Bursts and Shocks in a Continuum Shell Model

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March 21, 2022

Abstract

We study a “burst” event, i.e. the evolution of an initial condition having support only in a finite interval of k-space, in the continuum shell model due to Parisi. We show that the continuum equation without forcing or dissipation can be explicitly written in characteristic form and that the right and left moving parts can be solved exactly. When this is supplemented by the appropriate shock condition it is possible to find the asymptotic form of the burst.

Contribution to the proceedings of the Conference: Disorder and Chaos, in honour of Giovanni Paladin, September 22-24, 1997 in Rome.
1 Introduction

In the study of fully developed turbulence the so-called shell models have become an important tool. In these models the essential ingredients are the conservation laws and the assumption of locality on a discrete set of exponentially growing momentum shells. The models are constructed such that Kolmogorov’s 1941 scaling law is built in as a fixed point in the absence of forcing and viscosity.

It has recently been found (see e.g. [2] and references therein) that these models actually capture important features of the strongly intermittent behaviour seen in experiments, which gives rise to corrections to the simple Kolmogorov dimensional scaling. In order to understand how such corrections come about, it is necessary to study the strong “burst” events which destroy the homogeneity of, say, the dissipation field and presumably lead to the appearance of multiscaling.

In 1990 Parisi [1] suggested a continuum approximation of a shell model and showed that solitary wave excitations would exist on top of the Kolmogorov spectrum (a short presentation of his approach can be found in [2]). It has, however, recently been stressed [3] that the shell models have another (trivial) fixed point, where the fluid is quiescent, and that this state seems to be approached before a large burst. In the present paper we shall thus study how a single pulse (a disturbance initially localized in wave number space) will propagate into a motionless fluid. The details of how such a burst propagates are quite interesting and unexpected and although it remains to be seen, whether the shocks which are generated represent the bursts in real turbulent fluids, we believe that they are worth discussing since the methods that we use might be generalized to more realistic models. We feel sure that Giovanni Paladin would have enjoyed our story – indeed he was the one who introduced us to the shell models and later contributed and inspired so much of the work in this field.

2 The GOY shell model and its continuum limit

The basic idea behind the shell model is that the Navier-Stokes equation, when considered in Fourier space, receives contributions to the time deriva-
tive of the velocity from the velocities in a triangle of $k$-vectors. This condition is mimicked by taking $k$-vectors as the one-dimensional discrete set (“shells”) $k_n = k_0 r^n$, and introducing only a single complex field $u_n$ for each $k$ with interactions only between nearest and next nearest neighbours. The field $u_n$ represents the characteristic velocity differences across the $n$th shell, and, in the limit of vanishing viscosity and forcing, energy conservation is imposed in the form
\[ \sum |u_n|^2 = \text{const} \tag{1} \]

The most studied model is the so-called GOY model [2] (after Gledzer, Okhitani and Yamada)
\[ \left( \frac{du_n}{dt} + \nu k_n^2 u_n \right)^* = -ik_n (u_{n+1} u_{n+2} - \frac{\delta}{r} u_{n-1} u_{n+1} - \frac{1-\delta}{r^2} u_{n-1} u_{n-2}) + F_n \tag{2} \]
where $\nu$ is the viscosity and $F_n$ is the forcing. $\delta$ is a free parameter, but becomes fixed at $\delta = 1/2$ when the conservation of helicity is also satisfied.

### 2.1 The Parisi continuum limit

It is an interesting question how to define the shell model in the limit where the $k$-variable becomes continuous. A particular way of reaching this limit is to take $r \to 1$: By writing the distance between the shells as $r = 1 + \epsilon$ with $\epsilon \ll 1$, we have $k_n \approx \exp(ne)$, so with $n \sim \text{const.}/\epsilon$ a continuous range of values is obtained for the variable $k$.

To proceed we use a Taylor expansion of the type
\[ u_{n+1}(t) = u(k_{n+1}, t) = u(n \ln k_{n+1}, t) \approx u(n \ln(1 + \epsilon), t) + \ln(1 + \epsilon) \frac{\partial u_n}{\partial \ln k} \]
\[ \approx u(k, t) + \epsilon k \frac{\partial u(k, t)}{\partial k} \]
\[ = u(k, t) + \epsilon k \frac{\partial u(k, t)}{\partial k}, \text{ with } u(k, t) \equiv u(k_n, t) = u_n(t), \tag{3} \]
and similarly for $u_{n+2}, u_{n-1}$, and $u_{n-2}$. To first order one obtains [1]:
\[ u_t^* + \nu k^2 u^* = -ik(2 - \delta) \left( u^2 + 3k uu_k \right), \tag{4} \]
where $u_k \equiv \partial u/\partial k$ etc. The higher order corrections to (4) and the convergence to the discrete model will be examined in [4]. By rescaling time with
2 − δ we get the Parisi equation, which in its inviscid, unforced form is:

\[ u_t^* + 3ik^2 uu_k = -iku^2 \]  (5)

In terms of the real and imaginary part \( u = a + ib \) it can be written as:

\[ a_t - 3k^2 ab_k - 3k^2 ab_k = 2kab \]  (6)

and

\[ b_t - 3k^2 aa_k + 3k^2 bb_k = k(a^2 - b^2). \]  (7)

For the case of \( a = 0 \), a number of exact solutions to (7) have been found in [5].

### 3 Solution of the inviscid Parisi equation

We shall now study the inviscid, unforced Parisi equation (5) in the special case, where the initial condition is a single “burst”, i.e. the field \( u \) is only nonzero in some finite interval in \( k \)-space.

#### 3.1 Direct simulation

The Parisi-equation (5) or (6)-(7) is a hyperbolic equation [6], and hyperbolic equations are notoriously hard to solve numerically due to the appearance of shocks in the solution. These can be dealt with in a crude way by using upwind differencing of the flux terms [7], but to do this, one need to figure out in which direction a shock is moving. For the real part (6) there is a minus sign on the flux term, signifying that a shock moves to the left (provided both real and imaginary parts are positive). The imaginary part (7) is not so simple, since the flux term is composed of both a right and a left moving part. Never the less upwind differencing works there too, by assuming that the dominant direction is to the right.

An example of a numerical solution with semi-implicit update in time, is seen in figure [1]. The initial condition is the zero state, with a gaussian peak \( a(k, t = 0) = b(k, t = 0) = C \exp(-(k - k_0)^2) \), with \( k_0 = 40 \) and \( C = 0.25 \). In order to neglect viscosity, the inertial subrange must be to the right of the initial perturbation.
Figure 1: A numerical solution of the Parisi equations, without forcing and viscosity. The upper figure shows the solution at $t = 10^{-3}$ after having been started with a gaussian peak at $k = 40$. The inset on the uppermost figure shows the part moving to the left, and illustrate that the relation $a = \sqrt{3}b$ is fulfilled to a high accuracy. In the lower figure only the imaginary part is shown in a space-time plot. Here the splitting of the initial pulse into a left- and right-moving part is clearly visible. The initial phase is $\pi/4$. 

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A short while after the start, shocks are formed in both directions. As was seen from the equations, the real part is only able to move to the left, while the imaginary part moves both ways. Thus only the imaginary part is important in the inertial subrange. Setting $a = 0$ in the original equations, we get just a single real equation for the inertial range:

$$b_t + 3k^2 bb_k = -kb^2 \tag{8}$$

For the left-moving part of the pulse, the real and the imaginary parts are seen to become proportional. Inserting $b = Ca$ into the Parisi equation gives us two possibilities: $a = 0$ (which we already treated) and $a = \pm \sqrt{3}b$. For $a = \sqrt{3}b$ the equation for the imaginary part becomes:

$$b_t - 6k^2 bb_k = 2kb^2. \tag{9}$$

The explanation for the splitting of the pulse in this way will be given in the next section by writing (5) in characteristic form.

Another problem occurring when solving hyperbolic differential equations is to find the correct position and velocity of the shocks. Strictly speaking, the equations are not well-defined after the occurrence of a shock, because the solutions become multivalued. To solve past the time when the shocks first appear one has two choices: 1) to derive a shock condition by imposing conserved quantities (note that energy conservation alone is not enough, since two conditions are needed) or 2) include a higher order terms to smooth out the shock. When solving differential equations with finite differences, there is always a small amount of “numerical diffusion”, corresponding to a $\partial^2 u / \partial k^2$ term in the equations. This is not necessarily the correct higher order term, and therefore the velocity of the shocks as seen in figure 1 is not necessarily correct. We shall show later how to obtain the correct shock conditions by imposing energy conservation, and thereby the correct asymptotics for the solution.

### 3.2 Characteristic form for the Parisi equation

Surprisingly, the “source” term (i.e. the right hand side) of equation (5) can be removed by the simple substitution $u = k^{-1/3}v$ and $k = (2x)^{-3/2}$. This turns (5) into a complex equation reminding strongly of the Burgers equation

$$v_t^* - iuv_x = 0. \tag{10}$$
In terms of real and imaginary parts of $v$ (again denoted by $a$ and $b$) we get

$$a_t + ba_x + ab_x = 0$$  \hspace{1cm} (11)

and

$$b_t + aa_x - bb_x = 0$$  \hspace{1cm} (12)

The calculations turns out to become simpler if (10) is written in terms of modulus and phase, i.e.

$$v = re^{i\phi}$$  \hspace{1cm} (13)

whereby we find the equivalent equations

$$r_t + rr_x \sin \theta + \frac{1}{3} r^2 \theta_x \cos \theta = 0$$  \hspace{1cm} (14)

and

$$\theta_t + 3r_x \cos \theta - r \theta_x \sin \theta = 0$$  \hspace{1cm} (15)

where $\theta = 3\phi$. This can be written as a matrix equation

$$q_t + Aq_x = 0$$  \hspace{1cm} (16)

where $q = (r, \theta)$ and

$$A = \left( \begin{array}{cc} r \sin \theta & \frac{1}{3} r^2 \cos \theta \\ 3 \cos \theta & -r \sin \theta \end{array} \right)$$  \hspace{1cm} (17)

The characteristics are the eigenvalues of $A$, which are $\lambda = \pm r$ and the Riemann invariants can be found through the left eigenvectors of $A$ (or the right eigenvectors of the transpose of $A$). These eigenvectors are of the form $e = (\alpha, \beta)$, and satisfy

$$\beta = r \left( \frac{\pm 1 - \sin \theta}{3 \cos \theta} \right) \alpha$$  \hspace{1cm} (18)

The Riemann invariants satisfy

$$\frac{\partial J_\pm}{\partial r} = \alpha$$  \hspace{1cm} (19)

and

$$\frac{\partial J_\pm}{\partial \theta} = \beta$$  \hspace{1cm} (20)
and the Parisi equation (10) can be reformulated as the two equations

\[ J_{\pm} = \text{const} \text{ on the curve } \frac{dx}{dt} = \pm r(x, t) \]  

(21)

Only the ratio \( \beta/\alpha \) is uniquely defined. One solution for (19)-(20) is obtained by taking \( \alpha = 1/r \) and thus

\[ \alpha = \frac{\partial J_{\pm}}{\partial r} = \frac{1}{r} \]  

(22)

and

\[ \beta = \frac{\partial J_{\pm}}{\partial \theta} = \frac{\pm 1 - \sin \theta}{3 \cos \theta} \]  

(23)

which gives

\[ J_{\pm} = \log(r(1 \pm \sin \theta)^{1/3}) \]  

(24)

Now, the Riemann invariants are only defined by their invariance along the characteristics, and thus we can equally well take them to be defined without the logarithm, i.e.

\[ J_{\pm} = r(1 \pm \sin \theta)^{1/3} \]  

(25)

which of course is constant on the same locus.

To complete the solution we can express \( r \) and \( \theta \) through \( J_{\pm} \). We find

\[ r = \left( \frac{J_{\pm}^3 + J_{\mp}^3}{2} \right)^{1/3} \]  

(26)

and

\[ \sin \theta = \frac{J_{\pm}^3 - J_{\mp}^3}{J_{\pm}^3 + J_{\mp}^3} \]  

(27)

The formulation in terms of characteristics can now be used to understand the splitting of the pulse found in section 3.1. The important point is that, since \( r \) is nonnegative, one family of characteristics \( (J_{\pm}) \) can never move to the left while the other one \( (J_{\mp}) \) can never move to the right. If the initial condition has support in a limited region of \( x \), say \([x_{-}, x_{+}]\) the same is true of \( J_{\pm} \). They both vanish (in the initial condition) outside of this interval. For times \( t > 0 \) we compute the field values by finding where, on the \( x \)-axis (i.e. \( t = 0 \)) the characteristics going through the point \((x, t)\) emanate. Now, if \( x > x_{+} \), the \( J_{\mp} \)-characteristic going through this point must emanate from
some $x_0 > x_-$ and thus $J_-(x,t) = J_-(x_0,0) = 0$. This means that either $r = 0$ (which makes the entire field vanish) or $\sin \theta = 1$ which means that $\theta$ has the constant value $\pi/2$ or $\phi = \pi/6$. This corresponds exactly to the result of last section: that $a = \sqrt{3}b$ for the left moving pulse (which is moving right in the $x$-variable).

A completely similar argument, valid for $x < x_-$, shows that in this case $\sin \theta = -1$ or $\theta = 3\pi/2$ and $\phi = \pi/2$, which implies that $a = 0$, again in agreement with the result of last section for the right moving pulse. This shows that the regions outside $[x_-, x_+]$ are so-called simple wave regions [6].

When we said that $\theta = \pi/2 \Rightarrow \phi = \pi/6$ it is not entirely correct. The value $\phi = \pi/6$ is only one possibility, the two others being $\phi = 5\pi/6$ and $\phi = 3\pi/2$. Likewise, $\theta = 3\pi/2$ can mean $\phi = \pi/2$, $\phi = 7\pi/6$ or $\phi = 11\pi/6$. If the initial phase is in the interval between two of these six values of $\phi$, corresponding, consecutively, to $\sin \theta$ having the value $+1$ and $-1$, the values of $\phi$ selected will be precisely those two. The values $\phi = \pi/6$ and $\phi = \pi/2$ given above correspond to an initial phase around $\pi/4$.

### 3.3 Asymptotic analysis of the burst

The splitting of the pulse into a right-moving part with $a = 0$ and a left-moving part with $a = \sqrt{3}b$ makes it possible to give a complete solution for a single burst event. We shall divide the analysis into the two simplified cases; forward and backward moving parts. In each regime the field is described by a single scalar equation. It is therefore possible to find the correct shock condition using a single conservation law: energy conservation. We have chosen the case where the initial phase is around $\pi/4$, but the other cases can be handled similarly.

#### 3.3.1 Forward propagation

When the real part $a = 0$ the equation for the imaginary part $b$ (which we shall call $u$ in this section) is

$$u_t + 3k^2uu_k = -ku^2 \quad (28)$$

where we, for the sake of explicitness, use the original form instead of the “Burgers form” (10). Again, this is a hyperbolic equation, which can be
solved by the method of characteristics. The characteristic equations are

\begin{align}
    k'(t) &= 3k^2u \\
    u'(t) &= -ku^2
\end{align}

and they can be solved e.g. if initial conditions are specified in the form

\begin{equation}
    u(k, t = 0) = u_0(x)
\end{equation}

where \( x = k_0 = k(t = 0) \) (note that this \( x \) is different from the one used in section 3.2). Indeed one can note that for \( u \) as a function of \( k \) (which is what we want) (29-30) leads to

\begin{equation}
    \frac{du}{dk} = \frac{u'(t)}{k'(t)} = -\frac{1}{3} \frac{u}{k}
\end{equation}

which can be integrated to:

\begin{equation}
    u = u_0(x)(k/x)^{-1/3}
\end{equation}

This can now be used to solve (29-30) explicitly as:

\begin{align}
    u(x, t) &= u_0(x)(1 - 2f(x)t)^{1/2} \\
    k(x, t) &= x(1 - 2f(x)t)^{-3/2}
\end{align}

where \( f(x) = xu_0(x) \).

Two types of singularities occur in these solutions:

1. \( u(k) \) becomes multiple-valued when \( k'(x) = 0 \). This will be called a “turning point” and it implies the existence of a shock.

2. Both \( k(x) \) and \( u(x) \) cease to exist beyond a certain finite time, where \( k(x) \to \infty \). This will be referred to as the finite time singularity.

For given initial conditions we can determine the singularities as function of time and thus the asymptotics of \( u \). We assume that \( u_0(x) > 0 \) (the opposite case will be treated in the next section), say monotonically increasing up to \( x = x^* \) and monotonically decreasing for \( x > x^* \). An example is

\begin{equation}
    u_0(x) = xe^{-x^2}
\end{equation}
– a localized initial disturbance. Note that the initial conditions can always be scaled by a constant, say $u = A v$ if time is also scaled as $t = T \tau$ such that $T A = 1$.

The turning time $t = \alpha$ is found from

$$k'(x) = (1 - 2f(x)t)^{-5/2}(1 + (3xf' - 2f)t) = 0$$  \hspace{1cm} (37)

In particular the time $\alpha_1$ of the first turning point occurs at the minimal time for which

$$1 + (3xf'(x) - 2f(x))t = 1 + (3x^2u'_0 + xu_0)t = 0$$  \hspace{1cm} (38)

and in the special case $x = \sqrt{2} / 3$ the minimum occurs when $2 - 8x^2 + 3x^4 = 0$ or

$$x_1 = \sqrt{(4 + \sqrt{10})/3} \approx 1.54$$  \hspace{1cm} (39)

($x = \sqrt{(4 - \sqrt{10})/3}$ must be discarded since $t < 0$) and

$$\alpha_1 = \frac{1}{2f(x_1)(3x_1^2 - 2)} \approx 0.44$$  \hspace{1cm} (40)

As $t$ increases we see from (38) that the turning point $x$ has to occur in the regime where $u_0 < 0$ and thus $x \geq x^*$. For the case (36) we get as $t \to \infty$ we get $x_\infty = \sqrt{2}/3 \approx 0.816$.

The finite time singularity occurs when $k(x) \to \infty$, which by (35) is equivalent to

$$t = \beta = \frac{1}{2f(x)}$$  \hspace{1cm} (41)

The smallest value $\beta_1$ occurs at the maximum of $f(x)$. For the case (36) this is $x = 1$ and $\beta_1 = e/2 \approx 1.36$.

For very large times the solution makes sense up to

$$x_c(t) \approx (2t)^{-1/2}$$  \hspace{1cm} (42)

which is much smaller than $x_\infty \geq x^*$. At large times thus only the part $0 < x < x_c$ will contribute to the solution and the turning point becomes irrelevant. We can therefore use the form (33) expanded for small $x$ to find

$$u(k, t) = x_c(t)^{4/3}k^{-1/3} \approx (2t)^{-2/3}k^{-1/3}$$  \hspace{1cm} (43)
3.3.2 Shock conditions

Due to the multiple valuedness this solution is cut off by a shock, where it jumps approximately to 0. The appropriate shock conditions have to be found by using the appropriate conservation law. It is reasonable to assume that this is the conservation of energy, which is the fundamental ingredient in the shell model. In terms of the continuous wavenumber, the conservation of energy is

$$\frac{d}{dt} \int u^2 \frac{dk}{k} = 0$$  \hspace{1cm} (44)

which can be written as the local conservation law

$$q_t = -j_k$$  \hspace{1cm} (45)

i.e.

$$\left(\frac{u^2}{k}\right)_t = -(2u^3 k)_k$$  \hspace{1cm} (46)

As long as no shock occur (46) and (28) are equivalent, but crossing a shock the appropriate shock condition is

$$V = \frac{dk}{dt} = \frac{[j]}{[q]}$$  \hspace{1cm} (47)

where \([.]\) denotes the discontinuity across the jump and \(V\) is the velocity of the shock. Using energy conservation gives

$$V = \frac{[2u^3 k]}{[u^2 k^{-1}]} = 2uk^2$$  \hspace{1cm} (48)

3.3.3 Asymptotic forward pulse

Combining the shock condition (48) with the asymptotic form (43) gives for the edge of the shock \(k_e(t)\)

$$\frac{dk_e}{dt} = 2(2t)^{-2/3}(k_e)^{5/3}$$  \hspace{1cm} (49)

which leads to

$$k_e(t) = \left(A - ct^{1/3}\right)^{-3/2}$$  \hspace{1cm} (50)

(where \(c = 4 \times 2^{-2/3}\) and \(A\) is an unknown constant). Thus the shock position diverges at the finite time \(t^* = (A/c)^3\) where the spectrum becomes of the Kolmogorov form all the way to the largest \(k\) and only decays as \(t^{-2/3}\).
3.3.4 Backwards propagation

In the backwards direction we use the fact that

\[ a = \text{Re} u = \sqrt{3} \text{Im} u = \sqrt{3} b \]  

(51)

Then

\[ b_t - 6k^2 bb_k = 2kb^2 \]  

(52)

and now the substitution \( u = -b/2 \) gives us back (28) for \( u \). Thus solving the propagation to smaller \( k \) can be done by taking a negative initial condition for (28) and we thus have the solution as (34) - (35) with \( u_0 < 0 \), i.e. of the form

\[ u_0(x) = -xe^{-x^2} \]  

(53)

In this case the finite time singularity, where \( k(x) \to \infty \) never occurs since \( 1 - 2ft > 0 \) for all \( x, t > 0 \). The turning points, which create shocks are again found by \( k'(x) = 0 \). The analysis is very similar to the forward case, but now, for the initial condition (53), the first shock appears at \( x_1 = \sqrt{(4 - \sqrt{10})/3} \approx 0.53 \) with \( \alpha_1 \approx 2.04 \).

For \( t > \alpha_1 \) the turning points occur at two distinct values of \( x, x_a \) and \( x_b > x_a \). For \( t \to \infty \) we see from (38) that they are solutions of \( 3x^2u_0' + xu_0 = 0 \) which means that \( x_a \) approaches the left hand edge of the interval of support for \( u_0 \) (which is zero for (53)) and \( x_b < x^* \) since \( u'(x_b) < 0 \) (for (33), \( x_b \to \sqrt{2}/3 \)). (It can be noted in passing that the maximum for \( u \) also occurs very close to \( x_b \), but at slightly larger \( x \). The curve thus never crosses itself and only develops a cusp in the limit \( t \to \infty \).)

3.3.5 Asymptotic left-going pulse

Asymptotically the only relevant regimes of initial data are \( 0 < x < x_a \) and \( x_b < x < \infty \). For very small \( x \) we simply expand in \( x \) to get \( u(k) \). The interesting regime is \( x_b < x < \infty \) and here the main variation in \( u \) comes from the regime where \( -2ft \gg 1 \). There

\[ ukt = u_0(1 - 2ft)^{-3/2}x(1 - 2ft)^{1/2}t = ft(1 - 2ft)^{-1} \to -\frac{1}{2} \]  

(54)

or

\[ u(k, t) \approx -\frac{1}{2kt} \]  

(55)
Again, the shock condition (48) gives (since \( u \) jumps to a very small value)

\[
\frac{dk_e}{dt} = 2uk_e^2 = -\frac{k_e}{t}
\]

with the solution

\[
k_e(t) = \frac{C}{t}
\]

Asymptotically the solution thus looks like (55) down to \( k = k_e \) given by (57) where it jumps to a small value (\( O(1/t) \)). Note that the value of \( u \) on the edge of the shock remains constant

\[
u_e \approx -\frac{1}{2C}
\]

4 Conclusion

We have shown that an initial pulse on top of the zero solution of the inviscid Parisi equations splits into two parts, one for each direction. The asymptotical solutions for the two pulses are:

\[
\begin{align*}
 u_{\text{right}} &= (2t)^{-2/3}k^{-1/3} \\
u_{\text{left}} &= (tk)^{-1} \text{ down to } k \sim C/t
\end{align*}
\]

both decaying in time. Thus a burst created on top of the zero solution not only travels down the inertial range until it is dissipated by viscosity (here, by the continuum), it also travels upwards, to the smallest \( k \). The burst does not remain localized in \( k \)-space, but is distributed over the \( k \)-range, and then decays. It is interesting to note that the initial disturbance creates both a forward and an inverse cascade.

We would like to thank Jakob Langgaard Nielsen for helpful discussions.

References

[1] Parisi G., *A Mechanism for Intermittency in a Cascade Model for Turbulence*, Preprint of the University of Rome II (1990). Unpublished.

[2] Bohr T., Jensen M. H., Paladin G. and Vulpiani A., *Dynamical Systems Approach to Turbulence*, Cambridge, in press (1998)
[3] Okkels, F. and Jensen, M. H., Physical Review E, in press (1998).

[4] K. H. Andersen, T. Bohr, M. H. Jensen, J. L. Nielsen and P. Olesen, under preparation.

[5] M. H. Jensen and P. Olesen, Physica D 111, 243 (1998).

[6] Whitham, G. B., Linear and Nonlinear Waves, (John Wiley 1974).

[7] LeVeque, R. J., Numerical Methods for Conservation Laws, (Birkhäuser, 1992).

[8] Chorin, A. J. and Marsden, J. E., A Mathematical Introduction to Fluid Mechanics, (Springer 1990).