Some properties of the moment estimator of shape parameter for the gamma distribution

Piotr Nowak
Mathematical Institute, University of Wrocław
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail: nowak@math.uni.wroc.pl

Abstract

Exact distribution of the moment estimator of shape parameter for the gamma distribution for small samples is derived. Order preserving properties of this estimator are presented.

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1 Introduction and preliminaries

It is well known that the gamma distribution has wide application, for example in meteorology to describe the distribution of rainfall.

The moment estimators $\hat{\alpha}$ and $\hat{\lambda}$ of the parameters $\alpha$ and $\lambda$ of the gamma distribution with density

$$f(x; \alpha, \lambda) = \frac{1}{\Gamma(\alpha)\lambda^\alpha}x^{\alpha-1}\exp(-x/\lambda), \quad x > 0, \quad \alpha > 0, \lambda > 0$$

for a random sample $X_1, \ldots, X_n$ are

(1.1) 
$$\hat{\alpha} = \frac{\bar{X}^2}{S^2} \quad \text{and} \quad \hat{\lambda} = \frac{S^2}{\bar{X}},$$

where $\bar{X}$ and $S^2$ are sample mean and sample biased variance respectively.

In this paper we are mainly interested in order preserving property of the estimator $\hat{\alpha}$. In general, we say that the estimator $\hat{\theta}$ based on the sample $X_1, \ldots, X_n$ from population with density $f(\cdot; \theta)$ is increasing in $\theta$ with respect to the order $\prec$ if $\hat{\theta}_1 \prec \hat{\theta}_2$, whenever $\theta_1 < \theta_2$, where $\hat{\theta}_1 (\hat{\theta}_2)$ is the estimator based on sample from density $f(\cdot; \theta_1) (f(\cdot; \theta_2))$. We shall consider when $\prec$ is one of the following orders: $\leq_{st}, \leq_{disp}, \leq_*$. For details and definitions of these stochastic orders we refer the reader to Shaked and Shanthikumar [5].

Nowak [3] proved the following theorem.
Theorem 1. The estimators $\hat{\alpha}$ and $\hat{\lambda}$ are stochastically increasing respectively in $\alpha$ and $\lambda$.

Deriving exact distributions of moment estimators $\hat{\alpha}$ and $\hat{\lambda}$ is very tedious for $n > 2$. Now we find the distribution of $\hat{\alpha}$ for $n = 2$. First we prove the more general fact.

Lemma 1. Suppose that $X_1 \sim G(\lambda, \alpha_1)$ and $X_2 \sim G(\lambda, \alpha_2)$, $X_1$ and $X_2$ are independent random variables. Then $Z = \frac{S^2}{X^2}$ has density of the form

$$h(z; \alpha_1, \alpha_2) = \frac{1}{B(\alpha_1, \alpha_2)} z^{-1/2} \times \left[ (1 - \sqrt{z})^{\alpha_1-1}(1 + \sqrt{z})^{\alpha_2-1} + (1 + \sqrt{z})^{\alpha_1-1}(1 - \sqrt{z})^{\alpha_2-1} \right].$$

Corollary 1. Let $X_1, X_2 \sim G(\lambda, \alpha)$, $Y \sim Ex(1)$, $X_1, X_2$ and $Z$ are independent random variables. Then

$$(1.2) \quad \frac{X_1^2 + X_2^2}{(X_1 + X_2)^2} = \frac{1}{Z} \frac{(X_1 + Y)^2 + X_2^2}{(X_1 + Y + X_2)^2}.$$ 

Problem 1. Equation (1.2) yields some interesting property of the exponential distribution. There arises a question: Does this equation characterize the exponential distribution?

Corollary 2. For $n = 2$ the statistics $1/\hat{\alpha}$ has the beta distribution $B(1/2, \alpha)$. Hence $\hat{\alpha}$ has monotone likelihood ratio.

Now we examine dispersive ordering of the $\hat{\alpha}$ estimator. First we notice that $1/\hat{\alpha}$ is not dispersive monotone in $\alpha$. We conclude it from Theorem 3.B.14 of Shaked and Shanthikumar [5] since the support of the estimator $1/\hat{\alpha}$ is the finite interval $(0, n-1)$.

Theorem 2. The estimator $1/\hat{\alpha}$ is not dispersively monotone in $\alpha$.

We now consider dispersive ordering of the estimator $\hat{\alpha}$. From previous theorem it does not follow that $\hat{\alpha}$ is not dispersive monotone in $\alpha$, though the function $1/x$ is decreasing and convex for $x > 0$. We should notice that Theorem 3.B.10(b) of Shaked and Shanthikumar [5] is not valid here. For example, in the following theorem we prove that for $n = 2$ the estimator $\hat{\alpha}$ is dispersively increasing in $\alpha$.

Theorem 3. For $n = 2$ the estimator $\hat{\alpha}$ is dispersively increasing in $\alpha$.

Proof. Let $f^\alpha$ denotes density of the estimator $\hat{\alpha}$. We know that $1/\hat{\alpha}$ has the beta distribution $B(1/2, \alpha)$, so

$$f^\alpha(x) = \frac{1}{B(1/2, \alpha)} x^{-3/2}(1 - 1/x)^{\alpha-1}, \quad x > 1.$$
In order to prove that $\hat{\alpha}_1 \leq_{\text{disp}} \hat{\alpha}_2$ whenever $\alpha_1 < \alpha_2$ it suffices to prove that

$$S^-(f^\alpha_c(x + 1) - f^\alpha_2(x + 1)) \leq 2 \quad \text{for every } c > 0$$

with the sign sequence being $-, +, -$ in the case of equality and $\hat{\alpha}_1 \leq_{\text{st}} \hat{\alpha}_2$ (see Theorem 2.6 and Remark 2.1 of Shaked [4]).

Fix $c > 0$. When $x > c$ then

$$f^\alpha_c(x + 1) - f^\alpha_2(x + 1) = \frac{1}{B(1/2, \alpha_2)}(x + 1)^{-3/2}(1 - 1/(x + 1))^{\alpha_2 - 1} \times$$

$$\times [A(x + 1)^{3/2}(x - c + 1)^{-3/2}(1 - 1/(x - c + 1))^{\alpha_1 - 1}/(1 - 1/(x + 1))^{\alpha_2 - 1} - 1],$$

where $A = B(1/2, \alpha_2)/B(1/2, \alpha_1)$. Thus $S^-(f^\alpha_c - f^\alpha_2) = S^-(h(x))$, where

$$h(x) = \log A + 3/2 \log(x + 1) - 3/2 \log(x - c + 1) +$$

$$+ (\alpha_1 - 1) \log(1 - 1/(x - c + 1)) - (\alpha_2 - 1) \log(1 - 1/(x + 1)).$$

Now we show that $h$ has exactly one maximum when $\alpha_1 > 1$ and the sign sequence is $-, +, -$. In the other case $h$ is decreasing. After calculating we have that $h'(x) = 0$ if and only if $-w(x)/(2x(x + 1)(x - c)(x - c + 1)) = 0$, where

$$w(x) = (2\alpha_2 - 2\alpha_1 + 3c)x^2 + (2\alpha_2 - 2\alpha_1 + 4c - 4\alpha_2c - 3c^2)x + 2c - 2\alpha_2c - 2c^2 + 2\alpha_2c^2.$$

So we must show that $w$ has only one root in the interval $(c, \infty)$ if $\alpha_1 > 1$ and no roots when $\alpha_1 \in (0, 1]$. Define $\bar{w}(x) = w(x + c)$. Therefore

$$\bar{w}(x) = (2\alpha_2 - 2\alpha_1 + 3c)x^2 + (2\alpha_2 - 2\alpha_1 + 4c(1 - \alpha_1) + 3c^2)x + 2(1 - \alpha_1)(c + c^2).$$

It is easy to see that for $\alpha_1 > 1$, $\bar{w}$ has two different roots $x_1$ and $x_2$ that $x_1x_2 < 0$ and for $\alpha_1 \in (0, 1]$ $\bar{w}$ has no roots in $(0, \infty)$. At the end we must show that the sign sequence is $-, +, -$ when $S^-(h) = 2$. For $\alpha_1 > 1$

$$\lim_{x \to c^+} h(x) = -\infty, \quad \lim_{x \to \infty} h(x) = \log A < 0.$$ 

Combining these above facts with Theorem 1 we end the proof.

**Remark 1.** The above theorem does not hold in general for $n \geq 3$, but direct calculating is impossible due to occurrence of hyper elliptic integrals. For example, numerical calculations show, that for $n = 3$, $\hat{\alpha}_1 \not\leq_{\text{disp}} \hat{\alpha}_2$ if $\alpha_1 = 1/5$ and $\alpha_2 = 1/4$. Then $G^{-1}(x) - F^{-1}(x)$ has the local maximum at $x \approx 0.72$ and the local minimum at $x \approx 0.85$.

Deriving exact distribution when $n = 3$ is more tedious. We should add, that the estimator $\hat{\alpha}$ is closely related to the student statistic $\sqrt{n}\bar{X}/S$. The properties of this statistic was very intensively studied in the literature, especially under non-normal conditions. First, we find joint distribution of the vector $(\bar{X}, S)$. One can proof the following lemma, see for example Craig [2].
Lemma 2. If \( f \) is a density on \((0, \infty)\), then the joint distribution \((X, S)\) for \( n = 3 \) is given by formula

\[
F(\bar{x}, s) = \begin{cases} 
18s \int_{x-s\sqrt{2}}^{x+s\sqrt{2}} f(x) f(\frac{3x-x_1 + R}{2}) f(\frac{3x-x_1 - R}{2}) dx_1, & 0 \leq s \leq \bar{x}\sqrt{2}/2, \\
\frac{1}{R} \int_0^{\frac{3\sqrt{2} - 6s^2 - s^2}{2}} + \int_{\frac{3\sqrt{2} - 6s^2 - s^2}{2}}^{\frac{3\sqrt{2} + 6s^2 - s^2}{2}} \frac{1}{R} f(x_1) f(\frac{3x-x_1 + R}{2}) f(\frac{3x-x_1 - R}{2}) dx_1, & \bar{x}\sqrt{2}/2 \leq s \leq \bar{x}\sqrt{2},
\end{cases}
\]

where \( R = \sqrt{6s^2 - 3(x_1 - \bar{x})^2} \).

If we have the joint distribution \((X, S)\) it is easy to derive the distribution of the statistics \( T = \bar{X}/S = \sqrt{\alpha} \) from the formula

\[
f_T(t) = \int uF(ut, u)du.
\]

Proposition 1. For \( n = 3 \) the cumulative distribution function of \( T \) for the exponential distribution with density \( f(x) = e^{-x}, x > 0 \) is given by

\[
F(t) = \begin{cases} 
1 - \frac{2\pi}{3\sqrt{3}t^2}, & t \geq \sqrt{2}, \\
-\sqrt{2-t^2} + \frac{\pi}{3\sqrt{3}} - \frac{2\arcsin\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{\sqrt{3t^2}} + 1, & \sqrt{2}/2 \leq t \leq \sqrt{2}.
\end{cases}
\]

On the other hand it is not easy to derive in the general the distribution of \( T \). For example, if \( \alpha \in (0, 1) \) above integrals can be calculated only by numerical methods. For \( n = 3 \), after a bit algebra we can also derive the distribution of \( T \) for the gamma distribution with shape parameter \( \alpha = 2 \).

Proposition 2. For \( n = 3 \) the cumulative distribution function of \( T \) for the gamma distribution with density \( g(x) = xe^{-x}, x > 0 \) is given by

\[
G(t) = \begin{cases} 
1 - \frac{10\pi(4t^2-3)}{27\sqrt{3}t^4}, & t \geq \sqrt{2}, \\
\frac{\sqrt{2-t^2}(-33t^4-13t^2+8)+5\pi(4t^2-3)-30t(4t^2-3)\arcsin\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{27\sqrt{3}t^2} + 1, & \sqrt{2}/2 \leq t \leq \sqrt{2}.
\end{cases}
\]

From previous propositions it follows, that the statistic \( T \) is not monotone with respect to the star order \( \leq_s \), that is the function \( G^{-1}F(x)/x \) is not monotone in \( x > 0 \). To see it, let us calculate:

1) \( \frac{G^{-1}F(x)}{x} \bigg|_{x=\frac{11}{33}} = \frac{10}{33} \sqrt{\frac{1}{33} \left( \frac{11}{33} \right)} \approx 1.32686 \)

2) \( \frac{G^{-1}F(x)}{x} \bigg|_{x=\frac{2\pi}{2}} = \sqrt{\frac{40\pi + 2\sqrt{10\pi(22\pi - 27\sqrt{3})}}{27\sqrt{3} + 18\pi}} \approx 1.31502 \)

3) \( \frac{G^{-1}F(x)}{x} \bigg|_{x=\sqrt{2}} = \sqrt{\frac{40\pi + 2\sqrt{10\pi}}{6}} \approx 1.32081 \)
Since the star order is preserved under increasing function, we have the following corollary.

**Corollary 3.** For \( n = 3 \) the estimator \( \hat{\alpha} \) is not increasing in \( \alpha \) with respect to the star order.

The star order is closely related to the dispersive order, since

\[ X \leq_{\ast} Y \text{ iff } \log X \leq_{\text{disp}} \log Y. \]

Using the same technique as in Theorem 3 we can prove the following proposition.

**Proposition 3.** For \( n = 2 \) the estimator \( \hat{\alpha} \) is increasing in \( \alpha \) with respect to the star order.

Applying the property that for nonnegative random variables such \( X \leq_{\text{st}} Y \) and \( X \leq_{\ast} Y \) implies the ordering \( X \leq_{\text{disp}} Y \) (see, for example, Ahmed et al. [1]) we can also deduce from Theorem 1 and Proposition 3 that \( \hat{\alpha} \) is dispersively monotone for \( n = 2 \).

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