Chirally Constraining the $\pi\pi$ Interaction in Nuclear Matter

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Abstract

A general prescription for the construction of $\pi\pi$ interaction potentials which preserve scattering length constraints from chiral symmetry when iterated in scattering equations is derived. The prescription involves only minor modifications of typical meson-exchange models, so that coupling constants and cut-off masses in the models are not greatly affected. Calculations of $s$-wave $\pi\pi$ scattering amplitudes in nuclear matter for two models are compared with those for similar models which violate the chiral constraint. While the prescription tends to suppress the accumulation of the near sub-threshold strength of the $\pi\pi$ interaction, an earlier conjecture that amplitudes which satisfy chiral constraints will not exhibit an instability towards $\pi\pi$ $s$-wave pair condensation appears to be incorrect. At the same time, however, conventional $\pi\pi$ interaction models which fit scattering data well can readily be adjusted to avoid the instability in nuclear matter without recourse to exotic mechanisms.
1. INTRODUCTION

The role of correlated two-pion exchange in the interaction of two nucleons is well-established [1], with the bulk of the intermediate range attraction due to the scalar-isoscalar ($\sigma$-meson) channel. This same mechanism is presumably responsible for most of the attraction between nucleons in nuclei, despite the possibly strong influence of the nuclear medium on the $\pi\pi$ correlations [2], especially in the $\sigma$-channel [3]. There the coupling of pions to nucleons and $\Delta$’s enhances the attraction of the pions at energies near the $2\pi$ threshold, which can lead to quasi-bound states, and even to $\pi\pi$ s-wave pair condensation at relatively low densities [4].

In a recent study Aouissat et al. [5], hereafter referred to as I, demonstrated the unattractive possibility of $\pi\pi$ pair condensation at nuclear densities of $1.2-1.3\rho_0$ ($\rho_0 = 0.16$ fm$^{-1}$, normal nuclear matter density) for two phenomenological models [6,7] which fit $\pi\pi$ s-wave phases over a broad energy range. It was also found in I that the tendency towards $\pi\pi$ pair condensation appeared to be dramatically suppressed if the $\pi\pi$ amplitude were forced to satisfy a constraint required by chiral symmetry, which is that the amplitude must satisfy the so-called soft-pion theorems [8]. Implicit in this constraint is that the $\pi\pi$ s-wave scattering lengths be proportional to the pion mass as the pion mass approaches zero.

Their study immediately suggests two questions: (1) Does the imposition of the chiral constraint on the amplitude provide a guarantee against $\pi\pi$ pair condensation in the nuclear medium at low or moderate density? (2) How can one construct interactions based on meson-exchange models which are unitary and which satisfy the soft-pion theorems without recourse to reformulating the treatment of pions in nuclear matter [5]? We will address these questions in reverse order and first derive a simple prescription for how to construct pseudopotentials in which the scattering length constraint is guaranteed. Having done this, we will examine reasons why meson-exchange models cannot follow the prescription and then address the problem of sufficiency posed in the first question.
2. MINIMAL CHIRAL CONSTRAINTS

The difficulty in imposing the constraint on the scattering lengths, hereafter called the minimal chiral constraint, or MCC, is that the $\pi\pi$ amplitudes used in the in-medium calculations are generated from an interaction kernel which is unitarized by means of a scattering equation (e.g., the Blankenbecler-Sugar equation \[\text{[9]}\]). In that case, any symmetries of the interaction kernel are usually destroyed by iteration in the scattering equation. This is a well known result. In meson-exchange models, for example, the interaction kernel is the Born amplitude of an effective field-theoretic lagrangian, which is crossing-symmetric. Solving the scattering equation effectively sums a subset (ladder/bubble graphs) of the full set of Feynman diagrams, thereby destroying the crossing symmetry of the kernel. In the same way an interaction kernel which is chirally symmetric in the limit of zero pion mass results in an amplitude in which this property is lost through iteration in the scattering equation.

Were the objective simply to calculate scattering amplitudes for pions at zero density, the problem of preserving (broken) chiral (and crossing) symmetry is solved, in principle, by chiral perturbation theory. However calculations in chiral perturbation theory beyond one loop are overwhelming, so that unitarity cannot easily be enforced, nor can one readily adapt the method to the problem of pion interactions in the nuclear medium. For practical reasons we are led back to effective models used with scattering equations.

2.1 Enforcing the MCC

The problem of enforcing the MCC was solved in I by using a once-subtracted form of the 2-pion propagator in the Blankenbecler-Sugar (BbS) equation \[\text{[9]}, taking the subtraction point at $s = 0$ ($s$ is the square of the center-of-mass energy) and setting the subtraction constant to zero. For free particle scattering, this resulted in the replacement of the BbS propagator

$$G_{\pi\pi}(s, k) = \frac{1}{\omega_k(s - 4\omega_k^2 + i\eta)} \tag{1}$$
by the form
\[ \tilde{G}_{\pi\pi}(s, k) = \frac{s}{\omega_k(s - 4\omega_k^2 + i\eta)4\omega_k^2}, \] (2)
where \( k \) is the modulus of the c.m. 3-momentum of the pions and \( \omega_k = \sqrt{k^2 + m_{\pi}^2} \). The appearance of \( s \) in the numerator of \( \tilde{G}_{\pi\pi}(s, k) \) as an external variable in the integral equation ensures that the scattering amplitude generated by solving the BbS equation (quantum numbers suppressed)
\[ M(s; q, q') = V(s; q, q') + \int_0^\infty k^2 dk V(s; q', k) \tilde{G}_{\pi\pi}(s, k)M(s; k, q) \] (3)
will be \( \mathcal{O}(s) \) if \( V(s; q', q) \) is \( \mathcal{O}(s) \) as \( q, q' \to 0 \), assuming that the integration over intermediate momenta gives a factor of \( \mathcal{O}(1) \) or higher. Since the scattering length in the \( i \)-th channel (where \( i \) stands for \( (I = 0, J = 0) \) or \( (I = 2, J = 0) \)) is given by
\[ a^i = \lim_{q \to 0} \frac{M^i(s; q, q)}{32\pi\sqrt{s}}, \] (4)
with \( q = \sqrt{s - m_{\pi}^2} \), if \( M^i \propto s \), then \( a^i \propto \sqrt{s} \to 2m_{\pi} \) and the MCC is fulfilled. The difficulty with this solution is that the suppression of the rescattering terms at small momenta forces one to use large coupling constants and unrealistically high values of cutoffs, and to rely on heavy meson-exchange (\( e.g., \) the \( f_2(1270) \)), in meson exchange models such as the Jülich model \([4]\) in order to achieve a good fit to \( \pi\pi \) phase shifts, so that one loses contact with the underlying low-energy physics of the \( \pi\pi \) interaction. Here we wish to construct interaction kernels which will yield amplitudes which satisfy the MCC without having to resort to the subtraction scheme of I, and thereby avoid its unattractive features. In that sense, this is a continuation of the work begun in I. For that reason we will adopt the same model for renormalizing the single-pion propagator in nuclear matter as in that work. Since the model is described in detail in I, we will not repeat it here.

The solution to building the MCC into the interaction kernel is quite straightforward. We noted earlier that the reason the subtracted form of the 2-pion propagator preserved the scattering length constraint contained in the interaction kernel is that \( s \), the square of the
center-of-mass energy, appears as an external variable in the rescattering term in the integral equation, eq. (3). Thus, if $V(s; q, q)$ is $\mathcal{O}(s)$ as $q \to 0$, the same will hold for $M(s; q, q)$. If the kernel is such that it always contains external kinematic factors which are each $\mathcal{O}(s)$ or higher at each stage in the iterative solution of the scattering equation, then the MCC will be satisfied.

One simple way to accomplish this is through the use of separable potentials. If $V(s; q', q)$ is a sum of terms of the form $u_i(s, q')u_i(s, q)$ such that $u_i(s, q)$ is $\mathcal{O}(\sqrt{s})$ (or higher) as $q \to 0$, the amplitude will be $\mathcal{O}(s)$ and the MCC will be fulfilled. This property was exploited in I with the linear $\sigma$-model. They started from the $I = 0$ Born amplitude in the linear $\sigma$-model,

$$M_{B}^{I=0} = \frac{m_{\sigma}^2 - m_{\pi}^2}{f_{\pi}^2} \left( 3 \frac{s - m_{\pi}^2}{s - m_{\sigma}^2} + \frac{t - m_{\pi}^2}{t - m_{\sigma}^2} + \frac{u - m_{\pi}^2}{u - m_{\sigma}^2} \right), \quad (5)$$

where $s$, $t$, and $u$ are the usual Mandelstam variables for physical processes given by

$$s = (q_1 + q_2)^2 = (q_3 + q_4)^2$$
$$t = (q_1 - q_3)^2 = (q_4 - q_2)^2$$
$$u = (q_1 - q_4)^2 = (q_3 - q_2)^2$$

(6)

with $q_1, q_2$ the incoming and $q_3, q_4$ the outgoing pion 4-momenta.

In order to construct a separable potential, they took the off-shell continuation of $s$, $t$, and $u$ to be

$$s = E^2$$
$$t = 2m_{\pi}^2 - 2\omega_q\omega_{q'} + \mathbf{q} \cdot \mathbf{q}' \quad (7)$$
$$u = 2m_{\pi}^2 - 2\omega_q\omega_{q'} - \mathbf{q} \cdot \mathbf{q}'$$

corresponding to placing the pions on the mass shell ($\omega_q \equiv \sqrt{q^2 + m_{\pi}^2}$) in $t$ and $u$, but taking $s$ as an external variable in the scattering equation. Neglecting the $t$ and $u$ in the denominators as being small compared with $m_{\sigma}^2$, the interaction kernel becomes

$$V^{00}(E; q', q) = \frac{m_{\sigma}^2 - m_{\pi}^2}{f_{\pi}^2} \left( 3 \frac{E^2 - m_{\sigma}^2}{E^2 - m_{\sigma}^2} + \frac{4\omega_q\omega_{q'} - 2m_{\pi}^2}{m_{\sigma}^2} \right) v(q')v(q). \quad (8)$$
Here $q$ and $q'$ are the moduli of $q$ and $q'$ and $v(q)$ is a dipole-type form factor with $v(0) = 1$ to insure convergence of the integral. Clearly the interaction defined in eq. (8) satisfies the criteria for MCC set out above: Every term contains either an external factor (e.g., $E^2$ or $m^2_{\pi}$) or a factor which is a product of terms $(\omega_q \omega_{q'})$ which survive the integration and are $O(m^2_{\pi})$ as $q, q' \to 0$.

Notice, however, that separability of the potential is not a crucial part of the recipe for ensuring MCC. If it were, then all meson-exchange models would be irreparable. The important point is that all the terms in the potential have in their numerators products of factors which are of order $m_{\pi}$ in the chiral limit, one of which depends only on the initial momentum and the other only on the final momentum. Were the $t$- and $u$-dependence of the denominators retained in defining $V^{00}(E; q', q)$, the MCC would still be satisfied. This is evident if one writes the scattering equation in iterated form (schematically):

$$M = V + VGV + VGVGV + \ldots$$ (9)

Every term in the series will contain one factor which depends on the initial momentum, and one which depends on the final momentum. As long as the integrals implied in the equation are $O(1)$ or higher (the form factors used in the potential guarantee this), the MCC is satisfied by $M$. Thus, it is the choice of off-shell continuation of the kinematic variables, and not separability, which is crucial to preserving the scattering length constraint. Observe also that the choice of off-shell continuation is not unique. We could have taken $s = (q_1 + q_2)^\mu(q_3 + q_4)_\mu = 4\omega_q \omega_{q'}$ and still fulfilled the prescription for satisfying the MCC. The choice $s = E^2$ is made on physical grounds: we want the $\sigma$-meson to appear as a fixed pole in energy in the scattering channel.

A generalization of the prescription for construction of an MCC-preserving potential is now clear:

1. Start with an on-shell Born amplitude derived from a chirally symmetric lagrangian (broken by the pion mass term), expressing it in terms of the Mandelstam variables in crossing-symmetric form.
2. Make the off-shell continuation of $s$, $t$, and $u$ as indicated in eq. (4).

3. Include form factors, as necessary, to insure convergence of the rescattering term.

We will refer to potentials of this type as MCC quasi-separable.

Of course, the above prescription is sufficient, but not necessary. It is certainly possible that potentials can be constructed which are not of this type but which, through a delicate cancellation of terms, do fulfill the MCC. Nevertheless, the prescription allows us to define, for any underlying effective lagrangian which is (broken) chirally symmetric, a potential which will yield an amplitude that has the proper behavior of the scattering lengths.

2.2 Inconsistencies in the MCC Prescription

Although the prescription above guarantees MCC-compliance of the interaction kernel, it is neither unique, nor free from inconsistencies which seem to be unavoidable. We have already alluded indirectly to such inconsistencies in our discussion above of the relation of the continuation of the Mandelstam variables via eq. (14) to the BbS equation. These appear to arise whenever one uses an effective lagrangian to generate a Born term to use in a scattering equation; that is, the off-shell continuation needed to satisfy the MCC is not derivable from the evaluation of interaction vertices based on a lagrangian used consistently with a scattering equation.

We illustrate this with a common example: the exchange of a $\rho$-meson. The interaction term in the lagrangian is

$$\mathcal{L}_{\rho\pi\pi} = -g_\rho (\vec{\pi} \times \partial^\mu \vec{\pi}) \cdot \vec{\rho}_\mu. \tag{10}$$

The $I = 0$ and $I = 2$ Born amplitudes contain $t$- and $u$-channel exchange terms which are given by

$$M^{I=0,2}_{\rho,\bar{B}} = (2, -1)2g_\rho^2 \left[ \frac{(q_1 + q_3)^\mu(q_2 + q_4)_\mu}{(q_1 - q_3)^2 - m_\rho^2} + \frac{(q_1 + q_4)^\mu q_2 + q_3)_\mu}{(q_1 - q_4)^2 - m_\rho^2} \right]. \tag{11}$$
We have omitted the contributions arising from gauge terms in the $\rho$ propagator, since they vanish for the fully on-shell, equal-mass Born amplitude, and they are usually neglected in the meson-exchange potentials as well. In that case the amplitude, expressed in manifestly crossing-symmetric form, is

$$M_{\rho,B}^{I=0,2} = (2, -1)2g_\rho^2 \left( \frac{s-u}{t-m_\rho^2} + \frac{s-t}{u-m_\rho^2} \right).$$

Let us examine this potential as evaluated consistently in the BbS approach, and then in time-ordered perturbation theory.

In the BbS approach, the interaction is instantaneous, so that only 3-momentum is transferred in the exchange of the meson. Thus the scattering particles remain on the energy shell, but not on the mass shell. In the c.m. frame $q_{1,2} = (E/2, (+,-)q)$ and $q_{3,4} = (E/2, (+,-)q')$. In terms of these variables, eq. (11) becomes

$$M_{\rho,B}^{I=0,2} = (2, -1)2g_\rho^2 \left( E^2 + q^2 + q'^2 + 2q \cdot q' \left( \frac{2}{-(q-q')^2 - m_\rho^2} \right) + \frac{E^2 + q^2 + q'^2 - 2q \cdot q'}{-(q+q')^2 - m_\rho^2} \right).$$

This form can be obtained from the off-shell continuation of $s$, $t$, and $u$

$$s = E^2$$
$$t = -(q - q')^2 = -q^2 - q'^2 + 2q \cdot q'$$
$$u = -(q - q')^2 = -q^2 - q'^2 - 2q \cdot q'$$

applied to eq. (12). This, therefore, is the “natural” or consistent off-shell continuation of the Mandelstam variables in the BbS approach. The occurrence of $q^2$ and $q'^2$ standing separately violates the requirements of MCC quasi-separability. In time-ordered perturbation theory, the scattering particles are on the mass shell, but off the energy shell. The lagrangian gives one the interactions at the vertices (the numerators) and the denominators arise from the normalization of the $\rho$-meson wave function. In this case $q_{1,2} = (\omega_q, (+, -)q)$ and $q_{3,4} = (\omega_{q'}, (+, -)q')$, yielding

$$M_{\rho,B}^{I=0,2} = (2, -1)2g_\rho^2 \left( \frac{2(m_\pi^2 + \omega_q \omega_{q'}) + q^2 + q'^2 + q \cdot q'}{-(q - q')^2 - m_\rho^2} + \frac{2(m_\pi^2 + \omega_q \omega_{q'}) + q^2 + q'^2 - q \cdot q'}{-(q + q')^2 - m_\rho^2} \right).$$
As in the BbS approach, the interaction will not preserve the MCC.

Thus, in order to fulfill the MCC, one cannot simply use interaction vertices given by the consistent use of a meson-exchange lagrangian in conjunction with one or another scattering equation. One can see that the $\rho \pi \pi$ vertices for $t$- and $u$-channel exchange will always have terms in which $q^2$ and $q'^2$ appear as a sum, and therefore destroy any possibility for the interaction to be MCC quasi-separable. To construct one which is, one must start with the fully on-shell amplitude expressed in manifestly crossing-symmetric form, e.g. eq. (12), and continue the kinematic variables off shell according to the prescription of section 2.2 – or some similar one – in which each term in the numerator is either an external energy or mass squared, or is a product of an initial energy or momentum and a final one.

3. TWO MODELS: NUMERICAL RESULTS

In this section we will examine the results of two models in which the potential is constructed according to the prescription in section 2. The results will be compared with those for the same models when a frequently-used off-shell continuation which violates MCC quasi-separability is used.

3.1 The Linear $\sigma$-Model

We choose the linear $\sigma$-model as a first example. In this case we retain the $t$- and $u$-dependence arising from the exchange of $\sigma$ mesons in the crossed channels. The fit to the $\text{JI}=0\text{0} \pi \pi$ phase shifts is displayed in fig. 1. The parameters and form factors used for this model are shown in Table I. The in-medium results, shown in the upper panel of fig. 2, are quite similar to those of I and confirm the notion that separability is not the crucial property of the potential. (The definition of $M$ differs from that of I by a factor $4\pi^2$.) The amplitude shows no tendency towards pair condensation, and there is hardly any invasion of strength into the subthreshold region even at densities as high as $2\rho_0$. 
In order to demonstrate the consequences of violation of the MCC, we use the same model, but with the off-shell continuation given by eq. (14), the natural one for the BbS reduction of the Bethe-Salpeter equation, wherein the intermediate pions are on the energy shell, but off the mass shell. The occurrence in \( t \) and \( u \) of \( q^2 \) and \( q'^2 \) standing separately spoil the MCC quasi-separability of the potential and result in a contribution from the rescattering integral which is \( O(1) \) in the chiral limit. Results for this model are shown in the lower panel of fig. 2. Note that at zero density the amplitude for this model and for the previous one are very close without having changed the parameters. (This good agreement, however, relies on choosing a rather small bare \( \sigma \)-mass \( m_\sigma \), which allows for a small cutoff parameter \( \Lambda_\sigma \); for higher \( \sigma \) masses, which require higher cutoffs for fitting the \( s \)-wave phase shifts, the different off-shell behavior of eqs. (7) and (14) induces larger differences.) At higher densities, however, one clearly recognizes a much stronger accumulation of strength below the \( 2\pi \) threshold compared to the model including the MCC (compare upper panel of fig. 2), although there is no pair instability in either case at these densities.

### 3.2 The Jülich Model

While the linear \( \sigma \)-model is instructive, we wish to examine the results for a model which achieves a quantitatively better fit to elastic \( \pi\pi \) data: the Jülich model [7], which is based on explicit meson exchange. In order that the Born amplitude satisfies the soft-pion theorems we here include contact terms in the lagrangian that arise from a proper gauging procedure of the non-linear \( \sigma \) model [8]. To further ensure the MCC we employ the on-mass-shell prescription, eq. (7), for the off-shell continuation of the pseudopotentials. With some readjustment of the meson exchange parameters and typical cutoff parameters \( \Lambda_\varepsilon = 700 - 1000 \text{ MeV} \) for the additionally introduced contact terms (compare table 2) a good overall fit to the \( \pi\pi \) scattering data can be obtained for c.m. energies well beyond 1 GeV (see fig. 3). By construction, the \( s \)-wave scattering lengths vanish in the chiral limit (upper panel of fig. 4). The density dependence of the amplitude in the \( JJ=00 \)-channel is displayed in fig. 5 (upper
panel). In contrast to the BbS-Jülich model employed in I (without contact interactions and without MCC), in which condensation occurred just above $\rho_0$, our chirally improved model shows a somewhat weaker tendency towards instability; as a consequence of the additional repulsion induced by the contact interactions, the critical density for condensation is pushed up to approximately $1.4\rho_0$, with a moderate portion of strength absorbed in the condensing peak. The critical density can be increased to approximately $2\rho_0$ with a relatively small increase in the cutoff mass for the contact terms while retaining a good fit to the phase shift. Thus the conjecture that an interaction which produces an amplitude that satisfies the MCC would eliminate the pair condensation instability is not correct, even though it appears to work in that direction.

To examine the impact of the MCC in more detail, we replace the on-mass-shell prescription for the Mandelstam variables, eq. (7), by the BbS (on-energy-shell) identification, eq. (14). A slight readjustment of some parameters is necessary to obtain an overall fit to the $\pi\pi$ phase shifts of similar quality to that of fig. 3 (see also table 3), although the cutoffs for the contact terms are kept fixed. The results are presented in the lower panel of fig. 5. The contrast with the previous case is significant. Condensation density is slightly higher – about $1.5\rho_0$ – and can easily be increased to above $2\rho_0$ with a small increase of the contact term cutoffs. The accumulation of strength in the condensing peaks is very much smaller than in the previous case and similar to that found for the Jülich model with chiral constraints investigated in I. The scattering lengths as a function of pion mass (lower panel of fig. 4) show clearly the violation of the chiral constraints for this model. Therefore the conjecture in I that scattering length constraint insures against pair condensation at low-to-moderate density appears to be neither sufficient nor necessary.

That is not to say that chiral invariance is unimportant. The "improvements" in the Jülich model are just those which make the kernel satisfy the soft-pion theorems. What is clear now is that the suppression of the instability is due in large part to the change in the kernel; there is no necessity to redefine the 2-pion propagator in order to avoid the instability at near-nuclear density. One must simply be careful in constructing the kernel to build in
enough repulsion at sub-threshold energy to suppress the tendency towards $\pi\pi$ bound state formation. The results are somewhat sensitive to the off-shell continuation of the kinematic variables, but this sensitivity can largely be compensated by readjustment of parameters in standard models.

One other difference between the MCC-compliant and BbS cases is worthy of mention. Despite the earlier onset of the pairing instability, the MCC-compliant amplitude actually shows considerably less strength in the near sub-threshold region, $E \approx 1 - 2m_\pi$, than the BbS case. This means that at normal nuclear matter density the range of the in-medium nucleon-nucleon interaction will be less affected in the MCC case than in the BbS. Since the effects are highly non-linear with density, they are very model-dependent, not only on the $\pi\pi$ interaction, but also on the approximations underlying the many-body aspects of the calculation. Conclusions concerning the range of the nucleon-nucleon interaction must be made with extreme care. Nevertheless, the question of whether the chiral constraint in some way "protects" the range of the nucleon-nucleon force up to nuclear matter density is worthy of further study.

5. SUMMARY

We have given a general prescription for the construction of $\pi\pi$ potentials for scattering equations which enforces constraints on the scattering lengths in the limit of zero pion mass, in accordance with the requirements of chiral symmetry. When used in a particular scattering equation, the constrained amplitudes do not automatically ensure that the $\pi\pi$ interaction in nuclear matter will be stable against $\pi\pi$ bound state formation at low-to-moderate density. The imposition of the scattering length constraint on the kernel does delay the onset of the instability to higher densities, however; imposition of the constraint on the full amplitude appears especially effective in suppressing the $\pi\pi$ interaction strength in the near sub-threshold region.

Due to the complexity of the calculation and the non-linearity of the in-medium effects,
it is difficult to make very general quantitative statements about the interactions of pions in the nuclear medium. Based on our limited investigations it appears that commonly employed models of the $\pi\pi$ interactions, adjusted to satisfy the soft-pion theorems, do not necessarily lead to instabilities when carried over to in-medium calculations. Small adjustments in model parameters allow one simultaneously to fit $\pi\pi$ scattering phases and to avoid the pairing instability at densities below approximately $2\rho_0$. Reformulation of the 2-pion propagator, which is essentially a definition of a new scattering equation, is not necessary. Finally, there are numerical indications that chiral constraints play a significant part in our understanding of the nucleon-nucleon interaction in the nuclear medium, but exactly what that part is remains a subject for further investigation.

ACKNOWLEDGMENTS

We are grateful for productive conversations with J. Speth and K. Nakayama. One of us (JWD) wishes especially to thank Prof. Speth for his hospitality and support during JWD’s visits to the Forschungszentrum Jülich. One of us (RR) acknowledges financial support from the German Academic Exchange Service (DAAD) under program HSPII/AUFIE. This work is supported in part by the National Science Foundation under Grant No. NSF PHY94-21309.
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### TABLES

| meson | coupling | mass  | cutoff | form                           |
|-------|----------|-------|--------|--------------------------------|
| X     |          | $m_X$ [MeV] | $\Lambda_X$ [MeV] | factor                        |
| s-channel $\sigma^{(0)}$ | $(m_\sigma^2 - m_\pi^2)/f_\pi^2 \pi^2\sigma$ | 800   | 1250   | $(2\Lambda^2)^2/(2\Lambda^2 + 4q^2)^2$ |
| t-channel $\sigma$     |          |       |        | $(2\Lambda^2)^2/(2\Lambda^2 + (q - \tilde{q})^2)^2$ |

**TABLE I.** Coupling, parameters and form factor types used in the linear $\sigma$-model for both on-energy-shell and on-mass-shell prescription
### $\pi\pi$ channel

| meson  | coupling                                      | coupl. const. | mass [MeV] | cutoff [MeV] |
|--------|-----------------------------------------------|---------------|------------|--------------|
| $\epsilon^{(0)}(1400)$ | $(g_{\pi\pi}\epsilon/m_{\pi}) (\partial_{\mu}\vec{\pi} \cdot \partial^{\mu}\vec{\pi})\phi_{\epsilon}$ | 0.008         | 1585       | 1750         |
| $\rho^{(0)}(770)$ | $g_{\pi\pi}\rho(\vec{\pi} \times \partial_{\mu}\vec{\pi}) \cdot \vec{\rho}$ | 1.6           | 1045       | 3000         |
| $\rho(770)$ | '' | 3.0           | 770        | 1410         |
| $f_{2}^{(0)}(1270)$ | $(g_{\pi\pi}f_{2}/m_{\pi}) (\partial_{\mu}\vec{\pi} \cdot \partial^{\mu}\vec{\pi})\phi^{\mu\nu}_{f_{2}}$ | 0.015         | 1573       | 2285         |

Contact

| meson  | coupling                                      | coupl. const. | mass [MeV] | cutoff [MeV] |
|--------|-----------------------------------------------|---------------|------------|--------------|
| $f_{2}^{(0)}$ | $(m_{\pi}^{2}/8f_{\pi}^{2}) (\vec{\pi}^{2})^{2}$ | $f_{\pi}=93$ MeV | $m_{\pi}=139.57$ | 700 |

### $\pi\pi \rightarrow K\bar{K}$ channel

| meson  | coupling                                      | coupl. const. | mass [MeV] | cutoff [MeV] |
|--------|-----------------------------------------------|---------------|------------|--------------|
| $K^{*}(895)$ | $g_{\pi K^{*}}\partial^{\mu}\vec{\pi} \cdot (K\bar{\pi}K_{\mu})$ | 0.75         | 895        | 1410         |

### $K\bar{K}$ channel

| meson  | coupling                                      | coupl. const. | mass [MeV] | cutoff [MeV] |
|--------|-----------------------------------------------|---------------|------------|--------------|
| $\epsilon^{(0)}(1400)$ | $(g_{KK}\epsilon_0/m_{K})\partial_{\mu}K\partial^{\mu}K\phi_{f_0}$ | 0.002         | 1585       | 1750         |
| $\rho^{(0)}(770)$ | $g_{KK}\rho(K\bar{\pi}\partial_{\mu}\bar{K}) \cdot \vec{\rho}$ | 0.4           | 1045       | 3000         |
| $\rho(770)$ | '' | 0.75           | 770        | 2135         |
| $\omega(782)$ | $g_{KK}\omega(K\partial_{\mu}\bar{K})\omega_{\mu}$ | -0.75        | 782.6      | 2135         |
| $\phi(1020)$ | $g_{KK}\phi(K\partial_{\mu}\bar{K})\phi_{\mu}$ | -1.5          | 1020       | 2135         |
| $f_{2}^{(0)}(1270)$ | $(g_{KK}f_{2}/m_{K})\partial_{\mu}K\partial_{\nu}\bar{K}\phi^{\mu\nu}_{f_{2}}$ | 0.004         | 1573       | 2285         |
TABLE II. Couplings and parameters of the chirally improved $\pi\pi$/KK Jülich model employing the on-mass-shell prescription, eq. (7), for the off-shell continuation of the pseudopotentials; s-channel pole graphs are labelled with superscript '(0)'; form factors used are of dipole type: $F^{(s)}(q) = \frac{(2\Lambda^2 + m^2)^2}{(2\Lambda^2 + 4\omega_q^2)^2}$ for s-channel pole graphs, $F^{(t)}(q,q') = \frac{(2\Lambda^2 - m^2)^2}{(2\Lambda^2 + (\vec{q} - \vec{q'})^2)^2}$ for t-channel exchange graphs and $F^{(c)}(q) = \frac{(2\Lambda^2 - 4m_q^2)^2}{(2\Lambda^2 + 4q^2)^2}$ for contact interactions.
| meson | coupling | coupl. const. | mass | cutoff |
|-------|----------|---------------|------|--------|
| $\epsilon^{(0)}(1400)$ | $(g_{\pi\pi\epsilon}/m_\pi) (\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}) \phi_\epsilon$ | 0.016 | 1545 | 1900 |
| $\rho^{(0)}(770)$ | $g_{\pi\pi\rho}(\vec{\pi} \times \partial_\mu \vec{\pi}) \cdot \vec{\rho}$ | 1.6 | 1108 | 3500 |
| $\rho(770)$ | ” | 3.0 | 770 | 1450 |
| $f_2^{(0)}(1270)$ | $(g_{\pi\pi f_2}/m_\pi) (\partial_\mu \vec{\pi} \cdot \partial_\nu \vec{\pi}) \phi_{f_2}^{\mu\nu}$ | 0.015 | 1710 | 2770 |
| contact | $(m_\pi^2/8f_\pi^2) (\vec{\pi}^2)^2$ | $m_\pi=93$ MeV | $m_\pi=139.57$ | 700 |
| inter- | $(1/4f_\pi^2) (\vec{\pi}^2)(\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi})$ | ” | ” | ” |
| actions | $(g_{\pi\pi f_2}/2m_\rho^2) (\vec{\pi} \times \partial^\mu \vec{\pi})^2$ | 1.6 | $m_\rho=770$ | ” |

| meson | coupling | coupl. const. | mass | cutoff |
|-------|----------|---------------|------|--------|
| $K^*(895)$ | $g_{\pi K K^*} \partial^\mu \vec{\pi} \cdot (K \bar{\tau} K^*_\mu)$ | 0.75 | 895 | 1575 |

TABLE III. Same as table 2, but employing the BbS (on-energy-shell) prescription, eq. (14).
Figure Captions

**Figure 1**: Fit to the scalar-isoscalar $\pi\pi$ phase shifts with the linear $\sigma$-model. The dashed line is the result when employing the on-mass-shell prescription, eq. (7), whereas the full line corresponds to the on-energy-shell (BbS) prescription, eq. (14), using the same set of parameters (see table I).

**Figure 2**: Scalar-isoscalar invariant $\pi\pi$ $M$ amplitude with the linear $\sigma$ model in nuclear matter; long-dashed lines: $\rho/\rho_0=0.5$, short-dashed lines: $\rho/\rho_0=1.0$, dotted lines: $\rho/\rho_0=2.0$; the full lines correspond to the amplitude in free space. The upper panel shows the results using the on-mass-shell prescription, whereas the lower panel corresponds to the BbS prescription.

**Figure 3**: Fit to the $\pi\pi$ phase shifts with the Jülich model supplemented by contact interactions and using the on-mass-shell prescription; parameters used are listed in table II. A very similar fit is obtained using the BbS prescription with the slightly modified parameter set of table III (contact term parameters unchanged).

**Figure 4**: $S$-wave $\pi\pi$ scattering lengths in the chiral limit for various versions of the Jülich model; upper panel: on-mass-shell prescription including contact interactions (corresponding to the model of fig. 3 / table II); lower panel: BbS prescription, dashed lines: including contact interactions (corresponding to the model of table III), dotted lines: without contact interactions.

**Figure 5**: Scalar-isoscalar invariant $\pi\pi$ amplitude in nuclear matter for the Jülich model supplemented with contact interactions; upper panel: employing the on-mass-shell prescription (corresponding to table II), long-dashed line: $\rho/\rho_0=0.5$, short-dashed line: $\rho/\rho_0=1.0$, dotted line: $\rho/\rho_0=1.3$; lower panel: employing the BbS prescription, line identification as in upper panel ex-
cept that the dotted line is for $\rho/\rho_0=1.45$.

The full lines correspond to the amplitudes in free space.
