Stabilising periodic orbits in a chaotic system with hysteresis

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Abstract. This paper considers the implementation of a control algorithm based on the theory of split-hyperbolicity. Control of chaotic systems generally uses assumptions regarding the hyperbolicity of the underlying dynamical system. However this cannot apply to systems with hysteresis. Through the application to a “toy” model, control of a chaotic system with hysteresis is demonstrated, stabilising periodic orbits of various periods.

1. Introduction
It has been known for some time that chaotic systems possess some features which make them prime candidates for efficient control algorithms. The sensitivity of chaotic systems, which could be seen as detrimental to control, enables large changes to be affected by small perturbations. Further, since periodic orbits of many periods do exist (albeit unstable orbits), the system need not be “forced” down a path which it would otherwise avoid. Once a periodic orbit is approached, very small adjustments should be all that is required to keep the orbit stable. Indeed, control algorithms can actually be used to locate more precisely such periodic points, once approximate positions are found.

Existing chaotic control algorithms, the typical example being the OGY algorithm, see [1], assume certain conditions about the underlying systems — usually some degree of smoothness, and usually a finite-dimensional dynamics. In systems with hysteresis, however, neither of these conditions hold, so an alternative must be found.

Split-hyperbolicity is an extension of the notion of hyperbolicity to account for systems with “hidden” variables, which are not everywhere smooth. The theory was first proposed in [2], and a recent review of developments was presented in [3] (which includes results from this work). This paper will present some of the definitions and principal results from [2] related to control, and proceed to describe the implementation of a split-hyperbolic control algorithm. The model system used for the implementation is the hysteretic Kaldor model, a hysteretic modification of a “toy” model from an economic context.

1.1. The hysteretic Kaldor model
The Kaldor model of the trade cycle (see [4]), is regularly cited as one of the original nonlinear models in macroeconomics. In recent years the Kaldor model has reemerged as a proving ground for applying nonlinear techniques to economic models, see for example [5, 6, 7, 8, 9, 10] and as a basis for further models, for example [11, 12].
In Kaldor-type models, the nonlinear behaviour is introduced in the relationship between economic activity (quantified in most cases by the two variables Capital Stock, $K$, and Income, $Y$) and investment, $I$. An interesting modification is the inclusion of hysteretic effects in this investment relationship. This follows the justifications for economic hysteresis described in [13, 14, 15]. The details of this new hysteretic model are presented in [16, 17] and the online preprint [18] and are not included here. These analyses of this model have proven the existence of chaos for certain parameter values, the existence of an abundance of periodic orbits (specifically, orbits of all minimal periods greater than 57) and a positive lower bound on the topological entropy.

The details of the formulation in [16, 17] are not included here, but the resulting two-dimensional (partial) map is illustrated in fig. 1. A number of the results described in [16, 17] are used in the implementation of the control algorithm below.

![Figure 1. Iterates of the hysteretic Kaldor model, showing the chaotic attractor.](image)

2. Split-hyperbolicity and control
Control of dynamical systems consists of two separate procedures: targeting, where the object is to approach a desired region of the phase space; and stabilisation, in which the goal is to ensure the system remains on the desired trajectory. This article will only deal with the stabilisation component. As such, the algorithm presented here is a direct analogue of the OGY algorithm [1]. Before detailing the control algorithm, the basic concept of split-hyperbolicity is outlined.

2.1. Split-hyperbolicity in metric spaces
Informally, the dynamics of a split-hyperbolic system about a particular point can be described (loosely) as “contracting” along one particular direction, and “expanding” along another. These “directions” are analogous to the stable and unstable manifolds of hyperbolic systems, and will form the basis for the control algorithm. The following formal definition of a split-hyperbolic system is drawn from [2, 19, 3].

Let $M_n^s$, $M_n^u$ where $n \in \mathbb{Z}$ be complete metric spaces with the metrics $\rho_n^s$, $\rho_n^u$. Suppose that non-empty balls in $M_n^s$ and in $M_n^u$ are connected, i.e. can not be represented as a disjoint union of two nonempty sets each of which is both relatively open and relatively closed. Elements from
the Cartesian product $M_n = M_n^s \times M_n^u$ are treated as pairs $x = (x^s, x^u)$. The spaces $M_n$ are endowed with the usual metric

$$\rho_n(x, y) \triangleq \max \{\rho_n^s(x^s, y^s), \rho_n^u(x^u, y^u)\}. \quad (2.1)$$

Examine the bi-infinite sequence $\{f_i\}_{i \in \mathbb{Z}}$ of continuous mappings, which may be partial, from $M_n$ to $M_{n+1}$: $f_n(x^s, x^u) = (f_n^s(x^s, x^u), f_n^u(x^s, x^u))$ where $f_n^s : M_n^s \times M_n^u \mapsto M_{n+1}^s$ and $f_n^u : M_n^s \times M_n^u \mapsto M_{n+1}^u$. This sequence is denoted $f$.

Let $J \subset \mathbb{Z}$ be a set, which may be infinite, of consecutive numbers. Let the sequence $X = \{x_n\}_{n \in J}$ of the elements $x_n \in M_n$ be fixed, such that the images $f_n(x_n)$ are defined for all $n \in J$. Denote by $B_n^s[r]$ the closed $r$-ball in $M_n^s$ centered at $x_n^s \in M_n^s$; the balls $B_n^u[r]$ for $x_n^u \in M_n^u$, $r > 0$ are defined analogously. Let $\delta^s, \delta^u$ be some positive constants. Denote $U_n \triangleq B_n^s[\delta^s] \times B_n^u[\delta^u]$. Let $D_n$ be the set of those $y \in U_n$ which satisfy $f_n(y) \in U_{n+1}$.

A split is defined as a four-tuple $s = (\lambda^s, \lambda^u, \mu^s, \mu^u)$ with the condition that $\lambda^s < 1 < \lambda^u$ and $\Delta(s) \triangleq (1 - \lambda^s)/(\lambda^u - 1) - \mu^s\mu^u > 0$.

The sequence $f$ of the mappings $f_n$ is said to be split $(s)$-hyperbolic in the $(\delta^s, \delta^u)$-neighborhood of the sequence $X$ if it satisfies the following three conditions.

**C0.** $D_n$ is closed for all $n \in J$ and for each boundary point $y$ of $D_n$ either $y$ belongs to the boundary of $U_n$ or $f(y)$ belongs to the boundary of $U_{n+1}$ (whenever $n + 1 \in J$).

**C1.** The inequalities

$$\rho_{n+1}^s(f_n^s(y), f_n^s(z)) \leq \lambda^s \rho_n^s(y^s, z^s) + \mu^s \rho_n^u(y^u, z^u) \quad (2.2)$$

and

$$\rho_{n+1}^u(f_n^u(y), f_n^u(z)) \geq \lambda^u \rho_n^u(y^u, z^u) - \mu^u \rho_n^s(y^s, z^s) \quad (2.3)$$

hold for all $n, n + 1 \in J$ and for all $y, z \in D_n$.

**C2.** The mapping $w \mapsto f_n^u(v, w)$ is open as a mapping from $B_n^u[\delta^u]$ to $B_{n+1}^u[\delta^u]$ for each $v \in B_n^s[\delta^s]$, in the sense that the image $f_n^u(v, U)$ of an open subset $U$ of $B_n^s[\delta^s]$ is relatively open in $B_{n+1}^u[\delta^u]$ (whenever $n, n + 1 \in J$).

It is important to note that this definition does not depend on differentiability of the system. The product spaces $M^s$ and $M^u$ can be interpreted as local coordinates, along which the dynamics of the system are “almost” contracting (for $M^s$) and “almost” expanding ($M^u$). The most significant result in the theory of split-hyperbolicity is a shadowing theorem detailed in [2]. This theorem ensures that it is possible to consider approximate dynamics of a split-hyperbolic system (such as that entailed by a computer implementation, for example), and trust that an exact trajectory lies within a small neighbourhood. The purpose of the control algorithm is to approach the actual, “shadowing”, path, and to remain close to it.

### 2.2. Control algorithm

The hysteretic Kaldor model, described in section 1.1, is an example of a system which is suited to the use of split-hyperbolicity. The aim is to stabilise periodic orbits of the system which are embedded in the chaotic attractor. As mentioned earlier, the algorithm described here is only concerned with the stabilisation step of this control — gathering information about the system (specifically the location of quasi-periodic orbits and the local dynamics in the neighbourhood of these points) and targeting are not considered. The fundamental idea is to provide a controlling “kick” to the system at each time-step, based on information that is known about the next time-step (from a previous visit or initial information).
2.2.1. Foundations of the control algorithm Consider the following problem: let the sequence of maps \( f_0, f_1, \ldots, f_{n-1} \) be split-hyperbolic for the sequence of metric spaces \( M_0, M_1, \ldots, M_n = M_0 \) in some neighborhood of \( x = \{ x_0, \ldots, x_n = x_0 \} \), \( x_i \in M_i \). Then, by the above-mentioned shadowing theorem, there exists a unique periodic orbit \( y \) of period \( n \) in the neighborhood of \( x \).

Here we describe the procedure of locating the periodic orbit \( y \). For simplicity, let \( z_0 \) be close to \( x_0 \). We look for a discrete control impulse \( u_i \), which will be applied after each application of \( f_i \) without altering the form of \( f_i \). Thus \( z_1 = f(z_0) \), \( \tilde{z}_1 = z_1 + u_1 \) and, for \( i > 0 \)

\[
\begin{align*}
z_i &= f_{i-1}(\tilde{z}_{i-1}) \mod n, \\
\tilde{z}_i &= (z_i^u, z_i^u + u_i) = f_{i-1}(\tilde{z}_{i-1}) \mod n + (0, u_i)
\end{align*}
\]

Let \( \tilde{z}_i = x_{i+n} \) for \( -n \leq i < 0 \). It is possible to show that if \( \{ u_i \} \) are constructed as

\[
u_i = h^{-1}(\tilde{z}_{i+1-n}^u) - z_i^u, \quad h(\cdot) = f_{i-1}^{u} \mod n(\tilde{z}_i, \cdot),
\]

then \( \tilde{z}_i \to y_i \mod n \) as \( i \to +\infty \). Moreover, if \( \{ g_i \} \) are close approximations of \( f_i \) that are easy to compute (e.g., local linearisations), then the control sequence

\[
u_i = h^{-1}(\tilde{z}_{i+1-n}^u) - z_i^u, \quad h(\cdot) = g_i \mod n(\tilde{z}_i, \cdot),
\]

also results in \( \tilde{z}_i \to y_i \mod n \) as \( i \to +\infty \). A precise statement on how close an approximation \( g_i \) should be to \( f_i \) is given in Proposition 6.1 in [2].

3. Implementation and results

The control algorithm described above was implemented to control orbits of the hysteretic Kaldor model from [16, 17], here denoted \( T \). A quasi-periodic sequence of points, already identified, is used to initialise the algorithm. This nearly-periodic sequence of points is denoted \( x = \{ x_i \}, \quad i \in 0, \ldots, n-1, \) where \( n \) is the “period”. Also assumed is that an approximation to the \( n \)-step dynamics of \( T \) is known — namely local linearisations of \( T^n \) in the vicinity of the \( x_i \). These local linearisations form a sequence of (linear) mappings, denoted \( l = \{ l_i \}, \quad i \in 0, \ldots, n-1 \).

One of the eigenvalues of each \( l_i \) is expected to be less than 1, the corresponding eigenvector is denoted \( \xi_i^u \). The second eigenvalue should be greater than 1, its eigenvector is denoted \( \xi_i^s \).

The system is started with no control influence. The first iterate of \( T \) to approach a point in \( x \) is denoted \( z_0 \), and the point which it is closest to in \( x \) is set to be \( x_0 \) — thus \( x_1 \simeq T(x_0) \) and so on. At this stage the control algorithm is switched on, and the iterates of the controlled system become

\[
z_{i+1} = T(\tilde{z}_i), \quad \text{where} \quad \tilde{z}_i = z_i + u_i,
\]

with the control influence, \( u_i \), as given below. At this stage also a “history” of our system is defined using the points in \( c \):

\[
\begin{align*}
\tilde{z}_{i-n} &= x_i, \\
u_{i-n} &= T(x_{(i-1) \mod n}) - x_i, \quad i = 0, \ldots, n-1.
\end{align*}
\]

The action of the control algorithm on each traversal of the orbit is to improve the estimate of a point on the orbit, \( y_i \mod n \), using information from the previous approximation to \( y_i \mod n \) (i.e., \( \tilde{z}_{i-n} \)) and the previous approximation to the next point on the orbit (i.e., \( \tilde{z}_{i-n+1} \)).

A one-step linearisation of \( T \) in the neighbourhood of \( x_i \) is introduced here, and denoted \( A_i \). This linearisation implies that, for a point \( x \) in the neighbourhood of \( y_i \), \( T(x) \simeq x_{(i+1) \mod n} + A_i \cdot (x - x_i) \). The vector \( r \) is also introduced here, at each time step this vector is given by

\[
r = z_i - (\tilde{z}_{i-n} + A_{i \mod n} \cdot (u_{i-n+1})).
\]
This vector $r$ is the difference between the uncontrolled point $z_i$ and the point $\tilde{z}_{i-n} + A_{i \mod n}^{-1}(u_{i-n+1})$. This second point itself is composed of the previous approximation to $y_{k \mod n}$ — namely $\tilde{z}_{i-n}$ — and a correction term calculated from the previous approximation to the next point $y_{(i+1) \mod n}$. When the system last visited the neighbourhood of $y_{(i+1) \mod n}$, $u_{i-n+1}$ was used to improve the approximation to it. The correction term (approximately) applies this improvement one step earlier on the next pass.

The control impulse $u_i$ is then defined as

$$u_i = s \xi_{i \mod n}, \quad (3.4)$$

where the magnitude of the impulse is $s$ and the direction given by the eigenvector $\xi_{i \mod n}$ defined in a previous paragraph. The magnitude of the control impulse, $s$ is calculated from the components of $r$, $\xi_{i \mod n}$ and $\xi_{i \mod n}$ by

$$s = \frac{\xi_1^s r_2 - \xi_2^s r_1}{\xi_2^s \xi_1^s - \xi_1^s \xi_2^s}, \quad (3.5)$$

where the subscript $i \mod n$ has been dropped for clarity.

### 3.1. Results

As described in [16], two quasi-periodic sequences of points were available (previously identified) for use in the control algorithm — with periods of 5 and 11. These were used as initial approximations for the control algorithm, and the resulting controlled trajectories are shown in figures 2 and 3.

It was noted that any initial condition chosen converged to the same orbit (whether of period 5 or 11), and that this orbit differed from the initial approximation. To illustrate the convergence of the algorithm, fig. 4 shows the magnitude of the control impulse in a logarithmic plot, displaying exponential decrease in the required control, until numerical constraints dominate.

As proven in [16], the hysteretic Kaldor map $T$ contains unstable orbits of many periods. Some of these at least lie nearby the orbits of period 5 and 11 already tested, and indeed it was found that any orbit of period $k$ where $k = 5m + 11n$, $m, n \in \mathbb{Z}$, could be found as follows. Taking the period 5 orbit as the sequence \{x_{i}^{(5)}\} and the period 11 orbit as \{x_{i}^{(11)}\}, construct a new sequence $x_i$ by concatenating these smaller sequences. For example, an orbit of period 21 ($5 \times 2 + 11 \times 1$) can be constructed by taking

$$x_i = \begin{cases} 
  x_{i}^{(5)} & \text{for } i = 0, \ldots, 4, \\
  x_{i-5}^{(5)} & \text{for } i = 5, \ldots, 9, \\
  x_{i-10}^{(11)} & \text{for } i = 10, \ldots, 20.
\end{cases} \quad (3.6)$$

This procedure was tested for a number of such periods, and the control algorithm successfully stabilised the system in each case. The simplest example, with period 16, is shown in fig. 5. Again, the points settled on as the controlled trajectory were different from those of the initial pseudo-orbit $x_i$. The points were also different from those found when controlling trajectories of different periods.
**Figure 2.** Resulting trajectory of stabilised period 5 orbit, in red. The attractor is shown in grey, the transient prior to activation of control is in black.

**Figure 3.** Resulting trajectory of stabilised period 11 orbit, in red. The attractor is shown in grey, the transient prior to activation of control is in black.
Figure 4. Graph showing the logarithm of the control magnitude for each iteration. The rapid convergence of the control algorithm is clear.

Figure 5. Resulting trajectory of stabilised period 16 orbit, in red. The attractor is shown in grey, the transient prior to activation of control is in black.
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