Analytical Volume Analysis for the Finite-time Controllable Region of the Linear Discrete-time Systems

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Work supported by the National Natural Science Foundation of China (Grant No. 61273005)

Abstract
In this paper, the works on the analytical volume analysis for the controllable regions of the linear discrete-time (LDT) systems in papers [17] and [16] are discussed further and a new theorem on the analytical computing for the finite-time controllability zonotope (controllable region) of LDT systems are proven. And then, three analytical factors describing the control capability of the systems are deconstructed successfully from the analytical volume expression of the controllable region. Finally, the theorem is generalized to three cases: the narrow controllable region, the matrix $A$ with $n$ negative eigenvalues, the linear continuous-time systems.

Keywords: volume computation, control capability, analytical computation, controllable region, reachable region, zonotope, linear discrete-time systems

1. Introduction
Since the state controllability, a concept describing whether the state variables and space of the dynamical systems are controlled or not by the input variables, is put forward by R. Kalman, et al, in 1960's [11], the further studies on describing quantitatively the control capability of the input variables to the state space attract the attentions of many researchers in control theory field. Based on the analysis of the controllability Gramian matrix
and the mobile analysis of the system eigenvalues, some pioneering works on quantizing control capability (defined for describing the control ability and efficiency in paper [18]) are made [14], [6], [13], and [9]. In recent years, more systematicness and deep-going works about that are got [15], [18], [16], and [19], such that,

1. putting forth the definitions, computing methods and deconstruction of the control capability of the inputs to states;
2. proving relations among the control capability of the open-loop controlled plants, control Strategy Space of the controller, and the performance of the closed-loop control systems;
3. constructing relations between the control capability and the waste time\energe in the control process of the linear dynamcial systems.

These studies reveal profoundly the dynamical systems and their properties for being controlled, and then it lays a good foundation for better analysis and design of control systems.

In papers [18] and [16], a kind of control capability of the input variables with the minimum control time attribute is defined and discussed. Based on the computation and optimization of the control capability, the bigger control strategy space, the better closed-loop performace and the stronger robustness of the controller can be gotten. In fact, the time-attribute control capability is the control ability of the inputs with bounded amplitudes and then its computation and analysis can be regarded as the volume computation and analysis of a special zonotope, namely a controllable region, generated by a matrix pair \( \{A, B\} \), where the matrices \( A \) and \( B \) are respectively the system matrix and input matrix in the state space models of the linear systems. In paper [17], the definitions and the volume computation of the controllable region are discussed in detail, and recurssive equations for the volume computation of the finite-time controllable region, and then the analytical equations for the infinite-time controllable region when the eigenvalues \( \lambda_i \) of the matrix \( A \) satisfy \( \{\lambda_i \in (0, 1), i = 1, n\} \) are got. More significantly, by deconstrucing these volume computing equations [16], some analytical factors describe the control capability are gotten and more extensive and in-depth analysis and optimization of the control capability can be carried out.

In this paper, the analytical volume computation for the finite-time controllable zonotope (FTCZ) of the linear discrete-time (LDT) systems will be discussed further based on the results in paper [17] [16], and then some
analytical factors describe the control capability will be deconstructed from
the analytical computing equation of the zonotope volume.

2. The Definition and the Some Results on the Volume Computation for the Special Zonotope

2.1. The definition and the volume computation of the zonotope

In papers [12], [7], [1], and [17], a \(n\)-dimensional (\(n\)-D) zonotope spaned by a set of the \(n\)-D vectors is defined as follows.

**Definition 1.** The zonotope spanned by the \(n\)-D vectors of matrix \(Z_m = [z_1, z_2, \ldots, z_m] \in R^{n \times m}\) and the parameter set with a finite interval is defined as

\[
C_q(Z_m) = \left\{ \sum_{i=1}^{m} c_i z_i \bigg| \forall c_i \in [0, 1], i = 1, m \right\}
\]

(1)

where \(q = \text{rank}(A_m), c_i (i = 1, m)\) are the parameters representing the zonotope, and vectors \(z_i(i = 1, m)\) are called as the generators of the zonotopes.

According to the knowledge of the geometry and matrix algebra, the volume computing of the \(n\)-D zonotope can be deduced as the following matrix computation.

**Theorem 1.** For any full row rank matrix \(Z_m \in R^{n \times m}\), the volume of the \(n\)-D zonotope \(C_n(Z_m)\) spanned by the vectors of \(Z_m\) can be computed as

\[
V_n(C_n(Z_m)) = \sum_{(i_1, i_2, \ldots, i_n) \in \Omega_{i,m}^n} |\text{det} [z_{i_1}, z_{i_2}, \ldots, z_{i_n}]|\]

(2)

where the \(n\)-tuple set \(\Omega_{m_1,m_2}^n\) consists of all possible \(n\)-tuples \((i_1, i_2, \ldots, i_n)\) whose elements are picked from the set \(\{m_1, m_1 + 1, \ldots, m_2\}\) and are sorted by their values. The computational complexity of the volume-computation method, i.e., the times computing the \(n \times n\) determinant values, is

\[
\frac{m!}{(m - n)!n!}
\]

(3)

times, noted as the polynomial time \(O(m^n)\) on the vector number \(m\).
2.2. The definition of the controllable region

In the dynamics analysis and control theory fields, the LDT systems can be modeled as follows

\[ x_{k+1} = Ax_k + Bu_k, \quad x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^r, \]  

(4)

where \( x_k \) and \( u_k \) are the state variable and input variable, respectively, and matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times r} \) are the state matrix and input matrix, respectively, in the system models \([10], [3]\).

For many practical engineering systems, the input variables \( u_k \) are bounded or with the input-saturation property, and then the bounded and saturated expression can be normalized as follows

\[ \|u_k\|_{\infty} \leq 1 \]

Based on the above normalized expression, the controllable region can be defined clearly. The so-called controllable region here is a broad appellation and, in reality, can be divided into two cases, narrow controllable region and reachable region. The \( N \)-steps narrow controllable region is the state region in which all states can be stabled to the origin in the state space by the \( N \)-stpes bounded inputs and it also can be called as the recover region. Correspondingly, the \( N \)-steps reachable region is the state region in which all states can be reached from the origin by the \( N \)-stpes bounded inputs. In fact, as two gemmetries, the narrow controllable region and the reachable region can be converted to each other through linear transformation \([16]\). Therefore, the reachability region as a broad controllable region is discuss in detail and the obtained resulte can be generlized to the narrow controllable region.

For the state controllabilty analysis, the \( N \)-steps controllable region of the LDT systems, i.e., the state space generated by the \( N \)-stpes bounded inputs \( U_N = [u_0^T, u_1^T, \ldots, u_{N-1}^T]^T \) can be described as follows

**Definition 2.** The \( N \)-steps controllable region generated by the matrix pair \( \{A, B\} \) in the LDT models and the parameter set with a finite interval are defined as

\[
R_N^d(A) = \left\{ \sum_{k=0}^{N-1} A^k Bu_k, \quad \|u_k\|_{\infty} \leq 1 \right\}
\]

\[
= \left\{ \sum_{i=1}^{rN} c_i p_i, \quad \forall c_i \in [0, 1], i = 1, rN \right\}
\]

(5)
where \( P_d^N = [B, AB, \ldots, A^{N-1}B] = [p_1, p_2, \ldots, p_{rN}] \), \( c_i (i = 1, rN) \) are the parameters representing the zonotope, and the matrix pair \( \{A, B\} \) is called the generator pair of the zonotopes.

According to the above definition, the controllable region of the LDT systems is a special zonotope generated by the vectors of the matrix \( P_d^N \).

As discussed above, similar to the definition of the broad controllable region \( R_d^N(A) \), we can define the \( N \)-steps narrow controllable region \( R_c^N(A) \) as follows

**Definition 3.** The \( N \)-steps narrow controllable region generated by the matrix pair \( \{A, B\} \) in the LDT models and the parameter set with a finite interval are defined as

\[
R_c^N(A) = \left\{ \sum_{k=0}^{N-1} A^{-N+k}Bu_k, \quad \|u_k\|_\infty \leq 1 \right\}
\]  

(6)

As pointed in paper [16], the \( N \)-steps narrow controllable region \( R_c^N(A) \) and the reachable region \( R_d^N(A) \) for the LDT systems \( \Sigma(A, B) \) satisfy the following relations

\[
R_c^N(A) = A^{-1}R_d^N(A^{-1})
\]  

(7)

\[
R_d^N(A) = A^{-1}R_c^N(A^{-1})
\]  

(8)

And then, their volumes satisfy the following relations

\[
V_n(R_c^N(A)) = |\det A|^{-1}V_n(R_d^N(A^{-1}))
\]  

(9)

\[
V_n(R_d^N(A)) = |\det A|^{-1}V_n(R_c^N(A^{-1}))
\]  

(10)

where \( V_n \) means the volume of the \( n \)-D regions. Therefore, the analysis and computing results of the regions \( R_c^N(A) \) and \( R_d^N(A) \) can be generalized conveniently to each others.

2.3. The recursive volume computation of the finite-time controllable region

By the linear transformation \( \bar{x} = Wx \) between two state vectors \( x \) and \( \bar{x} \), the system models \( \Sigma(A, B) \) can be transformed as

\[
\Sigma(A, B) = \Sigma(WAW^{-1}, WB)
\]  

5
and then the volumes of two controllable regions $R_N(A)$ and $R_N(\overline{A})$ satisfy the following equation \[17\]

\[ V_n(R_N(\overline{A})) = |\det W| V_n(R_N(A)) \tag{11} \]

When there exists some multiple eigenvalues or not in the system matrix $A$, the matrix $A$ can be transformed as Jordan or diagonal matrix by the linear transformation. And then, based on two kinds of special structure matrices $A$ and the transforming equation (11), some more effective methods for the volume computation of the controllable regions can be got.

When all eigenvalue of the matrix $A$ for the singel-input LDT systems are real numbers and different from each other, the diagonal tranformed systems can be represented as follows \[10\], \[3\]

\[ \Lambda = W_d A W_d^{-1} = \text{diag} \{ \lambda_1, \lambda_2, \cdots, \lambda_n \} \tag{12} \]

\[ \Gamma = W_d B = [\beta_1, \beta_2, \cdots, \beta_n]^T \tag{13} \]

where $\beta_i = q_i b$ and $q_i$ is the $i$-th row of the diagonal transforming matrix $W_d$ and can be choosed as the unit left eigenvector corresponding to the $i$-th eigenvalue $\lambda_i$.

It can be proven that the volumes of the zonotopes $R_N(A)$ and $R_N(\Lambda)$ can be computed recursively with complexity $O(N)$, and the corresponding result can be determined by the following theorem \[17\].

**Theorem 2.** If $\Lambda$ is a diagonal matrix that all diagonal elements are differential each other and are with same signs, and $\Gamma$ is only a vector, the volume of the zonotope $R_N(\Lambda)$ generated by matrix pair $\{\Lambda, \Gamma\}$ can be computed with computational complexity $O(N)$ by the following equation:

\[ V_n(R_N(\Lambda)) = 2^n \left| \prod_{i=1}^{n} \beta_i \right| \left| V_N^{\lambda_1 \lambda_2 \cdots \lambda_n} \right| \tag{14} \]
where

\[ V^*_{\lambda_1 \lambda_2 \cdots \lambda_n} = \sum_{(i_1, i_2, \cdots, i_n) \in \Omega_{0, N-1}^n} F_{\lambda_1 \lambda_2 \cdots \lambda_n}^{i_1 i_2 \cdots i_n} \]

\[ = V_{N-1}^{\lambda_1 \lambda_2 \cdots \lambda_n} + \sum_{j=1}^{n} (-1)^{n+j} \lambda_j^{N-1} V_{N-1}^{\lambda_1 \lambda_2 \cdots \lambda_n \setminus \lambda_j}, \quad N > n \quad (15) \]

\[ V_N^{\lambda_i} = \frac{1 - \lambda_i^N}{1 - \lambda_i}, \quad i = 1, 2, \ldots, n \quad (16) \]

\[ V_n^{\lambda_1 \lambda_2 \cdots \lambda_n} = F_{\lambda_1 \lambda_2 \cdots \lambda_n}^{0, 1, \cdots, n-1} = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \quad (17) \]

\[ F_{\lambda_1 \lambda_2 \cdots \lambda_n}^{i_1 i_2 \cdots i_n} = \det \begin{bmatrix} \lambda_1^{i_1} & \lambda_1^{i_2} & \cdots & \lambda_1^{i_n} \\ \lambda_2^{i_1} & \lambda_2^{i_2} & \cdots & \lambda_2^{i_n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n^{i_1} & \lambda_n^{i_2} & \cdots & \lambda_n^{i_n} \end{bmatrix} \quad (18) \]

where \( \lambda_1 \lambda_2 \cdots \lambda_n \setminus \lambda_j \) means that \( \lambda_j \) is deleted from sequence \( \lambda_1 \lambda_2 \cdots \lambda_n \).

2.4. The analytical volume computation of the infinite-time controllable region for the matrix \( A \) with \( n \) different eigenvalues

For the needs of many analysis problems on the control capability of practical dynamic systems (4), our focus will be on the infinite-time controllable region \( R_c^{\infty}(A) \) and reachable region \( R_d^{\infty}(A) \). When the time variable \( N \to \infty \), the computational cost of these region volumes by Theorem 1 and Theorem 2 will approach infinity and will exceed the accepted time cost in the analysis, design and on-line control for the practical engineering systems. We now propose a theorem on an analytic computation method with complexity \( O(1) \) that has nothing to do with the time variable \( N \) as follows [17].

**Theorem 3.** When the \( n \) eigenvalues \( \lambda_i (i = 1, n) \) of the matrix \( A \) are different each other and are with same signs, the volume of the infinite-time \( R^{\infty}(A) \) is as

\[ V_n (R^{\infty}(A)) = 2^n |\det W_d|^{-1} \prod_{i=1}^{n} \beta_i |\Phi_{\lambda_1 \lambda_2 \cdots \lambda_n}| \quad (19) \]

where

\[ \Phi_{\lambda_1 \lambda_2 \cdots \lambda_n} = \left( \prod_{1 \leq j_1 < j_2 \leq n} \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \right) \left( \prod_{i=1}^{n} \frac{1}{1 - |\lambda_i|} \right) \quad (20) \]
2.5. Decoding the Controllable Region

According to the volume computing equation (19), some factors described the shape and size of the controllable region \( R_d^N(A) \), that is, the control capability of the dynamical systems, are deconstructed as follows.

\[
F_1 = \left| \prod_{1 \leq j_1 < j_2 \leq n} \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \right| \tag{21}
\]

\[
F_{i,j}^{j,j} = \left| \frac{\lambda_j - \lambda_i}{1 - \lambda_j \lambda_i} \right| \tag{22}
\]

\[
F_{(j_1 \ldots j_s)}^{(k_1 \ldots k_q)} = \prod_{j_i \in \{j_1, \ldots, j_s\}} \prod_{k_v \in \{k_1 \ldots k_q\}} \frac{1 - \lambda_{j_i} \lambda_{k_v}}{\lambda_{k_v} - \lambda_{j_i}} \tag{23}
\]

\[
F_{2,i} = \frac{|q_i b_i|}{1 - |\lambda_i|}, \quad i = 1, 2, \ldots, n \tag{24}
\]

\[
F_{3,i} = |q_i b_i|, \quad i = 1, 2, \ldots, n \tag{25}
\]

The above analytical factors can be called respectively as the shape(pole distribution factor) factor, the shape factor in the 2-D section of the region, the side length of the circumscribed rhombohedral, and the modal controllability. In fact, the shape factor \( F_1 \) is also the eigenvalue evenness factor of the linear system, and can describe the control capability caused by the eigenvalue distribution. In addition, the modal controllability factor \( F_{3,i} \) have been put forth by papers [2] [8] [5] [4], and will not be discussed here.

3. Some Key Lemmas

First, for that matrix pair, the following lemma about the sign of a class of quasi-Vandermonde matrices is proposed and proven.

**Lemma 1.** For any \( n > 0 \), if \( 0 \leq k_1 < k_2 < \cdots < k_n \) and \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \), we have

\[
F_{\lambda_1, \lambda_2, \ldots, \lambda_n}^{k_1, k_2, \ldots, k_n} = \det \begin{bmatrix} \lambda_1^{k_1} & \lambda_1^{k_2} & \cdots & \lambda_1^{k_n} \\ \lambda_2^{k_1} & \lambda_2^{k_2} & \cdots & \lambda_2^{k_n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n^{k_1} & \lambda_n^{k_2} & \cdots & \lambda_n^{k_n} \end{bmatrix} > 0 \tag{26}
\]
Lemma 2. For any variables $\lambda_i \in C (i = 1, n)$, the following equation holds.

\[
(1 - \Upsilon_n) \Phi_{\lambda_1 \cdots \lambda_n} = \sum_{k=1}^{n} (-1)^{1+k} \Upsilon_{1,k} \Phi_{\lambda_1 \cdots \lambda_n \setminus \lambda_k}
\]  

(27)

where \( \lambda_1 \cdots \lambda_n \setminus \lambda_j \) means that \( \lambda_j \) is deleted from sequence \( \lambda_1 \cdots \lambda_n \).

\[
\Upsilon_s = \prod_{i=1}^{s} \lambda_i, \quad \Upsilon_s|_k = \prod_{i=1,s}^{i=k} \lambda_i
\]

(28)

\[
\Phi_{\lambda_1 \cdots \lambda_n} = \left( \prod_{1 \leq j < k \leq n} \frac{\lambda_k - \lambda_j}{1 - \lambda_j \lambda_k} \right) \left( \prod_{i=1}^{n} \frac{1}{1 - \lambda_i} \right)
\]

(29)

Lemma 3. For the any set $\Lambda = \{ \lambda_{k_1}, \lambda_{k_2}, \ldots, \lambda_{k_m} \}$ and the distribution factor $\Phi_\Lambda$, the following three equations hold.

1. \[
\Phi_\Lambda|_{\lambda_j = \lambda_i} = \begin{cases} 
0 & \lambda_i, \lambda_j \in \Lambda \\
(-1)^{s-h-1} \Phi_{\lambda_i \cup \Lambda \setminus \lambda_j} & k_h < i < k_{h+1} \leq j = k_s, \quad \lambda_j \notin \Lambda 
\end{cases}
\]

(30)

2. \[
(1 - \lambda_i) \Phi_\Lambda|_{\lambda_i = 1} = \begin{cases} 
(-1)^{m-h} \Phi_{\Lambda \setminus \lambda_i} & i = k_h \quad \lambda_j \notin \Lambda \\
0 & \text{others}
\end{cases}
\]

(31)

3. \[
(1 - \lambda_i \lambda_j) \Phi_\Lambda|_{\lambda_j = \lambda_i} = \begin{cases} 
(-1)^{s-h-1} \Phi_{\Lambda \setminus \lambda_i \setminus \lambda_j} & \lambda_j \notin \Lambda \\
0 & \text{others}
\end{cases}
\]

(32)

Lemmas 1 and 2 are got in paper [17], Lemma 3 can be proven as follows.

Proof of Lemma 3. (1) Eq. (30) can be proven by the definition equation (29) as follows.

1) By Eq. (29), if $\lambda_i$ and $\lambda_j$ are in the sequence $K$, we have

\[
\Phi_\Lambda|_{\lambda_j = \lambda_i} = 0
\]

(33)

2) By Eq. (29), if $k_h < i < k_{h+1} \leq j = k_s$, we have

\[
\Phi_\Lambda = \left( \prod_{1 \leq q < r \leq m} \frac{\lambda_k - \lambda_q}{1 - \lambda_q \lambda_k} \right) \left( \prod_{q=1}^{m} \frac{1}{1 - \lambda_q} \right)
\]

\[
= \frac{\Phi_{\Lambda \setminus \lambda_j}}{1 - \lambda_j} \left( \prod_{q=1}^{s-1} \frac{\lambda_j - \lambda_q}{1 - \lambda_q \lambda_j} \right) \left( \prod_{q=s+1}^{m} \frac{\lambda_k - \lambda_j}{1 - \lambda_k \lambda_j} \right)
\]

(34)
And then, we have

\[
\Phi_A|_{\lambda_j = \lambda_i} = \frac{\Phi_{\Lambda \setminus \{\lambda_i\}}}{1 - \lambda_i} \left( \prod_{q=1}^{s-1} \frac{\lambda_i - \lambda_{k_q}}{1 - \lambda_i \lambda_{k_q}} \right) \left( \prod_{q=s+1}^{m} \frac{\lambda_{k_q} - \lambda_i}{1 - \lambda_{k_q} \lambda_i} \right)
\]

\[
= (-1)^{s-h-1} \frac{\Phi_{\Lambda \setminus \{\lambda_i\}}}{1 - \lambda_i} \left( \prod_{q=1}^{h} \frac{\lambda_i - \lambda_{k_q}}{1 - \lambda_i \lambda_{k_q}} \right) \left( \prod_{q=h+1}^{m} \frac{\lambda_{k_q} - \lambda_i}{1 - \lambda_{k_q} \lambda_i} \right)
\]

\[
= (-1)^{s-h-1} \Phi_{\Lambda \setminus \{\lambda_i\}} \Phi_{\Lambda \setminus \{\lambda_j\}}
\] (35)

3) if \( \lambda_j \) isn’t in the sequence \( K \), we have

\[
\Phi_A|_{\lambda_j = \lambda_i} = \Phi_A
\] (36)

(2) By Eq. (29) and (34), if \( i = k_h \), we have

\[
(1 - \lambda_i) \Phi_A|_{\lambda_i = 1} = \frac{\Phi_{\Lambda \setminus \{\lambda_i\}}}{1 - \lambda_i} \left( \prod_{q=1}^{h} \frac{\lambda_i - \lambda_{k_q}}{1 - \lambda_i \lambda_{k_q}} \right) \left( \prod_{q=h+1}^{m} \frac{\lambda_{k_q} - \lambda_i}{1 - \lambda_{k_q} \lambda_i} \right)_{\lambda_i = 1}
\]

\[
= (-1)^{s-h} \Phi_{\Lambda \setminus \{\lambda_i\}}
\] (37)

If \( \lambda_i \notin \Lambda \), we have,

\[
(1 - \lambda_i) \Phi_A|_{\lambda_i = 1} = 0
\] (38)

(3) By Eq. (29), if \( i = k_h < j = k_s \), we have

\[
\Phi_A = \frac{(\lambda_j - \lambda_i) \Phi_{\Lambda \setminus \{\lambda_j\}}}{(1 - \lambda_i) (1 - \lambda_j)} \left( \prod_{q=1}^{h} \frac{\lambda_i - \lambda_{k_q}}{1 - \lambda_i \lambda_{k_q}} \right) \left( \prod_{q=h+1}^{m} \frac{\lambda_{k_q} - \lambda_i}{1 - \lambda_{k_q} \lambda_i} \right)
\]

\[
\times \left( \prod_{q=1}^{s-1} \frac{\lambda_j - \lambda_{k_q}}{1 - \lambda_{k_q} \lambda_j} \right) \left( \prod_{q=s+1}^{m} \frac{\lambda_{k_q} - \lambda_j}{1 - \lambda_{k_q} \lambda_j} \right)
\] (39)

Therefore, we have

\[
(1 - \lambda_i \lambda_j) \Phi_A|_{\lambda_j = \frac{1}{\lambda_i}} = \frac{(1 - \lambda_i^2) \Phi_{\Lambda \setminus \{\lambda_i\}}}{(1 - \lambda_i)^2} \left( \prod_{q=1}^{h} \frac{\lambda_i - \lambda_{k_q}}{1 - \lambda_i \lambda_{k_q}} \right) \left( \prod_{q=h+1}^{m} \frac{\lambda_{k_q} - \lambda_i}{1 - \lambda_{k_q} \lambda_i} \right)
\]

\[
\times \left( \prod_{q=1}^{s-1} \frac{1 - \lambda_i \lambda_{k_q}}{\lambda_i - \lambda_{k_q}} \right) \left( \prod_{q=s+1}^{m} \frac{\lambda_i \lambda_{k_q} - 1}{\lambda_i - \lambda_{k_q}} \right)
\]

\[
= (-1)^{s-h} \frac{1 + \lambda_i}{1 - \lambda_i} \Phi_{\Lambda \setminus \{\lambda_i\} \setminus \{\lambda_j\}}
\] (40)
In addition, if $\lambda_i \notin \Lambda$ or $\lambda_j \notin \Lambda$, we have

$$\left(1 - \lambda_i \lambda_j\right) \Phi_{\Lambda | \lambda_j = \frac{1}{\lambda_i}} = 0$$  \hspace{1cm} (41)

\[\square\]

4. The analytical volume computation of the finite-time controllable region

Based on the above lemma, we have the following analytical theorem of the volume computation about the finite-time controllable region $R_N(A)$.

**Theorem 4.** If the $n$ eigenvalues $\lambda_i (i = 1, n)$ of the matrix $A$ are positive real numbers and satisfy

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n,$$

the volume of the finite-time controllable region $R_N(A)$ when $N \geq n$ can be computed analytical as follows

$$V_n \left(R_N(A)\right) = 2^n \left| \det \left(W_d^{-1}\right) \prod_{i=1}^{n} \beta_i \right| V_N^{\lambda_1 \ldots \lambda_n}$$  \hspace{1cm} (42)

$$V_N^{\lambda_1 \ldots \lambda_n} = \sum_{(i_1, \ldots, i_n) \in \Omega_{0, N-1}^n} F^{i_1 \ldots i_n}_{\lambda_1 \ldots \lambda_n}$$

$$= \Phi_{\lambda_1 \ldots \lambda_n} + \sum_{s=1}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_n^s} c_{(j_1 \ldots j_s)}^{(n)} \Psi_{(j_1 \ldots j_s)}^N \Phi_{(j_1 \ldots j_s)} \Phi_{n \setminus (j_1 \ldots j_s)}$$  \hspace{1cm} (43)

$$= \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_n^s} c_{n\setminus(j_1 \ldots j_s)}^{(n)} \Psi_{n\setminus(j_1 \ldots j_s)}^N \Phi_{n\setminus(j_1 \ldots j_s)}$$  \hspace{1cm} (44)

$$= \left\{ \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_n^s} c_{(j_1 \ldots j_s)}^{(n)} \Psi_{(j_1 \ldots j_s)}^N F_{(j_1 \ldots j_s)} \right\} \Phi_{(1 \ldots n)}$$  \hspace{1cm} (45)
where $F_{(j_1\ldots j_s)}^n$ is defined as Eq. (23); the $s$-tuple set $\Theta_s^n$ consists of all possible $s$-tuples $(j_1,\ldots, j_s)$ whose elements are picked from the set \{1, 2, \ldots, n\} and sorted by their values:

\[ c_{(j_1\ldots j_s)}^{(n)} = (-1)^{(n+1)s-\sum_{i=1}^s j_i} \tag{46} \]

\[ \Upsilon_{(j_1\ldots j_s)}^N = \prod_{k=1}^s \lambda_{jk}^N \tag{47} \]

\[ \Phi_{(j_1\ldots j_s)} = \left( \prod_{1\leq i<k\leq s} \frac{\lambda_{jk} - \lambda_{ji}}{1 - \lambda_{ji}\lambda_{jk}} \right) \left( \prod_{i=1}^s \frac{1}{1 - \lambda_{ji}} \right) \tag{48} \]

where $c_{(n)} = 1$, $\Upsilon_\infty = 1$, and $\Phi_\infty = 1$; "-" means that the number sequence is blank.

By the above theorem, if all eigenvalues $\lambda_i$ are in $[0, 1)$, the function $V_{(j_1\ldots j_s)}^{\lambda_1\ldots \lambda_n}$ can be represented as follows

\[ V_{(j_1\ldots j_s)}^{\lambda_1\ldots \lambda_n} = \Phi_{\lambda_1\ldots \lambda_n} + \sum_{i=0}^n O(\lambda_i^N) \tag{49} \]

where $\lim_{N \to \infty} O(\lambda_i^N) = 0$, that is, $V_{\infty}^{\lambda_1\ldots \lambda_n} = \Phi_{\lambda_1\ldots \lambda_n}$. Therefore, the analytical volume equation of the infinite-time controllable region $R^d_{(\infty)}(A)$ in papers [17] and [16] is a special example of the above theorem.

**Proof of Theorem 4** The theorem can be proven by the inductive method. Firstly, Eq. (43) when $N = n$ is proven by the first-time inductive method. And then, based on the proven result when $N = n$, Eq. (43) when $N > n$ is proven by the second-time inductive method.

(1) Next, Eq. (43) when $N = n$ is proven at first.
1.1) When \( n = 2 \), by the definition of \( V_{N}^{\lambda_{1} \lambda_{2}} \) and \( F_{\lambda_{1} \lambda_{2}}^{i_{1} i_{2}} \), we have

\[
V_{N}^{\lambda_{1} \lambda_{2}} = \sum_{(i,j) \in \Omega_{N}^{2,n-1}} F_{\lambda_{1} \lambda_{2}}^{i_{1} i_{2}} = \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} \left| \begin{array}{cc}
\lambda_{1}^{i} & \lambda_{1}^{j} \\
\lambda_{2}^{i} & \lambda_{2}^{j}
\end{array} \right|
\]

\[
= \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} [\lambda_{1}^{i} \lambda_{2}^{j} - \lambda_{1}^{j} \lambda_{2}^{i}]
\]

\[
= \sum_{i=0}^{N-2} \left[ \frac{\lambda_{1}^{i} \lambda_{2}^{i+1} - \lambda_{2}^{N} \lambda_{1}^{i+1} - \lambda_{1}^{N} \lambda_{2}^{i}}{1 - \lambda_{1} - \lambda_{2}} \right]
\]

\[
= \sum_{i=0}^{N-2} \frac{\lambda_{1}^{i} (\lambda_{2}^{i+1} - \lambda_{2}^{N}) (1 - \lambda_{1}) - \lambda_{1}^{i+1} (\lambda_{1}^{i+1} - \lambda_{1}^{N}) (1 - \lambda_{2})}{(1 - \lambda_{1}) (1 - \lambda_{2})}
\]

\[
= \sum_{i=0}^{N-2} \frac{[\lambda_{2} (1 - \lambda_{1}) - \lambda_{1} (1 - \lambda_{2})] \lambda_{1}^{i} \lambda_{2}^{j} - \lambda_{1}^{i} \lambda_{2}^{N} (1 - \lambda_{1}) + \lambda_{2}^{N} \lambda_{1}^{i} (1 - \lambda_{2})}{(1 - \lambda_{1}) (1 - \lambda_{2})}
\]

\[
= (\lambda_{2} - \lambda_{1}) \frac{(1 - \lambda_{1}^{N-1}) \lambda_{2}^{N} + (1 - \lambda_{2}^{N-1}) \lambda_{1}^{N}}{(1 - \lambda_{1}) (1 - \lambda_{2})}
\]

\[
(50)
\]

By Eqs. (46), (47) and (48), when \( n = 2 \), we have

\[
\sum_{s=0}^{2} \sum_{(j_{1},...,j_{s}) \in \Theta_{2}^{s}} c_{(j_{1}...j_{s})}^{(2)} \gamma_{(j_{1}...j_{s})}^{N} \Phi_{(j_{1}...j_{s})} \Phi_{2}(j_{1}...j_{s})
\]

\[
= c_{(2)}^{(2)} \Phi_{-} \Phi_{1} \lambda_{1} \lambda_{2} + c_{1}^{(2)} \gamma_{1}^{N} \Phi_{1} \Phi_{2} + c_{2}^{(2)} \gamma_{2}^{N} \Phi_{2} \Phi_{1} + c_{12}^{(2)} \gamma_{12}^{N} \Phi_{12} \Phi_{-}
\]

\[
= \frac{(\lambda_{2} - \lambda_{1}) (1 - \lambda_{1}^{N} \lambda_{2}^{N})}{(1 - \lambda_{1}) (1 - \lambda_{2}) (1 - \lambda_{1} \lambda_{2})} - \frac{\lambda_{2}^{N} - \lambda_{1}^{N}}{(1 - \lambda_{1}) (1 - \lambda_{2})}
\]

\[
(51)
\]

Therefore, by Eqs. (50) and (51), we can see, Eq. (43) holds when \( n = 2 \).

It is worth noting that, when \( N = n = 2 \), we have

\[
V_{2}^{\lambda_{1} \lambda_{2}} = \frac{(\lambda_{2} - \lambda_{1}) (1 - \lambda_{1}^{2} \lambda_{2}^{2})}{(1 - \lambda_{1}) (1 - \lambda_{2}) (1 - \lambda_{1} \lambda_{2})} - \frac{\lambda_{2}^{2} - \lambda_{1}^{2}}{(1 - \lambda_{1}) (1 - \lambda_{2})}
\]

\[
= \lambda_{2} - \lambda_{1}
\]

\[
(52)
\]
Considered that
\[ V_n^{\lambda_1 \cdots \lambda_n} = F_{\lambda_1 \cdots \lambda_n}^{01 \cdots n} = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \tag{53} \]

to prove that Eq. \[(43)\] holds when \(N = n\), it is needs to prove that for the rational function
\[ G_{\lambda_1 \cdots \lambda_n} = \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_n^s} c_{(j_1, \ldots, j_s)}^{(n)} \gamma_{(j_1, \ldots, j_s)}^{n} \Phi_{(j_1, \ldots, j_s)} \Phi_{n \setminus (j_1, \ldots, j_s)} \tag{54} \]
satisfies the following equation
\[ G_{\lambda_1 \cdots \lambda_n} = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \tag{55} \]

By the definition of the functions \(\Phi\) and Eq. \[(54)\], we know, the factors \(1 - \lambda_i\) and \(1 - \lambda_j \lambda_k\) for \(1 \leq i \leq n\) and \(1 \leq j < k \leq n\) are the all factors in the denominator polynomial of the rational function \(G_{\lambda_1 \cdots \lambda_n}\). Therefore, Eq. \[(55)\] is equal to the following equation
\[ G_{\lambda_1 \cdots \lambda_n} \prod_{i=0}^{n} (1 - \lambda_i) \times \prod_{1 \leq i < j \leq n} (1 - \lambda_i \lambda_j) = \prod_{i=0}^{n} (1 - \lambda_i) \times \prod_{1 \leq i < j \leq n} [(\lambda_j - \lambda_i) (1 - \lambda_i \lambda_j)] \tag{56} \]

Hence, to prove that Eq. \[(55)\] holds, it is needs only to prove the following equations hold for all possible \(i\) and \(j\).
\[ G_{\lambda_1 \cdots \lambda_n} |_{\lambda_j = \lambda_i} = 0 \tag{57} \]
\[ (1 - \lambda_i) G_{\lambda_1 \cdots \lambda_n} |_{\lambda_i = 1} = 0 \tag{58} \]
\[ (1 - \lambda_i \lambda_j) G_{\lambda_1 \cdots \lambda_n} |_{\lambda_j = \frac{1}{\lambda_i}} = 0 \tag{59} \]
In fact, the function $G_{\lambda_1 \ldots \lambda_n}$ can be represented as follows

$$G_{\lambda_1 \ldots \lambda_n} = \sum_{s=0}^{n-2} \sum_{(j_1, \ldots, j_s) \in \Theta^s \setminus \{j\}} \left[ c_{(i,j_1 \ldots j_s)}^{(n)} \Upsilon^n_{(i,j_1 \ldots j_s)} \Phi_{(i,j_1 \ldots j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} \right. $$

$$+ c_{(i,j_1 \ldots j_s)}^{(n)} \Upsilon^n_{(i,j_1 \ldots j_s)} \Phi_{(i,j_1 \ldots j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} $$

$$+ c_{(i,j_1 \ldots j_s)}^{(n)} \Upsilon^n_{(i,j_1 \ldots j_s)} \Phi_{(i,j_1 \ldots j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} $$

$$+ c_{(j_1, \ldots, j_s)}^{(n)} \Upsilon^n_{(j_1, \ldots, j_s)} \Phi_{(j_1, \ldots, j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} \right]$$

or

$$= \sum_{s=0}^{n-1} \sum_{(j_1, \ldots, j_s) \in \Theta^s \setminus \{i\}} \left[ c_{(i,j_1 \ldots j_s)}^{(n)} \Upsilon^n_{(i,j_1 \ldots j_s)} \Phi_{(i,j_1 \ldots j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} \right.$$

$$+ c_{(j_1, \ldots, j_s)}^{(n)} \Upsilon^n_{(j_1, \ldots, j_s)} \Phi_{(j_1, \ldots, j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} \right]$$

By Lemma 3 and Eq. (60), we have

$$G_{\lambda_1 \ldots \lambda_n} \vert_{\lambda_j = \lambda_i} = \sum_{s=0}^{n-2} \sum_{(j_1, \ldots, j_s) \in \Theta^s \setminus \{j\}} \left[ c_{(i,j_1 \ldots j_s)}^{(n)} \Upsilon^n_{(i,j_1 \ldots j_s)} \Phi_{(i,j_1 \ldots j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} \right.$$

$$+ c_{(j_1, \ldots, j_s)}^{(n)} \Upsilon^n_{(j_1, \ldots, j_s)} \Phi_{(j_1, \ldots, j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} \right] \lambda_j = \lambda_i$$

$$= \sum_{s=0}^{n-2} \sum_{(j_1, \ldots, j_s) \in \Theta^s \setminus \{j\}} \Upsilon^n_{(i,j_1 \ldots j_s)} \left[ (-1)^{(n+1)s-i} \sum_{k=1}^{s} j_k \times (-1)^{h-1} \right.$$

$$+ (-1)^{(n+1)s-j} \sum_{k=1}^{s} j_k \times (-1)^{j-i-h} \left. \Phi_{(i,j_1 \ldots j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} \right]$$

$$= \sum_{s=0}^{n-2} \sum_{(i,j_1 \ldots j_s) \in \Theta^s \setminus \{j\}} \Upsilon^n_{(i,j_1 \ldots j_s)} \left[ (-1)^{(n+1)s-i} \sum_{k=1}^{s} j_k \times (-1)^{i-h} \right.$$

$$+ (-1)^{-i-h} \left. \Phi_{(i,j_1 \ldots j_s)} \Phi_{n \setminus \{j_1 \ldots j_s\}} \right]$$

$$= 0$$

(62)

where $h$ is the position difference between the numbers $i$ and $j$ when $i$ and $j$ insert the ordered sequence $j_1 j_2 \ldots j_s$. Therefore, $j - i - h + 1$ is also the position difference between $i$ and $j$ in the ordered sequence $(12 \ldots n)$ \.
By Lemma 3 and Eq. (61), we have

\[
(1 - \lambda_i) G_{\lambda_1 \cdots \lambda_n} \bigg|_{\lambda_i=1} = \sum_{s=0}^{n-1} \sum_{(j_1, \ldots, j_s) \in \Theta_n^{* \setminus i} \setminus i} \left[ c_{(i,j_1j_s)}^{(n)} \gamma_{(i,j_1 \ldots, j_s)}^{n} (1 - \lambda_i) \Phi_{(i,j_1 \ldots, j_s)} \Phi_{n \setminus i \setminus \delta (j_1 \ldots j_s)} \right]_{\lambda_i=1} + c_{(j_1 \ldots j_s)}^{(n)} \gamma_{(j_1 \ldots j_s)}^{n} (1 - \lambda_i) \Phi_{(j_1 \ldots j_s)} \Phi_{n \setminus j_1 \ldots j_s} \right)
\]

\[
= \sum_{s=0}^{n-1} \sum_{(j_1, \ldots, j_s) \in \Theta_n^{* \setminus i} \setminus j} \gamma_{(j_1 \ldots j_s)}^{n} \left[ (-1)^{(n+1)(s+1) - i - \sum_{k=1}^{s} j_k} \times (-1)^{s+\delta - 1} \right] \Phi_{(j_1 \ldots j_s)} \Phi_{n \setminus j_1 \ldots j_s} \right)
\]

where \( \delta \) is the position of the number \( i \) when \( i \) inserts the ordered sequence \( j_1j_2 \ldots j_s \). Therefore, \( i - \delta + 1 \) is also the position of \( i \) in the ordered sequence \((12 \ldots n) \setminus j_1j_2 \ldots j_s \). By the above deduction, we can see, for any \( i \), \( 1 - \lambda_i \) must be the factors in the denominator polynomial of rational function \( G_{\lambda_1 \cdots \lambda_n} \).

By Lemma 3 and Eq. (60), when \( j > i \), we have,

\[
(1 - \lambda_i \lambda_j) G_{\lambda_1 \cdots \lambda_n} \bigg|_{\lambda_j=1} = \sum_{s=0}^{n-2} \sum_{(j_1, \ldots, j_s) \in \Theta_n^{* \setminus j} \setminus j} \left[ c_{(i,j_1j_s)}^{(n)} \gamma_{(i,j_1 \ldots, j_s)}^{n} (1 - \lambda_i \lambda_j) \Phi_{(i,j_1 \ldots, j_s)} \Phi_{n \setminus i \setminus j \setminus (j_1 \ldots j_s)} \right]
\]

\[
+ c_{(j_1 \ldots j_s)}^{(n)} \gamma_{(j_1 \ldots j_s)}^{n} (1 - \lambda_i \lambda_j) \Phi_{(j_1 \ldots j_s)} \Phi_{n \setminus j_1 \ldots j_s} \right)
\]

\[
= \sum_{s=0}^{n-2} \sum_{(j_1, \ldots, j_s) \in \Theta_n^{* \setminus j} \setminus j} \gamma_{(j_1 \ldots j_s)}^{n} \left[ \frac{1 - \lambda_i}{1 + \lambda_i} \left[ (-1)^{(n+1)(s+2) - i - j - \sum_{k=1}^{s} j_k} \times (-1)^h \right] \Phi_{(j_1 \ldots j_s)} \Phi_{n \setminus i \setminus j \setminus (j_1 \ldots j_s)} \right]
\]

\[
= 0
\]

where \( h \) is as in Eq. (62). By the above deduction, we can see, for any \( j > i \), \( 1 - \lambda_i \lambda_j \) must be the factors in the denominator polynomial of rational function \( G_{\lambda_1 \cdots \lambda_n} \).
Hence, by Eqs. (62), (63) and (64), Eq. (55) is true, and then Eq. (43) holds when $N = n$.

(3) It is assumed that when $N = k$, Eq. (43) holds for any $n$, that is, we have

$$V_{k}^{\lambda_1 \ldots \lambda_n} = \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_n^s} c_{(j_1, \ldots, j_s)}^{(n)} \Upsilon_k^{(j_1, \ldots, j_s)} \Phi_{(j_1, \ldots, j_s)} \Phi_n \setminus (j_1, \ldots, j_s)$$

(4) Next, according to the inductive method, based on the above step (1), (2) and (3), It is needs to prove when $N = k + 1$, Eq. (43) holds for any $n$, that is, the following equation will be needs to be proven true.

$$V_{k+1}^{\lambda_1 \ldots \lambda_n} = \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_n^s} c_{(j_1, \ldots, j_s)}^{(n)} \Upsilon_{k+1}^{(j_1, \ldots, j_s)} \Phi_{(j_1, \ldots, j_s)} \Phi_n \setminus (j_1, \ldots, j_s)$$
By the definition of \( V_N^{\lambda_1 \ldots \lambda_n} \) and \( F_{\lambda_1 \ldots \lambda_n}^{i_1 \ldots i_n} \), we have
\[
V_{k+1}^{\lambda_1 \ldots \lambda_n} = \sum_{(i_2, \ldots, i_n) \in \Omega_k^{n-1}} F_{\lambda_1 \ldots \lambda_n}^{0i_2 \ldots i_n} + \sum_{(i_1, \ldots, i_n) \in \Omega_k^n} F_{\lambda_1 \ldots \lambda_n}^{i_1 \ldots i_n}
\]
\[
= \sum_{(i_2, \ldots, i_n) \in \Omega_k^{n-1}} \sum_{q=1}^n (-1)^{1+k} \gamma_{n}^{i_2 \ldots i_n} F_{\lambda_1 \ldots \lambda_n \lambda_k}^{i_2 \ldots i_n} + \gamma_n \sum_{(i_1, \ldots, i_n) \in \Omega_k^n} F_{\lambda_1 \ldots \lambda_n}^{i_1 \ldots i_n}
\]
\[
= \sum_{q=1}^n (-1)^{1+q} \gamma_{n}^{q} V_k^{\lambda_1 \ldots \lambda_n \lambda_q} + \gamma_n V_k^{\lambda_1 \ldots \lambda_n}
\]
\[
= \sum_{q=1}^n (-1)^{1+q} \gamma_{n}^{q} \sum_{s=0}^{n-1} \sum_{(j_1, \ldots, j_s) \in \Theta_{n}^{s}} c_{(\hat{j}_1 \ldots \hat{j}_s)}^{n} \gamma_{(j_1 \ldots j_s)}^{k} \Phi_{(j_1 \ldots j_s)}^{(j_1 \ldots j_s)} \Phi_{n}^{q \setminus (j_1 \ldots j_s)}
\]
\[
\quad + \gamma_n \sum_{s=0}^{n-1} \sum_{(j_1, \ldots, j_s) \in \Theta_{n}^{s}} c_{(\hat{j}_1 \ldots \hat{j}_s)}^{n} \gamma_{(j_1 \ldots j_s)}^{k} \Phi_{(j_1 \ldots j_s)}^{(j_1 \ldots j_s)} \Phi_{n}^{q \setminus (j_1 \ldots j_s)}
\]
\[
= \sum_{q=1}^n (-1)^{1+q} \gamma_{n}^{q} \sum_{s=0}^{n-1} \sum_{(j_1, \ldots, j_s) \in \Theta_{n}^{s}} c_{(\hat{j}_1 \ldots \hat{j}_s)}^{n} \gamma_{n}^{q \setminus (j_1 \ldots j_s)} \gamma_{(j_1 \ldots j_s)}^{k+1} \Phi_{(j_1 \ldots j_s)}^{(j_1 \ldots j_s)} \Phi_{n}^{q \setminus (j_1 \ldots j_s)}
\]
\[
\quad + \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_{n}^{s}} c_{(\hat{j}_1 \ldots \hat{j}_s)}^{n} \gamma_{n}^{q \setminus (j_1 \ldots j_s)} \gamma_{(j_1 \ldots j_s)}^{k+1} \Phi_{(j_1 \ldots j_s)}^{(j_1 \ldots j_s)} \Phi_{n}^{q \setminus (j_1 \ldots j_s)}
\]
(67)

where \( \hat{j}_1 \ldots \hat{j}_s \) is a new ordinal number sequence because that the ordinal number sequence \( (j_1, \ldots, j_s) \) is produced from the space \( \Theta_{n}^{s} \), and then, we have
\[
\hat{j}_k = \begin{cases} 
  j_k & j_k < q \\
  j_k - 1 & j_k > q
\end{cases}
\]
(68)

Comparing between Eqs. (66) and (67), for proving Eq. (66), the following
equation should be proven at first.

\[
\sum_{q=1}^{n} (-1)^{1+q} \sum_{s=0}^{n-1} \sum_{(j_1, \ldots, j_s) \in \Theta_{n-q}^s} c_{(j_1, \ldots, j_s)}^{(n-1)} \left( \gamma_{n\setminus q}(j_1, \ldots, j_s) \Phi_{(j_1, \ldots, j_s)} \Phi_{n\setminus q}(j_1, \ldots, j_s) \right) \\
+ \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_{n}^s} c_{(j_1, \ldots, j_s)}^{(n)} \left( \gamma_{n\setminus (j_1, \ldots, j_s)} - 1 \right) \gamma_{k+1}(j_1, \ldots, j_s) \Phi_{(j_1, \ldots, j_s)} \Phi_{n\setminus (j_1, \ldots, j_s)} = 0
\]

(69)

In fact, the left side of the above equation can be classified and rearranged according to the factor \( \gamma_{k+1}(j_1, \ldots, j_s) \Phi_{(j_1, \ldots, j_s)} \) and then, if the sum of the factors accompanying with the factor \( \gamma_{k+1}(j_1, \ldots, j_s) \Phi_{(j_1, \ldots, j_s)} \) can be proven as zero, Eq. (69) can be proved successfully. The proving process can be exemplified as follows.

1) For the factor \( \gamma_{k+1} \Phi_1 = \lambda_{k+1} \Phi_1 \), by Lemma 2 and Eq. (46), the sum of the accompanying factors in the left side of Eq. (69) is as follows.

\[
\sum_{q=2}^{n} (-1)^{1+q} c_1^{(n-1)} \gamma_{n\setminus q\setminus 1} \Phi_{n\setminus q\setminus 1} + c_1^{(n)} \left( \gamma_{n\setminus 1} - 1 \right) \Phi_{n\setminus 1} \\
= (-1)^{n-1} \sum_{q=2}^{n} (-1)^{1+q} \gamma_{n\setminus q\setminus 1} \Phi_{n\setminus q\setminus 1} - (-1)^{n} \sum_{q=1}^{n-1} (-1)^{1+q} \gamma_{n\setminus q+1\setminus 1} \Phi_{n\setminus q+1\setminus 1} \\
= \sum_{q=2}^{n} (-1)^{n+q} \gamma_{n\setminus q\setminus 1} \Phi_{n\setminus q\setminus 1} - \sum_{q=2}^{n} (-1)^{n+q} \gamma_{n\setminus q\setminus 1} \Phi_{n\setminus q\setminus 1} \\
= 0
\]

(70)

2) For the factor \( \gamma_{k+1}(j_1, \ldots, j_s) \Phi_{(j_1, \ldots, j_s)} \), by Lemma 2 and Eq. (46), the sum
of the accompanying factors in the left side of Eq. (69) is as follows.

\[ n \backslash (j_1 \ldots j_s) \]
\[ \sum_{q=1}^{n \backslash (j_1 \ldots j_s)} (-1)^{1+q} c_{n \backslash (j_1 \ldots j_s)}^{(n-1)} \gamma_{n \backslash q \backslash (j_1 \ldots j_s)} \Phi_{n \backslash q \backslash (j_1 \ldots j_s)} \]
\[ + c_{n \backslash (j_1 \ldots j_s)}^{(n)} \left( \gamma_{n \backslash (j_1 \ldots j_s)} - 1 \right) \Phi_{n \backslash (j_1 \ldots j_s)} \]
\[ = \sum_{q=1}^{n \backslash (j_1 \ldots j_s)} (-1)^{1+q} c_{n \backslash (j_1 \ldots j_s)}^{(n-1)} \gamma_{n \backslash q \backslash (j_1 \ldots j_s)} \Phi_{n \backslash q \backslash (j_1 \ldots j_s)} \]
\[ - c_{n \backslash (j_1 \ldots j_s)}^{(n)} \sum_{q=1}^{n \backslash (j_1 \ldots j_s)} (-1)^{1+q} \gamma_{n \backslash q \backslash (j_1 \ldots j_s)} \Phi_{n \backslash q \backslash (j_1 \ldots j_s)} \]
\[ = \sum_{q=1}^{n \backslash (j_1 \ldots j_s)} \left[ (-1)^{1+q} c_{n \backslash (j_1 \ldots j_s)}^{(n-1)} - c_{n \backslash (j_1 \ldots j_s)}^{(n)} (-1)^{1+q} \right] \gamma_{n \backslash q \backslash (j_1 \ldots j_s)} \Phi_{n \backslash q \backslash (j_1 \ldots j_s)} \]

(71)

where \( \hat{q} \) is the position number of the number 'q' in the ordinal number sequence \((12 \ldots n) \backslash (j_1 \ldots j_s)\), that is, we have

\[ \hat{q} = q - k, \quad j_k < q < j_{k+1} \]

(72)

In addition, by Eq. (68), we have

\[ \sum_{k=1}^{s} j_k = \sum_{k=1}^{s} j_k - (s - k), \quad j_k < q < j_{k+1} \]

(73)

Therefore, by Eqs. (72) and (73), if \( j_k < q < j_{k+1} \), we have

\[ (-1)^{1+q} c_{n \backslash (j_1 \ldots j_s)}^{(n-1)} - c_{n \backslash (j_1 \ldots j_s)}^{(n)} (-1)^{1+\hat{q}} \]
\[ = (-1)^{n \hat{q}} - \sum_{k=1}^{s} j_k + 1 + q - (-1)^{n+1} s - \sum_{k=1}^{s} j_k + 1 + \hat{q} \]
\[ = (-1)^{n \hat{q}} - \sum_{k=1}^{s} j_k - (s - k) + 1 + q - (-1)^{n+1} s - \sum_{k=1}^{s} j_k + 1 + q - k \]
\[ = 0 \]

(74)

In summary, by Eqs. (71) and (73), Eqs. (69) and (66) are proved successively as true.

Synthesized the step (1) to (4) in the inductive process, the theorem has been proven. \( \square \)
Based on the above theorem, for \( n = 2, 3 \), we have

\[
V_N^{\lambda_1, \lambda_2} = \Phi_{\lambda_1 \lambda_2} + (\lambda_1^N - \lambda_2^N) \Phi_{\lambda_1} \Phi_{\lambda_2} - \lambda_1^N \lambda_2^N \Phi_{\lambda_1 \lambda_2} \\
= \frac{(\lambda_2 - \lambda_1) (1 - \lambda_1^N \lambda_2^N)}{(1 - \lambda_1) (1 - \lambda_2) (1 - \lambda_1 \lambda_2)} + \frac{\lambda_1^N - \lambda_2^N}{(1 - \lambda_1) (1 - \lambda_2)} \tag{75}
\]

\[
V_N^{\lambda_1, \lambda_2, \lambda_3} = \Phi_{\lambda_1 \lambda_2 \lambda_3} - \lambda_1^N \Phi_{\lambda_1} \Phi_{\lambda_2 \lambda_3} + \lambda_2^N \Phi_{\lambda_2} \Phi_{\lambda_1 \lambda_3} - \lambda_3^N \Phi_{\lambda_3} \Phi_{\lambda_1 \lambda_2} \\
- \lambda_1^N \lambda_2^N \Phi_{\lambda_1 \lambda_2} \Phi_{\lambda_3} + \lambda_1^N \lambda_3^N \Phi_{\lambda_1 \lambda_3} \Phi_{\lambda_2} - \lambda_2^N \lambda_3^N \Phi_{\lambda_2 \lambda_3} \Phi_{\lambda_1} \\
+ \lambda_1^N \lambda_2^N \lambda_3^N \Phi_{\lambda_1 \lambda_2 \lambda_3} \\
= (1 + \lambda_1^N \lambda_2^N \lambda_3^N) \Phi_{\lambda_1 \lambda_2 \lambda_3} - (\lambda_1^N + \lambda_2^N \lambda_3^N) \Phi_{\lambda_1} \Phi_{\lambda_2 \lambda_3} \\
+ (\lambda_2^N + \lambda_1^N \lambda_3^N) \Phi_{\lambda_2} \Phi_{\lambda_1 \lambda_3} - (\lambda_3^N + \lambda_1^N \lambda_2^N) \Phi_{\lambda_3} \Phi_{\lambda_1 \lambda_2} \tag{76}
\]

5. The analytical factors describing the finite-time control capability

By Theorem 4 the factor with the biggest absolute value in the sum expression (42) and (43) is as follows

\[
\left| \det (W^{-1}) \prod_{i=1}^{n} q_i b \right| \times \left| \gamma_{(j_1 \ldots j_s)}^N \Phi_{(j_1 \ldots j_s)} \Phi_{n \setminus (j_1 \ldots j_s)} \right| \tag{77}
\]

where \( \{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_s}\} \) are all eigenvalues of matrix \( A \) satisfying \( \lambda_{j_i} > 1 \). Therefore, some analytical factors describing the finite-time control capability of the LDTs can be extracted from the analytical expression of the volume of the controllable region as follows

\[
F_1 = F_1^+ \times F_1^- \\
= \left| \prod_{\forall j_1, j_2 \in P^+} \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \right| \times \left| \prod_{\forall k_1, k_2 \in P^-} \frac{\lambda_{k_2} - \lambda_{k_1}}{1 - \lambda_{k_1} \lambda_{k_2}} \right| \tag{78}
\]

\[
F_{2,i} = \frac{|q_i b| (1 - \lambda_i^N)}{1 - \lambda_i}, \quad i = 1, 2, \ldots, n \tag{79}
\]

\[
F_{3,i} = |q_i b|, \quad i = 1, 2, \ldots, n \tag{80}
\]

where \( P^+ \) and \( P^- \) are defined as follows

\[
P^+ = \{ \lambda_i : \lambda_i > 1, \forall i = 1, 2, \ldots, n \} \\
P^- = \{ \lambda_i : \lambda_i < 1, \forall i = 1, 2, \ldots, n \}
\]
As discussing in papers [15], [18] and [16], the above analytical factors can be called respectively as the shape (pole distribution factor), the side length of the circumscribed rhombohedron, and the modal controllability, and these factors are with same significances to these papers.

6. Three generalized cases of Theorem 4

6.1. The Narrow control capability and its analytical computing

Strictly speaking, as discussing above the control capability discussed above is a broad control capability (also called reachable capability). Based on Eq. (7) to Eq. (10), the above analytical analysis results on the reachable region $R^d_N(A)$ can be generalized to the narrow controllable region $R^c_N(A)$. Therefore, based on the above relations and Theorem 4 we have the following theorem about the analytical computation for the volume of the region $R^c_N(A)$.

**Theorem 5.** If the $n$ eigenvalues $\lambda_i (i = \overline{1, n})$ of the matrix $A$ are positive real numbers and satisfy

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n,$$

the volume of the finite-time narrow controllable region $R^c_N(A)$ when $N \geq n$ can be computed analytical as follows

$$V_n (R^c_N(A)) = 2^n \left| \det (W^{-1}) \prod_{i=1}^{n} \beta_i \right| V_N^{\lambda_1 \cdots \lambda_n}$$

(81)

$$V_N^{\lambda_1 \cdots \lambda_n} = \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta^s_N} c_{(j_1, \ldots, j_s)}^{(n)} \gamma^{\cdot N}_{(j_1, \ldots, j_s)} \Phi_{(j_1, \ldots, j_s)} \Phi_{n \setminus (j_1, \ldots, j_s)}$$

(82)

where

$$\gamma^{\cdot N}_{(j_1, \ldots, j_s)} = \prod_{k=1}^{s} \lambda_{j_k}^{-N}$$

(83)

By Theorem 5, the factor with the biggest absolute in the sum expression (81) and (82) is as follows

$$\left| \det (W^{-1}) \prod_{i=1}^{n} q_i b \right| \times \left| \gamma^{\cdot N}_{(j_1, \ldots, j_s)} \Phi_{(j_1, \ldots, j_s)} \Phi_{n \setminus (j_1, \ldots, j_s)} \right|$$

(84)
where \( \{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_s}\} \) are all eigenvalues of matrix \( A \) satisfying \( \lambda_{j_i} < 1 \). Therefore, some analytical factors describing the finite-time control capability of the LDTs can be extracted from the analytical expression of the volume of the controllable region as follows

\[
F_1 = F_1^+ \times F_1^-
\]

\[
= \prod_{\forall j_1, j_2 \in P^+} \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \times \prod_{\forall k_1, k_2 \in P^-} \frac{\lambda_{k_2} - \lambda_{k_1}}{1 - \lambda_{k_1} \lambda_{k_2}}
\]  

(85)

\[
F_{2,i} = \frac{|q_i b| \left(1 - \lambda_i^{-N}\right)}{1 - \lambda_i}, \quad i = 1, 2, \ldots, n
\]

(86)

\[
F_{3,i} = |q_i b|, \quad i = 1, 2, \ldots, n
\]

(87)

where \( P^+ \) and \( P^- \) are defined as follows

\[
P^+ = \{\lambda_i : \lambda_i > 1, \forall i = 1, 2, \ldots, n\}
\]

\[
P^- = \{\lambda_i : \lambda_i < 1, \forall i = 1, 2, \ldots, n\}
\]

The above analytical factors can be called respectively as the shape(pole distribution factor) factor, the side length of the circumscribed rhombohedron, and the modal controllability. In fact, the shape factor \( F_1 \) is also the eigenvalue evenness factor of the linear system, and can describe the control capability caused by the eigenvalue distribution. In addition, the modal controllability factor \( F_{3,i} \) have been put forth by papers [2] [8] [5] [4], and will not be discussed here.

6.2. All eigenvalues of the matrix \( A \) are negative

**Theorem 6**. If the \( n \) eigenvalues \( \lambda_i (i = 1, n) \) of the matrix \( A \) are negative real numbers and satisfy

\[
\lambda_1 < \lambda_2 < \cdots < \lambda_n < 0,
\]

the volume of the finite-time zonotope \( E_n(P_N) \) spaned by the columns of the matrix \( P_N = [b, Ab, \ldots, A^{N-1}b] \) when \( N \geq n \) can be computed analytical as
follows

\[
V_n(E_n(P_N)) = 2^n \det(W^{-1}) \prod_{i=1}^{n} \beta_i V_N^{\lambda_1 \ldots \lambda_n}
\]

(88)

\[
V_N^{\lambda_1 \ldots \lambda_n} = \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_n^s} c_{(j_1, \ldots, j_s)}^{(n)} \sum_{j \in (j_1, \ldots, j_s)} \sum_{i \in (i_1, \ldots, i_s)} \beta_i \det(W - I)
\]

(89)

where

\[
\Psi_{(j_1, \ldots, j_s)} = \prod_{k=1}^{s} \lambda_k
\]

(90)

\[
\Phi_{(j_1, \ldots, j_s)} = \prod_{1 \leq i < k \leq s} \frac{\lambda_k - \lambda_i}{1 - \lambda_i \lambda_j} \prod_{i=1}^{s} \frac{1}{1 + \lambda_i}
\]

(91)

6.3. The volume computing of the controllable region of the linear continuous-time systems

As discussing in the paper [19], the volume computing of the controllable regions of the LDT systems can be generalized to the linear continuous-time (LCT) systems \( \Sigma(A_c, B_c) \), and then we have the corresponding theorem to Theorem 4 as follows

**Theorem 7.** If the \( n \) eigenvalues \( \lambda_i, i = 1, n \) of the matrix \( A \) are real numbers and the smoothing zonotope \( R_T(A_c) \) generated by the matrix pair \( \Sigma(A_c, B_c) \) in the finite time \([0, T]\) is defined as follows

\[
R_T(A_c) = \left\{ x : x = \int_0^T \exp(A_c t) B_c u_t dt, \forall u_t \in [-1, 1] \right\}
\]

(92)

the volume of \( R_T(A_c) \) in the finite-time \( T \) can be computed analytical as follows

\[
V_n(R_T(A_c)) = 2^n \det(W^{-1}) \prod_{i=1}^{n} \beta_i V_T^{\lambda_1 \ldots \lambda_n}
\]

(93)

\[
V_T^{\lambda_1 \ldots \lambda_n} = \sum_{s=0}^{n} \sum_{(j_1, \ldots, j_s) \in \Theta_n^s} c_{(j_1, \ldots, j_s)}^{(n)} \Psi_{(j_1, \ldots, j_s)} \Phi_{(j_1, \ldots, j_s)} \Phi_{n \setminus (j_1, \ldots, j_s)}
\]

(94)
where

$$
\Upsilon_{(j_1,\ldots,j_s)}^T = \exp \left( \sum_{k=1}^{s} \lambda_{jk} T \right) 
$$

(95)

$$
\Phi_{(j_1,\ldots,j_s)} = \left| \prod_{1 \leq i < k \leq s} \frac{\lambda_{jk} - \lambda_{ji}}{\lambda_{ji} + \lambda_{jk}} \left( \prod_{i=1}^{s} \frac{1}{\lambda_{ji}} \right) \right|
$$

(96)

By the above theorem, the factor with the biggest absolute in the sum expression (93) and (94) is as follows

$$
\left| \det (W^{-1}) \prod_{i=1}^{n} q_i b \right| \times \left| \Upsilon_{(j_1,\ldots,j_s)}^T \Phi_{(j_1,\ldots,j_s)} \Phi_n \backslash (j_1,\ldots,j_s) \right|
$$

(97)

where \( \{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_s}\} \) are all eigenvalues of matrix \( A \) satisfying \( \lambda_{ji} > 0 \). Therefore, some analytical factors describing the finite-time control capability of the LDTs can be extracted from the analytical expression of the volume of the controllable region as follows

$$
F_1 = F_1^+ \times F_1^-
$$

$$
= \left| \prod_{\forall j_1,j_2 \in P^+}^n \frac{\lambda_{j_2} - \lambda_{j_1}}{\lambda_{j_1} + \lambda_{j_2}} \right| \times \left| \prod_{\forall k_1,k_2 \in P^-}^n \frac{\lambda_{k_2} - \lambda_{k_1}}{\lambda_{k_1} + \lambda_{k_2}} \right|
$$

(98)

$$
F_{2,i} = \left| q_i b \right| \left| \frac{1 - \exp (\lambda_{ii} T)}{\lambda_{ii}} \right|, \quad i = 1, 2, \ldots, n
$$

(99)

$$
F_{3,i} = \left| q_i b \right|, \quad i = 1, 2, \ldots, n
$$

(100)

where \( P^+ \) and \( P^- \) are defined as follows

\[ P^+ = \{ \lambda_i : \lambda_i > 0, \forall i = 1, 2, \ldots, n \} \]

\[ P^- = \{ \lambda_i : \lambda_i < 0, \forall i = 1, 2, \ldots, n \} \]

As discussing in papers [15], [18] and [16], the above analytical factors can be called respectively as the shape(pole distribution factor) factor, the side length of the circumscribed rhombohedral, and the modal controllability, and these factors are with same significances to these papers.
7. Numerical Experiments (Not available here)

8. Conclusions (Not available here)

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