Succinct representation of labeled trees

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Abstract

We give a representation for labeled ordered trees that supports labeled queries such as finding the \(i\)-th ancestor of a node with a given label. Our representation is succinct, namely the redundancy is small-o of the optimal space for storing the tree. This improves the representation of He et al. [8] which is succinct unless the entropy of the labels is small.

1 Introduction

A problem which was extensively studied in recent years is representing an ordered rooted tree using space close to the information-theoretic lower bound while supporting numerous queries on the tree, e.g. [3, 5, 7, 9]. Geary et al. [7] studied an extension of this problem, in which the nodes of the tree are labeled with characters from alphabet \([1, \ldots, \sigma]\). The tree queries now receive a character \(\alpha\) as an additional argument, and the goal of a query is to locate a certain node whose label is \(\alpha\) or to count the nodes satisfying some property and whose labels are \(\alpha\). The set of queries considered by Geary et al. is given in Table 1.

Geary et al. gave a representation that uses \(n \log \sigma + 2n + O(n \sigma \log \log n / \log \log n)\) bits, where \(n\) is the number of nodes and \(\sigma\) is the size of the alphabet, and answers queries in constant time. For \(\sigma = o(\log \log n / \log \log \log n)\), the space is \(n \log \sigma + 2n + o(n)\), namely, the space is \(o(n)\) more than the information-theoretic lower bound.

Barbay et al. [1] and Ferragina et al. [6] gave labeled tree representations that use space close to the lower bound for large alphabets, but the set of supported queries is more restricted. He et al. [8] improved the result of Geary et al. by showing a labeled tree representation based upon a rank-select structure on the string \(P_T\) that contains the labels of the nodes in preorder. Using the rank-select structure of Belazzougui and Navarro [2] the following results were obtained: (1) For \(\sigma = w^{O(1)}\), there is a representation that uses \(nH_0(P_T) + O(n)\) bits and answers queries in \(O(1)\) time, where \(w\) is the word size and \(H_0(P_T)\) is the zero-order entropy of \(P_T\). (2) For \(\sigma \leq n\), there is a representation that uses \(nH_0(P_T) + o(nH_0(P_T)) + O(n)\) bits. label queries are answered in \(O(1)\) time, and preorder, rank queries are answered in \(\omega(1)\) time (namely, for every function \(f\) satisfying \(f(n) = \omega(1)\), the time is \(O(f(n))\)), or vice versa. Other queries are answered in \(O(\log^{\log \sigma} \frac{\log n}{\log w})\) time. The representation of He et al. supports all the queries of Table 1 and additional queries. Note that the representation is succinct if \(H_0(P_T) = \Omega(1)\).

In this paper we give a fully succinct representation of labeled trees. Our result is given in the following theorem.
Table 1: Supported queries on a labeled tree. A node with label $\alpha$ is called an $\alpha$-node. A $\alpha$-child of a node is a child which is an $\alpha$-node. Other $\alpha$- terms are defined similarly.

| Query                     | Description                                                                 |
|----------------------------|-----------------------------------------------------------------------------|
| label($x$)                 | The label of $x$.                                                           |
| depth$_\alpha$($x$)        | The number of $\alpha$-nodes on the path from the root to $x$.              |
| level_ancestor$_\alpha$($x$, $i$) | The $\alpha$-ancestor $y$ of $x$ for which depth$_\alpha$($y$) = depth$_\alpha$($x$) $- i$. |
| parent$_\alpha$($x$)       | level_ancestor$_\alpha$($x$, 1).                                            |
| deg$_\alpha$($x$)          | The number of $\alpha$-children of $x$.                                    |
| child_rank$_\alpha$($x$)   | The rank of $x$ among its $\alpha$-siblings.                               |
| child_select$_\alpha$($x$, $i$) | The $i$-th $\alpha$-child of $x$.                                        |
| num_descendants$_\alpha$($x$) | The number of $\alpha$-descendants of $x$.                              |
| preorder_rank$_\alpha$($x$) | The preorder rank of $x$ among the $\alpha$-nodes.                      |
| preorder_select$_\alpha$($i$) | The $i$-th $\alpha$-node in the preorder.                                 |
| postorder_rank$_\alpha$($x$) | The postorder rank of $x$ among the $\alpha$-nodes.                   |
| postorder_select$_\alpha$($i$) | The $i$-th $\alpha$-node in the postorder.                          |

**Theorem 1.** Let $T$ be a labeled tree with $n$ nodes and labels from $\{1, \ldots, \sigma\}$.

1. For $\sigma = w^{O(1)}$, there is a representation of $T$ that uses $nH_0(P_T) + 2n + o(n)$ bits and answers the queries of Table 1 in $\omega(1)$ time.

2. For $\sigma \leq n$, there is a representation of $T$ that uses $nH_0(P_T) + 2n + o(nH_0(P_T)) + O(n)$ bits. label queries are answered in $O(1)$ time, and preorder_rank queries are answered in $\omega(1)$ time, or vice versa. The rest of the queries of Table 1 are answered in $O(\log \log \sigma \log w)$ time.

Note that our representation supports only the queries considered by Geary et al. and it does not support the additional queries considered by He et al.

2 Preliminaries

A rank-select structure stores a string $S$ over alphabet $\{1, \ldots, \sigma\}$ and supports the following queries: (1) rank$_\alpha(S, i)$ returns the number of occurrences of $\alpha$ in the first $i$ characters of $S$ (2) select$_\alpha(S, i)$ returns the $i$-th occurrence of $\alpha$ in $S$ (3) access$(S, i)$ returns the $i$-th character of $S$. The problem of designing a succinct rank-select structure with efficient query times was studied extensively. For our purpose, we use the following results.

**Theorem 2** (Belazzougui and Navarro [2]). A rank-select structure can be built on a string $S$ of length $n$ over alphabet $\{1, \ldots, \sigma\}$ such that (1) If $\sigma = w^{O(1)}$, the space is $nH_0(S) + o(n)$ bits, and the structure answers rank queries in $O(1)$ time (2) If $\sigma \leq n$, the space is $nH_0(S) + o(nH_0(S)) + o(n)$ bits, and the structure answers rank queries in $O(\log \log \frac{\sigma}{\log w})$ time. The structure answers access queries in $O(1)$ time and select queries in $\omega(1)$ time, or vice versa.

**Theorem 3** (Raman et al. [14]). A rank-select structure can be built on a binary string $S$ of length $n$ that contains $k$ ones such that the space is $O(k \log (n/k)) + o(n)$ bits, and the structure answers queries in $O(1)$ time.
Table 2: Supported queries on a weighted tree. BP denotes the balanced parenthesis representation of the tree.

| Query                  | Description                                                                 |
|------------------------|-----------------------------------------------------------------------------|
| \(w_a(x)\)            | The \(w_a\)-weight of \(x\).                                               |
| \(\text{depth}_a(x)\) | The \(w_a\)-weight of the nodes of the path from the root to \(x\).        |
| \(\text{level}_a(x,i)\) | The lowest ancestor \(y\) of \(x\) for which the \(w_a\)-weight of the nodes of the path from \(y\) to \(x\), excluding \(x\), is at least \(i\). |
| \(\text{parent}_a(x)\) | \(\text{level}_a(x,1)\).                                                 |
| \(\text{deg}_a(x)\)   | The \(w_a\)-weight of the children of \(x\).                             |
| \(\text{child\_rank}_a(x)\) | The \(w_a\)-weight of \(x\) and its left siblings.                 |
| \(\text{child\_select}_a(x,i)\) | The leftmost child \(y\) of \(x\) for which \(\text{child\_rank}_a(y) \geq i\). |
| \(\text{num\_descendants}_a(x)\) | The \(w_a\)-weight of the proper descendants of \(x\).  |
| \(\text{bp}(i)\)      | The \(i\)-th character of \(BP\).                                     |
| \(\text{bp\_open}(x)\) | The index of the ‘1’ character in \(BP\) that corresponds to \(x\).     |
| \(\text{bp\_close}(x)\) | The index of the ‘0’ character in \(BP\) that corresponds to \(x\).     |
| \(\text{bp\_node}(i)\) | The node that corresponds to the \(i\)-th character of \(BP\).         |
| \(\text{bp\_rank}_{a,b}(i)\) | The \(w_a\)-weight of the nodes that correspond to ‘1’ characters of \(BP\) with index at most \(i\), plus the \(w_b\)-weight of the nodes that correspond to ‘0’ characters of \(BP\) with index at most \(i\). |
| \(\text{bp\_select}_{a,b}(i)\) | The minimum index \(j\) for which \(\text{bp\_rank}_{a,b}(j) \geq i\). |

We also use the following result on representation of (unlabeled) ordered trees. We are interested in a representation that supports the unlabeled versions of the queries listed in Table 1 and additional queries such as lca queries.

**Theorem 4** (Navarro and Sadakane [12]). An ordered tree can be stored using \(2n + o(n)\) bits such that tree queries can be answered in \(O(1)\) time.

### 2.1 Representation of weighted trees

In this section we consider the problem of representing ordered trees with weights on the nodes. We will use weighted trees in our representation of labeled trees.

Let \(T\) be a tree with weights \(w_1(v), \ldots, w_s(v)\) for each node, where each weight is from \(\{0, \ldots, X - 1\}\). For a set of nodes \(U\), the \(w_a\)-weight of \(U\) is \(\sum_{v \in U} w_a(v)\). The weighted tree queries we need are described in Table 2. Throughout this section we assume that the balanced parenthesis string of a tree is a binary string, where open and close parenthesis are represented by 1 and 0, respectively.

**Lemma 5.** A weighted tree with \(n\) nodes and \(s = O(1)\) weight functions with weights from \(\{0, \ldots, X - 1\}\), where \(X = O(\log n)\), can be stored using at most \(2n \log(2X^s) + o(n)\) bits such that the queries in Table 2 are answered in \(O(1)\) time.

**Proof.** The representation is a variation of the unweighted tree representation of Navarro and Sadakane [12]. We first review the latter representation. Let \(T\) be an unweighted tree, and let \(P\) be its balanced parenthesis representation. Navarro and Sadakane showed that tree queries can be implemented by supporting a set of base queries that include (1) the queries deg, child_rank, and child_select (2) the following queries on \(P\) and a function...
\( f : \{0, 1\} \to \{-1, 0, 1\} \) from a fixed set of functions \( \mathcal{F} \):

\[
\text{sum}(P, f, i, j) = \sum_{k=i}^{j} f(P[k])
\]

\[
\text{fwd\_search}(P, f, i, d) = \min\{j \geq i : \text{sum}(P, f, i, j) \geq d\}
\]

\[
\text{bwd\_search}(P, f, i, d) = \max\{j \leq i : \text{sum}(P, f, i, j) \geq d\}
\]

\[
\text{RMQi}(P, f, i, j) = \arg\max\{\text{sum}(P, f, 1, k) : i \leq k \leq j\}
\]

We note that Navarro and Sadakane used a slightly different definition for fwd\_search and bwd\_search, but their technique is easily modified for the alternative definitions above.

The main idea of Navarro and Sadakane’s representation is to partition \( P \) into blocks of size \( N = 2^c \) for some constant \( c \). Each block of \( P \) is stored using an aB-tree \(^{13}\) which is able to support the base queries on the block in constant time. The space of an aB-tree is \( O(1) \) more than the information-theoretic lower bound. Since \( P \) is a binary string, the space of one aB-tree is \( N + O(1) \), and the space for all trees is \( 2n + o(n) \). In order to support base queries on the entire string \( P \), Navarro and Sadakane added additional data-structures. The space of the additional data-structures is \( O((n/N) \log^{O(1)} n) \). Thus, by choosing large enough \( c \), the additional space is \( o(n) \).

For example, for supporting fwd\_search for a function \( f \), a tree \( T_f \) is constructed with weights from \( \{0, \ldots, N\} \) on its edges, and a weighted ancestor data-structure is built over \( T_f \). The tree \( T_f \) is defined as follows. Let \( M_i \) be the maximum value of \( \text{sum}(P, f, 1, j) \) for an index \( j \) that belongs to the \( i \)-th block of \( P \). The nodes of \( T_f \) are \( \{0, 1, \ldots, n/N\} \). A node \( i > 0 \) is a child of node \( j \), where \( j \) is the minimum index for which \( j > i \) and \( M_j > M_i \). The edge between these nodes have weight \( M_j - M_i \). If no such index exists, \( i \) is a child of node 0. A fwd\_search\((P, f, i, d)\) query is answered by first checking whether the answer lies in the block of \( i \) (using the aB-tree that stores the block). If not, a weighted ancestor query on \( T_f \) finds the block in which the answer lies, and the location inside the block is found using the aB-tree storing this block.

We now describe our representation for weighted trees. Let \( BP \) denote the balanced parenthesis representation of \( T \). Define a string \( P \) of length \( 2n \) in which each character is a tuple of \( s + 1 \) elements. For an index \( i \), the tuple \( P[i] \) is \((w_1(v), w_2(v), \ldots, w_s(v), BP[i])\), where \( v \) is the node that corresponds to \( BP[i] \). As for the case of unweighted trees, it suffices to support (1) the weighted tree queries deg, child\_rank, and child\_select (2) sum, fwd\_search, bwd\_search and RMQi queries on the string \( P \) and a function \( f \) (from a fixed set of functions \( \mathcal{F} \)) which has the form

\[
\phi_{a,b}(x) = \begin{cases} x_a & \text{if } x_{s+1} = 1 \\ x_b & \text{if } x_{s+1} = 0 \end{cases} \quad \text{or} \quad \pi_a(x) = \begin{cases} x_a & \text{if } x_{s+1} = 1 \\ -x_a & \text{if } x_{s+1} = 0 \end{cases}
\]

where \( x_a \) denotes the \( a \)-th coordinate of \( x \) (we also denote \( x_0 = 0 \)). To support the base queries, \( P \) is partitioned into blocks and each block is stored using an aB-tree. The space of one aB-tree is now \( N \log(2X^*) + O(1) \). Thus, the total space of the aB-trees is \( 2n \log(2X^*) + o(n) \). Additionally, since now the range of a function \( f \) is \( \{-X, \ldots, X\} \) whereas the range is \( \{-1, \ldots, 1\} \) for unweighted trees, the space of the additional data-structures is increased by a factor of at most \( X \) (for example, the \( T_f \) trees have now edge weights from \( \{0, \ldots, NX\} \) and thus require more space). Since \( X = O(\log n) \), we can ensure the additional space is \( o(n) \) by increasing \( c \) by 1. ■
Figure 1: Example of tree decomposition using $L = 4$. Figure (a) shows the decomposition of Farzan and Munro, and Figure (b) shows a modified decomposition. Figure (c) shows the tree $T_L$.

2.2 Tree decomposition

In the next lemma we present a tree decomposition that we will use in our labeled tree representation. The decomposition is a slightly modified version of the decomposition of Farzan and Munro [5]. An example for the decomposition is given in Figure 1b.

Lemma 6. For a tree $T$ with $n$ nodes and an integer $L$, there is a collection $\mathcal{D}_{T,L}$ of subtrees of $T$ with the following properties.

1. Every edge of $T$ appears in exactly one tree of $\mathcal{D}_{T,L}$.

2. The size of each tree in $\mathcal{D}_{T,L}$ is at most $2L + 1$.

3. The number of trees in $\mathcal{D}_{T,L}$ is $O(n/L)$.

4. For every tree $T' \in \mathcal{D}_{T,L}$ there are two intervals of integers $I_1$ and $I_2$ such that a node $x \in T$ is a non-root node of $T'$ if and only if preorder_rank$(x) \in I_1 \cup I_2$, where preorder_rank$(x)$ is the preorder rank of $x$ in $T$. The node with preorder rank $\min(\max(I_1), \max(I_2))$ is called the special node of $T'$.

5. For every $T' \in \mathcal{D}_{T,L}$, only the root and the special node of $T'$ can appear in other trees of $\mathcal{D}_{T,L}$.

Proof. We first construct the decomposition of Farzan and Munro [5] (see Figure 1a for an example), which satisfied the properties of the lemma, except for property 1. It also satisfies stronger versions of properties 2 and 3. Each tree in the decomposition has size at most $2L$, and the common node of two trees in the decomposition can only be the root of both trees. Moreover, for a tree $T'$ in the decomposition, for every edge $(v, w)$ between a node $v \in T'$ and a node $w \in T'$, $v$ is the root of $T'$, except perhaps for one edge.

We now change the decomposition as follows (see Figure 1b). First, for every edge $(v, w)$ for which $v$ and $w$ are in different trees, if $v$ is not a root of a tree in the decomposition, add the node $w$ to the unique tree containing $v$. Note that in this case, the tree also contains the predecessor of $v$ in the preorder, so property 4 is maintained. Otherwise, add a new tree to the decomposition that consists of the nodes $v$ and $w$. After the first step is performed, remove from the decomposition all trees that consist of a single node. It is easy to verify that the new decomposition satisfied all the properties of the lemma. $\blacksquare$
For a tree $T$ and an integer $L$ we define a tree $T_L$ as follows (see Figure 1c). Construct a tree decomposition $D_{T,L}$ according to Lemma 6. If the root $r$ of $T$ appears in several trees of $D_{T,L}$, add to $D_{T,L}$ a tree that consists of $r$. The set of nodes of $T_L$ is the set $D_{T,L}$. For two trees $x', y' \in D_{T,L}$, $x'$ is the parent of $y'$ in $T_L$ if and only if the root of $y'$ is equal to the special node of $x'$ (or $x'$ is the tree that consists of $r$).

3 Representation of labeled trees

As in He et al. [8], we build trees $T^\alpha$ for every $\alpha \in \{1, \ldots, \sigma\}$. To build $T^\alpha$, we temporarily add to $T$ a new root with label $\alpha$. Then, let $X_\alpha$ be the set of all $\alpha$-nodes in $T$ and their parents. The nodes of $T^\alpha$ are the nodes of $X_\alpha$, and $x$ is a parent of $y$ in $T^\alpha$ if and only if $x$ is the lowest (proper) ancestor of $y$ that appears in $X_\alpha$. Unlike He et al., we do not store the tree $T^\alpha$. Instead, we store a weighted tree that contains only part of the information of $T^\alpha$. The weighted tree is the tree $T^\alpha_L$ obtained from the tree decomposition of Lemma 6. For the case $\sigma = w^{O(1)}$ the value of $L$ is $L = f(n)$, where $f$ is a function that satisfies $f(n) = \omega(1)$, and for large $\sigma$ set $L = \sqrt{\log \log \sigma}$ Denote by $n_\alpha$ the number of $\alpha$-nodes in $T$. We say that a character $\alpha$ is frequent if $n_\alpha \geq L$. We only construct the trees $T^\alpha_L$ for frequent characters.

For a node $x'$ in $T^\alpha_L$, let $V(x')$ be the set of all $\alpha$-nodes in the tree $x'$, excluding the root of $x'$. Let $C(x')$ be the leftmost $\alpha$-child of the root of $x'$ among the nodes of $V(x')$. We define the following weight functions for $T^\alpha_L$.

1. $w_1(x')$ is the number of nodes in $V(x')$.
2. $w_2(x')$ is the number of nodes in $V(x')$ whose preorder rank in $T^\alpha$ is less than or equal to the preorder rank the special node of $x'$.
3. $w_3(x')$ is equal to $w_1(x') - w_2(x')$.
4. $w_4(x')$ is the number of nodes in $V(x')$ which are ancestors of the special node of $x'$.
5. $w_5(x')$ is the number of $\alpha$-children of the root of $x'$.
6. $w_6(x')$ is the rank of $C(x')$ among the nodes of $V(x')$, when the nodes are sorted according to preorder (if $C(x')$ does not exist, $w_6(x') = 0$).
7. $w_7(x')$ is equal to 1 if the special node of $x'$ is an $\alpha$-node, and 0 otherwise.

Our representation of $T$ consists of the following data-structures. We store a rank-select structure on $P_T$, using one of the structures of Theorem 2 according to the size of the alphabet. We also store an unlabeled tree $T$ obtained from $T$ by removing the labels. $T$ is stored using Theorem 4. We also store the trees $T^\alpha_L$. In order to reduce the space, we do not store these trees individually. Instead, we merge them into a single tree $T'$. The tree $T'$ contains a new root node, on which the trees $T^\alpha_L$ are hanged, ordered by increasing values of $\alpha$. The tree $T'$ is stored using the representation of Lemma 3. In order to map a node of $T^\alpha_L$ to the corresponding node of $T'$, and vice versa, we store the rank-select structure of Theorem 3 on the string $N = 10^\alpha_1 10^\alpha_2 \cdots$, where $\alpha_1 < \alpha_2 < \cdots$ are the frequent characters. We also store the rank-select structure of Theorem 3 on a binary string $F$ of length $\sigma$ in which $F[\alpha] = 1$ if $\alpha$ is frequent. Since a mapping between nodes
of $T^α_L$ and nodes of $T'$ can be computed in constant time, in the following we shall assume that the trees $T^α_L$ are available.

We now analyze the space complexity of the representation. The space for $P_T$ is $nH_0(P_T) + o(n)$ bits for small alphabet, and $nH_0(P_T) + o(nH_0(P_T)) + o(n)$ bits for large alphabet. The space for $\hat{T}$ is $2n + o(n)$ bits. The tree $T'$ has $O(n/L)$ nodes, and the weight functions have ranges $\{0, \ldots, L\}$. Thus, the space for $T'$ is $O((n/L) \log L) + o(n) = o(n)$ bits. The strings $N$ and $F$ have at most $n$ zeros and at most $n/L$ ones. Thus, the space for the rank-select structures on these strings is $nH_0(P_T) + 2n + o(n)$ for small alphabet, and $nH_0(P_T) + 2n + o(nH_0(P_T)) + o(n)$ for large alphabet.

In the following, when we use an unlabeled tree operation (e.g., parent($x$)) we assume that the operation is performed on $\hat{T}$, and when we use a weighted tree operation we assume that the operation is performed on $T^α_L$.

### 3.1 Mapping from $T^α_L$ to $T$

Let $x'$ be a node of $T^α_L$. From property $[\ref{lem:tree-operation}]$ of Lemma $[\ref{lem:tree-operation}]$ there are two intervals $[l_1(x'), r_1(x')]$ and $[l_2(x'), r_2(x')]$ such that an $α$-node $x$ of $T$ is a non-root node of $x'$ if and only if the rank of $x$ among all the $α$-nodes of $T$, sorted in preorder, is in one of the intervals. These intervals can be computed as follows:

$$r_1(x') = \text{bp\_rank}_{2,3}(\text{bp\_open}(x'))$$

$$l_1(x') = r_1(x') - w_2(x') + 1$$

$$r_2(x') = \text{bp\_rank}_{2,3}(\text{bp\_close}(x'))$$

$$l_2(x') = r_2(x') - w_3(x') + 1$$

The set $V(x')$ can be computed with

$$V(x') = \{\text{preorder\_select}(\text{select}_α(P_T, s)) : s \in [l_1(x'), r_1(x')] \cup [l_2(x'), r_2(x')]\},$$

and $C(x')$ can be computed with preorder\_select(select$_α(P_T, k))$, where $k = l_1(x') + w_6(x') - 1$ if $w_6(x') \leq w_2(x')$, and $k = l_2(x') + w_6(x') - w_2(x') - 1$ otherwise.

### 3.2 Mapping from $T$ to $T^α_L$

Let $x$ be a node of $T$ and $α$ be a frequent character. If $x \in X_α$, we want to compute the node $x' \in T_L$ such that $x \in V(x')$. We denote this node by map($x$). Additionally, we compute whether $x$ is the special node of map($x$).

We consider two cases. The first case is when $x$ is an $α$-node. Then,

$$\text{map}(x) = \text{bp\_node}(\text{bp\_select}_{2,3}(\text{preorder\_rank}_α(x))).$$

Moreover, $x$ is the special node of map($x$) if and only if $w_7(x') = 1$ and preorder\_rank$_α(x) = r_1(x')$.

Now suppose that $x$ is not an $α$-node. The algorithm for computing map($x$) is as follows.

1. Let $u$ and $v$ be the $α$-descendants of $x$ with minimum and maximum preorder ranks, respectively ($u, v$ are computed using the structures on $P_T$ and $\hat{T}$). If $u, v$ do not exist return NULL.
2. Compute \( u' = \text{map}(u) \) and \( v' = \text{map}(v) \).

3. If \( x \neq \text{lca}(u,v) \)
   
   (a) If \( \text{parent}(u) \neq x \) return NULL.
   
   (b) Compute \( C(u') \). If \( C(u') = u \) then \( \text{map}(x) = \text{parent}(u') \) and \( x \) is the special node of \( \text{map}(x) \). Otherwise, \( \text{map}(x) = u' \) and \( x \) is not the special node.

4. If \( x = \text{lca}(u,v) \)
   
   (a) If \( u' \) is an ancestor of \( v' \) (including \( u' = v' \)), compute \( V(u') \). Scan the nodes of \( V(u') \) and check for each node whether it is a child of \( x \). If no child of \( x \) was found, return NULL. Compute \( C(u') \). If \( C(u') \) exists and \( \text{parent}(C(u')) = x \) then \( \text{map}(x) = \text{parent}(u') \) and \( x \) is the special node of \( \text{map}(x) \). Otherwise, \( \text{map}(x) = u' \) and \( x \) is not the special node.
   
   (b) If \( v' \) is a proper ancestor of \( u' \), handle this case analogously to the handling of the previous case.
   
   (c) If neither node \( u' \) or \( v' \) is an ancestor of the other, \( \text{map}(x) = \text{lca}(u',v') \) and \( x \) is the special node of \( \text{map}(x) \).

To see the correctness of the algorithm above, observe that if \( u,v \) do not exist then \( x \) does not have \( \alpha \)-children. Since \( x \) is not an \( \alpha \)-node, by definition \( x \notin T^\alpha \). Now suppose that \( u,v \) exist. If \( x \neq \text{lca}(u,v) \), the only possible \( \alpha \)-child of \( x \) is \( u \). Thus, if \( \text{parent}(u) \neq x \) then \( x \notin T^\alpha \). If \( \text{parent}(u) = x \) then \( x \) is a node of \( T^\alpha \). By the definition of \( \text{map}() \), \( u \) is not the root of \( u' \), and therefore the node \( x \) is also a node of \( u' \). Thus, \( \text{map}(x) = \text{parent}(u') \) if \( x \) is the root of \( u' \) and \( \text{map}(x) = u' \) otherwise. Since \( u \) is the only \( \alpha \)-child of \( x \), it follows that the former case occurs if and only if \( C(u') = u \).

Now consider the case \( x = \text{lca}(u,v) \). Denote by \( u_2 \) and \( v_2 \) the ancestors of \( u \) and \( v \) that are children of \( x \) in \( T^\alpha \). By the definition of \( u \) and \( v \), if \( x \) has \( \alpha \)-children then every \( \alpha \)-child \( y \) of \( x \) is between \( u_2 \) and \( v_2 \) in \( T^\alpha \). If \( u' \) is an ancestor of \( v' \) then \( u_2 \) and \( v_2 \) must belong to the tree \( u' \). Thus, all the \( \alpha \)-children of \( x \) (if there are any) belong to \( u' \). Therefore, it suffices to scan the nodes of \( V(u') \) in order to decide whether \( x \) has \( \alpha \)-children. If \( x \) has \( \alpha \)-children then these children are non-root nodes of \( u' \). It follows that \( x \) belongs to the tree \( u' \). Moreover, \( \text{map}(x) = \text{parent}(u') \) if \( x \) is the root of \( u' \), and \( \text{map}(x) = u' \) otherwise.

### 3.3 Answering queries

In this section describe how to implement the queries of Table [1]. The queries label, preorder.rank, preorder.select, num.descendants and postorder.rank are answered as in [8]. Answering the remaining queries is also similar to [8], but here additional steps are required as a weighted tree \( T_L^\sigma \) holds less information than \( T^\alpha \). The general idea is to use the tree \( T_L^\sigma \) to get an approximate answer to the query. Then, by enumerating the nodes of constant number of \( V() \) set, the exact answer is found. The time complexity of answering a query is thus \( O(L \cdot t_{\text{select}}) \), where \( t_{\text{select}} \) is the time for a select query on \( P_T \).

Since \( t_{\text{select}} = O(1) \) for small alphabet and \( t_{\text{select}} = o(\sqrt{\frac{\log \sigma}{\log w}}) \) for large alphabet, it follows that the time for answering a query is \( \omega(1) \) for small alphabet and \( O(\frac{\log \sigma}{\log w}) \) for large alphabet.

We assume that \( \alpha \) is a frequent character until the end of the section. Handling queries in which \( \alpha \) is non-frequent is done by enumerating all the \( \alpha \)-nodes in \( T \).
3.3.1 parent$_{\alpha}(x)$ query

We consider two cases. If $x$ is an $\alpha$-node the query is answered as follows.

1. Compute $x' = \text{map}(x)$.
2. Compute $V(x')$. Scan the nodes of $V(x')$ in reverse order, and check for each node $v$ whether $v$ is an ancestor of $x$. If an ancestor of $x$ is found, return it.
3. Compute $y' = \text{parent}_{\alpha}(x')$. If $y'$ does not exist return NULL.
4. Compute $V(y')$. Scan the nodes of $V(y')$ in reverse order, and check for each node $v$ whether $v$ is an ancestor of $x$. When an ancestor of $x$ is found, return it.

If $x$ is not an $\alpha$-node then the query is answered as in He et al. [8].

3.3.2 depth$_{\alpha}(x)$ query

1. Let $y = x$ if $x$ is an $\alpha$-node, and $y = \text{parent}_{\alpha}(x)$ otherwise.
2. Compute $y' = \text{map}(y)$.
3. Compute $V(y')$ and scan the nodes of $V(y')$. Count the number of nodes that are ancestors of $x$, and let $i$ denote this number.
4. Return $i + \text{depth}_{\alpha}(y') - w_{\alpha}(y')$.

3.3.3 level_ancestor$_{\alpha}(x, i)$ query

1. Compute $y = \text{parent}_{\alpha}(x)$. If $y$ does not exist return NULL.
2. If $i = 1$ return $y$.
3. Compute $y' = \text{map}(y)$.
4. Compute $V(y')$. Scan the nodes of $V(y')$ in reverse order. For each node $v$, if $v$ is an ancestor of $x$, decrease $i$ by 1. If $i$ becomes 0 return $v$, and otherwise continue with the scan.
5. Compute $z' = \text{level_ancestor}_{\alpha}(y', i)$. If $z'$ does not exist return NULL.
6. Set $i \leftarrow i - (\text{depth}_{\alpha}(\text{parent}(y')) - \text{depth}_{\alpha}(z'))$.
7. Compute $V(z')$. Scan the nodes of $V(z')$ in reverse order. For each node $v$, check whether $v$ is an ancestor of $x$. If $v$ is an ancestor, decrease $i$ by 1. If $i$ becomes 0 return $v$, and otherwise continue with the scan.
3.3.4 $\text{deg}_{\alpha}(x)$ query

1. Compute $x' = \text{map}(x)$. If $x'$ does not exist return 0.

2. If $x$ is not the special node of $x'$, compute $V(x')$ and scan the nodes of $V(x')$. Return the number of nodes that are children of $x$.

3. Return $\text{deg}_5(x')$.

We now explain the correctness of the algorithm above. If $x$ is not the special node of $x'$ then all the children of $x$ in $T_\alpha$ are in the tree $x'$. Thus, it suffices to scan $V(x')$. If $x$ is the special node, the set of $\alpha$-children of $x$ is precisely the set of $\alpha$-children of the roots of all trees $y'$ such that $y'$ is a child of $x'$ in $T_\alpha^L$. Therefore, $\text{deg}_\alpha(x) = \text{deg}_5(x')$.

3.3.5 $\text{child\_rank}_\alpha(x)$ query

If $x$ is an $\alpha$-node the query is answered as follows.

1. Compute $x' = \text{map}(x)$.

2. Compute $V(x')$ and scan the nodes of $V(x')$. Count the number of nodes that are left sibling of $x$, and let $i$ denote this number.

3. Compute $u = \text{parent}(x)$ and $u' = \text{map}(u)$.

4. If $u$ is not the special node of $u'$ return $i$.

5. Return $i + \text{child\_rank}_5(x') - w_5(x')$.

We now consider the case when $x$ is not an $\alpha$-node.

1. Compute $u = \text{parent}(x)$ and $u' = \text{map}(u)$. If $u'$ does not exist return 0.

2. If $u$ is not the special node of $u'$, compute $V(u')$ and scan the nodes of $V(u')$. Return the number of nodes that are left sibling of $x$.

3. Let $v$ be the $\alpha$-predecessor. If $v$ does not exist or if $\text{preorder\_rank}(v) \leq \text{preorder\_rank}(u)$ return 0.

4. Compute $v' = \text{map}(v)$.

5. Let $w'$ be the child of $u'$ which is an ancestor of $v'$.

6. Compute $V(w')$ and scan the nodes of $V(w')$. Count the number of nodes that are left sibling of $x$, and let $i$ denote this number.

7. Return $i + \text{child\_rank}_5(x')$. 

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3.3.6 \textit{child\_select}_\alpha(x, i) \text{ query}

1. Compute $x' = \text{map}(x)$. If $x$ does not exist return NULL.

2. If $x$ is not the special node of $x$, compute $V(x')$ and scan the nodes of $V(x')$. For each node $v$, check whether $v$ is a child of $x$. Return the $i$-th child.

3. Compute $y' = \text{child\_select}_5(x', i)$.

4. Set $i \leftarrow i - (\text{child\_rank}_5(y') - \text{w}_5(y'))$.

5. Compute $V(y')$ and scan the nodes of $V(y')$. For each node $v$, check whether $v$ is a child of $x$. Return the $i$-th child.

3.3.7 \textit{postorder\_select}_\alpha(i) \text{ query}

1. Return $\text{bp\_node}(\text{bp\_select}_{0,1}(i))$.

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