Extensions of the \((p,q)\)-Flexible-Graph-Connectivity model

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Abstract

We present approximation algorithms for network design problems in some models related to the \((p,q)\)-FGC model. Adjishvili, Hommelseim and Mühlenthaler [2, 1] introduced the model of Flexible Graph Connectivity that we denote by FGC. Boyd, Cherian, Haddadan and Ibrahimpur [4] introduced a generalization of FGC. Let \(p \geq 1\) and \(q \geq 0\) be integers. In an instance of the \((p,q)\)-Flexible Graph Connectivity problem, denoted \((p,q)\)-FGC, we have an undirected connected graph \(G = (V,E)\), a partition of \(E\) into a set of safe edges \(S\) and a set of unsafe edges \(U\), and nonnegative costs \(c \in \mathbb{R}_{\geq 0}^E\) on the edges. A subset \(F \subseteq E\) of edges is feasible for the \((p,q)\)-FGC problem if for any set \(F' \subseteq U\) with \(|F'| \leq q\), the subgraph \((V,F \setminus F')\) is \(p\)-edge connected. The algorithmic goal is to find a feasible edge-set \(F\) that minimizes \(c(F) = \sum_{e \in F} c_e\).

We introduce a generalization of the \((p,q)\)-FGC model, called \(\{(0,1,\ldots,p),(0,\ldots,q)\}\)-FGC, where a required level of edge-connectivity \(p_{ij} \in \{0,\ldots,p\}\) and a fault-tolerance level \(q_{ij} \in \{0,\ldots,q\}\) is specified for every pair of nodes \(\{i,j\}\). The goal is to find a subgraph \(H = (V,F)\) of minimum cost such that for any pair of nodes \(\{i,j\}\) and any set of at most \(q_{ij}\) unsafe edges \(F' \subseteq F\), the graph \(H - F'\) has \(p_{ij}\) edge-disjoint \((i,j)\)-paths. Assuming that \(p = 1\) or \(q = 1\) (i.e., when \(p_{ij} \in \{0,1\}\) or when \(q_{ij} \in \{0,1\}\)), we present \(\max(2(p+1), 2(q+1))\)-approximation algorithms for this model by reductions to the capacitated network design problem (Cap-NDP); we apply Jain’s iterative rounding method [10] to solve the Cap-NDP instances.

We also consider the Flexible Steiner Tree model, denoted FST. We present a straight-forward approximation algorithm for FST that achieves approximation ratio \(\approx 2.9\) via recent results of Ravi, Zhang & Zlatin [13].

Finally, we introduce the NC-FGC model. In an instance of this problem, we have an undirected connected graph \(G = (V,E)\), a partition of \(V\) into a set of safe nodes \(V^S\) and a set of unsafe nodes \(V^U\), and non-negative costs \(c \in \mathbb{R}_{\geq 0}^E\) on the edges; moreover, for every pair of nodes \(\{s,t\}\), a required level of connectivity \(r_{st} \in \mathbb{Z}_{\geq 0}\) is specified. The goal is to find a subgraph \(H = (V,F)\) of minimum cost such that for any pair of nodes \(\{s,t\}\), and any set of unsafe nodes \(U \subseteq V^U - \{s,t\}\), the graph \(H - \hat{U}\) has \(\min(0, r_{st} - |\hat{U}|)\) edge-disjoint \((s,t)\)-paths. For the (uniform connectivity) \(p\)-NC-FGC model, assuming that there is at least one safe node, we show that there is a 2-approximation algorithm via a result of Frank [9] Theorem 4.4].

1 Introduction

Adjishvili, Hommelseim & Mühlenthaler [2, 1] introduced the model of Flexible Graph Connectivity that we denote by FGC. Recently, Boyd et al. [4] introduced a generalization of FGC. Let \(p \geq 1\) and \(q \geq 0\) be integers. In an instance of the \((p,q)\)-Flexible Graph Connectivity problem, denoted \((p,q)\)-FGC, we have an undirected connected graph \(G = (V,E)\), a partition of \(E\) into a set of safe edges \(S\) and a set of unsafe edges \(U\), and nonnegative costs \(c \in \mathbb{R}_{\geq 0}^E\) on the edges. A subset \(F \subseteq E\) of edges is feasible for the \((p,q)\)-FGC problem if

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for any set $F' \subseteq U$ with $|F'| \leq q$, the subgraph $(V, F \setminus F')$ is $p$-edge connected. The algorithmic goal is to find a feasible solution $F$ that minimizes $c(F) = \sum_{e \in F} c_e$.

We present approximation algorithms for network design problems in some models related to the $(p, q)$-FGC model.

First, we present a straight-forward approximation algorithm for the Flexible Steiner tree problem, denoted FST, that achieves approximation ratio $\approx 2.9$ via recent results of Ravi, Zhang & Zlatin [14].

Next, we introduce the $\langle 0, 1, \ldots, p \rangle, \langle 0, \ldots, q \rangle$-FGC model, where a required level of edge-connectivity $p_{ij} \in \{0, 1, \ldots, p\}$ and a fault-tolerance level $q_{ij} \in \{0, \ldots, q\}$ is specified for every pair of nodes $\{i, j\}$. The goal is to find a subgraph $H = (V, F)$ of minimum cost such that for any pair of nodes $\{i, j\}$ and any set of at most $q_{ij}$ unsafe edges $F' \subseteq F$, the graph $H - F'$ has $p_{ij}$ edge-disjoint $(i, j)$-paths. When $q = 1$ (i.e., when each $q_{ij} \in \{0, 1\}$), we present a $2(p + 1)$-approximation algorithm. Additionally, when $p = 1$ (i.e., when each $p_{ij} \in \{0, 1\}$), we present a $2(q + 1)$-approximation algorithm.

Finally, we introduce the Node-Connectivity Flexible Graph Connectivity problem, denoted NC-FGC; in this model, the set of nodes is partitioned into a set of safe nodes (that never fail) and a set of unsafe nodes (there is no partition of the edge-set); see Section 4 for details. We observe that there is a simple reduction from the NC-FGC model to the well-known NC-SNDP model; the latter model has been studied for decades, see [11, 12]. Nevertheless, for the (uniform connectivity) $p$-NC-FGC model, assuming that there is at least one safe node, we show that there is a 2-approximation algorithm via a result of Frank [9, Theorem 4.4]. In contrast, for the well-known special case of (uniform connectivity) $p$-NC-SNDP, even with the assumption of $|V(G)| \geq p^3$, the best approximation ratio known is $4 + \epsilon$ due to Nutov [13].

Other models and results pertaining to Flexible Graph Connectivity are presented by Adjiashvili, Hommelsheim, Mühlenthaler & Schaudt [3] and by Chekuri & Jain [7].

2 Preliminaries

This section has definitions and preliminary results. Our notation and terms are consistent with [8, 15], and readers are referred to these texts for further information.

For a positive integer $k$, we use $[k]$ to denote the set $\{1, \ldots, k\}$.

Let $G = (V, E)$ be a (loop-free) multi-graph with non-negative costs $c \in \mathbb{R}^E_{\geq 0}$ on the edges. We take $G$ to be the input graph, and we use $n$ to denote $|V(G)|$. For a set of edges $F \subseteq E(G)$, $c(F) := \sum_{e \in F} c(e)$, and for a subgraph $G'$ of $G$, $c(G') := \sum_{e \in E(G')} c(e)$. For a graph $H$ and a set of nodes $S \subseteq V(H)$, $\delta_H(S)$ denotes the set of edges that have one end node in $S$ and one end node in $V(H) \setminus S$; moreover, $H[S]$ denotes the subgraph of $H$ induced by $S$, and $H - S$ denotes the subgraph of $H$ induced by $V(H) \setminus S$. For a graph $H$ and a set of edges $F \subseteq E(H)$, $H - F$ denotes the graph $(V(H), E(H) \setminus F)$. We may use relaxed notation for singleton sets, e.g., we may use $\delta_H(v)$ instead of $\delta_H(\{v\})$, etc.

A multi-graph $H$ is called $k$-edge connected if $|V(H)| \geq 2$ and for every $F \subseteq E(H)$ of size $< k$, $H - F$ is connected. A multi-graph $H$ is called $k$-node connected if $|V(H)| > k$ and for every $S \subseteq V(H)$ of size $< k$, $H - S$ is connected.

For any instance $H$, we use $\text{OPT}(H)$ to denote the minimum cost of a feasible subgraph (i.e., a subgraph that satisfies the requirements of the problem). When there is no danger of ambiguity, we use $\text{OPT}$ rather than $\text{OPT}(H)$. 
3 Extensions of the \((p, q)\)-FGC model with edge-connectivity requirements

In this section, we present approximation algorithms and reductions for some extensions of the \((p, q)\)-FGC model with non-uniform connectivity requirements.

First, in Section 3.1 we present a straightforward approximation algorithm for FST that achieves approximation ratio \(\approx 2.9\) via recent results of Ravi, Zhang & Zlatin [14].

Next, in Section 3.2 we introduce the \(\{0, 1, \ldots, p\}, \{0, 1, \ldots, q\}\)-FGC model, and we present approximation algorithms when \(q = 1\) (i.e., when each \(q_{ij} \in \{0, 1\}\)) or when \(p = 1\) (i.e., when each \(p_{ij} \in \{0, 1\}\)).

3.1 Simple approximation algorithm for FST

This section presents and analyzes a simple (and obvious) two-stage approximation algorithm for FST that achieves approximation ratio \(\approx 2.9\) via recent results of Ravi, Zhang & Zlatin [14].

The first-stage algorithm applies the best-known approximation algorithm for the Steiner tree problem, due to [5][6], to the input, and makes no distinction between safe edges and unsafe edges. Let \(H_1 = (V_1, F_1)\) denote the Steiner tree found by this algorithm; clearly, \(c(F_1) \leq 1.4\text{OPT}\).

The second-stage algorithm applies Jain’s iterative rounding 2-approximation algorithm, [10], to the following instance of SNDP (i.e., Survivable Network Design Problem): Let \(G_2 = G/(F_1 \cap S)\) be the graph obtained from \(G\) by contracting the safe edges of \(F_1\), and fix the edge-costs \(c''\) of \(G_2\) by fixing \(c''_e = 0\) for each edge \(e \in F_1 \cap U\) and fixing \(c''_e = c_e\) for the other edges \(e\); moreover, let the set of terminals be \(T_2 = T\) if \(V(F_1 \cap S)\) has no terminals, otherwise, let \(T_2 = \hat{T} \cup (T \setminus V(F_1 \cap S))\), where \(\hat{T}\) denotes the set of contracted nodes \(\hat{u}\) of \(G_2\) such that the set of edges of \(G\) whose contraction results in \(\hat{u}\) is incident to a terminal; the connectivity requirements are two for each pair of terminals (and zero for all other node pairs). (In this instance, there is no distinction between safe edges and unsafe edges.) Let \(H_2 = (V_2, F_2)\) denote the Steiner-2ECS found by this algorithm. We claim that (i) \(c(F_2) \leq 2\text{OPT}\), and, moreover, (ii) \((V_1 \cup V_2, F_1 \cup F_2)\) is a feasible subgraph of the FST instance.

To prove claim (i), consider the subgraph \(H_0\) formed by the edge-set \((F^* \cup F_1)/(F_1 \cap S)\), where \(F^*\) denotes the edge-set of an optimal subgraph of the FST instance; observe that for any edge \(e\), \(H_0 - e\) contains a Steiner tree on \(T_2\), that is, \(H_0\) has two edge-disjoint paths between each pair of nodes of \(T_2\) (in more detail, if \(e\) is a safe edge, then \(F_1/(F_1 \cap S) = F_1 \cap U\) contains a Steiner tree on \(T_2\), and if \(e\) is an unsafe edge, then \(F^*/(F_1 \cap S)\) contains a Steiner tree on \(T\), so \(F^*/(F_1 \cap S)\) contains a Steiner tree on \(T_2\)); clearly, \(c''(H_0) \leq \text{OPT}\). Claim (ii) follows easily from the fact that \(H_2\) has two edge-disjoint paths between each pair of nodes of \(T_2\).

We remark that the approximation guarantee of the above algorithm can be improved by using better algorithms for the second-stage. In particular the second-stage algorithm is equivalent to the problem of augmenting a Steiner tree. Recently, Ravi, Zhang & Zlatin [14] provided a \((1.5 + \epsilon)\) approximation algorithm for this problem. Hence, the two-stage approximation algorithm for FST achieves approximation ratio \(\approx 2.9\).

3.2 FGC with non-uniform edge-connectivity requirements and fault-tolerance

In this section, we study the \(\{0, 1, \ldots, p\}, \{0, 1, \ldots, q\}\)-FGC model. This model is a generalization of the \((p, q)\)-FGC model in [4] with non-uniform edge-connectivity requirements. The input consists of an undirected graph \(G = (V, E)\) with non-negative costs on the edges \(c \in \mathbb{R}^E_{\geq 0}\), and a partition of the edge-set \(E\) into a set \(S\) of safe edges and a set \(U\) of unsafe edges; additionally, for every pair of nodes \(\{i, j\}\), a required level of edge-connectivity \(p_{ij} \in \{0, 1, \ldots, p\}\) and a fault-tolerance \(q_{ij} \in \{0, 1, \ldots, q\}\) is specified. The goal is to find a subgraph \(H = (V, F)\) of minimum cost such that for any pair of nodes \(\{i, j\}\) and any set of at most \(q_{ij}\) unsafe edges \(F' \subseteq F\),
the graph $H \setminus F'$ has at least $p_{ij}$ edge-disjoint $(i,j)$-paths. We present a $2(p+1)$-approximation algorithm when $q = 1$ (i.e., when each $q_{ij} \in \{0,1\}$). We also provide a $2(q + 1)$-approximation algorithm when $p = 1$ (i.e., when each $p_{ij} \in \{0,1\}$). The main idea is to formulate the problems as a capacitated network design problem (Cap-NDP), and then apply Jain’s iterative rounding method, [10], to solve the resulting Cap-NDP instance.

3.2.1 ($\{0,1,\ldots,p\},\{0,\ldots,q\}$)-FGC with $q = 1$

First, we consider the ($\{0,1,\ldots,p\},\{0,\ldots,q\}$)-FGC model when $q = 1$ (i.e., when each $q_{ij} \in \{0,1\}$). We formulate a corresponding instance of Cap-NDP where the capacity of an edge $e$ is equal to $p+1$ if edge $e$ is safe and is equal to $p$ otherwise (thus, $u_e = p$ if $e$ is unsafe and $u_e = p+1$ if $e$ is safe); moreover, in the Cap-NDP instance, fix the demand between a pair of nodes $\{i,j\}$ to be $D_{ij} = (p + q_{ij})p_{ij}$. This gives rise to the following integer program.

$$
\begin{align*}
\min & \quad \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(S)} u_e x_e \geq \max_{(i,j): i \in S, j \notin S} D_{ij} & \forall \emptyset \neq S \neq V \\
 & \quad x_e \in \{0, 1\} & \forall e \in E
\end{align*}
$$

**Lemma 1.** A feasible solution $H = (V,F)$ of the ($\{0,1,\ldots,p\},\{0,1\}$)-FGC problem corresponds to a feasible solution of the above integer program.

**Proof.** A feasible solution $H = (V,F)$ is characterized by the following property: For any $(i,j)$-cut $\delta_H(S)$ of $H$, the cut contains either $p_{ij}$ safe edges or at least $p_{ij} + q_{ij}$ edges in total. We claim that feasible solutions to the above integer program are also characterized by the same property. Indeed if the cut $\delta_H(S)$ has at least $p_{ij} + q_{ij}$ edges in total, then the capacity across this cut is at least $(p_{ij} + q_{ij})p \geq p_{ij}p + q_{ij}p_{ij} = D_{ij}$. If the cut $\delta_H(S)$ has fewer than $p_{ij} + q_{ij}$ edges and exactly $p_{ij}$ edges, then $q_{ij} = 1$. We argue that in this case all edges in this cut must be safe edges, otherwise, the capacity across this cut will be strictly less than $p_{ij}(p+1) = D_{ij}$. Finally, if the cut $\delta_H(S)$ has fewer than $p_{ij}$ edges, then the capacity across this cut is at most $(p_{ij} - 1)(p + 1) < D_{ij}$. \qed

To solve (approximately) the above instance of Cap-NDP, we replace each edge $e \in E$ by $u_e$ parallel edges $e_1', \ldots, e_{u_e}'$ of capacity one and cost $c_e$ each, obtaining an instance of the survivable network design problem (SNDP). Suppose that $F^*$ is an optimal solution to the Cap-NDP instance. Then taking all the edges $\{e_1', \ldots, e_{u_e}'\}_{e \in F^*}$ gives us a feasible solution to the SNDP instance. The cost of this solution is at most $(p + 1)c(F^*)$. Using Jain’s iterative rounding method, [10], we can obtain a solution $F'$ to the SNDP instance with cost at most $2(p + 1)c(F^*)$. Finally, by picking edges $e \in E$ if $e_i \in F'$ for any $i = 1,\ldots, u_e$, we obtain a feasible solution to the Cap-NDP instance with cost at most $2(p + 1)c(F^*)$.

3.2.2 ($\{0,1,\ldots,p\},\{0,\ldots,q\}$)-FGC with $p = 1$

Next, we consider the ($\{0,1,\ldots,p\},\{0,\ldots,q\}$)-FGC model when $p = 1$ (i.e., when each $p_{ij} \in \{0,1\}$). We formulate a corresponding instance of Cap-NDP where the capacity of an edge $e$ is equal to $q+1$ if edge $e$ is safe and is equal to 1 otherwise (thus, $u_e = 1$ if $e$ is unsafe and $u_e = q + 1$ if $e$ is safe); moreover, in the Cap-NDP instance, fix the demand between a pair of nodes $\{i,j\}$ to be $D_{ij} = (q_{ij} + 1)p_{ij}$. This gives rise to the following integer program.
$$\min \sum_{e \in E} c_e x_e$$

s.t. $$\sum_{e \in \delta(S)} u_e x_e \geq \max_{(i,j):i \in S,j \notin S} D_{ij} \quad \forall \emptyset \neq S \neq V$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

Lemma 2. A feasible solution $$H = (V,F)$$ of the $$\{0,1\}$$-FGC problem corresponds to a feasible solution of the above integer program.

Proof. A feasible solution $$H = (V,F)$$ is characterized by the following property: For any $$(i,j)$$-cut $$\delta_H(S)$$ of $$H$$, the cut contains either $$p_{ij} \in \{0,1\}$$ safe edges or at least $$p_{ij} + q_{ij}$$ edges in total. We claim that feasible solutions to the above integer program are also characterized by the same property. Indeed if the cut $$\delta_H(S)$$ has at least $$p_{ij} + q_{ij}$$ edges in total, then the capacity across this cut is at least $$p_{ij}(q_{ij} + 1) = D_{ij}$$. If the cut $$\delta_H(S)$$ has at least $$p_{ij}$$ safe edges, then the capacity across this cut is at least $$(q + 1)p_{ij} \geq (q_{ij} + 1)p_{ij} = D_{ij}$$. Finally, if the cut $$\delta_H(S)$$ contains fewer than $$p_{ij} + q_{ij}$$ edges in total and fewer than $$p_{ij}$$ safe edges, then we must have $$p_{ij} = 1$$, so then, the cut contains no safe edges. Hence, the capacity across this cut is at most $$q_{ij} < D_{ij}$$.

We have already seen how to solve (approximately) the above instance of Cap-NDP and thus we can obtain a feasible solution with cost at most $$2(q + 1)c(F^*)$$, where $$F^*$$ is an optimal solution.

4 The NC-FGC model

In this section, we introduce the Node-Connectivity Flexible Graph Connectivity problem, denoted NC-FGC. We observe that there is a simple reduction from the NC-FGC model to the well-known NC-SNDP model; the latter model has been studied for decades, see [11, 12]. Nevertheless, for the (uniform connectivity) $$p$$-NC-FGC model, assuming that there is at least one safe node, we show that there is a 2-approximation algorithm via a result of Frank [9, Theorem 4.4]. In contrast, for the well-known special case of (uniform connectivity) $$p$$-NC-SNDP, even with the assumption of $$|V(G)| \geq p^3$$, the best approximation ratio known is $$4 + \epsilon$$ due to Nutov [13].

In an instance of the NC-FGC problem, we have an undirected connected graph $$G = (V,E)$$, a partition of $$V$$ into a set of safe nodes $$V^S$$ and a set of unsafe nodes $$V^U$$, and non-negative costs $$c \in \mathbb{R}^E$$ on the edges; moreover, for every pair of nodes $$\{s,t\}$$, a required level of connectivity $$r_{st} \in \mathbb{Z}_{\geq 0}$$ is specified. The goal is to find a subgraph $$H = (V,F)$$ of minimum cost such that for any pair of nodes $$\{s,t\}$$, and any set of unsafe nodes $$\bar{U} \subseteq V^U - \{s,t\}$$, the graph $$H - \bar{U}$$ has $$\min(0, r_{st} - |\bar{U}|)$$ edge-disjoint $$(s,t)$$-paths. The goal can be re-stated using the notion of q-connectivity, see Nutov [12]. Given node capacities $$q \in \mathbb{Z}^V_{\geq 0}$$, the q-connectivity of a pair of nodes $$\{s,t\}$$ of a graph $$H$$, denoted $$\lambda_H^{q}(s,t)$$, is the maximum number of pairwise edge disjoint $$(s,t)$$-paths such that each node $$v \in V - \{s,t\}$$ is in $$\leq q_v$$ of these paths. We fix $$q_v = \infty$$ for each safe node $$v$$, and we fix $$q_u = 1$$ for each unsafe node $$u$$. Then, the goal is to find a minimum cost subgraph $$H = (V,F)$$ such that $$\lambda_H^{q}(s,t) \geq r_{st}$$ for each pair of nodes $$\{s,t\}$$. For some special cases of the q-connectivity model, the known approximation algorithms for the well-known NC-SNDP model extend to the q-connectivity model with the same approximation ratio, see [12] and [11]. To see this directly for the NC-FGC model (with $$q \in \{1,\infty\}^V$$), one can “inflate” each safe node $$v$$ of $$G$$ to a complete graph $$K_v$$ on $$\deg_G(v)$$ nodes with edges of cost zero, and replace the edges incident to $$v$$ (in $$G$$) by edges incident to distinct nodes of $$K_v$$ (in the inflated graph) while
preserving the edge costs; thus, any \((s, t)\)-path of \(G\) maps to a \((K_s, K_t)\)-path of the inflated graph and vice versa.

Let \(p\)-NC-FGC denote the special case of the NC-FGC model with a uniform connectivity requirement of \(r_{st} = p\) for every pair of nodes \(\{s, t\}\). We apply a result of Frank, [9, Theorem 4.4], to present a 2-approximation algorithm for the \(p\)-NC-FGC model, assuming that there is at least one safe node.

**Proposition 3.** Given an instance \(G = (V, E), c \in \mathbb{R}^E_{\geq 0}\), \(p\) of \(p\)-NC-FGC such that \(V^S \neq \emptyset\), there is an (polynomial-time) algorithm that finds a feasible subgraph of cost \(\leq 2 \cdot \text{opt}\).

**Proof.** We construct a digraph \(D\) by replacing each edge \(e\) of \(G\) by a pair of anti-parallel arcs that each have cost \(c_e\). We pick the root \(s_0\) to be any safe node of \(D\). Then we assign node capacities to the nodes of \(D\): we fix \(q_v = p\) for each safe node \(v\), and we fix \(q_u = 1\) for each unsafe node \(u\). Consider the (directed) rooted \(q\)-connectivity problem for \(D, q, c, s_0\): the goal is to find a minimum-cost subgraph \(\overrightarrow{H}\) of \(D\) such that \(\lambda_{\overrightarrow{H}}(s_0, t) \geq p\) for each node \(t \in V(D) - s_0\), where \(\lambda_{\overrightarrow{H}}(s_0, t)\) denotes the maximum number of \((s_0, t)\)-dipaths in \(\overrightarrow{H}\) such that each node \(v \in V - \{s_0, t\}\) is in at most \(q(v)\) of these dipaths. Frank, [9, Theorem 4.4], presents a reduction from the (directed) rooted \(q\)-connectivity problem to weighted matroid intersection. Thus, we can find a minimum-cost rooted \(q\)-connected subgraph \(\overrightarrow{H}\) of \(D\).

Finally, we return the subgraph \(H\) of \(G\) that corresponds to \(\overrightarrow{H}\), where \(H\) has an edge \(ij\) if \(\overrightarrow{H}\) contains one of the arcs \((i, j)\) or \((j, i)\). We claim that \(H\) is a feasible subgraph of the instance of \(p\)-NC-FGC, that is, \(\lambda_{H}(s, t) \geq p\) for every pair of nodes \(\{s, t\}\), where \(q\) is as above. To verify this, consider a pair of nodes \(\{s, t\}\) and any set \(\hat{U} \subseteq V^U - \{s, t\}\) of size \(\leq p - 1\). Observe that \(\overrightarrow{H} - \hat{U}\) has \(p - |\hat{U}|\) arc disjoint \((s_0, t)\)-dipaths (such that these dipaths contain at most one arc from each pair of anti-parallel arcs); hence, \(H - \hat{U}\) has \(p - |\hat{U}|\) edge disjoint \((s_0, t)\)-paths. Similarly, it follows that \(H - \hat{U}\) has \(p - |\hat{U}|\) edge disjoint \((s_0, s)\)-paths. Hence, \(H - \hat{U}\) has \(p - |\hat{U}|\) edge disjoint \((s, t)\)-paths. This proves our claim that \(H\) is a feasible subgraph of the instance of \(p\)-NC-FGC.

Observe that \(H\) has cost \(\leq 2 \cdot \text{opt}\), because the optimal subgraph of the instance of \(p\)-NC-FGC corresponds to a subgraph of \(D\) of cost \(2 \cdot \text{opt}\) that is feasible for the above (directed) rooted \(q\)-connectivity problem. □
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