A syntactical proof of the operational equivalence of two
λ-terms

René DAVID and Karim NOUR

Abstract

In this paper we present a purely syntactical proof of the operational equivalence of
I = λxx and the λ-term J that is the η -infinite expansion of I.

1 Introduction

Two λ-terms M and N are operationnely equivalent (M ≃ oper N) iff for all context C : C[M]
is solvable iff C[N] is solvable.

Let I = λxx and J = (Y G) where Y is the Turing’s fixed point operator and G = λxλyλz(y (x z)).

J is the η-infinte expansion of I. His Böhm tree is in fact λxλx_1(x λx_2(x_1 λx_3(x_2 λx_4(x_3,....

The following Theorem is well known (see [1],[3]).

Theorem I ≃ oper J.

The usual proof is semantic : two λ-terms are operationnely equivalent iff they have the same
interpretation in the modele D_∞ .

We give below an elementary and a purely syntactical proof of this result. This proof analyses
in a fine way the reductions of C[I] and C[J] by distinguant the ”real” β -redex of ceux which
come of the η-expansion.

This proof may be generalize to prove (this result is also well known) the operationnely equiva-
lence of two λ-terms where the Böhm tree are equal à η - infinite expansion près. The necessary
technical tool is the directed λ-calculus (see [2]).

2 Definitions and notations

- λx U represents a sequence of abstractions.

- Let T,U,U_1,...,U_n be λ-terms, the application of T to U is denoted by (T U) or T U. In
the same way we write T U_1 U_n or T U instead of (...(T U_1)...U_n).

- Let us recall that a λ-term T either has a head redex [i.e. t = λx(λxU V ) V , the head
redex being (λxU V )], or is in head normal form [i.e. t =

lxx V ].
• The notation $U \rightarrow_t V$ (resp. $U \rightarrow_{t^*} V$) means that $V$ is obtained from $U$ by one head reduction (resp. some head reductions).

• A λ-term $T$ is said solvable iff the head reduction of $T$ terminates.

The following Lemma is well known.

**Lemma 2.1** $(U \mathbin{V})$ is solvable iff $U$ is solvable (and has $U'$ as head normal form) and $(U' \mathbin{V})$ is solvable.

### 3 Proof of the Theorem

The idea of the proof is the following : we prove that, if we assimilate the reductions where $I$ (resp $J$) are in head position, $C[I]$ and $C[J]$ reduce, by head reduction in the same way. For this we add a constante $H$ (which represente either $I$ or $J$). We define on those terms the $I$ (resp $J$) head reduction, corresponding to the case where $H = I$ (resp $J$). To prove that the reductions are equivalent we prove that the terms obtained by ”removing” the constante $H$ are equal. This is the role of the extraction fonction $E$.

#### 3.1 λH-calculus and the application $E$

• We add a new constante $H$ to the λ-calculus and we call λH-terms the terms which we obtain.

• We define (by induction) on the set of λH-terms the application $E :$

\[
E(x) = x ;
E(H) = H ;
E(\lambda x U) = \lambda x E(U) ;
E(U \mathbin{V}) = E(U)E(V) \text{ if } U \neq HU_1U_2...U_n ;
E(HU_1U_2...U_n) = E(U_1U_2...U_n) .
\]

• A λH-term is in head normal form if it is of the forme : $\lambda \overline{x} H$ or $\lambda \overline{x} x \overline{V}$.

**Lemma 3.1** If $T$ is a λH-term, then $E(T)$ is of the forme $\lambda \overline{x} H$ or $\lambda \overline{x} x \overline{V}$ or $\lambda \overline{x} (\lambda x U) \overline{V}$.

**Proof** By induction on $T$. □

**Lemma 3.2** If $T$ is a λH-term, then $E(E(T)) = E(T)$.

**Proof** By induction on $T$. □

**Lemma 3.3** Let $T, \overline{U}$ be λH-terms. $E(T \overline{U}) = E(E(T)E(\overline{U}))$.

**Proof** By induction on $T$. We distinguish the cases: $T \neq H \overline{V}$ and $T = H \overline{V}$. □
Lemma 3.4 Let \( U, V \) be \( \lambda H \)-terms and \( x \) a variable, \( E(U[V/x]) = E(E(U)[E(V)/x]) \).

Proof By induction on \( U \). The only interesting case is \( U = xU \). By Lemma 3.3, \( E(U[V/x]) = E(E(V)E(U)[E(V)/x]) \). Therefore, by induction hypothesis and Lemma 3.3, \( E(U[V/x]) = E(E(V)E(U)[E(V)/x]) = E(E(U)[E(V)/x]) \). \( \Box \)

Lemma 3.5 Let \( U_1, U_2, V_1, V_2 \) be \( \lambda H \)-terms such that \( E(U_1) = E(U_2) \) and \( E(V_1) = E(V_2) \). \( E(U_1[V_1/x]) = E(U_2[V_2/x]) \).

Proof By Lemma 3.4. \( \Box \)

Lemma 3.6 Let \( U_1, U_2, V_1, V_2 \) be \( \lambda H \)-terms. If \( U_1 \rightarrow_t V_1, U_2 \rightarrow_t V_2, \) and \( E(U_1) = E(U_2) \), then \( E(V_1) = E(V_2) \).

Proof By Lemmas 3.3 and 3.5. \( \Box \)

3.2 The I-reduction

- We define on the \( \lambda H \)-terms a new head reduction:
  \[ HU_1...U_n \rightarrow_I U_1U_2...U_n \]
- We denote by \( \rightarrow_{I^*} \) the reflexive and transitive closure of \( \rightarrow_I \).
- A \( \lambda H \)-term \( U \) is I-t-solvable iff a finite sequence of I-reductions and t-reductions of \( U \) gives a head normal form.

Lemma 3.7 Let \( U, V \) be \( \lambda H \)-terms. If \( U \rightarrow_{I^*} V \), then \( E(U) = E(V) \).

Proof By induction on the reduction of \( U \). \( \Box \)

Lemma 3.8 Each I-reduction is finite.

Proof The I-reduction decreases the complexity of a \( \lambda H \)-term. \( \Box \)

Lemma 3.9 Let \( U \) be \( \lambda H \)-term. \( U \) is I-t-solvable iff \( U[I/H] \) is solvable.

Proof Immediate. \( \Box \)
3.3 The J-reduction

- We define on the \(\lambda H\)-terms a new head reduction:
  \[HU_1...U_n \rightarrow_j U_1(H U_2)U_3...U_n\]

- We denote by \(\rightarrow_j\) the reflexive and transitive closure of \(\rightarrow_j\).

- A \(\lambda H\)-term \(U\) is \(J\)-t-solvable iff a finite sequence of \(J\)-reductions and \(t\)-reductions of \(U\) gives a head normal form.

**Lemma 3.10** Let \(U, V\) be \(\lambda H\)-terms. If \(U \rightarrow_j V\), then \(E(U) = E(V)\).

**Proof** It is enough to do the proof for one step of \(J\)-reduction. The only interesting case is \(U = (H)U_1U_2\). In this case \(U \rightarrow_j U_1(H U_2)\), and, by induction hypothesis, \(E((U_1(H U_2)) = E(V)\), therefore -by Lemma 3.3- \(E(U) = E(V)\). □

**Lemma 3.11** Let \(U, V\) be \(\lambda H\)-terms. If \(U \rightarrow_j V\), then, for each sequence \(\overline{W} = W_1...W_n\), there is a sequence \(\overline{W'} = W'_1...W'_n\) such that \(U\overline{W} \rightarrow_j V\overline{W'}\) and for, all \(1 \leq k \leq n\), \(W'_k \rightarrow_j W_k\).

**Proof** By induction on the reduction of \(U\). It enough to do the proof for one step of \(J\)-reduction. The only interesting case is \(U = HU'\) and \(\overline{W} = W_1\overline{W'}\). In this case \(V = U'\), \(UW_1\overline{W'} \rightarrow_j V(H W_1)\overline{W'}\) and \(HW_1 \rightarrow_j W_1\). □

**Lemma 3.12** Each \(J\)-reduction is finite.

**Proof** By induction on \(U\). The only interesting case is \(U = HV_1...V_n\) \((n \geq 2)\). We prove, by recurrence on \(n\), that if the reductions of \(V_1,...,V_n\) are finite, then so is for \(U = HV_1...V_n\). \(U \rightarrow_j V_1(H V_2) V_3...V_n\) and \(V_1 \rightarrow_j V'_1\). By Lemma 3.11, \(U \rightarrow_j V'_1 W_2 W_3...W_n\) where \(W_2 \rightarrow_j H V_2 \rightarrow_j V_2\) and \(W_1 \rightarrow_j W_1\), therefore the reductions of \(W_i\) are finite.

- If \(E(V_1) \neq H\). \(V'_1\) begin soit by \(\lambda\), soit by a \(\beta\)-redex, soit by a variable. Therefore, by Lemma 3.11, the \(J\)-reduction of \(U\) is finite.

- If \(E(V_1) = H\). By Lemma 3.11, \(U \rightarrow_j HW_2...W_n\) and the recurrence hypothesis allows to conclude. □

**Lemma 3.13** Let \(U\) be a \(\lambda H\)-term. \(U\) is \(J\)-t-solvable iff \(U[J/H]\) is solvable.

**Proof** The only difficulty is to prove that: if \(U\) is \(J\)-t-solvable, then \(U[J/H]\) is solvable. We prove that by induction on the reduction of \(U\). The only interesting case is \(U = \lambda x\overline{\tau} H V\). In this case, \(U \rightarrow_j \lambda x\overline{\tau} V\) and \(U[J/H] \rightarrow_t \lambda y V[J/H] (J y)\). By induction hypothesis \(V[J/H]\) is solvable, and, by Lemma 2.1, we may begin to reduce \(V[J/H]\) in \(\lambda x\overline{\tau} \lambda y V[J/H] (J y)\). If the head normal form of \(V[J/H]\) is not of the forme \(\lambda x\overline{\tau} x\overline{\tau} (J y)\overline{W}\), the result is true. If not the head reduction of \(U[J/H]\) gives \(\lambda x\overline{\tau} \lambda x\overline{\tau} (J y)\overline{W}\) which is solvable. □
3.4 The proof of the Theorem

$U \rightarrow_{(I^*, k)} V$ (resp. $U \rightarrow_{(J^*, k)} V$) means that $V$ is obtained from $U$ by $I$-reductions (resp. $J$-reductions) and $k$ $t$-reductions.

**Lemma 3.14** Let $U_1, U_2, V_1, V_2$ be $\lambda H$-terms. If $U_1 \rightarrow_{(I^*, k)} V_1$, $U_2 \rightarrow_{(J^*, k)} V_2$, and $E(U_1) = E(U_2)$, then $E(V_1) = E(V_2)$.

**Proof** Consequence of Lemmas 3.6, 3.7 and 3.10. □

**Lemma 3.15** Let $U$ be a $\lambda H$-term. $U$ is $I$-t-solvable iff $U$ is $J$-t-solvable.

**Proof** Consequence of Lemmas 3.8, 3.12 and 3.14. □

**Proof of the Theorem** Consequence of Lemmas 3.9, 3.13 and 3.15. □

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LAMA - Equipe de Logique - Université de Chambéry - 73376 Le Bourget du Lac
e-mail david,nour@univ-savoie.fr