$gl(N|N)$ Super-Current Algebras for Disordered Dirac Fermions in Two Dimensions

S. Guruswamy$^a$, A. LeClair$^b$, A.W.W. Ludwig$^c$

$^a$ Institute for Theoretical Physics, Valckenierstraat 65, 1018 XE Amsterdam, THE NETHERLANDS

$^b$ Newman Laboratory, Cornell University, Ithaca, NY 14853.

$^c$ Department of Physics, University of California, Santa Barbara, CA 93106.

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Abstract

We consider the non-hermitian 2D Dirac Hamiltonian with (A): real random mass, imaginary scalar potential and imaginary gauge field potentials, and (B): arbitrary complex random potentials of all three kinds. In both cases this Hamiltonian gives rise to a delocalization transition at zero energy with particle-hole symmetry in every realization of disorder. Case (A) is in addition time-reversal invariant, and can also be interpreted as the random-field XY Statistical Mechanics model in two dimensions. The supersymmetric approach to disorder averaging results in current-current perturbations of $gl(N|N)$ super-current algebras. Special properties of the $gl(N|N)$ algebra allow the exact computation of the beta-functions, and of the correlation functions of all currents. One of them is the Edwards-Anderson order parameter. The theory is ‘nearly conformal’ and possesses a scale-invariant subsector which is not a current algebra. For $N = 1$, in addition, we obtain an exact solution of all correlation functions. We also study the delocalization transition of case (B), with broken time reversal symmetry, in the Gade-Wegner (Random-Flux) universality class, using a $GL(N|N;C)/U(N|N)$ sigma model, as well as its $PSL(N|N)$ variant, and a corresponding generalized random XY model. For $N = 1$ the sigma model is shown to be identical to the current-current perturbation. For the delocalization transitions (case (A) and (B)) a density of states, diverging at zero energy, is found.

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I. INTRODUCTION

Quantum field theory in the presence of quenched disorder is an increasingly important subject in condensed matter physics. Unfortunately, even in two dimensions, fully solvable but non-trivial models are rather scarce. In this paper we consider, amongst others, what is perhaps the simplest model of disordered fermions that can be exactly solved but also exhibits some non-trivial behavior. We work in two spatial dimensions throughout. The model can be defined by the euclidean action

\[ S_0 = \int \frac{d^2x}{2\pi} \left( \psi_L^\dagger (\partial_z + A_x) \psi_L + \psi_R^\dagger (\partial_z + A_z) \psi_R + m(x) \psi_R^\dagger \psi_L + m^*(x) \psi_L^\dagger \psi_R \right) \] (1.1)

where \(d^2x = dx dy\), \(z = (x + iy)/2\), \(\overline{z} = (x - iy)/2\), and \(A_x = A_x - iA_y, A_\overline{z} = A_x + iA_y\). The subscripts \(L, R\) on the fermion fields denote left versus right movers in the conformal theory with \(m = m^* = 0\). Letting \(m = V + iM, m^* = V - iM\), we take \(V\) and \(M\) to have gaussian probability distributions of equal strength \(g\) and zero mean, and similarly for \(A_x, A_y\) with strength \(g_A\):

\[ P[m, m^*] = \exp \left( -\frac{1}{g} \int \frac{d^2x}{2\pi} m^*(x)m(x) \right) \] (1.2)

\[ P[A_\overline{z}, A_\overline{z}] = \exp \left( -\frac{1}{g_A} \int \frac{d^2x}{2\pi} A_\overline{z}(x)A_\overline{z}(x) \right) \]

(The couplings \(g, g_A\) represent twice the variance.) Disorder averaging corresponds to a gaussian functional integral:

\[ \langle O \rangle = \int Dm^* Dm DA_\overline{z} DA_\overline{z} P[m, m^*]P[A_\overline{z}, A_\overline{z}] \langle O \rangle \] (1.3)

In the context of two-dimensional statistical mechanics, the above model may be thought of as a random field XY model or equivalently the random field sine-Gordon model [1], at the free-fermion point. The usual sine-Gordon model has \(m\) constant, which now acquires a random phase, preventing the theory from becoming massive. Random field XY models were first studied by Cardy and Ostlund [2] using replicas [3]. The action in Eq.(1.1) can be related (see below) to a 2D Dirac hamiltonian \(H_2\) subject to a real random mass \(M\), a random imaginary potential \(−iV\) with strength \(g_M = gV = g\), and also in a random imaginary gauge-field

\[ H_2 = (-i\partial_x + iA_x)\sigma_x + (-i\partial_y + iA_y)\sigma_y + M(x, y)\sigma_3 - iV(x, y) \] (1.4)

where \(\sigma_i\) are the standard Pauli matrices. This hamiltonian is non-hermitian. This theory should be compared with the models described in Ref.’s [4], [5] which were argued to have a

\[ ^1\text{partition function } Z = \int \mathcal{D}[\psi^\dagger, \psi] \exp(-S) \]

\[ ^2\text{The difference of the variances of Gaussian random variables } V \text{ and } M \text{ renormalizes to zero.} \]
transition in the universality class of the quantum Hall plateau transition; the latter models have a real random mass and a real scalar potential with strength \( g_M \neq g_V \), and a real random gauge field, the corresponding 2D Dirac Hamiltonian being hermitian. The same model as in Eq. (1.4), also describes the continuum limit of electrons hopping on a square lattice with \( \pi \) flux, exhibiting a delocalization transition at zero energy, first discussed by Hatsugai, Wen and Kohmoto [HWK] in Ref. [8]. This involves a doubling of the number of degrees of freedom (see section II and Appendix B), leading to a random hermitian 4-component Hamiltonian. Applications of our results to this delocalization transition will be described in section V.

We perform the disorder averaging using the supersymmetric method. This leads to a left-right current-current perturbation of the \( gl(N|N) \) conformal supercurrent algebra for the \( N \)-species version, used to compute \( N \)th moment disorder averages. Conformal field theory methods were previously used by Bernard for the case where \( g_M (or \ g_V) = 0 \), which leads to the \( osp(2N|2N) \) algebra. Also, the case of \( g_V = g_M = 0 \), but \( g_A \neq 0 \), where the conformal symmetry is unbroken and the model is equivalent to free bosons, was studied in [4]. [1]. Though our model has a non-zero renormalization group \( \beta \) function, some special features of the \( gl(N|N) \) algebra, in particular the nilpotency of the fermionic generators of \( gl(N|N) \) and the existence of two quadratic Casimirs, lead to a complete solution for all correlation functions in the case \( N = 1 \), and to the solution of the correlation functions of all Noether currents in the case of general \( N \). The only non-vanishing \( \beta \) function is that for \( g_A \) which we compute exactly. It only depends on \( g \). Integration of the renormalization group equations allows us to compute for instance the density of states of the delocalization problem of [8].

Our results reveal some interesting features of the current-current perturbations of Lie superalgebras in comparison with the analogous perturbation of the ordinary bosonic Lie algebras. Our models are integrable quantum field theories for the usual reasons (Lax pair, etc.), and are not conformal. However the structure of the theory allows a solution which doesn’t rely on exact S-matrices, form-factors, etc. Regarding the difficulty of solution they lie somewhere between the conformal field theories and the massive integrable theories.

Delocalization transitions of ‘sublattice’ (or ‘random flux’) models, studied first by Hikami et al. [11], and by Gade and Wegner [12] perturbatively using replica sigma models, have been known for some time to exhibit R.G. beta functions of the kind we obtained for the current-current perturbations of \( gl(N|N) \) supercurrent algebras. These delocalization

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\(^3\)We point out that for this Quantum Hall setting, based on the beta functions computed in [1], the line \( g_M + g_V = 0 \) which is the subject of this paper, is a line of fixed points in the ultraviolet. One can check numerically that starting from \( g_M, g_V, g_A \) all positive, one can actually flow to the line \( g_M + g_V = 0 \) in the UV.

\(^4\)We were first made aware of the relationship between the model of Hatsugai et al. and the random XY model by M.P.A. Fisher [6] in the context of the replica field theory.

\(^5\)The current-current perturbation of \( su(2) \) at level 1 gives the Gross-Neveu model, which is a massive theory with factorizable S-matrix. See for instance [10].
transitions possess a special particle-hole symmetry, known to prevent localization due to disorder even in one dimension \[13\]. This is not an accident: In the last section of this paper we exhibit a random Dirac Hamiltonian giving rise to these theories (in 2D). From this Dirac Hamiltonian we derive a SUSY sigma model, which we find to live on a target manifold \(GL(N|N;C)/U(N|N)\). We solve this sigma model exactly for the case \(N = 1\), in the presence of a topological Wess-Zumino-Witten(WZW) term with arbitrary real coupling \(k\). We find that the \(GL(1|1;C)/U(1|1)\) sigma model with WZW term is the same theory as the current-current perturbation of \(gl(1|1)\) supercurrent algebra. One expects that such an equivalence should extend to the case of general \(N\).

An interesting and special feature of our current-current perturbations is that (for any value of \(N\)) the non-vanishing \(\beta\)-function is associated only with two of the generators of the \(gl(N|N)\) algebra. These can be separated out and a scale invariant fixed point theory results for any value of the disorder strength \(g\), i.e. a line of fixed points (for any value of \(N\)). After we obtained the current algebra results in section IV, the papers \[14\] \[15\] appeared on \(PSL(N|N)\) sigma models, exhibiting a line of fixed points as the sigma model coupling constant is varied. We have added a paragraph to section IV C explaining the connection of our results with these papers. Our bosonized formulation may provide a useful way of studying the \(PSL(N|N)\) sigma models. Also, a proposal for a relationship between the \(PSL\) sigma models and the integer quantum Hall plateau transition has appeared recently \[16\].

The outline of the paper is as follows. In section II we give a general symmetry-based overview of the models we discuss in this paper. Sections III and IV address the random field XY model. In section III we formulate the supersymmetric method and extend it to the \(N\)-species version. In section IV A we compute the \(\beta\)-function for the random field XY model to all orders, and the exact correlation functions using current algebra techniques. In section IV B some of these results are extended to the \(N\)-species case, possessing \(gl(N|N)\) symmetry. The scale invariant subsector of this theory is discussed in more detail in Section IV C. A generalized random field XY model, whose associated Dirac hamiltonian lacks time-reversal symmetry is discussed in Section IV D. (This Dirac Hamiltonian underlies the Gade-Wegner ‘sublattice’ models discussed later in section VI.) In section V we turn to delocalization transition of Hatsugai et al. \[8\] and compute the density of states as a function of energy. In section VI A we discuss the corresponding delocalization transition with \textit{broken time reversal} symmetry, which is in the Gade-Wegner universality class \[12\] and find that it is described by a \(GL(1|1;C)/U(1|1)\) sigma model, which we solve exactly. In section VI C we relate the sigma model to the current-current perturbation discussed earlier (in section IV A). In section VI C we conclude with a summary of the \(N\)-species generalization, the \(GL(N|N;C)/U(N|N)\) sigma model, and its scale invariant \(PSL(N|N)\) variant. A number of technical details are delegated to three Appendices. In Appendix A we relate the \(N = 1\) species theory discussed in section IV A, to the random field XY model. Appendix B gives the steps leading from the general non-hermitian Dirac hamiltonian and its hermitianization to a corresponding conformal field theory action. Finally, Appendix C contains details of the 1-loop RG equations of Section IV D.
II. THE MODELS AND THEIR SYMMETRIES: AN OVERVIEW

The quantum mechanics of non-interacting particles (electrons) subject to random potentials and their possible universality classes of localization/delocalization transitions are conventionally classified according to those symmetries which are present for a fixed realization of disorder. In this section we describe the class of models discussed in this paper, all of which exhibit particle-hole symmetry.

A. Time Reversal Symmetric Models

The fundamental model that we study in this paper can be formulated as the random field XY model. This is a 2D statistical mechanics model. At the free fermion point of the XY model, however, the XY spin operator can be represented, using abelian bosonization, by a bilinear $\psi_L^\dagger \psi_R$ of Dirac fermions governed by the action of Eq.(1.1). Alternatively, this action can also be related to the 2-component random Dirac Hamiltonian $H_2$ of Eq.(1.4) in two spatial dimensions, containing a random real Dirac mass $M$, a random imaginary scalar potential $V$ and purely imaginary gauge potential $\vec{A}$ term. (Details of this connection are laid out in Appendix A). The latter being a non-interacting system, this allows us to use the supersymmetry (SUSY) method for disorder averaging. This is the method we use in this paper to study the random XY models. An apparent difficulty in applying this method is the non-hermiticity \cite{17} of this quantum mechanical hamiltonian $H_2$, which seems to prevent the functional integral over the bosonic fields, appearing in the SUSY approach to disorder, from converging. This problem is however easily remedied by considering an associated hermitian 4-component hamiltonian (Appendix A)

$$H_4 \equiv \begin{pmatrix} 0 & H_2 \\ H_2^\dagger & 0 \end{pmatrix} \quad (2.1)$$

As discussed by HWK \cite{8}, the hamiltonian $H_4$ itself describes the quantum mechanics (in the continuum limit) of fermions hopping, with real hopping amplitudes, on a 2D square lattice with $\pi$-flux through each plaquette (see also \cite{18}). This theory possesses a delocalization transition at zero energy, the characteristic symmetries of which are particle-hole and time-reversal symmetries.\footnote{For the action of these symmetries in the continuum theory with hamiltonian $H_4$, see Eq. (B3,B2).} Hence there are two apparently unrelated problems linked to this 4-component hamiltonian, the random field XY model, and the lattice fermion hopping model. In the latter model, a small imaginary part has to be added to define the quantum mechanical Green’s functions. This is not necessary for the interpretation as a random XY statistical mechanics model. As discussed in more detail below (and in Appendix A), the theory describing averaged quantities of the random XY model has $gl(1|1)$ global SUSY, and $gl(N|N)$ SUSY when higher moment averages (e.g. of ‘Edwards-Anderson order parameter’ type) are considered. The corresponding delocalization transition \cite{8}, on the other hand, can be shown to be a $GL(2|2; R)/Osp(2|2)$ sigma model.
B. Models with Broken Time Reversal Symmetry

In the random field XY model discussed above, the complex XY spin, represented by $\psi_L^\dagger \psi_R$ is coupled to a random ordering field, which is a random phase $m(x) = me^{i\vartheta}$ with $\vartheta$ real. The corresponding random 2D Dirac Hamiltonian $H_4$ (of Eq. (1.4, 2.1)) is time reversal invariant. Time reversal symmetry breaking may for example be introduced by allowing, in the language of the Dirac Hamiltonian $H_2$, in addition to the purely imaginary gauge potential discussed above, also a real gauge potential. The addition of a real random gauge potential leads to the generation of both, real and imaginary parts of all three types of potentials, and the theory flows off to strong coupling. It is then more useful to employ a description in terms of a non-linear sigma model (section VI). The corresponding quantum mechanical Hamiltonian $H_4$, possessing particle-hole symmetry but not time-reversal symmetry, is now, by universality, in the class of the Gade-Wegner sublattice models, describing fermions hopping on the square lattice with complex hopping amplitudes. The corresponding random XY model has $GL(1|1; C)$ global SUSY. In section VI we describe the corresponding Gade-Wegner type delocalization transition in terms of a non-linear sigma model on a manifold which we denote by $GL(1|1; C)/U(1|1)$ (allowing also for a topological Wess-Zumino term.)

It is noteworthy to re-emphasize that both models, with or without time reversal symmetry, possess particle-hole symmetry at zero energy, a feature which can circumvent localization of all states due to disorder, at that energy, even in one dimension.

III. SUPERSYMMETRIC DISORDER AVERAGING

In this section and the following sections IV A, B, C, we discuss the random field XY model, described by the Hamiltonian of Eq. (1.4), which we can solve exactly.

To implement the supersymmetric method for disorder averaging, we augment the theory with bosonic ghosts $\beta_L^\dagger, \beta_{L,R}$, coupled to the disorder in the same way as the fermions, with action $S_0 = S_f^f (\psi \rightarrow \beta)$. The partition function of the fermions plus ghosts is now independent of the disordered potentials so the order of functional integration over the ‘matter-ghosts’ and disorder can be interchanged. Integrating over disorder, this yields an effective action for the fermions and ghosts:

$$S_{\text{eff}} = \int \frac{d^2 x}{2\pi} \left( (\psi_L^\dagger \partial_z \psi_L + \psi_R^\dagger \partial_z \psi_R + \beta_L^\dagger \partial_z \beta_L + \beta_R^\dagger \partial_z \beta_R) + gO_m + g_A O_A \right)$$

7 Properties of the most general Dirac Hamiltonian where random Dirac mass, scalar and gauge potentials have both real and imaginary parts, as well as the relationship with a corresponding SUSY field theory, are summarized in Appendix B.

8 See Eq. (4.84) of section IV D, with the notations of Appendix B.

9 There is a subtlety arising from the convergence of the bosonic functional integral. A careful discussion of this issue, done in Appendix A, results in the effective action described below.
where
\[
\mathcal{O}_m = - \left( \psi_R^\dagger \psi_L + \beta_R^\dagger \beta_L \right) \left( \psi_L^\dagger \psi_R + \beta_L^\dagger \beta_R \right)\] (3.2) \[
\mathcal{O}_A = - \left( \psi_L^\dagger \psi_R + \beta_L^\dagger \beta_R \right) \left( \psi_R^\dagger \psi_L + \beta_R^\dagger \beta_L \right)
\]

In the sequel we will need to introduce a multi-species version of the above model. Introducing a species index \( \psi_{La}, a = 1, \ldots, N \), we define the \( N \)-species model as
\[
S_f^0 = \int \frac{d^2x}{(2\pi)^N} \sum_{a=1}^{N} \left( \psi_L^a(\partial z + A^\perp) \psi_L^a + \psi_R^a(\partial z + A^\perp) \psi_R^a + m(x) \psi_L^a \psi_L^a + m^*(x) \psi_R^a \psi_R^a \right) (3.3)
\]

Introducing bosonic ghosts and integrating over disorder leads to the effective action:
\[
S_{\text{eff}}^{(N)} = \int \frac{d^2x}{(2\pi)^N} \left( \sum_{a=1}^{N} \left( \psi_L^a \partial \bar{z} \psi_L^a + \psi_R^a \partial \bar{z} \psi_R^a + \beta_L^a \partial \bar{z} \beta_L^a + \beta_R^a \partial \bar{z} \beta_R^a \right) + g \mathcal{O}_m + g_A \mathcal{O}_A \right) (3.4)
\]

where now
\[
\mathcal{O}_m = - \sum_{a,b} \left( \psi_R^a \psi_L^b + \beta_R^a \beta_L^b \right) \left( \psi_L^b \psi_R^a + \beta_L^b \beta_R^a \right) \] (3.5) \[
\mathcal{O}_A = - \sum_{a,b} \left( \psi_L^a \beta_L^b + \beta_L^a \beta_L^b \right) \left( \psi_R^b \beta_R^a + \beta_R^b \beta_R^a \right) \] (3.6)

IV. RANDOM XY MODELS VIA GL\((N|N)\) CURRENT ALGEBRAS

A. Random Field XY Model: \( N = 1 \) Species

In this section we consider the \( N = 1 \) model in detail. The bosonic ghosts \( \beta_L (\beta_R) \) have Lorentz spin 1/2 (−1/2). The conformal field theory for such a bosonic first order action was treated in ref. \[21\]. The Virasoro central charge for one copy of the \( \beta^\dagger, \beta \) system is \( c = -1 \), so that the total central charge of the matter plus ghosts is zero. This had to be the case since the partition function is one.

In the effective action eq. (3.1) for \( N = 1 \) species the operators \( \mathcal{O}_m \) and \( \mathcal{O}_A \) are bilinears in the dimension 1 currents:
\[
\begin{align*}
G_+ &= \beta_L^\dagger \psi_L, & G_- &= \beta_L \psi_L^\dagger, & J &= \psi_L \psi_L^\dagger, & J' &= \beta_L \beta_L^\dagger
\end{align*}
\] (4.1) \[
\begin{align*}
\overline{G}_+ &= \beta_R^\dagger \psi_R, & \overline{G}_- &= \beta_R \psi_R^\dagger, & \overline{J} &= \psi_R \psi_R^\dagger, & \overline{J'} &= \beta_R \beta_R^\dagger
\end{align*}
\] (4.2)

(We denote all left-right currents as \( (J_L, J_R) = (J, \overline{J}) \).) Namely,
\[
\begin{align*}
\mathcal{O}_m &= J \overline{J} - J' \overline{J'} - G_- \overline{G}_+ + G_+ \overline{G}_- \] (4.3) \[
\mathcal{O}_A &= -(\overline{J'} - J)(J' - J)
\]

In the conformal limit \( g = g_A = 0 \), the currents generate a \( gl(1|1) \) current algebra. The currents \( J, J' \) are bosonic \( U(1) \) currents for the fermions and ghosts respectively, and \( G_\pm \) are their fermionic superpartners. From the operator product expansions (OPE)
\[ \psi_L(z)\psi^\dagger_L(w) = \psi^\dagger_L(z)\psi_L(w) \sim \frac{1}{z-w}, \quad \beta^\dagger_L(z)\beta_L(w) = -\beta_L(z)\beta^\dagger_L(w) \sim -\frac{1}{z-w} \]  

(4.4)

one obtains the OPE’s:

\[ J(z)J(w) \sim \frac{1}{(z-w)^2}, \quad J'(z)J'(w) \sim -\frac{1}{(z-w)^2} \]  

(4.5)

\[ J(z)G_\pm(w) \sim \pm \frac{1}{z-w}G_\pm(w), \quad J'(z)G_\pm(w) \sim \pm \frac{1}{z-w}G_\pm(w) \]

\[ G_-(z)G_+(w) \sim \frac{1}{(z-w)^2} + \frac{1}{z-w}(J(w) - J'(w)) \]

Let \( j, j', g_\pm \) denote the zero modes of the currents, defined as \( j = \oint \frac{dz}{2\pi i} J(z) \), etc. These zero modes are generators for the global \( gl(1|1) \) algebra:

\[ [j, g_\pm] = [j', g_\pm] = \pm g_\pm, \quad \{g_+, g_-\} = j - j', \quad g_\pm^2 = 0 \]

(4.6)

The algebra \( gl(1|1) \) has two quadratic Casimirs:

\[ C = j^2 - j'^2 - g_- g_+ + g_+ g_-, \quad \bar{C} = (j - j')^2 \]

(4.7)

The operators \( \mathcal{O}_m \) and \( \mathcal{O}_A \) are left-right current-current perturbations with precisely the structure of the Casimirs \( C, \bar{C} \) respectively. This implies that the \( gl(1|1)_L \otimes gl(1|1)_R \) symmetry of the conformal field theory is broken to a diagonal \( gl(1|1) \) symmetry in \( S_{\text{eff}} \).

The model defined by \( S_{\text{eff}} \) is a perturbation of a \( c = 0 \) conformal field theory, and can be studied in the framework of Zamolodchikov [22]. The \( gl(1|1) \) conformal field theory was studied in [23]. Current-current perturbations are generically integrable in the case of the bosonic Lie algebras, which means there are an infinite number of conserved currents. It is likely that this feature also holds for the current-current perturbations of the superalgebras. However we do not pursue this here since, as we now show, the model can be solved by straightforward manipulations of the functional integral.

The manner in which conformal symmetry is broken in \( S_{\text{eff}} \) is contained in the \( \beta \)-functions. The OPE’s of the perturbing operators are the following:

\[ \mathcal{O}_m(z, \bar{z})\mathcal{O}_m(0) \sim -\frac{2}{z\bar{z}}\mathcal{O}_A(0), \quad \mathcal{O}_A(z, \bar{z})\mathcal{O}_m(0) \sim 0, \quad \mathcal{O}_A(z, \bar{z})\mathcal{O}_A(0) \sim 0 \]

(4.8)

Note that because of the supersymmetry, the \( 1/(z\bar{z})^2 \) terms vanish. One obtains to lowest order [24]

\[ \beta_A(g) = \frac{dg_A}{dl} = 2g^2 + ... \]

\[ \beta_g = 0 \]

(4.9)

10 For the complete set of modes the current algebra (1.6) defines the affine Lie superalgebra \( \hat{gl}(1|1) \).
where \( l \) is the \( \log \) of the rescaling factor. The ultraviolet limit corresponds to \( l \to -\infty \) where \( g_A \to -\infty \), whereas the infrared limit is \( l \to \infty \) with \( g_A \to +\infty \). This shows that if \( g_A \) disorder were not included from the start it would be generated under renormalization. Since the operator \( \mathcal{O}_m \) is never generated in the OPE at higher orders, it follows that \( \beta_g = 0 \) to all orders.

It will be useful to bosonize the theory. The \( U(1) \) fermion current is bosonized with a scalar field \( \phi \):

\[
J = i\partial_z \phi, \quad \overline{J} = -i\partial_{\overline{z}} \phi
\]  

(4.10)

In the conformal limit, \( \phi(z, \overline{z}) = \varphi_L(z) + \varphi_R(\overline{z}) \) and

\[
\psi_L = \exp(i\varphi_L), \quad \psi_L^\dagger = \exp(-i\varphi_L), \quad \psi_R = \exp(-i\varphi_R), \quad \psi_R^\dagger = \exp(i\varphi_R)
\]  

(4.11)

Bosonization of the \( \beta^\dagger - \beta \) system was described in [21]. The \( J' \) currents are expressed in terms of another scalar field \( \phi' \):

\[
J' = i\partial_z \phi', \quad \overline{J'} = -i\partial_{\overline{z}} \phi'
\]  

(4.12)

The scalar field \( \phi' \) has the opposite sign kinetic term compared to \( \phi \), which implies:

\[
\phi(z, \overline{z})\phi(0) \sim -\log z\overline{z}, \quad \phi'(z, \overline{z})\phi'(0) \sim +\log z\overline{z}
\]  

(4.13)

Since the \( \phi' \) field still contributes \( c = 1 \), in order to obtain the \( c = -1 \) of the \( \beta^\dagger - \beta \) system one needs an additional fermionic \( \eta - \xi \) system with \( c = -2 \). The result is

\[
\beta^\dagger_L = e^{-i\varphi_L}\partial_z \xi_L, \quad \beta_L = e^{i\varphi_L}\eta_L, \quad \beta^\dagger_R = e^{i\varphi_R}\partial_{\overline{z}} \xi_R, \quad \beta_R = e^{-i\varphi_R}\eta_R
\]  

(4.14)

where as before \( \phi' = \varphi'_L + \varphi'_R \).

The \( \eta - \xi \) system has the first order action

\[
S_{\eta-\xi} = \int \frac{d^2 x}{2\pi} \eta_L \partial_z \xi_L + \eta_R \partial_{\overline{z}} \xi_R
\]  

(4.15)

In order to have a more symmetric effective action it is convenient to formulate the \( \eta - \xi \) system in terms of a complex dimension zero fermion \( \chi \), with the action [25]

\[
S_{\chi} = \int \frac{d^2 x}{4\pi} \left( \partial_z \chi^\dagger \partial_{\overline{z}} \chi + \partial_{\overline{z}} \chi^\dagger \partial_z \chi \right)
\]  

(4.16)

The equation of motion \( \partial_z \partial_{\overline{z}} \chi = 0 \) implies that \( \chi \) separates into left and right moving parts: \( \chi = \chi_L(z) + \chi_R(\overline{z}) \). The \( \eta - \xi \) fields are related to \( \chi \) as follows:

\[
\partial_z \xi_L = i\partial_z \chi, \quad \partial_z \xi_R = i\partial_{\overline{z}} \chi, \quad \eta_L = i\partial_{\overline{z}} \chi^\dagger, \quad \eta_R = i\partial_z \chi^\dagger
\]  

(4.17)

In order to absorb the \(-J\overline{J} + J'\overline{J'}\) terms in \( \mathcal{O}_m \) into the kinetic energy, we define

\[
\phi_{\pm} = \frac{1}{\zeta} (\phi \pm \phi'), \quad \zeta \equiv \frac{1}{\sqrt{1 + 2g}}
\]  

(4.18)
Putting all of this together one obtains the effective action

\[ S_{\text{eff}} = \int \frac{d^2x}{4\pi} \left[ \partial_z \phi_+ \partial_z \phi_- - 2\zeta^2 g_A \partial_z \phi_\phi_- + \left( 1 + 2ge^{-ic\phi_-} \right) \partial_z \chi \partial_z \chi \right] \]

(4.19)

In order to compute correlation functions, we treat the effective theory in a number of steps. First note that

\[ S_{\text{eff}}[\phi_+, \phi_-, g_A] = S_{\text{eff}}[\phi_+ - 2\zeta^2 g_A \phi_-, g_A = 0] \]  

(4.20)

Let \( \Phi(\phi_+, \phi_-, \chi) \) be a composite field or product of such fields at different space-time points. Making a change of variables \( \phi_+ \rightarrow \phi_+ + 2\zeta^2 g_A \phi_- \) in the functional integral, one relates correlation functions with \( g_A \neq 0 \) to those in the \( g_A = 0 \) theory:

\[ \langle \Phi(\phi_+, \phi_-, \chi) \rangle_{g_A} = \langle \Phi(\phi_+ + 2\zeta^2 g_A \phi_-, \phi_-, \chi) \rangle_{g_A=0} \]  

(4.21)

Henceforth \( S_{\text{eff}} \) denotes the effective action (4.20) with \( g_A = 0 \).

The next step recognizes that \( \phi_+ \) acts as a Lagrange multiplier. Now let \( \Phi(\phi_-, \chi) \) denote a field or product of fields that is independent of \( \phi_+ \). Introducing a source \( \rho \) for \( \phi_+ \) then functionally integrating over \( \phi_+ \) gives a functional \( \delta \) function: one has

\[ \langle e^{-\int \frac{d^2x}{4\pi}\rho \Phi(\phi_-, \chi)} \rangle_{g_A=0} = \int D\chi \delta(\partial_z \chi) e^{-S_{\text{eff}}[\phi_+, \phi_-=0]} \Phi(\phi_-, \chi) \]  

(4.22)

where \( \hat{\rho} \) is the solution to

\[ \partial_z \partial_z \hat{\rho} = \rho \]  

(4.23)

(We have set the partition function to 1.)

Insertions of the operator \( \partial_z \partial_z \phi_+ \) in a correlation function can be obtained from functional derivatives with respect to \( \hat{\rho} \). For instance,

\[ \langle \partial_z \partial_z \phi_+(z, \bar{z}) \Phi(\phi_-, \chi) \rangle_{g_A=0} = -4\pi \left[ \frac{\delta}{\delta \hat{\rho}(z, \bar{z})} \int \frac{\delta}{\delta \hat{\rho}(z, \bar{z})} \int D\chi e^{-S_{\text{eff}}[\phi_+, \phi_-=\hat{\rho}]} \Phi(\phi_-=\hat{\rho}, \chi) \right]_{\hat{\rho}=0} \]  

(4.24)

One needs

\[ 4\pi \frac{\delta}{\delta \hat{\rho}} S_{\text{eff}}[\phi_+ = 0, \phi_- = \hat{\rho}] \bigg|_{\hat{\rho}=0} = 2ig\zeta \left( \partial_z \chi^\dagger \partial_z \chi - \partial_z \chi^\dagger \partial_z \chi \right) \]  

(4.25)

and

\[ S_{\text{eff}}[\phi_+ = \phi_- = 0] = (1 + 2g)S_\chi \]  

(4.26)

We define a rescaled \( \chi \) field,

\[ \hat{\chi} = \frac{1}{\zeta} \chi \]  

(4.27)
such that

\[ S_\chi = (1 + 2g)S_\chi = \int \frac{d^2x}{4\pi} \left( \partial_\tau \hat{\chi}^\dagger \partial_\tau \hat{\chi} + \partial_\tau \hat{\chi}^\dagger \partial_\tau \hat{\chi} \right) \]  

(4.28)

The above formulas imply that all correlation functions are reduced to a free-field correlators in the rescaled \( \chi \) system defined by \( S_\chi \). For instance:

\[ \langle \partial_\tau \partial_\tau \phi^+ (z, \overline{z}) \Phi(\phi^-, \chi) \rangle_{gA=0} = 2i g \zeta \left( \left( \partial_\tau \chi^\dagger \partial_\tau \chi - \partial_\tau \chi^\dagger \partial_\tau \chi \right) \Phi(\phi^- = 0, \chi) \right) \hat{\chi} \]

(4.29)

\[ - 4\pi \left( \left[ \frac{\delta \Phi(\phi^-, \chi)}{\delta \phi^-(z, \overline{z})} \right]_{\phi^-=0} \right) \hat{\chi} \]

(4.30)

All correlation functions \( \langle \rangle \hat{\chi} \) on the RHS of the above equation are computed with the free action \( \hat{\chi} \) (4.16). In particular

\[ \langle \hat{\chi}^\dagger (z, \overline{z}) \hat{\chi}(0) \rangle = - \langle \hat{\chi}(z, \overline{z}) \hat{\chi}^\dagger(0) \rangle = - \log(z\overline{z}) \]

(4.31)

The second term in eq. (4.29) generally involves \( \delta \) functions. More generally the insertion of \( \partial_\tau \partial_\tau \phi^+ \) at different space-time points is equivalent to the insertion of the operator on the RHS of eq. (4.25) at the same points in the free theory \( \hat{\chi} \) up to \( \delta \)-functions:

\[ \langle (\partial_\tau \partial_\tau \phi^+)^n \Phi(\phi^- \chi) \rangle_{gA=0} = (2i g \zeta)^n \left( \left( \partial_\tau \chi^\dagger \partial_\tau \chi - \partial_\tau \chi^\dagger \partial_\tau \chi \right)^n \Phi(\phi^- = 0, \chi) \right) \hat{\chi} + \delta - \text{functions} \]

(4.32)

The above formulas are sufficient to compute all correlation functions. First, for correlations not involving \( \phi^+ \), the field \( \phi^- \) may be set to zero:

\[ \langle \Phi(\phi^-, \chi) \rangle_{gA=0} = \langle \Phi(\phi^- = 0, \chi) \rangle_{gA=0} \]

(4.33)

Thus,

\[ \langle \phi^-(z, \overline{z}) \phi^-(0) \rangle = 0 \]

(4.34)

Equation (4.29) implies:

\[ \partial_\tau \partial_\tau \langle \phi^+(z, \overline{z}) \phi^- \rangle_{gA=0} = -4\pi \delta^2(x) \]

(4.35)

Thus to all orders in \( g \):

\[ \langle \phi^+(z, \overline{z}) \phi^-(0) \rangle_{gA=0} = -2 \log(z\overline{z}) \]

(4.36)

Correlation functions involving multiple \( \phi^+ \) insertions are more interesting. From (4.32) one finds

\[ \langle \partial_\tau \partial_\tau \phi^+(z, \overline{z}) \partial_\omega \partial_\omega \phi^+(w, \overline{w}) \rangle = -8g^2 \zeta^6 \frac{1}{(z - w)^2(z - \overline{w})^2} + \delta - \text{functions} \]

(4.37)

Integrating this, one finds a \( \log^2 \) contribution:
\[ \langle \phi(z, \bar{z}) \phi(0) \rangle_{g_A=0} = -4g^2 \xi^6 \log^2(z\bar{z}/a^2) \]  

(4.38)

where \( a \) is an ultraviolet cutoff scale.

Finally let us compute correlation functions of the 'matter' \( U(1) \) field \( \phi = \zeta (\phi_+ + \phi_-)/2 \) with \( g_A \neq 0 \). Using eq. (4.21) one obtains

\[ \langle \phi(z, \bar{z}) \phi(0) \rangle = -\zeta^2 (1 + 2\xi^2 g_A) \log(z\bar{z}/a^2) - g^2 \xi^8 \log^2(z\bar{z}/a^2) \]  

(4.39)

Similarly,

\[ \langle \phi'(z, \bar{z}) \phi'(0) \rangle = \zeta^2 (1 - 2\xi^2 g_A) \log(z\bar{z}/a^2) - g^2 \xi^8 \log^2(z\bar{z}/a^2) \]  

(4.40)

The \( \beta \)eta-function \( \beta_A(g) \) to all orders in \( g \) can be computed from (4.39) along with the renormalization group equation

\[ \left( \frac{\partial}{\partial \log a} + \beta_A(g) \frac{\partial}{\partial g_A} \right) \langle \phi(r) \phi(0) \rangle = 0 \]  

(4.41)

One finds

\[ \beta_A(g) = 2g^2 \xi^4 = \frac{2g^2}{1 + 2g^2}, \quad \beta_g = 0 \]  

(4.42)

Note that this agrees with the lowest order result (4.11).

We consider now correlation functions of exponentials of \( \phi \) and \( \phi' \). The 'free-field property' eq. (4.32) implies that correlation functions of exponentials can be computed using Wick’s theorem in the free theory \( S_{\chi} \). One has

\[ \langle \prod_i e^{i\alpha_i \phi(z_i, \bar{z}_i)} \rangle = \prod_{i<j} \exp \left(-\alpha_i \alpha_j \langle \phi(z_i, \bar{z}_i) \phi(z_j, \bar{z}_j) \rangle \right) \]  

(4.43)

\[ = \prod_{i<j} |z_i - z_j|^{2\alpha_i \alpha_j \xi^2 (1 + 2\xi^2 g_A)} \exp \left( \alpha_i \alpha_j g^2 \xi^8 \log^2|z_i - z_j|^2 \right) \]  

(4.44)

From this one deduces how the disorder affects the dimension of the exponential fields:

\[ \dim (e^{i\alpha \phi}) = \alpha^2 \xi^2 (1 + 2\xi^2 g_A), \quad \dim (e^{i\alpha \phi'}) = -\alpha^2 \xi^2 (1 - 2\xi^2 g_A) \]  

(4.45)

**B. Random Field XY Model: \( N \)-Species**

We now extend some of the results of the last section to \( N \)-species. The effective action eq. (3.4) has a \( gl(N|N) \) Lie superalgebra symmetry. To see this, define the left-moving currents

\[ J_{ab} = \psi_{Lb} \rho_{La}, \quad J'_{ab} = \beta_{Lb} \rho'_{La} \]  

(4.46)

\[ G_{ab} = \beta_{Lb} \rho_{La}, \quad G'_{ab} = \beta_{Lb} \rho'_{La} \]

and similarly for the right-moving currents \( \mathcal{J}_{ab} = \psi_{Rb} \rho_{Ra} \), etc. In the conformal field theory with \( g = g_A = 0 \), one has a \( gl(N|N) \) current algebra with the operator product expansions
\[ J_{ab}(z)J_{cd}(w) \sim k \frac{\delta_{ad} \delta_{bc}}{(z-w)^2} + \frac{1}{z-w} (\delta_{ad} J_{bc}(w) - \delta_{bc} J_{ad}(w)) \] (4.47)

\[ J'_{ab}(z)J'_{cd}(w) \sim -\frac{k \delta_{ad} \delta_{bc}}{(z-w)^2} - \frac{1}{z-w} (\delta_{ad} J'_{bc}(w) - \delta_{bc} J'_{ad}(w)) \] (4.48)

\[ J_{ab}(z)G_{cd}(w) \sim \frac{\delta_{ac}}{z-w} G_{bd}(w), \quad J'_{ab}(z)G_{cd}(w) \sim \frac{\delta_{bd}}{z-w} G_{ca}(w) \] (4.49)

\[ J_{ab}(z)G^\dagger_{cd}(w) \sim -\frac{\delta_{bc}}{z-w} G^\dagger_{ad}(w), \quad J'_{ab}(z)G^\dagger_{cd}(w) \sim -\frac{\delta_{ad}}{z-w} G^\dagger_{bc}(w) \] (4.50)

\[ G_{ab}(z)G^\dagger_{cd}(w) \sim -\frac{k \delta_{ac} \delta_{bd}}{(z-w)^2} + \frac{1}{z-w} (\delta_{ac} J'_{bd}(w) - \delta_{bd} J_{ca}(w)) \] (4.51)

where the level of the current algebra is \( k = 1 \).

The perturbing fields are current-current perturbations which preserve the global diagonal (left-right) \( gl(N|N) \) symmetry:

\[ \mathcal{O}_m = \sum_{a,b} \left( J_{ba} \mathcal{J}_{ab} - J'_{ba} \mathcal{J}_{ab} + G_{ab} \mathcal{G}_{ab} - G^\dagger_{ba} \mathcal{G}_{ba} \right) \] (4.52)

\[ \mathcal{O}_A = -\sum_{a,b} (J'_{aa} - J_{aa}) (\mathcal{J}_{bb} - \mathcal{J}_{bb}) \]

The structure of the perturbative expansion in \( g, g_A \) about the conformal \( gl(N|N) \) current algebra closely parallels the 1-species case. In particular, using the OPE’s (4.51), one can verify that the OPE’s (1.8) still hold. In particular, the \( N \) dependence cancels. Since the ultraviolet divergences arise from the singular terms in the OPE eq. (4.8), this implies that the \( \beta \)-eta-functions are independent of \( N \) and are the same as in eq. (4.10). Thus the \( N \)-species theory has the same \( \beta \)-eta function as the 1-species theory.

Since the perturbative expansion can be derived from (4.8), we expect that correlation functions in the \( gl(N|N) \) theory have the same structure as we found for \( gl(1|1) \). However, since the functional integral manipulations we used to solve the bosonized \( gl(1|1) \) theory are not straightforwardly extended to the \( gl(N|N) \) case, here we obtain expressions for the correlation functions by using properties of the current algebra. This can be viewed as an alternative method to the functional integral methods described in section IV A, and in fact explain the results we obtained there in a more covariant fashion.

Let \( J_\alpha \) denote an arbitrary left current, \( J_\alpha \in \{ J_{ab}, J'_{ab}, G_{ab}, G^\dagger_{ab} \} \). The perturbing operators \( \mathcal{O}_m, \mathcal{O}_A \) are \( gl(N|N) \) invariant and can be expressed as

\[ \mathcal{O}_m = C^{ab}_\alpha J_\alpha \mathcal{J}_\beta, \quad \mathcal{O}_A = -\bar{C}^{\alpha\beta} J_\alpha \mathcal{J}_\beta \] (4.53)

where \( C^{ab}_\alpha \) and \( \bar{C}^{\alpha\beta} \) define two independent Casimirs. In the abelian sub-vector space spanned by \( (J_{11}, J_{22}, ..., J_{NN}, J'_{11}, ..., J'_{NN}) \), one has

\[ \{C^{ab}_\alpha\} = \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix}, \quad \{\bar{C}^{\alpha\beta}\} = \begin{pmatrix} \{1\} & -\{1\} \\ -\{1\} & \{1\} \end{pmatrix} \] (4.54)

where \( 1_N \) is the \( N \times N \) identity matrix, and \( \{1\} \) is the \( N \times N \) matrix with 1’s in every entry. Since \( C \) is invertible (whereas \( \bar{C} \) is not), \( C \) defines a metric to be utilized to raise and lower indices. We will need \( \bar{C}^{\alpha\beta} = C_{\alpha\gamma} C^{\beta\gamma}_\beta \), where \( C_{\alpha\beta} = C^{ab}_\alpha \). In the subspace defined above,
\{\bar{C}_{\alpha\beta}\} = \left(\begin{array}{c} \{1\} \\
\{1\} \\
\{1\}\end{array}\right)

(4.55)

The OPE’s (4.51) can be expressed as

\[ J_\alpha(z) J_\beta(w) \sim \frac{C_{\alpha\beta}}{(z-w)^2} + \frac{1}{z-w} f^\gamma_{\alpha\beta} J_\gamma(w) \]

(4.56)

where \(f^\gamma_{\alpha\beta}\) are structure constants.

The \(U(1)\) currents can be bosonized as in (4.10, 4.12):

\[ J_{aa} = i\partial_z \phi_a, \quad J'_{aa} = -i\partial_z \phi'_a \]

(4.57)

By bosonizing the fields as in (4.11, 4.14), one can verify that the shift property (4.20) continues to hold:

\[ S_{\text{eff}}[\phi_a, \phi'_a, gA] = S_{\text{eff}}[\phi_a - \zeta gA \Sigma, \phi'_a - \zeta gA \Sigma, gA = 0] \]

(4.58)

where

\[ \Sigma \equiv \sum_a (\phi_a - \phi'_a) \]

(4.59)

and \(\zeta\) is defined in (4.18). This implies

\[ \langle \mathcal{O}(\phi_a, \phi'_a) \rangle_{gA} = \langle \mathcal{O}(\phi_a + \zeta^2 gA \Sigma, \phi'_a + \zeta^2 gA \Sigma) \rangle_{gA=0} \]

(4.60)

where \(\mathcal{O}\) is any operator.

The field \(\Sigma\) has very simple correlation functions because it doesn’t appear in any exponential interactions. Collecting the kinetic terms for the scalar fields coming from the conformal field theory and from the terms \(g(J_{aa} J'_{aa} - J'_{aa} J_{aa})\)

\[ S_{\text{kinetic}} = \frac{1}{\zeta^2} \int \frac{d^2x}{4\pi} \sum_a \partial_z \phi_a^+ \partial_z \phi_a^- \]

(4.61)

where

\[ \phi_a^\pm = \phi_a \pm \phi'_a \]

(4.62)

Since \(\Sigma\) doesn’t couple to any of the remaining interaction terms, one has the exact 2-point function

\[ \langle \phi_a^+(z, \bar{z}) \Sigma(0) \rangle = -2\zeta^2 \log z\bar{z} \]

(4.63)

The above properties, together with current conservation provide strong constraints on the 2-point functions of the currents. Euclidean rotational invariance, and the fact that \(J_\alpha, \overline{J}_\alpha\) are dimension 1 currents constrains the \(\langle JJ \rangle\) 2-point function to be of the form:

\[ \langle J_\alpha(z, \bar{z}) \overline{J}_\beta(0) \rangle_{gA=0} = \frac{1}{z\bar{z}} \left( C_{\alpha\beta} F(g, z\bar{z} \mu^2) + \bar{C}_{\alpha\beta} \bar{F}(g, z\bar{z} \mu^2) \right) \]

(4.64)
where we have used the unbroken $gl(N|N)$ symmetry. $F, \tilde{F}$ are scaling functions of $g$ and the dimensionless combination $z\bar{z}\mu^2$, where $\mu$ is the energy scale that enters through the renormalization group.

The field $\Sigma$ has the covariant description:

$$i \partial_z \Sigma = \bar{C}_\alpha^\beta J_\beta, \quad -i \partial_{\bar{z}} \Sigma = \bar{C}_\alpha^\beta \bar{J}_\beta, \quad \forall \alpha \quad (4.65)$$

From (4.63) we see that away from $z\bar{z} = 0$, one must have $\bar{C}_\alpha^\alpha \langle J_\alpha \bar{J}_\beta \rangle = 0$. Since $\bar{C}_\alpha^\alpha \bar{C}_\alpha^\beta \neq 0$, and $\bar{C}_\alpha^\alpha \bar{C}_\alpha^\beta = 0$, one deduces that $F = 0$.

The property (4.63), together with (4.60), leads to $g_A$ independence (up to $\delta$-functions) of the correlation function (4.64). Since the dependence on the scale $\mu$ must be compatible with the $\beta$-eta-function for $g_A$, the $g_A$ independence implies that $\tilde{F}$ is only a function of the dimensionless coupling $g$.

$$\langle J_\alpha(z, \bar{z}) \bar{J}_\beta(0) \rangle_{g_A=0} = \frac{1}{z\bar{z}} \bar{C}_\alpha^\beta \bar{F}(g) \quad (4.66)$$

Current conservation, $\partial_{\bar{z}} J_\alpha + \partial_z \bar{J}_\alpha = 0$, requires

$$\partial_{\bar{z}} \langle J_\alpha(z, \bar{z}) J_\beta(0) \rangle = -\partial_z \left( \langle J_\alpha(z, \bar{z}) \bar{J}_\beta(0) \rangle \right) \quad (4.67)$$

Integrating this,

$$\langle J_\alpha(z, \bar{z}) J_\beta(0) \rangle_{g_A=0} = \frac{1}{z^2} \left( \zeta^2 C_{\alpha\beta} + \bar{C}_{\alpha\beta} \bar{F}(g) \log z\bar{z} \right) \quad (4.68)$$

The coefficient $\zeta^2$ of the $C_{\alpha\beta}$ term was derived from (4.63).

To fix the one unknown function $\tilde{F}(g)$, we use current algebra equations of motion. The equations of motion to first order in $g$ can be obtained in conformal perturbation theory

$$\partial_{\bar{z}} J_\alpha(z, \bar{z}) = g \oint \frac{dw}{2\pi i} J_\alpha(w, \bar{z}) \mathcal{O}_m(z, \bar{z}) \quad (4.69)$$

Taking into account the $\zeta$-rescaling $\phi \rightarrow \phi/\zeta$ that gives the kinetic term the standard normalization, and using the OPE’s (4.56), one obtains

$$\partial_{\bar{z}} J_\alpha = g \zeta^2 f_\alpha^\gamma J_\beta J_\gamma = -\partial_z \bar{J}_\alpha \quad (4.70)$$

Using this in the correlation function $\langle \partial_{\bar{z}} J_\alpha \partial_z \bar{J}_\beta \rangle$, along with (4.64),

$$\frac{1}{z^2 \bar{z}^2} \bar{C}_{\alpha\beta} \bar{F} = g^2 \zeta^2 f_\alpha^\gamma J_\beta J_\gamma \langle \langle J_\beta \bar{J}_\gamma \rangle(z, \bar{z}) (J_\sigma \bar{J}_\gamma')(0) \rangle_{g_A=0} \quad (4.71)$$

The non-zero contribution to $\tilde{F}$ comes from

$$\langle \langle J_\beta \bar{J}_\gamma \rangle(z, \bar{z}) (J_\sigma \bar{J}_\gamma')(0) \rangle_{g_A=0} \sim \frac{\zeta^4}{z^2 \bar{z}^2} C_{\beta'\sigma} C_{\gamma'\gamma} \quad (4.72)$$

where we have used (4.68). Now, in terms of the structure constants, the OPE $\mathcal{O}_m(z, \bar{z}) \mathcal{O}_m(0) \sim -2 C_A / z\bar{z}$ implies
\[ 2\tilde{C}_{\alpha\beta} = f^\beta_\gamma f^{\gamma'}_\delta C_{\beta\alpha} C_{\gamma\delta} \]  \hspace{1cm} (4.73) 

The last three equations thus give
\[ \tilde{F}(g) = 2g^2\zeta^8 \]  \hspace{1cm} (4.74) 

Let us summarize the above results by integrating the current correlations to find the 2-point functions of the scalar fields:
\[ \langle \phi^+_a(z,\bar{z})\phi^+_b(0) \rangle_{gA=0} \sim -4g^2\zeta^8\log^2 z\bar{z}, \quad \forall \ a, b \]  \hspace{1cm} (4.75) 
\[ \langle \phi^+_a(z,\bar{z})\phi^-_b(0) \rangle_{gA=0} \sim -2\zeta^2\delta_{ab}\log z\bar{z} \]  \hspace{1cm} (4.76) 
\[ \langle \phi^-_a(z,\bar{z})\phi^-_b(0) \rangle_{gA=0} \sim 0, \quad \forall \ a, b \]  \hspace{1cm} (4.77) 

Finally, using the shift property (4.60),
\[ \langle \phi_a(z,\bar{z})\phi_b(0) \rangle = -4\log z\bar{z}, \quad \delta_{ab} = \pm \]  \hspace{1cm} (4.78) 
\[ \langle \phi'_a(z,\bar{z})\phi'_b(0) \rangle = \zeta^2 \left( \delta_{ab} + 2\zeta^2 g_A \right) \log z\bar{z} - g^2 \zeta^8 \log^2 z\bar{z} \]  \hspace{1cm} (4.79) 

C. Nearly Conformal Structure

We have seen in the previous section that the N-species theory (describing N copies of the random field XY model) is nearly conformal. This can be seen more clearly by separating out the non-conformal pieces (coupling holomorphic and anti-holomorphic coordinate dependences) by rewriting Eq.'s (4.66,4.68) as follows. Instead of the basis of currents as in Eq.(4.47), we extract \( U(1) \times U(1) \) from the bosonic \( U(N) \times U(N) \) subgroup. Explicitly, consider traceless generator matrices \( T_A \) of \( su(N) \), \( (A = 1, ..., N^2 - 1) \)
\[ J_A \equiv (T_A)_{ab}J_{ab}, \quad J'_A \equiv (T_A)_{ab}J'_{ab} \]
and
\[ J \equiv \sum_a J_{aa}, \quad J' \equiv \sum_a J'_{aa} \]
and similar for the fermionic currents (which we denote by \( J_A, A = 2N^2 + 1, ..., 4N^2 \)). Forming the combinations
\[ J_- \equiv J - J', \quad J_+ \equiv J + J', \]
and noting that all non-conformal terms in the current-current correlators are multiplied by the invariant \( \tilde{C}_{AB} \) whose only non-vanishing matrix elements are \( A, B = \pm \) in this basis, we see that the two point correlation functions of all the remaining currents are
\[ \langle J_A(z,\bar{z})J_B(0) \rangle = \frac{\zeta^2 C_{AB}}{z^2}, \quad \langle J_A(z,\bar{z})J_{\pm}(0) \rangle = 0, \quad (A, B \neq \pm) \]
Note that we see from Eq.(4.72) that these correlators are invariant under independent action of the symmetry group on the left- and right- chiral components of the currents.

The ability to separate off the $J_\pm$ currents (and the corresponding fields, $\phi_\pm$) can be seen from the hamiltonian, expressed as

$$H = H_0 + H_I$$

where $H_0$ is the non-interacting $gl(N|N)$-Sugawara Hamiltonian, and $H_I$ is the current-current interaction. Recalling that $C_{AB} = C_{AB}$ there are no terms in the hamiltonian $H$ which couple the generators $J_\pm, \overline{J}_\pm$ to the remaining ones, which allows to consistently set $J_\pm = \overline{J}_\pm = 0$, yielding a scale invariant theory.

Setting $J_\pm = \overline{J}_\pm = 0$ effectively sets $g_A = 0$. Since the non-zero beta-function is $\beta g_A(g)$, what is left is a conformal field theory since $\beta g = 0$. In terms of groups, setting $J_\pm = 0$ is equivalent to dividing by the $U(1) \otimes U(1)$ subgroup: $GL(N|N)/U(1) \otimes U(1) = PSL(N|N)$. Since the two $U(1)$ bosons give $c = 2$, and the total central charge for $GL(N|N)$ is zero, what is left is a $c = -2$ conformal field theory. The conformal $PSL(N|N)$ sigma model was recently studied in [14] [15] (see also section VI below).

In the $N = 1$-species case the emergence of conformal symmetry is easily understood: Bosonizing the non-interacting theory, there are only two generators $G_+ = i\partial \chi, G_- = i\partial \chi^\dagger$ left once the currents $J_\pm$ and thus the fields $\phi_\pm$ have been eliminated. The $PSL(1|1)$ sigma model is a free theory.

One proceeds similarly for the $N > 1$ cases. The $N = 2$ species case for example describes the second moments of observables in the random XY model. An interesting observable is the ‘Edwards-Anderson order parameter’, which is a bilinear of one of the ‘conformal’ currents:

$$q_{12} \equiv (\psi_{L1}^\dagger \psi_{R1})(\psi_{L2}^\dagger \psi_{R2}^\dagger) = J_{12}\overline{J}_{21},$$

We see from Eq.(4.72) that the two-point of this field decays algebraically,

$$<q_{12}(z, \overline{z})q_{21}(0)> \propto \frac{\zeta^4}{z^2 \overline{z}^2}$$

It was recently proposed by Zirnbauer that the $PSL(N|N)$ sigma models describe the integer Quantum Hall transition [16]. The kind of disorder considered in the present paper, which, as we have explained, leads to the $PSL(N|N)$ sigma models, is not generally believed to correspond to kind of disorder needed for the Quantum Hall effect, at least not in an obvious way. Nevertheless this is an interesting proposal that needs further investigation.

### D. Generalized Random XY Model (Broken Time-Reversal Symmetry)

In this subsection we discuss a more general random XY model (at the free fermion point). This model may be defined in terms of an associated 2-component Dirac hamiltonian of type $H_2$, where now all random potentials have both real $(A_x, A_y, V, M)$ and imaginary $(A'_x, A'_y, V', M')$ parts. As discussed in Appendix B, in this case the corresponding 4-component hamiltonian $H_4$ lacks in general time reversal symmetry (but still has particle-hole symmetry), and the corresponding field theory action possesses $GL(1|1; C)$
global SUSY. This subsection is devoted to the 1-loop RG equations of this theory. To summarize the result, we find that the theory flows off to strong coupling. The strong coupling physics should be captured by a non-linear sigma model of the kind discussed in Section VI.

Performing the disorder average over independent random variables \( m = V + iM \) and \( \mu = i(V' + iM') \) with variance \( g_m = \overline{mm^*} \) and \( g_\mu = \overline{\mu\mu^*} \) respectively, we arrive at the following lagrangian for the averaged theory

\[
\mathcal{L}_{\text{susy}}^g = g_1 \sum_{a=1,2} \left( J_{aa} \mathcal{T}_{aa} - J'_{aa} \mathcal{T}'_{aa} + G_{aa} \mathcal{G}^\dagger_{aa} - G'_{aa} \mathcal{G}'_{aa} \right)
+ g_2 \sum_{a,b=1,2} \left( J_{ab} \mathcal{T}_{ab} - J'_{ab} \mathcal{T}'_{ab} + G_{ab} \mathcal{G}^\dagger_{ab} - G'_{ab} \mathcal{G}'_{ab} \right)
\]

(4.80)

where

\[
g_1 \equiv g_m - g_\mu, \quad g_2 \equiv g_m + g_\mu
\]

(4.81)

The additional contribution from the random vector potentials is

\[
\mathcal{L}_{\text{susy}}^A = g_1^A (-1) \sum_{a=1,2} (J'_{aa} - J_{aa}) (\mathcal{T}'_{aa} - \mathcal{T}_{aa}) + g_2^A (-1) \sum_{a=1,2} (J'_{aa} - J_{aa}) (\mathcal{T}'_{bb} - \mathcal{T}_{bb})
\]

(4.82)

where

\[
g_1^A = g_A - g_A', \quad g_2^A = g_A + g_A'
\]

(4.83)

\([g_A \text{ and } g_A' \text{ are the variances of the imaginary and the real vector potential, respectively.}]

We recover the time-reversal symmetric case, Eq.(4.53), by letting \( g_\mu = g_A' = 0 \). We note that in the special case where only the potentials \( A_x, A_y, V', M' \) are non-vanishing, which corresponds to \( g_m = 0, g_\mu > 0 \), we recover time reversal symmetry [Eq. (B9) of Appendix B]. The RG equations for this case are identical to those of the case \( g_m > 0, g_\mu = 0 \), studied in the previous section, upon letting \( g_m \rightarrow -g_\mu \). This can be seen by replacing in Eq.(B8, B9) the left moving fields \( \psi_{L1}, \psi_{L2} \) by \( i\psi_{L1}, -i\psi_{L2} \) (and \( \psi_{L1}^\dagger \) by \( -i\psi_{L1}^\dagger \), and \( \psi_{L2}^\dagger \) by \( i\psi_{L2}^\dagger \)) and similarly for the \( \beta \)-fields. All right moving fields remain unchanged. With these redefinitions the coupling constants in Eq.(L80) become \( g_1 = g_2 = -g_\mu \). On the other hand, as these redefinitions do not change the kinetic term in Eq.(B11) of Appendix B, the OPE’s of these fields remain unchanged. This implies that the OPE’s of the \( gl(2|2) \) currents in Eq. (4.51) remain also unchanged, and the entire analysis of sections IV A, B remains unchanged upon replacing \( g_m \) by \( -g_\mu \).

\(^{11}\) \( GL(N|N;C) \) symmetry for the \( N \)th moment averages.

\(^{12}\) using the action of Eq.(B11) of Appendix B
Let us now return to the most general case. The 1-loop RG equations for the coupling constants of the action above are derived in Appendix C (Eq.’s (C2), (C3) with the result 13:

\[
\begin{align*}
\frac{dg_1}{dl} &= 0, \\
\frac{dg^A}{dl} &= 2[(g_m)^2 + (g_\mu)^2] \\
\frac{dg_2}{dl} &= Dg_A g_2, \\
\frac{dg^A}{dl} &= 4g_m g_\mu
\end{align*}
\] (4.84)

The coupling \(g_2\) flows according to

\[
\frac{d^2 \log(g_2)}{dl^2} = D \frac{dg^A}{dl} = D[(g_2)^2 - (g_1)^2]
\]

which means that \(g_2\) flows to large values. (Note from Eq. (4.81) that the expression \([(g_2)^2 - (g_1)^2]\) is always non-negative.) The strong coupling physics should be described via the sigma model technology of section VI.

**V. DENSITY OF STATES OF THE HATSUGAI ET AL. DELOCALIZATION TRANSITION FROM CURRENT ALGEBRAS**

In this section we apply some of the above results to the problem of localization of electrons hopping with real amplitudes on a square lattice with flux \(\pi\) per plaquette [8]. In the continuum limit the hamiltonian of this system is the 4-component (hermitian) Dirac hamiltonian \(H_4\),

\[
H_4 = \begin{pmatrix} 0 & H_2 \\ H_2^\dagger & 0 \end{pmatrix}
\] (5.1)

where \(H_2\) is defined in eq. (1.4). The reality of lattice hopping amplitudes implies that the hamiltonian \(H_4\) must have time-reversal symmetry, which is manifest with this form of \(H_2\) (see Eq. (B2)). The corresponding SUSY field theory action (Eq. (B1) of Appendix B with \(\mu \equiv 0\)) can be seen to possess \(GL(2|2;R)\) symmetry. (This model has also been studied numerically in [26].) In this subsection we compute the density of states of the hamiltonian \(H_4\). In particular, we will be interested in the eigenstates of the Schrödinger equation \(H_4 \Psi = E \Psi\), where \(\Psi\) is a 4-component wave function. The single particle Green functions are defined by the functional integral

\[
\int D\Psi^\dagger D\Psi \exp(-S_0^f),
\]

and \(S_0^f = \int d^2x \frac{x}{2\pi} \Psi^\dagger(x) i (H_4 - E) \Psi(x)
\] (5.2)

and \(E = E + i\epsilon\). For \(\epsilon = 0^+\), the functional integral defines the retarded Green function:

\[
G_R(x,x';E)_{ab} = \lim_{\epsilon \to 0^+} \langle x,a| \frac{1}{H_4 - (E + i\epsilon)} |x',b \rangle = \lim_{\epsilon \to 0^+} i \langle \Psi_a(x) \Psi^\dagger_{b}(x') \rangle
\] (5.3)

13 In terms of the notations of the Appendix, \(D \equiv (d_{22}^2 - d_{21}^2)\)
Making the identifications$^{14}$ (Appendix B)

\[ \Psi = (\psi_{L2}^\dagger, \psi_{R2}^\dagger, \psi_L, \psi_{R1})^t, \quad \Psi^\dagger = (\psi_{R1}^\dagger, \psi_{L1}^\dagger, \psi_{R2}, \psi_L) \]  

the action takes the form of an \( E \) perturbation of the \( N = 2 \) species model:

\[ S = \int \frac{d^2 x}{2\pi} \Psi^\dagger(x) \imath (H_4 - E) \Psi(x) \]  

\[ = \int \frac{d^2 x}{2\pi} \sum_{a=1}^2 \left( \psi_{La}^\dagger \partial_x \psi_{La} + \psi_{Ra}^\dagger \partial_x \psi_{Ra} + m(x) \psi_{Ra}^\dagger \psi_{La} + m^*(x) \psi_{La}^\dagger \psi_{Ra} \right) \]  

\[ + A_\tau \left( \psi_{L1}^\dagger \psi_{L1} + \psi_{L2}^\dagger \psi_{L2} \right) + A_z \left( \psi_{R1}^\dagger \psi_{R1} + \psi_{R2}^\dagger \psi_{R2} \right) - i \mathcal{E} \int \left( \psi_{L1}^\dagger \psi_{R2}^\dagger - \psi_{L2}^\dagger \psi_{R1}^\dagger + \text{h.c.} \right) \]  

For simplicity, we can set the physical \( A \)-couplings to zero from the beginning and then let the \( g_A \) coupling be generated dynamically under renormalization. Introducing ghosts and integrating over disorder leads to the effective action:

\[ S = S_{\text{eff}}^{(N=2)} - i \mathcal{E} \int \frac{d^2 x}{2\pi} \mathcal{O}_E \]  

where

\[ \mathcal{O}_E = \psi_{L1}^\dagger \psi_{R2}^\dagger - \psi_{L2}^\dagger \psi_{R1}^\dagger + \beta_{L1}^\dagger \beta_{R2}^\dagger + \beta_{L2}^\dagger \beta_{R1}^\dagger + \text{h.c.' terms} \]  

(see Appendix B).

Of interest is the density of states (DOS)

\[ \rho(E) = \frac{1}{V} \text{Tr} \delta(H_4 - E) = \frac{1}{V \pi} \lim_{\epsilon \to 0^+} \text{ImTr} \left( \frac{1}{H_4 - E - i\epsilon} \right) \]  

where \( V \) is the two-dimensional volume. The averaged density of states can be expressed in terms of the averaged retarded Green function

\[ \overline{\rho(E)} = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \text{tr} \, G_R(x, x; E) \]  

where \( \text{tr} \) denotes the trace over matrix indices of the Green’s function of Eq. (5.3).

Thus the density of states is related to the one-point function of the, say, fermionic part of the operator \( \mathcal{O}_E \) which couples to \( \mathcal{E} \):

\[ \overline{\rho(E)} = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \text{Re} \langle \psi_{L1}^\dagger \psi_{R2}^\dagger - \psi_{L2}^\dagger \psi_{R1}^\dagger \rangle E \]  

where the one-point function on the right hand side is computed using \( S_{\text{eff}} \) with the \( \mathcal{E} \) term.

Generally, the \( \mathcal{E} \)-term spoils the complete solvability of the theory. Our strategy will be to use our solution of the \( \mathcal{E} = 0 \) theory in conjunction with the renormalization group to learn something about the density of states. Let us view \( S_{\mathcal{E}} = -i \mathcal{E} \int \frac{d^2 x}{2\pi} \mathcal{O}_E \) as a perturbation of

\[ \text{superscript } t \text{ denotes the transposed vector.} \]
the nearly conformal field theory. The anomalous scaling dimensions of fields are controlled by $S_{\text{eff}}$ when $E = 0$.

We need the anomalous dimension $\Gamma$ of the field $O_E$ which contains:

$$\psi_{L1}^\dagger \psi_{R2}^\dagger := \exp(-i\varphi_{L1} + i\varphi_{R2}) := \exp\{-i(\phi_1 + \phi_2)/2\} \exp\{-i(\phi_1 - \phi_2)/2\}.$$ Here, $\tilde{\phi}_a = \varphi_{La} - \varphi_{Ra}$ denotes the dual field, whose 2-point function is found from Eq.(4.66), (4.74) to be

$$\langle \tilde{\phi}_a^+(z, \bar{z}) \tilde{\phi}_b^+(0) \rangle_{g_A = 0} \sim 4g^2\zeta^8 \log^2(z\bar{z}/a^2)$$

The requirement that for arbitrary $g_A$ this function is to satisfy Eq.(4.41) gives

$$\langle \tilde{\phi}_a^+(z, \bar{z}) \tilde{\phi}_b^+(0) \rangle_{g_A} \sim 8\zeta^4 g_A \log(z\bar{z}/a^2) + 4g^2\zeta^8 \log^2(z\bar{z}/a^2)$$

This together with

$$\langle \tilde{\phi}_a^+(z, \bar{z}) \tilde{\phi}_b^-(0) \rangle_{g_A} \sim -2\zeta^2 \delta_{ab} \log(z\bar{z}/a^2)$$

yields the anomalous dimension of the field $\psi_{L1}^\dagger \psi_{R2}^\dagger$:

$$\Gamma(g_A) = \frac{1}{2}\zeta^2 (1 - 4\zeta^2 g_A) + h(g)$$

where $h(g)$ is independent of $g_A$. Since the action is dimensionless,

$$\text{dim}(E) = 2 - \Gamma$$

ensuring that $\rho(E) dE$ has dimension 2, as it should in two dimensions.

The renormalization group equation for $\overline{\rho}(E, g_A)$

$$\left((2 - \Gamma)E \frac{\partial}{\partial E} + \beta_A(g) \frac{\partial}{\partial g_A} - \Gamma(g_A)\right)\overline{\rho}(E, g_A) = 0$$

implies that the renormalized density of states $\overline{\rho}(l) \equiv \overline{\rho}(E(l), g_A(l))$ (where $e^l \geq 1$ is the rescaling factor) is related to the unrenormalized DOS $\overline{\rho}(0)$ by

$$\overline{\rho}(0)/\overline{\rho}(l) = \exp\{-\int_0^l \Gamma(l') dl'\} = \exp\{\int_0^l [2 - \Gamma(l')] dl' - 2l\}$$

Here

$$\frac{dE(l)}{dl} = [2 - \Gamma]E, \quad \frac{dg_A(l)}{dl} = \beta_A$$

which permits to express the $l$-dependence in terms of the the renormalized coupling constant $E_R \equiv E(l)$ and its bare value $E \equiv E(0) \leq E_R$. This yields for the DOS

$$\frac{\rho(E)}{\rho(E_R)} = \frac{E_R}{E} \exp\left(-2[g_A(E_R) - g_A(E)]/\beta_A\right)$$

$^{15}$For $N \geq 2$ the free field property that leads to exact expressions such eq. (4.43) is not expected to hold. However if we assumed that for the purposes of computing the anomalous dimension we can parallel what we did for $N = 1$, we would obtain $h(g) = \zeta^2/2$. 22
(valid since $\beta_{g_A}(g, g_A)$ is independent of $g_A$). From this we find in the limit $E/E_R \to 0$

$$\frac{\mathcal{P}(E)}{\mathcal{P}(E_R)} \sim \frac{E_R}{E} \exp\left\{ -\frac{\sqrt{2}(1 + 2g)^2}{g} \sqrt{\log(E_R/E)} \right\}$$

This diverging (but integrable) density of states was not seen in the simulations of \cite{26}. Rather, a power law varying with the strength of disorder $g$ was found. Such power law behavior of the DOS is obtained from the R.G. analysis above for intermediate not asymptotically small energies.

**VI. GADE-WEGNER DELOCALIZATION TRANSITION VIA GL(N|N; C)/U(N|N) SIGMA MODEL**

**A. GL(1|1; C)/U(1|1) Sigma Model**

Localization problems are usually described in terms of sigma models, using replicas \cite{27} \cite{28} or supersymmetry \cite{20}. In this section we study the Gade-Wegner localization problem \cite{12}. A version of this model consists of electrons hopping on a 2-dimensional square lattice with arbitrary complex hopping amplitudes (and flux $\pi$ per plaquette). The continuum limit of the hamiltonian of this model is again, as for the model discussed in section V, of the form of a 4-component Dirac hamiltonian. In contrast to the latter model, however, the lattice hamiltonian of the present model, and thus also its continuum limit $H_4$ lacks time reversal invariance due to the complex hopping amplitudes. As discussed in more detail in Appendix B, the associated Dirac hamiltonian $H_4$ has now arbitrary complex random scalar potential (with real and imaginary parts $V'$ and $V$) and complex Dirac mass (with real and imaginary parts $M$ and $M'$) terms, which may be combined into two independent complex random variables $m$ and $\mu$, with variance $g_m$ and $g_\mu$ respectively. The explicit form of the hamiltonian is (from Eq.(B7) of Appendix B):

$$H_4 = (-i) \begin{pmatrix} \epsilon & 0 & m - \mu & \partial \\ 0 & \epsilon & m^* + \mu^* & \partial \\ -m^* + \mu^* & \partial & \epsilon & 0 \\ \partial & -m - \mu & 0 & \epsilon \end{pmatrix} \quad (6.1)$$

where

$$m \equiv (V + iM), \quad \mu \equiv i(V' + iM')$$

In general, an arbitrary complex vector potential will be generated upon RG flow in the corresponding SUSY field theory\cite{16}. We set the vector potentials to zero initially, and let it be generated upon renormalization group transformations later on. In order to study this system we use the conventional sigma-model technique.

The lagrangian is

\[16\] see Eq. (1.84).
\[ \mathcal{L} = \Psi^i (H_4 - \mathcal{E}) \Psi + \Phi^i (H_4 - \mathcal{E}) \Phi \]  \hspace{1cm} (6.2)

where \( \mathcal{E} = E + i \epsilon \) \((\epsilon = 0^+)\) and

\[ \Psi = \begin{pmatrix} \psi_1 \\ \phi_1 \\ \psi_2 \\ \phi_2 \\ \psi_3 \\ -\psi_2 \\ \psi_4 \\ -\phi_2 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \hspace{1cm} (6.3) \]

Let us define new fields

\[ \Upsilon = \begin{pmatrix} \psi_1 \\ \phi_1 \\ \psi_4 \\ \phi_4 \end{pmatrix} = \Upsilon^+ + \Upsilon^- \]

\[ \tilde{\Upsilon} = \begin{pmatrix} \psi_3 \\ \phi_3 \\ -\psi_2 \\ -\phi_2 \end{pmatrix} = (\tilde{\Upsilon}^+ + \tilde{\Upsilon}^-) \hspace{1cm} (6.4) \]

Using Eq. (B10) of the Appendix the lagrangian can be expressed as

\[ \mathcal{L} = \tilde{\Upsilon}^+ D \tilde{\Upsilon} - \Upsilon^+ D \Upsilon + m_i(x) \Upsilon_i^+ \tilde{\Upsilon}_i - m^*_i(x) \tilde{\Upsilon}_i^+ \Upsilon_i - i \mathcal{E} \left( \tilde{\Upsilon}^+ \tilde{\Upsilon} + \Upsilon^+ \Upsilon \right) \hspace{1cm} (6.7) \]

\[ m_{\pm} \equiv (m \pm \mu) \hspace{1cm} (6.8) \]

where a sum over the repeated index \( i = \pm 1 \) is implied and

\[ D = (\sigma_+ \otimes 1) \partial_z + (\sigma_- \otimes 1) \partial_{\bar{z}} = \begin{pmatrix} 0 & 0 & \partial_z & 0 \\ 0 & 0 & 0 & \partial_z \\ \partial_{\bar{z}} & 0 & 0 & 0 \\ 0 & \partial_{\bar{z}} & 0 & 0 \end{pmatrix} \hspace{1cm} (6.9) \]

When \( \mathcal{E} = 0 \), the lagrangian has the following symmetry:

\[ \Upsilon \rightarrow G \Upsilon = \begin{pmatrix} (G^{-1})^\dagger & 0 \\ 0 & G \end{pmatrix} \Upsilon, \quad \tilde{\Upsilon} \rightarrow \tilde{G} \tilde{\Upsilon} = \begin{pmatrix} G & 0 \\ 0 & (G^{-1})^\dagger \end{pmatrix} \tilde{\Upsilon} \hspace{1cm} (6.10) \]

where \( G \) is a 2 x 2 supermatrix with bosonic (fermionic) diagonal (off-diagonal) elements. That this is a symmetry follows from \( G^\dagger (\sigma_\pm \otimes 1) G = \tilde{G}^\dagger (\sigma_\pm \otimes 1) \tilde{G} = \sigma_\pm \otimes 1 \) and \( \tilde{G}^\dagger G = 1 \). Since \( G \) is only required to be invertible, \( G \) is an element of \( GL(1|1; C) \), i.e. the complexification of \( GL(1|1) \). In the presence of the \( \mathcal{E} \) term, the above transformation continues to be a symmetry only if \( G^\dagger G = \tilde{G}^\dagger \tilde{G} = 1 \), which implies \( G^\dagger G = 1 \). Thus the \( \mathcal{E} \) term breaks the symmetry to \( U(1|1) \).
Without loss of generality the random variables \((m_+, m_+^*)\) and \((m_-, m_-^*)\) may be taken to be statistically independent with the distribution \((1.3)\) each, and the same variance \(g\). Upon disorder averaging one then obtains the term in the effective Lagrangian:

\[
L_{eff} = g \sum_{i=\pm} (\bar{\Upsilon}^i_+ \Upsilon^i_+) (\bar{\Upsilon}^i_+ \Upsilon^i_-) \quad (6.11)
\]

Introduce a grade \([a]\) for the vector index \(a\), where \([1] = [3] = 1\) (fermions), \([2] = [4] = 0\) (bosons), such that

\[
\bar{\Upsilon}_a \Upsilon_b = (-1)^{|a|\delta} \Upsilon_a \bar{\Upsilon}_b \quad (6.12)
\]

Then using the fact that \((-1)^{|b|2+|a|\delta} = (-1)^{|b|\delta}\), one finds

\[
(\Upsilon^*_i \bar{\Upsilon}_i) \equiv \Upsilon^*_i \bar{\Upsilon}_i = \Upsilon^*_i \bar{\Upsilon}_i, \quad (M_i)_{pq} = \Upsilon^*_i \bar{\Upsilon}_i, \quad (\tilde{M}_i)_{pq} = \tilde{\Upsilon}_i \bar{\Upsilon}_i^* \quad p, q = 1, 2 \quad (6.13)
\]

Let us introduce two \(4 \times 4\) block diagonal supermatrices of fields \(Q, \tilde{Q}\)

\[
Q \equiv \begin{pmatrix} Q_+ & 0 \\ 0 & Q_- \end{pmatrix}, \quad \tilde{Q} \equiv \begin{pmatrix} \tilde{Q}_+ & 0 \\ 0 & \tilde{Q}_- \end{pmatrix}
\]

and define

\[
M \equiv \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \quad \tilde{M} \equiv \begin{pmatrix} \tilde{M}_+ & 0 \\ 0 & \tilde{M}_- \end{pmatrix}
\]

Consider the following Lagrangian:

\[
L = STr \log \left( (D - g \tilde{Q} - i\mathcal{E}) \tilde{M} - (D + g Q + i\mathcal{E}) M - g \tilde{Q} Q \right) \quad (6.15)
\]

A functional integral over the \(Q_\pm, \tilde{Q}_\pm\) fields gives the effective interaction \(L_{eff} = g \text{Str}(M_+ \tilde{M}_-) + g \text{Str}(M_- \tilde{M}_+)\). If instead we first perform the gaussian functional integral over the \(\bar{\Upsilon}, \bar{\Upsilon}\) fields, we obtain the effective action for the \(Q, \tilde{Q}\) fields:

\[
S_{Q, \tilde{Q}} = STr \log \left( (D - g \tilde{Q} - i\mathcal{E}) \tilde{M} - (D + g Q + i\mathcal{E}) M - g \tilde{Q} Q \right) + \int \frac{d^2x}{2\pi} \text{Str} \tilde{Q} Q \quad (6.16)
\]

In the above equation, \(\text{STr}\) incorporates the integral \(d^2x\): For a diagonal functional operator \(A(x, y) = A(x)\delta(x - y)\) we define \(\text{STr} A = \frac{1}{a} \int \frac{d^2x}{2\pi} \text{Tr} A(x)\), where \(a\) is an ultraviolet cutoff.

The symmetry transformations on \(M, \tilde{M}\) follow from \((6.10)\). The Lagrangian \((6.15)\) then has the following symmetry when \(\mathcal{E} = 0\):

---

\(^{17}\)For general values of \(g_m\) and \(g_\mu\), the variables \(m_+\) and \(m_-\) will be correlated and will have different variances. Changing the ratio \(g_m/g_\mu\) will however not affect the universality class of the sigma model. Therefore we may choose this ratio to be unity.
\[
M \rightarrow \mathcal{G}M\mathcal{G}^\dagger, \quad \tilde{M} \rightarrow \tilde{\mathcal{G}}\tilde{M}\tilde{\mathcal{G}}^\dagger
\]
\[
Q \rightarrow (\mathcal{G}^\dagger)^{-1}Q\mathcal{G}^{-1}, \quad \tilde{Q} \rightarrow (\tilde{\mathcal{G}}^\dagger)^{-1}\tilde{Q}\tilde{\mathcal{G}}^{-1}
\]

(6.17)

When \( \mathcal{E} \neq 0 \), the symmetry is as in (6.17) with \( G \) an element of \( U(1|1) \).

Due to the linear terms in \( S_{Q,\tilde{Q}} \), the fields \( Q, \tilde{Q} \) develop vacuum expectation values \( \langle Q \rangle \), \( \langle \tilde{Q} \rangle \). Let us take the vacuum expectation value (VEV) to be \( \langle (Q_{\pm})_{ab} \rangle = q_{\pm}\delta_{ab}, \langle (\tilde{Q}_{\pm})_{ab} \rangle = \tilde{q}_{\pm}\delta_{ab} \). Minimizing \( S_{Q,\tilde{Q}} \) with respect to \( Q, \tilde{Q} \), one finds that for \( \mathcal{E} \rightarrow 0 \), the VEV’s \( q = q_{\pm} \) and \( \tilde{q} = \tilde{q}_{\pm} \) must be solutions to a set of self-consistent equations which have the solution

\[
q = \tilde{q} = \frac{1}{a \ g}(\exp\left(\frac{4\pi}{g}\right) - 1)^{-1/2}
\]

which for small \( g \) goes to zero exponentially as follows:

\[
q = \tilde{q} \sim \frac{1}{a \ g} \exp\left(-\frac{2\pi}{g}\right)
\]

(6.18)

This follows from the saddle point equations

\[
Q = - <x|(D - g\tilde{Q})^{-1}|x>, \quad \tilde{Q} = <x|(D + gQ)^{-1}|x>
\]

which, when written in momentum space, lead to

\[
\frac{gq}{4\pi} \log(1 + \frac{1}{(a \ gq)^2}) = q
\]

(and a similar equation with \( q \leftrightarrow \tilde{q} \)).

Since the VEV breaks the \( GL(1|1;\ C) \) symmetry to \( U(1|1) \), there are massless Goldstone modes. These Goldstone modes as usual can be viewed as a symmetry transformation of the VEV. Thus, let us define the fields

\[
U = (\mathcal{G}^\dagger)^{-1}\langle Q \rangle\mathcal{G}^{-1} = \begin{pmatrix} qGG^\dagger & 0 \\ 0 & q(GG^\dagger)^{-1} \end{pmatrix} = \begin{pmatrix} qU & 0 \\ 0 & qU^{-1} \end{pmatrix}
\]

\[
\tilde{U} = (\tilde{\mathcal{G}}^\dagger)^{-1}\langle \tilde{Q} \rangle\tilde{\mathcal{G}}^{-1} = \begin{pmatrix} q(GG^\dagger)^{-1} & 0 \\ 0 & qGG^\dagger \end{pmatrix} = \begin{pmatrix} qU^{-1} & 0 \\ 0 & qU \end{pmatrix}
\]

(6.20)

The latter are expressed in terms of \( U \equiv GG^\dagger \). An element \( G \in GL(1|1;\ C) \) can be factorized as \( G = G_uG_a \) where \( G_u \) is real and \( G_a \) is \( G \). One has \( U = G_uG_aG_uG_a = G^2_u \), thus the field \( U \) lives on the coset space \( GL(1|1;\ C)/U(1|1) \).

A parametrization of the Goldstone modes \( U \) which manifests the required properties that \( U \) be invertible and \( U^\dagger = U \) is the following:

\[
U = \begin{pmatrix} e^{\phi_1} & 0 \\ 0 & e^{\phi_2} \end{pmatrix} \begin{pmatrix} 1 - \chi^\dagger \chi & \sqrt{2} \chi \\ \sqrt{2} \chi^\dagger & 1 - \chi \chi^\dagger \end{pmatrix} \begin{pmatrix} e^{\phi_1} & 0 \\ 0 & e^{\phi_2} \end{pmatrix}
\]

\[
U^{-1} = \begin{pmatrix} e^{-\phi_1} & 0 \\ 0 & e^{-\phi_2} \end{pmatrix} \begin{pmatrix} 1 - \chi^\dagger \chi & -\sqrt{2} \chi \\ -\sqrt{2} \chi^\dagger & 1 - \chi \chi^\dagger \end{pmatrix} \begin{pmatrix} e^{-\phi_1} & 0 \\ 0 & e^{-\phi_2} \end{pmatrix}
\]

(6.21)

The fields \( \phi_{1,2} \) are bosonic whereas \( \chi, \chi^\dagger \) are fermionic.
The low energy $\sigma$-model for the Goldstone modes $U$ is obtained by substituting $Q, \tilde{Q} \rightarrow U, \tilde{U}$ and performing a derivative expansion using $\text{STr} \log(A + D) = \text{STr} \log A + \text{STr} \log(1 + A^{-1}D)$. The zero-th order term in $E$ is

$$- \frac{1}{2a^2g^2} \left[ \text{Str} \left( D U^{-1} D U^{-1} + D \tilde{U}^{-1} D \tilde{U}^{-1} \right) \right] = -\frac{q^{-1}}{a^2 g^2} \text{Str} \partial_\mu U^{-1} \partial_\mu U$$

(6.22)

Keeping only the linear term in $E$, and expressing everything in terms of $U$, one finally obtains

$$S = \int d^2x \left[ -\frac{1}{8\pi \lambda^2} \text{Str} \left( \partial_\mu U^{-1} \partial_\mu U \right) - \frac{1}{2\pi \lambda^2} \left( \text{Str} U^{-1} \partial_\mu U \right)^2 - i\lambda_3 \text{Str} (U + U^{-1}) \right]$$

(6.23)

where for small $g$

$$\frac{1}{\lambda^2} = \frac{4}{a^2 g^2 q^2} \approx 4 \exp(\frac{4\pi}{g}) \gg 1, \quad \lambda_3 = \frac{2}{a^2 g q} \approx \frac{2}{a} \exp(\frac{2\pi}{g})$$

(6.24)

As explained below, the $\lambda_A/\lambda^2$ term is generated under renormalization.

In terms of the parametrization (6.21) we find the following expressions:

$$-\frac{1}{4} \text{Str} \left( \partial_\mu U^{-1} \partial_\mu U \right) = \{ \partial_\mu \phi_+ \partial_\mu \phi_- + \partial_\mu \chi^\dagger \partial_\mu \chi \} + \chi^\dagger \chi \left( \partial_\mu \phi_- \right)^2$$

(6.25)

$$\frac{1}{4} \left( \text{Str} U^{-1} \partial_\mu U \right)^2 = (\partial_\mu \phi_-)^2$$

(6.26)

$$\text{Str}(U + U^{-1}) = 2(1 - \chi \chi^\dagger) \cosh(\lambda \phi_2) - 2(1 - \chi^\dagger \chi) \cosh(\lambda \phi_1)$$

(6.27)

where $\phi_\pm \equiv (\phi_2 \pm \phi_1)$

(6.28)

(The effect of a Wess-Zumino topological term is discussed below.)

We may study the sigma model as a perturbed conformal field theory. As an example we will compute the density of states for the present model. Rescaling the fields $\phi_{1,2} \rightarrow \frac{1}{\lambda} \phi_{1,2}, \chi \rightarrow \frac{2}{\lambda} \chi$, the sigma model action can be written as

$$S = S_{CFT} + \int \frac{d^2x}{2\pi} \left( \lambda^2 \left( \frac{\lambda}{4} \mathcal{O}_g + \lambda \mathcal{O}_2 - i\mathcal{E} \mathcal{O}_E \right) \right)$$

(6.29)

where the conformal field theory has the same field content as our $gl(1|1)$ model, with action

$$S_{CFT} = \int \frac{d^2x}{8\pi} \left( (\partial_\mu \phi_2)^2 - (\partial_\mu \phi_1)^2 + \partial_\mu \chi^\dagger \partial_\mu \chi \right)$$

(6.30)

and

$$\mathcal{O}_g = \frac{1}{4} \chi^\dagger \chi \left( \partial_\mu \phi_- \right)^2$$

(6.31)

$$\mathcal{O}_2 = - (\partial_\mu \phi_-)^2$$

(6.32)

$$\mathcal{O}_E = 2\lambda_3 \left[ (1 - \frac{\lambda^2}{4} \chi \chi^\dagger) \cosh(\lambda \phi_2) - (1 - \frac{\lambda^2}{4} \chi^\dagger \chi) \cosh(\lambda \phi_1) \right]$$

(6.33)
Since $\chi^\dagger \chi$ is a logarithmic operator \[29\], it mixes under RG scale transformations with the identity operator, thereby generating the $\lambda_A$ coupling.

This can also be seen, alternatively, by making the following change of variables

$$\xi^\dagger \equiv e^{\phi - \chi^\dagger}, \quad \xi \equiv e^{\phi - \chi}$$

which yields

$$-\frac{1}{4} \text{Str} \left( \partial_\mu U^{-1} \partial_\mu U \right) = \left\{ \partial_\mu \phi^+ \partial_\mu \phi^- + e^{-2\phi^-} \partial_\mu \xi^\dagger \partial_\mu \xi \right\} - \left( \partial_\mu \phi^- \right) \partial_\mu \left[ e^{-2\phi^-} \xi^\dagger \xi \right]$$

(6.34)

The last term can be eliminated by shifting $\phi^+$,

$$\phi^+ \rightarrow \Phi^+ \equiv \phi^+ - e^{-2\phi^-} \xi^\dagger \xi$$

(6.35)

Let us now also consider a topological Wess-Zumino-Witten term with coupling $k$. To lowest (cubic) order this term is easily computed with the result

$$S_{WZW} = \frac{ik}{24\pi} \int_{M_3} d^3 x \epsilon^{ijk} \text{Str} \left\{ U^{-1} \partial_i U \left[ U^{-1} \partial_j U, U^{-1} \partial_k U \right] \right\} =$$

$$= \frac{ik}{2\pi} \int d^2 x \left( \partial_\mu \phi^- \right) \epsilon^{\mu\nu} \left[ (\partial_\nu \xi^\dagger) \xi - \xi^\dagger (\partial_\nu \xi) \right] + \text{quartic and higher terms}$$

(6.36)

where, as usual, $M_3$ is a three-dimensional manifold whose boundary is the two-dimensional space of interest and $\epsilon^{\mu\nu}$ the two-dimensional antisymmetric tensor.

The sigma model including the WZW term can be solved using the Lagrange multiplier method of section IV. In order to see this, note that the fundamental matrix field is of the form

$$U = e^{\Phi^+} V$$

(6.37)

where $V$ does not depend on $\Phi^+$. This implies that the commutator in Eq.(6.30) does not contain any $\Phi^+$-dependence. Since the supertrace of a commutator vanishes, the WZW term does not contain any $\Phi^+$ dependence either\[18\]. Therefore, as for the current-current perturbation of section IV, $\Phi^+$ acts as a Lagrange multiplier, and we can use the same steps to solve the present theory.

Making the same rescalings of the fields as above, the action for the sigma model with WZW term is

$$S = S_{CFT} + \int \frac{d^2 x}{2\pi} \left( \frac{\lambda}{2} O_g - \frac{1}{2} k^3 \lambda \right) O_{WZW} + \lambda_A O_2 - i E O_E \right) + ...$$

(6.38)

where\[19\]

---

\[18\] Note that this argument is equally valid for the $GL(N|N)/U(N|N)$ generalization of section VI B below.

\[19\] No confusion should arise between the fields $\Phi_{\pm}$, $\Phi_{1,2}$ defined here and those of Eq. (6.2,6.3) and in Appendix A,B.
\[
\Phi_+ \equiv (\Phi_2 + \Phi_1), \quad \Phi_- \equiv \phi_- \equiv (\Phi_2 - \Phi_1)
\]  

(6.39)

and only terms of linear order in \(\Phi_-\) have been written\(^{20}\). Here \(S_{\text{CFT}}\) is given as before by Eq.\((6.30)\) upon replacing, \(\phi_\pm \to \Phi_\pm\), \(\chi \to \xi\), and

\[
\begin{align*}
\mathcal{O}_g &= -\frac{1}{2} \Phi_- (\partial_\mu \xi^\dagger \partial_\mu \xi) \quad (6.40) \\
\mathcal{O}_{WZW} &= \frac{1}{2} \Phi_- \ i\epsilon^{\mu\nu} (\partial_\mu \xi^\dagger \partial_\nu \xi) \quad (6.41) \\
\mathcal{O}_2 &= - (\partial_\mu \Phi_-)^2 \quad (6.42) \\
\mathcal{O}_E &= 2 \lambda_3 \left[ \cosh(\lambda \Phi_2) - \cosh(\lambda \Phi_1) - \frac{\lambda^2}{4} \xi^\dagger \xi (e^{\lambda \Phi_1} + e^{\lambda \Phi_2}) \right]. \quad (6.43)
\end{align*}
\]

We proceed with the 1-loop R.G. equations. One has the OPE's:

\[
\begin{align*}
\mathcal{O}_g(z, \overline{z}) \mathcal{O}_g(0) &\sim -\frac{1}{4z\overline{z}} \mathcal{O}_2(0) + \ldots \quad (6.44) \\
\mathcal{O}_{WZW}(z, \overline{z}) \mathcal{O}_{WZW}(0) &\sim +\frac{1}{4z\overline{z}} \mathcal{O}_2(0) + \ldots \quad (6.45)
\end{align*}
\]

Thus, as in section IV, to lowest order the \(\beta\)eta-functions are

\[
\begin{align*}
\beta_\lambda &= 0, \quad \beta_k = 0, \quad \beta_{\lambda A} = \frac{\lambda^2}{16} [1 - (k\lambda^2)^2] + \ldots \quad (6.46)
\end{align*}
\]

The sigma model coupling constant \(\lambda\), and the WZW coupling \(k\) generate \(\lambda_A\) terms (denoted \(g_A\) in section IV) of opposite signs. The 1-loop R.G. equations have the same structure as those found for the \(gl(1|1)\) current-current perturbation model in section IV, where \(\mathcal{O}_2\) is analogous to \(\mathcal{O}_A\). In the presence of a WZW term, the sigma model becomes the conformal \(gl(1|1)\) WZW model at a particular finite value of the sigma model coupling \(1/\lambda^2 = k\).

We may solve the theory in Eq.\((6.38)\) exactly using the Lagrange multiplier method of section IV A. In order to compute correlators of \(\Phi_+\) one needs

\[
4\pi \frac{\delta S}{\delta \Phi_-(z, \overline{z})|_{\Phi_- = 0}} = \frac{\lambda}{2} \left( - (\partial_\mu \xi^\dagger \partial_\mu \xi) + (k\lambda^2) i\epsilon^{\mu\nu} (\partial_\mu \xi^\dagger \partial_\nu \xi) \right) = \\
= -\lambda \left( [1 - (k\lambda^2)] \partial_\overline{z} \xi^\dagger \partial_\overline{z} \xi + [1 + (k\lambda^2)] \partial_\overline{z} \xi^\dagger \partial_\overline{z} \xi \right)
\]

As in section IV A this gives the two point function of \(\Phi_+\) fields:

\[
< \partial_\overline{z} \partial_\overline{w} \Phi_+(z, \overline{z}) \partial_w \partial_\overline{w} \Phi_+(w, \overline{w}) >_{\lambda_A = 0} = -\frac{\lambda^2}{2} [1 - (k\lambda^2)^2] \frac{1}{(z-w)^2(\overline{z}-\overline{w})^2} + \delta - \text{functions}
\]

\(^{20}\)As in section IV, the term linear in \(\Phi_-\) will be sufficient in the following.
Furthermore, as in Eq.(4.35) we find

\[ < \Phi_+(z, \bar{z}) \Phi_-(0) > = -2 \log(z\bar{z}), \quad < \Phi_-(z, \bar{z}) \Phi_-(0) > = 0 \]

As in section IV A, this yields the correlation functions

\[ < \Phi_1(z, \bar{z}) \Phi_1(0) > = (1 - 4\lambda_A) \log(z\bar{z}/a^2) - \frac{\lambda^2}{16} [1 - (k\lambda^2)^2] \log^2(z\bar{z}/a^2) \]  \hspace{1cm} (6.47)

\[ < \Phi_2(z, \bar{z}) \Phi_2(0) > = -(1 + 4\lambda_A) \log(z\bar{z}/a^2) - \frac{\lambda^2}{16} [1 - (k\lambda^2)^2] \log^2(z\bar{z}/a^2) \]  \hspace{1cm} (6.48)

as well as the exact beta function (identical to the 1-loop result):

\[ \frac{d\lambda_A}{dl} = \beta_{\lambda_A} = \frac{\lambda^2}{16} [1 - (k\lambda^2)^2] \]  \hspace{1cm} (6.49)

We now proceed with the calculation of the density of states. The most relevant operator in \( \mathcal{O}_E \) is \( \cosh(\lambda\Phi_2) \), whose anomalous dimension is

\[ \Gamma(\lambda_A) = \dim(\cosh(\lambda\Phi_2)) = -\lambda^2 (1 + 4\lambda_A). \]  \hspace{1cm} (6.50)

The analysis of section V applies to the density of states with \( \lambda_A \) playing the role of \( g_A \), yielding

\[ \frac{\rho(E)}{\rho(E_R)} = \frac{E_R}{E} \exp \left( -2[\lambda_A(E_R) - \lambda_A(E)]/\beta_{\lambda_A} \right) \]  \hspace{1cm} (6.51)

(valid since \( \beta_{\lambda_A} \) is independent of \( \lambda_A \)). In the limit \( E \to 0 \) this gives a divergent (but integrable) density of states

\[ \overline{\rho}(E) \sim \frac{1}{E} \exp \left\{ -\frac{4\sqrt{2}}{\lambda^2 \sqrt{1 - (k\lambda^2)^2} \log(E_R/E)} \right\} \]

in agreement with the perturbative result obtained by Gade [12] using replicas when \( k = 0 \), i.e. in the absence of the WZW term.

We end this section by commenting on the connection with the scale invariant \( PSL(1|1) \) sigma model. Eliminating the two bosonic coordinates \( \phi_\pm \) the fundamental \( PSL(1|1) \) sigma model field becomes

\[ V = \left( \begin{array}{cc} (1 - \chi^\dagger \chi) & \sqrt{2} \chi \\ \sqrt{2} \chi^\dagger & (1 + \chi^\dagger \chi) \end{array} \right) \]  \hspace{1cm} (6.52)

whose dynamics is governed by the quadratic action obtained from \( S_{CFT} \) of Eq.(6.30) by eliminating \( \phi_1, \phi_2 \). In subsection C we briefly discuss the generalization to the less trivial \( PSL(N|N) \) case.
B. $GL(1|1)/U(1|1)$ Sigma Model as a Current-Current Perturbation

Upon comparing the correlation functions of the sigma model (Eq.(6.47) ) with those of the $N = 1$-species XY model (Eq.( 4.39, 4.40)), which is a current-current perturbation, one finds that the two models have identical correlation functions if one identifies the coupling constants as follows:

$$2\lambda A = \zeta^2 g_A$$

$$\frac{1}{4} \lambda^2 [1 - (k\lambda^2)^2] = (2g)^2 \zeta^6, \quad \text{(where } \zeta^2 = 1/(1 + 2g))$$

The identity of the sigma model with the current-current perturbation of the conformal current algebra is likely a special case of a more general statement (see next subsection).

C. $GL(N|N)/U(N|N)$ Sigma Model

The generalization of the sigma model of the previous section to $GL(N|N)$ is immediate. We start with a theory of $2N$ bosons and $2N$ fermions (in the previous section $N = 1$). The construction of the previous section provides us in the standard way with a field variable on a manifold that we denote by $U \in GL(N|N;C)/U(N|N)$. Denoting the elements of the Super-Lie algebra $gl(N|N)$ by $J_\pm$ and $J_A$, a parametrization of the fundamental matrix field, say of the form:

$$U[X_\pm, X_A] \equiv e^{\{X_+ J_+\}} \exp\{X_- J_- + \sum_{A \neq \pm} X_A J_A\}$$

yields an action generalizing Eq.(6.25). The $PSL(N|N)$ version is obtained by setting the coordinates $X_\pm$ to zero. This sigma model in the presence of a WZW term with coupling $k$ becomes the $gl(N|N)$ WZW conformal field theory at a finite value $\lambda^2 = 1/k$ of the sigma model coupling constant. The latter is a $gl(N|N)$ current algebra at level $k$. Generalizing the observation made in the preceding subsection for the $N = 1$ case, we suggest that the line of fixed points arising from the current-current perturbation of the $PSL(N|N)$ variant of the current algebra analyzed in section IV is an alternative and ‘dual’ description of the $PSL(N|N)$ sigma model with WZW term, as the sigma model coupling is varied away from the scale invariant point. We plan on reporting in more detail on this connection in a subsequent publication.

VII. CONCLUSIONS

In this paper we have studied current-current perturbations of $gl(N|N)$ super-current algebras. The existence of two quadratic Casimir invariants leads to a ‘nearly conformal’

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21 This follows as in the $N = 1$ case from the decomposition $G = G_a G_u$, see below Eq.(5.20).
theory for any value of the coupling. Current algebra techniques allowed us to compute exactly the correlation functions of all current operators in the perturbed theory, as well as the exact beta functions (in a particular RG scheme). The current correlators are conformal except for correlation functions involving the currents corresponding to the trace and the supertrace of the super-Lie algebra. In the case $N=1$ we have computed the correlation functions of all fields. We have applied this to two different, but related disordered models, the 2D random field XY Statistical Mechanics model, and the delocalization transition of electrons hopping on a 2D square lattice with $\pi$-flux per plaquette and real hopping amplitudes (HWK, [8]). Both theories can be formulated in terms of the 2D Dirac Hamiltonian subject to random real mass, imaginary scalar and imaginary vector potentials. This Hamiltonian is invariant under two discrete symmetries, charge-conjugation (particle-hole) and time-reversal, in every realization of disorder. For the random XY model we have computed all correlation functions involving 1st-moment averages (described by the ‘1-species theory’), as well as certain correlation functions involving Nth-moment averages (namely those involving the $gl(N|N)$ currents). In particular, we have computed the correlation function of the ‘Edwards-Anderson order parameter’, which is scale invariant and of scaling dimension $x=2$, in the 2nd moment ($N=2$-species) theory. For the random hopping model of Hatsugai et al. we have obtained the density of states which shows a divergence at zero energy. In section IV D we derived the 1-loop RG equations for a generalized random XY model: When formulated in terms of a random 2D Dirac Hamiltonian, the latter still exhibits particle-hole symmetry, but lacks time-reversal symmetry, as opposed to that of HWK. The delocalization transition exhibited by this more general random Dirac Hamiltonian is in the Gade-Wegner universality class, describing hopping of electrons on a 2D square lattice with general complex hopping amplitudes. In section VI we have derived a SUSY sigma model from the underlying random Dirac Hamiltonian, found to be defined on a target manifold which we denote by $GL(N|N;C)/U(N|N)$. We have solved this sigma model exactly for $N=1$ (including also a WZW term), and shown that it is identical to a current-current perturbation of a $GL(1|1)$ super-Lie current algebra. The density of states is obtained from the sigma model.

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APPENDIX A: SUSY APPROACH TO THE RANDOM FIELD XY-MODEL

In this Appendix we show that the SUSY approach to the random field XY model leads to the $N=1$ species action discussed in the bulk of the paper.

The non-hermitian Hamiltonian relevant for the 2D random field XY model is

$$H_2 = (-i\partial_x + iA_x)\sigma_1 + (-i\partial_y + iA_y)\sigma_2 + M\sigma_3 - iV1_2$$
In order to be able to integrate over bosonic variables, we need to consider the hermitian hamiltonian

\[ H_4 \equiv \begin{pmatrix} H_2 & \text{ } & 0 \\ \text{ } & \text{ } & H_2^\dagger \\ 0 & \text{ } & 0 \end{pmatrix} \]

\[ = -i\partial_x \sigma_1 \otimes \tau_1 + A_x \sigma_1 \otimes \tau_2 + A_y \sigma_2 \otimes \tau_2 + M \sigma_1 \otimes \tau_1 - V 1_2 \otimes \tau_2 \]

(A1)

\[ H_s = U^\dagger H_4 U \]

where \( U_{\sigma} \equiv (1 - i\sigma_1)/\sqrt{2}, \ U_{\tau} \equiv (1 - i\tau_3)/\sqrt{2} \)

yielding the real symmetric hamiltonian

\[ H_s^\dagger = U^\dagger H_4 U = -i\partial_x (\sigma_1 \otimes \tau_2) + i\partial_y (\sigma_3 \otimes \tau_2) - V (1 \otimes \tau_1) - M (\sigma_2 \otimes \tau_2) \]

(A3)

For integration over fermionic variables an antisymmetric form is needed. This can be obtained by conjugating by

\[ U' \equiv U_{\sigma} \otimes 1_2 \]

\[ H_4^a = U'^\dagger H_4 U' = i\partial_x (\sigma_1 \otimes \tau_1) + i\partial_y (\sigma_3 \otimes \tau_1) + V (1 \otimes \tau_2) - M (\sigma_2 \otimes \tau_1) \]

(A4)

All correlation functions relevant for the XY model can be obtained from an action constructed with a \((real) 4\)-component fermion field \( \chi \), and a \( real \) \( 4\)-component boson field \( \varphi \), defined as follows:

\[ \chi \equiv \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}, \quad \varphi \equiv \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}, \]

with Lagrangian (density)

\[ \mathcal{L} \equiv \mathcal{L}_f + \mathcal{L}_b, \quad \mathcal{L}_f \equiv i\chi^t U'^\dagger H_4 U' \chi, \quad \mathcal{L}_b \equiv i\varphi^t U'^\dagger H_4 U \varphi \]

It will prove convenient to introduce 4-component complex fields and their adjoints by

Fermions : \( \Psi \equiv \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix} \equiv U' \chi, \quad \overline{\Psi} \equiv (\overline{\Psi}^+, \ \overline{\Psi}^-) \equiv \chi^t U'^\dagger \]

Bosons : \( \Phi \equiv \begin{pmatrix} \Phi^+ \\ \Phi^- \end{pmatrix} \equiv U \varphi, \quad \Phi^\dagger \equiv (\Phi_+^\dagger, \ \Phi_-^\dagger) \equiv \varphi^t U^\dagger \]

so that\(^{22}\)

\[ \mathcal{L}_f = i\overline{\Psi} H_4 \Psi = 2i\overline{\Psi}_+ H_2 \Psi_- = 2i\overline{\Psi}_- H_2^\dagger \Psi_+, \quad \mathcal{L}_b = i\Phi^\dagger H_4 \Phi = 2i\Phi_+^\dagger H_2 \Phi_- = 2i\Phi_-^\dagger H_2^\dagger \Phi_+ \]

\(^{22}\) Note that the latter equalities involving \( \Psi_\pm, \Phi_\pm \) arise because \( H_4^a \) is symmetric, and because \( U' \) and \( U \) do not mix the \( 2 \times 2 \) blocks involving \( H_2 \).

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The so-defined fields are not independent from their adjoints. Rather, the above definitions immediately give the following relations:

$$(\Psi)^t = U^* \chi = U^* U^\dagger \Psi, \quad (\Phi^\dagger)^t = \Phi^* = U^* U^\dagger \Phi$$

Since $U^* U^\dagger = U^*_\sigma U^\dagger_\sigma = -i\sigma_1$ and $U^* U^\dagger = -\sigma_1 \otimes \tau_3$ this implies

$$
\begin{pmatrix}
\bar{\psi}_1 \\
\bar{\psi}_2 \\
\bar{\psi}_3 \\
\bar{\psi}_4
\end{pmatrix} = (i) 
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{pmatrix},
\quad 
\begin{pmatrix}
\bar{\varphi}_1^* \\
\bar{\varphi}_2^* \\
\bar{\varphi}_3^* \\
\bar{\varphi}_4^*
\end{pmatrix} = \begin{pmatrix}
-\varphi_2 \\
-\varphi_1 \\
\varphi_4^* \\
\varphi_3^*
\end{pmatrix}
$$

Therefore we may chose

$$\Psi_+ = (\bar{\psi}_1, \bar{\psi}_2) \equiv (\psi_R^\dagger, \psi_L^\dagger), \quad \text{and} \quad \Psi_- = (\psi_3, \psi_4) \equiv (\psi_L, \psi_R) \quad (A5)$$

as well as

$$\Phi_+^\dagger = (\varphi_1^*, -\varphi_1) \equiv (\beta_R^*, \beta_L^*), \quad \text{and} \quad \Phi_- = (\varphi_4^*, \varphi_4) \equiv (\beta_L, \beta_R) \quad (A6)$$

as our independent 4 fermionic and 4 (real) bosonic integration variables. The resulting supersymmetric action

$$L_{SUSY} = i (\psi_R^\dagger, \psi_L^\dagger) H_2 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} + i (\beta_R^\dagger, \beta_L^\dagger) H_2 \begin{pmatrix} \beta_L \\ \beta_R \end{pmatrix}$$

is the $N = 1$-species theory discussed in the bulk of the paper.

**APPENDIX B: THE MOST GENERAL PARTICLE-HOLE SYMMETRIC DIRAC HAMILTONIAN**

In this appendix we discuss in some detail the SUSY invariant action for the most general random particle-hole symmetric, but not necessarily time-reversal symmetric Dirac Hamiltonian. We start from the hermitian quantum mechanical Dirac Hamiltonian $H_4$ defined by

$$H_4 \equiv \begin{pmatrix}
-i\epsilon \mathbf{1}_2 \\
H_2 \\
H_2 \\
-i\epsilon \mathbf{1}_2
\end{pmatrix} \quad (B1)$$

where

$$H_2 \equiv (-i\partial_x + iA_x)\sigma_1 + (-i\partial_y + iA_y)\sigma_2 + M\sigma_3 - iV\mathbf{1}_2$$

(As usual, infinitesimals $\epsilon \to 0^+$ are needed to extract quantum mechanical Green’s functions). This model is time reversal invariant if all the potentials $(A_x, A_y, V, M)$ are real. In this case the time reversal operation is implemented by

$$\mathcal{T} H_4^\dagger \mathcal{T} = H_4, \quad \text{with} : \quad \mathcal{T} = \sigma_1 \otimes \tau_3 \quad (B2)$$
We may consider a more general (non-hermitian) model, where we add to any of the potentials an imaginary part, $A', A', M', V'$. Note that $A', A'$ corresponds to a real vector potential. The most general hermitian hamiltonian $H_4$ incorporating all these potentials is

$$H_4 = -i\partial_x \sigma_1 \otimes \tau_1 - i\partial_y \sigma_2 \otimes \tau_1 + A_x \sigma_1 \otimes \tau_2 + A_y \sigma_2 \otimes \tau_2 + M \sigma_3 \otimes \tau_1 - V \mathbf{1}_2 \otimes \tau_2 \quad (B3)$$

$$- A'_x \sigma_1 \otimes \tau_1 - A'_y \sigma_2 \otimes \tau_1 - M' \sigma_3 \otimes \tau_2 + V' \mathbf{1}_2 \otimes \tau_1 \quad (B4)$$

In general, this hamiltonian is no longer time reversal invariant\(^{23}\). However, it is always invariant under particle-hole transformations

$$C H_4 C = -H_4, \quad \text{with} \quad C = \mathbf{1} \otimes \tau_3 \quad (B5)$$

Time reversal symmetry is recovered for the special case where only $A_x, A_y, V', M'$ are non-vanishing. In this case the time reversal operation is

$$\mathcal{T}' H_4^* \mathcal{T}' = H_4, \quad \text{with}: \quad \mathcal{T}' = \sigma_2 \otimes \mathbf{1}_2 \quad (B6)$$

These time reversal symmetry properties are summarized in the following table:

| non-vanishing potentials | $A_x, A_y, V, M$ | $A_x, A_y, V', M'$ | others |
|--------------------------|-----------------|-------------------|--------|
| time reversal            | YES ($\mathcal{T}$) | YES ($\mathcal{T}'$) | NO     |

Let us now return to the most general situation, where time reversal symmetry is in general absent. The $4 \times 4$ matrix corresponding to the hamiltonian $H_4$ has the following block structure:

$$i H_4 = \begin{pmatrix}
\epsilon & 0 & m - \mu & \partial + A_z + iA'_z \\
0 & \epsilon & \bar{\partial} + A_z^* + iA'^*_z & m^* + \mu^* \\
-m^* + \mu^* & \partial - A_z + iA'_z & \epsilon & 0 \\
\bar{\partial} - A_z^* + iA'^*_z & -m - \mu & 0 & \epsilon
\end{pmatrix} \quad (B7)$$

where

$$m \equiv (V + iM), \quad (-\mu) \equiv (M' - iV')$$

and

$$A_z \equiv (A_x - iA_y), \quad A_T \equiv A_z^* \quad (-1)A'_z \equiv (A'_y - iA'_x), \quad A'_\bar{T} \equiv A'^*_z$$

This matrix is manifestly anti-hermitian as it should\(^{24}\).

\(^{23}\) the imaginary parts $A', A', M', V'$ break invariance under time reversal, using $\mathcal{T}$ defined above.

\(^{24}\) note that $\partial \propto [\partial_x - i\partial_y], \bar{\partial} \propto [\partial_x + i\partial_y]$, so that $\tilde{\partial} = [-\partial_x - i\partial_y] = -\bar{\partial}$. 

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Averaged Green’s functions of the hamiltonian $H_4$ can be obtained from the following action. We introduce a 4-component complex bosonic field $\Phi$ and its adjoint $\Phi^\dagger$,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \equiv \begin{pmatrix} \beta_{L2}^\dagger \\ \beta_{R2}^\dagger \\ \beta_{L1} \\ \beta_{R1} \end{pmatrix}$$

$$\Phi^\dagger \equiv (\phi_1^*, \phi_2^*, \phi_3^*, \phi_4^*) \equiv (\beta_{R1}^\dagger, \beta_{L1}^\dagger, -\beta_{R2}, -\beta_{L2})$$ (B8)

as well as a 4-component (complex) fermionic field $\Psi$ and $\Psi^\dagger$:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \equiv \begin{pmatrix} \psi_{L2}^\dagger \\ \psi_{R2}^\dagger \\ \psi_{L1}^\dagger \\ \psi_{R1}^\dagger \end{pmatrix}$$

$$\Psi^\dagger \equiv (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \equiv (\psi_{R1}^*, \psi_{L1}^*, \psi_{R2}, \psi_{L2})$$ (B9)

The SUSY lagrangian (density) reads, in the absence of the vector potentials:

$$\mathcal{L}_{susy} = \Psi^\dagger iH_4 \Psi + \Phi^\dagger iH_4 \Phi =$$

$$= \left[ (\psi_1^* \partial \psi_4 + \psi_4^* \overline{\partial} \psi_1) + (\phi_1^* \partial \phi_4 + \phi_4^* \overline{\partial} \phi_1) \right] - \left[ (\psi_3^* \partial (-\psi_2) + (-\psi_2^*) \overline{\partial} \psi_3) + (\phi_3^* \partial (-\phi_2) + (-\phi_2^*) \overline{\partial} \phi_3) \right] +$$

$$+ m_+ (\psi_1^* \psi_3 + \phi_1^* \phi_3) + m_- (\psi_1^* (-\psi_2) + \phi_1^* (-\phi_2)) - m_+^* (\psi_3^* \psi_1 + \phi_3^* \phi_1) + m_-^* (-\psi_2^* \psi_4 + (-\phi_2^*) \phi_4) +$$

$$+ \epsilon \left[ \psi_1^* \psi_1 + \phi_1^* \phi_1 + (-\psi_2^*)(-\psi_2) + (-\phi_2^*)(-\phi_2) + \psi_3^* \psi_3 + \phi_3^* \phi_3 + \psi_4^* \psi_4 + \phi_4^* \phi_4 \right]$$ (B10)

where

$$m_+ \equiv (m + \mu), \quad m_- \equiv (m - \mu).$$

This may also be expressed in terms of the $R$- and $L$-moving fields, defined above:

$$\mathcal{L}_{susy} =$$

$$= \sum_a (\psi_{Ra}^\dagger \partial \psi_{Ra} + \beta_{Ra}^\dagger \partial \beta_{Ra} + \psi_{La}^\dagger \overline{\partial} \psi_{La} + \beta_{La}^\dagger \overline{\partial} \beta_{La}) +$$

$$+ m \left( \sum_a (\psi_{Ra}^\dagger \psi_{La} + \beta_{Ra}^\dagger \beta_{La}) \right) + m^* \left( \sum_a (\psi_{La}^\dagger \psi_{Ra} + \beta_{La}^\dagger \beta_{Ra}) \right) +$$

$$+ \mu \left( (\psi_{R1}^\dagger \psi_{L1} + \beta_{R1}^\dagger \beta_{L1}) - (\psi_{R2}^\dagger \psi_{L2} + \beta_{R2}^\dagger \beta_{L2}) \right) - \mu^* \left( (\psi_{L1}^\dagger \psi_{R1} + \beta_{L1}^\dagger \beta_{R1}) - (\psi_{L2}^\dagger \psi_{R2} + \beta_{L2}^\dagger \beta_{R2}) \right) +$$

The symbols $\beta_{Ri}, \beta_{Li}, \beta_{Ri}^\dagger, \beta_{Li}^\dagger$ will be used to exhibit the chiral ($L/R$) nature of the corresponding fields (see below).
\[ + \epsilon (\psi_{R1}^\dagger \psi_{L2}^\dagger - \psi_{R2}^\dagger \psi_{L1}^\dagger + \psi_{L2} \psi_{R1} - \psi_{L1} \psi_{R2} + \beta_{R1}^\dagger \beta_{L2}^\dagger + \beta_{R2}^\dagger \beta_{L1}^\dagger - \beta_{R1} \beta_{L2} + \beta_{R2} \beta_{L1}) \] (B11)

The vector potentials give rise to an additional contribution to the action of the form:

\[ L_{\text{susy}}^A \equiv A_z \left( \sum_a (\psi_{Ra}^\dagger \psi_{Ra} + \beta_{Ra}^\dagger \beta_{Ra}) \right) + A_{\tau}\left( \sum_a (\psi_{La}^\dagger \psi_{La} + \beta_{La}^\dagger \beta_{La}) \right) \]

\[ + i A'_z \left( \psi_{R1}^\dagger \psi_{R1} + \beta_{R1}^\dagger \beta_{R1} - \psi_{R2}^\dagger \psi_{R2} - \beta_{R2}^\dagger \beta_{R2} \right) + i A'_{\tau}\left( \psi_{L1}^\dagger \psi_{L1} + \beta_{L1}^\dagger \beta_{L1} - \psi_{L2}^\dagger \psi_{L2} - \beta_{L2}^\dagger \beta_{L2} \right) = \]

\[ = A_z \left[ (J'_{11} - J_{11}) + (J'_{22} - J_{22}) \right] + A_{\tau}\left[ (J'_{11} - J_{11}) + (J'_{22} - J_{22}) \right] + \]

\[ + i A'_{\tau}\left[ (J'_{11} - J_{11}) - (J'_{22} - J_{22}) \right] \] (B12)

APPENDIX C: 1-LOOP RG EQUATIONS FOR THE GENERALIZED RANDOM XY MODEL (BROKEN TIME REVERSAL SYMMETRY)

In this Appendix we analyze the 1-loop RG equations for the random XY model without time reversal symmetry.

Let us first consider the RG equations for the couplings \( g_1 = g_m - g_\mu \) and \( g_2 = g_m + g_\mu \), defined in Eq.’s (4.80, 4.81). The most general form of the 1-loop RG is:

\[ \frac{dg_1}{dl} = a_1 g_1^2 + b_1 g_1 g_2 + c_1 g_2^2 + (d_{11}^1 g_1 + d_{21}^1 g_2) g_1^A + (d_{12}^1 g_1 + d_{22}^1 g_2) g_2^A \]

\[ \frac{dg_2}{dl} = a_2 g_1^2 + b_2 g_1 g_2 + c_2 g_2^2 + (d_{11}^2 g_1 + d_{21}^2 g_2) g_1^A + (d_{12}^2 g_1 + d_{22}^2 g_2) g_2^A \]

We proceed in the following steps, to simplify these equations:

**Step 1:** We know that both RG equations vanish when \( g_2 = g_2^A = 0 \) describing two decoupled 1-species theories. This gives \( a_1 = d_{11}^1 = a_2 = d_{11}^2 = 0 \).

**Step 2:** Adding and subtracting the two equations:

\[ \frac{d(g_1 + g_2)}{dl} = 2 \frac{dg_m}{dl} = (b_1 + b_2) g_1 g_2 + (c_1 + c_2) g_2^2 + (d_{21}^1 + d_{21}^2) g_2 g_1^A + [(d_{12}^1 + d_{12}^2) g_1 + (d_{22}^1 + d_{22}^2) g_2] g_2^A \]

\[ \frac{d(g_1 - g_2)}{dl} = (-2) \frac{dg_\mu}{dl} = (b_1 - b_2) g_1 g_2 + (c_1 - c_2) g_2^2 + (d_{21}^1 - d_{21}^2) g_2 g_1^A + [(d_{12}^1 - d_{12}^2) g_1 + (d_{22}^1 - d_{22}^2) g_2] g_2^A \]

\[ \] Note that we have omitted terms on the r.h.s. which are quadratic in the vector potential couplings, i.e. of the form \( g_i^A g_j^A \). The operators coupling to \( g_i^A \) are of the form \( J_{aa} \) and \( J'_{ab} \) (see Eq. (4.82)). Since there are no poles in the OPE of such operators with each other (see Eq.(4.51)) they do not generate 1-loop RG flows.
Setting $g_{\mu} = g_{A'} = 0$, both equations must vanish identically. In this case we have $g_1 = g_2 = g_m$ and $g_1^A = g_2^A = g_A$, which yields, when inserted into the RG equations:

\[b_1 + c_1 = d_{21}^1 + d_{12}^1 + d_{22}^1 = b_2 + c_2 = d_{21}^2 + d_{12}^2 + d_{22}^2 = 0\]

**Step 3:** Finally, setting $g_m = g_{A'} = 0$, again both equations must vanish. In this case we have $g_1 = -g_{\mu}, g_2 = g_{\mu}$ and $g_1^A = g_2^A = g_A$, which yields, when inserted into the RG equations:

\[-b_1 + c_1 = d_{21}^1 - d_{12}^1 + d_{22}^1 = -b_2 + c_2 = d_{21}^2 - d_{12}^2 + d_{22}^2 = 0\]

Combining Steps 1, 2 and 3 we find:

\[a_1 = d_{11}^1 = a_2 = d_{11}^2 = 0\]

\[b_1 = b_2 = c_1 = c_2 = 0, \quad d_{12}^1 = d_{12}^2 = 0,\]

\[(d_{21}^1 + d_{22}^1) = (d_{21}^2 + d_{22}^2) = 0\] (C1)

The 1-loop RG equation thus simplifies to

\[\frac{dg_1}{dl} = (d_{21}^1 g_1^A + d_{22}^1 g_2^A) g_2\]

\[\frac{dg_2}{dl} = (d_{21}^2 g_1^A + d_{22}^2 g_2^A) g_2\]

Using Eq.(4.83) this becomes

\[\frac{dg_1}{dl} = -(d_{21}^1 - d_{22}^1) g_{A'} g_2\]

\[\frac{dg_2}{dl} = -(d_{21}^2 - d_{22}^2) g_{A'} g_2\]

Let us now consider explicitly the relevant OPE’s: The coupling constant $g_{A'}$ couples to a left-right bilinear of a Cartan generator. Since the simple pole term in the OPE of a Cartan generator with any operator can only give back the same operator, times a number (= the eigenvalue of the Cartan generator when applied to this operator), we know that $g_{A'} g_2$ can only generate again the operator coupling to $g_2$. Therefore we know that $(d_{21}^1 - d_{22}^1) = 0$ and the RG equations simplifies further to:

\[\frac{dg_1}{dl} = 0\]

\[\frac{dg_2}{dl} = -(d_{21}^2 - d_{22}^2) g_{A'} g_2\] (C2)

(Recall that $g_2 = g_m + g_{\mu}$, from Eq. (4.81).)
The corresponding RG equations for the vector potential couplings are of the form

\[
\frac{dg_A}{dl} = \alpha_1 g_m^2 + \beta_1 g_m g_\mu + \gamma_1 g_\mu^2
\]

\[
\frac{dg_A'}{dl} = \alpha_2 g_m^2 + \beta_2 g_m g_\mu + \gamma_2 g_\mu^2
\]

Since \(g_A'\) is not generated when either \(g_\mu = 0\) or when \(g_m = 0\), we have \(\alpha_2 = \gamma_2 = 0\). In addition, the flow for \(g_A\) is the same in both cases, implying \(\alpha_1 = \gamma_1\). Adding and subtracting the resulting equations

\[
\frac{dg_1^A}{dl} = \frac{d(g_A - g_A')}{dl} = \alpha_1 (g_m^2 + g_\mu^2) + (\beta_1 - \beta_2) g_m g_\mu
\]

\[
\frac{dg_2^A}{dl} = \frac{d(g_A + g_A')}{dl} = \alpha_1 (g_m^2 + g_\mu^2) + (\beta_1 + \beta_2) g_m g_\mu
\]

When \(g_2 = 0\) (i.e.: \(g_\mu = -g_m\), \(g_1 = 2g_m\)) we know that

\[
\frac{dg_1^A}{dl} = \alpha_1 g_1^2, \quad \frac{dg_2^A}{dl} = 0
\]

This finally yields the following RG equations \((\beta_1 = 0, \beta_2 = 2\alpha_1):\)

\[
\frac{dg_A}{dl} = 2[(g_m)^2 + (g_\mu)^2], \quad \frac{dg_A'}{dl} = 4 g_m g_\mu
\]

where we have used that \(\alpha_1 = 2\) [see Eq.(4.10)].
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