CUCKER-SMALE TYPE FLOCKING MODELS ON A SPHERE

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Abstract. We present a Cucker-Smale (C-S) type flocking model on a sphere. We study velocity alignment on a sphere and prove the emergence of flocking for the proposed model. Our model includes three new terms: a centripetal force, multi-agent interactions on a sphere and inter-particle bonding forces. To compare velocity vectors on different tangent spaces, we introduce a rotation operator in our new interaction term. Due to the geometric restriction, the rotation operator is singular at antipodal points and the relative velocity between two agents located at these points is not well-defined. Based on an energy dissipation property of our model and a variation of Barbalat’s lemma, we show the alignment of velocities for an admissible class of communication weight functions. In addition, for sufficiently large bonding forces we prove time-asymptotic flocking which includes the avoidance of antipodal points.

1. Introduction

Collective behaviors are ubiquitous phenomena in nature. Many organisms employ collective behaviors to survive in nature: examples include the aggregation of bacteria, the flocking of birds and fish schools. Recently, these phenomena have been intensively studied in engineering communities due to their applications. In engineering, a model with a constant speed is preferred for practical reasons and control problems are often addressed. For instance, in [15, 16], Justh and Krishnaprasad considered a unit speed particle model satisfying a Frenet-Serret equation with curvature control. Generalizations of a discrete time Vicsek model with leadership and without leadership were discussed in [14]. A planar model that has a gyroscopic steering force with unit speed was studied in [26]. Their coupling depends on the position and angle of velocity. Leonard et al. designed a particle motion control to collect information in [19]. They focused on developing and solving an optimal control problem for cost functions.

In the mathematics and physics communities, many researchers have also studied these phenomena in various perspectives and in various forms. For examples, Topaz and Bertozzi considered a fluid type model describing social interaction in two spatial dimensions and studied swarming patterns in [31]. Their model contains a nonlocal velocity alignment term. In [10], Fetecau and Eftimie presented a discrete velocity model with a nonlocal force and turning rate. They considered the global existence and aggregation phenomena. Especially, after the mathematical model of Winfree and Kuramoto for collective dynamics [17, 18, 33], many researchers have proposed agent-based models to study the emergent behaviors analytically and numerically. Vicsek et al. in [32] proposed a second-order flocking model with discrete time scheme, the so-called Vicsek model. In this model, all agents in the system have a constant speed. We refer to [7, 9, 30] for noise and position dependent force, derivation of a macroscopic model with a diffusion coefficient, and a statistical view point, respectively.

In this paper, we continue to study the flocking model from Cucker and Smale [6], which is a kind of N-body system. Cucker and Smale [6] introduced a system of ordinary differential equations such
that acceleration is described by weighted internal relaxation forces:

\[
\frac{dx_i}{dt} = v_i, \\
\frac{dv_i}{dt} = \sum_{j=1}^{N} \frac{\psi_{ij}}{N} (v_j - v_i),
\]

(1.1)

where \(x_i\) and \(v_i\) are the position and velocity in \(\mathbb{R}^d\) of the \(i\)th agent, respectively, and \(\psi_{ij}\) is the communication rate between the \(i\)th and \(j\)th agents. Cucker and Smale \([6]\) also provided sufficient conditions for initial data leading to flocking configuration. We refer to \([1, 3, 12, 13, 24, 25]\) for results related to C-S models.

While the original C-S model (1.1) describes \(N\)-body agents in \(\mathbb{R}^d\), collective behaviors can occur on other manifolds. For example, all living organisms in nature are located in the Earth and the geometry of the Earth may not be negligible in long-distance travel. We provide a brief historical remark of flocking dynamics on manifolds. The Kuramoto model in \([17, 18]\) is a first-order differential equation type model, and the oscillators are arranged in \(S^1\). Lohe generalized the Kuramoto model to a matrix model in \([20, 21]\). From the matrix model, Lohe derived a dynamical model in \(S^4\) that was inspired by quantum information theory. The ellipsoid model was studied in \([34]\). Vicsek and his collaborators in \([32]\) considered the second-order discrete time model with \(v \in S^1\). See \([11]\) for the continuous time model and \([5]\) for general dimension cases.

The main objective of this paper is to derive a C-S type flocking model on a unit sphere. Additionally, we need a new definition of flocking for the model to describe the flocking phenomena on a sphere. The main difficulty comes from analysis of the velocity difference, a so-called relative velocity, on a manifold. The concept of a relative velocity on a manifold has been well-developed and widely used in general relativity (See \([4, 22, 28, 29]\)). The relative velocity can be considered as parallel transport along geodesics (See \([4, \text{Equation (3.109)}]\)). On a sphere, a geodesic is a part of a great circle and a parallel transport along a great circle is characterized by a rotation matrix, given in Definition 2.1. Furthermore, as our domain is not a Euclidean domain but a unit sphere, the classical Lyapunov functional approach which provides an exponential convergence rate in the previous articles cannot be applied directly. Instead, we use an integrable system property to obtain the flocking result.

The model proposed in this paper is an agent-based Newton’s equation type model as the original C-S model. We assume that each particle has unit mass. Consider an ensemble of agents on a unit sphere. Let \(x_i, v_i \in \mathbb{R}^3\) be the position and momentum of the \(i\)th agent respectively. Note that \(x_i\) is a unit vector corresponding to the position on the unit sphere. Because of the geometric restrictions, we need an extra term to conserve the modulus of \(x_i\), called the centripetal force. In the original C-S model, the sum of velocity differences between two particles is considered as acceleration. Since the surface is not flat in our model, we cannot obtain a relative velocity between the other two agents by a simple vector difference. More precisely, if we simply take the difference \(v_i - v_j\) between \(i\)th and \(j\)th agents as the relative velocity, then it may not be contained in the tangent space of the given position vector \(x_i\) at time \(t > 0\). Therefore, we need a new interaction rule between the two velocities \(v_i\) and \(v_j\).

The result of this paper are threefold. First, we derive a C-S type system on a unit sphere, based on the original C-S model (1.1), by using the centripetal force, a rotation operator, and an inter-particle bonding force. The following system of first-order ordinary differential equations is a counterpart of the C-S model on the unit sphere: for \(x_i, v_i \in \mathbb{R}^3\) and \(i = 1, \ldots, N\),

\[
\dot{x}_i = v_i, \\
\dot{v}_i = -\frac{\|v_i\|^2}{\|x_i\|^2}x_i + \sum_{j=1}^{N} \frac{\psi_{ij}}{N} (R_{x_j \rightarrow x_i} (v_j) - v_i) + \sum_{k=1}^{N} \frac{\sigma}{N} (\|x_i\|^2 x_k - \langle x_i, x_k \rangle x_i),
\]

(1.2)
where $\psi_{ij} = \psi(||x_i - x_j||)$ is the communication rate and $\sigma$ is the inter-particle bonding parameter. Here, $R_{x_1 x_2}(y)$ is the rotation operator, which consists of a matrix $R(x_1, x_2)$ and its matrix product (See Definition 2.1). The detailed definition and properties of the operator $R_{x_1 x_2}(y)$ are discussed in Section 2. We refer to [8, 23] for C-S models on $\mathbb{T}^d$.

In the original C-S model [6], the communication rate $\psi_{ij} = \psi(||x_i - x_j||)$ quantifies how the agents affect each other and $\psi$ is a decreasing function of distance $||x_i - x_j||$ between two agents $x_i$ and $x_j$. The main concern in system (1.2) is to determine $\psi_{ij}$ when two points $x_i$ and $x_j$ in $\mathbb{S}^2$ are antipodal. As there are infinitely many geodesics connecting two antipodal points in $\mathbb{S}^2$, it is unclear what effect the corresponding antipodal point has. Rather, it is natural to assume that the influence of one agent on another agent is negligible if their positions are antipodal. To illustrate this, we assume that $\psi_{ij} = \psi(||x_j - x_i||)$ and a decreasing $C^1$ function $\psi : [0, 2] \rightarrow [0, +\infty)$ satisfies $\psi(2) = 0$ and $\psi'(2) < 0$. (1.3)
Second, we develop flocking on a unit sphere as follows.

**Definition 1.1.** A system has time-asymptotic flocking on a sphere in \( \mathbb{R}^3 \) if and only if a solution \((x_i, v_i)_{i=1}^N\) of the system satisfies the following condition:

- (velocity alignment) the relative velocity on the unit sphere goes to zero time-asymptotically:
  \[
  \lim_{t \to +\infty} \max_{1 \leq i, j \leq N} \|x_i(t) + x_j(t)\| R_{x_i(t)\to x_j(t)}(v_j(t)) - v_i(t)\| = 0.
  \]

- (antipodal points avoidance) any two agent are not located at the antipodal points:
  \[
  \liminf_{t \geq 0} \min_{1 \leq i, j \leq N} \|x_i(t) + x_j(t)\| > 0.
  \]

It is worth noting that Definition 1.1 has the avoidance of antipodal points while the boundedness of position fluctuations is included in the original definition of flocking in \( \mathbb{R}^3 \). As the domain is compact in our case, the boundedness condition for the position difference \(x_i - x_j\) does not guarantee the formation of a group. Instead, the avoidance of antipodal points is required because antipodal points are as far away from each other as possible.

The rotation operator from one point to its antipodal point cannot be defined (See Remark 2.3(2)). For this reason, the relative velocity will be only considered if points are not antipodal. In fact, the conditions below

\[
\lim_{t \to +\infty} \|x_i(t) + x_j(t)\| R_{x_i(t)\to x_j(t)}(v_j(t)) - v_i(t)\|^k = 0
\]

are equivalent for any \(k > 0\) as long as \(v_i\) and \(v_j\) are uniformly bounded in time (See Lemma C.1). The simplest example of flocking on a sphere is a set of unit speed circular motions.

**Example 1.1.** For \(t \geq 0\) and \(1 \leq i \leq N\), consider

\[
x_i(t) := (\cos(t + \alpha_i), \sin(t + \alpha_i), 0), \quad v_i(t) := (-\sin(t + \alpha_i), \cos(t + \alpha_i), 0)
\]

and \(0 \leq \alpha_i < \pi\). (1.4)

By direct computation, it holds that for all \(t \geq 0\) and \(1 \leq i \leq N\),

\[
R_{x_i(t)\to x_j(t)}(y) = \begin{pmatrix}
\cos(\alpha_i - \alpha_j) & -\sin(\alpha_i - \alpha_j) & 0 \\
\sin(\alpha_i - \alpha_j) & \cos(\alpha_i - \alpha_j) & 0 \\
0 & 0 & 1
\end{pmatrix} y.
\]

Thus, we have \(R_{x_i(t)\to x_j(t)}(v_j(t)) = v_i(t)\) for all \(t \geq 0\), and \((x_i, v_i)_{i=1}^N\) given in (1.4) satisfies the conditions in Definition 1.1.

Third, we provide the global-in-time existence and flocking result for an admissible class of communication weight functions. To obtain the flocking estimate for the solution to (1.2), we consider the following total energy functional \(\mathcal{E}\) given by the sum of the kinetic energy \(\mathcal{E}_K\) and configuration energy \(\mathcal{E}_C\) motivated by [27]: for a given ensemble \((x_i, v_i)_{i=1}^N\), we define energy functional \(\mathcal{E}(x(t), v(t))\) such as

\[
\mathcal{E} := \mathcal{E}_K + \mathcal{E}_C, \quad \mathcal{E}_K(t) := \frac{1}{N} \sum_{k=1}^N \|v_k(t)\|^2, \quad \mathcal{E}_C(t) := \frac{\sigma}{2N^2} \sum_{k,l=1}^N \|x_k(t) - x_l(t)\|^2.
\]

If the bonding force rate \(\sigma\) is large enough comparing the differences of agents’ velocities and positions, then the following flocking result holds. What follows is a summary of our results from Theorem 3.1, Theorem 4.4 and Theorem 4.7.

**Theorem 1.** For \(\psi\) satisfying (1.3) and \(\sigma \geq 0\), there exists a unique global solution to (1.2) and the system in (1.2) has the velocity alignment on a sphere. Moreover, for \(\sigma > N^2\mathcal{E}(0)/2\), the solution to (1.2) has time-asymptotic flocking on a unit sphere. Here, \(\mathcal{E}(0)\) is the initial energy of the system given in (1.5).
We note that it remains open to show the emergence of flocking for $\sigma = 0$ and the complete position flocking for $\sigma > 0$. For the original C-S model defined in a flat space, the a priori assumption that the spatial diameter of agents is uniformly bounded yields an exponential decay for the maximum velocity differences between agents. Conversely, the exponential decay of the maximum velocity differences also leads to the uniform boundedness of the position difference. From this a priori estimate argument, the emergence of the flocking for the original C-S model is attained. However, this standard methodology is not applicable to our model on the sphere.

The main difficulty comes from estimating the position difference between two agents. We expect that the exponential decay of the relative velocity $\|R_{x_j-x_i}(v_j) - v_i\|$ yields the boundedness of the position difference. However, in our case since the sphere is a curved space, $R_{x_j-x_i}$ depends on agents' location. Thus, unlike the original C-S model, it is not clear whether there is a Gronwall-type dissipative differential inequality for the relative velocity, even assuming a priori that the position difference at $t = 0$ is small.

The rest of this paper is organized as follows. In Section 2, we present a derivation of the C-S type model (1.2) on the unit sphere from the original C-S model. In Section 3, we provide the global well-posedness of the solution to the derived model (1.2). In Section 4, we prove the asymptotic flocking theorem for the system. Finally, Section 5 is devoted to the summary of our main results.

Notation: For given $z \in \mathbb{R}^3$, we use the symbols $\|z\|$ and $\|z\|_\infty$ to denote the $\ell_2$-norm and $\ell_\infty$-norm, respectively. For three-dimensional vectors $y$ and $z$, we denote the standard inner product between $y$ and $z$ as $\langle y, z \rangle$.}

2. Motivation and derivation of the C-S type model on a sphere

2.1. Derivation. In this section, we present a derivation of the C-S type flocking model on a sphere. After normalization, we can assume that the domain is a unit sphere:

$$S^2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 + a_2^2 + a_3^2 = 1\}.$$ 

We first consider the classical form of the C-S model in $\mathbb{R}^3$.

$$\dot{x}_i = v_i,$$

$$\dot{v}_i = \sum_{j=1}^{N} \frac{\psi_{ij}}{N} (v_j - v_i).$$

Here, $x_i$ and $v_i$ represent the position and velocity of the $i$th agent, respectively, and $\psi_{ij} = \psi(\|x_i - x_j\|)$ is the communication rate or weight function for the interaction between $i$th and $j$th agents. In the original C-S model, the communication rate $\psi$ has a key role in the emergence of flocking as a control parameter. Note that to obtain global flocking result in the original C-S model, the power of denominator is important, i.e., if $\psi(x) = 1/(1 + |x|^2)^{\beta/2}$, the following statement holds.

(1) $\beta \in [0, 1]$, unconditional flocking occurs,

(2) $\beta \in (1, \infty)$, conditional flocking occurs.

Remark 2.1. Unlike in the case of $\mathbb{R}^3$, the flocking model on the unit sphere does not require the long-range communications for unconditional flocking. On the other hand, since it has singularity in the antipodal positions, a vanishing condition in the antipodal positions such as (1.3) is required.

In the following, we construct a C-S type flocking model on a unit sphere that shares the same structure as the original C-S model, including the communication rate and velocity difference. Essentially, the C-S flocking model consists of three components: (1) the classic relation between the position $x_i$ and velocity $v_i$ in the first equation, $\dot{x}_i = v_i$, (2) the velocity difference $v_j - v_i$ in the second equation, and (3) the communication rate $\psi_{ij}$.
For the first trial, we fix the first two components and we try to find $\psi_{ij}$ to conserve the modulus of $x_i$. To make dynamics on the sphere, we need to obtain

$$||x_i|| \equiv 1, \text{ for all time } t > 0. \quad (2.1)$$

If we have compatible initial data $x_i(0)$ and $v_i(0)$, an equivalent relation to the above is

$$\langle x_i, v_i \rangle \equiv 0, \text{ for all time } t > 0. \quad (2.2)$$

Under this compatible initial condition, we can also obtain another equivalent relation as follows:

$$\langle x_i(t), \dot{v}_i(t) \rangle + \|v_i(t)\|^2 \equiv 0, \text{ for all time } t > 0, \quad (2.3)$$

and

$$\langle x_i(0), v_i(0) \rangle = 0. \quad (2.4)$$

For the details, see Proposition 3.2.

**Proposition 2.2.** Let $\psi_{ij}$ be scalar functions depending on $\{x_1, \cdots, x_N\}$ for all $i, j \in \{1, \cdots, N\}$. Then a solution to the C-S flocking model (1.1) with communication rate $\psi_{ij}$ lying on the unit sphere does not exist.

**Proof.** We assume that each agent lies on a unit sphere, i.e., for all $1 \leq i \leq N$, $x_i \in \mathbb{S}^2$.

Assume that the solution $(x_i, v_i)_{i=1}^N$ to the following C-S type equation is as follows:

$$\dot{x}_i = v_i,$$

$$\dot{v}_i = \sum_{j=1}^N \frac{\psi_{ij}}{N} (v_j - v_i),$$

and take the inner product between $\dot{v}_i$ and $x_i$ to obtain

$$\langle \dot{v}_i, x_i \rangle = \sum_{j=1}^N \frac{\psi_{ij}}{N} \langle v_j - v_i, x_i \rangle.$$  

Note that from the argument in (2.1)-(2.4), we have

$$\langle x_i, v_i \rangle \equiv 0$$

and

$$\langle x_i, \dot{v}_i \rangle + \|v_i\|^2 \equiv 0.$$  

It follows that

$$-\|v_i\|^2 = \langle \dot{v}_i, x_i \rangle = \sum_{j=1}^N \frac{\psi_{ij}}{N} \langle v_j - v_i, x_i \rangle = \sum_{j=1}^N \frac{\psi_{ij}}{N} \langle v_j, x_i \rangle.$$  

As $\psi_{ij}$ does not depend on $\{v_1, \cdots, v_N\}$, the above equation holds for any $v_i$ such that $\langle v_i, x_i \rangle = 0$. Therefore, there is no such solution $\psi_{ij}$ to the equation. $\square$

If we do not change the first equation, the only possibility to construct a unit sphere model is modification of the $v_j - v_i$ terms for $i, j \in \{1, \cdots, N\}$. As mentioned in Section 1, from geometrical consideration, it is natural to consider a relative velocity to keep all agents’ positions within the given manifold. However, without interactions between each agent, the positions of each agent are not maintained in the manifold. Therefore, we need an additional external force term. In the sense of (2.1) - (2.4), the acceleration $\dot{v}_i$ in our model has to satisfy

$$\langle x_i, \dot{v}_i \rangle = -\|v_i\|^2.$$
We propose one possible model as follows. We add a self-consistency term, the so-called centripetal force, as $-\frac{\|v_i\|^2}{\|x_i\|^2} x_i$. Consider the following centripetal equation:

$$\dot{v}_i = -\frac{\|v_i\|^2}{\|x_i\|^2} x_i.$$ 

It is well known that the solution to the centripetal equation gives uniform circular motion on a sphere. Then, the remaining interaction term for the $i$th agent between agents must be orthogonal to $x_i$. Thus, we need an operator map from the tangent space of $x_j$ to the tangent space of $x_i$.

2.2. Model. Our proposed model is as follows: for $x_i, v_i \in \mathbb{R}^3$ and $i = 1, \ldots, N$,

$$\dot{x}_i = v_i,$$

$$\dot{v}_i = -\frac{\|v_i\|^2}{\|x_i\|^2} x_i + \sum_{j=1}^{N} \frac{\psi_{ij}}{N} (R_{x_j - x_i}(v_j) - v_i) + \sum_{k=1}^{N} \frac{\sigma}{N} (\|x_i\|^2 x_k - \langle x_i, x_k \rangle x_i).$$

Here, $R_{x_1 - x_2}(y) = R(x_1, x_2) \cdot y$ is the rotation operator defined below and $\sigma > 0$ is the rate of the inter-particle bonding force.

**Definition 2.1.** Let $x_1, x_2 \in S^2$ be column vectors with $x_1 \neq -x_2$. We define a $3 \times 3$ matrix $R(x_1, x_2) := \begin{cases} \langle x_1, x_2 \rangle I - x_1 x_2^T + x_2 x_1^T + (1 - \langle x_1, x_2 \rangle) \left( \frac{x_1 \times x_2}{||x_1 \times x_2||} \right) \left( \frac{x_1 \times x_2}{||x_1 \times x_2||} \right)^T, & \text{if } x_1 \neq x_2, \\ I, & \text{if } x_1 = x_2, \end{cases}$ where $I$ is the identity matrix in $\mathbb{R}^3$ and $M^T$ is the transpose of a matrix $M$. The operation $R_{x_1 - x_2}(y)$ is defined by the linear transform as $R_{x_1 - x_2}(y) = R(x_1, x_2) \cdot y$.

Here, $y$ is a column vector and $\cdot$ is the matrix product. Furthermore, we set $\|x_1 + x_2\| R(x_1, x_2) := 0$ at $x_1 = -x_2$.

**Remark 2.3.**

1. By direct calculation, we can rewrite the rotation matrix as $R(x_1, x_2) = \cos \theta I + \sin \theta [u] + (1 - \cos \theta) u \otimes u$, \hspace{1cm} (2.5)

where $u = \frac{x_1 \times x_2}{||x_1 \times x_2||}$ is the normalized cross product between the vectors $x_1$ and $x_2$, $\theta$ is the angle between $x_1$ and $x_2$, and $[u], u \otimes u$ are given by

$$[u] = \begin{pmatrix} 0 & -u_{e_3} & u_{e_2} \\ u_{e_3} & 0 & -u_{e_1} \\ -u_{e_2} & u_{e_1} & 0 \end{pmatrix}, \quad u \otimes u = \begin{pmatrix} u_{e_1}^2 & u_{e_1} u_{e_2} & u_{e_1} u_{e_3} \\ u_{e_1} u_{e_2} & u_{e_2}^2 & u_{e_2} u_{e_3} \\ u_{e_1} u_{e_3} & u_{e_2} u_{e_3} & u_{e_3}^2 \end{pmatrix}.$$ 

Here, $u_{e_1}, u_{e_2}$ and $u_{e_3}$ are the first, second and third components of the vector $u$, respectively.

2. For $x_1 = -x_2$, the third term in the right-hand side of (2.5) is not well-defined. Geometrically, if two points are located at opposite poles (for example, the north pole and south pole), then there are infinitely many geodesics connecting the two points. This means that we cannot determine a unique parallel transport from this property, we need to take $\psi_{ij} = \psi(\|x_1 - x_j\|)$ with some decay assumption at $x_1 = -x_2$. Thus, the model (1.2) is valid although $R(x_1, x_2)$ is not defined at $x_1 = -x_2$. Our assumptions on $\psi_{ij}$ and the well-posedness will be shown later.

3. For any $x_1 \in S^2$, it holds that $\lim_{x_2 \to x_1} R(x_1, x_2) = I$. 

(4) Note that \( R(x_1, x_2) \) is bounded for any \( x_1, x_2 \in \mathbb{S}^2 \) (See Lemma 2.5). Thus, although \( R(x_1, x_2) \) is not defined at \( x_1 = -x_2 \), it is natural to define \( \|x_1 + x_2\| R(x_1, x_2) = 0 \) at \( x_1 = -x_2 \).

Next, we provide elementary properties of the rotation matrix.

**Lemma 2.4.** For \( x_1, x_2 \in \mathbb{S}^2 \) with \( x_1 \neq -x_2 \), the following holds.

\[
R_{x_1 \rightarrow x_2}(x_1) = x_2, \quad R_{x_1 \rightarrow x_2}(x_2) = 2\langle x_1, x_2 \rangle x_2 - x_1 \quad \text{and} \quad R_{x_1 \rightarrow x_2}(x_1 \times x_2) = x_1 \times x_2.
\]

Furthermore, we have

\[
R_{x_1 \rightarrow x_2}^T = R_{x_2 \rightarrow x_1}, \quad R_{x_1 \rightarrow x_2}^T \circ R_{x_1 \rightarrow x_2} = I_{\mathbb{S}^2}.
\]

**Proof.** By definition of the rotation map \( R_{x_1 \rightarrow x_2} : \mathbb{S}^2 \rightarrow \mathbb{S}^2 \), it is enough to consider the equivalent properties for a matrix \( R(x_1, x_2) \) defined in Definition 2.1. For the case of \( x_1 = x_2 \), we can easily check that \( R(x_1, x_2) \) satisfies Lemma 2.4 since \( R(x_1, x_2) = I \).

Next, we consider the case, \( x_1 \neq x_2 \). As the two vectors \( x_1 \) and \( x_2 \) are perpendicular to \( x_1 \times x_2 \), direct computation shows that

\[
R(x_1, x_2) \cdot x_1 = x_2, \quad R(x_1, x_2) \cdot x_2 = 2\langle x_1, x_2 \rangle x_2 - x_1 \quad \text{and} \quad R(x_1, x_2) \cdot (x_1 \times x_2) = x_1 \times x_2.
\]

Furthermore, since we have

\[
R(x_1, x_2)^T = \langle x_1, x_2 \rangle I - x_2 x_1^T + x_1 x_2^T + (1 - \langle x_1, x_2 \rangle) \left( \frac{x_1 \times x_2}{|x_1 \times x_2|} \right)^T,
\]

we conclude that

\[
R(x_1, x_2)^T = R(x_2, x_1).
\]

We show that \( R(x_1, x_2) \) is an orthogonal matrix, that is

\[
R(x_1, x_2)^T R(x_1, x_2) = I.
\]

From (2.6) and (2.7), it holds that

\[
R(x_1, x_2)^T R(x_1, x_2) \cdot x_1 = R(x_2, x_1) \cdot x_2 = x_1
\]

and

\[
R(x_1, x_2)^T R(x_1, x_2) \cdot x_2 = R(x_2, x_1) \cdot (2\langle x_1, x_2 \rangle x_2 - x_1) = 2\langle x_1, x_2 \rangle x_1 - (2\langle x_1, x_2 \rangle x_1 - x_2) = x_2.
\]

Furthermore, the last equality in (2.6) implies that

\[
R(x_1, x_2)^T R(x_1, x_2) \cdot (x_1 \times x_2) = x_1 \times x_2.
\]

As \( x_1, x_2 \) and \( x_1 \times x_2 \) are linearly independent, we conclude that for any \( y \in \mathbb{R}^3 \)

\[
\left( R(x_1, x_2)^T R(x_1, x_2) - I \right) \cdot y = 0,
\]

and thus we conclude (2.8). □

As a consequence of Lemma 2.4, we have the following property.

**Lemma 2.5.** Let \( x_1, x_2 \in \mathbb{S}^2 \) with \( x_1 \neq -x_2 \). Then, we have

\[
\|v\| = \|R_{x_1 \rightarrow x_2}(v)\| \quad \text{for any} \ v \in \mathbb{R}^n,
\]

where \( R_{x_1 \rightarrow x_2}(y) \) is the rotation operator from Definition 2.1.

**Proof.** From Lemma 2.4 it holds that

\[
\|v\|^2 = \langle v, v \rangle = \langle v, R(x_1, x_2)^T R(x_1, x_2) \cdot v \rangle
\]

\[
= \langle R(x_1, x_2) \cdot v, R(x_1, x_2) \cdot v \rangle = \langle R_{x_1 \rightarrow x_2}(v), R_{x_1 \rightarrow x_2}(v) \rangle = \|R_{x_1 \rightarrow x_2}(v)\|^2.
\] □
Remark 2.6.  

(1) For \( x_1, x_2 \in S^2 \) with \( x_1 \neq -x_2 \), the rotation map \( R_{x_1,x_2} : S^2 \to S^2 \) is a linear bijection and isometry between two spheres. Furthermore, the differential of \( R_{x_1,x_2} \) at \( x_1 \) gives a linear map between the tangent spaces at \( x_1 \) and \( x_2 \), which contain the velocity vectors. Note that the differential of the map \( R_{x_1,x_2} \) at \( x_1 \) is the same as the matrix \( R(x_1,x_2) \) since the rotation map is linear.

(2) The rotation operator is a rather admissible choice. If we assume that \( \|x_i\| \equiv 1 \), then we have
\[
\langle v_i, x_i \rangle = 0.
\]
When the force equation has \(-\|v_i\|^2 x_i\) term, then we have to replace the \( v_j \) term by some of the tangential vectors of the sphere at \( x = x_i \). In this point of view, the rotation vector \( R_{x_j,x_i}(v_j) \) is the most natural choice for replacement.

Lastly, our model includes the inter-particle bonding force. For the original C-S model in \( \mathbb{R}^3 \), the definition of the flocking contains the uniform boundedness of position differences. This property has been proven for the unconditional and conditional cases using the corresponding initial data and means that the velocity alignment is faster than the dissipation of the ensemble. This is achieved by obtaining the exponential decay rate of velocity difference and from the exponential decay, the uniform boundedness of position differences was obtained. However, for \( S^2 \) case in this paper, we cannot control the diameter of agent’s positions, even with an exponential decay rate of velocity difference, due to the geometric property of \( S^2 \) and the rotation operator. Thus, the argument that used in the original C-S model cannot be applied to our model. To guarantee the antipodal points avoidance, we included the inter-particle bonding force similar to one for the augmented C-S model in [27],
\[
\frac{\sigma}{N} \sum_{k=1}^{N} (x_k - x_i).
\]
We note that tighter spatial configurations can be achieved by adding this term to the original C-S model [27]. As the agent needs to be located on the sphere for all time in our case, so we need some modification in the above terms and we develop the inter-particle bonding force on a sphere based on Lohe operator in [20, 21]:
\[
\sum_{k=1}^{N} \frac{\sigma}{N} (\|x_i\|^2 x_k - \langle x_i, x_k \rangle x_i).
\]
(2.9)
From this modification, we can prove that the ensemble \((x_i, v_i)_{i=1}^{N}\) satisfying (1.2) is located on the sphere and obtain an energy dissipation property that plays a crucial role in the proof of the flocking theorem. For the detailed, see Proposition 3.2 and 3.3.

3. The global well-posedness of a unit sphere model

In this section, we prove the global existence and uniqueness of the solution to (1.2).

Theorem 3.1. If \( \psi \) satisfies (1.3), then there exists a unique solution \((x_i, v_i)_{i=1}^{N}\) to the system (1.2) for all time. In particular, \((x_i)_{i=1}^{N}\) are located in a unit sphere for all time \( t > 0 \).

Note that in our model, the position of each agent is located on a unit sphere. The following natural conditions are required for this property to appear: We say that the initial data are admissible if it holds that
\[
\langle v_i(0), x_i(0) \rangle = 0 \quad \text{and} \quad \|x_i(0)\| = 1 \quad \text{for all} \quad i \in \{1, \ldots, N\}.
\]
(3.1)
The following proposition shows that the modulus of \( x_i \) is conserved and \( v_i \) is in the tangent space of a unit sphere at \( x_j \).
Proposition 3.2. Let \((x_i(t), v_i(t))_{i=1}^N\) be a solution to (1.2) and assume that the initial data are admissible and \(\psi_{ij}\) are nonnegative bounded functions for all \(i, j \in \{1, \ldots, N\}\). Then for all \(i \in \{1, \ldots, N\}\) and \(t > 0\),

\[
\langle v_i(t), x_i(t) \rangle = 0 \quad \text{and} \quad \|x_i(t)\| = 1.
\]

Proof. We take the inner product between \(\dot{x}_i\) and \(x_i\). From the first equation of (1.2), it follows that

\[
\frac{d}{dt} \|x_i\|^2 = 2\langle \dot{x}_i, x_i \rangle = 2(v_i, x_i).
\]

Thus, the necessary and sufficient condition for conservation of the modulus \(\|x_i(t)\| \equiv 1\) is

\[
\langle v_i(t), x_i(t) \rangle \equiv 0,
\]

since initial conditions satisfy \(\|x_i(0)\| = 1\) and \(\langle v_i(0), x_i(0) \rangle = 0, i \in \{1, \ldots, N\}\).

Note that

\[
0 = \frac{d}{dt} \langle v_i, x_i \rangle = \langle \dot{v}_i, x_i \rangle + \langle v_i, \dot{x}_i \rangle = \langle \dot{v}_i, x_i \rangle + \langle v_i, v_i \rangle.
\]

Thus, an equivalent relation to \(\langle v_i, x_i \rangle \equiv 0\) is

\[
\langle x_i, \dot{v}_i \rangle + \|v_i\|^2 \equiv 0.\tag{3.2}
\]

From the above argument, it suffices to prove that \(\langle x_i, \dot{v}_i \rangle + \|v_i\|^2 \equiv 0\). Taking the inner product between the second equation on the system and \(x_i\) leads that

\[
\langle \dot{v}_i, x_i \rangle = -\|v_i\|^2 + \sum_{j=1}^N \frac{\psi_{ij}}{N} (\langle R_{x_j \to x_i}(v_j), x_i \rangle - \langle v_i, x_i \rangle) + \sum_{k=1}^N \frac{\sigma}{N} (\|x_i\|^2 \langle x_k, x_i \rangle - \langle x_i, x_k \rangle \langle x_i, x_i \rangle)
\]

\[
= -\|v_i\|^2 + \sum_{j=1}^N \frac{\psi_{ij}}{N} (\langle R_{x_j \to x_i}(v_j), x_i \rangle - \langle v_i, x_i \rangle).
\]

By Lemma 2.4, the operator \(R_{x_j \to x_i}\) satisfies

\[
R_{x_j \to x_i}^T = R_{x_j \to x_i}^{-1} = R_{x_j \to x_i},
\]

and \(R_{x_i \to x_i}(x_i) = x_i\).

The above equalities yield

\[
\langle \dot{v}_i, x_i \rangle = -\|v_i\|^2 + \sum_{j=1}^N \frac{\psi_{ij}}{N} (\langle R_{x_j \to x_i}(v_j), x_i \rangle - \langle v_i, x_i \rangle)
\]

\[
= -\|v_i\|^2 + \sum_{j=1}^N \frac{\psi_{ij}}{N} (\langle v_j, R_{x_i \to x_i}(x_i) \rangle - \langle v_i, x_i \rangle)
\]

\[
= -\|v_i\|^2 + \sum_{j=1}^N \frac{\psi_{ij}}{N} (\langle v_j, x_j \rangle - \langle v_i, x_i \rangle).
\]

Thus, we have

\[
\langle \dot{v}_i, x_i \rangle + \|v_i\|^2 = \sum_{j=1}^N \frac{\psi_{ij}}{N} (\langle v_j, x_j \rangle - \langle v_i, x_i \rangle).\tag{3.2}
\]

\footnote{Therefore, we have (3.2).}
We sum up (3.2) with respect to index $i$ to obtain
\[
\frac{d}{dt} \sum_{i=1}^{N} |\langle v_i, x_i \rangle|^2 = 2 \sum_{i=1}^{N} (\langle \dot{v}_i, x_i \rangle + \langle v_i, \dot{x}_i \rangle) \langle v_i, x_i \rangle
\]
\[
= 2 \sum_{i=1}^{N} (\langle \dot{v}_i, x_i \rangle + \|v_i\|^2) \langle v_i, x_i \rangle
\]
\[
= 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\psi_{ij}}{N} \langle (v_j, x_j) - (v_i, x_i) \rangle \langle v_i, x_i \rangle
\]
\[
\leq 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\psi_{ij}}{N} \langle v_j, x_j \rangle \langle v_i, x_i \rangle.
\]

Then we have
\[
\frac{d}{dt} \sum_{i=1}^{N} |\langle v_i(t), x_i(t) \rangle|^2 \leq 2N \max_{1 \leq j, k \leq N} \frac{\psi_{jk}}{N} \sum_{i=1}^{N} |\langle v_i(t), x_i(t) \rangle|^2.
\]

Since we assume that initial data satisfy $\sum_{i=1}^{N} |\langle v_i(0), x_i(0) \rangle|^2 = 0$, the Gronwall inequality implies that
\[
\sum_{i=1}^{N} |\langle v_i(t), x_i(t) \rangle| \equiv 0, \text{ for } t > 0.
\]

As a consequence, the above argument shows that
\[
\|x_i(t)\| \equiv 1, \text{ for } t > 0, \ i \in \{1, \ldots N\}.
\]

□

As we mentioned before, we cannot use the Lyapunov functional approach to obtain the flocking estimate due to the geometric modification. Instead, the following integrability in Proposition 3.3 plays an important role in the proof of the flocking theorem.

**Proposition 3.3.** Let $(x_i(t), v_i(t))_{i=1}^{N}$ be a solution to (1.2). Assume that $(x_i(0), v_i(0))_{i=1}^{N}$ satisfies the admissible initial data condition in (3.1) and $\psi_{ij}$ are nonnegative bounded functions and $\psi_{ij} = \psi_{ji}$ for all $i, j \in \{1, \ldots, N\}$. Then we have
\[
\frac{d \mathcal{E}}{dt} = - \sum_{i,j=1}^{N} \frac{\psi_{ij}}{N^2} \|R_{x_j - x_i}(v_j) - v_i\|^2.
\]

Moreover, the following estimate holds:
\[
\mathcal{V}(t) \leq \sqrt{N \mathcal{E}(0)},
\]
where $\mathcal{V} : [0, +\infty) \to [0, +\infty)$ is the maximal speed given by
\[
\mathcal{V}(t) := \max_{1 \leq i \leq N} \|v_i(t)\|.
\]
Proof. We take the time-derivative of $\mathcal{E}$ and use (1.2) to obtain
\[
\frac{d\mathcal{E}}{dt} = \frac{2}{N} \sum_{i=1}^{N} \langle \dot{v}_i, v_i \rangle + \frac{\sigma}{N^2} \sum_{i,j=1}^{N} \langle x_i - x_j, v_i - v_j \rangle
\]
\[
= -\frac{2}{N} \sum_{i=1}^{N} \|v_i\|^2 \langle x_i, v_i \rangle + \frac{2}{N^2} \sum_{i,j=1}^{N} \psi_{ij}(R_{x_j \rightarrow x_i}(v_j) - v_i, v_i)
\]
\[
+ \frac{2\sigma}{N^2} \sum_{i,j=1}^{N} \langle x_j - x_i, v_i \rangle + \sigma \sum_{i,j=1}^{N} \langle x_i - x_j, v_i - v_j \rangle.
\]

By interchanging indices $i$ and $j$, we have
\[
2 \sum_{i,j=1}^{N} \langle x_j - x_i, v_i \rangle = \sum_{i,j=1}^{N} \langle x_j - x_i, v_i \rangle + \langle x_i - x_j, v_j \rangle = -\sum_{i,j=1}^{N} \langle x_i - x_j, v_i - v_j \rangle.
\]

From Proposition 3.2, it holds that $\langle x_i(t), v_i(t) \rangle = 0$ and thus
\[
\frac{d\mathcal{E}}{dt} = \frac{2}{N^2} \sum_{i,j=1}^{N} \psi_{ij}(R_{x_j \rightarrow x_i}(v_j) - v_i, v_i).
\]

(3.6)

From the symmetric assumption $\psi_{ij} = \psi_{ji}$, it follows that
\[
\frac{d\mathcal{E}}{dt} = \sum_{i,j=1}^{N} \psi_{ij} \left( (R_{x_i \rightarrow x_j}(v_i) - v_j, v_j) + (R_{x_j \rightarrow x_i}(v_j) - v_i, v_i) \right).
\]

Note that from the properties of operator $R_{x_i \rightarrow x_j}$ in Lemmas 2.4 and 2.5,
\[
\langle R_{x_i \rightarrow x_j}(v_i), v_j \rangle = \langle v_i, R_{x_j \rightarrow x_i}(v_j) \rangle, \quad \langle v_j, v_j \rangle = \langle R_{x_j \rightarrow x_i}(v_j), R_{x_j \rightarrow x_i}(v_j) \rangle.
\]

Therefore, we conclude (3.3).

Next, we consider (3.4). By the definition of the energy functional $\mathcal{E}$, for each time $t > 0$, we have
\[
\mathcal{V}^2(t) \leq N\mathcal{E}(t) \leq N\mathcal{E}(t),
\]
for $\mathcal{V}$ given in (3.5). From (3.3), we obtain
\[
\mathcal{V}^2(t) \leq N\mathcal{E}(0),
\]
we conclude (3.4). \qed

Remark 3.4. The energy dissipation in (3.3) holds if $R_{x_i \rightarrow x_j}$ satisfies
\[
R_{x_i \rightarrow x_j}^{-1} = R_{x_j \rightarrow x_i} = R_{x_j \rightarrow x_i},
\]
(3.7)

Therefore, even if we choose another sphere transformation $T$, if $T$ satisfies (3.7), then the energy dissipation in (3.3) also holds.

Lemma 3.5. For $Q := (\mathbb{R}^3 \setminus \{0\}) \times (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$, a function $T : Q \rightarrow \mathbb{R}^3$ defined by
\[
T(x_1, x_2, v) = \begin{cases} 
\psi \left( \frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_2\|} \right) R_{x_1 \rightarrow x_2} - \frac{s_{x_1}}{\|x_1\|}, & \text{if } \frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|} \neq 0, \\
0, & \text{if } \frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|} = 0 
\end{cases}
\]
(3.8)
is locally Lipschitz continuous in $Q$. Here, $R$ is the rotation operator given in Definition 2.1 and $\psi$ satisfies assumptions in Theorem 4.
Proof. Let
\[ A := \{ (x_1, x_2, v) \in Q : \frac{x_1}{\|x_1\|} = \frac{x_2}{\|x_2\|} \} \quad \text{and} \quad B := \{ (x_1, x_2, v) \in Q : \frac{x_1}{\|x_1\|} = -\frac{x_2}{\|x_2\|} \}. \]
We claim that \( T \) is Lipschitz in small neighborhood of \((x_1^*, x_2^*, v^*) \in Q\).

First, we consider the case that \((x_1^*, x_2^*, v^*) \in Q \setminus (A \cup B)\). Note that the first three terms of \( \psi R \) for the rotation operator \( R \) are Lipschitz continuous in \( Q \). It remains to show that the last term is locally Lipschitz continuous in \( Q \). We denote the last term of \( \psi R \) by \( VV^Tv \), where \( V : Q \to \mathbb{R}^n \) is given by
\[
V(x_1, x_2, v) = \begin{cases} \psi \left( \frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_2\|} \right) \left( 1 - \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \right) \frac{x_1 \times x_2}{\|x_1 \times x_2\|}, & \text{in } Q \setminus (A \cup B), \\ 0, & \text{in } A \cup B. \end{cases}
\]
In a small neighborhood of \((x_1^*, x_2^*, v^*) \in Q \setminus (A \cup B)\), \( \|x_1 x_2\| \) is nonzero and \( V(x_1, x_2) V(x_1, x_2)^Tv \) is Lipschitz continuous with respect to \( x_1, x_2 \) and \( v \).

Next, we consider the second case, \((x_1^*, x_2^*, v^*) \in A\). As the previous step, we focus on the last term of \( \psi R \) by \( VV^Tv \). By the definition of \( V \),
\[
\|V(x_1, x_2, v)\| \leq C_1 \left( 1 - \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \right) \frac{x_1 \times x_2}{\|x_1 \times x_2\|}, \quad \text{for some constant } C_1 > 0.
\]
Furthermore, \((3.9)\) and \( V(x_1^*, x_2^*, v^*) = 0 \) at \( x_1^*/\|x_1^*\| = x_2^*/\|x_2^*\| \) imply that
\[
\|V(x_1, x_2, v) - V(x_1^*, x_2^*, v^*)\| \leq C_2 \frac{x_1^*}{\|x_1^*\|} \frac{x_2^*}{\|x_2^*\|} + C_2 \frac{x_1}{\|x_1\|} \frac{x_2}{\|x_2\|}
\]
for some constant \( C_2 > 0 \). Here, we used the following simple inequality: for any \( a, b \in \mathbb{R}^n \setminus \{0\} \),
\[
\frac{a}{\|a\|} - \frac{b}{\|b\|} \leq \frac{a - b}{\|a\|} + \frac{b - a}{\|b\|} \leq \frac{2\|a - b\|}{\|a\|}.
\]
We choose a small ball of center \((x_1^*, x_2^*, v^*) \in Q \setminus B\). For any two points \((x_1, x_2, v)\) and \((y_1, y_2, w)\) in the ball, if either \((x_1, x_2, v)\) or \((y_1, y_2, w)\) in \( A \), we apply \((3.10)\) to obtain the Lipschitz constant
\[
C_2 \text{ or } C_2 \frac{\|x_1\|}{\min\{\|x_1\|, \|x_2\|\}}.
\]
If both \((x_1, x_2, v)\) or \((y_1, y_2, w)\) are in \( Q \setminus (A \cup B) \), we consider the line segment \( \alpha(x_1, x_2, v) + (1 - \alpha)(y_1, y_2, w) \) for all \( \alpha \in [0, 1] \). If the line segment intersects with \( A \), then we apply \((3.10)\) with the triangle inequality to obtain the Lipschitz constant. Thus, it is enough to consider
\[
\{ \alpha(x_1, x_2, v) + (1 - \alpha)(y_1, y_2, w) : 0 \leq \alpha \leq 1 \} \subset Q \setminus (A \cup B).
\]
We point out that \( V \) is differentiable at \((x_1, x_2, v)\) in \( Q \setminus (A \cup B) \) and
\[
\left\| \frac{\partial V}{\partial x_1} (x_1, x_2, v) \right\|, \quad \left\| \frac{\partial V}{\partial x_2} (x_1, x_2, v) \right\| \leq \frac{C_0}{\min\{\|x_1\|, \|x_2\|\}}
\]
for some constant \( C_0 > 0 \). By the fundamental theorem of calculus, we have
\[
\|V(x_1, x_2, v) - V(y_1, y_2, w)\| \leq 2C_0 \min \frac{\|x_1 \|}{\min_{i=1, 2 \text{ and } 0 \leq \alpha \leq 1} \alpha x_i + (1 - \alpha)y_i},
\]
and we conclude that \( V \) is Lipschitz in the small ball of center \((x_1^*, x_2^*, v^*)\).
Lastly, we consider the case that \((x_1^*, x_2^*, v^*) \in B\). We denote \((x_1^*, x_2^*) = (k_1a, -k_2a)\) for some \(a \in S^2\) and \(k_1, k_2 > 0\). Since \(\frac{1}{2}\|x_1 + x_2\|^2 = 1 + \langle x_1, x_2 \rangle\) and \(\|x_1 \times x_2\|^2 = \|x_1\|^2\|x_2\|^2 - \langle x_1, x_2 \rangle^2\), it holds that
\[
\left\| \left(1 - \frac{1}{\|x_1\|} x_1 \times x_2 \right) \right\| = \left\| \frac{1}{\|x_1\|} x_1 \times x_2 \right\| = \frac{\|x_1\|\|x_2\| - \langle x_1, x_2 \rangle}{\|x_1\|\|x_2\|} = \frac{\|x_1\|\|x_2\|}{\|x_1\|\|x_2\|} = 1 - \langle x_1, x_2 \rangle.
\]
Thus, we conclude that
\[
\psi(x_1, x_2, v) = \|x_1\|^2\|x_2\|^2 - \langle x_1, x_2 \rangle^2.
\]
Similarly, from (1.3), there exists \(C > 0\) such that \(\psi(x_1, x_2, v) > 0\) for all \(x, y \in S^2\).

From (3.12) and Lemma 3.6 below, we have
\[
\|V(x_1, x_2, v)\| \leq C_3 \left\| \frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|} \right\|.
\]
for some constant \(C_3 > 0\). Thus, \(V(k_1a, -k_2a) = 0\) and (3.13) imply
\[
\|V(k_1a, -k_2a, v) - V(x_1, x_2, v)\| \leq C_3 \left\| \frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|} \right\|
\leq C_4\|(k_1a, -k_2a) - (x, y)\|
\leq \frac{C_4}{\min\{k_1, k_2\}} \|(k_1a, -k_2a) - (x, y)\|
\]
for some constant \(C_4 > 0\), which implies that
\[
\|V(k_1a, -k_2a, v) - V(x_1, x_2, v)\| \to 0 \quad \text{as} \quad (x_1, x_2, v) \to (k_1a, -k_2a, v^*).
\]
Thus \(V\) is continuous at \((k_1a, -k_2a, v^*)\). As discussed in the second case, we consider a small ball of center \((x_1^*, x_2^*, v^*)\) in \(Q \setminus A\) and apply (3.11) and (3.14). This shows that \(\psi VVT\) and \(T\) are Lipschitz continuous in a small ball of center \((x_1^*, x_2^*, v^*)\).

**Lemma 3.6.** For \(\psi\) satisfying (1.3), there exists \(C > 0\) such that for all \(x, y \in S^2\)
\[
\frac{\|x + y\|^2}{C} \leq \psi(\|x - y\|) \leq C\|x + y\|^2.
\]

**Proof.** From the Lipschitzness of \(\psi\) and \(\psi(2) = 0\), we have \(\psi(a) \leq C_1(2 - a)\) for \(a \in [0, 2]\). On the other hand, as \(x, y \in S^2\),
\[
2 - \|x - y\| = \frac{\|x + y\|^2}{2 + \|x - y\|} \leq \frac{1}{2}\|x + y\|^2.
\]
As a consequence, we conclude that
\[
\psi(\|x - y\|) \leq C_1(2 - \|x - y\|) \leq C_1\|x + y\|^2.
\]
Similarly, from (1.3), there exists \(C_2 > 0\) such that \(\psi(a) \geq C_2(2 - a)\) for all \(a \in [0, 2]\). Note that
\[
2 - \|x - y\| = \frac{\|x + y\|^2}{2 + \|x - y\|} \geq \frac{1}{4}\|x + y\|^2.
\]
Thus, we conclude that
\[
\psi(\|x - y\|) \geq C_2(2 - \|x - y\|) \geq \frac{C_2}{4}\|x + y\|^2.
\]
We are ready to prove the existence and uniqueness theorem.

**Proof of Theorem 3.1** We consider the following system of ordinary differential equations:
\[
\dot{x}_i = v_i,
\]
\[
\dot{v}_i = -\frac{||v_i||^2}{||x_i||^2}x_i + \sum_{j=1}^{N} \psi_{ij} \frac{1}{N}(T(x_i, x_j, v_j) - v_i) + \sum_{k=1}^{N} \frac{\sigma}{N} ||x_i||^2 x_k - \langle x_i, x_k \rangle x_i,
\]
(3.16)
where \( T(\cdot, \cdot, \cdot) \) is given in (3.8). From the parallel argument of Proposition 3.2, if \((x_i, v_i)_{i=1}^{N}\) is a solution of (3.16), then \\{\(x_i\)\}_{i=1}^{N} \subset S^2. \) Therefore, \((x_i, v_i)_{i=1}^{N}\) is also a solution of (1.2). It remains to show that the solution to (3.16) uniquely exists for all time \( t > 0 \).

For the admissible initial data, Lemma 3.5 implies that the right-hand side of (3.16) is Lipschitz continuous with respect to \((x_i, v_i)_{i=1}^{N}\) in a small neighborhood of \((x_i(0), v_i(0))_{i=1}^{N}\) in \(\mathbb{R}^{6N}. \) From the Picard-Lindelöf Theorem (See [2, Theorem 2.2]), a solution of (3.16) exists in an interval \([0, \epsilon]\) for some \(\epsilon > 0.\)

We now prove that the solution of (3.16) uniquely exists for all \( t > 0. \) Suppose that \( I_M = [0, t_M) \) is the maximal interval of existence of a solution starting at \( t = 0. \) Proposition 3.2 implies that \( \langle x_i \rangle_{i=1}^{N} \subset S^2. \) Furthermore, from the energy inequality in Proposition 3.3, we conclude that \( \langle v_i \rangle_{i=1}^{N} \) are uniformly bounded for all time \( t > 0. \) As \( \langle x_i, v_i \rangle_{i=1}^{N} \) are uniformly bounded, we can apply the extensibility of solutions in [2, Corollary 2.2] and conclude that \( t_M = \infty. \)

\[\Box\]

4. Flocking theorem

In this section, we prove the flocking theorem for the unit sphere model. Because of the curved geometry, we adapt a vector difference \( v_j - v_i \) of the original C-S model to \( R_{x_j-x_i}(v_j) - v_i. \) Mainly, this new term causes difficulty in analyzing the asymptotic behavior of the solution to the C-S model on a unit sphere. As we mentioned before, we cannot use the Lyapunov functional approach that is used in the previous articles to obtain the flocking theorem for \(\mathbb{R}^d. \) Our plan for the proof of the main theorem contains the following three steps. First, we prove the modified version of Barbalat’s lemma in which an integrable function with bounded derivative converges to zero. Then, Proposition 3.3 implies that \( \psi_{ij} \| R_{x_j-x_i} (v_j) - v_i \|^2 \) is integrable with respect to time. Finally, from Lemma 4.3, we verify that the above quantity has a bounded time derivative, and this fact combined with the Barbalat type lemma yields the velocity alignment result for \( \sigma \geq 0 \) and the flocking estimate for \( \sigma > 0 \) under a sufficient condition for the initial data. We notice that we need to control the diameter of position difference to obtain flocking behavior. To control the diameter, we consider the additional bonding force under the initial data condition: \( 2\sigma > N^2 \mathcal{E}(0). \)

The following lemma provides our main framework for the velocity alignment.

**Lemma 4.1.** Suppose that a continuous nonnegative function \( f : [0, \infty) \rightarrow \mathbb{R} \) satisfies
\[
\lim_{b \to \infty} \int_{a}^{b} f(\tau)d\tau < \infty
\]
for some constant \( a \in \mathbb{R}. \) Assume that on the support \( \{ t \in [a, \infty) : f(t) > 0 \}, \) \( f \) has uniformly bounded time-derivative: that is, there is a constant \( C > 0 \) such that
\[
\left| \frac{df}{dt} \right| < C \quad \text{on} \quad \{ t \in [a, \infty) : f(t) > 0 \}.
\]
Then
\[
\lim_{t \to \infty} f(t) = 0.
\]
We postpone the proof until Appendix A. Next, we estimate the time-derivative of $R$.

**Lemma 4.2.** For any $(x_i, v_i)_{i=1}^N$ satisfying $\dot{x}_i = v_i$ for all $i \in \{1, 2, \cdots, N\}$, we assume that $\|x_i\| = 1$, $\langle v_i, x_i \rangle = 0$, $\|v_i\| \leq V_{\text{max}}$ and $x_i \neq -x_j$ for any $i, j \in \{1, 2, \cdots, N\}$. Then, the following holds for some constant $C > 0$.

$$
\left\| \frac{dR(x_j, x_i)}{dt} \right\| \leq C V_{\text{max}} + \frac{C}{\|x_i + x_j\|} V_{\text{max}}.
$$

We also postpone the proof until Appendix B, as it is technical.

**Lemma 4.3.** Let $(x_i, v_i)_{i=1}^N$ be the solution to system (1.2) subject to the admissible initial data $(x_i(0), v_i(0))_{i=1}^N$. We assume that $\psi_{ij} = \psi_{ji}$ and $x_i \neq -x_j$, for all $i, j \in \{1, 2, \cdots, N\}$.

Then, it holds that

$$
\left| \frac{d}{dt} R_{x_j \rightarrow x_i}(v_j) - v_i \right|^2 \leq C \left( (NE(0))^{3/2} + \frac{(NE(0))^{3/2}}{\|x_i + x_j\|} + \max_{1 \leq i, k \leq N} \psi_{ik} NE(0) \right).
$$

Here, $C > 0$ is a generic constant and $\mathcal{E}$ is given in (1.5).

**Proof.** Note that

$$
\frac{1}{2} \frac{d}{dt} \|R_{x_j \rightarrow x_i}(v_j) - v_i\|^2 = \left\langle \frac{d(R(x_j, x_i) \cdot v_j)}{dt} - \frac{dv_i}{dt}, R(x_j, x_i) \cdot v_j - v_i \right\rangle
$$

$$
= \left\langle \frac{dR(x_j, x_i)}{dt} \cdot v_j + R(x_j, x_i) \cdot \frac{dv_j}{dt}, R(x_j, x_i) \cdot v_j - v_i \right\rangle
$$

$$
= \left\langle \frac{dR(x_j, x_i)}{dt} \cdot v_j, R(x_j, x_i) \cdot v_j - v_i \right\rangle
$$

$$
+ \left\langle R(x_j, x_i) \cdot \frac{dv_j}{dt}, R(x_j, x_i) \cdot v_j - v_i \right\rangle
$$

$$
- \left\langle \frac{dv_i}{dt}, R(x_j, x_i) \cdot v_j - v_i \right\rangle =: I_1^{ij} + I_2^{ij} + I_3^{ij}.
$$

In the sequel, we will obtain estimates for $I_1^{ij}$, $I_2^{ij}$ and $I_3^{ij}$, separately. We first consider the $I_3^{ij}$ case. From the second equation of (1.2), we can rewrite $I_3^{ij}$ as follows:

$$
I_3^{ij} = -\left\langle \frac{dv_i}{dt}, R_{x_j \rightarrow x_i}(v_j) - v_i \right\rangle
$$

$$
= -\left\langle -\frac{\|v_i\|^2}{\|x_i\|^2} x_i + \sum_{k=1}^N \frac{\psi_{ik}}{N} (R_{x_k \rightarrow x_i}(v_k) - v_i), R_{x_j \rightarrow x_i}(v_j) - v_i \right\rangle. \tag{4.4}
$$

Note that by Proposition 3.2

$$
\text{ (4.5)}
$$

From the properties of $R_{x_j \rightarrow x_i}$ in Lemma 2.4, it follows that

$$
\langle x_i, R_{x_j \rightarrow x_i}(v_j) \rangle = \langle R_{x_j \rightarrow x_i}(x_i), v_j \rangle = \langle x_j, v_j \rangle = 0. \tag{4.6}
$$

By (4.5) and (4.6), we simplify (4.4) as follows:

$$
I_3^{ij} = -\left\langle \sum_{k=1}^N \frac{\psi_{ik}}{N} (R_{x_k \rightarrow x_i}(v_k) - v_i), R_{x_j \rightarrow x_i}(v_j) - v_i \right\rangle.
$$
We take the absolute value of the above and use the Cauchy and triangle inequalities to obtain

$$|I_{ij}^3| = \left| \sum_{k=1}^{N} \frac{\psi_{ik}}{N} (R_{\sigma_k \rightarrow x_i} (v_k) - v_i), R_{\sigma_j \rightarrow x_j} (v_j) - v_j) \right|$$

$$\leq \max_{1 \leq i, m \leq N} \psi_{im} \max_{1 \leq k \leq N} |(R_{\sigma_k \rightarrow x_i} (v_k) - v_i), R_{\sigma_j \rightarrow x_j} (v_j) - v_j)|$$

$$\leq \max_{1 \leq i, m \leq N} \psi_{im} \max_{1 \leq k \leq N} \|R_{\sigma_k \rightarrow x_i} (v_k) - v_i\| \|R_{\sigma_j \rightarrow x_j} (v_j) - v_j\|$$

$$\leq \max_{1 \leq i, m \leq N} \psi_{im} \max_{1 \leq k \leq N} (\|R_{\sigma_k \rightarrow x_i} (v_k)\| + \|v_i\|)(\|R_{\sigma_j \rightarrow x_j} (v_j)\| + \|v_j\|).$$

We note that by Lemma 2.3, the rotation operator $R_{\sigma_j \rightarrow x_i}$ conserves the modulus of velocities $v_j$.

Thus, we have

$$|I_{ij}^3| \leq \max_{1 \leq i, m \leq N} \psi_{im} \max_{1 \leq k \leq N} (\|v_k\| + \|v_i\|)(\|v_j\| + \|v_i\|).$$

By Proposition 3.3, the velocities have a uniform upper bound: $\|v_i(t)\| \leq \mathcal{V}(t) \leq \sqrt{N\mathcal{E}(0)}$, for any $i \in \{1, \ldots, N\}$. Thus, we obtain the desired result as follows:

$$|I_{ij}^3| \leq 4 \max_{1 \leq i, m \leq N} \psi_{im} \mathcal{N}\mathcal{E}(0). \quad (4.7)$$

For the $|I_{ij}^2|$ case, we use the $|I_{ij}^3|$ result. Since the rotation operator $R_{\sigma_i \rightarrow x_j}$ satisfies

$$R_{\sigma_i \rightarrow x_j}^{-1} = R_{x_j \rightarrow x_i}^{T} = R_{x_j \rightarrow x_i},$$

we reduce $I_{ij}^2$ to $I_{ij}^3$ by using the properties of the rotation operator as follows:

$$I_{ij}^2 = \left( R_{x_j \rightarrow x_i} \left( \frac{dv_j}{dt}, R_{x_j \rightarrow x_i} (v_j) - v_i \right), R_{x_j \rightarrow x_j} \left( R_{x_j \rightarrow x_j} (v_j) - v_i \right) \right)$$

$$= \left( \frac{dv_j}{dt}, v_j - R_{x_j \rightarrow x_j} (v_j) \right) = I_{ij}^3$$

Therefore, the previous result $(4.7)$ for $I_{ij}^3$ implies that

$$|I_{ij}^2| \leq 4 \max_{1 \leq i, m \leq N} \psi_{im} \mathcal{N}\mathcal{E}(0). \quad (4.8)$$

It now remains to obtain an estimate for $I_{ij}^1$:

$$I_{ij}^1 = \left( \frac{dR(x_j, x_i)}{dt}, v_j, R_{x_j \rightarrow x_i} (v_j) - v_i \right).$$

We again take the absolute value of $I_{ij}^1$ and use the Cauchy inequality and triangle inequality to obtain

$$|I_{ij}^1| \leq \left| \left( \frac{dR(x_j, x_i)}{dt}, v_j \right) \left( R_{x_j \rightarrow x_i} (v_j) - v_i \right) \right| \leq \left( \frac{dR(x_j, x_i)}{dt} \right) \left( \|R_{x_j \rightarrow x_i} (v_j)\| + \|v_i\| \right).$$

From the properties of the rotation operator and the Cauchy inequality, it follows that

$$\|R_{x_j \rightarrow x_i} (v_j)\| = \|v_j\| \text{ and } \left( \frac{dR(x_j, x_i)}{dt} \right) \leq \|R_{x_j \rightarrow x_i} (v_j)\| \|v_j\|.$$ 

Thus, it holds that

$$|I_{ij}^1| \leq \left( \frac{dR(x_j, x_i)}{dt} \right) \left( \|v_j\| + \|v_i\| \right) \leq 2 \left( \frac{dR(x_j, x_i)}{dt} \right) \mathcal{N}\mathcal{E}(0).$$

Here, we used the uniform upper bound of velocities in Proposition 3.3. By Lemma 1.2, we have

$$|I_{ij}^1| \leq C(N\mathcal{E}(0))^{3/2} + C \left( \frac{dR(x_j, x_i)}{dt} \right)^{3/2} \mathcal{N}\mathcal{E}(0). \quad (4.9)$$
where $C$ is a positive constant.

Finally, combining (4.7), (4.8) and (4.9), we obtain

$$
\left| \frac{d}{dt} \| R_{x_j \rightarrow x_i} (v_j) - v_i \| \right|^2 \leq 2 |I_1^{ij}| + 2 |I_2^{ij}| + 2 |I_3^{ij}|
$$

$$\leq C \left( (N\mathcal{E}(0))^{3/2} + \left(\frac{N\mathcal{E}(0))^{3/2}}{\|x_i + x_j\|} + \max_{1 \leq i, k \leq N} |\psi_{ik}|N\mathcal{E}(0) \right) \right).
\square
$$

We are ready to prove the velocity alignment result in the main theorem.

**Theorem 4.4.** For $\psi$ satisfying (1.3) and any given initial data $(x_i(0), v_i(0))_{i=1}^N$ satisfying the admissible condition in (3.1), the solution $(x_i, v_i)_{i=1}^N$ to (1.2) with $\sigma \geq 0$ satisfies

$$
\lim_{t \to \infty} \| x_i(t) + x_j(t) \| \| R_{x_j \rightarrow x_i}(v_j(t)) - v_i(t) \| = 0.
$$

(4.10)

**Proof.** We recall the identity in Proposition 3.3 for the case of $\sigma = 0$.

$$
\frac{d\mathcal{E}}{dt} = - \sum_{i,j=1}^N \psi_{ij}(t) \| R_{x_j \rightarrow x_i}(v_j) - v_i \|^2.
$$

Taking the integral of the above with respect to time on $(0, t)$ yields

$$
N^2 \mathcal{E}(t) - N^2 \mathcal{E}(0) = - \int_0^t \sum_{i,j=1}^N \psi_{ij}(\tau) \| R_{x_j(\tau) \rightarrow x_i(\tau)}(v_j(\tau)) - v_i(\tau) \|^2 d\tau.
$$

(4.11)

For $f_{ij}(t) := \psi_{ij}(t) \| R_{x_j(t) \rightarrow x_i(t)}(v_j(t)) - v_i(t) \|^2 \geq 0$, since $f_{ij}(t)$ is nonnegative for any $i, j \in \{1, \ldots, N\}$ and $t > 0$, (4.11) implies that

$$
\lim_{t \to \infty} \int_0^t f_{ij}(\tau) d\tau \leq \lim_{t \to \infty} \int_0^t \sum_{i,j=1}^N f_{ij}(\tau) d\tau = \lim_{t \to \infty} \left( N^2 \mathcal{E}(0) - N^2 \mathcal{E}(t) \right) \leq N^2 \mathcal{E}(0).
$$

(4.12)

Recall that if $x_i(t) + x_j(t) = 0$, we have $f_{ij}(t) = 0$.

We assume that $x_i(t) + x_j(t) \neq 0$ and take the absolute value of the derivative of $f_{ij}(t)$ with respect to $t$ to obtain

$$
\left| \frac{df_{ij}(t)}{dt} \right| = \left| \frac{\langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle}{\|x_i(t) - x_j(t)\|} \psi'(\|x_i(t) - x_j(t)\|) \| R_{x_i(t) \rightarrow x_j(t)}(v_i(t)) - v_j(t) \|^2 \right.
$$

$$+ \psi_{ij} \frac{d}{dt} \| R_{x_j(t) \rightarrow x_i(t)}(v_j(t)) - v_i(t) \|^2 \right|

\leq \left| \frac{\langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle}{\|x_i(t) - x_j(t)\|} \psi'(\|x_i(t) - x_j(t)\|) \| R_{x_i(t) \rightarrow x_j(t)}(v_i(t)) - v_j(t) \|^2 \right|

$$

$$+ \left| \psi_{ij} \frac{d}{dt} \| R_{x_j(t) \rightarrow x_i(t)}(v_j(t)) - v_i(t) \|^2 \right| =: K_1^{ij} + K_2^{ij}.
$$

We estimate $K_1^{ij}$ as follows:

$$
K_1^{ij} \leq \left| \frac{\langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle}{\|x_i(t) - x_j(t)\|} \right| \left| \psi'(\|x_i(t) - x_j(t)\|) \right| \| R_{x_i(t) \rightarrow x_j(t)}(v_i(t)) - v_j(t) \|^2.
$$

From Proposition 3.3, we have

$$
\left| \frac{\langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle}{\|x_i(t) - x_j(t)\|} \right| \leq \| v_i(t) - v_j(t) \| \leq 2 \mathbf{v}(t) \leq 2 \sqrt{N\mathcal{E}(0)}.
$$

Similarly, Proposition 3.3 and Lemma 2.5 yield that

$$
\| R_{x_i(t) \rightarrow x_j(t)}(v_i(t)) - v_j(t) \|^2 \leq \| R_{x_i(t) \rightarrow x_j(t)}(v_i(t)) \|^2 \leq 4N\mathcal{E}(0).
$$
From the assumption on $\psi$, we conclude that
\[ K_{ij}^2 \leq 8C(NE(0))^{3/2}. \tag{4.13} \]

Next, we estimate $K_{ij}^2$. Lemma 3.6 implies that
\[ K_{ij}^2 = \|\psi([x_i(\tau) - x_j(\tau)]) \|_2^2 \frac{d}{d\tau} \|R_{x_i(\tau)\to x_j(\tau)}(v_j(\tau)) - v_i(\tau)\|^2. \]
\[ \leq C \|x_i(\tau) - x_j(\tau)\|_2^2 \frac{d}{d\tau} \|R_{x_i(\tau)\to x_j(\tau)}(v_j(\tau)) - v_i(\tau)\|^2. \]

On the other hand, Lemma 4.3 and $\|\psi\|$ imply that
\[ \|x_i(\tau) + x_j(\tau)\| \leq 2 \text{ yield} \]
\[ \|x_i(\tau) + x_j(\tau)\| \leq \|R_{x_i(\tau)\to x_j(\tau)}(v_j(\tau)) - v_i(\tau)\|^2 \]
\[ \leq C \|x_i(\tau) + x_j(\tau)\| \left( (NE(0))^{3/2} + \max_{1 \leq i, k \leq N} \psi_{ik}NE(0) \right) \]
\[ \leq C_1. \]

Thus, it follows from the above estimates that
\[ K_{ij}^2 \leq 2CC_1. \tag{4.14} \]

From (4.13) and (4.14), for any $t > 0$, it holds that
\[ f(t) = 0 \text{ or } \left| \frac{df_{ij}(t)}{dt} \right| \leq C \tag{4.15} \]

where $C$ is a positive constant.

From (4.12) and (4.15), we can apply Lemma 4.1 to $f_{ij}(t)$ to obtain that
\[ \lim_{t \to \infty} \psi([x_i(t) - x_j(t)]) \|R_{x_i(t)\to x_j(t)}(v_j(t)) - v_i(t)\|^2 = 0. \]

From Lemma 3.6 we conclude (4.10).

Next, we focus on the flocking theorem. As a direct consequence of Proposition 3.3, we have the following inequalities.

**Lemma 4.5.** Let $(x_i, v_i)_{i=1}^N$ be the solution to (1.2) and $\sigma > 0$. For $t > 0$ and $1 \leq i, j \leq N$, it holds that
\[ \|v_i(t)\|^2 \leq NE(0) \quad \text{and} \quad \|x_i(t) - x_j(t)\|^2 \leq \frac{2N^2E(0)}{\sigma}. \tag{4.16} \]

**Proposition 4.6.** Let $(x_i, v_i)_{i=1}^N$ be the solution to (1.2). If
\[ 2\sigma > N^2E(0), \]
then $\frac{d}{dt}v_i$ and $\frac{d}{dt}R_{x_j\to x_i}(v_j)$ are bounded.

**Proof.** Since $x_i \in S^2$, by Lemma 4.5, $\frac{d}{dt}v_i$ is bounded. The orthogonality of the rotation operator in Lemma 2.4 and Lemma 4.3 imply that
\[ \|R_{x_j\to x_i}(v_j) - v_i\| \leq \|v_j\| \leq \|v_i\| \leq 2\sqrt{NE(0)}. \]

As $\psi_{ij}$ is bounded, we conclude that the second term of $\tilde{c}_i$ in (1.2) is bounded. The boundedness of the last term follows from $x_i \in S^2$.

Next, we prove that $\frac{d}{dt}R_{x_j\to x_i}(v_j)$ is bounded. By the chain rule, it holds that
\[ \frac{d}{dt}(R_{x_j\to x_i}(v_j)) = R_{x_j\to x_i} \frac{d}{dt}v_j + v_j \frac{d}{dt}R_{x_j\to x_i}. \tag{4.17} \]
Similar to $v_i$, the first term in (4.17) is uniformly bounded. From the direct computation of the second term in (4.17), it follows that

\[
\frac{d}{dt}(R_{x_i}x_j) = \frac{d}{dt}\left\{ (x_j,x_i)I - x_jx_i^T + x_ix_j^T \right\} + \frac{1}{1 + \langle x_j,x_i \rangle} \frac{d}{dt}\left\{ (x_j \times x_i)(x_j \times x_i)^T \right\}
\]

Moreover, by Lemma 4.5, we have

\[
1 + \langle x_i,x_j \rangle = \frac{1}{2} \| x_i + x_j \|^2 > 2 - \frac{N\mathcal{E}(0)}{\sigma} > 0.
\]

Combining this with uniform boundedness of $(x_i,v_i)_{i=1}^N$, we conclude that the second term and the third term in (4.18) are uniformly bounded. Therefore, $\frac{d}{dt}R_{x_i}x_j(v_j)$ is bounded. □

**Theorem 4.7.** Let $(x_i,v_i)_{i=1}^N$ be the solution to (1.2) and $\psi$ satisfy (1.3). If $2\sigma > N^2\mathcal{E}(0)$, then (1.2) has time-asymptotic flocking on a unit sphere.

**Proof.** From the assumption on $\psi$ and Lemma 4.5,

\[
\sup_{t \geq 0} \max_{1 \leq i,j \leq N} \| x_i(t) - x_j(t) \| < 2,
\]

and

\[
\frac{\psi_{ij}}{N} > \psi(\sqrt{2N^2\mathcal{E}(0)/\sigma}) =: C_\psi,
\]

for any $i,j \in \{1, \ldots, N\}$ and $t \geq 0$. By Proposition 3.3 it holds that

\[
C_\psi \int_0^t \| R_{x_j(s)}x_i(s)(v(j)(s)) - v_i(s) \|^2 ds \leq \mathcal{E}(0) \quad \text{for all } t \in [0, \infty).
\]

Therefore, $\sum_{j=1}^N \| R_{x_j(t)}x_i(t)(v(j)(t)) - v_i(t) \|^2$ is integrable in $[0, \infty)$.

As $\frac{d}{dt}v_i$ and $\frac{d}{dt}R_{x_i}x_j(v_j)$ are bounded in $[0, \infty)$ for any $i,j \in \{1, \ldots, N\}$ from Lemma 4.5 and Proposition 4.6, we conclude that $\sum_{i,j=1}^N \frac{d}{dt}\| R_{x_j(t)}x_i(t)(v(j)(t)) - v_i(t) \|^2$ is bounded. From Lemma 4.1, we conclude that

\[
\sum_{i,j=1}^N \| R_{x_j(t)}x_i(t)(v(j)(t)) - v_i(t) \|^2 \to 0 \quad \text{as } t \to \infty.
\]

5. Conclusion

In this paper, we consider a C-S type flocking model on the unit sphere $S^2$. To derive the flocking model on the unit sphere, we introduced the rotation operator $R$. The rotation operator has the modulus conservation property, but singularity occurs at antipodal points. Therefore, in order to cancel such singularity, we assumed that $\phi$ vanished at antipodal points. Moreover, we introduce a centripetal force term to obtain conservation of the modulus, $\| x_i \| \equiv 1$. For the velocity alignment, we employed energy dissipation and a Barbalat’s lemma type argument. To define flocking on a spherical surface, in addition to velocity alignment, we consider antipodal points avoidance that the positions of two agents are not located at antipodal points, which corresponds to the boundedness of the relative position of the flocking definition in the flat space. However, it is difficult to control the relative position by the geometric properties of the spherical surface. Therefore, in addition to
relative velocity, a bonding force term is added. Using the bonding force term and initial conditions, we controlled the relative positions of the particles.

**Appendix A. Proof of Lemma 4.1**

In this section, we present the proof of Lemma 4.1. Suppose that (4.3) does not hold. Then
\[
\limsup_{\tau \to \infty} f(\tau) > 0,
\]
and there is a sequence of real numbers \(\{b_n\}\) such that \(b_n \to \infty\) as \(n \to \infty\) and
\[
\lim_{n \to \infty} f(b_n) = d
\]
for some \(d > 0\). We can choose \(b_n\) on the support \(\{t \in [a, \infty) : f(t) > 0\}\) of \(f\) and there is \(N \in \mathbb{N}\) such that for \(n > N\),
\[
f(b_n) > \frac{d}{2}.
\]
From (4.2), it holds that
\[
0 < -C|\tau - b_n| + f(b_n) < f(\tau) \quad \text{for all } n > N \text{ and } \tau \in \left( b_n - \frac{d}{2C}, b_n + \frac{d}{2C} \right).
\]
By replacing \(\{b_n\}\) with its subsequence if necessary, we assume that
\[
b_{n+1} - b_n > \frac{d}{C}.
\]
We define a sequence \(\{a_n\}\) such that
\[
a_n := \int_{b_n - \frac{d}{C}}^{b_n + \frac{d}{C}} f(\tau)d\tau.
\]
Since \(f\) is nonnegative and \(\{b_n\}\) satisfies (A.2), we have
\[
a_n = \int_{b_n - \frac{d}{C}}^{b_n + \frac{d}{C}} f(\tau)d\tau > \int_{b_n - \frac{d}{C}}^{b_n + \frac{d}{C}} f(\tau)d\tau,
\]
and we obtain the lower bound of \(a_n\) by using (A.1).
\[
a_n > \int_{b_n - \frac{d}{C}}^{b_n + \frac{d}{C}} \left(-C|\tau - b_n| + f(b_n)\right)d\tau = \frac{4f(b_n)d - d^2}{4C} > \frac{d^2}{4C}.
\]
By definition of \(\{a_n\}\), the following holds.
\[
\lim_{b \to \infty} \int_{a}^{b} f(\tau)d\tau = \sum a_n > \lim_{n \to \infty} \frac{d^2}{4C}n = \infty,
\]
which contradicts (4.1) and thus we conclude (4.3).

**Appendix B. Proof of Lemma 4.2**

In this section, we prove Lemma 4.2. First, we show the following elementary estimates.

**Lemma B.1.** For any \((x_i, v_i)\) \(i = 1, 2, \ldots, N\) satisfying \(\dot{x}_i = v_i\) for all \(i \in \{1, 2, \ldots, N\}\), we assume that
\[
\|x_i\| = 1, \quad \langle v_i, x_i \rangle = 0, \quad \|v_i\| \leq V_{\text{max}}, \quad \text{and } x_i + x_j \neq 0 \text{ for any } i, j \in \{1, 2, \ldots, N\}.
\]
Then, the following estimate holds:
\[
\left\| \frac{d}{dt} \left[ \frac{1}{1 + \langle x_i, x_j \rangle} (x_i \times x_j)(x_i \times x_j)^T \right] \right\| \leq \frac{C}{\|x_i + x_j\|} V_{\text{max}},
\]
for a positive constant \(C\).
Proof. We take the time-derivative of the given term. Elementary differentiation shows that
\[
\frac{d}{dt} \left( \frac{1}{1 + \langle x_i, x_j \rangle} (x_i \times x_j)(x_i \times x_j)^T \right)
= \frac{\langle v_i, x_j \rangle + \langle x_i, v_j \rangle}{(1 + \langle x_i, x_j \rangle)^2} (x_i \times x_j)(x_i \times x_j)^T + \frac{1}{1 + \langle x_i, x_j \rangle} (v_i \times x_j)(x_i \times x_j)^T
+ \frac{1}{1 + \langle x_i, x_j \rangle} (x_i \times v_j)(x_i \times x_j)^T + \frac{1}{1 + \langle x_i, x_j \rangle} (x_i \times x_j)^T (v_i \times x_j)^T
+ \frac{1}{1 + \langle x_i, x_j \rangle} (x_i \times x_j)^T (x_i \times v_j)^T
= : J_1 + J_2 + J_3 + J_4 + J_5.
\]

We claim that
\[
|J_1| \leq \frac{C}{\|x_i + x_j\|} \mathcal{V}_{\text{max}}
\]  \hspace{1cm} (B.4)
for some constant $C > 0$. Note that for any vector $x \in \mathbb{R}^3$, $x \times x = 0$. Thus the following holds:
\[
x \times y = (x + y) \times y, \text{ for } x, y \in \mathbb{R}^3.
\]  \hspace{1cm} (B.5)
Moreover, for each $i \in \{1, \ldots, N\}$, the velocity and position are orthogonal, i.e.,
\[
\langle v_i, x_i \rangle = 0, \text{ for all } i \in \{1, \ldots, N\}. \hspace{1cm} (B.6)
\]
\[\text{[B.5] and [B.6] yield}
J_1 = \frac{\langle v_i, x_j \rangle + \langle x_i, v_j \rangle}{(1 + \langle x_i, x_j \rangle)^2} (x_i \times x_j)(x_i \times x_j)^T
= \frac{\langle v_i, x_j \rangle + \langle x_i, v_j \rangle}{(1 + \langle x_i, x_j \rangle)^2} ((x_i + x_j) \times x_j)(x_i + x_j) \times x_j)^T
= \frac{\langle v_i, x_j + x_i \rangle + \langle x_i + x_j, v_j \rangle}{(1 + \langle x_i, x_j \rangle)^2} ((x_i + x_j) \times x_j)(x_i + x_j) \times x_j)^T.
\]
The Cauchy inequality and triangle inequality show that
\[
\|J_1\| \leq \frac{C}{(1 + \langle x_i, x_j \rangle)^2} \left( \|v_i\| \|x_i + x_j\| + \|x_i + x_i\| \|v_j\| \right) \|x_i + x_j\| \|x_j\| \|x_i + x_j\| \|x_j\|.
\]
Here, $C$ is a positive constant. By assumption, $x_i$ and $x_j$ are unit vectors and $v_i$, $v_j$ are bounded such that
\[
\|v_i\|, \|v_j\| \leq \mathcal{V}_{\text{max}}.
\]
This implies the following estimate for $J_1$:
\[
\|J_1\| \leq \frac{C}{(1 + \langle x_i, x_j \rangle)^2} \left( \|v_i\| \|x_i + x_j\| + \|x_i + x_j\| \|v_j\| \right) \|x_i + x_j\| \|x_j\| \|x_i + x_j\| \|x_j\|
\leq \frac{C}{(1 + \langle x_i, x_j \rangle)^2} \|v_i\| + \|v_j\| \|x_i + x_j\|^3
\leq \frac{C}{(1 + \langle x_i, x_j \rangle)^2} \mathcal{V}_{\text{max}} \|x_i + x_j\|^3.
\]
Since we have $1 + \langle x_i, x_j \rangle = \frac{\|x_i + x_j\|^2}{2}$, we obtain the estimate for $J_1$ in (B.4).

Next, we show an estimate for $J_2$ as
\[
\|J_2\| \leq \frac{C}{\|x_i + x_j\|} \mathcal{V}_{\text{max}}.
\]  \hspace{1cm} (B.7)
Here, we take the matrix norm of $J_2$. By (B.5),
\[
J_2 = \frac{1}{1 + \langle x_i, x_j \rangle} (v_i \times x_j)(x_i \times x_j)^T = \frac{1}{1 + \langle x_i, x_j \rangle} (v_i \times x_j)((x_i + x_j) \times x_j)^T.
\]
Taking the matrix norm on the above equation, we obtain
\[
\|J_2\| = \left\| \frac{1}{1 + \langle x_i, x_j \rangle} (v_i \times x_j)((x_i + x_j) \times x_j)^T \right\| \leq \frac{C}{1 + \langle x_i, x_j \rangle} \|v_i\| \|x_i + x_j\| \|x_j\|.
\]
Then, (B.7) follows from boundedness of $x_i$, $x_j$, $v_i$ and $v_j$, and the simple fact that
\[
1 + \langle x_i, x_j \rangle = \frac{\|x_i + x_j\|^2}{2}.
\]
From the parallel argument, the following estimates hold for $\|J_3\|$, $\|J_4\|$ and $\|J_5\|$:
\[
\|J_3\|, \|J_4\|, \|J_5\| \leq \frac{C}{\|x_i + x_j\|} \frac{\|\|V\|\|}{\max_j}.
\]
We omit the proof of the above estimates. By adding (B.4), (B.7) and (B.8), we conclude (B.3). \[\square\]

**Example B.2.** Consider
\[
x_1(t) = \left( t^2 \sin \frac{1}{\sqrt{t}}, -t^2 \cos \frac{1}{\sqrt{t}}, -\sqrt{-\left( t^2 \sin \frac{1}{\sqrt{t}} + t^2 \cos \frac{1}{\sqrt{t}} + 1 \right) \right),
\]
\[
x_2(t) = (0, 0, 1).
\]
Then
\[
\frac{1}{1 + \langle x_i, x_j \rangle} (x_i \times x_j)(x_i \times x_j)^T = \begin{pmatrix}
(\sqrt{1 - t^4} + 1) \cos^2 \frac{1}{\sqrt{t}} & \frac{t^4 \sin \frac{2t}{\sqrt{t}}}{2 - 2\sqrt{1 - t^4}} & 0 \\
\frac{t^4 \sin \frac{2t}{\sqrt{t}}}{2 - 2\sqrt{1 - t^4}} & (\sqrt{1 - t^4} + 1) \sin^2 \frac{1}{\sqrt{t}} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
and
\[
\frac{d}{dt} \left[ \frac{1}{1 + \langle x_i, x_j \rangle} (x_i \times x_j)(x_i \times x_j)^T \right] = \begin{pmatrix}
\frac{(\sqrt{1 - t^4} + 1) \sin \frac{2t}{\sqrt{t}}}{2t^3} - \frac{2t^3 \cos^2 \frac{1}{\sqrt{t}}}{\sqrt{1 - t^4}} & m(t) & 0 \\
0 & \frac{m(t)}{\sqrt{1 - t^4}} - \frac{(\sqrt{1 - t^4} + 1) \sin \frac{2t}{\sqrt{t}}}{2t^3} & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
where
\[
m(t) = \frac{2t^3 (t^4 + \sqrt{1 - t^4} - 2) \sin \frac{2t}{\sqrt{t}} - t^{5/2} (t^4 + \sqrt{1 - t^4} - 1) \cos \frac{2t}{\sqrt{t}}}{2 \sqrt{1 - t^4} (\sqrt{1 - t^4} - 1)^2}.
\]
Note that $\|x_1(t) + x_2(t)\| = \sqrt{2 - 2\sqrt{1 - t^4}} = t^2 + O(t^4)$,
\[
\|v_1(t)\| = \frac{1}{2} \sqrt{ \frac{t (t^4 - 16t^3 - 1)}{t^4 - 1} } = \frac{\sqrt{t}}{2} + O(t^2),
\]
and
\[
\left\| \frac{d}{dt} \left[ \frac{1}{1 + \langle x_i, x_j \rangle} (x_i \times x_j)(x_i \times x_j)^T \right] \right\| = \left( \frac{1}{t^2} + O(t^2) \right) \frac{2}{\sqrt{t}} + \left( \frac{1}{t^2} + O(t^2) \right) \cos \frac{2}{\sqrt{t}}.
\]
Thus, we conclude that the result in Lemma [B.7] is optimal:

\[
\left\| \frac{d}{dt} \left[ \frac{1}{1 + \langle x_i, x_j \rangle} \langle x_i \times x_j \rangle \langle x_i \times x_j \rangle^T \right] \right\| \sim \frac{V_{\text{max}}}{\| x_i + x_j \|}.
\]

**Proof of Lemma 4.2** From Definition 2.1, the rotation matrix is given by

\[
R(x_j, x_i) = \langle x_i, x_j \rangle I + x_ix_j^T - x_jx_i^T + \frac{1}{1 + \langle x_i, x_j \rangle} \langle x_i \times x_j \rangle \langle x_i \times x_j \rangle^T.
\]

If we take the time-derivative of the above, then

\[
\frac{dR(x_j, x_i)}{dt} = \langle v_i, x_j \rangle I + \langle x_i, v_j \rangle I + v_ix_j^T + x_iv_j^T - v_jx_i^T - x_jv_i^T + \frac{d}{dt} \left[ \frac{1}{1 + \langle x_i, x_j \rangle} \langle x_i \times x_j \rangle \langle x_i \times x_j \rangle^T \right].
\]

Taking the matrix norm \( \| \cdot \| = \| \cdot \|_2 \) leads to

\[
\left\| \frac{dR(x_j, x_i)}{dt} \right\| = \left\| \langle v_i, x_j \rangle I + \langle x_i, v_j \rangle I + v_ix_j^T + x_iv_j^T - v_jx_i^T - x_jv_i^T + \frac{d}{dt} \left[ \frac{1}{1 + \langle x_i, x_j \rangle} \langle x_i \times x_j \rangle \langle x_i \times x_j \rangle^T \right] \right\|
\]

\[
\leq \left\| \langle v_i, x_j \rangle I \right\| + \left\| \langle x_i, v_j \rangle I \right\| + \left\| v_ix_j^T \right\| + \left\| x_iv_j^T \right\| + \left\| v_jx_i^T \right\| + \left\| x_jv_i^T \right\| + \left\| \frac{d}{dt} \left[ \frac{1}{1 + \langle x_i, x_j \rangle} \langle x_i \times x_j \rangle \langle x_i \times x_j \rangle^T \right] \right\|
\]

The six terms except for the last term on the right-hand side of the above inequality is bounded by \( V_{\text{max}} \):

\[
\left\| \langle v_i, x_j \rangle I \right\|, \left\| \langle x_i, v_j \rangle I \right\|, \left\| v_ix_j^T \right\|, \left\| x_iv_j^T \right\|, \left\| v_jx_i^T \right\|, \left\| x_jv_i^T \right\| \leq CV_{\text{max}}, \tag{B.9}
\]

where \( C \) is a positive constant. From Lemma [B.1], it follows that

\[
\left\| \frac{d}{dt} \left[ \frac{1}{1 + \langle x_i, x_j \rangle} \langle x_i \times x_j \rangle \langle x_i \times x_j \rangle^T \right] \right\| \leq \frac{C}{\| x_i + x_j \|} V_{\text{max}}. \tag{B.10}
\]

[B.9] and [B.10] imply the desired result.

\[\square\]

**Appendix C. Equivalence of the flocking definitions**

**Lemma C.1.** Let

\[
f_k(t) := \| x_i(t) + x_j(t) \| \| R_{x_j(t) - x_i(t)}(v_j(t)) - v_i(t) \|_k.
\]

Suppose that \( v_i \) and \( v_j \) are uniformly bounded in time and \( x_i, x_j \in \mathbb{S}^2 \). Then, the following conditions

\[
\lim_{t \to \infty} f_k(t) = 0
\]

are equivalent for any \( k > 0 \).

**Proof.** Choose \( k > m > 0 \). As

\[
f_k = \| R_{x_j - x_i}(v_j) - v_i \|_k^{k-m} \| R_{x_j - x_i}(v_j) - v_i \|_m \quad \text{and} \quad f_k^m = \| x_i + x_j \|^{\frac{k}{k-m}} f_k^\frac{k}{k-m}
\]

hold, it is enough to check that \( \| x_i + x_j \| \) and \( \| R_{x_j - x_i}(v_j) - v_i \| \) are uniformly bounded in time. As \( x_i \) and \( x_j \) are in \( \mathbb{S}^2 \), we have \( \| x_i + x_j \| \leq 2 \). On the other hand, from Lemma 2.5,

\[
\| R_{x_j - x_i}(v_j) - v_i \| \leq \| R_{x_j - x_i}(v_j) \| + \| v_i \| = \| v_j \| + \| v_i \|.
\]

As \( v_i \) and \( v_j \) are uniformly bounded in time, we conclude. \[\square\]
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