FREE ACTIONS ON HANDLEBODIES

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Abstract. The equivalence (or weak equivalence) classes of orientation-preserving free actions of a finite group $G$ on an orientable 3-dimensional handlebody of genus $g \geq 1$ can be enumerated in terms of sets of generators of $G$. They correspond to the equivalence classes of generating $n$-vectors of elements of $G$, where $n = 1 + (g - 1)/|G|$, under Nielsen equivalence (or weak Nielsen equivalence). For abelian and dihedral $G$, this allows a complete determination of the equivalence and weak equivalence classes of actions for all genera. Additional information is obtained for solvable groups and for the groups $\text{PSL}(2, 3^p)$ with $p$ prime. For all $G$, there is only one equivalence class of actions on the genus $g$ handlebody if $g$ is at least $1 + \ell(G)/|G|$, where $\ell(G)$ is the maximal length of a chain of subgroups of $G$. There is a stabilization process that sends an equivalence class of actions to an equivalence class of actions on a higher genus, and some results about its effects are obtained.

Introduction

The orientation-preserving free actions of a finite group $G$ on 3-dimensional orientable handlebodies have a close connection with a long-studied concept from group theory, namely Nielsen equivalence of generating sets. Indeed, as we observe in section 2 below, the free actions of $G$ on the handlebody of genus $g$, up to equivalence, correspond to the Nielsen equivalence classes of $n$-element generating sets of $G$, where $n = 1 + (g - 1)/|G|$. We will utilize this to prove a number of results about equivalence and weak equivalence of free actions. These results are summarized in concise form in section 4, which also contains precise definitions of equivalence, weak equivalence, and other concepts that we shall use.

A special feature of free actions on handlebodies is that there is a stabilization process relating actions on different genera. When a handlebody $V$ with a free $G$-action contains a $G$-invariant handlebody $U$ such that $V - U$ consists of disjoint 1-handles, the action on $V$ is called a stabilization of the action on $U$. Inequivalent actions can become equivalent after stabilization, indeed we do not know an example of actions that remain inequivalent after even an elementary stabilization (i.e. a stabilization for which $V - U$ consists of $|G|$ 1-handles). Such an example could not involve a solvable group,
since a result of M. Dunwoody implies that for solvable \( G \), any two actions on a handlebody of more than the minimum possible genus for a \( G \)-action are equivalent (see corollary 3.2 below). For an arbitrary \( G \), proposition 6.1 shows that any two actions become equivalent after \( \mu(G) \) elementary stabilizations, where \( \mu(G) \) is the minimum number of generators of \( G \). We remark that there is an interesting theory of stabilization of actions on 2-manifolds \([9, 18]\).

The connection between free actions on handlebodies and Nielsen equivalence is well-known in some circles, although we cannot find an explicit statement in the literature. It was known to J. Kalliongis and A. Miller and is a direct consequence of theorem 1.3 in their paper \([10]\) (for free actions, the graph of groups will have trivial vertex and edge groups, and the equivalence of graphs of groups defined there is readily seen to be the same as Nielsen equivalence on generating sets of \( G \)). Indeed, more delicate classifications of nonfree actions on handlebodies have been examined in considerable depth. A general theory of actions was given in \([16]\) and \([10]\), and the actions on very low genera were extensively studied in \([14]\). Actions with the genus small relative to the order of the group are investigated in \([19]\), and the special case of orientation-reversing involutions is treated in \([12]\). The first focus on free actions seems to be \([24]\), where it was proven that for a cyclic group, any free action on a handlebody of genus above the minimal one is the stabilization of an action on minimal genus, and that any two free actions on a handlebody are weakly equivalent. These results were generalized to dihedral and abelian groups in \([28, 29]\), whose results are reconfirmed and extended in section 4 below.

Some of the arguments in this paper can be shortened by invoking results from \([16]\). Since the general theory given there is much more elaborate than the elementary methods needed for the present work, we have chosen to make our arguments self-contained. After giving some more precise definitions and stating our main results in section 1, we develop the general theory relating free actions to Nielsen equivalence in section 2. We apply this in section 3 to treat the case when \( G \) is solvable, and in section 4 we examine the specific cases of abelian and dihedral groups. In section 5, we show that for \( p \) prime, two free actions of \( \text{PSL}(2, 3^p) \) on a handlebody of genus above the minimal one are equivalent. By work of Evans \([4]\) and Gilman \([8]\), it is known that the same is true for \( \text{PSL}(2, 2^m) \) (for all \( m \geq 2 \)) and \( \text{PSL}(2, p) \) (for \( p \) prime).

Section 6 gives some general results on stabilization, in particular, we prove that if the genus of \( V \) is at least \( 1 + \ell(G)|G| \), where \( \ell(G) \) is the maximum length of a decreasing chain of nonzero subgroups of \( G \), then any two free \( G \)-actions are equivalent. In section 7, we state some open problems. In particular, do there even exist inequivalent actions that are not minimal genus actions?
1. Statement of results

Two (effective) actions $\rho_1, \rho_2 : G \to \text{Homeo}(X)$ are said to be equivalent if they are conjugate as representations, that is, if there is a homeomorphism $h : X \to X$ such that $h\rho_1(g)h^{-1} = \rho_2(g)$ for each $g \in G$. They are weakly equivalent if their images are conjugate, that is, if there is a homeomorphism $h : X \to X$ so that $h\rho_1(G)h^{-1} = \rho_2(G)$. Said differently, there is some automorphism $\alpha$ of $G$ so that $h\rho_1(g)h^{-1} = \rho_2(\alpha(g))$ for all $g$. In words, equivalent actions are the same after a change of coordinates on the space, while weakly equivalent actions are the same after a change of coordinates on the space and possibly a change of the group by automorphism. If $X$ is homeomorphic to $Y$, then the sets of equivalence (or weak equivalence) classes of actions on $X$ and on $Y$ can be put into correspondence using any homeomorphism from $X$ to $Y$.

Henceforth the term action will mean an orientation-preserving free action of a finite group on a 3-dimensional orientable handlebody of genus $g \geq 1$ (only the trivial group can act freely on the handlebody of genus 0, the 3-ball). One may work with either piecewise-linear or smooth actions; we assume that one of these categories has been chosen, and that all maps, isotopies, etc. lie in the category.

For a finite group $G$ we denote by $\mu(G)$ the minimum number of generators in any generating set for $G$, and by $\ell(G)$ the maximum $\ell$ such that $G = G_1 \supset G_2 \supset \cdots \supset G_\ell \supset \{1\}$ is a properly descending chain of subgroups of $G$. When there is only one group $G$ under consideration, we often write $\mu$ for $\mu(G)$ and $\ell$ for $\ell(G)$.

Fix a finite group $G$, and consider an action of $G$ on a handlebody $V$. The quotient map $V \to V/G$ is a covering map, so the action corresponds to an extension

\[ 1 \longrightarrow \pi_1(V) \longrightarrow \pi_1(V/G) \overset{\phi}{\longrightarrow} G \longrightarrow 1. \]

A torsionfree finite extension of a finitely generated free group is free (by any finitely generated virtually free group is the fundamental group of a graph of groups with finite vertex groups, and if the group is torsionfree, the vertex groups must be trivial). So $\pi_1(V/G)$ is free. Since $V$ is irreducible, so is $V/G$, and theorem 5.2 of \ref{8} shows that $V/G$ is a handlebody. From equation (\ast), the genus of $V/G$ must be at least $\mu$, so its Euler characteristic is at most $1 - \mu$. Therefore the Euler characteristic of $V$ is at most $|G|(1 - \mu)$, and the genus of $V$ is at least $1 + |G|(\mu - 1)$. On the other hand, if $W$ is a handlebody of genus $\mu$, we may fix any surjective homomorphism from $\pi_1(W)$ to $G$, and the covering space $V$ corresponding to its kernel has genus $1 + |G|(\mu - 1)$ and admits an action of $G$. So $1 + |G|(\mu - 1)$ is the minimal genus among the handlebodies that admit a $G$-action. We denote this minimal genus by $\Psi(G)$, and we call an action of $G$ on a handlebody of genus $\Psi(G)$ a minimal genus action.

If $V$ is any handlebody that admits a $G$-action, then the genus of $V/G$ is $\mu + k$ for some integer $k \geq 0$, so the genus of $V$ is $1 + |G|(\mu + k - 1)$. Denote
by $E(k)$ the set of equivalence classes of actions of $G$ on a handlebody of genus $1 + |G| (\mu + k - 1)$, and by $W(k)$ the set of weak equivalence classes. In particular, $E(0)$ and $W(0)$ are the equivalence classes of minimal genus actions. There is a natural surjection from $E(k)$ to $W(k)$. In the introduction we explained that an action of $G$ on $V$ is a stabilization of an action on $U$ if and only if there is a $G$-equivariant imbedding of $U$ into $V$ such that $V - U$ consists of 1-handles. We now give a more convenient description. For $G$ acting on $V$, choose a disc $E \subset V$ which is disjoint from all of its $G$-translates. Attach a 1-handle $D^2 \times I$ to $V$ using an orientation-reversing imbedding $j : D^2 \times \partial I \to V$. For each $g \in G$ attach a 1-handle to $g(E)$ using the imbedding $g \circ j$. The $G$-action on $V$ extends to an action of $G$ on the union of $V$ with these 1-handles, and this action is called an elementary stabilization of the original action. Alternatively, we may think of this as attaching a 1-handle to a disc in the boundary of $V/G$ and extending the homomorphism $\phi : \pi_1(V/G) \to G$ that determines the action in the exact sequence $(\ast)$ to a homomorphism from $\pi_1(V/G) \ast \mathbb{Z}$ to $G$ by sending the generator of $\mathbb{Z}$ to 1. Since any two discs in $\partial V/G$ are isotopic, and any orientation-reversing imbeddings of $D^2 \times \partial I$ into a disc are isotopic, the equivalence class of the resulting action is well-defined. The result of applying some number of elementary stabilizations is called a stabilization of the original action. For each $k \geq 0$ and $m \geq 1$, stabilization gives well-defined functions from $E(k)$ to $E(k + m)$ and from $W(k)$ to $W(k + m)$.

Let $e(k)$ denote the cardinality of $E(k)$, and $w(k)$ the cardinality of $W(k)$. Clearly $e(k) \geq w(k)$, and $w(k) \geq 1$ for $k \geq 0$. Recall that the Euler $\varphi$-function is defined on positive integers by $\varphi(1) = 1$, and for $m > 1$, $\varphi(m)$ is the number of integers $q$ with $1 \leq q < m$ such that $\gcd(m, q) = 1$. Notice that for $m > 2$, $\varphi(m)$ is even, since if $\gcd(m, q) = 1$ then $\gcd(m, m - q) = 1$. Using these notations, we can give concise statements of our main results:

1. If $G$ is solvable, then $e(k) = 1$ for all $k \geq 1$ (corollary 3.2), while $w(0)$ can be arbitrarily large (theorem 4.3).

2. If $G$ is abelian, with $G \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_m$, where $d_{i+1} | d_i$ for $1 \leq i < m$, then $w(0) = 1$, and $e(0) = 1$ if $d_m = 2$, otherwise $e(0) = \varphi(d_m)/2$ (theorem 4.1).

3. If $G$ is dihedral of order $2m$, then $w(0) = 1$, and $e(0) = 1$ if $m = 2$, otherwise $e(0) = \varphi(m)/2$ (theorem 4.4).

4. If $G = \text{PSL}(2, 3^p)$ with $p$ prime, then $e(k) = 1$ for all $k \geq 1$ (corollary 5.2).

5. The smallest genus of handlebody admitting inequivalent actions of a group is $q = 11$, which has two equivalence classes of actions of the dihedral group of order 10 (corollary 4.7).

6. The smallest genus of handlebody admitting inequivalent actions of an abelian group is $g = 26$, which has two equivalence classes of $C_5 \times C_5$-actions (corollary 1.3).
7. For all \( G \) and all \( k \geq 0 \), \( E(k) \rightarrow E(k + \mu(G)) \) and \( W(k) \rightarrow W(k + \mu(G)) \) are constant (proposition 6.3).

8. For all \( G \) and all \( k > \ell(G) - \mu(G) \), \( e(k) = 1 \) (corollary 6.3).

In addition to these general results, we will see calculations for several specific groups.

The results given above indicate that there is a strong tendency for actions to become equivalent after stabilization, and as mentioned in the introduction, we do not know even a single example of inequivalent actions which do not become equivalent after one elementary stabilization. Even in the simplest of cases, however, the underlying topology of an equivalence between the stabilizations of two inequivalent actions can be surprisingly complicated. Figure 1 gives the steps of a visualization of an explicit equivalence between two actions of the cyclic group \( C_5 \) of order 5 on the handlebody of genus 6 which are the stabilizations of inequivalent actions of \( C_5 \) on the solid torus (the classification of the actions of cyclic groups is detailed in corollary 4.3).

2. Free actions and Nielsen equivalence

The main result of this section, theorem 2.3, gives the correspondence between actions of \( G \) and Nielsen equivalence classes of generating vectors of \( G \). The proof is elementary, requiring only the basic theory of covering spaces and some well-known facts about free groups. We deduce, in corollary 2.4, an algebraic criterion for an action to be a stabilization of another action. We close with a few examples that illustrate the theory.

Consider two actions of \( G \) on genus \( g \) handlebodies \( V_1 \) and \( V_2 \). Since the Euler characteristics of the quotients must be equal (to \( (1 - g)/|G| \)), we may choose diffeomorphisms from them to a single handlebody \( W \). The original actions are then equivalent to the actions of \( G \) by covering transformations on the covering spaces of \( W \) determined by two surjective homomorphisms \( \phi_1, \phi_2 : \pi_1(W) \rightarrow G \). That is, in classifying actions of a fixed group \( G \) on handlebodies of a fixed genus \( g \), up to equivalence or up to weak equivalence, we may assume that their quotients are the same handlebody \( W \).

A generating vector for a group is a tuple \( S = (s_1, \ldots, s_n) \) such that \( \{s_1, \ldots, s_n\} \) is a generating set. A generating vector \( T = (t_1, \ldots, t_n) \) of \( H \) is said to be obtained from \( S \) by a Nielsen move if there is a \( j \) so that \( s_i = t_i \) for all \( i \neq j \), and for some \( k \neq j \), \( t_j \) equals \( s_j s_k^\pm 1 \) or \( s_k^\pm 1 s_j \). Also, an interchange of two of the entries, or the replacement of an entry by its inverse, is a Nielsen move. Generating vectors are called Nielsen equivalent if there is a sequence of Nielsen moves that changes one to the other. Generating vectors \( S = (s_1, \ldots, s_n) \) and \( T = (t_1, \ldots, t_n) \) are called weakly Nielsen equivalent if there is an automorphism \( \alpha \) of \( G \) such that \( \alpha(S) \) is equivalent to \( T \), where \( \alpha(S) = (\alpha(s_1), \ldots, \alpha(s_n)) \). Note that equivalent or weakly equivalent generating vectors must have the same number of elements. A minimal generating vector for a free group is called a basis.
Figure 1. An equivalence between the stabilizations of inequivalent actions on the solid torus.
If $H$ is a free group with basis $S$, then any Nielsen move on $S$ induces an automorphism of $H$. Nielsen proved [22] that any two bases for $H$ are Nielsen equivalent, consequently the Nielsen moves generate the automorphism group of $H$. Associated to a given handlebody structure on a handlebody $W$ is a standard basis of the free group $\pi_1(W)$, where the $i^{th}$ generator corresponds to a loop that goes once around the $i^{th}$ 1-handle and not around any other handle. Any Nielsen move on this basis is induced by an orientation-preserving diffeomorphism of $W$ (see for example [17]), so any automorphism of $\pi_1(W)$ can be induced by an orientation-preserving diffeomorphism of $W$. This also shows that associated to any basis of $\pi_1(W)$ is a handlebody structure with respect to which the basis is standard.

**Lemma 2.1.** Let $W$ be a handlebody, and let $\phi_1, \phi_2 : \pi_1(W) \rightarrow G$ be surjective homomorphisms to a finite group $G$. Let $(X_1, \ldots, X_n)$ be a basis for $\pi_1(W)$, so that $S = (\phi_1(X_1), \ldots, \phi_1(X_n))$ and $T = (\phi_2(X_1), \ldots, \phi_2(X_n))$ are generating vectors for $G$. Then $S$ and $T$ are weakly Nielsen equivalent if and only if there are an isomorphism $\psi : \pi_1(W) \rightarrow \pi_1(W)$ and an isomorphism $\alpha$ of $G$ such that $\alpha \phi_1 = \phi_2 \psi$. They are Nielsen equivalent if and only if $\alpha$ can be taken to be the identity automorphism of $G$.

**Proof.** Suppose $\psi$ and $\alpha$ exist. Since $(\psi(X_1), \ldots, \psi(X_n))$ is a basis for $\pi_1(W)$, there is a sequence of Nielsen moves that carries $(\psi(X_1), \ldots, \psi(X_n))$ to $(X_1, \ldots, X_n)$. Applying $\phi_2$ shows that the corresponding Nielsen moves in $G$ carry $(\phi_2 \psi(X_1), \ldots, \phi_2 \psi(X_n))$ to $(\phi_2(X_1), \ldots, \phi_2(X_n))$. The latter is $T$, and since $\phi_2 \psi(X_i) = \alpha \phi_1(X_i)$, the former is $\alpha(S)$. Conversely, suppose the generating vectors are weakly Nielsen equivalent. For a sequence of Nielsen moves carrying $T$ to $\alpha(S)$, lifting the Nielsen moves to corresponding Nielsen moves starting from $(X_1, \ldots, X_n)$ yields an isomorphism $\psi$ of $\pi_1(W)$ carrying $(X_1, \ldots, X_n)$ to a basis $(\psi(X_1), \ldots, \psi(X_n))$ such that $\phi_2 \psi(X_i) = \alpha \phi_1(X_i)$. This proves the lemma for weak equivalence. The same argument, taking $\alpha$ to be the identity automorphism, gives it for equivalence. \hfill $\square$

To translate this algebraic information into statements about equivalence of actions, we use the following consequence of the theory of covering spaces.

**Lemma 2.2.** Let $\phi_1, \phi_2 : \pi_1(W) \rightarrow G$ determine actions on handlebodies. The actions are weakly equivalent if and only if there is an isomorphism $\psi : \pi_1(W) \rightarrow \pi_1(W)$ and an isomorphism $\alpha : G \rightarrow G$ such that $\alpha \phi_1 = \phi_2 \psi$. They are equivalent if and only if $\alpha$ may be taken to be the identity.

**Proof.** Suppose that $\psi$ and $\alpha$ exist, and fix $j = 1, 2$, let $V_j$ be the covering space of $W$ corresponding to the kernel of $\phi_j$. Choose a diffeomorphism $h : W \rightarrow W$ inducing $\psi$. Since $\psi$ must take the kernel of $\phi_1$ to the kernel of $\phi_2$, $h$ lifts to a diffeomorphism from $V_1$ to $V_2$. Also, $\psi$ induces $\alpha$ from $G = \pi_1(W)/\ker(\phi_1)$ to $G = \pi_1(W)/\ker(\phi_2)$. Using covering space theory, one can check that this determines a weak equivalence of the actions, and an equivalence when $\alpha$ is the identity automorphism. Conversely, if the weak
equivalence (or equivalence, when \( \alpha = 1 \)) \( H: V_1 \to V_2 \) exists, then since
\( H(gx) = \alpha(g)H(x) \), \( H \) induces a diffeomorphism \( h: W \to W \). Again by
covering space theory, the induced automorphism \( \psi \) of \( h \) on \( \pi_1(W) \) satisfies
\( \phi_2 \psi = \alpha \phi_1 \).

Putting these two lemmas together gives our main classification theorem.

**Theorem 2.3.** Let \( G \) be finite, let \( n \geq 1 \), and let \( g = 1 + |G|(n - 1) \).
Then the weak equivalence classes of actions of \( G \) on genus \( g \) handlebodies

correspond bijectively to the weak equivalence classes of \( n \)-element generating
sets for \( G \). The equivalence classes of actions correspond to the equivalence
classes of \( n \)-element generating sets.

**Proof.** For \( n < \mu \), both sets are empty, so we assume that \( n \geq \mu \). Let
\( W \) be a handlebody of genus \( n \) and fix a basis \((X_1, \ldots, X_n)\) for \( \pi_1(W) \).
Any action is equivalent to one determined by a surjective homomorphism
\( \phi: \pi_1(W) \to G \). By lemmas 2.2 and 2.1, sending the action determined by
\( \phi \) to the basis \((\phi(X_1), \ldots, \phi(X_n))\) determines a bijection from the set of
(equivalence or) weak equivalence classes to the set of (equivalence or) weak
equivalence classes of \( n \)-element generating vectors of \( G \).

Let \( \phi: \pi_1(W) \to G \) determine a \( G \)-action on a handlebody \( V \), and let
\( W' \) be obtained from \( W \) by attaching a single 1-handle. Then \( \pi_1(W') = \pi_1(W) \ast \mathbb{Z} \), where the generator of \( \mathbb{Z} \) corresponds to a loop that goes once
around the additional 1-handle. We have noted that the action resulting
from an elementary stabilization of the action on \( V \) is determined by the
homomorphism \( \phi': \pi_1(W') \to G \) which equals \( \phi \) on \( \pi_1(W) \) and sends the
generator of \( \mathbb{Z} \) to 1.

**Corollary 2.4.** Suppose \( G \) acts on handlebodies \( U \) and \( V \), where the genus
of \( U \) is less than the genus of \( V \). Let \((X_1, \ldots, X_m)\) be a basis for \( \pi_1(U/G) \)
and let \((Y_1, \ldots, Y_n)\) be a basis for \( \pi_1(V/G) \), and let \( \phi_U: \pi_1(U/G) \to G \)
and \( \phi_V: \pi_1(V/G) \to G \) determine the actions. Then the action on \( V \) is a
stabilization of the action on \( U \) if and only if the generating \( n \)-vectors
\((\phi_U(X_1), \ldots, \phi_U(X_m), 1, \ldots, 1) \) and \((\phi_V(Y_1), \ldots, \phi_V(Y_n))\) of \( G \) are equivalent.

**Proof.** Put \( W_U = U/G \) and \( W_V = V/G \). Since for any basis there is a
handlebody structure for which the basis is standard, there is an inclusion
\( j: W_U \to W_V \) so that \( \pi_1(W_U) \to \pi_1(W_V) \) carries \( X_i \) to \( Y_i \) for \( i \leq m \), and so that \( \overline{W_V - j(W_U)} \) is a disjoint union of 1-handles. Then, the
stabilized action is determined by the homomorphism \( \phi'_V: \pi_1(W_V) \to G \) for
which \( \phi'_V(Y_i) = \phi_U(X_i) \) for \( i \leq m \), and \( \phi'_V(Y_i) = 1 \) for \( i > m \). By theorem 2.3, the stabilized action and the action on \( V \) are equivalent if and
only if \((\phi_U(X_1), \ldots, \phi_U(X_m), 1, \ldots, 1) \) and \((\phi_V(Y_1), \ldots, \phi_V(Y_n))\) of \( G \) are Nielsen equivalent.

This provides a simple criterion for an action to be a stabilization. A
vector of elements of \( G \) is called redundant if it is Nielsen equivalent to a
vector with an entry equal to 1. Notice that a vector is redundant if it is even weakly Nielsen equivalent to a vector with an entry equal to 1.

**Corollary 2.5.** An action of \( G \) on a handlebody \( V \) is a stabilization if and only if a generating vector \((s_1, \ldots, s_n)\) of \( G \) corresponding to the action as in theorem 2.3 is redundant.

The literature contains a number of results on Nielsen equivalence of generating vectors. They were used to count \( G \)-defining subgroups of free groups in [7]. The action of the automorphism group of the free group on generating vectors appears to have been introduced in [20]. For infinite groups, there are quite a few instances of inequivalent generating vectors of cardinality greater than \( \mu \). The paper of M. Evans [5] gives general constructions of these, as well as a summary of earlier results. But for finite groups, no such example is known (see section 4 below). In the remainder of this section, we will collect some of the known calculations for specific finite groups, and give their consequences for group actions. Some important general results of M. Dunwoody for solvable groups will be stated and used in the next section.

For \( G = A_5 \), B. Neumann and H. Neumann [20] (see also [23]) showed that there are two weak equivalence classes of generating 2-vectors for \( A_5 \). Thus there are two weak equivalence classes of actions on the handlebody of genus \( \Psi(A_5) = 1 + 60(2 - 1) = 61 \). D. Stork [26] carried out similar calculations for \( \text{PSL}(2,7) \) and \( A_6 \). These show, for example, that there are 4 weak equivalence classes of \( A_6 \)-actions on the handlebody of genus 361, the minimal genus. Techniques developed by M. Lustig [14] using the Fox calculus yield additional examples of inequivalent generating systems for certain groups.

Gilman [6] proved that for \( p \) prime, all 3-element generating sets of \( \text{PSL}(2, p) \) are equivalent. That is, \( e(k) = 1 \) for all \( k \geq 1 \) for these groups. In particular, this holds for \( \text{PSL}(2, 5) \cong A_5 \), so the two equivalence classes of actions of \( A_5 \) on the handlebody of genus 61 become equivalent after a single elementary stabilization. In [6], M. Evans proved that when \( G \) is \( \text{PSL}(2, 2^m) \) or the Suzuki group \( \text{Sz}(2^{2m-1}) \) for \( m \geq 2 \), then for any \( n \geq 3 \), all \( n \)-element generating vectors are equivalent. Since \( \mu = 2 \) for any of these groups, this says that all actions of one of these groups on any genus above the minimal genus are equivalent. In section 4, we will prove a similar result for \( \text{PSL}(2, 3^p) \) with \( p \) prime. This includes the case of \( \text{PSL}(2, 9) \cong A_6 \), so the four inequivalent \( A_6 \)-actions on the handlebody of genus 361 all become equivalent after a single elementary stabilization.

### 3. Actions of solvable groups

In this section we use results of Dunwoody to examine the actions of solvable groups. They show that although there can be an arbitrarily large number of weak equivalence classes of minimal genus actions, all actions on a handlebody whose genus is above the minimal one are equivalent.
From [2], we have the following fact.

**Theorem 3.1.** Let $G$ be a solvable group, and let $n > \mu(G)$. Then any two $n$-element generating vectors are Nielsen equivalent.

Applying theorem 2.3, we have immediately:

**Corollary 3.2.** Let $G$ be solvable and let $g$ be greater than $\Psi(G)$. Then any two actions of $G$ on handlebodies of genus $g$ are equivalent. Consequently, any action of a solvable group on a handlebody not of minimal genus can be destabilized to any given action of minimal genus.

Some examples due to Dunwoody also have implications for free actions. The following examples are from [3]:

**Theorem 3.3.** For every pair of positive integers $n$ and $N$, there exists a $p$-group $G(n,N)$, nilpotent of length 2 and with $\mu(G(n,N)) = n$, which has at least $N$ weak Nielsen equivalence classes of $n$-element generating sets.

This yields immediately:

**Corollary 3.4.** For every pair of positive integers $n$ and $N$, there exists a $p$-group $G(n,N)$, nilpotent of length 2, with $\mu(G(n,N)) = n$ and $w(0) \geq N$.

Of course, by corollary 3.2, all of the actions in $E(0)$ become equivalent after a single stabilization.

We include here an elementary and transparent example of two weakly inequivalent $G$-actions of a nilpotent group of order $2^{12}$ on the handlebody of genus 8193. B. Neumann [21] gave a nilpotent group of order $2^{13}$ admitting weakly inequivalent actions on the handlebody of this same genus; his example is slightly more complicated, but requires only two generators, whereas ours requires 3. Dunwoody’s examples in [3] are considerably more sophisticated renderings of the one we give here.

Let $G$ be the group with presentation

\[
\langle x, y, z \mid x^8 = y^8 = z^{64} = 1, [x, z] = [y, z] = 1, [x, y] = z^8 \rangle.
\]

There is an extension

\[1 \rightarrow C_{64} \rightarrow G \rightarrow C_8 \oplus C_8 \rightarrow 1,\]

where $C_{64}$ is the subgroup generated by $z$, and the images of $x$ and $y$ generate $C_8 \oplus C_8$. Now $xyx^{-1} = yz^8$, from which it follows that $x^ay^bzx^{-a} = y^bz^{8ab}$ for all integers $a$ and $b$.

We will show that the generating vectors $(x, y, z)$ and $(x, y, z^3)$ are not weakly Nielsen equivalent. By theorem 2.3, these correspond to two actions of $G$ on the handlebody of genus $g = 1 + |G|(3 - 1) = 8193$ which are not weakly equivalent.

Every element of $G$ can be written in the form $x^ay^bz^c$, where $a$ and $b$ are integers mod 8, and $c$ is mod 64, and the inverse of $x^ay^bz^c$ is $x^{-a}y^{-b}z^{-c}z^{-8ab}$. Using this, we can calculate that $[x^py^qz^m, x^ry^sz^n] = z^{8(ps-rq)}$. 
Sending \( x \) to \((1, 0, 0)\), \( y \) to \((0, 1, 0)\), and \( z \) to \((0, 0, 1)\) defines a homomorphism \( G \to C_8 \oplus C_8 \oplus C_8 \). Regarding these as vectors of integers mod 8, any three-element generating set determines a \( 3 \times 3 \) matrix with entries mod 8. Nielsen moves on the generating set only change the determinant of the associated matrix by multiplication by \( \pm 1 \). The determinant of the matrix associated to \( \{x, y, z^3\} \) is 3.

The subgroup \( C_{64} \) is central. If \( x^ay^bz^c \) has \( 0 < a < 8 \), then \( yx^ay^bz^cy^{-1} = x^ay^bz^{c-8a} \), so \( x^ay^bz^c \) is not central. Similarly, if \( 0 < b < 8 \), the element is not central. Therefore the center of \( G \) is exactly \( C_{64} \), and any automorphism of \( G \) must carry \( z \) to \( z^d \) for some \( d \).

Now consider any automorphism \( \alpha \) of \( G \), with \( \alpha(x) = x^py^qz^m \), \( \alpha(y) = x^ry^sz^n \), and \( \alpha(z) = z^d \). From above, \( \alpha([x, y]) = z^{8(ps-rq)} \). Since this must equal \( \alpha(z^8) = z^{8d} \), it follows that \( ps - rq \) is congruent to \( d \) modulo 8. The matrix associated to the generating vector \((\alpha(x), \alpha(y), \alpha(z))\) is

\[
\begin{pmatrix}
p & q & m \\
r & s & n \\
0 & 0 & d
\end{pmatrix},
\]

which has determinant \((ps - rq)d\). This is congruent to \( d^2 \) (mod 8). Since the only squares modulo 8 are 1 and 4, it follows that \((\alpha(x), \alpha(y), \alpha(z))\) cannot be Nielsen equivalent to the generating set \((x, y, z^3)\). Therefore the generating vectors \((x, y, z)\) and \((x, y, z^3)\) are not weakly Nielsen equivalent.

### 4. Abelian and dihedral groups

In this section we examine the actions of abelian and dihedral groups. We will see that for either of these two kinds of groups, all actions on the minimal genus are weakly equivalent, but that there can be arbitrarily large numbers of equivalence classes.

**Theorem 4.1.** Let \( A \) be a finite abelian group, \( G \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_m \) where \( d_{i+1}|d_i \) for \( 1 \leq i < m \). Then \( \Psi(A) = 1 + |A|/(m - 1) \). If \( g > \Psi(A) \), then any two \( A \)-actions on a handlebody of genus \( g \) are equivalent. Any two \( A \)-actions on a handlebody of genus \( \Psi(A) \) are weakly equivalent. If \( d_m = 2 \), then all \( A \)-actions on the handlebody of genus \( \Psi(A) \) are equivalent, while if \( d_m > 2 \), then there are exactly \( \varphi(d_m)/2 \) equivalence classes, where \( \varphi \) denotes the Euler \( \varphi \)-function.

In preparation for the proof, we will prove a general lemma about vectors of elements in solvable groups. By a cyclic tower for a group \( G \) we mean a descending sequence of subgroups \( G = G_1 \supset G_2 \supset \cdots \supset G_m \supset \{1\} \) such that \( G_{i+1} \) is normal in \( G_i \) and \( G_i/G_{i+1} \) is cyclic. We allow \( G_i \) to equal \( G_{i+1} \) for some \( i \), also we define \( G_j = \{1\} \) for all \( j > m \).

**Lemma 4.2.** Let \( G \) be solvable and let \( G = G_1 \supset G_2 \supset \cdots \supset G_m \supset \{1\} \) be a cyclic tower for \( G \). Let \( T = (t_1, \ldots, t_n) \) be a vector of elements of \( G \) (not necessarily a generating vector). Then \( T \) is Nielsen equivalent to a vector with \( t_i \in G_i \) for all \( i \) (in particular, \( t_i = 1 \) for any \( i > m \)).
Proof. Let $H$ be the subgroup of $G$ generated by $T$. Replacing each $G_i$ by $H \cap G_i$, we may assume that $T$ is a generating set for $G$.

Suppose first that $G$ is a cyclic group $C_m$, generated by $t$, and that at least two of the $t_i$, say $t_1$ and $t_2$, are not equal to 1. Write $t_1 = t^a$ and $t_2 = t^b$, where $1 \leq a, b \leq m - 1$. If $a \geq b$, replace $t_1$ by $t_1t_2^{-1}$, then $t_1 = t^{a-b}$ and $t_2 = t^b$. If $a < b$, replace $t_2$ by $t_2t_1^{-1}$. Repeat this process until either $t_1 = 1$ or $t_2 = 1$. By an interchange, we may assume that $t_1 \neq 1$ and $t_2 = 1$. Repeating the process with the other elements, we eventually achieve that $t_i = 1$ for all $i \geq 2$.

In the general case, we may regard $T$ as a vector of elements in the cyclic group $G_1/G_2$. By the cyclic case, we may assume that $t_i \in G_2$ for all $i > 1$. The subgroup of $G$ generated by $\{t_2, \ldots, t_n\}$ has a cyclic tower of length $m - 1$. By induction on $m$, $(t_2, \ldots, t_n)$ is Nielsen equivalent to a vector with $t_i \in G_i$.

Proof of Theorem 4.1. We have $\mu = m$, since the minimal number of generators of $A \otimes (\mathbb{Z}/d_m) = (\mathbb{Z}/d_m)^m$ is $m$, so $\Psi(A) = 1 + |A|(m - 1)$. If $g > \Psi(A)$, then corollary 3.2 shows that any two $A$-actions on a handlebody of genus $g$ are equivalent.

Regard $G$ as solvable with cyclic tower given by $G_i = \mathbb{Z}/d_i \oplus \cdots \oplus \mathbb{Z}/d_m$. Let $S = (s_1, \ldots, s_n)$ be any generating vector. By lemma 4.2 we may assume that $s_i \in G_i$, and consequently $s_1$ generates $G_1/G_2$. Now $(s_2, \ldots, s_n)$ generate $G_2$, since otherwise the quotient of $A$ by the subgroup that it generates is of the form $\mathbb{Z}/d_1 \oplus A_2$ with $d_1$ and the order of $A_1$ not relatively prime, but this quotient could not be generated by $s_1$. So we may apply Nielsen moves changing $s_1$ by multiples of the other $s_i$ until $s_1 \in \mathbb{Z}/d_1$. Inductively, we may assume that $s_i \in \mathbb{Z}/d_i$ and generates $\mathbb{Z}/d_i$ for $i \leq m$, and $s_i = 1$ for $i > m$.

Let $T = (t_1, \ldots, t_m)$ be any another $n$-element generating vector for $A$. Again, we may assume that $t_i \in \mathbb{Z}/d_i$ and $t_i$ generates $\mathbb{Z}/d_i$, so $t_i = s_i^{p_i}$ and $s_i = t_i^{q_i}$, for some $p_i$ and $q_i$ relatively prime to $d_i$. By Nielsen moves, replace $t_2$ by $t_2t_1^{q_i}$, then $t_1$ by $t_1(t_2t_1^{q_1})^{-p_1} = t_1t_2^{-p_1}t_1^{-p_1q_1} = t_2^{-p_1}$. Since $p_1$ is relatively prime to $d_1$ and $d_2$ divides $d_1$, $p_1$ is also relatively prime to $d_2$. So there exists $r$ with $(t_2^{-p_1})^r = t_2$, and by Nielsen moves we may replace $t_2t_1^{q_i}$ by $t_1^{q_i} = s_1$. Interchanging $t_1$ and $t_2$, we have that $t_1 = s_1$ and $t_2$ still generates $\mathbb{Z}/d_2$. Continuing, we may assume that $t_i = s_i$ for all $i < n$. If $n > m$, then $t_i = s_i$ for all $i$ since both equal 1 for $i > m$. This proves that all actions on genera greater than $\Psi(A)$ are equivalent. If $n = m$, then we have only that $t_m = s_m^{p_m}$, with $p$ relatively prime to $d_m$.

If $d_m = 2$, then $T$ must be Nielsen equivalent to $S$. From now on, assume that $d_m > 2$. The automorphism $\alpha$ of $A$ defined by $\alpha(s_i) = s_i$ for $i < m$ and $\alpha(s_m) = s_m^{p_m}$ shows that all $m$-element generating vectors are weakly Nielsen equivalent. Since $(s_1, \ldots, s_{m-1}, s_m^{p_m})$ is Nielsen equivalent to $(s_1, \ldots, s_{m-1}, s_m^{p_m})$, there are at most $\phi(d_m)/2$ equivalence classes. To show that this is a lower bound for the number of equivalence classes, we define a
function from (ordered) generating sets to $m \times m$ matrices as follows. Regard $s_i$ as an $m$-tuple with all entries 0 except for a 1 in the $i^{th}$ place. Working mod $d_m$, any generating $m$-vector then determines an $m \times m$ matrix with $i^{th}$ row the vector corresponding to $t_i$. A Nielsen move corresponds to multiplying by an elementary matrix (that adds one row to another, or interchanges two rows, or multiplies one row by $-1$). These elementary matrices have determinant ±1, so Nielsen equivalent generating sets have determinants that are either equal or are negatives, as elements of $\mathbb{Z}/d_m$. The determinant of the matrix corresponding to $(s_1, \ldots, s_{m-1}, s_m^p)$ is $p$. This gives the lower bound of $\phi(d_m)/2$ on the number of equivalence classes.

Specializing to the case of a cyclic group, we have the following:

**Corollary 4.3.** For the cyclic group $C_k$, $\Psi(C_k) = 1$. If $g > 1$, then any two $C_k$-actions on a handlebody of genus $g$ are equivalent. On the solid torus, any two $C_k$-actions are weakly equivalent, any two $C_2$-actions are equivalent, and if $k > 2$, then there are exactly $\phi(k)/2$ equivalence classes of $C_k$-actions.

Explicitly, if $\phi_q$ is the action on $S^1 \times D^2$ defined by $\phi_q(t)(\exp(id), x) = (\exp(id + 2\pi q/k), x)$ where $t$ is a fixed generator of $C_k$ and $q$ is relatively prime to $k$, then $\phi_{q_1}$ and $\phi_{q_2}$ are equivalent if and only if $q_1 \equiv q_2 \pmod{k}$.

**Proof.** The corollary is immediate from the statement of theorem 4.1, except for the explicit description of the equivalence classes. The matrix corresponding to $\phi_q$ in the proof of theorem 4.1 is $[q]$, and the last paragraph of the proof shows the condition for equivalence of $\phi_{q_1}$ and $\phi_{q_2}$. \qed

**Theorem 4.4.** Let $D_{2m}$ be the dihedral group of order $2m$. Then $\Psi(D_{2m}) = 2m + 1$. Any two $D_{2m}$-actions on the handlebody of genus $2m + 1$ are weakly equivalent. If $m = 2$, all actions on the handlebody of genus $2m + 1$ are equivalent, and if $m > 2$ then there are exactly $\phi(m)/2$ equivalence classes.

**Proof.** Regard $D_{2m}$ as $\langle a, b \mid b^m = 1, ab^{-1}a = b^{-1}b \rangle$, and let $C_m$ be the cyclic subgroup of $D_{2m}$ generated by $b$. By lemma 4.2, applied to the tower $D_{2m} \supset C_m \supset \{1\}$, any two-element generating vector $S = (x, y)$ is Nielsen equivalent to one of the form $(ab^j, b^j)$. Since $ab^j$ has order 2, $b^j$ must generate $C_m$, so gcd($m, j$) = 1 and the vector is equivalent to $(a, b^j)$. For $m = 2$, the proof is complete. For $m > 2$, there are at most $\phi(m)/2$ equivalence classes, since $(a, b^j)$ is equivalent to $(a, b^{-j})$. On the other hand, it is essentially an observation of D. Higman (see [21]) that the pair (possibly equal) of conjugacy classes of $[x, y]$ and $[y, x]$ is an invariant of the Nielsen equivalence class of $(x, y)$. In our case, $[a, b^j]^{\pm 1} = b^{\pm 2j}$; since $b^{2j}$ is conjugate to $b^{\pm 2\ell}$ only when $j = \pm \ell$, there are exactly $\phi(m)/2$ equivalence classes. \qed

**Corollary 4.5.** The smallest genus of handlebody admitting two inequivalent actions is genus 11, which has two equivalence classes of $D_{10}$-actions. The smallest genus of handlebody admitting two inequivalent actions of an abelian group is genus 26, which has two equivalence classes of $C_5 \times C_5$-actions.
Proof. Theorem 4.1 verifies the assertion about abelian groups. By theorem 4.4, the smallest-genus inequivalent actions of dihedral groups are the two $D_{10}$-actions in the corollary. Only cyclic groups have $\mu(G) = 1$, so $\Psi(G) \geq |G| + 1$ for a noncyclic group. The only nonabelian and nondihedral group of order smaller than 11 is the quaternion group, which is easily checked to have only one equivalence class of generating pair. \hfill \Box

5. Free Actions of $\text{PSL}(2,3^p)$ ($p$ a prime number)

We have mentioned that Gilman [3] proved that for $p$ prime, all 3-element generating sets of $\text{PSL}(2,p)$ are equivalent, and Evans [4] proved the same for $\text{PSL}(2,2^m)$ and for the Suzuki groups $\text{Sz}(2^{2m-1})$ for $m \geq 2$. In this section, we prove the same for all $\text{PSL}(2,3^p)$ with $p$ prime. The main result is the following.

**Theorem 5.1.** Let $p$ be prime. If $n > 2$, then any $n$-element generating vector for $\text{PSL}(2,3^p)$ is redundant.

In particular, these groups include the case of $\text{PSL}(2,9) \cong A_6$ ([27], p. 412). Before proving theorem 5.1, we deduce a corollary.

**Corollary 5.2.** Let $p$ be prime. If $n > 2$, then any two $n$-element generating vectors for $\text{PSL}(2,3^p)$ are Nielsen equivalent. Consequently, for any handlebody of genus above the minimal one, $1 + 3^p(3^{2p} - 1)/2$, all $\text{PSL}(2,3^p)$-actions are equivalent.

Proof. We recall from [3] that a 2-generator group $G$ is of spread 2 when for any pair $h_1, h_2$ of nontrivial elements of $G$, there exists $y \in G$ such that $\langle y, h_1 \rangle = \langle y, h_2 \rangle = G$. When $p > 2$, theorem 4.02 of [3] shows that $\text{PSL}(2,3^p)$ has spread 2. For $p = 2$, $\text{PSL}(2,3^2)$ is isomorphic to $A_6$ ([27], p. 412), so it has spread 2 by proposition 3.02 of [3].

Let $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ be any two $n$-element generating sets. By repeated use of theorem 5.1, we may assume they are of the form $(s_1, s_2, 1, (1))$ and $(t_1, t_2, 1, (1))$, where (1) indicates a possibly empty string of 1’s. Choose $y$ such that $\langle y, s_1 \rangle = \langle y, t_1 \rangle = \text{PSL}(2,3^p)$. As in lemma 2.8 of [3], we have equivalences $(s_1, s_2, 1, (1)) \sim (s_1, s_2, y, (1)) \sim (s_1, y, 1, (1)) \sim (s_1, y, t_1, (1)) \sim (y, t_1, 1, (1)) \sim (y, t_1, t_2, (1)) \sim (t_1, t_2, 1, (1))$. \hfill \Box

To prepare for the proof of theorem 5.1, we will list several group-theoretic results. The first is lemma 4.10 of [3].

**Lemma 5.3.** Let $G$ be a simple group generated by involutions $g_1, \ldots, g_n$, $n \geq 3$. Then $(g_1, \ldots, g_n)$ is redundant.

By direct calculation, we have the following information.

**Lemma 5.4.** Let $x$ and $y$ be elements of $S_4$ such that $x \neq y$ and $x \neq y^{-1}$.

(a) If $|x| = |y| = 3$, then $|xy| = |yx| = 2 + t$ and $|yx^2| = |y^2x| = 3 - t$, where $t$ is either 0 or 1.
Notice that a Sylow $3$-subgroup $Q$ of $\text{PSL}(2,5)$, since $|\text{PSL}(2,5)| = 3^p(3^{2p}-1)/2$, and the group $K$ is its normalizer, since $K$ is maximal. We will refer to a subgroup $K$ as in theorem (5.8)(b) as a Sylow $3$-normalizer.
We will also need the following facts about centralizers, which follow from 3(6.5) and 3(6.8) of [27].

**Lemma 5.9.** The centralizer of any element in $\text{PSL}(2, 3^p)$ is a maximal abelian subgroup, and is either cyclic of order prime to $3$, isomorphic to $C_2 \times C_2$, or is a Sylow 3-subgroup of $\text{PSL}(2, 3^p)$. Distinct maximal abelian subgroups have trivial intersection.

In particular, any two distinct Sylow 3-subgroups have trivial intersection.

**Proof of theorem 5.1.** Fix an $n$-element generating vector $T = (t_1, \ldots, t_n)$ of $\text{PSL}(2, 3^p)$. Suppose first that $n > 3$. We will show that $T$ is redundant.

Put $H = \langle t_1, t_2, t_3 \rangle$. If $H = \text{PSL}(2, 3^p)$, then $T$ is redundant. If $H$ is cyclic, dihedral, $A_4$, or $S_4$, then $T$ is redundant by theorem 5.1, and if $H$ is $A_5$, then lemma 5.3 applies. If $H$ is contained in a Sylow 3-normalizer $K$ of a Sylow 3-subgroup $Q$, then $(t_1, t_2, t_3)$ is (Nielsen) equivalent to $(s_1, s_2, s_3)$ such that $s_1 \in Q$ (and $s_1 \neq 1$, otherwise $T$ is redundant). As $\text{PSL}(2, 3^p) \neq K$, there is a $j \geq 4$ for which $(s_1, s_2, t_j)$ is not a subgroup of $K$. Since the Sylow 3-subgroups intersect trivially, $(s_1, s_2, t_j)$ cannot be contained in any other Sylow 3-normalizer. So $(s_1, s_2, t_j)$ is redundant or $\text{PSL}(2, 3^p) = (s_1, s_2, t_j)$. In either case, $T$ is redundant.

In the rest of the proof, we may assume that $T = (u, v, w)$. We assume at every stage of the argument that no two elements of $T$ generate a cyclic group or all of $\text{PSL}(2, 3^p)$, since otherwise $T$ is clearly redundant. We will argue that $T$ either is redundant or is equivalent to $(x, y, z)$ with $x$, $y$, and $z$ all of order 2. This will prove the theorem, since lemma 5.3 shows that the latter is also redundant.

We first show that $T$ is (redundant or) equivalent to $(x, v, w)$ with $x$ of order 2. If none of $u$, $v$, or $w$ already has order 2, put $H = \langle u, v \rangle$. We may assume that $H$ is not contained in any Sylow 3-normalizer $K$. For if it is, then since $K/Q$ is cyclic, we may assume that $u \in Q$ (and $u \neq 1$). Interchanging $v$ and $w$ if necessary, we may assume that $H$ is not contained in $K$. Since the Sylow 3-subgroups intersect trivially, $H$ cannot be contained in any other Sylow 3-normalizer.

Since neither of its generators has order 2, $H$ cannot be dihedral. If $H$ is $A_4$ or $S_4$, then lemma 5.4 can be used to find $x$. If it is $A_5$, then lemma 5.6 can be used. So we may write $T$ as $(x, v, w)$ with $x$ of order 2.

Next, we will show that $(x, v, w)$ is equivalent to $(x, y, w)$ with $y$ also of order 2. Put $H = \langle v, w \rangle$. By lemma 5.4, we may assume that $H$ is nonabelian.

We may assume that $H$ is not contained in any Sylow 3-normalizer. For suppose it lies in $K$. Since $K/Q$ is cyclic, we may assume that $v \in Q$. Since $x$ cannot be in $K$, $xw$ cannot be in $K$. Since the Sylow 3-subgroups are disjoint, $(v, xw)$ cannot be in any Sylow 3-normalizer. If $H$ is dihedral, then one of $v$ or $w$ already has order 2 and will be $y$. If $H$ is one of $A_4$, $S_4$, or $A_5$, then as before, either lemma 5.4 or 5.6 produces $y$. 


We now have $T = (x, y, w)$ where $x$ and $y$ are elements of order 2. We may assume that $|w| > 2$ and, using lemma 5.7, that $[x, y] \neq 1$, so $|xy| > 2$. Put $H = \langle xy, w \rangle$. It cannot be dihedral, since neither generator is of order 2.

Suppose that $H$ is contained in a Sylow 3-normalizer $K$. If $xy$ is of order 3, then since $x$ inverts $xy$, $x \in K$ and hence $y \in K$, so $K = \text{PSL}(2, 3^p)$, a contradiction. So $xy \notin Q$.

Assume first that $p > 2$, so that $|K|$ is odd. Then $\langle x, w \rangle$ is not contained in a Sylow 3-normalizer. If it is dihedral, then we can take $z = xw$. Otherwise, it is isomorphic to $A_4$, so $w$ has order 3. Similarly, considering $\langle x, (xy)w \rangle$, we may assume that $(xy)w$ has order 3. Since this element lies in $K$, it must lie in $Q$, forcing the contradiction that $xy \notin Q$.

When $p = 2$, we have from p. 398 of [27] that $K$ is a split extension of $C_3 \times C_3$ by $C_4$. If $k$ generates $C_4$, then $q^2$ must act by inverting each element of $C_3 \times C_3$ (otherwise it would act trivially, but then the centralizer of an element of $Q$ would be larger than $Q$, contradicting lemma 5.9) so $k^2q$ has order 2 for any $q \in C_3 \times C_3$. Since $xy \notin Q$ and $|xy| > 2$, we have $|xy| = 4$. If $w$ has order 3, then $w(xy)^2$ is of order 2. If $w$ is of order 4, then $w(xy)$ is of order 2. So we may assume that $H$ is not contained in a Sylow 3-normalizer.

If $H$ is isomorphic to $A_4$ or $S_4$, then lemma 5.4 applies. The case of $p > 2$ is complete, so we may assume that $p = 2$ and $H \cong A_5$. Since $T$ is equivalent to $\langle x, xy, w \rangle$, we may apply lemma 5.6 to $\langle xy, w \rangle$ to obtain a new generating vector of the form $\langle x, y, w \rangle$ where $x$ and $y$ have order 2, $w$ has order 5, and $\langle y, w \rangle \cong A_5$. By the previous arguments, we may also assume that $\langle x, w \rangle$ and $\langle xy, w \rangle$ are isomorphic to $A_5$.

In the remainder of the proof, it is convenient to regard $\text{PSL}(2, 9)$ as $A_6$. Also, we may apply any automorphism to all elements of the generating vector, since any vector weakly equivalent to a redundant vector is redundant.

According to 3(2.19) of [27], there are two conjugacy classes of $A_5$-subgroups in $A_6$. One class consists of the six $A_5$-subgroups that stabilize one of the six letters, and the other class consists of six $A_5$-subgroups that act transitively on the six letters. Moreover, there exists an automorphism $\alpha$ of $A_6$ that interchanges the two classes. Applying $\alpha$ to each of the generating elements, if necessary, we may assume that $\langle xy, w \rangle$ fixes a letter. Since $w$ is a 5-cycle, this must be the unique letter fixed by $w$. Neither of $\langle x, w \rangle$ and $\langle y, w \rangle$ can fix a letter. For if so, it would fix the same letter as $\langle xy, w \rangle$, and one of $x$ or $y$ would be in $\langle xy, w \rangle$, forcing the latter to be all of $A_6$. Applying $\alpha$ to all generators, $\langle x, w \rangle$ and $\langle y, w \rangle$ become $A_5$-subgroups fixing the letter fixed by $w$, so are equal, achieving the same contradiction.

The principal difficulties in extending the proof of theorem 5.1 to more general cases $\text{PSL}(2, q^r)$ seem to be the more complicated Sylow $q$-normalizers, when $q \neq 3$, the presence of subgroups of the form $\text{PSL}(2, q^r)$ when $r$ is not prime, and analyzing the case when any two generators generate an $A_5$ subgroup.
6. Stabilization of actions

Throughout this section, $G$ is an arbitrary finite group, and as usual $\mu(G)$ or just $\mu$ will denote the minimal number of generators for $G$, and $\ell(G)$ or just $\ell$ the maximum length of a chain of strictly decreasing nonzero subgroups of $G$. Clearly $\mu \leq \ell$, and if $|G| = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then $\ell \leq \alpha_1 + \cdots + \alpha_r$.

**Proposition 6.1.** Any two actions of a group $G$ on handlebodies of the same genus become equivalent after at most $\mu(G)$ stabilizations.

**Proof.** Let $S = (s_1, \ldots, s_{\mu})$ be a generating vector of minimal length, and let $T = (t_1, \ldots, t_m)$ be any generating vector. Put $G_1 = \langle t_1, \ldots, t_i \rangle$, so $G = G_m \geq G_{m-1} \geq \cdots \geq G_1 \geq \{1\}$. Since $m > \ell$, $G_i = G_{i-1}$ for some $i > 0$, so $t_i$ can be written as a word in $t_1, \ldots, t_{i-1}$. Therefore $T$ is equivalent to $(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_m)$. So we may assume that $t_1 = 1$. Since $t_1 = 1$, $T$ is equivalent to $(s_1, t_1, \ldots, t_m)$. Let $G_1 = \langle s_1 \rangle$ and $G_i = \langle s_1, t_1, \ldots, t_{i-1} \rangle$ for $i > 1$. Since $s_1 \neq 1$, we must have $G_i = G_{i-1}$ for some $i > 1$. So $t_{i-1} \in \langle s_1, t_1, \ldots, t_{i-2} \rangle$, and therefore $T$ is equivalent to $(s_1, t_1, \ldots, t_{i-2}, 1, t_{i-1}, \ldots, t_m)$. We may reselect notation so that $T$ is equivalent to $(s_1, 1, t_3, \ldots, t_m)$. Inductively, suppose that $T$ is of the form $(s_1, \ldots, s_k, t_{k+1}, \ldots, t_m)$, for some $k < \mu$. Put $G_i = \langle s_1, \ldots, s_i \rangle$ for $i \leq k+1$ and to $(s_1, \ldots, s_k, s_{k+1}, t_{k+1}, \ldots, t_{i-1})$ for $i > k+1$. For $i \leq k+1$, since $S$ is a minimal generating set, we have $G_i \neq G_{i-1}$, so we must have $G_j = G_{j-1}$ for some $j > k+1$. So $t_{j-1} \in \langle s_1, \ldots, s_{k+1}, t_{k+1}, \ldots, t_{j-2} \rangle$. This implies that $T$ is equivalent to $(s_1, \ldots, s_{k+1}, t_{k+1}, \ldots, t_{j-2}, 1, t_j, \ldots, t_m)$, and, after reselecting notation, to $(s_1, \ldots, s_{k+1}, t_{k+2}, \ldots, t_m)$. So $T$ is equivalent to $(s_1, \ldots, s_{\mu}, t_{\mu+1}, \ldots, t_m)$, and hence to $(s_1, \ldots, s_{\mu}, 1, \ldots, 1)$.

Since $\ell(G) + 1$ is at most $1 + \log_2(|G|)$, proposition 5.2 improves theorem 3 of [6], which shows that any two generating vectors of length at least $2 \log_2(|G|)$ are equivalent.

**Corollary 6.3.** If $g \geq 1 + |G| \ell(G)$, then any two actions of $G$ on a handlebody of genus $g$ are equivalent.

If $A = (\mathbb{Z}/p)^k$, for $p$ prime, then $\mu(A) = \ell(A) = k$. Thus by theorem 4.1, if $p \geq 5$, there are inequivalent actions on the handlebody of genus $1 + (\ell(A) - 1)|A|$, showing that the estimate in corollary 5.3 is the best possible,
in general. On the other hand, it appears to be far from the best possible for many cases. Frequently there is a large gap between $\mu(G)$ and $\ell(G)$ (for example, all symmetric groups can be generated by two elements, but have values of $\ell(G)$ that are arbitrarily large).

7. Questions

Our results on free actions are far from complete. The most obvious question is:

**Question 1:** Are all actions on genera above the minimum genus equivalent?

That is, is $e(k) = 1$ for all $k \geq 1$? Algebraically, if $n > \mu(G)$ are any two $n$-element generating sets of $G$ Nielsen equivalent? According to p. 92 of [3], this algebraic version was first asked by F. Waldhausen. It has been resolved negatively for infinite groups. The first example appears to be due to G. A. Noskov [23]; and general constructions are given in [5]. An affirmative answer to Question 1 for finite groups would imply affirmative answers to the next two questions:

**Question 2:** Is every action the stabilization of a minimal genus action?

That is, is $E(0) \to E(k)$ surjective for all $k$? Algebraically, is every generating vector $(s_1, \ldots, s_n)$ Nielsen equivalent to a generating vector of the form $(t_1, \ldots, t_\mu, 1, \ldots, 1)$?

**Question 3:** Are all actions of a group $G$ on a handlebody of genus $g$ equivalent after an elementary stabilization?

That is, is $E(k) \to E(k+1)$ always constant? Algebraically, are all generating vectors of the forms $(s_1, \ldots, s_n, 1)$ and $(t_1, \ldots, t_n, 1)$ equivalent?

We can ask whether our example in section 3 is the smallest of its kind.

**Question 4:** Are there weakly inequivalent actions of a nilpotent group on a handlebody of genus less than 8193?

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