A FEW PROPERTIES OF DISORDERED CONDUCTORS

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In this brief tutorial review, I show how phase coherent properties of disordered conductors can be described in a simple and unified way. These properties include transport properties like weak-localization correction and universal conductance fluctuations, and thermodynamic properties like orbital magnetism and persistent currents. They can be related to the classical return probability for a diffusive particle. For a network with \( N \) nodes, the return probability can be related to the determinant of a \( N \times N \) "connectivity" matrix \( M \) so that the magnetization and the transport quantities can be directly written in term of \( \det M \).

1 Introduction

The goal of this short review is to show that transport and thermodynamic properties of phase coherent conductors can be described in a simple unified way in which the quantities of interest are related to the classical return probability for a diffusive particle. We consider weakly disordered conductors, for which the mean free path \( l_e \) is much larger than the distance between electrons: \( k_F l_e \gg 1 \), where \( k_F \) is the Fermi wave vector.

Let us first consider the effect of phase coherence on transport. The Drude-Boltzmann conductivity assumes that interference between electronic waves can be neglected. The structure of this classical conductivity is given by a sum of probability intensities \( \sigma_{cl} \propto \sum_j |A_j|^2 \) where \( A_j \) represents some amplitude related to a diffusion process. However, one knows that in quantum mechanics one must add amplitudes instead of intensities. Thus, the structure of the conductivity has to be \( \sigma \propto \sum_{i,j} A_i A_j^* \). Since the interference terms, of the form \( A_i A_j^* \), have random phases, they cancel in average so that the conductivity reduces to its classical value given by the diagonal terms in the sum. However, there is a class of contributions which may not cancel in average. They correspond to diffusive trajectories which form closed loops. Such a loop can be traveled in clockwise or anti-clockwise directions. If there is time reversal symmetry, both trajectories, \( j \) and its time-reversed \( j^T \), have the same action, so that they interfere constructively. As a result, in addition to the classical average conductivity, there is a correction of the form \( \Delta \sigma \propto \langle \sum_{j} A_j A_j^* \rangle \) where the sum extends over the closed trajectories. The sign of the correction is negative because the trajectories \( j \) and \( j^T \) have opposite momenta. The conductivity is thus reduced and the correction is called weak-localization correction. This is a phase coherent effect because it implies that the time reversed trajectories have the same action. This phase coherence is broken by inelastic events due to the coupling to other degrees of freedom or due to electron-electron interactions. Such coherence breakdown is temperature dependent and can be

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phenomenologically described by a phase coherence length \( L_\phi \): trajectories larger than \( L_\phi \) cannot contribute to the weak-localization corrections. The effect of a magnetic field which breaks time-reversal symmetry is to destroy this phase-coherent contribution to the conductance, leading to a negative magnetoresistance (in the absence of spin-orbit scattering) which is one of the most spectacular signature of phase coherence.

Another important signature of the coherent nature of quantum transport is the phenomenon of Universal Conductance Fluctuations. When a physical parameter is varied, such as the Fermi energy, the magnetic field or the disorder configuration, the conductance fluctuates around its average value. These fluctuations are reproducible and are the signature of the interference pattern associated to a given impurity configuration. The width of the distribution is universal and of the order of \( e^2/h \). The structure of the variance of the conductivity is

\[
\delta\sigma^2 \propto \sum_{i,j,k,l} [\langle A_i A_j^* A_k A_l^* \rangle - \langle A_i A_j^* \rangle \langle A_k A_l^* \rangle].
\]

Correlation terms imply pairs of diffusive trajectories.

These transport properties have been extensively studied in the 70’ and 80’. More recently, a wide interest emerged in the experimental studies of the equilibrium properties. Among them the search for the persistent current of an isolated mesoscopic metallic ring pierced by a magnetic Aharonov-Bohm flux \( \phi \). More generally, one is interested in the magnetization \( M \) of a phase coherent system. In the thermodynamic limit, its average value is nothing but the Landau magnetization. However the fluctuations of the magnetization depend on phase coherence. Moreover, it is known that considering electron-electron interactions leads to an additional phase coherent contribution to the magnetization which can become larger than the Landau magnetization. In the geometry of a ring, this magnetization corresponds to a current \( I \) flowing along the ring. This persistent current has been observed in single or arrays of isolated mesoscopic rings.

In this paper, I briefly review the derivation of these different transport and thermodynamic quantities. I show that they are related to the return probability for a diffusive particle. I treat a few examples and I present a formalism to calculate simply all these quantities on networks of any geometry.

2 Phase coherence and Diffusion in a disordered medium

Technically, the quantities of interest are either response functions or equilibrium quantities, which can be written in terms of products of two single particle Green functions \( G_\epsilon (r, r') \), solutions of the Schrödinger equation for a particle in a disordered potential \( V(r) \):

\[
[\epsilon + \frac{\hbar^2}{2m} \Delta - V(r)] G_\epsilon (r, r') = \delta (r - r')
\]  

By definition, the probability for a particle to evolve from one point \( r \) to another \( r' \) is also related to the product of two propagators. After disorder averaging, one can show that in the limit \( k_F l_\epsilon \gg 1 \) and for slow variations, the probability \( P(r, r', \omega) \) defined as:

\[
P(r, r', \omega) = \frac{1}{2\pi \rho_0} G^R_\epsilon (r, r') G^A_{\epsilon - \omega} (r', r)
\]  

where \( G^R \) and \( G^A \) are the retarded and advanced Green functions, is the solution of a classical diffusion equation:

\[
[-i\omega - D\Delta] P_{cl} (r, r', \omega) = \delta (r - r')
\]  

where \( D \) is the diffusion coefficient: \( D = v_F l_\epsilon / d \). \( v_F \) is the Fermi velocity and \( d \) is the space dimensionality.

\( G_\epsilon (r, r') \) describes the electronic propagation from \( r \) to \( r' \) and the probability \( P(r, r', \omega) \) is the sum of the contributions of pair of trajectories, each trajectory being characterized by an
amplitude and a phase proportional to its action. For most of these pairs, the phase difference is large so that in average, their contribution cancel. Then the probability is essentially given by the sum of the intensities corresponding to the modulus square of the contribution of trajectories. However, when \( r \approx r' \), the trajectories form closed loops. Both a trajectory and its time-reversed have the same action, so that they interfere constructively. As a result, when there is time reversal symmetry, the return probability is doubled compared to its classical value. This is a phase coherent effect because only trajectories of size smaller than the phase coherence length \( L_\phi \) will contribute to this additional contribution.

As a result, the return probability has two components, a purely classical one and an interference term which results from interferences between pairs of time-reversed trajectories. In the diagrammatic picture, they are related respectively to the diffuson and Cooperon diagrams. The interference term, \( P_{\text{int}}(r, r, \omega) \), is field dependent and, in the weak-field limit \( \omega \tau_e \ll 1 \), it is solution of the diffusion equation:

\[
[\gamma - i\omega - D(\nabla + \frac{2ieA}{\hbar c})^2]P_{\text{int}}(r, r', \omega) = \delta(r - r')
\]  

whose solution has to be taken at \( r' = r \). The scattering rate \( \gamma = 1/\tau_\phi = D/L_\phi^2 \) describes the breaking of phase coherence. \( L_\phi \) is the phase coherence length and \( \tau_\phi \) is the phase coherence time. \( \gamma \) has to be compared to the Thouless rate \( 1/\tau_D = D/L^2 \) where \( \tau_D \) is the diffusion time, typical time to diffuse through the system of linear size \( L \).

Finally, let us define the space integrated (dimensionless) return probability:

\[
P(t) = \int P(r, r, t)dr
\]

In a magnetic field, it is the sum of a classical term and an interference term:

\[
P(t, B) = P_{\text{cl}}(t) + P_{\text{int}}(t, B)
\]

and can be written as

\[
P(t) = \sum_n (e^{-E_n^{(cl)} t} + e^{-E_n^{(int)} t})
\]

where the \( E_n^{(cl)} \) and the \( E_n^{(int)} \) are the eigenvalues of the equations associated respectively to the equations (3) and (4).

\[-D\Delta_r \psi_n(r) = E_n^{(cl)} \psi_n(r) \quad \text{and} \quad -D(\nabla_r + \frac{2ieA}{\hbar c})^2 \psi_n(r) = E_n^{(int)} \psi_n(r)
\]

Note that these equations have the same structure as a Schrödinger equation, with the substitution \( \frac{\hbar}{2m} \rightarrow D \) and \( e \rightarrow 2e \). Let us now turn to the study of the physical quantities of interest and write them as functions of \( P(t) \).

3 Weak localization

The first quantum correction to the classical Drude-Boltzmann conductivity is called the weak-localization correction. Linear response theory shows that the d.c. \( T = 0K \) average conductivity can be written as:

\[
\langle \sigma \rangle = \frac{se^2\hbar^3}{2\pi m^2 V} \int d\mathbf{r}d\mathbf{r}' \langle \partial_x G^R_\epsilon(r, r')\partial_x G^A_\epsilon(r', r) \rangle
\]

where \( s = 2 \) is the spin degeneracy. Eq. (7) has a structure very similar to the return probability (2). As we have seen in the introduction, the amplitude of the correction is proportional to the
The number of loops of length $v_F t$ being proportional to the return probability $P(t)$, one deduces that the total correction is proportional to the time integrated interference part of the return probability. More precisely, it can be shown that:

$$\Delta \sigma = -\sigma \frac{\Delta}{\pi \hbar} \int_0^\infty P_{int}(t)[e^{-\gamma t} - e^{-t/\tau_e}]dt$$  \hspace{1cm} (8)$$

$\Delta$ is the interlevel spacing (for one spin direction), and $\sigma = e^2 D \rho_0$ where $\rho_0$ is the density of states (DOS) for one spin direction: $\rho_0 = 1/\Delta \Omega$, $\Omega$ being the volume of the system. The contribution of the return probability is integrated between $\tau_e$, the smallest time for diffusion, and $\tau_\phi = 1/\gamma$, the time after which the electron loses phase coherence. A magnetic field or an Aharonov-Bohm flux, by breaking the time-reversal symmetry, destroys the weak-localization correction.

4 Universal Conductance Fluctuations

An important signature of the coherent nature of quantum transport is the phenomenon of Universal Conductance Fluctuations. The averaged square of the conductance contains terms of the form $\langle G^R(r_1, r_1')G^A(r_2, r_2') \rangle$ (for clarity we omitted the gradients). After averaging, two contractions are possible: $r_1' = r_1, r_2' = r_2$ and $r_2 = r_1, r_2' = r_1'$. The first term a) is proportional to $\int \langle G^R(r_1, r_1, t)G^A(r_2, r_2, t) \rangle dt dr_1 dr_2$ where $r_1$ and $r_2$ belong to the same orbit of length $v_F t$. Therefore integration on $r_2$ gives a factor proportional to $v_F t$ and the corresponding contribution to the conductance fluctuation has the form:

$$\langle \delta g^2 \rangle = \left(\frac{\sigma}{\langle g \rangle} \right)^2 \int_0^\infty t P(t)dt \hspace{1cm} (9)$$

The second term b) is proportional to $\int P(r, r', t)P(r', r, \tau)dt d\tau dr dr'$. It can be also rewritten in the form $\int t P(t)dt$. It describes the contribution of the fluctuations of the diffusion coefficient to the conductance fluctuations. Finally, adding these two contributions, one can show that:

$$\langle \delta \sigma^2 \rangle = \sigma^2 \frac{3|\Delta|^2}{\beta \pi^2 \hbar^2} \int_0^\infty t P(t)e^{-\gamma t}dt \hspace{1cm} (10)$$

where $\beta = 1$ if there is time reversal symmetry and $\beta = 2$ in the absence of such symmetry.

The conductivity $\sigma$ is proportional to the diffusion coefficient $D$. Since the integral scales as the square of the characteristic time $\tau_D \propto 1/D$, one then concludes that the fluctuations are universal, in the sense that they do not depend on the disorder strength. Good and bad conductors have different conductivities. But the fluctuation of the conductivity when some parameter is varied (gate voltage, magnetic field or impurity configuration) is universal.

5 Orbital magnetism

I now consider the magnetic response of a disordered electron gas and more specifically the geometries of a 2D gas, of a quasi-1D ring. Since the system is disordered, the magnetic response has to be defined by its distribution. Like for the conductance, I will consider the average and the fluctuation of the magnetization.

\footnote{The lower bound of the time integrals is actually the mean collision time $\tau_e$ above which diffusion takes place.}
5.1 Average Magnetization

Non interacting particles

Let us first neglect the interactions between electrons, as we have done for the calculation of the conductance. The magnetization $M(B)$ is the derivative of the grand potential $A$ with respect to the magnetic field $B$: $M = -\frac{\partial A}{\partial B}$. Introducing the field dependent DOS for one spin direction, $\rho(\epsilon, B)$, the magnetization can be written at zero temperature as (taking the spin into account):

$$M = -2 \frac{\partial}{\partial B} \int_{-\epsilon_F}^{0} \epsilon \rho(\epsilon, B) d\epsilon$$

The origin of energies is taken at the Fermi energy. The average magnetization is thus related to the field dependence of the average density of states $\langle \rho(\epsilon, B) \rangle$. In a bulk system, this leads to the Landau diamagnetism. In the thin ring geometry where no field penetrates the conductor, the DOS only depends on the Aharonov-Bohm flux. Its average is flux independent because the flux modifies only the phase factors of the propagator which cancel in average.

Contribution due to interactions

In the Hartree-Fock approximation, the total energy $E_T$ is now

$$E_T = E_T^0 + \frac{1}{2} \sum_{i,j} \int U(r - r') |\psi_j(r')|^2 |\psi_i(r)|^2 dr dr'$$

$$- \frac{1}{2} \sum_{i,j} \delta_{\sigma_i, \sigma_j} \int U(r - r') \psi_j^*(r') \psi_j(r) \psi_i^*(r) \psi_i(r') dr dr'$$

where $E_T^0$ is the total energy in the absence of interaction. $U(r - r')$ is the interaction. The summation $\sum_{i,j}$ is on filled energy levels. $\sigma_i$ is the spin of an eigenstate $\psi_i$. We now assume that the Coulomb interaction is screened and that the states $\psi_i$ are those of the non-interacting system. This corresponds to the so-called RPA approximation.

$$U(r - r') = U \delta(r - r')$$

where $U = 4\pi e^2 / q_{TF}^2$, $q_{TF}$ being the Thomas-Fermi wave vector. For such a local interaction, the Fock term has the same structure as the Hartree term. Introducing the local density $n(r) = \sum_i |\psi_i(r)|^2$, one has finally

$$\langle M_{e-e} \rangle = -\langle \frac{\partial E_T}{\partial B} \rangle = -\frac{U}{4} \frac{\partial}{\partial B} \int \langle n^2(r) \rangle dr$$

We define the local DOS $\rho(r, \omega)$ so that $n(r) = 2 \int_{-\epsilon_F}^{0} \rho(r, \omega) d\omega$ (the factor 2 accounts for spin). The magnetization can be rewritten as

$$\langle M_{e-e} \rangle = -U \frac{\partial}{\partial B} \int \langle \rho(r, \omega_1) \rho(r, \omega_2) \rangle dr d\omega_1 d\omega_2$$

$$= -\frac{U}{2\pi^2} \frac{\partial}{\partial B} \int \langle G_{\omega_1}^{R}(r, r) G_{\omega_2}^{A}(r, r) \rangle dr d\omega_1 d\omega_2$$

The average product in the integral is nothing but the return probability defined in eq. (2). The interaction contribution to the average magnetization can thus be written as a function of field dependent part of the return probability.

$$\langle M_{e-e} \rangle = -\frac{\lambda_0}{\pi} \frac{\partial}{\partial B} \int_0^\infty P_{int}(t, B) \frac{e^{-\gamma t}}{t^2} dt$$
where $\lambda_0 = U \rho_0$ is the interaction parameter. Considering higher corrections in the Cooper channel leads to a ladder summation\footnote{In the limit $B \to 0$, one recovers the return probability for an infinite 2D system: $P(t) = S/(4\pi Dt)$}, so that $\lambda_0$ should be replaced by $\lambda(t) = \lambda_0/(1 + \lambda_0 \ln(\epsilon_F t)) = 1/\ln(T_0 t)$ where $T_0$ is defined as $T_0 = \epsilon_F e^{1/\lambda_0}$\footnote{In the limit $B \to 0$, one recovers the return probability for an infinite 2D system: $P(t) = S/(4\pi Dt)$}. We shall discuss later the contribution of this renormalization.

5.2 Typical Magnetization

We first calculate the typical magnetization $M_{typ}$, defined as $M_{typ}^2 = \langle M^2 \rangle - \langle M \rangle^2$. From eq.(11), it can be written as:

$$M_{typ}^2 = 4 \frac{\partial}{\partial B} \frac{\partial}{\partial B'} \int_{-\epsilon_F}^0 \epsilon' K(\epsilon - \epsilon', B, B') d\epsilon d\epsilon' |_{B' = B} \tag{17}$$

where $K$ is the correlation function of the DOS: $K(\epsilon - \epsilon', B, B') = \langle \rho(\epsilon, B) \rho(\epsilon', B') \rangle - \rho_0^2$. $K(\epsilon)$ has been calculated by Altshuler and Shklovskii\footnote{In the limit $B \to 0$, one recovers the return probability for an infinite 2D system: $P(t) = S/(4\pi Dt)$}. A very useful semiclassical picture has been presented by Argaman et al., which relates the form factor $\tilde{K}(t)$, the Fourier transform of $K(\epsilon)$, to the integrated return probability $P(t, r, t) = \int P(r, r, t) dr$ for a diffusive particle\footnote{In the limit $B \to 0$, one recovers the return probability for an infinite 2D system: $P(t) = S/(4\pi Dt)$}:

$$\tilde{K}(t) = \frac{\Delta^2}{4\pi^2} |t| P(|t|) \tag{18}$$

The return probability, and consequently the form factor, is the sum of a classical and an interference term:

$$P(t, B, B') = P_{cl}(t, \frac{B - B'}{2}) + P_{int}(t, \frac{B + B'}{2}) \tag{19}$$

Fourier transforming $K(\epsilon - \epsilon')$ and using the identity $\int_0^\infty e^{\epsilon t} \epsilon d\epsilon = -1/t^2$, one obtains straightforwardly

$$M_{typ}^2 = \frac{1}{2\pi^2} \int_0^{+\infty} \left[ P''_{int}(t, B) - P''_{cl}(t, 0) \right] e^{-\gamma t} \frac{e^{-\gamma t}}{\epsilon^3} dt \tag{20}$$

where $P''(t, B) = \partial^2 P(t, B)/\partial B^2$.

6 Simple examples

We have now an ensemble of quantities which are all simply written in terms of integrals of the return probability $P(t)$. I choose now a few examples where known results can be recovered straightforwardly with the use the general formula derived above.

6.1 2D electron gas

Consider an infinite 2D electron gas in a magnetic field. The solutions of the diffusion equation\footnote{In the limit $B \to 0$, one recovers the return probability for an infinite 2D system: $P(t) = S/(4\pi Dt)$} are the Landau levels $E_n(B) = (n + 1/2)4eDB/\hbar$ with degeneracy $g_n = 2eB/\hbar$ so that the return probability in a field is given by the sum\footnote{In the limit $B \to 0$, one recovers the return probability for an infinite 2D system: $P(t) = S/(4\pi Dt)$}:

$$P_{int}(t, B) = \frac{BS/\phi_0}{\sinh 4\pi BDt/\phi_0} \tag{21}$$

where $\phi_0 = h/e$ is the flux quantum. The integral (8) gives the weak-localization correction in a field\footnote{In the limit $B \to 0$, one recovers the return probability for an infinite 2D system: $P(t) = S/(4\pi Dt)$} (using the Einstein relation $\sigma = se^2D\rho_0$ and $\rho_0 = 1/(\Delta S)$):
\[
\Delta \sigma(B) = -\frac{e^2}{2\pi^2 h} \left[ \Psi\left(\frac{1}{2}\right) + \frac{\hbar}{4eDB\tau_e} \right] - \Psi\left(\frac{1}{2}\right) + \frac{\hbar}{4eDB\tau_\phi} \right] 
\]

where \( \Psi(x) \) is the digamma function.

Consider now the interaction contribution \( \langle \rangle \) to the weak-field magnetization. The low-field behavior of the return probability \( P(t, B) = P(t, 0) + \mathcal{S}\left[-\frac{2\pi Dt}{3\phi_0} B^2 + \frac{56\pi^3 D^3 e^2}{45\phi_0} B^4\right] \). The second term of the expansion gives immediately the electron-electron contribution to the susceptibility \( \chi_{ee} \). From eq. (16), one has, per unit area:

\[
\chi_{ee} = \frac{4D}{3\phi_0^2} \int_{r_e}^{r_\phi} dt \frac{1}{t \ln T_0t} = -\frac{4D}{3\phi_0^2} \ln \left[ \frac{\ln T_0r_\phi}{\ln T_0r_e} \right] \frac{2}{\pi} \chi_L k_F l_e \ln \left[ \frac{\ln T_0r_\phi}{\ln T_0r_e} \right]
\]

It is larger than the 2D Landau susceptibility \( \chi_L = -\frac{e^2}{4\pi m_e} \) by a factor \( k_F l_e \).

Using the next term of the expansion of \( P(t, B) \), one derives the variance \( \delta \chi^2 \) of the susceptibility, that one can write also in terms of the 2D Landau susceptibility:

\[
\frac{(\delta \chi^2)^{1/2}}{|\chi_L|} = \sqrt{\frac{84}{5\pi}} \frac{L_\phi}{\sqrt{S}} \frac{k_F l_e}{\phi_0}
\]

### 6.2 Persistent currents in rings

In a system where the diffusion is one-dimensional, the return probability writes \( L/\sqrt{4\pi Dt} \). In a ring geometry, there is an additional probability to reach the origin after \( 1, 2, \ldots, m \) turns around the loop. The accumulated phase is \( 4\pi m \phi \), where \( \phi = \phi / \phi_0 \), so that the total return probability writes:

\[
P_{\text{int}}(t, \phi) = \frac{L}{\sqrt{4\pi Dt}} \sum_{m=-\infty}^{\infty} e^{-m^2 L^2/4Dt} \cos(4\pi m \phi)
\]

By straightforward integration, this leads directly to the Fourier expansion of the average current \( \langle I_{e-e} \rangle \) and of the typical current \( I_{\text{typ}} \), where the current \( I \) is simply given by \( I = -\frac{\partial A}{\partial \phi} = M/S \) where \( S \) is the area of the ring. One gets, in the limit \( L_\phi \rightarrow \infty \):

\[
I_{\text{typ}}^2(\phi) = \frac{96E_c^2}{\phi_0^2} \sum_{m=1}^{\infty} \frac{\sin^2(2\pi m \phi)}{m^2}
\]

and

\[
\langle I_{e-e}(\phi) \rangle = \frac{16\lambda_0 E_c}{\phi_0} \sum_{m=1}^{\infty} \frac{\sin 4\pi m \phi}{m^2}
\]

as obtained for the first time in refs.\([20]\) and \([13]\). For discussion of these results and comparisons to experiments see for example ref.\([4]\).

### 6.3 Weak-localisation in cylinders

One of the most famous experiments showing phase coherence effect on transport, is the one performed by Sharvin and Sharvin who measured the magnetoresistance of a cylinder pierced by a magnetic flux \( \phi \). In this case the return probability \( P_{\text{int}}(t, \phi) \) is modulated by the flux through the cylinder. The time dependence of the return probability on the cylinder resembles the one on the ring (eq.\([24]\)), with a different power law:

\[
P_{\text{int}}(t, \phi) = \frac{S}{4\pi Dt} \sum_{m=-\infty}^{\infty} e^{-m^2 L^2/4Dt} \cos(4\pi m \phi)
\]
where $S$ is the area of the cylinder. By integration, one obtains immediately:

$$
\Delta \sigma(\phi) = -\frac{e^2 D}{\pi h} \left[ \ln \frac{L \phi}{l_c} + 2 \sum_{m=1}^{\infty} K_0(m \frac{L}{L \phi}) \cos(4\pi m \varphi) \right]
$$

(28)

This expression was first obtained by Altshuler, Aronov and Spivak.

6.4 Universal conductance fluctuations in a quasi-1D wire

Consider a quasi-1D wire of length $L$ connected to leads. The return probability in this case is $P(t) = \sum_q e^{-Dq^2 t}$ where the modes are quantized as $q = n\pi/L$, with $n > 0$. Eq. (10) gives, (using $\sigma = se^2 D \rho_0 = se^2 D/(\Delta L)$)

$$
\langle \delta \sigma^2 \rangle = \frac{e^2}{3} \frac{\beta \pi}{\bar{h}} \sum_{q \neq 0} \frac{1}{D^2 q^4} = \frac{12 s^2}{\beta \pi} L^2 \left( \frac{e^2}{h} \right)^2 \sum_{n>0} \frac{1}{n^4}
$$

(29)

so that the fluctuation of the conductance $G = \sigma/L$ is universal:

$$
\langle \delta G^2 \rangle = \frac{2s^2}{15\beta}
$$

7 Temperature effect

The above results which have been obtained for $T = 0K$. Thermal broadening can be taken into account straightforwardly. Let us take the example of the typical magnetization. In eq. (17), thermal functions must be inserted so that it has now the form:

$$
\int \int d\epsilon d\epsilon' K(\epsilon - \epsilon') F(\epsilon) F(\epsilon')
$$

where $F(\epsilon) = \frac{1}{\beta} \ln(1 + e^{\beta(\epsilon - \epsilon_F)})$ is the integral of the Fermi factor. By Fourier transform, this integral is simply rewritten as

$$
\int \frac{\pi^2 T^2}{(t \sinh \pi T t)^2} \tilde{K}(t) dt
$$

so that eq. (20) becomes:

$$
M_{\text{typ}}^2 = \frac{1}{2\pi^2} \int_0^{+\infty} \left[ P_{\text{int}}''(t, B) - P_{\text{cl}}''(t, 0) \right] \frac{\pi^2 T^2 t}{(t \sinh \pi T t)^2} e^{-\gamma t} dt
$$

(30)

Let us define a temperature characteristic length $L_T$ by $L_T^2 = D/T$ ($\hbar = k_B = 1$). In the limit $L_T < L\phi$, using the field expansion of $P(t, B)$ written in section 6.4, eq. (30) becomes:

$$
\delta \chi^2(T) = \frac{112\pi}{5} S \frac{D^3}{\phi_0^3} \int_0^{+\infty} \left( \frac{\pi T t}{\sinh \pi T t} \right)^2 dt
$$

(31)

where $\chi = \partial M/\partial B$. The integral is $\pi/(6T)$ so that

$$
\frac{(\delta \chi^2)^{1/2}}{|\chi L|} = \sqrt{\frac{14}{5}} \frac{L_T}{\sqrt{S}} k_F l_c
$$

as obtained in a different way by Raveh et al. Other temperature dependences of transport and thermodynamic coefficients can be obtained in the same way.
8 Diffusion on Graphs

The calculation of the above quantities can be extended to the case of any structure – called a network – made of quasi-one-dimensional wires. First, we note that the quantities of interest have all the same structure:

\[ \int t^\alpha P(t)e^{-\gamma t}dt \]  

(32)

where \( P(t) = \sum_n e^{-E_n t} \). The time integral of \( P(t) \) can be straightforwardly written in terms of a quantity called the spectral determinant \( S_d(\gamma) \):

\[ P \equiv \int_0^\infty dt P(t) = \sum_n E_n + \gamma = \frac{\partial}{\partial \gamma} \ln S_d(\gamma) \]  

(33)

where \( S_d(\gamma) \) is, within a multiplicative constant independant of \( \gamma \):

\[ S_d(\gamma) = \prod_n (\gamma + E_n) \]  

(34)

\( E_n \) being the eigenvalues of the diffusion equations (3) or 4). Using standard properties of Laplace transforms, the above time integrals can be rewritten in terms of the spectral determinant, so that the physical quantities described above read:

\[ \Delta \sigma = -\sigma \Delta \pi \frac{\partial}{\partial \gamma} \ln S_d(\gamma) \]  

(35)

\[ \langle \delta \sigma^2 \rangle = -\sigma^2 \frac{3\Delta^2}{\beta \pi^2} \frac{\partial^2}{\partial \gamma^2} \ln S_d(\gamma) \]  

(36)

\[ M_{\text{typ}}^2 = \frac{1}{2\pi^2} \int_0^\infty d\gamma_1 (\gamma - \gamma_1) \frac{\partial^2}{\partial B^2} \ln S_d(\gamma_1) \bigg|_0^B \]  

(37)

\[ \langle M_{ee} \rangle = \frac{\lambda_0}{\pi} \int_0^\infty d\gamma_1 \frac{\partial}{\partial B} \ln S_d(\gamma_1) \]  

(38)

These expressions are quite general, strictly equivalent to expressions (1) and (2). In the case of a ring or a graph geometry, the integral converges at the upper limit. For the case of a magnetic field in a bulk system, this limit should be taken as \( 1/\tau_e \) where \( \tau_e \) is the elastic time. The problem now remains to calculate the spectral determinant on graphs.

By solving the diffusion equation on each link, and then imposing Kirchoff type conditions on the nodes of the graph, the problem can be reduced to the solution of a system of \( N \) linear equations relating the eigenvalues at the \( N \) nodes. Let us introduce the \( N \times N \) matrix \( M \) given by:

\[ M_{\alpha\alpha} = \sum_\beta \coth(\eta_{\alpha\beta}) , \quad M_{\alpha\beta} = -\frac{e^{i\theta_{\alpha\beta}}}{\sinh \eta_{\alpha\beta}} \]  

(39)

The sum \( \sum_\beta \) extends to all the nodes \( \beta \) connected to the node \( \alpha \); \( l_{\alpha\beta} \) is the length of the link between \( \alpha \) and \( \beta \). \( \eta_{\alpha\beta} = l_{\alpha\beta}/L_\phi \). The off-diagonal coefficient \( M_{\alpha\beta} \) is non zero only if there is a link connecting the nodes \( \alpha \) and \( \beta \). \( \theta_{\alpha\beta} = (4\pi/\phi_0) \int_\alpha^\beta A.dl \) is the circulation of the vector potential between \( \alpha \) and \( \beta \). \( N_B \) is the number of links in the graph. It can then be shown that the integrated return probability can be rewritten as: \( P = \frac{\partial}{\partial \gamma} \ln S_d \) where the spectral determinant \( S_d \) is given by:

\[ S_d = \left( \frac{L_\phi}{L_0} \right)^{N_B-N} \prod_{(\alpha\beta)} \sinh \eta_{\alpha\beta} \det M \]  

(40)
$L_0$ is an arbitrary length independent of $\gamma$ (or $L_\phi$). We have thus transformed the spectral determinant which is an infinite product in a finite product related to $\det M$.

Let us come back to a network made of diffusive connected rings. Experimentally, the coherence length is of the order of the perimeter of one ring so that only a few harmonics of the flux dependence may be observed. It is then useful to make a perturbative expansion. We split the matrix as $M = D - N$, where $D$ is a diagonal matrix: $D_{\alpha\alpha} \approx z_{\alpha}$ to the lowest order in $L_\phi$ ($z_{\alpha}$ is the connectivity of the node $\alpha$); $N_{\alpha\beta} \approx 2 e^{-L_{\alpha\beta}/L_\phi} e^{i\theta_{\alpha\beta}}$. Expanding $\ln \det (I - D^{-1} N) = \text{Tr} \left[ \ln (I - D^{-1} N) \right]$, we have:

$$\ln \det M = \ln \det D - \sum_{n \geq 1} \frac{1}{n} \text{Tr} \left[ (D^{-1} N)^n \right]$$  \hspace{1cm} (41)

We call “loop” $l$, a set of $n$ nodes linked by $n$ wires in a closed loop. The length $L_l$ of a loop $l$ is the sum of the lengths of the $n(l)$ links. The flux dependent part of $\ln S$ can be expanded as:

$$\ln S = -2 \sum_{\{l\}} \frac{2}{z_1} \ldots \frac{2}{z_{n(l)}} e^{-L_l/L_\phi} \cos(4\pi\phi_l/\phi_0)$$  \hspace{1cm} (42)

$\phi_l$ is the flux enclosed by the loop $l$.

For example, we consider the different configurations of connected rings shown on Table. 8. The first harmonics of the total magnetization, to the first order in $\lambda_0$ is:

$$\langle M_{ee} \rangle = 2 G \frac{\lambda_0 e D}{\pi^2} \left( L/L_\phi + 1 \right) e^{-L/L_\phi}$$  \hspace{1cm} (43)

where $G$ is shown in Table. 8.

9 Mapping

We now wish to emphasize an interesting correspondence between the HF magnetization of a phase coherent interacting diffusive system and the grand canonical magnetization $M_0$ of the corresponding non-interacting clean system. The latter can also be written in term of a spectral determinant. The grand canonical magnetization $M_0$ is given quite generally by:

$$M_0 = - \frac{\partial \Omega}{\partial B} = - \frac{\partial}{\partial B} \int_{\epsilon_-}^{\epsilon_F} d\epsilon N(\epsilon)$$  \hspace{1cm} (44)

where the integrated DOS is

$$N(\epsilon) = - \frac{1}{\pi} \text{Im} \sum_{\epsilon_\mu} \ln(|\epsilon_\mu - \epsilon_+|) = - \frac{1}{\pi} \text{Im} \ln S(\epsilon_+)$$  \hspace{1cm} (45)
where $\epsilon_+ = \epsilon + i0$, $S(\epsilon) = \prod_{\epsilon, \mu} b_{\mu}(\epsilon_{\mu} - \epsilon) = S_d(\gamma = -\epsilon)$. where $\epsilon_{\mu}$ are the eigenvalues of the Schrödinger equation. For a clean system these eigenvalues are the same as those of the diffusion equation, with the substitutions $D \to \hbar/(2m)$ and $2e \to e$.

Comparing eqs. (44, 45) with eq. (38), we can now formally relate $M_0$ and the HF magnetization $\langle M_{ee} \rangle$ of the same diffusive system:

$$M_0 = -\lim_{\lambda_0 \to 0} \frac{1}{\lambda_0} \text{Im}[\langle M_{ee} \rangle(-\epsilon_F - i0)]$$

(46)

As a simple illustration, consider the orbital magnetic susceptibility of an infinite disordered plane. For a disordered conductor, it is given by eq. (23). After replacing $\gamma$ by $-\epsilon_F - i0$, taking the imaginary part of the logarithm and replacing $D$ and $2e$, we recover the Landau susceptibility for the clean system: $\chi_0 = -e^2/(24\pi m)$.

10 Conclusion

In conclusion, we have shown how to relate phase coherent transport and thermodynamic properties to the return probability for a diffusive particule. It is then possible to calculate straightforwardly these quantities by simple integrals of this return probability in simple geometries. For networks made of diffusive wires, we have developed a formalism which relates directly the persistent current, and the transport properties to the determinant of a matrix describing the connectivity of the graph. From a loop expansion of this determinant, simple predictions for the persistent current in any geometry can now be compared with forthcoming experiments on connected and disconnected rings. We have also found a correspondence between the phase coherent contribution to the orbital magnetism of a disordered interacting system and the orbital response of the corresponding clean non-interacting system.

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10. If the number of particle is fixed in each ring (so that $\epsilon_F$ has to be flux dependent), there is indeed a small contribution to the average persistent current, called the "canonical contribution". It is smaller that the contribution of the interactions discussed in the next paragraph. For a review see ref. 11.

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