COMMON COUPLED FIXED POINT THEOREM UNDER GENERALIZED MIZOGUCHI-TAKAHASHI CONTRACTION FOR HYBRID PAIR OF MAPPINGS GENERALIZED MIZOGUCHI-TAKAHASHI CONTRACTION

BHAVANA DESHPANDE\textsuperscript{a} AND AMRISH HANDA\textsuperscript{b,∗}

Abstract. We establish a common coupled fixed point theorem for hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction on a noncomplete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. An example is also given to validate our results. We improve, extend and generalize several known results.

1. Introduction

Let \((X, d)\) be a metric space. We denote by \(2^X\) the class of all nonempty subsets of \(X\), by \(CL(X)\) the class of all nonempty closed subsets of \(X\), by \(CB(X)\) the class of all nonempty closed bounded subsets of \(X\) and by \(K(X)\) the class of all nonempty compact subsets of \(X\). A functional \(H : CL(X) \times CL(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}\) is said to be the Pompeiu-Hausdorff generalized metric induced by \(d\) is given by

\[
H(A, B) = \left\{ \begin{array}{ll}
\max \{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}, & \text{if maximum exists,} \\
+\infty, & \text{otherwise,}
\end{array} \right.
\]

for all \(A, B \in CB(X)\), where \(D(x, A) = \inf_{a \in A} d(x, a)\) denote the distance from \(x\) to \(A \subset X\). For simplicity, if \(x \in X\), we denote \(g(x)\) by \(gx\).

Markin [23] initiated to study the existence of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric which was further studied by many authors under different contractive conditions. The theory of multivalued mappings has application in control theory, convex optimization, differential inclusions and economics.
Bhaskar and Lakshmikantham [6] established some coupled fixed point theorems and applied these results to study the existence and uniqueness of solution for periodic boundary value problems, which were later extended by Lakshmikantham and Cirić [19]. For more details, see [5, 7, 8, 9, 10, 16, 17, 21, 22, 27, 30].

Samet et al. [28] claimed that most of the coupled fixed point theorems for single valued mappings on ordered metric spaces are consequences of well-known fixed point theorems.

The concepts related to coupled fixed point theory for multivalued mappings were extended by Abbas et al. [2] and obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces. Very few researcher gave attention to coupled fixed point problems for hybrid pair of mappings including [1, 2, 11, 12, 13, 14, 15, 20, 29].

In [2], Abbas et al. introduced the following for multivalued mappings:

**Definition 1.1.** Let $X$ be a nonempty set, $F : X \times X \to 2^X$ (a collection of all nonempty subsets of $X$) and $g$ be a self-mapping on $X$. An element $(x, y) \in X \times X$ is called

1. a coupled fixed point of $F$ if $x \in F(x, y)$ and $y \in F(y, x)$.
2. a coupled coincidence point of hybrid pair $\{F, g\}$ if $gx \in F(x, y)$ and $gy \in F(y, x)$.
3. a common coupled fixed point of hybrid pair $\{F, g\}$ if $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$.

We denote the set of coupled coincidence points of mappings $F$ and $g$ by $C(F, g)$. Note that if $(x, y) \in C(F, g)$, then $(y, x)$ is also in $C(F, g)$.

**Definition 1.2.** Let $F : X \times X \to 2^X$ be a multivalued mapping and $g$ be a self-mapping on $X$. The hybrid pair $\{F, g\}$ is called $w-$compatible if $gF(x, y) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.

**Definition 1.3.** Let $F : X \times X \to 2^X$ be a multivalued mapping and $g$ be a self-mapping on $X$. The mapping $g$ is called $F-$weakly commuting at some point $(x, y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.

**Lemma 1.4** ([26]). Let $(X, d)$ be a metric space. Then, for each $a \in X$ and $B \in K(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b \in B} d(a, b)$. 
Nadler [25] extended the famous Banach Contraction Principle [4] from single-valued mapping to multivalued mapping. Mizoguchi and Takahashi [24] proved the following generalization of Nadler’s fixed point theorem for weak contraction:

**Theorem 1.5.** Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Assume that

$$H(Tx, Ty) \leq \psi(d(x, y))d(x, y),$$

for all $x, y \in X$, where $\psi$ is a function from $[0, \infty)$ into $[0, 1)$ satisfying

$$\limsup_{s \to t^+} \psi(s) < 1$$

for all $t \geq 0$. Then $T$ has a fixed point.

Suzuki [31] gave its very simple proof. Amini-Harandi and O’Regan [3] obtained a generalization of Mizoguchi and Takahashi’s fixed point theorem.

In [8], Ciric et al. proved coupled fixed point theorems for mixed monotone mappings satisfying a generalized Mizoguchi-Takahashi’s condition in the setting of ordered metric spaces. Main results of Ciric et al. [8] extended and generalized the results of Bhaskar and Lakshmikantham [6], Du [17] and Harjani et al. [18].

In this paper, we prove a common coupled fixed point theorem for hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction on a noncomplete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. We improve, extend and generalize the results of Amini-Harandi and O’Regan [3], Bhaskar and Lakshmikantham [6], Ciric et al. [8], Du [17], Harjani et al. [18] and Mizoguchi and Takahashi [24]. The effectiveness of our generalization is demonstrated with the help of an example.

2. **Main Results**

Let $\Phi$ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

(i) $\varphi$ is non-decreasing,

(ii) $\varphi(t) = 0 \iff t = 0$,

(iii) $\limsup_{t \to 0^+} \frac{t}{\varphi(t)} < \infty$.

Let $\Psi$ denote the set of all functions $\psi : [0, +\infty) \to [0, 1)$ which satisfies

$$\lim_{r \to t^+} \psi(r) < 1$$

for all $t \geq 0$. For example, if $\varphi(t) = \ln(t + 1)$ and $\psi(t) = \frac{\varphi(t)}{t}$.

Obviously, then $\varphi \in \Phi$ and $\psi \in \Psi$, because $\varphi$ is non-decreasing, positive in $(0, +\infty)$,
\[ \varphi(0) = 0 \text{ and } \limsup_{s \to +1} \frac{s}{\varphi(s)} = 1 < \infty. \] Also, \[ \lim_{r \to +1} \varphi(r) = \lim_{r \to +1} \frac{\varphi(r)}{r} = \lim_{t \to 0} \frac{\ln(r+1)}{t} = \lim_{t \to 0} \frac{\ln(t+1)}{t} = 1. \]

**Theorem 2.1.** Let \( (X, d) \) be a metric space, \( F : X \times X \to K(X) \) and \( g : X \to X \) be two mappings. Assume that there exist some \( z \in X \) such that for all \( x, y, u, v \) \( \forall \phi \leq H(F(x, y), F(u, v)) \)

\[ \leq \psi(\varphi(\max\{d(gx, gu), d(gy, gv)\})) \varphi(\max\{d(gx, gu), d(gy, gv)\}), \]

for all \( x, y, u, v \in X \). Furthermore assume that \( F(X \times X) \subseteq g(X) \) and \( g(X) \) is a complete subset of \( X \). Then \( F \) and \( g \) have a coupled coincidence point. Moreover, \( F \) and \( g \) have a common coupled fixed point, if one of the following conditions holds:

(a) \( F \) and \( g \) are \( w \)-compatible. \( \lim_{n \to \infty} g^n x = u \) and \( \lim_{n \to \infty} g^n y = v \) for some \( (x, y) \in C(F, g) \) and for some \( u, v \in X \) and \( g \) is continuous at \( u \) and \( v \).

(b) \( g \) is \( F \)-weakly commuting for some \( (x, y) \in C(F, g) \) and \( gx \) and \( gy \) are fixed points of \( g \), that is, \( g^2 x = gx \) and \( g^2 y = gy \).

(c) \( g \) is continuous at \( x \) and \( y \). \( \lim_{n \to \infty} g^n u = x \) and \( \lim_{n \to \infty} g^n v = y \) for some \( (x, y) \in C(F, g) \) and for some \( u, v \in X \).

(d) \( g(C(F, g)) \) is a singleton subset of \( C(F, g) \).

**Proof.** Let \( x_0, y_0 \in X \) be arbitrary. Then \( F(x_0, y_0) \) and \( F(y_0, x_0) \) are well defined. Choose \( gx_1 \in F(x_0, y_0) \) and \( gy_1 \in F(y_0, x_0) \), because \( F(X \times X) \subseteq g(X) \). Since \( F : X \times X \to K(X) \), therefore by Lemma 1.4, there exist \( z_1 \in F(x_1, y_1) \) and \( z_2 \in F(y_1, x_1) \) such that

\[ d(gx_1, z_1) \leq H(F(x_0, y_0), F(x_1, y_1)), \]
\[ d(gy_1, z_2) \leq H(F(y_0, x_0), F(y_1, x_1)). \]

Since \( F(X \times X) \subseteq g(X) \), there exist \( x_2, y_2 \in X \) such that \( z_1 = gx_2 \) and \( z_2 = gy_2 \). Thus

\[ d(gx_1, gx_2) \leq H(F(x_0, y_0), F(x_1, y_1)), \]
\[ d(gy_1, gy_2) \leq H(F(y_0, x_0), F(y_1, x_1)). \]

Continuing this process, we obtain sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that for all \( n \in \mathbb{N} \), we have \( gx_{n+1} \in F(x_n, y_n) \) and \( gy_{n+1} \in F(y_n, x_n) \) such that

\[ d(gx_n, gx_{n+1}) \leq H(F(x_{n-1}, y_{n-1}), F(x_n, y_n)), \]
\[ d(gy_n, gy_{n+1}) \leq H(F(y_{n-1}, x_{n-1}), F(y_n, x_n)). \]
which, by $(i_\varphi)$ and (2.1), implies
\[
\varphi (d(gx_n, gx_{n+1})) \\
\leq \varphi (H(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\
\leq \psi (\varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\})) \\
\times \varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}),
\]
which, by the fact that $\psi < 1$, implies
\[
\varphi (d(gx_n, gx_{n+1})) \leq \varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}).
\]
Similarly
\[
\varphi (d(gy_n, gy_{n+1})) \leq \varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}).
\]
Combining (2.2) and (2.3), we get
\[
\max\{\varphi (d(gx_n, gx_{n+1})), \varphi (d(gy_n, gy_{n+1}))\} \\
\leq \varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}).
\]
Since $\varphi$ is non-decreasing, it follows that
\[
\varphi (\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}) \\
\leq \varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}), \text{ for all } n \geq 0.
\]
Now (2.4) shows that $\{\varphi (\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\})\}$ is a non-increasing sequence. Therefore, there exists some $\delta \geq 0$ such that
\[
\lim_{n \to \infty} \varphi (\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}) = \delta.
\]
Since $\psi \in \Psi$, we have $\lim_{r \to \delta} \psi(r) < 1$ and $\psi(\delta) < 1$. Then there exists $\alpha \in [0, 1)$ and $\varepsilon > 0$ such that $\psi(r) \leq \alpha$ for all $r \in [\delta, \delta + \varepsilon)$. From (2.5), we can take $n_0 \geq 0$ such that $\delta \leq \varphi (\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}) \leq \delta + \varepsilon$ for all $n \geq n_0$. Then from (2.1), for all $n \geq n_0$, we have
\[
\varphi (d(gx_n, gx_{n+1})) \\
\leq \psi (\varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\})) \\
\times \varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) \\
\leq \alpha \varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}).
\]
Thus, for all $n \geq n_0$, we have
\[
\varphi (d(gx_n, gx_{n+1})) \leq \alpha \varphi (\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}).
\]
Similarly, for all \( n \geq n_0 \), we have

\[
\varphi(d(gy_n, gy_{n+1})) \leq \alpha \varphi\left(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}\right).
\]

Combining (2.6) and (2.7), for all \( n \geq n_0 \), we get

\[
\max\{\varphi(d(gx_n, gx_{n+1})), \varphi(d(gy_n, gy_{n+1}))\} \\
\leq \alpha \varphi\left(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}\right).
\]

Since \( \varphi \) is non-decreasing, it follows that, for all \( n \geq n_0 \),

\[
\varphi\left(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\right) \\
\leq \alpha \varphi\left(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}\right).
\]

Letting \( n \to \infty \) in (2.8) and using (2.5), we obtain that \( \delta \leq \alpha \delta \). Since \( \alpha \in [0, 1) \), therefore \( \delta = 0 \). Thus

\[
\lim_{n \to \infty} \varphi\left(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\right) = 0.
\]

Since \{\varphi\left(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\right)\} is a non-increasing sequence and \( \varphi \) is non-decreasing, then \{\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\} is also a non-increasing sequence of positive numbers. This implies that there exists \( \theta \geq 0 \) such that

\[
\lim_{n \to \infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} = \theta.
\]

Since \( \varphi \) is non-decreasing, we have

\[
\varphi\left(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\right) \geq \varphi(\theta).
\]

Letting \( n \to \infty \) in this inequality, by using (2.9), we get \( 0 \geq \varphi(\theta) \), which, by (i,\( \varphi \)), implies that \( \theta = 0 \). Thus, by (2.10), we get

\[
\lim_{n \to \infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} = 0.
\]

Suppose that \( \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} = 0 \), for some \( n \geq 0 \). Then, we have \( d(gx_n, gx_{n+1}) = 0 \) and \( d(gy_n, gy_{n+1}) = 0 \) which implies that \( gx_n = gx_{n+1} \in F(x_n, y_n) \) and \( gy_n = gy_{n+1} \in F(y_n, x_n) \), that is, \( (x_n, y_n) \) is a coupled coincidence point of \( F \) and \( g \). Now, suppose that \( \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} \neq 0 \), for all \( n \geq 0 \). Denote

\[
a_n = \varphi\left(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\right), \text{ for all } n \geq 0.
\]

From (2.8), we have

\[
a_n \leq \alpha a_{n-1}, \text{ for all } n \geq n_0.
\]
Then, we have

$$\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{n_0} a_n + \sum_{n=n_0+1}^{\infty} a^{n-n_0}a_{n_0} < \infty.$$  

On the other hand, by (iii), we have

$$\limsup_{n \to \infty} \max \left\{ d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}) \right\} < \infty.$$  

Thus, by (2.12) and (2.13), we have

$$\sum_{n=0}^{\infty} \max \left\{ d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}) \right\} < \infty.$$  

It means that \( \{gx_n\}_{n=0}^{\infty} \) and \( \{gy_n\}_{n=0}^{\infty} \) are Cauchy sequences in \( g(X) \). Since \( g(X) \) is complete, therefore there exist \( x, y \in X \) such that

$$\lim_{n \to \infty} gx_n = gx \quad \text{and} \quad \lim_{n \to \infty} gy_n = gy.$$  

Now, since \( gx_{n+1} \in F(x_n, y_n) \) and \( gy_{n+1} \in F(y_n, x_n) \), therefore by using condition (2.1) and (\( i. \)), we get

$$\varphi (D(gx_{n+1}, F(x, y))) \leq \varphi (H(F(x_n, y_n), F(x, y))) \leq \psi (\varphi (\max \{d(gx_n, gx), d(gy_n, gy)\})) \times \varphi (\max \{d(gx_n, gx), d(gy_n, gy)\}) \leq \varphi (\max \{d(gx_n, gx), d(gy_n, gy)\}).$$  

Since \( \varphi \) is non-decreasing, we have

$$D(gx_{n+1}, F(x, y)) \leq \max \{d(gx_n, gx), d(gy_n, gy)\}.$$  

Letting \( n \to \infty \) in (2.15), by using (2.14), we obtain

$$D(gx, F(x, y)) = 0.$$  

Similarly, we have

$$D(gy, F(y, x)) = 0.$$  

which implies that

$$gx \in F(x, y) \quad \text{and} \quad gy \in F(y, x),$$

that is, \( (x, y) \) is a coupled coincidence point of \( F \) and \( g \). Hence \( C(F, g) \) is nonempty.

Suppose now that \( (a) \) holds. Assume that for some \( (x, y) \in C(F, g) \),

$$\lim_{n \to \infty} g^n x = u \quad \text{and} \quad \lim_{n \to \infty} g^n y = v,$$

where \( u, v \in X \). Since \( g \) is continuous at \( u \) and \( v \), we have, by (2.16), that \( u \) and \( v \) are fixed points of \( g \), that is,

$$gu = u \quad \text{and} \quad gv = v.$$
As $F$ and $g$ are $w$–compatible, so
\[(g^n x, g^n y) \in C(F, g), \text{ for all } n \geq 1,\]
that is,
\[(2.18) \quad g^n x \in F(g^{n-1} x, g^{n-1} y) \text{ and } g^n y \in F(g^{n-1} y, g^{n-1} x), \text{ for all } n \geq 1.\]

Now, by using (2.1) and (2.18), we obtain
\[
\varphi(D(g^n x, F(u, v))) \\
\leq \varphi(H(F(g^{n-1} x, g^{n-1} y), F(u, v))) \\
\leq \psi(\varphi(\max\{d(g^n x, gu), d(g^n y, gv)\})) \\
\times \varphi(\max\{d(g^n x, gu), d(g^n y, gv)\}) \\
\leq \varphi(\max\{d(g^n x, gu), d(g^n y, gv)\}).
\]

Since $\varphi$ is non-decreasing, we have
\[D(g^n x, F(u, v)) \leq \max\{d(g^n x, gu), d(g^n y, gv)\}.\]

On taking limit as $n \to \infty$ in the above inequality, by using (2.16) and (2.17), we get
\[D(gu, F(u, v)) = 0. \text{ Similarly } D(gv, F(v, u)) = 0,\]
which implies that
\[(2.19) \quad gu \in F(u, v) \text{ and } gv \in F(v, u).\]

Now, from (2.17) and (2.19), we have
\[u = gu \in F(u, v) \text{ and } v = gv \in F(v, u),\]
that is, $(u, v)$ is a common coupled fixed point of $F$ and $g$.

Suppose now that (b) holds. Assume that for some $(x, y) \in C(F, g)$, $g$ is $F$–weakly commuting, that is $g^2 x \in F(gx, gy)$, $g^2 y \in F(gy, gx)$ and $g^2 x = gx$, $g^2 y = gy$. Thus $gx = g^2 x \in F(gx, gy)$ and $gy = g^2 y \in F(gy, gx)$, that is, $(gx, gy)$ is a common coupled fixed point of $F$ and $g$.

Suppose now that (c) holds. Assume that for some $(x, y) \in C(F, g)$ and for some $u, v \in X,$
\[\lim_{n \to \infty} g^n u = x \text{ and } \lim_{n \to \infty} g^n v = y.\]

Since $g$ is continuous at $x$ and $y$, then $x$ and $y$ are fixed points of $g$, that is,
\[gx = x \text{ and } gy = y.\]
Since \((x, y) \in C(F, g)\), so we obtain

\[ x = gx \in F(x, y) \text{ and } y = gy \in F(y, x), \]

that is, \((x, y)\) is a common coupled fixed point of \(F\) and \(g\).

Finally, suppose that \((d)\) holds. Let \(g(C(F, g)) = \{(x, x)\}\). Then \(\{x\} = \{gx\} = F(x, x)\). Hence \((x, x)\) is a common coupled fixed point of \(F\) and \(g\). \(\square\)

**Example.** Suppose that \(X = [0, 1]\), equipped with the metric \(d : X \times X \to [0, +\infty)\) defined as \(d(x, y) = \max\{x, y\}\) and \(d(x, x) = 0\) for all \(x, y \in X\). Let \(F : X \times X \to K(X)\) be defined as

\[ F(x, y) = \begin{cases} \{0\}, & \text{for } x, y = 1 \\ \left[0, \frac{x^4}{4}\right], & \text{for } x, y \in [0, 1), \end{cases} \]

and \(g : X \to X\) be defined as

\[ g(x) = x^2, \text{ for all } x \in X. \]

Define \(\varphi : [0, +\infty) \to [0, +\infty)\) by

\[ \varphi(t) = \begin{cases} \ln(t + 1), & \text{for } t \neq 1 \\ \frac{3}{4}, & \text{for } t = 1, \end{cases} \]

and \(\psi : [0, +\infty) \to [0, 1)\) defined by

\[ \psi(t) = \frac{\varphi(t)}{t}, \text{ for all } t \geq 0. \]

Now, for all \(x, y, u, v \in X\) with \(x, y, u, v \in [0, 1)\), we have

**Case (a).** If \(x = u\), then

\[
\begin{align*}
H(F(x, y), F(u, v)) &= \frac{u^4}{4} \\
&\leq \ln(u^2 + 1) \\
&\leq \ln(\max\{x^2, u^2\} + 1) \\
&\leq \ln(d(gx, gu) + 1) \\
&\leq \ln(\max\{d(gx, gu), d(gy, gv)\} + 1),
\end{align*}
\]
which implies that

\[
\varphi (H(F(x, y), F(u, v))) \\
= \ln (H(F(x, y), F(u, v)) + 1) \\
\leq \ln (\ln (\max \{d(gx, gu), d(gy, gv)\} + 1) + 1) \\
\leq \ln (\ln (\max \{d(gx, gu), d(gy, gv)\} + 1) + 1) \\
\times \ln (\max \{d(gx, gu), d(gy, gv)\} + 1) \\
\leq \psi (\varphi (\max \{d(gx, gu), d(gy, gv)\})) \\
\times \varphi (\max \{d(gx, gu), d(gy, gv)\}).
\]

Case (b). If \( x \neq u \) with \( x < u \), then

\[
H(F(x, y), F(u, v)) = \frac{u^4}{4} \\
\leq \ln (u^2 + 1) \\
\leq \ln (\max \{x^2, u^2\} + 1) \\
\leq \ln (d(gx, gu) + 1) \\
\leq \ln (\max \{d(gx, gu), d(gy, gv)\} + 1),
\]

which implies that

\[
\varphi (H(F(x, y), F(u, v))) \\
= \ln (H(F(x, y), F(u, v)) + 1) \\
\leq \ln (\ln (\max \{d(gx, gu), d(gy, gv)\} + 1) + 1) \\
\leq \ln (\ln (\max \{d(gx, gu), d(gy, gv)\} + 1) + 1) \\
\times \ln (\max \{d(gx, gu), d(gy, gv)\} + 1) \\
\leq \psi (\varphi (\max \{d(gx, gu), d(gy, gv)\})) \\
\times \varphi (\max \{d(gx, gu), d(gy, gv)\}).
\]

Similarly, we obtain the same result for \( u < x \). Thus the contractive condition (2.1) is satisfied for all \( x, y, u, v \in X \) with \( x, y, u, v \in [0, 1) \). Again, for all \( x, y, u, v \in X \) with \( x, y \in [0, 1) \) and \( u, v = 1 \), we have

\[
H(F(x, y), F(u, v))
\]
\[
\begin{align*}
&= \frac{x^4}{4} \\
&\leq \ln(x^2 + 1) \\
&\leq \ln(\max\{x^2, u^2\} + 1) \\
&\leq \ln(d(gx, gu) + 1) \\
&\leq \ln(\max\{d(gx, gu), d(gy, gv)\} + 1),
\end{align*}
\]
which implies that
\[
\begin{align*}
\varphi(H(F(x, y), F(u, v))) &= \ln(H(F(x, y), F(u, v)) + 1) \\
&\leq \ln(\ln(\max\{d(gx, gu), d(gy, gv)\} + 1) + 1) \\
&\leq \frac{\ln(\ln(\max\{d(gx, gu), d(gy, gv)\} + 1) + 1)}{\ln(\max\{d(gx, gu), d(gy, gv)\} + 1)} \\
&\times \ln(\max\{d(gx, gu), d(gy, gv)\} + 1) \\
&\leq \psi(\max\{d(gx, gu), d(gy, gv)\}) \\
&\times \varphi(\max\{d(gx, gu), d(gy, gv)\}).
\end{align*}
\]
Thus the contractive condition (2.1) is satisfied for all \(x, y, u, v \in X\) with \(x, y \in [0, 1)\) and \(u, v = 1\). Similarly, we can see that the contractive condition (2.1) is satisfied for all \(x, y, u, v \in X\) with \(x, y, u, v = 1\). Hence, the hybrid pair \(\{F, g\}\) satisfies the contractive condition (2.1), for all \(x, y, u, v \in X\). In addition, all the other conditions of Theorem 2.1 are satisfied and \(z = (0, 0)\) is a common coupled fixed point of hybrid pair \(\{F, g\}\). The function \(F : X \times X \to K(X)\) involved in this example is not continuous at the point \((1, 1) \in X \times X\).

**Remark 2.2.** We improve, extend and generalize the results of Ciric et al. [8] in the sense that

(i) We prove our result for hybrid pair of mappings.

(ii) We prove our result in the framework of non-complete metric space \((X, d)\) and the product set \(X \times X\) is not empowered with any order.

(iii) We prove our result without the assumption of continuity and mixed \(g\)-monotone property for mapping \(F : X \times X \to K(X)\).

(iv) The functions \(\varphi : [0, +\infty) \to [0, +\infty)\) and \(\psi : [0, +\infty) \to [0, 1)\) involved in our theorem and example are discontinuous.
If we put $g = I$ (the identity mapping) in Theorem 2.1, we get the following result:

**Corollary 2.3.** Let $(X, d)$ be a complete metric space, $F : X \times X \to K(X)$ be a mapping. Assume that there exist some $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(H(F(x, y), F(u, v))) \leq \psi(\varphi(\max\{d(x, u), d(y, v)\})) \varphi(\max\{d(x, u), d(y, v)\}),$$

for all $x, y, u, v \in X$.

If we put $\psi(t) = 1 - \frac{\tilde{\psi}(t)}{t}$ for all $t \geq 0$ in Theorem 2.1, then we get the following result:

**Corollary 2.4.** Let $(X, d)$ be a metric space, $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings. Assume that there exist some $\varphi \in \Phi$ and $\tilde{\psi} \in \Psi$ such that

$$\varphi(H(F(x, y), F(u, v))) \leq \varphi(\max\{d(gx, gu), d(gy, gv)\}) - \tilde{\psi}(\varphi(\max\{d(gx, gu), d(gy, gv)\})),$$

for all $x, y, u, v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of $X$. Then $F$ and $g$ have a coupled coincidence point. Moreover, $F$ and $g$ have a common coupled fixed point, if one of the following conditions holds:

(a) $F$ and $g$ are $w$-compatible. $\lim_{n \to \infty} g^n x = u$ and $\lim_{n \to \infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and $g$ is continuous at $u$ and $v$.

(b) $g$ is $F$-weakly commuting for some $(x, y) \in C(F, g)$ and $gx$ and $gy$ are fixed points of $g$, that is, $g^2 x = gx$ and $g^2 y = gy$.

(c) $g$ is continuous at $x$ and $y$. $\lim_{n \to \infty} g^n u = x$ and $\lim_{n \to \infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

If we put $g = I$ (the identity mapping) in the Corollary 2.4, we get the following result:

**Corollary 2.5.** Let $(X, d)$ be a complete metric space, $F : X \times X \to K(X)$ be a mapping. Assume that there exist some $\varphi \in \Phi$ and $\tilde{\psi} \in \Psi$ such that

$$\varphi(H(F(x, y), F(u, v))) \leq \varphi(\max\{d(x, u), d(y, v)\}) - \tilde{\psi}(\varphi(\max\{d(x, u), d(y, v)\})),$$

for all $x, y, u, v \in X$. Then $F$ has a coupled fixed point.
Corollary 2.6. Let \((X, d)\) be a metric space, \(F : X \times X \to K(X)\) and \(g : X \to X\) be two mappings. Assume that there exists some \(\psi \in \Psi\) such that

\[
H(F(x, y), F(u, v)) \\
\leq \psi\left(2 \max\{d(gx, gu), d(gy, gv)\}\right) \max\{d(gx, gu), d(gy, gv)\},
\]

for all \(x, y, u, v \in X\). Furthermore assume that \(F(X \times X) \subseteq g(X)\) and \(g(X)\) is a complete subset of \(X\). Then \(F\) and \(g\) have a coupled coincidence point. Moreover, \(F\) and \(g\) have a common coupled fixed point, if one of the following conditions holds:

(a) \(F\) and \(g\) are \(w\)-compatible. \(\lim_{n \to \infty} g^n x = u\) and \(\lim_{n \to \infty} g^n y = v\) for some \((x, y) \in C(F, g)\) and for some \(u, v \in X\) and \(g\) is continuous at \(u\) and \(v\).

(b) \(g\) is \(F\)-weakly commuting for some \((x, y) \in C(F, g)\) and \(gx\) and \(gy\) are fixed points of \(g\), that is, \(g^2 x = gx\) and \(g^2 y = gy\).

(c) \(g\) is continuous at \(x\) and \(y\). \(\lim_{n \to \infty} g^n u = x\) and \(\lim_{n \to \infty} g^n v = y\) for some \((x, y) \in C(F, g)\) and for some \(u, v \in X\).

(d) \(g(C(F, g))\) is a singleton subset of \(C(F, g)\).

If we put \(g = I\) (the identity mapping) in the Corollary 2.6, we get the following result:

Corollary 2.7. Let \((X, d)\) be a complete metric space, \(F : X \times X \to K(X)\) be a mapping. Assume that there exists some \(\psi \in \Psi\) such that

\[
H(F(x, y), F(u, v)) \leq \psi\left(2 \max\{d(x, u), d(y, v)\}\right) \max\{d(x, u), d(y, v)\},
\]

for all \(x, y, u, v \in X\). Then \(F\) has a coupled fixed point.

If we put \(\psi(t) = k\) where \(0 < k < 1\), for all \(t \geq 0\) in Corollary 2.6, then we get the following result:

Corollary 2.8. Let \((X, d)\) be a metric space. Assume \(F : X \times X \to K(X)\) and \(g : X \to X\) be two mappings satisfying

\[
H(F(x, y), F(u, v)) \leq k \max\{d(gx, gu), d(gy, gv)\},
\]

for all \(x, y, u, v \in X\), where \(0 < k < 1\). Furthermore assume that \(F(X \times X) \subseteq g(X)\) and \(g(X)\) is a complete subset of \(X\). Then \(F\) and \(g\) have a coupled coincidence point. Moreover, \(F\) and \(g\) have a common coupled fixed point, if one of the following conditions holds:
(a) $F$ and $g$ are $w$–compatible. $\lim_{n \to \infty} g^n x = u$ and $\lim_{n \to \infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and $g$ is continuous at $u$ and $v$.

(b) $g$ is $F$–weakly commuting for some $(x, y) \in C(F, g)$ and $gx$ and $gy$ are fixed points of $g$, that is, $g^2 x = gx$ and $g^2 y = gy$.

(c) $g$ is continuous at $x$ and $y$. $\lim_{n \to \infty} g^n u = x$ and $\lim_{n \to \infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

If we put $g = I$ (the identity mapping) in the Corollary 2.8, we get the following result:

**Corollary 2.9.** Let $(X, d)$ be a complete metric space. Assume $F : X \times X \to K(X)$ be a mapping satisfying

$$H(F(x, y), F(u, v)) \leq k \max \{d(x, u), d(y, v)\},$$

for all $x, y, u, v \in X$, where $0 < k < 1$. Then $F$ has a coupled fixed point.

**References**

1. M. Abbas, B. Ali & A. Amini-Harandi: Common fixed point theorem for hybrid pair of mappings in Hausdorff fuzzy metric spaces. *Fixed Point Theory Appl.* 2012, 225.
2. M. Abbas, L. Ciric, B. Damjanovic & M.A. Khan: Coupled coincidence point and common fixed point theorems for hybrid pair of mappings. *Fixed Point Theory Appl.* 1687-1812-2012-4.
3. A. Amini-Harandi & D. O’Regan: Fixed point theorems for set-valued contraction type mappings in metric spaces. *Fixed Point Theory Appl.* 7 (2010), Article ID 390183.
4. S. Banach: Sur les Operations dans les Ensembles Abstraits et leur. Applications aux Equations Integrales. *Fund. Math.* 3 (1922), 133-181.
5. V. Berinde: Coupled fixed point theorems for $\varphi$–contractive mixed monotone mappings in partially ordered metric spaces. *Nonlinear Anal.* 75 (2012), 3218-3228.
6. T.G. Bhaskar & V. Lakshmikantham: Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal.* 65 (2006), no. 7, 1379-1393.
7. B.S. Choudhury & A. Kundu: A coupled coincidence point results in partially ordered metric spaces for compatible mappings. *Nonlinear Anal.* 73 (2010), 2524-2531.
8. L. Ciric, B. Damjanovic, M. Jileli & B. Samet: Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications. *Fixed Point Theory Appl.* 2012, 51.
9. B. Deshpande & A. Handa: Nonlinear mixed monotone-generalized contractions on partially ordered modified intuitionistic fuzzy metric spaces with application to integral equations. *Afr. Mat.* 26 (2015), no. 3-4, 317-343.

10. _____: Application of coupled fixed point technique in solving integral equations on modified intuitionistic fuzzy metric spaces. *Adv. Fuzzy Syst.* (2014), Article ID 348069.

11. _____: Common coupled fixed point theorems for hybrid pair of mappings satisfying an implicit relation with application. *Afr. Mat.* DOI 10.1007/s13370-015-0326-7.

12. _____: Common coupled fixed point theorems for two hybrid pairs of mappings under \( \varphi - \psi \) contraction. *ISRN* (2014), Article ID 608725.

13. _____: Common coupled fixed point for hybrid pair of mappings under generalized nonlinear contraction. *East Asian Math. J.* 31 (2015), no.1, 77-89.

14. _____: Common coupled fixed point theorems for hybrid pair of mappings under some weaker conditions satisfying an implicit relation. *Nonlinear Analysis Forum* 20 (2015), 79-93.

15. B. Deshpande, S. Sharma & A. Handa: Common coupled fixed point theorems for nonlinear contractive condition on intuitionistic fuzzy metric spaces with application to integral equations. *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.* 20 (2013), no. 3, 159-180.

16. H.S. Ding, L. Li & S. Radenovic: Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces. *Fixed Point Theory Appl.* 2012, 96.

17. W.S. Du: Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi-Takahashi’s condition in quasi ordered metric spaces. *Fixed Point Theory Appl.* 2010, 9 (2010) Article ID 876372.

18. J. Harjani, B. Lopez & K. Sadarangani: Fixed point theorems for mixed monotone operators and applications to integral equations. *Nonlinear Anal.* 74 (2011), 1749-1760.

19. V. Lakshmikantham & L. Ciric: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Anal.* 70 (2009), no. 12, 4341-4349.

20. W. Long, S. Shukla & S. Radenovic: Some coupled coincidence and common fixed point results for hybrid pair of mappings in 0-complete partial metric spaces. *Fixed Point Theory Appl.* 2013, 145.

21. N.V. Luong & N.X. Thuan: Coupled fixed points in partially ordered metric spaces and application. *Nonlinear Anal.* 74 (2011), 983-992.

22. M. Jain, K. Tas, S. Kumar & N. Gupta: Coupled common fixed point results involving a \( \varphi - \psi \) contractive condition for mixed g-monotone operators in partially ordered metric spaces. *J. Inequal. Appl.* 2012, 285.

23. J.T. Markin: Continuous dependence of fixed point sets. *Proc. Ame. Math. Soc.* 38 (1947), 545-547.
24. N. Mizoguchi & W. Takahashi: Fixed point theorems for multivalued mappings on complete metric spaces. *J. Math. Anal. Appl.* **141** (1989), 177-188.

25. S.B. Nadler: Multi-valued contraction mappings. *Pacific J. Math.* **30** (1969), 475-488.

26. J. Rodriguez-Lopez & S. Romaguera: The Hausdorff fuzzy metric on compact sets. *Fuzzy Sets Syst.* **147** (2004), 273-283.

27. B. Samet: Coupled fixed point theorems for generalized Meir-Keeler contraction in partially ordered metric spaces. *Nonlinear Anal.* **72** (2010), 4508-4517.

28. B. Samet, E. Karapinar, H. Aydi & V. C. Rajic: Discussion on some coupled fixed point theorems. *Fixed Point Theory Appl.* 2013, 50.

29. N. Singh & R. Jain: Coupled coincidence and common fixed point theorems for set-valued and single-valued mappings in fuzzy metric space. *Journal of Fuzzy Set Valued Analysis* 2012, Article ID jfsva-00129.

30. W. Sintunavarat, P. Kumam & Y. J. Cho: Coupled fixed point theorems for nonlinear contractions without mixed monotone property. *Fixed Point Theory Appl.* 2012, 170.

31. T. Suzuki: Mizoguchi-Takahashi’s fixed point theorem is a real generalization of Nadler’s. *J. Math. Anal. Appl.* **340** (2008), no. 1, 752-755.

---

*aDepartment of Mathematics, Govt. Arts & Science P.G. College, Ratlam- 457001 (M.P.) India  
Email address: bhavnadespande@yahoo.com*

*bDepartment of Mathematics, Govt. P. G. Arts and Science College, Ratlam-457001 (MP), India  
Email address: amrishhanda83@gmail.com*