Scenario of the evolution of the universe with equation of state of the Weierstrass type gas

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Abstract. In this article, we examined the Starobinsky gravity model in homogeneous and isotropic space-time described by the Friedman-Robertson-Walker metric. In the framework of this model, cosmological parameters were described that explain it in the early and late evolutionary times, such as the slow roll parameters, perturbation of curvature, scalar-tensor ratio, equation of state parameter and deceleration parameter. For reconstruction, an ansatz of a special type was chosen, which can be considered as a generalization of the Chaplygin type of gas. Energy conditions are also calculated for this model. Dependencies of these parameters on cosmological time are plotted graphically.

1. Introduction
Over the past 20 years, a huge mass of information has accumulated in cosmology, due to which there have been changes in the understanding of the evolution of the universe. It was shown that the universe is 95% composed of substances unknown to us – dark matter and dark energy. And the description of the evolution of the early universe is based on the Big Bang hypothesis. During this time, theoretical works appeared in which it was shown that at the stage of dominant dark energy, finite-time singularities of the future will arise [1, 2, 3, 4]. There is also the likelihood of a singularity called Big Crunch that predicts the fatal evolutionary path of our universe. To avoid such singularities, there are various ways to circumvent them, such as the cyclic universe [5], the ekpyrotic scenario, and the pulsating universe. One of the possible cyclical scenarios of the evolution of the universe was presented in [6]. In [7, 8, 9] investigated modified theory of gravity by Noether symmetry, in [10] applies reconstruction method in $F(T)$ gravity. In [11, 12, 13, 14, 15, 16, 17] investigated problems of evolution of the universe $F(R)$ gravity framework. In [18] considered tachyon field with cosmological observation data. In works [19] early time acceleration and cosmological perturbations in Horndeski gravity framework were considered.

In this work, to set a cosmological model, we use an ansatz in the form of an elliptic Weierstrass function. The Weierstrass function is chosen as a generalization of the equation of state of matter such as the Chaplygin gas - Weierstrass gas. The choice of such an equation of state of matter ensures the cyclical evolution. Elliptic functions are quite successfully used by cosmologists to describe the evolution of the universe as a whole and its local objects. For
example, in [2] it is shown that elliptic functions can be solutions of the Einstein equations in the case of the Friedmann-Lemetre-Robertson-Walker cosmology with a cosmological constant, in [3] the Lemeter-Tolman-Bondi models in the presence of the cosmological constant Λ, in [6], is showed a model of a gas similar to Chaplygin’s gas, which can totally describe the accelerated expansion of the universe. Therefore, its useful to describe all kinds of processes in the universe and it is justified. We give the ordinary differential equations used in the work [20]

\[ \dot{\wp}^2(t; g_2, g_3) = 4\wp^3 - g_3\wp - g_3, \wp = 6\wp^2 - \frac{1}{2}g_2, \wp^{(3)} = 12\wp\wp, \wp^{(4)} = 120\wp^3 - 18g_2\wp - 12g_3, \]

\[ \wp^{-1}(t; g_2, g_3) = \wp^{-1}, \wp^{-1} = \frac{g_2 - 12t^2}{2}\wp^{-3}, (\wp^{-1})^{(3)} = \frac{3(92 - 12t^2)^2}{4}\wp^{-5} - 4t\wp^{-3}, \]

\[ (\wp^{-1})^{(4)} = \frac{15(92 - 12t^2)^3}{8}\wp^{-7} - 54t(92 - 12t^2)\wp^{-5} - 12\wp^{-3}, \]

\[ \wp^{-1} = \frac{105(92 - 12t^2)^4}{16}\wp^{-9} - 270t(92 - 12t^2)^2\wp^{-7} - 72(92 - 30t^2)^2\wp^{-5} \] (1)

where \(\wp(t; g_2, g_3)\) and \(\wp^{-1}(t; g_2, g_3)\) – Weierstrass elliptic function and its inverse function, respectively, \(g_2\) and \(g_3\) – invariants of the Weierstrass function, “dot” means the derivative with respect to the dimensionless time \(t\).

Real universe contains a huge amount of material fields; therefore, the energy-momentum tensor will be multicomponent. In this case, even if we would know an exact form of the contribution of each field individually and its equations of motion, it is difficult to take into account the contribution of each component to the energy-momentum tensor. Thus, it may seem that there is little hope of predicting the existence of singularities in the universe according to Einstein’s equations, since we do not know their right-hand side. However, apparently, there are reasons to require the fulfillment of certain inequalities for the energy-momentum tensor. It turns out that in many situations they are enough to prove the presence of singularities, regardless of the type of energy-momentum tensor. These inequalities were proposed by S. Hawking and called the conditions of energy dominance. In this paper, we study the obtained solution of the type [6] for fulfilling the conditions of energy dominance. Energy conditions for \(F(R)\) gravity models considered in [21, 22].

2. Starobinsky gravity model

Let us briefly outline the basics of the theory of gravity, in which the sum of the Ricci scalar and its square, known as a Starobinsky [1] model of gravity, acts as an invariant in action. We introduce the standard action for the gravitational field (\(k = 8\pi G = 1\)) We consider the conformal Einstein-Hilbert action in the following form

\[ S = \frac{1}{k} \int d^4x\sqrt{-g} \left( \frac{1}{2} \left( R + \frac{R^2}{6} \right) + L_m \right), \] (2)

together with the Friedman-Robertson-Walker metric, which describes the geometry of homogeneous and isotropic space-time

\[ ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \] (3)

Hereinafter \(R = 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = 6 \left( \dot{H} + 2H^2 \right)\) – Ricci scalar, \(\sqrt{-g} = a^3\), \(H = \frac{\dot{a}}{a}\) – Hubble parameter. We take as a matter a scalar field defined by the standard Lagrangian in the form

\[ L_m = \frac{\dot{\phi}^2}{2} - V(\phi). \] (4)
After substituting the Ricci scalar in (2) we get the Lagrangian in the following form

\[ L = -3a^2 \dot{a} + \frac{3}{2} \dot{a}^2 a + 3\ddot{a}a^2 + \frac{3}{2} \frac{\dot{a}^4}{a} + a^3 L_m. \] (5)

Next, substituting the Lagrangian (5) in the Euler-Poisson equation and the zero-energy condition

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} = 0, \] (6)

\[ \frac{\partial L}{\partial \dot{q}} \ddot{q} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \dddot{q} + \frac{\partial L}{\partial \dddot{q}} \dddot{q} - L = 0, \] (7)

where the notation is introduced \( q \equiv \{a, \phi\}, \dot{q} \equiv \{\dot{a}, \dot{\phi}\} \) and \( \ddot{q} \equiv \ddot{a} \). We get the system of equations of motion of gravitational and scalar fields in terms of the Hubble parameter

\[ -3\dot{H}^2 - 2\ddot{H} = \frac{9}{2} \dot{H}^2 + 9\dot{H}\ddot{H} + 6\dot{H} + H^{(3)} + p_\phi, \] (8)

\[ 3\dot{H}^2 = -3\dddot{H} - 9\dot{H}\ddot{H} + \frac{3}{2} \dot{H}^2 + \rho_\phi, \] (9)

\[ \dot{\phi} + 3H\dot{\phi} + V_\phi = 0. \] (10)

\[ \dot{\rho}_{eff} + 3H(p_{eff} + \rho_{eff}) = 0. \] (11)

Here \( p_{eff} = p_g + p_\phi \) and \( \rho_{eff} = \rho_g + \rho_\phi \); pressure \( p_\phi \) and energy density \( \rho_\phi \) of scalar fields are described by the energy-momentum tensor of an perfect fluid

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \] (12)

whose components are described by expressions

\[ p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi), \] (13)

\[ \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi). \] (14)

From (13) and (14) scalar field \( \dot{\phi}^2 \) and potential \( V(\phi) \) will take the form

\[ \dot{\phi}^2 = \rho_\phi + p_\phi, \] (15)

\[ V(\phi) = \frac{1}{2} (\rho_\phi - p_\phi). \] (16)

Form given (8), (9), (15) and (16) we get

\[ \dot{\phi}^2 = -H^{(3)} - 3\dddot{H} - 3\dot{H}^2 - 2\ddot{H}, \] (17)

\[ V = \frac{1}{2} H^{(3)} + \frac{9}{2} \dot{H}\ddot{H} + \frac{3}{2} \ddot{H}^2 + 9\dot{H}\dddot{H} + 3H^2. \] (18)

If we define the slow roll parameters [23] as follows

\[ \epsilon(\phi) = \frac{1}{2} \frac{(V')^2}{V}, \quad \eta(\phi) = \frac{V''}{V}, \]
or in case $\phi = \phi(t)$ and $V = V(t)$ then

$$
\epsilon(\phi) = \frac{1}{2\phi^2} \frac{\dot{V}}{V} \quad \text{and} \quad \eta(\phi) = \frac{\ddot{V}}{\phi^2} - \frac{\dot{V}^2}{\phi^3},
$$

(19)

where the first determines the slope of the potential, and the second determines the curvature. For the emergence and continuation of the inflationary stage, it is necessary that these parameters are in the region

$$
\epsilon(\phi) << 1, \quad |\eta(\phi)| << 1.
$$

(20)

(21)

It should be noted, that $\eta$ does not have to be small for inflation to take place. Inflation occurs when the condition (20) is true, regardless of the value of $\eta$. The fulfillment of conditions (20) and (21) means that the change of $\epsilon$ in the region under consideration is extremely small. When parameters of slow rolling take values $\epsilon(\phi) \approx 1$ and $\eta(\phi) \approx 1$, then the inflationary stage is exited.

Finally, for a canonical slowly rolling down scalar field, the spectral index of perturbations of curvature $n_s$ and the scalar tensor relation $r$ can be determined through parameters of slow rolling

$$
n_s \cong 1 - 6\epsilon + 2\eta, \quad r \cong 16\epsilon.
$$

(22)

(23)

The equation of state parameter and the deceleration parameter are

$$
\omega = \frac{p}{\rho}, \quad q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1.
$$

(24)

3. Reconstruction of cosmological model

We consider the reconstruction of a cosmological model with an anzatz in next form

$$
\dot{H} = \partial_t \left( A_1 \phi^{-1} \left( t^2 + A_2 \exp(t) \right) + A_3 t^{-\frac{3}{2}} \right),
$$

(25)

where $A_1, A_2, A_3$ – arbitrary constants of this model.

In this article we apply methods which described in [4, 5]. Dependence of $\dot{H}$ and $H$ shown on figure 1. From figure 1 we see, that this model is characterized by the initial moment of time and quasi-de Sitter behavior occurs, then the Hubble parameter grows and at the final stage enters the de Sitter behavior.

3.1. Early epoch of the universe

Reconstruction of an early epoch evolution of the universe leads to results shown on figure 2 and 2. The early epoch of the universe characterized by slow roll parameters and cosmological observational parameters scalar to tensor ratio $n_s$ and curvature perturbation $r$. From figure 2 follows, that slow roll parameter $\epsilon$ always less than 1. On the one hand, this provides inflationary statistics. On the other hand slow roll parameter $\eta$ at some point in time takes a value more than 1. This provides a way out of the inflationary stage.

The figure 3 shows scalar to tensor ratio and curvature perturbation versus cosmological time. According to cosmological observational data [24] scalar to tensor ratio $n_s \approx 0.96$ and curvature perturbation $r < 0.1$. In our case value of curvature perturbation agree with observational data.
3.2. Late epoch of the universe

One way to characterize an evolution of a late epoch of the universe is parameter equation of state and cosmography (deceleration parameter, jerk, snap et.al). Here we consider late epoch of the universe by parameter equation of state and deceleration parameter. Reconstruction of late epoch evolution of universe leads us to results shown on figure 4. From the figure follows, that our model on the late epoch behaves as ΛCDM model because parameter equation of state seek to $-1$. 

Figure 1. Dependence $\dot{H}$ and $H$ from cosmological time $t$ in (25) at $g_2 = g_3 = 1$ and following parameters of model $A_1 = 10$, $A_2 = 0.95$, $A_3 = 0.8$

Figure 2. Dependence of slow roll parameters $\epsilon$ and $\eta$ at $g_2 = g_3 = 1$ from cosmological time $t$

Figure 3. Scalar to tensor ratio $n_s$ and curvature perturbation $r$ from cosmological time $t$
Figure 4. Dependence of the parameter equation of state $\omega$ and deceleration parameter $q$ from cosmological time $t$

4. Energy conditions

Energy condition play an important role in the investigation of cosmological models and impose constraint on energy-momentum tensor. It has the following form

\[
NEC \Rightarrow q \geq -1, \quad (26)
\]
\[
SEC \Rightarrow q \geq 0, \quad (27)
\]
\[
DEC \Rightarrow q \leq 2. \quad (28)
\]

We analyze the energy conditions using the deceleration parameter $q$ shown in the figure 4. This figure can conditionally be divided into three areas: 1) $q > -1$; 2) $q < -1$ and 3) $q = -1$. Then we can get the following: NEC is not violated in areas 1 and 3. It provides acceleration expansion in early and late times of the universe and violated in area 2. It provides superacceleration regime in phantom area. SEC is violated in all areas. If SEC is not violated then the acceleration regime is turned off. DEC is not violated in all areas because NEC and DEC compatible and allow regimes in which $q < 0$.

5. Conclusion

In this article, we examined the Starobinsky gravity model in homogeneous and isotropic space-time described by the Friedman-Robertson-Walker metric. In the framework of this model, cosmological parameters were showed that they can describe it in the early and late evolutionary times, such as the slow roll parameters, perturbation of curvature, scalar-tensor ratio, equation of state parameter and deceleration parameter. For reconstruction, an ansatz of a special type was chosen, which can be considered as a generalization of the Chaplygin type of gas. This anzatz provides an exit from inflationary epoch and as $\Lambda$CDM model in late epoch of the universe. The energy conditions are also calculated and analyzed for this model.

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