Josephson junction transmission lines as tunable artificial crystals

Carsten Hutter,1 Kai Stammigel,1,2 Erik A. Tholén,3 Jack Lidmar,4 and David B. Haviland3

1Department of Physics, Stockholm University, AlbaNova University Center, SE–106 91 Stockholm, Sweden
2Institut für Theoretische Festkörperphysik, Universität Karlsruhe, D–76128 Karlsruhe, Germany
3Nanostructure Physics, Royal Institute of Technology, SE–106 91 Stockholm, Sweden
4Theoretical Physics, Royal Institute of Technology, SE–106 91 Stockholm, Sweden

(Dated: June 27, 2008)

We investigate one-dimensional Josephson junction arrays with generalized unit cells, beyond a single junction or SQUID, as a circuit approach to engineer band gaps. Within a specific frequency range, the order of the single junction plasma frequency, the dispersion relation becomes gapped and the impedance becomes purely imaginary. We derive the parameter dependence of this gap and suggest designs to lower it to microwave frequencies. The gap can be tuned in a wide frequency range by applying external flux, and persists in the presence of small imperfections. These arrays, which can be thought of as tunable artificial crystals, may find use in applications ranging from filters to the protection of quantum bits.

I. INTRODUCTION

The design of Josephson junction circuits in an appropriate electromagnetic environment is currently of great interest in the context of superconducting qubits. Such qubits are tunable artificial atoms. In analogy, one can envision long, periodic arrays of Josephson junctions, which are engineered metamaterials, or fully designable and tunable artificial crystals. In this context, we study the physics of regular, one-dimensional Josephson junction arrays (1d-JJAs) with nontrivial unit cells, beyond a single junction or superconducting quantum interference device (SQUID).

Regular JJAs have gained renewed interest: 1d-JJAs have been the subject of study for the development of the 10 Volt Josephson voltage standard, and were more recently investigated for a current standard, the transfer of quantum information, and the readout and amplification of qubit signals at the quantum noise level. 2d-JJAs have long been of interest in the context of quantum phase transitions, and were recently investigated as negative permeability metamaterials.

Here we focus on the exploration of dispersion relations and band gaps, well known from condensed matter and crystal physics. Very recently, crystal physics with 1d-JJAs was studied in the regime where the Josephson junctions are treated as charge qubits. We consider the opposite limit of junctions in the phase regime, where junctions have a comparatively large area and therefore the advantage that they can be fabricated with a low relative spread of parameters. Furthermore, we suggest two designs for which the band gap can be tuned by only one common applied magnetic field, which simplifies possible experiments. Junctions in the phase regime are approximately described by their linear behavior if the current flowing in the junctions is much less than the critical current. The array can then be regarded as a complex transmission line with a non-trivial, gapped dispersion relation. A resonator made with such a transmission line could find use in circuit cavity QED for read-out of qubit signals. Nonlinear corrections to the model presented here can be used to realize parametric amplification and quantum noise squeezing.

We restrict ourselves to the linear regime, and focus on the possibilities of more complex unit cells and the parameter regimes achievable in realistic experiments. We begin our studies with a unit cell consisting of two junctions (or SQUIDs) with different parameters, as shown in Fig. 1b. In this case, we find a gap in the dispersion relation as shown in Fig. 2, i.e., a frequency region in which no propagating modes appear and the real part of the impedance vanishes. This gap appears at frequencies of the order of the plasma frequencies of the individual junctions. We investigate the parameter dependency of the gap and possibilities to lower it to the experimentally accessible frequency region, appropriate for qubit design. Then we discuss the connection between the real part of the impedance and the density of states for arbitrary unit cells in the linear regime, and present a more general treatment which no longer requires the unit cells to be identical. We use this general approach to show that the gap persists in the presence of a small parameter spread, and present simulations for a transmission experiment with realistic boundary conditions.
II. JOSEPHSON JUNCTION ARRAYS WITH TWO-JUNCTION UNIT CELLS

A model of 1d-JJAs in the linear regime is shown in Fig. 1a, where each unit cell consists of a Josephson junction (with Josephson energy $E_J$ and capacitance $C_J$), and a capacitance to ground $C_0$. In the linear phase regime, one can introduce the plasma frequency $\omega_p = 1/\sqrt{L_1C_J}$, with linear Josephson inductance $L_J = \Phi_0^2/(4\pi^2E_J)$ and superconducting flux quantum $\Phi_0 = h/2e$. Here, $L_1$ and $C_J$ describe an intersite inductive and capacitive interaction, respectively, where we defined a site as the island between two junctions. The capacitance $C_0$ describes an onsite Coulomb interaction between Cooper pairs. We consider situations were resistive terms can be neglected. Also note that in the limit $C_J \to 0$ the model reduces to the discrete, lumped element model of a transmission line for transverse electromagnetic waves. We start our studies with the generalization that the array consists of a lattice of unit cells each consisting of a basis of two Josephson junctions, for which we introduce an additional index 1 or 2 to the parameters above, see Fig. 1b. Let us first discuss a special case included in the generalized model: If we choose $C_{01} = C_{12} = 0$ and regard $L_{12}$ as a geometric inductance instead of a Josephson inductance, we recover a model containing one junction and an additional inductance in series. This model was studied earlier, where the inductance $L_0$ was included in order to model the electromagnetic inductance of the JJA transmission line\cite{22}. This inductance led to a gap in the real part of the impedance, taken between input port and ground\cite{22}, which corresponds to a gap in the dispersion relation as shown in Fig. 2. However, for typical parameters the gap appeared at approximately $10^{11}$-$10^{14}$ Hz. While the lower frequency could, in principle, be reduced by lowering the plasma frequency of the junction, the upper frequency extended beyond the range of validity of the simple Josephson junction model used. With the two-junction model presented here, however, the upper band edge can be reduced in frequency by orders of magnitude.

FIG. 2: (Color online) (a) Real part of the impedance (for an infinite array) and (b) dispersion relation. Both show a gap in the same frequency range. The impedance is normalized by $Z_{01} = \sqrt{L_{11}/C_{11}}$ and the frequency by $\omega_{p1} = 1/\sqrt{L_{11}/C_{11}}$. The length of a unit cell is called $a$, and $k$ is the wave number of a traveling wave solution. Here we used parameters $C_{12} = C_{01} = 0$, where the model reduces to that of one junction and an additional inductance. Further, we chose parameters $C_{02}/C_{31} = L_{12}/L_{11} = 0.1$.

![Figure 2](image-url)

We shall now motivate the presence of this gap and then analyze its parameter dependence. First take an array with periodic boundary conditions and simple unit cells as in Fig. 1a, where each unit cell has only one degree of freedom due to loop constraints. Linearization of the equations of motion, in small value of phase difference across each junction, approximates the system as coupled harmonic oscillators. A traveling wave ansatz yields a single branch in the dispersion relation. This branch will have an upper cutoff frequency due to the discreteness of the model but no second branch and no gap. If we now consider new unit cells consisting of two original cells, each of the new cells has two independent degrees of freedoms. This results in a representation of the dispersion relation where the original branch is mirrored at half the Brillouin zone, and thus appears as two branches as shown in Fig. 3a. Here, we used the length $a$ for the new unit cell. If an asymmetry of parameters is introduced within each unit cell, a splitting in the dispersion relation occurs as shown in Fig. 3b.

A geometric inductance $L_0 \ll L_1$ introduces a much stronger asymmetry, which explains the wide gap and experimentally inaccessibly high frequency of the upper band edge (which tends to infinity for $L_0/L_1 \to 0$) in the model of Ref. 21. With the two-junction model, we have a wide range of accessible parameters so that we can engineer the band gap in an appropriate frequency range.

As shown in the appendix, the dispersion relation is given as

$$[\omega_{\pm}(k)]^2 = \frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A},$$

where

$$A = (C_{01} + C_{02})(C_{11} + C_{12}) + C_{01}C_{02} + C_{11}C_{12} \beta_k,$$
$$B = \frac{C_{01} + C_{02}}{L_{12}} + \frac{C_{11}}{L_{12}} + \frac{C_{12}}{L_{11}} \beta_k,$$
$$C = \frac{\beta_k}{L_{11}L_{12}},$$

with $1/L_{12} = 1/L_{11} + 1/L_{12}$ and $\beta_k = 2[1 - \cos(ka)]$.
Figs. 2b and 3b. More explicitly, we find at \( k_a = 0 \) and \( k_t \) the capacitances and inductances, for which the maximum of the respective. In the following we consider positive inductions and capacitances, for which the maximum of the \( 1 \)switching the position of the minima of \( \omega_{+}(k) \) between \( k = 0 \) and \( k = \pi/a \), which can be shown to be realized by a flat (constant in \( k \)) upper branch \( \omega_{+}(k) \) at these points.

Tunable inductances as in Fig. 1 can be achieved by replacing one or both junctions in each unit cell by a SQUID, which consists of two parallel junctions with Josephson energy \( E_{j0} \), and which is pierced by a magnetic field \( B \). The Josephson energy, which is inversely proportional to the Josephson inductance, can then be controlled as \( E_j = 2E_{j0}\cos(2\pi|B|A_8/\Phi_0) \). Here, \( A_8 \) is the effective area of the SQUID with respect to the \( B \)-field. A design with one junction and one SQUID per unit cell has the advantage that one of the two inductances can be tuned continuously without changing the other inductance. A design with two SQUIDs per unit cell has different advantages. Clearly, one can then tune both Josephson energies. If one chooses different areas \( A_{S1} \) and \( A_{S2} \) for the two SQUIDs, a change in magnetic field \( \Delta B = \Phi_0/A_{S1} \) leaves the Josephson energy of the first SQUID invariant, while it changes that of the second SQUID. In this sense, one can tune both Josephson energies independently with only one common magnetic field. Thus, this design is preferable in experiments which test many combinations of inductances \( (L_{11}, L_{12}) \), while the aforementioned design is better if one needs a continuous change of one inductance, which might become important for applications.

### III. More General Unit Cells

Here, we will clarify why the real part of the impedance and the dispersion relation are connected as presented in Fig. 2, and show that this connection is of more general nature. At the same time, we will provide formulas which can also be used in the case of more general types of unit cells (e.g. with three or more different Josephson junctions or SQUIDs) and for the investigation of parameter spreads.

If we allow an arbitrary combination of capacitances, inductances and Josephson junctions, the Lagrangian in the linear regime can always be written as

\[
L = \sum_{j,k=1}^{M} \left[ \Phi_j \left( \frac{C_{jk}}{2} \Phi_k - \frac{E_j (L^{-1})_{jk}}{2} \Phi_k \right) \right].
\]

Here, \( M \) is the total number of independent variables, and \( \Phi_j \) are independent flux variables. In the case of the two-junction unit cell we had \( M = 2N \), with \( N \) the number of unit cells, and as variables we used the integrated voltage at the capacitances to ground, \( \Phi_j = \int_{-\infty}^{t} V_j(t')dt' \). Further, \( C \) is the capacitance matrix and \( L^{-1} \) is the inverse inductance matrix, which can contain both the kinetic inductance due to Josephson junctions and the geometric inductance. This is a problem of coupled harmonic oscillators, which can be diagonalized by the transformation \( \Phi \rightarrow \tilde{\Phi} = U^{T}C^{1/2} \Phi \). Here we took into account that the matrices \( C \) and \( L \) can always be chosen symmetric, and defined \( U \) as the matrix which has columns consisting of the normalized, real eigenvectors of the matrix \( \Omega^2 = C^{-1/2}L^{-1}C^{-1/2} \). The trans-
formed Lagrangian is given as \( \tilde{L} = \frac{1}{2} \sum_{j=1}^{N} \left[ \dot{\Phi}_j^2 - \omega_j^2 \Phi_j^2 \right] \), where we introduced the eigenvalues \( \omega_j^2 \) of \( \Omega^2 \). These frequencies \( \omega_j \) resemble the dispersion relation for a regular array, and can still be calculated in the presence of imperfections as shown in section IV.

The equations of motion in this eigenbasis are decoupled and given as \( \ddot{\Phi}_j = -\omega_j^2 \Phi_j \). The Hamiltonian corresponding to the transformed Lagrangian is given as \( H = \frac{1}{2} \sum_j \left( \dot{Q}_j^2 + \omega_j^2 \Phi_j^2 \right) \), where \( \dot{Q}_j = \dot{\Phi}_j \) is the conjugate variable to \( \Phi \). For later convenience, this Hamiltonian can be rewritten in the standard form \( H = \sum_{j=1}^{N} \hbar \omega_j \left( a_j^\dagger a_j + \frac{1}{2} \right) \) when creation and annihilation operators are defined by the equations

\[
\dot{\Phi}_j = \sqrt{\hbar/(2\omega_j)} \left( a_j^\dagger + a_j \right), \\
\dot{Q}_j = i\sqrt{\hbar\omega_j/2} \left( a_j^\dagger - a_j \right).
\] (7)

Now we proceed to consider the connection between the real part of the impedance, the density of states and the dispersion relation. For the frequency range at which the dispersion relation shows a gap, there are no undamped, propagating modes and hence no transmission (in the limit of an infinite array). Then signals applied to the edge of the array having a frequency within this gap will be reflected back, and the impedance becomes purely imaginary. In general, the real part of the impedance can be expressed via the fluctuation dissipation theorem as

\[
\text{Re} \, Z(\omega) = \frac{1}{2\hbar \omega} (1 - e^{-\beta \hbar \omega}) S_V(\omega).
\] (8)

Here, the spectral density is given as \( S_V(\omega) = \int_{-\infty}^{\infty} \langle V(t)V(0) \rangle e^{i\omega t} dt \), where the voltage operator is given in the Heisenberg representation, \( V(t) = e^{iHt/\hbar} V e^{-iHt/\hbar} \), and we assume a Boltzmann distribution for the averaging, \( \langle O \rangle = Tr(e^{-\beta H} O) / Z \) with \( Z = Tr(e^{-\beta H}) \). Evaluating the trace in the energy eigenbasis \( H \langle n \rangle = E_n \langle n \rangle \) yields

\[
S_V(\omega) = \frac{1}{Z} \sum_{m,n} e^{-\beta E_n} \left| \langle m | V | n \rangle \right|^2 \delta(\omega - \frac{E_m - E_n}{\hbar}).
\] (9)

In the case of a Hamiltonian consisting of \( M \) coupled harmonic oscillators discussed above, the states \( | n \rangle \) and \( | m \rangle \) are product states, and the energies \( E_n \) and \( E_m \) depend on the occupation numbers of these states. For example, the energy of the state \( | n \rangle = \prod_{j=1}^{M} | n_j \rangle \) is

\[
E_n = \sum_{j=1}^{M} \epsilon_j (n_j + \frac{1}{2}).
\] (10)

We shall assume sufficient spread of capacitances and inductances in an experiment, such that we do not have to consider degeneracies of the energies \( \epsilon_j \equiv \hbar \omega_j \). If we further consider for the operator of voltage in Eq. (9) the voltage at one site, \( V_k = \hat{\Phi}_k \), we can rewrite the voltage operator as \( V_k = \sum_j \alpha_{kj} \sqrt{2\hbar \omega_j} (a_j^\dagger - a_j) \), with coefficients \( \alpha_{kj} = \frac{i}{\pi} \sum_l (C_l^{-1/2})_{kj} U_{lj} \).

It is then straightforward to calculate the matrix elements \( \langle m | V | n \rangle \) in Eq. (9) to get

\[
S_V(\omega) = \left( \frac{2\hbar \omega}{1 - e^{-\beta \hbar \omega}} \right) \sum_j |\alpha_{kj}|^2 \delta(\omega - \epsilon_j / \hbar).
\] (11)

Using Eq. (9), the real part of the impedance is given as

\[
\text{Re} \, Z_k(\omega) = \sum_j |\alpha_{kj}|^2 \delta(\omega - \epsilon_j / \hbar).
\] (12)

The delta-function on the right side of the equation picks out only terms in the sum over \( j \) for which \( \epsilon_j = \hbar \omega \). The real part of the impedance is thus given as the number of states per frequency interval [in the continuum limit \( \propto 1/|\partial \omega / \partial k| \)], multiplied by a weight factor \( |\alpha_{kj}|^2 \) which depends on the distribution of capacitances and inductances in the circuit.

### IV. EXPERIMENTAL CONSIDERATIONS

#### A. Range of validity and experimental parameters

The two branches in the dispersion relation and the associated gap in frequency, where no propagating modes exist in the JJA transmission line, could be a useful property for the design of quantum circuits in the microwave region. The essential ingredient for realizing this gap is an inequality of the parameters for each of two junctions in the basis of the periodic structure, such that Eq. (5) is not fulfilled. Here we examine realistic designs, subject to the constraints of fabrication, which can achieve this asymmetry. The designs naturally fall into two different parameter regimes depending on the transmission line geometry used, coplanar or stripline.

When the JJA is made in a coplanar waveguide (CPW) geometry, where the ground plane is on the sides of the JJA, the parameter regime \( C_{11}, C_{12} \gg C_{01}, C_{02} \) is easily realized. In this regime, the gap is vanishing or small when the plasma frequencies of the two junctions are the same, and one should thus aim for parameters \( \omega_{p1} \neq \omega_{p2} \) in order to observe the gap. When fabricating JJAs, typically all junctions are made in the same process step, resulting in a tunnel barrier which is nearly uniform across the entire chip or wafer. In this case, the Josephson capacitance \( C_j \) and inductance \( L_j \) will be proportional and inversely proportional to the junction area, respectively, and the plasma frequency \( \omega_p = 1/\sqrt{L_j C_j} \) will therefore be independent of the junction area. Thus, simply changing the junction area in the fabrication process will not achieve the desired goal.

Subject to the constraint of uniform tunnel barriers, there are two ways to bring down the plasma frequency. The first method is to increase \( L_j \) of one of the junctions...
by forming a SQUID loop of this junction and applying an external magnetic flux. This method is attractive because changing the external flux corresponds to tuning the frequency range of the transmission gap. However, with this approach, dropping the plasma frequency also drops the critical current of the transmission line, and therefore non-linear corrections will become important at much lower power. The second possibility to drop the plasma frequency is to fabricate an on-chip capacitance in parallel with each junction. This method will not cause a degradation of critical current, however, it does require more layers of lithography than the simple single layer process used in the shadow deposition technique. Fabrication with the Nb trilayer technique however provides this parallel capacitance naturally.

When designing an array in CPW geometry, one finds that the characteristic impedance of the array is not well matched to the termination impedance. When the array is terminated with a direct connection to an electrical lead, the termination of the array impedance at microwave frequencies will be approximate to an electrical lead, the termination of the array, which is the pure real impedance of an infinite JJA, which is the pure real impedance of an infinite array, is in the zero frequency limit given as

\[ Z_A(0) = \sqrt{(L_{11} + L_{12})/(C_{01} + C_{02})}. \]

It can be much larger than \( Z_0 \) in the CPW geometry, where \( Z_0 \) is relatively small, especially if \( L_1 \) is made large by suppressing the critical current. We desire that \( Z_A \ll R_Q = h/4e^2 = 6.45k\Omega \), in order to avoid quantum fluctuations of the phase which are not included in our model based on classical phase dynamics. When \( Z_A \gg R_Q \), one finds that large quantum fluctuations of the phase result in a Coulomb blockade, and our assumption of classical phase dynamics has completely broken down.

An alternative route to circuit design is based on the stripline geometry, where the array is fabricated on top of a ground plane with a thin, insulating (non-tunneling) barrier separating the array islands from the ground plane. For the stripline geometry, one easily realizes the regime \( C_{11}, C_{12} \lesssim C_{01}, C_{02} \). In this regime, it is not necessary to have different plasma frequencies of the two junctions, and we find that a considerable gap in transmission also occurs if \( C_{01} \neq C_{02} \). In this case the gap appears well below the plasma frequency. The condition of unequal capacitances to ground is easily realized in the stripline geometry, where the arrays are made as planar circuits, with overlapping films forming the tunnel barriers.

We have formulated a design for a JJA on an heavily oxidized Al ground plane, to be fabricated with the shadow evaporation technique. In our Al tunnel junction fabrication, we find that it is possible to achieve plasma frequencies as low as \( \omega_{p1}/2\pi = \omega_{p2}/2\pi = 33 \text{ GHz} \). A design with large area base electrodes (2.5 \( \mu \text{m} \times 20 \mu\text{m} \)) and long, narrow Dolan bridges (0.1 \( \mu \text{m} \times 2.5 \mu\text{m} \)) with small overlap (0.1 \( \mu\text{m} \)) after shadow evaporation can achieve the following parameters: \( C_{01} = 0.68 \text{ pF} \), \( C_{02} = 3.4 \text{ fF} \), \( C_{J1} = 2.0 \text{ pF} \), \( C_{J2} = 12 \text{ fF} \), with the array critical current being dominated by the smaller junction \( J \), \( I_{C2} = 170 \text{ nA} \). For this design, we find that the lower gap edge comes down in frequency to \( \omega_{gA}/2\pi = 8.5 \text{ GHz} \), in a frequency range accessible to present day qubit designs or broad band transmission measurements. For this design, the transmission line impedance of the array is \( Z_A(0) = 53\Omega \), which is well matched to the impedance of the input and output ports of an array with high frequency leads connected at each end. Such a design, with a rather low critical current and therefore strongly non-linear inductance, is ideal for the distributed parametric amplifier. In the low power regime, where linear behavior is expected, we find that such an array makes a good superconducting low pass filter, with a very sharp drop in transmission at 8.5 GHz in a design with only 20 unit cells in series.

### B. Influence of parameter spread and transmission

So far, we assumed that junctions can be fabricated identically if desired. However, in reality, there will be a spread of junction parameters in the fabrication. We expect that the gap in the spectrum, impedance, and transmission will persist provided that the disorder is weak enough and/or the array is short enough that localization effects can be ignored. We investigated this further as shown in Fig. 5 (obtained by classical circuit theory), where we simulated a random spread of Josephson inductances with normal distribution and standard deviation 5% in Fig. 5a,b,d,f. Parameter spread, or disorder, violates the condition of periodicity, and wave vectors are not well-defined. However, it is still possible to investigate the density of states, which is shown in Fig. 5a,b. Here we counted the number of states in a discretized frequency interval and considered 500 unit cells, as compared to 30 unit cells for Fig. 5c-f, in order to count a reasonable number of states. Despite the spread in parameters, a gap can still be clearly observed in the density of states.

In an experiment, it is easier to measure transmission than the dispersion itself. Furthermore, accurate boundary conditions on a finite length array become important for a real experiment. In Fig. 6a-d we show the transmission for a finite array with boundary conditions defined by input and output leads with transmission line impedances \( Z_{in} = Z_{out} = 50\Omega \). While this setup is not a cavity, it behaves like a resonator and supports standing waves because the input and output impedances \( Z_{in}, Z_{out} \) are not matched to the array impedance \( Z_A \). The standing waves can be seen by looking at the voltage at each site as in Fig. 6a-f. Note that because \( Z_A > Z_{in} = Z_{out} \), the voltage antinode occurs in the middle of the array for the fundamental mode, opposite to standing waves in resonators formed by a large point-like impedance at each end of a transmission line, where \( Z_{in}, Z_{out} > Z_A \). Each standing wave condition is associated with a peak in the
transmission as calculated in Fig. 6c,d for the case of no disorder, and 5% parameter spread, respectively. For frequencies inside the gap region, the transmission drops drastically which can be understood by an exponential decay of voltage amplitude from the edge of the array. When disorder or parameter spread is introduced, an investigation of the eigenmodes shows localized states in the gap region, near the gap edge (not shown). However, the broad gap in transmission remains essentially unaffected by a 5% spread in parameters. Thus, the design of such a gap appears to be a robust and useful feature for quantum circuit engineering.

V. SUMMARY AND OUTLOOK

We theoretically investigated regular Josephson junction arrays with generalized unit cells, e.g., unit cells consisting of two junctions with different parameters. In the linear approximation of the phase regime, we found a gap in the dispersion relation and in the real part of the impedance. This gap is not present in the linear regime of models with only one junction per unit cell, unless additional circuit elements in each unit cell are used. We derived the parameter dependence of the gap, and found that for a design with two different Josephson junctions, the gap appears at frequencies of the same order of magnitude as the plasma frequencies of the two junctions. We suggested how to lower these frequencies in an experimental setup in order to shift the gap to an accessible frequency range. By replacing one of the two junctions per unit cell with a SQUID, the gap can be tuned in situ. Furthermore, we derived the connection between the real part of the impedance, the density of states, and the dispersion relation for more general unit cells, and found expressions which can be used when the periodicity condition is lifted. Using these expressions we showed that the gap persists upon a realistic parameter spread of the junctions. We also suggested a transmission experiment which we modeled with realistic boundary conditions.

Our results could on the one hand be used for comparatively simple demonstration of tunable artificial crystals with Josephson junctions, which is interesting from the point of view of fundamental physics. On the other hand, such tunable artificial crystals, embedded in superconducting circuits, could become important for new types of applications, for example, for specific types of frequency filters.

One could also envision to protect qubits against decoherence with such JJAs, by placing a Cooper pair box qubit in the middle of an array. If the energy splitting of the qubit lies inside the region of the gap, where no traveling modes are available, we expect the relaxation of the qubit to be strongly suppressed. Decoherence of a qubit is composed not only of the relaxation but also of the pure dephasing, where the latter time scale is typically the critical, shorter one. However, recent experiments reached extremely high decoherence times, which, at least for part of the frequency range, appeared to be limited by the relaxation\textsuperscript{25}. In this case, suppression of the relaxation with a properly engineered gap would allow refined studies on remaining sources of decoherence.

The model we presented in this paper is a linear analysis of the JJA transmission line. The interesting effects we describe arise due to the plasma resonance of the Josephson junctions when they are arranged in a discrete periodic structure. Many analogies exist between the microwave phenomena presented here, and the emerging field of “plasmonics”, as represented by optical plasma resonances in metallic and semiconducting nanostructures\textsuperscript{26}. However, the linear approximation is valid only when the currents flowing in the junctions are
much less than the critical current. When this condition is violated, nonlinear effects will appear, which can be very strong in comparison with dissipative effects in JJAs. These nonlinear effects give rise to a host of interesting phenomena, such as parametric amplification. Here, the ability to match the JJA transmission line impedance for the case of the stripline geometry, with the electromagnetic transmission line impedance, leads to the possibility of a broad band parametric amplifier. Another interesting nonlinear effect is the trapping or localization of energy in discrete breather modes, where a gap in the dispersion relation is used to prevent the radiation damping of Josephson oscillations in a junction in the middle of the array. We hope that the analysis presented here will aid the development of future experiments in these directions.

Acknowledgments

We thank Hans Hansson for valuable discussions and comments on the manuscript. This work was supported by the Swedish Research Council (VR) and by NordForsk. DH gratefully acknowledges sabbatical support from the Wenner-Gren Foundation.

APPENDIX A: DERIVATION OF THE DISPERSION RELATION.

The Lagrangian for the model consisting of unit cells with two types of Josephson junctions (Fig. 1b) is given as

$$L_2 = \sum_{j=0}^{N-1} \left[ \frac{C_{11}}{2} \dot{\Phi}_{j,1}^2 + \frac{C_{12}}{2} \dot{\Phi}_{j,2}^2 \right] + \sum_{j=0}^{N-1} \left[ \frac{C_{11}}{2} (\dot{\Phi}_{j-1,2} - \dot{\Phi}_{j,1})^2 + \frac{C_{12}}{2} (\dot{\Phi}_{j,1} - \dot{\Phi}_{j,2})^2 \right] + \sum_{j=0}^{N-1} \left[ E_{11} \cos(\phi_{j-1,2} - \phi_{j,1}) + E_{12} \cos(\phi_{j,1} - \phi_{j,2}) \right] ,$$

(A1)

where we considered \( N \) unit cells, with periodic boundary conditions eliminated the Josephson phases \( \phi_{1,j} \), and defined fluxes and phases \( \Phi_{j,1/2} = \Phi_{0(\phi_{1,j})/2}/2\pi \) at the capacitors to ground.

From this Lagrangian, one can find the equations of motion, and, after the linear approximation, \( \phi_{1,j} \ll 1 \), make a traveling wave ansatz,

$$\begin{pmatrix} \Phi_{11} \\ \Phi_{12} \end{pmatrix} = \begin{pmatrix} u \\ v e^{i k a/2} \end{pmatrix} e^{i(k a - \omega t)} .$$

(A2)

Here, we introduced a length \( a \) for the total unit cell, which results in a factor \( a/2 \) for a single junction. The equations of motion can be rewritten as a matrix \( M \) multiplying the vector \((u, v)^T\) such that \( M(u, v)^T = 0 \). Non-trivial solutions exist only when the determinant of \( M \) is zero, which results in the dispersion relation stated in Eq. (I). The range of validity of the linear approximation is discussed in Section IV.A.

1. G. Schön and A. D. Zaikin, Phys. Rep. 198, 237 (1990).
2. Yu. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. 73, 357 (2001).
3. R. M. Bradley and S. Doniach, Phys. Rev. B 30, 1138 (1984).
4. R. L. Kautz, Rep. Prog. Phys. 59, 935 (1996).
5. J. Bylander, T. Duty, and P. Delsing, Nature 434, 361 (2005).
6. M. Cholascinski and R. W. Chhajlany, Phys. Rev. Lett. 98, 127001 (2007).
7. A. Romito, R. Fazio, and C. Bruder, Phys. Rev. B 71, 100501(R) (2005).
8. E. A. Tholén et al., Appl. Phys. Lett. 90, 253509 (2007).
9. M. A. Castellanos-Beltran and K. W. Lehnert, Appl. Phys. Lett. 91, 083509 (2007).
10. B. Yurke et al., Phys. Rev. Lett. 60, 764 (1988).
11. M. A. Castellanos-Beltran et al., arXiv:0806.0659.
12. R. Fazio, H. van der Zant, Phys. Rep. 355, 235 (2001).
13. C. Du, H. Chen, and S. Li, Phys. Rev. B 74, 113105 (2006).
14. N. Lazarides and G. P. Tsironis, Appl. Phys. Lett. 90, 163501 (2007).
15. A. L. Rakhmanov et al., Phys. Rev. B 77, 144507 (2008).
16. A. Wallraff et al., Phys. Rev. B 77, 144507 (2008).
17. D. I. Schuster et al., Nature 445, 515 (2007).
18. B. Yurke et al., Appl. Phys. Lett. 69, 3078 (1996).
19. B. Yurke and E. Buks, J. Lightw. Technol. 24, 5054 (2006).
20. R. Movshovich et al., Phys. Rev. Lett. 65, 1419 (1990).
21. B. D. Haviland, K. Andersson, and P. Agren, J. Low Temp. Phys. 118, 733 (2000).
22. H. Nyquist, Phys. Rev. 32, 110 (1928).
23. E. Chow, P. Delsing and D. B. Haviland, Phys. Rev. Lett. 81, 204 (1998).
24. R. Dolata et al., J. Appl. Phys. 97, 054501 (2005).
25. J. A. Schrieber et al., Phys. Rev. B 77, 180502(R) (2008).
26. E. Ozbay, Science 311, 189 (2006).
27. D. K. Campbell, S. Flach, and Y. S. Kivshar, Phys. Today 57, 43 (2004).
28. Note, however, that the analysis in Ref. 27 assumes a linear coupling between nonlinear oscillators, while in our case, the nonlinearity of the oscillators and of their coupling is always of same order.