Hydrodynamic Formulation of the Hubbard Model

Girish S, Sethur

November 13, 2018

Abstract

In this article, we show how to recast the Hubbard model in one dimension in a hydrodynamic language and use the path integral approach to compute the one-particle Green function. We compare with the Bethe ansatz results of Schulz and find exact agreement with the formulas for spin and charge velocities and anomalous exponent in weak coupling regime. These methods may be naturally generalized to more than one dimension by simply promoting wavenumbers to wavevectors.

1 Introduction

The Hubbard model is one of the most extensively studied models in Condensed Matter Physics. It is the simplest example of an interacting Fermi system on a lattice. In one dimension the ground state and excited states may be written down explicitly using Bethe ansatz[1]. The collection of reprints by Korepin and Essler[3] is particularly useful. The pioneering work of Schulz[2] in computing the spin and charge velocities and the anomalous exponent for all values of the onsite repulsion $U$ is quite significant. There have been other attempts notably by Weng et.al.[4] in recasting the Hubbard model in a path integral language. Unfortunately all these methods are restricted to one dimension. One notable exception is the work by Liu [5] that uses eigenfunctional theory. Our approach is quite different from all these and uses the hydrodynamic formulation recently developed by the author. It is simple and powerful as we shall see since it is naturally generalisable to more than one dimension and yields the important correlation functions quite easily. Our approach is able to give the right functional dependence of the various quantities such as spin and charge velocities and exponents (as a function of $U$) for small $U$. The formalism is naturally generalisable to more than one dimension by simply promoting wavenumbers to wavevectors. Unfortunately we are unable to probe the large $U$ case which is relevant to high $T_c$ since the formalism does not reproduce the right results in one dimension.
The program of quantizing hydrodynamics has a long and distinguished history. Landau [6] and his students were among the first to attempt this. Later on Sunakawa et.al. [7] and others - notably Rajagopal and Grest [8] took this program further. Dashen, Sharp, Menikoff and Goldin[9] in the seventies introduced many of these ideas. Recently, Jackiw and collaborators[10] have revived interest in this approach in the context of relativistic quarks. In our earlier work, we introduced the density phase variable ansatz for fermions[11]. We also note that Rajagopal and Grest[8] had already in the seventies pointed out the need for having a nonzero-phase functional found in the density phase variable ansatz. In our earlier work[11] we made a first pass at computing the phase functional. This attempt yielded an answer that in retrospect is quite wrong. Upon closer examination the $U_0(q)$ of our earlier work[11] is imaginary when it was postulated to be real(for small $q$). So far the author has avoided this issue by taking refuge under the the sea-boson approach that enables us to derive the momentum distribution, anomalous exponents, quasiparticle residue and so on without yielding the full dynamical propagator which is of interest only because it contains information about quantities just mentioned. If one is able to compute them without having to compute the full propagator so much the better. However, there are physical problems in which the full propagator is important. The X-ray edge problem[14] is one such. In fact we tried using the DPVA to compute the X-ray edge spectra in a preprint[13] and found that we obtain the right answers in one dimension but the answers in higher dimensions were inconsistent with Mahan’s exact results[14].

In an earlier preprint, after much reflection, we chose to dismiss the approach that only uses the hydrodynamic variables namely the density and its conjugate as ‘myopic’ (myopic bosonization). This is because a hamiltonian formulation in terms of the hydrodynamic variables is unable to distinguish between fermions and bosons. We have to further decompose these variables in terms of linear combination of oscillators in order to distinguish between the two statistics. However, the sea-boson approach is not without its share of problems. For one it does not generalise to finite temperatures easily. Also the full dynamical propagator is not reducible to quadratures due to a technical difficulty. Both these problems may be resolved in an approach that incorporates only the hydrodynamical variables. We show in this preprint, that the path integral approach is an avenue to distinguish between the statistics when using only the hydrodynamical variables. This approach that only uses hydrodynamic variables was not expected to work out for fermions, since one has to take into account the extended nature of the Fermi surface. However, this idea seems too important to pass up. In particular, the natural manner in which gauge theory may be studied in this approach[15] makes this effort for fermions worthwhile and urgent.

In this section, we introduce the hydrodynamic formulation that has been
developed by the author. It involves writing the field variable in terms of observables such as currents and densities. In the long-wavelength limit, it can be shown that current algebra (that is, the mutual commutation rules between currents and densities) is obeyed only if the current operator is expressible as shown below.

\[ J(x, t) = -\rho(x, t) \nabla \Pi(x, t) \] (1)

Here \( \Pi \) is the potential for the velocity and is conjugate to the density \( \rho \). Thus in the hydrodynamic limit the velocity operator is irrotational. The hydrodynamic part of the field variable may be written in a polar form.

\[ \psi_{\text{slow}}(x, t) = e^{i\Lambda(\rho; x, t)} e^{-i\Pi(x, t)} \sqrt{\rho(x, t)} \] (2)

For fermions, we expect \( \psi_{\text{slow}}(x, t) \) to be a Grassmann variable. But we shall take the point of view that the fermionic nature of the field is captured at the level of the propagator by the introduction of the phase functional \( \Lambda(\rho; x, t) \). The fermionic KMS boundary conditions is obeyed since a position independent global Klein factor is able to capture this\(^{12}\). Thus the program involves expanding \( \Lambda(\rho; x, t) \) in powers of the density fluctuations and making contact with the free theory and fixing the coefficients. The rest of the discussion is similar to our earlier work on bosons\(^{15}\), therefore we shall not dwell on those details. Suffice it to say that if we expand the action in powers of the density fluctuations and retain only the harmonic parts, in order to recover the right density-density correlation functions, we have to set,

\[ \lambda(\rho; q_n) = C(q_n) \rho_{-q, -n} \] (3)

and \( \Lambda(\rho; x, t) = \sum_{q, n} e^{iq \cdot x} e^{zn t} \lambda(\rho; q, n) \). The coefficient may be computed as follows.

\[ \beta z_n C(q_n) = \frac{1}{2} \langle \rho_{q, n} \rho_{-q, -n} \rangle_0 - \frac{\beta z_n^2}{4N_0 q^2} - \frac{\beta q^2}{4N_0} \] (4)

and \( \langle \rho_{q, n} \rho_{-q, -n} \rangle_0 \) is the density-density correlation function of the free theory obtained from elementary considerations. In one dimension, for spinless fermions we may write,

\[ C(q, n) = \frac{v_F^2}{4N_0 z_n} \] (5)

where \( v_F \) is the Fermi velocity and \( z_n = 2\pi n/\beta \) is the bosonic Matsubara frequency.
3 Hubbard Model

The Hubbard model in one dimension\cite{14} may be written down as follows (here $G = 2\pi/a$).

$$H = -2t \sum_{k\sigma} \cos(ka) \, c_{k\sigma}^\dagger c_{k\sigma} + \frac{U}{N} \sum_q \rho_{q\uparrow} \rho_{-q\downarrow} + \frac{U}{N} \sum_q \rho_{q\downarrow} \rho_{-q+G\downarrow} + \frac{U}{N} \sum_q \rho_{q\uparrow} \rho_{-q-G\downarrow}$$

(6)

First we replace the cosine dispersion by a parabolic one. $\epsilon_k = -2t \cos(ka) = c_0 + c_1 k^2$. From this we find $c_0 = -2t$ and we require that the slope of the dispersion at $k = \pm k_F$ be identical to the cosine dispersion. This means $c_1 = (ta/k_F) \sin(k_Fa) = 1/(2m)$.

Thus we may write a quadratic action for the one band Hubbard model in one dimension, including umklapp processes in the hydrodynamic language.

$$S_{Hubb} = \sum_{q\sigma,n} (-i\beta z_n) \rho_{q\sigma,n} X_{q\sigma,n} + \frac{i\beta N^0}{2} \sum_{q\sigma,n} q^2 X_{q\sigma,n} X_{-q\sigma,-n}$$

$$+ i\beta \sum_{\sigma\neq 0} z_n C(qn) \rho_{q\sigma,n} \rho_{-q\sigma,-n} + \frac{i\beta U}{N^0} \sum_{q\neq 0} \rho_{q\uparrow,n} \rho_{-q\downarrow,-n} + \frac{i\beta U}{N^0} \sum_{q\neq 0} \rho_{q\downarrow,n} \rho_{-q+G\downarrow,-n} + \frac{i\beta U}{N^0} \sum_{q\neq 0} \rho_{q\uparrow,n} \rho_{-q-G\downarrow,-n}$$

(7)

where $G = 2\pi/a$. The quadratic action implicitly ignores three body density correlation functions. It is not clear why this is valid except that it renders the path integrals tractable. However we may expect to find nontrivial results already at the harmonic level with umklapp processes for strong coupling. Since we are considering umklapp process which involves large momentum transfer, we have to make sure that the $C(qn)$ is evaluated in general for both small and large $q$. In our earlier work we showed,

$$\beta z_n C(q\sigma,n) = \frac{1}{2 \langle \rho_{q\sigma,n}\rho_{-q\sigma,-n} \rangle_0} - \frac{(2m)\beta z_n^2}{2N^0 q^2} - \frac{\beta q^2}{2N^0(2m)}$$

(8)

where $N^0$ is the total number of electrons including both spins and $\langle \rho_{q\sigma,n}\rho_{-q\sigma,-n} \rangle_0$ is the density-density correlation function of the free theory evaluated using elementary considerations. The field variable is now given by,

$$\psi_{\text{slow}}(x\sigma,t) = e^{-i\sum_q e^{iqx} e^{iz\tau} X_{q\sigma,n}} e^{i\eta} \sum_q e^{iqx} e^{iz\tau} C(q\sigma,n) \rho_{-q\sigma,-n}$$

(9)

Here $\psi_{\text{slow}}$ is the hydrodynamic (slow) part of the field. Also $\eta$ is a ‘fudge factor’ needed to make sure that we recover the right exponents. It is a numerical factor.
we have,\[ \gamma \]

that a quantum distribution is given by,\[ \gamma \]

The Bethe ansatz result of Schulz shows that Here, \[ v \]

For weak coupling we have,\[ v \]

If we set \( \eta = 2 \) we find,

\[
\begin{align*}
\langle T\psi(x,\uparrow,t)\psi^\dagger(x',\uparrow,t') \rangle &= e^{\sum_{q,n} \frac{4\beta U}{\lambda_{q,n} - q^2} \left( 2 - \frac{\beta U}{\lambda_{q,n} - q^2} \right)^2} \\
&= \prod_{j=1,2} \left( \frac{1}{(x - x')^\alpha} \right) \left( \frac{1}{(x' + t - t')^\alpha} \right)
\end{align*}
\]

Here, \( v_{F,j} = v_F \left( 1 \pm \frac{U}{2m^2}\right) \). The anomalous exponent related to the momentum distribution is given by,

\[
\gamma = 2 \alpha = U^2 \frac{1}{16(k_F^2/m)^2}
\]

The Bethe ansatz result of Schulz shows that \( \gamma = U^2/(4\pi^2v_F^2) \). In units such that \( a = 1 \) near half filling we have \( k_F = \pi/2 \) and \( m = \pi/4 \). Thus in our case we have, \( \gamma = U^2/(16\pi^2) \). From the result of Schulz also we find, \( \gamma = U^2/(16\pi^2) \).
Therefore this approach gives the right spin and charge velocities (especially close to half-filling), and the right anomalous exponent for small $U$.

However, for large $U$, this approach does not give us the right qualitative behaviour as it predicts $\gamma \sim \sqrt{U}$. According to the Bethe ansatz solution the anomalous exponent saturates to a value $\gamma = 1/8$. For strong coupling, we are unable to make progress. We have tried using the no-double occupancy constraint since this is easy to implement using the hydrodynamic formulation, but we have been unable to make it work. The umklapp terms are also unfortunately of little use. Thus we shall not advocate the use of this approach for strong coupling. In two spatial dimensions we expect to find that the system is a Landau Fermi liquid. We shall not carry out this calculation since it is not very interesting. The main purpose of this article is to highlight the usefulness of the hydrodynamic approach and its generality so that in future publications we may use this to study disordred systems and the like.

This work was supported by the Harish Chandra Research Institute.

References

[1] E.H. Lieb and F.Y. Wu, Phys. Rev. Lett., 20, 1445 (1998), Erratum, ibid. 21, 192 (1968).
[2] H.J. Schulz, Int. J. Mod. Phys., 5, 57 (1991).
[3] V.E. Korepin and F.H.L. Essler, Exactly Solvable Models of Strongly Correlated Electrons, World Scientific, Singapore, 1994.
[4] Z.Y. Weng, C.S. Ting and T.K. Lee, Phys. Rev. B 43, 3790 (1991); Z.Y. Weng, D.N. Sheng, C.S. Ting and Z.B. Su Phys. Rev. B 45, 7850 (1992).
[5] Y.L. Liu, Int. J. of Mod. Phys. B, 16 4127 (2002); 16, 773 (2002); 16, 2201 (2002).
[6] L.D. Landau, Collected Papers, Pergamon Press, Oxford, 1965.
[7] S. Sunakawa, S. Yamasaki and T. Kebukawa, Prog. Theor. Phys. 38, 804 (1967); 41, 919 (1969); 44 565 (1970); 49 1802 (1973).
[8] A. K. Rajagopal and G. S. Grest, Phys Rev A 10, 1837 (1974); Phys. Rev. A10, 1395 (1974); Prog. Theor. Phys. 52, 811 (1974); 52, 1719 (1974) Errata.; A.K. Rajagopal in Quantum Fluids, in Particular, Superfluid Helium, Three Lectures in the Bose Symposiums on Statistical Physics (July 15-27, 1974). Supplement to Journal of Ind. Inst. of Sci. 85 (1975).
[9] G. A. Goldin, R. Menikoff and D. H. Sharp, J. Math. Phys. 21 (1980); R. Menikoff, D. H. Sharp, J. Math. Phys. 18, 471 (1977); R.F. Dashen and D. H. Sharp, Phys. Rev. 165, 1857 (1968).
[10] R. Jackiw and A.P. Polychronakos, Phys. Rev. D, 62 085019 (2000) ; B. Bistrovic, R. Jackiw, H. Li, V. P. Nair and S.-Y. Pi, Phys. Rev. D, 67 025013 (2003).

[11] G.S. Setlur and Y.C. Chang, 57 , 15144 (1998).

[12] G.S. Setlur and D.S. Citrin, Phys. Rev. B, 65, 165111 (2002).

[13] G.S. Setlur, X-ray Edge Spectra from Sea-bosons, cond-mat/0111423.

[14] G.D. Mahan, Many Particle Physics, Plenum Press, New York, 1990.

[15] G.S. Setlur, Pramana, 62, 101, (2004).