Lie symmetries and similarity solutions for rotating shallow water

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Abstract

We study a nonlinear system of partial differential equations which describe rotating shallow water with an arbitrary constant polytropic index $\gamma$ for the fluid. In our analysis we apply the theory of symmetries for differential equations and we determine that the system of our study is invariant under a five dimensional Lie algebra. The admitted Lie symmetries form the $\{2A_1 \oplus 2A_1\} \oplus A_1$ Lie algebra for $\gamma \neq 1$ and $2A_1 \oplus 3A_1$ for $\gamma = 1$. The application of the Lie symmetries is performed with the derivation of the corresponding zero-order Lie invariants which applied to reduce the system of partial differential equations into integrable systems of ordinary differential equations. For all the possible reductions the algebraic or closed-form solutions are presented. Travel-wave and scaling solutions are also determined.

Keywords: Lie symmetries; invariants; shallow water; similarity solutions

1 Introduction

Lie symmetries is an essential tool for the study of nonlinear differential equations. The main characteristic of the Lie symmetry analysis is that invariant surfaces, in the space where the parameters of the nonlinear differential equation evolve, are determined which can be used to performed an extended analysis of the nonlinear differential equation [1–7], construct conservation laws [8–10] and when it is feasible to determine solutions of the differential equation [11–14]. In applied mathematics Lie symmetries cover a wide range of applications from physics, biology, financial mathematics and many others for instance see [15–28] and references therein.

In this work, we interest on the application of Lie’s theory on an system of partial differential equations (PDEs) describe one-dimensional rotating shallow water phenomena. The system of our consideration expressed in Lagrangian coordinates is [29]

$$h_t + h^2 u_x = 0$$ (1)
$$u_t + h^{\gamma-1} h_x - v = 0$$ (2)
$$v_t + u = 0$$ (3)

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where $h = h(t, x)$ denotes the height of the fluid surface, $u = u(t, x)$ denotes the velocity component in the $x$-direction and $v = v(t, x)$ is the other horizontal velocity component that is in the direction orthogonal to the $x$-direction [29]. Parameter $\gamma$ is the polytropic parameter of the fluid, where in this work is assumed to be $0 \leq \gamma$. The system (1)-(3) is important for the study of atmospheric phenomena like geostrophic adjustment and zonal jets, for more details of the physical properties of the above system we refer the reader in [30][32] and references therein.

The application of Lie symmetries in shallow-water theory is not new. Indeed, there are various studies in the literature [33][38] which has been provide important results with special physical interest. Recently, a detailed study of the nonlocal symmetries for a variable coefficient shallow water equation performed in [39]. However, the majority of these studies are for the case where the fluid has a specific polytropic exponent $\gamma$, or the shallow water equations describe non-rotating phenomena. The plan of the paper is as follows.

In Section 2 we present the basic properties and definitions of Lie symmetry analysis which is the main mathematical tool for our analysis. The main results of this work are presented in Section 3. More specifically, we reduce the system (1)-(3) into two equations for the variables $h$ and $v$. We derive that the latter system of two PDEs admits five Lie point symmetries and we study all the possible reductions in ordinary differential equations (ODEs) with the use of zero-order Lie invariants. We find that the reduce systems can be solve explicitly and we derive the algebraic solution or closed-form solutions for every possible reduction and every value of the parameter $\gamma$. The latter result is important because it shows how powerful is the method of Lie symmetry analysis for the study of shallow-water phenomena to prove the existence of solutions for the model of our study. Emphasis is given on the travel-wave and scaling solutions. Finally our discussion and conclusions are presented in Section 4.

2 Preliminaries

In this Section we briefly discuss the basic definitions and main steps for the determination of Lie point symmetries for differential equations.

Consider a system of PDEs

$$H^A (y^i, u^A, u^A_i, ...) = 0,$$

(4)

where $u^A$ denotes the dependent variables, $y^i$ are the independent variables and $u^A_i = \frac{\partial u^A}{\partial y^i}$.

We assume the one-parameter point transformation (1PPT) in the space of the independent and dependent variables

$$\bar{y}^i = y^i (y^j, u^B; \varepsilon),$$

$$\bar{u}^A = u^A (y^j, u^B; \varepsilon),$$

(5)

(6)

in which $\varepsilon$ is an infinitesimal parameter, the differential equation (4) remain invariant if and only if

$$\bar{H}^A (\bar{y}^i, \bar{u}^A, ...; \varepsilon) = H^A (y^i, u^A, ...),$$

(7)

or equivalently [12]

$$\lim_{\varepsilon \to 0} \frac{\bar{H}^A (\bar{y}^i, \bar{u}^A, ...; \varepsilon) - H^A (y^i, u^A, ...)}{\varepsilon} = 0.$$  

(8)

The latter conditions is expressed

$$\mathcal{L}_X (H^A) = 0,$$

(9)
where \( \mathcal{L} \) denotes the Lie derivative with respect the vector field \( X^{[n]} \) which is the \( n \)-th-extension of generator \( X \) of the infinitesimal transformation (5), (6) in the jet space \( \{ y^i, u^A, u^A_i, ... \} \)

\[
X^{[n]} = X + \eta^{[1]} \partial_{u^A} + ... + \eta^{[n]} \partial_{u^A_i ... i_n},
\]

(10)

with generator

\[
X = \frac{\partial y}{\partial \varepsilon} \partial_x + \frac{\partial u^A}{\partial \varepsilon} \partial_{u^A},
\]

(11)

and

\[
\eta^{[n]} = D_i \eta^{[n-1]} - u_{i_1 i_2 ... i_n} D_i \left( \frac{\partial y}{\partial \varepsilon} \right), \ i \geq 1.
\]

(12)

When condition (9) is satisfied for a specific 1PPT, the vector field \( X \) is called a Lie point symmetry for the system of PDEs (4). For an unknown 1PPT, in order to specify the generators \( X \) which are Lie point symmetries for a given differential equation, from the symmetry condition (9) we specify a system of PDEs with dependent variables the components of the generator \( X \). The solution of the latter system provides the generic symmetry vector and the number of independent solutions give the number of independent vector field and the dimension of the admitted Lie algebra.

### 3 Lie symmetry analysis

We write the system (11)-(12) as two second-order PDEs

\[
v_{tx} - h^{-2} h_t = 0,
\]

(13)

\[
v_{tt} - h^{-1} h_x + v = 0
\]

(14)

while the application of Lie’s theory provides a five dimensional Lie algebra consists by the following vector fields

\[
X_1 = \partial_t , \ X_2 = \partial_x , \\
X_3 = \cos(t) \partial_v , \ X_4 = \sin(t) \partial_v \\
X_5 = (\gamma + 1) x \partial_x + (\gamma - 1) v \partial_v + 2h \partial_h
\]

In table II the commutators of the Lie symmetries are presented. Consequently, from table II we can refer that the admitted Lie algebra is the \( \{ 2A_1 \oplus s \ A_1 \} \oplus s \ A_1 \) in the Morozov-Mubarakyanov Classification Scheme [40-43]. However, in the limit where \( \gamma = 1 \), the admitted Lie algebra is the \( 2A_1 \oplus s \ 3A_1 \).

In order to continue with the application of the Lie point symmetries it is important to determine the one-dimensional optimal system and invariants [44]. In order to do that the adjoint representations should be calculated. By definition, for every basis of the Lie symmetries \( X_i \), the adjoint representation is given by the following expression

\[
\text{Ad (exp } (\varepsilon X_i)) X_j = X_j - \varepsilon [X_i, X_j] + \frac{1}{2} \varepsilon^2 [X_i, [X_i, X_j]] + ... .
\]

(15)

For the admitted Lie point symmetries of the system (13), (14) the adjoint representation are given in [2]. In order to find the optimal system we consider the generic symmetry vector

\[
X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5
\]

(16)
and we find the equivalent vectors by considering the adjoint representation. At this point it is important to mention that the adjoint action admits two invariant functions, the $\phi_1(a_i) = a_1$ and $\phi_2(a_i) = a_5$ \[13\]. The invariants can be used to simplify the calculations on the derivation of the optimal system. Indeed we have to consider the cases $a_1 a_2 \neq 0$ and $a_1 a_2 = 0$.

Case 1: For $a_1 a_2 \neq 0$, we have that

$$X' = \text{Ad}(\exp(\varepsilon_4 X_4)) \text{Ad}(\exp(\varepsilon_3 X_3)) \text{Ad}(\exp(\varepsilon_2 X_2)) X$$

becomes

$$X' = a_1 X_1 + a_5 X_5$$

for specific values of the parameters $\varepsilon_2$, $\varepsilon_3$ and $\varepsilon_4$.

Case 2: For $a_1 a_2 = 0$, there are three subcases, (a) $a_1 = 0$, $a_5 \neq 0$; (b) $a_1 \neq 0$, $a_5 = 0$ and (c) $a_1 = a_5 = 0$.

Case 2a: For $a_1 = 0$ and $a_2 \neq 0$, and following the steps as before we find the optimal system $X_5$ where $\gamma \neq 1$. In the limit $\gamma = 1$, the generic optimal system is $a_3 X_3 + a_4 X_4 + X_5$.

Case 2b: For $a_1 = 0$ and $a_1 \neq 0$, the optimal system is derived

$$X' = \text{Ad}(\exp(\varepsilon_4 X_4)) \text{Ad}(\exp(\varepsilon_3 X_3)) X$$

which for specific values of $\varepsilon_3$ and $\varepsilon_4$ is simplified as

$$X' = a_1 X_1 + a_2 X_2$$

Parameter $a_2$ is not an invariant hence, it can be zero too. Hence, the two optimal systems are $a_1 X_1 + a_2 X_2$ and $X_1$.

Case 2c: For $a_1 = a_5 = 0$, we calculate the generic optimal systems $a_2 X_2 + a_3 X_3 + a_4 X_4$.

Hence, the one-dimensional optimal systems for $\gamma \neq 1$

$$X_1 \ , \ X_2 \ , \ X_5 \ , \ aX_1 + X_2 \ ,$$

$$aX_1 + X_5 \ , \ a_2 X_2 + a_3 X_3 + a_4 X_4$$

and for $\gamma = 1$

$$X_1 \ , \ X_2 \ , \ aX_1 + X_2 \ , \ aX_1 + X_5 \ ,$$

$$a_2 X_2 + a_3 X_3 + a_4 X_4 \ , \ a_3 X_3 + a_4 X_4 + X_5$$

There is a difference in the number of one-dimensional optimal systems which depends on the parameter $\gamma$, that is expected because the structure of the Lie algebra changes.

We proceed our analysis by applying the Lie symmetries to reduce the system of PDEs into a system of ODEs and solve the resulting ODEs by applying the method of Lie symmetries.

### 3.1 Static solution

The application of the symmetry vector $X_1$ in \[13\] provides with the static solution $h = H(t_0, x)$ and $v = V(t_0, x)$. The system of PDEs reduce to one first-order ODE

$$\frac{1}{\gamma} (H^\gamma)_x - V = 0, \quad (17)$$

which provides a constraint condition between the velocity $v$ and the height $h$. 

Table 1: Commutators of the admitted Lie point symmetries by system 13-14

|   | X₁ | X₂ | X₃ | X₄ | X₅ |
|---|----|----|----|----|----|
| X₁ | 0  | 0  | −X₄| X₃| 0  |
| X₂ | 0  | 0  | 0  | 0 | (γ + 1)X₂ |
| X₃ | X₄| 0  | 0  | 0 | (γ − 1)X₃ |
| X₄ | −X₃| 0  | 0  | 0 | (γ − 1)X₄ |
| X₅ | 0  | −(γ + 1)X₂| −(γ − 1)X₃| −(γ − 1)X₄| 0 |

Table 2: Adjoint representation for the Lie point symmetries of the system 13-14

| Ad (exp (εXᵢ)) Xⱼ | X₁ | X₂ | X₃ | X₄ | X₅ |
|---------------------|----|----|----|----|----|
| X₁ | X₁ | X₂ | cos (ε)X₃ + sin (ε)X₄ | cos (ε)X₄ − sin (ε)X₃ | X₅ |
| X₂ | X₁ | X₂ | X₃ | X₄ | X₅ − ε (γ + 1)X₂ |
| X₃ | X₁ − εX₄ | X₂ | X₃ | X₄ | X₅ − ε (γ − 1)X₃ |
| X₄ | X₁ + εX₄ | X₂ | X₃ | X₄ | X₅ − ε (γ − 1)X₄ |
| X₅ | X₁ | e^(ε(γ+1))X₂ | e^(ε(−1))X₃ | e^(ε(−1))X₄ | X₅ |

3.2 Point solution

The application of the symmetry vector X₂ provides with the time-dependent solution in a specific point, i.e. h = H (t, x₀) and v = V (t, x₀). The resulting system provides H (t, x₀) = H₀ and the second-order ODE

\[ V_{tt} + V = 0. \]  \(\text{(18)}\)

The later equation is nothing else than the oscillator which admits eight Lie point symmetries and it is maximally symmetric. The Lie symmetries X₁, X₃ and X₄ are inherit symmetries, while the rest five Lie point symmetries are Type II symmetries. The exact solution of equation (18) is

\[ V (t, x₀) = V₁ \cos (t) + V₂ \sin (t). \]  \(\text{(19)}\)

3.3 Travel-wave solution

The linear combination of X₁ + cX₂ provides travel-wave solutions h = H (x − ct) , v = V (x − ct) where parameter c describes the the wave speed. The reduced system is

\[ V_{xx} - H^{-2}H_{xx} = 0, \] \(\text{(20)}\)

\[ c²V_{xx} - H^{-1}H_{xx} + V = 0. \] \(\text{(21)}\)

in which the new independent parameter ξ is defined as ξ = x − ct.

From equation (20) we derive

\[ H^{-1} = H₀ - V_ξ, \] \(\text{(22)}\)

where by substitute in (21) it follows

\[ \left( \begin{array}{c} c² - \left( H₀ - V_ξ \right)⁻¹ \end{array} \right) V_{xx} + V = 0. \] \(\text{(23)}\)
Figure 1: Qualitative evolution of the functions $V(\xi)$ and $H(\xi)$ provided by the numerical simulation of the nonlinear differential equation (23). The plots are for $c = 1$ and $H_0 = 2$ and initial conditions $V(0) = 0.01$, $V_\xi(0) = -0.2$. Left figure is for $\gamma = 1.1$ while right figure is for $\gamma = 2$. We observe that $V(\xi)$ and $H(\xi)$ are periodic functions and have similar behaviour, the different values of $\gamma$ changes only the frequency of the oscillations. However, as we decreased the initial value $V_\xi(0)$ in values where $c^2 - (H_0 - V_\xi)^{-\gamma-1} \approx 1$ then the numerical simulation provided singular behaviour for the $V(\xi)$ which correspond to a shock.

The latter equation admits only one Lie point symmetry for $\gamma \geq 1$, the autonomous symmetry $\partial_\xi$. Recall that for $\gamma = -1$, equation (23) becomes a maximally symmetric equation but such value for parameter $\gamma$ is not physical accepted.

Application of the differential invariants of the autonomous symmetry vector $\partial_\xi$ in (23) lead to the nonlinear first-order ODE

$$\frac{w}{(H_0 - w)} \frac{dw}{dz} = z \left( (H_0 - w)^{-\gamma} - c^2 (H_0 - w) \right)^{-1},$$

with solution

$$\gamma (\gamma - 1) \left( z^2 + c^2 w^2 + w_0 \right) + (H_0 - w)^{-\gamma+1} = 0,$$

where the new variables $\{z, w(z)\}$ are defined as $z = V(\xi)$ and $w(z) = V_\xi$.

In the simplest case where the integration constant $H_0$ vanishes, and $\gamma = 1$, the generic solution is given in terms of the Lambert function

$$\ln(w(z)) = -\frac{1}{2} W \left( -c^2 \exp( z^2 + 2w_0 ) \right) + \frac{z^2}{2} + w_0.$$

In Fig. 1 we present a numerical simulation of the $H(\xi)$ and $V(\xi)$ as they provided by the differential equation (23). The plots which are presented are for $\gamma = 1.1$ and $\gamma = 2$. From the figure we observe a travelling-wave solution for $V(\xi)$, and for the variable $H(\xi)$. 

\[6\]
3.4 Scaling solution

The Lie invariants of the scaling symmetry vector $X_5$ are

$$h = H(t) x^{\frac{\gamma - 1}{\gamma + 1}} , v = V(t) x^{\frac{\gamma - 1}{\gamma + 1}}.$$  (27)

Hence, the reduced system consists by two second-order ODEs

$$\frac{\gamma - 1}{\gamma + 1} V_t + H^{-2} H_t = 0,$$  (28)

$$V_{tt} - \frac{2}{\gamma + 1} H^\gamma + V = 0.$$  (29)

From (28) and for $\gamma > 1$ we find

$$V(t) = -\frac{\gamma + 1}{\gamma - 1} H^{-1} + V_0,$$  (30)

and replacing in (29) we end up with one second-order ODE with dependent variable the $H(t)$, i.e.,

$$Z_{tt} + Z - \frac{(\gamma^2 - 1)}{(\gamma + 1)^2} V_0 + \frac{2(\gamma - 1)}{(\gamma + 1)^2} Z^{-\gamma} = 0$$  (31)

where we have replaced $H = Z^{-1}$ in order to simplify the form of the differential equation.

For arbitrary parameter $\gamma$, equation (31) admits only the autonomous symmetry vector $\partial_t$. In the special case where $V_0 = 0$ and $\gamma = 3$ equation (31) is invariant under the $sl(3, R)$ Lie algebra and reduce to the Ermakov-Pinney equation [46, 47]. We proceed with the application of the autonomous vector field.

The Lie invariants of the autonomous symmetry are $z = Z$ and $w = Z_t$, hence, equation (31) reduces to the first-order ODE

$$\frac{d}{dz} \left( \frac{w^2}{2} \right) = \frac{(\gamma^2 - 1)}{(\gamma + 1)^2} V_0 - 2\frac{(\gamma - 1)}{(\gamma + 1)^2} z^{-\gamma} - z.$$  (32)

with solution

$$w(z)^2 = 2\frac{(\gamma^2 - 1)}{(\gamma + 1)^2} V_0 z - 4\frac{z^{1-\gamma}}{(\gamma + 1)^2} z^2.$$  (33)

For $\gamma = 1$, the Lie invariants of the scaling symmetry $X_5$ are $h = H(t) x , v = V(t)$, where the reduced system gives, $H_t = 0$, i.e. $H = H_0$ and $V_{tt} + V = 0$.

3.5 Reduction with the vector fields $X_3$ & $X_4$

The existence of the two symmetry vectors $X_3, X_4$ or of the more general symmetry $\Gamma = \cos(t + t_0) \partial_v$, indicates that the system of our consideration [13]-[14] is invariant under the point transformation

$$v \rightarrow v + \cos(t + t_0).$$  (34)

Consequently, by taking any linear combination of the other symmetries with the vector field $\Gamma$ we determine the same reduced equations, where the invariants have been transformed according to the rule (34), except to the case of point reduction, i.e. $X_2$ and scaling solution for $\gamma = 1$, which are the two cases we present in the following lines.
3.5.1 Reduction with $X_2 + \beta \Gamma$

By considering the Lie invariants in (13)-(14) of the symmetry vector $X_2 + \beta \Gamma$ we find $h = H(t)$ and $v = \beta x \cos (t + t_0) + V(t)$ where

\begin{align*}
H^{-2}H_t + \beta \sin (t + t_0) &= 0, \quad \text{(35)} \\
V_{tt} + V &= 0. \quad \text{(36)}
\end{align*}

Therefore, the main difference with the point reduction solution is that the height $h$ is not a constant anymore, and it is given by

\begin{equation}
H(t) = -(H_0 - \beta \cos (t + t_0))^{-1}, \quad \text{(37)}
\end{equation}

while now

\begin{equation}
v = \beta x \cos (t + t_0) + V_1 \cos (t + t_1) \quad \text{(38)}
\end{equation}

It is important here to mention that in order to avoid singular behaviour in finite time, then the integration constants $H_0 \neq 0$ while $\beta < H_0$. The latter solution provide a linear spread of the velocities in the space.

3.5.2 Reduction with $X_5 + \beta \Gamma$ for $\gamma = 1$

The case of scaling solutions with $\gamma = 1$ when we consider the symmetry vector $\Gamma$ in the reduction is totally different from the case before. Indeed, the Lie invariants are

\begin{align*}
h = H(t) x, \quad v &= \frac{\beta}{2} \cos (t + t_0) \ln x + V(t), \quad \text{(39)}
\end{align*}

where $H(t)$ and $V(t)$ satisfy the reduced equations

\begin{align*}
2H^{-2}H_t + \beta \sin (t + t_0) &= 0, \quad \text{(40)} \\
V_{tt} + V - H &= 0. \quad \text{(41)}
\end{align*}

From (10) it follows

\begin{equation}
H(t) = \frac{2}{H_0 - \beta \cos (t + t_0)} \quad \text{(42)}
\end{equation}

where by replacing in (11) gives a maximally symmetric second-order ODE with generic solution

\begin{align*}
V(t) &= V_1 \cos (t + t_0 + t_1) + \frac{\beta}{2} 
\left( 2 \sin (t + t_0) \arctan \left( \frac{\cos (t + t_0) - 1}{\sin (t + t_0)} \right) - \cos (t + t_0) \ln (\beta \cos (t + t_0) - 2H_0) \right) + \\
&- 8 \left( \beta \sqrt{4H_0^2 - \beta^2} \right)^{-1} H_0 \sin (t + t_0) \arctan \left( \frac{2H_0 + \beta \cos (t + t_0) - 1}{\sqrt{4H_0^2 - \beta^2} \sin (t + t_0)} \right) \quad \text{(43)}
\end{align*}

while the conditions follows $\beta < H_0$ in order to avoid singularities at finite time.

In contrary with the previous reduction where the evolution of the speeds in the space is linear, in this case, the speed evolve with a logarithmic expansion which provide an initial singularity at $x = 0$.

3.6 Reduction with the vector fields $\alpha X_1 + X_5$

For the vector field $\alpha X_1 + X_5$, where $\alpha$ is nonzero constant the Lie invariants are derived to be

\begin{align*}
\sigma &= xe^{-\frac{\alpha}{\beta} t}, \quad h = H(\sigma) e^{\frac{\alpha}{\beta} t}, \quad v = V(\sigma) e^{\frac{\alpha}{\beta} t}, \quad \text{(44)}
\end{align*}
from where by replacing in (13)-(14) we find the reduced system

\[ \sigma H^2 \left( \gamma^2 - 1 \right) V_\sigma - \left( (\gamma - 1)^2 + \alpha^2 \right) H^2 V - (\gamma + 1)^2 \sigma^2 H_\sigma + \alpha^2 H^{\gamma+1} H_\sigma + 2\sigma H = 0, \]  
(45)

\[ (\gamma + 1) \sigma H^2 V_\sigma - (\gamma + 1) \sigma H_\sigma + 2H + 2H^2 V_\sigma = 0. \]  
(46)

An exact solution for the later system can be calculated by assuming a power-law behaviour for the functions \( H \) and \( V \). Indeed for any symmetry vector we considered the application of the zero-order Lie invariants and we rewrote the system (13)-(14) as follows

\[ \left\{ \begin{array}{l}
 Z_\lambda + \frac{\alpha^2 \lambda^3 + 24Z^2}{\lambda (\alpha^2 \lambda^2 - 4) Z^2} Z_\lambda + \frac{2 \left( \alpha^2 \lambda^2 + 24Z \right)}{\lambda (\alpha^2 \lambda^2 - 4)} = 0.
\end{array} \right. \]  
(51)

where now the special solution (47) becomes \( H (\sigma) = H_0 \sigma \), \( V (\sigma) = V_0 \). From the system (49), (50) we can identify the second-order ODE

\[ (\alpha^2 \lambda^2 - 4) Z_\lambda + \frac{2\alpha \lambda^2 \left( 1 + \lambda \sqrt{Z} \right) + 16Z}{\lambda} + Z_0 \lambda^2 = 0. \]  
(52)

4 Conclusions

In this work we studied a system of nonlinear PDEs which describe rotating shallow-water with the method of Lie symmetries. More specifically, we determine the Lie symmetries for the system (13)-(14) and we found that the system of PDEs is invariant under a five dimensional Lie algebra. The admitted Lie symmetries form the \( \{ 2A_1 \oplus_2 2A_1 \} \oplus_3 A_1 \) Lie algebra in the Morozov-Mubarakzyanov Classification Scheme for the parameter \( \gamma > 1 \) or the \( 2A_1 \oplus_3 3A_1 \) Lie algebra when \( \gamma = 1 \). That difference in the admitted Lie algebra between the two cases \( \gamma = 1 \) and \( \gamma \neq 1 \) are observable in the application of Lie symmetries and more specifically in the reduction process.

Indeed for any symmetry vector we considered the application of the zero-order Lie invariants and we rewrote the system of PDEs into a system ODEs which we were able to solve them explicitly in all cases by using the Lie symmetry analysis. Another important feature of the Lie symmetries is that we can transform solutions into solutions after the application of the invariant 1PPT.

From the Lie symmetry vectors \( X_1 - X_5 \) of the system (13)-(14) we determine the generic 1PPT to be

\[ \begin{array}{ll}
 \bar{i} & = c_1 (t + \varepsilon) \\
 \bar{x} & = c_2 (x + \varepsilon) + c_5 e^{(\gamma+1)} \varepsilon x \\
 \bar{h} & = c_3 h + c_5 e^{2\varepsilon} \bar{h} \\
 \bar{v} & = c_4 (v + \varepsilon \cos (t + t_0)) + c_5 e^{(\gamma-1)} \varepsilon v 
\end{array} \]
It is important to mention that someone could have started the present analysis by studying the original three-dimensional first-order differential equations (1)-(3). Either in that approach the results of the analysis will not change, except that the point transformation is defined in the space of variables $J = \{t, x, h, v, u\}$. In our analysis we consider to study the point transformations defined in the space of variables $I = \{t, x, h, v\}$. Now by extending the symmetry vectors obtained in the space $I$ in the (partial-) extension space $I' = \{t, x, h, v, \frac{\partial v}{\partial t}\}$, the symmetry vectors $X_1-5$ become

$$
X_1 = \partial_t + 0\partial_u, \quad X_2 = \partial_x + 0\partial_u,
X_3 = \cos(t)\partial_v - \sin(t)\partial_u, \quad X_4 = \sin(t)\partial_v + \cos(t)\partial_u,
X_5 = (\gamma + 1)x\partial_x + (\gamma - 1)v\partial_v + 2h\partial_h + (\gamma - 1)v_t\partial_v,
$$

where by replacing $u = -v_t$ are the Lie point symmetries of the original system (1)-(3).

We conclude that with the application of Lie symmetries we were able to prove the existence of solutions for the rotating shallow wave system (13)-(14). Another important observation is that the reduced differential equations were reduced into well-known first-order ODEs. Finally, we proved the existence of travel-wave and scaling solutions.

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A Lie symmetry analysis in the Euler coordinates

The dynamical system (1)-(3) in the Euler coordinates is written as

$$
\begin{align*}
h_t + (hu)_x &= 0, \\
(hu)_t + \left(hu^2 + \frac{1}{\gamma}h^\gamma\right)_x - hv &= 0, \\
(hv)_t + (hv)_x + hu &= 0.
\end{align*}
$$

The symmetry analysis provide the same results in the Euler coordinates. The applications of the symmetry condition (30) gives the symmetry vector fields

$$
\begin{align*}
Y_1 &= \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = \sin(t)(\partial_x - \partial_v) + \cos(t)\partial_u, \\
Y_4 &= \cos(t)(\partial_x - \partial_v) - \sin(t)\partial_u, \quad Y_5 = (\gamma - 1)(x\partial_x + u\partial_u + v\partial_v) + 2h\partial_h.
\end{align*}
$$

The latter symmetry vectors form the same Lie algebras with that of system (1)-(3). Because they are in a different representation with the vector fields $X_1 - X_5$, the invariant functions are different. Vector fields $Y^1$ and $Y^2$ provide static and stationary solutions while the linear combination $cY_1 + Y_2$ gives the similarity solution which describe traveling waves. For $\gamma \neq 1$, the vector field $Y_5$ provide the scaling invariants

$$
h = H(t)x^\frac{\gamma - 1}{\gamma}, \quad u = U(t)x, \quad v = V(t)x
$$

where functions $H(t), \, U(t)$ and $V(t)$ satisfy the following system of first-order ODEs

$$
\begin{align*}
(\gamma - 1)H_t + (\gamma + 1)HU &= 0, \\
(\gamma - 1)(HU)_t - HV + 2\gamma(HU^2 + H^\gamma) &= 0, \\
(\gamma - 1)(HV)_t + (\gamma - 1)UV + 2\gamma HUV &= 0.
\end{align*}
$$
Figure 2: Qualitative evolution of the parameters $H(t)$, $U(t)$ and $V(t)$ given by the system (57)-(59) for $\gamma = 2$ and $\gamma = 3/2$. For initial conditions we assumed $H(0) = 0.02$, $U(0) = 0.01$ and $V(0) = 0.01$. The numerical solutions provide oscillations for the dynamical parameters.

The latter system can be easily integrated, however its general solution it is not given by a closed-form expression. Numerical simulations of the latter system are presented in Fig. 2 for two values of the parameter $\gamma$. More specifically for $\gamma = 2$ and $\gamma = \frac{3}{2}$. From the figures we observe periodic behaviour for the dynamical parameters of the dynamical system.

Finally, the symmetry vector $aY_2 + Y_3$ provide the generic solution

$$h(t, x) = h_0 (a + \sin(t))^{-1}, \ u(t, x) = \frac{\cos(t)}{(a + \sin(t))} x + U(t), \ v(t, x) = -\frac{\sin(t)}{(a + \sin(t))} x + V(t),$$

(60)

where

$$U(t) = \frac{U_1 a \cos(t) + U_2 (a \sin(t) + 1)}{a (a + \sin(t))}, \ V(t) = \frac{U_2 \cos(t) - U_1 \sin(t)}{(a + \sin(t))}.$$  

(61)

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