Singularity of Full Scaling Limits of Planar Nearcritical Percolation

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Abstract

We consider full scaling limits of planar nearcritical percolation in the Quad-Crossing-Topology introduced by Schramm and Smirnov. We show that two nearcritical scaling limits with different parameters are singular with respect to each other. The results hold for percolation models on rather general lattices, including bond percolation on the square lattice and site percolation on the triangular lattice.

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1 Introduction

Percolation theory has attracted more and more attention since Smirnov’s proof of the conformal invariance of critical percolation interfaces on the triangular lattice. This was the missing link for the existence of a unique scaling limit of critical exploration paths. In the sequel, not only limits of exploration paths, but also limits of full percolation configurations have been explored. To obtain a scaling limit, one considers percolation on a lattice with mesh size $\eta > 0$ and lets $\eta$ tend to 0. In the case of the full configuration limit, it is a-priori not clear, in what sense, or in what topology, the limit $\eta \to 0$ shall be taken. There are several possibilities, which are explained in [SS-11, p. 1770ff]. For example, Camia and Newman established the full scaling limit of critical percolation on the triangular lattice as an ensemble of oriented loops, see [CN-06]. Schramm and Smirnov suggested to look at the set of quads which are crossed by the percolation configuration and constructed a nice topology for that purpose, the so-called Quad-Crossing-Topology, see [SS-11]. It has the advantage that it yields the existence of limit points for free (by compactness). Therefore we choose to work with their set-up.

They considered percolation models on tilings of the plane, rather than on lattices. Each tile is either coloured blue or yellow, independently of each other. All site or bond percolation models can be handled in this way using appropriate tilings. The results of [SS-11] hold on a wide range of percolation models. In fact, two basic assumptions on the one-arm event and on the four-arm event are sufficient. The results of the present article also hold on rather general tilings, but a bit stronger assumptions are needed. Basically, we require that the assumption of [SS-11] on the four-arm event holds and that the Russo-Seymour-Welsh (RSW) theory works. The exact conditions are presented below. In particular, we need the arm separation lemmas of [K-87] and [N-08]. They should hold on any graph which is invariant...
under reflection in one of the coordinate axes and under rotation around the origin by an angle $\phi \in (0, \pi)$, as stated in \cite{K-87} p. 112. But the proofs are written up only for bond or site percolation on the square lattice in \cite{K-87} and for site percolation on the triangular lattice in \cite{N-08}. Hence we choose to formulate the exact properties we need as conditions. We will first prove our results under that conditions and we will verify them for bond percolation on the square lattice and site percolation on the triangular lattice afterwards.

We want to consider nearcritical scaling limits. Nearcritical percolation is obtained by colouring a tile blue with a probability slightly different from the critical one. The difference depends on the mesh size, but converges to zero in a well-chosen speed. It includes – for each tile – one free real parameter. The main result of the present note is the following: We consider two (inhomogeneous) nearcritical percolations such that the difference of their parameters are uniformly bounded away from zero in a macroscopic region. Then we show that any corresponding sub-sequential scaling limits are singular with respect to each other.

Nolin and Werner showed in \cite{NW-09, Proposition 6} that – on the triangular lattice – any (sub-sequential) scaling limit of nearcritical exploration paths is singular with respect to an $\text{SLE}_6$ curve, i.e. to the limit of critical exploration paths. This was extended in \cite{A-12, Theorem 1}, where it is shown that the limits of two nearcritical exploration paths with different parameters are singular with respect to each other. The present result is somewhat different to those results. While in \cite{NW-09} and \cite{A-12} the singularity of exploration paths was detected, here it is the singularity of the full configurations in the Quad-Crossing-Topology. Note that this is not an easy corollary to the singularity of the exploration paths. While it is true that the trace of the exploration path can be recovered from the set of all crossed quads, this set does not provide any information about the behaviour of the exploration path at double points. Thus, in the limit, the exploration path as a curve is not a random variable of the set of crossed quads. Moreover, the results of \cite{NW-09} and \cite{A-12} hold only for site percolation on the triangular lattice, whereas the results of the present article hold under rather general assumptions on the lattice, which are, for instance, also fulfilled by bond percolation on the square lattice. Last, and indeed least, the percolation may also be inhomogeneous here. Since the restriction to homogeneous percolation in \cite{NW-09} and \cite{A-12} has only technical, but not conceptual reasons, this is only a minor difference.

The proofs use ideas from \cite{NW-09} and \cite{A-12}. In fact, the proofs of this article are technically simpler since there is no need to consider domains with fractal boundary. In section 2, we formally introduce the model and state all theorems and lemmas, which will be proved in section 3.

2 Results

As already mentioned, we use the set-up of \cite{SS-11}. Therefore we consider percolation on tilings of the plane rather than on lattices. A tiling is a collection of polygonal, topologically closed tiles such that the tiles may intersect each other only at their boundary and such that their union is the whole plane. We further require that the tilings are locally finite, i.e. any bounded set contains only finitely many tiles, and trivalent, i.e. any point belongs to at most three tiles.

For $\eta > 0$, let $H_\eta$ be a locally finite trivalent tiling such that the diameter of each tile is at most $\eta$. A percolation model is obtained by colouring every tile either blue or yellow. Some tiles may have a deterministic colour, while each tile $t \in H_\eta \subseteq H_\eta$ is coloured randomly blue with some probability $p(t) \in [0, 1]$ and otherwise yellow, independently of each other. Any site or bond percolation model can be realized using such a tiling, cf. \cite{SS-11} p. 1774f. Colouring some tiles deterministically en-
and define the set

$$\Omega_\eta := \{\text{blue,yellow}\}^{H'_\eta}, \quad A_\eta, \quad P_\eta := \bigotimes_{t \in H'_\eta} (p(t)\delta_{\text{blue}} + (1 - p(t))\delta_{\text{yellow}})$$

with product-\(\sigma\)-algebra \(A_\eta\) and \(p : H'_\eta \to [0, 1]\).

But we want to describe all discrete processes as well as the scaling limit by different probability measures on the same space. Thereto we use the space \(\mathcal{H}\) of all closed lower sets of quads introduced by Schramm and Smirnov in [SS-11] Section 1.3. As the exact construction is not important for understanding the present note (but it is important for the properties derived in [SS-11] we need), we explain it only very briefly. A quad \(Q\) is a homeomorphism \(Q : [0, 1]^2 \to Q([0, 1]^2) \subset \mathbb{C}\). A crossing of \(Q\) is a connected closed subset of \(Q([0, 1]^2)\) which intersects the images of the left and of the right side of \([0, 1]^2\). The question, whether every crossing of a quad contains a crossing of a second quad, provides a partial order on the quads. If a set of quads also contains all smaller quads (in the sense of the partial order), it is called a lower set of quads. Then \(\mathcal{H}\) is the space of all closed lower sets of quads.

For a quad \(Q\), we define the event \(\exists Q \subset \mathcal{H}\) that the quad \(Q\) is crossed: It is the set of all lower sets which contain \(Q\). The space \(\mathcal{H}\) is equipped with the so-called Quad-Crossing-Topology, which is the minimal topology containing all \((\exists Q)^c\) and other certain lower sets of quads. The induced Borel-\(\sigma\)-algebra \(\mathcal{B}(\mathcal{H})\) is generated by the events \(\exists Q\). For \(D \subset \mathcal{C}\), let \(\mathcal{B}_D\) be the restriction of \(\mathcal{B}(\mathcal{H})\) to lower sets of quads inside \(D\).

Any configuration \(\tilde{\omega}_\eta \in \Omega_\eta\) induces an element of \(\mathcal{H}\), namely the set \(\omega_\eta\) of all quads, which contain a blue crossing, i.e. a crossing which is a subset of the union of all blue tiles. Note that this is a lower set. Thus, for all \(\eta > 0\) and \(p : H'_\eta \to [0, 1]\), the measure \(P_\eta^p\) induces a probability measure \(P_\eta^p\) on \((\mathcal{H}, \mathcal{B}(\mathcal{H}))\). We will mainly work with these probability measures.

Now we define a special measure on \(\mathcal{H}\), namely the critical measure \(P_\eta^{\text{crit}}\). It is induced by \(\tilde{P}_\eta^p\) with \(p(t) = p_\eta^{\text{crit}}\) for all tiles \(t \in H'_\eta\). There \(p_\eta^{\text{crit}}\) is the critical probability of the tiling \(H_\eta\), i.e.

$$p_\eta^{\text{crit}} := \sup \{p \in [0, 1] \mid P_\eta^p[\text{There is an infinite blue cluster}] = 0, \ p(t) = p \ \forall t \in H'_\eta\}.$$ 

In fact, we do not use criticality. Thus \(p_\eta^{\text{crit}}\) could be any number in \((0, 1)\) such that the conditions below are satisfied. But they usually hold only if \(p_\eta^{\text{crit}}\) is indeed the critical probability.

For \(z \in \mathbb{C}\) and \(0 < \eta \leq r < R\), let \(A_\eta(z, r, R)\) be the event that there are four crossings of alternating colour inside the annulus centred at \(z\) with radii \(r\) and \(R\).

We fix some \(R_0, N_0 > 0\) and \(z_0 \in \mathbb{C}\) for the remainder of the article. We want to define the nearcritical models. We abbreviate

$$\alpha_\eta^4 := P_\eta^p[A_\eta(z_0, \eta, R_0)]$$

and define the set

$$\Pi_\eta := \{P_\eta^p \mid p(t) = (p_\eta^{\text{crit}} + \varsigma(t) : \eta^2 \alpha_\eta^4) \lor 0 \land 1, \ \varsigma(t) \in [-N_0, N_0], \ t \in H'_\eta\},$$

the set of all probability measures on \((\mathcal{H}, \mathcal{B}(\mathcal{H}))\) which are in the critical window. If we want to specify the chosen parameter \(\varsigma = (\varsigma(t))_{t \in H'_\eta}\), we write \(P_\eta^\varsigma\) for the corresponding measure. We therefore use the speed factor \(\eta^2 / \alpha_\eta^4\) for the convergence of the nearcritical probabilities to the critical one. This rate is inspired by [K-87 Theorem 4], [N-08 Proposition 32] and the results of [GPS-10]. From Lemma 4 below and [NW-09 Proposition 4], it follows that \(\eta^2 / \alpha_\eta^4\) is indeed the correct rate.
Conditions. We impose the following basic conditions on the tilings \( H_\eta \), \( \eta > 0 \). The constants \( \eta_0, c_1, c_2, c_3 > 0 \) as well as the functions \( \Delta_4 \) and \( \Delta_1 \) may depend on \( R_0 \) and \( N_0 \). The words in italic are only headings without any formal meaning.

1. The following multi-scale bound on the four arm event holds:
   There exists a positive function \( \Delta_4(r, R) \) such that for all fixed \( R \leq R_0 \)
   \[
   \lim_{r \to 0} \Delta_4(r, R) = 0
   \]
   and such that for all \( \eta \leq r < R \leq R_0 \)
   \[
   P_\eta[A_4(z_0, r, R)] \leq \frac{r}{R} \Delta_4(r, R).
   \]

2. The probabilities in the critical window are eventually strictly in between 0 and 1:
   There exists \( \eta_0 > 0 \) such that for all \( \eta \in (0, \eta_0) \):
   \[
   0 < p_{\eta \text{\ crit}} - N_0 \frac{\eta^2}{\alpha_4^2} < p_{\eta \text{\ crit}} + N_0 \frac{\eta^2}{\alpha_4^2} < 1.
   \]

3. The probabilities of the four-arm events are comparable on the whole plane over all (near)critical measures:
   There are constants \( c_1, c_2 > 0 \) such that for all \( \eta \leq r < R \leq R_0 \), \( z \in \mathbb{C} \) and \( P_\eta \in \Pi_\eta \) the following holds:
   \[
   c_1 P_\eta[A_4(z_0, \eta, R)] \leq P_\eta[A_4(z, \eta, R)] \quad \text{and} \quad P_\eta[A_4(z, r, R)] \leq c_2 P_\eta[A_4(z_0, r, R)].
   \]
   (Note that we need the first inequality for \( r = \eta \) only.)

4. The probability of the four arm event is uniformly comparable to the probability of the following modified four arm events:
   For \( R > 0 \) and \( z \in \mathbb{C} \), let \( Q(z, R) \) be the square with side length \( R \) centred at \( z \).
   For a tile \( t \) in \( Q(z, R) \) whose distance from \( z \) is at most \( R/4 \), let \( A_4'(t, \partial Q(z, R)) \) be event that there are four arms of alternating colour from \( t \) to the left, lower, right and upper boundary of \( Q(z, R) \).
   There exists a constant \( c_3 > 0 \) such that for all \( 4\eta \leq R \leq R_0 \), \( z \in \mathbb{C} \), \( P_\eta \in \Pi_\eta \) and all tiles \( t \) in \( Q(z, R) \) whose distance from \( z \) is at most \( R/4 \):
   \[
   P_\eta[A_4'(t, \partial Q(z, R))] \geq c_3 P_\eta[A_4(z, \eta, R)].
   \]

5. There is the following bound on the one arm event:
   There exists a positive function \( \Delta_1(r, R) \) such that for all fixed \( R \leq R_0 \)
   \[
   \lim_{r \to 0} \Delta_1(r, R) = 0
   \]
   and such that for all \( \eta \leq r < R \leq R_0 \), \( z \in \mathbb{C} \), \( P_\eta \in \Pi_\eta \) and \( \text{col} \in \{\text{blue, yellow}\} \)
   \[
   P_\eta[A_1^\text{col}(z, r, R)] \leq \Delta_1(r, R),
   \]
   where \( A_1^\text{col}(z, r, R) \) is the event that there exists a crossing of colour \( \text{col} \) inside the annulus centred at \( z \) with radii \( r \) and \( R \).
Corollary 2. Let the conditions of Theorem 1 be fulfilled. Let uniformly in each other.

Assume that there exist \( \sigma > 0 \) and an open, non-empty set \( D \subset \mathbb{C} \) such that

\[
\lambda_\eta(t) - \eta_\eta(t) \geq \sigma
\]

uniformly in \( \eta \in \{ \eta_n : n \in \mathbb{N} \} \) and all tiles \( t \in H'_n \) which are contained in \( D \).

Then the laws \( P^\mu \) and \( P^\lambda \) restricted to \( B_D \) are singular with respect to each other.

Similarly to [A-12, Corollary 2], we can even detect the asymmetry by only looking at an infinitesimal neighbourhood of a point inside \( D \), more precisely:

**Corollary 2.** Let the conditions of Theorem 1 be fulfilled. Let \( z \in D \). Let

\[
B_z := \bigcap_{n \in \mathbb{N}} B_{B^\lambda(z)}
\]

be the tail-\( \sigma \)-algebra of the restrictions of \( B(H) \) to lower sets of quads in the ball \( B^\lambda(z) \).

Then the laws \( P^\mu \) and \( P^\lambda \) restricted to \( B_z \) are singular with respect to each other.

We base the proof of Theorem 1 on the following two lemmas. The first one is specific for the model. The second one is rather abstract to detect the singularity.

**Lemma 3.** Under the conditions of Theorem 1 there exists a function \( \Delta_\delta : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \Delta_\delta(\delta) \to 0 \) as \( \delta \to 0 \) such that for any square \( Q \) of side length \( \delta \leq R_0 \) inside \( D \):

\[
P^\lambda[\square Q] - P^\mu[\square Q] \geq \frac{\delta}{\Delta_\delta(\delta)},
\]

where \( \square Q \) denotes the event that there exists a horizontal blue crossing of the square \( Q \).

**Lemma 4.** Let \( P \) and \( P' \) be two probability measures on a space \((\Omega, \mathcal{A})\). Let \( a, b > 0 \) and let \((\Delta_n)_{n \in \mathbb{N}}\) be a sequence converging to infinity. Set \( K_n := [an^2] \), \( n \in \mathbb{N} \). For large enough \( n \in \mathbb{N} \), let \( X^n_k, k \in \{1, \ldots, K_n\} \), be random variables which are uncorrelated in \( k \) with respect to \( P \) and \( P' \), absolutely bounded by \( b \), and satisfy

\[
E_P[X^n_k] - E_{P'}[X^n_k] \geq \frac{1}{n} \Delta_n, \quad k \in \{1, \ldots, K_n\}.
\]

Then \( P \) and \( P' \) are singular with respect to each other.
Using results of [SW-01], [SS-11] Appendix B, [N-08] and [K-87] as well as standard techniques, we can easily verify conditions 1-5 in the two most important cases:

**Lemma 5.** Conditions 1 to 5 are fulfilled by tilings representing site percolation on the triangular lattice or bond percolation on the square lattice.

Thereto we will need the following converse of [N-08, Proposition 32], which estimates the characteristic length. For the remainder of this section, we consider site percolation on the triangular lattice or bond percolation on the square lattice, each with mesh size 1. Let \( p_c = \frac{1}{2} \) be the critical probability. For \( \varepsilon \in (0, \frac{1}{2}) \) and \( p \in (0, 1) \), let \( L_\varepsilon(p) \) be the corresponding characteristic length as defined in [N-08, Section 3.1] or [K-87, Equation (1.21)], respectively, i.e.

\[
L_\varepsilon(p) := \begin{cases} 
\inf \{ n \in \mathbb{N} : P_p[\Xi(n \times n)] \leq \varepsilon \} & \text{if } p < p_c \\
\inf \{ n \in \mathbb{N} : P_p[\Xi(n \times n)] \geq 1 - \varepsilon \} & \text{if } p > p_c 
\end{cases}
\]

and \( L_\varepsilon(p_c) = \infty \), where \( P_p \) denotes the product measure with probability \( p \) for blue, and \( \Xi(m \times n) \) denotes the event that there is a horizontal blue crossing of a \( m \times n \) rectangle.

Moreover, for \( m < n \), let \( \alpha_4(m, n) \) be the probability that at critical percolation there exist four arms of alternating colour inside the annulus centred at the origin with radii \( m \) and \( n \). We abbreviate \( \alpha_4(n) := \alpha_4(1, n) \).

**Lemma 6.** For all \( \varepsilon \in (0, \frac{1}{2}) \) and \( C_1, C_2 > 0 \), there exist \( C_3, C_4 > 0 \) such that for all \( p \in (0, 1) \) and \( n \geq 1 \) the following implication holds:

\[
C_1 \leq |p - p_c| n^{2} \alpha_4(n) \leq C_2 \implies C_3 \leq \frac{n}{L_\varepsilon(p)} \leq C_4.
\]

Finally, we need the following lemma, which restates Remark 36 of [N-08]. Since the author is not aware of a formal statement in the literature, it is included here for the sake of completeness.

**Lemma 7.** For all \( \varepsilon_0 \in (0, \frac{1}{2}) \) and all \( K \geq 1 \), there exists an \( \varepsilon \in (0, \varepsilon_0) \) such that for all \( 0 < p < p_c \):

\[
L_\varepsilon(p) \geq K \cdot L_{\varepsilon_0}(p).
\]

### 3 Proofs

In this section, we give the proofs of all stated assertions.

**Proof of Lemma 3.** Let \( Q \) be a square of side length \( \delta \leq R_0 \) inside \( D \). Let \( \eta \in \{ \eta_n : n \in \mathbb{N} \} \) small enough such that \( 4\eta < \delta \) and \( \eta < \eta_0 \), where \( \eta_0 \) is chosen according to condition 2.

We construct a coupling \((\hat{\Omega}, \hat{A}, \hat{P})\) as follows. Let

\[
\hat{\Omega} := (\{ \text{blue,yellow} \} \times \{ \text{blue,yellow} \})^{H_n}
\]

with product-\(\sigma\)-algebra \( \hat{A} \). Informally, let \( \hat{P} \) be the probability measure which has marginal distributions \( \hat{P}_\eta \) and \( \hat{P}_\lambda \) such that the set of blue tiles in \( Q \) increases. More precisely, we define the random variables

\[
f_I : \hat{\Omega} \to \mathcal{H}, \quad I \in \{1, 2\}^{H_n},
\]

\[
f_I := \begin{cases} 
\eta \quad & \text{if } I = \{1\} \\
\lambda \quad & \text{if } I = \{2\}
\end{cases}
\]

and for \( I = \{1, 2\} \),

\[
f_I := \begin{cases} 
\eta \quad & \text{if } I = \{1\} \\
\lambda \quad & \text{if } I = \{2\}
\end{cases}
\]

with marginal distributions \( \hat{P}_\eta \) and \( \hat{P}_\lambda \).
For $\hat{\omega} = (\hat{\omega}_1(t), \hat{\omega}_2(t))_{t \in H_n^c} \in \hat{\Omega}$, let $f_1(\hat{\omega})$ be the set of all quads which contain a blue crossing if tile $t \in H_n^c$ is coloured with colour $\hat{\omega}_1(t)$). We abbreviate $I := (1, \ldots, 1)$ and $(2) := (2, \ldots, 2)$. Then let $\hat{P}$ be a probability measure on $\Omega$ such that

$$f_{(1)}(\hat{P}) = P^\mu_{\eta}, \quad f_{(2)}(\hat{P}) = P^\lambda_{\eta} \quad \text{and} \quad \hat{P}[\hat{\omega} : \hat{\omega}_1(t) = \text{blue}, \hat{\omega}_2(t) = \text{yellow} \text{ for some tile } t \in Q] = 0.$$ 

Such a coupling can be obtained, for example, from the standard monotone coupling out of switches from yellow to blue of some tiles in $T$. For $\hat{\omega} = (\hat{\omega}_1, \ldots, \hat{\omega}_k, \hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_1, \hat{\omega}_2)$, we restrict ourselves to the event that the crossing arises only for one independent events. Using the described disjointness and independence, we get

$$P^\mu_{\eta}[\Box Q] - P^\mu_{\eta}[\Box Q] = \hat{P}[f_{(1)}^{-1}[\Box Q] \setminus f_{(1)}^{-1}[\Box Q]) - \hat{P}[f_{(2)}^{-1}[\Box Q] \setminus f_{(2)}^{-1}[\Box Q]] = \hat{P}[f_{(1)}^{-1}[\Box Q] \cap f_{(2)}^{-1}[\Box Q]] - 0,$$

since $\hat{\omega} \in f_{(1)}^{-1}[\Box Q] \setminus f_{(2)}^{-1}[\Box Q]$ implies that there is a tile $t$ in $Q$ with $\hat{\omega}_1(t) = \text{blue}$ and $\hat{\omega}_2(t) = \text{yellow}$. Thus we have to estimate the probability of the event of all $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2)$ such that $\hat{\omega}_2$ induces a blue crossing of $Q$, but $\hat{\omega}_1$ does not.

Let $T = \{t_1, \ldots, t_k\}$ be the set of all tiles in $Q$ whose distance from the centre $z_Q$ of $Q$ is at most $\delta/4$ – arranged in any (but fixed) order. In order to prove the proposed estimate, we restrict ourselves to the event that the crossing arises out of switches from yellow to blue of some tiles in $T$. Thereto we change the coordinates of $\hat{\omega}$ we use for the tiles in $T$ one by one. Formally, for $k = 0, \ldots, K$, let $I_k = \{I_k(t) = 1 \text{ if } t \in H_n^c \setminus \{t_1, \ldots, t_k\}, \text{ and } I_k(t) = 2 \text{ if } t \in \{t_1, \ldots, t_k\}$.

Then

$$\hat{P}[f_{(1)}^{-1}[\Box Q] \cap f_{(2)}^{-1}[\Box Q]] \geq \hat{P}\left[\bigcup_{k=1}^K f_{I_k^{-1}[\Box Q] \cap f_{I_k^{-1}[\Box Q]}\right].$$

As the crossing event is increasing, the event $f_{I_k^{-1}[\Box Q] \cap f_{I_k^{-1}[\Box Q]}$ can happen only for one $k \in \{1, \ldots, K\}$. This is the case if and only if the following two events occur: first, the event $f_{I_k^{-1}[\Box Q] \cap f_{I_k^{-1}[\Box Q]}$ that there are four arms of alternating colour from $t_k$ to the left, lower, right and upper boundary of $Q$, which means that $t_k$ is pivotal for the crossing event; second, the event that the colour of $t_k$ switches from $\hat{\omega}_1(t_k) = \text{yellow}$ to $\hat{\omega}_2(t_k) = \text{blue}$, which we denote by $Sw(t_k)$. Note that they are independent events. Using the described disjointness and independence, we get

$$\hat{P}\left[\bigcup_{k=1}^K f_{I_k^{-1}[\Box Q] \cap f_{I_k^{-1}[\Box Q]}\right] = \sum_{k=1}^K \hat{P}[f_{I_k^{-1}[\Box Q] \cap f_{I_k^{-1}[\Box Q]}] \cdot \hat{P}[Sw(t_k)].$$

Now we estimate these probabilities. Elementary probability calculus and the construction of the coupling yield

$$\hat{P}[Sw(t_k)] = \hat{P}[\{\hat{\omega} : \hat{\omega}_2(t_k) = \text{blue} \} \setminus \{\hat{\omega} : \hat{\omega}_1(t_k) = \text{blue}\}] = \hat{P}[\{\hat{\omega} : \hat{\omega}_2(t_k) = \text{blue}\} - \hat{P}[\{\hat{\omega} : \hat{\omega}_1(t_k) = \text{blue}\}] + \hat{P}[\{\hat{\omega} : \hat{\omega}_1(t_k) = \text{blue}\} \setminus \{\hat{\omega} : \hat{\omega}_2(t_k) = \text{blue}\}] = P^\mu_{\eta}[t_k \text{ blue}] - P^\mu_{\eta}[t_k \text{ blue}] + \hat{P}[\hat{\omega}_1(t_k) = \text{blue}, \hat{\omega}_2(t_k) = \text{yellow}] = (\mu^\text{crit} + \lambda^\eta(t_k) \cdot \eta^2_{\alpha^\eta}) - (\mu^\text{crit} + \mu^\eta(t_k) \cdot \eta^2_{\alpha^\eta}) + 0 = (\lambda^\eta(t_k) - \mu^\eta(t_k)) \cdot \eta^2_{\alpha^\eta} \geq \sigma \cdot \eta^2_{\alpha^\eta},$$

where $\sigma$ is a constant chosen in such a way that

$$\hat{P}[Sw(t_k)] \geq \sigma \cdot \eta^2_{\alpha^\eta}.$$
because of \( \eta < \eta_0 \) (such that, by condition (2) the probabilities are given by the used formulas) and because of the assumption in Theorem (4).

Let \( P_{\eta}^k \) denote the image law of \( \hat{P} \) under \( f_{I_k} \). Then \( P_{\eta}^k \in \Pi_{\eta} \). Using conditions (4) and (3) we conclude

\[
\hat{P} \left[ f_{I_k}^{-1} [A'_4(t_k, \partial Q)] \right] \geq c_3 P_{\eta}^k [A_4(z_0, \eta, \delta)] \geq c_3 c_1 P_{\eta}^0 [A_4(z_0, \eta, \delta)].
\]

As there are \( K \geq c_4 (\delta/\eta)^2 \) tiles in \( T \) (for some numerical constant \( c_4 > 0 \)), the equations above imply

\[
P_{\eta}^k [\Xi \Xi] - P_{\eta}^k [\Xi | \Xi] \geq c_4 \frac{\eta^2}{\alpha^2} c_3 c_1 P_{\eta}^0 [A_4(z_0, \eta, \delta)], \quad \frac{\eta^2}{\alpha^4} = \sigma c_1 c_3 c_4 \cdot \frac{\delta^2 P_{\eta}^0 [A_4(z_0, \eta, \delta)]}{P_{\eta}^0 [A_4(z_0, \eta, R_0)]}.
\]

Using first \( A_4(z_0, \eta, R_0) \subseteq A_4(z_0, \eta, \delta) \cap A_4(z_0, \delta, R_0) \) and independence of the latter two events and then condition (1) we conclude

\[
P_{\eta}^k [\Xi \Xi] - P_{\eta}^k [\Xi | \Xi] \geq \sigma c_1 c_3 c_4 \cdot \frac{\delta^2 R_0}{\delta \Delta_4 (\delta, R_0)} = \frac{\delta}{\Delta_\sigma (\delta)}
\]

with \( \Delta_\sigma (\delta) := (\sigma c_1 c_3 c_4 R_0)^{-1} \Delta_4 (\delta, R_0) \). Condition (1) implies \( \Delta_\sigma (\delta) \to 0 \) as \( \delta \to 0 \).

For \( i \in \{ \mu, \lambda \} \), Lemma 5.1 of [SS-11] (implying \( P^0 [\partial \Xi \Xi] = 0 \)) and the weak convergence of \( P_{\eta_0}^k \) yield \( P_{\eta_0}^k [\Xi \Xi] \to P^0 [\Xi \Xi] \) as \( n \to \infty \), which concludes the proof.

**Proof of Lemma (4)**. We define for large enough \( n \in \mathbb{N} \)

\[
Z_n := \sum_{k=1}^{K_n} \left( X_{k}^n - E_P[X_{k}^n] \right).
\]

It follows that \( E_P[Z_n] = 0 \) and that

\[
E_P[Z_n] = \sum_{k=1}^{K_n} (E_P[X_{k}^n] - E_P[X_{k}^n]) \geq K_n \cdot \frac{1}{n} \Delta_n \geq a n \Delta_n,
\]

because of the assumption and \( K_n = [an^2] \). Since the random variables are uncorrelated and bounded, we can estimate the variance of \( Z_n \) under \( P \) or under \( P^0 \) as follows:

\[
\text{Var}[Z_n] = \sum_{k=1}^{K_n} \text{Var} [X_{k}^n - E_P[X_{k}^n]] = \sum_{k=1}^{K_n} \text{Var}[X_{k}^n] \leq K_n b^2 \leq (a + 1)b^2 n^2.
\]

Using Chebyshev’s Inequality, we estimate

\[
P[Z_n \geq \frac{a}{2} n \Delta_n] \leq \frac{4}{a^2 n^2 \Delta_n^2} \text{Var}_P[Z_n] \leq \frac{4(a + 1)b^2 n^2}{a^2 n^2 \Delta_n^2} = \frac{4(a + 1)b^2}{a^2} \cdot \Delta_n^{-2}
\]

and

\[
P[Z_n < \frac{a}{2} n \Delta_n] = P'[\left( E_P[Z_n] - Z_n \right) > (E_P[Z_n] - \frac{a}{2} n \Delta_n)] \\
\leq P'[\left| E_P[Z_n] - Z_n \right| > (an \Delta_n - \frac{a}{2} n \Delta_n)] \\
\leq \frac{4}{a^2 n^2 \Delta_n^2} \text{Var}_P[Z_n] \leq \frac{4(a + 1)b^2}{a^2} \cdot \Delta_n^{-2}.
\]
If we now choose a sparse enough sub-sequence $n_l, l \in \mathbb{N}$, i.e. such that $\sum_l \Delta_n^{-2} < \infty$, the Borel-Cantelli Lemma yields

$$P'[Z_{n_l} < \frac{\eta}{n_l} \Delta_n, \text{for infinitely many } l] = 0,$$

implying $$P'[Z_{n_l} \geq \frac{\eta}{n_l} \Delta_n, \text{for infinitely many } l] = 1,$$

while $$P[Z_{n_l} \geq \frac{\eta}{n_l} \Delta_n, \text{for infinitely many } l] = 0.$$

Therefore we detected an event which has $P$-probability zero, but $P'$-probability one.

\begin{proof}[Proof of Theorem \ref{thm:main}] We want to apply Lemma \ref{lem:main}. Let $P' = P^\lambda$ and $P = P^\mu$. We set $\delta_n = 1/n$, $n \in \mathbb{N}$, and choose an appropriate $a > 0$ (depending on the size of $D$) such that, for sufficiently large $n$, we can place $K_n = [an^2]$ disjoint squares $Q^n_1, \ldots, Q^n_{K_n}$ of size $\delta_n$ in $D$. We define the random variables $X^n_k : \mathcal{H} \to \mathbb{R}$ by

$$X^n_k = \mathbbm{1}_{\exists Q^n_k} , \quad k \in \{1, \ldots, K_n\}.$$ 

Since the disjointness of the squares yields independence of the crossing events for all $P'_{\mathcal{R}^n}$, since $P_{\mathcal{R}^n} \to P'$ weakly and since $P'[\exists Q^n_k] = 0$ by [SS-11, Lemma 5.1], the random variables $X^n_k$, $k \in \{1, \ldots, K_n\}$, are independent for $P^n$, $i \in \{\mu, \lambda\}$. Moreover, $|X^n_k| \leq 1$, and Lemma \ref{lem:independence} yields

$$E_{P^n}[X^n_k] - E_{P^\mu}[X^n_k] = P^\lambda[\exists Q^n_k] - P^\mu[\exists Q^n_k] \geq \frac{\delta_n}{\Delta_n(\delta_n)} = \frac{1}{n} \Delta_n$$

with $\Delta_n := \Delta_n(\delta_n)^{-1} \to \infty$ as $n \to \infty$. Thus Lemma \ref{lem:independence} yields that $P^\mu$ and $P^\lambda$ are singular with respect to each other. Since all random variables $X^n_k$ are $\mathcal{B}_D$-measurable, we can also apply Lemma \ref{lem:main} when $P^\mu$ and $P^\lambda$ are restricted to $\mathcal{B}_D$.

\begin{proof}[Proof of Corollary \ref{cor:main}] The proof is analogous to the proof of [A-12, Corollary 2]. Let $m_0 \in \mathbb{N}$ such that $B_{\frac{1}{m_0}}(z) \subseteq D$. Let $n \geq m_0$. By Theorem \ref{thm:main} – applied inside $B_{\frac{1}{m_0}}(z)$ – there are sets $B_n \in B_{\frac{1}{m_0}}(z)$ with $P^\mu[B_n] = 0$ and $P^\lambda[B_n] = 1$. We set

$$B_* := \bigcup_{m \geq m_0} \bigcap_{n \geq m} B_n.$$

Then $B_* \in \mathcal{B}_z$. Since countable unions or intersection of sets of probability zero respectively one have probability zero respectively one, it follows that $P^\mu[B_*] = 0$ and $P^\lambda[B_*] = 1$, which proves the corollary.

\begin{proof}[Proof of Lemma \ref{lem:independence}] As it is proven on the triangular lattice that the 4-arm-exponent is 5/4, see [SW-01, Theorem 4], condition \ref{thm:main} holds. For bond percolation on the square lattice, this condition is proven by Christophe Garban in [SS-11, Lemma B.1].

Now we claim that $R_0$ is below the characteristic length, i.e. there is some $\varepsilon \in (0, \frac{1}{2})$ such that $R_0/\eta \leq L_\varepsilon(p^\mu_{\epsilon N_0})$ for all $0 > 0$. Thereto we provisionally fix some $\varepsilon_0 \in (0, \frac{1}{2})$. Since

$$|p_{\eta}^{-N_0} - p_{\eta}^{\text{crit}}| (R_0/\eta)^2 P_{\eta}[A(z_0, \eta, R_0)] = R_0^2 N_0,$$

Lemma \ref{lem:independence} (for $n = (R_0/\eta)$ and $p = p_{\eta}^{-N_0}$) yields that $R_0/\eta \leq C_4 L_\varepsilon(p_{\epsilon N_0})$ for some $C_4 = C_4(R_0, N_0, \varepsilon_0) > 0$. By Lemma \ref{lem:main} we find an $\varepsilon \in (0, \varepsilon_0)$ such that the claim holds.

Now we fix this $\varepsilon > 0$. Since every $P_{\eta} \in \Pi_\eta$ is between $P_{\eta}^{-N_0}$ and $P_{\eta}^{+N_0}$, the claim above allows us to use arguments of RSW style and to apply most of the
results of \cite{N-08} and \cite{K-87} as long as we use radii \( R \leq R_0 \). In fact, all of the remaining conditions easily follow from the results of these papers.

The following reasoning is a standard technique. By RSW, there is a constant \( c > 0 \) such that for all \( \text{col} \in \{ \text{blue, yellow} \}, \ z \in \mathbb{C}, \ \eta \leq r \leq R_0/2 \) and \( P_\eta \in \Pi_\eta \)

\[
c \leq P_\eta[A^\text{col}_1(z, r, 2r)] \leq 1 - c.
\]

Let \( R \leq R_0 \) be fixed. For \( r \in (\eta, R/2) \), let \( K_r \in \mathbb{N} \) be the largest number such that \( 2^{K_r} \leq R/r \). Then \( K_r \to \infty \) as \( r \to 0 \). It follows that

\[
P_\eta[A^\text{col}_1(z, r, R)] \leq P_\eta[\forall k = 1, \ldots, K_r : A^\text{col}_1(z, r 2^{k-1}, r 2^k)] \leq \prod_{k=1}^{K_r} P_\eta[A^\text{col}_1(z, r 2^{k-1}, r 2^k)] \leq (1 - c)^{K_r} \to 0
\]

as \( r \to 0 \), which shows condition 5 (on both lattices).

By \cite{SSSt-10} Corollary A.8 (stating that the 5-arm-exponent is 2) and Reimer’s Inequality, it follows that (for some \( \tilde{c} > 0 \))

\[
\tilde{c} R_0^{-2} \eta^2 \leq P_\eta[A_0(z_0, \eta, R_0)] \leq P_\eta[A_4(z_0, \eta, R_0)] \cdot P_\eta[A_1(z_0, \eta, R_0)]. \tag{1}
\]

Thus condition 3 yields \( \eta^2 / \alpha_1^2 \to 0 \) as \( \eta \to 0 \), which, together with \( P_\eta^{\text{crit}} = \frac{1}{2} \), implies condition 2 (on both lattices).

Since the considered lattices are transitive, the estimates in conditions 3 and 4 hold uniformly in \( z \in \mathbb{C} \), if they hold for \( z = 0 \). Thus we consider only this case. Condition 3 on the triangular lattice is included in Theorem 26 of \cite{N-08}. On the square lattice, condition 3 is a consequence of \cite{K-87} Lemma 8 \cite{K-87} with \( v = 0 \) and \cite{K-87} Lemma 4 \cite{K-87}. These two lemmas (with \( \kappa = 0.5 \)) also imply condition 4 on the square lattice. On the triangular lattice, it is a special case of equation (4.20) in \cite{N-08}.

Note that we are considering site percolation on the triangular lattice or bond percolation on the square lattice with mesh size 1 in the remaining two lemmas.

**Proof of Lemma 2** We fix some \( \varepsilon \in (0, \frac{1}{2}) \) and abbreviate \( L(p) := L_{\varepsilon}(p) \). We will use the following facts. First,

\[
\exists C_1(\varepsilon), C_2(\varepsilon) > 0 \forall p \in (0, 1) : C_1 \leq |p - p_\varepsilon| L(p)^2 \alpha_4(L(p)) \leq C_2, \tag{i}
\]

which is \cite{N-08} Proposition 32 for the triangular lattice and \cite{K-87} Theorem 4 for the square lattice. Second, we need \cite{SSSt-10} Proposition 4, i.e. quasi-multiplicativity:

\[
\exists C_5 > 0 \forall m < n : \alpha_4(m) \cdot \alpha_4(m, n) \leq C_5 \alpha_4(n). \tag{ii}
\]

Finally, we need an estimate of the four arm event, namely

\[
\exists \beta, C_6 > 0 \forall m < n : \alpha_4(m, n) \geq C_6 \left( \frac{m}{n} \right)^{2 - \beta}. \tag{iii}
\]

Its proof is analogous to the proof of equation (1) above. Note that we can a-priori apply the RSW theory for (iii), since there we consider only critical percolation.

Let \( C_1, C_2 > 0 \). We define \( C_3, C_4 \geq 0 \) by

\[
C_4 := \max \left\{ \left( \frac{C_6 C_5}{C_1 C_2} \right)^{\frac{1}{\beta}}, 1 \right\} \quad \text{and} \quad \frac{1}{C_3} := \max \left\{ \left( \frac{C_6 C_5}{C_1 C_2} \right)^{\frac{1}{\beta}}, 1 \right\}.
\]

Let \( p \in (0, 1) \) and \( n \geq 1 \) with

\[
C_1 \leq |p - p_\varepsilon| n^2 \alpha_4(n) \leq C_2. \quad (\ast)
\]
First, we show that \(n/L(p) \leq C_4\). We can assume \(n > L(p)\), since otherwise \(n/L(p) \leq 1 \leq C_4\). Facts (ii) and (iii) imply
\[
\frac{\alpha_4(n)}{\alpha_4(L(p))} \geq \frac{1}{C_5} \alpha_4(L(p), n) \geq \frac{C_6}{C_5} \left( \frac{L(p)}{n} \right)^{2-\beta}.
\]
Combined with the left inequality of (i) and the right inequality of (**), we conclude
\[
\frac{C_8}{C_7} \geq \frac{|p_p|n^2\alpha_4(n)}{|p_p| L(p)^2 \alpha_4(L(p))} \geq \left( \frac{n}{L(p)} \right)^2 \frac{C_6}{C_5} \left( \frac{L(p)}{n} \right)^{-\beta} = \frac{C_6}{C_5} \left( \frac{n}{L(p)} \right)^{\beta}
\]
and therefore \(n/L(p) \leq C_4\).

The same reasoning yields the other estimate, i.e. \(L(p)/n \leq 1/C_3\), if we interchange the role of \(L(p)\) and \(n\), and if we use the other inequalities of (i) and (**), and if we replace \(\mathcal{C}\) by \(C_1\) and \(C_2\) by \(C_2\).

Proof of Lemma 7. Let \(\epsilon_0 \in (0, \frac{1}{2})\) and \(K \geq 2\). The RSW Theorem (see [N-08, Theorem 2], for instance) states that there is a universal positive function \(f_K(\cdot)\), such that, for all \(m \in \mathbb{N}\), if the probability of crossing an \(m \times m\) rectangle is at least \(\delta\), then the probability of crossing an \(Km \times m\) rectangle is at least \(f_K(\delta)\). We set \(\epsilon \coloneqq f_K(\epsilon_0)/2\). Then \(\epsilon \in (0, \epsilon_0)\) as \(f_K(\delta) \leq \delta\). Let \(p \in (0, p_c)\). We abbreviate \(L \coloneqq L_{\epsilon_0}(p)\). We have to show that \(L_{\epsilon_0}(p) \geq KL\). By the definition of \(L_{\epsilon_0}(p)\), it suffices to show that \(P_p[\exists (n \times n)] \geq \epsilon\) for all \(n < KL\). If \(n \leq L\), then \(P_p[\exists (n \times n)] \geq \epsilon_0 > \epsilon\) by the definition of \(L \coloneqq L_{\epsilon_0}(p)\). Now let \(n \in (L, KL)\). Since every crossing of a \(KL \times L\) rectangle induces a crossing of an \(n \times n\) rectangle (if the rectangles are matched on the upper left corner), it follows that \(P_p[\exists (n \times n)] \geq P_p[\exists (KL \times L)] \geq f_K(\epsilon_0) > \epsilon\), which completes the proof.

References

[A-12] Simon Aumann: Singularity of Nearcritical Percolation Exploration Paths, [arXiv:1110.4203v2], 2012

[CN-06] Federico Camia, Charles Newman: Two-Dimensional Critical Percolation: The Full Scaling Limit, Comm. Math. Phys. 268, p. 1-38, 2006

[GPS-10] Christophe Garban, Gabor Pete, Oded Schramm: Pivotal, cluster and interface measures for critical planar percolation, arXiv:1008.1378 v2, 2010

[K-87] Harry Kesten: Scaling relations for 2D-Percolation, Comm. Math. Phys. 109, p. 109-156, 1987

[N-08] Pierre Nolin: Near-critical percolation in two dimensions, Electron. J. Probab. 13 No. 55, 1562-1623, 2008 [arXiv:0711.4948v1]

[NW-09] Pierre Nolin, Wendelin Werner: Asymmetry of near-critical percolation interfaces, J. Amer. Math. Soc. 22 No. 3, 797-819, 2009 [arXiv:0710.1470v1]

[SSt-10] Oded Schramm, Jeff Steif: Quantitative noise sensitivity and exceptional times for percolation, Ann. of Math. 171 No. 2, 619-672 [arXiv:math/0504586]

[SS-11] Oded Schramm, Stanislav Smirnov: On the scaling limits of planar percolation, Ann. Prob. 39 No. 5, 1768-1814, 2011 [arXiv:1101.5829v2]

[SW-01] Stanislav Smirnov, Wendelin Werner: Critical Exponents for two-dimensional percolation, Math. Res. Lett. 8 No. 5-6, 729-744, 2001 [arXiv:math/0109120v2]