On Auxiliary Fields in BF Theories

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July 29, 2021

Abstract

We discuss the structure of auxiliary fields for non-Abelian BF theories in arbitrary dimensions. By modifying the classical BRST operator, we build the on-shell invariant complete quantum action. Therefore, we introduce the auxiliary fields which close the BRST algebra and lead to the invariant extension of the classical action.

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Recently, an approach for performing the quantization of the simple supergravity has been introduced, see the first reference of [1]. This allows to construct the full quantum action with its on-shell BRST symmetry through a modification of the classical BRST symmetry which follows from a geometrical BRST operator obtained by using a principal superfiber bundle, and of the gauge-fixing action written as in Yang-Mills type theories. Furthermore, the minimal set of auxiliary fields and the invariant extension of the classical action have been recovered. The approach was meant as an alternative to the study of supergravity relying on the superspace formalism (see e.g. Ref. [2]), on the Batalin-Vilkovisky quantization procedure [3], and on the BRST superspace approach [4]. In the second reference of [1] the above quantization scheme for simple supergravity has been generalized to general open gauge theories.

In this note we derive the structure of the auxiliary fields in the non-Abelian BF theory, as a model example for reducible gauge theories, by using the same procedure as discussed in Ref. [1]. To this purpose, we first determine the BRST transformations by an appropriate extension of the gauge transformations expressed à la BRST, so that we guarantee the invariance of the complete quantum action in which the gauge-fixing action is put in a form similar to that of Yang-Mills theories. We then introduce the auxiliary fields as combinations of the fields associated to the reducible symmetry, so that we realize the closedness of the BRST algebra and the determination of the invariant extension of the classical action.

Let us note that the quantization of BF theories in the framework of the Batalin-Vilkovisky procedure has been realized in Refs. [5] [11]. In Ref. [6], a quantization of the four-dimensional BF theory has been discussed in relation to the superfiber bundle formalism. While, in Ref. [7] auxiliary fields for BF theories have been introduced in terms of a BRST superspace formalism. For other work on the introduction of auxiliary fields in BF theories see Refs. [8] [9] [12]. In particular, let us remark that in Ref. [12] the off-shell nilpotency of the BRST charge for topological non-Abelian BF theories in D-dimensions is guaranteed by combinations of the ghost for ghost fields and their conjugate momenta. Aim of the present letter is to show that auxiliary fields in the non-Abelian BF theories may also be obtained by generalizing the approach developed for irreducible theories with open algebra [1].

Now, let us start with the following BRST transformations

\[
QA = - Dc, \quad Qc = - \frac{1}{2} [c, c], \quad Q = - \frac{1}{2} [c, c],
\]

\[
QB_{n-g} = - DB_{n-g+1} - [c, B_{n-g}^g] \quad (0 \leq g \leq n - 1),
\]

\[
Q B_{n+1}^0 = - [c, B_{n+1}^0],
\]

(1)

where \(A\) is the Yang-Mills gauge field and \(c\) its associated ghost, \(B = B_{n+1}^0\) is the rank-\(n\) antisymmetric tensor gauge field and \(B_{n-g}^g\) \((0 \leq g \leq n)\) its associated ghosts, with the lower (upper) index denotes the form degree (the ghost number). These transformations represent the symmetries of \((n - 1)\)-stage reducible BF theory in \((n + 2)\) dimensions (for a review see Ref. [10]). They
can be simply derived by writing à la BRST the Yang-Mills symmetry, \( \delta A = -D\omega, \delta B = -[B, \omega] \), and the reducible symmetry \( \delta_{red} A = 0, \delta_{red} B = -D\lambda \), of the classical action \( S_0 = BF \), where \( F = DA = DA + \frac{1}{g} [A, A] \) is the Yang-Mills field strength and the integration sign over the \((n+2)\)-dimensional spacetime as well as the trace over the indices of the gauge group are omitted for simplicity. A consequence of the \((n-1)\)-stage on-shell reducible symmetry \( \delta_{red} \) is that the BRST operator \( Q \) as defined by Eq. (1) is nilpotent on shell, we have

\[
Q^2 B_{n-g}^g = - \left[ F, B_{n-g-2}^{g+2} \right] \quad (0 \leq g \leq n - 2).
\]

Because of this on-shell nilpotency at the classical level, it is obvious that a \( Q \)-exact form of the gauge-fixing action \( S_{gf} \), i.e. \( S_{gf} = Q\psi \) is not suitable to build the full invariant quantum action, \( S_q = S_0 + S_{gf} \). We remark that \( \psi \) denotes a gauge fermion introduced to implement gauge constraints associated to all the invariances of the classical action \( S_0 \). It can be cast in the form (see Refs. [5] and [10])

\[
\psi = \pi Y_{n+2}^0 + \sum_{k=1}^{n} \sum_{g=0}^{n-k} B_{n-g-k}^{\gamma(k)} Z_{g+k+2}^{-\gamma(k)-1},
\]

where \( \pi \) is the antighost of the Yang-Mills symmetry and \( Y_{n+2}^0 \) the associated gauge constraint, and \( B_{n-g-k}^{\gamma(k)} \) (\( 1 \leq k \leq n, 0 \leq g \leq n - k, \gamma(k) = g \ (-g - 1) \) for \( k \) even (odd)) are the antighosts of the reducible symmetry and \( Z_{g+k+2}^{-\gamma(k)-1} \) the associated gauge constraints. The gauge-fixing functions \( Y_{n+2}^0 \) and \( Z_{g+k+2}^{-\gamma(k)-1} \) may depend only on \( A \) and \( B_{n-g-k}^{\gamma(k)} \) \( (0 \leq k \leq n, 0 \leq g \leq n - k) \) respectively.

For example, we can choose the following usual constraints:

\[
Y_{n+2}^0 = d \ast A \quad \text{and} \quad Z_{g+k+2}^{-\gamma(k)-1} = d \ast B_{n-g-k}^{\gamma(k-1)} - d \ast B_{n-g-k}^{(k-1)}.
\]

The antighosts allow to introduce the Stueckelberg auxiliary fields \( b \) and \( \pi^+_{n-g-k} \) which permit to find the gauge-fixing conditions in a consistent way. This is done through the action of the operator \( Q \) so that

\[
Q\pi = b, \quad Qb = 0,
\]

\[
Q B_{n-g-k}^{\gamma(k)} = \pi^+_{n-g-k}, \quad Q \pi^+_{n-g-k} = 0.
\]

Moreover, in order to find the complete quantum action with on-shell nilpotent BRST symmetry, we have to modify both the classical BRST operator \( Q \) and the gauge-fixing action \( S_{gf} = Q\psi \). We proceed in analogy to what is realized for the case of supergravity [1]. We define a quantum BRST operator \( \Delta \) by modifying the classical one \( Q \) as follows

\[
\Delta = Q + \bar{Q},
\]

and a gauge-fixing action \( S_{gf} \) written as in Yang-Mills type theories by replacing \( Q \) in \( S_{gf} = Q\psi \) by \( \left(Q + x\bar{Q}\right)\), i.e. we put the gauge-fixed quantum action in
the form

\[ S_q = S_0 + \left( Q + x \tilde{Q} \right) \psi. \]  \hspace{1cm} (6)

In Eqs. (5) and (6) the determination of the operator \( \tilde{Q} \) and of the numerical coefficient \( x \) is guaranteed by the requirements that \( S_q \) is \( \Delta \)-invariant and that \( \Delta \) is nilpotent on shell at the quantum level. We note that the operator \( \tilde{Q} \) has vanishing action on the fields \( X \) satisfying \( Q^2 X = 0 \), since in this case the nilpotency of \( \Delta \) is automatically guaranteed, i.e. \( \Delta^2 X = Q^2 X = 0 \). So, the action of \( \tilde{Q} \) is non-trivial only on the fields \( B^g \) \( (0 \leq g \leq n - 2) \).

According to Eq. (2), the \( \Delta \)-variation of \( S_q \) can be written in the following form

\[ \Delta S_q = \left( \tilde{Q}B - \sum_{g=0}^{n-2} \left[ \psi_g, B^{g+2}_{n-g-2} \right] \right) F + \sum_{g=0}^{n} \psi_g \tilde{Q}B^g + x \sum_{g=0}^{n-2} \psi_g \tilde{Q}B^g_n - \frac{1}{2} \sum_{h=0}^{n-g-2} \left[ \psi_g, \psi_h \right] QB_{n-g-h-2}^g \]  \hspace{1cm} (7)

where \( \sigma = \sum_{k=1}^{n} \sum_{g=0}^{n-k} \psi \gamma_{(k,g)} QB_{n-g-k}^g \) and \( X_{\gamma(g)} (X_{\gamma(0,g)} = X_n) \) denotes the variation of \( X \) with respect to \( B_{n-g-k}^g \).

After a straightforward calculation, we find that Eq. (7) acquires the form

\[ \Delta S_q = (2x - 1) \sum_{g=0}^{n-2} \left\{ \sigma_g \tilde{Q}B^g_{n-g} - \frac{1}{2} \sum_{h=0}^{n-g-2} \left[ \psi_g, \psi_h \right] QB_{n-g-h-2}^g \right\} \]  \hspace{1cm} (8)

by taking

\[ \tilde{Q}B^g_{n-g} = \sum_{h=0}^{n-g-2} \left[ \psi_h, B_{n-g-h-2}^g \right]. \]  \hspace{1cm} (9)

We remark that it is the first term on the right hand side of Eq. (7) which leads to choose the solution as given in Eq. (9). We note that, in deriving Eq. (8), we have used the fact that the second term of the right hand side of Eq. (7) can be put, modulo a total divergence, in the form

\[ \sum_{g=0}^{n} \psi_g \tilde{Q}B^g_n = \frac{1}{2} \sum_{g=0}^{n-2} \sum_{h=0}^{n-g-2} \left[ \psi_g, \psi_h \right] QB_{n-g-h-2}^g, \]  \hspace{1cm} (10)
in view of Eqs. (1) and (9) and of the Jacobi identity. However, inserting Eq. (9) into the third term, we get

\[ x \sum_{g=0}^{n-2} \psi_{g} \tilde{Q} B_{n-g}^{g} = x \sum_{g=0}^{n-2} \left\{ \sigma_{g} \tilde{Q} B_{n-g}^{g} - \sum_{h=0}^{n-g-2} \left[ \psi_{g}, \psi_{h} \right] Q B_{n-g-h-2}^{g+h+2} \right\}. \tag{11} \]

Finally, using Eq. (9) and the Jacobi identity and after some computations, we find that the last term on the right hand side of Eq. (7) vanishes,

\[ \sum_{g=0}^{n-2} \psi_{g} \tilde{Q} B_{n-g}^{g} = 0. \tag{12} \]

Obviously the invariance of the quantum action \( S_{q} \) with respect to the quantum BRST operator \( \Delta \) is totally ensured by taking, besides the operator \( \tilde{Q} \) as constructed in Eq. (9),

\[ x = \frac{1}{2}. \tag{13} \]

From the constructed quantum action \( S_{q} \) and its BRST symmetry operator \( \Delta \), it follows

\[ \Delta^{2} B_{n-g}^{g} = - \sum_{h=0}^{n-g-2} \left[ S_{q,h}, B_{n-g-h-2}^{g+h+2} \right], \tag{14} \]

i.e. the quantum BRST operator \( \Delta \) is nilpotent on shell. For simplicity we will not write down here the equations of motion \( S_{q,h} \) associated to the fields \( B_{n-h}^{h} \). In particular, we note that in deriving Eq. (14) the identity

\[ \sum_{h=2}^{n-g-2} \sum_{l=0}^{n-g-h-2} \left[ \left[ \psi_{l}, \psi_{(h-2)} \right], B_{n-g-h-l-2}^{g+h+l+2} \right] = \sum_{h=2}^{n-g-2} \sum_{l=0}^{n-g-2} \left[ \left[ \psi_{l}, \psi_{(h-1-2)} \right], B_{n-g-h-2}^{g+h+2} \right] \tag{15} \]

has been used.

Now, after having obtained the on-shell BRST invariant quantum action for arbitrary dimensional non-Abelian BF theory, we are in a position to derive the structure of the auxiliary fields for such a theory in analogy to what is done in simple supergravity [1]. This is to show that another way to close the BRST algebra via the introduction of auxiliary fields and to build the BRST invariant extension of the classical action for BF theory may be simply related to the on-shell BRST invariant quantum action.

For this purpose, let us first rewrite the action of the quantum BRST operator \( \Delta \) on the fields \( B_{n-g}^{g} (0 \leq g \leq n-2) \) and the quantum action \( S_{q} \) by replacing \( \psi_{h} \) with \( H_{h+2}^{g-h-1} \), we have
BRST transformations are determined by imposing the off-shell nilpotency of the quantum BRST operator $\Delta$, we obtain

$$\Delta B_{n-g}^g = QB_{n-g}^g + \sum_{h=0}^{n-g-2} \left[ H_{n-2}^{g-h-1}, B_{n-g-h-2}^{g+h+2} \right] + \sum_{h=0}^{n-g-2} \left[ H_{h+2}^{g-h-1}, H_{h+2}^{g-h-1} \right] B_{n-g-h-2}^{g+h+2} + \Delta \psi. \quad (16)$$

However, considering $H_{h+2}^{g-h-1}$ ($0 \leq h \leq n-2$) as true fields, it follows then that their equations of motion are algebraic, $\sum_{h=0}^{n-g-2} \left[ \psi, H_{h+2}^{g-h-1}, B_{n-g-h-2}^{g+h+2} \right] = 0$, and by inserting them into Eq. (16), which is equivalent to replace $H_{h+2}^{g-h-1}$ with $\psi$, again we obtain the quantum action and its on-shell BRST symmetry. Therefore, the fields $H_{h+2}^{g-h-1}$ are nondynamical, auxiliary fields. Their BRST transformations are determined by imposing the off-shell nilpotency of the quantum BRST operator $\Delta$, we obtain

$$\Delta H_{n-2}^{-1} = F - [c, H_{n-2}^{-1}],$$
$$\Delta H_{n-2}^{-1} = -DH_{n-2}^{-1} - [c, H_{n-2}^{-2}],$$
$$\Delta H_{h+2}^{-h-1} = -DH_{h+1}^{-h-1} - [c, H_{h+2}^{-h-1}] + \frac{1}{2} \sum_{l=0}^{h-2} \left[ H_{l+2}^{l+1}, H_{h-l}^{l+1} \right], 2 \leq h \leq n \quad (17)$$

where, in particular, the last term in the third transformation arises from the following identity

$$\sum_{h=0}^{n-g-2} \sum_{l=0}^{n-g-h-4} \left[ H_{h+2}^{g-h-1}, H_{l+2}^{l-l-1}, B_{n-g-h-l-1}^{g+h+l+4} \right] = \frac{1}{2} \sum_{h=0}^{n-g-2} \sum_{l=0}^{n-g-h-2} \left[ H_{h+2}^{h+1}, H_{h-l}^{h-l-1}, B_{n-g-h-l}^{g+h+2} \right]. \quad (18)$$

Furthermore, after a similar straightforward calculation, it is easy to show that

$$S_{inv} = BF - \frac{1}{2} \sum_{g=0}^{n-2} \sum_{h=0}^{n-g-2} \left[ H_{g+2}^{g-h-1}, H_{h+2}^{g-h-1} \right] B_{n-g-h-2}^{g+h+2} \quad (19)$$

represents the $\Delta$-invariant extension of the classical action.

In summary, we have realized the construction of the off-shell invariant quantum action for non-Abelian BF theories in arbitrary dimensions by introducing auxiliary fields. The obtained results are equivalent to those derived in terms of a superspace formalism [7]. In the present paper we have shown, analogous to the case of supergravity [1], how the auxiliary fields simply emerge as combinations of the ghosts related to the reducible symmetry and their anti-ghosts through the gauge-fixing fermion. In doing so it is particularly the invariant
extension of the classical action which is built automatically. This follows from
the cubic ghost interactions term in the on-shell invariant full quantum action in
which the gauge-fixing action has been written as in Yang-Mills type theories by
modifying the classical BRST operator. Finally, as discussed in Ref. [1] for the
case of open gauge theories, it would be interesting to extend the prescription
developed here and to find how to quantize general reducible gauge theories via
the introduction of auxiliary fields.

Acknowledgements

MT would like to thank Prof. W. Ruehl for his kind hospitality during his
stay at the University of Kaiserslautern. He acknowledges support from the
Alexander von Humboldt Foundation.

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