Gradient Descent Can Take Exponential Time to Escape Saddle Points

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Abstract

Although gradient descent (GD) almost always escapes saddle points asymptotically [Lee et al., 2016], this paper shows that even with fairly natural random initialization schemes and non-pathological functions, GD can be significantly slowed down by saddle points, taking exponential time to escape. On the other hand, gradient descent with perturbations [Ge et al., 2015, Jin et al., 2017] is not slowed down by saddle points—it can find an approximate local minimizer in polynomial time. This result implies that GD is inherently slower than perturbed GD, and justifies the importance of adding perturbations for efficient non-convex optimization. While our focus is theoretical, we also present experiments that illustrate our theoretical findings.

1 Introduction

Gradient Descent (GD) and its myriad variants provide the core optimization methodology in machine learning problems. Given a function $f(x)$, the basic GD method can be written as:

$$x(t+1) \leftarrow x(t) - \eta \nabla f(x(t)), \quad (1)$$

where $\eta$ is a step size, assumed fixed in the current paper. While precise characterizations of the rate of convergence GD are available for convex problems, there is far less understanding of GD for non-convex problems. Indeed, for general non-convex problems, GD is only known to find a stationary point (i.e., a point where the gradient equals zero) in polynomial time [Nesterov, 2013].

A stationary point can be a local minimizer, saddle point, or local maximizer. In recent years, there has been an increasing focus on conditions under which it is possible to escape saddle points (more specifically, strict saddle points as in Definition 2.4) and converge to a local minimizer. Moreover, stronger statements can be made when the following two key properties hold: 1) all local minima are global minima, and 2) all saddle points are strict. These properties hold for a variety of machine learning problems, including tensor decomposition [Ge et al., 2015], dictionary learning [Sun et al., 2017], phase retrieval [Sun et al., 2016], matrix sensing [Bhojanapalli et al., 2016, Park et al., 2017], matrix completion [Ge et al., 2016, 2017], and...
matrix factorization [Li et al., 2016]. For these problems, any algorithm that is capable of escaping strict saddle points will converge to a global minimizer from an arbitrary initialization point.

Recent work has analyzed variations of GD that include stochastic perturbations. It has been shown that when perturbations are incorporated into GD at each step the resulting algorithm can escape strict saddle points in polynomial time [Ge et al., 2015]. It has also been shown that episodic perturbations suffice; in particular, [Jin et al., 2017] analyzed an algorithm that occasionally adds a perturbation to GD (see Algorithm 1), and proved that not only does the algorithm escape saddle points in polynomial time, but additionally the number of iterations to escape saddle points is nearly dimension-independent. These papers in essence provide sufficient conditions under which a variant of GD has favorable convergence properties for non-convex functions. This leaves open the question as to whether such perturbations are in fact necessary. If not, we might prefer to avoid the perturbations if possible, as they involve additional hyper-parameters. The current understanding of gradient descent is silent on this issue. The major existing result is provided by [Lee et al., 2016], who show that gradient descent, with any reasonable random initialization, will always escape strict saddle points eventually—but without any guarantee on the number of steps required. This motivates the following question:

Does randomly initialized gradient descent generally escape saddle points in polynomial time?

In this paper, perhaps surprisingly, we give a strong negative answer to this question. We show that even under a fairly natural initialization scheme (e.g., uniform initialization over a unit cube, or Gaussian initialization) and for non-pathological functions satisfying smoothness properties considered in previous work, GD can take exponentially long time to escape saddle points and reach local minima, while perturbed GD (Algorithm 1) only needs polynomial time. This result shows that GD is fundamentally slower in escaping saddle points than its perturbed variant, and justifies the necessity of adding perturbations for efficient non-convex optimization.

The counter-example that supports this conclusion is a smooth function defined on \( \mathbb{R}^d \), where GD with random initialization will visit the vicinity of \( d \) saddle points before reaching a local minimum. While perturbed GD takes a constant amount of time to escape each saddle point, GD will get closer and closer to the saddle points it encounters later, and thus take an increasing amount of time to escape. Eventually, GD requires time that is exponential in the number of saddle points it needs to escape, thus \( e^{\Omega(d)} \) steps.

### 1.1 Related Work

Over the past few years, there have been many problem-dependent convergence analyses of non-convex optimization problems. One line of work shows that with smart initialization that is assumed to yield a coarse estimate lying inside a neighborhood of a local minimum, local search algorithms such as gradient descent or alternating minimization enjoy fast local convergence; see, e.g., [Netrapalli et al., 2013, Hardt, 2014, Candes et al., 2015, Sun and Luo, 2016, Bhojanapalli et al., 2016, Yi et al., 2016, Zhang et al., 2017]. On the other hand, [Jain et al., 2017] show that gradient descent can stay away from saddle points, and provide global convergence rates for matrix square-root problems, even without smart initialization. Although these results give relatively strong guarantees in terms of rate, their analyses are heavily tailored to specific problems and it is unclear how to generalize them to a wider class of non-convex functions.

For general non-convex problems, the study of optimization algorithms converging to minimizers dates back to the study of Morse theory and continuous dynamical systems ([Palis and De Melo, 2012, Yin and Kushner, 2003]); a classical result states that gradient flow with random initialization always converges to a

\[1 \text{Assuming that the smoothness parameters (see Definition 2.1-2.3) are all independent of dimension.}\]
minimizer. For stochastic gradient, this was shown by Pemantle [1990], although without explicit running

time guarantees. Lee et al. [2016] established that randomly initialized gradient descent with a fixed stepsize
also converges to minimizers almost surely. However, these results are all asymptotic in nature and it is
unclear how they might be extended to deliver explicit convergence rates. Moreover, it is unclear whether
polynomial convergence rates can be obtained for these methods.

Next, we review algorithms that can provably find approximate local minimizers in polynomial time. The
classical cubic-regularization [Nesterov and Polyak, 2006] and trust region [Curtis et al., 2014] algorithms
require access to the full Hessian matrix. A recent line of work [Carmon et al., 2016, Agarwal et al., 2016].
Carmon and Duchi [2016] shows that the requirement of full Hessian access can be relaxed to Hessian-vector
products, which can be computed efficiently in many machine learning applications. For pure gradient-based
algorithms without access to Hessian information, Ge et al. [2015] show that adding perturbation in each
iteration suffices to escape saddle points in polynomial time. When smoothness parameters are all dimension
independent, Levy [2016] analyzed a normalized form of gradient descent with perturbation, and improved
the dimension dependence to $O(d^3)$. This dependence has been further improved in recent work [Jin et al.,
2017] to polylog($d$) via perturbed gradient descent (Algorithm 1).

1.2 Organization

This paper is organized as follows. In Section 2 we introduce the formal problem setting and background. In
Section 3 we discuss some pathological examples and “un-natural” initialization schemes under which the
gradient descent fails to escape strict saddle points in polynomial time. In Section 4 we show that even under
a fairly natural initialization scheme, gradient descent still needs exponential time to escape all saddle points
whereas perturbed gradient descent is able to do so in polynomial time. We provide empirical illustrations in
Section 5 and conclude in Section 6. We place most of our detailed proofs in the Appendix.

2 Preliminaries

Let $\|\cdot\|_2$ denote the Euclidean norm of a finite-dimensional vector in $\mathbb{R}^d$. For a symmetric matrix $A$, let
$\|A\|_{op}$ denote its operator norm and $\lambda_{\text{min}}(A)$ its smallest eigenvalue. For a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, let $\nabla f(\cdot)$ and $\nabla^2 f(\cdot)$ denote its gradient vector and Hessian matrix.

Let $B_x(r)$ denote the $d$-dimensional $\ell_2$ ball centered at $x$ with radius $r$, $[-1,1]^d$ denote the $d$-dimensional cube centered at 0 with side-length 2, and $B_{\infty}(x, R) = x + [-R, R]^d$ denote the $d$-dimensional cube centered at $x$ with side-length $2R$. We also use $O(\cdot)$, and $\Omega(\cdot)$ as standard Big-O and Big-Omega notation, only hiding absolute constants.

Throughout the paper we consider functions that satisfy the following smoothness assumptions.

Definition 2.1. A function $f(\cdot)$ is $B$-bounded if for any $x \in \mathbb{R}^d$:

$$|f(x)| \leq B.$$  

Definition 2.2. A differentiable function $f(\cdot)$ is $\ell$-gradient Lipschitz if for any $x, y \in \mathbb{R}^d$:

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq \ell \|x - y\|_2.$$  

Definition 2.3. A twice-differentiable function $f(\cdot)$ is $\rho$-Hessian Lipschitz if for any $x, y \in \mathbb{R}^d$:

$$\|\nabla^2 f(x) - \nabla^2 f(y)\|_{op} \leq \rho \|x - y\|_2.$$  


Intuitively, definition 2.1 says function value is both upper and lower bounded; definition 2.2 and 2.3 state the gradients and Hessians of function can not change dramatically if two points are close by. Definition 2.2 is a standard assumption in the optimization literature, and definition 2.3 is also commonly assumed when studying saddle points and local minima.

Our goal is to escape saddle points. The saddle points discussed in this paper are assumed to be “strict” defined as follows (Ge et al., 2015):

Definition 2.4. A saddle point $x^*$ is called an $\alpha$-strict saddle point if there exists some $\alpha > 0$ such that $\| \nabla f(x^*) \|_2^2 = 0$ and $\lambda_{\text{min}}(\nabla^2 f(x^*)) \leq -\alpha$.

That is, a strict saddle point must have an escaping direction so that the eigenvalue of the Hessian along that direction is strictly negative. It turns out that for many non-convex problems studied in machine learning, all saddle points are strict (see Section 1 for more details).

To escape strict saddle points and converge to local minima, we can equivalently study the approximation of second-order stationary points. For $\rho$-Hessian Lipschitz functions, it is defined by Nesterov and Polyak (2006):

Definition 2.5. A point $\mathbf{x}$ is called a second-order stationary point if $\| \nabla f(\mathbf{x}) \|_2 = 0$ and $\lambda_{\text{min}}(\nabla^2 f(\mathbf{x})) \geq 0$. We also define its $\epsilon$-version, that is, an $\epsilon$-second-order stationary point for some $\epsilon > 0$, if point $\mathbf{x}$ satisfies $\| \nabla f(\mathbf{x}) \|_2 \leq \epsilon$ and $\lambda_{\text{min}}(\nabla^2 f(\mathbf{x})) \geq -\sqrt{\rho \epsilon}$.

Second-order stationary points must have a positive semi-definite Hessian in addition to a vanishing gradient. Note if all saddle points $x^*$ are strict, then second-order stationary points are exactly equivalent to local minima.

In this paper, we compare gradient descent and one of its variants—the perturbed gradient descent algorithm (Algorithm 1) proposed by Jin et al. (2017). We focus on the case where the step size satisfies $\eta < 1/\ell$, which is commonly required for finding a minimum even in the convex setting (Nesterov, 2013).

The following theorem shows that if GD with random initialization converges, then it will converge to a second-order stationary point almost surely.

Theorem 2.6 (Lee et al., 2016). Suppose that $f$ is $\ell$-gradient Lipschitz, has continuous Hessian, and step size $\eta < \frac{1}{\ell}$. Furthermore, assume that gradient descent converges, meaning $\lim_{t \to \infty} \mathbf{x}^{(t)}$ exists, and the initialization distribution $\nu$ is absolutely continuous with respect to Lebesgue measure. Then $\lim_{t \to \infty} \mathbf{x}^{(t)} = \mathbf{x}^*$ with probability one, where $\mathbf{x}^*$ is a second-order stationary point.

The assumption that gradient descent converges holds for many non-convex functions (including all the examples considered in this paper). This assumption is used to avoid the case when $\| \mathbf{x}^{(t)} \|_2$ goes to infinity, so $\lim_{t \to \infty} \mathbf{x}^{(t)}$ is undefined.

Note the Theorem 2.6 only provides limiting behavior without specifying the convergence rate. On the other hand, if we are willing to add perturbations, the following theorem not only establishes convergence but also provides a sharp convergence rate:

Theorem 2.7 (Jin et al., 2017). Suppose $f$ is $B$-bounded, $\ell$-gradient Lipschitz, $\rho$-Hessian Lipschitz. For any $\delta > 0$, $\epsilon \leq \frac{\ell B}{\rho}$, there exists a proper choice of $\eta, \tau, t_{\text{thres}}, g_{\text{thres}}$ (depending on $B, \ell, \rho, \delta, \epsilon$) such that Algorithm 1 will find an $\epsilon$-second-order stationary point, with at least probability $1 - \delta$, in the following number of iterations:

$$O \left( \frac{\ell B}{\epsilon^{2}} \log^4 \left( \frac{d \ell B}{\epsilon^2 \delta} \right) \right).$$
Algorithm 1 Perturbed Gradient Descent [Jin et al., 2017]

1: Input: $x^{(0)}$, step size $\eta$, perturbation radius $r$, time interval $t_{\text{thres}}$, gradient threshold $g_{\text{thres}}$.
2: $t_{\text{noise}} \leftarrow -t_{\text{thres}} - 1$.
3: for $t = 1, 2, \cdots$ do
4: if $\|\nabla f(x^{(t)})\|_2 \leq g_{\text{thres}}$ and $t - t_{\text{noise}} > t_{\text{thres}}$ then
5: $x^{(t)} \leftarrow x^{(t)} + \xi_t, \xi_t \sim \text{unif}(B_0(r)), t_{\text{noise}} \leftarrow t$.
6: end if
7: $x^{(t+1)} \leftarrow x^{(t)} - \eta \nabla f(x^{(t)})$.
8: end for

Figure 1: If the initialization point is in red rectangle then it takes GD a long time to escape the neighborhood of saddle point $(0, 0)$.

This theorem states that with proper choice of hyperparameters, perturbed gradient descent can consistently escape strict saddle points and converge to second-order stationary point in a polynomial number of iterations.

3 Warmup: Examples with "Un-natural" Initialization

The convergence result of Theorem 2.6 raises the question: can gradient descent find a second-order stationary point in polynomial number of iterations? In this section, we discuss two very simple and intuitive counter-examples for which gradient descent with random initialization requires an exponential number of steps to escape strict saddle points. We will also explain that, however, these examples are unnatural and pathological in certain ways, thus unlikely to arise in practice. A more sophisticated counter-example with natural initialization and non-pathological behavior will be given in Section 4.

Initialize uniformly within an extremely thin band. Consider a two-dimentional function $f$ with a strict saddle point at $(0, 0)$. Suppose inside the neighborhood $U = [-1, 1]^2$ of saddle point, function is locally
quadratic \( f(x_1, x_2) = x_1^2 - x_2^2 \). For GD with \( \eta = \frac{1}{4} \), the update equation can be written as

\[
x_1^{(t+1)} = \frac{x_1^{(t)}}{2} \text{ and } x_2^{(t+1)} = \frac{3x_2^{(t)}}{2}.
\]

If we initialize uniformly within \([-1, 1] \times [-\frac{3}{2} - \exp(\frac{1}{2}), \frac{3}{2} - \exp(\frac{1}{2})]\) then GD requires at least \( \exp(\frac{1}{2}) \) steps to get out of neighborhood \( U \), and thereby escape the saddle point. See Figure 1a for illustration. Note that in this case the initialization region is exponentially thin (only of width \( 2 \cdot (\frac{3}{2}) - \exp(\frac{1}{2}) \)). We would seldom use such an initialization scheme in practice.

**Initialize far away.** Consider again a two-dimensional function with a strict saddle point at \((0, 0)\). This time, instead of initializing in an extremely thin band, we construct a very long slope so that a relatively large initialization region necessarily converges to this extremely thin band. Specifically, consider a function in the domain \([-\infty, 1] \times [-1, 1]\) that is defined as follows:

\[
f(x_1, x_2) = \begin{cases} 
  x_1^2 - x_2^2 & \text{if } -1 < x_1 < 1 \\
  -4x_1 + x_2^2 & \text{if } x_1 < -2 \\
  h(x_1, x_2) & \text{otherwise},
\end{cases}
\]

where \( h(x_1, x_2) \) is a smooth function connecting region \([-\infty, -2] \times [-\frac{1}{2}, \frac{1}{2}] \) and \([-1, 1] \times [-\frac{1}{2}, \frac{1}{2}] \) while making \( f \) have continuous second derivatives.\(^2\) For GD with \( \eta = \frac{1}{4} \), when \(-1 < x_1 < 1\), the dynamics are

\[
x_1^{(t+1)} = \frac{x_1^{(t)}}{2} \text{ and } x_2^{(t+1)} = \frac{3x_2^{(t)}}{2},
\]

and when \( x_1 < -2 \) the dynamics are

\[
x_1^{(t+1)} = x_1^{(t)} + 1 \text{ and } x_2^{(t+1)} = x_2^{(t)}.
\]

Suppose we initialize uniformly within \([-R - 1, -R + 1] \times [-\frac{1}{2}, \frac{1}{2}]\), for \( R \) large. See Figure 1b for an illustration. Letting \( t \) denote the first time that \( x_1^{(t)} \geq -1 \), then approximately we have \( t \approx R \) and so \( x_2^{(t)} \approx x_2^{(0)} \cdot \left(\frac{1}{2}\right)^R \). From the previous example, we know that if \( \left(\frac{1}{2}\right)^R \approx \left(\frac{3}{2}\right) - \exp \frac{1}{2} \), that is \( R \approx \exp \frac{1}{2} \), then GD will need exponential time to escape from the neighborhood \( U = [-1, 1] \times [-\frac{1}{2}, \frac{1}{2}] \) of the saddle point \((0, 0)\). In this case, we require an initialization region leading to a saddle point at distance \( R \) which is exponentially large. In practice, it is unlikely that we would initialize exponentially far away from the saddle points or optima.

### 4 Main Result

In the previous section we have shown that gradient descent takes exponential time to escape saddle points under “un-natural” initialization schemes. Is it possible for the same statement to hold even under “natural” initialization schemes and non-pathological functions? The following theorem confirms this:

\(^2\)We can construct such function using splines. See Appendix B.
Theorem 4.1 (Uniform initialization over a unit cube). Suppose the initialization point is uniformly sampled from \([-1,1]^d\). There exists a function \(f\) defined on \(\mathbb{R}^d\) that is \(B\)-bounded, \(\ell\)-gradient Lipschitz and \(\rho\)-Hessian Lipschitz with parameters \(B, \ell, \rho\) at most \(\poly(d)\) such that:

1. with probability one, gradient descent with step size \(\eta \leq 1/\ell\) will be \(\Omega(1)\) distance away from any local minima for any \(T \leq e^{\Omega(d)}\).

2. for any \(\epsilon > 0\), with probability \(1 - e^{-d}\), perturbed gradient descent (Algorithm 1) will find a point \(x\) such that \(\|x - x^*\|_2 \leq \epsilon\) for some local minimum \(x^*\) in \(\poly(d, 1/\ell)\) iterations.

Remark: As will be apparent in the next section, in the example we constructed, there are \(2^d\) symmetric local minima at locations \((\pm c, \ldots, \pm c)\), where \(c\) is some constant. The saddle points are of the form \((\pm c, \ldots, \pm c, 0, \ldots, 0)\). Both algorithms will travel across \(d\) neighborhoods of saddle points before reaching a local minimum. For GD, the number of iterations to escape the \(i\)-th saddle point increases as \(\kappa^i\) (\(\kappa\) is a multiplicative factor larger than 1), and thus GD requires exponential time to escape \(d\) saddle points. On the other hand, PGD takes about the same number of iterations to escape each saddle point, and so escapes the \(d\) saddle points in polynomial time. Notice that \(B, \ell, \rho = O(\poly(d))\), so this does not contradict Theorem 2.7.

We also note that in our construction, the local minimizers are outside the initialization region. We note this is common especially for unconstrained optimization problems, where the initialization is usually uniform on a rectangle or isotropic Gaussian. Due to isoperimetry, the initialization concentrates in a thin shell, but frequently the final point obtained by the optimization algorithm is not in this shell.

It turns out in our construction, the only second-order stationary points in the path are the final local minima. Therefore, we can also strengthen Theorem 4.1 to provide a negative result for approximating \(\epsilon\)-second-order stationary points as well.

Corollary 4.2. Under the same initialization as in Theorem 4.1, there exists a function \(f\) satisfying the requirements of Theorem 4.1 such that for some \(\epsilon = 1/\poly(d)\), with probability one, gradient descent with step size \(\eta \leq 1/\ell\) will not visit any \(\epsilon\)-second-order stationary point in \(T \leq e^{\Omega(d)}\).

The corresponding positive result that PGD to find \(\epsilon\)-second-order stationary point in polynomial time immediately follows from Theorem 2.7.

The next result shows that gradient descent does not fail due to the special choice of initializing uniformly in \([-1,1]^d\). For a large class of initialization distributions \(\nu\), we can generalize Theorem 4.1 to show that gradient descent with random initialization \(\nu\) requires exponential time, and perturbed gradient only requires polynomial time.

Corollary 4.3. Let \(B_\infty(z, R) = \{z\} + [-R, R]^d\) be the \(\ell_\infty\) ball of radius \(R\) centered at \(z\). Then for any initialization distribution \(\nu\) that satisfies \(\nu(B_\infty(z, R)) \geq 1 - \delta\) for any \(\delta > 0\), the conclusion of Theorem 4.1 holds with probability at least \(1 - \delta\).

That is, as long as most of the mass of the initialization distribution \(\nu\) lies in some \(\ell_\infty\) ball, a similar conclusion to that of Theorem 4.1 holds with high probability. This result applies to random Gaussian initialization, \(\nu = \mathcal{N}(0, \sigma^2 I)\), with mean 0 and covariance \(\sigma^2 I\), where \(\nu(B_\infty(0, \sigma \log d)) \geq 1 - 1/\poly(d)\).

4.1 Proof Sketch

In this section we present a sketch of the proof of Theorem 4.1. The full proof is presented in the Appendix. Since the polynomial-time guarantee for PGD is straightforward to derive from Jin et al. [2017], we focus on showing that GD needs an exponential number of steps. We rely on the following key observation.
We can lower bound $t$ where $g, g$ (thereby escaping the first saddle point). By the dynamics of gradient descent, we have

$$
L
$$

The key observation is that the number of steps to escape the second saddle point is $(2 + \eta L \gamma)^{t_1} x_1^{(0)}$. Formally, we consider the following function,

$$
f(x_1, x_2) = \begin{cases} 
-\gamma x_1^2 + L x_2^2 & \text{if } x_1 \in [0, 1], x_2 \in [0, 1] \\
L(x_1 - 2)^2 - \gamma x_2^2 & \text{if } x_1 \in [1, 3], x_2 \in [0, 1] \\
L(x_1 - 2)^2 + L(x_2 - 2)^2 & \text{if } x_1 \in [1, 3], x_2 \in [1, 3]
\end{cases}
$$

(3)

Note that this function is not continuous. In the next paragraph we will modify it to make it smooth and satisfy the assumptions of the Theorem but useful intuition is obtained using this discontinuous function. The function has an optimum at $(2, 2)$ and saddle points at $(0, 0)$ and $(2, 0)$. We call $[0, 1] \times [0, 1]$ the neighborhood of $(0, 0)$ and $[1, 3] \times [0, 1]$ the neighborhood of $(2, 0)$. Suppose the initialization $(x^{(0)}, y^{(0)})$ lies in $[0, 1] \times [0, 1]$. Define $t_1 = \min_{x_1^{(0)} \geq 1} t$ to be the time of first departure from the neighborhood of $(0, 0)$ (thereby escaping the first saddle point). By the dynamics of gradient descent, we have

$$x_1^{(t_1)} = (1 + 2\eta \gamma)^{t_1} x_1^{(0)}, \quad x_2^{(t_1)} = (1 - 2\eta L \gamma)^{t_1} x_2^{(0)}.$$ 

Next we calculate the number of iterations such that $x_2 \geq 1$ and the algorithm thus leaves the neighborhood of the saddle point $(2, 0)$ (thus escaping the second saddle point). Letting $t_2 = \min_{x_2^{(0)} \geq 1} t$, we have:

$$x_2^{(t_2)} (1 + 2\eta \gamma)^{t_2 - t_1} = (1 + 2\eta \gamma)^{t_2 - t_1} (1 - 2\eta L \gamma)^{t_1} x_2^{(0)} \geq 1.$$ 

We can lower bound $t_2$ by

$$t_2 \geq \frac{2\eta (L + \gamma) t_1 + \log(\frac{1}{x_2^{(0)}})}{2\eta \gamma} \geq \frac{L + \gamma}{\gamma} t_1.$$ 

The key observation is that the number of steps to escape the second saddle point is $\frac{L + \gamma}{\gamma}$ times the number of steps to escape the first one.

**Spline: connecting quadratic regions.** To make our function smooth, we create buffer regions and use splines to interpolate the discontinuous parts of Equation (3). Formally, we consider the following function, for some fixed constant $\tau > 1$:

$$f(x_1, x_2) = \begin{cases} 
-\gamma x_1^2 + L x_2^2 & \text{if } x_1 \in [0, \tau], x_2 \in [0, \tau] \\
g(x_1, x_2) & \text{if } x_1 \in [\tau, 2\tau], x_2 \in [0, \tau] \\
L(x_1 - 4\tau)^2 - \gamma x_2^2 - \nu & \text{if } x_1 \in [2\tau, 6\tau], x_2 \in [0, \tau] \\
L(x_1 - 4\tau)^2 + g_1(x_2) - \nu & \text{if } x_1 \in [2\tau, 6\tau], x_2 \in [\tau, 2\tau] \\
L(x_1 - 4\tau)^2 + L(x_2 - 2\tau)^2 - 2\nu & \text{if } x_1 \in [2\tau, 6\tau], x_2 \in [2\tau, 6\tau],
\end{cases}
$$

(4)

where $g, g_1$ are spline polynomials and $\nu > 0$ is a constant defined in Lemma B.2. In this case, there are saddle points at $(0, 0)$, and $(4\tau, 0)$ and the optimum is at $(4\tau, 4\tau)$. Intuitively, $[\tau, 2\tau] \times [0, \tau]$ and $[2\tau, 6\tau] \times [\tau, 2\tau]$ are buffer regions where we use splines ($g$ and $g_1$) to transition between regimes and make $f$ a smooth function. Also in this region there is no stationary point and we still satisfy the smoothness assumptions in the theorem. Figure 2a shows the surface and stationary points of this function. We call the union of the regions defined in Equation (4) a *tube*.
Figure 2: Graphical illustrations of our counter-example. The blue points are saddle points and the red point is the minimum. The pink line is the trajectory of gradient descent.

**From two saddle points to d saddle points.** We can readily adapt our construction of the tube to d dimensions, such that the function is smooth, the location of saddle points are \((0, \ldots, 0), (4\tau, 0, \ldots, 0), \ldots, (4\tau, \ldots, 4\tau, 0)\), and optimum is at \((4\tau, \ldots, 4\tau)\). Let \(t_i\) be the number of step to escape the neighborhood of the \(i\)-th saddle point. We generalize our key observation to this case and obtain 
\[ t_{i+1} \geq \frac{L+\gamma}{\gamma} \cdot t_i \text{ for all } i. \]
This gives 
\[ t_d \geq \left(\frac{L+\gamma}{\gamma}\right)^d \] which is exponential time. Figure 2b shows the tube and trajectory of GD.

**Mirror trick: from tube to octopus.** In the construction thus far, the saddle points are all on the boundary of tube. To avoid the difficulties of constrained non-convex optimization, we would like to make all saddle points be interior points of the domain. We use a simple mirroring trick; i.e., for both coordinate every coordinate \(x_i\) we reflect \(f\) along its axis. See Figure 2c for an illustration in the case \(d = 2\).

**Extension: from octopus to \(\mathbb{R}^d\).** Up to now we have constructed a function defined on a closed subset of \(\mathbb{R}^d\). The last step is to extend this function to the entire Euclidean space. Here we apply the classical Whitney Extension Theorem (Theorem B.3) to finish our construction. We remark that the Whitney extension may lead to more stationary points. However, we will demonstrate in the proof that GD and PGD stay within the interior of “octopus” defined above, and hence cannot converge to any other stationary point.

5 **Experiments**

In this section we use simulations to verify our theoretical findings. The objective function is defined in (14) and (15) in the Appendix. In Figures 3 and Figure 4 GD stands for gradient descent and PGD stands for Algorithm 1. For both GD and PGD we let the stepsize \(\eta = \frac{1}{4L}\). For PGD, we choose \(t_{\text{thres}} = 1\), \(g_{\text{thres}} = \frac{\gamma^c}{100}\) and \(r = \frac{1}{100}\). In Figure 3 we fix dimension \(d = 5\) and vary \(L\) as considered in Section 4.1, similarly in Figure 4 we choose \(d = 10\) and vary \(L\). First notice that in all experiments, PGD converges faster than GD as suggested by our theorems. Second, observe the “horizontal” segment in each plot represents the number of iterations to escape a saddle point. For GD the length of the segment grows at a fixed rate, which coincides with the result mentioned at the beginning for Section 4.1 (that the number of iterations to escape a saddle point increase at each time with a multiplicative factor \(\frac{L+\gamma}{\gamma}\)). This phenomenon is also verified in the figures by the fact that as the ratio \(\frac{L+\gamma}{\gamma}\) becomes larger, the rate of growth of the number of iterations to
escape increases. On the other hand, the number of iterations for PGD to escape is approximately constant ($\sim \frac{1}{\eta \gamma}$).

6 Conclusion

In this paper we established the failure of gradient descent to efficiently escape saddle points for general non-convex smooth functions. We showed that even under a very natural initialization scheme, gradient descent can require exponential time to converge to a local minimum whereas perturbed gradient descent converges in polynomial time. Our results demonstrate the necessity of adding perturbations for efficient non-convex optimization.

We expect that our results and constructions will naturally extend to a stochastic setting. In particular, we expect that with random initialization, general stochastic gradient descent will need exponential time to escape saddle points in the worst case. However, if we add perturbations per iteration or the inherent randomness is non-degenerate in every direction (so the covariance of noise is lower bounded), then polynomial time is known to suffice [Ge et al., 2015]. Another interesting direction is to use our counter-example as a building block to prove a computational lower bound under an oracle model [Nesterov, 2013, Woodworth and Srebro, 2016].

This paper does not rule out the possibility for gradient descent to perform well for some non-convex functions with special structures. Indeed, for the matrix square-root problem, Jain et al. [2017] shows that with reasonable random initialization, gradient updates will stay away from all saddle points, and thus converge to a local minimum efficiently. It is an interesting future direction to identify other classes of non-convex functions that gradient descent can optimize efficiently and does not suffer from the negative results described in this paper.
7 Acknowledgment

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A Proofs for Results in Section 4

In this section, we provide proofs for Theorem 4.1 and Corollary 4.3. The proof for Corollary 4.2 easily follows from the same construction as in Theorem 4.1, so we omit it here. For Theorem 4.1, we will prove each claim individually.

A.1 Proof for Claim 1 of Theorem 4.1

Outline of the proof. Our construction of the function is based on the intuition in Section 4.1. Note the function $f$ defined in (3) is 1) not continuous whereas we need a $C^2$ continuous function and 2) only defined on a subset of Euclidean space whereas we need a function defined on $\mathbb{R}^d$. To connect these quadratic functions, we use high-order polynomials based on spline theory. We connect $d$ such quadratic functions and show that GD needs exponential time to converge if $x^{(0)} \in [0, 1]^d$. Next, to make all saddle points as interior point, we exploit symmetry and use a mirroring trick to create $2^d$ copies of the spline. This ensures that as long as the initialization is in $[-1, 1]^d$, gradient descent requires exponential steps. Lastly, we use the classical Whitney extension theorem [1934] to extend our function from a closed subset to $\mathbb{R}^d$.

Step 1: The tube. We fix four constants $L = e, \gamma = 1, \tau = e$ and $\nu = (\frac{17\tau}{6} + \frac{11L}{2})\tau^2$. We first construct a function $f$ and a closed subset $D_0 \subset \mathbb{R}^d$ such that if $x^{(0)}$ is initialized in $[0, 1]^d$ then the gradient descent dynamics will get stuck around some saddle point for exponential time. Define the domain as:

$$D_0 = \bigcup_{i=1}^{d+1} \{x \in \mathbb{R}^d : 6\tau \geq x_1, \ldots, x_{i-1} \geq 2\tau, 2\tau \geq x_i \geq 0, \tau \geq x_{i+1}, \ldots, x_d \geq 0\},$$  

(5)

where $i = 1$ means $0 \leq x_1 \leq 2\tau$ and other coordinates are smaller than $\tau$, and $i = d + 1$ means that all coordinates are larger than $2\tau$. See Figure 5a for an illustration. Next we define the objective function as follows. For a given $i = 1, \ldots, d - 1$, if $6\tau \geq x_1, \ldots, x_{i-1} \geq 2\tau, \tau \geq x_i \geq 0, \tau \geq x_{i+1}, \ldots, x_d \geq 0$, we have

$$f(x) = \sum_{j=1}^{i-1} L(x_j - 4\tau)^2 - \gamma x_j^2 + \sum_{j=i+1}^{d} Lx_j^2 - (i - 1)\nu \triangleq f_{i,1}(x),$$  

(6)

and if $6\tau \geq x_1, \ldots, x_{i-1} \geq 2\tau, 2\tau \geq x_i \geq \tau, \tau \geq x_{i+1}, \ldots, x_d \geq 0$, we have

$$f(x) = \sum_{j=1}^{i-1} L(x_j - 4\tau)^2 + g(x_i, x_{i+1}) + \sum_{j=i+2}^{d} Lx_j^2 - (i - 1)\nu \triangleq f_{i,2}(x),$$  

(7)

where the constant $\nu$ and the bivariate function $g$ are specified in Lemma B.2 to ensure $f$ is a $C^2$ function and satisfies the smoothness assumptions in Theorem 4.1. For $i = d$, we define the objective function as

$$f(x) = \sum_{j=1}^{d-1} L(x_j - 4\tau)^2 - \gamma x_d^2 - (d - 1)\nu \triangleq f_{d,1}(x),$$  

(8)

if $6\tau \geq x_1, \ldots, x_{d-1} \geq 2\tau$ and $\tau \geq x_d \geq 0$ and

$$f(x) = \sum_{j=1}^{d-1} L(x_j - 4\tau)^2 + g_1(x_d) - (d - 1)\nu \triangleq f_{d,2}(x)$$  

(9)
if $6\tau \geq x_1, \ldots, x_{d-1} \geq 2\tau$ and $2\tau \geq x_d \geq \tau$ where $g_1$ is defined in Lemma B.2. Lastly, if $6\tau \geq x_1, \ldots, x_d \geq 2\tau$, we define

$$f (x) = \sum_{j=1}^{d} L (x_j - 4\tau)^2 - d\nu \triangleq f_{d+1,1}(x).$$

(10)

Figure. 5 shows an intersection surface (a slice along the $x_i$-$x_{i+1}$ plane) of this construction.

**Remark A.1.** As will be apparent in Theorem B.2, $g$ and $g_1$ are polynomials with degrees bounded by five, which implies that for $\tau \leq x_i \leq 2\tau$ and $0 \leq x_{i+1} \leq \tau$ the function values and derivatives of $g(x_i, x_{i+1})$ and $g(x_i)$ are bounded by poly($L$); in particular, $\rho = \text{poly}(L)$.

**Remark A.2.** In Theorem B.2 we show that the norms of the gradients of $g$ and $g_1$ gradients are strictly larger than zero by a constant ($\geq \gamma \tau$), which implies that for $\epsilon < \gamma \tau$, there is no $\epsilon$-second-order stationary point in the connection region. Further note that in the domain of the function defined in Eq. (6) and (8), the smallest eigenvalue of Hessian is $-2\gamma$. Therefore we know that if $x \in D_0$ and $x_d \leq 2\tau$, then $x$ cannot be an $\epsilon$-second-order stationary point for $\epsilon \leq \frac{4\gamma^2}{\rho}$.

Now let us study the stationary points of this function. Technically, the differential is only defined on the interior of $D_0$. However in Steps 2 and 3, we provide a $C^2$ extension of $f$ to all of $\mathbb{R}^d$, so the above lemma should be interpreted as characterizing the critical points of this extended function $f$ in $D_0$. Using the analytic form of Eq. (6) - (10) and Remark A.2, we can easily identify the stationary points of $f$.

**Lemma A.3.** For $f : D_0 \rightarrow \mathbb{R}$ defined in Eq. (6) to Eq. (10), there is only one local optimum:

$$x^* = (4\tau, \ldots, 4\tau)^\top,$$

and $d$ saddle points:

$$(0, \ldots, 0)^\top, (4\tau, 0, \ldots, 0)^\top, \ldots, (4\tau, \ldots, 4\tau, 0)^\top.$$

Next we analyze the convergence rate of gradient descent. The following lemma shows that it takes exponential time for GD to achieve $x_d \geq 2\tau$.

**Lemma A.4.** Let $\tau \geq \epsilon$ and $x^{(0)} \in [-1, 1]^d \cap D_0$. GD with $\eta \leq \frac{1}{2\tau}$ and any $T \leq \left(\frac{L+\gamma}{\tau}\right)^{d-1}$ satisfies $x_d^{(T)} \leq 2\tau$.

**Proof.** Define $T_0 = 0$ and for $k = 1, \ldots, d$, let $T_k = \min\{t \mid x_k^{(t)} \geq 2\tau\}$ be the first time the iterate escapes the neighborhood of the $k$-th saddle point. We also define $T_k^\tau$ as the number of iterations inside the region

$$\{x_1, \ldots, x_{k-1} \geq 2\tau, \tau \leq x_k \leq 2\tau, 0 \leq x_{k+1}, \ldots, x_d \leq \tau\}.$$

First we bound $T_k^\tau$. Lemma B.2 shows $\frac{\partial g(x_k, x_{k+1})}{\partial x_k} \leq -2\gamma \tau$ so after every gradient descent step, $x_k$ is increased by at least $2\eta\gamma \tau$. Therefore we can upper bound $T_k^\tau$ by

$$T_k^\tau \leq \frac{2\tau - \tau}{2\eta\gamma \tau} = \frac{1}{2\eta\gamma}.$$

Note this bound holds for all $k$. 

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Next, we lower bound $T_1$. By definition, $T_1$ is the smallest number such that $x_1^{(T_1)} \geq 2\tau$ and using the definition of $T_1^\tau$, we know $x_1^{(T_1-T_1^\tau)} \geq \tau$. By the gradient update equation, for $t = 1 \ldots, T_1-T_1^\tau$, we have $x_1^t = (1 + 2\eta\gamma)^t x_1^0$. Thus we have:

\[
    x_1^{(0)} (1 + 2\eta\gamma)^{T_1-T_1^\tau} \geq \tau
\]

\[
    \Rightarrow T_1 - T_1^\tau \geq \frac{1}{2\eta\gamma} \log \left( \frac{\tau}{x_1^{(0)}} \right).
\]

Since $x_1^0 \leq 1$ and $\tau \geq e$, we know $\log(\tau/x_1^0) \geq 1$. Therefore $T_1 - T_1^\tau \geq \frac{1}{\eta\gamma} \geq T_1^\tau$.

Next we show iterates generated by GD stay in $D_0$. If $x^{(t)}$ satisfies $6\tau \geq x_1, \ldots, x_{k-1} \geq 2\tau, \tau \geq x_k \geq 0, \tau \geq x_{k+1} \ldots, x_d \geq 0$, then for $1 \leq j \leq k$,

\[
    x_j^{(t+1)} = (1 - \eta L) x_j^{(t)} - 4\eta L \tau \in [2\tau, 6\tau],
\]

for $j = k$,

\[
    x_k^{(t+1)} = (1 + 2\eta\gamma) x_k^{(t)} \in [0, 2\tau],
\]

and for $j \geq k + 1$

\[
    x_j^{(t+1)} = (1 - 2\eta L) x_j^{(t)} \in [0, \tau].
\]

Similarly, if $x^{(t)}$ satisfies $6\tau \geq x_1, \ldots, x_{k-1} \geq 2\tau, 2\tau \geq x_k \geq \tau, \tau \geq x_{k+1} \ldots, x_d \geq 0$, the above arguments still hold for $j \leq k - 1$ and $j \geq k + 2$. For $j = k$, note that

\[
    x_j^{(t+1)} = x_j^{(t)} - \frac{\partial g(x_j, x_{j+1})}{\partial x_j}
\]

\[
    \leq x_j^{(t)} + 2\eta\gamma \tau \leq 6\tau,
\]
where in the first inequality we have used Lemma B.2. For \( j = k + 1 \), by the dynamics of gradient descent, at \((T_k, T_k^r)\)-th iteration, \( x_{k+1}^{(T_k-T_k^r)} = x_{k+1}^{(0)} (1 - 2\eta L) \tau (T_k - T_k^r) \). Note Lemma B.2 shows in the region \( \{ x_1, \ldots, x_{k-1} \geq 2\tau, \tau \leq x_k \leq 2\tau, 0 \leq x_{k+1}, \ldots, x_d \leq \tau \} \), we have

\[
\frac{\partial f(x)}{\partial x_{k+1}} \leq -2\gamma x_{k+1}.
\]

Putting this together we have the following upper bounds for \( t = T_k - T_k^r + 1, \ldots, T_k \):

\[
x^{(t)}_{k+1} \leq x^{(0)}_{k+1} (1 - 2\eta L) \tau (T_k - T_k^r) \cdot (1 + 2\eta \gamma)^{t-1} (T_k - T_k^r) \leq \tau,
\]

which implies \( x^{(t)} \) is in \( D_0 \).

Next, let us calculate the relation between \( T_k \) and \( T_k+1 \). By our definition of \( T_k \) and \( T_k^r \), we have:

\[
x^{(T_k)}_{k+1} \leq x^{(0)}_{k+1} (1 - 2\eta L) T_k - T_k^r \cdot (1 + 2\eta \gamma)^{T_k - T_k^r}.
\]

For \( T_k+1 \), with the same logic we used for lower bounding \( T_1 \), we have

\[
x^{(T_k+1-T_k^r+1)}_{k+1} \geq \tau
\]

\[
x^{(T_k)}_{k+1} (1 + \eta \gamma) T_k - T_k^r \geq \tau
\]

\[
x^{(0)}_{k+1} (1 - 2\eta L) (T_k - T_k^r) \cdot (1 + 2\eta \gamma)^{T_k - T_k^r} \geq \tau.
\]

Taking logarithms on both sides and then using \( \log(1 - \theta) \leq -\theta, \log(1 + \theta) \leq \theta \) for \( 0 \leq \theta \leq 1 \), and \( \eta \leq \frac{1}{2\gamma} \), we have

\[
2\eta \gamma \left( T_k+1 - T_k^r + (T_k - T_k^r) \right) \geq \log \left( \frac{T_k - T_k^r}{x^{(0)}_{k+1}} \right) + 2\eta L \left( T_k - T_k^r \right)
\]

\[
\geq \frac{L + \gamma}{\gamma} \left( T_k - T_k^r \right)
\]

In last step, we used the initialization condition whereby \( \log \left( \frac{x^{(0)}_{k+1}}{T_k - T_k^r} \right) \geq 1 \geq 0 \). Since \( T_1 - T_1^r \geq \frac{1}{2\eta \gamma} \), to enter the region \( x_1, \ldots, x_d \geq 2\tau \) we need \( T_d \) iterations, which is lower bounded by

\[
T_d \geq \frac{1}{2\eta \gamma} \left( \frac{L + \gamma}{\gamma} \right)^{d-1} \geq \left( \frac{L + \gamma}{\gamma} \right)^{d-1}.
\]

\[ \square \]

**Step 2: From the tube to the octopus.** We have shown that if \( x^0 \in [-1,1]^d \cap D_0 \), then gradient descent needs exponential time to approximate a second order stationary point. To deal with initialization points in \([−1,1]^d − D_0\), we use a simple mirroring trick; i.e., for each coordinate \( x_i \), we create a mirror domain of \( D_0 \) and a mirror function according to \( i \)-th axis and then take union of all resulting reflections. Therefore, we end up with an “octopus” which has \( 2^d \) copies of \( D_0 \) and \([−1,1]^d \) is a subset of this “octopus.” Figure 5b shows the construction for \( d = 2 \).
The mirroring trick is used mainly to make saddle points be interior points of the region (octopus) and ensure that the positive result of PGD (claim 2) will hold.

We now formalize this mirroring trick. For \(a = 0, \ldots, 2^d - 1\), let \(a_2\) denote its binary representation. Denote \(a_2(0)\) as the indices of \(a_2\) with digit 0 and \(a_2(1)\) as those that are 1. Now we define the domain

\[
D_a = \bigcup_{i=1}^{d} \left\{ x \in \mathbb{R}^d : 6\tau \geq |x_1|, \ldots, |x_{i-1}| \geq 2\tau, |x_i| \leq 2\tau, |x_{i+1}|, \ldots, |x_d| \leq \tau \right\},
\]

\[
D = \bigcup_{a=0}^{2^d-1} D_a.
\]

Note this is a closed subset of \(\mathbb{R}^d\) and \([-1, 1]^d \subset D\). Next we define the objective function. For \(i = 1, \ldots, d-1\), if \(6\tau \geq |x_1|, \ldots, |x_{i-1}| \geq 2\tau, |x_i| \leq \tau, |x_{i+1}|, \ldots, |x_d| \leq \tau\):

\[
f(x) = \sum_{j \leq i-1, j \in a_2(0)} L(x_j - 4\tau)^2 + \sum_{j \leq i-1, j \in a_2(1)} L(x_j + 4\tau)^2 - \gamma x_i^2
\]

\[+ \sum_{j=i+1}^{d} Lx_j^2 - (i-1)\nu, \tag{14}\]

and if \(6\tau \geq |x_1|, \ldots, |x_{i-1}| \geq 2\tau, \tau \leq |x_i| \leq 2\tau, |x_{i+1}|, \ldots, |x_d| \leq \tau\):

\[
f(x) = \sum_{j \leq i-1, j \in a_2(0)} L(x_j - 4\tau)^2 + \sum_{j \leq i-1, j \in a_2(1)} L(x_j + 4\tau)^2 + G(x_i, x_{i+1})
\]

\[+ \sum_{j=i+2}^{d} Lx_j^2 - (i-1)\nu, \tag{15}\]

where

\[
G(x_i, x_{i+1}) = \begin{cases} 
g(x_i, x_{i+1}) & \text{if } i \in a_2(0) 
g(-x_i, x_{i+1}) & \text{if } i \in a_2(1). 
\end{cases}
\]

For \(i = d\), if \(6\tau \geq |x_1|, \ldots, |x_{i-1}| \geq 2\tau, |x_i| \leq \tau\):

\[
f(x) = \sum_{j \leq i-1, j \in a_2(0)} L(x_j - 4\tau)^2 + \sum_{j \leq i-1, j \in a_2(1)} L(x_j + 4\tau)^2 - \gamma x_i^2 - (i-1)\nu, \tag{16}\]

and if \(6\tau \geq |x_1|, \ldots, |x_{i-1}| \geq 2\tau, \tau \leq |x_i| \leq 2\tau\):

\[
f(x) = \sum_{j \leq i-1, j \in a_2(0)} L(x_j - 4\tau)^2 + \sum_{j \leq i-1, j \in a_2(1)} L(x_j + 4\tau)^2 + G_1(x_i) - (i-1)\nu, \tag{17}\]

where

\[
G_1(x_i) = \begin{cases} 
g_1(x_i) & \text{if } i \in a_2(0) 
g_1(-x_i) & \text{if } i \in a_2(1). 
\end{cases}
\]
Lastly, if $6\tau \geq |x_1|, \ldots, |x_d| \geq 2\tau$:

$$f(x) = \sum_{j \leq i-1, j \in a_2(0)} L(x_j - 4\tau)^2 + \sum_{j \leq i-1, j \in a_2(1)} L(x_j + 4\tau)^2 - dv. \quad (18)$$

Note that if a coordinate $x_i$ satisfies $|x_i| \leq \tau$, the function defined in Eq. (14) to (17) is an even function (fix all $x_j$ for $j \neq i$, $f(\ldots, x_i, \ldots) = f(\ldots, -x_i, \ldots)$) so $f$ preserves the smoothness of $f_0$. By symmetry, mirroring the proof of Lemma A.4 for $D_d$ for $a = 1, \ldots, 2^d - 1$ we have the following lemma:

**Lemma A.5.** Choosing $\tau = e$, if $x^{(0)} \in [-1, 1]^d$ then for gradient descent with $\eta \leq \frac{1}{2\tau}$ and any $T \leq \left(\frac{L+\gamma}{\gamma}\right)^{d-1}$, we have $x^{(T)} \leq 2\tau$.

**Step 3: From the octopus to $\mathbb{R}^d$.** It remains to extend $f$ from $D$ to $\mathbb{R}^d$. Here we use the classical Whitney extension theorem (Theorem B.3) to obtain our final function $F$. Applying Theorem B.3 to $f$ we have that there exists a function $F$ defined on $\mathbb{R}^d$ which agrees $f$ on $D$ and the norms of its function values and derivatives of all orders are bounded by $O(poly(d))$. Note that this extension may introduce new stationary points. However, as we have shown previously, GD never leaves $D$ so we can safely ignore these new stationary points. We have now proved the negative result regarding gradient descent.

### A.2 Proof for Claim 2 of Theorem 4.1

To show that PGD approximates a local minimum in polynomial time, we first apply Theorem 2.7 which shows that PGD finds an $\epsilon$-second-order stationary point. Remark A.2 shows in $D$, every $\epsilon$-second-order stationary point is $\epsilon$ close to a local minimum. Thus, it suffices to show iterates of PGD stay in $D$. We will prove the following two facts: 1) after adding noise, $x$ is still in $D$, and 2) until the next time we add noise, $x$ is in $D$.

For the first fact, using the choices of $g_{\text{thres}}$ and $r$ in Jin et al. [2017] we can pick $\epsilon$ polynomially small enough so that $g_{\text{thres}} \leq \frac{\tau}{10}$ and $r \leq \frac{\tau}{10}$. Without loss of generality, suppose that in the region

$$\{x_1, \ldots, x_{k-1} \geq 2\tau, 0 \leq x_k, \ldots, x_d \leq \tau\},$$

we have $\|\nabla f(x)\|_2 \leq g_{\text{thres}} \leq \frac{\tau}{10}$, which implies $|x_j - 4\tau| \leq \frac{\tau}{20}$ for $j = 1, \ldots, k-1$, and $x_j \leq \frac{\tau}{20}$ for $j = k, \ldots, d$. Therefore,

$$\left|(x + \xi)_j - 4\tau\right| \leq \frac{\tau}{10} \quad \text{for } j = 1, \ldots, k-1$$

$$\left|(x + \xi)_j\right| \leq \frac{\tau}{10} \quad \text{for } j = k, \ldots, d. \quad (19)$$

For the second fact suppose at the $t'$-th iteration we add noise. Now without loss of generality, suppose that after adding noise, $x^{(t')} \geq 0$, and by the first fact $x^{(t'')}$ is in the region

$$\{x_1, \ldots, x_{i-1} \geq 2\tau, 0 \leq x_i \leq \ldots, x_d \leq \frac{\tau}{10}\}. $$

Now we use the same argument for proving GD stays in $D$. Suppose at $t''$-th iteration we add noise again. Then for $t' < t < t''$, we have that if $x^{(t)}$ satisfies $6\tau \geq x_1, \ldots, x_{k-1} \geq 2\tau, \tau \geq x_k \geq 0, \tau \geq x_{k+1}, \ldots, x_d \geq 0$, then for $1 \leq j \leq k$,

$$x^{(t+1)}_j = (1 - \eta L) x^{(t)}_j - 4\eta L\tau \in [2\tau, 6\tau],$$

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for \( j = k \),
\[
x_j^{(t+1)} = (1 + 2\eta\gamma) x_j^{(t)} \in [0, 2\tau],
\]
and for \( j \geq k + 1 \)
\[
x_j^{(t+1)} = (1 - 2\eta L) x_j^{(t)} \in [0, \tau].
\]
Similarly, if \( x^{(t)} \) satisfies \( 6\tau \geq x_1, \ldots, x_{k-1} \geq 2\tau, 2\tau \geq x_k \geq \tau, \tau \geq x_{k+1}, \ldots, x_{d} \geq 0 \), the above arguments still hold for \( j \leq k - 1 \) and \( j \geq k + 2 \). For \( j = k \), note that
\[
x_j^{(t+1)} = x_j^{(t)} - \eta \frac{\partial g(x_j, x_{j+1})}{\partial x_j}
\leq x_j^{(t)} + 2\eta \gamma \tau \leq 6\tau,
\]
where the first inequality we have used Lemma B.2.

For \( j = k + 1 \), by the dynamics of gradient descent, at the \((T_k - T_k^\tau)\)-th iteration, \( x_{k+1}^{(T_k - T_k^\tau)} = x_{k+1}^{(t')} (1 - 2\eta L)^{(T_k - T_k^\tau) - t'} \). Note that Lemma B.2 shows in the region
\[
\{x_1, \ldots, x_{k-1} \geq 2\tau, \tau \leq x_k \leq 2\tau, 0 \leq x_{k+1}, \ldots, x_d \leq \tau\},
\]
we have
\[
\frac{\partial f(x)}{\partial x_{k+1}} \geq -2\gamma x_{k+1}.
\]
Putting this together we obtain the following upper bound, for \( t = T_k - T_k^\tau + 1, \ldots, T_k \):
\[
x_{k+1}^{(t)} \leq x_{k+1}^{(t')} (1 - 2\eta L)^{(T_k - T_k^\tau) - t'} (1 + 2\eta \gamma)^{t - (T_k - T_k^\tau)} \leq \tau,
\]
where the last inequality is because \( t - (T_k - T_k^\tau) \leq T_k^\tau - \frac{1}{2\eta \gamma} \). This implies \( x^{(t)} \) is in \( D_0 \). Our proof is complete.

A.3 Proof for Corollary 4.3

Define \( g(x) = f(\frac{x - z}{R}) \) to be an affine transformation of \( f \), \( \nabla g(x) = \frac{1}{R} \nabla f(\frac{x - z}{R}) \), and \( \nabla^2 g(x) = \frac{1}{R^2} \nabla^2 f(\frac{x - z}{R}) \). We see that \( \ell_g = \frac{\ell_f}{R^2}, \rho_g = \frac{\rho_f}{R^2}, \) and \( B_g = B_f \), which are \( poly(d) \).

Define the mapping \( h(x) = \frac{x - z}{R} \), and the auxiliary sequence \( y_t = h(x_t) \). We see that
\[
x^{(t+1)} = x^{(t)} - \eta \nabla g(x^{(t)})
\]
\[
h^{-1}(y^{(t+1)}) = h^{-1}(y^{(t)}) - \frac{\eta}{R} \nabla f(y^{(t)})
\]
\[
y^{(t+1)} = h(Ry^{(t)} + z - \frac{\eta}{R} \nabla f(y^{(t)})) = y^{(t)} - \frac{\eta}{R^2} \nabla f(y^{(t)}).
\]
Thus gradient descent with stepsize \( \eta \) on \( g \) is equivalent to gradient descent on \( f \) with stepsize \( \frac{\eta}{R^2} \). The first conclusion follows from noting that with probability \( 1 - \delta \), the initial point \( x^{(0)} \) lies in \( B_\infty(z, R) \), and then applying Theorem 4.1. The second conclusion follows from applying Theorem 2.7 in the same way as in the proof of Theorem 4.1.
B Auxiliary Theorems

The following are basic facts from spline theory. See Equation (2.1) and (3.1) of [Dougherty et al. 1989]

**Theorem B.1.** Given data points $y_0 < y_1$, function values $f(y_0), f(y_1)$ and derivatives $f'(y_0), f'(y_1)$, the cubic Hermite interpolant is defined by

$$p(y) = c_0 + c_1\delta_y + c_2\delta_y^2 + c_3\delta_y^3,$$

where

$$c_0 = f(y_0), \quad c_1 = f'(y_0), \quad c_2 = \frac{3S - f'(y_1) - 2f'(y_0)}{y_1 - y_0}, \quad c_3 = \frac{-2S - f'(y_1) - f'(y_0)}{(y_1 - y_0)^2}$$

for $y \in [y_0, y_1]$, $\delta_y = y - y_0$ and slope $S = \frac{f(y_1) - f(y_0)}{y_1 - y_0}$. $p(y)$ satisfies $p(y_0) = f(y_0), p(y_1) = f(y_1), p'(y_0) = f'(y_0)$ and $p'(y_1) = f'(y_1)$. Further, for $f(y_1) < f(y_0) < 0$, if

$$f'(y_1) \geq \frac{3(f(y_1) - f(y_0))}{y_1 - y_0}$$

then we have $p(y) \leq f(y_0)$ for $y \in [y_0, y_1]$.

We use these properties of splines to construct the bivariate function $g$ and the univariate function $g_1$ in Section A. The next lemma studies the properties of the connection functions $g(\cdot, \cdot)$ and $g_1(\cdot)$.

**Lemma B.2.** Let $\nu = (\frac{17\gamma}{6} + \frac{11L}{2})\tau^2$. Define $g(x_i, x_{i+1}) = g_1(x_i) + g_2(x_i)x_{i+1}^2$. There exist polynomial functions $g$ and $g_1$ such that for any $i = 1, \cdots, d$, for $f_{i,1}$ and $f_{i,2}$ defined in Eq. (6)-(10), $g(x_i, x_{i+1})$ ensures $f_{i,2}$ satisfies, if $x_i = \tau$, then

$$f_{i,2}(x) = f_{i,1}(x), \quad \nabla f_{i,2}(x) = \nabla f_{i,1}(x), \quad \nabla^2 f_{i,2}(x) = \nabla^2 f_{i,1}(x),$$

and if $x_i = 2\tau$ then

$$f_{i,2}(x) = f_{i+1,1}(x), \quad \nabla f_{i,2}(x) = \nabla f_{i+1,1}(x), \quad \nabla^2 f_{i,2}(x) = \nabla^2 f_{i+1,1}(x).$$

Further, $g$ satisfies for $\tau \leq x_i \leq 2\tau$ and $0 \leq x_{i+1} \leq \tau$

$$\frac{\partial g(x_i, x_{i+1})}{\partial x_i} \leq -2\gamma \tau \quad \frac{\partial g(x_i, x_{i+1})}{\partial x_{i+1}} \geq -2\gamma x_{i+1}.$$
and $g_1$ satisfies for $\tau \leq x_i \leq 2\tau$

$$\frac{\partial g_1(x_i)}{\partial x_i} \leq -2\gamma \tau.$$ 

**Proof.** Let us first construct $g_1$. Since we know for a given $i \in [1, \ldots, d]$, if $x_i = \tau$, $\frac{\partial f_{i,1}}{\partial x_i} = -2\gamma \tau$, $\frac{\partial^2 f_{i,1}}{\partial x_i^2} = -2\gamma$ and if $x_i = 2\tau$, $\frac{\partial f_{i,1}}{\partial x_i} = -4L\tau$ and $\frac{\partial^2 f_{i,1}}{\partial x_i^2} = 2L$. Note for $L > \gamma$, $0 > -2\gamma \tau > -4L\tau$ and $2L > \frac{-4L\tau - (-2\gamma \tau)}{2\tau - \tau}$. Applying Theorem [B.1] we know there exists a cubic polynomial $p(x_i)$ such that

$$p(\tau) = -2\gamma \tau \quad \text{and} \quad p(2\tau) = -4L\tau$$

and $p(x_i) \leq -2\gamma \tau$ for $\tau \leq x_i \leq 2\tau$. Now define

$$g_1(x_i) = (x_i - \tau)p(x_i) - \gamma \tau^2.$$ 

Note by this definition $g_1$ satisfies the boundary condition at $\tau$. Lastly we choose $\nu = \left(\frac{17\gamma}{6} + \frac{11L}{2}\right)\tau^2 = -g_1(2\tau) + 4L\tau^2$. It can be verified that this construction satisfies all the boundary conditions.

Now we consider $x_{i+1}$. Note when if $x_i = \tau$, the only term in $f$ that involves $x_{i+1}$ is $Lx_{i+1}^2$ and when $x_i = 2\tau$, the only term in $f$ that involves $x_{i+1}$ is $-\gamma x_{i+1}^2$. Therefore we can construct $g_2$ directly:

$$g_2(x_i) = -\gamma - \frac{10(L + \gamma)(x_i - 2\tau)^3}{\tau^3} - \frac{15(L + \gamma)(x_i - 2\tau)^4}{\tau^4} - \frac{6(L + \gamma)(x_i - 2\tau)^5}{\tau^5}.$$ 

Note

$$g'_2(x_i) = -\frac{30(L + \gamma)(x_i - 2\tau)^2(x_i - \tau)^2}{\tau^5}.$$ 

After some algebra, we can show this function satisfies for $\tau \leq x_i \leq 2\tau$

$$g_2(x_i) \geq -\gamma,$$

$$g'_2(x_i) \leq 0,$$

$$g_2(\tau) = L, \quad g_2(2\tau) = -\gamma$$

$$g''_2(\tau) = g''_2(2\tau) = 0.$$ 

Therefore it satisfies the boundary conditions related to $x_{i+1}$. Further note that at the boundary ($x_i = \tau$ or $2\tau$), the derivative and the second derivative are zero, so it will not contribute to the boundary conditions involving $x_i$. Now we can conclude that $g$ and $g_1$ satisfy the requirements of the lemma.

We use the following continuous extension theorem which is a sharpened result of the seminal Whitney extension theorem [Whitney, 1934].

**Theorem B.3** (Theorem 1.3 of [Chang, 2015]). Suppose $E \subseteq \mathbb{R}^d$. Let the $C^m(E)$ norm of a function $F : E \to \mathbb{R}$ be sup \{ $|\partial^\alpha| : x \in E, |\alpha| \leq m$ \}. If $E$ is a closed subset in $\mathbb{R}^d$, then there exists a linear operator $T : C^m(E) \to C^m(\mathbb{R}^d)$ such that if $f \in C^m(E)$ is mapped to $F \in \mathbb{C}^m(\mathbb{R}^d)$, then $F|_E = f$ and $F$ has derivatives of all orders on $E^c$. Furthermore, the operator norm $\|T\|_{op}$ is at most $Cd^5m/2$, where $C$ depends only on $m$. 

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