Asymptotic properties of extremal Markov chains of Kendall type

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Abstract

We consider a class of max-AR(1) sequences connected with the Kendall convolution. For a large class of step size distributions we prove that the one dimensional distributions of the Kendall random walk with any unit step distribution, are regularly varying. The finite dimensional distributions for Kendall convolutions are given. We prove convergence of a continuous time stochastic process constructed from the Kendall random walk in the finite dimensional distributions sense using regularly varying functions.
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1 Introduction and model description

In this paper we study a class of extremal Markovian sequences ([1], [2]) called Kendall random walks \( \{X_n: n \in \mathbb{N}_0\} \), where \( \mathbb{N}_0 \) is the set of natural numbers including zero, introduced in [12] and studied in [13]. In contrast to [2], the sequences in the present paper have some dependency relationship between factors. The structure of the processes considered here is similar to the first order autoregressive maximal Pareto processes ([3], [4], [17], [21]), minification processes ([16], [17]), the max-autoregressive moving average processes MARMA ([11]), pARMAX and pRARMAX processes ([10]).

The Kendall convolution that is the main building block in construction of the process \( \{X_n\} \) is an important example of generalized convolution introduced by Urbanik [18, 19]. The importance of the development of generalized convolutions was motivated by seminal work by Kingman [15] about spherically symmetric random walks. In that paper one should look for sources of theory of generalized convolutions, which was introduced by Urbanik (see [15]). We believe that the richness of the class of generalized convolutions gives us hope for their widespread applications. It is worth to know that two most important examples of the convolutions: the one corresponding to classical sum and to classical maximum that are widely used in applications are also generalized convolutions.

Now imagine that we use other binary operations to model phenomena. This is a natural turn of events, after all, not everything can be described with the addition or operation of the maximum. Bingham in [5] and [6] investigated generalized convolutions in the context of regularly varying functions. Here we use the same technique to investigate the finite dimensional convergence of continuous time stochastic processes constructed by Kendall random walks. Since the random walks form a class of extremal Markov chains, then studying them will be a significant contribution to extreme value theory ([9]).

Innovative thinking about future possible applications comes from the fact that generalized convolution of two point-mass probability measures can be non-degenerate probability measure. In this paper we focus on the Kendall convolution case. The Kendall convolution of two probability measures concentrated at 1 is the Pareto distribution with density \( \pi_{2\alpha}(dy) = 2\alpha y^{-2\alpha-1}1_{[1,\infty)}(y)dy \). It is the main reason why it produces heavy tailed distributions, what we prove at this paper. The Lévy processes under generalized convolutions were introduced in [8]. Kendall random walks ([12]), considered here, are discrete time Markov chains with transition probabilities defined by Kendall convolution. The main mathematical tool in the Kendall convolution algebra, corresponding to characteristic function, which we use here is the Williamson transform.

Notation and organization of the paper: In Section 2 we give definition and properties of the Kendall convolution. The main mathematical tool, which we use here is the Williamson transform that is generalized characteristic function of the probability measure in the generalized convolution algebra ([8],[20]).
Next, in Section 3, we recall the definition and construction of random walks under the Kendall convolution (see, e.g., [12, 13]). In our proofs we use the fact, that random walks under weak generalized convolutions are discrete time Markov processes (see [8]) such that transition probabilities are characterized by the Williamson transform of probability measure $\delta_1$, unit step distribution and its Williamson transform. The finite dimensional distributions of the Markov chains are given in Section 4 and are described by truncated $\alpha$-moment and the Williamson transform of the unit step distribution. We present here also some examples of cumulative distribution functions of the random walks. Section 5 deals with the limit theorems. We prove asymptotic properties of random walks under Kendall convolution using regularly varying functions. Limit theorem for Kendall random walks is given in the case of finite $\alpha$-moment as well as in the case of regularly varying tail of unit step distribution. We define continuous time stochastic process based on random walk under Kendall convolution and prove convergence of finite-dimensional distributions using regular variation. The basic properties of Kendall random walks necessary for proofs are given in Appendix.

Through this paper, the distribution of the random element $X$ is denoted by $\mathcal{L}(X)$. For a probability measure $\lambda$ and $a \in \mathbb{R}_+$ the rescaling operator is given by $T_a \lambda = \mathcal{L}(aX)$ if $\lambda = \mathcal{L}(X)$. By $\mathcal{P}_+$ we denote family of all probability measures on the Borel subsets of $\mathbb{R}_+$. For abbreviation the set of all natural numbers with zero is denoted by $\mathbb{N}_0$. Additionally $\pi_{2\alpha}$ denotes random measure with Pareto distribution defined by probability density function $\pi_{2\alpha}(dy) = 2\alpha y^{-2\alpha-1}1_{[1,\infty)}(y)dy$. Moreover, by $m_{\nu}^{(\alpha)}$ we denote the $\alpha$th moment of measure $\nu$. For all $n \in \mathbb{N}$, by $F_n$ we denote the cumulative distribution function of measure $\nu^{\triangle_{\alpha}n}$, that is $F_n(t) = \nu^{\triangle_{\alpha}n}(0,t]$ and $H_n(t) := \int_0^t y^\alpha F_n(dy)$ is truncated $\alpha$-moment of the measure $\nu^{\triangle_{\alpha}n}$.

Finally a measurable function $f$ is regularly varying at infinity and with index $\beta$ if, for all $x > 0$, it satisfies
\[
\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^\beta.
\]

2 Kendall convolution and Williamson transform

We start with the definition of Kendall generalized convolution (see, e.g., [8]).

**Definition 2.1** Binary operation $\triangle_{\alpha} : \mathcal{P}_+^2 \to \mathcal{P}_+$ defined for discrete measures by
\[
\delta_x \triangle_{\alpha} \delta_y := T_M (\varrho \pi_{2\alpha} + (1 - \varrho)\delta_1),
\]
where $M = \max\{x, y\}$, $m = \min\{x, y\}$, $\varrho = m/M$, is called the Kendall convolution. The extension of $\triangle_{\alpha}$ to the whole $\mathcal{P}_+$ is given by
\[
\nu_1 \triangle_{\alpha} \nu_2(A) = \int_0^\infty \int_0^\infty (\delta_x \triangle_{\alpha} \delta_y)(A) \nu_1(dx) \nu_2(dy),
\]
(1)
Note that in the analyzed algebra the convolution of two point mass measures is a continuous measure that turns out to be Pareto

$$\delta_1 \triangle_\alpha \delta_1 = \pi_{2\alpha}.$$  

This is strictly different from the classical convolution world or the maximum convolution algebra, where convolution of discrete measures yields also a discrete one. For all $n \in \mathbb{N}$, the n-th Kendall convolution of measure $\nu$ is denoted by $\nu^{\triangle \alpha n} := \nu \triangle_\alpha \cdots \triangle_\alpha \nu$ (n-times).

In the Kendall convolution algebra the main tool used in the analysis of a measure $\nu$ is the Williamson transform (see [14], [20]) that is generalized characteristic function of $\nu$ and plays the same role as the classical Laplace or Fourier transform for convolutions defined by addition of independent random elements. For detailed properties and connections with generalized convolutions see e.g. [8].

**Definition 2.2** The operation $G: \mathbb{R} \to \mathbb{R}$ given by

$$G(t) = \int_0^\infty \Psi \left( \frac{x}{t} \right) \nu(dx), \quad \nu \in \mathcal{P}_+,$$

where

$$\Psi \left( \frac{x}{t} \right) = \left( 1 - \left( \frac{x}{t} \right)^\alpha \right)_+, \quad (3)$$

$a_+ = \max(0, a)$, is called the Williamson transform of measure $\nu$.

Observe that $\Psi(t)$ is a Williamson transform of $\delta_1$.

We use the notation $G_n(t)$ for the Williamson transform of the measure $\nu^{\triangle \alpha n}$. By the definition of the Williamson transform we have

$$G_n(t) = G(t)^n.$$  

(4)

## 3 Stochastic representation and basic properties of the process

In this section stochastic representation and basic properties of the Kendall random walk $\{X_n\}$ based on the generalized convolution $\nu^{\triangle \alpha n}$ are presented. For the proof and general discussion of existence of the Markov processes under generalized convolutions see [8]. Below we present recurrence construction of this kind of process based on Kendall convolution.

**Definition 3.1** Stochastic process $\{X_n: n \in \mathbb{N}_0\}$ is a discrete time Kendall random walk with parameter $\alpha > 0$ and step distribution $\nu \in \mathcal{P}_+$ if there exist

1. $(Y_k)$ i.i.d. random variables with distribution $\nu$,

2. $(\xi_k)$ i.i.d. random variables with uniform distribution on $[0, 1]$,
3. $(\theta_k)$ i.i.d. random variables with Pareto distribution with the density

$$\pi_{2\alpha}(dy) = 2\alpha y^{-2\alpha-1}1_{[1,\infty)}(y)\,dy,$$

such that sequences $(Y_k), (\xi_k)$ and $(\theta_k)$ are independent and moreover

$$X_0 = 1, \quad X_1 = Y_1, \quad X_{n+1} = M_{n+1} \left[ I(\xi_n > \theta_{n+1}) + \theta_{n+1}I(\xi_n < \theta_{n+1}) \right],$$

where $\theta_{n+1}$ and $M_{n+1}$ are independent and

$$M_{n+1} = \max\{X_n, Y_{n+1}\}, \quad m_{n+1} = \min\{X_n, Y_{n+1}\}, \quad \theta_{n+1} = \frac{m_{n+1}^\alpha}{M_{n+1}^\alpha}.$$

**Proposition 3.2** The process $\{X_n\}$ with the stochastic representation given by the Definition 3.1 is a homogeneous Markov process with a transition probability kernel

$$P_{k,k+n}(x, A) := P_n(x, A) = \delta_x \Delta_\alpha \nu^{\Delta n}(A), \quad (5)$$

where $k, n \in \mathbb{N}, A \in \text{Bor}[0, \infty), x \geq 0, \alpha > 0$.

It follows directly from the Definition 3.1 that the process $\{X_n\}$ satisfies the Markov property. Now, we show that, for all $k, n \in \mathbb{N}, A \in \text{Bor}[0, \infty), x \geq 0, \alpha > 0$, the transition probability of the process $\{X_n\}$ is of the form (5). In order to do this we shall proceed by induction starting with $k = 1$. By the Definition 3.1 we arrive at

$$\mathbb{P}(X_n \in A|X_{n-1} = x) = \int_0^\infty \mathbb{P}(X_n \in A|X_{n-1} = x, Y_n = y) \nu(dy)$$

$$= \int_0^\infty \left\{ \mathbb{P}(\max(x,y)\theta_n \in A, \xi_n < \left( \min(x,y) / \max(x,y) \right)^\alpha) + \mathbb{I}_A(\max(x,y))\mathbb{P}\left(\xi_n > \left( \min(x,y) / \max(x,y) \right)^\alpha) \right) \right\} \nu(dy).$$

By the independence of random variables $\theta_n$ and $\xi_n$ the above expression is equivalent to

$$\int_0^\infty \left\{ \left( \min(x,y) / \max(x,y) \right)^\alpha \mathbb{P}(\max(x,y)\theta_n \in A) + \left( 1 - \left( \min(x,y) / \max(x,y) \right)^\alpha \right) \mathbb{I}_A(\max(x,y)) \right\} \nu(dy)$$

$$= \int_0^\infty T_{\max(x,y)} \left[ \left( \min(x,y) / \max(x,y) \right)^\alpha \pi_{2\alpha}(A) + \left( 1 - \left( \min(x,y) / \max(x,y) \right)^\alpha \right) \delta_1(A) \right] \nu(dy)$$

$$= \int_0^\infty \left( T_{\max(x,y)} \left( \delta_{\min(x,y) / \max(x,y)} \Delta_\alpha \delta_1 \right) \right)(A) \nu(dy)$$

$$= \int_0^\infty \left( \delta_x \Delta_\alpha \delta_y \right)(A) \nu(dy) = (\delta_x \Delta_\alpha \nu)(A). \quad (6)$$

where (6) is by the Definition 2.1.
Now assuming that $\mathbb{P}(X_{k+n} \in A | X_k = x) = \delta_x \Delta_{\alpha} \nu^{\Delta_n}(A)$ holds for $n \geq 2$ we establish its validity for $n + 1$, any $A \in \text{Bor}[0,\infty)$ and $\alpha > 0$. In order to do this, observe that due to the Chapman-Kolmogorov equation of the process $\{X_n\}$ we have

$$
\mathbb{P}(X_{n+k+1} \in A | X_n = x) = \int_0^\infty \int_A P_1(y, dz) P_k(x, dy)
$$

$$
= \int_0^\infty (\delta_y \Delta_{\alpha} \nu)(A) (\delta_x \Delta_{\alpha} \nu^{\Delta_n})(dy) = (\delta_x \Delta_{\alpha} \nu^{\Delta_n+1})(A).
$$

The next two results that give characterization of transition probabilities play important role in the analysis of finite-dimensional distributions of the process $\{X_n\}$.

**Lemma 3.3** For all $k, n \in \mathbb{N}, x, y, t \geq 0, \alpha > 0$ we have

$$
(\delta_x \Delta_{\alpha} \delta_y) ([0,t]) = \left[ \Psi\left(\frac{x}{t}\right) + \Psi\left(\frac{y}{t}\right) - \Psi\left(\frac{x}{t}\right) \Psi\left(\frac{y}{t}\right) \right] 1_{\{x \leq t, y \leq t\}},
$$

(7)

$$
P_n(x, ((0,t])) = \left[ \Psi\left(\frac{x}{t}\right) F_n(t) + \left(1 - \Psi\left(\frac{x}{t}\right)\right) G_n(t) \right] 1_{\{x \leq t\}},
$$

(8)

**Proof.** The first equality is an analogue of the formula given in Lemma 3.1 in [14]. To obtain (8) observe that by (1) we have

$$
(\delta_x \Delta_{\alpha} \nu^{\Delta_n})(0,t) = \int_0^t (\delta_x \Delta_{\alpha} \delta_y)(0,t) \nu^{\Delta_n}(dy)
$$

$$
= \int_0^t \left[ \Psi\left(\frac{x}{t}\right) + \Psi\left(\frac{y}{t}\right) - \Psi\left(\frac{x}{t}\right) \Psi\left(\frac{y}{t}\right) \right] 1_{\{x \leq t\}} \nu^{\Delta_n}(dy)
$$

$$
= \left[ \Psi\left(\frac{x}{t}\right) F_n(t) + \left(1 - \Psi\left(\frac{x}{t}\right)\right) G_n(t) \right] 1_{\{x \leq t\}},
$$

since

$$
\int_0^t \Psi\left(\frac{y}{t}\right) \nu^{\Delta_n}(dy) = G_n(t).
$$

\[\square\]

### 4 Finite dimensional distributions

In this section we present a formula for finite dimensional distributions of the process $\{X_n\}$. In order to formulate the result it is convenient to introduce the notation

$$
\mathcal{A}_k = \{0,1\}^k \setminus \{(0,0,\ldots,0)\}, \quad \text{for any } k \in \mathbb{N}.
$$
Moreover, we denote
\[ \bar{\epsilon}_1 = \min \{ i \in \{1, \ldots, k \} : \epsilon_i = 1 \}, \ldots, \bar{\epsilon}_m := \min \{ i > \bar{\epsilon}_{m-1} : \epsilon_i = 1 \}, \ m = 1, 2, \ldots, s, \ s = \sum_{i=1}^{k} \epsilon_i. \]

**Theorem 4.1** Let \( n_1 \leq n_2 \cdots \leq n_k \) where \( n_j \in \mathbb{N} \) for all \( j \in \mathbb{N} \) and \( y_0 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq x_{k+1} \). Let \{ \( X_n : n \in \mathbb{N} \) \} be Kendall random walk with parameter \( \alpha > 0 \) and unit step distribution \( \nu \in \mathcal{P}_+. \) Then
\[
\int_{x_1}^{x_2} \cdots \int_{x_0}^{x_k} \Psi \left( \frac{y_k}{x_{k+1}} \right) P_{n_k-n_{k-1}}(y_{k-1}, dy_k) P_{n_{k-1}-n_{k-2}}(y_{k-2}, dy_{k-1}) \cdots P_{n_1}(y_0, dy_1)
\]
\[ = \sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in \mathcal{A}_k} \Psi \left( \frac{y_0}{x_{\bar{\epsilon}_1}} \right) \Psi \left( \frac{x_{\bar{\epsilon}_i}}{x_{\bar{\epsilon}_i+1}} \right) \prod_{i=1}^{s-1} \Psi \left( \frac{x_{\bar{\epsilon}_i}}{x_{\bar{\epsilon}_i+1}} \right) \prod_{j=1}^{k} (G(x_j))^{n_j-n_{j-1}-\epsilon_j} \left( \frac{(n_j-n_{j-1})H_1(x_j)}{x_j^\alpha} \right)^{\epsilon_j}
\]
\[ + \Psi \left( \frac{y_0}{x_{k+1}} \right) \prod_{j=1}^{k} (G(x_j))^{n_j-n_{j-1}} \]
where \( n_0 = 0 \) and
\[
\prod_{i=1}^{s-1} \Psi \left( \frac{x_{\bar{\epsilon}_i}}{x_{\bar{\epsilon}_i+1}} \right) = 1 \quad \text{for} \quad s = 1.
\]

**Proof.** Let \( k = 1 \). Then by (vii) in Lemma 6.1 we have
\[
\int_{x_1}^{x_2} \Psi \left( \frac{y_1}{x_2} \right) P_{n_1}(y_0, dy_1) =
\]
\[ = \Psi \left( \frac{y_0}{x_2} \right) G^{n_1}(x_1) + \frac{n_1}{x_1^\alpha} G^{n_1-1}(x_1) H_1(x_1) \Psi \left( \frac{y_0}{x_1} \right) \Psi \left( \frac{x_1}{x_2} \right),
\]
which ends the first step of proof by induction.

Now let assume that the formula holds for \( k \in \mathbb{N} \). We shall establish its validity for \( k + 1 \). Let
\[ \tilde{\eta}_1 = \min \{ i \geq 2 : \epsilon_i = 1 \}, \ldots, \tilde{\eta}_m := \min \{ i > \tilde{\eta}_{m-1} : \epsilon_i = 1 \}, \ m = 1, 2, \ldots, s_2, \ s_2 = \sum_{i=2}^{k+1} \epsilon_i. \]

Additionally \( s_1 = \sum_{i=1}^{k+1} \epsilon_i \).

Moreover, we denote
\[
\mathcal{A}_{k+1}^0 = \{(0, \epsilon_2, \epsilon_3, \cdots, \epsilon_{k+1}) \in \{0, 1\}^{k+1} : (\epsilon_2, \epsilon_3, \cdots, \epsilon_{k+1}) \in \mathcal{A}_k \},
\]
\[
\mathcal{A}_{k+1}^1 = \{(1, \epsilon_2, \epsilon_3, \cdots, \epsilon_{k+1}) \in \{0, 1\}^{k+1} : (\epsilon_2, \epsilon_3, \cdots, \epsilon_{k+1}) \in \mathcal{A}_k \}.
\]
By splitting $A_{k+1}$ into four subfamilies of sets, $A_{k+1}^0, A_{k+1}^1, \{(1,0,\ldots,0)\}$, and $\{(0,\ldots,0)\}$ and using formula for $k$ and the first induction step we obtain

$$
\int_{x_0}^{x_1} \cdots \int_{y_k}^{y_{k+1}} \Psi \left( \frac{y_{k+1}}{x_{k+2}} \right) P_{n_{k+1} - n_k} (y_k, dy_{k+1}) \cdots P_{n_{2} - n_1} (y_1, dy_2) \right] P_{n_1} (y_0, dy_1)
$$

$\sum_{(\epsilon_2, \epsilon_3, \ldots, \epsilon_{k+1}) \in A_{k+1}} \Psi \left( \frac{y_0}{x_{\eta_1}} \right) \prod_{i=1}^{s_2-1} \Psi \left( \frac{x_{\eta_i}}{x_{\eta_i+1}} \right) \int_0^{x_1} \Psi \left( \frac{y_1}{x_{\eta_1}} \right) P_{n_1} (y_0, dy_1) + \sum_{j=2}^{k+1} \prod_{i=1}^{s_j} \Psi \left( \frac{x_{\eta_i}}{x_{\eta_i+1}} \right) \int_0^{x_1} \Psi \left( \frac{y_j}{x_{k+2}} \right) P_{n_1} (y_0, dy_1)
$}

$$
S[A_{k+1}^0] = S \left[ (A_{k+1}^0) \right] + S \left[ (1,0,\ldots,0) \right] + S \left[ (0,\ldots,0) \right],
$$

where

$$
S[A_{k+1}^0] = \sum_{(0, \epsilon_2, \epsilon_3, \ldots, \epsilon_{k+1}) \in A_{k+1}^0} \Psi \left( \frac{y_0}{x_{\eta_1}} \right) \prod_{i=1}^{s_2-1} \Psi \left( \frac{x_{\eta_i}}{x_{\eta_i+1}} \right) \int_0^{x_1} \Psi \left( \frac{y_1}{x_{\eta_1}} \right) P_{n_1} (y_0, dy_1) + \sum_{j=2}^{k+1} \prod_{i=1}^{s_j} \Psi \left( \frac{x_{\eta_i}}{x_{\eta_i+1}} \right) \int_0^{x_1} \Psi \left( \frac{y_j}{x_{k+2}} \right) P_{n_1} (y_0, dy_1)
$$

$$
S[A_{k+1}^1] = \sum_{(1, \epsilon_2, \epsilon_3, \ldots, \epsilon_{k+1}) \in A_{k+1}^1} \Psi \left( \frac{y_0}{x_{1}} \right) \prod_{i=1}^{s_2-1} \Psi \left( \frac{x_{\eta_i}}{x_{\eta_i+1}} \right) \int_0^{x_1} \Psi \left( \frac{y_1}{x_{1}} \right) \prod_{j=2}^{k+1} \Psi \left( \frac{x_{\eta_j}}{x_{\eta_j+1}} \right)
$$

$$
S[(1,0,\ldots,0)] = \prod_{j=2}^{k+1} \Psi \left( \frac{x_{\eta_j}}{x_{\eta_j+1}} \right) \Psi \left( \frac{y_0}{x_{1}} \right) \prod_{j=2}^{k+1} \Psi \left( \frac{x_{\eta_j}}{x_{\eta_j+1}} \right)
$$

Now observe that for any sequence $(0, \epsilon_2, \epsilon_3, \ldots, \epsilon_{k+1}) \in A_{k+1}^0$ we have $(\tilde{\epsilon}_1, \tilde{\epsilon}_2, \ldots, \tilde{\epsilon}_{s_1}) = (\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_{s_2})$, with $s_1 = s_2$, which implies that

$$
\Psi \left( \frac{y_0}{x_{\eta_1}} \right) \prod_{i=1}^{s_2-1} \Psi \left( \frac{x_{\eta_i}}{x_{\eta_i+1}} \right) = \Psi \left( \frac{y_0}{x_{\tilde{\epsilon}_1}} \right) \prod_{i=1}^{s_1-1} \Psi \left( \frac{x_{\tilde{\epsilon}_i}}{x_{\tilde{\epsilon}_{i+1}}} \right).
$$

Moreover

$$
\prod_{j=2}^{k+1} \Psi \left( \frac{x_{\eta_j}}{x_{\eta_j+1}} \right) \Psi \left( \frac{y_0}{x_{1}} \right) \prod_{j=2}^{k+1} \Psi \left( \frac{x_{\eta_j}}{x_{\eta_j+1}} \right)
$$

$$
\prod_{j=1}^{k+1} \Psi \left( \frac{x_{\eta_j}}{x_{\eta_j+1}} \right) \Psi \left( \frac{y_0}{x_{1}} \right) \prod_{j=1}^{k+1} \Psi \left( \frac{x_{\eta_j}}{x_{\eta_j+1}} \right)
$$
\[ S[A_{k+1}^0] = \sum_{(x_1, x_2, \ldots, x_{k+1}) \in A_{k+1}^0} \Psi \left( \frac{y_0}{x_{\tilde{e}_1}} \right) \Psi \left( \frac{x_{\tilde{e}_j}}{x_{j+2}} \right) \prod_{i=1}^{s_1-1} \Psi \left( \frac{x_{\tilde{e}_i}}{x_{\tilde{e}_{i+1}}} \right) \prod_{j=1}^{k+1} (G(x_j))^{n_{j-n_{j-1}}-\epsilon_j} \left( \frac{(n_j - n_{j-1})H_1(x_j)}{x_{x_{x_j}}} \right)^{\epsilon_j}. \]  

(10)

Analogously for any sequence \((1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{k+1})A_{k+1}^1\) we have \((\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_{s_1}) = (1, \tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_{s_2})\), with \(s_1 = s_2 + 1\) which implies that

\[ \Psi \left( \frac{y_0}{x_1} \right) \Psi \left( \frac{x_{\tilde{e}_j}}{x_{\tilde{\eta}_j}} \right) \prod_{i=1}^{s_2-1} \Psi \left( \frac{x_{\tilde{e}_i}}{x_{\tilde{\eta}_{i+1}}} \right) = \prod_{i=1}^{s_1-1} \Psi \left( \frac{x_{\tilde{e}_i}}{x_{\tilde{e}_{i+1}}} \right). \]

(11)

where (11) is the consequence of

\[ \Psi \left( \frac{x_{\tilde{e}_j}}{x_{\tilde{\eta}_j}} \right) \prod_{i=1}^{s_2-1} \Psi \left( \frac{x_{\tilde{e}_i}}{x_{\tilde{\eta}_{i+1}}} \right) = \prod_{i=1}^{s_1-1} \Psi \left( \frac{x_{\tilde{e}_i}}{x_{\tilde{e}_{i+1}}} \right). \]

We also have

\[ \frac{n_1}{x_{\alpha}} (G(x_1))^{n_1-1}H_1(x_1) \prod_{j=2}^{k+1} (G(x_j))^{n_{j-n_{j-1}}-\epsilon_j} \left( \frac{(n_j - n_{j-1})H_1(x_j)}{x_{x_{x_j}}} \right)^{\epsilon_j} = \prod_{j=1}^{k+1} (G(x_j))^{n_{j-n_{j-1}}-\epsilon_j} \left( \frac{(n_j - n_{j-1})H_1(x_j)}{x_{x_{x_j}}} \right)^{\epsilon_j}. \]

Hence

\[ S[A_{k+1}^1] = \sum_{(1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{k+1}) \in A_{k+1}^1} \Psi \left( \frac{y_0}{x_{\tilde{e}_1}} \right) \Psi \left( \frac{x_{\tilde{e}_j}}{x_{j+2}} \right) \prod_{i=1}^{s_1-1} \Psi \left( \frac{x_{\tilde{e}_i}}{x_{\tilde{e}_{i+1}}} \right) \prod_{j=1}^{k+1} (G(x_j))^{n_{j-n_{j-1}}-\epsilon_j} \left( \frac{(n_j - n_{j-1})H_1(x_j)}{x_{x_{x_j}}} \right)^{\epsilon_j}. \]

(12)

Moreover, observe that

\[ S[\{(1, 0, \ldots, 0)\}] = \Psi \left( \frac{y_0}{x_{\tilde{e}_1}} \right) \Psi \left( \frac{x_{\tilde{e}_j}}{x_{j+2}} \right) \prod_{i=1}^{s_1-1} \Psi \left( \frac{x_{\tilde{e}_i}}{x_{\tilde{e}_{i+1}}} \right) \prod_{j=1}^{k+1} (G(x_j))^{n_{j-n_{j-1}}-\epsilon_j} \]

(13)

and, as \(n_0 = 0\) we have

\[ S[\{(0, \ldots, 0)\}] = \Psi \left( \frac{y_0}{x_{k+2}} \right) \prod_{j=1}^{k+1} (G(x_j))^{n_{j-n_{j-1}}}. \]

(14)

Finally, by combining (10), (12), (13), and (14) with (9) we obtain that

\[ \int_{0}^{x_1} \left[ \int_{0}^{x_2} \cdots \int_{0}^{x_{k+1}} \Psi \left( \frac{y_{k+1}}{x_{k+2}} \right) P_{n_{k+1}-n_k}(y_k, dy_{k+1}) \cdots P_{n_{2-n_1}}(y_1, dy_2) \right] P_{n_1}(y_0, dy_1) \]

\[ = \sum_{(e_1, e_2, \ldots, e_{k+1}) \in A_{k+1}} \Psi \left( \frac{y_0}{x_{\tilde{e}_1}} \right) \Psi \left( \frac{x_{\tilde{e}_j}}{x_{j+2}} \right) \prod_{i=1}^{s_1-1} \Psi \left( \frac{x_{\tilde{e}_i}}{x_{\tilde{e}_{i+1}}} \right) \prod_{j=1}^{k+1} (G(x_j))^{n_{j-n_{j-1}}-\epsilon_j} \left( \frac{(n_j - n_{j-1})H_1(x_j)}{x_{x_{x_j}}} \right)^{\epsilon_j} \]

\[ + \Psi \left( \frac{y_0}{x_{k+1}} \right) \prod_{j=1}^{k+1} (G(x_j))^{n_{j-n_{j-1}}}. \]
This completes the induction argument and the proof.

\[ \square \]

**Theorem 4.2** Let \( n_1 \leq n_2 \leq \cdots \leq n_k \) where \( n_j \in \mathbb{N} \) for all \( j \in \mathbb{N} \) and \( y_0 \leq x_1 \leq x_2 \leq \cdots \leq x_k \). Let \( \{X_n : n \in \mathbb{N}\} \) be a Kendall random walk with parameter \( \alpha > 0 \) and unit step distribution \( \nu \in \mathcal{P}_+ \). Then

\[
\mathbb{P}(X_{n_k} \leq x_k, X_{n_{k-1}} \leq x_{k-1}, \cdots, X_{n_1} \leq x_1) = \sum_{(\epsilon_1,\epsilon_2,\cdots,\epsilon_k) \in \{0,1\}^k} s^{-1} \prod_{i=1}^{s-1} \Psi\left(\frac{x_{\epsilon_i}}{x_{\epsilon_{i+1}}}\right) \prod_{j=1}^{k} (G(x_j))^{n_j-n_{j-1}-\epsilon_j} \left(\frac{(n_j - n_{j-1})H_1(x_j)}{x_j^{\alpha}}\right)^{\epsilon_j},
\]

where \( n_0 = 0 \), and

\[
\prod_{i=1}^{s-1} \Psi\left(\frac{x_{\epsilon_i}}{x_{\epsilon_{i+1}}}\right) = 1 \quad \text{for} \quad s \in \{0,1\}.
\]

**Proof.** By the definition of \( \Psi(\cdot) \) we see that, for any \( a > 0 \), we have

\[
\lim_{x_{k+1} \to \infty} \Psi\left(\frac{a}{x_{k+1}}\right) = 1 \quad \text{and} \quad \Psi\left(\frac{0}{a}\right) = 1.
\]

Moreover, observe that

\[
\mathbb{P}(X_{n_k} \leq x_k, X_{n_{k-1}} \leq x_{k-1}, \cdots, X_{n_1} \leq x_1) = \int_{0}^{x_1} \int_{0}^{x_2} \cdots \int_{0}^{x_k} P_{n_k-n_{k-1}}(y_{k-1},dy_{k})P_{n_{k-1}-n_{k-2}}(y_{k-2},dy_{k-1}) \cdots L_{n_1}(0,dy_1).
\]

Now in order to complete the proof it is enough to apply Theorem 4.1 with \( y_0 = 0 \) and \( x_{k+1} \to \infty \). \( \square \)

**Example 4.1** Let \( \nu = U(0,1) \) be uniform distribution with the density \( \nu(dy) = 1_{(0,1)}(y)dy \). Then

\[
F_n(x) = \left(\frac{\alpha}{\alpha + 1}\right)^n \left(1 + \frac{n}{\alpha}\right) x^n 1_{[0,1)}(x)
\]

\[
+ \left(1 - \frac{1}{(\alpha + 1)x^\alpha}\right)^{n-1} \left(1 + \frac{n-1}{(\alpha + 1)x^\alpha}\right) 1_{[1,\infty)}(x).
\]

The characterization given in Proposition 6.2 is the following:

\[
G(x) = (x \wedge 1) - \frac{(x \wedge 1)^{\alpha+1}}{(\alpha + 1)x^\alpha}, \quad H_1(x) = \frac{(x \wedge 1)^{\alpha+1}}{(\alpha + 1)}.
\]

**Example 4.2** Let \( \nu = \delta_1 \). Family of probability measures connected with probability measure concentrated at one is given by cumulative distribution function:

\[
F_n(x) = \left(1 + \frac{n-1}{x^\alpha}\right)^{n-1} \left(1 - \frac{1}{x^\alpha}\right)^{n-1} 1_{[1,\infty)}(x)
\]
for each \( n \geq 2, \alpha \in (0, 1] \). By Proposition 6.2, we have unit step characterization:

\[
G(x) = \left(1 - \frac{1}{x^\alpha}\right)_+ \quad \text{and} \quad H_1(x) = 1_{[1, \infty)}(x).
\]

**Example 4.3** Let \( \nu = p\delta_1 + (1-p)\pi_p \), where \( p \in (0, 1] \) and \( \pi_p \) be Pareto distribution with the density

\[
\pi_p(dx) = \frac{p}{x^{p+1}}1_{[1, \infty)}(x)dx.
\]

In this case we have characterization:

\[
G(x) = \left(1 - \frac{\alpha(1-p)}{(\alpha-p)}x^{-p} + \frac{p(1-\alpha)}{(\alpha-p)}x^{-\alpha}\right)1_{[1, \infty)}(x),
\]

\[
H_1(x) = p + \frac{p(1-p)}{(\alpha-p)}(x^{\alpha-p-1})1_{[1, \infty)}(x).
\]

For each natural number \( n \geq 2 \) and \( \alpha \in (0, 1] \) we have:

\[
F_n(x) = \left[1 - \frac{\alpha(1-p)}{(\alpha-p)}x^{-p} + \frac{p(1-\alpha)}{(\alpha-p)}x^{-\alpha}\right]^{n-1} \cdot \left[1 + \frac{(1-p)(np-\alpha)}{\alpha-p}x^{-p} - \frac{p(1-\alpha)(n-1)}{\alpha-p}x^{-\alpha}\right]1_{[1, \infty)}(x).
\]

and characterization

**Example 4.4** For the Kendall convolution the distribution having the tail

\[
\nu(x, \infty) = (1 - x^\alpha)_+.
\]

has the lack of memory property \((13)\). Then

\[
G(x) = \frac{x^\alpha}{2}1_{[0,1)}(x) + \left(1 - \frac{1}{2x^\alpha}\right)1_{[1, \infty)}(x), \quad H_1(x) = \frac{x^{2\alpha}}{2}1_{[0,1)}(x) + \frac{1}{2}1_{[1, \infty)}(x)
\]

and

\[
F_n(x) = \begin{cases}
\frac{1}{2} \frac{n+1}{2^n} x^{\alpha n} & \text{for } x \in [0, 1];
\frac{1}{2} \left(1 - \frac{1}{2x^\alpha}\right)^{n-1} \left(1 + \frac{n-1}{2x^\alpha}\right) & \text{for } x > 1.
\end{cases}
\]

**Example 4.5** Let \( \nu = \gamma(a, b) \) be the Gamma distribution with the density

\[
\gamma_{a,b}(dy) = \frac{b^a}{\Gamma(a)} y^{a-1}e^{-by}1_{(0, \infty)}(y)dy.
\]

Then \( \gamma(a, b)_{\alpha \cdot n} \) has the cumulative distribution function characterized by the functions:

\[
G(x) = \frac{1}{\Gamma(a)} \gamma(a, bx) - \frac{1}{\Gamma(a)x^\alpha} \gamma(a + \alpha, bx), \quad H_1(x) = \frac{1}{\Gamma(a)b^\alpha} \gamma(a + \alpha, bx),
\]

where \( \gamma(s, z) = \int_0^z t^{s-1}e^{-t}dt \) is the lower incomplete gamma functions. In particular for \( \alpha = 1 \) and \( a = 1 \), that is for exponential \( \nu \), the distribution function of \( X_n \) is characterized by

\[
G(x) = 1 - \frac{(1 - e^{-bx})}{bx}, \quad H_1(x) = \int_0^x ybe^{-by}dy = \frac{1 - e^{-bx} - bxe^{-bx}}{b}.
\]

11
5 Limit theorems

The main goal of this section is to investigate asymptotic properties of Kendall random walks, in particular using regular variation. In this case most of proofs is based on the following Lemma:

**Lemma 5.1** Let $H_1 \in RV_\theta$ and $0 < \theta < \alpha$. Then we can choose a sequence $\{a_n\}_n$ such that

$$\frac{a_n^\alpha}{H_1(a_n)} = n(1 + o(1))$$

as $n \to \infty$.

**Proof.** To see this, observe that $W(x) = x^\alpha/H_1(x) \in RV_{\alpha-\theta}$. Then, due to Thm. 1.5.12. in [7], there exists a function $V(x)$ such that $W(V(x)) = x(1 + o(1))$, as $x \to \infty$. In order to complete the proof it is enough to take $a_n = V(n)$.

**Theorem 5.2** For any $\alpha > 0, \nu \in \mathcal{P}_+$ we have

$$F_n(x) = nF(x) + \frac{1}{2}n(n-1)(H_1(x))^2x^{-2\alpha}(1 + o(1))$$

as $x \to \infty$.

**Proof.** Due to (19) in Lemma 6.1 we have the following representation for cumulative distribution function of measure $\nu^{\Delta a_n}$

$$F_n(x) = \left( F(x) - \frac{1}{x^\alpha}H_1(x) \right)^{n-1} \left( F(x) + \frac{n-1}{x^\alpha}H_1(x) \right)$$

$$= (F(x))^n + n \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n-1}{k-1} \frac{k-1}{k} \left( \frac{H_1(x)}{x^\alpha} \right)^k (F(x))^{n-k}$$

$$+ (-1)^{n-1}(n-1) \left( \frac{H_1(x)}{x^\alpha} \right)^n,$$  \hspace{1cm} (15)

where (15) is by application of binomial formula combined with the observation that for any $a \geq 0$ we have

$$(1-a)^{n-1}(1+a(n-1)) = 1 + n \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n-1}{k-1} \frac{k-1}{k} a^k + (-1)^{n-1}(n-1)a^n.$$

Thus

$$F_n(x) = I_1 + I_2,$$  \hspace{1cm} (16)

where

$$I_1 = 1 - (F(x))^n = nF(x)(1 + o(1))$$
as \( x \to \infty \). and

\[
I_2 = n \sum_{k=1}^{n-1} (-1)^k \frac{n-1}{k-1} \frac{(H_1(x))^{k-1}}{x^\alpha} (F(x))^{n-k} + (-1)^n (n-1) \left( \frac{H_1(x)}{x^\alpha} \right)^n.
\] (17)

Note that \( \lim_{x \to \infty} \frac{H_1(x)}{x^\alpha} = 0 \) for any measure \( \nu \in \mathcal{P}_+ \) and hence

\[
n \sum_{k=3}^{n-1} (-1)^k \frac{n-1}{k-1} \left( \frac{H_1(x)}{x^\alpha} \right)^{k-1} (F(x))^{n-k} + (-1)^n (n-1) \left( \frac{H_1(x)}{x^\alpha} \right)^n = o \left( \frac{1}{2} n(n-1) \left( \frac{H_1(x)}{x^\alpha} \right)^2 \right),
\]
as \( x \to \infty \). This completes the proof.

The following Corollary is a direct consequence of the Theorem 5.2.

**Corollary 5.3** Let \( \alpha > 0 \) and

(i) \( \overline{F}(x) \in \text{RV}_{\theta-\alpha} \) for \( 0 < \theta < \alpha \). Then

\[\overline{F}_n(x) = n\overline{F}(x)(1+o(1)) \quad \text{as} \quad x \to \infty.\]

(ii) \( m_{\nu}^{(\alpha)} < \infty \). Then

\[\overline{F}_n(x) = n\overline{F}(x) + \frac{1}{2} n(n-1) \left( m_{\nu}^{(\alpha)} \right)^2 x^{-2\alpha}(1+o(1)) \quad \text{as} \quad x \to \infty.\]

(iii) \( \overline{F}(x) = o \left( x^{-2\alpha} \right) \) as \( x \to \infty \). Then

\[\overline{F}_n(x) = \frac{1}{2} n(n-1) \left( m_{\nu}^{(\alpha)} \right)^2 x^{-2\alpha}(1+o(1)) \quad \text{as} \quad x \to \infty.\]

The following theorem present the limit distribution for extremal Markov chains of Kendall type in case of finite \( \alpha \)-moment as well as for regularly varying tail of the unit step.

**Theorem 5.4** Let \( \{X_n : n \in \mathbb{N}\} \) be Kendall random walk with unit step distribution \( Y_1 \sim \nu \in \mathcal{P}_+ \).

(i) If \( E(Y_1^{\alpha}) = m_{\nu}^{(\alpha)} < \infty \), then as \( n \to \infty \),

\[n^{-1/\alpha} X_n \xrightarrow{d} X \sim \rho_{\nu,\alpha} \neq \delta_0,
\]

where

\[
\rho_{\nu,\alpha}(dy) = \alpha \left( m_{\nu}^{(\alpha)} \right)^2 y^{-2\alpha-1} \exp\{-m_{\nu}^{(\alpha)} y^{-\alpha}\} \mathbf{1}_{(0,\infty)}(y)dy.
\]

(ii) If \( \overline{F} \in \text{RV}_{\theta-\alpha} \), where \( 0 \leq \theta < \alpha \), then there exists a sequence \( \{a_n\} \), \( a_n > 0 \), such that

\[a_n^{-1} X_n \xrightarrow{d} X \sim \rho_{\nu,\alpha,\theta} \neq \delta_0,
\]

where the density of \( \rho_{\nu,\alpha,\theta} \) is given by

\[
\rho_{\nu,\alpha,\theta}(dy) = \alpha y^{-2(\alpha-\theta)-1} \exp\{-y^{-(\alpha-\theta)}\} \mathbf{1}_{(0,\infty)}(y)dy.
\]
Proof. At the beginning, notice that by (i) of Lemma 6.1 the Williamson transform for $a_n^{-1}X_n$ is given by
\[ G\left(\frac{a_n}{z}\right)^n = \left(F\left(\frac{a_n}{z}\right) - \frac{z^\alpha}{a_n^\alpha}H_1\left(\frac{a_n}{z}\right)\right)^n. \] (18)

In order to prove (i) observe that under assumption of the finiteness of the $\alpha$-moment of unit step we have
\[ \lim_{n \to \infty} H_1\left(\frac{n^{1/\alpha}}{z}\right) = m_{\nu}^{(\alpha)} \quad \text{and} \quad \lim_{n \to \infty} F\left(\frac{a_n}{z}\right) = 1, \]
which, by (18), yields
\[ \lim_{n \to \infty} G\left(\frac{n^{1/\alpha}}{z}\right)^n = e^{-m_{\nu}^{(\alpha)}z^\alpha}. \]

Using Lemma 6.1(i) we can calculate inverse Williamson transform obtaining the cumulative distribution function of the measure $\rho_{\nu,\alpha}$:
\[
\rho_{\nu,\alpha}(0, y] = \begin{cases} 
(1 + m_{\alpha}y^{-\alpha}) e^{-m_{\alpha}y^{-\alpha}} & \text{for } y > 0 \\
0 & \text{for } y = 0
\end{cases}
\]
and the corresponding density function, which completes the proof of the case (i).

Proof of (ii) we start by noticing that if $F \in RV_{\theta-\alpha}$, then we have $H_1 \in RV_{\theta}$. Hence, by Lemma 5.1 we can choose a sequence $(a_n)_n$ such that
\[ \frac{a_n^\alpha}{H_1(a_n)} \sim n, \quad n \to \infty. \]

Moreover
\[ H_1\left(\frac{a_n}{z}\right) = z^{-\theta}H_1(a_n)(1 + o(1)), \]
as $n \to \infty$, which implies that
\[ \lim_{n \to \infty} \left(F\left(\frac{a_n}{z}\right) - \frac{z^\alpha}{a_n^\alpha}H_1\left(\frac{a_n}{z}\right)\right)^n = e^{-z^{\alpha-\theta}}. \]

Finally, by Lemma 6.1(i) there exists a measure $\rho_{\nu,\alpha,\theta} \in \mathcal{P}_+$, with cumulative distribution function
\[
\rho_{\nu,\alpha,\theta}(0, y] = \begin{cases} 
(1 + y^{-(\alpha-\theta)}) e^{-y^{-(\alpha-\theta)}} & \text{for } y > 0 \\
0 & \text{for } y = 0
\end{cases}
\]
and density function
\[
\rho_{\nu,\alpha,\theta}(dz) = \alpha y^{-(\alpha-\theta)-1} \exp\{-y^{-(\alpha-\theta)}\}1_{(0,\infty)}(y)dy.
\]
\[ \square \]
Example 5.1 Let us consider unit step distribution \( Y_1 \sim \rho_{\nu,\alpha} \), which is stable probability measure for Kendall random walk with unit step distribution \( \nu \) having the following density:

\[
\rho_{\nu,\alpha}(dy) = \alpha \left( m_{\nu}^{(\alpha)} \right)^2 y^{-2\alpha - 1} \exp\{-m_{\nu}^{(\alpha)}y^{-\alpha}\} \mathbf{1}_{(0,\infty)}(y)dy.
\]

and cumulative distribution function

\[
F(y) = \left( 1 + m_{\nu}^{(\alpha)} y^{-\alpha} \right) \exp\{-m_{\nu}^{(\alpha)} y^{-\alpha}\} \mathbf{1}_{(0,\infty)}(y).
\]

Then

\[
E(Y_1^\alpha) = m_{\nu}^{(\alpha)} < \infty
\]

and it is evident that

\[
G(z) = \exp\{-z^{-\alpha}\}
\]

\[
F_n(z) = F(n^{-1/\alpha}z).
\]

We define a new stochastic process \( \{Z_n(t) : n \in \mathbb{N}_0\} \) connected with Kendall random walk \( \{X_n : n \in \mathbb{N}_0\} \) such that

\[
Z_n(t) := a_n^{-1}X_{[nt]},
\]

i.e. \( \mathcal{L}(Z_n(t)) = T_{a_n^{-1}n^{\alpha}[nt]} \), where \([\cdot]\) denotes integer part and \(a_n\) is chosen such that

\[
\frac{a_n^\alpha}{H_1(a_n)} \sim n, \quad n \to \infty
\]

under assumption that \( H_1 \in RV_\theta \), where \( 0 \leq \theta < \alpha \).

We can prove the following convergence theorem of finite-dimensional distributions:

**Theorem 5.5** Let \( \{X_n : n \in \mathbb{N}_0\} \) be Kendall random walk with unit step distribution \( Y_1 \sim \nu \in \mathcal{P} \) such that \( H_1 \in RV_\theta \), where \( 0 \leq \theta < \alpha \). Then

\[
Z_n(t) \overset{d}{\to} Z(t)
\]

where \( \overset{d}{\to} \) denotes the convergence of finite-dimensional distributions and

\[
P(Z(t_1) \leq z_1, Z(t_2) \leq z_2, \ldots, Z(t_k) \leq z_k)
= \sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in \{0, 1\}^k} \prod_{i=1}^{s-1} \Psi \left( \frac{\tilde{z}_{\epsilon_i}}{\tilde{z}_{\epsilon_{i+1}}} \right) \prod_{j=1}^{k} \left( t_j - t_{j-1} \right) \frac{1}{\alpha - \theta} \sum_{i=1}^{k} \left( \beta_j \right)^{\theta - \alpha} \exp \left\{ -\frac{\alpha}{\alpha - \theta} \sum_{i=1}^{k} \left( \beta_j \right)^{\theta - \alpha} \right\}.
\]

where \( t_0 = 0 \) and

\[
\tilde{\epsilon}_1 = \min \{ i : \epsilon_i = 1 \}, \ldots, \tilde{\epsilon}_m := \min \{ i > \tilde{\epsilon}_{m-1} : \epsilon_i = 1 \}, \quad m = 1, 2, \ldots, s, \quad s = \sum_{i=1}^{k} \epsilon_i.
\]
Proof. For convergence of the k-dimensional distributions let us take \(0 \leq t_1 \leq t_2 \leq \cdots \leq t_k\), where \(k \in \mathbb{N}\). Then by Theorem 4.2 we obtain the distribution of \((Z_n(t_1), Z_n(t_2), \cdots, Z_n(t_k))\) in the following way:

\[
P(Z_n(t_1) \leq z_1, Z_n(t_2) \leq z_2, \cdots, Z_n(t_k) \leq z_k) = \sum_{(\epsilon_1, \epsilon_2, \cdots, \epsilon_k) \in \{0,1\}^k} \prod_{i=1}^{s-1} \psi \left( \frac{z_{\tilde{e}_i}}{z_{\tilde{e}_{i+1}}} \right) \prod_{j=1}^{k} (G(a_n z_j))^{[nt_j]-[nt_{j-1}]-\epsilon_j} \left( \frac{([nt_j]-[nt_{j-1}])H_1(a_n z_j)}{a_n^{\alpha} z_j^\alpha} \right)^{\epsilon_j}
\]

where \(t_0 = 0\),

\[
\tilde{e}_1 = \min \{i : \epsilon_i = 1\}, \ldots, \tilde{e}_m := \min \{i > \tilde{e}_{m-1} : \epsilon_i = 1\}, \quad m = 1, 2, \ldots, s, \quad s = \sum_{i=1}^{k} \epsilon_i.
\]

Now, due to property (4) of Williamson transform, for any \(t_j, z_j > 0\) and \(1 \leq j \leq k\), we have

\[
\lim_{n \to \infty} G(a_n z_j)^{[nt_j]-[nt_{j-1}]-\epsilon_j} = \lim_{n \to \infty} \left( F(a_n z_j) - \frac{H_1(a_n z_j)}{(a_n z_j)^\alpha} \right)^{[nt_j]-[nt_{j-1}]-\epsilon_j} = \exp \left\{ -(t_j - t_{j-1}) \frac{z_j^{\theta-\alpha}}{n} \right\}.
\]

Since \(H_1(a_n z) \sim z^\theta H_1(a_n)\), when \(z \to \infty\), one can see that by Lemma 5.1 we have

\[
\frac{H_1(a_n z_j)}{a_n^{\alpha}} \sim \frac{z_j^\theta}{n}, \quad_{n} \to \infty.
\]

It follows

\[
\lim_{n \to \infty} \left( \frac{([nt_j]-[nt_{j-1}])H_1(a_n z_j)}{a_n^{\alpha} z_j^\alpha} \right)^{\epsilon_j} = (t_j - t_{j-1}) \frac{z_j^{\theta-\alpha}}{n}.
\]

Hence

\[
\lim_{n \to \infty} P(Z_n(t_1) \leq z_1, Z_n(t_2) \leq z_2, \cdots, Z_n(t_k) \leq z_k) = \sum_{(\epsilon_1, \epsilon_2, \cdots, \epsilon_k) \in \{0,1\}^k} \prod_{i=1}^{s-1} \psi \left( \frac{z_{\tilde{e}_i}}{z_{\tilde{e}_{i+1}}} \right) \prod_{j=1}^{k} \left( (t_j - t_{j-1}) \frac{z_j^{\theta-\alpha}}{n} \right)^{\epsilon_j} \exp \left\{ -\sum_{i=1}^{k} z_i^{\theta-\alpha} (t_i - t_{i-1}) \right\},
\]

where \(t_0 = 0\).

Corollary 5.6 If \(H_1(\infty) = m^{(\alpha)}_{\nu} < \infty\), then \(H_1 \in RV_0\) and for sequence \((a_n), a_n > 0\) such that

\[
a_n \sim n^{1/\alpha}, \quad n \to \infty,
\]

we have

\[
Z_n(t) \xrightarrow{d} Z(t),
\]

where \(\xrightarrow{d}\) denotes the convergence of finite-dimensional distributions and

\[
P(Z(t_1) \leq z_1, Z(t_2) \leq z_2, \cdots, Z(t_k) \leq z_k) = \sum_{(\epsilon_1, \epsilon_2, \cdots, \epsilon_k) \in \{0,1\}^k} \prod_{i=1}^{s-1} \psi \left( \frac{z_{\tilde{e}_i}}{z_{\tilde{e}_{i+1}}} \right) \prod_{j=1}^{k} \left( (t_j - t_{j-1}) \frac{z_j^{\theta-\alpha}}{n} m^{(\alpha)}_{\nu} \right)^{\epsilon_j} \exp \left\{ -m^{(\alpha)}_{\nu} \sum_{i=1}^{k} z_i^{\theta-\alpha} (t_i - t_{i-1}) \right\},
\]

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where $t_0 = 0$ and

$$
\tilde{\epsilon}_1 = \min \{ i : \epsilon_i = 1 \}, \ldots, \tilde{\epsilon}_m := \min \{ i > \tilde{\epsilon}_{m-1} : \epsilon_i = 1 \}, \ m = 1, 2, \ldots, s, \ s = \sum_{i=1}^{k} \epsilon_i.
$$

**Proof.** For a sequence $(a_n)$, $a_n > 0$, such that $a_n \sim n^{1/\alpha}$, $n \to \infty$ and $H_1(\infty) = m^{(\alpha)}_\nu < \infty$ we have

$$
\lim_{n \to \infty} \left( \frac{(nt_j) - (nt_{j-1})}{a_n^\alpha \epsilon_j} \right)^{\epsilon_j} = (t_j - t_{j-1}) z_j^{-\alpha} m^{(\alpha)}_\nu
$$

and

$$
\lim_{n \to \infty} G(a_n z_j)^{[nt_j]-[nt_{j-1}]-\epsilon_j} = \lim_{n \to \infty} \left( F(a_n z_j) - \frac{H_1(a_n z_j)}{(a_n z_j)^\alpha} \right)^{[nt_j]-[nt_{j-1}]-\epsilon_j} = \exp \left\{ -m^{(\alpha)}_\nu (t_j - t_{j-1}) z_j^{\theta-\alpha} \right\},
$$

which proves the theorem.

**6 Appendix**

In the next lemma we present connections between functions $F_n(t), G_n(t), H_1(t)$.

**Lemma 6.1** Let $\nu \in P_+$ and $G(t) = \tilde{\nu}(t^{-1})$ and $H_1(t) = \int_0^t x^{\alpha} \nu(dx)$ for the probability measure $\nu \in P_+$ with cumulative distribution function $F$, $\nu(\{0\}) = 0$. For each $n \geq 1, t \geq x \geq 0$ we have

(i) $F(t) = G(t) + t^{-\alpha} H_1(t)$ except, possibly, for countable many points $t \in \mathbb{R}$;

(ii) $G(t) = \alpha t^{-\alpha} \int_0^t x^{\alpha-1} F(x) dx$;

(iii) $F(t) = G(t) + \frac{t}{\alpha} G'(t)$;

(iv) $F_n(t) = \left( F(t) - \frac{1}{t^\alpha} H_1(t) \right)^{n-1} \left( F(t) + \frac{(n-1)}{t^\alpha} H_1(t) \right)$ (19)

(v) $\delta_x \Delta_\alpha \nu^{\Delta_\alpha}(0, t) = \left( G(t)^n + \frac{n}{t^\alpha} H_1(t) G(t)^{n-1} \Psi \left( \frac{x}{t} \right) \right) 1_{\{x < t\}}$ (20)

(vi) $\int_0^t w^{\alpha} \left( \delta_x \Delta_\alpha \nu^{\Delta_\alpha} \right)(dw) = \left( x^{\alpha} G(t)^n + n G(t)^{n-1} H_1(t) \Psi \left( \frac{x}{t} \right) \right) 1_{\{x < t\}}$, (21)
\[
\int_0^{x_{k-1}} \Psi \left( \frac{y_{k-1}}{x_k} \right) \left( \delta_{y_{k-2}} \Delta_\alpha \nu^{\Delta n} \right) (dy_{k-1}) = \\
= \left[ \Psi \left( \frac{y_{k-2}}{x_k} \right) G^{n}(x_k) + \frac{n}{x_{k-1}} G^{n-1}(x_k) H_1(x_k) \Psi \left( \frac{x_{k-1}}{x_k} \right) \Psi \left( \frac{y_{k-2}}{x_k} \right) \right] 1_{\{y_{k-2} < x_{k-1}\}},
\]

**Proof.** In order to prove (i) notice that since
\[
G(t) = \int_0^\infty \left( 1 - \left( \frac{x}{t} \right)^\alpha \right) \nu(dx) = \int_0^t \left( 1 - \left( \frac{x}{t} \right)^\alpha \right) \nu(dx) = F(t) - t^{-\alpha} \int_0^t x^\alpha \nu(dx),
\]
then we obtain
\[
H_1(t) = t^\alpha (F(t) - G(t)).
\]
Integrating by parts \( H_1(t) \) we arrive at
\[
H_1(t) = \int_0^t x^\alpha \nu(dx) = t^\alpha F(t) - \alpha \int_0^t x^{\alpha-1} F(x) dx,
\]
which implies that
\[
\int_0^t x^{\alpha-1} F(x) dx = \alpha^{-1} t^\alpha G(t)
\]
and ends the proof of (ii). The proof of (iii) one can find in [13].

Differentiating with respect to \( t \) we arrive at the first formula in the above lemma. This completes the proof. Since (i) we have \( G(t) = F(t) - t^{-\alpha} H_1(t) \), then by Proposition 6.2 we have
\[
F_n(t) = G(t)^{n-1} \left[ nt^{-\alpha} H_1(t) + G(t) \right] = \left( F(t) - \frac{1}{t^\alpha} H_1(t) \right)^{n-1} \left( F(t) + \frac{n-1}{t^\alpha} H_1(t) \right),
\]
which completes the proof of (iv).

Next, using (4) and Proposition 6.2 we prove (v) in the following way:
\[
\delta_x \Delta_\alpha \nu^{\Delta n} (0,t) = G_n(t) 1_{\{x < t\}} + \Psi \left( \frac{x}{t} \right) (F_n(t) - G_n(t)) = \left( G(t)^n + \frac{n}{t^\alpha} H_1(t)G(t)^{n-1}\Psi \left( \frac{x}{t} \right) \right) 1_{\{x < t\}}.
\]

In order to prove (vi) we integrate by parts the following expression
\[
\int_0^t \omega^\alpha (\delta_x \Delta_\alpha \delta_y) (dw) = \int_0^t \omega^\alpha (\delta_x \Delta_\alpha \delta_y) (0,t) - \int_0^t \alpha \omega^{\alpha-1} (\delta_x \Delta_\alpha \delta_y) (0,w) dw
\]
\[
= \left( x^\alpha - \frac{2x^\alpha y^\alpha + y^\alpha}{t^\alpha} \right) 1_{\{x < t \}}
\]
for all \(0 \leq x, y \leq t\). Now it is evident that
\[
\int_0^t w^\alpha \left( \delta_{x} \Delta_{\alpha} \nu^{\Delta_{\alpha} n} \right) \, (dw) = \int_0^\infty \int_0^t w^\alpha \left( \delta_{x} \Delta_{\alpha} \delta_y \right) \, (dw) \nu^{\Delta_{\alpha} n} (dy)
\]
\[= \int_0^t \left( x^\alpha - 2 \frac{x^\alpha y^\alpha}{t^\alpha} + y^\alpha \right) \nu^{\Delta_{\alpha} n} (dy) 1_{\{x < t\}}
\]
\[= \left( x^\alpha F_n(t) - 2 \frac{x^\alpha}{t^\alpha} H_n(t) + H_n(t) \right) 1_{\{x < t\}}
\]
\[= \left( x^\alpha G(t)^n + nG(t)^{n-1} H_1(t) \Psi \left( \frac{x}{t} \right) \right) 1_{\{x < t\}}.
\]
The last equation is a simple consequence of Proposition 6.2. Now we are able to prove (vii). We have
\[
\int_0^{x_{k-1}} \Psi \left( \frac{y_{k-1}}{x_k} \right) \left( \delta_{y_{k-2}} \Delta_{\alpha} \nu^{\Delta_{\alpha} n} \right) \, (dy_{k-1}) = \left( \delta_{y_{k-2}} \Delta_{\alpha} \nu^{\Delta_{\alpha} n} \right) (0, x_{k-1})
\]
\[= x_{k-1}^{-\alpha} \int_0^{x_{k-1}} y_{k-1}^{\alpha} \left( \delta_{y_{k-2}} \Delta_{\alpha} \nu^{\Delta_{\alpha} n} \right) \, (dy_{k-1})
\]
\[= \left[ \Psi \left( \frac{y_{k-2}}{x_k} \right) G^n(x_{k-1}) + \frac{n}{x_{k-1}^\alpha} G^{n-1}(x_{k-1}) H_1(x_{k-1}) \Psi \left( \frac{x_{k-1}}{x_k} \right) \Psi \left( \frac{y_{k-2}}{x_k} \right) \right] 1_{\{y_{k-2} < x_{k-1}\}},
\]
since the last equation is a consequence of (iv) and (v).

Using the Williamson transform we are able to compute cumulative distribution function \(F_n\):

**Proposition 6.2 (cf. Prop. 3.1, [12])** Let \(\nu \in \mathcal{P}\). For each natural number \(n \geq 2\) the cumulative distribution function \(F_n\) of measure \(\nu^{\Delta_{\alpha} n}\) is equal
\[
F_n(t) = G(t)^{n-1} \left[ nt^{-\alpha} H_1(t) + G(t) \right],
\]
where
\[
H_1(t) = \int_0^t x^\alpha \nu (dx).
\]

**Proof.** In order to prove the above proposition you only need to replace \(F(t)\) by \(F_n(t)\), \(G(t)\) by \(G_n(t)\) and \(H_1(t)\) by \(H_n(t)\) in Lemma 6.1(i) and notice that
\[
H_n(t) := \mathbb{E} \left( X_n^\alpha 1_{\{X_n < t\}} \right) = \int_0^t y^\alpha F_n(dy) = nt^\alpha G(t)^{n-1} (F(t) - G(t)) = nG(t)^{n-1} H_1(t).
\]

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