Mechanism for particle fractionalization and universal edge physics in quantum Hall fluids

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Advancing a microscopic framework that rigorously unveils the underlying topological hallmarks of fractional quantum Hall (FQH) fluids is a prerequisite for making progress in the classification of strongly-coupled topological matter. We present a second-quantization framework that reveals an exact fusion mechanism for particle fractionalization in FQH fluids, and uncovers the fundamental structure behind the condensation of non-local operators characterizing topological order in the lowest-Landau-level. We show the first exact analytic computation of the quasielectron Berry connections leading to its fractional charge and exchange statistics, and perform Monte Carlo simulations that numerically confirm the fusion mechanism for quasiparticles. We express the sequence of (bosonic and fermionic) Laughlin second-quantized states, highlighting the lack of local condensation, and present a rigorous constructive subspace bosonization dictionary for the bulk fluid. Finally, we establish universal long-distance behavior of edge excitations by formulating a conjecture based on the DNA, or root state, of the FQH fluid.
Fractional quantum Hall (FQH) fluids have long constituted the best-known paradigm of strongly-correlated topological systems. Nonetheless, several fundamental issues remain unresolved. These include the exact mechanism leading to the quasiparticle (or fractional electron) excitations and viable universal signatures in edge transport that are rooted in the topological characteristics of the bulk FQH fluid. This state of affairs is partially due to a dearth of rigorous microscopic approaches capable of dealing with these highly entangled systems. A case in point is a first-principles computation of the quasielectron exchange statistics. The Entangled Pauli Principle (EPP) was introduced as an organizing principle for FQH ground states.

The EPP provides information about the pattern of entanglement of the complete subspace of zero-energy modes, i.e., ground states, of quantum Hall parent Hamiltonians for both Abelian and non-Abelian fluids. Those fluid states are generated from the so-called “DNA”, or root states, which encode the elementary topological characteristics of the fluid.

In this work, we advance second-quantization many-body techniques that allow for new fundamental insights into the nature of quasiparticle excitations of FQH liquids. In particular, we present an exact fractionalization procedure that allows for a very natural fusion mechanism of quasiparticle generation. We determine the quasihole and quasiparticle operators that explicitly flesh out Laughlin’s flux insertion/removal mechanism and provide the associated quasielectron wave function. The quasielectron that we find differs from Laughlin’s original proposal. We determine the Berry connection of this quasielectron wave function, considered as an Ehresmann connection on a principal fiber bundle, and as a result, a natural fusion mechanism gets unfolded. This, in turn, leads to the exact determination of the quasielectron fractional charge. We perform Monte Carlo simulations to numerically confirm this fusion mechanism of fractionalization. Similarly, we address the problem of the quasielectron exchange statistics by calculating analytically the Berry connection for two quasielectrons.

In addition, we introduce an unequivocal diagnostic for characterizing and detecting the topological order of the FQH fluid in terms of condensation of a non-local operator and present a constructive subspace bosonization (fermionization) dictionary for the bulk fluid that highlights the topological nature of the underlying theory. Our organizing EPP and the corresponding fluid’s DNA encode universal features of the bulk FQH state and its edge excitations. Here we formulate a conjecture that enables a demonstration of the universal long-distance behavior of edge excitations in weak confining potentials. This is based on the exact computation of the edge Green’s function over the DNA or root state of the topological fluid. Although our main results are derived in a field-theoretical manner, we will reformulate some of our conclusions in a first quantization language, where states become wave functions. For clarity, we will occasionally use a mixed representation.

Results and discussion

States and operator algebra in the lowest-Landau-level. The lowest-Landau-level (LLL) is spanned by single-particle orbitals \( \phi_i(x, y) \) whose functional form depends on geometry. We consider genus zero manifolds such as those of the disk and the cylinder. Lengths are measured in units of the magnetic length \( \ell = \sqrt{\frac{\hbar c}{eB}} \), where \( B \) is the magnetic field strength, \( \hbar \) is the reduced Planck constant, \( c \) is the speed of light, and \( e \) is the magnitude of the elementary charge. For ease of presentation, we will primarily focus on the disk geometry. (We remark that one can always apply a similarity transformation to map to the cylinder.) Then, \( \phi_i(z = x + iy) = z^i/N_{r_i} \), \( N_{r_i} = \sqrt{2\pi 2r_i} \), with \( r \geq 0 \) a non-negative integer labeling the angular momentum and \( z \in \mathbb{C} \).

(Normalization is defined as \( \int D[z](z^*)^j z^j = N_i^2 \delta_{ij} \) with \( D[z] = \delta^2(x - iy) e^{-\frac{1}{2} z^2} \)). The many-particle state of the Hilbert space \( H_{LLL} \) belongs to either the totally symmetric (bosons) or antisymmetric (fermions) representations of the permutation group \( S_N \), with elements \( \sigma \in S_N \). Whenever results apply to either representation, we use second-quantized creation (annihilation) \( a_i^\dagger (a_i) \) operators instead of the usual \( c_i^\dagger (c_i) \), \( b_i^\dagger (b_i) \), for fermions and bosons, respectively.

The field operator

\[
\Lambda(z) = \sum_{r > 0} \phi_r(z) a_r, \tag{1}
\]

and its adjoint \( \Lambda^\dagger(z) \) satisfy canonical (anti-)commutation relations:

\[
[\Lambda(z), \Lambda^\dagger(z')]_\pm = 0, \quad [\Lambda(z), \Lambda^\dagger(z')]_\mp = (z' - z - i\epsilon),
\]

where \( z' = \pm e^{\pm i\epsilon} \) is a bilocal kernel satisfying \( \int D[z] \Lambda(z)\Lambda^\dagger(z') = \Lambda(z')^6 \).

Many-particle states \( |\psi\rangle \in H_{LLL} \) are characterized by the number of particles \( N \) and the maximal occupied orbital \( r_{\text{max}} \), defining a filling factor \( \nu = (N - 1)/r_{\text{max}} \). Given an antisymmetric holomorphic function \( \psi \), one can construct the states

\[
|\psi\rangle = \int \left( \prod_{i=1}^N D[z_i] \right) \psi(z_N) \Psi(z_1) \cdots \Psi(z_N) |0\rangle,
\]

where \( Z_N = \{z_1, z_2, \ldots, z_N\} \), in terms of the fermionic field operators \( \Phi(z) \). Similarly, one can construct states for bosons in terms of permanents and field operators \( \Phi(z) \).

We now introduce the operator algebra necessary for the LLL operator fractionalization and constructive bosonization. We first review the operator equivalents of the multivariate power-sum, \( p_d(z) \), and elementary, \( s_d(z) \), symmetric polynomials \( (d \geq 0) \) are, respectively:

\[
\begin{align*}
O_d &= \sum_{r>0} a_i^d a_r, \\
e_d &= \frac{1}{\mathcal{A}_d^d} \sum_{r_1>0} \sum_{r_2>0} \cdots \sum_{r_d>0} a^d_{r_1} a^d_{r_2} \cdots a^d_{r_d} \cdot \mathcal{A}_d^d,
\end{align*}
\]

(with \( a^d_i = N^d_i a_i^d, \ a_i = N^{-1} a_i \)). The operator Newton–Girard relations \( \mathcal{A}_d, s_d, e_d \) (with \( e_0 = 1 \)) link these operators with each other. The second-quantized extensions of the Newton–Girard relations are similar to dualities in that they appear twice in a row yields back the original operators. Interestingly, the operators \( O_d \) can be expressed in terms of Bell polynomials in \( e_d \)’s (see Supplementary Note 1). Consequently, any quantity expressible in terms of \( O_d \)’s can be also written in terms of \( e_d \)’s and vice versa. Both the \( O_d \) and \( e_d \) operators generate the same commutative algebra \( A \). Furthermore, they satisfy the commutation relations \( [O_d, a_i^j] = -a^j_{i+1}, [O_d, a_i^d] = a_i^d \) and \( [e_d, a_i] = a_i^d e_d - a_i^d e_{d-1} - e_{d-1} a_i^d \).

A set of first-quantized symmetric operators, of relevance to Laughlin’s quasielectron and conformal algebras, involves derivatives in \( z \). Similar to the operators defined above, we introduce symmetric polynomials \( p_d(\partial_z) \) and \( s_d(\partial_z) \) where their second-quantized representations are

\[
\begin{align*}
Q_d &= \sum_{r=0}^d r(r - 1) \cdots (r - d + 1) a_i^d a_r, \\
f_d &= \frac{1}{\mathcal{A}_d^d} \sum_{r_1>0} \sum_{r_2>0} \cdots \sum_{r_d>0} a^d_{r_1} a^d_{r_2} \cdots a^d_{r_d} \cdot \mathcal{A}_d^d,
\end{align*}
\]

and are Newton–Girard-related, \( df_d + \sum_{k=1}^d (1 - k) Q_d f_k = 0 \), with \( f_0 = 1 \). One can, analogously, define operators mixing polynomials and derivatives as in the positive \( (d, d \geq 0) \) Witt algebra \( \{\ell_d, \ell_d^\dagger\} = (d - d')\delta_{d,d'} \). These Witt algebra generators are \( \ell_d = -\sum_{i=1}^d e_i^d a_i^d \).

Their second-quantized version is
\[ \hat{d}_v = -\sum_{r>0} a^\dagger_{v, r} a_r. \]

Physically, the operators \( \mathcal{O}_d, \mathcal{e}_d, \hat{d}_v (\mathcal{Q}_d, \mathcal{f}_d) \) increase (decrease) the total angular momentum or "add (subtract) fluxes". Rigorous mathematical proofs appear in the Supplementary Note 1.

Symmetric operators stabilizing incompressible FQH fluids as their eigenvector with lowest eigenvalue are known as "parent FQH Hamiltonians". The EPP \(^2,14\) is an organizing principle for generating both Abelian and non-Abelian FQH states as zero modes (ground states) of frustration-free positive-semidefinite microscopic Hamiltonians. The Hamiltonian stabilizing Laughlin states of filling factor \( v = 1/M \), with \( M \) a positive integer, is known as the \( H_M = \sum_{r \neq m} \mathcal{O}_d \mathcal{e}_d \mathcal{Q}_d, \mathcal{f}_d \) Hamiltonian. Here \( H_M \) are the Haldane pseudopotentials and the sum is performed over all \( 0 \leq m < M \) sharing the even/odd parity of \( M \). As demonstrated before, the integrals over \( \eta_j, \eta_j, m) \) are geometry-dependent form factors. For odd (even) \( m \), the operator \( \hat{a} \) is \( \hat{e} \) \( \hat{b} \) respectively.

The space \( Z_M \) of all zero modes of \( H_M \) is generated by the states \( \psi_j \in \mathcal{H}_{LL\text{u}} \) satisfying \( T_{j,m} \psi_j = 0 \). This space contains the Laughlin state \( |\psi_M^N\rangle \) as its minimal total angular momentum, \( J = MN(N - 1)/2 \), state. All other zero modes are obtained by the action of some linear combination of products of \( \mathcal{O}_d, \mathcal{e}_d \) equivalent \( \mathcal{a}_b \) operators onto \( |\psi_M^N\rangle \).4,7 Inward squeezing is an angular-momentum preserving operation generated by

\[ A_{r'}^{\dagger} = a^\dagger_{r'} a_{r'} a_{r'} - d_{r'} + d_{r'} \quad \text{and} \quad d_{r'} > 0, \]

where the multiple actions on the root partition \( |\psi_M^N\rangle = \prod_{j=1}^{N} a^\dagger_{j-1} a_{j-1} d_{j-1} d_{j-1} = 0 \) generate all occupation number eigenstates \( |\lambda\rangle \) in the expansion of Laughlin state \( |\psi_M^N\rangle = \sum_{\lambda} b_\lambda \langle \lambda |\psi_M^N\rangle \) with integers \( b_\lambda \).4,7 By angular momentum conservation, \( \langle \psi_M^N | a_{r'}^\dagger a_{r'}^\dagger |\psi_M^N\rangle = a(r) \delta_{r, r'} ||\psi_M^N\rangle|^2 \). In the thermodynamic limit \( N, r \rightarrow \infty \), \( a \rightarrow N/(r + 1) \) we remain constant.

Operator fractionalization and topological order. Our next goal is to construct second-quantized quasihole and quasiparticle operators. Following Laughlin’s insertion/removal of magnetic fluxes, fractionalization is the notion behind that construction. Repeating this procedure \( M \) times should yield an object with quantum numbers corresponding to a hole or a particle. Surprisingly, as we shall show, the case of quasielectron excitations does not coincide with Laughlin’s proposal (nor other proposals).

As a byproduct, we will obtain a compact representation of Laughlin states (bosonic and fermionic) that emphasizes a sort of incompressibility Fermi-liquid non-conserving quasihole operator. Then, for both bosons and fermions.

\[ \langle \psi_M^N | a_{r'}^\dagger a_{r'}^\dagger |\psi_M^N\rangle = a(r) \delta_{r, r'} ||\psi_M^N\rangle|^2 \].

Although illuminating, this representation depends on \( |\psi_M^N\rangle \) itself through \( \tilde{U}_M(z) \) (see Supplementary Note 3). This condensation of non-local objects is hidden in the intrinsic topological order of Laughlin fluids. One can show this by studying the long-range order behavior of Read’s operator \(^{12}\). Before doing so, we need a result (see Supplementary Note 4) that justifies calling \( \tilde{U}_M(z) \) the quasihole operator. Had one created \( M \) quasiholes at position \( \eta \) one should generate an object with the quantum numbers of a hole. That is,

\[ \Lambda(\eta) |\psi_M^{N+1}\rangle \equiv |\psi_M^N\rangle |\psi_M^N\rangle \]

This relation was previously discussed in first-quantization language13,14. For a rigorous and general proof of Eq. (7) see the Supplementary Note 4.

Studying the long-range order of Read’s operator15 amounts to establishing that \( \langle \tilde{K}_M(z) |\tilde{K}_M(0)\rangle \) approaches a non-zero constant at large \( |z| \), or alternatively, the condensation of \( \tilde{K}_M(0) \) in the U(1) coherent state \( |\theta\rangle \equiv \sum_{N>0} \delta_{MN} e^{-i\eta N} |\psi_M^N\rangle \), where \( \delta_{MN} = \alpha_N = \alpha_N |\psi_M^N\rangle^{-1} \). We next expand on Read’s arguments12. Let us choose \( \alpha_N \) such that \( \psi_M^N = \alpha^N |\psi_M^N\rangle^{-1} \). In other words, expanding \( \langle \psi_M^N | \alpha^N |\psi_M^N\rangle \) represents a probability distribution concentrated around (an assumed large) \( N \). Using the operator fractionalization relation, \( \langle \theta |\tilde{K}_M(0)\rangle = \langle \theta |\psi_M^N\rangle \langle \psi_M^N |\lambda(\theta)\rangle \langle \psi_M^N |\psi_M^N\rangle \). Leading contributions to the sum come from terms with \( N \) close to \( N \), in which case \( \langle \psi_M^N |\lambda(\theta)\rangle \langle \lambda(\theta) |\psi_M^N\rangle \). Therefore, \( \langle \theta |\tilde{K}_M(0)\rangle \langle \theta |\psi_M^N\rangle \rightarrow \psi_M^N \psi_M^N \) for \( N \rightarrow \infty \). Obviously, \( \langle \theta |\tilde{K}_M(0)\rangle \) is not a local order parameter12.
Do we have a similar operator fractionalization relation for the quasiparticle operator $\hat{V}_N(\eta)$, which reduces to Laughlin’s quasielectron in the case of fermions? Since within the LLL one has $A(z)\hat{U}_N(\eta) = (2\Delta_z - \eta^2)^k \hat{U}_N(\eta)A(z)$ it seems natural, by analogy to the quasi-hole, to define quasiparticles as the second-quantized version of $W_N(\eta) = \prod_{i=1}^N (2\Delta_{z_i} - \eta^2)$, Laughlin’s original proposal. Note, though, that the second-quantized representation of this operator is $\hat{W}_N(\eta) = \sum_{d=d_0}^{\infty} (-\eta)^{d-d_0} \hat{d}_d$, and not $\hat{U}_N(\eta)$. This proposal does not satisfy the operator fractionalization relation $A^i(\eta)\psi^{N-1}_M = \hat{W}_N(\eta)^i\psi^{N}_M$ since total angular momenta do not match. A simple modification $A^i(\eta)\psi^{N-1}_M = \hat{W}_N(\eta)^i\psi^{N}_M$, can be made to match total angular momenta as can be easily verified by localizing the quasiparticle at $\eta = 0$. A proper embedding of the quasiparticle should satisfy

$$\Lambda^i(\eta)\psi^{N-1}_M = \hat{W}_N(\eta)^i\psi^{N}_M$$

as can be derived from the quasi-hole (i.e., hole fractionalization) relation. Indeed, this operator is well-defined when acting on the $N$-particle Laughlin state. Can $\hat{W}_N(\eta)^i\psi^{N}_M$ be written as the $M$-th power of another operator? Suppose that one wants to localize a quasiparticle at $\eta = 0$, then $\hat{U}_N(0)^i\psi^{N}_M = \hat{c}_0^i\psi^{N}_M$ and the problem reduces to proving that $\hat{c}_0^i\hat{c}_0^{-1}\hat{a}_0^i\hat{a}_0^{-1}$. Recall that any Laughlin state can be obtained by an inward squeezing process of a zeroth angular momentum state and that the Aharonov-Bohm effective charge $\eta$ differs significantly from prior proposals.

Wave functions of quasiparticles. The field-theoretic approach provides an elegant formalism to prove the exact mechanism behind particle fractionalization. We next illustrate how this mechanism is translated in a first-quantized language. To this end, we start using a mixed representation of the quasiparticle wave function. In this representation the corresponding quasiparticle (quasielectron) wave function, localized at $\eta \in \mathbb{C}$, is given by

$$\psi^{\text{qp}}_\eta(Z_N) = \Lambda^i(\eta)\psi^{(M-1)\text{qp}}_\eta(Z_N),$$

where $\Lambda^i(\eta)$ creates a particle in the state $\psi^{\text{qp}}_\eta(z) = N_0 e^{-i\delta z - \eta^2}$. (In first quantization the Gaussian factor is typically not included in the integration measure.)

$$\psi^{(M-1)\text{qp}}_\eta(Z_N) = N^{(M-1)\text{qp}} \prod_{k=1}^N (z_k - \eta)^{M-1}\psi^{(M-1)\text{qp}}_\eta(Z_N),$$

is the $M-1$-quasiholes, located at $\eta$, wave function for $N-1$ particles, and Laughlin’s (un-normalized) state

$$\psi^{\text{qp}}_\eta(Z_N) = \prod_{1 \leq i \neq j \leq N} (z_i - z_j)^M e^{-\frac{1}{2} \sum_{i=1}^N |z_i|^2}.$$

By the definition of the operator $\Lambda^i(\eta)$, then,

$$\psi^{\text{qp}}_\eta(Z_N) = \sqrt{N} \Lambda^i(\eta)\psi^{(M-1)\text{qp}}_\eta(Z_N).$$

This straightforwardly gives the first quantized wave function particle, for instance, in fermions

$$\hat{A}(\psi^\eta_\eta) = \frac{1}{N!} \sum_{\sigma_i \in \mathbb{S}_N} \text{sgn}(\sigma)(z_{\sigma(1)}, \ldots, z_{\sigma(N)}).$$

This straightforwardly gives the first quantized wave function particle, for instance, in fermions

$$\psi^{\text{qp}}_\eta(Z_N) = \sqrt{N} \Lambda^i(\eta)\psi^{(M-1)\text{qp}}_\eta(Z_N).$$

with all normalization factors included. We claim that this wave function is properly normalized. Indeed, we have

$$\langle \psi^{\text{qp}}_\eta^{(M-1)\text{qp}} | \Lambda^i(\eta) \psi^{\text{qp}}_\eta \rangle = \langle \psi^{(M-1)\text{qp}}_\eta | \Lambda^i(\eta) \psi^{\text{qp}}_\eta \rangle \Lambda^i(\eta).$$

Since the orbital $\psi^i_\eta$ is unoccupied in $\psi^{(M-1)\text{qp}}_\eta$, $\psi^{(M-1)\text{qp}}_\eta$ is an eigenstate of $\Lambda(\eta)\Lambda^i(\eta)$ with eigenvalue $1$. Therefore,

$$\langle \psi^{\text{qp}}_\eta | \psi^{\text{qp}}_\eta \rangle^{(M-1)\text{qp}} = \langle \psi^{(M-1)\text{qp}}_\eta | \psi^{(M-1)\text{qp}}_\eta \rangle,$$

and $\psi^{\text{qp}}_\eta$ is normalized if $\psi^{(M-1)\text{qp}}_\eta$ is normalized.

One can re-write the (un-normalized) quasiparticle (quasielectron) wave function $\psi^{\text{qp}}_\eta$ in an enlightening manner

$$\psi^{\text{qp}}_\eta(Z_N) = \Gamma^i(\eta)\psi^{\text{qp}}_\eta(Z_N),$$

with the quasiparticle (quasielectron) operator

$$\Gamma_i(\eta) = \sqrt{N} \prod_{j \neq i} e^{-\frac{1}{2} |z_i - z_j|^2},$$

which clearly shows that it differs significantly from prior proposals, see Supplementary Note 5. But this is not the whole story. It is even more illuminating to understand the precise mechanism leading to this remarkable quasiparticle, that we emphasize once more is not an Ansatz. Before doing so, we will first compute the charge of this excitation using the Berry connection idea advanced by Arovas et al. and further elaborated in the book by Stoner see Section 2.4 therein) for the quasihole, that is the Aharonov–Bohm effective charge coupled to magnetic flux. We will then show a remarkable exact
property of the charge density that will shed light on the underlying fractionalization mechanism.

**Berry connection for one quasiparticle.** For pedagogical reasons, we next focus on the fermionic (electron) case. Consider an adiabatic process (in time $t$) where the position of the quasiparticle, $\eta = \eta(t)$, is encircling an area enclosing a magnetic flux $\phi$. We will next show that the Berry connection decomposes into

$$\langle \Psi_{\eta} \left| \frac{d}{dt} \Psi_{\eta} \right| \rangle = iA_1 + iA_{M-1}. \quad (18)$$

As we will explain, $A_1$ describes the Berry phase contribution from a single particle (electron) and $A_{M-1}$ is the contribution from $M-1$ quasiholes. It is convenient to demonstrate this relation in second quantization, where only in the end, $A_{M-1}$ is computed from first quantization methods.\(^{22-24}\) So let $\Psi_{\eta}^{(M-1)qh} = \tilde{\Psi}_{\eta}^{(0)}|0\rangle$, where $\tilde{\Psi}_{\eta}^{(0)}$ is an element in the algebra generated by $c_j^\dagger$, where $c_j^\dagger$ creates a particle in the orbital $\psi_j(z)$. Thus,

$$\tilde{\Psi}_{\eta}^{(0)} = \sum_{j_1,...,j_{M-1}} F_{j_1,...,j_{M-1}} c_{j_1}^\dagger ... c_{j_{M-1}}^\dagger \quad (19)$$

with some coefficients $F_j$ dependent on $\eta$.

The statement made earlier that $\Psi_{\eta}^{(0)}(z)$ is not occupied in $\Psi_{\eta}^{(M-1)qh} = (1-M)^{-1} \Psi_{\eta}^{(0)} A(\eta)$ is equivalent to saying that

$$A(\eta) \Psi_{\eta}^{(0)} = (1-M)^{-1} \Psi_{\eta}^{(0)} A(\eta).$$

Thus, $A(\eta) = (1-M)^{-1} \Psi_{\eta}^{(0)} A(\eta)$ and $A(\eta)$ (or $\frac{d}{dt} A(\eta)$) with $\Psi_{\eta}^{(0)}$. From normalization,

$$\Lambda(\eta) A(\eta) |0\rangle = |0\rangle = \tilde{\Psi}_{\eta}^{(0)} |0\rangle.$$

Thus,

$$\langle \Psi_{\eta}^{(0)} \left| \frac{d}{dt} \Psi_{\eta}^{(0)} \right| \rangle = \langle 0 | \tilde{\Psi}_{\eta}^{(0)} A(\eta) \left( \frac{d}{dt} A(\eta) \right) \tilde{\Psi}_{\eta}^{(0)} |0\rangle$$

$$= \langle 0 | \tilde{\Psi}_{\eta}^{(0)} A(\eta) \left( \frac{d}{dt} A(\eta) \right) \tilde{\Psi}_{\eta}^{(0)} |0\rangle$$

$$= iA_1 + iA_{M-1}.$$

Therefore, the quasiparticle charge $e^* \equiv e$ has a contribution from a particle of charge $e$ and $M-1$ quasiholes of charge $-e/M$, i.e., $e^* = e - e(M-1)/M = e/M$, as expected. In simple terms, the channel fusing two quasiholes with one electron leads to a quasielectron of charge $e/3$ in an $v = 1/3$ Laughlin fluid. This is a very intuitive (and exact) mechanism that has been overlooked until now. Notice that we proved that the evaluation of the quasiparticle Berry connection is exact for any $N$, while the quasihole charge $e/M$ is only exact asymptotically in the limit $N \to \infty$ (see Section 2.4 of Stone's book).\(^*\)

**Charge density.** A consequence of this effective fusion mechanism manifests in the calculation of the quasiparticle charge density $\rho_{\eta^{(M-1)qh}}(z)$. We appeal once more to the fact that

$$\Lambda(\eta) \Psi_{\eta}^{(M-1)qh} = 0.$$  

This can be expressed as

$$\int d^2 r_j \Psi_{\eta}^{(0)}(z_j)^* \Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j) = 0,$$

for $j = 1, \ldots, N$. Here $d^2 r_j = \frac{1}{2} dx_j \wedge dz_j$ is the usual two dimensional measure on the complex plane.

We can write the quasielectron wave function as

$$\Psi_{\eta}^{(M-1)qh} = \frac{1}{\sqrt{N!}} \sum_{\{1\}^{N-1} \Psi_{\eta}^{(0)}(Z_{N-1}, z_j) \Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j),}$$

where $z_j$ means that coordinate $z_j$ is absent. We want to evaluate

$$\rho_{\eta^{(M-1)qh}}(z) = \frac{N}{\int d^2 r_j \Psi_{\eta}^{(0)}(z_j)^* \Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j) ^2}$$

$$= N \int d^2 r_j \cdots d^2 r_{N-1} \Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j) ^2$$

$$\times \Psi_{\eta}^{(0)}(z_j)^* \Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j) ^2 \right]_{z_{j} = z} - .$$

We now see that terms with $j \neq j'$ do not contribute. This is so since in such a case, at least one of them is not equal to $N$, say $j \neq N$, and then (26) gives zero. In the $j = j'$ case, we get

$$\Psi_{\eta}^{(0)}(z_j)^* \Psi_{\eta}^{(0)}(z_j) = \langle \Psi_{\eta}^{(0)}(z_j)^* | \Psi_{\eta}^{(0)}(z_j) \rangle,$$

where $\rho(z)$ is the density operator. For $j = j' \neq N$, the integral over the $j$th variable gives $1$, and the rest precisely gives $\langle \Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j) \Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j) \rangle$. We have just shown that

$$\rho_{\eta^{(M-1)qh}}(z) = \langle \Psi_{\eta}^{(0)}(z_j)^* | \Psi_{\eta}^{(0)}(z_j) \rangle + \langle \Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j) | \Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j) \rangle.$$

Here, the first term is just $\frac{1}{z_{j}^2} e^{-\frac{1}{2} z_{j}^2}$, while the second one is the local particle density at $z$ of $\Psi_{\eta}^{(M-1)qh}(Z_{N-1}, z_j)$, which is governed by a plasma analog.\(^*\)

This picture is physically appealing. On one hand, there is no (local) plasma analogy for a state such as $\Psi_{\eta}^{(M-1)qh}$, but certain properties such as the Berry connection or the quasiparticle charge density simplify because of the fusion mechanism of fractionalization for any finite $N$. On the other hand, this same mechanism facilitate numerical computations, such as Monte Carlo methods, of certain physical properties. For example, in Fig. 1 we
where, in a located at position \( \eta \) of an incompressible \( \nu = 1/3 \) Laughlin fluid with \( N = 7 \) particles, we get \( \delta \rho \) separated quasiparticles following form of the wave function for a system of \( N \) particles (adding the electron charge density \( \frac{1}{\pi} e^{-\frac{1}{2}z^2i} \) (with \( z = x + iy \)) leads to the exact same \( \rho \)). \( \Phi \) are contour plots of their 3D plots from \( a-c \), respectively. Monte Carlo simulations are averaged over more than \( 2 \times 10^{10} \) equilibrated configurations.

Berry connection for two quasiparticles: The problem of statistics. Our mechanism for particle fractionalization suggests the following form of the wave function for a system of \( N_{\text{qp}} \ll N \) well-separated quasiparticles

\[
\Psi_{\eta_1, \ldots, \eta_{N_{\text{qp}}}}^{N_{\text{qp}}} (Z_N) = N^{\nu} \prod_{i<j} \rho^{\nu} (\eta_i) \prod_{i\neq \eta} \rho^\nu (\eta_i) \cdot 
\times \Psi_{\eta_1, \ldots, \eta_{N_{\text{qp}}}}^{N_{\text{qp}}} (Z_{N_{\text{qp}}}) \quad (31)
\]

where \( \Psi_{\eta_1, \ldots, \eta_{N_{\text{qp}}}}^{N_{\text{qp}}} (Z_{N_{\text{qp}}}) \) denotes the state with \( M - 1 \) quasiholes at \( \eta_1, M - 1 \) quasiholes at \( \eta_2 \) and so on, up to \( M - 1 \) quasiholes at \( \eta_{N_{\text{qp}}} \). \( N^{\nu} \prod_{i<j} \rho^{\nu} (\eta_i) \prod_{i\neq \eta} \rho^\nu (\eta_i) \) is a normalization factor associated with the electron creation operators, as shown below.

To address the quasiparticle (a composite of one electron and \( M - 1 \) quasiholes) exchange statistics, we next focus on \( N_{\text{qp}} = 2 \), have checked numerically that the fusion mechanism works for the charge density of an \( N = 7 \) electron system and \( \nu = 1/3 \). In this way, we can simulate an arbitrary large system of electrons because there is an “effective plasma analogy” and the Monte Carlo updates become quite efficient. Figure 2 shows Monte Carlo simulations of the radial density for a system of \( N = 50 \) electrons. We can measure the charge of the quasiparticle by using the expression \( \delta \rho_{\text{qp}} = 2\pi \int_0^{\infty} \left[ \rho_{\text{qp}}(r) - \rho_i(r) \right] r \, dr \)

where, in a finite system, the cut-off radius \( R_{\text{cut-off}} \) must at least enclose completely the quasiparticle and, at the same time, be sufficiently far from the boundary to avoid boundary effects. Using the Monte Carlo data for \( N = 400 \) particles (see Supplementary Fig. 1 in the Supplementary Note 6) and choosing \( R_{\text{cut-off}} \leq 30 \), we get a saturation of the fractional charge at the value \( \delta \rho_{\text{qp}} = 0.3330(30) \). Similarly, for two quasiholes we get \( \delta \rho_{\text{2qh}} = -0.6634(30) \).

**Fig. 2** Quasiparticles and quasiholes for Laughlin fluids. \( a, b \) represent density profiles (in units of the uniform density \( \rho_0 = \frac{1}{\pi} \)) for one quasielectron located at position \( \eta \) of an incompressible \( \nu = 1/3 \) Laughlin fluid with \( N = 7 \) particles. \( c \) depicts two quasiholes in an otherwise \( \nu = 1/3 \) Laughlin fluid with \( N = 6 \) particles (adding the electron charge density \( \frac{1}{\pi} e^{-\frac{1}{2}z^2i} \) (with \( z = x + iy \)) leads to the exact same \( \rho \)). \( d-f \) are contour plots of their 3D plots from \( a-c \), respectively. Monte Carlo simulations are averaged over more than \( 2 \times 10^{10} \) equilibrated configurations.

**Fig. 2** Quasiparticle localized at \( \eta = 4 + 3i \) in which case we get

\[
\Psi_{\eta_1, \eta_2}^{N_{\text{qp}}} (Z_N) = \sqrt{N(N-1)} N^{2(M-1)\text{qh}} \prod_{i<j} \frac{e^{-\frac{1}{2} \sum_{k=1}^{N-2} |z_k|^2}}{\prod_{k=1}^{N-2} (z_k - \eta_1)^{M-1}} \prod_{1 \leq i < j \leq N-2} (z_i - z_j)^{M-1} \quad (32)
\]

Similar to the one-particle case, \( \Psi_{\eta_1, \eta_2}^{2(M-1)\text{qh}} (Z_{N-2}) \) has orbitals \( \psi_{\eta_1, \eta_2}^{i} \), \( i = 1, 2 \), unoccupied, owing to the presence of factors...
By a straightforward computation, in the mixed representation, we get
\[
\langle \psi_{(M-1)q}\psi_{(M-1)q}\rangle = |\psi_{(M-1)q}|^2, \quad i = 1, 2. \tag{33}
\]

A weaker notion may involve subspaces de

is that of a state of two clusters of

Constructive subspace bosonization

normalizes the quasihole cluster state \( |\psi_{(M-1)q}|^2 \) and cancels the second line. The latter is just the normalization of the 2-fermion state \( |\Lambda(x_i)\Lambda(x_j)|^2 |0\rangle \), so this choice of \( |\psi_{(M-1)q}|^2 \) can also be expressed as
\[
|\Lambda(x_i)\Lambda(x_j)|^2 |0\rangle = |0\rangle \tag{35}
\]

and/or its Hermitian adjoint, which will be useful in the following.

For the computation of the Berry connection, just as in the one quasiparticle case, one can write \( |\psi_{(M-1)q}|^2 = \psi_{(M-1)q} |0\rangle \) for some

N - 2 particle operator \( \psi_{(M-1)q} \) in the algebra generated by the \( |\Lambda(x_i)|^2 \), in terms of which (31) can be equivalently stated as
\[
|\Lambda(x_i)\psi_{(M-1)q}|^{(2)} \psi_{(M-1)q} |0\rangle = |0\rangle, \quad i = 1, 2. \tag{36}
\]

Then, utilizing the last two equations, the calculation of the Berry connection proceeds analogously to the single-particle case. In particular, one obtains two contributions
\[
\langle \psi_{(M-1)q}\psi_{(M-1)q}\rangle = \frac{d}{dt} |\psi_{(M-1)q}|^2 = \Lambda_2 + i\Lambda_2(2M-2), \tag{37}
\]

where
\[
\Lambda_2 = \langle \eta_i, \eta_j | \frac{d}{dt} |\eta_i, \eta_j\rangle \tag{38}
\]

is the Berry connection of a normalized 2-electron state \( |\eta_i, \eta_j\rangle = \Lambda_1(x) \Lambda_2(x) |0\rangle \), and
\[
\Lambda_2(2M-2) = \langle \psi_{(M-1)q}\psi_{(M-1)q}\rangle = \frac{d}{dt} |\psi_{(M-1)q}|^2 \tag{39}
\]

is that of a state of two clusters of \( M - 1 \) quasiholes each.

For large \( |\eta_i - \eta_j| \), both contributions are analytically under control, the 2-electron one \( \Lambda_2 \) trivially so, and the one from the quasihole cluster state \( \Lambda_2(2M-2) \), via methods along the lines of Arovas-Schrieffer-Wilkiewicz62,22. Dropping Aharonov-Bohm contributions, and defining the statistical phase as \( e^{i\eta} \), the contribution to \( y \) from the 2-electron state is 1 (assuming, for the time being, that the underlying particles are fermions with \( M \) odd), and that of the quasihole-cluster is \( (M - 1)^2/2M \). Thus,
\[
\eta y = \pi + (M - 1)^2 \cdot \frac{\pi}{M} \quad (\text{mod} 2\pi) = \frac{\pi}{M} \quad (\text{mod} 2\pi), \tag{40}
\]

as expected for a quasielectron. The same final result \( \eta y \) would be obtained for bosonic states and even \( M \).

Constructive subspace bosonization. A bosonization map is an example of a duality\(^6\). Typically, dualities are dictionaries constructed as isometries of bond algebras acting on the whole Hilbert space\(^5\). A weaker notion may involve subspaces defined from a prescribed vacuum and, thus, are Hamiltonian-dependent. This is the case of Luttinger’s bosonization\(^28\) that describes, in the thermodynamic limit, collective low energy excitations about a gapless fermion ground state. Our bosonization is performed with respect to a radically different vacuum - that of the gapped Laughlin state. Unlike most treatments, we will not bosonize the one-dimensional FQH edge (by assuming it to be a Luttinger system) but rather bosonize the entire two-dimensional FQH system. Contrary to gapless collective excitations about the one-dimensional Fermi gas ground state associated with the Luttinger bosonization scheme, our bosonization does not describe modes of arbitrarily low finite energy but rather only the zero-energy (topological) excitations\(^7\) that are present in the gapped Laughlin fluid. The zero-mode subspace \( Z = \bigoplus_{N=0}^{\infty} \mathbb{Z}_N \) is generated by the action of the commutative algebra \( A \) on the Laughlin state \( |\psi_M\rangle \) for different particle numbers \( N \). Yet another notable difference with the conventional Luttinger bosonization (and conjectured extensions to 2 + 1 dimensions\(^29\)) is, somewhat similar to earlier continuum renditions\(^30\) (as opposed to our discrete case) that the indices parameterizing our bosonic excitations, \( d \neq 0 \), are taken from the discrete positive half-line (angular momentum values) instead of the continuous full real line of the Luttinger system (or planck\(^29\)). Each zero-energy state in our original (fermionic/bosonic) Hilbert space has an image in the mapped bosonized Hilbert space. Consider the following bosonic creation (annihilation) operators
\[
b^L = \mathcal{O}_d/\sqrt{dK}, \quad b^L = \mathcal{O}_d/\sqrt{dK}. \tag{41}
\]

The commutator \([b^L, b^L]\) does not preserve total angular momentum when \( d \neq d. \) It follows that, in the thermodynamic limit, within the Laughlin state subspace, \([b^L, b^L]^\dagger = \delta_{d,d}^d\). The field operator \( \varphi(z) = \sum_{d \neq 0} \varphi_d(z) b^d + \text{adjoint} \) satisfies \( \varphi(z) \varphi(z') = 0 \) and \( \varphi(z) \varphi'(z') = |z' - z| \).

We next construct the operators connecting different particle sectors, that is, the Klein factors that commute with the bosonic degrees of freedom \( b^d, b^L \) and are \( N \)-independent. Since
\[
|\psi_M^{(N+1)} = \frac{1}{\sqrt{\pi N}} K_{MN} |\psi_M\rangle \text{ we define } F_{MN} = \frac{1}{\sqrt{\pi N}} K_{MN} \text{ and } F_M = \sum_{N \geq 1} F_{MN} |\psi_M\rangle. \tag{42}
\]

This illustrates the relation between the Klein factors of bosonization with the (non-local) Read operator. We then get \( \langle \psi_M^{(N+1)} | O_d, F_{MN} | \psi_M\rangle = 0 \) and \( \langle \psi_M^{(N+1)} | b^L, F_M | \psi_M\rangle = 0 \). One can prove a similar relation for \( F_M := (F_M^\dagger)^\dagger \) and, analogously, for \( b^L \) replaced by \( b^L \) (see Supplementary Note 7). Since the \( \mathcal{U}_N(\eta) \) operators can be expressed in terms of \( b^L \)'s, the fractionalization equations (both for quasiparticle as well as quasihole) can be thought of as the dictionary, at the field operator level, for our bosonization. We reiterate that this bosonization within the zero-mode subspace reflects its purely topological character. Indeed, the only Hamiltonian that commutes with the generators of \( A \) is the null operator.

Universal edge behavior. An understanding of the bulk-boundary correspondence in interacting topological matter is a long standing challenge. For FQH fluids, Wen’s hypothesis\(^31\) for using Luttinger physics for the edge compounded by further effective edge Hamiltonian descriptions\(^32,33\) constitutes our best guide for the edge physics. We now advance a conjecture enabling direct analytical calculations. We posit that the asymptotic long-distance behavior of the single-particle edge Green’s function may be calculated by evaluating it for the root state (the DNA) of the corresponding FQH state. As we next illustrate, our computed long-distance behavior shows remarkable agreement with Wen’s hypothesis. Our (root state) angular momentum basis calculations do not include the effects of boundary confining potentials
Consider the fermionic Green’s function

\[ -iG(z, z') = \rho(z, z') = \langle \psi_N^\dagger \mid \psi(z) \psi(z') \mid \psi_N \rangle \]

\times \frac{1}{\|\psi_N\|^2} e^{-\frac{1}{2}(|z|^2 + |z'|^2)},

(42)

and coordinates \( z = Re^{i\theta}, \quad z' = Re^{i\theta'} \), where \( R = \sqrt{r_{\text{max}} + 1} \) is the radius of the last occupied orbital and it can be identified with the classical radius of the droplet, i.e., it satisfies \( \pi R^2 = a = N \) with \( a = N/(r_{\text{max}} + 1) \) being the average density of the (homogeneous) droplet. Then,

\[ \rho(z, z') = \frac{e^{-2kF}}{2\pi \rho_{\text{crit}}} \frac{1}{\pi \rho_{\text{crit}}} \frac{1}{\pi \rho_{\text{crit}}} R^2)^{-\frac{1}{2}} g^k \theta - \theta' \frac{\psi_N^\dagger \mid \psi_N \mid}{\|\psi_N\|^2}. \]

(43)

Similarly, the edge Green’s function associated with the root partition \( \psi_N^\dagger \)

\[ \tilde{\rho}(z, z') = \frac{e^{-2k^2}}{2\pi \rho_{\text{crit}}} \sum_{k=0}^{N-1} \left( \frac{R^2}{2} \right)^{(N-k)M} g^k \sin \theta - \theta' \frac{\psi_N^\dagger \mid \psi_N \mid}{\|\psi_N\|^2}. \]

(44)

where we used \( \sum_{k=0}^{N-1} \left( \frac{R^2}{2} \right)^{(N-k)M} g^k \sin \theta - \theta' \frac{\psi_N^\dagger \mid \psi_N \mid}{\|\psi_N\|^2} = 1 \) for \( r = 0, M, \ldots, (N-1)M, \) and 0 otherwise. Thus far, our root partition calculation is exact. We next perform asymptotic analysis. For large \( k \), the largest phase oscillations appear when \( \cos(M(\theta - \theta')) = (-1)^k \), i.e., for \( |\theta - \theta'| = \theta \) near \( \pi \frac{kM}{2} \) with \( l = 0, \ldots, m \) and \( M = 2m + 1 \). This implies that the dominant contributions to the sum originate from small \( k \) values. We can then apply Stirling’s approximation

\[ (N - k)M! \approx \sqrt{\pi R^2}^{(N-k)M} e^{-R^2/2}, \]

where we used \( R^2 = 2N \) (valid since \( 1 - \nu \ll R^2 \) leading to

\[ |\tilde{\rho}(z, z')| \approx \frac{1}{4^{\pi^2/2} \sin \theta \frac{\theta'}{2}}. \]

(45)

Long distances correspond to \( \tilde{\theta} \) near \( \pi \).

\[ \sin \left( \frac{M\tilde{\theta}}{2} \right) = \sin \left( \frac{\theta}{2} \right)^M \]

(46)

the edge Green’s function

\[ |\tilde{\rho}(\tilde{\theta})| = \frac{1}{4^{\pi^2/2} \sin \theta \frac{\theta'}{2}} \left( 1 + O(|\tilde{\theta} - \pi|^4) \right), \]

(47)

or, equivalently, \( |\tilde{\rho}(\tilde{\theta})| \propto |z - z'|^{-M} \). This is only valid in the vicinity of \( \tilde{\theta} = \pi \) (e.g., demanding the corrections to be \( \leq \frac{1}{100} \)), for \( M = 3 \), restricts us to \( [0.86M, \pi] \), as Eq. (45) spans a broader range—see Fig. 3. The Green’s function was computed by using the tables of characters for permutation groups \( S_{N(N-1)} \) for \( M = 3 \) (up to \( N = 8 \) and then extrapolating the results), adjusting Dunne’s approach. The difference between \( |\rho| \) and \( |\tilde{\rho}| \) vanishes at \( \pi \) as \( N^{-1/2} \).

Nevertheless, the long-distance \( (\tilde{\theta} \sim \pi) \) behavior of the Green’s function, in the thermodynamic limit, cannot be reliably determined from small \( N \) calculations. For instance, by examining the slope \( \mu \) of \( \log |G(\tilde{\theta})| \) plotted as a function of \( \log |\sin(\tilde{\theta}/2)| \) for \( N = 8 \) (Fig. 3), we get \( \mu = -3.88 \) when using the range \( [0.967, 1] \) for \( \sin(\tilde{\theta}/2) \), while the value for \( N = 7 \) in the range \( [0.991, 1] \) is \( \mu = -6 \). The deduced numerical value is highly dependent on the range used in the fitting procedure, e.g., for \( N = 8 \) and the range \( [0.6, 1] \) we obtain \( \mu = -3.23 \) (for linear scale of \( \tilde{\theta} \)). We established that the asymptotic long-distance behavior of the edge Green’s function corresponding to the root state coincides with Wen’s conjecture31.

Beyond the LLL. The aforementioned behavior remains true also beyond the LLL that forms the focus of our work. Indeed, repeating the above calculation when using the DNA2,36 of the Jain’s 2/5 state, we found \( \mu = -3 \), in agreement with Wen’s hypothesis31. In this Jain’s state example, our computation captures the (EPP) entanglement structure of the root state2 not present in Laughlin states.

In this case we need the exact form of the following orbitals:

\[ \psi_{0r}(z) = \frac{z^r}{N_{0,r}}, \quad \psi_{1r}(z) = \frac{z^r z^{r+1} - 2(r+1)z^r}{N_{1,r}}, \]

(48)

with \( N_{0,r} = \sqrt{2} r^2 \pi^2 \) and \( N_{1,r} = \sqrt{2} r^2 \pi^2 (r+1)! \). The fermionic field operator is now \( \Psi(z) = \sum_{n,r} \psi_{n,r}(z) \epsilon_{n,r} \) which leads to the Green’s function of the following form:

\[ \rho(z, z') = \sum_{n,r} \sum_{n',r'} \psi_{n,r}(z) \psi_{n',r'}(z') \frac{\psi_{n',r'}^\dagger \mid \psi_{n,r}^\dagger \mid}{\|\psi\|^2} e^{-\frac{1}{2}(|z|^2 + |z'|^2)}, \]

(49)

where \( |\psi\rangle \) is the corresponding ground state. By the angular momentum conservation, \( r = r' \) under the above summation, so that

\[ \rho(z, z') = \sum_{n,r} \psi_{n,r}(z) \psi_{n,r}(z') \frac{\psi_{n,r}^\dagger \mid \psi_{n,r}^\dagger \mid}{\|\psi\|^2} e^{-\frac{1}{2}(|z|^2 + |z'|^2)}. \]

(50)

For the “DNA” of the ground state \( |\psi\rangle \) we get

\[ \tilde{\rho}(z, z') = \sum_{n,r} \sum_{n',r'} \psi_{n',r'}(z) \psi_{n,r}(z') \frac{\text{DNA}_{n,r}^\dagger \mid \text{DNA}_{n',r'}^\dagger \mid}{\|\text{DNA}\|^2} e^{-\frac{1}{2}(|z|^2 + |z'|^2)}. \]

(51)
The "DNA" associated to Jain's 2/5 state\textsuperscript{2,36} is of the form \(|D|\mathbb{A} = \prod_{k=0}^{\infty} \varphi_{3k+3}(0)\) with
\[
\varphi_r = \alpha_{0,0}(r) \left( \frac{\sqrt{r+2}}{2} \varphi_{0,r-1} \varphi_{1,r+1} + \frac{\sqrt{r-2}}{2} \varphi_{0,r-1} \varphi_{1,r+1} \right),
\] (52)
where \(\alpha_{0,0}\) is an \(r\)-dependent factor.

As a result,
\[
\tilde{\rho}(z, z') = e^{-2\pi i|z|^2 + |z'|^2} \left\{ \sum_{0 \leq k < \text{re}_{\text{sn}}} \frac{(z^2 z')^{5l+2}}{2^{5l+4} h_l(5l+2)} \left[ 1 + (5l+5)^2 \right] \right. \\
\left. + \frac{2(5l+3) - |z|^2 + (2(5l+3) - |z|^2)(5l+3)}{4} \right\} \\
\left. + \frac{2(5l+5) - |z|^2 + (|z|^2 - 2(5l+3)(5l+5))}{4} \right\}
\] (53)
where \(h_l = 1 + (5l+3)^2 + (5l+5)^2\).

Henceforth, we will focus on points \(|z| = R e^{\theta}\) and \(|z'| = R e^{\theta'}\) that lie on the edge. We next discuss the two contributions to \(\tilde{\rho}(z, z')\) in Eq. (53).

We start by discussing the contribution to \(\tilde{\rho}(z, z')\) of exponent \(5l + 2\). With the above polar substitution for the boundary points \(z\) and \(z'\), this becomes
\[
e^{-2\pi i N} \sum_{k=0}^{N} \left( \frac{R^2}{2} \right)^{5(N-k)+2} e^{-i(5(N-k)+2)(\theta - \theta')} h_{N-k}(5(N-k)+2) \left[ 1 + (5(N-k)+5)^2 \right] \\
\times \left[ 1 + (5(N-k)+3)^2 \right] + \frac{2(5(N-k)+3) - R^2}{4} \cdot (20(N-k)+13),
\] (54)
where \(N = \left\lfloor \frac{1}{2} \left( \frac{N}{2} - 1 \right) \right\rfloor + 1\) with \(\nu = \frac{1}{2}\), and we have used the same change of summation index as in the case of the LLL.

Assume now that only small integers \(k\) are of relevance in the above summation - we will check validity of this assumption later on. Then using the Stirling approximation, the fact that \(N \approx R^2 \gg 1\) and \(N - k \approx N\), we get
\[\left[ 5(N-k)! \right] \approx \sqrt{\pi R^2} \left( \frac{R^2}{2} \right)^{5(N-k)+2} e^{-\frac{R^2}{2}},\] (55)
and, as a result, for the part with the exponent 5l + 2, we end up with
\[
e^{-i5\pi/s(\theta - \theta')} \frac{k_2(N, R)}{2 R^{1/2}} \sum_{k=1}^{N} e^{i(5k-2)(\theta - \theta')},\] (56)
where \(k_2(N, R)\) is a certain rational function in \(N\).

Similar to the above, for the part having 5l + 4 as an exponent, we get
\[
e^{-i5\pi/s(\theta - \theta')} \frac{k_4(N, R)}{2 R^{1/2}} \sum_{k=1}^{N} e^{i(5k-4)(\theta - \theta')},\] (57)
with a rational, in \(N \approx \left\lfloor \frac{1}{2} \left( \frac{N}{2} - 4 \right) \right\rfloor + 1\) function \(k_4(N, R)\).

Next, we observe that for large radius \(R\) we can without the loss of generality assume that \(N \approx N\), so that \(e^{-i5\pi/s(\theta - \theta')}\)'s lead to an irrelevant global factor since at the very end we will be interested in the absolute value of the Green's function. We now argue that in the thermodynamic limit, \(\frac{5}{2} \approx 1\). Indeed, since \(R^2 \sim \frac{N}{5}\) and \(\nu = \frac{3}{2}\) we have \(R^2 \sim 5N\). Moreover, we know that \(N \sim \frac{N}{5} \approx \frac{N}{2}\).

Hence \(R^2 \sim 10N\). This shows that \(\frac{5}{2} \approx 1\). Furthermore, this also shows that, in the limit \(N \to \infty\), we have for \(\tilde{\rho}\):
\[
e^{-i5\pi/s(\theta - \theta')} \frac{\sum_{k=1}^{N} e^{i(5k-2)(\theta - \theta')}}{4 R^{1/2}} + \frac{\sum_{k=1}^{N} e^{i(5k-4)(\theta - \theta')}}{4 R^{1/2}} \right),\] (58)
since both \(k_2\) and \(k_4\) tend to the \(\frac{1}{2}\) in this limit.

We next explain why the assumption \(N - k \approx N\) is valid. Towards this end, one needs to verify that the only \(k\) values that matter are the small ones, i.e., that \(\cos(5k-2(\theta - \theta'))\) \(\approx (-1)^k\). This is indeed true (in particular around \(\theta = \theta' = \pi\), which is exactly our point of interest). Analogous considerations work also for the term of exponent 5l + 4. Therefore, the problem reduces to the evaluation of
\[
e^{-i5\pi/s(\theta - \theta')} \frac{\sum_{k=1}^{N} e^{i(5k-2)(\theta - \theta')}}{4 R^{1/2}} + \frac{\sum_{k=1}^{N} e^{i(5k-4)(\theta - \theta')}}{4 R^{1/2}} \right),\] (59)

To ascertain the long distance behavior, we examine angular differences \(|\theta - \theta'| = \theta\) close to \(\pi\), where this asymptotically becomes
\[
e^{-i5\pi/s(\theta - \theta')} \frac{\sum_{k=1}^{N} e^{i(5k-2)(\theta - \theta')}}{4 R^{1/2}} + \frac{\sum_{k=1}^{N} e^{i(5k-4)(\theta - \theta')}}{4 R^{1/2}} \right),\] (60)
The above derived result is in agreement with Wen's conjecture\textsuperscript{31} for the FQH Jain's 2/5 liquid.

**Conclusions.** Our approach sheds light on the elusive exact mechanism underlying fractionalized quasiexciton excitations in FQH fluids (and formalizes the fractionalization of quasihole excitations\textsuperscript{14}). By solving an outstanding open problem\textsuperscript{19,20}, our construct underscores the importance of a systematic operator-based microscopic approach complementing Laughlin’s original quasiparticle wave function Ansatz. The algebraic structure of the LLL is deeply tied to the Newton-Girard relations. We have shown that there are numerous pairs of "dual" operators that are linked to each other via these relations (including operators associated with the Witt algebra). The Newton-Girard relations typically convert a local operator into a non-local "dual" operator. The main message of the present work is that "derivative operations" on FQH vacua do not lead to exact quasiexciton excitations. The precise mechanism leading to charge fractionalization consists of the joint process of flux and (original) particle insertions. In other words, an elementary fusion channel of quasiholes and an electron generates a quasielectron excitation. To generate one quasielectron excitation in a \(v = 1/M\) Laughlin fluid one needs to insert \(M - 1\) fluxes, in an \(N - 1\)-electron fluid, and fuse them with an additional electron. Thus, for instance, two quasiholes plus one electron of charge \(e\) lead to an exact quasielectron of fractional charge \(e/3\), and exchange statistics 1/3, in a \(v = 1/3\) Laughlin fluid. A fundamental difference between quasihole and quasiparticle excitations can be traced back to their \(M\)-clustering properties\textsuperscript{3}. While quasiholes preserve the \(M\)-clustering property of the incompressible (ground state) fluid, quasiparticle states break it down. This is at the origin of the asymmetry between these two kinds of excitations. Equivalently, while quasiparticle wave functions sustain a (local) plasma analogy this is not the case for quasiexcitons.

We explicitly constructed the quasiparticle (quasielectron) wave function. Our found fusion mechanism of quasiparticle generation is not only the mathematically exact (for an arbitrary number of particles) field-theoretic operator procedure but it is also behind the exact analytic computation of other quasiparticle
properties, such as its charge density and Berry connections leading to the right fractional charge and exchange statistics.1,3,5,6,22,27,37. This is a truly unprecedented remarkable result that we have numerically confirmed via detailed Monte Carlo simulations.

Intriguingly, within our field-theoretical framework, we find that the Laughlin state is a condensate of a non-local Read type operator. Our approach allows for a constructive (zero-energy) subspace bosonization of the full two-dimensional system that further evinces the non-local topological character of the problem and, once again, cements links to Read’s operator. The constructed Klein operator associated with this angular momentum (and flux counting) root state based bosonization scheme is none other than Read’s non-local operator. We suspect that this angular momentum (flux counting) based mapping might relate to real-space flux attachment (and attendant Chern-Simons) type bosonization schemes.29,38. Lastly, we illustrated how the long-distance behavior of edge excitations associated with the root state component (DNA) of the bulk FQH ground state may be readily calculated. Strikingly, the asymptotic long-distance edge physics derived in this manner matches Wen’s earlier hypothesis in the cases that we tested. This agreement hints at a possibly general powerful computational recipe for predicting edge physics.

Data availability
The data that support the findings of this study are available from the corresponding author upon reasonable request

Code availability
The code that support the findings of this study are available from the corresponding author upon reasonable request

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Competing interests
The authors declare no competing interests.
