Baker-Akhiezer Modules on the Intersections of Shifted Theta Divisors

Koji Cho
Department of Mathematics, Kyushu University

Andrey Mironov†
Sobolev Institute of Mathematics and Novosibirsk State University

Atsushi Nakayashiki‡
Department of Mathematics, Kyushu University

Abstract
The restriction, on the spectral variables, of the Baker-Akhiezer (BA) module of a \(g\)-dimensional principally polarized abelian variety with the non-singular theta divisor to an intersection of shifted theta divisors is studied. It is shown that the restriction to a \(k\)-dimensional variety becomes a free module over the ring of differential operators in \(k\) variables. The remaining \(g - k\) derivations define evolution equations for generators of the BA-module. As a corollary new examples of commutative ring of partial differential operators with matrix coefficients and their non-trivial evolution equations are obtained.

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* e-mail: cho@math.kyushu-u.ac.jp
† e-mail: mironov@math.nsc.ru
‡ e-mail: 6vertex@math.kyushu-u.ac.jp


1 Introduction

The Baker-Akhiezer (BA) module was introduced in [5, 6] in order to extend the theory of the BA function due to Krichever [1] to higher dimensions. It is a geometric counterpart of the $D$-module generated by the wave operator in Sato’s theory of KP-hierarchy and universal Grassmann manifold.

A fundamental example of the BA function is a function on an elliptic curve of the form

$$\varphi(z; x) = \frac{\sigma(z + x)}{\sigma(z)\sigma(x)} e^{-x\zeta(z)},$$

where $\sigma(z), \zeta(z)$ are Weierstrass’ sigma and zeta functions. The corresponding BA-module is the $D$-module generated by $\varphi(z; x)$:

$$M = D\varphi(z; x) = \sum_{n=0}^{\infty} O\partial_x^n \varphi(z; x),$$

where $O$ is a suitable ring of functions such as the convergent power series ring, its quotient field etc. and $D = O[\partial_x]$ is the ring of differential operators in $x$ with the coefficients in $O$. It is a rank one free module over $D$.

Let $A = \mathbb{C}[\varphi(z), \varphi'(z)]$ be the affine ring of the elliptic curve. An important property of the BA-module is that it is not only a $D$ module but also an $A$-module. As a consequence $A$ is embedded in $D$ as a commutative subring.

Similarly, in the case of genus $g$ algebraic curves, the BA-module becomes a ($D_g, A$)-bimodule, where $D_g = O[\partial_1, ..., \partial_g]$ is the ring of differential operators in $g$ variables and $A$ is the affine ring of the curve. It becomes a rank one free module over the subring $D_1 = O[\partial_1]$ of operators in one variable and the affine ring $A$ is embedded in $D_1$. The action of the commuting derivations $\partial_2, ..., \partial_g$ specifies evolution equations of the BA-module, or the deformation of the image of $A$ in $D$. In this way solutions of integrable nonlinear equations such as KP equation, KdV equation are constructed [1].

In [5] the BA-module of a $g$ dimensional principally polarized Abelian variety $(X, \Theta)$ with a non-singular $\Theta$ is studied. It is proved that BA-module becomes a free $D$-module of rank $g!$, where $D$ is the ring of differential operators in $g$ variables. Consequently the affine ring $A$ of $X\setminus\Theta$ is embedded in the ring $Mat(g!, D)$ of differential operators with the coefficients in $g! \times g!$ matrices. However in this case there is no non-trivial deformation. To have deformations it is necessary to consider the BA-module of polarized subvarieties of $(X, \Theta)$.

We consider an intersection $Y^k$ of shifted theta divisors as a subvariety of $X$ and the intersection $Q^k$ of it with the theta divisor as a divisor, where $k$ denotes the codimension of $Y^k$ in $X$. We show that the restriction of the BA-module of $(X, \Theta)$ to $Y^k$ is a free $D_{g-k}$ module of rank $g!$, where $D_i$ is the ring of differential operators in $i$ variables. As a by-product we have the embedding of the affine ring of $Y^k \setminus Q^k$ in $Mat(g!, D_{g-k})$ and $k$ commuting derivations which specify the deformation of the image of it.

The simplest case of $g = 3$ and $k = 1$ is studied in [6]. The case of intersections of more general divisors are studied in [2]. Unfortunately the proof of the freeness
is incomplete in that paper. Other examples of BA-modules which have non-trivial
deformations are studied in [7].

The plan of the paper is as follows. In section 2 the definition of BA-modules and
the main result are given. The combinatorial properties of the restriction of the BA-
module is studied in section 3. It is shown that the character of the associated graded
module of the BA-module coincides with that of the free module. It means that as far
as the dimension is concerned the restriction of the BA-module becomes a free module.
In section 4 the proof of the main theorem is given based on the result of section 3.

2 Results

Let \( \Omega \) be a point of the Siegel upper half space of degree \( g \), \( X \) the corresponding
principally polarized Abelian variety

\[
X = \mathbb{C}^g/\Gamma, \quad \Gamma = \mathbb{Z}^g + \Omega \mathbb{Z}^g,
\]

\( \theta_{a,b}(z) \) Riemann’s theta function with the characteristic \( \tau(a,b), a, b \in \mathbb{R}^g \)

\[
\theta_{a,b}(z) = \theta_{a,b}(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i \tau(n + a)\Omega(n + a) + 2\pi i\tau(n + a)(z + b)),
\]

and \( \Theta \) the theta divisor on \( X \) defined by the zero set of \( \theta(z) = \theta_{0,0}(z) \). For \( c \in \mathbb{C}^g \), \( L_c \)
denotes the holomorphic flat line bundle on \( X \) which has \( \theta(z + c)/\theta(z) \) as a meromorphic
section. A meromorphic section of \( L_c \) can be considered as a meromorphic function \( f(z) \)
on \( \mathbb{C}^g \) which satisfies

\[
f(z + m + \Omega n) = \exp(-2\pi i\tau(nc)f(z), \quad m, n \in \mathbb{Z}^g. \tag{1}
\]

We denote \( L_c(n) \) the space of meromorphic sections of \( L_c \) whose poles are only on \( \Theta \)
of order at most \( n \). It is known that \( \dim L_c(n) = n^g \) and a linear basis is given by the
functions \( f_{n,a}(z + \frac{\zeta}{n})/\theta(z)^n, \quad a \in \mathbb{Z}^g/n\mathbb{Z}^g \), \[ ]

\[
f_{n,a}(z) = \theta_{\frac{\zeta}{n},0}(nz, n\Omega).
\]

We define

\[
L_c = \bigcup_{n=0}^{\infty} L_c(n).
\]

The subspaces \( L_c(n) \) define an increasing filtration of \( L_c \).

Set

\[
\zeta_i(z) = \frac{\partial}{\partial z_i} \log \theta(z).
\]

It satisfies, for \( m, n \in \mathbb{Z}^g \),

\[
\zeta_j(z + m + \Omega n) = \zeta_j(z) - 2\pi i n_j.
\]
Let $\mathcal{O}$ be the convergent power series ring in $x = (x_1, \ldots, x_g)$, $\mathcal{K}$ its quotient field, $\partial_{x_i} = \partial/\partial x_i$, $\mathcal{D} = \mathcal{K}[\partial_{x_1}, \ldots, \partial_{x_g}]$ the ring of differential operators with the coefficients in $\mathcal{K}$, $\mathcal{D}(n) = \{ \sum_{|\alpha| \leq n} a_\alpha \partial_\alpha \in \mathcal{D} \}$ the differential operators of order at most $n$ and $\text{gr} \mathcal{D} = \oplus \mathcal{D}(n)/\mathcal{D}(n-1)$ the commutative ring of principal symbols.

In general, for a module with an increasing filtration $M = \cup_n M(n)$, the associated graded module is defined by

$$\text{gr} M = \oplus_n \text{gr}_n M, \quad \text{gr}_n M = M(n)/M(n-1).$$

Let

$$M_c(n) = \sum_{a \in \mathbb{Z}/n\mathbb{Z}} \mathcal{K} \frac{f_{n,a}(z + \frac{c+z}{n})}{\theta(z)^n} e^{-\sum_{i=1}^g x_i \zeta_i(z)},$$

$$M_c = \cup_{n=0}^\infty M_c(n).$$

The space $M_c(n)$ is a $n^g$ dimensional vector space over $\mathcal{K}$ and the subspaces $\{M_c(n)\}$ specify an increasing filtration of $M_c$. Then, for $c \notin \Gamma$,

$$\dim_{\mathcal{K}} \text{gr}_n M_c = \dim_{\mathcal{K}} \text{gr}_n L_c = n^g - (n-1)^g, \quad n \geq 1. \quad (2)$$

As a function of the variables $z$ any element of $M_c$ satisfies the equation (1). The differentiation in $x_i$ preserves this equation. Thus $M_c$ becomes a $\mathcal{D}$-module. It is introduced in [5] and called the Baker-Akhiezer (BA) module of $(X, \Theta)$. Since

$$\partial_{x_i} M_c(n) \subset M_c(n+1),$$

$\text{gr} M_c$ becomes a $\text{gr} \mathcal{D}$-module.

Let $A$ be the affine ring of $X \setminus \Theta$. Analytically it is described as

$$A = \{ \frac{F(z)}{\theta(z)^n} \mid F(z) \text{ is holomorphic on } \mathbb{C}^g, \quad \frac{F}{\theta^a}(z + \gamma) = \frac{F}{\theta^a}(z) \text{ for any } \gamma \in \Gamma \}.$$
Theorem 1 Suppose that $\Theta$ is non-singular and $c \notin \mathbb{Z}^g + \Omega \mathbb{Z}^g$. Then $\text{gr} \ M_c$ is a free $\text{gr} \mathcal{D}$-module of rank $g!$ and $M_c$ is a free $\mathcal{D}$-module of rank $g!$. More precisely

$$
\text{gr} \ M_c = \bigoplus_{i=1}^g \bigoplus_{j=1}^{r_i} (\text{gr} \mathcal{D}) \psi_{ij}, \quad \psi_{ij} \in \text{gr}_i M_c,
$$

$$
M_c = \bigoplus_{i=1}^g \bigoplus_{j=1}^{r_i} \mathcal{D} \phi_{ij}, \quad \phi_{ij} \in M_c(i),
$$

where $\psi_{ij}$ is the projection of $\phi_{ij}$ in $\text{gr}_i M_c$.

Let $\Phi = (\phi_{ij})$ be the column vector of dimension $g!$ and $\text{Mat}(m, \mathcal{D})$ the ring of $m \times m$ matrices with the components in $\mathcal{D}$. Since $M_c$ is an $A$-module, for $a \in A$, there exists an element $\ell(a) \in \text{Mat}(g!, \mathcal{D})$ such that

$$
a \Phi = \ell(a) \Phi.
$$

It defines an embedding of $A$ into $\text{Mat}(g!, \mathcal{D})$ as a ring. Thus $A$ is realized as a commutative subring of the ring differential operators in $g$ variables with matrix coefficients.

In this paper we shall extend Theorem 1 to the BA-modules on the intersections of shifted theta divisors.

For $a \in \mathbb{C}^g$ we set

$$
\Theta_a = \{ \theta(z - a) = 0 \} \subset X.
$$

Take $a_1, ..., a_{g-1} \in \mathbb{C}^g$. Beginning from $(Y^0, Q^0) = (X, \Theta)$ we define $(Y^k, Q^k)$ for $k \geq 1$ by

$$
Y^k = \Theta_{a_1} \cap \cdots \cap \Theta_{a_k},
$$

$$
Q^k = Y^k \cap \Theta.
$$

We assume that, for any $k \leq g$, $\Theta_{a_1}, ..., \Theta_{a_k}$ intersect transversally and so does $\Theta, \Theta_{a_1}, ..., \Theta_{a_k}$. It means in particular that $Y^k$ and $Q^k$ are non-singular subvarieties of $X$ of dimensions $g - k$ and $g - k - 1$ respectively. It can be shown that for generic $a_1, ..., a_{g-1}$ the assumption is satisfied.

We denote the restriction of $\mathcal{L}_c$ to $Y^k$ by the same symbol for simplicity. Let $\mathcal{L}_c(nQ^k)$ be the sheaf of germs of meromorphic sections of $\mathcal{L}_c$ on $Y^k$ with poles only on $Q^k$ of order at most $n$. We set

$$
L^k_c(n) = H^0(Y^k, \mathcal{L}_c(nQ^k)).
$$

We define the space $\tilde{M}^k_c(n)$ as the set of functions of the form

$$
f(z; x) e^{-\sum_{i=1}^g x_i \gamma_i(z)}|_{Y^k},
$$

where $f(z; x)$ satisfies the following conditions.

There is an open neighborhood $U$ of $0 \in \mathbb{C}^g$, which can depend on $f$, with the following properties.

(i) For each $x \in U$ $f(z; x)$ belongs to $L^k_{c+x}(n)$ as a function of $z$.

(ii) As a function of $x$ $f(z; x)$ is analytic on $U$.

It is obvious that $\tilde{M}^k_c(n)$ is an $\mathcal{O}$-module.
Lemma 1 Let $k$ satisfy $0 \leq k \leq g - 1$. Suppose that $c + \sum_{i \in I} a_i \not\in \Gamma$ for any subset $I$ of $\{1, \ldots, k\}$. Then

(i) $H^i(Y^k, \mathcal{L}_c(nQ^k)) = 0$, $i \neq 0, g - k$, $n \in \mathbb{Z}$.

(ii) The restriction map $L_c^{k-1}(n) \rightarrow L_c^k(n)$ is surjective for any $n \in \mathbb{Z}$.

Proof. We have the following exact sequence and isomorphism:

$$0 \rightarrow \mathcal{L}_c(nQ^{k-1} - Y^k) \rightarrow \mathcal{L}_c(nQ^{k-1}) \rightarrow \mathcal{L}_c(nQ^k) \rightarrow 0,$$

(i4)

$$\mathcal{L}_c(nQ^{k-1} - Y^k) \cong \mathcal{L}_{c+a_{k-1}}((n-1)Q^{k-1}).$$

(i5)

The isomorphism (5) follows from $\mathcal{O}_X(\Theta_c) \cong \mathcal{O}_X(\Theta) \otimes \mathcal{L}_c$. The assertion (i) can be proved by induction on $k$ using the cohomology sequence of (4) and the vanishing [3]:

$$H^i(X, \mathcal{L}_c(n\Theta)) = 0,$$

for $i \geq 1$, $n \geq 1$ or $i \geq 0$, $n = 0$ or $i \neq g$, $n < 0$. The assertion (ii) follows from the cohomology sequence of (4) and (i).

Lemma 2 Assume the same conditions as in Lemma 1. Then $\dim L_c^k(n)$ does not depend on $c$ and satisfies

$$\dim L_c^k(n) = \dim L_c^{k-1}(n) - \dim L_c^{k-1}(n-1).$$

(i6)

Proof. The lemma can easily be proved by induction using the cohomology sequence of (4), the isomorphism (5), (i) of Lemma 1 and the fact that $\dim L_c^0(n) = n^g$ for $n \geq 0$ if $c \not\in \Gamma$.

Example. $\dim L_c^1(n) = n^g - (n-1)^g$ for $n \geq 1$.

$\dim L_c^2(n) = n^g - 2(n-1)^g + (n-2)^g$ for $n \geq 2$ and $\dim L_c^2(1) = 1$.

Lemma 3 Assume the same conditions as in Lemma 2. Then $\tilde{M}_c^k$ is a free $\mathcal{O}$-module of rank $\dim L_c^k(n)$.

Proof. We take some basis $\{f_i(z)\}$ of $L_c^k(n)$ and lift it to $\{f_i(z;x)\}$ such that the conditions (i),(ii) are satisfied and $f_i(z;0) = f_i(z)$. Then it gives a $\mathcal{O}$-free basis of $\tilde{M}_c^k(n)$. Such analytic lift can be constructed among the functions $\{f_{n,a}(z + \frac{c+k}{n})\}$ restricted to $Y^k$ since the restriction map $L_c^0(n) \rightarrow L_c^k(n)$ is surjective for $x$ sufficiently close to $0 \in \mathbb{C}^g$ by (ii) of Lemma 1.

We set

$$M_c^k(n) = \mathcal{K} \otimes_{\mathcal{O}} \tilde{M}_c^k, \quad M_c^k = \cup M_c^k(n).$$

The set of subspaces $\{M_c^k(n)\}$ defines an increasing filtration on $M_c^k$. 

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Let $\pi_k : M^k_c \to M^{k+1}_c$ be the restriction map. It is surjective for $k \geq 0$ as shown above. In particular $\pi_{0k} := \pi_k \pi_{k-1} \cdots \pi_0 : M_c \to M^k_c$ is surjective. Thus $M^k_c$ can be directly described as the restriction of $M_c$ to $Y^k$ with respect to the $z$ variables:

$$M^k_c = M_c|_{Y^k}.$$ 

It is obvious that the restriction in $z$ variables commutes with the action of $\partial_{x_i}$. Therefore $M^k_c$ becomes a $\mathcal{D}$-module. Moreover the action of $\partial_{x_i}$ satisfies $\partial_{x_i} M^k_c(n) \subset M^k_c(n + 1)$. Thus $\text{gr} \ M^k_c$ becomes a $\text{gr} \mathcal{D}$ module. The main result of this paper is

**Theorem 2** Suppose that $\Theta$ is non-singular and $c + \sum_{i \in I} a_i \notin \Gamma$ for any subset $I$ of $\{1, 2, ..., g - 1\}$. Then there exists a set of linear independent vector fields $D_i = \sum_{j=1}^g c_{ij} \partial_{x_j} \ 	ext{such that the following properties are valid.}$

(i) Let $\mathcal{D}_i = \mathcal{K}[D_1, ..., D_i]$. Then $M^k_c$ is a free $\mathcal{D}_{g-k}$ module of rank $g!$. More precisely it is of the form

$$M^k_c = \bigoplus_{i=1}^g \bigoplus_{j=1}^r \mathcal{D}_{g-k} \phi_{ij}^k, \quad \phi_{ij}^k \in M^k_c(i).$$

(ii) The module $\text{gr} \ M^k_c$ is a free $\text{gr} \mathcal{D}_{g-k}$ module of rank $g!$. More precisely it is of the form

$$\text{gr} \ M^k_c = \bigoplus_{i=1}^g \bigoplus_{j=1}^r (\text{gr} \mathcal{D}_{g-k}) \psi_{ij}^k, \quad \psi_{ij}^k \in \text{gr}_i M^k_c,$$

where $\psi_{ij}^k$ is the projection of $\phi_{ij}^k$ in $M^k_c$ and the filtration of $\mathcal{D}_{g-k}$ is specified by $\mathcal{D}_k(n) = \mathcal{D}_k \cap \mathcal{D}(n)$.

Let $\Phi^k = (\phi_{ij}^k)$ be the column vector and $A^{g-k}$ be the affine ring of $Y^k \setminus Q^k$. We have $\Phi^0 = \Phi$, $A^0 = A$. The ring $A^{g-k}$ acts on $M^k_c$. Thus for any $a \in A^{g-k}$ there exists a unique operator $\ell^k(a) \in \text{Mat}(g!, \mathcal{D}_{g-k})$ such that

$$a \Phi^k = \ell^k(a) \Phi^k.$$

It defines an embedding of $A^{g-k}$ in $\text{Mat}(g!, \mathcal{D}_{g-k})$. Since $M^k_c$ is a $\mathcal{D}$-module and $\mathcal{D} = \mathcal{D}_g$, for $D_i$ with $g - k + 1 \leq i \leq g$, there exists a unique operator $B^k_i \in \text{Mat}(g!, \mathcal{D}_{g-k})$ such that

$$D_i \Phi^k = B^k_i \Phi^k.$$ 

Those operators satisfy, for any $a, b, i, j$,

$$[\ell^k(a), \ell^k(b)] = 0,$$

$$[D_i - B^k_i, D_j - B^k_j] = 0,$$

$$[D_i - B^k_i, \ell^k(a)] = 0.$$
3 Combinatorial freeness

For a graded \( \mathcal{K} \)-vector space \( V = \bigoplus_{n \in \mathbb{Z}} V_n \) such that each \( V_n \) is finite dimensional we define the character \( \text{ch} V \) by

\[
\text{ch} V = \sum (\dim_\mathcal{K} V_n) t^n.
\]

Obviously

\[
\text{ch} \text{gr} D_j = \frac{1}{(1-t)^j}.
\]

We have

\[
\text{ch} \text{gr} M_c^0 = \sum_{n=1}^{\infty} (n^g - (n-1)^g) t^n = (1-t)^g \left( \frac{d}{dt} \right)^g (1-t)^{-1}.
\]

Let \( \{ \psi_{ij} \} \) be a \( \mathcal{D} \)-free basis of \( \text{gr} M_c \) as in Theorem 1 and \( F = \oplus \mathcal{K} \psi_{ij} \) the subspace of \( \text{gr} M_c \). By Theorem \( \square \) \( \text{gr} M_c \simeq (\text{gr} D_g) \otimes F \). The module \( F \) naturally inherits a grading from \( \text{gr} M_c \). Then

\[
\text{ch} \text{gr} M_c = (\text{ch} \text{gr} D_g) \cdot \text{gr} F = (1-t)^{-g} \sum_{i=1}^{g} r_i t^i.
\]

**Example** For \( g = 1, 2, 3, 4 \) \( \text{ch} \text{gr} M_c \) is given by

\[
\begin{align*}
\frac{t}{(1-t)^2}, & \quad \frac{t + t^2}{(1-t)^3}, & \quad \frac{t + 4t^2 + t^3}{(1-t)^4}, & \quad \frac{t + 11t^2 + 11t^3 + t^4}{(1-t)^5}.
\end{align*}
\]

**Lemma 4** Assume the same conditions for \( c, a_1, \ldots a_{g-1} \) as in Theorem 2. Then

(i) \( \dim_\mathcal{K} \text{gr}_n M_c^{j+1} = \dim_\mathcal{K} \text{gr}_n M_c^j - \dim_\mathcal{K} \text{gr}_{n-1} M_c^j \).

(ii) \( \text{ch} \text{gr} M_c^{j+1} = (1-t) \text{ch} \text{gr} M_c^j \).

**Proof.** The assertion (ii) follows from (i) and (i) follows from Lemma 2 and 3.

By Lemma 4 we have

\[
\text{ch} \text{gr} M_c^j = (1-t)^{-g+j} \sum_{i=1}^{g} a_i t^i = (\text{ch} D_{g-j}) \cdot \text{ch} F.
\]
4 Proof of Theorem 2

Notice that (i) of Theorem 2 follows from (ii) of Theorem 2. We shall prove

Proposition 1 Assume the same conditions as in Theorem 2. Set $y^{(0)} = (y_{1}^{(0)}, ..., y_{g}^{(0)}) = (x_{1}, ..., x_{g})$. Then, for each $k \geq 1$ there exist a linear change of the coordinates from $y^{(k-1)} = (y_{1}^{(k-1)}, ..., y_{g-k+1}^{(k-1)})$ to $y^{(k)} = (y_{1}^{(k)}, ..., y_{g-k+1}^{(k)})$ and $\psi_{ij}^{k} \in \text{gr}_i M_{c}^{k-1}$, $1 \leq i \leq g$, $1 \leq j \leq r_{i}$ such that the following properties hold. Let $D_{g-k} = K[\partial_{y_{1}^{(k)}}, ..., \partial_{y_{g-k}^{(k)}}]$, $D_{g-k}(n) = D_{g-k} \cap D(n)$ and $\xi^{(k)}_{i}$ the image of $\partial_{y_{i}^{(k)}}$ in $\text{gr}_{i}D$. Then

$$\xi_{i}^{(k)} = \sum_{j'} (\text{gr}_{j'}D_{g-k})\psi_{ij}',$$

$$\text{gr}_i M^{k} = \bigoplus_{i=1}^{g} \bigoplus_{j=1}^{r_{i}} (\text{gr}_{j}D_{g-k})\psi_{ij}.$$  

If we define, for $1 \leq k \leq g$, $D_{g-k+1} = \partial_{y_{g-k+1}}^{(k)}$, then Theorem 2 (ii) follows from this proposition.

We prove the proposition by induction on $k$, where the case of $k = 0$ is established by Theorem 1. We assume that the proposition is valid for $k$ if $c + \sum_{i \in I} a_{i} \notin \Gamma$ for any subset $I$ of $\{1, ..., k\}$. We shall prove that the proposition is true for $k + 1$ if $c + \sum_{i \in I} a_{i} \notin \Gamma$ for any subset $I$ of $\{1, ..., k+1\}$.

Let us set

$$\tilde{\psi}_{ij}^{k+1} = \psi_{ij}^{k} |_{y_{k+1}}.$$  

Lemma 5 For each $(ij)$ there exist a non-zero element $P_{ij} \in \text{gr}_{N^{ij}} D_{g-k}$ for some $N^{ij} \geq 0$ and a linear change of the coordinates from $(y_{1}^{(k)}, ..., y_{g-k}^{(k)})$ to $(y_{1}^{(k+1)}, ..., y_{g-k}^{(k+1)})$ such that the following properties hold.

(i) $P_{ij} \tilde{\psi}_{ij}^{k+1} = 0$ in $\text{gr} M_{c}^{k+1}$.

(ii) Let $\xi_{i} = \xi_{i}^{(k+1)}$. Then $P_{ij}$ is of the form

$$P_{ij} = \xi_{g-k}^{N_{ij}} + \sum_{|\alpha| = N^{ij}, \alpha_{g-k} < N^{ij}} a_{ij; \alpha} \xi_{1}^{\alpha_{1}} \cdots \xi_{g-k}^{\alpha_{g-k}},$$  

where $\alpha = (\alpha_{1}, ..., \alpha_{g-k})$, $|\alpha| = \sum_{i=1}^{g-k} \alpha_{i}$.

Proof. (i) By (7) the dimension of $\text{gr}_{n} M_{c}^{k+1}$ is a polynomial in $n$ of degree $g - k - 2$ for sufficiently large $n$. If there are no non-trivial linear relations among $\xi_{1}^{\alpha_{1}} \cdots \xi_{g-k}^{\alpha_{g-k}} \tilde{\psi}_{ij}^{k+1}$, $\sum_{i=1}^{g-k} \alpha_{i} = n - i$ for any $n$, then $\dim_{K} \left( (\text{gr}_{n-i} D_{g-k}) \tilde{\psi}_{ij}^{k+1} \right)$ is a polynomial in $n$ of degree $g - k - 1$ for sufficiently large $n$. Thus there should be a relation as in the assertion.
(ii) Let us write
\[ P_{ij} = \sum_{\alpha_1 = N^ij} q_{\alpha_1, \ldots, \alpha_{g-k}} \xi_{\alpha_1} \cdots \xi_{\alpha_{g-k}}. \]
Changing the name of the variables if necessary one can assume that \( q_{\alpha_1, \ldots, \alpha_{g-k}} \neq 0 \) for some \( (\alpha_1, \ldots, \alpha_{g-k}) \) with \( \alpha_{g-k} \neq 0 \). If \( q_{0, \ldots, 0, N^ij} \neq 0 \) then we get the desired element by dividing \( P_{ij} \) by \( q_{0, \ldots, 0, N^ij} \neq 0 \). If this is not the case, we consider the change of the variables of the form
\[ \xi_i^{(k)} = \sum_{l=1}^{g-k} c_{i,l} \xi_i^{(k+1)}. \]
Let \( c_i = c_{i,g-k} \). Then in the resulting expression of \( P_{ij} \) the coefficient of \((\xi_{g-k}^{(k+1)})^{N^ij}\) is
\[ \sum_{|\alpha| = N^ij} c_{i}^{\alpha_1} \cdots c_{g-k}^{\alpha_{g-k}} q_{\alpha_1, \ldots, \alpha_{g-k}}. \]  
This is a non-zero homogeneous polynomial in \( c_1, \ldots, c_{g-k} \). Thus it is non-zero on a non-empty open subset of \( C^{g-k} \). Take a point of it, make a change of the coordinates and dividing \( P_{ij} \) by (8) we get a desired result.

Let
\[ \text{gr} \pi_k : \text{gr} M_k \rightarrow \text{gr} M_{k+1} \]
be the restriction map induced by \( \pi_k \) and \( K^k = \oplus K_n^k \) the kernel of \( \text{gr} \pi_k \). Since \( \text{gr} \pi_k \) is a homomorphism of \( \text{gr} \mathcal{D} \)-modules, \( K^k \) is a \( \text{gr} \mathcal{D} \) submodule of \( \text{gr} M_k^k \). We denote by \( \tilde{K}^k = \oplus K_n^k \) the \( \text{gr} \mathcal{D} \) module obtained from \( K^k \) by shifting the grading by \(-1\), that is, \( \tilde{K}_n^k = K_n^{k+1} \).

**Lemma 6** The map
\[ \text{gr}_n M_{c+a_k+1} \rightarrow K_{n+1}^k, \]
\[ \phi(z) \mapsto \frac{\theta(z - a_k + 1)}{\theta(z)} \phi(z), \]
(9)
\[ \text{gives an isomorphism of} \ \text{gr} M_{c+a_k+1} \ \text{and} \ \tilde{K}^k \ \text{as gr} \mathcal{D}_{g-k} \text{-modules}. \]

**Proof.** We can assume \( g-k \geq 2 \). Using Lemma (i), (5) and the cohomology sequence of (4) we have
\[ \text{Ker}(\pi_k|_{M_k^k}) = \frac{\theta(z - a_k + 1)}{\theta(z)} |_{Y_k} M_{c+a_k+1}^k (n - 1). \]  
(10)
Let \( \text{gr}_n (\mathcal{L}_c(-mY^{k+1})|_{Y^k}) \) be the sheaf on \( Y^k \) defined by the exact sequence;
\[ 0 \rightarrow \mathcal{L}_c((n-1)Q^k - mY^{k+1}) \rightarrow \mathcal{L}_c(nQ^k - mY^{k+1}) \rightarrow \text{gr}_n (\mathcal{L}_c(-mY^{k+1})|_{Y^k}) \rightarrow 0. \]  
(11)
Then one can easily verify that the following is an exact sequence,

$$0 \longrightarrow \text{gr}_n (L_c(-Y^{k+1})|_{Y^k}) \longrightarrow \text{gr}_n (L_c|_{Y^k}) \longrightarrow \text{gr}_n (L_c|_{Y^{k+1}}) \longrightarrow 0. \quad (12)$$

By the cohomology sequence of (11) with $m = 0$ the natural map

$$\text{gr}_n L^k_c := L^k_c(n)/L^k_c(n-1) \longrightarrow H^0(Y^k, \text{gr}_n (L_c|_{Y^k})) \quad (13)$$

is always injective and becomes isomorphic if $g - k \geq 2$. Then we have

$$\text{Ker}(\text{gr}_n L^k_c \rightarrow \text{gr}_n L^{k+1}_c) \cong \frac{H^0(Y^k, \text{gr}_n (L_c(-Y^{k+1})|_{Y^k}))}{H^0(Y^k, L_c((nQ^k - Y^{k+1}))} \cong \frac{\theta(z - a_{k+1})}{\theta(z)} |_{Y^k \text{gr}_n L^k_c + 1} \in K^c_{i+1}.$$

where we use the cohomology sequences of (12), (11), Lemma 1 (i), (5), (10).

If $c + a_{k+1} + \sum_{i \in I} a_i \not\in \Gamma$ for any subset $I$ of $\{1, ..., k\}$ the induction hypothesis can be applied to $M^k_{c+a_{k+1}}$. By the lemma and the assumption of induction $K^k$ is described as

$$K^k = \bigoplus_{i=1}^g \bigoplus_{j=1}^{r_i} (\text{gr}D_{g-k})\varphi_{ij},$$

$$\varphi_{ij} = \frac{\theta(z - a_{k+1})}{\theta(z)} \psi^k_{ij}|_{c \rightarrow c + a_{k+1}} \in K^c_{i+1}.$$

Since $\varphi_{ij} \in \text{gr}_{i+1} M^k_c$, it can be written as a linear combination of $\{\psi^k_{ij}\}$ with the coefficients in $\text{gr}D_{g-k}$ as

$$\varphi_{ij} = \sum Q_{ij;i'j'} \psi^k_{i'j'}, \quad Q_{ij;i'j'} = \sum_{|a| + i' = i+1} q^\alpha_{ij;i'j'} \xi^\alpha. \quad (14)$$

By (i) of Lemma 5 we have $P_{ij} \psi^k_{ij} \in K^k_{i+N(i)}$. Thus it can be written as a linear combination of $\{\varphi_{ij}\}$ with the coefficients in $\text{gr}D_{g-k}$ as

$$P_{ij} \psi^k_{ij} = \sum R_{ij;i'j'} \varphi_{i'j'}, \quad R_{ij;i'j'} = \sum_{|a| + i' + 1 = i+N(i)} r^\alpha_{ij;i'j'} \xi^\alpha. \quad (15)$$

Composing these relations we get

$$P_{ij} \psi^k_{ij} = \sum R_{ij;i'j'} Q_{i'j';i''j''} \psi^k_{i''j''}. \quad (16)$$

In the matrix form it is written as

$$P = RQ,$$

where $P = (P_{ij})$ is the diagonal matrix and $R = (R_{ij;i'j'})$, $Q = (Q_{ij;i'j'})$ are $g! \times g!$ matrices.
We shall construct a basis \( \{ \tilde{\psi}^k_{ij} \} \) of \( \text{gr} M^k_c \) as a \( \text{gr} \mathcal{D}_{g-k} \)-module modifying \( \{ \psi^k_{ij} \} \) appropriately such that they satisfy

\[
\xi_{g-k} \tilde{\psi}^k_{ij} \in \sum_{i' \leq i+1} \mathcal{K}[\xi_1, ..., \xi_{g-k-1}] \tilde{\psi}^k_{im} + K^k. \tag{18}
\]

To this end we use the relation (14). Let us write it more explicitly as

\[
\varphi_{ij} = \sum q_{ij;i+1}^j \psi^k_{i+1,1}^j + \sum_{l=1}^{g-k} q_{ij;i+j}^l \xi_l \psi^k_{i+j} + \sum_{i' < i, \alpha+i' = i+1} q_{ij;i+j}^l \xi_\alpha \psi^k_{i+j}. \tag{19}
\]

For the sake of simplicity we identify \( q_{ij;j} \) with \( q_{0;0} \).

We construct \( \{ \tilde{\psi}^k_{ij} \} \) satisfying the property (18) by induction on \( i \).

Let us consider the case \( i = 1 \). Then the equation (19) becomes

\[
\varphi_{11} = \sum q_{11;2}^j \tilde{\psi}^k_{2j} + \sum_{l=1}^{g-k} q_{11;11}^l \xi_l \tilde{\psi}^k_{11}.
\]

We proceed by dividing the case.

(i) the case \( q_{11;11}^g = 0 \). In this case (18) holds for \( (ij) = (11) \) by defining \( \tilde{\psi}^k_{11} = \psi^k_{11} \) and \( \tilde{\psi}^k_{2j} = \psi^k_{2j} \).

(ii) the case \( q_{11;11}^g = 0 \) and \( q_{11;2j} = 0 \) for some \( j \). In this case we modify \( \psi^k_{2j} \) to \( \tilde{\psi}^k_{2j} = \psi^k_{2j} - \xi_{g-k} \tilde{\psi}^k_{11} \). Correspondingly the operators \( P_{ij} \) is changed. We can take the product \( P_{2j}P_{11} \) as the operator for \( \tilde{\psi}^k_{2j} \), since \( P_{2j}P_{11} \tilde{\psi}^k_{2j} = 0 \). We set \( \tilde{\psi}^k_{2j'} = \psi^k_{2j'} \) for \( j' \neq j \) and \( \tilde{\psi}^k_{11} = \psi^k_{11} \). Then (18) holds for \( (ij) = (11) \).

(iii) the case \( q_{11;11}^g = 0 \) and \( q_{11;2j} = 0 \) for all \( j \). This case is impossible. In fact, suppose that this is the case. We consider the equation (17) modulo the ideal

\[
I = \sum_{l=1}^{g-k-1} (\text{gr} \mathcal{D}_{g-k}) \xi_l.
\]

Then the right hand side is degenerate while the left hand side is non-degenerate.

As a whole we have constructed \( \tilde{\psi}^k_{ij} \), \( i \leq 2 \) such that (18) holds for \( (ij) = (11) \).

Assume that \( \tilde{\psi}^k_{ij} \), \( i' \leq i, 1 \leq j' \leq r' \), are constructed in such a way that (18) holds for \( (i', j') \), \( i' < i \). As a consequence of the change from \( \{ \psi^k_{ij} | i' \leq i \} \) to \( \{ \tilde{\psi}^k_{ij} | i' < i \} \) the matrices \( P, Q, R \) may be changed. However their properties that the equation (17) holds and \( P \) is non-degenerate remain valid. Therefore we use the same symbol \( P, Q, R \) and their components in the argument below for the sake of simplicity.

Using the relation (18) for \( \tilde{\psi}^k_{ij} \) for \( i' < i \), the equation (14) for \( \varphi_{ij} \) can be written as

\[
\tilde{\varphi}_{ij} = \sum q_{ij;i+1}^{j'} \tilde{\psi}^k_{i+1,j'} + \sum_{l=1}^{r} \sum_{j'=1}^{r} q_{ij;i+j'}^l \xi_l \tilde{\psi}^k_{ij'} + \sum_{i' < i, \alpha+i' = i+1, \alpha_g-k} q_{ij;i+j'}^l \xi_\alpha \tilde{\psi}^k_{ij'},
\]
for some $\tilde{\varphi}_{ij} \in R_{i+1}^k$ and some $\tilde{q}_{ij,j'}^i$, $i' \leq i$. Here $q_{ij,j'}^i$ changes to $\tilde{q}_{ij,j'}^i$ as a consequence of the use of (18). Notice that to use (18), as to the effect on the matrix $Q$, is to make fundamental transformations in rows of $Q$.

We have

$$\tilde{\varphi}_{ij} = \sum q_{iij+1j'}^k \psi_{i+1j'}^k + \sum_{j'=1}^{r_i} \tilde{q}_{ij,j'}^{g-k} \xi_{g-k} \psi_{ij'}^k \mod. I \operatorname{gr} M_c^k.$$ 

The rank of the $r_i \times (r_i + r_{i+1})$ matrix

$$\left((q_{iij+1j'})_{1 \leq j' \leq r_{i+1}}, (\tilde{q}_{ij,j'}^{g-k})_{1 \leq j' \leq r_i}\right)_{1 \leq j \leq r_i}$$

is maximal. For, otherwise it contradicts the non-degeneracy of the matrix $P$. Thus, as in the case of $i = 1$, modifying $\psi_{i+1j'}^k$ to $\tilde{\psi}_{i+1j'}^k = \psi_{i+1j'}^k - \xi_{g-k} \psi_{ij'}^k$, for some $j'$ and $j''$ if necessary, we get $\{\tilde{\psi}_{i,j'}^k | i' \leq i + 1\}$ such that (18) holds for $(i',j')$, $i' \leq i$. Notice that, in the last step $i = g$, $\tilde{q}_{g+1,i}^{g-k} \neq 0$ is automatic.

Thus a gr $D_{g-k}$-free basis $\{\tilde{\psi}_{ij}^k\}$ of gr $M_c^k$ satisfying the condition (18) is constructed. Set

$$\psi_{ij}^{k+1} = \tilde{\psi}_{ij}^k |_{Y_{k+1}}.$$

Then (18) implies that

$$\xi_{g-k} \psi_{ij}^{k+1} \in \sum \operatorname{gr} D_{g-k-1} \psi_{ij}^{k+1}. \quad (20)$$

**Lemma 7** If $c + \sum_{i \in I} a_i \notin \Gamma$ for any subset $I$ of $\{1, \ldots, k\}$, the restriction map gr $\pi_k$ is surjective.

**Proof.** The lemma can be proved by a similar argument to the proof of Lemma 6.

By the lemma and the assumption of induction on gr $M_c^k$, gr $M_c^{k+1}$ is generated by $\{\psi_{ij}^{k+1}\}$ over gr $D_{g-k}$. Then (20) implies that

$$\operatorname{gr} M_c^{k+1} = \sum (\operatorname{gr} D_{g-k-1}) \psi_{ij}^{k+1}.$$ 

It follows from (7) that the sum of the right hand side is a direct sum. This completes the proof of Proposition 1.

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