Mean-field Theory of the Localization Transition

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A mean field theory of the localization transition for bosonic systems is developed. Localization is shown to be sensitive to the distribution of the random site energies. It occurs in the presence of a triangular distribution, but not a uniform one. The inverse participation ratio, the single site Green’s function, the superfluid order parameter and the corresponding susceptibility are calculated, and the appropriate exponents determined. All of these quantities indicate the presence of a new phase, which can be identified as the Bose-glass.

The localization transition in disordered systems has been a focus of statistical and condensed matter physics ever since the famous paper of Anderson [1]. Though most of the work deals with the metal-insulator transition in fermionic systems, there is a recent surge of interest in bosonic models [2-3]. The superfluid-insulator transition in granular superconductors [4] or $^4$He in disordered media [5] are the paradigmatic experimental realizations of such systems. The theoretical understanding of phase transitions is typically based on a mean field description and subsequent fluctuation analysis. The generally held belief even today is that the spatial homogeneity of this “mean field” allows only for extended states, thus obliterating the localized phase [6]. Below we demonstrate that in a model of hard core bosons with random site energies and infinite range hopping this conventional wisdom does not hold, and with a suitable choice of the disorder distribution the localization transition can indeed be captured within the mean field technique.

To highlight the involved physics, consider the standard argument against a transition in infinite connectivity lattices. It is believed [2] that because of the infinite number of neighbors, every site will be connected to “virtually degenerate” sites for a continuous distribution of the disordered site energies. Thus hopping between these sites always gains kinetic energy with zero cost in potential energy, delocalizing the particles. However let us observe that for a finite system of size $N$, the gain from hopping between two sites is $O(1/N)$ and the potential energy difference to the energetically closest sites is of the same order, since one has $N$ site energies chosen independently from a finite interval. This means that there is a finite probability that no sites at all are available within the $1/N$ energy window set by the kinetic energy. When hopping from such a site, the potential energy cost certainly outweighs the kinetic energy gain.

A complex sum of gains and costs will decide whether a state will be localized or extended.

To demonstrate the nontriviality of this sum, consider the spectrum of the ordered array. The ground state is homogeneous, non-degenerate with an energy of -1, clearly maximally benefitting from the kinetic term. All of the other $N$ single particle excited states possess zero energy. This is so because in this model to ensure orthogonality to the ground state the wavefunctions of the excited states have fluctuating signs across the sample. This destructive quantum interference frustrates the kinetic term, preventing any gain from the hopping process. So for weak disorder there is more kinetic energy to be gained by staying extended, and only a sufficiently strong disorder can localize the ground state. For the excited states however staying localized at a suitable site offers the lowering of the total energy in the absence of a kinetic energy premium, thus the excited states will become localized for arbitrarily small disorder. As we will see, this physics can be brought out by a non-traditional choice of the disorder: we will focus on the triangular distribution of the site energies.

To establish the framework of the physics, consider the Hamiltonian

$$H = - \sum_{i,j} J_{ij} a_i^\dagger a_j$$ (1)

where $a_i^\dagger$ ($a_i$) creates (annihilates) a hard-core boson at site $i$ ($i = 1, \ldots, N$). We consider the case of infinite range hopping: $J_{ij} = N^{-1}$ for all pairs, where the mean-field approach is exact. $J_{ii} = \mu + \hbar_i$, where $\mu$ is the chemical potential and $\hbar_i$ is a random on-site energy. We recall that [2] in the absence of disorder there are two phases of the model: for generic fillings the bosons can propagate. At zero temperature they form a superfluid. On the other hand for precisely one boson per site, the particles localize in a Mott insulating phase. When we take away a single particle from this insulator, the resulting hole will behave exactly as the first particle added to the empty lattice. This can be proven by performing a particle-hole transformation on the Hamiltonian.

In what follows we approach the localized phase from two directions. First, from the Mott insulator by adding holes. In this case a one particle approach clearly suffices. We calculate the density of states (DOS) and the participation ratio, and show that above a critical disorder the ground state becomes localized. Critical exponents will be evaluated as well. Second, we approach the...
same phase by decreasing the density of holes from the extended phase and compute the superfluid order parameter and the susceptibility.

In studying the one particle problem, the quenched disorder averaging \( < \cdots >_{ave} \) is performed with the replica trick, allowing us to rewrite the DOS as:

\[
\rho(\lambda) = \frac{1}{N} < \sum_k \delta(\lambda + J_k) >_{ave}
\]

\[
= \lim_{n \to 0} \frac{2}{N\pi} \int dx_{ia} \frac{\partial}{\partial \lambda} \int dh \rho(h) \exp \left( -\frac{\lambda}{2} \sum_{i\alpha} z_{i\alpha}^2 - \frac{1}{2} \sum_{ij\alpha} J_{ij} x_{i\alpha} x_{j\alpha} \right)
\]

\[\lambda\] is an energy eigenvalue with a positive infinitesimal imaginary part and the \( J_{ij} \)'s are the eigenvalues of the \( J \) matrix. Next the off-diagonal terms are decoupled using an auxiliary field \( z_{i\alpha} \), and the \( x \) integrals are performed to transform the exponent into:

\[
S_{eff} = \frac{1}{2} \sum_{\alpha} z_{\alpha}^2 + N \ln < \exp \left[ -\frac{n}{2} \ln (\lambda - h) \right] > - \frac{1}{2N(\lambda - h)} \sum_{\alpha} z_{\alpha}^2 > ,
\]

where the averaging over the disorder sites energy \( \int dh \rho(h) A(h) \) is denoted by \( < A(h) > \). In the expectation value we use the inverse of the number of lattice sites \( N \) as a small parameter to carry out an exact expansion. We also keep only the terms linear in the replica number \( n \) to arrive at the exponent:

\[
S_{eff} = \frac{1}{2} \left( \ln < \frac{1}{\lambda - h} > \sum_{\alpha} z_{\alpha}^2 + Nn < \ln(\lambda - h) > \right) .
\]

Finally performing the integration over the auxiliary field \( z_{\alpha} \) yields:

\[
\rho(\lambda) = \frac{1}{\pi} \ln < \frac{1}{\lambda - h} > - \frac{1}{N\pi} \int dh \rho(h)\ln(1 + < \frac{1}{\lambda - h} >) = P(\lambda) + \frac{1}{N} \delta(\lambda + \lambda_0),
\]

where \( \lambda_0 \) is obtained from: \( < (\lambda_0 - h)^{-1} > = 1 \). We assume that \( P(h) \), the distribution of the site energies is non-zero on the interval \( (-\Delta, \Delta) \).

The DOS was easiest to obtain in the above path-integral framework, but the subsequent physical quantities can be determined by the simpler method of calculating the eigenvectors \( \varphi(\lambda) \) of the \( J \) matrix. One finds \( \varphi_i(\lambda) = m/(\lambda - h_i) \), where \( m \) is the superfluid order parameter: \( m = (1/N) \sum \varphi_i \). The self-consistency equation for \( m \) yields: \( 1 = 1/N \sum 1/(\lambda - h_i) \). If \( \lambda \) is inside \( P(h) \), then a few \( \varphi(\lambda)'s \) will be \( \sim O(1) \) and the rest of the \( \varphi(\lambda)'s \) will be \( \sim O(1/N) \) to satisfy the normalization condition, i.e. the states inside the continuum are localized, whereas for \( \lambda_0 \) outside the continuum all \( \varphi_i(\lambda_0) \sim O(1/\sqrt{N}), \) describing an extended state. This picture is in complete accordance with the above determined DOS. The ground state becomes localized if \( -\lambda_0 \) reaches the bottom of the band. This does not happen for a rectangular distribution since explicit calculation yields \( \lambda_0 = \Delta \coth \Delta > \Delta \). However for the triangular distribution \( P(h) = \Delta^{-2}(\Delta - |h|) \) the ground state eigenvalue \(-\lambda_0 = \lambda_0 \) is given implicitly by the equation

\[
\Delta^2 = (\lambda_0 - \Delta) \ln \frac{\lambda_0 - \Delta}{\lambda_0} + (\lambda_0 + \Delta) \ln \frac{\lambda_0 + \Delta}{\lambda_0} ,
\]

which is valid only for \( \lambda_0 > \Delta \). Taking the \( \lambda_0 \to \Delta + 0 \) limit in Eq. \( (\ref{eq:lambda0}) \) shows that \(-\lambda_0 \) reaches the bottom of the band at the critical disorder \( \Delta_c = 2 \ln 2 = 1.38 \). For \( \Delta > \Delta_c \) the DOS becomes identical to the distribution of the site energies, i.e. \( \rho(\lambda) = P(\lambda) \).

One measure of localization is the participation ratio \( P = (\sum_i |\varphi_i|^2)^2/(\sum_i |\varphi_i|^4) \). \( P \) is proportional to the system size \( N \) for extended states and remains \( \sim O(1) \) for localized ones. The expression for \( \varphi_i \) yields

\[
\frac{P}{N} = < (\lambda_0 - h)^{-2} > / < (\lambda_0 - h)^{-4} > ,
\]

for \( \Delta < \Delta_c \) and \( P/N = 0 \) for \( \Delta > \Delta_c \). For the uniform distribution there is no critical disorder, and the participation ratio is \( P/N = 3/(3\cosh^2 \Delta + \sinh^2 \Delta) \sim O(1) \), clearly indicating that the ground state remains extended for any finite disorder strength. For the triangular distribution the result of the integral \( (\ref{eq:lambda0}) \) is

\[
\frac{P}{N} = \frac{3\lambda_0^2}{3\lambda_0^2 - \Delta^2} \left( \frac{\lambda_0^2 - \Delta^2}{\Delta^2} \right)^2 \ln^2 \lambda_0^2 - \Delta^2 / \lambda_0^2 ,
\]

and \( \lambda_0 \) is obtained from Eq. \( (\ref{eq:lambda0}) \) for \( \Delta < \Delta_c \). As \( \lambda_0 \) approaches \( \Delta \), \( P/N \) disappears as \( \propto (\Delta_0 - \Delta)^2 \), indicating that for \( \Delta > \Delta_c \), the ground state becomes localized. From now on we will focus only on this more interesting case of the triangular distribution.

\[
\begin{figure}[h]
\begin{center}
\includegraphics[width=0.8\textwidth]{fig1}
\caption{Analytical and numerical results for the ground state participation ratio.}
\end{center}
\end{figure}
\]
One can independently test these results numerically by diagonalizing the matrix \( J \) and measuring \( \mathcal{P} \). Fig. 1 displays convincing agreement between analytic results (solid line) and numerical results (symbols). The crossing and subsequent decrease of \( \mathcal{P} \) with \( N \) (inset) provides compelling evidence for the localization of the ground state. To demonstrate how \( \mathcal{P} \) distinguishes between extended and localized states, one usually considers two possibilities, \( \varphi_i = 1/\sqrt{N} \) and \( \varphi_i = \delta_{i,0} \). These yield \( \mathcal{P} = N \) and \( \mathcal{P} = 1 \) respectively. One often equivalently phrases this analysis in terms of the inverse participation ratio \( \mathcal{P}^{-1} \) which takes on small values \( O(1/N) \) for extended states and values near unity for localized ones. However, if \( \varphi_i \) has a few large amplitudes on selected sites, and an extended background, \( \varphi_i = [1/2] \delta_{i,0} + 1/\sqrt{2(N-1)}(1-\delta_{i,0}) \), \( \mathcal{P} \) and \( \mathcal{P}^{-1} \) will be close to one and indicate, incorrectly, that the state is localized. To eliminate the possibility of misidentifying such “pseudo-localized” states, we computed \( \mathcal{P}^{-1}(n) \), which we define by systematically removing the \( n \) sites where the wavefunction assumes its largest values. We show the results of this calculation in Fig. 2.

When \( \Delta = 1.0 < \Delta_c \), the ground state is extended, and \( \mathcal{P}^{-1}(n) \) remains \( O(1/N) \) as sites are removed. Meanwhile, for all the excited states, \( \mathcal{P}^{-1} \) rapidly plunges to much smaller values than \( O(1/N) \), emphasizing that as a few sites are removed from the sum, the weight of the remaining sites is negligible. We checked this behavior on different lattice sizes, and found that \( \mathcal{P}^{-1}(n) \) fell slightly more rapidly with \( n \) for larger lattices, eliminating the possibility that the states were extended but over a small fraction of the lattice. We conclude that all states in the continuum are indeed localized. When \( \Delta = 1.7 > \Delta_c \) even the ground state is localized, as shown.

Returning to the analytic calculations, the transition can be parametrized by the chemical potential as well. If \( \mu \) exceeds a critical value \( \mu_c \), then the ground state has no holes. At \( \mu_c \) the first hole appears in the system. \( \mu_c \) is given by the lowest eigenvalue of the \(-J\) matrix, i.e. \( \mu_c = \lambda_0 \) for \( \Delta < \Delta_c \) and \( \mu_c = \Delta \) for \( \Delta > \Delta_c \). The transition is well captured by the imaginary time on-site Green’s function [8]

\[
g(\tau) \equiv \frac{1}{N} \sum_i \langle T_\tau a_i^\dagger(\tau) a_i(0) \rangle_0 = \int d\lambda \rho(\lambda) \exp[-\tau(\mu - \lambda)],
\]

for \( \tau > 0 \), where \( \langle \cdots \rangle_0 \) is the ground state expectation value, and \( T_\tau \) is the time ordering operator. For weak disorder the transition happens to the state at \(-\lambda_0 \). Combining Eqs. (9) and (8) one obtains at criticality:

\[
g(\tau) \approx \frac{1}{N} + \exp[-\tau(\lambda_0 - \Delta)] \quad \text{near the transition, for} \quad \tau \gg 1.
\]

On the other hand, for strong disorder \((\Delta > \Delta_c)\) the critical \( g(\tau) \) decays as: \( g(\tau) \propto \tau^{-2} \). This change of the critical behavior suggests that we enter into different phases for \( \Delta < \Delta_c \) and \( \Delta > \Delta_c \). It is worth noting that if the distribution \( P(h) \propto (\Delta - h)^\alpha \), then in the strong disorder regime \( g(\tau) \propto \tau^{-\alpha-1} \) right at the critical point.

This means that in the strong disorder regime the critical exponent depends on the distribution of the disorder.

If one approaches the same localized phase from higher densities of holes, these one particle techniques must be abandoned. We will concentrate on the free energy of the Hamiltonian [8]: \( f = k_B T \ln Z/N \), where \( Z = \text{Tr} \exp[-\beta H] \). Introducing the magnetization \( m \) as a Hubbard-Stratonovich field decouples the different sites to yield a single site problem:

\[
f = \frac{1}{4} m^2 - \frac{1}{\beta N} \sum_i \ln Q_i,
\]

where

\[
Q_i = \text{Tr} \exp \beta [(\mu + h_i) a_i^\dagger a_i + ma_i^\dagger a_i + \text{h.c.}]
\]

For hard-core bosons the Hilbert space is only two dimensional, as a site may be only empty or occupied by one particle. This allows the exact evaluation of \( Q \) and hence the free energy per site, which we give here only for \( T = 0 \):

\[
f = \frac{1}{4} m^2 - \frac{1}{2} < (\mu - h) + \sqrt{(\mu - h)^2 + m^2} > .
\]

\( m \) is determined by the saddle point condition: \( m = m < [(\mu - h)^2 + m^2]^{-1/2} > \). The order parameter \( m \) is proportional to \( \langle a_i \rangle \). The hard-core boson problem is equivalent to a spin 1/2 XY model in a transverse field, and \( m \) corresponds to the magnetization in the XY plane.

\( m = 0 \) is always a solution, but upon exiting the Mott insulator below a critical chemical potential \( \mu_c \) another solution with nonzero magnetization \( m \) appears. For
weak disorder $\Delta < \Delta_c$, $\mu_c = \lambda_0$. Expanding in $m$ near the critical point the magnetization behaves the same way as it does without disorder, i.e. $m \propto \sqrt{\mu_c - \mu}$, the well-known Landau result.

In the strong-disorder regime when increasing the density, the particles occupy localized states. Do we have a superfluid in this case? In order to get the answer, we write out the saddle-point equation for the triangular distribution:

$$
\Delta^2 = \mu_+ \ln \left( \frac{\mu_+ + \sqrt{\mu_+^2 + m^2}}{\mu + \sqrt{\mu^2 + m^2}} \right) + \mu_- \ln \left( \frac{\mu_- + \sqrt{\mu_-^2 + m^2}}{\mu + \sqrt{\mu^2 + m^2}} \right) + 2\sqrt{\mu^2 + m^2} - \sqrt{\mu_+^2 + m^2} - \sqrt{\mu_-^2 + m^2},
$$

where we have introduced the notation $\mu_+ = \mu + \Delta$ and $\mu_- = \mu - \Delta$. For strong disorder $\mu_c = \Delta$, because if $\mu$ is slightly below $\Delta$, i.e. $\mu_-$ is negative, one sees immediately from the second term of the right hand side (RHS) of Eq. (13) that $m$ must be different from zero, otherwise the logarithm 'blows up'. Clearly a superfluid is formed, even though the constituent particles possess localized wavefunctions. The physics is then that the phases of the localized wavefunctions lock up to yield the off-diagonal long range order. This situation is very reminiscent to the "localized superconductor" picture of Ma and Lee.

The above results clearly show that the truly localized phase occupies only a line in the $\mu - \Delta$ plane. However we think that this is indeed the seed of the "Bose-glass" phase, because for the finite range hopping model if one partitions the system into blocks of the size of the hopping length, each will support only one of these truly localized, non-superfluid states. There will be a macroscopic number of these blocks, thus the original line will expand into a finite region as a function of the density.

The one-particle states being localized, it is natural to assume that $m \ll (\mu_c - \mu) = |\mu_-|$. In this case the second term of the RHS of Eq. (13) behaves like $\mu_- \ln(m^2/|\mu_-|)$, and this term must be finite, so we find

$$
m \propto \sqrt{\mu_c - \mu} \exp \left( -\frac{a}{\mu_c - \mu} \right),
$$

where $a = \Delta(\Delta - \Delta_c)/2$. As we can see, below $\mu_c$ the system is superfluid, though the magnetization is much smaller than in the weak-disorder case. The critical behaviour is again different in the weak and strong disorder regime. It is remarkable that in the study of the corresponding one dimensional problem also essential singularities were found.

Finally we calculate the susceptibility by adding an infinitesimal in-plane field $B$ to $m$ and taking $\chi = -\partial^2 f/\partial B^2$. Approaching the transition from the $m > 0$ side for arbitrary disorder we get $\chi \propto (\mu - \mu_c)^{-1}$. On the other hand approaching from the non-superfluid side (i.e. $m = 0$), while for weak disorder one again obtains $\chi \propto (\mu - \mu_c)^{-1}$, for strong disorder $\chi$ remains finite even at the transition. This difference of the exponents again demonstrates that the transitions from the Mott insulator into the superfluid or into the localized region belong to different universality classes, further strengthening the argument that the localized region in fact is a separate phase. Just as for the Green’s function, the above exponents also depend on the asymptotics of the disorder distribution for strong disorder.

To summarize, we investigated the disordered hard core boson problem. We proved that, contrary to previous beliefs, the mean field theory is able to capture the localization transition. We approached the localized region both from the Mott insulator and from the superfluid phase, calculating the density of states, the inverse participation ratio, the on-site Green’s function, the magnetization and the susceptibility. We tested our theory with independent numerical investigations and found detailed agreement. The important observation is that the exponents of all of the above physical quantities differ for the direct insulator - superfluid and the insulator - glass or superfluid - glass transitions, clearly demonstrating that the glass transition belongs to a new universality class. We also gave a real space blocking argument why we expect the glass to expand into a region in the parameter space when the range of hopping is reduced to finite values.

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[1] P. W. Anderson, Phys. Rev. 109 1492 (1958).
[2] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B 40, 546 (1989).
[3] M. Wallin, E.S. Sorensen, S.M. Girvin and A.P. Young, Phys. Rev. B49, 12115 (1994); K. Singh and D.S. Rokhsar, Phys. Rev. B46, 3002 (1992); P. Nisamaneephong, L. Zhang and M. Ma, Phys. Rev. Lett. 71, 3803 (1993).
[4] A. F. Hebard and M. A. Palaarne, Phys. Rev. Lett. 65, 927 (1990); H. M. Jaeger, D. B. Haviland, A. M. Goldman, and B. G. Orr, Phys. Rev. B34, 4920 (1986).
[5] D. Finotello, K. A. Gillis, A. Wong, and M. H. W. Chan, Phys. Rev. Lett. 61, 1954 (1988); J. D. Reppy, J. Low Temp. Phys. 87, 205 (1992).
[6] F. Pázmándi and Z. Dománski, to appear in Phys. Rev. Lett.
[7] M. Ma and P. A. Lee, Phys. Rev. B32, 5658 (1985).
[8] T. Giamarchi and H. J. Schulz, Phys. Rev. B37, 325 (1988).