Computing degree of determinant via discrete convex optimization on Euclidean building

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Abstract

In this paper, we consider the computation of the degree of the Dieudonné determinant of a linear symbolic matrix $A = A_0 + A_1x_1 + \cdots + A_mx_m$, where each $A_i$ is an $n \times n$ polynomial matrix over $K[t]$ and $x_1, x_2, \ldots, x_m$ are pairwise “non-commutative” variables. This quantity is regarded as a weighed generalization of the non-commutative rank (nc-rank) of a linear symbolic matrix, and its computation is shown to be a generalization of several basic combinatorial optimization problems, such as weighted bipartite matching and weighted linear matroid intersection problems.

Based on the work on nc-rank by Fortine and Rautenauer (2004), and Ivanyos, Qiao, and Subrahmanyam (2018), we develop a framework to compute the degree of the Dieudonné determinant of a linear symbolic matrix. We show that the deg-det computation reduces to a discrete convex optimization problem on the Euclidean building for $\text{SL}(K(t)^n)$. To deal with this optimization problem, we introduce a class of discrete convex functions on the building. This class is a natural generalization of L-convex functions in discrete convex analysis (DCA). We develop a DCA-oriented algorithm (steepest descent algorithm) to compute the degree of determinants. Our algorithm works with matrix computation on $K$, and uses a subroutine to compute a certificate vector subspace for nc-rank, where the number of calls of the subroutine is sharply estimated. Our algorithm captures some of classical combinatorial optimization algorithms with new insights, and is also understood as a variant of the combinatorial relaxation algorithm, which was developed earlier by Murota for computing the degree of the (ordinary) determinant.

Keywords: non-commutative rank, Dieudonné determinant, skew field, discrete convex analysis, mixed matrix, combinatorial relaxation algorithm, submodular function, L-convex function, Euclidean building, uniform modular lattice.
1 Introduction

A linear symbolic matrix or linear matrix $A$ is a matrix each of whose entry is a linear (affine) function in variables $x_1, x_2, \ldots, x_m$. Namely $A$ admits the form of

$$A = A_0 + A_1 x_1 + \cdots + A_m x_m,$$

(1.1)

where each $A_i$ is a matrix not containing variables. In this paper, we address the symbolic rank computation of linear matrices and its generalization. This problem is a fundamental problem in discrete mathematics and computer science. In a classical paper [11], Edmonds noticed that the maximum matching number of a bipartite graph is represented as the rank of such a matrix: For each edge $e = ij$, consider variable $x_e$ and matrix $E_e$ having 1 for the $(i, j)$-entry and zero for the others. Then the rank of linear matrix $A = \sum_e E_e x_e$ (with $A_0 = 0$ in (1.1)) is equal to the maximum matching number of the graph. In the same paper, Edmonds asked for a polynomial time algorithm to compute the rank of a general linear matrix. Clearly the rank computation is easy if we substitute an actual number for each variable. According to random substitution, Lovász [35] developed a randomized polynomial time algorithm to compute the rank of linear matrices. Designing a deterministic polynomial time algorithm is a big challenge in theoretical computer science, since it would lead to a breakthrough in circuit complexity theory [33]. Currently, such deterministic algorithms are known for very restricted classes of linear matrices. Most of them are connected to polynomially-solvable combinatorial optimization problems, such as matching, matroid intersection, and their generalizations; see [36]. A mixed matrix due to Murota and Iri [42], which is a linear symbolic matrix including constant matrix $A_0$ in the above bipartite graph example, is also such an example; see [40] for theory of mixed matrix and its engineering application.

Recently there are significant developments in the rank computation of linear matrices. In the above paragraph, we implicitly assumed that $A$ is a matrix over polynomial ring $\mathbb{K}[x_1, x_2, \ldots, x_m]$ with field $\mathbb{K}$ and indeterminates $x_1, x_2, \ldots, x_m$, and the rank is considered in rational function field $\mathbb{K}(x_1, x_2, \ldots, x_m)$. However $A$ can also be viewed as a matrix over the free ring $\mathbb{K}\langle x_1, x_2, \ldots, x_m \rangle$ generated by $x_1, x_2, \ldots, x_m$, where variables are supposed to be pairwise non-commutative, i.e., $x_i x_j \neq x_j x_i$. It is shown by Amitsur [1] that there is a non-commutative analogue $\mathbb{K}\langle\langle x_1, x_2, \ldots, x_m \rangle \rangle$ of the rational function field, called the free skew field, to which $\mathbb{K}\langle x_1, x_2, \ldots, x_m \rangle$ is embedded. Now we can define the rank of $A$ in this skew field $\mathbb{K}\langle\langle x_1, x_2, \ldots, x_m \rangle \rangle$. This rank concept of $A$ is called the non-commutative rank (nc-rank) of $A$. Fortine and Rautenauer [12] proved a formula of nc-rank saying that it is equal to the optimal value of an optimization problem over the lattice of all vector subspaces of $\mathbb{K}^n$. Garg, Gurvits, Oliveira, and Wigderson [13] proved that the nc-rank of $A$ can be computed in deterministic polynomial time if $\mathbb{K} = \mathbb{Q}$. They showed that Gurvits’ operator scaling algorithm [16], which was earlier developed for the rank computation of a special class (called Edmonds-Rado class) of linear matrices, can be a polynomial time algorithm for nc-rank. Ivanyos, Qiao, and Subrahmanym [26, 27] developed a polynomial time algorithm to compute nc-rank in an arbitrary field. Their algorithm is a vector-space analogue of the augmenting path algorithm for bipartite matching, and utilizes an invariant theoretic result by Derksen and Makam [17] for the complexity estimate. Independent of this line of research, Hamada and Hirai [18] investigated the optimization problem for nc-rank which
they called the maximum vanishing subspace problem (MVSP). Their motivation comes
from a canonical form of a matrix under block-restricted transformations [23, 29]. They
also developed a polynomial time algorithm for nc-rank based on the fact that MVSP
is viewed as a submodular function optimization on the modular lattice of all vector
subspaces of $\mathbb{K}^n$.

In this paper, we consider a “weighted analogue” of nc-rank. Our principal motiva-
tion is to capture the weighted versions of combinatorial optimization problems from
the non-commutative points of view. Consider, for example, the weighted matching
problem on a bipartite graph, where two color classes are supposed to have the same
cardinality, and each edge $e$ has (integer) weight $c_e$. By introducing new indetermi-
nate $t$, modify the above bipartite-graph linear matrix $A$ as $A := \sum_{e} t^{c_e} E_e$. Then the
maximum weight of a perfect matching is equal to the degree of the determinant of $A$ with respect to $t$. This well-known example suggests that such a weighted analogue is
the degree of determinants. This motivates us to consider the degree of determinant of
linear matrices in the non-commutative setting.

The main contribution of this paper is to develop a computational framework for
degree of determinant in the non-commutative setting, which captures some of classi-
cal weighted combinatorial optimization problems. Our results and their feature are
summarized as follows:

- For a determinant concept for matrices over skew field $\mathbb{F}$, we consider Dieudonné
determinant [8]. Although the value of Dieudonné determinant is no longer an
element of the ground field, in the skew field $\mathbb{F}(t)$ of rational functions (Ore
quotient ring of polynomial ring $\mathbb{F}[t]$), its degree is well-defined, see e.g., [45]. Our
target is the degree $\deg \Det A$ of Dieudonné determinant $\Det A$ of a linear matrix
$A = A_0 + A_1 x_1 + \cdots + A_m x_m$, where each $A_i$ is a square polynomial matrix over
$\mathbb{K}[t]$ and $A$ is viewed as a matrix over the rational function skew field $\mathbb{F}(t)$ of the
free skew field $\mathbb{F} = \mathbb{K}\langle x_1, x_2, \ldots, x_m \rangle$.

- We establish a duality theorem for $\deg \Det$, which is a natural generalization
of the Fortin-Rautenauer formula for nc-rank. In fact, a weak duality relation
was previously observed by Murota [38] for $\deg \det$, and is now a strong duality
for $\deg \Det$. Analogously to the Fortine-Rautenauer formula saying that the nc-
rank is equal to the optimal value of an optimization problem (MVSP) over the
lattice of all vector subspaces of $\mathbb{K}^n$, our formula says that $\deg \Det$ is equal to the
optimal value of an optimization problem over the lattice of all full-rank $\mathbb{K}(t)^{-\infty}$
submodules of $\mathbb{K}(t)^n$, where $\mathbb{K}(t)^-$ is the valuation ring of $\mathbb{K}(t)$ with valuation
deg. In the literature of group theory, this lattice structure is known as the
Euclidean building for $\SL(\mathbb{K}(t)^n)$ [3], whereas the lattice of all vector subspaces is
the spherical building for $\SL(\mathbb{K}^n)$ [46].

- We approach this optimization problem on the building from Discrete Convex
Analysis (DCA) [11] with its recent generalization [20, 21]. Although DCA was
originally a theory of discrete convex functions on $\mathbb{Z}^n$ that generalizes matroids
and submodular functions, recent study [20, 21] shows that DCA-oriented concepts
and algorithm designs are effective and useful for optimization problems on
certain discrete structures beyond $\mathbb{Z}^n$. $L$-convexity, which is one of the central
concepts of DCA, is particularly important for us. $L$-convex functions are gener-
alization of submodular functions and arise naturally from representative combi-
natorial optimization problems such as minimum-cost network flow and weighted bipartite matching. L-convex functions admit a simple minimization algorithm, called the *steepest descent algorithm (SDA)*, on which our algorithm for deg Det will be built.

We introduce an analogue of L-convex function on the building. The previous work [20, 21] introduced L-convexity on Euclidean building of type C, whereas our building here is of type A. We show that the established formula of deg Det gives rise to an L-convex function, analogously to the submodular function in MVSP for nc-rank. Consequently deg Det is computed via an L-convex function minimization on the Euclidean building.

- We develop an algorithm to compute deg Det $A$ for linear polynomial matrix $A$ over $\mathbb{K}[t]$. Our algorithm requires a subroutine to solve MVSP over $\mathbb{K}$, and is described in terms of matrix computation over $\mathbb{K}$. However it can be viewed as the steepest descent algorithm (SDA) applied to the L-convex function on the Euclidean building. By utilizing the recent analysis on SDA [43], we show that the number of the subroutine calls is sharply estimated by the *Smith-McMillan form* of $A$.

Our algorithm can also be interpreted as a variant of the *combinatorial relaxation method*, which was developed earlier for deg det of matrices (without variables) by Murota [37, 38] and was further extended to mixed polynomial matrices by Iwata and Murota [30]; see [40, Section 7.1] and recent work [31, 32]. This interpretation sheds building-theoretic insights on the combinatorial relaxation algorithm.

- We study a class of linear matrices $A$ for which $\text{deg det } A = \text{deg Det } A$ holds. In the case of nc-rank, it is known from the results in [24, 35] that if each $A_i$ other than $A_0$ is a rank-1 matrix over $\mathbb{K}$, then it holds $\text{rank } A = \text{nc-rank } A$. We show a natural extension: if each $A_i$ other than $A_0$ is a rank-1 matrix over $\mathbb{K}(t)$, then it holds $\text{deg det } A = \text{deg Det } A$. This property implies that some of classical combinatorial optimization problems, represented as deg det, fall into our framework of deg Det. Examples include weighted bipartite matching and weighted linear matroid intersection. In these examples, the optimal value is interpreted as deg det as well as deg Det. A *mixed polynomial matrix* [40] is also such an example. We explain how our SDA framework works for these examples, and discuss connections to some of classical algorithms, such as the Hungarian method, the matroid greedy algorithm, and the matroid intersection algorithm by Lawler [34]. We also mention a possible application to mixed-matrix DAE analysis.

The rest of this paper is organized as follows. In Section 2 we summarize basic facts on skew field, nc-rank, Fortine-Rautenauer formula, MVSP, and Dieudonné determinant. In Section 3 we introduce L-convex functions on Euclidean buildings and show their basic properties. Instead of dealing with Euclidean buildings in the usual axiom system, we utilize an elementary lattice-theoretic equivalent concept, called *uniform modular lattices* [22]. This class of modular lattices admits L-convexity concept in a straightforward way. In Section 4 we establish a formula for the degree of the Dieudonné determinant of a linear matrix and present an algorithm. In Section 5 we study linear matrices with rank-1 summands.
2 Skew field

A skew field (or division ring) is a ring $\mathbb{F}$ such that every nonzero element $x \in \mathbb{F}$ has inverse element $x^{-1} \in \mathbb{F}$ with $x^{-1}x = xx^{-1} = 1$. The product $\mathbb{F}^n$ of $\mathbb{F}$ will be treated as a right $\mathbb{F}$-vector space of column vectors as well as a left $\mathbb{F}$-vector space of row vectors; which one we suppose will be clear in the context. The set of all $n \times n'$ matrices over $\mathbb{F}$ is denoted by $\mathbb{F}^{n\times n'}$. The rank of matrix $A \in \mathbb{F}^{n\times n'}$ is defined as the dimension of the right $\mathbb{F}$-vector space spanned by columns of $A$, which is equal to the dimension of the left $\mathbb{F}$-vector space spanned by rows of $A$. Let $\ker_R A$ denote the right kernel $\{x \in \mathbb{F}^{n'} \mid Ax = 0\}$, and let $\ker_L A$ denote the left kernel $\{x \in \mathbb{F}^n \mid xA = 0\}$. Then the rank of $A \in \mathbb{F}^{n\times n'}$ equals to $n - \dim \ker_R A = n' - \dim \ker_L A$. A square matrix $A \in \mathbb{F}^{n\times n}$ is called nonsingular if its rank is equal to $n$, or equivalently if it has the inverse, which is denoted by $A^{-1}$, i.e., $AA^{-1} = A^{-1}A = I$. These properties are easily seen from the Bruhat normal form of $A$; see Lemma 2.4 in Section 2.2.

An $n \times n'$ matrix $A \in \mathbb{F}^{n\times n'}$ is viewed as a map $\mathbb{F}^n \times \mathbb{F}^{n'} \to \mathbb{F}$ by

$$A(x, y) := xAy \quad (x \in \mathbb{F}^n, y \in \mathbb{F}^{n'}). \quad (2.1)$$

Then $A$ is bilinear in the sense that $A(\alpha x + \alpha' x', y) = \alpha A(x, y) + \alpha' A(x', y)$ and $A(x, y\beta + y'\beta') = A(x, y)\beta + A(x, y')\beta'$. Conversely, any bilinear map on the product of a left $\mathbb{F}$-vector space $U$ and a right $\mathbb{F}$-vector space $V$ is identified with a matrix over $\mathbb{F}$ by choosing bases of $U$ and $V$. Let $S_R(\mathbb{F}^n)$ and $S_L(\mathbb{F}^n)$ denote the families of all right and left $\mathbb{F}$-vector subspaces of $\mathbb{F}^n$, respectively. If $\mathbb{F}$ is commutative, then $R$ and $L$ are omitted, such as $S(\mathbb{F}^n)$.

2.1 Free field, nc-rank, and MVSP

Let $\mathbb{K}$ be a field. Let $\mathbb{K}[x_1, x_2, \ldots, x_m]$ and $\mathbb{K}(x_1, x_2, \ldots, x_m)$ denote the ring of polynomials and the field of rational functions over variables $x_1, x_2, \ldots, x_m$, respectively, where variables are supposed to commute each other, i.e., $x_ix_j = x_jx_i$. Let $\mathbb{K}\langle x_1, x_2, \ldots, x_m \rangle$ be the free ring generated by pairwise non-commutative variables $x_1, x_2, \ldots, x_m$ over $\mathbb{K}$. It is known that the free ring $\mathbb{K}\langle x_1, x_2, \ldots, x_m \rangle$ is embedded to the universal skew field of fractions, called the free field, which is denoted by $\mathbb{K}(\langle x_1, x_2, \ldots, x_m \rangle)$, or simply by $\mathbb{K}(\langle x \rangle)$. Elements of $\mathbb{K}(\langle x_1, x_2, \ldots, x_m \rangle)$ are equivalence classes of all rational expressions constructed from $x_i$, $x_i^{-1}$, and elements of $\mathbb{K}$, under an equivalence relation obtained by substitutions of nonsingular matrices for variables $x_i$; see [11 1 5] for details. We do not go into the actual construction of $\mathbb{K}(\langle x_1, x_2, \ldots, x_m \rangle)$.

As mentioned in the introduction, a linear symbolic matrix or linear matrix on $\mathbb{K}$ is a matrix of form $A = A_0 + A_1x_1 + A_2x_2 + \cdots + A_mx_m$, where each summand $A_i$ is a matrix over $\mathbb{K}$. A linear matrix is viewed as a matrix over $\mathbb{K}(x_1, x_2, \ldots, x_m)$ as well as over $\mathbb{K}\langle x_1, x_2, \ldots, x_m \rangle$. The (commutative) rank of $A$ is defined as the rank of $A$ regarded as a matrix over $\mathbb{K}(x_1, x_2, \ldots, x_m)$. The non-commutative rank (nc-rank) of $A$, denoted by nc-rank $A$, is defined as the rank of $A$ regarded as a matrix over $\mathbb{K}(\langle x_1, x_2, \ldots, x_m \rangle)$. The nc-rank is not less than the (commutative) rank.

Let $A = A_0 + A_1x_1 + A_2x_2 + \cdots + A_mx_m$ be a linear $n \times n'$ matrix. We consider an upper bound of nc-rank $A$. For nonsingular matrices $S \in \mathbb{K}^{n\times n}, T \in \mathbb{K}^{n'\times n'}$, if $SAT$ has a zero submatrix of $r$ rows and $s$ columns, then nc-rank $A$ is at most $n + n' - r - s$. 

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This gives rise to the following optimization problem (MVSP):

\[
\text{MVSP} : \quad \text{Max. } \quad r + s \\
\text{s.t. } \quad SAT \text{ has a zero submatrix of } r \text{ rows and } s \text{ columns,} \\
\quad S \in \mathbb{K}^{n \times n}, T \in \mathbb{K}^{n' \times n'}: \text{nonsingular.}
\]

This upper bound was observed by Lovász \[36\] for the usual rank. The name MVSP will be explained below. Fortin and Reutenauer \[12\] showed that this upper bound is actually tight for nc-rank \(A\).

**Theorem 2.1** (\[12\]). For a linear \(n \times n'\) matrix \(A = A_0 + A_1 x_1 + A_2 x_2 + \cdots + A_m x_m\), nc-rank \(A\) is equal to \(n + n'\) minus the optimal value of MVSP.

As mentioned in the introduction, Garg, Gurvits, Oliveira, and Wigderson \[13\] showed that the nc-rank \(A\) can be computed in polynomial time when \(\mathbb{K} = \mathbb{Q}\). One drawback of this algorithm is not to output optimal matrices \(S, T\) in MVSP. Such an algorithm was developed by Ivanyos, Qiao, and Subrahmanyam \[26, 27\] for an arbitrary field.

**Theorem 2.2** (\[26, 27\]). MVSP can be solved in polynomial time.

Hamada and Hirai \[18\] also gave a “polynomial time” algorithm based on submodular optimization (see Lemma \[23\]), though the bit-length required in the algorithm is not bounded if \(\mathbb{K} = \mathbb{Q}\).

As \[13, 19\] did, MVSP is formulated as the following optimization problem (maximum vanishing subspace problem) of the space of all vector subspaces of \(\mathbb{K}^n\):

\[
\text{MVSP} : \quad \text{Max. } \quad \dim X + \dim Y \\
\text{s.t. } \quad A_i(X, Y) = \{0\} \quad (i = 0, 1, 2, \ldots, m), \\
\quad X \in \mathcal{S}(\mathbb{K}^n), Y \in \mathcal{S}(\mathbb{K}^{n'}). \quad (2.2)
\]

Recall the notation \(\mathcal{S}(\mathbb{K}^n)\) for all vector subspaces of \(\mathbb{K}^n\), and that each matrix \(A_i\) is viewed as a bilinear form \(\mathbb{K}^n \times \mathbb{K}^{n'} \to \mathbb{K}\) as in \[23\]. We call an optimal \((X, Y)\) a maximum vanishing subspace or mv-subspace. Notice that we can eliminate variable \(X\) by substituting \(X = A(Y)^\perp\) (= the orthogonal complement of the image of \(Y\) by \(A\)), and obtain the formulation of \[13, 26, 27\] — the minimum shrunk subspace problem.

The nc-rank \(A\) is also obtained via MVSP over \(\mathbb{K}(\langle x \rangle)\):

\[
\text{MVSP} : \quad \text{Max. } \quad \dim X + \dim Y \\
\text{s.t. } \quad A(X, Y) = \{0\}, \\
\quad X \in \mathcal{S}_L(\mathbb{K}(\langle x \rangle)^n), Y \in \mathcal{S}_R(\mathbb{K}(\langle x \rangle)^{n'}). \quad (2.2)
\]

Indeed, MVSP has obvious optimal solutions \((\mathbb{K}(\langle x \rangle)^n, \ker R A)\) and \((\ker L A, \mathbb{K}(\langle x \rangle)^{n'})\). Notice that \(\mathbb{K}(\langle x \rangle)^n\) is a scalar extension of \(\mathbb{K}^n\), i.e., \(\mathbb{K}(\langle x \rangle)^n \simeq \mathbb{K}(\langle x \rangle) \otimes \mathbb{K}^n\). Therefore a feasible solution in MVSP is embedded into a feasible solution in \(\overline{\text{MVSP}}\) by \((X, Y) \mapsto (\mathbb{K}(\langle x \rangle) \otimes X, Y \otimes \mathbb{K}(\langle x \rangle))\). Then Theorem \([2, 1]\) also says that MVSP is an exact inner approximation of \(\overline{\text{MVSP}}\):

**Lemma 2.3.** Any mv-subspace of MVSP is also an mv-subspace of \(\overline{\text{MVSP}}\).
2.2 Dieudonné determinant

Here we introduce a determinant concept for matrices over skew field \(\mathbb{F}\), known as Dieudonné determinant \([8]\). Our reference of Dieudonné determinant is \([4]\) Section 11.2. Our starting point is the following normal form of matrices over skew field \(\mathbb{F}\).

**Lemma 2.4** (Bruhat normal form; see \([4]\) THEOREM 2.2 in Section 11.2)). Any matrix \(A\) over \(\mathbb{F}\) is represented as

\[ A = LDPU, \]

for diagonal matrix \(D\), permutation matrix \(P\), lower-unitriangular matrix \(L\), and upper-unitriangular matrix \(U\), where \(D\) and \(P\) are uniquely determined.

Here a lower(upper)-unitriangular matrix is a lower(upper)-triangular matrix such that all diagonals are 1. The proof is done by Gaussian elimination, and is essentially the LU-decomposition (without pivoting).

Let \(\mathbb{F}^\times\) denote the multiplicative group of \(\mathbb{F}\), and let \([\mathbb{F}^\times, \mathbb{F}^\times]\) denote the derived group of \(\mathbb{F}^\times\), i.e., \([\mathbb{F}^\times, \mathbb{F}^\times]\) is the group generated by all commutators \(aba^{-1}b^{-1}\). The abelization \(\mathbb{F}_{ab}\) of \(\mathbb{F}^\times\) is defined by \(\mathbb{F}_{ab} := \mathbb{F}^\times / [\mathbb{F}^\times, \mathbb{F}^\times]\). For a nonsingular matrix \(A\), the Dieudonné determinant \(\text{Det}A\) of \(A\) is defined by

\[ \text{Det}A := \text{sgn}(P)d_1d_2\cdots d_n \mod [\mathbb{F}^\times, \mathbb{F}^\times], \]

where \(A\) is represented as (2.3) for permutation matrix \(P\) and diagonal matrix \(D\) with nonzero diagonals \(d_1, d_2, \ldots, d_n\). If \(\mathbb{F}\) is a field, then \([\mathbb{F}^\times, \mathbb{F}^\times]\) = \(\{1\}\), \(\mathbb{F}_{ab} = \mathbb{F}^\times\), and \(\text{Det} = \det\).

**Example 2.5.**

\[ \text{Det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} a(d - ca^{-1}b) & \mod [\mathbb{F}^\times, \mathbb{F}^\times] \text{ if } a \neq 0, \\ -bc & \mod [\mathbb{F}^\times, \mathbb{F}^\times] \text{ if } a = 0. \end{cases} \]  

The Dieudonné determinant has desirable properties of determinant, though its value is no longer an element of \(\mathbb{F}\).

**Lemma 2.6** (\([8]\); see \([4]\) Theorem 2.6 in Section 11.2]). For nonsingular matrices \(A, B \in \mathbb{F}^{n \times n}\), it holds \(\text{Det}AB = \text{Det}A\text{Det}B\).

Next we introduce the polynomial ring \(\mathbb{F}[t]\) and its skew field \(\mathbb{F}(t)\) of fractions (or the Ore quotient ring of \(\mathbb{F}[t]\)), where indeterminate \(t\) commutes every element in \(\mathbb{F}\). Here \(\mathbb{F}[t]\) consists of polynomials \(\sum_{i=0}^{k} a_it^i\), where \(k \geq 0\) and \(a_i \in \mathbb{F}\). By using commuting rule \(xt = tx\), the addition and multiplication in \(\mathbb{F}[t]\) are naturally defined. The resulting ring \(\mathbb{F}[t]\) is called a polynomial ring over \(\mathbb{F}\) with indeterminate \(t\). In the notation of \([1] [3] [14]\), \(\mathbb{F}[t]\) is a skew polynomial ring \(\mathbb{F}[t; 1, 0]\). The degree \(\deg p\) of polynomial \(p = \sum_{i=0}^{k} a_it^i\) with \(a_k \neq 0\) is defined by \(\deg p := k\).

The polynomial ring \(\mathbb{F}[t]\) is an Ore domain, i.e., any two nonzero polynomials \(p, q \in \mathbb{F}[t]\) admit a common multiple \(pu = qv\) for some nonzero \(u, v \in \mathbb{F}[t]\). See \([4]\) Section 9.1 and \([14]\) Chapter 6 for details of Ore domain and its skew field of fractions. This enables us to introduce addition and multiplication for the set \(\mathbb{F}(t)\) of all fractions \(p/q\) for \(p \in \mathbb{F}[t], q \in \mathbb{F}[t] \setminus \{0\}\). Here \(p/q\) is an equivalence class of \((p, q) \in \mathbb{F}[t] \times \mathbb{F}[t] \setminus \{0\}\) under the equivalence relation \((p, q) \sim (p', q') \Leftrightarrow (pu, qu) = (p'v, q'v)\) for nonzero \(u, v \in \mathbb{F}[t]\).
Addition $p/q + p'/q'$ is defined as $(pu + p'v)/qu$ by choosing $u, v \in \mathbb{F}[t]$ with $qu = q'v$. Multiplication $(p/q)(p'/q')$ is defined as $pu/q'v$ by choosing $u, v \in \mathbb{F}[t]$ with $qu = p'v$. They are well-defined. The inverse of nonzero element $p/q$ (i.e., $p \neq 0$) is given by $q/p$.

The degree of $p/q$ is defined as $\deg p - \deg q$. As was observed by Taelman [45], the degree of Dieudonné determinant is well-defined, since the degree is zero on commutators.

**Example 2.7.** We see from Example [2.5] that the degree of the determinant of a $2 \times 2$ matrix over $\mathbb{F}(t)$ is similar to the commutative case:

$$\deg \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leq \max\{\deg(a) + \deg(d), \deg(b) + \deg(c)\}.$$  

The equality holds if $\deg(a) + \deg(d) \neq \deg(b) + \deg(c)$.

We let $\deg \det A := -\infty$ if $A$ is singular. From the definition and Lemma 2.6, we have:

**Lemma 2.8** (see [45]). For $A, B \in \mathbb{F}(t)^{n \times n}$, it holds $\deg \det AB = \deg \det A + \deg \det B$.

By using Dieudonné determinant, we can formulate the Smith-McMillan form of a matrix over $\mathbb{F}(t)$; see [10] Section 5.1.2 for the commutative case. An element $p/q \in \mathbb{F}(t)$ is said to be proper if $\deg(p/q) \leq 0$. Let $\mathbb{F}(t)^-$ denote the ring of proper elements of $\mathbb{F}(t)$. A matrix over $\mathbb{F}(t)^-$ is also called proper. A proper matrix is called biproper if it is nonsingular and its inverse is also proper. For integer vector $\alpha \in \mathbb{Z}^n$, let $(t^\alpha)$ denote the diagonal $n \times n$ matrix such that the $(i, i)$-entry is $t^{\alpha_i}$ for $i = 1, 2, \ldots, n$.

**Proposition 2.9** (Smith-McMillan form). For a nonsingular matrix $A \in \mathbb{F}(t)^{n \times n}$, there are biproper matrices $S, T$ and integer vector $\alpha \in \mathbb{Z}^n$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ such that

$$SAT = (t^\alpha).$$

The integers $\alpha_k$ are uniquely determined by

$$\alpha_k = \delta_k - \delta_{k-1} \quad (k = 1, 2, \ldots, n),$$

where $\delta_k$ is the maximum degree of Dieudonné determinants of $k \times k$ submatrices of $A$, and let $\delta_0 := 0$.

The part of the statement is given as an exercise in [4] pp. 459–460] in a general setting of valuation ring. The proof goes in almost the same way as in the commutative case, which is given in Appendix A.3.

**Lemma 2.10.** For $A \in \mathbb{F}(t)^{n \times n}$, it holds $\deg \det A \leq n\alpha_1$. In addition, if $A$ is a nonsingular polynomial matrix, then $\deg \det A \geq 0$.

The latter part is contained in [45] Theorem 1.1.]
Proof. The first statement follows from the Smith-McMillan form. Polynomial ring $\mathbb{F}[t]$ is a (left and right) Euclidean domain. Therefore, by row and column elementary operations on $\mathbb{F}[t]$ with row and column permutations, $A$ is diagonalized so that the diagonal entries are polynomials in $\mathbb{F}[t]$ (such as the Smith form); see [1] Section 9.2. Namely $PAQ$ is a diagonal matrix for some polynomial matrices $P, Q$ with $\deg \det P = \deg \det Q = 1$. This implies $\deg \det A = \deg \det PAQ \geq 0$.

Any proper element $p/q$ is written as $u + p'/q$, where $u \in \mathbb{F}$ and $\deg p' < \deg q$. Indeed, if $(at^k + p'')/(bt^k + q')$ with $a, b \in \mathbb{F}$, $b \neq 0$, $\deg p' < k$, and $\deg q' < k$, then $u = ab^{-1}$. This element $u$ is uniquely determined (independent of expression $p/q$). See [14] Exercise 6F. Thus any proper matrix $A$ is uniquely written as $A = A^0 + t^{-1}A'$, where $A^0$ is a matrix over $\mathbb{F}$ and $A'$ is proper.

Lemma 2.11. Let $A$ be a square proper matrix. Then the degree of each diagonal of the Smith-McMillan form of $A$ is nonpositive, and $\deg \det A \leq 0$. In addition, the following conditions are equivalent:

1. $\deg \det A = 0$.
2. $A^0$ is nonsingular over $\mathbb{F}$.
3. $A$ is biproper.
4. $A$ is the product of permutation matrices and proper unitriangular matrices.

Proof. The former part is immediate from Lemma 2.9 with the fact that $\alpha_i$ is the maximum degree of entries of $A$, and is now nonpositive. We show the equivalence.

(4) $\Rightarrow$ (3) follows from the fact that a proper unitriangular matrix is biproper.
(3) $\Rightarrow$ (2). If $B$ is the inverse of $A$ and is represented as $B = B^0 + t^{-1}B'$, then $B^0$ must be the inverse of $A^0$.
(2) $\Rightarrow$ (1). Consider the Smith-McMillan form $A = S(t^\alpha)T$. From $\alpha \leq 0$ and $S^0(t^\alpha)T^0 = A^0$, if $\deg \det A < 0$, i.e., $\alpha_i < 0$ for some $i$, then $A^0$ must be singular.
(1) $\Rightarrow$ (4). We see in the proof of Proposition 2.9 (Appendix A.3) that $S, T$ in the Smith-McMillan form of $A$ are taken as the product of permutation matrices and proper unitriangular matrices.

Finally we note a useful discrete convexity property of the degree of determinants. A valued matroid [9 10] on a set $E$ is a function $\omega : 2^E \to \mathbb{R} \cup \{-\infty\}$ satisfying the following condition:

(Exc) For any $X, Y \subseteq E$ with $\omega(X), \omega(Y) \neq -\infty$ and $e \in X \setminus Y$, there is $f \in Y \setminus X$ such that

$$\omega(X) + \omega(Y) \leq \omega(X \cup \{f\} \setminus \{e\}) + \omega(Y \cup \{e\} \setminus \{f\}).$$

(2.6)

It is well-known in the literature that the deg-det function gives rise to a valued matroid [9 10]; see [40, Chapter 5]. This is also the case for the deg-Det function.

Proposition 2.12. Let $A$ be an $n \times m$ matrix over $\mathbb{F}(t)$. The following function $\omega : 2^{\{1, \ldots, m\}} \to \mathbb{R} \cup \{-\infty\}$ is a valued matroid:

$$\omega(X) := \begin{cases} \deg \det A[X] & \text{if } |X| = n, \\
-\infty & \text{otherwise} \\
\end{cases} \quad (X \subseteq \{1, 2, \ldots, m\}),$$

(2.7)

where $A[X]$ denotes the submatrix of $A$ consisting of the $i$-th columns over $i \in X$. 

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Proof. We may assume that \( \omega(X) \neq -\infty \) for some \( X \). Since subsets \( X \) with \( \omega(X) \neq -\infty \) form the base family of a matroid (represented by column vectors of \( A \) in skew field \( F(t) \)), it suffices to verify (EXC) for pair \( X,Y \) with \( |X \setminus Y| = |Y \setminus X| = 2 \); see [40, Theorem 5.2.25]. By applying row elementary operations and column permutations, we can make \( A \) so that \( A[X \cap Y] \) is a diagonal \( n \times (n-2) \) matrix. This reduces the verification to the \( 2 \times 4 \) submatrix of columns in \( (X \setminus Y) \cup (Y \setminus X) \) and the last two rows of \( A \). Therefore it suffices to verify (EXC) for \( 2 \times 4 \) matrix \( A \). Then (EXC) is equal to:

\[
(4PT) \quad \text{the maximum of } \omega(12) + \omega(34), \omega(13) + \omega(24), \omega(14) + \omega(23) \text{ is attained at least twice,}
\]

where \( \omega(\{i, j\}) \) is simply written as \( \omega(ij) \).

We may assume that \( \omega(12) \neq -\infty \) and \( (1,1)\)-entry is nonzero (by column permutation). By row operations, we can make \( A \) so that the \( (1,2)\)-entry is zero. If the \( (2,3)\)-entry is nonzero, then make \( A \) so that \( (1,3)\)-entry is zero. Then we may consider two cases:

\[
\begin{pmatrix} a & c & d & e \\ 0 & b & 0 & f \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & c & 0 & e \\ 0 & b & d & f \end{pmatrix}.
\]

Recall Example [27]. For the former case, \( \omega(12) + \omega(34) = \omega(14) + \omega(23) = \deg(a) + \deg(b) + \deg(d) + \deg(f) \), and \( \omega(13) + \omega(24) = -\infty \). For the latter case, \( \omega(12) + \omega(34) = \deg(a) + \deg(b) + \deg(d) + \deg(e), \ \omega(14) + \omega(23) = \deg(a) + \deg(f) + \deg(c) + \deg(d) \), and \( \omega(13) + \omega(24) \leq \deg(a) + \deg(d) + \max\{\deg(c) + \deg(f), \deg(e) + \deg(b)\} \) with equality if \( \deg(c) + \deg(f) = \deg(e) + \deg(b) \). Thus (4PT) holds for all cases. \( \square \)

3 L-convex function on Euclidean building

In this section, we introduce L-convex function on Euclidean building. It will turn out in Section 4 that the \( \text{deg-Det} \) computation reduces to an L-convex function minimization on the Euclidean building for \( \text{SL}(K(t)^n) \). Our approach is lattice-theoretic. First we set up basic lattice terminologies. Then we introduce a uniform modular lattice, which is a lattice-theoretic counterpart of a Euclidean building of type A, and introduce L-convexity on it.

3.1 Lattice

A lattice is a partially ordered set \( \mathcal{L} \) such that every pair \( x,y \) of elements has the minimum common upper bound \( x \lor y \) and the maximum common lower bound \( x \land y \); the former is called the join and the latter is called the meet. The partial order is denoted by \( \preceq \), where \( x \prec y \) is meant as \( x \preceq y \) and \( x \neq y \). A totally ordered subset of \( \mathcal{L} \) is called a chain, which is written as \( x^0 \prec x^1 \prec \cdots \prec x^k \). The length of chain \( C \) is defined as \(|C| - 1 \). For \( x,y \in \mathcal{L} \) with \( x \preceq y \), the interval \([x,y]\) is defined as the set of elements \( z \) with \( x \preceq z \preceq y \). If \([x,y] = \{x,y\}\), we say that \( y \) covers \( x \). In this paper, we only consider lattices in which every chain of every interval has finite length. A sublattice \( \mathcal{L}' \) is a subset of \( \mathcal{L} \) such that \( x,y \in \mathcal{L}' \) implies \( x \land y, x \lor y \in \mathcal{L}' \). An interval is a sublattice. The opposite \( \mathcal{L}^* \) of lattice \( \mathcal{L} \) is the lattice obtained from \( \mathcal{L} \) by reversing the partial order of \( \mathcal{L} \). The direct product \( \mathcal{L} \times \mathcal{L}' \) of two lattices \( \mathcal{L}, \mathcal{L}' \) becomes a lattice by the product order: \( (x,x') \preceq (y,y') \iff x \preceq y \text{ and } x' \preceq y' \).
A modular lattice is a lattice $\mathcal{L}$ such that for every triple $x, y, z \in \mathcal{L}$ with $x \leq z$, it holds $x \lor (y \land z) = (x \lor y) \land z$. The opposite of a modular lattice is also a modular lattice. A useful criterion for modularity is given as follows. A valuation on a lattice $\mathcal{L}$ is a function $v : \mathcal{L} \to \mathbb{R}$ such that

- $v(x) + v(y) = v(x \land y) + v(x \lor y)$ for all $x, y \in \mathcal{L}$, and
- $v(x) < v(y)$ for all $x, y \in \mathcal{L}$ with $x < y$.

**Lemma 3.1** ([2]). If a lattice $\mathcal{L}$ admits a valuation, then $\mathcal{L}$ is a modular lattice.

A unit valuation is a valuation $v$ such that $v(x) = v(y) - 1$ provided $y$ covers $x$. A modular lattice (having the minimum element) is said to be complemented if every element is the join of atoms (= elements covering the minimum element).

**Example 3.2.** Let $\mathbb{F}$ be a skew field. The families $\mathcal{S}_R(\mathbb{F}^n)$ and $\mathcal{S}_L(\mathbb{F}^n)$ of vector subspaces of $\mathbb{F}^n$ are complemented modular lattices, where the partial order is the inclusion relation. The join and meet are given by $+$ and $\cap$, respectively. Also $X \mapsto \dim X$ is a unit valuation.

A function $f$ on lattice $\mathcal{L}$ is called submodular if it satisfies

$$f(x) + f(y) \geq f(x \land y) + f(x \lor y) \quad (x, y \in \mathcal{L}).$$

As was noticed in [18, 19, 23], MVSP is viewed as a submodular optimization on a modular lattice. For a matrix $A \in \mathbb{F}^{n \times n'}$ regarded as a bilinear form (2.1), define $r_A : \mathcal{S}_L(\mathbb{F}^n) \times \mathcal{S}_R(\mathbb{F}^{n'}) \to \mathbb{Z}$ by

$$r_A(X, Y) := \text{the rank of the restriction of } A \text{ to } X \times Y.$$ 

**Lemma 3.3** ([29]; also see [18, 19]). Let $A \in \mathbb{F}^{n \times n'}$ be a matrix over $\mathbb{F}$. Then $r_A$ is submodular on $\mathcal{S}_L(\mathbb{F}^n) \times \mathcal{S}_R(\mathbb{F}^{n'})$, i.e.,

$$r_A(X, Y) + r_A(X', Y') \geq r_A(X + X', Y \cap Y') + r_A(X \cap X', Y + Y').$$

The vanishing condition $A(X, Y) = \{0\}$ is equivalent to $r_A(X, Y) = 0$. By including $r_A$ in the objective as a penalty term, MVSP is formulated as an unconstrained submodular optimization over modular lattice $\mathcal{S}(\mathbb{K}^n) \times \mathcal{S}(\mathbb{K}^{n'})$. The approach by Hamada and Hirai [18] is based on this property.

### 3.2 Uniform modular lattice and Euclidean building

The ascending operator of a lattice $\mathcal{L}$ is a map $(\cdot)^+ : \mathcal{L} \to \mathcal{L}$ defined by

$$ (x)^+ := \bigvee \{y \in \mathcal{L} \mid y \text{ covers } x\} \quad (x \in \mathcal{L}). \quad (3.1) $$

A uniform modular lattice [22] is a modular lattice $\mathcal{L}$ such that the ascending operator is defined and is an automorphism on $\mathcal{L}$. Let $\mathcal{L}$ be a uniform modular lattice. The rank (= the length of a maximal chain) of $[x, (x)^+]$ is independent of $x$, and is called the uniform-rank of $\mathcal{L}$. The inverse of $(\cdot)^+$ is given by $x \mapsto$ the meet of elements covered by $x$. In particular, the opposite $\hat{\mathcal{L}}$ of $\mathcal{L}$ is also uniform modular. The product of two uniform modular lattices is also uniform modular.
Example 3.4. \( \mathbb{Z}^n \) becomes a lattice with respect to vector order \( \leq \), where \( x \lor y \) (componentwise maximum of \( x, y \)) and \( x \land y \) equals \( \min(x, y) \) (componentwise minimum of \( x, y \)). Now \( \mathbb{Z}^n \) is a uniform modular lattice, where \( z \mapsto \sum_{i=1}^n z_i \) is a valuation, the ascending operator is given by \( x \mapsto x + 1 \) for all one-vector \( 1 \), and the uniform-rank is equal to \( n \).

A \( \mathbb{Z}^n \)-skeleton of \( \mathcal{L} \) is a sublattice \( \Sigma \) such that \( \Sigma \) is isomorphic to \( \mathbb{Z}^n \), and the restriction of the ascending operator of \( \mathcal{L} \) to \( \Sigma \) is the same as the ascending operator of \( \Sigma \). A chain \( x_0 < x_1 < \cdots < x_m \) is said to be short if \( x_m \leq (x_0)^+ \).

Lemma 3.5 (\cite{22}). Let \( \mathcal{L} \) be a uniform modular lattice with uniform-rank \( n \).

(B1) For two short chains \( C, D \), there is a \( \mathbb{Z}^n \)-skeleton \( \Sigma \) containing them.

(B2) If two \( \mathbb{Z}^n \)-skeletons \( \Sigma, \Sigma' \) contain short chains \( C, D \), there is an order-preserving bijection from \( \Sigma \) to \( \Sigma' \) such that it is the identity on \( C \cup D \).

(B1) and (B2) are essentially the apartment axiom of Euclidean building of type A \cite{3}; see \cite{17}. The paper \cite{22} shows that the family of all short chains in a uniform modular lattice actually forms a Euclidean building of type A, and that every Euclidean building of type A is obtained in this way. An apartment system of \( \mathcal{L} \) is a family of \( \mathbb{Z}^n \)-skeletons such that a \( \mathbb{Z}^n \)-skeleton in (B1) can be chosen from the family. A \( \mathbb{Z}^n \)-skeleton in an apartment system is called an apartment.

Next we consider an important example of a uniform modular lattice arising from a skew field with discrete valuation. Let \( \mathbb{F} \) be a skew field, and let \( \mathbb{F}(t) \) be the skew field of rational functions. Let \( \mathbb{F}(t)^- \) be the ring of proper elements of \( \mathbb{F}(t) \). Consider \( n \)-product \( \mathbb{F}(t)^n \), which is regarded as a left \( \mathbb{F}(t)^- \)-module of row vectors as well as a right \( \mathbb{F}(t)^- \)-module of column vectors. Notice that \( \mathbb{F}(t)^- \) is a (right and left) principal ideal domain (PID). Then every submodule of free module \( \mathbb{F}(t)^n \) is free. Let \( \mathcal{L}_L(\mathbb{F}(t)^n) \) denote the family of all full-rank free \( \mathbb{F}(t)^- \)-submodules\(^1\) of \( \mathbb{F}(t)^n \), where \( \mathbb{F}(t)^n \) is regarded as a left \( \mathbb{F}(t)^- \)-module of row vectors. Let \( \mathcal{L}_R(\mathbb{F}(t)^n) \) be defined as the right analogue. By definition, an element \( L \in \mathcal{L}_R(\mathbb{F}(t)^n) \) is represented as \( \langle Q \rangle_L := \{ \lambda Q \mid \lambda \in (\mathbb{F}(t)^-)^n \} \) for a nonsingular matrix \( Q \) over \( \mathbb{F}(t) \). Similarly, an element \( L \in \mathcal{L}_R(\mathbb{F}(t)^n) \) is written as \( \langle P \rangle_R := \{ P \lambda \mid \lambda \in (\mathbb{F}(t)^-)^n \} \) for a nonsingular matrix \( P \) over \( \mathbb{F}(t) \). For \( L \in \mathcal{L}_L(\mathbb{F}(t)^n) \) or \( \mathcal{L}_R(\mathbb{F}(t)^n) \), define \( \deg L \) by

\[
\deg L := \deg \det P 
\]

(3.2)

for nonsingular matrix \( P \) with \( L = \langle P \rangle_L \) or \( \langle P \rangle_R \). This is well-defined; if \( \langle P \rangle_R = \langle P' \rangle_R \), then \( P' = PS \) for biproper matrix \( S \), and \( \deg \det P' = \deg \det P \) by Lemmas 2.8 and 2.11.

We give three lemmas on the family \( \mathcal{L}_R(\mathbb{F}(t)^n) \) below. They hold when \( R \) is replaced by \( L \). The first one is shown in \cite{22} for the case where \( \mathbb{F} \) is a field.

Lemma 3.6 (\cite{22}). \( \mathcal{L}_R(\mathbb{F}(t)^n) \) is a uniform modular lattices, where \( L \mapsto \deg L \) is a unit valuation, the uniform-rank is equal to \( n \), and the ascending operator is given by \( L \mapsto tL \).

\(^1\)In the literature of building, such a module is called a lattice. We do use this term for avoiding confusion.
We give in the appendix a proof by adapting the argument in \cite{22} for our non-commutative setting.

For an integer vector $z \in \mathbb{Z}^n$, recall that $(t^z)$ denotes the diagonal matrix with diagonals $t^{z_1}, t^{z_2}, \ldots, t^{z_n}$. For a nonsingular matrix $Q$, let $\Sigma_R(Q)$ denote the sublattice of $L_R(\mathbb{F}^n)$ consisting of $\langle (t^{z^t}) \rangle_R$ for all $z \in \mathbb{Z}^n$. Similarly, define $\Sigma_L(Q)$ by $\Sigma_L(Q) := \{ \langle (t^{z^t}) \rangle_L \mid z \in \mathbb{Z}^n \}$.

**Lemma 3.7** (see \cite{17} Chapter 19) for the commutative version. The family of sublattices consisting of $\Sigma_R(Q)$ for all nonsingular $Q \in \mathbb{F}(t)^{n \times n}$ forms an apartment system in $L_R(\mathbb{F}(t))$, where $z \mapsto \langle (t^{z^t}) \rangle_R$ is an isomorphism between $\mathbb{Z}^n$ and $\Sigma_R(Q)$.

The proof is given in the appendix. Next we study the lattice structure of interval $[L, (L)^+] = [L, tL]$, which is a complemented modular lattice. For $M \in L_R(\mathbb{F}(t)^n)$ with $L \subseteq M \subseteq tL$, quotient group $M/L$ becomes a right $\mathbb{F}$-vector space by $(u+L)\alpha := u\alpha + L$ for $\alpha \in \mathbb{F}$. For an $\mathbb{F}$-vector subspace $X$ of $tL/L$, define submodule $L \circ X$ of $tL$ by

$$L \circ X := \{ u \in tL \mid u + L \in X \}.$$  \hspace{1em} (3.3)

**Lemma 3.8.** Let $L \in L_R(\mathbb{F}(t)^n)$.

1. $tL/L$ is a right $\mathbb{F}$-vector space with dimension $n$.

2. $[L, tL]$ is isomorphic to $S_R(tL/L)$ by $M \mapsto M/L$ with inverse $X \mapsto L \circ X$.

3. For $X \in S_R(tL/L)$, it holds $\deg L \circ X = \deg L + \dim X$.

4. If $L = \langle P \rangle_R$, then $[L, tL]$ is given by

$$[L, tL] = \{ (PS(t^{1_{\leq k}}))_{R} \mid 0 \leq k \leq n, S \in \mathbb{F}^{n \times n} : \text{nonsingular} \},$$

where $1_{\leq k}$ denotes the $0,1$-vector such that the first $k$ elements are $1$ and others are zero.

**Proof.** (1). Suppose that $\{p_1, p_2, \ldots, p_n\}$ is a basis of $L$. Then $\{tp_1, tp_2, \ldots, tp_n\}$ is a basis of $tL$. We show that $\{tp_1 + L, tp_2 + L, \ldots, tp_n + L\}$ is a basis of $tL/L$. Every element $u \in tL$ is written as $u = \sum_{i=1}^{n} t^{p_i} \lambda_i$ for $\lambda_i \in \mathbb{F}(t)^-$. Here $\lambda_i$ is written as $\lambda_i = \lambda_1^0 + t^{-1} \lambda_i^0$ for $\lambda_1^0 \in \mathbb{F}$ and $\lambda_i^0 \in \mathbb{F}(t)^-$. This means that $u \in \sum_{i=1}^{n} t^{p_i} \lambda_1^0 + L$. Thus $\{tp_1 + L, tp_2 + L, \ldots, tp_n + L\}$ spans $tL/L$. We show linear independence. Suppose that $\sum_{i=1}^{n} t^{p_i} \alpha_i = u \in L$ for $\alpha_i \in \mathbb{F}$. Since $\{p_1, p_2, \ldots, p_n\}$ is a basis of $L$, it must be $\alpha_i t \in \mathbb{F}(t)^-$, and $\alpha_i = 0$.  

(2). It suffices to verify that $L \circ X$ is a full-rank free submodule. Indeed, $L \circ X$ is a free module containing $L$, and hence has rank $n$.

(3) follows from (2) and the fact that $\deg$ and $\dim$ are unit valuations.

(4). By the proof of (1), column vectors of $tP$ modulo $L$ become an $\mathbb{F}$-basis of $tL/L$. Therefore any vector subspace $X \subseteq tL/L$ is spanned by $\mathbb{F}$-linear combinations of column vectors of $tP$ modulo $L$. Thus, if $\dim X = k$, then for some nonsingular matrix $S$ over $\mathbb{F}$, $X$ is spanned by the first $k$ columns of $tPS$ (modulo $L$). Then $L \circ X = \langle (PS(t^{1_{\leq k}}))_{R} \rangle \mod L$ must hold. Indeed, $\supseteq$ is obvious, and the equality follows from $\deg \langle PS(t^{1_{\leq k}}) \rangle_{R} = \deg L + k = \deg L + \dim X$. \hfill $\square$
### 3.3 L-convex function

We first review L-convex functions on $\mathbb{Z}^n$; see [41, Chapter 7] for details. A function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ is called L-convex if it satisfies:

{(SUB) $g(x) + g(y) \geq g(\min(x, y)) + g(\max(x, y))$ for $x, y \in \mathbb{Z}^n$.}

{(LIN+1) There is $\alpha \in \mathbb{R}$ such that $g(x + 1) = g(x) + \alpha$ for $x \in \mathbb{Z}^n$.}

**Example 3.9.** The following function $g : \mathbb{Z}^n \to \mathbb{R}$ is known to be L-convex, where $\alpha_{ij} \in \mathbb{R}$:

$$g(x) = \max_{1 \leq i, j \leq n} x_i - x_j + \alpha_{ij} \quad (x \in \mathbb{Z}^n).$$

This L-convex function arises from the dual of minimum-cost network flow.

We are interested in minimization of an L-convex function. Note that $\alpha = 0$ in (LIN+1) is a necessary condition for the existence of a minimizer. The following optimality property is basic.

**Lemma 3.10 ([39]).** Let $g : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ be an L-convex function. A point $x \in \mathbb{Z}^n$ is a minimizer of $g$ if and only if

$$g(x) \leq g(x + u) \quad (u \in \{0, 1\}^n). \quad (3.4)$$

This property naturally leads to the following simple descent algorithm, called the steepest descent algorithm.

**Steepest Descent Algorithm (SDA($\mathbb{Z}^n$))**

**Input:** L-convex function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$.

**Output:** A minimizer of $g$.

**Step 0:** Choose $x^0 \in \mathbb{Z}^n$ with $g(x^0) < \infty$. Let $i := 0$.

**Step 1:** Find a minimizer $y$ of $g$ over $x^i + \{0, 1\}^n$.

**Step 2:** If $g(y) < g(x^i)$, then let $i \leftarrow i + 1$, $x^i \leftarrow y$, and go to step 1.

**Step 3:** Otherwise, output $x^i$ as a minimizer.

A point $y$ in Step 1 is called a steepest direction at $x^i$. The function $u \mapsto g(x + u)$ is submodular on Boolean lattice $\{0, 1\}^n$. Hence a steepest direction can be found by a submodular function minimization on the Boolean lattice.

An intriguing property of SDA is the following bound of the number of iterations.

**Theorem 3.11 ([33]).** Let $g : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ be an L-convex function, and let $k$ be the minimum $l_{\infty}$-distance between $x^0$ and minimizers $y \geq x^0$, i.e.,

$$k := \min \{ \|x^0 - y\|_{\infty} \mid y \text{ is a minimizer of } g \text{ with } y \geq x^0 \}. \quad (3.5)$$

In SDA, the $k$-th point $x^k$ is a minimizer of $g$. 

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Next we introduce L-convex functions on uniform modular lattice, and generalize the above properties. The argument goes in a straightforward way. Let \( \mathcal{L} \) be a uniform modular lattice with uniform-rank \( n \). A function \( g : \mathcal{L} \to \mathbb{R} \cup \{ \infty \} \) is called L-convex if it satisfies:

\[
\text{(SUB)} \quad g(x) + g(y) \geq g(x \wedge y) + g(x \vee y) \quad \text{for all} \quad x, y \in \mathcal{L}.
\]

\[
\text{(LIN)} \quad \text{There is } \alpha \in \mathbb{R} \text{ such that } g((x)^+) = g(x) + \alpha \quad \text{for all} \quad x \in \mathcal{L}.
\]

Fix an arbitrary apartment system of \( \mathcal{L} \). Every apartment of \( \mathcal{L} \) is a sublattice isomorphic to \( \mathbb{Z}^n \) and preserves the ascending operation. Thus L-convexity is characterized by L-convexity on each apartment.

**Lemma 3.12.** A function \( g : \mathcal{L} \to \mathbb{R} \cup \{ \infty \} \) is L-convex if and only if the restriction of \( g \) to every apartment \( \Sigma \) is L-convex, where \( \Sigma \) is identified with \( \mathbb{Z}^n \).

The optimality criterion (Lemma 3.10) is generalized as follows.

**Lemma 3.13.** Let \( g : \mathcal{L} \to \mathbb{R} \cup \{ \infty \} \) be an L-convex function. A point \( x \in \mathcal{L} \) is a minimizer of \( g \) if and only if

\[
g(x) \leq g(x + u) \quad (u \in [x, (x)^+]).
\]

**Proof.** Suppose that \( x \) is not a minimizer. Consider a minimizer \( y \) of \( g \). Choose an apartment \( \Sigma \) containing \( x \) and \( y \). By \( \Sigma \simeq \mathbb{Z}^n \) and Lemmas 3.10 and 3.12 there is \( u \in [x, x+1] \subseteq [x, (x)^+] \) with \( g(u) < g(x) \). \( \square \)

The steepest descent algorithm is formulated as:

**Steepest Descent Algorithm (SDA(\( \mathcal{L} \)))**

**Input:** L-convex function \( g : \mathcal{L} \to \mathbb{R} \cup \{ \infty \} \).

**Output:** A minimizer of \( g \).

**Step 0.** Choose \( x^0 \in \mathcal{L} \) with \( g(x^0) < \infty \). Let \( i := 0 \).

**Step 1.** Find a minimizer \( y \) of \( g \) over \( [x^i, (x^i)^+] \).

**Step 2.** If \( g(y) < g(x^i) \), then let \( i \leftarrow i + 1 \), \( x^i \leftarrow y \), and go to step 1.

**Step 3.** Otherwise, output \( x^i \).

The interval \( [x^i, (x^i)^+] \) is a complemented modular lattice, and \( g \) is submodular on \( [x^i, (x^i)^+] \). Hence Step 1 reduces to a submodular function minimization on complemented modular lattice. To generalize the iteration bound (Theorem 3.11), we introduce the \( l_\infty \)-distance on \( \mathcal{L} \). For two elements \( x, y \in \mathcal{L} \), choose an apartment \( \Sigma \) containing \( x, y \), identify \( \Sigma \) with \( \mathbb{Z}^n \), and define the \( l_\infty \)-distance \( d_\infty(x, y) \) by

\[
d_\infty(x, y) := \|x - y\|_\infty.
\]

One can see from the property (B2) in Lemma 3.5 that \( d_\infty \) is independent of the choice of the apartment.
Theorem 3.14. Let $g : \mathcal{L} \to \mathbb{R} \cup \{\infty\}$ be an $L$-convex function, and let $k$ be the minimum $l_\infty$-distance between $x^0$ and minimizers $y \geq x^0$, i.e.,
\[ k := \min \{ d_\infty(x^0, y) \mid y \text{ is a minimizer of } g \text{ with } y \geq x^0 \}. \] (3.8)

In SDA($\mathcal{L}$), the $k$-th point $x^k$ is a minimizer of $g$.

Proof. We may suppose that a minimizer exists. It suffices to show that the distance $k$ decreases by 1 on the update $x^0 \to x^1$. Let $y$ be a minimizer of $g$ with $y \geq x$ and $d_\infty(x^0, y) = k$. Choose apartment $\Sigma$ containing $y$ and short chain $\{x^0, x^1\}$. Identify $\Sigma$ with $\mathbb{Z}^n$. Then $x^0$ is not a minimizer of $g$ over $\Sigma$. Thus the update $x^0 \to x^1$ is viewed as the update of the first iteration of SDA($\mathbb{Z}^n$). Thus, by Theorem 3.11, the distance decreases on $\Sigma$ and on $\mathcal{L}$.

Let $\mathbb{F}$ be a skew field, and let $\mathbb{F}(t)$ be the skew field of rational functions. For (nonzero) $A \in \mathbb{F}(t)^{n \times n}$, define $\deg A : \mathcal{L}_L(\mathbb{F}(t)^n) \times \mathcal{L}_R(\mathbb{F}(t)^n) \to \mathbb{Z}$ by
\[ \deg(A(L, M)) := \max \{ \deg u \mid u \in A(L, M) \}. \] (3.9)
Notice that if $L = \langle P \rangle_L$ and $M = \langle Q \rangle_R$ then $\deg A(L, M)$ is equal to the maximum degree of an entry of $PAQ$. An affine analogue of Lemma 3.3 is the following:

Lemma 3.15. Let $A \in \mathbb{F}(t)^{n \times n}$. Then $\deg A$ is $L$-convex on $\mathcal{L}_L(\mathbb{F}(t)^n) \times \mathcal{L}_R(\mathbb{F}(t)^n)$.

Here $\mathcal{L}_L(\mathbb{F}(t)^n) \times \mathcal{L}_R(\mathbb{F}(t)^n)$ is viewed as a uniform modular lattice with ascending operator $(L, M) \mapsto (tL, t^{-1}M)$.

Proof. By Lemmas 3.12 and 3.7, it suffices to show $L$-convexity on apartment $\Sigma_L(P) \times \Sigma_R(Q)$ of $\mathcal{L}_L(\mathbb{F}(t)^n) \times \mathcal{L}_R(\mathbb{F}(t)^n)$. Suppose that row vectors of $P$ are $p_1, p_2, \ldots, p_n$ and column vectors of $Q$ are $q_1, q_2, \ldots, q_n$. Then apartment $\Sigma_L(P) \times \Sigma_R(Q)$ is isomorphic to $\mathbb{Z}^n \times \mathbb{Z}^n$ by $(z, w) \mapsto ((\langle t^i \rangle P)_L, \langle Q(t^{-w}) \rangle_R)$. Now $\deg A((\langle t^i \rangle P)_L, \langle Q(t^{-w}) \rangle_R) = \max_{1 \leq i \leq n} \deg (t^i p_i A t^{-w} q_j) = \max_{1 \leq i \leq n} \deg (z_i - w_j + \deg p_i A q_j)$. By Example 3.9, $(z, w) \mapsto \deg A((\langle t^i \rangle P)_L, \langle Q(t^{-w}) \rangle_R)$ is $L$-convex on $\mathbb{Z}^n \times \mathbb{Z}^n$. This means that $\deg A$ is $L$-convex on $\Sigma_L(P) \times \Sigma_R(Q)$.

4 Computing degree of determinant

The goal of this section is to establish a formula and algorithm for the degree of the Dieudonné determinant of a linear symbolic matrix. Let $A = A_0 + A_1 x_1 + \cdots + A_n x_n$ be a linear matrix over $\mathbb{K}(t)$. Now $A$ is viewed as a matrix over the skew field $\mathbb{K}(\langle x_1, x_2, \ldots, x_m \rangle)$ of rational functions over the free field $\mathbb{K}(\langle x_1, x_2, \ldots, x_m \rangle)$. As in the case of nc-rank, we first give an upper bound of $\deg \det A$. The following upper bound of $\deg \det A$ is observed by Murota [38] for $\deg \det A$, which was a basis of the combinatorial relaxation algorithm.

Lemma 4.1. For nonsingular matrices $P, Q$ over $\mathbb{K}(t)$, if $PA_0 Q$ is a proper matrix over $\mathbb{K}(t)$ for $i = 0, 1, 2, \ldots, m$, then $\deg \det A \leq -\deg \det P - \deg \det Q$.

Proof. $PA_0 Q$ is a proper matrix over $\mathbb{K}(\langle x \rangle)$ (t). Also $\det P = \det P$ and $\det Q = \det Q$. Thus the claim follows from Lemmas 2.6 and 2.11.
This gives rise to the following optimization problem (maximum vanishing submodule problem (MVMP)):

\[
\text{MVMP : Max. } \deg \det P + \deg \det Q \\
\text{s.t. } PA_i Q: \text{ proper } \quad (i = 0, 1, \ldots, m). \\
P, Q \in \mathbb{K}(t)^{n \times n} : \text{nonsingular.}
\]

Just as MVSP is formulated as an optimization over the lattice of vector subspaces of \(\mathbb{K}^n\) (see (2.2)), MVMP is reformulated as an optimization over the lattice of submodules of \(\mathbb{K}(t)^n\). Recall notions in Section 3.2. Then the above problem is rephrased as the following:

\[
\text{MVMP : Max. } \deg L + \deg M \\
\text{s.t. } \deg A_i(L, M) \leq 0 \quad (i = 0, 1, \ldots, m). \\
L \in \mathcal{L}_L(\mathbb{K}(t)^n), \ M \in \mathcal{L}_R(\mathbb{K}(t)^n).
\]

The following theorem states that this upper bound is tight for \(\deg \text{Det}\), which is an extension of the Fortine-Rautenauer formula (Theorem 3.1). The proof is given later.

**Theorem 4.2.** Let \(A = A_0 + A_1x_1 + A_2x_2 + \cdots + A_mx_m\) be an \(n \times n\) linear matrix over \(\mathbb{K}(t)\). Then \(\deg \text{Det} A\) is equal to the negative of the optimal value of MVMP.

We also give an algorithm to solve MVMP for a polynomial matrix \(A\).

**Theorem 4.3.** Let \(A = A_0 + A_1x_1 + A_2x_2 + \cdots + A_mx_m\) be an \(n \times n\) linear matrix over \(\mathbb{K}[t]\). MVMP can be solved in \(O(\ell n \gamma + \ell^2 mn^{\omega+2})\) time, and in \(O((\ell-\alpha_n)\gamma + (\ell-\alpha_n)^2 mn^{\omega})\) time if \(A\) is nonsingular, where

- \(\gamma\) is the time complexity of solving MVSP for an \(n \times n\) linear matrix over \(\mathbb{K}\),
- \(\ell(= \alpha_1)\) is the maximum degree of entries in \(A\), and
- \(\alpha_n\) is the minimum degree of the Smith-McMillan form of \(A\) in \(\mathbb{K}(\langle x \rangle)(t)\), and
- \(\omega\) is the exponent of the time complexity of matrix multiplication of \(n \times n\) matrices.

Here we make a strong assumption that arithmetic operations on \(\mathbb{K}\) can be done in constant time. The bit-length consideration of our algorithm is left to future work. The above results are proved in the next subsections.

**Remark 4.4.** The maximum degree of subdeterminants of \(n \times n'\) linear matrix \(A\) can be computed by combining the above result with the valuated-matroid property (Proposition 2.12). Indeed, consider an expanded matrix \(\hat{A} := (I A)\) and the valuated matroid \(\omega\) obtained from column vectors of \(A\). Then the maximum degree of subdeterminants of \(A\) is equal to the maximum value of \(\omega(X)\) over \(X \subseteq \{1, 2, \ldots, n + n'\}\) with \(|X| = n\). By the greedy algorithm [9] (see also [10, Section 5.2.4]), it is obtained by \(O((n + n')^2)\) evaluations of \(\omega\), where the evaluation is done by the algorithm in Theorem 4.3.
4.1 Optimality

Here we establish an optimality criterion for MVMP, and prove Theorem 4.2. We first note that MVMP can be viewed as L-convex function minimization on a uniform modular lattice:

\[
\begin{align*}
\text{Min. } & -\deg L - \deg M + \sum_{i=0}^{\infty} \cdot \deg A_i(L, M) \\
\text{s.t. } & (L, M) \in L(L(\mathbb{K}(t)^n)) \times L_R(\mathbb{K}(t)^n),
\end{align*}
\tag{4.1}
\]

where \(\infty \cdot c := \infty\) if \(c > 0\), and 0 otherwise. By Lemma 3.15, \(\infty \cdot \deg A_i\) is L-convex.

This fact and Lemma 3.10 motivate us to consider the restriction of MVMP to interval \([\hspace{1em} (L, M) \hspace{1em}, \hspace{1em} (L, M) + 1 \hspace{1em}]\) = \([\hspace{1em} (L, M) \hspace{1em}, \hspace{1em} (tL, tM - 1) \hspace{1em}]\) for a feasible solution \((L, M)\) of MVMP.

Since \(\deg A_i(L, M) \leq 0\), it holds \(\deg A_i(L', M') \leq 1\) for \((L', M') \in [(L, M), (L, M) + 1]\); see Lemma 3.8 (4). Therefore we may consider the coefficient of \(t\) in \(A_i(tu, v)\) for \(u \in L, v \in M\). For each \(A_i\), define a bilinear map \(A_{L,M}^i: tL/L \times M/t - 1 M \rightarrow \mathbb{K}\) by

\[
A_{L,M}^i(tu + L, v + t - 1 M) := A_i(u, v)^0 = \text{the coefficient of } t \text{ in } A_i(tu, v) \quad (u \in L, v \in M).
\]

This is well-defined (by \(A(L, M) \subseteq \mathbb{K}(t)^-\)). Define MVSP\(_{L,M}^L\) by

\[
\text{MVSP}_{L,M}^L: \quad \text{Max. } \dim X + \dim Y \\
\text{s.t. } & A_{i}^L(X, Y) = \{0\} \quad (i = 0, 1, \ldots, m), \\
& X \in S(tL/L), Y \in S(M/t - 1 M).
\]

Recall notation \(L \circ X\) for \(X \in S(tL/L)\). Also, for \(Y \in S(M/t - 1 M)\), define \(M \bullet Y := t - 1 M \circ Y\). Then \(\deg A(L \circ X, M \bullet Y) \leq 0\) if and only if \(A_{i}^{L,M}(X, Y) = \{0\}\). Thus, by Lemma 3.8, MVSP\(_{L,M}^L\) is the restriction of MVMP to \([\hspace{1em} (L, M) \hspace{1em}, \hspace{1em} (L, M) + 1 \hspace{1em}]\):

**Proposition 4.5.** Suppose that \((X, Y)\) is a feasible solution of MVSP\(_{L,M}^L\). Then \((L \circ X, M \bullet Y)\) is feasible to MVMP with \(\deg L \circ X + \deg M \bullet Y = \deg L + \deg M + (\dim X + \dim Y - n)\).

Suppose that \(L\) and \(M\) are given as \(L = \langle P \rangle_L\) and \(M = \langle Q \rangle_R\). Then MVSP\(_{L,M}^L\) is also written as

\[
\text{MVSP}_{P,Q}^L: \quad \text{Max. } \dim X + \dim Y \\
\text{s.t. } & (PA_iQ)^0(X, Y) = \{0\} \quad (i = 0, 1, \ldots, m), \\
& X \in S(\mathbb{K}^n), Y \in S(\mathbb{K}^n).
\]

Now we have the following optimality criterion.

**Proposition 4.6.** The following conditions are equivalent:

1. \((L, M)\) is optimal to MVMP.
2. The optimal value of MVSP\(_{L,M}^L\) is at most \(n\).
(3) The linear matrix
\[
(PAQ)^0 = (PA_0Q)^0 + (PA_1Q)^0x_1 + \cdots + (PA_mQ)^0x_m
\]
is nonsingular on \( \mathbb{K}(\langle x \rangle) \).

(4) \( \det \det A \) is equal to \(-\deg \det P - \deg \det Q\).

Proof. (4) \( \Rightarrow \) (1) follows from Lemma 4.1. (2) \( \Rightarrow \) (3) follows from Theorem 2.1. (3) \( \Rightarrow \) (4) follows from Lemma 2.11. (1) \( \Rightarrow \) (2) follows from Proposition 4.5. \( \Box \)

Proof of Theorem 4.2. We may assume that MVMP is bounded. Take a feasible solution \((L, M) = (\langle P \rangle_L, \langle Q \rangle_R)\). If \((PAQ)^0\) is nonsingular, then \(\deg \det A = -\deg L - \deg M\). Otherwise, we obtain another feasible solution \((L', M')\) of MVMP with \(\deg L' + \deg M' > \deg L + \deg M\). Let \((L, M) \leftarrow (L', M')\). Repeating this procedure finitely many times, we obtain \(\deg \det A = -\deg L - \deg M\).

4.2 Steepest descent algorithm

The above proof of Theorem 4.2 is algorithmic, and naturally leads to the following algorithm, which can be viewed as the steepest descent algorithm for the L-convex function in (4.1).

Steepest Descend Algorithm for \( \deg \det \) (coordinate-free version)

Input: Linear matrix \( A = A_0 + A_1x_1 + \cdots + A_mx_m \) over \( \mathbb{K}(t) \).

Output: The degree \( \deg \det A \) of Dieudonné determinant of \( A \).

Step 0: Choose a feasible solution \((L, M)\) of MVMP.

Step 1: Solve MVSP\(_{L,M}\) to obtain an mv-subspace \((X, Y)\).

Step 2: If \(\dim X + \dim Y \leq n\), then \((L, M)\) is optimal to MVMP and output \(\deg \det A := -\deg L - \deg M\).

Step 3: Let \((L, M) \leftarrow (L \circ X, M \bullet Y)\), and go to step 1.

In step 1, the algorithm chooses an mv-subspace \((X, Y)\), and therefore \((L \circ X, M \bullet Y)\) is actually a steepest direction at \((L, M)\). The input is allowed to be a linear rational matrix \( A \). If \( A \) is nonsingular, then the algorithm outputs the correct answer after finitely many iterations; the exact number of iterations will be given in Lemma 4.8 In the case of singular \( A \), we do not know when the algorithm should output \(-\infty\).

We next consider the case of a linear polynomial matrix, and prove Theorem 4.3. We specialize the above algorithm with a matrix form.

Steepest Descend Algorithm for \( \deg \det \) (matrix version)

Input: Linear matrix \( A = A_0 + A_1x_1 + \cdots + A_mx_m \) over \( \mathbb{K}[t] \), where \( \ell \) is the maximum degree of entries of \( A \).

Output: The degree \( \deg \det A \) of Dieudonné determinant of \( A \).
Step 0: Let $A_i \leftarrow A_i t^{-\ell}$ for $i = 0, 1, 2, \ldots, m$, and $D^* \leftarrow n\ell$.

Step 1: Solve MVSP in the matrix form

$$\begin{align*}
\text{Max.} \quad & r + s \\
\text{s.t.} \quad & SA^0_t T \text{ has a zero submatrix in first } r \text{ rows and first } s \text{ columns,} \\
& S, T \in \mathbb{K}^{n \times n} : \text{nonsingular,}
\end{align*}$$

and obtain optimal matrices $S, T$.

Step 2: If the optimal value $r + s$ is at most $n$, then output $\deg \det A = D^*$.

Step 3: Let $A_i \leftarrow (t^{1 \leq r}) SA_i T(t^{-1 \geq s})$ for $i = 0, 1, 2, \ldots, m$, and $D^* \leftarrow D^* - (r + s - n)$. If $D^* < 0$, then output $\deg \det A = -\infty$. Go to step 1 otherwise.

Here $1_{> s} := 1 - 1_{s \leq s}$. In step 0, we suppose feasible module $(L, M) = (\langle P \rangle_L, \langle Q \rangle_R) = (I, t^{-1})$ with $\deg L + \deg M = -D^* = -n\ell$. The update in step 3 can be understood as the movement from $(L, M) = (\langle P \rangle_L, \langle Q \rangle_R)$ to a steepest direction $((t^{1 \leq r}) SP)_L, \langle QT(t^{-1 \geq s}) \rangle_R)$ in $[(L, M), (tL, t^{-1}M)]$; see Lemma 3.8 (4). Since the input is a polynomial matrix, $\deg \det A$ is guaranteed to be nonnegative if $A$ is nonsingular (Lemma 4.1). Also $D^*$ is always an upper bound of $\deg \det A$. Thus $D^* < 0$ in step 3 implies $\deg \det A = -\infty$.

The following modification of step 3 is natural.

Step 3': Choose the minimum integer $\kappa \geq 1$ such that $((t^{1 \leq r}) SA(t^{-1 \geq s}))^0$ has nonzero submatrix in first $r$ rows and $s$ columns. Let $A_i \leftarrow (t^{1 \leq r}) SA_i R(t^{-1 \geq s})$ for $i = 0, 1, 2, \ldots, m$, and let $D^* := D^* - \kappa(r + s - n)$. If $\kappa$ is unbounded or $D^* < 0$, then output $\deg \det A = -\infty$. Go to step 1 otherwise.

The coordinate-free formulation cannot incorporate this modification, since it depends on basis matrices for the current $(L, M)$ and the mv-subspace in step 2. The modified SDA using step 3' is considered in Section 5.

We next estimate the number of iterations by using $L$-convexity (Theorem 3.14). For this purpose, we consider the master problem $\overline{\text{MVMP}}$ of MVMP:

$$\begin{align*}
\overline{\text{MVMP}} : \quad & \text{Max.} \quad \deg L + \deg M \\
& \text{s.t.} \quad \deg A(L, M) \leq 0, \\
& \quad L \in \mathcal{L}_L(\mathbb{K}(\langle x \rangle)(t)^n), \ M \in \mathcal{L}_R(\mathbb{K}(\langle x \rangle)(t)^n),
\end{align*}$$

where the linear matrix $A$ is regarded as bilinear form on $\mathbb{K}(\langle x \rangle)(t)^n \times \mathbb{K}(\langle x \rangle)(t)^n$. Solving $\overline{\text{MVMP}}$ is theoretically easy. Choose biproper matrices $P, Q$ so that $PAQ$ is the Smith-McMillan form. Now $PAQ$ is the diagonal matrix $(t^{\alpha})$ for $\alpha \in \mathbb{Z}$. Consider $L^* := (\langle t^{-\alpha} \rangle P)_L$ and $M^* := (\langle Q(t^{-\alpha}) \rangle)_R$ for $\alpha^* := \max(0, \alpha)$ and $\alpha^- := \min(0, \alpha)$. Then $(L^*, M^*)$ is feasible to $\overline{\text{MVMP}}$. Also $\deg L^* + \deg M^* = -\sum_{i=1}^n \alpha_i = -\deg \det A$, and hence $(L^*, M^*)$ is an optimal solution.

As for MVSP embedded to $\overline{\text{MVSP}}$, $\text{MVMP}$ is embedded to $\overline{\text{MVMP}}$ by scalar extension $(L, M) \mapsto (\mathbb{K}(\langle x \rangle)(t)^- \otimes L, M \otimes \mathbb{K}(\langle x \rangle)(t)^-)$. In particular $\text{MVMP}$ is an exact inner approximation of $\overline{\text{MVMP}}$. We further show that the steepest descent algorithm for MVMP is viewed as that for $\overline{\text{MVMP}}$. Let $(L, M) = (\langle P \rangle_L, \langle Q \rangle_R)$ be a feasible
solution of MVMP and of MVMP. Consider MVSP \( P,Q \), and then MVMP \( P,Q \), which is given by

\[
\text{MVSP}^{P,Q} : \quad \text{Max.} \quad \dim X + \dim Y \\
\text{s.t.} \quad (PAQ)^0(X,Y) = \{0\}, \quad X \in \mathcal{S}_L(\mathbb{K}(\langle x \rangle)^n), \quad Y \in \mathcal{S}_R(\mathbb{K}(\langle x \rangle)^n).
\]

By Lemma 2.3 any mv-subspace of MVSP \( P,Q \) is also an mv-subspace of MVSP \( P,Q \) = MVSP \( P,Q \). Thus we have:

**Lemma 4.7.** A steepest direction for MVMP at \((L, M)\) is also a steepest direction for MVMP at \((L, M)\).

We next show the exact number of iterations of SDA, where by the number of iterations we mean the number of updates of \((L, M)\) (or \(A\)).

**Lemma 4.8.** If \(A\) is nonsingular, then the number of iterations of the steepest descent algorithm is equal to \(\alpha_1 - \alpha_n\), where \(\alpha_1\) and \(\alpha_n\) are the maximum and minimum degrees, respectively, of the Smith-McMillan form of \(A\).

**Proof.** Notice that \(\ell = \alpha_1\). By the initial update \(A \leftarrow At^{-\alpha_1}\), we can assume that \(\alpha_1 = 0\) and the initial point \((L, M)\) is \((\langle I \rangle_L, \langle I \rangle_R)\). An optimal solution \((L^*, M^*)\) of MVMP with \((L^*, M^*) (\geq) (L, M)\) is given by \((\langle (t^{-\alpha}) \rangle_L, \langle I \rangle_R)\). Thus the \(\ell\)-distance from initial point to optimal solutions is at most \(\alpha_n\). By Theorem 3.14 the number of iteration is at most \(\alpha_n\). The algorithm terminates when \(\alpha_n \leq 0\). Thus it suffices to show that \(\alpha_n(< 0)\) increases by at most one in the update \(A \leftarrow (t^{1-\alpha})SAT(t^{-1-\alpha})\). This follows from the observation that \(\delta_n\) increases by \(r + s - n\) and \(\delta_{n-1}\) increases by \(r + s - n\) or \(r + s - 1 - n\).

**Proof of Theorem 4.3** We verify the time complexity of SDA (matrix form). After the initialization (step 0), each matrix \(A_i\) is kept in the form

\[
A_i^0 + A_i^{(1)}t^{-1} + \cdots + A_i^{(d)}t^{-d}, \quad (4.2)
\]

where \(A_i^{(j)}\) is a matrix over \(\mathbb{K}\), and \(d := \ell\). Step 1 can be done in \(\gamma\) time. The update of expression (4.2) in Step 3 can be done in \(O(dmn\omega)\) time. The total number of iterations is \(nl\) if \(A\) is singular, and \(\ell - \alpha_n\) if \(A\) is nonsingular (Lemma 4.8). In each iteration, \(d\) increases by one. Thus the total is \(O((\ell - \alpha_n)\gamma + (\ell - \alpha_n)^2mn\omega)\) if \(A\) is singular, and is \(O((\ell - \alpha_n)\gamma + (\ell - \alpha_n)^2mn\omega)\) if \(A\) is nonsingular.

### 4.3 Combinatorial relaxation algorithm

The steepest descent algorithm changes basis matrices \(P, Q\) in each iteration. It is a natural idea to optimize on apartment \(\Sigma_L(P) \times \Sigma_R(Q)\) in each iteration. This modification can be expected to reduce matrix operations, and leads to the following algorithm, which is viewed as a generalization of the combinatorial relaxation algorithm previously developed for deg det \(\text{deg det} \ [30, 31, 32, 37, 38]\).

**Combinatorial Relaxation Algorithm for deg Det**
**Input:** Linear matrix \( A = A_0 + A_1x_1 + \cdots + A_mx_m \) over \( \mathbb{K}[t] \), where \( \ell \) is the maximum degree of entries of \( A \).

**Output:** The degree \( \deg \det A \) of Dieudonné determinant of \( A \)

**Step 0:** Let \( A_i \leftarrow A_it^{-\ell} \) for \( i = 0, 1, 2, \ldots, m \), and \( D^* \leftarrow n\ell \).

**Step 1:** If \( A^0 \) is nonsingular, then output \( D^* = \deg \det A \).

**Step 2:** Find nonsingular matrices \( S, T \in \mathbb{K}^{n \times n} \) such that each \( SA_iT \) has a zero submatrix in first \( r \) rows and first \( s \) columns with \( r + s > n \).

**Step 3:** Solve the following problem:

\[
\text{MVMP}(\Sigma) : \quad \text{Max.} \quad \sum_i p_i - \sum_i q_i \\
\text{s.t.} \quad (t^p)SAT(t^{-q}) \text{ is proper,} \\
p, q \in \mathbb{Z}^n_+
\]

to obtain optimal vectors \( p, q \in \mathbb{Z}^n_+ \). Let \( A_i \leftarrow (t^p)SA_iT(t^{-q}) \) for \( i = 0, 1, 2, \ldots, m \), and let \( D^* \leftarrow D^* - \sum_i p_i + \sum_i q_i \). If \( D^* < 0 \) or MVMP(\( \Sigma \)) is unbounded, then output \( \deg \det A := -\infty \). Otherwise, go to step 1.

The condition \( r + s > n \) in step 1 guarantees that \( D^* \) strictly decreases. Hence the algorithm terminates after \( \ell n \) steps. If the current solution \( (L, M) \) is given by \( \langle P \rangle_L \langle Q \rangle_R \), then MVMP(\( \Sigma \)) is viewed as the restriction of MVMP to apartment \( \Sigma_L(SP) \times \Sigma_R(QT) \).

Moreover, MVSP(\( \Sigma \)) is the dual of the weighted matching problem in a bipartite graph. Indeed, the condition that \( (t^p)SAT(t^q) \) is proper is written as

\[
p_i - p_j + d_{ij} \leq 0 \quad (1 \leq i, j \leq n), \tag{4.3}
\]

where \( d_{ij}(\leq 0) \) is the maximum degree of the \((i, j)\)-entry of SAT. Thus MVMP(\( \Sigma \)) is the dual of the following weighted perfect matching problem:

\[
\text{Max.} \quad \sum_{i=1}^n d_{i\sigma(i)} \\
\text{s.t.} \quad \sigma : \text{permutation on } \{1, 2, \ldots, n\},
\]

which can be efficiently solved by the Hungarian method to obtain optimal solution \( p, q \) of the dual.

The combinatorial relaxation algorithm is seemingly more efficient than the steepest descent algorithm, although we do not know any nontrivial iteration bound. The meaning of “relaxation” is explained as follows. Step 3 can be viewed as a relaxation process that linear matrix \( A \) is “relaxed” into another linear matrix \( \hat{A} \) by replacing each leading term \( a_{ij}t^{d_{ij}} \) (\( a_{ij} \in \mathbb{K} \)) of \( A \) with \( x_{ij}t^{d_{ij}} \) for new variable \( x_{ij} \). The optimal value of MVMP(\( \Sigma \)) is the negative of \( \deg \det \hat{A} \), and \( \deg \det A \leq \deg \det \hat{A} \). Step 1 tests whether the relaxation is tight or not.
5 Linear symbolic matrix with rank-1 summands

In this section, we study a class of linear matrices $A = A_0 + A_1 x_1 + \cdots + A_m x_m$ for which $\deg \det A = \det \text{Det} A$ holds. In the case of (nc-)rank, Lovász [30] showed that if each summand $A_i$ is a rank-1 matrix, then the rank of $A$ is given by MVSP, i.e., rank $A = \text{nc-rank } A$. Ivanov, Karpinski, and Saxena [25] extended this result to the case where each $A_i$ other than $A_0$ has rank one.

Theorem 5.1 ([25]). Let $A = A_0 + A_1 x_1 + \cdots + A_m x_m$ be a linear matrix over field $\mathbb{K}$. If $A_1, A_2, \ldots, A_m$ are rank-1 matrices, then rank $A = \text{nc-rank } A$.

We remark that the rank computation of such a matrix reduces to linear matroid intersection [36, 44].

Remark 5.4. Observe from the Fortine-Rautenauer formula (Theorem 2.1) that the nc-rank $A$ is a property of the matrix vector subspace $A \subseteq \mathbb{K}^{n \times n'}$ spanned by $A_0, A_1, \ldots, A_m$ over $\mathbb{K}$. Therefore, rank $A = \text{nc-rank } A$ still holds if the matrix subspace $A' \subseteq \mathbb{K}^{n \times n'}$ spanned by $A_1, \ldots, A_m$ admits a rank-1 basis $B_1, B_2, \ldots, B_m$. Indeed, the constraint $A_i(X, Y) = \{0\}$ in MVSP can be replaced by $B_i(X, Y) = \{0\}$. See [16, 24] for the rank computation of such a linear matrix with a hidden rank-1 basis.

Remark 5.5. An important example of a linear matrix with possibly rank $< \text{nc-rank}$ is a skew-symmetric linear matrix $A = \sum_{i=1}^m x_i A_i$ with rank-2 skew-symmetric summands $A_i$. The problem of computing the (usual) rank of such a matrix is a generation of the nonbipartite matching problem, and is equivalent to the linear matroid parity problem; see [36]. Recently Iwata and Kobayashi [28] developed a polynomial time algorithm for the weighted linear matroid parity problem by considering deg det and using

Lemma 5.2. $\deg \det A \leq \det \text{Det } A$.

Proof. Let $(P)_L, (Q)_R$ be an optimal solution for MVMP. By Theorem 4.2 it holds $\deg \text{Det } A = -\deg \det P - \deg \det Q$. Now $PAQ$ is also a proper matrix over $\mathbb{K}(x)(t)$. By Lemma 2.11 we have $\deg \det A \leq -\deg \det P - \deg \det Q = \deg \text{Det } A$.

Theorem 5.3. Let $A = A_0 + A_1 x_1 + \cdots + A_m x_m$ be a linear matrix over $\mathbb{K}(t)$. If $A_1, A_2, \ldots, A_m$ are rank-1 matrices, then $\deg \det A = \deg \text{Det } A$.

Proof. Consider an optimal module $(P)_L, (Q)_R$ for MVMP. By Proposition 4.6, the linear matrix $(PAQ)^0$ is nonsingular as a matrix over $\mathbb{K}(\langle x \rangle)$. Notice that each $(PA_iQ)^0$ for $i = 1, 2, \ldots, m$ has rank one. Indeed, $PA_iQ$ is written as $t^c u^d v^\top$ for $u, v \in (\mathbb{K}(t)')^n$ with $c + d \leq 0$. Then $(PA_iQ)^0 = u^0(v^0)^\top$ if $c + d = 0$ and zero if $c + d < 0$. By Theorem 5.1, the linear matrix $(PAQ)^0$ is also nonsingular as a matrix over $\mathbb{K}(x)$. By Lemma 2.11 we have $\deg \det A = -\deg \det P - \deg \det Q = \deg \text{Det } A$.

Remark 5.6. If each summand $A_i$ is a rank-1 matrix, then the rank of $A$ is given by MVSP, i.e., rank $A = \text{nc-rank } A$. Ivanov, Karpinski, and Saxena [25] extended this result to the case where each $A_i$ other than $A_0$ has rank one.

Lemma 5.7. For $i = 1, 2, \ldots, m$, the problem of computing the (usual) rank of such a matrix is a generation of the nonbipartite matching problem, and is equivalent to the linear matroid parity problem; see [36]. Recently Iwata and Kobayashi [28] developed a polynomial time algorithm for the weighted linear matroid parity problem by considering deg det and using
the idea of the combinatorial relaxation method. It is an interesting future direction to
refine our non-commutative framework for skew-symmetric linear matrices to capture
nonbipartite matching and its generalizations.

5.1 Some classical examples in combinatorial optimization

As mentioned in the introduction, some of classical combinatorial optimization problems
are formulated as the computation of the degree of the determinant of a linear matrix
with the rank-1 property. Here we consider representative three examples (bipartite
matching, linear matroid greedy algorithm, linear matroid intersection), and explain
how the steepest descent algorithm works on these problems. This gives some new
insights on classical algorithms in combinatorial optimization.

For a subset \( J \subseteq \{1, 2, \ldots, n\} \), let \( \mathbb{Q}^J \subseteq \mathbb{Q}^n \) denote the coordinate subspace spanned
by unit vectors \( e_i \) for \( i \in J \), and let \( 1_J := \sum_{i \in J} e_i \).

5.1.1 Bipartite matching

Let \( G = (U, V; E) \) be a bipartite graph with color classes \( U, V \). Vertices of \( U \) (resp.
\( V \)) are numbered as 1, 2, \ldots, \( n \) (resp. 1, 2, \ldots, \( m \)). As mentioned in the introduction, the maximum size \( \nu(G) \) of a matching of \( G \) is written as the rank of an \( n \times m \) linear
matrix \( A = \sum_{e=ij \in E} x_e E_{ij} \), where \( E_{ij} \) is the matrix having 1 at \((i, j)\)-entry and zero at
others, and \( x_e \) (\( e \in E \)) are variables. Each \( E_{ij} \) of \( A \) has rank 1. It holds that rank \( A = \text{nc-rank} A \). By Theorem 2.1, \( \nu(G) \) is equal to \( n + m \) minus the dimension of an \( mV\)-subspace \( (X, Y) \). Observe that any feasible subspace \( (X, Y) \) of MVSP is of the form of \((\mathbb{Q}^J, \mathbb{Q}^K)\) for \( J \subseteq \{1, 2, \ldots, n\}, K \subseteq \{1, 2, \ldots, m\} \) such that there is no edge between \( J \) and \( K \), i.e., \( J \cup K \) is a stable set of \( G \), and is the complement of a vertex cover. Thus
Theorem 2.1 is nothing but König’s formula for the maximum matching.

Next we consider the weighted situation. Suppose \(|U| = |V| = n \) for simplicity, and
that each edge \( e \in E \) has weight \( c_e \in \mathbb{Z} \). Consider a linear matrix \( A := \sum_{e=ij \in E} t^e x_e E_{ij} \) over \( \mathbb{Q}(t) \). Then the maximum weight of a perfect matching of \( G \) is equal to \( \text{deg} \text{det} A \), and is equal to \( \text{deg} \text{Det} A \) (by Theorem 5.3). We explain how the steepest descent
algorithm works in this case. We use the modified step 3'. Suppose for explanation
that \( c_e \leq 0 \) for each \( e \in E \). Linear matrix \( A^0 \) corresponds to the subgraph \( G^0 \) consisting
of edges with \( c_e = 0 \). A steepest direction is given by \((\mathbb{Q}^J, \mathbb{Q}^K)\) for a maximum stable
set \( J \cup K \) of \( G^0 \). In step 3', \( \kappa \) is chosen as the maximum of \(-c_e(> 0)\) for edges \( e \in E \)
belonging to \( J \cup K \). Then \( A \) is updated to \((t^{\kappa + 1})A(t^{-\kappa(1-\kappa)}) \). SDA repeats this process,
which is viewed as a cut-canceling algorithm. The resulting optimal solution is a form
of \((\langle p \rangle, q)\) for \( p, q \in \mathbb{Z}^n \). Here vectors \( p, q \) are dual optimal solutions of the LP-
formulation of the weighted matching problem. If we always choose a maximum stable
set \( J \cup K \) with maximal \( J \) in each iteration, then SDA coincides with the Hungarian
method. Indeed, \( J \cup K \) is the complement of the reachable subset in the residual graph
of a maximum matching in \( G^0 \).

5.1.2 Maximum weight base in linear matroid

Let \( a_1, a_2, \ldots, a_m \) be \( n \)-dimensional vectors of \( \mathbb{Q}^n \). Consider \( m \) variables \( x_1, x_2, \ldots, x_m \),
and linear matrix \( A = \sum_{i=1}^m x_i a_i a_i^\top \). Then rank \( A = \text{nc-rank} \ A = \text{rank} (a_1 \ a_2 \ \cdots \ a_m) \).

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Let $W \subseteq \mathbb{Q}^n$ be the vector space spanned by $a_1, a_2, \ldots, a_m$. Then $(W^\perp, \mathbb{R}^n)$ is an mv-subspace, where $W^\perp$ denotes the orthogonal subspace of $W$.

As in Section 5.1.1, consider the weighted situation. Let $c_i \in \mathbb{Z}$ be the weight on $a_i$ for each $i$. Consider linear matrix $A = \sum_{i=1}^m t^i x_i a_i a_i^\top$. Then $\deg \det A = \deg \det A$ is equal to the maximum of $\sum_{i \in B} c_i$ over all $B \subseteq \{1, 2, \ldots, m\}$ such that $\{a_i \mid i \in B\}$ forms a basis of $\mathbb{Q}^n$. Namely $\deg \det A$ is equal to the maximum weight of a base of the matroid represented by vectors $a_1, a_2, \ldots, a_m$.

In this case, the steepest descent algorithm is viewed as the greedy algorithm. Suppose that each $c_i$ is nonpositive. Then $A^0$ is the linear matrix $\sum_{i \in I_0} x_i a_i a_i^\top$, where $I_0$ is the set of indices $i$ with $c_i = 0$. Consider the subspace $W_1$ spanned by $a_i$ $(i \in I_0)$. Then $(W_1^\perp, \mathbb{Q}^n)$ is an mv-subspace. Consider a nonsingular matrix $Q \in \mathbb{Q}^{n \times n}$ such that the first $k_1$ rows form a basis of $W_1$. In step 3, $A$ is updated as $(t^{1 \leq k_1})QA$, or feasible module $(L, M)$ moves from $((I)_L, (I)_R)$ to $((t^{1 \leq k_1})Q)_L, (I)_R)$. The exponent of term $t^{c_i}x_iQAa_i a_i^\top$ increases if and only if $a_i$ does not belong to $W_1$. Thus, in step 3, SDA can augment $A$ as $(t^{1 \leq k_1})QA$ until $c_i + \alpha_1$ for some $a_i \not\in W_1$ becomes zero. Then $I_0$ increases, and the next subspace $W_2$ spanned by $a_i$ $(i \in I_0)$ increases. Consequently $W_2^\perp \subseteq W_1^\perp$. We can modify $Q$ so that it also includes a basis of $W_2^\perp$. SDA moves $(L, M)$ to $((t^{1 \leq k_1}+\alpha_2^{1 \leq k_2})Q)_L, (I)_R)$, and obtain $W_3^\perp \subseteq W_2^\perp$ as above. Repeat the same process. Eventually SDA reaches an optimal module $((t^{1 \leq k_1}+\alpha_2^{1 \leq k_2})Q)_L, (I)_R)$, where $Q$ contains bases of vector spaces $W_1^\perp \supset W_2^\perp \supset \cdots \supset W_h^\perp$. This process simply chooses vectors $a_i$ from largest weights. It is nothing but the matroid greedy algorithm, where we need no explicit computation of $Q$. The obtained $\alpha_k$ can be interpreted as an optimal dual solution of the LP-formulation of the maximum weight base problem, where $\alpha_k$ is the dual variable corresponding to the flat $\{i \mid a_i \in W_k\}$.

### 5.1.3 Linear matroid intersection

In addition to $a_1, a_2, \ldots, a_m$ above, we are given vectors $b_1, b_2, \ldots, b_m \in \mathbb{Q}^n$. Consider a linear matrix $A = \sum_{i=1}^m x_i a_i b_i^\top$ with variables $x_1, x_2, \ldots, x_m$. We have $\text{rank } A = \text{nc-rank } A$, they are equal to the maximum cardinality of a subset $I \subseteq \{1, 2, \ldots, m\}$ such that both $\{a_i \mid i \in I\}$ and $\{b_i \mid i \in I\}$ are independent. Namely, $\text{rank } A$ is the maximum cardinality of a common independent set of two matroids $M_1$ and $M_2$ represented by $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$, respectively. For $I \subseteq \{1, 2, \ldots, m\}$, let $\rho(I)$ and $\rho'(I)$ denote the dimension of vector spaces spanned by $a_i$ $(i \in I)$ and by $b_i$ $(i \in I)$, respectively. By the matroid intersection theorem, $\text{rank } A$ is the minimum of $\rho(I) + \rho'(J)$ over all bi-partitions $I, J$ of $\{1, 2, \ldots, m\}$. Then an mv-subspace $(X, Y)$ is given by $X = \{a_i \mid i \in I\}^\perp$ and $Y = \{b_j \mid j \in J\}^\perp$ for bi-partition $I, J$ attaining the minimum. This fact is noted in [36].

Suppose that we are further given weights $c_i \in \mathbb{Z}$ for each $i = 1, 2, \ldots, m$. Consider a linear matrix $A = \sum_{i=1}^m t^i x_i a_i b_i^\top$ over $\mathbb{Q}(t)$. Then $\deg \det A = \deg \det A$ is equal to the maximum weight $\sum_{i \in B} c_i$ of a common independent set $B \subseteq \{1, 2, \ldots, m\}$ with $|B| = n$ of matroids $M_1$ and $M_2$. Namely, the problem of finding $\deg \det A$ is the weighted linear matroid intersection problem. Let us explain the behavior of the steepest descent algorithm applied to this case. Suppose that we are given a feasible module $(L, M)$ of form $L = \langle (t^a)S \rangle_L$ and $M = \langle T(t^b) \rangle_R$ for nonsingular matrices $S, T \in \mathbb{Q}^{n \times n}$ and integer vectors $a, b \in \mathbb{Z}^n$. It may appear that a naive choice of a steepest direction $(X, Y)$ at $(L, M)$ would violate this form in the next step, but, in fact, such a situation can naturally be avoided.
Let $R_1, R_2, \ldots, R_n$ be the partition of $\{1, 2, \ldots, n\}$ such that $i, j$ belong to the same part if and only if $\alpha_i = \alpha_j$. Similarly, let $C_1, C_2, \ldots, C_v$ be the partition such that $i, j$ belong to the same part if and only if $\beta_i = \beta_j$. Regard matrix SAT as a block matrix, where columns and rows are partitioned by $R_1, R_2, \ldots, R_n$ and $C_1, C_2, \ldots, C_v$. In $(t^t)SAT(t^\beta)$, the $(k, \ell)$-th block is uniformly multiplied by $t^{\alpha_i+\beta_j}$ for $i \in R_k, j \in C_\ell$. Consider the linear matrix $((t^t)SAT(t^\beta))^0$ (to obtain a steepest direction). Then each summand $((t^t)Sac_i t^T(t^\beta))^0x_i$ has at most one nonzero block, where the nonzero block (if it exists) has rank 1. Now $((t^t)SAT(t^\beta))^0$ is essentially in the situation of a partitioned matrix with rank-1 blocks \cite{19}. By the partition structure, any mv-subspace $(X, Y)$ is of the form of $(X_1 \oplus X_2 \oplus \cdots \oplus X_\mu, Y_1 \oplus Y_2 \oplus \cdots \oplus Y_\nu)$ for $X_k \subseteq \mathbb{Q}^{R_k}$ and $Y_\ell \subseteq \mathbb{Q}^{C_\ell}$ $(k = 1, 2, \ldots, \mu, \ell = 1, 2, \ldots, \nu)$. Then basis matrices $S', T'$ for $X, Y$ are taken as block diagonal form so that $S'(t^\alpha) = (t^\alpha)S'$ and $T'(t^\beta) = (t^\beta)T'$. In the next iteration, $(L, M)$ is $(((t^t)S'S)_L, (TT'(t^\beta))_R)$. Consequently the obtained optimal solution is of the form of $(((t^t)S)_L, (T(t^\beta))_R)$. In particular, exponent vectors $\alpha, \beta$ can be dealt with as numerical vectors. It is an interesting question whether this algorithm can be a polynomial time algorithm in $n$ and the bit-length of $c$. We leave this issue to future work.

We here note that this algorithm is viewed as a variant of the primal dual algorithm for weighted matroid intersection problem by Lawler \cite{34}. His algorithm keeps and updates chains of flats in $M_1$ and $M_2$ and their weights. Observe that module $(t^Tt)_f$ can be identified with a chain $\emptyset \neq X_1 \subset X_2 \subset \cdots \subset X_n = \mathbb{Q}^n$ of subspaces and coefficients $\lambda_i$ $(i = 1, 2, \ldots, n)$ such that $\lambda_i \geq 0$ for $i < n$. Indeed, arrange $\beta$ as $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, and define $X_i$ as the subspace spanned the first $i$ rows and $\lambda_i$ as $\beta_i - \beta_{i+1}$ (with $\beta_{n+1} = 0$). This correspondence is unique if subspaces $X_i$ with $\lambda_i = 0$ are omitted. In this way, module $(L, M)$ can be kept as a pair of weighted chains of subspaces. If these subspaces are orthogonal complements of the subspaces spanned by some subsets of $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$, then $(L, M)$ can further be kept by a pair of weighted chains of flats of matroids $M_1$ and $M_2$, as in Lawler’s algorithm.

### 5.2 Mixed polynomial matrix

A mixed polynomial matrix is a polynomial matrix $A = \sum_{k=0}^{\ell} Q_k + T_k$ with indeterminate $t$ such that $Q_k$ is a matrix over $\mathbb{Q}$, each entry of $T_k$ is zero or one of variables $x_1, x_2, \ldots, x_m$, and each variable $x_i$ appears as one entry of one of $T_1, T_2, \ldots, T_{\ell}$. In the case of $\ell = 0$, $A$ is called a mixed matrix. See \cite{19} for detail of mixed (polynomial) matrices. A mixed polynomial matrix is viewed as a linear matrix over $\mathbb{Q}(t)$ with rank-1 summands, since the coefficient matrix of $x_k$ is written as $E_{ij}$. Therefore it holds that $\deg \det A = \deg \det A$.

It is shown in \cite{31, 32} that the combinatorial relaxation algorithm computes $\deg \det A$ in $O(\ell^2 n^{\omega+2})$ time. This estimate seems very rough, since it is based on a trivial bound $\ell n$ of the number of iterations. In the case of the steepest descent algorithm, we obtain a sharper estimate.

**Theorem 5.6.** Let $A$ be an $n \times n$ mixed polynomial matrix with maximum degree $\ell$. By the steepest descent algorithm, $\deg \det A$ can be computed in $O(\ell^2 n^{\omega+2})$ time, and in $O((\ell - \alpha_n)n^3 \log n + (\ell - \alpha_n)^2 n^2)$ time if $A$ is nonsingular, where $\alpha_n$ is the minimum degree of diagonals of the Smith-McMillan form of $A$.  

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To prove Theorem 5.6, we will work on a mixed matrix of a special form, as in [31]. A \textit{layered mixed (polynomial) matrix} [41] is a mixed (polynomial) matrix $A$ of form

$$A = \begin{pmatrix} Q \\ T \end{pmatrix},$$

(5.1)

where $Q$ is a matrix over $\mathbb{Q}[t]$ and $T$ is a variable matrix as above. It is well-known in the mixed-matrix literature that the rank and deg-det computation of a mixed matrix $Q + T$ reduce to those of a layered one

$$\begin{pmatrix} Q & I \\ T & D \end{pmatrix},$$

(5.2)

where $D$ is a diagonal matrix of new variables.

To compute a steepest direction, we need an mv-subspace of a layered mixed (non-polynomial) matrix, which is naturally obtained from a min-max formula of the rank. Let $A$ in (5.1) be an $n \times n'$ layered mixed matrix. For $J \subseteq \{1, 2, \ldots, n'\}$, let $Q[J]$ denote the submatrix of $Q$ consisting of $j$-th columns for $j \in J$, and let $\Gamma(J)$ denote the set of indices $i$ with $T_{ij} \neq 0$ for some $j$.

\textbf{Theorem 5.7} ([40, Theorem 4.2.5]). For an $n \times n'$ layered mixed matrix $A$ in (5.1).

$$\text{rank } A = \min_{J \subseteq \{1,2,\ldots,n'\} \setminus \Gamma(J)} \{ \text{rank } Q[J] + |\Gamma(J)| - |J| \} + n'.$$

(5.3)

Let $R_Q$ and $R_T$ denote the sets of row indices of matrices $Q$ and $T$, respectively.

\textbf{Lemma 5.8.} Let $J$ be a minimizer of (5.3). Let $X := \ker_{\mathbb{L}} Q[J] \oplus Q^{R_T \setminus \Gamma(J)}$ and $Y := Q^J$. Then $(X, Y)$ is an mv-subspace.

\textbf{Proof.} This follows from $n + n' - \dim X - \dim Y = n + n' - |R_Q| + \text{rank } Q[J] - |R_T| + |\Gamma(J)| - |J| = \text{rank } Q[J] + |\Gamma(J)| - |J| + n'$.

(5.4)

\textbf{Proof of Theorem 5.6.} For a mixed matrix $Q + T$, we compute the degree of the determinant of the corresponding layered mixed matrix (5.2). As an initialization (step 0), $(P, Q)$ is defined as $P = I$ and $Q = (t^{\leq \ell_1 \leq n})$, and let $A \leftarrow PAQ$. In step 1, SDA computes an mv-subspace of layered mixed matrix $A^0$. A minimizer $J$ of (5.3) is obtained by Cunningham’s matroid intersection algorithm [6] in $O(n^3 \log n)$ time. Namely $\gamma = O(n^3 \log n)$. In step 3, the matrix multiplication is needed only for the $Q$-part of $A$, which eliminates $m$ in the time complexity of Theorem 4.3.

\textbf{Application to DAE.} A motivating application of mixed polynomial matrices is analysis of linear \textit{differential algebraic equations (DAE)} with constant coefficients, where each coefficient is an accurate number or one of (inaccurate) parameters $x_1, x_2, \ldots, x_m$, and no parameter appears as distinct coefficients; see [41, Chapter 6]. By the Laplace transformation, the analysis of such a DAE reduces to linear equation $Ax = b$, where $A$ is a mixed polynomial matrix over $\mathbb{R}[s]$ with variables $x_1, x_2, \ldots, x_m$. Suppose the case where matrix $A$ is a square matrix. The \textit{index} is a barometer of “difficulty” of DAE $Ax = b$, and is defined as $-\alpha_n + 1$, where $\alpha_n$ is the minimum degree of the Smith-McMillan form of $A$. A DAE with high index ($\geq 2$) is difficult to solve numerically, and suggests an inconsistency of the mathematical modeling in deriving
this DAE. Therefore it is meaningful to decide whether the index of given a DAE is at most the limit $\Delta$. Here $\Delta(\approx 2)$ is the allowable upper bound for the index of DAE-models of the system we want to analyze. The steepest descent algorithm can decide in $O((\ell + \Delta)n^2\log n + (\ell + \Delta)^2 n^\omega)$ time whether DAE $Ax = b$ has index at most $\Delta$. Indeed, apply SDA to A. Index $-\alpha_n + 1$ is obtained from the number $\ell - \alpha_n$ of required iterations (Lemma 4.8). If SDA terminates before $\ell + \Delta$ iterations, then the DAE has index within $\Delta$. Otherwise the index is over the limit $\Delta$.

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A Appendix

A.1 Proof of Lemma 3.6

Let \( \mathcal{L} := \mathcal{L}_R(\mathbb{F}(t)^n) \). We omit \( R \) from \( \langle \rangle_R \). We first consider the join and meet of two \( L, L' \in \mathcal{L} \). Here we can assume that \( L = \langle P \rangle, L' = \langle P' \rangle = \langle P(t^\alpha) \rangle \), where \( (t^\alpha) \) is the Smith-McMillan form of \( P^{-1}P' \). (If \( S^{-1}P^{-1}P'T = (t^\alpha) \) for biproper \( S, T \), then \( P \) and \( P' \) can be replaced by \( PS \) and \( P'T \), respectively, since \( \langle P \rangle = \langle PS \rangle \) and \( \langle P' \rangle = \langle P'T \rangle \).)

We show by induction on \( k \) that \( \deg \det L \leq \ell \). By Lemma 2.8, it holds that \( \deg \det L \leq \langle \rangle \). Since \( \langle R \rangle \in \mathcal{L} \) for biproper \( R \), \( \deg \det L \geq \langle R \rangle \). This concludes that \( \mathcal{L} \) is a uniform modular lattice.

A.2 Proof of Lemma 3.7

We use the above notation by omitting \( R \). Consider two short chains \( C, D \). By (B1) in Lemma 3.5, there is a \( \mathbb{Z}^n \)-skeleton \( \Sigma \) containing them. Identify \( \Sigma \) with \( \mathbb{Z}^n \). If \( L \in \Sigma \) corresponds to \( x \in \mathbb{Z}^n \), then we write \( L \equiv x \). We may assume that \( C, D \) belong to interval \( [0, k]^n \subseteq \mathbb{Z}^n \). It suffices to show that the interval \( [0, k]^n \) belongs to \( \Sigma(P) \) for some nonsingular matrix \( P \).

Let \( \langle Q \rangle \) with \( L = 0 \in \mathbb{Z}^n \). Suppose that columns of \( Q \) are \( q_1, q_2, \ldots, q_n \). We first show by induction on \( k \) that \( q_1 \) can be replaced by some \( p_1 \) so that \( \langle t^\ell p_1 q_2 \cdots q_n \rangle \equiv \ell e_1 \) for \( \ell \leq k \). By induction, we can assume that \( \langle t^\ell q_1, q_2, \ldots, q_n \rangle \equiv \ell e_1 \) for \( \ell \leq k - 1 \). Consider \( L' \in \Sigma \) with \( L' \equiv k e_1 \). By Lemma 3.8 (4), \( L' \) is generated by vectors obtained by replacing one of \( t^{k-1}q_1, q_2, \ldots, q_n \) with \( q_1' = t^{k-1}q_1 + \sum_{i=2}^n q_i \lambda_i \) for \( \lambda_i \in \mathbb{F} \). Now \( \lambda_1 \neq 0 \). Indeed, if \( \lambda_1 = 0 \) and \( \lambda_2 \neq 0 \) (say), then \( L' = \langle (t^{k-1}q_1 q_1' \cdots q_n) \rangle \subseteq t\langle (t^{k-2}q_1 q_1 \cdots q_n) \rangle \), implying a contradiction \( ke_1 \leq (k - 2)e_1 + 1 \). Let \( p_1 := t^{-k}q_1' \). Thus \( L' = \langle t^\ell p_1 q_2 \cdots q_n \rangle \). Also observe \( \langle t^\ell p_1 q_2 \cdots q_n \rangle = \langle t^\ell q_1 q_2 \cdots q_n \rangle \) for \( \ell \leq k - 1 \).

In this way, we obtain \( p_1, p_2, \ldots, p_n \) such that \( \langle p_1 \cdots t^\ell p_1 \cdots p_n \rangle \equiv \ell e_i \) for \( \ell \leq k \). Then \( P = (p_1 \ p_2 \ \cdots \ p_n) \) is a desired matrix; necessarily \( \langle P(t^\alpha) \rangle \equiv \alpha \) holds for \( \alpha \in [0, k]^n \).

A.3 Proof of Proposition 2.9

The degree of determinant \( \deg \det \) is a matrix valuation in the sense of [5, Section 9] (with min and max reversed); see Theorem 9.3.4 of the reference. In particular, the \( \deg \det \) function satisfies the following property ((MV.4) in [5, Section 9.3]):
(MV) For nonsingular \( A \in \mathbb{F}(t)^{n \times n} \) and a vector \( b \in \mathbb{F}(t)^n \) regarded as a row (or column) vector, let \( B \) be the matrix obtained from \( A \) by replacing the first row (or column) by \( b \), and let \( C \) be the matrix obtained from \( A \) by adding \( b \) to the first row (or column) vector. Then it holds

\[
\deg \det C \leq \max\{\deg \det A, \deg \det B\}.
\]

The strict inequality holds only if \( \deg \det A = \deg \det B \).

In \([5]\), only the column version is proved but the row version can be proved in the same way. Indeed, by column permutation, we can make \( A \) (and \( B, C \)) so that the cofactor \( A' \) of \((1,1)\)-entry is nonsingular. Then we have

\[
A = \begin{pmatrix} a_{11} & a' \\ 0 & A' \end{pmatrix} E, \quad B = \begin{pmatrix} b_{11} & b' \\ 0 & A' \end{pmatrix} E, \quad C = \begin{pmatrix} c_{11} & c' \\ 0 & A' \end{pmatrix} E,
\]

where \( E \) is the product of permutation matrices and upper unitriangular matrices, and \((c_{11}, c') = (a_{11}, a') + (b_{11}, b')\). Thus \( \deg \det A = \deg a_{11} + \deg \det A' \), \( \deg \det B = \deg b_{11} + \deg \det A' \), and \( \deg \det C = \deg c_{11} + \deg \det A' = \deg (a_{11} + b_{11}) + \deg \det A' \). From \( \deg (a_{11} + b_{11}) \leq \max\{\deg a_{11}, \deg b_{11}\} \), we obtain (MV).

Now let us start to prove Proposition \([2, 3]\). For \( u \in \mathbb{F}(t) \) and \( k, \ell \in \{1, 2, \ldots, n\} \) with \( k \neq \ell \), define \( E(k, \ell; u) \in \mathbb{F}(t)^{n \times n} \) by

\[
E(k, \ell; u)_{ij} = \begin{cases} 1 & \text{if } i = j, \\ u & \text{if } i = k, j = \ell, \\ 0 & \text{otherwise}. \end{cases}
\]

\( E(k, \ell; u) \) is called an elementary matrix. Observe that \( E(k, \ell; u) \) is nonsingular with \( E(k, \ell; u)^{-1} = E(k, \ell; -u) \). In particular, \( E(k, \ell; u) \) is biproper if and only if \( u \in \mathbb{F}(t)^- \).

A required diagonalization is obtained as follows. First, by multiplying permutation matrices to the left and the right of \( A \), modify \( A \) so that \( A_{11} \) has the maximum degree among all entries of \( A \). By multiplying elementary matrices \( E(1, \ell; u) \) from right and \( E(\ell', 1; u') \) from left, modify \( A \) so that all entries except \( A_{11} \) in the first row and column are zero. Here \( u, u' \) can be taken from \( \mathbb{F}(t)^- \) by the maximality. Therefore \( E(1, \ell; u) \) and \( E(\ell, 1'; u') \) are biproper, and the degree of entries of \( A \) does not increase. Now \( A_{11} \) is written as \( t^{a_1}v \) for \( a_1 = \deg A_{11} \) and \( v \in \mathbb{F}(t)^- \) with \( \deg v = 0 \). Multiply a biproper diagonal matrix whose \((1,1)\)-entry is \( v^{-1} \) and other diagonals are 1. Then \( A_{11} \) is now \( t^{a_n} \). Repeat the same process to the submatrix from the second row and column. Eventually \( A \) is diagonalized to \( (t^{a_n}) \) with \( a_1 \geq a_2 \geq \cdots \geq a_n \). By construction, \( P, Q \) are the product of proper elementary matrices and permutation matrices, and are biproper.

Next we show that \( \delta_k \) is invariant throughout the above procedure, which implies the latter part of the claim. It is obvious that \( \delta_k \) is invariant under any row and column permutation. Consider the case of multiplying elementary matrix \( E(i, j; u) \) from the right. This operation corresponds to adding the \( i \)-th columns multiplied by \( u \) to the \( j \)-th column. Consider a \( k \times (k + 1) \) submatrix having the \( i \)- and \( j \)-th columns, and consider the change of its \( k \times k \) minors by the multiplication of \( E(i, j; u) \). Obviously any \( k \times k \) minor containing the \( i \)-th column does not change. Consider the \( k \times k \) minor not containing the \( i \)-th column. From the property (MV), the degree of this minor is at most the degree of the original or \( \deg u (\leq 0) \) plus the degree of the minor not containing \( j \). From this, we see that the maximum degree of a \( k \times k \) minor of this matrix does not change. Consequently \( \delta_k \) does not change. The proof for the left multiplication is the same.