A New Type of Gröbner Basis and Its Complexity

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Abstract

The new type of ideal basis introduced herein constitutes a compromise between the Gröbner bases based on the Buchberger’s algorithm and the characteristic sets based on the Wu’s method. It reduces the complexity of the traditional Gröbner bases and subdues the notorious intermediate expression swell problem and intermediate coefficient swell problem to a substantial extent. The computation of an S-polynomial for the new bases requires at most \( O(m \ln^2 m \ln \ln m) \) word operations whereas \( O(m^6 \ln^2 m) \) word operations are requisite in the Buchberger’s algorithm. Here \( m \) denotes the upper bound for the numbers of terms both in the leading coefficients and for the rest of the polynomials. The new bases are for zero-dimensional polynomial ideals and based on univariate pseudo-divisions. However in contrast to the pseudo-divisions in the Wu’s method for the characteristic sets, the new bases retain the algebraic information of the original ideal and in particular, solve the ideal membership problem. In order to determine the authentic factors of the eliminant, we analyze the multipliers of the pseudo-divisions and develop an algorithm over principal quotient rings with zero divisors.

1 Introduction

The theory of Gröbner bases [1, 3, 8, 11, 14, 19, 21, 26] has been established as a standard tool in algebraic geometry and computer algebra and solves many significant problems in mathematics, science and engineering [4]. Nonetheless the computational complexity of Gröbner bases often demands an enormous amount of computing time and storage space even for problems of moderate sizes [8, P116] [19, P616]. The striking phenomena include the intermediate coefficient swell problem in the computation of Gröbner bases over the rational field with respect to the LEX ordering, as well as the intermediate expression swell problem referring to a generation of a huge number of intermediate polynomials during the implementation of the algorithm. These challenges stimulate decades of ardent endeavors in developing various methodologies such as the normal selection strategies and signatures [3, 6, 12, 17, 20, 37, 38, 22], the modular and \( p \)-adic techniques and Hensel lifting [2, 15, 23, 33, 40, 43], as well as the Gröbner basis conversion methods like the FGLM algorithm [16] and Gröbner Walk [7, 9, 39]. However albeit with all these endeavors over the decades, the high-level complexity associated with the Gröbner basis computations remains a conundrum.

The Wu’s method [44] is based on pseudo-divisions and thus more efficient than the method of Gröbner bases. However the pseudo-divisions only yield the zero locus or radical ideal of the original ideal and hence lose too much algebraic information to solve algebraic problems like the ideal membership problem.

The new type of Gröbner basis introduced herein is a compromise between the Gröbner bases and the characteristic sets based on the Wu’s method. We take the variable of the eliminant as the parametric variable and use univariate pseudo-divisions to reduce the computational complexity and retain the algebraic information of the original ideal simultaneously.

In Algorithm 3.6 we compute the pseudo-eliminant \( \chi_{\varepsilon} \) and pseudo-basis \( B_{\varepsilon} \) of the original ideal. Then we compare \( \chi_{\varepsilon} \) with the multipliers of the pseudo-divisions to discriminate its compatible and incompatible parts in Definition 4.1. Theorem 4.5 establishes that the compatible part of \( \chi_{\varepsilon} \) constitutes a bona fide factor of the eliminant \( \chi \).

We conduct a complete analysis of the incompatible part \( \text{IP}(\chi_{\varepsilon}) \) of \( \chi_{\varepsilon} \) based on modular algorithms whose moduli are the composite divisors of \( \text{IP}(\chi_{\varepsilon}) \) as in Definition 4.3. The principal quotient rings (PQR) thus
obtained might contain zero divisors and we have to redefine the $S$-polynomials in Definition 5.4 carefully. In Algorithm 5.7 we obtain the proper eliminants and proper bases by proper divisions as in Theorem 5.3. We prove rigorously in Theorem 5.10 that the nontrivial proper divisors as in Definition 6.1 that are obtained in Algorithm 5.7 are the bone fide factors of the eliminant $\chi$ of the original ideal.

The characterizations of the new type of basis $B_z$, $B_q$ and $B_p$ are in (6.3), (6.6) and (6.7) respectively. This new type of basis in (6.9) corresponds to a decomposition of the original ideal in (6.8).

A special scenario consisting of two basis elements in Lemma 7.1 reveals that the Buchberger’s algorithm contains the Extended Euclidean Algorithm computing the greatest common divisor of their leading coefficients and in particular, the Bézout coefficients that might swell to an enormous size. This might help to unveil the mystery of the high-level complexity associated with the traditional Gröbner basis computations such as the intermediate coefficient and expression swell problems. By contrast the computation of our new type of $S$-polynomial in (7.1) yields the above results in one step without the Bézout coefficients in a conspicuously moderate number of requisite word operations. Example 7.2 contains a specific example of the new type of basis.

For a ring $R$ we denote $R^* := R \setminus \{0\}$ and use $R^\times$ to denote the set of units in $R^*$.

2 A Pseudo-division Algorithm over PIDs

Let $R$ be a PID and $R[x]$ a polynomial algebra over $R$. Let us denote the set of monomials in $x = (x_1, \ldots, x_n)$ as $[x] := \{x^\alpha : \alpha \in \mathbb{N}^n\}$ with a monomial ordering denoted as $\succ$. A nonzero ideal $I \subset R[x]$ is called a monomial ideal if $I$ is generated by monomials in $[x]$.

Notation 2.1. Let $f = \sum_{\alpha} c_\alpha x^\alpha$ be a polynomial in $R[x]$. We denote the support of $f$ as supp$(f) := \{x^\alpha \in [x] : c_\alpha \neq 0\}$. In particular, we define $\text{supp}(f) := \{1\}$ when $f \in R^*$ and supp$(f) := \emptyset$ when $f = 0$.

Hereafter we use the following terminologies. The leading term of $f$ is a term $c_\beta x^\beta$ that satisfies $x^\beta := \max_{\succ} \{x^\alpha \in \text{supp}(f)\}$ and is denoted as $\text{lt}(f) := c_\beta x^\beta$. Here $\max_{\succ}$ denotes the maximal element with respect to the monomial ordering $\succ$. The leading monomial of $f$ is the monomial $x^\beta$ and denoted as $\text{lm}(f) := x^\beta$. The leading coefficient of $f$ is the coefficient $c_\beta$ and denoted as $\text{lc}(f) := c_\beta \in R^*$.

Let $B = \{b_j : 1 \leq j \leq s\}$ be a polynomial set in $R[x] \setminus \{0\}$. We denote the leading monomial set $\{\text{lm}(b_j) : 1 \leq j \leq s\}$ as $\text{lm}(B)$. Let us also denote the monomial ideal generated by $\text{lm}(B)$ in $R[x]$ as $\langle \text{lm}(B) \rangle$.

In what follows we use gcd$(a, b)$ and lcm$(a, b)$ to denote the greatest common divisor and least common multiple of $a, b \in R^*$ respectively.

Definition 2.2 (Term pseudo-reduction over a PID $R$).

For $f \in R[x] \setminus R$ and $g \in R[x] \setminus \{0\}$, suppose that $f$ has a term $c_\alpha x^\alpha$ such that $x^\alpha \in \text{supp}(f) \cap \langle \text{lm}(g) \rangle$. Then we can make a pseudo-reduction of the term $c_\alpha x^\alpha$ by $g$ as follows.

$$h = \mu f - \frac{m x^\alpha}{\text{lt}(g)} g$$

(2.1)

with the multipliers $m := \text{lcm}(c_\alpha, \text{lc}(g))$ and $\mu := m/c_\alpha \in R^*$. We call $h$ the remainder of the pseudo-reduction and $\mu$ the interim multiplier on $f$ with respect to $g$.

Definition 2.3 (Pseudo-reduced polynomial).

A polynomial $r \in R[x]$ is pseudo-reduced with respect to a polynomial set $B = \{b_j : 1 \leq j \leq s\} \subset R[x] \setminus R$ if supp$(r) \cap \langle \text{lm}(B) \rangle = \emptyset$. In particular, this includes the special case when $r = 0$ and hence supp$(r) = \emptyset$. We also say that $r$ is pseudo-reducible with respect to $B$ if it is not pseudo-reduced with respect to $B$, i.e., supp$(r) \cap \langle \text{lm}(B) \rangle \neq \emptyset$.

Theorem 2.4 (Pseudo-division over a PID $R$).

Suppose that $B = \{b_j : 1 \leq j \leq s\} \subset R[x] \setminus R$ is a polynomial set. For every $f \in R[x]$, there exist a multiplier $\lambda \in R^*$, a remainder $r \in R[x]$ and quotients $q_j \in R[x]$ for $1 \leq j \leq s$ such that

$$\lambda f = \sum_{j=1}^s q_j b_j + r,$$

(2.2)
where \( r \) is pseudo-reduced with respect to \( B \). Moreover, the polynomials in (2.2) satisfy the following condition:

\[
\text{LM}(f) = \max \{ \max_{1 \leq j \leq s} \{ \text{LM}(q_j b_j) \}, \text{LM}(r) \}.
\]

(2.3)

**Proof.** If \( f \) is not pseudo-reduced with respect to \( B \), we define \( x^\alpha := \max_{\varphi} \{ \text{supp}(f) \cap (\text{LM}(B)) \} \). There exists some \( j \) such that \( x^\alpha \) is divisible by \( \text{LM}(b_j) \). Let us make a pseudo-reduction of the term \( c_\alpha x^\alpha \) of \( f \) by \( b_j \) as in (2.1). We denote the remainder as \( h \) and it is easy to see that \( x^\alpha \succ x^\beta := \max_{\varphi} \{ \text{supp}(h) \cap (\text{LM}(B)) \} \). Such term pseudo-reductions terminate in finite steps until the remainder \( h \) is pseudo-reduced with respect to \( B \) since the monomial ordering \( \succ \) is a well-ordering. Hence follows the representation (2.2) in which the multiplier \( \lambda \in R^* \) is a product of such interim multipliers \( \mu \) as in (2.1).

To prove the equality (2.3), it suffices to prove that it holds for the term pseudo-reduction in (2.1). \( \square \)

We call the expression in (2.2) a pseudo-division of \( f \) by \( B \). More specifically, we name the polynomial \( r \) in (2.2) as a remainder of \( f \) and \( \lambda \in R^* \) in (2.2) as a multiplier of the pseudo-division. We say that \( f \) pseudo-reduces to the remainder \( r \) via the multiplier \( \lambda \in R^* \) with respect to \( B \).

## 3 Pseudo-eliminants of Zero-dimensional Ideals

In this section we consider the case when the PID \( R \) in Section 2 satisfies \( R = K[x_1] \) with \( K \) being a field and \( x_1 \) the least variable of \( x \). We always treat the algebra \( K[x] \) as the algebra \( (K[x_1])[\tilde{x}] \) with the variables \( \tilde{x} := (x_2, \ldots, x_n) \). With \( \alpha = (\alpha_2, \ldots, \alpha_n) \), we denote a monomial \( x_2^{\alpha_2} \cdots x_n^{\alpha_n} \) as \( \tilde{x}^\alpha \). Hence \( \text{LC}(f) \in (K[x_1])^* \) for \( f \in (K[x_1])[\tilde{x}] \). Let us use \( (g) \) to denote the principal ideal in \( K[x_1] \) that is generated by \( g \in K[x_1] \). Recall that \( (f) \) denotes a principal ideal in \( (K[x_1])[\tilde{x}] \) that is generated by \( f \in (K[x_1])[\tilde{x}] \).

In what follows let us suppose that \( I \) is a zero-dimensional ideal of \( K[x] = (K[x_1])[\tilde{x}] \).

**Definition 3.1 (Eliminant).**

For a zero-dimensional ideal \( I \subset (K[x_1])[\tilde{x}] \), we denote the generator of the principal ideal \( I \cap K[x_1] \) as \( \chi \) such that \( I \cap K[x_1] = (\chi) \). We call \( \chi \) the eliminant of the zero-dimensional ideal \( I \) henceforth.

**Definition 3.2 (S-polynomial).**

Suppose that \( f, g \in (K[x_1])[\tilde{x}] \setminus K[x_1] \). Let us denote \( m := \text{lcm}(\text{LC}(f), \text{LC}(g)) \in (K[x_1])^* \) and \( \tilde{x}^\gamma := \text{lcm}(\text{LM}(f), \text{LM}(g)) \in [\tilde{x}] \). Then the polynomial

\[
S(f, g) := \frac{m \tilde{x}^\gamma}{\text{LT}(f)} f - \frac{m \tilde{x}^\gamma}{\text{LT}(g)} g
\]

is called the S-polynomial of \( f \) and \( g \).

When \( g \in (K[x_1])^* \) and \( f \in (K[x_1])[\tilde{x}] \setminus K[x_1] \), we take \( \text{LM}(g) = 1 \) and \( m = \text{lcm}(\text{LC}(f), g) \). The S-polynomial in (3.1) is now defined as:

\[
S(f, g) := \frac{m}{\text{LC}(f)} f - m \cdot \text{LM}(f).
\]

(3.2)

By the identity \( m/g = \text{LC}(f)/d \) with \( d := \text{gcd}(\text{LC}(f), g) \in (K[x_1])^* \), we can easily deduce Lemma 3.3 as follows. The same for the proof of Lemma 3.4.

**Lemma 3.3.** When \( g \in (K[x_1])^* \) and \( f \in (K[x_1])[\tilde{x}] \setminus K[x_1] \), the S-polynomial in (3.2) satisfies:

\[
dS(f, g) = (f - \text{LT}(f)) \cdot g := f_1 g
\]

with \( f_1 := f - \text{LT}(f) \).

**Lemma 3.4.** For \( f, g \in (K[x_1])[\tilde{x}] \setminus K[x_1] \), suppose that \( \text{LM}(f) \) and \( \text{LM}(g) \) are relatively prime. Let us denote \( d := \text{gcd}(\text{LC}(f), \text{LC}(g)) \). Then their S-polynomial in (3.1) satisfies:

\[
dS(f, g) = f_1 g - g_1 f = f_1 \cdot \text{LT}(g) - g_1 \cdot \text{LT}(f)
\]

with \( f_1 := f - \text{LT}(f) \) and \( g_1 := g - \text{LT}(g) \). Moreover, we have:

\[
\text{LM}(S(f, g)) = \max\{\text{LM}(f_1 g), \text{LM}(g_1 f)\}.
\]

(3.5)
Lemma 3.5. If lcm(LM(f), LM(g)) ∈ ⟨LM(h)⟩ for f, g, h ∈ (K[x₁])[x] \ K[x₁], then we have the following triangular relationship among their S-polynomials:

\[ λS(f, g) = \frac{λ \cdot \text{lcm}(LT(f), LT(g))}{\text{lcm}(LT(f), LT(h))} S(f, h) - \frac{λ \cdot \text{lcm}(LT(f), LT(g))}{\text{lcm}(LT(g), LT(h))} S(g, h) \]

where the multiplier \( λ := \text{lcm}(h)/d \) with \( d := \gcd(m, \text{lcm}(h)) \in K[x₁] \) and \( m := \text{lcm}(\text{lc}(f), \text{lc}(g)) \). Henceforth let us also call the identity (3.6) the triangular identity of \( S(f, g) \) with respect to \( h \).

Proof. It suffices to write the numerator \( m\tilde{x}^γ \) in the definition of \( S \)-polynomial in (3.1) into \( m\tilde{x}^γ = \text{lcm}(LT(f), LT(g)) \). In fact, the identity (3.6) readily follows if we also write the \( S \)-polynomials \( S(f, h) \) and \( S(g, h) \) into this form. \( \square \)

Algorithm 3.6 (Pseudo-eliminant of a zero-dimensional ideal over \( K[x₁] \)).

Input: A finite polynomial set \( F \subset (K[x₁])[x] \setminus K \).

Output: A pseudo-eliminant \( \chi_ε \in (K[x₁])^* \), pseudo-basis \( B_ε \subset \langle F \rangle \setminus K[x₁] \) and multiplier set \( Λ \subset K[x₁] \setminus K \).

Initialization: A temporary basis set \( G := F \setminus K[x₁] \); a multiplier set \( Λ := \emptyset \); a temporary set \( Ψ := \emptyset \) of \( S \)-polynomials. We initialize \( f₀ := \gcd(F \cap K[x₁]) \) or \( f₀ := 0 \) depending on \( F \cap K[x₁] \neq \emptyset \) or not.

For each pair \( f, g \in G \) with \( f \neq g \), we invoke Procedure \( P \) as follows to compute their \( S \)-polynomial \( S(f, g) \).

Procedure \( P \):

Input: \( f, g \in (K[x₁])[x] \setminus K[x₁] \).

If \( \text{LM}(f) \) and \( \text{LM}(g) \) are relatively prime, we define \( d := \gcd(\text{lc}(f), \text{lc}(g)) \) as in (3.4). If \( d \in K[x₁] \setminus K \), we add \( d \) into the multiplier set \( Λ \), and we do nothing otherwise. Then we disregard the \( S \)-polynomial \( S(f, g) \).

If \( \text{lcm}(\text{LM}(f), \text{LM}(g)) \in \langle \text{LM}(h) \rangle \) for an \( h \in G \setminus \{ f, g \} \), and the triangular identity (3.6) has never been applied to the same triplet \( \{ f, g, h \} \) before, we compute the multiplier \( λ \) as in (3.6). If \( λ \in K[x₁] \setminus K \), we add \( λ \) into the multiplier set \( Λ \), and we do nothing otherwise. Then we disregard the \( S \)-polynomial \( S(f, g) \).

If neither of the above two cases is true, we compute their \( S \)-polynomial \( S(f, g) \) as in (3.1). Then we add \( S(f, g) \) into the set \( Ψ \).

End of \( P \)

We recursively repeat Procedure \( P \) as follows for the pseudo-reductions of all the \( S \)-polynomials in the set \( Ψ \).

Procedure \( P \):

For an \( S \in Ψ \), we invoke Theorem 2.4 to make a pseudo-reduction of \( S \) by the temporary basis set \( G \).

If the multiplier \( λ \in K[x₁] \setminus K \) in (2.2), we add \( λ \) into the multiplier set \( Λ \).

If the remainder \( r = 0 \), we do nothing and continue with the algorithm.

If the remainder \( r \in (K[x₁])[x] \setminus K[x₁] \), we add \( r \) into \( G \). For every \( f \in G \setminus \{ r \} \), we invoke Procedure \( Q \) to compute the \( S \)-polynomial \( S(f, r) \).

If the remainder \( r \in K[x₁] \setminus K \), we redefine \( f₀ := \gcd(r, f₀) \).

Then we delete \( S \) from the set \( Ψ \).

End of \( P \)

Finally we define \( \chi_ε := f₀ \) and \( B_ε := G \) respectively.

Procedure \( R \):

For every \( f \in B_ε \), if \( d := \gcd(\text{lc}(f), \chi_ε) \in K[x₁] \setminus K \), we add \( d \) into the multiplier set \( Λ \).

End of \( R \)

We output \( \chi_ε, B_ε \), and \( Λ \). \( \square \)

Definition 3.7 (Pseudo-eliminant \( \chi_ε \); pseudo-basis \( B_ε \); multiplier set \( Λ \)).

Henceforth we call the univariate polynomial \( \chi_ε \) obtained via Algorithm 3.6 a \emph{pseudo-eliminant} of the zero-dimensional ideal \( I \). We also call the basis set \( B_ε \) a \emph{pseudo-basis} of the ideal \( I \) and \( Λ \) its \emph{multiplier set}.

Lemma 3.8. Algorithm 3.6 terminates in finite steps.
Proof. The termination of the algorithm follows from \((K[x_1]|\tilde{a}|\) being Noetherian. In fact, in Procedure \(P\) of Algorithm 3.6, the monomial ideal \(\text{LM}(G)\) is strictly expanded each time we add the remainder \(r \in (K[x_1]|\tilde{a}| \setminus K[x_1]\) into \(G\) since \(r\) is pseudo-reduced with respect to \(G \setminus \{r\}\).

\end{proof}

\section{Pseudo-eliminant Divisors and Compatibility}

In this section we prove that the compatible part \(\text{CP}(\chi_\varepsilon)\) of the pseudo-eliminant \(\chi_\varepsilon\) is a bona fide factor of the eliminant \(\chi_\varepsilon\).

\begin{definition}[Composite divisor and parts]
For an irreducible factor \(p\) of \(\chi_\varepsilon\) with multiplicity \(i\), if \(p\) is relatively prime to every multiplier \(\lambda\) in \(\Lambda\), then \(p^i\) is called a \textit{compatible divisor} of \(\chi_\varepsilon\). Otherwise \(p^i\) is called an \textit{incompatible divisor} of \(\chi_\varepsilon\).

We name the product of all the compatible divisors of \(\chi_\varepsilon\) as the \textit{compatible part} of \(\chi_\varepsilon\) and denote it as \(\text{CP}(\chi_\varepsilon)\). The \textit{incompatible part} of \(\chi_\varepsilon\) is defined as \(\text{IP}(\chi_\varepsilon) := \chi_\varepsilon / \text{CP}(\chi_\varepsilon)\).

\end{definition}

\begin{algorithm}[Compatible part \(\text{CP}(\chi_\varepsilon)\) and squarefree decomposition of the incompatible part \(\text{IP}(\chi_\varepsilon)\)].
Input: The pseudo-eliminant \(\chi_\varepsilon \in (K[x_1])^*\) and multiplier set \(\Lambda \subset (K[x_1])^*\) that are obtained from Algorithm 3.6.

Output: Compatible part \(\text{CP}(\chi_\varepsilon)\) and a squarefree decomposition \(\{\Omega_i : 1 \leq i \leq s\}\) of the incompatible part \(\text{IP}(\chi_\varepsilon)\).

We initialize \(\Omega_i = \emptyset\) for \(1 \leq i \leq s\) and make a squarefree factorization of the pseudo-eliminant \(\chi_\varepsilon\) as \(\chi_\varepsilon = \prod_{i=1}^s q_i^\varepsilon_i\).

For every \(\lambda \in \Lambda\), we compute \(d_{\lambda i} := \gcd(\lambda, q_i)\). If \(d_{\lambda i} \in K[x_1] \setminus K\), we check whether \(d_{\lambda i}\) is relatively prime to every element \(\omega\) that is already in \(\Omega_i\). If not, we substitute \(d_{\lambda i}\) by \(d_{\lambda i} / \gcd(d_{\lambda i}, \omega)\). We also substitute the \(\omega\) in \(\Omega_i\) by both \(\gcd(d_{\lambda i}, \omega)\) and \(\omega / \gcd(d_{\lambda i}, \omega)\) if neither of them is in \(K^*\). Let us repeat the process until either \(d_{\lambda i} \in K^*\) or \(d_{\lambda i} \in K[x_1] \setminus K\) is relatively prime to every element in \(\Omega_i\). Then we add \(d_{\lambda i}\) into \(\Omega_i\) if \(d_{\lambda i} \in K[x_1] \setminus K\).

Finally, we output \(\chi_\varepsilon / \prod_{i=1}^s \prod_{\omega \in \Omega i} \omega^i\) as the compatible part \(\text{CP}(\chi_\varepsilon)\). We also output \(\{\Omega_i : 1 \leq i \leq s\}\) as a squarefree decomposition of the incompatible part \(\text{IP}(\chi_\varepsilon)\).

\end{algorithm}

\begin{definition}[Composite divisor \(\omega^i\)].
For an element \(\omega\) of the univariate polynomial set \(\Omega_i\) for \(1 \leq i \leq s\) obtained in Algorithm 4.2, we call its \(i\)-th power \(\omega^i\) a \textit{composite divisor} of the incompatible part \(\text{IP}(\chi_\varepsilon)\).

\end{definition}

\begin{lemma} Suppose that each \(f_j\) in \(F := \{f_j : 1 \leq j \leq s\} \subset (K[x_1]|\tilde{a}| \setminus K[x_1]\) has the same leading monomial \(\text{LM}(f_j) = \tilde{a}^\alpha \in [\tilde{a}]\). If \(f = \sum_{j=1}^s f_j\) satisfies \(\text{LM}(f) < \tilde{a}^\alpha\), then there exist multipliers \(b, b_j \in (K[x_1])^*\) for \(1 \leq j < s\) such that
\begin{equation}
bf = \sum_{1 \leq j < s} b_j S(f_j, f_s)
\end{equation}
with the \(S\)-polynomial \(S(f_j, f_s)\) being defined as in (3.1). Moreover, for each irreducible polynomial \(p \in K[x_1] \setminus K\), we can always relabel the subscripts of the polynomial set \(F\) such that the multiplier \(b\) of \(f\) in (4.1) is not divisible by \(p\).

\end{lemma}

\begin{proof}
Let us denote \(l_j := \text{LC}(f_j)\) for \(1 \leq j \leq s\) and \(m_j := \text{lc}(l_j, l_s)\) for \(1 \leq j < s\). From \(\text{LM}(f) < \tilde{a}^\alpha\), we can deduce that \(\sum_{j=1}^s m_j = 0\). Now the identity in (4.1) can be easily corroborated if we define the multipliers as follows:
\begin{equation}
b := \text{LC}(\frac{m_j}{l_j})\colon b_j := \frac{bl_j}{m_j} \quad (1 \leq j < s).
\end{equation}

Let us denote the multiplicity of \(p\) in \(l_j\) as \(\text{mult}_p(l_j)\). We relabel the subscripts of \(f_j\) and \(l_j\) for \(1 \leq j \leq s\) such that \(\text{mult}_p(l_s) = \min_{1 \leq j \leq s}(\text{mult}_p(l_j))\). Then \(\text{mult}_p(m_j/l_j) = \text{mult}_p(l_s / \gcd(l_j, l_s)) = 0\) for \(1 \leq j < s\). Thus the multiplier \(b\) in (4.2) is not divisible by \(p\).

\end{proof}

\begin{theorem} Let \(\chi\) and \(\chi_\varepsilon\) be the eliminant and pseudo-eliminant of a zero-dimensional ideal \(I \subset (K[x_1]|\tilde{a}|\) respectively. Then \(\chi\) is divisible by the compatible divisors of \(\chi_\varepsilon\) and hence by the compatible part \(\text{CP}(\chi_\varepsilon)\) of \(\chi_\varepsilon\).
\end{theorem}
Moreover, from (4.5) and (4.7) we have the following inequality for (4.8):

\[ \eta_{\alpha} \leq S \]

with \( \eta \leq S \), in which we have pseudo-reduced every element in the set \( \{ f_j \in B_\ast : \text{LM}(h_j f_j) = \tilde{x}^\beta, 0 \leq j \leq s \} \) into a new set \( B_t := \{ g_j : 1 \leq j \leq t \} \). And the subscripts of the functions \( \{ h_j \} \) are adjusted accordingly. In this way we have:

\[ \chi = \sum_{j=1}^{t} h_j g_j + \sum_{f_i \in \mathcal{F} \setminus B_t} h_i f_i \]  

(4.3)

If we denote \( \mathit{LT}(h_j) := c_j \tilde{x}^{\alpha_j} \) with \( c_j \in (K[x_1])^* \) for \( 1 \leq j \leq t \), then according to Lemma 4.4, there exist multipliers \( b, b_j \in (K[x_1])^* \) for \( 1 \leq j \leq t \) such that the polynomial \( g := \sum_{j=1}^{t} \mathit{LT}(h_j) \cdot g_j \) satisfies the following identity:

\[ b g = \sum_{1 \leq j < t} b_j S(c_j \tilde{x}^{\alpha_j} g_j, c_i \tilde{x}^{\alpha_i} g_i) \]  

(4.4)

Moreover, we can relabel the subscript set in (4.4) such that \( \text{mult}_p(b) = 0 \) by Lemma 4.4.

In the case of \( B_t \subset (K[x_1])[\tilde{x}] \setminus K[x_1] \), if we denote \( \tilde{x}^{\gamma_j} := \text{lcm}(\text{LM}(g_j), \text{LM}(g_i)) \), then we can simplify the \( S \)-polynomials in (4.4) as follows:

\[ S(c_j \tilde{x}^{\alpha_j} g_j, c_i \tilde{x}^{\alpha_i} g_i) = m_j \tilde{x}^{\beta - \gamma_j} S(g_j, g_i) \]  

(4.5)

with \( m_j := \text{lcm}(c_j \cdot \text{LC}(g_i), c_i \cdot \text{LC}(g_i))/\text{lcm}(\text{LC}(g_j), \text{LC}(g_i)) \) for \( 1 \leq j < t \).

Let \( B_\tau = \{ \tilde{g}_k : 1 \leq k \leq \tau \} \subset (K[x_1])[\tilde{x}] \setminus K[x_1] \) be the pseudo-basis of the ideal \( I \) obtained in Algorithm 3.6, in which we have pseudo-reduced every \( S \)-polynomial \( S(g_j, g_i) \) in (4.5) by \( B_\tau \). More specifically, as per Theorem 2.4, there exist a multiplier \( \lambda_j \in (K[x_1])^* \) as well as a remainder \( \rho_j \chi_\varepsilon \) with \( \rho_j \in K[x_1] \) and quotients \( \tilde{q}_j \in (K[x_1])[\tilde{x}] \) for \( 1 \leq k \leq \tau \) such that the following pseudo-reduction holds for \( 1 \leq j < t \):

\[ \lambda_j S(g_j, g_i) = \sum_{k=1}^{\tau} \tilde{q}_j \tilde{g}_k + \rho_j \chi_\varepsilon \]  

(4.6)

with \( \text{mult}_p(\lambda_j) = 0 \) for \( 1 \leq j < t \) since \( p^t \) is a compatible divisor. As per (2.3), it readily follows that for \( 1 \leq j < t \):

\[ \max_{1 \leq k \leq \tau} \{ \text{LM}(\tilde{q}_j \tilde{g}_k) \} = \text{LM}(S(g_j, g_i)) \prec \tilde{x}^{\gamma_j} \]  

(4.7)

Based on a combination of (4.5) and (4.6), it is straightforward to obtain a pseudo-reduction of the \( S \)-polynomial \( S(c_j \tilde{x}^{\alpha_j} g_j, c_i \tilde{x}^{\alpha_i} g_i) \) in (4.4) by the pseudo-basis \( B_\tau \) with the same multiplier \( \lambda_j \). This combined with (4.4) yield the following representation:

\[ b \lambda g = \sum_{k=1}^{\tau} q_k \tilde{g}_k + \eta \chi_\varepsilon \]  

(4.8)

with \( \eta, q_k \in (K[x_1])[\tilde{x}] \) for \( 1 \leq k \leq \tau \). The multiplier \( b \lambda \) is relatively prime to the compatible divisor \( p^t \). Moreover, from (4.5) and (4.7) we have the following inequality for (4.8):

\[ \max \{ \max_{1 \leq k \leq \tau} \{ q_k \tilde{g}_k, \eta \chi_\varepsilon \} \prec \tilde{x}^\beta \} \]  

(4.9)

Now we can rewrite the representation in (4.3) into a new one in terms of \( B_\tau \) and \( \chi_\varepsilon \) as follows.

\[ b \lambda \chi = \sum_{k=1}^{\tau} \mu_k \tilde{g}_k + \mu_0 \chi_\varepsilon \]  

(4.10)

with \( \mu_k \in (K[x_1])[\tilde{x}] \) for \( 0 \leq k \leq \tau \). And the leading monomials in (4.10) satisfy

\[ \max \{ \max_{1 \leq k \leq \tau} \{ \text{LM}(\mu_k \tilde{g}_k), \text{LM}(\mu_0) \} \prec \tilde{x}^\beta \} \]  

(4.11)
where $r \in F$ satisfies $f$ if there exists an $r_0 \in B$, as in (4.3), we can prove by (3.2) that the conclusion is still sound.

We repeat the above procedure of rewriting the representations of the eliminant $\chi$ so as to strictly reduce the orderings of their leading monomials. Moreover, the multipliers for the representations are always relatively prime to the compatible divisor $p^i$. Since the monomial ordering is a well-ordering, the above process halts after a finite number of repetitions. In this way we shall reach a representation bearing the following form:

$$\nu \chi = h \chi \epsilon,$$

(4.12)

where the multiplier $h \in (K[x_1])^\ast$. In particular, the multiplier $\nu \in (K[x_1])^\ast$ is relatively prime to the compatible divisor $p^i$. Hence follows the conclusion.

5 Analysis of Incompatible Divisors via Modular Method

Let $K$ be a field and $q \in K[x_1] \setminus K$. With $R = K[x_1]$, the quotient ring $R/(q)$ is called a Principal ideal Quotient Ring and abbreviated as a PQR henceforth. Consider the set $R_q := \{r \in K[x_1] : \deg(r) < \deg(q)\}$ with $\deg(r) = 0$ for $r \in K$. We redefine the two binary operations, the addition and multiplication, on $R_q$ such that it is isomorphic to the PQR $R/(q)$. We call $R_q$ a normal PQR and define an epimorphism $\sigma_q : R \to R_q$ as $\sigma_q(f) := r$ via the division $f = hq + r$ with the quotient $h \in K[x_1]$ and unique remainder $r \in R_q$. We can also define an injection $\iota_q : R_q \to R$ as $\iota_q(r) := r$ since $R_q \subseteq K[x_1]$. The epimorphism $\sigma_q$ can be extended to $\sigma_q : (K[x_1])[\tilde{x}] \to R_q[\tilde{x}]$ that is the identity map on the variables $\tilde{x}$. Similarly the injection $\iota_q$ can be extended to $\iota_q : R_q[\tilde{x}] \to (K[x_1])[\tilde{x}]$.

**Definition 5.1** (Term reduction in $R_q[\tilde{x}]$).

For $f \in R_q[\tilde{x}] \setminus R_q$ and $g \in (R_q[\tilde{x}])^\ast \setminus R_q^\times$, suppose that $f$ has a term $c_\alpha \tilde{x}^\alpha$ with $\tilde{x}^\alpha \in \text{supp}(f) \cap (\text{LM}(g))$. We define the multipliers $\mu := \sigma_q(\text{lcm}(l_\alpha, l_g)/l_\alpha)$ and $m := \sigma_q(\text{lcm}(l_\alpha, l_g)/l_g)$ with $l_\alpha := \iota_q(c_\alpha)$ and $l_g := \iota_q(\text{LC}(g))$. We can make a reduction of the term $c_\alpha \tilde{x}^\alpha$ by $g$ as follows.

$$h = \mu f - \frac{m \tilde{x}^\alpha}{\text{LM}(g)} g.$$  

(5.1)

We call $h$ the remainder of the reduction and $\mu$ the interim multiplier on $f$ with respect to $g$.

**Definition 5.2** (Properly reduced polynomial).

A nonzero term $c_\alpha \tilde{x}^\alpha \in R_q[\tilde{x}]$ is said to be properly reducible with respect to $F = \{f_1, \ldots, f_s\} \subseteq R_q[\tilde{x}] \setminus R_q$ if there exists an $f_j \in F$ such that $\tilde{x}^\alpha \in (\text{LM}(f_j))$ and the interim multiplier $\mu$ with respect to $f_j$ as in (5.1) satisfies $\mu \in R_q^\times$. We say that a polynomial $f \in R_q[\tilde{x}]$ is properly reduced with respect to $F$ if none of its terms is properly reducible with respect to $F$.

The proof of the following theorem is almost a verbatim repetition of that for Theorem 2.4.

**Theorem 5.3** (Proper division or reduction).

Suppose that $F = \{f_1, \ldots, f_s\}$ are polynomials in $R_q[\tilde{x}] \setminus R_q$. For every $f \in R_q[\tilde{x}]$, there exist a multiplier $\lambda \in R_q^\times$ as well as a remainder $r \in R_q[\tilde{x}]$ and quotients $q_j \in R_q[\tilde{x}]$ for $1 \leq j \leq s$ such that:

$$\lambda f = \sum_{j=1}^{s} q_j f_j + r,$$

(5.2)

where $r$ is properly reduced with respect to $F$. Moreover, the polynomials in (5.2) satisfy the following condition:

$$\text{LM}(f) = \max \{ \max_{1 \leq j \leq s} \{ \text{LM}(q_j) \cdot \text{LM}(f_j) \}, \text{LM}(r) \}.$$  

(5.3)
Definition 5.4 (S-polynomial over \( R_q \)).

Suppose that \( f \in R_q[\tilde{x}] \setminus R_q \) and \( g \in (R_q[\tilde{x}])^* \setminus R_q^\times \). Let us denote \( l_f := \iota_q(\text{LC}(f)) \) and \( l_g := \iota_q(\text{LC}(g)) \) in \((K[\tilde{x}])^*\) respectively. We also define the multipliers \( m_f := \sigma_q(\text{lcm}(l_f, l_g)/l_f) \) and \( m_g := \sigma_q(\text{lcm}(l_f, l_g)/l_g) \) as well as the monomial \( \tilde{x}^\gamma := \text{lcm}(\text{LM}(f), \text{LM}(g)) \) \in \([\tilde{x}] \). Then the following polynomial:

\[
S(f, g) := \frac{m_f \tilde{x}^\gamma}{\text{LM}(f)} f - \frac{m_g \tilde{x}^\gamma}{\text{LM}(g)} g
\]  

(5.4)

is called the S-polynomial of \( f \) and \( g \) in \( R_q[\tilde{x}] \).

In particular, when \( f \in R_q[\tilde{x}] \setminus R_q \) and \( g \in R_q^* \setminus R_q^\times \), we can take \( \text{LM}(g) = 1 \) and then the S-polynomial in (5.4) bears the following form:

\[
S(f, g) := m_f f - m_g g \cdot \text{LM}(f) = \sigma_q\left(\frac{l_g}{d}\right)(f - \text{LT}(f))
\]

(5.5)

with \( d := \gcd(l_f, l_g) \) and \( l_g := \iota_q(g) \).

When \( \text{LC}(f) \in R_q^* \setminus R_q^\times \) for \( f \in R_q[\tilde{x}] \setminus R_q \), there is another special kind of S-polynomial:

\[
S(f, q) := n_f f = n_f(f - \text{LT}(f))
\]

(5.6)

with \( n_f := \sigma_q(\text{lcm}(l_f, q)/l_f) \).

We can easily deduce the following lemma.

Lemma 5.5. For \( f, g \in R_q[\tilde{x}] \setminus R_q \), suppose that \( \text{LM}(f) \) and \( \text{LM}(g) \) are relatively prime. With the same notations as in Definition 5.4, let us also denote \( d := \gcd(l_f, l_g) \). Then their S-polynomial satisfies:

\[
\sigma_q(d) \cdot S(f, g) = f_1 \cdot \text{LT}(g) - g_1 \cdot \text{LT}(f) = f_1 g - g_1 f
\]

(5.7)

with \( f_1 := f - \text{LT}(f) \) and \( g_1 := g - \text{LT}(g) \).

Let us use the same notations as in Definition 5.4. For \( f, g \in (R_q[\tilde{x}])^* \setminus R_q^\times \) without both of them in \( R_q^* \setminus R_q^\times \), we define \( \text{cmr}(g, f) := m_f \tilde{x}^\gamma/\text{LM}(f) \). Then the S-polynomial \( S(f, g) = \text{cmr}(g, f) \cdot f - \text{cmr}(f, g) \cdot g \), by which we can deduce the following lemma.

Lemma 5.6. For \( f, g, h \in (R_q[\tilde{x}])^* \setminus R_q^\times \) with at most one of them in \( R_q^* \setminus R_q^\times \), if \( \text{lcm}(\text{LM}(f), \text{LM}(g)) \in (\text{LM}(h)) \), then we have the following relationship between their S-polynomials:

\[
\lambda S(f, g) = \frac{\lambda \cdot \text{cmr}(g, f)}{\text{cmr}(h, f)} S(f, h) - \frac{\lambda \cdot \text{cmr}(f, g)}{\text{cmr}(h, g)} S(g, h).
\]

(5.8)

Here the multiplier \( \lambda := \sigma_q(l_h/d) \in R_q^* \) with \( l_h := \iota_q(\text{LC}(h)) \) and \( d := \gcd(\text{lcm}(l_f, l_g), l_h) \).

For a multiplicity \( i \) satisfying \( 1 \leq i \leq s \) and composite divisor \( \omega^i \) with \( \omega \in \Omega_i \) as in Definition 4.3, let us denote \( \omega^i \) as the modulus \( q \) and consider the normal PQR \( R_q \) with \( R = K[x_1] \). Suppose that we have a unique factorization \( q = \omega^i = \prod_{k=1}^t p_k^t \) with \( t \in \mathbb{N}^* \) and \( u \in R^\times \). When \( t > 1 \) the irreducible factors \( \{p_k : 1 \leq k \leq t\} \subset R^* \setminus R^\times \) are pairwise relatively prime. Then every \( a \in R_q^* \) has a standard representation as follows:

\[
a \sim a^* := \prod_{k=1}^t p_k^{\beta_k}, \quad 0 \leq \beta_k \leq i; \quad a = a^* \cdot a^\times.
\]

(5.9)

Algorithm 5.7 (Proper eliminant and proper basis over a normal PQR \( R_q \)).

Input: A finite polynomial set \( F \subset R_q[\tilde{x}] \setminus R_q \).

Output: A proper eliminant \( e_q \in R_q \) and proper basis \( B_q \subset R_q[\tilde{x}] \setminus R_q \).

Initialization: A temporary set \( \mathcal{S} := \emptyset \) in \( R_q[\tilde{x}] \setminus R_q \) for S-polynomials; a temporary \( e \in R_q \) as \( e := 0 \).

For each pair \( f, g \in F \) with \( f \neq g \), we invoke Procedure \( \mathcal{R} \) as follows to compute their S-polynomial \( S(f, g) \).

Procedure \( \mathcal{R} \):
If \( \text{LM}(f) \) and \( \text{LM}(g) \) are relatively prime, we compute the multiplier \( \sigma_q(d) \) as in (5.7) with \( d := \gcd(t_q(\text{LC}(f)), t_q(\text{LC}(g))) \). If \( \sigma_q(d) \in R_q^* \setminus R_q^x \), we compute the \( S \)-polynomial \( S(f, g) \) as in (5.7) and then add it into the set \( \mathcal{S} \). If \( \sigma_q(d) \in R_q^x \), we disregard \( S(f, g) \).

If \( \text{lcm}(\text{LM}(f), \text{LM}(g)) \in \{ \text{LM}(h) \} \) for an \( h \in F \setminus \{ f, g \} \), and the triangular identity (5.8) has not been applied to the same triplet \( \{ f, g, h \} \) before, we compute the multiplier \( \lambda \) as in (5.8). If \( \lambda \in R_q^* \setminus R_q^x \), we compute the \( S \)-polynomial \( S(f, g) \) as in (5.4) and then add it into the set \( \mathcal{S} \). If \( \lambda \in R_q^x \), we disregard \( S(f, g) \).

If neither of the above two cases is true, we compute the \( S \)-polynomial \( S(f, g) \) as in (5.4) and then add it into the set \( \mathcal{S} \).

End of \( \mathcal{R} \)

We recursively repeat Procedure \( \mathcal{P} \) as follows for proper reductions of all the \( S \)-polynomials in \( \mathcal{S} \).

Procedure \( \mathcal{P} \):

For an \( S \in \mathcal{S} \), we invoke Theorem 5.3 to make a proper reduction of \( S \) by \( B \).

If the remainder \( r = 0 \), we do nothing and continue with the algorithm.

If the remainder \( r \in R_q^* \setminus R_q^x \), we halt the algorithm and output \( e_q = 1 \).

If the remainder \( r \in R_q[\bar{x}] \setminus R_q \), we add \( r \) into \( F \). For every \( f \in F \setminus \{ r \} \), we invoke Procedure \( \mathcal{R} \) to compute the \( S \)-polynomial \( S(f, r) \).

If the remainder \( r \in R_q^* \setminus R_q^x \) and \( e = 0 \), we redefine \( e := \sigma_q(\gcd(t_q(r), q)) \).

If the remainder \( r \in R_q^* \setminus R_q^x \) and \( e \in R_q^* \), we compute \( d = \sigma_q(\gcd(t_q(r), t_q(e))) \). If \( df \notin R_q \), we redefine \( e := d \).

Then we delete \( S \) from \( \mathcal{S} \).

End of \( \mathcal{P} \)

Next we recursively repeat Procedure \( \mathcal{Q} \) as follows for proper reductions of the special kinds of \( S \)-polynomials in (5.5) and (5.6).

Procedure \( \mathcal{Q} \):

If \( \mathcal{S} = \emptyset \) and \( e = 0 \), then for every \( f \in F \) with \( \text{LC}(f) \in R_q^* \setminus R_q^x \), we compute the \( S \)-polynomial \( S(f, q) \) as in (5.6) and add it into \( \mathcal{S} \) if this has not been done for \( f \) in a previous step.

Then we recursively repeat Procedure \( \mathcal{P} \).

If \( \mathcal{S} = \emptyset \) and \( e \in R_q^* \), then for every \( f \in F \) with \( \text{LC}(f) \in R_q^* \setminus R_q^x \), if \( \sigma_q(d) \in R_q^* \setminus R_q^x \) with \( d := \gcd(t_q(\text{LC}(f)), t_q(e)) \), we compute the \( S \)-polynomial \( S(f, e) \) as in (5.5) and add it into \( \mathcal{S} \) unless one of its associated has been added into \( \mathcal{S} \) in a previous step.

Then we recursively repeat Procedure \( \mathcal{P} \).

End of \( \mathcal{Q} \)

Finally we define and output \( e_q := e \) and \( B_q := F \) respectively.

**Definition 5.8** (Proper eliminator \( e_q \); proper basis \( B_q \); modular eliminator \( \chi_q \)).

With the ideal \( I_q := \sigma_q(I) \), we call the standard representation \( e_q^* \) as in (5.9) of \( e_q \in I_q \cap R_q \) obtained in Algorithm 5.7, whether it is zero or not, a proper eliminator of \( I_q \). In what follows let us simply denote \( e_q := e_q^* \). We also call the final polynomial set \( B_q \) obtained in Algorithm 5.7 a proper basis of \( I_q \). Moreover, \( \chi_q := \sigma_q(\chi) \) is called the modular eliminator of \( I_q \).

**Lemma 5.9.** Let \( F = \{ f_j : 1 \leq j \leq s \} \subset R_q[\bar{x}] \setminus R_q \) be a polynomial set. Suppose that for \( 1 \leq j \leq s \), each \( f_j \) has the same leading monomial \( \text{LM}(f_j) = \bar{x}^\alpha \). If \( f = \sum_{j=1}^{s} f_j \) satisfies \( \text{LM}(f) \prec \bar{x}^\alpha \), then there exist multipliers \( b, b_j \in R_q^* \) for \( 1 \leq j < s \) such that

\[
bf = \sum_{1 \leq j < s} b_j S(f_j, f_s) \tag{5.10}
\]

with the \( S \)-polynomial \( S(f_j, f_s) \) being defined as in (5.4). Moreover, for every irreducible factor \( p \) of the composite divisor \( q \), we can always relabel the subscripts of the polynomial set \( F = \{ f_j : 1 \leq j \leq s \} \) such that the multiplier \( b \in R_q^* \) in (5.10) is not divisible by \( p \).

**Proof.** Let us denote \( l_j := t_q(\text{LC}(f_j)) \) for \( 1 \leq j \leq s \). We define the multipliers \( m_j := \text{lcm}(l_j, l_s)/l_j \) for \( 1 \leq j < s \). Let us also define a multiplier \( b := \sigma_q(a) \) with \( a := \text{lcm}_{1 \leq j < s}(m_j) \). The identity (5.10) can be corroborated by the multipliers \( b_j := \sigma_q(a_j) \) with \( a_j := a/m_j \) for \( 1 \leq j < s \). Moreover, for an irreducible factor \( p \) of the composite divisor \( q \), we change the order of the elements in \( F \) so that \( \text{mult}_p(l_s) = \text{min}_{1 \leq j < s}(\text{mult}_p(l_j)) \).

Hence \( \text{mult}_p(\lambda_j) = 0 \) for \( 1 \leq j < s \). And thus \( \text{mult}_p(a) = 0 \).
**Theorem 5.10.** Let $q = \omega^i$ be a composite divisor and $\epsilon_q$ and $\chi_q$ denote the proper and modular eliminants respectively as in Definition 5.8.

1. If the proper eliminant $\epsilon_q = 0$, the eliminant $\chi$ is divisible by the composite divisor $q = \omega^i$ and the modular eliminant $\chi_q = 0$.

2. If the proper eliminant $\epsilon_q \in R_q^*$, the eliminant $\chi$ is divisible by $\iota_q(\epsilon_q)$ and the modular eliminant $\chi_q^* = \epsilon_q$.

**Proof.** The proof is similar to that of Theorem 4.5. After fixing an irreducible factor $\tau$ we define $\theta$ define a proper divisor $\chi$ as in Definition 6.1 and in particular, the leading monomials of the representations. Moreover, the multipliers for the representations are always $\nu \chi_q = h \epsilon_q$ (5.11) with $h \in R_q$ and in particular, the multiplier $\nu \in R_q^*$ not divisible by the irreducible factor $p$ of the composite divisor $q$. The conclusion follows from an analysis of (5.11). \hfill $\square$

6 A New Type of Basis for Zero-dimensional Ideals

**Definition 6.1 (Proper divisors $\theta_q$).**

For every composite divisor $q = \omega^i$, there corresponds to a proper eliminant $\epsilon_q$ as in Definition 5.8. We define a proper divisor $\theta_q \in K[x_1]$ in accordance with $\epsilon_q$ as follows. If $\epsilon_q \in R_q^*$, we define $\theta_q := q$; If $\epsilon_q = 0$, we define $\theta_q := \iota_q(\epsilon_q)$.

The following conclusion is straightforward.

**Theorem 6.2.** The eliminant $\chi$ is the product of the compatible part $CP(\chi_e)$ and all the proper divisors $\theta_q$.

With an almost verbatim repetition of the proof for Theorem 4.5, we can prove the following conclusion.

**Lemma 6.3.** Let $B_e = \{g_k: 1 \leq k \leq \tau\}$ be a pseudo-basis of a zero-dimensional ideal $I$ and $CP(\chi_e)$ the compatible part of the pseudo-eliminant $\chi_e$ associated with $B_e$. For every $f \in I$, there exist $\{v_k: 0 \leq k \leq \tau\} \subset (K[x_1])[\tilde{x}]$ and a multiplier $\lambda$ relatively prime to $CP(\chi_e)$ such that:

$$\lambda f = \sum_{k=1}^{\tau} v_k g_k + v_0 \chi_e. \quad (6.1)$$

Moreover, we have:

$$\text{LM}(f) = \max\{ \max_{1 \leq k \leq \tau} \text{LM}(v_k g_k), \text{LM}(v_0) \}. \quad (6.2)$$

**Lemma 6.4.** Let us treat the compatible part $CP(\chi_e)$ as the modulus $d$ and define the normal $PQR$ $R_d$ and the epimorphism $\sigma_d: (K[x_1])[\tilde{x}] \rightarrow R_d[\tilde{x}]$ as before. Then for $I_d := \sigma_d(I)$ and $B_d := \sigma_d(B_e)$, we have an ideal identity in $R_d[\tilde{x}]$ as follows.

$$\langle \text{LT}(I_d) \rangle = \langle \text{LT}(B_d) \rangle. \quad (6.3)$$

**Proof.** For every $g \in I_d$, there exists $f \in I$ such that $\sigma_d(f) = g$ and $\sigma_d(\text{LC}(f)) = \text{LC}(g) \in R_q^*$.

Both (6.1) and (6.2) hold for $f$. We apply $\sigma_d$ to the identity (6.1) and collect the subscript $k$ into a set $\Lambda$ if $\text{LM}(v_k) \cdot \text{LM}(g_k) = \text{LM}(f)$ and $\sigma_d(\text{LC}(v_k g_k)) = \sigma_d(\text{LC}(v_k) \cdot \text{LC}(g_k)) \in R_q^*$.

We have the following conclusions similar to Lemma 6.3 and Lemma 6.4 whose proofs are omitted.
Lemma 6.5. Let $q$ be a composite divisor and $e_q$ and $B_q = \{g_k: 1 \leq k \leq \tau\}$ be the proper eliminant and proper basis of $I_q = \sigma_q(I)$ respectively. For every $f \in I_q$, there exist a multiplier $\lambda \in R_q^\times$ and $\{v_k: 0 \leq k \leq \tau\} \subset R_q[\tilde{x}]$ such that:

$$\lambda f = \sum_{k=1}^{\tau} v_k g_k + v_0 e_q. \quad (6.4)$$

Moreover, we have:

$$\text{LM}(f) = \max \{ \max_{1 \leq k \leq \tau} \{\text{LM}(v_k) \cdot \text{LM}(g_k)\}, \text{LM}(v_0 e_q)\}. \quad (6.5)$$

In particular, the above conclusions are still sound when the proper eliminant $e_q = 0$.

Lemma 6.6. Let $q$ be a composite divisor and $I_q = \sigma_q(I)$. Let $e_q \in R_q \setminus R_q^\times$ and $B_q$ denote the proper eliminant and proper basis of $I_q$ obtained in Algorithm 3.7 respectively. If $e_q = 0$, then we have:

$$\langle \text{LT}(I_q) \rangle = \langle \text{LT}(B_q) \rangle. \quad (6.6)$$

If $e_q \in R_q^* \setminus R_q^\times$, let us treat $e_q$ as the modulus $p$ and define the normal PQR $R_p$. We also define $I_p := \sigma_p(I)$ and $B_p := \sigma_p(B_e)$. Then we have:

$$\langle \text{LT}(I_p) \rangle = \langle \text{LT}(B_p) \rangle. \quad (6.7)$$

In summary, we have the following new type of basis for a zero-dimensional ideal.

Theorem 6.7. Let $d = \text{cp}(\chi_e)$ be the compatible part and $\Theta$ the set of nontrivial proper divisors in Definition 6.1. Then we have the following decomposition of a zero-dimensional ideal $I$.

$$I = (I + \langle d \rangle) \cap \bigcap_{\theta_q \in \Theta} (I + \langle \theta_q \rangle). \quad (6.8)$$

We have a new type of basis in accordance with the above ideal decomposition:

$$(\iota_d(B_d) \cup \{d\}) \cup \bigcup_{e \in \Theta} (\iota_e(B_e) \cup \{e\}), \quad (6.9)$$

where $d$ and $\iota_d(B_d)$ are as in (6.3). Here $e = \theta_q \in \Theta$ and $\iota_e(B_e)$ denotes either $\iota_q(B_q)$ in (6.6) or $\iota_p(B_p)$ in (6.7).

7 Complexity Comparison and Example

Recall that the traditional $S$-polynomial of $f, g \in K[x] \setminus \{0\}$ over a field $K$ is defined as $s(f, g) := x^n(f/\text{lt}(f) - g/\text{lt}(g))$ with $x^n := \text{lcm}(\text{lm}(f), \text{lm}(g))$. Here $\text{lt}(f)$ denotes the leading term of $f$ over $K$ and the same for $\text{lt}(g)$.

Lemma 7.1. Suppose that $I = \langle f, g \rangle$ is a zero-dimensional ideal in $(K[x_1])[\tilde{x}]$ such that $\text{LT}(f) = a\tilde{x}^\alpha$ and $\text{LT}(g) = b\tilde{x}^\beta$ with $a, b \in (K[x_1])^\times$. The Buchberger’s algorithm for the traditional Gröbner bases computes the $S$-polynomial $S(f, g)$ in (3.1) for the new type of basis. In essence it implements the Extended Euclidean Algorithm to compute $\gcd(a', b')$ with $a' := a/\text{lcm}(a)$ and $b' := b/\text{lcm}(b)$. Moreover, the Bézout coefficients of $\gcd(a', b')$ are the coefficient factors of $S(f, g)$.

Proof. We compute the $S$-polynomial $S(f, g)$ as in (3.1) for the new type of basis over $K[x_1]$ as follows.

$$S(f, g) = \lambda \tilde{x}^\gamma - a \mu \tilde{x}^\beta g_1. \quad (7.1)$$

where $f_1 := f - \text{LT}(f)$ and $g_1 := g - \text{LT}(g)$. Here $\tilde{x}^\gamma := \text{lcm}(\tilde{x}^\alpha, \tilde{x}^\beta)$. And the multipliers $\lambda := m/a = b/\rho$ and $\mu := m/b = a/\rho$ with $m := \text{lcm}(a, b)$ and $\rho := \gcd(a, b)$.

With $\deg(a) \geq \deg(b)$, the traditional $S$-polynomial $s(f, g)$ and its further reductions in the Buchberger’s algorithm are equivalent to the polynomial division of $a'$ by $b'$. If $s(f, g)$ is reduced to $h$ with $\text{LT}(h) = r\tilde{x}^\gamma$, then $r$ is exactly the remainder of the division. The procedure of adding $h$ into the basis $\{f, g\}$
and reducing the traditional \( S \)-polynomial \( s(g, h) \) is equivalent to a polynomial division of \( b' \) by \( r/lc(r) \). We can continue to show that the Buchberger’s algorithm essentially implements the Extended Euclidean Algorithm to compute both \( \rho := \gcd(a', b') \) and its Bézout coefficients \( s \) and \( t \), which yields \( w := \rho x^\gamma + s\alpha x^\gamma f_1/lc(a) + t\alpha x^\gamma g_1/lc(b) \). A reduction of the traditional \( S \)-polynomials \( s(f, w) \) and \( s(g, w) \) by \( w \) leads to \( tS(f, g)/(lc(a)lc(b)) \) and \( -sS(f, g)/(lc(a)lc(b)) \) respectively.

We obtained the \( S \)-polynomial in (7.1) in one step without the Bézout coefficients. This substantially scales down the number of intermediate polynomials for the intermediate expression swell problem and the sizes of the intermediate coefficients for the intermediate coefficient swell problem.

Since the worst-case complexity associated with Buchberger’s algorithm is still an open problem, we do not address the problem here. Instead, a meticulous complexity analysis shows that the number of requisite word operations for the computation of the \( S \)-polynomial \( S(f, g) \) in (7.1) is \( O(m \ln^2 m \ln \ln m) \) whereas that of Buchberger’s reduction process in the proof of Lemma 7.1 is \( O(m^6 \ln^2 m) \). Here \( m := \max\{d, N\} \) with \( d := \max\{\deg(lc(f)), \deg(lc(g))\} \). And \( N = \max\{N(f_1), N(g_1)\} \) for \( f_1 = f - \text{LT}(f) \) and \( g_1 = g - \text{LT}(g) \) with \( N(f_1) \) denoting the number of elements in \( \text{supp}(f_1) \) and the same for \( N(g_1) \).

**Example 7.2.** Suppose the ideal \( I = \langle f, g, h \rangle \subset \mathbb{Q}[x, y, z] \) with
\[
\begin{align*}
    f &= -z^2(z + 1)^4 x + y; \quad g = z^4(z + 1)^3 x - y^2; \\
    h &= -x^2y + y^3 + z^4(z - 1)^5.
\end{align*}
\]

The eliminant \( \chi \) bears the following form:
\[
\begin{align*}
    \chi &= z^6(z - 1)^5(z^{13} + 9z^{12} + 36z^{11} + 84z^{10} + 126z^9 + 126z^8 \\
    &\quad + 85z^7 + 31z^6 + 19z^5 - 9z^4 + 4z^3 - 4z^2 - 3z - 1).
\end{align*}
\]

The multiplier set \( \Lambda = \{z^2(z + 1)^3, z^4(z + 1)^6 - 1\} \). The new type of basis is as follows. With the modulus \( p = z^6 \) and over the normal PQR \( R_p \simeq \mathbb{K}[z]/(z^6) \), the modular basis \( B_p \) of \( I_p \) bears the form:
\[
B_p \begin{cases}
    b_1 := z^2(z + 1)^3 y; & b_2 := y^2; \\
    b_3 := z^2(z + 1)^3 x - y; & b_4 := x^2y - z^4(z - 1)^5.
\end{cases}
\]

With the modulus being the compatible part \( q = cp(\chi z) = \chi/z^6 \) and over the normal PQR \( R_q \simeq \mathbb{K}[z]/(q) \), the modular basis \( B_q \) of \( I_q \) bears the form:
\[
B_q \begin{cases}
    a_1 := z^2(z + 1)^3(z^4(z + 1)^6 - 1)y + z^6(z + 1)^3(z - 1)^5; \\
    a_2 := z^4(z + 1)^6(z^4(z + 1)^6 - 1)x + z^6(z + 1)^3(z - 1)^5.
\end{cases}
\]

We list the traditional reduced Gröbner basis of \( I \) in the Appendix for comparison.

A direction for future research is to generalize the new type of basis to ideals of positive dimensions as well as to enhance its computational efficiency. A complexity analysis that is parallel to those in [10, 25, 28, 29, 30, 31] on the traditional Gröbner bases shall be interesting. Some inherent connections have been found between the Gröbner bases and characteristic sets [27, 41, 42]. We are curious whether the new type of basis can shed some new light on these connections.

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Appendix

\[ g_1 = 20253807 z^2 y + 26417412 z^{23} + 1185923612 z^{22} + 850814520 z^{21} - 3776379304 z^{20} - 6824277548 z^{19} + 1862876196 z^{18} + 12815317453 z^{17} + 3550475420 z^{16} + 2124010584 z^{15} - 35582561480 z^{14} + 42918431554 z^{13} - 41728834070 z^{12} + 3564944325 z^{11} - 17049238505 z^{10} + 33886599639 z^9 + 93024031 z^8 - 61146095 z^7 - 51831181 z^6. \]

\[ g_2 = 20253807 y^2 + 903303104 z^{23} + 4102316224 z^{22} + 3140448384 z^{21} - 12683487983 z^{20} - 23996669428 z^{19} + 4804720290 z^{18} + 43739947868 z^{17} + 349557551130 z^{16} + 9051639768 z^{15} - 121400613331 z^{14} + 139970660534 z^{13} - 138071007235 z^{12} + 118589702914 z^{11} - 55199680030 z^{10} + 11927452134 z^9 + 2021069107 z^8 - 38017822 z^7 - 1768266833 z^6; \]

\[ g_3 = 2592487296 x^2 + (777461888 z - 2592487296) y + 108083949263 z^{23} + 486376518055 z^{22} + 34555611130 z^{21} - 1558206505718 z^{20} - 282017901211 z^{19} + 788268739077 z^{18} + 535042098351 z^{17} + 1476923019345 z^{16} + 689330555757 z^{15} - 14602936038043 z^{14} + 1738612348761 z^{13} - 16350039201517 z^{12} + 13787524468420 z^{11} - 6235683207154 z^{10} + 786997920549 z^9 + 62830552934 z^8 - 64382649769 z^7 - 20653113875 z^6; \]

\[ g_4 = 20253807 x^2 y + 1037047036 z^{23} + 4686773132 z^{22} + 34555611112 z^{21} - 14868243976 z^{20} - 2747043972 z^{19} + 673144644 z^{18} + 51651585868 z^{17} + 16267315284 z^{16} + 7429467573 z^{15} - 141636109619 z^{14} + 16316836472 z^{13} - 155454190640 z^{12} + 135706468958 z^{11} - 62903516282 z^{10} + 11263865469 z^9 + 2500312823 z^8 + 197272975 z^7 - 1682438629 z^6 - 101269035 z^5 + 20253807 z^4. \]

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