Dimension free convergence rates for Gibbs samplers for Bayesian linear mixed models

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Abstract

The emergence of big data has led to a growing interest in so-called convergence complexity analysis, which is the study of how the convergence rate of a Monte Carlo Markov chain (for an intractable Bayesian posterior distribution) scales as the underlying data set grows in size. Convergence complexity analysis of practical Monte Carlo Markov chains on continuous state spaces is quite challenging, and there have been very few successful analyses of such chains. One fruitful analysis was recently presented by Qin and Hobert (2021b), who studied a Gibbs sampler for a simple Bayesian random effects model. These authors showed that, under regularity conditions, the geometric convergence rate of this Gibbs sampler converges to zero (indicating immediate convergence) as the data set grows in size. It is shown herein that similar behavior is exhibited by Gibbs samplers for more general Bayesian models that possess both random effects and traditional continuous covariates, the so-called mixed models. The analysis employs the Wasserstein-based techniques introduced by Qin and Hobert (2021b).

1 Introduction

Markov chain Monte Carlo (MCMC) methods have revolutionized the exploration of intractable probability distributions (see, e.g., Diaconis, 2009). This has affected many areas of science, but none more than Bayesian statistics. Indeed, MCMC allows for the exploration of intractable high dimensional posterior distributions that would have remained virtually impenetrable without these

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tools. The revolution has not come without a cost, however. Employing an MCMC algorithm in a principled manner requires information about the rate at which the underlying Markov chain converges to its target distribution (Flegal et al., 2008), and such information is notoriously difficult to ascertain. Substantial progress has been made over the last several decades, both on the development of general techniques to get bounds on convergence rates (mostly based on drift and minorization conditions), and on the application of said techniques to specific Monte Carlo Markov chains. Unfortunately, most of these results are not sharp enough to be useful in the emerging area of convergence complexity analysis, which is the (big data driven) study of how the convergence rate of a Monte Carlo Markov chain for a Bayesian problem scales with the size of the underlying data set (Rajaratnam and Sparks, 2015; Qin and Hobert, 2021). In fact, it has become clear that convergence complexity analysis requires new theory (see, e.g., Durmus and Moulines, 2015; Hairer et al., 2011; Qin and Hobert, 2021; Yang and Rosenthal, 2019). Qin and Hobert (2021) provide a new technique that entails bounding the Wasserstein distance to stationarity, and then converting the Wasserstein bounds into total variation bounds. These authors apply their method to a Gibbs sampler for a simple Bayesian random effects model and show that, under regularity conditions, the geometric convergence rate converges to zero (indicating immediate convergence) as the data set grows in size. This is, of course, even better than dimension free. In this paper, we show that similar behavior is exhibited by Gibbs samplers for more general Bayesian models that possess both random effects and traditional continuous covariates, the so-called mixed models. We now provide a detailed description of the Gibbs samplers that are the focus of our study.

Consider the linear mixed effects model given by

$$ y_{ij} = x_{ij}^T \beta + \eta_i + e_{ij}, $$

where $i = 1, \ldots, q$, $j = 1, \ldots, r_i$, $x_{ij}$ is a $p \times 1$ vector of known covariates associated with $y_{ij}$, $\beta$ is an unknown $p$-dimensional regression parameter, the $\eta_i$ are iid $N(\mu, \lambda^{-1})$, the $e_{ij}$ are iid $N(0, \tau^{-1})$ and independent of the $\eta_i$. A Bayesian version of this model is formed by placing prior distributions on the unknown parameters $\beta$, $\mu$, $\lambda$, and $\tau$. In the model we consider, the four parameters are assumed a priori independent, and we put flat priors on $\beta$ and $\mu$, and proper gamma priors on $\lambda$ and $\tau$. In particular, our prior density is proportional to

$$ \lambda^{a_1-1} e^{-b_1 \lambda} I_{\mathbb{R}^+}(\lambda) \tau^{a_2-1} e^{-b_2 \tau} I_{\mathbb{R}_+}(\tau), $$

where $a_1$, $a_2$, $b_1$, and $b_2$ are strictly positive hyperparameters and $\mathbb{R}_+ := (0, \infty)$. (Assume for the time being that the resulting posterior distribution is proper.) Denote the posterior density by
is the observed data, and, as usual, the random effects are included because they are unobserved. This posterior density, which has dimension $q+p+3$, is highly intractable. However, because the prior density is conditionally conjugate, there is a simple (two-block) Gibbs sampler that can be used to explore $\pi^*$. It turns out to be more convenient to work with a very simple transformation of $\pi^*$. Indeed, let $\eta_00 = \sqrt{q}\beta$ and $\eta_0 = \sqrt{q}\mu$, and define $\eta = (\eta_{00}, \eta_0, \eta_1, \ldots, \eta_q)^\top$. Denote the new posterior density by $\pi(\eta, \lambda, \tau \mid y)$. In this paper, we analyze the Markov chain $\Gamma = \{\eta(n)\}_{n=0}^\infty$ whose Markov transition density $k : \mathbb{R}^{q+p+1} \times \mathbb{R}^{q+p+1} \to (0, \infty)$ is given by

$$k(\eta, \tilde{\eta}) = \int_0^\infty \int_0^\infty \pi_1(\tilde{\eta} \mid \lambda, \tau, y) \pi_2(\lambda, \tau \mid \eta, y) \, d\lambda \, d\tau,$$

where $\pi_1(\eta \mid \lambda, \tau, y)$ and $\pi_2(\lambda, \tau \mid \eta, y)$ denote conditional densities associated with $\pi$. Of course, $\Gamma$ is the $\eta$-marginal chain of the two-block Gibbs sampler that alternates between $(\lambda, \tau)$ and $\eta$. It’s easy to see that $\Gamma$ is irreducible, aperiodic, and positive Harris recurrent with invariant density $\pi(\eta \mid y) := \int_0^\infty \int_0^\infty \pi(\eta, \lambda, \tau \mid y) \, d\lambda \, d\tau$. Furthermore, it is well known that, for two-block Gibbs samplers, the marginal chains have the same convergence rate as the underlying two-block Gibbs sampler (see, e.g., Roberts and Rosenthal, 2001; Diaconis et al., 2008). Thus, in order to learn about the convergence complexity of the two-block Gibbs sampler, it suffices to study the marginal chain $\Gamma$. It would clearly also suffice to study the other marginal chain whose Markov transition density $k' : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to (0, \infty)$ is given by

$$k'((\lambda, \tau), (\tilde{\lambda}, \tilde{\tau})) = \int_{\mathbb{R}_+^{q+p+1}} \pi_2(\tilde{\lambda}, \tilde{\tau} \mid \eta, y) \pi_1(\eta \mid \lambda, \tau, y) \, d\eta.$$

However, despite the fact that this chain’s dimension remains constant (at 2) as $p$ and $q$ increase, $\Gamma$ is more amenable to analysis.

Drawing from $\lambda, \tau \mid \eta, y$ is simple since, conditional on $(\eta, y)$, $\lambda$ and $\tau$ are independent, and each has a gamma distribution. Making draws from $\eta \mid \lambda, \tau, y$ is more complicated, but still straightforward. Our strategy is to sample sequentially. We first draw from $\eta_{00} \mid \lambda, \tau, y$, and then from $\eta_0 \mid \eta_{00}, \lambda, \tau, y$, and finally from $\eta_1, \ldots, \eta_q \mid \eta_0, \eta_{00}, \lambda, \tau, y$. All three of these conditional distributions are normal distributions, and the last of the three steps is easy since $\{\eta_i\}_{i=1}^q$ are independent given $(\eta_0, \eta_{00}, \lambda, \tau, y)$. The distributions of $\eta_{00} \mid \lambda, \tau, y$ and $\eta_0 \mid \eta_{00}, \lambda, \tau, y$ are derived in Appendix A.

In order to provide a precise statement of the algorithm, we must introduce a bit of notation. Let $N = \sum_{i=1}^q r_i$ (total sample size), and let $X$ denote what is traditionally called the design matrix, i.e., $X$ is the $N \times p$ matrix given by $X = (x_{11}, \ldots, x_{qr})^\top$. Let $\bar{X} = (\bar{x}_11_{r_1}, \ldots, \bar{x}_q1_{r_q})^\top$, where $\bar{x}_i = r_i^{-1} \sum_{j=1}^{r_i} x_{ij}$, and $1_{r_i}$ is an $r_i \times 1$ column vector of 1s. Note that $\bar{X}$ is also $N \times p$. Let
\[ \bar{Y} = (\bar{y}_1 1^\top, \ldots, \bar{y}_q 1^\top) \top, \text{ where } \bar{y}_i = r_i^{-1} \sum_{j=1}^{r_i} y_{ij}. \] Thus, we have

\[
X = \begin{bmatrix}
  x_{11}^\top \\
  \vdots \\
  x_{i1}^\top \\
  \vdots \\
  x_{q1}^\top \\
  \vdots \\
  x_{qr_q}^\top
\end{bmatrix}, \quad \tilde{X} = \begin{bmatrix}
  \tilde{x}_1^\top \\
  \vdots \\
  \tilde{x}_i^\top \\
  \vdots \\
  \tilde{x}_q^\top \\
  \vdots \\
  \tilde{x}_{r_q}^\top
\end{bmatrix}, \quad \text{and} \quad \bar{Y} = \begin{bmatrix}
  \bar{y}_1^\top \\
  \vdots \\
  \bar{y}_i^\top \\
  \vdots \\
  \bar{y}_q^\top \\
  \vdots
\end{bmatrix}.
\]

Let \( c_i = r_i \tau / (\lambda + r_i \tau) \), \( D_c = \oplus_{i=1}^q c_i I_{r_i} \), and

\[
M = D_c + \frac{(I - D_c) 11^\top (I - D_c)}{11^\top (I - D_c) 1}.
\]

(When we use 1 with no subscript, it is understood to mean \( 1_N \).) Finally, let \( Q = (X^\top X - X^\top M X)^{-1}, v = \sqrt{q} Q (X Y - X^\top M Y), t_i = \lambda + r_i \tau \), and \( z_i = t_i / (r_i \lambda \tau) \). We now state the algorithm for simulating \( \Gamma \). If the current state of the chain is \( \eta^{(n)} = \eta = (\eta_0, \eta_0, \eta_1, \ldots, \eta_q)^\top \), then we simulate the next state, \( \eta^{(n+1)} \) using the following procedure.

\begin{enumerate}
  \item Draw \( \lambda \sim \text{Gamma} \left( \frac{q}{2} + a_1, b_1 + \frac{1}{2} \sum_{i=1}^q (\eta_i - \eta_0 / \sqrt{q})^2 \right) \).
  \item Draw \( \tau \sim \text{Gamma} \left( \frac{N}{2} + a_2, b_2 + \frac{1}{2} \sum_{i=1}^q \sum_{j=1}^{r_i} (y_{ij} - x_{ij}^\top \eta_{00} / \sqrt{q} - \eta_i / \sqrt{q})^2 \right) \).
  \item Draw \( \eta_{00}^{(n+1)} \sim N_p \left( v, \frac{2}{\tau} Q \right) \).
  \item Draw \( \eta_0^{(n+1)} \sim N \left( \sqrt{q} \sum_{i=1}^q (\bar{y}_i - \bar{x}_i^\top \eta_{00}^{(n+1)} / \sqrt{q}) / z_i, \frac{q}{\sum_{i=1}^q 1/z_i} \right) \).
  \item For \( i = 1, 2, \ldots, q \), draw \( \eta_i^{(n+1)} \sim N \left( \frac{\lambda \eta_i^{(n+1)}}{\sqrt{q} z_i} + \frac{r_i \tau}{z_i} (\bar{y}_i - \bar{x}_i^\top \eta_{00}^{(n+1)} / \sqrt{q}), \frac{1}{z_i} \right) \).
\end{enumerate}

Steps 1 and 2 simulate from \( \pi_2(\lambda, \tau \mid \eta, y) \), while steps 3, 4, and 5 simulate from \( \pi_1(\eta \mid \lambda, \tau, y) \).

Let \( k^{(n)}(\eta, \cdot) \) denote the \( n \)-step Markov transition density (Mtd); i.e., the density of \( \eta^{(n)} \) given that \( \eta^{(0)} = \eta \). The chain \( \Gamma \) is said to be geometrically ergodic if there exist a function \( M : \mathbb{R}^{q+p+1} \rightarrow [0, \infty) \) and a constant \( \rho \in [0, 1) \) such that for all \( n \) and all \( \eta \), we have

\[
\frac{1}{2} \int_{\mathbb{R}^{q+p+1}} |k^{(n)}(\eta, \bar{\eta}) - \pi(\bar{\eta} \mid y)| d\bar{\eta} \leq M(\eta) \rho^n. \tag{2}
\]
The left-hand side of (2) is the total variation distance between the distribution of $\eta^{(n)}$ (given that $\eta^{(0)} = \eta$) and the invariant distribution. A more general development of these concepts is provided in Section 2. The geometric rate of convergence, $\rho_*$, is defined to be the smallest $\rho \in [0, 1]$ that satisfies (2) with some $M(\cdot)$. Results in Román and Hobert (2012) imply that, under mild regularity conditions, $\Gamma$ is indeed geometrically ergodic, but their result is qualitative in the sense that they do not provide an explicit upper bound for $\rho_*$. Our interest here centers on the deeper issue of convergence complexity, i.e., on an understanding of how $\rho_*$ scales as the underlying data set increases in size. More specifically, it is clear that $\rho_*$ depends on the underlying data, which is comprised of $y$, $X$, $q$, $p$, and the $r_i$s. We consider a sequence of data sets of increasing size, and show that, under regularity conditions concerning the relative rates at which $q$, $p$, and the $r_i$s grow, not only is $\rho_*$ bounded away from unity as the data set grows, but it actually converges to 0. Keep in mind that a convergence rate of 0 corresponds to immediate convergence, as if making iid draws from the target. Our main result (Proposition 3 in Section 3) is a generalization of Proposition 25 of Qin and Hobert’s (2021b), who consider the Gibbs sampler for a Bayesian random effects model that is a simplified version of our mixed model in which there are no covariates, and the data set is balanced, i.e., $r_i \equiv r$. Our proofs are similar in structure to those of Qin and Hobert (2021b), but there are substantial differences due to the fact that our mixed model is markedly more complex than the model that they considered.

The remainder of the paper is organized as follows. Section 2 contains general background on Markov chain convergence, as well as statements of results from Qin and Hobert (2021b) and Madras and Sezer (2010), which we use to analyze $\Gamma$. Section 3 contains a statement and proof of the main result. The proof relies on three preparatory results, also stated in Section 3, that are proven in subsequent sections. In particular, Section 4 contains a proof of Proposition 4, which allows us to convert Wasserstein bounds into total variation bounds. Sections 5 and 6 contain proofs of Proposition 5 (a drift condition) and Proposition 6 (a contraction condition), respectively. Proofs of some technical lemmas are relegated to the Appendix.

2 Markov Chain Background

Let $X \subset \mathbb{R}^d$ and let $\mathcal{B}$ denote its Borel $\sigma$-algebra. Suppose that $K : X \times \mathcal{B} \rightarrow [0, 1]$ is a Markov transition kernel (Mtk). For any $n \in \mathbb{N} := \{1, 2, 3, \ldots \}$, let $K^n$ be the $n$-step transition kernel, so that $K^1 = K$. For any probability measure $\nu : \mathcal{B} \rightarrow [0, 1]$, denote $\int_X \nu(dx)K^n(x, \cdot) \text{ by } \nu K^n(\cdot)$. If $\delta_x$ denotes a point mass at $x$, then $\delta_xK^n(\cdot) = K^n(x, \cdot)$, and we will abbreviate this to $K^n_x(\cdot)$. Assume
that the Markov chain corresponding to $K$ is irreducible, aperiodic, and positive Harris recurrent (see, e.g., Meyn and Tweedie, 2009), and let $\Pi$ denote its unique invariant probability distribution. The goal of convergence analysis is to understand how quickly $\nu K^n$ converges to $\Pi$ as $n \to \infty$ for a large class of $\nu$s. The difference between $\nu K^n$ and $\Pi$ is usually measured using the total variation (TV) distance, which is now defined. For two probability measures on $(X, B)$, $\nu$ and $\phi$, the TV distance between them is

$$d_{TV}(\nu, \phi) = \sup_{A \in B} [\nu(A) - \phi(A)].$$

(If $\nu$ and $\phi$ each have a density with respect to a common measure, then TV distance can be computed by integrating the absolute difference between the densities - see, e.g., [2].) The Markov chain defined by $K$ is geometrically ergodic if there exist $\rho < 1$ and $M : X \to [0, \infty)$ such that, for each $x \in X$ and $n \in \mathbb{N}$,

$$d_{TV}(K^n x, \Pi) \leq M(x) \rho^n. \quad (3)$$

Define the geometric convergence rate of the chain as

$$\rho_* = \inf \{ \rho \in [0, 1] : (3) \text{ holds for some } M : X \to [0, \infty) \}.$$

The chain is geometrically ergodic if and only if $\rho_* < 1$.

The standard technique for developing upper bounds on $\rho_*$ requires the construction of drift and minorization (d&m) conditions for the Markov chain under study (Rosenthal, 1995; Roberts and Rosenthal, 2004; Baxendale, 2005). Unfortunately, the d&m-based methods are often overly conservative, especially in high-dimensional situations (see, e.g., Rajaratnam and Sparks, 2015; Qin and Hobert, 2021a), and there is mounting evidence suggesting that convergence complexity analysis becomes more tractable when TV distance is replaced with Wasserstein distance (see, e.g., Hairer et al., 2011; Durmus and Moulines, 2015; Qin and Hobert, 2021b). In the remainder of this section, we describe a method of bounding $\rho_*$ indirectly using Wasserstein distance.

Let $\psi(\cdot, \cdot)$ denote the usual Euclidean distance on $\mathbb{R}^d$, i.e., $\psi(x, y) = \|x - y\|$, and assume that $(X, \psi)$ is a Polish metric space. For two probability measures on $(X, B)$, $\nu$ and $\phi$, their Wasserstein distance is defined as

$$d_W(\nu, \phi) = \inf_{\xi \in \tau(\nu, \phi)} \int_{X \times X} \|x - y\| \xi(dx, dy),$$

where $\tau(\nu, \phi)$ is the set of all couplings of $\nu$ and $\phi$; i.e., the set of all probability measures $\xi(\cdot, \cdot)$ on $(X \times X, B \times B)$ having marginals $\nu$ and $\phi$. Here is a result that provides a connection between Wasserstein distance and TV distance.
Theorem 1 (Madras and Sezer (2010)). Assume that $K_x(\cdot)$ has a density $k(x,\cdot)$ with respect to some dominating measure $\mu$ for all $x \in X$. If there exists a constant $C < \infty$ such that, for all $x, y \in X$,

$$\int_X |k(x, z) - k(y, z)| \mu(dz) \leq C \|x - y\|,$$

then, for all $n \in \{2, 3, 4, \ldots\}$, we have

$$d_{TV}(K^n_x, \Pi) \leq \frac{C}{2} d_W(K^{n-1}_x, \Pi).$$

Suppose that the hypothesis of Theorem 1 holds, and that the Markov chain driven by $K$ is geometrically ergodic with respect to Wasserstein distance, i.e., we have $\gamma \in [0, 1)$ and $M : X \to [0, \infty)$ such that $d_W(K^n_x, \Pi) \leq M(x) \gamma^n$ for all $x \in X$ and all $m \in \mathbb{N}$. Then it follows immediately from Theorem 1 that $\rho_* \leq \gamma$.

One way to bound Wasserstein distance is through coupling, and coupling is often achieved via random mappings, which we now describe. On a probability space $(\Omega, \mathcal{F}, P)$, let $\theta : \Omega \to \Theta$ be a random element, and let $\bar{f} : X \times \Theta \to X$ be a Borel measurable function. Define $f(x) = \bar{f}(x, \theta)$ for all $x \in X$. Then $f$ is called a random mapping on $X$. The evolution of a Markov chain can often be viewed as being driven by a random mapping. If $f(x) \sim K_x(\cdot)$ for all $x \in X$, then we say that $f$ induces $K$. For example, suppose that $X = \mathbb{R}$ and $K(x, dy) = (2\pi)^{-1/2} \exp\{- (y - x)^2 / 2\} dy$. Let $Z$ be standard normal, and define $\bar{f}(x, Z) = x/2 + Z$. Then the random mapping $f(x) = x/2 + Z$ induces $K$. A random mapping $f$ is called differentiable if, with probability 1, for each $x, y \in X$, $\frac{df(x + t(y - x))}{dt}$, as a function of $t \in [0, 1]$, exists and is integrable (Qin and Hobert, 2021b). The following result provides a constructive method of forming a bound on the Wasserstein distance to stationarity.

Theorem 2 (Qin & Hobert, 2021b). Suppose that $X$ is a convex subset of $\mathbb{R}^d$, and assume that the following three conditions hold.

(A1) There exist $c \in (0, \infty)$, $\zeta \in [0, 1)$, $L \in [0, \infty)$, and a function $V : X \to [0, \infty)$ such that

$$\int_X V(x') K(x, dx') \leq \zeta V(x) + L \quad (4)$$

for each $x \in X$, and

$$c^{-1}\|x - y\| \leq V(x) + V(y) + 1 \quad (5)$$

for each $(x, y) \in X \times X$. 

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(A2) There exist a differentiable random mapping $f$ that induces $K$, some $\xi > 2L/(1 - \zeta)$, $\gamma < 1$ and $\gamma_0 < \infty$ such that

$$\sup_{s \in [0,1]} E \left\| \frac{d}{ds} f(x + s(y - x)) \right\| \leq \begin{cases} \gamma \|x - y\| & \text{if } (x, y) \in C \\ \gamma_0 \|x - y\| & \text{otherwise} \end{cases},$$

where $C = \{(x, y) \in X \times X : V(x) + V(y) \leq \xi\}$.

(A3) Either $\gamma_0 \leq 1$ or

$$\frac{\log(2L + 1)}{\log(2L + 1) - \log(\gamma)} < \frac{-\log \left[ (\zeta \xi + 2L + 1)/(\xi + 1) \right]}{\log(\gamma_0) - \log \left[ (\zeta \xi + 2L + 1)/(\xi + 1) \right]}.$$

Then for each $x \in X$, $n \in \mathbb{N}$, and any real number $a$ such that

$$\frac{\log(2L + 1)}{\log(2L + 1) - \log(\gamma)} < a < \frac{-\log \left[ (\zeta \xi + 2L + 1)/(\xi + 1) \right]}{\log(\gamma_0 \vee 1) - \log \left[ (\zeta \xi + 2L + 1)/(\xi + 1) \right]},$$

we have

$$d_W(K^n_x, \Pi) \leq c \left( \frac{(\zeta + 1)V(x) + L + 1}{1 - \rho} \right) \rho^n_a,$$

where

$$\rho_a = \left[ \gamma_a (2L + 1)^{1-a} \right] \vee \left[ \gamma_0 \left( \frac{\zeta \xi + 2L + 1}{\xi + 1} \right)^{1-a} \right] < 1.$$

We refer to conditions (A1) and (A2) as the drift and contraction conditions, respectively. In the next section, we apply Theorems 1 and 2 to $\Gamma$.

3 Main Result

In order to state our main result, we must introduce a few definitions and some notation. Let $K(\eta, \cdot)$ denote the Mtk corresponding to the Mtd $k(\eta, \cdot)$ defined at (1), and let $K^n(\cdot)$ denote the probability measure associated with the $n$-step Mtd $k^{(n)}(\eta, \cdot)$. So, for a Borel set $B \in \mathbb{R}^{p+q+1}$, $K(\eta, B) = \int_B k(\eta, \tilde{\eta}) \, d\tilde{\eta}$ and $K^n(\cdot) = \int_B k^{(n)}(\eta, \tilde{\eta}) \, d\tilde{\eta}$. Also let $\Pi(\cdot)$ denote the invariant probability measure of $\Gamma$, i.e., $\Pi(\cdot)$ is the probability measure corresponding to the marginal posterior density $\pi(\eta | y)$. (The conditions that we will impose in our main result imply posterior propriety.) Define $V : \mathbb{R}^{p+q+1} \to [0, \infty)$ as follows:

$$V(\eta) = \frac{1}{r^q} \sum_{i=1}^q \sum_{j=1}^{r_i} \left( \bar{y}_i - x_{ij}^t \eta_{00} / \sqrt{q} \right)^2 + \frac{\eta_0^2}{q} + \frac{1}{r^q} \sum_{i=1}^q r_i (\eta_i + \bar{y} - \bar{y}_i)^2,$$

where $\bar{y} = q^{-1} \sum_{i=1}^q \bar{y}_i$. This will serve as our drift function.
For two non-negative functions $g(z)$ and $h(z)$, we write $g(z) = \mathcal{O}(h(z))$ if there exist positive numbers $c$ and $z_0$ such that $g(z) \leq ch(z)$ for all $z \geq z_0$. We write $g(z) = \Theta(h(z))$ if $g(z) = \mathcal{O}(h(z))$ and $h(z) = \mathcal{O}(g(z))$. Let $r_{\text{max}} = \max\{r_1, \ldots, r_q\}$ and $r_{\text{min}} = \min\{r_1, \ldots, r_q\}$. Let $\bar{r} = q^{-1} \sum_{i=1}^{q} r_i$, so that $N = q\bar{r}$. Let $\lambda_{\min}(\cdot)$ denote the smallest eigenvalue of the (square symmetric matrix) argument, and define $\lambda_{\max}(\cdot)$ analogously.

We envision a sequence of growing data sets with $q$, $p = p(q)$, and $r_{\text{min}} = r_{\text{min}}(q)$ all diverging. Our basic assumptions about the manner in which the data set grows are as follows.

(B1) There exists a positive constant $m$, not depending on $q$, such that for all $q$,

$$\frac{r_{\text{max}}}{r_{\text{min}}} \leq m.$$  
(B2) There exist positive constants $k_1$ and $k_2$, not depending on $q$, such that for all large $q$,

$$k_1 \leq \lambda_{\min}\left[\frac{1}{r q} (X^\top X - \bar{X}^\top \bar{X})\right] \leq \lambda_{\max}\left(\frac{1}{r q} X^\top X\right) \leq k_2.$$  
(B3) There exists a positive constant $\ell$, not depending on $q$, such that for all $q$,

$$\frac{1}{r q} \sum_{i=1}^{q} \sum_{j=1}^{r_i} y_{ij}^2 \leq \ell.$$  
(B4) $p = \mathcal{O}(q)$.
(B5) There exists a positive constant $\delta$, not depending on $q$, such that

$$\frac{\bar{r}}{q^{2+\delta}} \to \infty$$

as $q \to \infty$.

Here is our main result:

**Proposition 3.** Under conditions (B1) – (B5), there exist positive constants $C$, $C'$ and $L_0$ (all independent of $q$), a rate $\rho = \rho(q)$ satisfying $\rho(q) \to 0$ as $q \to \infty$, and a positive integer $q_0$ such that, for all $q \geq q_0$, all $\eta \in \mathbb{R}^{p+q+1}$, and all $n \in \{2, 3, 4, \ldots \}$,

$$d_{TV}(K^n_\eta, \Pi) \leq C q \bar{r}^{3/2} d_W(K^{n-1}_\eta, \Pi) \leq C' q^2 \bar{r}^{3/2} (V(\eta) + L_0) \rho^{n-1}.$$  

We will prove Proposition 3 using Theorems 1 and 2. We begin by stating three standalone results (each proved in a subsequent section) that we will use in the proof. The first allows for conversion (Theorem 1), and the other two concern drift and contraction (Theorem 2). Here is the conversion condition, which is established using Theorem 1 in Section 4.
Proposition 4. Under conditions \((B_1)\) and \((B_2)\), there exist a constant \(C > 0\) and a positive integer \(q_0\) such that, for all \(q \geq q_0\),
\[
d_{TV}(K^n_\eta, \Pi) \leq C q r^{3/2} d_W(K^{n-1}_\eta, \Pi)
\]
for all \(\eta \in \mathbb{R}^{p+q+1}\) and all \(n \in \{2, 3, 4, \ldots \}\).

Here is the drift condition, which is established in Section 5.

Proposition 5. Under conditions \((B_1)-(B_5)\), there exist \(\zeta = \zeta(q) = \mathcal{O}(q^{-1})\), \(L = L(q) = \mathcal{O}(1)\), and a positive integer \(q_0\) such that for all \(q \geq q_0\) and all \(\eta \in \mathbb{R}^{p+q+1}\), we have
\[
\int_{\mathbb{R}^{p+q+1}} V(\tilde{\eta}) K(\eta, d\tilde{\eta}) \leq \zeta V(\eta) + L.
\]

The contraction condition involves a random mapping that induces \(\Gamma\), and we now describe this mapping. (We note that this mapping is also exploited in the proof of Proposition 5.) Let \(N_{00} \sim N_p(0, I)\), \(N_i \sim N(0, 1)\), \(i = 0, \ldots, q\), \(J_1 \sim \text{Gamma}(q/2 + a_1, 1)\), and \(J_2 \sim \text{Gamma}(N/2 + a_2, 1)\), and assume these are all pairwise independent. Denote the current state of \(\Gamma\) as \(\eta = (\eta_0^T, \eta_0, \eta_1, \ldots, \eta_q)^T\). Then the next state \(\tilde{\eta} = (\tilde{\eta}_0^T, \tilde{\eta}_0, \tilde{\eta}_1, \ldots, \tilde{\eta}_p)^T\) can be expressed as the following random mapping:
\[
f(\eta) = \tilde{\eta}(\eta) = \begin{bmatrix}
\tilde{z}_0(\eta) \\
\tilde{\eta}_0 \\
\tilde{z}_1(\eta) \\
\vdots \\
\tilde{z}_q(\eta) \\
\tilde{\eta}_q
\end{bmatrix}
\]
where
\[
\tilde{z}_0(\eta) = z_0(\eta) + \sqrt{\frac{q}{\tau(\eta)}} (Q(\eta)_{00}^{1/2} N_{00})
\]
\[
\tilde{z}_i(\eta) = \sqrt{\frac{q}{\sum_{i=1}^{q} 1/z_i(\eta)}} (\bar{y}_i - \bar{x}_i \tilde{z}_0(\eta)/\sqrt{q} / z_i(\eta) + \sqrt{\frac{q}{\sum_{i=1}^{q} 1/z_i(\eta)}} N_0)
\]
\[
\tilde{z}_i(\eta) = \frac{\lambda_i(\eta)}{\tilde{t}_i(\eta)} \tilde{z}_0(\eta)/\sqrt{q} + \frac{r_i(\eta)}{\tilde{t}_i(\eta)} (\bar{y}_i - \bar{x}_i \tilde{z}_0(\eta)/\sqrt{q}) + \sqrt{\frac{1}{\tau_i(\eta)}} N_i, \quad i = 1, 2, \ldots, q.
\]
and

\[
\lambda^{(\eta)} = \frac{\lambda_1}{b_1 + \frac{1}{2} \sum_{i=1}^{q} (\eta_i - \eta_0/\sqrt{q})^2}, \\
\tau^{(\eta)} = \frac{\lambda_2}{b_2 + \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{r_i} (y_{ij} - x_{ij}^0/\sqrt{q} - \eta_i)^2}, \\
t_i^{(\eta)} = r_i \tau^{(\eta)} + \lambda^{(\eta)}, \quad i = 1, 2, \ldots, q \\
z_i^{(\eta)} = \frac{t_i^{(\eta)}}{r_i \tau^{(\eta)}}, \quad i = 1, 2, \ldots, q \\
D_c^{(\eta)} = \bigoplus_{i=1}^{q} \frac{r_i \tau^{(\eta)}}{t_i^{(\eta)}} I_i \\
M^{(\eta)} = D_c^{(\eta)} + \frac{(I - D_c^{(\eta)}) 1 1^\top (I - D_c^{(\eta)})}{1^\top (I - D_c^{(\eta)}) 1} \\
Q^{(\eta)} = (X^\top X - \bar{X}^\top M^{(\eta)} \bar{X})^{-1} \\
v^{(\eta)} = \sqrt{q} Q^{(\eta)} (X^\top Y - \bar{X}^\top M^{(\eta)} \bar{Y}).
\]

Here is the contraction condition, which is established in Section 6.

**Proposition 6.** Assume that \((B_1)-(B_5)\) hold, and define

\[
C = \{(\eta, \eta') \in \mathbb{R}^{p+q+1} \times \mathbb{R}^{p+q+1} : V(\eta) + V(\eta') \leq q^{\delta/3}\},
\]

where \(V(\cdot)\) is the drift function defined in \((6)\), and \(\delta\) is given in \((B_5)\). Let \(f\) be the random mapping defined above. There exist

\[
\gamma = \gamma(q) = O\left(\sqrt{q^{2+\delta}/r} \lor \frac{1}{\sqrt{q}}\right) \quad \text{and} \quad \gamma_0 = \gamma_0(q) = O(\sqrt{q})
\]

and a positive integer \(q_0\) such that for all \(q \geq q_0\), we have

\[
\sup_{s \in [0,1]} E \left\| \frac{d}{ds} f(\eta + s(\eta' - \eta)) \right\| \leq \begin{cases} 
\gamma \|\eta' - \eta\| & (\eta, \eta') \in C \\
\gamma_0 \|\eta' - \eta\| & \text{otherwise}.
\end{cases}
\]

**Remark 7.** Under \((B_5)\), \(\lim_{q \to \infty} \gamma(q) = 0\).

**Proof of Proposition.** We will show that there exist positive constants \(C''\) and \(L_0\) (both independent of \(q\)), a rate \(\rho = \rho(q)\) satisfying \(\rho(q) \to 0\) as \(q \to \infty\), and a positive integer \(q_0\) such that, for all \(q \geq q_0\), all \(\eta \in \mathbb{R}^{p+q+1}\), and all \(n \in \{2, 3, 4, \ldots\}\),

\[
d_W(K^n_\eta, \Pi) \leq C'' q (V(\eta) + L_0) \rho^n. \tag{7}
\]

\[
\end{align}
\]

\[
\]
The result then follows immediately from Proposition $4$. The Wasserstein bound in (7) will be established using Theorem $2$ in conjunction with Propositions $5$ and $6$. We begin by showing that assumption $(A_1)$ holds. Proposition $5$ provides the inequality (4), and shows that we can choose $\zeta = \zeta(q) = \Theta(q^{-1})$ and $L = L(q) = \Theta(1)$. However, we still need to establish the second inequality, (5), which relates the drift function to the Euclidean norm. Fix $(\eta, \eta') \in \mathbb{R}^{q+p+1}$. We have

$$\|\eta - \eta'\| \leq \|\eta - \eta'\|^2 + 2(q + p + 1)$$

$$= (\eta_0 - \eta'_0)\,^T(\eta_0 - \eta'_0) + (\eta_0 - \eta'_0)\,^T + \sum_{i=1}^{q}(\eta_i - \eta'_i)\,^2 + 2(q + p + 1).$$

Now, $X^T X - \tilde{X}^T \tilde{X} \preceq X^T X$, so $\lambda_{\min}(X^T X - \tilde{X}^T \tilde{X}) \leq \lambda_{\min}(X^T X)$. Hence,

$$\lambda_{\min}(X^T X - \tilde{X}^T \tilde{X}) \leq \lambda_{\min}(X^T X) \leq \lambda_{\min}(X^T X - \tilde{X}^T \tilde{X}).$$

where the third inequality follows from $(B_2)$. Continuing, we have

$$\sum_{i=1}^{q}(\eta_i - \eta'_i)\,^2 \leq 2 \sum_{i=1}^{q}(\eta_i + \tilde{y} - \tilde{y}_i)\,^2 + 2 \sum_{i=1}^{q}(\eta'_i + \tilde{y} - \tilde{y}_i)\,^2$$

$$= 2 \sum_{i=1}^{q}(\eta_i + \tilde{y} - \tilde{y}_i)\,^2 + 2 \sum_{i=1}^{q}(\eta'_i + \tilde{y} - \tilde{y}_i)\,^2$$

$$= 2m \sum_{i=1}^{q}(\eta_i + \tilde{y} - \tilde{y}_i)\,^2 + 2m \sum_{i=1}^{q}(\eta'_i + \tilde{y} - \tilde{y}_i)\,^2.$$

So we conclude that

$$\|\eta - \eta'\| \leq (\eta_0 - \eta'_0)\,^T(\eta_0 - \eta'_0) + (\eta_0 - \eta'_0)\,^T + \sum_{i=1}^{q}(\eta_i - \eta'_i)\,^2 + 2(q + p + 1)$$

$$\leq 2 \sum_{i=1}^{q}(\eta_i + \tilde{y} - \tilde{y}_i)\,^2 + 2 \sum_{i=1}^{q}(\eta'_i + \tilde{y} - \tilde{y}_i)\,^2$$

$$+ 2m \sum_{i=1}^{q}(\eta_i + \tilde{y} - \tilde{y}_i)\,^2 + 2m \sum_{i=1}^{q}(\eta'_i + \tilde{y} - \tilde{y}_i)\,^2 + 2(q + p + 1)$$

$$= \mathcal{O}(q)[V(\eta) + V(\eta') + 1].$$
Thus, (5) is satisfied when $q$ is large, and we can take $c$ in (5) to be $c_1 q$, where $c_1$ is a constant (not depending on $q$).

Now since $\zeta = \zeta(q) = \Theta(q^{-1})$ and $L = L(q) = \Theta(1)$, it’s clear that $q^{\delta/3} > \frac{2L}{\xi}$ for all large $q$. Thus, Proposition [5] implies that (A2) holds with $\xi = q^{\delta/3}$ when $q$ is large. We now show that (A3) holds when $q$ is large. If $\gamma_0 < 1$, then there’s nothing to prove, so we assume that $\gamma_0 \geq 1$. Define

$$LHS = \frac{\log(2L + 1)}{\log(2L + 1) - \log(\gamma)} \quad \text{and} \quad RHS = \frac{- \log \left( [\xi + 2L + 1]/(\xi + 1) \right)}{\log(\gamma_0) - \log \left( [\xi + 2L + 1]/(\xi + 1) \right)}.$$ 

We must show that, for large $q$, LHS < RHS. Since $L \geq 0$ and $L = \Theta(1)$, it follows that $\log(2L + 1) \geq 0$ and $\log(2L + 1) = \Theta(1)$. Remark [7] implies that $\log(\gamma) \to -\infty$ as $q \to \infty$. Hence, LHS $\to 0$ as $q \to \infty$. Next we show that RHS is bounded from below by a positive constant. It’s clear from (B5) that we can assume (without loss of generality) that $\delta < 3$, and it follows that

$$\frac{\zeta + 2L + 1}{\xi + 1} = \frac{\xi q^{\delta/3} + 2L + 1}{q^{\delta/3} + 1} = \Theta \left( \frac{1}{q^{\delta/3}} \right).$$

Then since $\gamma_0 = \mathcal{O}(\sqrt{q})$, we have that, for large $q$, RHS $\geq h(q)$, where $h(q) = \frac{2\ell}{3+2\delta}$ as $q \to \infty$. We conclude that, for large $q$, LHS < RHS, and condition (A3) is satisfied. According to Theorem 2, we can take the rate to be

$$\rho_a = [\gamma^a (2L + 1)^{1-a}] \vee [\gamma_0^a \left( \frac{\zeta + 2L + 1}{\xi + 1} \right)^{1-a}]$$

for any $a \in (\text{LHS}, \text{RHS})$. We now demonstrate that there exists an $a \in (\text{LHS}, \text{RHS})$ for which $\rho_a \to 0$ as $q \to \infty$. Clearly, for any fixed $a \in (0,1)$, $\gamma^a (2L + 1)^{1-a} \to 0$ as $q \to \infty$. Moreover,

$$\gamma_0^a \left( \frac{\zeta + 2L + 1}{\xi + 1} \right)^{1-a} = \left[ \mathcal{O}(\sqrt{q}) \right]^a \left[ \Theta \left( \frac{1}{q^{\delta/3}} \right) \right]^{1-a} = \mathcal{O} \left( q^{\frac{\delta}{3} + \frac{1}{2} a - \frac{1}{2}} \right).$$

Thus, as long as $a < \frac{2\ell}{3+2\delta}$, we have

$$\gamma_0^a \left( \frac{\zeta + 2L + 1}{\xi + 1} \right)^{1-a} \to 0$$

as $q \to \infty$. Thus, for such a value of $a$, $\rho_a \to 0$ as $q \to \infty$. Finally, for $q$ large enough, there exists a constant $L_0$ such that, for all $\eta \in \mathbb{R}^{p+q+1}$, we have

$$\left( \frac{(\zeta + 1)V(\eta) + L + 1}{1 - \rho_a} \right) \leq 2(V(\eta) + L_0).$$

This analysis, in conjunction with Theorem 2, shows that there exists a positive integer $q_0$ such that, for all $q \geq q_0$, all $\eta \in \mathbb{R}^{p+q+1}$, and all $n \in \{2,3,4,\ldots\}$,

$$d_W(K^n, \Pi) \leq c_1 q \left( \frac{(\zeta + 1)V(x) + L + 1}{1 - \rho} \right)^n \leq 2c_1 q(V(\eta) + L_0) \rho^n,$$

and $\rho = \rho(q) \to 0$ as $q \to \infty$. \qed
4 Conversion Condition: Proof of Proposition 4

In this section, we use Theorem 1 to prove Proposition 4, which is restated here for convenience.

**Proposition 8.** Under conditions \((B_1)\) and \((B_2)\), there exist a constant \(C > 0\) and a positive integer \(q_0\) such that, for all \(q \geq q_0\),

\[
d_{TV}(K^n_{\eta}, \Pi) \leq C \hat{r}^{3/2} q d_W(K^{n-1}_{\eta}, \Pi)
\]

for all \(\eta \in \mathbb{R}^{p+n+1}\) and all \(n \in \{2, 3, \ldots \}\).

**Proof.** According to Theorem 1 it suffices to find a \(C > 0\) and a positive integer \(q_0\) such that, for all \(q \geq q_0\), we have

\[
\int_{\mathbb{R}^{p+n+1}} |k(\eta, \tilde{\eta}) - k(\eta', \tilde{\eta})| \, d\tilde{\eta} \leq 2C \hat{r}^{3/2} q \|\eta - \eta'\|
\]

for all \(\eta, \eta' \in \mathbb{R}^{p+n+1}\). Fix \((\eta, \eta')\) and assume that \(\eta \neq \eta'\), otherwise the result is trivial. Let \(\pi_2(\lambda | \eta, y) = \pi_2(\lambda | \eta, y)\pi_22(\tau | \eta, y)\), and note that

\[
\begin{align*}
|\pi_1(\tilde{\eta} | \lambda, \tau, y)\pi_2(\lambda | \eta, y)\pi_22(\tau | \eta, y) - \pi_1(\tilde{\eta} | \lambda, \tau, y)\pi_21(\lambda | \eta', y)\pi_22(\tau | \eta', y)| & = \pi_1(\tilde{\eta} | \lambda, \tau, y)\pi_21(\lambda | \eta, y)\pi_22(\tau | \eta, y) - \pi_1(\tilde{\eta} | \lambda, \tau, y)\pi_2(\lambda | \eta', y)\pi_22(\tau | \eta', y) \\
& = \pi_1(\tilde{\eta} | \lambda, \tau, y)\pi_21(\lambda | \eta, y)\pi_22(\tau | \eta, y) - \pi_1(\tilde{\eta} | \lambda, \tau, y)\pi_21(\lambda | \eta', y)\pi_22(\tau | \eta', y) \\
& = \pi_1(\tilde{\eta} | \lambda, \tau, y)\pi_21(\lambda | \eta, y)\pi_22(\tau | \eta, y) - \pi_1(\tilde{\eta} | \lambda, \tau, y)\pi_21(\lambda | \eta', y)\pi_22(\tau | \eta', y).
\end{align*}
\]

It follows that

\[
\int_{\mathbb{R}^{p+n+1}} |k(\eta, \tilde{\eta}) - k(\eta', \tilde{\eta})| \, d\tilde{\eta} \leq A + B
\]

where

\[
A = \int_{\mathbb{R}^+} |\pi_21(\lambda | \eta, y) - \pi_21(\lambda | \eta', y)| \, d\lambda \quad \text{and} \quad B = \int_{\mathbb{R}^+} |\pi_22(\tau | \eta, y) - \pi_22(\tau | \eta', y)| \, d\tau.
\]

We begin with the second term, \(B\). Recall that \(\tau | \eta, y \sim \Gamma(\alpha, \beta)\) where \(\alpha = \frac{N}{2} + a_2\) and \(\beta = b_2 + \frac{1}{2} \sum_{i=1}^q \sum_{j=1}^{r_i} l_{ij}^2\), where

\[
l_{ij} = y_{ij} - \frac{\eta_{00}}{\sqrt{q}} - \eta_i.
\]

So, \(\tau | \eta', y \sim \Gamma(\alpha', \beta')\) where \(\beta' = b_2 + \frac{1}{2} \sum_{i=1}^q \sum_{j=1}^{r_i} (l_{ij}')^2\), where

\[
l_{ij}' = y_{ij} - \frac{\eta_{00}}{\sqrt{q}} - \eta_i'.
\]

Without loss of generality, assume that \(\beta' > \beta\). It’s easy to show that there is a (unique) number \(u \in (0, \infty)\) such that \(\pi_22(u | \eta', y) = \pi_22(u | \eta, y)\), and that \(\pi_22(\tau | \eta', y) \geq \pi_22(\tau | \eta, y)\) for \(\tau \in (0, u]\). It
follows that

\[ B = 2 \int_0^u \left[ \pi_{22}(\tau | \eta', y) - \pi_{22}(\tau | \eta, y) \right] d\tau \]
\[ = 2 \int_0^u \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} e^{-\beta \tau} \left[ \left( \frac{\beta'}{\beta} \right)^\alpha e^{(\beta' - \beta')\tau} - 1 \right] d\tau \]
\[ \leq 2 \int_0^u \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} e^{-\beta \tau} \left[ \left( \frac{\beta'}{\beta} \right)^\alpha - 1 \right] d\tau \]
\[ \leq 2 \left[ \left( \frac{\beta'}{\beta} \right)^\alpha - 1 \right] \]
\[ = 2 \left[ \left( \beta + \Delta \right)^\alpha - 1 \right], \]

where \( \Delta = \beta' - \beta \). Now,

\[ 2\Delta = 2(\beta' - \beta) \]
\[ = \sum_{i=1}^q \sum_{j=1}^{r_i} \left( (l_{ij}')^2 - l_{ij}^2 \right) \]
\[ = \sum_{i=1}^q \sum_{j=1}^{r_i} \left[ \left( y_{ij} - x_{ij} \eta_{00}/\sqrt{q} - \eta_i \right)^2 - \left( y_{ij} - x_{ij} \eta_{00}/\sqrt{q} - \eta_i \right)^2 \right] \]
\[ = \sum_{i=1}^q \sum_{j=1}^{r_i} \left[ \left( x_{ij} \eta_{00}/\sqrt{q} + \eta_i - x_{ij} \eta_{00}'/\sqrt{q} - \eta_i' \right)^2 \right. \]
\[ \quad + \left. 2 \left( y_{ij} - x_{ij} \eta_{00}/\sqrt{q} - \eta_i \right) \left( x_{ij} \eta_{00}/\sqrt{q} + \eta_i - x_{ij} \eta_{00}'/\sqrt{q} - \eta_i' \right) \right] \]
\[ \leq \sum_{i=1}^q \sum_{j=1}^{r_i} \left( x_{ij} \eta_{00}/\sqrt{q} + \eta_i - x_{ij} \eta_{00}'/\sqrt{q} - \eta_i' \right)^2 \]
\[ + 2 \sqrt{\sum_{i=1}^q \sum_{j=1}^{r_i} l_{ij}^2} \sqrt{\sum_{i=1}^q \sum_{j=1}^{r_i} \left( x_{ij} \eta_{00}/\sqrt{q} + \eta_i - x_{ij} \eta_{00}'/\sqrt{q} - \eta_i' \right)^2}, \]
where the inequality is Cauchy-Schwarz. Now, \( \sum_{i=1}^{q} \sum_{j=1}^{r_i} \left( x_{ij}^T \eta_0 / \sqrt{q} + \eta_i - x_{ij}^T \eta_0 / \sqrt{q} - \eta_i' \right)^2 \)

\[
\leq \sum_{i=1}^{q} \sum_{j=1}^{r_i} \left[ \frac{2}{q} (\eta_0 - \eta_0')^T x_{ij} x_{ij}^T (\eta_0 - \eta_0') + 2 (\eta_i - \eta_i')^2 \right]
\]

\[
= \frac{2}{q} (\eta_0 - \eta_0')^T \sum_{i=1}^{q} \sum_{j=1}^{r_i} (x_{ij} x_{ij}^T) (\eta_0 - \eta_0') + 2 \sum_{i=1}^{q} \sum_{j=1}^{r_i} (\eta_i - \eta_i')^2
\]

\[
= \frac{2}{q} (\eta_0 - \eta_0')^T X^T X (\eta_0 - \eta_0') + 2 \sum_{i=1}^{q} \sum_{j=1}^{r_i} (\eta_i - \eta_i')^2
\]

\[
\leq \frac{2Nk_2}{q} (\eta_0 - \eta_0')^T (\eta_0 - \eta_0') + 2r_{\text{max}} \sum_{i=1}^{q} (\eta_i - \eta_i')^2
\]

\[
\leq 2\bar{r}k_2 (\eta_0 - \eta_0')^T (\eta_0 - \eta_0') + 2m\bar{r} \sum_{i=1}^{q} (\eta_i - \eta_i')^2
\]

\[
\leq 2c_2\bar{r} \| \eta - \eta' \|^2 ,
\]

where the second inequality follows from assumption \((B_2)\), and \( c_2 := k_2 \vee m \). Thus,

\[
\Delta \leq c_2\bar{r} \| \eta - \eta' \|^2 + 2\sqrt{c_2\bar{r}} \beta \| \eta - \eta' \| ,
\]

and it follows that

\[
B \leq 2 \left[ 1 + 2\sqrt{\frac{c_2\bar{r}}{b_2}} \| \eta - \eta' \| + \frac{c_2\bar{r}}{b_2} \| \eta - \eta' \|^2 \right]^{\alpha} - 1.
\]

So if we define \( f : [0, \infty) \to [0, \infty) \) as

\[
f(x) = \left( 1 + 2\sqrt{\frac{c_2\bar{r}}{b_2}} x + \frac{c_2\bar{r}}{b_2} x^2 \right)^{\alpha} - 1,
\]

then \( B \leq 2 f(\| \eta - \eta' \|) \). Note that \( f(0) = 0 \), and

\[
f'(x) = \alpha \left( 1 + 2\sqrt{\frac{c_2\bar{r}}{b_2}} x + \frac{c_2\bar{r}}{b_2} x^2 \right)^{\alpha-1} \left[ 2\sqrt{\frac{c_2\bar{r}}{b_2}} + 2\frac{c_2\bar{r}}{b_2} x \right]
\]

is increasing on \((0, \infty)\). Now assume that \( \| \eta - \eta' \| < \bar{r}^{-3/2} q^{-1} \). Then by the mean-value theorem, we have

\[
B \leq 2 f' \left( \bar{r}^{-3/2} q^{-1} \right) \| \eta - \eta' \| = 2 \left[ \bar{r}^{-3/2} q^{-1} f' \left( \bar{r}^{-3/2} q^{-1} \right) \right]^{3/2} q \| \eta - \eta' \| . \tag{8}
\]

We now examine the behavior of the term in curly brackets as \( q \) gets large. First, straightforward manipulations show that

\[
\lim_{q \to \infty} \left( 1 + 2\sqrt{\frac{b_2}{c_2} \frac{1}{\bar{r}q}} + \frac{b_2}{c_2 \bar{r}^2 q^2} \right)^{rq/2+a_2-1} = \exp \left\{ \sqrt{\frac{b_2}{c_2}} \lim_{q \to \infty} \left[ \sqrt{\frac{b_2}{c_2} \frac{1}{\bar{r}q}} \log \left( 1 + 2\sqrt{\frac{b_2}{c_2} \frac{1}{\bar{r}q}} + \frac{b_2}{c_2 \bar{r}^2 q^2} \right) \right] \right\} .
\]
Then, letting $z = \sqrt{\frac{b_2 r_2}{c_2} z}$, we have
\[
\lim_{q \to \infty} \left(1 + 2 \sqrt{\frac{c_2}{b_2 r_2} + \frac{c_2}{b_2 r_2^2 q^2}} \right)^{r_2/2+\omega_2-1} = \exp \left\{ \sqrt{\frac{c_2}{b_2}} \lim_{z \to \infty} \left[ z \log \left(1 + \frac{1}{z} + \frac{1}{4z^2} \right) \right] \right\}.
\]
An application of L'Hôpital's rule shows that the limit on the right-hand side is unity. Thus,
\[
\lim_{q \to \infty} \left(1 + 2 \sqrt{\frac{c_2}{b_2 r_2} + \frac{c_2}{b_2 r_2^2 q^2}} \right)^{r_2/2+\omega_2-1} = \exp \left\{ \sqrt{\frac{c_2}{b_2}} \right\}.
\]
So we have
\[
\lim_{q \to \infty} \tilde{r}^{-3/2} q^{-1} f' \left(\tilde{r}^{-3/2} q^{-1} \right) = \lim_{q \to \infty} \tilde{r}^{-3/2} q^{-1} \left(\tilde{r} q / 2 + \omega_2 \right) \left(1 + 2 \sqrt{\frac{c_2}{b_2 r_2} + \frac{c_2}{b_2 r_2^2 q^2}} \right)^{r_2/2+\omega_2-1} \left[2 \sqrt{\frac{c_2 r_2}{b_2}} + \frac{2c_2}{b_2 r_2 q} \right] = \lim_{q \to \infty} \left(2 \sqrt{\frac{c_2}{b_2}} \right).\]
Thus, there exists $c_3 < \infty$ such that
\[
\max_{q > 2} 2 \left\{ \tilde{r}^{-3/2} q^{-1} f' \left(\tilde{r}^{-3/2} q^{-1} \right) \right\} = c_3.
\]
It now follows from $\{\|$ that
\[
B \leq c_3 \tilde{r}^{3/2} q \| \eta - \eta' \|
\]
whenever $\| \eta - \eta' \| < \tilde{r}^{-3/2} q^{-1}$.

A slightly simpler version of this same argument shows that there exists a $c_1 \in (0, \infty)$ such that
\[
A \leq c_1 q \| \eta - \eta' \| \quad \text{whenever} \quad \| \eta - \eta' \| \leq 1/q.
\]
Putting all of this together shows that, for all $q > 2$ and all pairs $(\eta, \eta')$ such that $\| \eta - \eta' \| < \tilde{r}^{-3/2} q^{-1}$, we have
\[
\int_{\mathbb{R}^{p+q+1}} \left| k(\eta, \tilde{\eta}) - k(\eta', \tilde{\eta}) \right| d\tilde{\eta} \leq A + B \leq c_1 q \| \eta - \eta' \| + c_3 \tilde{r}^{3/2} q \| \eta - \eta' \| \leq (c_1 + c_3) \tilde{r}^{3/2} q \| \eta - \eta' \|.
\]
Now all that remains is to extend this so it holds for all pairs $(\eta, \eta')$. For an arbitrary pair $(\eta, \eta')$, let $v = \frac{1}{m^*} (\theta' - \theta)$ where the integer $m^*$ is chosen such that
\[
\|v\| = \left\| \frac{1}{m^*} (\eta' - \eta) \right\| = \frac{1}{m^*} \| \eta' - \eta \| < \tilde{r}^{-3/2} q^{-1}.
\]
For $j = 0, 1, \ldots, m^*$, let $\xi_j = \eta + jv$. Then $\xi_0 = \eta$, $\xi_{m^*} = \eta'$, and $\|\xi_{j+1} - \xi_j\| = \|v\| < \tilde{r}^{-3/2} q^{-1}$.
Hence,
\[
\int_{\mathbb{R}^{q+1}} |k(\eta, \tilde{\eta}) - k(\eta', \tilde{\eta})| \, d\tilde{\eta} \leq \sum_{j=0}^{m^*-1} \int_{\mathbb{R}^{q+1}} |k(\xi_j + 1, \tilde{\eta}) - k(\xi_j, \tilde{\eta})| \, d\tilde{\eta}
\]
\[
\leq \sum_{j=0}^{m^*-1} (c_1 + c_3) \tilde{r}^{3/2} q \|\xi_{j+1} - \xi_j\|
\]
\[
= \sum_{j=0}^{m^*-1} (c_1 + c_3) \tilde{r}^{3/2} q \|v\|
\]
\[
= (c_1 + c_3) \tilde{r}^{3/2} q \|\eta - \eta'\|.
\]

\[\square\]

5 Drift Condition: Proof of Proposition 5

We begin by stating three results, all proven in Appendix B, that will be used to establish the drift and contraction conditions. The first lists a few simple facts about \(X\) & \(Y\), the second is a simple matrix result that we could not find stated elsewhere, and the third provides an upper bound on the largest eigenvalue of the square of the derivative of the square root of a matrix.

Lemma 9. The data, \(X\) and \(Y\), satisfy the following:

\(i\) \(\bar{X}^\top \bar{X} \preceq X^\top X\)

\(ii\) \(\sum_{i=1}^{q} \bar{x}_i \bar{x}_i^\top \preceq \frac{1}{r_{\min}} \bar{X}^\top \bar{X} \preceq \frac{1}{r_{\min}} X^\top X\)

\(iii\) \(\sum_{i=1}^{q} \bar{y}_i^2 \leq \frac{1}{r_{\min}} Y^\top Y\)

\(iv\) \(\bar{y}^2 \leq \frac{1}{q r_{\min}} Y^\top Y\)

Remark 10. Under \((B_1)-(B_3)\), the results in Lemma 9 imply that \(\sum_{i=1}^{q} \bar{x}_i \bar{x}_i^\top \preceq m q k_1^2 I\), \(\sum_{i=1}^{q} \bar{y}_i^2 = O(q)\), and \(\bar{y}^2 = O(1)\).

Lemma 11. If \(C, D\) are conformable matrices, then

\((C + D)^\top (C + D) \preceq 2(C^\top C + D^\top D)\).

Lemma 12. Let \(A = A(x)\) be a positive definite matrix that depends on a scalar, \(x\). If \(A^{1/2} = A^{1/2}(x)\) is the unique positive definite square root of \(A\), then

\[\lambda_{\max}\left\{ \left( \frac{dA^{1/2}}{dx} \right)^2 \right\} \leq \frac{\lambda_{\max}\left\{ \left( \frac{dA}{dx} \right)^2 \right\}}{4\lambda_{\min}(A)}.\]
Proof. We have
\[
V(\tilde{\eta}) = \frac{1}{N} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \left( \tilde{y} - x_{ij} \tilde{\eta}_{00}/\sqrt{q} \right)^2 + \frac{\tilde{\eta}_{0}^2}{q} + \frac{1}{N} \sum_{i=1}^{q} r_i (\tilde{\eta}_i + \tilde{y} - \tilde{y}_i)^2
\]
\[
\leq \frac{2}{N} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \tilde{y}^2 + \frac{2}{qN} \tilde{\eta}_{00} \left( \sum_{i=1}^{q} \sum_{j=1}^{r_i} x_{ij} x_{ij}^T \right) \tilde{\eta}_{00} + \frac{\tilde{\eta}_{0}^2}{q} + \frac{r_{\max}}{N} \sum_{i=1}^{q} (\tilde{\eta}_i + \tilde{y} - \tilde{y}_i)^2
\]
\[
\leq 2\tilde{y}^2 + \frac{2}{qN} \tilde{\eta}_{00}^T (X^T X) \tilde{\eta}_{00} + \frac{\tilde{\eta}_{0}^2}{q} + \frac{m}{q} \sum_{i=1}^{q} (\tilde{\eta}_i + \tilde{y} - \tilde{y}_i)^2,
\]
where, in the last line, we used the fact that $r_{\max}/\tilde{r} \leq m$. We now examine the last term in (9).

Using the definition of $\tilde{\eta}_i$ from the random mapping, we have
\[
\sum_{i=1}^{q} (\tilde{\eta}_i + \tilde{y} - \tilde{y}_i)^2 = \sum_{i=1}^{q} \left( \lambda_{t_i} \tilde{\eta}_{0}/\sqrt{q} + \frac{r_{i\tau}}{t_i} (\tilde{y}_i - \bar{x}_i \tilde{\eta}_{00}/\sqrt{q}) + \sqrt{\frac{1}{t_i}} N_i + \tilde{y} - \tilde{y}_i \right)^2
\]
\[
= \sum_{i=1}^{q} \left( \lambda_{t_i} \tilde{\eta}_{0}/\sqrt{q} - \frac{r_{i\tau}}{t_i} \bar{x}_i \tilde{\eta}_{00}/\sqrt{q} + \sqrt{\frac{1}{t_i}} N_i + \tilde{y} - \tilde{\eta}_{0} \tilde{y}_i \right)^2
\]
\[
\leq 5\tilde{\eta}_{0}^2 + 5\tilde{\eta}_{00}^2 \left( \sum_{i=1}^{q} \bar{x}_i \bar{x}_i^T \right) \tilde{\eta}_{00} + 5 \sum_{i=1}^{q} \frac{1}{t_i} N_i^2 + 5q \tilde{y}^2 + 5 \sum_{i=1}^{q} \tilde{y}_i^2
\]
\[
\leq 5\tilde{\eta}_{0}^2 + 5mk_2\tilde{\eta}_{00}^2 \tilde{\eta}_{00} + 5 \sum_{i=1}^{q} \frac{1}{r_{i\tau}} N_i^2 + O(q) + O(q)
\]
\[
= O(1) \tilde{\eta}_{0}^2 + O(1) \tilde{\eta}_{00}^2 \tilde{\eta}_{00} + O\left( \frac{1}{\tau} \right) \frac{1}{\tau} \sum_{i=1}^{q} N_i^2 + O(q),
\]
where the second inequality follows from Remark 10. Hence, by (B2) and Remark 10, we have
\[
V(\tilde{\eta}) \leq 2\tilde{y}^2 + \frac{2}{qN} \tilde{\eta}_{00}^T (X^T X) \tilde{\eta}_{00} + \frac{\tilde{\eta}_{0}^2}{q} + \frac{m}{q} \sum_{i=1}^{q} (\tilde{\eta}_i + \tilde{y} - \tilde{y}_i)^2
\]
\[
= O(1) + O\left( \frac{1}{q} \right) \tilde{\eta}_{0}^2 \tilde{\eta}_{00} + O\left( \frac{1}{q} \right) \tilde{\eta}_{00}^2 \tilde{\eta}_{00} + \frac{m}{q} \left( O(1) \tilde{\eta}_{0}^2 + O(1) \tilde{\eta}_{00}^2 \tilde{\eta}_{00} + O\left( \frac{1}{\tau} \right) \frac{1}{\tau} \sum_{i=1}^{q} N_i^2 + O(q) \right)
\]
\[
= O(1) + O\left( \frac{1}{q} \right) \tilde{\eta}_{0}^2 \tilde{\eta}_{00} + O\left( \frac{1}{q} \right) \tilde{\eta}_{00}^2 \tilde{\eta}_{00} + O\left( (qr)^{-1} \right) \frac{1}{\tau} \sum_{i=1}^{q} N_i^2.
\]
Now let $A = E(\tilde{\eta}_{00}^2 \tilde{\eta}_{00})$ and $B = E(\tilde{\eta}_{00}^2)$, where the expectation is, of course, with respect to the Mtk $K$. It follows that
\[
\int_{\mathbb{R}^p+q+1} V(\tilde{\eta}) K(\eta, d\tilde{\eta}) = O(1) + O\left( \frac{1}{q} \right) A + O\left( \frac{1}{q} \right) B + O\left( \frac{1}{\tau} \right) E\left( \frac{1}{\tau} \right).
\]
To bound \( A \), recall that \( \eta_{00} | \lambda, \tau, Y \sim N_p(v, \frac{4}{\tau} Q) \), where \( Q = (X^\top X - \bar{X}^\top M \bar{X})^{-1} \) and \( v = \sqrt{q} Q(X^\top Y \bar{X}^\top M Y) \). Let \( \text{tr}(\cdot) \) denote the trace of the (square matrix) argument. We have

\[
A = E(\eta_{00}^\top \eta_{00}) = E[v^\top v + \text{tr} \left( \frac{q}{\tau} Q \right)] = E(v^\top v) + \text{tr} \left[ E \left( \frac{q}{\tau} Q \right) \right].
\]  

(11)

In order to handle the trace term in (11), we need to look carefully at \( Q \). First,

\[
M = D_c + \frac{(I - D_c)11^\top (I - D_c)}{11^\top (I - D_c)1}
\]

\[
= D_c + (I - D_c) \frac{\frac{1}{2} 11^\top (I - D_c) \frac{1}{2} (I - D_c)^{\frac{1}{2}}}{11^\top (I - D_c)1} (I - D_c)^{\frac{1}{2}}
\]

\[
\leq D_c + (I - D_c) \frac{11^\top (I - D_c) 1}{11^\top (I - D_c)1} (I - D_c)^{\frac{1}{2}}
\]

\[
= D_c + (I - D_c)
\]

\[
= I,
\]

where the “inequality” is due to the fact that, for any vector \( u \), \( uu^\top \preceq u^\top u I \). It now follows from (B2) that

\[
Q = (X^\top X - \bar{X}^\top M \bar{X})^{-1} \preceq (X^\top X - \bar{X}^\top \bar{X})^{-1} \preceq \left( \frac{q r k_1}{\tau} \right)^{-1} \frac{1}{q r k_1} I.
\]  

(12)

Therefore,

\[
\text{tr} \left[ E \left( \frac{q}{\tau} Q \right) \right] \leq \text{tr} \left[ E \left( \frac{1}{\tau} \frac{1}{q r k_1} I \right) \right] = O \left( \frac{p}{\tau} \right) E \left( \frac{1}{\tau} \right).
\]  

(13)

We now go to work on the penultimate term in (11). As in Appendix B, let \( R = \bigoplus_{i=1}^r \frac{1}{r_i} J_{r_i} \). It’s clear that \( RX = \bar{X} \) and \( RY = \bar{Y} \), and that \( R \) is idempotent. Hence, \( R \bar{X} = R^2 X = RX = \bar{X} \).

Moreover, \( R \) and \( D_c \) commute, which implies that \( R \) and \( M \) also commute. It follows that

\[
v = \sqrt{q} Q(X^\top Y - \bar{X}^\top M \bar{Y}) = \sqrt{q} Q(X^\top Y - \bar{X}^\top M R Y) = \sqrt{q} Q(X^\top - \bar{X}^\top M R) Y = \sqrt{q} Q(X - M \bar{X})^\top Y.
\]

Since \( Q^2 \preceq \lambda_{\max}^2 (Q) I \preceq (q r k_1)^{-2} I \), we have

\[
v^\top v = q Y^\top (X - M \bar{X}) Q^2 (X - M \bar{X})^\top Y
\]

\[
\leq \frac{q}{(q r k_1)^2} Y^\top (X - M \bar{X}) (X - M \bar{X})^\top Y
\]

\[
\leq \frac{2 q}{(q r k_1)^2} Y^\top (X X^\top + M \bar{X} \bar{X}^\top M) Y
\]

\[
\leq \frac{2 q^2 r k_2}{(q r k_1)^2} Y^\top (I + M^2) Y
\]

\[
\leq \frac{4 k_2}{r k_1^2} Y^\top Y
\]

\[
\leq \frac{4 k_2}{r k_1^2} (q \bar{\ell})
\]

\[
= O(q),
\]  

(14)
where the second inequality follows from Lemma 11, the third inequality follows from \((B_2)\), the fourth inequality holds because \(M \preceq I\), and the fifth inequality follows from \((B_3)\). Combining \((13)\) and \((14)\), we have

\[
A = \mathcal{O}\left(\frac{p}{\tau}\right) E\left(\frac{1}{\tau}\right) + \mathcal{O}(q) .
\]  

We now move onto \(B\). Recall that

\[
\tilde{\eta}_0 = \sqrt{q} \frac{\sum_{i=1}^{q} (\tilde{y}_i - \bar{x}_i^T \tilde{\eta}_{00} / \sqrt{q}) / z_i}{\sum_{i=1}^{q} 1 / z_i} + \sqrt{\frac{q}{\sum_{i=1}^{q} 1 / z_i}} N_0 .
\]

Define \(t_{\text{min}} = \lambda + r_{\text{min}} \tau\) and \(z_{\text{min}} = t_{\text{min}} / (r_{\text{min}} \lambda \tau)\), and define \(t_{\text{max}}\) and \(z_{\text{max}}\) analogously. It’s important to note that while \(t_{\text{min}}\) is, in fact, the smallest \(t_i\), \(z_{\text{min}}\) is actually the largest \(z_i\), but we find this notation convenient. We have \(1 \leq \frac{z_{\text{min}}}{z_{\text{max}}} \leq m\). Now,

\[
B = E(\tilde{\eta}_0^2)
\]

\[
= E\left(\sqrt{q} \frac{\sum_{i=1}^{q} (\tilde{y}_i - \bar{x}_i^T \tilde{\eta}_{00} / \sqrt{q}) / z_i}{\sum_{i=1}^{q} 1 / z_i} + \sqrt{\frac{q}{\sum_{i=1}^{q} 1 / z_i}} N_0 \right)^2
\]

\[
= qE\left(\frac{\sum_{i=1}^{q} (\tilde{y}_i - \bar{x}_i^T \tilde{\eta}_{00} / \sqrt{q}) / z_i}{\sum_{i=1}^{q} 1 / z_i} \right)^2 + E\left(\frac{q}{\sum_{i=1}^{q} 1 / z_i} N_0 \right)
\]

\[
\leq qE\left(\frac{\sum_{i=1}^{q} (\tilde{y}_i - \bar{x}_i^T \tilde{\eta}_{00} / \sqrt{q})^2 / z_i}{\sum_{i=1}^{q} 1 / z_i} \right) + E\left(\frac{q}{\sum_{i=1}^{q} 1 / z_i} \right)
\]

\[
\leq qE\left(\frac{\sum_{i=1}^{q} (\tilde{y}_i - \bar{x}_i^T \tilde{\eta}_{00} / \sqrt{q})^2 / z_{\text{max}}}{\sum_{i=1}^{q} 1 / z_{\text{min}}} \right) + E(z_{\text{min}})
\]

\[
\leq mE\left(\sum_{i=1}^{q} (\tilde{y}_i - \bar{x}_i^T \tilde{\eta}_{00} / \sqrt{q})^2 \right) + E(z_{\text{min}})
\]

\[
\leq 2m \sum_{i=1}^{q} \tilde{y}_i^2 + \frac{2m}{q} E\left[\tilde{\eta}_{00}^T \left(\sum_{i=1}^{q} \bar{x}_i \bar{x}_i^T \right) \tilde{\eta}_{00}\right] + E\left(\frac{t_{\text{min}}}{r_{\text{min}} \lambda \tau}\right)
\]

\[
= \mathcal{O}(q) + \mathcal{O}(1) A + E\left(\frac{1}{\lambda} \right) + \mathcal{O}\left(\frac{1}{\tau}\right) E\left(\frac{1}{\tau}\right) ,
\]  

where the third equality follows from the fact that \(N_0\) has mean 0 and is independent of all of the other random vectors in the random mapping, the first inequality is Jensen’s, and the last line
follows from Remark 10. Combining (10), (15), and (16) yields

\[ \int_{\mathbb{R}^{p+q+1}} V(\tilde{\eta}) K(\eta, d\tilde{\eta}) \]

\[ = \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{q}\right) A + \mathcal{O}\left(\frac{1}{q}\right) B + \mathcal{O}\left(\frac{1}{r}\right) E\left(\frac{1}{r}\right) \]

\[ = \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{q}\right) A + \mathcal{O}\left(\frac{1}{q}\right) \left[ O(q) + \mathcal{O}(1) A + E\left(\frac{1}{r}\right) \right] + \mathcal{O}\left(\frac{1}{r}\right) E\left(\frac{1}{r}\right) \]

\[ = \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{q}\right) A + \mathcal{O}\left(\frac{1}{q}\right) E\left(\frac{1}{r}\right) + \mathcal{O}\left(\frac{1}{r}\right) E\left(\frac{1}{r}\right) \]

\[ = \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{q}\right) E\left(\frac{1}{r}\right) + \mathcal{O}\left(\frac{1}{r}\right) E\left(\frac{1}{r}\right), \quad (17) \]

where the last line follows from (B4). Now using the definition of \( \lambda \) from the random mapping, we have

\[ E\left(\frac{1}{\chi}\right) = \frac{2b_1 + \sum_{i=1}^{q} \eta_i - \eta_0/\sqrt{q})^2}{q + 2a_1 - 2} \]

\[ \leq \frac{2b_1 + 3 \sum_{i=1}^{q} \eta_i + \bar{y} - \bar{y}_i)^2 + 3 \sum_{i=1}^{q} (\bar{y}_i - \bar{y})^2 + \frac{3}{q} \sum_{i=1}^{q} \eta_i^2}{q + 2a_1 - 2} \]

\[ \leq \frac{2b_1 + 6q\bar{y}^2 + 6 \sum_{i=1}^{q} \bar{y}_i^2}{q + 2a_1 - 2} + \frac{3 \sum_{i=1}^{q} \eta_i + \bar{y} - \bar{y}_i)^2 + 3\eta_0^2}{q + 2a_1 - 2} \]

\[ \leq \frac{2b_1 + 6q\bar{y}^2 + 6 \sum_{i=1}^{q} \bar{y}_i^2}{q + 2a_1 - 2} + \frac{3 \sum_{i=1}^{q} \eta_i + \bar{y} - \bar{y}_i)^2 + 3r_{min}\eta_0^2}{r_{min}(q + 2a_1 - 2)} \]

\[ \leq \frac{2b_1 + 6q\bar{y}^2 + 6 \sum_{i=1}^{q} \bar{y}_i^2}{q + 2a_1 - 2} + \frac{3q\bar{y}V(\eta)}{r_{min}(q + 2a_1 - 2)} \]

\[ = \mathcal{O}(1) + \mathcal{O}(1)V(\eta). \quad (18) \]

Similarly,

\[ E\left(\frac{1}{\tau}\right) = \frac{2b_2 + \sum_{i=1}^{q} \sum_{j=1}^{r_i} (\eta_i - x_{ij}^{T}\eta_0)/\sqrt{q} - \eta_i)^2}{N + 2a_2 - 2}. \]

Now

\[ \sum_{i=1}^{q} \sum_{j=1}^{r_i} (\eta_i - x_{ij}^{T}\eta_0)/\sqrt{q} - \eta_i)^2 \]

\[ = \sum_{i=1}^{q} \sum_{j=1}^{r_i} \left[ (\eta_i + \bar{y} - \bar{y}_i) + (x_{ij}^{T}\eta_0)/\sqrt{q} - \bar{y} \right] (\eta_i - y_{ij})^2 \]

\[ \leq 3 \sum_{i=1}^{q} \sum_{j=1}^{r_i} (\eta_i + \bar{y} - \bar{y}_i)^2 + 3 \sum_{i=1}^{q} \sum_{j=1}^{r_i} (x_{ij}^{T}\eta_0)/\sqrt{q} - \bar{y})^2 + 3 \sum_{i=1}^{q} \sum_{j=1}^{r_i} (y_{ij} - \bar{y}_i)^2 \]

\[ \leq 3q\bar{y}V(\eta) + 3 \sum_{i=1}^{q} \sum_{j=1}^{r_i} (\eta_i - \bar{y}_i)^2. \]
Thus,
\[
\mathbb{E}\left( \frac{1}{\tau} \right) \leq \frac{2b_2 + 3q\bar{r}V(\eta) + 3\sum_{i=1}^{q} \sum_{j=1}^{q'} (y_{ij} - \bar{y}_i)^2}{N + 2a_2 - 2} = \mathcal{O}(1) + \mathcal{O}(1)V(\eta),
\]
where the last line follows from (B_3) and Remark 10. Combining (17), (18), and (19), we have
\[
\int_{\mathbb{R}^{p+q+1}} V(\tilde{\eta}) K(\eta, d\tilde{\eta}) = \mathcal{O}(1) + \mathcal{O}(1)\left[ E\left( \frac{1}{\lambda} \right) + E\left( \frac{1}{\tau} \right) \right]
\]
\[
= \mathcal{O}(1) + \mathcal{O}\left( \frac{1}{q} \right) \left[ E\left( \frac{1}{\lambda} \right) + E\left( \frac{1}{\tau} \right) \right]
\]
\[
= \mathcal{O}(1) + \mathcal{O}\left( \frac{1}{q} \right) \left[ \mathcal{O}(1) + \mathcal{O}(1)V(\eta) \right]
\]
\[
= \mathcal{O}(1) + \mathcal{O}\left( \frac{1}{q} \right) V(\eta),
\]
where the second equality follows from (B_5). Hence, there exist \( \zeta = \zeta(q) = \mathcal{O}(q^{-1}) \) and \( L = \mathcal{O}(1) \) such that
\[
\int_{\mathbb{R}^{p+q+1}} V(\tilde{\eta}) K(\eta, d\tilde{\eta}) \leq \zeta V(\eta) + L,
\]
and this completes the proof. \( \square \)

6 Contraction Condition: Proof of Proposition 6

We restate Proposition 6 for convenience.

**Proposition 14.** Assume that (B_1)-(B_5) hold, and define
\[
C = \left\{ (\eta, \eta') \in \mathbb{R}^{p+q+1} \times \mathbb{R}^{p+q+1} : V(\eta) + V(\eta') \leq q^{\delta/3} \right\},
\]
where \( V(\cdot) \) is the drift function defined in (6), and \( \delta \) is given in (B_5). Let \( f \) be the random mapping defined in Section 3. There exist
\[
\gamma = \gamma(q) = \mathcal{O}\left( \sqrt{\frac{q^{2+\delta}}{r}} \lor \frac{1}{\sqrt{q}} \right)
\]
and \( \gamma_0 = \gamma_0(q) = \mathcal{O}(\sqrt{q}) \)

and a positive integer \( q_0 \) such that for all \( q \geq q_0 \), we have
\[
\sup_{s \in [0,1]} \mathbb{E}\left\| \frac{df(\eta + s(\eta' - \eta))}{ds} \right\| \leq \begin{cases} \gamma\|\eta' - \eta\| & (\eta, \eta') \in C \\ \gamma_0\|\eta' - \eta\| & \text{otherwise}. \end{cases}
\]

**Proof.** We will shorten \( (\eta + s(\eta' - \eta)) \) to \( (\eta + s\alpha) \) where \( \alpha = (\eta' - \eta) = (\alpha_0, \alpha_1, \ldots, \alpha_q)^T \in \mathbb{R}^{p+q+1}. \)
The next step is to develop upper bounds on

\[ f(\eta + s\alpha) = \begin{bmatrix} \eta_{00}^{(\eta + s\alpha)} \\ \eta_{01}^{(\eta + s\alpha)} \\ \vdots \\ \eta_{q}^{(\eta + s\alpha)} \end{bmatrix}. \]

By Jensen’s inequality, we have

\[
\left( \mathbb{E} \left\| \frac{df(\eta + s\alpha)}{ds} \right\| \right)^2 \leq \mathbb{E} \left\| \frac{df(\eta + s\alpha)}{ds} \right\|^2 = \mathbb{E} \left\| \frac{d\eta_{00}^{(\eta + s\alpha)}}{ds} \right\|^2 = \mathbb{E} \left( \frac{d\eta_{00}^{(\eta + s\alpha)}}{ds} \right)^2 + \mathbb{E} \left( \frac{d\eta_{01}^{(\eta + s\alpha)}}{ds} \right)^2 + \sum_{i=1}^{q} \mathbb{E} \left( \frac{d\eta_{i}^{(\eta + s\alpha)}}{ds} \right)^2. \tag{20}
\]

The next part of the proof is an extremely long and tedious development of upper bounds for the three terms on the right-hand side of (20). We relegate these calculations to Appendix C and simply state the resulting bound here, but we must first introduce a bit of notation. (In order to simplify notation, we will sometimes omit the superscript \( \eta + s\alpha \), which should not cause any confusion.) Define \( \phi = \lambda/(\bar{r}\tau) \) and

\[ \Phi = \left( \frac{\lambda^3}{\bar{r}^2\tau^2J_1} + \frac{\lambda^2}{\bar{r}\tau J_2} \right)\|\alpha\|^2 = \left( \frac{\phi^2\lambda J_1}{J_1} + \frac{\phi\lambda J_2}{J_2} \right)\|\alpha\|^2. \]

Let \( e_i = t_i/(r_i\tau) \), and define \( e_{\min} = t_{\min}/(r_{\min}\tau) \) and \( e_{\max} = t_{\max}/(r_{\max}\tau) \). So \( e_{\min} \) and \( e_{\max} \) are actually the largest and smallest \( e_i \), respectively. Finally, define

\[ U_1 = \mathbb{E} \left( \frac{\Phi}{e_{\max}} \right), \quad U_2 = \mathbb{E} \left( \frac{\Phi}{r e_{\max}} \right), \quad U_3 = \mathbb{E} \left[ (\phi \land 1) \left( \frac{1}{J_1} + \frac{1}{J_2} \right) \right] \|\alpha\|^2, \quad \text{and} \quad U_4 = \mathbb{E} \left( \frac{\Phi}{\lambda e_{\max}} \right). \]

Here is the key bound, which is derived in Appendix C

\[
\left\{ \mathbb{E} \left\| \frac{df(\eta + s\alpha)}{ds} \right\| \right\}^2 \leq \mathcal{O}(q)U_1 + \mathcal{O}\left( \frac{p^2}{q} \right)U_2 + \mathcal{O}(q^2)U_3 + \mathcal{O}(1)U_4 + \mathcal{O}\left( \frac{1}{q} \right)\|\alpha\|^2. \tag{21}
\]

The next step is to develop upper bounds on \( U_i, i = 1, 2, 3, 4 \), in two different cases: \( (\eta, \eta') \in \mathcal{C} \) and \( (\eta, \eta') \notin \mathcal{C} \). Define \( g : \mathbb{R}^{q+p+1} \to [0, \infty) \) as follows

\[ g(w) = \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \left[ y_{ij} - x_{ij}^\top w_{00}/\sqrt{q} - w_i \right]^2, \]

where \( w \in \mathbb{R}^{q+p+1}, w_{00} \) denotes the first \( p \) elements of \( w \), and for \( i = 0, 1, \ldots, q, w_i \) denotes the \((p + 1 + i)\)th element. Recall that our drift function \( V : \mathbb{R}^{p+q+1} \to [0, \infty) \) is given by

\[ V(w) = \frac{1}{N} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \left( \bar{y} - x_{ij}^\top w_{00}/\sqrt{q} \right)^2 + \frac{w_{00}^2}{q} + \frac{1}{N} \sum_{i=1}^{q} r_i (w_i + \bar{y} - \bar{y}_i)^2. \]
We now show that \( g \) can be bounded by a linear function of \( V \). Indeed,

\[
g(w) = \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \left[ y_{ij} - x_{ij}^T w_{00} / \sqrt{q} - w_i \right]^2
\]

\[
= \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \left[ (w_i + \bar{y} - \bar{y}_i) + (x_{ij}^T w_{00} / \sqrt{q} - \bar{y}) + (\bar{y}_i - y_{ij}) \right]^2
\]

\[
\leq \sum_{i=1}^{q} \sum_{j=1}^{r_i} \left[ (w_i + \bar{y} - \bar{y}_i) + (x_{ij}^T w_{00} / \sqrt{q} - \bar{y}) \right]^2 + \sum_{i=1}^{q} \sum_{j=1}^{r_i} (\bar{y}_i - y_{ij})^2
\]

\[
\leq 2 \sum_{i=1}^{q} r_i (w_i + \bar{y} - \bar{y}_i)^2 + 2 \sum_{i=1}^{q} \sum_{j=1}^{r_i} (x_{ij}^T w_{00} / \sqrt{q} - \bar{y})^2 + 2 \sum_{i=1}^{q} \sum_{j=1}^{r_i} \bar{y}_i^2 + 2 \sum_{i=1}^{q} \sum_{j=1}^{r_i} y_{ij}^2
\]

\[
\leq 2NV(w) + 4mN\ell.
\]

Recall from the random mapping that

\[
\tau = \tau^{(q+sa)} = \frac{J_2}{b_2 + g(\eta + sa)},
\]

where \( J_2 \sim \text{Gamma}(N/2 + a_2, 1) \). Also recall that

\[
C = \left\{ (\eta, \eta') \in \mathbb{R}^{p+q+1} : \mathbb{E}(\Phi) \in \mathbb{R}^{p+q+1} : V(\eta) + V(\eta') \leq q^{\delta/3} \right\}.
\]

A straightforward calculation shows that \( V \) is a convex function. Now if \( (\eta, \eta') \in C \), then \( V(\eta) \leq q^{\delta/3} \) and \( V(\eta') \leq q^{\delta/3} \), and convexity implies that \( V(\eta + s(\eta' - \eta)) \leq q^{\delta/3} \) for all \( s \in [0, 1] \). Consequently, whenever \( (\eta, \eta') \in C \), we have

\[
g(\eta + sa) \leq 2NV(\eta + sa) + 4mN\ell \leq 2\bar{r} q^{1+\delta/3} + 4m\bar{r}q\ell = O(\bar{r} q^{1+\delta/3}).
\]

Note that this bound is free of \( \eta, \eta' \) and \( s \). Clearly, \( \lambda = \lambda^{(q+sa)} \leq J_1/b_1 \). Hence, for \( (\eta, \eta') \in C \), we have

\[
\phi = \frac{\lambda}{\bar{r} \tau} \leq \frac{J_1}{\bar{b}_1} \left[ b_2 + g(\eta + sa) \right] = \frac{J_1}{b_1} O(\bar{r} q^{1+\delta/3}) J_2 = O(\bar{r} q^{1+\delta/3}) \frac{J_1}{J_2},
\]

and

\[
\Phi = \left( \frac{\phi^2 \lambda}{J_1} + \frac{\phi \lambda}{J_2} \right) \|\alpha\|^2 = O(q^{2+2\delta/3}) \frac{J_1^2}{J_2^2} \|\alpha\|^2 + O(q^{1+\delta/3}) \frac{J_1^2}{J_2^2} \|\alpha\|^2 = O(q^{2+2\delta/3}) \frac{J_1^2}{J_2^2} \|\alpha\|^2.
\]

Continuing under the assumption that \( (\eta, \eta') \in C \), we have

\[
U_1 = E\left( \frac{\Phi}{\epsilon_{\text{max}}} \right) \leq E(\Phi) = O(q^{2+2\delta/3}) E\left( \frac{J_1^2}{J_2^2} \right) \|\alpha\|^2 = O(q^{2+2\delta/3}) O\left( \frac{q^2}{\bar{r}^2 q^2} \right) \|\alpha\|^2 = O\left( \frac{q^{2+\delta}}{\bar{r}^2} \right) \|\alpha\|^2,
\]

25
\[ U_2 = \mathbb{E}\left( \frac{\Phi}{\tau e_{\max}^4} \right) \leq \mathbb{E}\left( \frac{\Phi}{\tau} \right) = \mathcal{O}(q^{2+2\delta/3})\mathbb{E}\left( \frac{1}{\tau} \frac{J_1^2}{J_2^3} \right) \|\alpha\|^2 \]
\[ = \mathcal{O}(q^{2+2\delta/3})\mathbb{E}\left( \frac{b_2 + g(\eta s + \alpha)}{J_2} \right) \frac{J_1^2}{J_2^3} \|\alpha\|^2 \]
\[ = \mathcal{O}(q^{2+2\delta/3})\mathcal{O}(\bar{r}^{1+\delta/3})\mathbb{E}\left( \frac{J_1^2}{J_2^3} \right) \|\alpha\|^2 \]
\[ = \mathcal{O}(\bar{r}^{3+\delta})\mathbb{E}\left( \frac{J_1^2}{J_2^3} \right) \|\alpha\|^2 \]
\[ = \mathcal{O}(\bar{r}^{3+\delta})\mathcal{O}\left( \frac{q^2}{\bar{r}^3 q^3} \right) \|\alpha\|^2 \]
\[ = \mathcal{O}\left( \frac{q^{2+\delta}}{\bar{r}^2} \right) \|\alpha\|^2 , \]

\[ U_3 = \mathbb{E}\left[ (\phi \wedge 1) \frac{1}{J_1} + \frac{1}{J_2} \right] \|\alpha\|^2 \leq \mathbb{E}\left[ \frac{\phi}{J_1} + \frac{1}{J_2} \right] \|\alpha\|^2 \]
\[ = \mathbb{E}\left[ \mathcal{O}(q^{1+\delta/3}) \frac{1}{J_1} + \frac{1}{J_2} \right] \|\alpha\|^2 \]
\[ = \mathcal{O}(q^{1+\delta/3})\mathbb{E}\left( \frac{1}{J_2} \right) \|\alpha\|^2 \]
\[ = \mathcal{O}\left( \frac{q^{\delta/3}}{\bar{r}} \right) \|\alpha\|^2 , \]

and

\[ U_4 = \mathbb{E}\left( \frac{\Phi}{\lambda e_{\max}^3} \right) \leq \mathbb{E}\left( \frac{\Phi}{\lambda} \right) = \mathbb{E}\left[ \frac{1}{\lambda} \left( \frac{\phi^2 \lambda}{J_1} + \frac{\phi \lambda}{J_2} \right) \right] \|\alpha\|^2 \]
\[ = \mathbb{E}\left( \frac{\phi^2}{J_1} + \frac{\phi}{J_2} \right) \|\alpha\|^2 \]
\[ = \mathbb{E}\left( \mathcal{O}(q^{2+2\delta/3}) \frac{J_1}{J_2} + \mathcal{O}(q^{1+\delta/3}) \frac{J_1}{J_2} \right) \|\alpha\|^2 \]
\[ = \mathcal{O}(q^{2+2\delta/3})\mathbb{E}\left( \frac{J_1}{J_2} \right) \|\alpha\|^2 \]
\[ = \mathcal{O}(q^{2+2\delta/3})\mathcal{O}\left( \frac{q}{\bar{r}^2 q^2} \right) \|\alpha\|^2 \]
\[ = \mathcal{O}\left( \frac{q^{1+2\delta/3}}{\bar{r}^2} \right) \|\alpha\|^2 . \]
Therefore, using \((21)\) and \((B_5)\), when \((\eta, \eta') \in C\), we have

\[
\left\{ E \left\| \frac{df(\eta + s\alpha)}{ds} \right\| \right\}^2 = O(q) U_1 + O\left(\frac{p}{r}\right) U_2 + O(q^2) U_3 + O(1) U_4 + O\left(\frac{1}{q}\right) \|\alpha\|^2
\]

\[
= \left[ O\left(\frac{q^{3+\delta}}{r^2}\right) + O\left(\frac{q^{3+\delta}}{r^3}\right) + O\left(\frac{q^{2+\delta/3}}{r}\right) + O\left(\frac{q^{1+2\delta/3}}{r^2}\right) \right] \|\alpha\|^2 + O\left(\frac{1}{q}\right) \|\alpha\|^2
\]

\[
= O\left(\frac{q^{2+\delta}}{r}\right) \|\alpha\|^2 + O\left(\frac{1}{q}\right) \|\alpha\|^2
\]

\[
= O\left(\frac{q^{2+\delta}}{r} \vee \frac{1}{q}\right) \|\alpha\|^2 .
\]

(22)

Our final major task is to develop bounds for \(U_i, i = 1, 2, 3, 4\), in the case where \((\eta, \eta') \notin C\). First,

\[
e_{\text{max}} = 1 + \frac{\lambda}{r_{\text{max}}} \geq 1 + \frac{\lambda}{m r_{\text{max}}} \geq \frac{1 + \phi}{m} ,
\]

and it follows that

\[
U_1 = E\left(\frac{\Phi}{\epsilon_{\text{max}}^2}\right) \leq E\left[ \frac{m^2}{(1 + \phi)^2} \left( \frac{\phi^2 \lambda}{J_1} + \frac{\phi \lambda}{J_2} \right) \right] \|\alpha\|^2 \leq m^2 E\left(\frac{\lambda}{J_1} + \frac{\lambda}{J_2}\right) \|\alpha\|^2
\]

\[
\leq m^2 E\left(\frac{1}{b_1 + \frac{J_1}{b_1 J_2}}\right) \|\alpha\|^2
\]

\[
= m^2 O(1) \|\alpha\|^2
\]

\[
= O(1) \|\alpha\|^2 .
\]

Now,

\[
\frac{1}{r_{\text{max}} e_{\text{max}}} = \frac{1}{r_{\text{max}} t_{\text{max}}} \leq \frac{m}{t_{\text{max}}}
\]

and it follows that

\[
U_2 = E\left(\frac{\Phi}{r \epsilon_{\text{max}}^4}\right) = \bar{r} E\left(\frac{\Phi}{r \epsilon_{\text{max}}^4}\right) \leq \bar{r} E\left[ \frac{m}{t_{\text{max}}} \left( \frac{m}{1 + \phi} \right)^3 \left( \frac{\phi^2 \lambda}{J_1} + \frac{\phi \lambda}{J_2} \right) \right] \|\alpha\|^2
\]

\[
\leq \bar{r} m^4 E\left(\frac{1}{J_1} + \frac{1}{J_2}\right) \|\alpha\|^2
\]

\[
= \bar{r} O\left(\frac{1}{q}\right) \|\alpha\|^2
\]

\[
= O\left(\frac{\bar{r}}{q}\right) \|\alpha\|^2 .
\]

Clearly,

\[
U_3 = E\left[ (\phi \wedge 1) \frac{1}{J_1} + \frac{1}{J_2} \right] \|\alpha\|^2 \leq E\left(\frac{1}{J_1} + \frac{1}{J_2}\right) \|\alpha\|^2 = O\left(\frac{1}{q}\right) \|\alpha\|^2 .
\]

Finally,

\[
U_4 = E\left(\frac{\Phi}{\lambda \epsilon_{\text{max}}^3}\right) \leq E\left[ \frac{1}{\lambda} \left( \frac{m}{1 + \phi} \right)^3 \left( \frac{\phi^2 \lambda}{J_1} + \frac{\phi \lambda}{J_2} \right) \right] \|\alpha\|^2 \leq m^3 E\left(\frac{1}{J_1} + \frac{1}{J_2}\right) \|\alpha\|^2 = O\left(\frac{1}{q}\right) \|\alpha\|^2 .
\]
Thus, it follows from (21), (B4), and (B5) that

\[
\left\{ E \left\| \frac{df(\eta + s\alpha)}{ds} \right\| \right\}^2 = O(q)U_1 + O\left(\frac{p}{q}\right)U_2 + O(q^2)U_3 + O(1)U_4 + O\left(\frac{1}{q}\right)\|\alpha\|^2
\]

\[
= \left[ O(q) + O(1) + O(q) + O\left(\frac{1}{q}\right) \right] \|\alpha\|^2 + O\left(\frac{1}{q}\right)\|\alpha\|^2
\]

\[
= O(q)\|\alpha\|^2 .
\]

Combining (22) and (23), we have

\[
\left\{ E \left\| \frac{df(\eta + s\alpha)}{ds} \right\| \right\}^2 = \begin{cases} O\left(\sqrt{q^2 + \delta \bar{r}} \lor \frac{1}{\sqrt{q}}\right)\|\alpha\|^2 & \text{if } (\eta, \eta') \in C \\ O(q)\|\alpha\|^2 & \text{otherwise.} \end{cases}
\]

Therefore, there exist \( \gamma < 1 \) and \( \gamma_0 < \infty \) (both depending on \( q \)) such that when \( q \) is large enough,

\[
\sup_{s \in [0,1]} E \left\| \frac{df(\eta + s(\eta' - \eta))}{ds} \right\| \leq \begin{cases} \gamma \|\eta' - \eta\| & (\eta, \eta') \in C \\ \gamma_0 \|\eta' - \eta\| & \text{otherwise,} \end{cases}
\]

where

\[
\gamma = O\left(\sqrt{\frac{q^2 + \delta}{r}} \lor \frac{1}{\sqrt{q}}\right) \quad \text{and} \quad \gamma_0 = O(\sqrt{q}).
\]

\( \square \)
Appendix

A Derivations of Conditional Distributions

Here we derive the distributions of \( \eta_0 \mid \lambda, \tau, y \) and \( \eta_0 \mid \eta_{00}, \lambda, \tau, y \). Recall that \( t_i = \lambda + r_i \tau \). First, the conditional density of \( (\eta_{00}, \eta_0) \) given \( (\lambda, \tau, y) \) is given by

\[
\int_{\mathbb{R}^q} \pi(\eta \mid \lambda, \tau, y) \, d\eta_1 \cdots d\eta_q \\
\propto \int_{\mathbb{R}^q} \exp \left\{ -\frac{\lambda}{2} \sum_{i=1}^q (\eta_i - \eta_0/\sqrt{q})^2 - \frac{\tau}{2} \sum_{i=1}^q \sum_{j=1}^{r_i} (\eta_i + x_{ij}^T \eta_{00}/\sqrt{q} - y_{ij})^2 \right\} \, d\eta_1 \cdots d\eta_q \\
= \prod_{i=1}^q \int_{\mathbb{R}} \exp \left\{ -\frac{\lambda}{2} \left( \eta_i - \eta_0/\sqrt{q} \right)^2 - \frac{\tau}{2} \sum_{j=1}^{r_i} (\eta_i + x_{ij}^T \eta_{00}/\sqrt{q} - y_{ij})^2 \right\} \, d\eta_i \\
= \prod_{i=1}^q \exp \left\{ -\frac{\lambda \eta_i^2}{2q} - \frac{\tau}{2} \sum_{j=1}^{r_i} (x_{ij}^T \eta_{00}/\sqrt{q} - y_{ij})^2 \right\} \\
\times \prod_{i=1}^q \exp \left\{ -\frac{t_i}{2} \left[ \eta_i - \frac{\lambda \eta_i}{\sqrt{q}} + r_i \tau (\bar{y}_i - \bar{x}_i^T \eta_{00}/\sqrt{q}) \right]^2 + \frac{t_i}{2} \left( \frac{\lambda \eta_i}{\sqrt{q}} + r_i \tau (\bar{y}_i - \bar{x}_i^T \eta_{00}/\sqrt{q}) \right)^2 \right\} \, d\eta_i \\
= \prod_{i=1}^q \exp \left\{ -\frac{\lambda \eta_i^2}{2q} + \frac{\lambda^2 \eta_i^2}{2t_i q} + \frac{r_i \lambda \tau}{\sqrt{q} t_i} (\bar{y}_i - \bar{x}_i^T \eta_{00}/\sqrt{q}) \eta_i \right\} \\
\times \exp \left\{ -\frac{\tau}{2} \sum_{j=1}^{r_i} (x_{ij}^T \eta_{00}/\sqrt{q} - y_{ij})^2 + \frac{(r_i \tau)^2}{2t_i} (\bar{y}_i - \bar{x}_i^T \eta_{00}/\sqrt{q})^2 \right\}. \tag{24} \right.

Recall that \( z_i = t_i/(r_i \lambda \tau) \). It follows from (24) that the conditional density of \( \eta_0 \) given \( (\eta_{00}, \lambda, \tau, y) \) is proportional to

\[
\prod_{i=1}^q \exp \left\{ -\frac{\lambda}{2} \left( 1 - \frac{\lambda}{t_i} \right) \eta_i^2 + \frac{r_i \lambda \tau}{\sqrt{q} t_i} (\bar{y}_i - \bar{x}_i^T \eta_{00}/\sqrt{q}) \eta_i \right\} \\
= \prod_{i=1}^q \exp \left\{ -\frac{1}{2q z_i} \eta_i^2 + \frac{1}{\sqrt{q} z_i} (\bar{y}_i - \bar{x}_i^T \eta_{00}/\sqrt{q}) \eta_i \right\} \\
= \exp \left\{ -\eta_i^2 \sum_{i=1}^q \frac{1}{2q z_i} + \eta_0 \sum_{i=1}^q \frac{1}{\sqrt{q} z_i} (\bar{y}_i - \bar{x}_i^T \eta_{00}/\sqrt{q}) \right\}. 
\]

Thus,

\[
\eta_0 \mid \eta_{00}, \lambda, \tau, y \sim N \left( \frac{\sqrt{q} \sum_{i=1}^q (\bar{y}_i - \bar{x}_i^T \eta_{00}/\sqrt{q})/z_i}{\sum_{i=1}^q 1/z_i}, \frac{q}{\sum_{i=1}^q 1/z_i} \right). 
\]
The conditional density of \( \eta_{00} \) given \((\lambda, \tau, y)\) is proportional to the integral of (24) with respect to \( \eta_0 \), which is given by

\[
\exp \left\{ \frac{1}{2} \left( \sum_{i=1}^{q} \frac{1}{z_i} \right)^{-1} \left( \sum_{i=1}^{q} (y_i - \bar{x}_i^T \eta_{00}/\sqrt{\eta})/z_i \right)^2 \right\}
\times \exp \left\{ -\frac{\tau}{2} \sum_{i=1}^{q} \sum_{j=1}^{r_i} (x_{i j}^T \eta_{00}/\sqrt{\eta} - y_{i j})^2 + \sum_{i=1}^{q} \frac{(r_i \tau)^2}{2t_i} (y_i - \bar{x}_i^T \eta_{00}/\sqrt{\eta})^2 \right\}
= \exp \left\{ \frac{\tau}{2} \left( \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{\lambda y_i}{t_i} \right)^{-1} \left( \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{\lambda (y_i - \bar{x}_i^T \eta_{00}/\sqrt{\eta})}{t_i} \right)^2 - \frac{\tau}{2} (X \eta_{00}/\sqrt{\eta} - Y)^T (X \eta_{00}/\sqrt{\eta} - Y) \right. \\
\left. + \frac{\tau}{2} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{r_i \tau}{t_i} (y_i - \bar{x}_i^T \eta_{00}/\sqrt{\eta})^2 \right\}
= \exp \left\{ \frac{\tau}{2} \left( I^T (I - D_c) 1 \right)^{-1} \left( I^T (I - D_c) (\bar{Y} - \bar{X} \eta_{00}/\sqrt{\eta}) \right)^2 - \frac{\tau}{2} (X \eta_{00}/\sqrt{\eta} - Y)^T (X \eta_{00}/\sqrt{\eta} - Y) \right. \\
\left. + \frac{\tau}{2} (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y})^T D_c (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y}) \right\}
= \exp \left\{ \frac{\tau}{2} (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y})^T \left( \frac{I - D_c}{1^T (I - D_c) 1} \right) (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y}) \right. \\
\left. - \frac{\tau}{2} (X \eta_{00}/\sqrt{\eta} - Y)^T (X \eta_{00}/\sqrt{\eta} - Y) + \frac{\tau}{2} (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y})^T D_c (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y}) \right\}
= \exp \left\{ \frac{\tau}{2} (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y})^T \left( D_c + \frac{I - D_c}{1^T (I - D_c) 1} \right) (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y}) \right. \\
\left. - \frac{\tau}{2} (X \eta_{00}/\sqrt{\eta} - Y)^T (X \eta_{00}/\sqrt{\eta} - Y) \right\}.
\] (25)

Now recall that

\[
M = D_c + \frac{(I - D_c) 1 1^T (I - D_c)}{1^T (I - D_c) 1},
\]
so we can rewrite (25) as

\[
\exp \left\{ \frac{\tau}{2} (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y})^T M (\bar{X} \eta_{00}/\sqrt{\eta} - \bar{Y}) - \frac{\tau}{2} (X \eta_{00}/\sqrt{\eta} - Y)^T (X \eta_{00}/\sqrt{\eta} - Y) \right\}.
\]

A simple complete the square argument shows that

\[
\eta_{00} \mid \lambda, \tau, y \sim N_p \left( v, \frac{\tau}{\lambda} Q \right).
\]

where \( Q = (X^T X - \bar{X}^T M \bar{X})^{-1} \) and \( v = \sqrt{\lambda} Q (X^T Y - \bar{X}^T \bar{Y}) \).

**B  Proofs of Supporting Lemmas**

Proof of Lemma 2. Part (i): Let

\[
R = \bigoplus_{i=1}^{q} \frac{1}{r_i} J_{r_i},
\]

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where, as usual, \( J_{r_i} \) denotes an \( r_i \times r_i \) matrix of 1s. Note that \( R \) is symmetric and idempotent. It’s easy to see that \( \bar{X} = RX \) and \( \bar{Y} = RY \). Now

\[
X^\top X = X^\top (R + I - R)X = X^\top R^2 X + X^\top (I - R)^2 X \succ (RX)^\top (RX) = \bar{X}^\top \bar{X}.
\]

Part (ii): By part (i),

\[
\sum_{i=1}^{q} \bar{x}_i \bar{x}_i^\top \preceq \frac{1}{r_{\min}} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \bar{x}_i \bar{x}_i^\top = \frac{1}{r_{\min}} \bar{X}^\top \bar{X} \preceq \frac{1}{r_{\min}} X^\top X.
\]

Part (iii): We have

\[
\sum_{i=1}^{q} \bar{y}_i^2 \leq \frac{1}{r_{\min}} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \bar{y}_i^2 = \frac{1}{r_{\min}} \bar{Y}^\top \bar{Y} \leq \frac{1}{r_{\min}} Y^\top Y.
\]

Part (iv): Define \( a^\top = \left( \frac{1}{qr_1} I_{r_1}, \ldots, \frac{1}{qr_q} I_{r_q} \right) \), and note that \( \bar{y} = a^\top Y \). Now

\[
a^\top a = \sum_{i=1}^{q} r_i \left( \frac{1}{qr_i} \right)^2 \leq \frac{1}{q} \sum_{i=1}^{q} \frac{1}{r_{\min}^2} = \frac{1}{qr_{\min}}.
\]

Hence,

\[
\bar{y}^2 = (a^\top Y)^2 \leq (a^\top a)(Y^\top Y) \leq \frac{Y^\top Y}{qr_{\min}}.
\]

\( \square \)

**Proof of Lemma 11.** For any vector \( x \),

\[
x^\top C^\top Dx = x^\top D^\top Cx = (Cx)^\top Dx
\leq \sqrt{(Cx)^\top Cx} \sqrt{(Dx)^\top Dx}
= \sqrt{x^\top C^\top Cx} \sqrt{x^\top D^\top Dx}
\leq \frac{1}{2} (x^\top C^\top Cx + x^\top D^\top Dx).
\]

Therefore, for any \( x \),

\[
x^\top (C + D)^\top (C + D)x = x^\top C^\top Cx + x^\top D^\top Dx + x^\top C^\top Dx + x^\top D^\top Cx
\leq 2(x^\top C^\top Cx + x^\top D^\top Dx)
= x^\top 2(C^\top C + D^\top D)x.
\]

\( \square \)

**Proof of Lemma 12.** We use contradiction. Suppose that there exists an \( x \) such that

\[
\lambda_{\max} \left\{ \left( \frac{dA^\frac{1}{2}}{dx} \right)^2 \right\} > \frac{\lambda_{\max} \left\{ \left( \frac{dA}{dx} \right)^2 \right\}}{4\lambda_{\min}(A)}.
\]

Then either
1. \( \lambda_{\text{max}} \left( \frac{dA}{dx} \right) > \frac{1}{2\sqrt{\lambda_{\text{min}}(A)}} \sqrt{\lambda_{\text{max}} \left\{ \left( \frac{dA}{dx} \right)^2 \right\} } \), or

2. \( \lambda_{\text{min}} \left( \frac{dA}{dx} \right) < -\frac{1}{2\sqrt{\lambda_{\text{min}}(A)}} \sqrt{\lambda_{\text{max}} \left\{ \left( \frac{dA}{dx} \right)^2 \right\} } \).

Suppose that (1) holds. Then there exists \( x_0 \) such that

\[
\lambda_{\text{max}} \left( \frac{dA}{dx} \bigg|_{x=x_0} \right) > \frac{1}{2\sqrt{\lambda_{\text{min}}(A)}|_{x=x_0}} \sqrt{\lambda_{\text{max}} \left\{ \left( \frac{dA}{dx} \bigg|_{x=x_0} \right)^2 \right\} } .
\]

Let \( A_0 = A(x_0) \),

\[
D_0 = \frac{dA}{dx} \bigg|_{x=x_0} \quad \text{and} \quad R_0 = \frac{dA}{dx} \bigg|_{x=x_0} .
\]

Let \( \lambda_0 = \lambda_{\text{max}}(R_0) \), and let \( \xi_0 \) denote the corresponding (normalized) eigenvector. Then

\[
\lambda_0 > \frac{\sqrt{\lambda_{\text{max}}(D_0^2)}}{2\sqrt{\lambda_{\text{min}}(A_0)}} > 0 .
\]

Now since \( A_0^\frac{1}{2} \) and \( R_0 \) are both symmetric, we have

\[
\xi_0^T D_0 \xi_0 = \xi_0^T \left( \frac{dA}{dx} \bigg|_{x=x_0} \right) \xi_0
= \xi_0^T \left\{ \left( \frac{dA}{dx} \bigg|_{x=x_0} \right) A_0^\frac{1}{2} + A_0^\frac{1}{2} \left( \frac{dA}{dx} \bigg|_{x=x_0} \right) \right\} \xi_0
= \xi_0^T \{ R_0 A_0^\frac{1}{2} + A_0^\frac{1}{2} R_0 \} \xi_0
= 2 \xi_0^T R_0 A_0^\frac{1}{2} \xi_0
= 2 \lambda_0 \xi_0^T A_0^\frac{1}{2} \xi_0
\geq 2 \lambda_0 \sqrt{\lambda_{\text{min}}(A_0)} \xi_0^T \xi_0
> 2 \frac{\sqrt{\lambda_{\text{max}}(D_0^2)}}{2\sqrt{\lambda_{\text{min}}(A_0)}} \sqrt{\lambda_{\text{min}}(A_0)}
= \sqrt{\lambda_{\text{max}}(D_0^2)}
\geq \lambda_{\text{max}}(D_0) ,
\]

but this is a contradiction. An analogous argument shows that 2. also leads to a contradiction.

**C Development of the Upper Bound** \((21)\)

In this Appendix, we develop upper bounds for each of the three terms on the right-hand side of \((20)\), which together yield \((21)\). Of course, the components of \( f(\eta + s\alpha) \) depend on \( \lambda^{(\eta+s\alpha)} \) and
We will require upper bounds on $\tau^{(\eta+sa)}$, so we begin with these. According to the random mapping, we have

$$\lambda^{(\eta+sa)} = \frac{J_1}{b_1 + \frac{1}{2} \sum_{i=1}^{q} (\eta_i + sa_i - (\eta_0 + sa_0)/\sqrt{q})^2}$$

$$\tau^{(\eta+sa)} = \frac{J_2}{b_2 + \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{r_i} [y_{ij} - x_{ij}^T(\eta_{00} + sa_{00})]/\sqrt{q} - (\eta_i + sa_i)]^2}.$$

We will require upper bounds on $(\frac{d\lambda^{(\eta+sa)}}{ds})^2$ and $(\frac{d\tau^{(\eta+sa)}}{ds})^2$. We have

$$\frac{d\lambda^{(\eta+sa)}}{ds} = -\frac{J_1}{J_1^2} \sum_{i=1}^{q} (\eta_i + sa_i - (\eta_0 + sa_0)/\sqrt{q})^2 \sum_{i=1}^{q} (\alpha_i - \alpha_0/\sqrt{q})^2$$

$$\leq 4\left(\frac{\lambda^{(\eta+sa)}}{J_1}\right)^2 \left[ b_1 + \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \sum_{i=1}^{q} (\alpha_i^2 + \alpha_0^2/q) \right]$$

$$\leq 4\left(\frac{\lambda^{(\eta+sa)}}{J_1}\right)^2 \|\alpha\|^2.$$

Thus, by Cauchy-Schwarz,

$$(\frac{d\lambda^{(\eta+sa)}}{ds})^2 \leq 4\left(\frac{\lambda^{(\eta+sa)}}{J_1}\right)^2 \sum_{i=1}^{q} (\eta_i + sa_i - (\eta_0 + sa_0)/\sqrt{q})^2 \sum_{i=1}^{q} (\alpha_i - \alpha_0/\sqrt{q})^2$$

$$\leq 4\left(\frac{\lambda^{(\eta+sa)}}{J_1}\right)^2 \left[ b_1 + \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \sum_{i=1}^{q} (\alpha_i^2 + \alpha_0^2/q) \right] \sum_{i=1}^{q} (\alpha_i^2 + \alpha_0^2/q)$$

$$\leq 4\left(\frac{\lambda^{(\eta+sa)}}{J_1}\right)^4 \left[ \frac{J_1}{J_1^2} \right] \|\alpha\|^2.$$

A similar argument yields

$$(\frac{d\tau^{(\eta+sa)}}{ds})^2 \leq 4\left(\frac{\tau^{(\eta+sa)}}{J_2}\right)^3 \left\{ q \sum_{i=1}^{q} \sum_{j=1}^{r_i} \alpha_i^2 + \frac{1}{q} \alpha_{00} + \frac{q}{q} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \alpha_{00} \alpha_{00} \right\}.$$

Using Remark 10, we have

$$(\frac{d\tau^{(\eta+sa)}}{ds})^2 \leq 4\left(\frac{\tau^{(\eta+sa)}}{J_2}\right)^3 \left\{ q \sum_{i=1}^{q} \sum_{j=1}^{r_i} \alpha_i^2 + \frac{q}{q} \alpha_{00} + \frac{q}{q} \sum_{i=1}^{q} \sum_{j=1}^{r_i} \alpha_{00} \alpha_{00} \right\}.$$

It now follows from (26) and (27) that there exists a positive constant $\ell_1$ such that

$$(\frac{d\lambda^{(\eta+sa)}}{ds})^2 \leq \frac{\ell_1}{J_1} \|\alpha\|^2 \quad \text{and} \quad (\frac{d\tau^{(\eta+sa)}}{ds})^2 \leq \frac{\ell_1}{J_2} \|\alpha\|^2.$$

In order to simplify notation, we omit the superscript $\eta + sa$ in the remainder of the proof. This should not cause any confusion. We now develop an upper bound for $E\|\frac{d\theta_{00}}{ds}\|^2$. Recall that
\[ \tilde{\eta}_{00} = v + \sqrt{\frac{2}{\tau}} Q^{\frac{1}{2}} N_{00}, \text{ where } Q = (X^T X - \tilde{X}^T \tilde{X})^{-1} \text{ and } v = \sqrt{q} Q (X^T Y - \tilde{X}^T \tilde{Y}). \] Since \( RX = \tilde{X}, \) \( RY = \tilde{Y}, \) \( R^2 = R, \) and \( R \) and \( M \) commute, we have

\[ \tilde{\eta}_{00} = \sqrt{q} Q (X^T Y - \tilde{X}^T \tilde{Y}) + \sqrt{\frac{q}{\tau}} Q^{\frac{1}{2}} N_{00} = \sqrt{q} Q X^T (I - MR) Y + \sqrt{\frac{q}{\tau}} Q^{\frac{1}{2}} N_{00}. \]

So we have

\[ \frac{d\tilde{\eta}_{00}}{ds} = \sqrt{q} \frac{d}{ds} \left( Q X^T (I - MR) Y + \frac{1}{\tau} \frac{d}{ds} \right) Q^{\frac{1}{2}} N_{00} \]

\[ = \sqrt{q} \left[ \left( \frac{d Q}{ds} \right) X^T (I - MR) Y - Q X^T \left( \frac{d M}{ds} \right) R Y - \frac{1}{2 \tau^2} \left( \frac{d\tau}{ds} \right) Q^{\frac{1}{2}} N_{00} + \frac{1}{\tau^2} \left( \frac{d Q^{\frac{1}{2}}}{ds} \right) N_{00} \right]. \]

Thus,

\[ \left\| \frac{d\tilde{\eta}_{00}}{ds} \right\|^2 \leq 4q \left\| \left( \frac{d Q}{ds} \right) X^T (I - MR) Y \right\|^2 + 4q \left\| Q X^T \left( \frac{d M}{ds} \right) R Y \right\|^2 \]

\[ + \frac{q (\tau')^2}{\tau^3} \left\| Q^{\frac{1}{2}} N_{00} \right\|^2 + \frac{4q}{\tau} \left\| \left( \frac{d Q^{\frac{1}{2}}}{ds} \right) N_{00} \right\|^2, \]

where \( \tau' = \frac{d\tau}{ds} \). Now define

\[ T_0 = E \left\{ \frac{q (\tau')^2}{\tau^3} \left\| Q^{\frac{1}{2}} N_{00} \right\|^2 \right\}, \]

\[ T_1 = E \left\{ \frac{q}{\tau} \left\| \left( \frac{d Q^{\frac{1}{2}}}{ds} \right) N_{00} \right\|^2 \right\}, \]

\[ T_2 = E \left\{ q \left\| \left( \frac{d Q}{ds} \right) X^T (I - MR) Y \right\|^2 \right\}, \]

\[ T_3 = E \left\{ q \left\| Q X^T \left( \frac{d M}{ds} \right) R Y \right\|^2 \right\}. \]

So we have

\[ E \left\| \frac{d\tilde{\eta}_{00}}{ds} \right\|^2 \leq T_0 + 4T_1 + 4T_2 + 4T_3. \quad (29) \]

Using (28) and (12), we have

\[ T_0 = E \left\{ \frac{q (\tau')^2}{\tau^3} \left\| Q^{\frac{1}{2}} N_{00} \right\|^2 \right\} \leq \frac{\ell_1 \tilde{r} q}{\ell_2} E \left( \frac{1}{J_2} \right) E (N_{00}^T N_{00}) \| \alpha \|^2 \]

\[ \leq \frac{\ell_1}{k_1} \left( \frac{1}{J_2} \right) E (N_{00}^T N_{00}) \| \alpha \|^2 \]

\[ = \frac{2\ell_1 p}{k_1 (N + 2a_2 - 2)} \| \alpha \|^2 \]

\[ = O \left( \frac{p}{\tau q} \right) \| \alpha \|^2. \quad (30) \]

We now develop some bounds that will allow us to handle \( T_1, T_2 \) & \( T_3 \). Recall that \( e_i = c_i^{-1} = t_i / (r_i \tau) \), and that \( e_{\text{min}} = t_{\text{min}} / (r_{\text{min}} \tau) \) and \( e_{\text{max}} = t_{\text{max}} / (r_{\text{max}} \tau) \). So \( e_{\text{min}} \) and \( e_{\text{max}} \) are actually the
largest and smallest $e_i$, respectively. Now,
\[
\frac{dD_c}{ds} = \frac{d}{ds} \left( \bigoplus_{i=1}^{q} \frac{1}{e_i} I_{r_i} \right) = -\bigoplus_{i=1}^{q} \frac{1}{e_i^2} \left( \frac{de_i}{ds} \right) I_{r_i} . \tag{31}
\]
Recall that
\[
\Phi = \left( \frac{\lambda^3}{r^2 \tau^2 J_1} + \frac{\lambda^2}{r \tau J_2} \right) \|\alpha\|^2 .
\]
Then, for each $i = 1, 2, \ldots, q$, we have
\[
\left( \frac{de_i}{ds} \right)^2 = \left( \frac{d}{ds} \frac{\lambda}{r_i \tau} \right)^2 = \left( \frac{\lambda'}{r_i \tau} - \frac{\lambda \tau'}{r_i \tau^2} \right)^2 
\leq \frac{2(\lambda')^2}{(r_i \tau)^2} + \frac{2\lambda^2 (\tau')^2}{r_i \tau^4} 
\leq \frac{2\ell_1 \lambda^3}{r_i \min \tau^2 J_1} \|\alpha\|^2 + \frac{2\ell_1 \lambda^2}{r_i \min \tau J_2} \|\alpha\|^2 
\leq 2m^2 \ell_1 \left( \frac{\lambda^2}{r_i \tau^2 J_1} + \frac{\lambda \lambda}{r_i \tau J_2} \right) \|\alpha\|^2 
= \ell_2 \Phi , \tag{32}
\]
where the second inequality follows from (28), and $\ell_2 = 2m^2 \ell_1$. So, using (31), we have
\[
\left( \frac{dD_c}{ds} \right)^2 \succeq \bigoplus_{i=1}^{q} \frac{1}{e_i^4} \frac{\ell_2 \Phi}{e_i^4} I_{r_i} = \bigoplus_{i=1}^{\max} \frac{\ell_2 \Phi}{e_i^4} I_N .
\]
Now recall that $M = D_c + \frac{(I-D_c)11^\top (I-D_c)}{1^\top (I-D_c)1}$. Define
\[
w_1 = (I - D_c)1 \quad \text{and} \quad w_2 = \left( \frac{dD_c}{ds} \right) 1 .
\]
Then we have
\[
\frac{d}{ds} \left\{ \frac{(I-D_c)11^\top (I-D_c)}{1^\top (I-D_c)1} \right\} = \left\{ \frac{(I-D_c)11^\top (I-D_c)}{1^\top (I-D_c)1} \right\} 1^\top \left( \frac{dD_c}{ds} \right) 1 
- \left\{ \frac{1}{1^\top (I-D_c)1} \right\} \left\{ \left( \frac{dD_c}{ds} \right) 11^\top (I-D_c) + (I-D_c)11^\top \left( \frac{dD_c}{ds} \right) \right\} 
\]
which is equal to
\[
\left\{ \frac{w_1 w_1^\top}{1^\top w_1^2} \right\} (1^\top w_2) \quad \text{and} \quad \left\{ \frac{1}{1^\top w_1} \right\} \left( w_2 w_1^\top + w_1 w_2^\top \right) .
\]
Therefore,
\[
\frac{dM}{ds} = \frac{dD_c}{ds} + \left\{ \frac{w_1 w_1^\top}{1^\top w_1^2} \right\} (1^\top w_2) \quad \text{and} \quad \left\{ \frac{1}{1^\top w_1} \right\} \left( w_2 w_1^\top + w_1 w_2^\top \right) . \tag{33}
\]
Note that each of the three terms on the left-hand side of (33) is a symmetric matrix. Several applications of Lemma 11 leads to

\[
\left( \frac{dM}{ds} \right)^2 = \left( \frac{dD_c}{ds} + \frac{w_1 w_1^\top}{(1^\top w_1)^2} \right) (1^\top w_2)^2 \leq 2 \left( \frac{dD_c}{ds} + \frac{w_1 w_1^\top}{(1^\top w_1)^2} \right) (1^\top w_2)^2 + 2 \left( \frac{1}{1^\top w_1} \right) (w_2 w_1^\top + w_1 w_2^\top)^2 \\
\leq 4 \left( \frac{dD_c}{ds} \right)^2 + 4 \left( \frac{w_1 w_1^\top}{(1^\top w_1)^2} \right) (1^\top w_2)^2 + \frac{4}{(1^\top w_1)^2} \left[ (w_2 w_1^\top)^2 + (w_1 w_2^\top)^2 \right] \\
\leq 4 \left( \frac{dD_c}{ds} \right)^2 + 4 w_1^\top w_1 (1^\top w_2)^2 + \frac{8}{(1^\top w_1)^2} (w_1^\top w_1)(w_2^\top w_2) I. 
\]  

(34)

Now

\[
w_1^\top w_1 = 1^\top (I - D_c)^2 1 = \sum_{i=1}^q \sum_{j=1}^{r_i} \left( \frac{\lambda_{ij}}{t_{ij}} \right)^2 \leq \bar{\rho} q \left( \frac{\lambda_{i_{\min}}}{t_{i_{\min}}} \right)^2, \\
w_2^\top w_2 = 1^\top \left( \frac{dD_c}{ds} \right)^2 1 \leq 1^\top \left( \frac{\ell_2 \Phi}{e_{\max}} I_q \right) 1 = \ell_2 \bar{\rho} q \left( \frac{\Phi}{e_{\max}} \right), \\
(1^\top w_1)^2 = [1^\top (I - D_c) 1]^2 = \left( \sum_{i=1}^q \sum_{j=1}^{r_i} \lambda_{ij} \right)^2 \geq \left( \bar{\rho} q \frac{\lambda_{i_{\min}}}{t_{i_{\min}}} \right)^2, \\
(1^\top w_2)^2 \leq (w_2^\top w_2)(1^\top 1) \leq \ell_2 \bar{\rho} q^2 \left( \frac{\Phi}{e_{\max}} \right). 
\]

So, in conjunction with (34), we have

\[
\left( \frac{dM}{ds} \right)^2 \leq 4 \left( \frac{dD_c}{ds} \right)^2 + 4 w_1^\top w_1 (1^\top w_2)^2 + \frac{8}{(1^\top w_1)^2} (w_1^\top w_1)(w_2^\top w_2) I \\
\leq 4 \ell_2 \Phi \left( \frac{\Phi}{e_{\max}} \right) I + 4 \ell_2 m^4 \Phi \left( \frac{\Phi}{e_{\max}} \right) I + \frac{8 \ell_2 m^2 \Phi}{e_{\max}} I \\
\leq 16 \ell_2 m^4 \left( \frac{\Phi}{e_{\max}} \right) I. 
\]  

(35)

Recall that \( Q^{-1} = (X^\top X - \bar{X}^\top M \bar{X}) \). We have

\[
\left( \frac{dQ^{-1}}{ds} \right)^2 = \left( - \bar{X}^\top \left( \frac{dM}{ds} \right) \bar{X} \right)^2 \\
= \bar{X}^\top \left( \frac{dM}{ds} \right) \bar{X} \bar{X}^\top \left( \frac{dM}{ds} \right) \bar{X} \\
\leq \bar{\rho} k_2 \bar{X}^\top \left( \frac{dM}{ds} \right)^2 \bar{X} \\
\leq 16 \ell_2 m^4 \bar{\rho} k_2 \bar{X}^\top \left( \frac{\Phi}{e_{\max}} \right) I \bar{X} \\
\leq 16 \ell_2 m^4 (\bar{\rho} k_2)^2 \left( \frac{\Phi}{e_{\max}} \right) I_p \\
= \ell_3 (\bar{\rho})^2 \left( \frac{\Phi}{e_{\max}} \right) I_p. 
\]

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where the first and third inequalities follow from (B2) (and Lemma 9), the second inequality follows from (35), and $\ell_3 = 16\ell_2 m^4 k_2^2$. Recall that $Q^2 \preceq (q^\tau k_1)^{-2}I$. It follows that

$$\left( \frac{dQ}{ds} \right)^2 = -Q \left( \frac{dQ^{-1}}{ds} \right) Q \preceq \frac{1}{(q^\tau k_1)^2} Q \left( \frac{dQ^{-1}}{ds} \right)^2 Q \preceq \ell_3 \left( \frac{\Phi}{\epsilon_{\text{max}}} \right) Q^2 \preceq \ell_3 k_1^2 \left( \frac{\Phi}{\epsilon_{\text{max}}} \right) I_\rho \cdot \quad (36)$$

It’s clear that $X^\tau X - \tilde{X}^\tau M \tilde{X} \preceq X^\tau X$, and it follows that $\lambda_{\text{max}}(X^\tau X - \tilde{X}^\tau M \tilde{X}) \leq \lambda_{\text{max}}(X^\tau X)$. Furthermore, we have $\lambda_{\text{min}}[(X^\tau X - \tilde{X}^\tau M \tilde{X})^{-1}] = [\lambda_{\text{max}}(X^\tau X - \tilde{X}^\tau M \tilde{X})]^{-1}$ and $\lambda_{\text{min}}[(X^\tau X)^{-1}] = [\lambda_{\text{max}}(X^\tau X)]^{-1}$. It follows that

$$\lambda_{\text{min}}(Q) = \lambda_{\text{min}}[(X^\tau X - \tilde{X}^\tau M \tilde{X})^{-1}] \geq \lambda_{\text{min}}[(X^\tau X)^{-1}] \geq \frac{1}{\bar{r} q k_2} \cdot \quad (37)$$

Using Lemma 12, (36), and (37), we have

$$\lambda_{\text{max}} \left\{ \left( \frac{dQ^2}{ds} \right)^2 \right\} \leq \lambda_{\text{max}} \left\{ \left( \frac{dQ}{ds} \right)^2 \right\} \leq \frac{\ell_3 k_2}{4 \lambda_{\text{min}}(Q)} \leq \frac{\ell_3 k_2}{4 \bar{r} q k_1^2} \left( \frac{\Phi}{\epsilon_{\text{max}}} \right) \leq \left( \frac{\ell_3 k_2}{4 \bar{r} q k_1^2} \right) \left( \frac{\Phi}{\epsilon_{\text{max}}} \right).$$

We are now ready to attack $T_1, T_2,$ and $T_3$. We have

$$T_1 = E \left\{ \frac{q}{\tau} \left\| \left( \frac{dQ^2}{ds} \right) N_{00} \right\|^2 \right\}$$

$$= q E \left\{ \frac{1}{\tau} N_{00} \left( \frac{dQ^2}{ds} \right)^2 N_{00} \right\}$$

$$\leq q E \left\{ \frac{1}{\tau} \left( \frac{\ell_3 k_2}{4 \bar{r} q k_1^2} \right) \left( \frac{\Phi}{\epsilon_{\text{max}}} \right) N_{00}^T N_{00} \right\}$$

$$= O \left( \frac{p}{\tau} \right) E \left( \frac{\Phi}{\tau \epsilon_{\text{max}}} \right).$$

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Recall that \( T_2 = \mathbb{E}\{ q \| (\frac{dQ}{ds}) X^\top (I - MR) Y \|^2 \} \). So

\[
\| (\frac{dQ}{ds}) X^\top (I - MR) Y \|^2 = Y^\top (I - MR) X (\frac{dQ}{ds})^2 X^\top (I - MR) Y \\
\leq \left( \frac{\ell_3}{\sqrt{q^2 q_1 k_1}} \right) \left( \frac{\Phi}{\epsilon_{\max}} \right) Y^\top (I - MR) X X^\top (I - MR) Y \\
\leq \left( \frac{\ell_3}{\sqrt{q^2 q_1 k_1}} \right) \left( \frac{\Phi}{\epsilon_{\max}} \right) Y^\top Y \lambda_{\max} [(I - MR) XX^\top (I - MR)] \\
\leq \left( \frac{\ell_3}{\sqrt{q^2 q_1 k_1}} \right) \left( \frac{\Phi}{\epsilon_{\max}} \right) (\tilde{r} q \ell) \lambda_{\max} [X^\top (I - MR)^2 X] \\
\leq \left( \frac{\ell_3 \ell}{\tilde{r} q k_1^2} \right) \left( \frac{\Phi}{\epsilon_{\max}} \right) \lambda_{\max} [2(X^\top X + \tilde{X}^\top M^2 \tilde{X}^\top)] \\
\leq \left( \frac{\ell_3 \ell}{\tilde{r} q k_1^2} \right) \left( \frac{\Phi}{\epsilon_{\max}} \right) (4\tilde{r} q k_2) \\
= \mathcal{O}(1) \left( \frac{\Phi}{\epsilon_{\max}} \right),
\]

where the first inequality is from \( (35) \), the third uses \( (B_3) \), the fourth follows from Lemma \( \mathbb{I} \) and the fact that \( \tilde{X}^\top M^2 \tilde{X}^\top \preceq \tilde{X}^\top X \preceq X^\top X \) (recall that \( M \preceq I \)).

Now recall that \( T_3 = \mathbb{E}\{ q \| Q X^\top (\frac{dM}{ds}) R Y \|^2 \} \). We have

\[
\| Q X^\top (\frac{dM}{ds}) R Y \|^2 = \bar{Y} \left( \frac{dM}{ds} \right) X Q^2 X^\top \left( \frac{dM}{ds} \right) \bar{Y} \leq \frac{16\ell_2 k_2 m^4}{k_1^2} \left( \frac{\Phi}{\epsilon_{\max}} \right)^2 = \mathcal{O}(1) \left( \frac{\Phi}{\epsilon_{\max}} \right),
\]

where we have used \( Q^2 \preceq (q\tilde{r} k_1)^{-2} I, (B_2), (35) \), and \( (B_3) \). Recall that

\[
U_1 = \mathbb{E}\left( \frac{\Phi}{\epsilon_{\max}} \right) \quad \text{and} \quad U_2 = \mathbb{E}\left( \frac{\Phi}{\tau \epsilon_{\max}} \right).
\]

Since \( e_{\max} \geq 1, \frac{\Phi}{\epsilon_{\max}} \leq \frac{\Phi}{e_{\max}} \), so we have \( T_1 = \mathcal{O}(\frac{\Phi}{\tau}) U_2, T_2 = \mathcal{O}(q) U_1, \) and \( T_3 = \mathcal{O}(q) U_1 \). It now follows from \( (29) \) and \( (30) \) that

\[
\mathbb{E}\| \frac{d\tilde{\eta}_{00}}{ds} \|^2 \leq T_0 + 4T_1 + 4T_2 + 4T_3 = \mathcal{O}\left( \frac{\Phi}{\tau q} \right) \| \alpha \|^2 + \mathcal{O}(q) U_1 + \mathcal{O}\left( \frac{\Phi}{\tau} \right) U_2. \quad (38)
\]

We now develop an upper bound for \( \mathbb{E}(\frac{d\tilde{\eta}_{00}}{ds})^2 \), which is the second term on the right-hand side of \( (20) \). Recall that \( \tilde{\eta}_{00} = v + \sqrt{\frac{2}{7} Q^2 \tilde{N}_{00}} \). Hence,

\[
\tilde{\eta}_{00} = \sqrt{\frac{\sum_{i=1}^{q} (\tilde{y}_i - \tilde{x}_i^\top \tilde{\eta}_{00})/\sqrt{q}}{z_i}} + \sqrt{\frac{q}{\sum_{i=1}^{q} 1/z_i}} \tilde{N}_{00} \\
= \sqrt{q} \left( \frac{\sum_{i=1}^{q} (\tilde{y}_i - \tilde{x}_i^\top v/\sqrt{q} - \sqrt{\frac{2}{7} \tilde{x}_i^\top Q^2 \tilde{N}_{00} / \sqrt{q}})/z_i}{\sum_{i=1}^{q} 1/z_i} \right) + \sqrt{\frac{q}{\sum_{i=1}^{q} 1/z_i}} \tilde{N}_{00} \\
= \sqrt{q} \left[ \frac{\sum_{i=1}^{q} (\tilde{y}_i - \tilde{x}_i^\top v/\sqrt{q})/z_i}{\sum_{i=1}^{q} 1/z_i} - \left( \frac{2}{7} \sum_{i=1}^{q} \tilde{x}_i^\top Q^2 \tilde{N}_{00} / z_i \right) \right] + \sqrt{\frac{q}{\sum_{i=1}^{q} 1/z_i}} \tilde{N}_{00}.
\]
Using (32) and the fact that \( e \)

Now define

We have

So, we have

We have

where

\[ T_4 = qE \left[ \frac{d}{ds} \left( \sum_{i=1}^{q} \left( \frac{\bar{y}_i - \bar{x}_i^T v / \sqrt{q}}{z_i} \right) \right) \right]^2, \]

\[ T_5 = E \left[ \frac{d}{ds} \left( \sqrt{\sum_{i=1}^{q} \frac{x_i^T Q x_i}{z_i}} \right) \right]^2, \]

\[ T_6 = qE \left[ \frac{d}{ds} \left( \sqrt{\sum_{i=1}^{q} \frac{1}{z_i}} N_0 \right) \right]^2. \]

So, we have

\[ E \left( \frac{d\eta_0}{ds} \right)^2 \leq 3T_4 + 3T_5 + 3T_6. \]

We have

\[ T_4 = qE \left[ \frac{d}{ds} \left( \sum_{i=1}^{q} \left( \frac{\bar{y}_i - \bar{x}_i^T v / \sqrt{q}}{z_i} \right) \right) \right]^2 \]

\[ = qE \left[ \frac{d}{ds} \left( \sum_{i=1}^{q} \left( \frac{\bar{y}_i - \bar{x}_i^T v / \sqrt{q}}{e_i} \right) \right) \right]^2 \]

\[ = qE \left[ \sum_{i=1}^{q} \left( \frac{e_i / e_i^2}{\sum_{i=1}^{q} 1/e_i} \right) \right]^2 \]

\[ \sum_{i=1}^{q} \frac{\bar{y}_i - \bar{x}_i^T v / \sqrt{q}}{\sum_{i=1}^{q} 1/e_i} - \sum_{i=1}^{q} \frac{\bar{y}_i - \bar{x}_i^T v / \sqrt{q}}{\sum_{i=1}^{q} 1/e_i} \]

\[ \leq 3T_{41} + 3T_{42} + 3T_{43}. \]

where

\[ T_{41} = qE \left[ \sum_{i=1}^{q} \left( \frac{e_i / e_i^2}{\sum_{i=1}^{q} 1/e_i} \right) \right]^2 \]

\[ T_{42} = qE \left[ \sum_{i=1}^{q} \left( \frac{\bar{y}_i - \bar{x}_i^T v / \sqrt{q}}{\sum_{i=1}^{q} 1/e_i} \right) \right]^2 \]

\[ T_{43} = E \left[ \frac{\sum_{i=1}^{q} \bar{x}_i^T (d\frac{dx}{ds}) / e_i}{\sum_{i=1}^{q} 1/e_i} \right]^2. \]

Using (32) and the fact that \( e_{\text{min}} / e_{\text{max}} \leq m \) yields

\[ T_{41} = qE \left[ \sum_{i=1}^{q} \left( \frac{e_i / e_i^2}{\sum_{i=1}^{q} 1/e_i} \right) \right]^2 \]

\[ \leq qE \left\{ \frac{e_{\text{min}}}{q^4} \left( \sum_{i=1}^{q} \frac{1}{e_i} \right) \left( \sum_{i=1}^{q} (e_i^2) \right) \left( \sum_{i=1}^{q} \frac{1}{e_i^2} \right) \left( \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T v / \sqrt{q})^2 \right) \right\} \]

\[ \leq qE \left\{ \frac{e_{\text{min}}}{q^4} \left( \frac{q}{e_{\text{max}}^4} \right) (q\ell_2 \Phi) \left( \frac{q}{e_{\text{max}}^2} \right) \left( \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T v / \sqrt{q})^2 \right) \right\} \]

\[ = \mathcal{O}(1) E \left\{ \frac{\Phi}{e_{\text{max}}^2} \left( \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T v / \sqrt{q})^2 \right) \right\}. \]
But
\[\sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T v / \sqrt{q})^2 \leq 2 \sum_{i=1}^{q} \bar{y}_i^2 + \frac{2}{q} v^T \left( \sum_{i=1}^{q} \bar{x}_i \bar{x}_i^T \right) v \leq 2 \sum_{i=1}^{q} \bar{y}_i^2 + \frac{2mqk_2}{q} v^T v = O(q), \] (40)
where the second inequality follows from Remark 10 and the final equality follows from (14) and Remark 10. It follows that \( T_{41} = O(q)U_1 \). Similarly, we have
\[ T_{42} = qE \left[ \frac{\sum_{i=1}^{q} [e_i^2 e_i^2 (\bar{y}_i - \bar{x}_i^T v / \sqrt{q})]^2}{\sum_{i=1}^{q} 1/e_i} \right] \]
\[ \leq qE \left\{ \frac{e_i^2}{q^2} \left( \sum_{i=1}^{q} \frac{1}{e_i^2} \right) \left( \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T v / \sqrt{q})^2 \right) \right\} \]
\[ \leq qE \left\{ \frac{e_i^2}{q^2} \left( \frac{q^2}{e_i^2} \right) \left( \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T v / \sqrt{q})^2 \right) \right\} \]
\[ = O(q)U_1. \]

It follows from our work on \( \left\| \frac{d\bar{y}}{ds} \right\| \) that \( E \left\| \frac{dv}{ds} \right\|^2 \leq 2(T_2 + T_3) \). Thus,
\[ T_{43} = E \left[ \frac{\sum_{i=1}^{q} \bar{x}_i^T (\frac{dv}{ds}) / e_i}{\sum_{i=1}^{q} 1/e_i} \right]^2 \]
\[ \leq E \left\{ \frac{e_i^2}{q^2} \left( \sum_{i=1}^{q} \frac{1}{e_i^2} \right) \left( \sum_{i=1}^{q} \bar{x}_i \bar{x}_i^T \right) \left( \sum_{i=1}^{q} \frac{dv}{ds} \right) \right\} \]
\[ \leq E \left\{ \frac{e_i^2}{q^2} \left( \frac{q^2}{e_i^2} \right) \left( \sum_{i=1}^{q} \frac{dv}{ds} \right) \right\} \]
\[ \leq m^3 k_2 E \left\| \frac{dv}{ds} \right\|^2 \]
\[ \leq 2m^3 k_2 (T_2 + T_3) \]
\[ = O(q)U_1. \]

We conclude that
\[ T_4 = O(q)U_1. \] (41)

We now move on to \( T_5 \). We have
\[ T_5 = E \left[ \frac{d}{ds} \left( \sqrt{\frac{q}{q}} \sum_{i=1}^{q} \frac{\bar{x}_i^T Q \frac{1}{N_00}}{1/z_i} \right) \right]^2 \]
\[ = E \left[ \sum_{i=1}^{q} \left[ \frac{z_i}{z_i^2} \right] \left( \sqrt{\frac{q}{q}} \sum_{i=1}^{q} \frac{\bar{x}_i^T Q \frac{1}{N_00}}{1/z_i} \right) - \sqrt{\frac{q}{q}} \sum_{i=1}^{q} \left[ \frac{z_i}{z_i^2} \right] \bar{x}_i^T Q \frac{1}{N_00} \right] \frac{1}{\sum_{i=1}^{q} 1/z_i} \]
\[ \leq 3T_{51} + 3T_{52} + 3T_{53}, \]
where

\[
T_{51} = E \left[ \frac{\sum_{i=1}^{q} \left[ z_i' / z_i \right] \sqrt{\frac{2}{\tau} \sum_{i=1}^{q} x_i^t Q_{11} N_{00} / z_i}}{\left[ \sum_{i=1}^{q} 1 / z_i \right]^2} \right]^2,
\]

\[
T_{52} = E \left[ \frac{\sqrt{\frac{2}{\tau} \sum_{i=1}^{q} \left[ z_i' / z_i \right] x_i^t Q_{11}^{1/2} N_{00}}}{\sum_{i=1}^{q} 1 / z_i} \right]^2,
\]

\[
T_{53} = E \left[ \frac{\sum_{i=1}^{q} x_i^t \left( \frac{1}{\tau} \sqrt{\frac{2}{\tau} Q_{11}^{1/2} N_{00}} \right) / z_i}{\sum_{i=1}^{q} 1 / z_i} \right]^2.
\]

Now,

\[
T_{51} = E \left[ \frac{\left[ \sum_{i=1}^{q} \left[ z_i' / z_i \right] \right] \sqrt{\frac{2}{\tau} \sum_{i=1}^{q} x_i^t Q_{11}^{1/2} N_{00} / z_i}}{\left[ \sum_{i=1}^{q} 1 / z_i \right]^2} \right]^2
\leq E \left[ \frac{\left[ \sum_{i=1}^{q} \left[ z_i' / z_i \right] \right] \sqrt{\frac{2}{\tau} \sum_{i=1}^{q} x_i^t Q_{11}^{1/2} N_{00} / z_i}}{\left[ \sum_{i=1}^{q} 1 / z_i \right]^2} \right]^2
\leq E \left[ \frac{\left[ \sum_{i=1}^{q} \left[ z_i' / z_i \right] \right] \sqrt{\frac{2}{\tau} \sum_{i=1}^{q} x_i^t Q_{11}^{1/2} N_{00} / z_i}}{\left[ \sum_{i=1}^{q} 1 / z_i \right]^2} \right]^2
\leq \frac{m^4}{q} E \left[ \left( \sum_{i=1}^{q} \left[ z_i' / z_i \right] \right) \left( \frac{1}{\tau} \sum_{i=1}^{q} x_i^t Q_{11}^{1/2} N_{00} / z_i \right)^2 \right] .
\]

But we have

\[
\sum_{i=1}^{q} \left( x_i^t Q_{11}^{1/2} N_{00} / z_i \right)^2 = N_{00}^t Q_{11}^{1/2} \left( \sum_{i=1}^{q} x_i x_i^t \right) Q_{11}^{1/2} N_{00} \leq (mk_2) N_{00}^t Q_{11} N_{00} \leq \frac{mk^2}{r k_1} N_{00}^t N_{00},
\]

so that

\[
T_{51} \leq \frac{m^4}{q} E \left[ \left( \sum_{i=1}^{q} \left[ z_i' / z_i \right] \right) \left( \frac{1}{\tau} \sum_{i=1}^{q} x_i^t Q_{11}^{1/2} N_{00} / z_i \right)^2 \right]
\leq \frac{m^4}{q} E \left[ \left( \sum_{i=1}^{q} \left[ z_i' / z_i \right] \right) \left( \frac{1}{\tau} \sum_{i=1}^{q} x_i^t Q_{11}^{1/2} N_{00} / z_i \right)^2 \right] \leq \frac{m^5 k^2}{q r k_1} E \left[ \left( \sum_{i=1}^{q} \left[ z_i' / z_i \right] \right) \left( \frac{1}{\tau} \sum_{i=1}^{q} x_i^t Q_{11}^{1/2} N_{00} / z_i \right)^2 \right] \leq \frac{m^5 k^2}{q r k_1} E \left[ \left( \sum_{i=1}^{q} \left[ z_i' / z_i \right] \right) \right] \left( \frac{1}{\tau} \sum_{i=1}^{q} x_i^t Q_{11}^{1/2} N_{00} / z_i \right)^2 .
\]

(43)
Recall that $z_i = \frac{1}{r_i \lambda r_i} = \frac{1}{r_i r_i \lambda} + \frac{1}{\lambda}$. Thus,

$$(z_i')^2 = \left[ \frac{d}{ds} \left( \frac{1}{r_i \lambda r_i} + \frac{1}{\lambda} \right) \right]^2$$

$$= \left[ - \frac{1}{r_i^2 \lambda^2} \right]^2$$

$$\leq 2 \left( \frac{r_i'}{r_i^2} \right)^2 + 2 \left( \frac{\lambda'}{\lambda} \right)^2$$

$$\leq \frac{2m^2 \ell_1}{\tau} \| \alpha \|^2 + \frac{2m^2 \ell_1}{\lambda} \| \alpha \|^2$$

$$= 2m^2 \ell_1 \left( \frac{1}{\tau} \frac{1}{J_2} + \frac{1}{\lambda} \frac{1}{J_1} \right) \| \alpha \|^2.$$  \hfill (44)

Now

$$\frac{1}{\tau z_{\text{max}}} = r_{\text{max}} \left( \frac{\lambda}{\lambda + r_{\text{max}} \tau} \right) \leq r_{\text{max}} \left( \frac{\lambda}{r_{\text{max}} \tau} \wedge 1 \right) \leq r_{\text{max}} \left( \frac{\lambda}{\tau} \wedge 1 \right) \leq r_{\text{max}} (\phi \wedge 1),$$

where $\phi = \lambda/(r \tau)$. Thus,

$$T_{51} \leq \frac{m^5 k_2 p}{q r k_1} E \left[ \left( \sum_{i=1}^q (z_i')^2 \left( \frac{1}{\tau z_{\text{max}}} \right) \right) \right]$$

$$\leq \frac{2m^7 \ell_1 k_2 p}{\tau k_1} E \left[ \left( \frac{1}{\tau J_2} + \frac{1}{\lambda J_1} \right) \left( \frac{r_{\text{max}} (\phi \wedge 1)}{z_{\text{max}}} \right) \right] \| \alpha \|^2$$

$$\leq \frac{2m^8 \ell_1 k_2 p}{k_1} E \left[ \left( \frac{1}{\tau J_2} + \frac{1}{\lambda J_1} \right) \left( \frac{1}{z_{\text{max}}} (\phi \wedge 1) \right) \right] \| \alpha \|^2$$

$$\leq \frac{2m^8 \ell_1 k_2 p}{k_1} E \left[ \left( \frac{m \lambda}{r_{\text{max}} J_2} + \frac{r_{\text{max}} \tau}{J_1} \right) (\phi \wedge 1) \right] \| \alpha \|^2$$

$$\leq \frac{2m^9 \ell_1 k_2 p}{k_1} E \left[ \left( \frac{1}{J_1} + \frac{1}{J_2} \right) (\phi \wedge 1) \right] \| \alpha \|^2$$

$$\leq \frac{2m^9 \ell_1 k_2 p}{k_1} E \left[ \left( \phi \wedge 1 \right) \frac{1}{J_1} + \frac{1}{J_2} \right] \| \alpha \|^2$$

$$= \mathcal{O}(q) E \left[ \left( \phi \wedge 1 \right) \frac{1}{J_1} + \frac{1}{J_2} \right] \| \alpha \|^2,$$

where the last line follows from $(B_4)$. Recall that

$$U_3 = E \left[ \left( \phi \wedge 1 \right) \frac{1}{J_1} + \frac{1}{J_2} \right] \| \alpha \|^2,$$
then $T_{51} = \mathcal{O}(q)U_3$. Now

$$T_{52} = E \left[ \frac{\sqrt{q} \sum_{i=1}^q \frac{1}{z_i} \bar{x}_i^T Q_{\frac{q}{2}} N_{00}}{\sum_{i=1}^q 1/z_i} \right]^2$$

$$\leq E \left[ \frac{z_{\min}^2}{q} \left( \sum_{i=1}^q \frac{1}{z_i^2} \right) \left( \sum_{i=1}^q (\bar{x}_i^T Q_{\frac{q}{2}} N_{00})^2 \right) \right]$$

$$\leq E \left[ \frac{z_{\min}^2}{q} \left( \sum_{i=1}^q (\bar{x}_i^T Q_{\frac{q}{2}} N_{00})^2 \right) \right]$$

$$\leq \frac{m^3 k_2 p}{\tau k_1} E \left[ \left( \frac{1}{\tau z_{\max}^2} \right) \left( \sum_{i=1}^q (\bar{x}_i^T Q_{\frac{q}{2}} N_{00})^2 \right) \right]$$

$$\leq q \frac{m^3 k_2 p}{\tau k_1} E \left[ \left( \frac{1}{\tau z_{\max}^2} \right) \left( \sum_{i=1}^q (\bar{x}_i^T Q_{\frac{q}{2}} N_{00})^2 \right) \right]$$

but this is $q$ times (43), so $T_{52} = \mathcal{O}(q^2)U_3$. Continuing, we have

$$T_{53} = E \left[ \frac{\sum_{i=1}^q \frac{1}{z_i} \bar{x}_i^T \left( \frac{d}{ds} \sqrt{q \tau Q_{\frac{q}{2}} N_{00}} \right) / z_i}{\sum_{i=1}^q 1/z_i} \right]^2$$

$$\leq E \left[ \frac{z_{\min}^2}{q^2} \bar{z}_{\max} \sum_{i=1}^q \left( \bar{x}_i^T \left( \frac{d}{ds} \sqrt{q \tau Q_{\frac{q}{2}} N_{00}} \right) \right)^2 \right]$$

$$\leq \frac{m^2}{q} E \left[ \left( \frac{d}{ds} \sqrt{q \tau Q_{\frac{q}{2}} N_{00}} \right)^T \left( \sum_{i=1}^q \bar{x}_i \bar{x}_i^T \right) \left( \frac{d}{ds} \sqrt{q \tau Q_{\frac{q}{2}} N_{00}} \right) \right]$$

$$\leq m^3 k_2 E \left[ \frac{1}{q} \left( \frac{d}{ds} \sqrt{q \tau Q_{\frac{q}{2}} N_{00}} \right)^2 \right]$$

$$\leq 2m^3 k_2 \left\{ E \left[ \frac{q(\tau^2)}{\tau^3} \| Q_{\frac{q}{2}} N_{00} \|^2 \right] + E \left[ \frac{q}{\tau} \left\| \frac{d}{ds} \sqrt{Q_{\frac{q}{2}} N_{00}} \right\|^2 \right] \right\}$$

$$= 2m^3 k_2 (T_0 + T_1)$$

$$= \mathcal{O} \left( \frac{p}{r^2} \right) \| \alpha \|^2 + \mathcal{O} \left( \frac{p}{r} \right) U_2$$

Combining our bounds on $T_{51}$, $T_{52}$, and $T_{53}$ yields

$$T_5 \leq 3T_{51} + 3T_{52} + 3T_{53}$$

$$= \mathcal{O}(q)U_3 + \mathcal{O}(q^2)U_3 + \mathcal{O} \left( \frac{p}{r^2} \right) \| \alpha \|^2 + \mathcal{O} \left( \frac{p}{r} \right) U_2$$

$$= \mathcal{O}(q^2)U_3 + \mathcal{O} \left( \frac{p}{r^2} \right) \| \alpha \|^2 + \mathcal{O} \left( \frac{p}{r} \right) U_2 \quad (45)$$
We now move on to $T_6$. We have

$$T_6 = qE\left[ \frac{d}{ds} \left( \sqrt{\sum_{i=1}^{q} \frac{1}{z_i}} N_0 \right) \right]^2$$

$$= \frac{q}{4} E \left[ \left( \sum_{i=1}^{q} \frac{1}{z_i} \right)^{-3/2} \left( \sum_{i=1}^{q} \frac{z_i'}{z_i^2} \right) N_0 \right]^2$$

$$\leq \frac{q}{4} E \left[ \left( \sum_{i=1}^{q} \frac{1}{z_i} \right)^{-3} \left( \sum_{i=1}^{q} (z_i')^2 \right) \left( \sum_{i=1}^{q} \frac{1}{z_i} \right) \right] E[N_0^2]$$

$$\leq \frac{q}{4} E \left[ \left( \frac{z_{\text{min}}}{q} \right) \left( \frac{q}{z_{\text{max}}} \right) \left( 2 q m^2 \ell_1 \right) \left( \frac{1}{q^2} \frac{1}{J_2} + \frac{1}{\lambda J_1} \right) \right] \|\alpha\|^2$$

$$\leq \frac{m^5 \ell_1}{2} E \left[ \left( \frac{1}{z_{\text{max}}} \right) \left( \frac{1}{q^2} \frac{1}{J_2} + \frac{1}{\lambda J_1} \right) \right] \|\alpha\|^2$$

$$\leq \frac{m^5 \ell_1}{2} E \left( \frac{m \lambda}{t_{\text{max}} J_2} + \frac{r_{\text{max}} r}{t_{\text{max}} J_1} \right) \|\alpha\|^2$$

$$= \mathcal{O}\left( \frac{1}{q} \right) \|\alpha\|^2,$$

where the second inequality follows from (44), and the last line holds because $E(J_1^{-1}) = (q/2 + a_1 - 1)^{-1}$ and $E(J_2^{-1}) = (q\bar{r}/2 + a_2 - 1)^{-1}$. Combining (39), (41), (45), and (46), we have

$$E\left( \frac{d^2}{ds^2} \tilde{h}_i \right) \leq 3T_4 + 3T_5 + 3T_6$$

$$= \mathcal{O}(q) U_1 + \mathcal{O}(q^2) U_3 + \mathcal{O}\left( \frac{P}{\bar{r}q} \right) \|\alpha\|^2 + \mathcal{O}\left( \frac{P}{\bar{r}} \right) U_2 + \mathcal{O}\left( \frac{1}{q} \right) \|\alpha\|^2 \quad (47)$$

We now go to work on $\sum_{i=1}^{q} E\left( \frac{d^2}{ds^2} \tilde{h}_i \right)$, which is the third term on the right-hand side of (20).

First,

$$\tilde{h}_i = \lambda \frac{\tilde{y}_0}{\sqrt{q}} + \frac{r_i t}{t_i} (\tilde{y}_i - \bar{x}_i \tilde{y}_{00}/\sqrt{q}) + \sqrt{\frac{1}{t_i}} N_i$$

$$= \left( 1 - \frac{1}{e_i} \right) \frac{\tilde{y}_0}{\sqrt{q}} + \frac{1}{e_i} (\tilde{y}_i - \bar{x}_i \tilde{y}_{00}/\sqrt{q}) + \sqrt{\frac{1}{t_i}} N_i.$$

Now,

$$\left( \frac{d\tilde{h}_i}{ds} \right)^2 = \left\{ \frac{e_i}{e_i} \frac{\tilde{y}_0}{\sqrt{q}} + \left( 1 - \frac{1}{e_i} \right) \frac{d\tilde{y}_0}{ds} + \frac{e_i}{e_i} \left( \tilde{y}_i - \bar{x}_i \tilde{y}_{00}/\sqrt{q} \right) - \frac{1}{e_i} \bar{x}_i \frac{d\tilde{y}_{00}}{ds} \right\}^2$$

$$\leq \frac{5}{q} \left( \frac{e_i'}{e_i} \right)^2 \tilde{y}_0^2 + \frac{5}{q} \left( 1 - \frac{1}{e_i} \right)^2 \left( \frac{d\tilde{y}_0}{ds} \right)^2 + \frac{5}{q} \left( \frac{e_i}{e_i} \right)^2 \left( \tilde{y}_i - \bar{x}_i \tilde{y}_{00}/\sqrt{q} \right)^2 + \frac{5}{q} \left( \frac{1}{e_i} \right)^2 \left( \bar{x}_i \frac{d\tilde{y}_{00}}{ds} \right)^2 + \frac{5}{4} t_i^{-3} (e_i')^2 N_i^2$$

$$\leq \frac{5}{q} \frac{\ell_2 \Phi}{e_{\text{max}}} \tilde{y}_0^2 + \frac{5}{q} \left( \frac{d\tilde{y}_0}{ds} \right)^2 + \frac{5}{q} \left( \frac{\ell_2 \Phi}{e_{\text{max}}} (\tilde{y}_i - \bar{x}_i \tilde{y}_{00}/\sqrt{q})^2 + \frac{5}{q} \left( \bar{x}_i \frac{d\tilde{y}_{00}}{ds} \right)^2 + \frac{5}{4} t_i^{-3} (e_i')^2 N_i^2.$$

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We now bound \( \sum_{i=1}^{q} \mathbb{E}(\frac{d\tilde{y}_i}{ds})^2 \). First,

\[
\frac{5 \ell_2 \Phi}{e_{\text{max}}^4} \tilde{\eta}_0^2 = \frac{5 \ell_2 \Phi}{e_{\text{max}}^4} \left( \sqrt{q \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T \tilde{\eta}_00/\sqrt{q})/z_i} + \sqrt{\frac{q}{\sum_{i=1}^{q} 1/z_i} N_0} \right)^2 \\
\leq \frac{10 \ell_2 \Phi}{e_{\text{max}}^4} \left( q \left[ \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T \tilde{\eta}_00/\sqrt{q})/z_i \right]^2 + \frac{q}{\sum_{i=1}^{q} 1/z_i} N_0^2 \right) \\
\leq \frac{10 \ell_2 \Phi}{e_{\text{max}}^4} \left( q \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T \tilde{\eta}_00/\sqrt{q})^2 \sum_{i=1}^{q} (1/z_i)^2 \right) + z_{\min} N_0^2 \\
\leq \frac{10 \ell_2 m^2 \Phi}{e_{\text{max}}^4} \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T \tilde{\eta}_00/\sqrt{q})^2 + \frac{10 \ell_2 \Phi}{e_{\text{max}}^4} z_{\min} N_0^2 ,
\]

but

\[
\sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T \tilde{\eta}_00/\sqrt{q})^2 = \sum_{i=1}^{q} \left\{ \bar{y}_i - \bar{x}_i^T \left( v + \sqrt{\frac{q}{\tau} Q_{i0}^T N_00/\sqrt{q}} \right) \right\}^2 \\
\leq 2 \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T v) \leq 2 q \sum_{i=1}^{q} \left( \sqrt{\frac{q}{\tau} x_i^T Q_{i00}^T N_00} \right)^2 \\
= \mathcal{O}(q) + \mathcal{O}(\frac{1}{\tau}) \frac{N_{00}^T N_{00}}{\tau} ,
\]

where the last line follows from (10) and (12). So finally,

\[
\frac{5 \ell_2 \Phi}{e_{\text{max}}^4} \tilde{\eta}_0^2 = \mathcal{O}(q) \frac{\Phi}{e_{\text{max}}^4} + \mathcal{O}(\frac{1}{\tau}) \frac{\Phi}{\tau e_{\text{max}}^4} N_{00}^T N_{00} + \mathcal{O}(1) \frac{\Phi}{\lambda e_{\text{max}}^4} N_0^2 ,
\]

and we have used the fact that \( z_{\min}/e_{\text{max}} \leq m/\lambda \). Now,

\[
\frac{5 \ell_2 \Phi}{e_{\text{max}}^4} \sum_{i=1}^{q} (\bar{y}_i - \bar{x}_i^T \tilde{\eta}_00/\sqrt{q})^2 = \mathcal{O}(q) \frac{\Phi}{e_{\text{max}}^4} + \mathcal{O}(\frac{1}{\tau}) \frac{\Phi}{\tau e_{\text{max}}^4} N_{00}^T N_{00} ,
\]

and

\[
\frac{5 \ell_2}{q} \sum_{i=1}^{q} \left( \bar{x}_i^T \frac{d\tilde{y}_00}{ds} \right)^2 = \frac{5 \ell_2}{q} \left( \frac{d\tilde{y}_00}{ds} \right)^T \left( \sum_{i=1}^{q} \bar{x}_i \bar{x}_i^T \right) \left( \frac{d\tilde{y}_00}{ds} \right) = \mathcal{O}(1) \left\| \frac{d\tilde{y}_00}{ds} \right\|^2 .
\]

Finally, we have

\[
\frac{5}{4} \sum_{i=1}^{q} t_i^3 (t_i')^2 N_i^2 = \frac{5}{4} \sum_{i=1}^{q} \left( \frac{1}{r_i + \lambda} \right)^3 (\lambda' + r_i \tau')^2 N_i^2 \\
\leq \frac{5}{2} \left( \frac{1}{r_{\min} + \lambda} \right)^3 ((\lambda')^2 + r_{\max} (\tau')^2) \sum_{i=1}^{q} N_i^2 \\
\leq \frac{5m^3}{2(\tau + \lambda)^3} \left( \frac{\ell_1 \lambda^3}{J_1} \left\| \alpha \right\|^2 + \frac{m^2 l_1 \tau^3 r^3}{J_2} \left\| \alpha \right\|^2 \right) \sum_{i=1}^{q} N_i^2 \\
\leq \frac{5m^5 \ell_1}{2} \left( \frac{\lambda^3}{(\tau + \lambda)^3} \frac{1}{J_1} + \frac{\tau^3}{(\tau + \lambda)^3} \frac{1}{J_2} \right) \left\| \alpha \right\|^2 \sum_{i=1}^{q} N_i^2 \\
\leq \frac{5m^5 \ell_1}{2} \left( [\phi \wedge 1] \frac{1}{J_1} + \frac{1}{J_2} \right) \left\| \alpha \right\|^2 \sum_{i=1}^{q} N_i^2 ,
\]

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where the second inequality follows from [28] and the fourth follows from the fact that $\phi = \lambda/(\bar{r}\tau)$.

Combining all of these bounds yields

$$
\sum_{i=1}^{q} \left( \frac{d^2\eta_i}{ds} \right)^2 = \mathcal{O}(q)\frac{\Phi}{e_{\max}^4} + \mathcal{O}\left( \frac{1}{\tau e_{\max}^4} \right) N_0^\top N_{q0} + \mathcal{O}(1)\frac{\Phi}{\lambda e_{\max}^3} N_0^2 + \mathcal{O}(1)\left( \frac{d\eta_0}{ds} \right)^2
$$

$$
+ \mathcal{O}(1)\left\| \frac{d\eta_0}{ds} \right\|^2 + \mathcal{O}(1)\left( \left[ \phi \wedge 1 \right] \frac{1}{J_1} + \frac{1}{J_2} \right) \|\alpha\|^2 \sum_{i=1}^{q} N_i^2.
$$

Recall that

$$
U_4 = E\left( \frac{\Phi}{\lambda e_{\max}^3} \right),
$$

and that

$$
U_1 = E\left( \frac{\Phi}{e_{\max}^2} \right), \quad U_2 = E\left( \frac{\Phi}{\tau e_{\max}^2} \right), \quad \text{and} \quad U_3 = E\left[ \left( \phi \wedge 1 \right) \frac{1}{J_1} + \frac{1}{J_2} \right] \|\alpha\|^2.
$$

So, finally, we have

$$
\sum_{i=1}^{q} \mathbb{E}\left( \frac{d^2\gamma_i}{ds} \right)^2 = \mathcal{O}(q)U_1 + \mathcal{O}\left( \frac{p}{\tau} \right) U_2 + \mathcal{O}(q)U_3 + \mathcal{O}(1)U_4 + \mathcal{O}(1)\mathbb{E}\left( \frac{d\eta_0}{ds} \right)^2 + \mathcal{O}(1)\mathbb{E}\left\| \frac{d\eta_0}{ds} \right\|^2.
$$

Combining this with (20), (38), and (47) we have

$$
\left\{ \mathbb{E}\left\| \frac{df(\eta + s\alpha)}{ds} \right\|^2 \right\}^2 \leq \mathbb{E}\left\| \frac{d\eta_0}{ds} \right\|^2 + \mathbb{E}\left( \frac{d\eta_0}{ds} \right)^2 + \sum_{i=1}^{q} \mathbb{E}\left( \frac{d\eta_i}{ds} \right)^2
$$

$$
= \mathcal{O}(q)U_1 + \mathcal{O}\left( \frac{p}{\tau} \right) U_2 + \mathcal{O}(q)U_3 + \mathcal{O}(1)U_4 + \mathcal{O}(1)\mathbb{E}\left( \frac{d\eta_0}{ds} \right)^2 + \mathcal{O}(1)\mathbb{E}\left\| \frac{d\eta_0}{ds} \right\|^2
$$

$$
= \mathcal{O}(q)U_1 + \mathcal{O}\left( \frac{p}{\tau} \right) U_2 + \mathcal{O}(q^2)U_3 + \mathcal{O}(1)U_4 + \mathcal{O}\left( \frac{p}{qr} \right) \|\alpha\|^2 + \mathcal{O}\left( \frac{1}{q} \right) \|\alpha\|^2
$$

$$
= \mathcal{O}(q)U_1 + \mathcal{O}\left( \frac{p}{\tau} \right) U_2 + \mathcal{O}(q^2)U_3 + \mathcal{O}(1)U_4 + \mathcal{O}\left( \frac{1}{q} \right) \|\alpha\|^2,
$$

where the last line follows from (B4) and (B5). This is (21).

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