Local definitions of formations of finite groups

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Abstract
A problem of constructing of local definitions for formations of finite groups is discussed in the article. The author analyzes relations between local definitions of various types. A new proof of existence of an \(\omega\)-composition satellite of an \(\omega\)-solubly saturated formation is obtained. It is proved that if a non-empty formation of finite groups is \(X\)-local by Förster, then it has an \(X\)-composition satellite.

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1. Introduction

We consider only finite groups. So, all group classes considered are subclasses of the class \(\mathcal{E}\) of all finite groups. Recall that a formation is a group class closed under taking homomorphic images and subdirect products (see [1]). A formation \(\mathcal{F}\) is said to be \(p\)-saturated (\(p\) a prime) if the condition:

\[G/N \in \mathcal{F}\text{ for a }G\text{-invariant }p\text{-subgroup }N\text{ of }\Phi(G)\]

always implies \(G \in \mathcal{F}\). A formation \(\mathcal{F}\) is said to be \(\mathcal{N}_p\)-saturated if the condition

\[G/\Phi(N) \in \mathcal{F}\text{ for a normal }p\text{-subgroup }N\text{ of }G\]

always implies \(G \in \mathcal{F}\).

If a formation is \(p\)-saturated for any prime \(p\), then it is called saturated. Clearly, every \(p\)-saturated formation is \(\mathcal{N}_p\)-saturated. The converse is not true:
there is an extensive class of $\mathcal{N}_p$-saturated formations which are not $p$-saturated. However, as it is established in [2], between local definitions of these two types of formations there is a close connection.

The concept of local definitions of saturated formations was considered for the first time by W. Gaschütz [1]. Following [3], we formulate it in the general form.

A local definition is a map $f: \mathcal{E} \to \{\text{formations}\}$ together with a $f$-rule which decide whether a chief factor is $f$-central or $f$-eccentric in a group. In addition, we follow the agreement that the local definition $f$ does not distinguish between non-identity groups with the same (up to isomorphism) set of composition factors. Therefore, for any fixed prime $p$, $f$ is not distinguish between any two non-identity $p$-groups; we will denote through $f(p)$ a value of $f$ on non-identity $p$-groups.

If a class $\mathcal{F}$ coincides with the class of all groups all of whose chief factors are $f$-central, we say that $f$ is a local definition of $\mathcal{F}$. It generalises the concept of nilpotency. Thus, the problem of finding local definitions for group classes is equivalent to a problem of finding classes of generalized nilpotent groups.

In this paper we analyze relations between local definitions of different types and give a new proof of a theorem on a local definition of a formation which is $\mathcal{N}_p$-saturated for any $p$ in a set $\omega$ of primes.

2. Preliminaries

We use standard notations and definitions [4]. We say that a map $f$ does not distinguish between $\mathcal{H}$-groups if $f(A) = f(B)$ for any two groups $A$ and $B$ in $\mathcal{H}$. Following Gaschütz, the $\mathcal{F}$-residual $G^{\mathcal{F}}$ of a group $G$ is the least normal subgroup with quotient in $\mathcal{F}$. The Gaschütz product $\mathcal{F} \circ \mathcal{H}$ of formations $\mathcal{F}$ and $\mathcal{H}$ is defined as the class of all groups $G$ such that $G^{\mathcal{F}} \in \mathcal{F}$. If $\mathcal{F}$ is closed under taking of normal subgroups, then $\mathcal{F} \circ \mathcal{H}$ coincides with the class $\mathcal{F}\mathcal{H}$ of all extensions of $\mathcal{F}$-groups by $\mathcal{H}$-groups.

$\mathbb{P}$ is the set of all primes; $\text{Char}(\mathcal{X})$ is the set of orders of all simple abelian groups in $\mathcal{X}$. A group $G$ is called a $pd$-group if its order is divisible by a prime $p$; $C_p$ is a group of order $p$; if $\omega \subseteq \mathbb{P}$, then $\omega' = \mathbb{P} \setminus \omega$; an $\omega d$-group (a chief $\omega d$-factor) is a group (a chief factor) being $pd$-group for some $p \in \omega$; $G_{\omega d}$ is the largest normal subgroup all of whose $G$-chief factors are $\omega d$-groups ($G_{\omega d} = 1$ if all minimal normal subgroups in $G$ are $\omega'$-groups). If $\mathcal{H}$ is a class of groups, then $\mathcal{H}_\omega$ is the class of all $\omega$-groups in $\mathcal{H}$. A chief factor $H/K$ of $G$ is called a chief $\mathcal{H}$-factor if $H/K \in \mathcal{H}$. The socle $\text{Soc}(G)$ of a group $G \neq 1$ is the product
of all minimal normal subgroups of $G$.

$[A]B$ is a semidirect product with a normal subgroup $A$; $O_{\omega}(G)$ is the largest normal $\omega$-subgroup in $G$; $\pi(G)$ is the set of all primes dividing the order of a group $G$; $\pi(\mathfrak{F}) = \cup_{G \in \mathfrak{F}} \pi(G)$; $\mathfrak{A}$ is the class of all nilpotent groups; $\mathfrak{A}$ is the class of all abelian groups; $\text{Com}(G)$ is the class of all groups that are isomorphic to composition factors of a group $G$; $\text{Com}(\mathfrak{F}) = \cup_{G \in \mathfrak{F}} \text{Com}(G)$; $\text{Com}^+(\mathfrak{F})$ is the class of all abelian groups in $\text{Com}(\mathfrak{F})$; $\text{Com}^-(\mathfrak{F})$ is the class of all non-abelian groups in $\text{Com}(\mathfrak{F})$; $(G)$ is the class of all groups isomorphic to $G$; $\mathfrak{J}$ is the class of all simple (abelian and non-abelian) groups; if $\mathfrak{L}$ is a subclass in $\mathfrak{J}$, then $\mathfrak{L}' = \mathfrak{J} \setminus \mathfrak{L}$; $\mathfrak{L}^{+}$ is the class of all abelian groups in $\mathfrak{L}$, $\mathfrak{L}^{-} = \mathfrak{L} \setminus \mathfrak{L}^{+}$. $\text{E}_{\mathfrak{F}}$ is the class of all groups $G$ such that $\text{Com}(G) \subseteq \mathfrak{F}$; $G_{\text{E}_{\mathfrak{F}}}$ is the $\text{E}_{\mathfrak{F}}$-radical of $G$, the largest normal $\text{E}_{\mathfrak{F}}$-subgroup in $G$. If $S \in \mathfrak{J}$, then $C^{S}(G)$ is the intersection of centralizers of all chief $E(S)$-factors of $G$ ($C^{S}(G) = G$ if $S \notin \text{Com}(G)$); if $S = C_{p}$, we write $C^{p}(G)$ in place of $C^{S}(G)$.

**Lemma 2.1** (see [3], Lemmas 2-3). (a) If $S$ is a non-abelian simple group, then $C^{S}(G)$ is the $E(S)^{'+}$-radical of $G$, the largest normal subgroup not having composition factors isomorphic to $S$.

(b) Let $p$ be a prime, and $\mathfrak{F}$ be the class of all groups all of whose chief $p$-factors are central. Then $C^{p}(G)$ is the $G_{\mathfrak{F}}$-radical of $G$, for every group $G$.

The following three lemmas are reformulations of Lemmas IV.4.14–IV.4.16 in [4] whose proofs use only $p$-solubly saturation.

**Lemma 2.2.** Let $\mathfrak{F}$ be an $\mathfrak{N}_{p}$-saturated formation, $p$ a prime. If $C_{p} \in \text{Com}(\mathfrak{F})$, then $\mathfrak{N}_{p} \subseteq \mathfrak{F}$.

**Lemma 2.3.** Let $\mathfrak{F}$ be an $\mathfrak{N}_{p}$-saturated formation containing $\mathfrak{N}_{p}$, $p$ a prime. Let $N$ be an elementary abelian normal $p$-subgroup in $G$ such that $[N](G/N) \in \mathfrak{F}$. Then $G \in \mathfrak{F}$.

**Lemma 2.4.** Let $p$ be a prime, and let $\mathfrak{F}$ be an $\mathfrak{N}_{p}$-saturated formation containing $\mathfrak{N}_{p}$. Let $N$ be an elementary abelian normal $p$-subgroup in $G$ such that $G/N \in \mathfrak{F}$ and $[N](G/C_{G}(N)) \in \mathfrak{F}$. Then $G \in \mathfrak{F}$.

**Proof.** Set $M = [N](G/N)$, $C = C_{G}(N)$. Evidently, $C/N = C_{G/N}(N)$. In the group $M$ we have $C_{M}(N) = N \times C/N$ and $C/N$ is normal in $M$. Hence $M/(C/N) \cong [N](G/C) \in \mathfrak{F}$. Since $M/N \in \mathfrak{F}$, it follows that $M/N \cap (C/N) \cong M \in \mathfrak{F}$. Now we apply Lemma 2.3.

**Lemma 2.5.** Let $\mathfrak{F}$ be an $\mathfrak{N}_{p}$-saturated formation containing $\mathfrak{N}_{p}$, $p$ a prime.
Let $H \in \mathfrak{F}$ and let $C_p(H/L) \leq L \triangleleft H$. If $N$ is an irreducible $\mathbb{F}_p(H/L)$-module, then $[N](H/L) \in \mathfrak{F}$.

**Lemma 2.6** (see [4], Proposition IV.1.5). Let $\mathfrak{F}$ be a formation and $G \in \mathfrak{F}$. Let $S, R, K$ be normal subgroups in $G$ such that $S \subseteq R$ and $K \subseteq C_G(R/S)$. Then $[R/S](G/K) \in \mathfrak{F}$.

**Lemma 2.7** (see [5] or [6], Theorem 7.11). If $H/\Phi(G) = \text{Soc}(G/\Phi(G))$, then $C_G(H) \subseteq H$.

**Lemma 2.8** (see [4], Lemma IV.4.11). Let $p$ be a prime, $L = \Phi(O_p(G))$. Then $C_p(G/L) = C_p(G)/L$.

3. Local and $\omega$-local satellites

The following type of a local definition was proposed by W. Gaschütz [1].

**Definition 3.1.** Let $f$ be a local definition such that

$$f(A) = \bigcap_{p \in \pi(A)} f(p)$$

for any group $A \neq 1$. Let an $f$-rule be defined as follows: a chief factor $H/K$ of a group $G$ is $f$-central if $G/C_G(H/K) \in f(H/K)$. Then $f$ is called a local satellite.

**Definition 3.2** (see [4], p. 387). Let $A$ be a group of operators for a group $G$, and $f$ a local satellite.

(i) We say that $A$ acts $f$-centrally on an $A$-composition factor $H/K$ of $G$ if $A/C_A(H/K) \in f(p)$ for every prime $p \in \pi(H/K)$.

(ii) We say that $A$ acts $f$-hypercentrally on $G$ if $A$ acts $f$-centrally on every $A$-composition factor of $G$.

The convenient notation $LF(f)$ for a group class with a local satellite $f$ was introduced by Doerk and Hawkes [3]. Clearly, $LF(f)$ is a non-empty formation (we have always $1 \in LF(f)$).

The following proposition is evident.

**Proposition 3.1.** Let $f$ be a local satellite and $\pi = \{p \in \mathbb{P} \mid f(p) \neq \emptyset\}$. Then $LF(f)$ consists precisely of $\pi$-groups $G$ satisfying the following condition: $G/O_{\pi'}(p)(G) \in f(p)$ for any $p \in \pi(G)$. Thus, if $\pi = \emptyset$, we have $LF(f) = \{1\}$. If $\pi \neq \emptyset$, we have that

$$LF(f) = \mathcal{E}_\pi \bigcap_{p \in \pi} (\mathcal{E}_p \mathcal{E}_p f(p)).$$
We remember the reader that a formation $\mathfrak{F}$ is saturated if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$ (by definition, the empty set is a saturated formation). W. Gashütz has shown that every formation with a local satellite is saturated. This fact follows also from the following theorem of P. Schmid.

**Theorem 3.1** (see [4], Theorem IV.6.7). Let $f$ be a local satellite, and let $A$ be a group of operators for a group $G$. If $A$ acts $f$-hypercentrally on $G/\Phi(G)$, then $A$ acts likewise on $G$.

The following remarkable result is known as the Gaschütz–Lubeseder–Schmid theorem, see [4], Theorem IV.4.6.

**Theorem 3.2.** A non-empty formation has a local satellite if and only if it is saturated.

It is straightforward to verify that if $\mathfrak{F}$ is a non-empty formation, then $\mathfrak{N} \mathfrak{F}$ is a formation with a local satellite $f$ such that $f(p) = \mathfrak{F}$ for every prime $p$. Evidently, the formation $\mathfrak{A}_p \times \mathfrak{N}_p'$ of all nilpotent groups with an abelian Sylow $p$-subgroup is not saturated, but for every prime $q \neq p$, $G/(\Phi(G) \cap O_q(G)) \in \mathfrak{A}_p \times \mathfrak{N}_p'$ always implies $G \in \mathfrak{A}_p \times \mathfrak{N}_p'$. One more fact of the same sort is the following. Consider a saturated formation of the form $\mathfrak{M} \circ \mathfrak{F}$. Here $\mathfrak{F}$ can be non-saturated, but for every prime $p \in \mathbb{P} \setminus \mathfrak{P}(\mathfrak{M})$, $G/(\Phi(G) \cap O_p(G)) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. The facts of such kind lead to the concept of a $\omega$-saturated formation [11].

**Definition 3.3.** Let $\omega$ be a set of primes. A formation $\mathfrak{F}$ is called $\omega$-saturated if for every prime $p \in \omega$, $G/(\Phi(G) \cap O_p(G)) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$.

The problem of finding of local definitions of $\omega$-saturated formations was considered in [7] and [3]. While solving this problem the following concept of small centralizer was useful (see [8]).

**Definition 3.4.** Let $H/K$ be a chief factor of a group $G$. The *small centralizer* $c_G(H/K)$ of $H/K$ in $G$ is the subgroup generated by all normal subgroups $N$ of $G$ such that $\text{Com}(NK/K) \cap \text{Com}(H/K) = \varnothing$.

With the help of Definition 3.4 we can introduce the concept ‘$\omega$-saturated satellite’ as follows.

**Definition 3.5.** Let $\omega$ be a set of primes, and $f$ a local definition which does not distinguish between all non-identity $\omega'$-groups; if $\omega' \neq \varnothing$, we denote
through \( f(\omega') \) a value of \( f \) on non-identity \( \omega' \)-groups. In addition, we assume that
\[
 f(A) = \bigcap_{p \in \pi(A) \cap \omega} f(p)
\]
for any \( \omega \)-group \( A \). Let an \( f \)-rule be defined by the following way: a chief factor \( H/K \) of \( G \) is \( f \)-central in \( G \) if either \( H/K \) is an \( \omega \)-group and \( G/C_G(H/K) \in f(H/K) \) or else \( H/K \) is an \( \omega' \)-group and \( G/C_G(H/K) \in f(\omega') \). Then \( f \) is called an \( \omega \)-local satellite. We denote by \( LF_\omega(f) \) the class of all groups all of whose chief factors are \( f \)-central. By definition, \( 1 \in LF_\omega(f) \).

Clearly, if \( \omega = \mathbb{P} \), then an \( \omega \)-local satellite \( f \) is a local satellite, and \( LF_\omega(f) = LF(f) \). If \( \omega \neq \mathbb{P} \) and \( f(\omega') = \emptyset \), then \( LF_\omega(f) = LF(h) \) where \( h(p) = f(p) \) if \( p \in \omega \), and \( h(p) = \emptyset \) if \( p \in \omega' \).

**Lemma 3.1** (see [3], Lemma 1). Let \( \mathcal{E} \) be a subclass in \( \mathcal{F} \), and \( \{S_i \mid i \in I\} \) be the set of all \( E\mathcal{E} \)-factors of a group \( G \). Then \( \bigcap_{i \in I} c_G(S_i) \) is the \( E(\mathcal{E}') \)-radical \( G_{E(\mathcal{E}')} \) of \( G \).

**Remark 3.1.** In Lemma 3.1 the set \( \{c_G(S_i) \mid i \in I\} \) can be empty. We always follow the agreement that the intersection of an empty set of subgroups of \( G \) coincides with \( G \).

The following proposition is similar to Proposition 3.1.

**Proposition 3.2.** Let \( f \) be an \( \omega \)-local satellite, and \( \omega \) a proper subset in \( \mathbb{P} \). Let \( \pi = \{p \in \omega \mid f(p) \neq \emptyset\} \). Then:

1. if \( \pi = \emptyset \) and \( f(\omega') = \emptyset \), then \( LF_\omega(f) = (1) \);
2. if \( \pi = \emptyset \) and \( f(\omega') \neq \emptyset \), then \( LF_\omega(f) = \mathcal{E}_{\omega'} \cap f(\omega') \);
3. if \( f(\omega') \neq \emptyset \), then \( LF_\omega(f) \) consists precisely of groups \( G \) such that \( G/G_{\omega d} \in f(\omega') \) and \( G/O_{\omega',p}(G) \in f(p) \) for any \( p \in \pi(G) \cap \omega \).

**Proof.** Statements (1) and (2) are evident.

Prove (3). Assume that \( f(\omega') \neq \emptyset \), and let \( G \in LF_\omega(f) \). Let \( \mathcal{E} \) be the set of all chief \( \omega' \)-factors in \( G \). If a chief factor \( H/K \) of \( G \) is an \( \omega' \)-group, then \( G/c_G(H/K) \in f(\omega') \). Therefore, \( G/\bigcap_{H/K \in \mathcal{E}} c_G(H/K) \in f(\omega') \). By Lemma 3.1, \( \bigcap_{H/K \in \mathcal{E}} c_G(H/K) = G_{\omega d} \). So, \( G/G_{\omega d} \in f(\omega') \). If \( p \in \omega \) and \( H/K \) is an chief \( \omega \)-factor, then \( G/C_G(H/K) \in f(p) \), and we have \( G/O_{\omega',p}(G) \in f(p) \).

Conversely, let \( G \) be a group such that \( G/G_{\omega d} \in f(\omega') \) and \( G/O_{\omega',p}(G) \in f(p) \) for any \( p \in \pi(G) \cap \omega \). Clearly, we have that all \( G \)-chief \( \omega d \)-factors are \( f \)-central. Let \( H/K \) be a \( G \)-chief \( \omega' \)-factor of \( G \). Then \( G_{\omega d} H/K \subseteq c_G(H/K) \), and \( G/G_{\omega d} \in f(\omega') \) implies \( G/c_G(H/K) \in f(\omega') \). □
The following result extends Theorem 3.2 to \(\omega\)-saturated formations.

**Theorem 3.3** (see [7], Theorem 1). *Let \(\omega\) be a set of primes. A non-empty formation has a \(\omega\)-local satellite if and only if it is \(\omega\)-saturated.*

4. Composition and \(\mathcal{L}\)-composition satellites

Gaschütz’s main idea [1] was to study groups modulo \(p\)-groups, and he implemented it through local satellites of soluble formations. While considering non-soluble formations, we have to follow the following principle: study groups modulo \(p\)-groups and simple groups. That approach was proposed in the lecture [9] at the conference in 1973; in that lecture composition satellites were considered under the name ‘primarily homogeneous screens’.

**Definition 4.1.** Let \(f\) be a local definition, and let an \(f\)-rule be defined as follows: a chief factor \(H/K\) of a group \(G\) is \(f\)-central if \(G/C_G(H/K) \in f(H/K)\). Then \(f\) is called a *composition satellite*. We denote by \(CF(f)\) the class of all groups all of whose chief factors are \(f\)-central.

**Definition 4.2.** Let \(A\) be a group of operators for a group \(G\), and \(f\) a composition satellite.

(i) We say that \(A\) acts \(f\)-centrally on an \(A\)-composition factor \(H/K\) of \(G\) if \(A/C_A(H/K) \in f(H/K)\).

(ii) We say that \(A\) acts \(f\)-hypercentrally on \(G\) if it acts \(f\)-centrally on every \(A\)-composition factor of \(G\).

As an example, we consider the class \(\mathfrak{N}^*\) of all quasinilpotent groups (for the definition of a quasinilpotent group, see [12], Definition X.13.2). It is easy to check that \(\mathfrak{N}^* = CF(f)\) where \(f\) is a composition satellite such that \(f(p) = (1)\) for every prime \(p\), and \(f(S) = \text{form}(S)\) for every non-abelian simple group \(S\). Here \(\text{form}(S)\) is a least formation containing \(S\); it consists of all groups represented as a direct product \(A_1 \times \cdots \times A_n\) with \(A_i \simeq S\) for any \(i\). The formation \(\mathfrak{N}^*\) is non-saturated, but it is solubly saturated.

As pointed out in [4], formations with composition satellites were also considered—in different terminology—by R. Baer in his unpublished manuscript. By R. Baer, a formation \(\mathfrak{F}\) is called *solubly saturated* if the condition \(G/\Phi(G_\mathfrak{F}) \in \mathfrak{F}\) always implies \(G \in \mathfrak{F}\) (here \(G_\mathfrak{F}\) is the soluble radical of \(G\)). The question of the coincidense of the family of non-empty solubly saturated formations and the family of formations with composition satellites was solved by the following result due to R. Baer.
Theorem 4.1 (see [4], Theorem IV.4.17). A non-empty formation has a composition satellite if and only if it is solubly saturated.

A composition satellite $h$ is called integrated if $h(S) \subseteq CF(h)$ for any simple group $S$. If $\mathcal{F} = CF(f)$, then $\mathcal{F} = CF(h)$ where $h(S) = f(S) \cap \mathcal{F}$ for any simple group $S$. Thus, if a formation has a composition satellite, then it has an integrated composition satellite.

Remark 4.1. Let $\{CF(f_i) \mid i \in I\}$ be a family of formations having composition satellites. Let $f = \cap_{i \in I} f_i$ be a composition satellite such that $f(S) = \cap_{i \in I} f_i(S)$ for every $S \in \mathcal{F}$. Clearly, $CF(f) = \cap_{i \in I} CF(f_i)$.

Remark 4.2. Let $X$ be a set of groups. Let $\{\mathcal{F}_i \mid i \in I\}$ be the class of all formations $\mathcal{F}_i$ satisfying the following two conditions: 1) $X \subseteq \mathcal{F}_i$; 2) $\mathcal{F}_i$ has a composition satellite. Set $cform(X) = \cap_{i \in I} \mathcal{F}_i$. By Remark 4.1, $cform(X)$ has a composition satellite. In the subsequent we will use that notation $cform(X)$.

Remark 4.3. Assume that a non-empty formation $\mathcal{F}$ has an composition satellite. Let $\{f_i \mid i \in I\}$ be the class of all composition satellites of $\mathcal{F}$. Having in mind Remarks 4.1 and 4.2 we see that $f = \cap_{i \in I} f_i$ is a composition satellite of $\mathcal{F}$; $f$ is called the minimal composition satellite of $\mathcal{F}$.

Lemma 4.1. Let $X$ be a set of groups, and $S$ a simple group. Then $\mathcal{H} = Q(G/C^S(G) \mid G \in form(X))$ is a formation, and $Com(\mathcal{H}) \subseteq Com(X)$.

Proof. By Proposition IV.1.10 in [4], $\mathcal{H}$ is a formation. By Lemma II.1.18 in [4], $form(X) = QR_0 X$. Therefore, inclusion $Com(\mathcal{H}) \subseteq Com(X)$ is valid.

Lemma 4.2. Let $X$ be a non-empty set of groups, and $f$ be a composition satellite such that $f(S) = Q(G/C^S(G) \mid G \in form(X))$ if $S \in Com(X)$, and $f(S) = \emptyset$ if $S \in \mathcal{J} \setminus Com(X)$. Then $f$ is the minimal composition satellite of $cform(X)$.

Proof. Let $f_1$ be the minimal composition satellite of $\mathcal{F} = cform(X)$ (see Remark 4.3). We will prove that $f_1 = f$.

Since $X \subseteq \mathcal{F}$, $G/C^S(G) \in f_1(S)$ for any group $G \in X$ and any $S \in Com(G)$ and therefore $f(S) \subseteq f_1(S)$. So $CF(f) \subseteq \mathcal{F} \subseteq CF(f_1)$. On the other hand, $X \subseteq CF(f)$. Thus $\mathcal{F} = CF(f)$ and $f = f_1$.

The following theorem proved independently in [13] and [14] was the first important result on composition formations.
**Theorem 4.2.** Let $f$ be an integrated composition satellite. Let $A$ be a group of automorphisms of a group $G$. If $A$ acts $f$-hypercentrally on $G$, then $A \in CF(f)$.

Applying Theorem 4.2 to the formation $\mathfrak{U}$ of all supersoluble groups, we have the following result.

**Theorem 4.3** (see [13], Theorem 2.4). Let $A$ be a group of automorphisms of a group $G$. Assume that there exists a chain of $A$-invariant subgroups
\[ G = G_0 > G_1 > \cdots > G_n = 1 \]
with prime indices $|G_{i-1}:G_i|$. Then $A$ is supersoluble.

In 1968 S.A. Syskin tried to prove Theorem 4.3 in the soluble universe, but his proof [15] is false.

In [2] there has been begun studying of local definitions of $\omega$-solubly saturated formations.

**Definition 4.3.** Let $\omega$ be a set of primes. A formation $\mathfrak{F}$ is called:

1. **$\omega$-solubly saturated** if the condition
   \[ G/N \in \mathfrak{F} \text{ for } G\text{-invariant } \omega\text{-subgroup } N \text{ in } \Phi(G_{\omega,0}) \]
   always implies $G \in \mathfrak{F}$ (here $G_{\omega,0}$ is the $\omega$-soluble radical of $G$);

2. **$\mathfrak{M}_{\omega}$-saturated** if for every prime $p \in \omega$, the condition $G/\Phi(O_p(G)) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$.

Later we will establish that the $p$-solubly saturation is equivalent to the $\mathfrak{M}_p$-saturation, and therefore a formation $\mathfrak{F}$ is $\omega$-solubly saturated if and only if it is $p$-solubly saturated for every $p \in \omega$.

**Definition 4.4.** Let $\mathfrak{L}$ be a class of simple groups. Let $f$ be a local definition which does not distinguish between all non-identity $E(\mathfrak{L}')$-groups; if $\mathfrak{L}' \neq \emptyset$, we denote by $f(\mathfrak{L}')$ an value of $f$ on non-identity $E(\mathfrak{L}')$-groups. Let $f$-rule be defined as follows: a chief factor $H/K$ of a group $G$ is $f$-central in $G$ if either $H/K$ is an $E\mathfrak{L}$-group and $G/C_G(H/K) \in f(H/K)$ or $H/K$ is a $E(\mathfrak{L}')$-group and $G/c_G(H/K) \in f(H/K) = f(\mathfrak{L}')$. Then $f$ is called an $\mathfrak{L}$-composition satellite. We denote by $CF_\mathfrak{L}(f)$ the class of all groups all of whose chief factors are $f$-central. By definition, $1 \in CF_\mathfrak{L}(f)$.

Clearly, if $\mathfrak{L} = \mathfrak{F}$, then an $\mathfrak{L}$-composition satellite $f$ is a composition satellite, and $CF_\mathfrak{L}(f) = CF(f)$. If $\mathfrak{L} \neq \mathfrak{F}$ and $f(\mathfrak{L}') = \emptyset$, then $CF_\mathfrak{L}(f) = CF(h)$ where $h(S) = f(S)$ if $S \in \mathfrak{L}$, and $h(S) = \emptyset$ if $S \in \mathfrak{L}'$. 

\[9\]
Proposition 4.1. Let \( \mathfrak{L} \) be a class of simple groups, and \( f \) an \( \mathfrak{L} \)-composition satellite. Let \( \mathfrak{F} = \{ S \in \mathfrak{L} \mid f(S) \neq \varnothing \} \). Then:

1. if \( \mathfrak{F} = \varnothing \) and \( f(\mathfrak{L}') = \varnothing \), then \( \mathcal{CF}_\mathfrak{L}(f) = (1) \);
2. if \( \mathfrak{F} = \varnothing \) and \( f(\mathfrak{L}') \neq \varnothing \), then \( \mathcal{CF}_\mathfrak{L}(f) = \mathcal{E}(\mathfrak{L}') \cap f(\mathfrak{L}') \);
3. if \( f(\mathfrak{L}') \neq \varnothing \), then \( \mathcal{CF}_\mathfrak{L}(f) \) consists precisely of groups \( G \) such that \( G/\mathcal{G}_\mathfrak{L} \in f(\mathfrak{L}') \) and \( G/\mathcal{C}_S(G) \in f(S) \) for every \( S \in \text{Com}(G) \cap \mathfrak{L} \).

Proof. Statements (1) and (2) are evident.

Prove (3). Assume that \( f(\mathfrak{L}') \neq \varnothing \), and let \( G \in \mathcal{CF}_\mathfrak{L}(f) \). Let \( \mathfrak{T} \) be the set of all chief \( \mathcal{E}(\mathfrak{L}') \)-factors in \( G \). If a chief factor \( H/K \) of \( G \) is an \( \mathcal{E}(\mathfrak{L}') \)-group, then \( G/\mathcal{C}_G(H/K) \in f(\mathfrak{L}') \). Therefore, \( \mathcal{G} = \bigcap_{H/K \in \mathfrak{T}} \mathcal{C}_G(H/K) \in f(\mathfrak{L}') \). By Lemma 3.1, \( \mathcal{G} = G_{\mathfrak{L}'} \). So, \( G/\mathcal{G}_{\mathfrak{L}'} \in f(\mathfrak{L}') \). If \( S \in \mathfrak{L} \) and \( H/K \) is an chief \( \mathcal{E}(\mathfrak{L}) \)-factor, then \( G/\mathcal{C}_G(H/K) \in f(S) \), and we have \( G/\mathcal{C}_S(G) \in f(S) \).

Conversely, let \( G \) be a group such that \( G/\mathcal{G}_{\mathfrak{L}'} \in f(\mathfrak{L}') \) and \( G/\mathcal{C}_S(G) \in f(S) \) for every \( S \in \text{Com}(G) \cap \mathfrak{L} \). Clearly, all chief \( \mathcal{E}(\mathfrak{L}) \)-factors of \( G \) are \( f \)-central. Let \( H/K \) be a chief \( \mathcal{E}(\mathfrak{L}') \)-factor of \( G \). Then \( \mathcal{G}_{\mathfrak{L}'} \subseteq \mathcal{C}_G(H/K) \), and therefore \( G/\mathcal{G}_{\mathfrak{L}'} \in f(\mathfrak{L}') \) implies \( G/\mathcal{C}_G(H/K) \in f(\mathfrak{L}') \).

An \( \mathfrak{L} \)-composition satellite \( f \) is called integrated if \( f(S) \in \mathcal{CF}_\mathfrak{L}(f) \) for every \( S \in \mathfrak{F} \). If \( \mathfrak{F} = \mathcal{CF}_\mathfrak{L}(f) \), then \( \mathfrak{F} = \mathcal{CF}_\mathfrak{L}(h) \) where \( h(S) = f(S) \cap \mathfrak{F} \) for any simple group \( S \). Thus, if a formation has an \( \mathfrak{L} \)-composition satellite, then it has an integrated \( \mathfrak{L} \)-composition satellite.

Lemma 4.3. If \( \mathfrak{F} = \mathcal{CF}_\mathfrak{L}(f) \), then \( \mathfrak{F} = \mathcal{CF}_\mathfrak{L}(h) \) where \( h \) is an integrated \( \mathfrak{L} \)-composition satellite such that \( h(S) = \mathfrak{F} \) for every \( S \in (\mathfrak{L}^+) \).

Proof. We can assume without loss of generality that \( f \) is integrated. Let \( \mathfrak{F} = \mathcal{CF}_\mathfrak{L}(h) \) where \( h(S) = f(S) \) if \( S \in \mathfrak{L}^+ \), and \( h(S) = \mathfrak{F} \) if \( S \in (\mathfrak{L}^+) \). Evidently, \( \mathfrak{F} \subseteq \mathfrak{F} \). Assume that \( \mathfrak{F} \not\subseteq \mathfrak{F} \), and choose a group \( G \) of minimal order in \( \mathfrak{F} \). Then \( L = G^{\mathfrak{F}} \) is a unique minimal normal subgroup in \( G \), and \( L \) is not \( f \)-central. Clearly, \( c_G(L) = 1 \), and \( c_G(L) = 1 \) if \( L \) is non-abelian. Let \( A \in \text{Com}(L) \). Applying Definition 4.4 and considering the cases \( A \in \mathfrak{L}^+ \), \( A \in \mathfrak{L}^- \) and \( A \in \mathfrak{L} \), we arrive at a contradiction.

Theorem 4.4 (see [10], Theorem 2). Let \( \mathfrak{F} \) be a non-empty formation, \( \mathfrak{L} \) a class of simple groups. The following statements are equivalent:

1. \( \mathfrak{F} \) has an \( \mathfrak{L} \)-composition satellite;
2. \( \mathfrak{F} \) has an \( \mathfrak{L}^+ \)-composition satellite.

Proof. (1) \( \Rightarrow \) (2). Let \( \mathfrak{F} = \mathcal{CF}_\mathfrak{L}(f) \). Applying Lemma 4.3 we can suppose
that \( f \) is integrated and \( f(S) = \mathfrak{F} \) for every \( S \in (\mathfrak{L}^+)' \). Let \( \mathfrak{H} = CF_{\mathfrak{L}^+}(h) \) where \( h \) is an \( \mathfrak{L}^+ \)-composition satellite such that \( h(S) = f(S) \) if \( S \in \mathfrak{L}^+ \), and \( h(S) = \mathfrak{F} \) if \( S \in \mathfrak{L}^+ \cup \mathfrak{L}^- = (\mathfrak{L}^+)' \). We will prove that \( \mathfrak{F} = \mathfrak{H} \).

If \( G \) is a group of minimal order in \( \mathfrak{F} \setminus \mathfrak{H} \), then \( L = G^\mathfrak{F} \) is a unique minimal normal subgroup in \( G \), and \( L \) is not \( h \)-central. Clearly, \( c_G(L) = 1 \), and \( C_G(L) = 1 \) if \( L \) is non-abelian. Applying Definition 4.4 we see that \( L \) is \( h \)-central, a contradiction. Thus \( \mathfrak{F} \subseteq \mathfrak{H} \).

Let \( G \) be a group of minimal order in \( \mathfrak{H} \setminus \mathfrak{F} \). Then \( L = G^\mathfrak{F} \) is a unique minimal normal subgroup in \( G \), and \( L \) is not \( f \)-central. Clearly, \( c_G(L) = 1 \), and \( C_G(L) = 1 \) if \( L \) is non-abelian. Applying again Definition 4.4 we see that \( L \) is \( f \)-central, and we arrive at a contradiction. Thus \( \mathfrak{H} \subseteq \mathfrak{F} \).

(2) \( \Rightarrow \) (1). Let \( \mathfrak{F} = CF_{\mathfrak{L}^+}(f) \). Applying Lemma 4.3 we can suppose that \( f \) is integrated and \( f((\mathfrak{L}^+)') = \mathfrak{F} \). Let \( h \) be an \( \mathfrak{L} \)-composition satellite such that \( h(S) = f(S) \) if \( S \in \mathfrak{L}^+ \), and \( h(S) = \mathfrak{F} \) if \( S \in (\mathfrak{L}^+)' \). It is easy to see that \( \mathfrak{F} = CF_{\mathfrak{F}}(h) \). \( \square \)

**Remark 4.4.** It follows from Theorem 4.4 that every non-empty formation \( \mathfrak{F} \) with the property \( \text{Com}^+(\mathfrak{F}) \cap \mathfrak{L} = \emptyset \) has an \( \mathfrak{L} \)-composition satellite.

**Remark 4.5.** When \( \mathfrak{L} = \mathfrak{L}^+ \) and \( \omega = \pi(\mathfrak{L}) \), we usually use the term ‘\( \omega \)-composition satellite’ and the notations \( CF_\omega(f) \), \( f(\omega') \) in place of the term ‘\( \mathfrak{L} \)-composition satellite’ and the notations \( CF_{\mathfrak{L}}(f) \), \( f(\mathfrak{L}) \), respectively.

**Theorem 4.5** (see [2], Theorems 3.1 and 3.2). Let \( \mathfrak{F} \) be a non-empty formation, \( \omega \) a set of primes. The following statements are pairwise equivalent:

1. \( \mathfrak{F} \) is \( \mathfrak{R}_\omega \)-saturated;
2. \( \mathfrak{F} \) is \( \omega \)-solubly saturated;
3. \( \text{cform}(\mathfrak{F}) \subseteq \mathfrak{R}_\omega \mathfrak{F} \);
4. \( \mathfrak{F} = CF_\omega(f) \) where \( f \) is a \( \omega \)-composition satellite satisfying the following conditions:
   \begin{enumerate}
   \item \( f(p) = Q(G/C_p(G) \mid G \in \mathfrak{F}) \) if \( p \in \omega \) and \( C_p \in \text{Com}(\mathfrak{F}) \);
   \item \( f(p) = \emptyset \) if \( p \in \omega \) and \( C_p \notin \text{Com}(\mathfrak{F}) \);
   \item \( f(S) = \mathfrak{F} \) if \( S \in \mathfrak{F} \setminus \{C_p \mid p \in \omega\} \).
   \end{enumerate}

**Proof.** (1) \( \Rightarrow \) (3). Set \( \mathfrak{H} = \text{cform}(\mathfrak{F}) \). Fix \( p \in \omega \). Since \( \mathfrak{H} \subseteq \mathfrak{R}_\omega \mathfrak{F} \), it is sufficient to show that \( \mathfrak{H} \subseteq \mathfrak{R}_p \mathfrak{F} \). Let \( G \) be a group of minimal order in \( \mathfrak{H} \setminus \mathfrak{R}_p \mathfrak{F} \). Clearly, \( G \) is monolithic and \( L = \text{Soc}(G) \) is the \( \mathfrak{R}_p \mathfrak{F} \)-residual of \( G \). Since \( \mathfrak{F} \subseteq \mathfrak{R}_p \mathfrak{F} \), it follows that \( G^\mathfrak{F} \geq L \). Since \( G \in \mathfrak{H} \subseteq \mathfrak{R}_p \mathfrak{F} \), we have
\(G^3 \in \mathfrak{N}\). Since \(G\) is monolithic and \(G \not\in \mathfrak{N}_p \mathfrak{F}\), it follows that \(G^3\) is a \(p\)-group. From \(G/L \in \mathfrak{N}_p \mathfrak{F}\) it follows that \((G/L)^3 = G^3 / L\) is a \(p'\)-group. Therefore, \(G^3 = L = G^{3p\mathfrak{F}}\). By Lemma 4.2, \(\mathfrak{F}\) has a composition satellite \(h\) such that \(h(p) = Q(A/C^p(A) \mid A \in \mathfrak{F})\). Since \(L\) is a \(p\)-group, we have \(C_p \in \text{Com}(G)\). Now from Lemma 4.1 it follows that \(C_p \in \text{Com}(\mathfrak{F})\). Thus, applying Lemma 2.2, it follows that \(\mathfrak{N}_p \subseteq \mathfrak{F}\). Since \(L\) is a composition satellite of \(\mathfrak{F}\), we have that \(G/C_G(L) \in h(p)\). Therefore \([L](G/C_G(L)) \in \mathfrak{F}\), and \(G/C_G(L)\) acts fixed-point-free on \(L\). It follows that \(G/C_G(L) \simeq T/N, T = A/C^p(A), A \in \mathfrak{F}\). If \(C_p \notin \text{Com}(A)\), then \(A = C_p(A), T = 1\) and \(G = L \in \mathfrak{F}\). Assume that \(C_p \in \text{Com}(A)\). Since \(G/C_G(L) \simeq T/N\), we can consider \(L\) as an irreducible \(\mathbb{F}_p(T/N)\)-module. Then \(L\) becomes an irreducible \(\mathbb{F}_p T\)-module by inflation (see [H], p. 105). Since \(T = A/C^p(A)\), we have by Lemma 2.5 that \([L]T \in \mathfrak{F}\). By Lemma 2.6 it then follows that \([L](T/N) \in \mathfrak{F}\). From this and \(T/N \simeq G/C_G(L)\) we deduce that \([L](G/C_G(L)) \in \mathfrak{F}\). Hence, by Lemma 2.4 it follows that \(G \in \mathfrak{F}\).

\((3) \Rightarrow (2)\). It is sufficient to consider only the case \(\omega = \{p\}\). Let \(H\) be a \(p\)-soluble normal subgroup in \(G\), \(L = O_p(H) \cap \Phi(H)\), and \(G/L \in \mathfrak{F}\). We need to prove that \(G \in \mathfrak{F}\). If \(O_p(H) \neq 1\), then \(LO_p'(H)/O_p'(H) \leq \Phi(H/O_p'(H))\), and by induction we have \(G/O_p'(H) \in \mathfrak{F}\). From this and \(G/L \in \mathfrak{F}\) it follows that \(G \in \mathfrak{F}\). Assume that \(O_p(H) = 1\). By Lemma 4.2, \(\mathfrak{F} = \text{cform}(\mathfrak{F})\) has a composition satellite \(h\) such that \(h(p) = Q(A/C^p(A) \mid A \in \mathfrak{F})\). Let \(t\) be a local satellite such that \(t(p) = h(p)\) and \(t(q) = \mathfrak{F}\) for every prime \(q \neq p\). Since \(G/L \in \mathfrak{F}\), \(L = \Phi(H)\), \(G\) acts \(t\)-hypercentrally on \(H/\Phi(H)\). By Theorem 3.1, \(G\) acts \(t\)-hypercentrally on \(L = \Phi(H)\). But then \(G\) acts \(t\)-hypercentrally on \(L = \Phi(H)\), and we get \(G \in \mathfrak{F} \subseteq \mathfrak{N}_p \mathfrak{F}\). Thus, \(G^3 \in \mathfrak{N}_p \cap \mathfrak{N}_p = (1)\). So, \(G \in \mathfrak{F}\), as required.

\((1) \Rightarrow (4)\). Assume that \(\mathfrak{F}\) is \(\mathfrak{N}_\omega\)-saturated. Let \(h\) be the minimal composition satellite of \(\mathfrak{F} = \text{cform}(\mathfrak{F})\). Let \(\mathfrak{M} = \text{CF}_\omega(f)\) where \(f\) is an \(\omega\)-composition satellite satisfying the following conditions:

1. \(f(p) = h(p)\) if \(p \in \omega\);
2. \(f(S) = \mathfrak{F}\) if \(S \in \mathfrak{F} \setminus \{C_p \mid p \in \omega\}\).

Inclusion \(\mathfrak{F} \subseteq \mathfrak{M}\) is evident. Assume that the converse inclusion is false, and let \(G\) be a group of minimal order in \(\mathfrak{M} \setminus \mathfrak{F}\). Then \(L = G^3\) is a unique minimal normal subgroup in \(G\). If \(L\) is not an abelian \(\omega\)-group, it follows from \(G \in \mathfrak{M}\) and \(c_\omega(L) = 1\) that \(G \simeq G/c_\omega(L) \in \mathfrak{F}\). Therefore \(L\) is an \(\omega\)-group for some \(p \in \omega\), and we have \(G/C^p(G) \in f(p) = h(p)\). Thus \(G \in \mathfrak{F}\). Since \((1) \Rightarrow (3)\), we get \(G \in \mathfrak{N}_p \mathfrak{F}\), and therefore \(G^3 \in \mathfrak{N}_p \cap \mathfrak{N}_p = (1)\). So \(\mathfrak{F} = \mathfrak{M}\). We notice that by Lemma 4.2 we have \(f(p) = h(p) = \emptyset\) if \(p \in \omega\) and \(C_p \notin \text{Com}(\mathfrak{F})\).
(4) \Rightarrow (1). Let \( G/L \in \mathcal{F} \) and \( L = \Phi(O_p(G)), \ p \in \omega \). By Lemma 2.8, \( C^p(G)/L = C^p(G/L) \). Applying Proposition 4.1 to \( G/L \), we have \( G/O_p(G) \cong (G/L)/O_p(G/L) \in \mathcal{F} \) and \( G/C^p(G) \cong (G/L)/C^p(G/L) = (G/L)/C^p(G/L) \in f(p) \). But then by Proposition 4.1 we get \( G \in \mathcal{F} \).

**Corollary 4.5.1.** If a non-empty formation \( \mathcal{F} \) is \( p \)-solubly saturated and \( C_p \in \text{Com}(\mathcal{F}) \), then \( \mathcal{F} \) has a \( p \)-composition satellite \( f \) such that \( f(p') = \mathcal{F} \) and \( f(p) = Q(G/C^p(G) \mid G \in \mathcal{F}) \).

**Corollary 4.5.2.** If a non-empty formation \( \mathcal{F} \) is solubly saturated, then \( \mathcal{F} = CF(f) \) where \( f \) is a composition satellite satisfying the following conditions:

(i) \( f(p) = Q(G/C^p(G) \mid G \in \mathcal{F}) \) if \( p \in \omega \) and \( C_p \in \text{Com}(\mathcal{F}) \);

(ii) \( f(S) = \mathcal{F} \) for every \( S \in \text{Com}^- (\mathcal{F}) \);

(iii) \( f(S) = \emptyset \) for every \( S \in \mathcal{F} \setminus \text{Com}(\mathcal{F}) \).

**Theorem 4.6** (see [2], Theorem 3.1(b)). Let \( \mathcal{F} \) be a non-empty \( \omega \)-saturated formation, and \( h \) be the minimal composition satellite of \( \text{cform}(\mathcal{F}) \). Then \( \mathcal{F} = LF_\omega (f) \) where \( f \) is an \( \omega \)-local satellite such that \( f(p) = h(p) \) for every \( p \in \omega \).

**Proof.** We may suppose without loss of generality that \( \omega \subseteq \pi(\mathcal{F}) \). By Lemma 4.2, \( h(S) = Q(H/C^S(H) \mid H \in \mathcal{F}) \) if \( S \in \text{Com}(\mathcal{F}) \), and \( h(S) = \emptyset \) if \( S \in \mathcal{F} \setminus \text{Com}(\mathcal{F}) \).

Let \( p \) be a prime in \( \omega \), and \( S \) be a non-abelian \( pd \)-group in \( \text{Com}(\mathcal{F}) \). We will now prove that \( h(S) \subseteq h(p) \). Consider \( R = H/C^S(H), H \in \mathcal{F} \). By Lemma 2.1, \( C^S(H) \) is the largest normal subgroup not having composition factors isomorphic to \( S \). Clearly, \( O^{p'}_{p'}(R) = 1 \). Let \( A_p(R) \) be the \( p \)-Frattini module, i.e., the kernel of the universal Frattini, \( p \)-elementary \( R \)-extension:

\[
1 \to A_p(R) \xrightarrow{\mu} E \xrightarrow{\nu} R \to 1.
\]

Here \( E/A_p(R) \cong R, \) and \( A_p(R) \) is an elementary abelian \( p \)-group contained in \( \Phi(E) \). Let \( N_1, \ldots, N_t \) be all minimal normal subgroups in \( E \) contained in \( A_p(R) \). Since \( \mathcal{F} \) is \( p \)-saturated, we have \( E \in \mathcal{F} \subseteq \text{cform}(\mathcal{F}) \), and therefore \( E/\cap_i C_E(N_i) \in h(p) \). Since \( N_1, \ldots, N_t \) are simple submodules of the \( \mathbb{F}_pR \)-module \( A_p(R) \), it follows that \( R/\text{Ker}(R \text{ on } (N_1 \ldots N_t)) \in h(p) \). By theorem of Griess and P. Schmid, \( \text{Ker}(R \text{ on } (N_1 \ldots N_t)) = O^{p'}_{p'}(R) \) (see [17] or [4], p. 833). Since \( O^{p'}_{p'}(R) = 1 \), it follows that \( R \in h(p) \). Thus, \( h(S) = Q(H/C^S(H) \mid H \in \mathcal{F}) \subseteq h(p) \) if \( S \in \text{Com}(\mathcal{F}) \) and \( p \in \omega \cap \pi(S) \).

Let \( f \) be an \( \omega \)-local satellite such that \( f(p) = h(p) \) if \( p \in \omega \), and \( f(\omega') = \mathcal{F} \) if \( \omega' \neq \emptyset \). We will prove now that \( \mathcal{F} = LF_\omega (f) \).
Let $G$ be a group of minimal order in $\mathfrak{F} \setminus LF_\omega(f)$. Then $L = G^{LF_\omega(f)}$ is a unique minimal normal subgroup in $G$, and $L$ is not $f$-central in $G$. If $L$ is an $\omega'$-group, then $c_O(L) = 1$ and $G \cong G/c_O(L) \in f(\omega') = \mathfrak{F}$. If $L$ is a non-abelian $pd$-group for some $p \in \omega$ and $S \in \text{Com}(L)$, then $C_G(L) = 1$ and we have $G \cong G/C_G(L) \in h(S) \subseteq h(p) \subseteq \mathfrak{F}$. Assume that $L$ is a $p$-group, $p \in \omega$. Since $L$ is not $f$-central, $L \not\subseteq Z(G)$. By Lemma 2.1 we have $C_p(G) = 1$. So $G \in h(p) = Q(H/C_p(H) \mid H \in \mathfrak{F})$, i.e., $L$ is $f$-central, a contradiction. Thus, $\mathfrak{F} \subseteq LF_\omega(f)$.

Let $G$ be a group of minimal order in $LF_\omega(f) \setminus \mathfrak{F}$. Then $L = G^\mathfrak{F}$ is a unique minimal normal subgroup in $G$. Clearly, $c_G(L) = 1$, and $C_G(L) = 1$ if $L$ is non-abelian. Hence, it follows from $G \in LF_\omega(f)$ that if $L$ is an $\omega'$-group, then $G \in f(\omega') = \mathfrak{F}$, and if $L$ is a non-abelian $pd$-group for some $p \in \omega$, then $G \in f(p) = h(p) \subseteq \mathfrak{F}$, and we get a contradiction. Assume that $L$ is a $p$-group, $p \in \omega$. Evidently, $L$ is not contained in $\Phi(G)$ (recall that $\mathfrak{F}$ is $p$-saturated). By Lemma 2.7, $L = C_G(L)$. Since $L$ is $f$-central, we obtain that $G = [L:T]$ where $T \in f(p)$. Therefore, $T \simeq R/K$ where $R = H/C_p(H)$ for some $H \in \mathfrak{F}$. Now we can consider $L$ as an irreducible $F_pR$-module by inflation (see [1], p. 105). By Lemma 2.5 we have $[L]R \in \mathfrak{F}$. Since $K$ acts identically on $L$, it follows from Lemma 2.6 that $[L](R/K) \simeq LT = G \in \mathfrak{F}$, and we again arrive at a contradiction. So $LF_\omega(f) = \mathfrak{F}$.

**Corollary 4.6.1.** If a non-empty formation $\mathfrak{F}$ is $\omega$-saturated, then $\mathfrak{F}$ has an $\omega$-local satellite $f$ such that $f(p) = Q(G/C_p(G) \mid G \in \mathfrak{F})$ if $p \in \omega \cap \pi(\mathfrak{F})$, $f(p) = \emptyset$ if $p \in \omega \setminus \pi(\mathfrak{F})$, and $f(\omega') = \emptyset$ if $\omega' \neq \emptyset$.

**Corollary 4.6.2.** If a non-empty formation $\mathfrak{F}$ is saturated, then $\mathfrak{F} = LF(f)$ where $f$ is a local satellite such that $f(p) = Q(G/C_p(G) \mid G \in \mathfrak{F})$ for every $p \in \pi(\mathfrak{F})$, and $f(p) = \emptyset$ for every prime $p \notin \pi(\mathfrak{F})$.

### 4. $\mathfrak{X}$-local formations

In 1985 Förster [13] introduced the concept ‘$\mathfrak{X}$-local formation’ in order to obtain a common extension of Theorem 3.2 and 4.1.

**Definition 5.1.** Let $\mathfrak{X}$ be a class of simple groups such that $\text{Char}(\mathfrak{X}) = \pi(\mathfrak{X})$. Consider a map

$$f : \pi(\mathfrak{X}) \cup \mathfrak{X}' \rightarrow \{\text{formations}\}$$

which does not distinguish between any two non-identity isomorphic groups. Denote through $LF_\mathfrak{X}(f)$ the class of all groups $G$ satisfying the following conditions:
(i) if $H/K$ is a chief $E\mathcal{X}$-factor of a group $G$, then $G/C_G(H/K)$ belongs to $f(p)$ for any $p \in \pi(H/K)$;

(ii) if $G/L$ is a monolithic quotient of $G$ and $\text{Soc}(G/L) \in E(\mathcal{X}')$, then $G/L \in f(S)$ where $S \in \text{Com}(\text{Soc}(G/L))$.

The class $LF_X(f)$ is a formation; it is called an $\mathcal{X}$-local formation.

$\mathcal{X}$-local formations were investigated in [20] [19] [21] [22] [23]. In [24] it was proved with help of some lemmas in [22] that every $\mathcal{X}$-local formation has a $\mathcal{X}^+$-composition satellite. Now we give a direct proof of that fact.

**Theorem 5.1.** Let $\mathcal{F}$ be a non-empty formation, $\mathcal{X}$ a class of simple groups such that $\text{Char}(\mathcal{X}) = \pi(\mathcal{X})$. Let $\mathcal{L}$ be a class of simple groups such that $\mathcal{L}^+ = \mathcal{X}^+$.

1. If $\mathcal{F}$ is an $\mathcal{X}$-local formation, then $\mathcal{F}$ has an $\mathcal{L}$-composition satellite.

2. If $\mathcal{F}$ has an $\mathcal{L}$-composition satellite, then $\mathcal{F}$ is an $\mathcal{X}^+$-local formation.

**Proof.** Set $\omega = \text{Char}(\mathcal{X})$. Evidently, $\mathcal{L}^- \cup \mathcal{L}' = \mathcal{X}^- \cup \mathcal{X}' = (\mathcal{L}^+)' = (\mathcal{X}^+)'$.

1. Let $\mathcal{F}$ be a $\mathcal{X}$-local formation, $\mathcal{F} = LF_X(f)$. Consider an $\mathcal{L}$-composition satellite $h$ such that $h(p) = f(p) \cap \mathcal{F}$ if $p \in \omega$, and $h(S) = \mathcal{F}$ if $S \in \mathcal{L}^- \cup \mathcal{L}'$. We will prove that $\mathcal{F} = CF_\mathcal{L}(h)$.

Suppose that $\mathcal{F} \not\subseteq CF_\mathcal{L}(h)$. Let $G$ be a group of minimal order in $\mathcal{F} \setminus CF_\mathcal{L}(h)$. Then $G$ is monolithic, and $G/M \in CF_\mathcal{L}(h)$ where $M$ is the socle of $G$. Clearly, $M$ is the $CF_\mathcal{L}(h)$-residual of $G$, and every chief factor between $G$ and $L$ is $h$-central. Assume that $M$ is an $E(\mathcal{L}^- \cup \mathcal{L}')$-group. Since $G \in \mathcal{F}$, we have that $G \in h(S)$ where $S \in \text{Com}(M)$. Since $c_G(L) = 1$, we have that $M$ is $h$-central in $G$, and so $G \in CF_\mathcal{L}(h)$. Assume now that $M$ is a $p$-group, $p \in \omega$. Since $G \in \mathcal{F}$, we have $G/C_G(M) \in f(p) \cap \mathcal{F} = h(p)$, i. e., $M$ is $h$-central. We see that $G \in CF_\mathcal{L}(h)$, a contradiction. Thus, $\mathcal{F} \subseteq CF_\mathcal{L}(h)$.

Suppose now that $CF_\mathcal{L}(h) \not\subseteq \mathcal{F}$. Choose a group $G$ of minimal order in $CF_\mathcal{L}(h) \setminus \mathcal{F}$. Then $G$ is monolithic, and $G/M \in \mathcal{F}$ where $M = G^\mathcal{F}$ is the socle of $G$. Assume that $M$ is an $E(\mathcal{L}^- \cup \mathcal{L}')$-group. Then from $c_G(L) = 1$ and $h$-centrality of $L$ it follows that $G/c_G(M) \simeq G \in \mathcal{F}$. Assume that $M$ is a $p$-group, $p \in \omega$. Then

$$G/C_G(M) \in h(p) = \mathcal{F} \cap f(p) \subseteq f(p).$$

We see that all the chief factors and all the quotients of $G$ satisfies conditions (1) and (2) of Definition 5.1. So, $G \in \mathcal{F}$, a contradiction. Thus, $\mathcal{F} = CF_\mathcal{L}(h)$.

(2) Let $\mathcal{F}$ be a formation having an $\mathcal{L}$-composition satellite. By Lemma 4.3, $\mathcal{F} = CF_\mathcal{L}(f)$ where $f$ is an $\mathcal{L}$-composition satellite such that $f(S) = \mathcal{F}$.
for every $S \in \mathcal{L}^- \cup \mathcal{L}'$. Consider an $\mathcal{X}^+$-local formation $\mathfrak{F} = LF_{\mathcal{X}^+}(h)$ where $h(p) = f(p)$ for any $p \in \omega$, and $h(S) = \mathfrak{F}$ for every $S \in (\mathcal{X}^+)'$. It easy to check that $\mathfrak{F} = \mathfrak{H}$. 

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