Dirac-Born-Infeld action, Seiberg-Witten map, and Wilson Lines

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ABSTRACT

We write the recently conjectured action for transformation of the ordinary Born-Infeld action under the Seiberg-Witten map with one open Wilson contour in a manifestly non-commutative gauge invariant form. This action contains the non-constant closed string fields, higher order derivatives of the non-commutative gauge fields through the $\ast_N$-product, and a Wilson operator. We extend this non-commutative $D_9$-brane action to the action for $D_p$-brane by transforming it under T-duality. Using this non-commutative $D_p$-brane action we then evaluate the linear couplings of the graviton and dilaton to the brane for arbitrary non-commutative parameters. By taking the Seiberg-Witten limit we show that they reduce exactly to the known results of the energy-momentum tensor of the non-commutative super Yang-Mills theory. We take this as an evidence that the non-commutative action in the Seiberg-Witten limit includes properly all derivative correction terms.

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1 Introduction

Recent years have seen dramatic progress in the understanding of non-perturbative aspects of string theory – see, e.g., [1]. With these studies has come the realization that solitonic extended objects, other than just strings, play an essential role. An important object in these investigations has been Dirichlet branes [2]. D-branes are non-perturbative states whose perturbative excitations are described by the fundamental open string states on their world-volumes. They are also source of various perturbative closed string states including the Ramond-Ramond states.

The massless excitations of a single D-brane in type II theories are a U(1) vector, $A_a$, a set of scalars, $\Phi^i$, describing the transverse motion of the brane, and their superpartners [3]. The leading order low-energy effective action for the massless fields corresponds to a dimensional reduction of a ten dimensional U(1) Yang Mills theory. As usual in string theory, there are higher order $\alpha' = \ell_s^2$ corrections, where $\ell_s$ is the string length scale. As long as derivatives of the field strength (and second derivatives of the scalars) are small compared to $\ell_s$, then the action takes a Dirac-Born-Infeld form [4], in which the spacetime metric is the trivial Minkowskian metric and all other closed string fields are zero. One may naturally extend this flat space DBI action to the appropriate action in the curved space by adding the non-constant closed string fields, i.e., the metric, dilaton and Kalb-Ramond fields, to the D-brane action. The resulting world-volume action is [4]

$$S = -T_p \int d^{p+1}x \ e^{-\phi(\lambda \Phi)} \sqrt{-\text{det}(P[g_{ab}(\lambda \Phi) + B_{ab}(\lambda \Phi)]) + \lambda F_{ab}},$$

where we have defined $\lambda = 2\pi \alpha'$, and the D-brane tension is $T_p = (g_s (2\pi)^p \alpha'^{(p+1)/2})^{-1}$. Here, $F_{ab}$ is the abelian field strength of the world-volume gauge field, while the metric and antisymmetric tensors are the pull-backs of the bulk tensors to the D-brane world-volume, e.g.,

$$P[g_{ab}] = g_{ab} + 2\lambda g_{i(a} \partial_{b)} \Phi^i + \lambda^2 g_{ij} \partial_a \Phi^i \partial_b \Phi^j,$$

where we have used that fact that we are employing static gauge throughout the paper, i.e., $X^a(x) = x^a$ for world-volume and $\lambda \Phi^i(x^a)$ for transverse coordinates. The action [4] describes the dynamics of the open string fields, i.e., $A_a$ and $\Phi^i$, and their couplings with each other and with the closed string fields. Dynamics of the closed string fields, on the other hand, are described by the bulk supergravity action in which we are not interested in this paper. The derivatives of the gauge field strength, and the second derivatives of the scalars are not included in this action. However, the action includes the transverse derivatives of the closed string fields through the Taylor expansion of these fields [3, 5].

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1Our index conventions are that Greek indices take values in the entire ten-dimensional space-time, e.g., $\mu, \nu = 0, 1, \ldots, 9$; early Latin indices take values in the world-volume , e.g., $a, b = 0, 1, \ldots, p$; and middle Latin indices take values in the transverse space, e.g., $i, j = p + 1, \ldots, 9$. 

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\[\phi(\lambda \Phi) = \phi^0 + \lambda \Phi^i \partial_i \phi^0 + \frac{\lambda^2}{2} \Phi^i \Phi^j \partial_i \partial_j \phi^0 + \frac{\lambda^3}{3!} \Phi^i \Phi^j \Phi^k \partial_i \partial_j \partial_k \phi^0 + \cdots\]

where the subscript 0 means that the closed string field should be evaluated at the position of the brane, \(i.e., x^i = 0\). Appearance of the above derivative terms in the DBI action verified in [6] by explicit evaluation of string theory S-matrix elements. At the same time the string theory calculation confirms that there is no analogous world-volume derivative terms in the action.

On the other hand, the coordinates of the D\(_p\)-brane with background constant B-flux becomes non-commutative\([7, 8, 9, 10, 11]\). The leading order effective action of the D\(_p\)-brane in the SW limit is the non-commutative U(1) Yang-Mills theory. In this action the symmetric part of \((g + \lambda B)^{-1}\), where \(g\) and \(B\) are the background constant fields, plays the role of the world-volume metric and the antisymmetric part plays the role of the non-commutative parameters. Higher order \(\alpha'\) correction to this action for the case that the derivative of the non-commutative gauge field strength is small is the Born-Infeld generalization of the non-commutative Yang-Mills theory\([11]\). In this approximation, however, one should use the ordinary product as the multiplication rule between fields in the Born-Infeld action. Extension of this Born-Infeld action to the action which includes the couplings of the non-constant closed string fields to the D\(_p\)-brane is not trivial, see \([12]\) for a proposal.

The linear coupling of the graviton to the non-commutative gauge fields and scalar fields of D\(_p\)-brane was found in \([13]\) by explicit evaluation of the disk S-matrix elements of one graviton and infinite number of massless open string states in the SW limit. This calculation shows that the couplings in the superstring theory and in the bosonic theory are not identical. We shall find in the present paper a non-commutative action which is the transformation of the DBI action \([1]\) under the Seiberg-Witten map with one open Wilson contour (see eq.\((14)\)). This action includes linear and non-linear couplings of the massless closed string fields to the D-branes of the superstring theory. The linear couplings in the SW limit are exactly those that were found in \([13]\).

Our strategy to find the above mentioned action is that we start with expanding the action \((1)\) which properly includes the non-constant closed string fields. We then transform the open string commutative fields in this action to the non-commutative fields using the SW map. Since the SW map transforms each commutative open string field in the DBI action \((1)\) to infinite number of non-commutative fields that smear along one open Wilson contour\([14, 15, 16, 17]\), the resulting action has infinite number of open Wilson contours. On the other hand, we know that for evaluating the string theory S-matrix elements corresponding to the disk level effective action, the open string vertex operators must be inserted on the boundary of the disk world-sheet. Moreover, from the results in \([18, 19]\), we know that the boundary of the disk world-sheet corresponds to one open Wilson contour. Hence, to map the disk level effective action \((1)\) to another disk level action, we impose an extra operation
that reduces the infinite number of Wilson lines to only one line. The first attempt in this direction was made in [19]. In [19] we found that in the transformed action the closed string fields should be functional of the non-commutative gauge fields as well as the non-commutative scalar fields. This feature was then confirmed by explicit evaluation of disk $S$-matrix elements of one closed string and infinite number of open string states for the special value of the non-commutative parameters, i.e., finite $\alpha'$ and very large B-flux. It was hard to prove that the action found in [19] is invariant under the non-commutative gauge transformation. We shall show in this paper, using the result in [14, 20], that the transformed action can be written in a manifestly gauge invariant form.

Although the action (1) is valid only when the derivative of $F_{ab}$ is small, in the transformed action there is no such a limitation on $\hat{F}_{ab}$, the non-commutative gauge field strength. The transformed action includes derivatives of $\hat{F}_{ab}$ through the $\ast$-product between $\hat{F}$’s, couplings of the massless non-constant closed string fields to the $\hat{F}$’s, a Wilson line, and a Wilson operator. Interestingly, in the SW limit the new action describes properly the non-commutative super Yang-Mills theory. The part of the action which includes only the open string fields reduces to the non-commutative super YM action. And the linear coupling of the graviton to the D-brane reduces exactly to the energy-momentum tensor of the non-commutative super YM theory found in [13, 21]. In [13, 21], the energy-momentum tensor was calculated in string theory by explicit evaluation of the disk $S$-matrix element of one graviton and infinite number of open string gauge fields in the SW limit. In this calculation there is no limitation on $\hat{F}_{ab}$, and the derivatives of $\hat{F}_{ab}$ appear only through the $\ast$-product and the Wilson line. This indicates that the proposed non-commutative action has no further derivative correction terms in the SW limit.

The reminder of the paper is organized as follows: We begin in section 2 by reviewing the construction of the non-commutative $D_9$-brane action proposed in [19]. We then construct the action for $D_p$-brane using the T-duality transformation rules. In section 3, using the important result in [14, 20], we write the action in a manifestly non-commutative gauge invariant form. We extract the linear couplings of the graviton to the non-commutative gauge fields for arbitrary non-commutative parameters in section 4. In section 5, we take the SW limit of the graviton couplings and show that they are reduced to the energy-momentum tensor of the non-commutative super YM theory found in [13, 21]. In section 5.1, we write the non-commutative super YM theory in terms of the commutative fields, and compare it with the theory in terms of the non-commutative fields. We conclude with a brief discussion of our results in section 6.
2 Transformation of DBI action under the SW map with one Wilson line

In [19], we found an expression for transforming commutative DBI action of $D_p$-brane (1) under SW map with one open Wilson contour for any $p$. In that paper we did not check the consistency of the proposed action with T-duality and hence the commutators of two non-commutative scalar fields are not included in that action. To include properly these terms into the action, we start with the proposed non-commutative action for the $D_9$-brane which has no scalar field. We then transform it to the action for $D_p$-brane using the T-duality transformation rules. So we begin with a brief review of the construction of the non-commutative $D_9$-brane as follows:

1. Start with the commutative $D_9$-brane action in which the non-constant closed string fields are included, that is,

$$S = -T_9 \int d^{10}x e^{-\phi} \sqrt{-\det(g_{\mu\nu} + B_{\mu\nu} + \lambda F_{\mu\nu})} .$$

2. Expand the above action for non-constant quantum fluctuations around the constant background fields $g_{\mu\nu} + \lambda B_{\mu\nu}$. For example, the expansion for the linear dilaton is

$$\mathcal{L}(\phi, A) = T_9 \lambda c \phi \left( \frac{1}{2} \text{Tr}(VF) \right) ,$$

$$\mathcal{L}(\phi, 2A) = T_9 \lambda^2 c \phi \left( -\frac{1}{4} \text{Tr}(VVF) + \frac{1}{8} (\text{Tr}(VF))^2 \right) ,$$

$$\mathcal{L}(\phi, 3A) = T_9 \lambda^3 c \phi \left( \frac{1}{6} \text{Tr}(VVFVF) - \frac{1}{8} \text{Tr}(VVF) \text{Tr}(VVF) + \frac{1}{48} (\text{Tr}(VF))^3 \right) ,$$

where the constants $c$ and $V^{\mu\nu}$-matrix are defined as

$$c \equiv \sqrt{-\det(g_{\mu\nu} + \lambda B_{\mu\nu})} ; \quad V^{\mu\nu} \equiv \left( \frac{1}{g + \lambda B} \right)^{\mu\nu} .$$

3. Transform each commutative gauge field strength $F_{ab}$ to non-commutative gauge field $\hat{A}_a$ and $\hat{F}_{ab}$ according to the SW map [22, 18]

$$F_{\mu\nu} = \hat{F}_{\mu\nu} + \theta^{\alpha\beta} \left( \partial_{\beta}(\hat{A}_\alpha \hat{F}_{\mu\nu}) - \frac{1}{2} \hat{F}_{\mu\nu} \hat{F}_{\alpha\beta} - \hat{F}_{\mu\alpha} \hat{F}_{\nu\beta} \right) *_2$$

$$+ \frac{1}{2} \theta^{\alpha\beta \gamma \delta} \left( \partial_{\gamma}(\hat{F}_{\mu\nu} \hat{A}_{\alpha\beta}) - \partial_{\delta}(\hat{F}_{\alpha\beta} \hat{F}_{\mu\nu} \hat{A}_\delta) + 2 \partial_{\gamma}(\hat{F}_{\mu\alpha} \hat{F}_{\nu\beta} \hat{A}_\delta) \right)$$

$$- (\hat{F}_{\mu\alpha} \hat{F}_{\nu\beta} \hat{F}_{\gamma\delta}) + \frac{1}{4} (\hat{F}_{\mu\nu} \hat{F}_{\alpha\beta} \hat{F}_{\gamma\delta}) + \frac{1}{2} (\hat{F}_{\mu\nu} \hat{F}_{\beta\gamma} \hat{F}_{\alpha\delta}) - 2 (\hat{F}_{\alpha\gamma} \hat{F}_{\mu\beta} \hat{F}_{\nu\delta})) *_3 + O(\hat{A}^4) .$$

\footnote{Note that we have used the same symbols for the constant background fields and for the the whole closed string fields.}
Appearance of the $\ast_N$ above shows that the non-commutative fields on the left hand side are smeared along a Wilson line. In the above transformation, $\hat{A}_\mu$ is the non-commutative gauge field, and

$$
\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i \hat{A}_\mu \ast \hat{A}_\nu + i \hat{A}_\nu \ast \hat{A}_\mu
$$

$$
= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i [\hat{A}_\mu, \hat{A}_\nu]_M
$$

is the non-commutative gauge field strength. The $\ast$-product is

$$
f(x) \ast g(x) = e^{\frac{i}{2} \theta_{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} f(x_1) g(x_2)\big|_{x_1=x_2=x}}
$$

4. After transforming all the commutative fields according to the above SW map, one finds infinite number of Wilson lines. We perform an extra operation that reduces the infinite number of Wilson lines to only one line, i.e., insert the $\ast_N$ product between $N$ non-commutative gauge fields and $\hat{F}$'s. This stems from the fact that the disk world-sheet has only one boundary. After some manipulations, and benefit of the following identity

$$
\theta^{\mu\nu} \partial_\mu (f_1...f_N \partial_\nu g) \ast_N = i \sum_{j=1}^{N-1} (f_1...[f_j, g]_M...f_{N-1}) \ast_{N-1}
$$

one finds

$$
\hat{L}(\phi, \hat{A}) = T_9 c \partial_\mu \phi(\hat{A}_\mu),
$$

$$
\hat{L}(\phi, 2\hat{A}) = T_9 c \left( \frac{1}{2} \partial_\mu \partial_\nu \phi(\hat{A}_\mu \hat{A}_\nu) \ast_2 - \frac{\lambda^2}{4} \phi \text{Tr}(G\hat{F}G\hat{F}) \ast_2 \right),
$$

$$
\hat{L}(\phi, 3\hat{A}) = T_9 c \left( \frac{1}{3!} \partial_\mu \partial_\nu \partial_\alpha \phi(\hat{A}_\mu \hat{A}_\nu \hat{A}_\alpha) \ast_3 - \frac{\lambda^2}{4} \partial_\mu \phi \left( \hat{A}_\mu \text{Tr}(G\hat{F}G\hat{F}) \right) \ast_3 \right),
$$

where we have defined $\hat{A}_\mu = \theta^{\mu\nu} \hat{A}_\nu$, and

$$
\theta^{\mu\nu} = \lambda \left( \frac{1}{g + \lambda B} \right)^{\mu\nu}_A; \quad G^{\mu\nu} = \left( \frac{1}{g + \lambda B} \right)^{\mu\nu}_S,
$$

and $(\cdot)_A$ and $(\cdot)_S$ denote the antisymmetric and symmetric part of the $V$-matrix. In writing the terms in the above equations we have ignored some total derivative terms, e.g., we have written $(1/2)\partial_\mu \partial_\nu \phi(\hat{A}_\mu \hat{A}_\nu) \ast_2$ instead of $(1/2)\phi \partial_\mu \partial_\nu (\hat{A}_\mu \hat{A}_\nu) \ast_2$.

Note that in the above Lagrangians only the symmetric part of the $V$-matrix plays the role of metric for the open string field $\hat{F}_{\mu\nu}$. This is consistent with the fact that the $G$ is the open string metric. The terms that involve the antisymmetric part of the $V$-matrix, i.e., $\theta/\lambda$, combine in such a way that they make the closed string field to be functional of the non-commutative gauge field, i.e., functional of $\theta^{\mu\nu} \hat{A}_\nu$. If one does similar calculation for
graviton, one again finds that $G^{\mu \nu}$ appears only when both indices of the $V$-matrix contract with $\hat{F}$'s, otherwise the matrix $V^{\mu \nu}$ plays the role of raising the indices. At the same time the graviton, like the dilaton, becomes the functional of the non-commutative gauge field. All the resulting terms can be reproduced by the following prescribed action:

$$\hat{S} = -T_0 \int d^{10}x \left( e^{-\phi(\hat{A})} \sqrt{-\det \left( g_{\mu \nu}(\hat{A}) + B_{\mu \nu}(\hat{A}) + \lambda \hat{F}_{\mu \nu} \right)} \right) *_N .$$  \hspace{1cm} (6)

Expanding above action around the constant background fields for non-constant quantum fluctuations, one finds various couplings between open and closed string fields that matrix $V = (g + \lambda B)^{-1}$ plays the role of metric. The prescription is that when both indices of $V^{\mu \nu}$ contract with the open string fields, one must replace it with the symmetric part of the $V$-matrix, that is

$$V^{\mu \nu} \rightarrow G^{\mu \nu} \text{ when both indices contract with the open string fields.} \hspace{1cm} (7)$$

The resulting terms are then fully consistent with the terms in part 4. above. When there is no background B-flux, the above non-commutative action reduces to the ordinary BI action (3) with no B-flux. When there is no non-constant closed string field, only the symmetric part of the $V$-matrix appears in the expansion, and the $*_N$ product reduces to $*_{\text{product}}$[19]. Consequently the action (6) is consistent with the non-commutative BI action found in [11]. Hence it was conjectured in [19] that the action (6) is the correct transformation of the ordinary BI action (3) under the SW map with one Wilson line.

To extend this action to the action for $D_p$-brane, we use the familiar rules of T-duality. We now assume that the B-flux is non-zero only in $0, 1, \cdots p$ directions. To apply T-duality we also assume that all fields are independent of $p + 1, \cdots, 9$ directions. Since the non-commutative parameter $\theta^{ab}$ that appears in the definition of $*_N$ product (see eq.(11)) and in the functional dependence of the closed strings to $\hat{A}^a$, have no component in the $p + 1, \cdots, 9$ directions, these parts of action do not change under T-duality transformation. The transformation of the other parts of the action (6) under T-duality are exactly like the non-abelian commutative cases that were studied in details by Myers in [23]. The only change to that analysis is that every non-abelian commutator must be replaced by Moyal commutator. Therefore, using the result of [23], one will find the following T-dual action:

$$\hat{S} = -T_p \int d^{p+1}x \left( e^{-\phi(\hat{A})} \times \sqrt{-\det \left( P_{\theta}[E_{ab}(\hat{A}) + E_{ai}(\hat{A})(Q^{-1} - \delta)_{ij} E_{jb}(\hat{A})] + \lambda \hat{F}_{ab} \right) \det(Q_{ij})} \right) *_N , \hspace{1cm} (8)$$

where the closed string field $E_{\mu \nu} = g_{\mu \nu} + B_{\mu \nu}$, the transverse indices $i, j, \cdots$ are raised by the inverse of $E_{ij}$, i.e., $E^{ij}$ and

$$Q_{ij} = \delta_{ij} - i \lambda [\hat{\Phi}^i, \hat{\Phi}^k]_M E_{kj}(\hat{A}) .$$

\(^3\)Needless to mention that when both indices of $V^{\mu \nu}$ contract with one $\hat{F}$ the result is zero.
The definition of the pull-back $P_\theta$ is the extension of (2) in which ordinary derivative is replaced by its non-commutative covariant derivative, i.e., $\partial_i \Phi^j \rightarrow D_a \hat{\Phi}^i = \partial_a \hat{\Phi}^i - i[\hat{A}_a, \hat{\Phi}^i]_M$. In the action (8) one must also use the prescription (7).

Now that we have found the T-dual action, similar to the non-abelian commutative case, we assume that the closed string fields depend on the $p + 1, \cdots, 9$ directions, and further generalize it by assuming that the closed string fields are functional of the non-commutative scalar fields $\hat{\Phi}^i$, i.e., $\phi(\hat{A}) \rightarrow \phi(\hat{A}, \lambda \hat{\Phi})$ and $E_{\mu, \nu}(\hat{A}) \rightarrow E_{\mu, \nu}(\hat{A}, \lambda \hat{\Phi})$. The functional dependence of the closed string fields on both $\hat{A}^a$ and $\hat{\Phi}^i$ was also found in [19] by direct evaluation of the disk S-matrix elements, and by transforming the ordinary DBI action under the SW map with one Wilson line.

The action (8) may be extended to the non-abelian case by converting the open string fields to the matrix valued fields. Since the covariant derivative of the transverse scalar fields and commutator of two scalar fields are already included in the action in such a way that they are consistent with T-duality transformation rules, one needs only to define a prescribed trace over the matrices. The prescription for the trace should be in such a way that when $B = 0$ the result reduces to the non-abelian commutative case. In that case the prescription for the trace is the symmetrized trace [24, 23]. In particular, after expanding the action for quantum fluctuations, one should write each terms of the expanded action in a form that is symmetric between $\hat{F}^{ab}$, $D_a \hat{\Phi}^i$, $[\hat{\Phi}^i, \hat{\Phi}^j]_M$ and individual $\hat{\Phi}^i$. The later field coming from Taylor expansion of the closed string fields. Then take the trace of the resulting terms. Hence, in the non-abelian non-commutative case, our prescription is the symmetrized trace over $\hat{F}^{ab}$, $D_a \hat{\Phi}^i$, $[\hat{\Phi}^i, \hat{\Phi}^j]_M$, $\hat{A}^a$, and $\hat{\Phi}^i$ which are coming from the Taylor expansion of the closed string fields. The resulting action reduces to the commutative case when $B = 0$, because in this case $*_N$-product $\rightarrow$ ordinary product, $\hat{A}^a = \theta^{ab} \hat{A}_b \rightarrow 0$, and $V^{ab} \rightarrow G^{ab}$.

The above symmetrized trace prescription make sense only when the multiplication rules between $\hat{F}^{ab}$, $D_a \hat{\Phi}^i$, $[\hat{\Phi}^i, \hat{\Phi}^j]_M$, $\hat{A}^a$, and $\hat{\Phi}^i$ are symmetric under permutation of these fields. In fact it was shown in [14] that the $*_N$ satisfies this condition. Therefore in the non-abelian extension of the action (8), the $*_N$ must be the multiplication rule between fields $\hat{F}^{ab}$, $D_a \hat{\Phi}^i$, $[\hat{\Phi}^i, \hat{\Phi}^j]_M$, $\hat{A}^a$, and $\hat{\Phi}^i$. This is also consistent with the step 4 in our construction of the non-commutative D9-brane action.

3 Action in gauge invariant form

The transformed DBI action under the SW map with one Wilson line in the form appearing in (8) is not manifestly invariant under the non-commutative gauge transformation. In this section we would like to write it in a manifestly gauge invariant form. To this end, we will use the following observation [14, 20]. Given a collection of open string local operators
\( \hat{Q}_i(x^a) \) on the world-volume of the \( D_p \)-brane which transform in the adjoint representation of non-commutative \( U(1) \) gauge transformation, one can obtain a natural gauge invariant operator of fixed momentum \( k^a \) by smearing these local operators along the straight contour \( \zeta^a(\tau) = (\theta^{ab}k_b)\tau \) with \( 0 \leq \tau \leq 1 \), and multiplying the product by a Wilson operator along the same contour. Following this observation, we consider the Wilson operator

\[
W(k^\mu,x^a,C) = e^{i \int_0^1 d\tau (k_a \theta^{ab} \dot{A}_a(x+\zeta(\tau)) + \lambda k_i \hat{\Phi}_i(x+\zeta(\tau)))}, \tag{9}
\]

and the following collection of local operators \( \hat{Q}_i(x^a) \) that transform in the adjoint and local closed string fields \( O_j(x^\mu) \) that are scalar under the non-commutative gauge transformation:

\[
f(x^\mu) = \prod_{j=1}^m O_j(x^\mu) \prod_{i=1}^n \hat{Q}_i(x^a).
\]

According to the result in [14, 20], the following operator is gauge invariant

\[
\tilde{f}_W(x^a,k^\mu) = \int d^{10}y \prod_{j=1}^m O_j(x^a,y^i) \times P_* \left[ W(k^\mu,y^a,C) \prod_{i=1}^n \int_0^1 d\tau_i \hat{Q}_i(y^a + \zeta^a(\tau_i)) \right] * e^{ik \cdot y}
\]

\[
= \int d^{10}y \prod_{j=1}^m O_j(x^a,y^i)L_* \left[ W(k^\mu,y^a,C) \prod_{i=1}^n \hat{Q}_i(y^a) \right] * e^{ik \cdot y},
\]

where \( k \cdot y = k_\mu y^\mu \) and \( P_* \) denotes path-ordering with respect to the \(*\)-product, while \( L_* \) is an abbreviation for the combined path-ordering and integrations over \( \tau \)'s. In this formula the open string operators \( \hat{Q}_i \) are smeared over the straight contour of the Wilson line while the locations of the closed string operators \( O_j \) are independent of the contour. Note that the \( y \)-integral is over the world-volume position of the open string operators and over the transverse position of the closed string fields. There is no integral over the world-volume position of the closed string fields.

Now we expand the exponential in the Wilson operator, and then perform the line integrals over \( \tau \)'s. Following [17], one may Fourier transforms the open string operators

\[
\mathcal{O}_i(y^a + \zeta^a(\tau_i)) = \int \frac{dp^{p+1}k_i}{(2\pi)^{p+1}} \tilde{\mathcal{O}}_i(k_i)e^{-i(k_i)_a(y^a + \zeta^a(\tau_i))}
\]

\[
= \int \frac{dp^{p+1}k_i}{(2\pi)^{p+1}} \tilde{\mathcal{O}}_i(k_i)e^{-i(k_i)_a y^a - i(k_i \times k)_\tau_i},
\]

where \( k_i \times k = (k_i)_a \theta^{ab}k_b \), and \( \mathcal{O}_i \) is any of \( \hat{\Phi}_i \)'s, \( \hat{A}_i \)'s or \( \hat{Q}_i \)'s. The integral over \( \tau \)'s converts to an elementary exponential integral which yields the \(*_N\)-product in the momentum space [25].
\[ (\tilde{O}_1 \cdots \tilde{O}_N)_{\ast N} = \int_0^1 d\tau_1 \int_0^1 d\tau_2 \cdots \int_0^1 d\tau_N \exp \left( -i \sum_{i=1}^N (k_i \times k_i)\tau_i + \frac{i}{2} \sum_{i<j}^N (k_i \times k_j)\epsilon(\tau_{ij}) \right) (\tilde{O}_1 \cdots \tilde{O}_N), \]

where \( \epsilon(\tau_{ij}) = +1(-1) \) for \( \tau_{ij} > 0(\tau_{ij} < 0) \), and \( \tau_{ij} = \tau_i - \tau_j \). Note that the integral over \( y^a \) yields the condition \( k^a = \sum_i (k_i)^a \). Replacing all the momenta in the expansion with their appropriate derivatives and then using

\[ \int \frac{d^{p+1}k_i}{(2\pi)^{p+1}} \tilde{O}_i(k_i)e^{-i(k_i)_a y^a} = \mathcal{O}_1(y^a), \]

one will find

\[ \tilde{f}_W(x^a, k^\mu) = \int d^{10}y e^{ik \cdot y} \sum_{p=0}^\infty \frac{\lambda^p}{p!q!} (\partial_{y^{i_1}} \cdots \partial_{y^{i_p}}) \left( \prod_{j=1}^m O_j(x^a, y^i) \right) \times (\partial_{x^{a_1}} \cdots \partial_{x^{a_p}}) \left( \hat{\Phi}^{i_1} \cdots \hat{\Phi}^{i_p} \hat{A}^{a_1} \cdots \hat{A}^{a_q} \hat{Q}_1 \cdots \hat{Q}_n \right) \ast_{p+q+n}, \]

where now the \( \ast_N \) is in the position space (recall that \( \hat{A}^a = \theta^{ab} \hat{A}_b \)). The Fourier inverse of the above function is

\[ f_W(x^\mu) = \int \frac{d^{10}k}{(2\pi)^{10}} \tilde{f}_W(x^a, k^\mu)e^{-ik \cdot x} = \sum_{p=0}^\infty \frac{\lambda^p}{p!q!} (\partial_{x^{i_1}} \cdots \partial_{x^{i_p}}) \left( \prod_{j=1}^m O_j(x^a, x^i) \right) \times (\partial_{x^{a_1}} \cdots \partial_{x^{a_p}}) \left( \hat{\Phi}^{i_1} \cdots \hat{\Phi}^{i_p} \hat{A}^{a_1} \cdots \hat{A}^{a_q} \hat{Q}_1 \cdots \hat{Q}_n \right) \ast_{p+q+n} \]

\[ = \left( \prod_{j=1}^m O_j(\hat{A}, \lambda\hat{\Phi}) \prod_{i=1}^n \hat{Q}_i \right) \ast_{N} + \text{(total world volume derivative terms)}. \]

The total world-volume derivative terms above are exactly like the total derivative terms that we ignored in the step 4 in the section 2 for constructing the non-commutative action. The world-volume integral of \( f_W \) that we are interested in is

\[ \hat{S}' = \int d^{10}x f_W(x^\mu)\delta(x^i) \]

\[ = \int d^{p+1}x \left( \prod_{j=1}^m O_j(\hat{A}, \lambda\hat{\Phi}) \prod_{i=1}^n \hat{Q}_i \right) \ast_{N}. \]  

\(^4\)An alternative formula for the \( \ast_N \)-product was found in [19].
Therefore, expanding the Wilson operator and performing the Wilson line integral, one finds that the closed string fields become functional of the non-commutative scalar and gauge fields, and the $*_{N}$ operates as the multiplication rule between the open string fields.

On the other hand, if one does not perform the $\tau$ integral in (10), one will find that the Wilson line and operator appear in the action, that is

$$\hat{S}' = \int d^{p+1}x \frac{d^{10}k}{(2\pi)^{10}} d^{10}y e^{-ik_{a}x^{a}} \prod_{j=1}^{m} O_{j}(x^{a}, y^{i})$$

$$\times L_{*} \left[ W(k^{\mu}, y^{a}, C) \prod_{i=1}^{n} \hat{Q}_{i}(y^{a}) \right] * e^{ik_{a}y^{a}} .$$

(13)

One may wish to write the closed string fields in the above action in the momentum space, that is

$$O_{j}(x^{a}, y^{i}) = \int \frac{d^{10}p_{j}}{(2\pi)^{10}} \tilde{O}_{j}(p_{j}^{\mu}) e^{-i(p_{j})_{a}x^{a} - i(p_{j})_{y^{i}}} .$$

In this case, one can perform the integral over $x^{a}$, $y^{i}$ and then over $k^{\mu}$. The result simplifies to

$$\hat{S}' = \prod_{j=1}^{m} \int \frac{d^{10}p_{j}}{(2\pi)^{10}} \tilde{O}_{j}(p_{j}^{\mu}) \int d^{p+1}y L_{*} \left[ W(k^{\mu}, y^{a}, C) \prod_{i=1}^{n} \hat{Q}_{i}(y^{a}) \right] * e^{ik_{a}y^{a}} ,$$

where now the components of the momentum of the Wilson operator is $k^{a} = -\sum_{j}(p_{j})^{a}$ and $k^{i} = \sum_{j}(p_{j})^{i}$. From the $y^{a}$-integral, on the other hand, we find that the Wilson line momentum $k^{a} = \sum_{i}(k_{i})^{a}$. Hence, there is the momentum conservation in the world-volume directions, i.e., $\sum_{i}(k_{i})^{a} + \sum_{j}(p_{j})^{a} = 0$, whereas there is no momentum conservation in the transverse direction. This stems from the fact that we fixed the position of the $D_{p}$-brane at $x^{i} = 0$.

The structure in (12) is exactly the one appears in the proposed non-commutative action in (8), when one expands the square root in the action. Hence, an alternative way of writing the non-commutative action (8) is

$$\hat{S} = \int d^{p+1}x \frac{d^{10}k}{(2\pi)^{10}} d^{10}y e^{-ik_{a}x^{a}} L_{*} \left[ W(k^{\mu}, y^{a}, C) \hat{L}(x^{a}, y^{\mu}) \right] * e^{ik_{a}y^{a}} ,$$

(14)

where

$$\hat{L}(x^{a}, y^{\mu}) = -T_{p} e^{-\bar{\phi}} \sqrt{- \det \left( P_{\mu}E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij} E_{jb} \right) + \lambda \hat{F}_{ab} \det(Q^{ij})} ,$$

(15)

and

$$Q^{ij} = \delta^{ij} - i\lambda [\hat{\Phi}^{i}, \hat{\Phi}^{k}]_{M} E_{kj} .$$
The closed string fields in the above action are function of $x^a, y^i$, whereas, the open string fields are function of $y^a$. In (13), one must also take the prescription (7) into account. When expanding the square root in (13), one finds a tower of different terms. Each term may contain the open string operators $\hat{F}_{ab}, D_a\hat{\Phi}^i$ or $[\hat{\Phi}^i, \hat{\Phi}^j]_M$, as well as massless closed string fields. The position of the closed string fields are independent of the Wilson line, and the multiplication rule between them is also the ordinary product. Hence, the $L_*$ has no effect on the closed string fields in each term of the expansion, and its effect on the open string operators is that it smears them along the open Wilson line. Since the open string operators $\hat{F}_{ab}, D_a\hat{\Phi}^i$ and $[\hat{\Phi}^i, \hat{\Phi}^j]_M$ transform in the adjoint under the gauge transformation, the action (14), like (13), is manifestly non-commutative gauge invariant.

Extension of this form of action (14) to the non-abelian case is again straightforward. We need to change the fields to matrix valued fields and a prescribed trace over the matrices. The prescription is again the symmetrized trace over $\hat{F}_{ab}, D_a\hat{\Phi}^i, [\hat{\Phi}^i, \hat{\Phi}^j]_M$ which are coming from the expansion of the square root in (13), and $A^a, \Phi^i$ which are coming from the expansion of the Wilson operator in (14). In this case, however, the symmetrized trace is reproduced by the ordinary trace and the path ordering prescription. The path ordering means that after expanding the square root and the Wilson operator, one must integrate all different sequences of the open string operators along the Wilson line. Taking the trace of the resulting terms produces the symmetrized trace.

4 Linear Couplings

The action (14) includes all linear and non-linear couplings of the closed string fields to the non-commutative D$p$-branes. We would like to find the linear couplings of the dilaton to the branes. So we write the closed string fields as the classical constant background fields plus their non-constant quantum fluctuations, i.e., $g_{ab} = g_{ab} + 2\kappa h_{ab}$, $g_{ai} = 2\kappa h_{ai}$, $g_{ij} = g_{ij} + 2\kappa h_{ij}$, $\phi = 2\kappa\phi'$, $B_{ab} = \lambda B_{ab}$, and the other components of the B-flux are zero. Now we expand (13) around the background fields $g_{ab} + \lambda B_{ab}$. For the dilaton one finds,

$$\hat{\mathcal{L}}_\phi = 2T_p\kappa c \phi' \sqrt{\det (\delta^a \hat{\Phi}^i D_c D_b \hat{\Phi}^j + \lambda G^{ac} \hat{F}_{cb}) \det (Q^i_j)}$$

(16)

where

$$Q^i_j = \delta^i_j - i\lambda [\hat{\Phi}^i, \hat{\Phi}^j]_M$$

where we have factored out the constant $c = \sqrt{-\det (g_{ab} + \lambda B_{ab})}$ from the square root above, and used the prescription (7). The square root involves only the open string fields in this case, hence, one must replace the $V^{ab}$ matrix by the open string metric $G^{ab}$ in all terms. In the above and subsequent equations the indices of the transverse scalar fields are lowered by the background field $g_{ij}$.
For the graviton, because of the prescription (7), it is hard to find a closed form for its linear coupling to the D-brane. So we expand the square root in (13) and keep terms that involve linearly the graviton. The expansion is straightforward, and the result is,

\[
\hat{\mathcal{L}}_h = -T_{pK} \left( V^{ab} h_{ba} \left( 1 + \frac{\lambda^2}{2} G^{\cd} D_c \hat{\Phi}_i D_a \hat{\Phi}_i - \frac{\lambda^2}{4} G^{\cd} \hat{F}_{de} G^{ef} \hat{F}_{fc} + \frac{\lambda^2}{4} \left[ \hat{\Phi}_i, \hat{\Phi}_j \right]_M \left[ \hat{\Phi}_k, \hat{\Phi}_l \right]_M \right) 
- V^{ab} h_{ba} V^{cd} \left( \lambda \hat{F}_{da} + \lambda^2 D_d \hat{\Phi}_i D_a \hat{\Phi}_i - \lambda^2 \hat{F}_{de} G^{ef} \hat{F}_{fa} \right) + \lambda^2 V^{ab} h_{ia} (b D_a) \hat{\Phi}_i - 2 \lambda^2 V^{ab} h_{ia} (b D_c) \hat{\Phi}_i V^{cd} \hat{F}_{da} + 2i \lambda^2 V^{ab} h_{ia} [\hat{\Phi}_i, \hat{\Phi}_j]_M D_b \hat{\Phi}_j + \lambda^2 \left[ \hat{\Phi}_i, \hat{\Phi}_j \right]_M \left[ \hat{\Phi}_k, \hat{\Phi}_l \right]_M \hat{h}_{ij} + \cdots \right) ,
\]

we have used again the prescription (7). In this case though, the closed string graviton as well as the open string fields appear in the expansion, so one should not replace the \( V^{ab} \) by \( G^{ab} \) for all terms in the expansion above. In eq. (17), dots represent terms that would be of order \( \lambda^3 \) if the \( V \)-matrix and \( g_{ij} \) were independent of \( \lambda \). Writing \( V^{ab} = G^{ab} + \theta^{ab}/\lambda \), one finds

\[
\hat{\mathcal{L}}_h = -T_{pK} \left( G^{ab} h_{ba} \left( 1 + \frac{\lambda^2}{2} G^{\cd} D_c \hat{\Phi}_i D_a \hat{\Phi}_i - \frac{\lambda^2}{4} G^{\cd} \hat{F}_{de} G^{ef} \hat{F}_{fc} + \frac{\lambda^2}{4} \left[ \hat{\Phi}_i, \hat{\Phi}_j \right]_M \left[ \hat{\Phi}_k, \hat{\Phi}_l \right]_M \right) 
+ 2G^{ab} h_{bc} \theta^{cd} \hat{F}_{da} - \lambda^2 G^{ab} h_{bc} G^{cd} D_d \hat{\Phi}_i D_a \hat{\Phi}_i - \theta^{ab} h_{bc} \theta^{cd} D_d \hat{\Phi}_i D_a \hat{\Phi}_i + \lambda^2 G^{ab} h_{bc} G^{cd} \hat{F}_{de} G^{ef} \hat{F}_{fa} + \theta^{ab} h_{bc} \theta^{cd} \hat{F}_{de} G^{ef} \hat{F}_{fa} + \lambda^2 G^{ab} h_{ia} (b D_a) \hat{\Phi}_i + 2 \lambda G^{ab} h_{ia} (b D_c) \hat{\Phi}_i - 2 \lambda \theta^{ab} h_{ia} (b D_c) \hat{\Phi}_i G^{cd} \hat{F}_{da} + 2i \lambda \theta^{ab} h_{ia} [\hat{\Phi}_i, \hat{\Phi}_j]_M D_b \hat{\Phi}_j + \lambda^2 \left[ \hat{\Phi}_i, \hat{\Phi}_j \right]_M \left[ \hat{\Phi}_k, \hat{\Phi}_l \right]_M \hat{h}_{ij} + \cdots \right) .
\]

According to the prescription (7), each term in the expansion has at most two \( V \)-matrix, \textit{i.e.}, the indices of other \( V^{ab} \) contract with open string fields so they must be replaced by \( G^{ab} \). More specifically, terms that involve \( h_{ab} \), \( h_{ia} \) and \( h_{ij} \) has two, one and no \( V^{ab} \) matrix. Therefore, assuming that \( G^{ab}, \theta^{ab} \) and \( g_{ij} \) are independent of \( \lambda \), the dots above gives corrections of order \( O(\lambda) \), \( O(\lambda^2) \) and \( O(\lambda^3) \) to the terms that involve \( h_{ab} \), \( h_{ia} \) and \( h_{ij} \), respectively.

Now replacing (16) and (18) to the action (13), transforming the quantum closed string fields to the momentum space, and performing the integral over \( x^a, y^i \) and \( k^\mu \), one will find

\[
\hat{S}_\phi = \frac{2 \kappa}{g^2_{YM} \lambda^2} \int \frac{d^{10}p}{(2\pi)^{10}} \sqrt{-\det G} \left( \hat{T}_\phi(p) \right) ,
\]

\[
\hat{S}_h = -\frac{\kappa}{g^2_{YM} \lambda^2} \int \frac{d^{10}p}{(2\pi)^{10}} \sqrt{-\det G} \left( \hat{T}_{h^{\mu\nu}}(p) \right) ,
\]

where

\[
\hat{T}(p) = \int d^{10+1}y L_s \left[ W(k^\mu, y^a, C) T_\phi(y) \right] \cdot e^{ik_a y^a} ,
\]

\[
\hat{T}^{\mu\nu}(p) = \int d^{10+1}y L_s \left[ W(k^\mu, y^a, C) T_h^{\mu\nu}(y) \right] \cdot e^{ik_a y^a} ,
\]

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and

\[ T_\phi = \sqrt{\det \left( \delta_{ab} + \lambda^2 \langle Q^{-1} \rangle_{ij} G^{ac} D_c \hat{\phi}^i D_b \hat{\phi}^j + \lambda G^{ac} \hat{F}_{cb} \right) \det(Q^{-1})}, \]

\[ T_{ab}^h = G^{ab} \left( 1 + \frac{\lambda^2}{2} G^{cd} D_c \hat{\phi}_i D_d \hat{\phi}_i - \frac{\lambda^2}{4} G^{cd} \hat{F}_{de} G^{ef} \hat{F}_{fc} + \frac{\lambda^2}{4} \left[ \hat{\phi}_i \hat{\phi}_j \right]_{M} \left[ \hat{\phi}_j \hat{\phi}_i \right]_{M} \right) \]

\[ -2G^{ca} \theta^{bd} \hat{F}_{dc} - \lambda^2 G^{ca} G^{bd} D_d \hat{\phi}_i D_e \hat{\phi}_i - \theta^{ca} \theta^{bd} D_d \hat{\phi}_i D_e \hat{\phi}_i \]

\[ + \lambda^2 \theta^{ca} G^{bd} \hat{F}_{de} G^{ef} \hat{F}_{fc} + \theta^{ca} \theta^{bd} \hat{F}_{de} G^{ef} \hat{F}_{fc} + \cdots, \quad (20) \]

\[ T_{ai}^h = \lambda G^{ba} D_b \hat{\phi}_i - \lambda \theta^{ba} D_b \hat{\phi}_i G^{cd} \hat{F}_{db} + i \lambda \theta^{ab} \left[ \hat{\phi}_i \hat{\phi}_j \right]_{M} D_b \hat{\phi}_j + \cdots, \]

\[ T_{ij}^h = \lambda^2 G^{ab} D_a \hat{\phi}_i D_b \hat{\phi}_j + \lambda^2 \left[ \hat{\phi}_i \hat{\phi}_j \right]_{M} \left[ \hat{\phi}_k \hat{\phi}_j \right]_{M} + \cdots. \]

Components of the momentum of the Wilson operator are \( k^a = -p^a \) and \( k^i = p^i \). In the equation (19), using the definition of the effective non-commutative Yang-Mills coupling [11], we have written

\[ T_{p c} = \left( \frac{\sqrt{-\det(g_{ab} + \lambda B_{ab})}}{\sqrt{-\det G}} \right) \sqrt{-\det G} \]

\[ = \frac{1}{g_s^2 M \lambda^2} \sqrt{-\det G}. \]

The parameters in the energy-momentum tensors in (19) are arbitrary. These tensors are fully consistent with the disk S-matrix element of two massless open strings and one closed string state, and of two massless closed string states in the superstring theory [26, 27, 22, 12], for the arbitrary parameters. However, we would like to compare our result with the disk S-matrix element of one closed string and infinite number of the massless open string states. This calculation has been done in [13] for the special values of the parameters, i.e., the SW limit.

## 5 The Seiberg-Witten limit

In the Seiberg-Witten limit \( \lambda \sim \sqrt{\epsilon} \to 0, g_{ab} \sim \epsilon \to 0 \) and all other closed string background fields are finite. If the background B-flux has only space-space components, then the limit is \( \lambda \to 0 \) and \( G^{ab}, \theta^{ab} \), and \( g_{ij} \) are finite. In this limit, all the closed strings and the massive open string fields decouple from the D-brane, and the entire string dynamics is described by the non-commutative U(1) YM field theory [11]. In this limit, when there is

\[ \text{The minimal change to our notation to include the background B-flux with only space-space components is the following: We still use } \theta^{ab} \text{ for the whole world-volume directions. However, when one of the indices of } \theta^{ab} \text{ takes value in the commutative directions of the world-volume, it must be replaced by zero.} \]
no non-constant closed string field, the action (14) reduces to
\[ \hat{S} = -\frac{1}{g_{YM}^2} \int d^{p+1}x \sqrt{-\det G} \left( -\frac{1}{4} G^{cd} \hat{F}_{de} G^{ef} \hat{F}_{fc} + \frac{1}{2} G^{cd} D_c \hat{\Phi}_i D^i + \frac{1}{4} [\hat{\Phi}^i, \hat{\Phi}^j]_M [\hat{\Phi}_j, \hat{\Phi}_i]_M \right), \] (21)

where we have used the fact that the momentum of the Wilson operator (1) is summation of the closed string fields. No closed string field, no Wilson operator and no Wilson line (recall that the length of Wilson line is proportional to the momentum of the Wilson operator). This action is the non-commutative action introduced in [11].

The linear coupling of the non-commutative YM theory to graviton and to the dilaton can be read from the general result in (19)–(20) by taking the SW limit. It was argued in [13] that the on-shell condition of the closed string fields, in the SW limit, causes that the transverse components of the closed string momentum depend on \( \lambda \) as \( p_i \sim \lambda^{-1} \). So in this limit the Wilson operator (1) is independent of \( \lambda \) (recall that \( k^a = -p^a \) and \( k^i = p^i \)), and the equation (20) reduces to
\[
\begin{align*}
T_{\phi}(y) &= 1, \\
T^{ab}_{\bar{h}}(y) &= G^{ab} - 2G^{ca} \theta^{bd} \hat{F}_{dc} - \theta^{ca} \theta^{bd} D_a \hat{\Phi}_i D^i + \theta^{ca} \theta^{bd} \hat{F}_{de} G^{ef} \hat{F}_{fc}, \\
T^{ki}_{\bar{h}}(y) &= \lambda G^{ba} D_b \hat{\Phi}_i - \lambda \theta^{ba} D_c \hat{\Phi}_i G^{cd} \hat{F}_{db} + i \lambda \theta^{ab} [\hat{\Phi}^i, \hat{\Phi}^j]_M D_b \hat{\Phi}_j, \\
T^{ij}_{\bar{h}}(y) &= \lambda^2 G^{ab} D_a \hat{\Phi}_i D_b \hat{\Phi}_j + \lambda^2 [\hat{\Phi}^i, \hat{\Phi}^k]_M [\hat{\Phi}_k, \hat{\Phi}^j]_M. \end{align*}
\] (22)

The equation (14) with the above tensors and the previous footnote, reproduces exactly the result of the disk amplitude of one closed and infinite number of open superstrings in the SW limit [13, 21]. Note that in the limit \( \theta \to 0 \) the result in (20) reduces to the energy-momentum tensor of the commutative theory.

### 5.1 Non-commutative theory in terms of commutative fields

We have seen that, using the SW map with one open Wilson contour, the action (1) in the presence of the background B-flux can be mapped into the action (14) that is in terms of non-commutative fields. Even though the commutative action (1) is valid for the slowly varying open string fields, the fields in the non-commutative action (14) need not to be slowly varying fields. Comparing this action with the string theory calculations [13, 21], one concludes that there is no more derivative correction term to this action in the SW limit. On the other hand, the commutative action (1), can not describe properly the string theory in the SW limit. The derivative of \( F^{ab} \) and the second derivative of the scalar fields

\[ p_i \sim \lambda^{-1}. \]

It was argued in [13] that the on-shell condition on the closed string momentum, in the SW limit, causes the momentum in the non-commutative directions not to be arbitrary. Hence, one can not perform the inverse Fourier transform of (13) in those directions.
that are not included in it may have significant effects\textsuperscript{7}. However, when these derivative terms are small compared to $\lambda$, one may use the commutative action (1) for describing, in the SW limit, the non-commutative theory. When there is no closed string field, the action (1) in the SW limit reduces to the following commutative U(1) gauge theory:

\[
\hat{S} = -\frac{1}{g_{YM}^2} \int d^{p+1}x \sqrt{-G} \frac{1}{\det(\mathcal{F})} \left( -\frac{1}{4} \mathcal{F}^a_b G^{bc} F_{cd} \mathcal{F}^d/e G^{ef} F_{fa} \\
+ \frac{1}{2} \mathcal{F}^a_b G^{bc} \partial_c \Phi^i \partial_d \Phi^j - \frac{1}{2} \mathcal{F}^a_b \theta^{bc} \partial_c \Phi^i \partial_d \Phi^j \mathcal{F}^d/e G^{ef} F_{fa} \\
- \frac{1}{4} \mathcal{F}^a_b \theta^{bc} \partial_c \Phi^i \partial_d \Phi^j \mathcal{F}^d/e \theta^{ef} \partial_f \Phi^i \partial_a \Phi^j + \cdots \right), \tag{23}
\]

where dots represent the derivative terms, and we have defined

\[
\mathcal{F}^a_b = \left( \frac{1}{1 + \theta F} \right)^{a/b}.
\]

In expanding the square root in the action (1) around the background fields, one finds also a term of order $\lambda^{-2}$ which is $(\lambda^2 \sqrt{\det(\mathcal{F})})^{-1}$. However, by expanding this term, one can verify that it produces total derivative terms. For the special case that there is no background B-flux, $\mathcal{F}^a_b = \delta^a_b$, and so the action (23) reduces to the commutative YM theory. Expanding the action to linear order of $\theta^{ab}$, one finds the deformed abelian YM action found in \textsuperscript{28}\textsuperscript{8}.

The evaluation of the linear couplings of the dilaton and the graviton to the non-commutative D-brane in terms of the commutative fields is also straightforward. The results are

\[
S_\phi = \frac{2\kappa}{g_{YM}^2 \lambda^2} \int \frac{d^{10}p}{(2\pi)^{10}} \sqrt{-G} \left( \hat{\phi}'(p) \hat{T}_\phi(p) \right),
\]

\[
S_h = -\frac{\kappa}{g_{YM}^2 \lambda^2} \int \frac{d^{10}p}{(2\pi)^{10}} \sqrt{-G} \left( \hat{h}_{\mu\nu}(p) \hat{T}_{\mu\nu}(p) \right),
\]

where

\[
\hat{T}_\phi(p) = \int d^{p+1}y T_\phi(y^a) e^{-ip_0 y^a + i\lambda p_i \Phi^i},
\]

\[
\hat{T}_{\mu\nu}(p) = \int d^{p+1}y T_{\mu\nu}(y^a) e^{-ip_0 y^a + i\lambda p_i \Phi^i}.
\]

\textsuperscript{7}Note that the operation in the step 4 in section 2 which reduces the infinite number of Wilson lines to only one line, causes that the commutative action (1) and the non-commutative action (14) not to be identical.

\textsuperscript{8}The fact that the non-commutative super YM theory can be written in terms of the commutative BI action up to the derivative correction terms, was used in \textsuperscript{23} to study some symmetry of the non-commutative theory.
and
\[ T_\theta = \frac{1}{\sqrt{\det(F)}}(1 + \cdots), \]
\[ T_{ab}^i = \frac{1}{\sqrt{\det(F)}} \left( F^a_c G^{cb} \partial_c \Phi^i_\theta - F^a_c (\theta^{cd} \partial_d \Phi^i_\theta \partial_c \Phi^i_\theta + G^{cd} F_{de}) \Phi^e_j \Phi^j + \cdots \right), \]
\[ T_{ai}^i = \frac{\lambda}{\sqrt{\det(F)}} \left( F^a_b G^{bc} \partial_c \Phi^i_\theta - F^a_b (\theta^{bc} \partial_c \Phi^i_\theta \partial_d \Phi^i_\theta + G^{bc} F_{de}) \Phi^e_j \partial_j \Phi^i + \cdots \right), \]
\[ T_{ij}^{ij} = \frac{\lambda^2}{\sqrt{\det(F)}} \left( F^a_b G^{bc} \partial_c \Phi^i_\theta \partial_j \Phi^j - F^a_b (\theta^{bc} \partial_c \Phi^i_\theta \partial_d \Phi^i_\theta + G^{bc} F_{de}) \Phi^e_j \partial_j \Phi^i_\theta + \cdots \right). \]

Here again the dots represents the derivative terms. Since the transverse components of the closed string fields depend on \( \lambda \) as \( p_i \sim \lambda^{-1} \), the exponential factors above are independent of \( \lambda \). One expects that, when the B-flux is zero, the above results and the result in (23) reduce to the results in (22) and in (21) for \( B=0 \). So we recover the known fact that the derivative correction terms to the ordinary DBI action are such that, in the SW limit and for \( B=0 \), they all vanish.

In writing the abelian action (23) and the energy-momentum tensors above, we have taken the SW limit of the abelian DBI action (1). Extension of these results to the non-abelian case is straightforward. One should take the SW limit of the non-abelian extension of the DBI action [23].

**6 Discussion**

In this paper we have found an action for transformation of the ordinary DBI action, which includes the non-constant massless closed string fields, under the SW map with one open Wilson contour. This action contains derivatives of the non-commutative open string fields through the \( \ast_N \)-product, and one Wilson operator. The world-volume indices of the open string fields are raised by the open string metric \( G^{ab} \). They all stems from the SW map. The action also inherits the non-constant closed string fields from the commutative action. We have shown that the action is invariant under the non-commutative gauge transformation, and it is consistent with the T-duality rules. From the non-constant closed string fields, we have found the linear couplings of the dilaton and the graviton to the non-commutative gauge fields for arbitrary \( \lambda \) and background fields. In the SW limit, we have shown that these couplings reduce to the known results of the energy-momentum tensor of the non-commutative super YM theory. Hence, we reached to the conclusion that the non-commutative action (14) in the SW limit has no further derivative correction terms.

In sect. 5, we compared the proposed action (14) with the disk S-matrix elements of one closed string and infinite number of the open string states in the SW limit, and
found exact agreement between them. In this limit, the background fields are restricted to \( g_{aa} = B_{ai} = B_{ij} = 0 \). Using the action (14) it is easy to extend the results in section 5 to include these background fields as well. It would be interesting then to release that restriction in the disk amplitude calculation and check if still one finds agreement between the string theory calculation and the action (14).

The recipe for transforming ordinary \( D_p \)-brane action under SW map with one Wilson line in sect. 2 includes the transformations like
\[
(f_1 f_2 f_3) \ast (f_4 f_5 f_6) \ast \cdots \ast f_N \to (f_1 f_2 \cdots f_N)^* N,
\]
where \( f_1, \cdots f_N \) are any of \( \hat{A}^a \) or \( \hat{F}^a_{ab} \). Using the results in sect. 3, this means that we impose an extra operation that reduces the number of the open Wilson contours, \( n \), to one contour, \( i.e., L^*_n \to L_1 \). This stems from the fact that the world-sheet corresponding to the tree level effective action (disk) has one boundary. For one loop level effective action, the world-sheet (cylinder) has two boundaries. So in that case, one should impose an operation that reduces the number of the Wilson lines to two. It would be interesting, then, to impose this construction to the one loop effective action of the commutative theory to find the one loop effective action of the non-commutative theory[14].

In this paper, we have found the non-commutative action for \( D_p \)-brane by applying the familiar rules of T-duality on the \( D_9 \)-brane. One may wish to find this action directly from the commutative \( D_p \)-brane action and the prescriptions in sect.2. In this way the prescription is as follows:

1. Start with commutative \( D_p \)-brane.
2. Expand the square root for quantum fluctuations around the background \( g_{ab} + \lambda B_{ab} \) field.
3. Transform each commutative \( F_{ab}, \partial_a \Phi^i \) and \( \Phi^i \) to their non-commutative counterparts according to the SW map. The SW map for \( \partial_a \Phi^i \) and \( \Phi^i \) can be read from the corresponding map for \( F_{ab} \) and \( A_a \) by dimensional reduction [30, 19, 16].
4. Reduce the number of Wilson lines to only one line, \( i.e., \) replace the \( *_N \) between \( \hat{F}_{ab}, D_a \hat{\Phi}^i \), and any other field, and then simplify the result using the identity (5).

Let us work the above prescription for one simple example. Consider the coupling of one Kalb-Ramond closed string field with two transverse scalar fields, that is
\[
V^{ab} b_{ij} \partial_a \Phi^i \partial_b \Phi^j = \frac{1}{\lambda} b_{ij} \theta^{ab} \partial_a \Phi^i \partial_b \Phi^j.
\]
According to the above prescription it transforms to
\[
\frac{1}{\lambda} b_{ij} \theta^{ab} \left( D_a \hat{\Phi}^i D_b \hat{\Phi}^j \right) \ast_2 + \cdots = \frac{i}{\lambda} b_{ij} [\hat{\Phi}^i, \hat{\Phi}^j]_M + \cdots,
\]
where we have used the identity \( \theta^{ab} (\partial_a f \partial_b g) \ast_2 = i[f, g]_M \). The above term is exactly reproduced by the second square root in (3). It is not reproduced by the first square root because of the rule in (7). This term is also reproduced by the disk S-matrix elements of one closed and two open string states[22]. One needs more complicated identities to produce
the terms in the action (8) which have more than one commutator. Alternatively, one may compare the terms coming from the above prescription and the non-commutative action (8) to find the more complicated identities between the \(*_N\). As an example consider the following term which stems from the expansion of ordinary DBI action:

\[ V^{ab}\partial_b\Phi^i\partial_c\Phi_iV^{cd}\partial_d\Phi^j\partial_a\Phi_j = G^{ab}\partial_b\Phi^i\partial_c\Phi_iG^{cd}\partial_d\Phi^j\partial_a\Phi_j + \frac{1}{\lambda^2}\theta^{ab}\partial_b\Phi^i\partial_c\Phi_i\theta^{cd}\partial_d\Phi^j\partial_a\Phi_j. \]

According to the above prescription they transform to

\[ (G^{ab}\partial_b\hat{\Phi}^i\partial_c\hat{\Phi}_iG^{cd}\partial_d\hat{\Phi}^j\partial_a\hat{\Phi}_j) *_4 + \frac{1}{\lambda^2}(\theta^{ab}\partial_b\hat{\Phi}^i\partial_c\hat{\Phi}_i\theta^{cd}\partial_d\hat{\Phi}^j\partial_a\hat{\Phi}_j) *_4 + \cdots. \] (24)

However the corresponding terms from the action (8) is

\[ (G^{ab}\partial_b\hat{\Phi}^i\partial_c\hat{\Phi}_iG^{cd}\partial_d\hat{\Phi}^j\partial_a\hat{\Phi}_j) *_4 - \frac{1}{\lambda^2}[\hat{\Phi}_j, \hat{\Phi}_i]_M *_2 [\hat{\Phi}^i, \hat{\Phi}^j]_M + \cdots. \]

The first and second terms above are reproduced by the expansion of the first and the second determinant in the square root in (8), respectively. Comparing them with (24), one would expect the following identity

\[ (\theta^{ab}\partial_a f_1 \partial_b g_1 \theta^{cd}\partial_c f_2 \partial_d g_2) *_4 = -[f_1, g_1]_M *_2 [f_2, g_2]_M. \]

Examining the higher order terms, one would expect the following general identity

\[ (\theta^{a_1b_1}\partial_a f_1 \partial_b g_1 \cdots \theta^{a_nb_n}\partial_a f_n \partial_b g_n) *_{2n} = (i)^n([f_1, g_1]_M \cdots [f_n, g_n]_M) *_{n}. \]

One may try to prove this recursion formula between \(*_{2n}\) and \(*_n\) using the definition of \(*_N\)-product[14, 19, 31].

The commutative action (1) includes, among other things, the couplings of the closed string states to commutative gauge fields when derivative of \(F_{ab}\) is small compare to \(\lambda\). Whereas, the action (8) or (14) includes the similar couplings to non-commutative gauge fields with no restriction on \(\hat{F}_{ab}\) in the SW limit. When the derivative of \(\hat{F}_{ab}\) is small, one can replace the \(*_N\)-product with the ordinary product. In that case, the operation in the step 4 in the section 2 is identity operator. So one expects that the commutative and non-commutative actions to be identical. Therefore, one may compare the closed string couplings in these actions to extract some transformation between the commutative and the non-commutative fields under the SW map. For example, consider linear coupling of the dilaton to commutative gauge field on the world-volume of D9-brane (1), that is

\[ S = T_9\sqrt{-\det(\eta_{\mu\nu} + \lambda B_{\mu\nu})}\int d^{10}x \phi \sqrt{\det(1 + \lambda GF + \theta F)}, \]
where the $g_{\mu\nu} + \lambda B_{\mu\nu}$ is the constant background field. Now the same thing in terms of non-commutative fields is

$$\hat{S} = T_9 \sqrt{-\det(g_{\mu\nu} + \lambda B_{\mu\nu})} \int d^{10}x \left( \phi(\hat{A}) \sqrt{\det(1 + \lambda G\hat{F})} \right),$$

where we have used the prescription (7). Using the manipulation in sect. 3, one will find the following transformation

$$\sqrt{\det(1 + \lambda GF + \theta F)}(k) = \int d^{10}y \left[ W(k^\mu, y^\mu, C)\sqrt{\det(1 + \lambda G\hat{F})} \right] e^{ik\cdot y}$$

$$= \int d^{10}y \sqrt{\det(1 + \lambda G\hat{F})} e^{ik\cdot (y + \theta \hat{A})}.$$

Now we change to the new coordinate [14]

$$X(y) = y + \theta \hat{A}(y); \quad \int d^{10}y \longrightarrow \int \frac{d^{10}X}{\sqrt{\det(1 - \theta F)}}.$$

From this we find that

$$\sqrt{\det(1 + \lambda GF + \theta F)}(X) = \frac{\sqrt{\det(1 + \lambda G\hat{F})}}{\sqrt{\det(1 - \theta F)}}.$$

Doing some simple algebra, one finds

$$F_{\mu\nu}(X(y)) = \left( \hat{F}(y) \frac{1}{1 - \theta \hat{F}(y)} \right)_{\mu\nu}(y).$$

This is exactly the identity found in [14]. In [14] above identity was derived from the conjectured exact solution to the SW differential equation. That conjecture was then confirmed in [15, 16]. The solution to the SW differential equation found in [14, 15, 16] is valid only for $U(1)$ gauge theory. The reason is that if one directly integrates the SW differential equation for the case of non-abelian gauge fields, one will find that a new type of multiplication, the one which is not the $*_{N}$-product, appears in the solution [22]. So it would be interesting, using [14], to extend the perturbative solution found in [22] to an exact solution for the non-abelian cases. Other progress in this direction has been made in [32, 33].

We have seen that starting from the ordinary DBI action and mapping it to the non-commutative variables smeared along one Wilson line and then taking the SW limit, one finds exactly the known result in the non-commutative super YM theory, e.g., the energy-momentum tensor. However, the energy-momentum tensor for the D-branes of the bosonic theory is not the same as for the D-branes of the superstring theory [13, 21]. It would be interesting then to find a commutative action that reproduces the result for the bosonic case using the prescription given in this paper.
The Matrix theory is a special limit of the non-abelian non-commutative super YM theory. The dimensional reduction of the YM theory along the world-volume space, reduces it to the Matrix theory. In this case, the SW limit reduces to the DKPS limit\[34\]. As we discussed in the text, to extend the non-commutative action (14) to the non abelian case one should change the fields to the matrix valued fields, and at the same time should use the symmetrized trace prescription. Hence, the known form of the Matrix theory action is the dimensional reduction of the non-abelian form of the action [21], that is,

\[
S = -\frac{1}{g_{YM}^2} \int dt \sqrt{-\det G} \left( \frac{1}{2} G^{00} D_0 \Phi_i D_0 \Phi^i \Phi^j [\Phi^I, \Phi^J] \right),
\]

where now the indices \( I, J = 1, 2, \ldots, 9 \), and there is no non-commutative parameter. The energy-momentum tensor of the theory can be read from the results in (19)–(20), that is,

\[
S_{\phi} = \frac{2\kappa}{g_{YM}^2 \lambda^2} \int \frac{d^{10} p}{(2\pi)^{10}} \sqrt{-\det G} \left( \hat{\Phi}'(p) \hat{T}_\phi(p) \right),
\]

\[
S_h = -\frac{\kappa}{g_{YM}^2 \lambda^2} \int \frac{d^{10} p}{(2\pi)^{10}} \sqrt{-\det G} \left( \hat{h}_{\mu \nu}(p) \hat{T}_h^{\mu \nu}(p) \right),
\]

where

\[
\hat{T}_\phi(p) = \int dt \left( T_\phi(t) e^{-ip_0 t + i\lambda p_I \Phi^I} \right),
\]

\[
\hat{T}_h^{\mu \nu}(p) = \int dt \left( T_h^{\mu \nu}(t) e^{-ip_0 t + i\lambda p_I \Phi^I} \right),
\]

and

\[
T_\phi(t) = 1,
\]

\[
T_0^{00}(t) = G^{00},
\]

\[
T_0^{0I}(t) = \lambda G^{00} D_0 \Phi^I,
\]

\[
T_0^{IJ}(t) = \lambda^2 G^{00} D_0 \Phi^I D_0 \Phi^J + \lambda^2 [\Phi^I, \Phi^K] [\Phi_K, \Phi^J].
\]

These expressions are exactly the known results for the energy-momentum tensor of the Matrix theory [35, 36, 21]. Note that in this case there is no non-commutative parameter, no Wilson line and no path ordering prescription. Therefore, we have used the symmetrized trace prescription instead of taking the trace of the path ordered terms.

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