New Interaction between Dark Energy and Dark Matter Changes Sign during Cosmological Evolution

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Abstract

It is found by Cai and Su that the interaction between dark energy and cold dark matter is likely to change the sign during the cosmological evolution. Motivated by this, we suggest a new form of interaction between dark energy and dark matter, which changes from negative to positive as the expansion of our universe changes from decelerated to accelerated. We find that the interacting model is consistent with the second law of thermodynamics and the observational constraints. And, we also discuss the unified adiabatic-squared sound speed of the model.

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1 Introduction

Increasing astronomical observations [1, 2, 3] tell us that the present universe is dominated by the so-called dark energy (DE), which accounts for \(\simeq 70\%\) of the critical mass density and has been pushing the universe into accelerated expansion [4, 5]. The other main component in the universe is cold dark matter (CDM), which accounts for \(\simeq 30\%\) of the critical mass density and behaves as the pressureless dust. However, we have known little about the nature of dark energy and dark matter so far. The simplest candidate for dark energy is Einstein’s cosmological constant, which can fit the observations well so far. But, the cosmological constant is plagued with well-known fine-tuning and cosmic coincidence problems.

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To alleviate these problems, dynamical DE models have been considered in the literature. The simplest one is scalar-field dark energy models. So far, a wide variety of scalar-field dark energy models have been proposed, such as quintessence [6], phantom [7], k-essence [8], tachyon [9], quintom [10], hessence [11], etc. Other dynamical dark energy models include Chaplygin gas models [12], braneworld models [13], holographic models [14], agegraphic models [15], etc. A lot of efforts have been made to explore the nature of dark energy. Furthermore, since no known symmetry in nature prevents or suppresses a nonminimal coupling between DE and CDM, there may exist interactions between the two components. At the same time, from the observation side, no piece of evidence has been so far presented against such interactions. Indeed, possible interactions between the two dark components have been discussed intensively in recent years. It is found that a suitable interaction can help to alleviate the coincidence problem [16]. Different interacting models of dark energy have been investigated [17, 18].

In the literature, the model with interaction between DE and CDM is usually described by the Friedmann equation

\[ H^2 = \frac{\kappa^2}{3}(\rho_m + \rho_d), \quad \kappa^2 \equiv 8\pi G. \] (1)

and the two conservation laws

\[ \dot{\rho}_m + 3H\rho_m = Q, \] (2)
\[ \dot{\rho}_d + 3H(\rho_d + p_d) = -Q, \] (3)

where \( Q \) denotes the phenomenological interaction term; \( \rho_m \) and \( \rho_d \) are the energy densities of CDM and DE respectively; \( p_d \) is the pressure density of DE; \( H \equiv \dot{a}/a \) is the Hubble parameter; \( a \) is the scale factor in the Friedmann-Robertson-Walker (FRW) metric; a dot denotes the derivative with respect to the cosmic time \( t \). Usually, three forms of \( Q \) are used

\[ Q_1 = 3bH\rho_d, \] (4)
\[ Q_2 = 3bH(\rho_d + \rho_m), \] (5)
\[ Q_3 = 3bH\rho_m, \] (6)

where \( b \) is the coupling constant. Then positive \( b \) means that DE decays into CDM, while negative \( b \) means CDM decays into DE. In the cases of \( Q = Q_1 \) and \( Q = Q_2 \), negative \( b \) would lead \( \rho_m \) to be negative in the far future. For negative \( b \) in the case of \( Q = Q_3 \), no such difficulty exists. But in Ref.[19], from the thermodynamical view, it is argued that the second law of thermodynamics strongly favors that DE decays into CDM, i.e. \( b \) is positive (see Ref.[20] for a different view). So generally \( b \) is taken to be positive.

However, recently it was found that the observations may favor the decaying of CDM into DE [20, 21, 22]. Particularly, in Ref.[24], in a way independent of specific interacting...
forms the authors fitted the interaction term $Q$ with observations. They found that $Q$ was likely to cross the noninteracting line ($Q = 0$), namely the sign of interaction $Q$ changed, around $z = 0.5$. This raises a remarkable challenge to the interacting models, since the usual phenomenological forms of interaction, as shown in the last paragraph, do not change their signs during the cosmological evolution. As noted in [24], more general forms of interaction should be considered.

In our paper, we are interested in proposing such a new form of interaction. It is known that our universe changes from deceleration to acceleration around $z = 0.5$ [25]. Thus the interacting term proportional to $(\rho_d - \rho_m)$ would change its sign naturally around $z = 0.5$. So we may assume the new form of interaction to be

$$Q_4 = 3\sigma H(\rho_d - \alpha \rho_m),$$

where $\sigma$ is the coupling constant and $\alpha$ is a positive constant of order unity. For simplicity, we take $\alpha = 1$. So the new interaction term is assumed to be

$$Q_4 = 3\sigma H(\rho_d - \rho_m).$$

Henceforth, we will denote the new interacting model as $\sigma$. The parameter $\sigma$ is assumed to be positive, since negative $\sigma$ would lead to negative $\rho_m$ in the far future. Obviously, in the early stage, $Q_4$ is negative, since $\rho_m > \rho_d$. As the expansion of our universe changes from decelerated to accelerated, $Q_4$ changes from negative to positive.

Below, we first show that the interacting model with $Q = Q_4$ is consistent with the second law of thermodynamics by following the argument in [19]. Second, we calculate the unified adiabatic sound speeds of the interacting model. Then we compare the model with observations. Finally, conclusions are given.

### 2 New Interaction and The Second Law of Thermodynamics

We recall the thermodynamical description of DE and CDM in [19, 20, 23]. A perfect fluid is characterized by $(n, \rho, p, s, u^a)$, where $n$ is the particle number density, $\rho$ is the energy density, $p$ is the pressure density, $s$ is the entropy per particle and $u^a$ is the 4-velocity. In the FRW universe, we take $u^a = (\partial/\partial t)^a$ and $u^a_{;a} = 3H$. In this paper, a semicolon denotes the covariant derivative compatible with the FRW metric. The energy-momentum tensor may be assumed to be [23]

$$T^{ab} = \rho u^a u^b + (p + \Pi)(g^{ab} + u^a u^b).$$

Then the conservation law of the fluid $u_a T^{ab}_{;b} = 0$ gives us [20]

$$\rho + 3H(p + p) = -3H\Pi,$$
where $-3H\Pi$ represents the phenomenological interaction between the fluid and others. The balance equation for the particle number is assumed to be

$$\dot{n} + 3Hn = n\Gamma,$$

(11)

where $\Gamma$ is the rate of the change of the particle number of the fluid. The temperature $T$ of the fluid is defined via the Gibbs equation

$$Tds = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right),$$

(12)

so that the variation rate of the entropy per particle is

$$\dot{s} = -\frac{3H\Pi}{T} - \rho + p \frac{T}{n}\Gamma.$$

(13)

By defining the entropy flow vector as

$$S^a = snu^a,$$

(14)

we have

$$S^a ; a = (s - \rho + p \frac{nT}{n^2}) n\Gamma - \frac{3H\Pi}{T}.$$  

(15)

The entropy per particle is

$$s = \rho + p \frac{nT}{nT} - \frac{\mu}{T},$$

(16)

where $\mu$ is the chemical potential. Then, we have

$$S^a ; a = -\frac{\mu}{T} n\Gamma - \frac{3H\Pi}{T}.$$  

(17)

In the general case, the Gibbs equation can be rewritten as

$$ds = -\frac{\rho + p}{Tn^2} dn + \frac{1}{Tn} d\rho,$$

(18)

and the integrability condition

$$\frac{\partial^2 s}{\partial \rho \partial n} = \frac{\partial^2 s}{\partial n \partial \rho}$$

tells us

$$n \frac{\partial T}{\partial n} + (\rho + p) \frac{\partial T}{\partial \rho} = T \frac{\partial p}{\partial \rho}.$$  

(19)

Using Eqs. (10), (11), (13) and (19), we have

$$\frac{\dot{T}}{T} = -3H \frac{\partial p}{\partial \rho} + n\dot{s} \frac{\partial T}{\partial \rho} + \Gamma \frac{\partial p}{\partial \rho}.$$  

(20)
Generally, we should assume that at any event in spacetime, the thermodynamics-state of the fluid is close to the fictitious equilibrium-state at that event \([23]\). This implies that the right-hand side of Eq.\((20)\) is dominated by the first term, and we have approximately

\[
\frac{\dot{T}}{T} \simeq -3H \frac{\partial p}{\partial \rho}
\]  

(21)

Now we apply the results above to the model with interaction between DE and CDM. Usually the energy-momentum tensor (EMT) of CDM is taken as

\[
T_{m}^{ab} = \rho_{m}u_{1}^{a}u_{1}^{b} + p_{m}(g_{ab} + u_{1}^{a}u_{1}^{b}).
\]

(22)

Then the conservation law should be

\[
T_{m ; b}^{ab} = \left[\Pi(g_{ab} + u_{1}^{a}u_{1}^{b})\right]_{,b}.
\]

(23)

Here \(u_{1}^{a}\) is the four-velocity of CDM. In this paper, we use the subscripts \(m\) and \(d\) to denote the corresponding parameters of CDM and DE, respectively. With \(p_{m} = 0\) and choosing

\[
\Pi = \sigma(\rho_{d} - \rho_{m}),
\]

(24)

we can deduce Eq.\((2)\) with \(Q = Q_{4}\) by contracting Eq.\((23)\) with \(u_{1a}\). Then equivalently we can define the effective EMT of CDM as

\[
T_{me}^{ab} = \rho_{m}u_{1}^{a}u_{1}^{b} + (p_{m} - \Pi)(g_{ab} + u_{1}^{a}u_{1}^{b}).
\]

(25)

Obviously, the effective EMT of CDM is conserved

\[
T_{me ; b}^{ab} = 0.
\]

Similarly, although the EMT of DE

\[
T_{d}^{ab} = \rho_{d}u_{2}^{a}u_{2}^{b} + p_{d}(g_{ab} + u_{2}^{a}u_{2}^{b}),
\]

is not conserved

\[
T_{d ; b}^{ab} = -\left[\Pi(g_{ab} + u_{2}^{a}u_{2}^{b})\right]_{,b},
\]

we can define the effective EMT of DE as

\[
T_{de}^{ab} = \rho_{d}u_{2}^{a}u_{2}^{b} + (p_{d} + \Pi)(g_{ab} + u_{2}^{a}u_{2}^{b}),
\]

(26)

which is also conserved

\[
T_{de ; b}^{ab} = 0.
\]

Here \(u_{2}^{a}\) is the four-velocity of DE. We can recover Eq.\((3)\) with \(Q = Q_{4}\) from \(u_{2a}T_{de ; b}^{ab} = 0\) if \(\Pi\) is chosen as given in Eq.\((24)\).
The equation of state of DE is
\[ p_d = w \rho_d. \]  
(27)

In this paper, we only consider the model with constant \( w \). For CDM, approximately we have \[ \rho_m = n_m M + \frac{3}{2} n_m T_m, \quad p_m = n_m T_m \quad (k_B = 1), \]  
(28)

so long as \( T_m \ll M \). From Eq. (21), approximately we have
\[ T_m \propto a^{-2}, \quad T_d \propto a^{-3w}. \]  
(29)

The results tell us that as the universe expands, the temperature of CDM, \( T_m \), decreases and the temperature of DE, \( T_d \), increases. Thus, one may expect that at the present and in the future \( T_m < T_d \), while in the past \( T_m > T_d \). Following Ref. [19], we assume that both DE and CDM have null-chemical potentials. Thus from Eq. (17), we have
\[ S_{m,a}^a + S_{d,a}^a = \left( \frac{1}{T_m} - \frac{1}{T_d} \right) Q, \]  
(30)

The existence of interaction means that there is a transfer of energy between DE and CDM. It is a natural conclusion that nowadays the energy is transferred from DE to CDM, i.e. \( Q > 0 \), since currently \( T_d > T_m \). In addition, from Eq. (30), the second law of thermodynamics \( S_{m,a}^a + S_{d,a}^a \geq 0 \) and because \( T_m < T_d \) indicate that currently \( Q > 0 \).

The interaction term \( Q = Q_1 \) is used in the analysis in [19]; \( Q_1 \) cannot change its sign, and is still positive even when \( T_m > T_d \). It seems that at earlier, the second law of thermodynamics was violated, since \( T_m > T_d \), \( Q_1 > 0 \) and Eq. (30) indicate that \( S_{m,a}^a + S_{d,a}^a < 0 \) and the energy is being transferred from the lower temperature DE to the higher temperature CDM. To overcome the difficulty, the author in [19] argued that the thermodynamical description breaks-down at some point, both when \( a \ll 1 \) and when \( a \gg 1 \).

Now, we apply the analysis in [19] to the case of \( Q = Q_4 \) in Eq. (8). Obviously, nowadays the second law of thermodynamics is satisfied since \( Q_4 \) is positive. Earlier, since \( \rho_m > \rho_d \), \( Q_4 \) was negative, which indicates the energy is transferred from CDM to DE. In fact, this is just what is expected from the second law of thermodynamics when \( T_m > T_d \). So, the interacting model \( \sigma \) with \( Q = Q_4 \) is always consistent with the second law of thermodynamics even at early times.

3 Adiabatic Sound Speed

The squared sound speed \( c_s^2 \), defined as
\[ c_s^2 = \frac{\delta p}{\delta \rho}, \]
is an important quantity for the cosmological evolution, which determines the stability of the cosmological evolution \[26\]. The adiabatic-squared sound speed \(c_a^2\) is defined as

\[
c_a^2 = \frac{\dot{p}}{\dot{\rho}}.
\]  

(31)

In the interacting model \(\sigma\), we rewrite Eqs. (2) and (3), respectively as

\[
\dot{\rho}_m + 3H(\rho_m + p^{\text{eff}}_m) = 0, \\
\dot{\rho}_d + 3H(\rho_d + p^{\text{eff}}_d) = 0,
\]  

(32)

(33)

where

\[
p^{\text{eff}}_m = -\sigma(\rho_d - \rho_m), \\
p^{\text{eff}}_d = p_d + \sigma(\rho_d - \rho_m).
\]  

(34)

(35)

Naively, we may still define the squared sound speed of DE \(c_{sd}^2\) as

\[
c_{sd}^2 = \frac{\delta p}{\delta \rho}.
\]  

(36)

Actually, this is not the physical sound speed of DE. (The two physical sound speeds \(\lambda_{\pm}\) are shown in the Appendix.) However, we find that in the model \(\sigma\) the stabilities of CDM and DE under perturbations are still determined by the squared sound speed of DE \(c_{sd}^2\) (see the Appendix for details). The corresponding adiabatic sound speed is

\[
c_{ad}^2 = \frac{\dot{p}_d}{\dot{\rho}_d}.
\]  

(37)

Then, using Eqs. (27), we have

\[
c_{ad}^2 = w.
\]  

(38)

Negative \(w\) indicates that adiabatic instabilities exist. Yet, this is not very astonishing since it is well-known that even in the noninteracting model with constant \(w\), the adiabatic squared sound speed of DE \(c_{ad}^2\) is negative \((c_{ad}^2 = w < 0)\), leading to the adiabatic instabilities.

4 Comparison with Observational Data

In this section, first we will explore whether the interacting model \(\sigma\) is consistent with the results in Ref. \[27\]. Second, we will explore whether the model \(\sigma\) is consistent with the observational constraint on the position of the first peak of the cosmic microwave background power spectrum.
In order to explore whether the model $\sigma$ is consistent with the results in [27], we should calculate the dimensionless coordinate distance, $y(z) = H_0 a_0 \tilde{r}$, and the two first-derivatives with respect to redshift, and then compare the results with the observational data of supernovae-type Ia (SN Ia) and radio-galaxies between the redshift $z = 0$ and $z = 1.8$. Here $z \equiv \frac{a}{a_0} - 1$ is the cosmological redshift and $\tilde{r}$ is the radial coordinate in the FRW metric.

Since $dt = -a(t)d\tilde{r}$ for photons flying from their sources to observer in the flat-FRW universe, we have
\[
E(z) \equiv \frac{H(z)}{H_0} = \frac{1}{y'(z)},
\]
where $y' \equiv dy/dz$. We define
\[
\rho_m = \rho_{m0} \mathcal{S}_1(z), \quad \rho_d = \rho_{d0} \mathcal{S}_2(z).
\]

Then using Eqs. (1), (2), (3), (39) and (40), we have
\[
y'(z) = \frac{1}{[\Omega_{m0} \mathcal{S}_1 + \Omega_{d0} \mathcal{S}_2]^{1/2}},
\]
\[
y''(z) = -\frac{3}{2(1+z)} \left[ \frac{\Omega_{m0} \mathcal{S}_1 + (1+w)\Omega_{d0} \mathcal{S}_2}{\Omega_{m0} \mathcal{S}_1 + \Omega_{d0} \mathcal{S}_2} \right],
\]
where $y'' \equiv d^2y/dz^2$.

Now let us calculate $\mathcal{S}_1$ and $\mathcal{S}_2$. By using Eqs. (8), (40) and (27), the conservation laws (2) and (3) read respectively
\[
\frac{d\mathcal{S}_1}{da} + \frac{3(1+\sigma)}{a} \mathcal{S}_1 = \frac{3\sigma}{\alpha r_0} \mathcal{S}_2,
\]
\[
\frac{d\mathcal{S}_2}{da} + \frac{3(1+w+\sigma)}{a} \mathcal{S}_2 = \frac{3\rho_{d0}\sigma}{a} \mathcal{S}_1,
\]
where $r_0 \equiv \frac{\rho_{m0}}{\rho_{d0}}$. By solving the two equations, we have
\[
\mathcal{S}_1(z) = c(1+z)^s+ + (1-c)(1+z)^s-,
\]
\[
\mathcal{S}_2(z) = -\frac{r_0}{3\sigma}[c s_+(1+z)^s+ + (1-c) s_-(1+z)^s-],
\]
where
\[
s_\pm = \frac{3}{2}[2(1+\sigma) + w \pm \sqrt{4\sigma^2 + w^2}],
\]
\[
\tilde{s}_\pm = \frac{3}{2}(w \pm \sqrt{4\sigma^2 + w^2}),
\]
\[
c = \frac{1}{2} - \frac{w + 2\sigma r_0}{2\sqrt{w^2 + 4\sigma^2}}.
\]
Fig. 1: $y(z)$ versus $z$ in the interacting model $\sigma$ for fixed $\Omega_{m0} = 0.272$, $\sigma = 10^{-3}$ and different $w$. For comparison, the prediction of the $\Lambda$CDM model is also shown.

Fig. 2: $y'(z)$ versus $z$ in the interacting model $\sigma$ for fixed $\Omega_{m0} = 0.272$, $\sigma = 10^{-3}$ and different $w$. For comparison, the prediction of the $\Lambda$CDM model is also shown.
Fig. 3: $y''(z)$ versus $z$ in the interacting model $\sigma$ for fixed $\Omega_{m0} = 0.272$, $\sigma = 10^{-3}$ and different $w$. For comparison, the prediction of the $\Lambda$CDM Model is shown, too.

In the absence of interaction, the Eqs. (11) and (12) reduce to Eqs.(1) and (2) of Ref. [27].

The evolutions of $y$, $y'$ and $y''$ with respect to $z$ are depicted in Fig.1, Fig.2 and Fig.3. We have used the different values of the equation of state parameter $w$, and fixed the values of $\Omega_{m0} = 0.272$ [29] and $\sigma = 10^{-3}$. Comparisons of Figs.1, Fig.2 and Fig.3 with the corresponding figures in Ref. [27] reveal that the interacting model $\sigma$ is consistent with the analysis of Ref. [27].

Now, let us calculate the shift parameter $R$, which characterizes the position of the first peak of the cosmic microwave background spectrum and is defined as [28]

$$R = \sqrt{\Omega_m} \int_0^{z_\ast} \frac{dz}{E(z)}. \quad (50)$$

Here $z_\ast$ is the redshift of decoupling. The 7-year Wilkinson Microwave Anisotropy Probe (WMAP) observations tell us that $z_\ast = 1091.3 \pm 0.91$ at 1$\sigma$ confidence level [29]. In this paper, we fix $z_\ast = 1091$. The values of the shift parameter $R$ for different sets of parameters in the model $\sigma$ and the $\Lambda$CDM model are displayed in Table 1. We have fixed $\Omega_{m0} = 0.272$ and used the flat FRW metric. We first consider that the universe is filled with DE ($\Omega_{d0} = 0.728$) and CDM ($\Omega_{m0} = 0.272$). Second, we consider the case that the universe is filled with DE ($\Omega_{d0} = 0.728$), CDM ($\Omega_{dm0} = 0.2264$), and baryon matter ($\Omega_{b0} = 0.0456$) [29]. The total fractional-energy density of CDM and baryon matter is still fixed to be $\Omega_{m0} = 0.272$. The component of baryon matter is assumed to be evolving separately. In the $\Lambda$CDM model, the shift parameter $R$ is determined by the total fractional energy density of CDM and baryons. The 7-year WMAP observations tell us $R = 1.725 \pm 0.018$ at 1$\sigma$ confidence level. Then from the results displayed in Table 1 we know that the interacting model $\sigma$ is consistent with the 7-year WMAP observations.
TABLE 1: The values of the shift parameter for different sets of parameters in the model $\sigma$ and the $\Lambda$CDM model.

| Model | $\Omega_{m0}$ | $\Omega_{k0}$ | $w$ | $\sigma$ | R   |
|-------|---------------|---------------|-----|---------|-----|
| $\sigma$ | 0.272 | 0 | -0.9 | $10^{-3}$ | 1.712 |
| $\sigma$ | 0.272 | 0 | -1.05 | $10^{-3}$ | 1.738 |
| $\sigma$ | 0.272 | 0.0456 | -0.9 | $10^{-3}$ | 1.713 |
| $\sigma$ | 0.272 | 0.0456 | -1.05 | $10^{-3}$ | 1.739 |
| $\Lambda$CDM | 0.272 | -1 | $10^{-3}$ | 1.733 |

5 Conclusions

The dark energy models with interaction between DE and CDM have been investigated intensively. It is argued in [30] that the interacting models of dark energy may be key to solving the cosmic coincidence problem. Recently, it was found that the interaction is likely to cross the noninteracting line [24]. However, the usual forms of interaction used in the literature can not change their sign. The result in [24] raises a challenge to the interacting models of dark energy; more general interaction is needed to be considered. In [31], the interacting models with interaction proportional to the deceleration parameter are suggested.

In this paper, we suggest the new interacting model $\sigma$ with interaction $Q = Q_4$. Obviously the sign of $Q_4$ changes from negative to positive as the expansion of our universe changes from decelerated to accelerated. We found that the interacting model is consistent with the second law of thermodynamics. Then we also found the squared sound speed of DE, $c_{sd}^2$, to be crucial in determining the stabilities of DE and CDM in the model. As in the noninteracting model with constant $w$, there also exist adiabatic instabilities in the model $\sigma$ with constant $w$ due to $c_{sd}^2 = w < 0$. Furthermore, we compared the interacting model with observational data. And we found the interaction model to be consistent with the result in [27] and the 7-year WMAP observations [29]. Thus, we believe that the interacting model with $Q = Q_4$ is consistent with observational constraints. In the future, we plan to explore how to obtain the phenomenological interaction term $Q_4$ from an action principle.

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Appendix

Here, we try to show that the squared sound speed of DE determines the stabilities of the cosmological evolution in the interacting model. We adopt the conformal Newtonian gauge. Then the perturbed metric about a spatially flat, homogeneous and isotropic FRW background is given to be \[32\]

\[
ds^2 = a^2[-(1 + 2\phi)d\tau^2 + (1 - 2\psi)d\mathbf{x}^2].
\]

where \(a\) is the scale factor and the perturbations of the metric are characterized by two potentials, \(\phi\) and \(\psi\). The total EMT is

\[
T_{ab}^\text{me} = T_{ab}^\text{eff} + T_{ab}^\text{de}.
\]

The \(T_{ab}^\text{me}\) and \(T_{ab}^\text{de}\) are the effective EMT of CDM and DE, respectively,

\[
T_{ab}^\text{me} = \rho_m u_1^a u_1^b + p_m^\text{eff}(u_1^a u_1^b + g^{ab}),
\]

\[
T_{ab}^\text{de} = \rho_d u_2^a u_2^b + p_d^\text{eff}(u_2^a u_2^b + g^{ab}).
\]

Here \(p_m^\text{eff}\) and \(p_d^\text{eff}\) are defined in Eqs.(34) and (35) respectively, and \(T_{me:b}^\text{ab} = T_{me:b}^\text{ab} = 0\). To the first-order of perturbation, we have

\[
\rho_m(\tau, \mathbf{x}) = \rho_m(\tau)[1 + \delta_1(\tau, \mathbf{x})],
\]

\[
p_m^\text{eff}(\tau, \mathbf{x}) = p_m^\text{eff}(\tau)[1 + \delta_1(\tau, \mathbf{x})],
\]

\[
u_1^a(\tau, \mathbf{x}) = a^{-1}[(1 - \phi)(\partial/\partial \tau)^a + \partial_i v_1(\tau, \mathbf{x})(\partial/\partial x^i)^a],
\]

and

\[
\rho_d(\tau, \mathbf{x}) = \rho_d(\tau)[1 + \delta_2(\tau, \mathbf{x})],
\]

\[
p_d^\text{eff}(\tau, \mathbf{x}) = p_d^\text{eff}(\tau)[1 + \delta_2(\tau, \mathbf{x})],
\]

\[
u_2^a(\tau, \mathbf{x}) = a^{-1}[(1 - \phi)(\partial/\partial \tau)^a + \partial_i v_2(\tau, \mathbf{x})(\partial/\partial x^i)^a].
\]

Here we use the subscript \(b\) to denote the spatially homogeneous-background value of the corresponding quantity, and \(\delta_1 \equiv \delta_\rho_m\) and \(\delta_2 \equiv \delta_\rho_d\) are the fractional perturbations in the energy densities of CDM and DE, respectively; \(v_1\) and \(v_2\) are the peculiar velocity potentials of CDM and DE, respectively, with the same order of \(\delta_1\) and \(\delta_2\); \(\delta p_m^\text{eff}(t, \mathbf{x})\) and \(\delta p_d^\text{eff}(t, \mathbf{x})\) are the perturbations of \(p_m^\text{eff}(t, \mathbf{x})\) and \(p_d^\text{eff}(t, \mathbf{x})\), respectively,

\[
\delta p_m^\text{eff}(\tau, \mathbf{x}) = -\sigma[\rho_d(\tau)\delta_2(\tau, \mathbf{x}) - \rho_m(\tau)\delta_1(\tau, \mathbf{x})],
\]

\[
\delta p_d^\text{eff}(\tau, \mathbf{x}) = \delta p_d(\tau, \mathbf{x}) + \sigma[\rho_d(\tau)\delta_2(\tau, \mathbf{x}) - \rho_m(\tau)\delta_1(\tau, \mathbf{x})],
\]

where \(\delta p_d(\tau, \mathbf{x}) = p_d(\tau, \mathbf{x}) - p_d(\tau)\).
In the conformal Newtonian gauge, the first-order perturbed Einstein equations give us 

\[ 3 \mathcal{H} \psi' + k^2 \psi + 3 \mathcal{H}^2 \phi = -4\pi G a^2 (\delta_1 \rho_{\text{mb}} + \delta_2 \rho_{\text{db}}), \tag{63} \]

\[ k^2 \psi' + k^2 \mathcal{H} \phi = 4\pi G a^2 [ (\rho_{\text{mb}} + p_{\text{mb}}^\text{eff}) \theta_1 + (\rho_{\text{db}} + p_{\text{db}}^\text{eff}) \theta_2 ] \tag{64} \]

\[ \psi'' + \mathcal{H} (2 \psi' + \phi') + \left( \frac{2a''}{a} - \mathcal{H}^2 \right) \phi + \frac{k^2}{3} (\psi - \phi) = 4\pi G a^2 \delta \rho \tag{65} \]

\[ \psi - \phi = 0. \tag{66} \]

Hereafter, primes denote the derivatives with respect to the conformal time \( \tau \), \( \mathcal{H}' \equiv a'/a \), \( \theta_1 \equiv -k^2 v_1 \) and \( \theta_2 \equiv -k^2 v_2 \). The first-order equation of the conservation law of CDM \( T_{\text{mc,}b}^{ab} = 0 \) (in Fourier space) tells us 

\[ \delta_1' - 3 \mathcal{H} \frac{p_{\text{mb}}^\text{eff}}{\rho_{\text{mb}}} \delta_1 + 3 \mathcal{H} \frac{p_{\text{mb}}^\text{eff}}{\rho_{\text{mb}}} - 3(1 + \frac{p_{\text{mb}}^\text{eff}}{\rho_{\text{mb}}}) \psi' + (1 + \frac{p_{\text{mb}}^\text{eff}}{\rho_{\text{mb}}}) \theta_1 = 0, \tag{67} \]

\[ \theta_1' + \frac{\rho_{\text{mb}}'}{\rho_{\text{mb}} + p_{\text{mb}}^\text{eff}} \theta_1 + 4 \mathcal{H} \theta_1 - k^2 \phi - k^2 \frac{\delta p_{\text{mb}}^\text{eff}}{\rho_{\text{mb}} + p_{\text{mb}}^\text{eff}} = 0. \tag{68} \]

The first-order equation of the conservation law of DE \( T_{\text{de,}b}^{ab} = 0 \) tells us that 

\[ \delta_2' - 3 \mathcal{H} \frac{p_{\text{db}}^\text{eff}}{\rho_{\text{db}}} \delta_2 + 3 \mathcal{H} \frac{p_{\text{db}}^\text{eff}}{\rho_{\text{db}}} - 3(1 + \frac{p_{\text{db}}^\text{eff}}{\rho_{\text{db}}}) \psi' + (1 + \frac{p_{\text{db}}^\text{eff}}{\rho_{\text{db}}}) \theta_2 = 0, \tag{69} \]

\[ \theta_2' + \frac{\rho_{\text{db}}'}{\rho_{\text{db}} + p_{\text{db}}^\text{eff}} \theta_2 + 4 \mathcal{H} \theta_2 - k^2 \phi - k^2 \frac{\delta p_{\text{db}}^\text{eff}}{\rho_{\text{db}} + p_{\text{db}}^\text{eff}} = 0. \tag{70} \]

By differentiating Eq. (67) with respect to \( \tau \) and using Eqs (65), (66), (68) and (63), taking the geometric optic limit, finally we can get 

\[ \delta_1'' = -k^2 \frac{\delta p_{\text{mb}}^\text{eff}}{\rho_{\text{mb}}}, \tag{71} \]

Similarly, from Eq. (69), and using Eqs (65), (66), (70) and (63), taking the geometric optic limit, we can also get 

\[ \delta_2'' = -k^2 \frac{\delta p_{\text{db}}^\text{eff}}{\rho_{\text{db}}}, \tag{72} \]

By using Eqs (61) and (62), we can rewrite the above two equations as 

\[ \frac{d^2}{d\tau^2} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = -k^2 \mathcal{M} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}. \tag{73} \]

Here the matrix \( \mathcal{M} \) is 

\[ \mathcal{M} = \begin{pmatrix} \sigma & -\sigma r \\ -\sigma r^{-1} & c_{sd}^2 + \sigma \end{pmatrix}, \tag{74} \]
where
\[ r \equiv \frac{\rho_{db}}{\rho_{mb}}, \quad c_{sd}^2 \equiv \frac{\delta p_d}{\delta \rho_d}. \]

This matrix possesses two eigenvalues \( \lambda_+ \) and \( \lambda_- \),
\[
\lambda_{\pm} = \frac{c_{sd}^2 + 2\sigma \pm \sqrt{c_{sd}^4 + 4\sigma^2}}{2}.
\] (75)

Clearly, \( \lambda_{\pm} \) are crucial for determining the stability of the interacting model. In fact, \( \lambda_{\pm} \) are the physical sound speeds in the interacting model. If \( \lambda_+ \geq 0 \) and \( \lambda_- \geq 0 \), the model is stable. Otherwise, the model is unstable. Here we assume \( \sigma \geq 0 \) and it can be easily checked that \( c_{sd}^2 > 0 \) indicates \( \lambda_+ > 0 \) and \( c_{sd}^2 < 0 \) indicates \( \lambda_- < 0 \). So we know \( c_{sd}^2 \) is crucial for determining the stability of the interacting model, and negative \( c_{sd}^2 \) indicates the existence of instabilities.

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