Anomalies and Locality in Field Theories and M–theory

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Abstract

We review some basic notions on anomalies in field theories and superstring theories, with particular emphasis on the concept of locality. The aim is to prepare the ground for a discussion on anomalies in theories with branes. In this light we review the problem of chiral anomaly cancellation in M–theory with a 5–brane.

1 Introduction

Local chiral anomalies represent a breaking of classical infinitesimal gauge symmetries. They are an obstruction to defining path integrals involving chiral fermions or (anti)self–dual tensor fields. The corresponding theories are ill–defined. Therefore one of the important issues in constructing a theory is to verify that it is free of these anomalies. Once the absence of chiral perturbative anomalies is ascertained, there may still be global anomalies, i.e. breaking of the symmetry with respect to gauge transformations not connected to the identity. One has to make sure that the theory is free of these anomalies too. In this contribution we will be concerned with chiral local perturbative anomalies. Historically, these anomalies have been discovered in Yang–Mills theories, subsequently extended to gravity theories and finally studied in (super)string theories. In the latter context they became a fundamental tool in selecting, out of infinite many, five superstring theories in 10D (a type I, two type II and two heterotic).

After the second string revolution, anomalies are anew an important tool in discriminating among different theories. The new input, with respect to field theories or superstring theories, is the presence of branes, i.e. topological defects over which new degrees of freedom are defined, which have their own autonomous dynamics, but are also in interaction with the ambient theory. In this new situation we may have three new contributions to chiral anomalies, if the
brane field content is chiral: the anomalies of the brane in isolation, the anomalies induced via the embedding of the brane in the ambient theory, and the inflow anomalies, which come from the coupling of the brane to the ambient theory. The cancellation of the anomaly resulting from the sum of all the contributions is then a basic criterion for consistency.

At first sight it may look like we are improperly mixing things. After all we are talking about local perturbative anomalies in a non-local and non-perturbative setting such as the embedding of a topological defect in a given theory. However, first of all the term ‘perturbative’, referred to the chiral anomalies we are dealing with, is correct in the sense that they can be evaluated in perturbative field theory, but it may be misleading. Actually these anomalies represent an incurable disease of the theory, which is not limited to its perturbative consistency. Second, the branes we are considering preserve part of the supersymmetry, therefore they are stable as long as supersymmetry survives; this means in particular that their massless spectrum (on which the anomalies are calculated) is also stable, consequently if the corresponding theory is anomalous it is hopelessly inconsistent: there is no possibility that we have miscalculated the anomaly by considering an unstable massless spectrum.

In this paper we discuss one of these problems, perhaps the most relevant one, that is the anomaly of the 5-brane embedded in M-theory. Our aim here is to view this problem in the framework of the principle of locality in field theory. To this end we start from very basic notions of anomalies in field theory. Section 2 contains a pedagogical review of perturbative chiral anomalies in field and string theories with particular emphasis on locality. In section 3 we collect the information we need on M-theory and the M-theory 5-brane in order to discuss the relevant anomaly problem. The M-5-brane ‘naked eye’ anomaly does not identically vanish, so one has to set out to find some sophisticated cancellation mechanism. In section 4 we present two such cancellation mechanisms, one is of topological nature while the other is provided by a local counterterm. We discuss the problems that arise if we accept these mechanisms.

2 Chiral perturbative anomalies in field theories

Let us consider a gauge theory with gauge group $G$ in which chiral fermions $\psi$ are coupled to the gauge potential $A$. The fermions belong to a given representation of the Lie algebra $G$ with generators $T^a$, $a = 1, ..., \dim G$, and $A = \sum_a A^a T^a$. The classical action is of the form

$$S = \int \bar{\psi} \slashed{D} \chi (A) \psi,$$

where $\slashed{D} \chi (A)$ is the chiral (left or right) covariant Dirac operator. $S$ is invariant under the gauge transformations

$$\delta A = d\xi + [A, \xi], \quad \delta \psi = \xi \psi,$$

where $\xi = \sum_a \xi^a T^a$. Therefore we have the classical Ward–Takahashi identity $\delta S = 0$. At one-loop $S$ is replaced by the one-particle irreducible vertex operator $\Gamma_1$. $\Gamma_1$ may not satisfy
the Ward–Takahashi identity. In general one finds
\[ \delta \Gamma_1 = \hbar \mathcal{A} + O(\hbar^2). \]
\( \mathcal{A} \) turns out to be a local functional of the fields, i.e. an integrated polynomial of the fields and their derivatives, linear in \( \xi \). The functional operator \( \delta \) can be made into a coboundary operator: \( \delta^2 = 0 \). Then \( \mathcal{A} \) is seen to satisfy the Wess–Zumino consistency conditions [1]
\[ \delta \mathcal{A} = 0. \]

Now one is faced with two possibilities. Either \( \mathcal{A} = \delta \mathcal{C} \)
for some local functional of the fields \( \mathcal{C} \), or this is not possible for any \( \mathcal{C} \). In the first case the classical Ward–Takahashi identity can be reconstructed at one–loop by suitably redefining \( \Gamma_1 \), as follows:
\[ \hat{\Gamma}_1 = \Gamma_1 - \hbar \mathcal{C}, \quad \delta \hat{\Gamma}_1 = O(\hbar^2). \]
In the second case the classical gauge invariance is broken and we say we have an anomaly.

What the presence of an anomaly means is not only that we cannot retrieve a classical conservation law (this in itself may not be a problem), but that there is an obstruction to defining the determinant of the chiral Dirac operator \( D_\chi(A) \), i.e. we cannot properly define the functional integral of the corresponding theory.

It is therefore crucial to know whether a given theory is anomaly free or not. This takes two steps: the characterization of the anomalies and the method to calculate them.

2.1 Anomalies and locality

From the very definition we see that an anomaly corresponds to a non–trivial cohomology class of \( \delta \), i.e. an anomaly is only defined up to \( \delta \mathcal{C} \), for any local functional \( \mathcal{C} \). Let us make a few remarks on the issue of locality. The idea of locality in field theory is translated into the idea that the only allowed terms are integrals over the space–time manifold of polynomials of \( A \) and of the gauge transformations \( \xi \), as well as of their derivatives [2]. This definition based on polynomiality works well for Yang–Mills gauge theories, but it is not quite fit for gravity theories, where metric and vielbein fields are involved. We will shortly give a definition of locality that bypasses this difficulty. But, for the time being, let us concentrate on gauge theories without gravity. If the space–time manifold is \( M \) we can write
\[ \mathcal{A} = \int_M \omega^1_n, \quad n = \dim M, \]
where \( \omega^1_n \) is an \( n \)–form constructed as a polynomial in \( A \) and \( \xi \) and their derivatives, but linear in \( \xi \). Then we can translate the above conditions on \( \mathcal{A} \) as follows:
\[ \delta \mathcal{A} = 0 \quad \leftrightarrow \quad \delta \omega^1_n + d\omega^2_{n-1} = 0 \quad (1) \]
for some $\omega^1_n, \omega^2_{n-1}$, and

$$A \neq \delta C \iff \omega^1_n \neq \delta \omega^0_n + d\omega^1_{n-1}$$  \hspace{1cm} (2)

for any $\omega^1_{n-1}, \omega^0_n$. The subscript of $\omega$ denotes the form degree, while the superscript represents the number of $\xi$’s that appear in the expression.

The problem represented by (1, 2) was solved long ago [3, 4]. The solutions, i.e. the anomalies, are given in terms of (reducible or irreducible) symmetric polynomials $P_k$, with $k$ entries, which are invariant under the adjoint action of $G$ in $G$. Let $P_k$ be the polynomial that corresponds to a given anomaly, and let

$$F_t = tdA + \frac{1}{2}t^2[A, A], \quad 0 \leq t \leq 1, \quad F_1 \equiv F.$$ \hspace{1cm} (3)

Then we write the descent equations for $P_k$. The first is the Chern–Weil formula

$$P_k(F, ..., F) = d\left(k \int_0^1 dt \ P_k(A, F_t, ..., F_t)\right) \equiv d \left(TM_k(A)\right).$$ \hspace{1cm} (4)

The second determines the anomaly $\omega^1_n$

$$\delta TM_k(A) + d\omega^1_n = 0, \quad n = 2k - 2.$$ \hspace{1cm} (5)

The explicit expression of the anomaly can be given as

$$\omega^1_n = k(k-1) \int_0^1 dt \ P_k(d\xi, A, F_t, ..., F_t), \quad n = 2k - 2.$$ \hspace{1cm} (6)

### 2.2 Locality and universality

In the following we have to be a bit more precise. So we introduce the principal fiber bundle $P(M, G)$ where our gauge theory is defined. Then $A$ is a connection in $P$, $F$ its curvature, $P_k(F, ..., F)$ are basic forms while $TM_k(A)$ are not, and so on. Let us consider, for example, a reducible polynomial $P_k = P_{k_1}P_{k_2}$, $k = k_1 + k_2$, and suppose that $P_{k_2}(F, ..., F)$ is a trivial class in $H^{2k}(M)$, i.e.

$$P_{k_2}(F, ..., F) = d\alpha, \quad \text{where } \alpha \text{ is a basic form in } P(M, G).$$ \hspace{1cm} (7)

This happens if $P_{k_2}(F, ..., F)$ is in the kernel of the Weil homomorphism. Then the Chern–Simons form corresponding to $P_k$ can be written as $-d(TM_{k_1}(A)\alpha)$ since $P_{k_1}(F, ..., F)\alpha$ is a basic $n+1(=2k-1)$ form and so identically vanishes. This Chern–Simons term will give rise, via the descent equations, to an anomaly which is a non–trivial local anomaly (unless the field content of the theory is enlarged, see below) although it is trivial from a cohomological point of view. It was customary some time ago to call such an anomaly non–topological to make a distinction with the topological ones, which correspond either to irreducible polynomials or to reducible ones in which any factor is cohomologically non–trivial.
From the above example we see that the cohomology of $P(M,G)$ is not sufficient to tell us which are the local anomalies. For that we need a different kind of information, called the \textit{locality} or \textit{universality principle}, $[5]$.

Any principal fiber bundle, such as $P(M,G)$, can be obtained by pulling back the universal bundle $EG$ by a suitable classifying map $f$, according to the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\hat{f}} & EG \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & BG
\end{array}
$$

$BG$ is the classifying space, $\hat{f}$ is any map that projects to $f$. In $EG$ there exists a universal connection $A_u$, such that any connection $A$ in $P$ can be obtained as $A = \hat{f}^*A_u$, via some $\hat{f}$. Now, take any polynomial $P_k$ and construct $TP_k(A_u)$ in $EG$. Forms in $EG$, constructed like this out of $A_u$, will be called \textit{universal}. Now we are ready to state the \textit{universality principle for anomalies}: local anomalies are $n$–forms (modulo exact universal $n$–forms) characterized by (obtained from) universal $n+1$–forms $TP_k(A_u)$. Here $P_k$ can be a reducible or irreducible polynomial. This principle identifies local perturbative anomalies in Yang–Mills gauge theories and extends to gravity theories as well, $[5]$. It gives a rigorous formulation to the idea that in field theory local perturbative anomalies are \textit{local}, i.e. they do not depend on the global details of the space–time $M$.

What remains for us to recall is the method to compute anomalies once the massless fermionic spectrum of a given theory is specified. By far the simplest way is to use Atiyah–Singer’s family index theorem. This theorem provides, in a space $M$ of dimension $n$, the $n+2$–form from which the anomaly can be extracted via the descent equations, as above.

One may wonder why we worry about the locality principle when the index theorem gives us a precise expression for all the anomalies. The point is that in most theories we have different types of zero modes (chiral spin 1/2 and spin 3/2 fermions, selfdual or antiselfdual tensors, see below), each one characterized by its own index theorem. Moreover, in more complex theories, there may be inflow anomalies (see below). Of course what is important is the resulting anomaly, namely the sum of the anomalies corresponding to each zero mode. The result of this calculation has then to be compared with the locality principle: is the total anomaly a true local anomaly or can it be cancelled by a local counterterm?

### 2.3 Getting around universality

Up to now we have discussed the correct way to deal with anomalies in Yang–Mills and gravity theories (for a review, see $[3, 7]$). String theories carry something new in this panorama: the Green–Schwarz mechanism. The low energy effective field theory that represents the interactions of the massless modes of the type I superstring in 10D (let us concentrate on this theory for simplicity) is $N=1$ super–Yang–Mills coupled to supergravity theory. It is a chiral theory and it is anomalous for a generic gauge group $G$. However, if $G = SO(32)$ or $E_8 \times E_8$, $[5]$.
the expression of the anomaly gets drastically simplified, all the irreducible polynomials cancel and we are left with the following expression for the anomaly

\[ A = \int_{M_{10}} (\omega_{2L}^1 - \omega_{2G}^1) X_8. \]  

(9)

In order to explain this equation we introduce, beside the gauge connection \( A \) with curvature \( F \), also the spin connection \( \omega \) with curvature \( R \). Moreover let us call \( K \) the relevant polynomials with two entries – essentially the (suitably normalized) Killing forms in the corresponding groups. Then \( \omega_{2L}^1 \) is the 2–form anomaly that comes, via descent equations, from the 4–form polynomial \( K(R, R) \). Similarly \( \omega_{2G}^1 \) comes from \( K(F, F) \). The relative Chern–Simons terms are

\[ \omega_{3L} := TK(\omega) = 2 \int_0^1 dt K(\omega, R_t), \quad \text{and} \quad \omega_{3G} := TK(A) = 2 \int_0^1 dt K(A, F_t), \]

respectively. Finally, in eq.(9), \( X_8 \) is an 8–form constructed with \( R \) and \( F \), invariant under both gauge and Lorentz transformations. The suggestion of Green and Schwarz, \cite{8}, was that the 2–form field \( B \) in the theory and its modified curvature \( H \) incorporate the cancellation mechanism for the residual anomaly \( \omega_{2L}^1 \). More precisely,

\[ dH = K(F, F) - K(R, R), \quad H = dB + \omega_{3G} - \omega_{3L}. \]  

(10)

Therefore, if we assume \( \delta H = 0 \), we get \( \delta B = \omega_{2G}^1 - \omega_{2L}^1 \). Consequently the counterterm \( \int_{M_{10}} BX_8 \) cancels the residual anomaly.

We have reported this well–known construction here in order to relate it to the locality principle. What we have done here is to suppose first that

\[ K(F, F) - K(R, R) = d\gamma, \quad \text{where} \ \gamma \ \text{is a basic form}, \]  

(11)

i.e. that the LHS is cohomologically trivial as a form in \( M \). This is exactly the case considered at the beginning of section 2.2. We saw there that this by itself is not enough to trivialize the local anomaly \( \omega_{2L}^1 \). In fact what is understood in \( \omega_{3G}^1 \) is much more than that: it means that we have promoted \( \gamma \) to a local field and identified it with \( H \); at this point it is consistent to require that \( B \) is a local field with the properties described above\cite{1}, although its geometrical nature remains to be defined.

At first sight the properties of the local fields \( B \) and \( H \), although not inconsistent, may sound arbitrary. But the field theory we are considering here is the low energy effective field theory (LEEFT) of a superstring theory. One can therefore compute the anomaly in another way: just using superstring theory instead of its LEEFT. If one does the relevant computation, \cite{9}, one finds that the residual anomaly \( \omega_{2L}^1 \) actually does not appear. In other words \( \omega_{3G}^1 \) is

\footnote{Actually \( \omega_{3G}^1 \) is not accurately defined since \( \omega_{3G} \) and \( \omega_{3L} \) are forms in the total space of the respective principal fiber bundles, and are not basic, contrary to what \( B \) and \( H \) are supposed to be. However this can be taken care of by introducing suitable background connections \cite{5}. See also the remark at the end of section 3.}
an artifact of the LEEFT. Therefore if the latter is to represent the low energy dynamics of superstring theory, it is not only allowed but also compulsory to promote $\gamma$ to a local field and embody it in $H$, etc. We emphasize again that the anomaly (9) is a genuine non–trivial local anomaly that passes the universality test. However the ‘exotic’ properties of $H$ and $B$ ‘trivialize’ it.

The Green–Schwarz mechanism is paradigmatic. It suggests how in some theories we can get around the universality principle. Similar and generalized Green-Schwarz mechanisms have been applied to many other different circumstances. Below we are going to discuss a difficult case, the case of anomalies on 5–branes embedded in M–theory, in which the idea of locality is rather put under strain\(^2\). We will try, also in that context, to stick as much as possible to what has been said in this section.

## 3 The M–5–brane and its anomaly

The M–theory anomalies can be studied by looking at the massless fields of the LEEFT, i.e. 11D supergravity. These are the graviton, the gravitino and the components of the 3–form $C_3$. Of course there are no perturbative anomalies in odd dimensions and so M–theory in isolation is perturbative anomaly–free.

The world–volume massless content of the M–theory 5–brane fills up the tensor multiplet of (2, 0) supersymmetry in 6D. It contains 8 chiral spinors $\psi$, 5 bosons $X^i$, $i = 1, ..., 5$, which represent the directions transverse to the 5–brane and 3 components of an antiselfdual two–form $B_{ij}$. So the corresponding theory is potentially anomalous. However, since there are no inborn gauge field or metric on the 5–brane, the latter is anomaly free when in isolation. World volume dynamics of the M–5–brane has been determined in \[12, 13, 14\].

However the situation changes if the 5–brane is embedded in M–theory. Geometrically this corresponds to having a six–dimensional manifold $W$, the world–volume of the 5–brane, embedded in the eleven–dimensional manifold $Q$ of M–theory. The anomalies in question arise from a breakdown of invariance under the diffeomorphisms of $Q$ that preserve the embedding of $W$ in $Q$. More precisely, the tangent bundle $TQ$ of $Q$ restricted to $W$ decomposes according to $TQ|_W = TW \oplus N$, where $TW$ and $N$ are the tangent bundle and the normal bundle of $W$, respectively. A Riemannian metric on $Q$ induces a Riemannian metric on $W$ and a metric and an $SO(5)$ connection on $N$. Now the relevant diffeomorphisms of $Q$ are those that map $W$ to $W$. Any such diffeomorphism generates a diffeomorphism of $W$; if the diffeomorphism induced on $W$ is the identity, it determines a gauge transformation in $N$. Therefore the anomalies to be considered are the usual gravity anomalies in $TW$ and the gauge anomalies in $N$. They can be calculated by means of the index theorem. Their evaluation has been carried out in

\(^2\)Other cases of theories involving branes seem to offer unambiguous if not straightforward cancellation mechanisms for their anomalies, \[10, 11\].
The 8–form corresponding to the total anomaly from $\psi$ and $B^-$ is given by

$$I_{B^-} + I_\psi = -\frac{1}{8} L(W)|_8 + \frac{1}{2} \hat{A}(TW) \, ch(S(N)|_8$$

$$= \frac{1}{192} \left( p_1(TW) - p_1(N) \right)^2 + \frac{1}{48} \left( p_2(N) - p_2(TW) \right),$$

where $L$ is the $L$ polynomial, $\hat{A}$ is the arithmetic genus and $ch$ is the Chern character of the spin bundle $S(N)$ of which the $\psi$ are sections, i.e. the tensor product of the spin bundle over $W$ tensored with the vector bundle associated to the normal bundle $N$ via the representation 4 of SO(5). The symbol $|_8$ means that we extract the 8–form part of the corresponding expression. $p_1$ and $p_2$ denote the appropriate Pontrjagin classes. This is not the end of the story, as far as perturbative anomalies are concerned. In the 11D supergravity action one must include also the gravitational correction term [15, 16]

$$S_{\text{bulk}} = \int_Q C_3 \wedge I_8, \quad I_8 = \frac{1}{48} \left( -\frac{1}{4} p_1(TQ)^2 + p_2(TQ) \right).$$

Notice that this term (13) is not quite written in the proper way when a 5–brane is embedded. For supergravity in isolation we have

$$F_4 = dC_3,$$

and so

$$dF_4 = 0.$$  

When the 5–brane is present this is replaced by [18, 17]

$$dF_4 = \delta_W.$$  

Physically this means that the 5–brane is magnetically coupled to M–theory. From the mathematical point of view, $\delta_W$ is a 5–form which can be interpreted as the Poincarè dual of the submanifold $W$ of $Q$. It can be chosen to have (a delta–function–like) support on $W$. One can now write (13) in the appropriate form in the presence of M–5–brane

$$S_{\text{bulk}} = \int_Q F_4 \wedge TI_8,$$

Taking its variation with respect to the above mentioned gauge transformations and using (17), we find

$$\delta S_{\text{bulk}} = \delta \int_Q F_4 \wedge TI_8 = -\int_Q F_4 \wedge d\omega_{in} = \int_W \omega_{in},$$
where $\omega_{in}$ is the *inflow* anomaly generated via the descent equations from the 8–form $I_8$, which can be rewritten as

$$I_8 = -\frac{1}{192}\left(p_1(TW) - p_1(N)\right)^2 + \frac{1}{48}(p_2(N) + p_2(TW))$$

Adding up (12) and (18), we obtain the total perturbative anomaly of the 5–brane. It is generated by the 8–form

$$\frac{1}{24}p_2(N).$$

Let us introduce, for later use, the corresponding properly normalized Chern–Simons term and 6–form anomaly $\omega_7(N)$ and $A_6$, respectively:

$$\frac{1}{24}p_2(N) = d\omega_7(N),
0 = \delta\omega_7(N) + dA_6,$$

where $\delta$ represents the gauge transformations induced on the normal bundle.

It is shown in [17] that this residual anomaly can be canceled by means of a suitable 6D counterterm in the case in which $Q$ is the product of a ten dimensional manifold and a circle, i.e. when M–theory reduces to type IIA superstring theory (in the limit of small radius) and the 5–brane reduces to the solitonic 5–brane of type IIA. In the following we want to discuss the anomaly cancellation problem in the general M–theory set up, [19].

Before we start our analysis, let us recall another fact about anomalies. An anomaly, being the result of a local calculation, must be represented by a global basic form, therefore in eq.(20), $\omega_7(N)$ must also be a basic form. A global basic form $\omega_7(N)$ in $W$ can be written by introducing a reference connection. Let $A$ be the generic connection (with curvature $F$) in the principal bundle $P(W,SO(5))$ to which the normal bundle is associated, and $A_0$ the reference connection. Then

$$\omega_7(N) = 4\int_0^1 dt\ P_4(A - A_0, F_t, F_t, F_t)
= TP_4(A) - TP_4(A_0) + dS(A, A_0),$$

where $P_4$ is the fourth order symmetric $SO(5)$–ad–invariant polynomial corresponding to $p_2(N)/24$,

$$P_4(F^4) = \frac{1}{24}p_2(N).$$

$TP_4$ is the usual Chern–Simons form (11) and $S(A, A_0)$ is a suitable 6–form in $P$ (see [20]). Here $F_t$ is the curvature of $tA + (1 - t)A_0$ and $F_t = tdA + t^2/2[A, A]$.

The three forms in the RHS of (21) are all defined in the total space of $P(W,SO(5))$, while the LHS is basic. In general there is very little need to stress that anomalies are basic forms,
and up to now we have not done it ourselves. However this fact, which is usually understood, is crucial in the present case and must be brought to the foreground.

For instance we remark that $\omega_7$ being basic means that it vanishes identically in $W$, as $W$ is a 6–dimensional space. This fact is immaterial as long as we deal with descent equations (for example, in section 2 of [3] it is shown how to reconcile it with the descent equations), but becomes very important if one wants to use $\omega_7(N)$ to construct an action term. This will be one of our main concerns in the following.

4  Mechanisms for the anomaly cancellation

Let us start by recalling that a 5–brane magnetically couples to M–theory via eq.(15). As long as (15) is a defining equation for the 5–brane, it implies, by Poincarè duality, that $W$ is the boundary of some seven–dimensional manifold $Y$ in $Q$,

$$W = \partial Y.$$  \hspace{1cm} (23)

In the following we are going to exploit this fact in order to cancel the residual anomaly. We are going to study two mechanisms in particular [19]. Other mechanisms have been considered in [21],[19], but they fail in one respect or another.

4.1  A topological counterterm

We examine the possibility to cancel the anomaly of the M–5–brane by means of the counterterm,

$$S_{\text{top}} = \int_Y \omega_7.$$  \hspace{1cm} (24)

where eq.(23) is understood. Formally,

$$\delta S_{\text{top}} = - \int_Y dA_6 = - \int_W A_6.$$  

However we must exert some care in order for this to make sense. We have just recalled that $\omega_7(N)$, as defined in (20), is a basic form in $W$, and so it vanishes identically. Therefore in order to have a non–trivial counterterm, we must extend $\omega_7(N)$ to $Y$ in a non–canonical (constructive) way, because the only canonical extension of 0 is 0; this in turn requires that we extend the connection and, before that, the normal bundle.

In summary, we must make sure that:

1. the normal bundle over $W$ extends to a bundle over $Y$;
2. the connection on the normal bundle extends too;
3. the counterterm (24) is independent of the choice of $Y$. 
Conditions 1 and 2 are needed to ensure that the counterterm (24) makes sense. Let us analyze them first.

To this end the normal bundles need a more precise notation: \( N(X, Z) \) will denote the normal bundle of \( X \) embedded in \( Z \). For instance, according to this new notation, \( N \equiv N(W, Q) \). First we have

\[ i^*TQ \equiv TQ|_W = TW \oplus N(W, Q), \quad (25) \]

where \( i: W \to Q \) is the embedding and

\[ TQ|_W = TY|_W \oplus N(Y, Q). \]

As for \( Y \), it is not closed, still we have

\[ TY|_W = TW \oplus N(W, Y). \]

It follows that

\[ N(W, Q) = N(W, Y) \oplus N(Y, Q)|_W. \]

Notice that \( L \equiv N(W, Y) \) is a one-dimensional bundle. Since it is orientable, it is trivial and so we can write

\[ N = N' \oplus L, \quad N' \equiv N(Y, Q)|_W, \quad L = W \times \mathbb{R}, \quad (26) \]

where \( N' \) is an \( SO(4) \)-bundle, i.e. the gauge group \( SO(5) \) reduces to \( SO(4) \). The decomposition (26) implies in particular that \( p_2(N) = p_2(N') \). Since

\[ p_2(N') = p_2(N(Y, Q))|_W, \]

\( p_2(N) \) extends to \( 0 \in H^8(Y) \). As forms we can write

\[ p_2(N) = e(N(Y, Q))|_W^2, \]

where \( e \) denotes the Euler class.

After these preliminaries let us discuss the conditions 1 and 2 above. Using (26), one can trivially extend \( L \) and \( N' \) to bundles over \( Y \) as follows:

\[ N' = N(Y, Q)|_W \to N(Y, Q), \]
\[ L = W \times \mathbb{R} \to \tilde{L} = Y \times \mathbb{R}. \]

Consequently \( N \) extends in a natural way as

\[ N \to \tilde{N} = N(Y, Q) \oplus \tilde{L}. \quad (27) \]
We can now extend the connection over this bundle as follows. Let us construct a connection over $\tilde{\mathcal{N}}$ by taking the trivial connection in $\tilde{\mathcal{L}}$ plus the connection induced from $\mathcal{Q}$ in $\mathcal{N}(Y,\mathcal{Q})$. This is of course an extension of the connection induced from $\mathcal{Q}$ in $\mathcal{N}$. Therefore also the form $\omega_7$ extends, as well as all the operations on it. All this is made possible by (26).

It remains for us to discuss condition 3. By a standard argument the term (24) will not depend on the particular $Y$ if

$$\frac{1}{2\pi} \int_X \omega_7 \in \mathbb{Z}$$

for any compact 7–manifold $X$ without boundary. Looking at (24) it is actually easy to make a more precise statement about the integral in (28). We remark that $\omega_7$ vanishes when $A = A_0$. Now, a generic connection $A$ can be continuously joined to $A_0$. Therefore also the results of the integral in (28) when evaluated at $A_0$ and at $A$ should be continuously connected. This means that the only value of the integral compatible with (28) is zero.

This condition limits the possible embeddings of the 5–brane. It seems to be a rather strong one, but it is not easy to rewrite it in a simpler form. Moreover, since the anomalies we are dealing with are field theory anomalies, it is natural to ask what is the relation of the cancellation mechanism presented above with the locality principle. In fact the term (24) does not seem to be reducible to the locality principle, [19]. Therefore, if on one side we do not want to exclude a priori this mechanism, on the other side we can ask ourselves if the residual anomaly can be cancelled by some kind of local counterterm. This is indeed possible, as we will see in the next subsection.

### 4.2 A world–volume counterterm

We have seen at the beginning of the section that a 5–brane in M–theory is characterized not only by the usual data (in particular by eq.(13)), but also by specifying a seven–manifold $Y$ that bounds its world–volume. However, if the 5–brane is to be specified by some other data beside the traditional ones, there is a less stringent and more manageable way to do it than by specifying a bounding manifold $Y$: we can simply add the specification of a ‘collar’ or ‘framing’ attached to the 5–brane. The previous analysis told us that any 5–brane is framed, i.e. that its normal bundle splits, as in (29) – with rather non–standard terminology we call such a splitting a framing. If we consider a 5–brane with a fixed framing, we can construct a counterterm, akin to the one in [17], which, at some additional cost, cancels the anomaly. We stress again that the fixed framing in this section is part of the definition of the 5–brane and is a local property as opposed to the global existence of a bounding seven–manifold $Y$.

Therefore, let the normal bundle $\mathcal{N}$ split as

$$\mathcal{N} = \mathcal{N}' \oplus \mathcal{L},$$

where $\mathcal{N}'$ is an $SO(4)$-bundle and $\mathcal{L}$ is a trivial line bundle. We have

$$p_2(\mathcal{N}) = p_2(\mathcal{N}') = e(\mathcal{N}')^2$$

(30)
and
\[ \Phi(N) = pr_1^* \Phi(N') \wedge pr_2^* \Phi(L), \] (31)
where \( pr_i, i = 1, 2 \) are the projections in the decomposition (29) of \( N \) and \( \Phi(E) \) is the Thom class \([22]\) of a real vector bundle \( E \) over \( W \). \( \Phi \) and \( e \) are related by
\[ \sigma_0^* \Phi(N') = e(N'), \] (32)
where \( \sigma_0: W \to N' \) is the zero section of the bundle \( N' \). From now on, for simplicity, we will drop the pull–back symbols.

Let \( N_{\epsilon}(W) \subset N \) be a suitable tubular neighborhood of \( W \) of size \( \epsilon \) embedded in \( Q \). If we identify \( N_{\epsilon}(W) \) with the corresponding submanifold of \( Q \), then since the support of \( \Phi(N) \) can be taken to lie inside \( N_{\epsilon}(W) \), one can view \( \Phi(N) \) as a form on \( N_{\epsilon}(W) \) and identify it with the Poincaré dual of \( W \), represented by the form \( \delta_W \).

Let \( \omega_3 \) be the basic 3–form constructed out of the polynomial corresponding to \( e(N')/24 \) just like \( \omega_7 \) in eq.(21). Using the descent equations we get
\[ \delta \omega_3 + da = 0, \] (33)
says that the integrated anomaly for the framed 5–brane is given by
\[ \int_W A_6 = \int_W e' \wedge a. \]
with \( e' \equiv e(N') \).

Let \( v \) be a nonvanishing vertical vector field on \( L \) such that its contraction with \( \Phi(L) \) satisfies
\[ i_v \Phi(L)|_W = 1. \] (33)
The existence of such a field follows from the 5–brane being framed. If \( s \) is the coordinate along the fiber of \( L \), we can simply choose \( v = \alpha \frac{\partial}{\partial s} \) with the constant \( \alpha \) such that (33) be satisfied. If we now assume that the Lie derivative of \( F_4 \) in the \( L \)-direction is zero on \( W \),
\[ \mathcal{L}_v F_4|_W = 0, \] (34)
then, using (15), (31), (33) and (34), it is easy to see that
\[ di_v F_4|_W = -e'. \] (35)
Now, consider the counterterm
\[ S = \int_W i_v F_4 \wedge \omega_3. \] (36)
Its gauge variation is, using (35),

$$\delta S = - \int_W i_v F_4 \wedge da = \int_W d i_v F_4 \wedge a = - \int_W e' \wedge a.$$ 

Hence the counterterm cancels the anomaly.

The case of IIA 5–brane, in which \( Q = M \times S^1 \), follows as a particular situation of the above. In this case the framing is determined by the background geometry.

Eq.(34), i.e. the constancy of \( F_4 \) along \( L \) on \( W \), is the only condition required for this mechanism to work. It must be added to the defining eq.(17). We remark that the counterterm (36) depends on the vector field \( v \), i.e. on the framing. In this section we have picked a definite one. However a 5–brane can have in general several distinct framings. In the previous subsection we saw that the 5–brane can have several distinct \( Y \)'s that bound it. In this subsection we have seen that the global datum of a bounding seven–manifold can be replaced by the local datum consisting of \( v \), and the topological counterterm can be well replaced by a local one.

5 Discussion

In the above subsection we have seen that the residual anomaly of the M–5–brane can be cancelled by the local six–dimensional counterterm (36) – rather than with the rather unappealing topological counterterm (24). However the anomaly cancellation problem does not end here. We have added a new degree of freedom, the vector field \( v \). We have to ask ourselves whether this new degree of freedom is physical or not and whether it is dynamical or not. Unfortunately, unlike in the Green–Schwarz mechanism for the superstring case, we do not know at present the fundamental formulation of M–theory. Therefore we can only formulate a few hypotheses, which are not inconsistent with what we know about M–theory. If \( v \) is unphysical we should be able to soak it in some gauge symmetry. If it is physical, then the problem is how it should be understood. Perhaps the simplest interpretation is that \( v \) represents a thickening of the 5–brane. In a related case, in [23], it was shown how a five–brane can be seen as a limiting situation in F–theory, in which a squeezing takes place: in such a case the world volume of the 5–brane can be regarded as higher dimensional manifold which becomes six–dimensional in the limit. If, as it is reasonable, the 5–brane becomes anomalous only in the limit, the field \( v \) could be interpreted as the reminder that the theory must be non anomalous. In [24], the authors introduced a (non–dynamical) vector field which describes the background geometry of massive 11D supergravity and imposed conditions similar to (34).

In any case a more complete formulation of M–theory with 5–branes is necessary. Recently, on the wake of previous works, [14, 13], a classical duality symmetric formulation of 11D supergravity with couplings to M-branes has been achieved in [27]. In such treatments bounding seven–manifolds for the five–brane world volume seem to be essential, while the gravitational correction term (13) is not considered. It is clear that both issues are crucial for a comparison with our results.
Finally we should remark that the counterterm (36) breaks supersymmetry. This reminds us again of the Green–Schwarz mechanism in the LEEFT of the superstring; the Green–Schwarz counterterm breaks supersymmetry. It is possible to recover supersymmetry by adding an infinite number of terms to the equation of motions. However it is impossible to satisfy all local field theory axioms: in particular, if we wish supersymmetry, locality and absence of anomalies in a field theory with a finite number of fields, we are bound to find physical ghosts with a consequent breaking of unitarity, see [29] and references therein. Of course this conclusion is not unexpected since the theory in question is a LEEFT, and represents superstring theory only approximately. This fact is recalled here to suggest that even in the present case we may not be so lucky as to find a closed solution of all the problems in a local field theory action. After all M–theory with a 5–brane is much more complex than superstring theories.

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References

[1] J. Wess and B. Zumino, Consequences of anomalous Ward identities, Phys. Lett. B37 (1971) 95.
[2] C. Becchi, A. Rouet and R. Stora, Renormalization of Gauge Theories Ann.Phys. 98 (1976) 287.
[3] L. Bonora and P. Cotta-Ramusino, Consistent and covariant anomalies and local cohomology, Phys.Rev. D33 (1986) 3055.
[4] M. Dubois-Violette, M. Talon and C. Viallet, BRS algebra. Analysis of the consistency equations in gauge theories, Comm.Math.Phys. 102 (1985) 105.
[5] L. Bonora, P. Cotta-Ramusino, M. Rinaldi and J. Stasheff, The Evaluation Map in Field Theory, Sigma–Models and Strings I, Comm. Math. Phys. 112 (1987) 237.
[6] B. Zumino, Chiral anomalies and differential geometry, Lectures given at Les Houches Summer School on Theoretical Physics, Les Houches, France, Aug 8 - Sep 2, 1983. R. Stora, Algebraic structure and topological origin of anomalies, In Progress in gauge fields theory, G ’t. Hooft, A. Jaffe, H. Lehmann, P. K. Mitter, I.M. Singer, R. Stora (eds.), New York: Plenum Press 1984.
[7] L. Alvarez-Gaume and E. Witten, Gravitational anomalies, Nucl. Phys. B234 (1984) 269. L. Alvarez-Gaume and P. Ginsparg, The structure of gauge and gravitational anomalies, Ann. Phys. 161 (1985) 423.
[8] M.B. Green and J.H. Schwarz, Anomaly cancellations in supersymmetric $D = 10$ gauge theory and superstring theory, Phy. Lett. 149B (1985) 117.
[9] M.B. Green, J.H. Schwarz and E. Witten, Superstring theory, vol.II, Cambridge Univ. Press, Cambridge 1987.

[10] J. Mourad, Anomalies of the $SO(32)$ five-brane and their cancellation, hep-th/9709012.

[11] Y.K.E.Cheung and Z.Yin, Anomalies, Branes and Currents hepth/9710206.

[12] M. Aganagic, J. Park, C. Popescu, J. H. Schwarz, World-Volume Action of the M Theory Five-Brane, Nucl. Phys. B496 (1997) 191.

[13] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin and M. Tonin, Covariant Action for the Super-Five-Brane of M-Theory, Phys. Rev. Lett. 78 (1997) 4332.

[14] P.S. Howe and E. Sezgin $D = 11, p = 5$, Phys.Lett. B394 (1997) 62.

[15] C. Vafa and E. Witten, A one–loop test of string duality, Nucl. Phys. B447 (1995) 261.

[16] M. J. Duff, J.-T. Liu and R. Minasian, Eleventh–dimensional origin of string/string duality: a one loop test, Nucl. Phys. B452 (1995) 261.

[17] E. Witten, Five-Brane Effective Action in M–theory, hep-th/9610234.

[18] E. Witten, Five–branes and M–Theory on an Orbifold, Nucl. Phys. B463 (1996) 383.

[19] L. Bonora, C.S. Chu and M. Rinaldi, Perturbative anomalies of the M–5–brane, hep-th/9710063.

[20] S.S. Chern and J. Simons, Characteristic Forms and Geometric Invariants, Ann. of Math. 99 (1974) 48.

[21] S.P.de Alwis, Coupling of Branes and Normalization of Effective Actions in String/M–theory, hep-th/9705139.

[22] R. Bott and L.W. Tu, Differential Forms in Algebraic Topology, Springer Verlag, New York, Heidelberg, Berlin, 1982.

[23] P.S.Aspinwall M–theory versus F–theory Pictures of the Heterotic String, hepth/9707014.

[24] E.Bergshoeff, Y.Lozano and T. Ortin, Massive branes, hepth/9712115.

[25] I. Bandos, N. Berkovits and D. Sorokin, Duality–symmetric eleventh–dimensional supergravity and its couplings to M-branes, hep-th/9711055.

[26] L.Bonora, M.Bregola, K.Lechner, P.Pasti and M.Tonin, Anomaly-free supergravity and super–Yang–Mills theories in ten dimensions Nucl.Phys. B296 (1988) 877.