Massively Parallel Algorithms for Distance Approximation and Spanners

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Abstract

Over the past decade, there has been increasing interest in distributed/parallel algorithms for processing large-scale graphs. By now, we have quite fast algorithms—usually sublogarithmic-time and often poly(log log n)-time, or even faster—for a number of fundamental graph problems in the massively parallel computation (MPC) model. This model is a widely-adopted theoretical abstraction of MapReduce style settings, where a number of machines communicate in an all-to-all manner to process large-scale data. Contributing to this line of work on MPC graph algorithms, we present poly(log k) ∈ poly(log log n) round MPC algorithms for computing O(k^{1+o(1)})-spanners in the strongly sublinear regime of local memory. To the best of our knowledge, these are the first sublogarithmic-time MPC algorithms for spanner construction.

As primary applications of our spanners, we get two important implications, as follows:

- For the MPC setting, we get an O(log^2 log n)-round algorithm for O(log^{1+o(1)} n) approximation of all pairs shortest paths (APSP) in the near-linear regime of local memory. To the best of our knowledge, this is the first sublogarithmic-time MPC algorithm for distance approximations.
- Our result above also extends to the CONGESTED CLIQUE model of distributed computing, with the same round complexity and approximation guarantee. This gives the first sub-logarithmic algorithm for approximating APSP in weighted graphs in the CONGESTED CLIQUE model.

1 Introduction and Related Work

1.1 Massively Parallel Computation

Processing large-scale data is one of the indubitable necessities of the future, and one for which we will rely more and more on distributed/parallel computation. Over the past two decades, we have witnessed the emergence and wide-spread usage of a number of practical distributed data processing frameworks, including MapReduce [DG08], Hadoop [Whi12], Spark [ZCF+10] and Dryad [IBY+07]. More recently, there has also been increasing interest in building a corresponding algorithmic toolbox for such settings. By now, there is a de-facto standard theoretical abstraction of these frameworks, known as the Massively Parallel Computation (MPC) model. The model was introduced first by Karloff et al. [KSV10] and refined later by Beam et al. [BKS13] and Goodrich et al. [GSZ11].

MPC model. On a very high-level, the model assumes a number of machines, each with a memory capacity polynomially smaller than the entire data, which can communicate in an all-to-all fashion, in synchronous message passing rounds, subject to their memory constraints. More concretely, in the MPC model [KSV10, GSZ11, BKS13], we are given an input of size N
which is arbitrarily distributed among a number of machines. Each machine has a memory of size $S = N^\alpha$ for some $0 < \alpha < 1$, known as the local memory or memory per machine. Since the data should fit in these machines, the number of machines is at least $N/S$, and often assumed to be at most $O(N/S \poly(\log N))$. Hence, the global memory—that is, the summation of the local memories across the machines—is $\tilde{O}(N)$. The machines can communicate in synchronous message-passing rounds, subject to the constraint that the total amount of messages a machine can communicate per round is limited by its memory $S$.

In the case of graph problems, given a graph $G = (V,E)$ the total memory $N$ is $O(|E|)$ words. Ideally, we would like to be able to work with machines that have a small local memory, and still have only few rounds. However, the task gets more complex as one reduces the local memory. Depending on how the local memory compares with the number of vertices $n = |V|$, there are three regimes that have been studied in the MPC model:

- the strongly superlinear regime where $S \geq n^{1+\varepsilon}$ for a constant $\varepsilon > 0$,
- the near-linear regime where $S = \tilde{O}(n)$, and
- the strongly sublinear regime where $S = n^\gamma$ for a positive constant $\gamma < 1$.

We note that the algorithms that can work in the strongly sublinear memory regime are sometimes referred to as scalable massively parallel algorithms. Our focus will be on the more stringent, and also more desirable, regimes of near-linear and strongly sublinear memory.

### 1.2 Graph Problems and Massively Parallel Computation

At the center of the effort for building algorithmic tools and techniques for large-scale data processing has been the subarea of MPC algorithms for graph problems, e.g., [GSZ11, LMSV11, BKS13, ANOY14, BKS14, HP15, AG15, RVW16, IMS17, CLM18, Ass17, ABB19, GGK18, HLL18, BFU18, BEG18, ASS18, BBD19, BHH19, BDE19, ASS18, ASZ19, ABB19, ASW19, GKMS19, GLM19, GU19, LMOS20, ILMP19, CFG19, GKI19, GNT20, GN20]. We discuss a very brief overview here. Please see Appendix A for a more detailed overview, with quantitative bounds.

Early on [KSV10, BKS13], it was observed that MPC can simulate classic PRAM parallel algorithms (subject to conditions on the total amount of work) in the same time. This immediately led to poly($\log n$) round algorithms for a wide range of graph problems.

Since then, the primary objective in the study of MPC algorithms has been to obtain significantly faster algorithms, e.g., with constant or poly($\log \log n$) round complexity. This was achieved initially for the strongly super-linear memory regime, for many problems, and over the past few years, there has also been significant progress on near-linear and strongly sublinear memory regimes. In particular, there has been quite some progress for (A) global graph problems such as connected components, maximal forest, minimum-weight spanning tree, minimum cut, etc, [LMSV11, AGM12, ASS18, BDE19, GNT20, GN20] and (B) local graph problems such as maximum matching approximation, graph coloring, maximal independent set, vertex cover approximation, etc [LMSV11, HLL18, CLM18, GGK18, ABB19, ACK19, CFG19, CFG19, RG20].

**Distance Problems.** Despite the substantial progress on various graph problems, one fundamental category of graph problems for which the progress in MPC has been slower is distance computations and, more generally, distance-related graph problems. In particular, considering that a key criteria in the area is to obtain near constant (and especially $o(\log n)$) time algorithms, the following question has remained open.

**Question:** Are there poly($\log \log n$)-time MPC algorithms in the near-linear memory regime that compute all pairs shortest paths, or any reasonable approximation of them?
In fact, to the best of our knowledge, prior to our work, there was no known MPC algorithm even with a sublogarithmic round complexity, for any non-trivial approximation factor, and even for single-source shortest paths.

The known algorithms provide only \( \text{poly}(\log n) \) round complexity, which is considerably above our target. For instance, one can obtain a \( \text{poly}(\log n) \) round algorithm for \( 1 + \epsilon \) approximation of single source shortest paths (SSSP) in the sublinear memory regime of MPC, by adapting the PRAM algorithm of Cohen [Coh00]. While that algorithm requires \( m^{1+\Omega(1)} \) global memory, recent PRAM results reduce that to \( \tilde{O}(m) \) [Li20, ASZ20]. Among more recent work, Hajiaghayi et al. [HLSS19] give an all pairs shortest path (APSP) algorithm that runs \( O(\log^2 n) \) rounds in strongly sublinear local memory, for certain range of edge weights, and uses a large polynomial global memory, which depends on exponents of matrix multiplication.

To tackle the above question, we can naturally ask whether, in the allowed time, one can sparsify the graph considerably, without stretching the distances – as that sparse graph then can be potentially moved to one machine. This directly brings us to the notion of spanners, as introduced by Peleg and Schäffer [PS89], which are subgraphs with few edges that preserve distances, up to a certain multiplicative factor. A more detailed explanation follows.

**Spanners.** Given a graph \( G = (V, E) \), a \( k \)-spanner is a (sparse) spanning subgraph \( H \) of \( G \) such that the distance between each pair of nodes in \( V \) on this subgraph \( H \) is at most \( k \) times their distance in the original graph \( G \) [Pel00]. It is known that every graph admits a \( (2k - 1) \)-spanner of size \( O(n^{1+1/k}) \), for \( k \geq 1 \), and assuming Erdős girth conjecture, this tradeoff is also tight. Spanners have found many applications in various models, such as, in constructing sparsifiers in the streaming model [KW14], designing work-efficient PRAM algorithms [FL18], and transshipment based distance approximations in PRAM [Li20] and Congested Clique [BKKL17]. In the context of strongly sublinear memory MPC, [DN19] used spanners to obtain an exponential speed up in preprocessing of distance sketches (though, still requiring polylogarithmic rounds). In particular, they achieve this by using spanners in order to simulate non-work-efficient PRAM algorithms in MPC without using extra memory. In general, spanners can be applied to reduce use of resources such as communication and memory for distance-related computation on denser graphs at the expense of accuracy.

We are not aware of any \( \text{poly}(\log \log n) \) round MPC algorithm for computing spanners, in the sublinear or near-linear memory regime. Though one can obtain some (weaker) results from those of other computational models, as we will be discussed later, when mentioning our results.

### 1.3 Our Contribution

In this paper, we provide the first sublogarithmic time MPC algorithms for distance approximations and for constructing spanners. Our spanner construction works in the strongly sublinear memory regime, while for using them in distance approximation we need to move to the near-linear memory regime. Since, as stated, spanners are versatile tools that have found applications in various distance-related graph problems, we are hopeful that our spanner construction should also help in a wider range of problems.

**Spanner Constructions.** Our main technical result, as stated below, provides a family of algorithms which provide a general trade-off between the number of MPC rounds and the stretch of the resulting spanner.

**Theorem 1.1.** Given a weighted graph \( G \) on \( n \) vertices and \( m \) edges and a parameter \( t \), there is an algorithm that runs in \( O\left(\frac{1}{t^\gamma} \cdot \frac{t \log k}{\log(t+1)}\right) \) rounds of MPC and w.h.p. outputs a spanner of size \( O(n^{1+1/k} \cdot (t + \log k)) \), and stretch \( O(k^\gamma) \) when memory per machine is \( O(n^\gamma) \) for any constant \( \gamma > 0 \), and where \( s = \frac{\log(2t+1)}{\log(t+1)} \). This algorithm uses total memory of \( \tilde{O}(m) \).
Since the theorem statement in Theorem 1.1 might be complex due to the number of variables involved, we next state several interesting corollaries of this theorem.

**Corollary 1.2.** Given a total memory of $\tilde{O}(m)$ and memory per machine being $O(n^\gamma)$, for any constant $\gamma > 0$, there is an MPC algorithm that w.h.p. outputs a spanner with the following guarantees in terms of round complexity, stretch and size:

1. runs in $O(\log k)$ rounds, has $O(k^{\log 3})$ stretch and $O(n^{1+1/k} \cdot \log k)$ size;
2. runs in $O\left(2^{1/\varepsilon} \cdot \varepsilon \cdot \log k\right)$ rounds, has $O(k^{1+\varepsilon})$ stretch and $O(n^{1+1/k} \cdot (2^{1/\varepsilon} + \log k))$ size;
3. runs in $O\left(\frac{\log^2 k}{\log \log k}\right)$ rounds, has $O(k^{1+o(1)})$ stretch and $O(n^{1+1/k} \cdot \log k)$ size;
4. runs in $O\left(\frac{\log^2 \log n}{\log \log \log n}\right)$ rounds, has $O(\log^{1+o(1)} n)$ stretch and $O(n \cdot \log \log n)$ size.

Moreover, and crucially for our distance approximation applications, our algorithms are applicable to weighted graphs.

PRAM algorithms for spanners such as Baswana-Sen [BS07] have depth at least $\Omega(k)$, which leads to MPC algorithms with $O(k)$ complexity. Our algorithms are significantly faster, at the price of a small penalty in the stretch. Another relevant prior work that we are aware of is a one of Parter and Yogev in the CONGESTED CLIQUE model, on constructing graph spanners in unweighted graphs [PY18] of $O(k)$ stretch. In Appendix B, we show how by building on this work one can obtain the following result.

**Theorem 1.3.** Given an unweighted graph $G$ on $n$ vertices and $m$ edges, there is an algorithm that in $O(\log k)$ MPC rounds w.h.p. outputs a spanner of size $O(n^{1+1/k} \cdot k)$ and stretch $O(k)$ when memory per machine is $O(n^\gamma)$, for any constant $\gamma > 0$. This algorithm uses a total memory of $\tilde{O}(m + n^{1+\gamma})$.

It is worth noting that these round complexities are close to optimal, conditioned on a widely believed conjecture. This is because lower bounds in the distributed LOCAL model imply an $\Omega(\log k)$ conditional lower bound for the closely related problem of finding spanners with optimal parameters, i.e., $(2k-1)$-spanners with $O(n^{1+1/\varepsilon})$ edges. Specifically, there is an $\Omega(k)$ lower bound for the problem in the LOCAL model [DGPV08], and as shown in [GKU19], this implies $\Omega(\log k)$ conditional lower bound in MPC with sublinear memory, under the widely believed conjecture that connectivity requires $\Omega(\log n)$ rounds.$^1$ This conjecture is also called the cycle vs two cycles conjecture, as even distinguishing between one cycle of $n$ nodes from 2 cycles of $n/2$ nodes, is conjectured to require $\Omega(\log n)$ rounds. Our algorithms have complexity of poly($\log k$) for near-optimal (up to a factor of $k^{o(1)}$) stretch which nearly matches the lower bound.

It is an interesting question whether spanners with optimal parameters can be also constructed in poly($\log k$) time. While our main goal is to construct spanners in standard MPC in poly($\log k$) rounds, we also provide two results that lead to near optimal parameters (stretch $O(k)$ and size $\tilde{O}(n^{1+1/k})$) at the expense of computational resources. As mentioned above, Theorem 1.3 constructs $O(k)$-spanners with $O(n^{1+1/k} \cdot k)$ edges in $O(\log k)$ rounds. This however comes at a price of extra total memory, and also only works for unweighted graphs. Additionally, in Section 3, we provide an algorithm that runs in $O(\sqrt{k})$ rounds and computes an $O(k)$-spanner with $O(\sqrt{k} n^{1+1/k})$ edges. While the parameters of the spanner are near-optimal, this comes at a price of significant increase in the round complexity (this is still much faster compared to previous algorithms that require $O(k)$ time).

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$^1$The proof in [GKU19] is for MPC algorithms that are component-stable, see [GKU19] for the exact details.
Distance Approximations. As a direct but important corollary of our spanner construction in Corollary 1.2, in the near-linear memory regime we get an $O(\log^2 \log n)$-round algorithm for $O(\log^{1+o(1)} n)$ approximation of distance related problems, including all-pairs-shortest-paths. Alternatively, we can obtain a faster algorithm that runs in $O(\log \log n)$ rounds, if we relax the approximation factor to $O(\log^{1+\varepsilon} n)$ for a constant $\varepsilon > 0$. It is important to note that as the global memory in MPC is bounded by $\tilde{O}(m)$, we do not have enough space to store the complete output of APSP. Instead, we have one coordinator machine that stores the spanner, which implicitly stores approximate distances between all vertices. This machine can then compute distances locally based on the spanner. This gives the following (See Section 7 for full details).

Corollary 1.4. There is a randomized algorithm that w.h.p computes $O(\log s n)$-approximation for APSP in weighted undirected graphs, and runs in $O(\frac{\log \log n}{\log (t+1)})$ rounds of MPC, when memory per machine is $\tilde{O}(n)$, $s = \frac{\log(2t+1)}{\log(t+1)}$, and $t = O(\log \log n)$. This algorithm uses total memory of size $\tilde{O}(m)$.

Open Problem. Can we compute a constant, or perhaps even $1 + o(1)$, approximation of all pairs shortest paths in poly$(\log \log n)$ rounds in the near-linear local memory regime of MPC?

Extension to Other Models. In addition to these two models, we show the generality of our techniques by extending our results to the PRAM and Congested Clique models.

Congested Clique. We can extend our spanner construction to work also in the distributed Congested Clique model (see Section 8), leading to the following corollary for approximate shortest paths.

Corollary 1.5. There is a randomized algorithm in the Congested Clique model that w.h.p computes $O(\log^s n)$-approximation for APSP in weighted undirected graphs, and runs in $O(\frac{\log \log n}{\log(t+1)})$ rounds, where $s = \frac{\log(2t+1)}{\log(t+1)}$, and $t = O(\log \log n)$.

As two special cases, this gives $O((\log n)^{\log 3})$-approximation in $O(\log \log n)$ time, and $O(\log^{1+o(1)} n)$-approximation in $O((\log \log n)^2)$ time. It is important to note that these are the first sub-logarithmic algorithms that approximate weighted APSP in the Congested Clique model. Prior results take at least poly-logarithmic number of rounds or only work for unweighted graphs, as we review next.

Previous poly-logarithmic algorithms include the following. First, the spanner construction of Baswana-Sen [BS07] gives $O(\log n)$-approximation for weighted APSP in $O(\log n)$ rounds, by building $O(\log n)$-spanner of size $O(n \log n)$ and letting all vertices learn the spanner. Additionally, it is possible to get a $(3+\epsilon)$-approximation for weighted APSP in $O(\frac{\log^2 n}{\epsilon})$ time [CHDKL19]. In unweighted graphs, a recent work [DP20] shows that it is possible to get $(2+\epsilon)$-approximation for APSP in poly$(\log \log n)$ time, however the techniques only apply for the unweighted case.

We note that while there are fast constructions of spanners in the Congested Clique model [PY18], they work only for unweighted graphs, and also the size of the spanner is at least $\Omega(n \log n)$. The latter adds $\Omega(\log n)$ rounds to the complexity of APSP, for collecting the spanner, which means one cannot achieve a sublogarithmic round complexity for APSP using these spanner constructions.

PRAM. Our result also extends to the PRAM CRCW model. We would get the same PRAM depth as the MPC round complexity, with an additional multiplicative $\log^* n$ factor that arises from certain PRAM primitives (see Section 6). We note that $O(k \log^* n)$ depth
PRAM algorithms for $O(k)$ spanners were studied in [MPVX15], and [BS07]. To the best of our knowledge, before this work there were no known algorithms in PRAM with depth $o(k)$, even for suboptimal spanners.

2 Overview of Our Techniques

The main objective of our work is to design spanners that can be implemented in small parallel depth, e.g., $\text{poly}(\log \log n)$ rounds of MPC. For the sake of simplicity, we present our main ideas by considering unweighted graphs only. However, our main results apply to weighted graphs as well, and that’s one of the key strengths of our approach.

Our general approach will be to grow clusters of vertices by engulfing adjacent vertices and clusters. Additionally, each cluster will be associated with a rooted tree that essentially shows the “history” of the cluster’s growth, with the root representing the oldest vertex of the cluster. The tree edges will all be part of the spanner, and the maximum depth of the tree will be called the radius of the corresponding cluster.

A general paradigm when attaching a cluster to either a vertex or another cluster will be to keep only one edge between them (the one of minimum weight) in the spanner, and discarding the remaining edges. The discarded edges will be spanned by the single included edge and the tree edges within the cluster(s). Thus, (disregarding many details) we can generally upper bound the stretch of any discarded edge by four times the maximum radius of any cluster. Moving forwards, we will talk about stretch and cluster radius interchangeably since they are asymptotically equal.

2.1 Fast Algorithm using Cluster-Cluster Merging

As a starting point, we design an algorithm based on cluster-cluster merging, that can be summarized as follows.

| Cluster-cluster merging |
|------------------------|
| Given an unweighted $G = (V, E)$ and parameter $k \geq 2$: |
| - Let $C_0$ be a set of $n$ clusters ($|V| = n$), where each vertex of $V$ is a single cluster. |
| - For $i = 1 \ldots \log k$: |
| (A) Let $C_i$ be a set of clusters obtained by sampling each cluster from $C_{i-1}$ with probability $n(-2i^{-1}/k)$. |
| (B) For each cluster $c \in C_{i-1} \setminus C_i$: |
| (i) If $c$ has a neighbor $c' \in C_i$, add an edge between $c$ and $c'$ to the spanner, and merge $c$ to $c'$. |
| (ii) Otherwise, for every $c' \in C_{i-1} \setminus C_i$ that is a neighbor of $c$ add an edge between $c$ and $c'$ to the spanner. |
| - Add an edge between each pair of clusters remaining. |

We start by letting each vertex be a singleton cluster, and in each subsequent iteration, a set of clusters is sub-sampled. These sampled clusters are “promoted” to the next iteration (the remaining clusters will not be considered in future iterations). However, the sampled clusters grow to absorb the neighboring unsampled ones. Conversely, each unsampled cluster joins its nearest neighboring sampled cluster. If no such neighboring (sampled) cluster exists, then we can conclude that, with high probability, there are very few neighboring clusters, and this allows us to deal with these problematic clusters by including edges to all their neighbors in the spanner.
Intuition for the Analysis. The intuition behind this approach is as follows. As in each iteration clusters are merged, the number of clusters decreases significantly between iterations. Since we are aiming for a spanner of size $O(n^{1+1/k})$, if $C$ is the current set of clusters, we can allow each cluster in $C$ to add $2^{1+1/k}|C|$ adjacent edges to the spanner. This means that when the number of clusters reduces, we can allow each cluster to add significantly more edges to the spanner. This allows us to decrease the sampling probability $p$ between different iterations drastically. As a consequence, the number of clusters decreases rapidly, which results in a log $k$ complexity. We show that in each iteration the merging grows the stretch by a factor of 3, leading to an overall stretch of $O(3^\log k) = O(k \log 3)$.

2.2 Obtaining Better Stretch

We now discuss how to reduce the stretch, all the way to $k^{1+o(1)}$, while maintaining the same spanner size and increasing the number of iterations only by an extra log $k$ factor.

Intuitively, the spanner algorithm described has sub-optimal stretch of $k \log 3$ mainly since it is too aggressive in growing clusters in each iteration, and when two clusters are merged, the radius increases by a factor of three in one iteration. Thus a natural idea would be to grow the clusters much more gradually; that is, instead of merging a cluster $c$ to cluster $c'$ entirely in one iteration, $c'$ could consume parts of $c$ repeatedly (over multiple iterations). In other words, similar to the algorithm of [BS07] we can grow the clusters incrementally before performing a merge. We call this process cluster-vertex merging.

Cluster-vertex merging The main difference between cluster-cluster and cluster-vertex merging can be outlined as follows: Firstly, in Step A use a smaller sampling probability, e.g. $n^{-1/k}$, instead of $n^{-2^{i-1}/k}$; Secondly, in Step Bi instead of merging two clusters, merge to a sampled cluster $c'$, only the vertices incident to $c'$ that do not belong to any sampled cluster.

This essentially recovers the algorithm in [BS07], where it was shown that using only the cluster-vertex merging, one can construct a spanner of stretch $O(k)$, but this also requires $O(k)$ rounds. We investigate the interpolation between these two extremes by using a hybrid approach, which uses both incremental cluster growing (cluster-vertex merging) and cluster-cluster merging.

Combining the two approaches using cluster contractions. The basic idea is as follows. Instead of merging clusters in each iteration, we alternate between iterations where we apply the cluster-vertex merging approach and iterations where we apply the cluster-cluster merging approach. After a few iterations where we apply cluster-vertex merging, we merge clusters (an operation we also refer to as contraction). this allows to get an improved stretch but still keep a small running time.

Running the cluster-vertex merging procedure on a contracted graph also results in the cluster size growing much faster in each step. This accelerated cluster growth is the main reason for the speedup in our algorithm.

Intuitively, contractions result in loss of information about the internal structure of the cluster, and any time they are performed we incur extra stretch penalties. Our aim is to carefully balance out this loss with the cluster growth rate to get the best tradeoffs. We adjust this by tuning the sampling probabilities, and the intervals at which we perform contractions. This interpolation will allow us to reduce the stretch to $k^{1+o(1)}$, while still having $O(\text{poly log } k)$ round (iteration) complexity. More generally, our various tradeoffs are a consequence of how much we grow the clusters before each contraction.

2.3 The General Algorithm for Round-Stretch Tradeoffs

Now, we provide a more detailed overview of our general algorithm. We will then look at specific parameter settings that lead to various tradeoffs. The algorithm proceeds in a sequence
of epochs, where epoch $i$ consists of $t$ iterations as follows: In each iteration we subsample the clusters with probability $p_i$ (to be defined later). As before, we then grow each sampled cluster by adding all neighboring vertices that have not joined any other sampled cluster. At this point we contract the clusters, and move to the next epoch.

Our general algorithms (in Section 5.1) achieve a range of tradeoffs, parameterized by $t$, which is the number of growth iterations before a contraction, as we outline below.

(A) Assume that the number of super-nodes in the current graph is $n'$ (originally $n$).

(B) Repeat $t$ times (essentially performing $t$ iterations of [BS07] with adjusted sampling probabilities):

1. Perform cluster-sub-sampling with $p_i = \frac{n'}{n^{1+\frac{1}{k}}}$.
   Ensures that number of added edges $\frac{n'}{p_i} = O(n^{1+\frac{1}{k}})$.

2. Perform incremental cluster-vertex merging to increase the radius of each cluster by one unit in the current graph:
   Note that the current graph may contain contracted clusters as super-nodes. Assuming that the internal radius of the super-nodes is $r$, the actual radius increase in the original graph can be as large as $2r + 1$ units (see Fig. 1).

(C) Perform cluster-contraction on the most recent clusters (clusters become super-nodes).
   This has the effect of reducing the number of clusters in the graph by a factor of $p_i^t$ (probability of a specific cluster surviving for $t$ repititions of cluster-sub-sample). So, we get $n' \leftarrow n' \cdot p_i^t$.

Following are some example results we obtain for specific values of parameter $t$.

• ($t = k$): This is one extreme case where there is no contraction, and the cluster radius only increases by 1 for $k$ repetitions, thus recovering the [BS07] result. This is the slowest algorithm, but it achieves optimal stretch $2k - 1$.

• ($t = \sqrt{k}$): The immediate generalization involves exactly one contraction (analysis in Section 3), which occurs after the first set of $\sqrt{k}$ iterations. A new sampling probability is introduced after the contraction (reduced size of graph). According to this probability, the remainder of the algorithm is actually just a normal $\sqrt{k}$ stretch spanner construction. This algorithm attains $O(k)$ stretch, but it dramatically reduces the number of rounds to $O(\sqrt{k})$.

• ($t = 1$): This is the other extreme case (analyzed in Section 4), where the three procedures (sub-sample, grow, contract) are performed repeatedly one after another, i.e., the algorithm contracts immediately after a single grow step. Consequently, the cluster radius grows exponentially, and the algorithm terminates after $\log k$ repetitions, yielding Item 1 of Corollary 1.2. This is the fastest algorithm, but only achieves stretch $O(k^{\log 3})$.

One special interesting setting, which we also use for application in distance approximation, is when we set $t = \log k$. This leads to stretch $k^{1+o(1)}$ and requires only $O(\frac{\log^2 k}{\log \log k})$ rounds. The general tradeoffs can be found in Section 5.1.

2.4 Related Spanner Constructions

Our approach can also be seen as a new contraction-based spanner algorithm with a focus on parallel depth/round efficiency.

In the context of dynamic stream algorithms, another contraction-based algorithm was proposed by [AGM12], but only for unweighted graphs. Their contractions are based on a different
type of clustering formed based on vertex degrees. Algorithm of [AGM12] has resemblance to a special case of our algorithm described in Section 4.1, but it has a weaker stretch. In particular, for a spanner of size $O(n^{1+1/k})$ they obtain stretch $k^\log 5$ in log $k$ passes in streaming (where a pass corresponds to one round of communication in MPC) in unweighted graphs, whereas in the same time (pass/round) we obtain stretch $k^{\log 3}$ even for weighted graphs. Our general algorithm, as a special case, obtains the much stronger stretch of $k^{1+o(1)}$ in $O(\frac{\log^2 k}{\log \log k})$ iterations.

These contraction-based algorithms can be seen as an alternative approach to the well-known algorithm of [BS07]. Mainly, the goal in [BS07] is to compute optimal spanners of stretch $2k-1$, whereas our main goal is time efficiency. As a result our algorithm has a slightly weaker stretch/size tradeoff, but requires exponentially fewer iterations.

This general contraction-based framework may be of interest also in other related distance objects. A related work by [BLP20] focuses on $(\alpha, \beta)$-spanners and hopsets, and they use a similar type of clustering as one part of their construction. However, they connect the clusters differently and in since their main focus is not computation time, their algorithms run in polynomial time in most models. We hope that our fast clustering techniques also give insight into faster algorithms for these structures, perhaps at an extra cost in the stretch. Such improvements will have immediate implications for distance computation in various models.

After the submission of our initial manuscript in March 2020, we found out that an independent and concurrent work [FKN21] obtained similar bounds for spanner construction in dynamic stream settings. Concretely, [FKN21] (see Theorem 3, and set $g = \epsilon \cdot \log k$), obtained similar type of clustering formed based on vertex degrees. Algorithm of [AGM12] has resemblance to a special case, obtains the much stronger stretch of $k^{1+o(1)}$ in $O(\frac{\log^2 k}{\log \log k})$ iterations.

Mainly, the goal in [BS07] is to compute optimal spanners of size $\tilde{O}(n)$ in polynomial time in most models. We hope that our fast clustering techniques also give insight into faster algorithms for these structures, perhaps at an extra cost in the stretch. Such improvements will have immediate implications for distance computation in various models.

As a warm-up, in this section we discuss an algorithm that takes $O(\sqrt{k})$ rounds and constructs a spanner with stretch $O(k)$ and size $O(\sqrt{k}n^{1+1/k})$ in unweighted graphs. Note that this simple algorithm is already significantly faster compared to [BS07], requiring only $O(\sqrt{k})$ rounds instead of $O(k)$ rounds.

Given an unweighted graph $G = (V, E)$ we compute a spanner of size $O(\sqrt{k}n^{1+1/k})$ with stretch $O(k)$. The high-level idea is as follows. We run the algorithm of [BS07] twice: in the first phase we perform the first $t = \Theta(\sqrt{k})$ iterations of [BS07] and stop. We will form a supergraph $\hat{G} = (\hat{V}, \hat{E})$ defined by setting each cluster $C_i$ to be a supernode, and will add an edge between supernodes $c_1, c_2 \in \hat{V}$ if the corresponding clusters are connected with at least one edge in the original graph $G$. Now, for $t' = \Theta(\sqrt{k})$, we compute a $t'$-spanner on $\hat{G}$ by running the [BS07] algorithm on the graph $\hat{G}$ as a black-box, this requires only $t' = \sqrt{n}$ additional iterations. Next we describe these two phases in detail.

**First Phase:** Start with $R_0 = V$, and $V' = V, E = E$. We will have a sequence of clusterings $R_1 \supseteq \ldots \supseteq R_t$ for a parameter $t$ (we will set $t = \sqrt{n}$). $V'$ and $E$ will be the set of vertices and edges that are not yet settled. In each iteration of the first phase, $V'$ is the set of vertices with one endpoint in $E$.

1. Sample a set of clusters $R_i$ by choosing each cluster in $R_{i-1}$ with probability $n^{-1/k}$. Set $C_i = R_i$.
2. For all $v \in V'$:

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This general contraction-based framework may be of interest also in other related distance objects.
(i) If \( v \) is adjacent to a sampled cluster \( R_i \), then add \( v \) to the closest \( c \in C_i \) and add one edge from \( E(v, c) \) to the spanner. Discard (remove from \( E \)) all the edges in \( E(v, c) \).

(ii) If \( v \) is not adjacent to any sampled clusters in \( R_i \), then for each neighboring cluster \( c' \in C_i - 1 \) add a single edge from \( E(v, c') \) to the spanner, and discard all other edges between \( E(v, c') \).

(iii) Remove the intra-cluster edges: remove all the edges with both endpoints in \( C_i \).

Second Phase: Define a supergraph \( \hat{G} = (\hat{V}, \hat{E}) \) by setting each cluster \( C_t \) to be a node in \( \hat{G} \) and adding an edge in \( \hat{E} \) for each pair of adjacent clusters in \( C_t \). We then run a black-box algorithm for computing a \((2t' - 1)\)-spanner (e.g., by running the algorithm of [BS07]) on \( \hat{G} \), for \( t' = \sqrt{n} \).

Analysis Sketch. The high-level idea is that we are stopping the algorithm of [BS07] when there are \( O(n^{1 - 1/k}) \) (or more generally \( O(n^{1 - t/k}) \)) clusters in \( C_t \). This means that the supergraph \( \hat{G} \) is significantly smaller, and now we can afford to compute a spanner with a better stretch on \( \hat{G} \), for fixed size. The radius of the clusters at termination is \( O(\sqrt{kn^{1+t'/k}}) \), and thus the overall stretch is \( O(k) \). To formalize this argument, we start with the size analysis.

**Theorem 3.1.** The set of edges added by this algorithm is \( O(\sqrt{kn^{1+1/k}}) \).

**Proof.** In the first phase we only add as many edges as the [BS07] algorithm does and the total number of edges is \( O(tn^{1+1/k}) \). In the second phase we will add \( O(tn^{1-t/k}1+1/t') \) edges, which is \( O(n^{1-1/k}) \) for \( t = t' = \sqrt{n} \). \( \square \)

We provide a brief argument for the stretch. A similar and more formal argument for similar claims will be presented in Section 4.

We next provide the stretch analysis of the algorithm. We first argue that if an edge \((u, v)\) is not added to the spanner, or discarded by the end of iteration \( i \), then the endpoints \( u \) and \( v \) are in distinct clusters in \( C_i \). This would imply that we can only restrict our attention to the vertices that survive inside clusters. Otherwise, they are removed from \( V' \).

**Lemma 3.2.** Let \((u, v) \in E\) be an edge not added to the spanner. At the end of iteration \( i \) of the first phase, \((u, v)\) is either discarded (removed from \( E \)), or both its endpoints belong to clusters in \( C_i \).

**Proof.** Let \( j \leq t \) be the first iteration in which at least one of \( u \) or \( v \) does not belong to \( C_j \). In other words, their cluster does not get promoted (i.e., is not a cluster subsampled at level \( j \)). W.l.o.g assume that \( u \) is the endpoint that is not part of any cluster in \( C_j \). In other words, \( u \) does not have any neighboring sampled cluster in \( C_j \). By our assumption on \( j \), we know that there exists a cluster \( c_v \in C_{j-1} \) that \( v \) belongs to. Hence if edge \((u, v)\) is not added to the spanner, it must have been discarded in step 2 (ii). \( \square \)

We will also use the following property of the algorithm as shown in [BS07].

**Lemma 3.3.** At iteration \( i \) of the first phase, all clusters \( C_i \) have radius \( i \).

We now use the properties described to prove the overall stretch.

**Theorem 3.4.** For each edge \((u, v) \in E\) not added to the spanner there is a path of length \( O(k) \) between \( u \) and \( v \).
Proof. Let \((u, v) \in E\). From Lemma 3.2, by the end of the first phase, \((u, v)\) is either discarded, or its endpoints belong to clusters in \(C_i\). We start by discussing the case it is discarded, we consider the following two cases.

Case 1: the edge \((u, v)\) is removed in step 2 (i) or (ii). Assume w.l.o.g. that this edge was removed while processing node \(u\). In this case, an edge \((u, v')\) was added to the cluster \(c_v \in C_i\) that \(v\) belongs to. Given the fact that this cluster has radius \(i\), we know that there is a path of length \(2t + 1\) between \(u\) and \(v\).

Case 2: \((u, v)\) is removed in step (iii). Here \(u\) and \(v\) both are assigned to the same cluster, say \(c \in C_i\). This means (by Lemma 3.3) that there is a path of length at most \(i\) from \(u\) to center of \(c\), and a path of length at most \(i\) from \(v\) to center of \(c\). Thus there is a path of length at most \(2i\) between \(u\) and \(v\). Hence at the end of first phase all clusters have radius \(2t\).

We next consider the case that by the end of the first phase, both of the endpoints of \((u, v)\) belong to clusters in \(C_i\). If they belong to the same cluster by iteration \(t\), then we know that there is a path of length \(2t\) between \(u\) and \(v\), since this cluster has radius at most \(t\). Otherwise \(u\) and \(v\) belong to distinct clusters in \(C_i\), which we denote by \(c_u\) and \(c_v\) respectively. These clusters will be “supernodes” in \(\hat{V}\). In other words, any vertex that is not part of clusters in \(C_i\) is already processed and removed from \(V'\). Therefore we can restrict our attention to vertices that are part of clusters in \(\hat{V}\).

Since we construct a \(t\)'-spanner on \(\hat{G}\), at the end of the second phase, there will be a path of length \(t'\) in \(\hat{G}\) between \(c_u\) and \(c_v\). Since each of the clusters in \(\hat{G}\) have radius \(t\) at the time they get passed to the second phase, a path on \(\hat{G}\) with stretch \(O(t')\) corresponds to a path in \(G\) with stretch \(O(tt')\). The claim follows when \(t = t' = \sqrt{n}\).

\section{Cluster-merging Approach}

Following the general paradigm of Baswana and Sen [BS07], our algorithm proceeds in two phases. In the first phase, the algorithm creates a sequence of growing clusters, where initial clusters are singleton vertices. This is similar to the first phase in the [BS07] algorithm, but has the following crucial differences:

- In each epoch, a sub-sampled set of clusters (from the previous epoch) expand, by engulfing neighboring clusters that were not sub-sampled. In [BS07], the sub-sampled clusters only engulf neighboring vertices.
- Consequently, the radius of our clusters increase by a factor of 3 (roughly) in every epoch, and thus the radius after epoch \(i\) is \(O(3^i)\). On the other hand, the cluster radius in [BS07] increments by 1 in each epoch, leading to a radius of \(i\) at the end of epoch \(i\).
- The sub-sampling probability at epoch \(i\) is \(n^{-\frac{i+1}{k}}\), i.e., the probabilities decrease as a double exponential, as opposed to [BS07], where the probabilities are always the same.
- Our algorithm proceeds for \(O(\log k)\) epochs as opposed to \(k\) epochs in [BS07].
- The final stretch we achieve is \(O(k^{\log 3})\), instead of \(O(k)\) as in [BS07].

Based on the aforementioned sampling probabilities, the final number of clusters will be \(n^{1/k}\). Subsequently, we enter the second phase, where we add edges between vertices that still have un-processed edges and the final clusters. The final output is a set of edges \(ES\) that represent the spanner.

The main benefit of our approach, compared to [BS07], is that it provides a significantly faster way of constructing spanners in MPC. Namely, the algorithm of [BS07] inherently requires \(O(k)\) iterations and it is not clear how to implement it in \(o(k)\) MPC rounds while not exceeding a total memory of \(\hat{O}(m)\). On the other hand, in Section 6 we show that each epoch of our
algorithm can be implemented in $O(1)$ MPC rounds. This implies the above approach can be implemented in $O(\log k)$ MPC rounds.

4.1 Algorithm

In this section we describe the cluster-merging approach. We will use the following notation for inter-cluster edges.

**Definition 4.1.** We define $E(c_1, c_2)$ to be the set of edges in $E$ that have one endpoint in cluster $c_1$ and the other endpoint in cluster $c_2$. We will also abuse this notation, and use $E(v, c)$ to denote all edges between vertex $v$ and cluster $c$.

Before a formal description of our algorithm, we also need two more definitions.

**Definition 4.2.** A cluster $c$ is a set of vertices $V_c \in V$ along with a rooted tree $T(c) = (V_c, E_c)$. The “center” of the cluster is defined as the root of $T$ (the oldest member), and the “radius” of the cluster is the depth of $T$ (from the root).

**Definition 4.3.** A clustering of a graph $G = (V, E)$ is a partition of $V$ into a set of disjoint clusters $\mathcal{C}$, such that for all $c, c' \in \mathcal{C}$, we have $V_c \cap V_{c'} = \emptyset$.

We next describe the two phases of our algorithm.

**Phase 1:** The first phase proceeds through $\log k$ epochs. Let $C_0$ be the clustering of $V$ where each $v \in V$ is a cluster. At a high-level, during epoch $i$, we sub-sample a set of clusters $R^{(i)} \subseteq C^{(i-1)}$, we will connect the clusters to each other as follows: Consider a cluster $c$ that is not sampled at epoch $i$. If $c$ does not have a neighboring sampled cluster in $R^{(i)}$, we merge it with each of the (un-sampled) neighboring clusters in $C^{(i-1)}$, using the lowest weight edge. On the other hand, if $c$ has at least one neighboring sampled cluster, then we find the closest such cluster and merge it to $c$ using the lowest weight edge, say $e$. Additionally, in the weighted case, we also add an edge to each of the other neighboring clusters, that are adjacent to $c$ with an edge of weight strictly lower than that of $e$.

Throughout, we maintain a set $E$ (initialized to $E$) containing the unprocessed edges. During each epoch, edges are removed from $E$. During execution, some edges from $E$ are added to the set of spanner edges $E_S$, and some are discarded. Specifically, when we merge clusters $c_1$ and $c_2$, only the lowest weight edge in $E(c_1, c_2)$ is added to the spanner $E_S$, and all other edges in $E(c_1, c_2)$ are discarded from $E$ (the notation $E(\cdot, \cdot)$ is in Definition 4.1). We will use $E^{(i)}$ to denote the state of the set $E$ at the end of epoch $i$. During epoch $i$, we also construct a set of edges $E^{(i)} \subseteq E_S$, containing the edges that are used to connect (merge) sampled old clusters with un-sampled clusters to form the new ones $C^{(i)}$.

At epoch $i$, for $i = 1 \ldots \log k$, we perform the following steps:

1. Sample a set of clusters $R^{(i)} \subseteq C^{(i-1)}$, where each $c \in C^{(i-1)}$ is chosen to be a member of $R^{(i)}$ with probability $n^{-\frac{2^i}{k}}$. Initialize $E^{(i)}$ to the subset of edges in $E^{(i-1)}$ that are contained in some cluster $c \in R^{(i)}$.

2. Consider a cluster $c \in C^{(i-1)} \setminus R^{(i)}$ that has a neighbor in $R^{(i)}$. Let $N^{(i)}(c) \in R^{(i)}$ be the closest neighboring sampled cluster of $c$.

   a. Add the lowest weight edge $e \in E(c, N^{(i)}(c))$ to both $E^{(i)}$ and $E_S$ and remove the entire set $E(c, N^{(i)}(c))$ from $E$.

   b. For all clusters $c' \in C^{(i-1)}$ adjacent to $c$ with any edge of weight strictly less than $e$, add the lowest weight edge in $E(c, c')$ to $E_S$ and then discard all the edges in $E(c, c')$ from $E$.
Consider a cluster $c \in \mathcal{C}^{(i-1)} \setminus \mathcal{R}^{(i)}$ that has no neighbor in $\mathcal{R}^{(i)}$. Let $C' \subseteq \mathcal{C}^{(i-1)}$ be all the clusters of $\mathcal{C}^{(i-1)}$ in the neighborhood of $c$. For each $c_j \in C'$ move the lowest weight edge in $E(c, c_j)$ to $E_S$ and discard all edges in $E(c, c_j)$ from $E$.

The clustering $\mathcal{C}^{(i)}$ is formed by taking the clusters in $\mathcal{R}^{(i)}$, and then extending them using all the edges in $\mathcal{E}^{(i)}$ to absorb other clusters that are connected to $\mathcal{R}^{(i)}$ (using only edges in $\mathcal{E}^{(i)}$).

- Specifically, let $c \in \mathcal{R}^{(i)}$ be a sampled cluster, and let $\Delta^{(i-1)}(c) = \{ \tilde{c} \in \mathcal{C}^{(i-1)} \mid E(c, \tilde{c}) \cap \mathcal{E}^{(i)} \neq \emptyset \}$ be the set of adjacent clusters that will be absorbed. Each such $c$ results a new cluster $c' \in \mathcal{C}^{(i)}$, where $c'$ has the same root node as $c$, and the tree $T(c')$ is formed by attaching the trees $T(\tilde{c})$ (for each $\tilde{c} \in \Delta^{(i-1)}(c)$), to the corresponding leaf node of the tree $T(c)$, using the appropriate edge in $E(c, \tilde{c}) \cap \mathcal{E}^{(i)}$ (by construction, there is exactly one such edge).

Remove all edges $(u, v) \in E$ where $u$ and $v$ belong to the same cluster in $\mathcal{C}^{(i)}$. This set $E$ at the end of the $i$th epoch is denoted $E^{(i)}$. We then contract and form the new quotient graph (super-graph) and proceed to the next epoch.

**Phase 2:** In the second phase, let $V'$ be the set of all endpoints of the remaining edges $E^{(\log k)}$. For each $v \in V'$ and each $c \in \mathcal{C}^{(\log k)}$, we add the lowest edge in $E(v, c)$ to $E_S$ before discarding the edges in $E(v, c)$.

### 4.2 Analysis of Phase 1

We first show that, for each edge $e = (u, v)$ that is discarded (not added to $E_S$), there exists a path from $u$ to $v$ in $E_S$, of weight at most $k \cdot w_e$ (see Theorem 4.10), i.e., the edge $e$ is spanned by existing spanner edges in $E_S$. Next, in Section 4.2.2, we show that the number of edges added to the spanner $E_S$ during phase 1, is $O(n^{1+1/k} \log k)$ in expectation (see Theorem 4.13).

#### 4.2.1 Stretch Analysis

We begin by providing some definitions used throughout the analysis, and then get into a formal stretch analysis.

**Definition 4.4 (Weighted-Stretch Radius).** $\mathcal{C}$ is a clustering of weighted-stretch radius $r$ with respect to an edge set $E$ in a graph $G$ if and only if

(A) For all $c \in \mathcal{C}$, the cluster $c$ has radius at most $r$ (equivalently, $T_c$ has depth at most $r$).

(B) For each edge $e = (x, v) \in E$ such that $x \in c \in \mathcal{C}$, all edges on the path from $x$ to the root of $T_c$ have weight less than or equal to $w_e$.

**Definition 4.5 (Cluster of a vertex).** For a vertex $v$, $c^{(i)}(v)$ refers to the cluster of $\mathcal{C}^{(i)}$ containing $v$.

**Definition 4.6 (Cluster center).** For a vertex $v$, $\mathcal{F}^{(i)}(v)$ denotes the center of $c^{(i)}(v)$.

First, an inductive argument shows that all the remaining edges in $\mathcal{E}^{(i)}$ are between the current set of clusters $\mathcal{C}^{(i)}$.

**Lemma 4.7.** During the execution of Phase 1, any edge $e = (u, v) \in E$ at the end of epoch $i$ is such that both end-points are members of distinct clusters in $\mathcal{C}^{(i)}$. 

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Proof. We prove this statement by induction.

**Base case:** Before the first epoch, \( E = E \) and all the edges have endpoints in \( C_0 \) since this is the set of all vertices.

**Inductive hypothesis:** Assume that at the beginning of epoch \( i \), all edges in \( E \) have both endpoints in distinct clusters of \( C^{(i-1)} \).

**Induction:** Towards a contradiction, assume that there is an edge \( e = (u,v) \in E \) that survives to the end of epoch \( i \) and has at least one endpoint that is not in any cluster of \( C^{(i)} \). Without loss of generality, assume that this endpoint is \( v \). Note that \( c^{(i-1)}(v) \) and \( c^{(i-1)}(u) \) exist by the inductive hypothesis. (See Definition 4.5 for the definition of \( c^{(i-1)}(\cdot) \).)

If \( c^{(i-1)}(v) \) is adjacent to any cluster in \( R^{(i)} \), then it would have been processed in Step 2. Therefore, some edge between \( c^{(i-1)}(v) \) and \( r \in R^{(i)} \) was added to \( E^{(i)} \). In this case, \( c^{(i-1)}(v) \) will be absorbed into a new cluster in \( C^{(i)} \) (see Step 4), and consequently, \( v \) is also a member of \( C^{(i)} \). Hence, \( c^{(i-1)}(v) \) was not adjacent to any cluster in \( R^{(i)} \).

So, \( c^{(i-1)}(v) \) was processed in Step 3. In this case, all edges in \( E(c^{(i-1)}(v),c^{(i-1)}(u)) \) were discarded (one of the edges was added to the spanner \( E_S \), and hence this case could not happen neither. This now leads to a contradiction (as Step 2 or Step 3 has to occur) and implies that both \( u \) and \( v \) belong to some clusters in \( C^{(i)} \).

It remains to show that \( u \) and \( v \) belong to distinct clusters of \( C^{(i)} \). But this follows directly from Step 5.

Next, we argue inductively that in each epoch the cluster radius grows by a factor of 3.

**Theorem 4.8.** At the end of epoch \( i \), \( C^{(i)} \) is a clustering of weighted-stretch radius (see Definition 4.4) at most \( \frac{3^i - 1}{2^i} \) with respect to the current set \( E^{(i)} \).

Proof. We first prove Property (A) and then Property (B) of Definition 4.4. Each of the properties are proved by an inductive argument.

**Property (A) of Definition 4.4.** As a base case, notice that before the first epoch, each cluster has radius 0.

During epoch \( i \), each cluster \( c' \in C^{(i)} \) is a union of a cluster \( c \in R^{(i)} \subseteq C^{(i-1)} \), and some number of clusters in \( C^{(i-1)} \) that are adjacent to \( c \). The root of cluster \( c' \) remains the same as the root of \( c \). However, the radius of \( c' \) becomes the depth of the new rooted tree, which can be at most three times the old radius plus one (for the edge connecting the adjacent cluster). Thus the radius (not necessarily strong) of the new larger cluster is at most \( 3r + 1 \), where \( r \) is the radius of \( C^{(i-1)} \). For an illustration, see Fig. 1 and consider the distance between the center of cluster \( c \) and vertex \( x \). Assuming that the inductive hypothesis is satisfied for \( C^{(i-1)} \) i.e., \( r \leq \frac{3^{i-1} - 1}{2^{i-1}} \), we see that the radius of \( C^{(i)} \) is at most \( \frac{3^i - 1}{2^i} \).

Note that this proof implies that Property (A) holds independently of Property (B).

**Property (B) of Definition 4.4.** As an inductive hypothesis, assume that the clustering \( C^{(i-1)} \) has weighted-stretch radius \( \frac{3^{i-1} - 1}{2^{i-1}} \) with respect to \( E^{(i-1)} \). Now, consider an edge \( e = (x,y) \in E^{(i)} \), such that \( x \in c' \in C^{(i)} \) and \( y \notin c' \). Note that by Lemma 4.7, each edge \( e \in E^{(i)} \) is of this form, i.e., the endpoints of \( e \) belong to distinct clusters of \( C^{(i)} \). According to Step 4, cluster \( c' \) was formed from a sampled cluster \( c \in R^{(i)} \) that engulfed the adjacent clusters in \( \Delta^{(i-1)}(c) \).

**Case** \( x \in \bar{c} \in \Delta^{(i-1)}(c) \): Let \( w' \) be the weight of the edge \( (u,v) \in E^{(i)} \) that connects \( u \in T_c \) to \( v \in T_a \), and let \( w_e \) be the weight of the edge \( e = (x,y) \) (see Fig. 1). We construct a path from \( x \) to \( F^{(i)}(x) \), by concatenating the following paths \( x \rightarrow F^{(i-1)}(x) \rightarrow v \rightarrow u \rightarrow F^{(i)}(u) = F^{(i)}(x) \). Since we know that the edge \( e = (x,y) \) survived Step 2b, it must have weight at least as the edge \( (u,v) \), i.e., \( w' \leq w_e \). By the inductive hypothesis for level \( i-1 \), all edges on the first two segments of the above path have weight at most \( w' \). Similarly, the first two segments of the
path only contain edges with weight at most $w_c$ (again using the inductive hypothesis at $i - 1$). Furthermore, the number of edges on the path is at most $r + r + 1 + r = 3r + 1$, where $r$ is the weighted-stretch radius of clustering at the previous epoch.

Case $x \in c$: Since $e = (x, y) \in E(i) \implies (x, y) \in E(i-1)$, the inductive hypothesis implies that the path from $x$ to the root of $T_c$ (the center is $\mathcal{F}(i-1)(x) = \mathcal{F}(i)(x)$) only uses edges of weight at most $w_c$.

Corollary 4.9. The weighted-stretch radius of the final clustering $C^{\log k}$ is $\frac{k^{\log 3} - 1}{2}$.

Using Theorem 4.8, we show the following.

Theorem 4.10. For all edges $e = (u, v)$ removed in Phase 1, there exists a path between $u$ and $v$ in $E_S$ of weight at most $w_e \cdot k^{\log 3}$.

Proof. Depending on when $e = (u, v)$ was removed from $E$, there are three cases to consider: Step 2, Step 3 and Step 5. Before analyzing these cases, note that by Lemma 4.7 after a cluster is formed no edge is removed from it in the subsequent epochs. This fact will be important in the rest of this proof as when we show that there is a path between $u$ and $v$ of weight at most $k^{\log 3} \cdot w_e$, we will show that this path belongs to a cluster (or to two adjacent ones). Hence, once this path belongs to a cluster it also belongs to $E_S$.

Case $e$ was removed in Step 2 in epoch $i$. Let $e_1 = (x, y)$ be the edge that was kept between $c^{i-1}(u)$ and $c^{i-1}(v)$. Let $x \in c^{i-1}(u)$ and $y \in c^{i-1}(v)$. Then, by Theorem 4.8 there exists a path from $u$ to $x$ within $c^{i-1}(u)$ of weight at most $2^{3^{i-1} - 1} \cdot w_e$. Similarly, there is a path between $v$ and $y$ within $c^{i-1}(v)$ of weight at most $2^{3^{i-1} - 1} \cdot w_e$. Since $E_S$ also contains the edge $e_1$ and $w_{e_1} \leq w_e$ by Steps 2a and 2b, we have that $E_S$ contains a path between $u$ and $v$ of weight at most $(2(3^{i-1} - 1) + 1) \cdot w_e < 3^i \cdot w_e \leq k^{\log 3} w_e$.

Case $e$ was removed in Step 3 in epoch $i$. This case is analogous to the previous one.

Case $e$ was removed in Step 5 in epoch $i$. Let $c \in C^{(i)}$ be the cluster from which $e = (v_1, v_2)$ was removed. By construction, this edge is between two clusters $c_1$ and $c_2$ from $C^{(i-1)}$ that are merged with $c$ in this epoch. Assume that $v_1 \in c_1$. Let $e_1 = (x_1, x)$ be the edge via which $c_1$ was merged to $c$, and let $x_1 \in c_1$. Also, let $e_2 = (y_2, y)$ be the edge with which $c_2$ was merged with $c$, and let $y_2 \in c_2$. Since $e$ was not removed before Step 5, it means that when $c_1$ and $c_2$ got merged with $c$ the edge $e$ was not discarded in Step 2b. This in turn implies that $w_e \geq w_{e_1}$ and $w_e \geq w_{e_2}$.
Now, similar to the analysis of Step 2, using Theorem 4.8 we have that there is a path in $E_S$ between $v_1$ and $x_1$ of weight at most $(3^{i-1} - 1) \cdot w_e$. Also, there is a path between $x$ and $y$ in $E_S$ of weight at most $(3^{i-1} - 1) \cdot w_{c_i}$. Finally, there is a path between $v_2$ and $y_2$ in $E_S$ of weight at most $(3^{i-1} - 1) \cdot w_e$. Combining these together, $E_S$ contains a path between $v_1$ and $v_2$ of weight at most $(3 \cdot (3^{i-1} - 1) + 2) \cdot w_e = (3^i - 1) \cdot w_e \leq k^{\log 3} \cdot w_e$, as desired.

**Stretch Analysis of Phase 2.** Recall, that in the second phase, we let $V'$ be the set of all endpoints of the un-processed edges in $E^\log k$. Subsequently, we add the lowest edge in $E(v, c)$ to $E_S$ before discarding the edges in $E(v, c)$, for each $v \in V'$ and $c \in C^{\log k}$.

Using the weighted stretch radius of the final clustering from Corollary 4.9, we can prove the following lemma, using an argument similar to Theorem 4.10. We omit the formal proof to avoid repetition.

**Lemma 4.11.** For each edge $w_e = (v, c) \in V' \times C^{\log k}$ removed in Phase 2, there exists a path between $u$ and $v$ in $E_S$ of weight at most $w_e \cdot k^{\log 3}$.

**4.2.2 Size Analysis**

Next, we provide an upper-bound on $E_S$. First, we upper-bound the number of clusters in each $C^{(i)}$.

**Lemma 4.12.** For each $i \leq \log k$, in expectation it holds $|C^{(i-1)}| \in O\left(n^{1-2(i-1)/k}\right)$.

**Proof.** A cluster $c$ belongs to $C^{(i-1)}$ only if $c$ was sampled to $R_j$ in Step 1 for each $1 \leq j \leq i-1$. This happens with probability $\prod_{j=1}^{i-1} n^{-2^{j-1}/k} = n^{-2^{i-1}-1/k}$. Therefore, $\mathbb{E}[|C^{(i-1)}|] = n^{-2^{i-1}-1/k}$.

Building on Lemma 4.12 we obtain the following claim.

**Theorem 4.13.** During Phase 1, in expectation there are $O\left(n^{1+1/k} \cdot \log k\right)$ edges added to $E_S$.

**Proof.** Steps 1, 4 and 5 do not affect $E_S$. Hence, we analyze only the remaining steps.

Consider epoch $i$. Fix a cluster $c \in C^{(i-1)}$ (which might or might not be in $R^{(i)}$). We will upper-bound the number of edges that in expectation are added when considering $c$.

Let $p = n^{-2^{i-1}/k}$. Recall that each cluster $c'$ is added from $C^{(i-1)}$ to $R^{(i)}$ independently and with probability $p$. Order the clusters $c'$ of $C^{(i-1)}$ adjacent to $c$ in the non-decreasing order by the lowest-edge in $E(c, c')$. Consider the first $A$ among those sorted clusters.

Taking into account both Steps 2 and 3, an edge from $c$ to the $A$-th cluster is added to $E_S$ if and only if all previous clusters are not sampled, which happens with probability $(1 - p)^{A-1}$. Hence, the expected number of edges added by $c$ is upper-bounded by $\sum_{A=1}^n (1 - p)^{A-1} < \sum_{A=1}^\infty (1 - p)^{A-1} = \frac{1}{p}$. Hence, we have that in expectation the number of edges added to the spanner when considering $c$ is $O(1/p)$.

**Size Analysis of Phase 2.** Finally, As a corollary of Lemma 4.12, we see that in expectation $|C^{\log k}| \in O\left(n^{1/k}\right)$. Therefore, the number of edges added in Phase 2 is at most $|V'| \cdot |C^{\log k}| \in O\left(n^{1+1/k}\right)$. This concludes our spanner construction, and yields the main theorem.

**Theorem 4.14.** Given a weighted graph $G$, the cluster-merging algorithm builds a spanner of stretch $O(k^{\log 3})$ and expected size $O(n^{1+1/k} \cdot \log k)$, within $O(\log k)$ epochs.
General Trade-off between Stretch and Number of Rounds

In this section, we provide an overview of an algorithm that combines ideas of Section 3 and Section 4, with the cluster-vertex merging concept that we described earlier. This gives us a general tradeoff between number of rounds, and stretch. For instance, we can construct a spanner with stretch $k^{1+o(1)}$ in $\frac{\log^2(k)}{\log\log(k)}$ rounds. At a high-level, the algorithm runs in a sequence of epochs, and each epoch performs $t$ iterations of [BS07].

We can imagine the algorithm of [BS07] as being one extreme of this tradeoff (when $t = k$). The algorithm of Section 3 generalizes this, by splitting the $k$ iterations of [BS07] over two epochs, each with $\sqrt{k}$ iterations. After the first set of $\sqrt{k}$ iterations (the first epoch), we contract the most recent clusters, and then repeat $\sqrt{k}$ iterations (the second epoch) on the contracted graph. Importantly, the second epoch uses different sampling probabilities – as if we were actually trying to construct a stronger $O(\sqrt{k})$ stretch spanner.

Meanwhile, the algorithm of Section 4 occupies the other extreme of our tradeoff. This algorithm immediately contracts after a single “[BS07]-like step”. Consequently, each step now becomes an “epoch”, and they all use different sampling probabilities. With log $k$ epochs, this is the “fastest” algorithm in our tradeoff, and thus has the worst stretch.

We interpolate between these extremes by repeating the following two steps:

- In each epoch, we grow clusters, based on the cluster-vertex merging approach, till a certain radius $t$ on the quotient graph, where each super-node in the graph is a contracted cluster from the previous epoch.

- At the end of an epoch, we contract the clusters of radius $t$ to obtain the quotient graph for the next epoch, adjust the sampling probabilities and continue.

The parameter $t$ can be varied, thus resulting in a family of algorithms that achieve our trade-offs. For instance, $t = 1$ corresponds to Section 4, $t = \sqrt{k}$ corresponds to Section 3, and $t = k$ brings us back to the algorithm of [BS07].

The idea behind the generalization is that now, the stretch (equivalently, the radius of clusters), rather than increasing by a multiplicative factor of 3 in each epoch (as it did in Section 4), now it grows by a factor of $(2t + 1)$ in each epoch. We again use the intuition that after each contraction, since the remaining graph is smaller in size, we can afford to grow clusters faster, and we adjust this rate by decreasing the sampling probabilities.

5.1 General Trade-off Algorithm

Let us first formalize the concept of a quotient graph before getting into the algorithm details:

**Definition 5.1.** Given a graph $G = (V, E)$ and a clustering $C$, the quotient graph $\hat{G} = (C, \hat{E})$, is defined as follows. Its vertices are the clusters, and there is an edge between two clusters $c, c' \in C$, if there exist $v' \in c, v' \in c'$ such that $(v, v') \in E$.

We next give a formal description of the algorithm. As can be expected, many of the steps inside each epoch is similar to the algorithms of previous sections and [BS07].

**Phase 1:** The first phase of the algorithm proceeds in $l = \log k / \log(t+1)$ epochs. Through the epochs, we create a sequence of clusterings $\mathcal{D}^{(1)}, \mathcal{D}^{(2)}, \ldots, \mathcal{D}^{(l)}$, and graphs $G^{(0)} \supset G^{(1)} \supset \cdots \supset G^{(l)}$. Here $G^{(0)} = G$, and $G^{(i)} = (V^{(i)}, E^{(i)}) = G^{(i-1)}/\mathcal{D}^{(i)}$ is the quotient graph induced by the $i^{th}$ level clustering.

In what follows, we will use the term vertex to refer to a member of $V = V^{(0)}$ i.e. a vertex of the original graph. On the other hand, a super-node will refer to a member of any intermediate $V^{(i)}$ (including $V^{(0)}$).

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2The quotient graph is sometimes called a super-graph.
Throughout the execution, we maintain a set of edges $E$. The state of this set at the end of epoch $i$ is denoted as $E^{(i)}$, and $E^{(0)}$ is initialized to all the edges in $G$. During epoch $i$, edges are removed from $E^{(i-1)}$ to form $E^{(i)}$. Some of these removed edges are added to the set of spanner edges $E_S$, and the other ones are discarded.

At epoch $i$ we perform the following steps:

(A) Let $G^{(i-1)} = (V^{(i-1)}, E^{(i-1)})$ be the graph at this epoch, and let $D_0^{(i)}$ be the initial clustering comprising of all singleton super-nodes i.e. $D_0^{(i)} \overset{\text{def}}{=} \{\{v\} \mid v \in V^{(i)}\}$.

Next (in Step B), we perform $t$ iterations of Baswana-Sen\(^3\) on $G^{(i)}$, in order to obtain the final clustering $D_t^{(i)}$. We also maintain a set of edges $E_j^{(i)} \subseteq E_S$ at iteration $j$, where the edges in $E_j^{(i)}$ induce the clustering at the $j^{th}$ iteration. Henceforth, we will drop the superscript $(i)$ when it is clear from context.

We will use the set $E_j^{(i)}$, (where $E_0^{(i)}$ is initialized to $E^{(i-1)}$), to denote the set of unprocessed edges at the end of iteration $j$. When clear from context, we will abbreviate this as $E$.

(B) For $j = 1, 2, \ldots, t$, iteration $j$ proceeds as follows:

1. Sample a set of clusters $R_j \subseteq D_{j-1}$, where each $c \in D_{j-1}$ is chosen to be a member of $R_j$ with probability $n^{-\frac{(\log j + \log \log t+\epsilon)}{k}}$. We also initialize $E_j$ to the subset of edges in $E_{j-1}$ that are contained in some cluster $c \in R_j$.

2. Consider all (contracted) super-nodes $v \in G^{(i-1)} \setminus R_j$ that are not in any sampled cluster.

We define $\Gamma_j(v) \subseteq D_{j-1}$ to be all the clusters of $D_{j-1}$ in the neighborhood of $v$.

3. For each super-node $v$, such that $\Gamma_j(v)$ intersects $R_j$ (i.e. super-nodes with some neighboring sampled cluster), let $N_j(v) \in R_j \cap \Gamma_j(v)$ be the closest neighboring sampled cluster of $v$.

   - Add the lowest weight edge $e$ in $E(v, N_j(v))$ to both $E_j$ and $E_S$ and remove the entire set $E(v, N_j(v))$ from $E$.
   - For all clusters $c \in \Gamma_j(v)$ adjacent to $v$ with any edge of weight strictly less than $e$, Add the lowest weight edge in $E(v, c)$ to $E_S$ and then discard all the edges in $E(v, c)$ from $E$.

4. For all super-nodes $v$, such that $\Gamma_j(v) \cap R_j = \emptyset$ (i.e. super-nodes with no neighboring sampled cluster),

   - For each $c \in \Gamma_j(v)$ move the lowest weight edge in $E(v, c)$ to $E_S$ and discard all edges in $E(v, c)$ from $E$.

5. The clustering $D_j$ is formed by taking each cluster in $R_j$, and then extending it using the edges in $E_j$, to absorb other super-nodes that are connected to $R_j$ (using only edges in $E_j$).

Specifically, for each $c \in R_j$, we define $\Delta_j(c) = \{v \in V^{(i-1)} \mid E^{(i-1)}(v, c) \cap E_j \neq \emptyset\}$ as the set of adjacent super-nodes that will be absorbed. Each such $c$ results in a new cluster $c' \in D_j$, where $c'$ has the same root super-node as $c$, and the tree $T(c')$ is formed by attaching each super-node $v \in \Delta_j$ to the corresponding leaf node of $T(c)$, using the appropriate edge in $E(c, c) \cap E_j$ (by construction, there is exactly one such edge).

Note that this description is simpler than the one presented in Step 4 in Section 4.1, because we are working in the intermediate graph $G^{(i)}$.

6. Remove all edges $(u, v) \in E$ where $u$ and $v$ belong to the same cluster in $D_j$. This set $E$ at the end of the $j^{th}$ iteration (in the $i^{th}$ epoch) is denoted by $E_j^{(i)}$.

\(^3\) with fixed sampling probability $n^{-\frac{(\log j + \log \log t+\epsilon)}{k}}$
(C) Construct the quotient graph \( G^{(i+1)} = G^{(i)}/\mathcal{D}_{i}^{(i)} \). For all pairs of super-nodes \( u, v \in V^{(i)} \), we only keep the minimum weight edge between \( u \) and \( v \) in \( E \) (currently denoted by \( E_{i}^{(i)} \)). All other edges are discarded, and the final set at the end of the epoch is denoted by \( E^{(i)} \).

Phase 2: In the second phase, let \( V' \) be the set of all endpoints of the remaining edges \( E^{(i)} \). For each \( v \in V' \) and \( c \in \mathcal{D}_{i}^{(i)} \), we add the lowest edge in \( E(v, c) \) to \( E_{S} \) before discarding the edges in \( E^{(i)}(v, c) \).

5.2 Intermediate Graph Clusterings on Original Graph \( G \)

Notice that the clusterings \( \mathcal{D}_{j}^{(i)} \) are defined on the intermediate graphs \( G^{(i)} \). For the sake of clarity, we define the clustering \( \mathcal{C}_{j}^{(i)} \), which is just the same \( \mathcal{D}_{j}^{(i)} \), but on the original graph \( G = G^{(0)} \). We achieve this by defining an operation that is essentially the inverse of the contraction being performed in Step C. This operation allows us to define \( \mathcal{C}_{j}^{(i)} \) recursively as follows:

Definition 5.2 (Composition of Clusterings). Given a clustering \( \mathcal{C}^{(i-1)} \) defined on \( G^{(0)} \), and clustering \( \mathcal{D}_{j}^{(i)} \) defined on the contracted graph \( G^{(i)} = G/\mathcal{C}^{(i)} \), we can obtain the corresponding \( \mathcal{C}^{(i)} \), by replacing each super-node \( v \in c_{v} \in \mathcal{D}_{j}^{(i)} \) by the corresponding (older) cluster \( c_{v}^{(i-1)} \in \mathcal{C}^{(i-1)} \) (i.e. the super-node \( v \in G^{(i)} \) was formed by contracting a cluster \( c_{v}^{(i-1)} \in \mathcal{C}^{(i-1)} \) in the original \( G \)). Specifically, we modify the tree \( T(c_{v}) \), by replacing each internal super-node \( v \) by the corresponding tree \( T\left( c_{v}^{(i-1)} \right) \). Thus, the final tree (also the cluster) only contains vertices from \( G = G^{(0)} \).

We also need to specify how we obtain \( \mathcal{C}^{(i)} \), and the following observation suffices.

Definition 5.3. Since \( \mathcal{C}_{j}^{(i)} \) corresponds to \( \mathcal{D}_{j}^{(i)} \), and \( G^{(i+1)} = G^{(i)}/\mathcal{D}_{j}^{(i)} \), and therefore, we define \( \mathcal{C}^{(i)} \) to be the same as \( \mathcal{C}_{j}^{(i)} \). As a base case, \( \mathcal{C}^{(0)} \) def = \( \mathcal{D}^{(0)} \) (singleton clusters).

In the next section, we will see how the radius of the clusters is affected by the composition defined above. This should be reminiscent of Theorem 4.8. First, we start with some definitions towards proving an analog of Lemma 4.7.

Definition 5.4 (Cluster of a vertex). For a vertex \( v \) in the original vertex set \( V \), \( c_{j}^{(i)}(v) \) refers to the cluster of \( \mathcal{C}_{j}^{(i)} \) containing \( v \).

Definition 5.5 (Cluster center). For a vertex \( v \), \( \mathcal{F}_{j}^{(i)}(v) \) denotes the center of \( c_{j}^{(i)}(v) \).

Lemma 5.6. During the execution of Phase 1, for any un-processed edge \( e = (u, v) \in E_{j}^{(i)} \) at the end of iteration \( j \) in epoch \( i \), both end-points \( u, v \) are members of distinct clusters in \( \mathcal{C}_{j}^{(i)} \).

Proof. The proof is essentially a combination of Lemma 4.7 and Lemma 3.2, and we proceed by induction on \( j \). In the case that \( i \neq 0, j = 0 \), we note that the clusters in \( \mathcal{C}_{0}^{(i)} \) correspond exactly to the clusters in \( \mathcal{C}^{(i-1)} \). The edge set \( E_{0}^{(i)} \) is also just \( E_{i}^{(i-1)} \) under a re-labeling, and thus induction on \( j \) suffices.

Base case: Initially, \( E_{0}^{(0)} = E \), and all the edges have endpoints in \( \mathcal{C}_{0}^{(0)} \), since it is comprised of singleton clusters \( \{v\} \) for all vertices \( v \in V^{(0)} = V \) (see Step A).

Inductive hypothesis: Assume that at the beginning of iteration \( j \) within epoch \( i \), all edges in \( E_{j-1}^{(i)} \) have both endpoints in distinct clusters of \( \mathcal{C}_{j-1}^{(i)} \).
In what follows, we will be working on the graph $G^{(i)}$ during the $i$th epoch, and in general, we will drop the $(i)$ super-script from all symbols.

**Induction:** Towards a contradiction, assume that there is an edge $e = (u, v) \in E_j$ that survives to the end of iteration $j$, and has at least one endpoint that is not in any cluster of $C_j$. Without loss of generality, assume that $v$ is such a “problematic” endpoint. Note that since $E_j \subseteq E_{j-1} \implies (u, v) \in E_{j-1}$, the inductive hypothesis implies that both $c_{j-1}(v)$ and $c_{j-1}(u)$ exist (see Definition 5.4).

If the super-node containing the “problematic” $v$ is adjacent to any cluster in $R_j$, then it would have been processed in Step B3. Therefore, some edge between $c_{j-1}(v)$ and $N_j(v) \in R_j$ would have been added to $E_j$. In this case, the super-node containing $v$ would be absorbed into a new cluster in $C_j$, i.e. $v \in c \in C_j$. Hence, in order to maintain the assumption that $v$ is “problematic”, it must be the case that $v$ was not adjacent to any cluster in $R_j$.

What if the super-node containing $v$ was processed in Step B4? In this case, all edges in $E(v, c_{j-1}(u))$ were discarded (one of the edges was added to the spanner $E_S$), and hence $(u, v) \notin E_j$.

This implies that neither Step B3 nor Step B4 can have occurred, and therefore $v$ must have been a member of some cluster in $R_j$, which in turn implies that $v$ must be a member of some $c \in C_j$. This contradicts the assumption that $v$ was “problematic”.

Thus, we can conclude that both $u$ and $v$ belong to some clusters in $C_j$. It remains to show that $u$ and $v$ belong to distinct clusters of $C_j$, which follows directly from Step B6.

### 5.3 General Stretch Analysis

In this section, we provide a stretch analysis for our general algorithm. For this we need few more notations.

**Definition 5.7.** For any epoch $i \in [t]$ and any $j \in [t]$, we define $r_j^{(i)}$ to be the weighted-stretch radius of the clustering $C_j^{(i)}$ with respect to the edge set $E_j^{(i)}$. For the sake of convenience, we also define $r^{(i)} = r_0^{(i+1)} = r_1^{(i)}$.

**Lemma 5.8.** For any $i \in [t]$ and any $j \in [t]$, we have $r_j^{(i)} \leq r_{j-1}^{(i)} + 2r^{(i-1)} + 1$.

**Proof.** The proof proceeds analogous to the induction in Theorem 4.8, where we claimed that the new radius of clustering is upper bounded by $r + 2r + 1$ ($r$ being the radius of the previous clustering). This was illustrated in Fig. 1. It is strongly recommended that one reads the proof of Theorem 4.8 before this one, since we will be refer to it.

The difference in the general case, is that the clusters present at a certain iteration within the $i$th epoch, do not have a radius that only depends on $i$. Instead, we now also have a dependence on the iteration number $j$.

Specifically, following the proof of Theorem 4.8, we consider a cluster $c$ of radius $r_{j-1}^{(i)}$ that is about to engulf a super-node $\tilde{c}$. Recall that each such super-node refers to a cluster in $D_t^{(i-1)}$ (or rather in $C_t^{(i-1)}$, since we are working in the original graph), that was contracted at the end of the $(i-1)$th epoch (Fig. 2), and thus has radius $r^{(i-1)}$. The grey circles in Fig. 2 correspond to such super-nodes, each of which have weighted-stretch radius $r_i^{(i-1)} = r^{(i-1)}$ (this follows from inductive hypothesis).

To justify that $r_j^{(i)}$ satisfies the above recurrence, we have to show that the two properties in Definition 4.4 hold for the clustering $c_j^{(i)}$. The fact that Property (A) holds, should be clear from Fig. 2. Specifically, the radius of the cluster $C_j^{(i)}$ is at most one greater than the the sum of the radius of $C_j^{(i-1)}$ and twice the radius of $C^{(i-1)}$. 

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In order to show that Property (B) holds, we consider an edge \( e = (x, y) \in E_1 \) such that \( x \in c' \in C, \) and \( y \notin c'. \) We can use an argument similar to the one in Theorem 4.8 (Fig. 2 illustrates the analog of the first case in the proof of Theorem 4.8). We omit the details to avoid repetition.

\[ \text{Figure 2: A figure used in the proof of Lemma 5.8. Sampled super-node } c \text{ will engulf } \bar{c}, \text{ which is also a supernode (not sampled).} \]

**Corollary 5.9.** For any \( i \in [I], \) we have \( r(i) \leq \frac{(2t+1)^i - 1}{2}. \)

**Proof.** We solve the recurrence from Lemma 5.8 by noticing the base case \( r(0) = 0, \) and proceeding with induction.

\[
\begin{align*}
r(i) &= r(i) \
&\leq r(i) + 2r(i-1) + 1 \
&\leq r(i) + 2r(i-1) + 1 \
&\implies r(i) \leq r(i) + (2r(i-1) + 1) \
&\implies r(i) = (2t+1) \cdot r(i-1) + t \
&= (2t+1) \cdot \frac{(2t+1)^{i-1} - 1}{2} + t = \frac{(2t+1)^{i} - 1}{2}
\end{align*}
\]

\[ \blacksquare \]

**Corollary 5.10.** The final clustering \( C \) has weighted-stretch radius \( k^{s-1}/2 \) w.r.t. \( E \), where \( s = \frac{\log(2t+1)}{\log(t+1)} \).

**Proof.** Substituting the value of \( l = \frac{\log k}{\log(t+1)} \), we see that \( r(l) = (2t + 1)^l = \frac{\log k}{\log(t+1)} = k^{s-1}/2 \).

\[ \blacksquare \]

Since Definition 4.4 only requires an upper bound on the radius, the above Corollary 5.10 implies that any intermediate clustering \( C_j \) also has weighted-stretch radius \( k^{s-1}/2 \).

**Theorem 5.11.** For any edges \( e = \{u, v\} \) of weight \( w_e \) removed in Phase 1, there exists a path \( P_e \in E_S \) of total weight at most \( 2k^{s} \cdot w_e \), between \( u \) and \( v \), where \( s = \frac{\log(2t+1)}{\log(t+1)} \).

**Proof.** The proof proceeds similarly to that of Theorem 4.10. Edges are discarded in Step B3, Step B4, Step B6, or Step C.
Case $e(v,u)$ was removed in Step B3, Step B4, or Step C. While these removals are slightly different, they all share a common structure, which is illustrated in Fig. 3. Specifically, when removing an edge $e = (v,u) \in E^{(0)}$, the algorithm considers $c_v, c_u \in C^{(i)}$ such that $v \in c_v, u \in c_u$ and it adds edge $e' = (v',u') \in E(c_v, c_u)$ to the spanner $E_S$, before discarding all the edges in $E(c_v, c_u)$ (including $e$). Since $e'$ is always chosen to be the lowest weight edge, we have $w_{e'} \leq w_e$. We also define $F_v, F_u$ to be the centers of the $c_v, c_u$ respectively. Assuming that $c_v$ and $c_u$ have weighted-stretch radius $r_1, r_2$ respectively (see Fig. 3), we know that there is a path $F_v \rightarrow v' \rightarrow u' \rightarrow F_u$ of weight at most $w_e(r_1 + 1 + r_2)$ that use only edges in $E_S$. Similarly, we know that there are paths $v \rightarrow F_v$ and $F_u \rightarrow u$ of total weight at most $w_e(r_1 + r_2)$, again using only spanner edges. Fig. 3 illustrates this construction. So, we can conclude that the discarded edge of weight $w_e$ can be replaced by a path of total weight at most $w_e(2r_1 + 2r_2 + 1)$.

Finally, we can use a very similar argument to the stretch analysis in Lemma 4.11 for edges removed in Phase 2. It follows that the overall stretch of our spanner algorithm is $O(k^s)$.

5.4 General Size Analysis

We now upper-bound the size of a spanner constructed by the algorithm from this section. The analysis presented here is in many similar to the one given in Section 4.2.2. The following claim is similar to Lemma 4.12.

Lemma 5.12. At the end of epoch $i$, the number of super-nodes in the graph $G^{(i)}$ (which is the same as the number of clusters $D^{(i)}_i$), is $O\left(n^{1-\frac{(i+1)\cdot k-1}{k}} + \log n\right)$ in expectation.
Proof. A cluster $c \in \mathcal{D}_t^{(i-1)}$ belongs to $\mathcal{D}_t^{(i)}$ only if $c$ was sampled to $\mathcal{R}_j^{(i)}$ in Step B1 for each $1 \leq j \leq t$. This happens with probability $\prod_{j=1}^{t} n^{- \frac{(t+i)^{i-1}}{k}} = n^{- \frac{(t+i)^i - 1}{k}}$. So, the probability that a specific vertex $v \in V^{(0)}$ survives in $V^{(i)}$ is:

$$\Pr \left[ v \in V^{(i)} \right] = \prod_{i'=1}^{i} n^{- \frac{(t+i)^{i'-1}}{k}} = n^{- \frac{(t+i)^i - 1}{k}} \implies E \left[ |V^{(i)}| \right] = n^{- \frac{(t+i)^i - 1}{k}}$$

\[ \blacksquare \]

Corollary 5.13. The expected number of super-nodes in the final contracted graph $G_{t}^{(0)}$ is $O \left( n^{1/k} \right)$.

Lemma 5.14. During Phase 1, in expectation there are $O \left( n^{1+1/k} \cdot (l + t) \right)$ edges added to $E_S$.

Proof. We first analyze the contribution of Step A and then the contribution of Step B to the size of spanner across all the epochs.

**Analysis of Step A.** Let $n_i$ be the number of super-nodes at the beginning of epoch $i$. Then, by following [BS07] analysis, this step adds $O \left( n_i^{1+1/k} \cdot t \right)$ edges to the spanner. So, by Lemma 5.12 we have that across all the epochs this step adds the following number of edges to the spanner:

$$\sum_{i=1}^{l} O \left( n_i^{1+1/k} \cdot t \right) = \sum_{i=1}^{l} O \left( n^{(1- \frac{(t+i)^{i-1}-1}{k})(1+1/k)} \cdot t \right)$$

$$\in O \left( n^{1+1/k} \cdot t \right) \sum_{i=1}^{l} n^{- \frac{(t+i)^{i-1} - 1}{k}}$$

$$= O \left( n^{1+1/k} \cdot t \right).$$

**Analysis of Step B.** Fix an epoch $i$. Observe that by letting $t = 1$, then this step corresponds to the algorithm given in Section 4.1. Hence, the size analysis of this case will be similar to that in Theorem 4.13. We now discuss the differences.

Notice that the proof of Theorem 4.13 is parameterized by $p$. In our case, $p = n^{- \frac{(t+i)^{i-1}}{k}}$. Following the same analysis as in Theorem 4.13, a cluster $c$ considered at iteration $j$ of epoch $i$ in expectation adds $O(1/p)$ edges to $E_S$. Observe that $p$ is the same throughout the entire epoch. We now sum contributions of the clusters across all iterations. By applying Lemma 5.12 and the fact that a cluster proceeds from iteration $x$ to iteration $x + 1$ with probability $p$, we derive

$$O(1/p) \cdot \sum_{j=1}^{t} O \left( n^{- \frac{(t+i)^{i-1}-1}{k}} \cdot p^{j-1} \right) \in O(1/p) \cdot O \left( n^{1- \frac{(t+i)^{i-1}-1}{k}} \right) = O \left( n^{1+1/k} \right).$$

Since there are $l$ epochs, the claim for this step follows.

Finally, we analyze the number of edges added in the second phase to conclude the size argument. Recall that $l = \frac{\log k}{\log (t+1)}$. Re-phrasing Corollary 5.13 in the terms defined in Section 5.2, we see that the expected number of clusters in the final $C_{t}^{(0)}$ is:

$$|C_{t}^{(0)}| = O \left( n^{1- \frac{(t+i)^{i-1}}{k}} \right) = O \left( n^{1- \frac{k-1}{k}} \right) = O \left( n^{1/k} \right).$$
Hence, the number of edges added in this phase is clearly upper-bound by $O(n^{1+1/k})$. To conclude, we get the following.

**Theorem 5.15.** Given a weighted graph $G$ and a parameter $t$, there is an algorithm that takes $O(t \log k / \log(t+1))$ iterations and outputs a spanner of size $O(n^{1+1/k} \cdot (t + \log k))$ in expectation, and stretch $O(k^s)$, where $s = \log(2t+1) / \log(t+1)$.

### 6 Implementation in MPC and PRAM

The implementation of the algorithms described is straightforward when memory per machine is $\Theta(n)$. In this case, each node along with all of its incident edges can be assigned to one machine. We can maintain a label at each edge to determine whether it is discarded or added to the spanner, and a label for each node corresponding to the largest level cluster it belongs to. Nodes can maintain all of these information simply by communicating with their neighbors in each round, which is straightforward when memory per machine is $\Theta(n)$. Maintaining these labels and performing contractions and merges is somewhat more complicated in the low memory regime, since edges incident to a node may not fit into one machine. Hence, the low memory regime is our main focus for the rest of this section.

**Input Configuration.** We will create and maintain the following configuration of the input throughout the algorithm: Given a graph $G = (V,E)$, the goal is to store all the edges incident to each node $v$ in a contiguous group of machines (based on machine ID), which we denote by $M(v)$. We also denote a designated machine in the set $M(v)$ by the leader, that represents the node. This could for instance always be the first machine in the set.

Throughout the algorithm we will append each edge with certain labels (e.g., “corresponding to a cluster”, “corresponding to a supernode”, etc). In order to create such a setting we can use several subroutines from previous work. The following subroutines can all be implemented in $O(1/\gamma)$ rounds when memory per machine is $O(n^\gamma)$, for any constant $\gamma > 0$.

- **Sort ( [GSZ11]).** Given a set of $N$ comparable items, the goal is to have the items sorted on the output machines, i.e., the output machine with smaller ID holds smaller items.

- **Find Minimum ( $v$) (e.g., [DN19]).** Finds the minimum value over the set of machines $M(v)$, i.e., machines containing edges incident to node $v \in V$.

- **Broadcast ( $b,v$) (e.g., [DN19]).** Broadcasts a message $b$ to all machine in $M(v)$.

Most of these subroutines can be implemented using an (implicit) aggregation tree, with branching factor $n^\gamma$, where the machines in $M(v)$ are the leaves. In each round, the aggregate function is computed over a set of $n^\gamma$ nodes in this tree and moved up to the higher levels (e.g. in case of find minimum), or moved down the tree (in case of broadcast). For more details on these standard technique, see [GSZ11].

First, using sorting and indexing subroutines we can create the configuration described above. This can be done by sorting the edges of form $(u,v)$ based on ID of the first node in the tuple. We also assume w.l.o.g that pairs $(u,v)$ and $(v,u)$ are present in the input for each undirected edge. Any time one of these tuples, say $(u,v)$ changes (or adds) a label, this label is also added to the machine containing $(v,u)$ by using the sort subroutine again but based on the smallest ID of the two endpoints. In this case the tuples $(u,v)$ and $(v,u)$ will be place in the same machine (or machines next to each other) and the new labels can be updated for both tuples. After this process, we will sort the input again based on the ordered pairs to return to the initial configuration.

In addition to the standard subroutines described, for implementing our algorithm in MPC we also need the following subroutines:
- **Clustering**: Each node maintains a label corresponding to the highest level cluster it belongs to.

- **Merge**: Merging two clusters into one cluster.

- **Contraction**: Creating a supergraph by contracting a set of clusters to form supernodes and the corresponding superedges.

**Lemma 6.1.** Subroutines clustering, merge and contraction (as described above) can be implemented in \(O(1/\gamma)\) rounds of MPC, when memory per machine is \(O(n^\gamma)\), for any constant \(\gamma > 0\).

**Proof.** Each of these operations can be performed by creating and updating labels for each edge. In particular, we will maintain tuples of form \(((u, v), \mathcal{I}_u, \mathcal{I}_v, i)\), where \(i\) represents the highest level cluster center, and \(\mathcal{I}_u\) represents the ID of the cluster center node \(u\) belongs to.

**Clustering.** Whenever the level of a cluster increments, we first update the level label of the cluster center. Then we can sort all the tuples by the cluster center ID of the first endpoint and increment the corresponding label. We can then repeat the process again the second endpoint.

**Merge.** The merge operation can be performed similarly by relabeling the ID of the center for each edge that is merging into the another cluster.

**Contraction.** For this operation, we again use the sorting subroutine to group all the nodes in a cluster to a contiguous set of machines that are simulating a supernode. We can then relabel the edge tuples so that \((u, v)\) is replaced with \((\hat{u}, \hat{v})\), where \(\hat{u}\) and \(\hat{v}\) are IDs of the cluster centers that were contracted. Hence each edge is replaced with a new tuple corresponding to the two supernodes that its endpoints belong to.

At the end of each of these operations, we will sort the tuples again based on (possibly new) edge tuples to get back to the initial configuration, where all edges incident to a node (or supernode) are in adjacent. All of these steps can also be done in \(O(1/\gamma)\) rounds when memory per machine is \(O(n^\gamma)\).

We will now argue how using these subroutines, we can implement the algorithm of Section 5, and implementation of the algorithms of Section 4 and Section 3 will immediately follow.

**Theorem 1.1.** Given a weighted graph \(G\) on \(n\) vertices and \(m\) edges and a parameter \(t\), there is an algorithm that runs in \(O\left(\frac{1+\log k}{\log(t+1)}\right)\) rounds of MPC and w.h.p. outputs a spanner of size \(O(n^{1+1/k} \cdot (t + \log k))\), and stretch \(O(k^s)\) when memory per machine is \(O(n^\gamma)\) for any constant \(\gamma > 0\), and where \(s = \frac{\log(2t+1)}{\log(t+1)}\). This algorithm uses total memory of \(\tilde{O}(m)\).

**Proof.** We now describe how operation needed for a single iteration of the our algorithm can be implemented in MPC with \(O(n^\gamma)\) memory per machine using the subroutines described.

Sampling is straightforward: each cluster center will be subsampled with probability \(n^{-\frac{(i+1)(i-1)}{k}}\) (at iteration \(t\) of the \(i\)-th epoch), and using the Clustering subroutine, as described in Lemma 6.1, all other nodes in the subsampled cluster will update their labels.

Recall that we always maintain the configuration in which the edges of each node are grouped in a contiguous set of machines at any time for the quotient graph of each epoch. This allows us to discard edges or add edges to the spanner in a straightforward way by maintaining another label for each edge.

In Step B3 and Step B4, we use the subroutine Find Min\((v)\) to find the minimum weight edge among a subset of nodes incident to a given node \(v\).

After each iteration we will perform a Merge operation, and at the end of each epoch we will perform a Contraction operation. All of these operation can be performed in \(O(1/\gamma)\) rounds.
Finally, for the overall number of rounds, recall that the algorithm proceeds in \( \ell = \frac{\log k}{\log(t+1)} \) epochs. For each epoch we run \( t \) iterations of a set of operations all of which takes \( O(1/\gamma) \) rounds based on the subroutines explained. Finally, at an additional \( O(\log n) \) overhead on the overall memory, we can repeat the algorithm in parallel \( O(\log n) \) rounds to turn the expected size guarantee into a high probability bound using a simple Chernoff-bounds arguments. Hence, overall the algorithm takes \( O(\frac{1}{\gamma} \cdot t \log k \log(t+1)) \) rounds.

PRAM Using similar primitives as [BS07] we can obtain low depth CRCW PRAM algorithms. Specifically, we can use the primitives of hashing, semisorting, and generalized find min that were utilized in [BS07]. Additionally, we need a new primitive which has to merge two sets of vertices (clusters). The merge primitive can be implemented like a union find data structure, where each set has a “leader” node, and all other nodes point to the leader. The merge operation can be performed in \( O(1) \) steps, by identifying and changing all the leader pointers in parallel. The PRAM depth is the same as the MPC round complexity, with an additional multiplicative \( \log^* n \) factor that arises from the hashing, semisorting and find min primitives. We note that \( O(k) \) depth PRAM algorithms for spanners were also studied in [MPVX15], in addition to [BS07].

7 Application for Approximate Shortest Paths

While we focused on constructing spanners in a setting where each machine has a memory of sublinear size, the same algorithm also works for the less restrictive setting of MPC with linear memory, e.g., we can consider each machine as subdivided into multiple machines of less memory. The advantage in this model is that if we build a spanner of near-linear size, we can allow to store it on one machine which leads to a \( \text{poly}(\log n) \)-approximation for the all pairs shortest paths (APSP) problem. We next elaborate on it.

As a reminder, our algorithm builds a spanner of size \( O(n^{1+1/k} \cdot (t + \log k)) \) in \( \text{poly}(\log k) \) rounds. If we focus on \( t = O(\log \log n) \) and plug in \( k = \log n \), we get a construction of spanner of near-linear size and \( \text{poly}(\log n) \) stretch in \( \text{poly}(\log \log n) \) time. Since the spanner has size \( O(n \log \log n) \), if we allow each machine to have \( \tilde{O}(n) \) memory, we can just send the whole spanner to one machine. As the spanner has stretch \( \text{poly}(\log n) \) between all pairs of vertices, this gives \( \text{poly}(\log n) \) approximation for APSP, summarized as follows.

**Corollary 1.4.** There is a randomized algorithm that w.h.p computes \( \Omega(n) \)-approximation for APSP in weighted undirected graphs, and runs in \( O(t \log \log n) \) rounds of MPC, when memory per machine is \( \tilde{O}(n) \), \( s = \frac{\log(2t+1)}{\log(t+1)} \), and \( t = O(\log \log n) \). This algorithm uses total memory of size \( \tilde{O}(m) \).

As two special cases, this gives \( O(\log \log^3 n) \)-approximation in \( O(\log \log n) \) time, and \( O(\log^{1+o(1)} n) \)-approximation in \( O(\log^2 \log n) \) time.

8 Applications in Congested Clique

We next discuss applications for the closely related distributed CONGESTED CLIQUE model. In this model there are \( n \) nodes that communicate by sending \( \Theta(\log n) \) bit messages in synchronous rounds. It was shown in [BDH18] that a less restricted variant of MPC, called semi-MPC is equivalent to the CONGESTED CLIQUE model. This is a variant of MPC where there are \( n \) machines with \( \Theta(n) \) memory. Compared to the more standard MPC model, in this model the total memory is \( O(n^2) \) and not \( \tilde{O}(n) \). The spanner algorithm also works in this less restricted variant, with the following difference. The basic spanner algorithm gives a guarantee on the size of the spanner in expectation. To get this guarantee w.h.p, we can run it \( O(\log n) \) times in
parallel in the MPC model which adds logarithmic factor to the total size of memory. However, in the Congested Clique model, this approach is problematic because it would multiply the complexity by an \( O(\log n) \) factor, which is too expensive as we aim for a \( poly(\log \log n) \) complexity. To avoid this, we next sketch how to do parallel repetitions in the Congested Clique efficiently, which results in a similar spanner construction for the Congested Clique model.

**Theorem 8.1.** There is an algorithm that runs in \( O(\frac{t\log k}{\log(t+1)}) \) rounds of Congested Clique and w.h.p. outputs a spanner of size \( O(n^{1+1/k} \cdot (t+\log k)) \), and stretch \( O(k^s) \), where \( s = \frac{\log(2t+1)}{\log(t+1)} \).

**Proof.** As discussed above, we can run one run of the spanner algorithm in the semi-MPC model, and hence also in the Congested Clique model using the simulation described in [BDH18]. This gives \( O(k^s) \)-spanner of expected size \( O(n^{1+1/k} \cdot (t+\log k)) \) in \( O\left(\frac{t\log k}{\log(t+1)}\right) \) rounds. We next explain how to run \( O(\log n) \) runs of the algorithm in parallel, which would enable us to build a spanner of size \( O(n^{1+1/k} \cdot (t+\log k)) \) w.h.p.

The size analysis of the spanner is based on the following randomized process. In each iteration\(^4\) of the algorithm, we have a set of super-nodes \( C \), and we sample from them a set of clusters \( R \) such that each vertex in \( C \) is added to \( R \) with probability \( p \). The size analysis is based on the following:

1. The expected number of clusters is \( O(|C|/p) \).
2. The expected number of edges added to the spanner in this iteration is \( O(|C|/p) \).

A standard application of Chernoff bound shows that 1 actually holds w.h.p (as long as the number of clusters is \( \Omega(\log n) \), but when it is smaller it is easy to show that the total number of edges added to the spanner is small enough). Also, from Markov’s inequality, it follows that 2 happens with constant probability. It follows that if we simulate this randomized process \( O(\log n) \) times in parallel, we get w.h.p a run where both events hold, which implies that we get the bound on the number of edges w.h.p. Hence, we just need to simulate this part \( O(\log n) \) times (in each iteration), to guarantee a total of \( O(n^{1+1/k} \cdot (t+\log k)) \) edges w.h.p.

This is done as follows. In the algorithm, first, each super-node from \( C \) is added to \( R \) with probability \( p \). The sampling is a completely local process done by each super-node separately. Hence, each super-node can just simulate it \( O(\log n) \) times. Then, we would like each super-node to let all other nodes learn the outcome of the sampling process. This can be done by just sending a single \( O(\log n) \) bit message from each super-node \( v \) to all nodes, where the \( i \)’th bit of the message indicates if \( v \) is added to \( R \) in the \( i \)’th run of the algorithm. This takes only one round. Now, based on this information, each super-node in \( C \) knows exactly which edges adjacent to it are taken to the spanner in each run of the algorithm. To conclude, we choose \( O(\log n) \) nodes that would be responsible for the \( O(\log n) \) different runs. Node \( v_i \) would collect from all super-nodes the number of edges adjacent to them added to the spanner in run \( i \). In addition, all nodes know how many vertices were added to \( R \) in each one of the runs. Finally, we choose a run where the number of nodes in \( R \) is \( O(|C|/p) \), and the number of edges added to the spanner is \( O(|C|/p) \), which exists w.h.p as explained above. We only added a constant number of rounds for each iteration, hence the time complexity matches the complexity of the MPC algorithm.

**Approximate Shortest Paths.** The spanner algorithm results in a \( poly(\log n) \)-approximation for weighted APSP in \( poly(\log \log n) \) rounds in the Congested Clique model, as follows. Choosing \( k = \log n \) and \( t = O(\log \log n) \), results in a spanner of size \( O(n \log \log n) \). We can let all vertices learn the whole spanner in \( O(\log \log n) \) rounds using Lenzen’s routing [Len13], which results in the following corollary.

\(^4\)This also refers to iterations where we apply Baswana-Sen, as it is based on the same randomized process.
Corollary 1.5. There is a randomized algorithm in the Congested Clique model that w.h.p computes $O(\log^2 n)$-approximation for APSP in weighted undirected graphs, and runs in $O\left(\frac{\log \log n}{\log(t+1)}\right)$ rounds, where $s = \frac{\log(2t+1)}{\log(t+1)}$, and $t = O(\log \log n)$.

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A Related Work on MPC algorithms for Other Graph Problems

Here, we provide a brief and high-level overview of the recent progress on massively parallel algorithms for graph problems. We discuss these in two categories:

One category is the problem of connected components and other related problems such as maximal forest, minimum-weight spanning tree, and minimum cut (in unweighted graphs). For the super-linear and even near-linear memory regimes, there are $O(1)$ round algorithms \cite{LMSV11,AGM12} for connected components, maximal forest, and minimum weight spanning tree. For the minimum cut problem, there are now $O(1)$ round algorithms for both the unweighted \cite{GNT20} and also weighted \cite{GN20} graphs, using near-linear memory. In contrast, for the strongly sublinear memory regime, even for the connected components problem, the best known is an $O(\log n)$ time algorithm \cite{KSV10} that follows from Boruvka’s classic method \cite{NMN01}, and
this complexity is widely believed to be optimal.\textsuperscript{5} Recent work has improved the complexity for connected components to $O(\log D + \log \log n)$, where $D$ denotes the graph diameter (per connected component) [ASS+18, BDE+19].

Another category is problems with a local nature (in the style of those studied frequently in the LOCAL model as symmetry breaking tasks), including maximum matching approximation, graph coloring with $\Delta + 1$ colors where $\Delta$ denotes the maximum degree, maximal independent set, vertex cover approximation, etc. For all of these, there are $O(1)$ algorithms in the super-linear memory regime [LMSV11, HLL18]. A recent line of work brought the complexity down to $O(\log \log n)$ also in the near-linear memory regime, for maximal matching approximation, vertex cover approximation and maximal independent set [CLM+18, GGK+18, ABB+19], and to $O(1)$ for coloring [ACK19, CFG+19]. There has been also some progress in the strongly sublinear memory regime, with $\tilde{O}(\sqrt{\log n})$ complexity for all of these problems except coloring [GU19], and $\Omega(\log \log n)$ round conditional lower bounds are known for them, conditioned on the aforementioned $\Omega(\log n)$ complexity of the connectivity problem [GKU19]. Finally, for the $\Delta + 1$ coloring problem, the state of the art is much faster and stands at $O(\log \log \log n)$ rounds [CFG+19, RG20].

B Spanners with $O(k)$-stretch in Suboptimal Total Memory

In this section we discuss how one can extend the ideas\textsuperscript{6} of [PY18] to prove Theorem 1.3.

B.1 A Brief Overview of [PY18]

On a high-level, the algorithm of [PY18] splits the vertices into \textit{sparse} and \textit{dense}. A vertex $v$ is called sparse if its $(k/2)$-hop neighborhood has size $O(\sqrt{n})$, and otherwise it is called dense.

The idea now is to for each sparse vertex collect its $(k/2)$-hop neighborhood and simulate there any distributed algorithm, e.g., [BS07]. This effectively enables us to handle sparse vertices. To handle dense vertices, we contract them in such a way that after the contraction one obtains a graph on $n^{\alpha}$ vertices, for some constant $\alpha < 1$.

The entire process is now repeated on the contracted graph recursively for $O(1)$ times, until the graph becomes small enough. The authors show that in this way is obtained a spanner of size $O(kn^{1+1/k})$ and $O(k)$ stretch. Moreover, this approach requires $O(\log k)$ rounds in Congested Clique.

B.2 Our Adaptation to MPC

Collecting balls of size $O(\sqrt{n})$ around each vertex would require a total memory of $O(m + n^{3/2})$. To reduce it to $O(m + n^{1+\gamma})$, we collect balls of size $O(n^{\gamma/2})$ (including both edges and vertices) and define sparse and dense accordingly. We will ensure that for a dense vertex its ball has size $\Omega(n^{\gamma/2})$. This property is important as it directly implies that the collected ball of a dense vertex contains $\Omega(n^{\gamma/2})$ vertices.

\textbf{Sparse vertices.} Since our goal is to obtain a $O(k)$-stretch spanner, we can afford to collect larger than $(k/2)$-hop neighborhoods and simulate an algorithm locally there. In our modification, we collect $(4k)$-hop neighborhoods (or only $\Theta(n^{\gamma/2})$-size connected part of it if

\textsuperscript{5}In fact often used as a hardness assumption for other conditional lower bounds, see e.g., [GKU19, LMOS20].

It is known that this $\Omega(\log n)$ complexity lower bounds for many natural algorithmic approaches [BKS13] but that proving it as a general $\Omega(\log n)$ lower bound would lead to a significant breakthrough in circuit complexity, refuting that $P \subseteq NC^1$ [RVW16].

\textsuperscript{6}Mainly from Section 5.3 in \url{https://drops.dagstuhl.de/opus/volltexte/2018/9829/pdf/LIPIcs-DISC-2018-40.pdf}. 
the size exceeds $O(n^{\gamma/2})$.

Then, for each sparse vertex $v$ we simulate $k$ iterations of [BS07] for $v$ and for each vertex in its $(k + 1)$-hop neighborhood and add the corresponding edges to the spanner. Let $e$ be an edge incident to $v$. In this way, all the vertices which are at distance at most $k$ from $e$ will locally simulate [BS07] and hence correctly decide whether $e$ should be added to the spanner or not. If $e$ is not added to the spanner, then this approach will properly span $e$. Overall, this allows us to properly add to the spanner each edge incident to each sparse vertex. Notice that this simulation is local and requires no additional rounds. In Appendix B.2.1 we discuss how to collect $O(k)$-hop neighborhoods in MPC.

To have consistent simulation of [BS07] over distinct balls, we use the same randomness for each vertex. Notice that each vertex requires only $O(k \log n)$ random bits for simulation, which can be assigned to each vertex at the beginning of the algorithm. As long as $k \log n \in O(n^{\gamma/2})$, replicating the necessary randomness over all $n$ balls in total requires $O(n^{1+\gamma})$ memory.

**Dense vertices.** So, it remains to properly handle edges that connect dense vertices. We also use a hitting set similar to how it is used in [PY18]. Namely, we choose $\tilde{O}(n^{1-\gamma/4})$ vertices uniformly at random. Let that set be $Z$. This ensures that with high probability each dense ball is hit by at least one vertex from $Z$; recall that a dense ball contains $\Omega(n^{\gamma/4})$ vertices. Now, for each heavy vertex $v$ we add to the spanner a shortest path between $v$ and one of the vertices $w \in Z$ in its ball. This adds $O(kn)$ edges. We say that $v$ is assigned to $w$. Observe that in this way any edge connecting two heavy vertices that are assigned to the same vertex from $Z$ has stretch $O(k)$. So, it remains to handle edges that have heavy endpoints such that the endpoints are assigned to distinct vertices of $Z$.

We consider a graph only on the vertices of $Z$. Two vertices $u, v \in Z$ are connected only if there exists a pair of vertices $x$ and $y$ such that $x$ and $y$ are adjacent in $G$, and $x$ is assigned to $u$ while $y$ is assigned to $v$. This graph has $\tilde{O}(n^{1-\gamma/4})$ vertices. We now compute a $(4/\gamma)$-stretch spanner on that graph (by simulating iteration by iteration of [BS07]), which has size $\tilde{O}(n^{(1-\gamma/4)(1+\gamma/4)}) \in O(n)$. We properly map back to $G$ the edges in this spanner, which ensures $O(k/\gamma)$ stretch of edges with endpoints in heavy vertices which are assigned to distinct vertices of $Z$. Recall that we assume that $\gamma$ is a constant.

This concludes the algorithm description.

**B.2.1 MPC Implementation**

Next, we explain how the exponentiation is performed, while taking into account sparse and dense balls. A standard graph exponentiation was used in the context of MPC already, e.g., see [GLM19, GU19]. On a high-level, in step $i$ each vertex holds a ball of radius $2^i$, and in the next step attempts to obtain a ball of radius $2^{i+1}$. Let $B_i(v)$ be the set of vertices that vertex $v$ at the beginning of iteration $i$ has. To obtain $B_{i+1}(v)$, $v$ “requests” each vertex $w \in B_i(v)$ for $B_i(w)$.

Notice that here the number of requests for any ball $B_i(\cdot)$ is upper-bounded by the size of the largest ball. So, if we can fit each ball in the local memory, it means that the number of requests will be relatively small, e.g., $O(n^\gamma)$ or even $O(n^{\gamma/2})$ with a proper setting of parameters. However, this property does not hold when we have both dense and sparse vertices, and where our goal is to stop growing dense balls. For instance, consider the star graph with the center in $c$. Vertex $c$ becomes dense immediately (and $B_0(c)$ will contain $O(n^{\gamma/2})$ of some of the neighbors of $c$), while the leaves are sparse in the first iteration. However, already in the first iteration $c$ receives $n - 1$ requests, which is significantly larger than $n^{\gamma/2}$.

Nevertheless, the total number of requests in the whole graph is $O(n^{1+\gamma/2})$ and each request needs to fetch a ball of size $O(n^{\gamma/2})$. This in total requires $O(n^{1+\gamma})$ messages, which fits into our framework. Now, to satisfy all requests for a given vertex $w$, we first sort all the requests,

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7A similar idea was observed in [PY18], see Footnote 5 therein.
\(\Theta(n^{\gamma/2})\) requests per machine. All the requests asking for \(B(w)\) will appear on consecutive machines. Now, we can propagate \(B(w)\) to all those machine by building a \(\Theta(n^{\gamma/2})\)-ary tree on top of those machines. In this way, each machine receives \(O(n^{\gamma/2} \cdot |B(w)|) \in O(n^\gamma)\) messages. Once those machines receive \(B(w)\), they each send \(B(w)\) to machines/vertices that requested \(B(w)\).

As a reminder, our goal is also to ensure that balls of dense vertices have size \(\Theta(n^{\gamma/2})\), i.e., to ensure that such balls are not too small. To achieve that, consider a vertex \(v\) that receives balls for all the requests it created and it happens that they result in a dense ball. Then, \(v\) will start performing BFS within that dense aggregated ball, and terminate the exploration as soon as the size exceeds \(a \cdot n^{\gamma/2}\) for some fixed constant \(a\).

Other steps of the described algorithm are straightforward to implement in MPC by using some of the standard techniques. This concludes the MPC implementation.