ON THE NONTRIVIAL REAL ZERO OF THE REAL PRIMITIVE CHARACTERISTIC DIRICHLET L FUNCTION

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Abstract. This paper prove that the real primitive characteristic Dirichlet L function does not exist the non-trivial real zero.

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Set $s = \sigma + it$ is the complex number, suppose that $Re s > 1$, the definition of Riemann Zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$$

The definition of Dirichlet L function is

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$$

where $\chi(n)$ is Dirichlet characteristic of mod $q$.

The product of $\zeta(s)$ and $L(s, \chi)$ is

$$\zeta(s)L(s, \chi) = \sum_{n=1}^{+\infty} \frac{a_n}{n^s}$$

where $a_n = \sum_{d|n} \chi(n)$. If $n = p_1^{\alpha_1} \cdots p_u^{\alpha_u}$ is the standard prime factor expression of $n$, then from multiplicative of $a_n$, have
\[ a_n = \prod_{r=1}^{n} (1 + \chi(p_r) + \cdots + \chi(p_r^a)) \]

easy to see, if \( \chi \) is the real characteristic, then there must be \( a_n \geq 0 \).

**Lemma 1.** (1) When \( 0 < \sigma < 1 \), there is a constant \( 0 \leq \alpha \leq 1 \), we have the formula below established
\[
\sum_{1 \leq n \leq \xi} \frac{1}{n^\sigma} = \frac{\xi^{1-\sigma} - 1}{1 - \sigma} + \alpha + O\left(\frac{1}{\xi^\sigma}\right)
\]

(2) when \( 1 < \sigma \), \( 1 \leq \xi \), we have
\[
\sum_{\xi \leq n} \frac{1}{n^\sigma} = \frac{1}{(\sigma - 1)\xi^{\sigma-1}} + O\left(\frac{1}{\xi^\sigma}\right)
\]

The proof of the lemma see the theorem 2 of the page 101, and the example 4 of the page 103, of the references [1]

**Lemma 2.** Set \( s = \sigma + it \) is the complex number, when \( \sigma > 0 \), \( x \geq 4q(|t| + 2) \), for the arbitrary nonprincipal character \( \chi \) of mod \( q \), have the formula below established
\[
L(s, \chi) = \sum_{1 \leq n \leq x} \frac{\chi(n)}{n^s} + O\left(\sqrt{q} \log^2 q \right)
\]

The proof of the lemma see the theorem 1 of the page 447 of the references [2]

**Lemma 3.** Set \( \chi \) is the arbitrary real primitive character of mod \( q \), \( q \geq 3 \), \( L(s, \chi) \) is the corresponding Dirichlet L function.

(1) There exist a positive absolute constant \( c_1 \), such that
\[
L(1, \chi) \geq c_1 (\sqrt{q} \log^2 q)^{-1}
\]

(2) there exist a positive absolute constant \( c_2 \), the real zero \( \beta \) of the function \( L(s, \chi) \) satisfy
\[ \beta \leq 1 - c_2 (\sqrt[4]{q} \log q)^{-1} \]

The proof of the lemma see the theorem 2 of the page 296, and the theorem 3 of the page 299 of the references [2].

**Lemma 4.** Set \( \frac{1}{2} \leq \beta < 1 \) is a real zero of the real primitive character Dirichlet \( L(s, \chi) \) function, and \( a_n = \sum_{d|n} \chi(n) \), when positive integer \( x \geq q^6 \), have

(1)
\[
\sum_{1 \leq n \leq x} \frac{a_n}{n^\beta} = \frac{x^{(1-\beta)}}{1 - \beta} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} \log x q^{\frac{3}{2}} \log^4 q\right)
\]

(2)
\[
\sum_{n=x+1}^\infty \frac{a_n}{n^\beta} = \frac{L(1, \chi)}{2} (x+1)^{-2} + O\left(x^{-\frac{5}{2} \log \log q}\right)
\]

**Proof.** (1) When positive integer \( x \geq q^6 \), have

\[
\sum_{1 \leq n \leq x} \frac{a_n}{n^\beta} = \sum_{1 \leq n \leq x} \frac{1}{n^\beta} \sum_{d|n} \chi(d) = \sum_{1 \leq d \leq x} \frac{\chi(d)}{d^\beta} \sum_{m \leq \frac{x}{d}} \frac{1}{m^\beta}
\]

\[
= \sum_{1 \leq d \leq x^{\frac{1}{2}}} \frac{\chi(d)}{d^\beta} \sum_{m \leq \frac{x}{d}} \frac{1}{m^\beta} + \sum_{x^{\frac{1}{2}} \leq d \leq x} \frac{\chi(d)}{d^\beta} \sum_{m \leq \frac{x}{d}} \frac{1}{m^\beta} = \sum_1 + \sum_2
\]

From lemma 1 (1), lemma 2, and lemma 3 (2), have

\[
\sum_1 = \sum_{1 \leq d \leq x^{\frac{1}{2}}} \frac{\chi(d)}{d^\beta} \left(\frac{x^{1-\beta}}{(1-\beta)d^{1-\beta}} - \frac{1}{1 - \beta} + \alpha + O\left(d^\beta x^{-\beta}\right)\right)
\]

\[
= \frac{x^{1-\beta}}{1 - \beta} \sum_{1 \leq d \leq x^{\frac{1}{2}}} \frac{\chi(d)}{d} + \left(\alpha - \frac{1}{1 - \beta}\right) \sum_{1 \leq d \leq x^{\frac{1}{2}}} \frac{\chi(d)}{d^\beta} + O\left(x^{-\beta+\frac{1}{2}}\right)
\]
\[
\frac{x^{1-\beta}}{1-\beta} L(1, \chi) + O\left(\frac{x^{1-\beta}}{1-\beta} q\right) + O\left(\frac{x^{1-\beta}}{1-\beta} q\right) + O\left(x^{-\frac{1}{2}}\right)
\]

\[
= \frac{x^{1-\beta}}{1-\beta} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} q^2 \log^4 q\right)
\]

\[
\sum_2 = \sum_{1 \leq m \leq x^{\frac{1}{2}}} \frac{1}{m^3} \sum_{x^{\frac{1}{2}} \leq d \leq \frac{x}{m}} \frac{\chi(d)}{d^3} = O\left(\sum_{1 \leq m \leq x^{\frac{1}{2}}} \frac{1}{m^3} (x^{1-\beta} \log q)\right)
\]

\[
= O\left(x^{\frac{1}{2}(1-\beta)} x^{-\frac{1}{2}} \log x \sqrt{q} \log q\right) = O\left(x^{(1-\beta)-\frac{1}{2}} \log x \sqrt{q} \log q\right)
\]

(2) Set \( y = x + 1 \), and the positive integer \( x \geq q^6 \), then

\[
\sum_{y \leq n < \infty} \frac{a_n}{n^3} = \sum_{y \leq n < \infty} \frac{1}{n^3} \sum_{d \mid n} \chi(d) = \sum_{1 \leq d < \infty} \frac{\chi(d)}{d^3} \sum_{\frac{x}{2} \leq m < \infty} \frac{1}{m^3}
\]

\[
= \sum_{1 \leq d < y^{\frac{1}{2}}} \frac{\chi(d)}{d^3} \sum_{\frac{x}{2} \leq m < \infty} \frac{1}{m^3} + \sum_{y^{\frac{1}{2}} \leq d < \infty} \frac{\chi(d)}{d^3} \sum_{\frac{x}{2} \leq m < \infty} \frac{1}{m^3} = \sum_3 + \sum_4
\]

From lemma 1 (2) and lemma 2, have

\[
\sum_3 = \sum_{1 \leq d \leq y^{\frac{1}{2}}} \frac{\chi(d)}{d^3} \left(\frac{d^2}{2y^2} + O\left(\frac{d^3}{y^3}\right)\right) = \frac{1}{2y^2} \sum_{1 \leq d \leq y^{\frac{1}{2}}} \frac{\chi(d)}{d} + O\left(y^{-\frac{3}{2}}\right)
\]
\[ \sum_{1} = \sum_{1 \leq m \leq y^{\frac{1}{2}}} \frac{1}{m^3} \sum_{y \leq d < \infty} \frac{\chi(d)}{d^3} = O \left( \sum_{1 \leq m \leq y^{\frac{1}{2}}} \frac{1}{m^3} \left( \frac{m^2 \sqrt{q} \log q}{y^3} \right) \right) \]

\[ = O \left( y^{-\frac{5}{2}} \sqrt{q} \log q \right) \]

This completes the proof.

**Lemma 5.** Set \( \Gamma(s) \) is Euler Gamma function.

(1) When \( Re \, s > 0 \), have

\[ \Gamma(s) = \int_{0}^{\infty} e^{-u} u^{s-1} \, du \]

(2) When \( y > 0, \, b > 0 \), have

\[ e^{-y} = \frac{1}{2\pi i} \int_{(b)} y^{-s} \Gamma(s) \, ds \]

where \( \int_{(b)} = \int_{b-i\infty}^{b+i\infty} \)

(3) Set \( s \) is the arbitrary complex number, we have

\[ -\frac{\Gamma'(s)}{\Gamma(s)} = \frac{1}{s} + \gamma + \sum_{n=1}^{\infty} \left( \frac{1}{n + s} - \frac{1}{n} \right) \]

where \( \gamma \) is Euler constant.

The proof of the lemma see the page 20, the properties 4 of the page 45, and the properties 8 of the page 48, of the references [2]

**Lemma 6.** Set \( \Gamma(s) \) is Euler Gamma function.

(1) When \( \frac{1}{8} \leq \sigma \leq \frac{1}{2} \), have \( \Gamma'(\sigma) < 0 \), in other words, \( \Gamma(\sigma) \) decrease monotonically in this interval.
\( \Gamma(\frac{1}{3}) \geq 2.67, \Gamma(\frac{1}{6}) \leq 6. \)

**Proof.** (1) When \( \frac{1}{8} \leq \sigma \leq \frac{1}{2} \), from the lemma 5 (3), have

\[
- \frac{\Gamma'(\sigma)}{\Gamma(\sigma)} = \frac{1}{\sigma} + \gamma + \sum_{n=1}^{\infty} \left( \frac{1}{n+\sigma} - \frac{1}{n} \right)
\]

\[
\frac{\Gamma'(\sigma)}{\Gamma(\sigma)} = -\frac{1}{\sigma} - \gamma + \sigma \sum_{n=1}^{\infty} \frac{1}{(n+\sigma)n}
\]

\[
\leq -2 - \gamma + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = -2 - \gamma + \frac{\pi^2}{12} < -1
\]

Because when \( \frac{1}{8} \leq \sigma \leq \frac{1}{2} \), have \( \Gamma(\sigma) > 0 \), so \( \Gamma'(\sigma) < 0 \), namely, \( \Gamma(\sigma) \) decrease monotonically in this interval.

(2) From the current lemma (1), have \( \Gamma(\frac{1}{3}) \geq \Gamma(0.334) \), from the functional equation \( \Gamma(s+1) = s\Gamma(s) \), and consult the page 1312 of the references [3], have \( \Gamma(0.334) = \frac{\Gamma(1.334)}{0.334} \geq \frac{0.8929}{0.334} \geq 2.67 \). On the same score, \( \Gamma(\frac{1}{6}) \leq \Gamma(0.166) = \frac{\Gamma(1.166)}{0.166} \leq \frac{0.928}{0.166} \leq 6 \). This completes the proof.

**Lemma 7.** Set \( \chi \) is a real primitive characteristic of mod \( q \), and \( a_n = \sum_{d|n} \chi(d) \). When \( y > 0 \), \( \Re s > \frac{1}{3} \), we have

\[
\int_{0}^{y} \left( \sum_{n=1}^{+\infty} a_n e^{-n^3u} \right) u^{s-1} \, du = L(1, \chi) \Gamma\left( \frac{1}{3} \right) \frac{3y^{s-\frac{1}{3}}}{3s-1} + \delta L(0, \chi) \zeta(0) \frac{y^s}{s}
\]

\[
+ \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{y^{s-w}}{s-w} \, dw
\]

where \( \delta = \frac{1}{2}(1 - \chi(-1)) \)

**Proof.** From the lemma 5 (2), when \( u > 0 \), we have

\[
e^{-u} = \frac{1}{2\pi i} \int_{(2)} u^{-w} \Gamma(w) \, dw
\]
Set $n$ is a positive integer, now let us do the integral transformation in
the above formula: $u \rightarrow n^3 u$, we have

\[ e^{-n^3 u} = \frac{1}{2\pi i} \int_{(2)} \frac{u^{-w}}{n^{3w}} \Gamma(w) \, dw \]

\[ \sum_{n=1}^{+\infty} a_n e^{-n^3 u} = \frac{1}{2\pi i} \int_{(2)} \left( \sum_{n=1}^{+\infty} a_n \right) \frac{u^{-w}}{n^{3w}} \Gamma(w) \, dw \]

\[ = \frac{1}{2\pi i} \int_{(2)} L(3w, \chi) \zeta(3w) u^{-w} \Gamma(w) \, dw \]

When $Re \, w \geq -\frac{1}{3}$, integrand function in $w = \frac{1}{3}$ place has a pole of order 1. When $\delta = 1$, integrand function in $w = 0$ place has a pole of order 1.

from the residue theorem have the above formula

\[ = L(1, \chi) \Gamma\left(\frac{1}{3}\right) u^{-\frac{1}{3}} + \delta L(0, \chi) \zeta(0) + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \, dw \]

Suppose that $Re \, s > \frac{1}{3}$, $y > 0$ the above equation multiply by $u^{s-1}$ on both sides, then integral, we have

\[ \int_0^y \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} \, du = L(1, \chi) \Gamma\left(\frac{1}{3}\right) \int_0^y u^{s-\frac{4}{3}} \, du \]

\[ + \delta L(0, \chi) \zeta(0) \int_0^y u^{s-1} \, du + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \left( \int_0^y u^{-w+s-1} \, du \right) \, dw \]

\[ = L(1, \chi) \Gamma\left(\frac{1}{3}\right) \frac{3y^{s-\frac{4}{3}}}{3s-1} + \delta L(0, \chi) \zeta(0) \frac{y^s}{s} + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{y^{s-w}}{s-w} \, dw \]

This completes the proof.

**THEOREM.** Set $\chi$ is a real primitive characteristic of mod $q$, the corresponding Dirichlet L function $L(s, \chi)$ does not exist the nontrivial real zero.

**Proof.** Set $n$ is a positive integer and $Re \, s > \frac{1}{3}$, as variable transformation for below formula: $n^3 u \rightarrow u$, from the lemma 5 (1), have
\[ \int_0^\infty e^{-n^3 u} u^{s-1} \, du = \frac{1}{n^{3s}} \int_0^\infty e^{-u} u^{s-1} \, du = \frac{1}{n^{3s}} \Gamma(s) \]

so

\[ \Gamma(s) \sum_{n=1}^{\infty} \frac{a_n}{n^{3s}} = \int_0^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} \, du \]

\[ \Gamma(s) L(3s, \chi) \zeta(3s) = \int_0^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} \, du \]

\[ = \int_y^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} \, du + \int_0^y \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} \, du \]

where \( y > 0 \).

from the lemma 7, we have

\[ \Gamma(s) L(3s, \chi) \zeta(3s) = \int_y^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} \, du \]

\[ + L(1, \chi) \Gamma(\frac{1}{3}) \frac{3y^{\frac{s}{3} - \frac{1}{3}}}{3s-1} + \delta L(0, \chi) \zeta(0) \frac{y^s}{s} + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{y^{s-w}}{s-w} \, dw \]

From the above formula, we put the function \( \Gamma(s) L(3s, \chi) \zeta(3s) \) analytic continuation to the half plane \( Re \, s > -\frac{1}{3} \), this function have a pole of order 1 on \( s = \frac{1}{3} \), and when \( \delta = 1 \), have a pole of order 1 on \( s = 0 \).

Now assume, \( L(s, \chi) \) exists a nontrivial real zero \( \beta \), from the functional equation of \( L(s, \chi) \), we can assume that \( \frac{1}{2} \leq \beta < 1 \).

Now take \( s = \frac{\beta}{3} \), we have

\[ 0 = \int_y^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{\frac{\beta}{3}-1} \, du + L(1, \chi) \Gamma(\frac{1}{3}) \frac{3y^{\frac{\beta}{3} - 1}}{\beta - 1} \]

\[ + \delta L(0, \chi) \zeta(0) \frac{3y^{\frac{\beta}{3}}}{\beta} + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{3y^{\frac{\beta}{3} - w}}{\beta - 3w} \, dw \]
in other words

\[ L(1, \chi) \Gamma\left(\frac{1}{3}\right) \frac{3y^{\beta-1}}{1-\beta} = \int_y^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3u} \right) u^{\frac{\beta}{3}-1} \, du \]

\[ + \delta L(0, \chi) \zeta(0) \frac{3y^\beta}{\beta} + \frac{1}{2\pi i} \int_{(-\frac{1}{2})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{3y^{\beta-w}}{\beta-3w} \, dw = I_1 + I_2 + I_3 \]

Now take \( y = x^{-3} \), \( x \) is a positive integer, and \( x \geq q^6 \). we calculated the above formula every one on the right

\[ I_2 = O\left( \delta L(1, \chi) q^\frac{1}{2} y^\frac{1}{3} \right) = O\left( x^{-\frac{1}{2}} q^\frac{1}{2} \log q \right) \]

From the functional equation of \( L(s, \chi) \), \( \zeta(s) \), and the asymptotic formula of \( \Gamma(s) \), we have

\[ |I_3| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| L(-1+i\nu, \chi) \right| \left| \zeta(-1+i\nu) \right| \left| \Gamma\left(-\frac{1}{3}+i\nu\right) \right| \left| \frac{3y^{\frac{1+\beta}{3}}}{\beta + 1 - 3i\nu} \right| \, d\nu \]

\[ = O\left( q^\frac{1}{3} y^{\frac{1+\beta}{3}} \right) = O\left( x^{-\frac{1}{2}} q^\frac{1}{2} \right) \]

Now calculate \( I_1 \)

\[ I_1 = \sum_{n=1}^{x} a_n \int_{y}^{+\infty} e^{-n^3u} u^{\frac{\beta}{3}-1} \, du + \sum_{n=x+1}^{+\infty} a_n \int_{y}^{+\infty} e^{-n^3u} u^{\frac{\beta}{3}-1} \, du = J_1 + J_2 \]

Now calculate \( J_1 \)

\[ J_1 \leq \sum_{n=1}^{x} a_n \int_{0}^{\infty} e^{-n^3u} u^{\frac{\beta}{3}-1} \, du \]

as variable transformation for above formula : \( n^3u \to u \), have the above formula
\[
\sum_{n=1}^{x} \frac{a_n}{n^\beta} \int_{0}^{\infty} e^{-u} u^{\beta-1} \, du = \Gamma\left(\frac{\beta}{3}\right) \sum_{n=1}^{x} \frac{a_n}{n^\beta}
\]

From the lemma 4 (1), have the above formula

\[
= \Gamma\left(\frac{\beta}{3}\right) \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} q^\frac{3}{2} \log x \log^4 q\right)
\]

From the lemma 6 (1) and (2), have the above formula

\[
\leq \Gamma\left(\frac{1}{6}\right) \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} q^\frac{3}{2} \log x \log^4 q\right)
\]

\[
\leq 6 \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} q^\frac{3}{2} \log x \log^4 q\right)
\]

Now calculate \(J_2\)

\[
J_2 \leq y^{\beta-1} \sum_{n=x+1}^{+\infty} a_n \int_{y}^{\infty} e^{-n^3 u} \, du
\]

as variable transformation for above formula: \(n^3 u \rightarrow u\), have the above formula

\[
= y^{\beta-1} \sum_{n=x+1}^{+\infty} \frac{a_n}{n^3} \int_{y}^{\infty} e^{-u} \, du \leq y^{\beta-1} \sum_{n=x+1}^{+\infty} \frac{a_n}{n^3} \int_{1}^{\infty} e^{-u} \, du = \frac{y^{\beta-1}}{e} \sum_{n=x+1}^{+\infty} \frac{a_n}{n^3}
\]

From the lemma 4 (2), have the above formula

\[
= \frac{y^{\beta-1}}{e} \frac{L(1, \chi)}{2} (x+1)^{-2} + O\left(y^{\beta-1} x^{-\frac{5}{2}} q\right) \leq \frac{x^{(1-\beta)}}{2e} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} q\right)
\]

we synthesize the above calculation, have

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\[ L(1, \chi) \Gamma \left( \frac{1}{3} \right) \frac{3x^{(1-\beta)}}{1-\beta} \leq 6 \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + \frac{x^{(1-\beta)}}{2e} L(1, \chi) + O(x^{(1-\beta) - \frac{3}{2} q^{\frac{3}{2}}} \log x \log^4 q) \]

From the lemma 6 (2), have

\[ 8 L(1, \chi) \frac{x^{(1-\beta)}}{1-\beta} \leq 6 \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + \frac{x^{(1-\beta)}}{2e} L(1, \chi) + O(x^{(1-\beta) - \frac{3}{2} q^{\frac{3}{2}}} \log x \log^4 q) \]

so

\[ 2 L(1, \chi) \frac{x^{(1-\beta)}}{1-\beta} \leq \frac{x^{(1-\beta)}}{2e} L(1, \chi) + O(x^{(1-\beta) - \frac{3}{2} q^{\frac{3}{2}}} \log x \log^4 q) \]

Divided by \( 2 L(1, \chi) x^{(1-\beta)} \) on both sides, and from the lemma 3 (1), have

\[ \frac{1}{1-\beta} \leq \frac{1}{4e} + O \left( x^{-\frac{1}{2}} q^{2} \log x \log^6 q \right) \]

Make \( x \to +\infty \), we have

\[ \frac{1}{1-\beta} \leq \frac{1}{4e} \leq \frac{1}{10} \]

so \( \beta \leq -9 \), with \( \frac{1}{2} \leq \beta < 1 \) contradictions.

This completes the proof.

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