Critical quantum chaos and the one dimensional Harper model

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We study the quasiperiodic Harper’s model in order to give further support for a possible universality of the critical spectral statistics. At the mobility edge we numerically obtain a scale-invariant distribution of the bands $S$, which is closely described by a semi-Poisson $P(S) = 4S \exp(-2S)$ curve. The $\exp(-2S)$ tail appears when the mobility edge is approached from the metal while $P(S)$ is asymptotically log-normal for the insulator. The multifractal critical density of states also leads to a sub-Poisson linear number variance $\Sigma$.

During the past few years the statistical description of the energy levels in quantum systems has emerged as a new field of study, bringing together the areas of mesoscopic physics and quantum chaos. In disordered metals, the states are extended with correlated energy levels characterized by level-repulsion found in appropriate random matrix ensembles. This universal limit known as Wigner statistics also applies to quantum systems with chaotic classical dynamics. In disordered insulators, the states are localized and the energy spectra obey Poisson statistics, which is the generic limit for integrable systems. At the metal-insulator transition, another scale invariant limit applies for the spectral fluctuations, associated with the multifractal nature of the critical eigenstates, which are neither extended nor localized. However, the analysis is complicated by boundary effects which persist in the thermodynamic limit. But after taking specific boundary conditions, or averaging over a combination of them, the critical statistics resembles that obtained in certain weakly chaotic quantum systems, such as pseudo-integrable rational triangle billiards. Similar statistics also appears in certain many body spectra (e.g. two particles in a disordered chain with on-site interactions), where $P(S)$ is Wigner-like (linear) for small $S$ and Poisson-like (exponential) at large $S$, overall described by the semi-Poisson curve $P(S) = 4S \exp(-2S)$. This result is accompanied by a sub-Poisson linear number variance $\Sigma_2(E) \sim \chi E$ with $\chi < 1$. Semi-Poisson $P(S)$ and sub-Poisson $\Sigma_2(E)$ seem to be a common feature of different critical systems, raising the question of a possible universality. The semi-Poisson $P(S)$ can be exactly derived from a short range plasma model for the joint probability distribution of the energy levels, once the logarithmic pairwise interaction is truncated to nearest neighbours. Screened interactions were also found to be consistent with the non universal long range part of the spectral fluctuations in disordered metals and characterize solvable models proposed by Gaudin and Yukawa. Our aim is to show that the signatures of “critical chaos” appear at the mobility edge of the 1d Harper model, supporting the idea of possible universality.

The quasiperiodic Harper’s equation has a mobility edge in one dimension with no Wigner statistics for the metal, due to the quasi-ballistic nature of the extended states. The Poisson law is also never reached for the insulator, due to the correlated localized states. At the critical point the spectrum is multifractal with gaps $G$ of all sizes distributed according to the inverse power-law $P(G) \sim G^{-(D+1)}$ and $D \approx 0.5$, the spectral fractal dimension. The distribution of gaps was interpreted as nearest-level spacing distribution characterized by strong level-clustering $P(G \to 0) \to \infty$, which is very different from the level-repulsion $P(S \to 0) \to 0$ of the critical $P(S)$. This apparently contradicts the idea of universality of the critical fluctuations and led us to revisit the issue. In order to obtain a possibly universal behaviour, we should (i) unfold the fractal spectrum to get rid of the average variation of the density of states $\rho(E)$ by keeping the rescaled local mean level spacing $1/\rho(E)$ fixed, (ii) identify the crucial role of boundary conditions, and (iii) connect the critical level statistics to the observed multifractality.

The difficulty of (i) lies on the wildly fluctuating multifractal spectrum and was circumvented by mapping the system onto a ring threaded by a flux and averaging over all possible fluxes (or equivalently by choosing different boundary conditions in the chain ends). This could suggest to average over boundary conditions in order to achieve universality. Moreover, it leads us to argue that the distribution of bandwidths $S$ is more appropriate than the distribution of gapwidths $G$ for probing level statistics since we show that the normalized $P(S)$ at the mobility edge is close to the scale-invariant semi-Poisson curve observed in other critical systems. On the basis of these findings we suggest a certain universality of the critical fluctuations and reveal their connection to multifractality.

We analyse the Harper’s equation

$$E\psi_n = \lambda \cos(2\pi n + \nu) \psi_n + \psi_{n+1} + \psi_{n-1},$$

(1)
where \( \psi_n \) indicates the particle wavefunction of energy \( E \) on site \( n \), with \( \nu = 0 \). Eq. (1) for \( \lambda = 2 \) also describes a particle in a two-dimensional lattice in a uniform magnetic field with \( \sigma \) flux quanta per elementary plaquette. When \( \sigma \) is an irrational number the period of the potential is incommensurate with the lattice period. We can consider generic irrationals which cannot be approximated “too well” by rationals taking \( \sigma \) as the limit of successive rationals \( M/N \), so that the potential becomes commensurate with the lattice with period \( N \). Then the problem reduces to the diagonalization of \( N \times N \) matrices with a spectrum which consist of \( N \) bands separated by \( N - 1 \) gaps. This also defines a scaling procedure where the incommensurate limit \( N \to \infty \) becomes equivalent with the thermodynamic limit. The states of Eq. (1) are extended when \( \lambda < 2 \), with non-zero Lebesgue spectral measure \( 2|2 - \lambda| \) independent of \( N \). For \( \lambda > 2 \) the states are localized and the spectral measure falls off exponentially fast. The most interesting case is the critical point \( \lambda = 2 \) where the spectrum and the wavefunctions are known to have hierarchical multifractal structure \([13,19,22]\). Thouless showed that the total band measure (sum of bandwidths) is proportional to 4.664974644...\( N \)^{-1}, which defines a universal number \( 22 \). The critical spectrum is multifractal which implies that the various bandwidths \( S \) fall off with different power laws \( N^{-1/\alpha} \), where \( \alpha \) are singularity strengths having density \( f(\alpha) \). At the edges of the spectrum the bands scale as \( N^{-1/\alpha_{\min}} \), the central band as \( N^{-1/\alpha_{\max}} \) and the mean band is 4.664974644...\( N^{-1/\alpha_{0}} \), where \( \alpha_{\min} \approx 0.421, \alpha_{\max} \approx 0.547 \) and \( \alpha_{0} = 0.5 \) are the most prominent spectral dimensions \([13,19]\).

We compute the bandwidths \( S_{i} \) for long-\( N \) chains by finding the eigenvalues of the corresponding matrices for periodic and anti-periodic boundary conditions using the symmetry of the potential \( \lambda \cos(2\pi \sigma n) \) in order to diagonalize the symmetric and antisymmetric tridiagonal matrices suggested in \([22]\). We find the non-overlapping bandwidths \( S_{i} \), \( i = 1, 2, ..., N \) and the gaps \( G_{i} \), \( i = 1, 2, ..., N - 1 \) for the inverse golden mean \( \sigma = (\sqrt{5} - 1)/2 \), which is the limit of the ratios of successive Fibonacci numbers \( N \). At the mobility edge \( \lambda = 2 \) the obtained normalized bandwidth distribution \( P(S) \) is displayed in Fig. 1 for various sizes \( N \). It shows linear-level-repulsion and exponential tails, overall being described by the semi-Poisson \( P(S) = 4S \exp(-S) \) curve, in sharp contrast with the power-law \( P(G) \sim G^{-3/2} \) obtained from the distribution of gaps \([14,17]\). This suggests to examine whether the obtained \( P(S) \), which gives the distribution of the energy shifts when the boundary conditions are twisted for a commensurate potential of periodicity \( N \), can be also understood as a meaningful spacing distribution for the incommensurate limit.
fixed ($\sim 1/N$) since to a given $\phi$ correspond $N$ levels, one in every band. Moreover, each band has a smooth density of states which can be made constant independently of its form, by scaling the bandwidths $S_i$ with the fixed $N$. Thus, every bandwidth contributes a delta function to $P(S)$ centered at $S_i$, similarly to a delta function $P(S)$ at the mean spacing obtained for a smooth periodic band and the level-spacing statistics can be associated with the bandwidth distribution, essentially without unfolding. Viewed this way $P(S)$ concerns the distribution of the mean spacings, each one of them arising from the levels filling every band. The shown scale-invariant $P(S)$ described by the semi-Poisson curve (Fig. 1) can also justify the results at the metal-insulator transition the levels filling every band. The shown scale-invariant $P(S)$ was obtained by averaging over boundary conditions [3].

A remarkable connection was recently made between level-statistics and the multifractality of critical wavefunctions [5]. At the mobility edge the long range spectral fluctuations are described by a linear number of functions [18]. At the mobility edge the long range level-statistics and the multifractality of critical wavefunctions [5,6].

For up to $N = 313897$ it gives the spectral exponents $D_q = \tau(q)/(q - 1)$ with $D_{-1} = 0.5$, $D_0 = 0.498$, $D_1 \approx 0.496$, $D_2 \approx 0.493$, etc. Their Legendre transform gives the singularities $\alpha = \frac{d\tau(q)}{dq}$ and their density $f(\alpha) = q_0 - \tau(q)$ with the extreme dimensions $D_{+\infty} = \alpha_{\min}$, $D_{-\infty} = \alpha_{\max}$. The position of the maximum lies at $\alpha_0 = 0.5$ with $f(\alpha_0) = D$ (see inset of Fig. 2).

\[
\sum_{i=1}^{N} S_i^{-\tau(q)} \sim N^q.
\]

In the formalism of multifractals a spectral scaling function $\tau(q)$ is usually evaluated by box counting in energy, where an exact partitioning of boxes is suggested by the bandwidths $S_i$ since no levels fall in the gaps. The scaling dimensions are obtained by scaling the moments of the computed distribution of $S_i$ [4] via

\[
\mathcal{N}(E_{in} + E) - \mathcal{N}(E_{in}) \sim E^\alpha,
\]

where $\mathcal{N}(E)$ is the number of levels from the lowest energy up to $E$.

**FIG. 2.** The number variance at the mobility edge $\lambda = 2$. Inset: the $\alpha$-$f(\alpha)$ spectral dimensions.

**FIG. 3.** The nearest level spacing distribution function in the metallic phase $\lambda < 2$ for $N = 121393$. The continuous line is the semi-Poisson limit approached when $\lambda = 2$.

The level number variance

\[
\Sigma_2(E) = \int dE_{in} \left( \mathcal{N}(E_{in} + E) - \mathcal{N}(E_{in}) \right)^2
\]

\[
- \left( \int dE_{in} \left( \mathcal{N}(E_{in} + E) - \mathcal{N}(E_{in}) \right) \right)^2,
\]

is defined by taking energy boxes of size $E$ over all possible $E_{in}$ in the spectrum. In the box multifractal formalism the knowledge of $\alpha - f(\alpha)$ makes Eq. (4) strictly equivalent to

\[
\Sigma_2(E) = \int_{\alpha_{\min}}^{\alpha_{\max}} \alpha f(\alpha) E^{2\alpha} - \left( \int_{\alpha_{\min}}^{\alpha_{\max}} \alpha f(\alpha) E^{2\alpha} \right)^2,
\]

which is approximately linear $\Sigma_2 \approx 0.012 + 0.041 E$ as shown in Fig. 2, leading to level-compressibility $\chi \approx$
are simply $\lambda$. The computed value of $\chi$ applied to this formula gives $D_2^\psi \approx 0.918$, which can be compared with $D_2^\psi \approx 0.82$ obtained for states near the band center [21]. Since all the levels in the band correspond to critical states with variable $D_2^\psi$, we have also computed directly the average exponent for all critical states which gave the much lower value $D_2^\psi \approx 0.537$.

The level statistics of Harper’s equation for $\lambda < 2$ or $\lambda > 2$ can be simply understood from a duality argument [13]. In the metallic regime we follow $P(S)$ for increasing $\lambda$ in Fig. 3. The Poisson tail $\exp(-2S)$ is seen to emerge when approaching the critical point $\lambda = 2$. The spectral dimensions in the metallic limit are simply $D_q = 1$, for $q < 2$, and $D_q = 1/(2-2/q)$, for $q \geq 2$. For example, for $\lambda = 0$ we trivially obtain $S_j = (2\pi/N)|\sin(\pi M/N)|$, $j = 1,2,...,N$, which leads to $P(S) = \pi^{\sqrt{\pi/2}}$ shown in Fig. 3, with only one singularity $\alpha = 0.5$, $f(\alpha = 0.5) = 0$. More and more singularities develop as we approach the critical point $\lambda = 2$ where $\alpha$ lies in $(\alpha_{\min}, \alpha_{\max})$ with finite density $f(\alpha)$. The obtained critical semi-Poisson $P(S)$ gives another hint towards the fascinating problem of the analytical computation of the spectral multifractal dimensions, possibly from finite-$N$ corrections to strings in the recent formulation via Bethe-ansatz equations [22]. In the localized regime $\lambda > 2$ the bands become exponentially small and $P(S)$ asymptotically log-normal since the localized levels are spatially correlated. In the extended phase almost all $\alpha = 1$ and in the localized phase $\alpha = 0$, which lead to trivial $\Sigma_2(E) = 0$ in the two limits.

The main result of our study, which is able to treat numerically very long chains, is the appearance of a semi-Poisson $P(S) = 4S \exp(-2S)$ distribution at the mobility edge in one dimension. This associates the mobility edge with level-repulsion for small spacings, rather than level-clustering, in agreement with $3d$ critical disordered systems. At the mobility edge both the multifractal spectrum and the nearest-level statistics are seen to arise from the semi-Poisson distributed bandwidths $S_j$, whose moment scaling describes a fractal spectrum with infinite many singularities $\alpha - f(\alpha)$. A second result directly obtained from the multifractal spectrum is a sub-Poisson linear number variance $\Sigma_2(E) \sim \chi E$, in addition to the approximate validity of the proposed formula for $\chi$ in terms of the multifractal wavefunctions. In conclusion, the obtained universal critical statistics for the Harper’s model verifies numerically the “critical chaos” scenario summarized in the semi-Poisson $P(S)$ and the sub-Poisson linear number variance $\Sigma_2(E)$.

“Critical chaos” is seen to be intimately connected to spectral and wavefunction multifractality. Our study can also shed light on the average over boundary conditions in the critical statistics of disordered systems.

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