A general condition for Monge solutions in the multi-marginal optimal transport problem

Young-Heon Kim† and Brendan Pass‡

May 11, 2014

Abstract

We develop a general condition on the cost function which is sufficient to imply Monge solution and uniqueness results in the multi-marginal optimal transport problem. This result unifies and generalizes several results in the rather fragmented literature on multi-marginal problems. We also provide a systematic way to generate new examples from old ones.

1 Introduction

In this paper, we establish a general Monge solution and uniqueness result for the multi-marginal Monge-Kantorovich problem, under a natural analogue of the twist condition. We call this condition twist on splitting sets.

Given compactly supported Borel probability measures \( \mu_1, \ldots, \mu_m \) on smooth manifolds \( M_1, M_2, \ldots, M_m \), respectively, and a continuous cost function \( c : M_1 \times M_2 \times \cdots \times M_m \to \mathbb{R} \), the multi-marginal optimal transport problem is to minimize

\[
\int_{M_1 \times \cdots \times M_m} c(x_1, x_2, \ldots, x_m) d\gamma,
\]

among probability measures \( \gamma \) on \( M_1 \times \cdots \times M_m \) which project to the \( \mu_i \). When an optimal measure \( \gamma \) is concentrated on the graph \( \{ (x, T(x)) \} \) of a function \( T : M_1 \to M_2 \times \cdots \times M_m \), it is said to induce a Monge solution. When \( m = 2 \),

---

\*Y.-H.K. is supported in part by Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grants 371642-09 as well as Alfred P. Sloan research fellowship. B.P. is pleased to acknowledge the support of a University of Alberta start-up grant and National Sciences and Engineering Research Council of Canada Discovery Grant number 412779-2012. Part of this work was done while Y.-H.K was visiting University Paris-Est Créteil (UPECE) and Korea Advanced Institute of Science and Technology (KAIST) and he thanks for their hospitality.

†Department of Mathematics, University of British Columbia, Vancouver BC Canada V6T 1Z2 yhkim@math.ubc.ca

‡Department of Mathematical and Statistical Sciences, 632 CAB, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1 pass@ualberta.ca
reduces to the classical Monge-Kantorovich problem, which remains a very active area with a wide variety of applications (see [25] for a comprehensive review). Recently, applications for the $m \geq 3$ case have arisen in such diverse areas as matching in economics [5] [7], electronic correlations in physics [8] [2], monotonicity relationships among vector fields [9] [15] [14] and model free pricing of derivatives in finance [10] [1] [17].

Under reasonable conditions on the cost and marginals, existence of an optimal measure $\gamma$ is not hard to show. Two natural open questions are: “when is the optimal measure $\gamma$ unique?” and “when does the optimal measure induce a Monge solution?”

In the $m = 2$ case, the well known twist condition, dictating that the mapping $x_2 \mapsto D_{x_1} c(x_1, x_2)$ is injective for fixed $x_1$, ensures the uniqueness and Monge structure of the optimal $\gamma$ [11] [12] [20] [2]. For larger $m$, these questions are still largely open. Examples of special cost functions for which the optimal measure has this structure are known [13] [16] [4] [21] [19], as well as several examples for which uniqueness and Monge solutions fail [23] [6]. There are also strong differential conditions on the cost which are known to imply Monge solutions and uniqueness [22]; however, these conditions are not sharp, as some of the positive examples do not satisfy them. What seems to be missing is an analogue of the twist condition; that is, a general condition implying Monge solution and uniqueness results, which unifies the scattered, previously established results.

In this paper we propose such a condition on the cost, which we call twist on $c$-splitting sets (or simply, twist on splitting sets), and show that it is indeed sufficient for Monge solutions and uniqueness. We require the mapping $(x_2, ..., x_m) \mapsto D_{x_1} c(x_1, x_2, ..., x_m)$ to be injective along certain subsets, which we call splitting sets (see Definition 2.1 below); splitting sets, roughly speaking, are multi-marginal analogues of $c$-super differentials of $c$-concave functions.

We also consider a natural extension of the concept of $c$-cyclical monotonicity to multi-marginal problems. As we show, any splitting set is automatically $c$-cyclically monotone. The converse, when $m = 2$, is a well known theorem of Rüschendorf [24]; whether the converse holds for $m \geq 3$ remains an interesting open question. As an immediate corollary, we obtain Monge solution and uniqueness results whenever $c$ is twisted on $c$-cyclically monotone sets, which in practice may be more direct to check for a given cost than twistedness on splitting sets.

An important conceptual contribution of this paper is that it unifies and extends known Monge solution results for multi-marginal problems. For example, Monge solution and uniqueness results for a class of costs called matching costs (due to their application in economics) was established in [21]. These costs may not satisfy the differential conditions in [22]; in turn, there are costs satisfying the differential conditions which are not of matching form. However, both the differential conditions and the matching structure imply twist on splitting sets; indeed, we show that the conditions imposed in [22] are in fact sufficient (but not necessary) differential conditions for twist on splitting sets. For cost functions of the form in [21], we show twist in $c$-monotonicity is satisfied as long
as the $c_i$ are twisted, and therefore other conditions on the derivatives of the $c_i$, required for the argument in [21], are not needed here. Indeed, we are able to extend, in a systematic way, this type of example to a more general class, namely, costs defined as infimums of functions of less variables (see Section 5).

Note that we work with semi-concave (not necessarily smooth) cost functions here. This makes some of the definitions and proofs slightly more complicated and less elegant looking; on the other hand, we require the semi-concave framework to handle natural examples where the cost is not everywhere smooth (as in Section 5).

In the next section, we introduce the key conditions we will use in this paper. In the third section we state and prove our main theorem, while the final two sections are reserved for two key types of examples. It is in these final sections that we show the results in [22] [21] [13] [16] [19] fit into our framework.

2 Preliminaries

We now formulate the main concepts used in the paper.

Definition 2.1. A set $S \subseteq M_1 \times M_2 \times \ldots \times M_m$ is a c-splitting set if there exists Borel functions $u_i : M_i \to \mathbb{R}$ such that for all $(x_1, x_2, \ldots, x_m)$

$$\sum_{i=1}^{m} u_i(x_i) \leq c(x_1, x_2, \ldots, x_m)$$

(2)

with equality whenever $(x_1, x_2, \ldots, x_m) \in S$. We will call the $u_i$ c-splitting functions for $S$.

Definition 2.2. A set $S \subseteq M_1 \times M_2 \times \ldots \times M_m$ is c-cyclically monotone if for any finite subset \{$(x_1^1, x_2^1, \ldots, x_m^1), (x_1^2, x_2^2, \ldots, x_m^2), \ldots, (x_1^N, x_2^N, \ldots, x_m^N)$\} $\subseteq S$ and any $m$ permutations $\sigma_1, \sigma_2, \ldots, \sigma_m$ on $N$ letters, we have

$$\sum_{i=1}^{N} c(x_1^1, x_2^1, \ldots, x_m^1) \leq \sum_{i=1}^{N} c(x_1^{\sigma_1(i)}, x_2^{\sigma_2(i)}, \ldots, x_m^{\sigma_m(i)}).$$

Note that, by considering the permutations $\sigma_i \circ \sigma_1^{-1}$, we can always take $\sigma_1 = Id$ (or $\sigma_j = Id$, for any other fixed $j$) in the above definition. The following result relates these concepts to optimal measures $\gamma$ in (1).

Proposition 2.3. A probability measure $\gamma$ on $M_1 \times \ldots \times M_m$ is optimal in (1) for its marginals if and only if its support is a c-splitting set. Any c-splitting set is c-cyclically monotone.

Proof. The equivalence of the optimality of $\gamma$ and the splitting set property of its support follows easily from a classical duality theorem of Kellerer [18]. We now prove that any c-splitting set $S$ is c-cyclically monotone. Let $(u_1, u_2, \ldots, u_m)$ be c-splitting functions for $S$. Then, for any \{$(x_1^1, x_2^1, \ldots, x_m^1), (x_1^2, x_2^2, \ldots, x_m^2), \ldots, (x_1^N, x_2^N, \ldots, x_m^N)$\} $\subseteq S$, all the conditions of Definition 2.2 hold.
it is clear that we have
\[ \sum_{i=1}^{N} c(x^i_1, x^i_2, \ldots x^i_m) = \sum_{i=1}^{N} \sum_{j=1}^{m} u_j(x^i_j). \] (3)

On the other hand, by the definition of splitting functions, we have, for any permutations \(\sigma_2, \sigma_3, \ldots, \sigma_m\) (and setting \(\sigma_1 = id\))
\[ \sum_{i=1}^{N} \sum_{j=1}^{m} u_j(x^i_j) = \sum_{i=1}^{N} \sum_{j=1}^{m} u_j(\sigma_j^{\sigma(i)}) \leq \sum_{i=1}^{N} c(x^i_1, x^i_2(\sigma(i)), \ldots x^i_m(\sigma(i))). \] (4)

**Definition 2.4.** Let \(c\) be a continuous, semi-concave cost function. We say \(c\) is twisted on \(c\)-splitting sets (respectively, twisted in \(c\)-cyclical monotonicity) whenever for each fixed \(x^i_1 \in M_1\) and \(c\)-splitting set (respectively, \(c\)-cyclically monotone set) \(S \subseteq \{x^i_1\} \times M_2 \times \cdots \times M_m\), the map
\[(x^i_2, \ldots, x^i_m) \mapsto D_{x^i_1} c(x^i_1, x^i_2, \ldots, x^i_m)\]
is injective on the subset of \(S\) where \(c\) is differentiable with respect to \(x^i_1\) (i.e, the subset where \(D_{x^i_1} c(x^i_1, x^i_2, \ldots, x^i_m)\) exists).

When \(m = 2\), any set \(S \subseteq \{x^i_1\} \times M_2\) is trivially both \(c\)-cyclically monotone and a \(c\)-splitting set, so both twist on splitting sets and twist in \(c\)-cyclical monotonicity reduce to the standard twist condition. For higher \(m\), twist in \(c\)-cyclical monotonicity clearly implies twist on splitting sets, by Proposition 2.3. When \(m = 3\), a set \(S \subseteq \{x^i_1\} \times M_2 \times M_3\) is \(c\)-cyclical monotone if and only if it is splitting set, by Rüschendorf’s theorem, and so twist in \(c\)-cyclical monotonicity and twist on splitting sets are equivalent. We do not know whether this equivalence holds for larger \(m\).

## 3 Monge solution and uniqueness

We are now ready to state and prove the main result.

**Theorem 3.1.** Assume \(c\) is twisted on splitting sets and the measure \(\mu_1\) is absolutely continuous with respect to local coordinates. Then the solution \(\gamma\) in (1) induces a Monge solution and is unique.

**Proof.** We first prove the Monge solution assertion. The key observation is that the twist on splitting sets condition is enough to extend a standard argument from the two marginal case (found in, for example, [12]) to the multimarginal case. By Proposition 2.3 there exist splitting functions \((u_1, u_2, \ldots, u_m)\) for \(spt(\gamma)\), and it is well known that they can be taken to be \(c\)-conjugate [18] [13] [22]: that is, for each \(i\),
\[ u_i(x_i) = \inf_{x_j, j \neq i} c(x_1, x_2, \ldots, x_m) - \sum_{j \neq i} u_i(x_j). \] (5)
In particular, as an infimum of semi-concave functions, $u_1$ is itself semi-concave and therefore differentiable almost everywhere with respect to local coordinates (and hence $\mu_1$ almost everywhere by absolute continuity).

Fix $x_1 \in spt(\mu_1)$ where $u_1$ is differentiable. We must show that there exists a unique $(x_2,...,x_m)$ such that $(x_1,x_2,...,x_m) \in spt(\gamma)$: that is, the set

$$S = spt(\gamma) \cap \{x_1\} \times M_2 \times \cdots \times M_m$$

is a singleton. Non-emptiness of $S$ follows immediately, as the support of $\mu_1$ must project to the support of $\gamma$ from Proposition 2.3. By (5), then, for each $(x_1,x_2,...,x_m) \in S$, we must have

$$\partial x_1 c(x_1,...,x_m) \subseteq \partial u_1(x_1) = \{Du_1(x_1)\}$$

where $\partial x_1 c(x_1,...,x_m)$ denotes the superdifferential of $c$ with respect to $x_1$; note that the last equality follows by the differentiability of $u_1$ at $x_1$. It follows that $c$ is differentiable with respect to $x_1$ at $(x_1,...,x_m)$ and we have

$$Du_1(x_1) = D_x c(x_1,x_2,...,x_m).$$

As $(x_2,...,x_m) \mapsto D_x c(x_1,x_2,...,x_m)$ is injective on $S$ by the twist on splitting sets condition, this immediately implies that $S$ must be a singleton.

This shows that every solution to $\gamma$ is concentrated on a graph over the first variable. Uniqueness follows by a standard argument; as the functional (1) is linear, the convex interpolant $\frac{1}{2}(\gamma + \hat{\gamma})$ of any two solutions must also be a solution. However, if $\gamma$ and $\hat{\gamma}$ are concentrated on graphs $T$ and $\hat{T}$, respectively, then $\frac{1}{2}(\gamma + \hat{\gamma})$ is concentrated on the union of the graphs of $T$ and $\hat{T}$; this set itself cannot be a graph unless $T = \hat{T}$ almost everywhere. Uniqueness of the optimal measure $\gamma = (ID,T)|\#\mu_1$ follows immediately.

The following result now follows easily from Theorem 3 and Proposition 2.3.

**Corollary 3.2.** Assume $c$ satisfies the twist in $c$-monotonicity condition and $\mu_1$ is absolutely continuous with respect to local coordinates. Then the solution $\gamma$ in (1) induces a Monge solution and is unique.

### 4 Differential conditions

We now exhibit several example classes of cost functions that satisfy the twist on splitting sets condition. First, in this section, we show that the differential conditions in [22] imply twist on splitting sets. Let us recall those conditions:

Let $M_i \subseteq \mathbb{R}^n$, $i = 1,...,m$. We will assume throughout this section that $c$ is $(1,m)$-twisted; that is, the map

$$x_m \mapsto D_x c(x_1,x_2,...,x_m)$$
is injective for fixed $x_1, \ldots, x_{m-1}$. We will also assume that $c$ is $(1, m)$-non-degenerate; that is, the matrix

$$D_{x_1x_m}^2 c = \left( \frac{\partial^2 c}{\partial x_i^j \partial x_m^l} \right)_{i,j}$$

is everywhere non-degenerate.

The most restrictive condition in [22] is based on the following tensor.

**Definition 4.1.** Suppose $c$ is $(1, m)$-non-degenerate. Let $\vec{y} = (y_1, y_2, \ldots, y_m) \in M_1 \times M_2 \times \ldots \times M_m$. For each $i := 2, 3, \ldots, m - 1$ choose a point $y(i) = (y_1(i), y_2(i), \ldots, y_m(i)) \in M_1 \times M_2 \times \ldots \times M_m$ such that $y(i) \neq y_i$. Define the following bi-linear maps on $T_{y_2}M_2 \times T_{y_3}M_3 \times \ldots \times T_{y_{m-1}}M_{m-1}$:

$$S_{\vec{y}} = -\sum_{i=2}^{m-1} \sum_{j=2}^{m-1} D_{x_i x_j}^2 c(y) + \sum_{i,j=2}^{m-1} (D_{x_i, x_m}^2 c) (D_{x_1 x_m}^2 c)^{-1} D_{x_i x_j}^2 c(y)$$

$$H_{\vec{y}, y(2), y(3), \ldots, y(m-1)} = \sum_{i=2}^{m-1} (\text{Hess}_{x_i} c(y(i)) - \text{Hess}_{x_i} c(y))$$

$$T_{\vec{y}, y(2), y(3), \ldots, y(m-1)} = S_{\vec{y}} + H_{\vec{y}, y(2), y(3), \ldots, y(m-1)}$$

The main condition required for Monge solutions in [22] is negative definiteness of the tensor $T$, for all choices of the $\vec{y}$, $\vec{y}(2)$, $\vec{y}(3)$, ..., $\vec{y}(m - 1)$.

A geometric condition on the domains, defined in terms of the following set, is also required.

**Definition 4.2.** Let $x_1 \in M_1$ and $p_1 \in T_{x_1}^* M_1$. We define $Y_{x_1, p_1}^c \subseteq M_2 \times M_3 \times \ldots \times M_{m-1}$ by

$$Y_{x_1, p_1}^c = \{(x_2, x_3, \ldots, x_{m-1}) \mid \exists x_m \in M_m \text{ s.t. } D_{x_m} c(x_1, x_2, \ldots, x_m) = p_1\}$$

These conditions are discussed in more detail in [22]. Although they are fairly restrictive, several examples of cost functions satisfying these conditions are exhibited in [22], including the Gangbo-Swiech [13] cost, $\sum_{i \neq j} |x_i - x_j|^2$ on $\mathbb{R}^n$, and perturbations thereof, the cost function considered by Heimich [14], $h(\sum_{i=1}^m x_i)$ for a strictly concave $h : \mathbb{R}^n \to \mathbb{R}$, and three marginal functions of the form $x_1 \cdot x_2 + x_2 \cdot x_3 + g(x_1, x_3)$, whenever $D_{x_1 x_3}^2 g > 0$.

Below, we prove that these conditions imply the twist on splitting sets condition.

**Proposition 4.3.** *(Sufficient differential conditions)*

Suppose that:

1. $c$ is $(1, m)$-non-degenerate.
2. $c$ is $(1, m)$-twisted.

3. For all choices of $\bar{y} = (y_1, y_2, ..., y_m) \in M_1 \times M_2 \times ... \times M_m$ and of $\bar{y}(i) = (y_1(i), y_2(i), ..., y_m(i)) \in \overline{M}_1 \times \overline{M}_2 \times ... \times \overline{M}_m$ such that $y_i(i) = y_i$ for $i = 2, ..., m - 1$, we have

$$T_{\bar{y}, \bar{y}(2), \bar{y}(3), ..., \bar{y}(m-1)} < 0.$$ 

4. For all $x_1 \in M_1$ and $p_1 \in T_{x_1}^* M_1$, $Y_{x_1, p_1}^c$ is geodesically convex.

Then $c$ is twisted on splitting sets.

**Proof.** Fix $x_1 \in M_1$ and let $S \subseteq \{x_1\} \times M_2 \times ... \times M_m$ be a $c$-splitting set. Take points $(x_2, ..., x_m)$ and $(\bar{x}_2, ..., \bar{x}_m)$ such that $p_1 := D_{x_1} c(x_1, x_2, x_3) = D_{x_1}(c(x_1, \bar{x}_2, ..., \bar{x}_m))$. It suffices to show that at most one of these two points can be in $M_1$. Without loss of generality, assume $(x_2, ..., x_m) \in S$; we must show that $(\bar{x}_2, ..., \bar{x}_m) \notin S$.

For $i = 2, 3, ..., m - 1$, choose geodesics joining $\gamma_i(t) = x_i(0)$ and $\bar{x}_i = \gamma_i(1)$. The geodesic convexity of $Y_{x_1, p_1}^c$ implies that for each $t$, there exists an $x_m(t) \in M_m$ such that $p_1 = D_{x_1}(c(x_1, \gamma_2(t), ..., \gamma_m(t), x_m(t)))$. Twistedness ensures the uniqueness of $x_m(t)$. Note that $x_m(0) = x_m$, and $x_m(1) = \bar{x}_m$.

Now, as $S$ is a splitting set, we have splitting functions $u_i : M_1 \to \mathbb{R}$ such that

$$\sum_{i=2}^{m} u_i(x_i) \leq c(x_1, x_2, ..., x_m),$$

with equality on $S$. By a standard convexification trick and compactness of the $M_i$, we can assume that

$$u_i(x_i) = \min_{x_j, j \neq i} c(x_1, ..., x_m) - \sum_{j=2, j \neq i}^{m} u_j(x_j).$$

The functions $u_i$ are semi-concave and have superdifferentials everywhere. Furthermore, for any $(x_2, ..., x_m) \in S$, we have

$$D_{x_i} c(x_1, x_2, ..., x_m) \in \partial u_i(x_i)$$

for $i = 2, 3, ..., m$, where $\partial u_i(x_i)$ denotes the super-differential of $u_i$.

Take a measurable selection of covectors $V_i(t) \in \partial u_i(\gamma_i(t))$, and set

$$f(t) := \sum_{i=2}^{m-1} \langle V_i(t) - D_{x_i} c(x_1, \gamma_2(t), ..., \gamma_{m-1}(t), x_m(t)) \rangle \frac{d\gamma_i}{dt}.$$

Now, note that we can take $V_i(0) = D_{x_i} c(x_1, x_2, ..., x_m)$, as $(x_2, ..., x_m) \in S$, in which case $f(0) = 0$. Similarly, if $(\bar{x}_2, ..., \bar{x}_m) \in S$, we can choose $V_i(1) = D_{x_i} c(x_1, \bar{x}_2, ..., \bar{x}_m)$, in which case $\overline{f}(1) = 0$.

However, the calculation in [22] (in the proof of Theorem 3.1) it is shown that under the conditions 1, 2, 3 and 4, $f(1) < f(0)$, for any selection of covectors $V_i(t) \in \partial u_i(\gamma_i(t))$. This implies that we cannot have $(\bar{x}_2, ..., \bar{x}_m) \in S$, completing the proof.
5 Infimal convolution examples

In this section we consider a sort of infimal convolution of several cost functions; that is, cost functions defined by

\[ c(x_1, \ldots, x_m) = c(X_1, X_2, \ldots, X_k) = \min_{y \in Y} \sum_{j=1}^{k} c_j(X_j, y). \]  

(6)

Here, for notational convenience, we decomposed the \( m \)-tuple \((x_1, \ldots, x_m)\) into \( k \) smaller tuples \((X_1, \ldots, X_k) = (x_1, \ldots, x_m)\) with \( X_j = (x_{m_j-1+1}, \ldots, x_{m_j}) \), with \( 0 = m_0 < m_1 < m_2 < \cdots < m_k = m \); in particular, \( X_1 = (x_1, \ldots, x_{m_1}) \). We also assume that \( Y \) is a smooth manifold without boundary, and we are implicitly assuming the existence of a minimizing \( y \) for all \((x_1, x_2, \ldots, x_m)\) (which holds, for example, whenever \( Y \) is compact). We also assume that the functions \( c_j \) are semi-concave, so that \( c \) is also semi-concave.

As a special case, when each \( X_j \) is a singleton, we have

\[ c(x_1, \ldots, x_m) := \min_{y \in Y} \sum_{i=1}^{m} c_i(x_i, y). \]  

(7)

These cost functions, called matching costs, have important applications in matching problems in economics \[5\] \[7\]. In \[21\], one of the present authors proved a Monge solution and uniqueness result for costs of this form. The argument was completely different than the one here, and required additional conditions on the \( c_i \), including nondegeneracy of various matrices of mixed second-order partials and uniqueness of the minimizing \( y \). In a recent preprint, we studied the special case when each \( c_i \) is the distance squared on a Riemannian manifold \[19\]; in this case, the smoothness and non-degeneracy conditions required in \[21\] may fail, and the techniques developed there are closer to those used in this paper.

Our main result in this direction is the following, which gives a systematic way to generate new multi-marginal cost functions which ensure the Monge solution structure and uniqueness of the optimal measure.

**Theorem 5.1.** Assume \( c_1 \) satisfies twist in \( c_1 \)-cyclical monotonicity and \( c_j \), \( j = 2, \ldots, k \), satisfies twist in \( c_j \)-cyclical monotonicity with respect to \( y \), i.e., the map \( X_j \in S \mapsto D_y c_j(X_j, y) \) is injective along \( c \)-monotone subsets \( S \subseteq M_{m_j-1+1} \times \cdots \times M_{m_j} \times \{y\} \). Then the cost \( c(x_1, x_2, \ldots, x_m) \) defined by (6) satisfies twist in \( c \)-cyclical monotonicity.

**Remark 5.2.** In this theorem, in fact, a slightly stronger result holds, namely, one can replace twist in \( c \)-cyclical monotonicity in the conclusion, with twist in \( c \)-monotonicity of order two, which is defined exactly as in Definition 2.2, except the number \( N \) there is fixed to be \( N = 2 \). This will be obvious by examining the proof.
The proof of this result is based on the same essential idea as our argument in [19] and is divided into several Lemmas. The first two of these show that a cyclically monotone set $S$ projects, in a certain sense, to $c_j$-cyclically monotone sets.

**Lemma 5.3.** Suppose the set $S \subseteq M_1 \times \ldots \times M_m$ is $c$-cyclically-monotone. Use the notation given in the beginning of this section. Then the set

$$S := \{(X_1, y) : \exists (X_2, \ldots, X_k) \text{ such that } (X_1, \ldots, X_k) \in S \text{ and } y \in \arg\min\left[\sum_{j=1}^{k} c_j(X_j, y)\right]\} \subseteq M_1 \times \ldots \times M_m \times Y.$$ 

is $c_1$-cyclically-monotone.

**Proof.** The proof is straightforward and we include it here for the reader’s convenience. Given

$$(X^i_1, y^i) = (x^i_1, x^i_2, \ldots x^i_m, y^i) \in \bar{S},$$

for $i = 1, 2, \ldots, l$, permutations $\sigma_j$ on $l$ letters for $j = 2, 3, \ldots, m_1$ and a permutation $\eta$ on $l$ letters (corresponding to the $y$ argument in $c_1$), we need to show

$$\sum_{i=1}^{l} c_1(x^i_1, x^i_2, \ldots x^i_m, y^i) \leq \sum_{i=1}^{l} c_1(x^i_1, x^i_2, \ldots x^i_m, y^i(\sigma_i)).$$

Now, for each $i$ we can choose $(X^i_2, \ldots, X^i_k)$ so that $(X^i_1, X^i_2, \ldots, X^i_k) \in S$ and

$$y^i \in \arg\min\left[\sum_{j=1}^{k} c_j(X^i_j, y^i)\right].$$

Now, for $j = m_1 + 1, \ldots, m$, choose $\sigma_j = \eta$. Then, from $c$-cyclical monotonicity, we have

$$\sum_{i=1}^{l} c_1(x^i_1, x^i_2, \ldots x^i_m, y^i) + c_2(x^i_{m_1+1}, \ldots x^i_{m_2}, y^i) + \cdots + c_k(x^i_{m_{k-1}+1}, \ldots x^i_m, y^i)$$

$$= \sum_{i=1}^{l} c(x^i_1, x^i_2, \ldots x^i_m)$$

$$\leq \sum_{i=1}^{l} c(x^i_1, x^i_2, \ldots x^i_m, y^i(\sigma_i)).$$

$$\leq \sum_{i=1}^{l} c_1(x^i_1, x^i_2, \ldots x^i_m, y^i(\sigma_i)) + c_2(x^i_{m_1+1}, \ldots x^i_{m_2}, y^i(\eta(i))$$

$$+ \cdots + c_k(x^i_{m_{k-1}+1}, \ldots x^i_m, y^i(\eta(i)),$$

$$\leq \sum_{i=1}^{l} c_1(x^i_1, x^i_2, \ldots x^i_m, y^i(\sigma_i)).$$
Noting that
\[
\sum_{i=1}^{l} c_2(x_{m_1}^{i+1}, \ldots, x_{m_2}^i, y^i) + \cdots + c_k(x_{m_k}^{i+1}, \ldots, x_m^i, y^i) \\
= \sum_{i=1}^{l} c_2(x_{m_1}^{-i}, \ldots, x_{m_2}^{-i}, y^{-i}) + \cdots + c_k(x_{m_k}^{-i}, \ldots, x_m^{-i}, y^{-i}),
\]
we have
\[
\sum_{i=1}^{l} c_1(x_1, x_2^i, \ldots, x_k^i, y^i) \leq \sum_{i=1}^{l} c_1(x_1, x_2^{\sigma_2^i}, \ldots, x_k^{\sigma_k^i}, y^{\sigma(i)}),
\]
which completes the proof.

Similarly, we have

**Lemma 5.4.** Suppose the set \( S \subseteq M_1 \times \ldots \times M_m \) is \( c \)-cyclically-monotone. Fix \( j \). Then the set
\[
\tilde{S} := \{(X_j, y) : \exists X_i \text{ for } i \neq j \text{ such that } (X_1, \ldots, X_k) \in S \text{ and } y \in \arg\min \left[ \sum_{i=1}^{k} c_i(X_j, y) \right], \quad y \in M_{m_j-1+1} \times \ldots \times M_{m_j} \times Y.
\]
is \( c_j \)-cyclically-monotone.

**Proof.** The proof is very similar to the proof of the preceding lemma and is skipped.

**Lemma 5.5.** Fix \( x_1 \in M_1 \) and suppose the set \( S \subseteq \{x_1\} \times M_2 \times \ldots \times M_m \) is \( c \)-cyclically-monotone and \( c_1 \) satisfies the twist in cyclical monotonicity condition. Choose \((x_2, \ldots, x_m)\) and \((\bar{x}_2, \ldots, \bar{x}_m)\) in \( S \). Let
\[
y \in \arg\min \sum_{j=1}^{k} c_j(X_j, y)
\]
and
\[
\bar{y} \in \arg\min \sum_{j=1}^{k} c_j(\bar{X}_j, y),
\]
where \((X_1, \ldots, X_k) = (x_1, x_2, \ldots, x_m)\) and \((\bar{X}_1, \ldots, \bar{X}_k) = (x_1, \bar{x}_2, \ldots, \bar{x}_m)\). If
\[
D_{x_1} c(x_1, x_2, \ldots, x_m) = D_{x_1} c(x_1, \bar{x}_2, \ldots, \bar{x}_m),
\]
then we must have \( y = \bar{y} \), and \( x_i = \bar{x}_i \), for \( i = 2, 3, \ldots, m_1 \).
Proof. Note that existence of the derivative \( D_{x_1} c_1(x_1, x_2, ..., x_m, y) \) (respectively, \( D_{x_1} c_1(x_1, \bar{x}_2, ..., \bar{x}_m, \bar{y}) \)) implies the existence of \( D_{x_1} c(x_1, x_2, ..., x_m) \) (respectively \( D_{x_1} c(x_1, \bar{x}_2, ..., \bar{x}_m, \bar{y}) \)) by standard arguments, and we have
\[
D_{x_1} c_1(x_1, x_2, ..., x_m, y) = D_{x_1} c(x_1, x_2, ..., x_m) = D_{x_1} c(x_1, \bar{x}_2, ..., \bar{x}_m, \bar{y}).
\]
The result now follows, since \( c_1 \) is twisted in \( c_1 \)-cyclic monotonicity and the projection of \( S \) to \( M_1 \times \cdots \times M_m \) is \( c_1 \)-cyclically monotone from Lemma 5.3. \( \square \)

Proof of Theorem 5.1. Fix \( x_1 = \bar{x}_1 \in M_1 \). Let \( S \) be \( c \)-cyclically-monotone in \( \{x_1\} \times M_2 \times \cdots \times M_m \). Let \((x_1, ..., x_m), (\bar{x}_1, ..., \bar{x}_m) \in S \) such that
\[
D_{x_1} c(x_1, x_2, ..., x_m) = D_{x_1} c(\bar{x}_1, \bar{x}_2, ..., \bar{x}_m),
\]
we need to show \( x_i = \bar{x}_i \) for all \( i = 2, 3, ..., m \). (Here, the existence of these derivatives is part of the assumption.) For \( i = 2, ..., m_1 \), this follows immediately from Lemma 5.3.

To take care of the other \( i \), let us use the notation
\[
X_j = (x_{m_{j-1}+1}, ..., x_{m_j});
\]
\[
\bar{X}_j = (\bar{x}_{m_{j-1}+1}, ..., \bar{x}_{m_j}).
\]

From the same Lemma 5.3 we obtain the existence of a \( y \) such that
\[
y \in \left[ \arg\min_{j=1}^{k} c_j(X_j, y) \right] \cap \left[ \arg\min_{j=1}^{k} c_j(\bar{X}_j, y) \right].
\]
We then obtain, by minimality of \( \sum_{j=1}^{k} c_j(X_j, y) \) and \( \sum_{j=1}^{k} c_j(\bar{X}_j, y) \) at \( y \),
\[
D_y \sum_{j=1}^{k} c_j(X_j, y) = 0, \quad D_y \sum_{j=1}^{k} c_j(\bar{X}_j, y) = 0
\]
Here, the differentiability of these derivatives follows from the semi-concavity of \( c_j \)'s together with the minimality at \( y \): a semi-concave function \( f \) should be differentiable at a minimum point. This last fact can be seen easily by considering the superdifferential of the function \( f \) (i.e. the subdifferential of \( -f \)), because, if the superdifferential \( \partial f \) at a point \( x_0 \) has an element other than \( 0 \), then \( x_0 \) cannot be a minimum point by the definition of superdifferential.

Now, for fixed \( l \) with \( 1 \leq l \leq k \) let \( X'_l = X_l \) and \( X'_j = X_j \) for \( j \neq l \). Similarly, set \( \bar{X}'_l = \bar{X}_l \) and \( \bar{X}'_j = \bar{X}_j \) for \( j \neq l \). Note that by the definition of \( c \),
\[
c(X_1, X'_2, ..., X'_k) + c(X_1, \bar{X}_2, ..., \bar{X}_k) \leq \sum_{j=1}^{k} c_j(X'_j, y) + \sum_{j=1}^{k} c_j(\bar{X}'_j, y)
\]
\[
= \sum_{j=1}^{m} c_j(X_j, y) + \sum_{j=1}^{m} c_j(\bar{X}_j, y)
\]
\[
= c(X_1, X_2, ..., X_k) + c(X_1, \bar{X}_2, ..., \bar{X}_k).
\]
On the other hand, by $c$-monotonicity, we must have
\[ c(X_1, X_2, \ldots, X_k) + c(X_1, \bar{X}_2, \ldots, \bar{X}_k) \leq c(X_1, \bar{X}_2, \ldots, X_k) + c(X_1, \bar{X}_2, \ldots, \bar{X}_k). \]

In light of the preceding series of inequalities, this implies that
\[ c(X_1, X_2', \ldots, X'_k) + c(X_1, \bar{X}_2', \ldots, \bar{X}'_k) = \sum_{j=1}^{k} c_j(X'_j, y) + \sum_{j=1}^{k} c_j(\bar{X}'_j, y), \]

and $y \in \text{argmin} \sum_{j=1}^{k} c_j(X'_j, y)$, so that $\sum_{j=1}^{m} D_y c_j(X'_j, y) = 0$, or
\[ \sum_{j \neq l}^{k} D_y c_j(X_j, y) = -D_y c_l(\bar{X}_l, y). \]

(Here again, the differentiability of these functions follows from the semi-concavity of $c_j$’s together with the minimality at $y$.) Now, from (9),
\[ \sum_{j \neq l}^{k} D_y c_j(X_j, y) = -D_y c_l(X_l, y). \]

We therefore conclude that
\[ D_y c_l(X_l, y) = D_y c_l(\bar{X}_l, y). \quad (10) \]

By Lemma 5.4 and the twist in $c_l$-monotonicity with respect to $y$, we obtain $X_l = \bar{X}_l$. Since $l$ is arbitrary, this shows $x_i = \bar{x}_i$ for $1 \leq i \leq m$, completing the proof.

\[ \square \]

References

[1] M. Beiglböck, P. Henry-Labordere, and F. Penkner. Model independent bounds for option prices: a mass transport approach. Preprint available at http://www.mat.univie.ac.at/~mathias/BeHePe11.pdf.

[2] Giuseppe Buttazzo, Luigi De Pascale, and Paola Gori-Giorgi. Optimal-transport formulation of electronic density-functional theory. Phys. Rev. A, 85:062502, Jun 2012.

[3] L. Caffarelli. Allocation maps with general cost functions. In Partial Differential Equations and Applications, volume 177 of Lecture Notes in Pure and Applied Math, pages 29–35. Dekker, New York, 1996.

[4] G. Carlier. On a class of multidimensional optimal transportation problems. J. Convex Anal., 10(2):517–529, 2003.
[5] G. Carlier and I. Ekeland. Matching for teams. *Econ. Theory*, 42(2):397–418, 2010.

[6] G. Carlier and B. Nazaret. Optimal transportation for the determinant. ESAIM Control Optim. Calc. Var., 14(4):678–698, 2008.

[7] P-A. Chiapporri, R. McCann, and L. Nesheim. Hedonic price equilibria, stable matching and optimal transport; equivalence, topology and uniqueness. *Econ. Theory*, 42(2):317–354, 2010.

[8] C. Cotar, G. Friesenke, and C. Klüppelberg. Density functional theory and optimal transportation with Coulomb cost. Preprint available at arXiv:1104.0603.

[9] A. Galichon and N. Ghoussoub. Variational representations for N-cyclically monotone vector fields. Preprint.

[10] A. Galichon, P. Henry-Labordere, and N. Touzi. A stochastic control approach to non-arbitrage bounds given marginals, with an application to Lookback options. Preprint available at https://sites.google.com/site/alfredgalichon/research.

[11] W. Gangbo. Habilitation thesis, Universite de Metz, available at http://people.math.gatech.edu/˜gangbo/publications/habilitation.pdf, 1995.

[12] W. Gangbo and R. McCann. The geometry of optimal transportation. *Acta Math.*, 177:113–161, 1996.

[13] W. Gangbo and A. Święch. Optimal maps for the multidimensional monge-kantorovich problem. *Comm. Pure Appl. Math.*, 51(1):23–45, 1998.

[14] N. Ghoussoub and B Maurey. Remarks on multi-marginal symmetric monge-kantorovich problems. to appear in *Discrete and Continuous Dynamical Systems-A, special issue on “Optimal Transport and Applications” (2012).*

[15] N. Ghoussoub and A. Moameni. Symmetric monge-kantorovich problems and polar decompositions of vector fields. Preprint, (February 10, 2013) 23pp.

[16] H. Heinich. Probleme de Monge pour n probabilities. *C.R. Math. Acad. Sci. Paris*, 334(9):793–795, 2002.

[17] P. Henry-Labordere and N. Touzi. An explicit martingale version of Brenier’s theorem. Preprint at http://www.cmap.polytechnique.fr/˜touzi/MartingaleBrenier-discret.pdf.

[18] H.G. Kellerer. Duality theorems for marginal problems. *Z. Wahrsch. Verw. Gebiete*, 67:399–432, 1984.
[19] Y.-H. Kim and B. Pass. Multi-marginal optimal transport on a Riemannian manifold. Preprint. Currently available at arXiv:1303.6251.

[20] V. Levin. Abstract cyclical monotonicity and Monge solutions for the general Monge-Kantorovich problem. Set-Valued Analysis, 7(1):7–32, 1999.

[21] B. Pass. Multi-marginal optimal transport and multi-agent matching problems: uniqueness and structure of solutions. Preprint. Currently available at arXiv:1210.7372.

[22] B. Pass. Uniqueness and monge solutions in the multimarginal optimal transportation problem. SIAM Journal on Mathematical Analysis, 43(6):2758–2775, 2011.

[23] B. Pass. On the local structure of optimal measures in the multimarginal optimal transportation problem. Calculus of Variations and Partial Differential Equations, 43:529–536, 2012. 10.1007/s00526-011-0421-z.

[24] L Rüschendorf. On c-optimal random variables. Statist. Probab. Lett., 27:267–270, 1996.

[25] C. Villani. Optimal transport: old and new, volume 338 of Grundlehren der mathematischen Wissenschaften. Springer, New York, 2009.