Abstract

In this paper, we first show that the irreducible characters of a quotient table algebra modulo a normal closed subset can be viewed as the irreducible characters of the table algebra itself. Furthermore, we define the character products for table algebras and give a condition in which the products of two characters are characters. Thereafter, as a main result we state and prove the Burnside-Brauer Theorem on finite groups for table algebras.

Key words: table algebra, character product.
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1 Introduction

One of important results in the character theory of finite groups is the Burnside-Brauer Theorem. This theorem states that if a finite group $G$ has a faithful character $\chi$ which takes $k$ values on $G$, then every irreducible character of $G$ is a constituent of one of the characters $\chi^i$ for $0 \leq i < k$. One of important results in this paper is to state and prove an analog of the Burnside-Brauer Theorem for table algebras. Therefore, we deal with products of characters in table algebras. We mention that products of characters in table algebras need not be characters in general. In order to provide a condition in which the products of characters of a given table algebra are characters,
we need to observe the relationship between the characters of a table algebra and the characters of its quotient.

In section 2 some elementary facts about table algebras are given. Section 3 deals with the characters of the quotient table algebras. More precisely, for a given table algebra \((A, B)\) and a normal closed subset \(C\) of \(B\), we show that the set of irreducible complex characters of \((A/C, B/C)\) can be embedded in the set of irreducible complex characters of \((A, B)\). This is a generalization of [6, Section 3] for association schemes to table algebras.

Another interesting problem on characters of table algebras is character products. Since table algebras are not Hopf algebras in general, character products need not be characters. In [7], Hanaki defined character products for association schemes and gave a condition which implies that character products are characters. In section 4 we define the character products for table algebras and by using the results in Section 3 we obtain a condition for which character products are characters. Finally, we prove the Burnside-Brauer Theorem for table algebras which is a well-known theorem in the theory of finite groups.

## 2 Preliminaries

Throughout this paper we follow from [11] for the definition of non-commutative table algebras and related notions. Hence we deal with non-commutative table algebras as the following:

A non-commutative table algebra \((A, B)\) is a finite dimensional algebra \(A\) over the complex field \(\mathbb{C}\) and a distinguished basis \(B = \{b_1 = 1_A, \ldots, b_d\}\) for \(A\), where \(1_A\) is a unit, such that the following properties hold:

(I) The structure constants of \(B\) are nonnegative real numbers, i.e., for \(a, b \in B\):

\[ ab = \sum_{c \in B} \lambda_{abc} c, \quad \lambda_{abc} \in \mathbb{R}^+ \cup \{0\}. \]

(II) There is a semilinear involutory anti-automorphism (denoted by \(^*\)) of \(A\) such that \(B^* = B\).

(III) For \(a, b \in B\) the equality \(\lambda_{ab1_A} = \delta_{ab^*} |a|\) holds where \(|a| > 0\) and \(\delta\) is the Kronecker symbol.

(IV) The mapping \(b \rightarrow |b|, b \in B\) is a one-dimensional linear representation of the algebra \(A\) such that \(|b| = |b^*|\) for all \(b \in B\) which is called the degree map.

Throughout this paper a table algebra means a non-commutative table algebra.

Let \((A, B)\) be a table algebra. Then [2, Theorem 3.11] implies that \(A\) is semisimple. The value \(|b|\) is called the degree of the basis element \(b\). From condition (IV) we see that \(|b| = |b^*|\) for all \(b \in B\). Therefore, for an arbitrary element \(\sum_{b \in B} x_b b \in A\), we have \(|\sum_{b \in B} x_b b| = \sum_{b \in B} x_b |b|\).
For each \( a = \sum_{b \in B} x_b b \), we set \( a^* = \sum_{b \in B} \overline{x_b} b^* \), where \( \overline{x_b} \) means the complex conjugate of \( x_b \). For any \( x = \sum_{b \in B} x_b b \in A \), denote by \( \text{Supp}(x) \) as the set of all basis elements \( b \in B \) such that \( x_b \neq 0 \). If \( E, D \subseteq B \), then we set \( ED = \bigcup_{e \in E, d \in D} \text{Supp}(ed) \).

A nonempty subset \( C \subseteq B \) is called a closed subset, if \( C^* C \subseteq C \). We denote by \( \mathcal{C}(B) \) the set of all closed subsets of \( B \). In addition, \( C \in \mathcal{C}(B) \) is said to be normal in \( B \) if \( bC = Cb \) for every \( b \in B \), and denote it by \( C \trianglelefteq B \).

Let \((A,B)\) be a table algebra with basis \( B \) and let \( C \in \mathcal{C}(B) \). From \([2, \text{Proposition 4.7}]\), it follows that \( \{Cb \mid b \in B\} \) is a partition of \( B \). A subset \( Cb \) is called a \( C \)-double coset or double coset with respect to the closed subset \( C \). Let

\[
\frac{b}{C} := |C^+|^{-1} (Cb)^+ = |C^+|^{-1} \sum_{x \in CbC} x
\]

where \( C^+ = \sum_{c \in C} c \) and \( |C^+| = \sum_{c \in C} |c| \). Then the following theorem is an immediate consequence of \([2, \text{Theorem 4.9}]\):

**Theorem 2.1.** Let \((A,B)\) be a table algebra and let \( C \in \mathcal{C}(B) \). Suppose that \( \{b_1 = 1_A, \ldots, b_k\} \) be a complete set of representatives of \( C \)-double cosets. Then the vector space spanned by the elements \( b_i/C, 1 \leq i \leq k \), is a table algebra (which is denoted by \( A/C \)) with a distinguished basis \( B/C = \{b_i/C \mid 1 \leq i \leq k\} \). The structure constants of this algebra are given by the following formula:

\[
\gamma_{ijk} = |C^+|^{-1} \sum_{r \in Cb_iC, s \in Cb_jC} \lambda_{rst}
\]

where \( t \in Cb_kC \) is an arbitrary element.

The table algebra \((A/C,B/C)\) is called the quotient table algebra of \((A,B)\) modulo \( C \).

We refer the reader to \([10]\) for the background of association schemes.

## 3 Embedding of \( \text{Irr}(A/C) \) into \( \text{Irr}(A) \)

In this section we show that there is an embedding from \( \text{Irr}(A/C) \) into \( \text{Irr}(A) \) where \( C \trianglelefteq B \). We mention that such an embedding is given for association schemes in \([6]\).

Let \((A,B)\) be a table algebra and \( C \in \mathcal{C}(B) \). Set \( e = |C^+|^{-1} C^+ \). Then \( e \) is an idempotent for the table algebra \( A \) and the subalgebra \( eAe \) is equal to the quotient table algebra \((A/C,B/C)\) modulo \( C \), see \([2]\).

**Lemma 3.1.** Let \( C \in \mathcal{C}(B) \). Then \( C \trianglelefteq B \) if and only if \( e = |C^+|^{-1} C^+ \) is a central idempotent of \( A \).

**Proof.** Let \( C \) be a closed subset of \( B \). Clearly \( e^2 = e \). We first assume that \( C \trianglelefteq B \). In order to prove that \( e \) is central, it is enough to show that for every \( b \in B \), \( bC^+ = C^+b \). From the normality of \( C \) it follows that \((bC)^+ = (Cb)^+ \). From \([2, \text{Proposition 4.8(ii)}]\), we have \((Cb)^+ = \alpha^{-1} C^+ b \) and \((bC)^+ = \beta^{-1} bC^+ \), for some suitable...
\( \alpha, \beta \in \mathbb{R}^* \). Therefore \( \alpha^{-1}C^+b = \beta^{-1}bC^+ \). But \( |C^+b| = |C^+||b| = |b||C^+| = |bC^+| \) and so \( \alpha = \beta \) which forces \( bC^+ = C^+b \), as desired.

Conversely let \( e \) be a central idempotent. Then

\[
Cb = \text{Supp}(C^+b) = \text{Supp}(bC^+) = bC
\]

which means that \( C \leq B \), and we are done. \( \square \)

**Lemma 3.2.** Let \( C \leq B \). Then the map \( \pi : A \to A/C \) defined by \( \pi(b) = \alpha_b(b/C) \) where \( \alpha_b \in \mathbb{R} \) such that \( C^+b = \alpha_b(b/C)^+ \), is an algebra homomorphism.

**Proof.** Set \( e = |C^+|^{-1}C^+ \). Then from Lemma 3.1 \( e \) is a central idempotent. We define the map \( \pi : A \to eAe \) where \( \pi(b) = ebe \). It is easily seen that \( \pi \) is an algebra homomorphism. From [2, Proposition 4.8(ii)] there is \( \alpha_b \in \mathbb{R}^* \) such that \( C^+b = \alpha_b(b/C)^+ \). From this fact along with the obvious equality \( ebe = |C^+|^{-1}C^+b \) we deduce that \( ebe = \alpha_b(C^+)^{-1}(Cb)^+ = \alpha_b(b/C) \). Thus \( \pi : A \to A/C \) where \( \pi(b) = \alpha_b(b/C) \) is an algebra homomorphism and we are done. \( \square \)

**Theorem 3.3.** Let \( C \leq B \) and let \( \psi : A/C \to \text{Mat}_s(\mathbb{C}) \) be a representation of \( A/C \). Then \( \overline{\psi} : A \to \text{Mat}_s(\mathbb{C}) \) defined by \( \overline{\psi}(b) = \alpha_b\psi(b/C) \), where \( C^+b = \alpha_b(b/C)^+ \), is a representation of \( A \).

**Proof.** Put \( \overline{\psi} = \psi \circ \pi \), where \( \pi \) is defined in Lemma 3.2. Then \( \overline{\psi} \) is an algebra homomorphism and \( \overline{\psi}(b) = \psi \circ \pi(b) = \psi(\alpha_b(b/C)) = \alpha_b\psi(b/C) \), as desired. \( \square \)

**Remark 3.4.** (1) By Theorem 3.3 we may embed \( \text{Irr}(A/C) \) into \( \text{Irr}(A) \), indeed if \( \chi \) and \( \psi \) are distinct characters of \( A/C \), then \( \chi(b/C) \neq \psi(b/C) \), for some \( b/C \in B/C \) and so \( \overline{\chi}(b) \neq \overline{\psi}(b) \), where \( \overline{\chi} \) is a representation of \( A \) defined by \( \overline{\chi}(b) = \alpha_b\chi(b/C) \).

(2) In Theorem 3.3, the correspondence preserves the irreducibility of representations. Furthermore, if \( D \) is an irreducible representation of \( A \) such that \( \text{rank } D(e) \neq 0 \), then \( D = \overline{E} \) where \( E \) is appropriately explained.

In Theorem 3.3, the value of \( \alpha_b \) is not be given precisely. But if we consider another property on \( C \) which is stronger than normality, namely strongly normal closed subset, then it is possible to give the value of \( \alpha_b \) in precise form. The rest of this section deals with the quotient of table algebras modulo \( C \), where \( C \) is a strongly normal closed subset.

**Definition 3.5.** A closed subset \( C \) of \( B \) is said to be strongly normal and denoted by \( \leq' B \), if for each \( b \in B \)

\[
b^*Cb \subset C.
\]

In the following we show that a strongly normal closed subset is a normal closed subset.

**Lemma 3.6.** Every strongly normal closed subset is a normal closed subset.
Proof. Let $C \leq' B$. Then for $b \in B$ we have $b^*C b \subseteq C$, and so $bC^*Cb \subseteq bC$ which implies that $Cb \subseteq bC$. On the other hand, from $bC^* \subseteq C$ it follows that $bC^*Cb \subseteq Cb$, and so $bC \subseteq Cb$. Thus $Cb = bC$, as desired. 

The following example shows that the converse of the above lemma is not true, i.e., a normal closed subset is not necessarily strongly normal closed subset.

Example 3.7. Let $q \geq 2$ and $B = \{r_0, r_1, \ldots, r_{q+1}\}$ be a basis for a complex vector space $A$ of dimension $q + 2$. We define multiplication

$$r_ir_j = \begin{cases} (q-1)r_0 + (q-2)r_i, & \text{if } i = j \neq 0 \\ \sum_{k \neq 0,i,j} r_k, & \text{if } i \neq j \end{cases}$$

for all $i, j$ with $1 \leq i, j \leq q + 1$, and $r_ir_0 = r_i$ for all $i$. This extends to a multiplication in $A$ which is commutative and association with unit element $r_0 = 1_A$. A direct computation shows that $(A, B)$ is a table algebra where $r_i^* = r_i$ for all $i$; and $|r_i| = q - 1$ for $i > 0$. Clearly the set $\{r_0, r_i\}$ for every $i \neq 0$ is a normal closed subset but it is not a strongly normal closed subset. In fact, the construction of this table algebra is given in [7].

Theorem 3.8. Let $b \in B$ and $C \in C(B)$. Then $|b/C| = 1$ if and only if $b^*Cb \subseteq C$.

Proof. Let $T = \{b_1 = 1_A, b_2, \ldots, b_t\}$ be a complete set of representatives of $C$-double cosets and let $b = b_i$ for some $i, 1 \leq i \leq t$.

Suppose that $|b/C| = 1$ and let $d \in b^*Cb$. Since $1_A \in C$, we have $d \in (Cb^*C)(CbC)$. Then there exists $r \in Cb^*C$ and $s \in CbC$ such that $\lambda_{rsd} \neq 0$. As $B = \bigcup_{j=1}^t Cb_jC$, we may assume that $d \in Cb_kC$ for some $k, 1 \leq k \leq t$. So $\gamma_{i*ik} \neq 0$ where

$$\gamma_{i*ik} = |C^+|^{-1} \sum_{r \in Cb^*_kC, s \in Cb_kC} \lambda_{rsd}. $$

But from the assumption we conclude that $(b/C)(b^*/C) = \{1_A/C\}$. Hence $\gamma_{i*11} \neq 0$ and so that $k = 1$. Thus $d \in C$ and so $b^*Cb \subseteq C$.

Conversely, let $b^*Cb \subseteq C$. Then $(Cb^*C)(CbC) \subseteq C$ and so

$$\gamma_{i*11} \leq |C^+|^{-1} \sum_{r,s \in C} \lambda_{rsd} \quad (1)$$

for $d \in C$. Now from (1) and the equalities $\sum_{r,s \in C} \lambda_{rsd} = \sum_{r \in C} |r| = |C^+|$ (see [2] Proposition 2.3(i))) we deduce that $\gamma_{i*11} = 1$. Thus $|b/C| = 1$, as desired.

Corollary 3.9. Let $(A, B)$ be a table algebra and $C \in C(B)$. Then $(A/C, B/C)$ is a group algebra if and only if $C \leq' B$.

Proof. This follows immediately from Theorem 3.8.

Theorem 3.10. Let $C \leq' B$ and $\psi$ be a representation of $A/C$. Then the mapping $\overline{\psi} : b \mapsto |b|\psi(b/C)$ for every $b \in B$, is a representation for $A$. 

Proof. Using the notation in the proof of Theorem 3.3 it suffices to show that \( \alpha_b = |b| \). Since \( C \subseteq B \), it follows that \( bb^*C \subseteq C \). Then \( \text{Supp}(bb^*) \subseteq C \) and from [2] Proposition 4.8(iv) the equality \( |(Cb)^+| = |C^+| \) follows. On the other hand, from [2] Proposition 4.8(ii) there exists \( \alpha_b \in \mathbb{R}^* \) such that \( |C^+||b| = \alpha_b(|Cb|) \). Thus \( |C^+||b| = \alpha_b|C^+| \) and so \( \alpha_b = |b| \), as desired.

4 Character products

For an associative algebra \( A \), the tensor product \( V \otimes W \) of two \( A \)-modules \( V \) and \( W \) is a vector space, but not necessarily an \( A \)-module. In order to make an \( A \)-module on \( V \otimes W \), there must be a linear binary operation \( \Delta : A \rightarrow A \otimes A \) which is also an algebra homomorphism. This is an important property for the algebra \( A \) becomes a Hopf algebra. For instance, in group theory the tensor products of two \( G \)-modules \( V \) and \( W \) gives us a module, indeed the group algebra \( \mathbb{C}G \) is a Hopf algebra with \( \Delta : g \rightarrow g \otimes g \). So if \( \chi \) and \( \psi \) are afforded by two \( G \)-modules, then their tensor product affords the character \( \chi\psi(g) := \chi(g)\psi(g) \) which is called the character product of \( \chi \) and \( \psi \).

In general, a table algebra \((A, B)\) is not a Hopf algebra and so it is not generally possible to define the structure of an \( A \)-module on \( V \otimes W \). In [4] Doi introduced a generalization of Hopf algebras and defined a binary linear operation \( \Delta : b \rightarrow \frac{1}{|b|}b \otimes b \), \( b \in B \). By considering this binary linear operation, we define the character products of \( \chi \) and \( \psi \) by:

\[
\chi\psi(b) := \frac{1}{|b|}\chi(b)\psi(b), \quad b \in B.
\]

(2)

Since \( \Delta \) is not necessarily an algebra homomorphism, a character products in a table algebra is not generally a character. It might be mentioned that this is an analog of association schemes which has already done by Hanaki in [7].

Through this section we assume that \((A, B)\) is a table algebra with a strongly normal closed subset \( C \) and \( e = |C^+|^{-1}C^+ \).

Let \( V \) and \( W \) be \( A/C \)-module and \( A \)-module, respectively. We define a multiplication of \( A \) on \( V \otimes W \) as the following:

\[
b(v \otimes w) := (b/C)v \otimes bw, \quad v \in V, \ w \in W, \ b \in B.
\]

(3)

Lemma 4.1. Let \( V \) be an irreducible \( A \)-module with \( \dim_{\mathbb{C}}(eV) \neq 0 \) and let \( W \) be an \( A \)-module. Then \( V \otimes W \) is an \( A \)-module given by the multiplication in (3).

Proof. We first claim that \( \mu : A \rightarrow A/C \otimes A \) by \( \mu(b) = b/C \otimes b \), for any \( b \in B \) is an algebra homomorphism. Let \( b, c \in B \) be given. Then

\[
\mu(bc) = \mu(\sum_{d \in B} \lambda_{bcd}d) = \sum_{d \in B} \lambda_{bcd}\mu(d) = \sum_{d \in B} \lambda_{bcd}d/C \otimes d.
\]

(4)
On the other hand,
$$
\mu(b)\mu(c) = (b/C \otimes b)(c/C \otimes c) = \sum_{d \in B} \lambda_{bcd}(b/C)(c/C) \otimes d.
$$

Now in the above equality, if \( \lambda_{bcd} \neq 0 \) then \( \gamma_{b/C,c/C,d/C} \neq 0 \) and so \( (b/C)(c/C) = d/C \), indeed by Corollary 3.9 \( B/C \) is a group. Thus \( \mu(b)\mu(c) = \sum_{d \in B} \lambda_{bcd}d/C \otimes d \) which is equal to \( \mu(bc) \) by (4), and the claim is proved. Since \( \dim_{\mathbb{C}}(eV) \neq 0 \), then from [3, Corollary 3.6], \( V \) is an \( A/C \)-module. Let \( \text{dim}_{\mathbb{C}}(eV) \neq 0 \). W e first note that under our assumptions \( V \otimes W \) is an \( A/C \)-module. Now let \( D_1 : A/C \to \text{Mat}_{d_1}(\mathbb{C}) \) and \( D_2 : A \to \text{Mat}_{d_2}(\mathbb{C}) \) be representations of \( A/C \) and \( A \) corresponding to \( V \) and \( W \), respectively. Then \((D_1 \otimes D_2) \circ \mu : A \to \text{Mat}_{d_1d_2}(\mathbb{C})\) is a representation of \( A \) corresponding to \( V \otimes W \), where \( \mu : A \to A/C \otimes A \) is the algebra homomorphism given in the proof of Lemma 4.1. Now we conclude that the trace of \((D_1 \otimes D_2) \circ \mu)(b)\) is equal to \( \chi(b/C)\psi(b) \), for \( b \in B \). But from Theorem 3.10 and Remark 3.4 (2) it follows that \( \chi(b/C) = \frac{1}{|B|}\chi(b) \), and we are done.

**Lemma 4.2.** Let \( \chi \) be an irreducible character afforded by \( A \)-module \( V \) such that \( \text{dim}_{\mathbb{C}}(eV) \neq 0 \). Then for every \( A \)-module \( W \) the tensor product \( V \otimes W \) is an \( A \)-module which affords the character \( \frac{1}{|B|}\chi(b)\psi(b) \).

**Proof.** We first note that under our assumptions \( V \otimes W \) is an \( A \)-module, by Lemma 4.1. Since \( \text{dim}_{\mathbb{C}}(eV) \neq 0 \), from Corollary 3.6 it follows that the module \( V \) can be considered as an \( A/C \)-module. Now let \( D_1 : A/C \to \text{Mat}_{d_1}(\mathbb{C}) \) and \( D_2 : A \to \text{Mat}_{d_2}(\mathbb{C}) \) be representations of \( A/C \) and \( A \) corresponding to \( V \) and \( W \), respectively. Then \((D_1 \otimes D_2) \circ \mu : A \to \text{Mat}_{d_1d_2}(\mathbb{C})\) is a representation of \( A \) corresponding to \( V \otimes W \), where \( \mu : A \to A/C \otimes A \) is the algebra homomorphism given in the proof of Lemma 4.1. Now we conclude that the trace of \((D_1 \otimes D_2) \circ \mu)(b)\) is equal to \( \chi(b/C)\psi(b) \), for \( b \in B \). But from Theorem 3.10 and Remark 3.4 (2) it follows that \( \chi(b/C) = \frac{1}{|B|}\chi(b) \), and we are done.

**Theorem 4.3.** Let \( \chi \) be an irreducible character of \( A \) such that \( \chi(e) \neq 0 \). Then \( \chi \psi \) is a character of \( A \), where \( \psi \) is a character of \( A \).

**Proof.** Since \( \chi(e) \neq 0 \), it follows from [3, Corollary 3.5] that \( \chi \) is an irreducible character of \( A/C \). Thus from Lemma 4.2 we conclude that the character product \( \chi \psi \) is a character, as desired.

**Theorem 4.4.** Let \( \chi, \psi \in \text{Irr}(A) \). If \( \chi(e) = 1 \), then \( \chi \psi \in \text{Irr}(A) \).

**Proof.** Let \( V \) and \( W \) be two irreducible \( A \)-modules which afford \( \chi \) and \( \psi \) respectively. The equality \( \chi(e) = 1 \) implies that \( \text{dim}_{\mathbb{C}}(V) = \chi(e) = 1 \). From [3, Corollary 3.5] and Theorem 4.3 it follows that \( \chi \) is a linear character of \( A/C \) and \( \chi \psi \) is a character, respectively. Moreover, \( \text{dim}_{\mathbb{C}}(V) = 1 \) implies that every \( A \)-submodule of \( V \otimes W \) is of the form \( V \otimes W' \) where \( W' \) is an \( A \)-submodule of \( W \). Therefore, by irreducibility of \( W \) the \( A \)-module \( V \otimes W \) is irreducible and so \( \chi \psi \in \text{Irr}(A) \). This completes the proof.

Let \( A \) be a finite dimensional algebra with a basis \( w_1, \ldots, w_r \) over a field \( F \). Let \( \zeta \) be a non-degenerate feasible trace on \( A \). Then from [8], \( \zeta \) induces a dual form \([\cdot, \cdot]\) on \( \text{Hom}_F(A, F) \) in which for every \( \chi, \varphi \in \text{Hom}_F(A, F) \) we have

$$
[\chi, \varphi] = \sum_{i=1}^r \chi(w_i)\varphi(\hat{w}_i)
$$

where \( \hat{w}_1, \ldots, \hat{w}_r \) is the dual basis defined by \( \zeta(w_i\hat{w}_j) = \delta_{i,j} \).
Now let \((A, B)\) be a table algebra. The linear function \(\zeta\) on \(A\) is defined in [3] by setting \(\zeta(b) = \delta_{b, 1_A}|B^+|\), for \(b \in B\). Then \(\zeta\) is a non-degenerate feasible trace on \(A\) and it follows that the dual form \([\cdot, \cdot]\) on \(\text{Hom}_C(A, C)\) is as follows:

\[
[\chi, \varphi] = \frac{1}{|B^+|} \sum_{b \in B} \frac{1}{|b|} \chi(b) \varphi(b^*)
\]  

for every \(\chi, \varphi \in \text{Hom}_C(A, C)\).

Let \(\chi\) be a character of table algebra \((A, B)\). The following subset of \(B\)

\[
K(\chi) = \{b \in B : \chi(b) = |b|\chi(1)\}
\]

is a close subset of \(B\). The proof of this result can be found in [5, Theorem 3.2] for the case of association scheme, but the proof also works for table algebras.

Below we give our main result which proves the Burnside-Brauer Theorem on finite groups for table algebras.

**Theorem 4.5.** Let \((A, B)\) be a table algebra. Suppose that \(A\) has a character \(\chi\) with \(K(\chi) = \{1_A\}\) such that \(\chi(b) = \frac{1}{|b|}\chi(1)\) takes on exactly \(k\) different values for \(b \in B\). If all powers of \(\chi\) is a character, then each irreducible character of \(A\) appears as an irreducible component of one of \(\chi^i\), where \(0 \leq i \leq k - 1\) and \(\chi^0 = \rho\).

**Proof.** Let \(\alpha_1, \ldots, \alpha_k\) be the distinct values taken by \(\chi(b) = \frac{1}{|b|}\chi(1)\), \(b \in B\). Define \(B_t = \{b \in B|\chi(b) = \frac{1}{|b|}\alpha_t\}\). Assume that \(\alpha_1 = \chi(1)\) so that \(B_1 = K(\chi)\). Fix \(\psi \in \text{Irr}(A)\) and let \(\beta_i = \sum_{b \in B_i} \psi(b^*)\) for \(1 \leq i \leq k\). Since \(\chi^2(b) = \frac{1}{|b|^2}\chi(b)^2\), it follows that \(\chi^j(b) = \frac{1}{|b|^j}\chi(b)^j\).

Hence

\[
[\chi^j, \psi] = \frac{1}{|B^+|} \sum_{b \in B} \frac{1}{|b|} \chi^j(b) \psi(b^*)
\]

\[
= \frac{1}{|B^+|} \sum_{i=1}^{k} \sum_{b \in B_i} \frac{1}{|b|^j} \chi^j(b) \psi(b^*)
\]

\[
= \frac{1}{|B^+|} \sum_{i=1}^{k} (\alpha_i)^j \beta_i.
\]

Therefore, if \(\psi\) is not a constituent of \(\chi^j\) for all \(0 \leq j \leq k - 1\), then

\[
\sum_{i=1}^{k} (\alpha_i)^j \beta_i = 0, \quad j = 0, 1, \ldots, k - 1.
\]  

Let \(M := (a_{i,j})\) be a \(k \times k\) matrix whose \(i\)th row and \(j\)th column is \((\alpha_i)^j\) and let \(X = (\beta_1, \beta_2, \ldots, \beta_k)\). Therefore [3] shows that \(XM = 0\). But the determinant of \(M\) is Vandermonde determinant and is equal to \(\pm \Pi_{i<j}(\alpha_i - \alpha_j) \neq 0\). It follows that \(X = 0\). But \(\beta_1 = \psi(1_A) \neq 0\), which is a contradiction. This completes the proof of the theorem.
Remark 4.6. By using Theorem 4.5 for \((A, B) = (\mathbb{C}G, G)\), where \(G\) is a finite group, we get the Burnside-Brauer Theorem on finite groups.

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References

[1] Z. Arad, E. Fisman, M. Muzychuk, *On a Product of Two Elements in Non-commutative C-Algebras*, Algebra Colloquium, 5:1, 85-97, 1998.

[2] Z. Arad, E. Fisman, M. Muzychuk, *Generalized Table Algebras*, Israel J. Math. 114, 29-60, 1999.

[3] J. Bagherian, A. Rahnamai Barghi, *Standard Character Condition for C-algebras*, arXiv: 0810.5305v1, Submitted to Algebra Colloquium.

[4] Y. Doi, *Bi-Frobenius Algebras and Group-Like Algebra*, Hopf algebras in: Lecture Note in Pure and Appl.Math, Vol. 237, Dekker, New York, pp.143-155, 2004.

[5] A. Hanaki, *Characters of association schemes and normal closed subsets*, Graphs Combin. 19, 363-369, 2003.

[6] A. Hanaki, *Representations of Association Schemes and Their Factor Schemes*, Graphs Combin. 19, 195-201, 2003.

[7] A. Hanaki, *Character Products of Association Schemes*, J. Algebra 283, 596-603, 2005.

[8] D. G. Higman, *Coherent Algebras*, J. Linear Algebra and it’s Applications, 93, 209-239, 1987.

[9] I. Ponomarenko, A. Rahnamai Barghi, *On Amorphic C-algebras*, Journal of Mathematical Sciences, Vol. 145, No. 3, 2007.

[10] P-H. Zieschang, *An Algebraic Approach to Association Schemes*, Lecture Notes in Math., vol. 1628, Springer-Verlag, Berlin, 1996.