On the next-to-minimal weight of projective Reed-Muller codes

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Abstract. In this paper we present several values for the next-to-minimal weights of projective Reed-Muller codes. We work over $\mathbb{F}_q$ with $q \geq 3$ since in [3] we have determined the complete values for the next-to-minimal weights of binary projective Reed-Muller codes. As in [3] here we also find examples of codewords with next-to-minimal weight whose set of zeros is not in a hyperplane arrangement.

1 Introduction

Reed-Muller codes were introduced in 1954 by D.E. Muller ([11]) as codes defined over $\mathbb{F}_2$, and a decoding algorithm for them was devised by I.S. Reed ([12]). In 1968 Kasami, Lin, and Peterson ([7]) extended the original definition to a finite field $\mathbb{F}_q$, where $q$ is any prime power, and named these codes “generalized Reed-Muller codes”. They also presented some results on the weight distribution, the dimension of the codes being determined in later works. In coding theory one is always interested in the values of the higher Hamming weights of a code because of their relationship with the code performance, but usually this is not a simple problem. For the generalized Reed-Muller codes, the complete determination of the second lowest Hamming weight, also called next-to-minimal weight, was only completed in 2010, when Bruen ([2]) observed that the value of these weights could be obtained from unpublished results in the Ph.D. thesis of D. Erickson ([4]) and Bruen’s own results from 1992 and 2006. Now that Bruen called the attention the Erickson’s thesis we know that the complete list of the next-to-minimal weights for the generalized Reed-Muller codes may be obtained by combining results from Erickson’s thesis with results by Geil (see [6]) or with results by Rolland (see [16]).

In 1990 Lachaud introduced the class of projective Reed-Muller codes (see [8]). The parameters of this codes were determined by Serre ([19]), for some cases, and by Sørensen ([20]) for the general case. As for the determination of the next-to-minimal weight for these codes, there are some results (also about higher Hamming weights) on this subject by Rodier and Sboui ([14], [15], [18]) and also by Ballet and Rolland ([1]). Recently

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the authors of this note completely determined the next-to-minimal weight of projective Reed-Muller codes defined over \( \mathbb{F}_2 \) (see [3]). In this paper we present several results on the next-to-minimal Hamming weights for projective Reed-Muller codes, including the complete determination of the next-to-minimal weights for the case of projective Reed-Muller codes defined over \( \mathbb{F}_3 \). In the next section we recall the definitions of the generalized and projective Reed-Muller codes, and some results of geometrical nature that will allow us to determine many cases of higher Hamming weights for the projective Reed-Muller codes, which we do in the last section.

2 Preliminary results

Let \( \mathbb{F}_q \) be a finite field and let \( I_q = (X_1^q - X_1, \ldots, X_n^q - X_n) \subset \mathbb{F}_q[X_1, \ldots, X_n] \) be the ideal of polynomials which vanish at all points \( P_1, \ldots, P_q \) of the affine space \( \mathbb{A}^n(\mathbb{F}_q) \). Let \( \varphi : \mathbb{F}_q[X_1, \ldots, X_n]/I_q \to \mathbb{F}_q^n \) be the \( \mathbb{F}_q \)-linear transformation given by \( \varphi(g + I_q) = (g(P_1), \ldots, g(P_q)) \).

**Definition 2.1** Let \( d \) be a nonnegative integer. The generalized Reed-Muller code of order \( d \) is defined as \( \text{RM}(n,d) = \{ \varphi(g + I_q) \mid g = 0 \text{ or } \deg(g) \leq d \} \).

It is not difficult to prove that \( \text{RM}(n,d) = \mathbb{F}_q^n \) if \( d \geq n(q - 1) \), so in this case the minimum distance is 1. Let \( d \leq n(q - 1) \) and write \( d = a(q - 1) + b \) with \( 0 < b \leq q - 1 \), then the minimum distance of \( \text{RM}(n,d) \) is

\[
W_{\text{RM}}^{(1)}(n,d) = (q - b)q^{n-a-1}.
\]

According to [4] and [2] (see also [1, Thm. 9]) the next-to-minimal weight \( W_{\text{RM}}^{(2)}(n,d) \) of \( \text{RM}(n,d) \) is equal to

\[
W_{\text{RM}}^{(2)}(n,d) = W_{\text{RM}}^{(1)}(n,d) + cq^{n-a-2} = \left( 1 + \frac{c}{(q - b)q} \right) (q - b)q^{n-a-1}\quad (2.1)
\]

where

\[
c = \begin{cases} 
q & \text{if } a = n - 1; \\
b - 1 & \text{if } a < n - 1 \text{ and } 1 < b \leq (q + 1)/2; \\
q & \text{or } a < n - 1 \text{ and } b = q - 1 \neq 1; \\
q - 1 & \text{if } a = 0 \text{ and } b = 1; \\
q - 1 & \text{if } q < 4, 0 < a < n - 2, \text{ and } b = 1; \\
q - 1 & \text{if } q = 3, 0 < a = n - 2 \text{ and } b = 1; \\
q & \text{if } q = 2, a = n - 2 \text{ and } b = 1; \\
q & \text{if } q \geq 4, 0 < a \leq n - 2 \text{ and } b = 1; \\
b - 1 & \text{if } q \geq 4, a \leq n - 2 \text{ and } (q + 1)/2 < b.
\end{cases}
\]
We will need some specific values of \( W_{\text{RM}}^{(2)}(n, d) \) in the next section.

Let \( Q_1, \ldots, Q_N \) be the points of \( \mathbb{P}^n(\mathbb{F}_q) \), where \( N = q^n + \ldots + q + 1 \). From e.g. [13] or [10] we get that the homogeneous ideal \( J_q \subset \mathbb{F}_q[X_0, \ldots, X_n] \) of the polynomials which vanish in all points of \( \mathbb{P}^n(\mathbb{F}_q) \) is generated by \( \{ X_j^q X_i - X_i^q X_j \mid 0 \leq i < j \leq n \} \). We denote by \( \mathbb{F}_q[X_0, \ldots, X_n]_d \) (respectively, \( (J_q)_d \)) the \( \mathbb{F}_q \)-vector subspace formed by the homogeneous polynomials of degree \( d \) (together with the zero polynomial) in \( \mathbb{F}_q[X_0, \ldots, X_n] \) (respectively, \( J_q \)).

**Definition 2.2** Let \( d \) be a positive integer and let \( \psi : \mathbb{F}_q[X_0, \ldots, X_n]_d/(J_q)_d \to \mathbb{F}_q^N \) be the \( \mathbb{F}_q \)-linear transformation given by \( \psi(f + (J_q)_d) = (f(Q_1), \ldots, f(Q_N)) \), where we write the points of \( \mathbb{P}^n(\mathbb{F}_q) \) in the standard notation, i.e. the first nonzero entry from the left is equal to 1. The projective Reed-Muller code of order \( d \), denoted by \( \text{PRM}(n, d) \), is the image of \( \psi \).

In [20] Sørensen determined all values for the minimum distance \( W_{\text{PRM}}^{(1)}(n, d) \) of \( \text{PRM}(n, d) \) and proved that

\[
W_{\text{PRM}}^{(1)}(n, d) = W_{\text{RM}}^{(1)}(n, d - 1).
\]

One may wonder if there is a similar relation between the next-to-minimal weight of projective Reed-Muller codes and the next-to-minimal weight of generalized Reed-Muller codes. A hint that this might be true comes from the following reasoning. Let \( \omega \) be the Hamming weight of \( \varphi(g + I_q) \), where \( g \in \mathbb{F}_q[X_1, \ldots, X_n] \) is a polynomial of degree \( d - 1 \), and let \( g^{(h)} \) be the homogenization of \( g \) with respect to \( X_0 \). Then the degree of \( g^{(h)} \) is \( d - 1 \) and the weight of \( \psi(X_0 g^{(h)} + (J_q)_d) \) is \( \omega \). This shows that, denoting by \( W_{\text{PRM}}^{(2)}(n, d) \) the next-to-minimal weight of \( \text{PRM}(n, d) \), we have

\[
W_{\text{PRM}}^{(2)}(n, d) \leq W_{\text{RM}}^{(2)}(n, d - 1). \tag{2.2}
\]

In [3] we determined all values for the next-to-minimal weights of projective Reed-Muller codes defined over \( \mathbb{F}_2 \), and from those results we get that there are cases for which equality does not hold in (2.2). In the next section we will determine several values for the next-to-minimal weights of projective Reed-Muller codes defined over \( \mathbb{F}_q \) with \( q \geq 3 \), and we will also find some cases where equality does not hold in (2.2). We recall from [3] a definition and some results that we will need to prove the main results.

**Definition 2.3** Let \( f \in \mathbb{F}_q[X_0, \ldots, X_n]_d \). The set of points of \( \mathbb{P}^n(\mathbb{F}_q) \) which are not zeros of \( f \) is called the support of \( f \), and we denote its cardinality by \( |f| \) (hence \( |f| \) is the weight of the codeword \( \psi(f + (J_q)_d) \)).
In what follows the integers \( k \) and \( \ell \) will always be the ones uniquely defined by the equality
\[
d - 1 = k(q - 1) + \ell
\]
with \( 0 \leq k \leq n - 1 \) and \( 0 < \ell \leq q - 1 \).

**Theorem 2.4** ([3, Lemma 2.4, Lemma 2.5, Prop. 2.6 and Prop. 2.7]) Let \( f \in \mathbb{F}_q[X_0, \ldots, X_n]_d \) be a nonzero polynomial, and let \( S \) be its support, which we assume to be nonempty. Then:

i) if there exists a hyperplane \( H \subset \mathbb{P}^n(\mathbb{F}_q) \) such that \( S \cap H = \emptyset \) and \( |f| > W^{(1)}_{PRM}(n, d) \) then \( |f| \geq W^{(2)}_{RM}(n, d - 1) \);

ii) if \( |S| < \left(1 + \frac{1}{q}\right)(q - \ell)q^{n-k-1} \) then there exists a hyperplane \( H \subset \mathbb{P}^n(\mathbb{F}_q) \) such that \( S \cap H = \emptyset \);

iii) if \( |S| \leq \left(1 + \frac{1}{(q - \ell)}\right)(q - \ell)q^{n-k-1} = (q - \ell + 1)q^{n-k-1} \) then there exists \( r \geq k \) and a linear subspace \( H_r \subset \mathbb{P}^n(\mathbb{F}_q) \) of dimension \( r \) such that \( S \cap H_r = \emptyset \).

### 3 Main results

In this section we determine the next-to-minimal weight for most cases of projective Reed-Muller codes. Recall that we are assuming \( q \geq 3 \). We start by treating the case where \( k = n - 1 \).

**Proposition 3.1** If \( d - 1 = (n - 1)(q - 1) + \ell \), with \( 0 < \ell \leq q - 1 \) then
\[
W^{(2)}_{PRM}(n, d) = W^{(2)}_{RM}(n, d - 1) = q - \ell + 1.
\]

**Proof:** Let \( f \in \mathbb{F}_q[X_0, \ldots, X_n]_d \) be a nonzero polynomial such that
\[
0 \neq |f| \leq W^{(2)}_{RM}(n, d - 1) = q - \ell + 1
\]
(such a polynomial exists because \( W^{(1)}_{PRM}(n, d) = W^{(1)}_{RM}(n, d - 1) = q - \ell \). Let \( S \) be the support of \( f \), from Theorem 2.4 (iii) there exists a hyperplane \( H \) such that \( S \cap H = \emptyset \). From Theorem 2.4 (i) we get that \( |f| = W^{(1)}_{PRM}(n, d) \) or \( |f| \geq W^{(2)}_{RM}(n, d - 1) \), so that \( W^{(2)}_{PRM}(n, d) \geq W^{(2)}_{RM}(n, d - 1) \), and from inequality (2.2) we get \( W^{(2)}_{PRM}(n, d) = W^{(2)}_{RM}(n, d - 1) \). \( \square \)

Now we start to study the case where \( k < n - 1 \).
Proposition 3.2 Let \( d - 1 = k(q - 1) + \ell \) be such that \( k < n - 1 \). If either \( 1 < \ell \leq \frac{q + 1}{2} \), or \( q = 3, k > 0 \) and \( \ell = 1 \), we get

\[
W_{PRM}^{(2)}(n, d) = W_{RM}^{(2)}(n, d - 1) = \begin{cases} (q - 1)(q - \ell + 1)q^{n-k-2} & \text{if } 1 < \ell \leq \frac{q + 1}{2}, \\ 8 \cdot 3^{n-k-2} & \text{if } q = 3, k > 0 \text{ and } \ell = 1. \end{cases}
\]

Proof: From (2.1) we get that

\[
W_{RM}^{(2)}(n, d - 1) = \left(1 + \frac{c}{(q - \ell)q}\right)(q - \ell)q^{n-k-1},
\]

where, when \( q = 3 \), \( 0 < k \leq n - 2 \) and \( \ell = 1 \) we have \( c = q - 1 = 2 \) (and a fortiori \( \frac{c}{(q - \ell)q} = \frac{1}{q} \)), and when \( q \geq 3 \) and \( 1 < \ell \leq (q + 1)/2 \) we have \( c = \ell - 1 \) (and a fortiori \( \frac{c}{(q - \ell)q} \leq \frac{1}{q} \)). Assume, by means of absurd, that there exists \( f \in \mathbb{F}_q[X_0, \ldots, X_n]_d \) such that

\[
W_{PRM}^{(1)}(n, d) < |f| < W_{RM}^{(2)}(n, d - 1) \leq \left(1 + \frac{1}{q}\right)(q - \ell)q^{n-k-1}
\]

and let \( S \) be the support of \( f \). From Theorem 2.4 (ii) and (i) we get \( |f| \geq W_{RM}^{(2)}(n, d - 1) \), a contradiction. Hence \( W_{PRM}^{(2)}(n, d) \geq W_{RM}^{(2)}(n, d - 1) \) and a fortiori \( W_{PRM}^{(2)}(n, d) = W_{RM}^{(2)}(n, d - 1) \). □

Next we treat the case where \( n \geq 3, 0 \leq k < n - 2 \) and \( \ell = 1 \). We observe from (2.1) that \((q^2 - 1)q^{n-k-2} \leq W_{RM}^{(2)}(n, d - 1)\) (actually we have equality when \( q = 3 \) and an strict inequality when \( q > 3 \)).

Proposition 3.3 Let \( n \geq 3 \), \( 0 \leq k < n - 2 \) and \( \ell = 1 \), then

\[
W_{PRM}^{(2)}(n, d) = \left(1 + \frac{1}{q}\right)(q - 1)q^{n-k-1} = (q^2 - 1)q^{n-k-2}.
\]

Proof: In case \( 1 \leq k < n - 2 \) let \( f = X_1 X_{k+3} g + X_0 X_{k+2} h \), where

\[
g = \prod_{i=2}^{k+1} (X_i^{q-1} - X_1^{q-1}) \quad \text{and} \quad h = \prod_{i=2}^{k+1} (X_i^{q-1} - X_0^{q-1}),
\]

or let \( f = X_1 X_3 + X_0 X_2 \) in case \( k = 0 \). We claim that \( |f| = (q^2 - 1)q^{n-k-2} \), and let \( \alpha = (\alpha_0 : \ldots : \alpha_n) \) be a point in the support of \( f \). If \( \alpha_0 = 0 \) then we may take \( \alpha_1 = 1 \),
and we have \( f(\alpha) = \alpha_{k+3} \prod_{i=2}^{k+1} (\alpha_i^{q-1} - 1) \) (or \( f(\alpha) = \alpha_3 \) in case \( k = 0 \)). Thus we must have \( \alpha_{k+3} \in \mathbb{F}_q^* \) and \( \alpha_2 = \cdots = \alpha_{k+1} = 0 \), so we get \((q - 1)q^{n-k-2}\) points of this form in the support. If \( \alpha_0 = 1 \) then

\[
f(\alpha) = \alpha_1 \alpha_{k+3} \prod_{i=2}^{k+1} (\alpha_i^{q-1} - 1) + \alpha_{k+2} \prod_{i=2}^{k+1} (\alpha_i^{q-1} - 1)
\]

in case \( 1 \leq k < n - 2 \), or \( f(\alpha) = \alpha_1 \alpha_3 + \alpha_2 \) in case \( k = 0 \). We see that for any \( \alpha_1 \in \mathbb{F}_q \) we get \( f(\alpha) = (\alpha_1 \alpha_{k+3} + \alpha_{k+2}) \prod_{i=2}^{k+1} (\alpha_i^{q-1} - 1) \), thus we must have \( \alpha_{k+2} \neq -\alpha_1 \alpha_{k+3} \) and \( \alpha_2 = \cdots = \alpha_{k+1} = 0 \), so we get \((q - 1)q^{n-k-1}\) points of this form in the support, and we get

\[
|f| = (q - 1)q^{n-k-2} + (q - 1)q^{n-k-1} = (q^2 - 1)q^{n-k-2},
\]

this proves that \( W_{\text{PRM}}^{(2)}(n, d) \leq (q^2 - 1)q^{n-k-2} \). Assume, by means of absurd, that there exists \( a \in \mathbb{F}_q[X_0, \ldots, X_n]_d \) such that

\[
W_{\text{PRM}}^{(1)}(n, d) < |a| < (q^2 - 1)q^{n-k-2} = \left(1 + \frac{1}{q}\right)(q - 1)q^{n-k-1} \leq W_{\text{RM}}^{(2)}(n, d - 1)
\]

and let \( S \) be the support of \( a \). From Theorem 2.3 (ii) and (i) we get \(|a| \geq W_{\text{RM}}^{(2)}(n, d - 1)\), a contradiction, and we conclude that \( W_{\text{PRM}}^{(2)}(n, d) = (q^2 - 1)q^{n-k-2} \). \( \square \)

The above proof shows that the next-to-minimal weight of projective Reed-Muller codes is not, in all cases, attained only by evaluating polynomials that split completely as a product of degree one factors. For example, in the case \( k = 0 \) of the above result (hence \( n = 2 \) and \( d = 2 \)) the next-to-minimal weight is attained from the evaluation of an irreducible quadric. This is in contrast with what happens with polynomials that attain the minimum distance of \( \text{RM}(n, d) \) and \( \text{PRM}(n, d) \) which must be product of degree one polynomials (see [5] and [1] respectively). Also, in [9], the author proves that codewords of next-to-minimal weight in \( \text{RM}(n, d) \), when \( q \geq 3 \), always come from the evaluation of polynomials which are products of degree one polynomials.

The following result deals with the case where \( n = 2 \) and \( d = 2 \) (so that \( k = 0 \) and \( \ell = 1 \)).

**Proposition 3.4** Let \( n = 2 \) and \( d = 2 \) then

\[
W_{\text{PRM}}^{(2)}(2, 2) = q^2.
\]

**Proof:** Let \( g \in \mathbb{F}_q[X_0, X_1, X_2]_2 \) be such that

\[
W_{\text{PRM}}^{(1)}(2, 2) < |g| < (1 + \frac{1}{q})W_{\text{PRM}}^{(1)}(2, 2) = q^2 - 1.
\]
From Theorem 2.4 (ii) and (i) we get \(|g| \geq W_{\text{RM}}^{(2)}(2,1) = q^2\), a contradiction, hence from (2.2) we get \(W_{\text{PRM}}^{(2)}(2,2) \in \{q^2 - 1, q^2\}\). Let \(f \in \mathbb{F}_q[X_0, X_1, X_2]_2\) be such that \(|f| \in \{q^2 - 1, q^2\}\) and let \(S\) be its support. From Theorem 2.4 (i) if there exists a line in \(\mathbb{P}^2(\mathbb{F}_q)\) not intersecting \(S\) then \(|f| = q^2\), so let’s assume that there is no such line.

After a projective transformation we may assume that \(P = (0 : 0 : 1)\) is not in \(S\), and let \(L_0, \ldots, L_q\) be the lines that contain \(P\). Let \(i \in \{0, \ldots, q\}\), since \(\deg(f) = 2\) we have \(|S \cap L_i| \geq q - 1\) and from \(|S| \in \{q^2 - 1, q^2\}\) we must have at most one line intersecting \(S\) in \(q\) points. After a projective transformation that fixes \(P\) we may assume that the line \(X_0 = 0\) intersects \(S\) in \(q - 1\) points, and that \(P\) and \(Q = (0 : 1 : 0)\) are the points of the line missing \(S\), so we may write \(f = X_0(a_0X_0 + a_1X_1 + a_2X_2) + X_1X_2\). Observe that there are \(q - 1\) points of the form \((0 : 1 : \alpha)\) in the support of \(f\). Let \(\alpha, \beta \in \mathbb{F}_q\), then \(f(1 : \alpha : \beta) = a_0 + a_1\alpha + (a_2 + \alpha)\beta\) and for each \(\alpha \neq -a_2\) we have \(q - 1\) values for \(\beta\) such that \(f(1 : \alpha : \beta) \neq 0\). On the other hand, if \(\alpha = -a_2\) then either \(f(1 : -a_2 : \beta) = 0\) or \(f(1 : -a_2 : \beta) \neq 0\) for \(\beta \in \mathbb{F}_q\), so that \(|f| = (q - 1) + (q - 1)^2 = (q - 1)q\) or \(|f| = (q - 1) + (q - 1)^2 + q = q^2\), which proves \(|f| = q^2\).

We summarize the results for \(W_{\text{PRM}}^{(2)}(n, d)\) obtained in [3] and in this paper in the following tables, where we also list the corresponding values of \(W_{\text{RM}}^{(2)}(n, d - 1)\) for comparison.

| \(n\) | \(k\) | \(\ell\) | \(W_{\text{RM}}^{(2)}(n, d - 1)\) | \(W_{\text{PRM}}^{(2)}(n, d)\) |
|---|---|---|---|---|
| \(n \geq 3\) | \(k = 0\) | \(\ell = 1\) | \(2^n\) | \(3 \cdot 2^{n-2}\) |
| \(n \geq 4\) | \(1 \leq k < n - 2\) | \(\ell = 1\) | \(3 \cdot 2^{n-k-2}\) | \(3 \cdot 2^{n-k-2}\) |
| \(n \geq 2\) | \(k = n - 2\) | \(\ell = 1\) | \(4\) | \(4\) |
| \(n \geq 2\) | \(k = n - 1\) | \(\ell = 1\) | \(2\) | \(2\) |

Table 1: Next-to-minimal weights for \(\text{RM}(n, d)\) and \(\text{PRM}(n, d)\) when \(n \geq 2\) and \(q = 2\)

| \(n\) | \(k\) | \(\ell\) | \(W_{\text{RM}}^{(2)}(n, d - 1)\) | \(W_{\text{PRM}}^{(2)}(n, d)\) |
|---|---|---|---|---|
| \(n = 2\) | \(k = 0\) | \(\ell = 1\) | \(3^2\) | \(3^2\) |
| \(n \geq 3\) | \(k = 0\) | \(\ell = 1\) | \(3^n\) | \(8 \cdot 3^{n-k-2}\) |
| \(n \geq 3\) | \(1 \leq k \leq n - 2\) | \(\ell = 1\) | \(8 \cdot 3^{n-k-2}\) | \(8 \cdot 3^{n-k-2}\) |
| \(n \geq 2\) | \(0 \leq k \leq n - 2\) | \(\ell = 2\) | \(4 \cdot 3^{n-k-2}\) | \(4 \cdot 3^{n-k-2}\) |
| \(n \geq 1\) | \(k = n - 1\) | \(\ell = 1, 2\) | \(4 - \ell\) | \(4 - \ell\) |

Table 2: Next-to-minimal weights for \(\text{RM}(n, d)\) and \(\text{PRM}(n, d)\) when \(n \geq 1\) and \(q = 3\)
Table 3: Next-to-minimal weights for $RM(n, d)$ and $PRM(n, d)$ when $n \geq 1$ and $q \geq 4$

| $n$   | $k$   | $\ell$ | $W_{RM}^{(2)}(n, d-1)$ | $W_{PRM}^{(2)}(n, d)$ |
|-------|-------|--------|------------------------|------------------------|
| $n = 2$ | $k = 0$ | $\ell = 1$ | $q^2$ | $q^2$ |
| $n \geq 3$ | $k < n - 2$ | $\ell = 1$ | $q^{n-k}$ | $q^{n-k} - q^{n-k-2}$ |
| $n \geq 3$ | $k = n - 2$ | $\ell = 1$ | $q^2$ | ??? |
| $n \geq 2$ | $k \leq n - 2$ | $1 \leq \ell \leq \frac{q+1}{2}$ | $(q-1)(q-\ell+1)q^{n-k-2}$ | $(q-1)(q-\ell+1)q^{n-k-2}$ |
| $n \geq 2$ | $k \leq n - 2$ | $\frac{q+1}{2} < \ell \leq q - 1$ | $(q-1)(q-\ell+1)q^{n-k-2}$ | ??? |
| $n \geq 1$ | $k = n - 1$ | $1 \leq \ell \leq q - 1$ | $q - \ell + 1$ | $q - \ell + 1$ |

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