On Unbalanced Optimal Transport: An Analysis of Sinkhorn Algorithm

Khiem Pham *1 Kang Le *1 Nhat Ho 2 Tung Pham 3 Hung Bui 1

Abstract

We provide a computational complexity analysis for the Sinkhorn algorithm that solves the entropic regularized Unbalanced Optimal Transport (UOT) problem between two measures of possibly different masses with at most $n$ components.

We show that the complexity of the Sinkhorn algorithm for finding an $\varepsilon$-approximate solution to the UOT problem is of order $O(n^2/\varepsilon^2)$, which is near-linear time. To the best of our knowledge, this complexity is better than the complexity of the Sinkhorn algorithm for solving the Optimal Transport (OT) problem, which is of order $O(n^2/\varepsilon)$. Our proof technique is based on the geometric convergence of the Sinkhorn updates to the optimal dual solution of the entropic regularized UOT problem and some properties of the primal solution. It is also different from the proof for the complexity of the Sinkhorn algorithm for approximating the OT problem since the UOT solution does not have to meet the marginal constraints.

1. Introduction

The Optimal Transport (OT) problem has a long history in mathematics and operation research, originally used to find the optimal cost to transport masses from one distribution to the other (Villani, 2003). Over the last decade, OT has emerged as one of the most important tools to solve interesting practical problems in statistics and machine learning (Peyrè & Cuturi, 2019). Recently, the Unbalanced Optimal Transport (UOT) problem between two measures of possibly different masses has been used in several applications in computational biology (Schiebinger et al., 2019), computational imaging (Lee et al., 2019), deep learning (Yang & Uhler, 2019) and machine learning and statistics (Froger et al., 2015; Janati et al., 2019).

The UOT problem is a regularized version of Kantorovich formulation which places penalty functions on the marginal distributions based on some divergence (Liero et al., 2018). When the two measures are the probability distributions, the standard OT is a limiting case of the UOT. Under the discrete setting of the OT problem where each probability distribution has at most $n$ components, the OT problem can be recast as a linear programming problem. The benchmark methods for solving the OT problem are interior-point methods of which the most practical complexity is $O(n^3)$ developed by (Pele & Werman, 2009). Recently, (Lee & Sidford, 2014) used Laplacian linear system algorithms to improve the complexity of interior-point methods to $O(n^{5/2})$. However, the interior-point methods are not scalable when $n$ is large.

To deal with the scalability of computing the OT, (Cuturi, 2013) proposed to regularize its objective function by the entropy of the transportation plan, which results in the entropic regularized OT. One of the most popular algorithms for solving the entropic regularized OT is the Sinkhorn algorithm (Sinkhorn, 1974), which was shown by (Altschuler et al., 2017) to have a complexity of $O(n^2/\varepsilon^3)$ when used to approximate the OT within an $\varepsilon$-accuracy. In the same article, (Altschuler et al., 2017) developed a greedy version of the Sinkhorn algorithm, named the Greenkhorn algorithm, that has a better practical performance than the Sinkhorn algorithm. Later, the complexity of the Greenkhorn algorithm was improved to $O(n^2/\varepsilon^2)$ by a deeper analysis in (Lin et al., 2019b). To accelerate Sinkhorn and Greenkhorn algorithms, (Lin et al., 2019a) introduced Randkhorn and Gandkhorn algorithms that have complexity upper bounds of $O(n^{7/3}/\varepsilon)$. These complexities are better than those of Sinkhorn and Greenkhorn algorithms in terms of the desired accuracy $\varepsilon$. A different line of algorithms for solving the OT problem is based on primal-dual algorithms. These algorithms include accelerated primal-dual gradient descent algorithm (Dvurechensky et al., 2018), accelerated primal-dual mirror descent algorithm (Lin et al., 2019b), and accelerated primal-dual coordinate descent algorithm (Guo et al., 2019). These primal-dual algorithms all have complexity upper bounds of $O(n^{2.5}/\varepsilon)$, which are better than those of Sinkhorn and Greenkhorn algorithms in terms of $\varepsilon$. Recently, (Jambulapati et al., 2019; Blanchet et al., 2018)

1VinAI Research 2Department of EECS, University of California, Berkeley 3Faculty of Mathematics, Mechanics and Informatics, Hanoi University of Science, Vietnam National University.

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developed algorithms with complexity upper bounds of 
\( O(n^2/\varepsilon) \), which are believed to be optimal, based on either a dual extrapolation framework with area-convex mirror mapping or some black-box and specialized graph algorithms. However, these algorithms are quite difficult to implement. Therefore, they are less competitive than Sinkhorn and Greenkhorn algorithms in practice.

**Our Contribution.** While the complexity theory for OT has been rather well-understood, that for UOT is still nascent. In the paper, we establish the complexity of approximating UOT between two discrete measures with at most \( n \) components. We focus on the setting when the penalty functions are Kullback-Leibler divergences. Similar to the entropic regularized OT, in order to account for the scalability of computing UOT, we also consider an entropic regularization of order \( \log n \) to the entropic regularized UOT, in order to account for the most components. We focus on the setting when the\( \epsilon \)-approximation algorithm for the UOT has a complexity upper bound of \( O(\log n) \) (see Lemma 4); however, its complexity for approximating the UOT has not been studied. Our contribution is to prove that the Sinkhorn algorithm has a complexity of

\[
O \left( \frac{n^2}{\varepsilon^2} \log(n) \left[ \log(\|C\|_{\infty}) + \log(\log(n)) + 3 \log \left( \frac{1}{\varepsilon} \right) \right] \right).
\]

This complexity is close to the probably optimal one by a factor of logarithm of \( n \) and \( 1/\varepsilon \).

The main difference between finding an \( \varepsilon \)-approximation solution for OT and UOT by the Sinkhorn algorithm is that the Sinkhorn algorithm for OT knows when it is close to the solution because of the constraints on the marginals, while the UOT does not have that advantage. Despite lacking that useful property, the UOT enjoys more freedom resulting in some interesting equations that relate the optimal value of the primal function to the masses of two measures (see Lemma 4). Those equations together with the geometric convergence of the dual solution prove the almost linear time convergence to an \( \varepsilon \)-approximation solution of the UOT.

**Organization.** The remainder of the paper is organized as follows. In Section 2, we provide a setup for the regularized UOT in primal and dual forms, respectively. Based on the dual form, we show the dual solution has a geometric convergence rate in Section 3. We also show in Section 3 that the Sinkhorn algorithm for the UOT has a complexity of order \( O(n^2/\varepsilon) \). Section 4 presents some empirical results confirming the complexity of the Sinkhorn algorithm. Finally, we conclude with Section 5.

**Notation.** We let \([n]\) stand for the set \( \{1, 2, \ldots, n\} \) while \( \mathbb{R}^n_+ \) stands for the set of all vectors in \( \mathbb{R}^n \) with nonnegative components for any \( n \geq 2 \). For a vector \( x \in \mathbb{R}^n \) and \( 1 \leq p \leq \infty \), we denote \( \|x\|_p \) as its \( \ell_p \)-norm and \( \text{diag}(x) \) as the diagonal matrix with \( x \) on the diagonal. \( \mathbf{1}_n \) stands for a vector of length \( n \) with all of its components equal to 1. \( \partial_x f \) refers to a partial gradient of \( f \) with respect to \( x \).

Lastly, given the dimension \( n \) and accuracy \( \varepsilon \), the notation \( a = \mathcal{O}(b(n, \varepsilon)) \) stands for the upper bound \( a \leq C \cdot b(n, \varepsilon) \) where \( C \) is independent of \( n \) and \( \varepsilon \). Similarly, the notation \( a = \mathcal{O}(b(n, \varepsilon)) \) indicates the previous inequality may depend on the logarithmic function of \( n \) and \( \varepsilon \), and where \( C > 0 \).

### 2. Unbalanced Optimal Transport with entropic regularization

In this section, we present the primal and dual form of the entropic regularized UOT problem and define an \( \varepsilon \)-approximation for the solution of the unregularized UOT.

For any two positive vectors \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n_+ \) and \( b = (b_1, \ldots, b_n) \in \mathbb{R}^n_+ \), the UOT problem takes the form

\[
\min_{X \in \mathbb{R}^{n \times n}} f(X) \quad \text{where}
\]

\[
f(X) = \langle C, X \rangle + \tau \mathbf{KL}(X \mathbf{1}_n || a) + \tau \mathbf{KL}(X^\top \mathbf{1}_n || b),
\]

\( C \) is a cost matrix, \( \tau > 0 \) is a given regularization parameter and the KL divergence between vectors \( x \) and \( y \) is defined as

\[
\mathbf{KL}(x||y) = \sum_{i=1}^n x_i \log \left( \frac{x_i}{y_i} \right) - x_i + y_i,
\]

and \( X \) is called the transportation plan. When \( a^\top \mathbf{1}_n = b^\top \mathbf{1}_n \) and \( \tau \to \infty \), the UOT problem becomes the standard OT problem. Similar to the original OT problem, the exact computation of UOT is expensive and not scalable in terms of dimension \( n \). Inspired by the recent success of the entropic regularized OT problem as an efficient approximation of OT problem, we also consider the entropic version of the UOT problem (Frogner et al., 2015) of finding

\[
\min_{X \in \mathbb{R}^{n \times n}} g(X),
\]

where

\[
g(X) := \langle C, X \rangle - \eta H(X) + \tau \mathbf{KL}(X \mathbf{1}_n || a) + \tau \mathbf{KL}(X^\top \mathbf{1}_n || b),
\]

\( \eta > 0 \) is the regularization parameter and \( H(X) \) is the entropic regularization defined by

\[
H(X) := -\sum_{i,j=1}^n X_{ij} (\log(X_{ij}) - 1).
\]

For each \( \eta > 0 \), the entropic regularized UOT problem is strongly convex.

**Definition 1.** For any \( \varepsilon > 0 \), we call \( X \) an \( \varepsilon \)-approximation transportation plan if the following holds

\[
\langle C, X \rangle + \tau \mathbf{KL}(X \mathbf{1}_n || a) + \tau \mathbf{KL}(X^\top \mathbf{1}_n || b) \leq \langle C, \widehat{X} \rangle + \tau \mathbf{KL}(\widehat{X} \mathbf{1}_n || a) + \mathbf{KL}(\widehat{X}^\top \mathbf{1}_n || b) + \varepsilon,
\]
where \( \tilde{X} \) is an optimal transportation plan for the UOT problem \((1)\).

We aim to develop an algorithm to obtain \( \varepsilon \)-approximation transportation plan for the UOT problem \((1)\). In order to do that, we consider the Fenchel-Legendre dual form of entropic regularized UOT, which is given by

\[
\max_{u,v \in \mathbb{R}^n} -F^*(u) - G^*(v) - \eta \sum_{i,j} \exp \left( \frac{u_i + v_j - C_{ij}}{\eta} \right),
\]

where

\[
F^*(u) = \max_z z^\top u - \tau \mathbf{KL}(z||a) = \tau \left( e^{u/\tau}, a \right) - a^\top 1_n,
\]

\[
G^*(v) = \max_x x^\top v - \tau \mathbf{KL}(x||b) = \tau \left( e^{v/\tau}, b \right) - b^\top 1_n.
\]

Since \(a\) and \(b\) are given non-negative vectors, finding the optimal solution for the above objective is equivalent to finding the optimal solution for the following objective

\[
\min_{u,v \in \mathbb{R}^n} h(u,v) := \eta \sum_{i,j=1}^n \exp \left( \frac{u_i + v_j - C_{ij}}{\eta} \right) + \tau \left( e^{-u/\tau}, a \right) + \tau \left( e^{-v/\tau}, b \right).
\]

Problem \((4)\) is referred to as dual entropic regularized UOT.

### 3. Complexity analysis of approximating unbalanced optimal transport

In this section, we provide a complexity analysis of the Sinkhorn algorithm for approximating UOT solution. We start with some notations and useful quantities followed by the lemmas and main theorems.

#### 3.1. Notations and assumptions

We first denote \( \sum_{i=1}^n a_i = \alpha, \sum_{j=1}^n b_j = \beta \). For each \( u, v \in \mathbb{R}^n \), its corresponding optimal transport in the dual form \((4)\) is denoted by \( B(u,v) \), where \( B(u,v) := \text{diag}(e^{u/\eta}) e^{-u/\eta} \text{diag}(e^{v/\eta}) \). The corresponding solution in \((2)\) is denoted by \( X = B(u,v) \). Let \( a = B(u,v) 1_n, b = B(u,v)^\top 1_n \) and \( \sum_{i,j=1}^n X_{ij} = x \).

Let \( (u^k, v^k) \) be the solution returned at the \( k \)-th iteration of the Sinkhorn algorithm and \((u^*, v^*)\) be the optimal solution of \((4)\). Following the above scheme, we also define \( X^k, a^k, b^k, x^k \) and \( X^*, a^*, b^*, x^* \) correspondingly. Additionally, we define \( \tilde{X} \) to be the optimal solution of the unregularized objective \((1)\) and \( \sum_{i,j=1}^n \tilde{X}_{ij} = \tilde{x} \).

Different from the balanced OT, the optimal solutions of the entropic regularized UOT and our complexity analysis overall also depend on the masses \( \alpha, \beta \) and the KL regularization parameter \( \tau \). We will assume the following simple regularity conditions throughout the paper.

#### Regularity Conditions:

- \((A1)\) \( \alpha, \beta, \tau \) are positive constants.
- \((A2)\) \( C \) is a matrix of non-negative entries.

Before presenting the main theorem and analysis, for convenience, we define some quantities that will be used in our analysis and quantify their magnitudes under the regularity conditions.

#### List of quantities:

\[
R = \max \left\{ \|\log(a)\|_{\infty}, \|\log(b)\|_{\infty} \right\}
\]

\[
\Delta_k = \max \left\{ \|u^k - u^*\|_{\infty}, \|v^k - v^*\|_{\infty} \right\},
\]

\[
\Lambda_k = \left( \frac{\tau}{\tau + \eta} \right)^k \times \tau \times R,
\]

\[
S = \frac{1}{2}(\alpha + \beta) + \frac{1}{2} + \frac{1}{4 \log(\alpha)},
\]

\[
T = \left( \frac{\alpha + \beta}{2} \right) \left[ \log \left( \frac{\alpha + \beta}{2} \right) + 2 \log(\alpha) - 1 \right]
\]

\[
+ \log(\alpha) + \frac{5}{2}
\]

\[
U = \max \left\{ S + T, 2\varepsilon, \frac{4\varepsilon \log(\alpha)}{\tau}, \frac{4\varepsilon(\alpha + \beta) \log(\alpha)}{\tau} \right\}.
\]

As we shall see, the quantities \( \Delta_k \) and \( \Lambda_k \) are used to establish the convergence rate of \((u_k, v_k)\). We now consider the order of \( R, S, T \) and \( C \). Since the order of the penalty function \( \eta H(X) \) is \( O(\eta \log(n)) \) and should be small for a good approximation, \( \eta \) is often chosen such that \( \eta \log(n) \) is sufficiently small. Hence, we can assume the dominant factor in the second term of \( R \) is \( \frac{1}{\eta} \|C\|_{\infty} \). If \( \alpha = \sum_{i=1}^n a_i \) is a positive constant, then we can expect that \( a_i \) be as small as \( O(n^{-\kappa}) \) for a constant \( \kappa \geq 1 \). In this case, \( \|\log(a)\|_{\infty} = O(\log(n)) \). Overall, we can assume that \( R = O\left( \frac{1}{\eta} \|C\|_{\infty} \right) \) and if \( \alpha, \beta \) and \( \tau \) are positive constants, then \( S = O(1) \) and \( T = O(\log(n)) \).

#### 3.2. Sinkhorn algorithm

The Sinkhorn algorithm (Chizat et al., 2016) alternatively minimizes the dual function in \((4)\) with respect to \( u \) and \( v \). Suppose we are at iteration \( k + 1 \) for \( k \geq 0 \) and \( k \) even, by setting the gradient to 0 we can see that given fixed \( v^k \), the update \( u^{k+1} \) that minimizes the function in \((4)\) satisfies

\[
\exp \left( \frac{u_i^{k+1}}{\eta} \right) \sum_{j=1}^n \exp \left( \frac{v_j^k - C_{ij}}{\eta} \right) = \exp \left( \frac{u_i^{k+1}}{\tau} \right) a_i.
\]
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Algorithm 1 UNBALANCED_SINKHORN(C, ε)

Input: k = 0 and u^0 = v^0 = 0 and η = ε/U

while k ≤ \left(\frac{\tau}{\eta} + 1\right)\left[\log(8nR) + \log(\tau(\tau + 1)) + 3\log\left(\frac{\tau}{\eta}\right)\right] do

a_k = B(u^k, v^k) 1_n,

b_k = B(u^k, v^k) 1_n.

if k is even then

u^{k+1} = \left[\frac{u^k}{\eta} + \log(a^k) - \log\left(\frac{a_k}{\eta}\right)\right] \eta \tau

v^{k+1} = \left[\frac{v^k}{\eta} + \log(b^k) - \log\left(\frac{b_k}{\eta}\right)\right] \eta \tau

else

u^{k+1} = \left[\frac{v^k}{\eta} + \log(b^k) - \log\left(\frac{b_k}{\eta}\right)\right] \eta \tau

v^{k+1} = \left[\frac{u^k}{\eta} + \log(a^k) - \log\left(\frac{a_k}{\eta}\right)\right] \eta \tau

end if

k = k + 1.

end while

Output: B(u^k, v^k).

The next corollary sums up the complexity of Algorithm 1.

Corollary 1. Under conditions (A1-A2) and assume that R = O\left(\frac{1}{n}\|C\|_\infty\right), S = O(1) and T = O(\log(n)). Then the complexity of Algorithm 1 is

O\left(\frac{n^2}{\epsilon} \log(n) \left[\log(\|C\|_\infty) + \log(\log(n)) + \log\left(\frac{1}{\epsilon}\right)\right]\right),

which is also \tilde{O}(n^2/\epsilon).

Proof of Corollary 1. By the assumptions on the order of R, S, T and the definition of U in (10), we have

U = O(S) + O(T) + \epsilon O(\log(n)) = O(\log(n)).

Overall, we obtain

k = O\left(\log(n) \times \left[\log(\|C\|_\infty) + \log(\log(n)) + 3 \log\left(\frac{1}{\epsilon}\right)\right]\right).

Multiplying with \tilde{O}(n^2) arithmetic operations per iteration, we obtain the final complexity.

Remark 2. In comparison to the best well-known OT’s complexity of the similar order of n, i.e. (Dvurechensky et al., 2018), our complexity for the OT is better by a factor of \epsilon. Meanwhile, among the practical algorithms for OT which have similar order of \epsilon, i.e. Gankhorn and Randkhorn, our bound is better by a factor of n^{1/3}.

3.3. Analysis of the Sinkhorn algorithm

The analysis for Unbalanced Optimal Transport is different from that of Optimal Transport, since a and b are no longer probability measures. The proof of Theorem 1 requires the convergence rate of (u^k, v^k) and an upper bound on the supremum norm of the optimal dual solution (u^*, v^*), the later of which is presented in Lemma 3.

Lemma 1. The optimal solution (u^*, v^*) of (4) satisfies the following equations:

\frac{u^*}{\tau} = \log(a) - \log(a^*), \text{ and } \frac{v^*}{\tau} = \log(b) - \log(b^*).

Proof. Since (u^*, v^*) is a fixed point of the update in the Algorithm 1, we get

u^* = \left[\frac{u^*}{\eta} + \log(a) - \log(a^*)\right] \eta \tau

This directly leads to the stated equality for \frac{u^*}{\tau}, and that for \frac{v^*}{\tau} can be obtained similarly.
Lemma 2. Assume the regularity conditions (A1-A2) hold, the following are true

(a) \[ \log \left( \frac{a_i^+}{a_i^-} \right) - \frac{u_i^+ - u_i^-}{\eta} \leq \max_{1 \leq j \leq n} \left| \frac{v_j^+ - v_j^-}{\eta} \right|, \]

(b) \[ \log \left( \frac{b_j^+}{b_j^-} \right) - \frac{v_j^+ - v_j^-}{\eta} \leq \max_{1 \leq i \leq n} \left| \frac{u_i^+ - u_i^-}{\eta} \right|. \]

The proof is given in the appendix.

Lemma 3. The sup norm of the optimal solution \( \| u^* \|_\infty \) and \( \| v^* \|_\infty \) is bounded by:

\[ \max\{\| u^* \|_\infty, \| v^* \|_\infty\} \leq \tau R, \]

where \( R \) is defined in (5).

Proof. We start with the equations for the solution \( u^* \) in Lemma 1, i.e.

\[ u_i^* = \log(a_i) - \log \left( \sum_{j=1}^{n} \exp \left( \frac{u_j^* + v_j^* - C_{ij}}{\eta} \right) \right), \]

which can be rewritten as

\[ u_i^* \left( \frac{1}{\tau} + \frac{1}{\eta} \right) = \log(a_i) - \log \left( \sum_{j=1}^{n} \exp \left( \frac{v_j^* - C_{ij}}{\eta} \right) \right). \]

The second term can be bounded as follows

\[ \log \left( \sum_{j=1}^{n} \exp \left( \frac{v_j^* - C_{ij}}{\eta} \right) \right) \geq \log(n) + \min_{1 \leq j \leq n} \left\{ \frac{v_j^* - C_{ij}}{\eta} \right\} \]

\[ \geq \log(n) - \max_{1 \leq j \leq n} \left\{ \frac{\| v^* \|_\infty}{\eta} - \frac{\| C \|_\infty}{\eta} \right\}, \]

and

\[ \log \left( \sum_{j=1}^{n} \exp \left( \frac{v_j^* - C_{ij}}{\eta} \right) \right) \leq \log(n) + \max_{1 \leq j \leq n} \left\{ \frac{v_j^* - C_{ij}}{\eta} \right\} \]

\[ \leq \log(n) + \frac{\| v^* \|_\infty}{\eta}, \]

thus leading to

\[ \log \left( \sum_{j=1}^{n} \exp \left( \frac{v_j^* - C_{ij}}{\eta} \right) \right) \]

\[ \leq \frac{\| v^* \|_\infty}{\eta} + \max \left\{ \log(n), \frac{\| C^* \|_\infty}{\eta} - \log(n) \right\}. \]

Hence,

\[ \left| u_i^* \right| \left( \frac{1}{\tau} + \frac{1}{\eta} \right) \leq |\log(a_i)| + \frac{\| v^* \|_\infty}{\eta} + \]

\[ \max \left\{ \log(n), \frac{\| C^* \|_\infty}{\eta} - \log(n) \right\}. \]

Choosing \( i \) such that \( |u_i^*| = \| u^* \|_\infty \), combining with the fact that \( |\log(a_i)| \leq \max\{\| \log(a) \|_\infty, \| \log(b) \|_\infty\} \), we have

\[ \| u^* \|_\infty \left( \frac{1}{\tau} + \frac{1}{\eta} \right) \leq \| v^* \|_\infty + R. \]

WLOG assume that \( \| u^* \|_\infty \geq \| v^* \|_\infty \), we can easily obtain the stated bound.

Proof of Theorem 1. We first consider the case when \( k \) is even. From the update of \( u^{k+1} \) in Algorithm 1, we have:

\[ u_i^{k+1} = \left( \frac{u_i^k + \log(a_i) - \log(a_i^k)}{\eta} \right) \frac{\eta}{\tau + \eta} \]

\[ = \left\{ \frac{u_i^k}{\eta} + [\log(a_i) - \log(a_i^k)] + [\log(a_i^*) - \log(a_i^k)] \right\} \frac{\eta}{\tau + \eta}. \]

Using Lemma 1, the above equality is equivalent to

\[ u_i^{k+1} - u_i^* = \left[ \eta \log \left( \frac{a_i^k}{a_i^*} \right) - (u_i^* - u_i^k) \right] \frac{\tau}{\tau + \eta}. \]

Using Lemma 2, we get

\[ \left| u_i^{k+1} - u_i^* \right| \leq \max_{1 \leq j \leq n} \left| \frac{v_j^* - v_j^k}{\eta} \right| \frac{\tau}{\tau + \eta}. \]

This leads to \( \| u^{k+1} - u^* \|_\infty \leq \frac{\tau}{\tau + \eta} \| v^k - v^* \|_\infty \). Similarly, we obtain \( \| v^k - v^* \|_\infty \leq \frac{\tau}{\tau + \eta} \| u^{k-1} - u^* \|_\infty \). Combining the two inequalities yields

\[ \| u^{k+1} - u^* \|_\infty \leq \left( \frac{\tau}{\tau + \eta} \right)^2 \| u^{k-1} - u^* \|_\infty. \]

Repeating all the above arguments alternatively, we have

\[ \| u^{k+1} - u^* \|_\infty \leq \left( \frac{\tau}{\tau + \eta} \right)^{k+1} \| v^0 - v^* \|_\infty = \left( \frac{\tau}{\tau + \eta} \right)^{k+1} \| v^* \|_\infty. \]

Note that \( u^{k+1} = v^k \) for \( k \) even, then

\[ \| v^{k+1} - v^* \|_\infty \leq \frac{\tau}{\tau + \eta} \| u^{k-1} - u^* \|_\infty \leq \left( \frac{\tau}{\tau + \eta} \right)^k \| v^* \|_\infty. \]

These two results lead to \( \Delta_{k+1} \leq \left( \frac{\tau}{\tau + \eta} \right)^k \| v^* \|_\infty. \)

Similarly, for \( k \) odd we obtain \( \Delta_{k+1} \leq \left( \frac{\tau}{\tau + \eta} \right)^k \| v^* \|_\infty. \)

Thus the above inequality is true for all \( k \). Using the fact that \( \| v^* \|_\infty \leq \max\{\| u^* \|_\infty, \| v^* \|_\infty\} \) and Lemma 3, we obtain the conclusion.

3.4. Proof of the main theorem

The proof is based on the upper bound for the convergence rate in Theorem 1 and an upper bound for the solutions \( \bar x \) and \( z^* \) of (1) and (2), respectively, which are direct consequences of the following lemma.
Lemma 4. Assume that the function $g(X)$ attains its minimum at $X^*$, then
\[ g(X^*) + (2\tau + \eta)x^* = \tau(\alpha + \beta). \] (12)

Similarly, assume that $f(X)$ attains its minimum at $\tilde{X}$, then
\[ f(\tilde{X}) + 2\tau\tilde{x} = \tau(\alpha + \beta). \] (13)

Both equations in Lemma 4 establish the relationships between the optimal solutions of (1) and (2) with other parameters. Those relationships are very useful for analysing the behaviour of the optimal solution of UOT, because the UOT does not have any conditions on the marginals as the OT does. Consequences of Lemma 4 include Corollary 2 which provides upper bounds for $\tilde{x}$ and $x^*$ of (1) and (2) as well as bounds for the entropic functions in the proof of Theorem 2. The key idea of the proof surprisingly comes from the fact that the UOT solution does not have to meet the marginal constraints. We now present the proof of Lemma 4 and defer the proof of Corollary 2 to the Appendix.

Proof. Consider the function $g(tX^*)$, where $t \in \mathbb{R}^+$,
\[ g(tX^*) = (C_t X^*) + \tau KL(tX^*_n \| a) + \tau KL((tX^*)^T1_n \| b) - \eta H(tX^*). \]

For the KL term of $g(tX^*)$, we have:
\[
\begin{aligned}
KL(tX^*_n \| a) & = \sum_{i=1}^n (ta^*_i) \log \frac{ta^*_i}{a_i} - \sum_{i=1}^n (ta^*_i) + \sum_{i=1}^n a_i \\
& = \sum_{i=1}^n (ta^*_i) \left( \log \frac{a_i^*}{a_i} + \log(t) \right) - tx^* + \alpha \\
& = t \sum_{i=1}^n \left( a^*_i \log \frac{a_i^*}{a_i} - x^* + a_i \right) + (1-t)\alpha + x^* t \log(t) \\
& = t KL(X^*_n \| a) + (1-t)\alpha + x^* t \log(t).
\end{aligned}
\]

Similarly, we get
\[KL((tX^*)^T1_n \| b) = t KL((X^*)^T1_n \| b) + (1-t)\beta + x^* t \log(t).\]

For the entropic penalty term,
\[-H(tX^*) = \sum_{i,j=1}^n t_{ij} \left( \log(tX^*_{ij}) - 1 \right) = \sum_{i,j} t_{ij} \left( \log(X^*_{ij}) - 1 \right) + x^* t \log(t) = -tH(X^*) + x^* t \log(t).\]

Putting all results together, we obtain
\[ g(tX^*) = tg(X^*) + \tau(1-t)(\alpha + \beta) + (2\tau + \eta)x^* t \log(t). \]

Taking the derivative of $g(tX^*)$ with respect to $t$,
\[
\frac{\partial g(tX^*)}{\partial t} = g(X^*) - \tau(\alpha + \beta) + (2\tau + \eta)x^* (1 + \log(t)).
\]

The function $g(tX^*)$ is well-defined for all $t \in \mathbb{R}^+$. We know that $g(tX^*)$ attains its minimum at $t = 1$. Replace $t = 1$ into the above equation, we obtain
\[ g(X^*) - \tau(\alpha + \beta) + (2\tau + \eta)x^* = 0 \]
\[ g(X^*) + (2\tau + \eta)x^* = \tau(\alpha + \beta). \]

The second claim is proved in the same way. \(\square\)

Corollary 2. Assume that condition (A1-A2) hold and $\eta \log(n)$ is sufficiently small. We have the following bounds on $x^*$ and $\tilde{x}$:

(a) $x^* \leq \left(\frac{1}{2} + \frac{\eta \log(n)}{2\tau - 2\eta \log(n)}\right) (\alpha + \beta) + \frac{1}{6 \log(n)}$,

(b) $\tilde{x} \leq \frac{\alpha + \beta}{2}$.

Next, we use the condition for $k$ in Theorem 2 to bound some relevant quantities at the $k$-th iteration of the Sinkhorn algorithm.

Lemma 5. Assume that the regularity conditions (A1-A2) hold and $k$ satisfies the inequality in Theorem 2. The following are true

(a) $\Lambda_{k-1} \leq \frac{\eta^2}{8(\tau + 1)}$,

(b) $|x^k - x^*| \leq \frac{3}{\eta} \Delta_k \min \{x^*, x^k\}$,

(c) $x^k \leq S$,

where $S$ is defined in (8).

We are now ready to construct a proof for Theorem 2.

Proof of Theorem 2. From the definitions of $f$ and $g$, we have
\[
\begin{aligned}
f(X^k) - f(\tilde{X}) &= g(X^k) + \eta H(X^k) - g(\tilde{X}) - \eta H(\tilde{X}) \\
&= g(X^k) + \eta H(X^k) - g(\tilde{X}) - \eta H(\tilde{X}) - g(X^*) + g(X^*) \\
&\leq \left[ g(X^k) - g(X^*) \right] + \eta \left[ H(X^k) - H(\tilde{X}) \right],
\end{aligned}
\] (14)

since $g(X^*) - g(\tilde{X}) \leq 0$, as $X^*$ is the optimal solution of (2). The above two terms can be bounded separately as follows:
Upper bound of $H(X^k) - H(\tilde{X})$.

We first show the following inequalities

$$x - x\log(x) \leq H(X) \leq 2x\log(n) + x - x\log(x)$$  \hfill (15)  

for any $X$ that $X_{ij} \geq 0$ and $x = \sum_{i,j} X_{ij}$.

Indeed, rewriting $-H(X)$ as

$$-H(X) = x \left[ \sum_{i,j=1}^{n} \frac{X_{ij}}{x} \log \left( \frac{X_{ij}}{x} \right) - 1 \right] + x\log(x).$$

and using $-2\log(n) \leq \sum_{i,j=1}^{n} \frac{X_{ij}}{x} \log \left( \frac{X_{ij}}{x} \right) \leq 0$, we thus obtain (15).

Now apply the lower bound of (15) to $-H(\tilde{X})$

$$-H(\tilde{X}) \leq \hat{x}\log(\hat{x}) - \hat{x}$$

$$\leq \max \left\{ 0, \frac{\alpha + \beta}{2} \left[ \log \left( \frac{\alpha + \beta}{2} \right) - 1 \right] \right\}$$

$$\leq \frac{\alpha + \beta}{2}\log \left( \frac{\alpha + \beta}{2} \right) - \frac{\alpha + \beta}{2} + 1,$$

where the second inequality is due to $x \log x - x$ being convex and $0 \leq \hat{x} \leq \frac{1}{2}(\alpha + \beta)$ by Corollary 2 and the third inequality is due to $\frac{\alpha + \beta}{2} \left[ \log \left( \frac{\alpha + \beta}{2} \right) - 1 \right] + 1 \geq 0$.

Similarly, apply the upper bound of (15) to $H(X^k)$

$$H(X^k) \leq 2x^k\log(n) + x^k - x^k\log(x^k)$$

$$\leq 2x^k\log(n) + 1$$

$$\leq \left( \alpha + \beta + 1 + \frac{1}{2}\log(n) \right) \log(n) + 1.$$

By combining the two results, we have

$$H(X^k) - H(\tilde{X}) \leq T,$$  \hfill (16)  

where $T$ is defined in (9).

Upper bound of $g(X^k) - g(X*)$.

WLOG we assume that $k$ is odd. At step $k-1$ of Algorithm 1, we find $u^k$ by minimizing the dual function (4) given $a$ and fixed $v^{k-1}$, and simply keep $v^k = v^{k-1}$. Hence, $X^k = B(u^k, v^k)$ is the optimal solution of

$$\min_{X \in \mathbb{R}^{n \times n}_+} g^k(X) := \langle C, X \rangle - \eta H(X)$$

$$+ \tau \text{KL}(X1_n || a) + \tau \text{KL}(X^\top 1_n || b^k),$$

where $b^k = \exp \left( \frac{u^k}{\tau} \right) \odot [(X^k)^\top 1_n]$ with $\odot$ denoting element-wise multiplication.

Denote $\sum_{i=1}^{n} b_i^k = \beta^k$. By Lemma 4,

$$g^k(X^k) = \tau (\alpha + \beta k) - (2\tau + \eta)x^k,$$

$$g(X^*) = \tau (\alpha + \beta) - (2\tau + \eta)x^*.$$

Writing $g(X^k) - g(X^*) = [g(X^k) - g^k(X^k)] + [g^k(X^k) - g(X^*)]$, following some derivations using the above equations of $g^k(X^k)$ and $g(X^*)$ and the definitions of $g(X^k)$ and $g^k(X^k)$, we get

$$g(X^k) - g(X^*) = \left[ -(2\tau + \eta)(x^k - x^*) \right]$$

$$+ \tau \left[ \sum_{i=1}^{n} b_i^k \log \left( \frac{b_i^k}{b_j} \right) \right].$$  \hfill (17)  

By part (b) of Lemma 5, the first term is bounded by $(2\tau + \eta) \frac{\Delta_k}{2} x^k$.

Note that $b_j^k = \exp \left( \frac{v_j^k}{\tau} \right) b_j^*$ and $b_j^* = \exp \left( \frac{v_j^*}{\tau} \right)$. Use part (b) of Lemma 2

$$\log \left( \frac{b_j^k}{b_j^*} \right) \leq \left( \frac{2}{\eta}\Delta_k \right) + \left( \frac{1}{\tau}\Delta_k \right) = \left( \frac{2}{\eta} + \frac{1}{\tau} \right) \Delta_k.$$

Note that $b_j^* \geq 0$ for all $j$. The above inequality leads to

$$\left| \sum_{j=1}^{n} b_j^k \log \left( \frac{b_j^k}{b_j^*} \right) \right| \leq \left( \sum_{j=1}^{n} b_j^k \right) \max_{1 \leq j \leq n} \left| \log \left( \frac{b_j^k}{b_j^*} \right) \right|$$

$$\leq x^k \left( \frac{2}{\eta} + \frac{1}{\tau} \right) \Delta_k.$$

We have

$$g(X^k) - g(X^*) \leq \left[ (2\tau + \eta) + 3(2\tau + \eta) \right] x^k \times \frac{\Delta_k}{\eta}$$

$$\leq 8(\tau + 1) \times \eta \times \frac{\Lambda_k}{\eta},$$

where the first inequality is obtained by combining the bounds for two terms of (17) while the second inequality results from the fact that $\eta = \frac{\tau}{2} \leq \frac{\tau}{2\tau + \eta}$ with $U$ defined in (10), part (c) of Lemma 5, and Theorem 1.

Using part (a) of Lemma 5, this leads to

$$g(X^k) - g(X^*) \leq \eta S.$$  \hfill (18)  

Combining (14), (16), (18) and the fact that $\eta = \frac{\tau}{\tau + \eta}$, we get

$$f(X^k) - f(\tilde{X}) \leq \eta S + \eta T \leq \varepsilon.$$
4. Experiments

In this section, we provide empirical evidence to illustrate our proven complexity on both synthetic data and real images. In both examples, we vary $\varepsilon$ such that it is small relative to the minimum of the unregularized UOT function in (1) which is computed in advance by using the cvxpy library (Agrawal et al., 2018) with the splitting conic solver option. We then report the two $k$ values:

- The first $k$, denoted by $k_f$, follows the stopping rule in Algorithm 1.
- The second, denoted by $k_c$, is defined as the minimal $k_c$ such that for all later known iterations $k' \geq k_c$ in the experiment, Algorithm 1 returns an $\varepsilon$-approximation solution of the UOT problem.

4.1. Synthetic data

For the simulated example, we choose $n = 10$ and $\tau = 5$. The elements of the cost matrix $C$ are drawn uniformly in $[1, 50]$ while those of the marginal vectors $a$ and $b$ are drawn uniformly in $[0.1, 1]$ and then normalized to have masses 2 and 4, respectively. By varying $\varepsilon$ from 1.0 to 0.05, we follow the scheme presented in the beginning of the section, and report values of $k_f$ and $k_c$ in Figure 1.

Figure 1 shows the log values of $k_f$, $k_c$ stated above when varying $\varepsilon$. As can be seen from the left plot, the gap between the two values is not small, it gets smaller as $\varepsilon$ decreases to 0.05. This is more apparent in the right plot, where the ratio between the two values significantly decreases as $\varepsilon$ becomes smaller, indicating that the theoretical bound is getting better when decreasing $\varepsilon$.

4.2. MNIST

For the MNIST dataset\(^1\), we follow similar settings in (Dvurechensky et al., 2018; Altschuler et al., 2017): The

\[^1\]http://yann.lecun.com/exdb/mnist/

\[\text{Figure 2. Comparison between } \log(k_f) \text{ and } \log(k_c) \text{ on MNIST when varying } \varepsilon \text{ from 5 to 0.5. We used higher } \varepsilon \text{ to keep the relative error similar to the first experiment, due to a higher optimal value (among 10 chosen pairs, the minimum was 117.524 and the maximum was 459.297.)}\]

Figure 3. The ratios as observed empirically remain close to our geometric factor for most of the iterations.

\[\text{Figure 3. The ratios as observed empirically remain close to our geometric factor for most of the iterations.}\]

For the MNIST dataset, we use two flattened images in a pair and the cost matrix $C$ is the matrix of $\ell_1$ distances between pixel locations. We also add a small constant $10^{-6}$ to each pixel with intensity 0, except we do not normalize the marginals. We average the results over 10 randomly chosen image pairs and plot the results in Figure 2. The results on MNIST dataset confirm our theoretical results on the bound of $k$. It also shows that the smaller $\varepsilon$ in the approximation, the closer the empirical result to the theoretical result.

4.3. A further analysis for synthetic data

In order to investigate how challenging it is to improve the theoretical bound for the number of required iterations, we carry out a deeper analysis on the synthetic example. In particular, we set $\eta = 0.5$, $\tau = 5$ and compute the ratios $\frac{\|u^k - u^*\|_\infty}{\|u^{k+1} - u^{k}\|_\infty}$ and $\frac{\|\hat{u}^k - \hat{u}^*\|_\infty}{\|\hat{u}^{k+1} - \hat{u}^{k}\|_\infty}$ for $k$ even in range $[0, 100]$ and plot them in Figure 3. As has been proved in Theorem 1, these ratios are no less than $\frac{2 + \eta}{\tau}$. The main reason for this choice is that these differences are used to construct bounds for many key quantities in lemmas and theorems. These ratios, which are extremely close to 1.1 for most of the iterations, are consistent with the ratio $\frac{2 + \eta}{\tau} = 1.1$. Consequently, it is difficult to improve our inequalities.
5. Discussion

In this paper, we have proved that the complexity of the Sinkhorn algorithm for approximating the UOT problem is better than that for the OT problem. In our analysis, some inequalities might not be tight, since we prefer to keep them in simple forms for easier presentation. These suboptimalities perhaps lead to the inclusion of the logarithmic terms of $\varepsilon$ and $n$ in our complexity upper bound of the Sinkhorn algorithm. We now discuss a few future directions that can serve as natural follow-ups of our work. First, our analysis could be used in the multi-marginal case of UOT by applying Algorithm 1 repeatedly to every pair of marginals. Second, since the UOT barycenter problem has found several applications in recent years (Janati et al., 2019; Schiebinger et al., 2019), it is desirable to establish the complexity analysis of algorithms for approximating it.
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Supplement to “On Unbalanced Optimal Transport: An Analysis of Sinkhorn algorithm”

6. Proofs of remaining results

Before proceeding with the proofs, we state the following simple inequalities first for convenience:

**Lemma 6.** The following inequalities are true for all positive \( x_i, y_i, x, y \) and \( 0 \leq z < \frac{1}{2} \):

\[
\begin{align*}
(a) \quad & \min_{1 \leq i \leq n} \frac{x_i}{y_i} \leq \sum_{i=1}^{n} \frac{x_i}{y_i} \leq \max_{1 \leq i \leq n} \frac{x_i}{y_i}, \\
(b) \quad & \exp(z) \leq 1 + |z| + |z|^2, \\
(c) \quad & \text{If } \max \left\{ \frac{x}{y}, \frac{y}{x} \right\} \leq 1 + \delta, \text{ then } |x - y| \leq \delta \min \{x, y\}, \\
(d) \quad & \left(1 + \frac{1}{x}\right)^{x+1} \geq e.
\end{align*}
\]

*Proof of Lemma 6.*

(a) Given \( x_i \) and \( y_i \) positive

\[
\begin{align*}
\min_{1 \leq i \leq n} \frac{x_i}{y_i} \leq \sum_{i=1}^{n} \frac{x_i}{y_i} \leq \max_{1 \leq i \leq n} \frac{x_i}{y_i},
\end{align*}
\]

Taking the sum over \( j \),

\[
\begin{align*}
\sum_{j=1}^{n} y_j \times \min_{1 \leq i \leq n} \frac{x_i}{y_i} & \leq \sum_{j=1}^{n} x_j \leq \sum_{j=1}^{n} y_j \times \max_{1 \leq i \leq n} \frac{x_i}{y_i} \times y_j, \\
\min_{1 \leq i \leq n} \frac{x_i}{y_i} & \leq \sum_{j=1}^{n} \frac{x_j}{y_j} \leq \max_{1 \leq i \leq n} \frac{x_i}{y_i}.
\end{align*}
\]

(b) For the second inequality, \( \exp(x) \leq 1 + |x| + |x|^2 \), we have to deal with the case \( x > 0 \). Since \( x \leq \frac{1}{2} \),

\[
\begin{align*}
\exp(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + x^2 - \frac{x^2}{2} + \sum_{n=3}^{\infty} \frac{x^n}{n!} & \leq 1 + x + x^2 - \frac{x^2}{2} + \frac{x^3}{6} \sum_{n=3}^{\infty} x^{n-3} \\
& \leq 1 + x + x^2 - \frac{x^2}{2} + \frac{x^3}{6} \left(1 - \frac{1}{1-x}\right) \leq 1 + x + x^2 - \frac{x^2}{2} + \frac{x^3}{3} \leq 1 + x + x^2.
\end{align*}
\]

(c) For the third inequality, WLOG assume \( x > y \)

\[
\frac{x}{y} \leq 1 + \delta \Rightarrow x \leq y + y\delta \Rightarrow |x - y| \leq y\delta.
\]

(d) For the fourth inequality, taking the log of both sides, it is equivalent to \( (x + 1) \left[ \log(x + 1) - \log(x) \right] \geq 1 \). By the mean value theorem, there exists a number \( y \) between \( x \) and \( x + 1 \) such that \( \log(x + 1) - \log(x) = 1/y \), then \( (x + 1)/y \geq 1 \). □

By the choice of \( \eta = \frac{\epsilon}{2} \) and the definition of \( U \), we also have the following conditions on \( \eta \):

\[
\eta \leq \frac{1}{2}; \quad \frac{\eta}{2} \leq \frac{1}{4 \log(n) \max \{1, \alpha + \beta\}}
\]
Now we come to the lemmas and the corollary in the main text.

**Proof of Lemma 2.**

(a) + (b): From the definitions of $a_i^k$ and $a_i^*$, we have

\[
\log \left( \frac{a_i^*}{a_i^k} \right) = \left( \frac{u_i^* - u_i^k}{\eta} \right) + \log \left( \frac{\sum_{j=1}^{n} \exp(v_j^*-C_{ij})}{\sum_{j=1}^{n} \exp(v_j^*-C_{ij})} \right).
\]

The required inequalities are equivalent to an upper bound and a lower bound for the second term of the RHS. Apply part (a) of Lemma 6, we obtain

\[
\min_{1 \leq j \leq n} \frac{v_j^*-v_j^k}{\eta} \leq \log \left( \frac{a_i^*}{a_i^k} \right) - \frac{u_i^* - u_i^k}{\eta} \leq \max_{1 \leq j \leq n} \frac{v_j^*-v_j^k}{\eta}.
\]

Part (b) follows similarly. \qed

**Proof of Corollary 2.** Recall that we have proved in Lemma 4:

\[
g(X^*) + (2\tau + \eta)x^* = \tau(\alpha + \beta)
\]

\[
f(\hat{X}) + 2\tau \hat{x} = \tau(\alpha + \beta).
\]

From the second equality and the fact that $f(\hat{X}) \geq 0$ (it is easy to see that for $X$, $x_i \geq 0$, the KL terms and $\langle C, X \rangle$ are all non-negative), we immediately have $\hat{x} \leq \frac{\alpha + \beta}{2}$, proving the second inequality. For the first inequality, we have $g(X^*) \geq -\eta H(X^*) \geq -2\eta x^* \log(n) - \eta x^* + \eta x^* \log(x^*)$, therefore

\[
\tau(\alpha + \beta) - (2\tau + \eta)x^* \geq \eta x^* \log(x^*) - 2\eta x^* \log(n) - \eta x^*
\]

\[
\tau(\alpha + \beta) \geq \eta x^* \log(x^*) + 2\left(\tau - \eta \log(n)\right)x^*.
\]

It follows from the inequality $z \log(z) \geq z - 1$ that:

\[
\eta x^* - \eta + 2(\tau - \eta \log(n))x^* \leq \tau(\alpha + \beta)
\]

\[
x^* (2\tau - 2\eta \log(n) + \eta) \leq \tau(\alpha + \beta) + \eta.
\]

By inequality (19), $4\eta \log(n) \leq \tau$. Then

\[
x^* \leq \frac{\tau(\alpha + \beta) + \eta}{2\tau - 2\eta \log(n) + \eta} \leq \frac{\tau(\alpha + \beta) - (\alpha + \beta)\eta \log(n)}{2\tau - 2\eta \log(n)} + \frac{2(\alpha + \beta)\eta \log(n) + \eta}{2\tau - 2\eta \log(n)}
\]

\[
\leq \frac{\alpha + \beta}{2} + \frac{\eta \log(n)}{2\tau - 2\eta \log(n)} + \frac{4\log(n)}{\eta} \leq \frac{1}{2} + \frac{\eta \log(n)}{2\tau - 2\eta \log(n)} (\alpha + \beta) + \frac{1}{6\log(n)}.
\]

\qed

**Proof of Lemma 5.**

(a) We prove that $\Lambda_k \leq \frac{\eta^2}{8(\tau+1)}$ for $\eta = \frac{\tau}{s}$ and $k \geq \left( \frac{1}{\eta} + 1 \right) \times \left( \log(8\eta R) + \log(\tau(\tau + 1) + 3\log(\frac{1}{\eta}) \right)$ (note that the stated bound can be obtained by replacing $k$ by $k-1$).

Denote $\frac{8\eta R(\tau+1)}{\tau^2} = D$ and $\frac{\tau}{\rho} = s > 0$. From inequality (19), we have $s < 1$. The required inequality is equivalent to

\[
\left( \frac{\eta}{\tau + \eta} \right)^k \tau R \leq \frac{\eta^2}{8(\tau+1)}
\]

\[
\iff \left( \frac{\tau + \eta}{\tau} \right)^k \eta^3 \geq \frac{8\eta R(\tau + 1)}{\tau^2}
\]

\[
\iff (1 + s)^k s^3 \geq D.
\]
Let $t = 1 + \frac{\log(D)}{3\log(\frac{1}{s})}$. By definition (5), $R \geq \log(n)$, thus $D \geq \frac{8n\log(n)(\tau+1)}{\tau^2} > \frac{n^3}{\tau^7} = s^3$ and $t > 1 + \frac{3s^3}{3\log(\frac{1}{s})} = 0$. We claim the following chain of inequalities

$$s^3(1 + s)^k \geq s^3(1 + s)^{\frac{1}{2} + \frac{33}{2}s^3(\frac{1}{s})t} \geq s^3e^{3\log(\frac{1}{s})t}$$

The first inequality results from $k \geq (\frac{3U}{\tau} + 1) \times [\log(8nR) + \log(\tau(\tau+1)) + 3\log(\frac{U}{\tau})] = (1 + \frac{1}{2})s^3(\frac{1}{s})t > 0$ (using the definitions of $D$, $s$, the choice of $t$ and $\eta = \frac{s}{\tau}$). The second inequality is due to part (d) of Lemma 6. The last equality is

$$s^3e^{3\log(\frac{1}{s})t} = \frac{1}{s^{3t-3}} = \frac{1}{s^{3\log(D)/\log(1/s)}} = \frac{1}{s^{3\log(D)/\log(s)}} = D$$

We have thus proved our claim.

(b) We need to prove $|x^k - x^*| \leq \frac{3}{\eta} \min\{x^*, x^k\} \Delta_k$. From the definition of $x^k$ and $x^*$ and note that they are non-negative:

$$x^k = \sum_{i,j=1}^{n} \exp \left( \frac{u^k_{ij} + v^k_{ij} - C_{ij}}{\eta} \right) \quad \text{and} \quad x^* = \sum_{i,j=1}^{n} \exp \left( \frac{u^*_{ij} + v^*_{ij} - C_{ij}}{\eta} \right).$$

We have

$$\exp \left( \frac{u^k_{ij} + v^k_{ij} - C_{ij}}{\eta} \right) = \exp \left( \frac{u^k_{ij} - u^*_i}{\eta} \right) \times \exp \left( \frac{v^k_{ij} - v^*_j}{\eta} \right) \leq \max_{1 \leq i \leq n} \exp \left( \frac{|u^k_{ij} - u^*_i|}{\eta} \right) \times \max_{1 \leq j \leq n} \exp \left( \frac{|v^k_{ij} - v^*_j|}{\eta} \right).$$

Note that each of $x^k$ and $x^*$ is the sum of $n^2$ elements and the ratio between $\exp \left( \frac{u^k_{ij} + v^k_{ij} - C_{ij}}{\eta} \right)$ and $\exp \left( \frac{u^*_{ij} + v^*_{ij} - C_{ij}}{\eta} \right)$ is bounded by $\max_{1 \leq i \leq n} \exp \left( \frac{|u^k_{ij} - u^*_i|}{\eta} \right) \times \max_{1 \leq j \leq n} \exp \left( \frac{|v^k_{ij} - v^*_j|}{\eta} \right)$ for all pairs $i, j$. Apply part (a) of Lemma 6,

$$\max \left\{ \frac{x^k}{x^*} \right\} \leq \max_{1 \leq i \leq n} \exp \left( \frac{|u^k_{ij} - u^*_i|}{\eta} \right) \times \max_{1 \leq j \leq n} \exp \left( \frac{|v^k_{ij} - v^*_j|}{\eta} \right).$$

We have proved from part (a) that $\Delta_{k-1} \leq \frac{\eta^2}{6(\tau^2+1)} \leq \frac{\eta^2}{8}$. From Theorem 1 we get $\Delta_k \leq \Delta_{k-1}$. It means that

$$\max_{i,j} \left\{ \frac{|u^k_{ij} - u^*_i|}{\eta}, \frac{|v^k_{ij} - v^*_j|}{\eta} \right\} = \frac{\Delta_k}{\eta} \leq \frac{\Delta_{k-1}}{\eta} \leq \frac{\eta}{8} \leq \frac{1}{8}.$$

Apply part (b) of Lemma 6,

$$\exp \left( \frac{|u^k_{ij} - u^*_i|}{\eta} \right) \leq 1 + \frac{|u^k_{ij} - u^*_i|}{\eta} + \left( \frac{|u^k_{ij} - u^*_i|}{\eta} \right)^2, \quad \text{and} \quad \exp \left( \frac{|v^k_{ij} - v^*_j|}{\eta} \right) \leq 1 + \frac{|v^k_{ij} - v^*_j|}{\eta} + \left( \frac{|v^k_{ij} - v^*_j|}{\eta} \right)^2.$$

Then

$$\max \left( \frac{x^k}{x^*} \right) \leq \left( 1 + \frac{1}{\eta} \Delta_k + \frac{\Delta^2_k}{\eta^2} \right) \left( 1 + \frac{1}{\eta} \Delta_k + \frac{\Delta^2_k}{\eta^2} \right) = 1 + 2 \frac{\Delta_k}{\eta} + 3 \frac{\Delta^2_k}{\eta^2} + 2 \frac{\Delta^3_k}{\eta^3} + 3 \frac{\Delta_k}{\eta^4} \leq 1 + \frac{2\Delta_k}{\eta} + \frac{3\Delta^2_k}{\eta^2} + \frac{2\Delta^3_k}{\eta^3} + \frac{3\Delta_k}{\eta^4} \leq 1 + \frac{\Delta_k}{\eta} \left( 2 + \frac{3\Delta_k}{\eta^2} + 2 \frac{\Delta^2_k}{\eta^3} + \frac{\Delta^3_k}{\eta^4} \right) \leq 1 + \frac{\Delta_k}{\eta} \left( 2 + \frac{3}{8} + \frac{1}{8^2} + \frac{1}{8^3} \right) \leq 1 + 3 \frac{\Delta_k}{\eta}.$$ 

Apply part (c) of Lemma 6, we get

$$|x^k - x^*| \leq \frac{3}{\eta} \Delta_k \min\{x^k, x^*\}.$$
(e) From Lemma 5 part (a) and Theorem 1 we have \( \frac{\Delta_k}{\eta} \leq \frac{\Delta_k}{\eta} \leq \frac{\eta}{8} \leq \frac{1}{12} \). By part (b) of Lemma 5, we have \( x^k \leq x^* + \frac{3}{\eta} \Delta_k x^* \leq \frac{3}{2}x^* \). Then

\[
x^k \leq x^* + \frac{3}{\eta} \Delta_k x^* \leq \left[ (\alpha + \beta) \left( \frac{1}{2} + \frac{\eta \log(n)}{2\tau - 2\eta \log(n)} \right) + \frac{1}{6 \log(n)} \right] \left( 1 + 3 \frac{\Delta_k}{\eta} \right) \\
\leq (\alpha + \beta) \left( \frac{1}{2} + \frac{\eta \log(n)}{2\tau - 2\eta \log(n)} \right) \left( 1 + 3 \frac{\Delta_k}{\eta} \right) + \frac{1}{4 \log(n)} \\
\leq \frac{1}{2} (\alpha + \beta) + (\alpha + \beta) \frac{3}{2} \frac{\Delta_k}{\eta} + (\alpha + \beta) \frac{\eta \log(n)}{\tau} + \frac{1}{4 \log(n)} \\
\leq \frac{1}{2} (\alpha + \beta) + \frac{1}{4} + (\alpha + \beta) \frac{3\eta}{12\tau} + \frac{1}{4 \log(n)} \\
\leq \frac{1}{2} (\alpha + \beta) + \frac{1}{2} + \frac{1}{4 \log(n)}.
\]