Matrix Elements of $U_q(su(2)_k)$ Vertex Operators via Bosonization

A.H. BOUGOURZI$^1$ and ROBERT A. WESTON$^2$

Centre de Recherche Mathématiques, Université de Montréal
C.P. 6128-A, Montréal (Québec) H3C 3J7, Canada.

Abstract

We construct bosonized vertex operators (VOs) and conjugate vertex operators (CVOs) of $U_q(su(2)_k)$ for arbitrary level $k$ and representation $j \leq k/2$. Both are obtained directly as two solutions of the defining condition of vertex operators - namely that they intertwine $U_q(su(2)_k)$ modules. We construct the screening charge and present a formula for the n-point function. Specializing to $j = 1/2$ we construct all VOs and CVOs explicitly. The existence of the CVO allows us to place the calculation of the two-point function on the same footing as $k = 1$; that is, it is obtained without screening currents and involves only a single integral from the CVO. This integral is evaluated and the resulting function is shown to obey the q-KZ equation and to reduce simply to both the expected $k = 1$ and $q = 1$ limits.

$^1$ Email: bougourz@ere.umontreal.ca
$^2$ Email: westonr@ere.umontreal.ca
1 Introduction

In the last decade much effort has been expended in the study of two dimensional exactly solvable models. The use of infinite-dimensional conformal and chiral symmetries has proved eminently successful in the task of solving and classifying those models which are critical [1]. Following the success of this program attention has shifted in recent years to massive off-critical models. There are several approaches to studying these models including the Bethe Ansatz [2], the QISM program of the Leningrad school [3] and the exact S-Matrix bootstrap technique [4]. The method that is most similar to that used in the critical case is to make use of quantum or quantum-affine symmetries [5]. In analogy to CFT, the hope of this method is that it will enable one to describe the energy levels, physical states, correlation functions, and in addition the S-matrix, of a lattice model or massive quantum field theory purely in terms of the representation theory of the quantum algebras. In the recent works of the Kyoto school [6, 7, 8, 9, 10], many of these goals have been realised for a class of quantum-spin chains, or related vertex models, that are invariant under the action of $U_q(su(2)_k)$.

The Hamiltonian of the spin 1/2 XXZ Heisenberg quantum spin chain is [1, 2]

$$H_{XXZ} = -\frac{1}{2} \sum_{i=-\infty}^{\infty} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right).$$

(1.1)

As a function of the anisotropy parameter $\Delta$, the phase diagram of this model has three regions: for $\Delta < 1$ there is a massive antiferromagnetic phase; for $-1 \leq \Delta \leq 1$ a massless phase, and for $\Delta > 1$ a massive ferromagnetic phase [11, 12]. It is the massive antiferromagnetic region that is considered in references [6, 7]. In (1.1) the $\sigma_i$ are the Pauli matrices which act on the $i$th component of the infinite tensor product

$$\cdots \otimes V \otimes V \otimes V \cdots, \quad V = C^2.$$  

(1.2)

When $\Delta = (q + q^{-1})/2$ the Hamiltonian is invariant under the $U_q(su(2)_1)$ symmetry in the sense that

$$[\Delta^\infty(x), H] = 0,$$  

(1.3)

where $\Delta^\infty(x)$ is the infinite coproduct which defines formally how $x \in U_q(su(2)_k)$ acts on $\mathbb{C}^2$. Following [13], Davis et al. introduced the following vertex operators that intertwine
$U_q(su(2)_k)$ modules \[3\] 
\[
\Phi^{j_2}_{j_1}: V(j_1) \rightarrow V(j_2) \otimes V^j(z). \quad (1.4)
\]
Here $V(j_i)$ is a spin $j_i$ $U_q(su(2)_{1})$ module, $z$ is a spectral parameter and $V^j(z)$ is the $k = 0$ ‘evaluation representation’ of $U_q(su(2)_k)$ which is isomorphic to $V^j \otimes \mathbb{C}[z, z^{-1}]$. Constructing the vertex operators explicitly through bosonization allowed Jimbo et al. to produce an integral formula for n-point correlation functions of local fields \[7\]. In particular, they calculated explicitly the correlation function $\langle \sigma^z \rangle$ (on alternate sites), which triumphantly reproduced the known formula for the staggered polarization of the 6-vertex model \[12\] (the spin 1/2 chain and the 6-vertex model are equivalent in that their transfers matrices commute\[12\]).

For $k > 1$ the above procedure can be repeated. The corresponding physical systems are the spin $k/2$ XXZ chain \[14\] or the spin $k/2$ generalisation of the 6-vertex model \[15\]. However the bosonization procedure is more complex for $k > 1$ \[16, 17, 18, 19, 20, 21, 22, 23\]. A trivial reason is that three deformed bosonic fields are now necessary, as is the case for the Wakimoto bosonization of $su(2)_k$ \[24\]. For $k = 1$ \[6\], the Frenkel-Jing bosonization \[25\] was used which, like its undeformed ancestor the Frenkel-Kac bosonization \[26\], requires only one field. A more serious difficulty is that in calculating matrix elements of vertex operators one must balance the charge of the bosonic out vacuum (this charge is zero for $k = 1$ as in CFT). This can be done formally by inserting screening currents directly into matrix elements of vertex operators \[27, 18\]. However on its own this approach maximises the number of integrals in the expression for the n-point function. The technique employed in CFT to overcome this, developed by Dotsenko, Fateev and others \[28, 29, 30\], is to introduce conjugate vertex operators. Inserting conjugate vertex operators, as well as vertex operators, into the n-point function decreases the number of screening currents required.

In this paper, we introduce the vertex operators and conjugate vertex operators relevant to $U_q(su(2)_k)$. We are able to construct the bosonized forms of both of them. They appear at the same time, as two possible solutions of the intertwining requirement. This construction is carried out for arbitrary $k$ and $j \leq k/2$. After obtaining the screening charges we present the formula for n-point matrix elements. Specialising to $j = 1/2$, we construct the vertex operators and conjugate vertex operators explicitly. The existence of the conjugate vertex
operator allows us to place the calculation of the two-point function on the same footing as the \( k = 1 \) case - namely that the function can be obtained in its simplest form, which involves no screening currents and therefore no integrals. We show that this two-point function is a solution of the q-KZ equation.

The layout of the paper is as follows. In Section 2, we define our bosonization of \( \mathcal{U}_q(su(2)_k) \) and the properties of vertex operators. We present the intertwining condition and solve it for arbitrary \( k \) and \( j \) to give the vertex operators and their conjugates. In Section 3, we define the Fock space and the dual Fock space, and discuss the role of BRST charges. In Section 4, we introduce the screening charges, and present the formula for n-point matrix elements. In Section 5, we specialise to \( j = 1/2 \), and construct the two vertex operators and two conjugates vertex operators explicitly. We calculate the two-point function, and show that it obeys the q-KZ equation. Finally we present some conclusions and discuss the connection with quantum spin-chains.

## 2 Bosonization of \( \mathcal{U}_q(su(2)_k) \)

The associative \( \mathcal{U}_q(su(2)_k) \) algebra is generated by the operators \( \{E_n^\pm (n \in \mathbb{Z}), H_m (m \in \mathbb{Z} \neq 0), q^{\pm \sqrt{2}H_0}, \gamma^{1/2}\} \). In the Cartan-Weyl basis it reads \([22, 31, 32]\)

\[
\begin{align*}
[H_n, H_m] &= \frac{[2n]}{2n} \frac{\gamma^{nk} - \gamma^{-nk}}{q - q^{-1}} \delta_{n+m,0}, \quad n \neq 0, \\
[q^{\pm \sqrt{2}H_0}, H_m] &= 0, \\
[H_n, E_n^\pm] &= \pm \sqrt{2} \frac{\gamma^{n|n/2|}[2n]}{2n} E_{n+m}^\pm, \quad n \neq 0, \\
q^{\sqrt{2}H_0} E_n^\pm q^{-\sqrt{2}H_0} &= q^{\pm 2} E_n^\pm, \\
[E_n^+, E_m^+] &= \frac{\gamma^{k(n-m)/2} \psi_{n+m} - \gamma^{k(m-n)/2} \varphi_{n+m}}{q - q^{-1}}, \\
E_{n+1}^+ E_m^\pm - q^{\pm 2} E_m^\pm E_{n+1}^+ &= q^{\pm 2} E_n^\pm E_{m+1}^\pm - E_{m+1}^\pm E_n^\pm,
\end{align*}
\]

where \( \gamma^{1/2} \) is in the centre of the algebra, and as usual \([x] = (q^x - q^{-x})/(q - q^{-1})\). \( \psi_n \) and \( \varphi_n \) are the modes of fields \( \psi(z) \) and \( \varphi(z) \) defined by

\[
\begin{align*}
\psi(z) &= \sum_{n \geq 0} \psi_n z^{-n} = q^{\sqrt{2}H_0} \exp\{\sqrt{2}(q - q^{-1}) \sum_{n > 0} H_n z^{-n}\}, \\
\varphi(z) &= \sum_{n \leq 0} \varphi_n z^{-n} = q^{-\sqrt{2}H_0} \exp\{-\sqrt{2}(q - q^{-1}) \sum_{n < 0} H_n z^{-n}\}.
\end{align*}
\]
The above algebra is in fact only a subalgebra of $U_q(su(2)_k)$ which includes in addition the grading operator and the Serre relations. In what follows it will be clear from the context which algebra is being considered. The defining relations (2.6) are the Drinfeld realization of the usual $U_q(su(2)_k)$ which is given in the Chevalley basis as:

\[ t_it_j = t_jt_i, \]
\[ t_i e_i t_i^{-1} = q^2 e_i, \quad t_i e_j t_i^{-1} = q^{-2} e_j, \quad i \neq j, \]
\[ t_i f_i t_i^{-1} = q^{-2} f_i, \quad t_i f_j t_i^{-1} = q^2 f_j, \quad i \neq j, \]
\[ [e_i, f_j] = \delta_{ij}\frac{t_i - t_i^{-1}}{q - q^{-1}}, \]

where $i, j = 0, 1$. Here the Chevalley generators $\{e_i, f_i, t_i\}$ are related to $\{q, e_i, f_i\}$ through the relations

\[ t_0 = \gamma^k q^{-\sqrt{2}H_0}, \quad t_1 = q^{\sqrt{2}H_0}, \]
\[ e_0 = E_1 q^{-\sqrt{2}H_0}, \quad e_1 = E_0^+, \]
\[ f_0 = q^{\sqrt{2}H_0} E_1^{-}, \quad f_1 = E_0^{-}. \]

The above algebra is a Hopf algebra with the following comultiplication:

\[ \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \]
\[ \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \]
\[ \Delta(t_i) = t_i \otimes t_i. \]

This comultiplication gives rise to the following comultiplication of the $U_q(su(2)_k)$ algebra as realized in (2.10):

\[ \Delta(E_+^n) = E_+^n \otimes \gamma^{kn} + \gamma^{2kn} q^{\sqrt{2}H_0} \otimes E_+^n + \sum_{i=0}^{n-1} \gamma^{k(n+i)/2} q^{\psi_{i-n} \otimes \gamma^{k(n+i)} E_+^i} \mod N_- \otimes N_+^2, \]
\[ \Delta(E_-^m) = E_-^m \otimes \gamma^{-km} + q^{\sqrt{2}H_0} \otimes E_-^m + \sum_{i=0}^{m-1} \gamma^{k(m+i)/2} \varphi_{-m+i} \otimes \gamma^{k(m-i)} E_-^i \mod N_+ \otimes N_-^2, \]
\[ \Delta(E_n) = E_n \otimes \gamma^{-kn} q^{-\sqrt{2}H_0} + \gamma^{-kn} \otimes E_n + \sum_{i=0}^{n-1} \gamma^{k(n-i)} E_n \otimes \gamma^{k(n+i)} \varphi_{n-i} \mod N_2 \otimes N_+, \]
\[ \Delta(H_m) = H_m \otimes \gamma^{km/2} + \gamma^{3km/2} \varphi H_m \mod N_+ \otimes N_-, \]
\[ \Delta(H_n) = H_n \otimes \gamma^{-km/2} + \gamma^{-3km/2} \varphi H_n \mod N_+ \otimes N_-, \]
\[ \Delta(q^\pm \sqrt{2}H_0) = q^\pm \sqrt{2}H_0 \otimes q^\pm \sqrt{2}H_0, \]
\[ \Delta(\gamma^{\pm \frac{1}{2}}) = \gamma^{\pm \frac{1}{2}} \otimes \gamma^{\pm \frac{1}{2}}, \]

(2.10)

where $m > 0, n \geq 0$, and $N_\pm$ and $N_\pm^2$ are left $Q(q)[\gamma^\pm, \psi_m, \varphi_{-n}; m, n \in \mathbb{Z}_{\geq 0}]$ modules generated by $\{E_+^m; m \in \mathbb{Z}\}$ and $\{E_-^m, E_+^m; m, n \in \mathbb{Z}\}$ respectively [33, 34]. This comultipli-
ocation will be useful in deriving the intertwining properties of the vertex operators. For the purpose of bosonization, it is convenient to rewrite this quantum algebra in terms of OPEs (the quantum current algebra). The $U_q(su(2)_k)$ QCA then reads \[22, 17, 18, 35\]

$$\psi(z) . \varphi(w) = \frac{(z - wq^{2+k})(z - wq^{-2-k})}{(z - wq^{-2-k})(z - wq^{2+k})} \varphi(w) . \psi(z), \quad (2.11)$$

$$\psi(z) . E^\pm(w) = q^{\pm 2} \frac{(z - wq^{\mp (2+k/2)})}{z - wq^{\pm (2-k/2)}} E^\pm(w) . \psi(z), \quad (2.12)$$

$$\varphi(z) . E^\pm(w) = q^{\pm 2} \frac{(z - wq^{\mp (2-k/2)})}{z - wq^{\pm (2+k/2)}} E^\pm(w) . \varphi(z), \quad (2.13)$$

$$E^+(z) . E^-(w) \sim \frac{1}{w(q - q^{-1})} \left\{ \psi(wq^{k/2}) - \frac{\varphi(wq^{-k/2})}{z - wq^{-k}} \right\}, \quad |z| > |wq^\pm|, \quad (2.14)$$

$$E^\pm(z) . E^\pm(w) = \frac{(zq^{\pm 2} - w)}{z - wq^{\pm 2}} E^\pm(w) . E^\pm(z). \quad (2.15)$$

### 2.1 Bosonization of the currents

In order to bosonize the “classical” $su(2)_k$ current algebra for arbitrary $k$, it is necessary to introduce three sets of bosonic oscillators \[16\]. This is known as the Wakimoto bosonization \[24\]. For $k = 1$ however, the Frenkel-Kac bosonization is also available - which uses only one set of oscillators \[26\]. For the quantum $U_q(su(2)_k)$, the situation is analogous. For $k = 1$ there is the Frenkel-Jing bosonization which requires only one set of deformed oscillators \[25\], and for general $k$ there are (at least five) q-deformations of the Wakimoto bosonization available \[16, 17, 18, 19, 22, 21, 23\]. (See \[16\] for a detailed discussion of the equivalence of these different bosonizations.) We shall use the fifth bosonization introduced in \[16\].

The three deformed Heisenberg algebras required are

\[
[a^j_n, a^\ell_m] = (-1)^{j-1} n I_j(n) \delta^{ij} \delta_{n+m,0}, \quad [a^j_n, a^\ell_0] = (-1)^{j-1} i \delta^{ij, \ell}, \quad j, \ell = 1, 2, 3, \quad (2.16)
\]

where

\[
I_1(n) = \frac{[2n][nk]}{2kn^2}, \quad I_2(n) = \frac{[nk][n(2+k)]}{n^2k(2+k)} q^{nk}, \quad I_3(n) = \frac{[2n]^2}{4n^2}. \quad (2.17)
\]

Inspired by the notation of Kato et al. \[27\], we choose to express all of our fields in terms of
three generic fields $\Omega^j(L, M, N|\sigma, \alpha, \beta| \pm |z)$ \( \{j = 1, 2, 3\} \) defined as follows:

\[
\Omega^j(L, M, N|\sigma, \alpha, \beta| \pm |z) = \alpha^j - i\alpha_0^j \ln (\pm zq^n) + i\frac{L_M}{N} \sum_{n>0} \frac{[N_n]}{[M_n]} a^n q^{\sigma n} z^{-n} + i\frac{L_M}{N} \sum_{n<0} \frac{[N_n]}{[M_n]} a^n q^{\beta n} z^{-n}.
\]

where $L, M, N, \sigma, \alpha$ and $\beta$ are parameters associated with the $q$-deformation. These deformed fields are normalized so that they coincide with usual free bosonic field in the limit $q \rightarrow 1$. All OPEs are then given in terms of expression $[\Lambda, 7]$ in the appendix.

Let us define the deformed bosonic fields

\[
\begin{align*}
\chi^{1, \pm}(z) &= \Omega^1(k, 1, 1|0, \mp k/2, \pm k/2| + |z), \\
\chi^2(z) &= \Omega^2(k, 1, 1|0, -k/2, k/2| + |z), \\
\chi^3(z) &= \Omega^3(2, 1, 1|0, k + 2, k + 2| + |z).
\end{align*}
\]

In terms of these fields the bosonization of the currents is

\[
\begin{align*}
\psi(z) &= \exp \left\{ i\sqrt{2}\left( \chi^{1,+} (zq^{k/2}) - \chi^{1,-} (zq^{-k/2}) \right) \right\} \\
&= q^{k/2} \chi^1(z) \exp \left\{ \sqrt{2}k (q - q^{-1}) \sum_{n>0} \chi^1_n z^{-n} \right\}, \\
\varphi(z) &= \exp \left\{ i\sqrt{2}\left( \chi^{2,-} (zq^{-k/2}) - \chi^{2,+} (zq^{k/2}) \right) \right\} \\
&= q^{-k/2} \chi^2(z) \exp \left\{ -\sqrt{2}k (q - q^{-1}) \sum_{n<0} \chi^2_n z^{-n} \right\}, \\
E^+(z) &= \frac{\exp \left\{ i\sqrt{2}\chi^{1,+}(z)+i\frac{2+k}{k} \chi^2(z) \right\}}{z(q-q^{-1})} \left( \exp \{-i\chi^3(zq^{-1})\} - \exp \{-i\chi^3(zq)\} \right) \\
&\equiv \frac{1}{z(q-q^{-1})} (E^+_1(z) - E^+_2(z)), \\
E^-(z) &= \frac{\exp \left\{ -i\sqrt{2}\chi^{1,-}(z) \right\}}{z(q-q^{-1})} \left( \exp \{-i\frac{2+k}{k} \chi^2(zq^{-1}) \} - \exp \{-i\frac{2+k}{k} \chi^2(zq) \} \right) \\
&\equiv \frac{1}{z(q-q^{-1})} (E^-_1(z) - E^-_2(z)).
\end{align*}
\]

In these, and in all other expressions in this paper, operators and products of operators defined at the same point $z$ are understood to be normal ordered with respect to the Heisenberg generators.

### 2.2 The vertex operators and their conjugates

The vertex operators relevant to this discussion are the intertwiners of Section 6 of reference $[6]$. They are defined as maps between $U_q(su(2)_k)$ modules in the following way

\[
\Phi^j_{j_1, j_2}(z) : V(j_1) \rightarrow V(j_2) \otimes V^j(z).
\]
Here $V(j) \equiv V(\Lambda_j)$ are left highest weight $U_q(su(2)_k)$ modules, with \{$\Lambda_j = (k - 2j)\lambda_0 + 2j\lambda_1, \ j = 0, \ldots, k/2$\} and \{$\lambda_0, \lambda_1$\} denoting the sets of $U_q(su(2)_k)$ dominant highest weights and fundamental weights respectively. $V^j(z)$ is the $k = 0$ ‘evaluation representation’ of $U_q(su(2)_k)$. It is isomorphic to $V^j \otimes \mathbb{C}[z, z^{-1}]$, where $V^j$ is the $2j + 1$ dimensional representation with the basis \{$v^j_m, \ -j \leq m \leq j$\}. $V^j(z)$ is equipped with the following $U_q(su(2)_k)$ module structure [27]:

$$
\begin{align*}
\gamma^\pm v^j_m &= v^j_m, \\
q^{\mathfrak{h}_0}v^j_m &= q^{-2m}v^j_m, \\
E^+_n v^j_m &= z^n q^{2n(l-m)}[j-m+1]v^j_{m-1}, \\
E^-_n v^j_m &= z^n q^{-2nm}[j+m+1]v^j_{m+1}, \\
H_n v^j_m &= \frac{z^n}{\sqrt{2n}}\{2nj - q^{n(j-m+1)}(q^n + q^{-n})[n(j+m)]\}v^j_m,
\end{align*}
$$

(2.22)

where is it understood that $v^j_m$ is identically zero if $m > j$ or $m < -j$. We also define vertex operators $\tilde{\Phi}^{j_2 j_1}(z)$ as

$$
\Phi^{j_2 j_1}(z) = z^{(\Delta_j - \Delta_{j_1})} \tilde{\Phi}^{j_2 j_1}(z),
$$

(2.23)

where $\Delta_j = j(j+1)/(k+2)$. By definition these vertex operators obey the intertwining condition [13, 6]

$$
\tilde{\Phi}^{j_2 j_1}(z) \circ x = \Delta(x) \circ \tilde{\Phi}^{j_2 j_1}(z) \quad \forall \ x \in U_q(su(2)_k).
$$

(2.24)

Here $x$ denotes a generator in the Drinfeld realization, which is appropriate for explicitly constructing the vertex operators in terms of free bosons. It is also convenient to define components of these vertex operators through

$$
\tilde{\Phi}^{j_2 j_1}(z) = g^{j_2 j_1}(z) \sum_{m=-j}^j \phi^j_m(z) \otimes v^j_m,
$$

(2.25)

where the normalisation function $g^{j_2 j_1}(z)$ is given by

$$
g^{j_2 j_1}(z) = (-zq^{k+2})^{(\Delta_j + \Delta_{j_1} - \Delta_{j_2})}.
$$

(2.26)

Using the above relation 2.24, the comultiplication 2.10, and the fact that $N_+ V^j_j = N_- V^j_j = 0$, $N^j_m \in F[z, z^{-1}]V^j_{m+1}$, we arrive at the following commutation relations:
\[ [E^+(w), \phi_j^j(z)] = 0, \quad (2.27) \]
\[ [H_n, \phi_j^j(z)] = j\sqrt{2} \left\{ \delta_{n,0} + q^n \left[ \frac{[2jn]}{2jn} \right] z^n \phi_j^j(z) \right\}, \quad (2.28) \]
\[ \phi_j^j(z) = \frac{1}{[j-m]} \ldots [\phi_j^j(z), E_0^{-}], \quad (2.29) \]

In (2.29) there are \((j-m)\) quantum commutators, where the quantum commutator \([A,B]_{q^x}\) is defined by
\[ [A,B]_{q^x} = AB - q^x BA. \quad (2.30) \]

Contrary to the case of the Frenkel-Jing realization (i.e. \(k = 1\))\(^7\), the system of equations (2.24) has two independent solutions \(\phi_j^j(z)\) and \(\hat{\phi}_j^j(z)\) given in terms of the above bosonization by
\[ \phi_j^j(z) = \exp \left\{ j\sqrt{2} i\xi^1(z) + \frac{2j}{\sqrt{k(k+2)}} i\xi^2(z) \right\}, \]
\[ \hat{\phi}_j^j(z) = X_j^j(z) \phi_j^j(z), \quad (2.31) \]
\[ X_j^j(z) = \exp \left\{ \frac{-k(k+1-2j)}{\sqrt{k(k+2)}} i\hat{\xi}^2(z) + (k-2j) i\hat{\xi}^3(z) \right\}, \]

where
\[ \xi^1(z) = \Omega^1(2, k, 2j | k+2, -2 - k/2, -2 - 3k/2 | - | z), \]
\[ \xi^2(z) = \Omega^2(k+2, k, 2j | k+2, -2 - 3k/2, -2 - k/2 | - | z), \]
\[ \hat{\xi}^2(z) = \Omega^2(k+2, 1, k+1 - 2j | k+2, -2 - 3k/2, -2 - k/2 | - | z), \]
\[ \hat{\xi}^3(z) = \Omega^2(2, 1, k-2j | k+2, 0, 0 | - | z). \quad (2.32) \]

\(\phi_m^j(z)\) and \(\hat{\phi}_m^j(z)\) are derived from \(\phi_j^j(z)\) and \(\hat{\phi}_j^j(z)\) through (2.29). From now on we shall refer to \(\phi_m^j(z)\) and \(\hat{\phi}_m^j(z)\) as the vertex operators (VOs) and conjugate vertex operators (CVOs) respectively.

### 3 Fock modules and Fock spaces

We define the Fock module \(F(n_1, n_2, n_3)\) and its dual \(F^+(n_1, n_2, n_3) : F(n_1, n_2, n_3) \rightarrow \mathbb{C}\) as follows:
\[ F(n_1, n_2, n_3) = F_-|n_1, n_2, n_3>, \]
\[ F_+|n_1, n_2, n_3> = 0. \quad (3.33) \]
where $F_\mp$ is a free $Q(q)$ module generated by \{a_{\pm n}^1, a_{\pm n}^2, a_{\pm n}^3, \ n > 0\}$, and the states $|n_1, n_2, n_3>$, labelled by integers $n_1, n_2$ and $n_3$, are defined by

$$|n_1, n_2, n_3> = \exp\left\{\frac{n_1}{\sqrt{2k}}ia_1 - \frac{n_2}{\sqrt{k(k+2)}}ia_2 - n_3ia^3\right\}|0>, \quad (3.34)$$

where $|0>= |0,0,0>$ is the ‘in’ vacuum and it is annihilated by $\{a_{n}^1, a_{n}^2, a_{n}^3; \ n \geq 0\}$. As in the classical ($q = 1$) case [29], there will be an asymmetry between the ‘in’ and the ‘out’ vacua due to the existence of a background charge. Guided as always by this example, we define the out vacuum (and our normalisation) by

$$<0, k(k+1), -k|0> = 1, \quad (3.35)$$

where

$$<n_1, n_2, n_3| = <0|\exp\left\{-\frac{n_1}{\sqrt{2k}}ia_1 - \frac{n_2}{\sqrt{k(k+2)}}ia_2 - n_3ia^3\right\}. \quad (3.36)$$

The dual Fock module $F^+(n_1, n_2, n_3)$ will then be given formally by

$$F^+(n_1, n_2, n_3) = <n_1, n_2 + k(k+1), n_3 - k|F_+. \quad (3.37)$$

The currents $E^\pm(z)$, VOs and CVOs define the following mappings on the Fock modules:

$$E^\pm(z) : \quad F(n_1, n_2, n_3) \rightarrow F(n_1 \pm 2, n_2 \pm (k + 2), n_3 \mp 1), \quad (3.38)$$

$$\phi^j_m(z) : \quad F(n_1, n_2, n_3) \rightarrow F(n_1 + 2m, n_2 + m(k+2) - jk, n_3 + j - m), \quad (3.39)$$

$$\hat{\phi}^j_m(z) : \quad F(n_1, n_2, n_3) \rightarrow F(n_1 + 2m, n_2 + m(k+2) - k(k+1-j), n_3 + k - j - m). \quad (3.40)$$

One can show that the action of the currents is well defined (single valued) on the Fock modules $F(n_1, n_2, n_3)$ and $F^+(n_1, n_2, n_3)$ provided that the following condition is satisfied:

$$n_1 - n_2 \in k\mathbb{Z}. \quad (3.41)$$

From (3.38) it is clear that the representations of the currents are the complete Fock spaces

$$\mathcal{F}(n_1, n_2, n_3) = \bigoplus_{r \in \mathbb{Z}} F(n_1 + 2r, n_2 + r(k+2), n_3 - r). \quad (3.42)$$

Let us now briefly discuss the embedding of the left $U_q(su(2)_k)$ highest weight modules $V^j \equiv V(\Lambda_j)$ in the above Fock space $\mathcal{F}(n_1, n_2, n_3)$. The same analysis will apply to the
embedding of the right modules $V^\dagger(\Lambda_j)$ in the Fock space $\mathcal{F}^\dagger(n_1, n_2, n_3)$. The $U_q(su(2)_k)$ highest weight states $|j> \equiv |\Lambda_j>$ must satisfy the conditions:

$$e_i|\Lambda_j> = 0, \quad i = 0, 1.$$  \hspace{1cm} (3.43)

It can be easily checked that the state $|j>$ can be identified with the highest weight state $|2j, 2j, 0>$ of the Fock module $F(2j, 2j, 0)$. As noted in [18], and as is well known in the classical case, one has to study the BRST cohomology structure of the Fock space $\mathcal{F}(2j, 2j, 0)$ in order to single out the irreducible highest weight $V(j)$, with $0 \leq j \leq k/2$, and show that the vertex operators are acting in the latter, which then ensures the nonvanishing of their matrix elements. This analysis has been partially carried out in Ref [18]. Here we will conjecture that the classical result (described fully in [36]) still holds in the quantum case - namely that we have the isomorphism:

$$V^j \simeq \delta_{s,0} \text{Ker}Q^2j+1_s/\text{Im}Q^{k-2j+1}_s, \quad 0 \leq j \leq k/2, \quad s \in \mathbb{Z},$$  \hspace{1cm} (3.44)

where $Q^2j+1_s$ and $Q^{k-2j+1}_s$ are BRST charges acting in a graded complex of the quantum analogue of the Fock spaces introduced in [36]. Let us just mention here that contrary to the Fock spaces of [36], the cohomology structure of the Fock spaces $\mathcal{F}(n_1, n_2, n_3)$ given by 3.42 is much more involved and requires the introduction of at least one extra screening charge [18]. The BRST charges of 3.44 are constructed as integer powers of the screening charge $Q$, which is itself constructed in terms of a screening current that will be introduced below. We also conjecture that as in the classical case the VOs and CVOs are constructed as BRST invariant combinations of free fields (see [36]).

### 4 The Screening Charge and Screening Current

As is well know in CFT, a screening charge $Q$ is a dimensionless operator that commutes with the underlying symmetry algebra [1, 28, 29]. Using the powerful technics of OPEs it can be systematically constructed as a closed contour integral of a dimension 1 operator, called a screening current $S(z)$, which commutes with the currents generating the algebra up to total derivatives. In the case of $U_q(su(2)_k)$ one expects that $S(z)$ will commute with
the currents \( E^\pm(z), \varphi(z) \) and \( \psi(z) \) up to total quantum derivatives. A quantum derivative of a function \( f(z) \) is defined by:

\[
_k D_z f(z) = \frac{f(zq^k) - f(zq^{-k})}{z(q - q^{-1})}.
\] (4.45)

In fact one can easily show that the screening current

\[
S(z) = _k D_z (\exp\{ -i\eta^3(z) \}) \exp\{ i\sqrt{\frac{k}{k+2}} \eta^2(z) \}
\] (4.46)

with

\[
\eta^2(z) = \Omega^2(k + 2, 1, 1) - k - 2, -k/2, k/2 + |z|,
\]

\[
\eta^3(z) = \Omega^3(2, 1, 1) - k - 2, k + 2, k + 2 + |z|,
\] (4.47)

satisfies the following commutation relations:

\[
[\varphi(z), S(w)] = 0,
\] (4.48)

\[
[\psi(z), S(w)] = 0,
\] (4.49)

\[
[E^+(z), S(w)] = 0,
\] (4.50)

\[
[E^-(z), S(w)] = -_k D_w (h(w) - w),
\] (4.51)

where \( h(w) \) is regular except at \( w = 0 \). Because of relation (4.51) the screening charge \( Q \) is constructed as a Jackson integral of the screening current, that is,

\[
Q = \int_0^{\infty} d_p z S(z),
\] (4.52)

with \( p = q^{2(k+2)} \). Here the Jackson integral of a function \( f(z) \) is defined by [13]:

\[
\int_0^{\infty} d_p z f(z) = s(1 - p) \sum_{n \in \mathbb{Z}} f(sp^n)p^n.
\] (4.53)

The action of the screening charges [4.52] on \( F(n_1, n_2, n_3) \) is given by

\[
Q : F(n_1, n_2, n_3) \rightarrow F(n_1, n_2 + k, n_3 - 1).
\] (4.54)

The two BRST charges introduced in the previous section are given in terms of this screening charge by \( Q^x_s = Q^x \), where \( x = 2j + 1 \) or \( x = k - 2j + 1 \) depending on whether \( Q^x_s \) is acting in \( \{\mathcal{F}(j + s(k + 2)), s \in \mathbb{Z}\} \) or \( \{\mathcal{F}(-j - 1 + s(k + 2)), s \in \mathbb{Z}\} \) respectively. These particular values of \( x \) ensure that the BRST charges are single valued when acting on these Fock spaces.
The screening charges are necessary in order to calculate matrix elements. They must be inserted in the correct number in order to balance the charge of the vacuum. The n-point matrix element \( \langle \hat{\Phi}^n(z_n)\Phi^{n-1}(z_{n-1})\cdots\Phi^j(z_1) \rangle \) of the original vertex operators of \( \Phi^j \) is given by

\[
\langle \hat{\Phi}^n(z_n)\Phi^{n-1}(z_{n-1})\cdots\Phi^j(z_1) \rangle = \sum_{\{m_1,\ldots,m_n\}} \left( \prod_{i=1}^n z_i^{\Delta_i-\Delta_{i-1}} g_i^{j_i} f_i^{j_{i-1}}(z_i) \right) \langle 0| Q^L \hat{\Phi}^n(z_n)\phi^{n-1}_{m_{n-1}}(z_{n-1})\cdots\phi^j_{m_1}(z_1)|0 \rangle \otimes \cdots \otimes \nu^j_{m_1},
\]

where the \( J_i \) indices are specified by \( \{J_0 = 0, J_i = \sum_{\ell=1}^i j_\ell, J_n = J_{n-1} - j_n \} \). \( J_i \) and \( J_{i-1} \) label the highest weight modules that are intertwined by a VO or CVO with the argument \( z_i \). They are fixed by the action of \( 3.39 \) and \( 3.40 \) and are suppressed from the left hand side above. The requirement that one must balance the vacuum charge is equivalent to the condition that after acting with the VO, CVOs and Qs one arrives at a product

\[
\langle n_1, n_2, n_3 - k|n_1, n_2, n_3 \rangle = C
\]

for some \( n_i \). This dictates that firstly the sum extends over only those \(-j_i \leq m_i \leq j_i \) such that \( \sum_{i=1,n} m_i = 0 \), and secondly that the number of screening currents is \( L = -j_n + \sum_{i=1,n-1} j_i \). Additional constraints are provided by demanding that the correlation function be independent of the position at which the CVO is inserted. If it is placed in the \( \ell \)th position instead of the first, then \( L_\ell = -j_\ell + \sum_{i\neq\ell} j_i \) screening currents are required. The requirement that \( (L_i \geq 0; i = 1,\ldots,N) \) fixes the “fusion rules” for the vertex operators \( \Phi^j \).

As an example of the above constraints, consider the two-point function. Depending on the position of the CVO the numbers of screening currents required are \( L_1 = -j_1 + j_2 \) and \( L_2 = -j_2 + j_1 \). The conditions \( L_i \geq 0 \) imply that \( j_1 = j_2 \) and \( L_1 = L_2 = 0 \). Thus the only non-vanishing terms in \( 4.55 \) are of the form

\[
\langle \hat{\phi}^j_m(z)\phi^{-j}_{-m}(w) \rangle.
\] (4.56)

5 \( j=1/2 \) Vertex Operators and Matrix Elements

The explicit solutions to the intertwining conditions \( 2.24 \) for \( j = 1/2 \) are

\[
\phi^{1/2}_j(z) = \exp \left( \frac{1}{\sqrt{2k}} j \xi^1(z) + \frac{1}{\sqrt{k(k+2)}} j \xi^2(z) \right),
\]

(5.57)
the contour now winding around the pole that each of 5.58 and 5.61 can be expressed in the classical case as a single integral but with the two integrals from the pole at because the two integrands differ. Furthermore one cannot simply evaluate the residue of the pole at function. So these integrals must be left until we come to evaluate a specific correlation operator. So these integrals must be left until we come to evaluate a specific correlation function.

The two-point matrix element corresponding to the left hand side of (5.55) is

$$\langle \hat{\phi}_{1}^{\frac{1}{2}}(z) \hat{\phi}_{-1}^{\frac{1}{2}}(w) > =$$

$$\frac{(w/z)^{\frac{1}{2}} g_{+}^{\frac{1}{2}}(w) g_{0}^{\frac{1}{2}}(z) g_{-}^{\frac{1}{2}}(w)}{q(w - zq^{k+1})} (w - z)$$

$$\langle 0 | \phi_{-1}^{\frac{1}{2}}(z) \phi_{0}^{\frac{1}{2}}(w) | 0 > v_{+} \otimes v_{-} +$$

$$\langle 0 | \phi_{-1}^{\frac{1}{2}}(z) \phi_{0}^{\frac{1}{2}}(w) | 0 > v_{-} \otimes v_{+} ,$$

where $v_{\pm}$ correspond to $v_{\pm}^{1/2}$. Carrying out the integrals we evaluate this expression as

$$f(w/z)(v_{-} \otimes v_{+} - qv_{+} \otimes v_{-}) ,$$

where $f(z) = z^{-\frac{3}{2}} \prod_{n=0}^{\infty} \frac{(p^{n+1}z)^{\frac{1}{2}}(p^{n+1}q^{4}z)^{\frac{1}{2}}}{(p^{n+1}q^{-2}z)^{\frac{1}{2}}(p^{n+1}q^{6}z)^{\frac{1}{2}}} , (5.63)$$

and $p = q^{2(k+2)}$. Here we have used the conventional notation

$$(a)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^{4n}) . (5.64)$$
It can be easily shown that the function \( f(z) \) is a solution of the q-KZ equation\(^{[13]}\)
\[
f(pz) = q^{\frac{k}{2}} \frac{(pq^{-2}z)_{\infty}(pq^6z)_{\infty}}{(pz)_{\infty}(pq^4z)_{\infty}} f(z).
\]
Moreover when we set \( k = 1 \), \( f(z) \) becomes
\[
f(z) = z^{\frac{k}{4}} \frac{(q^6z)_{\infty}}{(q^4z)_{\infty}}.
\]
This expression is that found by Davies et al.\(^{[6]}\) using the Frenkel-Jing bosonization for \( k = 1 \). The function \( f(z) \) can be expressed in the appealing exponential form
\[
f(z) = z^{\frac{k}{2(k+2)}} \exp \sum_{n>0} \frac{z^n}{n} \frac{[n][3n]}{[2n][n(k+2)]} q^{n(k+2)}.
\]
The limits \( k = 1 \) and \( q \to 1 \) can both be read off simply from this form. Setting \( k = 1 \) gives the expression above. Taking the ‘classical’ limit \( q \to 1 \) yields the expression
\[
z^{\Delta_{\frac{k}{2}}} \frac{1}{(1 - z)^{2\Delta_{\frac{k}{2}}}},
\]
where \( \Delta_{\frac{k}{2}} = 3/(4(k+2)) \) is the conformal weight expected for the \( su(2)_k \) CFT\(^{[38]}\).

6 Conclusions

To summarise: In this paper we have constructed vertex operators and conjugate vertex operators at the same level, as solutions of the intertwining conditions\(^{[2,24]}\). We have constructed the screening charge, and given an expression for a general n-point matrix element function for arbitrary level \( k \). By demanding that the vacuum charge be balanced, and that the n-point matrix elements should be invariant under changing the position of the CVO we have obtained the fusion rules. Specialising to \( j = 1/2 \) we have constructed the VOs and CVOs explicitly. The existence of the CVOs has allowed us place the calculation of the two-point function on the same footing as the \( k = 1 \) case\(^{[3,7]}\) - namely that the two-point function is obtained without screening currents and involves only a single classical integral from the CVO. This integral has been carried out to produce a function\(^{[5,63]}\) that obeys the q-KZ equation, and reduces simply to both the \( k = 1 \) and \( q = 1 \) limits.

What of the spin \( k/2 \) quantum spin chain, or related vertex model? In order to calculate thermodynamic quantities such as the polarisation or susceptibility of relevance to these
models it is necessary to calculate correlation functions of local fields. These fields can be considered as acting on the $V_j^j(z)$ representation; and n-point correlation functions of local fields can be expressed in terms of 2n-point matrix elements of the VOs. For $k = 1$, a formula for general n-point correlation functions was obtained in this manner by Jimbo et al. The results described above allow for a rather natural generalisation of the $k = 1$ case. Details will be published elsewhere.

Acknowledgements

We wish to thank Amine El Gradechi for many illuminating discussions. We are also grateful to Luc Vinet for providing us with his notes on q-hypergeometric functions. RAW thanks CRM for providing him with a research fellowship.
References

[1] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.

[2] L.A. Takhtadzhan and L.D. Faddeev: Russ. Math. Surveys 34:5 (1979) 11-68; Zapiski
Nauch. Sem. LOMI 109 (1981) 134; J. Soviet Math. 24 (1984) 241.

[3] E.K. Sklyanin, L.A. Takhtadzhyan and L.D. Faddeev, Theor. Math. 40 (1980) 688.

[4] A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. (N.Y.) 120 (1979) 253;
A.B. Zamolodchikov, “Integrable Field Theory from Conformal Field Theory”, Pro-
cedings of the Taniguchi Symposium, Kyoto (1988).

[5] V. G. Drinfeld, Proc. ICM (Am. Math. Soc., Berkeley, CA, 1986).

[6] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki, Comm. Math. Phys. 151
(1993) 89.

[7] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, “Correlation Functions of the XXZ
model for $\Delta < -1$”, RIMS preprint, (1992).

[8] M. Jimbo, T. Miwa and Y. Ohta, Int. J. Mod. Phys. A8 (1993) 1457.

[9] M. Idzumi, T. Tokihiro, K. Iohara M. Jimbo, T. Miwa and T. Nakashima, Int. J. Mod.
Phys. A8 (1993) 1479.

[10] O. Foda and T. Miwa, Int. J. Mod. Phys A7 supplement 1A (1992) 279.

[11] I. Affleck, “Field Theory Methods and Quantum Critical Phenomena”, E. Brézin and J.
Zinn-Justin, eds., Les Houches, Session XLIX, 1988, “Champs, Cordes et Phénomènes
Critiques”, Elsevier 1989.

[12] R. J. Baxter, Exactly Solved Models in Statistical Mechanics, (Academic, London, 1982).

[13] I. B. Frenkel and N. Yu Reshetikhin, Comm. Math. Phys. 146 (1992) 1.

[14] N. Reshetikhin, J. Phys, A: Math. Gen. 24, (1991) 3299.
[15] P. P. Kulish and N. Reshetikhin, Zapiski Nauch. Sem. LOMI 101 (1981) 101.

[16] A. H. Bougourzi, preprint CRM-1852, in press Nucl. Phys. B (93).

[17] A. Matsuo, Nagoya University preprints Aug. (1992).

[18] A. Matsuo, Nagoya University preprints Dec. (1992).

[19] J. Shiraishi, preprint UT-617 (1992).

[20] A. H. Bougourzi, Phys. Lett. B286 (1992) 279.

[21] A.H. Bougourzi and M.A. El Gradechi, preprint CRM-1827 (1992), in press J. Group Theory Phys.

[22] A. Abada, A.H. Bougourzi and M.A. El Gradechi, preprint CRM-1829 (1992), in press Mod. Phys. Lett. A.

[23] K. Kimura, Kyoto University preprint (1992).

[24] M. Wakimoto, Comm. Math. Phys. 104 (1986) 605.

[25] I. B. Frenkel and N. H. Jing, Proc. Nat’l. Acad. Sci. (USA) 85 (1988) 9373.

[26] I. B. Frenkel and V. G. Kac, Invent. Math. 62 (1980) 23.

[27] A. Kato, Y.-H. Quano and J. Shiraishi, preprint UT-618 (1992).

[28] VI.S. Dotsenko and V.A. Fateev, Nucl. Phys. B240 (1984) 312.

[29] VI.S. Dotsenko: Nucl. Phys. B338 (1990) 747; Nucl. Phys. B358 (1991) 547.

[30] A. V. Marshakov, Phys. Lett. B224 (1989) 141.

[31] V. G. Drinfeld, Soviet Math. Doklady 32 (1985) 254; 36 (1988) 212.

[32] M. Jimbo, Lett. Math. Phys. 10 (1985) 63.

[33] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, RIMS preprint, (1992).

[34] V. Chari and A. Pressesley, Comm. Math. Phys. 142 (1991) 261.
[35] D. Bernard, Lett. Math. Phys. 17 (1989) 239.

[36] D. Bernard and G. Felder, Comm. Math. Phys. 127 (1990) 145.

[37] L. Alvarez-Gaumé, C. Gomez and G. Sierra, Topics in conformal field theory, Knizhnik Memorial Volume (World Scientific, Singapore, 1990).

[38] V.G. Knizhnik and A.B. Zamolodchikov, Nuc. Phys. B247 (1984) 83.
A Appendix

Correlation functions and OPEs

In calculating the OPE of two vertex operators one uses the usual expression

\[ e^A := e^A e^B := e^{<AB>} . \]  

(A.69)

For strings of operators this generalises to ‘Wick’s theorem for vertex operators’. That is

\[ \prod_{i=1}^N e^{A_i} := \prod_{i=1}^N e^{A_i} : \exp(\sum_{i<j} <A_i B_j>) . \]  

(A.70)

Thus to calculate any of the OPEs and matrix elements used in this paper it is sufficient to know the two-point function of our generic field \( \Omega^j \). This is given by the following expression

\[
\exp\left( <\Omega^j(L, M, N|\sigma, \alpha, \beta|\ell|z)\Omega^{j'}(L', M', N'|\sigma', \alpha', \beta'|\ell'|z')> \right) = \\
(\ell z q^\sigma)^{(-1)^j} \exp\left( (-1)^{j-1} \frac{LM L' M'}{NN'} \sum_{n>0} \frac{[Nn][N'n][nI(n)]}{[Ln][Mn][Ln'][M'n']} q^{(\sigma-\beta) n} \left( \frac{z'}{z} \right)^n \right). 
\]  

(A.71)

Using the two identities

\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1, 
\]  

(A.72)

and

\[
\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \text{ for } |x| < 1, 
\]  

(A.73)

in various combinations then allows the sum to be carried out.