Sections of Lagrangian fibrations
on holomorphically symplectic manifolds
and degenerate twistorial deformations

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Abstract
Let $(M, I, \Omega)$ be a holomorphically symplectic manifold equipped with a holomorphic Lagrangian fibration $\pi : M \rightarrow X$, and $\eta$ a closed form of Hodge type $(1,1)+(2,0)$ on $X$. We prove that $\Omega' := \Omega + \pi^* \eta$ is again a holomorphically symplectic form, for another complex structure $I'$, which is uniquely determined by $\Omega'$. The corresponding deformation of complex structures is called “degenerate twistorial deformation”. The map $\pi$ is holomorphic with respect to this new complex structure, and $X$ and the fibers of $\pi$ retain the same complex structure as before. Let $s$ be a smooth section of $\pi$. We prove that there exists a degenerate twistorial deformation $(M, I', \Omega')$ such that $s$ is a holomorphic section.

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1 Introduction

1.1 Complex structure obtained from a complex symplectic form

A complex structure $I$ on a real vector space $V$ is uniquely determined by a complex-linear symplectic form $\Omega$. Indeed, $\Omega$ has Hodge type $(2,0)$ because it is complex-linear. The corresponding map $\Omega : V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow V^* \otimes_{\mathbb{R}} \mathbb{C}$ is non-degenerate on the Hodge component $V^{1,0}$, and vanishes on $V^{0,1}$. Since the complex structure on $V$ is uniquely determined by $V^{0,1} \subset V \otimes_{\mathbb{R}} \mathbb{C}$, $\Omega$ determines $I$ uniquely.

This description is very beneficial on a manifold, when $\Omega$ is a complex-valued differential form. In that case, $d\Omega = 0$ implies that the almost complex structure determined by $I$ is integrable (Proposition 2.12). In [V2], this observation was used to construct deformations of hyperkähler manifolds admitting a Lagrangian fibration. It turns out that a holomorphic symplectic form can be characterized intrinsically in terms of its rank and exterior powers (Proposition 2.5).

More precisely, a complex-valued exterior 2-form $\Omega$ on a $4n$-dimensional real vector space is called c-symplectic if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is non-degenerate. By Proposition 2.5, this is equivalent to being a non-degenerate complex linear 2-form for some complex structure on $V$.

Recall that holomorphic Lagrangian fibration on a holomorphically symplectic manifold $(M, \Omega)$ is a holomorphic map with fibers which are Lagrangian with respect to $\Omega$.

Given a Lagrangian fibration $\pi : M \longrightarrow B$ on a holomorphic symplectic manifold $(M, \Omega)$ and a closed 2-form $\eta$ on $B$ of Hodge type $(2,0) + (1,1)$, the sum $\Omega + \pi^* \eta$ is c-symplectic, hence determines an almost complex structure $I_\eta$ on $M$ (Proposition 2.5). This almost complex structure is integrable when $\eta$ is closed (Proposition 2.12). The corresponding deformation of complex structures is uniquely determined by the cohomology class of $\eta$; it is called degenerate twistor deformation ([V2]).

As shown in [V2], the map $\pi : M \longrightarrow B$ is a holomorphic Lagrangian fibration with respect to $\eta$, and the complex structure on the fibers of $\pi$ does not change; in dimension 1, this deformation gives “the Tate-Shafarevich twist” of an elliptic fibration (that is, a deformation of a fibration which retains the complex structures on the base and on the fibers).
1.2 Lagrangian fibrations on hyperkähler manifolds

When $M$ is a compact hyperkähler manifold of maximal holonomy\footnote{This is the same as “IHS”, irreducibly holomorphic symplectic manifold.}, essentially all non-trivial holomorphic maps $M \to B$ are Lagrangian fibrations, by Matsushita’s theorem.

**Theorem 1.1:** Let $M$ be a compact hyperkähler manifold of maximal holonomy, and $\pi : M \to B$ a surjective holomorphic map, with $\dim M > \dim B > 0$. Then $\pi$ is a holomorphic Lagrangian fibration.

**Proof:** [Mat1]. ■

Applying the Stein factorization theorem, we can factorize a given holomorphic Lagrangian fibration through a fibration with connected fibers. Further on, we shall tacitly assume that all Lagrangian fibrations we consider have connected fibers. In this case the base $B$ is known to have the same rational cohomology as a complex projective space, by a theorem of D. Matsushita (Mat2). J.-M. Hwang has shown that $B$ is a complex projective space whenever it is smooth ([Hw]). It was conjectured that $B$ is biholomorphic to $\mathbb{C}P^n$ when it is normal ([CMSB]).

Since the degenerate twistor deformation is uniquely determined by the cohomology class of $\eta$, and $b_2(B) = 1$, all degenerate twistor deformations belong to a 1-dimensional holomorphic family determined by $\Omega + t\pi^*\eta$, with $t \in \mathbb{C}$. This family is obtained as a limit of twistor families, which explains the term “degenerate twistor deformation”. Like in the usual twistor case, a generic fiber of a degenerate twistor family is non-algebraic. However, the Moishezon fibers are dense in the degenerate twistor family.

Much is still unknown about the degenerate twistor families. For example, it is unknown whether all fibers of the degenerate twistor deformation are Kähler. It is unclear when a degenerate twistor deformation of a Lagrangian fibration admits a section.

In this paper we prove that existence of a section is essentially a topological condition.

**Theorem 1.2:** Let $\pi : M \to B$ be a Lagrangian fibration on a holomorphic symplectic manifold $(M, \Omega)$, and $S : B \to M$ its smooth section. Then there exists a closed form $\eta \in \Lambda^{2,0}(B) + \Lambda^{1,1}(B)$ such that $S$ is holomorphic Lagrangian with respect to the complex structure $I_\eta$ induced by $\Omega + \pi^*\eta$.

**Proof:** [Theorem 3.5] ■
Existence of holomorphic Lagrangian fibrations is a standard conjecture, sometimes referred to as “hyperkähler SYZ conjecture”; see [V1] for the history and a precise formulation. It is known that holomorphic Lagrangian fibrations exist for all known classes of hyperkähler manifolds (see e. g. [KV, Claim 1.20]). Any Lagrangian torus on a hyperkähler manifold \( M \) is a fiber of a rational Lagrangian fibration; moreover, this fibration becomes holomorphic after replacing \( M \) with another holomorphically symplectic birational model ([HW, GLR]).

Existence of holomorphic sections would follow if we construct a smooth section, which seems to be doable in many (or all) cases. This would bring us closer to understanding the holomorphic Lagrangian fibrations, and (hopefully) bring new classification results.

There is an obvious topological obstruction to existence of a smooth section: a fibration with multiple fibers cannot have sections. It is easy to construct holomorphic Lagrangian fibrations with multiple fibers if the general fiber is not connected. However, this problem can be rectified by using the Stein factorization theorem. Further on, we shall assume that all holomorphic Lagrangian fibrations we consider have connected fibers.

In that case (as far as we know) there are no examples of Lagrangian fibrations with multiple fibers on compact, maximal holonomy hyperkähler manifolds. We conjecture that multiple fibers in this situation don’t occur.

For K3 surface non-existence of multiple fibers is implied by Kodaira’s classification of singular elliptic fibers (it follows directly from the canonical bundle formula, [BHPV, Thm 12.1]). We give a direct proof of this fact in Section 4.

Multiple fibers of Lagrangian fibrations were classified in [HO] and [Mat3]. In particular, it is shown that the multiplicity of a general fiber is at most 6 ([HO, Theorem 1.1]).

It is clear from our construction that for any smooth section \( S \) of a Lagrangian fibration \( \pi \), there exists a unique degenerate twistorial deformation of \( \pi \) such that \( S \) is holomorphic. We don’t know the number of degenerate twistorial deformations of \( \pi \) admitting a holomorphic section; it is unknown even whether this number is finite or infinite.

2 Complex structures via symplectic forms

2.1 C-symplectic structures

Definition 2.1: Let \( V \) be a real vector space equipped with an operator
$I \in \text{End}(V)$, $I^2 = -\text{Id}_V$. We say that $I$ is a \textbf{complex structure operator} on $V$, and $V$ is a \textbf{complex vector space}. The operator $I$ can be understood as imaginary unit $\sqrt{-1} \in \mathbb{C}$ acting on $V$. Complex-linear maps of complex vector spaces are the same as maps which commute with the complex structure.

Suppose $W$ is a complex vector space and $\Omega \in \Lambda^2 W^*$ a complex symplectic form. Let us forget about the complex structure on $W$, and consider its underlying real vector space $W_\mathbb{R}$ with a 2-tensor $\Omega \in \Lambda^2 W^* \otimes \mathbb{C}$ with complex coefficients. Knowing just this tensor, one can uniquely reconstruct the complex structure operator $I_\Omega : W_\mathbb{R} \to W_\mathbb{R}$. In the present section we determine which forms can occur as complex symplectic forms for some complex structure on $W_\mathbb{R}$.

**Definition 2.2:** Let $V$ be an $4n$-dimensional real vector space. A 2-tensor $\Omega \in \Lambda^2 V^* \otimes \mathbb{C}$ is called a \textbf{c-symplectic form}, if for any nonzero vector $v \in V$ one has $\iota_v \Omega \neq 0 \in V^* \otimes \mathbb{C}$, and $\ker \Omega \subset V \otimes \mathbb{C}$ has rank $2n$. A pair $(V, \Omega)$ in such situation is called a \textbf{c-symplectic vector space}.

**Remark 2.3:** The kernel rank $\ker \Omega = 2n$ is maximal possible for a form which does not vanish on real vectors. Indeed, if $\dim_{\mathbb{C}} \ker \Omega > 2n$, equivalently, $\dim_{\mathbb{R}} \ker \Omega > 4n$ this space would intersect $V \subset V \otimes \mathbb{C}$ of real dimension $4n$, and this is impossible.

**Definition 2.4:** Let $(V, \Omega)$ be a c-symplectic vector space. A complex structure operator $I : V \to V$ is called an \textbf{induced} by the c-symplectic form $\Omega$, if it makes $\Omega$ a complex linear symplectic form in $\Lambda^2_{\mathbb{C}} V^*$, where the complex exterior power is taken w. r. t. the complex structure $I$.

**Proposition 2.5:** For any c-symplectic vector space $(V, \Omega)$ there exists a unique induced complex structure $I$ on $V$, that is, a complex structure operator $I \in \text{End}_{\mathbb{R}}(V)$ such that $\Omega$ is a non-degenerate form of Hodge type (2,0)

**Proof:** Since $\ker \Omega \subset V \otimes \mathbb{C}$ has maximal possible dimension, any real vector $v \in V$ can be represented as $v = v^{1,0} + v^{0,1}$, where $v^{0,1} \in \ker \Omega$ and $v^{1,0} \in \ker \Omega$. Moreover, this representation is unique, since $\ker \Omega$ contains no real vectors and hence does not intersect its complex conjugate subspace. One can define an operator $I$ as multiplication by $-\sqrt{-1}$ on $\ker \Omega$ and by $\sqrt{-1}$ on $\overline{\ker \Omega}$. Since it is self-conjugate, it is defined...
over reals, it is well-defined because of the existence and uniqueness of the above decomposition, and it obviously squares to $-\text{Id}_V$. Moreover, one has $\Omega(Iu,v) = \Omega(\sqrt{-1}u^{1,0} - \sqrt{-1}u^{0,1}, v) = \sqrt{-1}\Omega(u^{1,0}, v) = \sqrt{-1}\Omega(u, v)$. Therefore $I$ is an induced complex structure; in particular, at least one induced complex structure exists.

On the other hand, any induced complex structure must have $\ker\Omega$ as its $-\sqrt{-1}$-eigenspace, and this determines a complex structure operator in a unique way. ■

We obtain that any c-symplectic vector space is of nature prescribed above: it is the underlying real space of some complex symplectic space. In what follows, we denote the induced complex structure of a c-symplectic vector space $(V, \Omega)$ by $I_\Omega$.

**Definition 2.6:** Let $(V, \Omega)$ be a c-symplectic vector space. A real subspace $U \subset V$ is called c-**isotropic** if for any $u, u' \in U$ one has $\Omega(u, u') = 0$, and c-**Lagrangian** if it is c-isotropic and is not contained in any proper c-isotropic supersubspace.

**Proposition 2.7:** ([HI Proposition 1]) Any c-Lagrangian subspace of a c-symplectic vector space is preserved by the induced complex structure.

**Proof:** Let $L \subset V$ be a c-Lagrangian subspace, and $u \in L$ be any vector. Consider the linear span of the space $L$ and the vector $I_\Omega u$, denote it by $L_u$. What is the restriction of the form $\Omega$ on this span? By definition of a c-isotropic subspace, one has $\Omega(u, v) = 0$ for any $v \in L$, so it is completely determined by the 1-form $(I_\Omega u \Omega) |_{L_u}$. Moreover, since the form $\Omega$ has Hodge type $(2, 0)$ w. r. t. $I_\Omega$, one has $0 = \sqrt{-1} \cdot 0 = \sqrt{-1}\Omega(u, v) = \Omega(I_\Omega u, v)$, and since $\Omega$ is skew-symmetric, one has $\Omega(I_\Omega u, I_\Omega u) = 0$. Therefore the form $\Omega$ vanishes identically on the subspace $L_u$, i. e. $L_u$ is c-isotropic, and since $L$ is contained in no proper c-isotropic supersubspace, it must be equal to $L$. This implies that $I_\Omega u \in L$, and since $u$ is arbitrary, it means that the operator $I_\Omega$ maps the subspace $L$ to itself. ■

In particular, quotient by a c-Lagrangian vector subspace inherits the complex structure. Note that this complex structure on the quotient is not in general determined by any c-symplectic form: indeed, the quotient may have odd complex dimension. In what follows, we shall refer to the complex structures on quotients of c-symplectic vector spaces by their c-Lagrangian
subspaces which makes the projection map complex linear as to \textit{inherited} ones.

**Proposition 2.8:** (existence of a c-symplectic basis). Let $V$ be a c-symplectic space of real dimension $4n$. Then $V$ possesses a basis in which its c-symplectic form is given by a block diagonal matrix with equal blocks $Q$ on the diagonal, where $Q$ stands for a $4 \times 4$-block given by

$$Q = \begin{pmatrix} 0 & 0 & 1 & \sqrt{-1} \\ 0 & 0 & \sqrt{-1} & -1 \\ -1 & -\sqrt{-1} & 0 & 0 \\ -\sqrt{-1} & 1 & 0 & 0 \end{pmatrix}.$$  

**Proof:** The Proposition is proved by running an analogue of the symplectic Gram–Schmidt process. Indeed, $I_\Omega$ be the induced complex structure, and $z_1, \ldots, z_{2n} \in V$ a basis in $(V^*, I)$, considered as a complex vector space, such that $\Omega = \sum_{i=1}^{n} z_{2i-1} \wedge z_{2i}$. Then $z_1, I_\Omega(z_1), \ldots, z_{2n}, I_\Omega(z_{2i})$ is a real basis in $V$ such that $\Omega$ is written as the above block matrix. For the convenience of the reader, we give a direct argument constructing such a basis explicitly.

For the first two vectors in the basis we may choose an arbitrary nonzero vector $u_1$ and its image under the induced complex structure, $I_\Omega u_1$. Orthogonals of these vectors (i. e. the kernels of the forms $\iota_{u_1} \Omega$ and $\iota_{I_\Omega u_1} \Omega$) coincide by the definition of the induced complex structure. If one picks a vector $u_2$ outside this orthogonal, then, by outstretching it and adding to it a multiple of the vector $I_\Omega u_2$ in case of necessity, we can make $u_2$ such that

$$\Omega(u_1, u_2) = 1.$$  

One can deduce from complex linearity of $\Omega$ w. r. t. $I_\Omega$ and the above relation that the restriction of the form $\Omega$ onto the four-dimensional subspace $U \subset V$ spanned by $\{u_1, I_\Omega u_1, u_2, I_\Omega u_2\}$ is given by the matrix $Q$ in these coordinates.

Now we can proceed, exercising the same procedure in the orthogonal to $U$ (i. e. the set $U^\perp$ of vectors $w \in V$ s. t. $\langle \iota_w \Omega \rangle |_U = 0$), because $U \cap U^\perp = \{0\}$.  

**Claim 2.9:** Let $V$ be a real vector space of real dimension $4n$, and $\Omega \in \Lambda^2 V^* \otimes \mathbb{C}$ be a complex-valued skew-symmetric 2-form. The following are equivalent:

1. $\dim_{\mathbb{C}} \ker \Omega |_{V \otimes \mathbb{C}} = 2n,$
2. $(\Omega \wedge \overline{\Omega})^n$ is nonzero and $\Omega^{n+1} = 0$.

**Proof:** Suppose that $\dim \ker \Omega = 2n$. Then for any decomposable polyvector $a \in \Lambda^{2n+2}V \otimes \mathbb{C}$ the corresponding subspace in $V$ needs to intersect $\ker \Omega$, so $i_a \Omega^{n+1}$ vanishes. Hence $\Omega^{n+1} = 0$. An easy direct calculation shows that, in notation of the Proposition 2.8, $(Q \wedge Q)\sigma(u_1, v_1, u_2, v_2) = 4$, so the top power of $\Omega \wedge \overline{\Omega}$ cannot be zero.

Suppose that $\Omega^{n+1} = 0$. If the top power of a skew-symmetric form on some vector space is zero, then this form has a nontrivial kernel. That’s why the form $\Omega$ has nontrivial kernel when restricted to any $(2n+2)$-plane inside $V \otimes \mathbb{C}$ (and, moreover, any $(2n + 2k)$-plane for any $k > 0$). Restriction of a symplectic form onto a maximal subspace transversal to its kernel is non-degenerate, so $\dim \ker \Omega \geq 2n$. If $(\Omega \wedge \overline{\Omega})^n \neq 0$, it cannot be greater, because in this case $\ker \Omega$ needs to intersect $\ker \overline{\Omega}$, thus giving a real vector in $\ker \Omega$, substitution of which would vanish $(\Omega \wedge \overline{\Omega})^n$.

### 2.2 Sections of Lagrangian projections of c-symplectic vector spaces

In this subsection, we will deal with the following situation. Let $(V, \Omega)$ be a $4n$-dimensional real vector space space equipped with a c-symplectic structure, and $L \subset V$ a c-Lagrangian subspace. The complex linear surjective map $V \twoheadrightarrow V/L$ is called a c-Lagrangian projection. This is a linearization of the Lagrangian fibrations which are common in holomorphic symplectic geometry (Subsection 1.2).

**Proposition 2.10:** Let $V$ be a real vector space, $\Omega$ a c-symplectic form on it, $L$ a c-Lagrangian subspace and $\sigma: V/L \rightarrow V$ a real section (not necessarily complex linear w. r. t. the induced complex structures). Define the form $\Omega_\sigma \in \Lambda^2(V/L)^* \otimes \mathbb{C}$ by the rule $\Omega_\sigma(u_1, u_2) = \Omega(\sigma(u_1), \sigma(u_2))$. In other words, $\Omega_\sigma$ is a restriction of $\Omega$ onto the subspace $\sigma(V/L)$ after the identification $\pi|_{\sigma(V/L)}: \sigma(V/L) \rightarrow V/L$. Then $\Omega_\sigma$ has Hodge type $(2, 0) + (1, 1)$ w. r. t. the inherited complex structure on the quotient $V/L$.

**Proof:** Suppose that $\sigma = \sigma_0$ is complex linear. Then the form $\Omega_\sigma$ is of type $(2, 0)$, since the type is preserved by complex linear maps.

Now let $\sigma = \sigma_0 + \tau$, where $\sigma_0$ is a complex linear section and $\tau: V/L \rightarrow L$
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a real perturbation. For any vectors \( u, v \in (V/L) \),

\[
\Omega_\sigma(u, v) = \Omega(\sigma(u), \sigma(v)) = \\
= \Omega(\sigma_0(u), \sigma_0(v)) + \Omega(\sigma_0(u), \tau(v)) + \\
+ \Omega(\tau(u), \sigma_0(v)) + \Omega(\tau(u), \tau(v)). \tag{2.1}
\]

Since one has \( \tau(u), \tau(v) \in L \) and \( L \) is a c-Lagrangian subspace w. r. t. \( \Omega \), the last term vanishes. The first term is a \((2, 0)\) form, since \( \sigma_0 \) is a complex linear section. The term \( \Omega(\sigma_0(u), \tau(v)) + \Omega(\tau(u), \sigma_0(v)) \) vanishes if \( u, v \) are both of type \((0, 1)\), since \( \sigma_0(u) \) and \( \sigma_0(v) \) are vectors of type \((0, 1)\) in this case, and therefore annihilate the form \( \Omega \). Therefore, all terms of (2.1) are of type \((2, 0) + (1, 1)\).

Fiberwise application of the above construction allows one to obtain an almost complex structure on a \( 4n \)-dimensional manifold \( X \) from a non-degenerate complex-valued 2-form \( \Omega \) on it such that \((\Omega \wedge \Omega)^n\) is nowhere zero and \( \Omega^{n+1} = 0 \); by [Claim 2.9] this condition gives a c-symplectic form on the tangent bundle.

**Definition 2.11:** An almost c-symplectic form on a manifold \( X \) of real dimension \( 4n \) is a form \( \Omega \in \Gamma(\Lambda^2 T^*X \otimes \mathbb{C}) \) such that the top degree form \((\Omega \wedge \Omega)^n\) is nowhere zero and \( \Omega^{n+1} = 0 \). A c-symplectic form is a closed almost c-symplectic form.

We shall denote the induced almost complex structure obtained from the form \( \Omega \) as the induced complex structure on each fiber by \( I_\Omega \).

**Proposition 2.12:** [V2], Theorem 3.5
Let \( \Omega \) be a c-symplectic form on a manifold, and \( I_\Omega \in \text{End}(TM) \) the corresponding almost complex structure. Then \( I_\Omega \) is integrable.

**Proof:** From Cartan’s formula it follows immediately that for any closed \( k \)-form \( \Phi \) and two vectors \( X, Y \) such that the contractions of \( \Phi \) with \( X, Y \) vanish, \( \Phi \lrcorner X = \Phi \lrcorner Y = 0 \), one also has \( \Phi \lrcorner ([X, Y]) = 0 \). However, the space \( T^{0,1}_{I_\Omega} M \subset TM \otimes \mathbb{C} \) is defined as

\[
T^{0,1}_{I_\Omega} M = \{X \in TM \otimes \mathbb{C} \mid \Omega \lrcorner X = 0\},
\]

which gives \([T^{0,1}_{I_\Omega}, T^{0,1}_{I_\Omega}] \subset T^{0,1}_{I_\Omega} M\). By Newlander-Nirenberg, this condition is equivalent to integrability of \( I_\Omega \).
Note that converse is not generally true: if \( \Omega \) is a c-symplectic form and \( f \) is a non-vanishing function, then \( f\Omega \) is not closed unless \( f \) is constant; however, \( I_{f\Omega} = I_\Omega \).

### 3 Degenerate twistorial deformation

Though this section is self-contained, its results generalise and simplify the results of [V2].

**Proposition 3.1:** Let \( \Omega \in \Lambda^2 V^* \otimes \mathbb{C} \) be a c-symplectic form on a real vector space \( V \), \( L \) a c-Lagrangian subspace in \( V \) and \( \pi: V \to V/L \) the projection. Then for any complex valued form \( \gamma \in \Lambda^2 (V/L)^* \otimes \mathbb{C} \) of Hodge type \((2,0) + (1,1)\) w. r. t. the inherited complex structure on \( V/L \), the form \( \Omega_\gamma = \Omega + \pi^* \gamma \) is c-symplectic.

**Proof:** We shall proceed in two steps: first, we prove that \( \dim \ker \Omega_\gamma \geq \frac{1}{2} \dim V \). Clearly, this would follow if we prove that \( \Omega^{n+1} = 0 \), where \( 4n = \dim \mathbb{R} V \). Let \( V_\mathbb{C} := V \otimes \mathbb{C} \) be the complexification of \( V \), \( L \oplus K = V \) a direct sum decomposition, with \( K, L \subset V \) complex vector spaces, and \( L_\mathbb{C}, K_\mathbb{C} \) complexifications of these spaces. Denote by \( L_\mathbb{C} = L_{1,0}^C \oplus L_{0,1}^C \) and \( K_\mathbb{C} = K_{1,0}^C \oplus K_{0,1}^C \) their Hodge decompositions. Consider a linear automorphism \( T_\lambda: V_\mathbb{C} \to V_\mathbb{C} \) acting as multiplication by a scalar \( \lambda \in \mathbb{C} \) on \( K_{1,0}^C \) and as identity on \( L_\mathbb{C} \) and \( K_{0,1}^C \). Since \( \Omega \) is a pairing between \( K_{1,0}^C \) and \( L_{1,0}^C \), one has \( T_\lambda(\Omega) = \lambda \Omega \). Since \( \pi^* \gamma \) vanishes on \( L \), one has \( T_\lambda(\pi^* \gamma) = \lambda^2 (\pi^* \gamma)^{2,0} + \lambda (\pi^* \gamma)^{1,1} \). Using this, we write the weight decomposition for the action of \( T_\lambda \) on \( \Omega^{n+1}_\gamma \) as follows:

\[
T_\lambda(\Omega^{n+1}_\gamma) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \lambda^{n+1+j} \Omega^i \wedge (\gamma^{2,0})^j \wedge (\gamma^{1,1})^{n-i-j+1}
\]

However, \( T_\lambda \) cannot act on \( 2n + 2 \)-forms with weight \( \geq n + 1 \) because \( \dim \mathbb{C} L_{1,0}^C = n \). Therefore, \( \Omega^{n+1}_\gamma = 0 \).

**Assertion 2:** \( \dim \ker \Omega_\gamma \leq \frac{1}{2} \dim V \), or, equivalently, \( \text{rk} \Omega_\gamma \geq \text{rk} \Omega \). Clearly, the point \( \Omega \in \Lambda^2 (V_\mathbb{C}) \) has a neighbourhood \( U \) such that all 2-forms \( y \in U \) have rank \( \geq \text{rk} \Omega \). Assertion 2 is trivial when \( \Omega_\gamma \in U \). Consider, as
in the previous step, the decomposition
\[ V_C = L_C^{1,0} \oplus L_C^{0,1} \oplus K_C^{1,0} \oplus K_C^{0,1}, \]
and let \( R_\lambda \in \text{Aut}(V_C) \) act on \( K_C^{0,1} \) and \( L_C^{0,1} \) as identity, on \( L_C^{1,0} \) as multiplication by \( \lambda^{-1} \) and on \( K_C^{1,0} \) as multiplication by \( \lambda \). Since \( \Omega \) is a pairing between \( L_C^{1,0} \) and \( K_C^{1,0} \), one has \( R_\lambda(\Omega) = \Omega \). Since \( \gamma \) vanishes on \( L \), one has \( R_\lambda(\gamma) = 2\lambda\gamma + \lambda \gamma \).

Then \( \lim_{\lambda \to 0} R_\lambda(\Omega \gamma) = \Omega \), hence for \( \lambda \) sufficiently small, the form \( R_\lambda(\Omega \gamma) \) belongs to \( U \), and satisfies \( \text{rk} \Omega \gamma \geq \text{rk} \Omega \). We proved Proposition 3.1.

**Proposition 3.2:** Let \( \Omega \in \Lambda^2 V^* \otimes \mathbb{C} \) be a c-symplectic form on a real vector space \( V \), \( L \) a c-Lagrangian subspace in \( V \) and \( \pi: V \to V/L \) the projection. Consider a \((2,0)+(1,1)\)-form \( \gamma \) on \( V/L \), and let \( \Omega_t := \Omega + t\pi^* \gamma \), for some \( t \in \mathbb{C} \). Then the subspace \( L \) is c-Lagrangian w. r. t. all the forms \( \Omega_t \), and restrictions \( \left. I_{\Omega_t} \right|_L \) of induced complex structures \( I_{\Omega_t} \) coincide. Moreover, the complex structures on the quotient \( V/L \) inherited from \( I_{\Omega_t} \) also coincide, for all \( t \in \mathbb{C} \).

**Proof:** The first claim is obvious from the construction of \( \Omega_t \). The prove the second claim, take \( v \in (V/L)^{0,1}_{t_0} \), and let \( u \in \pi^{-1}(v) \) be a vector in its preimage. Then \( \Omega_t \.Imp u = t\pi^* (\gamma \Imp v) \). Therefore,
\[ u \in V^{0,1}_{t_0} \iff \forall z \in V, \quad \Omega(u, z) = -t\pi^* (\gamma \Imp v)(z) \tag{3.1} \]
The 1-form \( z \mapsto t\pi^* (\gamma \Imp v)(z) \) vanishes on \( L \) and has type \((1,0)\) on \((V, I_{\Omega})\), because \( \gamma \) is of type \((2,0)+(1,1)\).

Since \( \Omega \) is a non-degenerate pairing between \( L^{0,1}_{t_0} \) and \((V/L)^{0,1}_{t_0} \), for any \((1,0)\)-form \( \xi \) on \( V/L \) there exists a vector \( x \in L \) such that \( \xi = \Omega \Imp x \).

This gives a vector \( \zeta_t \in L^{0,1}_{t_0} \) such that such that
\[ \Omega(u + \zeta_t, z) = -t\pi^* (\gamma \Imp v)(z) \]
for all \( z \in V \). By (3.1), \( u + \zeta_t \in V^{0,1}_{t_0} \) is a vector which projects to \( v \).

This implies that the space of \((0,1)\)-vectors in \((V/L)^{0,1}_{t_0} \) is independent from \( t \). ■

**Theorem 3.3:** Suppose that \( X \xrightarrow{\pi} B \) is a Lagrangian fibration on a holomorphically symplectic manifold \((X, \Omega)\), and \( \eta \in \Omega_\cl(B) \) a closed \((2,0)+(1,1)\)-form on the base. Then the forms \( \Omega_t = \Omega + t\pi^* \eta \) on \( X \) are...
c-symplectic, and this deformation (called degenerate twistorial deformation) preserves the Lagrangian fibration and the base.

**Proof:** From Proposition 3.1 it follows that \( \Omega_t \) is c-symplectic. The fibers stay Lagrangian, and the complex structure thereof, as well as that of the base, remains unchanged by Proposition 3.2.

**Remark 3.4:** This result was proven for compact manifolds in \([V2, \text{Theorem 1.10}]\).

For a hyperkähler manifold \( X \), the degenerate twistorial deformation produces an entire curve in the period space of \( X \). In terms of the oriented 2-plane Grassmannian (see e. g. Section 3 in \([De]\)), the plane corresponding to \((X, \Omega_x + \sqrt{-1}y)\) is spanned by \((\Omega + \Omega) + 2x\eta\) and \(\sqrt{-1}(\Omega - \Omega) - 2y\eta\). Thus one can define degenerate twistorial curves in an abstract situation: namely, for a 2-plane \( \tau \in \text{Gr}^+ (V, q) \) and a vector \( e \in \tau^\perp \subset V \) with \( q(e, e) = 0 \), the subvariety \( \text{Deg}_\tau(e) = \text{Gr}^+ (\text{span}(\tau, e), q|_{\text{span}(\tau, e)}) \subseteq \text{Gr}^+ (V, q) \) is an entire curve, and in the case when \( V = H^2(X, \mathbb{R}) \), \( q \) is the Bogomolov–Beauville–Fujiki form and \( e = \pi^*[\omega] \) is the inverse image of the Kähler class on the base, this curve is exactly the base of the degenerate twistorial deformation.

**Theorem 3.5:** Let \( X \xrightarrow{\pi} B \) be a holomorphic Lagrangian fibration on a holomorphically symplectic manifold \((X, \Omega)\), and \( \sigma : B \to X \) a smooth section of \( \pi \). Then there exists a degenerate twistorial deformation \((X, \Omega')\) of \((X, \Omega)\) s. t. the fibers stay Lagrangian, the complex structure on the fibers and the base stays the same, and \( \sigma \) is a holomorphic map.

**Proof:** Consider the form \( \eta = \sigma^* \Omega \in \Omega^2(B) \). By Proposition 2.10 it is of type \((2, 0) + (1, 1)\). Then by Proposition 3.3 the forms \( \eta \) gives rise to a deformation with desired properties. For \( t = -1 \) one has \( \Omega_t|_{\sigma(B)} = (\Omega - \pi^* \sigma^* \Omega)|_{\sigma(B)} = \Omega|_{\sigma(B)} - \Omega|_{\sigma(B)} = 0 \). By Hitchin’s lemma (Proposition 2.7), this means that the submanifold \( \sigma(B) \) is a complex submanifold. Since the projection is a holomorphic map, the section \( \sigma \) is also holomorphic.
4 Holomorphic Lagrangian fibrations on a K3 surface

In this section we apply Theorem 3.5 to obtain sections of holomorphic Lagrangian fibrations on K3 surfaces.

It is not hard to see that the elliptic fibrations on a K3 surface $X$ are in bijective correspondence with primitive numerically effective classes $e \in \text{NS}(X)$ with $(e, e) = 0$. Indeed, its linear system has no basepoints and establishes a map to $\mathbb{CP}^1$ with generic fiber elliptic curve. Cross-sections of the fibration determined by $e$ correspond to classes $s \in \text{NS}(X)$ with $(s, e) = 1$. Our first goal is to find an effective class $s \in H^2(X, \mathbb{Z})$ with $(s, e) = 1$ and a deformation in which $s$ would have type $(1,1)$.

**Claim 4.1:** For any isotropic primitive vector $e$ in an even unimodular lattice $\Lambda$ there exists a vector $a \in \Lambda$ such that one has $(a, e) = 1$ and $(a, a) = -2$.

**Proof:** Since the lattice $\Lambda$ is unimodular, one can pick some vector $b \in \Lambda$ with $(b, e) = 1$. The number $(b, b)$ is even since $\Lambda$ is even, so the vector $a = b - (1 - (b, b)/2) e$ is integral. One has $(a, e) = (b, e) - (1 + (b, b)/2)(e, e) = 1$ and $(a, a) = (b - (1 + (b, b)/2)e, b - (1 + (b, b)/2)e) = (b, b) - 2(1 + (b, b)/2)(b, e) = (b, b) - 2 - (b, b) = -2$. ■

The following claim is trivial.

**Claim 4.2:** Suppose that $X$ is a K3 surface with an elliptic fibration determined by an isotropic class $e$. Then the pullback of the Fubini–Study form from the base represents $e$.

**Proof:** The Fubini–Study form on the base represents a Poincaré dual to the class of a point. Then its inverse image represents the Poincaré dual to the class of the inverse image of the point, i.e. of the fiber. ■

**Lemma 4.3:** Suppose that $X$ is a K3 surface with elliptic fibration $X \to \mathbb{CP}^1$ determined by an isotropic class $e$, $\omega_{\text{FS}}$ the Fubini–Study form on its base, and $s \in H^2(X, \mathbb{Z})$ is such that one has $(s, e) \neq 0$. Then there exists a unique degenerate twistorial deformation $X'$ of $(X, \pi)$ in such way that $s$ belongs to $H^{1,1}(X')$. 

\[ -13 - \]
Proof: The forms $\Omega - t\pi^*\omega_{FS}$, $t \in \mathbb{C}$, are $c$-symplectic by Proposition 3.3. Denote by $I_t$ the complex structure induced by $\Omega - t\pi^*\omega_{FS}$ (Proposition 2.5). The homology class of $s$ is of type $(1, 1)$ if and only if $(s, [\Omega - t\pi^*\omega_{FS}]) = 0$. Using Claim 4.2 one can rewrite this equation as $(s, [\Omega]) = t(s, e)$. It has a unique solution $t$ whenever $(s, e) \neq 0$.

**Proposition 4.4:** Let $X$ be a K3 surface with elliptic fibration $X \to \mathbb{C}P^1$. Then there exists a degenerate twistorial deformation $X'$ of $(X, \pi)$ admitting a holomorphic section.

**Proof:** Since the lattice $H^2(X, \mathbb{Z})$ for a K3 surface $X$ is even and unimodular, by Claim 4.1 one can find a vector $s \in H^2(X, \mathbb{Z})$ such that $(s, e) = 1$ and $(s, s) = -2$. By Lemma 4.3 we can deform the complex structure on $X$ in such way that $s$ would have type $(1, 1)$. By Lefschetz theorem on $(1, 1)$-classes, there exists a line bundle $L \to X$ with $c_1(L) = s$. Riemann–Roch theorem for this bundle $L$ reads $\chi(L) = \chi(\mathcal{O}_X) + \frac{L \cdot (L \otimes K_X)}{2}$. By Serre’s duality one has $h^2(L) = h^0(L^* \otimes K_X)$, and, as soon as $K_X$ is trivial and $\chi(\mathcal{O}_X) = 2$, we can rewrite it as $h^0(L) - h^1(L) + h^0(L^*) = 2 + \frac{-2}{2} = 1$. Since $L$ is nontrivial, either $h^0(L)$ or $h^0(L^*)$ does not vanish, and either $L$ or $L^*$ is effective. If $L^*$ is, then $-s$ is represented by a curve, and $0 < (-s, e) = -(s, e) = -1$, hence $L$ is effective and $s$ is represented by a curve. Suppose that $s = \sum_i s_i$, where $s_i$ are the classes of irreducible curves. One has $1 = (s, e) = \sum_i (s_i, e) = \sum_i (s_i, e)$. All the numbers $(s_i, e)$ are positive integers with sum $1$, so exactly one of them, say $(s_0, e)$, equals $1$. The curve represented by the class $s_0$ intersects each fiber at one point, i. e. is a section of the fibration.

In particular, this implies that elliptic fibrations on K3 surfaces cannot have multiple fibers.

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