SO(3) GAUGED SOLITON OF AN O(4) SIGMA MODEL ON $R^3$

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Abstract

Vector SO(3) gauged O(4) sigma models on $R^3$ are presented. The topological charge supplying the lower bound on the energy and rendering the soliton stable coincides with the Baryon number of the Skyrmion. These solitons have vanishing magnetic monopole flux. To exhibit the existence of such solitons, the equations of motion of one of these models is integrated numerically. The structure of the conserved Baryon current is briefly discussed.

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1 Introduction

The problem of gauging a sigma model has been considered in the past, first by Fadde’ev\cite{1} and also in the works of Witten\cite{2} and Rubakov\cite{3} for the Skyrme\cite{4} model on $\mathbb{R}^3$, which is essentially the $O(4)$ sigma model. Rubakov\cite{3} in particular considers the properties of the soliton in the $SU(2)$ gauged Skyrme model. The purpose of the present work is to construct a topologically stable soliton in a particular $SO(3)$ gauged Skyrme model.

The problem of gauging a Sigma model such that the resulting system supports topologically stable finite action or energy solutions was recently considered in \cite{5} and \cite{6}, the first for the $U(1)$ gauged $CP^1$ Grassmanian model on $\mathbb{R}_2$ and the second for $SO(2n)$ gauged Grassmanian models on $\mathbb{R}_{2n}$.

The only limitation of these models is that their definitions are restricted to even dimensions and the physically important problem of the soliton of the Skyrme model\cite{4}, which is the $O(4)$ Sigma model defined on $\mathbb{R}_3$. Indeed with this problem in mind, a peculiar gauging of the $O(3)$ model on $\mathbb{R}_2$ was proposed in \cite{3}. Denoting the Sigma model fields $\phi^a = (\phi^\alpha, \phi^3)$ with $\alpha = 1, 2$, and $\phi^a \phi^a = 1$, the gauging prescription proposed was stated via the minimal coupling in terms of the covariant derivative $D_i \phi^a = (D_i \phi^\alpha, D_i \phi^3)$ as:

$$D_i \phi^\alpha = \partial_i \phi^\alpha + A_i \varepsilon^{\alpha\beta} \phi^\beta,$$

$$D_i \phi^3 = \partial_i \phi^3. \quad (1)$$

The topological invariants we sought to employ in \cite{3} were the integrals of densities which were total divergences. Since the topological charge density of the $O(3)$ sigma model is not a total divergence but is only locally a total divergence\cite{7}, we ignored the $O(3)$ model gauged with the $U(1)$ field according to (1) and proceeded\cite{8} instead to gauge the $CP^1$ model. Subsequently however the $O(3)$ model on $\mathbb{R}_2$, gauged according to (1) was shown by Schroers\cite{8} to support topologically stable and even self-dual solutions.

It is our aim here to gauge the $O(4)$ gauge model\cite{4} on $\mathbb{R}_3$ employing an extended version of the minimal coupling prescription given by (1), where
the $U(1)$ is replaced by the $SO(3)$ gauge group, and to show that the corresponding gauged system supports stable solitons. While we shall restrict our considerations here to this 3 dimensional case, it should be noted that all our considerations in the present work can be extended to the $d$ dimensional case systematically, with the $SO(d)$ gauging of the $O(d + 1)$ Skyrme-Sigma model\[9\]. The starting point in that case would be the generalisation of the minimal coupling prescription (1) to

$$D_i \phi^\alpha = \partial_i \phi^\alpha + (T^\alpha)^{\beta\gamma} A_\beta^i \phi^\gamma, \quad D_i \phi^{d+1} = \partial_i \phi^{d+1}$$

(2)

where $(T)$ are the generators of $SO(d)$ in the vector representation, with $\alpha = 1, 2, ..., d$, and $\phi^\alpha \phi^\alpha = 1$. In what follows, we shall restrict ourselves to the case of $d = 3$. In that case, the minimal coupling given by (2) corresponds to gauging the vector $SU(2)$ subgroup of the chiral $SU(2) \times SU(2) \sim O(4)$.

Note that the above gauging (2) for $d = 3$ differs from that employed in [3]. This is easily seen by identifying the $SU(2)$ group valued field $U$ as $U = \phi^\alpha \tau^\alpha, U^{-1} = \phi^\alpha \bar{\tau}^\alpha$, with $\tau^\alpha = (i\sigma^\alpha, 1)$, $\bar{\tau}^\alpha = (-i\sigma^\alpha, 1)$, and $A_i = -i A_i^\alpha \sigma^\alpha$. In components then, it turns out that $(D_i U)^\alpha = D_i \phi^\alpha - A_i^\alpha \phi^4$ and $(D_i U)^4 = D_i \phi^4 + A_i^\alpha \phi^\alpha$, in the notation of (2).

## 2 The topological Invariant and the Model

The cornerstone in our construction of the gauged Skyrme model on $\mathbb{R}_3$ is the selection of a suitable topological charge density which could be exploited to give a lower bound on the energy density of a suitably modified version of the 3 dimensional Skyrme model. Since we are dealing with a Skyrme-Sigma model featuring a constrained field and a global $O(4)$ invariance, we expect that the relevant topological charge density will be only a locally total derivative quantity as was realised in [8]. The topological charge density of the $O(4)$ Skyrme\[3\] model is given by
\[ \varrho_0 = \varepsilon_{ijk} \varepsilon^{abcd} \partial_i \phi^a \partial_j \phi^b \partial_k \phi^c \phi^d \]  
where the index \( a = \alpha, 4, \) and \( \alpha = 1, 2, 3, \) etc., and the volume integral of (3) yields the winding number provided that the fields exhibit the appropriate asymptotic behaviour.

The gauged version of the density (3), namely
\[ \varrho_1 = \varepsilon_{ijk} \varepsilon^{abcd} D_i \phi^a D_j \phi^b D_k \phi^c \phi^d. \]  
is given in terms of the \( d = 3 \) version of the covariant derivative (2)
\[ D_i \phi^\alpha = \partial_i \phi^\alpha + \varepsilon^{\alpha\beta\gamma} A_\beta^i \phi^\gamma, \quad D_i \phi^4 = \partial_i \phi^4. \]  

The two densities (3) and (4) are related as follows
\[ \varrho_1 = \varrho_0 + 3 \varepsilon_{ijk} \phi^4 \mathcal{F}_{ij}^\alpha D_k \phi^\alpha + \partial_i \Omega_i \]  
where \( \mathcal{F}_{ij}^\alpha \) is the \( SO(3) \) curvature of the connection defined by the first member of (2), and the density \( \Omega_i \) is given by
\[ \Omega_i = 3 \varepsilon_{ijk} \phi^\alpha (2 A_j^\alpha \partial_k \phi^4 - \phi^4 \mathcal{F}_{jk}^\alpha). \]  

The latter is a gauge variant quantity, like the density \( \varrho, (3) \). As we shall see below, the topological charge density will turn out to be
\[ \varrho = \varrho_0 + \partial_i \Omega_i = \varrho_1 - 3 \varepsilon_{ijk} \phi^4 \mathcal{F}_{ij}^\alpha D_k \phi^\alpha, \]  
which is manifestly gauge invariant as it should be.

The volume integral of the topological charge density \( \varrho \) of (8) in fact reduces to the volume integral of the winding number density \( \varrho_0 \) of (3), provided that the surface integral of \( \Omega_i \) vanishes. This will be verified below for the spherically symmetric field configurations. Here we proceed to explain our criteria by means of stating the following Belavin inequality:
\[ (\lambda_1 D_i \phi^\alpha - \frac{k_2}{2} \varepsilon_{ijk} \varepsilon^{abcd} D_{[j} \phi^b D_{k]} \phi^c \phi^d)^2 \geq 0 \]
where the square brackets\([\ldots]\) on the indices imply antisymmetrisation, the constant \(\kappa_2\) has the dimension of length and the constant \(\lambda\) is dimensionless. The inequality (9) is the first member of the two pairs, for the second member of which there are two options. These are, respectively,

\[
(\kappa_0 F_{ij}^\alpha + \frac{1}{2} g(\phi^4) \varepsilon_{ijk} D_k \phi^\alpha)^2 \geq 0 \tag{10}
\]

\[
(\kappa_0 h(\phi^4) F_{ij}^\alpha + \frac{1}{2} \varepsilon_{ijk} D_k \phi^\alpha)^2 \geq 0, \tag{11}
\]

in both of which the constant \(\kappa_0\) has the dimension of length and the yet undetermined functions \(g\) and \(h\) depend only on \(\phi^4\), which according to the second member of (4) is the gauge invariant component of \(\phi^a\).

As usual, the energy density will be the sum of the square terms of (9) and (10), and respectively (11), when these are expanded. Choosing the functions \(g(\phi^4)\) and \(h(\phi^4)\) appropriately in each case, results in the cross term coinciding with the topological charge density \(\varrho\) defined by (8). The topological charge, which is the volume integral of the latter, will be equal to the volume integral of \(\varrho_0\) since the surface integral of \(\Omega_i\) vanishes as will be verified explicitly in the case of the spherically symmetric field configuration below. It then follows that the lower bound on the energy is the degree of the map, namely the winding number of the Hedgehog given by the volume integral of \(\varrho_0\), provided that suitable asymptotic conditions are satisfied. These are stated as usual to be

\[
\lim_{|\vec{x}| \to 0} \phi^4 = -1, \quad \lim_{|\vec{x}| \to \infty} \phi^4 = 1. \tag{12}
\]

The appropriate choices for the functions \(g\) and \(h\) turn out to be the same, and namely,

\[
g(\phi^4) = h(\phi^4) = 3\lambda \left( \frac{\kappa_2}{\kappa_0} \right) \phi^4
\]

respectively, leading to the two alternative Hamiltonian densities given by
\[ \mathcal{H}_1 = \kappa_0^2 (F^\alpha_{ij})^2 + \frac{9 \lambda_1^2}{2} \left( \kappa_0^2 \phi^4 \right)^2 + \frac{\lambda_1^2}{2} (D_i \phi^a)^2 + \frac{1}{2} \kappa_2^2 (D_i \phi^a D_j \phi^b)^2 \]  

(13)

and

\[ \mathcal{H}_2 = 9 \lambda_1^2 \kappa_2^2 (\phi^4)^2 + \frac{1}{2} (D_i \phi^a)^2 + \frac{\lambda_1^2}{2} (D_i \phi^a)^2 + \frac{1}{2} \kappa_2^2 (D_i \phi^a D_j \phi^b)^2. \]  

(14)

We notice that both Hamiltonian densities break the global \( O(4) \) symmetry of the corresponding ungauged sigma model, namely the Skyrme model. This is manifested through the appearance of \( |\phi^\alpha|^2 \) and \( |D_i \phi^a|^2 \) instead of \( |\phi^a|^2 \) and \( |D_i \phi^a|^2 \) respectively. This situation can easily be altered by adding suitable positive definite terms to each of (13) and (14) without invalidating the respective topological inequalities. In the case of (13) for example, the quantity to be added is \( \frac{9 \lambda_1^2 \kappa_2^2}{2 \kappa_0} \) times \( (D_i \phi^a)^2 + (D_i \phi^4)^2 \). Both (13) and (14) then take the form

\[ \mathcal{H}_0 = \eta_0^2 (F^\alpha_{ij})^2 + \frac{\tau_1^2}{2} (D_i \phi^a)^2 + \frac{\eta_2^2}{8} (D_i \phi^a D_j \phi^b)^2, \]  

(15)

in which the constants \( \eta_0 \) and \( \eta_2 \) have the dimensions of length, and the constant \( \tau_1 \) is dimensionless. The Hamiltonian density (15) coincides with the system proposed by Fadde’ev in Ref. [1].

The volume integrals of (13), (14) and (15) are bounded from below by the winding number according to

\[ \int d^3 x \mathcal{H} \geq \int d^3 x \varrho = \int d^3 x \varrho_0, \]  

(16)

provided that the field configurations satisfy at least the asymptotic conditions (12).

It is not possible to saturate the inequality (16) by saturating the inequalities (9) and (10) separately to minimise absolutely the energy corresponding to (13), and, (3) and (11) separately to minimise the energy of (14).
absolutely. By minimising absolutely we mean solving the system by some first order Bogomol’nyi equations, and in the case of (15) this is not possible even in principle. For the models given by (13) and (14), while in principle possible, the saturated versions of (9) and (10), (11) respectively, are overdetermined as is the case also for the (ungauged) Skyrme model[4]. This contrasts with the corresponding situation for the 2 dimensional $O(3)$ model where both the ungauged[10] and the gauged[8] models support self-dual solutions. In the following therefore, we are concerned only with solutions of the second order Euler-Lagrange equations of the models (13), (14) and (15), and not with the solutions of some first order Bogomol’nyi equations.

3 Spherically Symmetric Fields

The spherically symmetric fields are given by the Ansatz

$$\phi^\alpha = \hat{x}_i \sin f(r), \quad \phi^4 = \cos f(r)$$

$$A_i^\alpha = \frac{a(r) - 1}{r} \varepsilon_{i\alpha\beta} \hat{x}_\beta$$

yielding the following field strengths

$$D_i \phi^\alpha = \frac{a}{r} \hat{x}_i \sin f + (f' \cos f - \frac{a}{r} \sin f) \hat{x}_i \hat{x}^\alpha$$

$$F_{ij}^\alpha = \frac{a'}{r} \varepsilon_{ija} - \left( \frac{a'}{r} - \frac{a^2}{r^2} - 1 \right) \varepsilon_{ij\beta} \hat{x}_\beta \hat{x}_\alpha.$$
(20) into (13), (14) and (15) respectively, by

\[ \int d\rho H[f, f_\rho; a, a_\rho] = \frac{1}{4\pi} \int d^3x \mathcal{H} \]  

(21)

where the dimensionless radial variable \( \rho \) is defined in the two cases as \( \rho = \frac{r}{\kappa_0} \) and \( \rho = \frac{r}{\kappa_2} \) respectively, and \( f_\rho = \frac{df}{d\rho} \). The definition (21) is made, up to an unimportant constant multiple in each case.

Since the alternative models (13), (14) and (15) are qualitatively similar, we shall restrict ourselves in the following to the detailed asymptotic and numerical study of one of these models only. We find it more natural to prefer models (13) and (14) to (15) because the former satisfy the minimal topological inequality (16), albeit without saturating it, while model (15) satisfies an inequality derived from (16) itself. Next, we eschew model (14) because of its unconventional Yang-Mills term. Thus we restrict our considerations below to the model given by (13).

In terms of the dimensionless parameter \( \lambda_2 = \left( \frac{\kappa_2}{\kappa_0} \right) \), the resulting one dimensional Hamiltonian density for the models given by (13) is

\[
H_1 = 2[2a_\rho^2 + \left( \frac{a^2 - 1}{\rho^2} \right)^2] + \frac{9\lambda_1^2\lambda_2^2}{2} \cos^2 f [\rho^2 f_\rho^2 \cos^2 f + 2a^2 \sin^2 f] \\
+ \frac{\lambda_2^2}{2} [\rho^2 f_\rho^2 + 2a^2 \sin^2 f] + 2\lambda_2^2a^2 \sin^2 f [2f_\rho^2 + \frac{a^2 \sin^2 f}{\rho^2}] 
\]  

(22)

We first check that the topological charge, namely the volume integral of \( \varrho \) given by (8) reduces to the usual winding number, namely the volume integral of \( \varrho_0 \) for the spherically symmetric field configuration (17) and (18) when the appropriate asymptotic conditions for the function \( f(r) \)

\[
\lim_{r \to \infty} f(r) = 0, \quad \lim_{r \to 0} f(r) = \pi 
\]  

(23)

are satisfied. In that case the volume integral of \( \varrho_0 \) is guaranteed to be the unit topological charge of the Hedgehog.
Concerning the asymptotic behaviour required of the function \(a(r)\) in the region \(r \ll 1\), this is determined by regularity at the origin, while in the region \(r \gg 1\) there are several possibilities consistent with a power decay of the function \(a(r)\) at infinity, which is a necessary condition for finite energy solutions. These asymptotic values are \(a(\infty) = \pm 1\) and \(a(\infty) = 0\). Unlike in \(SU(2)\) Higgs theory\(^\text{[11]}\), the behaviour of the gauge field and hence of the function \(a(r)\) at infinity is not directly relevant to the topological stability of the soliton. In the latter case\(^\text{[11]}\), the topological charge coincides with the magnetic flux of the monopole field, while here the topological charge is the degree of the map, which is the unit winding number for the spherically symmetric fields under consideration. Using the magnetic flux density \(B_i = \frac{1}{2} \varepsilon_{ijk} \phi^\alpha F_{jk}^\alpha\), for the spherically symmetric field configuration, we calculate the surface integral for the magnetic flux

\[
\Phi = \int \vec{B} \cdot d\vec{S} = 4\pi [(a^2 - 1) \sin f]|_{r=\infty}.
\]  

Irrespective of the value of \(a(\infty)\), the flux \(\Phi\) vanishes by virtue of the second member of (23). The models at hand therefore do not describe magnetic monopoles.

Before stating our asymptotic conditions for the function \(a(r)\), we note that for the topological charge density \(\varrho\) given by (9) to reduce to the usual winding number density \(\varrho_0\), the surface integral of \(\Omega_i\) must vanish. This is seen by calculating the relevant one dimensional integrand, namely the quantity \(r^2\hat{x}_i\Omega_i\), from its definition (7),

\[
r^2\hat{x}_i\Omega_i = (a^2 - 1) \sin 2f
\]  

which on the infinite 2-sphere clearly vanishes irrespective of which of the asymptotic values \(a(\infty) = \pm 1\) or \(a(r) = 0\) holds, provided that the corresponding condition stated by the second member of (23) does hold.

Anticipating the results of our numerical integration to be carried out below, we state the asymptotic values of \(a(r)\) as

\[
\lim_{r \to 0} a(r) = 1, \quad \lim_{r \to \infty} a(r) = 0.
\]  

The Euler-Lagrange equations for the model given by (22), with respect to the arbitrary variations of the functions \(f(r)\) and \(a(r)\) are respectively,
\[ \lambda_1^2(1 + 9\lambda_2^2 \cos^4 f)(\rho^2 f_\rho)_\rho + 8\lambda_2^2 \sin^2 f \ (a^2 f_\rho)_\rho \]

\[ -18\lambda_1^2 \lambda_2^2 \sin f \cos f [\rho^2 f_\rho^2 \cos^2 f + \rho^2(\cos^2 f - \sin^2 f)] \]

\[ + 8\lambda_2^2 a^2 \sin f \cos f [f_\rho^2 - \frac{a^2 \sin^2 f}{\rho^2} - 2\lambda_1^2 a^2 \sin f \cos f = 0 \quad (27) \]

and

\[ a_{\rho\rho} - \frac{a}{\rho^2} (a^2 - 1) - a \sin^2 f \left[ \frac{9}{4} \lambda_1^2 \lambda_2^2 \cos^2 f + \frac{1}{4} \lambda_1^2 + \lambda_2^2 (f_\rho^2 + \frac{a^2 \sin^2 f}{\rho^2}) \right] = 0, \quad (28) \]

The asymptotic values in the \( r \ll 1 \) region are given by the first members of (23) and (26), and for the model (13) we find the following behaviours

\[ f(\rho) = \pi + A\rho + o(\rho^3) \quad (29) \]

\[ a(\rho) = 1 + B\rho^2 + o(\rho^4), \quad (30) \]

which lead, as expected, to differentiable fields (19) and (18) at the origin.

The asymptotic behaviours of the solution in the \( r \gg 1 \) region are also power decays like for the usual (ungauged) Skyrmion\(^4\) (Exponential decay can be obtained by incorporating a suitable gauge invariant \( O(4) \) breaking potential\(^9\)). Since we shall integrate the field equations with the asymptotic conditions (23) and (26), we give the corresponding asymptotic solutions in this region

\[ f(\rho) = \frac{C}{\rho}, \quad a(\rho) = \frac{D}{\rho^\beta} \quad (31) \]

in which \( \beta = \frac{1}{2}[\sqrt{(C^2\lambda_1^2(\lambda_1^2 + 9\lambda_2^2) - 3)} - 1] \), and the constants \( C \) and \( D \) will not be computed.

Having solved the relevant Euler-Lagrange equations (24) and (27) in the asymptotic regions \( \rho << 1 \) and \( \rho >> 1 \), we proceed to integrate them numerically, subject to the asymptotic conditions (18) and (19). The constants \( A \) and \( B \) in (29) and (30) are fixed by the numerical integration.
The numerical integrations have been performed for the values of the
dimensionless coupling constants \( \lambda_1 = \lambda_2 = 1.5 \), and \( \lambda_1 = \lambda_2 = 1.4 \). The values of the pair of constants \( \{A, B\} \) were fixed in each of these cases respectively to be

\[
\{A, B\} = \{-1.9606513707554, -5.6289634247230\}
\]

\[
\{A, B\} = \{-1.8663921477326, -4.7278588179841\}
\]

The profiles of the function \( f(\rho) \) are given in Figure 1, and the profiles of the function \( a(\rho) \) in Figure 2. The profiles of the energy densities pertaining to each of these solutions are plotted in Figure 3, and they correspond to the total energies \( E_1 = 29.924879981245 \) and \( E_2 = 27.463710758882 \) respectively.

4 Discussion and Summary

Before proceding to summarise our results and making some qualitative comments, we give a brief quantitative description of the Baryonic current that the topological charge employed above pertains to. The latter is the volume integral of the density \( \varrho \) given by (8), which we identify with the fourth, time-like, component of this current \( j^\mu \). Accordingly, the full Minkowskian vector current is

\[
j^\mu = \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} [D_\nu \phi^a D_\rho \phi^b D_\sigma \phi^c D_\sigma \phi^d - \frac{3}{4} F^{cd}_{\rho\sigma} D_\nu \phi^a \phi^b]. \tag{32}
\]

The curvature field strength in (32) consists only of an \( SU(2) \) field, say \( F_{\mu\nu} = \varepsilon^{\alpha\beta\gamma} F_{\mu\nu}^{\alpha\beta} \), with \( F_{\mu\mu}^{\alpha\beta} = 0 \). Accordingly, the second term in (32) can be re-expressed using

\[
\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} F^{cd}_{\rho\sigma} D_\nu \phi^a \phi^b = 2 \varepsilon^{\mu\nu\rho\sigma} [2 F_{\rho\sigma}^{\alpha\beta} D_\nu \phi^\alpha \phi^\beta - 2 \partial_\nu (F_{\rho\sigma}^{\alpha\beta} \phi^\alpha \phi^\beta)] \tag{33}
\]

The second term on the right hand side of (33) being a total divergence, its volume integral vanishes and hence can be neglected. It follows from (33) that the divergence of the current \( j^\mu \) given by (32) is
\[ \partial_\mu j^\mu = -4\varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} D_\mu \phi^\alpha D_\nu \phi^\beta D_\rho \phi^\gamma \partial_\sigma \phi^4. \] (34)

The right hand side of (34) can be shown to be \textit{locally total divergence} which means that its volume integral vanishes and hence can be ignored, leading to a conserved Baryonic current, \( \partial_\mu j^\mu = 0 \). This is exactly what we expect, since the \textit{vector} gauging (5) does not lead to the divergence of the current being equal to an anomaly. Our current (32) can be compared to that of Goldstone and Wilczek\cite{14}, where the \( O(4) \) sigma model has been gauged in the usual way according to \( D_\mu \phi^a = \partial_\mu \phi^a + A_{[ab]}^\mu \phi^b \), and contrasted with the corresponding current of D’Hoker and Farhi\cite{15} where the Skyrme model featuring the \( SU(2) \) valued field \( U \) has been gauged with the \((V - A) \) \( SU(2) \) field according to \( D_\mu U = \partial_\mu U - i \tilde{A}_\mu \sigma U \). In the latter case\cite{15}, the divergence of the Baryonic current equals the anomaly.

We have unfortunately \textit{not} succeeded to adapt the constructions employed in the \( O(4) \) sigma model with field \( \phi^a \), to the analogous case where the \( SU(2) \) valued Skyrme field \( U \) is used, which is the physically more interesting case as it leads to a non-conserved Baryonic current featuring the anomaly. Technically, this has come about because of our inability to reduce the topological charge density \( \varsigma \) in this case, analogous to \( \varrho \) used in the above, to the form

\[ \varsigma = \varsigma_0 + \partial_\dot{i} \tilde{\Omega}_i \] (35)

in which \( \varsigma_0 = \varepsilon_{ijk} Tr U^{-1} \partial_i U \partial_j U^{-1} \partial_k U \) is equal to \( \varrho_0 \) defined by (3) with \( U = \phi^a \tau^a \), \( U^{-1} = \phi^a \bar{\tau}^a \) defined at the end of Section 1. Had it turned out possible to establish (35), then the topological charge would have been related to \( \varsigma_1 \), analogous to \( \varrho_1 \) in (4), which would have been the time-like component of the topological Baryonic current

\[ j^\mu = \varepsilon^{\mu\nu\rho\sigma} Tr[U^{-1} D_\nu U U^{-1} D_\rho U U^{-1} D_\sigma U + \frac{3}{4} U^{-1} F_{\rho\sigma} D_\nu U]. \] (36)

used in \cite{15}, whose divergence \( \partial_\mu j^\mu \) does not vanish but is equal to the chiral
$SU_+ \pm$ anomaly. It would be very interesting if some other version of (35) could be found, which would lead to a lower bound on the static Hamiltonian.

To summarise, we have constructed three $SO(3)$ gauged versions of the $O(4)$ sigma model on $\mathbb{R}^3$, characterised by the static Hamiltonians (13), (14) and (15). These are equivalent to the corresponding gauged versions of the Skyrme model [4]. Models (13) and (14) have the feature of breaking the global $O(4)$ symmetry of the sigma model, but the latter can be restored by adding suitable positive definite terms to the Hamiltonian densities resulting in the model (15) which was first proposed by Fadde’ev [1]. The large $r$ asymptotic field configuration of the soliton presented here is $SO(3)$ symmetric and the magnetic flux of the corresponding field configuration vanishes. Accordingly this soliton model differs from the corresponding $SU(2)$ Higgs model [11], in which the asymptotic field configuration is $SO(2)$ symmetric and exhibits magnetic monopole flux.

These models admit finite energy topologically stable soliton solutions, whose energy is bounded from below by the winding number, which can be interpreted as the Baryon number. This topological bound is saturated by Bogomol’nyi equations, which however are overdetemined as in the case of the usual (ungauged) Skyrme model, and hence likewise our solitons are solutions to the full second order Euler-Lagrange equations. These were solved analytically only in the asymptotic regions $r << 1$ and $r >> 1$, and the full integrations were performed numerically.

In the present work, we have restricted ourselves to the spherically symmetric case. As such the soliton in question carries Baryon number 1. It would be interesting to find the Baryon number 2 axially symmetric solutions, analogous to the corresponding axially symmetric solutions of the (ungauged) Skyrme model [12]. Furthermore, since we know [9] that for sigma models in odd dimensional spaces there are spherically symmetric solitons of arbitrary Baryon number $N$, it would be interesting to study these in the present model. These higher degree field configurations are characterised by their asymptotic values, which for small $r$ differ from (23) according to
\[
\lim_{r \to \infty} f(r) = 0, \quad \lim_{r \to 0} f(r) = N\pi.
\] (37)

It would be interesting to integrate the Euler-Lagrange equations with the
asymptotic conditions (37), say with N=2, and to see whether the energy of that soliton is greater than twice the energy of the $N = 1$ soliton, as is the case for the usual (ungauged) Skyrme model [13]. All these detailed questions are deferred to future investigations.

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Figure Captions:

Figure 1. Profiles of the function $f(\rho)$ for $\lambda = \lambda_1 = \lambda_2 = 1.5$ and $\lambda = 1.4$. The higher curve pertains to $\lambda = 1.5$.

Figure 2. Profiles of the function $a(\rho)$ for $\lambda = \lambda_1 = \lambda_2 = 1.5$ and $\lambda = 1.4$. The higher curve pertains to $\lambda = 1.4$.

Figure 3. Profiles of the energy densities corresponding to the solutions with $\lambda = \lambda_1 = \lambda_2 = 1.5$ and $\lambda = 1.4$. The higher curve pertains to $\lambda = 1.5$. 
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