Analytical holographic superconductors in $AdS_N$-Lifshitz topological black holes

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Abstract

We present the analytic Lifshitz solutions for a scalar field model minimally coupled with the abelian gauge field in $N$ dimensions. We also consider the presence of cosmological constant $\Lambda$. The Lifshitz parameter $z$ appearing in the solution plays the role of the Lorentz breaking parameter of the model. We investigate the thermodynamical properties of the solutions and discuss the energy issue. Furthermore, we study the hairy black hole solutions in which the abelian gauge field breaks the symmetry near to the horizon. In the holographic picture, it is equivalent to a second order phase transition. Explicitly we show that there exists a critical temperature which is a function of the Lifshitz parameter $z$. The system below the critical temperature becomes superconductor, but the critical exponent of the model remains the same of the usual holographic superconductors without the higher order gravitational corrections, in agreement with Ginzburg-Landau theories.

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1 Introduction

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1] has many useful applications in condensed matter physics, especially for studying scale-invariant strongly-coupled systems, for example, and low temperature systems near quantum criticality [2]. In brief, this conjecture states that there is a direct duality between any classical solution of the gravitational bulk action in $d$ dimensional asymptotically $AdS_d$ spacetime to the quantum objects on the boundary. These quantum objects are described as well by a conformal field theory. At the first, it seems this following deep conjecture supported by the relativistic essence of the bulk action. By the classical solutions in the bulk we mean those locally Lorentz invariant theories of the gravity (or modified gravity but with more precise view) not with the semi quantum corrections. So the bulk must be completely classical from this point of view. Also, the quantum object who lives on the boundary must be unitary and real operator to support a qualified quantum theory without any serious problem. Different kinds of the dualities can be addressed. One possibility is the correspondence between the gravity solutions and the fluid mechanics equations. By some techniques which are inspired directly from the holographic picture [3] it is possible to find some more information about the exact solutions of the Navier-Stokes equations in some strictly imitated cases. Also the scale invariant condensation phenomena in the condensed matter provides another landscape for this conjecture. There are many attempts to relate the condensed matter problems to their gravitational duals. Since the high-$T_c$ superconductors are shown to be in the strong coupling regime, the BCS theory fails and one expects that the holographic method could give some insights into the pairing mechanism in the high-$T_c$ superconductors. From the ($d$ dimensional) field theory point of view, superconductivity is characterized by condensation of a generally composite charged operator $\hat{O}$ in low temperatures $T < T_c$. In the gravitational dual ($d + 1$ dimensional) description of the system, the transition to the super conductivity is observed as a classical instability of a black hole in an anti-de Sitter (AdS) space against perturbations by a charged scalar field $\psi$. The AdS/CFT correspondence relates the quantum dynamics of the boundary operator $\hat{O}$ to a simple classical dynamics of the bulk scalar field $\psi$ [4,5]. Various holographic superconductors have been studied in Einstein theory [6,7] or extended versions as Gauss-Bonnet (GB) [8,9], Weyl corrected ones [10–13], with magnetic field in the bulk action [14–20], with non linear Maxwell’s fields [21], as a toy model of two dimensional superconductors using $AdS_3/cft_2$ [22], in modified gravity scenario [23] and even in the non relativistic model of gravity as, for example, in Horava-Lifshitz theory [24,25]. The analytical methods have been used in the description of the phase transition phenomena in the superconductors of type II [26]. In recent years, holographic method have been used to study non-relativistic system [27]. In the framework of condensed mater theory, different
systems show a dynamical scaling near fixed points:

\[ t \to \lambda^{\frac{z+2}{2}} t, \quad x_i \to \lambda x_i, \quad z \neq 0. \]  

(1.1)

As a consequence, instead obeying the conformal scale invariance \( t \to \lambda t, x_i \to \lambda x_i \), the temporal and the spatial coordinates scale anisotropically. The Lifshitz topological black holes and charged Lifshitz black holes have previously been discussed in Refs. \[28, 29\]. Also the Einstein-Maxwell-dilaton system has been used to construct Lifshitz spacetime \[30\].

In the present paper we would like to study holographic superconductors in a new background. This set up of the s-wave holographic superconductors uses the AdS-Lifshitz black hole as the gravitational bulk metric in the probe limit. In the Section 2, we introduce a scalar field model non minimally coupled with the abelian gauge field in the presence of cosmological constant in \( N \) dimensions and we derive static, (pseudo-)spherically symmetric (SSS) solutions with various topologies. In particular, we will be interested in the AdS-Lifshitz black hole (AdS-BH) solutions. In the Section 3, we study the thermodynamical properties of the solutions and obtain the quasi-local generalized Misner-Sharp mass as a Killing conserved charge. We also verify the validity of the Gibbs equation by using the Kodama-Hayward temperature. In Section 4 we study the hairy black hole solutions in which near the horizon the abelian gauge field breaks the symmetry and in Section 5 we explore the scalar condensation in our Lifshitz black hole solutions by analytical approaches. The matching solutions and the critical temperature will be found. Finally, conclusions are given in the last Section.

2 Bulk asymptotic \( \text{AdS}_N \)-Lifshitz solution

We will consider the \( N \)-dimensional action of the following model where the scalar field \( \phi \) is non minimally coupled with electromagnetic potential,

\[
I = \int_{\mathcal{M}} d^N x \sqrt{-g} \left[ \frac{(R - 2\Lambda)}{2\kappa^2} - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + V(\phi) - \xi e^{\lambda \phi} (F^{\mu\nu} F_{\mu\nu}) \right].
\]

(2.1)

Here, \( g \) is the determinant of metric tensor, \( g_{\mu\nu} \), \( \mathcal{M} \) is the space-time manifold, \( \Lambda \) is a "non effective" cosmological constant, namely \( \Lambda = -(N-1)(N-2)/(2L^2) \), \( L \) being a length size, and \( F_{\mu\nu} \) is the electromagnetic field strength coupled with scalar field \( \phi \) as \( \xi \exp[\lambda \phi](F^{\mu\nu} F_{\mu\nu}) \), \( \xi \) and \( \lambda \) being generic constants (for example, in the four dimensional Einstein-Maxwell action one has \( \xi = 1/4 \) and \( \lambda = 0 \)). The field is also subjected to a potential \( V(\phi) \). Here, we use units of \( k_B = c = \hbar = 1 \) and denote the gravitational constant \( \kappa^2 = 8\pi G_N^N \equiv 8\pi (1/M_P^2)^N \) with the Planck

\[5\text{In this paper we use a different notation with respect to the one usually adopted in parameterizing the Lifshitz solution. In the seminal paper [55], \((z + 2)/2 \to z\). The reason of our choice is a more suitable form of Lifshitz solution for our derivation.} \]
mass of $M_{PL} = G_N^{-1/2} = 1.2 \times 10^{19}$ GeV.

We look for static, (pseudo-)spherically symmetric (SSS) solutions with various topologies, and write the metric element as

$$ds^2 = -e^{2\alpha(r)}B(r)dt^2 + \frac{dr^2}{B(r)} + r^2d\sigma_{N-2,k}^2,$$

(2.2)

where $\alpha(r)$ and $B(r)$ are function of $r$ only and $d\sigma_{N-2,k}^2$ represents the metric of a topological $(N-2)$-dimensional surface parametrized by $k = 0, \pm 1$, such that the manifold will be either a sphere $S_{N-2}$ (for $k = 1$), a torus $T_{N-2}$ (for $k = 0$) or a compact hyperbolic manifold $Y_{N-2}$ (for $k = -1$). In particular, we will be interested in the Lifshitz solutions, where $\alpha(r) \propto \log r^{z/2}$, being $z$ the redshift parameter. In this case, for power counting renormalizability in $N$ dimension, if we assume that the interaction potential can be expanded as

$$V(\phi) = \Sigma_{i=0}^K g_i \phi^i,$$

where $\phi^i$ are the polynomial terms of the series and $g_i$ suitable coefficients, by the dimensional engineering we get [31]

$$[g_i] = [m]^{\frac{(N+z-1-m(N-1-z))}{z}}.$$

Such kind of theory is renormalizable if the couplings have non-negative momentum (here mass) dimension $[m]$, so that we have two possibilities, namely

$$K = \begin{cases} \frac{2(N-1+z)}{N-1-z}, & z < N-1, \\ \infty, & z \geq N-1. \end{cases}$$

The above constraints are valid for any scalar theory under the Lifshitz scaling of the coordinates.

We propose the potential in the following form,

$$V(\phi) = V_0 e^{\gamma \phi},$$

(2.3)

where $V_0$ and $\gamma$ are generic parameters. Now, by comparing the exponential potential (2.3) with the polynomial form, we observe that in fact $K = \infty$, so that the theory (2.1) is renormalizable for $z \geq N-1$.

With the metric Ansatz (2.2), the scalar curvature reads

$$R = -3B\alpha' - 2B\alpha'^2 - B'' - 2B\alpha'' - (N-2)\left[\frac{2}{r}B' + \frac{2B\alpha'}{r} - \frac{(N-3)}{r^2}(k-B)\right],$$

(2.4)

where the prime index denotes the derivative with respect to $r$. Where is not necessary, the argument of the functions $\alpha(r)$ and $B(r)$ will be dropped.
Moreover, due to the $SO(N-2)$ symmetry and also demanding the parity symmetry, it is easy to see that the only non vanishing components of the electromagnetic field in $N$-dimension are

$$F_{01} = -\frac{dA_0}{dr}, \quad F^{01} = g^{00}g^{11}F_{01} = e^{-2\alpha}\frac{dA_0}{dr},$$

$$F_{10} = \frac{dA_0}{dr}, \quad F^{10} = g^{11}g^{00}F_{10} = -e^{-2\alpha}\frac{dA_0}{dr},$$  \hspace{1cm} (2.5)

$A_0$ being the electric potential. Now, in order to find the EOMs we can use the reduced action \[32\]. By assuming $\phi = \phi(r)$ and by plugging the above expressions into the action \[2.1\], making a partial integration, we finally get the following effective Lagrangian,

$$L_{\text{eff}} = e^\alpha r^{N-2} \left[ \frac{(N-2)(N-3)(k-B)}{r^2} - \frac{(N-2)B'}{r} - 2\Lambda - \frac{B\tilde{\phi}^2}{2} + \tilde{V}_0 e^{-\phi} + 2(2\kappa^2)\xi e^{\lambda \phi} e^{-2\alpha} \left( \frac{dA_0}{dr} \right)^2 \right],$$ \hspace{1cm} (2.6)

where, for simplicity, we have putted $\tilde{\phi} = 2\kappa^2 \phi$ and $\tilde{V}_0 = 2\kappa^2 V_0$.

The field equation for electromagnetic field coupled with the scalar field $\phi$ are derived from the action \[2.1\] and read

$$\frac{d}{dr} \left( r^{N-2} e^{\lambda \phi} e^{-\alpha} \frac{dA_0}{dr} \right) = 0,$$

namely

$$\frac{dA_0}{dr} = \frac{e^{\alpha} e^{-\lambda \phi} Q}{r^{N-2}},$$ \hspace{1cm} (2.7)

$Q$ being an integration constant of the electromagnetic field. The identification of $Q$ with the classical electric charge (eventually multiplied to some suitable dimensional parameter) is recovered in the flat limit $\alpha = 0$ and in the absence of the coupling with the field ($\lambda = 0$) as a consequence of the Gauss theorem and of the vanishing of electric potential at large distances.

By using the Euler-Lagrangian equations

$$\frac{d}{dr} \left( \frac{\partial L}{\partial Z_A'} \right) = \frac{\partial L}{\partial Z_A},$$

where, in our case, $Z_A = \{\alpha(r), B(r)\}$, we also obtain the following equations of motion (EOMs):

$$B' r - (N-3)(k-B) - \frac{r^2}{(N-2)} \left[ -2\Lambda - \frac{\tilde{\phi}^2 B}{2} + \tilde{V}_0 e^{\gamma \phi} - \frac{2\xi e^{-\lambda \phi} \tilde{Q}^2}{r^2(N-2)} \right] = 0,$$ \hspace{1cm} (2.8)

$$\alpha' - \frac{\tilde{\phi}^2 r}{2(N-2)} = 0.$$ \hspace{1cm} (2.9)
Here, after the derivation, we have used Eq. (2.7) in (2.8) and we have putted $\tilde{Q}^2 = 2\kappa^2 Q^2$.

Finally, the equation for $\phi(r)$ reads

$$\tilde{\phi}''B + \left[\alpha'B + B' + \frac{N - 2}{r}B\right]\tilde{\phi}' + \left[\gamma\tilde{V}_0 e^{\gamma\phi} + \frac{2\xi\lambda e^{-\lambda\phi}\tilde{Q}^2}{r^{2(N-2)}}\right] = 0,$$

(2.10)

where we have used (2.7) again after the derivation.

As we stated above, we are interested in the following solutions,

$$\alpha(r) = \log[(r/r_0)^{z/2}],$$

(2.11)

which correspond to the important class of Lifshitz solutions parameterized by red shift $z$ parameter. Here, $r_0$ is a dimensional constant. From Eq. (2.9) we get

$$\tilde{\phi}(r) = \sqrt{z(N - 2)} \log[r/\tilde{r}_0],$$

(2.12)

so that for renormalizable theory ($z > N - 1$) we deal with real fields, since in general we take $N > 3$. Here, $\tilde{r}_0$ is a new scale constant, which is in principle different from $r_0$ introduced above, and $\tilde{\phi}(r)$ has been taken with positive sign (an other possible solution is given by $\tilde{\phi}(r) = -\sqrt{z(N - 2)} \log[r/\tilde{r}_0]$). By using this result, we can solve Eq. (2.8) as

$$B(r) = \frac{2k(N - 3)}{z + 2N - 6} + Cr^{3-N+z/2} - \frac{2r^4\tilde{V}_0}{(N - 2)(6 - 2A - 2N - z)} + \frac{4\xi\tilde{Q}^2 r B}{(N - 2)(6 - 2B - 2N - z)} - \frac{4\Lambda r^2}{(N - 2)(2N - 2 + z)}.$$

(2.13)

In this equation, $C$ is a free integration constant of the solution and

$$A = \gamma\sqrt{z(N - 2) + 2},$$

$$B = -\lambda\sqrt{z(N - 2) + 6 - 2N}.$$ 

(2.14)

For simplicity, in the above expression we have also redefined $r_0^{-\gamma\sqrt{z(N-2)}} \tilde{V} \rightarrow \tilde{V}$ and $r_0^{\lambda\sqrt{z(N-2)}} \tilde{Q}^2 \rightarrow \tilde{Q}^2$. Note that if we turn off the scalar field potential and then take $\phi = 0$, we recover the Reissner-Norstrom solution with cosmological constant for $z = 0$,

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2d\sigma^2_{(N-2),k},$$

$$B(r) = k + Cr^{3-N} - \frac{2\Lambda r^2}{(N - 2)(N - 1)} + \frac{2\xi\tilde{Q}^2 r^{6-2N}}{(N - 2)(N - 3)}.$$

(2.15)

On the other hand, in the presence of the scalar field, the solution (2.13) is acceptable only if the
Klein Gordon equation (2.10) for $\phi$ is also satisfied. In this case, the generic form of the metric is given by

$$
\text{d}s^2 = -\left(\frac{r}{r_0}\right)^z B(r)\text{d}t^2 + \frac{\text{d}r^2}{B(r)} + r^2d\sigma_{N-2,k}^2. 
$$

(2.16)

Let us see for two different cases.

- **Absence of cosmological constant.** If $\Lambda = 0$, one possible solution of Eq. (2.10) is found by choosing

$$
\gamma = -\frac{2}{\sqrt{z(N-2)}},
\lambda = -\frac{2(N-3)}{\sqrt{z(N-2)}},
\tilde{V}_0 = \frac{k(N-3)(N-2)z - 2\tilde{Q}^2(2N - 6 + z)\xi}{(2-z)}. 
$$

(2.17)

The solution becomes

$$
B(r) = -\frac{4\left[k(N-3) - 2\tilde{Q}^2\xi\right]}{(z-2)(2N-6+z)} + Cr^{3-N-z/2}. 
$$

(2.18)

One remark is in order. In this and in the next case, the solution is not unique and depends on the choice of the parameters which must satisfy Eq. (2.10). As a consequence, we added some (suitable) additional constrains on the parameters $\gamma$ and $\lambda$. Thus, the constrain on $\tilde{V}_0$ follows from Eq. (2.10).

- **Cosmological constant $\Lambda \neq 0$.** One simple solution is given by

$$
\lambda = -\frac{2(N-2)}{\sqrt{z(N-2)}},
\gamma = -\frac{2}{\sqrt{z(N-2)}},
\tilde{V}_0 = \frac{k(N-3)(N-2)^2\tilde{Q}^2\xi}{(2N-2+\xi) + \Lambda}. 
$$

(2.19)

The solution reads

$$
B(r) = \frac{2}{(2N-6+z)} \left[ k(N-3) + \frac{\tilde{V}_0}{(N-2)} \right] + Cr^{3-N-z/2}
-\frac{4r^2(\Lambda + \tilde{Q}^2\xi)}{(N-2)(2N-2+z)}, 
$$

(2.20)
which asymptotically is a de Sitter/Anti de Sitter (dS/AdS) solution in the case of \( z = 0 \). Note that if \( \Lambda = 0 \), we get \( z = -2(N - 2) \), and, for \( N > 2 \), the solution is not acceptable, being \( \phi \) in Eq. (2.12) imaginary, and the theory becomes non renormalizable. In principle, for any dimension \( N \) and for any choice of \( z \) we can obtain the corresponding Lifshitz solution by setting the values of \( \gamma \), \( \lambda \), \( Q \) and \( V_0 \) appearing in the field lagrangian.

The solution (2.20) can be asymptotically Lifshitz AdS and will furnish our background in studying holographic superconductors. For a planar horizon \( k = 0 \), our solution turns out to be the one of Ref. [33], while the solutions for \( k = -1, 1 \) are novel.

### 3 Black hole solutions and thermodynamics

The solutions derived in the previous Section may describe charged black holes (BHs) in \( N \)-dimensional manifolds in the presence of scalar field non minimally coupled with electrodynamic potential. We recall that event horizon exists as soon as there exists a positive solution \( r_+ \) of

\[
B(r_+) = 0, \quad B'(r_+) \geq 0. \tag{3.1}
\]

We require \( B(r_+) \neq 0 \) to avoid the extremal BHs.

In the case of solution (2.18) one has

\[
r_+ = \left[ \frac{4k(N - 3) - 8\tilde{Q}^2\xi}{C(z - 2)(2N - 6 + z)} \right]^{\frac{1}{3-N-z/2}}. \tag{3.2}
\]

Since \( z > 0 \), we must require \( k/C > 2\tilde{Q}^2\xi(z - 2)/(N - 3) \) when \( N > 3 \). However, since in general \( \xi > 0 \), in order to have \( B'(r_+) > 0 \), we see that only in the topological case \( k = -1 \) we obtain a BH solution.

Concerning the solutions (2.15) and (2.20), it is always possible to describe topological BHs by making an appropriate choice of the parameters. For example, in the case of solution (2.20) with

\[
\frac{2}{(2N - 6 + z)} \left( k(N - 3) + \frac{\tilde{V}_0}{(N - 2)} \right) > 0, \quad \frac{4(\Lambda + \tilde{Q}^2\xi)}{(N - 2)(2N - 2 + z)} < 0,
\]

the equation \( B(r) = 0 \) has two roots, namely \( r_\pm \), the first one corresponding to the event horizon of the black hole \( (B'(r_+) > 0) \) and the second one to the (Anti-de Sitter) horizon of the cosmological background where the black hole is immersed \( (B'(r_-) < 0) \).

In the following, we will assume to deal with solutions whose parameters satisfy the conditions (3.1) for some value of \( r = r_+ \).

Let us study some physical propriety of these black holes. Since the field equations of the theory
are second order differential equations, we can easily derive a conserved current whose charge may be identified with the mass of the BHs. We will follow the approach proposed by Wheeler for Lovelock theories \cite{34} in Ref. \cite{35}. For simplicity, we denote with $G_{\mu\nu}$ the Einstein tensor plus the contribute of cosmological constant, namely

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}. \quad (3.3)$$

By means of time-like Killing vector field $K^\nu = (1, \vec{0})$ in $N$-dimension, in the case of static metric \eqref{2.2}, one can construct the conserved current

$$J_\mu := G_{\mu\nu}K^\nu, \quad (3.4)$$

such that

$$\nabla_\nu J^\nu = 0, \quad (3.5)$$

being $G_{\mu\nu}$ a conserved quantity. A direct evaluation of $J_0$ via \eqref{3.3} leads to

$$J_0 := G_{00}K^0 = (e^{2\alpha(r)}B(r))\left(\frac{N-2}{2r^{N-2}}\int d[r^{N-1}W(r)]\right), \quad (3.6)$$

where

$$W(r) = [k - B(r)]r^{-2} - \frac{2\Lambda}{(N-2)(N-1)}. \quad (3.7)$$

The current $J_\mu$ gives rise to a Killing conserved charge. This corresponds to the quasi-local generalized Misner-Sharp \cite{36} mass which reads

$$E_{MS}(r) \equiv -\frac{1}{\kappa^2} \int_{\Sigma} J^\mu d\Sigma^\mu = \frac{(N-2)V_{N-2,k}}{2\kappa^2} \int_0^r d\rho \frac{d(\rho^{N-1}W)}{d\rho} = \frac{(N-2)V_{N-2,k}}{2\kappa^2} r^{N-1}W(r), \quad (3.8)$$

where $\Sigma$ is a spatial volume at fixed time, $d\Sigma^\mu = (d\Sigma, \vec{0})$, and $V_{N-2,k}$ is the $N-2$ dimensional volume depending on the topology. For example, in the case of the sphere with $k = 1$, one has $V_{N-2,1} = 2\pi^{(N-1)/2}/\Gamma((N-1)/2)$, with $\Gamma(z)$ the Euler-Gamma function.

In particular, on shell, that is at the horizon $r = r_+$ such that $B(r_+) = 0$, the quasi local energy is identified with the black hole energy $E$ which reads

$$E := E_{MS}(r_+) = \frac{(N-2)V_{N-2,k}}{2\kappa^2} \left( k r_+^{N-3} - \frac{2\Lambda}{(N-2)(N-1)} r_+^{N-1} \right). \quad (3.9)$$
For example, in the vacuum case of Eq. (2.15) with $\tilde{Q} = 0$ one has

$$r^{N-1}W(r) = -C,$$  \hspace{1cm} (3.10)

and the black hole energy reads

$$E = \frac{(N - 2)V_{N-2,k}}{2\kappa^2}C,$$  \hspace{1cm} (3.11)

so that the constant of integration is related with the mass of the BH.

We note that expression (3.9) correctly returns the Misner-Sharp mass for asymptotically flat solutions ($\Lambda = 0$) in vacuum or in the presence of matter. In particular, for $N = 4$ and $k = 1$, by explicitly writing the Newton Constant, we get the familiar result $E = r_+/(2G_N)$, which corresponds, in the vacuum case, to $E = -C/(2G_N)$.

Now, let us show that the Gibbs equation $TdS = dE - pdV$ holds true for the black holes described by the model (2.1), with the Killing energy $E$ obtained below, and the pressure $p$ given by electromagnetic and scalar fields. $T$ and $S$ are the temperature and the entropy of the black hole, and $V$ is the volume enclosed by the horizon in $N - 1$ dimensional space.

For Lovelock gravity the validity of the First Law of black hole thermodynamics has been investigated in several places [37–39]. For our static non vacuum case we present a simple derivation from the first EOM (2.8) evaluated on the horizon $r = r_+$,

$$\frac{(N - 2)V_{N-2,k}B'(r_+)r_+^{N-3}}{2\kappa^2} - \frac{(N - 2)V_{N-2,k}}{2\kappa^2} \left( kr_+^{N-3} - \frac{2\Lambda}{(N - 2)(N - 1)}r_+^{N-1} \right)$$

$$- \frac{V_{N-2,k}r_+^{N-2}}{2\kappa^2} \left( \tilde{V}_0 e^{\gamma\phi} - \frac{2\xi e^{-\lambda\phi}\tilde{Q}^2}{r_+^{2(N-2)}} \right) = 0.$$  \hspace{1cm} (3.12)

Here, we have used the fact that $B(r_+) = 0$.

All thermodynamical quantities associated with black holes solutions can be computed by standard methods. The entropy can be calculated by the Wald method [40–42] and reads

$$S_W = \frac{2\pi V_{N-2,k}r_+^{N-2}}{\kappa^2}.$$  \hspace{1cm} (3.13)

Furthermore, for the static metric (2.2) it is possible to find a characteristic temperature related to the event horizon. A natural choice is to take the so called Killing/Hawking temperature [43]

$$T_K := \frac{\kappa_K}{2\pi} = \frac{e^{\alpha(r_+)}B'(r_+)}{4\pi},$$  \hspace{1cm} (3.14)

whose validity may be justified making use of derivations of Hawking radiation [44] or by eliminating the conical singularity in the corresponding Euclidean metric [45] or making use of the tunneling method [46,47]. In the above expression, $\kappa_K$ denotes the Killing surface gravity, namely
\( \kappa_K = e^{\alpha(r_+)}B'(r_+)/2 \), derived from the relation \( K^\mu \nabla_\mu K^\nu = \kappa_K K^\nu \), where \( K^\nu = (1, \vec{0}) \) is the time-like Killing vector field.

However, we would like to remind that in the spherical symmetric, dynamical case, the real geometric object which generalizes the Killing vector field is the Kodama field \[48\] with a related conserved current and a related Kodama surface gravity \( \kappa_H \). In such a case, a natural definition of the temperature for dynamical black holes reads as \( T_H = \kappa_H/2\pi \), where \( T_H \) is the Kodama/Hayward temperature, in analogy with the static case. This temperature permits to find the Gibbs relation in the dynamical case (see the seminal work of Hayward in Ref. \[49\] and the Appendix A), when the Killing surface gravity cannot be defined being the time-like Killing vector field absent. The interesting point is that in the static case the Kodama vector field still exists but does not coincide with the time-like Killing vector field and differs from it as \( K^\nu = e^{-\alpha(r)}K^\nu \), namely

\[
K^\mu = \left(e^{-\alpha(r)}, \vec{0}\right), \quad (3.15)
\]

such that the Kodama/Hayward surface gravity reads

\[
\kappa_H = \frac{B'(r_+)}{2}. \quad (3.16)
\]

As a consequence, one finds

\[
T_H := \frac{\kappa_H}{2\pi} = \frac{1}{4\pi} B'(r_+), \quad (3.17)
\]

namely \( T_H = e^{-\alpha(r)}T_K \) and in principle definitions (3.14) and (6) are different. In vacuum case where \( \alpha(r) = 0 \), the two temperatures coincide, but for ”dirty” BHs (i.e., in the presence of matter) as the ones we are considering, \( T_K \neq T_H \). This is related with the fact that the Killing vector cannot be defined unambiguously when the space-time is not asymptotically flat. We stress that all derivations of Hawking radiation lead to a semi-classical expression for the black hole radiation rate \( \Gamma \),

\[
\Gamma \equiv e^{-\frac{\Delta E_K}{T_K}}, \quad (3.18)
\]

in terms of the change \( \Delta E_K \) of the Killing energy \( E_K \), but if one uses the Kodama energy \( E_H \) for the emitted particle, one has

\[
\Gamma \equiv e^{-\frac{\Delta E_H}{T_H}}. \quad (3.19)
\]

This fact derives by the relationship \( \Delta E_H = e^{-\alpha(r)}\Delta E_K \). From the Eqs. (3.18)-(3.19), one arrives at the identity

\[
\frac{\Delta E_H}{T_H} = \frac{\Delta E_K}{T_K}, \quad (3.20)
\]

so that the tunneling probability is invariant under different choices of the temperature.

In Ref. \[50\] an attempt to identify the mass of static BHs in modified theories of gravity as the integration constant which appears in the vacuum solutions has been done. This result has
been derived by the EOMs and seems in favor of the Killing temperature with respect to the Kodama-Hayward one, but here, for our non vacuum solutions, Eq. (3.12) suggests the use of the Kodama temperature (6), in the attempt to recover the Gibbs relation. In fact, by making use of the BH entropy (3.13), one can rewrite Eq. (3.12) as

\[ T_H dS_W = dE + pdV, \]  

(3.21)

where \( E \) is the BH energy (3.9), \( V \) is the volume enclosed by the horizon in \( N - 1 \)-dimensional space, \( V = V_{N-2,k}r^{N-1}/(N - 1) \), and \( p = p_\phi + p_{EM} \) is the working term given by the radial scalar field pressure (\( p_\phi \)) and the radial pressure of electromagnetic field coupled with scalar field (\( p_{EM} \)) on the horizon.\(^6\)

\[ p_\phi = -V(\phi) = -V_0 e^{\gamma \phi}, \]
\[ p_{EM} = -\xi e^{\lambda \phi} F^{\mu\nu} F_{\mu\nu} = \frac{2\xi e^{-\lambda \phi} Q^2}{r^{2(N-2)}}. \]  

(3.22)

Here, we have reintroduced \( V_0 = \tilde{V}_0/(2\kappa^2) \) and \( Q^2 = \tilde{Q}^2/(2\kappa^2) \). In this case, the Gibbs equation holds true.

We prefer to use the Hayward temperature \( T_H \) (3.17) in analogy with therodynamic, by starting from the robust definitions of the energy as the charge of a conserved current and the entropy via Wald method.

About the possible phase transitions, we would like also to mention that, at least when \( k = 1 \) and \( z = 0 \), which is the well-known AdS-solution with a spherical horizon, the system should exhibit a Hawking-Page phase transition, namely a first-order phase transition between thermal AdS-space and a Schwarzschild-AdS black hole. In our AdS-Lifshitz black hole case, this transition is a Hawking-Page like phase transition between large Lifshitz black holes at high temperature and thermal Lifshitz (pure Lifshitz space with compact Euclidean time) at low temperature [51]. This phase transition corresponds to the confinement/deconfinement phase transition in dual theory. In Ref. [52], the AdS-soliton solution from the planar black hole and black brane metric, by a double Wick rotation, are investigated and it is shown that exists a critical temperature, where both solutions have the same free energy: at this point, a first order phase transition between the soliton and black hole occurs. This effect is the analogous to the Hawking-Page transition, and in the dual field theory this is a confinement/deconfinement transition.

\(^6\)The stress energy tensor of our model is given by \( T_{\mu\nu} = \mathcal{L} g_{\mu\nu} - 2\partial \mathcal{L}/\partial g_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi/2 + V g_{\mu\nu} + 4\xi e^{\lambda \phi} [F^\mu_\nu F^\nu_\alpha - g_{\mu\nu} F^\beta_\gamma F^\beta_\gamma]/4 \). Given our SSS metric and the static electromagnetic field, the radial pressure is derived as \( p = -T^r_r = -B(r)\phi'(r)^2/2 + V(r) - \xi e^{\lambda \phi} F^{\mu\nu} F_{\mu\nu} \). On the horizon, the dependence on \( \phi'(r) \) drops down and we recover the expressions in the formula.
4 s-wave Holographic superconductors in probe limit

Our goal in the following sections are to apply the black hole solution with the Lifshitz scaling which it has been obtained before to study the holographic picture of superconductor via gauge/gravity duality. In brief, by holographic superconductor we mean a condensed matter system under second order phase transition whose physical properties can be described by studying the dynamics of gauge field on the black hole background in the bulk. It is proved that by direct applying the AdS/CFT conjecture one can interpret the asymptotic behavior of the gauge fields on the AdS boundary as the expectation values of some physical operators. The expectation values of such scalar operators are dual to the super current in the s-wave high temperature type II superconductors. Here, s-wave refers to a scalar order parameter, whose expectation value breaks the $U(1)$ but not the rotational symmetry. The gauge field can be $SU(2)$ and corresponds to the Yang-Mills fields. Such holographic models are called p-wave, because super current is a vector and the condensation happens usually for one homogeneous component of it. To apply the gravitational model to the superconductors, we will modify our gravitational model \eqref{2.1} by adding a new matter field subjected to some abelian gauge field. By starting from the background metric of gravitational action previously studied, we will investigate the scalar condensation of the new field and we will show that some phase transition occurs.

We easily see that Eq. \eqref{2.20} may describe an Lifshitz (dS/AdS) black hole solution for our non minimally coupled gravity model in $N$ dimensional bulk. We are interested in Anti de Sitter solutions. If we want to relate our gravitational system with a strongly correlated system in the dual quantum theory, we need to describe the dual quantum operators via CFT. The condensed matter system dual to our classical BH solution can be addressed by holographic superconductors. In what follows, we will study the formation of the hairy BHs. The phenomenon is given by a second order phase transition and can be described by the holographic methods of the AdS/CFT.

At first, we note that the solution \eqref{2.20} with $\Lambda \neq 0$ may be asymptotically topological Lifshitz and it can be considered as the gravitational part of the holographic superconductor in the bulk. In fact, we take the gravity bulk as the charged topological black hole with a non zero, negative effective cosmological constant

$$\Lambda_{eff} = -\frac{12(\Lambda + \tilde{Q}^2 \xi)}{(N - 2)(2N - 2 + z)} < 0 .$$ \hspace{1cm} (4.1)

It is very interesting that the $U(1)$ reduced charge $\tilde{Q}$ is combined with cosmological constant in the solution, producing the AdS background. It is useful to rewrite the solution \eqref{2.20} as

$$B(r) = \frac{2}{(2N - 6 + z)}(k(N - 3) + \frac{\tilde{V}_0}{(N - 2)}) + C r^{3 - N / 2} + \frac{r^2}{l_{eff}^2} .$$ \hspace{1cm} (4.2)
where
\[
    l_{\text{eff}} = \frac{1}{2} \sqrt{\frac{(N-2)(2N-2+z)}{Q^2 \zeta + \Lambda}}
\]
is the effective length scale. We note that for \( N = 4 \) and \( z = 0 \), solution (4.3) reduces to the usual Schwarzschild-AdS form with \( l_{\text{eff}} = \sqrt{3/\Lambda} \). In fact, when we turn off the electromagnetic field and also we preserve the Lorentz invariance by \( z = 0 \), the form of the effective length scale is the same as the one of an AdS uncharged black hole. We want to find the second order phase transition in the bulk theory by studying the boundary operators. In particular, we want to study the role of the Lifshitz scaling \( z \) related to the critical temperature \( T_c \) and the condensation of the dual operators \( \langle \mathcal{O}_\pm \rangle \) (see also Refs. [53–55]). As a starting point, in order to discuss the superconducting phase via holographic picture, we need a scalar field \( \psi(r) \), with mass above the Breitenlohner-Freedman (BF) bound [57], and an abelian gauge field \( A_\mu \), which is minimally coupled with the scalar field, so that we modified the (2.1) by introducing a new matter Lagrangian in the following form [58]
\[
    \mathcal{L}_m = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - |D_\mu \psi|^2 - m^2 \psi^2,
\]
where \( D_\mu \) is the covariant derivative, \( D_\mu = \partial_\mu - iq A_\mu \), and \( F_{\mu\nu} \) is the electromagnetic field strength related to the abelian field. We note that in the probe limit and in the normal phase, when \( \psi = 0 \), the electromagnetic field satisfies the bulk Maxwell’s equations. As a consequence, the total action results to be
\[
    I_{\text{total}} = -\int d^N x \sqrt{-g} \mathcal{L}_m + \int d^N x \sqrt{-g} \left[ \frac{(R - 2\Lambda)}{2\kappa^2} - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + V(\phi) - \xi e^{\lambda \phi} (F^\mu_{\nu} F^\nu_{\mu}) \right].
\]
(4.5)

In the above expressions, \( q \) plays the role of bulk electric charge because \( q \) appears in the covariant derivative exactly as a standard electric charge. The matter action (4.4) is different from the bulk action. In fact, we add here the minimally coupled Maxwell field in order to break the \( U(1) \) symmetry of the abelian field near the BH horizon. Thus, we will work in the so called probe limit \( q \to \infty \), ignoring the back reaction, in the normal phase \( \psi = 0 \). In this case, the gravity sector decouples from the abelian one and the background metric can be derived from Eq. (2.16) and Eq (2.20).

In the probe limit the EOMs for \( \psi(r) \) and \( A_\mu = \phi(r) \delta_\mu t, \delta_\mu \nu \) being the Kroenecker delta function and \( \phi(r) \) a general function of \( r \), read in the following forms
\[
    D_\mu D^\mu \psi - m^2 \psi = 0,
\]
(4.6)
\[
    \nabla^\mu F_{\mu\nu} = iq [\psi^* D_\nu \psi - \psi D_\nu^* \psi^*].
\]
(4.7)
Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and we take $\psi^* = \psi$, motivated by the fact that we are free to choose our gauge. The above equations read in terms of the background metric as

$$\phi'' + \left(\frac{N - 2 - \frac{z}{y}}{r}\right)\phi' - \frac{2q^2\psi^2}{B}\phi = 0, \quad (4.8)$$

$$\psi'' + \left(\frac{N - 2 + \frac{z}{y}}{r} + \frac{B'}{B}\right)\psi' + \left(-\frac{m^2}{B} + \frac{q^2 r^{-z} \phi^2}{B^2}\right)\psi = 0. \quad (4.9)$$

For $N = 4$ and $z = 0$ the model is perfectly described by numerical methods [58]. We are interesting in the cases of $z \neq 0$ and $N > 3$.

We need to introduce in Eqs. (4.8)-(4.9) the horizon radius $r_+$. We rewrite the solution in the following form,

$$B(r) = \tilde{k} \left[1 - \left(\frac{r}{r_+}\right)^{3-N-\frac{z}{y}} \right] + \left(\frac{r_+}{l_{eff}}\right)^2 \left[\left(\frac{r}{r_+}\right)^2 - \left(\frac{r}{r_+}\right)^{3-N-\frac{z}{y}}\right], \quad (4.10)$$

where

$$\tilde{k} = \frac{2}{2N - 6 + z} \left(k(N - 3) + \frac{\tilde{V}_0}{N - 2}\right). \quad (4.11)$$

It is more convenient to work in terms of the dimensionless parameter $y(r) = r_+/r$, such that $y(r_+) = 1$ and at the infinity $y(r \to +\infty) \to 0^+$. In this case, the equations of motion (4.8) and (4.9) can be expressed as:

$$\phi'' - \left(\frac{\frac{z}{y} + N - 4}{y}\right)\phi' - \frac{2r_+^2 \psi^2}{y^4 B}\phi = 0, \quad (4.12)$$

$$\psi'' - \left(\frac{\frac{z}{y} + N - 4}{y} - \frac{B'}{B}\right)\psi' - \frac{r_+^2}{y^4} \left(\frac{m^2}{B} - \frac{y^z \phi^2}{r_+^2 B^2}\right)\psi = 0, \quad (4.13)$$

and the metric function reads

$$B(y) = \tilde{k} \left(1 - y^{N-3+\frac{z}{y}}\right) + \left(\frac{r_+}{l_{eff}}\right)^2 \left(y^{-2} - y^{N-3+\frac{z}{y}}\right). \quad (4.14)$$

Now, the prime denotes the derivative with respect to $y = y(r)$. We have obtained the basic set up for the holographic superconductors.

\footnote{In the literature usually the symbol is $z$, but here we kept $z$ for Lifshitz scaling and we introduced $y$ as the dimensionless radial coordinate.}
5 Critical temperature and condensation values by matching method

We are going to calculate the condensate $\langle O \rangle$ for fixed charge density.

Regularity at the horizon, namely $y = 1$, requires

$$\phi(1) = 0, \quad \psi'(1)B'(1) = r_+^2 m^2 \psi(1). \quad (5.1)$$

We want to find approximate solutions around the horizon and asymptotically AdS limit, $y = 1$ and $y = 0$, using Taylor’s expansion, then we want to connect these solutions in an arbitrary matching point $y_m$ between $y = 1$ and $y = 0$. In principle, we may do the computations by numerical methods using shooting algorithm, but in this paper we would like to keep the level of the analytical approach. At first, we calculate $B'(1) (= (dB(r_+)/dy)$ directly from Eq. (4.14),

$$B'(1) = -\tilde{k}(N - 3 + \frac{z}{2}) + \left(\frac{r_+}{l_{eff}}\right)^2 \left(-N + 1 - \frac{z}{2}\right). \quad (5.2)$$

It is easy to see that Eq. (5.2) is related to the Kodama temperature (6) by $B'(1) = -4\pi r_+ T_H$. Consequently, by solving this equation for $r_+$ and using (5.2) we have

$$\frac{r_+}{l_{eff}} = \frac{4 T_H \pi l_{eff} + \sqrt{16 T_H^2 \pi^2 l_{eff}^2 + 8 z\tilde{k} + 16 N\tilde{k} - 12\tilde{k} - z^2 \tilde{k} - 4 z N\tilde{k} - 4 N^2 \tilde{k}}}{2 N + z - 2} \quad (5.3)$$

We will use (5.2,5.3) to construct the series solutions for our topological holographic superconductor in the probe limit. Specially, because now the radius of the horizon is a function of $(l_{eff}, T_H, \tilde{k})$, as a result we will show that the critical temperature depends on the topological parameter $\tilde{k}$ as well as the Lifshitz scaling parameter $z$. This is one of the most important results of our calculations which we demonstrate that in the Lifshitz backgrounds, the critical temperature depends on the topological parameter $\tilde{k}$.

5.1 Solution near the BH horizon

We expand $\phi$ and $\psi$ near the black hole horizon at $y = 1$ as

$$\phi(y) = \phi(1) - \phi'(1)(1 - y) + \frac{1}{2} \phi''(1)(1 - y)^2 + \cdots, \quad (5.4)$$

$$\psi(y) = \psi(1) - \psi'(1)(1 - y) + \frac{1}{2} \psi''(1)(1 - y)^2 + \cdots. \quad (5.5)$$

From the boundary condition, we know that $\phi(1) = 0$ and for simplicity we put $a := -\phi'(1) < 0$ and $b := \psi(1) > 0$ for the positivity of $\phi(y)$ and $\psi(y)$. 

15
In order to discuss phase transition near the critical points, we need just to keep the second order terms in those series.

First, we compute the 2nd order coefficient of $\phi$ by using Eq. (4.12),

$$
\phi''(1) = -\phi'(1) \left[ -\left( \frac{z}{2} + N - 4 \right) + \frac{2r^2 b^2}{B'(1)} \right].
$$

(5.6)

In this case Eq. (5.4) reads

$$
\phi(y) = a \left[ (1 - y) + \frac{1}{2} \left[ -\left( \frac{z}{2} + N - 4 \right) + \frac{2r^2 b^2}{B'(1)} \right] (1 - y)^2 \right].
$$

(5.7)

We can calculate the 2nd derivative of $\psi$ from (4.13) in the same way,

$$
\psi''(1) = -\frac{ba^2 (r_\pm)^2 - z}{(B'(1))^2},
$$

(5.8)

and write the following series solution for scalar field $\psi$ (5.3)

$$
\psi(y) = b \left[ 1 - \frac{m^2 r_\pm^2}{B'(1)} (1 - y) - \frac{a^2 r_\pm^{2-z}}{2(B'(1))^2} (1 - y)^2 \right].
$$

(5.9)

In the above equations, $B'(1)$ is given by Eq. (5.2).

### 5.2 Solution near the asymptotic AdS region

The asymptotic regime of the metric function $B(y)$ in the AdS boundary is completely independent on the dimension of the spacetime $N$. It is trivial to recover the following asymptotic behavior of the metric

$$
B(y) \sim y^{-2}.
$$

(5.10)

Here we take $z > 0$ and $N > 3$ to avoid the problems of logarithmic divergence in $AdS_3/CFT_2$. Consider now the weak field behavior ($r \to +\infty$) of $\phi$ which depends on the value of dimension $N$. It is easy to show that this behavior completely changes at the critical dimension $N_c = 3 - \frac{z}{2}$, namely

$$
\phi(y) = \begin{cases} 
\frac{c_1}{\frac{z}{2} + N - 3} y^{\frac{z}{2} + N - 3} + c_2, & N \neq N_c, \\
\log(y) + c_2, & N = N_c.
\end{cases}
$$

(5.11)

(5.12)
We are interested in the fields with fall off behaviors near $y = 0$, so that we take $N \neq N_c$. It is useful to write (5.11) in terms of the radial coordinate $r$,

$$\phi(r) = \frac{c_1 r_+^{\frac{N}{2}+N-3}}{(\frac{r}{2} + N - 3)r_+^{\frac{N}{2}+N-3}} + c_2. \quad (5.13)$$

By writing this solution in terms of the dual physical quantities chemical potential $\mu$ and charge density $\rho$, we obtain

$$\phi(y) = \mu - \frac{\rho}{r_+^{\frac{N}{2}+N-3}y^{\frac{N}{2}+N-3}}, \quad (5.14)$$

where

$$\rho = -\frac{c_1 r_+^{\frac{N}{2}+N-3}}{(\frac{r}{2} + N - 3)}, \quad \mu = \frac{\rho}{r_+^{\frac{N}{2}+N-3}}. \quad (5.15)$$

The second condition derived from the fact that $\phi(r_+) = 0$. Also for the scalar field, near the AdS boundary we can write

$$\psi(r) = \frac{< \mathcal{O}_\pm >}{r^{\Delta_{\pm}}} + \ldots, \quad (5.16)$$

where $\mathcal{O}_+$ and $\mathcal{O}_-$ are the operators on the boundary and the conformal dimension is

$$\Delta_{\pm} = \frac{1}{2}[(N - 1) \pm \sqrt{(N - 1)^2 + 4m^2l_{\text{eff}}^2}]. \quad (5.17)$$

Here, the mass square must satisfies the following relation

$$m^2l_{\text{eff}}^2 > -\frac{(N - 1)^2}{4}. \quad (5.18)$$

A simple check shows that under this mass bound, only $\Delta_+$ has an enough rapid fall off and the scalar field behaves as

$$\psi(r) \rightarrow r^{-(\Delta_+ + 2)} + r^{-\Delta_+} < \mathcal{O}_+ >. \quad (5.19)$$

It means that the scalar field is dual to a quantum operator on the boundary with conformal dimension $\Delta_+$ and we can ignore $\mathcal{O}_-$. This is not the unique possible choice. It is easy to find that if

$$-\frac{(N - 1)^2}{4} < m^2l_{\text{eff}}^2 < 1 - \frac{(N - 1)^2}{4}, \quad (5.20)$$

both of the terms with conformal dimensions $\Delta_+$ fall off and we can keep they. In conclusion, the quantization scheme is a valid procedure. In any case, the scalar field is asymptotic to $< \mathcal{O}_\pm >$.
and these are dual to operators with dimension $\Delta_\pm$. In fact, it is possible to write this quantization scheme in terms of the $z$ parameter as it has been proposed Ref. [53]. However, in this work we assume that $-\frac{(N-1)^2}{4} \leq m^2 c_{eff}^2$, so that we limit ourselves to the fall off with $\Delta_+$. 

### 5.3 Matching and phase transition

In this Section, we will connect the solutions (5.7) and (5.9) with (5.14) and (5.16) at some completely arbitrary matching point $y = y_m$. In order to connect those solutions smoothly, we require the following four conditions:

$$
\mu - \frac{\rho}{r_+^{z/2 + N - 3}} r_m^{z/2 + N - 3} = a \left[ (1 - y_m) + \frac{1}{2} \left[ - \left( \frac{z}{2} + N - 4 \right) + \frac{2r_+^2 b^2}{B'(1)} \right] (1 - y_m)^2 \right],
$$

(5.21)

$$
- \rho \left( \frac{z}{2} + N - 3 \right) r_+^{z/2 + N - 3} y_m^{z/2 + N - 4} = a \left[ 1 - \left[ - \left( \frac{z}{2} + N - 4 \right) + \frac{2r_+^2 b^2}{B'(1)} \right] (1 - y_m) \right],
$$

(5.22)

$$
\frac{\langle \mathcal{O}_{\pm} \rangle}{r_+^{\Delta_+} y_m^{\Delta_+}} = b \left[ 1 - \frac{m^2 r_+^2}{B'(1)} (1 - y_m) - \frac{a^2 r_+^{2-z}}{2(B'(1))^2} (1 - y_m)^2 \right],
$$

(5.23)

$$
\frac{\Delta_+}{r_+^{\Delta_+} y_m^{\Delta_+}} = b \left[ \frac{m^2 r_+^2}{B'(1)} + \frac{a^2 r_+^{2-z}}{(B'(1))^2} (1 - y_m) \right].
$$

(5.24)

By combining Eqs. (5.21)-(5.22) we can eliminate $ab^2$ and one has

$$
\mu = \frac{2 \rho \left( \left( \frac{z}{2} + N - 3 \right) y_m^{\frac{z}{2} + N - 3} - \left( -5 + \frac{z}{2} + N \right) y_m^{\frac{z}{2} + N - 2} \right) r_+^{-\frac{z}{2} - N + 3} + 2 (1 - y_m) y_m a}{4y_m},
$$

(5.25)

$$
b = \frac{\sqrt{2}}{2r_+} \sqrt{\left[ \rho y_m^{\frac{z}{2} + N - 3} \left( \frac{z}{2} + N - 3 \right) - \left( \frac{z}{2} + N - 4 \right) y_m - \frac{z}{2} - N + 5 \right] y_m a r_+^{\frac{z}{2} + N - 3} B'(1)}/y_m (1 - y_m).
$$

(5.26)

The above relations allude to the phase transition, namely, given $\rho, \mu$ has a maximum value when we assume the non-trivial solution $b \neq 0$. Now we can reveal the phase transition in our simple system. In order to evaluate the expectation value of the operator $\langle \mathcal{O}_{\pm} \rangle$, we eliminate the $a^2 b$ term from (5.23) and (5.24) and obtain

$$
\langle \mathcal{O}_{\pm} \rangle = - \frac{r_+^{\Delta_+} b (m^2 r_+^2 (y_m - 1) + 2) y_m^{1-\Delta}}{(\Delta - 2) y_m - \Delta}. \quad (5.27)
$$

For non-vanishing $b$, we can compute $\langle \mathcal{O}_{\pm} \rangle$ from Eqs. (5.23) and (5.24) and one gets

$$
a = |B'(1)| \sqrt{\frac{2\Delta^2 - m^2 r_+^2 \left( y_m + 2\Delta (1 - y_m) \right)}{r_+^{2-z} \left[ \Delta (1 - y_m)^2 + 2y_m (1 - y_m) \right]}},
$$

(5.28)
By plugging this result in Eq. (5.26) we derive

\[
b = \frac{1}{2r_+} \sqrt{\frac{\sqrt{2}B'(1)}{(1 - y_m)\Sigma r_+^{\frac{2}{3} + N - 3}}} \left[ \left( \frac{z}{2} + N - 3 \right) \rho y_m^{\frac{3}{2} + N - 4} + \sqrt{2r_+^{\frac{4}{3} + N - 3}} \Sigma \left( \frac{z}{2} + N - 4 \right)(1 - y_m) - 1 \right],
\]

(5.29)

\[
\Sigma = |B'(1)| \sqrt{\frac{\left( m^2r_+^2(\Delta - 1)y_m - \Delta (-1 + m^2r_+^2) \right)}{r_+^{2-z}(y_m - 1)((\Delta - 2)y_m - \Delta)}}.
\]

(5.30)

By using the density (5.15), one has that \( \langle O_+ \rangle \) can be expressed as

\[
\langle O_+ \rangle = -\frac{y_m^{1-\Delta}(r_+^2(-1 + y_m)m^2 + 2)r_+^\Delta}{((\Delta - 2)y_m - \Delta)r_+} \sqrt{\Gamma},
\]

(5.31)

where the new function \( \Gamma \) is defined as

\[
\Gamma = \left( -\sqrt{2}(\frac{z}{2} + N - 4)y_m - \frac{z}{2} - N + 5)\Sigma r_+^{\frac{2}{3} + N - 3} + \left( \frac{z}{2} + N - 3 \right)y_m^{\frac{4}{3} + N - 4}\rho \right)^2 B'(1) \sqrt{2B'(1)} \frac{\Sigma r_+^{\frac{2}{3} + N - 3}(1 - y_m)}{r_+^{2-z}(y_m - 1)((\Delta - 2)y_m - \Delta)}. \]

(5.32)

Now we can write \( \Gamma \) in the following equivalent form

\[
\Gamma = A \frac{T_c - T_H}{\sqrt{T_H}},
\]

(5.33)

where \( T_H \) is the Kodama temperature \([11]\) and \( A \) is a function of \( \{r_+, \Delta, y_m\} \). The expression in (5.31) is complicated and it is hard to extrapolate the critical temperature from it. Furthermore, the values of the parameters depend on the matching point \( y_m \) in the bulk and this equation is not well written in terms of our physical parameters \( \rho, T_H \) instead the horizon radius \( r_+ \). In what follows, we will reduce and examine these expression in some simple but physically important cases. The mass of the scalar field can be set to zero, without loss of the generality. In addition, the location of the radial matching point is fixed at \( y_m = \frac{1}{2} \). As the first step, we will furnish the expression of \( r_+ \) in terms of \( T_H, l_{eff} \) from (5.3). However, the derived expression still remains so complicated. For this reason, we will discuss the cases of different values of \( z, N, \tilde{k} \) separately. In any case, we will write the critical temperature \( T_c \) as a function of \( \rho \). For the sake of simplicity, we also fix \( V_0 = 1, l_{eff} = 1 \).
5.4 The Lorentz invariance case in $N = 4$: $z = 0$

By setting $z = 0$, $N = 4$ and $k = 0$ (planar case) in (5.31), we get

$$\langle O_+ \rangle \sim 0.0063304(12.566T_H + \sqrt{157.92T_H^2 - 6})^2$$
$$\times \sqrt{5189.8T_H + 412.97(157.92T_H^2 - 6)^{1/2} + 90\rho}$$

(5.34)

We have that this scalar operator vanishes at

$$T_c = -277.9132603\rho + 0.0009828667268\sqrt{7.994700000 \times 10^{10} \rho^2 - 6.30253302 \times 10^8},$$

(5.35)

such that we can expand (5.35) near $T_c$ as

$$\langle O_+ \rangle \sim (T_c - T_H)^{1/2}.$$  

(5.36)

It is very interesting to observe that in such a case, in order to have a real critical temperature, $\rho \geq 0.0887$. It means there exists a lower bound on $\rho$ in which below it no condensation happens.

For $k = -1$ we obtain

$$\langle O_+ \rangle \sim 0.0063304(12.566T_H + \sqrt{157.92T_H^2 + 6})^2$$
$$\times \sqrt{5189.8T_H + 412.97\sqrt{157.92T_H^2 + 6} + 90\rho},$$

such that

$$T_c = -277.9132603\rho + 0.0009828667268\sqrt{7.994700000 \times 10^{10} \rho^2 + 6.30253302 \times 10^8}.$$  

(5.38)

We observe that the behavior of (5.38) near the critical temperature is the same of (5.35), but now we do not recover a minimal value for $\rho$.

Finally, for $k = 1$, we get

$$\langle O_+ \rangle \sim 0.025321 \left(6.2832T_H + \sqrt{39.479T_H^2 - 3.0} \right)^2$$
$$\times \sqrt{5189.7T_H + 825.98\sqrt{39.479T_H^2 - 3.0} + 90.0\rho},$$

$$T_c = 363.8366637\rho + 0.0001286830142\sqrt{7.994497500 \times 10^{12} \rho^2 + 9.628077237 \times 10^{10}}.$$  

(5.40)

Also in this case $\langle O_+ \rangle \sim (T_c - T_H)^{1/2}$ and the critical temperature behaves as $T_c \sim \sqrt{\rho}$, according with the literature about s-wave holographic superconductors and it increases monotonically. This behavior will change only in the presence of the higher order corrected backgrounds like Weyl’s models for s-wave. For example, for Gauss-Bonnet and Weyl corrections, it reads as $T_c = \sqrt[3]{\rho}$. 

[50]
5.5 Lorentz invariance breaking in $N = 4$: $z = 1$

Any Lifshitz redshift parameter $z \neq 0$ breaks the Lorentz symmetry by breaking the footing of the space and time coordinates. In this Subsection, in order to clarify the effect of the $z$ on condensation, we consider the case of $z = 1$ in $N = 4$, in order to recover a more realistic three dimensional holographic superconductor in the absence of the Lorentz symmetry.

We start by the case $k = 0$ and we get

$$
\langle O_+ \rangle \sim 0.0076809(12.566 T_H + \sqrt{157.92 T_H^2 - 7})^2 \times \frac{30743 T_H^2 + 2446.5 T_H \sqrt{157.92 T_H^2 - 7} - 681.41 + 245 \rho}{12.566 T_H + \sqrt{157.92 T_H^2 - 7}},
$$

and

$$
T_c = 86019000000 + 3.5945 \times 10^{16} \rho + 23351000000 \sqrt{2369800000000 \rho^2 - 951480000 \rho + 1480900000}.
$$

In this case $\rho$ is unbounded, since the argument of the root always is positive.

For $k = -1$ we have

$$
\langle O_+ \rangle \sim 0.0076809 \left(12.566 T_H + \sqrt{157.92 T_H^2 + 7}\right)^{3/2} \times \sqrt{30743 T_H^2 + 2446.5 T_H \sqrt{157.92 T_H^2 + 7} + 681.41 + 245 \rho},
$$

and the critical temperature reads

$$
T_c = 0.5238514818 \times 10^{-7} \sqrt{-8.6 \times 10^{11} + 3594546957000000.0 \rho + 23351084080.0 \eta},
$$

$$
\eta = \sqrt{236978700000.0 \rho^2 + 951482000.0 \rho + 1480876313.0}.
$$

Finally, for $k = 1$, we get

$$
\langle O_+ \rangle \sim 0.0076809 \left(12.566 T_H + \sqrt{157.92 T_H^2 - 21}\right)^2 \times \frac{30743.0 T_H^2 + 2446.5 T_H \sqrt{157.92 T_H^2 - 21} - 2044.2 + 245 \rho}{12.566 T_H + \sqrt{157.92 T_H^2 - 21}},
$$

$$
T_c = 0.02619257409 \times 10^{-6} \sqrt{279282401700 + 1437818783000000.0 \rho + \zeta},
$$

$$
\zeta = 4670216817 \sqrt{94791480000 \rho^2 - 90963600 \rho + 533102913}.
$$
We can see that, in all the topological cases, $\langle O_+ \rangle$ is zero at a specific value of $T_H = T_c$, namely the critical point, and condensation occurs at $T_H < T_c$. The important point is that the values of the critical temperature and the condensation scheme depend on the topological parameter $k$ and on the Lifshitz scaling $z$. Furthermore, we showed that the behavior of $\langle O_+ \rangle \propto (1 - T_H/T_c)^{1/2}$ always is recovered, in agreement with the literature. The system below this critical temperature becomes superconductor, but the critical exponent of the model remains the same of the usual holographic superconductor without the higher order gravitational corrections.

6 Discussions

In this paper we have considered a holographic model for a non-relativistic system showing superconductivity. We have used a black hole background which comes from a scalar field model minimally coupled with the Abelian gauge field in the presence of cosmological constant $\Lambda$ in N dimensions, and we have studied analytically holographic superconductors in this new kind of asymptotic AdS solutions. We have considered static, (pseudo-)spherically symmetric (SSS) solutions with various topologies in two different cases, $\Lambda = 0$ and $\Lambda \neq 0$. We have obtained the quasi-local generalized Misner-Sharp mass as a Killing conserved charge. This quasi-local energy at the horizon $r = r_+$ is identified with black hole energy. Then we have derived the Wlad entropy, Killing-Hawking temperature and Kodama-Hayward temperature of black hole solutions. These temperatures are in principle different. We have shown that for our non-vacuum solution, the first law of black hole thermodynamics holds true by making use of Kodama temperature: this argument substantiates our proposed temperature which is different with respect to the Hawking one generally chosen in literature. After that we have studied the hairy black hole solutions in which near the horizon the Abelian gauge field breaks the symmetry. In the holographic picture, this symmetry breaking is equivalent to a second order phase transition near the horizon. We also have analytically solved the system in the probe limits, near horizon and asymptotic region. We have found that there is also a critical temperature which is a function of the Lifshitz parameter $z$ and under such temperature a condensation field appears. The value of this critical temperature also depends on the topology. In some special topological cases, the critical temperature appears only for $\rho \in (\rho^*, \infty)$, where $\rho^*$ is a fixed (minimal) value of the charge density, so that $T_c(\rho^*)$ corresponds to the best configuration for the holographic superconductor.

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Appendix A. The Kodama/Hayward temperature

In this Appendix, we would like to give a brief derivation of the Kodama/Hayward temperature. In four dimensions, any spherically symmetric (dynamical) metric can be expressed in the following form,

\[ ds^2 = \gamma_{ij}(x^i)dx^i dx^j + R^2(x^i)d\Omega^2_2, \quad i, j \in \{0, 1\}, \]

where the two-dimensional metric

\[ d\gamma^2 = \gamma_{ij}(x^i)dx^i dx^j, \]

is referred to as the normal one with the related coordinates \( \{x^i\} \), while \( R(x^i) \) is the areal radius, considered as a scalar field in the two-dimensional normal space. A relevant scalar quantity in the reduced normal space is given by

\[ \chi(x^i) = \gamma^{ij}(x^i)\partial_i R(x^i)\partial_j R(x^i), \]

since the dynamical trapping horizon, if exists, is located in correspondence of

\[ \chi(x^i)\big|_H = 0, \quad \partial_i \chi(x^i)\big|_H \geq 0. \]

In the static case, if we refer to the metric (2.2), we have \( \chi(r) = B(r) \) and in General Relativity, by means of the time-like Killing vector field \( K^\mu = (1, 0, 0, 0) \), we have that the Misner Sharp mass corresponds to the charge of the conserved current \( J^\mu = G^\mu\nu K^\nu \), where \( G^\mu\nu \) is the Einstein tensor.

In the dynamical case, where we do not have the time-like Killing vector field, in order to define a conserved current, we need the Kodama vector field \[48\]

\[ K^i(x^i) := \frac{1}{\sqrt{-\gamma}} \varepsilon^{ij} \partial_j R(x^i), \quad i = 0, 1; \quad K^i := 0, \quad i \neq 0, 1, \]

where \( \varepsilon^{ij} \) is the completely antisymmetric Levi-Civita tensor on the normal space and \( \gamma \) the determinant of \( \gamma_{ij} \) metric tensor. Thus, the Kodama/Hayward surface gravity associated with dynamical horizon is given by the normal-space scalar

\[ \kappa_H := \frac{1}{2} \Delta\gamma R(x^i)\big|_H, \]

where \( \Delta\gamma \) is the Laplacian corresponding to the \( \gamma_{ij} \) metric. In Ref. \[49\], Hayward showed that on the dynamical, trapping horizon the following identity holds true

\[ \Delta E = \frac{\kappa_H}{2\pi} \Delta S - pd\nu_H, \]
where \( E \) is the Misner Shiarp mass of the black hole, \( S \) the entropy and \( p \) the matter pressure. Thus, in order to find the Gibbs relation of thermodynamic, we define the Hayward/Kodama temperature as

\[
T_H = \frac{\kappa_H}{2\pi}.
\]

In the static case, it simply results to be \( T_H = B'(r_H)/4\pi \).

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