Primary decomposable subspaces of $k[t]$ and Right ideals of the first Weyl algebra $A_1(k)$ in characteristic zero

M. K. KOUAKOU  
Université de Cocody  
UFR-Mathématiques et Informatique  
22 BP 582 Abidjan 22  
Côte d’Ivoire

cw1kw5@yahoo.fr

A. TCHOUDJEM  
Institut Camille Jordan  
UMR 5208  
Université Lyon 1  
43 bd du 11 novembre 1918  
69622 Villeurbanne cedex  
France

tchoudjem@math.univ-lyon1.fr

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In the classification of right ideals $A_1 := k[t, \partial]$ the first Weyl algebra over a field $k$, R. Cannings and M.P. Holland established in [3, Theorem 0.5] a bijective correspondence between primary decomposable subspaces of $R = k[t]$ and right ideals $I$ of $A_1 := k[t, \partial]$ the first Weyl algebra over $k$ which have non-trivial intersection with $k[t]$:

$$
\Gamma : V \mapsto \mathcal{D}(R, V) \ , \ \Gamma^{-1} : I \mapsto I \star 1
$$

This theorem is a very important step in this study, after Stafford’s theorem [1, Lemma 4.2]. However, the theorem had been established only when the field $k$ is an algebraically closed and of characteristic zero.

In this paper we define notion of primary decomposable subspaces of $k[t]$ when $k$ is any field of characteristic zero, particulary for $\mathbb{Q}, \mathbb{R}$, and we show that R. Cannings and M.P. Holland’s correspondence theorem holds. Thus right ideals of $A_1(\mathbb{Q}), A_1(\mathbb{R})\ldots$ are also described by this theorem.
1 Cannings and Holland’s theorem

1.1 Weyl algebra in characteristic zero and differential operators

Let $k$ be a commutative field of characteristic zero and $A_1 := A_1(k) = k[t, \partial]$ where $\partial, t$ are related by $\partial t - t \partial = 1$, be the first Weyl algebra over $k$.

$A_1$ contains the subring $R := k[t]$ and $S := k[\partial]$. It is well known that $A_1$ is an integral domain, two-sided noetherian and since the characteristic of $k$ is zero, $A_1$ is hereditary (see [2]). In particular, $A_1$ has a quotient division ring, denoted by $Q_1$.

For any right (resp: left ) submodule of $Q_1$, $M^*$ the dual as $A_1$-module will be identified with the set $\{ u \in Q_1 : uM \subset A_1 \}$ (resp:$\{ u \in Q_1 : Mu \subset A_1 \}$) when $M$ is finitely generated (see [1]).

$Q_1$ contains the subrings $D = k(t)[\partial]$ and $B = k(\partial)[t]$. The elements of $D$ are $k$-linear endomorphisms of $k(t)$. Precisely, if $d = a_n \partial^n + \cdots + a_1 \partial + a_0$ where $a_i \in k(t)$ and $h \in k(t)$, then

$$d(h) := a_n h^{(n)} + \cdots + a_1 h^{(1)} + a_0 h$$

where $h^{(i)}$ denotes the $i$-th derivative of $h$ and $a_i h^{(i)}$ is a product in $k(t)$. One checks that:

$$(dd')(h) = d(d'(h)) \text{ for } d, d' \in k(t)[\partial], \ h \in k(t)$$

For $V$ and $W$ two vector subspaces of $k(t)$, we set :

$$\mathcal{D}(V, W) := \{ d \in k(t)[\partial] : d(V) \subset W \}$$

$\mathcal{D}(V, W)$ is called the set of differential operators from $V$ to $W$.

Notice that $\mathcal{D}(R, V)$ is an $A_1$ right submodule of $Q_1$ and $\mathcal{D}(V, R)$ is an $A_1$ left submodule of $Q_1$. If $V \subseteq R$, one notes that $\mathcal{D}(R, V)$ is a right ideal of $A_1$. When $V = R$, then $\mathcal{D}(R, R) = A_1$.

If $I$ is a right ideal of $A_1$, we set

$$I \star 1 := \{ d(1), d \in I \}$$

Clearly, $I \star 1$ is a vector subspace of $k[t]$ and $I \subseteq \mathcal{D}(R, I \star 1)$.

Inclusion $A_1 \subset k(\partial)[t]$ and $A_1 \subset k(t)[\partial]$ show that it can be defined on $A_1$ two notions of degree: the degree associated to ”$t$” and the degree associated to ”$\partial$”. Naturally, those degree notions extend to $Q_1$. 

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1.2 Stafford’s theorem

Let $I$ be a non-zero right ideal of $A_1$. By J. T. Stafford in [1, Lemma 4.2], there exist $x, e \in Q_1$ such that:

(i) $xI \subset A_1$ and $xI \cap k[t] \neq \{0\}$, (ii) $eI \subset A_1$ and $eI \cap k[\partial] \neq \{0\}$

By (i) one sees that any non-zero right ideal $I$ of $A_1$ is isomorphic to another ideal $I'$ such that $I' \cap k[t] \neq \{0\}$, which means that $I'$ has non-trivial intersection with $k[t]$. We denote $\mathcal{I}_t$ the set of right ideals $I$ of $A_1$ the first Weyl algebra over $k$ such that $I \cap k[t] \neq \{0\}$

Stafford’s theorem is the first step in the classification of right ideals of the first Weyl algebra $A_1$.

1.3 The bijective correspondence theorem

Let $c$ be an algebraically closed field of characteristic zero. Cannings and Holland have defined primary decomposable subspace $V$ of $c[t]$ as finite intersections of primary subspaces which are vector subspaces of $c[t]$ containing a power of a maximal ideal $m$ of $c[t]$. Since $c$ is an algebraically closed field, maximal ideals of $c[t]$ are generated by one polynomial of degree one: $m = (t - \lambda)c[t]$. So, a vector subspace $V$ of $c[t]$ is primary decomposable if:

$$V = \bigcap_{i=1}^{n} V_i$$

where each $V_i$ contains a power of a maximal ideal $m_i$ of $c[t]$.

They have established the nice well-known bijective correspondence between primary decomposable subspaces of $c[t]$ and $\mathcal{I}_t$ by:

$$\Gamma : V \mapsto \mathcal{D}(R, V) \ , \ \Gamma^{-1} : I \mapsto I \ast 1$$

Since $V = \bigcap_{i=1}^{n} V_i$ and $m_i =< (t - \lambda_i)^{r_i}> \subseteq V_i$, one has $(t - \lambda_1)^{r_1} \cdot \cdot \cdot (t - \lambda_n)^{r_n}k[t] \subseteq V$. So, easily one sees that

$$(t - \lambda_1)^{r_1} \cdot \cdot \cdot (t - \lambda_n)^{r_n}k[t] \subseteq \mathcal{D}(R, V) \cap c[t]$$

However it is not clear that $I \ast 1$ must be a primary decomposable subspace of $c[t]$.  

Cannings and Holland’s theorem use the following result, which holds even if the field is just of characteristic zero:

**Lemma 1**: Let $I \in \mathcal{I}$ and $V = I \star 1$. One has:

$I = \mathcal{D}(R, V)$ and $I^* = \mathcal{D}(V, R)$.

For the proof of Cannings and Holland’s theorem one can see [3].

We note that, since $< (t-\lambda_i)^{r_i} > \subseteq V_i$, for any $s$ in the ring $c+(t-\lambda_i)^{r_i}c[t]$, one has:

$s \cdot V_i \subseteq V_i$

It is this remark which will allow us to give general definition of primary decomposable subspaces of $k[t]$ for any field $k$ of characteristic zero, not necessarily algebraically closed.

## 2 Primary decomposable subspaces of $k[t]$

Here we give a general definition of primary decomposable subspaces of $k[t]$ when $k$ is any field of characteristic zero not necessarily algebraically closed and we keep the bijective correspondence of Cannings and Holland.

### 2.1 Definitions and examples

- **Definitions**

Let $b, h \in R = k[t]$ and $V$ a $k$-subspace of $k[t]$. We set:

$O(b) = \{a \in R : a' \in bR\}$ and $O(b, h) = \{a \in R : a' + ah \in bR\}$

where $a'$ denotes the formal derivative of $a$.

$S(V) = \{a \in R : aV \subseteq V\}$ and $C(R, V) = \{a \in R : aR \subseteq V\}$

Clearly $O(b)$ and $S(V)$ are $k$-subalgebras of $k[t]$. If $b \neq 0$, the Krull dimension of $O(b)$ is $\dim_K(O(b)) = 1$. The set $C(R, V)$ is an ideal of $R$ contained in both $S(V)$ and $V$.

- A $k$-vector subspace $V$ of $k[t]$ is said to be primary decomposable if $S(V)$ contains a $k$-subalgebra $O(b)$, with $b \neq 0$. 


• Examples
  ◦ Easily one sees that $O(b) \subseteq S(O(b, h))$ and $C(R, O(b, h)) = C(R, O(b))$, in particular $O(b, h)$ is a primary decomposable subspace when $b \neq 0$.

Following lemmas and corollary show that classical primary decomposable subspaces are primary decomposable in the new way.

**Lemma 2:*** Let $k$ be a field of characteristic zero and $\lambda_1, .., \lambda_n$ finite distinct elements of $k$. Suppose that $V_1, .., V_n$ are $k$-vector subspaces $k[t]$, each $V_i$ contains $(t - \lambda_i)^{r_i} k[t]$ for some $r_i \in \mathbb{N}^*$. Then

$$O((t - \lambda_1)^{r_1-1} \cdots (t - \lambda_n)^{r_n-1}) \subseteq S(\bigcap_{i=1}^n V_i)$$

**Proof:** One has $O((t - \lambda_i)^{r_i-1}) = k + (t - \lambda_i)^{r_i} k[t]$ and

$$O((t - \lambda_1)^{r_1-1} \cdots (t - \lambda_n)^{r_n-1}) = \bigcap_{i=1}^n O((t - \lambda_i)^{r_i-1})$$

An immediate consequence of this lemma is:

**Corollary 3:** In the above hypothesis of lemma 2, let

$$V = \bigcap_{i=1}^n V_i$$

If $q \in C(R, V)$, then $O(q) \subseteq S(V)$.

**Proof:** First one notes that if $q \in pk[t]$, then $O(q) \subseteq O(p)$. Let $b = (t - \lambda_1)^{r_1} \cdots (t - \lambda_n)^{r_n}$.

In the above hypothesis, one has

$$C(R, V) = \bigcap_{i=1}^n C(R, V_i) = \bigcap_{i=1}^n (t - \lambda_i)^{r_i} k[t] = (\prod_{i=1}^n (t - \lambda_i)^{r_i}) k[t] = bk[t]$$

Since $b \in (t - \lambda_1)^{r_1-1} \cdots (t - \lambda_n)^{r_n-1} k[t] = b_0 k[t]$, one has $O(b_0) V_i \subseteq V_i$ for all $i$, so

$$O(b_0) \subseteq S(V) \text{ and } O(q) \subseteq O(b) \subseteq O(b_0)$$

**An opposite-example:**

Suppose the field $k$ is of characteristic zero and one can find $q \in k[t]$ such that: $q$ is irreducible and $\deg(q) \geq 2$. Then the vector subspace $V = k + qk[t]$ is not primary decomposable.
2.2 Classical properties of primary decomposable subspaces

Here we prove that when the field $k$ is algebraically closed of characteristic zero, those two definitions are the same.

**Lemma 4**: Let $k$ be an algebraically closed field of characteristic zero and $V$ be a $k$-vector subspace of $k[t]$ such that $S(V)$ contains a $k$-subalgebra $O(b)$ where $b \neq 0$. Then $V$ is a finite intersections of subspaces which contains a power of a maximal ideal of $k[t]$.

**Proof**: Since $k$ is algebraically closed field and $b \neq 0$, one can suppose $b = (t - \lambda_1)^{r_1} \cdots (t - \lambda_n)^{r_n}$. Let $b^* = (t - \lambda_1) \cdots (t - \lambda_n)$. One has

$$O(b) = \bigcap_{i=1}^{n} (k + (t - \lambda_i)^{r_i+1}R)$$

If we suppose that $V$ is not contained in any ideal of $R$, one has $V.R = R$. Clearly

$$bb^*R = \prod_{i=1}^{n} (t - \lambda_i)^{r_i+1}R \subset O(b)$$

so $bb^*R = (bb^*)(RV) = (bb^*)V = bb^*R \subset V$ (1). One also has

$$O(b) \cap (t - \lambda_i)R \neq O(b) \cap (t - \lambda_j)R \text{ for all } i \neq j$$

in particular one has

$$O(b) = [O(b) \cap (t - \lambda_i)R]^{r_i+1} + [O(b) \cap (t - \lambda_j)R]^{r_j+1} \quad (2)$$

With (1) and (2) one gets inductively:

$$V = \bigcap_{i=1}^{n} (V + (t - \lambda_i)^{r_i+1}R) \quad \diamond$$

One also obtains usual properties of primary decomposable subspaces.

**Lemma 5**: Let $k$ be a field of characteristic zero, $V$ and $W$ be primary decomposable subspaces of $k[t]$

(1) then $V + W$ and $V \cap W$ are primary decomposable subspaces.
(2) If \( q \in k(t) \) such that \( qV \subseteq k[t] \), then \( qV \) is a primary decomposable subspace.

**Proof:** One notes that \( O(ab) \subseteq O(a) \cap O(b) \) for all \( a, b \in k[t] \).

Let us recall basic properties on the subspace \( O(a, h) \).

**Lemma 6:**

1. \( O(a) \subseteq S(O(a, h)) \)
2. \( C(R, O(a)) = C(R, O(a, h)) \)
3. \( a^2k[t] \subseteq O(a) \cap O(a, h) \)
4. \( D(R, O(a, h)) = A_1 \cap (\partial + h)^{-1}aA_1 \)
5. the subspace \( O(a, h) \) is not contained in any proper ideal of \( R \).
6. For all \( q \in O(a, h) \) such that \( hcf(q, a) = 1 \), one has \( O(a, h) = qO(a) + C(R, O(a)) \)

**Proof:** One obtains (1), (2), (3), (4) by a straightforward calculation.

Suppose \( O(a, h) \subseteq gk[t] \). Then \( D(R, O(a, h)) \subseteq gA_1 \), and applying the \( k \)-automorphism \( \sigma \in Aut_k(A_1) \) such that \( \sigma(t) = t \) and \( \sigma(\partial) = \partial - h \), one obtains \( D(R, O(a)) \subseteq gA_1 \). Clearly the element \( f = \partial^{-1}a\partial^{m+1} \) where \( deg_t(a) = m \) belongs to \( D(R, O(a)) = A_1 \cap \partial^{-1}aA_1 \). When one writes \( f \) in extension, one gets exactly

\[
f = a\partial + a_{m-1}\partial^m + \cdots + a_1\partial + (-1)^m a!
\]

Since \( f \in gA_1 \), \((-1)^m a! \) must belong to \( gR \). Hence \( g \in k^* \) and one gets (5).

Let \( q \) be an element of \( O(a, h) \) such that \( hcf(q, a) = 1 \). One has also \( hcf(q, a^2) = 1 \), and by Bezout theorem there exist \( u, v \in k[t] \) such that:

\[
uq + va^2 = 1 \quad (\ast)
\]

The inclusion \( qO(a) + C(R, O(a)) \subseteq O(a, h) \) is clear since \( q \in O(a, h) \) and one has properties (1) and (2). Conversely let \( p \in O(a, h) \). Using (\( \ast \)), one gets

\[
p = (pu)q + a^2pv \quad (\ast\ast)
\]

One notes that \( p(uq) = p - pva^2 \in O(a, h) \), so \( (p(uq))' + (p(uq))h \in aR \). One has \( (p(uq))' + (p(uq))h = p'(uq) + p(uq)' + p(uq)h = p(uq)' + uq(p' + ph) \).
Since \( q \) is chosen in \( O(a,h) \), one has \( p' + ph \in aR \). Then \( q(\nu') \in aR \), and at the end, because of \( hcf(q,a) = 1 \), it follows that \( (\nu') \in aR \). Now, \( \nu \in O(a) \) and (**) shows that \( p \in qO(a) + C(R,O(a)) \).

**Proposition 7**: Let \( k \) be a field of characteristic zero and \( V \) a \( k \)-vector subspace of \( k[t] \) such that \( S(V) \) contains a \( k \)-subalgebra \( O(b) \). Then

\[
D(R,V) \ast 1 = V
\]

**Proof**:

- Suppose \( V = O(b) \). One has \( D(R,O(b)) = A_1 \cap \partial^{-1}bA_1 \).

  Suppose \( b = \beta_0 + \beta_1 t + \cdots + \beta_m t^m \), \( \beta_m \neq 0 \). Then \( f = \partial^{-1}b\partial^{m+1} \in A_1 \cap \partial^{-1}bA_1 \). Let us show that \( f(R) = O(b) \). For an integer \( 0 \leq p \leq m \), one has:

\[
\partial^{-1}t^p\partial^{m+1} = (t\partial - 1) \cdot (t\partial - 2) \cdots (t\partial - p)\partial^{m-p}
\]

and so

\[
f = \beta_0\partial^m + \sum_{p=1}^{m} \beta_p(t\partial - 1) \cdot (t\partial - 2) \cdots (t\partial - p)\partial^{m-p}
\]

In particular one sees that:

1. \( f(1) = \beta_m(1)m! \neq 0 \)
2. \( f(t^j) = 0 \) if \( 1 \leq j < m \)
3. \( f(t^m) = \beta_0 m! \)
4. \( \deg(f(t^j)) = j \) when \( j \geq m + 1 \)

It follows that

\[
\dim \frac{R}{f(R)} = m = \dim \frac{R}{O(b)}
\]

and since \( f(R) \subseteq O(b) \), one gets \( f(R) = O(b) \)

- Suppose that \( O(b) \subseteq S(V) \). One has \( VO(b) = V \) and then

\[
[V \mathcal{D}(R,O(b))] \ast 1 = V[D(R,O(b)) \ast 1] = VO(b) = V
\]

By lemma 1 the equality \( V \mathcal{D}(R,O(b)) = D(R,V) \) holds, so

\[
\mathcal{D}(R,V) \ast 1 = V.
\]

Next theorem is the main result of this paper.

**Theorem 8**: Let \( k \) be a field of characteristic zero and \( V \) a \( k \)-vector subspace of \( k[t] \) such that: \( C(R,V) = qk[t] \) with \( q \neq 0 \) and \( \mathcal{D}(R,V) \ast 1 = V \). Then \( S(V) \) contains some \( k \)-subalgebra \( O(b) \) with \( b \neq 0 \).
Proof: One has $qk[t] \subseteq V$, and there exist $v_0, v_1, \ldots, v_m$ in $V$ such that

$$V = \langle v_0, v_1, \ldots, v_m \rangle \oplus qk[t]$$

where $\langle v_0, v_1, \ldots, v_m \rangle$ denotes the vector subspace of $V$ generated by $\{v_0, v_1, \ldots, v_m\}$. For each $v_i$, there exist $f_i \in \mathcal{D}(R, V)$ such that $f_i(1) = v_i$. Let $r = \max\{\deg_\partial(f_i), 0 \leq i \leq m\}$, we prove that $O(q^r) \cdot V \subseteq V$.

Since the ideal $qk[t]$ of $R = k[t]$ is contained in $V$, we have only to prove that:

$$O(q^r) \cdot v_i \subseteq V \quad \forall 0 \leq i \leq m$$

We need the following lemma

**Lemma 9**: Let $d = a_p \partial^p + \cdots + a_1 \partial + a_0 \in A_1(k)$ where $p \in \mathbb{N}$, $b \in k[t]$ and $s \in O(b^p)$. Then $[d, s] = d \cdot s - s \cdot d \in bA_1$.

**Proof**: One has $[d, s] = [d_1 \partial, s] = [d_1, s] \partial + d_1[\partial, s]$, where $d_1 \in A_1$ and $d = d_1 \partial + a_0$. By induction on the $\partial$-degree of $d$, one has $[d_1, s] \partial \in bA_1$. Since $\deg_\partial(d_1) = p - 1$, it is also clear that $d_1 b^p \in bA_1$. Finally $[d, s] \in bA_1$.

By lemma 9 above, one has $f_i \cdot s \in \mathcal{D}(R, V)$ and $[f_i, s] \in qA_1$ for each $i$.

$$s \cdot v_i = s \cdot (f_i(1)) = (s \cdot f_i)(1) = (f_i \cdot s + [f_i, s])(1)$$

One has $(f \cdot s)(1) \in V$, $[f_i, s](1) \in qk[t]$, it follows that $s \cdot v_i \in V$ and that ends the proof of theorem 8.

Next lemma justify the definition we gave for primary decomposable subspaces.

**Lemma 10**: Let $k$ be a field of characteristic zero and suppose there exist $q$ an irreducible element of $k[t]$ with $\deg(q) \geq 2$. If $V = k + qk[t]$, then $\mathcal{D}(R, V) = qA_1$. In particular $V$ is not primary decomposable subspace.

**Proof**: Since $q$ is irreducible, one shows by a straightforward calculation that the right ideal $qA_1$ is maximal. Clearly one has $qA_1 \subseteq \mathcal{D}(R, V)$, and $\mathcal{D}(R, V) \neq A_1$ since $1 \notin \mathcal{D}(R, V)$. So one has $qA_1 = \mathcal{D}(R, V)$.

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