Electromagnetic Wavelets as Hertzian Pulsed Beams in Complex Spacetime

To Professor Jerzy Plebański on his 75th birthday

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Abstract

Electromagnetic wavelets are a family of $3 \times 3$ matrix fields $W_z(x')$ parameterized by complex spacetime points $z = x + iy$ with $y$ timelike. They are translates of a basic wavelet $W(z)$ holomorphic in the future-oriented union $\mathcal{T}$ of the forward and backward tubes. Applied to a polarization vector $p = p_m - ip_e$, $W(z)$ gives an anti-selfdual solution $W(z)p$ derived from a selfdual Hertz potential $\tilde{Z}(z) = -iS(z)p$, where $S$ is the Synge function acting as a Whittaker-like scalar Hertz potential. Resolutions of unity exist giving representations of sourceless electromagnetic fields as superpositions of wavelets. With the choice of a branch cut, $S$ splits into a difference $S^+(z) - S^-(z)$ of retarded and advanced pulsed beams whose limits as $y \to 0$ give the propagators of the wave equation. This yields a similar splitting of the wavelets and leads to their complete physical interpretation as pulsed beams absorbed and emitted by a disk source $D(y)$ representing the branch cut. The choice of $y$ determines the beam’s orientation, collimation and duration, giving beams as sharp as pulses as short as desired. The sources are computed as spacetime distributions of electric and magnetic dipoles supported on $D(y)$. The wavelet representation of sourceless electromagnetic fields now splits into representations with advanced and retarded sources. These representations are the electromagnetic counterpart of relativistic coherent-state representations previously derived for massive Klein-Gordon and Dirac particles.

1 Hertz Potentials

Acoustic and electromagnetic wavelets were introduced in [K92, K94, K94a] as generalized frames of localized solutions of the homogeneous wave equation and Maxwell’s equations, giving wavelet-like representations of general sourceless solutions. They were defined in Fourier space, and explicit space-time expressions were found for the acoustic but not the
electromagnetic wavelets. Here we compute the electromagnetic wavelets in spacetime and show that they have a simple interpretation as pulsed beams emitted and absorbed by localized electric and magnetic dipole distributions. By separating their advanced and retarded parts, the wavelet representation of electromagnetic fields is generalized to include localized sources, interpreted as polarization distributions generating Whittaker-like Hertz potentials.

We begin by reviewing the derivation of electromagnetic fields by Hertz potentials. Our presentation is essentially a translation of [N55] to the language of differential forms [AMT88, T96]. An electromagnetic field is represented by two 2-forms \( F, G \) satisfying Maxwell’s equations,

\[
\begin{align*}
    dF &= 0 \quad (F = E \cdot dxdt + B \cdot s) \quad (1) \\
    dG &= J \quad (G = D \cdot s - H \cdot dxdt). \quad (2)
\end{align*}
\]

Here \( J \) is the current 3-form, products of forms are wedge products, \( dx = (dx^1, dx^2, dx^3) \), and \( s \) is the vector-valued spatial area form \( s = \ast dxdt = (dx^2dx^3, dx^3dx^1, dx^1dx^2) \), where \( \ast \) is the Hodge duality operator in Minkowski space, defined [R] so that \( \ast \ast = -1 \).

In addition to (1) and (2), we need constitutive relations, given here in terms of the electric and magnetic polarization densities \( P_e, P_m \) of the medium:

\[
\ast F + G = P \quad (P = P_e \cdot s + P_m \cdot dxdt), \quad (3)
\]

where \( \ast F = -E \cdot s + B \cdot dxdt \) is the Hodge dual of \( F \). Without specifying initial and boundary conditions, the best we can hope for is general (or local) solutions up to an arbitrary sourceless field. We now show to derive these using Hertz potentials.

Since \( F \) is closed, it can be derived from a potential 1-form \( A \) subject to gauge transformations:

\[
F = dA = dA', \quad A = A_\mu dx^\mu, \quad A' = A - d\chi. \quad (4)
\]

Applying the co-differential operator \( \delta = \ast d \ast \) gives

\[
\delta A' = \delta A - \delta d\chi = \delta A - \Box \chi, \quad (5)
\]

where \( \Box = d\delta + \delta d \) is the (Hodge) d’Alembertian operator on differential forms, reducing to the wave operator on functions. Locally at least, \( \chi \) can be chosen to satisfy \( \Box \chi = \delta A \), so that \( \delta A' = 0 \) and \( A' \) is in the Lorenz gauge.\(^1\) By the Poincaré lemma, \( \ast A' \) can be derived from a potential 2-form:

\[
d \ast A' = 0 \Rightarrow \ast A' = -dZ \quad (Z = Z_e \cdot s + Z_m \cdot dxdt). \quad (6)
\]

This shows that the original potential, which need not be in Lorenz gauge, can be written locally as

\[
A = \ast dZ + d\chi, \quad F = d \ast dZ. \quad (6)
\]

\(^1\)Due to L.V. Lorenz, not H.A. Lorentz; see Penrose and Rindler [PR84].
$Z$ is called a Hertz potential with electric and magnetic vectors $Z_e, Z_m$. The relation between the components of $F$ and $Z$ can be expressed compactly in terms of the anti-selfdual form $F^- = i \ast F^-$ and the selfdual form $Z^+ = -i \ast Z^+$, defined by

$$
2F^- = F + i \ast F = 2F \cdot \omega \quad 2F = E + iB \quad \omega = dxdt - is \\
2Z^+ = Z - i \ast Z = 2Z \cdot \bar{\omega} \quad 2Z = Z_m - iZ_e \quad \bar{\omega} = dxdt + is.
$$

(7)

Namely,

$$
2F^- = d \ast dZ + i \delta dZ = d \ast dZ - i d \delta Z + i \square Z \\
= d \ast dZ - i d \ast dZ + i \square Z = 2d \ast dZ^+ + i \square Z,
$$

which translates to the complex vector equation

$$
2F = 2iLZ - \square Z_e
$$

(8)

where $L$ is the operator

$$
L Z \equiv \nabla \times (\nabla \times Z) + i \partial_t \nabla \times Z.
$$

To obtain a wave equation for $Z$ in terms of the sources, we need a stream potential [N55] for $J$, i.e., a 2-form $G^*$ such that

$$
J = -dG^* \quad (G^* \equiv - D^* \cdot s + H^* \cdot dxdt).
$$

The resemblance $G^* \sim -G$ is intentional, since one possibility is $G^* = -G$. But this would be circular since the field is unknown. A simple stream potential $G_0^*$ can be defined as a time integral of the current. Then $G$ differs from $-G'_0$ at most by a homogeneous solution of (2), i.e., an arbitrary exact 2-form:

$$
G = -G_0^* + d\alpha, \quad \alpha \in \Lambda^1.
$$

(9)

The constitutive relation (3) implies

$$
P = *F + G = \delta dZ - G_0^* + d\alpha = \square Z - d\delta Z - G_0^* + d\alpha,
$$

or

$$
\square Z = P + G^*
$$

(10)

where

$$
G^* = G_0^* - d\beta, \quad \beta = \alpha - \delta Z
$$

is another stream potential. The dependence of $\beta$ on $Z$ is not a problem since $\beta$, like $\alpha$, is arbitrary. The selfdual representation of (10) is

$$
\square Z = P + G^*, \quad 2P = P_m - iP_e, \quad 2G^* = H^* + iD^*.
$$

(11)

Thus $Z$ is produced by a unified source composed of material polarization and stream potential; the latter may be viewed as an effective classical vacuum polarization induced by the current. If the medium is polarizable, magnetizable or conducting, these sources in turn depend on the field. In that case it is better to use the common multiplicative form of the constitutive relations for the induced sources (assuming linearity), which gives a modified version of (10). For a thorough study of Hertz potentials and their gauge theory, see [N55, N57].
2 Hertz potentials in Fourier space

Initially, the construction of electromagnetic wavelets will be based on holomorphic fields obtained by extending sourceless anti-selfdual solutions to complex spacetime. Eventually, sources will be introduced that preserve the holomorphy of the fields locally, outside the sources. But for now, we specialize to a homogeneous field in vacuum,

\[ J = P = 0 \Rightarrow dF^- = 0 \Rightarrow i\partial_t F = \nabla \times F, \quad \nabla \cdot F = 0. \]  

(12)

Solutions are given in terms of selfdual Hertz potentials by (8):

\[ \square Z(x) = 0, \quad F(x) = i\mathcal{L}Z(x). \]  

(13)

The Fourier solution of the wave equation is

\[ Z(x) = \int_C d\tilde{k} \ e^{ikx} \hat{Z}(k) = \int_{C_+} d\tilde{k} \ e^{ikx} \hat{Z}(k) + \int_{C_-} d\tilde{k} \ e^{ikx} \hat{Z}(k), \]  

(14)

where \( x = (x, t), \ k = (k, k_0), \ kx = k_0 t - k \cdot x, \ C_\pm \) are the positive and negative-frequency light cones

\[ C_\pm = \{ k : \pm k_0 = \omega > 0 \}, \quad \omega \equiv |k|, \quad \text{and} \quad d\tilde{k} = d^3k/(16\pi^3\omega) \]

is the Lorentz-invariant measure on the double cone \( C = C_+ \cup C_- \). The coefficient function \( \hat{F}(k) \) is the restriction of the Fourier transform of \( F(x) \) to \( C \). Inserting the definition of \( \mathcal{L} \),

\[ F(x) = \int_C d\tilde{k} \ e^{ikx} \hat{F}(k) = i\int_C d\tilde{k} \ e^{ikx} \left[ -(k \times (k \times \hat{Z})) + ik_0 k \times \hat{Z} \right]. \]  

(15)

Note that (12) requires

\[ \hat{F} = i\mathbf{n} \times \hat{F}, \quad \text{where} \quad \mathbf{n}(k) = k/k_0, \quad \mathbf{n}^2 = 1 \quad \text{on} \quad C. \]  

(16)

Thus for every \( k \in C \), \( \hat{F}(k) \) is an eigenvector with eigenvalue 1 of

\[ \mathbb{S}(k) : \mathbb{C}^3 \to \mathbb{C}^3 \quad \text{defined by} \quad \mathbb{S}v = i\mathbf{n} \times v. \]

But

\[ \mathbb{S}^2v = v - \mathbf{n}(\mathbf{n} \cdot v) \Rightarrow \mathbb{S}^3 = \mathbb{S}, \]

so \( \mathbb{S} \) has the nondegenerate spectrum \( \{-1, 0, 1\} \) and the orthogonal projection to the eigenspace with \( \mathbb{S} = 1 \) is

\[ \mathbb{P}(k) = \frac{1}{2} [\mathbb{S}^2 + \mathbb{S}] \Rightarrow \mathbb{S}\mathbb{P} = \mathbb{P} = \mathbb{P}^* = \mathbb{P}^2. \]  

(17)

We can now write (15) in the form

\[ F(x) = \int_C d\tilde{k} \ e^{ikx} \hat{F}(k), \quad \hat{F} = 2i\omega^2\mathbb{P} \hat{Z} = \mathbb{P} \hat{F}. \]  

(18)
3 Extension to complex spacetime

Sourceless waves are “boundary values” of holomorphic functions in a sense to be explained. Denote the future and past cones at \( x = 0 \) by

\[ V_\pm = \{ y = (y, s) \in \mathbb{R}^4 : \pm s > |y| \}. \]

The forward and backward tubes [SW64, PR84] are the complex domains

\[ T_\pm = \{ x + iy \in \mathbb{C}^4 : y \in V_\pm \}. \]

Since they are disjoint, we are free to give them independent orientations. We orient them oppositely, considering \( T_+ \) to have a positive and \( T_- \) a negative orientation. Specifically, let \( T_+ \) be oriented by its volume form \( \tau = d^4x d^4y \) and \( T_- \) by the time-reversed form \( -\tau \) (\( dx \to dx, dt \to -dt, dy^\mu \to -dy^\mu \)). The oriented union (chain)

\[ T = T_+ - T_- \]

will be called the causal tube.

We will extend sourceless fields such as \( F(x) \) to \( T \), then interpret these extensions \( \tilde{F}(z) \) physically by examining their behavior on real spacetimes defined by slices

\[ \mathbb{R}^4_y = \{ x + iy : x \in \mathbb{R}^4 \}, \quad y \in V_\pm, \]

in the spirit of Newman, Plebański, Penrose, Rindler, Robinson, Trautman and others [N65, NJ65, N73, NW74, N02, PR78, PR84, T62]. As \( y \to 0 \), the extended fields converge to their “boundary values” on \( \mathbb{R}^4 \) in a sense to be made specific. (It is only in this sense that \( \tilde{F}(z) \) is to be regarded an extension of \( F(x) \).) But note that \( \mathbb{R}^4 \) is not the topological boundary of \( T_\pm \), which would be the 7-dimenional set \( \{ x + iy : \pm y_0 > 0, \ y^2 = 0 \} \), but its Shilov boundary [H73].

Solutions of homogeneous wave equations, such as \( Z(x) \) in (14), can be extended to \( T \) by the analytic-signal transform

\[ \tilde{Z}(z) = \hat{s} \int_{C_s} d\tilde{k} \ e^{ikz} \tilde{Z}(k), \quad z = x + iy \in T, \quad (19) \]

where

\[ \hat{s} = \text{sgn} s, \quad C_s = \begin{cases} C_+ & \text{if } s > 0 \\ C_- & \text{if } s < 0. \end{cases} \quad (s \equiv y_0). \]

Note that the positive and negative frequency parts of \( Z \) are extended to the forward and backward tubes, respectively. Since \( Z \) is complex to begin with, these parts are independent. The exponential factor \( e^{-ky} \) in (19) decays as \( |k| \to \infty \) whether \( y \in V_\pm \), so if \( \tilde{Z}(k) \) is reasonable (of polynomial growth, say), then the integral defines \( \tilde{Z}(z) \) as a holomorphic function. Since \( e^{-ky} \) decays least rapidly along those rays \( k \in C_\pm \) that are “nearly parallel” to \( y \in V_\pm \), it follows that such rays are favored by the extension (19) if \( y \) is “nearly lightlike.”
Since $Z(x)$ is the sum of its positive and negative frequency parts, the sign $\hat{s}$ in (19) tells us that it can be recovered (in a distributional sense) as the difference of the (Shilov) boundary values

$$Z(x) = \tilde{Z}(x + i\varepsilon_0) - \tilde{Z}(x - i\varepsilon_0),$$  \hspace{1cm} (20)

where

$$\tilde{Z}(x \pm i\varepsilon) = \lim_{\varepsilon \downarrow 0} \tilde{Z}(x \pm i\varepsilon y), \quad y \in V_+$$

and the limit can be shown to be independent of $y \in V_+$.

It is also possible to define $\tilde{Z}$ more suggestively as

$$\tilde{Z}(x + iy) = \frac{\hat{s}}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} Z(x + \tau y),$$  \hspace{1cm} (21)

which is defined (but not holomorphic) for all $y \in \mathbb{R}^4$ and can be shown to reduce to (19) when $y \in V_+$. This definition was used in [K90, KS92, K94] without the orientation $\hat{s}$. Only recently have I understood the value of this orientation, which connects the transform to hyperfunction theory and locality (see below).

The electromagnetic field can now be extended similarly as

$$\tilde{F}(z) = \hat{s} \int_{C_\pm} d\tilde{k} e^{ikz} \hat{F}(k) = i\mathcal{L}\tilde{Z}(z) = 2i\hat{s} \int_{C_\pm} d\tilde{k} e^{ikz} \omega^2 \mathbb{P}(k) \hat{Z}(k),$$  \hspace{1cm} (22)

where $\mathcal{L}$ differentiates with respect to $x$ (so as to be defined in $\mathbb{R}^4_y$). The positive and negative-frequency parts of $\hat{F}$ also have positive and negative helicities [K94], so the restrictions of $\tilde{F}(z)$ to $T_\pm$ are positive and negative-helicity solutions.

In one dimension, the above extension is trivial: the positive and negative frequency parts extend to the upper and lower complex half-planes, and a quite general distribution can be written in the form (20). (An elementary form of this idea was applied to communication theory by Dennis Gabor, who called the extensions analytic signals; hence the name.) In dimension $n > 1$, we need something to replace the half-lines of positive and negative frequencies. When the support in Fourier space is contained in a double cone (like the convex hulls of $C_\pm$), these become the appropriate replacements and the tubes over the open dual cones (in this case $V_\pm$) replace the upper and lower half-planes.

A hyperfunction $H(x)$ in $\mathbb{R}^n$ [K88, KS99] is a generalized distribution defined, roughly, by differences in the boundary values of a set of holomorphic functions $\tilde{H}_k(z)$ in a set of complex domains $D_k$ enveloping the support of $H$. The functions $\tilde{H}_k(z)$ are called generating functions for $H(x)$.\(^2\) Equation (20) suggests thinking of the restrictions of $\tilde{Z}(z)$ and $\tilde{F}(z)$ to $T_\pm$ as generating functions for $Z(x)$ and $F(x)$. We will not attempt to make rigorous use of hyperfunction theory; rather, it will serve mainly as a guide. This will not only free us

\(^2\)More accurately, differences in boundary values must in general be replaced by sheaf cohomology. Fortunately, for solutions of homogeneous relativistic equations (Klein-Gordon, Dirac, etc.), simple differences like (20) suffice.
from its highly technical requirements but also allow us to go beyond it, as some of our constructions (like the extended delta functions \( \tilde{\delta}_3(z) \) and \( \tilde{\delta}_4(z) \) below) do not fit neatly into the theory. Also, since our goal is to understand physics directly in \( T \), we do not view \( \tilde{F}(z) \) and \( \tilde{Z}(z) \) merely as tools for the analysis of the boundary distributions \( F(x) \) and \( Z(x) \), as they would be in hyperfunction theory.

4 Electromagnetic wavelets

To construct the wavelets, we need a Hilbert space of solutions. Initially the inner product will be defined in Fourier space. It is uniquely determined (up to a constant factor) by the requirement of Lorentz invariance to be

\[
\langle F_1, F_2 \rangle = \int_C \frac{d\tilde{k}}{\omega^2} \tilde{F}_1^* \tilde{F}_2 = 4 \int_C d\tilde{k} \omega^2 \tilde{Z}_1^* \tilde{Z}_2.
\]

Note that the measure \( d\tilde{k}/\omega^2 \propto d^3k/\omega^3 \) is invariant under scaling. In fact, an equivalent inner product has been shown to be invariant under the conformal group \( C \) of Minkowski space [Gr64]. Therefore the Hilbert space of anti-selfdual solutions

\[
\mathcal{H} = \{ F : \|F\|^2 = \langle F, F \rangle < \infty \}
\]

carries a unitary representation of \( C \). We want to construct the wavelets so as to preserve the connection with the conformal group. We therefore define the wavelets in the same spirit as the relativistic coherent states for the free Klein-Gordon and Dirac fields [K77, K78, K87, K90]. For any fixed \( z \in T \), consider the evaluation map

\[
\mathcal{E}_z : \mathcal{H} \to \mathbb{C}^3 \text{ defined by } \mathcal{E}_z F = \tilde{F}(z).
\]

This is a bounded operator between Hilbert spaces (where \( \mathbb{C}^3 \) is given its standard inner product). Its adjoint is, by definition, the electromagnetic wavelet

\[
\mathbb{W}_z = \mathcal{E}_z^* : \mathbb{C}^3 \to \mathcal{H}. \tag{23}
\]

\( \mathbb{W}_z \) maps any polarization vector \( p \in \mathbb{C}^3 \) to a solution \( \mathbb{W}_z p \in \mathcal{H} \). Therefore \( \mathbb{W}_z(x') \) must be a matrix-valued solution of Maxwell’s equations (three columns, each a solution). But

\[
\mathbb{W}_z^* F = \mathcal{E}_z F = \tilde{F}(z) = s \int_{C_s} d\tilde{k} e^{ikz} \cdot \mathcal{P} \tilde{F}(k) \quad \therefore \mathcal{P} \tilde{F} = \tilde{F},
\]

therefore, remembering the measure \( d\tilde{k}/\omega^2 \) for the inner product in \( \mathcal{H} \), the expressions for \( \mathbb{W}_z \) in the Fourier and spacetime domains are

\[
\mathbb{W}_z(k) = s e^{-ikz} \omega^2 \mathcal{P}(k), \quad \mathbb{W}_z(x') = s \int_{C_s} d\tilde{k} e^{ik(x'-z)} \omega^2 \mathcal{P}(k). \tag{24}
\]

\[ \text{We are really using a vectorial version of the Riesz representation theorem. If } \mathcal{H} \text{ were a Hilbert space of sufficiently nice scalar functions, } \mathcal{E}_z \text{ would be a bounded linear functional and } \mathcal{E}_z^*(1) \text{ its representation by the unique element of } \mathcal{H} \text{ guaranteed to exist by the Riesz theorem.} \]
The reproducing kernel is defined as
\[ K(z', \bar{z}) = \mathbb{W}^*_{z'} \mathbb{W}_z = \theta(y'y) \int_{C_s} d\tilde{k} \, e^{ikz} \omega^2 \mathbb{P}(k) \equiv \theta(y'y) \mathbb{W}(z' - \bar{z}), \tag{25} \]
where \( \theta \) is the Heaviside step function, the factor \( \theta(y'y) \) enforces the mutual orthogonality of wavelets parameterized by in the forward and backward tubes, and the holomorphic matrix function
\[ \mathbb{W}(z) = \int_{C_s} d\tilde{k} \, e^{ikz} \omega^2 \mathbb{P}(k) \tag{26} \]
generates the entire wavelet family by translations:
\[ \mathbb{W}_{z'}(x') = \hat{s} \mathbb{W}(x' - \bar{z}), \quad z \in \mathcal{T}. \tag{27} \]
We now compute \( \mathbb{W}(z) \) explicitly. Applying it to a vector \( p \in \mathbb{C}^3 \) gives
\[ 2\mathbb{W}(z)p = 2 \int_{C_s} d\tilde{k} \, e^{ikz} \omega^2 \mathbb{P}(k)p = i\mathcal{L} \tilde{R}(z)p, \tag{28} \]
where
\[ i\tilde{R}(z) = \int_{C_s} d\tilde{k} \, e^{ikz} = \hat{s} \int_{C_s} d\tilde{k} \, \tilde{k}_0 e^{ikz}, \quad \tilde{k}_0 = \text{sgn} \, k_0. \tag{29} \]
The second equality, which holds because \( s \) and \( k_0 \) have the same sign, shows that \( \tilde{R}(z) \) is the analytic-signal transform of
\[ R(x) = -i \int_C d\tilde{k} \, \tilde{k}_0 e^{ikx}. \tag{30} \]
It is easily checked that this integral gives the (unique) solution to the following initial-value problem:
\[ \Box R(x) = 0, \quad R(x, 0) = 0, \quad \partial_t R(x, 0) = \delta^3(x). \]
\( R \) is known as the Riemann function of the wave equation [T96]. The integral is readily found to be the difference between the retarded and advanced propagators,
\[ R(x) = R^+(x) - R^-(x), \quad R^\pm(x) = \frac{\delta(t \mp r)}{4\pi r}. \tag{31} \]
On the other hand, we find
\[ S(z) \equiv i\tilde{R}(z) = \int_{C_s} d\tilde{k} \, e^{ikz} = -\frac{1}{4\pi z^2}, \quad z^2 = z_\mu z^\mu = z_0^2 - z^2. \tag{32} \]
This is nothing but Synge’s “elementary solution” [S65, p. 360] of the wave equation! Among other things, it has been used by Trautman [T62] to construct interesting (null, curling) analytic solutions of the Maxwell and linearized Einstein equations.
I believe this connection between the Riemann and Synge functions confirms that the transform (19) is the “right” extension of solutions to $\mathcal{T}$. In the absence of the orientation $\hat{s}$ (with (20) now a sum), the Synge function is no longer the analytic signal of $iR(x)$. In particular, we lose the connection with Huygens’ principle in the limit $y \to 0$.

The Synge function is holomorphic on the complement of the complex null cone

$$\mathcal{N} = \{ z \in \mathbb{C}^4 : z^2 = 0 \}.$$ 

In particular, note that

$$x + iy \in \mathcal{N} \cap \mathcal{T} \implies x^2 = y^2 > 0, \quad \text{and} \quad xy = 0.$$ 

But the second equality is impossible since, by the first, $x$ and $y$ are both timelike. Thus $S(z)$ is holomorphic in $\mathcal{T}$.

According to (13), $2\mathbb{W}(z)p$ can thus be computed from the selfdual Hertz potential

$$2\tilde{Z}(z) = \tilde{R}(z)p,$$

which gives explicit expressions to the entire wavelet family.

• All the wavelets are translations of $\mathbb{W}(z)$. This can be further reduced by using the symmetries of $\mathbb{W}(z)$, which reflect those of Maxwell’s equations — i.e., the conformal group. For example, $\mathbb{W}(z)$ is homogeneous of degree $-4$, so the Lorentz norm of $y$ can be interpreted as a scale parameter:

$$\mathbb{W}_z(x') = \lambda^{-4} \mathbb{W}((x' - z)/\lambda), \quad \lambda = \sqrt{y^2} > 0. \quad (34)$$

• It is easily shown that

$$\mathbb{W}(z)^* = \mathbb{W}(\bar{z}), \quad \text{hence} \quad \mathbb{W}_z(x')^* = \mathbb{W}_{\bar{z}}(x').$$

• There exist (many equivalent) resolutions of unity [K94], obtained by integrating over various subsets $D \subset \mathcal{T}$ with appropriate measures $d\mu_D$:

$$\int_D d\mu_D(z)\mathbb{W}_z\mathbb{W}_z^* = I_{\mathcal{H}}. \quad (35)$$

This is a “completeness relation” dual to the (non-) “orthogonality” relation (25). Each resolution gives a representation of solutions as superpositions of wavelets,

$$\mathbb{F}(x') = \int_D d\mu_D(z)\mathbb{W}_z(x')\mathbb{W}_z^*F = \int_D d\mu_D(z)\mathbb{W}_z(x')\tilde{F}(z), \quad (36)$$

with the analytic signal $\tilde{F}(z)$ on $D$ as the “wavelet transform.” One natural subset for a resolution is the Euclidean region of real space and imaginary time,

$$\mathcal{E} = \{(x, is) : x \in \mathbb{R}^3, \ s \neq 0\}, \quad d\mu_E(x, is) = d^3x \ ds.$$
Since \( \lambda = |s| \) on \( \mathcal{E} \), the wavelets are now parameterized by their location and scale, just as in standard wavelet analysis [D92], with the Euclidean time as scale parameter.

- Applying the analytic-signal transform to (36) gives

\[
\tilde{F}(z') = \int_D d\mu_D(z) \mathbb{W}_z \mathbb{W}_z \tilde{F}(z) = \int_D d\mu_D(z) \mathcal{K}(z', \bar{z}) \tilde{F}(z),
\]

(37)

which explains the term “reproducing kernel.”

- By construction, the wavelets \( \mathbb{W}_z \) transform covariantly under the Poincaré group, which endows their parameters \( z \) with physical significance [K94]. In particular, any resolution (35) can be boosted, rotated or translated to give another resolution. Furthermore, since \( \mathcal{H} \) carries a unitary representation of the conformal group \( \mathcal{C} \), for which \( T \) is a natural domain, it is useful to study the action of \( \mathcal{C} \) on wavelets. To some extent this has been done in [K94]. A connection has been discovered recently with work on twistor-like transforms by Iwo Bilynicki-Birula [B02], and we have begun a joint project together with Simonetta Frittelli.

5 Interpretation as Hertzian pulsed beams

Recall that the absence of sources was the price we paid at the outset for the holomorphy used to construct wavelets. But now that we have the wavelets, it turns out that we can make them physical (causal!) by introducing sources in a way that preserves their holomorphy everywhere outside these sources – which is the most that can be expected!

The introduction of sources will be based on a causal splitting of \( \tilde{R}(z) \) similar to that of \( R(x) \). But whereas the splitting of \( R(x) \) is Lorentz-invariant [the hyperplane \( \{ t = 0 \} \) can be tilted without affecting \( R^\pm(x) \)], the corresponding splitting of \( \tilde{R}(z) \) is frame-dependent. We must choose a 4-velocity \( v (\dot{v}^2 = 1, v_0 > 0) \) and split \( z \) into orthogonal temporal and spatial parts

\[
z = \tau v + z_s, \quad \tau = vz, \quad z^2 = z_0^2 - z_s^2 = \tau^2 + z_s^2.
\]

Without loss of generality, we take \( v = (0, 1) \) in the following. The general case is recovered by letting

\[
z_0 \to \tau, \quad z^2 \to \tau^2 - z^2.
\]

Begin with the factorization

\[
z_0^2 - z^2 = (z_0 - \tilde{r})(z_0 + \tilde{r}),
\]

where

\[
\tilde{r}(z) = \sqrt{z^2} = \sqrt{r^2 - a^2 + 2i \overline{x} \cdot y}, \quad z = x + iy, \quad r = |x|, \quad a = |y|.
\]

For motivation, think of \( \tilde{r} \) as the complex distance from a “point source” at \(-iy\) to the observer at \( x \). A study of potential theory based on complex distances in \( \mathbb{C}^n \), and the
connections they furnish between elliptic and hyperbolic equations, is in progress; see [K00]; see also the related papers [K01, K01a, N02]. Given \( y \neq 0 \), the branch points of \( \tilde{r} \) form a circle in the plane orthogonal to \( y \),

\[
C(y) = \{ x \in \mathbb{R}^3 : r = a, \ x \cdot y = 0 \},
\]

and following a loop that threads \( C \) changes the sign of \( \tilde{r} \). We make \( \tilde{r} \) single-valued by choosing the “physical” branch defined by

\[
\text{Re } \tilde{r} \geq 0, \quad \text{so that } y \to 0 \Rightarrow \tilde{r} \to +r.
\]

Complex distance: The real part (left) and imaginary part (right) of \( \tilde{r}(x + iy) \) as functions of \( x = (x_1, 0, x_3) \), with \( y = (0, 0, 1) \). The real part is a pinched cone, with the branch disk in the \( x_1-x_2 \) plane projected to the interval \([-1, 1]\) of the \( x_1 \) axis.

The resulting branch cut is a disk\(^4\) bounded by \( C \),

\[
D(y) = \{ x : r \leq a, \ x \cdot y = 0 \}.
\]

The extended spatial delta function is defined as the distributional Laplacian with respect to \( x \) of the holomorphic Coulomb potential:

\[
\tilde{\delta}^3(z) \equiv -\frac{1}{4\pi \tilde{r}(z)}.
\]

As a distribution in \( x \), it is a natural extension of the point source \( \delta^3(x) \). Roughly speaking, displacing the singularity to \(-iy\) opens up a “light cone” in \( \mathbb{R}^3 \) with the \( y \)-axis as “time,” of which the branch circle is a “wave front” and \( D \) its interior. Similar remarks apply to extended delta functions in \( \mathbb{C}^n \), which have provided an intriguing connection [K00] between fundamental solutions of Laplace’s equation in Euclidean \( \mathbb{R}^n \) and the initial-value problem for wave equations in Lorentzian \( \mathbb{R}^n \).

\(^4\)An equivalent branch cut is obtained by continuously deforming the disk to an arbitrary membrane \( M \) bounded by \( C \).
Returning to spacetime, \( \tilde{R}(z) \) decomposes into partial fractions:

\[
\tilde{R}(z) = \frac{i}{4\pi^2(z_0 - \tilde{r})(z_0 + \tilde{r})} = \tilde{R}^+(z) - \tilde{R}^-(z),
\]

where

\[
\tilde{R}^\pm(z) = \frac{i}{8\pi^2\tilde{r}(z_0 \mp \tilde{r})}.
\]

Note that although \( \tilde{R}(z) = -iS(z) \) is holomorphic in \( \mathcal{T} \), its retarded and advanced parts \( \tilde{R}^\pm(z) \) are not. Viewed on the slice \( \mathbb{R}^4_y \), they are singular on the world-tube \( \tilde{D} \) traced out by the branch cut \( D \). Everywhere in \( \mathcal{T} \) outside this source region, they are still holomorphic.

The boundary values of \( \tilde{R}^\pm \) are given by the Plemelj formulas

\[
\tilde{R}^\pm(x + i0) = \frac{i}{8\pi^2r(t \mp r + i0)} = \delta(t \mp r) + \frac{i}{8\pi^2r} \mathcal{P} \frac{1}{t \mp r},
\]

\[
\tilde{R}^\pm(x - i0) = \frac{i}{8\pi^2r(t \mp r - i0)} = -\frac{i}{8\pi^2r} \mathcal{P} \frac{1}{t \mp r},
\]

where \( \mathcal{P} \) denotes the Cauchy principal value. Therefore the jumps across \( \mathbb{R}^4 \) are

\[
\tilde{R}^\pm(x + i0) - \tilde{R}^\pm(x - i0) = \frac{\delta(t \mp r)}{4\pi r} = R^\pm(x).
\]

We now show that \( \tilde{R}^\pm(z) \) have very interesting physical interpretations even when \( y \neq 0 \), by looking at their behavior in slices \( \mathbb{R}^4_y \).

Guided by the successful definition (38) of the extended spatial source, define the extended spacetime delta function

\[
\tilde{\delta}^4(z) = \Box \tilde{R}^\pm(z),
\]

where \( \Box \) is the distributional d’Alembertian acting on \( x \). Since \( \Box \tilde{R}^+ - \Box \tilde{R}^- = \Box \tilde{R} = 0 \), there is no sign ambiguity on the left. A detailed study of \( \tilde{\delta}^4(x + iy) \) is somewhat involved and will be given elsewhere [K03]. Here we note the following.

- \( \tilde{\delta}^4(x + iy) \) is a well-defined Schwartz distribution in \( \mathbb{R}^4_y \).
- By (41),

\[
\tilde{\delta}^4(x + i0) - \tilde{\delta}^4(x - i0) = \frac{\delta(t \mp r)}{4\pi r} = R^\pm(x).
\]

- It is easy to show that \( \Box \tilde{R}^\pm(x + iy) = 0 \) at all points of regularity. Thus \( \tilde{\delta}^4(x + iy) \) is supported on the world tube \( \tilde{D} \) representing the evolution of the source disk in \( \mathbb{R}^4_y \).
- In spite of (43), the restrictions of \( \tilde{\delta}^4(z) \) to \( \mathcal{T}_\pm \) are not generating functions for \( \delta^4(x) \) since they are not holomorphic in any neighborhood of \( \mathbb{R}^4 \). (This is what I meant earlier by saying that not all our constructions fit neatly into hyperfunction theory.)
• In the far zone, we have
\[ r \gg a \Rightarrow \tilde{r} = \sqrt{r^2 - a^2 + 2iarcos\theta} \approx r + ia\cos\theta \]
\[ (z_0 = t - is, \ |x| = r, \ |y| = a, \ x \cdot y = ra\cos\theta) , \]
giving a simple expression for the far field from which the pulsed-beam interpretation can be read off easily:
\[ r \gg a \Rightarrow \tilde{R}^\pm (z) \approx \frac{1}{8\pi^2r} \cdot \frac{i}{t \mp r + i(s \mp a\cos\theta)} . \]

**Time-lapse plots** at \( t = 1, 5, 10, 15, 20 \) of \( |\tilde{R}^+(x + iy)| \) with \( y = (0, 0, a, 1) \) as functions of \( x = (x_1, 0, x_3) \). Clockwise from upper left: \( a = .5, .9, .99, .999 \). The pulses propagate along \( y = (0, 0, a) \), getting sharper and sharper as \( a \to s = 1 \).

• \( \tilde{R}^- (x + iy) \) is an advanced pulsed beam converging toward \( D \) along \( y/s \) and absorbed in \( D \) around \( t = 0 \).
• \( \tilde{R}^+ (x + iy) \) is a retarded pulsed beam emitted from \( D \) around \( t = 0 \) and propagating along \( y/s \).
• \( D \) acts like an antenna dish, simultaneously absorbing \( \tilde{R}^- \) and emitting \( \tilde{R}^+ \), resulting in the sourceless pulsed beam \( \tilde{R} (x + iy) \) focused at \( x = 0 \).
Both beams have duration $|s| - a$ along the beam axis and are focused as sharply as desired by letting $(y/s)^2 \to 1$. As $(y/s)^2 \to 0$, they become spherical pulses of duration $|s|$.

Inserting (40) into (33) and (28) gives

$$W(z) = W^+(z) - W^-(z),$$

where

$$W^\pm(z)p = i\mathcal{L}\hat{Z}^\pm(z), \quad 2\hat{Z}^\pm(z) = \hat{R}^\pm(z)p.$$  \hfill (44)

By (42), the polarization associated with this Hertz potential is

$$2\hat{P}(z) = 2\Box\hat{Z}^\pm(z) = p\delta^4(z).$$  \hfill (45)

By (43), the polarization on $\mathbb{R}^4$ is an impulsive dipole

$$P_m(x) - iP_e(x) = 2P(x) = 2\hat{P}(x + i0) - 2\hat{P}(x - i0) = p\delta^4(x),$$

giving an interpretation of $p$ in terms of magnetic and electric dipole moments

$$p = p_m - iP_e.$$  \hfill (46)

Of course, the extension $P(x) \to \hat{P}(z)$ mixes the electric and magnetic dipoles since $\delta^4(z)$ is complex. Nevertheless, we are free to define real polarizations and causal/anticausal Hertz potentials and fields in the slice $\mathbb{R}^4_y$ by

$$P_m(z) - iP_e(z) = p\delta^4(z)$$

$$Z^\pm_m(z) - iZ^\pm_e(z) = \hat{R}^\pm(z)p$$

$$E^\pm(z) + iB^\pm(z) = i\mathcal{L}[\hat{R}^\pm(z)p].$$

The left-hand sides are then solutions of Maxwell’s equations inheriting the pulsed-beam interpretations derived for $\hat{R}^\pm(z)$, but now in the context of electrodynamics instead of scalar wave equations.

Whittaker [W04, N55] has shown that given any constant, nonzero vector $v \in \mathbb{R}^3$, a general Hertz potential can be reduced to two scalar potentials $\Pi_e, \Pi_m$ in the form

$$Z_e(x) = \Pi_e(x)v \quad \text{and} \quad Z_m(x) = \Pi_m(x)v,$$

or $2Z(x) = (\Pi_m - i\Pi_e)v$.

Comparison with (44) shows that $\hat{R}^\pm(z)$ are scalar Hertz potentials of Whittaker type for $2W^\pm(z)p$.

Finally, the wavelet decomposition (36) of sourceless anti-selfdual fields splits into

$$F(x') = F^+(x') - F^-(x'), \quad F^\pm(x') = \int_D d\mu_P(z)W^\pm_z(x')\hat{F}(z).$$

$F^+$ and $F^-$ are fields absorbed and emitted, respectively, by source disks distributed in $D$ according to the coefficient function $\hat{F}(z)$. This may be used for analyzing and synthesizing fields with general sources.

Pulsed beams similar to the above have also been studied and applied in the engineering literature, from a different point of view; see [HF01] for a recent review.
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