THE CONTINUUM LIMIT OF FOLLOW-THE-LEADER MODELS
— A SHORT PROOF

Helge Holden
Department of Mathematical Sciences
NTNU Norwegian University of Science and Technology
NO–7491 Trondheim, Norway

Nils Henrik Risebro
Department of Mathematics
University of Oslo
Blindern, NO–0316 Oslo, Norway

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We dedicate this paper to the memory of Hans Petter Langtangen (1962–2016)

ABSTRACT. We offer a simple and self-contained proof that the Follow-the-Leader model converges to the Lighthill–Whitham–Richards model for traffic flow.

1. Introduction. The problem of convergence of particle models to continuum models is fundamental. We here study it in the context of traffic flow. In this case there are two fundamentally different models: The first one is based on individual vehicles whose dynamics is determined by the behavior of the vehicle immediately in front of it. This gives the Follow-the-Leader (FtL) model, which constitutes a system of ordinary differential equations describing the dynamics of individual vehicles. The other model is based on the assumption of heavy traffic where the individual vehicles are represented by a density. Assuming that the number of vehicles is conserved, we get the classical Lighthill–Whitham–Richards (LWR) model [11, 12], which is nothing but a scalar hyperbolic conservation law. The question that we address in this paper is in what sense the FtL model approaches or approximates the LWR model in the case of dense traffic.

The principal assumption in FtL models is that the velocity $V$ of any given vehicle is a function of the distance to the vehicle in front of it. We shall write this function as

$$V\left(\frac{\Delta Z}{\ell}\right),$$

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* Corresponding author: Helge Holden.

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where $\Delta Z$ denotes the distance to nearest vehicle in front, and $\ell$ the length of each vehicle. For obvious reasons, $\Delta Z \geq \ell$. It is commonly assumed that $V$ is an increasing positive function defined in $[1, \infty)$, such that $\lim_{y \to \infty} V(y) = v_{\text{max}} < \infty$. Consider $N$ vehicles with length $\ell$ and position $Z_1(t) < \cdots < Z_N(t)$ on the real axis with dynamics given by

$$\frac{d}{dt} Z_i = V \left( \frac{Z_{i+1} - Z_i}{\ell} \right) \quad \text{for } i = 1, \ldots, N - 1. \quad (1)$$

To close this system, we must prescribe the velocity of the first vehicle at $Z_N$. It is natural to model this by letting $\dot{Z}_N = v_{\text{max}}$.

In this paper we analyze the limit of this system of ordinary differential equations when $N \to \infty$ and $\ell \to 0$. We show that

$$\frac{\ell}{Z_{i+1}(t) - Z_i(t)} \to \rho(t, z),$$

where intuitively $Z_{i+1}, Z_i \to z$, and where $\rho$ is an entropy solution to the scalar conservation law

$$\rho_t + f(\rho)_x = 0, \quad f(\rho) = \rho V \left( \frac{1}{\rho} \right). \quad (2)$$

This problem has also been addressed by several other researchers. We here mention [1, 2, 4, 5, 7, 8]. The long and technically demanding paper [6] shows this convergence, while in [3, 13], the convergence of the discrete system is assumed rather than proved. The approach here resembles [10] where FtL models are viewed as a numerical approximation of the LWR model, and the proof of convergence depends on classical results by Crandall–Majda and Wagner for a grid approximation.

Here we offer a simple and straightforward proof of the continuum limit.

Solutions to scalar conservation laws are in general not continuous, and (2) must be considered in the weak sense; furthermore weak solutions to the Cauchy problem are not unique, and in order for the Cauchy problem to have a unique solution, one must impose the Kružkov entropy condition [9]: A function $\rho \in C([0, \infty); L^1(\mathbb{R}))$ is called an entropy solution to the Cauchy problem for (2) if for all constants $k \in \mathbb{R}$ and all non-negative test functions $\varphi \in C^0_0((0, \infty) \times \mathbb{R})$, one has

$$\int_0^\infty \int_\mathbb{R} (|\rho - k| \varphi_t + \operatorname{sign}(\rho - k)(f(\rho) - f(k)) \varphi_z) \, dz \, dt + \int_\mathbb{R} |\rho(0, z) - k| \varphi(0, z) \, dz \geq 0. \quad (3)$$

More precisely, we show the following result. Assume that the velocity function satisfies the reasonable assumptions (4), and the initial data $\rho(0, \cdot) \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Let $\rho_t(t, z)$ be the density of vehicles as defined by the FtL model, see (16). Then we show that $\lim_{\ell \to 0} \rho_t = \rho \in C([0, \infty); L^1(\mathbb{R}))$, where $\rho$ is the unique solution to (2) satisfying the entropy condition (3) such that $\rho(0, z) = \rho_0(z)$.

The rest of this note is organized as follows: In Section 2 we define the discrete model and prove some simple bounds on its solutions, and in Section 3 we give the elementary proof of convergence.

2. The model. We use units such that $v_{\text{max}} = 1$. Let $v(\rho)$ be a continuously differentiable function $v: [0, 1] \to [0, 1]$, such that $v' \leq 0$, $v(0) = 1$ and $v(1) = 0$. We use the notation $V(y) = v(1/y)$, assume that

$$V(y) \geq 1 - \frac{1}{y^{\sigma - 1}}, \quad \text{for some constant } \sigma > 1, \quad (4a)$$
Lemma 2.2. Define \( y^2 V'(y) \leq M \), for \( y \geq 1 \), and for some constant \( M \).\(^\text{(4b)}\)

Since \( V(y) = v(1/y) \), in terms of \( v \), these assumptions read
\[
v(\rho) \geq 1 - \rho^{\sigma-1}, \quad \text{for some constant } \sigma > 1, \quad \text{\( (5a) \)}
\]
\[
-v'(\rho) \leq M, \quad \text{for } 0 \leq \rho \leq 1, \text{ and for some constant } M. \quad \text{\( (5b) \)}
\]

In particular, the assumptions are satisfied for the commonly used Greenshield’s velocity function \( v(\rho) = 1 - \rho \).

Define the forward difference
\[
D_+ h_i = \frac{1}{\ell} (h_{i+1} - h_i).
\]

Let \( \{y_i(t)\}_{i=1}^{N-1} \) satisfy
\[
y_i(t) = D_+ V_i, \quad i = 1, \ldots, N - 1, \quad t > 0, \quad \text{\( (6) \)}
\]
where \( V_i = V(y_i) \) and \( V_N = 1 \). Later we will also need \( y_N = \infty \). Regarding the initial values, we assume that there is a function \( \rho_0 : \mathbb{R} \to [0, 1] \) normalized such that \( \int_{\mathbb{R}} \rho_0 \, dz = 1 \). Define \( \{z_{i+1/2}(0)\}_{i=0}^{N-1} \) inductively as
\[
\int_{z_{i-1/2}(0)}^{z_{i+1/2}(0)} \rho_0(z) \, dz = \frac{1}{N+1} = \ell, \quad i = 0, \ldots, N - 1. \quad \text{\( (7) \)}
\]

Thus with the current scaling, the length \( \ell \) of each vehicle is \( \ell = 1/(N + 1) \). We will also need \( z_{-1/2}(0) = -\infty \). Here we choose the infimum of possible values for \( z_{i+1/2}(0) \) satisfying \( \text{(7)} \). Set
\[
y_i(0) = \frac{1}{\ell} \left( z_{i+1/2}(0) - z_{i-1/2}(0) \right), \quad i = 1, \ldots, N - 1. \quad \text{\( (8) \)}
\]

Observe that it follows from \( \text{(7)} \) that \( y_i(0) \geq 1 \) for \( i = 1, \ldots, N - 1 \), since \( \rho_0 \in [0, 1] \).

**Lemma 2.1.** Assume that \( V \) satisfies \( \text{(4a)} \) and that \( \{y_i\}_{i=1}^{N-1} \) solves the system \( \text{(6a)} \) with initial values \( \text{(8)} \). Then
\[
1 \leq y_i(t) \leq \left( y_i(0)^\sigma + \frac{\sigma t}{\ell} \right)^{1/\sigma}, \quad i = 1, \ldots, N - 1. \quad \text{\( (9) \)}
\]

In particular,
\[
\lim_{\ell \to 0} (\ell^\kappa y_i(t)) = 0, \quad t \in (0, \infty), \quad \kappa > 1/\sigma, \quad i = 1, \ldots, N - 1. \quad \text{\( (10) \)}
\]

**Proof.** If \( y_i(t) = 1 \), then \( V(y_i(t)) = 0 \) and hence \( \dot{y}_i(t) \geq 0 \). This gives the lower bound on \( y_i \).

Using \( \text{(4a)} \) and the bound \( V_{i+1} \leq 1 \), we get
\[
\dot{y}_i = \frac{1}{\ell} (V_{i+1} - V_i) \leq \frac{1}{\ell y_i^{\sigma-1}}.
\]

By integrating this inequality, we see that the estimate \( \text{(9)} \) holds, and the limit \( \text{(10)} \) then follows trivially. \( \square \)

**Lemma 2.2.** Define \( \rho_i(t) = 1/y_i(t) \). Write \( V_i(t) = V(y_i(t)) \). We have that
\[
\sum_{i=1}^{N-1} |V_{i+1}(t) - V_i(t)| \leq \sum_{i=1}^{N-1} |V_{i+1}(0) - V_i(0)| \quad \text{\( (11) \)}
\]
and
\[ \sum_{i=1}^{N-1} |p_{i+1}(t) - p_i(t)| \leq \sum_{i=1}^{N-1} |p_{i+1}(0) - p_i(0)|. \] (12)

**Proof.** We find that
\[
\frac{d}{dt} |V_{i+1} - V_i| = \text{sign}(V_{i+1} - V_i) V'(y_{i+1}) |D+V_{i+1} - V'(y_i) |D+V_i|
\leq V'(y_{i+1}) |D+V_{i+1} - V'(y_i) |D+V_i|,
\]
since \( V' \geq 0 \). For \( i = N \) we recall the conventions that \( y_N = \infty \) and \( V_N = 1 \), and thus \( V'(y_N) = 0 \). We have that \( 0 \leq V_i \leq 1 \). This means that
\[
\frac{d}{dt} \sum_{i=1}^{N-1} |V_{i+1} - V_i| \leq \frac{1}{\ell} (V'(y_N) |V_N - V_{N-1}| - V'(y_1) |V_2 - V_1|)
= -\frac{1}{\ell} V'(y_1) |V_2 - V_1| \leq 0,
\]
since \( V'(\infty) = 0 \), which shows (11). Since \( \text{sign}(V_{i+1} - V_i) = -\text{sign}(p_{i+1} - p_i) \), we could also carry out these estimates for \( \rho_i \), proving (12). \( \square \)

Note that
\[ \sum_i |p_{i+1}(0) - p_i(0)| \leq |\rho_0|_{BV}, \]
where \( |\cdot|_{BV} \) denotes the bounded variation seminorm. For \( t > 0 \), define \( z_{i+1/2}(t) \) by
\[ z_{i-1/2} = V_i, \quad i = 1, \ldots, N, \] (13)
with initial values given by (7). Combining (6), (8), and (13) we conclude that
\[ y_i(t) = \frac{1}{\ell} (z_{i+1/2}(t) - z_{i-1/2}(t)), \quad t \in [0, \infty), i = 1, \ldots, N - 1. \] (14)
In particular,
\[ (z_{i+1/2}(t) - z_{i-1/2}(t)) \rho_i(t) = \ell, \quad i = 1, \ldots, N - 1. \] (15)
Note that \( z_{i-1/2} \) coincides with the position of the \( i \)th vehicle from the left, given by (1). Thus \( Z_i = z_{i+1/2} \).

Furthermore, define the functions
\[
\rho(t, z) = \sum_{i=1}^{N-1} p_i(t) \chi_{[z_{i-1/2}(t), z_{i+1/2}(t)]}(z),
\]
(16)
\[
V(t, z) = \sum_{i=1}^{N-1} V_i(t) \chi_{[z_{i-1/2}(t), z_{i+1/2}(t)]}(z),
\]
where \( \chi_I \) denotes the characteristic function of an interval \( I \). Observe that Lemma 2.2 implies that
\[ |\rho(t)|_{BV} \leq |\rho(0)|_{BV}, \quad |V(t)|_{BV} \leq |V(0)|_{BV}. \] (17)
3. The continuum limit.

**Theorem 3.1.** Assume that the function $V$ satisfies (4), and that $\rho_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Let $\rho_t$ be as defined above. Then $\lim_{t \to 0} \rho_t = \rho \in C([0, \infty); L^1(\mathbb{R}))$, where $\rho$ is the unique entropy solution to (2) such that $\rho(0, z) = \rho_0(z)$.

**Proof.** Observe that $\rho_t$ is in $L^1(\mathbb{R})$, since it is positive and

$$\frac{d}{dt} \int_{\mathbb{R}} \rho_t(t, z) \, dz = \sum_{i=1}^{N-1} \frac{d}{dt} \int_{z_{i-1/2}}^{z_{i+1/2}} \rho_t \, dz = \sum_{i=1}^{N-1} \frac{d}{dt} (z_{i+1/2} - z_{i-1/2}) \rho_t = 0.$$ 

Hence $\|\rho(t)\|_{L^1} \leq 1$. For any $\{h_i\}$ define

$$D^+_i h_i = \frac{h_{i+1} - h_i}{z_{i+1/2} - z_{i-1/2}} = \rho_i D_+ h_i.$$ 

Let $\varphi = \varphi(t, z)$ be a smooth test function with compact support in $\mathbb{R} \times (0, \infty)$. We calculate

$$\int_0^\infty \int_{\mathbb{R}} \rho_t \varphi_t \, dz \, dt = \int_0^\infty \sum_i \rho_i \int_{z_{i-1/2}}^{z_{i+1/2}} \varphi_t \, dz \, dt$$

$$= \int_0^\infty \sum_i \left[ \rho_i \frac{\partial}{\partial t} \int_{z_{i-1/2}}^{z_{i+1/2}} \varphi \, dz \right] \, dt$$

$$= \int_0^\infty \sum_i \left[ \int_{z_{i-1/2}}^{z_{i+1/2}} \rho_i \varphi \, dz - \ell D_+ (\rho_i) V_i \varphi_{i+1/2} \right] \, dt$$

$$= \int_0^\infty \sum_i \left[ \int_{z_{i-1/2}}^{z_{i+1/2}} \left( \rho_i^2 D_+ (V_i) \varphi + D^+_i (\rho_i) V_i \psi_{i+1/2} \right) \, dz \right] \, dt,$$  \hspace{1cm} (18)

where we have used that $\dot{\rho}_i = -\rho_i^2 D_+ V_i$, and introduced the notation $\psi_{i+1/2} = \varphi(t, z_{i+1/2})$. Similarly,

$$\int_0^\infty \int_{\mathbb{R}} \rho_t V_i \varphi_z \, dz \, dt = \int_0^\infty \sum_i \rho_i V_i \int_{z_{i-1/2}}^{z_{i+1/2}} \varphi_z \, dz \, dt$$

$$= \int_0^\infty \sum_i \ell D_+ (\rho_i V_i) \psi_{i+1/2} \, dt$$

$$= \int_0^\infty \sum_i \left[ \int_{z_{i-1/2}}^{z_{i+1/2}} (\rho_i D^+_i (V_i) \psi + D^+_i (\rho_i) V_i \psi_{i+1/2}) \, dz \right] \, dt.$$  \hspace{1cm} (19)

Using (18) with $\varphi(t, z) = \chi_{[t_1, t_2]}(t) \psi(z)$ for $0 < t_1 < t_2 < \infty$ and a smooth test function $\psi$ with $|\psi| \leq 1$, we formally get with $\psi_{i+1/2} = \psi(z_{i+1/2})$ that

$$\left| \int_{\mathbb{R}} (\rho_t(t_2, z) - \rho_t(t_1, z)) \psi(z) \, dz \right|$$

$$= \left| \int_{t_1}^{t_2} \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} \left( \rho_i D^+_i (V_i) \psi + D^+_i (\rho_i) V_i \psi_{i+1/2} \right) \, dz \, dt \right|$$

$$\leq \int_{t_1}^{t_2} \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} \left( |\rho_i| |D^+_i (V_i)| \psi + |D^+_i (\rho_i)| |V_i| \psi_{i+1/2} \right) \, dz \, dt.$$  

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\[ \leq \ell \int_{t_1}^{t_2} \sum_i \left( |\rho_i| |D_+(V_i)| + |D_+(\rho_i)| |V_i| \right) dt \]
\[ \leq \ell \int_{t_1}^{t_2} \sum_i \left( |D_+(V_i)| + |D_+(\rho_i)| \right) dt \]
\[ \leq (t_2 - t_1) \sum_i \left( |V_i+1(0) - V_i(0)| + |\rho_i+1(0) - \rho_i(0)| \right) \]
\[ \leq (t_2 - t_1) \left( |V_0(0)|_{BV} + |\rho_0(0)|_{BV} \right), \]

using first that \(|\psi|, |\rho_0|, |V_i| \leq 1 \) and subsequently Lemma 2.2. By approximating the characteristic function \( \chi_{[t_1,t_2]} \) with a smooth function, and taking the limit, we still obtain the above estimate. This implies
\[ \|\rho(\ell) - \rho_{t_1}\|_{L^1} = \sup_{|\psi| \leq 1} \left| \int (\rho(\ell, z) - \rho(t_1, z)) \psi(z) dz \right| \]
\[ \leq (t_2 - t_1) \left( |\rho_0(0)|_{BV} + |V_0(0)|_{BV} \right). \]

Thus, recalling (17), we can apply [9, Theorem A.11] to conclude that the set \( \{\rho_\ell\}_{\ell \geq 0} \) is compact in \( C([0, \infty), L^1(\mathbb{R})) \), and there exists a sequence \( \{\ell_j\}_{j=1}^\infty, \ell_j \to 0 \) as \( j \to \infty \), and a function \( \rho \) such that
\[ \rho_{\ell_j} \to \rho \quad \text{in} \quad C([0, \infty), L^1(\mathbb{R})), \quad \text{as} \quad j \to \infty. \]

To simplify the notation, we henceforth write \( \ell = \ell_j \). Furthermore, since \( v(\rho_\ell) = V_\ell \), \( V_\ell \to v(\rho) \). Adding (18) and (19), we get
\[ \left| \int_0^\infty \int_\mathbb{R} (\rho_\ell \varphi_t + \rho_\ell v_\ell \varphi_z) \, dz \, dt \right| = \left| \int_0^\infty \sum_i \rho_i (V_i+1 - V_i) \right| \]
\[ \quad \times \int_{z_{i-1/2}}^{z_{i+1/2}} (\varphi(t, z) - \varphi(t, z_{i+1/2})) \, dz \, dt \]
\[ \leq \frac{1}{2} \int_0^\infty \sup_i (\ell_i y_i(t))^2 \| \varphi_z(t, \cdot) \|_{L^\infty} \sum_i |V_i+1 - V_i| \, dt \]
\[ \to 0, \quad \text{as} \quad \ell \to 0, \]

and thus \( \rho \) is a weak solution. To show that \( \rho \) is an entropy solution, let \( \eta \) be a twice differentiable convex function. Since \( V' \geq 0 \), we get
\[ \frac{d}{dt} \eta(y_i) = \eta'(y_i) D_+ V_i = \frac{1}{\ell} \eta'(y_i) \int_{y_i}^{y_{i+1}} V'(y) \, dy \]
\[ \leq \frac{1}{\ell} \int_{y_i}^{y_{i+1}} \eta'(y) V'(y) \, dy = \frac{1}{\ell} \int_{y_i}^{y_{i+1}} Q'(y) \, dy = D_+ Q_i, \]

where \( Q' = \eta' V' \) and \( Q_i = Q(y_i) \). Introduce \( q(\rho) = Q(1/\rho) \) with \( q_i = q(\rho_i) \). Define \( \mu = \mu(\rho) \) by \( \mu(\rho) = \rho q(1/\rho) \). As usual we write \( \mu_i = \mu(\rho_i) \) and \( \eta_i = \eta(y_i) \). Then \( \mu \) is a convex function of \( \rho \), and if \( \mu \) is a convex function of \( \rho \), then \( \eta \) is a convex function of \( y \). We have that
\[ \frac{d}{dt} \mu(\rho_i) = -\mu_i^2 (D_+ V_i) \eta_i + \rho_i \frac{d}{dt} \eta_i. \]

Set \( \mu(t, z) = \sum \mu_i(t) \chi_{[z_{i-1/2},z_{i+1/2}]}(z) \) with \( \mu_i(t) = \mu(\rho_i(t)) \), and define \( q(t, z) \) similarly. As when establishing (18), we find for a non-negative test function \( \varphi \) with
support in \( \mathbb{R} \times (0, \infty) \) that
\[
\int_0^\infty \int_0^\infty \mu \ell \varphi_t \, dz \, dt
\]
\[
= \int_0^\infty \sum_i \mu_i \int_{z_i-1/2}^{z_i+1/2} \varphi_t \, dz \, dt
\]
\[
= \int_0^\infty \sum_i \left[ \mu_i \frac{\partial}{\partial t} \left( \int_{z_i-1/2}^{z_i+1/2} \varphi \, dz \right) - \mu_i \ell D_+ (z_i-1/2 \varphi_t) \right] dt
\]
\[
= \int_0^\infty \sum_i \left[ \int_{z_i-1/2}^{z_i+1/2} \mu_i \varphi \, dz - \ell D_+ (\mu_i) V_i+1 \varphi_t \right] dt
\]
\[
\geq \int_0^\infty \sum_i \int_{z_i-1/2}^{z_i+1/2} \left[ (\eta \rho_i^2 D_+ (V_i) - \rho_i D_+ (q_i)) \varphi + D_+^z (\mu_i) V_i+1 \varphi_t \right] dz \, dt
\]
\[
= \int_0^\infty \sum_i \int_{z_i-1/2}^{z_i+1/2} \left[ (\mu_i D_+^z (V_i) - D_+^z (q_i)) \varphi + D_+^z (\mu_i) V_i+1 \varphi_t \right] dz \, dt.
\]
Similarly
\[
\int_0^\infty \int_0^\infty (V_i \mu \ell - q_i) \varphi_z \, dz \, dt
\]
\[
= \int_0^\infty \sum_i (V_i \mu_i - q_i) \int_{z_i-1/2}^{z_i+1/2} \varphi_z \, dz \, dt
\]
\[
= -\int_0^\infty \sum_i \ell [D_+ (\mu_i V_i) - D_+ (q_i)] \varphi_t \, dt
\]
\[
= -\int_0^\infty \sum_i \int_{z_i-1/2}^{z_i+1/2} \left[ \mu_i D_+^z (V_i) + V_i+1 D_+^z (\mu_i) - D_+^z (q_i) \right] \varphi_t \, dt
\]
Therefore,
\[
\int_0^\infty \int_0^\infty \left( \mu \ell \varphi_t + (\mu \ell V_t - q_\ell) \varphi_z \right) \, dz \, dt \geq r_\ell,
\]
where
\[
|r_\ell| = \left| \int_0^\infty \sum_i \int_{z_i-1/2}^{z_i+1/2} \left( \mu_i D_+^z (V_i) - D_+^z (q_i) \right) (\varphi - \varphi_t) \, dz \, dt \right|
\]
\[
\leq \int_0^\infty \sum_i (|\mu_i| |V_i+1 - V_i| + |q_{i+1} - q_i|) \int_{z_i-1/2}^{z_i+1/2} |\varphi(t, z) - \varphi(t, z+1/2)| \, dz \, dt
\]
\[
\leq C \int_0^\infty \sup \ell_y(t) \| \varphi_z(t, \cdot) \| \sum_i (|V_i+1 - V_i| + |q_{i+1} - q_i|) \, dt.
\]
If \( q_\ell (t, \cdot) \) is of bounded variation, then \( r_\ell \rightarrow 0 \). We now assume that \( \mu \) is a smooth approximation to the Kružkov entropy \( \mu (\rho) = |\rho - k| \). A short computation yields that
\[
\mu (\rho) V \left( \frac{1}{\rho} \right) - q(\rho) = \operatorname{sign} (\rho - k) \left( \rho V \left( \frac{1}{\rho} \right) - k V \left( \frac{1}{k} \right) \right),
\]
which is consistent with (3). Then \( \eta (y) = y |1/y - k| \), and \( |\eta'(y)| \leq |k| \). If \( V \)

satisfies (4b), the mapping \( \rho \mapsto q(\rho) \) is Lipschitz, since
\[
|q'(\rho)| = \left| \frac{d}{d\rho} Q \left( \frac{1}{\rho} \right) \right| = \left| \eta' \left( \frac{1}{\rho} \right) V' \left( \frac{1}{\rho} \right) \frac{1}{\rho^2} \right| \leq M |k|.
\]
Hence \( q_\ell(t, \cdot) \) is of bounded variation, since \( \rho_\ell \) is in \( BV \).

A similar argument with a test function whose support include the initial data on \( t = 0 \), will show (3). We conclude that \( \rho \) is an entropy solution. Since the entropy solution is unique, we also conclude that the whole sequence, rather than just a subsequence, converges.

REFERENCES

[1] B. Argall, E. Cheleshkin, J. M. Greenberg, C. Hinde and P.-J. Lin, A rigorous treatment of a follow-the-leader traffic model with traffic lights present, SIAM J. Appl. Math., 63 (2002), 149–168.

[2] A. Aw, A. Klar, T. Materne and M. Raschle, Derivation of continuum traffic flow models from microscopic follow-the-leader models, SIAM J. Appl. Math., 63 (2002), 259–278.

[3] R. M. Colombo and E. Rossi, On the micro-macro limit in traffic flow, Rend. Sem. Math. Univ. Padova, 131 (2014), 217–235.

[4] E. Cristiani and S. Sahu, On the micro-to-macro limit for first-order traffic flow models on networks, Networks and Heterogeneous Media, 11 (2016), 395–413.

[5] M. Di Francesco, S. Fagioli and M. D. Rosini, Deterministic particle approximation of scalar conservation laws, Boll. Unione Mat. Ital., 10 (2017), 487–501.

[6] M. Di Francesco and M. D. Rosini, Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit, Arch. Ration. Mech. Anal., 217 (2015), 831–871.

[7] P. Goatin and F. Rossi, A traffic flow model with non-smooth metric interaction: well-posedness and micro-macro limit, Commun. Math. Sci., 15 (2017), 261–287.

[8] K. Han, T. Yao and T. L. Friesz, Lagrangian-based hydrodynamic model: Freeway traffic estimation, Preprint, arXiv:1211.4619v1, 2012.

[9] H. Holden and N. H. Risebro, Front Tracking for Hyperbolic Conservation Laws, Springer-Verlag, New York, 2015, Second edition.

[10] H. Holden and N. H. Risebro. Follow-the-leader models can be viewed as a numerical approximation to the Lighthill–Whitham–Richards model for traffic flow. Preprint, arXiv:1702.01718, 2017.

[11] M. J. Lighthill and G. B. Whitham, Kinematic waves. II. A theory of traffic flow on long crowded roads, Proc. Roy. Soc. (London), Series A, 229 (1955), 317–345.

[12] P. I. Richards, Shock waves on the highway, Operations Research, 4 (1956), 42–51.

[13] E. Rossi, A justification of a LWR model based on a follow the leader description, Discrete Cont. Dyn. Syst. Series S, 7 (2014), 579–591.

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E-mail address: helge.holden@ntnu.no
E-mail address: nilshr@math.uio.no