BOUNDARY REGULARITY AND EMBEDDED SOLUTIONS
FOR THE ORIENTED PLATEAU PROBLEM

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Any fixed $C^2$ Jordan curve $\Gamma$ in $\mathbb{R}^3$ is known to span an orientable minimal surface in several different senses. In the work of Douglas, Rado and Courant (see e.g. [3, IV, §4]) the minimal surface occurs as an area-minimizing mapping from a fixed orientable surface of finite genus and may possibly have self-intersections. In the work of Federer and Fleming (see e.g. [4, §5]) the minimal surface, which occurs as the support of an area-minimizing rectifiable current, is necessarily embedded (away from $\Gamma$) but was not previously known even to have finite genus. Our work in [7], which establishes complete boundary regularity for the latter surface, thus implies that there exists an orientable embedded minimal surface with boundary $\Gamma$. In fact:

\textbf{Theorem 1.} For any compact orientable $n - 1$ dimensional $C^2$ embedded submanifold $N$ of $\mathbb{R}^{n+1}$, there exists an orientable bounded stable minimal embedded $C^{1,\alpha}$ (for all $0 < \alpha < 1$) hypersurface $M$ with boundary $N$ so that the closure of $M$ in $\mathbb{R}^{n+1}$ equals $M \cup S$ for some compact set $S \subset \mathbb{R}^{n+1} \sim N$ of Hausdorff dimension $< n-7$.

Using the existence theory for area minimizing rectifiable currents [4, 5.1] and their interior regularity theory [5, Theorem 1], Theorem 1 follows from our boundary regularity result [7, 11.1]:

\textbf{Theorem 2.} If $U$ is an open subset of $\mathbb{R}^{n+1}$, $T$ is an $n$ dimensional absolutely area minimizing locally rectifiable current in $U$, and $\partial T$ is an oriented embedded $C^2$ submanifold of $U$, then, for some open neighborhood $V$ of $\text{spt} \partial T$ in $U$, $V \cap \text{spt} T$ is an embedded $C^{1,\alpha}$ hypersurface with boundary for all $0 < \alpha < 1$.

W. K. Allard [1, §5] has proven such regularity near points on the boundary of the convex hull of $\text{spt} T$. Boundary regularity in $n = 2$ for the unoriented problem [4, 5.3.21] (and so the existence of possibly nonorientable embedded minimal surfaces with boundary) also follows from his work. For $k \geq 2$, $C^{k,\alpha}$

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smoothness (analyticity) in Theorems 1, 2 for $C^{k,\alpha}$ (analytic) boundaries follows from [8, 1.10]. In proving Theorem 2, we obtain:

**Theorem 3.** Any compact orientable $n - 1$ dimensional embedded minimal submanifold of $S^n = \mathbb{R}^{n+1} \cap \{x : |x| = 1\}$ with boundary $S^n \cap \{(x_1, \ldots, x_{n+1}) : x_n = 0 = x_{n+1}\}$ must be a great hemisphere.

[2, Theorem A] shows that $\text{spt} \, T$ above may have an $n - 7$ dimensional interior singular set and that the analogue of Theorem 3 for submanifolds without boundary is false. For $n = 2$, Theorem 2 implies:

**Theorem 4.** For any $C^2$ Jordan curve $\Gamma$ in $\mathbb{R}^3$, there exists a nonnegative integer $G_\Gamma$ so that:

1. The Douglas-Courant type, genus $g$ least-area problem [3, IV, 4.1, 4.4] for $\Gamma$ has no solution whenever $g > G_\Gamma$.
2. There exists a Douglas-Courant type genus $G_\Gamma$ least-area solution for $\Gamma$ and any such solution is embedded.
3. The number of such solutions is finite if $\Gamma$ is $C^4$.

There are also a priori bounds on $G_\Gamma$, the number of solutions, and the absolute value of the Gaussian curvature of any solution.

**Sketch of Proof of Theorem 2.** To obtain regularity near a point $a \in \text{spt} \, \partial T$, we assume $a = 0$ and first prove that the support of some oriented tangent cone at 0 is contained in a hyperplane. For $n = 2$, this follows from the monotonicity formula [1, 3.4], interior regularity [5], and the planar nature of geodesics on $S^2$. For $n > 2$, an inductive argument using linear barriers is required. Letting $H_{\pm} = \mathbb{R}^n \cap \{(y_1, \ldots, y_n) : \pm y_n > 0\}$ and rotating, we assume that for some positive integer $m$, the oriented tangent cone is the sum of $m$ times $H_+ \times \{0\}$ and $m$ times $H_- \times \{0\}$, both taken with the usual orientation $e_1 \wedge \cdots \wedge e_m$. Since the case $m = 1$ has been treated by Allard [1, §5], we henceforth assume $m \geq 2$.

Using [4, 5.4.2], we now see that the normalized height

$$h(r) = \sup \{ |x_n + 1|/r : (x_1, \ldots, x_n + 1) \in \text{spt} \, T, |(x_1, \ldots, x_n)| \leq r \}$$

has lower limit 0 as $r \downarrow 0$. After establishing that $h(r)$ is comparable (except for a boundary curvature term and a slight change in $r$) with the cylindrical excess $\text{Exc}(T, 0, r)$ of [4, 5.3], we may apply the interior regularity theorem [4, 5.3.14] in vertical circular cylinders which do not meet $\text{spt} \, \partial T$. From this, one finds $C^1$ domains $\Omega_+ \subset H_+$ which are mutually tangent at the origin so that over $\Omega_+ \cup \Omega_-$, $\text{spt} \, T$ separates into graphs of real analytic minimal-surface-equation solutions:

1. $u_1^+ \leq u_2^+ \leq \cdots \leq u_m^+$ on $\Omega_+$,
2. $u_1^- \leq u_2^- \leq \cdots \leq u_m^-$. on $\Omega_-$. 

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Concerning the boundary behavior of each \( u_i^\pm \), one may, at this stage, only conclude that

\[
\lim_{\Omega_\pm \ni y \to 0} |u_i^\pm(y)| + |Du_i^\pm(y)| = 0.
\]

The goal of the middle third of [7] is the specific estimate

\[
\limsup_{r \to 0} r^{-\frac{1}{2}} h(r) < \infty.
\]

Besides involving many well-known concepts of geometric measure theory (monotonicity, excess, blowing-up) and well-known nonparametric regularity estimates (DeGiorgi-Nash, Schauder), the work here includes a new estimate on the radial derivative of each \( u_i^\pm \) and a new comparison between spherical and cylindrical excess.

Using (3), we verify that \( \Omega_{\pm}, u_i^\pm \) may be chosen so that

\[
\Omega_{\pm} \text{ is a } C^{1,1/10} \text{ domain, } u_i^\pm \in C^{1,1/4}(\text{Clos } \Omega_{\pm}).
\]

Under conditions (1), (2), and (4), the \( C^{1,\alpha} \) Hopf-type boundary point lemma of Finn and Gilbarg [6, Lemma 7] implies that \( u_1^+ = \cdots = u_m^+, u_1^- = \cdots = u_{m-1}^- \). For a small open ball \( B \) about 0, we then subtract off the oriented component, which meets the graph of \( u_1^+ \), of the regular points of \( B \cap (\text{spt } T) \sim \partial T \) to obtain an area minimizing \( S \in R_n^{\text{loc}}(B) \) with \( \partial S = 0 \) and \( \text{spt } S = B \cap \text{spt } T \). The proof is completed by using the interior regularity theorem [4, 5.3.18] which implies that (since \( h(r) \to 0 \) as \( r \downarrow 0 \)) \( \text{spt } S \) is, near 0, an embedded real analytic minimal submanifold.

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