Generalization Error Bounds for Extreme Multi-class Classification

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Abstract

Extreme multi-class classification is multi-class learning using an extremely large number of label classes. We show generalization error bounds with a mild dependency (up to logarithmic) on the number of classes, making them suitable for extreme classification. The bounds generally hold for empirical multi-class risk minimization algorithms using an arbitrary norm as regularizer. This setting comprises several prominent learning methods, including multinomial logistic regression, top-k multi-class SVM, and several classic multi-class SVMs. We show both data-dependent and data-independent bounds. Key to the data-dependent analysis is the introduction of the multi-class Gaussian complexity, which is able to preserve the coupling among classes. Furthermore, we show a data-independent bound that, choosing appropriate loss, can exhibit a tighter dependency on the number of classes than the data-dependent bound. We give a matching lower bound, showing that the result is tight, up to a constant factor. We apply the general results to several prominent multi-class learning machines, exhibiting a tighter dependency on the number of classes than the state of the art. For instance, for the multi-class SVM by Crammer and Singer (2002), we obtain a data-dependent bound with square root dependency (previously: linear) and a data-independent bound with logarithmic dependency (previously: square root).

Keywords: Multi-class SVMs, Generalization error bounds, Extreme Classification, Covering numbers, Rademacher complexities, Gaussian complexities.

1 Introduction

Multi-class learning is a classical problem in machine learning [1]. The outputs here stem from a finite set of categories (classes), and the aim is to classify each input into one of several possible target classes [2][4]. Classical applications of multi-class classification include handwritten optical character recognition, where the system learns to automatically interpret handwritten characters [5], part-of-speech tagging, where each word in a text is annotated with part-of-speech tags [6], and image
categorization, where predefined categories are associated with digital images [7, 8], to name only a few.

Providing a theoretical framework of multi-class learning algorithms is a fundamental task in statistical learning theory [1]. Statistical learning theory aims to ensure formal guarantees safeguarding the performance of learning algorithms, often in the form of generalization error bounds [9]. Such bounds may lead to improved understanding of commonly used empirical practices and spur the development of novel learning algorithms (“Nothing is more practical than a good theory” [1]).

Classic generalization bounds for multi-class learning scale rather unfavorably (like quadratic, linear, or square root at best) in the number of classes [9–11]. This may be because the standard theory has been constructed without the need of having a large number of label classes in mind as many classic multi-class problems consist only of a small number of classes, indeed. For instance, the historically first multi-class dataset—Iris—[12]—contains solely three classes, the MNIST dataset [13] consists of 10 classes, and most of the datasets in the popular UCI corpus [14] contain up to several dozen classes.

However, with the advent of the big data era, multi-class problems can involve tens or hundreds of thousands of classes. As examples consider the following: 1. We are continuously monitoring the internet for new webpages, which we would like to categorize [15]. 2. We have data from an online biomedical bibliographic database that we want to index for quick access to clinicians [16]. 3. We are collecting data from an online feed of photographs that we would like to classify into image categories [7]. 4. We add new articles to an online encyclopedia and intend to predict the categories of the articles [16].

The subarea of machine learning studying classification problems involving an extremely large number of classes (such as the ones mentioned above) is called eXtreme Classification (XC)—a term that goes back to the NIPS 2013 Workshop on eXtreme Classification [17]. The workshop spurred vivid research activity and follow-up workshops at NIPS (2013, 2015, and 2016) and ICML (2015). Not surprising, several algorithms have recently been proposed to speed up the training or improve the prediction accuracy in extreme classification problems [15, 18–27].

However, there is still a discrepancy between algorithms and theory in extreme classification, as standard statistical learning theory is void in the large number of classes scenario [28]. With the present paper we want to contribute toward a better theoretical understanding of extreme classification. Theoretical understanding of extreme multi-class learning can provide theoretical grounds to the commonly used empirical practices in extreme classification and lead to insights that may be used to guide the design of new learning algorithms.

Note that the present paper focuses on multi-class learning. Recently, there has been a growing interest in multi-label learning. The difference in the two scenarios is that each instance is associated with exactly one label class (in the multi-class case) or multiple classes (in the multi-label case), respectively. While the present analysis is tailored to the multi-class learning scenario, it may serve as a starting point for subsequent analysis of the multi-label learning scenario.

### 1.1 Contributions in a Nutshell

We build the present journal article upon our previous conference paper published at NIPS 2015 [29], where we propose a multi-class support vector machine (MC-SVM) using block $\ell_2,p$-norm regularization, for which we prove data-dependent generalization bounds based on Gaussian complexities.

In the present article, we generalize the proof technique, showing more general data-dependent bounds that apply to a wide range of multi-class classification methods. Furthermore, we show novel data-independent generalization bounds that, choosing appropriate loss, exhibit faster convergence than the data-dependent bounds.

As application of our theory, we show generalization error bounds for several prominent multi-class learning algorithms: regularized multinomial logistic regression [30], top-$k$ multi-class SVM [31], and several classic MC-SVMs [32–34]. For all of these methods, we show generalization bounds with an improved dependency on the number of classes. For instance, the best known bounds for multinomial logistic regression and the MC-SVM by Crammer and Singer [32] scale square root in the number of

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1. e.g., [https://www.ncbi.nlm.nih.gov/pubmed](https://www.ncbi.nlm.nih.gov/pubmed)
classes. We improve this dependency to logarithmic. This gives strong theoretical grounds for using these methods in extreme classification.

2 Related Work and Contributions

In this section, we discuss related work and outline the main contributions of this paper.

2.1 Related Work

In this section, we recapitulate the state of the art in multi-class learning theory.

2.1.1 Related Work on Data-dependent Bounds

Existing generalization error bounds for multi-class learning can be classified into two groups: data-dependent and data-independent error bounds. Both types of bounds are often based on the assumption that the data is realized from independent and identically distributed random variables. However, this assumption can be relaxed to weakly dependent time series, for which Mohri and Rostamizadeh [35] and Steinwart et al. [36] show data-dependent and data-independent generalization bounds, respectively.

Data-dependent generalization error bounds refer to bounds that can be evaluated on training samples and thus can capture properties of the distribution that has generated the data [9]. Often these bounds built on the empirical Rademacher complexity [37–39], which can be used in model selection and for the construction of new learning algorithms [40].

The investigation of data-dependent error bounds for multi-class learning was initiated, to our best knowledge, by Koltchinskii and Panchenko [10], who give the following structural result on Rademacher complexities: given a set $H = \{h = (h_1, \ldots, h_c)\}$ of vector-valued functions and training samples $x_1, \ldots, x_n$, it holds

$$\mathbb{E}_\epsilon \sup_{h \in H} \sum_{i=1}^{n} \epsilon_i \max \{h_1(x_i), \ldots, h_c(x_i)\} \leq \sum_{j=1}^{c} \mathbb{E}_\epsilon \sup_{h \in H} \sum_{i=1}^{n} \epsilon_i h_j(x_i). \quad (1)$$

Here, $\epsilon_1, \ldots, \epsilon_n$ denote independent Rademacher variables (i.e., taking values +1 or −1, with equal probability), and $\mathbb{E}_\epsilon$ denotes the conditional expectation operator removing the randomness coming from the variables $\epsilon_1, \ldots, \epsilon_n$.

In many subsequent theoretical work on multi-class learning, the above result is used as a starting point, by which the maximum operator involved in multi-class hypothesis classes (Eq. (1) left-hand side) can be removed [9, 32]. Applying the result leads to a simple sum of $c$ many Rademacher complexities (Eq. (1), right-hand side), each of which can be bounded using standard theory [38]. This way, Koltchinskii and Panchenko [10], Cortes et al. [41], and Mohri et al. [9] derive multi-class generalization error bounds that exhibit a quadratic dependency on the number of classes, which Kuznetsov et al. [42] improve to a linear one.

However, the reduction (1) comes at the expense of at least a linear dependency on the number of classes $c$, coming from the sum in Eq. (1) (right-hand side), which consists of $c$ many terms. We show in this paper that this linear dependency can oftentimes be suboptimal because (1) does not take into account the coupling among the classes. To understand why, it is illustrative to consider the example of the multi-class SVM (MC-SVM) by Crammer and Singer [32], which uses an $\ell_2$-norm constraint

$$\|(h_1, \ldots, h_c)\|_2 \leq \Lambda \quad (2)$$

to couple the components $h_1, \ldots, h_c$. The problem with Eq. (1) is that it decouples the components, resulting in the constraint $\|(h_1, \ldots, h_c)\|_\infty \leq \Lambda$, which—as illustrated in Figure 1—is a poor approximation of (2).

In our previous work [29], we give a structural result addressing this shortcoming and tightly preserving the constraint defining the hypothesis class. Our result is based on the so-called Gaussian complexity [38], a notion similar but yet different to the Rademacher complexity. The difference in
Figure 1: Illustration why Eq. (1) is loose. Consider a 1-dimensional binary classification problem with hypothesis class $H$ consisting of functions mapping $x \in \mathbb{R}$ to $\max(h_1(x), h_2(x))$, where $h_j(x) = w_j x$ for $j = 1, 2$. Assume the class is regularized through the constraint $\| (w_1, w_2) \|_2 \leq 1$, so the left-hand side of the inequality (1) involves a supremum over the $\ell_2$-norm constraint $\| (w_1, w_2) \|_2 \leq 1$. In contrast, the right-hand side of (1) has individual suprema for $w_1$ and $w_2$ (no coupling anymore), resulting in a supremum over the $\ell_\infty$-norm constraint $\| (w_1, w_2) \|_\infty \leq 1$. Thus applying Eq. (1) enlarges the size of constraint set by the area that is shaded in the figure, which grows as $O(\sqrt{c})$. In the present paper, we show a proof technique to elevate this problem, thus resulting in an improved bound (tighter by a factor of $\sqrt{c}$).

the two notions is that the Rademacher and Gaussian complexity is the supremum of a Rademacher and Gaussian process, respectively.

The core idea of our analysis is that we exploit a comparison inequality for the suprema of Gaussian processes known as Slepian’s Lemma [43], by which we can remove, from the Gaussian complexity, the maximum operator that occurs in the definition of the hypothesis class, thus preserving the above mentioned coupling—we call the supremum of the resulting Gaussian process the multi-class Gaussian complexity.

Using our structural result, we obtain in [29] a data-dependent error bound for [32] that exhibits—for the first time—a sublinear (square root) dependency on the number of classes. When using a block $\ell_{2,p}$-norm constraint (with $p$ close to 1), rather than an $\ell_2$-norm one, one can reduce this dependency to logarithmic, making the analysis appealing for extreme classification.

We note that, addressing the same need, the following structural result [44, 45] appear since publication of our previous work [29]:

$$E_\varepsilon \sup_{h \in H} \sum_{i=1}^{n} \varepsilon_i f_i(h(x_i)) \leq \sqrt{2} L E_\varepsilon \sup_{h \in H} \sum_{i=1}^{n} \sum_{j=1}^{c} \varepsilon_{ij} h_j(x_i),$$

(3)

where $f_1, \ldots, f_n$ are $L$-Lipschitz continuous with respect to (w.r.t.) the $\ell_2$ norm.

For the MC-SVM by Crammer and Singer [32], the above result leads to the same favorable square root dependency on the number of classes as our previous result in [29]. We note, however, that the structural result (3) requires $f_i$ to be Lipschitz continuous w.r.t. the $\ell_2$ norm, while some multi-class loss functions [31, 33] are Lipschitz continuous with a substantially smaller Lipschitz constant, when choosing a more appropriate norm. In these cases, the analysis given in the present paper improves not only the classical results obtained through (1), but also the results obtained through (3).

2.1.2 Related Work on Data-independent Bounds

Data-independent generalization bounds refer to classical theoretical bounds that hold for any data, with a certain probability over the draw of the samples [1, 46]. In their seminal contribution On the Uniform Convergence of Relative Frequencies of Events to Their Probabilities, Vapnik and Chervonenkis [47] show historically one of the first bounds of that type—introducing the notion of VC dimension.
Several authors consider data-independent bounds for multi-class learning. By controlling entropy numbers of linear operators with Maurey’s theorem, Guermeur [1] derive generalization error bounds with a linear dependency on the number of classes. Which is improved to square-root by Zhang [48] using \( \ell_\infty \)-norm covering numbers. Pan et al. [49] consider a multi-class Parzen window classifier and derive an error bound with a quadratic dependency on the number of classes. Several authors show data-independent generalization bounds based on combinatorial dimensions, including the graph dimension, the Natarajan dimension \( d_{nat} \), and its scale-sensitive analog \( d_{nat, \gamma} \) for margin \( \gamma \) [50–54].

Guermeur [50, 51] shows a generalization bound decaying as \( O(\log c \sqrt{d_{nat, \gamma} \log n}) \). When using an \( \ell_\infty \)-norm regularizer \( d_{nat, \gamma} \) is bounded by \( O(c^2 \gamma^{-2}) \), and the generalization bounds reduces to \( O(c \log c \sqrt{\log n}) \). The authors do not give a bound for an \( \ell_2 \)-norm regularizer as this is more challenging to deal with, due to the above mentioned coupling of the hypothesis components.

Daniely et al. [52] give a bound decaying as \( O\left(\sqrt{d_{nat}(H) \log c}\right) \), which transfers to \( O\left(\sqrt{d_{nat} \log c}\right) \) for multi-class linear classifiers since the associated Natarajan dimension grows as \( O(dc) \) [53].

Guermeur [53] recently establish an \( \ell_p \)-norm Sauer-Shelah lemma for large-margin multi-class classifiers, based on which error bounds with a square-root dependency on the number of classes are derived. This consists comprises the MC-SVM by Crammer and Singer [32].

What is common to all of the above mentioned bounds is their super logarithmic dependency (square root at best) on the number of classes. As notable exception, Kontorovich and Weiss [56] show a bound exhibiting a logarithmic dependency on the number of classes. However, their bound holds only for the specific nearest-neighbor-based algorithm that they propose, so their analysis does not cover the commonly used multi-class learning machines mentioned in the introduction (like multinomial logistic regression and classic MC-SVMs). Furthermore, their bound is of the order \( c \log c \sqrt{\log n} \) and thus has an exponential dependence on the doubling dimension \( D \) of the metric space where the learning takes place. For instance, for linear learning methods with input space dimension \( D \), the doubling dimension \( D \) grows linearly in \( d \), so the bound in [56] grows exponentially in \( d \). For kernel-based learning using an infinite doubling dimension (e.g., Gaussian kernels) the bound is void.

2.2 Contributions of This Paper

This paper aims to contribute to a solid theoretical foundation for learning with an extremely large number of class labels by presenting data-dependent and data-independent generalization error bounds with relaxed (up to logarithmic) dependencies on the number of classes. Below we summarize the main results of this paper.

2.2.1 Tighter Data-dependent Generalization Bounds

The heart of our data-dependent analysis is the novel structural result on Gaussian complexities (Lemma 1), which—as our previous result in [29]—preserves the correlation among different components. Similar to Maurer [41] and Cortes et al. [42], our structural result applies to function classes induced by operators satisfying a Lipschitz continuity. However, we measure the Lipschitz continuity with respect to a specially crafted variant of the \( \ell_2 \) norm (cf. Definition 2), which—as we show—can lead to smaller Lipschitz constants.

We show a data-dependent generalization error bound for regularized multi-class empirical risk minimization algorithms using an arbitrary norm as regularizer. As instantiations of our general bound, we compute specific bounds for \( \ell_2,p \)-norm and Schatten \( p \)-norm regularizers. We demonstrate that some commonly used loss functions satisfy our definition of Lipschitz continuity, including the margin-based loss function \( \ell_2 \), multinomial logistic loss [30], top-\( k \) hinge loss [31], and the loss functions used in [51] and [32], respectively.

Our data-dependent generalization bound enjoys a square-root dependency on the number of classes for regularized multinomial logistic regression [30] and the classic MC-SVM by Crammer and Singer [32]. For both of which the dependency can be dropped to logarithmic by using an block \( \ell_2,p \)-norm regularizer and \( p \) approaching 1 [29].
Our general analysis yields also the first generalization error bound for top-
$k$ MC-SVM, as a decreasing function in $k$. When setting $k$ proportional to $c$, the bound does not depend at all on the number of classes, which makes this algorithm appealing for extreme classification. For comparison, error bounds based on existing structural results [44, 45, 57] fail to shed insights on the influence of $k$ on the generalization performance because the involved Lipschitz constant is dominated by a constant.

For the MC-SVM by Weston and Watkins [32], our analysis yields a data-independent generalization bound exhibiting a linear dependency on the number of classes, while, when applying the structural results by Lei et al. [29], Maurer [44], Cortes et al. [45], the dependency is $O(c^{1/2})$.

### 2.2.2 Tighter Data-independent Generalization Bounds

While the data-dependent bounds require the associated loss function to be Lipschitz continuous w.r.t. the $\ell_2$ norm or a variant thereof, some multi-class loss functions allow for a substantially smaller Lipschitz constant when measuring the Lipschitz continuity w.r.t. the $\ell_\infty$ norm, rather than the $\ell_2$ norm. Motivated by this observation, we give a data-independent analysis based on $\ell_\infty$ covering numbers. We show that this can lead to a milder dependency of the bounds on the number of classes.

Key to our analysis is that we connect the Rademacher complexity of multi-class loss function classes with the so-called worst-case Rademacher complexity of an appropriately defined linear scalar-valued function class. The latter is much more convenient to deal with, due to its linearity. Our main result is a data-independent generalization bound for multi-class learning using a general regularizer, expressed in terms of the worst-case Rademacher complexity of a certain linear function class. We show also a matching lower bound, showing that the upper bound is tight, up to a constant.

Applying our bound to specific learning algorithms, yields data-independent generalization error bounds scaling linearly in the number of classes for the MC-SVMs by Weston and Watkins [32] and Lee et al. [34] and logarithmically for top-$k$ MC-SVM [33], trace-norm regularized MC-SVM [58], and the MC-SVM by Crammer and Singer [32]. Note that the previously best known result for Crammer and Singer [32] scales only square root in the number of classes [48].

### 3 Main Results

#### 3.1 Problem Setting

In multi-class classification with $c$ classes, we are given training examples $S = \{z_i = (x_i, y_i)\}_{i=1}^n \subseteq \mathcal{Z} := \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} \subseteq \mathbb{R}^d$ is the input space and $\mathcal{Y} = \{1, \ldots, c\}$ the output space. We assume that $z_1, \ldots, z_n$ are independently drawn from a probability measure $P$ defined on $\mathcal{Z}$.

Our aim is to learn, from a hypothesis space $\mathcal{H}$, a hypothesis $h = (h_1, \ldots, h_c) : \mathcal{X} \to \mathbb{R}^c$ used for prediction via the rule $y \to \arg \max_{y \in \mathcal{Y}} h_y(x)$. We consider prediction functions of the form $h^w_j(x) = \langle w_j, \phi(x) \rangle$, where $\phi$ is a feature map associated to a Mercer kernel $K$ defined over $\mathcal{X} \times \mathcal{X}$, $w_j$ belongs to the reproducing kernel Hilbert space $H_K$ induced from $K$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $H_K$.

We consider hypothesis spaces of the form

$$
H_\tau = \{h^w = (\langle w_1, \phi(x) \rangle, \ldots, \langle w_c, \phi(x) \rangle) : w = (w_1, \ldots, w_c) \in H_K^c, \tau(w) \leq \Lambda\},
$$

where $\tau$ is a functional defined on $H_K^c := H_K \times \cdots \times H_K$ and $\Lambda > 0$. Here we omit the dependency on $\Lambda$ for brevity.

We consider a general problem setting with $\Psi_y(h_1(x), \ldots, h_c(x))$ used to measure the prediction quality of the model $h$ at $(x, y)$ [48, 59], where $\Psi_y : \mathbb{R}^c \to \mathbb{R}_+$ is a real-valued function taking a $c$-component vector as its argument. This general loss function $\Psi_y$ is widely used in many existing MC-SVMs, including the models by Crammer and Singer [32], Weston and Watkins [32], Lee et al. [34], Zhang [48], and Lapin et al. [51].
| notation | meaning | page |
|----------|---------|------|
| $\mathcal{X}, \mathcal{Y}$ | the input space and output space, respectively | 6 |
| $z_i = (x_i, y_i)$ | the $i$-th example with $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$ | 6 |
| $c$ | number of classes | 6 |
| $K$ | reproducing kernel | 6 |
| $\phi$ | feature map associated to a kernel $K$ | 6 |
| $H_K$ | the reproducing kernel Hilbert space induced by a kernel $K$ | 6 |
| $\Psi_y$ | multi-class loss function for class label $y$ | 6 |
| $\| \cdot \|_p$ | $\ell_p$ norm defined on $\mathbb{R}^c$ | 8 |
| $\| \cdot \|_{2,p}$ | $\ell_{2,p}$ norm defined on $H_K^c$ | 8 |
| $\langle w, v \rangle$ | inner product on $H_K^c$ as $\sum_{j=1}^c \langle w_j, v_j \rangle$ | 8 |
| $p^*$ | dual exponent of $p$ satisfying $1/p + 1/p^* = 1$ | 8 |
| $\text{card}(v)$ | number of non-vanishing components of $v$ in $H_K^c$ | 8 |
| $\mathbb{E}_u$ | the expectation w.r.t. random $u$ | 8 |
| $B_K$ | the constant $\sup_{x \in \mathcal{X}} \sqrt{K(x,x)}$ | 8 |
| $B$ | the constant $B_K \sup_{w: \tau(w) \leq \Lambda} \| w \|_{2,\infty}$ | 8 |
| $A_\tau$ | supremum deviation between risk and empirical risk over $H_{\tau}$ minus a term | 8 |
| $I_y$ | indices of examples with class label $y$ | 8 |
| $N_n$ | the set $\{1, \ldots, n\}$ | 8 |
| $\| \cdot \|_S$ | Schatten-$p$ norm of a matrix | 8 |
| $|S|$ | cardinality of a set $S$ | 8 |
| $\mathcal{R}_S(H)$ | empirical Rademacher complexity of $H$ w.r.t. sample $S$ | 8 |
| $\mathcal{G}_S(H)$ | empirical Gaussian complexity of $H$ w.r.t. sample $S$ | 8 |
| $\mathcal{R}_n(H)$ | worst-case Rademacher complexity of $H$ w.r.t. $n$ examples | 8 |
| $F_{\tau,\Lambda}$ | loss function classes for MC-SVM | 10 |
| $H_{\tau}$ | class of scalar-valued linear functions defined on $H_K^c$ | 10 |
| $\rho_h(x, y)$ | margin of $h$ at $(x, y)$ | 12 |
| $\mathcal{N}_\infty(\epsilon, F, S)$ | empirical covering number of $F$ w.r.t. sample $S$ | 21 |
| $\mathcal{N}_\infty(\epsilon, F, n)$ | worst-case covering number of $F$ w.r.t. $n$ examples | 21 |
| $\text{fat}_\epsilon(F)$ | fat-shattering dimension of $F$ | 21 |
3.2 Notations

We collect some notations used throughout this paper (see also Table 1). We say that a function $f : \mathbb{R}^c \to \mathbb{R}$ is $L$-Lipschitz w.r.t. a norm $\| \cdot \|$ in $\mathbb{R}^c$ if

$$
|f(t) - f(t')| \leq L \| (t_1 - t'_1, \ldots, t_c - t'_c) \|, \quad \forall t, t' \in \mathbb{R}^c.
$$

The $\ell_p$-norm of a vector $t = (t_1, \ldots, t_c)$ is defined as $\|t\|_p = \left[ \sum_{j=1}^c |t_j|^p \right]^{\frac{1}{p}}$. For any $v = (v_1, \ldots, v_c) \in H^c_{K}$, we define the block $\ell_2,p$-norm $\|v\|_{2,p} = \left[ \sum_{j=1}^c \|v_j\|_p^2 \right]^{\frac{1}{2}}$, $\forall p \geq 1$, and denote by card($v$) the number of non-vanishing entries of $v$. Here, for brevity, we denote by $\|v_j\|_2$ the norm of $v_j$ in $H_K$. For any $w = (w_1, \ldots, w_c), v = (v_1, \ldots, v_c) \in H^c_{K}$, we denote $\langle w, v \rangle = \sum_{j=1}^c (w_j, v_j)$. For any $p \geq 1$, we denote by $p^*$ the dual exponent of $p$ satisfying $1/p + 1/p^* = 1$. For any norm $\| \cdot \|$ we use $\| \cdot \|_p$ to mean its dual norm. Furthermore, we define $B_\phi = \sup_{(x,y) \in \mathbb{R}^c \times H, \tau | x \rangle \in H} \Psi_y(h(x)), B_K = \sup_{t \in (0,1)} \sqrt{K(x,x)}$, and $B = B_K \sup_{w, \tau | (w) \leq A} \|w\|_{2,\infty}$. For brevity, for any functional $\tau$ over $H^c_{K}$, we introduce

$$
A_\tau := \sup_{h \in H, \tau | x \rangle \in H} \left[ \mathbb{E}_{x,y} \Psi_y(h(x)) - \frac{1}{n} \sum_{i=1}^n \Psi_y(h(x_i)) \right] - 3B_\phi \left[ \log \frac{2}{2n} \right]^{\frac{1}{4}},
$$

which we will use to write our bounds more compactly. Note that, for any random $u$, the notation $\mathbb{E}_u$ denotes the expectation w.r.t. $u$. The index $u$ is omitted if the expectation is taken over all involved random variables. For any $n \in \mathbb{N}$, we introduce the notation $\mathbb{N}_n := \{1, \ldots, n\}$. For any $y \in \mathcal{Y}$, we use $I_y =\{i \in \mathbb{N}_n : y_i = y\}$ to mean the indices of examples with label $y$.

If $\phi$ is the identity map, then the hypothesis $h^w$ can be compactly represented by a matrix $W = (w_1, \ldots, w_c) \in \mathbb{R}^{d \times c}$. For any $p \geq 1$, the Schatten-$p$ norm of a matrix $W \in \mathbb{R}^{d \times c}$ is defined as the $\ell_p$ norm of the vector of singular values $\sigma(W) := (\sigma_1(W), \ldots, \sigma_d(W))^\top$ (singular values assumed to be sorted in descending order): $\|W\|_{S_p} := \|\sigma(W)\|_p$.

3.3 Data-dependent Bounds

Our discussion on data-dependent generalization error bounds is based on the established methodology of Rademacher and Gaussian complexities [38].

**Definition 1** (Empirical Rademacher and Gaussian complexities). Let $H$ be a class of real-valued functions defined over a space $\tilde{Z}$ and $S = \{z_i\}_{i=1}^n \subseteq \tilde{Z}$. The empirical Rademacher and Gaussian complexities of $H$ with respect to $S$ are respectively defined as

$$
\mathcal{R}_S(H) = \mathbb{E}_z \left[ \sup_{h \in H} \frac{1}{n} \sum_{i=1}^n \epsilon_i h(z_i) \right], \quad \mathcal{G}_S(H) = \mathbb{E}_z \left[ \sup_{h \in H} \frac{1}{n} \sum_{i=1}^n g_i h(z_i) \right],
$$

where $\epsilon_1, \ldots, \epsilon_n$ are independent Rademacher variables, and $g_1, \ldots, g_n$ are independent $N(0,1)$ random variables. We define the worst-case Rademacher complexity as $\mathcal{R}_n(H) = \sup_{S \subseteq \tilde{Z}, |S| = n} \mathcal{R}_S(H)$ with $|S|$ being its cardinality.

Existing data-dependent analyses build on either the structural result [11] or [37], which either ignores the correlation among predictors associated to individual class labels or requires $f_i$ to be Lipschitz continuous w.r.t. the $\ell_2$ norm. Below we introduce a new structural complexity result based on the following Lipschitz property of vector-valued function classes composed by Lipschitz operators $f_i, i \in \mathbb{N}$.

**Definition 2** (Lipschitz continuity w.r.t. a variant of the $\ell_2$-norm). We say a function $f : \mathbb{R}^c \to \mathbb{R}$ is Lipschitz continuous w.r.t. a variant of the $\ell_2$-norm involving Lipschitz constants ($L_1, L_2$) and index $r$ if

$$
|f(t) - f(t')| \leq L_1 \| (t_1 - t'_1, \ldots, t_c - t'_c) \|_2 + L_2 |t_r - t'_r|, \quad \forall t, t' \in \mathbb{R}^c.
$$

(5)
The advantage of the above Lipschitz property over the previously used \(\ell_2\) Lipschitz continuity is that its definition yields smaller Lipschitz constants for some loss functions, including the ones used by Weston and Watkins [33] and Lapin et al. [31]. For example, the loss function used in the top-\(k\) SVM [31] satisfies [5] with \(L_1 = \frac{1}{\sqrt{\pi}}\) and \(L_2 = 1\). The Lipschitz constant \(L_1\) in front of the dominating term in the upper bound [6] will allow us to derive data-dependent bounds with no dependencies on the number of classes by setting \(k\) proportional to \(c\). In comparison, the structural complexity result [3] by Cortes et al. [43] and Maurer [44] requires us to set the involved constant to \(L = \frac{1}{\sqrt{\pi}} + 1\), which does not capture the special structure of the top-\(k\) loss function and leads to a square-root dependency for top-\(k\) SVM.

We now present our first core result of this paper, the following structural lemma. Proofs of results in this section are given in Section 5.1.

**Lemma 1** (Structural Lemma). Let \(H\) be a class of functions mapping from \(X\) to \(\mathbb{R}^c\). Let \(L_1, L_2 \geq 0\) be two constants and \(r : \mathbb{N} \to \mathcal{Y}\). Let \(f_1, \ldots, f_n\) be a sequence of functions from \(\mathbb{R}^c\) to \(\mathbb{R}\). Suppose that for any \(i \in \mathbb{N}_n\), \(f_i\) is Lipschitz w.r.t. a variant of \(\ell_2\)-norm involving Lipschitz constants \((L_1, L_2)\) and index \(r(i)\). Let \(g_1, \ldots, g_n, g_{11}, \ldots, g_{nc}\) be a sequence of independent \(N(0, 1)\) random variables. Then, for any sample \(S = \{x_i\}_{i=1}^n \subseteq X\) we have

\[
\mathbb{E}_g \sup_{h \in H} \sum_{i=1}^n g_i f_i(h(x_i)) \leq \sqrt{2} L_1 E_g \sup_{h \in H} \sum_{i=1}^n g_i h_j(x_i) + \sqrt{2} L_2 E_g \sup_{h \in H} \sum_{i=1}^n g_i h_{r(i)}(x_i). \tag{6}
\]

Based on the Gaussian (rather than the Rademacher) complexity, the above Lemma 1 upper bounds the Gaussian complexity of the initial hypothesis class, thereby removing the dependency on the potentially cumbersome operation \(f_i\) in the definition of the class (for instance for Crammer and Singer [32], \(f_i\) would be the component-wise maximum). The above lemma is based on a comparison result (Slepian’s lemma, Lemma 18) among the suprema of Gaussian processes.

Equipped with Lemma 1, we are now able to present our main result on data-dependent error bounds.

**Theorem 2** (General data-dependent generalization bound for general regularizer and Lipschitz continuous loss w.r.t. Def. 2). Consider the hypothesis space \(H_{r,\Lambda}\) in 4 with \(\tau(w) = \|w\|\), where \(\|\cdot\|\) is a norm defined on \(H_{\tau,\Lambda}\). Suppose that there exist \(L_1, L_2 \in \mathbb{R}_+\) such that \(\Psi_{y,\Lambda}\) is Lipschitz continuous w.r.t. a variant of \(\ell_2\) norm involving Lipschitz constants \((L_1, L_2)\) and index \(y\) for all \(y \in \mathcal{Y}\). Then, for any \(0 < \delta < 1/2\), with probability at least \(1 - 2\delta\), we have

\[
A_{\tau} \leq \frac{2\Lambda \sqrt{\pi}}{n} \left[ L_1 E_g\left( \sum_{i=1}^n g_i h \phi(x_i) \right) \right] + \frac{L_2 E_g\left( \sum_{i \in J} g_i \phi(x_i) \right)}{n}, \tag{7}
\]

where \(g_1, \ldots, g_n, g_{11}, \ldots, g_{nc}\) are independent \(\mathcal{N}(0, 1)\) random variables.

We now consider two applications of Theorem 2 by considering \(\tau(w) = \|w\|_{2,p}\) defined on \(H_{\tau,\Lambda}\) and \(\tau(w) = \|w\|_{p'}\) defined on \(\mathbb{R}^{d \times c}\), respectively.

**Corollary 3** (Data-dependent generalization bound for \(\ell_2,p\)-norm regularizer and Lipschitz continuous loss w.r.t. Def. 2). Consider the hypothesis space \(H_{p,\Lambda} := H_{\tau,\Lambda}\) in 4 with \(\tau(w) = \|w\|_{2,p}, p \geq 1\). If there exist \(L_1, L_2 \in \mathbb{R}_+\) such that \(\Psi_{y,\Lambda}\) is Lipschitz continuous w.r.t. a variant of \(\ell_2\) norm involving Lipschitz constants \((L_1, L_2)\) and index \(y\) for all \(y \in \mathcal{Y}\), then for any \(0 < \delta < 1/2\), the following inequality holds with probability at least \(1 - 2\delta\) (we use the abbreviation \(A_{p} = A_{\tau}\) with \(\tau(w) = \|w\|_{2,p}\))

\[
A_{p} \leq \frac{2\Lambda \sqrt{\pi}}{n} \left[ \sum_{i=1}^n K(x_i, x_i) \right] \frac{1}{\sqrt{p}} \inf_{q \geq p} \left[ L_1(q^*) + L_2(q^*) \frac{c}{\sqrt{p}} + \max(c, \frac{c}{p}) \right], \tag{8}
\]
Corollary 4 (Data-dependent generalization bound for Schatten-$p$ norm regularizer and Lipschitz continuous loss w.r.t. Def. 2). Let $\phi$ be the identity map and represent $w$ by a matrix $W \in \mathbb{R}^{d \times c}$.

Consider the hypothesis space $H_{\Psi, p} := H_{\tau, A}$ in (H) with $\tau(W) = \|W\|_{S_p, p} \geq 1$. If there exist $L_1, L_2 \in \mathbb{R}_+$ such that $\Psi_y$ is Lipschitz continuous w.r.t. a variant of $\ell_2$ norm involving Lipschitz constants $(L_1, L_2)$ and index $y$ for all $y \in \mathcal{Y}$, then for any $0 < \delta < 1/2$ with probability at least $1 - 2\delta$, we have (we use the abbreviation $A_{S_p} = A_{\tau}$ with $\tau(W) = \|W\|_{S_p}$)

$$A_{S_p} \leq \begin{cases} \frac{2 - \pi}{n} \inf_{x \in \mathcal{X}} \left\{ \left( L_1 \tau(x) + L_2 \right) \left[ \sum_{i=1}^{n} \|x_i\|_2^2 \right] + L_1 \left( \sum_{i=1}^{n} x_i w_i^{\top} \right)_S \right\}, & \text{if } p \leq 2, \\ 2 \pi \left( L_1 c_{\tau} + L_2 \right) \min_{c, d} \left[ \frac{1}{c} \right] \left( \sum_{i=1}^{n} \|x_i\|_2^2 \right)^{\frac{1}{2}}, & \text{otherwise.} \end{cases}$$

(9)

In comparison to Corollary 3, the error bound of Corollary 4 involves an additional term $O\left( \frac{c n^{-1}}{\sum_{i=1}^{n} x_i w_i^{\top} S \right)}$ for the case $p \leq 2$, which is due to the need of applying non-commutative Khintchine-Kahane inequality [48]. This mismatch, between the norms w.r.t. which the Lipschitz continuity is measured, requires an additional step of controlling the Rademacher complexity of the loss function class $F_{\tau, \lambda}$, which is due to the need of applying non-commutative Khintchine-Kahane inequality.

### 3.4 Data-independent Bounds

The data-dependent generalization bounds given in Section 3.3 assume the loss function $\Psi_y$ to be Lipschitz continuous w.r.t. a variant of $\ell_2$ norm. However, typical loss functions used in the multi-class setting are Lipschitz continuous w.r.t. the $\ell_\infty$ norm with a smaller Lipschitz constant. This mismatch, between the norms w.r.t. which the Lipschitz continuity is measured, requires an additional step of controlling the $\ell_\infty$ norm of vector-valued predictors by the $\ell_2$ norm in the application of Theorem 2, rendering a sublinear loss in the resulting dependency on the class size. This section aims to avoid this loss in the class-size dependency by giving instead data-independent bounds, using the tool of $\ell_\infty$-norm covering numbers.

The key step in this approach lies in estimating the covering numbers (and thus the resulting Rademacher complexity) of the loss function class

$$F_{\tau, \lambda} := \{ (x, y) \rightarrow \Psi_y(h^w(x)) : h^w \in H_{\tau} \}.$$  

(10)

A difficulty towards this purpose consists in the non-linearity of $F_{\tau, \lambda}$ and the fact that $h^w \in H_{\tau}$ takes vector-valued outputs, while standard analyses are limited to scalar-valued and essentially linear function classes [60, 62]. The key step in this approach lies in estimating the covering numbers (and thus the resulting Rademacher complexity) of the loss function class

$$\tilde{H}_{\tau} := \{ v \rightarrow (w, v) : w, v \in H_{\tilde{\tau}}, \tau(w) \leq \Lambda, \text{card}(v) \leq 1, \|v\|_{S_\infty} \leq B_K \}.$$  

(11)

Here, the constraint card($v$) $\leq 1$ implies that at most one of the $v_j$ in ($v_1, \ldots, v_c$) is non-zero. A key motivation in introducing $\tilde{H}_{\tau}$ is that the covering number of $F_{\tau, \lambda}$ w.r.t. $x_1, \ldots, x_n$ (covering numbers are defined in Section 5.2) can be related to the covering number of the function class

$$\{ v \rightarrow (w, v) : \tau(w) \leq \Lambda \},$$  

with an enlarged set of $nc$ examples ($\tilde{v}^1, \ldots, \tilde{v}^{nc}$) with card($v$) $\leq 1$. The latter of which can be conveniently tackled since it is a linear and scalar-valued function class, to which standard arguments apply. Theorem 5 reduces the estimation of $R_S(F_{\tau, \lambda})$ to bounding

$$R_{\lambda_{nc}}(\tilde{H}_{\tau})$$  

based on which the data-independent error bounds are given in Theorem 6. We present the proofs of Theorems 5 and 6 and Corollaries 9 and 10 in Section 5.2.

**Theorem 5** (Data-independent (worst-case) Rademacher complexity bound). Suppose that $\Psi_y$ is $L$-Lipschitz continuous w.r.t. the $\ell_\infty$ norm for any $y \in \mathcal{Y}$ and assume that $B_{\Psi} \leq 2cBnL$. Then the Rademacher complexity of $F_{\tau, \lambda}$ can be bounded by

$$R_S(F_{\tau, \lambda}) \leq 27 \sqrt{c} L R_{\lambda_{nc}}(\tilde{H}_{\tau}) \left( 1 + \left( \frac{3cBn \sqrt{c}}{10 R_{\lambda_{nc}}(\tilde{H}_{\tau})} \right)^2 \right).$$

(10)
**Theorem 6** (General data-independent generalization bound in terms of worst-case Rademacher complexity). Under the condition of Theorem 3 for any $0 < \delta < 1$, with probability at least $1 - \delta$, we have

$$A_\tau \leq 54\sqrt{c}LR_{\text{nc}}(H_\tau) \left( 1 + \left( \log \frac{3eBn\sqrt{c}}{10R_{\text{nc}}(H_\tau)} \right)^{\frac{3}{2}} \right).$$

The application of Theorem 6 requires to control the worst-case Rademacher complexity of the linear function class $H_\tau$ from both below and above, to which the following two propositions give respective tight estimates for $\tau(w) = \|w\|_{2,p}$ defined on $H^K_\tau$ [29] and $\tau(W) = \|W\|_{S_p}$ defined on $\mathbb{R}^{d_{\times c}}$ [54].

**Proposition 7** (Lower and upper bound on worst-case Rademacher complexity for block $\ell_{2,p}$-norm regularizer). For $\tau(w) = \|w\|_{2,p}, p \geq 1$ in (11), the function class $H_\tau$ becomes

$$\tilde{H}_p := \{v \rightarrow \langle w, v \rangle : w, v \in H^K_\tau, \|w\|_{2,p} \leq \Lambda, \operatorname{card}(v) \leq 1, \|v\|_{2,\infty} \leq B_K \}.$$

The Rademacher complexity of $\tilde{H}_p$ can be upper and lower bounded by

$$\Lambda B_K (2n)^{-\frac{1}{2}} c^{-\frac{1}{2\min(1, p)}} \leq R_{\text{nc}}(\tilde{H}_p) \leq \Lambda B_K n^{-\frac{1}{2}} c^{-\frac{1}{2\min(1, p)}}, \quad (12)$$

**Proposition 8** (Lower and upper bound on worst-case Rademacher complexity for Schatten-$p$ norm regularizer). Let $\phi$ be the identity map and represent $w$ by a matrix $W \in \mathbb{R}^{d_{\times c}}$. For $\tau(W) = \|W\|_{S_p}, p \geq 1$ in (11), the function class $H_\tau$ becomes

$$\tilde{H}_{S_p} := \{V \rightarrow \langle W, V \rangle : W, V \in \mathbb{R}^{d_{\times c}}, \|W\|_{S_p} \leq \Lambda, \operatorname{card}(V) \leq 1, \|V\|_{2,\infty} \leq B_K \}.$$

The Rademacher complexity of $\tilde{H}_{S_p}$ can be upper and lower bounded by

$$\begin{cases} \Lambda B_K (2nc)^{-\frac{1}{2}} \leq R_{\text{nc}}(\tilde{H}_{S_p}) \leq \Lambda B_K (nc)^{-\frac{1}{2}}, & \text{if } p = 2, \\ \Lambda B_K \min(c,d)^{-\frac{1}{2}} \leq R_{\text{nc}}(\tilde{H}_{S_p}) \leq \Lambda B_K \min(c,d)^{-\frac{1}{2}}, & \text{otherwise.} \end{cases} \quad (14)$$

The proofs of Propositions 7 and 8 which exploit the fact that $H_\tau$ is a class of functions defined on the set $\{v \in H^K_\tau : \operatorname{card}(v) \leq 1, \|v\|_{2,\infty} \leq B_K \}$, are given in Section 5.3 and Appendix B, respectively.

The associated data-independent error bounds, given in Corollary 9 and Corollary 10, are then immediate.

**Corollary 9** (Data-independent generalization bound for block $\ell_{2,p}$-norm regularizer and Lipschitz continuous loss w.r.t. $\ell_\infty$ norm). Consider the hypothesis space $H_{p,\Lambda} := H_{\tau,\Lambda}$ in (11) with $\tau(w) = \|w\|_{2,p}, p \geq 1$. Assume that $\Psi_y$ is $L$-Lipschitz continuous w.r.t. $\ell_\infty$ norm for any $y \in \mathcal{Y}$ and $B_\Psi \leq 2eB\text{nc}L$. Then, for any $0 < \delta < 1$, with probability $1 - \delta$, we have

$$A_p \leq \frac{54L\Lambda B_K c^{\frac{1}{2}}}{\sqrt{n}} \left( 1 + \log^\frac{3}{2} \left( en^{\frac{1}{2}} c^{\frac{1}{2}} \frac{\max(1, p)}{\max(2, p)} \right) \right).$$

**Corollary 10** (Data-independent generalization bound for Schatten-$p$ norm regularizer and Lipschitz continuous loss w.r.t. $\ell_\infty$ norm). Let $\phi$ be the identity map and represent $w$ by a matrix $W \in \mathbb{R}^{d_{\times c}}$. Consider the hypothesis space $H_{S_p,\Lambda} := H_{\tau,\Lambda}$ in (11) with $\tau(W) = \|W\|_{S_p}, p \geq 1$. Assume that $\Psi_y$ is $L$-Lipschitz continuous w.r.t. $\ell_\infty$ norm for any $y \in \mathcal{Y}$ and $B_\Psi \leq 2eB\text{nc}L$. Then, for any $0 < \delta < 1$, with probability $1 - \delta$, we have

$$A_{S_p} \leq \begin{cases} \frac{54L\Lambda B_K}{\sqrt{n}} \left( 1 + \log^\frac{3}{2} \left( \frac{\max(c,d)}{\sqrt{2}} \right) \right), & \text{if } p \leq 2, \\ \frac{54L\Lambda B_K \min(c,d)^{\frac{1}{2}} \frac{1}{c}}{\sqrt{n}} \left( 1 + \log^\frac{3}{2} \left( en^{\frac{1}{2}} c \right) \right), & \text{otherwise.} \end{cases}$$
To make a clear comparison between data-dependent and data-independent error bounds, we first set the parameter $p$ in Corollaries 3, 4, 9, and 10 to $p = 2$. In this case, the data-dependent bounds of Corollaries 3 and 4 exhibit a square-root dependency on the class size, while the data-independent bounds of Corollaries 9 and 10 enjoy a much tighter, logarithmic dependency on the class size.

For the case $p = \infty$, the data-dependent bounds of Corollaries 3 and 4 enjoy a linear dependency on $c$, while the data-independent bounds of Corollaries 9 and 10 enjoy a square-root dependency on $c$ (ignoring the logarithmic factor). This mismatch between data-dependent and data-independent bounds comes from the fact that the former analysis uses the Lipschitz continuity of the loss function w.r.t. a variant of the $\ell_2$ norm, while the latter makes a better use of the Lipschitz continuity w.r.t. the $\ell_\infty$ norm.

4 Applications

This section applies the general results in Sections 3.3 and 3.4 to study both data-dependent and data-independent error bounds for some prominent multi-class learning methods.

4.1 Classic MC-SVMs

We first apply the results from the previous section to several classic MC-SVMs. For this purpose, we need to show that the associated loss functions satisfy Lipschitz conditions. To this end, for any $h : \mathcal{X} \rightarrow \mathbb{R}^c$, we denote by

$$\rho_h(x, y) := h_y(x) - \max_{y' : y' \neq y} h_{y'}(x)$$

the margin of the model $h$ at $(x, y)$. It is clear that the prediction rule $h$ makes an error at $(x, y)$ if $\rho_h(x, y) < 0$. In Examples 1, 3, and 4 below, we assume that $\ell : \mathbb{R} \rightarrow \mathbb{R}_+$ is a decreasing and $L_{\ell}$-Lipschitz function.

**Example 1** (Multi-class margin-based loss [32]). The loss function defined as

$$\Psi^\ell_y(t) := \max_{y' : y' \neq y} \ell(t_y - t_{y'}) \quad \forall t \in \mathbb{R}^c$$

is $(2L_{\ell})$-Lipschitz continuous w.r.t. $\ell_\infty$ norm and $\ell_2$ norm. Furthermore, we have $\ell(\rho_h(x, y)) = \Psi^\ell_y(h(x))$.

The loss function $\Psi^\ell_y$ defined above in Eq. (16) is a margin-based loss function widely used in multi-class classification [32] and structured prediction [9].

Next, we study the multinomial logistic loss $\Psi^m_y$, which is defined in Eq. (17) below, and which is used in multinomial logistic regression [30].

**Example 2** (Multinomial logistic loss). The multinomial logistic loss $\Psi^m_y(t)$ defined as

$$\Psi^m_y(t) := \log \left( \sum_{j=1}^c \exp(t_j - t_y) \right) \quad \forall t \in \mathbb{R}^c$$

is 2-Lipschitz continuous w.r.t. the $\ell_\infty$ norm and the $\ell_2$ norm.

Finally the loss $\bar{\Psi}^\ell_y$ defined in Eq. (18) below is used in [33] to make pairwise comparisons among components of the predictor, and the loss $\tilde{\Psi}^\ell_y$ defined in Eq. (19) is used in [34] based on constrained comparisons.

**Example 3** (Loss function used in [33]). The loss function defined as

$$\tilde{\Psi}^\ell_y(t) = \sum_{j=1}^c \ell(t_y - t_j) \quad \forall t \in \mathbb{R}^c$$
is Lipschitz continuous w.r.t. a variant of \( \ell_2 \)-norm involving the Lipschitz constant \((L_\ell \sqrt{c}, L_\ell c)\) and index \(y\). Furthermore, it is also \(2L_\ell c\)-Lipschitz continuous w.r.t. the \(\ell_\infty\) norm.

**Example 4** (Loss function used in [34]). The loss function defined as

\[
\Psi(t) = \sum_{j=1, j\neq y}^c \ell(-t_j), \quad \forall t \in \Omega = \{ \hat{t} \in \mathbb{R}^c : \sum_{j=1}^c \hat{t}_j = 0 \}
\]

is \((L_\ell \sqrt{c})\)-Lipschitz continuous w.r.t. the \(\ell_2\) norm. Furthermore, it is also \((L_\ell c)\)-Lipschitz continuous w.r.t. the \(\ell_\infty\) norm.

The following data-dependent and data-independent error bounds are immediate by plugging the Lipschitz conditions established in Examples 1, 2, 3 and 4 into Corollaries 3, 4, 9 and 10, respectively.

**Corollary 11** (Generalization bounds for Crammer and Singer MC-SVM). Consider the MC-SVM in [32] with the loss function \(\Psi_y\) (16) and the hypothesis space \(H_\tau\) with \(\tau(w) = \|w\|_{2,2}\). Let \(0 < \delta < 1/2\). Then,

(1) with probability at least \(1 - 2\delta\), we have \(A_2 \leq \frac{4L_\ell L \sqrt{c} \mathbb{E}}{n} \left[ \sum_{i=1}^n K(x_i, x_i) \right]^{1/2} \) (data-dependent);

(2) with probability at least \(1 - \delta\), we have \(A_2 \leq \frac{108L_\ell B c L}{\sqrt{n}} (1 + \log \frac{2}{\delta} (ecn^2))\) (data-independent).

Analogous to Corollary 11, we have the following corollary on error bounds for multinomial logistic regression in [30].

**Corollary 12** (Generalization bounds for multinomial logistic regression). Consider the multinomial logistic regression with the loss function \(\Psi_y\) (17) and the hypothesis space \(H_\tau\) with \(\tau(w) = \|w\|_{2,2}\). Let \(0 < \delta < 1/2\). Then,

(1) with probability at least \(1 - 2\delta\), we have \(A_2 \leq \frac{4\Lambda \sqrt{c} \mathbb{E}}{n} \left[ \sum_{i=1}^n K(x_i, x_i) \right]^{1/2} \) (data-dependent);

(2) with probability at least \(1 - \delta\), we have \(A_2 \leq \frac{108\Lambda B c L}{\sqrt{n}} (1 + \log \frac{2}{\delta} (ecn^2))\) (data-independent).

The following two corollaries give error bounds for MC-SVMs in [33, 34].

**Corollary 13** (Generalization bounds for Weston and Watkins MC-SVM). Consider the MC-SVM in Weston and Watkins [33] with the loss function \(\Psi_y\) (18) and the hypothesis space \(H_\tau\) with \(\tau(w) = \|w\|_{2,2}\). Let \(0 < \delta < 1/2\). Then,

(1) with probability at least \(1 - 2\delta\), we have \(A_2 \leq \frac{4L_\ell \Lambda c \mathbb{E}}{n} \left[ \sum_{i=1}^n K(x_i, x_i) \right]^{1/2} \) (data-dependent);

(2) with probability at least \(1 - \delta\), we have \(A_2 \leq \frac{108L_\ell B c L}{\sqrt{n}} (1 + \log \frac{2}{\delta} (ecn^2))\) (data-independent).

**Corollary 14** (Generalization bounds for Lee et al. MC-SVM). Consider the MC-SVM in Lee et al. [32] with the loss function \(\Psi_y\) (19) and the hypothesis space \(H_\tau\) with \(\tau(w) = \|w\|_{2,2}\). Let \(0 < \delta < 1/2\). Then,

(1) with probability at least \(1 - 2\delta\), we have \(A_2 \leq \frac{2L_\ell \Lambda c \mathbb{E}}{n} \left[ \sum_{i=1}^n K(x_i, x_i) \right]^{1/2} \) (data-dependent);

(2) with probability at least \(1 - \delta\), we have \(A_2 \leq \frac{54L_\ell B c L}{\sqrt{n}} (1 + \log \frac{2}{\delta} (ecn^2))\) (data-independent).
It is interesting to compare the above generalization bounds with the best known results in the literature. To start with, the data-dependent error bounds of Corollary 11 exhibit a square-root dependency on the class size, matching the state of the art result from the conference version of this paper [29]. On the other hand, the corresponding data-independent bound of Corollary 11 enjoys even a logarithmic dependency on the class size, which improves the previously best bound by Zhang [48], which scales as $O(n^{-\frac{1}{2}}c^\frac{5}{2})$.

Now consider the generalization bound of Corollary 13 for the MC-SVM by Weston and Watkins [33], which scales linear in $c$. On the other hand, according to Example 3 it is evident that $\Psi_k^f$ is $(c + \sqrt{c})L_2$-Lipschitz continuous w.r.t. $\ell_2$ norm, for any $y \in \mathcal{Y}$. Therefore, when using the recent structural result [3] by Maurer [44] and Cortes et al. [45] to derive a data-dependent error bound for Weston and Watkins [32], one obtains a scaling in $c$ of $O(c^\frac{7}{2})$, which is worse than our bound given by Corollary 13.

Note that for the MC-SVMs by Weston and Watkins [33], Lee et al. [34], both the data-dependent and data-independent bounds enjoy a linear dependency (ignoring logarithmic factors) on the class size.

### 4.2 Top-$k$ MC-SVM

Motivated by the ambiguity in the class labels caused by the rapid increase in class size in modern computer vision benchmarks, Lapin et al. [31, 63] introduce the top-$k$ MC-SVM by using the top-$k$ hinge loss to allow for $k$ predictions for each object $x$. For any $t \in \mathbb{R}^c$, let the bracket $[\cdot]$ denote a permutation such that $[j]$ is the index of the $j$-th largest score, i.e., $t_{[1]} \geq t_{[2]} \geq \cdots \geq t_{[c]}$.

**Example 5** (Top-$k$ hinge loss [31]). The top-$k$ hinge loss defined by

$$
\Psi_k^f(t) = \max \left\{ 0, \frac{1}{k} \sum_{j=1}^{k} (1_{y \neq 1} + t_y - t_{y_j}) \right\}, \quad \forall t \in \mathbb{R}^c
$$

(20)

is Lipschitz continuous w.r.t. a variant of $\ell_2$-norm involving Lipschitz constants $(\frac{1}{\sqrt{k}}, 1)$ and index $y$. Furthermore, it is also 2-Lipschitz continuous w.r.t. $\ell_\infty$ norm.

With the Lipschitz conditions established in Example 5, we are now able to give generalization error bounds for top-$k$ MC-SVM [31].

**Corollary 15** (Generalization bounds for top-$k$ MC-SVM). Consider the top-$k$ MC-SVM with the loss function (20) and the hypothesis space $H_T$ with $\tau(w) = \|w\|_{2,2}$. Let $0 < \delta < 1/2$. Then,

1. with probability at least $1 - 2\delta$, we have $A_2 \leq \frac{2A_1\sqrt{\pi}}{\sqrt{k}}(c^{\frac{4}{5}}k^{-\frac{1}{5}} + 1)\sum_{i=1}^{n} K(x_i, x_i))^{\frac{1}{2}}$ (data-dependent);
2. with probability at least $1 - \delta$, we have $A_2 \leq \frac{10B_2\sqrt{\pi}}{\sqrt{k}}(1 + \log^2(en^2c))$ (data-independent).

An appealing property of Corollary 15 is the involvement of the factor $k^{-\frac{4}{5}}$ in the data-dependent error bounds. Note that we even can get class-size independent error bounds if we choose $k > \hat{C}c$ for a universal constant $\hat{C}$.

Comparing our result to the state of the art, it follows again from Example 5 that $\Psi_k^f$ is $(1 + k^{-\frac{4}{5}})$-Lipschitz continuous w.r.t. the $\ell_2$ norm for all $y \in \mathcal{Y}$, so using the recent structural result [3] by Maurer [44], Cortes et al. [45], one can derive a data-dependent error bound decaying as $O(n^{-\frac{1}{2}}c^\frac{5}{2} \sum_{i=1}^{n} K(x_i, x_i))^{\frac{5}{2}}$. This error bound is suboptimal to Corollary 15 since it does not shed insights on how the parameter $k$ would affect the generalization performance. On the other hand, our data-dependent error bound enjoys a logarithmic dependency on the class size.

### 4.3 $\ell_p$-norm MC-SVMs

In our previous work [29], we introduce the $\ell_p$-norm MC-SVM as an extension of the MC-SVM by Crummer and Singer [32]. The model is obtained by replacing the associated $\ell_{2,2}$-norm regularizer with
a general block $\ell_{2,p}$-norm regularizer \[29\]. We recover data-dependent error bounds in \[29\], showing a logarithmic dependency on the class size as $p$ decreases to 1. The present analysis yields the following bounds.

**Corollary 16** (Generalization bounds for $\ell_p$-norm MC-SVM). Consider the $\ell_p$-norm MC-SVM with loss function \[16\] and the hypothesis space $H_r$ with $\tau(\mathbf{w}) = ||\mathbf{w}||_{2,p}, p \geq 1$. Let $0 < \delta < 1/2$. Then,

1. with probability at least $1 - 2\delta$, we have:
   \[ A_p \leq \frac{4L_A \sqrt{\pi}}{n} \left( \sum_{i=1}^{n} K(\mathbf{x}_i, \mathbf{x}_i) \right) \frac{1}{2} \inf_{q \geq p} \left( \frac{1}{q} \right)^{1/p} c^{1/p} \] (data-dependent);

2. with probability at least $1 - \delta$, we have:
   \[ A_p \leq \frac{108L_AB_K \frac{4}{p^{\frac{1}{p} - \frac{1}{p^2}}}}{\sqrt{n}} \left( 1 + \log \frac{2}{\log c} \left( \frac{2c}{n} c^{\frac{1}{p}} \right) \right) \] (data-independent).

**Remark 1** (Analysis of the case $p \approx 1$). The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $f(t) = \frac{1}{t^{\frac{1}{p}}}$ is monotonically decreasing on the interval $(0, 2 \log c)$ and increasing on the interval $(2 \log c, \infty)$. Therefore, the data-dependent error bounds in Corollary \[16\] transfers to

\[ A_p \leq \begin{cases} 
\frac{4L_A \sqrt{\pi} n^{1-1/p} c^{1/p}}{n} \left( \sum_{i=1}^{n} K(\mathbf{x}_i, \mathbf{x}_i) \right) \frac{1}{2}, & \text{if } p < 2 \log c, \\
\frac{4L_A \sqrt{\pi} n^{1-1/p} c^{1/p}}{n} \left( \sum_{i=1}^{n} K(\mathbf{x}_i, \mathbf{x}_i) \right) \frac{1}{2}, & \text{otherwise}.
\end{cases} \]

That is, the dependency on the class size would be polynomial with exponent $1/p$ if $p < 2 \log c$ (or equivalently $p > \frac{2 \log c}{2 \log c - 1}$) and logarithmic otherwise. As a comparison, the data-independent error bounds would enjoy a logarithmic dependency on the class size if $p \leq 2$ and a polynomial dependency with exponent $\frac{1}{2} - \frac{1}{p}$ otherwise (ignoring logarithmic factors).

### 4.4 Schatten-$p$ norm MC-SVMs

Amit et al. \[58\] propose to use trace-norm regularization in multi-class classification to uncover shared structures always existing in the learning regime with a huge number of classes. Here we consider error bounds for the more general Schatten-$p$ norm MC-SVM.

**Corollary 17** (Generalization bounds for Schatten-$p$ norm MC-SVM). Let $\phi$ be the identity map and represent $\mathbf{w}$ by a matrix $W \in \mathbb{R}^{d \times c}$. Consider Schatten-$p$ norm MC-SVM with loss functions \[16\] and the hypothesis space $H_r$ with $\tau(W) = ||W||_{S_p}, p \geq 1$. Then,

1. with probability at least $1 - 2\delta$, we have the data-dependent bound
   \[ A_{S_p} \leq \begin{cases} 
\frac{2^{\frac{2}{3} + \frac{3}{2}}}{n^{1/2}} \inf_{q \leq 2} \left( q^{1/p} \right) \left[ \sum_{i=1}^{n} \|x_i\|_2^2 \right] \frac{1}{2} + c^{1/p} \left[ \sum_{i=1}^{n} x_i x_i^\top \right] \frac{1}{2}, & \text{if } p \leq 2, \\
\frac{2^{\frac{2}{3} + \frac{3}{2}}}{n^{1/2}} \inf_{q \leq 2} \left( q^{1/p} \right) \left[ \sum_{i=1}^{n} \|x_i\|_2^2 \right] \frac{1}{2}, & \text{otherwise}.
\end{cases} \]

2. with probability at least $1 - \delta$, we have the data-independent bound
   \[ A_{S_p} \leq \begin{cases} 
\frac{108L_A B_K}{\sqrt{n}} \left( 1 + \log \frac{2}{\log c} \left( \frac{2c}{n} c^{1/p} \right) \right), & \text{if } p \leq 2, \\
\frac{108L_A B_K \frac{2}{\log c}}{\sqrt{n}} \left( 1 + \log \frac{2}{\log c} (en c) \right), & \text{otherwise}.
\end{cases} \]
Remark 2 (Analysis of the case \( p \approx 1 \)). Analogous to Remark 1, the data-dependent bounds of Corollary 17 transfer to

\[
O(n^{-1}(p^*)^{\epsilon^*} \left( c^* \| \sum_{i=1}^n |x_i|^2 \|^{2 \over 2} + c^* \| \sum_{i=1}^n x_i \sum_{i=1}^n x_i^T \|^{2 \over 2} \right), \quad \text{if } 2 \leq p^* \leq 2 \log c,
\]

\[
O(n^{-1} \sqrt{\log c} \left( c^* \| \sum_{i=1}^n |x_i|^2 \|^{2 \over 2} + c^* \| \sum_{i=1}^n x_i \sum_{i=1}^n x_i^T \|^{2 \over 2} \right), \quad \text{if } 2 < 2 \log c < p^*,
\]

\[
O(n^{-1} \epsilon^* \| \sum_{i=1}^n |x_i|^2 \|), \quad \text{if } p > 2.
\]

The associated data-independent bounds would decay as \( O(n^{-{1 \over 2}} \log^{2 \over 2} (n^3 \epsilon c)) \) if \( p \leq 2 \) and \( O(n^{-{1 \over 2}} \min(c, d) \log^{2 \over 2} (n^3 \epsilon c)) \) otherwise.

5 Proofs

In this section, we present the proofs of the results presented in the previous sections.

5.1 Proof of Data-dependent Bounds

This section presents the proofs for data-dependent bounds in Section 3.3. The proof of Lemma 1 requires to use a comparison result (Lemma [18] on Gaussian processes attributed to Slepian [43], while the proof of Theorem 2 is based on a concentration inequality in [64].

Lemma 18. Let \( \{ \mathcal{X}_\theta : \theta \in \Theta \} \) and \( \{ \mathcal{Y}_\theta : \theta \in \Theta \} \) be two mean-zero separable Gaussian processes indexed by the same set \( \Theta \) and suppose that

\[
E[(\mathcal{X}_\theta - \mathcal{X}_\bar{\theta})^2] \leq E[(\mathcal{Y}_\theta - \mathcal{Y}_\bar{\theta})^2], \quad \forall \theta, \bar{\theta} \in \Theta.
\]

Then, \( E[\sup_{\theta \in \Theta} \mathcal{X}_\theta] \leq E[\sup_{\theta \in \Theta} \mathcal{Y}_\theta] \).

Lemma 19 (McDiarmid inequality). Let \( Z_1, \ldots, Z_n \) be independent random variables taking values in a set \( \mathcal{Z} \), and assume that \( f : \mathcal{Z}^n \to \mathbb{R} \) satisfies

\[
\sup_{z_1, \ldots, z_n, z_i \in \mathcal{Z}} |f(z_1, \ldots, z_i, \ldots, z_n) - f(z_1, \ldots, z_i, \ldots, z_n)| \leq c_i
\]

for \( 1 \leq i \leq n \). Then, for any \( 0 \leq \delta \leq 1 \), with probability at least \( 1 - \delta \), we have

\[
f(Z_1, \ldots, Z_n) \leq \mathbb{E} f(Z_1, \ldots, Z_n) + \sqrt{\sum_{i=1}^n c_i^2 \log(1/\delta)}.
\]

Proof of Lemma 18 Define two mean-zero separable Gaussian processes indexed by the finite dimensional Euclidean space \( \{ (h(x_1), \ldots, h(x_n)) : h \in H \} \)

\[
\mathcal{X}_h := \sum_{i=1}^n g_i f_i(h(x_i)), \quad \mathcal{Y}_h := \sqrt{2} L_1 \sum_{i=1}^n \sum_{j=1}^c g_{ij} h_j(x_i) + \sqrt{2} L_2 \sum_{i=1}^n g_i h_{r(i)}(x_i).
\]

For any \( h, h' \in H \), the independence among the \( g_i, g_{ij} \) and \( \mathbb{E} g_i^2 = 1, \mathbb{E} g_{ij}^2 = 1, \forall i \in \mathbb{N}_n, j \in \mathbb{N}_c \) imply that

\[
E[(\mathcal{X}_h - \mathcal{X}_{h'})^2] = E \left[ \left( \sum_{i=1}^n g_i (f_i(h(x_i)) - f_i(h'(x_i))) \right)^2 \right] = \sum_{i=1}^n [f_i(h(x_i)) - f_i(h'(x_i))]^2
\]

\[
\leq \sum_{i=1}^n \left[ L_1 \left( \sum_{j=1}^c [h_j(x_i) - h'(x_i)]^2 \right)^{1 \over 2} + L_2 [h_{r(i)}(x_i) - h'_{r(i)}(x_i)]^2 \right]
\]

\[
\leq 2L_1^2 \sum_{i=1}^n \sum_{j=1}^c [h_j(x_i) - h'(x_i)]^2 + 2L_2^2 \sum_{i=1}^n [h_{r(i)}(x_i) - h'_{r(i)}(x_i)]^2
\]

\[= E[(\mathcal{Y}_h - \mathcal{Y}_{h'})^2],
\]

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where we have used the Lipschitz continuity of \( f_i \) w.r.t. a variant of \( \ell_2 \) norm in the first inequality, the elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\) in the second inequality. Therefore, the condition (21) holds and Lemma 13 can be applied here to give

\[
\mathbb{E}_g \sup_{h \in H} \sum_{i=1}^{n} g_i f_i(h(x_i)) \leq \mathbb{E}_g \sup_{h \in H} \left[ \sqrt{2} L_1 \sum_{i=1}^{n} \sum_{j=1}^{c} g_{ij} h_j(x_i) + \sqrt{2} L_2 \sum_{i=1}^{n} g_i h_{ri}(x_i) \right]
\leq \sqrt{2} L_1 \mathbb{E}_g \sup_{h \in H} \sum_{i=1}^{n} \sum_{j=1}^{c} g_{ij} h_j(x_i) + \sqrt{2} L_2 \mathbb{E}_g \sup_{h \in H} \sum_{i=1}^{n} g_i h_{ri}(x_i).
\]

The proof of Lemma 13 is complete.

**Proof of Theorem 3.** According to McDiarmid’s inequality (Lemma 19) and the symmetrization technique (e.g., Theorem 4.4 in [9]), with probability at least 1 – 2\( \delta \), we have

\[
\sup_{h^w \in H_r} \mathbb{E}_{x,y} \mathbb{E}_g \Psi_g(h^w(x)) - \frac{1}{n} \sum_{i=1}^{n} \Psi_g(h^w(x_i)) \leq 2 \mathcal{R}_S(\{\Psi_g(h^w(x)) : h^w \in H_r\}) + 3B \Phi \sqrt{\frac{\log 2}{2n}}. \tag{23}
\]

Furthermore, according to the following relationship between Gaussian and Rademacher processes for any function class \( \bar{H} \) [e.g., 38]

\[
\mathcal{R}_S(\bar{H}) \leq \sqrt{\frac{\pi}{2}} \mathcal{G}_S(\bar{H}) \leq 3 \sqrt{\frac{\pi \log |S|}{2}} \mathcal{G}_S(\bar{H}),
\]

we derive

\[
\mathcal{R}_S(\{\Psi_g(h^w(x)) : h^w \in H_r\}) \leq \sqrt{\frac{\pi}{2}} \mathcal{G}_S(\{\Psi_g(h^w(x)) : h^w \in H_r\})
\leq \frac{L_1 \sqrt{\pi}}{n} \mathbb{E}_g \sup_{h^w \in H_r} \sum_{i=1}^{n} \sum_{j=1}^{c} g_{ij} h_j^w(x_i) + \frac{L_2 \sqrt{\pi}}{n} \mathbb{E}_g \sup_{h^w \in H_r} \sum_{i=1}^{n} g_i h_{ri}^w(x_i),
\]

where the last step follows from Lemma 13 with \( f_i = \Psi_{y_i} \) and \( r(i) = y_i, \forall i \in \mathbb{N}_n \). Plugging the above Rademacher complexity bound into (23) gives the following inequality with probability at least 1 – 2\( \delta \)

\[
A_r \leq \frac{2L_1 \sqrt{\pi}}{n} \mathbb{E}_g \sup_{h^w \in H_r} \sum_{i=1}^{n} \sum_{j=1}^{c} g_{ij} h_j^w(x_i) + \frac{2L_2 \sqrt{\pi}}{n} \mathbb{E}_g \sup_{h^w \in H_r} \sum_{i=1}^{n} g_i h_{ri}^w(x_i). \tag{24}
\]

It remains to estimate the two terms on the right-hand side of (24). According to the definition of dual norm, we derive

\[
\mathbb{E}_g \sup_{h^w \in H_r} \sum_{i=1}^{n} \sum_{j=1}^{c} g_{ij} h_j^w(x_i) = \mathbb{E}_g \sup_{h^w \in H_r} \sum_{i=1}^{n} \sum_{j=1}^{c} g_{ij} \langle w_j, \phi(x_i) \rangle = \mathbb{E}_g \sup_{h^w \in H_r} \sum_{j=1}^{c} \langle w_j, \sum_{i=1}^{n} g_{ij} \phi(x_i) \rangle
\leq \mathbb{E}_g \sup_{h^w \in H_r} \| w \| \| (\sum_{i=1}^{n} g_{ij} \phi(x_i))_{j=1}^{c} \|_* = \Lambda \mathbb{E}_g \left\| \sum_{i=1}^{n} g_{ij} \phi(x_i) \right\|_*. \tag{25}
\]

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Analogously, we also have
\[
\mathbb{E}_g \sup_{h^w \in H} \sum_{i=1}^n g_i h^{w_i} (x_i) = \mathbb{E}_g \sup_{k^w \in H^r} \sum_{i=1}^n g_i (w_{g_i}, \phi(x_i)) = \mathbb{E}_g \sup_{h^w \in H^r} \sum_{j=1}^c (w_j, \sum_{i \in I_j} g_i \phi(x_i))
\]
\[
= \mathbb{E}_g \sup_{h^w \in H^r} \left( w_i \left( \sum_{i \in I_j} g_i \phi(x_i) \right)_{j=1}^c \right)
\]
\[
\leq \Lambda \mathbb{E}_g \left\| \left( \sum_{i \in I_j} g_i \phi(x_i) \right)_{j=1}^c \right\|_*. 
\]
Plugging the above two inequalities back into (24) gives the stated bound (7).

Proof of Corollary 3. Let \( q \geq p \) be any real number. It follows from Jensen’s inequality and Khintchine-Kahane inequality (A.1) that
\[
\mathbb{E}_g \left\| \left( \sum_{i=1}^n g_i \phi(x_i) \right)_{j=1}^c \right\|_2^{q^*} = \mathbb{E}_g \left[ \sum_{j=1}^c \left\| \sum_{i=1}^n g_i \phi(x_i) \right\|_2^{q^*} \right]^{\frac{1}{q^*}} \leq \left[ \sum_{j=1}^c \mathbb{E}_g \left\| \sum_{i=1}^n g_i \phi(x_i) \right\|_2^{q^*} \right]^{\frac{1}{q^*}} \leq \left[ \sum_{j=1}^c \sum_{i=1}^n \left\| \phi(x_i) \right\|_2^{q^*} \right]^{\frac{1}{q^*}} = c^{\frac{1}{q^*}} \left[ \sum_{i=1}^n K(x_i, x_i) \right]^{\frac{1}{q^*}}. 
\]
(25)
Applying again Jensen’s inequality and Khintchine-Kahane inequality (A.1), we get
\[
\mathbb{E}_g \left\| \left( \sum_{i \in I_j} g_i \phi(x_i) \right)_{j=1}^c \right\|_2^{q^*} \leq \mathbb{E}_g \left( \sum_{j=1}^c \left\| \sum_{i \in I_j} g_i \phi(x_i) \right\|_2^{q^*} \right)^{\frac{1}{q^*}} \leq \sqrt{q^*} \left[ \sum_{j=1}^c \left\| \phi(x_i) \right\|_2^{q^*} \right]^{\frac{1}{q^*}}. 
\]
(26)
We now control the last term in the above inequality by distinguishing whether \( q \geq 2 \) or not. If \( q \leq 2 \), we have \( 2^{-1} q^* \geq 1 \) and it follows from the elementary inequality \( (a + b)^s \leq (a + b)^s, \forall a, b \geq 0, s \geq 1 \) that
\[
\sum_{j=1}^c \left[ \sum_{i \in I_j} K(x_i, x_i) \right]^{\frac{1}{q^*}} \leq \left[ \sum_{j=1}^c \sum_{i \in I_j} K(x_i, x_i) \right]^{\frac{1}{q^*}} = \left[ \sum_{i=1}^n K(x_i, x_i) \right]^{\frac{1}{q^*}}. 
\]
(27)
Otherwise we have \( 2^{-1} q^* \leq 1 \) and Jensen’s inequality implies
\[
\sum_{j=1}^c \left[ \sum_{i \in I_j} K(x_i, x_i) \right]^{\frac{1}{q^*}} \leq c \left[ \sum_{j=1}^c \frac{1}{c} \sum_{i \in I_j} K(x_i, x_i) \right]^{\frac{1}{q^*}} = c^{1-\frac{1}{q^*}} \left[ \sum_{i=1}^n K(x_i, x_i) \right]^{\frac{1}{q^*}}. 
\]
(28)
Combining (26), (27) and (28) together implies
\[
\mathbb{E}_g \left\| \left( \sum_{i \in I_j} g_i \phi(x_i) \right)_{j=1}^c \right\|_2^{q^*} \leq \max(c^{\frac{1}{q^*}-\frac{1}{q^*}}, 1) \left[ q^* \sum_{i=1}^n K(x_i, x_i) \right]^{\frac{1}{q^*}}. 
\]
(29)
According to the monotonicity of \( \| \cdot \|_{2,p} \) w.r.t. \( p \), we have \( H_{p, \Lambda} \subset H_{q, \Lambda} \) if \( p \leq q \). Plugging the complexity bound established in Eqs. (25), (29) into the generalization bound given in Theorem 2 we get the following inequality with probability at least \( 1 - 2\delta \)
\[
A_f \leq \frac{2\Lambda \sqrt{p}}{n} \left[ L_1 c^{\frac{1}{q^*}} \left[ q^* \sum_{i=1}^n K(x_i, x_i) \right]^{\frac{1}{q^*}} + L_2 \max(c^{\frac{1}{q^*}-\frac{1}{q^*}}, 1) \left[ q^* \sum_{i=1}^n K(x_i, x_i) \right]^{\frac{1}{q^*}} \right], \forall q \geq p. 
\]
The proof is complete.

Remark 3 (Tightness of the Rademacher Complexity Bound). Eq. (25) gives an upper bound on \( \mathbb{E}_g \left\| \left( \sum_{i=1}^n g_i \phi(x_i) \right)_{j=1}^c \right\|_2^{q^*} \). We now show that this bound is tight up to a constant of factor. Indeed, according to the elementary inequality
\[
(a_1 + \cdots + a_c)^{\frac{1}{q^*}} \geq c^{\frac{1}{q^*}-1} (a_1^{\frac{1}{q^*}} + \cdots + a_c^{\frac{1}{q^*}}),
\]
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we derive
\[
\left\| \left( \sum_{i=1}^{n} g_{ij} \phi(x_i) \right)^c \right\|_{2,q^*}^c = \left[ \sum_{j=1}^{c} \left\| \sum_{i=1}^{n} g_{ij} \phi(x_i) \right\|_{2}^{q^*} \right]^{\frac{c}{q^*}} \geq c^{\frac{c}{q^*} - 1} \sum_{j=1}^{c} \left\| \sum_{i=1}^{n} g_{ij} \phi(x_i) \right\|_{2}^{q^*}.
\]
Taking expectations on both sides, we get that
\[
\mathbb{E}_g \left\| \left( \sum_{i=1}^{n} g_{ij} \phi(x_i) \right)^c \right\|_{2,q^*}^c \geq c^{\frac{c}{q^*} - 1} \sum_{j=1}^{c} \mathbb{E}_g \left\| \sum_{i=1}^{n} g_{ij} \phi(x_i) \right\|_{2} \geq 2^{-\frac{1}{q^*}} c^{\frac{c}{q^*}} \left\| \sum_{j=1}^{n} K(x_i, x_i) \right\|_{q^*},
\]
where the second inequality is due to \(\text{A.2}\). The above lower bound coincides with the upper bound \(\text{A.25}\) up to a constant factor. Specifically, the above upper and lower bounds show that \(\mathbb{E}_g \left\| \left( \sum_{i=1}^{n} g_{ij} \phi(x_i) \right)^c \right\|_{2,q^*}^c\) enjoys exactly a square-root dependency on the class size if \(q = 2\) and a linear dependency if \(q = \infty\).

**Proof of Corollary 2** We first consider the case \(1 \leq p < q \leq 2\). Let \(q \in \mathbb{R}\) satisfy \(p \leq q \leq 2\). Denote \(\tilde{X}_i^j = (0, \ldots, 0, x_i, 0, \ldots, 0)\) with the \(j\)-th column being \(x_i\). Then, we have
\[
\left( \sum_{i=1}^{c} \sum_{j=1}^{n} \sigma_{q} g_{ij} \tilde{X}_i^j \right)^c = \sum_{i=1}^{c} \sum_{j=1}^{n} \sigma_{q} g_{ij} \tilde{X}_i^j \quad \text{and} \quad \left( \sum_{i \in I_1} \sum_{j=1}^{n} g_{ij} \tilde{X}_i^j \right) = \sum_{i \in I_1} \sum_{j=1}^{n} g_{ij} \tilde{X}_i^j.
\]
Since \(q^* \geq 2\), we can apply Jensen’s inequality and Khintchine-Kahane inequality \(\text{A.3}\) to derive (recall \(\sigma_{q}(X)\) denotes the \(r\)-th singular value of \(X\))
\[
\mathbb{E}_g \left\| \sum_{i=1}^{n} \sum_{j=1}^{c} g_{ij} \tilde{X}_i^j \right\|_{S_{q^*}} \leq \left[ \mathbb{E}_g \sum_{i=1}^{c} \sigma_{q} \left( \sum_{j=1}^{n} g_{ij} \tilde{X}_i^j \right) \right] \leq 2^{-\frac{1}{q^*}} \sqrt{\frac{n}{c}} \max \left\{ \left\| \sum_{i=1}^{n} \left( \sum_{j=1}^{c} \tilde{X}_i^j \right)^{\top} \tilde{X}_i^j \right\|_{S_{q^*}}^2, \left\| \sum_{i=1}^{c} \sum_{j=1}^{n} \tilde{X}_i^j (\tilde{X}_i^j)^{\top} \right\|_{S_{q^*}}^2 \right\}. \quad (31)
\]
For any \(u = (u_1, \ldots, u_c) \in \mathbb{R}^c\), we denote by \(\text{diag}(u)\) the diagonal matrix in \(\mathbb{R}^{c \times c}\) with the \(j\)-th diagonal element being \(u_j\). The following identities can be directly checked
\[
\sum_{i=1}^{n} \sum_{j=1}^{c} (\tilde{X}_i^j)^{\top} (\tilde{X}_i^j) = \sum_{i=1}^{n} \sum_{j=1}^{c} \| \tilde{X}_i^j \|_{2}^2 \text{diag}(\epsilon_j) = \sum_{i=1}^{n} \| \tilde{x}_i \|_{2}^2 I_{c \times c},
\]
\[
\sum_{i=1}^{n} \sum_{j=1}^{c} (\tilde{X}_i^j)(\tilde{X}_i^j)^{\top} = \sum_{i=1}^{n} \sum_{j=1}^{c} \tilde{x}_i \tilde{x}_i^{\top} = c \sum_{i=1}^{n} \tilde{x}_i \tilde{x}_i^{\top},
\]
where \(\epsilon_j\) is the \(j\)-th unit vector with the \(j\)-th component being 0 and \(I_{c \times c}\) is the identity matrix in \(\mathbb{R}^{c \times c}\). Therefore,
\[
\left\| \sum_{i=1}^{n} \sum_{j=1}^{c} (\tilde{X}_i^j)^{\top} (\tilde{X}_i^j) \right\|_{S_{q^*}}^2 = \left\| \left( \sum_{i=1}^{n} \| \tilde{x}_i \|_{2}^2 \right) I_{c \times c} \right\|_{S_{q^*}}^2 = c \left\| \sum_{i=1}^{n} \| \tilde{x}_i \|_{2}^2 \right\|_{q^*}^2, \quad (32)
\]
and
\[
\left\| \sum_{i=1}^{n} \sum_{j=1}^{c} (\tilde{X}_i^j)(\tilde{X}_i^j)^{\top} \right\|_{S_{q^*}}^2 = \sqrt{c} \left\| \left( \sum_{i=1}^{n} \tilde{x}_i \tilde{x}_i^{\top} \right) \right\|_{S_{q^*}}^2 = \sqrt{c} \left\| \sum_{i=1}^{n} \sigma_{q^*} \left( \sum_{i=1}^{n} \tilde{x}_i \tilde{x}_i^{\top} \right) \right\|_{S_{q^*}}^2 = \sqrt{c} \left\| \sum_{i=1}^{n} \tilde{x}_i \tilde{x}_i^{\top} \right\|_{S_{q^*}}^2 = \left[ c \left\| \sum_{i=1}^{n} \tilde{x}_i \tilde{x}_i^{\top} \right\|_{S_{q^*}}^2 \right]^{\frac{1}{2}}. \quad (33)
\]
Plugging (32) and (33) into (31) gives
\[
\mathbb{E}_g \left\| \sum_{j=1}^c g_i \tilde{X}_j^T \right\|_{S_q^n} \leq 2^{-\frac{1}{2} \sqrt{\frac{\pi q^*}{e}}} \max \left\{ c^{\frac{1}{2}}, \left( \sum_{i=1}^n \| x_i \|^2 \right)^{\frac{1}{2}}, c^{\frac{1}{2}} \left( \sum_{i=1}^n x_i x_i^T \right)^{\frac{1}{2}} \right\}. \tag{34}
\]

Applying again Jensen’s inequality and Khintchine-Kahane inequality [A33] gives
\[
\mathbb{E}_g \left\| \sum_{j=1}^c g_i \tilde{X}_j^T \right\|_{S_q^n} \leq 2^{-\frac{1}{2} \sqrt{\frac{\pi q^*}{e}}} \max \left\{ \left\| \sum_{j=1}^c \left( \sum_{i=1}^c \sum_{i \in I_j} g_i \tilde{X}_j^T \right) \right\|_{S_q^n}, \left\| \sum_{j=1}^c \sum_{i \in I_j} \tilde{X}_j^T \right\|^2_{S_q^n} \right\}. \tag{35}
\]

It can be directly checked that
\[
\sum_{j=1}^c \sum_{i \in I_j} (\tilde{X}_j^T)^\tau \tilde{X}_j^T = \sum_{j=1}^c \sum_{i \in I_j} \| x_i \|^2 \text{diag}(e_j) = \text{diag} \left( \sum_{i \in I_1} \| x_i \|^2, \ldots, \sum_{i \in I_c} \| x_i \|^2 \right),
\]
from which and \( q^* \geq 2 \) we derive
\[
\left\| \left( \sum_{j=1}^c \sum_{i \in I_j} (\tilde{X}_j^T)^\tau \tilde{X}_j^T \right)^\frac{1}{2} \right\|_{S_q^n} = \left\| \sum_{j=1}^c \sum_{i \in I_j} \| x_i \|^2 \right\|_{S_q^n} \leq \left\| \sum_{j=1}^c \sum_{i \in I_j} x_i x_i^T \right\|_{S_q^n} = \sum_{i=1}^n \| x_i \|^2_{S_q^n},
\]
where we use deductions analogous to (33) in the last identity. Plugging the above two inequalities back into (35) we get
\[
\mathbb{E}_g \left\| \sum_{j=1}^c g_i \tilde{X}_j^T \right\|_{S_q^n} \leq 2^{-\frac{1}{2} \sqrt{\frac{\pi q^*}{e}}} \left( \sum_{i=1}^n \| x_i \|^2 \right)^{\frac{1}{2}}. \tag{36}
\]

Plugging (34) and (36) into Theorem 2 and noting that \( H_{S_q^n} \subset H_{S_q} \) we get the following inequality with probability at least \( 1 - 2\delta \)
\[
A_{S_p} \leq \frac{2^{2\pi \Lambda}}{n \sqrt{c}} \inf_{p \geq q \leq 2} (q^*) \frac{1}{2} \left\{ L_1 \max \left\{ c^{\frac{1}{2}}, \left( \sum_{i=1}^n \| x_i \|^2 \right)^{\frac{1}{2}}, c^{\frac{1}{2}} \left( \sum_{i=1}^n x_i x_i^T \right)^{\frac{1}{2}} \right\} + L_2 \left( \sum_{i=1}^n \| x_i \|^2 \right)^{\frac{1}{2}} \right\}. \tag{37}
\]

This finishes the proof for the case \( p < 2 \).

We now consider the case \( p > 2 \). For any \( W \) with \( \| W \|_{S_p} \leq \Lambda \), we have \( \| W \|_{S_q} \leq \min(c, d) \frac{1}{2} + \Lambda \). The stated bound (31) for the case \( p > 2 \) then follows by recalling the established generalization bound (37) for \( p = 2 \).

\section{5.2 Proof of Data-independent Bounds}

We use the tool of empirical \( \ell_\infty \)-norm covering numbers to establish data-independent error bounds. The key observation to prove with the proof is that the empirical \( \ell_\infty \)-norm covering numbers of \( F_{\tau, \Lambda} \) w.r.t. the training examples can be dominated by that of \( H_{\tau} \) w.r.t. an enlarged dataset of cardinality \( nc \), due to the Lipschitz continuity of multi-class loss functions w.r.t. the \( \ell_\infty \)-norm. A similar idea is used to study the generalization ability of learning to rank [65]. The remaining problem is to estimate the covering numbers of \( H_{\tau} \), which, by the universal relationship between fat-shattering dimension and
covering numbers (Part (a) of Lemma 20), can be further transferred to the estimation of fat-shattering dimension. Finally, the problem of estimating fat-shattering dimension reduces to the estimation of worst case Rademacher complexity (Part (b) of Lemma 20). We summarize this deduction process in the proof of Theorem 21.

**Definition 3** (Covering number). Let $F$ be a class of real-valued functions defined over a space $\tilde{Z}$ and $S := \{z_1, \ldots, z_n\} \subset \tilde{Z}$ a set of cardinality $n$. For any $\epsilon > 0$, the empirical $\ell_\infty$-norm covering number $N_\infty(\epsilon, F, S)$ w.r.t. $S$ is defined as the minimal number $m$ of a collection of vectors $v^1, \ldots, v^m \in \{(f(z_1), \ldots, f(z_n)) : f \in F\}$ such that $(v^j_i$ is the $i$-th component of the vector $v^j)$

$$\sup_{f \in F} \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} |f(z_i) - v^j_i| \leq \epsilon.$$ 

In this case, we call $\{v^1, \ldots, v^m\}$ an $(\epsilon, \ell_\infty)$-cover of $F$ w.r.t. $S$. Denote $N_\infty(\epsilon, F, n) := \sup_{S \subset \tilde{Z}} |S| = n N_\infty(\epsilon, F, \tilde{Z})$ as the worst-case $\ell_\infty$-norm covering number.

**Definition 4** (Fat-Shattering Dimension). Let $F$ be a class of functions defined over a space $\tilde{Z}$. We define the fat-shattering dimension $\text{fat}_\tau(F)$ at scale $\tau > 0$ as the largest $D \in \mathbb{N}$ such that there exist $D$ points $z_1, \ldots, z_D \in \tilde{Z}$ and witnesses $s_1, \ldots, s_D \in \mathbb{R}$ satisfying: for any $\delta_1, \ldots, \delta_D \in \{\pm 1\}$ there exists $f \in F$ with

$$\delta_i(f(z_i) - s_i) \geq \epsilon/2, \quad \forall i = 1, \ldots, D.$$ 

**Lemma 20** \cite{52}. Let $F$ be a class of real-valued functions.

(a) If functions in $F$ take values in $[-B, B]$, then for any $\epsilon > 0$ with $\text{fat}_\tau(F) < n$ we have

$$\log_2 N_\infty(\epsilon, F, n) \leq \text{fat}_\tau(F) \log_2 \frac{2eBn}{\epsilon}.$$ 

(b) For any $\epsilon > 2\mathcal{R}_n(F)$, we have $\text{fat}_\tau(F) \leq \frac{16n}{\epsilon} \mathcal{R}_n^2(F)$.

(c) For any data points $S = \{x_1, \ldots, x_n\}$, we have

$$\mathcal{R}_S(F) \leq \inf_{\epsilon > 0} \left[ \epsilon + \frac{10}{\sqrt{n}} \sup_{f \in F} \sqrt{\frac{P_n f^2}{\epsilon}} \log N_\infty(\epsilon, F, S) \right], \quad (38)$$

where $P_n f^2 = \frac{1}{n} \sum_{i=1}^n f^2(x_i)$.

**Theorem 21** (Covering number bounds). Assume that, for any $y \in \mathcal{Y}$, the function $\Psi_y$ is $L$-Lipschitz w.r.t. the $\ell_\infty$ norm. Then, for any $\epsilon > 4L \mathcal{R}_n(\tilde{H}_\tau)$, the covering number of $F_{\tau, \Lambda}$ w.r.t. $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ can be bounded by

$$\log_2 N_\infty(\epsilon, F_{\tau, \Lambda}, S) \leq \frac{16ncL^2 \mathcal{R}_n^2(\tilde{H}_\tau)}{\epsilon^2} \log_2 \frac{2eBncL}{\epsilon}.$$ 

**Proof.** We introduce a sample $\tilde{X}_{1,nc}$ of cardinality $nc$ as follows

$$\tilde{X}_{1,nc} := \{\tilde{\phi}_1(x_1), \ldots, \tilde{\phi}_n(x_1), \ldots, \tilde{\phi}_1(x_2), \ldots, \tilde{\phi}_n(x_2), \ldots, \tilde{\phi}_1(x_n), \ldots, \tilde{\phi}_n(x_n)\},$$

where, for any $x \in X$, $\tilde{\phi}_j(x)$ denotes

$$\tilde{\phi}_j(x) := (0, \ldots, 0, \phi(x), 0, \ldots, 0) \in H^c_{\Lambda}, \quad j \in \mathbb{N}_c.$$ 

It is clear that $\text{card}(\tilde{\phi}_j(x)) \leq 1$, $\|\tilde{\phi}_j(x)\|_{2,\infty} \leq B_K$, which motivates us to define $\tilde{H}_\tau$ as a class of functions on $\{v \in H^c_K : \text{card}(v) \leq 1, \|v\|_{2,\infty} \leq B_K\}$. We proceed with the proof in three steps.
(1) We first estimate the covering number of $\tilde{H}_r$ w.r.t. $\tilde{X}_1^{nc}$. For any $\epsilon > 4\mathcal{R}_{nc}(\tilde{H}_r)$, Part (b) of Lemma 20 implies that
\[
fat_\epsilon(\tilde{H}_r) \leq \frac{16nc}{\epsilon^2} \mathcal{R}_{nc}^2(\tilde{H}_r) \leq nc. \tag{39}
\]
Also, the restriction $\text{card}(v) \leq 1$ in (11) immediately implies the following inequality for any function $v \mapsto \langle w, v \rangle$ in $\tilde{H}_r$
\[
|\langle w, v \rangle| = \left| \sum_{j=1}^{c} \langle w_j, v_j \rangle \right| = \max_{j=1,\ldots,c} |\langle w_j, v_j \rangle| \leq \max_{j=1,\ldots,c} \|w_j\|_2B_K
\]
\[
\leq B_K \sup_{w: \tau(w) \leq \Lambda} \|w\|_{2,\infty} = B,
\]
where we have used the definition of $B_K$ and $B$ in the above deduction. Then, the conditions of Part (a) in Lemma 20 are satisfied and we can apply it to control the covering numbers for any $\epsilon > 4\mathcal{R}_{nc}(\tilde{H}_r)$ (note $\text{fat}_\epsilon(\tilde{H}_r) < nc$ in (39))
\[
\log_2 \mathcal{N}_{\infty}(\epsilon, \tilde{H}_r, nc) \leq \text{fat}_\epsilon(\tilde{H}_r) \log \frac{2\epsilon Bnc}{\epsilon} \leq \frac{16nc\mathcal{R}_{nc}^2(\tilde{H}_r)}{\epsilon^2} \log_2 \frac{2\epsilon Bnc}{\epsilon}, \tag{40}
\]
where the second inequality is due to (39).

(2) We now relate the $\ell_\infty$-norm covering numbers of $\tilde{H}_r$ to that of $F_{2,\Lambda}$. Specifically, suppose that the following projection of functions $\{v \mapsto \langle w^j, v \rangle : j = 1,\ldots,N\}$ onto the examples $\{\tilde{\phi}_j(x_i) : i \in \mathbb{N}_n, j \in \mathbb{N}_c\}$
\[
\left\{ \left( \langle w^1, \tilde{\phi}_1(x_1) \rangle, \ldots, \langle w^j, \tilde{\phi}_j(x_1) \rangle, \langle w^1, \tilde{\phi}_1(x_2) \rangle, \ldots, \langle w^j, \tilde{\phi}_j(x_2) \rangle, \ldots, \langle w^1, \tilde{\phi}_1(x_n) \rangle, \ldots, \langle w^j, \tilde{\phi}_j(x_n) \rangle \right) : j = 1,\ldots,N \right\}\text{ related to } x_i \text{ related to } x_2 \text{ related to } x_n
\]
is an $(\epsilon, \ell_\infty)$-cover of $\tilde{H}_r$ w.r.t. $\tilde{X}_1^{nc}$ with $N$ less than the right-hand side of (10). Then, we can show that the projection of functions $\{(x, y) \mapsto \Psi_y(h^w(x)) : j = 1,\ldots,N\}$ onto the examples $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ as follows (note $h^w(x) = (\{w_1, \phi(x)\}, \ldots, \{w_c, \phi(x)\})$)
\[
\left\{ \left( \Psi_{y_1}(h^w(x_1)), \Psi_{y_2}(h^w(x_2)), \ldots, \Psi_{y_n}(h^w(x_n)) \right) : j = 1,\ldots,N \right\} \subset \mathbb{R}^n \tag{42}
\]
would be an $(L\epsilon, \ell_\infty)$-cover of the set
\[
\left\{ \left( \Psi_{y_1}(h^w(x_1)), \Psi_{y_2}(h^w(x_2)), \ldots, \Psi_{y_n}(h^w(x_n)) \right) : \tau(w) \leq \Lambda \right\} \subset \mathbb{R}^n.
\]
Indeed, for any $w \in H_K^c$ with $\tau(w) \leq \Lambda$, the construction of the cover in Eq. (11) guarantees the existence of $j(w) \in \{1,\ldots,N\}$ such that
\[
\max_{1 \leq i \leq n} \max_{1 \leq k \leq c} |\langle w^i, \tilde{\phi}_k(x_i) \rangle - \langle w, \tilde{\phi}_k(x_i) \rangle| \leq \epsilon. \tag{43}
\]
Then, the Lipschitz condition of $\Psi_y$ w.r.t. the $\ell_\infty$ norm implies that
\[
\max_{1 \leq i \leq n} |\Psi_{y_i}(h^w(x_i)) - \Psi_{y_i}(h^w(x_i))| \leq L \max_{1 \leq i \leq n} \|h^w(x_i) - h^w(x_i)\|_\infty
\]
\[
= L \max_{1 \leq i \leq n} \max_{1 \leq k \leq c} |\langle w^i_k, \phi(x_i) \rangle - \langle w^i_k, \phi(x_i) \rangle| \leq L \epsilon,
\]
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Proof of Theorem 6.

For brevity, denote $t = \phi_k(x) = \left(\left(\phi_{k_1}(x_1), \ldots, \phi_{k_{c-1}}(x_{c-1})\right), 0, \ldots, 0\right)$ where in the last step we have used the assignment $t$.

The stated result follows directly if we plug the complexity bound of $F_{r,A}$ w.r.t. $(x_1, y_1), \ldots, (x_n, y_n)$. Therefore,

$$ \log_2 N_\infty(\epsilon, F_{r,A}, n) \leq \log_2 N_\infty(\epsilon/L, \tilde{H}_r, nc), \quad \forall \epsilon > 0. \quad (44) $$

(3) The stated result follows directly if we plug the complexity bound of $\tilde{H}_r$ established in (40) into (44).

We can now apply the standard entropy integral to control the Rademacher complexity of $F_{r,A}$ in terms of $R_{nc}(\tilde{H}_r)$.

**Proof of Theorem 5** Plugging the covering number bounds established in Theorem 21 into the entropy integral, we derive the following inequality for any $\epsilon_0 \geq 4L R_{nc}(\tilde{H}_r)$

$$ R_{S}(F_{r,A}) \leq 4\epsilon_0 + \frac{10}{\sqrt{n}} \int_{\epsilon_0}^{B_\Phi} \sqrt{\log_2 N_\infty(\epsilon, F_{r,A}, S)} d\epsilon 
\leq 4\epsilon_0 + 40\sqrt{\epsilon L R_{nc}(\tilde{H}_r)} \int_{\epsilon_0}^{B_\Phi} \frac{1}{\epsilon} \sqrt{\log_2 \frac{2eBnL}{\epsilon}} d\epsilon. \quad (45) $$

For brevity, denote $t = 2eBnL$. Then with a change of variable $\epsilon' = \frac{t}{\epsilon}$ we have

$$ \int_{\epsilon_0}^{B_\Phi} \frac{1}{\epsilon} \sqrt{\log_2 \frac{t}{\epsilon}} d\epsilon = \int_{\frac{\epsilon_0}{t}}^{\frac{B_\Phi}{t}} \frac{1}{\epsilon'} \sqrt{\log_2 \epsilon'} d\epsilon' = \frac{2}{3} \int_{\frac{\epsilon_0}{t}}^{\frac{B_\Phi}{t}} d(\log \epsilon')^\frac{3}{2} = \frac{2}{3} \left[ (\log \frac{t}{\epsilon_0})^\frac{3}{2} - (\log \frac{t}{B_\Phi})^\frac{3}{2} \right]. $$

Plugging the above inequality into (45) and noting $t \geq B_\Phi$, we have

$$ R_{S}(F_{r,A}) \leq 4\epsilon_0 + \frac{80}{3} \sqrt{\epsilon L R_{nc}(\tilde{H}_r)} \left( \log \frac{2eBnL}{\epsilon_0} \right)^\frac{3}{2} 
\leq \frac{80}{3} \sqrt{\epsilon L R_{nc}(\tilde{H}_r)} \left( 1 + \left( \log \frac{3eBnL}{10R_{nc}(\tilde{H}_r)} \right)^\frac{3}{2} \right), $$

where in the last step we have used the assignment $\epsilon_0 = \frac{20}{3} \sqrt{\epsilon L R_{nc}(\tilde{H}_r)}$. 

The proof on data-independent error bounds is now immediate.

**Proof of Theorem 6** McDiarmid’s inequality implies the following inequality with probability at least $1 - \delta$

$$ \sup_{h \in \tilde{H}_r} \left[ \mathbb{E}_{x,y} \Psi_y(h^x(y)) - \frac{1}{n} \sum_{i=1}^{n} \Psi_y(h^x(x_i)) \right] 
\leq \mathbb{E} \left[ \sup_{h \in \tilde{H}_r} \mathbb{E}_{x,y} \Psi_y(h^x(y)) - \frac{1}{n} \sum_{i=1}^{n} \Psi_y(h^x(x_i)) \right] + B_\Phi \sqrt{\frac{\log \frac{2}{\delta}}{2n}} 
\leq 2\mathbb{E}[R_{S}(F_{r,A})] + B_\Phi \sqrt{\frac{\log \frac{2}{\delta}}{2n}}, \quad (46) $$

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The proof of Corollary 9 is complete.

Proof of Corollary 10: Consider any \( W = (w_1, \ldots, w_c) \in \mathbb{R}^{d \times c} \). If \( 1 < p \leq 2 \), then

\[
\|W\|_{S_p} \geq \|W\|_{S_2} = \|W\|_{2,2} \geq \|W\|_{2,\infty}.
\]

Otherwise, according to the following inequality for any semi-definite positive matrix \( A = (a_{jj})_{j,j=1}^c \) [e.g., (1.67) in 67]

\[
\|A\|_{S_p} \geq \left[ \sum_{j=1}^c |a_{jj}|^\frac{c}{p} \right]^{\frac{p}{c}}, \quad \forall p \geq 1,
\]

we derive

\[
\|W\|_{S_p} = \|(W^T W)\|_{S_p} = \left\| \left[ (w_j^T w_j)^c \right]_{j,j=1}^c \right\|_{S_p} = \left\| (w_j^T w_j)^c \right\|_{S_p} \geq \max_{j=1,\ldots,c} \|w_j\|_{2} = \|W\|_{2,\infty}.
\]

Thereby, for the specific choice \( \tau(W) = \|W\|_{S_p}, p \geq 1 \), we have

\[
B = B_K \sup_{W: \|W\|_{S_p} \leq \Lambda} \|W\|_{2,\infty} \leq B_K \Lambda.
\] (47)

We now consider two cases. If \( 1 < p \leq 2 \), plugging the Rademacher complexity bounds of \( \tilde{H}_{S_p} \) given in [14] into Theorem 6 gives the following inequality with probability at least \( 1 - \delta \)

\[
A_{S_p} \leq \frac{54LKBK}{\sqrt{n}} \left( 1 + \left( \log \frac{3\sqrt{2}eBn^{\frac{1}{2}p}}{10LKBK} \right)^\frac{1}{p} \right) \leq \frac{54LKBK}{\sqrt{n}} \left( 1 + \log \frac{2}{75} \left( \frac{3en^{\frac{1}{2}p}}{5\sqrt{2}} \right) \right),
\]

where the last step follows from (47). If \( p > 2 \), analyzing analogously yields the following inequality with probability at least \( 1 - \delta \)

\[
A_{S_p} \leq \frac{54LKBK \min(c,d)^{\frac{1}{2} + \frac{1}{p}}}{\sqrt{n}} \left( 1 + \log \frac{2}{75} \left( \frac{3en^{\frac{1}{2}p}}{5\sqrt{2}} \right) \right).
\]

The stated error bounds follow by combining the above two cases together. \( \square \)
5.3 Proofs on Data-independent Rademacher Complexities

We introduce the following function class
\[ \mathcal{V} = \{ \mathbf{v} = (v_1, \ldots, v_c) \in H^n : \text{card}(\mathbf{v}) \leq 1, \|\mathbf{v}\|_2 \leq B_K \}. \] (48)

Proof of Proposition 4. We proceed with the proof by distinguishing two cases according to the value of \( p \).

1. We first consider the case \( 1 \leq p \leq 2 \), for which the Rademacher complexity can be lower bounded by

\[
\mathcal{R}_{nc}(\tilde{H}_p) = \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{1}{nc} \mathbb{E}_\varepsilon \sup_{\|\mathbf{w}\|_2, \|\mathbf{v}\|_2 \leq A} \sum_{i=1}^{nc} \epsilon_i \langle \mathbf{w}, \mathbf{v}^i \rangle
\]

\[
= \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{1}{nc} \mathbb{E}_\varepsilon \sup_{\|\mathbf{w}\|_2 \leq A} \langle \mathbf{w}, \mathbf{v}^1 \rangle
\]

\[
= \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{1}{nc} \mathbb{E}_\varepsilon \left\| \sum_{i=1}^{nc} \epsilon_i \mathbf{v}^i \right\|_{2,p}^2
\]

where the equality (49) follows from the definition of dual norm and the inequality follows by taking \( \mathbf{v}^1 = \cdots = \mathbf{v}^{nc} \). Applying the Khitchine-Kahane inequality (A.2) and using card(\( \mathbf{v} \)) \leq 1, we then derive

\[
\mathcal{R}_{nc}(\tilde{H}_p) \geq \frac{\Lambda}{\sqrt{2nc}} \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \|\mathbf{v}^1\|_{2,p}^2 = \frac{\Lambda B_K}{\sqrt{2nc}}
\]

Furthermore, according to the subset relationship \( \tilde{H}_p \subseteq \tilde{H}_2(1 \leq p \leq 2) \) due to the monotonicity of \( \|\cdot\|_p \), the term \( \mathcal{R}_{nc}(\tilde{H}_p) \) can also be upper bounded by (\( \mathbf{v}_j^i \) denotes the \( j \)-th component of \( \mathbf{v}^i \))

\[
\mathcal{R}_{nc}(\tilde{H}_p) \leq \mathcal{R}_{nc}(\tilde{H}_2) = \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{\Lambda}{nc} \mathbb{E}_\varepsilon \left\| \sum_{i=1}^{nc} \epsilon_i \mathbf{v}_j^i \right\|_{2,2}^2
\]

\[
\leq \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{\Lambda}{nc} \sum_{j=1}^c \mathbb{E}_\varepsilon \left\| \sum_{i=1}^{nc} \epsilon_i \mathbf{v}_j^i \right\|_2^2 = \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{\Lambda}{nc} \sum_{j=1}^c \sum_{i=1}^{nc} \|\mathbf{v}_j^i\|_2^2
\]

\[
= \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{\Lambda}{nc} \sum_{i=1}^{nc} \|\mathbf{v}^1\|_{2,\infty}^2 = \frac{\Lambda B_K}{\sqrt{nc}}
\]

where the first step is due to (49), the second inequality is due to Jensen’s inequality and the last second identity is due to the constraint card(\( \mathbf{v} \)) \leq 1.

2. We now turn to the case \( p > 2 \). In this case, we have

\[
\mathcal{R}_{nc}(\tilde{H}_p) = \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{1}{nc} \mathbb{E}_\varepsilon \sup_{\|\mathbf{w}\|_2, \|\mathbf{v}\|_2 \leq \Lambda} \sum_{j=1}^{c} \epsilon_j \sum_{i=1}^{nc} \langle \mathbf{w}_j, \mathbf{v}_j^i \rangle
\]

\[
\geq \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{1}{nc} \mathbb{E}_\varepsilon \sup_{\|\mathbf{w}\|_2 \leq \Lambda} \sum_{j=1}^{c} \epsilon_j \sum_{i=1}^{nc} \langle \mathbf{w}_j, \mathbf{v}_j^i \rangle
\]

\[
= \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{1}{nc} \sum_{j=1}^{c} \mathbb{E}_\varepsilon \sup_{\|\mathbf{w}\|_2 \leq \Lambda} \sum_{i=1}^{nc} \epsilon_i \langle \mathbf{w}_j, \mathbf{v}_j^i \rangle
\]

\[
= \sup_{\mathbf{v}^{nc} \in \mathcal{V}} \frac{1}{nc} \sum_{j=1}^{c} \mathbb{E}_\varepsilon \langle \mathbf{w}_j, \sum_{i=1}^{nc} \epsilon_i \mathbf{v}_j^i \rangle,
\]
where we can exchange the summation over \( j \) with the supremum in the second identity since the constraint \( \|w_j\|_2^2 \leq \frac{\Lambda}{c}, j \in \mathbb{N}_c \) are decoupled. According to the definition of dual norm and the Khitchine-Kahane inequality (A.2), \( \mathfrak{R}_{nc}(\tilde{H}_p) \) can be further controlled by

\[
\mathfrak{R}_{nc}(\tilde{H}_p) \geq \sup_{v^i \in \mathcal{V}, i \in \mathbb{N}_{nc}} \frac{1}{nc} \sum_{j=1}^{c} \frac{\Lambda}{c^{\frac{p}{2}}} \sum_{i=1}^{nc} \|v_j^i\|_2 \geq \sup_{v^i \in \mathcal{V}, i \in \mathbb{N}_{nc}} \frac{1}{nc} \sum_{j=1}^{c} \frac{\Lambda}{\sqrt{2c}^\frac{p}{2}} \left[ \sum_{i=1}^{nc} \|v_j^i\|_2^2 \right]^\frac{p}{2}.
\]

(50)

We can find \( \tilde{v}^1, \ldots, \tilde{v}^{nc} \in \mathcal{V} \) such that for each \( j \in \mathbb{N}_c \), there are exactly \( n \) \( \tilde{v}^i \) with \( \|v_j^i\|_2 = B_K \). Then, \( \sum_{i=1}^{nc} \|v_j^i\|_2^2 = n(B_K)^2 \), \( \forall j \in \mathbb{N}_c \), which, coupled with (50), implies that

\[
\mathfrak{R}_{nc}(\tilde{H}_p) \geq \frac{1}{nc} \sum_{j=1}^{c} \frac{\Lambda}{\sqrt{2c}^\frac{p}{2}} \left[ \sum_{i=1}^{nc} \|v_j^i\|_2^2 \right]^\frac{p}{2} \geq \Lambda B_K (2n)^{-\frac{p}{2}} c^{-\frac{p}{4}}.
\]

On the other hand, according to (49) and Jensen’s inequality, we derive

\[
\frac{ne \mathfrak{R}_{nc}(\tilde{H}_p)}{\Lambda} = \sup_{v^i \in \mathcal{V}, i \in \mathbb{N}_{nc}} \mathbb{E} \left[ \left( \sum_{i=1}^{nc} \|v_j^i\|_2 \right)^{2p} \right]^{\frac{1}{2p}} \leq \sup_{v^i \in \mathcal{V}, i \in \mathbb{N}_{nc}} \mathbb{E} \left[ \left( \sum_{i=1}^{nc} \|v_j^i\|_2^p \right)^{\frac{1}{2}} \right]^{\frac{1}{2p}}.
\]

By the Khitchine-Kahane inequality (A.1) with \( p^* \leq 2 \) and the following elementary inequality

\[
\sum_{j=1}^{c} |t_j|^p \leq c^{1-p} (\sum_{j=1}^{c} |t_j|)^{\tilde{p}}, \forall 0 < \tilde{p} \leq 1,
\]

we get

\[
\frac{ne \mathfrak{R}_{nc}(\tilde{H}_p)}{\Lambda} \leq \sup_{v^i \in \mathcal{V}, i \in \mathbb{N}_{nc}} \left( \sum_{j=1}^{c} \left( \sum_{i=1}^{nc} \|v_j^i\|_2^p \right)^{\frac{1}{2}} \right)^{\frac{1}{2p}} \leq \sup_{v^i \in \mathcal{V}, i \in \mathbb{N}_{nc}} \left( c^{1-\frac{p}{2}} (\sum_{j=1}^{c} \sum_{i=1}^{nc} \|v_j^i\|_2^{p^*})^{\frac{1}{2}} \right)^{\frac{1}{2p}} \leq \sqrt{nc} c^{-\frac{1}{2}} B_K = \sqrt{nc}^{1-\frac{1}{p}} B_K,
\]

where the last inequality is due to the definition of \( \mathcal{V} \).

The above upper and lower bounds in the two cases can be written compactly as (12). The proof is complete.

\[\square\]

5.4 Proofs on Applications

\textit{Proof of Example 1.} According to the monotonicity of \( \ell \), there holds

\[
\ell(\rho(h(x), y)) = \ell\left( \min_{y' \neq y} (h_y(x) - h_{y'}(x)) \right) = \max_{y' \neq y} \ell(h_y(x) - h_{y'}(x)) = \Psi_y^L(h(x)).
\]

It remains to show the Lipschitz continuity of \( \Psi_y^L \). Indeed, for any \( \mathbf{t}, \mathbf{t}' \in \mathbb{R}^c \), we have

\[
|\Psi_y^L(\mathbf{t}) - \Psi_y^L(\mathbf{t}')| = |\max_{y' \neq y} \ell(t_y - t_{y'}) - \max_{y' \neq y} \ell(t'_y - t'_{y'})| \leq \max_{y' \neq y} |\ell(t_y - t_{y'}) - \ell(t'_y - t'_{y'})| \leq \max_{y' \neq y} |t_y - t_{y'} - (t'_y - t'_{y'})| \leq 2L \max_{y' \neq y} |t_{y'} - t'_{y'}| \leq 2L \|\mathbf{t} - \mathbf{t}'\|_2,
\]

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where in the first inequality we have used the elementary inequality
\[ |\max\{a_1, \ldots, a_c\} - \max\{b_1, \ldots, b_c\}| \leq \max\{|a_1 - b_1|, \ldots, |a_c - b_c|\}, \quad \forall a, b \in \mathbb{R}^c \]  
and the second inequality is due to the Lipschitz continuity of \( \ell \).

**Proof of Example 3.** Define the function \( f^m : \mathbb{R}^c \to \mathbb{R} \) by \( f^m(t) = \log \left( \sum_{j=1}^c \exp(t_j) \right) \). For any \( t \in \mathbb{R}^c \), the partial gradient of \( f^m \) with respect to \( t_k \) is
\[ \nabla f^m(t) = \frac{\exp(t_k)}{\sum_{j=1}^c \exp(t_j)}, \quad \forall k = 1, \ldots, c, \]
from which we derive that \( \|\nabla f^m(t)\|_1 = 1, \forall t \in \mathbb{R}^c \). Here \( \nabla \) denotes the gradient operator. For any \( t, t' \in \mathbb{R}^c \), according to the mean-value theorem we know the existence of \( \alpha \in [0, 1] \) such that
\[ |f^m(t) - f^m(t')| = |\langle \nabla f^m(\alpha t + (1 - \alpha)t'), t - t' \rangle| \]
\[ \leq \|\nabla f^m(\alpha t + (1 - \alpha)t')\|_1 \|t - t'\|_\infty = \|t - t'\|_\infty. \]
It then follows that
\[ |\Psi_y^m(t) - \Psi_y^m(t')| = |f^m(t_{\ell}) - f^m(t'_{\ell})| \]
\[ \leq \|t_{\ell} - t'_{\ell}\|_\infty \leq 2\|t - t'\|_\infty. \]
That is, \( \Psi_y^m \) is 2-Lipschitz w.r.t. the \( \ell_\infty \) norm.

**Proof of Example 4.** For any \( t, t' \in \mathbb{R}^c \), we have
\[ |\tilde{\Psi}_y^\ell(t) - \tilde{\Psi}_y^\ell(t')| = \left| \sum_{j=1}^c \ell(t_y - t_j) - \sum_{j=1}^c \ell(t'_y - t'_j) \right| \leq \sum_{j=1}^c |\ell(t_j - t_y) - \ell(t'_j - t'_y)| \]
\[ \leq L_{\ell} c|t_y - t'_y| + L_{\ell} \sum_{j=1}^c |t_j - t'_j| 
\leq L_{\ell} c|t_y - t'_y| + L_{\ell} \sqrt{c} \|t - t'\|_2. \]
The Lipschitz continuity of \( \tilde{\Psi}_y^\ell(t) \) w.r.t. \( \ell_\infty \) norm is also clear.

**Proof of Example 5.** For any \( t, t' \in \Omega \), we have
\[ |\Psi_y^\ell(t) - \Psi_y^\ell(t')| = \left| \sum_{j=1}^c [\ell(t) - \ell(t')| \right| \leq L_{\ell} \sum_{j=1}^c |t_j - t'_j| \]
\[ \leq L_{\ell} \sqrt{c} \|t - t'\|_2 \leq L_{\ell} c\|t - t'\|_\infty. \]
This establishes the Lipschitz continuity of \( \tilde{\Psi}_y^\ell \).

**Proof of Example 6.** It is clear that
\[ \sum_{j=1}^k t_{(j)} = \max_{1 \leq i_1 < i_2 < \cdots < i_k \leq c} \left[ t_{i_1} + \cdots + t_{i_k} \right], \quad \forall t \in \mathbb{R}^c. \]  
(52)
For any \( t, t' \in \mathbb{R}^c \), we have
\[
|\Psi_y^k(t) - \Psi_y^k(t')| \\
\leq \frac{1}{K} \left| \sum_{j=1}^k (1_{y \neq i} + t_y - y_{\hat{i}}) - \sum_{j=1}^k (1_{y \neq i} + t'_y - y_{\hat{i}}) \right| \\
= \frac{1}{K} \max_{1 \leq i_1 < i_2 < \cdots < i_k \leq c} \left| \sum_{r=1}^k (t_{i_r} - t'_y) - \sum_{r=1}^k (t_{i_r} - t'_y) \right| \\
\leq \frac{1}{K} \max_{1 \leq i_1 < i_2 < \cdots < i_k \leq c} \left| \sum_{r=1}^k (t_{i_r} - t'_y) \right| + |t_y - t'_y| \\
\leq \frac{1}{\sqrt{K}} \max_{1 \leq i_1 < i_2 < \cdots < i_k \leq c} \left[ \sum_{r=1}^k (t_{i_r} - t'_y)^2 \right]^{\frac{1}{2}} + |t_y - t'_y| \\
\leq \frac{1}{\sqrt{K}} \left[ \sum_{j=1}^k (t_j - t'_j)^2 \right]^{\frac{1}{2}} + |t_y - t'_y|, \tag{53}
\]
where the first and the second inequality are due to (51) and the first identity is due to (52). This establishes the Lipschitz continuity w.r.t. a variant of \( \ell_2 \) norm. The 2-Lipschitz continuity of \( \Psi_y^k \) w.r.t. \( \ell_\infty \) norm is clear from (53). The proof is complete. \( \square \)

6 Conclusion

Motivated by the ever-growing number of classes in the extreme classification, this paper shows both data-dependent and data-independent generalization error bounds exhibiting a mild dependency on the number of classes.

For the data-dependent analysis, we establish a novel structural result on Gaussian complexities of function classes induced by vector-valued function classes composed by Lipschitz operators measured by a variant of \( \ell_2 \) norm. This structural result is an extension of the related result in [29]. We demonstrate its effectiveness by applying it to establish data-dependent bounds for several popular MC-SVMs, which, as compared to those from existing structural results [29, 44, 45], enjoy milder dependencies on the class size for Top-\( k \) MC-SVM by Crammer and Singer [32] and Lapin et al. [31] as well as multinomial logistic regression, a sublinear dependency for \( \ell_p \)-norm MC-SVM [29] and Schatten-\( p \) norm MC-SVM [58], and a linear dependency for the MC-SVMs by Weston and Watkins [33] and Lee et al. [34].

Observing the Lipschitz continuity of several multi-class loss functions w.r.t. the \( \ell_\infty \) norm with a substantially smaller Lipschitz constant, we also use the tool of \( \ell_\infty \)-norm covering numbers to derive data-independent error bounds with further relaxed dependencies on the class size. We establish a general data-independent error bound for MC-SVMs in terms of worst-case Rademacher complexities of related linear function classes, towards which we give tight upper and lower bounds matching up to a constant factor for two specific instantiations. The resulting data-independent error bounds enjoy a logarithmic dependency on the class size for the MC-SVMs by Crammer and Singer [32] and Lapin et al. [31], as well as multinomial logistic regression, a sublinear dependency for \( \ell_p \)-norm MC-SVM [29] and Schatten-\( p \) norm MC-SVM [58], and a linear dependency for the MC-SVMs by Weston and Watkins [33] and Lee et al. [34].

There exists a gap between the derived data-dependent and data-independent error bounds. As a future direction, it would be very interesting to bridge this gap between these two types of error bounds. A key challenge towards this purpose consists in establishing structural results for Rademacher (Gaussian) complexities of function classes induced by Lipschitz operators measured w.r.t. the \( \ell_\infty \) norm.

Furthermore, research in extreme classification increasingly focuses on extreme multi-label classification with each output \( y_i \) taking values in \( \{0, 1\}^c \) [10, 23, 68]. It would be interesting to transfer
the results obtained in the present analysis to the multi-label case. To this aim, it suffices to check
the Lipschitz continuity of loss functions in multi-label learning, which, as in the present work, are
also typically of the form $\Psi_y(h(\mathbf{x}))$ [68, 69], (e.g., Hamming loss, subset zero-one loss, and ranking
loss [60]).

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A Khintchine-Kahane Inequality

The following Khintchine-Kahane inequality [70, 71] provides a powerful tool to control the $p$-th
norm of the summation of Rademacher (Gaussian) series.

**Lemma A.1.** (a) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space with $\| \cdot \|$ being the associated norm.

Let $\epsilon_1, \ldots, \epsilon_n$ be a sequence of independent Rademacher variables. Then, for any $p \geq 1$ there holds

$$\min(\sqrt{p - 1}, 1) \left[ \sum_{i=1}^{n} \|\epsilon_i \mathbf{v}_i\|^p \right]^\frac{1}{p} \leq \left[ \mathbb{E}_\epsilon \| \sum_{i=1}^{n} \epsilon_i \mathbf{v}_i \|^p \right]^\frac{1}{p} \leq \max(\sqrt{p - 1}, 1) \left[ \sum_{i=1}^{n} \|\mathbf{v}_i\|^2 \right]^\frac{1}{2}, \quad (A.1)$$

$$\mathbb{E}_\epsilon \| \sum_{i=1}^{n} \epsilon_i \mathbf{v}_i \| \geq 2^{-\frac{1}{4}} \left[ \sum_{i=1}^{n} \|\mathbf{v}_i\|^2 \right]^\frac{1}{2}. \quad (A.2)$$

The above inequalities also hold when the Rademacher variables are replaced by $N(0, 1)$ random
variables.

(b) Let $X_1, \ldots, X_n$ be a set of matrices of the same dimension and let $g_1, \ldots, g_n$ be a sequence of
independent $N(0, 1)$ random variables. For all $q \geq 2$,

$$\left( \mathbb{E}_g \| \sum_{i=1}^{n} g_i X_i \|_{S_q}^q \right)^\frac{1}{q} \leq 2^{-\frac{1}{4}} \sqrt{\frac{q \pi}{e}} \max \left\{ \| (\sum_{i=1}^{n} X_i^\top X_i)^{\frac{1}{2}} \|_{S_q}, \| (\sum_{i=1}^{n} X_i X_i^\top)^{\frac{1}{2}} \|_{S_q} \right\}. \quad (A.3)$$

**Proof.** For Part (b), the original Khintchine-Kahane inequality for matrices is stated for Rademacher
random variables, i.e, the Gaussian variables $g_i$ are replaced by Rademacher variables $\epsilon_i$. We now
show that it also holds for Gaussian variables. Let $\psi_i^{(k)} = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \epsilon_{ik+j}$ with $\epsilon_{ik+j}$ being a sequence of
independent Rademacher variables, then we have

$$\left( \mathbb{E}_\epsilon \left\| \sum_{i=1}^{n} \psi_i^{(k)} X_i \|_{S_q}^q \right\|^{\frac{1}{q}} \right)^\frac{1}{q} = \left( \mathbb{E}_\epsilon \| \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{1}{\sqrt{k}} \psi_i \mathbf{X}_i \|_{S_q}^q \right)^\frac{1}{q}$$

$$\leq \sqrt{\frac{q \pi}{2^2 e}} \max \left\{ \| (\sum_{i=1}^{n} \sum_{j=1}^{k} \frac{1}{k} X_i^\top X_j)^{\frac{1}{2}} \|_{S_q}, \| (\sum_{i=1}^{n} \sum_{j=1}^{k} \frac{1}{k} X_j X_i^\top)^{\frac{1}{2}} \|_{S_q} \right\},$$

where the first inequality is due to the Khintchine-Kahane inequality for matrices involving Rademacher
random variables [71]. The proof is complete if we take $k$ to $\infty$ and use central limit theorem. \qed
B Proof of Proposition 8

We present the proof of Proposition 8 in the appendix due to its similarity to the proof of Proposition 7. Since \( \phi \) is the identity map, the space \( \mathcal{V} \) in (16) becomes

\[
\mathcal{V} = \{ V \in \mathbb{R}^{d \times c} : \text{card}(V) \leq 1, \| V \|_{2,\infty} \leq B_K \}.
\]

Here card(\( V \)) reduces to the number of non-vanishing columns in \( V \).

1. We first consider the case \( 1 \leq p \leq 2 \). Since the dual norm of \( \| \cdot \|_{S_p} \) is \( \| \cdot \|_{S_p^*} \), we have the following lower bound on Rademacher complexity in this case

\[
\mathcal{R}_{nc}(\tilde{H}_{S_p}) = \sup_{\tilde{V} \in \mathcal{V} : \text{card}(\tilde{V}) \leq 1} \frac{1}{\sqrt{2nc}} \sum_{i=1}^{nc} \epsilon_i \langle W, V^i \rangle
\]

where the last identity follows from the following identity for any \( W \in \mathbb{R}^{d \times c} \)

\[
\| W \|_{S_p^*} \leq \| W \|_{2,\infty} \leq \| W \|_{2,\infty}.
\]

Taking \( V^1 = \cdots = V^{nc} \) and applying the Khintchine-Kahane inequality (A.2) further imply

\[
\mathcal{R}_{nc}(\tilde{H}_{S_p}) \geq \sup_{\tilde{V} \in \mathcal{V} : \text{card}(\tilde{V}) \leq 1} \frac{\Lambda}{\sqrt{2nc}} \sum_{i=1}^{nc} \epsilon_i \| V^i \|_{S_p^*} \geq \frac{\Lambda}{\sqrt{2nc}} \| V \|_{S_p^*} = \frac{\Lambda B_K}{\sqrt{2nc}},
\]

where the last identity follows from the following identity for any \( V \in \mathbb{R}^{d \times c} \) satisfying card(\( V \)) \( \leq 1 \)

\[
\| V \|_{S_p^*} = \| V \| _{S_2} = \| V \|_{2,2} = \| V \|_{2,\infty}.
\]

We now turn to the upper bound. It follows from the relationship \( \tilde{H}_{S_p} \subset \tilde{H}_{S_2}, \forall 1 \leq p \leq 2 \) and (B.1) that \( \langle \text{tr}(A) \rangle \) denotes the trace of \( A \)

\[
\mathcal{R}_{nc}(\tilde{H}_{S_p}) \leq \mathcal{R}_{nc}(\tilde{H}_{S_2}) = \sup_{\tilde{V} \in \mathcal{V} : \text{card}(\tilde{V}) \leq 1} \frac{\Lambda}{\sqrt{2nc}} \sum_{i=1}^{nc} \epsilon_i \| V^i \|_{S_2} = \frac{\Lambda}{\sqrt{2nc}} \| V \|_{S_2} \leq \frac{\Lambda B_K}{\sqrt{2nc}}.
\]

where the second identity follows from the identity between Frobenius norm and \( \| \cdot \|_{S_2} \), the second inequality follows from the Jensen’s inequality and the last identity is due to (B.2).

2. We now consider the case \( p > 2 \). We denote by \( \mathbb{D}^{d \times c} \) the class of rectangular diagonal matrix in \( \mathbb{R}^{d \times c} \) and introduce

\[
\mathcal{V} = \{ v \in \mathbb{R}^b : \text{card}(v) \leq 1, \| v \|_{\infty} \leq B_K \},
\]

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where \( b := \min(c, d) \). In this case, the Rademacher complexity can be lower bounded by

\[
\mathcal{R}_{nc}(\tilde{H}_{Sp}) \geq \sup_{\|v\|_2 \leq B_K, \text{card}(V') \leq 1} \frac{1}{nc} E_x \sup_{\|w\|_p \leq \Lambda, w \in \mathbb{R}^b} \sum_{i=1}^{nc} \epsilon_i (W(V'))
\]

\[
= \sup_{v^i \in \mathcal{V}, i \in [nc]} \frac{1}{nc} E_x \sup_{\|w\|_p \leq \Lambda, w \in \mathbb{R}^b} \sum_{i=1}^{nc} \epsilon_i (w, v^i)
\]

\[
\geq \sup_{v^i \in \mathcal{V}, i \in [nc]} \frac{1}{nc} E_x \sup_{\|w\|_\infty \leq \Lambda b} \sum_{j=1}^{b} \epsilon_i w_j v^i_j,
\]

where the identity holds since every \( V \in \mathbb{D}^b \) corresponds to a unique vector \( v \in \mathbb{R}^b \) consisting of diagonal elements of \( V \). We can exchange the summation over \( j \) with supremum since the constraints \( \|w\|_\infty \leq \Lambda b \) are decoupled, yielding

\[
\mathcal{R}_{nc}(\tilde{H}_{Sp}) \geq \sup_{v^i \in \mathcal{V}, i \in [nc]} \sum_{j=1}^{b} \frac{1}{nc} E_x \sup_{\|w\|_\infty \leq \Lambda b} \sum_{i=1}^{nc} \epsilon_i w_j v^i_j
\]

\[
= \sup_{v^i \in \mathcal{V}, i \in [nc]} \sum_{j=1}^{b} \frac{\Lambda}{ncb^2} E_x \left| \sum_{i=1}^{nc} \epsilon_i v^i_j \right|
\]

\[
\geq \sup_{v^i \in \mathcal{V}, i \in [nc]} \sum_{j=1}^{b} \frac{\Lambda}{\sqrt{2ncb^2}} \left[ \sum_{i=1}^{nc} \left| v^i_j \right|^2 \right]^{\frac{1}{2}},
\]

where the last inequality is due to (A.2). We can find \( V^1, \ldots, V^{nc} \in \mathbb{V} \) such that \( \sum_{i=1}^{nc} |v^i_j|^2 \geq \frac{\Lambda}{b} (2K)^2, \forall j \in [b] \). Plugging this inequality back into the above inequality then gives

\[
\mathcal{R}_{nc}(\tilde{H}_{Sp}) \geq \sum_{j=1}^{b} \frac{\Lambda}{\sqrt{2ncb^2}} \left[ \sum_{i=1}^{nc} \left| v^i_j \right|^2 \right]^{\frac{1}{2}} \geq \frac{\Lambda b}{\sqrt{2ncb^2}} \sqrt{\frac{nc}{b}}.
\]

We now turn to the upper bounds of Rademacher complexities. For any \( W \) with \( \|W\|_{Sp} \leq \Lambda \), we have \( \|W\|_{S_2} \leq b^{\frac{1}{2}} + \frac{\Lambda}{\sqrt{2ncb^2}} \), which, combined with (B.3), implies that

\[
\mathcal{R}_{nc}(\tilde{H}_{Sp}) \leq \sup_{V^i \in \mathcal{V}, i \in [nc]} \frac{1}{nc} E_x \sup_{\|w\|_2 \leq \Lambda b^{\frac{1}{2}} + \frac{\Lambda}{\sqrt{2ncb^2}}} \sum_{i=1}^{nc} \epsilon_i (W(V')) \leq \frac{\Lambda b^{\frac{1}{2}} + \frac{\Lambda}{\sqrt{2ncb^2}}}{\sqrt{nc}}.
\]

The proof is complete.

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