) \otimes V(\tilde{\sigma}^w_{-}) \otimes \text{Hom}_M(V_\sigma, V_\gamma) \otimes \text{Hom}_M(V_{\tilde{\sigma}^w}, V_{\tilde{\gamma}}).$$

The space of invariant sections of $V_{\sigma_{-}} \otimes V_{\tilde{\sigma}^w_{-}}$ over $\partial X \times \partial X \setminus \partial \Delta$ is spanned by the distribution kernel $\tilde{j}_{\sigma_{-}}^{w}$ of the Knapp-Stein intertwining operator $\tilde{J}_{\sigma_{-}}^{w}$. We conclude that for each $\sigma \in \tilde{M}(\gamma)$ there is a meromorphic family $A_\sigma(z) \in \text{Hom}_M(V_\sigma, V_\gamma) \otimes \text{Hom}_M(V_{\tilde{\sigma}^w}, V_{\tilde{\gamma}})$ such that for $\tau = \mu(\sigma, -), (\tilde{\sigma}^w, -)(z)$ we have $\beta_\tau(r(z)) = \tilde{j}_{\sigma_{-}}^{w}(z) \otimes A_\sigma(z)$ under the identifications above.
Let $\tau = \mu_{(\sigma,-),(\tilde{\sigma}^w,-)}(z)$. We now employ Lemma 6.5 which states that for $\text{Re}(z) \ll 0$ we have $d(z) = \sum_{1 \neq \gamma \in \Gamma} (L_\gamma \otimes 1) r(z)$. The sum converges locally uniformly and thus in the space of smooth section over $X \times X \setminus S$. We further see that convergence holds locally uniformly in a neighbourhood of $\Omega \times \Omega \setminus \partial S$. Thus by [10] we can consider distribution boundary values, and by continuity of the boundary value map we have on $\Omega \times \Omega \setminus \partial S$

$$
\beta_\tau(d(z)) = \sum_{1 \neq \gamma \in \Gamma} (\pi^{\sigma-\lambda}(z) (\gamma) \otimes 1) \beta_\tau(d(z)) = \sum_{1 \neq \gamma \in \Gamma} (\pi^{\sigma-\lambda}(z) (\gamma) \otimes 1) \hat{S}^{w}_{\sigma^w,\lambda}(z) \otimes A_\sigma(z) = (\hat{S}^{w}_{\sigma^w,\lambda}(z) - \hat{S}^{w}_{\sigma^w,\lambda}(z)) \otimes A_\sigma(z),
$$

where $\hat{S}^{w}_{\sigma^w,\lambda}(z)$ is the distribution kernel of the scattering matrix $\hat{S}^{w}_{\sigma^w,\lambda}(z)$, Here we use the identity $\pi_* \circ \hat{J}^{w}_{\sigma^w,\lambda}(z) = \hat{S}^{w}_{\sigma^w,\lambda}(z) \circ \pi_*$ which implies that the distribution kernel of the scattering matrix $\hat{S}^{w}_{\sigma^w,\lambda}$ can be obtained by averaging the distribution kernel of the Knapp-Stein intertwining operator $\hat{J}^{w}_{\sigma^w,\lambda}$ for $\text{Re}(\lambda) \gg 0$.

It follows from the results of [3] that $\hat{S}^{w}_{\sigma^w,\lambda}(z) - \hat{S}^{w}_{\sigma^w,\lambda}(z)$ extends to a meromorphic family of smooth sections on all of $a_n^\gamma$. It follows from [4], Lemma 2.19, 2.20, that it is indeed a meromorphic family of real analytic sections. Strictly speaking, in [4] we only considered the spherical $M$-type for $G = SO(1,n)$, but the same arguments can be applied in the general case. We conclude that $\beta_\tau(d(z))$ is real analytic as required.

A similar reasoning shows that $\beta_\tau(d(z)) = 0$ for all $\tau$ which are not of the form $\mu_{(\sigma,-),(\tilde{\sigma}^w,-)}(z)$ for some $\sigma \in \hat{M}(\gamma)$. \hfill \Box

**Lemma 6.8** We have an asymptotic expansion (for generic $z$)

$$
d(z)(k_1a,k_2a) \overset{a \to \infty}{\sim} \sum_{\sigma \in \hat{M}(\gamma)} \sum_{n=0}^{\infty} a^{-2\rho-2\lambda(z)-n_\sigma} p_{z,\sigma,n}(k_1,k_2)
$$

(45)

which holds locally uniformly for $k \in \Omega \times \Omega \setminus \partial S$, and where the real analytic sections $p_{z,\sigma,n}(k_1,k_2)$ of $V(\gamma(\sigma) - \lambda) \otimes V(\gamma(\tilde{\sigma}^w) - \lambda)$ depend meromorphically on $z$.

**Proof.** Since the boundary value of $d(z)$ along $\Omega \times \Omega \setminus \partial S$ is real analytic we can employ [10], Prop. 2.16, in order to conclude that $d(z)$ has an asymptotic expansion with coefficients which depend meromorphically on $z$. The formula follows from an inspection of the list of leading exponents Lemma 6.7. \hfill \Box
Lemma 6.8 has the following consequence. For generic $z$ we have
\[
\operatorname{tr} d(z)(ka,ka) \xrightarrow{a \to \infty} \sum_{\sigma \in \hat{M}(\gamma)} \sum_{n=0}^{\infty} a^{-2\rho - 2\lambda_\sigma(z) - n\alpha} p_{z,\sigma,n}(k)
\]
which holds locally uniformly for $k \in \Omega$, and where the real analytic functions $p_{z,\sigma,n}$ depend meromorphically on $z$.

### 6.4 The regularized trace of the resolvent

**Lemma 6.9** The integral $Q_\gamma(z) := \int_X \chi^\Gamma(x) \operatorname{tr} d(z) dx$ converges for $\Re(z) \ll 0$ and admits a meromorphic continuation to all of $\mathbb{C}_\gamma$.

**Proof.** Convergence for $\Re(z) \ll 0$ follows from Lemma 6.3. Fix $R \in A$. We write $Q_\gamma(z) = Q_1(z,R) + Q_2(z,R)$, where
\[
Q_1(z,R) := \int_1^R \int_K \chi^\Gamma(ka) \operatorname{tr} d(z)(ka) dk v(a) da ,
\]
where $v$ is such that $dk v(a) da$ is the volume measure on $X$. Note that $v(a) \sim a^{2\rho} (\omega_X + a^{-\alpha} c_1 + a^{-2\alpha} c_2 + ...) \text{ as } a \to \infty$.

It is clear that $Q_1(z,R)$ admits a meromorphic continuation. We have an asymptotic expansion as $a \to \infty$.
\[
u(z,a) := \int_K \chi^\Gamma(ka) \operatorname{tr} d(z)(ka) dk v(a) \sim \sum_{\sigma \in \hat{M}(\gamma)} \sum_{n=0}^{\infty} a^{-2\lambda_\sigma(z) - n\alpha} q_{z,\sigma,n} ,
\]
where $q$ depends meromorphically on $z$. For $m \in \mathbb{N}$ let
\[
u_m(z,a) := \nu(z,a) - \sum_{\sigma \in \hat{M}(\gamma)} \sum_{n=0}^{m} a^{-2\lambda_\sigma(z) - n\alpha} q_{z,\sigma,n} .
\]
Given a compact subset $W$ of $\mathbb{C}_\gamma$ we can choose $m \in \mathbb{N}_0$ such that $\int_R^{\infty} \nu_m(z,a) da$ converges (for generic $z$) and depends meromorphically on $z$ for all $z \in W$. We further have
\[
\sum_{\sigma \in \hat{M}(\gamma)} \sum_{n=0}^{m} \int_R^{\infty} a^{-2\lambda_\sigma(z) - n\alpha} q_{z,\sigma,n} da = \sum_{\sigma \in \hat{M}(\gamma)} \sum_{n=0}^{m} \frac{R^{-2\lambda_\sigma(z) - n\alpha} q_{z,\sigma,n}}{2\lambda_\sigma(z) + n\alpha} ,
\]
and this function extends meromorphically to $C_\gamma$. Since we can choose $W$ arbitrary large we conclude that $Q_\gamma(z)$ admits a meromorphic continuation to all of $\mathbb{C}_\gamma$. \qed
6.5 A functional equation

Let \( L(\gamma) := \{ c_\sigma \mid \sigma \in \hat{M}(\gamma) \} \) be the set of ramification points of \( \mathbb{C}_\gamma \), define \( \mathbb{C}_\gamma^\sharp := \mathbb{C} \setminus L(\gamma) \), and let \( \mathbb{C}_\gamma^\sharp \subset \mathbb{C}_\gamma \) be the preimage of \( \mathbb{C}_\gamma^\sharp \) under the projection \( \mathbb{C}_\gamma \to \mathbb{C} \). Then \( \mathbb{C}_\gamma^\sharp \to \mathbb{C}_\gamma^\sharp \) is a Galois covering with group of deck transformations \( \Pi := \oplus_{L(\gamma)} \mathbb{Z}_2 \). The action of \( \Pi \) extends to \( \mathbb{C}_\gamma \) such \( \mathbb{C}_\gamma \setminus \mathbb{C}_\gamma^\sharp \) consists of fixed points. For \( l \in L(\gamma) \) let \( q_l \in \Pi \) be the corresponding generator. Then we have \( \lambda_\sigma(q_lz) = -\lambda_\sigma(z) \) for all \( \sigma \in \hat{M}(\gamma) \) with \( c_\sigma = l \) and \( \lambda_{\sigma'}(q_lz) = \lambda_{\sigma'}(z) \) else.

Lemma 6.10 For \( l \in L(\gamma) \) we have

\[
Q_\gamma(q_lz) - Q_\gamma(z) = \sum_{\sigma \in \hat{M}(\gamma), c_\sigma = l} -\frac{[\gamma : \sigma]}{2\lambda_\sigma(z)} \pi_\sigma(\lambda_\sigma(z)) .
\]

Proof. In the proof of Lemma 6.1 we have seen that (using the notation introduced there)

\[
\langle r_Y(q_lz) - r_Y(z), \phi \otimes \psi \rangle = \sum_{\sigma \in \hat{M}(\gamma), c_\sigma = l} \frac{-1}{2\omega_\lambda \lambda_\sigma(z)} \sum_j \langle \phi, 0 E_{T_j(z)}^\sigma \circ 0(E_{-\lambda_\sigma(z)}^{T_j(\sigma)})^* \psi \rangle .
\]

We conclude that

\[
\langle r_Y(q_lz) - r_Y(z), \phi \otimes \psi \rangle = \sum_{\sigma \in \hat{M}(\gamma), c_\sigma = l} \frac{-1}{2\omega_\lambda \lambda_\sigma(z)} \sum_j 0 E_{T_j(z)}^\sigma \circ 0(E_{-\lambda_\sigma(z)}^{T_j(\sigma)})^* .
\]

The same reasoning applies to the trivial group \( \Gamma \), where the Eisenstein series get replaced by the Poisson transformations. Thus we can write

\[
d(q_lz) - d(z) = \sum_{\sigma \in \hat{M}(\gamma), c_\sigma = l} \frac{-1}{2\omega_\lambda \lambda_\sigma(z)} \sum_j 0 P_{T_j(z)}^\sigma \circ (\text{ext} \circ \pi_e - 1) \circ (0 P_{-\lambda_\sigma(z)}^{T_j(\sigma)})^* .
\]

The proof of Lemma 6.9 shows that \( Q_1(z, R) \) has an asymptotic expansion

\[
Q_1(z, R) \sim Q_\gamma(z) + \sum_{\sigma \in \hat{M}(\gamma), c_\sigma = l} \int_{\gamma} \frac{R^{-2\lambda_\sigma(z) - n} q_{z, \sigma, n}}{2\lambda_\sigma(z) + n} .
\]

In particular, if \( 2\lambda_\sigma(z) \not\in -\mathbb{N}_0 \alpha \) for all \( \sigma \), then \( Q_\gamma(z) \) is the constant term in the asymptotic expansion of \( Q_1(z, R) \).

We can now apply Proposition 4.4 which can be interpreted as the determination of the constant term (as \( R \to \infty \)) of

\[
\int_{\epsilon}^R \int_{K} \chi(ka) \text{tr} \left[ 0 P_{T_j(z)}^\sigma \circ (\text{ext} \circ \pi_e - 1) \circ (0 P_{-\lambda_\sigma(z)}^{T_j(\sigma)})^* \right] (ka, ka) dv(a) da
\]

as a distribution on \( i\mathbb{R}^+ \setminus \{0\} \). This shows the desired equation first on \( z_\sigma(i\mathbb{R}^+) \) and then everywhere by meromorphic continuation. \( \square \)
6.6 Selberg zeta functions

For a detailed investigation of Selberg zeta functions associated to bundles (for cocompact $\Gamma$) we refer to [2]. We assume that $\sigma$ is irreducible and Weyl invariant, or that it is of the form $\sigma' \oplus (\sigma')^w$ for some non-Weyl invariant irreducible $M$-type $\sigma'$. In the latter case we define $L_\Gamma(\pi_{\sigma',\lambda}) := L_\Gamma(\pi_{\sigma',\lambda}) + L_\Gamma(\pi_{(\sigma')^w,\lambda})$.

Let $P = MAN$ be a parabolic subgroup of $M$. If $\gamma \in \Gamma$, then it can be conjugated in $G$ to an element $m_g a_g \in MA$ with $a_g > 1$. Let $\mathfrak{n}$ be the negative root space of $(\mathfrak{g},\mathfrak{a})$. For $\text{Re}(\lambda) > \rho$ we can define the Selberg zeta function $Z_\mathcal{S}(\lambda,\sigma)$ by the converging infinite product

$$Z_\mathcal{S}(\lambda,\sigma) := \prod_{1 \neq [g] \in CT} \prod_{k=0}^{\infty} \det \left( 1 - \sigma(m_g) \otimes S^k(\text{Ad}(m_g a_g)_{|\mathfrak{n}}) a_g^{-\lambda - \rho} \right).$$

In the case of cocompact $\Gamma$ it was shown by [5] that $Z_\mathcal{S}(\lambda,\sigma)$ has a meromorphic continuation to all of $\mathbb{C}$. In [11] it was explained that the argument of [5] extends to the case of convex cocompact subgroups since it is the compactness of the non-wandering set of the geodesic flow of $Y$ that matters and not the compactness of $Y$. Strictly speaking, [11] deals with the spherical case of $SO(1,2n)$, but the argument extends to the general case.

There is a virtual representation $\gamma$ of $K$ (i.e. an element of the integral representation ring of $K$) such that $\gamma|_M = \sigma$ in the integral representation ring of $M$ (see [9], [2]). We call $\gamma$ a lift of $\sigma$. Note that $\gamma$ is not unique. We can extend the material developed above to virtual $K$-types by taking the traces with corresponding signs. Because of the factor $[\gamma : \sigma]$ Lemma 6.10 has the following corollary.

**Corollary 6.11** If $\gamma$ is a lift of $\sigma$, then $Q_\gamma(z)$ extends to a twofold branched cover of $\mathbb{C}$ associated with $\lambda_\sigma(z)$.

**Theorem 6.12** The Selberg zeta function satisfies

$$\frac{Z_\mathcal{S}(\lambda,\sigma)}{Z_\mathcal{S}(-\lambda,\sigma)} = \exp \int_0^\lambda L_\Gamma(\pi_{\sigma,\lambda}) d\lambda.$$

In particular, the residues of $L_\Gamma(\pi_{\sigma,\lambda})$ are integral.

**Proof.** By [2] Prop. 3.8. we have

$$Q_\gamma(z_\sigma(\lambda)) = \frac{1}{2\lambda} Z_\gamma'(\lambda,\sigma)/Z_\gamma(\lambda,\sigma)$$

(46) for $\text{Re}(\lambda) \gg 0$. Indeed, $Q_\gamma(z_\sigma(\lambda))$ is just what is called in [2] the hyperbolic contribution associated to the resolvent. So Corollary 6.11 and Lemma 6.10 yields the functional equation of
the logarithmic derivative of the Selberg zeta function
\[ \frac{Z'_S(-\lambda, \sigma)}{Z_S(-\lambda, \sigma)} + \frac{Z'_S(\lambda, \sigma)}{Z_S(\lambda, \sigma)} = L_{\Gamma}(\pi^{\sigma,\lambda}) . \]

Integrating and employing the apriori information that \( Z_S(\lambda, \sigma) \) is meromorphic we obtain the desired functional equation.

Remarks:

1. As explained in the introduction it is known (from the approach to \( Z_S \) using symbolic dynamics and Ruelles thermodynamic formalism) that \( Z_S(\lambda) \) is a meromorphic function of finite order. It follows that \( L_{\Gamma}(\pi^{\sigma,\lambda}) \), as a function of \( \lambda \), grows at most polynomially.

2. In order to describe the singularities of \( Z_S(\lambda, \sigma) \) we assume that \( \gamma \) is an admissible lift of \( \sigma \) (see [2]). Let \( n_{\lambda,\sigma} \) denote the (virtual) dimension of the subspace of the \( L^2 \)-kernel of \( \Omega_Y - z_\sigma(\lambda) \) on \( V_Y(\gamma) \) which is generated by non-discrete series representations of \( G \). It follows from Theorem 6.12 that
\[ \text{ord}_{\lambda=\mu} Z_S(\lambda, \sigma) = \begin{cases} \text{res}_{\lambda=\mu} L_{\Gamma}(\pi^{\sigma,\lambda}) + n_{-\mu,\sigma} & \text{Re}(\mu) < 0 \\ n_{\mu,\sigma} & 0 < \text{Re}(\mu) \end{cases} \]

3. If \( \mu \) is non-integral and if \( ext \) has a pole at \( \mu \) of at most first order, then one has
\[ \text{res}_{\lambda=\mu} L_{\Gamma}(\pi^{\sigma,\lambda}) = \dim \Gamma_{C^{-\infty}}(\Lambda, V(\sigma, \mu)) , \]
where \( \Gamma_{C^{-\infty}}(\Lambda, V(\sigma, \mu)) \) is the space of invariant distributions with support on the limit set \( \Lambda \). If \( ext \) has higher order singularity, then the residue has a similar interpretation (see [4], Prop. 5.6) This provides an independent argument for the integrality of the residues of \( L_{\Gamma}(\pi^{\sigma,\lambda}) \).

### 6.7 Integrality of \( N_{\Gamma}(\pi) \) for discrete series representations

Let \( \pi \) be a discrete series representation of \( G \) containing the \( K \)-type \( \tilde{\gamma} \). There are embeddings \( M_\pi \otimes V_\pi(\tilde{\gamma}) \hookrightarrow L^2(\Gamma \backslash G)(\tilde{\gamma}) \) and \( V_\pi \otimes V_\pi(\tilde{\gamma}) \hookrightarrow L^2(G) \). Let \( A \in \text{End}_K(V_\pi(\tilde{\gamma})) \) be given. We extend \( A \) by zero to the orthogonal complement of \( V_\pi(\tilde{\gamma}) \) thus obtaining an operator in \( \text{End}_K(V_\pi) \) which we will still denote by \( A \). The operator \( A \) induces operators \( \hat{R}_{\Gamma}(h_A) \) and \( \hat{R}_\Gamma(h_A) \) on \( L^2(\Gamma \backslash G) \) and \( L^2(G) \), where \( h_A \) is supported on \( \{ \pi \} \subset \hat{G} \) and \( h_A(\pi) := A \).

**Lemma 6.13** We have
\[ K_{\hat{R}_{\Gamma}(h_A)}(g, g) - K_{\hat{R}(h_A)}(g, g) \in L^1(\Gamma \backslash G) . \]
6.7 Integrality of $N_{\Gamma}(\pi)$ for discrete series representations

Proof. Let $D(G, \gamma)$ be the algebra of invariant differential operators on $V(\gamma)$. It is isomorphic to $(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{b}) \text{End}(V_\gamma))^K$. If $\pi'$ is an admissible representation of $G$, then $D(G, \gamma)$ acts in a natural way on $(V_{\pi'} \otimes V_\gamma)^K$. If $\pi'$ is irreducible, then $(V_{\pi'} \otimes V_\gamma)^K$ is an irreducible representation of $D(G, \gamma)$. The correspondence $\pi' \mapsto (V_{\pi'} \otimes V_\gamma)$ provides a bijection between the sets of equivalence classes of irreducible representations of $G$ containing the $K$-type $\gamma$ and irreducible representations of $D(G, \gamma)$.

Note that $\text{End}_K(V_{\pi'}(\gamma)) \cong \text{End}((V_{\pi'} \otimes V_\gamma)^K)$. We conclude that there is $D_A \in D(G, \gamma)$ that induces the endomorphism $A$ on $V_{\pi'}(\gamma)$. Let $z_0$ be the eigenvalue of the Casimir operator on $\pi$ and $Z$ be the finite set of irreducible representations of $G$ containing the $K$-type $\gamma$ such that $\Omega$ acts with eigenvalue $z_0$. Then we can choose $D_A$ such that it vanishes on all $(V_{\pi'} \otimes V_\gamma)^K$ for $\pi' \in Z$, $\pi' \neq \pi$.

For simplicity we assume that $z_0$ is not a branching point of $R(z)$. In the latter case the following argument can easily be modified. The operators $D_AR_Y$ and $D_AR(z)$ have poles at $z_0$ with residues $K_{R_A}(h_A)$ and $K_{R_A}(h_A)$. The difference $(D_A)1d(z):=(D_A)1r_Y(z)-(D_A)1r(z)$ of distribution kernels is still a meromorphic family of joint eigenfunctions with real analytic boundary values along $\Omega \times \Omega \setminus \partial S$. We have the asymptotic expansion

$$(D_A)1d(z)(k_1a, k_2a) \xrightarrow{a \to \infty} \sum_{\sigma \in M(\gamma)} \sum_{n=0}^{\infty} a^{-2\rho-2\lambda_\sigma(z)-na}p_{z, \sigma, n, A}(k_1, k_2).$$

The residue of $(D_A)1d(z)$ at $z_0$ can be computed by integrating $(D_A)1d(z)$ along a small circle counter-clockwise surrounding $z_0$. If we insert the asymptotic expansion (47) into this integral, then we obtain an asymptotic expansion

$$\text{res}_{z=z_0}(D_A)1d(z)(k_1a, k_2a) \xrightarrow{a \to \infty} \sum_{\sigma \in M(\gamma)} \sum_{n=0}^{\infty} \text{res}_{z=z_0}a^{-2\rho-2\lambda_\sigma(z)-na}p_{z, \sigma, n, A}(k_1, k_2)$$

$$\sim \sum_{\sigma \in M(\gamma)} \sum_{n=0}^{\infty} \sum_{m=0}^{\text{finite}} \log(a)^m a^{-2\rho-2\lambda_\sigma(z_0)-na}p_{z_0, \sigma, n, m, A}(k_1, k_2),$$

where $p_{z_0, \sigma, n, m, A}$ is a real analytic section on $\Omega \times \Omega \setminus \partial S$. Since $K_{R_V(h_A)}$ and $K_{R(h_A)}$ project onto eigenspaces of square integrable sections we have for $k_1 \neq k_2$

$$K_{R(h_A)}(k_1a_1, k_2a_2) \xrightarrow{a \to \infty} \sum_{n_1, n_2=0}^{\text{finite}} \log(a_1)^{n_1} \log(a_2)^{n_2} a_1^{-\lambda_\sigma(z_0)-n_1\alpha}a_2^{-\lambda_\sigma(z_0)-n_2\alpha}p_{z_0, \sigma, n_1, n_2, m_1, m_2, A}(k_1, k_2)$$

with $p_{z_0, \sigma, n_1, n_2, m_1, m_2, A}(k_1, k_2) = 0$ as long as $-\lambda_\sigma(z_0) - n_1\alpha \geq 0$, $-\lambda_\sigma(z_0) - n_2\alpha \geq 0$, and similar for $K_{R_V(h_A)}$. We conclude that $\text{Re}(-2\lambda_\sigma(z) - n\alpha) < 0$ if $p_{z_0, \sigma, n, m, A} \neq 0$. The assertion of the
Proposition 6.14 If $\pi$ be a representation of the discrete series of $G$, then $N_{\Gamma}(\pi) \in \mathbb{Z}$.

Proof. There exists an invariant generalized Dirac operator $D$ acting on a graded vector bundle $E \to X$, $E = E^+ \oplus E^-$, such that $V_\pi \oplus \{0\}$ is the kernel of $D$. If $\gamma$ is the graded $K$-type associated to $E$, then

$$\gamma|_{\mathcal{M}} = 0.$$ \hfill (48)

Let $D_Y$ be the induced operator on $Y$. The distribution kernels of $r(z) := (z - D^2)^{-1}$ and $r_Y(z) := (z - D_Y^2)^{-1}$ have meromorphic continuations to a branched covering of $\mathbb{C}$. Their difference goes into the functional $\Psi'$. The function

$$Q(z) := \int_{\Gamma \setminus G} \text{tr} (r_Y(z) - r(z))(g,g)dg$$

has a meromorphic continuation to all of $\mathbb{C}$ by (48) and Lemma 6.10. Its residue at $z = 0$ is given by

$$\text{res}_{z=0} Q(z) = N_{\Gamma}(\pi) + \sum_{\pi' \in \hat{G}_c} n(\gamma, \pi')N_{\Gamma}(\pi') ,$$

where the sum reflects the fact that a finite number of representations belonging to $\hat{G}_c$ may contribute to the kernel of $D_Y$. Note that $n(\gamma, \pi') \in \mathbb{Z}$ and $N_{\Gamma}(\pi') \in \mathbb{N}_0$. We show that $\text{res}_{z=0} Q(z) = 0$ in order to conclude that $N_{\Gamma}(\pi) = - \sum_{\pi' \in \hat{G}_c} n(\gamma, \pi')N_{\Gamma}(\pi') \in \mathbb{Z}$.

It suffices to show that $Q(z) \equiv 0$ for $\text{Re}(z) \ll 0$. This follows from (48), but we will give an independent argument. We can write

$$r_Y(z) - r(z) = \frac{1}{z}((D_Y^2)^1 r_Y(z) - (D^2)^1 r(z)) .$$

Integrating the restriction of this difference to the diagonal over $\Gamma \setminus G$ we obtain

$$Q(z) = \frac{1}{z} \int_{\Gamma \setminus G} \text{tr} ((D_Y^2)^1 r_Y(z) - (D^2)^1 r(z))(g,g)dg$$

$$= \frac{1}{z} \int_{G} \chi^\Gamma(g) \text{tr} ((D_Y^2)^1 r_Y(z) - (D^2)^1 r(z))(g,g)dg$$

$$= -\frac{1}{z} \int_{G} \chi^\Gamma(g) \text{tr} ((D_Y^1)(D_Y)^2 r_Y(z) - D_1 D_2 r(z))(g,g)dg$$

$$= -\frac{1}{z} \int_{G} \chi^\Gamma(g) \text{tr} ((D_Y^1)(D_Y)^2 r_Y(z) - D_1 D_2 r(z))(g,g)dg$$

$$= -\frac{1}{z} \int_{G} \chi^\Gamma(g) \text{tr} ((D_Y^1)(D_Y)^2 r_Y(z) - D_1 D_2 r(z))(g,g)dg$$

$$= -\frac{1}{z} \int_{G} \chi^\Gamma(g) \text{tr} ((D_Y^1)(D_Y)^2 r_Y(z) - D_1 D_2 r(z))(g,g)dg$$

$$= -\frac{1}{z} \int_{G} \chi^\Gamma(g) \text{tr} ((D_Y^1)(D_Y)^2 r_Y(z) - D_1 D_2 r(z))(g,g)dg$$
where \( c(d\chi^\Gamma) \) denotes Clifford multiplication. We conclude that
\[
Q(z) = -\frac{1}{2z} \int_G \text{tr} c(d\chi^\Gamma) ((D_Y)_1 r_Y(z) - D_1 r(z))(g,g) dg .
\]
The right hand side of this equation vanishes as a consequence of \( \sum_{\gamma \in \Gamma} \gamma^* \chi^\Gamma \equiv 1 \) and the \( \Gamma \)-invariance of \( ((D_Y)_1 r_Y(z) - D_1 r(z))(g,g) \). This finishes the proof of the proposition.

References

[1] J. N. Bernstein. On the support of Plancherel measure. \textit{J. of Geom. and Phys.}, 5(1988), 663–710.

[2] U. Bunke and M. Olbrich. \textit{Selberg Zeta and Theta Functions}. Akademie Verlag, 1995.

[3] U. Bunke and M. Olbrich. The spectrum of Kleinian manifolds. Submitted to J. Funct. Anal., Preprint available at \texttt{http://www.uni-math.gwdg.de/bunke/spzerl.dvi}, 1996.

[4] U. Bunke and M. Olbrich. Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group. \textit{Annals. of Math.}, 149 (1999), 627–689.

[5] D. Fried. The zeta functions of Ruelle and Selberg I. \textit{Ann. scient. éc. norm. sup. 4e Série}, 19 (1986), 491–517.

[6] L. Guillopé. Fonctions zéta de Selberg et surface de géométrie finie. \textit{Adv. Stud. Pure Math.}, 21(1992), 33–70.

[7] Harish-Chandra. Harmonic analysis on real reductive groups III. The Maass-Selberg relations and the Plancherel formula. \textit{Ann. of Math.}, 104 (1976), 117–201.

[8] M. Kashiwara and Oshima. Systems of differential equations with regular singularities and their boundary value problems. \textit{Ann. Math.}, 106(1977), 145–200.

[9] R. J. Miatello and J. A. Vargas. On the distribution of the principal series in \( L^2(\Gamma\backslash G) \). \textit{Trans. AMS}, 279 (1983), 63–75.

[10] T. Oshima and J. Sekiguchi. Eigenspaces of invariant differential operators on an affine symmetric space. \textit{Invent. math.}, 57(1980), 1–81.

[11] S. J. Patterson and P. A. Perry. The Divisor of Selberg’s zeta function for Kleinian groups. Preprint, 1999.