HOMOLOGY UNDER MONOTONE MAPS BETWEEN FINITE TOPOLOGICAL SPACES

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Abstract. It is shown that a surjective monotone map \( X \to Y \) between finite \( T_0 \)-spaces induces a surjective map on homology. As such a map turns out to be a sequence of edge contractions in the Hasse diagram of \( X \), followed by a homeomorphism, this leads to an explicit relation between the Betti numbers of \( X \) to those of \( Y \) and the cokernels of the edge contraction maps on the order complexes.

1. Introduction

According to [2, Ch. V §46.I], a continuous map \( X \to Y \) between topological spaces is called monotone, if the pre-image of every connected subset of \( Y \) is connected. This generalises an earlier definition of monotone in which the pre-image of points are connected, and has the advantage that it defines morphisms on topological spaces, as the composition of monotone maps is monotone. It turns out that Kuratowski’s definition of monotone is amenable to homology computations of finite topological spaces. Namely, we will prove first that any surjective monotone map \( X \to Y \) between finite \( T_0 \)-spaces is a sequence of edge contractions \( \kappa_e \) in the Hasse diagram, followed by a homeomorphism. Then we will see that for an edge contraction, the induced map on homology is surjective. This allows to compute the Betti numbers of \( X \) from those of \( Y \) and the Betti numbers of the cokernels \( \mathcal{K}_e \) of the chain maps induced by the edge contractions \( \kappa_e \). Generators for \( \mathcal{K}_e \) can be explicitly given. The homology of \( X \) itself is an iterated extension of the homology groups of \( Y \) and \( \mathcal{K}_e \).

2. A homomorphism theorem for Alexandroff spaces

An Alexandroff space is a topological space for which arbitrary intersections of open sets are open, or equivalently, every point has a minimal neighbourhood. Finite topological spaces are Alexandroff spaces. For every Alexandroff space \( X \) there is an associated pre-order \( \leq \) on \( X \) such that

\[
x \leq y \iff U_x \subseteq U_y,
\]

where \( U_a \) is the minimal neighbourhood of \( a \in X \). Conversely, every pre-order \( \leq \) on a set \( X \) defines an Alexandrov topology by taking \( U_x = \{ y \in X \mid y \leq x \} \) for \( x \in X \) as a base [1].

A \( T_0 \)-space is a topological space in which for each pair of distinct points there is an open set containing one point but not the other. An Alexandroff space is a \( T_0 \)-space if and only if the associated pre-order is a partial order.

Date: December 4, 2013.
Let $f : X \to Y$ be a map between topological spaces. Then there is an equivalence relation $\sim_f$ on $X$, where

$$x \sim_f y :\iff f(x) = f(y),$$

and let $\pi_f : X \to X/\sim_f$ be the canonical map, where $X/\sim_f$ is the quotient space.

**Theorem 2.1 (Homomorphism Theorem).** Let $f : X \to Y$ be monotone and surjective with $X, Y$ Alexandroff spaces and $Y$ a $T_0$-space. Then the diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\pi_f \downarrow & & \downarrow h \\
X/\sim_f & \xrightarrow{h} & Y
\end{array}$$

commutes, where $h$ is a homeomorphism.

**Proof.** The canonical map $\pi_f$ is continuous, and there is a continuous bijection $h$ which makes the diagram commutative.

1. $\pi_f$ is monotone. Let $A \subseteq X/\sim_f$ be connected. Then $h(A)$ is connected. Hence, $\pi_f^{-1}(A) = f^{-1}(h(A))$ is connected.

2. $h$ is monotone. Let $A \subseteq Y$ be connected. Then $h^{-1}(A) = \pi_f(f^{-1}(A))$ is connected, because $f^{-1}(A)$ is connected, and $\pi_f$ is continuous.

The assertion now follows from the following Lemma 2.2.

**Lemma 2.2.** Any bijective and monotone map $h : Z \to Y$ with $Z, Y$ Alexandroff spaces and $Y$ a $T_0$-space is a homeomorphism.

**Proof.** Let $x < y$ in $Y$. We will show that $h^{-1}(x) < h^{-1}(y)$ in $Z$. As $\{x, y\}$ is connected, it follows that $h^{-1}(x) < h^{-1}(y)$ or $h^{-1}(y) < h^{-1}(x)$, because $h$ is monotone and bijective. If $h^{-1}(y) < h^{-1}(x)$, then $y < x$ by continuity of $h$. But this contradicts the $T_0$-property of $Y$. Hence, $h^{-1}(x) < h^{-1}(y)$. □

3. **Edge contractions and homology**

Let $X$ be a finite $T_0$-space, and let $\mathcal{H}(X)$ be the Hasse diagram of $X$. This is a directed acyclic graph which is the transitive reduction of the associated partial order $\leq$ on $X$. An edge contraction is a map $\kappa_e : X \to X_e$ which identifies the two endpoints of an edge $e = (a, b)$ of $\mathcal{H}(X)$. Notice that the quotient space $X_e$ is also a $T_0$-space, and that the map $\kappa_e$ is surjective, and monotone. In fact, $X_e = X/\sim_{\kappa_e}$

The following result is a consequence of Theorem 2.1

**Theorem 3.1.** Let $f : X \to Y$ be a surjective and monotone map between finite $T_0$-spaces. Then $f$ is a sequence of edge contractions, followed by a homeomorphism.

**Proof.** By Theorem 2.1 the canonical map $\pi_f : X \to X/\sim_f$ is monotone. Hence, the equivalence classes $[x]$ under $\sim_f$ are connected, and $\pi_f$ contracts the edges of $\mathcal{H}(X)$ belonging to equivalence classes $[x]$ consisting of at least two points. □

To each finite $T_0$-space $X$ is associated a simplicial complex $K(X)$ whose simplexes are the chains $\{x_0 < \cdots < x_n\}$ of $X$. It is called the order complex of $X$ and is weak homotopy equivalent to $X$. In particular, the singular homology groups of $X$ and $K(X)$ are isomorphic.
Theorem 3.1 motivates the study of edge contractions. Let \( \kappa_e : X \to X_e \) be an edge contraction of a finite \( T_0 \)-space \( X \). This map induces a simplicial map \( \kappa_{e*} : K(X) \to K(X_e) \) between the corresponding order complexes. There are two induced short exact sequences of chain complexes which will be studied:

\[
\begin{align*}
(1) & \quad 0 \to \mathcal{E}_e \to \mathcal{K}(X) \to \kappa_{e*}(\mathcal{K}(X)) \to 0 \\
(2) & \quad 0 \to \kappa_{e*}(\mathcal{K}(X)) \to \mathcal{K}(X_e) \xrightarrow{\gamma} \mathcal{K}_e \to 0
\end{align*}
\]

Here, \( \mathcal{K}(X) \) is the simplicial chain complex associated to \( K(X) \).

As an abelian group, the chain complex \( \mathcal{E}_e \) of (1) is generated by the chains containing \( e \) and their simplicial boundaries. It is also the kernel of the map \( \kappa_* : \mathcal{K}(X(e)) \to \mathcal{K}(X(e)_e) \) induced by the same edge contraction applied to \( X(e) := \{ x \in X \mid x \leq b \lor a \leq x \} \) with \( e = (a, b) \). Notice that \( X(e)_e = U_{\bar{e}} \cup F_{\bar{e}} \) is the star of the point \( \bar{e} = \kappa_e(e) \) in \( X(e)_e \).

**Lemma 3.2.** The chain complex \( \mathcal{E}_e \) is acyclic.

**Proof.** Observe that the chain map \( \kappa_* \) is surjective. Namely, if \( \sigma \) is a chain of \( X(e)_e \), then there are two possibilities: either \( \bar{e} \notin \sigma \), then \( \sigma = \kappa(\sigma) \), or \( \sigma = \{ x_0 < \cdots < \bar{e} < \cdots < x_n \} \), then \( \sigma = \kappa(\{ x_0 < \cdots < a < \cdots < x_n \}) \). The assertion now follows from the long exact homology sequence for

\[
0 \to \mathcal{E}_e \to \mathcal{K}(X(e)) \to \mathcal{K}(X(e)_e) \to 0
\]
as both \( X(e) \) and \( X(e)_e \) are contractible spaces. \( \square \)

Consequently, the homology of \( X \) is isomorphic to the homology of \( \pi_{e*}(\mathcal{K}(X)) \).

**Theorem 3.3.** Let \( \kappa_e : X \to X_e \) be an edge contraction. Then there is a short exact sequence

\[
0 \to H_{r+1}(\mathcal{K}_e) \to H_r(X) \to H_r(X_e) \to 0
\]

for every \( r \geq 0 \).

**Proof.** The map \( \gamma \) in (2) fits into a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{K}(X_e) & \xrightarrow{\gamma} & \mathcal{K}_e \\
\pi \downarrow & & \downarrow g \\
\mathcal{K}(C_{\bar{e}}) & & \\
\end{array}
\]

where \( \pi \) is the projection onto \( \mathcal{K}(C_{\bar{e}}) \subseteq \mathcal{K}(X_e) \), \( C_{\bar{e}} \) is the star of \( \bar{e} \) inside \( X_e \), and \( g \) is the restriction of \( \gamma \) to \( \mathcal{K}(C_{\bar{e}}) \). In order to show that the diagram is indeed commutative, we need to show that \( g \) is surjective. For this, observe that \( \mathcal{K}_e \) is generated by the classes of simplices \( \{ x_0 < \cdots < \bar{e} < \cdots < x_n \} \) and their faces modulo \( \pi_{e*}(\mathcal{K}(X)) \). But those simplices and their faces also generate \( \mathcal{K}(C_{\bar{e}}) \). As \( C_{\bar{e}} \) is contractible, it follows that the induced map \( \gamma_* : H_r(X_e) \to H_r(\mathcal{K}_e) \) is the zero map for \( r \geq 1 \). As \( \mathcal{K}_{e,0} = 0 \), the map \( \gamma_* \) is also the zero map for \( r = 0 \). From this it follows that the long exact sequence for (2) breaks into the asserted short exact sequences. \( \square \)

The following example illustrates the effect of an edge contraction to \( \mathcal{K}_e \).
Example 3.4. A subgraph in the Hasse diagram $\mathcal{H}(X)$ of a space $X$ of the form

\[
\begin{array}{c}
  \mathcal{H}(X) \\
  a \\
  b \\
  \mathcal{H}(X_e) \\
  s \\
  t
\end{array}
\]

becomes, after contracting the edge $e = (a, b)$, the subgraph

\[
\begin{array}{c}
  \mathcal{H}(X_e) \\
  s \\
  \bar{e} \\
  t
\end{array}
\]

in $X_e$. The new chain $\{t < \bar{e} < s\}$ is non-zero in $K_{e_\mathcal{H}}$, and the chain $\{t < s\}$ is non-zero in $K_{e_\mathcal{H}}$ if and only if $t \not< s$ in $X$. These are special cases of the generating chains of $K_{e_\mathcal{H}}$ given in the proof of Theorem 3.3.

An immediate consequence of Theorem 3.3 is that the possible increase in dimension does not increase the homological dimension:

Corollary 3.5. The homological dimension of $X_e$ is bounded from above by the homological dimension of $X$.

We now give a criterion when the contraction of an edge $e = (a, b)$ is a weak homotopy equivalence. Let $U_x = \{z \in X \mid z \leq x\}$ be the minimal neighbourhood of $x$ in $X$, and $F_x = \{z \in X \mid x \leq z\}$ the closure of $x$ in $X$. The latter are a basis for the topological space $X^{op}$ where the open sets are the closed sets of $X$. A continuous map $f: X \to Y$ of finite spaces then defines a continuous map $f^{op}: X^{op} \to Y^{op}$.

Let $\hat{U}_x := U_x \setminus \{x\}$ resp. $\hat{F}_x := F_x \setminus \{x\}$. The point $x \in X$ is an up beat point if $\hat{F}_x = F_y$, and is a down beat point if $\hat{U}_x = U_y$ for some $y \in X$. A beat point is a point which is an up beat point or a down beat point.

Proposition 3.6. If for all $x \in \hat{F}_b$ the set $\hat{U}_x \cap U_a$ is contractible, or for all $y \in \hat{U}_a$ the set $F_y \cap \hat{F}_b$ is contractible, then the edge contraction $\kappa_e: X \to X_e$ is a weak homotopy equivalence.

Proof. Denote $U_x^e$ resp. $F_x^e$ the minimal neighbourhood resp. the closure of $x$ in $X_e$. Then for $x \in X$ we have

\[
\kappa_e^{-1}(U_{\kappa_e(x)}) = \begin{cases} 
U_x \cup U_a, & \bar{e} \leq \kappa_e(x) \\
U_x & \text{otherwise}
\end{cases}
\]

Likewise,

\[
\kappa_e^{-1}(F_{\kappa_e(y)}) = \begin{cases} 
F_y \cup F_b, & \kappa_e(y) \leq \bar{e} \\
F_y & \text{otherwise}
\end{cases}
\]

for $y \in X$. The assertion now follows from [3, Thm. 6] applied to $\kappa_e: X \to X_e$ resp. $\kappa_e^{op}: X^{op} \to X_e^{op}$. \qed

Corollary 3.7. If $a$ is a down beat point or $b$ is an up beat point of $X$, then $\kappa_e$ is a weak homotopy equivalence.

Proof. If $a$ is a down beat point, then $\hat{U}_a = U_b$, and $F_y \cap \hat{F}_b = F_b$ is contractible for all $y \in \hat{U}_a$. If $b$ is an up beat point, then $\hat{F}_b = F_a$, and $U_x \cap U_a = U_a$ is contractible for all $y \in \hat{F}_b$. \qed
A finite $T_0$-space $X$ is called $g$-minimal, if any surjective and monotone map $X \to Y$ which is a quasi-isomorphism, is in fact a homeomorphism.

**Corollary 3.8.** A finite $T_0$-space which is $g$-minimal is also minimal, i.e. has no beat points.

**Proof.** Assume that $X$ has a down beat point $a$. Then $\hat{U}_a$ has a unique maximum $b$. By Corollary 3.7, the edge contraction $\kappa_e: X \to X_e$ with $e = (a,b)$ is a weak homotopy equivalence. In particular, it is a quasi-isomorphism. But $X$ and $X_e$ are not homeomorphic. If $X$ has an up beat point, the argument is similar. □

### 4. Monotone maps and homology

Let $f: X \to Y$ be a surjective and monotone map between finite $T_0$-spaces. By Theorem 3.1, $f$ is a sequence of edge contractions

$$X \xrightarrow{\kappa_1} X_{e_1} \longrightarrow \cdots \longrightarrow X_{e_1 \ldots e_n} \xrightarrow{\kappa_n} X_{e_1 \ldots e_n} \xrightarrow{g} Y$$

where $g: X_{e_1 \ldots e_n} \to Y$ is a homeomorphism. In each step there is a short exact sequence

$$0 \to \kappa_i(\mathcal{K}(X_{e_1 \ldots e_{i-1}})) \to \mathcal{K}(X_{e_1 \ldots e_i}) \to \mathcal{K}_{e_i} \to 0$$

with $i = 1, \ldots, n$ (if $i = 1$, then $X_{e_1 \ldots e_{i-1}} = X$). By Theorem 3.3 we have short exact homology sequences:

$$0 \to H_{r+1}(X_{e_i}) \to H_r(X_{e_1 \ldots e_{i-1}}) \to H_r(X_{e_1 \ldots e_i}) \to 0$$

for $r \geq 0$ and $i = 1, \ldots, n$. There are also the exact sequences

$$0 \to \mathcal{C}_{e_1 \ldots e_i} \to \mathcal{K}(X) \to \kappa_{1\ldots i}(\mathcal{K}(X)) \to 0$$

$$0 \to \kappa_{1\ldots i}(\mathcal{K}(X)) \to \mathcal{K}(X_{e_1 \ldots e_i}) \to \mathcal{K}_{e_1 \ldots e_i} \to 0$$

where $\kappa_{1\ldots i} = \kappa_i \circ \cdots \circ \kappa_1: X \to X_{e_1 \ldots e_i}$ for each $i = 1, \ldots, n$.

**Lemma 4.1.** The chain complex $\mathcal{C}_{e_1 \ldots e_i}$ is acyclic for all $i = 1, \ldots, n$.

**Proof.** This follows by induction. For $i = 1$, this the statement of Lemma 3.2. Assume now that $\mathcal{C}_{e_1 \ldots e_i}$ is acyclic. Then the short exact sequence of kernels

$$0 \to \mathcal{C}_{e_1 \ldots e_i} \to \mathcal{C}_{e_1 \ldots e_{i+1}} \to \mathcal{C}_{e_i} \to 0$$

associated to the commutative diagram

$$\begin{align*}
\mathcal{K}(X) & \xrightarrow{\kappa_1 \cdots \kappa_{i+1}} \mathcal{K}(X_{e_1 \ldots e_i}) \\
\mathcal{K}(X_{e_1 \ldots e_{i+1}}) & \xrightarrow{\kappa_{1\ldots i+1}} \mathcal{K}(X_{e_1 \ldots e_i})
\end{align*}$$

induced by (3), where $\mathcal{C}_{e_1 \ldots e_i} = \ker(\kappa_{1\ldots i+1})$ is as in (3), yields that $\mathcal{C}_{e_1 \ldots e_{i+1}}$ is acyclic, because $\mathcal{C}_{e_i}$ is acyclic and $\mathcal{C}_{e_1 \ldots e_i}$ is acyclic by the induction hypothesis. □

**Corollary 4.2.** Let $f: X \to Y$ be a surjective and monotone map between finite $T_0$-spaces. Then there is a short exact sequence

$$0 \to H_{r+1}(X_{e_1 \ldots e_n}) \to H_r(X) \to H_r(Y) \to 0$$

for every $r \geq 0$, where $\mathcal{K}_{e_1 \ldots e_n}$ is the cokernel of the map $\kappa_{1\ldots n} = \kappa_n \circ \cdots \circ \kappa_1$ appearing in the decomposition of $f$ given by (3).
Proof. Consider the composition of maps

\[ \kappa_1 \ldots \iota : H_r(X) \rightarrow H_r(\kappa_1 \ldots \iota (X(X))) \rightarrow H_r(X_{\varepsilon_1 \ldots \varepsilon_i}) \]

induced by \( \kappa_1 \ldots \iota \). The first map is an isomorphism by Lemma \[4.1\]. By induction it follows that \( \kappa_1 \ldots \iota \) is surjective. Namely, the case \( i = 1 \) is given by Theorem \[3.3\]. Assume now that \( \kappa_1 \ldots \iota \) is surjective, then the commutative diagram

\[
\begin{array}{ccc}
H_r(X) & \xrightarrow{\kappa_1 \ldots \iota} & H_r(X_{\varepsilon_1 \ldots \varepsilon_i}) \\
\downarrow{\kappa_1 \ldots \iota + 1} & & \downarrow{\kappa_{i+1}} \\
H_r(X_{\varepsilon_1 \ldots \varepsilon_{i+1}}) & & \\
\end{array}
\]

yields that \( \kappa_{1 \ldots i+1} \) is surjective, because \( \kappa_i \) is. Hence, the long exact sequence of homology breaks into short pieces

\[ 0 \rightarrow H_{r+1}(X_{\varepsilon_1 \ldots \varepsilon_i}) \rightarrow H_r(X) \rightarrow H_r(X_{\varepsilon_1 \ldots \varepsilon_i}) \rightarrow 0 \]

from which the assertion follows by taking \( i = n \), because of the homeomorphism \( g : X_{\varepsilon_1 \ldots \varepsilon_n} \rightarrow Y \) as in \[3\].

Diagram \[7\] also induces a short exact sequence of cokernels

\[ 0 \rightarrow \mathcal{K}_{\varepsilon_1 \ldots \varepsilon_i} \rightarrow \mathcal{K}_{\varepsilon_1 \ldots \varepsilon_{i+1}} \rightarrow \mathcal{K}_{\varepsilon_{i+1}} \rightarrow 0 \]

for each \( i = 1, \ldots, n \) with \( \mathcal{K}_{\varepsilon_1 \ldots \varepsilon_\nu} \) as in \[6\].

**Lemma 4.3.** There is a short exact sequence

\[ 0 \rightarrow H_{r+1}(X_{\varepsilon_1 \ldots \varepsilon_i}) \rightarrow H_{r+1}(\mathcal{K}_{\varepsilon_1 \ldots \varepsilon_{i+1}}) \rightarrow H_{r+1}(\mathcal{K}_{\varepsilon_i}) \rightarrow 0 \]

for all \( i = 1 \ldots n - 1 \) and \( r \geq 0 \).

**Proof.** Consider the following diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \kappa_1 \ldots \iota* (\mathcal{K}(X)) & \rightarrow & \mathcal{K}(X_{\varepsilon_1 \ldots \varepsilon_i}) & \rightarrow & \mathcal{K}_{\varepsilon_1 \ldots \varepsilon_i} & \rightarrow & 0 \\
\downarrow{\phi} & & \downarrow{\kappa_i} & & \downarrow{\kappa_{i+1}} & & \downarrow{\iota} & & \\
0 & \rightarrow & \kappa_1 \ldots \iota + 1* (\mathcal{K}(X)) & \rightarrow & \mathcal{K}(X_{\varepsilon_1 \ldots \varepsilon_{i+1}}) & \rightarrow & \mathcal{K}_{\varepsilon_1 \ldots \varepsilon_{i+1}} & \rightarrow & 0 \\
\end{array}
\]

where \( \phi \) is the natural surjection induced by \( \kappa_\nu \) in the diagram \[7\], and \( \iota \) is the inclusion of cokernels of \[6\]. By taking the long exact homology sequences associated to the rows of this diagram, one obtains the following commutative square

\[
\begin{array}{ccc}
H_{r+1}(\mathcal{K}_{\varepsilon_1 \ldots \varepsilon_i}) & \xrightarrow{\delta_{1 \ldots \iota}} & H_r(\kappa_1 \ldots \iota* (\mathcal{K}(X))) \\
\downarrow{\iota*} & & \downarrow{\phi*} \\
H_{r+1}(\mathcal{K}_{\varepsilon_1 \ldots \varepsilon_{i+1}}) & \xrightarrow{\delta_{1 \ldots \iota + 1}} & H_r(\kappa_{1 \ldots \iota + 1* (\mathcal{K}(X))) \\
\end{array}
\]

where \( \delta_{1 \ldots \nu} \) is the connecting homomorphism of the long exact homology sequence.

**Claim.** \( \phi* \) is an isomorphism.
Proof of Claim. There is a diagram with exact rows

\[ 0 \to C_{e_1 \cdots e_i} \to \mathcal{K}(X) \to \kappa_{1 \cdots i}(\mathcal{K}(X)) \to 0 \]

where \( j \) is the inclusion of kernels induced by (7). The long exact sequences of homology then induce the commutative square:

\[ H_r(X) \to H_r(\kappa_{1 \cdots i}(\mathcal{K}(X))) \]
\[ \text{id} \downarrow \quad \quad \quad \quad \quad \quad \downarrow \phi_* \]
\[ H_r(X) \to H_r(\kappa_{1 \cdots i+1}(\mathcal{K}(X))) \]

where \( \text{id}_* \) is the identity map, and the horizontal maps are isomorphisms because \( C_{e_1 \cdots e_i} \) and \( C_{e_1 \cdots e_{i+1}} \) are acyclic. Hence, \( \phi_* \) is an isomorphism. □

As a consequence, the map \( \iota_* \) in (9) is injective for all \( r \geq 0 \), which proves the assertion. □

As a result, the homology of \( X \) is completely, but not necessarily uniquely, determined by the homology of a monotone image and the chain complexes \( \mathcal{K}_{e_i} \) associated to the intermediate edge contractions. This leads to a formula for the Betti numbers.

**Corollary 4.4.** Let \( f : X \to Y \) be a surjective and monotone map between finite \( T_0 \)-spaces. Then the Betti numbers \( b_r(X) \), \( b_r(Y) \), \( b_r(\mathcal{K}_{e_i}) \) satisfy the following equation

\[ b_r(X) = b_r(Y) + \sum_{i=1}^{n} b_{r+1}(\mathcal{K}_{e_i}) \]

where the edges \( e_1, \ldots, e_n \) are of the decomposition (9).

**Proof.** This is an immediate consequence of Corollary 4.2 and Lemma 4.3. □

In particular, if \( Y \) is acyclic, then \( b_r(X) \) can be computed from the Betti numbers of the chain complexes \( \mathcal{K}_{e_1}, \ldots, \mathcal{K}_{e_n} \). In case \( X \) is connected, this can be accomplished by contracting all edges of \( \mathcal{H}(X) \) one by one.

As an application, let \( f : X \to Y \) be a surjective continuous map of finite \( T_0 \)-spaces. By contracting the connected components of the fibres, one arrives at a decomposition

\[ X \xrightarrow{g} Z \]
\[ \quad \xrightarrow{h} Y \]

with \( g \) surjective and monotone, and \( h \) continuous with discrete fibres.
Corollary 4.5. Let $f : X \to Y$ be a surjective continuous map of finite $T_0$-spaces. Then the induced map $f_* : H_*(X) \to H_*(Y)$ factors

$$
\begin{array}{ccc}
H_*(X) & \xrightarrow{g_*} & H_*(Z) \\
\downarrow{f_*} & & \downarrow{h_*} \\
H_*(Y) & &
\end{array}
$$

with $g_*, h_*$ induced by $g, h$ in (10), and $g_*$ surjective.

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