Rotating Charged Solutions to Einstein-Maxwell-ChernSimons Theory in 2+1 Dimensions

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Abstract

We obtain a class of rotating charged stationary circularly symmetric solutions of Einstein-Maxwell theory coupled to a topological mass term for the Maxwell field. These solutions are regular, have finite mass and angular momentum, and are asymptotic to the uncharged extreme BTZ black hole.

1 Introduction

Gravity in 2+1 dimensions has received a great deal of attention in the recent past. The rationale for this is that it is technically simpler so that it may be used as a theoretical laboratory for studying certain aspects of gravity in 3+1 dimensions. In particular, it might shed some light on the quantum gravity problem. The technical simplicity of the 2+1 dimensional theory stems from the fact that in 2+1 dimensions the components of Riemann curvature tensor can be completely expressed in terms of Einstein tensor. This implies that Einstein’s theory is trivial in the absence of matter. The space time will be flat outside the matter sources with no propagating degrees of freedom and no Newtonian limit to the field equations. This means that the gravitational influence of the matter sources will be topological in character. For example, a point source will produce a conical structure to the space-time.
The situation changes when a cosmological term of anti-de Sitter (AdS) variety is added to Einstein’s equations. In this case, as pointed out by Banadoz-Teitelboim-Zanellie [2], the field equations admit black hole solutions with finite mass and finite angular momentum [3]. Various properties of this BTZ black hole such as its thermodynamic, statistical, and quantum properties have been studied extensively [4]. There are also a number of works in which the corresponding charged black holes have been investigated. The first of these was static charged black holes constructed by Banadoz et al. in [2]. It was then found that due to the logarithmic nature of the electromagnetic potential, the quasi local mass of this solution diverges [5]. Later, a horizonless static solution with magnetic charge was constructed by Hirshmann et al. [5]. In this solution, again the quasi local mass diverges as demonstrated by Chan [6].

A rotating charged black hole solution was found by Kamata et al. [7]. They obtained their solution by imposing a self (anti-self) duality condition on the electromagnetic field. The resulting solutions were asymptotic to an extreme BTZ black hole solution but again had diverging mass and angular momentum [8].

From the brief summary of some of the works in recent literature given above, it is clear that solutions in 2 + 1 dimensions with finite mass and angular momentum are a small minority. In this paper, we present a general class of solutions of Einstein-Maxwell theory, in which the electromagnetic potential is “screened” and in the simplest case falls as $1/r$ at large distances. This kind of behavior for the electromagnetic potential was achieved by introducing a topological mass term for the electromagnetic field [8]. Since we begin with the coupled field equations and use standard techniques, the issues of generality of the solutions and their uniqueness are transparent at every stage. Moreover, we explicitly compute the mass and the angular momentum of our solutions and show that they are finite. The plan of the paper is as follows. In section 2 we will derive the field equations for the coupled gravitational and electromagnetic fields. In section 3 we will look for stationary circularly symmetric solutions, first in general and then in a more specialized form. These solutions will still have undetermined parameters. The quasilocal mass and angular momentum of the solutions are computed in section 4. The properties of the rotating charged solutions are summarized and further discussed in section 5. Section 6 is devoted to comparison with other works.
2 The Equations of motion

In odd dimensional spaces, it is possible to introduce topologically non trivial gauge invariant terms which give mass to the gauge fields \[8\]. In 2+1 dimensions, an abelian gauge field becomes massive if the Lagrangian is modified by a Chern Simon term

\[
L = \frac{m_p}{2} \varepsilon^{\mu \nu \alpha \beta} F_{\alpha \beta} A_\mu \tag{2.1}
\]

where the gauge field, the corresponding field strength tensor, and the mass for the gauge field are given, respectively, by \(A_\mu\), \(F_{\mu \nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu\), and \(m_p\). In this work, we couple the Maxwell theory modified by a topological mass term to Einstein’s theory in 2+1 dimensions and look for its general solutions. The corresponding action can be written as

\[
I = \int d^3 x L
\]

where

\[
L = \sqrt{|g|} \left[ \frac{1}{2\pi} (R - 2\Lambda) - \frac{1}{4} g^{\mu \nu} g_{\rho \sigma} F_{\mu \rho} F_{\nu \sigma} \right] + \frac{m_p}{2} \varepsilon^{\nu \alpha \beta} F_{\alpha \beta} A_\nu \tag{2.2}
\]

In this expression, the Newtonian constant has been chosen to be \(\frac{1}{8}\), \(\Lambda\) is the cosmological constant, and \(m_p\) is the topological mass. We assume \(\Lambda\) to be negative. It follows that we can write down the gravitational field equations in the form

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + \Lambda g_{\mu \nu} = \pi T_{\mu \nu} \tag{2.3}
\]

where the energy-momentum stress energy tensor \(T_{\mu \nu}\) given by:

\[
T_{\mu \nu} = F_{\mu \rho} F_{\nu}^\rho - \frac{1}{4} g_{\mu \nu} F_{\lambda \sigma} F^{\lambda \sigma} \tag{2.4}
\]

Being a topological, the Chern Simons term does not contribute to the energy momentum tensor. The electromagnetic field equations takes the form:

\[
\partial_\mu (\sqrt{|g|} F^{\mu \nu}) + m_p \varepsilon^{\mu \nu \alpha \beta} F_{\alpha \beta} = 0 \tag{2.5}
\]

To look for solutions, we assume that the three dimensional space time is stationary and circularly symmetric, i.e., it has two commuting Killing
vectors. Writing the coordinates as the triple \((t, \phi, r)\), the two Killing vectors can be represented as \(\frac{\partial}{\partial \phi}\) and \(\frac{\partial}{\partial t}\). Then the line element can be put in the form

\[
d s^2 = -N^2 dt^2 + L^{-2} dr^2 + K^2 (d\phi + N^\phi dt)^2
\]

(2.6)

The functions \(N, L, K\) and \(N^\phi\) depend only on radial coordinate \(r\). We assume that the electromagnetic field is also circularly symmetric, so that its only non-zero components are \(E_r\) and \(B\).

To obtain solutions, it is simplest to use the tetrad formalism and Cartan structure equations. We transform to the orthonormal basis

\[
\theta^0 = N dt; \quad \theta^1 = K(d\phi + N^\phi dt); \quad \theta^2 = L^{-1} dr
\]

(2.7)

We will use the indices \(a, b = 0, 1, 2\) for the orthonormal basis and \(\mu, \nu = 0, 1, 2\) for the coordinate basis:

\[
x^0 = t; \quad x^1 = \phi; \quad x^2 = r;
\]

(2.8)

The non-vanishing components of the electromagnetic field tensor in the coordinate basis are given by

\[
F_{tr} = E; \quad F_{r\phi} = B;
\]

(2.9)

whereas in the orthonormal basis, they are given by

\[
F_{02} = \tilde{E}; \quad F_{21} = \tilde{B};
\]

(2.10)

Being a 2-form, an arbitrary field strength tensor can written in the two bases as

\[
F = F_{ab} \theta^a \wedge \theta^b = F_{\mu\nu} dx^\mu \wedge dx^\nu
\]

(2.11)

Specialized to circularly symmetric case, these expressions reduce to

\[
F = \tilde{E} \theta^0 \wedge \theta^2 + \tilde{B} \theta^2 \wedge \theta^1 = E \ dx^0 \wedge dx^2 + B \ dx^2 \wedge dx^1
\]

(2.12)

It then follows that the components in the two bases are related to each other by

\[
E = \frac{(\tilde{E}N - \tilde{B}KN^\phi)}{L}
\]
\[ B = \frac{\tilde{B}K}{L} \quad \text{(2.13)} \]

We will use the Cartan’s method to calculate the curvature components relative to the orthonormal basis. In this basis, the line element is simply given by

\[ ds^2 = \theta^i \theta^j \eta_{ij} = -\theta^0^2 + \theta^1^2 + \theta^2^2 \quad \text{(2.14)} \]

where the one form components \( \theta^i \) is given by (2.7) and \( \eta_{ij} = ( - + + ) \). Let us write the connection one form

\[ \Omega^\alpha_\beta = \gamma^\alpha_\beta \theta^\rho \quad \text{(2.15)} \]

with the anti symmetry property

\[ \Omega_{\alpha\beta} + \Omega_{\beta\alpha} = 0 \quad \text{(2.16)} \]

Then the curvature two form in this basis is given by

\[ \tilde{R}^\alpha_\beta = \tilde{R}^\alpha_\beta \theta^\mu \wedge \theta^\nu \quad \text{(2.17)} \]

The Cartan structure equations will relate the connection one forms and the curvature two forms as follows.

\[ d\theta^\alpha + \Omega^\alpha_\beta \wedge \theta^\beta = 0 \quad \text{(2.18)} \]

\[ \tilde{R}^\alpha_\beta = d\Omega^\alpha_\beta + \Omega^\alpha_\mu \wedge \Omega^\mu_\beta \quad \text{(2.19)} \]

Equations (2.18) and (2.19) can be used to compute all the unknown tensor components \( \gamma^\alpha_\beta \) and \( \tilde{R}^\alpha_\beta \) in the orthonormal basis and then the corresponding expressions in the parameterization given by equation (2.6). Then, after computing the components of Ricci and Einstein tensors, one can write down the gravitational field equation in the following form:

\[ \tilde{G}_{00} = -\frac{L(LK')'}{K} - \left( \frac{KLN^\phi'}{2N} \right)^2 = \Lambda + \frac{\pi}{2}(\tilde{B}^2 - \tilde{E}^2) \quad \text{(2.20)} \]

\[ \tilde{G}_{11} = \frac{L(LN')'}{N} - 3\left( \frac{KLN^\phi'}{2N} \right)^2 = -\Lambda + \frac{\pi}{2}(\tilde{B}^2 + \tilde{E}^2) \quad \text{(2.21)} \]
\[ G_{01} = -\frac{L}{K^2} \left( \frac{K^3 LN \phi'}{2N} \right)' = -\pi \tilde{B} \tilde{E} \quad (2.22) \]
\[ G_{22} = L^2 \left( \frac{N' K'}{NK} \right) + \left( \frac{KLN \phi'}{2N} \right)^2 = -\Lambda + \frac{\pi}{2} (\tilde{B}^2 - \tilde{E}^2) \quad (2.23) \]

3 Exact Solutions

We will look for exact solutions in which the electromagnetic field is self-(antiself-)dual. Then we will have from (2.10)
\[ \tilde{E} = \epsilon \tilde{B} = u(r) \quad (3.1) \]
where \( \epsilon = 1(-1) \) corresponds to self (anti-self) duality condition. Then from (2.13), the non-zero components of the electromagnetic field in the coordinate basis can expressed in the form
\[ E = F_{tr} = \frac{u(N - \epsilon N \phi K)}{L} \]
\[ B = F_{r\phi} = \epsilon \frac{uK}{L} \quad (3.2) \]

To solve the field equations, let us first define a function \( \rho(r) \) such that
\[ \frac{1}{L} = \frac{d\rho}{dr} \quad (3.3) \]
Then, using (3.2), the field equations (2.5) can be solved to get,
\[ u(\rho) = C_1 \exp[\epsilon m \rho L] \]
\[ C_1 \left( -N^\phi + \epsilon \frac{N}{K} \right) = \epsilon C_2 \quad (3.4) \]

Using the gravitational field equations (2.20) - (2.23), and the conditions given by (3.4), we find that a general stationary, circularly symmetric solution to the Einstein-Maxwell-Chern-Simons theory subject to (anti)self-duality condition (3.1) is given by
\[ \frac{1}{L} = \frac{d\rho}{dr} \quad (3.5) \]
\[ K^2 = A_1 + A_2 \exp[2|\Lambda|^{1/2} \rho] - r_m^2 \exp[2\epsilon m_p \rho]. \tag{3.6} \]

\[ N = C_0 \frac{\exp[2|\Lambda|^{1/2} \rho]}{K} \tag{3.7} \]

\[ N^\psi = \epsilon \left( \frac{N}{K} - \frac{C_2}{C_1} \right) \tag{3.8} \]

Here,
\[ r_m^2 = \frac{\pi C_1^2}{2m_p (m_p - \epsilon|\Lambda|^{1/2})} \tag{3.9} \]

and \( C_0, C_1, C_2, A_1 \) and \( A_2 \) are integrating constants. At this point, the solution depends on seven parameters two of which are the topological mass and the cosmological constants. We will see below how various physical requirements can be used to reduce this number to three. We note that these solutions represent the most general solutions of our field equations except for the special cases \( m_p = 0 \) and \( m_p = \epsilon|\Lambda|^{1/2} \).

These solutions is that they determine the functions \( L(r) \) and \( K(r) \) only up to a choice of parametrization (of coordinates). So, we will study these solutions subject to the additional requirement \( K = r \). Moreover, to simplify the notation and without loss of generality, we set \( m_p = -\epsilon|\Lambda|^{1/2} \). This choice also leads to a potential for the electromagnetic field with interesting properties which will be highlighted in a later section. Under these conditions, we have, from (3.6)

\[ r^2 = A_1 + A_2 \exp[2|\Lambda|^{1/2} \rho] - r_m^2 \exp[-2|\Lambda|^{1/2} \rho]. \tag{3.10} \]

We can solve this expression for \( \rho \) to get

\[ \rho = \frac{1}{2|\Lambda|^{1/2}} \ln[X] \tag{3.11} \]

where

\[ X = \frac{1}{2A_2} \left[ r^2 - A_1 + \sqrt{(r^2 - A_1)^2 + 4A_2 r_m^2} \right] \tag{3.12} \]

So, the functions representing the stationary charged solution are given by the following expressions:

\[ L = \frac{|\Lambda|^{1/2}}{r} \sqrt{[(r^2 - A_1)^2 + 24A_2 r_m^2]} \tag{3.13} \]
\[ N = \frac{C_0|\Lambda|^{1/2}}{2A_2r} \left[ r^2 - A_1 + \sqrt{(r^2 - A_1)^2 + 4A_2r^2m} \right] \] (3.14)

\[ N^\phi = \epsilon \left( \frac{N}{K} - |\Lambda|^{1/2} \right) \] (3.15)

where

\[ r_m^2 = \frac{\pi C_1^2}{4|\Lambda|} \] (3.16)

To see the significance of the constant \( C_1 \), we note that for large \( r \) and in the limit of vanishing topological mass, the electric and the magnetic fields, which are given by \( \tilde{E} = \epsilon \tilde{B} = C_1 \exp[\epsilon m_{\rho}\rho]/r \) approach the flat space space expression \( C_1/r \). Therefore, the constant \( C_1 \) can be identified with the electric (magnetic) charge \( Q_e \). Moreover, requiring that the shift function \( N^\phi \) remain finite as \( r \to \infty \), we must choose the constant \( C_2 \) as follows:

\[ C_2 = C_1|\Lambda|^{1/2} \] (3.17)

In the limit \( Q_e \to 0 \), the above solution reduces to

\[ L = |\Lambda|^{1/2} \frac{(r^2 - A_1)}{r} \] (3.18)

\[ N = \frac{C_0|\Lambda|^{1/2}}{2A_2} \frac{(r^2 - A_1)}{r} \] (3.19)

and

\[ r^2 = A_1 + A_2(r^2 - A_1) \] (3.20)

To satisfy equation (3.20), we must set \( A_2 = 1 \). We are thus left with two undetermined parameters \( C_0 \) and \( A_1 \). If we wish to cast these solutions in the BTZ form, we must choose \( C_0 = 1 \).

Then the only parameter left to be determined is \( A_1 \). We will see below that it can be expressed in terms of asymptotic values of mass or angular momentum. To demonstrate this, we shall use the quasilocal formalism developed by Brown, York, Creighton and Mann. 

8
4 Quasilocal Mass and Angular Momentum

In this section we will briefly review the formalism developed by Brown et al. [10][11], for computing the expressions for quasilocal energy and other conserved charges of a gravitating system. The conserved quasilocal charges such as the energy, $E$, are defined as the total charges within a region of space with boundary, $B$, whenever that boundary admits a corresponding Killing vector. The conserved charge associated with a timelike Killing vector defines the quasilocal Mass, $M$, and the conserved charge associated with the rotational Killing vector defines the quasilocal Angular momentum, $J$.

Let the spacetime manifold under consideration be $\mathcal{M}$. We take the topology of $\mathcal{M}$ to be the product of a spacelike hypersurface and a real line interval, i.e., $\Sigma \times I$. The boundary of $\Sigma$ is then given by $B$. The boundary of $\mathcal{M}$, denoted by $\partial \mathcal{M}$, consists of initial and final spacelike hypersurfaces $t'$ and $t''$, respectively, and a timelike hypersurface $T = B \times I$ joining the space-like hypersurfaces. The induced metric on the hypersurfaces $t'$ and $t''$ is denoted by $h_{ij}$, and the induced metric on $T$ is defined by $\gamma_{ij}$. The boundary element $T$ is foliated into one dimensional hypersurfaces $B$ with the induced parameter $\sigma$. The two-metric $\gamma_{ij}$ on $T$ can be decomposed, using the notation of equation (2.6), as follows:

$$\gamma_{ij}dx^idx^j = -N^2dt^2 + r^2(d\phi + N\phi dt)(d\phi + N\phi dt) \quad (4.1)$$

Then on $B$ the proper energy surface density $\varepsilon$ and proper angular momentum surface density $j_a$, on $B$ could be defined as follows [10][11]:

$$\varepsilon = \frac{k}{k_o} - \varepsilon_o \quad (4.2)$$

$$j_i = \frac{-2}{\sqrt{h}}\sigma_{ij}P^{jk}n_k - (j_o)_i \quad (4.3)$$

In these expressions, $k$ is the extrinsic curvature of $B$ considered as the boundary $B = \partial \Sigma$. For our metric, it has the value $-L/r$. The quantities $P^{ij}$ denote the gravitational momentum conjugate to $h_{ij}$ which are, in turn, the components of metric on the spacelike hypersurfaces $t'$ and $t''$. In our case, the only non-zero component of $P^{ij}$ is $P^{r\phi} = -rLN\phi/4\pi N$. The quantity $\vec{n}$ is the unit normal to $T$ with $n_\mu = L\delta^1_\mu$ for the given metric. Also $\sigma = r^2$, and $h = Det(h_{ij}) = L^{-1}r$.

The values $\varepsilon_o$ and $j_o$ corresponds to the “zero point configuration” which arises due to the freedom of choice in the definition of $E$, $M$, and $J$. In the
(2+1) dimensions, the zeropoint configuration can be chosen to be, e.g., a stationary slice of the zero mass-black hole solution.

With these preliminaries, the total quasilocal energy for the system is defined by,

\[ E = \int_B dx \sqrt{\sigma} \varepsilon \]  \hspace{1cm} (4.4)

For other charges, when there is a Killing vector field \( \xi \) in \( \mathcal{T} \), the corresponding conserved charge is given by

\[ Q_\xi = \int_B dx \sqrt{\sigma} (\varepsilon u^i + j^i) \xi^i \]  \hspace{1cm} (4.5)

The vector \( \vec{u} \) is the unit normal to \( \Sigma \), with \( u_\mu = -N \delta^0_\mu \). Thus, for a system with rotational (circular) symmetry and the Killing vector field \( \xi \) in \( \mathcal{T} \), the conserved charge is the angular momentum

\[ J = \int_B dx \sqrt{\sigma} j_i \xi^i \]  \hspace{1cm} (4.6)

Similarly, for the stationary spacetimes considered in this paper, the conserved anti-de Sitter mass, \( M \), corresponding to the timelike Killing vector has the form

\[ M = \int_B dx \sqrt{\sigma} (N \varepsilon - N^\phi j_i) \]  \hspace{1cm} (4.7)

Using the above formalism, it is easy to show that for the metric (2.6), with \( K^2 = r^2 \), the quasilocal angular momentum at radial distance \( r \) is given by

\[ J(r) = \frac{LN^\phi r^3}{N}. \]  \hspace{1cm} (4.8)

Using equations (3.14)-(3.17), this takes the form

\[ J(r) = 2|\Lambda|^{1/2} \epsilon \left[ r^2 - \sqrt{(r^2 - A_1)^2 + 4r_m^2} \right]. \]  \hspace{1cm} (4.9)

In the limit \( r \to \infty \), we obtain the asymptotic observable \( J \) given by

\[ J = J(\infty) = 2|\Lambda|^{1/2} \epsilon A_1 \]  \hspace{1cm} (4.10)

Similarly, using the same formalism [10], one can compute the expressions for quasilocal mass and quasilocal energy at the radius \( r \). The result is
The asymptotic observables \( J \) and \( M \), correspond to the Casimir invariants of the anti-de Sitter group. Since there are only two such invariants, it is clear that the quantity \( E \) cannot be an anti-de Sitter invariant. The significance of this quantity becomes clear if we perform a group contraction on anti-de Sitter to obtain the Poincare’ group. In that case, looking at the square of the quantity \( M(r) \) given by equation (4.11), we can see that \( E^2 \) will be the only part of \( M^2 \) left over after group contraction. So, it has the significance of mass (or energy) in the sense of Poincare’ group. We also note in passing that the second Casimir invariant of the anti-de Sitter group does not change under group contraction. Thus the notion of “spin” as the square root of the eigenvalue of the second Casimir operator has the same meaning in the two groups.

In equation (4.12), the quantity \( L_0 \) is the reference value of the function \( L(r) \) which determines the zero of the energy. Such a reference spacetime can be chosen by setting certain integration constants in the solution to zero. If we choose \( Q_e = 0, M = 0 \), the background metric approaches

\[
ds^2 = -|\Lambda| r^2 dt^2 + \frac{dr^2}{|\Lambda| r^2} + r^2 d\phi^2
\]

This is identical to the vacuum anti-de Sitter spacetime or the “zero mass black hole configuration” given in [3]. For such a reference spacetime,

\[
L_0 = |\Lambda|^{1/2} r.
\]

Then, using (4.10) and (4.12), the expression for quasilocal anti-de Sitter mass takes the form

\[
M(r) = 2|\Lambda| \left[r^2 - \left((r^2 - A_1)^2 + 4r_m^2\right)^{1/2}\right].
\]

Again in the limit \( r \to \infty \), we get the asymptotic observable \( M \):

\[
M = M(\infty) = 2|\Lambda| A_1
\]

This determines the last of our integration constants to be
Then,

\[ J = \frac{\epsilon M}{|\Lambda|^{1/2}} \]  

(4.18)

5 Rotating Charged Solutions

Compiling the results obtained in the previous sections, we now present a family of rotating charged solutions for Einstein-Maxwell theory in which the photon has a Chern-Simons mass term. They are characterized by mass \( M \), the angular momentum \( J \), and charge \( Q_e \). The spacetime metric of these solutions has the form,

\[ ds^2 = -N^2 dt^2 + L^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2. \]  

(5.1)

where

\[ L^2 = \frac{|\Lambda|}{r^2} \left[ \left(r^2 - \frac{M}{2|\Lambda|}\right)^2 + \frac{\pi Q_e^2}{|\Lambda|} \right]. \]  

(5.2)

\[ N = \frac{|\Lambda|^{1/2}}{2r} \left[ r^2 - \frac{M}{2|\Lambda|} + \sqrt{(r^2 - \frac{M}{2|\Lambda|})^2 + \frac{\pi Q_e^2}{|\Lambda|}} \right]. \]  

(5.3)

\[ N^\phi = \frac{J}{2r^2} \left[ \frac{|\Lambda|}{M} \sqrt{(r^2 - \frac{M}{2|\Lambda|})^2 + \frac{\pi Q_e^2}{|\Lambda|} - \frac{r^2|\Lambda|}{M} - \frac{1}{2}} \right]. \]  

(5.4)

These solutions are regular and horizonless. They have finite mass \( M \) and finite angular momentum \( \epsilon M|\Lambda|^{-1/2} \). Since the electromagnetic fields are (anti)self-dual, the trace of the energy momentum tensor \( T^\mu_\mu = F^{\mu\nu} F_{\mu\nu} = 0 \), leading to the same constant negative scalar curvature \( R = 6\Lambda \) as for the uncharged BTZ black hole solution. Thus, locally, the space-time geometry would have an anti-de Sitter structure.

The electromagnetic potential for these solutions is given by,

\[ A_\mu dx^\mu = \frac{Q_e}{\sqrt{2}} \left[ r^2 - \frac{M}{2|\Lambda|} + \sqrt{(r^2 - \frac{M}{2|\Lambda|})^2 + \frac{\pi Q_e^2}{|\Lambda|}} \right]^{-1/2} \left(|\Lambda|^{-1/2}d\phi - dt\right). \]  

(5.5)
Thus for large \( r \), the Electromagnetic potential behaves as

\[ A_\mu(r) \to r^{-1} \]  

(5.6)

This implies that the field strengths behave asymptotically as

\[ \vec{E} = e \vec{B} \to r^{-2} \]  

(5.7)

It is interesting to note that the potential in (5.6) has the behavior of the Coulomb field in flat space in 3 + 1 dimensions. In fact, had we not related the topological mass to the cosmological constant, i.e., for arbitrary \( m_p \), the potential would behave as \( r^{\frac{m_p}{|\Lambda|^{1/2}}} \). We may recall that in the absence of mass term, the electromagnetic potential in 2 + 1 dimensions behaves logarithmically. In our solutions, by introducing a topological mass term which behaves as an infrared regulator, we have been able to modify the long range behavior of the electromagnetic potential. In flat space, the addition of the topological mass term will result in an exponential modification of the field strengths:

\[ \vec{E} = e \vec{B} = Q e \frac{e^{m_p r}}{r} \]  

(5.8)

In curved space, the effect is more subtle and, of course, coordinate dependent.

It is also interesting to note that when \( Q_e \to 0 \), the our charged solutions are asymptotic to the extreme uncharged BTZ blackhole with \( |J| = M|\Lambda|^{-1/2} \). To see this, we note that for \( \frac{M}{2|\Lambda|} \gg Q_e^2 \), and large \( r \),

\[ N = L \to |\Lambda|^{1/2} \frac{(r^2 - \frac{M}{2|\Lambda|})}{r} \]  

(5.7)

\[ N^\phi \to \frac{-J}{2r^2} \]  

(5.8)

These expressions are identical to the corresponding expressions for the extreme uncharged BTZ black hole solution with mass \( M \) and angular momentum \( J \).

6 Comparison with previous charged solutions.

The static charged black hole solution discussed in [3], was specified by three parameters: the mass \( M \), the charge \( Q_e \), and the “radial parameter” \( r_o \).
Depending on the values of these parameters, a charged BTZ solution can have one, two, or no horizons. The charged solution with one horizon was identified as an extreme charged black hole. However, the quasilocal mass for this solution is given by

\[ M_{ql} = M + Q^2 \ln \left( \frac{r}{r_0} \right) \]

In contrast to what happens in our solutions, this quantity diverges for large \( r \) \[5\]. Consider next the rotating charged black hole solution for self-(antiself-) dual Einstein-Maxwell presented by Kamata et al. \[7\]. This solution has a horizon at \( r = r_0 \). Its angular momentum and the quasilocal mass for large \( r \) were computed by Chan \[6\]:

\[ j(r) = \frac{2Q^2}{|\Lambda|} \ln \left| \frac{r^2 - r_0^2}{r_0^2} \right| \quad (6.1) \]

\[ M(\infty) = 2|\Lambda| \left( \frac{Q^2}{\Lambda} + r_0^2 \ln \left| \frac{r^2}{r_0^2} \right| \right) \quad (6.2) \]

They both diverge for large \( r \). In both of these solutions, the divergences are due to the logarithmic behavior of the electromagnetic potential in 2+1 dimensions as we mentioned before. In the our solutions, the “screening” effect of the topological mass term is responsible for the finite values of mass and angular momentum.

After the completion of this work, we learned that using a dimensional reduction method, general solutions for Einstein-Maxwell-Chern Simons theory equivalent to ours have also been obtained by Clément \[12\]. Since the method of obtaining these solutions in Clément’s work are completely independent of our methods, the two works together provide a more convincing proof of the generality of these solutions. This is, in particular, evident in our direct computation of the masses and angular momenta.

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