Multiply robust estimators in longitudinal studies with missing data under control-based imputation

Siyi Liu, Shu Yang *
Department of Statistics, North Carolina State University
Yilong Zhang, Guanghan (Frank) Liu
Merck & Co., Inc.

Summary. Longitudinal studies are often subject to missing data. The ICH E9(R1) addendum addresses the importance of defining a treatment effect estimand with the consideration of intercurrent events. Jump-to-reference (J2R) is one classically envisioned control-based scenario for the treatment effect evaluation using the hypothetical strategy, where the participants in the treatment group after intercurrent events are assumed to have the same disease progress as those with identical covariates in the control group. We establish new estimators to assess the average treatment effect based on a proposed potential outcomes framework under J2R. Various identification formulas are constructed under the assumptions addressed by J2R, motivating estimators that rely on different parts of the observed data distribution. Moreover, we obtain a novel estimator inspired by the efficient influence function, with multiple robustness in the sense that it achieves $n^{1/2}$-consistency if any pairs of multiple nuisance functions are correctly specified, or if the nuisance functions converge at a rate not slower than $n^{-1/4}$ when using flexible modeling approaches. The finite-sample performance of the proposed estimators is validated in simulation studies and an antidepressant clinical trial.

Keywords: Longitudinal clinical trial, longitudinal observational study, semiparametric theory, sensitivity analysis

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1 Introduction

Missing data is a major concern in clinical studies, especially in longitudinal settings. Participants are likely to deviate from the current treatment due to the loss of follow-ups or a shift to certain rescue therapy. To estimate the treatment effect precisely, additional assumptions for the missing components are needed. It calls for the importance of defining an estimand that can reflect the key clinical questions of interest and take into account the intercurrent events such as the discontinuation of the treatment according to ICH (2021).

One strategy to construct the estimand is the hypothetical strategy, where the treatment effect is assessed under a hypothetical scenario.

One commonly used hypothetical strategy is to assume the participants who discontinue the treatment are in compliance, i.e., they still take the assigned drug throughout the entire study period. This strategy is often applied in primary analysis for the treatment efficacy (Carpenter et al., 2013), and it is related to the unverifiable missing at random (MAR; Rubin, 1976) assumption. However, the defined estimand under MAR may not be realistic, if there exist some additional unobserved covariates that affect both the future clinical outcomes and protocol deviation, leading to the missing not at random (MNAR; Rubin, 1976) assumption. Under MNAR, a different type of hypothetical estimand arises, targeting the treatment effect evaluation under a scenario in which participants who drop out do not take any other medications. This estimand is often used in the sensitivity analysis (e.g., Carpenter et al., 2013; Liu and Pang, 2016) to explore the robustness of results to alternative assumptions about the missingness in the violation of MAR. Moreover, it has been receiving growing attention in primary analysis (Tan et al., 2021) and in observational studies (Lee et al., 2021). Throughout the paper, we focus on this estimand connected to MNAR to assess the average treatment effect (ATE) in longitudinal studies.

With the concentration on the MNAR-related estimand under the hypothetical strategy, the control-based imputation (CBI; Carpenter et al., 2013) approaches assess the treatment effect under the envisioned settings in which the participants in the treatment group will have similar performance as those in the control group after the missingness occurs, based on the pattern-mixture model (PMM; Little, 1993) framework. In the paper, we focus on one specific model of CBI called jump-to-reference (J2R) as a specific case of MNAR, which
has been used in the US Food and Drug Administration statistical review and evaluation reports (e.g., US Food and Drug Administration 2016). Although we only explore one specific hypothetical condition, the idea of our developed framework can be extended readily to other imputation models, in both primary and sensitivity analysis.

The likelihood-based method and multiple imputation (Rubin 2004) are two major parametric approaches to handle missing data. However, they will result in a biased estimate of the ATE if any component of the likelihood function is misspecified. When the parametric modeling assumptions are untenable, semiparametric estimators of the ATE based on the weighted estimating equations can be applied. Robins et al. (1994) propose a doubly robust estimator for the regression coefficients under MAR. Bang and Robins (2005) further develop a doubly robust estimator in longitudinal data with a monotone missingness pattern with the use of sequential regressions. While the doubly robust estimators under MAR have been well-studied in literature, it remains uncultivated in the area of longitudinal clinical studies under the MNAR-related hypothetical conditions.

We develop a semiparametric framework to evaluate the ATE in longitudinal studies under J2R. As the estimand is defined under a hypothetical condition where the outcomes have not been observed, a potential outcomes framework is proposed to describe the counterfactuals. The assumptions regarding the treatment ignorability and partial ignorability of missingness with causal consistency in the context of J2R are put forward for identification. We first consider cross-sectional studies, a special case of longitudinal studies with one follow-up time, as a stepping stone. We discover three identification formulas for the ATE, each of which invokes an estimator that relies on two of the three models:

(a) the propensity score, as the model of the treatment conditional on the observed historical covariates;

(b) the response probability, as the model of the response status conditional on the observed historical covariates and the treatment;

(c) the outcome mean, as the model of the mean outcomes conditional on the observed historical covariates and the treatment.

The three different estimators assess the ATE in distinct aspects of the models, motivating us to construct a new estimator that combines all the modeling features. Drawn
on semiparametric theory (Bickel et al., 1993), we obtain the efficient influence function (EIF) and use it to prompt a novel estimator with the incorporation of models (a)–(c). The proposed estimator has a remarkable property of triple robustness in the sense that it is consistent if any two of the three models are correctly specified when using parametric models or achieves $n^{1/2}$-consistency if the models converge at a rate not slower than $n^{-1/4}$ when using flexible models such as semiparametric or machine learning models. Extending to longitudinal clinical studies, an additional model is needed for identification:

(d) the pattern mean, as the model of the mean outcomes adjusted by the response probability conditional on the observed historical covariates and the treatment for any missingness pattern.

Even under MAR, the derivation of the EIF for longitudinal data is notoriously hard. The complexity is escalated in our context under J2R, where the treatment group involves additional outcome mean information from the control group, resulting in an unexplored territory to date. Our major theoretical contribution is to obtain the EIF in longitudinal studies, which enables us to construct a multiply robust estimator with the guaranteed $n^{1/2}$-consistency and asymptotic normality if the models in (a)–(d) have convergence rates not slower than $n^{-1/4}$. To mitigate the impact of the potential extreme values involved in the estimator, we seek alternative formations to obtain more stabilized estimators via normalization (Lunceford and Davidian, 2004) and calibration (e.g., Hainmueller, 2012). Moreover, a sequential estimation procedure that is analogous to the steps in Bang and Robins (2005) but under the more complex MNAR-related setting is provided as a typical way to obtain the estimator in practice. Inspired by the semiparametric efficiency bound the estimator attains, we provide an EIF-based variance estimator.

The rest of the paper proceeds as follows. Section 2 constructs the semiparametric framework under J2R in cross-sectional studies. Section 3 extends it to longitudinal data. Section 4 assesses the finite-sample performance of the proposed estimator via simulations. Section 5 uses antidepressant trial data to further validate the novel estimator. Conclusions and remarks are presented in Section 6. Supplementary material contains technical details, additional simulation and real-data application results.
2 Cross-sectional studies

To ground ideas, we first focus on cross-sectional studies. Let $A_i$ be the binary treatment, $X_i$ the baseline covariates, $Y_{1,i}$ the outcome, and $R_{1,i}$ the response indicator where $R_{1,i} = 1$ indicates the outcome is observed and $R_{1,i} = 0$ otherwise, for unit $i = 1, \ldots, n$. Note that the subscript involved 1 indicates the first post-baseline time point, as we will extend the notations to the longitudinal setting. Assume $\{X_i, A_i, R_{1,i}, Y_{1,i} : i = 1, \ldots, n\}$ are independent and identically distributed. For simplicity of notation, omit the subscript $i$ for the subject. Let $V = (X, A, R_1 Y_1, R_1)$ be the random vector of all observed variables and follow the observed data distribution $P$. To define the treatment effect unambiguously, we introduce the potential outcomes framework and define $R_1(a)$ as the potential response indicator received treatment $a$, $Y_1(a, r)$ as the potential outcome received treatment $a$ with response status $r$.

**Assumption 1 (Treatment ignorability)** $A \perp \{R_1(a), Y_1(a, r)\} \mid X$, for all $a$ and $r$.

Assumption 1 is the classic treatment ignorability in observational studies (Rosenbaum and Rubin, 1983). In a randomized clinical trial, the treatment ignorability holds naturally.

**Assumption 2 (Partial ignorability of missingness)** $R_1 \perp Y_1 \mid (X, A = 0)$.

We distinguish Assumption 2 from the conventional MAR assumption, as it only requires conditional independence between the actual outcome and response status in the control group. However, we do not impose MAR in the treatment group.

**Assumption 3 (Causal consistency)** $R_1 = R_1(A)$, and $Y_1 = Y_1 \{A, R_1(A)\}$.

Assumption 3 is the stable unit treatment value assumption proposed by Rubin (1980).

**Assumption 4 (J2R for the outcome mean)** $E\{Y_1(1, 0) \mid X, R_1(1) = 0\} = E\{Y_1(0, 0) \mid X, R_1(0) = 0\} = E(Y_1 \mid X, A = 0)$.

Assumption 4 specifies the outcome model under J2R. For an individual who has missing outcomes in either group, the conditional outcome mean is the same as the conditional mean of the actual outcome in the control group given the same baseline covariates.
Assumptions 2 and 4 jointly characterize MNAR related to the hypothetical J2R setting, as the missingness in the treatment group relates to the future outcomes by following a similar profile of the participants in the control group given the same baseline covariates.

2.1 Three identification formulas under J2R

The ATE can be expressed under the potential outcomes framework as $\tau_{J2R} = E[Y_{1\{1, R(1)\}} - Y_{1\{0, R(0)\}}]$. Define the propensity score as $e(X) = P(A = 1 | X)$, the response probability as $\pi_1(a, X) = P(R_1 = 1 | X, A = a)$, the outcome mean as $\mu^a_1(X) = E(Y_1 | X, R_1 = 1, A = a)$. The following theorem provides three identification formulas.

**Theorem 1** Under Assumptions 2–4 and assume there exists $\varepsilon > 0$, such that $\varepsilon < e(X) < 1 - \varepsilon$ and $\pi_1(0, X) > \varepsilon$ for all $X$, the following identification formulas hold for the ATE:

(a) Based on the response probability and outcome mean,

$$\tau_{J2R} = E[\pi_1(1, X) \left\{ \mu_1^1(X) - \mu_0^1(X) \right\}].$$

(b) Based on the propensity score and outcome mean,

$$\tau_{J2R} = E\left[\frac{2A - 1}{e(X)^A \{1 - e(X)\}^{1-A}} \left\{ R_1 Y_1 + (1 - R_1) \mu_0^1(X) \right\} \right].$$

(c) Based on the propensity score and response probability,

$$\tau_{J2R} = E\left\{ \frac{A}{e(X)} R_1 Y_1 - \frac{1 - A}{1 - e(X)} \frac{\pi_1(1, X)}{\pi_1(0, X)} R_1 Y_1 \right\}.$$  

Theorem 1 requires the positivity assumption of the treatment assignment (Rosenbaum and Rubin, 1983). It means that each participant has a nonzero probability to be assigned to the control or treatment group. When missingness is involved, a positivity assumption regarding the response probability is also imposed, indicating that each individual assigned to the control group has a chance to be observed at the study endpoint.

We give some intuition of the identification formulas below. The intuition also helps when we extend our framework to the longitudinal setting. Theorem 1 (a) describes that for any subject in the target population, the individual treatment effect will be zero when missingness is involved, as J2R entails that the individual will always take the control treatment and thus have the same outcome mean regardless of the assigned treatment; if
the outcome is fully observed, the individual treatment effect given the baseline covariates will be \( \mu_1(X) - \mu_0(X) \). Taking the expectation over the response status in the treatment group results in the overall marginal treatment effect.

Theorem 1 (b) describes the treatment effect as the difference in means. The weights \( A/e(X) \) and \( (1 - A)/(1 - e(X)) \) correspond to the propensity score weights and are of use to adjust for the group difference. The missing outcomes are replaced with their imputed outcome means under J2R. In this way, the imputed outcomes are the combination of the observed and imputed values, distinguished by the observed indicator \( R_1 \).

In Theorem 1 (c), the first term adjusted by \( A/e(X) \) targets the participants who are still observed in the assigned treatment group, which corresponds to \( E\{\pi_1(1, X)\mu_1(X)\} \). The second term marginalizes the multiplication between \( \pi_1(0, X) \) and the transformed outcome \( (1 - A)R_1Y_1/\{1 - e(X)\} \pi_1(0, X) \) which measures the conditional control group mean \( \mu_0(X) \), and matches \( E\{\pi_1(1, X)\mu_0(X)\} \) in Theorem 1 (a).

### 2.2 Estimation based on the identification formulas

We introduce additional notations for convenience. Let \( P_n \) be the empirical average, i.e., \( P_n(U) = n^{-1}\sum_{i=1}^n U_i \) for any variable \( U \). Under the parametric modeling framework, let \( e(X; \alpha), \mu_1^a(X; \beta) \) and \( \pi_1(a, X; \gamma) \) be the working model of \( e(X), \mu_1^a(X) \), and \( \pi_1(a, X; \gamma) \), where \( \alpha, \beta, \gamma \) are model parameters. Suppose the model parameter estimates \( (\hat{\alpha}, \hat{\beta}, \hat{\gamma}) \) converge to their probability limits \( (\alpha^*, \beta^*, \gamma^*) \). Denote the true model parameters \( (\alpha_0, \beta_0, \gamma_0) \) and the true models \( \{e(X), \mu_1^a(X), \pi_1(a, X) : a = 0, 1\} \) for shorthand. To illustrate model specifications, we use \( M \) with the subscripts “ps”, “om”, “rp” to denote the correctly specified propensity score, outcome mean, and response probability, respectively. Under \( M_{ps} \), \( e(X; \alpha^*) = e(X) \); under \( M_{om} \), \( \mu_1^a(X; \beta^*) = \mu_1^a(X) \); under \( M_{rp} \), \( \pi_1(a, X; \gamma^*) = \pi_1(a, X) \).

We use + to indicate the correct specification of more than one model and \( \cup \) to indicate that at least one model is correctly specified, e.g., \( M_{rp+om} \cup M_{ps} \) implies that the response probability and outcome mean are correct or the propensity score is correct.

The estimators are obtained by replacing the underlying nuisance functions \( \{e(X), \pi_1(a, X), \mu_1^a(X) : a = 0, 1\} \) with the estimated models \( \{e(X; \hat{\alpha}), \pi_1(a, X; \hat{\gamma}), \mu_1^a(X; \hat{\beta}) : a = 0, 1\} \), and the expectation with the empirical average.

**Example 1** The estimators motivated by the identification formulas in Theorem 1 are:
(a) The response probability-outcome mean (rp-om) estimator:

\[ \hat{\tau}_{rp-om} = \mathbb{P}_n \left[ \pi_1(1, X; \hat{\gamma}) \left\{ \mu_1^1(X; \hat{\beta}) - \mu_0^0(X; \hat{\beta}) \right\} \right]. \]

The estimator is consistent under \( M_{rp+om} \).

(b) The propensity score-outcome mean (ps-om) estimator:

\[ \hat{\tau}_{ps-om} = \mathbb{P}_n \left[ \frac{2A - 1}{e(X; \hat{\alpha})^4 \{1 - e(X; \hat{\alpha})\}^{1-A}} \left\{ R_1Y_1 + (1 - R_1)\mu_1^0(X; \hat{\beta}) \right\} \right]. \]

The estimator is consistent under \( M_{ps+om} \).

(c) The propensity score-response probability (ps-rp) estimator:

\[ \hat{\tau}_{ps-rp} = \mathbb{P}_n \left\{ \frac{A}{e(X; \hat{\alpha})} R_1Y_1 - \frac{1 - A}{1 - e(X; \hat{\alpha})} \frac{\pi_1(1, X; \hat{\gamma})}{\pi_1(0, X; \hat{\gamma})} R_1Y_1 \right\}. \]

The estimator is consistent under \( M_{ps+rp} \).

The estimators \( \hat{\tau}_{ps-om} \) and \( \hat{\tau}_{ps-rp} \) involve taking the inverse of the estimated propensity score or response probability, which may produce extreme values when they are close to 0 or 1. To mitigate the issue, we seek an alternative version of the inverse probability weighting estimators by normalizing the weights (Lunceford and Davidian, 2004).

Example 2 The normalized version of the ps-om and ps-rp estimators are as follows:

(a) The normalized ps-om estimator:

\[ \hat{\tau}_{ps-om-N} = \mathbb{P}_n \left[ \frac{A}{e(X; \hat{\alpha})} \frac{R_1Y_1 + (1 - R_1)\mu_1^0(X; \hat{\beta})}{\mathbb{P}_n \left\{ \frac{A}{e(X; \hat{\alpha})} \right\}} \right] - \mathbb{P}_n \left[ \frac{1 - A}{1 - e(X; \hat{\alpha})} \frac{R_1Y_1 + (1 - R_1)\mu_1^0(X; \hat{\beta})}{\mathbb{P}_n \left\{ \frac{1 - A}{1 - e(X; \hat{\alpha})} \right\}} \right]. \]

The normalized estimator is consistent under \( M_{ps+om} \).

(b) The normalized ps-rp estimator:

\[ \hat{\tau}_{ps-rp-N} = \mathbb{P}_n \left\{ \frac{A}{e(X; \hat{\alpha})} R_1Y_1 \right\} / \mathbb{P}_n \left\{ \frac{A}{e(X; \hat{\alpha})} \right\} - \mathbb{P}_n \left\{ \frac{1 - A}{1 - e(X; \hat{\alpha})} \frac{\pi_1(1, X; \hat{\gamma})}{\pi_1(0, X; \hat{\gamma})} R_1Y_1 \right\} / \mathbb{P}_n \left\{ \frac{1 - A}{1 - e(X; \hat{\alpha})} \frac{R_1}{\pi_1(0, X; \hat{\gamma})} \right\}. \]

The normalized estimator is consistent under \( M_{ps+rp} \).

8
2.3 EIF and the EIF-based estimators

Based on the three different identification formulas and the motivated estimators, it is possible to combine the three sets of model components in one identification formula. In the subsection, we first compute the EIF for the ATE under J2R to get a new identification formula and then give the resulting EIF-based estimators.

**Theorem 2** Under Assumptions \( \mathcal{A} \) and suppose that there exists \( \varepsilon > 0 \), such that \( \varepsilon < e(X) < 1 - \varepsilon \) and \( \pi_1(0, X) > \varepsilon \) for all \( X \), the EIF for \( \tau_{1J2R} \) is

\[
\varphi_{1J2R}(V; \mathbb{P}) = \left\{ \begin{array}{l}
\frac{A}{e(X)} - \frac{1 - A}{1 - e(X)} \pi_1(1, X) \\
\end{array} \right\} R_1 \left\{ Y_1 - \mu_1^0(X) \right\} R_1 \left\{ \begin{array}{l}
Y_1 - \mu_0^1(X) \\
\end{array} \right\} - \tau_{1J2R}.
\]

By the fact that the mean of the EIF is zero, we can obtain another identification formula by solving \( \mathbb{E}\left\{ \varphi_{1J2R}(V; \mathbb{P}) \right\} = 0 \) as presented in the following corollary.

**Corollary 1** Under the assumptions in Theorem 2,

\[
\tau_{1J2R} = \mathbb{E} \left[ \left\{ \frac{A}{e(X)} - \frac{1 - A}{1 - e(X)} \pi_1(1, X) \right\} R_1 \left\{ Y_1 - \mu_1^0(X) \right\} - \tau_{1J2R} \right].
\]

Corollary 1 motivates the EIF-based estimator \( \hat{\tau}_{tr} \) and its corresponding normalized estimator \( \hat{\tau}_{tr-N} \) to reduce the impact of extreme weights, which are provided in S3.1 in the supplementary material. We also consider employing calibration (e.g., Hainmueller, 2012, Zhao, 2019) to improve the covariate balance and mitigate the existence of outliers. Using the logistic link function, we estimate the weights by solving the optimization problem

\[
\min_{w_i \geq 0} \sum_{i=1}^{n} (w_i - 1) \log(w_i - 1) - w_i
\]

subject to \( \sum_{i:A_i=1} w_{a_1,i} h(X_i) = n^{-1} \sum_{i=1}^{n} h(X_i) \) to compute the calibrated weights \( w_{a_1} \) when \( A = 1 \); subject to \( \sum_{i:A_i=0} w_{a_0,i} h(X_i) = n^{-1} \sum_{i=1}^{n} h(X_i) \) to compute the calibrated weights \( w_{a_0} \) when \( A = 0 \); and subject to \( \sum_{i:R_i=1} w_{r_1,i} h(X_i) = n^{-1} \sum_{i=1}^{n} h(X_i) \) to compute the calibrated weights \( w_{r_1} \) when \( R_1 = 1 \). Here, \( h(X) \) is any function of covariates. For example, one may incorporate the first moment of the covariates to achieve a balance in
means or add the first two moments to achieve a balance in both means and variances. The calibration-based estimator $\hat{\tau}_{tr-C}$ is also given in S3.1 in the supplementary material.

Interestingly, the $n^{1/2}$-consistency of the EIF-motivated estimators does not require the correct specification of all three models. As we explain in the next subsection, the estimators reach a $n^{1/2}$-consistency if any two of the three models are correct when using parametric modeling strategy, or if the convergence rate of any model is not less than $n^{-1/4}$. We call this property triple robustness.

### 2.4 Triple robustness

We focus on investigating the asymptotic properties of $\hat{\tau}_{tr}$ since the three EIF-based estimators are asymptotically equivalent. Theorem 3 explores the triple robustness and semi-parametric efficiency of $\hat{\tau}_{tr}$ under a parametric modeling strategy on the nuisance functions.

**Theorem 3** Under Assumptions 7 and 8, and suppose that there exists $\varepsilon > 0$, such that $\varepsilon < \{e(X; \alpha^*), e(X; \hat{\alpha})\} < 1 - \varepsilon, \{\pi_1(0, X; \gamma^*), \pi_1(0, X; \hat{\gamma})\} > \varepsilon$ for all $X$ almost surely, the estimator $\hat{\tau}_{tr}$ is triply robust in the sense that it is consistent for $\tau_{1J2R}$ under $M_{rp+om} \cup M_{ps+om} \cup M_{ps+rp}$. Moreover, $\hat{\tau}_{tr}$ achieves the semiparametric efficiency bound under $M_{ps+rp+om}$.

Theorem 3 requires that the true and estimated propensity score and response probability are bounded away from 0 and 1 to reduce the involvement of extreme values (Robins and Rotnitzky, 1995). Given that the EIF-based estimators and the estimators in Examples 1 and 2 are asymptotically linear, their variance estimations can be computed by the nonparametric bootstrap.

When the models for the nuisance functions are difficult to obtain parametrically, one can turn to more flexible modeling strategies such as semiparametric models like generalized additive models (GAM; Hastie and Tibshirani, 2017) or machine learning models to get the estimated models $\{\hat{e}(X), \hat{\pi}_1(a, X, \hat{\alpha}), \hat{\mu}_a(X) : a = 0, 1\}$. To illustrate the convergence rate, denote $\|U\| = \{E(U^2)\}^{1/2}$ as the $L_2$-norm of the random variable $U$. Suppose the convergence rates are $\|\hat{e}(X) - e(X)\| = o_P(n^{-c_e}), \|\hat{\mu}_a(X) - \mu_a(X)\| = o_P(n^{-c_\mu})$ and $\|\hat{\pi}_1(a, X) - \pi_1(a, X)\| = o_P(n^{-c_\pi})$. Denote $\hat{P}$ as the estimated distribution of the observed data. Theorem 4 illustrates the asymptotic distribution of the EIF-based estimator.
Theorem 4 Under Assumptions 1–4, and suppose that there exists $\varepsilon > 0$, such that $\varepsilon < \{e(X), \hat{e}(X)\} < 1 - \varepsilon$, $\{\pi_1(0, X), \hat{\pi}_1(0, X)\} > \varepsilon$ for all $X$ almost surely, and the nuisance functions and their estimators take values in Donsker classes. Assume $\|\varphi_{1}^{1\text{ JR}}(V; \hat{\mathbb{P}}) - \varphi_{1}^{1\text{ JR}}(V; \mathbb{P})\| = \mathcal{O}_\mathbb{P}(1)$. Then,

$$\hat{\tau}_\text{tr} = \tau_1^{1\text{ JR}} + \frac{1}{n} \sum_{i=1}^{n} \varphi_{1}^{1\text{ JR}}(V; \mathbb{P}) + \text{Rem}(\hat{\mathbb{P}}, \mathbb{P}) + \mathcal{O}_\mathbb{P}(n^{-1/2}),$$

where

$$\text{Rem}(\hat{\mathbb{P}}, \mathbb{P}) = \mathbb{E} \left[ \left\{ \frac{e(X)}{\hat{e}(X)} - 1 \right\} \left\{ \pi_1(1, X)\mu_1^1(X) - \hat{\pi}_1(1, X)\hat{\mu}_1^1(X) \right\} ight. + \left\{ 1 - \frac{1 - e(X)\pi_1(0, X)}{1 - \hat{e}(X)\hat{\pi}_1(0, X)} \right\} \hat{\pi}_1(1, X) \left\{ \mu_1^0(X) - \hat{\mu}_1^0(X) \right\} + \left\{ \pi_1(1, X) - \hat{\pi}_1(1, X) \right\} \left\{ \mu_1^0(X) - \frac{e(X)}{\hat{e}(X)}\hat{\mu}_1^0(X) \right\} \right].$$

If $\text{Rem}(\hat{\mathbb{P}}, \mathbb{P}) = \mathcal{O}_\mathbb{P}(n^{-1/2})$, then $\sqrt{n}(\hat{\tau}_\text{tr} - \tau_1^{1\text{ JR}}) \xrightarrow{d} \mathcal{N}(0, \mathbb{V}\{\varphi_{1}^{1\text{ JR}}(V; \mathbb{P})\})$, where the asymptotic variance of $\hat{\tau}_\text{tr}$ reaches the semiparametric efficiency bound.

The requirement of Donsker classes controls the complexity of the nuisance functions and their estimators (Kennedy, 2016), which can be further relaxed using cross-fitting (Chernozhukov et al., 2018). Theorem 4 invokes the triple robustness in terms of rate convergence when using flexible models, presented by the following corollary.

Corollary 2 Under the assumptions in Theorem 4, suppose $\|\varphi_{1}^{1\text{ JR}}(V; \hat{\mathbb{P}}) - \varphi_{1}^{1\text{ JR}}(V; \mathbb{P})\| = \mathcal{O}_\mathbb{P}(1)$, and further suppose that there exists $0 < M < \infty$, such that $\mathbb{P}\left( \max \left\{ |\hat{\mu}_1^0(X)|, |\hat{\mu}_1^1(X)| \right\} \right) \leq M = 1$, then $\hat{\tau}_\text{tr} - \tau_1^{1\text{ JR}} = \mathcal{O}_\mathbb{P} \left( n^{-1/2} + n^{-c} \right)$, where $c = \min(c_e + c_\mu, c_e + c_\pi, c_\mu + c_\pi)$.

The additional bounding condition for the estimated outcome mean models along with the ratio $\{1 - e(X)\}/\{1 - \hat{e}(X)\}$, which holds in most clinical studies, guarantees an upper bound of the remainder term $\text{Rem}(\hat{\mathbb{P}}, \mathbb{P})$. Corollary 2 provides alternative approaches to reach a $n^{1/2}$-rate consistency of the estimator. The nuisance functions can converge at a slower rate no less than $n^{-1/4}$ using flexible models.
3 Longitudinal data with monotone missingness

Next, we focus on the longitudinal setting and introduce additional notations. Suppose the longitudinal data contain $t$ time points. Let $Y_{s,i}$ be the outcome at time $s$, $H_{s-1,i} = (X^T_i, Y_{1,i}, \cdots, Y_{s-1,i})^T$ be the historical information at time $s$ for $s = 1, \cdots, t$, and $H_{0,i} = X_i$. When missingness is involved, denote $R_{s,i}$ as the response indicator at time $s$ and $D_i$ as the dropout time. Let $R_{0,i} = 1$, indicating the baseline covariates $H_{0,i}$ are always observed. We assume a monotone missingness pattern, i.e., if the individual drops out at time $s$, we would expect $R_{s,i} = \cdots = R_{t,i} = 0$. By monotone missingness, there exists a one-to-one relationship between the dropout time $D_i$ and the vector of response indicators $(R_{0,i}, \cdots, R_{t,i})$ as $D_i = \sum_{s=0}^{t} R_{s,i}$ for all $i$. Assume the full data $\{X_i, A_i, R_{1,i}, Y_{1,i}, \cdots, R_{t,i}, Y_{t,i} : i = 1, \cdots, n\}$ are independent and identically distributed. For simplicity of notation, omit the subscript $i$ for the subject. Let $V = (X, A, R_1, Y_1, \cdots, R_t, Y_t)$ be the vector of all observed variables and follow the observed data distribution $\mathbb{P}$. We extend the proposed potential outcomes framework to the longitudinal setting. Define $R_s(a)$ as the potential response indicator if the subject received treatment $a$ at time $s$, $D(a)$ as the potential dropout time if the subject received treatment $a$, $Y_s(a,d)$ as the potential outcome if the subject received treatment $a$ at time $s$ with the occurrence of dropout at time $d$. Due to the natural constraint that future dropouts do not affect the current and past outcomes, we have $Y_s(a, t + 1) = Y_s(a, s')$ for any $s < s' < t + 1$ and $D(a) = \sum_{s=0}^{t} R_s(a)$. We extend Assumptions 1–4 to the context of longitudinal data with monotone missingness.

**Assumption 5 (Treatment ignorability)** $A \perp \! \! \! \perp \{R_s(a), D(a), Y_s(a,d)\} \mid X$, for all $a, s$ and $d$.

**Assumption 6 (Partial ignorability of missingness)** $R_s \perp \! \! \! \! \perp Y_{s'} \mid (H_{s-1}, A = 0)$, for all $s' \geq s$.

**Assumption 7 (Causal consistency)** $R_s = R_s(A)$, $D = D(A)$, and $Y_s = Y_s\{A, D(A)\}$, for all $s$.

**Assumption 8 (J2R for the outcome mean)** $\mathbb{E}\{Y_s(1,d) \mid D(1) = d, H_{d-1}\} = \mathbb{E}\{Y_s(0,d) \mid D(0) = d, H_{d-1}\} = \mathbb{E}(Y_s \mid H_{d-1}, R_{d-1} = 1, A = 0)$, for all $s \geq d$. 

12
White et al. (2020) develop a similar potential outcomes framework for CBI in longitudinal clinical trials. However, their assumptions of the causal model are much stronger, as they assume a linear relationship between the future and historical outcomes. Our proposed framework does not rely on any modeling assumptions and is more flexible in practice. In this section, all results degenerate to the ones in cross-sectional studies when \( t = 1 \).

3.1 Three identification formulas under J2R

In most longitudinal clinical studies, the endpoint of interest is the ATE measured by the mean difference at the last time point between the two groups. Therefore, the ATE can be expressed using the potential outcomes framework as \( \tau_{t}^{\text{J2R}} = \mathbb{E}[Y_{t}\{1,D(1)\} - Y_{t}\{0,D(0)\}] \).

Define the propensity score \( e(H_{s-1}) = \mathbb{P}(A = 1 \mid H_{s-1}, R_{s-1} = 1) \), the response probability \( \pi_{s}(a, H_{s-1}) = \mathbb{P}(R_{s} = 1 \mid H_{s-1}, R_{s-1} = 1, A = a) \), the longitudinal outcome mean \( \mu_{s}^{a}(H_{s-1}) = \mathbb{E}\{\mu_{t}^{a}(H_{s}) \mid H_{s-1}, R_{s} = 1, A = a\} \) with \( \mu_{t}^{a}(H_{s}) = Y_{t} \), and the pattern mean \( g_{s+1}^{l}(H_{l-1}) = \mathbb{E}\{\pi_{l+1}(1,H_{l})g_{s+1}^{1}(H_{l}) \mid H_{l-1}, R_{l} = 1, A = 1\} \) for \( l = 1, \cdots, s-1 \) with \( g_{s+1}^{1}(H_{s-1}) = \mathbb{E}\{1 - \pi_{s+1}(1,H_{s})\} \mathbb{E}\{\mu_{t}^{0}(H_{s}) \mid H_{s-1}, R_{s} = 1, A = 1\} \) if we let \( \pi_{s+1}(1,H_{s}) = 0 \).

In addition, denote \( \bar{\pi}_{s}(a, H_{s-1}) = \prod_{k=1}^{s} \pi_{k}(a, H_{k-1}) \) as the cumulative response probability for the individual observed at time \( s \), for \( s = 1, \cdots, t \). Under Assumptions 5 and 6, Assumption 8 is equivalent to \( \mathbb{E}\{Y_{s}(1,d) \mid D(1) = d, H_{d-1}\} = \mathbb{E}\{Y_{s}(0,d) \mid D(0) = d, H_{d-1}\} = \mu_{s}^{0}(H_{d-1}) \), which is identifiable. The following theorem provides three identification formulas for longitudinal data with a monotone missingness pattern under J2R.

**Theorem 5** Under Assumptions 5–8, and suppose that there exists \( \varepsilon > 0 \), such that \( \varepsilon < e(H_{s-1}) < 1 - \varepsilon \) and \( \pi_{s}(0,H_{s-1}) > \varepsilon \) for all \( H_{s-1} \) with \( s = 1, \cdots, t \), the following identification formulas hold for the ATE under J2R:

(a) Based on the response probability and pattern mean,

\[
\tau_{t}^{\text{J2R}} = \mathbb{E}\left[ \bar{\pi}_{1}(1,H_{0}) \left\{ \sum_{s=1}^{t} g_{s+1}^{1}(H_{0}) - \mu_{t}^{0}(H_{0}) \right\} \right].
\]

(b) Based on the propensity score and outcome mean,

\[
\tau_{t}^{\text{J2R}} = \mathbb{E}\left[ \frac{2A - 1}{e(H_{0})^{A} \{1-e(H_{0})\}^{1-A}} \left\{ R_{t}Y_{t} + \sum_{s=1}^{t} R_{s-1}(1-R_{s})\mu_{t}^{0}(H_{s-1}) \right\} \right].
\]
Similar to the cross-sectional setting, the estimators can be obtained by replacing the functions

\[ \{ e(H_{s-1}), \pi_s(a, H_{s-1}), \mu_t^a(H_{s-1}), g^1_{s+1}(H_{t-1}) : l = 1, \ldots, s \} \]

with the estimated functions \( \{ \hat{e}(H_{s-1}), \hat{\pi}_s(a, H_{s-1}), \hat{\mu}_t^a(H_{s-1}), \hat{g}^1_{s+1}(H_{t-1}) : l = 1, \ldots, s \} \) and the expectation with the empirical average. Compared to the cross-sectional case, obtaining the ATE estimator here involves fitting sequential models at each time point. However, the complex iterated form of \( g^1_{s+1}(H_{t-1}) \) is infeasible to model parametrically. We consider using more flexible models such as semiparametric or machines learning models. Denote \( \hat{\mathbb{P}} \) as the estimated distribution of the observed data \( V \). Suppose the nuisance functions have convergence rates \( ||\hat{e}(H_{s-1}) - e(H_{s-1})|| = o_P(n^{-c_0}), ||\hat{\mu}_t^a(H_{s-1}) - \mu_t^a(H_{s-1})|| = o_P(n^{-c_0}), ||\hat{\pi}_s(a, H_{s-1}) - \pi_s(a, H_{s-1})|| = o_P(n^{-c_0}) \) for any \( H_{s-1} \), and \( ||\hat{g}^1_{s+1}(H_{t-1}) - g^1_{s+1}(H_{t-1})|| = o_P(n^{-c_0}) \) for any \( H_{t-1} \), when \( l = 1, \ldots, s; s = 1, \ldots, t \) and \( a = 0, 1 \).

**Example 3** The estimators motivated by the identification formulas in Theorem 3 are:

(a) The response probability-pattern mean (rp-pm) estimator:

\[
\hat{\tau}_{rp-pm} = \mathbb{P}_n \left[ \hat{\pi}_1(1, H_0) \left\{ \sum_{s=1}^t \hat{g}^1_{s+1}(H_0) - \hat{\mu}_t^0(H_0) \right\} \right],
\]

where \( \hat{g}^1_{s+1}(H_{t-1}) = \mathbb{E} \{ \hat{\pi}_{t+1}(1, H_t) \hat{g}^1_{s+1}(H_t) | H_{t-1}, R_t = 1, A = 1 \} \) for \( l = 1, \ldots, s - 1 \) and \( \hat{g}^1_{s+1}(H_{s-1}) = \mathbb{E} \{ (1 - \hat{\pi}_{s+1}(1, H_s)) \hat{\mu}_t^0(H_s) | H_{s-1}, R_s = 1, A = 1 \} \) if let \( \hat{\pi}_{t+1}(1, H_t) = 0 \).
(b) The ps-om estimator:
\[
\hat{\tau}_{ps-om} = \mathbb{P}_n \left[ \frac{2A - 1}{\hat{e}(H_0)^A \{1 - \hat{e}(H_0)\}^{1-A}} \left\{ \sum_{s=1}^{t} R_s Y_s + R_{t} Y_{t} + \sum_{s=1}^{t} R_{s-1} (1 - R_s) \hat{\mu}_s^0 (H_{s-1}) \right\} \right].
\]

(c) The ps-rp estimator:
\[
\hat{\tau}_{ps-rp} = \mathbb{P}_n \left( \frac{A}{\hat{e}(H_0)} R_t Y_t + \frac{1 - A}{1 - \hat{e}(H_0)} \left\{ \sum_{s=1}^{t} \hat{\pi}_{s-1} (0, H_s) \{1 - \hat{\pi}_s (1, H_s)\} \hat{\delta} (H_{s-1}) \right\} - 1 \right) \hat{\pi}_t (0, H_{t-1}).
\]
where \( \hat{\delta} (H_{s-1}) = \{\hat{e}(H_{s-1})/\hat{e}(H_0)\} / \{1 - \hat{e}(H_{s-1})\} / \{1 - \hat{e}(H_0)\} \).

The impact of extreme propensity score and response probability weights is more pronounced in the longitudinal setting with an extended long period of follow-up. To mitigate the influence, we consider the normalized estimators and present the expressions of \( \hat{\tau}_{ps-om-N} \) and \( \hat{\tau}_{ps-rp-N} \) in S3.2 in the supplementary material.

The estimation procedure is similar to the one introduced in [Bang and Robins (2005)], which involves fitting the models recursively. The propensity score \( \{e(H_{s-1}) : s = 1, \ldots, t\} \) and response probability \( \{\pi_s (a, H_{s-1}) : s = 1, \ldots, t\} \) incorporate all the available information \( H_{s-1} \) at each time point. For the outcome mean \( \{\mu_t^0 (H_{s-1}) : s = 1, \ldots, t\} \), we begin from the observed data at the last time point and use the predicted values to regress on the observed data recursively in backward order. For the pattern mean \( \{g^1_{s+1} (H_{t-1}) : l = 1, \ldots, s \} \) and \( s = 1, \ldots, t \}, the product of the predicted values \( \{1 - \hat{\pi}_{s+1} (1, H_s)\} \) and \( \hat{\mu}_t^0 (H_s) \) is regressed on the historical information \( H_{s-1} \) at time \( s \). The resulting predicted value \( \hat{g}_{s+1}^1 (H_{s-1}) \) multiplied by the predicted response probability \( \hat{\pi}_s (1, H_{s-1}) \) then severs as the outcome in the model \( g^1_{s+1} (H_{s-2}) \), with the incorporation of the observed data at time \( s - 1 \), recursively. Note that the estimated pattern mean will have good performance only if both the response probability and the outcome mean are well-approximated. We give the detailed steps for estimating \( \hat{\tau}_{rp-pm-N} \), \( \hat{\tau}_{ps-rp-N} \) when \( t = 2 \) in S3.3 in the supplementary material. Extensions to the estimators when \( t > 2 \) is straightforward.

**Remark 1** Modeling the pattern mean \( \{g^1_{s+1} (H_{s-1}) : l = s, \ldots, t\} \) and the outcome mean \( \mu_t^0 (H_{s-1}) \) involves fitting their corresponding predicted values on the same historical information \( H_{s-1} \) for the subjects observed at the current time \( s \). As addressed in [Bang and Robins (2005)].
those models may be incompatible since they use the same model to predict different terms. For example, when estimating \( \hat{\tau}_{rp-pm} \) with \( t = 2 \), incompatibility occurs in the model fitting of \( g_2^1(H_0), g_3^1(H_0) \) and \( \mu_2^0(H_0) \). However, the issue can be resolved by specifying the distribution function \( f(Y_s \mid H_{s-1}, R_s = 1, A = a) \) given the observed data.

3.3 EIF and the EIF-based estimators

Similar to cross-sectional studies, we derive the EIF for \( \tau_t^{JR} \) to motivate a new estimator.

**Theorem 6** Under Assumptions 5–8 and suppose that there exists \( \varepsilon > 0 \), such that \( \varepsilon < e(H_{s-1}) < 1 - \varepsilon \) and \( \pi_s(0, H_{s-1}) > \varepsilon \) for all \( H_{s-1} \) with \( s = 1, \cdots, t \), the EIF for \( \tau_t^{JR} \) is

\[
\varphi_t^{JR}(V; \mathbb{P}) = \frac{A}{e(H_0)} \left\{ R_t Y_t + \sum_{s=1}^t R_{s-1}(1 - R_s)\mu_t^0(H_{s-1}) \right\} - \tau_t^{JR}
\]

\[
+ \left\{ 1 - \frac{A}{e(H_0)} \right\} \left[ \tau_1(1, H_0) \sum_{s=1}^t g_{s+1}^1(H_0) + \left\{ 1 - \pi_1(1, H_0) \right\} \mu_t^0(H_0) \right] - \mu_t^0(H_0)
\]

\[
+ \frac{1 - A}{1 - e(H_0)} \sum_{s=1}^t \left\{ \sum_{k=1}^{s} \bar{\pi}_{k-1}(0, H_{k-2}) \left\{ 1 - \pi_k(1, H_{k-1}) \right\} \delta(H_{k-1}) \right\} - 1 \frac{R_s}{\pi_s(0, H_{s-1})} \left\{ \mu_t^0(H_s) - \mu_t^0(H_{s-1}) \right\}.
\]

Solving \( \mathbb{E}\{\varphi_t^{JR}(V; \mathbb{P})\} = 0 \) yields another identification formula in the following corollary.

**Corollary 3** Under the assumptions in Theorem 6,

\[
\tau_t^{JR} = \mathbb{E} \left( \frac{A}{e(H_0)} \left\{ R_t Y_t + \sum_{s=1}^t R_{s-1}(1 - R_s)\mu_t^0(H_{s-1}) \right\} \right)
\]

\[
+ \left\{ 1 - \frac{A}{e(H_0)} \right\} \left[ \tau_1(1, H_0) \sum_{s=1}^t g_{s+1}^1(H_0) + \left\{ 1 - \pi_1(1, H_0) \right\} \mu_t^0(H_0) \right] - \mu_t^0(H_0)
\]

\[
+ \frac{1 - A}{1 - e(H_0)} \sum_{s=1}^t \left\{ \sum_{k=1}^{s} \bar{\pi}_{k-1}(0, H_{k-2}) \left\{ 1 - \pi_k(1, H_{k-1}) \right\} \delta(H_{k-1}) \right\} - 1 \frac{R_s}{\pi_s(0, H_{s-1})} \left\{ \mu_t^0(H_s) - \mu_t^0(H_{s-1}) \right\}.
\]

Corollary 3 motivates the EIF-based estimator \( \hat{\tau}_{mr} \) by plugging in the estimated nuisance functions. The estimation combines the estimating steps for \( \hat{\tau}_{rp-pm}, \hat{\tau}_{ps-om} \) and \( \hat{\tau}_{ps-rp} \). In addition, one can consider the normalized estimator \( \hat{\tau}_{mr-N} \), or the calibration-based estimator \( \hat{\tau}_{mr-C} \) to mitigate the impact of extreme weights. We give the three EIF-based estimators and the detailed estimation steps in S3.4 in the supplementary material.
3.4 Multiple robustness

To simplify the notations, let $E_{0,l-1}(;H_s) := \mathbb{E}\{\cdots \mathbb{E}(\cdot \mid H_{l-1}, R_l = 1, A = 0) \cdots \mid H_s, R_{s+1} = 1, A = 0\}$ be the function of ($l-s$) layers conditional expectations, with the conditions beginning from $(H_{l-1}, R_l = 1, A = 0)$ to $(H_s, R_{s+1} = 1, A = 0)$, and $E_{1,s-1}(;H_0) := \mathbb{E}\{\cdots \mathbb{E}(\cdot \mid H_{s-1}, R_s = 1, A = 1) \cdots \mid H_0, R_1 = 1, A = 1\}$ be the function of $s$ layers conditional expectations, with the conditions beginning from $(H_{s-1}, R_s = 1, A = 1)$ to $(H_0, R_1 = 1, A = 1)$. Denote $g^1_{\mu,s+1}(H_{l-1}) = \mathbb{E}\{\pi_{l+1}(1, H_l)g^1_{\mu,s+1}(H_l) \mid H_{l-1}, R_l = 1, A = 1\}$ for $l = 1, \cdots, s-1$, and $g^1_{\mu,s+1}(H_{s-1}) = \mathbb{E}\{1 - \pi_{s+1}(1, H_s)\} \hat{\mu}^0(H_s) \mid H_{s-1}, R_s = 1, A = 1$ for $s = 1, \cdots, t$, i.e., we only estimate the outcome mean in the joint sequential pattern mean model $g^1_{s+1}(H_{l-1})$. The asymptotic properties of $\hat{\tau}_{mr}$ obtained via flexible models are presented in the following theorem.

**Theorem 7** Under Assumptions [18] and suppose that there exists $\varepsilon > 0$, such that $\varepsilon < \{c(H_{s-1}), \hat{e}(H_{s-1})\} < 1 - \varepsilon$, \{\pi_s(0, H_{s-1}), \hat{\pi}_s(0, H_{s-1})\} > \varepsilon$ for all $H_{s-1}$ with $s = 1, \cdots, t$, and the nuisance functions and their estimators take values in Donsker classes. Suppose $\|\varphi^J_{t,J2R}(V; \hat{P}) - \varphi^J_{t,J2R}(V; \mathbb{P})\| = o_{\mathbb{P}}(1)$, then

$$\hat{\tau}_{mr} = \tau^J_{t,J2R} + \frac{1}{n} \sum_{i=1}^{n} \varphi^J_{t,J2R}(V_i; \mathbb{P}) + \text{Rem}(\hat{P}, \mathbb{P}) + o_{\mathbb{P}}(n^{-1/2}),$$

where

$$\text{Rem}(\hat{P}, \mathbb{P}) = \mathbb{E}\left\{\left\{\frac{e(H_0)}{\hat{e}(H_0)} - 1\right\} \left\{\pi_1(1, H_0)g^1_{l+1}(H_0) - \hat{\pi}_1(1, H_0)\hat{g}^1_{l+1}(H_0)\right\}ight.$$

$$+ \left\{\frac{e(H_0)}{\hat{e}(H_0)} - 1\right\} \sum_{s=1}^{t-1} \left\{\pi_1(1, H_0)g^1_{\mu,s+1}(H_0) - \hat{\pi}_1(1, H_0)\hat{g}^1_{s+1}(H_0)\right\}
$$

$$+ \sum_{s=1}^{t-1} \sum_{l=s+1}^{t} E_{0,l-1} \left\{E_{0,l-1} \left[\pi_s(1, H_{s-1}) \left\{1 - \frac{e(H_0)}{1 - \hat{e}(H_0)} \right\} \left\{\hat{\pi}_{s+1}(1, H_{s-1}) \frac{\hat{\mu}^0(H_{s-1})}{\hat{\delta}(H_{s-1})} - \{1 - \pi_{s+1}(1, H_s)\} \hat{\mu}^0(H_l) - \hat{\mu}^0(H_{l-1})\right\} ; H_s\right) \hat{H}_{l-1}\right\}
$$

$$+ \{\hat{\pi}_1(1, H_0) - \pi_1(1, H_0)\} \left\{\frac{e(H_0)}{\hat{e}(H_0)} \hat{\mu}^0(H_0) - \mu^0(H_0)\right\}
$$

$$+ \hat{\pi}_1(1, H_0) \sum_{s=1}^{t} E_{0,s-1} \left\{\left\{1 - \frac{1 - e(H_0)}{1 - \hat{e}(H_0)} \pi_s(0, H_{s-1})\right\} \hat{\mu}^0(H_s) - \hat{\mu}^0(H_{s-1})\right\} ; H_0\right\}.
$$

If $\text{Rem}(\hat{P}, \mathbb{P}) = o_{\mathbb{P}}(n^{-1/2})$, then $\sqrt{n}(\hat{\tau}_{mr} - \tau^J_{t,J2R}) \overset{d}{\rightarrow} \mathcal{N}(0, \mathbb{V} \{\varphi^J_{t,J2R}(V; \mathbb{P})\})$, where the asymptotic variance of $\hat{\tau}_{mr}$ reaches the semiparametric efficiency bound.
Based on the semiparametric efficiency bound, we get the EIF-based variance estimator of $\hat{\tau}_{\text{mr}}$ as
\[
\hat{V}(\hat{\tau}_{\text{mr}}) = \frac{1}{n^2} \sum_{i=1}^{n} \left\{ \varphi_{J2R}^{\text{ir}}(V_i; \hat{P}) - \hat{\tau}_{\text{mr}} \right\}^2.
\]
In practice, the Wald-type confidence interval (CI) tends to have narrower intervals which can be anti-conservative (Boos and Stefanski, 2013). Symmetric t bootstrap CI (Hall, 1988) is considered to improve the coverage. In each bootstrap iteration from $b = 1, \cdots, B$, where $B$ is the total number of bootstrap replicates, we compute
\[
T^*(b) = \frac{\hat{\tau}(b) - \hat{\tau}}{\hat{V}^{1/2}(\hat{\tau}(b))}
\]
to get the estimated bootstrap distribution. The 95% symmetric t bootstrap CI of $\tau_{J2R}$ is obtained by $(\hat{\tau} - c^*\hat{V}^{1/2}(\hat{\tau}), \hat{\tau} + c^*\hat{V}^{1/2}(\hat{\tau}))$, where $c^*$ is the 95% quantile of $\{|T^*(b)| : b = 1, \cdots, B\}$.

Theorem 7 motivates the following corollary, which addresses the multiple robustness of $\hat{\tau}_{\text{mr}}$ in terms of the convergence rate.

**Corollary 4** Under the assumptions in Theorem 7, suppose
\[
\|\varphi_{J2R}^{\text{ir}}(V; \hat{P}) - \varphi_{J2R}^{\text{ir}}(V; P)\| = o_P(1),
\]
and there exists $0 < M < \infty$, such that
\[
P \left( \max \left\{ \left| \frac{e(H_0)}{\hat{e}(H_0)} \right|, \left| \mu_l(0) \right|, \left| \hat{g}_{s+1}(H_0) \right|, \left| \frac{1 - e(H_0)}{1 - \hat{e}(H_0)} \right| \delta(H_{s-1}) \right\} \leq M \right) = 1
\]
for $s = 1, \cdots, t$, then $\hat{\tau}_{\text{mr}} - \tau_{J2R} = O_P \left( n^{-1/2} + n^{-c} \right)$, where $c = \min \{ c_e + c_\mu, c_\mu + c_\pi, c_\mu + c_\pi, c_\mu + c_\pi, c_e + c_\pi \}$.

Similar to the cross-sectional setting, even if the nuisance functions converge at a lower rate, we can still obtain a $n^{1/2}$-rate consistency if the convergence rate of each function is not slower than $n^{-1/4}$. Additional nuisance function $\{g_{s+1}(H_{l-1}) : l = 1, \cdots, s$ and $s = 1, \cdots, t\}$ is involved in the longitudinal setting, whose convergence rate may be harder to control as it incorporates the estimation of both the outcome mean and response probability.

4 Simulation study

4.1 Cross-sectional setting

We first conduct the simulation in a cross-sectional setting to evaluate the finite-sample performance of the proposed estimators under J2R. Set the sample size as 500. The covariates $X \in \mathbb{R}^5$ are generated by $X_j \sim N(0.25, 1)$ for $j = 1, \cdots, 4$ and $X_5 \sim \text{Bernoulli}(0.5)$. Consider a nonlinear transformation of the covariates and denote $Z_j = \{X_j^2 + 2\sin(X_j) -
1.5}/√2 for \( j = 1, \ldots, 4 \) and \( Z_5 = X_5 \). We generate \( A \mid X \sim \text{Bernoulli}\{e(X)\} \), where logit\{e(X)\} = 0.1 \( \sum_{j=1}^{4} Z_j \); \( R_1 \mid (X, A = a) \sim \text{Bernoulli}\{\pi_1(a, X)\} \), where logit\{\pi_1(a, X)\} = \((2a - 1) \sum_{j=1}^{5} Z_j / 6\); and \( Y_1 \mid (X, A = a, R_1 = 1) \sim N\{\mu_1^a(X), 1\} \), where \( \mu_1^a(X) = (2 + a) \sum_{j=1}^{5} Z_j / 6 \). To evaluate the robustness of the EIF-based estimators, we consider two model specifications of the propensity score, response probability, and outcome mean. Specifically, we fit the corresponding parametric models with the covariates \( Z \) as the correctly specified models, or with the covariates \( X \) as the misspecified models.

We compare the estimators from Examples 1 and 2 with the three EIF-based estimators. The first moment of the covariates \( Z \) is incorporated for calibration. The estimators are assessed in terms of the point estimation, coverage rates of the 95% CI and mean CI lengths under 8 scenarios, each of which relies on whether the propensity score, response probability, or outcome mean is correctly specified. Here, “yes” denotes the correct model with the nonlinear covariates \( Z \), “no” denotes the wrong model with the linear covariates \( X \). We compute the variance estimates \( \hat{V} \) of the estimators by the nonparametric bootstrap with \( B = 100 \) and use the 95% Wald-type CI estimated by \((\hat{\tau} - 1.96\hat{V}^{1/2}, \hat{\tau} + 1.96\hat{V}^{1/2})\).

Figure 1 shows the point estimation results based on 1000 Monte Carlo simulations under 8 different model specifications. When three models are correctly specified, all the estimators are unbiased. For the estimators without triple robustness, they are biased when at least one of their required models is misspecified; while the three EIF-based estimators verify triple robustness since they are unbiased when any two of the three models are correct. Normalization mitigates the impact of extreme weights and results in smaller variations. Moreover, calibration produces a more steady estimator. The coverage rates and mean CI lengths are presented in Table 1 which match the observations we make from Figure 1. All estimators have satisfying coverage rates when their required models are correct. The coverage rates of the EIF-based estimators are close to the empirical value when any two of the three models are correct, with the smallest mean CI length produced by \( \hat{\tau}_{\text{tr-C}} \).

4.2 Longitudinal setting

We further evaluate the performance of the proposed estimators in longitudinal studies under J2R. Consider the data with two follow-up time points. We choose the same sample size as \( n = 500 \), generate the same covariates \( X \in \mathbb{R}^5 \) and use the same transformation on the
Figure 1: Performance of the estimators in the cross-sectional setting under 8 different model specifications.

covariates to construct $Z \in \mathbb{R}^5$. The treatment is generated by $A \mid X \sim \text{Bernoulli}(e(X))$, where $\text{logit}(e(X)) = 0.1 \sum_{j=1}^{4} Z_j$. The observed indicators and the longitudinal outcomes are generated in time order. Specifically, at the first time point, we generate $R_1 \mid (X, A = a) \sim \text{Bernoulli}(\pi_1(a, X))$, where $\text{logit}(\pi_1(a, X)) = 5(2a - 1) \sum_{j=1}^{4} Z_j/9$, and $Y_1 \mid (X, R_1 = 1, A = a) \sim N(\mu_1^a(X), 1)$, where $\mu_1^a(X) = (2 + a) \sum_{j=1}^{5} Z_j/6$; at the second time point, we generate $R_2 \mid (X, Y_1, R_1 = 1, A = a) \sim \text{Bernoulli}(\pi_2(a, X, Y_1))$, where $\text{logit}(\pi_2(a, X, Y_1)) = (2a - 1)(\sum_{j=1}^{5} Z_j + 0.1Y_1)/6$, and $Y_2 \mid (X, Y_1, R_2 = 1, A = a) \sim N(\mu_2^a(X, Y_1), 1)$, where $\mu_2^a(X, Y_1) = (2 + a)(\sum_{j=1}^{5} Z_j + Y_1)/3$. Since the models are infeasible to be obtained parametrically, we apply GAM using smooth splines. We consider different modeling approximations by incorporating different covariates in the four models and employ the calibration by incorporating the first two moments of the historical information sequentially, as elaborated in S5.2 in the supplementary material.

We compare the performance of the point estimation, coverage rates of the 95% CI and mean CI lengths for the 8 proposed estimators. For variance estimation, we compute the EIF-based variance estimates for the three EIF-based estimators and the corresponding 95%
Table 1: Coverage rates and mean CI lengths in the cross-sectional setting under 8 different model specifications.

| Model specification | Coverage rate (%) | (Mean CI length, %) |
|---------------------|-------------------|---------------------|
|                     | \( \hat{\tau}_{tr} \) | \( \hat{\tau}_{tr-N} \) | \( \hat{\tau}_{tr-C} \) | \( \hat{\tau}_{ps-rp} \) | \( \hat{\tau}_{ps-rp-N} \) | \( \hat{\tau}_{ps-om} \) | \( \hat{\tau}_{ps-om-N} \) | \( \hat{\tau}_{rp-om} \) |
| yes yes yes         | 94.7              | 94.7               | 94.4              | 95.7              | 95.5              | 94.9              | 94.9              | 94.3              |
|                     | (30.9)            | (29.5)             | (28.5)            | (59.9)            | (41.8)            | (29.1)            | (29.0)            | (28.2)            |
| yes yes no          | 95.3              | 94.8               | 94.3              | 95.7              | 95.5              | 80.6              | 80.6              | 57.6              |
|                     | (41.8)            | (36.1)             | (33.7)            | (59.9)            | (41.8)            | (33.1)            | (33.1)            | (34.0)            |
| yes no yes          | 94.1              | 94.1               | 94.2              | 79.7              | 80.0              | 94.9              | 94.9              | 93.5              |
|                     | (28.8)            | (28.3)             | (28.2)            | (36.7)            | (35.3)            | (29.1)            | (29.0)            | (27.7)            |
| no yes yes          | 94.4              | 94.4               | 94.4              | 85.8              | 86.0              | 72.8              | 72.9              | 94.3              |
|                     | (29.5)            | (29.1)             | (28.5)            | (45.7)            | (40.7)            | (37.9)            | (37.9)            | (28.2)            |
| yes no no           | 83.0              | 82.9               | 93.1              | 79.7              | 80.0              | 80.6              | 80.6              | 53.4              |
|                     | (32.7)            | (32.3)             | (33.8)            | (36.7)            | (35.3)            | (33.1)            | (33.1)            | (34.1)            |
| no yes no           | 84.1              | 83.9               | 94.3              | 85.8              | 86.0              | 53.8              | 53.8              | 57.6              |
|                     | (37.4)            | (35.9)             | (33.7)            | (45.7)            | (40.7)            | (34.6)            | (34.7)            | (34.0)            |
| no no yes           | 94.6              | 94.6               | 94.2              | 56.1              | 56.1              | 72.8              | 72.9              | 93.5              |
|                     | (29.2)            | (29.2)             | (28.2)            | (38.0)            | (37.4)            | (37.9)            | (37.9)            | (27.7)            |
| no no no            | 61.3              | 61.3               | 93.1              | 56.1              | 56.1              | 53.8              | 53.8              | 53.4              |
|                     | (35.1)            | (34.9)             | (33.8)            | (38.0)            | (37.4)            | (34.7)            | (34.7)            | (34.1)            |

symmetric t bootstrap CIs with \( B = 500 \). Notice that the number of bootstrap replicates is larger now since it is used to obtain CIs. For other estimators, since there is no multiple robustness guaranteed, we use nonparametric bootstrap to obtain their variance estimates and the bootstrap percentile intervals. Figure 2 shows the point estimation results based on 1000 Monte Carlo simulations. All the EIF-based estimators are unbiased, and the one involving calibration has the smallest variation, alleviating the impact of extreme values. Other estimators suffer from different levels of bias. Table 2 supports the superiority of the EIF-based estimators in terms of coverage rates and mean CI lengths. The detailed simulation results are given in S5.2 in the supplementary material.

5 Application

We apply our proposed estimators to analyze the data from an antidepressant clinical trial prepared by Mallinckrodt et al. (2014). The data consists of the longitudinal HAMD-17 score at baseline and weeks 1, 2, 4, 6, and 8. The endpoint of interest is the change of the
HAMD-17 score at the last time point from the baseline. The data involves 100 randomly assigned participants in both the control and the treatment groups, with the fully-observed baseline and week 1 HAMD-17 score, and a categorical variable indicating the investigation sites. We select the high dropout dataset for analysis, where 39 participants in the control group and 30 participants in the treatment group drop out during the trial.

We fit GAM for the nuisance functions sequentially, with the detailed procedure explained in S6 in the supplementary material. To handle the extreme weights, calibration is applied, where the first two moments of the historical information are included. We compute the EIF-based variance estimates along with the 95% symmetric t bootstrap CIs for the three EIF-based estimators, and the nonparametric bootstrap variance estimates along with the 95% bootstrap percentile intervals for other estimators, with $B = 500$.

Table 3 presents the analysis results of the HAMD-17 dataset for the ATE under J2R. All the estimators have similar point estimates. However, we detect a relatively obvious difference in the values between $\hat{\tau}_{ps-rp}$ and $\hat{\tau}_{ps-rp-N}$, indicating the existence of the extreme weights. The weight distributions in S6 in the supplementary material validate the presence of outliers at weeks 4, 6, and 8 in the control group. Calibration stabilizes the estimation results and results in a smaller CI compared to the other two EIF-based estimators. Although
Table 3: Analysis of the HAMD-17 data for the ATE under J2R.

| Estimator   | Point estimate | 95% CI      | CI length |
|-------------|----------------|-------------|-----------|
| \( \hat{\tau}_{mr} \) | -1.93          | (-3.63, -0.24) | 3.39      |
| \( \hat{\tau}_{mr-N} \) | -1.93          | (-3.62, -0.25) | 3.37      |
| \( \hat{\tau}_{mr-C} \) | -1.71          | (-3.25, -0.16) | 3.09      |
| \( \hat{\tau}_{ps-rp} \) | -2.05          | (-4.08, -0.50) | 3.57      |
| \( \hat{\tau}_{ps-rp-N} \) | -1.61          | (-3.74, -0.07) | 3.67      |
| \( \hat{\tau}_{ps-om} \) | -1.74          | (-3.20, -0.25) | 2.95      |
| \( \hat{\tau}_{ps-om-N} \) | -1.75          | (-3.18, -0.22) | 2.96      |
| \( \hat{\tau}_{rp-pm} \) | -1.78          | (-3.18, -0.25) | 2.93      |

\( \hat{\tau}_{ps-om} \) and \( \hat{\tau}_{rp-pm} \) have similar point estimates and narrower CIs compared to the EIF-based estimators, they rely on a good approximation of their corresponding two models, which however may not be guaranteed in practice. The EIF-based estimators are preferred with a trade-off between bias and precision. All the resulting 95% CIs indicate a statistically significant treatment effect.

6 Conclusion

Evaluating the treatment effect under a hypothetical condition regarding the missing components has been receiving growing interest in both primary analysis and sensitivity analysis in longitudinal studies. We propose a potential outcomes framework to describe the pre-specified hypothetical condition under appropriate assumptions to identify the ATE under J2R as one specific CBI model. The new estimator is constructed with the help of the EIF, combining the propensity score, response probability, outcome mean, and pattern mean. It can be obtained via a recursive estimation procedure, and allows flexible modeling strategies such as semiparametric or machine learning models, with the good property of multiple robustness in that it achieves \( n^{1/2} \)-consistency and asymptotic normality even when the models converge at a slower rate such as \( n^{-1/4} \). The proposed estimators can be applied in a wide range of clinical studies including randomized trials and observational studies, and are extendable to other MNAR-related hypothetical conditions.

The model assumptions are relaxed in the established semiparametric framework. However, standard untestable assumptions under J2R are imposed for the identification of the
ATE. The assumed outcome mean for the participants who drop out under J2R produces a conservative evaluation of the treatment effect if the treatment is supposed to be superior (Liu and Pang 2016), therefore more appealing to regulatory agencies in clinical studies.

Our framework relies on a monotone missingness pattern for the longitudinal data, which however may not always be the case in reality. Sun and Tchetgen Tchetgen (2018) provides an inverse probability weighting approach to deal with the MAR data with non-monotone missing patterns. It is possible to extend our method to handle intermittent missing data using their proposed approaches. We leave it as a future research direction.

The construction of the multiply robust estimators is based on the continuous longitudinal outcomes. Possibilities exist in the extension of the proposed framework to broader types of outcomes under appropriate hypothetical conditions put forward by the regulatory agencies. For example, Yang et al. (2020) consider the δ-adjusted and control-based models to evaluate the treatment effect for the survival outcomes in sensitivity analysis; Tang (2018) extends CBI to binary and ordinal longitudinal outcomes using sequential generalized linear models. These extensions shed light on establishing new multiply robust estimators with the use of our idea.

SUPPLEMENTARY MATERIAL

The supplementary material contains the proofs and more technical details.

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Supplementary Material for “Multiply robust estimators in longitudinal studies with missing data under control-based imputation” by Liu et al.

The supplementary material contains technical details, additional simulation, and real-data application results. Section S1 provides proofs for the identification formulas provided in Theorems 1 and 5. Section S2 presents detailed derivations of the EIFs in Theorems 2 and 6. Section S3 gives additional estimators and the detailed estimation steps. Section S4 consists of the proofs regarding multiple robustness. Section S5 contains additional simulation results. Section S6 shows additional notes on the real-data application.

S1  Proof of the identification formulas

S1.1  Proof of Theorem 1

We first prove the equivalence of the three identification formulas, then prove the validity of the identification formula (a) in Theorem 1.

Denote

\[ E_{1,1} = \mathbb{E}\left[ \pi_1(1, X) \{ \mu_1(X) - \mu_0(X) \} \right]; \]
\[ E_{2,1} = \mathbb{E}\left[ \frac{A}{e(X)} \{ R_1Y_1 + (1 - R_1)\mu_1^0(X) \} - \frac{1 - A}{1 - e(X)} \{ R_1Y_1 + (1 - R_1)\mu_0^0(X) \} \right]; \]
\[ E_{3,1} = \mathbb{E}\left[ \frac{A}{e(X)} R_1Y_1 - \frac{1 - A}{1 - e(X)} \pi_1(1, X) R_1Y_1 \right]. \]

Note that \( E_{1,1} = E_{2,1} \) holds since

\[
\mathbb{E}\left[ \frac{A}{e(X)} \{ R_1Y_1 + (1 - R_1)\mu_1^0(X) \} \right] = \mathbb{E}\left[ \frac{A}{e(X)} \mathbb{E}\{ R_1Y_1 + (1 - R_1)\mu_1^0(X) \mid X, A = 1 \} \right] = \mathbb{E}\left( \frac{A}{e(X)} \mathbb{E}\{ R_1 \mid X, A = 1 \} \mu_1^1(X) + (1 - \mathbb{E}\{ R_1 \mid X, A = 1 \}) \mu_1^0(X) \right) = \mathbb{E}\left( \frac{A}{e(X)} \mathbb{E}\{ \pi_1(1, X) \mu_1^1(X) + (1 - \pi_1(1, X)) \mu_1^0(X) \} \right),
\]

And similarly,

\[
\mathbb{E}\left[ \frac{1 - A}{1 - e(X)} \{ R_1Y_1 + (1 - R_1)\mu_0^0(X) \} \right] = \mathbb{E}\left[ \frac{1 - A}{1 - e(X)} \{ R_1\mu_0^0(X) + (1 - R_1)\mu_1^0(X) \} \right].
\]
Then, we have
\[ E_1 = \{ \frac{1 - A}{1 - e(X)} \mu_1^0(X) \} \]
\[ = \mathbb{E}\{ \mu_1^0(X) \}. \]

Also note that \( E_{1,1} = E_{3,1} \) holds since
\[ E_{3,1} = \mathbb{E} \left\{ \frac{E(A \mid X)}{e(X)} E(R_1 \mid X, A = 1) \mathbb{E}(Y_1 \mid X, R_1 = 1, A = 1) \right\} \]
\[ - \mathbb{E} \left\{ \frac{E(1 - A \mid X)}{1 - e(X)} \pi_1(1, X) \right\} \mu_1^1(X) \]
\[ = \mathbb{E} \left\{ \pi_1(1, X) \mu_1^1(X) - \pi_1(1, X) \mu_1^0(X) \right\} = E_{1,1}. \]

We proceed to prove the validity of the identification formula (a) in Theorem 1. Denote \( \tau_{1,1} = \mathbb{E}[Y_1 \{1, R(1)\}] \) and \( \tau_{0,1} = \mathbb{E}[Y_1 \{0, R(0)\}] \). Note that
\[ \tau_{1,1} = \mathbb{E}[R_1(1)Y_1(1, 1) + \{1 - R_1(1)\}Y_1(1, 0)] \]
\[ = \mathbb{E} \{ \mathbb{E} \{ R_1(1) \mid X \} \mathbb{E} \{ Y_1(1, 1) \mid X, R_1(1) = 1 \} + \{1 - R_1(1)\} \mathbb{E} \{ Y_1(1, 0) \mid X, R_1(1) = 0 \} \} \]
\[ = \mathbb{E} \{ \mathbb{E} \{ R_1 \mid X, A = 1 \} \mathbb{E} \{ Y_1(1, 1) \mid X, R_1(1) = 1, A = 1 \} + \{1 - R_1\} \mathbb{E} \{ Y_1(1, 0) \mid X, R_1(1) = 0 \} \} \]
\[ \quad \text{(By A1, A3)} \]
\[ = \mathbb{E} \{ \pi_1(1, X) \mathbb{E} \{ Y_1 \mid X, R_1 = 1, A = 1 \} + \{1 - \pi_1(1, X)\} \mathbb{E} \{ Y_1 \mid X, A = 1 \} \} \]
\[ \quad \text{(By A3, A4)} \]
\[ = \mathbb{E} \{ \pi_1(1, X) \mu_1^1(X) + \{1 - \pi_1(1, X)\} \mu_1^0(X) \} \]
\[ \quad \text{(By A2).} \]

and
\[ \tau_{0,1} = \mathbb{E}[R_1(0)Y_1(0, 1) + \{1 - R_1(0)\}Y_1(0, 0)] \]
\[ = \mathbb{E} \{ \mathbb{E} \{ R_1(0) \mid X \} \mathbb{E} \{ Y_1(0, 1) \mid X, R_1(0) = 1 \} + \{1 - R_1(0)\} \mathbb{E} \{ Y_1(0, 0) \mid X, R_1(0) = 0 \} \} \]
\[ = \mathbb{E} \left[ \mathbb{E} \{ R_1 \mid X, A = 0 \} \mathbb{E} \{ Y_1(0, 1) \mid X, R_1(0) = 1, A = 0 \} \right] \]
\[ + \mathbb{E} \left[ \{1 - R_1\} \mathbb{E} \{ Y_1(0, 0) \mid X, R_1(0) = 0, A = 0 \} \right] \]
\[ \quad \text{(By A1, A3)} \]
\[ = \mathbb{E} \left[ \pi_1(0, X) \mathbb{E} \{ Y_1 \mid A = 0, R_1 = 1, X \} + \{1 - \pi_1(1, X)\} \mu_1^0(X) \right] \]
\[ = \mathbb{E} \left[ \pi_1(0, X) \mu_1^1(X) + \{1 - \pi_1(1, X)\} \mu_1^0(X) \right] \]
\[ = \mathbb{E} \{ \mu_1^0(X) \}. \]

Combine the two parts, we have
\[ \tau_{1,1}^{2R} = \tau_{1,1} - \tau_{0,1} = \mathbb{E} \{ \pi_1(1, X) \mu_1^1(X) - \pi_1(1, X) \mu_1^0(X) \} = E_{1,1}. \]
S1.2 Proof of Theorem 5

We first prove the equivalence of the three identification formulas, then prove the validity of the identification formula (a) in Theorem 5.

Denote
\[ E_{1,t} = \mathbb{E} \left[ \pi_1(1, H_0) \left\{ \sum_{s=1}^{t} g_{s+1}^l(H_0) - \mu_t^0(H_0) \right\} \right]; \]
\[ E_{2,t} = \mathbb{E} \left[ \frac{2A - 1}{e(H_0)^A \{1 - e(H_0)\}^{1-A}} \left\{ R_t Y_t + \sum_{s=1}^{t} R_{s-1}(1 - R_s) \mu_t^0(H_{s-1}) \right\} \right]; \]
\[ E_{3,t} = \mathbb{E} \left( \frac{A}{e(H_0)} R_t Y_t + \frac{1 - A}{1 - e(H_0)} \left[ \sum_{s=1}^{t} \bar{\pi}_{s-1}(0, H_{s-2}) \{1 - \pi_s(1, H_{s-1})\} \delta(H_{s-1}) - 1 \right] \frac{R_t Y_t}{\bar{\pi}_t(0, H_{t-1})} \right). \]

To simplify the proof, we first introduce relevant lemmas.

**Lemma S1** Under MAR, the group mean can be identified using the sequential outcome means, i.e., \( \mathbb{E} \{ Y_t(0, D(0)) \} = \mathbb{E} \{ \mu_0^0(H_0) \}. \)

**Proof:** Similar to the notations in the main text, we define the pattern mean in the control group as \( g_{s+1}^l(H_{l-1}) = \mathbb{E} \{ \pi_{t+1}(0, H_l) g_{s+1}^0(H_l) \} \) for \( l = 1, \cdots, s-1 \) with \( g_{s+1}^0(H_{s-1}) = \mathbb{E} \{ \{1 - \pi_{s+1}(0, H_s)\} \mu_t^0(H_s) \} \) if we let \( \pi_{t+1}(0, H_t) = 0 \).

Based on the PMM framework, we express the potential outcome \( Y_t(0, D(0)) \) based on its potential dropout pattern as \( Y_t(0, D(0)) = \sum_{s=1}^{t+1} I \{ D(0) = s \} Y_t(0, s) \) and compute the expectation. For any \( s \in \{ 2, \cdots, t+1 \} \), \( \mathbb{E} \{ I \{ D(0) = s \} Y_t(0, s) \} \) is calculated as
\[ = \mathbb{E} \{ R_1(0) \cdots R_{s-1}(0) \{1 - R_s(0)\} Y_t(0, s) \} \]  
(By the definition of \( D \))
\[ = \mathbb{E} \{ \mathbb{E} (R_1(0) | H_0) \mathbb{E} [R_2(0) \cdots R_{s-1}(0) \{1 - R_s(0)\} Y_t(0, s) | H_0, R_1(0) = 1] \} \]
\[ = \mathbb{E} \left\{ \mathbb{E} (R_1(0) | H_0) \mathbb{E} \left[ \mathbb{E} \{ R_2(0) | H_1, R_1(0) = 1 \} \mathbb{E} \{ R_3(0) \cdots R_{s-1}(0) \{1 - R_s(0)\} Y_t(0, s) | H_1, R_2(0) = 1 \} \right] \right\} | H_0, R_1(0) = 1 \}
\[ = \cdots \text{(keep using the iterated expectation until the condition is } (H_{s-1}, R_{s-1}(0) = 1)) \]
\[ = \mathbb{E} \left\{ \mathbb{E} (R_1(0) | H_0) \mathbb{E} \left[ \cdots \mathbb{E} \{ \mathbb{E} (R_{s-1}(0) | H_{s-2}, R_{s-2}(0) = 1) \mathbb{E} \{ \{1 - R_s(0)\} Y_t(0, s) | H_{s-1}, R_{s-1}(0) = 1\} \right] \right\} | H_{s-2}, R_{s-2}(0) = 1 | \cdots | H_0, R_1(0) = 1 \}
\[ = \mathbb{E} \left\{ \mathbb{E} (R_1(0) | H_0) \mathbb{E} \left[ \mathbb{E} \{ R_2(0) | H_1, R_1(0) = 1 \} \cdots \mathbb{E} \{ R_{s-1}(0) | H_{s-2}, R_{s-2}(0) = 1 \} \right] \right\} \]

S3
The propensity score ratio \( \hat{f} = \frac{E[Y_t(0, s) | H_{s-1}, R_{s-1}(0) = 1]}{E[Y_t(0, s) | H_{s-1}, R_{s-1}(0) = 1]} \) holds since

\[
E \left[ \frac{1 - R_s(0)}{R_s(0)} | H_{s-1}, R_{s-1}(0) = 1 \right] \frac{1}{E \left[ Y_t(0, s) | H_{s-1}, R_{s-1}(0) = 1 \right] | H_{s-2}, R_{s-1}(0) = 1}
\]

\[
\cdots | H_0, R_1(0) = 1 \right] \right] \quad \text{(By A5, } R_s(0) \perp Y_t(0, s) \mid (H_{s-1}, R_{s-1}(0) = 1)\text{)}
\]

\[
= \mathbb{E} \left\{ \mathbb{E} (R_1 | H_0, A = 0) \mathbb{E} \left( \cdots \mathbb{E} \left[ \mathbb{E} (R_{s-1} | H_{s-2}, R_{s-2} = 1, A = 0) \mathbb{E} (1 - R_s | H_{s-1}, R_{s-1} = 1, A = 0) \right] \cdots | H_0, R_1 = 1, A = 0 \right) \right\}
\]

\[
= \mathbb{E} \left[ \pi_1(0, H_0) \mathbb{E} \left( \cdots \mathbb{E} \left( \pi_{s-1}(0, H_{s-2}) \mathbb{E} \left[ \{1 - \pi_s(0, H_{s-1})\} \mu^0_t(H_{s-1}) | H_{s-2}, R_{s-1} = 1, A = 0 \right] \cdots \mathbb{E} (1 - \pi_s(0, H_{s-1})\} \mu^0_t(H_{s-1}) | H_{s-2}, R_{s-1} = 1, A = 0 \right) \right] \cdots | H_0, R_1 = 1, A = 0 \right) \right]\]

\[
= \mathbb{E} \left[ \pi_1(0, H_0) \mathbb{E} \left( \cdots \mathbb{E} \left( \pi_{s-1}(0, H_{s-2}) g^0_s(H_{s-2}) | H_{s-3}, R_{s-2} = 1, A = 0 \right) \cdots | H_0, R_1 = 1, A = 0 \right) \right] \mid H_{s-3}, R_{s-2} = 1, A = 0
\]

\[
= \mathbb{E} \left[ \pi_1(0, H_0) g^0_s(H_0) \right] \quad \text{(By the definition of the pattern mean).}
\]

When \( s = 1 \), using the same calculation technique, we have \( \mathbb{E} \left[ \mathbb{I} \{D(0) = s\} Y_t(0, s) \right] = \mathbb{E} \left[ \{1 - \pi_1(0, H_0)\} \mu^0_t(H_0) \right] \). Note that under MAR, \( \sum_{s=1}^{t} \pi_1(0, H_0) g^0_s(H_0) + \{1 - \pi_1(0, H_0)\} \mu^0_t(H_0) = \mu^0_t(H_0) \), which completes the proof.

**Lemma S2** The propensity score ratio \( \delta(H_s) \) have the following expression:

\[
\delta(H_s) = \frac{\pi_s(1, H_{s-1})}{\pi_s(0, H_{s-1})} \prod_{j=1}^{s} \frac{f(Y_j | H_{j-1}, R_j = 1, A = 1)}{f(Y_j | H_{j-1}, R_j = 1, A = 0)}
\]

**Proof:** For any \( j \in \{1, \cdots, s\} \), we have

\[
\frac{f(Y_j | H_{j-1}, R_j = 1, A = 1)}{f(Y_j | H_{j-1}, R_j = 1, A = 0)} = \frac{f(Y_j, A = 1 | H_{j-1}, R_j = 1)/f(A = 1 | H_{j-1}, R_j = 1)}{f(Y_j, A = 0 | H_{j-1}, R_j = 1)/f(A = 0 | H_{j-1}, R_j = 1)}
\]

(by conditional probability)

\[
= \frac{f(A = 1 | H_{j-1}, R_j = 1)f(Y_j | H_{j-1}, R_j = 1)/f(A = 1 | H_{j-1}, R_j = 1)}{f(A = 0 | H_{j-1}, R_j = 1)f(Y_j | H_{j-1}, R_j = 1)/f(A = 0 | H_{j-1}, R_j = 1)}
\]

\[
= \frac{f(A = 1 | H_{j-1}, R_j = 1)/f(A = 1 | H_{j-1}, R_j = 1)}{f(A = 0 | H_{j-1}, R_j = 1)/f(A = 0 | H_{j-1}, R_j = 1)}
\]

\[
= \frac{f(A = 1 | H_{j-1}, R_j = 1)}{f(A = 0 | H_{j-1}, R_j = 1)}
\]

\[
= \frac{f(A = 1 | H_{j-1}, R_j = 1)}{f(A = 1 | H_{j-1}, R_j = 1)} \pi_j(0, H_{j-1})
\]

\[
= \frac{f(A = 0 | H_{j-1}, R_j = 1)}{f(A = 0 | H_{j-1}, R_j = 1)} \frac{f(A = 1 | H_{j-1}, R_j = 1)}{f(A = 1 | H_{j-1}, R_j = 1)} \pi_j(1, H_{j-1})
\]

The last equality holds since

\[
\frac{f(A = 0 | H_{j-1}, R_j = 1)}{f(A = 1 | H_{j-1}, R_j = 1)} = \frac{f(A = 0, R_j = 1 | H_{j-1}, R_j = 1)}{f(R_j = 1 | H_{j-1}, R_j = 1)}
\]

\[
= \frac{f(A = 1, R_j = 1 | H_{j-1}, R_j = 1)}{f(R_j = 1 | H_{j-1}, R_j = 1)}
\]

\[
= \frac{f(A = 0, R_j = 1 | H_{j-1}, R_j = 1)}{f(A = 1, R_j = 1 | H_{j-1}, R_j = 1)}
\]

\[
= \frac{f(R_j = 1 | H_{j-1}, R_j = 1, A = 0)}{f(R_j = 1 | H_{j-1}, R_j = 1, A = 1)}
\]

\[
= \frac{f(R_j = 1 | H_{j-1}, R_j = 1, A = 1)}{f(R_j = 1 | H_{j-1}, R_j = 1, A = 1)}
\]

\[
= \frac{f(A = 1 | H_{j-1}, R_j = 1, A = 1)}{f(A = 1 | H_{j-1}, R_j = 1, A = 1)}
\]

S4
we have

Then, we have components:

\[ E = E_1 \]

Taking the cumulative product for \( j \) from 1 to \( s \), we have

\[
\prod_{j=1}^{s} \frac{f(Y_j \mid H_{j-1}, R_j = 1, A = 1)}{f(Y_j \mid H_{j-1}, R_j = 1, A = 0)} = \prod_{j=1}^{s} \frac{f(A = 1 \mid H_j, R_j = 1)}{f(A = 0 \mid H_j, R_j = 1)} \frac{f(A = 0 \mid H_{j-1}, R_{j-1} = 1)}{f(A = 1 \mid H_{j-1}, R_{j-1} = 1)} \pi_j(0, H_{j-1}) \pi_j(1, H_{j-1})
\]

\[
= \frac{\bar{\pi}_s(0, H_{s-1})}{\bar{\pi}_s(1, H_{s-1})} \frac{f(A = 1 \mid H_s, R_s = 1)}{f(A = 1 \mid H_0)} / f(A = 0 \mid H_s, R_s = 1) / f(A = 0 \mid H_0)
\]

which completes the proof. \(\square\)

We proceed to prove for the equivalence of the three identification formulas. Note that

\[ E = E_1 = E_2 = E_3 \]

\[ E_1 = E_2, t \]

holds since

\[ \mathbb{E} \left[ A \left\{ R_t Y_t + \sum_{s=1}^{t} R_{s-1} (1 - R_s) \mu_t^0(H_{s-1}) \right\} / e(H_0) \right] = \mathbb{E} \left[ \frac{A}{e(H_0)} \left\{ R_t Y_t + \sum_{s=1}^{t} R_{s-1} (1 - R_s) \mu_t^0(H_{s-1}) \mid A = 1 \right\} \right] = \mathbb{E} \left[ \frac{A}{e(H_0)} \left\{ \sum_{s=1}^{t} \pi_1(1, H_0) g_{s+1}^1(H_0) + \{1 - \pi_1(1, H_0)\} \mu_t^0(H_0) \right\} \right] \]

(follow the proof in Lemma \[s1\])

\[
= \mathbb{E} \left[ \sum_{s=1}^{t} \pi_1(1, H_0) g_{s+1}^1(H_0) + \{1 - \pi_1(1, H_0)\} \mu_t^0(H_0) \right].
\]

Similarly, follow the proof in Lemma \[s1\]

\[
\mathbb{E} \left[ \frac{1 - A}{1 - e(H_0)} \left\{ R_t Y_t + \sum_{s=1}^{t} R_{s-1} (1 - R_s) \mu_t^0(H_{s-1}) \right\} \right] = \mathbb{E} \left\{ \frac{1 - A}{1 - e(H_0)} \mu_t^0(H_0) \right\} = \mathbb{E} \left\{ \mu_t^0(H_0) \right\}.
\]

Then, we have

\[ E_2, t = E_3, t \]

holds since for the first term in \( E_{3,t} \),

\[ \mathbb{E} \left\{ AR_t Y_t / e(H_0) \right\} = \mathbb{E} \left\{ \pi_1(1, H_0) g^1_{t+1}(H_0) \right\}. \]

We focus on the second term and consider separate it into two components:

\[
= \mathbb{E} \left[ \frac{1 - A}{1 - e(H_0)} \left\{ \sum_{s=1}^{t} \bar{\pi}_s(0, H_{s-2}) \{1 - \pi_s(1, H_{s-1})\} \delta(H_{s-1}) \right\} \frac{R_t Y_t}{\bar{\pi}_t(0, H_{t-1})} \right] - \mathbb{E} \left\{ \frac{1 - A}{1 - e(H_0)} \frac{R_t Y_t}{\bar{\pi}_t(0, H_{t-1})} \right\}.
\]

The second component can be easily obtained using the similar strategy in Lemma \[s1\] which results in \( \mathbb{E} \{ \mu_t^0(H_0) \} \). For the first components, apply Lemma \[s2\] for \( s \in \{2, \cdots, t\} \),

we have

\[
\mathbb{E} \left[ \frac{1 - A}{1 - e(H_0)} \bar{\pi}_s(0, H_{s-2}) \{1 - \pi_s(1, H_{s-1})\} \delta(H_{s-1}) \frac{R_t Y_t}{\bar{\pi}_t(0, H_{t-1})} \right]
\]
For components, we have formula (a) in Theorem 1, 1−πs(1, Hs−1)δ(0, Ht−1) 1−πt(0, Ht−1) E(Yt | Ht−1, Rt = 1, A = 0)

= ... (keep using the iterated expectation, conditional on Ht−2, · · · , Hs−1 in backward order)

= -E 1−A 1−e(H0)πs−1(0, Hs−2). (1−πs(1, Hs−1))δ(Hs−1) 1−πs−1(0, Hs−2) R0−t π(R0−t, Hs−1) E(Yt−1 | Ht−1, Rt−1 = 1, A = 1)

= -E 1−A 1−e(H0)πs−1(0, Hs−2) R1−s s 1 j=1 f(Yj | Hj−1, Rj = 1, A = 1) E(Yj | Hj−1, Rj = 1, A = 0)

= -E 1−A 1−e(H0)πs−1(0, Hs−2) R1−s s 1 j=1 f(Yj | Hj−1, Rj = 1, A = 1) E(Yj | Hj−1, Rj = 1, A = 0)

= ... (keep using the iterated expectation, conditional on Hs−4, · · · , H0 in backward order)

= -E {π1(1, H0)g1 s(H0)}.

For s = 1, use the same technique and can get E [1−π1(1, H0) µt(0(H0)). Combine those components, we have

E3,t = E π1(1, H0)g1 t+1(H0) + 1−π1(1, H0) µt(0(H0)) = E1,t.

We proceed to prove the validity of the identification formula (a) in Theorem 5. Denote τ1,t = E[Yt{1, D(1)}] and τ0,t = E[Yt{0, D(0)}]. For the first part of the identification formula (a) in Theorem 1

τ1,t = -E t+1 t s=1 11(D(1) = s) Yt−1(1, s)

= -E t t 11(D(1) = s) Yt(1, s) + 11(D(1) = t + 1) Yt(1, t + 1)

= t s=1 E[R1(1) ... Rs−1(1) {1 − Rs(1)} Yt(1, s)] + E[R1(1) ... Rt(1)Yt(1, t + 1)] (By the definition of D).

For s ∈ {2, · · · , t}, E[R1(1) ... Rs−1(1) {1 − Rs(1)} Yt(1, s)] can be computed by iterative
expectations as

\[ \mathbb{E} (\mathbb{E} \{ R_1(1) \mid H_0 \} \mathbb{E} \{ R_2(1) \cdots R_{s-1}(1) \{ 1 - R_s(1) \} Y_t(1, s) \mid H_0, R_1(1) = 1 \}) \]

\[ = \mathbb{E} \left\{ \mathbb{E} \{ R_1(1) \mid H_0 \} \mathbb{E} \left( \mathbb{E} \{ R_2(1) \mid H_1, R_1(1) = 1 \right) \right. \]

\[ \left. \mathbb{E} \{ R_3(1) \cdots R_{s-1}(1) \{ 1 - R_s(1) \} Y_t(1, s) \mid H_1, R_1(1) = 1 \} \mid H_0, R_1(1) = 1 \right\} \]

\[ = \cdots \text{ (keep using the iterated expectation, use similar steps in the proof of Lemma S1)} \]

\[ = \mathbb{E} \left\{ \mathbb{E} \{ R_1(1) \mid H_0 \} \mathbb{E} \left( \mathbb{E} \{ R_2(1) \mid H_1, R_1(1) = 1 \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-1}, R_{s-1}(1) = 1 \} \right) \right. \]

\[ \left. \mathbb{E} \{ Y_t(1, s) \mid H_{s-1}, D(1) = s \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-2}, R_{s-1}(1) = 1 \} \cdots \mid H_0, R_1(1) = 1 \right\} \]

\[ = \mathbb{E} \left\{ \mathbb{E} \{ R_1(1) \mid H_0 \} \mathbb{E} \left( \mathbb{E} \{ R_2(1) \mid H_1, R_1(1) = 1 \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-1}, R_{s-1}(1) = 1 \} \right) \right. \]

\[ \left. \mathbb{E} \{ Y_t(1, s) \mid H_{s-1}, D(1) = s \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-2}, R_{s-1}(1) = 1 \} \cdots \mid H_0, R_1(1) = 1 \right\} \]

\[ = \mathbb{E} \left\{ \mathbb{E} \{ R_1(1) \mid H_0 \} \mathbb{E} \left( \mathbb{E} \{ R_2(1) \mid H_1, R_1(1) = 1 \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-1}, R_{s-1}(1) = 1 \} \right) \right. \]

\[ \left. \mathbb{E} \{ Y_t(1, s) \mid H_{s-1}, D(1) = s \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-2}, R_{s-1}(1) = 1 \} \cdots \mid H_0, R_1(1) = 1 \right\} \]

\[ \text{(By A8)} \]

\[ = \mathbb{E} \left\{ \mathbb{E} \{ R_1(1) \mid H_0, A = 1 \} \mathbb{E} \left( \mathbb{E} \{ R_2(1) \mid H_1, R_1(1) = 1, A = 1 \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-1}, R_{s-1}(1) = 1, A = 1 \} \right) \right. \]

\[ \left. \mathbb{E} \{ Y_t(1, s) \mid H_{s-1}, D(1) = s \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-2}, R_{s-1}(1) = 1, A = 1 \} \cdots \mid H_0, R_1(1) = 1, A = 1 \right\} \]

\[ \text{(By A5)} \]

\[ = \mathbb{E} \{ \pi_1(1, H_0) \mathbb{E} \{ \pi_2(1, H_0) \cdots \mathbb{E} \{ 1 - \pi_s(1, H_{s-1}) \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-1}, R_{s-1}(1) = 1, A = 1 \} \cdots \mid H_0, R_1(1) = 1, A = 1 \} \}

\[ \text{(By A7)} \]

\[ = \mathbb{E} \{ \pi_1(1, H_0) g_s^1(1) \} \] (By the definition of the pattern mean).

Similarly, \( \mathbb{E} \{ R_1(1) \cdots R_t(1) Y_t(1, t + 1) \} \)

\[ = \mathbb{E} \left\{ \mathbb{E} \{ R_1(1) \mid H_0 \} \mathbb{E} \{ R_2(1) \cdots R_t(1) Y_t(1, t + 1) \mid H_0, R_1(1) = 1 \} \right\} \]

\[ = \mathbb{E} \left\{ \mathbb{E} \{ R_1(1) \mid H_0 \} \mathbb{E} \left( \mathbb{E} \{ R_2(1) \mid H_1, R_1(1) = 1 \right) \right. \]

\[ \left. \mathbb{E} \{ R_3(1) \cdots R_t(1) Y_t(1, t + 1) \mid H_1, R_2(1) = 1 \} \mid H_0, R_1(1) = 1 \right\} \]

\[ = \cdots \text{ (keep using the iterated expectation, use similar steps in the proof of Lemma S1)} \]

\[ = \mathbb{E} \left\{ \mathbb{E} \{ R_1(1) \mid H_0 \} \mathbb{E} \left( \mathbb{E} \{ R_2(1) \mid H_1, R_1(1) = 1 \} \mathbb{E} \{ R_3(1) \cdots R_t(1) Y_t(1, t + 1) \mid H_1, R_2(1) = 1 \} \right. \right. \]

\[ \left. \left. \mathbb{E} \{ Y_t(1, t + 1) \mid H_{t-1}, D(1) = t + 1 \} \mathbb{E} \{ 1 - R_s(1) \mid H_{s-2}, R_{s-1}(1) = 1 \} \cdots \mid H_0, R_1(1) = 1 \right\} \right. \]

\[ \text{(S7)} \]
When $s = 1$, we have $\mathbb{E} \{\{1 - R_s(1)\} Y_t(1, 1)\}$ as

$$
\mathbb{E} \{\{1 - R_1(1)\} \mathbb{E} \{Y_t(1, 1) | H_0, D(1) = 0\}\} = \mathbb{E} \{\{1 - R_1(1) | H_0, A = 1\} \mu^0_t(H_0)\}
$$

$$
= \mathbb{E} \{\{1 - \pi_1(1, H_0)\} \mu^0_t(H_0)\}.
$$

Therefore, $\tau_{1,t} = \mathbb{E} \{\pi_1(1, H_0) \sum_{s=1}^{t} g_{s+1}(H_0) + \{1 - \pi_1(1, H_0)\} \mu^0_t(H_0)\}$. For the second part, by Lemma S1 we know that $\tau_{0,t} = \mathbb{E} \{\mu_t^0(H_0)\}$. Combine the two parts, we have

$$
\tau_{12R} = \tau_{1,t} - \tau_{0,t} = \mathbb{E} \left[\pi_1(1, H_0) \sum_{s=1}^{t} g_{s+1}(H_0) + \{1 - \pi_1(1, H_0)\} \mu^0_t(H_0) - \mu^0_t(H_0)\right] = E_{1,t}.
$$

S1.3 Interpretations of Theorem 5

We give some intuition of the identification formulas in the longitudinal setting. Theorem 5 (a) describes the treatment effect in terms of the response probability and pattern mean. Under J2R, if the individual in the treatment group is not fully observed, we would expect its missing outcome will follow the same outcome model as the control group with the same missing pattern given the observed data. The treatment group mean is then expressed as the weighted sum over the missing patterns as $\mathbb{E} \left[\pi_1(1, H_0) \sum_{s=1}^{t} g_{s+1}(H_0) + \{1 - \pi_1(1, H_0)\} \mu^0_t(H_0)\right]$ under the PMM framework. For the control group, the group mean is $\mathbb{E} \{\mu^0_t(H_0)\}$ under MAR.

Theorem 5 (b) describes the treatment effect as the difference in means between the treatment and control groups over the missing patterns, in terms of the propensity score and outcome mean. Similar to the cross-sectional setting, after adjusting for the covariate balance with the use of propensity score weights, the outcomes at the last time point
are combinations of the observed outcomes and the conditional outcome means given the observed data, distinguished by distinct dropout patterns.

Theorem 5 (c) describes the treatment effect over the missing patterns in terms of the propensity score and response probability. The first term \( AR_t Y_t/e(H_0) \) characterizes the participants who stay in the assigned treatment throughout the entire study period identified by \( R_t \) after the adjustment for the group difference by \( A/e(H_0) \), which is parallel to \( \mathbb{E} \{ \pi_1(1, H_0) g_{l+1}(H_0) \} \) in Theorem 3 (a). The transformed outcome \( (1 - A) R_t Y_t / \{ [1 - e(H_0)] \bar{\pi}_t(0, H_{t-1}) \} \) measures the outcome mean \( \mu_t^0(H_0) \) given the baseline covariates, for the participants who complete the trial in the control group. Notice that

\[
\delta(H_{s-1}) = \frac{\bar{\pi}_{s-1}(1, H_{s-2}) \prod_{l=1}^{s-1} f(Y_l | H_{l-1}, R_l = 1, A = 1)}{\bar{\pi}_{s-1}(0, H_{s-2}) \prod_{l=1}^{s-1} f(Y_l | H_{l-1}, R_l = 1, A = 0)}
\]

is the cumulative product of the density ratios of the current outcome given the observed historical information, multiplied by a ratio of the cumulative response probability in the treatment and control group. Therefore, with the transformed outcome involved, the term \( \bar{\pi}_{s-1}(0, H_{s-2}) \{1 - \pi_s(1, H_{s-1})\} \delta(H_{s-1}) \) implicitly shifts the participants with the same observed information, who drop out at time \( s \) in the treatment group, to the control group, which matches \( \mathbb{E} \{ \pi_1(1, H_0) g_s^1(H_0) \} \) when \( s = 2, \cdots, t \) and \( \mathbb{E} \{ [1 - \pi_1(1, H_0)] \mu_t^0(H_0) \} \) when \( s = 1 \) after marginalizing the history. Therefore, the second term in the identification formula is equivalent to \( \mathbb{E} \left[ \bar{\pi}_1(1, H_0) \left\{ \sum_{s=1}^{t-1} g_{s+1}^1(H_0) - \mu_t^0(H_0) \right\} \right] \) in Theorem 5 (a).

**S2 Proof of the EIFs**

Let \( V = (X, A, R_1 Y_1, R_1, \cdots, R_t Y_t, R_t) \) with \( R_0 = 1 \) be the vector of all observed variables with the likelihood factorized as

\[
f(V) = f(X) f(A | X) \prod_{s=1}^{t} \left\{ f(Y_s | H_{s-1}, R_s = 1, A) f(R_s | H_{s-1}, R_{s-1} = 1, A) \right\}
\]

We will use the semiparametric theory in \textit{Bickel et al.} (1993) to derive the EIF of \( \tau_t^{\text{AR}} \).

To derive the EIFs, we consider a one-dimensional parametric submodel, \( f_\theta(V) \), which contains the true model \( f(V) \) at \( \theta = 0 \), i.e., \( f_\theta(V) |_{\theta=0} = f(V) \), where \( \theta \) consists of the nuisance model parameters. We use \( \theta \) in the subscript to denote the quantity evaluated with respect to the submodel, e.g., \( \mu_t^{a,\theta} \) is the value of \( \mu_t^a \) with respect to the submodel. We use dot to denote the partial derivative with respect to \( \theta \), e.g., \( \dot{\mu}_t^{a,\theta} = \partial \mu_t^a / \partial \theta \), and use \( s(\cdot) \)
to denote the score function. From formula (S1), the score function of the observed data can be decomposed as
\[ s_θ(V) = s_θ(X) + s_θ(A \mid X) + \sum_{s=1}^{t} \{ s_θ(Y_s \mid H_{s-1}, R_s = 1, A) + s_θ(R_s \mid H_{s-1}, R_{s-1} = 1, A) \}, \]
where \( s_θ(X) = \partial \log f_θ(X)/\partial θ \), \( s_θ(A \mid X) = \partial \log \mathbb{P}_θ(A \mid X)/\partial θ \), \( s_θ(Y_s \mid H_{s-1}, R_s = 1, A) = \partial \log f_θ(Y_s \mid H_{s-1}, R_s = 1, A)/\partial θ \), and \( s_θ(R_s \mid H_{s-1}, R_{s-1} = 1, A) = \partial \log \mathbb{P}_θ(R_s \mid H_{s-1}, R_{s-1} = 1, A)/\partial θ \) are the score functions corresponding to the \((2t + 2)\) components of the likelihood. Because \( f_θ(V)|_{θ=0} = f(V) \), we can simplify \( s_θ(\cdot)|_{θ=0} \) as \( s(\cdot) \).

From the semiparametric theory, the tangent space
\[ Λ = B_1 \oplus B_2 \oplus B_{3,1} \oplus B_{4,1} \oplus \cdots \oplus B_{3,t} \oplus B_{4,t} \]
is the direct sum of
\[
B_1 = \{ u(X) : E\{u(X)\} = 0 \}, \\
B_2 = \{ u(A, X) : E\{u(A, X) \mid X\} = 0 \}, \\
B_{3,s} = \{ u(H_s, A) : E\{u(H_s, A) \mid A, H_{s-1}\} = 0 \}, \\
B_{4,s} = \{ u(R_s, A, H_{s-1}) : E\{u(R_s, A, H_{s-1}) \mid A, H_{s-1}\} = 0 \},
\]
for \( s = 1, \cdots, t \), where \( B_1, B_2, B_{3,s} \) and \( B_{4,s} \) are orthogonal to each other, and \( u(\cdot) \) is some functions. The EIF for \( \tau_{t}^{JR} \), denoted by \( \varphi_{t}^{JR}(V; \mathbb{P}) \in Λ \), must satisfy
\[ \tau_{t,θ}^{JR}|_{θ=0} = E\{\varphi_{t}^{JR}(V; \mathbb{P})s(V)\}. \]

We will derive the EIFs in both cross-sectional and longitudinal settings. To simplify the proof, we first provide some lemmas with their proofs.

**Lemma S3** For any function \( u(V) \) that does not depend on \( θ \), \( \partial E_θ \{u(V)\}/\partial θ|_{θ=0} = E \{u(V)s(V)\} \).

**Proof:** By the definition
\[
\frac{\partial E_θ \{u(V)\}}{\partial θ}|_{θ=0} = \frac{\partial}{\partial θ} \int u(V)f_θ(V)dν(V)|_{θ=0} \\
= \int u(V)\frac{\partial}{\partial θ} \log f_θ(V)|_{θ=0}f(V)dν(V) \\
= E \{u(V)s(V)\}.
\]

□

S10
Lemma S4 For $s = 1, \ldots, t$, we have

$$\hat{\pi}_{s, \theta}(1, H_{s-1}) \big|_{\theta=0} = \mathbb{E} \left[ \frac{A}{e(X) \pi_{s-1}(1, H_{s-2})} \{ R_s - \pi_s(1, H_{s-1}) \} s(V) \mid H_{s-1} \right],$$

$$\hat{\pi}_{s, \theta}(0, H_{s-1}) \big|_{\theta=0} = \mathbb{E} \left[ \frac{1 - A}{1 - e(X) \pi_{s-1}(0, H_{s-2})} \{ R_s - \pi_s(0, H_{s-1}) \} s(V) \mid H_{s-1} \right].$$

**Proof:** Note that

$$\hat{\pi}_{s, \theta}(1, H_{s-1}) \big|_{\theta=0} = \frac{\partial}{\partial \theta} \mathbb{E}_{\theta} (R_s \mid H_{s-1}, R_{s-1} = 1, A = 1) \big|_{\theta=0}$$

$$= \mathbb{E} \{ R_s s(R_s \mid H_{s-1}, R_{s-1} = 1, A = 1) \mid H_{s-1}, R_{s-1} = 1, A = 1 \} \text{ (by Lemma S3)}$$

$$= \mathbb{E} \{ \{ R_s - \pi_s(1, H_{s-1}) \} s(R_s \mid H_{s-1}, R_{s-1} = 1, A = 1) \mid H_{s-1}, R_{s-1} = 1, A = 1 \}$$

$$= \mathbb{E} \left[ \frac{A}{e(X) \pi_{s-1}(1, H_{s-2})} \{ R_s - \pi_s(1, H_{s-1}) \} s(R_s \mid H_{s-1}, R_{s-1}, A) \mid H_{s-1} \right] \text{ (by Bayes’ rule)}$$

$$= \mathbb{E} \left[ \frac{A}{e(X) \pi_{s-1}(1, H_{s-2})} \{ R_s - \pi_s(1, H_{s-1}) \} s(V) \mid H_{s-1} \right],$$

where the last equality holds since $B_{3,s'}$, $B_{4,s'}$ for $s' > s$ are orthogonal to the spaces $B_1, B_2, B_{3,1}, B_{4,1}, \ldots, B_{3,s}, B_{4,s}$. Similarly, we can prove the result for $\hat{\pi}_{s, \theta}(0, H_{s-1}) \big|_{\theta=0}$. □

Lemma S5 For $s = 1, \ldots, t$, we have

$$\bar{\mu}_{t, \theta}^1(H_{t-1}) \big|_{\theta=0} = \mathbb{E} \left[ \frac{A}{e(X) \bar{\pi}_t(1, H_{t-1})} \{ Y_t - \mu_t(1)(H_{t-1}) \} s(V) \mid H_{t-1} \right],$$

$$\bar{\mu}_{t, \theta}^0(H_1) \big|_{\theta=0} = \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \sum_{k=s}^{t} \frac{R_k}{\pi_k(0, H_{k-1})} \{ \mu_t^0(H_k) - \mu_t^0(H_{k-1}) \} s(V) \mid H_{s-1} \right]. \quad \text{(S2)}$$

**Proof:** Note that

$$\bar{\mu}_{t, \theta}^1(H_{t-1}) \big|_{\theta=0} = \frac{\partial}{\partial \theta} \mathbb{E}_{\theta} (Y_t \mid H_{t-1}, R_t = 1, A = 1) \big|_{\theta=0}$$

$$= \mathbb{E} \{ Y_t s(Y_t \mid H_{t-1}, R_t = 1, A = 1) \mid H_{t-1}, R_t = 1, A = 1 \} \text{ (by Lemma S3)}$$

$$= \mathbb{E} \left\{ \frac{A}{e(X) \bar{\pi}_t(1, H_{t-1})} Y_t s(Y_t \mid H_{t-1}, R_t = 1, A = 1) \mid H_{t-1} \right\} \text{ (by Bayes’ rule)}$$

$$= \mathbb{E} \left[ \frac{A}{e(X) \bar{\pi}_t(1, H_{t-1})} \{ Y_t - \mu_t^1(H_{t-1}) \} s(Y_t \mid H_{t-1}, R_t, A) \mid H_{t-1} \right]$$

$$= \mathbb{E} \left[ \frac{A}{e(X) \bar{\pi}_t(1, H_{t-1})} \{ Y_t - \mu_t^1(H_{t-1}) \} s(V) \mid H_{t-1} \right].$$

The last equality holds by the orthogonality of the spaces.

For the condition involves $A = 0$, we prove it by induction in backward order since it involves iteratively taking the derivative with respect to $\theta$. 

S11
For $s = t$, we can obtain $\hat{\mu}_{t,\theta}^0(H_{t-1})|_{\theta=0}$ using the similar procedure as the one involves $A = 1$, and get

$$\hat{\mu}_{t,\theta}^0(H_{t-1})|_{\theta=0} = \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \frac{R_t}{\pi_t(0, H_{t-1})} \left\{ \mu_t^0(H_t) - \mu_t^0(H_{t-1}) \right\} s(V) \mid H_{t-1} \right],$$

which matches the right hand side of Equation (S2) when $s = t$.

Suppose Equation (S2) holds at time $(s + 1)$ when $s < t$, i.e.,

$$\hat{\mu}_{t,\theta}^0(H_s)|_{\theta=0} = \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \frac{R_k}{\pi_k(0, H_{k-1})} \left\{ \mu_k^0(H_k) - \mu_k^0(H_{k-1}) \right\} s(V) \mid H_s \right].$$

Then for the time point $s$, based on the sequential expression of $\mu_s^0(H_{s-1}) = \mathbb{E}\{\mu_t^0(H_s) \mid H_{s-1}, R_s = 1, A = 0\},$

$$\hat{\mu}_{t,\theta}^0(H_s)|_{\theta=0} = \frac{\partial}{\partial \theta} \mathbb{E}_\theta \left\{ \mu_t^0(H_s) \mid H_{s-1}, R_s = 1, A = 0 \right\} |_{\theta=0}
= \mathbb{E} \{\hat{\mu}_{t,\theta}^0(H_s) \mid H_{s-1}, R_s = 1, A = 1\}
+ \mathbb{E} \{\mu_t^0(H_s) s(Y_t \mid H_{s-1}, R_s = 1, A = 1) \mid H_{s-1}, R_s = 1, A = 1\} \quad \text{(by chain rule)}
= \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \sum_{k=s+1}^t \frac{R_k}{\pi_k(0, H_{k-1})} \left\{ \mu_k^0(H_k) - \mu_k^0(H_{k-1}) \right\} s(V) \mid H_{s-1} \right]
+ \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \frac{R_s}{\pi_s(0, H_{s-1})} \left\{ \mu_s^0(H_s) - \mu_s^0(H_{s-1}) \right\} s(Y_s \mid H_{s-1}, R_s, A) \mid H_{s-1} \right] \quad \text{(by Bayes’ rule)}
= \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \sum_{k=s+1}^t \frac{R_k}{\pi_k(0, H_{k-1})} \left\{ \mu_k^0(H_k) - \mu_k^0(H_{k-1}) \right\} s(V) \mid H_{s-1} \right] \quad \text{(by double expectation)}
+ \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \frac{R_s}{\pi_s(0, H_{s-1})} \left\{ \mu_s^0(H_s) - \mu_s^0(H_{s-1}) \right\} s(V) \mid H_{s-1} \right] \quad \text{(by orthogonality)}
= \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \sum_{k=s}^t \frac{R_k}{\pi_k(0, H_{k-1})} \left\{ \mu_k^0(H_k) - \mu_k^0(H_{k-1}) \right\} s(V) \mid H_{s-1} \right],
$$

which completes the proof. \hfill \Box

Denote the marginal mean for the longitudinal outcomes at the last time point in the control group as $\tau_{0,t}$, i.e., $\tau_{0,t} = \mathbb{E}[Y_t^{J2R} \{0, D(0)\}]$. Under J2R, the missing values in the control group is MAR. The following lemma provides the EIF for the control group mean $\tau_{0,t}$ under MAR.

**Lemma S6** Under MAR, the EIF for $\tau_{0,t}$ is

$$\varphi_{0,t}(V;IP) = \frac{1 - A}{1 - e(X)} \sum_{s=1}^t \frac{R_s}{\overline{\pi}_s(0, H_{s-1})} \left\{ \mu_t^0(H_s) - \mu_t^0(H_{s-1}) \right\} + \mu_t^0(H_0) - \tau_{0,t}.$$
\textbf{Proof:} From the proof of Theorem 5, \( \tau_{0,t} = \mathbb{E} \{ \mu_t^0(H_0) \} \). Then

\[
\dot{\tau}_{0,t} \big|_{\theta=0} = \frac{\partial}{\partial \theta} \mathbb{E}_\theta \{ \mu_t^0(H_0) \} \big|_{\theta=0} = \mathbb{E} \{ \mu_t^0(H_0) \} + \mathbb{E} \{ \mu_t^0(H_0) s(H_0) \} \text{ (by chain rule)}
\]

\[
= \mathbb{E} \left( \mathbb{E} \left[ \frac{1}{1-e(X)} \sum_{k=1}^t \pi_k(0,H_{k-1}) \{ \mu_t^0(H_k) - \mu_t^0(H_{k-1}) \} s(V) \mid H_0 \right] \right) + \mathbb{E} \left[ \{ \mu_t^0(H_0) - \tau_{0,t} \} s(V) \right] \text{ (by Lemma S5 and orthogonality)}
\]

\[
= \mathbb{E} \left( \left[ \frac{1}{1-e(X)} \sum_{k=1}^t \frac{R_k}{\pi_k(0,H_{k-1})} \{ \mu_t^0(H_k) - \mu_t^0(H_{k-1}) \} \mu_t^0(H_0) - \tau_{0,t} \right] s(V) \right).
\]

Therefore, the proof is completed by the definition of the EIF as \( \dot{\tau}_{0,t} \big|_{\theta=0} = \mathbb{E} \{ \varphi_{0,t}(V; \mathbb{P}) s(V) \} \).
\( \square \)

To proceed the proof in the longitudinal setting, we give the following lemma for \( \dot{g}_{s+1,\theta}(H_{l-1}) \) when \( l = 1, \cdots, s-1 \) and \( s = 1, \cdots, t \).

\textbf{Lemma S7} For any \( s \in \{1, \cdots, t\} \), when \( l = 1, \cdots, s-1 \), we have

\[
\dot{g}_{s+1,\theta}(H_{l-1}) \big|_{\theta=0} = \mathbb{E} \left\{ \left( \frac{A}{e(X)} \frac{R_l}{\pi_l(1,H_{l-1})} \left[ R_s \{ 1 - R_{s+1} \} \mu_t^0(H_s) - g_{s+1}^1(H_{l-1}) \right] \right. \right.
\]

\[
\left. \left. + \frac{1-A}{1-e(X)} \prod_{j=l+1}^s \pi_j(0,H_{j-1}) f(Y_{j-1} \mid H_{j-2}, R_{j-1} = 1, A = 1) \right) \left( \frac{f(Y_1 \mid H_{s-1}, R_{s-1} = 1, A = 0) D_{s+1}}{f(Y_s \mid H_{s-1}, R_{s-1} = 1, A = 0) D_{s+1}} \right) \right\} s(V) \mid H_{l-1} \}
\]

\[
\dot{g}_{s+1,\theta}(H_{s-1}) \big|_{\theta=0} = \mathbb{E} \left( \frac{A}{e(X)} \frac{R_s}{\pi_s(1,H_{s-1})} \{ (1 - R_{s+1}) \mu_t^0(H_s) - g_{s+1}^1(H_{s-1}) \} + \frac{1-A}{1-e(X)} D_{s+1} \right) s(V) \mid H_{s-1} \},
\]

where for the simplicity of notations, we denote

\[
D_{s+1} := \{ 1 - \pi_{s+1}(0,H_s) \} f(Y_s \mid H_{s-1}, R_{s} = 1, A = 1) \left[ \sum_{k=s+1}^t \frac{R_k}{\pi_k(0,H_{k-1})} \{ \mu_t^0(H_k) - \mu_t^0(H_{k-1}) \} \right],
\]

and let \( D_{t+1} = 0 \).

\textbf{Proof:} We first compute \( \dot{g}_{s+1,\theta}(H_{s-1}) \big|_{\theta=0} \), and use the iterated relationship \( \dot{g}_{s+1}^1(H_{l-1}) = \mathbb{E} \{ \pi_l(1,H_l) g_{s+1}^1(H_l) \mid H_{l-1}, R_l = 1, A = 1 \} \) for \( l = 1, \cdots, s-1 \) and proceed by induction in backward order beginning from \( l = s-1 \) to get \( \dot{g}_{s+1,\theta}(H_{s-1}) \big|_{\theta=0} \). For \( \dot{g}_{s+1,\theta}(H_{s-1}) \big|_{\theta=0} \),

\[
\dot{g}_{s+1,\theta}(H_{s-1}) \big|_{\theta=0} = \frac{\partial}{\partial \theta} \mathbb{E}_\theta \left\{ \{ 1 - \pi_{s+1}(1,H_s) \} \mu_t^0(H_s) \mid H_{s-1}, R_{s} = 1, A = 1 \right\} \big|_{\theta=0}
\]

\[
= \mathbb{E} \left[ -\pi_{s+1,\theta}(1,H_s) \right|_{\theta=0} \mu_t^0(H_s) \mid H_{s-1}, R_{s} = 1, A = 1]
\]

\[
+ \mathbb{E} \left\{ \{ 1 - \pi_{s+1}(1,H_s) \} \mu_t^0(H_s) \right\} \big|_{\theta=0} \mid H_{s-1}, R_{s} = 1, A = 1
\]

S13
\[
+ \mathbb{E} \left[ \{1 - \pi_{s+1}(1, H_s)\} \mu_t^0(H_s)s(Y_s \mid H_{s-1}, R_s = 1, A = 1) \mid H_{s-1}, R_s = 1, A = 1 \right]

= \mathbb{E} \left( -\mathbb{E} \left[ \frac{A}{e(X)} \frac{R_s}{\pi_s(1, H_s-1)} \{R_{s+1} - \pi_{s+1}(1, H_s)\} \mu_t^0(H_s)s(V \mid H_s) \mid H_{s-1} \right] \right) \quad \text{(Lemma S4)}

+ \mathbb{E} \left( \left[ \frac{1 - A}{1 - e(X)} \{1 - \pi_{s+1}(1, H_s)\} \sum_{k=s+1}^{t} \frac{R_k}{\pi_k(0, H_{k-1})} \{\mu_t^0(H_k) - \mu_t^0(H_{k-1})\} \right] s(V \mid H_s) \mid H_{s-1}, R_s = 1, A = 1 \right) \quad \text{(Lemma S5)}

+ \mathbb{E} \left[ \frac{A}{e(X)} \frac{R_s}{\pi_s(1, H_s-1)} \{1 - \pi_{s+1}(1, H_s)\} \mu_t^0(H_s)s(Y_s \mid H_{s-1}, R_s, A) \mid H_{s-1} \right]

= \mathbb{E} \left[ -\frac{A}{e(X)} \frac{R_s}{\pi_s(1, H_s-1)} \{R_{s+1} - \pi_{s+1}(1, H_s)\} \mu_t^0(H_s)s(V \mid H_s) \mid H_{s-1} \right] \quad \text{(by double expectation)}

+ \mathbb{E} \left[ \left[ \frac{1 - A}{1 - e(X)} \{1 - \pi_{s+1}(1, H_s)\} \sum_{k=s+1}^{t} \frac{R_k}{\pi_k(0, H_{k-1})} \{\mu_t^0(H_k) - \mu_t^0(H_{k-1})\} \right] s(V \mid H_s) \right] f(Y_s \mid H_{s-1}, R_s = 1, A = 1) \quad \text{by definition of expectation)

+ \mathbb{E} \left[ \frac{A}{e(X)} \frac{R_s}{\pi_s(1, H_s-1)} \{1 - \pi_{s+1}(1, H_s)\} \mu_t^0(H_s)s(V) \mid H_{s-1} \right]

= \mathbb{E} \left[ \left[ \frac{1 - A}{1 - e(X)} \{1 - \pi_{s+1}(1, H_s)\} \sum_{k=s+1}^{t} \frac{R_k}{\pi_k(0, H_{k-1})} \{\mu_t^0(H_k) - \mu_t^0(H_{k-1})\} \right] s(V \mid H_s) \right] \quad \text{(by double expectation)}

+ \mathbb{E} \left[ \left[ \frac{1 - A}{1 - e(X)} \{1 - \pi_{s+1}(1, H_s)\} \sum_{k=s+1}^{t} \frac{R_k}{\pi_k(0, H_{k-1})} \{\mu_t^0(H_k) - \mu_t^0(H_{k-1})\} \right] s(V) \mid H_{s-1} \right],

which completes the proof of the first part regarding \( \hat{g}^1_{s+1, \theta}(H_{s-1}) \mid \theta = 0 \).

For the second part of the proof, we derive it by induction backward starting from \( l = s - 1 \). For \( l = s - 1 \),

\[
\hat{g}^1_{s+1, \theta}(H_{s-2}) \mid \theta = 0 = \frac{\partial}{\partial \theta} \mathbb{E}_\theta \left[ \pi_s(1, H_{s-1})g^1_{s+1}(H_{s-1}) \mid H_{s-2}, R_{s-2} = 1, A = 1 \right] \mid \theta = 0
\]

\[
= \mathbb{E} \left[ \pi_s, \theta(1, H_{s-1}) \mid \theta = 0 g^1_{s+1}(H_{s-1}) \mid H_{s-1}, R_s = 1, A = 1 \right]

+ \mathbb{E} \left[ \pi_s(1, H_{s-1}) \hat{g}^1_{s+1, \theta}(H_{s-1}) \mid \theta = 0 \mid H_{s-2}, R_{s-1} = 1, A = 1 \right]

+ \mathbb{E} \left[ \pi_s(1, H_{s-1})g^1_{s+1}(H_{s-1})s(Y_{s-1} \mid H_{s-2}, R_{s-2} = 1, A = 1) \mid H_{s-2}, R_{s-1} = 1, A = 1 \right]

= \mathbb{E} \left( \left[ \frac{A}{e(X)} \frac{R_{s-1}}{\pi_{s-1}(1, H_{s-2})} \{R_s - \pi_s(1, H_{s-1})\} g^1_{s+1}(H_{s-1})s(V \mid H_{s-1}) \mid H_{s-2} \right] \right) \quad \text{(Lemma S4)}

+ \mathbb{E} \left[ \frac{A}{e(X)} \frac{R_s}{\pi_s(1, H_{s-1})} \pi_s(1, H_{s-1}) \{1 - R_{s+1}\} \mu_t^0(H_s) - g^1_{s+1}(H_{s-1}) \right] s(V \mid H_{s-1}) \mid H_{s-2})

+ \mathbb{E} \left[ \left[ \frac{1 - A}{1 - e(X)} \pi_s(1, H_{s-1})D_{s+1}s(V) \mid H_{s-1} \right] \mid H_{s-2}, R_{s-1} = 1, A = 1 \right]

+ \mathbb{E} \left[ \frac{A}{e(X)} \frac{R_{s-1}}{\pi_{s-1}(1, H_{s-2})} \pi_s(1, H_{s-1})g^1_{s+1}(H_{s-1})s(Y_{s-1} \mid H_{s-2}, R_{s-1}, A) \mid H_{s-2} \right] \quad \text{S14}
\]
\[ \dot{g}_{s+1,\theta}(H_t) \big|_{\theta = 0} = \frac{1}{1 - e(X)} \prod_{j=l+2}^s \pi_j(0, H_{j-1}) \frac{f(Y_{j-1} | H_{j-2}, R_{j-1} = 1, A = 1)}{f(Y_{j-1} | H_{j-2}, R_{s-1} = 1, A = 0)} D_{s+1} s(V) | H_t \bigg] \]

Then for \( l \), we apply chain rule on the iterated formula:

\[ \dot{g}_{s+1,\theta}(H_{l-1}) \big|_{\theta = 0} = \frac{\partial}{\partial \theta} \mathbb{E}_\theta \{ \pi_{l+1}(1, H_t) g_{s+1}^l(H_t) | H_{l-1}, R_t = 1, A = 1 \} \big|_{\theta = 0} \]

\[ = \mathbb{E} \{ \pi_{l+1,\theta}(1, H_t) g_{s+1}^l(H_t) | H_{l-1}, R_t = 1, A = 1 \} \]

\[ + \mathbb{E} \{ \pi_{l+1}(1, H_t) \dot{g}_{s+1,\theta}^l(H_t) | H_{l-1}, R_t = 1, A = 1 \} \]

\[ + \mathbb{E} \{ \pi_{l+1}(1, H_t) g_{s+1}^l(H_t) s(Y_t | H_{l-1}, R_t = 1, A = 1) | H_{l-1}, R_t = 1, A = 1 \} \]

\[ = \mathbb{E} \left[ \frac{A}{e(X)} \frac{R_{l+1}}{\bar{\pi}_{l+1}(1, H_{l-1})} \{ R_{l+1} - \pi_{l+1}(1, H_t) \} g_{s+1}^l(H_t) s(V) | H_{l-1} \right] \]

\[ + \mathbb{E} \left[ \frac{A}{e(X)} \frac{R_l}{\bar{\pi}_l(1, H_{l-1})} \{ R_s (1 - R_{s+1}) \mu_0^l(H_s) - g_{s+1}^l(H_{l-1}) \} s(V) | H_{l-1} \right] \]

\[ + \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \pi_{l+1}(1, H_t) \prod_{j=l+2}^s \pi_j(0, H_{j-1}) \frac{f(Y_{j-1} | H_{j-2}, R_{j-1} = 1, A = 1)}{f(Y_{j-1} | H_{j-2}, R_{s-1} = 1, A = 0)} D_{s+1} s(V) | H_t \bigg] \]

matches the right hand side when \( l = s - 1 \). Suppose the equality holds for \((l + 1)\) when \( l < s - 1 \), i.e.,

\[ \dot{g}_{s+1,\theta}(H_t) \big|_{\theta = 0} = \mathbb{E}_\theta \{ \pi_{l+1}(1, H_t) g_{s+1}^l(H_t) | H_{l-1}, R_t = 1, A = 1 \} + \mathbb{E} \{ \pi_{l+1}(1, H_t) \dot{g}_{s+1,\theta}^l(H_t) | H_{l-1}, R_t = 1, A = 1 \} + \mathbb{E} \{ \pi_{l+1}(1, H_t) g_{s+1}^l(H_t) s(Y_t | H_{l-1}, R_t = 1, A = 1) | H_{l-1}, R_t = 1, A = 1 \} \]

\[ = \mathbb{E} \left[ \frac{A}{e(X)} \frac{R_{l+1}}{\bar{\pi}_{l+1}(1, H_{l-1})} \{ R_{l+1} - \pi_{l+1}(1, H_t) \} g_{s+1}^l(H_t) s(V) | H_{l-1} \right] \]

\[ + \mathbb{E} \left[ \frac{A}{e(X)} \frac{R_l}{\bar{\pi}_l(1, H_{l-1})} \{ R_s (1 - R_{s+1}) \mu_0^l(H_s) - g_{s+1}^l(H_{l-1}) \} s(V) | H_{l-1} \right] \]

\[ + \mathbb{E} \left[ \frac{1 - A}{1 - e(X)} \pi_{l+1}(1, H_t) \prod_{j=l+2}^s \pi_j(0, H_{j-1}) \frac{f(Y_{j-1} | H_{j-2}, R_{j-1} = 1, A = 1)}{f(Y_{j-1} | H_{j-2}, R_{s-1} = 1, A = 0)} D_{s+1} s(V) | H_t \bigg] \]

S15
Lemma S7 implies that when $l = 1$, 

$$
\partial \frac{\partial \mathbb{E} \{ \pi_1(1, H_0)g^1_1(H_0) \}}{\partial \theta} \bigg|_{\theta=0} = \mathbb{E} \left( \left[ \frac{A}{e(X)} \right] \left\{ \sum_{k=s+1}^{l} \mathbb{E} \left[ \frac{R_i}{e(X)} \pi_t(1, H_0) \right] \left\{ R_s (1 - R_s+1) \mu_t^0(H_s) - g^1_{s+1}(H_0) \right\} \right\} \frac{1}{\bar{\pi}_{s+1}(1, H_0)} \sum_{j=1}^{s} f(Y_j \mid H_{j-1}, R_j = 1, A = 1) s(V) \right),
$$

completes the proof. \(\square\)

From Lemma S7, we proceed to obtain $\partial \mathbb{E}_\theta \{ \pi_1(1, H_0)g^1_{s+1}(H_0) \} / \partial \theta \big|_{\theta=0}$ in the following lemma.

**Lemma S8** For any $s \in \{1, \cdots, t\}$, we have

$$
\frac{\partial \mathbb{E}_\theta \{ \pi_1(1, H_0)g^1_{s+1}(H_0) \}}{\partial \theta} \bigg|_{\theta=0} = \mathbb{E} \left( \left[ \frac{A}{e(X)} \right] \left\{ \sum_{k=s+1}^{l} \mathbb{E} \left[ \frac{R_i}{e(X)} \pi_t(1, H_0) \right] \left\{ R_s (1 - R_s+1) \mu_t^0(H_s) - \pi_1(1, H_0)g^1_{s+1}(H_0) \right\} \right\} \frac{1}{\bar{\pi}_{s+1}(1, H_0)} \sum_{j=1}^{s} f(Y_j \mid H_{j-1}, R_j = 1, A = 1) s(V) \right),
$$

where $W_{s+1} = \sum_{k=s+1}^{t} R_k \{ \mu^0_k(H_k) - \mu^0_{k-1}(H_{k-1}) \} / \bar{\pi}_k(0, H_{k-1})$ and $W_{t+1} = 0$.

**Proof:** Lemma S7 implies that when $l = 1$,

$$
\hat{g}^1_{s+1, \theta}(H_0) \bigg|_{\theta=0} = \mathbb{E} \left( \left[ \frac{A}{e(X)} \right] \left\{ \sum_{k=s+1}^{l} \mathbb{E} \left[ \frac{R_i}{e(X)} \pi_t(1, H_0) \right] \left\{ R_s (1 - R_s+1) \mu_t^0(H_s) - g^1_{s+1}(H_0) \right\} \right\} \frac{1}{\bar{\pi}_{s+1}(1, H_0)} \sum_{j=1}^{s} f(Y_j \mid H_{j-1}, R_j = 1, A = 1) s(V) \right),
$$

Then we have $\partial \mathbb{E}_\theta \{ \pi_1(1, H_0)g^1_{s+1}(H_0) \} / \partial \theta \big|_{\theta=0}$

$$
= \mathbb{E} \left\{ \hat{g}^1_{s+1, \theta}(H_0) \right\} + \mathbb{E} \left\{ \pi_1(1, H_0)g^1_{s+1, \theta}(H_0) \big|_{\theta=0} \right\} 
+ \mathbb{E} \left\{ \pi_1(1, H_0)g^1_{s+1}(H_0) s(H_0) \right\} 
+ \mathbb{E} \left( \left[ \frac{A}{e(X)} \right] \left\{ \sum_{k=s+1}^{l} \mathbb{E} \left[ \frac{R_i}{e(X)} \pi_t(1, H_0) \right] \left\{ R_s (1 - R_s+1) \mu_t^0(H_s) - \pi_1(1, H_0)g^1_{s+1}(H_0) \right\} \right\} \frac{1}{\bar{\pi}_{s+1}(1, H_0)} \sum_{j=1}^{s} f(Y_j \mid H_{j-1}, R_j = 1, A = 1) s(V) \right) 
+ \mathbb{E} \left\{ \pi_1(1, H_0)g^1_{s+1}(H_0) s(V) \right\}.
$$

Note that $\delta(H_s) = \prod_{j=1}^{s} \left\{ f(Y_j \mid H_{j-1}, R_j = 1, A = 1) / f(Y_j \mid H_{j-1}, R_j = 1, A = 0) \right\} \bar{\pi}_s(1, H_{s-1}) / \bar{\pi}_s(0, H_{s-1})$ by Lemma S2, which completes the proof. \(\square\)
S2.1 Proof of Theorem 2

We compute the EIF by rewriting the identification formula in Theorem 1(a) as \( \tau_{1j2R} = \tau_{1,1} - \tau_{0,1} \), where \( \tau_{1,1} = \mathbb{E}[\pi_1(1, X) \mu_1^1(X) + \{1 - \pi_1(1, X)\} \mu_1^0(X)] \) and \( \tau_{0,1} = \mathbb{E}\{\mu_1^0(X)\} \) based on the proof in S1.1. By Lemma S6,

\[
\varphi_{0,1}(V; \mathbb{P}) = \frac{1 - A}{1 - e(X)} \frac{R_1}{\pi_1(0, X)} \left\{ Y_1 - \mu_1^0(X) \right\} + \mu_1^0(X) - \tau_{0,1}.
\]

We proceed to compute \( \hat{\tau}_{1,1, \theta} |_{\theta=0} \). Note that,

\[
\hat{\tau}_{1,1, \theta} |_{\theta=0} = \frac{\partial}{\partial \theta} \mathbb{E}_\theta \left[ \pi_1(1, X) \mu_1^1(X) + \{1 - \pi_1(1, X)\} \mu_1^0(X) \right] |_{\theta=0}
\]

\[
= \mathbb{E}\{\hat{\pi}_1, \theta(1, X) |_{\theta=0} \mu_1^1(X) + \pi_1(1, X) \hat{\mu}_1^1, \theta(X) |_{\theta=0} + \pi_1(1, X) \hat{\mu}_1^1(X)s(X)\}
\]

\[
+ \mathbb{E}\left[ \pi_1(1, X) \mu_1^0(X) + \{1 - \pi_1(1, X)\} \hat{\mu}_1^0(X) |_{\theta=0} + \{1 - \pi_1(1, X)\} \mu_1^0(X)s(X) \right]
\]

\[
= \mathbb{E}\left( \mathbb{E} \left[ \frac{A}{e(X)} \pi_1(1, X) \left\{ Y_1 - \mu_1^0(X) \right\} s(V) | X \right] \right) \text{ (by Lemma S4)}
\]

\[
+ \mathbb{E}\left( \mathbb{E} \left[ \frac{A}{e(X)} \pi_1(1, X) \left\{ Y_1 - \mu_1^0(X) \right\} s(V) | X \right] \right) \text{ (by Lemma S5)}
\]

\[
+ \mathbb{E}\left( \pi_1(1, X) \mu_1^1(X) + \{1 - \pi_1(1, X)\} \mu_1^0(X) - \tau_{1,1} \right) s(V) \text{ (by orthogonality)}
\]

\[
= \mathbb{E}\left( \mathbb{E} \left\{ \left( \{1 - \pi_1(1, X)\} \mu_1^1(X) + \{1 - \pi_1(1, X)\} \mu_1^0(X) \right) - \frac{A}{e(X)} \pi_1(1, H_0) \{\mu_1^1(X) - \mu_1^0(X)\} - \tau_{1,1} \right\} s(V) \right).
\]

Then we can get \( \varphi_{1,1}(V; \mathbb{P}) \) based on the definition of the EIF as \( \hat{\tau}_{1,1, \theta} |_{\theta=0} = \mathbb{E}\{\varphi_{1,1}(V; \mathbb{P})s(V)\} \).

The EIF of \( \tau_{1j2R} \) can then be obtained:

\[
\varphi_{1j2R}(V; \mathbb{P}) = \varphi_{1,1}(V; \mathbb{P}) - \varphi_{0,1}(V; \mathbb{P})
\]

\[
= \left\{ \frac{A}{e(X)} R_1 + \frac{1 - A}{1 - e(X)} \frac{R_1}{\pi_1(0, X)} \{1 - \pi_1(1, X)\} \right\} \left\{ Y_1 - \mu_1^0(X) \right\}
\]

\[
+ \left\{ \pi_1(1, X) \mu_1^1(X) + \{1 - \pi_1(1, X)\} \mu_1^0(X) \right\} - \frac{A}{e(X)} \pi_1(1, H_0) \{\mu_1^1(X) - \mu_1^0(X)\} - \tau_{1,1}
\]

\[
- \frac{1 - A}{1 - e(X)} \frac{R_1}{\pi_1(0, X)} \{Y_1 - \mu_1^0(X)\} - \mu_1^0(X) + \tau_{0,1}
\]

\[
= \left\{ \frac{A}{e(X)} \pi_1(1, X) \right\} R_1 \left\{ Y_1 - \mu_1^0(X) \right\} + \left\{ \frac{1 - A}{e(X)} \right\} \pi_1(1, X) \left\{ \mu_1^1(X) - \mu_1^0(X) \right\} - \tau_{1j2R}.
\]

S2.2 Proof of Theorem 6

We compute the EIF based on the identification formula in Theorem 5(a) as \( \tau_{tj2R} = \tau_{1,t} - \tau_{0,t} \), where

\[
\tau_{1,t} = \mathbb{E}\left[ \pi_1(1, H_0) \sum_{s=1}^t g_s^1(H_0) + \{1 - \pi_1(1, H_0)\} \mu_1^0(H_0) \right]
\]

and \( \tau_{0,t} = \)
\[ \mathbb{E} \{ \mu^0_t(H_0) \} \] based on the proof in S1.2 By Lemma S6, we can obtain \( \varphi_{0,t}(V; \mathbb{P}) \). We only need to calculate the EIF for \( \tau_{1,t} \). Note that for any \( s \in \{1, \cdots, t\}, \partial \mathbb{E}_{\theta} \{ \pi_1(1, H_0) g_{s+1}^1(H_0) \} / \partial \theta|_{\theta=0} \) is obtained by Lemma S8. The part \( \partial \mathbb{E}_{\theta} \{1 - \pi_1(1, H_0)\} \mu_t^0(H_0) / \partial \theta|_{\theta=0} \) can be derived using chain rule and Lemmas S4 and S5 as

\[
\mathbb{E}_{\theta}\left[ \{1 - \pi_1(1, H_0)\} \mu_t^0(H_0) \right]_{\theta=0} = \mathbb{E}\left[ -\pi_{1,\theta}(1, H_0) \right]_{\theta=0} + \mathbb{E}\left[ \{1 - \pi_1(1, H_0)\} \mu_{t,\theta}^0(H_0) \right]_{\theta=0}
+ \mathbb{E}\left[ \{1 - \pi_1(1, H_0)\} \mu_t^0(H_0) s(H_0) \right] 
= \mathbb{E}\left( -\frac{A}{e(X)} \left\{ R_1 - \pi_1(1, H_0) \right\} \mu_t^0(H_0) + \{1 - \pi_1(1, H_0)\} \mu_t^0(H_0) \right) s(V) 
+ \mathbb{E}\left[ \{1 - \pi_1(1, H_0)\} \varphi_{0,t}(V; \mathbb{P}) s(V) \right].
\]

Combine all terms together and by the definition of the EIF, we have

\[
\varphi_{1,t}(V; \mathbb{P}) = \frac{A}{e(X)} \sum_{s=0}^{t} R_s (1 - R_{s+1}) \mu_t^0(H_s) - \frac{A}{e(X)} \sum_{s=1}^{t} \pi_1(1, H_0) g_{s+1}^1(H_0)
+ \frac{1 - A}{1 - e(X)} \sum_{s=1}^{t-1} \pi_s(1, H_{s-1}) \{1 - \pi_{s+1}(0, H_s)\} \delta(H_s) W_{s+1} \text{ (since } D_{t+1} = 0) 
+ \pi_1(1, H_0) \sum_{s=1}^{t} g_{s+1}^1(H_0) + \{1 - \pi_1(1, H_0)\} \varphi_{0,t}(V; \mathbb{P}) + \mu_t^0(H_0) - \tau_{1,t}. 
\]

Apply Lemma S6 the EIF \( \varphi_{t}^{J2R}(V; \mathbb{P}) \) of \( \tau_{t}^{J2R} \) is

\[
\varphi_{t}^{J2R}(V; \mathbb{P}) = \varphi_{1,t}(V; \mathbb{P}) - \varphi_{0,t}(V; \mathbb{P})
= \frac{A}{e(X)} \sum_{s=0}^{t} R_s (1 - R_{s+1}) \mu_t^0(H_s) - \frac{A}{e(X)} \sum_{s=1}^{t} \pi_1(1, H_0) g_{s+1}^1(H_0)
+ \frac{1 - A}{1 - e(X)} \sum_{s=1}^{t-1} \pi_s(1, H_{s-1}) \{1 - \pi_{s+1}(0, H_s)\} \delta(H_s) W_{s+1} 
+ \pi_1(1, H_0) \sum_{s=1}^{t} g_{s+1}^1(H_0) - \pi_1(1, H_0) \varphi_{0,t}(V; \mathbb{P}) + \mu_t^0(H_0) - \tau_{t}^{J2R}
= \frac{A}{e(X)} \left\{ R_t Y_t + \sum_{s=1}^{t} R_{s-1} (1 - R_s) \mu_t^0(H_{s-1}) \right\} - \tau_{t}^{J2R} 
+ \left\{ 1 - \frac{A}{e(X)} \right\} \left[ \pi_1(1, H_0) \sum_{s=1}^{t} g_{s+1}^1(H_0) + \{1 - \pi_1(1, H_0)\} \mu_t^0(H_0) \right] - \mu_t^0(H_0)
+ \frac{1 - A}{1 - e(X)} \left( \sum_{s=1}^{t-1} \pi_s(1, H_{s-1}) \{1 - \pi_{s+1}(0, H_s)\} \delta(H_s) W_{s+1} + \{1 - \pi_1(1, H_0)\} W_1 - W_1 \right). 
\]
Simplify the last term, note that $\sum_{s=1}^{t} \sum_{k=1}^{s} \pi_{k-1}(1, H_{k-2}) \{1 - \pi_{k-1}(0, H_{k-2})\} \delta(H_{k-1}) W_{s} + \{1 - \pi_{k-1}(0, H_{k-1})\} W_{1} = \sum_{s=0}^{t-1} \sum_{k=s+1}^{t} \pi_{k}(0, H_{k-1}) \{\mu_{t}^{0}(H_{k}) - \mu_{t}^{0}(H_{k-1})\} = \sum_{k=1}^{k-1} \sum_{s=0}^{t} \pi_{s}(1, H_{s-1}) \{1 - \pi_{s+1}(0, H_{s})\} \delta(H_{s}) R_{k} \pi_{k}(0, H_{k-1}) \{\mu_{t}^{0}(H_{k}) - \mu_{t}^{0}(H_{k-1})\} \right) (change the order of k and s) = \sum_{s=0}^{t-1} \sum_{k=s+1}^{t} \pi_{k}(0, H_{k-1}) \{\mu_{t}^{0}(H_{k}) - \mu_{t}^{0}(H_{k-1})\} \right) (change s to s + 1) = \sum_{s=1}^{t} \sum_{k=1}^{s} \pi_{k-1}(1, H_{k-2}) \{1 - \pi_{k-1}(0, H_{k-1})\} \delta(H_{k-1}) \right) \pi_{s}(0, H_{s-1}) \{\mu_{t}^{0}(H_{s}) - \mu_{t}^{0}(H_{s-1})\} (interchange k and s). Then the last term becomes

$$\frac{1 - A}{e(X)} \left( \sum_{s=1}^{t} \sum_{k=1}^{s} \pi_{k-1}(1, H_{k-2}) \{1 - \pi_{k-1}(0, H_{k-1})\} \delta(H_{k-1}) \right) \frac{R_{s}}{\pi_{s}(0, H_{s-1})} \{\mu_{t}^{0}(H_{s}) - \mu_{t}^{0}(H_{s-1})\} - W_{1}$$

$$= \frac{1 - A}{e(X)} \left( \sum_{s=1}^{t} \sum_{k=1}^{s} \pi_{k-1}(1, H_{k-2}) \{1 - \pi_{k-1}(0, H_{k-1})\} \delta(H_{k-1}) \right) \frac{R_{s}}{\pi_{s}(0, H_{s-1})} \{\mu_{t}^{0}(H_{s}) - \mu_{t}^{0}(H_{s-1})\} - W_{1}$$

Therefore, the EIF $\varphi_{t}^{2R}(V; \mathbb{P})$ of $\tau_{t}^{2R}$

$$= \frac{A}{e(X)} \left\{ R_{t} Y_{t} + \sum_{s=1}^{t} R_{s} (1 - R_{s}) \mu_{t}^{0}(H_{s-1}) \right\} - \tau_{t}^{2R}$$

$$+ \left\{ 1 - \frac{A}{e(X)} \right\} \left\{ \pi_{1}(1, H_{0}) \sum_{s=1}^{t} g_{s+1}(H_{0}) + \{1 - \pi_{1}(1, H_{0})\} \mu_{t}^{0}(H_{0}) \right\} - \mu_{t}^{0}(H_{0})$$

$$+ \frac{1 - A}{e(X)} \left( \sum_{s=1}^{t} \sum_{k=1}^{s} \pi_{k-1}(1, H_{k-2}) \{1 - \pi_{k-1}(0, H_{k-1})\} \delta(H_{k-1}) \right) \frac{R_{s}}{\pi_{s}(0, H_{s-1})} \{\mu_{t}^{0}(H_{s}) - \mu_{t}^{0}(H_{s-1})\}$$

which matches the expression given in Theorem 6

S3  Estimation

S3.1  EIF-based estimators motivated from Corollary 1

We provide the EIF-based estimator $\hat{\tau}_{tr}$ and its normalized estimator $\hat{\tau}_{tr-N}$ in the cross-sectional studies as follows.

$$\hat{\tau}_{tr} = \mathbb{P} \left\{ \frac{A}{e(X; \hat{\alpha})} - \frac{1 - A}{e(X; \hat{\alpha})} \pi_{1}(1, X; \hat{\gamma}) \right\} R_{1} \left\{ Y_{1} - \mu_{1}(X; \hat{\beta}) \right\} - \frac{A - e(X; \hat{\alpha})}{e(X; \hat{\alpha})} \pi_{1}(1, X; \hat{\gamma}) \left\{ \mu_{1}(X; \hat{\beta}) - \mu_{1}(X; \hat{\beta}) \right\}$$

S19
The normalized version of the ps-om and ps-rp estimators are as follows:

**Example 4**

We give the normalized version of the ps-om and ps-rp estimators in the longitudinal setting below.

---

**S3.2 Normalized estimators motivated from Theorem [5]**

We give the normalized version of the ps-om and ps-rp estimators in the longitudinal setting below.

---

(a) The normalized ps-om estimator:

\[
\hat{\tau}_{\text{ps-om-N}} = \mathbb{P}_n \left( \frac{A}{\hat{e}(H_0)} \left[ R_t Y_t + \sum_{s=1}^{t} R_{s-1} (1 - R_s) \hat{\mu}_s^0(H_{s-1}) \right] \right) / \mathbb{P}_n \left( \frac{A}{\hat{e}(H_0)} \right)
\]

\[
- \mathbb{P}_n \left[ \frac{1 - A}{1 - \hat{e}(H_0)} \left( R_t Y_t + \sum_{s=1}^{t} R_{s-1} (1 - R_s) \hat{\mu}_s^0(H_{s-1}) \right) \right] / \mathbb{P}_n \left( \frac{1 - A}{1 - \hat{e}(H_0)} \right).
\]

(b) The normalized ps-rp estimator:

\[
\hat{\tau}_{\text{ps-rp-N}} = \mathbb{P}_n \left( \frac{A}{\hat{e}(H_0)} R_t Y_t \right) / \mathbb{P}_n \left( \frac{A}{\hat{e}(H_0)} \right)
\]

\[
- \mathbb{P}_n \left[ \frac{1 - A}{1 - \hat{e}(H_0)} \left( \sum_{s=1}^{t} \hat{\pi}_s(0, H_{s-2}) \{1 - \hat{\pi}_s(1, H_{s-1})\} \delta(H_{s-1}) - 1 \right) \frac{R_t Y_t}{\hat{\pi}_t(0, H_{t-1})} \right] / \mathbb{P}_n \left( \frac{1 - A}{1 - \hat{e}(H_0)} \frac{R_t}{\hat{\pi}_t(0, H_{t-1})} \right).
\]

S20
S3.3 Estimation procedure in the longitudinal setting

We consider the case when \( t = 2 \), and give detailed steps to estimate \( \hat{\tau}_{\text{rp-pm}}, \hat{\tau}_{\text{ps-om}} \) and \( \hat{\tau}_{\text{ps-rp}} \) as an example for a straightforward illustration. Extend the estimation procedure to the setting when \( t > 2 \) is straightforward. Based on Example 3(a),

\[
\hat{\tau}_{\text{rp-pm}} = \mathbb{P}_n \left\{ \hat{\pi}_1(1, H_0) \left( \mathbb{E} \left\{ \hat{\pi}_2(1, H_1) \mu_2^0(H_1) \mid H_0, R_1 = 1, A = 1 \right\} + \mathbb{E} \left\{ \{1 - \hat{\pi}_2(1, H_1)\} \mu_2^0(H_1) \mid H_0, R_1 = 1, A = 1 \right\} - \hat{\mu}_2^0(H_0) \right) \right\}.
\]

The steps of estimating the rp-pm estimator when \( t = 2 \) are summarized as follows:

**Step 1.** For subjects with \( R_2 = 1 \), obtain the fitted outcome mean \( \hat{\mu}_2^a(H_1) \) for \( a = 0, 1 \).

**Step 2.** For subjects with \( R_1 = 1 \), obtain the following estimated nuisance functions:

(a) The estimated pattern mean \( \hat{g}_2^1(H_0), \hat{g}_3^1(H_0) \): Fit \( g_2^1(H_0) = \mathbb{E} \left\{ \{1 - \pi_2(1, H_1)\} \mu_2^0(H_1) \mid H_0, R_1 = 1, A = 1 \right\} \) and \( g_3^1(H_0) = \mathbb{E} \left\{ \pi_2(1, H_1) \mu_2^1(H_1) \mid H_0, R_1 = 1, A = 1 \right\} \) using the predicted values \( \{1 - \hat{\pi}_2(1, H_1)\} \hat{\mu}_2^0(H_1) \) and \( \hat{\pi}_2(1, H_1) \hat{\mu}_2^1(H_1) \) against \( H_0 \) in the group with \( R_1 = 1 \) and \( A = 1 \), respectively.

(b) The estimated response probability \( \hat{\pi}_2(a, H_1) \).

(c) The estimated outcome mean \( \hat{\mu}_2^0(H_0) \): Fit \( \mu_2^0(H_0) = \mathbb{E} \left\{ \mu_2^0(H_1) \mid H_0, R_1 = 1, A = 0 \right\} \) using the predicted values \( \hat{\mu}_2^0(H_1) \) against \( H_0 \) in the group with \( R_1 = 1 \) and \( A = 0 \).

**Step 3.** For all the subjects, obtain the estimated response probability \( \hat{\pi}_1(a, H_1) \).

**Step 4.** Get \( \hat{\tau}_{\text{rp-pm}} \) by the empirical average.

Based on Example 3(b),

\[
\hat{\tau}_{\text{ps-om}} = \mathbb{P}_n \left[ \frac{2A - 1}{\hat{\epsilon}(X)^2 \{1 - \hat{\epsilon}(X)\}^{1 - A}} \left\{ R_2 Y_2 + R_1 (1 - R_2) \hat{\mu}_2^0(H_1) + (1 - R_1) \hat{\mu}_2^0(H_0) \right\} \right].
\]

The steps of estimating the ps-om estimator is as follows:

**Step 1.** For subjects with \( R_2 = 1 \), obtain the fitted outcome mean model \( \hat{\mu}_2^0(H_1) \).

**Step 2.** For subjects with \( R_1 = 1 \), obtain the fitted outcome mean model \( \hat{\mu}_2^0(H_0) \), by fitting \( \mu_2^0(H_0) = \mathbb{E} \left\{ \mu_2^0(H_1) \mid H_0, R_1 = 1, A = 0 \right\} \) using the predicted values \( \hat{\mu}_2^0(H_1) \) against \( H_0 \) in the group with \( R_1 = 1 \) and \( A = 0 \).
**Step 3.** For all the subjects, obtain the fitted propensity score model \( \hat{e}(X) \).

**Step 4.** Get \( \hat{\tau}_{ps-om} \) by the empirical average.

Based on Example 3 (c),

\[
\hat{\tau}_{ps-rp} = \mathbb{P}_n \left( \frac{A}{\hat{e}(X)} R_2 Y_2 + \frac{1 - A}{1 - \hat{e}(X)} \left[ \hat{\pi}_1(0, H_0) \{1 - \hat{\pi}_2(1, H_1)\} \hat{\delta}(H_1) - \hat{\pi}_1(1, H_0) \right] \frac{R_2 Y_2}{\hat{\pi}_2(0, H_1)} \right).
\]

The steps of estimating the ps-rp estimator is as follows:

**Step 1.** For subjects with \( R_1 = 1 \), obtain the following models:

(a) The fitted propensity score model \( \hat{e}(H_1) \).

(b) The fitted response probability model \( \hat{\pi}_2(a, H_1) \).

**Step 2.** For all the subjects, obtain the following models:

(a) The fitted propensity score model \( \hat{e}(X) \).

(b) The fitted response probability model \( \hat{\pi}_1(a, H_0) \).

**Step 4.** Obtain \( \hat{\delta}(H_1) = \{\hat{e}(H_1)/\hat{e}(H_0)\} / \{1 - \hat{e}(H_1)\} / \{1 - \hat{e}(H_0)\} \) for the subjects with \( R_1 = 1 \), and get \( \hat{\tau}_{ps-rp} \) by the empirical average.

**S3.4 Multiply robust estimators motivated from Corollary 3**

From the EIF, one can motivated new estimators of \( \tau_{J2R}^t \). We present the expression of \( \hat{\tau}_{mr} \) below.

\[
\hat{\tau}_{mr} = \mathbb{P}_n \left( \frac{A}{\hat{e}(H_0)} \left( R_t Y_t + \sum_{s=1}^{t} R_{s-1} (1 - R_s) \hat{\mu}_t^0 (H_{s-1}) \right) \right.
\]

\[
+ \left\{ \frac{1 - A}{1 - \hat{e}(H_0)} \right\} \left[ \hat{\pi}_1(1, H_0) \sum_{s=1}^{t} \hat{g}_{s+1}^1 (H_0) + \{1 - \hat{\pi}_1(1, H_0)\} \hat{\mu}_t^0 (H_0) \right] - \hat{\mu}_t^0 (H_0)
\]

\[
+ \left. \frac{1 - A}{1 - \hat{e}(H_0)} \sum_{s=1}^{t} \sum_{k=1}^{s} \hat{\pi}_{k-1}(0, H_{k-2}) \{1 - \hat{\pi}_k(1, H_{k-1})\} \hat{\delta}(H_{k-1}) - 1 \right] \frac{R_s}{\hat{\pi}_s(0, H_{s-1})} \left\{ \hat{\mu}_s^0 (H_s) - \hat{\mu}_s^0 (H_{s-1}) \right\} \right).
\]

Now, we provide the normalized version of \( \hat{\tau}_{mr} \) as follows. The normalized estimator is less influenced by the extreme weights compared to \( \hat{\tau}_{mr} \).

\[
\hat{\tau}_{mr-N} = \mathbb{P}_n \left( \frac{A}{\hat{e}(H_0)} \left( R_t Y_t + \sum_{s=1}^{t} R_{s-1} (1 - R_s) \hat{\mu}_t^0 (H_{s-1}) - \hat{\pi}_1(1, H_0) \sum_{s=1}^{t} \hat{g}_{s+1}^1 (H_0) \right) \right.
\]

S22
The calibration-based estimator expresses as follows.

\[- \{1 - \hat{\pi}_1(1, H_0)\} \hat{\mu}_1^0(H_0)] / \mathbb{P}_n \left\{ \frac{A}{\hat{e}(H_0)} \right\} \]

\[+ \mathbb{P}_n \left\{ \hat{\pi}_1(1, H_0) \sum_{s=1}^t \hat{g}_{s+1}^1(H_0) - \hat{\pi}_1(1, H_0) \hat{\mu}_1^0(H_0) \right\} \]

\[+ \sum_{s=1}^t \mathbb{P}_n \left\{ \frac{1 - A}{1 - \hat{e}(H_0)} \left( \left[ \sum_{k=1}^s \hat{\pi}_{k-1}(0, H_{k-2}) \{1 - \hat{\pi}_k(1, H_{k-1})\} \hat{\delta}(H_{k-1}) - 1 \right] \right. \]

\[\frac{R_s}{\hat{\pi}_s(0, H_{s-1})} \{ \hat{\mu}_t^0(H_s) - \hat{\mu}_t^0(H_{s-1}) \} \right\} / \mathbb{P}_n \left\{ \frac{1 - A}{1 - \hat{e}(H_0)} \frac{R_s}{\hat{\pi}_s(0, H_{s-1})} \right\}.

In addition, one can conduct calibration to further reduce the impact of the outliers. The calibration-based estimator expresses as follows.

\[\hat{\tau}_{mr-C} = \mathbb{P}_n \left( Aw_{a_1} \left[ R_t Y_t + \sum_{s=1}^t R_{s-1}(1 - R_s) \hat{\mu}_t^0(H_{s-1}) - \hat{\pi}_1(1, H_0) \sum_{s=1}^t \hat{g}_{s+1}^1(H_0) \right. \]

\[- \{1 - \hat{\pi}_1(1, H_0)\} \hat{\mu}_1^0(H_0) \right) / \mathbb{P}_n (Aw_{a_1}) \]

\[+ \mathbb{P}_n \left\{ \pi_1(1, H_0; \hat{\gamma}) \sum_{s=1}^t \hat{g}_{s+1}^1(H_0) - \pi_1(1, H_0; \hat{\gamma}) \hat{\mu}_t^0(H_0) \right\} \]

\[+ \mathbb{P}_n \left\{ (1 - A) R_s w_{a_0} w_{r_1} \cdots w_{r_s} \left[ \sum_{k=1}^s \hat{\pi}_{k-1}(0, H_{k-2}) \{1 - \hat{\pi}_k(1, H_{k-1})\} \hat{\delta}(H_{k-1}) - 1 \right] \right. \]

\[\left. \{ \hat{\mu}_t^0(H_s) - \hat{\mu}_t^0(H_{s-1}) \} \right\} / \mathbb{P}_n \{(1 - A) R_s w_{a_0} w_{r_1} \cdots w_{r_s}\}.

We present the detailed estimation steps for the calibration-based estimator \(\hat{\tau}_{mr-C}\) when \(t = 2\) below for illustration.

**Step 1.** For subjects with \(R_2 = 1\), obtain the fitted outcome mean models \(\hat{\mu}_2^0(H_1)\) for \(a = 0, 1\).

**Step 2.** For subjects with \(R_1 = 1\), obtain the following quantities:

(a) The fitted propensity score model \(\hat{e}(H_1)\).

(b) The fitted response probability model \(\hat{\pi}_2(a, H_1)\).

(c) The fitted outcome mean model \(\hat{\mu}_2^0(H_0)\), by fitting \(\mu_2^0(H_0) = \mathbb{E} \{\mu_2^0(H_1) \mid H_0, R_1 = 1, A = 0\}\) using the predicted values \(\hat{\mu}_2^0(H_0)\) against \(H_0\) in the group with \(R_1 = 1\) and \(A = 0\).
(d) The fitted models \( \hat{g}_2^1(H_0), \hat{g}_3^1(H_0) \): Fit \( g_2^1(H_0) = \mathbb{E} \{ \pi_2(1, H_1) \mu_2^1(H_1) \mid H_0, R_1 = 1, A = 1 \} \) and \( g_3^1(H_0) = \mathbb{E} \{ (1 - \pi_2(1, H_1)) \mu_2^0(H_1) \mid H_0, R_1 = 1, A = 1 \} \) using the predicted values \( \hat{\pi}_2(1, H_1) \hat{\mu}_2^1(H_1) \) and \( (1 - \hat{\pi}_2(1, H_1)) \hat{\mu}_2^0(H_1) \) against \( H_0 \) in the group with \( R_1 = 1 \) and \( A = 1 \), respectively.

(e) The calibration weights \( w_{r_2} \) associated with the response indicator \( R_2 \): Solve the optimization problem \( \text{[1]} \) subject to \( \sum_{i:R_2,i=1} w_{r_2,i} h(X_i) = \sum_{i:R_1,i=1} h(X_i) / (\sum_{i=1}^n R_{1,i}) \).

**Step 3.** For all the subjects, obtain the following models:

(a) The fitted propensity score model \( \hat{c}(H_0) \) and the ratio \( \hat{d}(H_1) \) for the subjects with \( R_1 = 1 \).

(b) The fitted response probability model \( \hat{\pi}_1(a, H_0) \).

(c) The calibration weights \( w_{r_1} \) associated with the response indicator \( R_1 \): Solve the optimization problem \( \text{[1]} \) subject to \( \sum_{i:R_1,i=1} w_{r_1,i} h(X_i) = n^{-1} \sum_{i=1}^n h(X_i) \).

(d) The calibration weights \( w_{a_1}, w_{a_0} \) associated with the treatment: Solve the optimization problem \( \text{[1]} \) subject to \( \sum_{i:A_i=1} w_{a_1,i} h(X_i) = n^{-1} \sum_{i=1}^n h(X_i) \) to get \( w_{a_1} \); subject to \( \sum_{i:A_i=0} w_{a_0,i} h(X_i) = n^{-1} \sum_{i=1}^n h(X_i) \) to get \( w_{a_0} \).

**Step 4.** Get the calibration-based estimator as

\[
\hat{\tau}_{\text{mr-c}} = \mathbb{P}_n \left[ A w_{a_1} \left( R_2 Y_2 + R_1 (1 - R_2) \hat{\mu}_2^0(H_1) + (1 - R_1) \hat{\mu}_2^0(H_0) \right) \right. \\
- \hat{\pi}_1(1, H_0) \mathbb{E} \left\{ \hat{\pi}_2(1, H_1) \hat{\mu}_2^1(H_1) \mid H_0, R_1 = 1, A = 1 \right\} \\
- \hat{\pi}_1(1, H_0) \mathbb{E} \left\{ (1 - \hat{\pi}_2(1, H_1)) \hat{\mu}_2^0(H_1) \mid H_0, R_1 = 1, A = 1 \right\} \\
- \left\{ 1 - \hat{\pi}_1(1, H_0) \right\} \hat{\mu}_2^0(H_0) \] / \mathbb{P}_n(A w_{a_1}) \\
+ \mathbb{P}_n \left( \hat{\pi}_1(1, H_0) \mathbb{E} \left\{ \hat{\pi}_2(1, H_1) \hat{\mu}_2^1(H_1) \mid H_0, R_1 = 1, A = 1 \right\} \\
+ \hat{\pi}_1(1, H_0) \mathbb{E} \left\{ (1 - \hat{\pi}_2(1, H_1)) \hat{\mu}_2^0(H_1) \mid H_0, R_1 = 1, A = 1 \right\} - \left\{ 1 - \hat{\pi}_1(1, H_0) \right\} \hat{\mu}_2^0(H_0) \right) \\
+ \mathbb{P}_n \left[ (1 - A) R_2 w_{a_0} w_{r_1} w_{r_2} \left\{ \hat{\pi}_1(1, H_0) \{1 - \hat{\pi}_2(1, H_1)\} \hat{\delta}(H_1) \\
- \hat{\pi}_1(1, H_0) \right\} \{ Y_2 - \hat{\mu}_2^0(H_1) \} \right] / \{ \mathbb{P}_n (1 - A) R_2 w_{a_0} w_{r_1} w_{r_2} \} \\
- \mathbb{P}_n \left[ (1 - A) R_1 w_{a_0} w_{r_1} \hat{\pi}_1(1, H_0) \{ \hat{\mu}_2^0(H_1) - \hat{\mu}_2^0(H_0) \} \right] / \mathbb{P}_n \left\{ (1 - A) R_1 w_{a_0} w_{r_1} \right\}.
\]
S4 Proof of the multiple robustness

We prove the multiple robustness and semiparametric efficiency for the EIF-based estimators. For the cross-sectional data, we prove the triple robustness in two aspects: consistency when using parametric models and rate convergence when using flexible models. For the longitudinal outcomes, we focus on the multiple robustness in terms of the rate consistency when using parametric models and rate convergence when using flexible models.

S4.1 Proof of Theorem 3

Proof of the triply robustness: Suppose the model estimators $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})^T$ converges to $\theta^* = (\alpha^*, \beta^*, \gamma^*)^T$ in the sense that $\|\hat{\theta} - \theta^*\| = o_p(1)$, where at least one component of $\hat{\theta}$ needs to converge to the true value. As the sample size $n \to \infty$, we would expect $\hat{\tau}_{tr}$ converges to

$$\mathbb{E} \left\{ \frac{A}{e(X; \alpha^*)} - \frac{1 - A}{1 - e(X; \alpha^*)} \frac{\pi_1(1, X; \gamma^*)}{\pi_1(0, X; \gamma^*)} R_1 \{ Y_1 - \mu_1^0(X; \beta^*) \} \right\}$$

(S3)

$$- \mathbb{E} \left\{ \frac{A - e(X; \alpha^*)}{e(X; \alpha^*)} \pi_1(1, X; \gamma^*) \{ \mu_1^1(X; \beta^*) - \mu_1^0(X; \beta^*) \} \right\}$$

(S4)

Rearrange (S3), we have

$$\mathbb{E} \left\{ \frac{A}{e(X; \alpha^*)} - \frac{1 - A}{1 - e(X; \alpha^*)} \frac{\pi_1(1, X; \gamma^*)}{\pi_1(0, X; \gamma^*)} R_1 \{ Y_1 - \mu_1^0(X; \beta^*) \} \right\} = \mathbb{E} \left[ \frac{E(A | X)}{e(X; \alpha^*)} E(R_1 | X, A = 1) \{ E(Y_1 | X, R_1 = 1, A = 1) - \mu_1^0(X; \beta^*) \} \right]$$

$$- \mathbb{E} \left[ \frac{E(1 - A | X)}{1 - e(X; \alpha^*)} \pi_1(1, X; \gamma^*) E(R_1 | X, A = 0) \{ E(Y_1 | X, R_1 = 1, A = 0) - \mu_1^0(X; \beta^*) \} \right]$$

$$= \mathbb{E} \left[ \frac{e(X)}{e(X; \alpha^*)} \pi_1(1, X) \{ \mu_1^1(X) - \mu_1^0(X; \beta^*) \} - \frac{1 - e(X)}{1 - e(X; \alpha^*)} \frac{\pi_1(0, X) \pi_1(1, X; \gamma^*)}{\pi_1(0, X; \gamma^*)} \{ \mu_1^1(X) - \mu_1^0(X; \beta^*) \} \right]$$

$$= \mathbb{E} \left[ \pi_1(1, X) \{ \mu_1^1(X) - \mu_1^0(X) \} + \pi_1(1, X) \mu_1^1(X) \left\{ \frac{e(X)}{e(X; \alpha^*)} - 1 \right\} + \pi_1(1, X) \left\{ \mu_1^0(X) - \frac{e(X)}{e(X; \alpha^*)} \mu_1^0(X; \beta^*) \right\} \right]$$

$$- \mathbb{E} \left[ \frac{1 - e(X)}{1 - e(X; \alpha^*)} \pi_1(0, X) \pi_1(1, X; \gamma^*) \{ \mu_1^0(X) - \mu_1^0(X; \beta^*) \} \right]$$

$$= \tau_{tr}^{CR} + \mathbb{E} \left[ \pi_1(1, X) \mu_1^1(X) \left\{ \frac{e(X)}{e(X; \alpha^*)} - 1 \right\} + \pi_1(1, X) \left\{ \mu_1^0(X) - \frac{e(X)}{e(X; \alpha^*)} \mu_1^0(X; \beta^*) \right\} \right]$$

$$- \mathbb{E} \left[ \left\{ \frac{1 - e(X)}{1 - e(X; \alpha^*)} \pi_1(0, X) \pi_1(1, X; \gamma^*) - \frac{e(X)}{e(X; \alpha^*)} \pi_1(1, X) \right\} \mu_1^0(X; \beta^*) \right].$$
Rearrange (S4), we have
\[
\mathbb{E} \left[ \frac{A-e(X)}{e(X; \alpha^*)} \pi_1(1, X; \gamma^*) \{ \mu_1(X; \beta^*) - \mu_1^0(X; \beta^*) \} \right]
\]
\[
= \mathbb{E} \left[ \frac{e(X) - e(X; \alpha^*)}{e(X; \alpha^*)} \pi_1(1, X; \gamma^*) \{ \mu_1(X; \beta^*) - \mu_1^0(X; \beta^*) \} \right].
\]

Combine the two parts together, (S3) + (S4)
\[
= \tau_{1 \text{CR}} + \mathbb{E} \left[ \pi_1(1, X) \left\{ \frac{e(X)}{e(X; \alpha^*)} - 1 \right\} \mu_1^0(X) + \left\{ \pi_1(1, X) - \frac{1 - e(X)}{1 - e(X; \alpha^*)} \pi_1(0, X) \pi_1(1, X; \gamma^*) \right\} \mu_1^0(X) \right]
\]
\[
+ \mathbb{E} \left[ \left\{ 1 - \frac{1 - e(X)}{1 - e(X; \alpha^*)} \frac{\pi_1(0, X) \pi_1(1, X; \gamma^*)}{\pi_1(0, X; \gamma^*)} \right\} \pi_1(1, X; \gamma^*) \{ \mu_1^0(X) - \mu_1^0(X; \beta^*) \} \right]
\]
\[
+ \mathbb{E} \left[ \pi_1(1, X) - \pi_1(1, X; \gamma^*) \right\{ \mu_1^0(X) - \frac{e(X)}{e(X; \alpha^*)} \mu_1^0(X; \beta^*) \right\}
\]

Therefore, the bias of \( \hat{\tau}_{1\text{r}} \) converges to
\[
\mathbb{E} \left[ \left\{ \frac{e(X)}{e(X; \alpha^*)} - 1 \right\} \left\{ \pi_1(1, X) \mu_1^0(X) - \pi_1(1, X; \gamma^*) \mu_1^0(X; \beta^*) \right\} \right] \quad (S5)
\]
\[
+ \mathbb{E} \left[ \left\{ 1 - \frac{1 - e(X)}{1 - e(X; \alpha^*)} \frac{\pi_1(0, X) \pi_1(1, X; \gamma^*)}{\pi_1(0, X; \gamma^*)} \right\} \pi_1(1, X; \gamma^*) \left\{ \mu_1^0(X) - \mu_1^0(X; \beta^*) \right\} \right] \quad (S6)
\]
\[
+ \mathbb{E} \left[ \left\{ \pi_1(1, X) - \pi_1(1, X; \gamma^*) \right\} \left\{ \mu_1^0(X) - \frac{e(X)}{e(X; \alpha^*)} \mu_1^0(X; \beta^*) \right\} \right] \quad (S7)
\]

Note that (S5) = 0 under \( \mathcal{M}_{\text{rp+om}} \cup \mathcal{M}_{\text{ps}} \), (S6) = 0 under \( \mathcal{M}_{\text{ps+rp}} \cup \mathcal{M}_{\text{om}} \), (S7) = 0 under \( \mathcal{M}_{\text{ps+om}} \cup \mathcal{M}_{\text{rp}} \). Thus, \( \hat{\tau}_{1\text{r}} \) is consistent for \( \tau_{1 \text{JR}} \) under \( \mathcal{M}_{\text{rp+om}} \cup \mathcal{M}_{\text{ps+om}} \cup \mathcal{M}_{\text{ps+rp}} \). The triple robustness holds.

**Proof of semiparametric efficiency:** We follow the proof in [Kennedy (2016)](Kennedy2016). To simplify the notations, denote \( \mathbb{P} \{ N(V; \theta_0) \} = \tau_{1 \text{JR}} \), where
\[
N(V; \theta_0) = \left\{ \frac{A}{e(X)} - \frac{1 - A}{1 - e(X)} \frac{\pi_1(1, X)}{\pi_1(0, X)} \right\} R_1 \left\{ Y_1 - \mu_1^0(X) \right\} - \frac{A - e(X)}{e(X)} \pi_1(1, X) \left\{ \mu_1^0(X) - \mu_1^0(X) \right\} .
\]

Then \( \mathbb{P} \{ N(V; \theta^*) \} = \mathbb{P} \{ N(V; \theta_0) \} = \tau_{1 \text{JR}} \). Consider the decomposition
\[
\hat{\tau}_{1\text{r}} - \tau_{1 \text{JR}} = (\mathbb{P}_n - \mathbb{P}) N(V; \hat{\theta}) - \mathbb{P} \left\{ N(V; \hat{\theta}) - N(V; \theta^*) \right\} . \quad (S8)
\]

Using empirical process theory, if the nuisance functions take values in Donsker classes, and satisfy the positivity assumption, i.e., there exists \( \varepsilon > 0 \), such that \( \varepsilon < e(X) < 1 - \varepsilon \),
and \( \pi_1(0, X) > \varepsilon \) for all \( X \), then \( N(V; \hat{\theta}) \) takes values in Donsker classes, and the first term can be written as
\[
(\mathbb{P}_n - \mathbb{P}) N(V; \hat{\theta}) = (\mathbb{P}_n - \mathbb{P}) N(V; \theta_0) + o_P(n^{-\frac{1}{2}}).
\]

For the second term \( \mathbb{P}\left\{ N(V; \hat{\theta}) - N(V; \theta^*) \right\} \), by computing the expectations, we have
\[
\mathbb{P}\left\{ N(V; \hat{\theta}) - N(V; \theta^*) \right\} = \mathbb{P}\left[ \left\{ \frac{e(X)}{e(X; \hat{\alpha})} - 1 \right\} \left\{ \pi_1(1, X) \mu_1^X(X) - \pi_1(1, X; \hat{\gamma}) \mu_1^X(\hat{\beta}) \right\} \right]
+ \mathbb{P}\left[ \left\{ 1 - \frac{1 - e(X)}{1 - e(X; \hat{\alpha})} \pi_1(0, X; \hat{\gamma}) \right\} \pi_1(1, X; \hat{\gamma}) \left\{ \mu_0^X(X) - \mu_0^X(\hat{\beta}) \right\} \right]
+ \mathbb{P}\left[ \left\{ \pi_1(1, X) - \pi_1(1, X; \hat{\gamma}) \right\} \left\{ \mu_0^X(X) - \frac{e(X)}{e(X; \hat{\alpha})} \mu_0^X(\hat{\beta}) \right\} \right].
\]

Under the positivity assumptions, we apply Cauchy-Schwarz inequality \((\mathbb{P}(fg) \leq \|f\|\|g\|)\) and obtain a upper bound for the second term as
\[
\mathbb{P}\left\{ N(V; \hat{\theta}) - N(V; \theta^*) \right\} \leq \left\| \frac{e(X)}{e(X; \hat{\alpha})} - 1 \right\| \left\{ \pi_1(1, X) \mu_1^X(X) - \pi_1(1, X; \hat{\gamma}) \mu_1^X(\hat{\beta}) \right\}
+ \left\| 1 - \frac{1 - e(X)}{1 - e(X; \hat{\alpha})} \pi_1(0, X; \hat{\gamma}) \right\| \left\{ \pi_1(1, X; \hat{\gamma}) \left\{ \mu_0^X(X) - \mu_0^X(\hat{\beta}) \right\} \right\}
+ \left\| \pi_1(1, X) - \pi_1(1, X; \hat{\gamma}) \right\| \left\{ \mu_0^X(X) - \frac{e(X)}{e(X; \hat{\alpha})} \mu_0^X(\hat{\beta}) \right\}
\leq \left\{ \frac{e(X)}{e(X; \hat{\alpha})} - 1 \right\} \left\{ \mu_1^X(X) - \mu_1^X(\hat{\beta}) \right\} \left\{ \pi_1(1, X) \right\}_\infty
+ \left\{ 1 - \frac{1 - e(X)}{1 - e(X; \hat{\alpha})} \right\} \left\{ \mu_0^X(X) - \mu_0^X(\hat{\beta}) \right\} \left\{ \pi_1(1, X; \hat{\gamma}) \right\}_\infty
+ \left\{ \pi_1(1, X) - \pi_1(1, X; \hat{\gamma}) \right\} \left\{ \mu_0^X(X) - \mu_0^X(\hat{\beta}) \right\} \left\{ \pi_1(1, X) \right\}_\infty
\]

The second inequality holds by the triangle inequality and Holder’s inequality, and the last inequality holds by Cauchy-Schwarz. Under \( \mathcal{M}_{ps+rp+om} \), we would expect \( \mathbb{P}\left\{ N(V; \hat{\theta}) - N(V; \theta^*) \right\} = O_P(n^{-1/2}) \cdot o_P(1) = o_P(n^{-1/2}) \). Therefore, the EIF-based estimator \( \hat{\tau}_t \) satisfies \( \hat{\tau}_t - \tau_1^{J2R} = \ldots \)
\((P_n - P)V(\theta_0) + o_P(n^{-\frac{1}{2}})\) and its influence function \(N(V; \theta_0) + \tau_1^{J2R}\), which is the same as the EIF in Theorem 2 and completes the proof.

### S4.2 Proof of Theorem 4 and Corollary 2

**Proof of Theorem 4**: When using flexible models, we let \(\theta\) consist of all the nuisance functions \(\{e(X), \pi_1(a, X), \mu_1^a(X) : a = 0, 1\}\), and \(\hat{\theta}\) be its limit. We use the same notations in S4.1 and consider the same decomposition as formula \(\text{(S8)}\). Using empirical process theory, if the nuisance functions take values in Donsker classes, and satisfy the positivity assumption, i.e., there exists \(\varepsilon > 0\), such that \(\varepsilon < e(X) < 1 - \varepsilon\) and \(\pi_1(0, X) > \varepsilon\) for all \(X\), then \(N(V; \hat{\theta})\) takes values in Donsker classes, and the first term can be written as

\[
(P_n - P)V(\hat{\theta}) = (P_n - P)V(\theta_0) + o_P(n^{-\frac{1}{2}}).
\]

For the second term \(P\left\{N(V; \hat{\theta}) - N(V; \theta^*)\right\}\), by computing the expectations, we have

\[
P\left\{N(V; \hat{\theta}) - N(V; \theta^*)\right\} = P\left\{\left\{e(X)\right\} - 1\right\} \left\{\pi_1(1, X)\mu_1^1(X) - \hat{\pi}_1(1, X)\hat{\mu}_1^1(X)\right\}
\]

\[+ P\left\{1 - \frac{1 - e(X)}{1 - \hat{e}(X)} \pi_1(0, X)\right\} \hat{\pi}_1(1, X) \left\{\mu_1^0(X) - \hat{\mu}_1^0(X)\right\}\]

\[+ P\left\{\pi_1(1, X) - \hat{\pi}_1(1, X)\right\} \left\{\mu_1^0(X) - e(X)\hat{\mu}_1^0(X)\right\} = \text{Rem}(\hat{P}, P)
\]

Therefore, \(\hat{\tau}_{tr} - \tau_1^{J2R} = (P_n - P)V(\theta_0) + \text{Rem}(\hat{P}, P) + o_P(n^{-\frac{1}{2}}) = P_n\left\{\phi_1^{J2R}(V_i; P)\right\} + \text{Rem}(\hat{P}, P) + o_P(n^{-1/2})\). If \(\text{Rem}(\hat{P}, P) = o_P(n^{-1/2})\), then \(\hat{\tau}_{tr} - \tau_1^{J2R} = n^{-1}\sum_{i=1}^n \phi_1^{J2R}(V_i; P) + o_P(n^{-1/2})\). Apply central limit theorem and we complete the proof.

**Proof of Corollary 2**: For the remainder term, based on the uniform bounded condition, apply Cauchy-Schwarz and Holder’s inequality, we have

\[
P\left\{N(V; \hat{\theta}) - N(V; \theta^*)\right\} \leq M\left\|\frac{e(X)}{\hat{e}(X)} - 1\right\| \left\{\left\|\mu_1^1(X) - \hat{\mu}_1^1(X)\right\| + \left\|\pi_1(1, X) - \hat{\pi}_1(1, X)\right\|\right\}
\]

\[+ M\left\|\mu_1^0(X) - \hat{\mu}_1^0(X)\right\| \left\{\left\|1 - \frac{1 - e(X)}{1 - \hat{e}(X)}\right\| + \left\|1 - \frac{\pi_1(0, X)}{\hat{\pi}_1(0, X)}\right\|\right\}
\]

\[+ M\left\|\pi_1(1, X) - \hat{\pi}_1(1, X)\right\| \left\{\left\|\frac{e(X)}{\hat{e}(X)} - 1\right\| + \left\|\mu_1^0(X) - \hat{\mu}_1^0(X)\right\|\right\}.
\]

With the convergence rate \(\|\hat{e}(X) - e(X)\| = o_P(n^{-\varepsilon_0}), \|\mu_1^a(X) - \mu_1^a(X)\| = o_P(n^{-\varepsilon_0}), \|\pi_1(a, X) - \pi_1(a, X)\| = o_P(n^{-\varepsilon_0})\), and by Theorem 4 based on the central limit theorem, we have
\( \hat{\tau}_{IR} - \tau_{IR}^{JR} = O_P(n^{-1/2} + n^{-c}) \), where \( c = \min(r_x, r_\pi, r_\mu, r_\pi + r_\mu) \), which completes the proof.

S4.3 Proof of Theorem 7 and Corollary 4

Proof of Theorem 7: When using flexible models, we let \( \theta \) consist of all the nuisance functions \( \{e(H_{s-1}), \pi_s(a, H_{s-1}), \mu_s^0(H_{s-1}), g_{s+1}^1(H_{t-1}) : l = 1, \ldots, s \text{ and } s = 1, \ldots, t; a = 0, 1 \} \), and \( \hat{\theta} \) be its limit. We use the same notations in S4.1 and denote \( N(V; \theta) := \phi_{IR}^*(V; \mathbb{P}) + \tau_{IR}^{JR} \).

Consider the same decomposition as formula (S8) functions \( \theta \) and satisfy the positivity assumption, i.e., there exists \( \varepsilon > 0 \), such that \( \varepsilon < \{e(H_{s-1}), \hat{e}(H_{s-1})\} < 1 - \varepsilon \) and \( \{\pi_s(0, H_{s-1}), \hat{\pi}_s(0, H_{s-1})\} > \varepsilon \) for all \( H_{s-1} \) when \( s = 1, \ldots, t \), then \( N(V; \hat{\theta}) \) takes values in Donsker classes, and the first term can be written as

\[
(\mathbb{P}_n - \mathbb{P}) N(V; \hat{\theta}) = (\mathbb{P}_n - \mathbb{P}) N(V; \theta_0) + o_P(n^{-\frac{1}{2}}).
\]

For the second term \( \mathbb{P}\{N(V; \hat{\theta}) - N(V; \theta^*)\} \), we proceed by deriving the expectations of \( N(V; \hat{\theta}) - N(V; \theta^*) \). Note that \( \mathbb{P}\{N(V; \hat{\theta})\} \) equals to

\[
\mathbb{P}\left\{ \frac{A}{\hat{e}(H_0)} \left\{ R_tY_t + \sum_{s=1}^{t} R_{s-1}(1 - R_s)\hat{\mu}_t^0(H_{s-1}) \right\} \right. \\
+ \left\{ 1 - \frac{A}{\hat{e}(H_0)} \right\} \left[ \hat{\pi}_1(1, H_0) \sum_{s=1}^{t-1} \hat{g}_{s+1}^1(H_0) + \left\{ 1 - \hat{\pi}_1(1, H_0) \right\} \hat{\mu}_t^0(H_0) \right] - \hat{\mu}_t^0(H_0) \\
+ \frac{1 - A}{1 - \hat{e}(H_0)} \left( \sum_{s=1}^{t} \sum_{k=1}^{s} \hat{\pi}_k(0, H_{k-2}) \{ 1 - \hat{\pi}_k(1, H_{k-1}) \} \hat{\delta}(H_{k-1}) - 1 \right) \left\{ \frac{R_s}{\hat{\pi}_s(0, H_{s-1})} \{ \hat{\mu}_t^0(H_s) - \hat{\mu}_t^0(H_{s-1}) \} \right\} \right\}. \\
\]

(S11)

By iterated expectations, the first term (S9) and the second term (S10) equal to

\[
\mathbb{E}\left( \frac{e(H_0)}{\hat{e}(H_0)} \left[ \hat{\pi}_1(1, H_0) g_{s+1}^1(H_0) + \sum_{s=1}^{t-1} \hat{\pi}_1(1, H_0) g_{s+1}^1(H_0) + \{ 1 - \hat{\pi}_1(1, H_0) \} \hat{\mu}_t^0(H_{s-1}) \right] \right. \\
+ \left\{ 1 - \frac{e(H_0)}{\hat{e}(H_0)} \right\} \left[ \hat{\pi}_1(1, H_0) \sum_{s=1}^{t-1} \hat{g}_{s+1}^1(H_0) + \left\{ 1 - \hat{\pi}_1(1, H_0) \right\} \hat{\mu}_t^0(H_0) \right] - \hat{\mu}_t^0(H_0) \right),
\]

using the notations in the main text.

For the third term (S11), for \( s = 1, \ldots, t \), we have

\[
\mathbb{E}\left( \frac{1 - A}{1 - \hat{e}(H_0)} \left[ \hat{\pi}_{k-1}(0, H_{k-2}) \{ 1 - \hat{\pi}_k(1, H_{k-1}) \} \hat{\delta}(H_{k-1}) - 1 \right] \frac{R_s}{\hat{\pi}_s(0, H_{s-1})} \{ \hat{\mu}_t^0(H_s) - \hat{\mu}_t^0(H_{s-1}) \} \right.
\]

S29
\[
| H_{s-1}, R_{s-1} = 1, A = 0 \\
= \mathbb{E} \left( \frac{1 - A}{1 - \hat{c}(H_0)} \left[ \hat{\pi}_{k-1}(0, H_{k-2}) \{1 - \hat{\pi}_k(1, H_{k-1})\} \hat{\delta}(H_{k-1}) - 1 \right] \left[ \mathbb{E} \left\{ \hat{\mu}_t^0(H_s) \mid H_{s-1}, R_s = 1, A = 0 \right\} - \hat{\mu}_t^0(H_{s-1}) \right] \right) \\
\times \frac{R_{s-1}}{\hat{\pi}_{s-1}(0, H_{s-2}) \hat{\pi}_s(0, H_{s-1})} \mid H_{s-1}, R_{s-1} = 1, A = 0 \right). \\
\]

And for \( k = 1, \ldots, s \), apply iterated expectations to the above formula and use the notation in the main text, we have

\[
\mathbb{E} \left( \frac{1 - A}{1 - \hat{c}(H_0)} \left[ \hat{\pi}_{k-1}(0, H_{k-2}) \{1 - \hat{\pi}_k(1, H_{k-1})\} \hat{\delta}(H_{k-1}) - 1 \right] \frac{R_k}{\hat{\pi}_k(0, H_{k-1})} \prod_{l=k+1}^{s} \frac{R_l}{\hat{\pi}_l(0, H_{l-1})} \left[ \mathbb{E} \left\{ \hat{\mu}_t^0(H_s) \mid H_{s-1}, R_s = 1, A = 0 \right\} - \hat{\mu}_t^0(H_{s-1}) \right] \mid H_{s-1}, R_{s-1} = 1, A = 0 \right) \\
\mathbb{E} \left( \frac{\pi_{k+1}(0, H_k)}{\hat{\pi}_{k+1}(0, H_k)} \ldots \right. \\
\mathbb{E} \left( \frac{\pi_s(0, H_{s-1})}{\hat{\pi}_s(0, H_{s-1})} \left[ \mathbb{E} \left\{ \hat{\mu}_t^0(H_s) \mid H_{s-1}, R_s = 1, A = 0 \right\} - \hat{\mu}_t^0(H_{s-1}) \right] \mid H_{s-2}, R_{s-1} = 1, A = 0 \right) \\
\mathbb{E} \left( \frac{\pi_{k-1}(0, H_{k-2})}{\hat{\pi}_{k-1}(0, H_{k-2})} \prod_{l=k}^{s} \frac{\pi_l(0, H_{l-1})}{\hat{\pi}_l(0, H_{l-1})} \left[ \mathbb{E} \left\{ \hat{\mu}_t^0(H_s) \mid H_{s-1}, R_s = 1, A = 0 \right\} - \hat{\mu}_t^0(H_{s-1}) \right] ; H_{k-1} \right) \right] \\
\mathbb{E} \left( \frac{1 - A}{1 - \hat{c}(H_0)} \left[ R_{k-1} \{1 - \hat{\pi}_k(1, H_{k-1})\} \hat{\delta}(H_{k-1}) - 1 \right] \frac{R_{k-1}}{\hat{\pi}_{k-1}(0, H_{k-2})} \prod_{l=k}^{s} \frac{\pi_l(0, H_{l-1})}{\hat{\pi}_l(0, H_{l-1})} \left[ \mathbb{E} \left\{ \hat{\mu}_t^0(H_s) \mid H_{s-1}, R_s = 1, A = 0 \right\} - \hat{\mu}_t^0(H_{s-1}) \right] ; H_{k-1} \right) \\
\mathbb{E} \left[ \frac{1 - A}{1 - \hat{c}(H_0)} R_{k-1} \{1 - \hat{\pi}_k(1, H_{k-1})\} \hat{\delta}(H_{k-1}) \delta(H_{k-1}) G_{\hat{\mu}, \breve{\pi}, s-2}(H_{k-1}) \right] \\
- \mathbb{E} \left[ \frac{1 - A}{1 - \hat{c}(H_0)} \frac{R_{k-1}}{\hat{\pi}_{k-1}(0, H_{k-2})} G_{\hat{\mu}, \breve{\pi}, s-2}(H_{k-1}) \right] \\
\text{(S12)} \\
\text{(S13)}
\]

if we denote

\[
G_{\hat{\mu}, \breve{\pi}, s-2}(H_{k-1}) = E_{0,s-2} \left( \prod_{l=k}^{s} \pi_l(0, H_{l-1}) \left[ \mathbb{E} \left\{ \hat{\mu}_t^0(H_s) \mid H_{s-1}, R_s = 1, A = 0 \right\} - \hat{\mu}_t^0(H_{s-1}) \right] / \hat{\pi}_l(0, H_{l-1}) ; H_{k-1} \right) \\
\]

to indicate the involvement of the estimated nuisance function \( \hat{\mu}_t^0(H_{l-1}) \) and \( \hat{\pi}_l(0, H_{l-1}) \) for \( l = k, \ldots, s \).
For the first term \([S12]\), by Bayes’ rule,

\[
\delta(H_{s-1}) = \frac{\hat{\pi}_{s-1}(1, H_{s-2})}{\hat{\pi}_{s-1}(0, H_{s-2})} \prod_{t=1}^{s-1} \frac{f(Y_t | H_{t-1}, R_t = 1, A = 1)}{f(Y_t | H_{t-1}, R_t = 1, A = 0)}.
\]

Take iterated expectations conditional on the historical information, it equals to

\[
\mathbb{E}\left[ \frac{1 - A}{1 - \hat{e}(H_0)} \frac{R_{k-1}}{\hat{\pi}_{k-1}(0, H_{k-2})} \hat{\pi}_{k-1}(1, H_{k-2}) \{1 - \hat{\pi}_k(1, H_{k-1})\} \frac{\hat{\delta}(H_{k-1})}{\delta(H_{k-1})} \prod_{t=1}^{k-2} f(Y_t | H_{t-1}, R_t = 1, A = 1) f(Y_t | H_{t-1}, R_t = 1, A = 0) \right] 
\]

\[
= \mathbb{E}\left[ \frac{1 - A}{1 - \hat{e}(H_0)} \frac{R_{k-1}}{\hat{\pi}_{k-1}(0, H_{k-2})} \hat{\pi}_{k-1}(1, H_{k-2}) \prod_{t=1}^{k-2} f(Y_t | H_{t-1}, R_t = 1, A = 1) f(Y_t | H_{t-1}, R_t = 1, A = 0) \right] 
\]

\[
\mathbb{E}\left[ \{1 - \hat{\pi}_k(1, H_{k-1})\} \frac{\hat{\delta}(H_{k-1}) f(Y_{k-1} | H_{k-2}, R_{k-1} = 1, A = 1)}{\delta(H_{k-1}) f(Y_{k-1} | H_{k-2}, R_{k-1} = 1, A = 0)} G_{\bar{\mu}, \bar{\pi}, s-2}(H_{k-1}) \right] 
\]

\[
= \mathbb{E}\left[ \frac{1 - A}{1 - \hat{e}(H_0)} \frac{R_{k-2}}{\hat{\pi}_{k-2}(0, H_{k-1})} \hat{\pi}_{k-1}(1, H_{k-2}) \prod_{t=1}^{k-2} f(Y_t | H_{t-1}, R_t = 1, A = 1) f(Y_t | H_{t-1}, R_t = 1, A = 0) \right] 
\]

\[
\{1 - \hat{\pi}_k(1, H_{k-1})\} \frac{\delta(H_{k-1})}{\delta(H_{k-1})} G_{\bar{\mu}, \bar{\pi}, s-2}(H_{k-1}) \right) 
\]

\[
: = E_{1,k-2} \left[ \frac{1 - e(H_0)}{1 - \hat{e}(H_0)} \hat{\pi}_{k-1}(1, H_{k-2}) \{1 - \hat{\pi}_k(1, H_{k-1})\} \frac{\hat{\delta}(H_{k-1})}{\delta(H_{k-1})} G_{\bar{\mu}, \bar{\pi}, s-2}(H_{k-1}) ; H_0 \right].
\]

For the second term \([S13]\), again by iterated expectations,

\[
\mathbb{E}\left[ \cdots \mathbb{E}\left[ \{1 - e(H_0) \hat{\pi}_{k-1}(0, H_{k-2}) \frac{G_{\bar{\mu}, \bar{\pi}, s-2}(H_{k-1})}{\delta(H_{k-1})} \} \right] \cdot \frac{G_{\bar{\mu}, \bar{\pi}, s-2}(H_{k-1})}{\delta(H_{k-1})} \right] 
\]

\[
\mathbb{E}_{0,0} \left[ \{1 - e(H_0) \hat{\pi}_{k-1}(0, H_{k-2}) \frac{G_{\bar{\mu}, \bar{\pi}, s-2}(H_{k-1})}{\delta(H_{k-1})} \} \right] \text{ (by the definition of } G_{\bar{\mu}, \bar{\pi}, s-2}(H_0)).
\]

Therefore, as the sample size \( n \to \infty \), the multiply robust estimator \( \hat{\tau}_{nr} \) converges to

\[
\mathbb{E}\left( \frac{e(H_0)}{\hat{e}(H_0)} \left[ \pi_1(1, H_0) g_{1+1}^{1}(H_0) + \sum_{s=1}^{t-1} \pi_1(1, H_0) g_{s+1}^{1}(H_0) \{1 - \pi(1, H_0)\} \hat{\mu}_s^0(H_0) \right] \right.
\]

\[
+ \left\{ 1 - \frac{e(H_0)}{\hat{e}(H_0)} \right\} \left[ \sum_{s=1}^{t} \hat{\pi}_1(1, H_0) \hat{\delta} g_{s+1}^{1}(H_0) + \{1 - \hat{\pi}_1(1, H_0)\} \hat{\mu}_s^0(H_0) \right] - \hat{\mu}_t^0(H_0)
\]

\[
+ \frac{1 - e(H_0)}{1 - \hat{e}(H_0)} \sum_{s=1}^{t} \left\{ \sum_{k=1}^{s} E_{1,k-2} \left[ \hat{\pi}_{k-1}(1, H_{k-2}) \{1 - \hat{\pi}_k(1, H_{k-1})\} \hat{\delta}(H_{k-1}) G_{\bar{\mu}, \bar{\pi}, s-2}(H_{k-1}) ; H_0 \right] - G_{\bar{\mu}, \bar{\pi}, s-2}(H_0) \right\}.
\]
Rearrange the terms, we can get the formula for \( \mathbb{P} \left\{ N(V; \hat{\theta}) = N(V; \theta^*) \right\} \) as

\[
\mathbb{E} \left\{ \frac{e(H_0)}{\bar{e}(H_0)} - 1 \right\} \pi_1(1, H_0) g_{t+1}^1(H_0) + \frac{e(H_0)}{\bar{e}(H_0)} \pi_1(1, H_0) \sum_{s=1}^{t-1} g_{s+1}^1(H_0) - \pi_1(1, H_0) \sum_{s=1}^{t-1} g_{s+1}^1(H_0) \\
+ \frac{e(H_0)}{\bar{e}(H_0)} \{ 1 - \pi_1(1, H_0) \} \bar{\mu}_t^0(H_0) + \pi_1(1, H_0) \mu_t^0(H_0) \\
+ \{ 1 - \frac{e(H_0)}{\bar{e}(H_0)} \} \left[ \sum_{s=1}^{t} \bar{\pi}_1(1, H_0) \bar{g}_{s+1}^1(H_0) + \{ 1 - \bar{\pi}_1(1, H_0) \} \bar{\mu}_t^0(H_0) \right] - \bar{\mu}_t^0(H_0) \\
+ \frac{1 - e(H_0)}{1 - \bar{e}(H_0)} \sum_{s=1}^{t} \left[ \bar{\pi}_{k-1}(1, H_{k-2}) \{ 1 - \bar{\pi}_k(1, H_{k-1}) \} \frac{\delta(H_{k-1})}{\delta(H_{k-1})} G_{\mu, \pi, s-2}(H_{k-1}; H_0) - G_{\mu, \pi, s-2}(H_0) \right] 
\]

For the terms related to \( g_{t+1}^1(H_0) \), we have

\[
\mathbb{E} \left\{ \frac{e(H_0)}{\bar{e}(H_0)} - 1 \right\} \pi_1(1, H_0) g_{t+1}^1(H_0) + \left\{ 1 - \frac{e(H_0)}{\bar{e}(H_0)} \right\} \bar{\pi}_1(1, H_0) \bar{g}_{t+1}^1(H_0) 
\]

\[
= \mathbb{E} \left\{ \frac{e(H_0)}{\bar{e}(H_0)} - 1 \right\} \{ \pi_1(1, H_0) g_{t+1}^1(H_0) - \bar{\pi}_1(1, H_0) \bar{g}_{t+1}^1(H_0) \} 
\]

For the terms with \( s \) layers of expectations and the condition \( A = 1 \) for \( s = 1, \ldots, t \), we have

\[
\mathbb{E} \left\{ \frac{e(H_0)}{\bar{e}(H_0)} \pi_1(1, H_0) g_{\mu,s+1}^1(H_0) - \pi_1(1, H_0) g_{s+1}^1(H_0) \right\} \\
\]

\[
+ \left\{ 1 - \frac{e(H_0)}{\bar{e}(H_0)} \right\} \pi_1(1, H_0) \bar{g}_{s+1}^1(H_0) + \sum_{l=s}^{t} E_{1,s-1} \left[ \bar{\pi}_s(1, H_{s-1}) \{ 1 - \bar{\pi}_{s+1}(1, H_s) \} \frac{\delta(H_s)}{\delta(H_{s-1})} G_{\mu, \pi, l}(H_s; H_0) \right] 
\]

\[
= \mathbb{E} \left\{ \frac{e(H_0)}{\bar{e}(H_0)} - 1 \right\} \{ \pi_1(1, H_0) g_{\mu,s+1}^1(H_0) - \bar{\pi}_1(1, H_0) \bar{g}_{s+1}^1(H_0) \} \\
\]

\[
+ \sum_{l=s}^{t} \mathbb{E} \left\{ \cdots \mathbb{E} \left[ \bar{\pi}_s(1, H_{s-1}) \left\{ 1 - \frac{e(H_0)}{1 - \bar{e}(H_0)} \right\} \{ 1 - \bar{\pi}_{s+1}(1, H_s) \} \frac{\delta(H_s)}{\delta(H_{s-1})} \prod_{k=s+1}^{t} \pi_l(0, H_{l-1}) \right] 
\]

\[
- \{ 1 - \bar{\pi}_{s+1}(1, H_s) \} \{ \bar{\mu}_l^0(H_l) - \bar{\mu}_k^0(H_{l-1}) \} \mid H_{l-1}, R_l = 1, A = 0 \} \cdots \mid H_s, R_{s+1} = 1, A = 0 \} 
\]

\[
= \mathbb{E} \left\{ \frac{e(H_0)}{\bar{e}(H_0)} - 1 \right\} \{ \pi_1(1, H_0) g_{\mu,s+1}^1(H_0) - \bar{\pi}_1(1, H_0) \bar{g}_{s+1}^1(H_0) \} \\
\]

\[
+ \sum_{l=s}^{t} E_{1,s-1} \left[ E_{0,l-1} \left( \bar{\pi}_s(1, H_{s-1}) \left\{ 1 - \frac{e(H_0)}{1 - \bar{e}(H_0)} \right\} \{ 1 - \bar{\pi}_{s+1}(1, H_s) \} \frac{\delta(H_s)}{\delta(H_{s-1})} \prod_{k=s+1}^{t} \pi_l(0, H_{l-1}) \right] 
\]

\[
- \{ 1 - \bar{\pi}_{s+1}(1, H_s) \} \{ \bar{\mu}_l^0(H_l) - \bar{\mu}_k^0(H_{l-1}) \} \mid H_s, H_0, R_{l+1}, A = 1 \} \}
\]

For the rest terms with the condition \( A = 0 \), we have

\[
\mathbb{E} \left\{ \frac{e(H_0)}{\bar{e}(H_0)} \{ 1 - \pi_1(1, H_0) \} \bar{\mu}_t^0(H_0) + \pi_1(1, H_0) \mu_t^0(H_0) \right\} 
\]
\[
\sum_n \mathbb{E} \left[ \frac{1}{1 - \hat{e}(H_0)} \left( \frac{\pi_1(0, H_0)}{\pi_1(0, H_0)} \sum_{s=1}^{t} G_{\hat{\mu}, \hat{\pi}, s-2}(H_0) \right) \right]
\]

\[
= \mathbb{E} \left[ \{\hat{\pi}_1(1, H_0) - \pi_1(1, H_0)\} \{\hat{e}(H_0) \hat{\mu}_t(H_0) - \hat{\mu}_t(H_0)\} + \hat{\pi}_1(1, H_0) \{\mu_t^0(H_0) - \mu_t^0(H_0)\} \right] 
\]

\[
- \frac{1 - e(H_0)}{1 - \hat{e}(H_0)} \hat{\pi}_1(1, H_0) \frac{\pi_1(0, H_0)}{\hat{\pi}_1(0, H_0)} \left( \sum_{s=1}^{t} \mathbb{E} \left[ \cdots \right] \right) 
\]

\[
E \left[ \prod_{l=2}^{s} \frac{\pi_l(0, H_{l-1})}{\hat{\pi}_l(0, H_{l-1})} \left\{ \hat{\mu}_t^0(H_s) - \hat{\mu}_t^0(H_{s-1}) \right\} \mid H_{s-1}, R_s = 1, A = 0 \right] \cdots \mid H_0, R_1 = 1, A = 0 \right) \right] 
\]

\[
= \mathbb{E} \left[ \{\hat{\pi}_1(1, H_0) - \pi_1(1, H_0)\} \{\hat{e}(H_0) \hat{\mu}_t(H_0) - \hat{\mu}_t(H_0)\} + \hat{\pi}_1(1, H_0) \left( \sum_{s=1}^{t} \mathbb{E} \left[ \cdots \right] \right) \right] 
\]

which matches the remainder term in Theorem 7.

Therefore, \( \hat{\tau}_{mr} - \tau_t^{2R} = (\mathbb{P}_n - \mathbb{P}) N(V; \theta_0) + \text{Rem}(\hat{\mathbb{P}}, \mathbb{P}) + o_p(n^{-1/2}) = \mathbb{P}_n \{ \varphi_t^{2R}(V; \mathbb{P}) \} + \text{Rem}(\hat{\mathbb{P}}, \mathbb{P}) + o_p(n^{-1/2}) \). If \( \text{Rem}(\hat{\mathbb{P}}, \mathbb{P}) = o_p(n^{-1/2}) \), then \( \hat{\tau}_{mr} - \tau_t^{2R} = n^{-1} \sum_{i=1}^{n} \varphi_t^{2R}(V_i; \mathbb{P}) + o_p(n^{-1/2}) \). Apply the central limit theorem and we complete the proof.
Proof of Corollary 4: For the remainder term, based on the uniform bounded condition, we proceed to apply Cauchy-Schwarz and Holder’s inequality to obtain the upper bound for each component. For the first term that corresponds to (S14), we have

\[
\leq \left\| \frac{e(H_0)}{\hat{e}(H_0)} - 1 \right\| \cdot \left\| \pi_1(1, H_0) \tilde{g}_{t+1}(H_0) - \hat{\pi}_1(1, H_0) \tilde{g}_{t+1}(H_0) \right\|
\]

\[
+ \left\| \frac{e(H_0)}{\hat{e}(H_0)} - 1 \right\| \cdot \left\{ \sum_{s=1}^{t-1} \left\{ \left\| \pi_1(1, H_0) \tilde{g}_{t+1}(H_0) - \hat{\pi}_1(1, H_0) \tilde{g}_{t+1}(H_0) \right\| \right\}
\]

\[
\leq \left\| \pi_1(1, H_0) \right\| \cdot \left\| \frac{e(H_0)}{\hat{e}(H_0)} - 1 \right\| \cdot \left\{ \left\| g_{t+1}(H_0) - \tilde{g}_{t+1}(H_0) \right\| + \left\| \pi_1(1, H_0) - \hat{\pi}_1(1, H_0) \right\| \right\}
\]

\[
+ \left\| \frac{e(H_0)}{\hat{e}(H_0)} - 1 \right\| \cdot \left\{ \sum_{s=1}^{t-1} \left\{ \left\| g_{t+1}(H_0) - \tilde{g}_{t+1}(H_0) \right\| + \left\| \pi_1(1, H_0) - \hat{\pi}_1(1, H_0) \right\| \right\}
\]

\[
\leq \left\| \frac{e(H_0)}{\hat{e}(H_0)} - 1 \right\| \cdot \left\{ \left\| g_{t+1}(H_0) - \tilde{g}_{t+1}(H_0) \right\| + \sum_{s=1}^{t-1} \left\| g_{t+1}(H_0) - \tilde{g}_{t+1}(H_0) \right\| + (t-1) \left\| \pi_1(1, H_0) - \hat{\pi}_1(1, H_0) \right\| \right\} \quad \text{(since } \pi_1(1, H_0) \leq 1) \]

The second inequality holds by Holder’s inequality and triangle inequality. Based on the derived upper bound, the bound of this term is \(O_{\mathbb{P}}(n^{-\min(c, c_0, c_0, c_0, c_0)})\).

For the second term that corresponds to (S15), we have

\[
\leq \sum_{s=1}^{t-1} \sum_{l=s+1}^{t} \left\| \pi_s(1, H_{s-1}) \right\| \cdot \left\| \frac{1 - e(H_0)}{1 - \hat{e}(H_0)} \{1 - \pi_{s+1}(1, H_s)\} \right\| \delta(H_{s-1}) \prod_{k=s+1}^{l} \frac{\pi_l(0, H_{l-1})}{\overline{\pi}_l(0, H_{l-1})}
\]

\[
- \{1 - \pi_{s+1}(1, H_s)\} \cdot \left\| \mathbb{E} \left\{ \tilde{\mu}_l^0(H_l) \mid H_{l-1}, R_l = 1, A = 0 \right\} - \tilde{\mu}_l^0(H_{l-1}) \right\| \right\}
\]

\[
\leq \sum_{s=2}^{t} \sum_{l=s+1}^{t} \left\| \frac{1 - e(H_0)}{1 - \hat{e}(H_0)} \{1 - \pi_{s+1}(1, H_s)\} \right\| \delta(H_{s-1}) \prod_{k=s+1}^{l} \frac{\pi_l(0, H_{l-1})}{\overline{\pi}_l(0, H_{l-1})} - \{1 - \pi_{s+1}(1, H_s)\} \right\| \right\} \cdot \left\{ \mathbb{E} \left\{ \tilde{\mu}_l^0(H_l) \mid H_{l-1}, R_l = 1, A = 0 \right\} - \tilde{\mu}_l^0(H_{l-1}) \right\} \right\} \right\}\right(\text{since } \pi_s(1, H_{s-1}) \leq 1) \right\} \right\}
\]

\[
\leq M \sum_{s=1}^{t-1} \sum_{l=s+1}^{t} \left\| \mathbb{E} \left\{ \tilde{\mu}_l^0(H_l) \mid H_{l-1}, R_l = 1, A = 0 \right\} - \tilde{\mu}_l^0(H_{l-1}) \right\| \right\}
\]

\[
\left\| \{1 - \pi_{s+1}(1, H_s)\} \prod_{k=s+1}^{l} \frac{\pi_l(0, H_{l-1})}{\overline{\pi}_l(0, H_{l-1})} - \{1 - \pi_{s+1}(1, H_s)\} \right\| + \left\| \frac{1 - e(H_0)}{1 - \hat{e}(H_0)} \right\| \delta(H_{s-1}) - 1 \right\| \right\}
\]

S34
The second and the third inequalities hold by Holder’s inequality and triangle inequality. The term is \( o_p(n^{-\min(c_e+c_\mu+c_\pi+c_g)}) \).

For the third term that corresponds to \( S16 \), we have

\[
\leq \left\| \hat{\pi}_1(1, H_0) - \pi_1(1, H_0) \right\| \cdot \left\| \frac{e(H_0)}{\tilde{e}(H_0)} \mu_t^0(H_0) - \mu_t^0(H_0) \right\| \\
\leq \left\| \hat{\pi}_1(1, H_0) - \pi_1(1, H_0) \right\| \cdot \left\{ \left\| \frac{e(H_0)}{\tilde{e}(H_0)} \right\| \left\| \mu_t^0(H_0) - \mu_t^0(H_0) \right\| + \left\| \frac{e(H_0)}{\tilde{e}(H_0)} - 1 \right\| \right\} \\
\leq M \left\| \hat{\pi}_1(1, H_0) - \pi_1(1, H_0) \right\| \cdot \left\{ \left\| \mu_t^0(H_0) - \mu_t^0(H_0) \right\| + \left\| \frac{e(H_0)}{\tilde{e}(H_0)} - 1 \right\| \right\}.
\]

The second inequality holds by Holder’s inequality and triangle inequality. The term is \( o_p(n^{-\min(c_e+c_\mu+c_\pi)}) \).

For the fourth term that corresponds to \( S17 \), we have

\[
\leq \sum_{s=1}^{t} \left\| 1 - \frac{1 - e(H_0)}{1 - \tilde{e}(H_0)} \right\| \cdot \left\| \mathbb{E} \{ \hat{\mu}_t^0(H_s) \mid H_{s-1}, R_t = 1, A = 0 \} - \mu_t^0(H_{s-1}) \right\| \\
\leq \sum_{s=1}^{t} \left\| \mathbb{E} \{ \hat{\mu}_t^0(H_s) \mid H_{s-1}, R_t = 1, A = 0 \} - \mu_t^0(H_{s-1}) \right\| \cdot \left\{ \left\| 1 - \frac{1 - e(H_0)}{1 - \tilde{e}(H_0)} \right\| + \left\| 1 - \frac{\tilde{\pi}_s(0, H_{s-1})}{\tilde{\pi}(0, H_{s-1})} \right\| \right\} \\
\leq M \sum_{s=1}^{t} \left\| \mathbb{E} \{ \hat{\mu}_t^0(H_s) \mid H_{s-1}, R_t = 1, A = 0 \} - \mu_t^0(H_{s-1}) \right\| \cdot \left\{ \left\| 1 - \frac{1 - e(H_0)}{1 - \tilde{e}(H_0)} \right\| + \left\| 1 - \frac{\tilde{\pi}_s(0, H_{s-1})}{\tilde{\pi}(0, H_{s-1})} \right\| \right\}.
\]

The term is \( o_p(n^{-\min(c_e+c_\mu+c_\pi)}) \). Therefore, based on Theorem \( 7 \) and apply central limit theorem, we have \( \hat{\pi}_m - \tau_{j2R}^2 = O_p \left( n^{-\frac{1}{2}} + n^{-c} \right) \), where \( c = \min(c_e+c_\mu, c_e+c_\pi, c_\mu+c_\pi, c_\pi+c_g) \), which completes the proof.

## S5 Additional results from simulation

### S5.1 Cross-sectional setting

Table \( 1 \) shows the simulation results of the eight estimators for single-time-point outcomes under 8 different model specifications in terms of the bias and the Monte Carlo standard deviation (denoted as SD) based on 1000 simulated datasets. It quantifies Figure \( 1 \). The proposed triply robust estimators are unbiased if any two of the three models are correct. The calibration-based estimator has the smallest variation among the three triply robust
estimators. Under the correct specification of all the models, the calibration-based triply robust estimator has a comparable SD compared to $\hat{\tau}_{ps-om}$ and $\hat{\tau}_{rp-om}$.

Table 4: Point estimation in the cross-sectional setting under 8 different model specifications.

| Correct specification | Estimators | $\hat{\tau}_{tr}$ | $\hat{\tau}_{tr-N}$ | $\hat{\tau}_{tr-C}$ | $\hat{\tau}_{ps-rp}$ | $\hat{\tau}_{ps-rp-N}$ | $\hat{\tau}_{ps-om}$ | $\hat{\tau}_{ps-om-N}$ | $\hat{\tau}_{rp-om}$ |
|-----------------------|------------|-------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| PS RP OM              | Bias (%)   | SD (%)            |                     |                     |                     |                     |                     |                     |                     |
| yes yes yes           | -0.04      | 7.40              | 11.68               | 7.10                | 7.09                | 7.03                |                     |                     |                     |
| yes yes no            | 0.59       | 9.53              | 9.22                | 8.59                | 8.09                | 8.09                |                     |                     |                     |
| yes no yes            | -0.11      | 7.24              | 9.23                | -0.10               | 7.09                | 6.91                |                     |                     |                     |
| yes no no             | 0.08       | 7.25              | 7.10                | 10.34               | 9.84                | 7.03                |                     |                     |                     |
| no yes yes            | 8.16       | 8.16              | 9.23                | 9.29                | 8.09                | 8.09                |                     |                     |                     |
| no no no              | 8.30       | 9.18              | 8.70                | 8.09                | 8.71                | 8.72                |                     |                     |                     |
| no no yes             | 0.06       | 7.39              | 9.34                | 9.85                | 9.84                | 6.91                |                     |                     |                     |
| no no no              | 14.87      | 8.83              | 9.34                | 8.71                | 8.72                | 8.65                |                     |                     |                     |

S5.2 Longitudinal setting

We consider different modeling approximation strategies by incorporating different covariates in the four models. For the propensity score, the input covariates are $Z_{i1}, Z_{i2}, \cdots, Z_{i5}$; for the response probability model, the input covariates are $X_1, \cdots, X_5$; for the outcome mean and pattern mean model, the input covariates are $Z_{1}, \cdots, Z_{5}$ in each group. For calibration, we incorporate the first two moments of the covariates $Z$ and all the interactions to calibrate the propensity score weights, and use the first two moments of the historical information and all the interactions to calibrate the response probability weights sequentially.

Table 5 shows the simulation results of the eight estimators for longitudinal outcomes under J2R in detail. The SD in the table refers to the Monte Carlo standard deviation, and SE refers to the estimated standard error. As we can see from the table, the multiply robust estimators produce unbiased point estimations, while other estimators suffer from deviations from the true value. Using the EIF-based variance estimate tends to underestimate
Table 5: Simulation results of the estimators in the longitudinal setting.

| Estimator       | Bias (%) | SD (%)  | SE (%)  | Coverage rate (%) | Mean CI length (%) |
|-----------------|----------|---------|---------|-------------------|--------------------|
| \(\hat{\tau}_{mr}\)   | 0.76     | 12.48   | 9.69    | 94.2              | 51.61              |
| \(\hat{\tau}_{mr-N}\) | 0.76     | 12.48   | 9.79    | 94.2              | 51.23              |
| \(\hat{\tau}_{mr-C}\) | 0.14     | 12.25   | 8.77    | 94.1              | 48.35              |
| \(\hat{\tau}_{ps-rp}\) | 57.59    | 21.74   | 26.76   | 20.3              | 82.85              |
| \(\hat{\tau}_{ps-rp-N}\) | 58.02    | 21.82   | 21.98   | 19.5              | 78.99              |
| \(\hat{\tau}_{ps-om}\) | 14.03    | 16.01   | 20.86   | 86.7              | 65.90              |
| \(\hat{\tau}_{ps-om-N}\) | 14.04    | 15.98   | 17.46   | 86.6              | 65.30              |
| \(\hat{\tau}_{rp-pm}\)  | -13.50   | 10.79   | 11.11   | 76.1              | 43.00              |

The true variance for the data with a finite sample size in practice. The underestimation issue is also addressed by Chan et al. (2016). However, using symmetric bootstrap t CI eases the anti-conservative issue, which results in satisfying coverage rates. Applying calibration tends to improve efficiency, as we observe a smaller Monte Carlo variation compared to the other two multiply robust estimators.

S6 Additional results from application

The antidepressant clinical trial data is available on https://www.lshtm.ac.uk/research/centres-projects-groups/missing-data#dia-missing-data prepared by Mallinckrodt et al. (2014). The longitudinal outcomes in the data suffer from missingness at weeks 2, 4, 6, and 8. All the missingness in the control group follows a monotone missingness pattern, while 1 participant in the treatment group has intermittent missing data. We first delete three individuals with unobserved investigation site numbers, and one individual with intermittent missing data for simplicity, since our proposed framework is only valid under a monotone missingness pattern. We fit models of the propensity score, response probability and outcome mean sequentially in backward order, starting from the last time point. For outcome mean models, we regress the observed outcome \(Y_4\) at the last time point on the historical information \(H_3\) in the group with \(A = a\) to get \(\hat{\mu}_4^a(H_3)\), and then regress the predicted value \(\hat{\mu}_4^a(H_s)\) at time \(s\) on the historical information \(H_{s-1}\) using the subset of the data with \((R_{s-1} = 1, A = a)\) to get \(\mu_4^a(H_{s-1})\) for \(s = 1, \ldots, 3\), recursively. For
response probability, we fit the observed indicator $R_s$ with the incorporation of the historical information $H_{s-1}$ on the data with $(R_{s-1} = 1, A = a)$ to get $\hat{\pi}_s(a, H_{s-1})$ for $s = 1, \cdots, 4$ sequentially. For propensity score models, the treatment indicator $A$ is regressed on $H_{s-1}$ using the subset of the data with $R_{s-1} = 1$ to get $\hat{e}(H_{s-1})$. For the pattern mean models $\{g_{s+1}^l(H_{l-1}) : l = 1, \cdots, s \text{ and } s = 1, \cdots, 4\}$ that rely on both the response probability and outcome mean models, we regress the predicted value on the historical information $H_{s-1}$ on the subset of the data with $(R_{s-1} = 1, A = 1)$.

The distributions of the normalized estimated weights involved in the multiply robust estimators are visualized in Figure 3 (type = “original”). The weights that correspond to weeks 4 ($A = 0$ and $R_2 = 1$), 6 ($A = 0$ and $R_3 = 1$) and 8 ($A = 0$ and $R_4 = 1$) suffer from extreme outliers. The existence of outliers explains a distinct difference in the point estimation of $\hat{\tau}_{ps-rp}$ and $\hat{\tau}_{ps-rp-N}$ in Table 3. Therefore, we consider using calibration to mitigate the impact. The distributions of calibrated weights are also presented in Figure 3. As shown by the figure, calibration tends to scatter the concentrated estimated weights when no outstanding outliers exist in the original weights, for weights when $A = 1$ and $(A = 0, R_1 = 1)$, . However, it stabilizes the extreme weights at weeks 4, 6, and 8, which explains the narrower CI produced by $\hat{\tau}_{mr-C}$ compared to the other two multiply robust estimators.
Figure 3: Weight distributions of the HAMD-17 data