The Third Boundary Value Problem of Potential Theory for the Exterior Ball and the Approximation behaviour of the solution; a Novel Open Problem

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Abstract

The paper is concerned with the interconnection of the boundary behaviour of the solutions of the exterior Dirichlet and Neumann problems of harmonic analysis for the unit ball in $\mathbb{R}^3$ with the corresponding behaviour of the associated ergodic inverse problems for the punched unlimited space. The basis is the theory of semigroups of linear operators mapping a Banach space $X$ into itself. The rates of approximation play a basic role.

Another tool is a Drazin-like inverse operator $B$ for the infinitesimal generator $A$ of a semigroup that arises naturally in ergodic theory. This operator $B$ is a closed, not necessarily bounded, operator. It was introduced in a paper with U. Westphal (1970/71) [1] and extended to a generalized setting with J. J. Koliha (2009) [2].

The novel open problem concerns the third or Robin’s problem of potential theory, the solution of which is not a semigroup of operators. Hence, the semigroup methods applied to Dirichlet’s or Neumann’s problem cannot be applied. The authors give several hints how to overcome these difficulties.

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1. Dirichlet’s problem for the three dimensional unit ball and its interconnections with the associated inverse problem in unlimited space

Dirichlet’s problem, also known as the first problem of potential theory, is to determine a function $w(\varphi, \theta, r)$ twice continuously differentiable on its domain $[0, \pi] \times \mathbb{R} \times (0, \infty)$ which is $2\pi$-periodic with respect to $\theta$, and satisfies Laplace’s equation (in spherical coordinates)

$$
\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} + \frac{\cos \varphi}{r^2 \sin \varphi} \frac{\partial w}{\partial \varphi} + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2 w}{\partial \theta^2} = 0 \quad (1)
$$

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together with the boundary conditions
\[
\lim_{r \to 1^+} \|w(\varphi, \theta, r) - h(\varphi, \theta)\|_{L^2(S)} = 0, \quad (1)
\]
\[
\lim_{r \to \infty} w(r, \varphi, \theta) = 0 \quad (\varphi \in [0, \pi], \theta \in [-\pi, \pi]), \quad (2)
\]
where \(h(\varphi, \theta)\) is a given function in \(L^2(S)\), and \(S\) denotes the unit sphere in \(\mathbb{R}^3\).

For the spherical coordinates \(\varphi, \theta, r\) we use the convention
\[
x = r \sin \varphi \cos \theta, \quad y = r \sin \varphi \sin \theta, \quad z = r \cos \varphi,
\]
where \(x, y, z\) are the rectangular coordinates in \(\mathbb{R}^3\), and \(-\pi < \theta \leq \pi\) is the longitude, while \(0 \leq \varphi \leq \pi\) is the latitude on the sphere of radius \(r = \sqrt{x^2 + y^2 + z^2}\).

The solution of the above problem can be written as a Fourier expansion with respect to the so-called (complex) spherical harmonics (see [3], [4], [5], [16, 31, 32], [6], [7], [8])
\[
Y^m_k(\varphi, \theta) := \sqrt{(2k+1)/(4\pi(k+|m|)!)} P^{|m|}_k(\cos \varphi)e^{im\theta} \quad (k = 0, 1, 2, \ldots; m = 0, \pm 1, \ldots, \pm k),
\]
where
\[
P^m_k(t) = (1-t^2)^{m/2} \frac{d^m}{dt^m} P_k(t) = (-1)^k \frac{(1-t^2)^{m/2}}{2^k k!} \frac{d^{k+m}}{dk+m} (1-t^2)^k \quad (m = 0, 1, \ldots, k),
\]
and \(P_k = P^0_k\) are the Legendre polynomials defined by
\[
P_k(t) := (-1)^k \frac{d^k}{dt^k} (1-t^2)^k.
\]

The \(P^m_k\) are known as associated Legendre functions. They are polynomials for \(m\) even, and polynomials multiplied by a factor \((1-t)^{1/2}\) for \(m\) odd.

It is well known that the spherical harmonics form a complete orthonormal system in the Hilbert space \(L^2(S)\) with respect to the scalar product and the norm
\[
\langle g_1, g_2 \rangle := \int_S g_1 g_2 \sin \varphi \, dS = \int_{-\pi}^\pi \int_0^\pi g_1(\varphi, \theta) g_2(\varphi, \theta) \sin \varphi \, d\varphi \, d\theta, \quad ||g||_{L^2(S)} := \sqrt{\langle g, g \rangle},
\]
d\(S\) being the surface area element of the unit sphere. In particular, there holds
\[
\langle Y^m_k, Y^l_j \rangle = \delta_{k,l} \delta_{m,j} \quad (k = 0, 1, 2, \ldots; m = 0, \pm 1, \pm 2, \ldots, \pm k).
\]
It follows that every function \(g \in L^2(S)\) can be expanded into a Fourier series with respect to the complex spherical harmonics,
\[
g(\varphi, \theta) = \sum_{k=0}^\infty \sum_{m=-k}^k \hat{g}(m, k) Y^m_k(\varphi, \theta),
\]
the series being convergent in \(L^2(S)\)-norm. The Fourier coefficients \(\hat{g}(m, k)\) are given by
\[
\hat{g}(m, k) := \langle g, Y^m_k \rangle = \int_{-\pi}^\pi \int_0^\pi g(\varphi, \theta) Y^m_k(\varphi, \theta) \sin \varphi \, d\varphi \, d\theta.
\]
Now the unique solution of Dirichlet's boundary value problem (1), (2), (3) is given by
\[
w(\varphi, \theta, r) = \sum_{k=0}^{\infty} r^{-(k+1)} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta) \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}, r > 1).
\]

Setting now \( r = e^t \), i.e.,
\[
w(\varphi, \theta, r) = w(\varphi, \theta, e^t) = \sum_{k=0}^{\infty} e^{-(k+1)t} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta),
\]
it follows that \( T_A(t)h(\varphi, \theta) := w(\varphi, \theta, e^t) = w_A(\varphi, \theta, e^t) \) defines a semigroup of class \( (C_0) \) with infinitesimal generator \( A \) given by
\[
Ah(\varphi, \theta) = -\sum_{k=0}^{\infty} (k+1) \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta).
\]

Its domain \( D(A) \) can be characterized in terms of the Fourier coefficients, namely,
\[
D(A) = \left\{ g \in L^2(S); \sum_{k=0}^{\infty} (k+1)^2 \sum_{m=-k}^{k} |\hat{g}(m, k)|^2 < \infty \right\},
\]
whereas the nullspace \( N(A) \) is trivial, i.e., \( A \) is injective. The inverse \( A^{-1} \) is defined on all of \( L^2(S) \) and has the Fourier series representation
\[
A^{-1}h(\varphi, \theta) = -\sum_{k=0}^{\infty} (k+1)^{-1} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta).
\]

Furthermore, the semigroup generated by \( A^{-1} \) is given by
\[
V_{A^{-1}}(t)h(\varphi, \theta) = w_{A^{-1}}(e^t, \varphi, \theta) = \sum_{k=0}^{\infty} e^{-t(k+1)} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta).
\]

We also need the resolvent \( R(\lambda; A) \) of the operator \( A \), which has the representation in terms of the associated semigroup \( T_A(t) \),
\[
R(\lambda; A)h(\varphi, \theta) = \int_0^{\infty} e^{-\lambda u} T_A(u)h(\varphi, \theta) \, du \quad (\lambda \in \mathbb{C}, \Re \lambda > 0),
\]
and likewise for \( R(\lambda; A^{-1}) \). It follows easily from (1) that
\[
\operatorname{s-lim}_{\lambda \to \infty} \hat{r}_A(\varphi, \theta; \lambda; h) := \operatorname{s-lim}_{\lambda \to \infty} \lambda R(\lambda; A)h(\varphi, \theta) = \operatorname{s-lim}_{t \to 0^+} T_A(t)h(\varphi, \theta) = h(\varphi, \theta).
\]
See [9, Section 1.3] in this respect.

In the following, we often write \( w_A(r, \varphi, \theta; h) \) and \( w_{A^{-1}}(r, \varphi, \theta; h) \) for \( w_A(r, \varphi, \theta) \) and \( w_{A^{-1}}(r, \varphi, \theta) \), respectively, in order to indicate the dependence of the boundary value \( h \).

Concerning the order of approximation of \( w(\varphi, \theta, r; h) \) towards \( h(\varphi, \theta) \) as well as the ergodic behaviour of \( w_{A^{-1}}(\rho, \varphi, \theta; h) \) one has by a general theorem of semigroup theory (see [10, 11, 12]),
Theorem 1.1. Let \( w_A(r, \varphi, \theta; h) := V_A(t)h(\varphi, \theta) \), \( w_{A^{-1}}(r, \varphi, \theta; h) := V_{A^{-1}}(t)h(\varphi, \theta) \) with \( r = e^t \).

a) For \( \alpha \in (0, 1] \), the following six assertions are equivalent:

\begin{align*}
(i) \quad & \|w_A(\varphi, \theta; r; h) - h(\varphi, \theta)\|_{L^2(S)} = \begin{cases} o(\log r) & (r \to 1+), \\ \mathcal{O}((\log r)^\alpha) & (r \to \infty), \end{cases} \\
(ii) \quad & \|r_A(\varphi, \theta; \lambda; h) - h(\varphi, \theta)\|_{L^2(S)} = \begin{cases} o(\lambda^{-1}) & (\lambda \to \infty), \\ \mathcal{O}(\lambda^{-\alpha}) & (\lambda \to 0+), \end{cases} \\
(iii) \quad & \left\| \frac{1}{\log r} \int_1^r w_{A^{-1}}(\rho, \varphi, \theta; h) \frac{d\rho}{\rho} \right\|_{L^2(S)} = \begin{cases} o(\log r)^{-1} & (r \to \infty), \\ \mathcal{O}((\log r)^{-\alpha}) & (r \to 0+), \end{cases} \\
(iv) \quad & \|r_{A^{-1}}(\varphi, \theta; \lambda; h)\|_{L^2(S)} = \begin{cases} o(\lambda) & (\lambda \to 0+), \\ \mathcal{O}(\lambda^{-\alpha}) & (\lambda \to 0+), \end{cases} \\
(v) \quad & K(t, h; L^2(S), D(A)) = \begin{cases} o(t) & (t \to 0+), \\ \mathcal{O}(t^\alpha) & \text{if } \alpha \in (0, 1], \end{cases}
\end{align*}

\( \left\{ \begin{array}{l}
h = 0 \text{ a. e.,} \\
h \in D(A) \text{ if } \alpha = 1.
\end{array} \right. \)

b) There exist elements \( h_\alpha, h_\alpha^*, h_\alpha^{**} \in L^2(S) \), such that

\begin{align*}
(i) \quad & \left\| \frac{1}{\log r} \int_1^r w_{A^{-1}}(\rho, \varphi, \theta; h_\alpha) \frac{d\rho}{\rho} \right\|_{L^2(S)} = \begin{cases} \mathcal{O}((\log r)^{-\alpha}) & (r \to \infty), \\ \neq o((\log r)^{-\alpha}) & (r \to 0+), \end{cases} \\
(ii) \quad & \|w_{A^{-1}}(\rho, \varphi, \theta; h_\alpha^*) - h_\alpha^*(\varphi, \theta)\|_{L^2(S)} = \begin{cases} \mathcal{O}(\log \rho)^\alpha & (\rho \to 1+), \\ \neq o((\log \rho)^\alpha) & (\rho \to 0+). \end{cases} \\
(iii) \quad & \|r_{A^{-1}}(\varphi, \theta; \lambda^{-1}; h_\alpha^{**})\|_{L^2(S)} = \|r_A(\varphi, \theta; \lambda; h_\alpha^{**}) - h_\alpha^{**}\|_{L^2(S)} = \begin{cases} \mathcal{O}(\lambda^\alpha) & (\lambda \to 0+), \\ \neq o(\lambda^\alpha) & (\lambda \to 0+). \end{cases}
\end{align*}

2. Neumann’s boundary value problem

Neumann’s problem, or second problem of potential theory, is to determine a function \( w(\varphi, \theta; r) \) satisfying conditions (1) and (3) above, but with the boundary condition (2) replaced by

\[ \lim_{r \to 1^+} \left\| \frac{\partial}{\partial r} w(\varphi, \theta; r) - h(\varphi, \theta) \right\|_{L^2(S)} = 0. \]
Its unique solution is given by (see, e.g., [5, §§ 16, 31, 32], [3], and for an exact formulation in a two-dimensional setting [13, p. 287])

\[ w(\varphi, \theta, r) = -\sum_{k=0}^{\infty} \frac{r^{-(k+1)}}{k+1} \sum_{m=-k}^{r} \hat{h}(m, k) Y^m_k(\varphi, \theta) \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}, r > 1). \quad (4) \]

In order to examine the order of approximation in (2), observe that

\[ \frac{\partial}{\partial r} w(\varphi, \theta, r) \bigg|_{r=e^t} = \sum_{k=0}^{\infty} e^{-(k+2)t} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta) \bigg|_{r=e^t} \]

\[ = \sum_{k=0}^{\infty} e^{-(k+2)t} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta) \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}, r > 0). \]

The right-hand side defines a $C_0$-semigroup $S(t)$ on $L^2(S)$, and its generator $A$ is given by

\[ Ah(\varphi, \theta) = -\sum_{k=0}^{\infty} (k+2) \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta) \]

with domain

\[ D(A) = \left\{ g \in L^2(S); \sum_{k=0}^{\infty} (k+2)^2 \sum_{m=-k}^{k} |\hat{g}(m, k)|^2 < \infty \right\}. \]

The associated resolvent operator is given by

\[ R(\lambda; A) h(\varphi, \theta) = \sum_{k=0}^{\infty} \frac{1}{\lambda + k + 2} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta) \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}, t > 0). \]

The generator $A$ is injective and the inverse $A^{-1}$ is defined on all of $L^2(S)$ and has the Fourier series representation

\[ A^{-1} h(\varphi, \theta) = -\sum_{k=0}^{\infty} \frac{1}{k+2} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta). \]

Furthermore, the semigroup generated by $A^{-1}$ is given by

\[ V_{A^{-1}}(t) h(\varphi, \theta) = w_{A^{-1}}(\varphi, \theta, e^t) = \sum_{k=0}^{\infty} e^{-t/(k+2)} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta), \quad (5) \]

having the resolvent operator

\[ R(\lambda; A^{-1}) h(\varphi, \theta) = \sum_{k=0}^{\infty} \frac{k+2}{\lambda(k+2) + 1} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta) \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}, t > 0). \]

The counterpart of Theorem 1.1 now reads.
Theorem 2.1. Let $w_A(\varphi, \theta; r; h) := w(\varphi, \theta, r)$ be defined by (4), and $w_{A^{-1}}(\varphi, \theta; r; h) := w_{A^{-1}}(\varphi, \theta, r)$ as in (5) with $r = \log t$. Further, let $r_A(\varphi, \theta; \lambda; h) = \lambda R(\lambda; A^{-1}) h(\varphi, \theta)$, and $r_{A^{-1}}(\varphi, \theta; \lambda; h) = \lambda R(\lambda; A^{-1}) h(\varphi, \theta)$.

a) The following six assertions are equivalent:

(i) \[ \| \frac{\partial}{\partial r} w_A(\varphi, \theta; r; h) - h(\varphi, \theta) \|_{L^2(S)} = \| \sum_{k=0}^{\infty} r^{-(k+2)} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta) - h(\varphi, \theta) \|_{L^2(S)} = \begin{cases} o(\log r) & (r \to 1^+), \\ O((\log r)^{\alpha}) & (r \to \infty), \end{cases} \]

(ii) \[ \| r_A(\varphi, \theta; \lambda; h) - h(\varphi, \theta) \|_{L^2(S)} \]

(iii) \[ \| \frac{1}{\log r} \int_1^r w_{A^{-1}}(\varphi, \theta; \rho; h) \frac{d\rho}{\rho} \|_{L^2(S)} \]

(iv) \[ \| w_{A^{-1}}(\varphi, \theta; r; h) \|_{L^2(S)} \]

(v) $K(t, h; L^2(S), D(A)) = \begin{cases} o(t) & (t \to 0^+), \\ O(t^\alpha) & \text{if } \alpha = 1. \end{cases}$

b) For any $\alpha \in (0, 1]$ there exist elements $h_\alpha, h_\alpha^*, h_\alpha^{**} \in L^2(S)$, such that

(i) \[ \| \frac{1}{\log r} \int_1^r w_A(\varphi, \theta; \rho; h_\alpha) \frac{d\rho}{\rho} \|_{L^2(S)} = \begin{cases} O((\log r)^{-\alpha}) & (r \to \infty), \\ \neq o((\log r)^{-\alpha}) & (r \to \infty), \end{cases} \]

(ii) \[ \| w_{A^{-1}}(\varphi, \theta; \rho; h_\alpha^*) - h_\alpha^*(\varphi, \theta) \|_{L^2(S)} = \begin{cases} O((\log \rho)^{\alpha}) & (\rho \to 1^+), \\ \neq o((\log \rho)^{\alpha}) & (\rho \to 1^+), \end{cases} \]
\[ \| r_{A^{-1}}(\varphi, \theta; \lambda^{-1}; h^*_\alpha) \|_{L^2(S)} = \| r_A(\varphi, \theta; \lambda) - h^*_\alpha(\varphi, \theta) \|_{L^2(S)} = \begin{cases} \mathcal{O}(\lambda^\alpha) & (\lambda \to 0^+) \\ \neq o(\lambda^\alpha) & \end{cases} \]

3. The third or Robin’s problem of potential theory

This problem of potential theory, also named after V. G. Robin, is to determine a function \( w(\varphi, \theta, r) \) defined on \((0, \pi) \times \mathbb{R} \times (0, \infty) \) which is \(2\pi\)-periodic with respect to \( \theta \), twice continuously differentiable on its domain, and satisfies Laplace’s equation (1) together with the boundary conditions

\[ \lim_{r \to 1^+} \alpha w(\varphi, \theta, r) + \beta \frac{\partial}{\partial r} w(\varphi, \theta, r) - h(\varphi, \theta) = 0, \]  
\[ \lim_{r \to \infty} w(\varphi, \theta, r) = 0 \quad (\varphi \in [0, \pi], \theta \in [-\pi, \pi]), \]

where \( h(\varphi, \theta) \) is a given function in \( L^2(S) \), \( S \) being the unit ball in \( \mathbb{R}^3 \).

The general solution of equation (1) satisfying the boundary conditions (6) and (7) is given by

\[ w(\varphi, \theta, r) = \sum_{k=0}^{\infty} r^{-(k+1)} \sum_{m=-k}^{k} c_{m,k} Y_k^m(\varphi, \theta). \]

For the coefficients \( c_{m,k} \), one has by (6),

\[ \lim_{r \to 1^+} (\alpha r^{-k+1} + \beta r^{-k+1}) c_{m,k} = (\alpha - \beta(k+1)) c_{m,k} = \hat{h}(m, k) \]

This yields the unique solution

\[ w(\varphi, \theta, r) = \sum_{k=0}^{\infty} \frac{r^{-(k+1)}}{\alpha - \beta(k+1)} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta) \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}, r > 1), \]

provided \( \alpha \) and \( \beta \) are such that the denominator \( \alpha - \beta(k+1) \) does not vanish for any \( k \), which is, e.g., the case if they have different sign; see, e.g., [3, pp. 181, 182].

Since we are interested in the order of approximation in (6), we may try to proceed as in Sections 1 and 2, and consider

\[ \alpha w(\varphi, \theta, r) + \beta \frac{\partial}{\partial r} w(\varphi, \theta, r) \bigg|_{r=e^t} = \sum_{k=0}^{\infty} \frac{\alpha e^{-(k+1)t} - \beta(k+1) e^{-(k+2)t}}{\alpha - \beta(k+1)} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta) \quad (t > 0). \]

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Footnote 1: Victor Gustave Robin (1855–1897), who was professor at the Sorbonne in Paris, is especially known for the Robin boundary conditions. He was awarded the Prix Francœur for 1893 and 1897, the Prix Poncelet for 1895; see [14, 15].
Unfortunately, the family of operators, defined by (8) does not possess the semigroup property, apart from the cases $\beta = 0$ (Dirichlet problem), or $\alpha = 0$ (Neumann problem). This means that one cannot apply the general theory developed in [12] in order to deduce results corresponding to those of Theorems 1.1 and 2.1.

A counterpart of Theorem 1.1 (i)$\Leftrightarrow$(v) or Theorem 2.1 (i)$\Leftrightarrow$(v), thus a direct and inverse approximation theorem for the operators (8), may be proved by classical approximation theoretic methods including $K$-functional methods, where $D(A)$ has to be replaced by a suitable subspace of $L^2(S)$. Setting

$$U(r)h(\varphi, \theta) := w(\varphi, \theta; r, h)$$

the operators $(U(r))_{r>1}$ mapping the Hilbert space $L^2(S)$ into $\text{span}\{Y^m_k; k \in \mathbb{Z}, |m| \leq k\}$ are clearly commutative, i.e., $U(r)U(s) = U(s)U(r)$. Hence the new approach presented in [16, 17] can be applied. For the Bernstein-type inequality needed see, e.g., [18, 19]. See also, [20, pp. ], [21, pp. ]. For the corresponding Jackson-type inequality see [16, 17, 22], [20, pp. ], [21, pp. ], [6, Chapter 5]. For the basic spherical harmonic theory in question see, e.g., [6, 7, 19, 23, 3].

For the other equivalent assertions of Part a), however, one has, first of all, to find a replacement for the infinitesimal generator $A$. With methods of spectral theory (see, e.g., [24, 25, 26]), which enables one to investigate the resolvent of $A$, one may be able to deduce an equivalence of type (i)$\Leftrightarrow$(ii).

Concerning (iii) and (iv) of Theorems 1.1 or 2.1, the fundamental tool is the so-called interconnection theorem, a link between the resolvents of $A$ and $A^{-1}$, where $A$ is the infinitesimal generator of a semigroup. It reads (see [12]),

$$\lambda R(\lambda; A^{-1})f = f - \lambda^{-1}R(\lambda^{-1}; A)f. \tag{9}$$

The question is, whether there exists something similar, when $A$ is not the generator of a semigroup of operators?

The situation becomes even more complicated, when $A$ is not injective. In this case $A^{-1}$ has to be replaced by the generalized Drazin inverse $A^{\text{gd}}$, introduced by Butzer, Westphal and Koliha (see [1, 27, 2]). Here $A$ is a closed not necessarily bounded operator. If $A$ is the generator of a semigroup, then (9) holds for $A^{\text{gd}}$ instead of $A^{-1}$ in a slightly modified form; see [10], [11, p. 37].

Alternative approaches to counterparts of (iii) and (iv) of Theorems 1.1 or 2.1 are using Shaw’s theory of $A$-ergodic nets [28] or via Hille’s pseudoresolvents (see [29, p. 215 ff.], [30, p. 521 ff.]).

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