Lyapunov dimension formula
of attractors in the Tigan and Yang systems

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Abstract

In latter days the technique of attractors dimension estimate of Lorenz type systems is actively developed. In this work the Lyapunov dimension of attractors of the Tigan and Yang systems is estimated.

Keywords: Lorenz-like system, Lorenz system, Yang system, Tigan system, Kaplan-Yorke dimension, Lyapunov dimension, Lyapunov exponents.

1. Introduction

Nowadays various estimates of Lyapunov dimension of attractors of generalized Lorenz systems

\[
\begin{align*}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= rx - dy - xz, \\
\dot{z} &= -bz + xy,
\end{align*}
\]

(1)

are actively developed [1, 2]. One particular case of (1) system is the Yang system [3]:

\[
\begin{align*}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= rx - xz, \\
\dot{z} &= -bz + xy,
\end{align*}
\]

(2)

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where \( \sigma, b \) are positive, \( r \) is arbitrary real number. In [4] an estimation of Lyapunov dimension for the generalized Lorenz system (1) was obtained (see also remarks in [5]), which for the case \( d = 0 \) (i.e. for the Yang system (2)) is as follows:

**Theorem 1.** [4] Let \( K \) be an invariant compact set of the Yang system (2). Assume \( \sigma > 0, b > 0, r > 0, b(\sigma - b) + \sigma r > 0 \), and if one of the following conditions holds

1. \( 2\sigma \geq b, \sigma^2(3b + r) - b^3 - \sigma b(6b + r) > 0 \)
2. \( 2\sigma < b, 3\sigma^2(b + r) - 6\sigma b^2 - b^3 > 0, \)

then \( \text{dim}_L K \leq 3 - \frac{2(\sigma + b)}{\sigma + \sqrt{\sigma^2 + 4\sigma r}}. \)

In our work we extend the domain of parameters, where the above estimation is valid.

2. **Lyapunov functions in the dimension theory**

Consider an autonomous differential equation

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad (3)
\]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is smooth. Suppose that solution \( x(t, x_0), x(0, x_0) = x_0 \) exist for all \( t \geq 0 \) and consider corresponding linearized system along the solution:

\[
\dot{u} = J(x(t, x_0)) u, \quad u \in \mathbb{R}^n, \quad t \in \mathbb{R}_+,
\]

where

\[
J(x(t, x_0)) = \left[ \frac{\partial f_i(x)}{\partial x_j} \bigg|_{x=x(t, x_0)} \right]
\]

is the \((n \times n)\) Jacobian matrix. By a linear variable change \( y = Sx \) with a nonsingular \((n \times n)\)-matrix \( S \) system (3) is transformed into

\[
\dot{y} = S \dot{x} = S f(S^{-1}y) = \tilde{f}(y).
\]

Consider the linearization along the corresponding solution \( y(t, y_0) = Sx(t, S^{-1}x_0) \):

\[
\dot{v} = \tilde{J}(y(t, y_0)) v, \quad v \in \mathbb{R}^n.
\]

Here the Jacobian matrix is as follows

\[
\tilde{J}(y(t, y_0)) = S J(x(t, x_0)) S^{-1}.
\]
Suppose that $\lambda_1(x, S) \geq \cdots \geq \lambda_n(x, S)$ are eigenvalues of the matrix
\[
\frac{1}{2} \left( S J(x(t, x_0)) S^{-1} + (S J(x(t, x_0)) S^{-1})^* \right).
\tag{7}
\]

**Theorem 2** ([1, 6]). Given an integer $j \in [1, n - 1]$ and $s \in [0, 1]$, suppose that there are a continuously differentiable scalar function $\vartheta : \mathbb{R}^n \rightarrow \mathbb{R}$ and a nonsingular matrix $S$ such that
\[
\lambda_1(x, S) + \cdots + \lambda_j(x, S) + s\lambda_{j+1}(x, S) + \dot{\vartheta}(x) < 0, \quad \forall x \in K.
\tag{8}
\]
Then $\dim L K \leq j + s$.

Here $\dot{\vartheta}$ is the derivative of $\vartheta$ with respect to the vector field $f$:
\[
\dot{\vartheta}(x) = (\text{grad}(\vartheta))^* f(x).
\]

The introduction of the matrix $S$ can be regarded as a change of the space metric.

**Theorem 3** ([1, 7–9]). Assume that there are a continuously differentiable scalar function $\vartheta$ and a nonsingular matrix $S$ such that
\[
\lambda_1(x, S) + \lambda_2(x, S) + \dot{\vartheta}(x) < 0, \quad \forall x \in \mathbb{R}^n.
\tag{9}
\]
Then any solution of system (3) bounded on $[0, +\infty)$ tends to an equilibrium as $t \to +\infty$.

Thus, if (9) holds, then the global attractor of system (3) coincides with its stationary set.

3. Main result

We can prove the following result for the Yang system.

**Theorem 4.** 1. Assume $r = 0$ and the following inequalities $b(\sigma - b) > 0$, $\sigma - \frac{(\sigma + b)^2}{4(\sigma - b)} \geq 0$ are satisfied. Then any bounded on $[0; +\infty)$ solution of system (2) tends to a certain equilibrium as $t \to +\infty$.

2. Assume $r < 0$ and $r \sigma + b(\sigma - b) > 0$. Then any bounded on $[0; +\infty)$ solution of system (2) tends to a certain equilibrium as $t \to +\infty$. 3
3. Assume $r > 0$ and there are two distinct real roots $\gamma^{(II)} > \gamma^{(I)}$ of equation

$$4br\sigma^2(\gamma + 2\sigma - b)^2 + 16b\gamma(r\sigma^2 + b(\sigma + b)^2 - 4\sigma(\sigma r + \sigma b - b^2)) = 0,$$

such that $\gamma^{(II)} > 0$.

In this case

(a) if $b(b - \sigma) < r\sigma < b(\sigma + b)$,

then any bounded on $[0; +\infty)$ solution of system (2) tends to a certain equilibrium as $t \rightarrow +\infty$.

(b) if $r\sigma > b(\sigma + b)$,

then

$$\dim L K = 3 - \frac{2(\sigma + b)}{\sigma + \sqrt{\sigma^2 + 4\sigma r}},$$

(12)

where $K \supset (0, 0, 0)$ is bounded invariant set of system (2).

For proving Theorem 4 we use Theorems 2 and 3 with the matrix $S$ of the form

$$S = \begin{pmatrix}
-\rho^{-1} & 0 & 0 \\
-\frac{b}{\sigma} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},$$

and the function

$$\vartheta(x, y, z) = \frac{(1 - s)V(x, y, z)}{\sigma^2 + 4\sigma r}^{\frac{1}{2}},$$

where

$$\rho = \frac{\sigma}{\sigma r + b(\sigma - b)}.$$

$$V(x, y, z) = \gamma_4x^2 + (\sigma\gamma_2 + \gamma_3)y^2 + \gamma_3z^2 - \frac{1}{4\sigma}\gamma_2x^4 + \gamma_2x^2z + \gamma_1\gamma_2xy - \frac{\sigma}{b}z.$$
and the greater root of corresponding equation (10) is equal

\[ \gamma_{1(II)} = \frac{6358}{135} + \frac{14}{135} \sqrt{178309}. \]

In the second case inequality (11) is as follows

\[ 1225 > 114 \]

and the greater root of (10) is equal

\[ \gamma_{1(II)} = \frac{193391}{1225} + \frac{2}{1225} \sqrt{7665943314}. \]

Thus, classical chaotic parameters satisfy the requirement of Theorem 4 and the formula (12) is valid.

4. T-system

Consider the Tigan system (T-system) [10, 11]:

\[
\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= (c - a)x - axz, \\
\dot{z} &= -bz + xy.
\end{align*}
\]  \hspace{1cm} (13)

By the transformation

\[ x \rightarrow \frac{x}{\sqrt{a}}, y \rightarrow \frac{y}{\sqrt{a}}, z \rightarrow \frac{z}{a} \]

one has

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= (c - a)x - xz \\
\dot{z} &= -bz + xy.
\end{align*}
\]  \hspace{1cm} (14)

So the T-system can be transformed to the Yang system with the following parameters \( \sigma = a, r = c - a \). Both systems were independently considered in 2008 year. But a particular case of the T-system was considered in 2004 [12].

The T-system has classical chaotic parameters \((a, b, c) = (2.1, 0.6, 30)\). Conditions of Theorem 4 are valid in the case \( \sigma = \frac{21}{10}, b = \frac{6}{10}, r = \frac{279}{10} \) and takes the form:
(1) parameter \( r = 27.9 > 0 \),
(2) inequality (11) is transformed to
\[
\frac{5859}{100} > \frac{81}{50},
\]
and, thus, is valid,
(3) the roots of the corresponding equation (10) are
\[
\gamma^{(I)} = \frac{9883}{1085} - \frac{1}{1085} \sqrt{82416853}
\]
and \( \gamma^{(II)} = \frac{9883}{1085} + \frac{1}{1085} \sqrt{82416853} \), and, thus, the greater root is positive.

Therefore, if \( K \supset (0,0,0) \) is a bounded invariant set of system (13) with the parameters \( \sigma = 2.1, b = 0.6, r = 27.9 \), then
\[
\dim L K = 3 - \frac{2(\sigma + b)}{\sigma + \sqrt{\sigma^2 + 4\sigma r}} \approx 2.692345984.
\]

5. Conclusions

The Yang system (2) by the transformation
\[
x \rightarrow \sigma x, \quad y \rightarrow \sigma y, \quad z \rightarrow \sigma z, \quad t \rightarrow \frac{t}{\sigma}, \quad \sigma \neq 0
\]
(15)
takes the form
\[
\begin{cases}
\dot{x} = y - x, \\
\dot{y} = \frac{r}{\sigma}x - xz, \\
\dot{z} = -\frac{b}{\sigma}z + xy.
\end{cases}
\]
(16)
For \( \sigma = 0 \) the Yang system becomes linear and its dynamics has minor interest. Thus, without loss of generality, one can assume that \( \sigma = 1 \). Below we compare the domains of parameters in Theorem 4 and Theorem 1 for \( \sigma = 1 \).

The points \( P1 \) and \( P2 \) are the parameters which correspond the classical chaotic self-excited attractors\(^1\) in the Yang system, \( P3 \) — classical chaotic self-excited attractor in the T-system, \( P4, P5 \) — parameters, for which we have also found chaotic self-excited attractors by numerical analysis.

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\(^1\)An attractor is called a self-excited attractor if its basin of attraction intersects with any open neighborhood of an equilibrium, otherwise it is called a hidden attractor [13–16].
Figure 1: Domains of parameters for $\sigma = 1$, $r < 15$. (a) Theorem 4. (b) Theorem 1.

Figure 2: Self-excited attractors in the Yang system: (a) $P_4$ — $\sigma = 1$, $b = 0.8$, $r = 9.1$. (b) $P_5$ — $\sigma = 1$, $b = 0.8$, $r = 9.7$.

Up to now in such Lorenz-like systems as Lorenz, Chen, Lu and Tigan systems self-excited chaotic attractors have been found only. In such Lorenz-like systems as Glukhovsky-Dolghansky and Rabinovich systems both self-excited and hidden attractors can be found [16–18]. Recent examples of hidden attractors can be found in The European Physical Journal Special Topics: Multistability: Uncovering Hidden Attractors, 2015 (see [19–30]). Note that while coexisting self-excited attractors can be found by the standard computational procedure, there is no regular way to predict the existence or coexistence of hidden attractors.
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