Clifford-Wolf Homogeneous Randers Spaces

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Abstract. A Clifford-Wolf translation of a connected Finsler space is an isometry which moves all points the same distance. A Finsler space \((M,F)\) is called Clifford-Wolf homogeneous if for any two points \(x_1,x_2 \in M\) there is a Clifford-Wolf translation \(\rho\) such that \(\rho(x_1) = x_2\). In this paper, we give a complete classification of connected simply connected Clifford-Wolf homogeneous Randers spaces.

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1. Introduction

Let \((M,F)\) be a connected Finsler space and \(d\) its distance function. An isometry \(\rho\) of \((M,F)\) is called a Clifford-Wolf translation (CW-translation for short) if the function \(d(x,\rho(x))\) is constant on \(M\). The Finsler space \((M,F)\) is called Clifford-Wolf homogeneous (CW-homogeneous) if for any two points \(x_1,x_2 \in M\), there is a CW-translation \(\rho\) such that \(\rho(x_1) = x_2\).

These definitions are natural generalizations of the related notions in Riemannian geometry. CW-translations have been studied extensively in Riemannian geometry, due to its relevance in the classification of space forms. Let \((N,Q_1)\) be a connected complete Riemannian manifold with constant sectional curvature \(k\). Then \((N,Q_1)\) is isometric to a quotient manifold \((M,Q)/\Gamma\), where \((M,Q)\) is a connected simply connected complete Riemannian manifold with constant sectional curvature \(k\), and \(\Gamma\) is a discrete discontinuous subgroup of the full group of isometries of \((M,Q)\) which acts freely on \(M\). It is well-known that a connected simply connected complete Riemannian manifold of constant curvature is homogeneous. However, the quotient manifold \((M,Q)/\Gamma\) is no longer homogeneous in general. J. A. Wolf proved in [16] that it is homogeneous if and only if \(\Gamma\) consists of CW-translations. Thus the classification of homogeneous space form is reduced to the study of CW-subgroups of the full group of isometries. This formulation was generalized to symmetric Riemannian spaces by J. A. Wolf in [17]; see [11, 12] for other proofs.

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Recently, connected simply connected Riemannian CW-homogeneous manifolds were classified by Berestovskii and Nikonorov in [2, 3, 4]. Their list consists of the euclidean spaces, the odd-dimensional spheres with constant curvature, the connected simply connected compact simple Lie groups with bi-invariant Riemannian metrics, and the direct products of the above Riemannian manifolds. The purpose of this paper is to give a classification of connected simply connected CW-homogeneous Randers spaces. Our main result is the following

**Theorem 1.1.** A connected simply connected Randers space \((M, F)\) is CW-homogeneous if and only if \(M\) is a product of odd dimensional spheres, connected simply connected compact simple Lie groups and a euclidian space, and the Finsler metric \(F\) has the navigation data \((h, W)\) such that \((M, h)\) is CW-homogeneous, i.e., \(h\) is a Riemannian product of Riemannian metrics on spheres of constant curvature, bi-invariant metrics on Lie groups and a flat metric on the euclidian space with respect to the product decomposition of \(M\), and \(W\) is a Killing vector field of constant length with \(\|W\|_h < 1\).

This result presents many examples of non-reversible CW-homogeneous Finsler spaces. It follows easily from the main results of Bao-Robles-Shen [6] that, on an odd dimensional sphere, a CW-homogeneous Randers space must be of constant flag curvature. However, the converse statement is not true. In fact, the Randers space \((S^{2n-1}, F)\) of constant flag curvature is CW-homogeneous if and only if in its navigation data \((h, W)\), \(h\) is the standard Riemannian metric and \(W\) is a Killing field of constant length with \(\|W\|_h < 1\) ([9]). When \(W\) is not zero and of constant length, it defines a complex structure on \(\mathbb{R}^{2n}\), and correspondingly the sphere can be denoted as \(U(2n)/U(2n-1)\). It is easily seen that \(W\) is the Killing vector field generating the center of \(U(2n)\) ([15]).

In view of the above result, it would be an interesting problem to classify CW-homogeneous Finsler spaces in general.

In Section 2, we recall the preliminary knowledge about Finsler geometry, CW-translation and CW-homogeneity. In Section 3, we prove an interrelation theorem between CW-translations and Killing vector fields of constant length on a homogeneous Finsler space. In Section 4, we prove the main theorem.

## 2. Preliminaries

Let \(M\) be a connected manifold. A Finsler metric is a continuous function \(F : TM \to \mathbb{R}^+\), which is smooth on the slit tangent bundle \(TM\setminus 0\). In a standard local coordinates \((x^i, y^j)\) for \(TM\), where \(x = (x^i)\) is the local coordinates for \(M\), and \(y = y^j \partial_{x^j}\) is the linear coordinates for \(y \in T_x(M)\), the Finsler metric \(F\) is required to satisfy the following properties:

1. \(F(x, y) > 0\) for any \(y \neq 0\).
2. \(F(x, \lambda y) = \lambda F(x, y)\) for any \(y \in T_x(M)\) and \(\lambda > 0\).
3. The Hessian matrix defined by \(g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}\) is positive definite.
A Randers metric on $M$ is a Finsler metric of the form $F = \alpha + \beta$, where $\alpha$ is a Riemannian metric and $\beta$ is a smooth 1-form on $M$ whose length with respect to $\alpha$ is everywhere less than 1. This kind of metric was introduced by G. Randers in 1941 ([13]) in the context of general relativity.

The above expression of a Randers metric is called the defining form in the literature. Using standard local coordinates $(x^i, y^j)$ for $TM$, a Randers metric can be presented as

$$F = \alpha + \beta = \sqrt{a_{ij}(x)y^i y^j + b_i(x)y^i}.$$  \hspace{1cm} (1)

There is another method introduced by Shen [14] to express a Randers metric. The main idea is that the Randers metric $F$ can also be uniquely written as

$$F(x, y) = \sqrt{h(y, W)^2 + \lambda h(y, y)} - \frac{h(y, W)}{\lambda},$$

where $h$ is a Riemannian metric, $W$ is a vector field on $M$ with $h(W, W) < 1$ and $\lambda = 1 - h(W, W)$. The pair $(h, W)$ is called the navigation data of the Randers metric $F$. The navigation data is convenient when handling some problems concerning the flag curvatures and Ricci scalar of a Randers space. For example, using navigation data, Bao, Robles and Shen presented a very explicit description of Einstein-Randers metrics and Randers spaces of constant flag curvature; see [5, 6].

In a local coordinate system, the transformation laws between the defining form and navigation data can be described as the following. If $$F = \alpha + \beta = \sqrt{a_{ij}(x)y^i y^j + b_i(x)y^i},$$

then the navigation data has the form

$$h_{ij} = (1 - ||\beta||^2)(a_{ij} - b_i b_j), \quad W^i = -\frac{a^{ij}b_j}{1 - ||\beta||^2_a}.$$  \hspace{1cm} (1.2)

Conversely, the defining form can also be expressed by the navigation data through the formula:

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i W_j}{\lambda}, \quad b_i = -\frac{W_i}{\lambda},$$

where $W_i = h_{ij}W^j$ and $\lambda = 1 - W^i W_i = 1 - h(W, W)$. All these formulas can be found in [5].

A Finsler metric $F$ defines the length of a tangent vector, and the integration along a piece-wise smooth path defines the arc length of curves. Taking the infimum of arc lengths for all piece-wise smooth paths, one defines a distance function $d(\cdot, \cdot)$ on $(M, F)$, which satisfies all the conditions of a distance in the general sense except the reversibility. Based on the distance function, we have generalized the concept of a Clifford-Wolf translation (CW-translation) to Finsler geometry.

**Definition 2.1** Let $(M, F)$ be a connected Finsler space. An isometry $\rho$ of $(M, F)$ is called a Clifford-Wolf translation if the function $d(x, \rho(x))$ is a constant on $M$. 
One can also define the notions of Clifford-Wolf homogeneous (CW-homogeneous) spaces and of restrictively Clifford-Wolf homogenous (restrictively CW-homogeneous) spaces.

**Definition 2.2** A Finsler manifold \((M, F)\) is called Clifford-Wolf homogeneous if for any two points \(x, x' \in M\), there is a CW-translation \(\rho\) such that \(\rho(x) = x'\). It is called restrictively Clifford-Wolf homogeneous if for any point \(x \in M\) there is a neighborhood \(V\) of \(x\), such that for any two points \(x_1, x_2 \in V\) there is a CW-translation \(\rho\) such that \(\rho(x_1) = x_2\).

From the above definitions, it is obvious that a Finsler manifold \((M, F)\) is restrictively CW-homogeneous if it is CW-homogeneous. Moreover, a connected restrictively CW-homogeneous Finsler space \((M, F)\) must be a homogeneous Finsler space, i.e., its isometry group \(G = I(M, F)\) acts transitively on \(M\). In fact, if it is connected and restrictively CW-homogeneous, then each \(G\)-orbit in \(M\) is an open as well as a closed subset.

### 3. Killing vector fields of constant length

CW-translations of a connected Finsler manifold \((M, F)\) have a natural interrelation with the Killing vector fields of constant length, which has been proved by Berestovskii and Nikonorov ([4]) in the Riemannian case, and by the authors in Finsler geometry ([8]).

**Theorem 3.1.** ([8]) Let \((M, F)\) be a complete Finsler manifold with a positive injectivity radius. If \(X\) is a Killing vector field of constant length, then the flow \(\varphi_t\), generated by \(X\), is a CW-translation for all sufficiently small \(t > 0\).

**Theorem 3.2.** ([8]) Let \((M, F)\) be a compact Finsler manifold. Then there is a \(\delta > 0\), such that any CW-translation \(\rho\) with \(d(x, \rho(x)) < \delta\) is generated by a Killing vector field of constant length.

Theorem 3.2 is not convenient when dealing with non-compact Finsler spaces. We now prove the following theorem which will be useful in proving the main theorem of this paper.

**Theorem 3.3.** Let \((M, F)\) be a connected homogeneous Finsler space. Then there is a \(\delta > 0\), such that any CW-translation \(\rho\) with \(d(x, \rho(x)) \leq \delta\) is generated by a Killing vector field of constant length.

**Proof.** The connected isometry group \(G = I_0(M, F)\) is Lie group ([7]) which acts transitively on \(M\). Let \(g\) be its Lie algebra. The exponential map \(\exp : g \rightarrow G\) is a diffeomorphism when restricted to a small round open disk neighborhood \(B\) of \(0 \in g\). When the positive number \(\delta\) is small enough, all the CW-translations \(\rho\) with \(d(x, \rho(x)) \leq \delta\) is contained in \(\exp(B)\). If we denote the set of all CW-translations \(\rho\) with \(d(x, \rho(x)) \leq \delta\) as \(C_\delta\), then both \(C_\delta\) and \(\exp^{-1}(C_\delta) \cap B\) are compact. As the homogeneous space \((M, F)\) has a positive injectivity radius \(r > 0\), we can choose \(B\) to be small enough such that for any \(X \in B\), \(F(X(x)) < r\).
Now any CW-translation $\rho \in \mathcal{C}_\delta$ is generated by a unique Killing vector field $X \in \mathcal{B}$, such that $\rho = \varphi_{1;X} = \exp X$, where $\varphi_{t;X}$ is the local one-parameter subgroup of diffeomorphisms generated by $X$. We need to prove that $F(X)$ is a constant function on $M$. Let $g_n$ be a sequence in $G$ such that $F(X(g_n^{-1}x))$ converge to $\sup F(X)$. The supremum of $F(X)$ is not required to be finite, but we can see from the later argument that $\sup F(X) \leq r$. We have a sequence of CW-translations $\rho_n = \text{Ad}(g_n)\rho$ in $\mathcal{C}_\delta$. Each $\rho_n$ is generated by the Killing vector field $X_n = g_n X$, i.e., $\rho_n = \varphi_{1;X_n} = \exp X_n$, and we have $F(X(g_n^{-1}x)) = F(X_n(x))$. By the connectedness of $G$ one easily sees that all the Killing vector fields $X_n$ is contained $\exp^{-1}(\mathcal{C}_\delta) \cap \mathcal{B}$. Choosing subsequence if necessary, one can assume that the sequence $\{\rho_n\}$ converges to $\rho' \in \mathcal{C}_\delta$, with $d(\cdot, \rho'(-)) \equiv d(\cdot, \rho(\cdot))$, that $\{X_n\}$ converges to a Killing vector field $X' \in \exp^{-1} \mathcal{C}_\delta \cap \mathcal{B}$ and that $\rho' = \varphi_{1;X'} = \exp X'$. At each point, we have

$$X'(-) = \lim_{n \to \infty} X_n(-).$$

In particular, at the point $x$, the function

$$F(X'(-)) = \lim_{n \to \infty} F(X_n(-))$$

reaches its maximum. If $F(X'(x)) = 0$, then $X = 0$ and $\rho$ is the identity map. Otherwise $\nabla^X_X X = -\frac{1}{2} F(X) F(X)^2$ vanishes along the flow curve $\varphi_{t;X'}(x)$, $t \geq 0$. Then by Corollary 3.2 of [8], the flow curve $\varphi_{t;X'}(x)$, $t \geq 0$ is a geodesic. Because $F(X'(\varphi_{t;X'}(x)))) = F(X'(x)) \leq r$ for all $t \geq 0$, the geodesic $\varphi_{t;X'}(x)$, $t \in [0, 1]$ is minimizing. Thus

$$F(X'(x)) = \sup(F(X)) = d(x, \rho'(x)) = d(x, \rho(x)).$$

A similar argument for $\inf(F(X))$ (which must be positive when the CW-translation is not the identity map) gives $\inf(F(X)) = d(x, \rho(x))$. Therefore $X$ is a Killing vector field of constant length.

As a corollary, we can give a description of restrictive CW-homogeneity by Killing vector fields of constant length, which will be used in the proof of the main theorem.

**Theorem 3.4.** Let $(M, F)$ be a connected homogeneous Finsler space. Then it is restrictively CW-homogeneous if and only if the Killing fields of constant length can exhaust all tangent directions, or equivalently, any geodesic starting from any point is the flow curve of a Killing field of constant length.

**Proof.** If $(M, F)$ is homogeneous, it has a positive injectivity radius $r > 0$. At each point, there is a small neighborhood, in which for any points $x$ and $x'$, $d(x, x') < r$. Then the proof of Theorem 3.1 indicates the existence of a CW-translation $\rho$ which maps $x$ to $x'$. So $(M, F)$ is restrictive CW-homogeneous.

If $(M, F)$ is connected and restrictively homogeneous, then it is homogeneous. Then Theorem 3.3 indicates that all tangent vectors are exhausted by Killing vector fields of constant length.

In the Riemannian case, the above arguments give an alternative proof for Theorem 7 in [4].
4. Proof of Theorem 1.1

We first prove that, if \((M, F)\) is a connected simply connected restrictively CW-homogeneous Randers space, then \(M\) has a product form with a metric \(F\) as described in the theorem.

Let \((h, W)\) be the navigation data of \(F\) and \(\varphi_{t,W}\) be the flow generated by the vector field \(W\). By Theorem 3.4, for any \(y \in TM_x\), there is a Killing vector field \(X\) of constant length for \(F\), such that \(X(x) = y\). Let \(x' \in M\) and \(t \geq 0\), such that \(\varphi_{t,W}(x) = x'\). Since \(X\) is a Killing vector field for \(F\), we have \(L_X W = [X, W] = 0\). Moreover, \(y = X(x)\) has the same \(F\)-length as 
\[ y' = X(x') = \varphi_{t,W}(y). \]
It follows that all \(\varphi_{t,W}\)'s are isometries and \(W\) is a Killing vector field of \(F\). For any Killing vector field \(X\) of constant length \(1\) of \(F\), \(X + W\) is a Killing vector field for \(h\). It is easily seen that \(X + W\) has constant length \(1\) with respect to \(h\). Since \((M, F)\) is restrictively CW-homogeneous, the set of Killing vector fields \(X\)'s of constant length \(1\) of \(F\) exhaust all tangent directions. Because the length of \(W\) with respect to \(F\) is less than \(1\), the set of all the vector fields \(X + W\) exhaust all tangent directions too. So \((M, h)\) is restrictively CW-homogeneous. By the classification theorem in [4], \((M, h)\) is a Riemannian product of odd dimensional spheres with constant curvature metrics, compact connected simply connected Lie groups with bi-invariant metrics, and a flat Euclidean space.

The vector field \(W\) is of constant length with respect to \(F\), so it is of constant length with respect to \(h\).

Next we prove that, if \(M\) is the product manifold with a Randers metric \(F\) as described in the theorem, then \((M, F)\) is restrictively CW-homogeneous.

If \((M, F)\) is the product manifold as described in the theorem, then it is homogeneous. Let \(h^2 = \sum_{i=1}^{n} h_i^2\) be the decomposition for the symmetric metric, with respect to the decomposition \(M = M_1 \times \cdots \times M_n\), and denote \(W = \sum_{i=1}^{n} W_i\). Then each \(W_i\) is a Killing field of constant length for \(h_i\). A diffeomorphism \(\psi = \psi_1 \times \cdots \times \psi_n \in I(M, h)\) in which each \(\psi_i \in I(M_i, h_i)\) is an isometry for \((M, F)\) if and only if \(\psi_i\) keeps \(W_i\) invariant for each \(i\). We only need to check that it acts transitively for each factor. If \(W_i = 0\), then the argument for the corresponding factor is trivial. Thus we assume \(W_i \neq 0\). If \((M_i, h_i)\) is a compact connected simply connected simple Lie group and \(h_i\) is bi-invariant, then \(W_i\) must belong to the Lie algebra for \(L(M)\) or \(R(M)\). Without losing generality, we assume \(W_i\) belongs to the Lie algebra for \(L(M_i)\), then all right translations acts transitively on \(M_i\). If \((M_i, h_i)\) is an odd dimensional sphere \(S^{2k-1}\) with constant curvature metric, then \(W_i\) defines a complex structure \(J\) such that \(W_i = cJ\) with \(c \neq 0\), when both are regarded as matrices in \(so(2k)\). Then the isometries keeping \(W_i\) invariant are just the group \(U(k)\) with respect to the complex structure \(J\), and it is obvious that \(U(k)\) act transitively on \(M_i\). If \((M_i, h_i)\) is a flat euclidean space, then \(W_i\) is a constant vector field, and \(\psi_i\) can be any parallel translation, which also acts transitively on \(M_i\).

We now prove that the set of Killing fields of constant length for \(h\) which commute with \(W\) exhaust all tangent directions at each point. Any Killing fields \(X\) of constant length for \(h\) can be decomposed as \(X = \sum_{i=1}^{n} X_i\), in which each \(X_i\) is a Killing field of constant length for \(h_i\). The condition \([X, W] = 0\) is equivalent to \([X_i, W_i] = 0\) for each \(i\). So the discussion breaks down to a case by
Lemma 4.1. Let \((M, h_i)\) be a compact connected simply connected simple Lie group with bi-invariant metric or a flat euclidian space, the assertion is obvious because the group of right translations or the parallel translations are CW-translations for \(h_i\), and its Lie algebra provides Killing fields of constant length for \(h_i\) which commutes with \(W_i\). If \((M, h_i)\) is an odd dimensional sphere with constant curvature, the assertion follows immediately from the fact ([9]) that the Randers space with navigation data \((h_i, \lambda W_i)\) is CW-homogeneous, where \(\lambda\) is any constant such that \(||\lambda W_i||_{h_i} < 1\).

When \(||W||_{h} < 1\), the set of the vector fields \(X - W\), where \(X\) is a Killing vector field of constant length 1 with respect to \(h\) commuting with \(W\), exhaust all tangent directions. It is easily seen that any vector field \(X - W\) as above is a Killing vector field of constant length 1 for \(F\). By Theorem 3.4, \((M, F)\) is restrictively CW-homogeneous.

Finally, we prove the CW-homogeneity for a Randers space \((M, F)\) as described in the theorem. If \((M, F)\) is not CW-homogeneous, then we can find a Killing vector field of the form \(X - W\) of constant length 1 with respect to \(F\), where \(X\) is a Killing vector field of constant length 1 for \(h\) and \([X, W] = 0\). Moreover, there is a positive constant \(t_0\) and two pairs of points \(x_i\) and \(x_i'\), \(i = 0, 1\), such that the diffeomorphisms \(\varphi_{X - W; t}\) generated by \(X - W\) satisfy the condition \(\varphi_{X - W; t_0}(x_i) = x_i', i = 0, 1\), and the geodesic flow curve of \(X - W\) is minimizing from \(x_0\) to \(x_0'\) but not minimizing from \(x_1\) to \(x_1'\). Since \([X, W] = 0\), the diffeomorphisms \(\varphi_{X; t}\) commute with \(\varphi_{W; t'}\). Furthermore, we have

\[
\varphi_{X - W; t} = \varphi_{X; t'} \varphi_{W; -t} = \varphi_{W; -t} \varphi_{X; t'}.
\]

(2)

Since the flow curve from \(x_1\) to \(x_1'\) is not minimizing, there is a Killing vector field \(X' - W\) of constant length 1 for \(F\), where \(X'\) is a Killing vector field of constant length 1 for \(h\) with \([X', W] = 0\), and a constant \(t' \in [0, t_0)\) such that

\[
\varphi_{W; -t_0} \varphi_{X; t_0}(x_1) = \varphi_{W; -t'} \varphi_{X'; t'}(x_1).
\]

(3)
i.e., \(\varphi_{X; t_0}(x_1) = \varphi_{W; t_0 - t'} \varphi_{X'; t'}(x_1)\). Since \(h\) is a symmetric Riemannian metric, and since \(X\) is a Killing field of constant length 1 with respect to \(h\), the centralizer of \(X\) in the full group of isometries of \(h\) acts transitively on \(M\) (see [17, 12]). Thus there is an isometry \(g\) of \(h\), which sends \(x_1\) to \(x_0\), such that \(\text{Ad}(g)X = X\). Then we have

\[
\varphi_{\text{Ad}(g)X; t_0}(x_0) = \varphi_{X; t_0}(x_0) = \varphi_{\text{Ad}(g)W; t_0 - t'} \varphi_{\text{Ad}(g)X'; t'}(x_0).
\]

(4)

Since the right side of the above equality gives a path with arc length \(t' + (t_0 - t')||W||_h < t_0\), the geodesic \(\varphi_{X; t}(x_0)\) from \(t = 0\) to \(t = t_0\) is not minimizing with respect to \(h\). Now the next lemma asserts that the above equality still holds with \(g\) changed to \(e\), for some other \(X'\) and \(t'\) satisfying the same properties as above. This implies that the geodesic \(\varphi_{X - W; t}(x_0)\) for \(F\) is not minimizing between \(t = 0\) to \(t = t_0\), which is a contradiction. Therefore \((M, F)\) must be CW-homogeneous.

**Lemma 4.1.** Let \((M, h)\) be connected simply connected Riemannian CW-homogeneous space, \(X\) be a Killing vector field of constant length 1 for \(h\), and \(W\) be
a Killing field of constant length less than 1. Assume the geodesic \( \varphi_{X:t}(x_0) \) of \( h \) from \( t = 0 \) to \( t = t_0 \) is not minimizing. Then there is a \( t' \in [0, t_0) \) and a Killing vector field \( X' \) of constant length 1 for \( h \) with \([X', W] = 0\), such that

\[
\varphi_{X:t_0}(x_0) = \varphi_{W:t_0-t'X':t'}(x_0).
\]  

(5)

\[\text{Proof.} \quad \text{Let } f(t) \text{ be the distance from } x_0 \text{ to } \varphi_{W:t}\varphi_{X:t_0}(x_0) \text{ with respect to } h. \text{ Obviously } f(t) \text{ is continuous and non-negative for all } t \geq 0 \text{ and } f(0) < t_0 \text{ because the geodesic } \varphi_{X:t}(x_0) \text{ from } t = 0 \text{ to } t = t_0 \text{ is not minimizing. There exists a } t' \in [0, t_0) \text{ such that } f(t_0 - t') = t'. \text{ The completeness of } M \text{ implies there is a minimizing geodesic from } x_0 \text{ to } \varphi_{W:t_0-t'}x_{X:t_0}(x_0). \text{ Our earlier arguments indicate that this minimizing geodesic is the flow curve of a Killing vector field } X' \text{ of constant length } 1 \text{ with respect to } h \text{ with } [X', W] = 0, \text{ i.e.}
\]

\[
\varphi_{X:t_0}(x_0) = \varphi_{W:t_0-t'X':t'}(x_0).
\]  

(6)

This completes the proof of the lemma as well as of the theorem. \[\Box\]

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