Numerical Construction of LISS Lyapunov Functions under a Small Gain Condition

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Abstract

In the stability analysis of large-scale interconnected systems it is frequently desirable to be able to determine a decay point of the gain operator, i.e., a point whose image under the monotone operator is strictly smaller than the point itself. The set of such decay points plays a crucial role in checking, in a semi-global fashion, the local input-to-state stability of an interconnected system and in the numerical construction of a LISS Lyapunov function. We provide a homotopy algorithm that computes a decay point of a monotone operator. For this purpose we use a fixed point algorithm and provide a function whose fixed points correspond to decay points of the monotone operator. The advantage to an earlier algorithm is demonstrated. Furthermore an example is given which shows how to analyze a given perturbed interconnected system.

Keywords: homotopy algorithm, monotone operator, LISS Lyapunov function, interconnected system, small gain condition

In recent years large-scale systems have received renewed attention with applications in formation control, logistics, consensus dynamics, networked control systems and further applications. While stability conditions for such large-scale systems have already been studied in [20, 26, 32] based on linear gains and Lyapunov techniques, nonlinear approaches are more recent. The groundbreaking concept that has proven fruitful is the notion of input-to-state stability (ISS) as introduced in [28].

For large-scale nonlinear systems it may be difficult to prove ISS directly, but if a large-scale system is defined through the interconnection of a number of smaller components, which are ISS, then there exist small gain type conditions guaranteeing the ISS property for the interconnected system. For the case of two subsystems this result was obtained in [15, 14] both in a trajectory based as well as a Lyapunov formulation. Recently, there has been a substantial effort to extend these results to the case of a greater number of subsystems, see [7, 8, 12, 18, 22, 4]. It is the purpose of this paper to provide numerical methods that make some of the available results applicable for practical problems.

The general setting is here to consider a number of systems that are input-to-state stable with respect to external and internal inputs. The effect of the subsystems, described by comparison functions, is collected in the gain matrix $\Gamma$. The special structure of the interconnected system now leads to a monotone operator $\Gamma_\mu$ on the positive orthant $\mathbb{R}_+^N$. So-called monotone
aggregation functions can be used to formulate the effect of several inputs on a system in a general manner. Standard examples of such functions are summation and maximization, but in [8] some examples are provided that also other types of aggregation functions may be useful depending on the system under consideration. We would like to point out that the particular relation of the maximization and summation formulation of small gain conditions is analyzed in [5]. In [16] the authors study interconnections where small gain conditions are satisfied after certain transient periods and derive stability results.

Many available small gain results state that input-to-state stability for the overall system follows from the existence of a so-called $\Omega$-path with respect to $\Gamma_\mu$, [4, 5, 6, 7, 8, 21]. Furthermore an ISS Lyapunov function for the interconnected system can be constructed using this path and the ISS Lyapunov functions of the subsystems. Note that also for other small gain type formulations as the cycle condition in the maximization case or the spectral radius condition in the linear summation case, it may be seen that these conditions can be equivalently formulated in terms of $\Omega$-paths.

In [8] the construction of an $\Omega$-path is described. The crucial ingredient that usually cannot be obtained in a straightforward manner is a decay point of $\Gamma_\mu$, that is a point $s \in \mathbb{R}_+^N$ for which $\Gamma_\mu(s) \ll s$ in the order induced by the cone $\mathbb{R}_+^N$. Once such a point is found there are straightforward numerical procedures for the construction of Lyapunov functions or for checking the ISS property. There are two particular cases in which a straightforward way is known to compute decay points: (i) If the gains are linear and summation is used, then the problem becomes one of checking whether the spectral radius of $\Gamma$ is below 1 and finding an appropriate eigenvector. As $\Gamma$ is nonnegative this problem is particularly easy and well studied; (ii) If the maximization formulation of ISS is used then a very nice observation of [17] is that, provided a small gain-condition holds, for $s^* := \max\{s, \Gamma(s), \ldots, \Gamma^{N-1}(s)\}$, we have $\Gamma(s^*) \leq s^*$ for all $s \in \mathbb{R}_+^N$, which is almost a decay point. The methods presented in this paper are suitable for the cases that the problem at hand is not within one of the two classes described above. In this paper we provide numerical procedures for computing such points and thus also for local $\Omega$-paths. We call the approach semi-global because it does not require a priori restrictions. In particular, if a small gain condition is satisfied globally, then the design variables of the algorithm can in principle be chosen so that the numerically guaranteed region of stability is arbitrarily large.

As we compute a decay point numerically the overall construction of Lyapunov functions as well as the verification of the ISS property is only performed locally. Indeed, the approach relies on local results of the small gain type. Local small gain results have been considered in [3, 2, 13] in an input-output operator context, resp. for discrete-time systems. In [6] local ISS (LISS) definitions and local small gain theorems within the framework considered here. In this work the knowledge of a decay point leads to the local input-to-state stability of the interconnected system and to the construction of a LISS Lyapunov function.

The algorithm developed here, that computes a decay point for a given monotone operator $\Gamma_\mu$, is a particular simplicial fixed point algorithm (SFP-algorithm) customized in such a way that we obtain a decay point of $\Gamma_\mu$. To ensure the convergence of the SFP-algorithm we require irreducibility of the gain matrix $\Gamma$. This is no significant restriction because by standard graph theoretic algorithms the irreducible components of the system can be obtained efficiently, [30]. The paper is organized as follows. In Section 1 we provide the necessary notions and a short introduction to comparison functions and graphs. In Section 2 we recall the Lyapunov formulation of ISS for interconnected systems, give a local small gain theorem and outline the
construction of a LISS Lyapunov function for the overall systems. Section 3 contains the main results of this paper. First we recall some facts about homotopy algorithms and introduce the SFP-algorithm where we mainly follow the book of [33]. In subsection 3.4 we state some sufficient conditions on $\Gamma_\mu$ and prove that the SFP-algorithm converges to a decay point of $\Gamma_\mu$. At the end of this section some improvements of the algorithm are discussed. We conclude this work in Section 4 where we discuss two examples. The first one shows that this new algorithm improves on an earlier algorithm that is due to a homotopy algorithm of Eaves [9] (cf. [24]) where we revisit a nonlinear example from [23]. In the second example we use our techniques to show numerically that a particular perturbed interconnected system is LISS.

1 Preliminaries

1.1 Notation and conventions

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{R}^N$ (resp. $\mathbb{R}_+^N$) the vector space of (nonnegative) real column vectors of length $N$. Then $\mathbb{R}^N$ induces a partial order for vectors $v, w \in \mathbb{R}^N$. We denote $v \geq w \iff v_i \geq w_i$, $v > w \iff v_i > w_i$ and $v \neq w$, $v \gg w \iff v_i > w_i$, each for $i = 1, \ldots, N$, where $v_i$ denotes the $i^{th}$ component of the vector $v$. Let $v, w \in \mathbb{R}_+^N$ be given. Then we define the order intervals $[v, w] := \{x \in \mathbb{R}_+^N : v \leq x \leq w\}$ if $v \leq w$, $(v, w) := \{x \in \mathbb{R}_+^N : v \ll x \ll w\}$ if $v \ll w$, and analogously the order intervals $(v, w]$ and $[v, w)$. For $x \in \mathbb{R}^N$ we use the Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^N |x_i|^2}$. The space of measurable and essentially bounded functions is denoted by $L^\infty = L^\infty([0, \infty); \mathbb{R}^M)$ with norm $\|\cdot\|_\infty$.

1.2 Comparison functions and induced monotone operators

To state the stability definitions that we are interested in, three sets of comparison functions are used. We call a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a function of class $\mathcal{K}$, if it is strictly increasing, continuous, and satisfies $\alpha(0) = 0$. If $\alpha \in \mathcal{K}$ is unbounded, it is said to be of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a function of class $\mathcal{KL}$, if it is of class $\mathcal{K}_\infty$ in the first argument and strictly decreasing to zero in the second argument. It is easy to see that if $\rho \in \mathcal{K}_\infty$, then its inverse $\rho^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ exists and is also of class $\mathcal{K}_\infty$.

To formulate general small gain conditions we need the following definition, see [8].

**Definition 1.1** A continuous function $\mu : \mathbb{R}_+^N \to \mathbb{R}_+$ is called a monotone aggregation function, if the following properties hold:

(i) positivity: $\mu(s) \geq 0$ for all $s \in \mathbb{R}_+^N$ and $\mu(s) = 0$, if and only if $s = 0$;

(ii) strict increase: $\mu(s) < \mu(t)$, if $s \ll t$;

(iii) unboundedness: $\mu(s) \to \infty$, if $\|s\| \to \infty$.

The space of monotone aggregation functions is denoted by MAF$_N$.

The properties in Definition 1.1 can be extended to vectors in the sense that $\mu = (\mu_1, \ldots, \mu_N)^T \in$ MAF$_N^N$, $\mu_i \in$ MAF$_N$, $i = 1, \ldots, N$, defines a mapping from $\mathbb{R}^{N \times N}$ to $\mathbb{R}^N$ by $\mu(A)_i = \mu_i(a_{i1}, \ldots, a_{iN})$ for $A = (a_{ij})_{i,j=1}^N \in \mathbb{R}_+^{N \times N}$.
We want to generalize this to matrices of the form \( \Gamma = (\gamma_{ij})_{i,j=1}^{N} \in (\mathcal{K}_{\infty} \cup \{0\})^{N \times N} \), where 0 denotes the zero function. This leads to an operator \( \Gamma_{\mu} : \mathbb{R}_{+}^{N} \rightarrow \mathbb{R}_{+}^{N} \) defined by

\[
\Gamma_{\mu}(s) := (\mu \circ \Gamma)(s) := \begin{pmatrix}
\mu_{1}(\gamma_{11}(s_{1}), \ldots, \gamma_{1N}(s_{N})) \\
\vdots \\
\mu_{N}(\gamma_{N1}(s_{1}), \ldots, \gamma_{NN}(s_{N}))
\end{pmatrix} \in \mathbb{R}_{+}^{N} \quad \text{for } s \in \mathbb{R}_{+}^{N}.
\]

For the \( k \) times composition of this operator we write \( \Gamma_{\mu}^{k} \). We call \( \Gamma_{\mu} \)

(i) **monotone**, if \( \Gamma_{\mu}(v) \leq \Gamma_{\mu}(w) \) for all \( v, w \in \mathbb{R}_{+}^{N} \) with \( v \leq w \);

(ii) **strictly increasing**, if \( \Gamma_{\mu}(v) \ll \Gamma_{\mu}(w) \) for all \( v, w \in \mathbb{R}_{+}^{N} \) with \( v \ll w \).

**Remark 1.2** Note that if \( \Gamma \in (\mathcal{K}_{\infty} \cup \{0\})^{N \times N} \) and \( \mu \in \text{MAF}_{N}^{+} \), then \( \Gamma_{\mu} \) is monotone and satisfies \( \Gamma_{\mu}(0) = 0 \).

The next definition is fundamental in the following.

**Definition 1.3** For a given function \( T : \mathbb{R}_{+}^{N} \rightarrow \mathbb{R}_{+}^{N} \) we define the set of decay \( \Omega \) by

\[
\Omega(T) := \{ s \in \mathbb{R}_{+}^{N} : T(s) \ll s \}.
\]

For short we just write \( \Omega \), if the reference to \( T \) is clear from the context. Points in \( \Omega \) are called decay points.

### 1.3 Graphs and matrices

A directed graph \( G(V, E) \) consists of a finite set of vertices \( V \) and a set of edges \( E \subset V \times V \). If \( G(V, E) \) consists of \( N \) vertices, then we may identify \( V = \{1, \ldots, N\} \). So if \( (j, i) \in E \), then there is an edge from \( j \) to \( i \). The **adjacency matrix** \( A_{G} = (a_{ij}) \) of this graph is defined by \( a_{ij} = 1 \), if \( (j, i) \in E \) and \( a_{ij} = 0 \) else. We call the graph \( G(V, E) \) **strongly connected**, if for each pair \( (i, j) \) there exists a path \( (e_{i_{0},i_{1}}, e_{i_{1},i_{2}}, \ldots, e_{i_{k-1},i_{k}}) \) with \( i = i_{0}, j = i_{k} \) such that \( e_{i_{l-1,i_{l}}} \in E \) for all \( i = 1, \ldots, k \). It is well known that the graph \( G(V, E) \) is strongly connected, if and only if the adjacency matrix \( A_{G} \) is irreducible, i.e., there exists no permutation matrix \( P \) such that

\[
A = P^{T} \begin{pmatrix}
B & C \\
0 & D
\end{pmatrix} P
\]

for suitable, square matrices \( B \) and \( D \). These definitions can be carried over to matrices \( \Gamma \in (\mathcal{K}_{\infty} \cup \{0\})^{N \times N} \). To this end we define the matrix \( A_{\Gamma} = (a_{ij})_{i,j=1}^{N} \) by \( a_{ij} = 1 \), if \( \gamma_{ij} \in \mathcal{K}_{\infty} \), and \( a_{ij} = 0 \), if \( \gamma_{ij} \equiv 0 \). We call \( \Gamma \) irreducible, if the matrix \( A_{\Gamma} \) is.

### 2 Input-to-state stability and small gain theorems

Consider the control system

\[
\dot{x}(t) = f(x(t), u(t)), \quad t \in \mathbb{R}_{+},
\]

where \( u \in \mathbb{R}^{m} \) is the **input** and \( x \in \mathbb{R}^{n} \) is the **state**. We assume that \( f : \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n} \) is continuous and locally Lipschitz in \( x \) uniformly for \( u \) in compact; by this we mean that for every compact \( K_{1} \subset \mathbb{R}^{n} \) and compact subset \( K_{2} \subset \mathbb{R}^{m} \) there is some constant \( c > 0 \) such that \( \|f(x, u) - f(z, u)\| \leq c\|x - z\| \) for all \( x, z \in K_{1} \) and all \( u \in K_{2} \). Further we assume \( f(0, 0) = 0 \) and all solutions can be extended to \([0, \infty)\).
Definition 2.1 Consider the system (2) and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be continuous and locally Lipschitz continuous on $\mathbb{R}^n \setminus \{0\}$. Then $V$ is called an ISS Lyapunov function, if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$

and if there exist $\gamma \in \mathcal{K}$ and a positive definite function $\alpha_3$ such that for all $u \in \mathbb{R}^m$ and almost all $x \in \mathbb{R}^n$,

$$V(x) \geq \gamma(\|u\|) \quad \Rightarrow \quad \nabla V(x)f(x, u) \leq -\alpha_3(\|x\|). \quad (4)$$

Note, that we only assume Lipschitz continuity of the ISS Lyapunov function $V$. By Rademacher’s Theorem, see e.g. [10], this implies that $V$ is differentiable almost everywhere and we consider the decay condition (4) only at points where $V$ is differentiable. An equivalent formulation can be given in terms of Clarke subdifferentials but we refrain from doing so, since this will play no further role in the paper, see also [7, 8].

System (2) is called input-to-state stable (ISS), if it has an ISS Lyapunov function. There is another, trajectory-based definition of ISS which is equivalent to the existence of an ISS Lyapunov function (cf. [29] for smooth Lyapunov functions and [8, Theorem 2.3] for continuous and locally Lipschitz continuous functions).

Now we want to generalize this stability definition to networks. Let $N \in \mathbb{N}$ and consider the $N$ interconnected systems given by

$$\begin{align*}
\dot{x}_1 &= f_1(x_1, \ldots, x_N, u) \\
& \vdots \\
\dot{x}_N &= f_N(x_1, \ldots, x_N, u)
\end{align*} \quad (5)$$

Assume that $x_i \in \mathbb{R}^{n_i}, u \in \mathbb{R}^m$ and the functions $f_i : \mathbb{R}^{\sum_{j=1}^N n_j + m} \to \mathbb{R}^{n_i}$ are continuous and locally Lipschitz in $x = (x_1^T, \ldots, x_N^T)^T$ uniformly for $u$ in compacts. Let $x_i$ denote the state of the $i^{th}$ subsystem and assume $u$ as an external control variable. Without loss of generality we may assume to have the same input for all systems, since we may consider $u$ as partitioned $u = (u_1^T, \ldots, u_N^T)^T$, such that each $u_i$ is the input for subsystem $i$ only. Then each $f_i$ is of the form $f_i(\ldots, u) = \tilde{f}_i(\ldots, \pi_i(u)) = \tilde{f}_i(\ldots, u_i)$ with a projection $\pi_i$.

If we consider individual systems, we treat the state $x_j, j \neq i$, as an independent input for $x_i$. Assume that for each subsystem $i \in \{1, \ldots, N\}$ there exists a continuous and locally Lipschitz continuous function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ such that for suitable $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$

$$\alpha_1(\|x_i\|) \leq V_i(x_i) \leq \alpha_2(\|x_i\|) \quad \text{for all } x_i \in \mathbb{R}^{n_i}. \quad (6)$$

We call the function $V_i$ an ISS Lyapunov function for the subsystem $i$, if there exist $\mu_i \in \text{MAF}_{N+1}, \gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}, j \neq i, \gamma_{iu} \in \mathcal{K} \cup \{0\}$ and a positive definite function $\alpha_i$ such that

$$\begin{align*}
V_i(x_i) &\geq \mu_i(\gamma_{i1}(V_1(x_1)), \ldots, \gamma_{iN}(V_N(x_N)), \gamma_{iu}(\|u\|)) \\
& \Rightarrow \nabla V_i(x_i)f_i(x, u) \leq -\alpha_i(\|x_i\|). \quad (7)
\end{align*}$$

The functions $\gamma_{ij}$ and $\gamma_{iu}$ are called ISS Lyapunov gains. We distinguish between the internal inputs $x_j$ and the external input $u$ of the $i^{th}$ subsystem. These gains indicate the influence of the inputs on the state. This is why we set $\gamma_{ij} \equiv 0$, if $f_i$ does not depend on $x_j$ and we collect the internal inputs into the gain matrix $\Gamma := (\gamma_{ij})_{i,j=1}^N$. Note that $\Gamma$ and the $\mu_i$ define a monotone operator $\Gamma_{\mu} : \mathbb{R}_+^N \to \mathbb{R}_+^N$ as in (1) (cf. Remark 1.2).
2.1 A local small gain theorem

In this section we assume that the interconnected system (5) satisfies an ISS condition of the form (7) for ISS Lyapunov functions $V_i$, $i = 1, \ldots, N$. Denote the corresponding gain operator by $\Gamma_\mu$ as in (1). We assume that $\Gamma$ is irreducible, so that $\Gamma_\mu$ is strictly increasing (cf. [21, Lemma 2.7]). A local ISS Lyapunov function for the overall system given by

$$\dot{x} = f(x,u)$$

and $x = (x_1^\top, \ldots, x_N^\top)^\top, f = (f_1^\top, \ldots, f_N^\top)^\top$ may now be constructed as follows.

Assume there exists a $w \gg 0$ with

$$\Gamma_\mu(w) \ll w. \quad (9)$$

Then the sequence $\Gamma_\mu^k(w), k = 1, 2, \ldots$ is strictly decreasing and so $\lim_{k \to \infty} \Gamma_\mu^k(w)$ exists. If

$$\lim_{k \to \infty} \Gamma_\mu^k(w) = 0, \quad (10)$$

then we define the linear interpolation of the points $\{\Gamma_\mu^k(W)\}_{k \in \mathbb{N}}$ by $\sigma : [0,1] \to \mathbb{R}_+^N$:

$$\sigma(r) = \begin{cases} 0, & \text{if } r = 0 \\ (k^2 + k) \left( \frac{1}{k} r \Gamma_\mu^k(w) + \left[ r - \frac{1}{k+1} \right] \Gamma_\mu^{k-1}(w) \right), & \text{if } r \in \left( \frac{1}{k+1}, \frac{1}{k} \right], k \in \mathbb{N}. \end{cases} \quad (11)$$

Note that $\sigma$ is continuous on $[0,1]$ by (10) and strictly increasing in all component functions as $\Gamma_\mu$ is assumed to be irreducible. With this construction local ISS Lyapunov functions can be constructed using the following summary of existing results (cf. [6, Theorem 5.5]).

**Theorem 2.2** Assume that system (5) satisfies ISS conditions of the form (7) for all $i = 1, \ldots, N$, and that the gain matrix $\Gamma$ is irreducible. If there exists a $w \gg 0$ so that (9) and (10) hold, then a local ISS Lyapunov function for the overall system (8) is given by

$$V(x) = \max_{i=1,\ldots,N} \sigma_i^{-1}(V_i(x_i)). \quad (12)$$

In particular, the implication

$$V(x) \geq \gamma(\|u\|) \Rightarrow \nabla V(x) \cdot f(x,u) \leq -\alpha(V(x)) \quad (13)$$

holds locally with $\gamma \in \mathcal{K}_\infty$ given by [6, Proposition 4.3].

**Remark 2.3** (i) By "local" we mean "in an open neighborhood of the origin $(x^*,u^*) = (0,0)$". In particular, [6, Theorem 5.5] shows that the assured domain of stability increases with the choice of $w$. In particular, if the small gain condition holds globally the domain where (13) holds can be made arbitrarily large.

(ii) By (9) and (10) we have the small gain condition $\Gamma_\mu(s) \not\geq s$ for all $s \in [0,w]$.

(iii) Note that $\sigma(r) \in \Omega(\Gamma_\mu)$ for all $r \in [0,1]$ and $\sigma$ belongs to the class of $\Omega$-paths (cf. [8, Definition 5.1]).
Remark 2.4 (i) Theorem 2.2 is the starting point for our numerical considerations. If we find the decay point \( w \), then the problem of constructing Lyapunov functions or checking small gain conditions becomes easy. In the remainder of the paper we concentrate on giving numerically tractable solutions to this problem.

(ii) In the linear case with \( \mu = \Sigma \) we have \( \Gamma_{\Sigma}(s) = \Gamma s \) with \( \Gamma \in \mathbb{R}^{N \times N}_+ \). Here the existence of a decay point \( w \gg 0 \) with \( \Gamma w \ll w \) is equivalent to the spectral radius of \( \Gamma \) being less than one, i.e., \( 1 > \rho(\Gamma) = \{ |\lambda| : \lambda \) is an eigenvalue of \( \Gamma \} \) (cf. [21, Lemma 1.1]). So finding a decay point is just an eigenvalue problem. This is why we assume \( \Gamma_\mu \) to be nonlinear.

3 A homotopy algorithm for computing a decay point \( w \in \Omega(\Gamma_\mu) \)

In this section we want to develop an algorithm that computes a decay point \( w \in \Omega(\Gamma_\mu) \) for a given continuous and monotone operator \( \Gamma_\mu : \mathbb{R}^N_+ \rightarrow \mathbb{R}^N_+ \). We know that such a point exists for any norm, if the small gain condition

\[
\Gamma_\mu(s) \not\geq s \quad \text{for all } s \in \mathbb{R}^N_+ \setminus \{0\}
\]  

is satisfied (cf. [7, Proposition 5.3]).

To find such a point we will extend a homotopy algorithm that was also used by Merrill [19] to compute fixed points of upper-semicontinuous (u.s.c.) point-to-set mappings. Note that since a continuous single-valued function is in particular an u.s.c. point-to-set mapping our problem falls in the class of problems that can be treated by homotopy algorithms. However, Merrill’s condition introduced in [19] is not sufficient for convergence in our case, as the domain of the mapping is only the nonnegative orthant. The idea to the design of a convergent algorithm is to construct a function \( \phi : \mathbb{R}^N_+ \rightarrow \mathbb{R}^N \), which has the property that its fixed points are decay points of \( \Gamma_\mu \), and to show that the homotopy algorithm will converge to approximate fixed points of \( \phi \), which are also decay points of \( \Gamma_\mu \). This algorithm is semi-global since by choosing design variables appropriately we end up in a decay point with arbitrarily large norm.

In Section 3.1 we present the triangulation we need for the computation of fixed points. Before introducing the homotopy algorithm in Section 3.3 we first provide some facts about homotopy algorithms in Section 3.2. In Section 3.3 we mainly follow the book of Yang [33, Section 4.3]. In Section 3.4 we will give the function \( \phi \) mentioned above and prove the convergence of the SFP-algorithm. Finally in Section 3.5 we give approaches for improving the algorithm and further give suggestions for the choice of the design variables used in the mapping \( \phi \).

3.1 Simplices and triangulations

We briefly recall facts about covering convex sets by triangulations. A set \( C \subset \mathbb{R}^N \) is called convex, if for all \( a, b \in C \) it holds \( S_{a,b} := \{ \lambda a + (1 - \lambda) b : \lambda \in [0,1] \} \subset C \). The convex hull \( \text{co}\{M\} \) of a set \( M \subset \mathbb{R}^N \) is the smallest convex set containing \( M \). If \( M \) is finite, we also say that \( \text{co}\{M\} \) is spanned by \( M \) and denote this by \( \langle \{v^i \in M\} \rangle \). The dimension of a convex set is equal to the dimension of the smallest affine subspace \( U \subset \mathbb{R}^N \) containing \( C \).

Definition 3.1 An \( N \)-simplex \( S \) is an \( N \)-dimensional, convex polytope spanned by \( N + 1 \) vectors \( v^1, \ldots, v^{N+1} \) in \( \mathbb{R}^M \), \( M \geq N \), i.e.,

\[
S = \langle v^1, \ldots, v^{N+1} \rangle.
\]
A subsimplex $\zeta$ of $S$ is a simplex spanned by a subset of the set of vertices of $S$, i.e., $\zeta = \{v^i : i \in I_\zeta\}$ with $I_\zeta \subset \{1, \ldots, N+1\}$. Zero-dimensional subsimplices are just the vertices of the simplex, one-dimensional subsimplices are called edges between the vertices and $(N - 1)$-subsimplices are called facets. The subsimplex $S(j) = \langle \{v^i : i \neq j\} \rangle$ is called the facet opposite $v^j$.

Clearly, since any $N$-simplex is $N$-dimensional, $N$ of the $N+1$ vertices are linearly independent and it holds $v^i \neq v^j$ for $i \neq j$. Simplices can be used to cover convex sets in $\mathbb{R}^N$ as follows.

**Definition 3.2** Let $C$ be an $m$-dimensional convex set in $\mathbb{R}^N$. A set $\mathcal{T}$ of $m$-simplices is called a triangulation of $C$, if

(i) $C$ is the union of all simplices in $\mathcal{T}$;

(ii) for any $\eta_1, \eta_2 \in \mathcal{T}$, $\eta_1 \neq \eta_2$, the intersection $\eta_1 \cap \eta_2$ is either the empty set or a common facet of both;

(iii) every $x \in C$ has an open neighborhood intersecting only a finite number of $\eta \in \mathcal{T}$.

By $\mathcal{T}^k$ we denote the set of all $k$-subsimplices of $\mathcal{T}$. It is easy to see that $\mathcal{T}^N = \mathcal{T}$ and $\mathcal{T}^0$ describes the set of the vertices of the simplices in $\mathcal{T}$. To distinguish simplices, or triangulations, we introduce the diameter of a simplex $\eta \in \mathcal{T}$ by

$$\text{diam}(\eta) = \max\{\|x - y\| : x, y \in \eta\}$$

and the mesh size of a triangulation $\mathcal{T}$ by

$$\text{mesh}(\mathcal{T}) = \sup\{\text{diam}(\eta) : \eta \in \mathcal{T}\}.$$ 

There is one special triangulation of $\mathbb{R}^N$, which will be used to compute decay points. Let $e_i$ denote the $i^{th}$ unit vector in $\mathbb{R}^N$. The $K_1$-triangulation is defined as the set of all $N$-simplices with vertices $x^1, \ldots, x^{N+1}$ such that

$$x^1 \in \mathbb{Z}^N \text{ and } x^{i+1} = x^i + e_{\pi_N(i)} \text{ for all } i \in \{1, \ldots, N\},$$

where $\pi_N = (\pi_N(1), \ldots, \pi_N(N))$ is a permutation of the elements of the set $\{1, \ldots, N\}$. We denote these simplices by $\eta(x^1, \pi_N)$. See [33, Theorem 1.4.8] for a proof that $K_1$ is a triangulation in the sense of Definition 3.2. An illustration of this triangulation is given in Figure 1.

Defining $\delta C = \{\delta x : x \in C\}$ for $C \subset \mathbb{R}^N$, $\delta > 0$, and $\delta F = \{\delta C : C \in F\}$ for a family $F$ of subsets of $\mathbb{R}^N$ we obtain that if $\mathcal{T}$ is a triangulation of $C$ and $\delta > 0$, then $\delta \mathcal{T}$ is a triangulation of $\delta C$. In this way we get the $\delta K_1$-triangulation of $\mathbb{R}^N$ for which mesh($\delta K_1$) = $\delta \sqrt{N}$ for $\delta > 0$. Let $\mathcal{T}$ be a triangulation of $\mathbb{R}^N \times [0, 1]$ with the restriction $\mathcal{T}^0 \subset \mathbb{R}^N \times \{0, 1\}$, i.e., the vertices only lie in $\mathbb{R}^N \times \{0, 1\}$. Then we call this triangulation two-layered. Let $\tilde{K}_1$ denote the restriction of the $K_1$-triangulation of $\mathbb{R}^{N+1}$ to $\mathbb{R}^N \times [0, 1]$. Then $\tilde{K}_1$ is two-layered. Further define the $(N + 1) \times (N + 1)$-matrix $P = [\delta e_1, \ldots, \delta e_N, e_{N+1}]$ for given $\delta > 0$. Define

$$\tilde{K}_1(\delta) = \{(P y^1, \ldots, P y^{N+2}) : (y^1, \ldots, y^{N+2}) \in \tilde{K}_1\},$$

then $\tilde{K}_1(\delta)$ is a two-layered triangulation of $\mathbb{R}^N \times [0, 1]$. 
3.2 Some facts about homotopy algorithms

In this section we want to provide the basic principles of homotopy algorithms.

**Definition 3.3** Let $f, g : C \to D$ be two continuous mappings from the topological space $C$ to the topological space $D$. We call $f, g$ homotopic, if there exists a continuous mapping $\vartheta : C \times [0,1] \to D$, $(s,t) \mapsto \vartheta(s,t)$ with $\vartheta(s,0) = f(s)$ and $\vartheta(s,1) = g(s)$ for all $s \in C$. We call $\vartheta$ the homotopy from $f$ to $g$.

Let $C$ be a nonempty, compact and convex subset of $\mathbb{R}^N$ and assume that $f : C \to C$ is continuous. Then it follows by Kakutani’s fixed point theorem (cf. [1, p.174]) that there exists at least one fixed point of $f$. To determine any fixed point we use the idea of the classical homotopy. Define the continuous mapping $f_t : C \to C$ by

$$f_t(s) := (1 - t)s_0 + tf(s), \quad t \in [0,1]$$

with $s_0 \in C$. Then by a further application of Kakutani’s fixed point theorem, there exists a fixed point of $f_t$ for every $t \in [0,1]$. We start with the constant mapping $f_0(s) = s_0$ and its fixed point $s_0$. Assume that $t_k \to 1$ for $k \to \infty$, then the sequence of functions $(f_{t_k}()_{k \in \mathbb{N}}$ converges even uniformly to $f_1(\cdot) = f(\cdot)$. Now one can show that the cluster points of the set of fixed points $s_{t_k}$ of $f_{t_k}$ are just the fixed points of $f$. Note that in this approach we have to extend the dimension of this problem, i.e., we now work in the space $C \times [0,1]$.

The numerical procedure for nonempty, compact and convex $C \subseteq \mathbb{R}^N$ is the following. We decompose the space $C \times [0,1]$ in simplices using a suitable triangulation $\mathcal{T}$. Under certain conditions there exists a path in this triangulation from an $N$-simplex $\tau^0 \in C \times \{0\}$ to an $N$-simplex $\tau^* \in C \times \{1\}$ which yields an approximate fixed point of the function $f$.

The algorithm that we use here, denoted by SFP-algorithm (*simplicial fixed point algorithm*) for short, follows the path by using the lexicographic pivoting rule from linear programming. The advantage is that the so-called degeneration problem (i.e., the path ends up in a circuit) cannot occur. We don’t want to enlarge on that fact and will only give the definition of lexicographically positive matrices. For a detailed description we refer to [31, Chapters 2&3].

**Definition 3.4** A row vector is called lexicographically positive, if its first nonzero entry is positive. A matrix $W$ is called lexicographically positive denoted by $W \succ 0$, if every row vector is lexicographically positive.
3.3 The SFP-algorithm

To compute a fixed point of a continuous function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ the SFP-algorithm uses a suitable homotopy $\vartheta : \mathbb{R}^N \times [0, 1] \to \mathbb{R}^N$, and a pivoting method to get from an $N$-simplex $\tau^0 \subset \mathbb{R}^N \times \{0\}$ to an $N$-simplex $\tau^1 \subset \mathbb{R}^N \times \{1\}$ which yields an approximate fixed point of $\phi$. For this purpose we have to triangulate the set $\mathbb{R}^N \times [0, 1]$ suitably.

Let $T$ be a triangulation of $\mathbb{R}^N \times [0, 1]$ with the restriction $T^0 \subset \mathbb{R}^N \times \{0\}$, i.e., $T$ is two-layered. We denote elements of $\mathbb{R}^N \times [0, 1]$ by $y = (v_1, \ldots, v_N, t)$ with $v \in \mathbb{R}^N$ and $t \in [0, 1]$ and define the projection onto the first factor $p_1 : \mathbb{R}^N \times [0, 1] \to \mathbb{R}^N$; $p_1(v, t) = v$. Suppose that the $N$-simplex $\tau = (y^1, y^2, \ldots, y^{N+1}) \in T^N$. We define the diameter of the projection of $\tau$ by

$$\text{diam}_p(\tau) := \max\{||p_1(y^i) - p_1(y^j)|| : i, j \in \{1, \ldots, N + 1\}\}.\]$$

Moreover, the mesh size of the projection of $T$ is defined by

$$\text{mesh}_p(T) := \sup\{\text{diam}_p(\tau) : \tau \in T^N\}.$$}

If $\tau = (y^1, \ldots, y^{N+1}) \in T^N$ and $\tau \subset \mathbb{R}^N \times \{i\}$, $i \in \{0, 1\}$, then $\tau_p := (p_1(y^1), \ldots, p_1(y^{N+1}))$ is an $N$-simplex in $\mathbb{R}^N$. The collection of all such simplices $\tau_p$ is denoted by $\mathcal{T}_i$.

We choose an arbitrary point $(c, 0) \in \mathbb{R}^N \times [0, 1]$ such that $(c, 0)$ lies in the interior of an $N$-simplex $\tau^0 \in \mathcal{T}_0$. Consider the following homotopy mapping $\vartheta : \mathbb{R}^N \times [0, 1] \to \mathbb{R}^N$ defined by

$$\vartheta(v, t) = (1 - t)c + t\phi(v). \quad (15)$$

A point $y$ is called a fixed point of $\vartheta$, if $p_1(y) = \vartheta(y)$. Clearly, $(c, 0)$ is the only fixed point of $\vartheta$ in $\mathbb{R}^N \times \{0\}$ and any fixed point $y$ of $\vartheta$ in $\mathbb{R}^N \times \{1\}$ projects to a fixed point of $\phi$, i.e., $p_1(y) = \phi(p_1(y))$. The concept of labelings establishes a way of studying the relation of the triangulation with approximate fixed points of $\phi$.

**Definition 3.5** Let $T$ be a two-layered triangulation of $\mathbb{R}^N \times [0, 1]$. Then we define the labeling rule $l : \mathbb{R}^N \times [0, 1] \to \mathbb{R}^N$ by

$$l(y) = \vartheta(y) - p_1(y). \quad (16)$$

Let the $N$-simplex $\tau = (y^1, \ldots, y^{N+1}) \subset T^N$ be given. Then we call the $(N + 1) \times (N + 1)$ matrix

$$L(\tau) := \begin{pmatrix} 1 & \cdots & 1 \\ l(y^1) & \cdots & l(y^{N+1}) \end{pmatrix} \quad (17)$$

the labeling matrix of $\tau$.

The $N$-simplex $\tau$ is called complete, if the system

$$L(\tau)W = I_{N+1}, \quad W > 0 \quad (18)$$

has a solution $W^* \in \mathbb{R}^{(N+1)\times(N+1)}$. Complete simplices play an important role in the following since a complete $N$-simplex $\tau \subset \mathbb{R}^N \times \{1\}$ contains an approximate fixed point of $\phi$. In addition, by choosing the mesh size of the triangulation small enough we can claim any accuracy of the approximate fixed point.
Proposition 3.6 Let $D$ be compact and $\phi : D \subset \mathbb{R}^N \to \mathbb{R}^N$ be continuous. For $\epsilon > 0$ let $\delta > 0$ be such that for all $x, y \in D$ we have the implication $\|x - y\| < \delta \Rightarrow \|\phi(x) - \phi(y)\| < \epsilon$. Let $T$ be a two-layered triangulation of $\mathbb{R}^N \times [0, 1]$ with mesh($T$) $< \delta$ and $\tau = \langle y^1, \ldots, y^{N+1} \rangle \subset \mathbb{R}^N \times \{1\}$ a complete simplex in $T$ with $y^j = (v^j, t_j)$ for all $j = 1, \ldots, N + 1$. Let $\lambda \in \mathbb{R}_+^N$ be the solution of the system

$$L(\tau)\tilde{\lambda} = e_1, \quad \tilde{\lambda} \in \mathbb{R}^{N+1}_+. \quad (19)$$

Then $v^* := \sum_{j=1}^{N+1} \lambda_j v^j$ is an approximate fixed point of $\phi$, i.e., $\|\phi(v^*) - v^*\| < \epsilon$.

Proof. Since $\tau \subset \mathbb{R}^N \times \{1\}$ we have $t_j = 1$ for all $j = 1, \ldots, N + 1$ and so $l(y^j) = \theta(y^j) - p_1(y^j) = \phi(v^j) - v^j$ by (15) and (16). Thus (19) is equivalent to

$$(i) \quad \sum_{j=1}^{N+1} \lambda_j = 1 \quad \text{and} \quad (ii) \quad \sum_{j=1}^{N+1} \lambda_j \phi(v^j) = \sum_{j=1}^{N+1} \lambda_j v^j. \quad (20)$$

By (20)(i) $v^*$ is a convex combination of the $v^1, \ldots, v^{N+1}$, i.e., $v^* \in \tau$. But then we have $\|v^* - v^j\| < \delta$ for all $j = 1, \ldots, N + 1$ and by continuity of $\phi$ we have $\|\phi(v^*) - \phi(v^j)\| < \epsilon$ for all $j = 1, \ldots, N + 1$. Together this yields

$$\|\phi(v^*) - v^*\| \overset{(20)(i)}{=} \|\left(\sum_{j=1}^{N+1} \lambda_j \phi(v^*) - \sum_{j=1}^{N+1} \lambda_j v^j\right)\|$$

$$= \left(\sum_{j=1}^{N+1} \lambda_j \|\phi(v^*) - \phi(v^j)\| \overset{(20)(i)}{<} \epsilon. \right.$$

To obtain a complete simplex in $\mathbb{R}^N \times \{1\}$ we first characterize the complete simplices. To this end we define the graph $G(V, E)$ of all complete simplices as follows. An $(N + 1)$-simplex $\eta$ of $T$ is a node, if it has at least one complete facet $\tau$. Two nodes are adjacent and connected by an edge, if they share a common complete facet. The degree of a node $\eta$ is the number of nodes adjacent to $\eta$, denoted by $\text{deg}(\eta)$.

Recall that $\tau^0$ is the $N$-simplex lying on $\mathbb{R}^N \times \{0\}$ and containing $(c, 0)$ in its interior. Let $\eta^0$ be the unique $(N + 1)$-simplex of $T$ having $\tau^0$ as its facet. Then we have (cf. [33, Lemma 4.3.3, Lemma 4.3.4 and Theorem 4.3.5]).

Lemma 3.7 The $N$-simplex $\tau^0$ is the only complete simplex on $\mathbb{R}^N \times \{0\}$.

Lemma 3.8 Given the graph $G(V, E)$ defined as above, for each node $\eta$ of $G(V, E)$, we have

(i) if $\eta$ has a complete facet lying on $\mathbb{R}^N \times \{0\}$ or $\mathbb{R}^N \times \{1\}$, then $\text{deg}(\eta) = 1$;

(ii) in all other cases, $\text{deg}(\eta) = 2$.

Theorem 3.9 For the graph $G(V, E)$ defined as above, each connected component of $G(V, E)$ has one of the following five forms.
(i) a simple circuit (i.e. a path \( e_{0,1}, e_{1,2}, \ldots, e_{k-1,k} \), \( k \in \mathbb{N} \) with \( e_{0,1} = e_{k-1,k} \) and \( e_{i,i+1} \neq e_{j,j+1} \) for \( i \neq j \) and \( i, j \in \{1, \ldots, k-1\} \));

(ii) a finite simple path (i.e. a path without circuits) whose two end nodes all have a complete facet lying on \( \mathbb{R}^N \times \{1\} \);

(iii) an infinite simple path starting with an \((N+1)\)-simplex which has a complete facet lying on \( \mathbb{R}^N \times \{1\} \);

(iv) a finite simple path which starts with the \((N+1)\)-simplex \( \eta^0 \) and ends with another \((N+1)\)-simplex having a complete facet on \( \mathbb{R}^N \times \{1\} \);

(v) an infinite simple path starting with the \((N+1)\)-simplex \( \eta^0 \).

From the point of view of computation we are interested in case (iv). In this case we can algorithmically go from \( \eta^0 \) to a simplex \( \eta^* \in \mathbb{R}^N \times \{1\} \) containing an approximate fixed point of \( \phi \) by Proposition 3.6. A schematic description is given in Figure 2.

**The Simplicial Fixed Point Algorithm**

**Step (0)** Set \( \text{mesh}_p(T) < \delta \). Let \( \tau^0 \) be the unique \( N \)-simplex of \( T^N \) containing \( (c,0) \) in its interior. Let \( \eta^0 \) be the unique \((N+1)\)-simplex in \( T \) which has \( \tau^0 \) as its facet. Let \( y^+ \) be the vertex of \( \eta^0 \) that is not a vertex of \( \tau^0 \). Set \( k = 0 \).

**Step (1)** Compute \( W = L^{-1}(\tau^k) \) with \( L \) from (17). Let \( W_i \) denote the \( i \)th row of \( W \). Compute \( l(y^+) \) and let \( q = (1,l^T(y^+))^T \). Let \( p = (p_1, \ldots, p_{n+1})^T = Wq \) denote the coefficient vector of the linear combination \( l(y^+) = \sum_{i=1}^{N+1} p_i l(y^i) \). Compute \( \zeta \in \{1, \ldots, N+1\} \) so that the quotient

\[
\frac{W_{h}}{p_{h}} = \min_{h} \left\{ \frac{W_{h}}{p_{h}} : p_{h} > 0, \; h = 1, \ldots, N+1 \right\}
\]

is lexicographically positive minimal. Note that \( \zeta \) is unique (cf. [33, Theorem 4.2.7]). Let \( \tau^{k+1} \) be the facet of \( \eta^k \) opposite \( y^\zeta \). If \( \tau^{k+1} \) lies on \( \mathbb{R}^N \times \{1\} \), this facet yields an approximate fixed point of \( \phi \) and stop. If \( \tau^{k+1} \) does not lie on \( \mathbb{R}^N \times \{1\} \), go to Step (2).

**Step (2)** Find a simplex \( \eta^{k+1} \) sharing the facet \( \tau^{k+1} \) with \( \eta^k \) (which is unique by Lemma 3.8), and let \( y^+ \) be the vertex \( \eta^{k+1} \) not being a vertex of \( \tau^{k+1} \). Set \( k = k + 1 \) and return to Step (1).

Figure 2: The SFP-algorithm

**Remark 3.10** In order to guarantee case (iv) in Theorem 3.9 Merrill (cf. [33]) gave a contraction condition that is sufficient for the convergence of the algorithm for a u.s.c. point-to-set mapping \( \phi : \mathbb{R}^N \rightarrow \mathbb{R}^N \), in particular for a continuous single-valued function \( \phi : \mathbb{R}^N \rightarrow \mathbb{R}^N \), see [33, Theorem 4.3.6]. The method of proof is to show that there is only a compact subset \( D \subset \mathbb{R}^N \) yielding complete simplices, so the path must be finite. Note that this condition is not
sufficient for convergence, if we choose \( \phi : \mathbb{R}_+^N \to \mathbb{R}^N \). In particular the function \( \phi \) defined in (21) satisfies Merrill’s condition (cf. [11, Satz 4.28]) but we have to impose other conditions to guarantee convergence.

### 3.4 Using the SFP-algorithm for computing decay points

Now we want to use the SFP-algorithm to compute a decay point \( w \in \Omega(\Gamma_\mu) \) of the monotone operator \( \Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N \) which satisfies \( \Gamma_\mu(0) = 0 \).

In the following the aim is to find a suitable function \( \phi \) whose fixed points \( w = \phi(w) \) correspond to decay points \( w \in \Omega(\Gamma_\mu) \), and to show that the SFP-algorithm converges for this choice of \( \phi \). Since \( \Gamma_\mu(0) = 0 \) and this point yields no information, we have to exclude 0 from being a fixed point of \( \phi \). Also, in order to show that complete simplices can only lie in a compact subset of \( \mathbb{R}_+^N \) it is desirable to have \( \phi \) small for large \( v \).

Consider the function \( \phi : \mathbb{R}_+^N \to \mathbb{R}^N \) defined by

\[
\phi(v) = \Gamma_\mu(v) \left( 1 + \min \left\{ 0, \frac{\kappa - 2\|v\|}{\|v\| + \kappa_0} \right\} \right) + \max \left\{ 0, \kappa_h - 2\|v\| \right\} e. \tag{21}
\]

Here let \( \kappa_0 > 0, \kappa_G > \kappa_h > 0 \) and \( e := \sum_{i=1}^N e_i \) the \( N \)-dimensional vector of ones. We illustrate the components of \( \phi \) in Figure 3.

![Figure 3: The components of the function \( \phi \)](image)

Some properties of \( \phi \) are as follows:

(i) \( \phi \) is continuous on \( \mathbb{R}_+^N \) since \( \Gamma_\mu \) is continuous on \( \mathbb{R}_+^N \).

(ii) For large \( v \in \mathbb{R}_+^N \) it holds \( \phi(v) < 0 \).

(iii) It holds \( \phi(0) = \kappa_h e \gg 0 \), i.e., the origin cannot be a fixed point of \( \phi \).

In Figure 4 we illustrate the definition of \( \phi \) on the positive orthant. To this end we partition the positive orthant in five regions:

- \( I = \{ v \in \mathbb{R}_+^N : \|v\| \in [0, \kappa_h/2) \} \), \quad \( I' = I \times [0, 1] \),
- \( II = \{ v \in \mathbb{R}_+^N : \|v\| \in [\kappa_h/2, \kappa_G/2) \} \), \quad \( II' = II \times [0, 1] \),
- \( III = \{ v \in \mathbb{R}_+^N : \|v\| \in [\kappa_G/2, \kappa_G + \kappa_0) \} \), \quad \( III' = III \times [0, 1] \),
- \( IV = \{ v \in \mathbb{R}_+^N : \|v\| \in [\kappa_G + \kappa_0, \kappa_G + \kappa_0 + \delta] \} \), \quad \( IV' = IV \times [0, 1] \),
- \( V = \{ v \in \mathbb{R}_+^N : \|v\| \in (\kappa_G + \kappa_0 + \delta, \infty) \} \), \quad \( V' = V \times [0, 1] \).
The next proposition indicates the relation between fixed points of $\phi$ and decay points of $\Gamma_\mu$.

**Proposition 3.11** Let $\phi : \mathbb{R}_+^N \to \mathbb{R}_+^N$ be defined as in (21) and assume that $\Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is monotone and satisfies the small gain condition (14). Let $s \in \mathbb{R}_+^N$ be a fixed point of the function $\phi$, i.e., $s = \phi(s)$. Then $s$ lies in the set of decay of the function $\Gamma_\mu$, i.e., $s \in \Omega(\Gamma_\mu)$. Moreover, $s \in I$.

**Proof.** We distinguish between the following two cases for $s \in \mathbb{R}_+^N$:

(i) $0 \leq \|s\| < \frac{\kappa h}{2}$: In this case $s = \phi(s) = \Gamma_\mu(s) + (\kappa h - 2\|s\|)e \gg \Gamma_\mu(s)$. It follows that $s \gg \Gamma_\mu(s)$, so $s \in \Omega(\Gamma_\mu)$. In particular, $s \in I$.

(ii) $\frac{\kappa h}{2} \leq \|s\|$: We have $s = \phi(s) = \Gamma_\mu(s)(1 + \min\{0, \frac{\kappa h - 2\|s\|}{\|s\| + \kappa_0}\}) \leq \Gamma_\mu(s)$ but this is a contradiction to the small gain condition (14), so this case cannot occur.

In the following we will always use the $\tilde{K}_1(\delta)$-triangulation. This triangulation has the essential advantage that the vertices of an $N$-simplex $\tau = (y^1, \ldots, y^{N+1})$ are in the order of $\mathbb{R}_+^{N+1}$, i.e., it holds $y^1 < \ldots < y^{N+1}$. Note that $y = (v, t) \in \mathbb{R}_+^N \times \{0, 1\}$.

Again, the SFP-algorithm starts with the $(N + 1)$-simplex $\eta^0$ which has the $N$-simplex $\tau^0 \in \mathbb{R}_+^N \times \{0\}$ as a facet containing $(c, 0)$ in its interior, where $c$ determines the homotopy mapping $\vartheta$ in (15). Here we choose $c \in I \cup II$ and any approximate fixed point $c'$ will also lie in $I \cup II$, see Theorem 3.14. Then the algorithm follows the path of complete $N$-simplices. If we can show that this path is finite and inside of the positive orthant, then we get, by Theorem 3.9, that the SFP-algorithm ends up with a $(N + 1)$-simplex containing a complete facet on $\mathbb{R}_+^N \times \{1\}$. Proposition 3.6 now tells us that this simplex contains an approximate fixed point of $\phi$. 

Figure 4: The definition of $\phi$ illustrated on the positive orthant
A first rough estimation where the path of complete simplices can run is given in the next proposition.

**Proposition 3.12** Let \( \phi : \mathbb{R}_+^N \to \mathbb{R}^N \) be defined as in (21) and assume that \( \Gamma_\mu : \mathbb{R}_+^N \to \mathbb{R}_+^N \) is monotone. Assume that the constant \( c \) used in (15) satisfies \( c \in I \cup \Pi \) and let \( \tau = \langle y^1, \ldots, y^{N+1} \rangle \) be an \( N \)-simplex in \( \mathcal{V} \). Then \( \tau \) is not complete.

**Proof.** We prove this by contradiction. Assume that \( \tau \) is complete. Then the linear system

\[
L(\tau)W = I_{N+1}, \quad W > 0
\]

with \( L \) defined as in (17), has a lexicographically positive solution \( W \). We have \( y^j = (v^j, t_j) \in \mathbb{R}_+^N \times \{0, 1\} \) and by \( \tau \subset \mathcal{V}' \) we have \( \|v^j\| > \kappa_\Gamma + \kappa_0 + \delta \), i.e., \( \phi(v^j) < 0 \) for all \( j = 1, \ldots, N+1 \). So we have the following two cases for

\[
l(y^j) = \phi(v^j, t_j) - v^j = (1 - t_j)c + t_i \phi(v^j) - v^j.
\]

(i) If \( t_j = 1 \), then \( l(y^j) < 0 \);

(ii) If \( t_j = 0 \), then \( l(y^j) = c - v^j \). Since \( c \in I \cup \Pi \) and \( v^j \in \mathcal{V} \) for all \( j = 1, \ldots, N+1 \) there exists a component \( i^* \in \{1, \ldots, N\} \) with \( c_{i^*} < v^j_{i^*} \leq v^j_{\ast} \) for all \( j = 2, \ldots, N+1 \).

Together it follows

\[
l(y^j)_{i^*} < 0 \quad \text{for all } j = 1, \ldots, N+1.
\]

Let \( L_l \) denote the \( l \)-th row of \( L \) and let \( W^m \) denote the \( m \)-th column of \( W \). Then we have \( W^1 \in \mathbb{R}_+^N \setminus \{0\} \) since \( W \) is lexicographically positive. By (23) we have \( L_{i^*+1}W^1 < 0 \). But then \( L_{i^*+1}W^1 < 0 \) in contradiction to (22). So \( \tau \) cannot be complete. \( \square \)

Note that this does not show that the path starting in \( \eta^0 \) is inside the positive orthant. To prove this we have to look at the boundary of the positive orthant. Here we need some additional assumptions. Note that for \( \Gamma_\mu, \Gamma \in (\mathcal{K}_\infty \cup \{0\})^{N \times N} \) is the underlying gain matrix (cf. Section 1.3), and by Remark 1.2 the operator \( \Gamma_\mu \) is monotone.

**Theorem 3.13** Let \( \phi : \mathbb{R}_+^N \to \mathbb{R}^N \) be defined as in (21) and assume that the underlying gain matrix \( \Gamma \) is irreducible. Let \( \tau = \langle y^1, \ldots, y^{N+1} \rangle \) be an \( N \)-simplex on the boundary of the positive orthant. If \( \|v^{N+1}\| = \|p_1(y^{N+1})\| < \kappa_0 + \kappa_\Gamma \) then \( \tau \) is not complete.

**Proof.** If \( \tau \) is an \( N \)-simplex on the boundary of the positive orthant then there exists an index \( i^* \in \{1, \ldots, N\} \) with \( v^j_{i^*} = 0 \) for all \( j = 1, \ldots, N+1 \). We prove by contradiction that \( \tau \) cannot be complete, if \( \|v^{N+1}\| < \kappa_0 + \kappa_\Gamma \). So assume

\[
L(\tau)W = I_{N+1}, \quad W > 0
\]

has the solution \( W^* \) and let \( \lambda \in \mathbb{R}_+^{N+1} \) be the first column of \( W^* \). Then it follows by (24) and using (17) that

\[
\sum_{j=1}^{N+1} \lambda_j l(y^j) = 0, \quad \sum_{j=1}^{N+1} \lambda_j = 1.
\]
The case \( ||v^{N+1}|| < \frac{\kappa_\delta}{2} \) yields, using (16), \( l(y^j)_i^\tau = (1 - t_j)c_i + t_j\phi(v^j)_i^\tau > 0 \) for all \( j \in \{1, \ldots, N + 1\} \), so \( \sum_{j=1}^{N+1} \lambda_j l(y^j)_i^\tau > 0 \) since \( \lambda \neq 0 \) and \( c > 0 \) (since \( c \) lies in the interior of a simplex \( \tau \)). But this is a contradiction to (25).

Now assume \( ||v^{N+1}|| < \frac{\kappa_\delta}{2} \). Then it holds \( l(y^j)_i^\tau \geq 0 \). In particular, \( l(y^j)_i^\tau = (1 - t_j)c_i + t_j\Gamma(v^j)_i^\tau + t_j\kappa_h - 2\|v^j\| > 0 \) for \( \|v^j\| < \frac{\kappa_\delta}{2} \). Let \( r \in \{1, \ldots, N + 1\} \) with \( t_1 = \ldots = t_r = 0 \) and \( t_{r+1} = \ldots = t_{N+1} = 1 \) as well as \( \rho \in \{1, \ldots, N + 1\} \) with \( \|v^\rho\| < \frac{\kappa_\delta}{2} \) and \( \|v^{\rho+1}\| \geq \frac{\kappa_\delta}{2} \).

Then equation (25) is equivalent to

\[
\sum_{j=1}^r \lambda_j c + \sum_{j=r+1}^{N+1} \lambda_j \Gamma(v^j) = \sum_{j=r+1}^{N+1} \lambda_j v^j, \quad \sum_{j=1}^{N+1} \lambda_j = 1. \tag{26}
\]

Since \( v^j_\rho = 0 \) for all \( j = 1, \ldots, N + 1 \) it follows \( \lambda_j = 0 \) for \( j = 1, \ldots, \hat{r} \) with \( \hat{r} := \max\{r, \rho\} \) and equation (26) is equivalent to

\[
\sum_{j=\hat{r}+1}^{N+1} \lambda_j \Gamma(v^j) = \sum_{j=\hat{r}+1}^{N+1} \lambda_j v^j, \quad \sum_{j=\hat{r}+1}^{N+1} \lambda_j = 1. \tag{27}
\]

Now there exists a largest index \( \tilde{r} \in \{1, \ldots, N + 1\} \) with \( \Gamma(v^{\tilde{r}})_i^\tau = 0 \) and \( \Gamma(v^{\tilde{r}})_i^\tau > 0 \). Since the \( v^j \) are ordered by the \( \tilde{K}_1(\delta) \)-triangulation, it follows by monotonicity of \( \Gamma \mu \),

\[
\Gamma\mu(v^1) < \ldots < \Gamma\mu(v^{\tilde{r}}) < \ldots < \Gamma\mu(v^{N+1}), \tag{28}
\]

and thus \( \lambda_{\tilde{r}+1} = \ldots = \lambda_{N+1} = 0 \) which leads to

\[
\sum_{j=\tilde{r}+1}^{\hat{r}} \lambda_j \Gamma(v^j) = \sum_{j=\tilde{r}+1}^{\hat{r}} \lambda_j v^j, \quad \sum_{j=\tilde{r}+1}^{\hat{r}} \lambda_j = 1. \tag{29}
\]

Without loss of generality assume \( v^{\tilde{r}} = [v^1_{\tilde{r}}, \ldots, v_{\tilde{r}}^l, 0, \ldots, 0]^{\top} \) with \( v^j_{\tilde{r}} > 0 \) for \( j = 1, \ldots, l, \)

\( l \leq N \). Equation (29) implies \( \Gamma\mu(v^\tilde{r}) = [*, \ldots, *, 0, \ldots, 0]^{\top} \) with \( * \geq 0, j = 1, \ldots, l \). But then \( \Gamma \) is of the form

\[
\Gamma = \begin{pmatrix}
\Gamma_{11} & \Gamma_{12} \\
0 & \Gamma_{22}
\end{pmatrix} \tag{30}
\]

with \( \Gamma_{11} \in \mathbb{R}_{+}^{l \times l} \) and \( \Gamma_{22} \in \mathbb{R}_{+}^{(N+1-l)\times(N+1-l)} \). This means that \( \Gamma \) is reducible, a contradiction to the assumption. So this case cannot occur.

Now assume \( \|v^{N+1}\| < \kappa_0 + \kappa_\Gamma \). Define \( \tilde{\Gamma}(v^j) := \Gamma\mu(v^j) \left( 1 + \frac{\kappa_0 - 2\|v^j\|}{\|v^j\| + \kappa_\Gamma} \right) \) if \( \|v^j\| > \frac{\kappa_\delta}{2} \). For \( \|v^j\| < \kappa_0 + \kappa_\Gamma \) it holds \( \tilde{\Gamma}(v^j)_k \geq 0 \) and \( \Gamma\mu(v^j)_k = 0 \), if and only if \( \tilde{\Gamma}(v^j)_k = 0 \). The same argumentation as above provides

\[
\sum_{j=\tilde{r}+1}^{\hat{r}} \lambda_j \tilde{\Gamma}(v^j) = \sum_{j=\tilde{r}+1}^{\hat{r}} \lambda_j v^j, \quad \sum_{j=\tilde{r}+1}^{\hat{r}} \lambda_j = 1. \tag{31}
\]

With \( v^\tilde{r} = [v^1_{\tilde{r}}, \ldots, v^l_{\tilde{r}}, 0, \ldots, 0]^{\top} \) it follows \( \tilde{\Gamma}(v^\tilde{r}) = [*, \ldots, *, 0, \ldots, 0]^{\top} \) with \( * \geq 0, j = 1, \ldots, l \). All in all we get \( \tilde{\Gamma}(v^\rho) < \Gamma\mu(v^\rho) = [*, \ldots, *, 0, \ldots, 0]^{\top} \) with \( * > 0, j = 1, \ldots, l \). But then \( \Gamma \) is of the form (30), a contradiction to the assumption. \( \Box \)
In other words Theorem 3.13 provides that no $N$-simplex $\tau \in I' \cup II' \cup III'$ lying on the boundary of the positive orthant can be complete. So it remains to show that the path starting in $\eta^0$ cannot enter the set $IV'$. For this purpose we show in the next theorem that the path of complete simplices runs inside of the region which is painted dark grey in Figure 5. To prove this we demand an upper bound for the feasible size of $\delta$.

![Figure 5: $\tilde{K}_1(\delta)$-triangulation and the maximum region of the path ($\tau$-simplices are 1-dimensional and $\eta$-simplices are 2-dimensional)](image)

**Theorem 3.14** Let $\phi : \mathbb{R}^N_+ \rightarrow \mathbb{R}^N$ be defined as in (21) and assume that $\Gamma_\mu : \mathbb{R}^N_+ \rightarrow \mathbb{R}^N_+$ satisfies the small gain condition (14). Furthermore assume that the underlying gain matrix $\Gamma$ does not contain any zero row, i.e., $\Gamma_\mu(e)_i \neq 0$ for all $i = 1, \ldots, N$. Then there exists a $\delta > 0$ such that for all simplices $\tau = \langle y^1, \ldots, y^{N+1} \rangle \subset III'$ with $y^{N+2} := y^1 + [\delta, \ldots, \delta, 1]^\top \in IV'$ it follows that $\tau$ cannot be complete. In particular, any approximate solution $c'$ satisfies $c' \in I \cup II$.

**Proof.** Simplices satisfying $\tau \subset III'$ with $y^{N+2} := y^1 + [\delta, \ldots, \delta, 1]^\top \in IV'$ are marked as black dotted lines in Figure 5. We show that such a simplex $\tau$ cannot be complete, i.e., the system

$$L(\tau)W = I_{N+1}, \quad W > 0$$

has no solution. First it holds for all $s \in \mathbb{R}^N_+$ with $\|s\| < \kappa_G + \kappa_0$ that

$$\Gamma_\mu(s) < \max \{\Gamma_\mu(v) : v \in \mathbb{R}^N_+ \wedge \|v\| = \kappa_G + \kappa_0\}$$

$$\ll \Gamma_\mu([\kappa_G + \kappa_0, \ldots, \kappa_G + \kappa_0]^\top) =: \Gamma_{\mu}^{\max}.$$ 

Choose $k \in \mathbb{N}$ such that

$$\frac{1}{2k-1} \Gamma_{\mu}^{\max} < c$$

(33)
and set $\delta > 0$ such that
\[
\delta < \min \left\{ \frac{\kappa r - \kappa h}{2\sqrt{N}}, \frac{\kappa r + 2\kappa_0}{2k}\right\}.
\]
(34)

Now any approximate solution $c'$ can only be in $I \cup II$ and not in $III$ since $\delta < \frac{\kappa r - \kappa h}{2\sqrt{N}}$. This follows by Proposition 3.11 because any fixed point $s^*$ of $\phi$ has the property $\|s^*\| < \frac{\kappa r}{2}$ and then
\[
\|c'| < \|s^*\| + \text{mesh}_p(\tilde{K}_1(\delta)) = \|s^*\| + \delta \sqrt{N} < \frac{\kappa r}{2} + \frac{\kappa r - \kappa h}{2\sqrt{N}} \sqrt{N} = \frac{\kappa r}{2}.
\]

The idea now is the following: Let $r \in \{1, \ldots , N\}$ with $t_r = 0$ and $t_{r+1} = 1$. Then it holds
\[
L(\tau) = \left( \begin{array}{ccc}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & \phi(v^{r+1}) - v^{r+1}
\end{array} \right).
\]

Moreover, $v^1 \in III$. So there exists at least one index $l \in \{1, \ldots , N + 1\}$ with $(c - v^l)_l < 0$. Then
\[
(c - v^j)_l < 0 \quad \text{for} \quad j = 1, \ldots , r.
\]
(35)

Now $\phi(v)$ converges to zero, if $v$ tends to the boundary of $IV$, i.e., if $v \to \tilde{v}$ with $\|\tilde{v}\| = \kappa r + \kappa_0$. So the aim is to get $\phi(v)$ as small as $(\phi(v^j) - v^j)_l < 0$ holds for all $j = r + 1, \ldots , N + 1$.

Note that the function
\[
g : \mathbb{R}^+ \to \mathbb{R}, \quad a \mapsto 1 + \frac{\kappa r - 2a}{a + \kappa_0}
\]
is strictly decreasing for $\kappa_0 > 0, \kappa r > 0$. From the relation $v^{N+2} = v^1 + [\delta, \ldots , \delta]^T$ we get
\[
\kappa r + \kappa_0 < \|v^{N+2}\| = \|v^1 + [\delta, \ldots , \delta]^T\| \leq \|v^1\| + \delta \sqrt{N}.
\]

Under this assumption it follows for all $a > \kappa r + \kappa_0 - \delta \sqrt{N}$
\[
g(a) < 1 + \frac{\kappa r - 2(\kappa r + \kappa_0 - \delta \sqrt{N})}{\kappa r + \kappa_0 + \delta \sqrt{N} + \kappa_0} = \frac{\kappa r + 2\kappa_0 - \delta \sqrt{N} + \kappa r - 2\kappa r - 2\kappa_0 + 2\delta \sqrt{N}}{\kappa r + 2\kappa_0 - \delta \sqrt{N}} < \frac{\delta \sqrt{N}}{\kappa r + 2\kappa_0 - \delta \sqrt{N}} < \frac{\delta \sqrt{N}}{\kappa r + 2\kappa_0 - \delta \sqrt{N}} < \frac{\delta \sqrt{N}}{2k\delta \sqrt{N} - \delta \sqrt{N}} = \frac{1}{2k - 1}.
\]

Together with (33) it follows for $j = r + 1, \ldots , N + 1$
\[
\phi(v^j) - v^j = \Gamma_{\mu}(v^j) \left(1 + \frac{\kappa r - 2\|v^j\|}{\|v^j\| + \kappa_0}\right) - v^j \ll \Gamma_{\mu}^{\min} \left(1 + \frac{\kappa r - 2\|v^j\|}{\|v^j\| + \kappa_0}\right) - v^j
\]
\[
\ll \Gamma_{\mu}^{\min} \frac{1}{2k - 1} - v^j \ll \frac{33}{c - v^j}.
\]

Altogether with (35) it follows for all $j = 1, \ldots , N + 1$
\[
l(y^i)_1 = \left((1 - t_j)c + t_j\phi(v^j)\right)_1 - v^j_l \leq c_1 - v^j_l < 0.
\]

Let $L_l$ denote the $l$th row of $L$ and let $W_m$ denote the $m$th column of $W$. From the above consideration it follows $(L_l)^T \ll 0$ and from $L_1W^1 = 1$ we get $W^1 > 0$. But then it follows $L_1W^1 < 0$, a contradiction to $L_1W^1 = 0$ according to equation (32). So $\tau$ is not complete.

Now we can deduce the following main theorem.
Theorem 3.15 Let $\phi$ be defined as in (21) and assume that $\Gamma$ is irreducible and that the operator $\Gamma_{\mu} : \mathbb{R}^+_N \to \mathbb{R}^+_N$, deduced from the gain matrix $\Gamma$, satisfies the small gain condition (14). Let $\delta > 0$ be chosen as in (34) with $k \in \mathbb{N}$ according to (33). Then the simple path starting with $\eta^0$ is finite and the SFP-algorithm converges to a decay point $s \in \Omega(\Gamma_{\mu})$.

Proof. The dark grey painted region in Figure 5 is compact. The path of complete simplices starts in the interior of this region. Theorem 3.13 and Theorem 3.14 now show that under the above assumptions the path starting with $\eta^0$ cannot leave this region. So the path remains in this region. Since the region is compact there exist only finitely many simplices in this region and we are in the situation of Theorem 3.9 (iv). So the path is finite and ends up in a simplex $\tau \in \mathbb{R}^+_N \times \{1\}$ which contains an approximate fixed point of $\phi$ by Proposition 3.6. So refining the triangulation leads to the convergence of the SFP-algorithm to a fixed point $s$ of $\phi$. Since $\Gamma_{\mu}$ satisfies the small gain condition (14) it follows by Proposition 3.11 that the fixed point $s$ of $\phi$ lies in the set of decay $\Omega(\Gamma_{\mu})$. So the SFP-algorithm converges to a decay point $s \in \Omega(\Gamma_{\mu})$. □

3.5 Improvement of the algorithm

To summarize implementation details we give some suggestions for the choice of $c$, $\delta$, and the constants $\kappa_0, \kappa_h$ and $\kappa_\Gamma$ for a given function $\Gamma_{\mu}$ of dimension $N$.

Suggestions for the choice of $\kappa_h, \kappa_\Gamma, \kappa_0$

Theorem 3.11 says that a fixed point $s = \phi(s)$ can only lie in region I. Since $\phi(s) = \Gamma_{\mu}(s) + (\kappa_h - 2\|s\|)e$ for $s \in I$ we may expect the fixed point $s$ to have a norm near $\kappa_h/2$. So we choose $\kappa_h$ as the double size of the norm of the desired fixed point.

Several computational experiments have shown that values for $\kappa_\Gamma$ near $\kappa_h$ and $\kappa_0$ small will probably lead to small computing times. So we give the suggestions $\kappa_\Gamma = \kappa_h + 1$ and $\kappa_0 = 1$.

Note that for smaller values the regions II' and III' are small and so the path tends to leave the region I' $\cup$ II' $\cup$ III' more often. This leads to more pivoting steps and so to longer computing times.

Suggestion for the choice of $c$

If we have no advance information about the location of the fixed point we choose $c$ by default as $c = 0.99 \frac{\kappa_h}{2\sqrt{N}} [1 \ldots 1]^\top$. The norm of $c$ then is $\|c\| = 0.99\kappa_h/2$, thus near $\kappa_h/2$ where we expect the fixed point. In addition no direction is preferred.

In some cases we have some information about the approximate location of the decay point. Then we can use this information by using this expected point as $c$ (if it lies in I $\cup$ II) to arrive smaller computing times.

Suggestion for $\delta$ and the refinement of $\delta$

The choice of $\delta > 0$ as in (34) is one that leads to provable convergence but we have seen in experiments that this choice leads to longer computing times. So we will ignore the choice of $\delta$ as in (34) and give another suggestion. To ensure that the algorithm converges stop the iteration, if the path leaves the region I' $\cup$ II' $\cup$ III', and start again with the same starting point and a refined, i.e., smaller $\delta$. Since the dimension $N$ of the operator $\Gamma_{\mu}$ can get large it is advisable not to choose $\delta$ too small. Since the simplex has a diameter of $\sqrt{N}\delta$ we also have
to include the dimension $N$ into the choice of $\delta$ such that the path doesn’t leave the region $I' \cup II' \cup III'$. Our suggestion therefore is $\delta = \frac{\kappa h}{N}$.

The algorithm refines $\delta$ until the desired accuracy is reached. So it is important that we do not only choose $\delta$ suitably, but even determine the refining sequence $\{\delta_k\}_{k \in \mathbb{N}}$ such that the approximation is quite good. Saigal [25, Section 5] gave such a sequence and showed that the algorithm converges quadratically, if we assume in addition that $\phi$ is continuously differentiable and its derivative is Lipschitz continuous.

In our case we are content with the refining factor $\frac{1}{2}$, i.e., $\delta_{k+1} = \frac{\delta_k}{2}$. The simple reason is that the algorithm stops, if it finds a point in the set of decay. Again numerical experiments suggest that we do not have to refine often to find a decay point.

**Summary**

We want to summarize our suggestions. Let $\text{norm} > 0$ denote the desired norm of the decay point. Then we have the suggested values to be computed as

$$
\kappa_h = 2\text{norm}, \quad \kappa_\Gamma = \kappa_h + 1, \quad \kappa_0 = 1, \quad c = 0.99 \frac{\kappa_h}{2\sqrt{N}} e, \quad \delta = \frac{\kappa_h}{N}
$$

**Remark 3.16** Note that if the SFP-algorithm does not converge for $\delta$ as in (34), then the small gain condition $\Gamma \mu \preceq \text{id}$ cannot hold on the whole region $I \cup II \cup III \cup IV$. So we have to choose a smaller $\text{norm} > 0$.

**4 Examples**

In this section we give two examples. First note that in [24] an algorithm is developed to compute decay points, which is derived from a homotopy algorithm due to Eaves [9]. In [23] an example is given and decay points are computed with the algorithm from [24], referred to as Eaves algorithm. We state the principle ideas of this article, pick up the results given by Eaves algorithm, and compare them to those of our SFP-algorithm.

The second example concerns about a biochemical control circuit model which leads to a non-linear gain matrix $\Gamma$. We give a general example of monod kinetics and state some conclusions about the input-to-state stability of this control circuit model. At the end we consider a perturbed system and use the methods presented here to check the local input-to-state stability numerically.

**4.1 Quasi-monotone systems**

The motivation for this example was the article of Rüffer et al. [23]. Therein a nonlinear system is given and decay points are computed with Eaves algorithm from [24]. For this purpose a nonnegative matrix $P \in \mathbb{R}^N \times N$ with spectral radius $\rho(P) < 1$ is constructed for given dimension $N$. By Perron-Frobenius theory it follows that the matrix $A := -I_N + P$ then has spectral abscissa $\alpha(A) := \max \{\Re \lambda : \lambda$ is an eigenvalue of $A\} = -1 + \rho(P) < 0$. So the matrix $A$ is Hurwitz with negative diagonal entries and nonnegative off-diagonal entries.

Now we define a smooth coordinate transformation $S: \mathbb{R}^N \to \mathbb{R}^N$ by

$$
S(v)_i = \begin{cases} 
  e^{v_i - 1} & \text{if } v_i > 1 \\
  v_i & \text{if } v_i \in [-1, 1] \\
  -e^{-v_i - 1} & \text{if } v_i < -1
\end{cases}
$$
It holds $S(0) = 0$ and $S(\mathbb{R}^N_+) = \mathbb{R}^N_+$. The mapping $S : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ is a monotone operator.

Then the systems
\begin{equation}
\dot{v} = S'(S^{-1}(v))A S^{-1}(v) =: g(v)
\end{equation}

and
\begin{equation}
\dot{z} = Az =: h(z)
\end{equation}

are equivalent under a nonlinear change of coordinates. Let $v^*$ be any decay point for the function $g$ in equation (36). With it $z^* := S^{-1}(v^*)$ is a decay point for the function $h$ in equation (37). We want to pick up the associated run times and numbers of iterations and compare them with those of the SFP-algorithm.

The following results correspond to matrices $P \in \mathbb{R}^{N \times N}_+$ with positive entries in $[0, 1]$ generated by a numerical approximation of the uniform distribution, and 30\% of those are set to zero. Then $\alpha(A) = -0.2$. The numbers are averages over 100 simulations. Here we assumed $\text{norm} = 10$, i.e., the norm of the desired decay point $v^*$ is $\|v^*\| \approx 10$. In Table 1 the results of [23] are listed. In Table 2 we give the results of the SFP-algorithm. In addition, we tested the SFP-algorithm even for large $N$.

| $N$ | run time | # iterations |
|-----|----------|--------------|
| 5   | 0.11465s | 267.62       |
| 10  | 0.64855s | 2059.65      |
| 15  | 1.7833s  | 5505.78      |
| 25  | 7.987s   | 19742.84     |

Table 1: Results of Eaves (K1)-algorithm from [23] for norm = 10

| $N$ | run time | # iterations | simulations |
|-----|----------|--------------|-------------|
| 5   | 0.0277s  | 20.9         | 100         |
| 10  | 0.0415s  | 34.5         | 100         |
| 15  | 0.0618s  | 72.3         | 100         |
| 25  | 0.1710s  | 187.8        | 100         |

| $N$ | run time | # iterations | simulations |
|-----|----------|--------------|-------------|
| 50  | 1.180s   | 688.4        | 100         |
| 100 | 13.22s   | 2711.9       | 50          |
| 150 | 78.35s   | 6614.3       | 10          |
| 200 | 273.6s   | 11243.8      | 10          |

Table 2: Results of the SFP-algorithm for norm = 10

Note that the run times and iterations can only be compared relatively since the simulations are executed on different computers. Nevertheless, our run times are considerably lower and even for relatively large dimensions we are able to compute decay points in a quite acceptable run time.

In Table 3 we give run times and iteration numbers for norm = 1000. One can see that we have a relatively small increase of iteration steps and therefore of run times despite a quite larger norm. This is a consequence of the fact that we choose the size of $\delta$, and with it the
| N | run times | # iterations | simulations |
|---|----------|--------------|-------------|
| 5 | 0.0451s  | 61.6         | 10          |
| 10| 0.0680s  | 62.5         | 10          |
| 15| 0.0879s  | 106.6        | 10          |
| 25| 0.2977s  | 317.0        | 10          |
| 50| 2.141s   | 1097.1       | 10          |
| 100| 25.07s   | 3991.3       | 10          |
| 150| 253.2s   | 9214.9       | 10          |
| 200| 542.0s   | 16252.1      | 10          |

Table 3: Results of the SFP-algorithm for $\text{norm} = 1000$

mesh size of the starting triangulation $\delta \sqrt{N}$, in dependency of the norm $\text{norm}$ ($\delta = \frac{2\text{norm}}{N}$). That is why the algorithm gets close to the desired decay point in few steps.

4.2 A biochemical control circuit model

We consider the following biochemical control circuit model similar to [27],

$$\dot{x}_1(t) = g(x_N(t)) - a_1x_1(t) + u(t)$$
$$\dot{x}_i(t) = x_{i-1}(t) - a_ix_i(t), \quad i = 2, \ldots, N,$$
$$x(t) = [x_1(t), \ldots, x_N(t)]^\top \in \mathbb{R}_+^N,$$

with $a_i > 0$ constant for all $i = 1, \ldots, N$, $u \in L^\infty([0, \infty); \mathbb{R})$ and $g: \mathbb{R}_+ \to \mathbb{R}_+$ a continuously differentiable function with $g(x) > 0$ for all $x > 0$. In contrast to [27] we added an external input $u$ and do not assume $g$ to be bounded, but we demand the following assumption, which was introduced in [17].

**Assumption 4.1** There exist $x_\star_N > 0, K > 0$ and $\lambda \in (0, 1)$ with $ax_\star_N = g(x_\star_N)$ and $a = \prod_{j=1}^N a_j$ such that

$$\frac{K + x_\star_N}{K + x} x \leq a^{-1}g(x) \leq x_\star_N + \lambda|x - x_\star_N| \quad \text{for all } x \geq 0. \quad (39)$$

**Remark 4.2** The function

$$g(x) = \frac{bx}{c + x}, \quad x \geq 0, \quad b, c > 0 \quad (40)$$

models the growth rate of cells or micro-organism and is further known as monod kinetics. Let $a > 0$ be arbitrary and $b, c > 0$ such that $b > ac$. Then assumption 4.1 is satisfied. This can easily be seen by setting $x_\star_N := \frac{b - ac}{a} > 0, K = c > 0$ and $\lambda := \frac{c}{c + x_\star_N}$.

Following similar calculations as in [17] we get the following result by setting the gains as

$$\gamma_{1,j}(s) \equiv 0 \quad \text{for } j \neq N \quad \text{and} \quad \gamma_{1,N}(s) := \frac{1}{2} \left( \ln \left( 1 + \theta(\exp(\sqrt{2}s) - 1) \right) \right)^2,$$
$$\gamma_{i,j}(s) \equiv 0 \quad \text{for } j \neq i - 1, \quad \gamma_{i,i-1}(s) := \frac{1}{2} \left( \ln \left( 1 + \zeta(\exp(\sqrt{2}s) - 1) \right) \right)^2,$$

with $\Theta \in \left( \max\left\{ \frac{K}{K + x_\star_N}, 1 \right\}, 1 \right)$ and $\zeta \in (1, \Theta^{-\frac{1}{N-1}})$. 22
Theorem 4.3 [11, Satz 5.5] Consider the system (38) with \( g \) defined as in (40) with \( b, c > 0 \).
If \( b > ac \) with \( a = \prod_{j=1}^{N} a_j > 0 \) then the equilibrium solution \( x^* := [x_1^*, \ldots, x_N^*]^{\top} \gg \mathbf{0} \) with \( x_N^* := \frac{b-ac}{a} \) and \( x_i^* := \left( \prod_{j=i}^{N-1} a_{j+1} \right) x_N^* \) for \( i = 1, \ldots, N - 1 \) is ISS on \( \mathbb{R}^N_+ \setminus \{0\} \).

4.2.1 A perturbed biochemical control circuit model

Theorem 4.3 is a nice theoretical result. But in applications we are always faced with perturbed systems. In this section we want to check the local input-to-state stability of a perturbed system with the methods developed in this work.

For this purpose consider the graph in Figure 6. The black arcs describe the real couplings of the biochemical control circuit model and the grey arcs describe the perturbations. The underlying system to this coupling graph is given by

\[
\begin{align*}
\dot{x}_1(t) &= g(x_3(t)) - a_1 x_1(t) + u(t) + \tilde{\varepsilon}_{11}(x_1(t)) \\
\dot{x}_2(t) &= x_1(t) - a_2 x_2(t) + \tilde{\varepsilon}_{22}(x_2(t)) \\
\dot{x}_3(t) &= x_2(t) - a_3 x_3(t) + \tilde{\varepsilon}_{33}(x_3(t)) + \tilde{\varepsilon}_{31}(x_1(t)).
\end{align*}
\]

Figure 6: The coupling graph of a perturbed control circuit model

For this system the functions \( \tilde{\varepsilon} \) describe the perturbations. Further let \( a_1 = 2, a_2 = 1, a_3 = 3 \) and \( a = a_1 \cdot a_2 \cdot a_3 = 6 \), and \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) be the monod function given in (40) with \( b = 8 \) and \( c = 1 \). Then the associated gain matrix \( \Gamma \) is of the form

\[
\Gamma = \begin{pmatrix}
\hat{\gamma}_{11} & 0 & \hat{\gamma}_{13} \\
\hat{\gamma}_{21} & \hat{\gamma}_{22} & 0 \\
\hat{\gamma}_{31} & \hat{\gamma}_{32} & \hat{\gamma}_{33}
\end{pmatrix}.
\]

Here let

\[
\begin{align*}
\hat{\gamma}_{13}(s) &:= \frac{1}{2} \left( \ln(1 + \theta (\exp(\sqrt{2s}) - 1)) \right)^2 \\
\hat{\gamma}_{21}(s) &:= \frac{1}{2} \left( \ln(1 + \zeta (\exp(\sqrt{2s}) - 1)) \right)^2 \\
\hat{\gamma}_{32}(s) &:= \frac{1}{2} \left( \ln(1 + \zeta (\exp(\sqrt{2s}) - 1)) \right)^2
\end{align*}
\]

be the gain functions from the previous subsection with

\[
\theta \in \left( \max \left\{ \frac{K}{K + x_N^*}, \lambda \right\}, 1 \right) \quad \text{and} \quad \zeta \in \left( 1, \theta^{-1/2} \right).
\]
By Remark 4.2 it follows $K = 1$, $\lambda = \frac{3}{4}$ and $x_N^* = 1/3$, i.e., $\theta \in \left(\frac{3}{4}, 1\right)$. Here we assume $\theta = 0.8$ and $\zeta = 1.1$, where (42) is satisfied. Since $b = 8 > 6 = ac$ and $a = 6 > 0$, it follows by Theorem 4.3 that the equilibrium solution $x^* = [2 1 1/3]^\top$ of the unperturbed system $(\tilde{\varepsilon}_{11} \equiv \tilde{\varepsilon}_{22} \equiv \tilde{\varepsilon}_{31} \equiv \tilde{\varepsilon}_{33} \equiv 0$ respectively $\tilde{\gamma}_{11} \equiv \tilde{\gamma}_{22} \equiv \tilde{\gamma}_{31} \equiv \tilde{\gamma}_{33} \equiv 0$) is ISS on $\mathbb{R}^N_+\setminus\{0\}$.

The perturbed gain functions of the system are given as

$$
\tilde{\gamma}_{11}(s) := 0.001s, \quad \tilde{\gamma}_{31}(s) := 0.005s^2, \quad \tilde{\gamma}_{22}(s) := 0.001s^{0.9}, \quad \tilde{\gamma}_{33}(s) := 0.001s^2.
$$

For the monotone aggregation function $\mu = \sum$ we get the monotone operator $\Gamma_\mu$. Now we apply the SFP-algorithm with the suggestions in section 3.5 and $\text{norm} = 12$ to $\Gamma_\mu$ and get the decay point as

$$
w = \begin{bmatrix} 6.54 \\ 6.90 \\ 7.33 \end{bmatrix} \gg \begin{bmatrix} 6.527 \\ 6.886 \\ 7.325 \end{bmatrix} \approx \Gamma_\mu(w).
$$

Since $\lim_{k \to \infty} \Gamma^k_\mu(w) = 0$, Theorem 2.2 is applicable, so the perturbed system (41) is locally ISS.

Finally we illustrate the first and third components of the path $\sigma$ starting in $w$ as defined in (11) in Figure 7. Recall that $\sigma$ is obtained as a linear interpolation (——) of the points $\Gamma^k_\mu(w)$ (●), where we plot this for $k = 1, \ldots, 1000$. Indeed, straightforward calculations show that the line from 0 to $w$ (——) is not contained in the decay set $\Omega(\Gamma_\mu)$. Although the difference between the path and the straight line appears to be negligible we see that without the numerical effort of computing the $\Gamma^k_\mu(w)$ no $\Omega$-path is obtained. For a better view we enlarged one region.

![Graph of the 1. and 3. component of the path σ from w](image)

Figure 7: Components of the path $\sigma$ from 0 to $w$

### 4.2.2 A higher dimensional test

In this section we consider system (38) and want to study the computational effort for higher dimensions. Let $N \in \mathbb{N}$ denote the dimension of system (38) with $a_i = \frac{i+1}{i}$, $i = 1, \ldots, N$, and
the function \( g \) be defined as in (40) with \( c = 1 \) and \( b = 2N \), then Assumption 4.1 is satisfied with

\[
a = \prod_{i=1}^{N} a_i = N + 1, \quad x_N^* = \frac{N - 1}{N + 1}, \quad K = 1, \quad \lambda = \frac{N + 1}{2N}.
\]

Now by Theorem 4.3 this system is ISS on \( \mathbb{R}^N_+ \setminus \{0\} \), if

\[
\theta \in (\lambda, 1) = \left( \frac{N + 1}{2N}, 1 \right) \text{ and } \zeta \in (1, \frac{1}{N - 1}).
\]

In Table 4 we tested the computational effort for higher dimensions for \( \text{norm}=12 \). Note that for large \( N \) we have \( \theta^{-\frac{1}{N-1}} \) near 1, so we have only a small range in choosing \( \zeta \) such that the system (38) still is ISS on \( \mathbb{R}^N_+ \setminus \{0\} \). So we guess that the decay set will be very thin and so the decay points are harder to reach, resulting in longer computing times. This can be seen in Table 4. Note that the run times include checking that the sequence \( \Gamma^k_\mu(w) \) is a zero sequence.

The counter \( k_{\text{step}} \) indicates the first \( \tilde{k} \in \mathbb{N} \) such that \( \|\Gamma^k_\mu(w)\| < 10^{-9} \).

| \( N \) | \( \theta \) | \( \zeta \) | run times | \# iterations | \( k_{\text{step}} \) |
|---|---|---|---|---|---|
| 10 | 0.75 | 1.020 | 0.30s | 134 | 1215 |
| 50 | 0.75 | 1.003 | 4.99s | 1405 | 4501 |
| 70 | 0.75 | 1.002 | 1.72s | 74 | 5911 |
| 90 | 0.75 | 1.002 | 57.61s | 8426 | 10257 |
| 110 | 0.70 | 1.002 | 105.25s | 9632 | 9888 |
| 150 | 0.70 | 1.001 | 532.43s | 22856 | 8961 |
| 200 | 0.70 | 1.001 | 2168.18s | 52752 | 12656 |

Table 4: Results of the SFP-algorithm for \( \text{norm} = 12 \)

5 Conclusions

In this paper we have presented a homotopy algorithm, that is suitable for the computation of decay points of gain operators which are crucial in checking the local input-to-state stability. The algorithm is proved to converge in a semi-global fashion, provided the mesh size of the underlying triangulation is sufficiently small, but experiments suggest that the result is conservative and that larger mesh sizes are frequently sufficient. The algorithm improves on a previous simplicial algorithm. The advantage of such algorithms is that they can be used to analyze networks with quite general small gain formulations whereas other approaches rely on special structure like linearity of the gain operator or the use of maximization as the monotone aggregation function. In future research we intend to further develop numerical techniques for small gain results and explore relevant examples.

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