Does Leading $\ln x$ Resummation Predict the Rise of $g_1$ at Small $x$?

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Abstract

We numerically analyse the evolution of the flavor non-singlet $g_1$ structure function taking into account the all-order resummation of $\alpha_s \ln^2 x$ terms which is expected to have much stronger effects than the DGLAP evolution in the small $x$ region. We include a part of the next-to-leading logarithmic corrections coming from the resummed “coefficient function” which are not considered in the calculation of Blümlein and Vogt to respect the factorization scheme independence. It is pointed out that the resummed coefficient function gives unexpectedly large suppression factor over the experimentally accessible range of $x$ and $Q^2$. This fact implies that the next-to-leading logarithmic contributions are very important for the $g_1$ structure function.

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1 Introduction

Recent new data for the unpolarized Deep Inelastic Scattering from HERA provide us with much information on the quark-gluon structure of Nucleon. The HERA experiments cover much broader kinematical regions than before. Especially, the behavior of the structure function at small values of the Bjorken variable \(x\) receives much attention of the physicists [1]. The small \(x\) region corresponds to the Regge limit. So we naively expect that the soft physics (Regge theory) may explain the small \(x\) behavior of the structure function. However the steep rise of the structure function in this region observed by the HERA experiments contradicts with this naive expectation. The physics at small \(x\) is now one of the most interesting subjects and many people believe that this problem could be handled in the context of the QCD perturbation theory [1]. The approach based on the Balitskii-Fadin-Kuraev-Lipatov (BFKL) [2] equation or on the high-energy factorization [3] seems to be very promising and it is important to reveal the role of the so-called BFKL Pomeron.

In the case of the polarized structure function \(g_1\), we have not yet had data at very small \(x\). However the recent data show some rise of \(g_1\) in the small \(x\) region [1]. This behavior again seems to contradict with naive Regge prediction \((g_1 \sim x^\alpha, \ 0 \leq \alpha \leq 0.5)\) [1]. In fact, to explain the rise of \(g_1\) at small \(x\) in the framework of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [3] approach, it is required to choose a steep function as an input parton density [4] as far as the \(Q^2\) evolution starts at the order of 1 \(GeV^2\) because the evolution effect in the DGLAP equation does not produce enough enhancement in the small \(x\) region [8]. So it is interesting to see whether the all-order resummation of \(\ln x\) terms which appear in the perturbative calculations reconcile the experimental behavior of \(g_1\) with the naive Regge prediction.

Some time ago, Kirschner and Lipatov [9] considered the all order resummation of \(\alpha_s \ln^2 x\) series in the case of quark-quark forward scattering process. Recently Bartels
et al. [10] have given the resummed expression for the $g_1$ structure function by using the Infra-Red Evolution Equation. They claim that the resummation effects may lead to 10 times larger results than the DGLAP ones. This analysis suggests the above possibility that the small $x$ behavior of $g_1$ is explained naturally by combining a flat input (non-perturbative) density expected from the naive Regge theory (it is reasonable at low $Q^2,x$) with the perturbatively resummed results of $\alpha_s \ln^2 x$ series.

On the other hand, the recent numerical analysis which has been done by Bl"umlein and Vogt [11] shows that there are no significant contributions to the evolution of $g_1$ from the resummation of the leading logarithmic (LL) corrections at the HERA kinematical region ($x \sim 10^{-3}$). The controversial aspect between their numerical analysis and the assertion by Bartels et al. might be coming from the fact that the resummed part of the “coefficient function” is considered in Ref. [10] but not in Ref. [11]. Bl"umlein and Vogt did not include the resummed part of the coefficient function because this part turns out to fall in the next-to-leading logarithmic (NLL) corrections and depends on the factorization scheme adopted. It is also to be noted that the evolution, in general, strongly depends on the input parton densities. If one chooses a steep input function, the perturbative contribution will be completely washed away since the structure function is given as the convolution integral of the parturbative part and the input density. So it will be interesting to see the sensitivity of the results to the choice of the input densities.

In the present paper, we numerically reanalyse the flavor non-singlet part of $g_1$ by taking into account the $\ln x$ resummation. We consider three different input densities: one is a flat density corresponding to the naive Regge prediction and others are steep ones in the small $x$ region. The coefficient function can not be included consistently at present since the anomalous dimension has been calculated only at the LL order. However we consider also the effects of the coefficient function. The reason is because
we could firstly clarify the above controversial aspect and secondly get some idea about the magnitude of the NLL order corrections in the resummation approach.

This paper is organized as follows. In section 2, we make a brief review on the resummation of $\ln x$ series and present an explicit expression for $g_1^{NS}$. In section 3, we show our numerical results and discuss the effects of the NLL corrections. The interpretation of the numerical results and summary will be given in section 4.

## 2 Resummation of $\ln x$ terms

The flavor non-singlet part of the polarized structure function $g_1^{NS}$ is given by the formula,

$$g_1^{NS}(Q^2, x) = \frac{\langle e^2 \rangle}{2} \int_x^1 \frac{dy}{y} C^{NS}(\alpha_s(Q^2), x/y) \Delta q^{NS}(Q^2, y) ,$$

where $\Delta q^{NS}$ is the flavor non-singlet combination of the polarized parton densities,

$$\Delta q^{NS}(Q^2, x) = \sum_{i=1}^{n_f} e_i^2 \frac{\langle e^2 \rangle}{\langle e^2 \rangle} (\Delta q_i(Q^2, x) - \Delta q_i(Q^2, y)) ,$$

and $C^{NS}$ is the coefficient function. $n_f$ is the number of active flavors with electric charge $e_i$, $\langle e^2 \rangle = \sum e_i^2 / n_f$. The perturbative evolution of the parton density is controlled by the DGLAP equation,

$$Q^2 \frac{\partial}{\partial Q^2} \Delta q(Q^2, x) = \int_x^1 \frac{dy}{y} P(\alpha_s(Q^2), x/y) \Delta q(Q^2, y) .$$

In the above equation and in the following, we suppress the superscript $NS$ which means the flavor non-singlet part. The coefficient function $C(\alpha_s, y)$ and the splitting function $P(\alpha_s, y)$ are both calculable in the QCD perturbation theory. When $x$ is finite, it may be enough to compute them to the fixed-order of perturbation. In the small $x$ region, however, the fixed-order calculation becomes questionable since there appear $\ln^m x$ corrections in the higher orders of the strong coupling constant $\alpha_s$. If these $\ln^m x$ terms compensate the smallness of $\alpha_s$, we must resum the perturbative series to the all orders to get a reliable prediction.
To see what terms show up at small $x$, it will be convenient to take the Mellin
transform of Eq.(1).

$$g_1(Q^2, N) \equiv \int_0^1 dx x^{N-1} g_1(Q^2, x) = \frac{\langle e^2 \rangle}{2} C(\alpha_s(Q^2), N) \Delta q(Q^2, N),$$

where

$$C(\alpha_s(Q^2), N) \equiv \int_0^1 x^{N-1} C(\alpha_s(Q^2), x),$$
$$\Delta q(Q^2, N) \equiv \int_0^1 x^{N-1} \Delta q(Q^2, x).$$

The DGLAP evolution equation Eq.(2) becomes,

$$Q^2 \frac{\partial}{\partial Q^2} \Delta q(Q^2, N) = -\gamma(\alpha_s(Q^2), N) \Delta q(Q^2, N).$$

(3)

Here the anomalous dimension $\gamma$ is the moment of the splitting function,

$$\gamma(N, \alpha_s(Q^2)) \equiv -\int_0^1 dx x^{N-1} P(\alpha_s(Q^2), x).$$

Eq.(3) is easily solved to give,

$$\Delta q(Q^2, N) = \Delta q(Q_0^2, N) \exp \left( -\int_{\alpha_s(Q_0^2)}^{\alpha_s(Q^2)} \frac{d\alpha}{\beta(\alpha)} \gamma(\alpha, N) \right),$$

where $\beta$ is the beta function,

$$\beta(\alpha) = \frac{\partial \alpha_s}{\partial \ln Q^2} = \alpha_s \left[ -\beta_0 \frac{\alpha_s}{4\pi} - \beta_1 \left( \frac{\alpha_s}{4\pi} \right)^2 - \cdots \right].$$

The first two coefficients of the $\beta$ function are,

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_R n_f, \quad \beta_1 = \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_R n_f - \frac{4}{3} C_F T_R n_f,$$

with $C_F = (N_c^2 - 1)/2N_c$ and $C_A = N_c$ for the SU($N_c$) color group and $T_R = 1/2$. 

The coefficient function \( C(\alpha_s, N) \) and the anomalous dimension \( \gamma(\alpha_s, N) \) may be expanded in the powers of \( \alpha_s \),

\[
C(\alpha_s, N) = 1 + \sum_{k=1}^{\infty} c_k(N) \tilde{\alpha}_s^k,
\]

\[
\gamma(\alpha_s, N) = \sum_{k=1}^{\infty} \gamma_k(N) \tilde{\alpha}_s^k.
\]

where (and in the following) we use the abbreviation,

\[
\tilde{\alpha}_s \equiv \frac{\alpha_s}{4\pi}.
\]

The singular behaviors of the coefficient and splitting functions as \( x \to 0 \) appear as the pole singularities at \( N = 0 \) in the moment space. The explicit next-to-leading order (NLO) calculations of the coefficient function \[12\] and the anomalous dimension \[13\] in the \( \overline{\text{MS}} \) scheme show a strong singularity at \( N = 0 \),

\[
c^1(N) = 2C_F \frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N}\right),
\]

\[
\gamma^2(N) = 4(3C_F^2 - 2C_AC_F) \frac{1}{N^3} + \mathcal{O}\left(\frac{1}{N^2}\right),
\]

whereas the leading order anomalous dimension looks like,

\[
\gamma^1(N) = -2C_F \frac{1}{N} - C_F + \mathcal{O}(N),
\]

at small \( N \). These strong singularities (double logarithmic corrections) will persist to all orders of perturbative series. Indeed, at the \( k \)-th loop, the anomalous dimension and the coefficient function are expected to behave as,

\[
\gamma^k(N) \sim N \left(\frac{1}{N^2}\right)^k, \quad c^k(N) \sim \left(\frac{1}{N^2}\right)^k.
\]

Since the singularities in \( N^{-m} \) correspond to the \( \ln^{m-1}\left(\frac{1}{z}\right) \) singularities, our task is to resum these terms to all-orders in the perturbative expansion.

Before discussing the resummed results, it may be worth mentioning the difference between the polarized (unpolarized flavor non-singlet) and the unpolarized flavor singlet
structure functions \[14\]. Naively one expects that the anomalous dimension behaves like \( \gamma \sim \alpha_s^k/N^{2k-1} \) at the \( k \)-th loop because there exist extra infrared and collinear singularities. In the case of the unpolarized flavor singlet structure functions, however, many of them are canceled and the true behavior at the \( k \)-th loop is \( \gamma \sim (\alpha_s/N)^k \).

These terms can be resummed by the BFKL equation. On the other hand, above strong singularities survive in the polarized structure function. This fact suggests that the polarized structure function will receive large perturbative corrections at small \( x \).

The resummation of \( \ln x \) singularities for the \( g_1 \) structure function has been done in Refs. \[9\] \[10\]. The result for the “parton (quark) ” target with the fixed coupling constant \[^†\] reads,

\[
g_1^{\text{parton}}(x, Q^2) = \frac{e_i^2}{2} \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} x^{-N} \left( \frac{Q^2}{\mu^2} \right)^{f_0^{-}(N)/8\pi^2} \frac{N}{N - f_0^{-}(N)/8\pi^2},
\]

where \( \mu \) is an arbitrary mass scale which regularizes the infrared and/or mass singularities. From this expression we could identify the resummed anomalous dimension \( \hat{\gamma} \) and the coefficient function \( \hat{C} \) to be,

\[
\hat{\gamma}(\alpha_s, N) \equiv \lim_{N \to 0} \gamma(\alpha_s, N) = -f_0^{-}(N)/8\pi^2,
\]

\[
\hat{C}(\alpha_s, N) \equiv \lim_{N \to 0} C(\alpha_s, N) = \frac{N}{N - f_0^{-}(N)/8\pi^2}.
\]

Here \( f_0^- \), which corresponds to the odd-signature quark-quark scattering amplitude in the color singlet channel, satisfies the equation,

\[
f_0^-(N) = 16\pi^2C_F \frac{\bar{\alpha}_s}{N} - 8C_F \frac{\bar{\alpha}_s}{N^2} f_8^+(N) + \frac{1}{8\pi^2N}(f_0^{-}(N))^2.
\]

The even-signature quark-quark scattering amplitude \( f_8^+ \) with the color octet quantum number in the t-channel is the solution of the equation,

\[
f_8^+(N) = -8\pi^2 \frac{1}{N_c N} \bar{\alpha}_s + 2N_c \frac{\bar{\alpha}_s}{N} dN f_8^+(N) + \frac{1}{8\pi^2N}(f_8^+(N))^2,
\]

\[^*\]We follow the convention of Ref. \[3\] for the moment.

\[^†\]In the genuine LL approximation, the strong coupling constant should be taken as a fixed parameter.
and given by,

\[ f_8^+(N) = 16\pi^2 N_c \tilde{\alpha}_s \frac{d}{dN} \ln(e^{z^2/4} D_{-1/2N_c^2}(z)) \] with \( z = \frac{N}{\sqrt{2 N_c \tilde{\alpha}_s}} \).

\( D_p(z) \) is the parabolic cylinder function \[5\]. Finally we reach,

\[ f_0^-(N) = 4\pi^2 N \left( 1 - \sqrt{1 - 8 C_F \frac{\tilde{\alpha}_s}{N_c} \left[ 1 - \frac{1}{2\pi^2 N} f_8^+(N) \right]} \right). \]

Now it will be instructive to re-expand Eqs.(6,7) in terms of \( \alpha_s \) to see whether these formulae sum up the most singular terms of the perturbative series. The expressions expanded up to \( \mathcal{O}(\alpha_s^5) \) read,

\[
\hat{\gamma} = -N \left[ 2C_F \left( \frac{\tilde{\alpha}_s}{N_c^2} \right) + 4C_F \left( C_F + \frac{2}{N_c} \right) \left( \frac{\tilde{\alpha}_s}{N_c^2} \right)^2 
\right.
\]

\[
+ 16C_F \left( C_F^2 + 2 \frac{C_F}{N_c} - \frac{1}{2N_c^2} - 1 \right) \left( \frac{\tilde{\alpha}_s}{N_c^2} \right)^3 
\]

\[
+ 16C_F \left( 5C_F^3 + 12 \frac{C_F^2}{N_c} + 2 \frac{C_F}{N_c^2} - 4C_F + \frac{1}{N_c^3} + \frac{5}{N_c} + 6N_c \right) \left( \frac{\tilde{\alpha}_s}{N_c^2} \right)^4 
\]

\[ + \ldots \]

\[ = \sum_{k=1}^{\infty} \hat{\gamma}^k \left( \frac{\tilde{\alpha}_s}{N_c^2} \right)^k, \tag{8} \]

\[
\hat{C} = 1 + 2C_F \left( \frac{\tilde{\alpha}_s}{N_c^2} \right) + 8C_F \left( C_F + \frac{1}{N_c} \right) \left( \frac{\tilde{\alpha}_s}{N_c^2} \right)^2 
\]

\[
+ 8C_F \left( 5C_F^3 + 8 \frac{C_F}{N_c} - \frac{1}{2N_c^2} - 2 \right) \left( \frac{\tilde{\alpha}_s}{N_c^2} \right)^3 
\]

\[
+ 32C_F \left( 7C_F^3 + 15 \frac{C_F^2}{N_c} + 2 \frac{C_F}{N_c^2} - 4C_F + \frac{1}{2N_c^3} + \frac{5}{2N_c} + 3N_c \right) \left( \frac{\tilde{\alpha}_s}{N_c^2} \right)^4 
\]

\[ + \ldots \]

\[ = \sum_{k=0}^{\infty} \hat{C}^k \left( \frac{\tilde{\alpha}_s}{N_c^2} \right)^k. \tag{9} \]

These results coincide with the previous expectation of Eq.(5). Furthermore, noting the relation,

\[ 2C_A - 3C_F = \frac{2}{N_c} + C_F, \]
which holds in SU($N_c$), we can see that the resummed expressions Eqs. (6,7) reproduce the known NLO results Eqs. (4) in the $\overline{\text{MS}}$ scheme. Therefore, it is quite plausible that Eqs. (6,7) correctly sum up the “leading” singularities to all orders.

Here a comment is in order concerning the scheme dependence. It is well-known that the anomalous dimension and the coefficient function individually depend on the factorization scheme and only an appropriate combination of them becomes scheme independent. When one considers the higher order corrections in the perturbation theory, therefore, one must specify the scheme adopted. Unfortunately we do not have by now any appropriate factorization theorems to the problem discussed in this paper. This means that we must be careful when considering the resummed quantities. In particular, the resummed “coefficient function” does not have any physical meaning until the scheme dependent part of the anomalous dimension is calculated in the same scheme. To clarify this issue, it is convenient to write the above results in the form which corresponds to the so-called DIS scheme [16]. The DIS scheme is defined so that the naive parton model relation is true to all orders in perturbation theory. The polarized parton densities become physical observables in this scheme. The parton densities and anomalous dimension in the DIS scheme are obtained by making the transformations,

$$\Delta q \rightarrow \Delta q^{\text{DIS}} \equiv C \Delta q,$$

$$\gamma^{\text{DIS}} \equiv C \gamma C^{-1} - \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \ln C.$$  

Using the resummed $\hat{\gamma}$ and $\hat{C}$ Eqs. (8,9), we get the resummed part of the anomalous dimension in the DIS scheme,

$$\hat{\gamma}^{\text{DIS}} = N \sum_{k=1}^{\infty} \hat{\gamma}^k \left( \frac{\bar{\alpha}_s}{N^2} \right)^k + \beta_0 N^2 \sum_{k=2}^{\infty} \hat{d}^k \left( \frac{\bar{\alpha}_s}{N^2} \right)^k + O \left( N^3 \left( \frac{\bar{\alpha}_s}{N^2} \right)^k \right), \quad (10)$$

where the second terms come from the resummed coefficient function and $\hat{d}^k$ are numerical numbers independent of $N$. The above equation tells us that the resummed
coefficient function belongs to the NLL order corrections in the context of the resummation approach. Then, one must include the NLL order anomalous dimension which has not yet been available to see the effects of the coefficient function. This is the reason why the authors in Ref. [11] throw away the coefficient function.

3 Numerical Analysis

Numerical analysis of the spin structure function \( g_{1}^{NS} \) in the small \( x \) region was done in the context of the small \( x \) resummation approach in Ref. [11]. They obtained the result that the small \( x \) resummation effect is not significant despite of a naive expectation discussed in Ref. [10]. In this section, we numerically reanalyze the behavior of \( g_{1}^{NS} \) structure function to show how the final results are sensitive to the choice of the input parton densities. In conjunction with the claim in Ref. [10], we also consider the effects from the resummed coefficient function. As already discussed in section 2, we can not include the coefficient function in a theoretically consistent way. However we believe that the inclusion of the coefficient function could shed some light on the size of the NLL corrections in the resummation approach.

At first, we explain our method to estimate the \( g_{1}^{NS} \) structure function numerically. Our starting point is the expression,

\[
g_{1}^{NS}(Q^2, x) = \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} x^{-N} \exp \left( - \int_{\alpha_s(Q_0^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{\beta} \gamma^{DIS} \right) g_{1}^{NS}(Q_0^2, N). \tag{11}
\]

The anomalous dimension \( \gamma^{DIS} \) which includes the resummation of \( \ln^n x \) terms is organized as follows,

\[
\gamma^{DIS}(N) = \bar{\alpha}_s \gamma^1(N) + \bar{\alpha}_s^2 \gamma^2(N) + K(N, \alpha_s) - \beta \frac{\partial}{\partial \alpha_s} \ln \left( 1 + \bar{\alpha}_s c^1 + H(N, \alpha_s) \right), \tag{12}
\]

where \( \gamma^{1,2} \) and \( c^1 \) are respectively the usual anomalous dimension and coefficient function at the one and two-loop fixed order perturbation theory. \( K(N, \alpha_s) \) (\( H(N, \alpha_s) \)) is

\[\text{This fact implies the LL resummed anomalous dimension} \, \hat{\gamma} \text{ being scheme independent.}\]
the resummed anomalous dimension Eq.(8) (Eq.(9)) with $k = 1, 2 (k = 0, 1)$ terms being subtracted because those terms have already been included in the usual anomalous dimension and coefficient function.

$$K(N, \alpha_s) \equiv \hat{\gamma}(N) - \hat{\gamma}^1 \frac{\alpha_s}{N} - \hat{\gamma}^2 \frac{\alpha_s^2}{N^3},$$

$$H(N, \alpha_s) \equiv \hat{C}(N) - 1 - c^1 \frac{\alpha_s}{N^2}.$$ 

It should be noted here that the anomalous dimension at $N = 1$ plays a special role for the non-singlet $g_1$ structure function. In a language of the operator product expansion, $\gamma(N = 1)$ is the anomalous dimension of the (non-singlet) axial vector current. Since the (non-singlet) axial vector current is conserved, the corresponding anomalous dimension should vanish. The perturbation theory guarantees this symmetry order by order in the $\alpha_s$ expansion. However, the resummation of the leading singularities in $N$ does not respect this symmetry. Therefore, we need to restore this symmetry “by hand”. In this paper, we multiply $K(N, \alpha_s)$ by $(1 - N) \ [17],$

$$K(N, \alpha_s) \rightarrow K(N, \alpha_s)(1 - N).$$

Of course, this is not a unique prescription and one can choose other procedure\[ which satisfies the condition of $\lim_{N \rightarrow 1} K(N, \alpha_s) = 0.$

Now let us explain how to perform the Mellin inversion Eq.(11) which is the integral in the complex $N$-plane. At first, we must know the Mellin transform of the input function $g_1(Q_0^2, N)$. It is easy to obtain an analytical form for it in the complex $N$-plane since we assume a simple function (see below) for the input density. Next we need an analytically continued expression of the anomalous dimension $\gamma^{DIS}$ in the complex $N$-plane. For the $g_1$ structure function, only odd moments are defined. So we replace $(-1)^N$ by $(-1)$ in the expression of the anomalous dimension obtained in Ref. [13]. The integration contour in the Mellin inversion should be on the right of the

\[\text{Our final conclusion remains the same qualitatively if we choose other prescription.}\]
rightmost singularity of the integrand. The contour integration along the imaginary axis from \( c - i\infty \) to \( c + i\infty \) is numerically inconvenient due to the slow convergence of the integral in the large \(|N|\) region. To get rid of this problem, we deformed the contour to the line which have an angle \( \phi (\phi > \pi/2) \) from the real \( N \) axis. By this change of the contour, we have a damping factor \( \exp(|N|\ln(1/x)\cos\phi) \) which strongly suppresses the contribution from the large \(|N|\) region. In the integration along this new contour, we will be able to cut the large \(|N|\) region. Finally we have checked the stability of results by changing the contour parameter. One can find the details of this technique in Ref. \([18]\).

We choose the starting value of the evolution to be \( Q^2_0 = 4 GeV^2 \). We calculate the \( Q^2 \) evolution for three types of the input densities A, B and C: A is a function which is flat at small \( x \) \((x^\alpha, \alpha \sim 0)\), B is slightly steep \( (\alpha \sim -0.2) \) and C rises more steeply \( (\alpha \sim -0.7) \). The explicit parametrization used in this paper is \([7]\),

\[
\Delta q(Q^2_0, x) = N(\alpha, \beta, a)\eta x^\alpha (1-x)^\beta (1+ax),
\]

where \( N \) is a normalization factor such that \( \int dx N x^\alpha (1-x)^\beta (1+ax) = 1 \) and \( \eta = \frac{1}{6} g_A \) \( (g_A = 1.26) \) in accordance with the Bjorken sum rule. A, B and C correspond to the following values of parameters,

- **A**: \( \alpha = +0.0, \beta = 3.09, a = 2.23 \),
- **B**: \( \alpha = -0.2, \beta = 3.15, a = 2.72 \),
- **C**: \( \alpha = -0.5, \beta = 2.41, a = 0.02 \).

In our analysis we put the flavor number \( n_f = 4 \) and \( \Lambda_{QCD} = 0.23 GeV \).

First we estimate the case which includes only the LL correction \( \hat{\gamma} \). The evolution kernel in this case is obtained by dropping \( H(N, \alpha_s) \) in Eq.(12). This is a consistent

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\footnote{This choice is essentially the same as one in Ref. \([11]\).}

\footnote{We take into account the fixed-order NLO corrections exactly.}
approximation in the resummation approach. Fig.1a (1b, 1c) shows the results (dashed curves) after evolving to $Q^2=10, 10^2, 10^4 GeV^2$ from the A (B, C) input density (dot-dashed line). The solid curves are the predictions of the NLO-DGLAP evolution. These results show a tiny enhancement compared with the NLO-DGLAP analysis and are consistent with those in Ref. [11]**. In the case of C, we can not discriminate a difference between the LL and DGLAP results. The enhancement is, as expected, bigger when the input density is flatter. However any significant differences are not seen between the results from different input densities.

Next, we include the NLL corrections coming from the resummed “coefficient function”. We show the results in Fig.2 by the dashed curves. (Other curves are the same as in Fig.1.) The results are rather surprising. The inclusion of the coefficient function leads to a strong suppression on the evolution of the structure function at small $x$. Since the effects from the coefficient function fall in the NLL level, the LL terms are expected to (should) dominate in the small $x$. However our results imply that the LL approximation is not sensible in the small $x$ region we are interested in. As the resummed coefficient function is only a part of the NLL correction, we can not present a definite conclusion on the (full) NLL correction. But it is obvious that the NLL correction is very important at the experimentally accessible region of $x$. In the following section, we explain why the coefficient function leads to such suppression.

4 Discussion and Summary

In the previous section, we have shown that although the LL resummed effect is very small at the experimentally accessible region of $x$, a part of the NLL resummed contribution from the coefficient function drastically changes the predictions.

To understand these numerical results, it will be helpful to remember the per-

**We have also calculated $g_1$ with the input function used by Blümlein and Vogt and could reproduce their results.
turbative expansion of the resummed anomalous dimension and coefficient function Eqs. (8,9). By using the explicit values \( N_C = 3, \ C_F = 4/3 \), we obtain for the anomalous dimension in the DIS scheme Eq. (10),

\[
\hat{\gamma}^{\text{DIS}} = N \left[ -0.212 \left( \frac{\alpha_s}{N^2} \right) - 0.068 \left( \frac{\alpha_s}{N^2} \right)^2 - 0.017 \left( \frac{\alpha_s}{N^2} \right)^3 - 0.029 \left( \frac{\alpha_s}{N^2} \right)^4 + \cdots \right] \\
+ N^2 \left[ 0.141 \left( \frac{\alpha_s}{N^2} \right)^2 + 0.119 \left( \frac{\alpha_s}{N^2} \right)^3 + 0.069 \left( \frac{\alpha_s}{N^2} \right)^4 + \cdots \right] \\
+ \cdots .
\] (13)

Here note that: (1) the perturbative coefficients of the LL terms (the first part of Eq. (13)) are negative and those of the higher orders are rather small number. This implies that the LL corrections push up the structure function compared to the fixed-order DGLAP evolution, but the deviations are expected to be small. (2) the perturbative ones from the NLL terms (the second part of Eq. (13)), however, are positive and somehow large compared with those of the LL terms. This positivity of the NLL terms has the effect of decreasing the structure function. This fact that the coefficients with both sign appear in the anomalous dimension should be contrasted with the case of the unpolarized structure function [19].

Now it might be also helpful to assume that the saddle-point dominates the Mellin inversion Eq. (11). We have numerically estimated the approximate position of the saddle-point and found that the saddle-point stays around \( N_{\text{SP}} \sim 0.31 \) in the region of \( x \sim 10^{-5} \) to \( 10^{-2} \). (Of course the precise value of the saddle-point depends on \( x, Q_0^2 \) and \( Q^2 \).) By looking at the explicit values of the coefficients in Eq. (13), the position of the saddle-point seems to suggest that the NLL terms can not be neglected. Since the coefficients from the higher order terms are not so large numerically, it is also expected that the terms which lead to sizable effects on the evolution may be only first few terms in the perturbative series in the region of \( x \) we are interested in. We
have checked that the inclusion of the first few terms in Eq. (13) already reproduces the results of section 3. Fig. 3a (3b) shows the numerical results of the contribution from each terms of the NLL corrections in Eq. (13) at $Q^2 = 10^2 \text{GeV}^2$ with the A (B) type input density. The solid (dot-dashed) line corresponds to the NLL (LL) result. The long-dashed, dashed and dotted lines correspond respectively to the case in which the terms up to the order $\alpha_s^2$, $\alpha_s^3$, $\alpha_s^4$, are kept in the NLL contributions. One can see that the dotted line already coincides with the full NLL (solid) line. These considerations could help us to understand why the NLL corrections turns out to give large effects on the evolution of the $g_1$ structure function.

The final discussion concerns the convergence issue of the perturbative series. As discussed in Refs. [20] [21], one must be careful when applying the perturbative approach to small $x$ evolution. The integrand in Eq. (11) has a singularity in the moment space. This (rightmost) singularity is equal to that of $f_0^-(N)$. The numerical value $N_0$ of the singularity position is $N_0 \sim 0.304$. This means that the $N$ can not become so small. On the other hand, the approximation scheme in the resummation approach is sensible only for small $N$. This apparent contradiction will be solved by analyzing the evolution in $x$ space [20]. By explicitly solving the evolution in $x$ space, it has been pointed out [21] that the saddle-point method is not a good approximation in the case of the unpolarized structure function. Although we have not used the saddle-point approximation to solve the evolution, the previous explanation relying on this method can be misleading. So according to Refs. [20] [21], we have tried to solve the evolution in $x$ space with first several terms of the perturbative expansion being kept and what we found is that the conclusion does not change. The numerical results are essentially the same as Fig. 3.

In summary, we have performed numerical studies for the flavor non-singlet $g_1$ structure function at small $x$ by incorporating the all-order resummed anomalous di-
mension and a part of the NLL corrections from the resummed coefficient function. Our results show that the resummed coefficient function has an effect which suppresses the structure function at small $x$. Including only the resummed coefficient part is not theoretically consistent, and so one should take into account also the anomalous dimension at the NLL level. However, our results suggest that the LL analysis is unstable, in the sense that a large suppression effect comes from the resummed coefficient function which should be NLL correction. We have explained why the inclusion of a part of the NLL corrections leads to such unexpected results. We need a full NLL analysis to make a definite conclusion on whether the all-order resummation approach predicts a rise of the flavor non-singlet $g_1$ structure function in the experimentally accessible small $x$ region.

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Figure Captions

Fig. 1
The LL evolution as compared to the DGLAP results with the flat input A (Fig. 1a) and steep ones B (Fig. 1b) and C (Fig. 1c).

Fig. 2
The NLL evolution as compared to the DGLAP results with the flat input A (Fig. 2a) and steep ones B (Fig. 2b) and C (Fig. 2c).

Fig. 3
Contributions from the fixed order terms in the NLL resummation with the flat input A (Fig. 3a) and steep one B (Fig. 3b).
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1a.pdf}
\caption{Graph showing various curves for different values of $Q_0^2$.}
\end{figure}

- \textbf{Q}_0^2 = 4\text{GeV}^2
- \textbf{DGLAP}
- \textbf{LL}

\begin{align*}
10^2 \\
10^4 \text{GeV}^2
\end{align*}
$Q_0^2 = 4 \text{GeV}^2$

DGLAP

LL

Fig. 1b
Fig. 1c

- $Q_0^2 = 4\text{GeV}^2$
- DGLAP
- LL

- $10^4 \text{GeV}^2$
- $10^2 \text{GeV}^2$
- $10 \text{GeV}^2$
$Q_0^2 = 4 \text{GeV}^2$

DGLAP

NLL

Fig. 2a
$\frac{Q_0^2}{\mathrm{GeV}^2} = 4 \mathrm{GeV}^2$

- DGLAP
- NLL

Fig. 2b
Fig. 2c
$Q^2 = 10^2 \, \text{GeV}^2$

- NLL
- $o(\alpha_s^4)$
- $o(\alpha_s^3)$
- $o(\alpha_s^2)$
- LL

Fig. 3a
$Q^2 = 10^2 \text{ GeV}^2$

- NLL
- $o(\alpha_s^4)$
- $o(\alpha_s^3)$
- $o(\alpha_s^2)$
- LL

Fig. 3b