NUMERICAL METHODS PRESERVING MULTIPLE HAMILTONIANS FOR STOCHASTIC POISSON SYSTEMS

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Abstract. In this paper, we propose a class of numerical schemes for stochastic Poisson systems with multiple invariant Hamiltonians. The method is based on the average vector field discrete gradient and an orthogonal projection technique. The proposed schemes preserve all the invariant Hamiltonians of the stochastic Poisson systems simultaneously, with possibility of achieving high convergence orders in the meantime. We also prove that our numerical schemes preserve the Casimir functions of the systems under certain conditions. Numerical experiments verify the theoretical results and illustrate the effectiveness of our schemes.

1. Introduction. A Poisson system is a system of autonomous differential equations in the form of

\[ dy = B(y) \nabla H(y) dt. \]

(1)

Here \( y \in \mathbb{R}^d \), \( H(y) \) is a smooth function of \( y \), and the structural matrix \( B(y) = (b_{ij}(y)) \in \mathbb{R}^{d \times d} \) is a smooth matrix-valued function of \( y \) which satisfies

\[ b_{ij}(y) = -b_{ji}(y), \]

\[ \sum_{l=1}^{d} \left( \frac{\partial b_{ij}(y)}{\partial y_l} b_{lk}(y) + \frac{\partial b_{jk}(y)}{\partial y_l} b_{li}(y) + \frac{\partial b_{ki}(y)}{\partial y_l} b_{lj}(y) \right) = 0, \]

(2)

for all \( i, j, k \in \{1, 2, \ldots, d\} \).

When \( d = 2n \) (\( n \in \mathbb{N} \)) and \( B(y) \equiv J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \), where \( I_n \) is an \( n \)-dimensional identity matrix, the Poisson system (1) is reduced to a Hamiltonian system which has been studied extensively in literature. What is noticeable about Poisson systems is that they can be both even and odd dimensional systems, which makes them more suitable for practical applications.

Poisson systems are well-known in physics and practical applications ([8, 11, 12, 13]). The history of Poisson systems dates back to 1888 when Sophus Lie.
discovered the Poisson structure in [22]. Up to now, such systems find applications in a vast variety of fields such as rigid bodies, quantum mechanics, satellite orbits, magnetization fluid dynamics, and so on ([31]).

A number of numerical methods for Poisson systems (1) have been developed during the last decades (see [10]). Among them, the so called structure-preserving numerical methods have drawn special attentions to researchers. For instance, in [5], the authors constructed a class of Hamiltonian (or energy) preserving numerical methods, since the Hamiltonian $H(y)$ is an invariant of the Poisson system (1), due to

$$dH(y) = \nabla H(y)^T dy = \nabla H(y)^T B(y) \nabla H(y) dt = 0. \quad (3)$$

Relative work can also be found in [1, 3, 24, 27, 28, 30], where the main techniques used for energy-preservation include the discrete gradient methods, the Hamiltonian BVMs methods ([1]), the Runge–Kutta Munthe–Kaas type methods, the exponentially-fitted integrators, etc. As was shown by numerous theoretical and numerical analysis, structure-preserving methods are superior to other general-purpose methods, especially in long time simulations (see e.g. [7, 10, 29] and references therein).

In this paper, we consider the following stochastic Poisson system in the Stratonovich stochastic differential equation (SDE) form ([15])

$$dy = B(y)(\nabla H^0(y) dt + \sum_{i=1}^{s} \nabla H^i(y) \circ dW^i(t)), \quad y(0) = y_0, \quad (4)$$

where $y \in \mathbb{R}^d$, $t \in [0, T]$, $B(y)$ is a smooth skew-symmetric matrix-valued function as in (1) with properties (2), the Hamiltonians $H^i(y) : \mathbb{R}^d \to \mathbb{R}^d$ ($i = 0, \ldots, s$) are smooth functions of $y$, $(W^1, \ldots, W^s)$ is an $s$-dimensional standard Wiener process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the normalization conditions, and the functions $B(y)$, $H^i(y)$ ($i = 0, \ldots, s$), as well as $y_0$ satisfy the conditions guaranteeing the almost sure existence and uniqueness of the solution of the system.

As in the deterministic case, we are interested in the Hamiltonians-preserving properties of the stochastic Poisson system (4). In the deterministic case, this is obvious as shown in (3). For the stochastic Poisson system, the application of the Stratonovich chain rule gives us

$$dH^j(y) = \nabla H^j(y)^T dy = \nabla H^j(y)^T B(y) \left( \nabla H^0(y) dt + \sum_{i=1}^{s} \nabla H^i(y) \circ dW^i(t) \right)$$

$$=: \{H^j, H^0\}(y) dt + \sum_{i=1}^{s} \{H^j, H^i\}(y) \circ dW^i(t) \quad (6)$$

for $j = 0, \cdots, s$, where the Poisson bracket $\{F, G\}(y)$ of two smooth functions $F(y)$ and $G(y)$ is defined as

$$\{F, G\}(y) := \nabla F(y)^T B(y) \nabla G(y).$$

Thus, for stochastic Poisson systems, $H^j(y)$, $j \in \{0, 1, \ldots, s\}$ are all invariants if and only if
\{H^j, H^i\}(y) \equiv 0, \quad \forall i, j = 0, \ldots, s. \tag{7}

Besides, for considering system (4) with \(s + 1\) invariant Hamiltonians satisfying the \(s + 1\) equations \(H^i(y) \equiv H^i(y_0) \quad (j = 0, \ldots, s)\), we need to assume
\[d > s + 1,\]  
\tag{8}
since otherwise it could be overdetermined, or without non-trivial solution.

The goal of this paper is constructing numerical approximations of system (4) where all the \(H^i(y) \quad (j = 0, \ldots, s)\) are invariants, such that \(H^i(y)\) are preserved when the input \(y\) is replaced by its numerical approximations. When \(B(y) \equiv J\), and \(H^i(y) = \sigma_i H^0(y) \quad (i = 1, \ldots, s)\), where \(\sigma_i\) are constants, namely, a special class of stochastic Hamiltonian systems, energy-preserving \((H^0\text{-preserving})\) numerical methods were investigated in e.g. \([6, 21, 26]\), where the energy-preserving technique in \([6]\) permits also certain special skew-symmetric constant matrices beyond \(J\) in the diffusion coefficients. In case of general skew-symmetric \(B(y)\) with property (2), instead of the particular \(J\), and also when \(H^i(y) = \sigma_i H^0(y) \quad (i = 1, \ldots, s)\), \([2]\) proposed a class of numerical methods that can preserve the Hamiltonian \(H^0(y)\) which is an invariant of the system, and proved the root mean-square convergence order 1 of the schemes under certain conditions. \([19]\) also constructed \(H^0\text{-preserving}\) numerical methods for stochastic Poisson systems of the form as in \([2]\), which can achieve arbitrarily high convergence orders of the schemes based on explicit parametric stochastic Runge–Kutta methods.

In this study, we use the discrete gradient to construct a class of Hamiltonians-preserving numerical schemes. Discrete gradient methods were first used for deterministic cases in \([9, 23]\). In recent years, they have also become effective tools for designing numerical schemes preserving conserved quantities of stochastic differential equations (see e.g. \([16, 20]\) and references therein). In this paper, based on discrete gradient and a projection technique, we propose a class of numerical methods for the stochastic Poisson system (4) where all the \(H^i(y) \quad (j = 0, 1, \ldots, s)\) are invariants of the system, namely (7) (with (8)) holds. In the sequel, we call the proposed methods the Hamiltonians-preserving methods. In addition to Hamiltonians-preservation, the proposed methods can also enjoy high convergence orders. The underlying idea is to apply the discrete gradient together with an orthogonal projection to modify existing high order schemes (e.g. those in \([18, 17, 25]\)) to obtain high order \(H^i\text{-preserving}\) schemes for \(i = 0, \ldots, s\). This is the stochastic extension of the deterministic methods introduced in \([4]\).

The content of the paper is organized as follows. In Section 2, we construct the Hamiltonians-preserving schemes and prove that they preserve all the invariant Hamiltonians of the stochastic Poisson system (4) with properties (7)–(8). A sufficient condition for the schemes to preserve the Casimir functions is given as well. Section 3 analyzes the root mean-square convergence orders of the Hamiltonians-preserving schemes. Numerical experiments are performed in Section 4 to verify the theoretical analysis and illustrate the numerical behavior of the proposed schemes, followed by a few concluding remarks in Section 5.

2. The Hamiltonians-preserving schemes. In this section, we first construct the Hamiltonians-preserving schemes for the stochastic Poisson system (4) satisfying (7)–(8) and then prove that they preserve all the invariant Hamiltonians. In the last part of this section, we give a sufficient condition for their preservation of the Casimir functions

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2.1. Construction of the schemes. Before constructing our schemes, we introduce the concept of discrete gradients.

**Definition 2.1.** [23] For a differentiable function \( H = H(y) : \mathbb{R}^d \to \mathbb{R}^d, \nabla H(v, u) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) is said to be a discrete gradient of \( H(y) \) if it is continuous and satisfies:

\[
\begin{align*}
\nabla H(v, u)^T (u - v) &= H(u) - H(v), \\
\nabla H(u, u) &= \nabla H(u).
\end{align*}
\]

Furthermore, if \( \nabla H(v, u) = \nabla H(u, v), \forall \, u, v \in \mathbb{R}^d, \nabla H \) is called a symmetric discrete gradient (SDG). Several different kinds of discrete gradients are given in ([23]). The symmetric discrete gradient adopted in our numerical schemes is the averaged vector field (AVF) gradient defined as

\[
\nabla H_{\text{AVF}}(v, u) = \int_0^1 \nabla H(\xi u + (1 - \xi)v) d\xi, \quad \forall \, u, v \in \mathbb{R}^d.
\]

Given a numerical scheme for the stochastic Poisson system (4) with properties (7)–(8)

\[
y_{n+1} = y_n + \phi(y_n, y_{n+1}, h, \mathcal{W}(\Delta_n, \omega)), \quad n = 0, 1, 2, \ldots,
\]

where \( \mathcal{W} := (W^1, \ldots, W^s), \Delta_n := [t_n, t_{n+1}], \omega \in \Omega, \) and \( \mathcal{W}(\Delta_n, \omega) := \{ \mathcal{W}(t, \omega), t \in [t_n, t_{n+1}] \} \). For convenience, we denote \( \phi(y_n, y_{n+1}, h, \mathcal{W}(\Delta_n, \omega)) =: \phi_{n, \omega} \) in certain places in the sequel. Moreover, we choose \( h \) such that the numerical solution from (11) exists and is unique. We say (11) preserves all Hamiltonians, if \( \forall n \in \mathbb{N}, \, i = 0, 1, \ldots, s \)

\[
H^i(y_{n+1}) - H^i(y_n) = 0 \quad \text{almost surely.}
\]

According to the definition of the discrete gradients (9), (12) is equivalent to

\[
\nabla H_{\text{AVF}}(y_n, y_{n+1})^T (y_{n+1} - y_n) = \nabla H_{\text{AVF}}(y_n, y_{n+1})^T \phi(y_n, y_{n+1}, h, \mathcal{W}(\Delta_n, \omega)) = 0, \quad \forall n \in \mathbb{N}, \, i = 0, 1, \ldots, s \quad (13)
\]

almost surely, that is, for Hamiltonians-preservation, \( \phi(y_n, y_{n+1}, h, \mathcal{W}(\Delta_n, \omega)) \) is expected to be perpendicular to the vector space:

\[
\text{span}\{\nabla H_{\text{AVF}}^i(y_n, y_{n+1}), \ldots, \nabla H_{\text{AVF}}^s(y_n, y_{n+1})\} =: \overline{\nabla}_{t_{n+1}},
\]

almost surely. However, for a general-purpose numerical approximation, the increment function \( \phi \) of the method may fail to satisfy (13), which means that the method can not preserve the invariant Hamiltonians.

Our idea of constructing Hamiltonians-preserving schemes is to modify the given numerical scheme (11) by projecting at each step its increment vector \( \phi_{n, \omega} \) onto the space perpendicular to \( \overline{\nabla}_{t_{n+1}} \), i.e. \( \overline{\nabla}^\perp_{t_{n+1}} \). To this end, we first perform a reduced QR-decomposition of the \( \mathbb{R}^d \times \mathbb{R}^{s+1} \) matrix whose \( i \)-th column vector is \( \nabla H_{\text{AVF}}^i(y_n, y_{n+1}) \) \( (i = 1, \ldots, s + 1) \). These vectors span the space \( \overline{\nabla}_{t_{n+1}} \). Then the matrix \( I - Q(y_n, y_{n+1})Q(y_n, y_{n+1})^T \) becomes a projector from \( \mathbb{R}^d \) to \( \overline{\nabla}^\perp_{t_{n+1}} \), as can be seen from the proof of Theorem 2.2.

Now we present our algorithm based on \( \nabla H_{\text{AVF}}(v, u) \) and the projector through the following procedure.

- Define \( Y(v, u) \in \mathbb{R}^d \times \mathbb{R}^{s+1} \) whose \( i \)-th column is

\[
\nabla H_{\text{AVF}}^{i-1}(v, u), \quad i = 1, 1, \ldots, s + 1.
\]
Theorem 2.2. For stochastic Poisson system specify the reference scheme \((16)\) in the following analysis.

1) Compute a reduced QR-decomposition (note that \(d > s + 1\))

\[
Y(v, u) = Q(v, u)R(v, u),
\]

(14)

where \(Q(v, u) \in \mathbb{R}^d \times \mathbb{R}^{s+1}\) satisfies \((v, u)^TQ(v, u) = I\), and \(R(v, u) \in \mathbb{R}^{s+1} \times \mathbb{R}^{s+1}\) is an upper triangular matrix. Then let

\[
P(v, u) = I - Q(v, u)Q(v, u)^T.
\]

(15)

2) The reduced Hamiltonians-preserving methods for SPSS are well-defined if the algorithm (17) is specified in the following analysis.

3) (16) can be explicit or implicit method. If implicit, \(\Delta_{n, i}\) should be first solved by fixed-point iterations and then substituted into (17).

Remark 1.

1) The accuracy of (16), which we call the reference scheme, will determine the accuracy of the final algorithm. The modification via the projector \((17)\) makes the final convergence order of the overall scheme \((16)\)–\((17)\) will be analyzed in Section 3.

2) The reduced QR-decomposition can be achieved by using the MATLAB command \(QR(\cdot, 0)\).

2.2. Preservation of Hamiltonians. Now we verify the Hamiltonians-preserving property of the scheme \((16)\)–\((17)\). It is worth mentioning that, we do not need to specify the reference scheme \((16)\) in the following analysis.

Theorem 2.2. For stochastic Poisson system \((4)\) with properties \((7)\)–\((8)\), the numerical scheme \((16)\)–\((17)\) preserves all the Hamiltonians \(H^i(y)\) \((i = 0, \ldots, s)\), namely \(H^i(y_n) = \text{Const} \) almost surely, for all \(n \in \mathbb{N}, i = 0, 1, \ldots, s\).

Proof. In what follows, the appearing equalities between random variables are in the ‘almost surely’ sense.

We need to show that

\[
H^i(Y_{n+1}^n) - H^i(Y_{n}^n) = 0, \quad \forall n \in \mathbb{N}, i = 0, \ldots, s.
\]

According to the definition of the discrete gradient,

\[
H^i(Y_{n+1}^n) - H^i(Y_{n}^n) = \nabla H^i_{AVF}(y_{n+1}, y_{n})^T(y_{n+1} - y_{n}), \quad i = 0, 1, \ldots, s.
\]

(18)

Since \(\nabla H^i_{AVF}(y_{n}, y_{n+1})\) is \((i+1)\)-th column of \(Y(y_{n}, y_{n+1})\), \((18)\) is equivalent to

\[
Y(y_{n}, y_{n+1})^T(y_{n+1} - y_{n}) = 0.
\]

According to the scheme \((17)\), we have

\[
Y(y_{n}, y_{n+1})^T(y_{n+1} - y_{n})
= Y(y_{n}, y_{n+1})^T P(y_{n}, y_{n+1}) \phi(y_{n}, y_{n+1}, h, W(\Delta_n, \omega))
= (Q(y_{n}, y_{n+1}) R(y_{n}, y_{n+1})^T (I - Q(y_{n}, y_{n+1}) Q(y_{n}, y_{n+1})^T) \phi(y_{n}, y_{n+1}, h, W(\Delta_n, \omega))
= R(y_{n}, y_{n+1})^T Q(y_{n}, y_{n+1})^T (I - Q(y_{n}, y_{n+1}) Q(y_{n}, y_{n+1})^T) \phi(y_{n}, y_{n+1}, h, W(\Delta_n, \omega))
= 0.
\]
The last equality is due to $Q(y_n, y_{n+1})^T Q(y_n, y_{n+1}) = I$. The proof is complete. □

It can be seen that all the Hamiltonians $H^i(y)$ ($i = 0, \ldots, s$) are preserved simultaneously by the method (16)–(17), no matter what $\phi$ is.

### 2.3. Preservation of Casimir functions.

**Definition 2.3.** ([10, 15]) A function $C = C(y)$ is called a Casimir function of the stochastic Poisson system (4), if

$$\nabla C(y)^T B(y) = 0 \quad \text{for all } y.$$  

Along solutions of (4) we have $C(y(t)) = Const$ almost surely, since

$$dC(y(t)) = \nabla C(y(t))^T B(y(t))(\nabla H^0(y(t))dt + \sum_{i=1}^{s} \nabla H^i(y(t)) \circ dW^i(t)) = 0.$$  

That is to say, a Casimir function of the stochastic Poisson system (4) is an invariant of the system. Next we give a sufficient condition under which our scheme (16)–(17) preserves the Casimir functions of the system (4) with (7)–(8).

**Theorem 2.4.** Assume that $C(y)$ is a Casimir function of the stochastic Poisson system (4) with (7)–(8). If the AVF discrete gradient of $C(y)$ satisfies

$$\nabla C_{AVF}(y_n, y_{n+1}) \in \text{span}\{\nabla H^0_{AVF}(y_n, y_{n+1}), \ldots, \nabla H^s_{AVF}(y_n, y_{n+1})\}, \forall n \in \mathbb{N},$$  

then the numerical scheme (16)–(17) preserves the Casimir function $C(y)$, i.e.,

$$C(y_n) = Const, \quad \text{almost surely, } \forall n \in \mathbb{N}.$$  

**Proof.** By the proof of Theorem 2.2 we see that $\forall n \in \mathbb{N}$, almost surely,

$$\nabla H^i_{AVF}(y_n, y_{n+1})^T (y_{n+1} - y_n) = 0, \quad i = 0, 1, \ldots, s,$$

hence, by (19),

$$C(y_{n+1}) - C(y_n) = \nabla C_{AVF}(y_n, y_{n+1})^T (y_{n+1} - y_n) = 0,$$

almost surely. □

**Remark 2.** Here we give an example where (19) is satisfied and thus the scheme (16)–(17) preserves the Casimir functions. Consider a stochastic Poisson system (4) with (7)–(8) that has quadratic Hamiltonians, i.e. $H^i(y) = y^T C^i y/2$, where $C^i$ are symmetric matrices ($i = 0, 1, \ldots, s$). Then it is easy to see that

$$\nabla H^i_{AVF}(y_n, y_{n+1}) = C^i(y_n + y_{n+1})/2.$$  

If the system has a linear Casimir function $C(y) = c^T y$, then $\nabla C_{AVF}(y_n, y_{n+1}) = c$. If there exists a vector $\alpha$ such that $\alpha^T \in \text{span}\{C^0, \ldots, C^s\}$, and if $\alpha^T (y_n + y_{n+1}) \neq 0 \ (\forall n \geq 0)$ a.s., then the numerical scheme (16)–(17) preserves the Casimir function $C(y)$, since there exist $a^0, \ldots, a^s$ such that

$$\alpha^T = \sum_{i=0}^{s} a^i C^i,$$

$$\alpha^T (y_n + y_{n+1}) = \sum_{i=0}^{s} a^i C^i (y_n + y_{n+1})/2,$$

$$c = \sum_{i=0}^{s} \alpha^T (y_n + y_{n+1}) C^i (y_n + y_{n+1})/2.$$
If the system has a quadratic Casimir function $C(y) = y^T D y / 2$, where $D$ is a symmetric matrix, then $\nabla C_{\text{AVF}}(y_n, y_{n+1}) = D(y_n + y_{n+1}) / 2$. If $D \in \text{span}\{C^0, \ldots, C^n\}$, it is then obvious that the numerical scheme (16)–(17) preserves the Casimir function $C(y)$ according to Theorem 2.4.

3. Root mean-square convergence order. For the error analysis, we assume $y_0$ and the coefficients of the stochastic Poisson system (4) with (7)–(8) satisfy the conditions:

\begin{enumerate}
  \item $y_0$ is $\mathcal{F}_0$-measurable with $E|y_0|^2 < \infty$, \hfill (20)
  \item there exists $D > 0$, such that $\forall x, y \in \mathbb{R}^d$,
  \begin{equation}
  |a(x) - a(y)| + \sum_{i=1}^s |a_i(x) - a_i(y)| + |b_i(x) - b_i(y)| \leq D |x - y|,
\end{equation}
  \item where $a(y) := B(y) \nabla H^0(y)$, $b_i(y) := B(y) \nabla H^1(y)$, $a_i(y) := \frac{1}{2} \frac{\partial b_i(y)}{\partial y} b_i(y)$, \hfill (21)
\end{enumerate}

such that the following fundamental theorem on mean-square convergence ([25]) can be applied.

**Theorem 3.1.** ([25]) Given a $d$-dimensional stochastic differential equation system

\begin{equation}
\begin{aligned}
dZ &= a(t, Z)dt + \sum_{i=1}^s b^i(t, Z)dW^i(t), \\
Z(t_0) &= Z_0,
\end{aligned}
\end{equation}

where $t \in [t_0, t_0 + T]$, $Z_0$ is $\mathcal{F}_{t_0}$-measurable with $E|Z_0|^2 < \infty$. Assume that for any $t \in [t_0, t_0 + T]$, $a(t, Z)$ and $b^i(t, Z)$ ($i = 1, \ldots, s$) satisfy the global Lipschitz condition with respect to $Z$ on $\mathbb{R}^d$. Let $\tilde{Z}_{t,y}(t+h)$ denote the one-step approximation, and $Z_{t,y}(t+h)$ the exact solution at time $t + h$ starting from $(t, y)$. Suppose the one-step approximation $\tilde{Z}_{t,y}(t+h)$ has order of accuracy $p_1$ for the mathematical expectation of the deviation and order of accuracy $p_2$ for the mean-square deviation; more precisely, for arbitrary $t_0 \leq t \leq t_0 + T - h, y \in \mathbb{R}^d$, the following inequalities hold:

\begin{equation}
\begin{aligned}
|E[Z_{t,y}(t+h) - \tilde{Z}_{t,y}(t+h)]| &\leq K(1 + |y|^2)^{1/2} h^{p_1} \\
\left[ E|Z_{t,y}(t+h) - \tilde{Z}_{t,y}(t+h)|^2 \right]^{1/2} &\leq K(1 + |y|^2)^{1/2} h^{p_2}
\end{aligned}
\end{equation}

Also, suppose that

\begin{equation}
p_2 \geq \frac{1}{2}, \quad p_1 \geq p_2 + \frac{1}{2}.
\end{equation}

Then for any $N$ ($N = \frac{T}{h}$) and $k = 0, 1, \ldots, N$, the following inequality holds:

\begin{equation}
\left[ E|Z_{t_0, y_0}(t_k) - \tilde{Z}_{t_0, y_0}(t_k)|^2 \right]^{1/2} \leq K(1 + E|Z_0|^2)^{1/2} h^{p_2 - 1/2},
\end{equation}

i.e. the order of accuracy of the method constructed using the one-step approximation $\tilde{Z}_{t,y}(t+h)$ is $p = p_2 - \frac{1}{2}$.
Note that the constant $K > 0$ in the above inequalities may vary in different context, but are all independent of $Z_0$ and $N$.

Based on Theorem 3.1, we are able to prove the following result about the root mean-square convergence order of our Hamiltonians-preserving scheme (16)–(17).

**Theorem 3.2.** Suppose the reference scheme (16) applied to the stochastic Poisson system (4) with properties (7)–(8) and (20)–(21) satisfies the condition (23), i.e., there exist constants $p_1$ and $p_2 := p + \frac{1}{2}$, $p_1 \geq p_2 + \frac{1}{2}$, and a positive constant $K$, which makes

$$|E(y(t_{n+1}) - \overline{y}_{n+1})| \leq K(1 + |y(t_n)|^2)^{1/2}h^{p_1},$$

$$\left| E \left|y(t_{n+1}) - \overline{y}_{n+1}\right|^2 \right|^{1/2} \leq K(1 + |y(t_n)|^2)^{1/2}h^{p_2},$$

such that the root mean-square convergence order of (16) is $p_2 - \frac{1}{2} = p$. Moreover we assume $p > \frac{1}{2}$. Then the root mean-square convergence order of the Hamiltonians-preserving scheme (16)–(17) applied to the same system is at least $p - \frac{1}{2}$.

**Proof.** We use the shorthand notation $P$ for $P(y(t_n), y(t_{n+1}))$, and the appearing equalities between random variables are in the ‘almost surely’ sense in this proof.

Let $V_{t_{n+1}} = \text{span}\{\nabla H^0_{AVF}(y(t_n), y(t_{n+1})), \ldots, \nabla H^n_{AVF}(y(t_n), y(t_{n+1}))\}$. From the definition of $P$ in (15) we know that $P$ is an orthogonal projection from $\mathbb{R}^d$ to $V_{t_{n+1}}^\perp$, and $I - P$ is an orthogonal projection from $\mathbb{R}^d$ to $V_{t_{n+1}}$.

Now we estimate the local truncation error of the scheme (16)–(17). We have

$$y(t_{n+1}) - y(t_n) - P\phi(y(t_n), \overline{y}_{n+1}, h, W(\Delta_n, \omega))$$

$$= y(t_{n+1}) - y(t_n) - P(\overline{y}_{n+1} - y(t_n))$$

$$= P(y(t_{n+1}) - \overline{y}_{n+1}) + (I - P)(y(t_{n+1}) - y(t_n)).$$

(27)

Since $H^i(y)$ $(i = 0, \ldots, s)$ are invariants of the system, it holds for $i = 0, \ldots, s$ that

$$\nabla H^i_{AVF}(y(t_n), y(t_{n+1}))(\overline{y}_{n+1} - y(t_n)) = H^i(y(t_{n+1})) - H^i(y(t_n)) = 0,$$

(28)

which implies $y(t_{n+1}) - y(t_n) \in V_{t_{n+1}}^\perp$. Thus $(I - P)(y(t_{n+1}) - y(t_n)) = 0$, which, together with (27), yields

$$y(t_{n+1}) - y(t_n) - P\phi(y(t_n), \overline{y}_{n+1}, h, W(\Delta_n, \omega)) = P(y(t_{n+1}) - \overline{y}_{n+1}).$$

(29)

According to the property of orthogonal projections, for arbitrary $u \in \mathbb{R}^d$,

$$|u|^2 = u^T u = (Pu + (I - P)u)^T(Pu + (I - P)u)$$

$$= (Pu)^T(Pu) + 2((I - P)u)^T(Pu) + ((I - P)u)^T((I - P)u)$$

$$= |Pu|^2 + |(I - P)u|^2$$

$$\geq |Pu|^2.$$
Hence \(|P(y(t_{n+1}) - y_{n+1})| \leq |y(t_{n+1}) - y_{n+1}|\). Then, with considering (26), we obtain
\[
E(y(t_{n+1}) - y(t_n) - P\phi(y(t_n), y_{n+1}, h, W(\Delta_n, \omega)))
\leq E(P(y(t_{n+1}) - y_{n+1}))
\leq E(y(t_{n+1}) - y_{n+1})
\leq \left(E|y(t_{n+1}) - y_{n+1}|^2\right)^{1/2}
\leq K(1 + |y(t_n)|^2)^{1/2}h^{p+\frac{1}{2}}.
\] (30)

On the other hand, we have
\[
\left(E|y(t_{n+1}) - y(t_n) - P\phi(y(t_n), y_{n+1}, h, W(\Delta_n, \omega)))|\right)^{1/2}
\leq \left(E|y(t_{n+1}) - y_{n+1}|^2\right)^{1/2}
\leq K(1 + |y(t_n)|^2)^{1/2}h^{p+\frac{1}{2}}.
\] (31)

By Theorem 3.1, we get the result of the Theorem 3.2. □

4. **Numerical experiments.** In this section, we illustrate the performance of the proposed Hamiltonians-preserving scheme (16)–(17) via numerical tests.

4.1. **A linear stochastic Poisson system.** Consider the linear stochastic Poisson system
\[
dy(t) = 
\begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 3 \\
1 & -3 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
2 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
y dt + \frac{1}{4}
\begin{pmatrix}
11 & 4 & 4 \\
4 & 2 & 1 \\
4 & 1 & 2
\end{pmatrix}
y \circ dW(t)
\],
y(0) = y_0.
\] (32)

Here \(y = (y^1, y^2, y^3)^T, t \in [0, T], B(y) = \begin{pmatrix} 0 & 1 & -1 \\
-1 & 0 & 3 \\
1 & -3 & 0 \end{pmatrix}, \) and the Hamiltonians of (32) are
\[
H^0(y) = \frac{1}{2}y^T
\begin{pmatrix}
2 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
y =: \frac{1}{2}y^TC^0y,
\]
\[
H^1(y) = \frac{1}{8}y^T
\begin{pmatrix}
11 & 4 & 4 \\
4 & 2 & 1 \\
4 & 1 & 2
\end{pmatrix}
y =: \frac{1}{2}y^TC^1y.
\]

It is easy to check that
\[
\{H^0, H^1\}(y) = \nabla H^0(y)^T B(y) \nabla H^1(y) = -\{H^1, H^0\}(y) = 0,
\]
which implies that \(H^0(y)\) and \(H^1(y)\) are both invariants of the system (32). Moreover, the function \(C(y) = 3y^1 + y^2 + y^3\) is a Casimir function of the system.

We use the order 1 Milstein scheme ([25]), the order 1.5 Kloeden scheme ([17]), and the Hamiltonians-preserving scheme (16)–(17) based on the above two schemes.
respectively to approximate the solution of (32). In the sequel, we call the (16)–(17) scheme taking the Milstein scheme as its reference scheme (16) the P-Milstein scheme, and that taking the Klöden scheme as its reference scheme (16) the P-Klöden scheme.

We can see from Figure 1 that the root mean-square order of the P-Milstein scheme is 1, and that of the P-Klöden scheme is 1.5, both coincide with the order of their reference schemes, respectively. Here we take $T = 1$, initial value $y_0 = [1, 2, 3]$, and time steps $h = [0.01, 0.02, 0.05, 0.1]$. The expectation is approximated by taking average over 1000 sample paths.

Figure 2 observes the evolution of the Hamiltonians $H^0(y)$, $H^1(y)$ over a long time interval $t \in [0, 1000]$, produced by the Milstein scheme and P-Milstein scheme, respectively, with initial value $y_0 = [1, 1, 1]$. It is clear that the Milstein scheme does not preserve the Hamiltonians, while the P-Milstein scheme does. Here we take $h = 0.01$.

Figure 3 shows two sample trajectories of $y^1$ on the long time interval $[0, 1000]$, produced by the Milstein scheme and the P-Milstein scheme, respectively. They are compared with the exact solution of $y^1$ (which has an analytical form). It can be seen that after a long time, the Milstein scheme produces large deviation from the exact solution, while the P-Milstein scheme exhibits good long-time behavior. The input parameters here are the same with those for Figure 2.

We see that $\nabla C_{AVF}(\cdot, \cdot) \equiv (3, 1, 1)^T$, and we can find the vector $\alpha = (3, 1, 1)^T$, and the numbers $\lambda_0 = -1$, $\lambda_1 = 4$, such that

$$\nabla C_{AVF} \cdot \alpha^T = \lambda_0 C^0 + \lambda_1 C^1. \quad (33)$$

Then by Theorem 2.4 and Remark 2, the Hamiltonians-preserving schemes preserve the Casimir function $C(y)$. 

**Figure 1.** Root mean-square convergence orders of the Milstein scheme, the Klöden scheme, the P-Milstein scheme, and the P-Klöden scheme.
Figure 2. Evolution of $H^0(y), H^1(y)$ by the Milstein scheme and the P-Milstein scheme for system (32)

Figure 3. Evolution of $y^1$ by the Milstein scheme and the P-Milstein scheme

Figure 4 displays the evolution of the computed Casimir function $C(y_n) = [3, 1, 1]^T y_n$ arising from the Milstein scheme and P-Milstein scheme, respectively, with initial value $y_0 = [1, 1, 1]$. We take $T = 10, h = 0.01$. As can be seen from the figure, both the P-Milstein scheme and the Milstein scheme can preserve the Casimir function $C(y)$. 
4.2. A nonlinear stochastic Poisson system. Now we consider the following nonlinear stochastic Poisson system

\[ dy(t) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \cos y^1 \\ \cos y^2 \\ \cos y^3 \end{pmatrix} (dt + c \circ dW(t)), \]

\[ y(0) = y_0, \]  

(34)

where \( y = (y^1, y^2, y^3)^T \), \( B(y) \equiv \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \), \( c \neq 0 \) is a constant, and the Hamiltonians are

\[ H^0(y) = \sin(y^1) + \sin(y^2) + \sin(y^3), \]

\[ H^1(y) = c(\sin(y^1) + \sin(y^2) + \sin(y^3)). \]

Obviously \( \{H^0, H^1\}(y) = \{H^1, H^0\}(y) = 0 \), and \( H^0(y), H^1(y) \) are invariants of the system.

We use the order 0.5 Euler scheme, the order 1 Milstein scheme, and the Hamiltonians-preserving scheme (16)–(17) based on the above two schemes, namely the P-Euler and the P-Milstein scheme, respectively, to approximate the solution of (34).

Figure 5 shows that the root mean-square order of the P-Euler scheme is 0.5, and that of the P-Milstein scheme is 1, both coinciding with the order of their reference schemes, respectively. Here we take \( T = 1 \), \( c = 1 \), initial value \( y_0 = [1, 2, 3] \), and time steps \( h = [0.01, 0.02, 0.05, 0.1] \). The reference solution is computed using the Milstein scheme with \( h = 10^{-4} \), and the expectation is approximated by taking average over 1000 sample paths.
Figure 5. Root mean-square convergence orders of the Euler scheme, the Milstein scheme, the P-Euler scheme, and the P-Milstein scheme

Figure 6. Evolution of $H^0(X)$, $H^1(X)$ by the Milstein scheme and the P-Milstein scheme for the system (34)

Figure 6 observes the evolution of the Hamiltonians $H^0(y)$, $H^1(y)$ over the time interval $t \in [0, 200]$, produced by the Milstein scheme and the P-Milstein scheme, respectively, with $y_0 = [1, 2, 3]$, $c = 2$, $h = 0.01$. It can be observed that the
Hamiltonians do not remain constant along the numerical solution produced by the Milstein scheme, while the P-Milstein scheme preserves the Hamiltonians exactly.

Figure 7 shows a sample path of $y^1$ of the system (34) arising from the Milstein scheme and the P-Milstein scheme, respectively, compared with the reference exact solution simulated by the Milstein scheme with time step $10^{-4}$. It can be seen that the P-Milstein scheme is more accurate than the Milstein scheme. The input parameters here are the same with those for Figure 6.

4.3. The stochastic rigid body system. In this example, we consider the stochastic rigid body system ([2, 15]) given by

$$\begin{align*}
    dy &= B_1(y) \nabla H^0(y)(dt + c_1 \circ dW(t)), \quad y(0) = y_0, \\
    B_1(y) &= \begin{pmatrix}
        0 & -y^3 & y^2 \\
        y^3 & 0 & -y^1 \\
        -y^2 & y^1 & 0
    \end{pmatrix},
\end{align*}$$

where $t \in [0, T]$, $y = (y^1, y^2, y^3)^	op$, $B_1(y)$, and the Hamiltonians of the system are

$$\begin{align*}
    H^0(y) &= \frac{1}{2} \left( \frac{(y^1)^2}{I_1} + \frac{(y^2)^2}{I_2} + \frac{(y^3)^2}{I_3} \right), \\
    H^1(y) &= \frac{c_1}{2} \left( \frac{(y^1)^2}{I_1} + \frac{(y^2)^2}{I_2} + \frac{(y^3)^2}{I_3} \right),
\end{align*}$$

where $c_1, I_1, I_2, I_3$ are non-zero constants. We suppose at least two of $I_1, I_2, I_3$ are different to avoid the trivial case $dy \equiv 0$.

Obviously $\{H^0, H^1\} = 0$ since $H^0$ and $H^1$ also differ only by a constant, which implies that $H^0(y)$ is invariant (so does $H^1(y)$ naturally). Therefore the conditions (7)–(8) holds, and we assume $y_0$ satisfies (20).
The well-posedness of the stochastic rigid body system (35) was given in e.g. [15]. Note that the coefficients
\[ f(y) := B_1(y)\nabla H^0(y) + \frac{1}{2} \frac{\partial g(y)}{\partial y} g(y) \]
\[ = \begin{pmatrix} y^2 y^3 a_{32} \\ y^1 y^3 a_{13} \\ y^1 y^2 a_{21} \end{pmatrix} + \frac{c_1}{2} \begin{pmatrix} y^1 (y^3)^2 a_{32} a_{13} + y^1 (y^2)^2 a_{32} a_{21} \\ y^2 (y^3)^2 a_{13} a_{32} + y^2 (y^1)^2 a_{13} a_{21} \\ y^3 (y^2)^2 a_{32} a_{21} + y^3 (y^1)^2 a_{13} a_{21} \end{pmatrix}, \]
\[ g(y) := c_1 B_1(y)\nabla H^0(y) = c_1 \begin{pmatrix} y^2 y^3 a_{32} \\ y^1 y^3 a_{13} \\ y^1 y^2 a_{21} \end{pmatrix} \]
do not satisfy the global Lipschitz condition (21), which is a condition of Theorem 3.2.

We apply the Milstein scheme and the P-Milstein scheme to the stochastic rigid body system to observe and compare their numerical behaviors.

\[ \text{Figure 8. Root mean-square convergence orders of the Milstein scheme and the P-Milstein scheme} \]

From Figure 8 we see that the root mean-square order of the P-Milstein scheme is 1, coinciding with the order of its reference scheme, the Milstein scheme. Here \( T = 1, c = 0.2, I_1 = 0.345, I_2 = 0.653, I_3 = 1 \), the initial value is \( y_0 = [0.8, 0.6, 1] \), and the time steps are \( h = [0.01, 0.02, 0.025, 0.05, 0.1] \). We use the Milstein scheme with \( h = 10^{-5} \) to simulate the reference exact solution, and the expectation is approximated by taking the average over 1000 sample paths.

Figure 9 shows the evolution of the Hamiltonians \( H^0(y), H^1(y) \) produced by the Milstein scheme and the P-Milstein scheme over the long time interval \( t \in [0, 200] \). Here \( h = 0.01 \) and the other input parameters are the same as those for Figure 8. Obviously the Milstein scheme fails to preserve the Hamiltonians, while the P-Milstein scheme preserves the Hamiltonians exactly.
Figure 9. Evolution of $H^0(y)$ and $H^1(y)$ by the Milstein scheme and the P-Milstein scheme for system (35).

Figure 10. Evolution of $y^2$ by the Milstein scheme and the P-Milstein scheme for the system (35).

Figure 10 illustrates the evolutions of the numerical solutions for $y^2$ arising from the Milstein scheme and the P-Milstein scheme on the time interval $[0, 200]$. As comparison, a reference exact solution of $y^2$ is also simulated by the Milstein scheme with time step $h = 10^{-5}$. As can be seen from the figure, The P-Milstein scheme
is more accurate than the Milstein scheme after a long time evolution. The input parameters are the same as those for Figure 9.

**Remark 3.** From this example, we see that although the coefficients do not satisfy the global Lipschitz condition, the convergence order of the overall scheme is still the same as that of its reference scheme. This implies that the global Lipschitz condition may not be necessary for the validity of the result of Theorem 3.2.

**Remark 4.** As is shown in the numerical tests, the root mean-square convergence order of the Hamiltonians-preserving scheme (16)–(17) can be the same as that of the reference scheme (16), without any loss of accuracy by the projection. It might be possible to verify this by a more accurate theoretical estimate.

5. **Conclusion.** Our rigorous analysis and numerical experiments demonstrated that the Hamiltonians-preserving schemes based on the AVF discrete gradient and the orthogonal projection technique are effective numerical methods for stochastic Poisson systems with multiple invariant Hamiltonians. These Hamiltonians-preserving schemes can also achieve high convergence orders. In addition, they preserve the Casimir functions under certain situations. As expected, the stable long time behavior of these structure-preserving schemes is also observed. Future works include error estimates of the proposed schemes for Hamiltonians-preserving stochastic Poisson systems with non-globally Lipchitz continuous coefficients.

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