A Chern-Simons transgression formula for supersymmetric path integrals on spin manifolds

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Abstract

Earlier results show that the $N=1/2$ supersymmetric path integral $\mathcal{Z}^g$ on a closed even dimensional Riemannian spin manifold $(X,g)$ can be constructed in a mathematically rigorous way via Chen differential forms and techniques from non-commutative geometry, if one considers $\mathcal{Z}^g$ as a current on the loop space $LX$, that is, as a linear form on differential forms on $LX$. This construction admits a Duistermaat-Heckman localization formula. In this note, fixing a topological spin structure on $X$, we prove that any smooth family $g_\bullet = (g_t)_{t \in [0,1]}$ of Riemannian metrics on $X$ canonically induces a Chern-Simons current $C^g_\bullet$ which fits into a transgression formula for the supersymmetric path integral. In particular, this result entails that the supersymmetric path integral induces a differential topological invariant on $X$, which essentially stems from the $\hat{A}$-genus of $X$.

1 Motivation

Let $X$ be a compact even dimensional topological spin manifold$^1$. The fixed topological spin structure induces an orientation (cf. Corollary E in [17]) on the Fréchet manifold $LX$ of smooth loops $\gamma : \mathbb{T} := S^1 \to X$, whose tangent space $T_\gamma LX$ at a fixed loop $\gamma \in LX$ is given by the space of vector fields on $X$ along $\gamma$, that is, smooth maps $A : \mathbb{T} \to TX$ with $\dot{\gamma}(s) \in T_{\gamma(s)}X$ for all $s \in \mathbb{T}$. Given a Riemannian metric $g$ on $X$ let $E^g_\gamma \in C^\infty(LX)$ and $\omega^g \in \Omega^2(LX)$ denote the energy functional and, respectively, the presymplectic form

$$E^g_\gamma := (1/2) \int_\mathbb{T} g(\dot{\gamma},\dot{\gamma}), \quad \omega^g(A,B) := \int_\mathbb{T} g(\nabla_\dot{\gamma} A,B),$$

(1.1)

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1We work exclusively in the category of smooth manifolds without boundary.
where we will occasionally identify $\mathbb{T} = [0,1]/\sim$. The following $N = 1/2$ supersymmetric path integral plays a crucial role in the context of Duistermaat-Heckman localization on $LX$: with

$$\hat{\Omega}(LX) := \prod_{j=0}^{\infty} \Omega^j(LX)$$

the space of smooth differential forms on $LX$, one formally sets

$$\mathcal{J}^g : \hat{\Omega}(LX) \to \mathbb{C}, \quad \mathcal{J}^g[\sigma] := \int_{LX} e^{-E^g - \omega^g} \wedge \sigma. \quad (1.2)$$

Note that even though $LX$ is oriented, as it stands, the definition of $\mathcal{J}^g$ does not make sense for (at least) the following reasons:

- there exists no infinite dimensional Lebesgue measure;
- the integral of an inhomogeneous differential form (which are the ones of interest) should by definition be the integral of its top degree part, however, $LX$ is infinite dimensional;
- $LX$ is noncompact, so even if one finds a natural way to integrate differential forms on $LX$, some care has to be taken concerning the question of finding a class of 'integrable' (smooth) differential forms.

As we are going to explain in a moment, the mathematical solution of these problems is tied together and manifests itself in a construction of $\mathcal{J}^g$ via Chen integrals and the differential graded Chern character on $(X,g)$. However, in order to motivate our main results, let us continue with our heuristic observations for the moment.

With $\iota$ the contraction by the vector field $K$ on $LX$ given by $\gamma \mapsto \dot{\gamma}$, which generates the natural $\mathbb{T}$-action on $LX$ given by rotating loops, and

$$\hat{\Omega}_\mathbb{T}(LX) := \{ \sigma \in \hat{\Omega}(LX) : \mathcal{L}_K \sigma = 0 \}$$

the space of $\mathbb{T}$-invariant differential forms, there is a supercomplex

$$\cdots \xrightarrow{d-\iota} \hat{\Omega}^+_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \hat{\Omega}^-(LX) \xrightarrow{d-\iota} \hat{\Omega}^+_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \cdots, \quad (1.3)$$

and (with a slight abuse of notation) the dual supercomplex

$$\cdots \xrightarrow{d-\iota} \hat{\Omega}^+_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \hat{\Omega}^-(LX) \xrightarrow{d-\iota} \hat{\Omega}^+_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \cdots, \quad (1.4)$$

i.e. $\hat{\Omega}^\mathbb{T}_{\pm}(LX)$ stands for the linear forms on $\hat{\Omega}^\mathbb{T}_{\pm}(LX)$ and $d-\iota$ acts dually.

Note that these complexes are actually well-defined within the differential calculus of Fréchet manifolds. Now, supersymmetry takes the form $(d-\iota)\mathcal{J}^g = 0$. Moreover, $\mathcal{J}^g$ is an even current, as $LX$ is formally even-dimensional, so that $\mathcal{J}^g$ determines an even
homology class in the homology of (1.4). Finally, one can derive the following infinite dimensional analogue of the Duistermaat-Heckman localization formula,

$$\mathcal{J}^g[\sigma] = \int_X \hat{A}(X,g) \wedge \sigma|_X \quad \text{for all } \sigma \in \hat{\Omega}(LX) \text{ with } (d - \iota)\sigma = 0,$$

where $\hat{A}(X,g)$ is the Chern-Weil representative of the $\hat{A}$-genus of $X$. This leads to a simple and differential geometric 'proof' of the Atiyah-Singer index theorem [3, 2, 1], which was in fact, the main motivation that lead to the discovery of $\mathcal{J}^g$.

The aim of this paper is to examine the dependence of $\mathcal{J}^g$ on $g$. To this end, let $g_\bullet = (g_t)_{t \in [0,1]}$ be a smooth family of Riemannian metrics on $X$ and define for every fixed $t \in [0,1]$ a differential form

$$\beta^g_t \in \Omega^1(LX), \quad \beta^g_t(A) := \frac{1}{2} \int_X (dg_t/dt)(\dot{\gamma}, A),$$

and the induced odd current

$$c_t^g : \hat{\Omega}(LX) \longrightarrow \mathbb{C}, \quad c_t^g(\sigma) := \mathcal{J}^g(\beta^g_t \wedge \sigma).$$

In the appendix, we are going to derive the formula

$$(d/dt)\mathcal{J}^g_t = (d - \iota)c_t^g \quad \text{for all } t \in [0,1]. \quad (1.5)$$

This equality has an important consequence: defining the (odd) Chern-Simons current $c^g_\bullet$ by

$$c^g_\bullet := \int_0^1 c^g_t dt : \hat{\Omega}(LX) \longrightarrow \mathbb{C},$$

one gets the transgression formula

$$\mathcal{J}^{g_1} - \mathcal{J}^{g_0} = (d - \iota)c^g_\bullet.$$

These heuristic observations dictate that any mathematically rigorous definition of $\mathcal{J}^g$ should admit a Chern-Simons type transgression formula, and that the homology class induced by $\mathcal{J}^g$ in the homology of (1.4) should not depend on a particular choice of a Riemannian metric $g$ on $X$. Let us denote this homology class with $\mathcal{J}$. Using Stokes formula it is easy to check that the current

$$\hat{A}(X,g) : \Omega(LX) \longrightarrow \mathbb{C}, \quad \sigma \mapsto \int_X \hat{A}(X,g) \wedge \sigma|_X,$$

satisfies $(d - \iota)\hat{A}(X,g) = 0$, and by a standard transgression argument one finds that the induced homology class does not depend on $g$. In fact, the Duistermaat-Heckman formula dictates that this homology class $\hat{A}(X)$ should be equal to $\mathcal{J}$. 

3
2 Main results

Let us explain now how these heuristic considerations can be verified in a mathematically rigorous way. To this end, we first explain the natural class of (smooth) integrable differential forms on $LX$: we turn $\hat{\Omega}(LX)$ into a complete locally convex Hausdorff space by equipping $\Omega^j(LX)$ with the family of seminorms $\nu_f(\sigma) := \nu(f^*\sigma)$, where $f$ is a smooth map from a finite dimensional manifold $Y$ to $LX$, and $\nu$ is a continuous seminorm on the Fréchet space $\Omega^j(Y)$, and by equipping $\hat{\Omega}(LX)$ with the product topology. Given $\sigma \in \Omega(LX)$ and $t \in T$ one defines $\sigma(t) \in \Omega(LX)$ to be the pullback of $\sigma$ with respect to the evaluation $\gamma \mapsto \gamma(t)$.

Consider the Fréchet space of $T$-invariant differential forms $\Omega_T(X \times T)$ on $X \times T$, with $T$ acting on the second slot. With $\vartheta_T \in \Omega(T)$ the volume form, any $\theta \in \Omega_T(X \times T)$ can be uniquely written in the form $\theta = \theta^' + \vartheta_T \wedge \theta^''$ with $\theta^', \theta^'' \in \Omega(X)$.

Associated to this construction, there is the space of entire chains $C^\epsilon_T(X)$ which is defined as the completion of $C_T(X) := \bigoplus_{N=0}^{\infty} \Omega_T(X \times T) \otimes \Omega_T(X \times T)^\otimes N$, with $\Omega_T(X \times T)^\otimes N := \Omega_T(X \times T)^\otimes N / (C \cdot 1)$

and where $C_T(X)$ is equipped with the following family of seminorms: given any continuous seminorm $\nu$ on $\Omega_T(X \times T)$, one gets the induced projective tensor norm

$\pi_{\nu,N}$ on $\Omega_T(X \times T) \otimes \Omega_T(X \times T)^\otimes N$,

and then a seminorm $\epsilon_{\nu}$ on $C_T(X)$ by setting

$\epsilon_{\nu}(c) := \sum_{N=0}^{\infty} \frac{\pi_{\nu,N}(c_N)}{[N/2]!}$, (2.1)

if $c = \sum_{N=0}^{\infty} c_N \in C_T(X), \text{ with } c_N \in \Omega_T(X \times T) \otimes \Omega_T(X \times T)^\otimes N$ for all $N$.

The required family of seminorms is now given by $\epsilon_{\nu}$, where $\nu$ is a continuous seminorm on $\Omega_T(X \times T)$.

There exists a uniquely determined continuous map $\hat{\Omega}(LX)$, the equivariant Chen iterated integral map,

$\text{Chen}_T: C^\epsilon_T(X) \rightarrow \hat{\Omega}(LX)$.

such that for all $N \in \mathbb{N}_{\geq 0}$, $\theta_0, \ldots, \theta_N \in \theta \in \Omega_T(X \times T)$, one has

$\text{Chen}_T(\theta_0 \otimes \cdots \otimes \theta_N) = \int_{\{0 \leq t_1 \leq \cdots \leq t_N \leq 1\}} \theta_0(0) \wedge (t \theta'_1(t_1) - \theta'_1(t_1)) \wedge \cdots \wedge (t \theta'_N(t_N) - \theta'_N(t_N)) \, dt_1 \cdots dt_N$. (2.3)
Definition 2.1. The space of integrable Chen forms \( \tilde{\Omega}(LX) \subset \hat{\Omega}(LX) \) is defined as the image of \( \text{Chen}_T \).

Set

\[
\tilde{\Omega}_T(LX) := \tilde{\Omega}(LX) \cap \hat{\Omega}_T(LX).
\]

The following result follows essentially from calculations made in [6]. A detailed proof will be given in Section 3.

Proposition 2.2. There is a well-defined supercomplex

\[
\cdots \xrightarrow{d-i} \tilde{\Omega}_T^+(LX) \xrightarrow{d-i} \tilde{\Omega}_T^-(LX) \xrightarrow{d-i} \tilde{\Omega}_T^+(LX) \xrightarrow{d-i} \cdots.
\]

(2.4)

The associated dual supercomplex will be denoted with

\[
\cdots \xrightarrow{d-i} \tilde{\Omega}_T^-(LX) \xrightarrow{d-i} \tilde{\Omega}_T^+(LX) \xrightarrow{d-i} \tilde{\Omega}_T^-(LX) \xrightarrow{d-i} \cdots.
\]

(2.5)

Let us now give the formula for \( \mathfrak{J}^g \). Recall that we have fixed a topological spin structure on \( X \). Consider the (super) spinor bundle \( \Sigma_g \to X \) induced by \( g \), with its (essentially self-adjoint) Dirac operator \( D_g \) on the super Hilbert space of \( L^2 \)-spinors \( \Gamma_{L^2}(X, \Sigma_g) \), and the (natural extension to differential forms of all degrees of the) Clifford multiplication

\[
c_g : \Omega(X) \to \Gamma_{C^\infty}(X, \text{End}(\Sigma_g)).
\]

Let \( \Psi(X, \Sigma_g) \) denote the super algebra of pseudodifferential operators in \( \Sigma_g \to X \). With \( H_g := D^2_g \), we define a linear map

\[
F_g : B_T(X) := \bigoplus_{N=0}^{\infty} \Omega_T(X \times \mathbb{T})^\otimes N \to \Psi(X, \Sigma_g),
\]

\[
F_g^{(0)} := H_g,
\]

\[
F_g^{(1)}(\theta) = -c_g(d\theta') + [D_g, c_g(\theta')],
\]

\[
F_g^{(2)}(\theta_1, \theta_2) = (-1)^{|\theta_1|} \left( c_g(\theta_1'\theta_2') - c_g(\theta_1')c_g(\theta_2') \right),
\]

\[
F_g^{(N)}(\theta_1, \ldots, \theta_N) = 0, \quad \text{if } N \geq 3,
\]

where here and in the sequel all commutators are super-commutators.

For \( M \leq N \) denote with \( P_{M,N} \) all tuples \( I = (I_1, \ldots, I_M) \) of subsets of \( \{1, \ldots, N\} \) with \( I_1 \cup \cdots \cup I_M = \{1, \ldots, N\} \) and with each element of \( I_a \) smaller than each element of \( I_b \) whenever \( a < b \). Given

\[
\theta_1, \ldots, \theta_N \in \Omega_T(X \times \mathbb{T}), \quad I = (I_1, \ldots, I_M) \in P_{M,N}, \quad 1 \leq a \leq M,
\]

set

\[
\theta_{I_a} := (\theta_{i+1}, \ldots, \theta_{i+m}), \quad \text{if } I_a = \{j \mid i < j \leq i + m\} \text{ for some } i, m.
\]
We finally define a linear map

\[ \Phi^g : \mathcal{B}_T(X) \longrightarrow \Psi(X, \Sigma_g), \]

\[ \Phi^g(\theta_1, \ldots, \theta_N) = \sum_{M=1}^{N} \left(-1\right)^M \sum_{I \in P_{M,N}} \int_{\{0 \leq t_1 \leq \cdots \leq t_M \leq 1\}} e^{-t_1 H_g} F_g(\theta_{I_1}) e^{-\left(t_2 - t_1\right) H_g} F_g(\theta_{I_2}) \cdots \]

\[ \cdots e^{-\left(t_M - t_{M-1}\right) H_g} F_g(\theta_{I_M}) e^{-\left(1 - t_M\right) H_g} \ dt_1 \cdots dt_M. \]

The linear map

\[ \alpha : \mathcal{C}_T(X) \longrightarrow \mathcal{B}_T(X), \]

\[ \alpha(\theta_0 \otimes \cdots \otimes \theta_N) := \sum_{k=1}^{N} (-1)^{m_k-n_k} \left(\theta_{k+1} \otimes \cdots \otimes \theta_N \otimes \cdots \otimes \theta_k\right), \]

where \( n_j := |\theta_1| + \cdots + |\theta_j| - j \), induces a linear map

\[ \alpha^g : \text{Hom}(\mathcal{B}_T(X), \Psi(X, \Sigma_g)) \longrightarrow \text{Hom}(\mathcal{C}_T(X), \Psi(X, \Sigma_g)), \]

given explicitly by

\[ [\alpha^g](\theta_0, \ldots, \theta_N) = \sum_{k=1}^{N+1} (-1)^{m_{k-1} - (m_k - 1)} l(\theta_k, \ldots, \theta_N; \theta_T \wedge \theta_0, \theta_1, \ldots, \theta_{k-1}); \]

where \( m_k := |\theta_0| + \cdots + |\theta_k| - k \). With \( \text{Str}_g \) the supertrace in \( \Gamma_{L^2}(X, \Sigma_g) \), the following is the main result of [7]:

**Theorem 2.3.** There exists a uniquely determined current \( \mathcal{J}^g : \tilde{\Omega}(LX) \rightarrow \mathbb{C} \) such that for all \( N \in \mathbb{N}_{\geq 0}, \theta_0, \ldots, \theta_N \in \Omega_T(X \times \mathbb{T}) \) one has

\[ \mathcal{J}^g \left[ \int_{\{0 \leq t_1 \leq \cdots \leq t_N \leq 1\}} \theta'_0(0) \wedge (i \theta'_1(t_1) - \theta''_1(t_1)) \wedge \cdots \wedge (i \theta'_N(t_N) - \theta''_N(t_N)) \ dt_1 \cdots dt_N \right] \]

\[ = -\text{Str}_g \left( [\alpha^g \Phi^g](\theta_0, \ldots, \theta_N) \right). \]

Moreover, \( \mathcal{J}^g \) is even and \((d - i) \mathcal{J}^g = 0\), so that \( \mathcal{J}^g \) defines an even homology class in the homology of \( (2.3) \), and one has the localization formula

\[ \mathcal{J}^g[\sigma] = \int_X \tilde{A}(X,g) \wedge \sigma|_X \quad \text{for all } \sigma \in \tilde{\Omega}(LX) \text{ with } (d - i)\sigma = 0. \]

That this definition of \( \mathcal{J}^g \) is natural, in the sense that it really serves as an *implementation* of the right hand side of (1.2), has been indicated in [9] using the Pfaffian line bundle. A probabilistic representation of \( \mathcal{J}^g \) has been derived in [10], generalizing the earlier result from [8] for \( N = 1 \) to all orders.
Assume \( g_t = (g_t)_{t \in [0,1]} \) is a smooth family of Riemannian metrics on \( X \). We briefly recall the Bourguignon-Gauduchon machinery for metric changes of the Dirac operator [4]. For any \( t \in [0,1] \), define a section \( \mathcal{A}^g_t \) of \( \text{End}(TX) \) by
\[
g_0(u,v) = g_t(\mathcal{A}^g_t u, v) \quad \text{for all} \quad x \in X, u,v \in T_x X.
\]
Then \( \mathcal{A}^g_t \) is strictly positive w.r.t. \( g_t \) and \( g_0 \) and \( (\mathcal{A}^g_t)^{-1/2} \) is a pointwise isometry \((TX, g_t) \to (TX, g_0)\). It therefore lifts canonically to an \( \text{SO}(n) \)-equivariant bundle map
\[
\mathcal{A}^{g_t \cdot SO} : \text{SO}(X, g_t) \to \text{SO}(X, g_0),
\]
where \( \text{SO}(X, g_t) \) denotes the bundle of oriented orthonormal frames of \( X \) w.r.t. the Riemannian metric \( g_t \).

Now recall that we have fixed a topological spin structure. This implies that every Riemannian metric \( g_t \) canonically induces a Riemannian spin structure on \( X \), i.e., a \( \text{Spin}(n) \)-principal fibre bundle \( P_{g_t} \) over \( X \) together with a \( \xi \)-equivariant map \( \pi_{g_t} : P_t \to \text{SO}(X, g_t) \) such that \( (P_t, \pi_{g_t}) \) is a \( \xi \)-reduction of \( \text{SO}(X, g_t) \). Here, \( \xi : \text{Spin}(n) \to \text{SO}(n) \) is the canonically given double cover. Furthermore, \((P_{g_t}, \pi_{g_t})\) being associated with a fixed topological spin structure, the map \( \mathcal{A}^{g_t \cdot SO} \) lifts to an equivariant bundle map \( \mathcal{A}^{g_t \cdot SO} : P_{g_t} \to P_{g_0} \) and through the associated vector bundle construction, we obtain a fibrewise isometric vector bundle isomorphism
\[
\mathcal{A}^{g_t \cdot \Sigma} : \Sigma_{g_t} \to \Sigma_{g_0},
\]
which moreover satisfies
\[
\mathcal{A}^{g_t \cdot \Sigma}(c_{g_t}(\vartheta)(\varphi)) = c_{g_0}(\sqrt{(\mathcal{A}^{g_t \cdot \Sigma})'(\vartheta)}(\mathcal{A}^{g_t \cdot \Sigma})(\varphi)) \quad \text{for all} \quad x \in X, \vartheta \in T^*_x X, \varphi \in (\Sigma_{g_t})_x,
\]
where \((\mathcal{A}^{g_t})' \in C^\infty(X, \text{End}(TX^*))\) denotes the section fibrewise dual to \( \mathcal{A}^{g_t} \).

With
\[
0 < \rho_t^{g_t} = d\mu_{g_t}/d\mu_{g_0} \in C^\infty(X)
\]
the Radon-Nikodym density of \( \mu_{g_0} \) w.r.t. \( \mu_{g_t} \), we obtain the unitary operator
\[
U_t^{g_t} : \Gamma_{L^2}(X, \Sigma_{g_t}) \to \Gamma_{L^2}(X, \Sigma_{g_0})
\]
\[
U_t^{g_t}\varphi(x) = (\rho_t^{g_t})^{-1/2}\mathcal{A}^{g_t \cdot \Sigma}((\varphi(x))),
\]
which we use to define a family \( M^{g_t} \) of \( \nu \)-summable Fredholm modules over \( \Omega(X) \) in the sense of Definition 2.1 in [7], by
\[
M^{g_t} := \left( \Gamma_{L^2}(X, \Sigma_{g_0}), \mathcal{E}^{g_t}, Q_t^{g_t} \right) := \left( \Gamma_{L^2}(X, \Sigma_{g_0}), U_t^{g_t} c_{g_t} U_t^{g_t \cdot \Sigma}, U_t^{g_t \cdot D_{g_t} U_t^{g_t \cdot \Sigma}} \right).
\]

Next, define
\[
\Xi_{g_t} : B_T(X) \to \Psi(X, \Sigma_{g_0})
\]
by
\[
\Xi_{g_t}^{(0)} := Q_t^{g_t}, \quad \Xi_{g_t}^{(1)}(\vartheta) = c_t^{g_t}(\vartheta'), \quad \Xi_{g_t}^{(N)}(\vartheta_1, \ldots, \vartheta_N) = 0, \quad \text{if} \quad N \geq 2,
\]
\[
\Xi_{g_t}^{(N)}(\vartheta_1, \ldots, \vartheta_N) = 0, \quad \text{if} \quad N \geq 2,
\]
and

\[ \Phi^\bullet_{t,r} : B_T(X) \rightarrow \Psi(X,\Sigma_{g_0}) \]

with \( H^\bullet_t := (Q^\bullet_t)^2 \) for \( 0 \leq r \leq 1 \),

\[ \Phi^\bullet_{t,r} : B_T(X) \rightarrow \Psi(X,\Sigma_{g_0}), \]

\[ \Phi^\bullet_{t,r}(\theta_1, \ldots, \theta_N) = \sum_{M=1}^N (-1)^M \sum_{l \in P_M} \int_{\{0 \leq s_1 \leq \cdots \leq s_M \leq r\}} e^{-s_1 H^\bullet_t} F_{g_{s,t}}(\theta_{l_1}) e^{-(s_2-s_1)H^\bullet_t} F_{g_{s,t}}(\theta_{l_2}) \cdots e^{-(s_{M-1}-s_M)H^\bullet_t} F_{g_{s,t}}(\theta_{l_M}) e^{-(s_M-s_{M-1})H^\bullet_t} ds_1 \cdots ds_M. \]

and

\[ F_{g_{s,t}} : B_T(X) \rightarrow \Psi(X,\Sigma_{g_0}), \]

\[ F_{g_{s,t}}(0) := H^\bullet_t, \]

\[ F_{g_{s,t}}(\theta) = -c^\bullet_t(d\theta') + [Q^\bullet_t, c^\bullet_t(\theta')] + c^\bullet_t(\theta''), \]

\[ F_{g_{s,t}}(\theta_1, \theta_2) = (-1)^{|\theta_1|} (c^\bullet_t(\theta_1') - c^\bullet_t(\theta_1) c^\bullet_t(\theta_2')), \]

\[ F_{g_{s,t}}(\theta_1, \ldots, \theta_N) = 0, \] if \( N \geq 3 \).

The space \( \text{Hom}(B_T(X),\Psi(X,\Sigma_g)) \) is turned into a super algebra by means of the product

\[ [l_1 l_2](\theta_1, \ldots, \theta_N) = \sum_{k=0}^N (-1)^{|l_1|(|\theta_1| + \cdots + |\theta_k| - k)} l_1(\theta_1, \ldots, \theta_k) l_2(\theta_{k+1}, \ldots, \theta_N). \]

The following Chern-Simons type transgression formula is the main result of this paper:

**Theorem 2.4.** Assume \( g_\bullet = (g_t)_{t \in [0,1]} \) is a smooth family of Riemannian metrics on \( X \). Then there exists a uniquely given odd current \( \mathcal{C}^\bullet : \Omega(LX) \rightarrow \mathbb{C} \) such that for all \( N \in \mathbb{N}_{\geq 0}, \theta_0, \ldots, \theta_N \in \Omega_T(X \times \mathbb{T}) \) one has

\[ \mathcal{C}^\bullet \left[ \int_{\{0 \leq t_1 \leq \cdots \leq t_N \leq 1\}} \theta'_0(0) \wedge (\theta'_1(t_1) - \theta'_1(t_1)) \wedge \cdots \wedge (\theta'_N(t_N) - \theta'_N(t_N)) \ dt_1 \cdots dt_N \right] \]

\[ = -\text{Str}_{g_\bullet} \left[ \alpha_{g_\bullet} \int_0^1 \int_0^1 \Phi^\bullet_{g_{s,r}}(d\Xi_{g_{s,r}}/dt) \Phi^\bullet_{g_{s,1-r}} \ dr \, ds \right](\theta_0, \ldots, \theta_N). \]

One has \( \mathcal{F}^{g_\bullet} - \mathcal{F}^{g_0} = (d - \iota) \mathcal{C}^\bullet \); in particular, the homology class induced by \( \mathcal{F}^\bullet \) in the homology of \( \Omega(LX) \) does not depend on a particular choice of a Riemannian metric \( g \) on \( X \).

**Remark 2.5.** The formula for \( \mathcal{C}^\bullet \) can be further evaluated by noting that

\[ Q^\bullet_t = \frac{1}{2}(\rho^\bullet_t)^{-1} c_{g_\bullet}(\mathcal{A}^\bullet_t)^{-1/2} \text{grad} \rho^\bullet_t + \mathcal{A}^\bullet_t \Sigma D_{g_\bullet} \left( \mathcal{A}^\bullet_t \Sigma \right)^{-1}, \]

\[ c^\bullet_t(\theta) = c_{g_\bullet}(\sqrt{(\mathcal{A}^\bullet_t \Sigma)}(\theta)), \]

\[ dc^\bullet_t/ dt(\theta) = c_{g_\bullet}(d \sqrt{(\mathcal{A}^\bullet_t \Sigma)}/dt)(\theta). \]
A local formula for the elliptic first-order differential operator $A_t^{g \cdot \Sigma} D_{g_t} A_t^{g \cdot \Sigma}^{-1}$ can be found in [4, Théorème 20]. From the above expression for $Q_t^{g \cdot \Sigma}$, one can derive an expression for the, in general nonelliptic, first-order differential operator $(d/dt)Q_t^{g \cdot \Sigma}$. The needed $t$-derivative of $A_t^{g \cdot \Sigma} D_{g_t} A_t^{g \cdot \Sigma}^{-1}$ is recorded in [4, Théorème 21].

As a consequence we get:

**Corollary 2.6.** Let $X$ and $Y$ be compact even-dimensional, oriented spin manifolds with fixed topological spin-structures. Assume there exists a diffeomorphism $f : X \rightarrow Y$ preserving orientations and topological spin-structures. Then, for any choice of Riemannian metrics $g$ and $h$ on $X$ resp. on $Y$, the homology class induced by $J_g^X$ in the homology of (2.5) equals the homology class of $f^* J_h^Y$.

**Proof.** Setting $g_1 := f^* h$, the diffeomorphism $f$ becomes an orientation and metric spin-structure preserving isometry $f : (X,g_1) \rightarrow (Y,h)$ furnishing unitary equivalences between Clifford multiplications and Dirac operators on $(X,g_1)$ and $(Y,h)$. Formula (2.6) shows that $J_X^g$ and $f^* J_Y^h$ are equal, and Theorem 2.4 establishes the claim. \[ \square \]

We denote the homology class of $J^g$ for some/any Riemannian metric $g$ on $X$ by $J_g$, which by the previous corollary is a differential topological invariant of $X$. Let us identify this invariant: for every Riemannian metric $g$ on $X$, using Stokes formula, it is easily seen that the current

$$ \hat{A}(X,g) : \hat{\Omega}(LX) \rightarrow \mathbb{C}, \quad \sigma \mapsto \int_X \hat{A}(X,g) \wedge \sigma|_X $$

satisfies $(d - \iota)\hat{A}(X,g) = 0$, and by Theorem E in combination with Lemma 9.3 from [7], the homology class of $\hat{A}(X,g)$ in (2.5) equals that of $J^g$. Moreover, by a standard transgression argument, the homology class of $\hat{A}(X,g)$ does not depend on $g$. Putting everything together, it follows that this class $\hat{A}(X)$ equals $J_g$.

## 3 Proof of Proposition 2.2

We have to show that $d - \iota$ maps

$$ \hat{\Omega}_T(LX) = \hat{\Omega}(LX) \cap \hat{\Omega}_T(LX) $$

to itself. We give $\Omega_T(X \times \mathbb{T})$ the $\mathbb{Z}$-grading

$$ \theta' + \partial_T \wedge \theta'' \in \Omega_T(X \times \mathbb{T})^j \Leftrightarrow \theta' \in \Omega^j(X), \theta'' \in \Omega^{j+1}(X) $$

and turn it into a locally convex DGA using the differential $d - \iota_{\partial_T}$ with $\partial_T$ the canonical vector field on $\mathbb{T}$. Then $C_T(X)$ inherits the $\mathbb{Z}$-grading induced by

$$ C_T(X) = \bigoplus_{N=0}^{\infty} \Omega_T(X \times \mathbb{T}) \otimes \Omega_T(X \times \mathbb{T})[1]^{\otimes N}, $$

9
where $\Omega_T(X \times \mathbb{T})[1]$ denotes $\Omega_T(X \times \mathbb{T})$ as a set with the shifted grading

$$\Omega_T(X \times \mathbb{T})[1] := \Omega_T(X \times \mathbb{T})^{j+1}.$$ 

With $b$ the Hochschild differential and $B$ the Connes differential in the $\mathbb{Z}$-graded category, the space $C_T(X)$ becomes a supercomplex with the differential $d - \iota \partial_T + b - B$. By continuity, the same holds true for $C_T^\epsilon(X)$.

Let $\mathbb{A} : \Omega(LX) \rightarrow \Omega(LX)$, $\sigma \mapsto \int_T \varphi^*\sigma$

be the idempotent linear operator obtained by averaging the $T$-action on $LX$, where

$$\varphi_s : LX \rightarrow LX, \quad \gamma \mapsto \gamma(\bullet + s), \quad s \in T.$$ 

Note that it is implicitly used here that $\mathbb{A}$ preserves the image of $\text{Chen}_T$, which follows from a simple calculation. Then, as shown in [6], one has the formulae

$$\mathbb{A}\text{Chen}_T(d - \iota \partial_T + b - B) = (d - \iota A)\mathbb{A}\text{Chen}_T,$$

noting that $\iota A = A\iota$.

Assume that $\sigma \in \Omega(LX)$ is $T$-invariant. This means that $\sigma = \text{Chen}_T(\theta)$ for some $\theta \in C_T^\epsilon(X)$ and that $A\text{Chen}_T(\theta) = \text{Chen}_T(\theta)$. Then we have

$$(d - \iota)\sigma = dA\text{Chen}_T(\theta) - \iota A^2\text{Chen}_T(\theta) = (d - \iota A)A\text{Chen}_T(\theta) = \mathbb{A}\text{Chen}_T((d - \iota \partial_T + b - B)\theta),$$

which shows that $(d - \iota)\sigma$ is $T$-invariant and also that $(d - \iota)\sigma$ is a Chen form because $\mathbb{A}$ preserves $\Omega(LX)$. This completes the proof.

### 4 Proof of Theorem 2.4

First of all, recall definition (2.7). We are going to omit $g_\bullet$ everywhere in the notation. Consider the Chern character

$$\text{Ch}_{g_\bullet} : C_T^\epsilon(X) \rightarrow \mathbb{C},$$

whose value at

$$\theta_0 \otimes \cdots \otimes \theta_N \in C_T^\epsilon(X)$$

is given by the RHS of (2.6) for $g = g_t$. Then $\text{Ch}_{g_\bullet}$ vanishes on the kernel of $\text{Chen}_T$ and this defines $\mathbb{J}^{g_\bullet}$. If we can show that $M^{g_\bullet}$ satisfies the axioms of Definition 6.1 in [7], then (using that Chern characters are invariant under unitary transformations) it follows that the (odd) Chern-Simons form

$$\text{CS}(M^{g_\bullet}) : C_T^\epsilon(X) \rightarrow \mathbb{C}$$

constructed on page 31 in [7] satisfies

$$\text{Ch}_{g_1} - \text{Ch}_{g_0} = (d - \iota \partial_T + b - B)\text{CS}(M^{g_\bullet})$$
and vanishes on the kernel of $\text{Chen}_T$, too. It follows that

$$C^g(\text{Chen}_T(\theta)) := \text{CS}(M^g_T)(\theta), \quad \theta \in C_T(X),$$

is well-defined and, being invariant under $A$ (which follows from its very construction), has the desired properties, in view of

$$A\text{Chen}_T(d - i\partial_t + b - B) = (d - i)A\text{Chen}_T.$$

It remains to show (H1), (H2) and (H3) from Definition 6.1 in [7], where (H1) is the condition

$$\sup_{t \in [0,1]} \text{tr} \left(e^{-Q_t^2}\right) < \infty,$$

(H2) is the condition

$$\sup_{t \in [0,1]} \left\| \dot{Q}_t(Q_t^2 + 1)^{-1/2} \right\| + \sup_{t \in [0,1]} \left\| (Q_t^2 + 1)^{-1/2} \dot{Q}_t \right\| < \infty,$$

and (H3) is the condition that for all $\theta \in \Omega_T(X \times T)$ the map

$$t \mapsto c_t^g(\theta) \in \{\text{bounded operators in } \Gamma_{L^2}(X,\Sigma_{g_0})\}$$

is strongly $C^1$.

Here, (H1) can be seen as follows: one can appeal to the Lichnerowicz formula for $D_t^2$ and semigroup domination (cf. Theorem 3.1 in [11]) to get

$$\text{tr} \left(e^{-Q_t^2}\right) \leq \text{rank}(\Sigma_0)e^{-\min_{x \in X}(1/4)\text{scal}_g(x)}\text{tr} \left(e^{-\Delta_g}\right),$$

which entails (H1), as $t \mapsto \min_{x \in X}(1/4)\text{scal}_g(x)$ is clearly continuous, and $t \mapsto \text{tr} \left(e^{-\Delta_g}\right)$ is smooth by Proposition 6.1 from [14].

To see (H2) note that by elliptic regularity, each $Q_t := U_tD_tU_t^*$ has the same domain of definition $W^{1,2}(X)$. Furthermore, $\dot{Q}_t := (d/dt)Q_t$ is a first order differential operator, which we consider as acting on smooth spinors. The proof of (H2) is based on the following lemma, which is a modification of Lemma 4.17 in [8]:

**Lemma 4.1.** Let $S$ be a densely defined, closed linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$, and let $T$ be a self-adjoint bounded linear operator in $\mathcal{H}_1$ with $T \geq -\lambda$ for some $\lambda \geq 0$. Assume that $S^*S + T \geq 0$. Then one has

$$\|S(S^*S + T + 1)^{-1/2}\| \leq \sqrt{\lambda + 1}.$$

**Proof.** By assumption we have

$$S^*S + 1 \leq S^*S + T + \lambda + 1,$$
which means
\[
\| (S^* S + 1)^{1/2} f \| \leq \| (S^* S + T + \lambda + 1)^{1/2} f \| \quad \text{for all } \ f \in \text{dom}(S^* S)^{1/2}.
\]
From this we obtain
\[
\| (S^* S + 1)^{1/2} (S^* S + T + 1)^{-1/2} h \| \leq \| (S^* S + T + \lambda + 1)^{1/2} (S^* S + T + 1)^{-1/2} h \|
\]
for all \( h \in \mathcal{H}_1 \). Using the functional calculus associated with the operator \( S^* S + T \), we calculate the norm of the operator appearing on the right hand side to be
\[
\| (S^* S + T + \lambda + 1)^{1/2} (S^* S + T + 1)^{-1/2} h \| \leq \| \mathcal{S} (S^* S + T + 1)^{-1/2} \|
\]
where we have used the polar decomposition \( S = U (S^* S)^{1/2} \) with a partial isometry \( U \) on the third line and the functional calculus associated with the operator \( S^* S \) on the fourth line.

Using this lemma, we are going to prove that one has (H2): first of all, note that \( Q_t \) acting on \( \Gamma_{C^\infty}(X, \Sigma_{g_0}) \) is a first order differential operator whose coefficients depend smoothly on \( t \in [0,1] \). Since \( X \) is compact, it follows that
\[
\sqrt{t + \lambda + 1} \sup_{t \geq 0} \left| \frac{d}{dt} \langle Q_t \varphi, \psi \rangle \right| < \infty.
\]

Now we can estimate
\[
\| S (S^* S + T + 1)^{-1/2} \| = \| S (S^* S + 1)^{-1/2} (S^* S + T + 1)^{-1/2} \|
\]
\[
\leq \sqrt{\lambda + 1} \| S (S^* S + 1)^{-1/2} \|
\]
\[
\leq \sqrt{\lambda + 1} \| (S^* S + T + 1)^{-1/2} \|
\]
\[
\leq \sqrt{\lambda + 1} \sup_{t \geq 0} \sqrt{\frac{t}{t + 1}}
\]
\[
\leq \sqrt{\lambda + 1},
\]
where we have used the polar decomposition \( S = U (S^* S)^{1/2} \) with a partial isometry \( U \) on the third line and the functional calculus associated with the operator \( S^* S \) on the fourth line.

Using this lemma, we are going to prove that one has (H2): first of all, note that \( Q_t \) acting on \( \Gamma_{C^\infty}(X, \Sigma_{g_0}) \) is a first order differential operator whose coefficients depend smoothly on \( t \in [0,1] \). Since \( X \) is compact, it follows that
\[
\langle Q_t \varphi, \psi \rangle = (d/dt) \langle Q_t \varphi, \psi \rangle = (d/dt) \langle \varphi, Q_t \psi \rangle = \langle \varphi, \dot{Q}_t \psi \rangle
\]
for all \( \varphi, \psi \in \Gamma_{C^\infty}(X, \Sigma_{g_0}) \), i.e., \( \dot{Q}_t \) is symmetric.

Secondly, the operator \( Q_t^2 + 1 \) being elliptic, it follows from a classical result of Seeley that \( (Q_t^2 + 1)^{-1/2} \) is a pseudo-differential operator. In particular, it maps \( \Gamma_{C^\infty}(X, \Sigma_{g_0}) \) to itself.

Turning to operator norms, note that \( \dot{Q}_t (Q_t^2 + 1)^{-1/2} \) is bounded if and only if
\[
\sup \left\{ \| \langle \dot{Q}_t (Q_t^2 + 1)^{-1/2} \varphi, \varphi \rangle \| : \varphi \in \Gamma_{C^\infty}(X, \Sigma_{g_0}) \right\} < \infty.
\]
The operators $\dot{Q}_t$ and $(Q_t^2 + 1)^{-1/2}$ being symmetric this, in turn, is equivalent to $(Q_t^2 + 1)^{-1/2}Q_t$ being bounded. Hence, it suffices to show that

$$\sup_{t \in [0,1]} \left\| \dot{Q}_t (Q_t^2 + 1)^{-1/2} \right\| < \infty. \quad (4.1)$$

To this end, we first use the unitary invariance of the functional calculus to compute

$$\left\| \dot{Q}_t (Q_t^2 + 1)^{-1/2} \right\| = \left\| \dot{Q}_t ((U_t D_{g_t} U_t^*)^2 + 1)^{-1/2} \right\| = \left\| \dot{Q}_t U_t (D_{g_t}^2 + 1)^{-1/2} U_t^* \right\| = \left\| U_t^* \dot{Q}_t U_t (D_{g_t}^2 + 1)^{-1/2} \right\|. \quad (4.2)$$

Next, we decompose

$$U_t^* \dot{Q}_t U_t = a_t \circ \nabla_t + \tau_t,$$

with $\nabla_t$ the spinor connection of $\Sigma_{g_t}$, and

$$a_t \in \Gamma_{\mathcal{C}^\infty(X, \text{Hom}(T^* X \otimes \Sigma_{g_t}, \Sigma_{g_t}))}, \quad \tau_t \in \Gamma_{\mathcal{C}^\infty(X, \text{End}(\Sigma_{g_t}))},$$

so that by the Lichnerowicz formula we have

$$U_t^* \dot{Q}_t U_t (D_{g_t}^2 + 1)^{-1/2} = a_t \nabla \left( \nabla^* \nabla + \frac{1}{4} \text{scal}_{g_t} + 1 \right)^{-1/2} + \tau_t \left( D_{g_t}^2 + 1 \right)^{-1/2}. \quad (4.2)$$

Because $\| (D_{g_t}^2 + 1)^{-1/2} \| \leq 1$, the operator norm of the second term on the right hand side is bounded by $\| \tau_t \|$, which is continuous in $t$. Hence,

$$\sup_{t \in [0,1]} \left\| \tau_t \left( D_{g_t}^2 + 1 \right)^{-1/2} \right\| < \infty.$$

Regarding the first term on the right hand side of (4.2), we appeal to the above lemma with

$$S = \nabla, \quad T = (1/4) \text{scal}_{g_t}, \quad \lambda_t := \frac{1}{4} \max_{x \in X} |\text{scal}_{g_t}(x)|,$$

to see that

$$\| a_t \nabla \left( \nabla^* \nabla + \frac{1}{4} \text{scal}_{g_t} + 1 \right)^{-1/2} \| \leq \| a_t \| \sqrt{\lambda_t + 1},$$

which is also continuous in $t$, thereby completing the proof of (4.1). The remaining condition (H3) is evident from the last two formulae in Remark 2.5. Note also that for each $\theta \in \Omega_T(X \times \mathbb{T})$ and $t \in [0,1],$

$$\text{the operators } \left( (Q_t^{\theta*})^2 + 1 \right)^{\pm 1/2} \left( dc_t^{\theta*} / dt(\theta) \right) (Q_t^{\theta*})^2 \left( (Q_t^{\theta*})^2 + 1 \right)^{\mp 1/2}$$

are densely defined and bounded. This completes the proof of Theorem 2.4.
Appendix: formal proof of formula (1.5)

We start by calculating the derivative of $\mathcal{I}^{g_t}$ w.r.t. $t$,

$$(d/dt)\mathcal{I}^{g_t}[\sigma] = \int_{LX} (d/dt)e^{-E_t^{g_t} - \omega^{g_t}} \wedge \sigma = \int_{LX} e^{-E_t^{g_t} - \omega^{g_t}} \wedge (d/dt) (-E_t^{g_t} - \omega^{g_t}) \wedge \sigma.$$  

Let $\nabla(t)$ denote the Levi-Civita connection for $g_t$, and let $\gamma \in LX$, $Y, Z \in T_s LX$. Recalling the definition of the energy functional and the presymplectic form (1.1), the $t$-derivative appearing in the integrand on the right-hand side is

$$(d/dt) \left( -E_t^{\gamma_t} - \omega_t^{\gamma_t} \right)(Y, Z) = -\frac{1}{2} \int_T g_t'(\dot{\gamma}_t, \dot{\gamma}_t) - \int_T g_t'(Y, \nabla_t^{(t)} Z) - \int_Y g_t \left( Y, \nabla_t^{(t)} Z \right), \quad (4.3)$$

where we have used primes to denote derivatives w.r.t. $t$ and dots to denote derivatives w.r.t. the loop parameter $s$.

Using that the covariant derivative commutes with every contraction, the second integral in (4.3) is equal to

$$\frac{1}{2} \int_T g_t'(Y, \nabla_t^{(t)} Z) = \frac{1}{2} \int_T \left\{ g_t'(Y, Z) - \nabla_t^{(t)} (g_t'(Y, \cdot))(Z) \right\}$$

$$= \frac{1}{2} \int_T g_t'(Y, \nabla_t^{(t)} Z) - \frac{1}{2} \int_T \nabla_t^{(t)} (g_t'(Y, \cdot))(Z)$$

$$= \frac{1}{2} \int_T \left\{ g_t'(Y, \nabla_t^{(t)} Z) - g_t' \left( Z, \nabla_t^{(t)} Y \right) \right\} - \frac{1}{2} \int_T (\nabla(t) \gamma_t g_t)(Y, Z).$$

For the third term on the right-hand side of (4.3), we use the well-known formula (see, e.g., [16, Proposition 2.3.1]) for the time derivative of the Levi-Civita connection,

$$\int_T g_t \left( Y, \nabla_t^{(t)} Z \right) = \frac{1}{2} \int_Y \left\{ (\nabla(t) z g_t'(t))(Y, \dot{\gamma}_t) + (\nabla(t) z g_t')(Y, Z) - (\nabla(t) Y g_t')(Z, \dot{\gamma}_t) \right\}.$$  

Putting the above together, we obtain

$$(d/dt) \left( -E_t^{\gamma_t} - \omega_t^{\gamma_t} \right)(Y, Z) = -\frac{1}{2} \int_T g_t'(\dot{\gamma}_t, \dot{\gamma}_t) - \frac{1}{2} \int_T \left\{ g_t'(Y, \nabla_t^{(t)} Z) - g_t' \left( Z, \nabla_t^{(t)} Y \right) \right\}$$

$$+ \frac{1}{2} \int_T \left\{ (\nabla(t) Y g_t')(\dot{\gamma}_t, Z) - (\nabla(t) z g_t')(\dot{\gamma}_t, Y) \right\}. \quad (4.4)$$

On the other hand, defining the 1-form $\beta_t^{\gamma_t}$ on $LX$ by

$$(\beta_t^{\gamma_t})(Y) = \frac{1}{2} \int_T g_t'(\dot{\gamma}_t, Y),$$

its exterior derivative $d\beta_t^{\gamma_t}$ is defined by the Cartan formula [12, 33.12],

$$d(\beta_t^{\gamma_t})(Y, Z) = Y \beta_t^{\gamma_t}(Z) - Z \beta_t^{\gamma_t}(Y) - \beta_t^{\gamma_t}([Y, Z]), \quad (4.5)$$
where $\tilde{Y}$ and $\tilde{Z}$ are local extensions of $Y, Z$, i.e., vector fields defined on a neighborhood of $\gamma \in LX$ with $\tilde{Y}_\gamma = Y$ and $\tilde{Z}_\gamma = Z$ (this definition is independent of the extensions $\tilde{Y}, \tilde{Z}$), and where we have used the usual identification of tangent vectors with the derivations they induce on the algebra of smooth functions on $LX$.

To compute the right hand side of (4.5), fix $\gamma \in LX$ and let $\eta, \xi : (-\varepsilon, \varepsilon) \to LX$ be smooth with $\eta(0) = \xi(0) = \gamma$ and $\dot{\eta}(0) = Y$, $\dot{\xi}(0) = Z$. Then

$$Y_{\beta_t} \cdot \tilde{Z} = \frac{1}{2} \frac{d}{d\tau} |_{\tau=0} \int_{T} (g_t'(\eta(\tau)(s) \left( \frac{\partial}{\partial s} \eta(\tau)(s), \tilde{Z}_{\eta(\tau)}(s) \right) ds$$

$$= \frac{1}{2} \int_{T} \left\{ (\nabla(t)Y(s)g_t' \langle \dot{\gamma}(s), Z(s) \rangle + g_t' \left( \frac{\nabla(t)}{\partial s} Y(s), Z(s) \right) + g_t' \left( \dot{\gamma}(s), \frac{\nabla(t)}{\partial t} \tilde{Z}_{\eta(\tau)}(s) \right) \right\} |_{\tau=0} ds$$

where the last equality comes from the well-known identity

$$\frac{\nabla(t)}{\partial s} \eta(\tau)(s) = \frac{\nabla(t)}{\partial s} \eta(\tau)(s).$$

Analogously, we have

$$Z_{\beta_t} \beta_t(Y) =$$

$$= \frac{1}{2} \int_{T} \left\{ (\nabla(t)Z(s)g_t' \langle \dot{\gamma}(s), Y(s) \rangle + g_t' \left( \frac{\nabla(t)}{\partial s} Z(s), Y(s) \right) + g_t' \left( \dot{\gamma}(s), \frac{\nabla(t)}{\partial s} \tilde{Y}_{\eta(\tau)}(s) \right) \right\} |_{\tau=0} ds.$$

To calculate $[\tilde{Y}, \tilde{Z}]_\gamma(s)$, we use that the space of smooth vector fields on $LX$ forms a Lie subalgebra of the space of bounded derivations [12, Theorem 32.8]. To this end, fix $s \in T$, let $f \in C^\infty(X)$, denote by $ev_s : LX \to X$ the smooth evaluation map $\gamma \mapsto \gamma(s)$, and define $\tilde{f} := f \circ ev_s \in C^\infty(LX)$. Then

$$\tilde{Z}_\gamma \tilde{f} = df_{\tilde{Z}_\gamma} = df_{\gamma(s)} d(ev_s)_{\gamma} \tilde{Z}_\gamma = df_{\gamma(s)} \tilde{Z}_\gamma(s),$$

so that

$$\tilde{Y}_\gamma \tilde{Z} \tilde{f} = \frac{d}{d\tau} |_{\tau=0} df_{\eta(\tau)(s)} \tilde{Z}_{\eta(\tau)}(s) \left( \nabla(t)Y(s)df(Z(s)) + df \frac{\nabla(t)}{\partial \tau} \tilde{Z}_{\eta(\tau)}(s) \right),$$

showing

$$[\tilde{Y}, \tilde{Z}]_\gamma(s) f = [\tilde{Y}, \tilde{Z}]_\gamma \tilde{f} = \text{Hess} f(Y(s), Z(s)) + \text{Hess} f(Z(s), Y(s))$$

$$= \text{Hess} f(Y(s), Z(s)) - \text{Hess} f(Z(s), Y(s)) = \left( \frac{\nabla(t)}{\partial \tau} \tilde{Z}_{\eta(\tau)}(s) |_{\tau=0} - \frac{\nabla(t)}{\partial \tau} \tilde{Y}_{\eta(\tau)}(s) |_{\tau=0} \right) f.$$

We have proved

$$d(\beta_t^{\bullet})_\gamma(Y, Z) = (d/dt) \left( -E_\gamma^\bullet - \omega_\gamma^\bullet \right) (Y, Z) + t \beta_t^{\bullet}.$$
Hence, for any differential form $\sigma$ on $LX$ we have
\[ \frac{d}{dt} J_{g^t}^g[\sigma] = \int_{LX} e^{-E_{g^t} - \omega_{g^t}} \wedge (d - \iota) \beta_{g^t}^g \wedge \sigma = \int_{LX} e^{-E_{g^t} - \omega_{g^t}} \wedge \beta_{g^t}^g \wedge (d - \iota) \sigma, \]
where the last equality follows from the fact that by definition one has
\[ (d - \iota) J_{g^t}^g[\sigma] = J_{g^t}^g[(d - \iota) \sigma] = 0. \]

Defining
\[ \mathcal{C}_{g^t}^g(\sigma) := \int_{LX} e^{-E_{g^t} - \omega_{g^t}} \wedge \beta_{g^t}^g \wedge \sigma, \]
we end up with
\[ \frac{d}{dt} J_{g^t}^g = (d - \iota) \mathcal{C}_{g^t}^g, \]
formally proving \((1.5)\).

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