An antiperiodic dynamical six-vertex model: I. Complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic eight-vertex model

G Niccoli
YITP, Stony Brook University, New York, NY 11794-3840, USA
E-mail: niccoli@max2.physics.sunysb.edu

Received 17 July 2012, in final form 25 November 2012
Published 4 February 2013
Online at stacks.iop.org/JPhysA/46/075003

Abstract
The spin-1/2 highest weight representations of the dynamical six-vertex and the standard eight-vertex Yang–Baxter algebra on a finite chain are considered in this paper. In particular, the integrable quantum models associated with the corresponding transfer matrices under antiperiodic boundary conditions for the dynamical six-vertex case and periodic boundary conditions for the eight-vertex case are analyzed here. For the antiperiodic dynamical six-vertex transfer matrix defined on chains with an odd number of sites, we adapt Sklyanin’s quantum separation of variable (SOV) method and explicitly construct the SOV representations from the original space of the representations. In this way, we provide the complete characterization of the eigenvalues and the eigenstates proving also the simplicity of its spectrum. Moreover, we characterize the matrix elements of the identity on separated states of this model by determinant formulae. The matrices entering these determinants have elements given by sums over the SOV spectrum of the product of the coefficients of the separate states. This SOV analysis is done without any need to be reduced to the case of the so-called elliptic roots of unit, and the results derived here define the required setup to extend to the dynamical six-vertex model the approach recently developed by the author and collaborators to compute the form factors of the local operators in the SOV framework. For the periodic eight-vertex transfer matrix, we prove that its eigenvalues have to satisfy a fixed system of equations. In the case of a chain with an odd number of sites, this system of equations is the same entering in the SOV characterization of the antiperiodic dynamical six-vertex transfer matrix spectrum. This implies that the set of the periodic eight-vertex eigenvalues is contained in the set of the antiperiodic dynamical six-vertex eigenvalues. A criterion is introduced to find simultaneous eigenvalues of these two transfer matrices and associate with any of such eigenvalues one nonzero eigenstate of the periodic eight-vertex transfer matrix by using the SOV results. Moreover, a preliminary discussion
on the degeneracy occurring for odd chains in the periodic eight-vertex transfer matrix spectrum is also presented.

PACS numbers: 02.30.Ik, 75.10.Jm, 05.50.+q

1. Introduction

In this paper, we analyze two classes of lattice integrable quantum models characterized in the quantum inverse scattering method (QISM) [1–14] by monodromy matrices which are solutions of the dynamical Yang–Baxter equation w.r.t. the six-vertex dynamical $R$-matrix and the (standard) Yang–Baxter equation w.r.t. the eight-vertex $R$-matrix. The representation theory of the dynamical six-vertex Yang–Baxter algebra was introduced by Felder in [15] by the so-called theory of the elliptic quantum groups; see also [16]. There, it was recognized that the known Boltzmann weights defining the SOS (solid-on-solid) statistical models [17] when opportunely reorganized in a $4 \times 4$ matrix define the $R$-matrix solution of the dynamical six-vertex Yang–Baxter algebra. The prototypical elements in these classes of integrable quantum models are constructed by defining representations of the corresponding monodromy matrices on chains of two-dimensional representations (spin-1/2 quantum chains). Under homogeneous limits, these representations define in the dynamical six-vertex case the SOS model [17–21], while in the eight-vertex case the spin-1/2 XYZ quantum chain [22–30]. It is worth recalling that the monodromy matrices of these models are related by Baxter’s intertwining vectors and the spectral problems of the transfer matrices under periodic boundary conditions have been analyzed by Bethe ansatz and $Q$-operator techniques in the case of chains with an even number of sites. In [18, 17, 19], Baxter has introduced the intertwining vectors, also called gauge transformations, in order to be able to use Bethe ansatz techniques to analyze the spectral problem (eigenvalues and eigenstates) of the eight-vertex transfer matrix reducing it to one of the six-vertex types. The use of gauge transformations allows us in particular to overcome the problem of the absence of reference states opening the possibility of analyzing the eight-vertex spectral problem by using the algebraic Bethe ansatz (ABA) [1, 2], as pioneered by Faddeev and Takhtajan in [3], while ABA analysis for the SOS model with periodic boundary conditions was developed in [20]. However, it is worth remarking that, a part of the general problem related to the proof of the completeness of the spectrum description, the analysis of the spectrum of these models by Bethe ansatz methods leads to the introduction of two constraints. The first constraint is on the number of sites of the quantum chains, which has to be even. This is required to obtain the commutativity of the dynamical six-vertex transfer matrix, which holds only for the reduction to the total spin-zero eigenspace under periodic boundary conditions. The second constraint is on the allowed values of the coupling constant $\eta$ of the eight-vertex transfer matrix, which have to be restricted to the so-called cyclic values or elliptic roots of unit (i.e. when $\eta$ belongs to an integer square lattice with steps the periods of the theta functions). This is required in order to construct eight-vertex transfer matrix eigenstates by finite sums of the dynamical six-vertex ones. In addition to the previously described constraints in the algebraic Bethe ansatz framework, the lack of a scalar

---

1 For the $Q$-operator construction, see [17, 31] and also the series of papers [32–37].

2 Let us comment that historically Baxter has used a vectorial representation for these transformations, e.g. see equation (3.3) of [17], which explains the original use of the terminology intertwining vectors. Here, we use the terminology gauge transformations, also used in [3], to refer to the matrix representation of the same transformations presented in (3.39).

3 In fact, for the periodic eight-vertex transfer matrices, the completeness of the spectrum description is verified only by some numerical analysis [38].
product analogue to the Slavnov formula [39–41] is the first fundamental missing step toward the computation of matrix elements of local operators. It is then clear that there is the need to overcome these problems in order to compute correlation functions.

In this paper, we implement a modified version of Sklyanin’s quantum separation of variables (SOVs) [42–44]. In particular, we derive the complete characterizations of the spectrum of the antiperiodic\textsuperscript{4} dynamical six-vertex transfer matrix defined on chains with an odd number of sites. Moreover, we compute the matrix elements of the identity for general separate states\textsuperscript{5} which apply in particular for the eigenstates of the antiperiodic dynamical six-vertex transfer matrix. Let us comment that the existing results [46, 47] for the antiperiodic dynamical six-vertex model are mainly restricted to the construction of the functional separation of variables of Sklyanin. In this functional version, an SOV representation of the dynamical six-vertex Yang–Baxter algebra is defined on a space of symmetric functions leading only to the description of the wavefunctions of the transfer matrix eigenstates. In fact, the explicit construction of the SOV representation as well as the transfer matrix eigenstates in the original representation space of the quantum chain is so far missing.

For the periodic eight-vertex transfer matrix, we prove that the set of all its eigenvalues is contained in the set of the solutions to an inhomogeneous system of $N$ quadratic equations in $N$ unknowns, where $N$ is the number of sites of the chain. In the case when $N$ is odd, this system coincides with the one entering in the SOV characterization of the antiperiodic dynamical six-vertex transfer matrix spectrum and so the set of the periodic eight-vertex eigenvalues is proven to be contained in the antiperiodic dynamical six-vertex one. Let us recall that the analysis of the odd $N$ case is of particular interest for the periodic eight-vertex transfer matrix as in this case the Bethe ansatz analysis of [18, 17, 19, 3] does not apply. We use the Baxter gauge transformations to further relate the periodic eight-vertex and the antiperiodic dynamical six-vertex transfer matrices, allowing us to analyze the eight-vertex spectral problem in terms of the SOV characterization derived here. It is worth stressing that in the dynamical quantum space $\mathcal{D}_{(6VD),N}$, characterized by the condition that the antiperiodic dynamical six-vertex transfer matrix is a one-parameter family of commuting operators, these gauge transformations are not invertible operators. Nevertheless, we are able to use them to get a sufficient criterion which allows us to select simultaneous eigenvalues of the antiperiodic dynamical six-vertex and the periodic eight-vertex transfer matrices and to associate with any one of these eigenvalues one corresponding nonzero eight-vertex eigenstate. In this paper we will explain that the non-invertible character of the gauge transformations on the dynamical quantum space $\mathcal{D}_{(6VD),N}$ is a natural requirement as these transformations link transfer matrices with different degeneracy properties. Indeed, while the antiperiodic dynamical six-vertex transfer matrix is proven here to have simple spectrum, the periodic eight-vertex one has degeneracy even for completely general inhomogeneities. A preliminary analysis of this degeneracy issue is presented here by analyzing explicitly the periodic eight-vertex transfer matrix spectrum for chains with one and three sites.

The results derived in this paper represent the first fundamental step in an approach to solve integrable quantum models, which can be considered as the generalization to the SOV framework of the Lyon group method [41, 48–63]. The use of SOV is a strong point of our approach as it works for a large class of integrable quantum models; under simple conditions, it leads to the complete construction of both the eigenvalues and the eigenstates of the transfer matrix, and the simplicity of the spectrum can be easily shown in this framework. Moreover, the analysis developed in this paper and that implemented previously in [69–73] suggest that

\textsuperscript{4} This quantum integrable model has been introduced in [45].

\textsuperscript{5} See section 2.4 for the definition.
this approach can lead to a universal representation of both the spectrum and the dynamics of a class of integrable quantum models which were not entirely solvable with other methods. Indeed, this is the case for all the key integrable quantum models analyzed so far in [69–72]. In more detail, in [71] and [72], the XXZ spin-1/2 quantum chain [22–30] and the higher spin-XXX quantum chain, both under antiperiodic boundary conditions, have been characterized and the form factors of the local spin operators have been represented in a determinant form. Similar results for the form factors have been obtained previously by this approach in [69, 70] for the lattice quantum sine-Gordon model [2, 14], the chiral Potts model [78–89] and the $\tau_2$-model [90]. These results are obtained by using as background the complete SOV spectrum characterization constructed in [91–93] for the lattice quantum sine-Gordon model and in [94–98] for the general cyclic representations of the six-vertex Yang–Baxter algebra corresponding to the $\tau_2$-model and the chiral Potts model. Moreover, in [73] the SOV setup has been implemented and the matrix elements of some interesting quasi-local string of local operators have been computed for the integrable quantum model associated with the spin-1/2 representations of the reflection algebra [99–106], under quite general non-diagonal boundary conditions. In all these models, the matrix elements of local operators on separate states are characterized by determinant formulae written as simple modifications of those of the identity. The main differences in all these formulae are only due to model-dependent features, like the nature of the spectrum of the quantum separate variables and the form of the SOV reconstruction of local operators. Let us comment that in the literature there exist previous results on matrix elements of local operators which even if developed by different approaches made use of quantum separation of variables. The results in Smirnov’s paper [107] are of special interest; there, for the quantum integrable Toda chain [42], the form factors of a conjectured basis of local operators are derived in Sklyanin’s SOV framework by determinant formulae which confirm the universal picture outlined. It is also worth pointing out that the form factors of the restricted sine-Gordon model at the reflectionless points in the $S$-matrix formulation [119, 120] admit once again determinant representations and the connection with SOV is established on the basis of the semi-classical analysis of [119], used there also as a tool to overcome the problem of the local field identification.

6 Like the algebraic Bethe ansatz, the coordinate Bethe ansatz [23, 64, 65], the Baxter $Q$-operator method [64] and the analytic Bethe ansatz [66, 67].

7 Let us comment that previous results on this model with antiperiodic boundary conditions were mainly given by the $Q$-operator construction [68] and the functional separation of variables [43, 74]. Moreover, see [75] for the eigenvalue analysis of the XXZ spin-1/2 chain with general antiperiodic boundary conditions by a functional method based on the Yang–Baxter algebra; the method also applies to open chain for general integrable boundary conditions.

8 Instead, in the periodic chain matrix, elements of local operators were computed in the ABA framework in [76, 77].

9 See [94–97] for the first analysis by the SOV method of the $\tau_2$-model and some results on form factors in the restricted case of the generalized Ising model.

10 There this conjecture is required by the absence of a direct SOV reconstruction of local operators. Later, a reconstruction was given in [108] w.r.t. a new set of quantum separate variables defined by a change of variables on the original Sklyanin ones.

11 Form factors which have been rederived in [109] also by exploiting previous results established in the series of papers [110–113] for the infinite volume limit of the XXZ spin-1/2 chain.

12 See [114–118] and references therein.

13 This is a longstanding problem in the $S$-matrix formulation which has been so far addressed by exploiting the description [121–124] of massive IQFTs as (super-)renormalizable perturbations of conformal field theories [125–129] by relevant local fields. Several results are known which allow us to classify the local fields of massive theories (i.e. the solutions to the form factor equations [130–145]) in terms of those of the ultraviolet conformal field theories, see for example [146–149] and the series of works [150–155].
2. The dynamical six-vertex models: spectrum and elementary matrix elements

2.1. The dynamical six-vertex models

In the following, we introduce an operator \( \tau \) whose eigenvalues on the space of the representation coincide with the dynamical parameter \( t \). The aim is to recover a separate description for the dynamical parameter which will be particularly useful in the SOV description of the antiperiodic dynamical six-vertex spectral problem.

2.1.1. Representation spaces of dynamical and spin operators. Let us introduce a couple of dynamical operators \( \tau \) and \( T_x^\pm \) which satisfy the following commutation relations:

\[
T_x^+ \tau = (\tau \pm \eta)T_x^+, \tag{2.1}
\]

and \( N \) copies of (local spin) \( sl(2) \) generators \( S^a_n \) and let us impose the following commutation relations:

\[
\left[ S^a_n, S^b_m \right] = \left[ S^a_n, T_x^\pm \right] = 0 \quad \forall n, m \in \{1, \ldots, N\} \text{ and } a, b = x, y, z. \tag{2.2}
\]

Then the space of the representation of these dynamical and spin operators can be chosen as follows:

\[
\mathbb{D}_{(\text{RVD})N} \equiv \mathbb{D}_N \otimes \bigotimes_{n=1}^N \mathbb{C}^2 \otimes \bigotimes_{n=1}^N \mathbb{C}^2, \tag{2.3}
\]

where \( \mathbb{D}_N \) is the space of the representation of the dynamical operators which is infinite dimensional in our definition.\(^{14}\) Here, we have chosen for all the \( sl(2) \) generators the spin-1/2 representation, i.e. a two-dimensional local quantum space \( \mathbb{C}^2 \) is associated with any site of the chain and the local spin generators are represented by the \( 2 \times 2 \) Pauli matrices \( \sigma^a_n \).

Moreover, we introduce the following definition of left (covecctors) and right (vectors) \( \tau \)-eigenbasis of \( \mathbb{D}_N^L \) and \( \mathbb{D}_N^R \), respectively:

\[
|t(a)\rangle \equiv |t(0)\rangle|T_x^+\rangle, \quad |\tau(t(a))\rangle \equiv T_x^{-a}|t(0)\rangle, \quad \forall a \in \mathbb{Z}, \tag{2.4}
\]

with

\[
|t(a)|\tau = t(a)|t(a)\rangle, \quad \tau|t(a)\rangle = t(a)|t(a)\rangle, \quad t(a) \equiv -\frac{\eta}{2}a \quad \forall a \in \mathbb{Z}, \tag{2.5}
\]

where we fix \( |t(a)|\tau = t(a)|t(a)\rangle, \quad \tau|t(a)\rangle = t(a)|t(a)\rangle, \quad t(a) \equiv -\frac{\eta}{2}a \quad \forall a \in \mathbb{Z} \). Moreover, defining the following left and right spin bases:

\[
|n, h_n\rangle|\sigma^+ - 2h_n\rangle|n, h_n\rangle, \quad |\sigma^- n, h_n\rangle = (1 - 2h_n)|n, h_n\rangle, \quad h_n \in \{0, 1\}, \tag{2.6}
\]

with \( |n, h_n\rangle|n, h_n\rangle' = \delta_{h_n, h_n'} \), in each local quantum spin chain of the representation, we can introduce the left and right dynamical-spin bases in \( \mathbb{D}_{(\text{RVD})N}^L \) and \( \mathbb{D}_{(\text{RVD})N}^R \), respectively:

\[
\otimes_{n=1}^N |n, h_n\rangle \otimes |t(a)\rangle, \quad \otimes_{n=1}^N |n, h_n\rangle \otimes |t(a)\rangle, \tag{2.7}
\]

composed of common eigenstates of the commuting operators \( \tau \) and \( \sigma^\pm \). A scalar product is introduced in the space \( \mathbb{D}_{(\text{RVD})N}^R \) by defining its action on the elements of the dynamical-spin basis:

\[
( \otimes_{n=1}^N |n, h_n\rangle \otimes |t(a)\rangle, \otimes_{n=1}^N |n, h_n'\rangle \otimes |t(a')\rangle) = \delta_{a,a'} \prod_{n=1}^N \delta_{h_n, h_n'}. \tag{2.8}
\]

\(^{14}\) Note that in the root of unit case, we can also define a cyclic representation for the operator \( \tau \). This point in the present formalism will be described elsewhere; anyhow, for the antiperiodic chain this cyclicity condition does not play a fundamental role as it did in the periodic case for the application of ABA to the eight-vertex model.
Note that we have defined the representation in a way that the spectrum (eigenvalues) of the operator $\tau$ contains that of $-\eta S/2$, where
\[
S = \sum_{n=1}^{N} \sigma_n^z
\]  
(2.9)
is the total $z$-component of the spin; the reason for that will be clear in the following.

2.1.2. Representations of the dynamical six-vertex models. Let us define the elliptic dynamical six-vertex $R$-matrix.\(^{15}\)
\[
R^{(\text{EV})}_{\omega}(\lambda|\tau) = \begin{pmatrix}
a(\lambda) & 0 & 0 & 0 \\
0 & b(\lambda|\tau) & c(\lambda|\tau) & 0 \\
0 & c(\lambda|\tau) & b(\lambda|\tau) & 0 \\
0 & 0 & 0 & a(\lambda)
\end{pmatrix},
\]  
(2.10)
where $a(\lambda)$, $b(\lambda|\tau)$ and $c(\lambda|\tau)$ are defined by
\[
a(\lambda) = \theta(\lambda + \eta)/\theta(\tau), \quad b(\lambda|\tau) = \frac{\theta(\lambda)\theta(\tau + \eta)}{\theta(\tau)} , \quad c(\lambda|\tau) = \frac{\theta(\eta)\theta(\tau + \lambda)}{\theta(\tau)} ;
\]  
(2.11)
here and in the following, we use the notation\(^{16}\)
\[
\theta(\lambda) = \theta_1(\lambda | \omega),
\]  
(2.12)
where $\theta_1(\lambda | \omega)$ is the standard theta-1 elliptic function of the modular parameter $\omega$. Then, the $R$-matrix $R^{(\text{EV})}_{1,2}(\lambda | \tau)$ is the solution of the following dynamical Yang–Baxter equation:
\[
R^{(\text{EV})}_{1,2}(\lambda_{12} | \tau + \eta \sigma_3^1)R^{(\text{EV})}_{2,1}(\lambda_1 | \tau)R^{(\text{EV})}_{2,1}(\lambda_2 | \tau + \eta \sigma_3^2) = R^{(\text{EV})}_{2,1}(\lambda_2 | \tau)R^{(\text{EV})}_{2,1}(\lambda_1 | \tau + \eta \sigma_3^1)R^{(\text{EV})}_{1,2}(\lambda_{12} | \tau),
\]  
(2.13)
where $\lambda_{12} \equiv \lambda_1 - \lambda_2$. It is possible to introduce the following dynamical six-vertex monodromy matrix:
\[
M^{(\text{EV})}_{0}(\lambda | \tau) \equiv R^{(\text{EV})}_{0,1}(\lambda - \xi_0 | \tau + n \sum_{a=1}^{N-1} \sigma_a^3) \cdots R^{(\text{EV})}_{0,1}(\lambda - \xi_1 | \tau) \equiv \begin{pmatrix} A(\lambda | \tau) & B(\lambda | \tau) \\ C(\lambda | \tau) & D(\lambda | \tau) \end{pmatrix},
\]  
(2.14)
where $\xi_n$ for $n \in \{1, \ldots, -N\}$ are the parameters of the model called inhomogeneities. Then this monodromy matrix is a solution of the same type of dynamical Yang–Baxter equation:
\[
R^{(\text{EV})}_{1,2}(\lambda_{12} | \tau + \eta S)M^{(\text{EV})}_{1}(\lambda_1 | \tau)M^{(\text{EV})}_{2}(\lambda_2 | \tau + \eta \sigma_3^1) = M^{(\text{EV})}_{2}(\lambda_2 | \tau)M^{(\text{EV})}_{1}(\lambda_1 | \tau + \eta \sigma_3^2)R^{(\text{EV})}_{1,2}(\lambda_{12} | \tau),
\]  
(2.15)
where $S$ is the total $z$-component of the spin defined in (2.9). Moreover, it is worth remarking that the following commutation relations hold:
\[
[A(\lambda | \tau), \tau] = [B(\lambda | \tau), \tau] = [C(\lambda | \tau), \tau] = [D(\lambda | \tau), \tau] = 0
\]  
(2.16)
and
\[
[A(\lambda | \tau), S] = [D(\lambda | \tau), S] = 0, \quad [C(\lambda | \tau), S] = -2C(\lambda | \tau), \quad [B(\lambda | \tau), S] = 2B(\lambda | \tau).
\]  
(2.17)
\(^{15}\)The presentation of our results will be given directly in this elliptic case; however, it is interesting to remark also that the trigonometric case corresponding to the following choice of dynamical $R$-matrix,
\[
a(\lambda) = \sinh(\lambda + \eta), \quad b(\lambda|\tau) = \frac{\sinh \lambda \sinh(\tau + \eta)}{\sinh \tau}, \quad c(\lambda|\tau) = \frac{\sinh \eta \sinh(\tau + \lambda)}{\sinh \tau},
\]  
can be similarly described in our approach.
\(^{16}\)In this paper, $\theta_i(\lambda | \omega)$ for $i \in \{1, \ldots, 4\}$ are the standard theta functions as defined for example on page 877 of [188], where we used for the argument of these functions $(\lambda | \omega)$ instead of $(\omega | \tau)$.
Note that defining
\[ T^a_{\tau} \equiv \begin{pmatrix} T^a_{\tau} & 0 \\ 0 & T^a_{\tau} \end{pmatrix}, \]
we can rewrite the dynamical Yang–Baxter equations in the following form:
\[ R^{(6VD)}_{1,2}(\lambda_{12} | \tau + \eta S) M^{(6VD)}_{1}(\lambda_1 | \tau) T^a_{\tau} M^{(6VD)}_{2}(\lambda_{2} | \tau) T^{-a}_{\tau} = M^{(6VD)}_{2}(\lambda_{2} | \tau) T^a_{\tau} M^{(6VD)}_{1}(\lambda_1 | \tau) T^{-a}_{\tau} R^{(6VD)}_{1,2}(\lambda_{12} | \tau). \] (2.19)
Let us remark that while the dynamical six-vertex generators \( B(\lambda | \tau) \) and \( C(\lambda | \tau) \) are nilpotent operators of order \( N + 1 \), for a chain of size \( N \), the generators \( A(\lambda | \tau) \) and \( D(\lambda | \tau) \) are not nilpotent operators. So, from the form of the dynamical six-vertex Yang–Baxter relations, it is clear that the set spanned by the dynamical parameter \( t \) (the eigenvalues of \( \tau \)) is always an infinite lattice of step \( \eta \). We can also define the following monodromy matrix:
\[
\begin{pmatrix}
A(\lambda | \tau) \\ C(\lambda | \tau) \\ D(\lambda | \tau)
\end{pmatrix}
= M^{(6VD)}_{0}(\lambda | \tau) \equiv M^{(6VD)}_{0}(\lambda | \tau) T^a_\tau,
\]
note that \( X(\lambda | \tau) \) (\( X = A, B, C \) and \( D \)) are the operator functions of \( \tau \) and \( T^a_\tau \), but for simplicity we omit the explicit dependence from \( T^a_\tau \) in their arguments. For this monodromy matrix, the dynamical Yang–Baxter equation reads
\[
R^{(6VD)}_{1,2}(\lambda_{12} | \tau + \eta S) M^{(6VD)}_{1}(\lambda_1 | \tau) M^{(6VD)}_{2}(\lambda_{2} | \tau) = M^{(6VD)}_{2}(\lambda_{2} | \tau) M^{(6VD)}_{1}(\lambda_1 | \tau) R^{(6VD)}_{1,2}(\lambda_{12} | \tau),
\] (2.21)
where we have used that
\[
T^{-a}_{\tau} T^{-a}_{\tau} R^{(6VD)}_{1,2}(\lambda_{12} | \tau) T^a_{\tau} = R^{(6VD)}_{1,2}(\lambda_{12} | \tau). \] (2.22)
Finally, let us comment that in [16] it was shown that the dynamical six-vertex Yang–Baxter equations (2.13) are just the rewriting of the Baxter star–triangle equations for the Boltzmann weights
\[
W\begin{bmatrix} t & t + 1 & \lambda \\ t + 1 & t + 2 & \lambda \\ t & t + 1 & \lambda \end{bmatrix} = a(\lambda), \quad W\begin{bmatrix} t & t + 1 & \lambda \\ t + 1 & t & \lambda \\ t & t + 1 & \lambda \end{bmatrix} = b(\lambda | t),
\]
\[
W\begin{bmatrix} t & t + 1 & \lambda \\ t - 1 & t & \lambda \\ t - 1 & t & \lambda \end{bmatrix} = c(\lambda | t),
\]
of the SOS model,\(^{17}\) where on the rhs of the above equations there are the entries of the dynamical six-vertex \( R \)-matrix.

### 2.1.3. Quantum determinant
A fundamental object to define in the dynamical six-vertex Yang–Baxter algebra is the so-called quantum determinant.\(^{18}\) In particular, it plays a fundamental role in the construction of the quantum separation of variables for these algebra as we will explain in the following.

\(^{17}\) This model of statistical mechanics is defined on a square lattice and with each site \( n \) a ‘height’ \( h_n \) is associated and the interactions are defined around each face (composed by four adjacent sites) of the lattice. These interactions are nonzero only for adjacent heights which differ by 1 and are described by the Boltzmann weights \( W(\{ \lambda \}) \) of equations (2.23) and (2.24).

\(^{18}\) See [156] and the historical note [157] for the first proof of the centrality of the quantum determinant in the Yang–Baxter algebra.
Proposition 2.1. In the dynamical six-vertex Yang–Baxter algebra, we can introduce the following central quantum determinant:

\[ \det_q M(\lambda) = \frac{\theta(\tau + \eta S)}{\theta(\tau)} (A(\lambda|\tau)D(\lambda - \eta|\tau + \eta) - B(\lambda|\tau)C(\lambda - \eta|\tau - \eta)) \]  

(2.25)

\[ = \frac{\theta(\tau + \eta S)}{\theta(\tau)} (D(\lambda|\tau)A(\lambda - \eta|\tau - \eta) - C(\lambda|\tau)B(\lambda - \eta|\tau + \eta)) \]  

(2.26)

\[ = \lambda(\lambda)D(\lambda - \eta). \]  

(2.27)

where

\[ \lambda(\lambda) \equiv \prod_{n=1}^{N} a(\lambda - \xi_n), \quad D(\lambda) \equiv \lambda(\lambda - \eta). \]  

(2.28)

Moreover, the following inversion formula holds:

\[ M_0^{(\text{VVD})}(\lambda|\tau) \left( \begin{array}{cc} D(\lambda - \eta|\tau + \eta) & -B(\lambda - \eta|\tau + \eta) \\ -C(\lambda - \eta|\tau - \eta) & A(\lambda - \eta|\tau - \eta) \end{array} \right) \frac{\theta(\tau + \eta S)/\theta(\tau)}{\det_q M(\lambda)} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right). \]  

(2.29)

Proof. The proof follows by proving the statement for the generic quantum site \( n \) and then showing that the product of the local quantum determinants reproduces the complete one. Let us introduce the notation

\[ \det_q R_{0n}^{(\text{VVD})}(\lambda|\tau) = \left( \begin{array}{cc} R_{0n}^{(\text{VVD})} & 0 \\ 0 & R_{0n}^{(\text{VVD})} \end{array} \right)_{11}(\lambda|\tau) \left( \begin{array}{cc} R_{0n}^{(\text{VVD})} & 0 \\ 0 & R_{0n}^{(\text{VVD})} \end{array} \right)_{22}(\lambda - \eta|\tau + \eta) \]  

(2.30)

and

\[ S_n \equiv \sum_{a=1}^{n} \sigma_a^z. \]  

(2.31)

Then it is a simple exercise to verify the identity

\[ \det_q R_{0n}^{(\text{VVD})}(\lambda - \xi_n|\tau + \eta S_{n-1}) = a(\lambda - \xi_n)\theta(\tau - \eta + 2n) \frac{\theta(\tau + \eta S_{n-1})}{\theta(\tau + \eta S_n)}, \]  

(2.32)

once we use the formula\(^{19}\)

\[ \theta_1(x + y)\theta_1(x - y)\theta_2^x(0) = \theta_2^x(x)\theta_2^x(y) - \theta_2^y(x)\theta_2^y(y) = \theta_2^y(x)\theta_2^y(y) - \theta_2^x(x)\theta_2^x(y). \]  

(2.33)

Now, we get the quantum determinant formula by using the previous results to rewrite the following product:

\[ \lambda(\lambda)D(\lambda - \eta) = \prod_{n=1}^{N} a(\lambda - \xi_n)\theta(\tau - \eta + 2n) \]  

(2.34)

\[ = \prod_{n=1}^{N} \left[ \frac{\theta(\tau + \eta S_n)}{\theta(\tau + \eta S_{n-1})} \right] \det_q R_{0,n}^{(\text{VVD})}(\lambda|\tau + \eta S_{n-1}) \]  

\[ = \frac{\theta(\tau + \eta S)}{\theta(\tau)} \det_q R_{0,n}^{(\text{VVD})}(\lambda|\tau + \eta S_{n-1}) \cdots \det_q R_{0,2}^{(\text{VVD})} \]  

\[ \times (\lambda|\tau + \eta S_1) \det_q R_{0,1}^{(\text{VVD})}(\lambda|\tau) \]  

\[ = \det_q M(\lambda). \]  

19) See for example equation (7) on page 881 of [158].
Finally, the inversion formula (2.29) follows from the quantum determinant formulae and from the identities
\[ A(\lambda | \tau )B(\lambda - \eta | \tau + \eta ) - B(\lambda | \tau )A(\lambda - \eta | \tau - \eta ) = 0, \]
(2.35)
\[ D(\lambda | \tau )C(\lambda - \eta | \tau - \eta ) - C(\lambda | \tau )D(\lambda - \eta | \tau + \eta ) = 0, \]
(2.36)
which directly follow from the dynamical Yang–Baxter equations (2.15).

2.1.4. Antiperiodic dynamical six-vertex representations. As we will show in this paper, it is of particular interest to introduce an antiperiodic version of the dynamical six-vertex model by introducing the following monodromy matrix:
\[
\mathcal{M}^{(6VD)}_0(\lambda_1 | \tau ) \equiv \sigma_0^{(6VD)}(\lambda_1 | \tau );
\]
(2.37)
then the dynamical Yang–Baxter equation reads
\[
R^{(6VD)}_{1,2}(\lambda_{12}| \tau - \eta \mathcal{S}) \mathcal{M}^{(6VD)}_1(\lambda_1 | \tau ) \mathcal{T}_t = \mathcal{M}^{(6VD)}_2(\lambda_2 | \tau ) \mathcal{T}_t \mathcal{T}^{-\sigma_1}_t = \mathcal{M}^{(6VD)}_2(\lambda_2 | \tau ) \mathcal{T}_t \mathcal{T}^{-\sigma_1}_t \mathcal{M}^{(6VD)}_1(\lambda_1 | \tau ) \mathcal{T}_t \mathcal{T}^{-\sigma_1}_t R^{(6VD)}_{1,2}(\lambda_{12}| \tau ),
\]
(2.38)
being
\[
\sigma_1^{(6VD)} \otimes \sigma_2^{(6VD)}(\lambda_1 | y) \equiv R^{(6VD)}_{1,2}(\lambda_1 | y) \sigma_1^{(6VD)} \otimes \sigma_2^{(6VD)}(\lambda_1 | y).
\]
(2.39)
It is also worth defining the following monodromy matrix:
\[
\mathcal{M}^{(6VD)}_0(\lambda | \tau ) \equiv \mathcal{M}^{(6VD)}_0(\lambda | \tau ) \mathcal{T}_t^{\sigma_1};
\]
(2.40)
For this, the dynamical Yang–Baxter equation reads
\[
R^{(6VD)}_{1,2}(\lambda_{12}| \tau - \eta \mathcal{S}) \mathcal{M}^{(6VD)}_1(\lambda_1 | \tau ) \mathcal{M}^{(6VD)}_1(\lambda_2 | \tau ) = \mathcal{M}^{(6VD)}_1(\lambda_2 | \tau ) \mathcal{M}^{(6VD)}_1(\lambda_1 | \tau ) R^{(6VD)}_{1,2}(\lambda_{12}| \tau ).
\]
(2.41)

2.1.5. Invariant subspace under antiperiodic 6VD-generators. Let we define the operator:
\[
\mathcal{S}_t \equiv \eta \mathcal{S} + 2 \tau,
\]
(2.42)
in \( \mathcal{D}^{(6VD)}_{(6VD),N} \); then we denote with \( \mathcal{D}^{(6VD)}_{(6VD),N} \) the 2\( N \)-dimensional linear eigenspace corresponding to the eigenvalue zero \( \mathcal{S}_t \), i.e. \( \mathcal{D}^{(6VD)}_{(6VD),N} \) is the linear space defined by the condition that the eigenvalues of the commuting operators \( 2 \tau \) and \( \eta \mathcal{S} \) are coinciding. In terms of the dynamical-spin basis, the linear (covector) space \( \mathcal{D}^{(2)}_{(6VD),N} \) is generated by the elements
\[
\sum_{n=1}^N | n, h_n \rangle \otimes | h_n \rangle, \text{ where } t_n \equiv - \eta s_n, \text{ } s_n \equiv \sum_{k=1}^N (1 - 2h_k) \text{ and } h \equiv (h_1, \ldots, h_N),
\]
(2.43)
and the linear (vector) space \( \mathcal{D}^{(2)}_{(6VD),N} \) is generated by the elements
\[
\sum_{n=1}^N | n, h_n \rangle \otimes | h_n \rangle.
\]
(2.44)
In the 2\( N \)-dimensional linear space \( \mathcal{D}^{(2)}_{(6VD),N} \), it is central to remark that zero is a \( \tau \)-eigenvalue for a chain with \( N \) even, while it is not for a chain with \( N \) odd. This simple observation implies that \( \mathcal{D}^{(2)}_{(6VD),N} \) is not a well-defined representation space of the dynamical Yang–Baxter algebra for the presence of divergences in (2.10) for the zero \( \tau \)-eigenvalue. In contrast, in the case of an odd chain it holds.

20 Note that we use the simplified notation \( t_n \) instead of \( t(s_n) \).
As previously extracted:

**Theorem 2.1.** On the linear spaces $\tilde{D}^L_{(\omega V D), N}$ are well-defined left/right finite-dimensional representations of the operators

$$A(\lambda|\tau), \quad D(\lambda|\tau), \quad B(\lambda|\tau), \quad C(\lambda|\tau).$$

Moreover, the antiperiodic dynamical six-vertex transfer matrix

$$\mathcal{T}^{(\omega V D)}(\lambda|\tau) \equiv \tau \mathcal{M}_0^{(\omega V D)}(\lambda|\tau) = B(\lambda|\tau) + C(\lambda|\tau),$$

defines a one-parameter family of commuting operators on $\tilde{D}^L_{(\omega V D), N}$.

**Proof.** To prove the first statement in the theorem, we have to prove that the linear spaces $\tilde{D}^L_{(\omega V D), N}$ are invariant under the action of the operators

$$A(\lambda|\tau), \quad D(\lambda|\tau), \quad B(\lambda|\tau), \quad C(\lambda|\tau).$$

(2.47)

All what we need are the following commutation relations:

$$[A(\lambda|\tau), S] = 0, \quad [A(\lambda|\tau), \tau] = 0,$$

$$[A(\lambda|\tau), S] = 0, \quad [A(\lambda|\tau), \tau] = 0,$$

$$[B(\lambda|\tau), S] = 2B(\lambda|\tau) \quad [B(\lambda|\tau), \tau] = -\eta B(\lambda|\tau),$$

$$[C(\lambda|\tau), S] = -2C(\lambda|\tau), \quad [C(\lambda|\tau), \tau] = \eta C(\lambda|\tau),$$

from which it follows that

$$[A(\lambda|\tau), S] = [D(\lambda|\tau), S] = [B(\lambda|\tau), S] = [C(\lambda|\tau), S] = 0,$$

and then $\tilde{D}^L_{(\omega V D), N}$ are invariant under the action of these operators; then $\tilde{D}^L_{(\omega V D), N}$ are invariant also w.r.t. the action of the transfer matrix $\mathcal{T}^{(\omega V D)}(\lambda|\tau)$. Let us now take the trace of (2.41):

$$\mathcal{T}^{(\omega V D)}(\lambda_1|\tau) \mathcal{T}^{(\omega V D)}(\lambda_2|\tau)$$

$$= \tau \mathcal{M}_1^{(\omega V D)}(\lambda_1|\tau) \mathcal{M}_2^{(\omega V D)}(\lambda_2|\tau)$$

$$= \tau \mathcal{M}_1^{(\omega V D)}(\lambda_1|\tau) A(\lambda_2|\tau) \mathcal{M}_2^{(\omega V D)}(\lambda_2|\tau)$$

$$= \tau \mathcal{M}_1^{(\omega V D)}(\lambda_1|\tau) [D(\lambda_2|\tau) \mathcal{M}_2^{(\omega V D)}(\lambda_2|\tau)]$$

$$= \tau \mathcal{M}_1^{(\omega V D)}(\lambda_1|\tau) [D(\lambda_2|\tau) \mathcal{M}_2^{(\omega V D)}(\lambda_2|\tau)]$$

$$= \tau \mathcal{M}_1^{(\omega V D)}(\lambda_1|\tau) [D(\lambda_2|\tau) \mathcal{M}_2^{(\omega V D)}(\lambda_2|\tau)]$$

(2.53)

which in $\tilde{D}^L_{(\omega V D), N}$ coincides with

$$\tau \mathcal{M}_1^{(\omega V D)}(\lambda_1|\tau) \mathcal{M}_2^{(\omega V D)}(\lambda_2|\tau) = \mathcal{T}^{(\omega V D)}(\lambda_1|\tau) \mathcal{T}^{(\omega V D)}(\lambda_2|\tau),$$

i.e. the commutativity.

From now on, we will implicitly assume that $N$ is odd when representations in the linear spaces $\tilde{D}^L_{(\omega V D), N}$ will be considered.
2.2. SOV representations for $\mathcal{T}^{(6VD)}$-spectral problem

2.2.1. SOV representations. Here, we will show that the standard method to define quantum SOV representations introduced by Sklyanin [42–44] for the transfer matrix of six-vertex Yang–Baxter algebra can be adapted for the dynamical case. In particular, SOV representations for the spectral problem of the antiperiodic $\mathcal{T}^{(6VD)}(\lambda|\tau)$ can be defined as the representations where the commutative family of operators $D(\lambda|\tau)$ (or $A(\lambda|\tau)$) is pseudo-diagonal and with simple spectrum in the (left/right) $2^N$-dimensional spaces $D_{L/R}^{(6VD),N}$. Here, we mean that we can construct explicitly left/right basis of the spaces $D_{L/R}^{(6VD),N}$ in terms of pseudo $D(\lambda|\tau)$-eigenstates.\(^{21}\)

In order to construct these bases, some preparation is needed; let us define the left and right reference states:\(^{22}\)

$$\langle 0 \rangle \equiv \otimes_{n=1}^{N} |n, h_n = 0 \rangle \otimes |\eta \rangle, \quad |1 \rangle \equiv \otimes_{n=1}^{N} |n, h_n = 1 \rangle \otimes |\tau \rangle,$$  \hspace{1cm} (2.55)

where we use the notations $\equiv (h_1 = 0, \ldots, h_N = 0)$ and $\equiv (h_1 = 1, \ldots, h_N = 1)$, so that $\equiv \langle -\eta N/2 \rangle/2$ and $\equiv \langle \eta N/2 \rangle/2$ by the definition (2.43) of $\eta$. Then, we can define the following sets of the left and right states:

$$\langle h_1, \ldots, h_N \rangle = \frac{1}{N} \langle 0 \rangle \prod_{n=1}^{N} \left( \frac{\mathcal{C} ( \xi_n | \tau )}{D ( \xi_n - \eta )} \right)^{h_n} |1\rangle,$$  \hspace{1cm} (2.56)

and

$$| h_1, \ldots, h_N \rangle = \frac{1}{N} \prod_{n=1}^{N} \left( \frac{\mathcal{C} ( \xi_n - \eta | \tau )}{D ( \xi_n - \eta )} \right)^{1-h_n} |1\rangle,$$  \hspace{1cm} (2.57)

where $h_n \in \{0, 1\}$, $n \in \{1, \ldots, N\}$. Note that the normalization $N$ has been introduced to simplify the form of the coupling of the above left and right states. Let us define first the following theta functions with characteristic:

$$\vartheta_j (\lambda) = \sum_{n \in \mathbb{Z}} \exp \left[ 2i\pi u N \left( n + \frac{1}{2} - j \frac{1}{N} \right) + 2i\pi N \left( n + \frac{1}{2} - j \frac{1}{N} \right) \left( \lambda + \frac{1}{2N} \right) \right].$$  \hspace{1cm} (2.58)

$N \in \mathbb{N}$ and $j \in \{0, \ldots, N - 1\}$, which satisfies the periodicity conditions:

$$\vartheta_j (\lambda + 1/N) = -e^{2\pi i (j/N)} \vartheta_j (\lambda), \quad \vartheta_j (\lambda + 2w) = -e^{2\pi i (w+N) \lambda} \vartheta_j (\lambda).$$  \hspace{1cm} (2.59)

Then we can fix

$$N^2 \equiv \det \left[ \Theta_{ij}^{(h)} \right]$$

where \(\Theta_{ij}^{(h)}\) is the $N \times N$ matrix of elements:

$$\Theta_{ij}^{(h)} \equiv \vartheta_{j-1} (\xi_i^{(h_i)}), \quad \xi_i^{(h_i)} = \xi^{(h_i)} + \frac{\eta}{2} + \frac{N - 1}{2N} \sum_{i=1}^{N} \xi_a, \quad \xi_i^{(h_i)} = \xi_i - \eta h_i.$$

It is simple to verify that the states (2.56) and (2.57) are simultaneous (left/right) eigenstates of $\tau$ and $\mathcal{S} \equiv \sum_{\sigma_a} \sigma_a$:

$$\langle h_1, \ldots, h_N | \tau \rangle = \eta \langle h_1, \ldots, h_N \rangle,$$  \hspace{1cm} (2.62)

\(^{21}\) What we mean by pseudo-eigenstates will be clarified in the following.

\(^{22}\) The left and right states of $D_{L/R}^{(6VD),N}$ with all spin up and down, respectively.
Moreover, under the following condition on the \(N\)-tuple of inhomogeneities \([\xi_1, \ldots, \xi_N] \in \mathbb{C}^N\), \(\xi_a \neq \xi_b^{(h_a)} \mod(2w) \ \forall h_b \in \{0, 1\} \text{ and } a < b \in \{1, \ldots, N\}\), the following theorem holds.

**Theorem 2.2.**

(I) **Left SOV representations.** Under the condition (2.66), the states (2.56) define a basis of pseudo \(D(\lambda|\tau)\)-eigenstates in \(\mathbb{D}_C^{(\lambda|\tau),N}\); indeed, it holds:

\[
\langle h_1, \ldots, h_N | D(\lambda|\tau) = d_h^C(\lambda) \left( \frac{1}{N} \prod_{n=1}^{N} \left( C(\xi_n|\tau-\eta) \right)^{h_n} \right),
\]

where

\[
d_h^C(\lambda) \equiv \frac{\theta(t_0 - \eta) \theta(t_1)}{\theta(t_0 + \eta) \theta(t_0)} d_h(\lambda), \quad d_h(\lambda) \equiv \prod_{n=1}^{N} \theta(\lambda - \xi_n^{(h_n)}).
\]

The action of the remaining generators on the generic state \(\langle h_1, \ldots, h_N|\rangle\) reads

\[
\langle h_1, \ldots, h_N| C(\lambda|\tau) = \sum_{a=1}^{N} \theta(\tau - \lambda + \xi_a^{(h_a)}) \prod_{b \neq a} \theta(\lambda - \xi_b^{(h_b)} - \xi_a^{(h_a)}) \tilde{D}(\xi_a^{(1-h_a)}) \langle h_1, \ldots, h_N| T_a^+, \quad (2.69)
\]

\[
\langle h_1, \ldots, h_N| B(\lambda|\tau) = \sum_{a=1}^{N} \theta(\tau - \lambda + \xi_a^{(h_a)}) \prod_{b \neq a} \theta(\lambda - \xi_b^{(h_b)} + \xi_a^{(h_a)}) \tilde{A}(\xi_a^{(1-h_a)}) \langle h_1, \ldots, h_N| T_a^-, \quad (2.70)
\]

where

\[
\langle h_1, \ldots, h_N| T_a^{\pm} = \langle h_1, \ldots, h_n \pm 1, \ldots, h_N|,
\]

and \(A(\lambda|\tau)\) is uniquely defined by the quantum determinant relation.

(II) **Right SOV representations.** Under the condition (2.66), the states (2.57) define a basis of pseudo \(D(\lambda|\tau)\)-eigenstates in \(\mathbb{D}_R^{(\lambda|\tau),N}\); indeed, it holds:

\[
D(\lambda|\tau + \eta)| h_1, \ldots, h_N \rangle = \left( \frac{1}{N} \prod_{n=1}^{N} \left( C(\xi_n - \eta|\tau + \eta) \right)^{(1-h_n)} \right) \tilde{D}_h^R(\lambda), \quad (2.72)
\]

where

\[
\tilde{D}_h^R(\lambda) \equiv \frac{\theta(t_0 + \eta)}{\theta(t_1 + \eta)} d_h(\lambda).
\]

The action of the remaining generators on the generic state \(| h_1, \ldots, h_N \rangle\) reads

\[
C(\lambda|\tau)| h_1, \ldots, h_N \rangle = \sum_{a=1}^{N} \theta(\tau - \lambda + \xi_a^{(h_a)}) \prod_{b \neq a} \theta(\lambda - \xi_b^{(h_b)} - \xi_a^{(h_a)}) \tilde{D}(\xi_a^{(h_a)}), \quad (2.74)
\]
Indeed, to prove that (2.67) on the fact that the left and right reference states are dynamical Yang–Baxter commutation relations.

\[ B(\lambda | \tau) | h_1, \ldots, h_N \rangle = \sum_{a=1}^{N} T_a^+ | h_1, \ldots, h_N \rangle \frac{\theta(\tau - \lambda + \xi_a^{(h)})}{\theta(\tau)} \prod_{b \neq a} \frac{\theta(\lambda - \xi_b^{(h)})}{\theta(\xi_b^{(h)} - \xi_a^{(h)})} A(\xi_a^{(h)}). \]

(2.75)

where

\[ T_a^+ | h_1, \ldots, h_N \rangle = | h_1, \ldots, h_a \pm 1, \ldots, h_N \rangle, \]

and \( A(\lambda | \tau) \) is uniquely defined by the quantum determinant relation.

**Proof.** The proof of the theorem is based on the dynamical Yang–Baxter commutation relations and on the fact that the left and right reference states are \( D(\lambda | \tau) \)-eigenstates:

\[ \langle 0 | A(\lambda | \tau) = \lambda(\lambda) | 0 \rangle, \quad \langle 0 | D(\lambda | \tau) = D(\lambda | \eta) | 0 \rangle, \quad \langle 0 | B(\lambda | \tau) = 0, \quad \langle 0 | C(\lambda | \tau) \neq 0, \]

(2.77)

and similarly

\[ D(\lambda | \tau) | 1 \rangle = | 1 \rangle \lambda(\lambda), \quad A(\lambda | \tau) | 1 \rangle = | 1 \rangle D(\lambda | \eta), \quad B(\lambda | \tau) | 1 \rangle = 0, \quad C(\lambda | \tau) | 1 \rangle \neq 0, \]

(2.78)

where

\[ D(\lambda | \eta) \equiv D(\lambda) \frac{\theta(\eta - + h_0)}{\theta(\eta + h_0)}. \]

(2.79)

Indeed, to prove that (2.56) and (2.57) are left and right pseudo-eigenstates of \( D(\lambda | \tau) \) as stated in (2.67) and (2.72), we have just to repeat the standard computations in algebraic Bethe ansatz\[9\] as done in \[71\]. More in detail, considering the action of \( D(\lambda | \tau) \) on the left states \( | h_1, \ldots, h_N \rangle \) and following the steps given in the proof of theorem 3.2 of \[71\] by using here the dynamical six-vertex commutation relation

\[ C(\mu | \tau) D(\lambda | \tau) = \left[ D(\lambda | \tau) C(\mu | \tau) \theta(\lambda - \mu + \eta) \theta(\tau) - D(\mu | \tau) C(\lambda | \tau) \theta(\eta) \theta(\tau + \lambda - \mu) \right] \]

\[ \times \frac{1}{\theta(\lambda - \mu) \theta(\tau + \eta)}, \]

(2.80)

we obtain

\[ \langle h_1, \ldots, h_N | D(\lambda | \tau) \rangle = \mathcal{D}_h^C(\lambda) \left[ \frac{1}{N} \left( \langle 0 | T_\tau^+ \right) \prod_{n=1}^{N} \left( \frac{C(\xi_n | \tau)}{D(\xi_n - \eta)} \right)^{h_n} \right]. \]

(2.81)

Similarly, by using the dynamical six-vertex commutation relation:

\[ D(\lambda | \tau) C(\mu | \tau) = \frac{1}{\theta(\mu - \lambda) \theta(\tau + \eta)} \left[ \theta(\mu - \lambda + \eta) \theta(\tau) C(\mu | \tau) D(\lambda | \tau) \right. \]

\[ - \theta(\eta) \theta(\tau + \mu - \lambda) C(\lambda | \tau) D(\mu | \tau), \]

(2.82)

we obtain

\[ D(\lambda | \tau) | h_1, \ldots, h_N \rangle = \left( \frac{1}{N} \prod_{n=1}^{N} \left( \frac{C(\xi_n - \eta | \tau)}{D(\xi_n - \eta)} \right)^{(1-h_n)} \right)^{(1-h_0)} \mathcal{D}_h^C(\lambda). \]

(2.83)

From formulae (2.81) and (2.83) by using the commutation relations (2.1), \( D(\lambda | \tau) = D(\lambda | \tau) T_\tau^+ \) and \( D(\lambda | \tau + \eta) = T_\tau^+ D(\lambda | \tau) \); then formulae (2.67) and (2.72) simply follow.

Let us prove now that the states \( | h_1, \ldots, h_N \rangle \) form a set of \( 2^N \) independent states, i.e. a basis of \( \mathcal{D}_h^C \). Similarly, we can prove that the states \( | h_1, \ldots, h_N \rangle \) form a basis of \( \mathcal{D}_h^R \). By definition, we have to prove that their linear combination to zero,

\[ \sum_{h_1, \ldots, h_N=0}^{1} c_b | h_1, \ldots, h_N \rangle = 0, \]

(2.84)

13
holds only if all the coefficients are zero. Let us denote with \( \vec{h} = [\vec{h}_1, \ldots, \vec{h}_N] \) the generic N-tuple in \([0, 1]^N\), then by applying to both sides of (2.84) the operator product

\[
\prod_{n=1}^{N} \mathcal{D}(\xi^{(\vec{h}_n)}_{\vec{h}_n}|\tau) \text{ with } \vec{h}_n = \vec{h}_n + 1 \text{ mod } 2 \in [0, 1],
\]

we obtain

\[
c_{\vec{h}} \prod_{n=1}^{N} \mathcal{D}(\xi^{(\vec{h}_n)}_{\vec{h}_n}) \left[ \left( \langle 0 | T_{\vec{h}_n}^{\vec{h}_n} \right) \prod_{n=1}^{N} \left( \frac{\mathcal{C}(\xi^{(\vec{h}_n)}_{\vec{h}_n}|\tau)}{\mathcal{D}(\xi^{(\vec{h}_n)}_{\vec{h}_n}-\eta)} \right) \right] \vec{h}_n = 0,
\]

which implies \( c_{\vec{h}} = 0 \) being

\[
\prod_{n=1}^{N} \mathcal{D}(\xi^{(\vec{h}_n)}_{\vec{h}_n}) \neq 0, \quad \left[ \left( \langle 0 | T_{\vec{h}_n}^{\vec{h}_n} \right) \prod_{n=1}^{N} \left( \frac{\mathcal{C}(\xi^{(\vec{h}_n)}_{\vec{h}_n}|\tau)}{\mathcal{D}(\xi^{(\vec{h}_n)}_{\vec{h}_n}-\eta)} \right) \right] \vec{h}_n \neq 0.
\]

The action of \( \mathcal{B}(\xi^{(\vec{h}_n)}_{\vec{h}_n}|\tau) \) and \( \mathcal{C}(\xi^{(\vec{h}_n)}_{\vec{h}_n}|\tau) \) on the left and right states (2.56) and (2.57) follows by imposing the dynamical Yang–Baxter commutation relations and the quantum determinant relations

\[
\langle h_1, \ldots, h_N \rangle \text{ det}_q \mathcal{M}(\lambda) = \lambda(\lambda) \mathcal{D}(\lambda-\eta) \langle h_1, \ldots, h_N \rangle, \\
\text{ det}_q \mathcal{M}(\lambda)|h_1, \ldots, h_N\rangle = \lambda(\lambda) \mathcal{D}(\lambda-\eta)|h_1, \ldots, h_N\rangle,
\]

where we have used that

\[
\frac{\theta(\tau + \eta S)}{\theta(\tau)} \bigg|_{\{h_1, \ldots, h_N\}} = \frac{\theta(-\bar{h})}{\theta(h)} = -1,
\]

and in the quantum determinant \( \text{ det}_q \mathcal{M}(\lambda) \) we use the identities

\[
\mathcal{A}(\lambda|\tau) \mathcal{D}(\lambda-\eta|\tau) - \mathcal{B}(\lambda|\tau) \mathcal{C}(\lambda-\eta|\tau) = \mathcal{A}(\lambda|\tau) \mathcal{D}(\lambda-\eta|\tau + \eta) - \mathcal{B}(\lambda|\tau) \mathcal{C}(\lambda-\eta|\tau + \eta),
\]

(2.90)

\[
\mathcal{D}(\lambda|\tau) \mathcal{A}(\lambda-\eta|\tau) - \mathcal{C}(\lambda|\tau) \mathcal{B}(\lambda-\eta|\tau) = \mathcal{D}(\lambda|\tau) \mathcal{A}(\lambda-\eta|\tau - \eta) - \mathcal{C}(\lambda|\tau) \mathcal{B}(\lambda-\eta|\tau - \eta).
\]

(2.91)

Finally, the left (2.69)–(2.70) and right (2.74)–(2.75) representations of \( \mathcal{B}(\lambda|\tau) \) and \( \mathcal{C}(\lambda|\tau) \) are just interpolation formulae in the special points \( \{\xi^{(\vec{h}_1)}_{\vec{h}_1}, \ldots, \xi^{(\vec{h}_N)}_{\vec{h}_N}\} \) which hold for elliptic polynomials as illustrated for example in appendix A of [159].

\[\square\]

2.2.2. SOV decomposition of the identity:

The previous results allow us to write the following spectral decomposition of the identity \( I \):

\[
I \equiv \sum_{h_1, \ldots, h_N=0}^{1} \mu_{\vec{h}} |h_1, \ldots, h_N\rangle \langle h_1, \ldots, h_N|,
\]

(2.92)

in terms of the left and right SOV bases. Here,

\[
\mu_{\vec{h}} \equiv \frac{1}{|h_1, \ldots, h_N\rangle \langle h_1, \ldots, h_N|}
\]

is the analogue of the so-called Sklyanin measure, which is discrete in these representations and defined by the following proposition.

**Proposition 2.2.** Let \( |h_1, \ldots, h_N\rangle \) be the generic covector (2.56) and \( |k_1, \ldots, k_N\rangle \) be the generic vector (2.57); then it holds

\[
\langle h_1, \ldots, h_N|k_1, \ldots, k_N\rangle = \frac{\prod_{n=1}^{N} \delta_{h_n,k_n}}{\text{ det}_{\text{det}_N} \Theta^{(\vec{h})}_{\vec{k}}},
\]

(2.94)
where $\Theta_{ij}^{(b)}$ is the $N \times N$ matrix defined in (2.61). Then, the SOV decomposition of the identity explicitly reads
\[ \mathbb{I} = \sum_{h_1, \ldots, h_N = 0}^{N} \det \Theta_{ij}^{(b)} |h_1, \ldots, h_N\rangle \langle h_1, \ldots, h_N|. \] (2.95)

**Proof.** The fact that $\langle h_1, \ldots, h_N| and $|k_1, \ldots, k_N\rangle$ are simultaneous eigenstates of $\tau$ with eigenvalues $\tau_h$ and $\tau_k$, respectively, implies that the lhs of (2.94) is zero unless
\[ \sum_{c=1}^{N} h_c = \sum_{c=1}^{N} k_c. \] (2.96)
Let us assume now that $\mathbf{h} \neq \mathbf{k}$ but that they satisfy the condition (2.96). Under these conditions, it is easy to understand that there exists at least one $n \in \{1, \ldots, N\}$ such that $h_n = 1$ and $k_n = 0$ and then the lhs of (2.94) contains the product of operators $\mathcal{C}(\xi_n^0) \mathcal{C}(\xi_n^0 - \eta \tau)$ which is zero for the standard six-vertex annihilation identities. Then, as stated in (2.94), $\langle h_1, \ldots, h_N|k_1, \ldots, k_N\rangle$ is zero for $\mathbf{h} \neq \mathbf{k}$; so we are left with the computations for $\mathbf{h} = \mathbf{k}$. In order to compute them, let us compute the matrix elements
\[ x_a = \langle h_1, \ldots, h_a = 0, \ldots, h_N| \mathcal{C}(\xi_a^0) \mathcal{C}(\xi_a^0 - \eta \tau)|h_1, \ldots, h_a = 1, \ldots, h_N\rangle, \] (2.97)
where $a \in \{1, \ldots, N\}$. Then using the left action of the operator $\mathcal{C}(\eta \tau)$, we obtain
\[ x_a = D(\xi_a^0 - \eta)(h_1, \ldots, h_a = 1, \ldots, h_N|h_1, \ldots, h_a = 1, \ldots, h_N), \] (2.98)
while using the right action of the operator $\mathcal{C}(\eta \tau)$ and the orthogonality of right and left pseudo $D$-eigenstates corresponding to different eigenvalues, we obtain
\[ x_a = \prod_{b \neq a} \frac{\theta(\xi_a^0 - \xi_b^0 + \eta h_b)}{\theta(\xi_a^0 - \eta - \xi_b^0 + \eta h_b)} D(\xi_a^0 - \eta)(h_1, \ldots, h_a = 0, \ldots, h_N|h_1, \ldots, h_a = 0, \ldots, h_N) \times \frac{\theta(h_a h_{a+1} - \eta)}{\theta(h_a h_{a+1})}, \] (2.99)
and so
\[ \langle h_1, \ldots, h_a = 1, \ldots, h_N|h_1, \ldots, h_a = 1, \ldots, h_N\rangle = \frac{\theta(h_a h_{a+1} = 0)}{\theta(h_a h_{a+1})} \prod_{b \neq a, b \neq a} \frac{\theta(\xi_a^0 - \xi_b^0 + \eta h_b)}{\theta(\xi_a^0 - \xi_b^0)}, \] from which the proposition simply follows when we use the identity
\[ \det \Theta_{ij}^{(b)} = c_n \theta \left( \sum_{a=1}^{N} \xi_a^{(h)} - \frac{N_1}{2} \right) \prod_{1 \leq a < b \leq N} \theta(\xi_a^{(h)} - \xi_b^{(h)}), \] (2.100)
and we recall that
\[ h_n = -\sum_{a=1}^{N} \xi_a^{(h)} + \frac{N_1 - 1}{2}, \] (2.101)
and that the choice of the normalization $N$ implies
\[ \langle 0|0 \rangle = \frac{1}{\det N \Theta_{ij}^{(0)}}, \] (2.102)
\[ \Box \]

---

23 See for example proposition 4 of [159] for a proof.
2.3. SOV characterization of the $T^{(VD)}$-spectrum

Let us denote with $\Sigma_{\tau^{\text{SOV}}}$ the set of the eigenvalue functions $t_{\text{VD}}(\lambda)$ of the transfer matrix $T^{(VD)}(\lambda|\tau)$; then the following characterization of the $T^{(VD)}$-spectrum (eigenvalues and eigenstates) in quantum separation of variables holds:

**Theorem 2.3.** For any fixed $N$-tuple of inhomogeneities $\{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N$ satisfying (2.66), the spectrum of $T^{(VD)}(\lambda|\tau)$ in $\mathbb{C}^{C/R}_{(\text{VD})N}$ is simple and $\Sigma_{\tau^{\text{SOV}}}$ coincides with the set of functions of the form

$$t_{\text{VD}}(\lambda) = \sum_{a=1}^{N} \frac{\theta(t_{a} - \lambda - \xi_{a})}{\theta(t_{a})} \prod_{\beta \neq a} \frac{\theta(\lambda - \xi_{\beta})}{\theta(\xi_{a} - \xi_{\beta})} t_{\text{VD}}(\xi_{a})$$

(2.103)

which are the solutions of the discrete system of equations

$$t_{\text{VD}}(\xi_{a}^{(1)}) t_{\text{VD}}(\xi_{a}^{(1)}) = \lambda(t_{a}^{(0)}) d(t_{a}^{(1)}), \quad \forall a \in \{1, \ldots, N\}.$$  

(2.104)

- (I) The right $T^{(VD)}$-eigenstate corresponding to $t_{\text{VD}}(\lambda) \in \Sigma_{\tau^{\text{SOV}}}$ is characterized by

$$|t_{\text{VD}}\rangle = \sum_{n_1, \ldots, n_N = 0}^{1} \prod_{a=1}^{N} Q_a(h_{a}) |h_1, \ldots, h_N\rangle,$$

(2.105)

up to an overall normalization, where the coefficients are characterized by

$$Q_a(h_{a}) / Q_a(h_{a}) = t_{\text{VD}}(\xi_{a}^{(1)}) / d(t_{a}^{(1)}).$$

(2.106)

- (II) The left $T^{(VD)}$-eigenstate corresponding to $t_{\text{VD}}(\lambda) \in \Sigma_{\tau^{\text{SOV}}}$ is characterized by

$$|t_{\text{VD}}\rangle = \sum_{n_1, \ldots, n_N = 0}^{1} \prod_{a=1}^{N} \tilde{Q}_a(h_{a}) |h_1, \ldots, h_N\rangle,$$

(2.107)

up to an overall normalization, where the coefficients are characterized by

$$\tilde{Q}_a(h_{a}) / \tilde{Q}_a(h_{a}) = t_{\text{VD}}(\xi_{a}^{(1)}) / \lambda(t_{a}^{(0)}).$$

(2.108)

**Proof.** Let $|t_{\text{VD}}\rangle$ be a $T^{(VD)}$-eigenstate corresponding to the $T^{(VD)}$-eigenvalue $t_{\text{VD}}(\lambda)$, then the coefficients (wavefunctions)

$$\Psi_{i}(h) \equiv \langle t_{\text{VD}}| h_{1}, \ldots, h_{N}\rangle$$

(2.109)

of $|t_{\text{VD}}\rangle$ in the right SOV-basis satisfy the equations

$$t_{\text{VD}}(\xi_{a}^{(1)}) \Psi_{i}(h) = \lambda(\xi_{a}^{(1)}) \Psi_{i}(T_{a}^{+}(h)) + d(\xi_{a}^{(1)}) \Psi_{i}(T_{a}^{-}(h)),$$

(2.110)

for any $n \in \{1, \ldots, N\}$ and $h \in \{0, 1\}^N$, where we have denoted

$$T_{a}^{\pm}(h) \equiv (h_{1}, \ldots, h_{N} \pm 1, \ldots, h_{N}).$$

(2.111)

These equations are obtained by computing the matrix elements:

$$\langle t_{\text{VD}}| T^{(VD)}(\xi_{a}^{(1)}|\tau)|h_1, \ldots, h_N\rangle.$$  

(2.112)

In particular, acting with $T^{(VD)}(\xi_{a}^{(1)}|\tau)$ on $|t_{\text{VD}}\rangle$, the lhs of (2.110) is reproduced, while acting on $|h_1, \ldots, h_N\rangle$ by using the SOV representation of $T^{(VD)}(\xi_{a}^{(1)}|\tau)$ the rhs of (2.110) is reproduced. Moreover, the representation (2.103) for the $T^{(VD)}$-eigenvalue functions $t_{\text{VD}}(\lambda)$ follows by computing the matrix element

$$\langle t_{\text{VD}}| T^{(VD)}(\lambda|\tau)|0\rangle$$

(2.113)

and by using (2.110) to rewrite the rhs in the desired form.
Then in the SOV representations, the spectral problem for $\mathcal{T}^{\text{6VD}}(\lambda|\tau)$ is reduced to a discrete system of $2^N$ Baxter-like equations (2.110) in the class of function of the form (2.103). Taking into account the identities

$$\Lambda(\xi_n^{(1)}) = d(\xi_n^{(0)}) = 0,$$

(2.114)

this system coincides with a system of homogeneous equations

$$
\begin{pmatrix}
t_{\text{6VD}}(\xi_n^{(0)}) & -\Lambda(\xi_n^{(0)}) \\
-d(\xi_n^{(1)}) & t_{\text{6VD}}(\xi_n^{(1)})
\end{pmatrix}
\begin{pmatrix}
\Psi_l(h_1, \ldots, h_n = 0, \ldots, h_1) \\
\Psi_l(h_1, \ldots, h_n = 1, \ldots, h_1)
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix},
$$

(2.115)

for any $n \in \{1, \ldots, N\}$ with $h_{n\neq n} \in \{0, 1\}$. The condition $t_{\text{6VD}}(\lambda) \in \Sigma_{\text{vvo}}$ is then equivalent to the requirement that the determinants of the $2 \times 2$ matrices in (2.115) must be zero for any $n \in \{1, \ldots, N\}$, i.e. equation (2.104). On the other hand, with

$$\Lambda(\xi_n^{(0)}) \neq 0 \quad \text{and} \quad d(\xi_n^{(1)}) \neq 0,$$

(2.116)

the rank of the matrices in (2.115) is 1 and then up to an overall normalization the solution is unique:

$$
\frac{\Psi_l(h_1, \ldots, h_n = 1, \ldots, h_1)}{\Psi_l(h_1, \ldots, h_n = 0, \ldots, h_1)} = \frac{t_{\text{6VD}}(\xi_n^{(0)})}{\Lambda(\xi_n^{(0)})},
$$

(2.117)

for any $n \in \{1, \ldots, N\}$ with $h_{n\neq n} \in \{0, 1\}$. This implies that given $t_{\text{6VD}}(\lambda) \in \Sigma_{\text{vvo}}$ there exists (up to normalization) one and only one corresponding $\mathcal{T}^{\text{6VD}}$-eigenstate ($t_{\text{6VD}}$) with coefficients which have the factorized form given in (2.107) and (2.108) and then the $\mathcal{T}^{\text{6VD}}$-spectrum is simple. The proof for the right eigenstates is given in a similar way.

Let us remark that the previous theorem completely characterizes the spectrum of the transfer matrix $\mathcal{T}^{\text{6VD}}(\lambda|\tau)$ in $\mathbb{D}(\mathcal{C}/\mathbb{R})$. However, a reformulation of the SOV characterization of the $\mathcal{T}^{\text{6VD}}$-spectrum by functional equations can be important for practical aims. One standard way to accomplish this result is by the construction of a Baxter $Q$-operator whose functional equation reduces to the finite system of Baxter-like equations (2.110) when computed in the eigenvalues of the quantum separate variables, i.e. the operator zero of $\mathcal{D}(\lambda|\tau)$. This construction is currently under analysis and it can be developed along the same lines presented in [68] for the antiperiodic XXZ spin-1/2 chain for general values of the coupling $\eta$. For the elliptic roots of unit case, we can construct the functional equation directly by using cyclic representations for the operator $\tau$; these interesting issues will be developed in a forthcoming paper. Finally, let us remark that once the $Q$-operator is constructed, its eigenvalues can be used to completely characterize not only the eigenvalues but also the eigenstates of the antiperiodic dynamical six-vertex transfer matrix in the SOV framework. Indeed, it is enough to introduce in formula (2.105) for the $Q_l(\xi_n^{(h)})$ the eigenvalue of the $Q$-operator computed in $\xi_n^{(h)}$ to obtain the corresponding simultaneous eigenstate.

2.4. Action of left separate states on right separate states

As for the other quantum integrable models analyzed by SOV in [69–73], a special role is played by the left and right separate states in the SOV representations. These are the states which have factorized coefficients in the SOV representations similar to those of the transfer matrix eigenstates; more in detail, we say that a covector $|\alpha\rangle \in \bar{\mathbb{D}}^{L}_{(\text{6VD}),N}$ and a vector $|\beta\rangle \in \bar{\mathbb{D}}^{R}_{(\text{6VD}),N}$ are separate states if they admit the following SOV decompositions:

$$
|\alpha\rangle = \sum_{h_1, \ldots, h_N = 0}^{1} \prod_{a=1}^{N} \alpha_a(\xi_n^{(h)}) \det \delta^{(h)}_{ij} |h_1, \ldots, h_N|,
$$

(2.118)
\[ |\beta\rangle = \sum_{h_1,\ldots,h_N=0}^{1} \prod_{a=1}^{N} \beta_a(\xi_a^{(h_a)}) \det N_i \Theta_{ij}^{(h_i)} |h_1,\ldots,h_N\rangle. \] (2.119)

The main interest toward these states is the simple determinant form for the action of left separate states on the right ones characterized by the following.

**Proposition 2.3.** The left \(|\alpha\rangle\) and the right \(|\beta\rangle\) separate states satisfy the identities

\[ \langle \alpha|\beta\rangle = \det_{N} \mathcal{T}_{a,b}^{(\alpha,\beta)} \] with \(\mathcal{T}_{a,b}^{(\alpha,\beta)} = \sum_{h=0}^{1} \alpha_a(\xi_a^{(h)}) \beta_b(\xi_b^{(h)}) \delta_{b-1}(\xi_b^{(h)}) \). (2.120)

Then the matrix elements of the left and right \(T^{(\theta\psi)}\)-eigenstates corresponding to generic eigenvalues,

\[ t_{\theta\psi}(\lambda) \in \Sigma_{\theta\psi}, \quad t'_{\theta\psi}(\lambda) \in \Sigma_{\theta\psi}, \] (2.121)

have the following simple form:

\[ \langle t_{\theta\psi} | t'_{\theta\psi} \rangle = \delta_{\tau t} \det_{N} \mathcal{T}_{a,b}^{(\tau,t)}, \] \[ \mathcal{T}_{a,b}^{(\tau,t)} = \sum_{l=0}^{1} \tilde{Q}_{l}(\xi_a^{(l)}) Q_{l}(\xi_b^{(l)}) \delta_{b-1}(\xi_b^{(l)}) \]. (2.122)

where we have used the notation \(\delta_{\tau t} = \{0 \text{ for } t_{\theta\psi}(\lambda) \neq t'_{\theta\psi}(\lambda) \in \Sigma_{\theta\psi}, 1 \text{ for } t_{\theta\psi}(\lambda) = t'_{\theta\psi}(\lambda) \in \Sigma_{\theta\psi} \} \).

**Proof.** From the SOV decomposition, we have

\[ \langle \alpha|\beta\rangle = \sum_{h_1,\ldots,h_N=0}^{1} \det_{N} \Theta_{ij}^{(h_i)} \prod_{a=1}^{N} \alpha_a(\xi_a^{(h_a)}) \beta_b(\xi_b^{(h_b)}) ; \] (2.123)

from this formula, by using the multilinearity of the determinant w.r.t. the rows, we prove the first identity in the proposition. The presence of the delta in (2.122) simply follows from the identities

\[ t_{\theta\psi}(\lambda) | t'_{\theta\psi} \rangle = (t_{\theta\psi} T^{(\theta\psi)}(\lambda|\tau)) | t'_{\theta\psi} \rangle = t'_{\theta\psi}(\lambda) | t_{\theta\psi} \rangle, \] (2.124)

where the lhs is obtained by acting on the left state with \(T^{(\theta\psi)}(\lambda|\tau)\) and the rhs is obtained by acting on the right state with \(T^{(\theta\psi)}(\lambda|\tau)\). Indeed, for \(t_{\theta\psi}(\lambda) \neq t'_{\theta\psi}(\lambda)\), the identity implies

\[ \langle t_{\theta\psi} | t'_{\theta\psi} \rangle = 0. \] (2.125)

Finally, for \(t_{\theta\psi}(\lambda) = t'_{\theta\psi}(\lambda)\) the form of the matrix elements just follows by recalling that the eigenstates of the transfer matrix are indeed separate states. \(\square\)

2.5. Decomposition of the identity in the left and right separate bases

The results of the previous section allow us to write the decomposition of the identity in the left and right bases of separate states. In order to define such a basis, let us introduce the following natural isomorphism between the sets \(\{0, 1\}^{N}\) and \(\{1, \ldots, 2^{N}\}\):

\[ \chi : h \equiv \{h_1, \ldots, h_N\} \in \{0, 1\}^{N} \rightarrow \chi(h) \equiv 1 + \sum_{a=1}^{N} 2^{(a-1)} h_a \in \{1, \ldots, 2^{N}\}, \] (2.126)
and let us denote with \( \chi^{-1} (i) \) (i) \( \in \{0, 1\} \) \( \in \{1, \ldots, N\} \) \( \chi^{-1} \) with any integer \( i \in \{1, \ldots, 2N\} \). Then, under the conditions

\[
\det \left| M_{ij}^{(\alpha)} \right| \neq 0, \quad M_{ij}^{(\alpha)} = \prod_{a=1}^{N} \alpha_a^{(j)\left(\chi^{-1}(i)\right)} \forall i, j \in \{1, \ldots, 2N\},
\]

\[
\det \left| M_{ij}^{(\beta)} \right| \neq 0, \quad M_{ij}^{(\beta)} = \prod_{a=1}^{N} \beta_a^{(j)\left(\chi^{-1}(i)\right)} \forall i, j \in \{1, \ldots, 2N\},
\]

the sets of covectors \( \langle \alpha_j \rangle \) and vector \( |\beta_j\rangle \) defined by

\[
\langle \alpha_j \rangle = \sum_{h_1, \ldots, h_N=0}^{1} \prod_{a=1}^{N} \alpha_a^{(j)\left(\chi^{-1}(i)\right)} \det \Theta^{(h_1, \ldots, h_N)} j \forall j \in \{1, \ldots, 2N\},
\]

\[
|\beta_j\rangle = \sum_{h_1, \ldots, h_N=0}^{1} \prod_{a=1}^{N} \beta_a^{(j)\left(\chi^{-1}(i)\right)} \det \Theta^{(h_1, \ldots, h_N)} j \forall j \in \{1, \ldots, 2N\},
\]

generate separate bases of \( \tilde{D}_{(6VD),N}^{(GVD)} \) and \( \tilde{D}_{(6VD),N}^{(R)} \), respectively. Moreover, define

\[
N_k = \{ k \in \{0, 1\}^N : \det \left| F_{a,b}^{(\alpha_k, \beta_k)} \right| \neq 0 \};
\]

then the following decomposition of the identity is implied on these separate basis:

\[
\mathbb{1} = \sum_{k \in \{0, 1\}^N} \sum_{k \in \{0, 1\}^N} \left( \det \left| F_{a,b}^{(\alpha_k, \beta_k)} \right| \right)^{-1} |\alpha_k\rangle \langle \alpha_k|,
\]

which reads

\[
\mathbb{1} = \sum_{(\ell, \tau) \in \Sigma_{6VD}} \left( \det \left| F_{a,b}^{(\ell, \tau)} \right| \right)^{-1} |\ell_{6VD}\rangle \langle \ell_{6VD}|.
\]

for the representations for which the antiperiodic transfer matrix \( \tilde{T}^{(6VD)}(\lambda, \tau) \) is proven to be diagonalizable.

### 3. On the periodic eight-vertex spectrum and connection with SOV

In this section, we will analyze the connection between the spectral problem of the periodic eight-vertex transfer matrix on chains with an odd number of quantum sites and one of the antiperiodic dynamical six-vertex transfer matrix. The Baxter gauge transformations are used together with the functional characterization of the eight-vertex transfer matrix to get central information on the spectrum (eigenvalues and eigenstates) of this model by our SOV results.

#### 3.1. The eight-vertex model

Let us recall the characterization in terms of QISM of the XYZ spin-1/2 quantum chain. The eight-vertex \( R \)-matrix reads

\[
R_{\alpha\beta}^{(8V)}(\lambda) = \begin{pmatrix}
  a(\lambda) & 0 & 0 & d(\lambda) \\
  0 & b(\lambda) & c(\lambda) & 0 \\
  0 & c(\lambda) & b(\lambda) & 0 \\
  d(\lambda) & 0 & 0 & a(\lambda)
\end{pmatrix},
\]

(3.1)
where
\[
a(\lambda) = \frac{2\theta_1(\eta|2\omega)\theta_4(\lambda|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}, \quad b(\lambda) = \frac{2\theta_4(\eta|2\omega)\theta_1(\lambda|2\omega\theta_4(\lambda + \eta|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)},
\]
\[
c(\lambda) = \frac{2\theta_1(\eta|2\omega)\theta_4(\lambda|2\omega\theta_4(\lambda + \eta|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}, \quad d(\lambda) = \frac{2\theta_1(\eta|2\omega)\theta_4(\lambda + \eta|2\omega\theta_1(\lambda|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}
\]
(3.2)

is the solution of the Yang–Baxter equation
\[
R_{12}^{(BV)}(\lambda_{12})R_{1a}^{(BV)}(\lambda_1)R_{2a}^{(BV)}(\lambda_2) = R_{2a}^{(BV)}(\lambda_2)R_{1a}^{(BV)}(\lambda_1)R_{12}^{(BV)}(\lambda_{12}).
\]
(3.4)

Then the monodromy matrix of the spin-1/2 representations is defined by
\[
M^{(BV)}_0(\lambda) \equiv R^{(BV)}_0(\lambda - \xi_0) \cdots R^{(BV)}_{01}(\lambda - \xi_1) \equiv \begin{pmatrix} A^{(BV)}(\lambda) & B^{(BV)}(\lambda) \\ C^{(BV)}(\lambda) & D^{(BV)}(\lambda) \end{pmatrix},
\]
(3.5)

with the parameters \(\xi_i\) which are the inhomogeneities. The monodromy matrix \(M^{(BV)}_0(\lambda)\) is itself the solution of the Yang–Baxter equation
\[
R_{12}^{(BV)}(\lambda_{12})M_1^{(BV)}(\lambda_1)M_2^{(BV)}(\lambda_2) = M_2^{(BV)}(\lambda_2)M_1^{(BV)}(\lambda_1)R_{12}^{(BV)}(\lambda_{12}),
\]
(3.6)

and then the corresponding transfer matrix
\[
T^{(BV)}(\lambda) = \text{tr}_0 M^{(BV)}_0(\lambda)
\]
(3.7)

defines a one-parameter family of commuting operators; the Hamiltonian of the XYZ spin-1/2 quantum chain is obtained in the homogeneous limit by
\[
H_{XYZ} = 2 \sinh \eta \left. \frac{\partial \ln T^{(BV)}(\lambda)}{\partial \lambda} \right|_{\lambda=0,\xi_i=0} - N \cosh \eta.
\]
(3.8)

3.2. Elementary properties of the periodic eight-vertex transfer matrix

Let us describe some elementary properties of the periodic eight-vertex transfer matrix which allow the first characterization of the spectrum, evidencing its connection to the antiperiodic dynamical six-vertex spectrum in the case of odd chains.

**Lemma 3.1.** In the eight-vertex Yang–Baxter algebra, we can introduce the following central quantum determinant:

\[
\det_q M^{(BV)}(\lambda) \equiv \left( A^{(BV)}(\lambda)D^{(BV)}(\lambda - \eta) - B^{(BV)}(\lambda)C^{(BV)}(\lambda - \eta) \right)
\]
\[
= \left( D^{(BV)}(\lambda)A^{(BV)}(\lambda - \eta) - C^{(BV)}(\lambda)B^{(BV)}(\lambda - \eta) \right)
\]
\[
= \lambda(\lambda)D(\lambda - \eta),
\]
(3.9) (3.10) (3.11)

and the following inversion formula holds:
\[
\left[M^{(BV)}_0(\lambda)\right]^{-1} = \frac{\sigma_0^\tau \left[M^{(BV)}_0(\lambda - \eta)\right]^{-1} \sigma_0^\nu}{\lambda(\lambda)D(\lambda - \eta)}.
\]
(3.12)
Proof. The centrality of the quantum determinant is a well-known property in the six-vertex Yang–Baxter algebra and it is possible to extend it to the eight-vertex case. The proof is given by proving the statement for the generic quantum site \( n \) and then showing that the product of the local quantum determinants reproduce the complete one. Let us introduce the notation

\[
\text{det}_q R_{0n}^{(8V)}(\lambda) = (R_{0n}^{(8V)})_{11}(\lambda)(R_{0n}^{(8V)})_{22}(\lambda - \eta) - (R_{0n}^{(8V)})_{12}(\lambda)(R_{0n}^{(8V)})_{21}(\lambda - \eta);
\]

its explicit form reads

\[
\text{det}_q R_{0n}^{(8V)}(\lambda - \xi_n) = \begin{pmatrix}
a(\lambda)b(\lambda - \eta) - d(\lambda)d(\lambda - \eta) & 0 \\
0 & b(\lambda)a(\lambda - \eta) - c(\lambda)c(\lambda - \eta)
\end{pmatrix};
\]

then all we need is to prove the following identities:

\[
a(\lambda)b(\lambda - \eta) - d(\lambda)d(\lambda - \eta) = b(\lambda)a(\lambda - \eta) - c(\lambda)c(\lambda - \eta) = a(\lambda - \xi_n)a(\lambda - \xi_n - 2\eta)
\]

which trivially follow once we use the formulae

\[
\theta_1(x + y)\theta_1(x - y)\theta_2(x + y)\theta_2(x - y)\theta_2(y)\theta_2(0) = \theta_2^2(x)\theta_2^2(y)\theta_2^2(0) - \theta_2^2(x)\theta_2^2(y)\theta_2^2(0),
\]

\[
\theta_1(x + y)\theta_2(x - y)\theta_2(x + y)\theta_2(x - y)\theta_2(y)\theta_2(0) = \theta_2^2(x)\theta_2^2(y)\theta_2^2(0) - \theta_2^2(x)\theta_2^2(y)\theta_2^2(0),
\]

\[
\theta_1(x)\theta_2(x)\theta_2(x) = \theta_1(x + y)\theta_2(x - y)\theta_2(x + y)\theta_2(x - y)\theta_2(y)\theta_2(0) + \theta_4(x + y)\theta_4(x - y)\theta_1(x - y)\theta_2(0).
\]

Finally, the inversion formula (3.12) follows from the quantum determinant formulae and from the identities

\[
\mathbf{A}^{(8V)}(\lambda)\mathbf{B}^{(8V)}(\lambda - \eta) - \mathbf{B}^{(8V)}(\lambda)\mathbf{A}^{(8V)}(\lambda - \eta) = 0,
\]

\[
\mathbf{D}^{(8V)}(\lambda)\mathbf{C}^{(8V)}(\lambda - \eta) - \mathbf{C}^{(8V)}(\lambda)\mathbf{D}^{(8V)}(\lambda - \eta) = 0,
\]

which directly follows from the eight-vertex Yang–Baxter equations (3.6).

Moreover, it holds the following.

Lemma 3.2. The products \((\mathbf{M}^{(8V)}(\xi_n^{(0)}))_{b,j} (\mathbf{M}^{(8V)}(\xi_n^{(1)}))_{k,j}\) and \((\mathbf{M}^{(8V)}(\xi_n^{(0)}))_{j,k} (\mathbf{M}^{(8V)}(\xi_n^{(0)}))_{j,k}\) of the elements of the eight-vertex monodromy matrix vanish for any \( n \in \{1, \ldots, N\} \) if \( h = k \) and the following identities hold:

\[
\mathbf{A}^{(8V)}(\xi_n^{(0)})\mathbf{D}^{(8V)}(\xi_n^{(1)}) = -\mathbf{C}^{(8V)}(\xi_n^{(0)})\mathbf{B}^{(8V)}(\xi_n^{(1)}),
\]

\[
\mathbf{D}^{(8V)}(\xi_n^{(0)})\mathbf{A}^{(8V)}(\xi_n^{(1)}) = -\mathbf{B}^{(8V)}(\xi_n^{(0)})\mathbf{C}^{(8V)}(\xi_n^{(1)}).
\]

Proof. This lemma is a trivial generalization to the eight-vertex Yang–Baxter algebra of the results known in the six-vertex case, see for example [61]. In both the eight- and six-vertex cases, these results are simple consequences of the reconstruction formulae of local operators in terms of matrix elements of the monodromy matrix first proven for the six-vertex case in [41] and then extended also to the eight-vertex case in [48]. In the eight-vertex case, the reconstructions read

\[
X_n = \prod_{b=1}^{n-1} T^{(8V)}(\xi_b^{(0)})\text{Tr}_0(\mathbf{M}^{(8V)}(\xi_n^{(0)}))X_0 \prod_{b=1}^{n} \frac{T^{(8V)}(\xi_b^{(1)})}{\text{det}_q \mathbf{M}^{(8V)}(\xi_b^{(0)})},
\]

\[
= \prod_{b=1}^{n} T^{(8V)}(\xi_b^{(0)}) \frac{\text{Tr}_0(\mathbf{M}^{(8V)}(\xi_n^{(1)}))\sigma_0^{(y)}X_0^{(y)}\sigma_0^{(y)}}{\text{det}_q \mathbf{M}^{(8V)}(\xi_n^{(0)})} \prod_{b=1}^{n-1} \frac{T^{(8V)}(\xi_b^{(1)})}{\text{det}_q \mathbf{M}^{(8V)}(\xi_b^{(0)})}.
\]

24 Respectively, equations (7), (10) and (2) on page 881 of [158].
where $X_n$ is a local operator on the quantum space $n$, i.e. it acts as the identity on any quantum space associated with a site $m \neq n$ and as the $2 \times 2$ matrix $X$ on the quantum space in the site $n$, while $X_0$ is the $2 \times 2$ matrix $X$ on the auxiliary space. Let us consider for example the following identity:

$$
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}_n = X_nX_n = Y_nZ_n,
$$

(3.24)

where

$$
X_n = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}_n, Y_n = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}_n \text{ and } Z_n = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}_n;
$$

(3.25)

then the identity (3.21) simply follows by using for the first $X_n$ and $Y_n$ the reconstruction (3.22), while for the second $X_n$ and $Z_n$ the reconstruction (3.23). All the other identities in this lemma are proven similarly by taking the product of a couple of local operators and using for them the two reconstructions. □

### 3.3. On the periodic eight-vertex transfer matrix eigenvalues

The previous two lemmas allow us to prove some preliminary characterization of the periodic eight-vertex eigenvalues as presented in the following proposition.

**Proposition 3.1.** The periodic eight-vertex transfer matrix of a chain with $N$ quantum sites satisfies the following properties:

$$
T^{(BV)}(\xi_n^{(0)})T^{(BV)}(\xi_n^{(1)}) = \det_\theta M^{(BV)}(\xi_n^{(0)}) \forall n \in \{1, \ldots, N\}
$$

(3.26)

and

$$
T^{(BV)}(\lambda + \pi) = (-1)^N T^{(BV)}(\lambda), \quad T^{(BV)}(\lambda + \pi \omega) = (-e^{-i(2\lambda + \pi \omega)})^N e^{-2i(\lambda - \sum_{n=1}^N \xi_n)} T^{(BV)}(\lambda).
$$

(3.27)

Then the eight-vertex eigenvalues are elliptic polynomials (or theta functions) of degree $N$ and character $e^{-2i(\omega - \sum_{n=1}^N \xi_n)}$ and they admit the following interpolation formula:

$$
t_{BV}(\xi_n) = \frac{\sum_{a=1}^N \theta(\theta - \lambda + \xi_a)}{\theta(\theta) \prod_{b \neq a} \theta(\xi_a - \xi_b)} t_{BV}(\xi_a),
$$

(3.28)

where $t_{BV}(\xi_a)$ are solutions of the discrete system of equations:

$$
t_{BV}(\xi_a^{(0)})t_{BV}(\xi_a^{(1)}) = \lambda(\xi_a^{(0)})d(\xi_a^{(1)}), \quad \forall a \in \{1, \ldots, N\}.
$$

(3.29)

**Proof.** Lemma 3.2 in particular implies the annihilation identities

$$
A^{(BV)}(\xi_n^{(0)})A^{(BV)}(\xi_n^{(1)}) = D^{(BV)}(\xi_n^{(0)})D^{(BV)}(\xi_n^{(1)}) = 0,
$$

(3.30)

from which we can write

$$
T^{(BV)}(\xi_n^{(0)})T^{(BV)}(\xi_n^{(1)}) = A^{(BV)}(\xi_n^{(0)})D^{(BV)}(\xi_n^{(1)}) + D^{(BV)}(\xi_n^{(0)})A^{(BV)}(\xi_n^{(1)})
$$

(3.31)

and then eliminating in the above equation $D^{(BV)}(\xi_n^{(0)})A^{(BV)}(\xi_n^{(1)})$ by using (3.20) or $A^{(BV)}(\xi_n^{(0)})D^{(BV)}(\xi_n^{(1)})$ by using (3.21), we obtain the identity (3.26).

Let us observe now that by using the identities, we can write

$$
\theta_1(x + \pi|2\omega) = -\theta_1(x|2\omega), \quad \theta_1(x + \pi \omega|2\omega) = i e^{-i(\lambda + \pi \omega)/2} \theta_2(x|2\omega)
$$

(3.32)

See equations (8.182-1), (8.182-3) and (8.183-5), (8.183-6) on page 878 of [158].

22
Then we can prove the following

\[ \theta_4(x + \pi|2\omega) = \theta_4(x|2\omega), \quad \theta_4(x + \pi|\omega) = i e^{-i(\lambda + \pi|\omega/2)} \theta_1(x|2\omega), \]  \hspace{1cm} (3.33)

it is simple to show that the coefficients of the eight-vertex \( R \)-matrix satisfy the following transformation properties:

\[ a(\lambda + \pi|\omega) = e^{-i(2\lambda + \pi|\omega)} e^{-i\theta} b(\lambda), \quad a(\lambda + \pi) = -a(\lambda), \quad d(\lambda + \pi) = d(\lambda), \]  \hspace{1cm} (3.34)

\[ c(\lambda + \pi|\omega) = e^{-i(2\lambda + \pi|\omega)} e^{-i\theta} d(\lambda), \quad b(\lambda + \pi) = -b(\lambda), \quad c(\lambda + \pi) = c(\lambda), \]  \hspace{1cm} (3.35)

which are equivalent to the following identities of the eight-vertex \( R \)-matrix:

\[ R_{\lambda\delta}^{(8V)}(\lambda + \pi|\omega) = -e^{-i(2\lambda + \pi|\omega)} e^{-i\theta} \sigma_{\lambda\delta}^{(a)} R_{\lambda\delta}^{(8V)}(\lambda) \sigma_{\lambda\delta}^{(a)}, \quad R_{\lambda\delta}^{(8V)}(\lambda + \pi) = -\sigma_{\lambda\delta}^{(a)} R_{\lambda\delta}^{(8V)}(\lambda) \sigma_{\lambda\delta}^{(a)}. \]  \hspace{1cm} (3.36)

Then, the monodromy matrix satisfies the identities

\[ M_0^{(8V)}(\lambda + \pi|\omega) = (-e^{-i(2\lambda + \pi|\omega)} )^N e^{-2i(\theta - \sum_{\alpha=1}^{N-1} \xi_\alpha)} \sigma_{\lambda\delta}^{(c)} M_0^{(8V)}(\lambda) \sigma_{\lambda\delta}^{(c)}, \]  \hspace{1cm} (3.37)

\[ M_0^{(8V)}(\lambda + \pi) = (-1)^N \sigma_{\lambda\delta}^{(c)} M_0^{(8V)}(\lambda) \sigma_{\lambda\delta}^{(c)}, \]  \hspace{1cm} (3.38)

from which (3.26) follows by the cyclicity of the trace. Formula (3.28) is the interpolation formula [159] for elliptic polynomials of degree \( N \) and character \( e^{-2i(\theta - \sum_{\alpha=1}^{N-1} \xi_\alpha)} \), while (3.29) is the rewriting of (3.26) for the periodic eight-vertex eigenvalues. \( \square \)

**Remark 1.** Let us note that the above results hold for both the even and the odd quantum chains. In the odd case, these results imply that the set of the periodic eight-vertex transfer matrix eigenvalues is contained in the set of the antiperiodic dynamical six-vertex transfer matrix eigenvalues. As we have proven that the antiperiodic dynamical six-vertex transfer matrix has a simple spectrum for general values of the inhomogeneities, differences in these sets of eigenvalues can only be produced from a degeneracy of the periodic eight-vertex transfer matrix eigenvalues. A preliminary analysis based on direct diagonalization shows that the periodic eight-vertex transfer matrix spectrum is double degenerate for \( N = 1, 3 \). These cases are considered in the appendix where a direct verification of the statement proven in theorem 2.3 that the system of equations defined by (2.103) and (2.104) characterizes the complete set of the antiperiodic dynamical six-vertex transfer matrix eigenvalues is also given.

### 3.4. Gauge transformation from eight-vertex to dynamical six-vertex models

In the case of an even chain, the spectral problem of the eight-vertex transfer matrix \( T^{(8V)}(\lambda) \) has been reduced to one of the periodic dynamical six-vertex transfer matrices by the gauge transformations introduced by Baxter in [17]. In detail, the following gauge transformation exists:

\[ R_{\lambda\delta}^{(8V)}(\lambda_1|\tau) S_0(\lambda_1|\tau) S_\delta(\lambda_2|\tau + \eta \sigma_0) = S_\delta(\lambda_2|\tau) S_0(\lambda_1|\tau + \eta \sigma_0) R_{\lambda\delta}^{(6VD)}(\lambda_1|\tau), \]  \hspace{1cm} (3.39)

which for the monodromy matrices reads

\[ M_0^{(8V)}(\lambda) S_0(\lambda|\tau) S_\delta(\tau + \eta \sigma_0) = S_\delta(\tau) S_0(\lambda|\tau + \eta \sigma_0) M_0^{(6VD)}(\lambda|\tau), \]  \hspace{1cm} (3.40)

where

\[ S_0(\lambda|\tau) \equiv \begin{pmatrix} \theta_2(\lambda + \tau|2\omega) & \theta_3(\lambda + \tau|2\omega) \\ \theta_3(-\lambda + \tau|2\omega) & \theta_2(-\lambda + \tau|2\omega) \end{pmatrix}, \]  \hspace{1cm} (3.41)

and

\[ S_\delta(\tau) \equiv S_{\delta(1|\tau)} \cdots S_{\delta(N|\tau)} \left( \xi_\delta|\tau + \eta \sum_{\alpha=1}^{N-1} \sigma_\alpha \right). \]  \hspace{1cm} (3.42)

Then we can prove the following

23
Lemma 3.3. In a chain with an odd number of quantum sites, the periodic eight-vertex transfer matrix has the following right action on the states of $\bar{D}^{(8V)}_{0,N}$:

$$T^{(8V)}(\lambda)S_0(\tau) = S_0(\tau - \eta)C(\lambda|\tau - \eta) + S_0(\tau + \eta)B(\lambda|\tau + \eta).$$

(3.43)

Proof. To prove (3.43), let us first rewrite the gauge transformation (3.40) as follows:

$$S_0(\lambda + \eta|\tau)S_0(\tau - \eta\sigma_0^c)\begin{pmatrix} D(\lambda|\tau - \eta) & -B(\lambda|\tau + \eta) \\ -C(\lambda|\tau - \eta) & A(\lambda|\tau + \eta) \end{pmatrix} \frac{\theta(\tau + \eta S)}{\theta(\tau)}
= \begin{pmatrix} D^{(8V)}(\lambda) & -B^{(8V)}(\lambda) \\ -C^{(8V)}(\lambda) & A^{(8V)}(\lambda) \end{pmatrix} S_0(\tau)S_0(\lambda + \eta|\tau + \eta S),$$

(3.44)

obtained by multiplying both sides of (3.40) from the right by the inverse of $M^{(8V)}_{\nu}(\lambda|\tau)$, as defined in (2.29), and from the left by the inverse of $M^{(8V)}_{0}(\lambda)$, as defined in (3.12), and finally changing the variable $\lambda \rightarrow \lambda + \eta$. The gauge transformation (3.44) can be further rewritten as follows:

$$S_0(\lambda + \eta|\tau - \eta)S_0(\tau - \eta\sigma_0^c)\begin{pmatrix} -C(\lambda|\tau - \eta) & A(\lambda|\tau + \eta) \\ D(\lambda|\tau + \eta) & -B(\lambda|\tau + \eta) \end{pmatrix} \frac{\theta(\tau + \eta S)}{\theta(\tau)}
= \begin{pmatrix} D^{(8V)}(\lambda) & -B^{(8V)}(\lambda) \\ -C^{(8V)}(\lambda) & A^{(8V)}(\lambda) \end{pmatrix} S_0(\tau)S_0(\lambda + \eta|\tau + \eta S),$$

(3.45)

by using the identity

$$S_0(\lambda - \tau) = S_0(-\lambda|\tau) = S_0(\lambda|\tau)\sigma_0^c.$$  

(3.46)

We can take now the trace w.r.t. the auxiliary space 0 and obtain

$$T^{(8V)}(\lambda)S_0(\tau) = tr_0 \left\{ S_0(\tau - \eta\sigma_0^c)\begin{pmatrix} -C(\lambda|\tau - \eta) & A(\lambda|\tau + \eta) \\ D(\lambda|\tau + \eta) & -B(\lambda|\tau + \eta) \end{pmatrix} \frac{\theta(\tau + \eta S)}{\theta(\tau)} \right\} [S_0(\lambda + \eta|\tau + \eta S)]^{-1} S_0(\lambda + \eta|\tau - \tau),$$

(3.47)

where we have used the commutativity

$$[M^{(8V)}_{\nu}(\lambda|\tau), \tau] = 0$$

(3.48)

and the cyclicity of the trace to move $S_0(\lambda + \eta|\tau - \tau)$. It is central to remark that $S_0(\lambda + \eta|\tau + \eta S)$ is an invertible matrix in the auxiliary space on any state of $\bar{D}^{(8V)}_{0,N}$ and so the identity (3.47) is well defined on $\bar{D}^{(8V)}_{0,N}$ and using it we obtain our result (3.43) being the eigenvalues of $\tau$ and $-\tau - \eta S$ coinciding on any left state of $\bar{D}^{(8V)}_{0,N}$. 

\[ \square \]

3.5. On the periodic eight-vertex transfer matrix spectrum by SOV

3.5.1. Connections between periodic eight-vertex and antiperiodic dynamical six-vertex spectrum. Let us define $\bar{D}^{R}_N = \sum_{n=-N}^{N} \langle t|s \rangle$; by the definition of the scalar product given in section 2.1.1, the action of $\bar{D}^{R}_N$ reduces the dynamical-spin vector space $D^{R}_{0,N}$ to the 2N-dimensional spin vector space $\bar{D}^{R}_N = \bar{D}^{R}_N \bar{D}^{R}_{0,N}$ and relates the generic vectors of their basis by $\bar{D}^{R}_N \otimes_{n=1}^{N} |n, h_n\rangle \oplus |n, h_n\rangle = \bar{D}^{R}_N |n, h_n\rangle$. Moreover, let us introduce the pure spin operator $S^R_n \in \text{End}(\bar{D}^{R}_N)$ by the following actions:

$$S^R_n \otimes_{n=1}^{N} |n, h_n\rangle \equiv S_1 \left( \xi_1| - \frac{n}{2} \sum_{a=1}^{N} \sigma_a^x \right) \cdots S_N \left( \xi_N| \frac{n}{2} \sum_{a=1}^{N} \sigma_a^x - \frac{n}{2} \sum_{a=1}^{N} \sigma_a^z \right).$$

(3.49)

on the spin basis of the pure spin quantum space $\bar{D}^{R}_N$, then we have the following.
Proposition 3.2.

(i) On any vector of $\mathbb{D}^{R}_{(6VD), N}$, it holds

$$\bar{\bar{p}}_{N}^{R}S_{q}(\tau) = S_{q}^{R}\bar{\bar{p}}_{N}^{R}. \quad (3.50)$$

(ii) The following identities hold

$$\bar{\bar{p}}_{N}^{R}S_{q}(\tau - \eta)C(\lambda|\tau - \eta) = S_{q}^{R}\bar{\bar{p}}_{N}^{R}C(\lambda|\tau), \quad \bar{\bar{p}}_{N}^{R}S_{q}(\tau + \eta)B(\lambda|\tau + \eta) = S_{q}^{R}\bar{\bar{p}}_{N}^{R}B(\lambda|\tau), \quad (3.51)$$

on any vector of $\mathbb{D}^{R}_{(6VD), N}$ and so it also follows

$$T^{(6V)}(\lambda)S_{q}^{R}\bar{\bar{p}}_{N}^{R} = S_{q}^{R}\bar{\bar{p}}_{N}^{R}T^{(6VD)}(\lambda|\tau). \quad (3.52)$$

Proof. The identity (3.50) follows by computing the action on the generic elements of the dynamical spin basis of $\mathbb{D}^{R}_{(6VD), N}$ and using the orthonormality of these states. Let us prove now (3.51); we use first the identities

$$C(\lambda|\tau) = C(\lambda|\tau)T_{\tau}^{+} = T_{\tau}^{+}C(\lambda|\tau - \eta), \quad (3.53)$$

$$B(\lambda|\tau) = B(\lambda|\tau)T_{\tau} = T_{\tau}B(\lambda|\tau + \eta), \quad (3.54)$$

to write

$$S_{q}(\tau - \eta)C(\lambda|\tau - \eta) = T_{\tau}S_{q}(\tau)C(\lambda|\tau), \quad S_{q}(\tau + \eta)B(\lambda|\tau + \eta) = T_{\tau}B(\lambda|\tau)S_{q}(\tau). \quad (3.55)$$

Then the following identities hold:

$$\bar{\bar{p}}_{N}^{R}T_{\tau}S_{q}(\tau)C(\lambda|\tau)[h_{1}, \ldots, h_{N}] = \sum_{s=-N}^{N} \langle \tau(s) - \eta|S_{q}(h_{s} - \eta)C(\lambda|h_{s})|h_{1}, \ldots, h_{N}\rangle \quad (3.56)$$

$$= \sum_{s=-N}^{N} \langle \tau(s)|S_{q}(h_{s} - \eta)C(\lambda|h_{s})|h_{1}, \ldots, h_{N}\rangle \quad (3.57)$$

$$= \bar{\bar{p}}_{N}^{R}S_{q}(\tau)C(\lambda|\tau)[h_{1}, \ldots, h_{N}] \quad (3.58)$$

$$= [h_{1}, \ldots, h_{N}]C(\lambda|\tau)\bar{\bar{p}}_{N}^{C}\left[S_{q}^{C}\right]^{-1}, \quad (3.59)$$

where (3.57) is equal to (3.56) being $C(\lambda|\tau)|h_{1} = 0, \ldots, h_{N} = 0\rangle$ zero, while the last equality follows directly by (3.50) being $C(\lambda|\tau)|h_{1}, \ldots, h_{N}\rangle \in \mathbb{D}^{R}_{(6VD), N}$; similarly, one can prove the second identity in (3.51).

The identity (3.52) is then proven by the following set of equalities which hold on $\mathbb{D}^{R}_{(6VD), N}$:

$$T^{(6V)}(\lambda)S_{q}^{R}\bar{\bar{p}}_{N}^{R} = \bar{\bar{p}}_{N}^{R}T^{(6V)}(\lambda)S_{q}(\tau) \quad (3.50)$$

$$= \bar{\bar{p}}_{N}^{R}S_{q}(\tau - \eta)C(\lambda|\tau - \eta) + \bar{\bar{p}}_{N}^{R}S_{q}(\tau + \eta)B(\lambda|\tau + \eta) \quad (3.51)$$

$$= S_{q}^{R}\bar{\bar{p}}_{N}^{R}T^{(6VD)}(\lambda|\tau). \quad (3.52)$$

$\square$
3.5.2. On the periodic eight-vertex transfer matrix spectrum by SOV in odd chains. It is central to remark that the operator \( S^R_q \in \text{End}(S^R_N) \) is not invertible in \( S^R_N \). This statement is simply verified observing that the subspace of \( S^R_N \) generated by the vectors:

\[
\otimes_{\nu=1}^{N-1} |n, h_n⟩ \odot (|N, h_N = 1⟩ - |N, h_N = 0⟩) \quad \text{for which} \quad \sum_{\nu=1}^{N-1} h_n = \frac{N-1}{2},
\]

belongs to the kernel of \( S^R_q \) as all these states are clearly annihilated by the action of the operator \( S_N (\xi|n⟩ \otimes \sum_{\nu=1}^{N-1} σ^z_\nu - \frac{q}{a} σ^+_n) \). Then, it is clear that we cannot use the identity (3.52) to completely reconstruct the spectrum of the periodic eight-vertex transfer matrix \( T^{(8V)}(λ) \) by using the SOV characterization of the spectrum of the antiperiodic dynamical six-vertex transfer matrix \( \bar{T}^{(6VD)}(λ|τ) \). Nevertheless, we can use (3.52) to define a first criterion to select eigenvalues of \( \bar{T}^{(6VD)}(λ|τ) \) which are also eigenvalues of \( T^{(8V)}(λ) \) and to associate with any one of these eigenvalues just one eigenvector of \( T^{(8V)}(λ) \). In particular, we can prove the following lemma.

**Lemma 3.4.** Let us consider \( η_{6VD}(λ) ∈ \Sigma_{T^{(6VD)}} \) and let \( |η_{6VD}⟩ \) be the corresponding eigenvector of \( \bar{T}^{(6VD)}(λ|τ) \); then if the vector

\[
\hat{ρ}_{N}^{R}|η_{6VD}⟩ = \sum_{h_1, \ldots, h_N=0}^{1} \prod_{a=1}^{N} Q_{a}(\bar{h}_{a}) \text{det} (θ_{ij})_N |h_1, \ldots, h_N⟩, \quad \text{with} |h_1, \ldots, h_N⟩ = |h_1⟩|h_1, \ldots, h_N⟩
\]

(3.64)

does not belong to the kernel of \( S^R_q \), then \( η_{6VD}(λ) \) is eigenvalue of \( T^{(8V)}(λ) \) and \( S^R_q \hat{ρ}_{N}^{R}|η_{6VD}⟩ \) is one corresponding eigenvector.

**Proof.** Under the condition

\[
S^R_q \hat{ρ}_{N}^{R}|η_{6VD}⟩ \neq 0 ∈ S^R_N,
\]

(3.65)
the proposition is a simple consequence of the identity (3.52); indeed, it holds

\[
T^{(8V)}(λ)S^R_q \hat{ρ}_{N}^{R}|η_{6VD}⟩ = S^R_q \hat{ρ}_{N}^{R}\bar{T}^{(6VD)}(λ|τ)|η_{6VD}⟩ = S^R_q \hat{ρ}_{N}^{R}|η_{6VD}⟩ η_{6VD}(λ).
\]

(3.66)

**Remark 2.** It is worth remarking that we need this criterion as, from the results in subsection 3.3, we only know that the set \( Σ_{T^{(6VD)}} \) of the eigenvalues of \( T^{(8V)}(λ) \) is contained in the set \( Σ_{T^{(6VD)}} \) of the eigenvalues of \( \bar{T}^{(6VD)}(λ|τ) \) in the case of a chain with an odd number of quantum sites. Moreover, it is important to clarify that currently we have only proven that the above lemma defines a criterion, i.e., a sufficient condition for an element of \( Σ_{T^{(6VD)}} \) to be also an element of \( Σ_{T^{(8V)}} \). It will be fundamental to understand if this is also a necessary condition as in this last case, we will obtain a complete characterization of \( Σ_{T^{(6VD)}} \) and one eigenstate of \( T^{(8V)}(λ) \) for any element of \( Σ_{T^{(6VD)}} \) just using the SOV characterization of the spectrum of \( \bar{T}^{(6VD)}(λ|τ) \). Finally, let us point out that the fact that \( S^R_q ∈ \text{End}(S^R_N) \) is not invertible in \( S^R_N \) is just required to make the identity (3.52) compatible with the observed degeneracy of the spectrum of \( T^{(8V)}(λ) \) for the cases \( N = 1 \) and 3 explicitly analyzed in the appendix. As in the presence of this degeneracy, the proven simplicity of the spectrum of \( \bar{T}^{(6VD)}(λ|τ) \) implies that \( Σ_{T^{(6VD)}} \) must be properly contained in \( Σ_{T^{(8V)}} \).
4. Conclusion

In this paper, we have focused our attention on the highest weight representations of the dynamical six-vertex Yang–Baxter algebra on a generic spin-1/2 quantum chain with an odd number \(N\) of sites. We have studied the integrable quantum model associated with the antiperiodic boundary conditions in the framework of the SOV method. For these integrable quantum models, we have derived

- The complete SOV description of transfer matrix eigenvalues and eigenstates and the simplicity of the spectrum.
- Matrix elements of the identity on separate states expressed by one determinant formulae of \(N \times N\) matrices with elements given by sums over the eigenvalues of the quantum separate variables of the product of the coefficients of the left/right separate states, which hold in particular for the eigenstates of the antiperiodic dynamical six-vertex transfer matrix.

The results derived in this paper provide the required setup to compute matrix elements on transfer matrix eigenstates of local operators. The analysis of the steps

- local operator reconstructions in terms of Sklyanin’s quantum separate variables,
- form factors of the local operators on the transfer matrix eigenstates in the determinant form

will be presented in a paper in collaboration with Levy-Bencheton and Terras [160], which is currently in preparation. The study of correlation functions done in [161] for the periodic dynamical six-vertex chain even if developed in the framework of the algebraic Bethe ansatz is relevant also for the current analysis. Indeed, the reconstruction of local operators of [161] can be adapted to the antiperiodic dynamical six-vertex chain to obtain reconstruction of local operators in terms of the quantum separate variables. Then the computation of the form factors for the antiperiodic dynamical six-vertex proceed in a similar way to that of the standard six-vertex quantum chain with antiperiodic boundary conditions as derived in [71] in the SOV framework. It is also worth mentioning that the knowledge of the form factors of local operators represents also an efficient tool for controlled numerical analysis of correlation functions. Indeed, the decomposition of the identity (2.133) allows us to rewrite the correlation functions in terms of form factors and then it is a priori possible to apply the same kind of approach developed in [162] in the ABA framework26 also in our SOV framework to get numerical evaluations of correlation functions.

We have moreover shown that the existence of gauge transformations allows us to use the antiperiodic dynamical six-vertex transfer matrix as a tool to further analyze the spectral problem of the periodic eight-vertex transfer matrix in the case of a chain with \(N\) odd and for general values of the coupling constant \(\eta\) (non restricted to the elliptic roots of unit). The potential relevance of this analysis is made clear observing that the standard Bethe ansatz analysis developed in [18, 17, 19, 3] does not apply to this case. In more detail, we have shown that the gauge transformations allow us to define a criterion to select eigenvalues of the antiperiodic dynamical six-vertex transfer matrix which are also eigenvalues of the periodic eight-vertex transfer matrix, moreover associating with any one of them one nonzero periodic eight-vertex eigenstate. Finally, let us stress the importance to understand if this criterion also defines a necessary condition for an antiperiodic dynamical six-vertex eigenvalue to be also a periodic eight-vertex eigenvalue. Indeed, in this case the SOV characterization of the antiperiodic dynamical six-vertex spectrum will also allow the complete characterization of the

26 See the series of papers [162–168] where the dynamical structure factors, observable by neutron scattering experiments [169–174], were numerically evaluated.
periodic eight-vertex eigenvalues and the construction of one of its eigenstates for any of its eigenvalues. The answer to this fundamental question requires a systematic and simultaneous analysis of the degeneracy of the periodic eight-vertex spectrum and of the dimension of the kernel of the operator $S^R_q$, which we will try to address elsewhere.

Acknowledgments

The author gratefully acknowledges B McCoy for the many stimulating discussions on the eight-vertex model and the interesting questions on quantum separation of variables which have strongly inspired and motivated the author to develop the present paper. The author would also like to thank N Kitanine, K K Kozlowski and J M Maillet for their interest and D Levy-Bencheton and V Terras for their interest, attentive reading and remarks on a first draft of this paper. The author is supported by National Science Foundation grants PHY-0969739 and gratefully acknowledges the YITP Institute of Stony Brook for the opportunity to develop his research programs. The author would also like to thank for their hospitality the Theoretical Physics Group of the Laboratory of Physics at ENS-Lyon and the Mathematical Physics Group at IMB of the Dijon University (under support ANR-10-BLAN-0120-04-DIADEMS).

Appendix

Here we analyze explicitly the spectral problem of the transfer matrices $T^{(8V)}(\lambda)$ and $T^{(6VD)}(\lambda|\tau)$ for the trivial cases of chains with $N=1$ and $N=3$. The aim is to make some of the above statements clear and to have some basic analysis about the degeneracy of the $T^{(8V)}(\lambda)$ spectrum.

A.1. Spectrum of $T^{(8V)}(\lambda)$ and $T^{(6VD)}(\lambda|\tau)$ for $N=1$

In the one site case, it holds

$$T^{(8V)}(\lambda) = (a(\lambda) + b(\lambda)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ on } S^R_{N=1}, \quad (A.1)$$

and from

$$T^{(6VD)}(\lambda|\tau) = T^+_\tau C(\lambda|\tau - \eta) + T^-_\tau B(\lambda|\tau + \eta), \quad (A.2)$$

it holds

$$T^{(6VD)}(\lambda|\tau) = c(\lambda|\eta/2) \begin{pmatrix} 0 & T^+_\tau \\ T^-_\tau & 0 \end{pmatrix} \text{ on } D^R_{(6VD),N=1}, \quad (A.3)$$

Then, we have that the spectrum of $T^{(6VD)}(\lambda|\tau)$ is simple and characterized by

$$T^{(6VD)}(\lambda|\tau)|_{_{(\pm)VD}}^{(\pm)} = |_{_{(\pm)VD}}^{(\pm)} T^{(\pm)}_{VD}(\lambda) \quad (A.4)$$

where we have defined

$$t^{(\pm)}_{VD}(\lambda) \equiv \pm c(\lambda|\eta/2), \quad |_{_{(\pm)VD}}^{(\pm)} = \left( \begin{array}{c} \pm |t(1)) \\ |t(-1)) \end{array} \right) \in D^R_{(6VD),N=1}, \quad (A.5)$$

and $|t(\alpha))$ is the $\tau$ eigenstate with eigenvalue $t(\alpha) \equiv -\eta \alpha/2$. Instead, the spectrum of $T^{(8V)}(\lambda)$ is double degenerate with eigenvalue $t_{VD}(\lambda) = a(\lambda) + b(\lambda);$ then the identity

$$a(\lambda) + b(\lambda) = c(\lambda|\eta/2) \quad (A.6)$$
imply the proper set inclusion $\Sigma_\gamma^{(\theta)} \subset \Sigma_\gamma^{(\bar{\theta})}$ for the $N = 1$ case. Let us now observe that

$$S^R_q = S_1(\lambda - \eta/2\sigma_1^i) \equiv \left( \begin{array}{c} \theta_1(\lambda + \eta/2|2w) \\ \theta_2(\lambda + \eta/2|2w) \\ \theta_3(\lambda + \eta/2|2w) \\ \theta_4(\lambda + \eta/2|2w) \end{array} \right)_0 \quad (A.7)$$

and

$$S^R_{q - \text{N-1}}(\pm i) = \left( \begin{array}{c} \theta_2(\lambda + \eta/2|2w)(1 \pm 1) \\ \theta_3(\lambda + \eta/2|2w)(1 \pm 1) \end{array} \right) \in \mathbb{R}^n. \quad (A.8)$$

Then, in agreement with the lemma 3.4, we have that $t^{(+)}_{\text{avd}}(\lambda)$ is $T^{(\theta)}(\lambda)$ eigenvalue and

$$S^R_{q - \text{N-1}}(\pm i) = 2 \left( \begin{array}{c} \theta_2(\lambda + \eta/2|2w) \\ \theta_3(\lambda + \eta/2|2w) \end{array} \right) \in \mathbb{R}^n \quad (A.9)$$

is one corresponding eigenstate. It is also interesting to remark that the eigenvalue $t^{(-)}_{\text{avd}}(\lambda)$ of $T^{(\theta)}(\lambda)$ for which it holds $S^R_{q - \text{N-1}}(\pm i) = \mathbf{0}$ is not an eigenvalue of $T^{(\bar{\theta})}(\lambda)$.

A.2. Spectrum of $T^{(\theta)}(\lambda)$ and $T^{(\theta)}(\lambda|\tau)$ for $N = 3$

A.2.1. General statements on the spectrum for $N = 3$. The system of equations which completely characterize the spectrum (eigenvalues and eigenvectors) of $T^{(\theta)}(\lambda|\tau)$ reads

$$x_n \left( \sum_{a=1}^{N} J_{n,a} x_a \right) - q_n = 0 \quad \forall n \in \{1, \ldots, N\}. \quad (A.10)$$

It is an inhomogeneous system of $N$ quadratic equations in the $N$ unknowns $x_n = t_{\text{avd}}(\xi_n)$ with coefficients characterized by

$$J_{n,a} = \frac{\theta(a - \xi_a + \xi_n + \eta)}{\theta(\eta)} \prod_{b \neq a} \frac{\theta(\xi_n - \xi_b - \eta)}{\theta(\xi_n - \xi_b)} \cdot \quad q_n = \lambda(\xi_n^{(0)}) \delta(\xi_n^{(1)}), \quad \forall n \in \{1, \ldots, N\}, \quad (A.11)$$

as it is simple to derive substituting the interpolation formula (2.104) for $t_{\text{avd}}(\xi_n^{(1)})$ into (2.103). It is trivial to observe that the set of the solutions to this system has a $Z_2$ symmetry, i.e. if $z_n^{(+)} \equiv \{x_1, \ldots, x_n, \ldots, x_N\}$ is a solution of it, then also $z_n^{(-)} \equiv \{-x_1, \ldots, -x_n, \ldots, -x_N\}$ is a solution. So from the interpolation formula (2.104), it follows that if $t^{(+)}_{\text{avd}}(\lambda) \in \Sigma_\gamma^{(\bar{\theta})}$, then also $t^{(-)}_{\text{avd}}(\lambda) \equiv (-t^{(+)}_{\text{avd}}(\lambda)) \in \Sigma_\gamma^{(\theta)}$.

In the case $N = 3$, we have verified that the system (A.10) has $2^3$ distinct solutions. Then $\Sigma_\gamma^{(\theta)}$ is composed by $2^3$ distinct elliptic polynomials $t^{(\pm,a)}_{\text{avd}}(\lambda)$ with $a \in \{1, \ldots, 4\}$ and so $T^{(\theta)}(\lambda|\tau)$ has a simple spectrum as proven in this paper. Moreover, in the case of $N = 3$, we have studied the spectrum of $T^{(\theta)}(\lambda)$ and we have observed that it has four distinct eigenvalues each one being double degenerate. We have observed that the set $\Sigma_\gamma^{(\theta)} \equiv \{t^{(1)}_{\text{avd}}(\lambda), t^{(2)}_{\text{avd}}(\lambda), t^{(3)}_{\text{avd}}(\lambda), t^{(4)}_{\text{avd}}(\lambda)\}$ coincide with the set of elliptic polynomials which are generated by using four distinct solutions $z_n^{(a)}$ of the system (A.10) and the interpolation formula (2.104). Moreover, the solutions $z_n^{(a)}$ with $a \in \{1, \ldots, 4\}$ used to construct $\Sigma_\gamma^{(\theta)}$ appear not to be related by the $Z_2$ symmetry of the system (A.10); in fact, it holds

$$z_n^{(a)} \neq -z_n^{(b)} \quad \text{for any } a, b \in \{1, \ldots, 4\}. \quad (A.12)$$

Let us fix the notation

$$t^{(\pm,a)}_{\text{avd}}(\lambda) \equiv \pm t^{(a)}_{\text{avd}}(\lambda) \quad \text{for any } a \in \{1, \ldots, 4\}; \quad (A.13)$$

then we can summarize the above observations on the spectrum of $T^{(\theta)}(\lambda|\tau)$ and $T^{(\bar{\theta})}(\lambda)$ by saying that with the two nondegenerate eigenvalues $t^{(\pm,a)}_{\text{avd}}(\lambda)$ and $t^{(\pm,-a)}_{\text{avd}}(\lambda)$ of $T^{(\theta)}(\lambda|\tau)$
it associates just the double degenerate eigenvalue $t_{\lambda\nu}^\omega(\lambda)$ of $T^{(8V)}(\lambda)$. Our analysis shows that this statement holds for $N = 1$ and $N = 3$ to be able to verify if it persists for a generic odd $N$ can be one central step toward the characterization of the spectrum of $T^{(8V)}(\lambda)$ in terms of the SOV characterization of the $T^{(8V)}(\lambda|\tau)$ spectrum.

A.2.2. Some numerical data for $N = 3$. For the convenience of the reader, we write explicitly the periodic eight-vertex transfer matrix $T^{(8V)}(\lambda)$ in the $N = 3$ case:

$$
\begin{array}{cccccccc}
 a_1 b_2 a_3 + b_3 b_2 + 0 & 0 & b_2 d_2 + a_1 c_2 d_3 & 0 & d_1 c_2 + c_1 b_2 d_3 & d_1 c_2 + c_1 b_2 d_3 & 0 \\
 0 & b_2 b_3 + b_1 a_3 + a_3 c_1 + b_3 d_1 & 0 & 0 & 0 & 0 & 0 \\
 a_1 d_2 c_3 + b_1 a_3 & 0 & 0 & b_2 a_3 + a_3 b_1 & 0 & c_1 c_2 b_3 + d_1 a_3 & 0 & 0 \\
 d_1 b_2 + c_1 c_2 a_3 & 0 & 0 & c_1 a_2 + d_1 b_3 & 0 & 0 & 0 & a_1 b_3 + b_3 a_2 + a_2 b_1 \\
 0 & d_1 c_2 + c_1 c_2 d_3 & 0 & d_1 c_2 b_3 + c_1 b_2 & 0 & b_1 b_3 + a_1 c_2 d_3 & c_1 c_2 b_3 + d_1 a_3 & 0 \\
 0 & d_1 d_2 + c_1 c_2 b_3 & 0 & b_1 d_2 + d_1 a_3 & 0 & a_1 d_2 c_3 + b_1 a_3 & 0 & a_1 d_2 c_3 + b_1 a_3 \\
 0 & d_1 d_2 + c_1 c_2 b_3 & 0 & b_1 d_2 + d_1 a_3 & 0 & a_1 d_2 c_3 + b_1 a_3 & 0 & a_1 d_2 c_3 + b_1 a_3 \\
 \end{array}
$$

where we have used the notations

$$
a_n = a(\lambda - \xi_n), \quad b_n = b(\lambda - \xi_n), \quad c_n = c(\lambda - \xi_n), \quad d_n = d(\lambda - \xi_n).
$$

To verify the statements given in the previous subsection, it is only needed to write in Mathematica the previous $8 \times 8$ matrix and the system of equations (A.10) for $N = 3$ and solve the eigenvalue problem and the system for generic values of the five parameters $(\xi_1, \xi_2, \xi_3, \eta, t = e^{i\tau u})$. Here we report just a few numerical data as a confirmation that we have really implemented this numerical exercise. Defining $w_{\lambda}^{(\nu)} = \{t_{\lambda\nu}^\omega(\xi_1), t_{\lambda\nu}^\omega(\xi_2), t_{\lambda\nu}^\omega(\xi_3)\}$, it holds

1. for $\xi_1 = 5.7, \xi_2 = 1.5, \xi_3 = 0.22, \eta = 0.7, t = 0.26$:

$$
\begin{align*}
 z_{\lambda}^{(1)} & = \pm [2.464 897 113 338 4494, 0.526 366 061 329 1964, -0.046 164 676 253 6026] \\
z_{\lambda}^{(2)} & = \pm [0.167 463 779 433 676 66, 0.094 385 846 960 007 17, -3.789 384 759 881 3264] \\
z_{\lambda}^{(3)} & = \pm [0.156 978 384 285 468 23, 0.512 457 412 943 1847, -0.744 558 515 987 6167] \\
z_{\lambda}^{(4)} & = \pm [0.025 681 586 506 628 99, 3.433 163 601 035 1154, -0.679 328 947 667 353] \\
w_{\lambda}^{(1)} & = [2.464 897 113 338 45, 0.526 366 061 329 1976, -0.046 164 676 253 6022] \\
w_{\lambda}^{(2)} & = [0.167 463 779 423 851, 0.094 385 846 966 4461, -3.789 384 759 881 33] \\
w_{\lambda}^{(3)} & = [0.156 978 384 285 472 73, 0.512 457 412 943 1814, -0.744 558 515 987 6165] \\
w_{\lambda}^{(4)} & = [0.025 681 586 506 630 664, 3.433 163 601 035 1154, -0.679 328 947 667 3527].
\end{align*}
$$

2. for $\xi_1 = 2.5, \xi_2 = 3.1, \xi_3 = 1.33, \eta = 0.3, t = 0.45$:

$$
\begin{align*}
 z_{\lambda}^{(1)} & = \pm [-2.367 205 288 538 7806, -0.034 216 835 533 282 85, 0.560 404 707 906 603] \\
z_{\lambda}^{(2)} & = \pm [0.106 722 021 706 9632, 7.959 749 585 813 279, 0.035 481 563 434 309 41] \\
z_{\lambda}^{(3)} & = \pm [0.143 444 596 414 061 13, 0.565 560 364 274 6968, 0.559 518 410 685 0913] \\
z_{\lambda}^{(4)} & = \pm [0.009 963 704 747 040 916, 0.503 953 663 224 0319, 9.039 099 021 589 0408] \\
w_{\lambda}^{(1)} & = [-2.367 052 885 234 99, -0.034 216 835 529 656 396, 0.560 404 707 906 5965] \\
w_{\lambda}^{(2)} & = [0.106 722 021 706 9637, 7.959 749 585 813 29, 0.035 481 563 434 310 45] \\
w_{\lambda}^{(3)} & = [0.143 444 596 399 125 85, 0.565 560 364 271 1194, 0.559 518 410 685 0958] \\
w_{\lambda}^{(4)} & = [0.009 963 750 993 033 916, 0.503 953 666 929 1063, 0.559 518 410 685 0958].
\end{align*}
$$
(3) for \( \xi_1 = 1.7, \xi_2 = 3.5, \xi_3 = 5.22, \eta = 4.7, t = 0.05 \):

\[
\begin{align*}
&z_1^{(1)} = \pm (0.907 144 750 766 911 9, 0.001 035 513 079 854 361, -0.616 390 386 876 6624) \\
&z_1^{(2)} = \pm (-0.186 027 247 837 570 33, -0.028 888 526 505 729 82, -0.107 742 261 240 702 94) \\
&z_1^{(3)} = \pm (0.137 254 238 579 344 35, -0.024 752 594 653 532 196, 0.170 428 233 662 1456) \\
&z_1^{(4)} = \pm (-0.047 402 553 972 947 48, 0.891 975 300 592 1505, 0.013 694 099 141 681 645) \\
&w_1^{(1)} = (0.907 144 750 766 913, 0.001 035 513 079 898 853, -0.616 390 386 876 6655) \\
&w_1^{(2)} = (-0.186 027 247 837 570 13, -0.028 888 526 505 732 478, -0.107 742 261 240 703 06) \\
&w_1^{(3)} = (0.137 254 238 579 343 46, -0.024 752 594 653 526 73, 0.170 428 233 662 146 16) \\
&w_1^{(4)} = (-0.047 402 553 972 947 48, 0.891 975 300 592 1487, 0.013 694 099 141 681 883),
\end{align*}
\]

(4) for \( \xi_1 = 49.7, \xi_2 = 10.5, \xi_3 = 12.22, \eta = 5.87, t = 0.726 \):

\[
\begin{align*}
&z_3^{(1)} = \pm (0.158 866 785 906 656, -0.002 317 414 600 871 322, 0.004 665 001 427 754 174) \\
&z_3^{(2)} = \pm (0.004 163 560 745 980 359, -0.133 525 049 970 415 53, 0.003 089 306 362 663 934) \\
&z_3^{(3)} = \pm (0.002 757 237 077 268 236, -7.693 461 066 977 195, 0.000 080 964 151 684 248 51) \\
&z_3^{(4)} = \pm (-0.001 396 539 108 516 703, -0.133 525 049 980 064 34, -0.009 210 278 823 835 091) \\
&w_3^{(1)} = (0.158 866 785 906 665 17, -0.002 317 414 600 954 629 76, 0.004 665 001 427 754 2385) \\
&w_3^{(2)} = (0.004 163 560 745 980 831, -0.133 525 049 970 420 03, 0.003 089 306 362 663 317) \\
&w_3^{(3)} = (0.002 757 237 007 726 877, -7.693 461 066 977 227, 0.000 080 964 151 684 24613) \\
&w_3^{(4)} = (-0.001 396 539 108 516 455, -0.133 525 049 979 987, -0.009 210 278 823 835 037),
\end{align*}
\]

(5) for \( \xi_1 = 11.2, \xi_2 = 1.1, \xi_3 = 0.82, \eta = 3.3, t = 0.096 \):

\[
\begin{align*}
&z_3^{(1)} = \pm (-0.138 450 986 679 049 34, -0.042 793 563 986 298 22, 0.017 867 992 946 492 404) \\
&z_3^{(2)} = \pm (0.123 505 394 487 378 66, 0.022 662 651 149 136 445, 0.037 822 797 196 118 43) \\
&z_3^{(3)} = \pm (0.114 828 517 972 111 38, -0.028 228 540 362 138 41, -0.032 659 693 368 688 764) \\
&z_3^{(4)} = \pm (-0.101 673 008 729 622 27, 0.052 191 832 632 450 655, -0.019 949 961 538 809 933) \\
&w_3^{(1)} = (-0.138 450 986 679 050 43, -0.042 793 563 986 298 37, 0.017 867 992 946 492 41) \\
&w_3^{(2)} = (0.123 505 394 487 358 99, 0.022 662 651 149 137 868, 0.037 822 797 196 118 53) \\
&w_3^{(3)} = (0.114 828 517 972 115 88, -0.028 228 540 362 138 98, -0.032 659 693 368 688 944) \\
&w_3^{(4)} = (-0.101 673 008 729 622 39, 0.052 191 832 632 450 88, -0.019 949 961 538 809 934),
\end{align*}
\]

In all these cases (up to small numerical errors), it is possible to observe that any eigenvalue of \( T^{(8V)}(\lambda) \) is double degenerate and that the following identities hold:

\[
z_3^{(a)} = w_3^{(a)} \text{ for any } a \in \{1, \ldots, 4\}. \quad (A.14)
\]

References

[1] Sklyanin E K and Faddeev L D 1978 Sov. Phys.—Dokl. 23 902
[2] Sklyanin E K, Takhtajan L A and Faddeev L D 1980 Theor. Math. Phys. 40 688
[3] Takhtajan L A and Faddeev L D 1979 Russ. Math. Surv. 34 11
[4] Sklyanin E K 1979 Dokl. Akad. Nauk SSSR 244 1337
[5] Sklyanin E K 1979 Sov. Phys.—Dokl. 24 107
[6] Kulish P P and Sklyanin E K 1979 Phys. Lett. A 70 461
[7] Faddeev L D 1980 Sov. Sci. Rev. Math. C1 107
[8] Sklyanin E K 1982 J. Sov. Math. 19 1546
[9] Faddeev L D 1984 Les Houches Lectures of 1982 (Amsterdam: Elsevier) p 563
[10] Faddeev L D 1996 How algebraic Bethe Ansatz works for integrable model arXiv:hep-th/9605187v1
[11] Jimbo M 1990 Advanced Series in Mathematical Physics vol 10 (Singapore: World Scientific)
[12] Kulish P P and Sklyanin E K 1982 Lect. Notes Phys. 151 61
[111] Boos H, Jimbo M, Miwa T and Smirnov F 2009 Hidden Grassmann structure in the XXZ model IV: CFT limit arXiv:0911.3731
[112] Jimbo M, Miwa T and Smirnov F 2011 Nucl. Phys. B 852 390
[113] Jimbo M, Miwa T and Smirnov F 2011 Fermionic screening operators in the sine-Gordon model arXiv:1103.1534
[114] Zamolodchikov A B 1977 Pis'ma Zh. Eksp. Teor. Fiz. 25 499
Zamolodchikov A B 1977 Commun. Math. Phys. 55 183
[115] Karowski M and Thun H J 1977 Nucl. Phys. B 130 295
[116] Korepin V E 1980 Commun. Math. Phys. 76 165
[117] Mussardo G 1992 Phys. Rep. 218 215
[118] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 Nucl. Phys. B 241 333
[119] Ginsparg P 1989 Applied conformal field theory Fields, Strings and Critical Phenomena (Les Houches Lecture Notes 1988) ed E Brézin and J Zinn-Justin (New York: Elsevier)
[120] Cardy J L 1990 Conformal invariance and statistical mechanics Fields, Strings and Critical Phenomena (Les Houches (1988), Session XLIX) ed E Brézin and J Zinn-Justin (Amsterdam: North-Holland) p 169
[121] Delfino G and Niccoli G 2005 J. Stat. Mech P04004
[122] Delfino G and Niccoli G 2006 J. High Energy Phys. JHEP05(2006)035
[123] Delfino G and Niccoli G 2008 Nucl. Phys. B 799 364
[124] Delfino G 2009 Nucl. Phys. B 807 455
[125] Izergin A G and Korepin V E 1981 Dokl. Akad. Nauk SSSR 259 76
[126] Izergin A G and Korepin V E 2009 A lattice model related to the nonlinear Schrödinger equation arXiv:0910.2095
[127] Pakuliak S, Rubtsov V and Silantyev A 2008 J. Phys. A: Math. Theor. 41 295204
[160] Levy-Bencheton D, Niccoli G and Terras V 2013 Antiperiodic dynamical six-vertex model II: form factors by separation of variables (in preparation)

[161] Levy-Bencheton D and Terras V 2012 Algebraic Bethe Ansatz approach to form factors and correlation functions of the cyclic eight-vertex solid-on-solid model arXiv:1212.0246

[162] Caux J-S and Maillet J M 2005 Phys. Rev. Lett. 95 077201

[163] Caux J-S, Hagemans R and Maillet J M 2005 J. Stat. Mech P09003

[164] Pereira R G, Sirker J, Caux J-S, Hagemans R, Maillet J M, White S R and Affleck I 2006 Phys. Rev. Lett. 96 257202

[165] Hagemans R, Caux J-S and Maillet J M 2006 Proc. 10th Training Course in the Physics of Correlated Electron Systems and High-tc Superconductors (Salerno, Oct. 2005); AIP Conf. Proc. 846 245

[166] Pereira R G, Sirker J, Caux J-S, Hagemans R, Maillet J M, White S R and Affleck I 2007 J. Stat. Mech. P08022

[167] Sirker J, Pereira R G, Caux J-S, Hagemans R, Maillet J M, White S R and Affleck I 2008 Physica B 403 1520

[168] Caux J S, Calabrese P and Slavnov N A 2007 J. Stat. Mech. P01008

[169] Bloch F 1936 Phys. Rev. 50 259

[170] Schwinger J S 1937 Phys. Rev. 51 544

[171] Halpern O and Johnson M H 1938 Phys. Rev. 55 898

[172] Van Hove L 1954 Phys. Rev. 95 249

[173] Van Hove L 1954 Phys. Rev. 95 1374

[174] Marshall W and Lovesey S W 1971 Theory of Thermal Neutron Scattering (Oxford: Academic)

[175] Balescu R 1975 Equilibrium and Nonequilibrium Statistical Mechanics (New York: Wiley)