On a Local Mean Oscillation Decomposition

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The “local mean oscillation” decomposition of Lerner has proven particularly useful in recent literature. Such a functional decomposition was first proved for averages by Garnett and Jones [3], then suggested for medians by Fujii [2] and proved by Lerner [5], [6]. In this note we generate two local median oscillation decompositions using medians rather than rearrangements, so they apply to arbitrary measurable functions.

In what follows, we adopt the notations of [8] and [9]. In particular, for a cube $Q \subset \mathbb{R}^n$ and $0 < t < 1$, we say that $m_{f,t}(Q) = \sup\{M : |\{y \in Q : f(y) < M\}| \leq t|Q|\}$ is the (maximal) median of $f$ over $Q$ with parameter $t$. For a cube $Q_0 \subset \mathbb{R}^n$ and $0 < s \leq 1/2$, the local sharp maximal function restricted to $Q_0$ of a measurable function $f$ at $x \in Q_0$ is

$$M_{0,s,Q_0}^f(x) = \sup_{x \in Q \subset Q_0} \inf_{c} \inf_{\alpha \geq 0} |\{y \in Q : |f(y) - c| > \alpha\}| < s|Q|.$$ 

Additionally, for a cube $Q \subset \mathbb{R}^n$, let $\mathcal{D}(Q)$ denote the family of dyadic subcubes relative to $Q$; that is to say, those formed by repeated dyadic subdivision of $Q$ into $2^n$ congruent subcubes. We will consider the maximal function

$$m_{Q_0}^{t,\Delta} f(x) = \sup_{Q \in \mathcal{D}(Q_0), x \in Q} |m_g(t, Q)|.$$ 

Finally, let $\hat{Q}$ denote the dyadic parent of a cube $Q$.

The first local median oscillation decomposition holds for a range of indices, and so it extends Lerner’s decomposition, which corresponds to the case $t = 1/2$, $s = 1/4$ in Theorem 1. On the other hand, the bound below is larger than his, but is sufficient for some applications such as those in [5].

**Theorem 1.** Let $f$ be a measurable function on a fixed cube $Q_0 \subset \mathbb{R}^n$, $0 < s < 1/2$, and $1/2 \leq t < 1 - s$. Then there exists a (possibly empty) collection of subcubes $\{Q_j^v\} \subset \mathcal{D}(Q_0)$ and a family of collections $\{I_2^v\}_v$ such that

(i) For a.e. $x \in Q_0$,

$$|f(x) - m_f(t, Q_0)| \leq 4M_{0,s,Q_0}^f(x) + \sum_{v=1}^{\infty} \sum_{j \in I_2^v} \left(10n \inf_{y \in Q_j^v} M_{0,s,Q_j^v}^f(y) + 2 \inf_{y \in Q_j^v} M_{0,s,Q_j^v}^f(y)\right) 1_{Q_j^v}(x).$$

(ii) For fixed $v$, $\{Q_j^v\}$ are nonoverlapping.
(iii) If $\Omega^v = \bigcup_j Q_j^v$, then $\Omega^{v+1} \subset \Omega^v$.

(iv) For all $j$, $|\Omega^{v+1} \cap Q_j^v| \leq \frac{s}{1-t}|Q_j^v|$.

Proof. Let $E^1 = \{x \in Q_0 : |f(x) - m_f(t, Q_0)| > 2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y)\}$. If $|E^1| = 0$, the decomposition halts - trivially, for a.e. $x \in Q_0$,

\[|f(x) - m_f(t, Q_0)| \leq 2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y).\]  

(1)

So suppose that $|E^1| > 0$. Recall that by Lemma 4.1 in [5], for $\eta > 0$,

\[|\{x \in Q_0 : |f(x) - m_f(t, Q_0)| \geq 2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y) + \eta\}| < s|Q_0|.

Thus, picking $\eta_k \rightarrow 0^+$, by continuity from below it readily follows that

\[|\{x \in Q_0 : |f(x) - m_f(t, Q_0)| \geq 2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y)\}| \leq s|Q_0|.

(2)

Now let $f^0 = (f - m_f(t, Q_0)) 1_{Q_0}$, and

\[\Omega^1 = \{x \in Q_0 : m_{\Omega^0} (f^0)(x) > 2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y)\}.

Then by Theorem 2.1 in [5], $E^1 \subset \Omega^1$ and $|\Omega^1| > 0$ as well. Write $\Omega^1 = \bigcup_j Q_j^1$ where the $Q_j^1$’s are nonoverlapping maximal dyadic subcubes of $Q_0$ such that

\[|m_{f^0}(t, Q_j^1)| > 2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y), \quad |m_{f^0}(t, Q_j^1)| \leq 2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y).

(3)

Since $m_{f^0}(t, Q_0) = 0$, $Q_j^1 \neq Q_0$ for any $j$.

Now since $t \geq 1/2$, from (1.10) in [5] it follows that

\[2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y) < |m_{f^0}(t, Q_j^1)| \leq m_{f^0}(t, Q_j^1),

(4)

and therefore by the definition of median

\[|\{x \in Q_j^1 : |f^0(x)| > 2 \inf_{y \in Q_0} M^y_{0,s,Q_0}\}| \geq (1-t)|Q_j^1|.

(5)

When these are summed, we have by (2) that

\[(1-t) \sum_j |Q_j^1| \leq \sum_j |\{x \in Q_j^1 : |f^0(x)| > 2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y)\}|

\[\leq |\{x \in Q_0 : |f^0(x)| > 2 \inf_{y \in Q_0} M^y_{0,s,Q_0} f(y)\}| \leq s|Q_0|,

so that

\[\sum_j |Q_j^1| \leq \frac{s}{1-t}|Q_0|,

(6)

where by the choice of $s$ and $t$, $s/(1-t) < 1$.\)
Let \( \alpha_j = m_{f_0}(t, Q_j^1) \). By Lemma 4.3 in [8], we see that
\[
|m_{f_0}(t, Q_j^1) - m_{f_0}(t, \hat{Q}_j^1)| \leq 10n \inf_{y \in Q_j^1} M_{0, s, Q_j}^2 f(y),
\] (7)
and therefore by (3) and (7)
\[
|\alpha_j| \leq |m_{f_0}(t, Q_j^1) - m_{f_0}(t, \hat{Q}_j^1)| + |m_{f_0}(t, \hat{Q}_j)| \\
\leq 10n \inf_{y \in Q_j^1} M_{0, s, Q_j}^2 f(y) + 2 \inf_{y \in Q_0} M_{0, s, Q_0}^2 f(y). \] (8)

Then the first iteration of the local median oscillation decomposition of \( f \) when \( |E^1| > 0 \) is as follows: for a.e. \( x \in Q_0 \), with \( g^1 = f^0 \mathbb{1}_{Q_0 \setminus \Omega^1} \),
\[
f^0(x) = g^1(x) + \sum_j \alpha_j \mathbb{1}_{Q_j^1}(x) + \sum_j \left( f^0(x) - m_{f_0}(t, Q_j^1) \right) \mathbb{1}_{Q_j^1}(x).
\]
Note that \( g^1 \) has support off \( \Omega^1 \), and clearly for a.e. \( x \in Q_0 \)
\[
|g^1(x)| \leq 2 \inf_{y \in Q_0} M_{0, s, Q_0}^2 f(y).
\]
Now focus on the second sum. Since \( f^0(x) - m_{f_0}(t, Q) = f(x) - m_f(t, Q) \) for all cubes \( Q \) and functions \( f \) supported in \( Q \), this sum equals
\[
\sum_j \left( f(x) - m_f(t, Q_j^1) \right) \mathbb{1}_{Q_j^1}(x).
\]
The idea is to repeat the above argument for each function \( f_j^1 = \left( f - m_f(t, Q_j^1) \right) \mathbb{1}_{Q_j^1} \), and so on.

We now describe the iteration. Assuming that \( \{Q_j^{k-1}\} \) are the dyadic cubes corresponding to the \((k-1)\)st generation of subcubes of \( Q_0 \) obtained as above, let
\[
f_j^{k-1} = \left( f - m_f(t, Q_j^{k-1}) \right) \mathbb{1}_{Q_j^{k-1}},
\]
and
\[
E_j^k = \{ x \in Q_j^{k-1} : f_j^{k-1}(x) > 2 \inf_{y \in Q_j^{k-1}} M_{0, s, Q_j}^2 f(y) \}.
\]
If \( |E_j^k| = 0 \), as in (1), we write \( s_j^k = f_j^{k-1} \) which satisfies
\[
|s_j^k(x)| \leq 2 \inf_{y \in Q_j^{k-1}} M_{0, s, Q_j}^2 f(y) \] (9)
for a.e. \( x \in Q_j^{k-1} \). These are the “s” functions since the decomposition “stops” at \( Q_j^{k-1} \): clearly \( s_j^k \) has its support on \( Q_j^{k-1} \), and \( Q_j^{k-1} \) contains no further subcubes of the decomposition.

If \( |E_j^k| > 0 \), we define
\[
\Omega_j^k = \{ x \in Q_j^{k-1} : m_{\Delta, q_j^{k-1}}^\Delta f_j^{k-1}(x) > 2 \inf_{y \in Q_j^{k-1}} M_{0, s, Q_j}^2 f(y) \}.
\]
Note that the $E_j^k$, and thus the $\Omega_j^k$, are nonoverlapping. Then $|\Omega_j^k| > 0$ as well, and

$$\Omega_j^k = \bigcup_i Q_i^k,$$

where the $Q_i^k$'s are nonoverlapping maximal dyadic subcubes of $Q_j^{k-1}$ such that

$$|m_{f_{j_k}^{-1}}(t, Q_i^k)| > 2 \inf_{y \in Q_i^{k-1}} M_y^2 0_{s, Q_i^{k-1}} f(y), \quad |m_{f_{j_k}^{-1}}(t, \widehat{Q_i^k})| \leq 2 \inf_{y \in Q_i^{k-1}} M_y^2 0_{s, Q_i^{k-1}} f(y). \quad (10)$$

Then define

$$\Omega^k = \bigcup_j \Omega_j^k.$$

Let $\alpha_i^{k,j} = m_{f_{j_k}^{-1}}(t, Q_i^k)$, and note that by (10)

$$|\alpha_i^{k,j}| \leq |m_{f_{j_k}^{-1}}(t, Q_i^k) - m_{f_{j_k}^{-1}}(t, \widehat{Q_i^k})| + |m_{f_{j_k}^{-1}}(t, \widehat{Q_i^k})| \leq 10n \inf_{y \in Q_i^k} M_y^2 0_{s, Q_i^k} f(y) + 2 \inf_{y \in Q_i^{k-1}} M_y^2 0_{s, Q_i^{k-1}} f(y). \quad (11)$$

We then have

$$f_{j_k}^{-1}(x) = g_j^k(x) + \sum_i \alpha_i^{k,j} 1_{Q_i^k}(x) + \sum_i \left( f(x) - m_f(t, Q_i^k) \right) 1_{Q_i^k}(x),$$

for a.e. $x \in Q_j^{k-1}$, where $g_j^k = f_{j_k}^{-1} 1_{Q_j^{k-1} \setminus \Omega_j^k}$ clearly satisfies

$$|g_j^k(x)| \leq 2 \inf_{y \in Q_j^{k-1}} M_y^2 0_{s, Q_j^{k-1}} f(y) \quad (12)$$

for a.e. $x \in Q_j^{k-1}$. These are the “$g$” functions since the decomposition “goes on” or continues, into $Q_j^{k-1}$; $g_j^k$ has support on $Q_j^{k-1}$ away from $\Omega_j^k$, which are the next subcubes in the decomposition.

We separate the $Q_j^{k-1}$ into two families. One family, indexed by $I_1^k$, contains those cubes where the decomposition stops, and the other, indexed by $I_2^k$, where it continues. Specifically, let

$$I_1^k = \{ j : \Omega_j^k \cap Q_j^{k-1} = \emptyset \}, \quad I_2^k = \{ j : \Omega_j^k \cap Q_j^{k-1} \neq \emptyset \}.$$

Now we group the $Q_i^k$ based on which $Q_j^{k-1}$ contains them: if $j \in I_2^k$, let

$$J_j^k = \{ i : Q_i^k \subset Q_j^{k-1} \}.$$

These definitions then give that

$$\Omega_j^k = \bigcup_{i \in J_j^k} Q_i^k.$$

Note that, as in (6),

$$|\Omega_j^k \cap Q_j^{k-1}| = \sum_{i \in J_j^k} |Q_i^k| \leq \frac{s}{1 - t} |Q_j^{k-1}|$$
so that

\[ |\Omega^k| = \sum_j |\Omega_j^k \cap Q_j^{k-1}| \leq \left( \frac{s}{1-t} \right) \sum_j |Q_j^{k-1}| = \left( \frac{s}{1-t} \right) |\Omega^{k-1}| \leq \left( \frac{s}{1-t} \right)^k |Q_0|. \quad (13) \]

The \( k \)th iteration of the local median oscillation decomposition of the function \( f \) is as follows: for a.e. \( x \in Q_0 \),

\[
f(x) - m_f(t, Q_0) = \sum_{v=1}^{k} \left( \sum_{j \in I_1^v} s_j^v + \sum_{j \in I_2^v} g_j^v \right) + \sum_{v=1}^{k} \sum_{j \in I_2^v} \alpha_j^{v,j}(x),
\]

where

\[
\psi^k = \sum_{j \in I_1^k} \sum_{i \in I_j^k} \left( f - m_f(t, Q_i^k) \right) \mathbb{1}_{Q_i^k}.
\]

Since \( \psi^k \) is supported in \( \Omega^k \), by (13) it readily follows that \( \psi^k \to 0 \) a.e. in \( Q_0 \) as \( k \to \infty \), and therefore

\[
f(x) - m_f(t, Q_0) = \sum_{v=1}^{\infty} \left( \sum_{j \in I_1^v} s_j^v + \sum_{j \in I_2^v} g_j^v \right) + \sum_{v=1}^{\infty} \sum_{j \in I_2^v} \alpha_j^{v,j}(x) = S_1(x) + S_2(x),
\]

say.

In order to bound \( |f(x) - m_f(t, Q_0)| \), consider first \( S_1 \). Of course, for all \( v \) and \( j \) the \( s_j^v \)'s have nonoverlapping support. This is also true for the \( g_j^v \)'s. Furthermore, the support of any \( g_j^v \) is nonoverlapping with that of any \( s_j^v \). So for every \( v, j \), and a.e. \( x \in Q_0 \), by (9) and (12)

\[
\left| \sum_{v=1}^{\infty} \left( \sum_{j \in I_1^v} s_j^v + \sum_{j \in I_2^v} g_j^v \right) \right| \leq \max \left\{ \sup_{j \in I_1^v} \left\| f_j^{v-1} \right\|_{L_\infty}, \sup_{j \in I_2^v} \left\| f_j^{v-1} \mathbb{1}_{Q_j^{v-1} \setminus \Omega_j^v}(x) \right\|_{L_\infty} \right\}
\]

\[
\leq \max \left\{ \sup_{j \in I_1^v} \left( 2 \inf_{y \in Q_j^{v-1}} M^2_{0,s,Q_j^{v-1}} f(y) \right), \sup_{j \in I_2^v} \left( 2 \inf_{y \in Q_j^{v-1}} M^2_{0,s,Q_j^{v-1}} f(y) \right) \right\}
\]

\[
\leq 2M^2_{0,s,Q_0} f(x). \quad (14)
\]

Let's now consider \( S_2 \). The summand for \( v = 1 \) is distinguished, so we deal with it separately. By (8) above,

\[
\left| \sum_j \alpha_j^1 \mathbb{1}_{Q_j^1}(x) \right| \leq \sum_j |\alpha_j^1| \mathbb{1}_{Q_j^1}(x)
\]

\[
\leq \sum_j \left( 10n \inf_{y \in Q_j^1} M^2_{0,s,Q_j^1} f(y) + 2 \inf_{y \in Q_0} M^2_{0,s,Q_0} f(y) \right) \mathbb{1}_{Q_j^1}(x)
\]

\[
\leq \sum_j \left( 10n \inf_{y \in Q_j^1} M^2_{0,s,Q_j^1} f(y) \right) \mathbb{1}_{Q_j^1}(x) + 2 \inf_{y \in Q_0} M^2_{0,s,Q_0} f(y). \quad (15)
\]
Thus the conclusion holds.

Theorem 2. For any weight \( w \), \( 0 < s < 1/2 \), \( 1/2 \leq t < 1 - s \), \( 0 < \delta \leq 1 \), and \( f \) measurable on a cube \( Q_0 \subset \mathbb{R}^n \),

\[
\int_{Q_0} |f(x) - m_f(t, Q_0)| w(x) dx \leq c_{n,s,t} \int_{\mathbb{R}^n} \left( M^{\frac{s}{s}}_{0,s,Q_0} f(x) \right)^{\delta} M \left( \left( M^{\frac{s}{s}}_{0,s,Q_0} f(\cdot) \right)^{1-\delta} w(\cdot) \right)(x) dx.
\]
If \( f \) is measurable on \( \mathbb{R}^n \) such that \( m_f(t, Q_0) \to 0 \) as \( Q_0 \to \mathbb{R}^n \), then
\[
\int_{\mathbb{R}^n} |f(x)| w(x) dx \leq c_{n,s,t} \int_{\mathbb{R}^n} \left( M_{0,s}^2 f(x) \right)^\delta M \left( \left( M_{0,s}^2 f(\cdot) \right)^{1-\delta} w(\cdot) \right) (x) dx.
\]

**Proof.** Applying Theorem 1, we have
\[
\int_{Q_0} |f(x) - m_f(t, Q_0)| w(x) dx \leq 4 \int_{Q_0} M_{0,s, Q_0}^2 f(x) w(x) dx
\]
\[
+ \sum_{v=1}^{\infty} \sum_{j \in I_2} \left( 10 n \inf_{y \in Q_j} M_{0,s, Q_j}^2 f(y) + 2 \inf_{y \in Q_j} M_{0,s, Q_j}^2 f(y) \right) \int_{Q_j} w(x) dx.
\]

Now it easily follows that
\[
\int_{Q_0} M_{0,s, Q_0}^2 f(x) w(x) dx \leq \int_{\mathbb{R}^n} \left( M_{0,s, Q_0}^2 f(x) \right)^\delta M \left( \left( M_{0,s, Q_0}^2 f(\cdot) \right)^{1-\delta} w(\cdot) \right) (x) dx.
\]

By construction, the \( Q_j^v \) are nonoverlapping over fixed \( v \), but not over all \( v \); indeed, the \( Q_j^v \) are subcubes of some \( Q_j^{v-1} \). So we define \( F_j^y = Q_j^y \setminus Q_j^{v+1} \), which are pairwise disjoint over all \( v \) and \( j \). And since
\[
|Q_j^{v+1} \cap Q_j^v| \leq \frac{s}{1-t} |Q_j^v|,
\]
we know that
\[
|F_j^v| \geq \left( 1 - \frac{s}{1-t} \right) |Q_j^v| = c_{s,t} |Q_j^v|.
\]

We can then compute
\[
\inf_{y \in Q_j^v} M_{0,s, Q_j^v}^2 f(y) \int_{Q_j^v} w(x) dx \leq c_{s,t} \frac{|F_j^v|}{|Q_j^v|} \inf_{y \in Q_j^v} M_{0,s, Q_0}^2 f(y) \int_{Q_j^v} w(x) dx
\]
\[
= c_{s,t} \left( \inf_{y \in Q_j^v} M_{0,s, Q_0}^2 f(y) \right) \frac{|F_j^v|}{|Q_j^v|} \int_{Q_j^v} \left( M_{0,s, Q_0}^2 f(x) \right)^{1-\delta} w(x) dx
\]
\[
\leq c_{s,t} \left( \int_{F_j^v} \left( M_{0,s, Q_0}^2 f(x) \right)^{\delta} dx \right) \inf_{y \in F_j^v} M \left( \left( M_{0,s, Q_0}^2 f(\cdot) \right)^{1-\delta} w(\cdot) \right) (y).
\]

Summing, we see that
\[
\sum_{v=1}^{\infty} \sum_{j \in I_2} \inf_{y \in Q_j^v} M_{0,s, Q_j^v}^2 f(y) \int_{Q_j^v} w(x) dx \leq c_{s,t} \frac{1}{|Q_j^v|} \int_{Q_j^v} \left( M_{0,s, Q_0}^2 f(x) \right)^{1-\delta} w(x) dx.
\]

Similarly for \( \sum_{v=1}^{\infty} \sum_{j \in I_2} \inf_{y \in Q_j^v} M_{0,s, Q_j^v}^2 f(y) \).

Lerner’s application of Theorem 2 is to Calderón-Zygmund singular integral operators via the inequality \( M_{s}^2(Tf)(x) \leq cMf(x) \). Similar results follow for multipliers using inequalities derived in [10] (e.g., p.171).

A different estimate is needed in order for a local median oscillation decomposition to be applicable in Lerner’s proof of the \( A_2 \) conjecture [7]. This new estimate leads to a similar local median oscillation decomposition, which we develop below.
Theorem 3. Let $f$ be a measurable function on a fixed cube $Q_0 \subset \mathbb{R}^n$, $0 < s < 1/2$, and $1/2 \leq t < 1 - s$. Then there exists a (possibly empty) collection of subcubes $Q_j^v \in D(Q_0)$ and a family of collections $\{I^v_j\}_v$ such that

(i) For a.e. $x \in Q_0$,
$$|f(x) - m_f(t, Q_0)| \leq 8M^s_{0,s,Q_0}f(x) + \sum_{v=1}^{\infty} \sum_j \left(m_{[f-m_f(t,\hat{Q}_j^v)]}(1 - (1-t)/2^n, \hat{Q}_j^v) + m_{[f-m_f(t,Q^v_j)]}(t, Q^v_j)\right) \mathbb{1}_{Q^v_j}(x).$$

(ii) For fixed $v$, $\{Q^v_j\}$ are pairwise disjoint families.

(iii) If $\Omega^v = \bigcup_j Q^v_j$, then $\Omega^{v+1} \subset \Omega^v$.

(iv) For all $j$, $|\Omega^{v+1} \cap Q^v_j| \leq \frac{s}{1-t}|Q^v_j|$.

Proof. We follow the proof of Theorem 1 in form, with only a few definitional changes. First note that for any cube $Q$,
$$m_{[f-m_f(t,Q)]}(t, Q) \leq 4 \inf_{y \in Q} M^{\sharp}_{0,s,Q}f(y).$$

To see this, from (1.6), (1.7), (1.5), and (4.3) in [3],
$$m_{[f-m_f(t,Q)]}(t, Q) \leq m_{[f-m_f(1-s,Q)]+m_{f(1-s,Q)}-m_f(t,Q)]}(t, Q) \leq 2m_{[f-m_f(1-s,Q)]}(t, Q) \leq 2m_{[f-m_f(1-s,Q)]}(1-s, Q) \leq 4 \inf_{y \in Q} M^{\sharp}_{0,s,Q}f(y).$$

We define $E^1 = \{x \in Q_0 : |f(x) - m_f(t, Q_0)| > m_{[f-m_f(t,Q_0)]}(t, Q_0)\}$. If $|E^1| = 0$, the decomposition halts, just as in Theorem 1. So we suppose $|E^1| > 0$. We then define
$$\Omega^1 = \{x \in Q_0 : m^{\Delta}_{Q_0,f^0}(f^0) > m_{[f-m_f(t,Q_0)]}(t, Q_0)\}.$$

Proceeding as above, we have that $\Omega^1 = \bigcup_j Q^1_j$ so that (as in (3))
$$|m_{f_0(t,Q^1_j)}| > m_{f_0}(t, Q_0), \ |m_f(t, \hat{Q}^1_j)| \leq m_{f_0}(t, Q_0).$$

Furthermore, we also have that
$$\sum_j |Q^1_j| \leq \frac{s}{1-t}|Q_0|.$$

Before proceeding, observe that
$$m_f(t, Q) \leq m_f \left(1 - (1-t)/2^n, \hat{Q}\right).$$

To see this,
$$|\{y \in \hat{Q} : f(y) \geq m_f(t, Q)\}| \geq |\{y \in Q : f(y) \geq m_f(t, Q)\}| \geq (1-t)|Q| = \frac{1-t}{2^n} |\hat{Q}|,$$
so taking complements in $\hat{Q}$ we have

$$|\{y \in \hat{Q} : f(y) < m_f(t, Q)}| \leq \left(1 - \frac{1 - t}{2^n}\right)|\hat{Q}|.$$

Let $\alpha_j^1 = m_{f_0}(t, Q_j^1)$. By (19) and (21) we have

$$|\alpha_j^1| \leq |m_{f_0}(t, Q_j^1) - m_{f_0}(t, \hat{Q}_j^1)| + |m_{f_0}(t, \hat{Q}_j^1)|
\leq m_{|f_0 - m_{f_0}(t, \hat{Q}_j^1)}(t, Q_j^1) + m_{f_0}(t, Q_0)
\leq m_{|f - m_f(t, \hat{Q}_j^1)}(1 - (1-t)/2^n, \hat{Q}_j^1) + m_{f_0}(t, Q_0).$$

This gives the first iteration of the local median oscillation decomposition of $f$ when $|E^1| > 0$: for a.e. $x \in Q_0$, with $g^1 = f^0 1_{Q_0 \setminus \Omega^1}$,

$$f^0(x) = g^1(x) + \sum_j \alpha_j^1 1_{Q_j^1}(x) + \sum_j (f^0(x) - m_{f_0}(t, Q_j^1)) 1_{Q_j^1}(x).$$

Clearly by (18)

$$|g^1(x)| \leq m_{|f - m_f(t, Q_0)}(t, Q_0) \leq 4 \inf_{y \in Q_0} M_{0, s, Q_0}^f f(y) \leq 4 M_{0, s, Q_0}^f f(x)$$
a.e. on $Q_0 \setminus \Omega^1$.

Proceeding as above, we assume that $\{Q_k^{k-1}\}$ are the dyadic cubes corresponding to the $(k-1)$st generation of subcubes of $Q_0$ obtained as above. Let

$$f_j^{k-1} = \left(f - m_f(t, Q_j^{k-1})\right) 1_{Q_j^{k-1}}$$

and

$$E_j^k = \{x \in Q_j^{k-1} : f_j^{k-1}(x) > m_{|f_j^{k-1}|}(t, Q_j^{k-1})\}.$$ 

If $|E_j^k| = 0$, we write

$$s_j^k = f_j^{k-1},$$

so that by (18), for a.e. $x \in Q_j^{k-1},$

$$|s_j^k(x)| \leq m_{|f_j^{k-1}|}(t, Q_j^{k-1}) \leq 4 \inf_{y \in Q_j^{k-1}} M_{0, s, Q_j^{k-1}}^f f(y) \leq 4 M_{0, s, Q_0}^f f(x).$$

(23)

If $|E_j^k| > 0$, we define

$$\Omega_j^k = \{x \in Q_j^{k-1} : m_{Q_j^{k-1}}^{t, \Delta} f_j^{k-1}(x) > m_{|f_j^{k-1}|}(t, Q_j^{k-1})\}.$$ 

Then $|\Omega_j^k| > 0$ and

$$\Omega_j^k = \bigcup_i Q_i^k$$
where the $Q^k_j$'s are nonoverlapping maximal dyadic subcubes of $Q^{k-1}_j$ such that

$$|m_{f^k_j(t, Q^k_i)}| > m_{f^{k-1}_j(t, Q^{k-1}_j)}, \quad |m_{f^k_j(t, \hat{Q}^k_i)}| \leq m_{f^{k-1}_j(t, Q^{k-1}_j)}. \tag{24}$$

Then define

$$\Omega^k = \bigcup_j \Omega^k_j.$$ 

Let $\alpha^{k,j}_i = m_{f^{k-1}_j(t, Q^k_i)}$, and just as above

$$|\alpha^{k,j}_i| \leq m_{|f-m_f(t, \hat{Q}^k_i)|} \left( 1 - (1-t)/2^n, \hat{Q}^k_i \right) + m_{f^{k-1}_j(t, Q^{k-1}_j)}. \tag{25}$$

We then have for a.e. $x \in Q^{k-1}_j$,

$$f^{k-1}_j(x) = g^k_j(x) + \sum_i \alpha^{k,j}_i 1_{Q^k_i}(x) + \sum_i \left( f(x) - m_f(t, Q^k_i) \right) 1_{Q^k_i}(x),$$

where, by (24) and (18), $g^k_j = f^{k-1}_j 1_{Q^{k-1}_j \setminus \Omega^k_j}$ satisfies

$$|g^k_j(x)| \leq m_{f^{k-1}_j(t, Q^{k-1}_j)} \leq 4 \inf_{y \in Q^{k-1}_j} M^{f^2}_{0,s, \alpha^{k-1}_j} f(y) \leq 4M^2_{0,s, Q_0} f(x) \tag{26}$$

for a.e. $x \in Q^{k-1}_j \setminus \Omega^k_j$. As in (13) we have that

$$|\Omega^k| \leq \left( \frac{8}{1-t} \right)^k |Q_0|. \tag{27}$$

Furthermore, we have the $k$th iteration of the local median oscillation decomposition of $f$: for a.e. $x \in Q_0$,

$$f(x) - m_f(t, Q_0) = \sum_{v=1}^k \left( \sum_{j \in I^v_i} s^v_j + \sum_{j \in I^v_{i+}} g^v_j \right) + \sum_{v=1}^k \sum_{j \in I^v_i} \sum_{i \in I^v_j} \alpha^{v,j}_i 1_{Q^v_i}(x) + \psi^k(x),$$

where

$$\psi^k = \sum_{j \in I^v_i} \sum_{i \in I^v_j} \left( f - m_f(t, Q^v_i) \right) 1_{Q^v_i}.$$ 

By (27), $\psi^k \to 0$ a.e. in $Q_0$ so that

$$f(x) - m_f(t, Q_0) = \sum_{v=1}^\infty \left( \sum_{j \in I^v_i} s^v_j + \sum_{j \in I^v_{i+}} g^v_j \right) + \sum_{v=1}^\infty \sum_{j \in I^v_i} \sum_{i \in I^v_j} \alpha^{v,j}_i 1_{Q^v_i}(x)$$

$$= S_1(x) + S_2(x),$$

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say. As for $S_1$, by the same reasoning as in Theorem 1, (23) and (26) give that for a.e. $x \in Q_0$,

$$\left| \sum_{j \in I_N^0} s_j^0 + \sum_{j \in I_N^1} g_j^0 \right| \leq \max \left\{ \max_{j \in I_N^0} \| f_j^{n-1} \|_{L^\infty}, \max_{j \in I_N^1} \| f_j^{n-1} \|_{L^\infty} \right\}
\leq 4M_{0,s,Q_0}^f(x). \tag{28}$$

Now for $S_2$ and the $v = 1$ summand, by (22),

$$\left| \sum_{j} \alpha_j^1 \mathbb{1}_{Q_j^1}(x) \right| \leq \sum_{j} m_{|f-m_f(t,Q_j^1)|} \left( 1 - (1-t)/2^n, \tilde{Q}_j^1 \right) \mathbb{1}_{Q_j^1}(x) + m_{|f_0|(t,Q_0)} \mathbb{1}_{Q_0}. \tag{29}$$

As for the other terms, by (25) we have

$$\left| \sum_{v=2}^{\infty} \sum_{j \in I_N^0} \sum_{i \in J^v_j} \alpha_i^{v,j} \mathbb{1}_{Q_i^v}(x) \right| \leq \sum_{v=2}^{\infty} \sum_{j \in I_N^0} \sum_{i \in J^v_j} m_{|f-m_f(t,Q_i^v)|} \left( 1 - (1-t)/2^n, \tilde{Q}_i^v \right) \mathbb{1}_{Q_i^v}(x)
+ \sum_{v=2}^{\infty} \sum_{j \in I_N^0} m_{|f_j^{v-1}|}(t,Q_j^{v-1}) \mathbb{1}_{Q_j^{v-1}}(x). \tag{30}$$

Combining (29) and (30), and by the same reasoning as in Theorem 1,

$$\left| \sum_{v=1}^{\infty} \sum_{j \in I_N^0} \sum_{i \in J^v_j} \alpha_i^{v,j} \mathbb{1}_{Q_i^v}(x) \right| \leq \sum_{v=1}^{\infty} \sum_{j \in I_N^0} \sum_{i \in J^v_j} m_{|f-m_f(t,Q_i^v)|} \left( 1 - (1-t)/2^n, \tilde{Q}_i^v \right) \mathbb{1}_{Q_i^v}(x)
+ \sum_{v=1}^{\infty} \sum_{j \in I_N^0} m_{|f-j^{v-1}|}(t,Q_j^{v-1}) \mathbb{1}_{Q_j^{v-1}}(x)
\leq \sum_{v=1}^{\infty} \sum_{j \in I_N^0} \left( m_{|f-m_f(t,Q_j^v)|} \left( 1 - (1-t)/2^n, \tilde{Q}_j^v \right) + m_{|f-m_f(t,Q_j^v)|}(t,Q_j^v) \right) \mathbb{1}_{Q_j^v}
+ m_{|f-m_f(t,Q_0)|}(t,Q_0). \tag{31}$$

Thus (18), (28), and (31) together give

$$|f - m_f(t,Q_0)| \leq 8M_{0,s,Q_0}^f(x) + \sum_{v=1}^{\infty} \sum_{j \in I_N^0} \left( m_{|f-m_f(t,Q_j^v)|} \left( 1 - (1-t)/2^n, \tilde{Q}_j^v \right) + m_{|f-m_f(t,Q_j^v)|}(t,Q_j^v) \right) \mathbb{1}_{Q_j^v}$$

for a.e. $x \in Q_0$.

\[ \square \]

**References**

[1] D. Cruz-Uribe, J. M. Martell, and C. Perez, *Sharp weighted estimates for approximating dyadic operators*, Electron. Res. Announc. Math. Sci. **17** (2010), 12–19.
[2] N. Fujii, *A condition for the two-weight norm inequality for singular integral operators*, Studia Math. **98** (1991), no. 3, 175–190.

[3] J.B. Garnett and P.W. Jones, *BMO from dyadic BMO*, Pacific J. Math. **99** (1982), no. 2, 351–371.

[4] B. Jawerth and A. Torchinsky, *Local sharp maximal functions*, J. Approx. Theory (1985), no. 3, 231–270.

[5] A. Lerner, *A pointwise estimate for the local sharp maximal function with applications to singular integrals*, Bull. Lond. Math. Soc. **42** (2010), no. 5, 843–856.

[6] ———, *A “Local Mean Oscillation” Decomposition and Some of its Applications*, u.math.biu.ac.il/lernera/paseky.pdf (2012).

[7] ———, *A Simple Proof of the $A_2$ Conjecture*, arxiv.org/abs/1202.2824 (2012).

[8] J. Poelhuis and A. Torchinsky, *Medians, Continuity, and Oscillation*, arxiv.org/abs/1207.2710 (2012).

[9] J.-O. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J. **28** (1979), no. 3, 511–544.

[10] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Mathematics, vol. 1381, Springer-Verlag, Berlin, 1989.