Complete weight enumerators of two classes of linear codes

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Abstract

Recently, linear codes with few weights have been constructed and extensively studied. In this paper, for an odd prime \( p \), we determined the complete weight enumerator of two classes of \( p \)-ary linear codes constructed from defining set. Results show that the codes are at almost seven-weight linear codes and they may have applications in secret sharing schemes.

Index Terms

Linear code, complete weight enumerator, Gauss sum.

I. INTRODUCTION

Throughout this paper, let \( p \) be an odd prime and let \( q = p^e \) for some positive integer \( e \). Let \( \mathbb{F}_p \) denote the finite field with \( p \) elements. An \([n,k,d]\) linear code \( C \) is over \( \mathbb{F}_p \) is a \( k \)-dimensional subspace of \( \mathbb{F}_p^n \) with minimum distance \( d \) [19].

Let \( A_i \) be the number of codewords of weight \( i \) in \( C \) of length \( n \). The (Hamming) weight enumerator of \( C \) is defined by [19]

\[
1 + A_1 x + A_2 x^2 + \cdots + A_n x^n.
\]

For \( 0 \leq i \leq n \), the list \( A_i \) is called the weight distribution or weight spectrum of \( C \). A code \( C \) is said to be a \( t \)-weight code if the number of nonzero \( A_i \) with \( 1 \leq i \leq n \) is equal to \( t \). Clearly, the minimum distance of \( C \) can be derived from the weight distribution of the code \( C \). By error detection and error correction algorithms [21], the weight distribution of a code can be applied to compute the error probability of error detection and correction. Thus, weight distribution is a significant research topic in coding theory and was studied in [2], [4], [7], [8], [9], [10], [14], [30], [31].

Let \( \mathbb{F}_p = \{0, 1, \ldots, p-1\} \). For a codeword \( c = (c_0, c_1, \ldots, c_{n-1}) \in C \), the complete weight enumerator of \( c \) is the monomial

\[
w(c) = w_0^{t_0} w_1^{t_1} \cdots w_{p-1}^{t_{p-1}}
\]

in the variables \( w_0, w_1, \ldots, w_{p-1} \). Here \( t_i \) \((0 \leq i \leq p-1)\) is the number of components of \( c \) which equal to \( i \). Then the complete weight enumerator of the linear code \( C \) is the homogeneous polynomial

\[
\text{CWE}(C) = \sum_{c \in C} w(c)
\]

of degree \( n \) (see [27], [28]).
From definition, the complete weight enumerators of binary linear codes are just the weight enumerators. For nonbinary linear codes, the (Hamming) weight enumerators, which have been extensively investigated [15], [16], [34], [15], [35], can be obtained from the complete weight enumerators. Further more, the complete weight enumerators are closely related to the deception of some authentication codes constructed from linear codes [11], and used to compute the Walsh transform of monomial functions over finite fields [18]. Thus, a great deal of research is devoted to the computation of the complete weight distribution of specific codes [1], [3], [20], [22], [23], [25].

Let $F_q$ be the finite field with $q$ elements and $D = \{d_1, d_2, \ldots, d_n\}$ be a nonempty subset of $F_q$. A generic construction of a linear code of length $n$ is given by

$$C_D = \{c_x = (\text{Tr}(x d_1), \text{Tr}(x d_2), \ldots, \text{Tr}(x d_n)) : x \in F_q\},$$

(1.1)

where $\text{Tr}$ denote the trace function from $F_q$ onto $F_p$ [26]. The set $D$ is called the defining set of $C_D$. This construction technique is employed in lots of researches to get linear codes with few weights. The readers are referred to [17], [29], [35], [12], [24], [29] for more details.

Naturally, a generalization of the code $C_D$ of (1.1) is defined by

$$C_D^\alpha = \{(\text{Tr}(x d_1) + u, \text{Tr}(x d_2) + u, \ldots, \text{Tr}(x d_n) + u) : u \in F_p, x \in F_q\}.$$  

(1.2)

The objective of this paper is to present linear codes over $F_p$ with at most seven weights using the above two construction methods. Further more, the complete weight enumerators of the two proposed linear codes are also calculated. The codes in this paper may have applications in authentication codes [13], secret sharing schemes [33] and consumer electronics.

II. THE MAIN RESULTS

In this section, we only present the $p$-ary linear codes and introduce their parameters. The proofs of their parameters will be given later.

For $a \in F_p$, the defining set is given by

$$D_a = \{x \in F_q^* : \text{Tr}(x^{p^\alpha + 1}) = a\},$$

(2.1)

where $\alpha$ is any natural number. It should be remarked that, for $p = 2$, the weight enumerator of $C_{D_0}$ of (1.1) have been determined in [17]. In this paper, for $a \in F_p$ the complete weight enumerators of $C_{D_a}$ of (1.1) and $C_{D_a}^\alpha$ of (1.2) have been explicitly presented by using exponential sums.

Lemma 1 ([5], Lemma 2.6): Let $d = \gcd(\alpha, e)$ and $p$ be odd. Then

$$\gcd(p^d + 1, p^e - 1) = \begin{cases} 2, & \text{if } e/d \text{ is odd}, \\ p^d + 1, & \text{if } e/d \text{ is even}. \end{cases}$$

Note that $\gcd(p^\alpha + 1, p^e - 1) = 2$ leads to

$$\{x^{p^\alpha + 1} : x \in F_q^*\} = \{x^2 : x \in F_q^*\}$$

which means that

$$D_0 = \{x \in F_q^* : \text{Tr}(x^{p^\alpha + 1}) = 0\}$$

$$= \{x \in F_q^* : \text{Tr}(x^2) = 0\}.$$  

By Lemma [1] if $e/d$ is odd, the code $C_{D_0}$ of (1.1) and the code $C_D$ in [16] are the same. Hence, we will assume $e/d$ is even and $e = 2m$ for a positive integer $m$. The main results of this paper are given below.
Theorem 2: Let \( m \geq 2 \). If \( m/d \equiv 1 \mod 2 \), then the code \( C_{D_0} \) of (1.1) is a \([p^{e-1} - (p-1)p^{m-1} - 1, e]\) linear code with weight distribution in Table I and its complete weight enumerator is

\[
w_0^{p^{e-1} - (p-1)p^{m-1} - 1} + (p^{e-1} - (p-1)p^{m-1} - 1)w_0^{p^{e-2} - (p-1)p^{m-1} - 1} \prod_{i=1}^{p-1} w_i^{p^{e-2}}
\]

\[
+ (p-1)(p^{e-1} + p^{m-1})w_0^{p^{e-2} - p^{m-1}} \prod_{i=1}^{p-1} w_i^{p^{e-2} - p^{m-1}}.
\]

Corollary 3: Let the symbols and conditions be the same as Theorem 2. The code \( \overline{C}_{D_0} \) of (1.2) is a \([p^{e-1} - (p-1)p^{m-1} - 1, e+1]\) linear code with weight distribution in Table II and its complete weight enumerator is

\[
\sum_{i=0}^{p-1} w_i^{p^{e-1} - (p-1)p^{m-1} - 1}
\]

\[
+ (p^{e-1} - (p-1)p^{m-1} - 1) \sum_{i=0}^{p-1} w_i^{p^{e-2} - (p-1)p^{m-1} - 1} \prod_{j \neq i} w_j^{p^{e-2}}
\]

\[
+ (p-1)(p^{e-1} + p^{m-1}) \sum_{i=0}^{p-1} w_i^{p^{e-2} - p^{m-1}} \prod_{j \neq i} w_j^{p^{e-2} - p^{m-1}}.
\]

Theorem 4: Let \( g \) be a generator of \( \mathbb{F}_p^* \) and \( a \in \mathbb{F}_p^* \). If \( m/d \equiv 1 \mod 2 \), then the code \( C_{D_a} \) of (1.1) is a \([p^{e-1} + p^{m-1}, e]\) linear code with weight distribution in Table III and its complete weight enumerator

| Weight \( w \) | Multiplicity \( A \) |
|----------------|------------------|
| 0              | 1                |
| \((p-1)p^{e-2}\) | \(p^{e-1} - (p-1)p^{m-1} - 1\) |
| \((p-1)(p^{e-2} - p^{m-1})\) | \((p-1)(p^{e-1} + p^{m-1})\) |
| \((p-1)p^{e-2} - (p-1)(p^{e-1} + p^{m-1})\) | \((p-1)(p^{e-1} - (p-1)p^{m-1} - 1)\) |
| \((p-1)p^{e-2} - (p-2)p^{m-1} - 1\) | \((p-1)^2(p^{e-1} + p^{m-1})\) |

| Weight \( b \) | Multiplicity \( A \) |
|----------------|------------------|
| 0              | 1                |
| \((p-1)p^{e-2}\) | \(p^{e-1} - (p-1)p^{m-1} - 1\) |
| \((p-1)(p^{e-2} - p^{m-1})\) | \((p-1)(p^{e-1} + p^{m-1})\) |
| \((p-1)p^{e-2} - (p-1)p^{m-1} - 1\) | \((p-1)(p^{e-1} - (p-1)p^{m-1} - 1)\) |
| \((p-1)p^{e-2} - (p-2)p^{m-1} - 1\) | \((p-1)^2(p^{e-1} + p^{m-1})\) |

| Weight \( b \) | Multiplicity \( A \) |
|----------------|------------------|
| 0              | 1                |
| \((p-1)p^{e-2} + 2p^{m-1}\) | \((p-1)(p^{e-1} + p^{m-1})\) |
| \((p-1)p^{e-2}\) | \(p^{e-1} - (p-1)(p^{e-1} + p^{m-1}) - 1\) |
where $(\cdot)$ denotes the Legendre symbol.

**TABLE IV:** The weight distribution of the codes of Corollary 5

| Weight $b$ | Multiplicity $A$ |
|------------|------------------|
| 0          | 1                |
| $(p - 1)p^{e-2} + 2p^{m-1}$ | $\frac{1}{2}(p - 1)(p - 2)(p^{e-1} + p^{m-1})$ |
| $(p - 1)p^{e-2}$ | $p^2 + \frac{1}{2}(p - 1)(p - 2)(p^{e-1} + p^{m-1}) - 1$ |
| $p^{e-1} + p^{m-1}$ | $p - 1$ |
| $(p - 1)p^{e-2} + p^{m-1}$ | $(p - 1)(2p^{e-1} - (p - 2)p^{m-1} - 1)$ |

**Corollary 5:** Let the symbols and conditions be the same as Theorem 4. The code $\overline{C}_{D_a}$ of (1.2) is a $[p^{e-1} + p^{m-1}, e + 1]$ linear code with weight distribution in Table IV and its complete weight enumerator is

\[
\begin{align*}
&\sum_{i=0}^{p-1} w_i^{p^{e-1}+p^{m-1}} + (p^{e-1} - (p - 1)p^{m-1} - 1)\sum_{i=0}^{p-1} w_i^{p^{e-2}+p^{m-1}} \prod_{j \neq i} w_j^{p^{e-2}} \\
&+ (p^{e-1} + p^{m-1})\sum_{\beta=1}^{n-1} \sum_{i=0}^{p-1} w_i^{p^{e-2} - \left(\frac{i}{p}\right)p^{m-1}} w_i^{p^{e-2} - \left(\frac{i}{p}\right)p^{m-1}} \prod_{j \neq i, i \pm 2^\beta} w_j^{p^{e-2} - \left(\frac{2^\beta + 2^\beta + 1}{p}\right)p^{m-1}} \\
&+ (p^{e-1} + p^{m-1})\sum_{\beta=1}^{n-1} \sum_{i=0}^{p-1} w_i^{p^{e-2} + \left(\frac{i}{p}\right)p^{m-1}} \prod_{j \neq i} w_j^{p^{e-2} - \left(\frac{2^\beta + 2^\beta + 1}{p}\right)p^{m-1}} .
\end{align*}
\]

**Example 1:** Let $(p, m, \alpha) = (3, 3, 1)$. If $a = 0$, then the code $C_{D_0}$ has parameters $[224, 6, 144]$ with complete weight enumerator

\[
w_0^{224} + 224w_0^{62} \prod_{i=1}^{2} w_i^{81} + 504w_0^{80} \prod_{i=1}^{2} w_i^{72} ,
\]

and the $\overline{C}_{D_0}$ has parameters $[224, 7, 143]$ with complete weight enumerator

\[
\sum_{i \in F_3} w_i^{224} + 224 \sum_{i \in F_3} w_i^{62} \prod_{j \neq i} w_j^{81} + 504 \sum_{i \in F_3} w_i^{80} \prod_{j \neq i} w_j^{72} .
\]

If $a = 1$, then the code $C_{D_1}$ has parameters $[252, 6, 162]$ with complete weight enumerator

\[
w_0^{252} + 476w_0^{90} \prod_{i=1}^{2} w_i^{81} + 252w_0^{72}w_1^{90}w_2^{90} ,
\]
and the code $\mathcal{C}_{D_1}$ has parameters $[252, 7, 162]$ with complete weight enumerator

$$
\sum_{i \in \mathbb{F}_3} w_i^{252} + 476 \sum_{i \in \mathbb{F}_3} w_i^{90} \prod_{j \neq i} w_j^{81} + 252 \sum_{i \in \mathbb{F}_3} w_i^{72} \prod_{j \neq i} w_j^{90}.
$$

**TABLE V:** The weight distribution of the codes of Theorem 6

| Weight $b$ | Multiplicity $A$ |
|------------|------------------|
| $0$        | $1$              |
| $(p - 1)p^{e-2} - (p - 1)^2p^{m+d-2}$ | $p^e - p^{e-2d}$ |
| $(p - 1)p^{e-2}$ | $p^{e-2d-1} - (p - 1)p^{m-d} - 1$ |
| $(p - 1)p^{e-2} - (p - 1)p^{m+d+1}$ | $(p - 1)(p^{e-2d-1} + p^{m-d-1})$ |

**Theorem 6:** Let $m > d + 1$. If $m/d \equiv 0 \mod 2$, then the code $C_{D_0}$ of (1.1) is a $[p^{e-1} - (p - 1)p^{m + d} - 1, e]$ linear code with weight distribution in Table V and its complete weight enumerator is

$$
\begin{align*}
&\sum_{i=1}^{p-1} w_i^{p^{e-1} - (p - 1)p^{m + d} - 1} + (p - 1) \left( p^{e-2d-1} + p^{m-d} - 1 \right) \prod_{i=1}^{p-1} w_i^{p^{e-2} - p^{m+d-1}} \\
&\quad + \left( p^e - p^{e-2d} \right) \sum_{i=1}^{p-1} w_i^{p^{e-2} - (p - 1)p^{m+d} - 1} \prod_{j \neq i} w_j^{p^{e-2} - (p - 1)p^{m+d-1}} \\
&\quad + \left( p^{e-2d-1} - (p - 1)p^{m-d} - 1 \right) \sum_{i=1}^{p-1} w_i^{p^{e-2} - (p - 1)p^{m+d} - 1} \prod_{j \neq i} w_j^{p^{e-2}}.
\end{align*}
$$

**TABLE VI:** The weight distribution of the codes of Corollary 7

| Weight $b$ | Multiplicity $A$ |
|------------|------------------|
| $0$        | $1$              |
| $(p - 1)p^{e-2} - (p - 1)^2p^{m+d} - 2$ | $p^e - p^{e-2d}$ |
| $(p - 1)p^{e-2}$ | $p^{e-2d-1} - (p - 1)p^{m-d} - 1$ |
| $(p - 1)p^{e-2} - (p - 1)p^{m+d+1}$ | $(p - 1)(p^{e-2d-1} + p^{m-d-1})$ |
| $p^{e-1} - (p - 1)p^{m+d+1}$ | $p^e - p^{e-2d}$ |
| $(p - 1)p^{e-2} - (p - 2)p^{m+d} - 1$ | $(p - 1)(p^{e-2d-1} + p^{m-d})$ |
| $(p - 1)(p^{e-2} - (p - 1)p^{m+d+1}) - 2$ | $(p - 1)(p^{e-2d-1} - (p - 1)p^{m-d} - 1)$ |

**Corollary 7:** Let the symbols and conditions be the same as Theorem 6. The code $C_{D_0}$ of (1.2) is a $[p^{e-1} - (p - 1)p^{m + d} - 1, e + 1]$ linear code with weight distribution in Table VI and its complete weight enumerator is

$$
\begin{align*}
&\sum_{i=0}^{p-1} w_i^{p^{e-1} - (p - 1)p^{m+d} - 1} \\
&\quad + (p - 1) \left( p^{e-2d-1} + p^{m-d} - 1 \right) \sum_{i=0}^{p-1} w_i^{p^{e-2} - 1} \prod_{j \neq i} w_j^{p^{e-2} - p^{m+d} - 1} \\
&\quad + \left( p^e - p^{e-2d} \right) \sum_{i=0}^{p-1} w_i^{p^{e-2} - (p - 1)p^{m+d} - 1} \prod_{j \neq i} w_j^{p^{e-2} - (p - 1)p^{m+d}-2} \\
&\quad + \left( p^{e-2d-1} - (p - 1)p^{m-d} - 1 \right) \sum_{i=0}^{p-1} w_i^{p^{e-2} - (p - 1)p^{m+d} - 1} \prod_{j \neq i} w_j^{p^{e-2}}.
\end{align*}
$$
Theorem 8: Let \(a \in \mathbb{F}_p^*\). If \(m/d \equiv 0 \mod 2\), then the code \(C_{D_a}\) of (1.1) is a \([p^{e-1} + p^{m+d-1}, e]\) linear code with weight distribution in Table VII and its complete weight enumerator is

\[
w_0^{p^{e-1}+p^{m+d-1}} + \left(p^e - p^{e-2d}\right) \prod_{i=0}^{p-1} w_i^{p^{e-2}+p^{m+d-2}} + \left(p^{e-2d-1} - (p-1)p^{m-d-1} - 1\right) w_0^{p^{e-2}+p^{m+d-1}} \prod_{i=1}^{p-1} w_i^{p^{e-2}} + \left(p^{e-2d-1} + p^{m-d-1}\right) \sum_{\beta=1}^{\frac{p-1}{2}} w_0^{p^{e-2} - \left(\frac{1}{p}\right)p^{m+d-1}} w_0^{p^{e-2}} w_i^{p^{e-2}} \prod_{j \neq 0, \pm 2g^\beta} w_j^{p^{e-2} - \left(\frac{2-42\beta+1}{p}\right)p^{m+d-1}} + \left(p^{e-2d-1} + p^{m-d-1}\right) \sum_{\beta=1}^{\frac{p-1}{2}} w_0^{p^{e-2} + \left(\frac{1}{p}\right)p^{m+d-1}} \prod_{i=1}^{p-1} w_i^{p^{e-2}} \prod_{j \neq i} w_j^{p^{e-2} - \left(\frac{2-42\beta+1}{p}\right)p^{m+d-1}},
\]

where \((\cdot)\) denote the Legendre symbol.

Corollary 9: Let the symbols and conditions be the same as Theorem 8. The code \(\overline{C}_{D_a}\) of (1.2) is a \([p^{e-1} + p^{m+d-1}, e + 1]\) linear code with weight distribution in Table VIII and its complete weight enumerator is

\[
\sum_{i=0}^{p-1} w_i^{p^{e-1}+p^{m+d-1}} + \left(p^e - p^{e-2d}\right) \prod_{i=0}^{p-1} w_i^{p^{e-2}+p^{m+d-2}} + \left(p^{e-2d-1} - (p-1)p^{m-d-1} - 1\right) \sum_{i=0}^{p-1} w_i^{p^{e-2}+p^{m+d-1}} \prod_{j \neq i} w_j^{p^{e-2}} + \left(p^{e-2d-1} + p^{m-d-1}\right) \sum_{\beta=1}^{\frac{p-1}{2}} \sum_{i=0}^{p-1} w_i^{p^{e-2} - \left(\frac{1}{p}\right)p^{m+d-1}} w_0^{p^{e-2}} w_i^{p^{e-2}} \prod_{j \neq 0, \pm 2g^\beta} w_j^{p^{e-2} - \left(\frac{2-42\beta+1}{p}\right)p^{m+d-1}} + \left(p^{e-2d-1} + p^{m-d-1}\right) \sum_{\beta=1}^{\frac{p-1}{2}} \sum_{i=0}^{p-1} w_i^{p^{e-2} + \left(\frac{1}{p}\right)p^{m+d-1}} \prod_{j \neq i} w_j^{p^{e-2} - \left(\frac{2-42\beta+1}{p}\right)p^{m+d-1}},
\]
Example 2: Let \((p, m, \alpha) = (3, 4, 1)\). If \(a = 0\), then the corresponding code \(C_{D_0}\) has parameters \([2024, 8, 1296]\) with complete weight enumerator

\[
  w_0^{2024} + 504w_0^{728} \prod_{i=1}^{2} w_i^{648} + 5832w_0^{674} \prod_{i=1}^{2} w_i^{675} + 224w_0^{566} \prod_{i=1}^{2} w_i^{729},
\]

and the code \(\overline{C}_{D_0}\) has parameters \([2024, 9, 1295]\) with complete weight enumerator

\[
  \sum_{i \in F_3} w_i^{2024} + 504 \sum_{i \in F_3} w_i^{728} \prod_{j \neq i} w_j^{648} + 5832 \sum_{i \in F_3} w_i^{674} \prod_{j \neq i} w_j^{675} + 224 \sum_{i \in F_3} w_i^{566} \prod_{j \neq i} w_j^{729}.
\]

If \(a = 1\), then the code \(C_{D_1}\) has parameters \([2268, 8, 1458]\) with complete weight enumerator

\[
  w_0^{2268} + 5832 \prod_{i=0}^{2} w_i^{756} + 476w_0^{810} \prod_{i=1}^{2} w_i^{729} + 252w_0^{648} \prod_{i=1}^{2} w_i^{810},
\]

and the code \(\overline{C}_{D_1}\) has parameters \([2268, 9, 1458]\) with complete weight enumerator

\[
  \sum_{i \in F_3} w_i^{2268} + 17496 \prod_{i \in F_3} w_i^{756} + 496 \prod_{i \in F_3} w_i^{810} \prod_{j \neq i} w_j^{729} + 252 \prod_{i \in F_3} w_i^{648} \prod_{j \neq i} w_j^{810}.
\]

### III. The Proofs of the Main Results

Our task in this section is to prove results in Section 2. Firstly, we review some basic notations and results of group characters and present some lemma which are needed for the proofs of the main results.

#### A. Preliminaries

We start with the additive character. A group homomorphism \(\chi\) from \(F_q\) into the complex numbers is called an additive character of \(F_q\). Let \(b \in F_q\), the mapping

\[
  \chi_b(c) = \zeta_p^{Tr(bc)} \text{ for all } c \in F_q,
\]

defines an additive character of \(F_q\), where \(\zeta_p = e^{2\pi i/p}\). The additive character \(\chi_0\) is called trivial and the other characters \(\chi_b\) with \(b \notin F_q^*\) are called nontrivial. The character \(\chi_1\) is called the canonical additive character of \(F_q\). And \(\chi_b(x) = \chi_1(bx)\) for all \(x \in F_q\).

By the orthogonal property of additive characters, we have ([26], Theorem 5.4),

\[
  \sum_{x \in F_q} \chi(x) = \begin{cases} 0, & \text{if } \chi \text{ is trivial,} \\ q, & \text{if } \chi \text{ is nontrivial.} \end{cases}
\]

The multiplicative characters of \(F_q\) are Characters over \(F_q^*\), which are given by

\[
  \psi_j(g^k) = \zeta_p^{2\pi \sqrt{-1} j/(q-1) k} \text{ for } k = 0, 1, \ldots, q - 2, \ 0 \leq j \leq q - 2.
\]

Here \(g\) is a generator of \(F_q^*\) ([26]). The multiplicative character \(\psi_{(q-1)/2}\) is called the quadratic character of \(F_q\), which is denoted by \(\eta\). And we assume that \(\eta(0) = 0\) in this paper.

We define the quadratic Gauss sum \(G = G(\eta, \chi_1)\) over \(F_q\) by

\[
  G(\eta, \chi_1) = \sum_{x \in F_q^*} \eta(x)\chi_1(x),
\]

and the quadratic Gauss sum \(\overline{G} = G(\overline{\eta}, \overline{\chi_1})\) over \(F_q^*\) by

\[
  G(\overline{\eta}, \overline{\chi_1}) = \sum_{x \in F_q^*} \overline{\eta(x)}\overline{\chi_1(x)},
\]
where $\eta$ and $\chi_1$ denote the quadratic and canonical character of $\mathbb{F}_p$.

The explicit values of quadratic Gauss sums are given as follows.

**Lemma 10 ([26], Theorem 5.15):** Let the symbols be the same as before. Then

$$G(\eta, \chi_1) = (-1)^{(m-1)}\sqrt{-1} \frac{(-1)^{e/d}}{\sqrt{q}}, \quad G(\eta, \overline{\chi_1}) = \sqrt{-1} \frac{(-1)^{e/d}}{\sqrt{p}}.$$

**Lemma 11 ([16], Lemma 7):** Let the symbols be the same as before. Then

1) if $m \geq 2$ is even, then $\eta(y) = 1$ for each $y \in \mathbb{F}_p^*$;
2) if $m$ is odd, then $\eta(y) = \overline{\eta}(y)$ for each $y \in \mathbb{F}_p^*$.

**Lemma 12 ([26], Theorem 5.33):** Let $\chi$ be a nontrivial additive character of $\mathbb{F}_q$, and let $f(x) = a_2 x^2 + a_1 x + a_0 \in \mathbb{F}_q[x]$ with $a_2 \neq 0$. Then

$$\sum_{x \in \mathbb{F}_q} \chi(f(x)) = \chi(a_0 - a_1^2(4a_2)^{-1}) \eta(a_2) G(\eta, \chi).$$

For $a \in \mathbb{F}_p^*$, $b \in \mathbb{F}_q$, define

$$A(a) = \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \sum_{x \in \mathbb{F}_q} \zeta_p^{yTr(x^{p+1})} \quad (3.1)$$

and

$$S(a, b) = \sum_{x \in \mathbb{F}_q} \chi(ax^{p+1} + bx). \quad (3.2)$$

For $a, c \in \mathbb{F}_p$ and $b \in \mathbb{F}_q^*$, define

$$B(a, c) = \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-ay-cz} \sum_{x \in \mathbb{F}_q} \chi_1(yx^{p+1} + bxz). \quad (3.3)$$

**Lemma 13 ([5], Theorem 2):** Let $e/d$ be even with $e = 2m$. Then

$$S(a, 0) = \begin{cases} 
p^m, & \text{if } a \frac{q-1}{p+1} \neq 1, \\
-p^{m+d}, & \text{if } a \frac{q-1}{p+1} = 1, \\
-p^m, & \text{if } a \frac{q-1}{p+1} \neq -1, \\
p^{m+d}, & \text{if } a \frac{q-1}{p+1} = -1,
\end{cases}$$

**Lemma 14 ([6], Theorem 1):** Let $q$ be odd and suppose $f(X) = aX^{p^m} + aX$ is a permutation polynomial over $\mathbb{F}_q$. Let $x_0$ be the unique solution of the equation $f(X) = -b^{p^m} (b \neq 0)$. The evaluation of $S(a, b)$ partitions into the following two cases:

1) If $e/d$ is odd, then

$$S(a, b) = \begin{cases} 
(-1)^{e-1} q^{-1} \sqrt{-a} \chi_1(ax_0^{p+1}), & \text{if } p \equiv 1 \pmod{4}, \\
(-1)^{e-1} q^{-1} \zeta_q^{-2} \chi_1(ax_0^{p+1}), & \text{if } p \equiv 3 \pmod{4}.
\end{cases}$$

2) If $e/d$ is even, then $e = 2m$, $a \frac{q-1}{p+1} \neq (-1)^m$ and

$$S(a, b) = (-1)^{e} p^m \chi_1(ax_0^{p+1}).$$

**Lemma 15 ([6], Theorem 2):** Let $q$ be odd and suppose $f(X) = aX^{p^m} + aX$ is not a permutation polynomial over $\mathbb{F}_q$. Then for $b \neq 0$ we have $S(a, b) = 0$ unless the equation $f(X) = -b^{p^m}$ is solvable. If the equation is solvable, with some solution $x_0$ say, then

$$S(a, b) = (-1)^{m} p^{m+d} \chi_1(ax_0^{p+1}).$$

**Lemma 16:** Let the symbols be the same as before.

1) If $a = 0$, then $A(0) = \begin{cases} 
-(p-1)p^{m+d}, & \text{if } m/d \equiv 0 \pmod{2}, \\
-(p-1)p^m, & \text{if } m/d \equiv 1 \pmod{2}.
\end{cases}$
2) If \( a \neq 0 \), then \( A(a) = \begin{cases} p^{m+d}, & \text{if } m/d \equiv 0 \mod 2, \\ p^m, & \text{if } m/d \equiv 1 \mod 2. \end{cases} \)

**Proof:** Notice that \( \sum_{x \in \mathbb{F}_q} \zeta_p^{y \operatorname{Tr}(x^{m+1})} = S(y, 0) \). For \( y \in \mathbb{F}_p^* \), we know \( y^{\frac{m-1}{d}} = 1 \). Then the results can be obtained directly from Lemma 13.

**Lemma 17:** Let the symbols be the same as before. If \( m/d \equiv 1 \mod 2 \), then \( X^{p^2 \alpha} + X = -b^{p\alpha} \) has a solution \( \gamma \) in \( \mathbb{F}_q \) and

1) \[
B(0, 0) = \begin{cases} -p^m(p-1)^2, & \text{if } \operatorname{Tr}(\gamma^{p^{\alpha+1}}) = 0, \\ p^m(p-1), & \text{if } \operatorname{Tr}(\gamma^{p^{\alpha+1}}) \neq 0; \end{cases}
\]

2) for \( c \neq 0 \), we have \[
B(0, c) = \begin{cases} (p-1)p^m, & \text{if } \operatorname{Tr}(\gamma^{p^{\alpha+1}}) = 0, \\ -p^m, & \text{if } \operatorname{Tr}(\gamma^{p^{\alpha+1}}) \neq 0; \end{cases}
\]

3) for \( a \neq 0 \), we have \[
B(a, 0) = \begin{cases} (p-1)p^m, & \text{if } \operatorname{Tr}(\gamma^{p^{\alpha+1}}) = 0, \\ -p^m, & \text{if } \operatorname{Tr}(\gamma^{p^{\alpha+1}}) \neq 0; \end{cases}
\]

4) for \( ac \neq 0 \), we have \[
B(a, c) = \begin{cases} -p^m, & \text{if } \operatorname{Tr}(\gamma^{p^{\alpha+1}}) = 0, \\ -p^m, & \text{if } \operatorname{Tr}(\gamma^{p^{\alpha+1}}) = c^2/(4a), \\ -p^{m+1}\gamma(c^2 - 4a\operatorname{Tr}(\gamma^{p^{\alpha+1}})) - p^m, & \text{otherwise.} \end{cases}
\]

**Proof:** If \( m/d \equiv 1 \mod 2 \), by Theorem 4.1 in [4], we get the equation \( X^{p^{2\alpha}} + X = 0 \) has no solution in \( \mathbb{F}_q^* \). Then \( X^{p^{2\alpha}} + X \) is a permutation polynomial over \( \mathbb{F}_q \) and \( X^{p^{2\alpha}} + X = -b^{p\alpha} \) has a unique solution \( \gamma \) in \( \mathbb{F}_q \). Thus \( y^{-1}z\gamma \) is the unique solution in \( \mathbb{F}_q \) of \( y^{p\alpha}X^{p^{2\alpha}} + yX = -(bz)^{p\alpha} \). By Lemma 14, \( S(y, bz) = -p^m\gamma \) \((y(y^{-1}z\gamma)^{p^{\alpha+1}}) \). Therefore,

\[
B(a, c) = -p^m\sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{y^{-1}z\gamma c^2 - 4a\operatorname{Tr}(\gamma^{p^{\alpha+1}})} \frac{z^{2\operatorname{Tr}(\gamma^{p^{\alpha+1}})}}{y}.
\]

If \( \operatorname{Tr}(\gamma^{p^{\alpha+1}}) = 0 \), the corresponding results follow from the orthogonal property of additive characters easily.

If \( \operatorname{Tr}(\gamma^{p^{\alpha+1}}) \neq 0 \), we have

\[
B(0, 0) = -p^m\sum_{z \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{2\operatorname{Tr}(\gamma^{p^{\alpha+1}})} \frac{z^{2\operatorname{Tr}(\gamma^{p^{\alpha+1}})}}{y} = -p^m\sum_{z \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p^*} \zeta_p^y = (p-1)p^m.
\]

\[
B(0, c) = -p^m\sum_{z \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{cz} \frac{z^{2\operatorname{Tr}(\gamma^{p^{\alpha+1}})}}{y} = -p^m\sum_{z \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{cz} \sum_{y \in \mathbb{F}_p^*} \zeta_p^y = -p^m.
\]
By Lemma 12, we get

$$B(a, 0) = -p^m \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \sum_{z \in \mathbb{F}_p} \zeta_p^{-\frac{2\text{Tr}(\gamma^{p^0+1})}{y}}$$

$$= -p^m \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \sum_{z \in \mathbb{F}_p} \chi_1 \left( -\frac{\text{Tr}(\gamma^{p^0+1})}{y} z^2 \right)$$

$$= -p^m \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \sum_{z \in \mathbb{F}_p} \chi_1 \left( -\frac{\text{Tr}(\gamma^{p^0+1})}{y} z^2 \right) + p^m \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay}$$

$$= -p^m \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \chi_1(0) \eta \left( -\frac{\text{Tr}(\gamma^{p^0+1})}{y} \right) \overline{G} - p^m$$

$$= -p^m \eta(a \text{Tr}(\gamma^{p^0+1})) \overline{G} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \eta(-ay) - p^m$$

$$= -p^{m+1} \eta(-a \text{Tr}(\gamma^{p^0+1})) - p^m.$$

Also by Lemma 12, we have

$$B(a, c) = -p^m \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay-cz} \sum_{z \in \mathbb{F}_p} \zeta_p^{-\frac{2\text{Tr}(\gamma^{p^0+1})}{y}}$$

$$= -p^m \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \sum_{z \in \mathbb{F}_p} \chi_1 \left( -\frac{\text{Tr}(\gamma^{p^0+1})}{y} z^2 - cz \right) + p^m \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay}$$

$$= -p^m \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \chi_1 \left( \frac{yc^2}{4\text{Tr}(\gamma^{p^0+1})} \right) \eta \left( -\frac{\text{Tr}(\gamma^{p^0+1})}{y} \right) \overline{G} - p^m$$

$$= -p^m \eta(-\text{Tr}(\gamma^{p^0+1})) \overline{G} \sum_{y \in \mathbb{F}_p^*} \frac{2^{2-4\text{Tr}(\gamma^{p^0+1})}}{4\text{Tr}(\gamma^{p^0+1})^2} \eta(y) - p^m.$$

(3.4)

If \(\text{Tr}(\gamma^{p^0+1}) = c^2/(4a)\), by (3.4), we have \(B(a, c) = -p^m\). Otherwise, we obtain

$$B(a, c) = -p^m \eta \left( 4a \text{Tr}(\gamma^{p^0+1}) - c^2 \right) \overline{G} \sum_{y \in \mathbb{F}_p^*} \frac{c^2 - 4a \text{Tr}(\gamma^{p^0+1})}{4\text{Tr}(\gamma^{p^0+1})} \eta \left( \frac{c^2 - 4a \text{Tr}(\gamma^{p^0+1})}{4\text{Tr}(\gamma^{p^0+1})} y \right) - p^m$$

$$= -p^m \eta \left( - \left( c^2 - 4a \text{Tr}(\gamma^{p^0+1}) \right) \right) \overline{G}^2 - p^m$$

$$= -p^{m+1} \eta \left( c^2 - 4a \text{Tr}(\gamma^{p^0+1}) \right) - p^m.$$

The proof of this lemma is completed.

**Lemma 18:** Let the symbols be the same as Lemma 17. Let \(m/d \equiv 0 \mod 2\). If \(X^{p^2a} + X = -b^{p^a}\) has no solution in \(\mathbb{F}_q\), then

$$B(a, c) = 0.$$  

Suppose \(X^{p^2a} + X = -b^{p^a}\) has a solution \(\gamma\) in \(\mathbb{F}_q\), we have

1)

$$B(0, 0) = \begin{cases} 
-p^{m+d}(p-1)^2, & \text{if } \text{Tr}(\gamma^{p^0+1}) = 0, \\
-p^{m+d}(p-1), & \text{if } \text{Tr}(\gamma^{p^0+1}) \neq 0.
\end{cases}$$
2) if \( c \neq 0 \), then
\[
B(0, c) = \begin{cases} 
(p - 1)p^{m+d}, & \text{if } \text{Tr}(\gamma^{p^a+1}) = 0, \\
-p^{m+d}, & \text{if } \text{Tr}(\gamma^{p^a+1}) \neq 0.
\end{cases}
\]

3) if \( a \neq 0 \), then
\[
B(a, 0) = \begin{cases} 
(p - 1)p^{m+d}, & \text{if } \text{Tr}(\gamma^{p^a+1}) = 0, \\
-p^{m+d+1}\eta(-a\text{Tr}(\gamma^{p^a+1})) - p^{m+d}, & \text{if } \text{Tr}(\gamma^{p^a+1}) \neq 0,
\end{cases}
\]

4) if \( ac \neq 0 \), then
\[
B(a, c) = \begin{cases} 
-p^{m+d}, & \text{if } \text{Tr}(\gamma^{p^a+1}) = 0, \\
-p^{m+d+1}\eta(c^2 - 4a\text{Tr}(\gamma^{p^a+1})) - p^{m+d}, & \text{if } \text{Tr}(\gamma^{p^a+1}) = e^2/(4a), \\
-p^m \text{ otherwise.}
\end{cases}
\]

**Proof:** If \( m/d \equiv 0 \mod 2 \), by Theorem 4.1 in [7], we know that \( X^{p^{2a}} + X \) is not a permutation polynomial. Then using Lemma [15], the proof is similar to that of Lemma [17]. We omit the details.

**Lemma 19:** Set \( f(X) = X^{p^{2a}} + X \) and
\[
S = \{ b \in \mathbb{F}_q : f(x) = -b^{p^a} \text{ is solvable in } \mathbb{F}_q \}.
\]
If \( m/d \) is even, then \( |S| = p^{e-2d} \).

**Proof:** Note that both \( e/d \) and \( m/d \) are even. By Theorem 4.1 in [7], it is easy to see that \( f(X) = 0 \) has \( p^{2d} \) solutions in \( \mathbb{F}_q \). So does the equation \( f(X) = -b^{p^a} \) with \( b \in S \). For \( b_1, b_2 \in \mathbb{F}_q \) and \( b_1 \neq b_2 \), it is not hard to know that the two equations \( f(X) = -b_1^{p^a} \) and \( f(X) = -b_2^{p^a} \) have no common solutions. Furthermore, for each \( a \in \mathbb{F}_q \), there must exist \( b \in \mathbb{F}_q \) such that \( f(a) = -b^{p^a} \), since \( X^{p^a} \) is a permutation polynomial over \( \mathbb{F}_q \). So the desired result can be concluded with the above discussions.

**B. Proofs of theorems in Section A**

For \( a \in \mathbb{F}_p \), recall that we set
\[
D_a = \{ x \in \mathbb{F}_q^* : \text{Tr}(x^{p^a+1}) = a \}.
\]
Let
\[
n_a = \begin{cases} 
|D_a \cup \{0\}|, & \text{if } a = 0, \\
|D_a|, & \text{if } a \neq 0.
\end{cases}
\]
From the definition of \( n_a \), we know
\[
n_a = \frac{1}{p} \sum_{x \in \mathbb{F}_q} \left( \sum_{y \in \mathbb{F}_p} \zeta_p^{y\text{Tr}(x^{p^a+1})-ax} \right) = p^{e-1} + \frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{-ay} \sum_{x \in \mathbb{F}_q} \zeta_p^{y\text{Tr}(x^{p^a+1})}.
\]
Therefore,
\[
n_a = p^{e-1} + p^{-1} A(a),
\]
(3.5)
where \( A(a) \) is defined by (3.1).

For \( a, c \in \mathbb{F}_p \) and \( b \in \mathbb{F}_q^* \), define
\[
N_b(a, c) = \{ x \in \mathbb{F}_q : \text{Tr}(x^{p^a+1}) = a \text{ and } \text{Tr}(bx) = c \}.
\]
Let \( \text{wt}(c_b) \) denote the Hamming weight of the codeword \( c_b \ (b \in \mathbb{F}_q^*) \) of the code \( C_{D_a} \). It is not difficult to see that
\[
\text{wt}(c_b) = n_a - |N_b(a, c)|.
\]
(3.6)
From definition, for \( b \in \mathbb{F}_q^* \), and \( a, c \in \mathbb{F}_p \) we have

\[
|N_b(a, c)| = p^{-2} \sum_{x \in \mathbb{F}_q} \left( \sum_{y \in \mathbb{F}_p} \zeta_p^{y \text{Tr}(x b^{\alpha+1} - ay)} \right) \left( \sum_{z \in \mathbb{F}_p} \zeta_p^{z \text{Tr}(bx) - cz} \right)
\]

\[
= p^{-2} \sum_{x \in \mathbb{F}_q} \left( 1 + \sum_{y \in \mathbb{F}_p^*} \zeta_p^{y \text{Tr}(x b^{\alpha+1} - ay)} \right) \left( 1 + \sum_{z \in \mathbb{F}_p^*} \zeta_p^{z \text{Tr}(bx) - cz} \right)
\]

\[
= p^{e-2} + p^{-2} \sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \zeta_p^{y \text{Tr}(x b^{\alpha+1} - ay)} + p^{-2} \sum_{z \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \zeta_p^{z \text{Tr}(bx) - cz}
\]

\[
= p^{e-2} + p^{-2} A(a) + p^{-2} B(a, c)
\]

(3.7)

where \( A(a) \) and \( B(a, c) \) are defined by (3.1) and (3.3), respectively.

Our task in the sequel is to calculate \( n_a, |N_b(a, c)| \) and give the proofs of the main results.

1) The first case: \( a = 0 \) and \( m/d \equiv 1 \mod 2 \).

Let \( \gamma \) be the unique solution of equation \( X^{b^{\alpha}} + X = -b^{\alpha} \). By (3.7), Lemmas \( \text{16 and 17} \) we have the following two lemmas.

**Lemma 20:** Let \( b \in \mathbb{F}_q^* \), then

\[
|N_b(0, 0)| = \begin{cases} 
  p^{e-2} - (p - 1)p^{m-1}, & \text{if } \text{Tr}(\gamma b^{\alpha+1}) = 0, \\
  p^{e-2}, & \text{if } \text{Tr}(\gamma b^{\alpha+1}) \neq 0.
\end{cases}
\]

**Lemma 21:** For \( b \in \mathbb{F}_q^* \) and \( c \in \mathbb{F}_p^* \), we have

\[
|N_b(0, c)| = \begin{cases} 
  p^{e-2}, & \text{if } \text{Tr}(\gamma b^{\alpha+1}) = 0, \\
  p^{e-2} - p^{m-1}, & \text{if } \text{Tr}(\gamma b^{\alpha+1}) \neq 0.
\end{cases}
\]

Now it comes to prove Theorem \( \text{2} \).

**Proof:** By (3.6) and Lemma \( \text{16} \) we get \( n_0 = p^{e-1} - (p - 1)p^{m-1} \). Using Lemma \( \text{20} \) we have \( \text{wt}(c_b) \in \{(p - 1)p^{e-2}, (p - 1)p^{e-2} - (p - 1)p^{m-1}\} \). Because \( \text{wt}(c_b) \neq 0 \) for each \( b \in \mathbb{F}_q^* \), the dimension of \( \mathcal{C}_{D_0} \) is \( e \). Suppose

\[
b_1 = (p - 1)p^{e-2},
\]

\[
b_2 = (p - 1)p^{e-2} - (p - 1)p^{m-1}.
\]

Also by Lemma \( \text{20} \) and the value of \( n_0 \), we know that

\[
A_{b_1} = p^{e-1} - (p - 1)p^{m-1} - 1,
\]

\[
A_{b_2} = (p - 1)(p^{e-1} - p^{m-1}).
\]

Hence, we get the Table 1. By the above two lemma, it is easy to get the complete weight enumerator of \( \mathcal{C}_{D_0} \). And we complete the proof of Theorem \( \text{2} \). \hfill \blacksquare
2) The second case: \( a \neq 0, \ m/d \equiv 1 \mod 2 \).

By (3.5), (3.7) and Lemmas 16 and 17 it is easy to get the values of \( n_a, |N_b(a, 0)| \) and \(|N_b(a, c)|\).

Lemma 22: For \( a \in \mathbb{F}_q^* \), if \( m/d \equiv 1 \mod 2 \), then \( n_a = p^{e-1} + p^{m-1} \).

Lemma 23: For \( b \in \mathbb{F}_q^* \) and \( a \in \mathbb{F}_p^* \), if \( m/d \equiv 1 \mod 2 \), then
\[
|N_b(a, 0)| = \begin{cases} 
  p^{e-2} + p^{m-1}, & \text{if } \text{Tr}(\gamma^{p^a + 1}) = 0, \\
  p^{e-2} - p^{m-1} \gamma(-a \text{Tr}(\gamma^{p^a + 1})), & \text{if } \text{Tr}(\gamma^{p^a + 1}) \neq 0.
\end{cases}
\]

Lemma 24: For \( b \in \mathbb{F}_q^* \), \( a \) and \( c \in \mathbb{F}_p^* \), if \( m/d \equiv 1 \mod 2 \), then
\[
|N_b(a, c)| = \begin{cases} 
  p^{e-2}, & \text{if } \text{Tr}(\gamma^{p^a + 1}) = 0, \\
  p^{e-2} - p^{m-1} \gamma(c^2 - 4a \text{Tr}(\gamma^{p^a + 1})), & \text{otherwise}.
\end{cases}
\]

Now we begin to prove Theorem 4.

Proof: In this case, using the above three lemma, as the proof of Theorem 2 one can prove Theorem 4 similarly. The details are omitted. \quad \blacksquare

3) The third case: \( a = 0 \) and \( m/d \equiv 0 \mod 2 \).

By (3.5), (3.7) and Lemmas 16 and 18 we get the following two lemmas.

Lemma 25: Let \( b \in \mathbb{F}_q^* \) and \( m/d \equiv 0 \mod 2 \). If \( X^{p^{2a}} + X = -b^{p^a} \) has no solution in \( \mathbb{F}_q \), then
\[
|N_b(0, 0)| = p^{e-2} - (p - 1)p^{m+d-2}.
\]

If \( X^{p^{2a}} + X = -b^{p^a} \) has a solution \( \gamma \) in \( \mathbb{F}_q \), then
\[
|N_b(0, 0)| = \begin{cases} 
  p^{e-2} - (p - 1)p^{m+d-1}, & \text{if } \text{Tr}(\gamma^{p^a + 1}) = 0, \\
  p^{e-2}, & \text{if } \text{Tr}(\gamma^{p^a + 1}) \neq 0.
\end{cases}
\]

Lemma 26: Let \( b \in \mathbb{F}_q^* \), \( c \in \mathbb{F}_p^* \) and \( m/d \equiv 0 \mod 2 \). If \( X^{p^{2a}} + X = -b^{p^a} \) has no solution in \( \mathbb{F}_q \), then
\[
|N_b(0, c)| = p^{e-2} - (p - 1)p^{m+d-2}.
\]

If \( X^{p^{2a}} + X = -b^{p^a} \) has a solution \( \gamma \) in \( \mathbb{F}_q \), then
\[
|N_b(0, c)| = \begin{cases} 
  p^{e-2}, & \text{if } \text{Tr}(\gamma^{p^a + 1}) = 0, \\
  p^{e-2} - p^{m+d-1}, & \text{if } \text{Tr}(\gamma^{p^a + 1}) \neq 0.
\end{cases}
\]

Now we give the proof of Theorem 6.

Proof: By (3.5) and Lemma 16 we get \( n_0 = p^{e-1} - (p - 1)p^{m+d-1} \). Together with Lemma 25 we have that \( \text{wt}(c) \) has three nonzero values. Suppose
\[
\begin{align*}
  b_1 &= (p - 1)p^{e-1} - (p - 1)^2 p^{m+d-2}, \\
  b_2 &= (p - 1)p^{e-2}, \\
  b_3 &= (p - 1)p^{e-2} - (p - 1)p^{m+d-1}.
\end{align*}
\]

By Lemma 19 \( A_{b_1} = p^e - p^{e-2d} \). By the first two Pless Power Moments([19], p. 260) the frequency \( A_{b_i} \) of \( w_i \) satisfies the following equations:
\[
\begin{cases} 
  A_{b_1} + A_{b_2} + A_{b_3} = p^m - 1, \\
  b_1 A_{b_1} + b_2 A_{b_2} + b_3 A_{b_3} = p^{e-1}(p - 1)n,
\end{cases}
\]
(3.8)

where \( n = p^{e-1} - (p - 1)p^{m+d-1} - 1 \). Solving the equations gives the weight distribution of Table 5. By Lemma 26 together the definition of complete weight enumerator of codes, we can obtain the complete weight enumerator if \( C_{D_b} \). The proof is completed. \quad \blacksquare
4) The fourth case: \( a \neq 0 \) and \( m/d \equiv 0 \) mod 2.

By (3.5), (3.7), Lemmas 16 and 18 we have the following three lemmas.

**Lemma 27:** Let \( a \in \mathbb{F}_p^* \) and \( m/d \equiv 0 \) mod 2. Then \( n_\alpha = p^{e-1} + p^{m+d-1} \).

**Lemma 28:** Let \( b \in \mathbb{F}_q^* \), \( a \in \mathbb{F}_p^* \) and \( m/d \equiv 0 \) mod 2. If \( X^{p^{2\alpha}} + X = -b^{\alpha} \) has no solution in \( \mathbb{F}_q \), then
\[ |N_b(a,0)| = p^{e-2} + p^{m+d-2}. \]

If \( X^{p^{2\alpha}} + X = -b^{\alpha} \) has a solution \( \gamma \) in \( \mathbb{F}_q \), then
\[ |N_b(a,0)| = \begin{cases} p^{e-2} + p^{m+d-1}, & \text{if } \text{Tr}(\gamma^{p^{\alpha}+1}) = 0, \\ p^{e-2} - p^{m+d-1} \eta \left(-a \text{Tr}(\gamma^{p^{\alpha}+1})\right), & \text{if } \text{Tr}(\gamma^{p^{\alpha}+1}) \neq 0. \end{cases} \]

**Lemma 29:** Let \( a, c \in \mathbb{F}_p^* \) and \( b \in \mathbb{F}_q^* \). Let \( m/d \equiv 0 \) mod 2. If \( X^{p^{2\alpha}} + X = -b^{\alpha} \) has no solution in \( \mathbb{F}_q \), then
\[ |N_b(a,c)| = p^{e-2} + p^{m+d-2}. \]

If \( X^{p^{2\alpha}} + X = -b^{\alpha} \) has a solution \( \gamma \) in \( \mathbb{F}_q \), then
\[ |N_b(a,c)| = \begin{cases} p^{e-2}, & \text{if } \text{Tr}(\gamma^{p^{\alpha}+1}) = 0, \\ p^{e-2}, & \text{if } \text{Tr}(\gamma^{p^{\alpha}+1}) = c^2/(4a), \\ p^{e-2} - p^{m+d-1} \eta \left(c^2 - 4a \text{Tr}(\gamma^{p^{\alpha}+1})\right), & \text{otherwise}. \end{cases} \]

Now we begin to prove Theorem 8.

**Proof:** In this case, by the three lemmas above, we can get the results in Theorem 6. And we omit the details.

By the definition of \( \overline{C}_{D_n} \), Corollaries 3 5 7 and 9 follow directly from Theorems 2 4 6 and 8 respectively. We omit the proofs.

**IV. CONCLUDING REMARKS**

In this paper, the complete weight enumerators of two families of linear code with few weights are determined. Let \( w_{\min} \) and \( w_{\max} \) denote the minimum and maximum nonzero weight of a linear code, respectively. As introduced in [33], any linear code over \( \mathbb{F}_p \) can be employed to construct secret sharing schemes with interesting access structures if \( \frac{w_{\min}}{w_{\max}} > \frac{b-1}{p} \). It can be verified that the linear code in Theorem 2, Corollary 3 and Theorem 4 have the property \( \frac{w_{\min}}{w_{\max}} > \frac{b-1}{p} \) if \( m \geq 3 \).

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