Braiding and Tangling the Chessboard Complex

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Abstract

We describe a series of complexes that relate to the braid groups as the matching complexes relate to the symmetric groups. A modified construction applies as well to other complexes based on edge sets in graphs. We show that our constructions will yield Cohen-Macauley complexes provided the underlying complexes are Cohen-Macauley.

Finally, we discuss a related series of complexes to provide some positive evidence that the braided Houghton groups $H_{m}^{br}$, introduced by F. Degenhardt, are of type $F_{m-1}$ but not of type $F_{m}$.

1 Preliminaries

A space $X$ is called $d$-connected if $\pi_i(X) = 0$ for $0 \leq i \leq d$. It is called $(-1)$-connected, if $X$ is non-empty. A space is called $d$-spherical if it of dimension $d$ and $(d-1)$-connected. Note that a $d$-spherical complex is homotopy equivalent to a (possibly infinite and possibly empty) wedge of $d$-spheres. A CW-complex is spherical if it is $d$-spherical for some dimension $d$.

The CW-complexes we are concerned with in this paper are mostly $\Delta$-complexes [Hat01]. These can be viewed as piece-wise Euclidean cell complexes whose cells have the shapes of regular simplices. The metric information, however, is not really essential. You might want to think of them just as very nice CW-complexes whose cells look like simplices. Note that every simplicial complex is a $\Delta$-complex.

Links in $\Delta$-complexes, in general, do not equal the boundaries of stars: A bigon is a perfectly valid $\Delta$-complex, and each of its vertices will have a link that consists of precisely two points. However, if a $\Delta$-complex happens to be a simplicial complex then the two notions of links coincide.

A poset $P$ is called $d$-connected or $d$-spherical if its geometric realization enjoys said property. For talking about posets, we find a topological language convenient: For any element $\alpha \in P$, we call

- $\overline{\alpha} := \{ \beta \in P \mid \beta \preceq \alpha \}$ the closure of $\alpha$,
- $\partial(\alpha) := \{ \beta \in P \mid \beta \prec \alpha \}$ the boundary of $\alpha$,
• \(\text{St}(\alpha) := \{\beta \in P \mid \alpha \preceq \beta\}\) the star of \(\alpha\), and

• \(\text{Lk}(\alpha) := \{\beta \in P \mid \alpha \prec \beta\}\) the link of \(\alpha\). The intersection

• \((\alpha, \beta) := \text{Lk}(\alpha) \cap \partial(\beta) = \{\xi \in P \mid \alpha \prec \xi \prec \beta\}\) is the open interval from \(\alpha\) to \(\beta\).

The geometric realization

• \(|P|\) is the simplicial complex of finite \(\prec\)-chains in \(P\). The dimension

• \(\text{dim}(P)\) of a poset \(P\) is the dimension of its geometric realization. If the dimension of \(P\) is finite, a maximum length chain in \(P\) has length \(\text{dim}(P)\). The height

• \(h(\alpha)\) of an element \(\alpha\) is the dimension of its closure.

This terminology is, of course, inspired by the poset of cells in a regular CW-complex where \(\prec\) is given by the face relation. We mention that people who view posets from a more algebraic angle might prefer a different terminology: The closure of an element \(\alpha\) is often referred to as the principal order ideal generated by \(\alpha\) and the star of \(\alpha\) is often called the principle filter generated by \(\alpha\). In this note, however, a topological terminology seems to be more appropriate.

We follow Quillen’s influential paper \cite{Quil78} and call a simplicial complex \(K\) Cohen-Macauley if it is spherical and every simplex \(\sigma\) has a link of dimension \(\text{dim}(K) - \text{dim}(\sigma) - 1\) that is spherical, as well. A poset \(P\) is Cohen-Macauley if its geometric realization (i.e., the associated simplicial complex of chains in \(P\)) is Cohen-Macauley. Quillen observes \cite[Proposition 8.6]{Quil78} that \(P\) is Cohen-Macauley if and only if it is spherical and all links, boundaries, and open intervals in \(P\) are spherical, too.\(^1\)

**Example 1 (The Solomon-Tits Theorem).** Every spherical building (i.e., a building with finite Weil group) is a spherical simplicial complex. Since all links in spherical buildings are spherical buildings, we see that spherical buildings are Cohen-Macauley.

**Example 2 (\cite[Theorem 12.4]{Quil78}).** Let \(p\) be a prime number and assume that the field \(k\) has characteristic \(\neq p\) and contains a \(p\)th root of unity. Then the poset \(A_p(\text{GL}_n(k))\) of non-trivial elementary Abelian subgroups of \(\text{GL}_n(k)\) is Cohen-Macauley of dimension \(n - 1\).

We call a \(\Delta\)-complex Cohen-Macauley if its associated poset of simplices is Cohen-Macauley. Equivalently, a \(\Delta\)-complex is Cohen-Macauley if its barycentric subdivision is Cohen-Macauley. Note that the barycentric subdivision of a Cohen-Macauley simplicial complex is Cohen-Macauley, too. Thus for \(\Delta\)-complexes that are already simplicial, the two notions of being Cohen-Macauley coincide.

\(^1\)It is necessary to point out that those posets are often called homotopy Cohen-Macauley in the literature.
Observation 3. We call a $\Delta$-complex strict if every closed simplex has an injective attaching map, i.e., no faces of an individual simplex are identified. In this case, the associated poset locally looks like the poset of a simplicial complex: the boundary of any element is isomorphic to the poset of strict subsets of a finite set. It follows that a strict $\Delta$-complex is Cohen-Macauley if it is spherical and has spherical links only. q.e.d.

All complexes discussed below are strict $\Delta$-complexes.

2 The Chessboard Complex and its Braided Version

The $n \times m$-chessboard complex $\text{CBC}_n^m$ is the simplicial complex whose vertex set is the set of squares of an $n \times m$ chessboard and whose simplices are configurations of non-threatening rooks on said chessboard. Thus, formally, this simplicial complex is the collection of those subsets $\sigma \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\}$ such that the projections $\pi_i (i = 1, 2)$ restrict to injective maps on $\sigma$: we have at most one rook in each row and each column of the chessboard.

The complexes $\text{CBC}_n^m$ have been studied intensively. In particular, a good deal is known about their connectivity properties:

Citation 4 ([BLVZ94]). Put $\nu := \min \left(n, m, \left\lfloor \frac{n+m+1}{3} \right\rfloor \right)$. Then the chessboard complex $\text{CBC}_n^m$ is $(\nu - 2)$-connected. In fact, the $(\nu - 1)$-skeleton of $\text{CBC}_n^m$ is Cohen-Macauley.

In particular, if $\min (n, m) \leq \left\lfloor \frac{n+m+1}{3} \right\rfloor$ then the complex $\text{CBC}_n^m$ is Cohen-Macauley.

There is another description of $\text{CBC}_n^m$ in terms of matchings in the complete $(n, m)$-bipartite graph. Recall that a matching in a graph is a subgraph that consists of disjoint edges, i.e., every vertex is contained in at most one edge of the subgraph. Table 1 illustrates how a non-threatening configuration of rooks and a matching in the complete bipartite graph represent the same subset of $\{1, \ldots, n\} \times \{1, \ldots, m\}$.
The matching picture suggest the following construction: Embed the vertex set of the \((n, m)\)-bipartite graph into the boundary of a cube so that the \(n\) blue vertices lie in the bottom square and the \(m\) red vertices lie in the top square. A partial braid is a braid running vertically through the cube all of whose strands connect a red to a blue vertex (see table 2). Of course, two braids are equal if one can be deformed into the other by an ambient homotopy fixing the boundary of the cube pointwise.

The set \(B^m_n\) of partial braids carries a natural poset structure: The face relation \(\prec\) is defined by deleting strands. We call the poset thus defined the braided chessboard poset. This set also is a strict \(\Delta\)-complex in an obvious way: a partial braid on \(d + 1\) strands is a \(d\)-simplex. The poset \(B^m_n\) is just the poset of cells in this complex, and we will silently identify the braided chessboard poset and the braided chessboard complex. Note that the braided chessboard complex \(B^m_n\) is not a simplicial complex: all cells have the shape of a simplex, but such a simplex is not determined uniquely by its set of vertices.

We have a natural projection

\[
\pi : B^m_n \to CBP^m_n
\]

from the braided chessboard poset to the poset \(CBP^m_n\) of simplices in \(CBC^m_n\) which is given by viewing the braid as a matching. This projection is a height-preserving morphism of posets. Table 3 illustrates that the face relation and the projection are compatible.

We remark that the braid groups \(B_n\) and \(B_m\) act from opposite sides on the braided chessboard complex \(B^m_n\) just as the symmetric groups \(Perm_n\) and \(Perm_m\) act on the chessboard complex.

**Observation 5.** A very useful property of chessboard complexes is that links in chessboard complexes are themselves chessboard complexes of smaller size: the link
Table 3: Face relation and projection

of a simplex of dimension $d$ in $\mathrm{CBC}_n^m$ is isomorphic to the chessboard complex $\mathrm{CBC}_{n-d-1}^m$. q.e.d.

Unfortunately, this does not hold for the restricted class of braided chessboard complexes that we have defined so far. To make our arguments amenable to induction, we therefore generalize our construction slightly: Let $\Theta$ be a graph embedded into the cube avoiding the red and blue vertices. A partial braid relative to $\Theta$ is a braid running vertically through the cube not meeting a small, closed regular neighborhood of $\Theta$ (see table 4). We consider relative partial braids equal if we can deform one into the other by an ambient isotopy that leaves the boundary of the cube and the regular neighborhood of $\Theta$ fixed pointwise. By $\mathcal{B}_n^m(\Theta)$, we denote the poset ($\Delta$-complex) of partial braids relative to $\Theta$. Note that we still have a natural projection

$$\pi : \mathcal{B}_n^m(\Theta) \to \mathrm{CBP}_n^m.$$

Now, we can describe links rather easily:

**Observation 6.** Let $\sigma$ be a simplex of dimension $d$ in $\mathcal{B}_n^m(\Theta)$. Then $\sigma$ is itself realized as a graph embedded in the cube. Its link is isomorphic to $\mathcal{B}_{n-d-1}^{m-d-1}(\sigma \cup \Theta)$. q.e.d.

Our goal is to understand the connectivity of the posets $\mathcal{B}_n^m(\Theta)$ and, in order to derive their connectivity properties, we will first study the fibers of the projection $\pi : \mathcal{B}_n^m(\Theta) \to \mathrm{CBP}_n^m$.

**Proposition 7.** Let $\tau$ be a simplex of dimension $d$ in the chessboard complex $\mathrm{CBC}_n^m$. Then the closed fiber over $\tau$

$$\mathcal{F}(\tau) := \pi^{-1}(\tau)$$

is Cohen-Macauley of dimension $(d - 1)$. 5
The proof is by induction. The case $d = 0$ just states that the fiber over a vertex is non-empty and discrete. Both claims are clearly true: any graph consisting of a single edge can be lifted, and any two such lifts are not joined by higher dimensional material. In the proofs of the following two lemmas, we will assume that the proposition holds for all simplices of dimension strictly less than $d$ and all graphs $\Theta$. In particular, we will assume the proposition for all strict faces of the simplex $\tau$.

**Lemma 8.** Let $\sigma$ be a strict face of the simplex $\tau$ in the chessboard complex $\text{CBC}_{m}^{n}$. Then the inclusion $\mathfrak{F}(\sigma) \hookrightarrow \mathfrak{F}(\tau)$ is trivial in homotopy.

**Proof.** Fix a sphere $S$ inside $\mathfrak{F}(\sigma)$. We have to show that this sphere can be contracted inside $\mathfrak{F}(\tau)$.

First note that, by compactness, the sphere $S$ involves only finitely many cells in $\mathfrak{F}(\sigma)$. Each of these cells is represented by some partial braid in $\mathcal{B}_{m}^{n}(\Theta)$ that lies above $\sigma$. Let $v$ be a vertex in $\tau - \sigma$. This vertex represents an edge in the matching $\tau$ connecting a red and a blue vertex. In the cube, we can find a strand $s$ connecting this red vertex to this blue vertex, a strand that does not braid with any of the partial braids used in the sphere $S$: there is only a finite number of them and we can always work around finitely many partial braids by staying sufficiently close to the boundary of the cube. The strand $s$, therefore, can be added to all the partial braids in a way that is compatible with face relations. Thus, this strand $s$ represents a vertex in $\mathfrak{F}(\tau)$ that serves as a cone point for a contraction of the sphere $S$. \textbf{q.e.d.}

**Remark 9.** It is not really important that the strand $s$ does not braid with any of the finitely many given partial braids used to build $S$. This is just the easiest way to ensure that $s$ braids with all the given braids in a consistent way: Since braids allow for deformation, a common face to two partial braids can look very different in its cofaces. We have to add $s$ in such a way that face relations are preserved.
We now turn to links inside the closed fiber $\mathcal{F}(\tau)$. The slogan is: links in a fiber are fibers in a link:

**Observation 10.** Let $v$ be a vertex in $\tau$. Note that the fiber over $v$ is a discrete set: every element is one way of embedding a single strand in the complement of $\Theta$. Fix a strand $s$ representing a vertex in the fiber $\mathcal{F}(v) \subset \mathcal{F}(\tau)$. The projection $\pi : \mathcal{B}^m_n(\Theta) \to \mathrm{CBP}_n^m$ is a strictly monotonic poset-map (i.e., it does not collapse cells). It restricts to the link of $s$ as follows:

\[
\begin{align*}
\mathrm{Lk}_{\mathcal{B}^m_n(\Theta)}(s) &= \mathcal{B}^{m-1}_{n-1}(\Theta \cup s) \\
\mathrm{Lk}_{\mathrm{CBP}_n^m}(v) &= \mathrm{CBP}^{m-1}_{n-1}
\end{align*}
\]

Moreover, the link of $s$ inside $\mathcal{F}(\tau)$ is the part in the link that maps to $\tau - \{v\}$. Thus:

\[
\mathrm{Lk}_{\mathcal{F}(\tau)}(s) = \mathcal{F}_{\pi_{\Theta \cup s}}(\tau - \{v\}).
\]

**Lemma 11.** Let $\sigma$ be a codimension 1 face of the simplex $\tau$ in the chessboard complex $\mathrm{CBC}_n^m$. Then the inclusion $\mathcal{F}(\sigma) \hookrightarrow \mathcal{F}(\tau)$ induces epimorphisms $\pi_i(\mathcal{F}(\sigma)) \twoheadrightarrow \pi_i(\mathcal{F}(\tau))$ in homotopy for $i < \dim(\tau)$.

**Proof.** Let $v$ be the vertex opposite to $\sigma$ in $\tau$ and let $s$ be a strand representing a vertex in $\mathcal{F}(v)$. We have just observed that the relative link $\mathrm{Lk}_{\mathcal{F}(\tau)}(s)$ is the fiber above $\sigma$ in the complex $\mathcal{B}^{m-1}_{n-1}(\Theta \cup s)$. Since $\dim(\sigma) < \dim(\tau)$ we can apply Proposition 7 by induction. It follows that the relative link of $s$ is $(\dim(\tau) - 2)$-connected. Thus, for $i \leq \dim(\tau) - 1$, every $i$-sphere that passes through $s$ can be homotoped off the vertex $s$. We can do this simultaneously for all vertices above $v$ and push any $i$-sphere into the fiber $\mathcal{F}(\sigma)$. Therefore, this fiber carries all of $\pi_i(\mathcal{F}(\tau))$. \hfill \textbf{q.e.d.}

**Remark 12.** A more formal proof can be based on combinatorial Morse theory for piecewise Euclidean complexes: Let $f : \tau \to [0, 1]$ be the affine map sending $v$ to 1 and $\sigma$ to 0. The composition $f \circ \pi$ is a Morse function on $\mathcal{F}(\tau)$ in the sense of [BeBr97]. The preceding argument establishes that the descending links are $(\dim(\tau) - 2)$-connected. Now the lemma follows from [BeBr97, Lemma 2.5 and Corollary 2.6].

**Proof of Proposition 7 (finish).** The preceding two lemmas state that the map

\[
\pi_i(\mathcal{F}(\sigma)) \twoheadrightarrow \pi_i(\mathcal{F}(\tau))
\]

is trivial and onto for $i < \dim(\tau)$. Thus we know that the fiber $\mathcal{F}(\tau)$ is $(d-1)$-connected. Thus fibers are spherical.

Invoking Observation 10 again, we conclude that all links of cells in $\mathcal{F}(\tau)$ are spherical, too. Since $\mathcal{F}(\tau)$ is a strict $\Delta$-complex, we infer that it is Cohen-Macaulay by Observation 8. \hfill \textbf{q.e.d.}
Now that we understand the fibers of the projection, we can apply the tools provided by D. Quillen:

Citation 13 ([Quil78, Corollary 9.7]). Suppose \( f : P \to Q \) is a strictly increasing morphism of posets. Assume that \( Q \) is Cohen-Macauley of dimension \( d \) and that for every \( \beta \in Q \), the preimage \( f^{-1}(\beta) = \{ \alpha \in P \mid f(\alpha) \preceq \beta \} \) of the closure \( \overline{\beta} \) is Cohen-Macauley of dimension \( h(\beta) \). Then \( P \) is Cohen-Macauley of dimension \( d \).

Theorem 14. Put \( \nu := \min(n, m, \left\lfloor \frac{n+m+1}{3} \right\rfloor) \). Then, for any graph \( \Theta \) in the cube, the braided chessboard complex \( \mathcal{B}_n^m(\Theta) \) has a Cohen-Macauley \((\nu - 1)\)-skeleton. In particular, the complex is \((\nu - 2)\)-connected; and if \( \min(n, m) \leq \left\lfloor \frac{n+m+1}{3} \right\rfloor \) then \( \mathcal{B}_n^m(\Theta) \) is Cohen-Macauley.

Proof. Since the projection \( \pi : \mathcal{B}_n^m(\Theta) \to \text{CBP}_n^m \) does not crush cells, the \((\nu - 1)\)-skeleton of the braided chessboard complex is the preimage of the \((\nu - 1)\)-skeleton of the chessboard complex \( \text{CBC}_n^m \), which is Cohen-Macauley by Citation 13.

By Proposition 7, the projection \( \pi \) satisfies the hypotheses of Citation 13. The theorem follows immediately.

3 Complexes Based on Collections of Edges

Let \( \mathcal{V} \) be a finite set, and let \( \mathcal{G} \) be a family of graphs sharing \( \mathcal{V} \) as their vertex sets. Suppose that \( \mathcal{G} \) is subgraph-closed, i.e., if \( \Gamma \in \mathcal{G} \) then every subgraph of \( \Gamma \) also belongs to \( \mathcal{G} \). The graph poset

- \( P(\mathcal{G}) \) induced by \( \mathcal{G} \) is the poset of non-empty graphs in \( \mathcal{G} \) ordered by inclusion.
- \( C(\mathcal{G}) \) induced by \( \mathcal{G} \) is the simplicial complex whose \( d \)-simplices are those graphs in \( \mathcal{G} \) containing precisely \( d + 1 \) edges. The graph poset is the poset of cells in this complex.

Example 15 (The Complex of Not i-Connected Graphs). Consider the family of non-i-connected simplicial graphs over the vertex set \( \mathcal{V} \). The corresponding graph complexes have been studied in [BBLSW99]. In particular, it is shown that the complex of not 2-connected graphs on \( \mathcal{V} \) is homotopy equivalent to a wedge of \( (|\mathcal{V}| - 2)! \) spheres of dimension \( 2|\mathcal{V}| - 5 \).

Example 16 (The Forest Complex). Let \( \Gamma \) be a fixed graph over the vertex set \( \mathcal{V} \) with \( k \) components, and let \( \mathcal{G} \) be the family of forests in \( \Gamma \), i.e., circle-free subgraphs of \( \Gamma \). Then the complex

\[ \mathcal{F}(\Gamma) := C(\mathcal{G}) \]

is the complex of forest in \( \Gamma \).
Proposition 17. The forest complex $\mathcal{F}(\Gamma)$ is Cohen-Macaulay of dimension $|\mathcal{V}| - k - 1$.

This was proved independently by several people. The earliest source, I am aware of is the thesis of J.S. Provan [Prov77]. The forest complex is the independence complex of a matroid and hence shellable by [Bjor92 Theorem 7.3.3]. For those who are scared by matroids and shellability, we include a down to earth proof base on the version given in [Vogt90 Proposition 2.2].

Proof. Every spanning forest of $\Gamma$ contains precisely $|\mathcal{V}| - k$ edges. Thus each maximal simplex in $\mathcal{F}(\Gamma)$ has dimension $|\mathcal{V}| - k - 1$.

A simplex $\sigma$ in $\mathcal{F}(\Gamma)$ is a sub-forest of $\Gamma$. Collapsing this sub-forest yields a new graph $\Gamma/\sigma$ that has the same number of connected components. However, each edge in the forest $\sigma$ connects two vertices, whence crushing this edge reduces the number of vertices by 1. It follows that $\mathcal{F}(\Gamma/\sigma)$ has dimension $\dim(\mathcal{F}(\Gamma)) - \dim(\sigma) - 1$. The link of $\sigma$ in $\mathcal{F}(\Gamma)$ is isomorphic to $\mathcal{F}(\Gamma/\sigma)$. Thus the complex will be Cohen-Macaulay, provided that $\mathcal{F}(\Gamma)$ is spherical for all graphs $\Gamma$. Since we already established the dimension of $\mathcal{F}(\Gamma)$, it remains to show that $\mathcal{F}(\Gamma)$ is $(|\mathcal{V}| - k - 2)$-connected.

Let $e$ represent a vertex in $\mathcal{F}(\Gamma)$, i.e, $e$ is a non-loop edge in $\Gamma$. If $e$ is a separating edge, then it serves as a cone point in $\mathcal{F}(\Gamma)$, in which case the forest complex is contractible and a fortiori $(|\mathcal{V}| - k - 2)$-connected.

If $e$ is non-separating, we can remove $e$ without increasing the number of components. Thus the graph $\Gamma - e$ has $|\mathcal{V}|$ vertices and $k$ components. By induction on the number of edges, we may assume that $\mathcal{F}(\Gamma - e)$ is spherical of dimension $(|\mathcal{V}| - k - 2)$. From this subcomplex, we obtain $\mathcal{F}(\Gamma)$ by coning of the link of $e$, which is isomorphic to $\mathcal{F}(\Gamma/e)$, which is spherical of dimension $(|\mathcal{V}| - 1 - k - 2)$ again by induction on the number of edges. It follows that $\mathcal{F}(\Gamma)$ is spherical of dimension $(|\mathcal{V}| - k - 2)$. q.e.d.

Another family of examples arises as follows: Fix a graph $\Gamma$ with vertex set $\mathcal{V}$, and let $\mathcal{M}(\Gamma)$ be the family of subgraphs satisfying the condition that each vertex is contained in at most one edge. Such subgraphs are called matchings in $\Gamma$. We denote the graph poset associated to the family of matchings by $\text{MP}(\Gamma)$ and its graph complex by $\text{MC}(\Gamma)$.

Example 18 (The Chessboard Complex). If $K_{n,m}$ is the complete bipartite graph on $m$ red and $n$ blue vertices, non-empty, edge-disjoint subgraphs correspond to partial matchings between the set of red vertices and the set of blue vertices. Thus, we recover the chessboard complex:

$$\text{MC}(K_{n,m}) = \text{CBC}_{m}^{n}.$$ 

We mention that $\text{CBC}_{m}^{n}$ is sometimes called the matching complex. We prefer, however to use this name for the following:
Table 5: A graph, a lift, and a face

Example 19 (The Matching Complex). Let $K_n$ be the complete graph on $n$ vertices. The elements of $MP(K_n)$ are collections of disjoint edges. The corresponding graph complex $MC(K_n)$ is called the matching complex.

Some connectivity properties of these complexes are known: Put

$$\nu := \left\lfloor \frac{n + 1}{3} \right\rfloor.$$ 

Then the $(\nu - 1)$-skeleton of $MC(K_n)$ is Cohen-Macauley [BLVZ94, Corollary 4.2].

Finally, we will have a use for the most basic subgraph-closed family:

Example 20 (The Simplex). Let $S(\Gamma)$ be the family of subgraphs of a given graph $\Gamma$, then $C(S(\Gamma))$ is nothing but a big simplex whose vertices are the edges in $\Gamma$. A single simplex is Cohen-Macauley.

4 The Tangling Construction

In the case of the chessboard complex, the underlying graph was bipartite. Thus, we could put the two kinds of vertices into opposite faces, top and bottom, of the cube and require that strands pass through the cube vertically. In general, we cannot arrange for this. Thus we will replace braids by tangles to make the construction applicable to the complexes discussed above.

Let $\mathcal{G}$ be a family of graphs over the vertex set $\mathcal{V}$ and assume $\mathcal{G}$ is subgraph-closed. Let $B$ be a closed 3-ball. Chose an embedding of $\mathcal{V}$ into its boundary sphere $S := \partial(B) \subset B$. A lift of a graph $\Gamma \in \mathcal{G}$ is an embedding of $\Gamma$ into $B$ that (a) extends the embedding of $\mathcal{V}$ and (b) maps interior points of edges to interior of $B$ such that (c) each edges lifts to an un-knotted curve in $B$ (see table 5). We will also call these lifts tangles. The term lift will be preferred when we want to stress the relation to the underlying graph in $\mathcal{G}$, whereas the term tangle emphasizes the
geometric structure of the embedded graph upstairs in the 3-ball. We consider two
tangles equal if there is an ambient homotopy between the two fixing the boundary
sphere of \( B \) (and, therefore, in particular the set \( \mathcal{V} \)) pointwise. The set of equivalence
classes of tangles forms the tangle poset

- \( P^t(G) \) where the face relation is given by deletion of strands: a strand is the
  lift of an edge. The tangle complex

- \( C^t(G) \) is the \( \Delta \)-complex whose \( d \)-simplices are indexed by tangles with \( d + 1 \)
  strands.

As we did with the braided chessboard complex, we will generalize this construc-
tion by allowing that a given embedded graph \( \Theta \) be removed from the 3-ball from
the start. This way, we will make sure that the class of \( \Delta \)-complexes we define is
closed with respect to taking links: Let \( \Theta \) be a graph embedded in \( B \). An \( \Theta \)-lift
of \( \Gamma \) (or a \( \Theta \)-tangle) is an embedding of \( \Gamma \) into \( B \) satisfying the conditions (a) to (c)
above such that the interiors of edges do not meet a fixed regular neighborhood of
\( \Theta \). Again, two \( \Theta \)-tangles are equivalent if there is an ambient homotopy from one to
the other fixing the boundary sphere and the regular neighborhood of \( \Theta \) pointwise.
Equivalence classes of \( \Theta \)-tangle form the poset

- \( P^t_\Theta(G) \), the face relation being deletion of strands.

**Observation 21.** Let \( \Gamma \in G \) be a graph with \( d + 1 \) edges, i.e., an element of height \( d \)
in the graph poset \( P(G) \). The link \( \text{Lk}_{P(G)}(\Gamma) \) consists of those graphs in \( G \) that contain
\( \Gamma \) as a proper subgraph. Removing the edges from \( \Gamma \) yields an isomorphism

\[ \text{Lk}_{P(G)}(\Gamma) \cong P(G - \Gamma) \]

where \( G - \Gamma \) is the family of those graphs in \( G \) that do not share any edges with \( \Gamma \).

Similarly, for any \( \Theta \)-lift \( \tilde{\Gamma} \) of \( \Gamma \), we have an isomorphism

\[ \text{Lk}_{P^t_\Theta(G)}(\tilde{\Gamma}) \cong P^t_{\Theta,\tilde{\Gamma}}(G - \Gamma). \]

The isomorphism is not given by removing strands in \( \tilde{\Gamma} \), but by “freezing” them.

q.e.d.

Our goal is to prove:

**Theorem 22.** If \( P(G) \) is Cohen-Macaulay, then so is \( P^t_\Theta(G) \).

This applies in particular to the forest complex, the chess board complex, and to
some skeleton in the matching complex.

**Remark 23.** The condition (c) above requiring strands to be un-knotted can be
dropped without affecting the theorem. The proof presented here applies to the
altered construction without change.
Note that there are canonical, strictly monotonic, height-preserving projections

\[ P^{tn}(G) \to P(G) \]
\[ C^{tn}(G) \to C(G) \]
defined by “ignoring the entanglement”.

We will closely follow the argument given for the braided chessboard complex. Thus, we have to understand fibers over closed simplices.

**Lemma 24.** Let \( \Gamma_0 \subset \Gamma_1 \) be a strict inclusion of graphs. The induced inclusion

\[ C^{tn}_\Theta(S(\Gamma_0)) \hookrightarrow C^{tn}_\Theta(S(\Gamma_1)) \]

is trivial in homotopy, i.e., any sphere in \( C^{tn}_\Theta(S(\Gamma_0)) \) can be crushed inside \( C^{tn}_\Theta(S(\Gamma_1)) \).

**Proof.** Assume first that \( \Theta \) is empty. W.l.o.g., we can assume that \( \Gamma_0 \) is obtained from \( \Gamma_1 \) by removing precisely one edge \( e \) connecting, say, the vertices \( v \) and \( w \). We choose a path \( p \) inside the boundary sphere \( \partial(B) \) connecting \( v \) and \( w \). Any sphere in \( C^{tn}(S(\Gamma_0)) \) involves only finitely many simplices. By compactness of \( \Theta \)-lifts, we can push the path \( p \) slightly into the interior of \( B \) without meeting any strands used by the sphere. After pushing it into the ball, the strand \( p \) represents a vertex in \( C^{tn}(S(\Gamma_1)) \) that allows us to cone off the sphere. Thus all homotopy of \( C^{tn}(S(\Gamma_0)) \) dies in \( C^{tn}(S(\Gamma_1)) \).

The argument works as well for non-empty \( \Theta \) – we just observe that in pushing \( p \) we can also avoid the neighborhood of \( \Theta \) since it is compact. \( \text{q.e.d.} \)

**Proposition 25.** Let \( \Gamma \) be a graph with \( d + 1 \) edges. Then the poset \( C^{tn}_\Theta(S(\Gamma)) \) is spherical of dimension \( d \).

**Proof.** We use induction on \( d \). The case \( d = 0 \) is obvious. So let \( \tilde{e} \) be a strand representing vertex in \( C^{tn}_\Theta(S(\Gamma)) \), and let \( e \) be the edge in \( \Gamma \) corresponding to \( \tilde{e} \). Note that the complex \( C^{tn}_\Theta(S(\Gamma - e)) \) is a subcomplex of \( C^{tn}_\Theta(S(\Gamma)) \). Moreover, observe that \( C^{tn}_\Theta(S(\Gamma)) \) is obtained from the subcomplex \( C^{tn}_\Theta(S(\Gamma - e)) \) by coning off

\[ \text{Lk}(\tilde{e}) \cong C^{tn}_{\Theta \cup \tilde{e}}(S(\Gamma) - e) = C^{tn}_{\Theta \cup \tilde{e}}(S(\Gamma - e)) \]

along the canonical map

\[ C^{tn}_{\Theta \cup \tilde{e}}(S(\Gamma - e)) \to C^{tn}_\Theta(S(\Gamma - e)) \]

given by deleting \( \tilde{e} \). Both, the link \( \text{Lk}(\tilde{e}) \) and the subcomplex \( C^{tn}_\Theta(S(\Gamma - e)) \) are \((d - 1)\)-spherical by induction. Thus, \( C^{tn}_\Theta(S(\Gamma)) \) is obtained from an \((d - 1)\)-spherical complex by conning off an \((d - 1)\)-spherical space. This process does not alter homotopy groups in dimensions \( \leq d - 2 \), and it can only kill but not introduce homotopy in dimension \( d - 1 \). Thus \( \pi_i(C^{tn}_\Theta(S(\Gamma))) = 0 \) for all \( i < d - 1 \), and

\[ \pi_{d-1}(C^{tn}_\Theta(S(\Gamma - e))) \to \pi_{d-1}(C^{tn}_\Theta(S(\Gamma))) \]

is onto. However, this map is trivial by Lemma 24. \( \text{q.e.d.} \)
Table 6: A pinched braid with two regular strands

**Corollary 26.** Let $\Gamma$ be a graph with $d+1$ edges, then $C_\Theta^{\text{tn}}(S(\Gamma))$ is Cohen-Macauley of dimension $d$.

**Proof.** By Observation 21 the link of a simplex $\sigma$ in $C_\Theta^{\text{tn}}(S(\Gamma))$ is isomorphic to $C_{\Theta,\sigma}^{\text{tn}}(S(\Gamma'))$ where $\Gamma'$ is obtained from $\Gamma$ by deleting all edges that have lifts in $\sigma$. Thus, the preceding proposition applies to those links as well, and all links in $C_\Theta^{\text{tn}}(S(\Gamma))$ are spherical. Thus the strict $\Delta$-complex $C_\Theta^{\text{tn}}(S(\Gamma))$ is Cohen-Macauley. q.e.d.

**Proof of Theorem 22.** The claim follows at once from Quillen’s result 13 and Corollary 26 because the closed fiber of the projection $\pi_\Theta : P_\Theta^{\text{tn}}(G) \to P(G)$ above the simplex represented by $\Gamma \in G$ is isomorphic to $P_\Theta^{\text{tn}}(S(\Gamma))$. q.e.d.

5 **My Motivation: The Complex of Pinched Braids**

Finally, I would like to present another $\Delta$-complex that also projects onto the chessboard complex. It is my struggle with this complex that motivated the study of the (seemingly more natural) constructions discussed above.

Fix two numbers $n$ and $m$. Embed a row of $n$ blue vertices in the bottom face of the cube labelled from left to right by $n, n - 1, \ldots, 2, 1$. Embed a row of $m$ red vertices in the top face and label them from left to right by $1, 2, \ldots, m - 1, m$. In front of the red row add a black vertex in the top face. We think of this vertex as being labelled by $\infty$. A pinched braid is a collection of disjoint strands running vertically through the interior of the cube connecting top vertices to bottom vertices such that the following conditions are met:
1. Every bottom vertex is hit by precisely one strand.
2. Every red vertex is hit by at most one strand.
3. At least one red vertex is hit by a strand.

Strands hitting red vertices are called regular. The strands issuing from $\infty$ are called singular. The conditions imply, that generically, there will be several singular strands, i.e., the black vertex will issue multiple strands. We consider two pinched braids equal if they can be deformed into one another by ambient homotopies fixing the boundary of the cube pointwise. Table 6 shows a pinched braid with two regular strands.

The set $X^m_n$ of all pinched braids forms a poset with the following face relation: every pinched braid has one immediate face for each regular strand obtained by sliding the red end of the strand along a straight line to the black vertex, thus turning the regular strand into a singular strand. Table 7 shows a $\prec$-chain of length two.

The poset $X^m_n$ thus defined is the poset of cells in a $\Delta$-complex whose vertex set is the set of pinched braids that have precisely one regular strand. Table 8 shows a 2-simplex in this complex with a complete labelling of all its faces by pinched braids.

We want to provide some evidence for the following:

**Conjecture 27.** $X^m_n$ is $(m - 1)$-spherical provided $n$ is large enough.

**Remark 28.** Deleting all singular strands defines a height-preserving poset map

$$X^m_n \to B^m_n.$$  

Thus, a natural idea is to use Quillen’s result. This thought led me to consider the braided chessboard complex in the first place. Unfortunately, the fibers of this projection seem to be difficult to analyze.

**Remark 29.** F. Degenhardt [Dege00] introduced the series $H^\text{br}_m$ of braided Houghton groups, and proved:
Table 8: A triangle in $X_6^4$ and its geometric realization
1. $H^\text{br}_1$ is not finitely generated.

2. $H^\text{br}_2$ is finitely generated but not finitely presented.

3. $H^\text{br}_3$ is finitely presented but not of type $\text{FP}_3$.

4. $H^\text{br}_m$ is of type $\text{F}_3$ for $m \geq 4$.

He conjectures that $H^\text{br}_m$ is of type $\text{F}_{m-1}$ but not of type $\text{F}_m$. In an attempt to prove his conjecture, I constructed a contractible cube complex upon which $H^\text{br}_m$ acts with cell stabilizers of type $\text{F}_\infty$. The complexes $X^m_n$ occur as relative links in a cocompact filtration by invariant subspaces. Thus, by standard arguments, Conjecture 27 implies Degenhardt’s conjecture on the finiteness properties of $H^\text{br}_m$.

We remark that the Houghton groups $H_m$ are groups of certain infinite permutations, and braided Houghton groups are groups of certain infinite braids. Ignoring the braiding defines a group homomorphism $H^\text{br}_m \to H_m$. K. Brown [Brow89, Section 5] derived the finiteness properties of $H_m$ from a filtration where the chessboard complexes $\text{CBC}^m_n$ arose as relative links.

We will show that Conjecture 27 holds “in the limit”: Adding an unused red vertex $m+1$ to the right of the top row induces an inclusion

$$X^m_n \subset X^{m+1}_n.$$ 

Adding a blue vertex $n+1$ to the left of the bottom row, we define an embedding

$$X^m_n \subset X^{m}_{n+1}$$

as follows: We fix a path in the boundary of the cube from the black vertex to the new blue vertex. For any pinched braid in $X^m_n$ we define its image by pushing the boundary path into the cube, thereby creating a singular strand to the new blue vertex. This process is compatible with the face relation in $X^m_n$ and, therefore, defines a poset morphism.

Put

$$X^\infty_m := \bigcup_{n=1}^{\infty} X^m_n$$

and

$$X^\infty := \bigcup_{m=1}^{\infty} X^\infty_m = \bigcup_{m,n \in \mathbb{N}} X^m_n.$$ 

**Theorem 30.** $X^m_\infty$ is $(m - 1)$-spherical.
A vertex $v \in X^\infty$ involves precisely one regular strand. Let

$$\text{top}(v)$$

be the label of its top slot and let

$$\text{bot}(v)$$

be the label of its bottom slot. Extending affinely to simplices, we define two height functions

$$\text{top} : X^\infty \to \mathbb{N}$$

and

$$\text{bot} : X^\infty \to \mathbb{N}.$$  

Since there are no horizontal edges, these height functions are Morse functions as defined in [BeBr97]. Note that $X^m_\infty$ is the sublevel set

$$\{x \in X^\infty | \text{top}(x) \leq m\}.$$  

**Observation 31.** Consider a sphere in $X^m_n \subset X^{m+1}_{n+1}$. In all its simplices, we can slide the top end of the singular strand based at the bottom vertex $n+1$ to the top slot $m+1$. The regular strand thus created serves defines the same vertex in all simplices of the given sphere and, therefore, serves as a cone point from which the whole sphere can be contracted. Thus, the inclusion

$$X^m_n \hookrightarrow X^{m+1}_{n+1}$$

is trivial in homotopy.  

**Observation 32.** Since any sphere in $X^\infty_\infty$ involves only finitely many cells, the argument just given also implies that the inclusion

$$X^m_\infty \hookrightarrow X^{m+1}_\infty$$

is trivial in homotopy.

In particular, all homotopy groups of $X^\infty_\infty$ vanish, i.e., $X^\infty_\infty$ is contractible.  

**q.e.d.**

We need to generalize Theorem 30 a little to make it amenable to an inductive argument. Let $X^m_n(k)$ be the poset of pinched braids with $n$ blue bottom vertices, $m$ red top vertices, one $\infty$-slot in front of the top row, and $k$ green fixed disjoint vertical strands connecting $k$ additional pairs of vertices. These green strands are not involved in the definition of the face relation, they stay put. This generalization now describes a class of complexes closed with respect to taking links:

**Observation 33.** The link of a vertex in $X^{m+1}_{n+1}(k)$ is isomorphic to $X^m_n(k+1)$.  

**q.e.d.**
Note that our previous observation carries over to the more general setting:

**Observation 34.** The inclusion

\[ X^m_\infty(k) \hookrightarrow X^{m+1}_\infty(k) \]

is trivial in homotopy. \[\textbf{q.e.d.}\]

The following includes Theorem 30:

**Theorem 35.** The map

\[ \pi_i(X^m_\infty(k)) \rightarrow \pi_i(X^{m+1}_\infty(k)) \]

induced by the inclusion is an isomorphism for \( i < m - 2 \) and onto for \( i = m - 2 \). In particular, the space \( X^m_\infty(k) \) is \( (m - 2) \)-connected in view of Observation 34.

**Proof.** This is combinatorial Morse theory and induction on \( m \): Consider the height function

\[ \text{bot} : X^\infty_\infty(k) \rightarrow \mathbb{N}. \]

The descending links of vertices of height \( m \) are isomorphic to \( X^{m-1}_\infty(k+1) \). This complex is \( (m - 3) \)-connected by induction. Thus the statement follows from [BeBr97, Lemma 2.5 and Corollary.6]. \[\textbf{q.e.d.}\]

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