PUFFINI–VIDEV MODELS AND MANIFOLDS

P. GILKEY, E. PUFFINI, AND V. VIDEV

1. Introduction

The study of commutativity and of spectral properties for natural operators in differential geometry has received much attention in recent years. Probably the seminal paper in the subject is due to Osserman [5] who proposed a characterization of Riemannian rank 1-symmetric spaces in terms of the spectrum of the Jacobi operator. There are, however, many other crucial works which should be cited – papers by Ivanova and Stanilov [6], by Stanilov [7], by Stanilov and Videv [9], by Szabó [10], and by Tsankov [8] are central. However, as the literature is a vast one, we must limit ourselves and shall refer to the bibliographies in [2, 3] for further information.

We shall work in both the geometric and in the algebraic contexts; the Jacobi operator will form the focus of our study. We begin by introducing some notational conventions. We say that \( \mathcal{M} := (V, \langle \cdot, \cdot \rangle, A) \) is a 0-model if \( \langle \cdot, \cdot \rangle \) is a non-degenerate inner product of signature \((p, q)\) on a finite dimensional vector space \( V \) of dimension \( m = p + q \) and if \( A \in \otimes^4 V^* \) is an algebraic curvature tensor, i.e. if \( A \) is a 4-tensor which satisfies the symmetries of the Riemann curvature tensor:

\[
A(v_1, v_2, v_3, v_4) = A(v_3, v_4, v_1, v_2) = -A(v_2, v_1, v_3, v_4),
A(v_1, v_2, v_3, v_4) + A(v_2, v_3, v_1, v_4) + A(v_3, v_1, v_2, v_4) = 0.
\]

The associated curvature operator \( A \) and Jacobi operator \( \mathcal{J} \) are then characterized, respectively, by the identities:

\[
\langle A(v_1, v_2)v_3, v_4 \rangle = A(v_1, v_2, v_3, v_4) \quad \text{and} \quad \langle \mathcal{J}(v_1)v_2, v_3 \rangle = A(v_2, v_1, v_3).
\]

The Jacobi operator \( \mathcal{J}(v) \) is quadratic in \( v \). It is convenient to polarize and to set

\[
\mathcal{J}(v_1, v_2) : v_3 \rightarrow \frac{1}{2}(A(v_1, v_2)v_3 + A(v_1, v_3)v_2).
\]

This operator was first introduced to study the geometry of the Jacobi operator by Videv [12]. If \( \vec{v} := (v_1, ..., v_k) \) is a basis for a non-degenerate \( k \)-plane \( \pi \), let \( \xi_{ij} := \langle v_i, v_j \rangle \) give the components of the metric restricted to \( \pi \) relative to the given basis. If \( \xi^{ij} \) denotes the inverse matrix, then one defines the higher order Jacobi operator by setting:

\[
\mathcal{J}(\vec{v}) := \sum_{i=1}^{k} \sum_{j=1}^{k} \xi^{ij} \mathcal{J}(v_i, v_j).
\]

Key words and phrases. 0-model, algebraic curvature tensor, Einstein manifold, Higher order Jacobi operator, Jacobi operator, Ricci operator.

2000 Mathematics Subject Classification. 53C20

1 Corresponding author.
Let $\pi(\vec{v}) := \text{Span}\{v_1, \ldots, v_k\}$. Then $J(\pi) := J(\pi(\vec{v}))$ is independent of the particular basis chosen for $\pi$; this operator was first introduced in this context by Stanilov and Videv in the Riemannian setting and latter extended by Gilkey, Stanilov, and Videv to the pseudo Riemannian setting. Note that $\rho := J(V)$ is the Ricci operator:

$$\rho : y \to \sum_{i=1}^{m} \sum_{j=1}^{m} \xi^{ij} A(y, e_i)e_j.$$ 

Thus the higher order Jacobi operator can also be thought of as a generalization of the Ricci operator to lower dimensional subspaces.

Let $\text{Gr}_{r,s}(V, \langle \cdot, \cdot \rangle)$ be the Grassmannian of all non-degenerate linear subspaces of $V$ which have signature $(r, s)$; the pair $(r, s)$ is said to be admissible if and only if $\text{Gr}_{r,s}(V, \langle \cdot, \cdot \rangle)$ is non-empty and does not consist of a single point or, equivalently, if the inequalities $0 \leq r \leq p$, $0 \leq s \leq q$, and $1 \leq r + s \leq m - 1$ are satisfied. Let $[A, B] := AB - BA$ denote the commutator of two linear maps. We shall establish the following result in Section 2.

**Theorem 1.1.** Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, A)$ be a 0-model. The following assertions are equivalent; if any is satisfied, then we shall say that $\mathcal{M}$ is a Puffini–Videv 0-model.\footnote{This notation was suggested by the first author}

1. There exists $(r_0, s_0)$ admissible so that $J(\pi)J(\pi^+) = J(\pi^+)J(\pi)$ for all $\pi \in \text{Gr}_{r_0,s_0}(V, \langle \cdot, \cdot \rangle)$.
2. $J(\pi)J(\pi^+) = J(\pi^+)J(\pi)$ for every non-degenerate subspace $\pi$.
3. $[J(\pi), \rho] = 0$ for every non-degenerate subspace $\pi$.

We say that $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, A)$ is decomposable if there exists a non-trivial orthogonal decomposition $V = V_1 \oplus V_2$ which decomposes $A = A_1 \oplus A_2$; in this setting, we shall write $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ where $\mathcal{M}_i := (V_i, \langle \cdot, \cdot \rangle|_{V_i}, A_i)$. One says that $\mathcal{M}$ is indecomposable if $\mathcal{M}$ is not decomposable. We say $\mathcal{M}$ is Einstein if the Ricci operator $\rho$ is a scalar multiple of the identity. By Theorem 1.1, any Einstein 0-model is Puffini–Videv. More generally, the direct sum of Einstein Puffini–Videv models is again Puffini–Videv; the converse holds in the Riemannian setting:

**Theorem 1.2.** Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, A)$ be a Riemannian 0-model. Then $\mathcal{M}$ is Puffini–Videv if and only if $\mathcal{M} = \mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_k$ where the $\mathcal{M}_i$ are Einstein.

In the pseudo-Riemannian setting, a somewhat weaker result can be established. One says that a 0-model is pseudo-Einstein either if the Ricci operator $\rho$ has only one real eigenvalue $\lambda$ or if the Ricci operator $\rho$ has two complex eigenvalues $\lambda_1, \lambda_2$ with $\lambda_1 = \lambda_2$. This does not imply that $\rho$ is diagonalizable in the higher signature setting and hence $\mathcal{M}$ need not be Einstein.

**Theorem 1.3.** Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, A)$ be a 0-model of arbitrary signature. If $\mathcal{M}$ is Puffini–Videv, then we may decompose $\mathcal{M} = \mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_k$ as the direct sum of pseudo-Einstein 0-models $\mathcal{M}_i$. 

We restrict to the Riemannian context henceforth. Theorem 1.2 yields the characterization that indecomposable Riemannian 0-model is Einstein if and only if it is Puffini–Videv. However, in passing to the geometric situation, things become a bit more complicated. Let $\mathcal{M} = (M, g)$ be a Riemannian manifold of dimension $m \geq 3$; any Riemann surface is automatically Puffini–Videv by Theorem 1.1. Let $\mathcal{M}_k(M, P) = (T_PM, \langle \cdot, \cdot \rangle|_P, \nabla_{\cdot}P, \nabla_{\cdot}P, \ldots, \nabla^{k}_{\cdot}P)$ be a purely algebraic object which encodes the geometric information about $\mathcal{M}$ up to the $(k + 2)^{th}$ order jets of the metric at a point $P$; $\mathcal{M}(M, P) = \mathcal{M}_0(M, P)$. The notion of indecomposability of a k-model is defined as above. One says that $\mathcal{M}$ is Puffini–Videv if $\mathcal{M}(M, P)$ is Puffini–Videv at each point of the manifold. One says $\mathcal{M}$ is locally reducible at a point...
P ∈ M if there exists a neighborhood O of P so that (O, g_M) = (O × O_2, g_1 ⊕ g_2) decomposes as a Cartesian product. We say M is locally irreducible at P if this does not happen. Clearly if \mathfrak{M}_k(M, P) is indecomposable for some k, then M is locally irreducible at P. Let

\[ \tau_M = \sum_{ijkl} g^{jk} g^{il} R_{ijkl} \]

be the scalar curvature of M. One says M exhibits scalar curvature blowup if there is a geodesic in M defined on a finite interval (0, T) so that \( \lim_{t \to 0} |\tau_M(\gamma(t))| = \infty \). Such a manifold is necessarily geodesically incomplete and cannot be embedded isometrically in a geodesically complete manifold.

One has the following examples as we shall discuss further in Section 3.

**Theorem 1.4.** Let \( N := (N, ds_N^2) \) be a Riemann surface. Assume \( \tau_N(P_0) \neq 2 \) for some \( P_0 \in N \). Let \((t, x_1, x_2)\) be local coordinates on \( M := (0, \infty) \times N \). Give M the metric \( g_M \) with non-zero components:

\[ g_M(\partial_t, \partial_t) = 1 \quad \text{and} \quad g_M(\partial_{x_i}, \partial_{x_i}) = t^2 g_N(\partial_{x_i}, \partial_{x_i}). \]

1. \( M := (M, g_M) \) is a 3-dimensional Riemannian Puffini–Videv manifold.
2. \( \mathfrak{M}_1(M, P) \) is indecomposable and M is locally irreducible at \( P = (t, P_0) \).
3. M exhibits scalar curvature blowup and is not Einstein.

**Theorem 1.5.** Let \((x_1, x_2, x_3, x_4)\) be coordinates on \( M := (0, \infty) \times (0, \infty) \times \mathbb{R}^2 \). Let \( \beta > 0 \). Give M the metric \( g_\beta \) whose non-zero components are:

\[ g_\beta(\partial_{x_1}, \partial_{x_1}) = g_\beta(\partial_{x_2}, \partial_{x_2}) = 1, \quad g_\beta(\partial_{x_3}, \partial_{x_3}) = x_1^2, \]
\[ g_\beta(\partial_{x_4}, \partial_{x_4}) = x_1(x_1 + \beta x_2). \]

1. \( \mathcal{M}_\beta := (M, g_\beta) \) is a 4-dimensional Riemannian Puffini–Videv manifold.
2. \( \mathfrak{M}_2(M_\beta, P) \) is indecomposable and M is locally irreducible for all \( P \in M \).
3. \( \mathcal{M}_\beta \) is not locally isometric to \( \mathcal{M}_\gamma \) for \( \beta \neq \gamma \).
4. \( \mathcal{M}_\beta \) exhibits scalar curvature blowup and is not Einstein.

### 2. Puffini–Videv 0-models

We begin our study in the algebraic context by establishing Theorem 1.1. Let \( \rho \) be the Ricci operator. Since \( \rho = \mathcal{J}(\pi) + \mathcal{J}(\pi^-) \), we have:

\[ \rho \mathcal{J}(\pi) = \mathcal{J}(\pi) \mathcal{J}(\pi^-) + \mathcal{J}(\pi^-) \mathcal{J}(\pi) - \mathcal{J}(\pi^-) \mathcal{J}(\pi^-) \mathcal{J}(\pi) \]

Thus establishes the equivalence of Assertions (2) and (3) in Theorem 1.1. It is immediate that Assertion (2) implies Assertion (1). We complete the proof of Theorem 1.1 by showing Assertion (1) implies Assertion (2). Assume there exists \((r_0, s_0)\) admissible so that

\[ \mathcal{J}(\pi) \mathcal{J}(\pi^-) = \mathcal{J}(\pi^-) \mathcal{J}(\pi) \quad \forall \pi \in Gr_{r_0, s_0}(V, \langle \cdot, \cdot \rangle). \]

Let \( 1 \leq \kappa := r_0 + s_0 < m := \dim(V) \). Let \( \{e_1, ..., e_\kappa, e_{\kappa+1}, ..., e_m\} \) be an orthonormal basis for V where \( \{e_1, ..., e_\kappa\} \) spans a non-degenerate plane \( \pi \) of signature \((r_0, s_0)\). Let \( \varepsilon_i := \langle e_i, e_i \rangle \). Then

\[ \mathcal{J}(\pi) := \sum_{i=1}^{\kappa} \varepsilon_i \mathcal{J}(e_i). \]

We distinguish two cases. Suppose first that \( \varepsilon_1 = \varepsilon_{\kappa+1} \). Set

\[ e_1(\theta) := \cos(\theta)e_1 + \sin(\theta)e_{\kappa+1}. \]
Then \( \{ e_1(\theta), e_2, ..., e_n \} \) is an orthonormal basis for a non-degenerate plane \( \pi(\theta) \) of signature \((r_0, s_0)\). One has
\[
0 = [\rho, \mathcal{J}(\pi(\theta)) - \mathcal{J}(\pi)] = 0
\]
\[
= [\rho, (\cos^2 \theta - 1)\mathcal{J}(e_1) + 2 \sin \theta \cos \theta \mathcal{J}(e_1, e_{\kappa+1}) + \sin^2 \theta \mathcal{J}(e_{\kappa+1})].
\]
This identity for all \( \theta \) implies
\[
[\rho, \mathcal{J}(e_1) - \mathcal{J}(e_{\kappa+1})] = 0 \quad \text{if} \quad \varepsilon_1 = \varepsilon_{\kappa+1}.
\]
Suppose next that \( \varepsilon_1 = -\varepsilon_{\kappa+1} \). Set \( e_1(\theta) := \cosh(\theta)e_1 + \sinh(\theta)e_{\kappa+1} \). A similar computation, after paying attention to the signs involved, yields:
\[
0 = [\rho, (\cosh^2 \theta - 1)\mathcal{J}(e_1) - 2 \sinh \theta \cosh \theta \mathcal{J}(e_1, e_{\kappa+1}) + \sinh^2 \theta \mathcal{J}(e_{\kappa+1})]
\]
which yields the identity
\[
0 = [\rho, \mathcal{J}(e_1) + \mathcal{J}(e_{\kappa+1})].
\]
We combine these two calculations to see that for all \( 1 \leq i, j \leq m \) we have that
\[
\varepsilon_i[\rho, \mathcal{J}(e_i)] = \varepsilon_j[\rho, \mathcal{J}(e_j)].
\]
We use Equation (2.c) to see that
\[
0 = [\rho, \mathcal{J}(\pi)] = \sum_{i=1}^{\kappa} \varepsilon_i[\rho, \mathcal{J}(e_i)] = \kappa \varepsilon_1[\rho, \mathcal{J}(e_1)]
\]
and thus \([\rho, \mathcal{J}(e_1)] = 0\). This shows that \([\rho, \mathcal{J}(v)] = 0\) for every unit spacelike vector if \( s_0 > 0 \) and for every unit timelike vector if \( r_0 > 0 \). We can rescale to conclude \([\rho, \mathcal{J}(v)] = 0\) on a non-empty open subset of \( V \) and hence, as this is a polynomial identity, conclude \([\rho, \mathcal{J}(v)] = 0\) for all \( v \in V \). It then follows from Equation (1.a) that \([\rho, \mathcal{J}(\pi)] = 0\) for every non-degenerate \( k \)-plane \( \pi \). This completes the proof of Theorem 1.1.

Let \( \mathfrak{M} \) be a Puffini–Videv 0-model. In the pseudo-Riemannian setting, the Ricci operator need not be diagonalizable. However, we can take the Jordan decomposition to decompose:
\[
V = \oplus_{\lambda} V_{\lambda} \quad \text{and} \quad \rho = \oplus_{\lambda} \rho_{\lambda}
\]
where one restricts to \( \lambda \) with non-negative imaginary parts and where \( \rho_{\lambda} \) has only the eigenvalue \( \lambda \) on \( V_{\lambda} \) if \( \lambda \) is real and the eigenvalues \( \{ \lambda, \bar{\lambda} \} \) on \( V_{\lambda} \) if \( \lambda \) is complex. As \([\mathcal{J}(x), \rho] = 0\),
\[
\mathcal{J}(x)V_{\lambda} \subset V_{\lambda} \quad \forall x, \lambda.
\]
Let \( x_i \in V_{\lambda_i} \) and let \( \xi \) be arbitrary. If \( \lambda_1 \neq \lambda_4 \), then \( \mathcal{J}(\xi)x_i \in V_{\lambda_1} \). Since \( V_{\lambda_1} \perp V_{\lambda_4} \), \( \langle \mathcal{J}(\xi)x_1, x_4 \rangle = 0 \). Setting \( \xi = x_2 + \varepsilon x_3 \) and letting \( \varepsilon \) vary then yields \( \langle \mathcal{J}(x_2, x_3)x_1, x_4 \rangle = 0 \). Consequently
\[
A(x_1, x_2, x_3, x_4) = -A(x_1, x_3, x_2, x_4) \quad \text{if} \quad \lambda_1 \neq \lambda_4.
\]
Suppose that \( \lambda_1 \neq \lambda_4 \) and that \( \lambda_2 \neq \lambda_4 \). We may then compute
\[
A(x_1, x_2, x_3, x_4) = -A(x_1, x_3, x_2, x_4) \quad \text{(Equation 2.d) as } \lambda_1 \neq \lambda_4
\]
\[
= A(x_3, x_2, x_1, x_4) + A(x_2, x_1, x_3, x_4)
\]
\[
= -A(x_2, x_3, x_1, x_4) + A(x_2, x_1, x_3, x_4) \quad \text{(the Bianchi identity)}
\]
\[
= A(x_2, x_1, x_3, x_4) + A(x_2, x_1, x_3, x_4) \quad \text{(curvature symmetries)}
\]
\[
= -2A(x_1, x_2, x_3, x_4) \quad \text{(Equation 2.d) as } \lambda_2 \neq \lambda_4
\]
\[
= -2A(x_1, x_2, x_3, x_4) \quad \text{(curvature symmetries)}.
\]
This shows
\[ (2.e) \quad A(x_1, x_2, x_3, x_4) = 0 \quad \text{if} \quad \lambda_1 \neq \lambda_4 \text{ and } \lambda_2 \neq \lambda_4. \]

Suppose that \( A(x_1, x_2, x_3, x_4) \neq 0 \) and that \( \lambda_2 \neq \lambda_4 \). Then we may use Equation (2.e) to see that \( \lambda_1 = \lambda_4 \) and \( \lambda_2 = \lambda_3 \). Since \( \lambda_2 \neq \lambda_4 \), we may apply Equation (2.e) to see
\[ A(x_1, x_2, x_3, x_4) = -A(x_2, x_1, x_3, x_4) = A(x_2, x_3, x_1, x_4). \]
This vanishes by Equation (2.e) since \( \lambda_2 \neq \lambda_4 \) and \( \lambda_3 \neq \lambda_4 \) which is a contradiction.
Consequently \( A(x_1, x_2, x_3, x_4) \neq 0 \) implies \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \). Thus we may decompose
\[ A = \oplus_A A_\lambda \quad \text{for} \quad A_\lambda \in \otimes^4 V^*_\lambda. \]
This completes the proof of Theorem 1.3. If \( \mathfrak{M} \) is Riemannian, then necessarily \( \lambda \) is real. Since \( \rho \) is self-adjoint with respect to a positive definite metric, \( \rho \) is diagonalizable. This implies \( \rho \) is a scalar multiple of the identity and hence \( \mathfrak{M} \) is Einstein. Theorem 1.2 now follows. \( \square \)

3. Irreducible Riemannian Puffini–Videv Non-Einstein Manifolds

In the geometric setting, matters are a bit more complicated. The work of Tsankov [11] can be used to construct examples of 3-dimensional manifolds which are irreducible, which are Puffini–Videv, and which are not Einstein. This construction has been generalized by Brozos-Vázquez et al. [11]. Let \( \mathcal{M} \) be as in Theorem 1.3. Let \( (x_1, x_2) \) be local coordinates on \( \mathcal{N} \) so the metrics take the form
\[ ds^2_\mathcal{N} = e^{2\alpha}(dt^2 + dx_1^2 + dx_2^2) \quad \text{and} \quad ds^2_\mathcal{M} = dt^2 + t^2 e^{2\alpha}(dx_1^2 + dx_2^2). \]
The curvature tensor of \( \mathcal{N} \) is then given by:
\[ R_\mathcal{N}(\partial_{x_1}, \partial_{x_2}, \partial_{x_2}, \partial_{x_1}) = e^{2\alpha} \left( \frac{\partial^2 \alpha}{\partial x_1^2} + \frac{\partial^2 \alpha}{\partial x_2^2} \right) \]
and work of [11] shows the only non-zero component of the curvature tensor is:
\[ R_\mathcal{M}(\partial_{x_1}, \partial_{x_2}, \partial_{x_2}, \partial_{x_1}) = -t^{-2} e^{2\alpha} \left( \frac{\partial^2 \alpha}{\partial x_1^2} + \frac{\partial^2 \alpha}{\partial x_2^2} + e^{2\alpha} \right) \]
Let \( V_1 := \text{Span}\{\partial_{x_1}, \partial_{x_2}\} \) and \( V_2 := \text{Span}\{\partial_1\} \). We then have an orthogonal direct sum decomposition \( TM = V_1 \oplus V_2 \). Furthermore, \( A = A_1 \oplus A_2 \) where \( A_2 \) is trivial and \( A_1 \) is Einstein. Thus \( \mathcal{M} \) is Puffini–Videv.

The scalar curvatures on \( \mathcal{M} \) and on \( \mathcal{N} \) are related by the identity:
\[ \tau_\mathcal{M} = t^{-2}(\tau_\mathcal{N} - 2). \]
The curves \( t \to (t, P_0) \) are unit speed geodesics and clearly \( \tau_\mathcal{M} \) blows up as \( t \to 0 \). Furthermore \( \mathcal{M} \) is not Einstein since \( \tau_\mathcal{N} - 2 \) does not vanish identically. The 1-model is indecomposable since Range\{\( \mathcal{R}_\mathcal{M} \) = \( \text{Span}\{\partial_1, \partial_2\} \) and since \( \tau_\mathcal{M} \) exhibits non-trivial dependence on \( t \). This completes the proof of Theorem 1.3. \( \square \)

The proof of Theorem 1.5 is similar. Again, the only non-vanishing curvature is \( R(\partial_{x_2}, \partial_{x_1}, \partial_{x_4}, \partial_{x_3}) \). If we set \( V_1 := \text{Span}\{\partial_{x_1}, \partial_{x_2}\} \) and \( V_2 := \text{Span}\{\partial_{x_3}, \partial_{x_4}\} \), then \( TM = V_1 \oplus V_2 \) is an orthogonal direct sum. We set \( A_1 = 0 \) and let \( A_2 = R|_{V_2} \). We then have \( R = A_1 \oplus A_0 \) so \( \mathcal{M} \) is Puffini–Videv. The scalar curvature of \( \mathcal{M} \) is:
\[ \tau_\mathcal{M} = x_1^{-1}(x_1 + \beta x_2)^{-1}. \]
From this it follows that \( \tau_\mathcal{M} \) exhibits blowup along the geodesic \( t \to (t, 0, 0, 0) \) as \( t \downarrow 0 \). Let \( \mathcal{E} := \text{Range}(R) = \text{Span}\{\partial_{x_1}, \partial_{x_2}\} \). Then \( \mathcal{E}^\perp = \text{Span}\{\partial_{x_1}, \partial_{x_2}\} \). Let \( \Psi := -\ln |\tau_\mathcal{M}| \). The Hessian of \( \Psi \) restricted to \( \mathcal{E}^\perp \) takes the form:
\[ H = \begin{pmatrix} -x_1^{-2} + (x_1 + \beta x_2)^{-2} & \beta (x_1 + \beta x_2)^{-2} \\ \beta (x_1 + \beta x_2)^{-2} & \beta^2 (x_1 + \beta x_2)^{-2} \end{pmatrix} \]
It now follows that
\[ \det(H|_{\mathcal{E}}) = \frac{1}{4} \beta \gamma^2. \]
This shows that \( \beta \) is an isometry invariant; in particular \( \mathcal{M}_\beta \) is not locally isometric to \( \mathcal{M}_\gamma \) if \( \beta \neq \gamma \). Since \( \det(H|_{\mathcal{E}}) \) has rank 2, it follows easily that the 2-model is indecomposable. We refer to [1] for further information concerning the geometry of these manifolds which were first discovered in a different context.

\[ \square \]

Acknowledgments

The research of P. Gilkey was partially supported by the Max Planck Institute for the Mathematical Sciences (Leipzig, Germany). The research of E. Puffini was supported by a grant from the K. I. T. It is a pleasure to acknowledge helpful conversations with C. Dunn and with Z. Zhelev concerning these matters.

References

[1] M. Brozos-Vázquez and P. Gilkey, The global geometry of Riemannian manifolds with commuting curvature operators, preprint.
[2] E. García–Ríoa, D. Kupeli, and R. Vázquez-Lorenzo, Osserman Manifolds in Semi-Riemannian Geometry, Lecture Notes in Math., 1777, Springer-Verlag, Berlin, 2002.
[3] P. Gilkey, Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor, World Scientific (2002).
[4] P. Gilkey, G. Stanilov, and V. Videv, Pseudo Riemannian Manifolds whose Generalized Jacobi Operator has Constant Characteristic Polynomial, J. Geom. 62 (1998) 144–153.
[5] R. Osserman, Curvature in the eighties, Amer. Math. Monthly 97 (1990), 731–756.
[6] R. Ivanova and G. Stanilov, A skew-symmetric curvature operator in Riemannian geometry, Symposia Gaussiana (1994), Conf. A: Mathematics, Eds. Behara, Fritsch and Lintz (1995), 391–395.
[7] G. Stanilov, Higher order skew-symmetric and symmetric curvature operators, C. R. Acad. Bulgare Sci. 57 (2004), 9–12.
[8] G. Stanilov and V. Videv, On a generalization of the Jacobi operator in the Riemannian geometry, Annaire Univ. Sofia Fac. Math. Inform. 86 (1992), 27–34.
[9] G. Stanilov and V. Videv, On the commuting of curvature operators, Mathematics and Education in Mathematics (Proc. of the 33rd Spring Conference of the Union of Bulgarian Mathematicians Borovtes, April 1–4, 2004), Sofia (2004), 176–179.
[10] Z. I. Szabó, A simple topological proof for the symmetry of 2 point homogeneous spaces, Invent. Math. 106 (1991), 61–64.
[11] Y. Tsankov, A characterization of n-dimensional hypersurface in \( \mathbb{R}^{n+1} \) with commuting curvature operators, Banach Center Publ. 69 (2005), 205–209.
[12] V. Videv, A characteristic of the real space forms by a linear operator, Plovdiv University [Paistí Hilendarski], Bulgaria, Scientific Works-Mathematics, vol 39, Book 3 (1993), 5–8.

PG: Mathematics Department, University of Oregon, Eugene, OR 97403, USA. Email: gilkey@uoregon.edu.

EP: K. I. T., Malvinas. Email: ekaterinapuffin@yahoo.com.

VV: Veselin Videv, Mathematics Department, Thracian University, University Campus, 6000 Stara Zagora Bulgaria. Email: videv@uni-sz.bg.
Let $J(\pi)$ be the higher order Jacobi operator. We study algebraic curvature tensors where $J(\pi)J(\pi^\perp) = J(\pi^\perp)J(\pi)$. In the Riemannian setting, we give a complete characterization of such tensors; in the pseudo-Riemannian setting, partial results are available. We present non-trivial geometric examples of Riemannian manifolds with this property.

1. Introduction

The study of commutativity and of spectral properties for natural operators in differential geometry has received much attention in recent years. Probably the seminal paper in the subject is due to Osserman [5] who proposed a characterization of Riemannian rank 1-symmetric spaces in terms of the spectrum of the Jacobi operator. There are, however, many other crucial works which should be cited—papers by Ivanova and Stanilov [6], by Stanilov [7], by Stanilov and Videv [9], by Szabó [10], and by Tsankov [8] are central. However, as the literature is a vast one, we must limit ourselves and shall refer to the bibliographies in [2, 3] for further information.

We shall work in both the geometric and in the algebraic contexts; the Jacobi operator will form the focus of our study. We begin by introducing some notational conventions. We say that $M := (V, \langle \cdot, \cdot \rangle, A)$ is a 0-model if $\langle \cdot, \cdot \rangle$ is a non-degenerate inner product of signature $(p, q)$ on a finite dimensional vector space $V$ of dimension $m = p + q$ and if $A \in \otimes^4 V^*$ is an algebraic curvature tensor, i.e. if $A$ is a 4-tensor which satisfies the symmetries of the Riemann curvature tensor:

$$A(v_1, v_2, v_3, v_4) = A(v_3, v_4, v_1, v_2) = -A(v_2, v_1, v_3, v_4),$$

$$A(v_1, v_2, v_3, v_4) + A(v_2, v_3, v_1, v_4) + A(v_3, v_1, v_2, v_4) = 0.$$ 

The associated curvature operator $A$ and Jacobi operator $J$ are then characterized, respectively, by the identities:

$$\langle A(v_1, v_2)v_3, v_4 \rangle = A(v_1, v_2, v_3, v_4) \quad \text{and} \quad \langle J(v_1)v_2, v_3 \rangle = A(v_2, v_1, v_1, v_3).$$

The Jacobi operator $J(v)$ is quadratic in $v$. It is convenient to polarize and to set

$$J(v_1, v_2) : v_3 \to \frac{1}{2} \{ A(v_1, v_2)v_3 + A(v_1, v_3)v_2 \}.$$ 

This operator was first introduced to study the geometry of the Jacobi operator by Videv [12]. If $\vec{v} := (v_1, ..., v_k)$ is a basis for a non-degenerate $k$-plane $\pi$, let $\xi_{ij} := \langle v_i, v_j \rangle$ give the components of the metric restricted to $\pi$ relative to the given basis. If $\xi^{ij}$ denotes the inverse matrix, then one defines the higher order Jacobi operator by setting:

$$(1.a) \quad J(\vec{v}) := \sum_{i=1}^{k} \sum_{j=1}^{k} \xi^{ij} J(v_i, v_j).$$

Key words and phrases. 0-model, algebraic curvature tensor, Einstein manifold, Higher order Jacobi operator, Jacobi operator, Ricci operator.

2000 Mathematics Subject Classification. 53C20

1 Corresponding author.
Let \( \pi(\bar{v}) := \text{Span}\{v_1, ..., v_k\} \). Then \( \mathcal{J}(\pi) := \mathcal{J}(\pi(\bar{v})) \) is independent of the particular basis chosen for \( \pi \); this operator was first introduced in this context by Stanilov and Videv [8] in the Riemannian setting and later extended by Gilkey, Stanilov, and Videv to the pseudo Riemannian setting [4]. Note that \( \rho := \mathcal{J}(V) \) is the Ricci operator:

\[
\rho : y \rightarrow \sum_{i=1}^{m} \sum_{j=1}^{m} \xi^{ij} A(y, e_i)e_j.
\]

Thus the higher order Jacobi operator can also be thought of as a generalization of the Ricci operator to lower dimensional subspaces.

Let \( \text{Gr}_{r,s}(V, \langle \cdot, \cdot \rangle) \) be the Grassmannian of all non-degenerate linear subspaces of \( V \) which have signature \((r, s)\); the pair \((r, s)\) is said to be admissible if and only if \( \text{Gr}_{r,s}(V, \langle \cdot, \cdot \rangle) \) is non-empty and does not consist of a single point or, equivalently, if the inequalities \( 0 \leq r \leq p, 0 \leq s \leq q \), and \( 1 \leq r + s \leq m - 1 \) are satisfied. Let \([A, B] := AB - BA \) denote the commutator of two linear maps. We shall establish the following result in Section 2:

**Theorem 1.1.** Let \( \mathcal{M} = (V, \langle \cdot, \cdot \rangle, A) \) be a 0-model. The following assertions are equivalent; if any is satisfied, then we shall say that \( \mathcal{M} \) is a Puffini–Videv 0-model.1

1. There exists \((r_0, s_0)\) admissible so that 
   \[ \mathcal{J}(\pi) \mathcal{J}(\pi^\perp) = \mathcal{J}(\pi^\perp) \mathcal{J}(\pi) \] 
   for all \( \pi \in \text{Gr}_{r_0,s_0}(V, \langle \cdot, \cdot \rangle) \).

2. \[ \mathcal{J}(\pi) \mathcal{J}(\pi^\perp) = \mathcal{J}(\pi^\perp) \mathcal{J}(\pi) \] for every non-degenerate subspace \( \pi \).

3. \[ [\mathcal{J}(\pi), \rho] = 0 \] for every non-degenerate subspace \( \pi \).

We say that \( \mathcal{M} = (V, \langle \cdot, \cdot \rangle, A) \) is decomposable if there exists a non-trivial orthogonal decomposition \( V = V_1 \oplus V_2 \) which decomposes \( A = A_1 \oplus A_2 \); in this setting, we shall write \( \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \) where \( \mathcal{M}_i := (V_i, \langle \cdot, \cdot \rangle|_{V_i}, A_i) \). One says that \( \mathcal{M} \) is indecomposable if \( \mathcal{M} \) is not decomposable. We say \( \mathcal{M} \) is Einstein if the Ricci operator \( \rho \) is a scalar multiple of the identity. By Theorem 1.1, any Einstein 0-model is Puffini–Videv. More generally, the direct sum of Einstein Puffini–Videv models is again Puffini–Videv; the converse holds in the Riemannian setting:

**Theorem 1.2.** Let \( \mathcal{M} = (V, \langle \cdot, \cdot \rangle, A) \) be a Riemannian 0-model. Then \( \mathcal{M} \) is Puffini–Videv if and only if \( \mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k \) where the \( \mathcal{M}_i \) are Einstein.

In the pseudo-Riemannian setting, a somewhat weaker result can be established. One says that a 0-model is pseudo-Einstein either if the Ricci operator \( \rho \) has only one real eigenvalue \( \lambda \) or if the Ricci operator \( \rho \) has two complex eigenvalues \( \lambda_1, \lambda_2 \) with \( \lambda_1 = \lambda_2 \). This does not imply that \( \rho \) is diagonalizable in the higher signature setting and hence \( \mathcal{M} \) need not be Einstein.

**Theorem 1.3.** Let \( \mathcal{M} = (V, \langle \cdot, \cdot \rangle, A) \) be a 0-model of arbitrary signature. If \( \mathcal{M} \) is Puffini–Videv, then we may decompose \( \mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k \) as the direct sum of pseudo-Einstein 0-models \( \mathcal{M}_i \).

We restrict to the Riemannian context henceforth. Theorem 1.2 yields the characterization that indecomposable Riemannian 0-model is Einstein if and only if it is Puffini–Videv. However, in passing to the geometric situation, things become a bit more complicated. Let \( \mathcal{M} = (M, g) \) be a Riemannian manifold of dimension \( m \geq 3 \); any Riemann surface is automatically Puffini–Videv by Theorem 1.1. Let \( \mathcal{M}_k(M, P) = (T_PM, \langle \cdot, \cdot \rangle, \nabla^R P, ..., \nabla^k R_P) \) be a purely algebraic object which encodes the geometric information about \( \mathcal{M} \) up to the \((k+2)\)th order jets of the metric at a point \( P \); \( \mathcal{M}(M, P) = \mathcal{M}_0(M, P) \). The notion of indecomposibility of a \( k \)-model is defined as above. One says that \( \mathcal{M} \) is Puffini–Videv if \( \mathcal{M}(M, P) \) is Puffini–Videv at each point of the manifold. One says \( \mathcal{M} \) is locally reducible at a point

---

1This notation was suggested by the first author.
We distinguish two cases. Suppose first that \( \varepsilon \)thonormal basis for \( V \)
\( (2.b) \). Immediate that Assertion (2) implies Assertion (1). We complete the proof of
This establishes the equivalence of Assertions (2) and (3) in Theorem 1.1. It is
\( \varepsilon \) is a geodesic in \( M \) isometrically in a geodesically complete manifold.
Such a manifold is necessarily geodesically incomplete and can not be embedded
isometrically in a geodesically complete manifold.

One has the following examples as we shall discuss further in Section 3:

**Theorem 1.4.** Let \( N := (N, ds_N^2) \) be a Riemann surface. Assume \( \tau_N(P_0) \neq 2 \) for
some \( P_0 \in N \). Let \( (t, x_1, x_2) \) be local coordinates on \( M := (0, \infty) \times N \). Give \( M \) the
metric \( g_M \) with non-zero components:
\[
g_M(\partial_t, \partial_t) = 1 \quad \text{and} \quad g_M(\partial_{x_1}, \partial_{x_2}) = t^2 g_N(\partial_{x_1}, \partial_{x_2}).
\]
(1) \( M := (M, g_M) \) is a 3-dimensional Riemannian Puffini–Videv manifold.
(2) \( M_1(M, P) \) is indecomposible and \( M \) is locally irreducible at \( P = (t, P_0) \).
(3) \( M \) exhibits scalar curvature blowup and is not Einstein.

**Theorem 1.5.** Let \( (x_1, x_2, x_3, x_4) \) be coordinates on \( M := (0, \infty) \times (0, \infty) \times \mathbb{R}^2 \).
Let \( \beta > 0 \). Give \( M \) the metric \( g_\beta \) whose non-zero components are:
\[
g_\beta(\partial_{x_1}, \partial_{x_1}) = g_\beta(\partial_{x_2}, \partial_{x_2}) = 1, \quad g_\beta(\partial_{x_3}, \partial_{x_3}) = x_1^2,
\]
\[
g_\beta(\partial_{x_4}, \partial_{x_4}) = x_1(x_1 + \beta x_2).
\]
(1) \( M_\beta := (M, g_\beta) \) is a 4-dimensional Riemannian Puffini–Videv manifold.
(2) \( M_2(M_\beta, P) \) is indecomposible and \( M \) is locally irreducible for all \( P \in M \).
(3) \( M_\beta \) is not locally isometric to \( M_\gamma \) for \( \beta \neq \gamma \).
(4) \( M_\beta \) exhibits scalar curvature blowup and is not Einstein.

2. PUFFINI–VIDEV 0-MODELS

We begin our study in the algebraic context by establishing Theorem 1.1. Let \( \rho \) be the Ricci operator. Since \( \rho = \mathcal{J}(\pi) + \mathcal{J}(\pi^\perp) \), we have:
\[
[\rho, \mathcal{J}(\pi)] = \{\mathcal{J}(\pi) + \mathcal{J}(\pi^\perp)\} \mathcal{J}(\pi^\perp) - \mathcal{J}(\pi^\perp) \{\mathcal{J}(\pi) + \mathcal{J}(\pi^\perp)\}
\]
\[
= \mathcal{J}(\pi) \mathcal{J}(\pi^\perp) + \mathcal{J}(\pi^\perp) \mathcal{J}(\pi^\perp) - \mathcal{J}(\pi^\perp) \mathcal{J}(\pi) - \mathcal{J}(\pi^\perp) \mathcal{J}(\pi^\perp)
\]
\[
= [\mathcal{J}(\pi^\perp), \mathcal{J}(\pi)].
\]
This establishes the equivalence of Assertions (2) and (3) in Theorem 1.1. It is
immediate that Assertion (2) implies Assertion (1). We complete the proof of
Theorem 1.1 by showing Assertion (1) implies Assertion (2). Assume there exists
\( r_0, s_0 \) admissible so that
\[
(2.b) \quad \mathcal{J}(\pi) \mathcal{J}(\pi^\perp) = \mathcal{J}(\pi^\perp) \mathcal{J}(\pi) \forall \pi \in Gr_{r_0,s_0}(V, \langle \cdot, \cdot \rangle).
\]
Let \( 1 \leq \kappa := r_0 + s_0 < \mu := \dim(V) \). Let \( \{e_1, ..., e_\kappa, e_{\kappa+1}, ..., e_\mu\} \) be an orthonormal basis for \( V \) where \( \{e_1, ..., e_\kappa\} \) spans a non-degenerate plane \( \pi \) of signature
\( r_0, s_0 \). Let \( \varepsilon_i := (e_i, e_i) \). Then
\[
\mathcal{J}(\pi) := \sum_{i=1}^{\kappa} \varepsilon_i \mathcal{J}(e_i).
\]
We distinguish two cases. Suppose first that \( \varepsilon_1 = \varepsilon_{\kappa+1} \). Set
\[
e_1(\theta) := \cos(\theta)e_1 + \sin(\theta)e_{\kappa+1}.
\]
Then \( \{e_1(\theta), e_2, ..., e_n\} \) is an orthonormal basis for a non-degenerate plane \( \pi(\theta) \) of signature \((r_0, s_0)\). One has
\[
\begin{align*}
0 &= [\rho, \mathcal{J}(\pi(\theta)) - \mathcal{J}(\pi)] = 0 \\
&= [\rho, (\cos^2 \theta - 1)\mathcal{J}(e_1) + 2 \sin \theta \cos \theta \mathcal{J}(e_1, e_{\kappa+1}) + \sin^2 \theta \mathcal{J}(e_{\kappa+1})].
\end{align*}
\]
This identity for all \( \theta \) implies
\[
[\rho, \mathcal{J}(e_1) - \mathcal{J}(e_{\kappa+1})] = 0 \quad \text{if} \quad \varepsilon_1 = \varepsilon_{\kappa+1}.
\]
Suppose next that \( \varepsilon_1 = -\varepsilon_{\kappa+1} \). Set \( e_1(\theta) := \cosh(\theta)e_1 + \sinh(\theta)e_{\kappa+1} \). A similar computation, after paying attention to the signs involved, yields:
\[
0 = [\rho, (\cosh^2 \theta - 1)\mathcal{J}(e_1) - 2 \sin \theta \cosh \theta \mathcal{J}(e_1, e_{\kappa+1}) + \sinh^2 \mathcal{J}(e_{\kappa+1})]
\]
which yields the identity
\[
0 = [\rho, \mathcal{J}(e_1) + \mathcal{J}(e_{\kappa+1})].
\]
We combine these two calculations to see that for all \( 1 \leq i, j \leq m \) we have that
\[(2.c) \quad \varepsilon_i[\rho, \mathcal{J}(e_i)] = \varepsilon_j[\rho, \mathcal{J}(e_j)].\]

We use Equation (2.c) to see that
\[
0 = [\rho, \mathcal{J}(\pi)] = \sum_{i=1}^k \varepsilon_i[\rho, \mathcal{J}(e_i)] = k \varepsilon_1[\rho, \mathcal{J}(e_1)]
\]
and thus \([\rho, \mathcal{J}(e_1)] = 0\). This shows that \([\rho, \mathcal{J}(v)] = 0\) for every unit spacelike vector if \( s_0 > 0 \) and for every unit timelike vector if \( r_0 > 0 \). We can rescale to conclude \([\rho, \mathcal{J}(v)] = 0\) on a non-empty open subset of \( V \) and hence, as this is a polynomial identity, conclude \([\rho, \mathcal{J}(v)] = 0\) for all \( v \in V \). It then follows from Equation (1.a) that \([\rho, \mathcal{J}(\pi)] = 0\) for every non-degenerate \( k \)-plane \( \pi \). This completes the proof of Theorem 1.1.

Let \( \mathfrak{M} \) be a Puffini–Videv 0-model. In the pseudo-Riemannian setting, the Ricci operator need not be diagonalizable. However, we can take the Jordan decomposition to decompose:
\[
V = \oplus_{\lambda} V_{\lambda} \quad \text{and} \quad \rho = \oplus_{\lambda} \rho_{\lambda}
\]
where one restricts to \( \lambda \) with non-negative imaginary parts and where \( \rho_{\lambda} \) has only the eigenvalue \( \lambda \) on \( V_{\lambda} \) if \( \lambda \) is real and the eigenvalues \( \{\lambda, \bar{\lambda}\} \) on \( V_{\lambda} \) if \( \lambda \) is complex. As \([\mathcal{J}(x), \rho] = 0\),
\[
\mathcal{J}(x)V_{\lambda} \subset V_{\lambda} \quad \forall \ x, \lambda.
\]
Let \( x_i \in V_{\lambda_1} \) and let \( \xi \) be arbitrary. If \( \lambda_1 \neq \lambda_4 \), then \( \mathcal{J}(\xi)x_i \in V_{\lambda_1} \). Since \( V_{\lambda_1} \perp V_{\lambda_4} \), \( \langle \mathcal{J}(\xi)x_1, x_4 \rangle = 0 \). Setting \( \xi = x_2 + \varepsilon x_3 \) and letting \( \varepsilon \) vary then yields \( \langle \mathcal{J}(x_2, x_3)x_1, x_4 \rangle = 0 \). Consequently
\[(2.d) \quad A(x_1, x_2, x_3, x_4) = -A(x_1, x_3, x_2, x_4) \quad \text{if} \quad \lambda_1 \neq \lambda_4.
\]
Suppose that \( \lambda_1 \neq \lambda_4 \) and that \( \lambda_2 \neq \lambda_4 \). We may then compute
\[
\begin{align*}
A(x_1, x_2, x_3, x_4) &= -A(x_1, x_3, x_2, x_4) \quad \text{(Equation (2.d) as} \ \lambda_1 \neq \lambda_4) \\
&= A(x_3, x_2, x_1, x_4) + A(x_2, x_1, x_3, x_4) \quad \text{(the Bianchi identity)} \\
&= -A(x_2, x_3, x_1, x_4) + A(x_2, x_1, x_3, x_4) \quad \text{(curvature symmetries)} \\
&= A(x_2, x_1, x_3, x_4) + A(x_2, x_1, x_3, x_4) \quad \text{(Equation (2.d) as} \ \lambda_2 \neq \lambda_4) \\
&= -2A(x_1, x_2, x_3, x_4) \quad \text{(curvature symmetries)}.
\end{align*}
\]
This shows
\[ A(x_1, x_2, x_3, x_4) = 0 \] if \( \lambda_1 \neq \lambda_4 \) and \( \lambda_2 \neq \lambda_4 \).

Suppose that \( A(x_1, x_2, x_3, x_4) \neq 0 \) and that \( \lambda_2 \neq \lambda_4 \). Then we may use Equation (2.e) to see that \( \lambda_1 = \lambda_4 = \lambda_2 = \lambda_3 \). Since \( \lambda_2 \neq \lambda_4 \), we may apply Equation (2.d) to see
\[ A(x_1, x_2, x_3, x_4) = -A(x_2, x_1, x_3, x_4) = A(x_2, x_3, x_1, x_4). \]
This vanishes by Equation (2.e) since \( \lambda_2 \neq \lambda_4 \) and \( \lambda_3 \neq \lambda_4 \) which is a contradiction.

Consequently \( A(x_1, x_2, x_3, x_4) \neq 0 \) implies \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \). Thus we may decompose
\[ A = \oplus_{\lambda} A_{\lambda} \quad \text{for} \quad A_{\lambda} \in \otimes^4 V_{\lambda}^*. \]
This completes the proof of Theorem 1.3. If \( \mathcal{M} \) is Riemannian, then necessarily \( \lambda \) is real. Since \( \rho \) is self-adjoint with respect to a positive definite metric, \( \rho \) is diagonalizable. This implies \( \rho \) is a scalar multiple of the identity and hence \( \mathcal{M} \) is Einstein. Theorem 1.2 now follows. \( \square \)

3. Irreducible Riemannian Puffini–Videv Non-Einstein Manifolds

In the geometric setting, matters are a bit more complicated. The work of Tsankov [11] can be used to construct examples of 3-dimensional manifolds which are irreducible, which are Puffini–Videv, and which are not Einstein. This construction has been generalized by Brozos-Vázquez et al. [1]. Let \( \mathcal{M} \) be as in Theorem 1.4. Let \((x_1, x_2)\) be local coordinates on \( \mathcal{N} \) so the metrics take the form
\[ ds_{\mathcal{N}}^2 = e^{2\alpha}(dx_1^2 + dx_2^2) \quad \text{and} \quad ds_{\mathcal{M}}^2 = dt^2 + t^2 e^{2\alpha}(dx_1^2 + dx_2^2). \]
The curvature tensor of \( \mathcal{N} \) is then given by:
\[ R_{\mathcal{N}}(\partial_{x_1}, \partial_{x_2}, \partial_{x_2}, \partial_{x_1}) = e^{2\alpha} \left( \frac{\partial^2 \alpha}{\partial x_1^2} + \frac{\partial^2 \alpha}{\partial x_2^2} \right) \]
and work of [1] shows the only non-zero component of the curvature tensor is:
\[ R_{\mathcal{M}}(\partial_{x_1}, \partial_{x_2}, \partial_{x_2}, \partial_{x_1}) = -t^{-2}e^{2\alpha} \left( \frac{\partial^2 \alpha}{\partial x_1^2} + \frac{\partial^2 \alpha}{\partial x_2^2} + e^{2\alpha} \right) \]
Let \( V_1 := \text{Span}\{\partial_{x_1}, \partial_{x_2}\} \) and \( V_2 := \text{Span}\{\partial_t\} \). We then have an orthogonal direct sum decomposition \( TM = V_1 \oplus V_2 \). Furthermore, \( A = A_1 \oplus A_2 \) where \( A_2 \) is trivial and \( A_1 \) is Einstein. Thus \( \mathcal{M} \) is Puffini–Videv.

The scalar curvatures on \( \mathcal{M} \) and on \( \mathcal{N} \) are related by the identity:
\[ \tau_{\mathcal{M}} = t^{-2}(\tau_{\mathcal{N}} - 2). \]
The curves \( t \to (t, P_0) \) are unit speed geodesics and clearly \( \tau_{\mathcal{M}} \) blows up as \( t \to 0 \). Furthermore \( \mathcal{M} \) is not Einstein since \( \tau_{\mathcal{N}} - 2 \) does not vanish identically. The 1-model is indecomposable since Range\( \{R_{\mathcal{M}}\} = \text{Span}\{\partial_t, \partial_2\} \) and since \( \tau_{\mathcal{M}} \) exhibits non-trivial dependence on \( t \). This completes the proof of Theorem 1.4. \( \square \)

The proof of Theorem 1.5 is similar. Again, the only non-vanishing curvature is \( R(\partial_{x_1}, \partial_{x_4}, \partial_{x_1}, \partial_{x_3}) \). If we set \( V_1 := \text{Span}\{\partial_{x_1}, \partial_{x_2}\} \) and \( V_2 := \text{Span}\{\partial_{x_3}, \partial_{x_4}\} \), then \( TM = V_1 \oplus V_2 \) is an orthogonal direct sum. We set \( A_1 = 0 \) and let \( A_2 = R|_{V_2} \). We then have \( R = A_1 \oplus A_0 \) so \( \mathcal{M} \) is Puffini–Videv. The scalar curvature of \( \mathcal{M} \) is:
\[ \tau_{\mathcal{M}} = x_1^{-1}(x_1 + \beta x_2)^{-1}. \]
From this it follows that \( \tau_{\mathcal{M}} \) exhibits blowup along the geodesic \( t \to (t, 0, 0, 0) \) as \( t \downarrow 0 \). Let \( \mathcal{E} := \text{Range}(R) = \text{Span}\{\partial_{x_1}, \partial_{x_2}\} \). Then \( \mathcal{E}^\perp = \text{Span}\{\partial_{x_3}, \partial_{x_4}\} \). Let \( \Psi := -\ln|\tau_{\mathcal{M}}| \). The Hessian of \( \Psi \) restricted to \( \mathcal{E}^\perp \) takes the form:
\[ H = \begin{pmatrix} -x_1^{-2} + (x_1 + \beta x_2)^{-2} & \beta(x_1 + \beta x_2)^{-2} \\ \beta(x_1 + \beta x_2)^{-2} & \beta^2(x_1 + \beta x_2)^{-2} \end{pmatrix} \]
It now follows that
\[ \det(H|_E) = \frac{1}{4}\beta r_\beta^2. \]
This shows that \( \beta \) is an isometry invariant; in particular \( M_\beta \) is not locally isometric to \( M_\gamma \) if \( \beta \neq \gamma \). Since \( \det(H|_E) \) has rank 2, it follows easily that the 2-model is indecomposable. We refer to [1] for further information concerning the geometry of these manifolds which were first discovered in a different context.

\begin{acknowledgments}

The research of P. Gilkey was partially supported by the Max Planck Institute for the Mathematical Sciences (Leipzig, Germany). The research of E. Puffini was supported by a grant from the K. I. T. It is a pleasure to acknowledge helpful conversations with C. Dunn and with Z. Zhelev concerning these matters.

\end{acknowledgments}

\begin{thebibliography}
[1] M. Brozos-Vázquez and P. Gilkey, The global geometry of Riemannian manifolds with commuting curvature operators, preprint.
[2] E. García–Río, D. Kupeli, and R. Vázquez-Lorenzo, Osserman Manifolds in Semi-Riemannian Geometry, Lecture Notes in Math., 1777, Springer-Verlag, Berlin, 2002.
[3] P. Gilkey, Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor, World Scientific (2002).
[4] P. Gilkey, G. Stanilov, and V. Videv, Pseudo Riemannian Manifolds whose Generalized Jacobi Operator has Constant Characteristic Polynomial, J. Geom. 62 (1998) 144–153.
[5] R. Osserman, Curvature in the eighties, Amer. Math. Monthly 97 (1990), 731–756.
[6] R. Ivanova and G. Stanilov, A skew-symmetric curvature operator in Riemannian geometry, Symposia Gaussiana (1994), Conf. A: Mathematics, Eds. Behara, Fritsch and Lintz (1995), 391–395.
[7] G. Stanilov, Higher order skew-symmetric and symmetric curvature operators, C. R. Acad. Bulgare Sci. 57 (2004),9–12.
[8] G. Stanilov and V. Videv, On a generalization of the Jacobi operator in the Riemannian geometry, Annuaire Univ. Sofia Fac. Math. Inform. 86 (1992), 27–34.
[9] G. Stanilov and V. Videv, On the commuting of curvature operators, Mathematics and Education in Mathematics (Proc. of the 33rd Spring Conference of the Union of Bulgarian Mathematicians Borovets, April 1-4, 2004), Sofia (2004), 176–179.
[10] Z. I. Szabó, A simple topological proof for the symmetry of 2 point homogeneous spaces, Invent. Math. 106 (1991), 61–64.
[11] Y. Tsankov, A characterization of n-dimensional hypersurface in \( \mathbb{R}^{n+1} \) with commuting curvature operators, Banach Center Publ. 69 (2005), 205–209.
[12] V. Videv, A characteristic of the real space forms by a linear operator, Plovdiv University [Paisii Hilendarski], Bulgaria, Scientific Works-Mathematics, vol 30, Book 3 (1993), 5–8.

PG: Mathematics Department, University of Oregon, Eugene, OR 97403, USA. Email: gilkey@uoregon.edu.

EP: K. I. T., Malevina. Email: ekaterinopuffin@yahoo.com.

VV: Veselin Videv, Mathematics Department, Thracian University, University Campus, 6000 Stara Zagora Bulgaria. Email: videv@uni-sz.bg.