THE RIGID-FLEXIBLE VALUE FOR SYMPLECTIC
EMBEDDINGS OF FOUR-DIMENSIONAL ELLIPSOIDS INTO
POLYDISCS

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Abstract. In previous work of Cristofaro-Gardiner, Frenkel, and Schlenk,
the embedding function \( c_b(a) \) describing the problem of symplectically embed-
ding an ellipsoid \( E(1,a) \) into the smallest scaling of the polydisc \( P(1,b) \) was
determined for all integers \( b \geq 2 \). As in McDuff’s work, Cristofaro-Gardiner,
Frenkel, and Schlenk found staircases associated with the embedding function.
More recently, Usher’s work shows that the appearance of infinite staircases as
\( b \) varies is hard to predict. The intricate structure found there suggests that
determining the entirety of the graph of \( c_b(a) \) for all \( b \) is intractable.

In contrast with this result, we show that for every polydisc \( P(1,b) \) with
\( b > 2 \), there is an explicit formula for the minimum \( a \) such that the embedding
problem is determined only by volume. That is, when the ellipsoid is suf-
ficiently stretched, there is a symplectic embedding of \( E(1,a) \) fully filling an
appropriately scaled polydisc \( P(\lambda, \lambda b) \). We call this value of \( a \) the rigid-flexible
value at \( b \), or the RF-value. This formula is piecewise smooth in \( b \).

To further describe the function \( RF(b) \), we investigate its behavior as \( b \to 1 \).
Frenkel and Müller showed that the RF-value at \( b = 1 \) is \( 7 \frac{1}{32} \) and Cristofaro-
Gardiner, Frenkel, and Schlenk showed that the RF-value at \( b = 2 \) is \( 8 \frac{1}{32} \).

By exhibiting a sequence of obstructive classes for \( b_n = \frac{n+1}{n} \) at \( a = 8 \), we
show that \( c_{b_n}(8) \) is above the volume constraint. So, in combination with the
Frenkel-Müller result, it follows that \( RF \) is discontinuous at \( b = 1 \).

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References

1. Introduction, Statement of Results

The problem of embedding one symplectic manifold into another touches on a wide variety of topics in symplectic geometry, and in this work we focus in particular on embeddings of ellipsoids $E(a, b)$ into polydiscs $P(a, b)$. Here, a polydisc

$$P(a, b) := \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi \|z_1\|^2 < a, \pi \|z_2\|^2 < b\}$$

is the 4-dimensional open symplectic manifold $B_2(a) \times B_2(b) \subset \mathbb{C}^2$, where each factor is a 2-disc of fixed radius centered at $0 \in \mathbb{C}$. Similarly the ellipsoid $E(a, b)$ is given by

$$E(a, b) := \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi \|z_1\|^2 + \pi \|z_2\|^2 < 1\}.$$

McDuff and Schlenk [12] consider the problem of embedding a symplectic ellipsoid $E(1, a)$ into a ball $B(r)$: for fixed $a$, what is the smallest $r$ such that $E(1, a) \hookrightarrow B(r)$ symplectically? One has the immediate necessary condition that the volume of $E(1, a)$ must be no greater than that of $B(r)$ since the embedding is symplectic, but in fact this is far from sufficient. In [12], the authors determine the function $c(a)$ whose output is the minimal $r$ guaranteeing such an embedding, and show that the problems fluctuate between the flexibility seen in volume-preserving geometry and the rigidity often seen in symplectic geometry. For example, they show $c(a)$ is continuous but contains a so-called “infinite staircase,” meaning that there is a convergent sequence of $a$-values $a_n \to a_\infty$ such that $c(a)$ is non-decreasing and linear or constant on each $[a_i, a_{i+1}]$. On the other hand, [12] also find that if $a \geq \left(\frac{17}{6}\right)^2$, then $c(a) = \sqrt{a}$, meaning that the only restriction to a symplectic embedding for sufficiently elongated ellipsoids is the volume. This bound is sharp, and records the $a$-value past which symplectic obstructions vanish. We call this number the rigid-flexible value, or the RF-value.

A related problem studies embeddings of an ellipsoid into a polydisc, i.e. the function $c_b(a)$ whose value at $a$ is the smallest $\lambda$ such that

$$E(1, a) \hookrightarrow P(\lambda, \lambda b).$$

Here we allow both the source and target in the embedding problem to become elongated, and ask how this affects the embedding function

$$c_b(a) := \inf\{\lambda > 0 \mid E(1, a) \hookrightarrow P(\lambda, \lambda b)\}.$$

Varying the extra parameter $b$, previous work uncovers more delicate structure in the functions $c_b(a)$. Cristofaro-Gardiner, Frenkel, and Schlenk [3] find that for $b \in \mathbb{Z}^+$, $c_b(a)$ has no infinite staircases, and the existence of embeddings is governed by two infinite sequences of \textit{“exceptional classes,”} which are homology classes in blow-ups of $\mathbb{C}P^2$ represented by embedded $J$-holomorphic $-1$ spheres. On the other hand, Usher [13] finds infinite sequences of (irrational) $b$ such that for each such $b,
THE RF-VALUE FOR $E(1, a) \to P(\lambda, \lambda b)$

$c_b(a)$ contains an infinite staircase. While [13] shows that “perfect” classes which contribute to $c_b(a)$ for some $b$ remain obstructive for nearby $b$, determining the entirety of $c_b(a)$ appears to be quite difficult. This complicated structure in the embedding function $c_b(a)$ cannot be described as straightforwardly as in [3], where there are explicit formulae for $c_b(a)$.

Thus, in the context of other work on this subject, our results for $b > 2$ are an interesting counterpoint to [13]. The computation of the RF-value appears to be a tractable problem, akin to the computation in [3] of the embedding functions $c_b(a)$ for integral $b$.

In this paper, we focus on the RF-value in the polydisc problem (1.1) for non-integral $b$. Burkhart, Panescu, and Timmons establish a lower bound on RF for all $b > 1$, and conjecture that it is sharp [1, Conj. 6.2]. Our result proves their conjecture for $b > 2$.

Theorem 1.1. For all real $b \geq 2$, the RF-value of $c_b(a)$ is given by

$$RF(b) = 2b \left( \frac{2|b| + 2\sqrt{2b + |b|} - 1}{b + |b| + \sqrt{2b + |b|} - 1} \right)^2.$$  

In particular, the function $RF(b)$ is increasing and piecewise smooth in $b$.

However, the behavior of $RF(b)$ is more delicate for $1 < b < 2$. In the case $b = 1$, Frenkel and Müller [4] show that $RF(1) = 7\frac{1}{12}$, determined by the exceptional class $(4, 4; 2 \times 6)$, whereas [3] show that $RF(2) = 8\frac{1}{18}$, determined by the exceptional class $(6, 3; 3, 2 \times 7)$. We show that for the sequence $\beta_n = \frac{n+1}{n}$, converging to $b = 1$, $\lim_{n \to \infty} RF(\beta_n) \neq RF(1)$.

Theorem 1.2. For the sequence $\beta_n = \frac{n+1}{n}$, with $n \geq 5$,

$$RF(\beta_n) = \frac{2(n+1)}{n} \left( \frac{8n^2 + 8n + 1}{2(2n+1)(n+1)} \right).$$

When $n \to \infty$, (1.3) approaches 8, while $RF(1) = 7\frac{1}{12}$. Hence the function $RF : [1, \infty) \to \mathbb{R}$ is not continuous at $b = 1$. The exceptional classes $(d, c; m)$ determining the RF-values for $\beta_n$ are of the form

$$(2n+1)(n+1), (2n+1)n; n^2 + n + 1, (n^2 + n)^{\times 7}$$

but do not affect $c_{\beta_n}(a)$.

It is an interesting question whether this sequence of classes is only associated to $\beta_n = \frac{n+1}{n}$, or if it determines the embedding functions or RF-values for other $b$.

1.1. Outline of Paper. In Sections 4 and 5, we give a proof of Theorem 1.1 by applying the reduction method for two distinct intervals of $a$. We first consider $a \leq RF(b)$, showing that $c_b(a)$ on this interval is strictly above the volume constraint. Then, using Proposition 3.5 (ii) in [3], $RF(b) \leq (\sqrt{2b} + 1)^2$, so we show again using the reduction method that $c_b(a)$ in the interval $[RF(b), (\sqrt{2b} + 1)^2]$ equals the volume constraint.

Then in Section 5, we proceed with the proof of Theorem 1.2. We establish an upper bound $RF \leq 9$ for $\beta_n$ using the reduction method. Section 6 shows that $RF(\beta_n) \geq 8$ by explicitly exhibiting a sequence of obstructive classes $R_n$ and showing they determine $c_{\beta_n}(8)$. An argument bounding the values of $d, e$ in possible obstructive classes $(d, c; m)$ on $(8, 9)$ then establishes the necessary result.
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2. Preliminaries: Three Methods for Finding Symplectic Embeddings

Here, we review three methods for detecting symplectic embeddings. More detailed expositions are in [3] [12], so we review only what we use in the sequel. The following definition is central to both methods. Fix \( b \geq 1 \). Since the function \( c_b(a) \) is continuous in \( a \), it suffices to compute it for \( a \geq 1 \) rational.

**Definition 2.1.** The weight expansion \( w(a) \) of such an \( a \) is the finite decreasing sequence
\[
w(a) = (1 \times \ell_0, w_1 \times \ell_1, \ldots, w_n \times \ell_n),
\]
where \( w_1 = a - \ell_0 < 1, w_2 = 1 - \ell_1 w_1 < w_1 \), and so on.

The three methods we describe here draw on two more general and powerful tools which are in and of themselves interesting objects of study: holomorphic curves and symplectic capacities.

The theory of holomorphic curves relates in our problem in the following way. In [9, Thm. 1.1], it is shown that there is a canonical decomposition of any ellipsoid \( E(a, b) \) into a collection of balls
\[
B(a, b) := \bigcup_i B(w_i)
\]
where the \( w_i \) are terms in the weight sequence of \((a, b)\).

**Theorem 2.2 ([4]).** Let \( a, b, c, d \in \mathbb{Q} \) be positive. There exists a symplectic embedding \( E(1, a) \hookrightarrow P(\lambda, \lambda b) \) if and only if there is a symplectic embedding
\[
B(a, b) \bigcup B(\lambda) \bigcup B(\lambda b) \hookrightarrow B(\lambda(b + 1)).
\]

This reduces the polydisc problem to a ball-packing problem of embedding balls of radius \( e_i \) into a ball of radius \( \mu \):
\[
(2.1) \quad \bigcup_i B(e_i) \hookrightarrow B(\mu).
\]

This relates to holomorphic curves via the correspondence between symplectic embeddings of balls and blow-ups. Briefly, if one can embed a ball \( B^{2n}(V + \epsilon) \) of volume \( V + \epsilon \) into a symplectic manifold \( M^{2n} \), then there is a symplectic manifold \( M \) obtained from the union of \( M \setminus B^{2n}(V + \epsilon) \) and a neighborhood of the zero-section in \( O_{\mathbb{P}^{n-1}}(-1) \). This neighborhood has total volume \( \epsilon \) where the zero-section has symplectic volume \( V \) [11 Ch. 7]. When \( n = 2 \), this corresponds to an embedded pseudoholomorphic curve, called the exceptional sphere. Specific to this four-dimensional case, let \( X_n \) denote the blow-up of \( \mathbb{C}P^2 \) in \( n \) points.

The purpose of introducing these constructions is that balls can be embedded symplectically precisely when the associated blow-ups of \( \mathbb{C}P^2 \) carry symplectic forms. We denote by \( \overline{C}_K(X_n) \) the set of cohomology classes represented by symplectic forms for which the anticanonical class is \( K = -3L + \sum_i E_i \), where \( L \) is Poincaré dual to a line in \( \mathbb{C}P^2 \) and each \( E_i \) is dual to the \( i \)th exceptional sphere. By [10], the embedding (2.1) exists when the following cohomology class is in the symplectic cone:
\[
(2.2) \quad \mu L - \sum_i e_i E_i \in \overline{C}_K(X_n).
\]
The above fact gives a sufficient criterion for a class $\alpha$ to lie in $\mathcal{T}_K(X_n)$. If there is a symplectic form in a given class, it must have non-negative intersection with certain holomorphic curves. By [8], this is also sufficient: if $E_k(X_n) := \{e \in H_2(X_n) \mid \langle e, e \rangle = -1, \langle K, e \rangle = -1\}$, then we may characterize the symplectic cone referenced in (2.2) as

$$\mathcal{T}_K(X_n) := \{\omega \in H^2(X_n) \mid \langle \omega, e \rangle \geq 0 \forall e \in E_k(X_n)\}.$$  

We verify this positivity of intersection with respect to a particular basis for cohomology. For the ball-packing problem in four dimensions, it is natural to consider the compactification of $B^4$ with volume $V$ by $\mathbb{C}P^2$ into a $\mathbb{C}P^4$ with volume $V$. This closed 4-manifold has $H_2$ of rank 1, with generator the boundary divisor $\mathbb{C}P^1$. We take the dual generator of $H^2$ to be the form giving $\mathbb{C}P^1$ area $\sqrt{V}$. Subsequent blow-ups of this manifold have homology bases given by this same generator along with the classes of exceptional spheres. The same process applies to ellipsoids.

For the problem of embedding ellipsoids into polydiscs, the more natural compactification of $P(a, b)$ adds a single point to each disc, yielding $S^2 \times S^2$. This manifold is in fact diffeomorphic to the 1-point blow-up of $\mathbb{C}P^2$, so then $n$-fold blow-up of $S^2 \times S^2$ (denoted $Y_n$) can be identified with $X_{n+1}$. The induced isomorphism on cohomology $\psi: H^2(Y_n) \to H^2(X_{n+1})$ is given by

$$\psi((d, e; m_1, m_2, m_3, ..., m_n)) := (d + e - m_1; d - m_1, e - m_1, m_2, ..., m_n)$$

The end result is that we may express any class in $H^2$ with respect to this basis, so that verifying positivity of intersection with effective classes of spheres reduces to checking positivity of entries of the vector (2.3).

This positivity condition completely characterizes the symplectic cone, so we now require a condition for determining when homology classes are represented by effective spheres. With respect to the basis of $H^2(X_n; \mathbb{R}) \simeq \mathbb{R}^{n+1}$ given above, a Cremona transform is the map given by

$$(d; m_1, ..., m_n) \mapsto (2d - m_1 - m_2 - m_3; d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2, m_4, ..., m_n)$$

We will use Cremona transforms in two different contexts below. First, we can state the condition for a weight vector to lie in the symplectic cone, which is proven in [12] based on work of [7, 8].

**Theorem 2.3.** A class $(d; m_1, ..., m_n) \in H_2(X_n; \mathbb{Z})$ is in $E_k(X_n)$ if and only if its entries satisfy the Diophantine equations

$$3d - 1 = \sum_i m_i$$
$$d^2 + 1 = \sum_i m_i^2$$

and $(d; m_1, ..., m_n)$ reduces to $(0; -1, 0, ..., 0)$ after a sequence of Cremona transformations.

**2.1. Obstructive Classes.** We can now describe one of the methods we employ to find symplectic embeddings. [4] Prop. 3.9] establishes the correspondence between the embedding problems (1.1) and (2.1), so that (not necessarily symplectic) embeddings correspond to cohomology classes $(\lambda b, \lambda; w(a))$ in $H^2(Y_n; \mathbb{R})$ for some $n$. The method given here verifies whether this class has a symplectic representative, i.e. represents a symplectic embedding.
Briefly, consider the embedding problem \( E(1, a) \hookrightarrow P(\lambda, \lambda b) \). If \( A := (d, e; m) \) encodes the coefficients of a homology class \( H_*(S^2 \times S^2; \mathbb{R}) \) with the basis given previously, the condition

\[
\lambda = \frac{\langle w(a), m \rangle}{d + e} > 0
\]

is equivalent to

\[
\lambda(d + e) - \langle w(a), m \rangle > 0,
\]

which is the statement that \( A \) has positive symplectic area. The first sum is the evaluation of \( A \) on the ruling lines of \( S^2 \times S^2 \) and \( w(a) \) comes from the equivalence of the ellipsoid-polydisc embedding problem to the embedding of a collection of balls with prescribed radii into a larger ball (c.f. [12]). The explicit statement is that, for \( w_i \) the term of the weight expansion of \( a \),

\[
\bigcap_i B(w_i) \hookrightarrow B(d + e)
\]

symplectically if and only if

\[
(d + e)L - \sum_i w_iE_i
\]

is in the symplectic cone of the blowup of \( \mathbb{CP}^2 \). Then [3] Thm 3.9] establishes how to translate this into a method for detecting embeddings of ellipsoids into polydiscs.

Note that (2.5) requires that the longer axis of the ellipsoid is at least as big as the the number \( \lambda \). The number \( a \) is independent of the homology class \( A \); this pairing measures whether the blowup determined by the weight sequence of \( a \) has sufficient volume for \( A \) to be represented by an exceptional sphere in that blowup. We state this as a theorem below.

**Theorem 2.4.** An embedding (1.1) exists iff \( \lambda \geq \sqrt{\frac{a}{2b}} \) and

\[
\mu_b(d, e; m) := \frac{\sum_i m_i \cdot w_i(a)}{d + e} \leq \lambda
\]

for every \( (d, e; m_1, ..., m_n) \in H_2(Y_n; \mathbb{Z}) \) which satisfies equations (2.4) and reduces to \((0; -1)\) after some sequence of Cremona transformations.

We say that a class \( A \) is **obstructive at** \( a > 0 \) if the infimum

\[
c_b(a) = \inf\{\lambda > 0 \mid E(1, a) \hookrightarrow P(\lambda, \lambda b)\}
\]

is larger than the volume constraint \( \sqrt{\frac{a}{2b}} \).

### 2.2. Reduction at a Point

Though the previous method is a necessary and sufficient condition for the embedding of an ellipsoid into a polydisc, it is far from efficient in that one might in principle need to check more classes than is computationally feasible. The following method provides an alternative condition which, although restricted in its statement to a single value of \( a \), can often be applied to infer the existence of embeddings over larger intervals.

**Definition 2.5.** The **defect** \( \delta \) of an ordered vector \( (d; m_1, ..., m_n) \) is the sum \( d - m_1 - m_2 - m_3 \).
The following is established in [2, 6].

**Theorem 2.6.** An embedding \( E(1, a) \hookrightarrow P(\lambda, \lambda b) \) exists if there exists a finite sequence of Cremona moves that transforms the ordered vector
\[
((b + 1)\lambda; b\lambda, \lambda, W(1, a))
\]
to an ordered vector with non-negative entries and defect \( \delta \geq 0 \).

We will apply this repeatedly in the proofs of both Theorem 1.1 and Theorem 1.2.

### 2.3. ECH Capacities.

The third method we use for obstructing symplectic embeddings is the computation of ECH capacities.

**Definition 2.1.** A **capacity** is a function which assigns to a symplectic manifold a sequence of real numbers
\[
0 = c_0(X, \omega) < c_1(X, \omega) \leq c_2(X, \omega) \leq \cdots \leq \infty
\]
such that if \((X, \omega) \hookrightarrow (N, \eta)\) symplectically, then \(c_k(X, \omega) \leq c_k(N, \eta)\) for all \(k\), and the inequality is strict if \(c_k(X, \omega)\) is finite.

Embedded contact homology, which is roughly the homology of a chain complex generated by Reeb orbits in a contact 3-manifold, is used to define these capacities for bounded star-shaped domains in \( \mathbb{R}^4 \), compact exact symplectic 4-manifolds with boundary with a contact form \( \lambda \) on the boundary such that \( d\lambda = \omega|_{\partial X} \).

Heuristically, these capacities measure how much symplectic action is needed to represent a given class in ECH. There is a filtration on the generators of ECH given by its action functional. Taking the subcomplex consisting of elements of action less that some \(L\), this includes in the full complex. Then \(c_k\) is the least \(L\) such that the image of the \(L\)-subcomplex is \(k\)-dimensional.

The technical details of this construction can be elided by applying the following result from [5]. To set notation, let \((a, b)_k\) denote the \(k\)th smallest entry in the matrix of real numbers \((am + bn)_{m, n \in \mathbb{N}}\), counted with repetitions so that \(c_k = c_{k+1}\) when they repeat. The convention here is that \(\mathbb{N}\) includes 0.

**Theorem 2.2.** The ECH capacities of an ellipsoid are given by \(c_k(E(a, b)) = (a, b)_{k+1}\), and the ECH capacities of a polydisc are given by \(c_k(P(a, b)) = \min\{am + bn \mid m, n \in \mathbb{N}, (m + 1)(n + 1) \geq k + 1\}\).

In fact, it is a result of McDuff that these ECH capacities are sharp for ellipsoids; the interior of \(E(a, b)\) embeds symplectically in \(E(c, d)\) if and only if \((a, b)_k \leq (c, d)_k\) for every \(k\). That is, ECH capacities form a complete list of obstructions, in the sense that if the capacities satisfy the inequality \(c_k(E(1, a)) \leq c_k(E(1, 2b))\) for all \(k\) then \(E(1, a) \hookrightarrow E(1, 2b)\).

Consequently, since the problems \(E(1, a) \hookrightarrow P(\lambda, \lambda b)\) and \(E(1, a) \hookrightarrow E(\lambda, \lambda 2b)\) are equivalent, checking ECH capacities suffices to guarantee the existence of an embedding. This allows us to reformulate \(c_k(a)\) as
\[
c_k(a) = \sup_{k \geq 1} \left\{ \frac{c_k(E(1, a))}{c_k(E(1, 2b))} \right\}.
\]
3. The Rigid-Flexible Value for Real $b > 2$: The Interval $[2n_b + 1, RF]$

In this section we begin by examining the RF-value in the case of non-integers, showing that the relevant classes for finding the RF-value when $b \geq 2$ are the $E_n$ from [3]. The general strategy follows Conjecture 6.2 of [1]; their conjectured formula for $RF$ arises from the definition

$$\sqrt{\frac{RF}{2b}} = \mu_b(d, e; m) = \frac{(m, w(a))}{d + be}.$$  

This identity and the formula (2.8) for the obstruction function suggests a sequence of obstructive classes depending on $b$ which we show determines the RF-value. Let

$$n_b = \lfloor b \rfloor + \lfloor \sqrt{2b} + \{b\} \rfloor - 1.$$  

The associated obstructive class is given by

$$(n_b, 1; 1 \times 2n_b + 1),$$

which corresponds to the class $E_{n_b}$ in [3].

With these classes in hand, we prove they are the only obstructions by first identifying the interval on which they are obstructive, i.e. where they are always above the volume constraint. We show, using ECH capacities, that for values of $a$ less than the claimed RF-value, these classes determine $c_b(a)$. Then, for values of $a$ larger than the claimed RF-value, we show that the reduction method guarantees that $c_b(a) = \sqrt{\frac{RF}{2b}}$.

3.1. Finding $c_b(2n_b + 1)$ via ECH Capacities. The following establishes what happens to the left of RF using ECH capacities.

**Lemma 3.1.** Fix $b > 2$, and $n_b$ the value given in [3]. Then $c_b(2n_b + 1) = \frac{2n_b + 1}{n_b + b}$.

**Proof.** We start with the ECH capacities of the ellipsoid $E := E(1, 2n_b + 1)$. The definition of $c_b(E(1, a))$ is the $k$th smallest entry of the matrix whose $(i, j)$th entry is $i + ja$. As $a \geq 1$, the smallest entries occur when $j = 0$. When $a = 2n_b + 1$, the ECH capacities $c_0(E), ..., c_{2n_b}(E)$ correspond to multiples of 1 by $i, j$ and range from 0 to $2n_b$. Following that, we see $2n_b + 1$ as the next two possibilities, arising as $(2n_b + 1) + 0$ and $(2n_b + 1)$. Hence $c_{2n_b + 1}(E) = 2n_b + 1$.

Now consider the polydisc

$$P := P\left(\frac{2n_b + 1}{n_b + b}, \frac{2n_b + 1}{n_b + b} \cdot b\right).$$

Here $\lambda = \frac{2n_b + 1}{n_b + b}$, and we wish to show that $c_{2n_b + 1}(P) \leq 2n_b + 1$ so that an embedding exists. We also show this is the minimal possible value of $\lambda$, so it determines $c_b(a)$. Recall that the $k$th ECH capacity of $P$ is the minimum entry of the matrix with $(i + 1, j + 1)$th entry $i \left(\frac{2n_b + 1}{n_b + b}\right) + j \left(\frac{2n_b + 1}{n_b + b} \cdot b\right)$, considering only those entries for which $(i + 1)(j + 1) \geq k + 1$. In particular, for fixed $k$, $i, j \leq k$.

As in the ellipsoid case, to understand the entry determining the capacity $c_{2n_b + 1}(P)$, we begin by looking along the first row and first column. Along the first column, $j = 0$, and since $b > 1$, multiples of $\frac{2n_b + 1}{n_b + b}$ are strictly smaller than multiples of...
\(\frac{2n_e+1}{n_b} \cdot b\) along the first column, so we begin our search here.

For each \(k\), we only consider those multiples satisfying the condition

\[
\frac{k + 1}{j + 1} - 1 \leq i.
\]

Since \(i\) must be an integer quantity, we find that \(i\) is no smaller than

\[
i_{\text{min}}(j, k) = \left\lfloor \frac{k + 1}{j + 1} - 1 \right\rfloor.
\]

For fixed \(k\), the minimum entries in column \(j\) will occur at row \(i_{\text{min}}\) as we take positive linear combinations of positive numbers. The capacity \(c_k(P(c, d))\) will be smallest element of \(\{jc + i_{\text{min}}d\}\), where \(j \in \{0, \ldots, k\}\).

To find \(c_{2n_b+1}\) we set \(k = 2n_b + 1, j \in \{0, \ldots, k = 2n_b + 1\}\), and take the minimum of the list

\[
\left( i_{\text{min}} : \frac{2n_b + 1}{b + n_b} + j \frac{(2n_b + 1)b}{b + n_b} \right).
\]

Now when \(j = 1, i_{\text{min}} = n_b\) and and the corresponding item of the above list is 

\(2n_b + 1\). We claim that all other items in the list are strictly larger. For \(j = 0\), the corresponding item is \(\frac{(2n_b+1)b}{b+n_b}\), which is larger than \(2n_b + 1\) as \(\frac{(2n_b+1)b}{b+n_b} \geq 1\). For \(j \geq 2\), we wish to verify that

\[
\left\lfloor \frac{2n_b + 2}{j + 1} - 1 \right\rfloor \geq \frac{2n_b + 1}{b + n_b} + j \frac{b(2n_b + 1)}{b + n_b} \geq 2n_b + 1\]

which holds exactly when

\[
\left\lfloor \frac{2n_b + 2}{j + 1} - 1 \right\rfloor \geq b(1 - j) + n_b\]

and since the left-hand side is non-negative, the inequality holds if \(n_b - 2b \leq 0\), which holds precisely when \(b \geq 2\).

Thus

\[
c_{2n_b+1}(E(1, 2n_b + 1)) = 2n_b + 1 = c_{2n_b+1}(P(\frac{2n_b + 1}{n_b + b} \cdot \frac{2n_b + 1}{n_b + b} \cdot b))
\]

so the ECH capacities of the ellipsoid and polydisc coincide. Note that this also shows this value is sharp, so no smaller \(\lambda\) will provide an embedding otherwise the ECH capacities no longer satisfy the required monotonicity property.

It remains to show that all subsequent capacities satisfy the monotonicity property. For this, the argument of [1 Prop. 3.1] carries over mutatis mutandis. By estimating the growth rate of \(c_k(P)\) and \(c_k(E)\), one finds that \(c_k(P)\) grows faster in \(k\) than \(c_k(E)\) for \(b > 2\) and \(k \geq 2n_b + 1\), so all subsequent capacities satisfy the necessary inequality.

With this in hand, we know that the class \(E_n\), determines the graph of \(c_h(a)\) at \(a = 2n_b + 1\). We now need to show that this remains the case up until where its obstruction function meets the volume constraint.

3.2. The Reduction Method on \([2n_b + 1, RF]\). To show that \(E_{n_b}\) is the only relevant class to the RF-value, note the obstruction function \(\mu_{E_{n_b}}(a)\) is constant for \(a > 2n_b + 1\). We claim that the point where the volume constraint \(\sqrt{\frac{RF}{2b}}\) equals \(\mu_{E_{n_b}}(2n_b + 1)\) is on the graph of \(c_h(a)\). This holds if there is an embedding of \(E(1, RF) \hookrightarrow P(\sqrt{\frac{RF}{2b}}, b \cdot \sqrt{\frac{RF}{2b}})\). If this embedding exists, it must determine the
graph of \( c_\delta(a) \) as \( c_\delta(a) \geq \sqrt{\frac{b}{a^2}} \).

To show that this embedding exists, we straightforwardly apply the reduction algorithm at the claimed \( RF \)-value. First, we require a computation justifying the title of this section. Recall that by \( 3 \), we have the bound

\[(3.4) \quad RF_b \leq (\sqrt{2b} + 1)^2\]

so this limits the length of the interval we need to consider.

Lemma 3.2. For \( b > 2 \), the interval \([2n_b + 1, RF]\) has nonzero length except at \( b = a_k \) where \( a_k \) are the positive roots of \( \sqrt{2b} + \{b\} - [\sqrt{2b} + \{b\}] \). In fact \( 0 \leq |(\sqrt{2b} + 1)^2 - (2n_b + 1)| \leq 2.\)

Proof. A straightforward computation establishes the first claim. For the second claim, using the formula for \( n_b \) we have

\[
(\sqrt{2b} + 1)^2 - (2n_b + 1) = 2b + 2\sqrt{2b} + 1 - 2n_b - 1
\]

\[
= 2(b - [b]) + 2\sqrt{2b} - 2[\sqrt{2b} + \{b\}] + 2
\]

\[
= 2(\sqrt{2b} + \{b\}) - 2[\sqrt{2b} + \{b\}] + 2.
\]

To bound this quantity above, \( \sqrt{2b} + \{b\} - [\sqrt{2b} + \{b\}] \leq 0 \), so the above difference is at most 2. Similarly \( \sqrt{2b} + \{b\} - [\sqrt{2b} + \{b\}] > -1 \) gives a lower bound of 0. \( \square \)

In fact, \( |RF - 2n_b - 1| \to 0 \) as \( b \to \infty \), but we will not need this here.

Lemma 3.3. If \( b > 2 \) and \( a \in [2n_b + 1, (\sqrt{2b} + 1)^2] \), then \( 1 < \sqrt{\frac{b}{a}} < \frac{4}{3} \).

Proof. Observe that \( \sqrt{2b} - \{b\} - [\sqrt{2b} + \{b\}] \) is bounded below by 1 for positive \( b \), whereas its derivative (defined almost everywhere) is strictly negative. Thus \( \sqrt{2b} - \{b\} - [\sqrt{2b} + \{b\}] \) approaches 1 from above starting at slightly less than \( \frac{4}{3} \) at \( b = 2. \) \( \square \)

We are now ready to apply the reduction method. For simplicity of notation, let \( \lambda = \sqrt{\frac{b}{a}} \). Consider the weight vector

\[
((b + 1)\lambda; b\lambda, \lambda, w(RF_b, 1))
\]

By Lemma 3.3, the entries of the vector above are ordered correctly. In particular, \( w(RF_b, 1) \) contains at least \( 1 \times 2n_b + 1 \), so we have

\[
((b + 1)\lambda; b\lambda, \lambda, 1 \times 2n_b + 1 || w(RF_b - 2n_b - 1, 1))
\]

with defect \(-1\), so we apply a Cremona to get the unordered vector

\[
((b + 1)\lambda - 1; b\lambda - 1, \lambda - 1, 1 \times 2n_b || w(RF_b - 2n_b - 1, 1))
\]

Remark 3.4. The symbol \( || \) means that the terms preceding it are ordered, but the terms after it are possibly not ordered. All terms before \( || \) are not less than the terms after \( || \).

Positivity of the entries follows from \( b > 1 \) and Lemma 3.3. Also by Lemma 3.3, the \( \lambda - 1 \) terms do not contribute to the defect, but we will have \( b\lambda - 1 > 1 \) by Lemma 3.3 and the condition that \( b \geq 2 \). So, we have the ordering

\[
((b + 1)\lambda - 1; b\lambda - 1, 1 \times 2n_b || \lambda - 1, w(RF_b - 2n_b - 1, 1))
\]

with defect

\[
\delta = (b + 1)\lambda - 1 - b\lambda + 1 - 2 = \lambda - 2,
\]
which is negative. Applying another Cremona yields

\[(3.5) \quad (b + 2)\lambda - 3; (b + 1)\lambda - 3, 1^{x2n_b-2}|(\lambda - 1)^x2, w(RF_b - 2n_b - 1, 1)\].

At this point we require the following lemma.

**Lemma 3.5.** For \(b > 2, (b + 1)\lambda - 3 > 1\).

**Proof.** Observe that for fixed \(b\), we may bound \(\lambda\) above and below using the formulas for \(RF\) and \(2n_b + 1\). At \(b = 2\) we see that \((b + 1)\lambda - 3 = 1.5\), and the derivative of \((b + 1)\lambda - 3\) is increasing where it exists. Where the derivative does not exist, \((b + 1)\lambda - 3\) still increases with \(b\) as both floor and ceiling functions are non-decreasing. Hence \((b + 1)\lambda - 3\) is strictly increasing so the inequality follows. \(\square\)

Thus the ordering in (3.5) is correct and the defect is

\[
\delta = (b + 2)\lambda - 3 - (b + 1)\lambda + 3 - 2
\]

\[
= \lambda - 2
\]

which is again negative. Note that in further Cremona transformations, as long as we have at least \(2\) \(1\)'s remaining, the defect will remain the same. For example, one more Cremona gives \((b + 3)\lambda - 5\) as the head, and \((b + 2)\lambda - 5\) as the first entry of the tail, and the defect is again \(\lambda - 2\). Note that this also disposes of \(2\) copies of \(1\).

So we apply \(n_b - 1\) more Cremonas to eliminate the \(1\)s and obtain

\[
((b + 1 + n_b)\lambda - (2n_b + 1); (b + n_b)\lambda - (2n_b + 1), |(\lambda - 1)^x2n_b+1, w(RF_b - 2n_b - 1, 1))
\]

To verify positivity of all entries, the first entry of the tail is positive since \(a \leq RF\), and the head is positive for the same reason (it is strictly larger than the first tail entry). Now the ordering of the last two terms is important for calculating the defect.

It turns out that both possibilities occur, so we will need to treat both cases. Let \(d\) be the first term in the weight sequence \(W(RF - 2n_b - 1, 1)\), so \(d < 1\).

**3.2.1. Case 1:** \((\lambda - 1) \geq d\). In this case we have the ordering

\[
((b + 1 + n_b)\lambda - (2n_b + 1); (b + n_b)\lambda - (2n_b + 1), (\lambda - 1)^x2n_b+1, w(RF_b - 2n_b - 1, 1))
\]

so the defect is

\[
(b + 1 + n_b)\lambda - (2n_b + 1) - (b + n_b)\lambda + (2n_b + 1) - 2(\lambda - 1)
\]

\[
= 2 - \lambda > 0.
\]

So, we have an embedding in this case.

**3.2.2. Case 2:** \((\lambda - 1) \leq d\). Either \(d > \frac{1}{b}\) or not. If not, let \(d'\) be the next term in the weight sequence. Now we could also have \(d' > \lambda - 1\) or not. Suppose that \(d' > \lambda - 1\). Then the weight vector has the form

\[
((b + 1 + n_b)\lambda - (2n_b + 1); (b + n_b)\lambda - (2n_b + 1), d, d', (\lambda - 1)^x2n_b+1, w(d', 1 - d))
\]

This has defect

\[
\delta = (b + 1 + n_b)\lambda - (2n_b + 1) - (b + n_b)\lambda + (2n_b + 1) - d - d'
\]

\[
= \lambda - d - d'
\]

Since \(d + d' < 1\) by properties of the weight sequence, this quantity is positive.

In the case where \(d' < \lambda - 1\), we have

\[
((b + 1 + n_b)\lambda - (2n_b + 1); (b + n_b)\lambda - (2n_b + 1), d, (\lambda - 1)^x2n_b+1, d', w(d, 1 - d'))
\]
This has defect
\[
\delta = (b + 1 + n_b)\lambda - (2n_b + 1) - (b + n_b)\lambda + (2n_b + 1) - 2\lambda + 2 \\
= 2 - \lambda
\]
which is positive by Lemma 3.3.

Now, assume we may take out at least 2 copies of \(d\) in the weight sequence. This looks like
\[
((b + 1 + n_b)\lambda - (2n_b + 1); (b + n_b)\lambda - (2n_b + 1), d, \lambda - 1)^{×2n_b + 1}, W(d, 1 - 2d)
\]
assuming that the terms of \(W(d, 1 - 2d)\) are all less than \(\lambda - 1\). This has defect
\[
\delta = (b + 1 + n_b)\lambda - (2n_b + 1) - (b + n_b)\lambda + (2n_b + 1) - 2d \\
= \lambda - 2d.
\]
The defect \(\lambda - 2d\) is nonnegative when \(\lambda \geq 2d\). This is equivalent to \(\lambda - 1 \geq 2d - 1\), but \(2d - 1\) is negative while \(\lambda - 1 > 0\). Thus, the defect is always nonnegative and the result follows. We have determined the graph of \(c_b(a)\) for \(a \in [2n_b + 1, RF]\) and \(b > 2\).

4. Beyond the RF-Value to the Sharp Bound

We now use the reduction method to show that on the interval \([RF, (\sqrt{2b} + 1)^2]\), the function equals the volume constraint. That is, we show that for any \(a \in [RF, (\sqrt{2b} + 1)^2]\), the corresponding weight vector reduces to one with positive defect, as we did to the left of the RF-value.

So, we begin with the same vector
\[
((b + 1)\lambda; b\lambda, \lambda, W(1, a))
\]
which as always has defect \(-1\). We apply a Cremona to get
\[
C_1 = ((b + 1)\lambda - 1; b\lambda - 1, \lambda - 1, 1^{×2n_b + 1})|W(1, a - 2n_b - 1)).
\]
Now, the ordering becomes important. The first tail term is certainly larger than 1 since \(b > 2\), and similarly is larger than the second term. However, the second term may not be larger than 1; indeed, note that \(\lambda - 1 > 1\) implies \(a > 8b\) and for \(a\) in the range we consider here, this is never true. Thus, the ordering becomes
\[
C_1 = ((b + 1)\lambda - 1; b\lambda - 1, 1^{×2n_b})|\lambda - 1, W(1, a - 2n_b - 1))
\]
which has defect
\[
\delta_1 = (b + 1)\lambda - 1 - b\lambda + 1 - 2 = \lambda - 2,
\]
Applying another Cremona and re-ordering gives
\[
C_2 = ((b + 1)\lambda - 1 + \lambda - 2; b\lambda - 1 + \lambda - 2, 1 + \lambda - 2, 1 + \lambda - 2, 1^{×2n_b - 2})|W(1, a - 2n_b - 1))
\]
which has defect
\[
\delta_2 = (b + 2)\lambda - 3 - (b + 1)\lambda + 3 - 2 = \lambda - 2.
\]
At this point, we note that the defect is the same but there are 2 fewer 1s in the vector. We apply \(n_b - 1\) more Cremonas, for a total of \(n_b\), as this gets rid of 2\(n_b\)
Figure 1. Relative ordering of terms in the tail of the weight vector in Equation (3.21), restricted to \(a \in [RF, (\sqrt{2b} + 1)^2]\) and an interval where each such term is smooth in \(b\). Black line segments are intersections of the graphs of these terms as functions of \(a, b\).

Total copies of 1.

Thus we have

\[
C_{n_b} : ((b + n_b + 1)\lambda - (2n_b + 1); (b + n_b)\lambda - (2n_b + 1))\]

\[
(\lambda - 1) \times 2^{n_b + 1}, W(1, a - 2n_b - 1)\).
\]

Computing the defect here is again dependent on ordering. By Lemma 3.2, we can pull out at most 1 copy of \(RF - 2n_b - 1\) from 1. If we increase \(a\), there is a point at which \(a - 2n_b - 1\) becomes larger than 1, in which case we can pull out more 1’s. We will need to distinguish some cases, i.e. whether the next term is the \(\lambda - 1\) term, the \((b + n_b)\lambda - (2n_b + 1)\) or the first term of \(W(1, a - 2n_b - 1)\). This variation in \(a\) and \(b\) is depicted in Figure 1.

Given the fact that the relative orderings of these terms can change, we organize our argument according to these orderings. Each section with multiple cases to consider begins with a flowchart to describe the possibilities.

4.1. Weight Sequence Terms Dominate and \(d \geq 1\). Note that by Lemma 3.2, if \(a \in [RF, (\sqrt{2b} + 1)^2]\), then \(a - 2n_b - 1 < 2\). Letting \(d' = (a - 2n_b - 1) - 1\), we consider the case when \(d'\) is greater than \(m = (b + n_b)\lambda - (2n_b + 1)\), which in turn is larger than \(\lambda - 1\). This means that \(a - 2n_b - 1 > 1\); the case where \(d < m\) will be treated in the following section.

We will also require the following lemma.
Figure 2. Flowchart for Subsection 4.1 when terms from the weight sequence \( W(1, a - 2n_b - 1) \) are largest. We call the first such unique term \( d \), and the second term \( d' \). Here \( m = (b + n_b)\lambda - (2n_b + 1) \) and \( \lambda = \sqrt{2b} \).

**Lemma 4.1.** For all \( b > 2 \) and \( a \in [RF, (\sqrt{2b} + 1)^2] \), \( m = 0 \) identically when \( a = RF \), and \( m < 1 \).

**Proof.** The first claim is a straightforward computation. For the second, as \( m \) depends on \( \lambda \), take the largest possible value for \( \lambda \) given by \( RF(b) \leq (\sqrt{2b} + 1)^2 \). Then at most,

\[
m \leq (b + n_b)\sqrt{2b + 1} - (2n_b + 1)
\]

and we show this is bounded above by 1. Note that when \( RF = (\sqrt{2b} + 1)^2 \), \( m = 0 \).

We first show that \( m(x_n) = 0 \) occurs for a sequence \( x_n \) diverging to \( \infty \) where \( C' < |x_{n+1} - x_n| < C'' \) and \( C', C'' \) are independent of \( n \). Then, we show that where \( m \) is differentiable, \( \frac{\partial m}{\partial b} < \frac{1}{C''} \), so the result follows.

Note first that the sequence \( \{x_n\} \) consists of those points where

\[
(b + n_b)\sqrt{2b + 1} - (2n_b + 1) = 0
\]

which is equivalent to the condition

\[
\sqrt{2x} + \{x\} \in \mathbb{Z}.
\]

Where this function is differentiable, the derivative is \( \frac{\sqrt{2x} + 1}{\sqrt{2x} + 1} \). Thus

\[
\frac{2}{3} \leq \frac{\sqrt{2x_n}}{\sqrt{2x_n} + 1} < |x_{n+1} - x_n| < \frac{\sqrt{2x_{n+1}}}{\sqrt{2x_{n+1}} + 1} < 1
\]

when \( b > 1 \), so this proves the first claim.

For the second claim, differentiating \( m \) with respect to \( b \) gives

\[
\frac{1}{\sqrt{2b}} \left(\frac{(\sqrt{2b} + 1)2b - b - n_b}{2b}\right)
\]

as \( \frac{\partial m}{\partial b} = 0 \). Now this is bounded above by 1; one can show it is equivalent to \( b < n_b \), which holds when \( b > 1 \). \( \hfill \Box \)
If an extra 1 appears in the vector and \( d' > m \), additional terms in the weight sequence may contribute in the following ways:

1. \( d' \leq \frac{1}{2} \), so there are \( k \geq 2 \) copies of \( d' \).
2. \( d' > \frac{1}{2} \) and \( d'' \geq m \), so there is 1 copy of \( d' \), 1 copy of \( d'' \).
3. \( d' > \frac{1}{2} \) and \( d'' \leq m \), so there is 1 copy of \( d' \), 1 copy of \( d'' \) with \( m \) between.

4.1.1. Case 1: \( d' \leq \frac{1}{2} \). We begin with Case (1). When this occurs, the vector has the form

\[
C_{n_b+1} : ((b + n_b + 1) \lambda - (2n_b + 1); 1, d', (b + n_b) \lambda - (2n_b + 1)
\]

\[
(\lambda - 1)^{\times 2n_b+1}, |W(1 - 2d', d')|
\]

with defect

\[
\delta_{n_b+1} = (b + n_b + 1) \lambda - (2n_b + 1) - 1 - 2d'
\]

\[
m + \lambda - 1 - 2d'.
\]

This is certainly negative given the assumptions. We continue with another Cremona:

\[
C_{n_b+2} : (2m + 2\lambda - 1 - 2d'; m + \lambda - 2d', m + \lambda - 1 - d', m + \lambda - 1 - d',
\]

\[
m, (\lambda - 1)^{\times 2n_b+1}, |W(1 - 2d', d')|.
\]

To ensure all terms are positive, we must have the inequality \( |m - d'| < \lambda - 1 \).

Lemma 4.2. When \( d' > m \), \( d' - m < \lambda - 1 \).

Proof. It suffices to show that when \( a = (\sqrt{2b} + 1)^2 \) and \( b \in x_n \) for \( x_n \) the points of discontinuity of \( n_b \), we have \( d' - m = \lambda - 1 \). To see this, note \( \frac{\partial m}{\partial b} > \frac{2d'}{\partial b} > 0 \) and \( \frac{\partial \lambda}{\partial b} < 0 \) where defined. Hence for fixed \( a \), the inequality above holds. Taking into account the partials with respect to \( a \), \( \frac{\partial d'}{\partial a} > \frac{\partial m}{\partial a} \geq 0 \), so the inequalities still hold.

So, we show the given identity for \( d' - m \).

\[
d' - m = \frac{(\sqrt{2b} + 2\sqrt{2b} + 1)\sqrt{2b} - \sqrt{2b} - (\sqrt{2b} + 1)(b + n_b)}{\sqrt{2b}}
\]

\[
= \frac{b\sqrt{2b} + 3b + \sqrt{2b} + 1 - (\sqrt{2b} + 1)|b| - (\sqrt{2b} + 1)[\sqrt{2b} + \{b\}]}{\sqrt{2b}}
\]

which will be equal to \( \lambda - 1 = \frac{\sqrt{2b} + 1}{\sqrt{2b}} - 1 = \frac{1}{\sqrt{2b}} \) (keeping in mind that \( a = (\sqrt{2b} + 1)^2 \)) precisely when

\[
\sqrt{2b} + b\sqrt{2b} + 3b - (\sqrt{2b} + 1)|b| - (\sqrt{2b} + 1)[\sqrt{2b} + \{b\}] = 0
\]

Factoring out \( (\sqrt{2b} + 1) \), this is true when

\[
\sqrt{2b} + \{b\} - [\sqrt{2b} + \{b\}] = 0
\]

which is precisely when \( n_b \) is discontinuous. \( \square \)

Now to determine the ordering, since \( d' \leq \frac{1}{2} \), we have the unordered vector

\[
C_{n_b+2} : (2m + 2\lambda - 1 - 2d'; m, m + \lambda - 2d', (m + \lambda - 1 - d')^{\times 2},
\]

\[
(\lambda - 1)^{\times 2n_b+1}, (\lambda - 1 - d')^{\times 2}|W(1 - 2d', d')|.
\]
There are two possibilities: either \( m + \lambda - 1 - d' > \lambda - 1 \) or \( m + \lambda - 1 - d' \leq \lambda - 1 \).

In the first case, the vector has the form

\[
C_{n_k+2} : (2m + 2\lambda - 1 - 2d'; m, m + \lambda - 2d', (m + \lambda - 1 - d')^2, \\
(\lambda - 1)^{2n_k+1}, (\lambda - 1 - d')^2||W(1 - 2d', d'))
\]

with defect

\[
\delta_{n_k+2} = 2m + 2\lambda - 1 - 2d' - m - m - \lambda + 2d' - m - \lambda + 1 + d' \\
= 1 - m + d' > 0
\]

which is positive as \( d' > m \) by assumption, so an embedding exists.

In the second case, the vector has the form

\[
C_{n_k+2} : (2m + 2\lambda - 1 - 2d'; m, m + \lambda - 2d', (\lambda - 1)^{2n_k+1}, \\
(\lambda - 1 - d')^2, (\lambda - 1 - d')^2||W(1 - 2d', d'))
\]

with defect

\[
\delta_{n_k+2} = 2m + 2\lambda - 1 - 2d' - m - m - \lambda + 2d' - \lambda + 1 \\
= 0
\]

so again in this case we have an embedding.

It is also possible that there are more than 2 copies of \( d' \), in which case we may have a vector of the form. Then

\[
C_{n_k+1} : ((b + n_b + 2)\lambda - (2n_b + 2 + 2d'); (b + n_b)\lambda - (2n_b + 1), d''^k, \\
(\lambda - 2d', (\lambda - 1)^{2n_k+1}, (\lambda - 1 - d')^2||W(1 - 2d', d'))
\]

with defect

\[
\delta_{n_k+1} = (b + n_b + 2)\lambda - (2n_b + 2 + 2d') - (b + n_b)\lambda + (2n_b + 1) - 2d' \\
= 2\lambda - 1 - 4d'
\]

Since \( \lambda > 1 \), this is positive when \( 4d' \leq 1 \), so that \( d' < \frac{1}{4} \) and we see at least 4 copies of \( d' \). We must also deal with the possibility that there are exactly 3 copies of \( d' \), in which case we have

\[
C_{n_k+2} : ((b + n_b + 2)\lambda - (2n_b + 2 + 2d'); (b + n_b)\lambda - (2n_b + 1), d', \\
(\lambda - 2d', (\lambda - 1)^{2n_k+1}, (\lambda - 1 - d')^2||W(1 - 2d', d'))
\]

with defect

\[
\delta_{n_k+2} = (b + n_b + 2)\lambda - (2n_b + 2 + 2d') - (b + n_b)\lambda + (2n_b + 1) - d' - \lambda + 2d' \\
= \lambda - 1 - d'
\]

which is negative since \( d' > \lambda - 1 \) by assumption. So we apply a further Cremona to obtain

\[
C_{n_k+3} : ((b + n_b + 3)\lambda - (2n_b + 3 + 3d'); (b + n_b + 1)\lambda - (2n_b + 2 + d'), \lambda - 1, \\
(\lambda - 2d', (\lambda - 1)^{2n_k+1}, (\lambda - 1 - d')^2||W(1 - 2d', d'))
\]
which reorders to
\[ C_{n_b+3} : (b + nb + 3)\lambda - (2nb + 3 + 3d') \lambda - (2nb + 2 + d'), \lambda - 2d', (\lambda - 1)^x2n_b+2, (\lambda - 1 - d')^x2||W(1 - 2d', d')) \]
where possibly the first two terms are switched. Either way, this has defect
\[
\delta_{n_b+3} = (b + nb + 3)\lambda - (2nb + 3 + 3d') - (b + nb + 1)\lambda + (2nb + 2 + d') - \lambda + 2d' - \lambda + 1 = 0.
\]
So again an embedding exists.

In the other case where \( \lambda - 2d' \leq m \), it is again possible that there are more than 2 copies of \( d' \), in which case we may have a vector of the form
\[
C_{n_b+2} : ((b + nb + 2)\lambda - (2nb + 2 + 2d'); \lambda - 2d', (b + nb)\lambda - (2nb + 1), \\
(\lambda - 1)^x2n_b+1, (\lambda - 1 - d')^x2||W(1 - 2d', d')) \]
with defect
\[
\delta_{n_b+1} = (b + nb + 2)\lambda - (2nb + 2 + 2d') - 3d' = (b + nb)\lambda - (2nb + 1) + 2\lambda - 5d'.
\]
Now since \( \lambda > 1 \), this is positive when \( 5d' < 2 \), so when \( d' < \frac{2}{5} \) an embedding exists. So what remains is when we see 3 or 2 copies of \( d' \).

When we see 2 copies of \( d' \) at the beginning, this has the form
\[
C_{n_b+2} : ((b + nb + 2)\lambda - (2nb + 2 + 2d'); \lambda - 2d', (b + nb)\lambda - (2nb + 1), \\
(\lambda - 1)^x2n_b+1, (\lambda - 1 - d')^x2||W(1 - 2d', d')) \]
with defect
\[
\delta_{n_b+1} = (b + nb + 2)\lambda - (2nb + 2 + 2d') - \lambda + 2d' - (b + nb)\lambda + (2nb + 1) - \lambda + 1 = 0
\]
and we have an embedding. Lastly, if there are 3 copies of \( d' \) to begin with, we have the vector
\[
C_{n_b+2} : ((b + nb + 2)\lambda - (2nb + 2 + 2d'); d', \lambda - 2d', (b + nb)\lambda - (2nb + 1), \\
(\lambda - 1)^x2n_b+1, (\lambda - 1 - d')^x2||W(1 - 2d', d')) \]
with defect
\[
\delta_{n_b+1} = (b + nb + 2)\lambda - (2nb + 2 + 2d') - d' - \lambda + 2d' - (b + nb)\lambda + (2nb + 1) = \lambda - 1 - d'
\]
which is negative by assumption. Thus we need another Cremona, which gives
\[
C_{n_b+2} : ((b + nb + 3)\lambda - (2nb + 3 + 3d'); \lambda - 1, 2\lambda - 1 - 3d', (b + nb + 1)\lambda - (2nb + 2 + d'), \\
(\lambda - 1)^x2n_b+1, (\lambda - 1 - d')^x2||W(1 - 2d', d')) \]
and upon ordering we see
\[
C_{n_b+2} : ((b + nb + 3)\lambda - (2nb + 3 + 3d'); 2\lambda - 1 - 3d', (b + nb + 1)\lambda - (2nb + 2 + d'), \\
(\lambda - 1)^x2n_b+2, (\lambda - 1 - d')^x2||W(1 - 2d', d')) \]
with possibly the first two terms switched. Either way, the defect is
\[
\delta_{n_b+2} = (b + n_b + 3)\lambda - (2n_b + 3 + 3d') - (b + n_b + 1)\lambda + (2n_b + 2 + d')
- 2\lambda + 1 + 3d' - \lambda + 1
= 2\lambda - 1 - 2d' - 3\lambda + 2 + 3d'
= 1 + d' - \lambda
\]
and this last term is \(d' - (\lambda - 1)\), which was assumed positive. This covers Case (1).

4.1.2. Case 2: \(d' > \frac{1}{2}\) and \(d'' \geq m\). Next, we deal with Case (2). The vector has the form
\[
C_{n_b+1} : ((b + n_b + 1)\lambda - (2n_b + 1);1,d',d'',(b + n_b)\lambda - (2n_b + 1)
\]
\[(\lambda - 1)^{\times 2n_b+1}, ||W(1 - 2d', d')||\]
and with the assumed ordering, we have defect
\[
\delta_{n_b+1} = (b + n_b + 1)\lambda - (2n_b + 1) - 1 - d' - d''
= (b + n_b + 1)\lambda - (2n_b + 2 + d' + d'')
\]
This could certainly be negative, so we apply a further Cremona:
\[
C_{n_b+2} : ((b + n_b + 1)\lambda - (2n_b + 1) + (b + n_b + 1)\lambda - (2n_b + 2 + d' + d'')
1 + (b + n_b + 1)\lambda - (2n_b + 2 + d' + d''), d' + (b + n_b + 1)\lambda - (2n_b + 2 + d' + d''),
d'' + (b + n_b + 1)\lambda - (2n_b + 2 + d' + d''), (b + n_b)\lambda - (2n_b + 1)
(\lambda - 1)^{\times 2n_b+1}, ||W(1 - 2d', d')||\]
\[
= (2(b + n_b + 1)\lambda - (4n_b + 3 + d' + d''); (b + n_b + 1)\lambda - (2n_b + 2 + d''),
b + n_b + 1)\lambda - (2n_b + 2 + d''), (b + n_b)\lambda - (2n_b + 1)
(\lambda - 1)^{\times 2n_b+1}, ||W(1 - 2d', d')||\]
\]
and we must determine the new ordering. This turns out to be straightforward; consider the inequality
\[(b + n_b + 1)\lambda - (2n_b + 2 + d') \geq \lambda - 1.
\]
This is equivalent to
\[(b + n_b)\lambda - (2n_b + 1) \geq d'
which we assumed was false for this case. A similar argument shows that the remaining two new terms introduced by the defect must be smaller than \(\lambda - 1\). Hence we have a new ordering
\[
C_{n_b+1} : (2(b + n_b + 1)\lambda - (4n_b + 3 + d' + d''); (b + n_b)\lambda - (2n_b + 1), (\lambda - 1)^{\times 2n_b+1},
||((b + n_b + 1)\lambda - (2n_b + 1 + d' + d''), (b + n_b + 1)\lambda - (2n_b + 2 + d''),
(b + n_b + 1)\lambda - (2n_b + 2 + d'), W(1 - 2d', d'))||\]
with defect
\[
\delta_{n_b+1} = 2(b + n_b + 1)\lambda - (4n_b + 3 + d' + d'') - (b + n_b)\lambda
+(2n_b + 1) - 2(\lambda - 1)
= (b + n_b)\lambda - (2n_b + d' + d'')
\]
Now we may rearrange this as

$$((b + n_b)\lambda - (2n_b + 1)) + 1 - d' - d''$$

and observe that the first term in parentheses is positive, while the last three terms sum to 0. Hence the overall sum is positive, and we have an embedding.

4.1.3. Case 3: $d' \geq \frac{1}{2}$ and $d'' \leq m$. Lastly we deal with Case (3). The extra weight sequence terms may or may not be larger than $-1$, but we deal with this only when necessary. This ordering looks like

$$C_{n_b+1} : ((b + n_b + 1)\lambda - (2n_b + 1); 1, d', (b + n_b)\lambda - (2n_b + 1)$$

$$||((\lambda - 1)^{2n_b+1}, W(1 - d', d'))$$

with defect

$$\delta_{n_b+1} = (b + n_b + 1)\lambda - (2n_b + 1) - 1 - d' - (b + n_b)\lambda + (2n_b + 1)$$

$$= \lambda - 1 - d'$$

This is negative by assumption on the ordering, so we must apply a Cremona.

$$C_{n_b+2} : ((b + n_b + 1)\lambda - (2n_b + 1) + \lambda - d' - 1;$$

$$1 + \lambda - d' - 1, d' + \lambda - d' - 1, m + \lambda - d' - 1$$

$$||((\lambda - 1)^{2n_b+1}, W(1 - d', d'))$$

$$= ((b + n_b + 2)\lambda - (2n_b + 2 + d'); \lambda - d', \lambda - 1, m + \lambda - d' - 1$$

$$||((\lambda - 1)^{2n_b+2}, W(1 - d', d'))$$

By reassociating terms as in previous arguments, these entries are non-negative; the first tail term is bounded below since $d' < 1$ and $\lambda > 2$ using Lemmata ??, similarly for the second term. The third is positive under these assumptions by Lemma 4.2. Also by the same lemma, the third term is smallest, so we can rearrange as

$$C_{n_b+2} : ((b + n_b + 1)\lambda - (2n_b + 1) + \lambda - d' - 1;$$

$$1 + \lambda - d' - 1, d' + \lambda - d' - 1, m + \lambda - d' - 1$$

$$||((\lambda - 1)^{2n_b+1}, W(1 - d', d'))$$

$$= ((b + n_b + 2)\lambda - (2n_b + 2 + d'); \lambda - d', (\lambda - 1)^{2n_b+2},$$

$$||m + \lambda - d' - 1, W(1 - d', d'))$$

So we can compute another defect:

$$\delta_{n_b+2} = m + 2\lambda - d' - 1 - \lambda + d' - 2\lambda + 2$$

$$= m + 1 - \lambda = m - (\lambda - 1)$$

which is positive by assumption.

4.2. Weight Sequence Terms Dominate and $d \leq 1$. In the previous section we assumed that we could take an extra copy of 1 out of the weight sequence of the above terms, but it is still possible to have $d \leq 1$ but all the same relative orderings of the variable terms. In this section, we account for these possibilities. Again, assuming $d' > m, \lambda - 1$, additional terms in the weight sequence may contribute in the following ways:

1. $d' \leq \frac{1}{2}$, so there are $k \geq 2$ copies of $d'$.
2. $d' > \frac{1}{2}$ and $d'' \geq m$, so there is 1 copy of $d'$, 1 copy of $d''$.
3. $d' > \frac{1}{2}$ and $d'' \leq m$, so there is 1 copy of $d'$, 1 copy of $d''$ with $m$ between.
the possibility that there are exactly 2, 3, or 4 copies of $d$. We call the first such unique term $d$, and the second term $d'$. Here $m = (b + n_b)\lambda - (2n_b + 1)$ and $\lambda = \sqrt{2n}$. 

![Figure 3. Flowchart for Subsection 4.2 when terms from the weight sequence $W(1, a - 2n_b - 1)$ are largest but $d \leq 1$. We call the first such unique term $d$, and the second term $d'$. Here $m = (b + n_b)\lambda - (2n_b + 1)$ and $\lambda = \sqrt{2n}$.

4.2.1. Case 1: $d' \leq \frac{1}{2}$. We begin with Case (1). When this occurs, the vector has the form

$$C_{n_b+1} : (b + n_b + 1)\lambda - (2n_b + 1), d', d', (b + n_b)\lambda - (2n_b + 1),$$

$$\lambda - 1)^{\lambda - 2n_b + 1}, ||W(1 - 2d', d'))$$

with defect

$$\delta_{n_b+1} = (b + n_b + 1)\lambda - (2n_b + 1) - 2d' - (b + n_b)\lambda + (2n_b + 1)$$

$$\lambda - 2d'.$$

This is positive as $\lambda - 1 > 0$ and $2d' \leq 1$ by assumption. So in this case, we see that an embedding exists.

It is also possible that there are more than 2 copies of $d'$, in which case we may have a vector of the form

$$C_{n_b+2} : (b + n_b + 2)\lambda - (2n_b + 2 + 2d'); d^k, (b + n_b)\lambda - (2n_b + 1),$$

$$\lambda - 2d', (\lambda - 1)^{\lambda - 2n_b + 1}, (\lambda - 1 - d')^{\lambda - 2} ||W(1 - 2d', d'))$$

with defect

$$\delta_{n_b+2} = (b + n_b + 2)\lambda - (2n_b + 2 + 2d') - 3d'$$

$$= (b + n_b)\lambda - (2n_b + 1) + 2\lambda - 1 - 5d'.$$

As the first quantity is strictly positive, since $\lambda > 1$ this defect is necessarily positive when $5d' \leq 1$, so that $d' < \frac{1}{5}$ and we see at least 5 copies of $d'$. We now deal with the possibility that there are exactly 2, 3, or 4 copies of $d'$.

In the case where we have 3 or 4 copies, it is still possible to perform the previous Cremona, and another iteration gives

$$C_{n_b+3} : (2b + 2n_b + 2)\lambda - (4n_b + 3 + 2d'); (b + n_b)\lambda - (2n_b + 1) + 2\lambda - 1 - 5d', d^{k-3},$$

$$(b + n_b)\lambda - (2n_b + 1), \lambda - 2d', (\lambda - 1)^{\lambda - 2n_b + 1}, (\lambda - 1 - d')^{\lambda - 2} ||W(1 - 2d', d')).$$

If $k = 4$ then note that one copy of $d'$ remains and

$$(b + n_b)\lambda - (2n_b + 1) + 2\lambda - 1 - 5d' \geq d'.$$
if $2\lambda - 1 - 6d' \geq 0$, i.e. if $d' \leq \frac{1}{6}$, which holds by assumption. So the ordering remains, and we have defect

$$
\delta_{n_3+3} = (2b + 2n_b + 2)\lambda - (2n_b + 2 + 2d') - (b + n_b)\lambda + (2n_b + 1) - 2\lambda + 1 + 5d' - d' - (b + n_b)\lambda + (2n_b + 1)
$$

$$= 6d' + 6$$

which is positive.

Continuing to the case where $k = 3$, this has defect

$$
\delta_{n_3+3} = (2b + 2n_b + 2)\lambda - (2n_b + 3 + 2d') - (b + n_b)\lambda + (2n_b + 1) - 2\lambda + 1 + 5d' - (b + n_b)\lambda + (2n_b + 1) - \lambda + 2d'
$$

$$= 5d'.$$

Thus, this quantity is positive, so again an embedding exists.

In the last case where we see precisely 2 copies of $d'$ (so $\frac{1}{3} < d' \leq \frac{1}{2}$), we have a vector of the form

$$C_{n_3+3} = ((b + n_b + 2)\lambda - (2n_b + 2 + 2d'); d', d'; (b + n_b)\lambda - (2n_b + 1) \lambda - 2d', (\lambda - 1)^{2n_b+1}, (\lambda - 1 - d')^2 \|W(1 - 2d', d'))$$

with defect

$$\delta_{n_3+3} = (b + n_b + 2)\lambda - (2n_b + 2 + 2d') - 2d' - (b + n_b)\lambda + (2n_b + 1)$$

$$= 2\lambda - 1 - 4d'.$$

Now since $2\lambda > 4$ and $\frac{1}{3} < 1 + 4d \leq 3$, this is positive as needed. This covers Case (1).

4.2.2. Case 2: $d' \geq \frac{1}{2}$ and $d'' \geq m$. Next, we deal with Case (2). The vector has the form

$$C_{n_4+1} = ((b + n_b + 1)\lambda - (2n_b + 1); d', d'', (b + n_b)\lambda - (2n_b + 1) \lambda - 1)^{2n_b+1}, \|W(1 - 2d', d'))$$

and with the assumed ordering, we have defect

$$\delta_{n_4+1} = (b + n_b + 1)\lambda - (2n_b + 1) - d' - d'' - (b + n_b)\lambda + (2n_b + 1)$$

$$= \lambda - d' - d''.$$

Now $\lambda > 2$ and both $d', d'' < 1$ by properties of the weight sequence, so this quantity is positive and we have an embedding.

4.2.3. Case 3: $d' \geq \frac{1}{2}$ and $d'' \leq m$. Lastly, we deal with Case (3). It now matters whether or not $\lambda - 1$ appears as the third term in the defect. We begin with $\lambda - 1 > d''$. This looks like

$$C_{n_3+1} = ((b + n_b + 1)\lambda - (2n_b + 1); d', (b + n_b)\lambda - (2n_b + 1) \|\lambda - 1)^{2n_b+1}, W(1 - d', d'))$$

with defect

$$\delta_{n_3+1} = (b + n_b + 1)\lambda - (2n_b + 1) - d' - (b + n_b)\lambda + (2n_b + 1) - \lambda + 1$$

$$= -d' + 1.$$
This is non-negative as \( d' \leq 1 \), so we have an embedding.

Similarly if \( d'' > \lambda - 1 \), we have

\[
C_{n_k+1} = ((b + n_k + 1)\lambda - (2n_k + 1); (b + n_k)\lambda - (2n_k + 1) \mid (\lambda - 1)^{2n_k+1}, W(1 - d', d'))
\]

with defect

\[
\delta_{n_k+1} = (b + n_k + 1)\lambda - (2n_k + 1) - d' - (b + n_k)\lambda + (2n_k + 1) - d''
\]

\[
\lambda - d' - d''.
\]

Since \( \lambda > 2 \), this is positive, so the result follows.

4.3. The term \( m := (b + n_k)\lambda - (2n_k + 1) \) dominates. It will turn out that the ordering

\[
C_{n_k} : ((b + n_k + 1)\lambda - (2n_k + 1); (b + n_k)\lambda - (2n_k + 1) \mid (\lambda - 1)^{2n_k+1}, W(1, a - 2n_k - 1))
\]

never happens for any \( a,b \) in the intervals of interest.

Lemma 4.3. For any \( a \in [RF, (\sqrt{2}b + 1)^2] \) and \( b \) in the interval where function is smooth, \( (b_n + b)\lambda - (2nb + 1) < a - 2n_k - 1 \) where \( \lambda = \sqrt{\frac{2}{2b + 1}} \).

Proof. At \( a = RF \) and \( b = x_n \) for some \( n \), the two quantities coincide, and moreover at \( a = RF \), \( m \) vanishes identically. We show that \( \frac{\partial m}{\partial a} > \frac{\partial d}{\partial a} \), and also \( \frac{\partial d}{\partial b} > 0 \) for all \( a \neq RF \). It then follows that \( m \leq d \), with equality only at \((a,b) = (RF,x_n)\).

The first claim follows immediately from the formulas for \( m,d \):

\[
m(a,b) = (b + n_k)\sqrt{\frac{a}{2b} - 2n_k - 1}
\]

\[
d(a,b) = a - 2n_k - 1
\]

so we see that

\[
\frac{\partial m}{\partial a} = (b + n_k)\frac{1}{2 \cdot \sqrt{2ab}}
\]

\[
\frac{\partial d}{\partial a} = 1.
\]

Testing the smallest possible value \( a = RF(b) \), we have

\[
\frac{\partial m}{\partial a}_{a=RF(b)} = (b + n_k)\frac{1}{2 \cdot \sqrt{2b(2n_k + 1)}}
\]

\[
= \frac{(b + n_k)^2}{4b(2n_k + 1)}.
\]

To see that this last term is less than 1, note it is equivalent to

\[
b^2 - 6bn_b - 4b + n_b^2 < 0
\]

Now at \( b = 2 \), this holds. \( \frac{\partial m}{\partial b} \) of the left-hand side gives \( 2b - 6n_b - 4 \). To see that this is strictly negative for the range of \( b \) in question, it is equivalent to

\[
b + 1 < 3|b| + 3|\sqrt{2b} + b|
\]
which holds since \(b + 1 < 3|b|\) once \(b > 1\).

Lastly, we verify that \(\frac{\partial d}{\partial b} > 0\), so the inequality established for fixed \(b = x_n\) persists. Let \(n' = \frac{2n_b}{b}\). We have

\[
\frac{\partial d}{\partial b} = 2 \left( \frac{2n_b + 1}{b + n_b} \right)^2 + 4b \left( \frac{2n_b + 1}{b + n_b} \right) \left( \frac{2n'(b + n_b) - (2n_b + 1)(1 + n')}{(b + n_b)^2} \right) - 2n'.
\]

We verify the inequality for the numerator

\[2(2n_b + 1)^2(b + n_b) + 4b(2n_b + 1)(-2n_b - 1) > 0\]

which simplifies to \(b < n_b\), a fact verified in Lemma 4.1. \(\square\)

It follows that for any \(a, b\) where \((b + n_b)\lambda - (2n_b + 1) \neq a - 2n_b - 1\), the ordering in \([1]\) does not occur.

However, when \(a - 2n_b - 1 > 1\), we may take \(d = 1\) in the weight sequence of \(W(1, a - 2n_b - 1)\). As before, \(d'\) is the next term in this weight sequence. Now it can certainly occur that \(d' < (b + n_b)\lambda - (2n_b + 1)\), and we must account for this option. So, returning to \(C_n\), we treat the case where \((b + n_b)\lambda - (2n_b + 1)\) is largest, so we see the ordering

\[
C_{n+1}^b : ((b + n_b + 1)\lambda - (2n_b + 1); 1, (b + n_b)\lambda - (2n_b + 1), d' \times k
\]

\[\| (\lambda - 1)^{\times 2n_b + 1}, W(1 - kd', d') \]

with defect

\[
\delta_{n+1}^b = (b + n_b + 1)\lambda - (2n_b + 1) - 1 - (b + n_b)\lambda + (2n_b + 1) - d'
\]

\[= \lambda - 1 - d'
\]

which is negative since by assumption \(d' > \lambda - 1\). So we apply another Cremona transformation.

\[
C_{n+2}^b : ((b + n_b + 1)\lambda - (2n_b + 1) + \lambda - 1 - d'; 1 + \lambda - 1 - d', (b + n_b)\lambda - (2n_b + 1) + \lambda - 1 - d', d' \times k - 1\| (\lambda - 1)^{\times 2n_b + 1}, W(1 - kd', d')
\]

\[= ((b + n_b + 2)\lambda - (2n_b + 2 + d'); \lambda - d', (b + n_b + 1)\lambda - (2n_b + 2 + d'), d' \times k - 1\| (\lambda - 1)^{\times 2n_b + 2}, W(1 - kd', d')).
\]
The ordering now changes to
\[ ((b + n_b + 2)\lambda - (2n_b + 2 + d'); \lambda - d', (b + n_b + 1)\lambda - (2n_b + 2 + d'), d'^{1-k-1} |\]
\[ (\lambda - 1)^{2n_b+2}, W(1 - kd', d')). \]

Thus the defect is
\[ \delta_{n_b+2} = (b + n_b + 2)\lambda - (2n_b + 2 + d') - (b + n_b + 1)\lambda + (2n_b + 2 + d') - \lambda + d' - d' \]
\[ = 0 \]
which guarantees an embedding. So this case works so long as \( k \) is at least 2.

When \( d' > \frac{1}{2} \), we cannot guarantee the above form of the weight vector, so we must consider this possibility separately. We begin with the un-ordered vector \( C_4 \).

Thus the defect depends on the relative orderings of \( \lambda - 1 \) and elements of \( W(1, a - 2n_b - 2) \), giving the cases:

1. \( \lambda - 1 \) larger than \( d', d'' \).
2. \( d' > \lambda - 1 \). We pull out at most 1 term, and it will suffice.
3. \( d' > d'' > \lambda - 1 \).

4.3.1. Case 1: \( \lambda - 1 > d' > d'' \). This ordering must look like
\[ C_{n_b+1} : \left((b + n_b + 1)\lambda - (2n_b + 1); 1, (b + n_b)\lambda - (2n_b + 1)|\right] \]
\[ d', (\lambda - 1)^{2n_b+1}, W(1 - d', d')). \]

Thus the defect depends on the relative orderings of \( \lambda - 1 \) and elements of \( W(1, a - 2n_b - 2) \), giving the cases:

\[ \delta_{n_b+2} = (b + n_b + 1)\lambda - (2n_b + 1) - 1 - (b + n_b)\lambda + (2n_b + 1) - \lambda + 1 \]
\[ = 0 \]
so an embedding exists.

4.3.2. Case 2: \( d' > \lambda - 1 \). Now for sub-case 2, the ordering is
\[ C_{n_b+2} : \left((b + n_b + 1)\lambda - (2n_b + 1); 1, (b + n_b)\lambda - (2n_b + 1), d', (\lambda - 1)^{2n_b+1}, d'^{1-k} \right] \]
\[ W(1, a - 2n_b - 2)) \]

with defect
\[ \delta_{n_b+2} = (b + n_b + 1)\lambda - (2n_b + 1) - 1 - (b + n_b)\lambda + (2n_b + 1) - d' \]
\[ = \lambda - 1 - d' \]
which may be negative. So we apply another Cremona
\[ C_{n_b+3} : \left((b + n_b + 2)\lambda - (2n_b + 2 + d'); \lambda - d', (b + n_b + 1)\lambda - (2n_b + 2 + d'), \right] \]
\[ (\lambda - 1)^{2n_b+2}, d'^{1-k}, W(1, a - 2n_b - 2)) \]
and the above vector is ordered. The ensuing defect is
\[ \delta_{n_b+3} = (b + n_b + 2)\lambda - (2n_b + 2 + d') - \lambda + d' - (b + n_b + 1)\lambda + (2n_b + 2 + d') - \lambda + 1 \]
\[ = d' - (\lambda - 1) \]
which by assumption is positive.
4.3.3. Case 3: \( d' > d'' > \lambda - 1 \). Lastly, we treat sub-case 3, where both terms of the weight sequence are larger than \( \lambda - 1 \). The first term is \( 2n_b + 1 - a \), and the second term is \( 2n_b + 1 - a + 2n_b + 2 = 4n_b + 3 - 2a \).

\[
C_{n_b+1} : \quad \begin{aligned} & (b + n_b + 2)\lambda - (2n_b + 3); (b + n_b + 1)\lambda - (2n_b + 3), (\lambda - 1)^x^2 \\ & ||(\lambda - 1)^x^{2n_b+1}, W(1, a - 2n_b - 2) \end{aligned} \\
= & \quad ((b + n_b + 2)\lambda - (2n_b + 3); (b + n_b + 1)\lambda - (2n_b + 3) \\
& 2n_b + 1 - a, 4n_b + 3 - 2a, ||(\lambda - 1)^x^{2n_b+3}, W(2n_b + 1 - a, 4n_b + 3 - 2a) \\
\end{aligned}
\]

with resulting defect

\[
\delta_{n_b+1} = (b + n_b + 2)\lambda - (2n_b + 3) - (b + n_b + 1)\lambda + (2n_b + 3) - 2n_b - 1 + a - 4n_b - 3 + 2a \\
= \lambda + 3a - 6n_b - 4.
\]

To verify the sign, since \( a \in [RF, (\sqrt{2b} + 1)^2] \), in principle \( a \) could be \( 2n_b + 1 \). But after rearranging, the defect looks like

\[(\lambda - 1) + 3(a - 2n_b - 1)\]

and both terms are strictly positive, so the embedding exists.

4.4. \((\lambda - 1)\) Dominates. Now suppose \( \lambda - 1 \) is the largest term. In this case, the weight vector looks like

\[
C_{n_b} : \quad \begin{aligned} & (b + n_b + 1)\lambda - (2n_b + 1); (\lambda - 1)^x^{2n_b+1}, \\ & (b + n_b)\lambda - (2n_b + 1), W(1, a - 2n_b - 1) \end{aligned} 
\]

This has defect

\[
\delta_{n_b} = (b + n_b + 1)\lambda - (2n_b + 1) - 3\lambda + 3 \\
= (b + n_b - 2)\lambda - (2n_b - 2) = m - 2\lambda + 3.
\]

To check the sign of this defect, we re-arrange into \( m - 2(\lambda - 1) + 1 > 0 \).

Lemma 4.4. For \( b > 2 \), \( m - 2(\lambda - 1) + 1 > 0 \).

Proof. This inequality is equivalent to \( m + 1 > 2(\lambda - 1) \). When \( a = RF \), we have \( m = 0 \), so this reduces to \( \frac{1}{2} > \lambda - 1 \) which holds for all \( b > 2 \). So it suffices to show that \( \frac{\partial m}{\partial a} > 2\frac{\partial \lambda}{\partial a} \), so the inequality still holds.

We computed \( \frac{\partial m}{\partial a} \) in Lemma 4.3, and we can now compute

\[
\frac{\partial \lambda}{\partial a} = \frac{1}{2\sqrt{2ab}}
\]

so the claim becomes

\[
(b + n_b) \cdot \frac{1}{2 \cdot \sqrt{2ab}} > 2 \cdot \frac{1}{2\sqrt{2ab}}
\]

which is in fact true for all \( b > 1 \). \( \square \)
5. THE RF-VALUE WHEN $1 < b < 2$: The Sequence $b_n = \frac{n+1}{n}$

At this point, we turn our attention towards the smaller values of $b$, with an eye towards the proof of Theorem 1.2. Our argument essentially follows the lines of [3], which first finds a sequence of exceptional classes which are obstructive at $a = 8$, one for each $b_n = \frac{n+1}{n}$. That is, for the embedding problem $E(1,8) \hookrightarrow P(\lambda, \lambda \cdot \frac{n+1}{n})$, we find for every $n$ solutions $(d,e;m)$ of non-negative integers to the Diophantine equations

\begin{align*}
\sum_i m_i &= 2(d + e) - 1 \\
\sum_i m_i^2 &= 2de + 1
\end{align*}

subject to the constraint given by Theorem 2.3.

5.1. Eliminating Possible Classes on $(8,9)$. We begin by restricting the set of possibilities for each $d$. This is accomplished by an analogue of [12, Prop. 5.2.1]. We first require a number of preliminary lemmas that help to bound the possible values of $d,e$ and the sequence $m$. The following quantities will be relevant in the sequel: let $\ell_0$ denote the number of 1’s in the weight expansion of $a$ and subsequently let $\ell_i$ denote the lengths of subsequent blocks. Recall that $M$ denotes the length of the weight sequence of $a$. When a class $(d,e;w(a))$ is obstructive, we have a vector of error terms $\epsilon$ defined as

\begin{equation}
m = \frac{d + be}{\sqrt{2a}}w(a) + \epsilon.
\end{equation}

Each contribution to this error, thought of as the difference between $m$ and $\frac{d + be}{\sqrt{2a}}w(a)$, will be important, so we define $v_i = \frac{d + be}{\sqrt{2a}}w_i$ for $i = 0, \ldots, M$. Here, $M$ is the length of the weight vector.

Also let

\begin{equation}
\sigma = \sum_{\ell_0 + 1}^{M} \epsilon_i
\end{equation}

denote the “residual error,” the contribution to the error vector coming from non-integer terms of $w(a)$. Also, define a related quantity

\begin{equation}
\sigma' = \sum_{\ell_0 + 1}^{M - \ell_N} \epsilon_i
\end{equation}

where $\ell_N$ is the length of the last block that ignores the contribution to the error from the smallest part of the weight expansion $w(a)$.

With these terms established, the following results (proven in [12, 3]) will be used repeatedly.

**Theorem 5.1.** Suppose that $(d,e;m)$ is obstructive for the embedding problem (1.1).

1. $\mu_b(d,e;m) > \frac{2}{\sqrt{2b}}$ iff the error vector satisfies $(\epsilon,w(a)) > 0$
2. If $\mu_b(d,e;m) > \frac{2}{\sqrt{2b}}$ then $d = be + h$ where $|h| < \sqrt{2b}$, and $||\epsilon||^2 < 1 - \frac{h^2}{2b}$.

We begin with the observation that if the graph of $c_b(a)$ does not follow the volume constraint, then it must be given by the obstruction function of some class
(d, e; m). Restricting to the interval where this class determines the graph, Lemma 2.1.3 states that there is a particular a-value on the interval whose weight expansion coincides with the number of positive entries of m, the tail of the obstructive class.

**Lemma 5.2.** Suppose that (d, e; m) is obstructive and effective, and let I be the maximal open interval on which this class remains obstructive (i.e. above the volume constraint). Then there is a unique a₀ ∈ I such that ℓ(a₀) = ℓ(m). Moreover ℓ(a) ≥ ℓ(a₀) for all a ∈ I.

We will also need a way to estimate the range of coefficients of obstructive classes, given a range of a-values. This is accomplished by understanding a particular function (denoted y(a)) which arises from the ball-packing problem.

Using this function, we state some analogues of propositions from the ellipsoid case. This version is in [3].

**Lemma 5.3.** For all (d, e; m) ∈ E, suppose that a ∈ Q such that ℓ(a) = ℓ(m) and µ(d, e; m)(a) > \sqrt{\frac{2b}{q}}. Then we have

1. \( \mu_b(d, e; m)(a) \leq \sqrt{\frac{2d + 1}{(d + be)^2}} \).
2. \( \mu_b(d, e; m)(a) > \sqrt{\frac{2b}{q}} \) if and only if \( \epsilon \cdot w > 0 \).
3. If \( \mu_b(d, e; m)(a) > \sqrt{\frac{2b}{q}} \) then \( \langle \epsilon, \epsilon \rangle < 1 - \frac{b}{\sqrt{2b}} \).
4. Let \( y(a) = a + 1 - \frac{b + 1}{\sqrt{2b}} \sqrt{a} \), where a = p/q is rational. Then

\[
- \sum_i \epsilon_i = 1 + \frac{d}{\sqrt{a}} \left( y(a) - \frac{1}{q} \right).
\]

Following [3] we have an adaptation of their Lemma 3.7.

**Lemma 5.4.** Let (d, e; m) be an exceptional class such that ℓ(a) = ℓ(m) and \( \mu_b(d, e; m) > \sqrt{\frac{2b}{q}} \), and b ∈ [1, 2]. Set \( v_M := \frac{(d + be)\sqrt{2b}}{q(b + 1)\sqrt{a}} \) where q is the last denominator in the weight expansion w(a). Then

1. \( \sum_i \epsilon_i \leq \sqrt{\sigma L} \).
2. If \( v_M < 1 \), then \( \sum_i \epsilon_i \leq \sqrt{\sigma'} L \).
3. If \( v_M \leq \frac{1}{2} \), then \( v_M > \frac{1}{2} \) and \( \sigma' \leq \frac{1}{2} \). If \( v_M \leq \frac{2}{3} \), then \( \sigma' \leq \frac{2}{3} \).
4. With \( \delta = y(a) - \frac{1}{q} \) and \( y(a) = a + 1 - \frac{b + 1}{\sqrt{2b}} \sqrt{a} \), we have

\[
2be + h \leq \frac{2be + h}{\delta} \left( \frac{\sigma \delta v_M}{\delta} - (1 - h(1 - \frac{1}{b})) \right) \leq \frac{\sqrt{2ba}}{\delta} \left( \frac{\sigma}{\delta v_M} - (1 - h(1 - \frac{1}{b})) \right).
\]

**Proof.** The first three claims are proven exactly as in [12 Lemma 5.1.2]. To show (4), we have from (1) – (3) that

\[
- \sum_i \epsilon_i = \frac{2be + h}{\sqrt{2ba}} \left( a + 1 - \frac{b + 1}{b} \sqrt{2ba} - \frac{1}{q} \right) + (1 - h(1 - \frac{1}{b})).
\]
Using $q \geq L$ and (1) of this lemma,
\[
\sqrt{\sigma L} \leq \sqrt{qL} \geq \frac{2be + h}{\sqrt{2ba}}(a + 1 - b + 1) + (1 - h(1 - \frac{1}{b}))
\]
\[
= \frac{2be + h}{\sqrt{2ba}}\delta + (1 - h(1 - \frac{1}{b}))
\]
\[
> \delta v_M q
\]
where the first line is by (1), the second is by definition of $\delta$, and the last comes from $b \in [1, 2)$.

It follows that $\sqrt{q} < \sqrt{\sigma \delta v_M}$. Rearranging the above inequality we see
\[
2be + h \leq \frac{\sqrt{2ba}}{\delta} \left( \sqrt{\sigma q} - (1 - h(1 - \frac{1}{b})) \right)
\]
\[
< \frac{\sqrt{2ba}}{\delta} \left( \frac{\sigma}{\delta v_M} - (1 - h(1 - \frac{1}{b})) \right).
\]

\[\square\]

We will use (4) to bound the value of $e$, and deduce that there are no obstructive classes satisfying this condition, using computer programs from [12, 4, 1].

The following lemma is [12, Lem. 2.1.8].

**Lemma 5.5.** Assume that $(d,e;m)$ is an exceptional class such that $\mu(d,e;m) > \sqrt{\frac{\sigma}{2b}}$ for some $a$. Let $J := \{k, ..., k + s - 1\}$ be a block of $s - 1$ consecutive integers for which $w(a_i), i \in J$ is constant. Then $(m_1, ..., m_{k+1})$ is of the form
\[
(m, ..., m)
\]
\[
(m - 1, m, ..., m)
\]
\[
(m, ..., m + 1).
\]

Moreover, there is at most one block of length $s \geq 2$ on which the $m_i$ are not all equal, and if $m_1 \neq m_{k+1}$, then \(\sum_{i=k}^{k+s-1} c_i^2 \geq \frac{1}{s}\).

Our process for detecting possible obstructive classes looks roughly as follows:

1. Using (4), calculate $q$ such that the inequality fails.
2. For each positive integer $q$ less than that, find the allowable $e$-values.
3. Test whether the possibilities satisfy equations (5.1) (5.2), ignoring possibilities which differ too much from the weight expansion of $a$.

This process rules out all possibilities for solutions to (5.1) (5.2) which might be obstructive for the given $a$-values.

**Lemma 5.6.** For $b_n = \frac{a + 1}{n}$ and $a \in (8, 9)$, there are no exceptional classes $(d,e;m)$ with obstruction function above the volume constraint on $(8,9)$ such that $\ell(a) = \ell(m)$.

**Proof.** If such classes exist, the same argument as in [3] using our Lemma 5.4(3) shows the following estimates. Note that while $v_m$ depends on $b$, the intervals below cover all possibilities, and $\sigma, \sigma'$ are independent of $b$. So the following estimates are
independent of $b$ (and hence $n$).

\[
\begin{align*}
v_m &\geq \frac{2}{3} \quad \Rightarrow \quad \frac{\sigma}{v_M} \leq \frac{3}{2}, \\
v_m &\in \left[\frac{1}{2}, \frac{2}{3}\right] \quad \Rightarrow \quad \frac{\sigma'}{v_M} \leq \frac{14}{9}, \\
v_m &\in \left[\frac{1}{3}, \frac{1}{2}\right] \quad \Rightarrow \quad \frac{\sigma'}{v_M} \leq \frac{3}{2}.
\end{align*}
\]

Then for fixed $q$ and $h$, we define the following functions from Lemma 5.4 (4) as in [3]:

\[
\begin{align*}
F(a,q,h) &:= \frac{\sqrt{2a(n+1)}}{n} \left(\sqrt{\sigma q} - (1 - h(1 - \frac{n}{n+1}))\right), \\
G(a,q,h) &:= \frac{\sqrt{2a(n+1)}}{n} \left(\frac{\sigma}{\delta v_M} - \left(1 - h(1 - \frac{n}{n+1})\right)\right).
\end{align*}
\]

Then by Lemma 5.4

\[
\frac{2(n+1)}{n} e + h \leq f(q,h) \leq g(q,h)
\]

where $f(q,h) = F(8\frac{1}{q},q,h)$ and $g(q,h) = G(8\frac{1}{q},q,h)$. Notice also that these functions are decreasing on $a \in (8,9)$ (recall that $\delta$ depends on $a$ and $q$).

**Lemma 5.7.** For $b \in (1,1.5]$ and $|h| < \sqrt{2b}$, let $q_0$ be the $q$-value where $f(q,h) = g(q,h)$. Then

1. $1 < q_0 < 4$,
2. $\frac{\partial f}{\partial q} > 0$ and $\frac{\partial g}{\partial q} < 0$ for $q > 1$, and
3. $g(2,h) < 9$.
4. $\frac{\partial f}{\partial h}, \frac{\partial g}{\partial h} > 0$.

**Proof.** (2) is a straightforward computation. (4) is immediate as its coefficients in $f,g$ are strictly positive for all $q,b$. Moreover, $\frac{\partial f}{\partial h}, \frac{\partial g}{\partial h}$ are constant in $h$, so as $h$ changes, $f,g$ only increase.

For (1), Note that $f(q,h)$ and $g(q,h)$ are equal if and only if $\sqrt{q} = \frac{14}{9} \frac{1}{\sigma(q,h)}$, which amounts to

\[
\sqrt{q} = \frac{14}{9} \frac{1}{\sqrt{2(\frac{2}{3})}}.
\]

This is a quadratic in $q$, and since $b \in (1,1\frac{1}{2})$, we find that $q$ ranges between 3.33 and 3.654. Hence the intersection point $q_0$ is no larger than 3.

For (3), by (4) we simply evaluate at the largest possible value of $h$, which is $h = \sqrt{2b} = \sqrt{2\frac{1}{2}} < 1.6$. This gives the bound.

Thus, if we have an obstructive class in this interval of the form $(d,e,m)$, this lemma shows that $q \in \{2,3\}$ and $d = be + h \leq 8$, since we must have $f \leq g$ and this is impossible past $q_0$ given the above properties. Using the Solutions program of [12], we can generate all possible solutions to the Diophantine equations determining
potential obstructive classes with \( d \leq 7 \). That is, if \( q = 2 \), then the length of such a class must be 10, as this is the length of the weight sequence of \( a = 8 + \frac{1}{q} \). We can make a complete list of such classes. Examining this list, one sees that they all violate Lemma 5.5 (recalling that for \( a = 8 + \frac{1}{q} \), the first 8 entries of \( W(1, 8 + \frac{1}{q}) \) are identically 1). Hence, none can be obstructive.

Repeating this process for \( q = 3 \), we find similarly that no classes found using this procedure can be obstructive at \( 8 + \frac{1}{3} \). □

5.2. Obstructive Classes at \( a = 8, b_n = \frac{n+1}{n} \). To establish the RF-value of \( b_n = \frac{n+1}{n} \), we now show that for each \( n \) there is only one possible obstructive class at \( a = 8 \). We define the following infinite family of homology classes

\[
R_n := ((2n+1)(n+1), (2n+1)n; \frac{1}{8}(2(2n+1)^2 + 6), \frac{1}{8}[(2(2n+1)^2 + 6) - 1]^7)
\]

Changing coordinates to those of \( X_n \), this becomes:

\[
(3n^2 + 3n; n^2 + 2n, n^2 + 1, (n^2 + n)^7).
\]

If this vector can be reduced to \((0; -1, 0, \ldots)\) via Cremona transformations, then the class will be effective.

**Lemma 5.8.** After \( 4n + 3 \) Cremona transformations, the obstructive class \( R_n \) takes the form:

\[
(3k^2 - (6n + 1)k + \sum_{j=0}^{n-1} (6j + 4); ((k - n)^2)^5, (k - n)^2 - 1, (k - n)(k - (n + 1))^3)
\]

where this vector is reduced for all \( n \geq 2 \) satisfying \( k = n \).

**Proof.** Letting \( c_p \) denote the \( p \)th Cremona’d vector, we note that applying 11 Cremonas yields

\[
c_{11} = (3k^2 - 13k + 24; ((k - 2)^2)^5, (k - 2)^2 - 1, (k - 2)(k - 3)^3).
\]

Plugging in \( k = 2 \) yields the reduced vector \((0; -1)\). Applying four Cremonas inductively yields the claim. □

Now, Lemma 5.6 applied to the interval \((8, 9)\) excludes the possibility of obstructive classes on this interval.

**Proposition 5.9.** The only exceptional class with \( \mu_{b_n}(8) > \sqrt{\frac{8}{2 + 6}} \) is

\[
((2n+1)n, (2n+1)(n+1); n^2 + n + 1, (n^2 + n)^7).
\]

**Proof.** As in [3] Lemma 3.10], the strategy is to examine the Diophantine equations for Chern number 1 and self-intersection \(-1\), and show that these equations have no solutions for the given parameter values. The fact that our parameter values are variable is only a technical complication.

Recall that by [Theorem 5.1 (2)] for an obstructive class \((d, e; m)\) we have

\[
d = \frac{n + 1}{n} e + h
\]
where \( d, e \in \mathbb{N} \) and \(|h| < \sqrt{\frac{2n+2}{n}}\). It follows that \( h \in \frac{1}{n} \mathbb{Z} \), so we write
\[
d = \frac{(n+1)e}{n} + \frac{k}{n}
\]
or, writing as a Diophantine equation,
(5.6) \[nd - (n+1)e = k.\]
Since \( \gcd(n, n+1) = 1 \) there are integer solutions of the following form. For the specific equation
\[xn + y(n+1) = \gcd(n, n+1) = 1\]
we have \( x = -1, y = 1 \) as solution, and more generally a particular solution \((d, e)\) to (5.6) is
\[n(-k) - (-k)(n+1) = k.
\]
It follows from general theory of linear Diophantine equations that all integer solutions can be constructed from this particular solution as
(5.7) \[d = -k + \frac{-(n+1)l}{\gcd(n, n+1)} = -k - (n+1)l\]
(5.8) \[e = -k - \frac{nl}{\gcd(n, n+1)} = -k - nl\]
with \( l \in \mathbb{Z} \). We will show that \( k = 0 \) necessarily for an obstructive class of this form. Using again the fact that we are at \( a = 8 \), we apply Lemma 5.5 to show that the sum of the \( m_i \) for any tail \( \mathbf{m} \) must satisfy
\[2de + 1 = \begin{cases} 8m^2 + 2m + 1 & \text{if } (m+1, m^7) \\ 8m^2 - 2m + 1 & \text{if } (m^7, m-1) \end{cases}\]
and similarly
\[2(d+e) - 1 = \begin{cases} 8m + 1 & \text{if } (m+1, m^7) \\ 8m - 1 & \text{if } (m^7, m-1) \end{cases}\]
so we combine these to obtain pairs of Diophantine equations. We treat the case \((m-1, m^7)\) first. Now we can substitute (5.7) and (5.8) to see
\[2(-k - l(n+1) - k - n) - 1 = 8m - 1\]
\[2(k^2 - k(2n+1) + n(n+1)) + 1 = 8m^2 - 2m + 1.\]
Solving for \( m \) in the first equation and substituting into the first gives the following polynomial in \( l \):
\[l^2 \left(n(n+1) - \frac{1}{4}(2n+1)^2\right) + l \left(-\frac{1}{4}(2n+1)\right) + \frac{1}{2}k = 0.\]
This quadratic has roots
\[l = -n - \frac{1}{2} \pm \sqrt{(2n+1)^2 + \frac{1}{2}k}.\]
If \( l \) is to be an integer, at the very least \( k = 0 \) as \(|k| < \sqrt{2n(n+1)}\) from Lemma 5.3 (3). But then
\[l = n - \frac{1}{2} \pm (2n+1)\]
which is not an integer. Thus, we cannot see obstructive classes of this form.

On the other hand, we can consider classes of the form \((m + 1, m \times 7)\). The same substitution gives the quadratic in \(l\)

\[
l^2 \left( n(n + 1) - \frac{1}{4}(2n + 1)^2 \right) + l \left( \frac{1}{4}(2n + 1) \right) - \frac{1}{2}k = 0.
\]

Again, since \(l\) must be an integer, we see that \(k = 0\), in which case this polynomial has roots

\[
l = \frac{(2n + 1) \pm 4 \sqrt{\frac{1}{16}(2n + 1)^2 - \frac{1}{4}k}}{2} = 0, 2n + 1.
\]

Hence the only possible obstructive classes at \(a = 8\) are of the form

\[
((2n + 1)n, (2n + 1)(n + 1); n^2 + n + 1, (n^2 + n) \times 7)
\]
as needed.

\[
\square
\]

5.3. The Reduction Method for \(b_n = \frac{n+1}{n}, n \in [9, \infty)\). To establish the shape of \(c_{b_n}(a)\) near the RF-value, we show that on \([9, \infty), c_{b_n}(a) = \sqrt{\frac{a}{2b}}\) the volume constraint. It will follow from the reduction method that if \(n \geq 8\) then \(RF \leq 9\).

As usual, our beginning weight vector has the form

\[
((b + 1)\lambda; b \cdot \lambda, \lambda, 1^{x9}, w(a - 9, 1)).
\]

and \(\delta = -1\) in the first step, so we see:

\[
((b + 1)\lambda - 1; b \cdot \lambda - 1, \lambda - 1, 1^{x8}, w(a - 9, 1)).
\]

Now if \(\lambda - 1 = \sqrt{\frac{a}{2b}} - 1 > 1\), which occurs for \(a \geq 9\) if \(n \geq 8\) in \(b_n = \frac{n+1}{n}\), then the defect is

\[
\delta = (b + 1)\lambda - 1 - b\lambda + 1 - \lambda + 1 - 1 = 0
\]
guaranteeing an embedding. So, we have that the \(RF\)-value for \(b_n = \frac{n+1}{n}, n \geq 8\) is no larger than 9 as needed.

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