Local existence of scalar wave equation on the Robertson-Walker universe as a background with $k = 0$

M. Iqbal$^{1,*}$, F. T. Akbar$^{1,**}$, and B. E. Gunara$^{1,2,†}$

$^1$Theoretical Physics Laboratory, Theoretical High Energy Physics and Instrumentation Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha no. 10 Bandung, Indonesia, 40132

$^2$Indonesian Center for Theoretical and Mathematical Physics

E-mail: *muhammad.iqbal7@students.itb.ac.id, ‡ftakbar@fi.itb.ac.id, ‡bobby@fi.itb.ac.id

Abstract. In this paper, we study about the wellposedness of scalar wave equation on Robertson-Walker universe as a background with zero spatial curvature, $k = 0$. We start from non-minimal Lagrangian for scalar field on curved background with potential turned on. Then we derive the equations of motion and tensor energy-momentum. After that we specify our case to $k = 0$. Finally, we prove the local existence and uniqueness of the solution of the equation of motion.

1. Introduction

1.1. Background: Robertson-Walker universe

Based on general cosmological principle, the metric that describe the spacetime of the universe is given by

$$ds^2 = -dt^2 + f^2(t)d\sigma^2,$$

where $d\sigma^2$ is 3 dimensional metric of spaces that have maximal symmetry property, i.e 3-sphere, 3-pseudosphere, and 3 dimensional Euclidean space. Spacetime described by this metric is called (Friedman-) Robertson-Walker spacetime [1].

Define a new time coordinate by

$$\frac{d\tau}{dt} = \frac{1}{f(t)},$$

then the metric (1) transforms into

$$ds^2 = f^2(\tau) (-d\tau^2 + d\sigma^2),$$

which is conformal to some metric depending on its spatial curvature $k$. The function $f(\tau)$ is one of the following, depending on $k$

$$f(\tau) = \begin{cases} \cosh \tau - 1 & k = -1, \\ \tau^2 & k = 0, \\ 1 - \cos \tau & k = 1 \end{cases}$$
Note that $k = -1$ for 3-psudosphere, $k = 0$ for 3 dimensional Euclidean space, and $k = 1$ for 3-sphere [2]. By a little calculation, the Ricci tensor and the scalar curvature related to the conformal metric are given by

$$R_{\mu\nu} = \frac{6}{\tau^2} (\tau_{\mu\nu} + 2\delta^0_\mu \delta^0_\nu) ; \quad R = \frac{12}{\tau^6},$$

respectively.

### 1.2. Action and equation of motion

The action of scalar field equation on spacetime background with non-minimal coupling is given by

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} (\partial_{\mu}\phi)(\partial^{\mu}\phi) + \frac{\xi}{2} R\phi^2 - V(\phi) \right\},$$

where $R$ is scalar curvature, $V$ is the scalar potential function, and $g$ denotes the determinant of the metric $g_{\mu\nu}$. The second term of (6) denotes the non-minimal coupling between $R$ and scalar field $\phi$ with coupling constant $\xi$ which we choose to be positive real number. The equation of motion related to (6) is given by

$$\nabla_\mu \nabla^\mu \phi - \xi R\phi + V'(\phi) = 0.$$  

In this paper, we consider the case that the spatial space has zero curvature, $k = 0$. Therefore, due to the metric (3), the wave equation (7) becomes

$$\frac{\partial^2 \phi}{\partial \tau^2} - \Delta \phi = F(\phi, \partial_\tau \phi),$$

where

$$F(\phi, \partial_\tau \phi) = \frac{12\xi}{\tau^2} \phi + \frac{8}{\tau} (\partial_\tau \phi) - \tau^4 V'(\phi).$$

We see that the wave equation of the form (8) is non linear wave equation with the source depending on scalar field and its derivative (in this case, only the first time derivative).

We define the $k$-order linear energy as,

$$E_k[u] = \frac{1}{2} \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} \left\{ (\partial^\alpha \partial_\tau u)^2 + |\nabla \partial^\alpha u|^2 \right\} dx.$$  

Note that $E_k$ is preserved for a solution to the homogeneous wave equation [3].

### 2. Iteration and Energy inequality

Our goal is to prove local existence and uniqueness of solution of equation of motion with initial data

$$\phi_{\tau\tau} - \Delta \phi = F(\phi, \partial_\tau \phi)$$

$$\phi(\tau_0, \cdot) = f$$

$$\phi_\tau(\tau_0, \cdot) = g,$$

where we choose $f \in H^{k+1}(\mathbb{R}^3)$ and $g \in H^k(\mathbb{R}^3)$. The expression of function $F(\phi, \partial_\tau \phi)$ is given by (9).
In order to prove the local existence, we set up an iteration argument. First, consider following iteration

\[ \partial^2 \phi_0 - \Delta \phi_0 = 0 \]
\[ \phi_0(\tau_0, \cdot) = f_0 \]
\[ \partial_t \phi_0(\tau_0, \cdot) = g_0 \],

and for \( l \geq 0 \),

\[ \partial^2 \phi_{l+1} - \Delta \phi_{l+1} = F(\phi_l, \partial_r \phi_l) \]
\[ \phi_l(\tau_0, \cdot) = f_l \]
\[ \partial_t \phi_l(\tau_0, \cdot) = g_l \].

Since Schwartz space is dense in Sobolev space, then we can choose the sequence \( \{ f_l \} \) and \( \{ g_l \} \) such that \( f_l, g_l \in \mathcal{S}(\mathbb{R}^3) \), \( f_l \to f \) with respect to \( H^{k+1}(\mathbb{R}^3) \) and \( g_l \to g \) with respect to \( H^k(\mathbb{R}^3) \).

Let \( \tau \in [\tau_0, \tau_0 + T] \) for some value of \( T > 0 \). Since \( \phi_0 \) is solution of homogeneous wave equation, then we have

\[ E_k[\phi_0](\tau) = E_k[\phi_0](\tau_0) \leq C_1 ; \quad E_k[\phi_l](\tau_0) \leq C_1 \quad \forall l \in \mathbb{N} \]  \hspace{1cm} (14)

where \( C_1 \) only depends on initial data. Using inductive argument, we prove that

\[ E_k[\phi_l](\tau) \leq C_1 + 1 \]  \hspace{1cm} (15)

for \( |\tau - \tau_0| \leq T \). For \( l = 0 \), this is true due to (14). Assume it is true for some \( l \), then if \( k > 3/2 \), we will get a bound for

\[ \sum_{j=1}^{3} \| \partial_j \phi_l(\tau, \cdot) \|_{C^0(\mathbb{R}^3)} \}
\]
\[ \| \partial_t \phi_l(\tau, \cdot) \|_{C^0(\mathbb{R}^3)} \]

(16)

due to Sobolev embedding theorem. Now, consider

\[ \left| \partial \int_{\mathbb{R}^3} \phi_l^2 \, dx \right| = 2 \left| \int_{\mathbb{R}^3} \phi_l \partial_r \phi_l \, dx \right| \leq 2 \left( \int_{\mathbb{R}^3} \phi_l^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^3} (\partial_r \phi_l)^2 \, dx \right)^{1/2} \]  \hspace{1cm} (17)

where we have used Hölder inequality. By taking integration of (17) and using Grönwall lemma and the induction, and also we assume that \( |\tau - \tau_0| \leq 1 \), we obtain

\[ \| \phi_l(\tau, \cdot) \|_{L^2(\mathbb{R}^3)} \leq \| \phi_l(\tau_0, \cdot) \|_{L^2(\mathbb{R}^3)} + \int_{\tau_0}^{\tau} \| \partial_r \phi_\sigma(\sigma, \cdot) \|_{L^2(\mathbb{R}^3)} \, d\sigma \leq C \]  \hspace{1cm} (18)

with constant \( C \) depending on initial data. Therefore, \( \phi_l \) is in \( L^2(\mathbb{R}^3) \). To conclude, we have

\[ \| \phi_l(\tau, \cdot) \|_{H^{k+1}(\mathbb{R}^3)} + \| \partial_r \phi_l(\tau, \cdot) \|_{H^k(\mathbb{R}^3)} \leq C \]  \hspace{1cm} (19)

for \( |\tau - \tau_0| \leq T \), and \( C \) only depends on initial data. By assuming the scalar potential to be smooth, we have bound on

\[ \| \left[ (\partial^B F)(\phi_l, \partial_r \phi_l) \right](\tau, \cdot) \|_{C^0(\mathbb{R}^3)} \]  \hspace{1cm} (20)

for multiindex \( |B| \leq k \) which depends on the data. Note that \( B \) here is order derivative respect to \( (\phi_l, \partial_r \phi_l) \).
Let us consider
\[ \frac{dE_k[\phi_{l+1}]}{d\tau} = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} \partial^\alpha F(\phi_l, \partial_\tau \phi_l) \partial_\tau \phi_{l+1} d\mathbf{x}. \] (21)

Using Schwartz inequality and Hölder inequality, we obtain
\[ \frac{dE_k[\phi_{l+1}]}{d\tau} \leq \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} [\partial^\alpha F(\phi_l, \partial_\tau \phi_l)]^2 d\mathbf{x} \right)^{1/2} \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} (\partial^\alpha \partial_\tau \phi_{l+1})^2 d\mathbf{x} \right)^{1/2} \] (22)
The last factor can be bounded by a constant times \( E_1^{1/2}[\phi_{l+1}] \). To get a bound of the first factor, we divide into two cases, \(|\alpha| > 0\) and \(|\alpha| = 0\), for multiindex \( \alpha \).

For \(|\alpha| > 0\), using (19), we get
\[ \|F(\phi_l, \partial_\tau \phi_l)\|_{H^k(\mathbb{R}^3)} \leq C \left[ 1 + \|V'(\phi_l)\|_{H^k(\mathbb{R}^3)} \right]. \] (23)
We can write the last term as the sum of terms that consists of a constant times an expression of the form
\[ \partial^B[V'(\phi_l)](\partial^\gamma_1 \phi_l)(\partial^\gamma_2 \phi_l) \cdots (\partial^\gamma_l \phi_l), \] (24)
where \( \gamma_1 + \gamma_2 + \cdots + \gamma_l = \alpha \). If we take \( k > 3/2 \), by using Sobolev embedding theorem, we get
\[ \|V'(\phi_l)\|_{H^k(\mathbb{R}^3)} \leq C \|\phi_l\|_{H^k(\mathbb{R}^3)} \leq C. \] (25)

For the case \(|\alpha| = 0\), by assuming \( V'(0) = 0 \), we have estimate
\[ |V'(\phi_l)| \leq C|\phi_l| \] (26)
Thus, we have bound for
\[ |F(\phi_l, \partial_\tau \phi_l)| \leq C[|\phi_l| + |\partial_\tau \phi_l|], \] (27)
where \( C \) depends on the data. Since \( \phi_l \) and \( \partial_\tau \phi_l \) can be estimated in \( L^2(\mathbb{R}^3) \), we obtain
\[ \int_{\mathbb{R}^3} |F(\phi_l, \partial_\tau \phi_l)|^2 d\mathbf{x} \leq C. \] (28)
where \( C \) depends on the data. Thus we have bound for
\[ \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} [\partial^\alpha F(\phi_l, \partial_\tau \phi_l)]^2 d\mathbf{x} \right)^{1/2} \leq C \] (29)
where \( C \) depends on the data. Therefore, (22) becomes
\[ \left| \frac{dE_k[\phi_{l+1}]}{d\tau} \right| \leq CE_k^{1/2}[\phi_{l+1}] \] (30)
By integrating this and then using Grönwall lemma, we obtain
\[ E_k^{1/2}[\phi_{l+1}](\tau) \leq E_k^{1/2}[\phi_{l+1}](\tau_0) + \frac{1}{2}CT \] (31)
We conclude that, for appropriate choice of \( T \), the inductive assumption holds for all \( l \in \mathbb{N} \).
3. Local Existence

In this section, we show the final step to prove the local existence. Similar with (21), we can obtain

\[ \left| \frac{dE_k[\hat{\phi}_l]}{d\tau} \right| \leq \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} (\partial^{\alpha} \hat{F}_l)^2 dx \right)^{1/2} \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} (\partial^{\alpha} \partial_\tau \hat{\phi}_l)^2 dx \right)^{1/2}, \]  

with

\[ \hat{\phi}_l = \phi_{l+1} - \phi_l, \quad \hat{F}_l = F(\phi_l, \partial_\tau \phi_l) - F(\phi_{l-1}, \partial_\tau \phi_{l-1}). \]  

The first factor of (32) can be written as

\[ \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} (\partial^{\alpha} \hat{F}_l)^2 dx \right)^{1/2} = \| \hat{F}_l \|_{H^k(\mathbb{R}^3)} \leq C \left( \| \hat{\phi}_{l-1} \|_{H^k(\mathbb{R}^3)} + \| \partial_\tau \hat{\phi}_{l-1} \|_{H^k(\mathbb{R}^3)} + \| V'(\phi_l) - V'(\phi_{l-1}) \|_{H^k(\mathbb{R}^3)} \right). \]  

Consider the following calculation for the third term

\[ V'(\phi_l) - V'(\phi_{l-1}) = \int_0^1 \partial_\sigma V'[\sigma \phi_l + (1 - \sigma) \phi_{l-1}] d\sigma \]

\[ = \int_0^1 \partial_\sigma V'[\sigma \phi_l + (1 - \sigma) \phi_{l-1}] d\sigma \cdot \phi_{l-1}. \]

Thus, we have an estimate for this expression in \( H^k(\mathbb{R}^3) \)-norm

\[ \| V'(\phi_l) - V'(\phi_{l-1}) \|_{H^k(\mathbb{R}^3)} \leq C \| \hat{\phi}_{l-1} \|_{H^k(\mathbb{R}^3)}. \]  

Therefore, we have an estimate for (34)

\[ \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^3} (\partial^{\alpha} \hat{F}_l)^2 dx \right)^{1/2} \leq C \left( \| \hat{\phi}_{l-1} \|_{L^2(\mathbb{R}^3)} + E_k^{1/2} \| \hat{\phi}_{l-1} \| \right). \]  

Let us define

\[ e_l := \sup_{|\tau - \tau_0| \leq T} E_k^{1/2} \| \hat{\phi}_{l-1} \| + \sup_{|\tau - \tau_0| \leq T} \| \hat{\phi}_{l-1}(\tau, \cdot) \|_{L^2(\mathbb{R}^3)}, \]  

Then, we have

\[ \left| \frac{dE_k[\hat{\phi}_l]}{d\tau} \right| \leq C e_l E_k^{1/2} \| \hat{\phi}_l \| \]  

for \( |\tau - \tau_0| \leq T \). Using Grönwall inequality, we get

\[ E_k^{1/2} \| \hat{\phi}_l \| \leq E_k^{1/2} \| \hat{\phi}_l \|_{\tau_0} + 2 C e_l T \]  

(40)
for $|\tau - \tau_0| \leq T$. From the expression of (37), we need to know an estimate to $\hat{\phi}_l$ in $L^2$-norm. Similar to what we have done in (17), we get the following estimate

$$
\left\| \hat{\phi}_l(\tau, \cdot) \right\|_{L^2(\mathbb{R}^3)} \leq \left\| \hat{\phi}_l(\tau_0, \cdot) \right\|_{L^2(\mathbb{R}^3)} + 2^{l/2} \left| \int_{\tau_0}^{\tau} E_k^{1/2}[\hat{\phi}_l](\tau) \, d\tau \right| \quad (41)
$$

Assuming $T < 1/2$ and combining this observation with (40), we get

$$
e_{l+1} \leq C' \left[ \left\| \hat{\phi}_l(\tau_0, \cdot) \right\|_{L^2(\mathbb{R}^3)} + 2E_k^{1/2}[\hat{\phi}_l](\tau_0) + 2Ce_lT \right]
$$

for some positive constant $C'$. By a little further exploration, we can show that the first term is bounded using the fact that the sequence $f_l$ is Cauchy in $H^k(\mathbb{R}^3)$. For the second term, we can bound it due to inductive assumption in the early.

Using inductive, we will show that there is $C_0 > 1$ which depends on the data such that

$$
e_l \leq \frac{C_0}{2^l} \quad (42)
$$

for all $l \in \mathbb{N}$. For $l = 1$, this is true due to our definition in (38) and a assumption that $C_0$ is large enough. Let us assume it is true for some $l$. Thus, by using this assumption and the following fact

$$
\left\| \hat{\phi}_{l+1}(\tau_0, \cdot) \right\|_{L^2(\mathbb{R}^3)} + 2E_k^{1/2}[\hat{\phi}_l](\tau_0) + 1 \leq \frac{1}{2^{l+2}} \quad (43)
$$

and taking $C'CT \leq 1/8$, we obtain

$$
e_{l+1} \leq \frac{1}{2^{l+2}} + \frac{C_0}{2^{l+2}} \leq \frac{C_0}{2^{l+1}} \quad (44)
$$

Therefore, (42) holds for all $l \in \mathbb{N}$.

Finally, we prove that $\phi_l$ and $\partial_x u_l$ are Cauchy sequence in $H^{k+1}(\mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$, respectively. Here we only demonstrate for $\phi_l$ where for $\partial_x u_l$ it follows with a similar way. From the expression of $E_k[\hat{\phi}_l]$, for all possible value of $|\alpha| \leq k$, we have

$$
\left\| \nabla^{\alpha} \hat{\phi}_{l-1}(\tau, \cdot) \right\|_{L^2(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} \left| \nabla^{\alpha} \hat{\phi}_{l-1}(\tau, \cdot) \right| \, dx \right)^{1/2} \leq CE_k[\hat{\phi}_{l-1}](t) \leq Ce_l \quad (45)
$$

In other word, we have

$$
\left\| \hat{\phi}_{l-1}(\tau, \cdot) \right\|_{H^{k+1}(\mathbb{R}^3)} \leq Ce_l. \quad (46)
$$

Thus, by using the inductive assumption (42), we conclude that $\phi_l$ is Cauchy sequence in $H^{k+1}(\mathbb{R}^3)$. However, using the induction above, we have

$$
\sup_{\tau \in [\tau_0, \tau_0 + T]} \left\| \hat{\phi}_{l-1}(\tau, \cdot) \right\|_{H^{k+1}(\mathbb{R}^3)} \leq Ce_l \quad (47)
$$

which show that $\phi_l$ is Cauchy sequence in $C^0 \left\{ [\tau_0, \tau_0 + T], H^{k+1}(\mathbb{R}^3) \right\}$. Similar way with this, we can find that $\partial_x \phi_l$ is Cauchy sequence in $C^0 \left\{ [\tau_0, \tau_0 + T], H^k(\mathbb{R}^3) \right\}$, and also that $\partial_i\partial_j \phi_l$ and
∂t∂xφ1 are Cauchy sequences in $C^0\left([\tau_0, \tau_0 + T], H^{k-1}(\mathbb{R}^3)\right)$. What we haven’t find yet is for $\partial_t^2 \phi$. For the sake of this, we use the wave equation to obtain

$$
\left\|\partial_t^2 (\phi_{l+1} - \phi_{l})\right\|_{H^{k-1}(\mathbb{R}^3)} \leq \left\|F(\phi_l, \partial_t \phi_l) - F(\phi_{l-1}, \partial_t \phi_{l-1})\right\|_{H^{k-1}(\mathbb{R}^3)} + \left\|\partial_t \partial^j (\phi_{l+1} - \phi_{l})\right\|_{H^{k-1}(\mathbb{R}^3)} \tag{48}
$$

We can prove, using (37)-(38) and assumption (42), that the first norm in the right hand side is bounded, while the second term are bounded due to (46). Therefore, we conclude that $\partial_t^2 \phi$ is Cauchy sequence in $H^{k-1}(\mathbb{R}^3)$ and, consequently, it is also Cauchy in $C^0\left([\tau_0, \tau_0 + T], H^{k-1}(\mathbb{R}^3)\right)$.

Let $(\tau_i, x_i) \rightarrow (\tau, x)$, where $\tau_0 \leq \tau, \tau_i \leq T$. Consider this estimate

$$
|\phi(\tau, x) - \phi(\tau, x_i)| \leq |\phi(\tau, x) - \phi(\tau, x_i)| + |\phi(\tau, x_i) - \phi(\phi(\tau, x))| \tag{49}
$$

By Sobolev embedding theorem, for $k > 3/2$ we have $\phi(\tau, \cdot)$ is continuous function in $x$. Thus, the first term goes to zero when $i$ tends to infinity. The second term, using Sobolev embedding theorem, can be estimated by

$$
C \left\|\partial_t (\phi - \phi_{i})\right\|_{H^k(\mathbb{R}^3)} \tag{50}
$$

Since we have proved that $\phi \in C^0\left([\tau_0, \tau_0 + T], H^{k-1}(\mathbb{R}^3)\right)$, this also converge to zero. Therefore, $\phi \in C^0\left([\tau_0, \tau_0 + T] \times \mathbb{R}^3\right)$. Similar with this, we can prove that $\partial_t^2 \phi, \partial_t \phi, \partial_t^2 \phi, \partial_t \partial_t \phi, \partial_t \partial_t \phi$, and $\partial_t^3 \phi$ are in $C^0\left([\tau_0, \tau_0 + T] \times \mathbb{R}^3\right)$. Thus, we have a solution $\phi \in C^2\left([\tau_0, \tau_0 + T] \times \mathbb{R}^3\right)$.

For proving uniqueness, we consider $\phi_1, \phi_2$ be a solutions of equation (3.3) with same initial data.

$$
\frac{d}{d\tau} E_k[\phi_1 - \phi_2] \leq C \left\|F(\phi_1, \partial_t \phi_1) - F(\phi_2, \partial_t \phi_2)\right\|_{H^k(\mathbb{R}^3)} \tag{51}
$$

Using Grönwall lemma and fact that $\phi_1, \phi_2$ have same initial data, then for all $\tau \in [\tau_0, \tau_0 + T]$, we conclude that $\phi_1(\tau) = \phi_2(\tau)$ and uniqueness follows.

Thus, we have the final theorem,

**Theorem 3.1** Let $f \in H^{k+1}((R)^3)$ and $g \in H^{k+1}((R)^3)$ be an initial data with compact support. Assume that scalar potential is a smooth function satisfying $V(0) = 0$ and $V'(\phi) = 0$ and that $k > 3/2$. Then, there exist $T > 0$ and a unique $\phi \in \left\{[\tau_0, \tau_0 + T] \times \mathbb{R}^3\right\}$ local solution to the equation (11) such that

$$
\phi \in C^0\left([\tau_0, \tau_0 + T], H^{k-1}(\mathbb{R}^3)\right) \cap C^1\left([\tau_0, \tau_0 + T], H^{k}(\mathbb{R}^3)\right) \tag{52}
$$

**Acknowledgement**

This work of this paper is supported by PDUPT Kemenristekdikti 2017.

**References**

[1] S. Klainerman and P. Sarnak, *Explicit solution of $\Box u = 0$ on the Friedman-Robertson-Walker space-times*, Ann. Inst. Henri Poincaré, Vol. XXXV, n° 4, 1981, p. 253-257

[2] Hawking S and Ellis G 1973 *The Large Scale Structure of Space Time* (Cambridge Monographs on Mathematical Physics)

[3] Ringström H 2009 *The Cauchy Problem in General Relativity* (European Mathematical Society Publishing House)

[4] Christopher D. Sogge, *Lectures on nonlinear wave equations*, Monograph in Analysis, II. International Press, 1995.