Abstract

The discrete symmetries of the Lorentz group are on the one hand a ‘complex’ interplay between linear and anti-linear operations on spinor fields and on the other hand simple linear reflections of the Minkowski space. We define operations for $T$, $CP$ and $CPT$ leading to both kinds of actions. These operations extend the action of $SL(2, \mathbb{C})$, representing the action of the proper orthochronous Lorentz group $SO^+(1, 3)$ on the Weyl spinors, to an action of the full group $O(1, 3)$. But it is more instructive to reverse the arguments. The action of $O(1, 3)$ is the natural way how $SL(2, \mathbb{C})$ together with its conjugation structure acts on Minkowski space.

Focusing on the symmetries of these (anti-)linear operations we can for example distinguish between $CP$-invariant and $CP$-violating symmetries. This is important if gauge symmetries are included. It turns out that, contrary to the general belief, $CP$ and $T$ are not compatible with $SU(n)$ for $n \geq 3$, especially with $SU(3)_{\text{colour}}$ or with the $U(3)$-Cabibbo-Kobayashi-Maskawa matrix.

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The history of the discrete symmetries was a history of surprises. For example when C.S. Wu discovered parity violation (after theoretical advice given by Lee and Yang), Wolfgang Pauli wrote to his former assistant Viktor Weisskopf: "What shocks me is not the fact that 'God is just left-handed' but the fact that in spite of this He exhibits Himself as left/right symmetric when He expresses Himself strongly. In short, the real problem now is why the strong interactions are left/right symmetric. How can the strength of an interaction produce or create symmetry groups, invariances or conservation laws? This question prompted me to my premature and wrong prognosis. I don’t know any good answer to that question but one should consider that already there exists a precedent: the rotational group in isotopic spin-space, which is not valid for the electromagnetic field. One does not understand either why it is valid at all. It seems that there is a certain analogy here!"  

[1]. Even more unexpected was the discovery of CP-violation by Finch et al.

Why was there such a surprise? Beginning with the discovery of spin by Stern and Gerlach and with the theoretical work of Dirac and Weyl the ‘real version’ of the Lorentz group, i.e. O(1, 3), lost more and more of its fundamental meaning and should have been replaced by SL(2, C). But therein parity is not defined. Only after the discovery of parity violation the Weyl theory became familiar. Nowadays the Standard Model is written in Weyl spinors. Is there a similar hint for CP-violation? In order to answer this question one has to take a look at the representation structure of the Lorentz group on the Weyl spinors. Is it possible to understand the discrete part of the Lorentz group like the continuous part? In the first three sections we define the discrete symmetries as (anti-)linear operations within the different kinds of Weyl spinors. Their action on the Cartan (bispinor) representation of the Minkowski space is the familiar action of the discrete symmetries on Minkowski space. Next we show that these operations lead also to the discrete symmetry operations on Dirac spinor fields.

The discrete symmetry operations on the Weyl spinors are deeply connected with the spin and boost structure of SL(2, C'). Including also inner symmetries we show that the discrete symmetries are not compatible with every representation of the inner symmetry groups. This holds especially for CP and T in an SU(3) gauge theory. It is interesting that for the same reason the U(3)-Cabibbo-Kobayashi-Maskawa matrix breaks CP invariance.
1 The Discrete Factors of the Lorentz Group $O(1, 3)$

The Minkowski time-space translations $M$ form a 4-dimensional real vector space with bilinear form of signature $(1,3)$. This bilinear form $\eta$ is usually called the Lorentz metric. It is left invariant by the action of the Lorentz group $O(1, 3)$. The Lorentz group is not simply connected. This is expressed in its semidirect and direct product structure of the sign group $I_2 \cong \{\pm \mathbb{1}\}$ with the proper orthochronous Lorentz group $SO^+(1, 3)$:

$$O(1, 3) \cong I^T_2 \times_s \left(I^{CPT}_2 \times SO^+(1, 3)\right).$$

(1)

Here $\times$ denotes the direct product and $\times_s$ the semidirect product$^2$.

The discrete factors are labelled according to their representation on the Minkowski space

$$I^{CPT}_2 \cong \{\pm \mathbb{1}_4\},$$

(2)

$$I^T_2 \cong \{\mathbb{1}_4, \begin{pmatrix} -1 & \mathbb{0} \\ \mathbb{0} & \mathbb{1}_3 \end{pmatrix}\}.$$  

(3)

The element $-\mathbb{1}_4$ of the representation of $I^{CPT}_2$ is called the strong reflection and is the representation of CPT on the Minkowski space. The factor $I^T_2$ contains the operation of the time reversal $T$. However, there is no canonical way to decompose the Minkowski space $M$ into space $\mathcal{T}$ and space $\mathcal{S}$, i.e. $\mathcal{T} \oplus \mathcal{S}$; this is like the Sylvester form of the Lorentz metric, $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1}_3 \end{pmatrix}$, which is only one possible form and in no way distinguished if there is no rest system. But without such a decomposition, there is no representation of $I^T_2$ possible like in (3). When given one rest system, then in a different (boosted) system, the space-time decomposition is different and therefore the representation of $I^T_2$ is different. This is due to the semidirect product structure, since $I^T_2$ commutes with the rotations $SO(3) \subset SO^+(1, 3)$, but not with the boosts $SO^+(1, 3)/SO(3)$.

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$^2$Contrary to the usual mathematical notation the normal subgroup is written as second factor in the semidirect product. In this notation the Poincaré group reads $O(1, 3) \times_s \mathbb{R}^4$. This notation reflects the action of the first factor group onto the normal subgroup as second factor, whereas there is no reverse action.
The symmetry group of the Euclidean space is $O(3)$, the direct product group of rotations and space reflection (parity $P$)

$$O(3) = \mathbb{I}_2^P \times SO(3), \quad \mathbb{I}_2^P \cong \{\pm \mathbb{I}_3\}. \quad (4)$$

The group $O(3)$ can be embedded in the Lorentz group in two different ways

$$O(3) = \mathbb{I}_2^P \times SO(3) \subset \mathbb{I}_2^{CPT} \times SO^+(1,3) = SO(1,3) \quad (5)$$
$$O(3) = \mathbb{I}_2^P \times SO(3) \subset \mathbb{I}_2^{CP} \times_s SO^+(1,3) \cong O^+(1,3) \quad (6)$$

with $\mathbb{I}_2^{CP} \subset \mathbb{I}_2^T \times_s \mathbb{I}_2^{CPT}$, $\mathbb{I}_2^{CP} \cong \{\mathbb{I}_4, \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\mathbb{I}_3 \end{pmatrix}\}$. Thus the embedding of parity is not unique. As far as no embedding is distinguished one should not identify the operation $\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\mathbb{I}_3 \end{pmatrix}$ with the representation of $P$. What we will show in the following is that this operation can be identified with CP. We will also show which structure, additional to the Lorentz group, is needed to define CP and T and how this is related to a space-time decomposition.

### 2 Spinor Representation of Minkowski Space

The proper orthochronous Lorentz group $SO^+(1,3)$ is the $D(4|1\frac{1}{2})$ representation of its covering group\footnote{In the following the group $SL(2,\mathcal{C})$ is always regarded as a 6-dimensional real Lie group: $SL(2,\mathcal{C}) = SL(2,\mathcal{C})_R$. As 6-dimensional real Lie group we refer to it also as Lorentz group.} $SL(2,\mathcal{C})$, a tensor product representation of the two fundamental $SL(2,\mathcal{C})$-representations. What follows in this section is the basis-independent definition of this representation and its representation space – the Cartan (or bispinor) representation of the Minkowski space $\mathbb{E}$. The reader familiar with basis-independent complex representations of real Lie-groups can read this section only for notations.

In general a complex vector space appears in a fourfold way. Every vector space $V$ has its dual space $V^T$, the linear forms on $V$. In addition to each complex vector space with action of the field $\mathcal{C}$ (scalar multiplication)

$$z \cdot v = zv, \quad z \in \mathcal{C}, \quad v \in V, \quad (7)$$
there is a complex conjugate or anti-space $\overline{\mathbf{V}}$ with complex conjugate action
\[ z \cdot v = \overline{z} v \quad z \in \mathbb{C}, \quad v \in \mathbf{V}. \tag{8} \]

The two vector spaces $V$ and $\overline{V}$ are identical when regarded as additive groups, they have to be distinguished when regarded as vector spaces.

All together we have a quartet of complex vector spaces, $V, V^T, \overline{V}$ and $\overline{V}^T$, where the anti-spaces and spaces are related to each other by the complex conjugation.

\[
\begin{array}{c}
V \\
\downarrow\text{co}_V \\
\overline{V}
\end{array}
\xrightarrow{\xi}
\begin{array}{c}
V^T \\
\downarrow\text{co}_{V^T} \\
\overline{V}^T
\end{array}
\]

The complex conjugation $\text{co}_V$ acts on the additive groups underlying $V$ and $\overline{V}$ as the identity, but on the vector spaces it has the anti-linear property
\[ \text{co}_V(\alpha v) = \overline{\alpha} \text{co}_V(v), \quad v \in V, \alpha \in \mathbb{C}. \tag{9} \]

We refer to it as the canonical conjugation of the vector spaces.

In addition, there may exist an isomorphism $\xi$ between the dual vector spaces $V$ and $V^T$. The corresponding isomorphism $\overline{\xi}$ between $\overline{V}$ and $\overline{V}^T$ is given by the canonical conjugation
\[ \overline{\xi} = \text{co}_{V^T} \xi \text{co}_V^{-1}. \tag{10} \]

There is an analogue fourfold structure of in general inequivalent representations of a group on these vector spaces. The group $GL(n, \mathbb{C})$,
\( n > 1 \), regarded as a real \( 2n^2 \)-dimensional Lie-group, has four complex \( n \)-dimensional fundamental representations. With the defining representation \( D_V(g) = D(g) = g \in GL(n, \mathbb{C}) \) given on \( V \cong \mathbb{C}^n \) one has the three partners:

\[
D_{V^T}(g) = D(g)^{-1T} =: \tilde{D}(g) \quad (11)
\]

\[
D_{\nabla}(g) = \text{cov}_V D(g) \text{cov}^{-1}_V =: \bar{D}(g) \quad (12)
\]

\[
D_{\nabla^T}(g) = \bar{D}(g)^{-1T} =: \tilde{\bar{D}}(g). \quad (13)
\]

In the case of \( s \in SL(2, \mathbb{C}) \), the dual representations on \( V \) and \( V^T \) or \( \nabla \) and \( \nabla^T \) are equivalent with the volume form\(^5\) or spinor metric,

\[
\varepsilon : V \longrightarrow V^T,
\]

\[
\varepsilon(e^A) = \varepsilon^{AB} \epsilon_B, \quad (14)
\]

\( e^A \) and \( \epsilon_A \) being dual bases of \( V \) and \( V^T \), resp., and \( \varepsilon^{AB} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \) the matrix representation of the dual isomorphism\(^6\) in this basis. The equivalence of the \( SL(2, \mathbb{C}) \) representations is expressed by

\[
\tilde{D}(s) = \varepsilon D(s) \varepsilon^{-1} \quad (15)
\]

\[
\tilde{\bar{D}}(s) = \bar{\varepsilon} \bar{D}(s) \bar{\varepsilon}^{-1}. \quad (16)
\]

The two fundamental representations are chosen as

\[
D(s) =: D^{(1\,0)}(s) \quad (17)
\]

\[
\tilde{D}(s) =: D^{(0\,1)}(s), \quad (18)
\]

which are the left-handed and right-handed Weyl representations, resp. They are connected by the action of the canonical conjugation together with the transposition \((\cdot)^\times := (\cdot)^T\). With equations \((\text{11})\) to \((\text{13})\) this gives

\[
D(s)^\times = \tilde{\bar{D}}(s)^{-1}. \quad (19)
\]

Therefore we refer to this operation as canonical conjugation on the representations.

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\(^5\) In general the \( SL(n, \mathbb{C}) \)-invariant volume form is multi-linear and totally antisymmetric. Only in the special case \( n = 2 \) it is a bilinear form and therewith it is equivalent to a dual isomorphism.

\(^6\) It is sometimes denoted as \( i\sigma_2 \).
The Dirac vector space $V_D := V \oplus \overline{V}^T \cong \mathbb{C}^4$ contains the two fundamental Weyl vector spaces. Therewith the Dirac representation of the Lorentz group is the direct sum of both fundamental Weyl representations: $D_D(s) = D(s) \oplus \tilde{D}(s)$. This representation lies in the complex 16-dimensional endomorphism algebra of the Dirac vector space, called the Dirac endomorphisms.

$$\text{end}(V_D) \cong V_D \otimes V_D^T \cong \begin{pmatrix} V \otimes V^T & V \otimes \overline{V} \\ \overline{V}^T \otimes V^T & \overline{V}^T \otimes \overline{V} \end{pmatrix}$$ (20)

The canonical conjugation of the underlying Weyl quartet gives a conjugate linear reflection of the Dirac endomorphisms. Its invertible elements with the property $f^\times = f^{-1}$ are elements of the indefinite unitary group $U(2, 2)$. Remembering eq.(19) one can see, that the Dirac representation is an embedding of the Lorentz group into $U(2, 2)$. The quotient group $U(2, 2)/U(1) \cong SO(2, 4)$ - the conformal group - contains the whole Poincaré group. Therefore in the Dirac endomorphisms there is a vector subspace with the properties of the Minkowski translations being anti-symmetric with respect to the canonical conjugation. We identify

$$\mathcal{M} = \{x \in V \otimes \overline{V} | x^\times = -x\}$$ (21)

as the Cartan representation of the Minkowski space. This is a real subspace of the linear mappings from $\overline{V}^T$ (right-handed Weyl spinors) to $V$ (left-handed Weyl spinors) $x : \overline{V}^T \to V$.

For these $2 \times 2$ dimensional mappings we can choose an appropriate basis $e^\mu, \mu = 0, \ldots, 3$ with $(e^\mu)^\times = -e^\mu$. One possible matrix representation is given with the Pauli matrices by $i\rho^\mu := (i\mathbb{1}, i\sigma^i)$, referred to as Weyl basis. In this basis an element of the Minkowski space

$$\mathcal{M} = \{x_\mu e^\mu | x_\mu \in \mathbb{R}\}$$ (22)

has the matrix representation

$$x \cong x_\mu i\rho^\mu = i \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} $$ (23)
which was first introduced by E.Cartan [2] (further developments are in [7, 8]). This matrix representation should always be regarded as embedded in the Dirac endomorphisms

\[ x_\mu e^\mu \cong \begin{pmatrix} 0 & x_\mu i \rho^\mu \\ 0 & 0 \end{pmatrix} . \]

Hence considered as Dirac endomorphisms the Minkowski space has a nilpotent product.

We have to emphasize the basis dependence of the definition of the Minkowski space in the representation (23). The Weyl basis is already a basis in which the Lorentz metric has the form \( \eta = \begin{pmatrix} 1 & -I_3 \end{pmatrix} \). (see app. A.) This basis anticipates a space-time decomposition and is appropriate to define the \( CP \) and \( T \) operations. The space-time decomposition without an anticipating basis is given in sect. 4.

To obtain the action of \( SL(2,C) \) on the Cartan representation of Minkowski space we use a method called induced action: In general when there are two \( G \)-sets \( S_1 \) and \( S_2 \), defined as two sets with an action \( \rho_1(g) \) and \( \rho_2(g) \) of a group \( g \in G \) resp., \( S_1, S_2 \in \text{set}_G \), then there is an induced \( G \)-action on the mappings \( S_2^{S_1} \) between these two sets, \( f : S_1 \rightarrow S_2 \), i.e. \( S_2^{S_1} \in \text{set}_G \). The induced action can be characterized by the commutative diagram

\[
\begin{array}{ccc}
S_1 & \xrightarrow{\rho_1(g)} & S_1 \\
\downarrow f & & \downarrow g \circ f = f_g \\
S_2 & \xrightarrow{\rho_2(g)} & S_2 
\end{array}
\]

where \( g \circ f \) denotes the action of \( g \in G \) on the mapping \( f \):

\[ g \circ f = f_g = \rho_2(g) \circ f \circ \rho_1(g)^{-1}. \tag{24} \]

This inducing construction is necessary to obtain the action of the discrete symmetry operations on the Minkowski space in the next section. It can

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9This induced action is more general than the method of induced representations given by Mackey [9]. The Mackey theory can be formulated in this language.
also be used for the action of \( SL(2, \mathbb{C}) \) on the Cartan representation of the Minkowski space: Substitute the sets \( S_i \) by the left- and right-handed Weyl vector spaces, \( V \) and \( V^T \), resp., the actions \( \rho_i(g) \) by the representations \( D(s) \) and \( \tilde{D}(s) \) and the mappings \( f \) by elements of the Minkowski space \( x \). Then the action of \( SL(2, \mathbb{C}) \) on Minkowski space is given with eq.(24) by

\[ s \cdot x = D^{(\frac{1}{2}, 0)}(s) \circ x \circ D^{(0, \frac{1}{2})}(s)^{-1} = D^{(\frac{1}{2}, \frac{1}{2})}(s).x . \]  

(25)

Because of the isomorphism

\[ SO^+(1, 3) \cong SL(2, \mathbb{C})/I_2, \]  

(26)

\[ I_2 \cong \{ \pm I_2 \}, \]  

the above representation is a faithful representation of \( SO^+(1, 3) \) only. In the Weyl basis of the Minkowski space this representation leads to the familiar Lorentz matrices \( \Lambda(s)_{\mu}^{\nu} \) as matrix representation

\[ D^{(\frac{1}{2}, \frac{1}{2})}(s).x \cong x_{\mu} \Lambda(s)^{\mu}_{\nu} i \rho^\nu. \]  

(27)

3 The Full Lorentz Group from Actions on the Weyl Spinors

The actions of \( SL(2, \mathbb{C}) \) can be regarded as the action of \( SO^+(1, 3) \) on the Weyl spinor spaces. Or vice versa, \( SO^+(1, 3) \) is the natural action of \( SL(2, \mathbb{C}) \) on the Cartan representation of the Minkowski space. This picture seems to fail for the discrete parts of the full group \( O(1, 3) \). However, there are additional operations within the complex quartet of the Weyl vector spaces (both linear and anti-linear) acting on the Cartan representation of the Minkowski space as (linear) automorphisms.

According to Wigner the operations of \( P \) and \( C \) in quantum field theory are linear and so is \( CP \). On the other hand \( T \) and therewith \( CPT \) are anti-linear operations \([11, 12]\).

One anti-linear action within the Weyl quartet reversing the Minkowski space is the canonical conjugation. With the property of the \( SL(2, \mathbb{C}) \) representation acting conjugation compatible on the Minkowski space, calculated from equations (19) and (25),

\[ \left(D^{(\frac{1}{2}, \frac{1}{2})}(s).x\right)^* = D^{(\frac{1}{2}, \frac{1}{2})}(s).x^*, \]  

(28)
i.e. the action of the canonical conjugation ‘commuting’ with the action of the Lorentz group

\[ \text{co}(s \cdot x) = s \cdot \text{co}(x), \]

the ‘product group’ generated by \( \text{co} \) and \( SL(2, \mathbb{C}) \) acts on the Minkowski space as the direct product group \( I \times I^{\text{CPT}} \times SO^+(1,3) \cong SO(1,3) \). In this context the canonical conjugation can be regarded as the action of \( CPT \) on the spinor spaces. In sect.3 we will show, that this action is also the \( CPT \) action on spinor fields in quantum field theory.

According to the remarks at the end of sect.1 we need for the representation of \( T \) in addition to the \( SL(2, \mathbb{C}) \) compatible structures \( \text{co} \) and \( \varepsilon \) a structural element being invariant with the \( SU(2) \) subgroup of \( SL(2, \mathbb{C}) \), but not with the boosts. This new structure is the anti-linear euclidian conjugation \( \delta \), which in general is the invariant dual isomorphism of the positive unitary group \( U(n) \).

\[ \delta : V \rightarrow V^T, \quad V \cong \mathbb{C}^2 \]

\[ \delta(a_A e^A) = \bar{a}_A \delta^{AB} \bar{e}_B \] (29)

\[ \bar{D}(u) = \delta D(u) \delta^{-1}, \quad u \in U(n) \] (30)

or

\[ D^*(u) := \delta^{-1} D(u)^T \delta = D(u)^{-1}. \] (31)

Within the Weyl quartet the totality of all \( SU(2) \) compatible operations is characterized by the following diagram:

Together with the volume form and the euclidian conjugation we can construct an anti-linear automorphism on the Weyl vector space, \( \delta^{-1} \circ \varepsilon : V \rightarrow V \), compatible with \( SU(2) \). The corresponding actions on the other partners of the quartet is given by \( \varepsilon \delta^{-1}, \bar{\delta}^{-1} \varepsilon \) and \( \bar{\varepsilon} \bar{\delta}^{-1} \).
With the concept of induced action we construct the action of these anti-linear operations on the Cartan representation of Minkowski space

\[ V^T \xrightarrow{\mathcal{E}\mathcal{D}^{-1}} V^T \]

\[ x \]

\[ (\mathcal{D}^{-1}\mathcal{E}) \bullet x = (\mathcal{D}^{-1}\mathcal{E}) \circ x \circ (\mathcal{E}\mathcal{D}^{-1})^{-1} \]

This abstract operation can be concretized in the matrix representations of equations (14), (23) and (29)

\[ (\mathcal{D}^{-1}\mathcal{E}) \bullet x \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_0 - x_3 \end{pmatrix}. \]

Hence the action of the combination of the volume form and the euclidian conjugation implements the time reversal

\[ (\mathcal{D}^{-1}\mathcal{E}) \bullet x = x_\mu \Lambda(T)^\mu_\nu e_\nu \cong (-x_0, \vec{x}), \quad (33) \]

with \( \Lambda(T)^\mu_\nu = \begin{pmatrix} -1 \\ \mathbf{1}_3 \end{pmatrix} \).

Since the operation \( \mathcal{D}^{-1}\mathcal{E} \) commutes with \( SU(2) \) in \( SL(2, \mathcal{C}) \), but not with the boosts \( SL(2, \mathcal{C})/SU(2) \), the action of the operations \( \mathcal{D}^{-1}\mathcal{E} \), \( co \) and \( s \in SL(2, \mathcal{C}) \) generates the semidirect product structure \( I^T_2 \times_s (I^{CP}_2 \times SO^+(1,3)) \cong O(1,3) \) on the Minkowski space. We take the operation of \( \mathcal{D}^{-1}\mathcal{E} \) and its three partners as the action of \( T \) on the Weyl spinor spaces.

It should be remarked that \( \mathcal{E}^{-1}\mathcal{D} = -\mathcal{D}^{-1}\mathcal{E} \) and therefore \( T^2 \cong -\mathbf{1}_V \) on the spinors, but \( T^2 \cong \mathbf{1}_M \) on the Minkowski space.

Finally the combination \( coV \circ \mathcal{E}^{-1}\mathcal{D} \) is an \( SU(2) \) compatible linear isomorphism between the vector space and the anti-space. Its induced action on the Minkowski space is given by

\[ (co\mathcal{E}^{-1}\mathcal{D}) \bullet x = x_\mu \Lambda(P)^\mu_\nu e_\nu \]

with \( \Lambda(P)^\mu_\nu = \begin{pmatrix} 1 \\ -\mathbf{1}_3 \end{pmatrix} \). According to the remarks at the end of sect.11 and for later consistency we identify this operation with \( CP \).
The operations dual to the $x_\mu$ are the momenta $ip^\mu = \partial^\mu$. Their Cartan representation lies in the dual space of the Minkowski space. Thus the momenta are symmetric with respect to the canonical conjugation:

$$ p \in iM^T = \left\{ p \in V T \otimes V^T | p^x = p \right\}. $$ (35)

The transformation properties of the momenta are equal to those of space-time for the linear operation $CP$,

$$ CP \cdot p = p^\mu \Lambda(P)^\nu_\mu e_\nu \cong (p^0, -\vec{p}), $$ (36)

and different for the anti-linear operations,

$$ CPT \cdot p = p, $$ (37)

$$ T \cdot p \cong (p^0, -\vec{p}). $$ (38)

The anti-linear operations guarantee the positivity of the energy component $p^0$ even in the case when time is reversed. This feature (on the level of the Schrödinger theory) was the starting point for Wigner to define the time reversal operation to be anti-linear [11, 12].

## 4 Euclidian Conjugation and Space-Time Decomposition

A decomposition of Minkowski space into space and time is given when there is a distinct time-like (basis) vector. Again it is the euclidian conjugation which provides this basis vector for the time translations: Notice that the operation $\bar{\delta} \circ co_V$ is a linear mapping between left- and right-handed Weyl vector spaces. Thus it is an element of $V T \otimes V^T$. Its inverse, as an element of $V \otimes \bar{V}$, multiplied with an $i$ can be defined as the time-like basis vector

$$ e^0 := ico_V^{-1} \bar{\delta}^{-1}. $$ (39)

In the Weyl basis this vector is given by

$$ e^0 \cong i\rho^0. $$ (40)
The properties of a time-like basis vector of Minkowski space

\[(e^0) \times = -e^0 \quad (41)\]

\[<e^0|e^0> = 1 \quad (42)\]

have to be proven without using any basis. This more technical part is done in app. 3. The time translations are given by \( T = IR \cdot e^0 \) and space is its orthogonal complement with respect to the Lorentz bilinear form \( S = T^\perp \). But there is no basis distinguished within position space.

5 Discrete Symmetry Operations on the Weyl Spinor Fields

We show in this section how the associations of the discrete symmetries in sect 3 lead to the well known operations on Weyl- and Dirac spinor fields.

The left and right handed Weyl spinors are elements of the complex Weyl quartet: \( l \in V, l^\dagger \in \overline{V}, r \in \overline{V}^T, r^\dagger \in V^T \). The spinor fields are mappings from Minkowski space \( \mathcal{M} \) into these vector spaces, e.g. \( l(\cdot) \in V^M \), carrying a positive unitary representation of the Poincaré group. Massive Weyl spinor fields have as harmonic analysis in the Wigner representation [13]

\[
l^A(x) = \int \frac{d^3q}{(2\pi)^{3/2}} \sqrt{\frac{m}{q_0}}s(q, m)^A_B e^{iqx}u^B(q) + e^{-iqx}a^B(q) \quad (43)
\]

\[
l^\dagger_A(x) = \int \frac{d^3q}{(2\pi)^{3/2}} \sqrt{\frac{m}{q_0}}s(q, m)^B_A e^{-iqx}u^*_B(q) + e^{iqx}a_B(q) \quad (44)
\]

\[
r^\dagger_A(x) = \int \frac{d^3q}{(2\pi)^{3/2}} \sqrt{\frac{m}{q_0}}s(q, m)^B_A e^{-iqx}u^*_B(q) - e^{iqx}a_B(q) \quad (45)
\]

\[
r^A(x) = \int \frac{d^3q}{(2\pi)^{3/2}} \sqrt{\frac{m}{q_0}}s(q, m)^B_A e^{iqx}u^*_B(q) - e^{-iqx}a_B(q) \quad (46)
\]

Here \( s(q, m) = D(s) \) is the matrix representation for a representative \( s \) of a boost coset \( SL(2, \mathbb{C})/SU(2) \) parametrized by the momenta \( q \) of the mass \( q^2 = m^2, m > 0 \) [14]:

\[
s^A_B(q, m) = \sqrt{\frac{q_0 + m}{2m}} \left[ \mathbb{1}_2 + \frac{\vec{\sigma} \cdot \vec{q}}{q_0 + m} \right]
\]

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\[ u(q), u^*(q) \text{ and } a(q), a^*(q) \text{ are the creation and annihilation operators of particles and antiparticles, resp. According to Wigner they carry only a finite dimensional positive unitary representation of the little group } SU(2), \text{ which for massive spinor fields is } SU(2). \text{ Therefore these operators map into the complex representation quartet of } SU(2), u \in V, u^* \in V^T, a \in \overline{V}, a^* \in \overline{V}^T. \text{ As mappings they have the discrete transformation properties induced by the discrete transformations of the momenta and of the Weyl spinors. For the corresponding basis this is}

\begin{align*}
\text{CPT} & : \quad co \bullet a_B(p) = \delta^{AB} a_B(p) \\
T & : \quad (\delta^{-1} \varepsilon) \bullet a_B(p) = \varepsilon^{AB} \delta_{BC} u^C(-p) \\
CP & : \quad (co \varepsilon^{-1} \delta) \bullet a_B(p) = \delta^{AB} \varepsilon_{BC} \delta^{CD} a_D(-p),
\end{align*}

with \( \varepsilon^{AB} \varepsilon_{BC} = \delta^A_C \). Calculating the action of the discrete symmetry operations on the representations of the boosts

\begin{align*}
co \bullet s(q,m) & = \bar{s}(q,m) \\
(\delta^{-1} \varepsilon) \bullet s(q,m) & = s(-q,m) \\
(co \varepsilon^{-1} \delta) \bullet s(q,m) & = \bar{s}(-q,m)
\end{align*}

one obtains the action of the discrete symmetries on quantized spinor fields (compare, e.g., [10]):

\begin{align*}
\text{CPT} & : \quad \{ \begin{array}{l}
 l^A(x)^\times := co \bullet l^A(x) = \delta^{AB} l_B^\dagger(-x) \\
 r^A(x)^\times := co \bullet r^A(-x) = -\delta^{AB} l_B^\dagger(-x)
\end{array} \\
T & : \quad \{ \begin{array}{l}
 (\delta^{-1} \varepsilon) \bullet l^A(x) = \varepsilon^{AB} \delta_{BC} l^C(-t,-\vec{x}) \\
 (\delta^{-1} \varepsilon) \bullet r^A(x) = \varepsilon^{AB} \delta_{BC} r^C(-t,-\vec{x})
\end{array} \\
CP & : \quad \{ \begin{array}{l}
 (co \varepsilon^{-1} \delta) \bullet l^A(x) = -\delta^{AB} \varepsilon_{BC} \delta^{CD} l^C_D(t,-\vec{x}) \\
 (co \varepsilon^{-1} \delta) \bullet r^A(x) = \delta^{AB} \varepsilon_{BC} \delta^{CD} r^C_D(t,-\vec{x})
\end{array}
\end{align*}

Again these are induced actions. For example the CPT operation on left-handed Weyl spinor fields is given by the commutative diagram.

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10 The star \( \star \) denotes the euclidian conjugation. The assignment of the creation and the annihilation operators is reversed compared to the standard notation.
In the Weyl representation of the Dirac field\(^\text{11}\),

\[
\psi(x) = l(x) \oplus r(x) \cong \begin{pmatrix} l(x) \\ r(x) \end{pmatrix},
\]

\[
\psi^\dagger(x) = l^\dagger(x) \oplus r^\dagger(x) \cong \begin{pmatrix} l^\dagger(x) & r^\dagger(x) \end{pmatrix},
\]

the action of the linear and anti-linear discrete operations can be expressed with the Dirac matrices given naturally in the chiral or Weyl representation,

\[
\text{CPT} \bullet \psi(x) \cong \psi^\dagger(-x)\gamma^5 \tag{57}
\]

\[
T \bullet \psi(x) \cong \gamma^1\gamma^3\psi(-t,\vec{x}) \tag{58}
\]

\[
\text{CP} \bullet \psi(x) \cong \psi^\dagger(t,\vec{x})\gamma^0\gamma^2. \tag{59}
\]

So we recover the discrete symmetry operations of quantum field theory on the Dirac spinor fields\(^\text{14}\).

\section{Discrete Symmetry Operations and Inner Symmetries}

The spinor fields of the Standard Model have nontrivial inner (gauge) symmetries. They are mappings onto a tensor product space \(V \otimes U\) of a representation space \(V\) for the Lorentz symmetry and a representation space \(U\) for the inner symmetry. To extend the discrete symmetries on this tensor product space we have to continue the operations \(\text{co}, \delta\) and \(\varepsilon\) on the inner

\(^{11}\)Because \(l^\dagger\) is dual to \(r\) and \(r^\dagger\) is dual to \(l\), the dual to \(\psi\) is the Dirac adjoint \(\bar{\psi}\).
symmetry space compatible with the inner symmetry. This is trivial for the canonical conjugation $c_0$, since its canonical construction does not depend on the symmetry structure. The euclidian conjugation $\delta$ can be continued on every inner symmetry space, because the inner symmetries are positive unitary groups, the invariance groups of the euclidian conjugation (eq.\((30)\)). In general, however, this is not possible for the $SU(2)$ and $SL(2,\mathbb{C})$ invariant volume form $\varepsilon$, because for more than two complex dimensions $\varepsilon$ is not bilinear.

To be more explicit, let us focus on some fields used in the Standard Model. The left-handed leptons carry a left-handed Weyl representation of the Lorentz group and the fundamental representation of $SU(2)$ weak isospin. They are elements of a tensor product space $\mathbf{l} \in V_l = V \otimes U_2 \cong \mathfrak{c}^2 \otimes \mathfrak{c}^2$. The canonical conjugation

$$c_{V_l} : V_l \longrightarrow \overline{V_l} = \overline{V} \otimes \overline{U}_2$$

(60)
defines the representations of the Lorentz group and the weak isospin on the anti-space. Thus $CPT$ is per definition compatible with the symmetry structure. For the time reversal we need an $SU(2)_{\text{spin}} \times SU(2)_{\text{isospin}}$ compatible anti-linear automorphism. This is possible if we use the weak isospin volume form $\varepsilon_{U_2}$, with the action

$$T \cdot \mathbf{l} = \left( \delta_{V_l}^{-1} \circ \varepsilon_{V} \otimes \varepsilon_{U_2} \right) (\mathbf{l})$$

(61)

$$CP \cdot \mathbf{l} = \left( c_{V_l} \circ (\varepsilon_{V} \otimes \varepsilon_{U_2})^{-1} \circ \delta_{V_l} \right) (\mathbf{l}).$$

(62)

These operations include an interchange between the two weak isospin components (after spontaneous symmetry breaking they can be identified for example with the neutrino and the electron). This introduction of $\varepsilon_{U_2}$ is equivalent to the introduction of $G$-parity for the strong isospin $[19, 20, 21]$. Hence eq.(62) defines a weak isospin $GP$ operation.

The situation changes drastically if we include symmetries $SU(n)$ with $n \geq 3$, like colour-$SU(3)$. As the simplest example we use the right-handed quarks. They carry a right-handed Weyl representation of the Lorentz group and a fundamental triplet representation of $SU(3)_{\text{colour}}$. They are elements of a tensor product space $\mathbf{q} \in V_q = \overline{V^T} \otimes U_3 \cong \mathfrak{c}^2 \otimes \mathfrak{c}^3$. The canonical conjugation again defines $\overline{V_q}$ and its representation structure, thus $CPT$ is defined. Since there is no $SU(3)$ invariant bilinear form on $U_3$ with which
one could extend the operations $T$ and $CP$ on the inner symmetry space, $T$ and $CP$ cannot be defined compatibly with colour-$SU(3)$.

To be even more explicit, take the fundamental matrix representations of the gauge groups $SU(2)$ and $SU(3)$ given by the Pauli- and Gell-Mann-matrices, resp.

\[
D(u_2).1 = e^{\frac{i}{2} \alpha_j \tau_j} 1, \quad j = 1, \ldots, 3, \quad \alpha_j \in \mathbb{IR}
\]
\[
D(u_3).\mathbf{q} = e^{\frac{i}{2} \beta_a \lambda_a} \mathbf{q}, \quad a = 1, \ldots, 8, \quad \beta_a \in \mathbb{IR}.
\]

Then the representation of $SU(2)$ and $SU(3)$ on the anti-spaces (in this case equivalent to the dual representation) is given by

\[
\bar{D}(u_2).1^\dagger = e^{-\frac{i}{2} \alpha_j \tau_j} 1^\dagger, \quad (65)
\]
\[
\bar{D}(u_3).\mathbf{q}^\dagger = e^{-\frac{i}{2} \beta_a \lambda_a} \mathbf{q}^\dagger, \quad (66)
\]

with e.g. $\bar{\tau}_j$ denotes the conjugation of the entries in the matrix without transposition. The action of $CP$ is a linear operation between these two representation spaces. A compatible $CP$ operation has to fulfill

\[
CP \cdot (D(u).\psi) = \bar{D}(u).\psi_{CP} \quad (67)
\]

For the $SU(2)$ representation this is possible with the matrix representation of $\varepsilon_{U_2}$ usually denoted by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (for simplicity the Lorentz structure is omitted)

\[
1_{CP} = i\tau_2 1^\dagger \quad (68)
\]
\[
\Rightarrow e^{-\frac{i}{2} \alpha_j \tau_j} i\tau_2 1^\dagger = i\tau_2 \left(e^{\frac{i}{2} \alpha_j \tau_j} 1^\dagger\right), \quad (69)
\]

but there is no linear operation for the $SU(3)$ representation corresponding to $\tau_j = i\tau_2 \bar{\tau}_j i\tau_2$.

### 7 Treatment of $CP$ in the Standard Model

The interaction Lagrangian of a gauge theory is believed to be invariant under $CP$ transformation. The $CP$ violating part in the Standard Model

\footnote{This sloppy notation seems to distinguish a basis. Its use is justified only by the identical matrix of the dual isomorphism $\varepsilon_{U_2}$ and the endomorphism $i\tau_2$.}
is provided by the mixing of the three families via the Cabibbo-Kobayashi-Maskawa matrix \[\mathbf{22}\]. Whereas the invariance of the interaction Lagrangian seems to contradict our analysis, we can show that it agrees with the KM-theory. For the standard treatment of CP we follow \[\mathbf{18,23,24}\].

From the action of CP on Dirac spinor fields, eq.(59), one can calculate the action on all bilinear products, especially for the $U(1)$-current

$$j^\mu = \frac{1}{2} [\bar{\psi}, \gamma^\mu \psi],$$

\[CP \cdot j^\mu(x) = CP \cdot \frac{1}{2} [\bar{\psi}(x), \gamma^\mu \psi(x)] = -\frac{1}{2} [\bar{\psi}(t, \vec{x}), \gamma^\mu \psi(t, \vec{x})] = -j^\mu(t, \vec{x}). \tag{70}\]

Including inner degrees of freedom the currents $j^\mu_{ij} = \frac{1}{2} [\bar{\psi}_i, \gamma^\mu \psi_j]$, with $\psi_i$ is a (basis) vector in the inner symmetry space and $i, j$ are the inner indices, transform according to

$$CP \cdot j^\mu_{ij}(x) = CP \cdot \frac{1}{2} [\bar{\psi}_i(x), \gamma^\mu \psi_j(x)]
\begin{align*}
&= -\frac{1}{2} [\bar{\psi}_j(t, -\vec{x}), \gamma^\mu \psi_i(t, -\vec{x})] \\
&= -j^\mu_{ji}(t, -\vec{x}). \tag{71}\end{align*}$$

This current couples to the Lie-algebra valued gauge fields

$$G^\mu_{ij}(x) = G^{a\mu}(x) \mathcal{D}(l_a)_{ij}.$$ 

with $G^{a\mu}(x)$ the gauge field and $\mathcal{D}(l_a)$ a matrix representation of the gauge Lie-algebra. Choosing the transformation properties of the gauge fields in such a way that

$$CP \cdot G^\mu_{ij}(x) = -G^\mu_{ji}(t, -\vec{x})
\begin{align*}
&= -G_{\mu}^{a}(t, -\vec{x}) \mathcal{D}(l_a)_{ji} \\
&=: G_{\mu}^{a}(t, -\vec{x}) \mathcal{D}(l_a)_{ij}. \tag{72}\end{align*}$$

e.g. for the representation of $su(2)$ with the Pauli matrices

$$CP : \left( G^{1\mu}, G^{2\mu}, G^{3\mu} \right)(x) \mapsto - \left( G^{1\mu}_{\mu}, -G^{2\mu}_{\mu}, G^{3\mu}_{\mu} \right)(t, -\vec{x}), \tag{73}\$$

and for the representation of $su(3)$ with the Gell-Mann matrices

$$CP : \left( G^{1\mu}, G^{2\mu}, G^{3\mu}, G^{4\mu}, G^{5\mu}, G^{6\mu}, G^{7\mu}, G^{8\mu} \right)(x)$$

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This leaves the Lagrange density formally invariant.

There are two points of criticism.

First, the $CP$ transformations for the gauge bosons are basis-dependent: the transformations (73) and (74) are given only with the representation of $su(2)$ and $su(3)$ by the Pauli- or the Gell-Mann matrices, resp. Another representation would yield another transformations of the gauge fields.

Whereas in the elektroweak sector via the Higgs field there exists a distinguished basis (asymptotically we know the difference between electron and neutrino), this is believed not to be the case for the quarks. The assumption of a distinguished basis in the colour-space contradicts the concept of an unbroken $SU(3)$ gauge theory.

Secondly, an operation on the Lie-algebra representation without referring to its action on the vector space of the representation is, at least from an algebraic point of view, unsatisfactory. Vice versa, the action on the endomorphisms are uniquely determined by the action on the representation space via the inducing construction.

The problem of $CP$-violation is treated in the Standard Model in terms of the three families and their mixing by the Cabibbo-Kobayashi-Maskawa matrix. The CKM matrix is an unitary basis transformation of a three dimensional family space $F \cong \mathcal{Q}^3$. Neglecting the inner symmetries the down-type quarks for example are elements of a tensor product space $V_d = V \otimes F \cong \mathcal{Q}^2 \otimes \mathcal{Q}^3$. The mathematical structure is similar to the case of the right-handed quarks with the only difference, that in $F$ there are fixed bases - one for to the mass eigenstates and one for to the weak interaction (current eigenstates) - correlated by the CKM matrix. Just as $CP$ is not compatible with $SU(3)_{colour}$, it is not compatible with the $U(3)$ CKM matrix, either. Because $SO(3)$ has an invariant dual isomorphism $CP$ would not be violated if the CKM matrix would be orthonormal. In this sense our analysis coincides with the Kobayashi-Maskawa theory.

\[ A \text{ basis-independent operation for the Lie algebra representation of } su(2) \text{ for } G_{\mu} \mapsto G_{\mu}^T \text{ is given only by the volume form } \varepsilon_{U_2} \]

\[ \varepsilon_{U_2} \circ \mathcal{D}(l) \circ \varepsilon_{U_2}^{-1} = \mathcal{D}(l) = -\mathcal{D}(l)^T. \]

This is the action of the $GP$-operation we introduced in (62) on the endomorphisms.
8 Conclusion

Without a separation of space and time there is a $CPT$-action but no $CP$ and $T$-actions.

Therefore in a theory without massive asymptotic states like pure QCD there is no need in defining operations for $CP$ and $T$. Hence, the impossibility of defining a $CP$ operation compatible with $SU(3)$ might be of no phenomenological consequence. For the colourless asymptotic particles of the strong interaction, the massive hadrons, $CP$ is well defined. It would be of interest whether the impossibility of defining an $SU(3)$-compatible $CP$ operation leads, via the Higgs mechanism or the confinement, to $CP$ violation given by the family-mixing of the KM-theory. A hint may be the similarity in the mathematical structure of the three colours and the three families.

The (mathematical) difference of the $SU(3)$ gauge theory and the KM-theory with respect to $CP$ is the nonexistence of a basis in the colour-space whereas there are two distinct bases in the family-space. This situation can be visualized easier in the elektroweak sector. Here in the pure $U(1) \times SU(2)$ gauge theory there is no basis given in the representation space, i.e. there is no difference between electron- and neutrino fields. Spontaneous symmetry breaking distinguishes bases for the Higgs field in the inner symmetry space. With the breaking of $SU(2)_L \times U(1)_Y$ to $U(1)_Q$ comes along the concept of mass and therewith the difference of electron- and neutrino particles. Hence the basis depending operation of $\left[ \begin{array}{c} \mathbb{R} \\ \mathbb{R} \end{array} \right]$ becomes possible after spontaneous symmetry breaking, i.e. for the asymptotic states.

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A The Lorentz Bilinear Form of the Cartan Representation of Minkowski space

Every dual isomorphism is equivalent to a bilinear form. The action of the spinor metric $\varepsilon$ on the Cartan representation of Minkowski space

$$\varepsilon \bullet x = \varepsilon x^T \varepsilon \in M^T$$

is again a dual isomorphism. We show that this is equivalent to the Lorentz bilinear form by calculating the action of this dual vector on the Minkowski vectors:

$$< x | y >_\varepsilon = (\varepsilon \bullet x)(y) = \frac{1}{2} \text{tr} \varepsilon x^T \varepsilon y$$

Especially in the Weyl basis this gives the familiar form of the Lorentz bilinear form

$$< x | y >_\varepsilon = \frac{1}{2} \text{Tr} \left[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] i \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}^T$$

$$\times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} i \begin{pmatrix} y_0 + y_3 & y_1 - iy_2 \\ y_1 + iy_2 & y_0 - y_3 \end{pmatrix}$$

$$= x_\mu \eta^{\mu\nu} y_\nu,$$

with $\eta = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

B The Time-like Basis Vector $e^0$

In sect.4 we defined as time-like basis vector $e^0 = i \sigma_5 \delta^{-1}$. Here we prove the properties

1. $\left(e^0\right)^\times = -e^0$  \hspace{1cm} (77)

2. $< e^0 | e^0 > = 1$. \hspace{1cm} (78)

by referring only to the properties of $\sigma_5, \delta$ and $\varepsilon$.  

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The first equation writes

\[
(e^0)^x = (ico_V^{-1}\delta^{-1})^x \\
= co_V^{-1}(ico_V^{-1}\delta^{-1})^T co_V^{-1} \\
= -ico_V^{-1}\delta^{-1} co_V T co_V^{-1} = -e^0.
\]

For the second equation we have to compute the square of \(e^0\) with respect to the Lorentz bilinear form in its basis independent definition of eq.(76):

\[
\langle e^0 | e^0 \rangle = \frac{1}{2} \text{tr} \bar{\epsilon} e^0 T \bar{\epsilon} e^0 \\
= \frac{1}{2} \text{tr} \bar{\epsilon}(ico_V^{-1}\delta^{-1})^T \bar{\epsilon} ico_V^{-1}\delta^{-1} \\
= -\frac{1}{2} \text{tr} \bar{\epsilon}\delta^{-1} co_V T \bar{\epsilon} co_V^{-1}\delta^{-1} \\
= -\frac{1}{2} \text{tr} \bar{\epsilon}\delta^{-1} \bar{\epsilon}\delta^{-1} \\
= -\frac{1}{2} \text{tr} (-\mathbb{1} V_T) = 1.
\]

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