THE CONSISTENT NEWTONIAN LIMIT of EINSTEIN’S GRAVITY WITH a COSMOLOGICAL CONSTANT

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Abstract

We derive the ‘exact’ Newtonian limit of general relativity with a positive cosmological constant $\Lambda$. We point out that in contrast to the case with $\Lambda = 0$, the presence of a positive $\Lambda$ in Einstein’s equations enforces, via the condition $|\Phi| \ll 1$ on the potential $\Phi$, a range $R_{\text{max}}(\Lambda) \gg r \gg R_{\text{min}}(\Lambda)$, within which the Newtonian limit is valid. It also leads to the existence of a maximum mass, $M_{\text{max}}(\Lambda)$. As a consequence we cannot put the boundary condition for the solution of the Poisson equation at infinity. A boundary condition suitably chosen now at a finite range will then get reflected in the solution of $\Phi$ provided the mass distribution is not spherically symmetric.

1 Introduction

The cosmological constant $\Lambda$ can, in principle, enter the gravitational equations determining the metric $g_{\mu\nu}$. It has been introduced by Einstein to save the universe from expanding, and rejected by him after expansion had been discovered by Hubble. Historically, the next stage of development regarding this constant can be characterized as the search for a mechanism which would allow to put the effective cosmological constant, being the sum of contributions from vacuum fluctuations and a constant from gravity, to zero [1] (for other reviews see [2] and [3]). Such an explanation is desirable to avoid fine-tuning problems. No generally accepted mechanism has been found. However, something unexpected emerged from the recent measurement of cosmological parameters whose values seem to favour an open accelerating universe [4, 5]. One of the theoretical models, in agreement with such a scenario, is Einstein’s gravity with a positive cosmological constant [4, 5, 6]. The theoretical efforts shifted then to finding an explanation for the actual value of $\Lambda$. One approach invokes the anthropic principle [7] i.e. by considering an ensemble of possible universes (be it by inflation [8] or quantum cosmology [9]) with different cosmological constants; the latter characterized by probability distributions [10].

Since for a long time the bias has been towards $\Lambda = 0$ and, if at all, the applications of models containing a non-zero $\Lambda$ were thought to be important only
at cosmological scales, not much attention was paid to the Newtonian limit in the presence of \( \Lambda \). We will show, however, that some new results of the Newtonian limit with \( \Lambda \) being non-zero can be derived, and that given the presently favoured value of \( \Lambda \), the vacuum force induced by it is non-negligible at astrophysical scale of one Mpc and bigger. Hence astrophysical applications of such a vacuum force are not excluded (indeed we will show examples where it can be quite important). A careful re-examination of the Newtonian limit, especially when it reveals some new insights, is therefore welcome.

The Newtonian limit of Einstein's gravity and its generalizations is given in the form of Poisson equation for the potential \( \Phi \) connected with the metric by the equation

\[
g_{00} \simeq -(1 + 2\Phi),
\]

which follows from the equation of geodesics for a weak static field produced by non-relativistic mass distribution. Needless to say, the Poisson equation has to be supplemented by some boundary condition which in the simplest case, general relativity with zero cosmological constant, is more or less obvious. This is partly so, because we know of course what to expect from a Newtonian theory, it being chronologically before Einstein's gravity. In general, however, we have to make a decision about the kind of boundary condition (i.e. Dirichlet, von Neumann or mixed), where in space we want to put our boundary condition and what value the boundary condition is supposed to assume. Obviously with \( \Lambda = 0 \) we have Dirichlet boundary conditions which we implement as \( \Phi|_R = \text{const} \) with \( R \to \infty \). Note that we can choose safely the point \( R \to \infty \) as it is indeed mathematically consistent (i.e. there is no other information following from the Newtonian limit which would contradict such a choice) as long as we keep in mind that from the physics point of view we should not extend the validity of the Newtonian limit to a scale of the size of the whole universe [11]. Another point worth stressing is that general relativity offers yet another information on the Newtonian potential \( \Phi \). It is the Schwarzschild solution from which we know, via equation (1.1), the potential already for a point-like and/or spherically symmetric object. This is a valuable information since we then also know, without actually solving the Poisson equation, what the potential of any object will look like from far enough distances. We can use this as the value of the boundary condition at, say, the point \( R \) with \( R \to \infty \) in case \( \Lambda = 0 \) as we said before, but not so in the case \( \Lambda > 0 \) as we will show below.

All these simple and rather obvious considerations might change in more sophisticated cases. It is a curious fact that this already happens when a positive cosmological constant is switched on. What remains is the need for a suitable boundary condition which, one might be inclined to think, could be chosen also at \( R \to \infty \) regardless of the fact that the potential assumes an infinite value there (we can treat this infinity as a constant and subtract it). But, as we will demonstrate in this work, the very same condition for the Newtonian approximation to hold, namely \( |\Phi| \ll 1 \) implies now also a restriction on the range of the distance \( r \). In other words, there
exist now a $R_{\text{max}}$ and $R_{\text{min}}$ dependent on $\Lambda$ such that

$$R_{\text{min}}(\Lambda) \ll r \ll R_{\text{max}}(\Lambda).$$

(1.2)

Therefore, it would be mathematically inconsistent to put our boundary condition at infinity. Similarly, the same requirement ($|\Phi| \ll 1$) leads also to the existence of a maximum mass $M_{\text{max}}$. Any mass $M$ considered in the Newtonian limit with $\Lambda > 0$ has to satisfy

$$M \ll M_{\text{max}}(\Lambda)$$

(1.3)

Both inequalities, (1.2) and (1.3), follow strictly from $|\Phi| \ll 1$.

2 The Newtonian limit of Einstein’s equations with $\Lambda \neq 0$

The weak field expansion in gravity starts by expressing the metric $g_{\mu\nu}$ through the Minkowski metric $\eta_{\mu\nu}$ and spin-2 field $h_{\mu\nu}$ (up to a multiplicative constant).

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

(2.1)

For a weak field we have

$$|h_{\alpha\beta}| \ll 1$$

(2.2)

This condition translates via (1.1) into

$$|\Phi| \ll 1$$

(2.3)

In this limit Einstein’s equation with $\Lambda \neq 0$ (we are using the conventions of [12])

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu}$$

(2.4)

reduces, in the first approximation, to the Poisson equation of the form [13]

$$\triangle \Phi = (4\pi G_N)\rho - \Lambda$$

(2.5)

where $\rho$ is the mass density function.

We mentioned already in the introduction that there is yet another information on $\Phi$. It is contained explicitly in the Schwarzschild solution, now with a non-zero cosmological constant. Hence for point-like and/or spherically symmetric objects we deduce using (1.1) that (see e.g. [14])

$$\Phi(r) = -\frac{G_N M}{r} - \frac{1}{6}\Lambda r^2$$

(2.6)
which is valid outside the mass distribution. To examine the consequences of (2.3), it suffices for the moment to use $\Phi$ as given in the last equation. We then have

$$G_N M \ll d(r) \equiv r - \frac{1}{6} \Lambda r^3$$

(2.7)

Several important conclusions can be drawn from (2.7). The very first one follows from the fact that we are only interested in values of $r$ bigger than or equal to zero and that the function $d(r)$ has a local maximum at

$$r_+ = \sqrt{\frac{2}{\Lambda}}$$

(2.8)

Hence a necessary condition to satisfy (2.7) is

$$M_{\text{max}}(\Lambda) \equiv \frac{2 \sqrt{2}}{3} \frac{1}{G_N \sqrt{\Lambda}} \gg M$$

(2.9)

which is the explicit version of (1.3) announced in the Introduction. It follows that there exists a restriction on the absolute value of a mass which can be used in the Newtonian limit.

Further insight, now into the restrictions on the distance $r$ entering the Newtonian limit, can be obtained by considering the equation $G_N M = d(r)$ whose explicit form reads

$$r^3 - \left(\frac{6}{\Lambda}\right) r + 6 \frac{G_N M}{\Lambda} = 0$$

(2.10)

On account of (2.9) there will be three real solutions of this equation, one negative and the other two positive. Consequently the discriminant $D$ of the cubic equation (2.10) is negative:

$$D = \frac{1}{\Lambda^2} \left[ 9 \left( G_N M \right)^2 - \frac{8}{\Lambda} \right] < 0$$

(2.11)

which, of course, is nothing else but equation (2.3) in a weaker form, $M < M_{\text{max}}$. The fact that $D < 0$ allows us now to find the three solutions using an auxiliary angle $\sigma$ or $\sigma_0$ given by

$$\cos \sigma = \cos \left( \sigma_0 + \frac{\pi}{2} \right) = \frac{M}{M_{\text{max}}}$$

(2.12)

In view of (2.9) we can actually write

$$\sigma_0 \approx - \frac{M}{M_{\text{max}}}$$

(2.13)

We then obtain for the three solutions the following parameterized expressions

$$R_0 = -2r_+ \cos \left( \frac{\sigma_0}{3} + \frac{\pi}{6} \right)$$

$$R_1 = -2r_+ \cos \left( \frac{\sigma_0}{3} + \frac{5\pi}{6} \right)$$

$$R_2 = -2r_+ \cos \left( \frac{\sigma_0}{3} + \frac{3\pi}{2} \right)$$

(2.14)
where \( r_+ \) is given in (2.8). One can easily check that \( R_0 < 0 \) and \( R_{1,2} > 0 \) as well as \( R_1 > R_2 \). Obviously we can now identify \( R_1 \) with \( R_{\text{max}}(\Lambda) \) and \( R_2 \) with \( R_{\text{min}}(\Lambda) \) in equation (1.2). More explicitly we obtain

\[
R_{\text{max}} \simeq \sqrt{\frac{6}{\Lambda}} \left[ 1 - \frac{1}{3\sqrt{3}} \frac{M_{\text{max}}}{M} \right] \gg r \gg R_{\text{min}} \simeq G_N M \left[ 1 - \frac{1}{54} \left( \frac{M}{M_{\text{max}}} \right)^2 \right] \tag{2.15}
\]

where in the expansion for \( R_{\text{max,min}} \) we kept only the first corrections to the leading terms. Of course, the right-hand side of this inequality is well known. We keep it here only for completeness and to display corrections due to non-zero \( \Lambda \).

To compare the curious fact given by (2.15) which puts an upper and lower bound on possible distances used in the Newtonian limit, let us briefly repeat the steps for the case \( \Lambda < 0 \). Since in this case we pick up a relative minus between the two terms in (2.6), we have to introduce a third scale defined by

\[
\bar{R} \equiv \left( \frac{6G_N M}{|\Lambda|} \right)^{1/3} \tag{2.16}
\]

Then in analogy to (2.7), we have,

\[
G_N M \ll \bar{d}(r) \equiv r + \frac{1}{6} |\Lambda| r^3 \quad \text{if} \quad \bar{R} > r
\]

\[
G_N M \ll d(r) = r - \frac{1}{6} |\Lambda| r^3 \quad \text{if} \quad \bar{R} < r \tag{2.17}
\]

Let us first concentrate on the first case i.e. we assume that we are with our \( r \) below the value \( \bar{R} \). The function \( \bar{d}(r) \) does not have a local maximum as it was the case with \( d(r) \) in (2.7). No restriction on the value of the mass follows (the use of \( M_{\text{max}}(|\Lambda|) \) below is only for algebraic convenience). The solutions of the corresponding cubic equation give us only one real solution which can be identified with \( \bar{R}_{\text{min}}(\Lambda) \). The discriminant is now positive and correspondingly our parameterized solution is

\[
\bar{R}_{\text{min}}(\Lambda) = \frac{2\sqrt{2}}{\sqrt{|\Lambda|}} \sinh \frac{\bar{\sigma}}{3}
\]

\[
\sinh \bar{\sigma} \simeq \bar{\sigma} = \frac{M}{M_{\text{max}}} \tag{2.18}
\]

where in the last equation we assumed, strictly speaking, that \( M/M_{\text{max}} \ll 1 \) as this does not follow stringently anymore. This assumption we will keep in the following for simplicity. It implies in particular that

\[
\bar{R} \gg \bar{R}_{\text{min}} \simeq R_{\text{min}}(|\Lambda|)
\]

\[
R_{\text{max}}(|\Lambda|) \gg \bar{R} \tag{2.19}
\]
Our restriction on the distance \( r \) reads in the case \( \Lambda < 0 \) as

\[
r \gg \mathcal{R}_{\text{min}} \simeq G_N M \left[ 1 + \frac{M}{2M_{\text{max}}(|\Lambda|)} \right]
\]  

(2.20)

which up to small corrections is the same as in the case with zero cosmological constant, provided that we are with our \( r \) below \( \mathcal{R} \). A priori, however, there could exist universes with \( \Lambda < 0 \) for which the crucial mass ratio \( M/M_{\text{max}} \) need not be small. Then we would have to work with the full solution in (2.18) without using the expansion of \( \sinh \tilde{\sigma} \). The coincidence which we had before with the \( \Lambda = 0 \) case would vanish.

On the other hand choosing to go beyond \( \mathcal{R} \) implies that a consistent Newtonian limit is only possible for

\[
r \ll \mathcal{R}_{\text{max}}(|\Lambda|) 
\]  

(2.21)

Coming back to the situation where \( \Lambda > 0 \) some comments on the actual present values of \( M_{\text{max}} \) and \( \mathcal{R}_{\text{min,max}} \) are in order. It is convenient to express the value of \( \Lambda \) through a constant density called vacuum density \( \rho_{\text{vac}} \):

\[
\Lambda = 8\pi G_N \rho_{\text{vac}}
\]  

(2.22)

Such a vacuum density is then best compared to the critical density of the universe

\[
\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G_N}
\]  

(2.23)

where \( H_0 \) is the Hubble constant given by \( H_0 = 100h_0\text{km}^{-1}\text{Mpc}^{-1} \). We can then write

\[
\mathcal{M}_{\text{max}} = 3.41 \times 10^{22}h_0^{-1}M_\odot \left( \frac{\rho_{\text{crit}}}{\rho_{\text{vac}}} \right)^{1/2}
\]

\[
\mathcal{R}_{\text{max}} = 4.24 \times 10^3\text{Mpc} \left[ h_0^{-1} \left( \frac{\rho_{\text{crit}}}{\rho_{\text{vac}}} \right)^{1/2} - 5.64 \times 10^{-24} \left( \frac{M}{M_\odot} \right) \right]
\]

\[
\mathcal{R}_{\text{min}} = 1.48\text{km} \left( \frac{M}{M_\odot} \right)
\]  

(2.24)

where we have neglected corrections of the order \((M/M_{\text{max}})^2\). With the recent measurements of crucial cosmological parameters [4], one of the favoured models in agreement with these measurements is an open universe with \( \Lambda > 0 \). The actual preferred value is \( \rho_{\text{vac}} \simeq (0.7 - 0.8)\rho_{\text{crit}} \). Then equation (2.13) together with (2.24) can be interpreted in two ways: (i) Even in a universe void of matter, but with \( \Lambda > 0 \), the validity of the Newtonian limit for a test mass \( M \) would be restricted by \( \mathcal{M}_{\text{max}} \) and \( \mathcal{R}_{\text{max,min}} \). As it happens \( \mathcal{R}_{\text{max}} \) is close to the size of the universe at present and \( \mathcal{M}_{\text{max}} \) to its mass. (ii) We could then equally well say, that the restrictions on the Newtonian limit in form of equations (2.13) and (2.24) taken with \( \rho_{\text{vac}} \sim \rho_{\text{crit}} \)
simply tell us that we should not apply the Newtonian limit to the whole universe. This actually does not follow from $|\Phi| \ll 1$ when we take $\Lambda = 0$ as noted already in [11]. Nevertheless the last two points remain curious facts concerning, say, any astrophysical applications at present in the universe we are living in. We emphasize that this is so, because of the small value of $\rho_{\text{vac}}$. This need not be so in other realistic investigations: (iii) We mention here the possibility of a time dependent $\Lambda$ which could have been larger in the past [13]. (iv) An anthropic principle, modeling other possible universes ("real" or hypothetical) would necessarily have to rely on (2.13) when examining via the Newtonian limit structure formation in universes with a sizeable positive cosmological constant. (v) Extrapolating the fate of an open universe [16] with $\rho_{\text{vac}} \sim \rho_{\text{crit}}$ (as our own universe seems to be) into a far future with $R_{\text{univ.}} \gg R_{\text{max}}$, the restricting equations (2.15) and (2.24) have e.g. the virtue to tell us that even for a diluted conglomeration of clusters of the size $R_{\text{max}}$, the Newtonian limit is not applicable. It is then legitimate to ask whether the fact that at present $R_{\text{univ.}} \simeq R_{\text{max}}$ is merely a coincidence.

3 Boundary condition and the solution of the Poisson equation

We choose Dirichlet boundary conditions setting the potential to a constant value at a distance $R$. This is almost as in the standard case with $\Lambda = 0$ save for the fact that in view of (2.13) we cannot let the distance $R$ go to infinity. Hence some finite effects of this finite value of $R$ are to be expected in the solution for $\Phi$. This is unlike the case with zero cosmological constant where the boundary condition $\Phi(x)|_{R \to \infty} = \text{const}$ does not leave any $x$ dependent terms in the solution $\Phi(x)$.

The general solution in the case under consideration in a space region $V$ reads

$$
\Phi(x) = -\int_V d'x' G(x, x') \left[ G_N \rho(x') - \frac{\Lambda}{4\pi} \right] - \frac{1}{4\pi} \int_{\partial V} dS' \Phi|_{\partial V} \hat{n} \cdot \nabla x' G(x, x')|_{\partial V}
$$

(3.1)

where $\Phi|_{\partial V}$ is the Dirichlet boundary condition chosen at the surface $\partial V$ of the volume $V$. To be more specific we opt in this case for

$$
\Phi|_{\partial V} = \Phi|_R = -\frac{G_N M}{R} - \frac{1}{6} \Lambda R^2
$$

(3.2)

i.e. the potential of a point mass as seen from the large distance $R$. $G(x, x')$ in (3.1) is the Green’s function which we give here in terms of spherical harmonics.

$$
G(x, x') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)}{2l + 1}
$$

(3.1)
\[ r_\prec = \min(|x|, |x'|) \]
\[ r_\succ = \max(|x|, |x'|) \]  

(3.4)

Solving the surface integral in (3.1) and using the identity
\[ \frac{1}{|x - x'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i \frac{r_\prec}{r_\succ^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \]  

(3.5)

the solution can be conveniently rewritten as
\[ \Phi(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{G_N}{2l + 1} \frac{r_\prec^{l+1}}{r_\succ^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \]  

(3.6)

where \( V' \subset V \) is the space region in which \( \rho(x') \neq 0 \) and

\[ G'(x, x') \equiv 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) r_\prec^{l+1} r_\succ^{l+1}}{2l + 1} \frac{1}{R^{2l+1}} \]  

(3.7)

The last two equations are what we would like to call the ‘exact’ Newtonian limit of Einstein’s gravity with a positive cosmological constant. The first two terms in (3.6) are what we would naively expect were we allowed to put the boundary condition at \( R \to \infty \). The third term represents exactly the effect of the boundary condition taken at finite distance \( R \). The name ‘exact’ Newtonian limit simply refers to the fact that in view of (2.15) mathematical consistency forces us to put the boundary condition at a finite range which in turn implies (3.6). The last term in (3.6) is a constant and as such of no further importance.

It might appear that we have neglected at the boundary (i.e. postulating our boundary condition) terms of the order \( 1/R^n \) and at the same time retain terms of the same order in the third term in (3.6). This is not a contradiction, however. Despite the physical appeal of the argument that far away of the matter distribution the potential has point-like structure, our choice of the boundary condition (3.2) has been quite arbitrary. Indeed, we could have opted for any other constant value for \( \Phi|_R \) without changing the solution (3.6), up to an unessential constant. This shows that actually there is no such contradiction.

Evidently we are now confronted with the question which value of \( R \) we should be using in the general solution (3.6). This is not as unique as one might wish, but
for any practical purposes we will offer a solution to this problem. First, with a given mass distribution $\rho$ and our solution in form of equation (3.6), we could copy some of the steps in the last section which led us to the existence of $R_{\text{max,min}}$ and $M_{\text{max}}$ and the restricting equations (2.13) and (2.9). This is, however, in general not necessary. Note that the third ‘new’ term in (3.6) becomes a constant for a spherically symmetric body. This is as it should be since in the Newtonian limit gravity with a cosmological constant shares with Newtonian gravity ($\Lambda = 0$) the virtue that in both cases a spherically symmetric object has the same potential as the point-mass (see e.g. [7]). Our solution, of course, reduces up to a constant to (2.4) for spherical objects. For bodies which are not spherically symmetric, new parameters of the dimension of length describing the off-sphericity of the object will appear. These parameters will be much smaller than $R_{\text{max}}$ in (2.15), independent of the numerical value of the latter. Then the equations (2.9) and (2.15) restricting the mass and the distance used in the Newtonian limit hold also for the general solution (3.6). If so, the only possible large distance is some $R$ which is one or two orders of magnitude smaller than $R_{\text{max}}$. Note that (2.3) does not allow us to identify $R = R_{\text{max}}$. It is also worth pointing out that, since we correlate $R$ with $R_{\text{max}}$ (i.e. $R$ is a fraction of $R_{\text{max}}$), to recover the standard Newtonian result from (3.6) it suffices to take $\Lambda \to 0$ as $R_{\text{max}} \propto 1/\sqrt{\Lambda}$ which then implies $R \to \infty$. This seems a reasonable solution so far. Whether it is stringent remains to be investigated in more detail as right now a certain amount of ambiguity stays with us. We think, however, for reasons outlined above that (2.15) and (2.9) are valid for any mass distribution and its gravitational potential.

In any case, it is clear that the third term in (3.6) will be suppressed by powers of $1/R$ and therefore normally small. Whether to retain it in an actual calculation depends, of course, on the requirements of accuracy of our calculation. The most promising case we can imagine is a galaxy at the outskirts of some supercluster where we could hope to see some effects of the boundary condition. Be it as it may, we think that it is of some importance to point out the mathematically consistent Newtonian limit of Einstein gravity with positive cosmological constant.

The discussion of a rather subtle point we have left till the end of this section. It is about the choice of the origin of the sphere with the radius $R$ at whose boundary we have put our boundary conditions. First note that the origin of our coordinate system coincides with the origin of this sphere. This has been done on purpose to simplify the formulae and is not unlike the coordinate system used in (2.6). The choice to put the origin of the coordinate system into the origin of the $R$-sphere is also the reason why terms of the form $|x - R_0|^2$, where $R_0$ would be a vector between the two origins, do not appear in (3.6). Equipped with these remarks we can now make the following statements a.) the choice of the origin of the $R$-sphere in space (now together with the coordinate system) is arbitrary as no distinguished point exists and b.) the repulsion of the test particle due to the $\Lambda$ term will be always away from the center of this sphere. Taking these two statements together we conclude that any imaginary point in space will experience a repulsion from any other point due to the $\Lambda$ term. We can
interpret this as a kinematical effect of the expansion of the universe (due to $\Lambda$) which will now remain even in the Newtonian limit. This has been noticed actually long ago in a different context in [17] which we will briefly touch now. The arbitrariness of the origin of the spheres requires that there exist a transformation law between two possible choices. This transformation is not the Galilean one. The reason is that the Minkowski metric is not a solution of the Einstein’s equations in the presence of $\Lambda$. The symmetry group of the tangent space-time is rather the de Sitter group (anti de Sitter if $\Lambda$ negative). By a non-relativistic group contraction ($c \to \infty$) of the Poincare group we obtain the Galilean transformation. Similarly now making the same group contraction in the de Sitter group we arrive at what is called the Newton group [17]. The corresponding space-time has been coined Newton-Hooke space-time to distinguish it from the normal euclidean space with Galilean transformations [18]. The transformations of the Newton-Hooke space-time read now [17]
\[
\begin{align*}
x' &= Ux + v\tau \sinh \frac{t}{\tau} + a \cosh \frac{t}{\tau} \\
t' &= t + b
\end{align*}
\]
where $b$ is time translation, $a$ space translation, $v$ a constant velocity and $U$ a rotation matrix. The parameter $\tau$ is proportional to $1/\sqrt{\Lambda}$. One can see from these equations that starting with a displacement $a$, one ends up with a growing distance between the two vectors $x$ and $x'$ which from the point of view of non-relativistic physics is a kinematical effect due to $\Lambda$. We can therefore state that the interpretation of the two Newtonian limits, one of the Einstein’s equations and the second one of the symmetry group via group contraction, agree as it should be since the latter is the symmetry of the former (dynamics).

4 Astrophyysical applications

With the presently favoured value of $\rho_{\text{vac}} \sim \rho_{\text{crit}}$ one might be inclined to think that the presence of a positive cosmological constant is only of relevance for cosmology where we are not allowed to use the Newtonian limit, but the full set of Einstein’s equations. However, by examining some simple examples below we can convince ourselves of just the opposite; the cosmological constant can be of relevance in astrophysical applications where the Newtonian limit plays a role. We will discuss three examples, one of field galaxies and the other two for larger structures like galaxy clusters and superclusters.

(a) Two field galaxies at a far away distance from any cluster could, in principle, display effects of ‘anti-gravity’ due to the cosmological constant. Consider the ratio
\[
\left(\frac{F_{\text{vac}}}{F_{\text{Newton}}}\right)_{\text{field galaxies}} = \frac{(Ar/3)}{(G_NM/r^2)} = 2.32 \times 10^{12}h_0^2 \left(\frac{r}{\text{Mpc}}\right)^3 \left(\frac{M}{M_\odot}\right) \left(\frac{\rho_{\text{vac}}}{\rho_{\text{crit}}}\right) \] (4.1)
where $F_{\text{vac}}$ is the force induced by the cosmological constant and $F_{\text{Newton}}$ the standard $1/r^2$ Newtonian force. To pick up a specific example let us fix $M_\odot/M \sim 10^{-11}$ and $r \sim 1 \text{ Mpc}$ with $\rho_{\text{vac}} \sim 0.8 \rho_{\text{crit}}$. The ratio in (4.1) becomes

$$\left( \left| \frac{F_{\text{vac}}}{F_{\text{Newton}}} \right| \right)_{\text{field galaxies}} \simeq 18.6 h_0^2 \sim 4.6$$

the last estimate with $h_0 \sim 0.5$. Field galaxies less massive and at a larger distance (than in our example above) would certainly “repel” each other (the repulsion is, of course, not due to the galaxies since any two space points will experience repulsion). A thorough survey of such field galaxies and their possible peculiar velocities would be a worthwhile task.

The example we have been considering is not unlike the system of our own galaxy and Andromeda. However, with the difference that the latter are embedded in the Local Group. For clusters or even bigger objects a comparison of densities is then more suitable. (b) Hence, considering now such clusters [20, 21], and assuming them to be roughly spherical we can write for the ratio

$$\left( \left| \frac{F_{\text{vac}}}{F_{\text{Newton}}} \right| \right)_{\text{cluster}} = \left( \frac{\Lambda r^3/3}{G N \rho_{\text{cl}} V(r)} \right) = 2 \left( \frac{\rho_{\text{crit}}}{\rho_{\text{cl}}} \right) \left( \frac{\rho_{\text{vac}}}{\rho_{\text{crit}}} \right)$$

where $V$ is the volume of the cluster and $\rho_{\text{cl}}$ its mass density. For a galaxy in the cluster of a typical density $\rho_{\text{cl}} \sim 3 \times 10^{-28}\text{g cm}^{-3}$ the ratio becomes

$$\left( \left| \frac{F_{\text{vac}}}{F_{\text{Newton}}} \right| \right)_{\text{cluster}} \sim 0.13 h_0^2 \left( \frac{\rho_{\text{vac}}}{\rho_{\text{crit}}} \right) \sim 0.025$$

with the assumption of $\rho_{\text{vac}} = 0.8 \rho_{\text{crit}}$. This is actually not too small especially if ‘diluted’ clusters with density less than $3 \times 10^{-28}\text{g cm}^{-3}$ exist. It might be again worthwhile to hunt for such diluted objects. In [21] very low density clusters of $M = 10^{12.5} M_\odot$ and radius $R = 1.5 \text{ Mpc}$ are mentioned. This would correspond to a density of $1.5 \times 10^{-29}\text{g cm}^{-3}$ and our ratio becomes now

$$\left( \left| \frac{F_{\text{vac}}}{F_{\text{Newton}}} \right| \right)_{\text{cluster}} \sim 0.5$$

Certainly, for rich clusters the cosmological constant does not play any role.

(c) It is well known that the densities of superclusters are of the order of $\rho_{\text{crit}}$ or even less [22, 23]. This, in our case, is a very important observation. In this case the ‘vacuum force’ $F_{\text{vac}}$ could become important or even dominant for a galaxy (or cluster) at the edge of such a conglomeration. To be specific we quote the case for the Local Supercluster below. Using the values of the mass $M = 5 \times 10^{48}\text{g}$ and radius $R = 3.16 \times 10^{25}\text{cm}$ as given in ref. [22] for the Local Supercluster, we get,

$$\left( \left| \frac{F_{\text{vac}}}{F_{\text{Newton}}} \right| \right)_{\text{Local}} = \left( \frac{\Lambda r^3/3}{G N \rho_{\text{supercl}} V(r)} \right) = \frac{8\pi}{3} \frac{\rho_{\text{vac}}}{M_{\text{supercl}}(r)} = 0.2$$
The density of the above supercluster is $4 \times 10^{-29} \text{g cm}^{-3}$. For the numerical estimate we consider a test body again at the edge of such a supercluster and use the same value for $\rho_{\text{vac}}$ as in (4.4). We can see that the importance of the vacuum force already increased a lot for the supercluster over typical clusters (see eq. (4.4)) with a density which is an order of magnitude smaller than that of the cluster in eq. (4.4). For superclusters with even smaller densities like those quoted by the Lick Observatory survey [24], namely, $\rho_{\text{supcl}} = 2.5 \times 10^{-30} \text{g cm}^{-3}$ and $\rho_{\text{supcl}} = 3.16 \times 10^{-31} \text{g cm}^{-3}$ in ref. [25], the effect of the vacuum force would be much larger, of the order

$$\left( \frac{|F_{\text{vac}}|}{|F_{\text{Newton}}|} \right)_{\text{supercluster}} \sim 2 - 20 \quad (4.7)$$

We have seen that in the Newtonian limit there are realistic chances that the force induced by a positive cosmological constant plays a non-negligible role in astrophysics. A detailed survey of pairs of field galaxies, non-rich (diluted) clusters and superclusters would indeed help us to shed some light on the cosmological constant. Questions concerning the applicability of the virial theorem to such objects are indeed valid questions. It is not at all clear, e.g., whether superclusters represent gravitationally bound systems. We hope to come to such questions in a more systematic way in a future publication.

5 Conclusions

The Newtonian limit of Einstein’s equations requires that the gravitational potential should satisfy the strong inequality $|\Phi| \ll 1$. We have shown that in the case of a positive cosmological constant $\Lambda$ this inequality leads to the existence of an upper bound on the mass ($M_{\text{max}}$) and the distance ($R_{\text{max}}$) to be used in the Newtonian limit. Consequently, we cannot put the boundary condition for the potential at infinity (which we do if $\Lambda = 0$ and can do if $\Lambda < 0$). In the solution of the Poisson equation there will appear then a term reflecting this boundary condition at a finite distance. We found these facts curious enough to merit a note.

With the presently favoured value of $\Lambda$, $R_{\text{max}}$ and $M_{\text{max}}$ come very close to the values possessed at present by our universe. Both maximum values are independent of epoch. We find it then a strange coincidence that $R_{\text{max}} \sim R_{\text{univ}}$ and $M_{\text{max}} \sim M_{\text{univ}}$. After all, we could have been living in a universe whose mass and radius are larger or smaller than the maximum values coming from the analysis of the Newtonian limit with positive $\Lambda$. The last statement could equally well be posed as a question. A closer inspection seems worthwhile.

We have also pointed out some possible astrophysical application of the vacuum force induced by $\Lambda$. Possible candidates being inflicted by the $\Lambda$-force, range from field galaxies at a distance of more than one Mpc to clusters and superclusters. Especially for the latter it is certainly interesting to revise the virial theorem.
Acknowledgments. Discussions with N. G. Kelkar, M. Drees, J. V. Narlikar, E. Gorbar, Y. Shtanov, R. Rosenfeld, G. Matsas and J. C. Sanabria are gratefully acknowledged. This work has been partly done at Universidad Estadual Paulista supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and Programa de Apoio a Núcleos de Excelência (PRONEX).

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$$\Box h_{\mu\nu} = -16\pi G_N \tilde{S}_{\mu\nu}$$

with

$$\tilde{S}_{\mu\nu} = \tilde{T}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \tilde{T}^\lambda_{\lambda}$$

and

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{\Lambda}{8\pi G_N} g_{\mu\nu}$$

Assuming now the static case we get for $h_{00}$

$$\Delta h_{00} = -16\pi G_N \tilde{S}_{00}$$

which reduces to (2.5) after inserting in the above equation $\tilde{S}_{00} \simeq (1/2) \rho - \Lambda/(8\pi G_N)$ and $h_{00} = -2\Phi$

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