1st February 2022

Abstract

The system of hydrodynamic-type equations, derived by two-side distribution function for a stratified gas in gravity field is applied to a problem of ultrasound propagation and attenuation. The background state and linearized version of the obtained system is studied and compared with the Navier-Stokes one at arbitrary Knudsen numbers. The WKB solutions for ultrasound in a stratified medium are constructed in explicit form. The problem of a generation by a moving plane in a rarefied gas is explored and used as a test while compared with experiment.

1 Introduction

Recently the problems of Kn regime wave propagation was revisited in connection with general fluid mechanics and nonsingular perturbation method development [1,2,3,4]. A generalized Boltzman theories [5,6] also contributed in a progress with respect to this important problem.

In [7] the propagation of one-dimension disturbance was studied on the base of the method of a piecewise continuous distribution function launched in a pioneering paper of Lees [8] and applied for a gas in gravity field in [11,9]. We derived hydrodynamic-type equations for a gas perturbations in gravity field so that the Knudsen number depends on the (vertical) coordinate. The generalization to three dimensions is given at [12].

The derivation of the hydrodynamic-type equations is based on kinetic equation with the model integral of collisions in BGK (Bhatnagar - Gross - Krook ) form which collision term is modelled as $\nu (f_l - f)_l$, via local-equilibrium distribution function $f_l$ and the
non-equilibrium one is expressed as \( f^+ \) at \( v_z \geq 0 \), and as \( f^- \) at \( v_z \leq 0 \)

\[
f^\pm = \frac{n^\pm}{\pi^{3/2} v_T^{\pm3}} \exp\left(-\frac{(\vec{V} - \vec{U}^\pm)^2}{v_T^{\pm2}}\right),
\]

the \( v_T = \sqrt{2kT/m} \) denotes the average thermal velocity of particles of gas, \( \nu = \nu(z) \) is the effective frequency of collisions between particles of gas at height \( z \). It is supposed, that density of gas \( n \), its average speed \( \vec{U} = (u_x, u_y, u_z) \) and temperature \( T \) are functions of time and coordinates enter the local-equilibrium \( f^{\pm} \). The resulting system is

\[
\begin{align*}
\partial_t \rho + \partial_{z}(\rho U) &= 0, \\
\partial_t U + U \partial_{z} U + \frac{1}{\rho} \partial_{z} P &= 0, \\
3k m \partial_t \rho T + \frac{3k}{2m} U \partial_{z} \rho T + \left(3k \rho T + P_{zz}\right) \partial_{z} U + \frac{1}{\rho} \partial_{z} q &= 0, \\
\partial_t P_{zz} + U \partial_{z} P_{zz} + 3P_{zz} \partial_{z} U + \frac{2}{\rho} \partial_{z} q &= -\nu(z)(P_{zz} - \frac{\rho}{m} kT), \\
\partial_t q + U \partial_{z} q + 2(q_z + \vec{q}) \partial_{z} U - \left(\frac{3k}{2m} T + \frac{1}{\rho} P_{zz}\right) \partial_{z} P_{zz} + \frac{1}{\rho} \partial_{z} J_1 &= -\nu(z)q_z, \\
\partial_t \vec{q} + U \partial_{z} \vec{q} + 4\vec{q} \partial_{z} U - \left(\frac{3}{2\rho} P_{zz} \partial_{z} P_{zz} + \frac{1}{\rho} \partial_{z} J_2 \right) &= -\nu(z)\vec{q}_z,
\end{align*}
\]

where

\[
J_1 = \frac{m}{2} < (V_z - U)^2(\vec{V} - \vec{U})^2 >, \quad J_2 = \frac{m}{2} < (V_z - U)^4 >.
\]

The increase of the number of parameters of distribution function results in that the distribution function differs from a local-equilibrium one and describes deviations from hydrodynamical regime. In the range of small Knudsen numbers \( l \ll L \) we automatically have \( n^+ = n^- \), \( U^+ = U^- \), \( T^+ = T^- \) and distribution function reproduces the hydrodynamics of Euler and at the small difference of the functional "up" and "down" parameters - the Navier-Stokes equations. In the range of big Knudsen numbers the theory gives solutions of collisionless problems [9].

We used a set of linearly independent eigen functions of the linearized Boltzmann operator, that in the case of the BGK equation is:

\[
\begin{align*}
\varphi_1 &= m, \quad \varphi_4 = \frac{m(V_z - U_z)^2}{\pi^{3/2} v_T^{\pm3}}, \\
\varphi_2 &= mV_z, \quad \varphi_5 = \frac{1}{2} m(V_z - U_z)^2 |\vec{V} - \vec{U}|^2, \\
\varphi_3 &= \frac{1}{2} m |\vec{V} - \vec{U}|^4, \quad \varphi_6 = \frac{1}{2} m (V_z - U_z)^3.
\end{align*}
\]

Let’s define a scalar product in velocity space:

\[
< \varphi_n, f >= \int d\vec{V} \varphi_n(t, z, \vec{V}) f(t, z, \vec{V}),
\]

where \( f \) is the distribution function.
\[ \langle \varphi_1 \rangle = \rho(t, z), \quad \langle \varphi_2 \rangle = \rho U, \quad \langle \varphi_3 \rangle = \frac{3}{2} \frac{e}{m} kT, \quad \langle \varphi_4 \rangle = P_{zz}, \quad \langle \varphi_5 \rangle = q_z, \quad \langle \varphi_6 \rangle = \bar{q}_z. \] (5)

Here \( \rho \) is mass density, \( P_{zz} \) is the diagonal component of the pressure tensor, \( q_z \) is a vertical component of a heat flow, \( \bar{q}_z \) is a parameter having dimension of the heat flow.

The system (1) of the equations according to the derivation scheme is valid at all frequencies of collisions and within the limits of the high frequencies should transform to the hydrodynamic equations.

If we estimate the functions \( \frac{U}{v_T} \) as small, that corresponds to small Mach numbers \( M = \text{max} | \frac{U}{v_T} | \). We shall base here on an expansion in \( M \), up to the first order. In this approach the functional parameters of the two-fold distribution function

\[ n^+ = n_0(1 + \alpha n_1^+), \quad n^- = n_0(1 + \alpha n_1^-), \quad \rho = nm \]
\[ V^+ = V_0(1 + \alpha V_1^+), \quad V^- = V_0(1 + \alpha V_1^-), \]
\[ U^+ = \alpha V_0 U_1^+, \quad U^- = \alpha V_0 U_1^- \]

Let's evaluate the integrals (2) and (5) directly, plugging the two-side distribution function. In the first order by Mach number \( \alpha \)

\[ n = n_0 + \left( \frac{n_0 U_1^+}{\sqrt{\pi}} + \frac{1}{2} n_0 n_1^+ + \frac{1}{2} n_0 n_1^- - \frac{n_0 U_1^-}{\sqrt{\pi}} \right) \alpha \]
\[ U = \left( -\frac{1}{2} \frac{V_0 V_1^-}{\sqrt{\pi}} - \frac{1}{2} \frac{V_0 n_1^-}{\sqrt{\pi}} + \frac{1}{2} V_0 U_1^+ + \frac{1}{2} V_0 n_1^+ + \frac{1}{2} V_0 U_1^- + \frac{1}{2} \frac{V_0 V_1^+}{\sqrt{\pi}} \right) \alpha \]
\[ \frac{3}{2} \frac{k_B}{m} \rho T = \frac{3}{4} n_0 V_0^2 + \left( \frac{3}{8} n_0 V_0^2 V_1^+ + \frac{3}{8} n_0 V_0^2 n_1^+ + \frac{3}{8} n_0 V_0^2 n_1^- + \frac{n_0 V_0^2 U_1^+}{\sqrt{\pi}} + \frac{1}{2} n_0 V_0^2 V_1^- + \frac{n_0 V_0^2 U_1^-}{\sqrt{\pi}} \right) \alpha \]
\[ \frac{1}{m} P_{zz} = \frac{1}{2} n_0 V_0^2 + \left( \frac{1}{4} n_0 V_0^2 n_1^+ + \frac{n_0 V_0^2 U_1^-}{\sqrt{\pi}} + \frac{1}{4} n_0 V_0^2 V_1^- + \frac{n_0 V_0^2 n_1^-}{\sqrt{\pi}} - \frac{1}{2} n_0 V_0^2 U_1^+ - \frac{1}{2} n_0 V_0^2 V_1^+ \right) \alpha \]
\[ \frac{1}{m} q_z = \left( \frac{3}{8} n_0 V_0^3 U_1^+ - \frac{5}{4} n_0 V_0^2 U - \frac{5}{8} n_0 V_0^3 U_1^- \right) \alpha \]
\[ \frac{1}{m} q_x = \left( \frac{5}{8} n_0 V_0^3 U_1^+ + \frac{5}{4} n_0 V_0^4 n_1^+ + \frac{5}{4} n_0 V_0^4 V_1^- \right) \alpha \]
\[ \frac{1}{m} \beta = \left( \frac{3}{16} n_0 V_0^4 n_1^+ + \frac{3}{4} n_0 V_0^4 V_1^- + \frac{3}{16} n_0 V_0^4 n_1^- - \frac{n_0 V_0^4 U_1^+}{\sqrt{\pi}} + \frac{3}{4} n_0 V_0^4 V_1^+ - \frac{n_0 V_0^4 U_1^-}{\sqrt{\pi}} \right) \alpha \]

Solving the system (6), we obtain for the parameters of the two-fold distribution
Let's linearize the system (1) this way:

\[
\begin{align*}
n_1^+ &= -\frac{3U\sqrt{\pi}}{2V_o} + \frac{n}{n_0} - 1 - 3 \frac{P_{zz}m_n}{n_0m^2V_0^2} + 3 \frac{k\rho T}{n_0m^2V_0^2} - 7 \frac{\bar{q}_z\sqrt{\pi}}{mn_nV_0^3} + 3 \frac{q_z\sqrt{\pi}}{mn_nV_0^3} + 3 \frac{nU\sqrt{\pi}}{2n_0V_0}, \\
n_1^- &= \frac{3U\sqrt{\pi}}{2V_o} + \frac{n}{n_0} - 1 - 3 \frac{P_{zz}m_n}{n_0m^2V_0^2} + 3 \frac{n_0m^2V_0^2}{n_0m^2V_0^2} + 7 \frac{\bar{q}_z\sqrt{\pi}}{mn_nV_0^3} - 3 \frac{q_z\sqrt{\pi}}{mn_nV_0^3} - 3 \frac{nU\sqrt{\pi}}{2n_0V_0}, \\
V_1^+ &= -\frac{1}{2} \frac{V_0n_0}{nU\sqrt{\pi}} - 1 \frac{P_{zz}m_n}{n_0m^2V_0^2} + 3 \frac{k\rho T}{n_0m^2V_0^2} - \frac{1}{2} \frac{n}{n_0} - \frac{q_z\sqrt{\pi}}{mn_nV_0^3} + 3 \frac{n_0V_0^2}{n_0V_0^2} - \frac{1}{2} \frac{V_0}{\sqrt{\pi}} - \frac{\bar{q}_z\sqrt{\pi}}{mn_nV_0^3}, \\
V_1^- &= \frac{1}{2} \frac{V_0n_0}{nU\sqrt{\pi}} - \frac{1}{2} \frac{P_{zz}m_n}{n_0m^2V_0^2} + 3 \frac{n_0m^2V_0^2}{n_0m^2V_0^2} - \frac{1}{2} \frac{n}{n_0} - \frac{q_z\sqrt{\pi}}{mn_nV_0^3} - 3 \frac{n_0V_0^2}{n_0V_0^2} + \frac{1}{2} \frac{V_0}{\sqrt{\pi}} + \frac{\bar{q}_z\sqrt{\pi}}{mn_nV_0^3}, \\
U_1^+ &= \frac{U}{V_0} + 3 \frac{\sqrt{\pi}P_{zz}}{2m_0V_0^2} - \frac{3}{2} \frac{\sqrt{\pi}k\rho T}{2m_0V_0^2} + 8 \frac{\bar{q}_z}{mn_nV_0^3} - 4 \frac{q_z}{mn_nV_0^3}, \\
U_1^- &= \frac{U}{V_0} - \frac{3}{2} \frac{\sqrt{\pi}P_{zz}}{2m_0V_0^2} + \frac{3}{2} \frac{\sqrt{\pi}k\rho T}{2m_0V_0^2} + 8 \frac{\bar{q}_z}{mn_nV_0^3} - 4 \frac{q_z}{mn_nV_0^3}.
\end{align*}
\]

The values of integrals \( \frac{2}{5} \) as functions of thermodynamic parameters of the system \( \mathbf{1} \) are linked to the thermodynamic variables as:

\[
\begin{align*}
J_1 &= -\frac{5}{2} \rho \left( \frac{kT_0}{m} \right)^2 + \frac{11}{4} \frac{kT_0P_{zz}}{m} + \frac{9}{4} \left( \frac{k}{m} \right)^2 \rho T_0 T, \\
J_2 &= -\frac{5}{2} \rho \left( \frac{kT_0}{m} \right)^2 + \frac{9}{4} \frac{kT_0P_{zz}}{m} + \frac{3}{4} \left( \frac{k}{m} \right)^2 \rho T_0 T.
\end{align*}
\] (8)

So we have closed the system \( \mathbf{1} \), hence a modification of the procedure for deriving fluid mechanics (hydrodynamic-type) equations from the kinetic theory is proposed, it generalizes the Navier-Stokes at arbitrary density (Knudsen numbers).

Our method gives a reasonable agreement with the experimental data in the case of homogeneous gas \( \mathbf{7} \). In the paper \( \mathbf{7} \) the expressions for \( (J_{1,2}) \) are obtained with account some nonlinear terms, that finally lead to more exact results.

## 2 Stationary case (undisturbed atmosphere).

Let's linearize the system \( \mathbf{1} \) this way:

\[
\begin{align*}
\rho &= \rho_0(z)(1 + \varepsilon \rho_1(t, z)), \\
P(t, z) &= P_0(z)(1 + \varepsilon P_1(t, z)), \\
q_z &= q_z_0(z)(1 + \varepsilon q_z_1(t, z)), \\
T_0(z)(1 + \varepsilon T_1(t, z)), \\
\varepsilon &<< 1.
\end{align*}
\]
We obtain in the zero order:
\[
\frac{d}{dz}P_0(z) \rho_0(z) + g = 0 ,
\]
\[
\frac{d}{dz}q_0(z) = 0 ,
\]
\[
2 \frac{d}{dz}q_{z0}(z) + \nu(z)(P_0(z) - \frac{k \rho_0(z) T_0(z)}{m}) = 0 ,
\]
\[
- \frac{1}{4 \pi n^2} T_0(z) \frac{d}{dz} \rho_0(z) - \frac{1}{4 \pi n^2} T_0(z) \frac{d}{dz} T_0(z) + \nu(z) q_0(z) + 
\]
\[
 \frac{3}{4 m} T_0(z) \frac{d}{dz} P_0(z) - \frac{3}{4 m} T_0(z) \frac{d}{dz} P_0(z) + \frac{11}{4} k \frac{d}{dz} T_0(z) = 0 ,
\]
\[
\nu(z) q_{z0}(z) + \frac{9 k}{4 m} P_0(z) \frac{d}{dz} T_0(z) - \frac{3}{2 m^2} \rho_0(z) T_0(z) \frac{d}{dz} T_0(z) - 
\]
\[
- \frac{3}{4 m^2} T_0(z) \frac{d}{dz} \rho_0(z) + \frac{9 k}{4 m} T_0(z) \frac{d}{dz} P_0(z) - \frac{3}{2} P_0(z) \frac{d}{dz} P_0(z) = 0 .
\]

Some version of such system that leads to a non-exponential density dependence on height was studied in [9, 10], the paradox was discussed at [11].

Let’s solve the zero order system.

\[ q_{z0} = C_1 . \] If \( P_0 = \frac{k}{m} \rho_0 T_0 \), then \( q_{z0} = C_2 = \frac{3}{5} C_1 \). If \( C_1 = 0 \), then \( T_0 = C_3 = \text{const} \) and we’ll have exponential density dependence on height.

We obtain in the first order:
\[
\frac{\partial}{\partial t} \rho_1 + V_T \frac{\partial}{\partial z} U_1 - \frac{V_T}{H} U_1 = 0 ,
\]
\[
\frac{\partial}{\partial t} U_1 + \frac{V_T}{2} \frac{\partial}{\partial z} P_1 + \frac{V_T}{2H} (\rho_1 - P_1) = 0 ,
\]
\[
\frac{\partial}{\partial t} T_1 + \frac{5}{3} V_T \frac{\partial}{\partial z} U_1 + \frac{2}{3} V_T \frac{\partial}{\partial z} q_1 - \frac{V_T}{H} (U_1 + \frac{2}{3} q_1) = 0 ,
\]
\[
\frac{\partial}{\partial t} P_1 + 3 V_T \frac{\partial}{\partial z} U_1 + 2 V_T \frac{\partial}{\partial z} q_1 - \frac{V_T}{H} (2 q_1 + U_1) + \nu (P_1 - \rho_1 - T_1) = 0 ,
\]
\[
\frac{\partial}{\partial t} q_1 + \frac{1}{8} V_T \frac{\partial}{\partial z} P_1 + \frac{9}{8} V_T \frac{\partial}{\partial z} T_1 - \frac{1}{8} V_T \frac{\partial}{\partial z} P_1 - \frac{3}{8} V_T (P_1 + T_1 - P_1 + \nu q_1 = 0 ,
\]
\[
\frac{\partial}{\partial t} \frac{\partial}{\partial z} U_1 + \frac{3}{8} V_T (\frac{\partial}{\partial z} P_1 + \frac{\partial}{\partial z} T_1 - \frac{\partial}{\partial z} T_1) + \frac{3}{8} V_T (P_1 - \rho_1 - T_1) + \nu \bar{q}_1 = 0 .
\]

3 Construction of solutions of the fluid dynamics system by WKB method.

In this section we apply the method WKB to the system (10). We shall assume, that on the bottom boundary at \( z = 0 \) a wave with characteristic frequency \( \omega_0 \) is generated. Next we choose the frequency \( \omega_0 \) to be large enough, to put characteristic parameter \( \xi = \frac{3 \omega_0 H}{v_r} \gg 1 \). We shall search for the solution in the form:
\[
M_n = \psi_n \exp(i \omega_0 t) + c.c . ,
\]
where, for example, $\psi_1$, corresponding to the moment $M_1$, is given by the expansion:

$$
\psi_1 = \sum_{k=1}^{6} \sum_{m=1}^{\infty} \frac{1}{(i\xi)^m} A^{(k)}_m \exp(i\xi \varphi_k(z)) ,
$$

(12)

here $\varphi_k(z)$ - the phase functions corresponding to different roots of dispersion relation. For other moments $M_n$, $n = 2, \ldots, 6$ corresponding functions $\psi_n$ are given by similar to (12) expansion. The appropriate coefficients of the series we shall designate by corresponding $B^{(k)}_m C^{(k)}_m D^{(k)}_m E^{(k)}_m F^{(k)}_m$. Substituting the series (12) at the system (10) one arrives at algebraic equations for the coefficients of (12) in each order. The condition of solutions existence results in the mentioned dispersion relation:

$$
\frac{54}{125} \eta^3 + \left( -\frac{12}{5} iu - \frac{63}{25} + \frac{3}{5} u^2 \right) \eta^2 + \left( -iu^3 + \frac{37}{5} iu - \frac{24}{5} u^2 + \frac{18}{5} \right) \eta - 1 - 3iu + 3u^2 + iu^3 = 0
$$

(13)

Here for convenience the following designations are introduced:

$$
\left( \frac{\partial \varphi_k}{\partial \bar{z}} \right)^2 = \frac{2}{15} \eta_k , \quad u = \nu_0 \exp(-\bar{z}) ,
$$

where $\bar{z} = \frac{z}{H}$. For the coefficients $A^{(k)}_1 B^{(k)}_1 \ldots$ the algebraic relations are obtained:

$$
B^{(k)}_1 = \frac{\pm \sqrt{30} A^{(k)}_1}{6 \sqrt{\eta_k}} , \quad C^{(k)}_1 = \frac{1}{3} A^{(k)}_1 \left( -25 + 20iu + 3\eta \right) , \quad D^{(k)}_1 = \frac{5}{3} \frac{A^{(k)}_1}{\eta_k},
$$

$$
E^{(k)}_1 = \frac{\pm 5}{12} \frac{A^{(k)}_1}{\sqrt{\eta}(-10 + 10iu + 9\eta)} ,
$$

$$
F^{(k)}_1 = \frac{\pm 1}{36} \frac{\sqrt{30} (50 - 100iu - 135\eta - 50u^2 + 190iu\eta + 81\eta^2 + 50u^2\eta - 30i\eta^2 u)}{\eta(-10 + 10iu + 9\eta)} A^{(k)}_1.
$$

The dispersion relation (13) represents the cubic equation with variable coefficients, therefore the exact analytical solution by formula Cardano looks very bulky and inconvenient for analysis. We study the behavior of solutions at $\nu \to 0$ (free molecular regime) and $\nu \to \infty$ (a hydrodynamical regime).

At the limit of collisionless gas $\nu = 0$ the dispersion relation becomes:

$$
\frac{54}{125} \eta^3 - \frac{63}{25} \eta^2 + \frac{18}{5} \eta - 1 = 0
$$

The roots are:

$$
\eta_1 \approx 3.80 , \quad \eta_2 \approx 0.37 , \quad \eta_3 \approx 1.67 .
$$

In a limit $\nu \to \infty$ (a hydrodynamical limit) for specifying roots (13) by the theory of perturbations up to $u^3$ for the three solutions branches it is obtained:

$$
\eta_1 = 1.00 - 2.32u^{-2} + i(1.20u^{-1} - 4.88u^{-3}) ,
$$

$$
\eta_2 = i 1.67u + 2.33 + 0.64u^2 ,
$$

$$
\eta_3 = -1.39u^2 + 2.50 + i(3.89u - 1.20u^{-1}) .
$$
The first root relates to the acoustic branch. Accordingly, for the \( k_{i,\pm} = \pm \sqrt{\eta_i} \) we have:

\[
k_1,\pm \approx 1.00 - 0.98u^{-2} + i(0.60u^{-1} - 1.85u^{-3}) ,
\]

\[
k_2,\pm \approx \sqrt{u}(1 - i)(0.64u^{-1} + 0.019u^{-3}) + \sqrt{u}(1 + i)(0.91 + 0.22u^{-2}) ,
\]

\[
k_3,\pm \approx 1.65 - 0.64u^{-2} + i(1.18u + 0.094u^{-1}) .
\]

The solution of the equation (13) at any \( u \) is evaluated numerically. As an illustration let us consider a problem of generation and propagation of a gas disturbance, by a plane oscillating with a given frequency \( \omega_0 \). We restrict ourselves by the case of homogeneous gas, because it is the only case of existing experimental realization. We evaluate numerically the propagation velocity and attenuation factor of a linear sound.

![Graph](image)

**Fig. 1.** The inverse non-dimensional phase velocity as a function of the inverse Knudsen number. The results of this paper-1 are compared to Navier-Stokes, previous our work [7]-2 and the experimental data of Meyer-Sessler [13]-circle.
4 Conclusion

In this paper we propose a one-dimensional theory of linear disturbances in a gas, stratified in gravity field, hence propagating through regions with crucially different Kn numbers. The regime of the propagation dramatically changes from a typically hydrodynamic to the free-molecular one. We also studied three-dimensional case [12]. Generally the theory is based on Gross-Jackson kinetic equation, which solution is built by means of locally equilibrium distribution function with different local parameters for molecules moving ”up” and ”down”. Equations for six moments yields in the closed fluid mechanics system. For the important generalizations of the foundation of such theory see the recent review of Alexeev [5].

5 Acknowledgements

We would like to thank Vereshchagin D.A. for important discussions.

References

[1] D.A. Vereshchagin, S.B. Leble. 1996, Nonlinear Acoustics in Perspective, ed. R. Wei, 142-146.
[2] X. Chen, H. Rao, and E. A. Spiegel, 2000, Physics Letters A 271, 87-91
[3] X. Chen, H. Rao, and E. A. Spiegel, 2001, Phys. Rev. E 64, 046309.
[4] E. A. Spiegel and J.-L. Thiffeault, 2003, Physics of Fluids 15(11), P. 3558-3567.
[5] B. V. Alexeev, 2004, Generalized Boltzmann Physical Kinetics, Elsevier.

[6] Elizarova, T. G.; Chetverushkin, B. N. Kinetically consistent difference schemes for the modeling of flows of a viscous heat conducting gas. (Russian) Zh. Vychisl. Mat. i Mat. Fiz. 28 (1988), no. 11, 1695–1710, 1759; translation in U.S.S.R. Comput. Math. and Math. Phys. 28 (1988), no. 6, 64–75 (1990)

[7] D. A. Vereshchagin, S. B. Leble, M. A. Solovchuk. 2006, Piecewise continuous distribution function method in the theory of wave perturbances of inhomogeneous gas. Physics Letters A, 348, 326-334.

[8] L. Lees, 1965, J. Soc. Industr. and Appl. Math., 13, No. 1. P. 278-311.

[9] D. A. Vereshchagin, S. B. Leble, 2005, Piecewise continuous distribution function method: Fluid equations and wave disturbances at stratified gas, physics/0503233.

[10] Leble S. Roman F. Vereshchagin D. White J-A., 1996, Molecular Dynamics and Momenta BGK Equations for Rarefied Gas in Gravity field. in Proceedings of 8th Joint EPS-APS International Conference Physics Computing CYFRONET-KRAKOW, Ed. P. Borcherds, M. Bubak, A. Maksymowicz, p. 218-221.

[11] Roman F. White J-A, Velasco S. 1995, On a paradox concerning the temperature distribution of an ideal gas in a gravitational field. Eur. J. Phys. 16 83-90.

[12] Solovchuk M.A., Leble S.B. 2005, The kinetic description of ultrasound propagation in a rarefied gas: from a piecewise continuous distribution to fluid equations. Proceedings of International Conference "Forum Acusticum 2005" (Budapest, 2005) L235-L240.

[13] E. Meyer, G. Sessler, Z. Physik. 149. (1957). P. 15-39.