Well-posed first-order reduction of the characteristic problem of the linearized Einstein equations

Simonetta Frittelli

Department of Physics, Duquesne University, Pittsburgh, PA 15282 and
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260
(Dated: March 24, 2022)

A choice of first-order variables for the characteristic problem of the linearized Einstein equations is found which casts the system into manifestly well-posed form. The concept of well-posedness for characteristic problems invoked is that there exists an a priori estimate of the solution of the characteristic problem in terms of the data. The notion of manifest well-posedness consists of an algebraic criterion sufficient for the existence of the estimates, and is to characteristic problems as symmetric hyperbolicity is to Cauchy problems. Both notions have been made precise elsewhere.

PACS numbers: 04.20.Ex, 04.25.Dm

I. INTRODUCTION

Well-posedness is a concept naturally associated with initial-value problems, whereby the norm of the solution at some time in the evolution is bounded by the norm of the initial data, independently of the data. The norm of the solution is defined by integrating on a time-slice a positive-definite quadratic expression in the fields. The norm of the data is the same positive-definite quadratic expression integrated on the initial slice. A well-posed initial-value problem turns out solutions that are stable with respect to small variations of the initial data, and is therefore naturally interpreted as a necessary condition for the existence of a stable numerical scheme (of the finite-difference kind) for the problem.

The well-posedness of the initial-value problem of the Einstein equations has received much attention since the pioneering work by Choquet-Bruhat [2], and has turned out a rich spectrum of equivalent formulations of the Einstein equations with differing properties, such as number of variables, number and nature of the associated characteristic speeds and allowed gauge choices [3]. A strong motivation for such attention is the expectation that well-posed formulations might play a main role in extending the run time of numerical simulations.

The understanding of the characteristic problem of the Einstein equations seems to have been evolving in the opposite direction. With no obvious issues of gauge and with a reduced number of fields, numerical simulations of spacetimes via the characteristic formulation have been performing at long running times with comparatively high efficiency [4, 5], and the attention seems to have centered around improved accuracy [6]. But, because little if anything is known about the stability of solutions of characteristic problems of partial differential equations, interest in the concept of a well-posed characteristic formulation of the Einstein equations has been rather limited.

The main obstacle to the concept of well-posedness of a characteristic problem seems to be the peculiar split of the data set into a “normal” subset which is usually interpreted as initial data, and a “null” subset which has usually been thought to have no analog in the initial-value framework. The “null” subset has been interpreted sometimes as a flow of information through the boundary of the region where the solution is being sought. So if one were to ask if there exists an a priori estimate for the solution of a characteristic problem, how should the answer even be framed? What is the information available a priori of the solution? The question is confusing because the null subset of data must be prescribed on a surface that is transverse to the characteristic initial slice and that contains the direction of evolution. One must then know such data variables at all evolution times, therefore it seems hardly appropriate to call it a priori information. But if the a priori information is confined to only the normal data, then it is impossible to obtain a truly a priori estimate because the solution will be affected by the “flow of information”. One must then include the null data somehow in the estimate, and one way to do so for the characteristic problem of the wave equation has been proposed by Balean and Bartnik [7, 8, 9].

But in essence, a priori information is any data that are known before the solution is known. There is really no issue of “evolution time”. If we allow ourselves to regard the null subset as a set of a priori information as good as the normal subset, then we have the same number of data as the initial value problem of the same equations, and the issue is simply to find a way to compare the size of the solution within the region of interest with the size of the data on both data surfaces. More precisely, if the $n$ dependent variables of the solution of the problem are represented as a vector $v$, the $m(<n)$ null data are represented by $w^0$ and the $n - m$ normal data are represented by $q^0$, then the estimate would read

\[ ||v||^2 \leq K(||w^0||^2 + ||q^0||^2) \]

where $K$ would be a number that would not depend on $(w^0, q^0)$. The norms would have to be defined appropri-
ately in each space. Clearly an inequality of this kind would ensure that vanishing values of \((u^0, q^0)\) on their respective data surfaces would return a vanishing solution \(v\), and the size of small perturbations of the data \((w^0, q^0)\) on their respective data surfaces would control the variation of the solution. The term \(||w^0||^2\) would be the integral of a quadratic expression in the null data on the corresponding data surface. Whether we refer to it as data or as a flow of energy is quite immaterial, but the parallel with initial value problems is maintained if we choose to call it \(data\).

With this viewpoint, a notion of well-posedness for generic linear homogeneous characteristic problems can be formulated in reference to the problem’s almost-canonical form \[10\]. Even though the choice of coordinates is not relevant to the essence of a characteristic problem, let’s assume that the equations are written in terms of coordinates adapted to the characteristic surfaces, so we have \((u, x, x^i)\) with \(i = 1 \ldots r - 2\) and with \(u\) labeling the characteristic surfaces. Then one can always find a choice of dependent variables \(v = (w, q)\) with \(w = w_1, \ldots, w_m\) and \(q = q_1, \ldots, q_n-m\) so that the system of equations takes the form

\[ 
\begin{align*}
N^u q_{iu} + N^x q_{ix} + N^v v_{i} + N^0 v &= 0 \quad (1a) \\
w_{ix} + L^v v_{i} + L^0 v &= 0 \quad (1b)
\end{align*}
\]

where the \(n-m\) square matrix \(N^u\) is non-degenerate. We refer to this as the almost-canonical form of the characteristic problem (the canonical form has the identity matrix in the place of \(N^u\) and is achieved by the transformation \(q \rightarrow N^u q\), which always exists because \(N^u\) is non-degenerate). This system of equations requires the values of \(q\) to be prescribed at \(u = 0\) and the values of \(w\) to be prescribed at \(x = 0\) in order for a unique solution to exist in the region to the future of \(u = 0\) and to the right of \(x = 0\), that is: a total of \(n\) functions of \(r-1\) variables \[11\]. We will say that the problem is \textit{well posed} if there exists a number \(\kappa\) independent of \((q^0, w^0)\) and a positive-definite \(n\)-dimensional symmetric matrix \(H\) such that

\[
||v||^2_T \leq e^{\kappa T}(||w^0||^2 + ||q^0||^2) \tag{2}
\]

where

\[
\begin{align*}
||v||^2_T &\equiv \int_{u+x=T} (v, Hv) \\
||q^0||^2 &\equiv \int_{u=0} (q^0, q^0) \\
||w^0||^2 &\equiv \int_{x=0} (w^0, w^0)
\end{align*}
\]

In each case, the notation \((, )\) denotes the standard Euclidean scalar product of the appropriate dimension; e.g., \((w, w) \equiv \sum_{i=1}^m w_i^2\). The notion of well-posedness thus consists of the norm of the solution with respect to a spacelike surface \(u+x = T\) being bounded by the norm of the data, irrespectively of the data. The surface \(u+x = T\) intersects the data surfaces \(u = 0\) and \(x = 0\), and the fixed value of \(T\) is arbitrary, but finite, and may be not expected to be large if the problem has non-constant coefficients and non-vanishing undifferentiated terms.

An equivalent representation of an almost-canonical characteristic problem is the form

\[
C^a v_{am} + Dv = 0, \tag{4}
\]

where \(a = u, x, i\), the principal matrix \(C^a\) has a block diagonal form \(\text{diag}(N^u, 0)\), the principal matrix \(C^x\) has a block diagonal form \(\text{diag}(N^x, 1)\), and the remaining principal matrices \(C^i\) are completely arbitrary. We say that the characteristic problem is \textit{manifestly well posed} if the following two conditions are satisfied:

i) All the principal matrices \(C^a\) are symmetric

ii) The normal block of the principal-\(u\) matrix, \(N^u\), is positive-definite and the normal block of the principal-\(x\) matrix, \(N^x\), is non-positive definite but such that \(N^u + N^x\) is positive definite (that is, \(-N^u < N^x \leq 0\)).

If these conditions are satisfied, then one can take \(H = C^u + C^x\) and there exists \(\kappa\) such that \[2\] holds. The proof of this statement is developed elsewhere \[1\], and parallels an argument first used to determine \textit{a priori} estimates for the characteristic problem of the wave equation \[12\]. The notion of manifest well-posedness of characteristic problems as defined here is a close analog of the notion of symmetric hyperbolicity for initial-value problems \[1\]. The conditions (i) and (ii) together constitute a sufficient algebraic criterion for well-posedness of characteristic problems. Clearly, the criterion is not a necessary condition (just as symmetric hyperbolicity is not necessary for well-posedness).

In Section \[11\] we introduce the characteristic problem of the Einstein equations in second-order form – the form in which the equations are most commonly used for numerical implementation– and linearize them around Schwarzschild spacetime. Subsequently, we find a set of first-order variables that cast the characteristic problem into manifestly well-posed form in the sense introduced in the previous paragraph. We conclude that the linearized characteristic problem of the Einstein equations admits a well-posed formulation. Section \[111\] offers concluding remarks regarding the reach of the result and some remaining open questions.

II. THE CHARACTERISTIC PROBLEM IN GENERAL RELATIVITY

A. Original problem

The line element \(ds^2 = g_{uu} du^2 + g_{ur} dudr + g_{uu} dudx^a + g_{ab} dx^a dx^b\) in coordinates adapted to a slicing by null hypersurfaces labeled by a coordinate \(u\) is

\[
\begin{align*}
ds^2 = -\left(\frac{e^{2\beta} V}{r} - r^2 h_{ab} U^a U^b \right) du^2 - 2e^{2\beta} dudr \\
-2r^2 h_{ab} U^b dudx^a + r^2 h_{ab} dx^a dx^b,
\end{align*}
\]

(5)
where \( x^a = (\theta, \phi) \) label null geodesics in the constant-
\( u \) slice, and \( \det(h_{ab}) = \sin^2 \theta \), so that \( r \) is the lumin-
nosity distance and functions as a parameter along the
geodesics. We define \( h^{ab} \) as the 2-inverse of \( h_{ab} \), via
\( h^{ab}h_{bc} = \delta^a_c \). Enough coordinate conditions have thus
been imposed to leave only six metric functions to deter-
mine by means of the Einstein equations.

Sachs’ approach \[13\] to the Einstein equations adapted to a null slicing considers the components of the Einstein equations \( G_{\mu \nu} = 0 \) in a frame adapted to the null slicing.
Such a frame has the direction \( \ell^\mu \) normal to the slicing as its main leg. With the current signature convention,
(\( -, +, +, + \)), the second leg of the frame is another null
vector \( n^\alpha \) such that \( \ell^\alpha n_\alpha = -1 \). The frame is completed by a complex null vector \( m^\alpha \) normalized by \( m^\alpha \bar{m}_\alpha = 1 \)
that is perpendicular to both \( \ell^\alpha \) and \( n^\alpha \). The complex
vector \( m^\alpha \) is equivalent to two real orthonormal space-
like vectors \( e_1^\alpha, e_2^\alpha \) tangent to the surface of fixed value of
\( u \), but the complex combination \( m^\alpha = (e_1^\alpha + ie_2^\alpha)/\sqrt{2} \) is
usually preferred because it facilitates some calculations
while leading to more compact expressions. Up to scaling,
the normal direction is \( \ell^\alpha = g^{\alpha \beta} u_\beta = g^{\alpha u} \), and is
tangent to the null geodesics of fixed value of \( x^a \) on each
slice, being thus proportional to \( \partial / \partial \theta \).

In the current notation, Sachs’ approach consists in
splitting the Einstein equations into three sets. The
first set contains six main equations, which themselves
split into the four hypersurface equations, \( G_{\mu \nu} \ell^\mu \ell^\nu =
G_{\mu \nu} \ell^\mu n^\nu = G_{\mu \nu} m^\mu \bar{m}_\nu = 0 \), and the two main equations,
\( G_{\mu \nu} m^\mu \bar{m}_\nu = 0 \). The second set has only one equation,
the trivial equation, \( G_{\mu \nu} \ell^\mu n^\nu = 0 \). The third set con-
tains the three supplementary conditions, \( G_{\mu \nu} m^\mu m^\nu = \)
\( G_{\mu \nu} m^\mu n^\nu = 0 \). The six main equations are the equations
that determine the six unknown metric functions
\( h_{ab}, \beta, V, U^a \). Sachs argues that by virtue of the Bianchi
identities \( \nabla \mu G_{\nu \mu} = 0 \), the remaining equations can be
thought of as constraints on the data for the unknown
metric functions and can be ignored for our purposes.

The main equations have been used for numerical sim-
ulations for some time, starting with cases with high sym-
metry \[14 \[15 \[16 \[17 \[18 \[19\] and moving to full general-
ity by 1996 \[20\] (see \[21\] for citations and an up-to-date
review of the numerical implementation of Sachs char-
acteristic approach to the Einstein equations). To my
knowledge, the hypersurface equations appeared explicit-
lly for the first time in \[22\], taking the following form:

\[
0 = \beta,_{\tau} - \frac{r^4}{16} h^{ac} h_{bd} h_{ab,r} h_{cd,r} \tag{6a}
\]

\[
0 = (r^4 e^{-2\beta} h_{ac} U^c,_{\tau} - 2r^4 \left( \frac{1}{r^2} \beta,_{\tau} \right) + r^2 h_{cd} D_c h_{ad,r} \tag{6b}
\]

\[
0 = 2V,_{\tau} - e^{2\beta} \mathcal{R} + 2e^{2\beta} (D^c D_c \beta + D^\alpha \beta D_\alpha \beta) + \frac{1}{r^2} D_c (r^4 U^c,_{\tau} + r e^{-2\beta} - \frac{2}{2} h_{cd U^c,_{\tau} U^b,_{\tau}}. \tag{6c}
\]

Here \( D \) is the covariant derivative associated with \( h_{ab} \)
(i.e., such that \( D_c h_{ab} = 0 \)), and \( \mathcal{R} \) is the curvature scalar
of the 2-surfaces given by the intersection of the constant-
\( u \) and constant-\( r \) hypersurfaces. The evolution equations as they appeared in \[22\] are

\[
0 = m^a m^b \left(r (r h_{ab,u}),_{\tau} - \frac{1}{2} \left( r V h_{ab,r} \right),_{\tau} \right)
- 2e^{2\beta} (D_a D_b \beta + D_a \beta D_b \beta) + h_{c(a} D_b) (r^2 U^c,_{\tau} + r e^{-2\beta} h_{ac} h_{bd U^c,_{\tau} U^d,_{\tau}} + r^2 U^c D_c h_{ab,r} - \frac{2}{2} D_e (U^c,_{\tau} h_{ac}),_{\tau} \right) \tag{7}
\]

Since we are interested in a real (as opposed to complex)
version of the evolution equations explicitly in terms of
the metric variables, rather than implicitly through the
vector \( m^a \), we wish to “un-contract” the vectors \( m^a m^b \).
Given a generic tensor quantity \( H_{ab} \), the contraction
\( m^a m^b H_{ab} = 0 \) kills both its skew-symmetric part
and its trace, so the “uncontracted” equivalent of \[7\] is sym-
metric in \( a, b \) and traceless:

\[
0 = r (r h_{ab,u}),_{\tau} - \frac{1}{2} (r V h_{ab,r}),_{\tau} \right.
- 2e^{2\beta} (D_a D_b \beta + D_a \beta D_b \beta) + h_{c(a} D_b) \left( 2 \right) + r^2 U^c D_c h_{ab,r} - \frac{2}{2} D_e (U^c,_{\tau} h_{ac}),_{\tau} \right) \tag{8}
\]

In the following, we linearize the equations around a
Schwarzschild background. In our current notation, Schwarzschild spacetime is given by \( U^a = \beta = 0, V = r - 2m \) and \( h_{ab} = \text{diag}(1, \sin^2 \theta) \equiv g_{ab} \). For spacetimes
that are close to Schwarzschild we can define first-order
variables \( \tilde{h}_{ab} \) and \( \tilde{V} \) via

\[
h_{ab} = g_{ab} + \tilde{h}_{ab} \tag{9}
\]

\[
V = r - 2m + \tilde{V}. \tag{10}
\]

This implies that the covariant derivative \( D_a \) is equal to
the covariant derivative on the sphere \( \nabla_a \) up to terms
of first order. Keeping only first-order terms in \( \tilde{h}_{ab}, \tilde{V}, \beta 
\text{ and } U^a \) in the equations one obtains their linearized
version:

\[
\frac{(r-2m)}{2} (r \tilde{h}_{ab},_{\tau}) = - \frac{2m}{r} (\tilde{h}_{ab},_{\tau} - h_{ab}) + 2 \nabla_a \nabla_b \beta
- q_{ab} \nabla_c \beta - q_{c(a} \nabla_b (r^2 U^c,_{\tau})\right) + \frac{1}{2} q_{ab} \nabla_c (r^2 U^c,_{\tau}) \tag{11a}
\]
\[ \beta_r = 0 \quad (11b) \]
\[ r(q_{ac}(r^2U^c),r) = 2q_{ac}U^c - 4\beta_a - \nabla^b(\tilde{h}_{ab})_r + \nabla^b\tilde{h}_{ab} \quad (11c) \]
\[ \tilde{V}_r = \frac{1}{2} q^{ac} q^b_{bd} \tilde{h}_{ab,cd} - \nabla^c \nabla_c \beta + \frac{1}{2} \nabla_c (r^2U^c)_r + r \nabla_c U^c + 2\beta + \frac{3 \cos \theta}{2 \sin \theta} \tilde{h}_{\theta\theta,\theta} - \tilde{h}_{\theta\theta}. \quad (11d) \]

In the process to obtain (11d) we used

\[ \mathcal{R} = 2 + \frac{2}{\sin^2 \theta} \tilde{h}_{\theta\phi,\theta\phi} + \tilde{h}_{\phi\theta,\theta\phi} + \frac{1}{\sin^4 \theta} \tilde{h}_{\phi\phi,\phi\phi} + \frac{3 \cos \theta}{2 \sin \theta} \tilde{h}_{\theta\theta,\theta} - \tilde{h}_{\theta\theta} \quad (12) \]

together with the fact that

\[ q^{ac} \tilde{h}_{ab} = 0 \quad (13) \]

by virtue of \( \det (h_{ab}) = \det(q_{ab}) \).

### B. Well-posed first-order form of the linearized characteristic problem

We define now a set of variables that casts the original system into first-order form, namely, such that no second-derivatives appear. Clearly there is an infinite number of ways to achieve this purpose, one of which has been used previously. However, the following choice is particularly convenient for our current purposes:

\[ P_{ab} \equiv (\tilde{h}_{ab})_r, \quad (14a) \]
\[ Q_a \equiv q_{ac}(r^2U^c)_r - 2\beta_a, \quad (14b) \]
\[ T_a \equiv \beta_a, \quad (14c) \]
\[ J_a \equiv \nabla^b \tilde{h}_{ab} + q_{ac}(r^2U^c)_r, \quad (14d) \]

It is clear that \( \tilde{h}_{ab,r} \) is encoded in \( P_{ab} \) and that the set \( (\beta_a, U^a, r) \) is encoded in the set \( (Q_a, T_a) \), since given \( (Q_a, T_a) \) we can solve uniquely for \( (\beta_a, U^a, r) \). In terms of these variables, equations (11a), (11c) and (11d) read

\[ rP_{ab,u} - \frac{(r - 2m)}{2} P_{ab,r} + \nabla^a Q_a - 1 \frac{1}{2} q_{ab} \nabla_c Q_c = - \frac{2m}{r}(P_{ab} - \tilde{h}_{ab}) \quad (15a) \]
\[ rQ_{a,r} + \nabla^b P_{ab} = 2 q_{ac} U^c - 4T_a + J_a \quad (15b) \]
\[ \tilde{V}_r + q^{bd} T_{bd} - \frac{1}{2} q^{bd} J_{bd} - \frac{3 \cos \theta}{2 \sin \theta} \tilde{h}_{\theta\theta,\theta} - r \nabla^c U^c + q^{ac} \left( q^{bd} \tilde{h}_{ab,cd} - q^{bd} \Gamma_{ad}^c \tilde{h}_{eb,c} - q^{bd} \Gamma_{bd}^c \tilde{h}_{ae,c} \right) = 2 \beta - \tilde{h}_{\theta\theta} - \frac{1}{2} q^{ad} \Gamma_{ad}^c Q_c + q^{ac} \left( q^{bd} \Gamma_{ad}^c \right) \tilde{h}_{eb} + q^{ac} \left( q^{bd} \Gamma_{bd}^c \right) \tilde{h}_{ae} \quad (15c) \]

The quantities \( \Gamma_{ab}^c \) that appear in the right-hand side of (15c) are the Christoffel symbols of the metric of the 2-sphere, \( q_{ab} \), thus representing given functions of \( \theta \), but not of the dependent variables. As it stands, Eq. (15c) is sufficiently explicit for our purposes, as will be made clear shortly. Additionally, we have

\[ T_{a,r} = 0 \quad (15d) \]
\[ rJ_{a,r} = Q_a - 2T_a + 2rU_a, \quad (15e) \]

as a consequence of (11b) and of (11c). Furthermore, by definition, we have the additional equations

\[ r\tilde{h}_{ab,r} = P_{ab} - \tilde{h}_{ab} \quad (15f) \]
\[ r^2U^a = Q^a + 2T^a - 2rU^a. \quad (15g) \]

Equations (11b), (11c), (14a), (14b), (14c), (14d), (15b), (15d) and (15e) constitute a linear homogeneous characteristic system of equations for the variables \( (\tilde{h}_{ab}, \beta, U^a, \tilde{V}, P_{ab}, Q_a, T_a, J_a) \) in canonical form. The two variables \( P_{ab} \) are the normal variables of the problem, whereas the remaining twelve variables are all null variables. The system has a unique solution for normal data \( P_{ab}^{(0)} \equiv (q_0^0, q^0_1) \equiv q^0 \) given on the slice \( u = 0 \), and null data \( (\tilde{h}^{(0)}, \beta^{(0)}, U^{(0)}, \tilde{V}^{(0)}, Q_a^{(0)}, T_a^{(0)}, J_a^{(0)}) \equiv (w_0, \ldots, w_{11}) \equiv w_0 \) given on the worldtube \( r = r_0 \).

For our purposes, however, we point out the following fact, particular to the linearized regime of the characteristic equations. The variable \( \tilde{V} \) appears only in Eq. (15b). The rest of the equations in the system yield the values of the remaining variables, which can be taken as a source for \( \tilde{V} \) in Eq. (15c). Thus the variable \( \tilde{V} \) can be thought of as decoupled from the system, in the linearized regime. In other words, the system can be solved for \( (P_{ab}, Q_a, T_a, J_a, \tilde{h}_{ab}, \tilde{V}, \beta, U^a) \) and then the solutions can be used to integrate \( \tilde{V} \) from equation (15c). The subsystem that yields \( (P_{ab}, Q_a, T_a, J_a, \tilde{h}_{ab}, \tilde{V}, \beta, U^a) \) is also a characteristic system in canonical form, with a two-dimensional normal space, and an eleven-dimensional null space. In the following, this is the point of view that we take in order to argue that the linearized system is well posed in the sense of Section B.

The subsystem obtained by leaving out Eq. (15c) has the form

\[ C^a v_u + C^r v_r + C^a v_a + Dv = 0 \quad (16) \]

where \( v = (P_{ab}, Q_a, T_a, J_a, \tilde{h}_{ab}, \tilde{V}, \beta, U^a) \). The matrix \( C^a \) has the form \( \text{diag}(1, 1, 0, \ldots, 0) \), whereas the matrix \( C^r \) has the form \( \text{diag}(- (r - 2m)/2r, 1, \ldots, 1) \). Additionally, the matrices \( C^a \) are symmetric, coupling only the variables \( Q_a \) and \( P_{ab} \). The system is linear and homogeneous and meets the criterion for well posedness outside of the event horizon, since the normal block of \( C^r \) is negative definite for all \( r > 2m \), and the normal block of \( C^u + C^r \) is positive definite for all values of \( r \). This implies that there exists a constant \( \kappa \) independent of the data \( v \equiv (q^0, w_0) \).
such that $||v||^2_T \leq e^{CT} (||q^0||^2 + ||w^0||^2)$ where the norms are defined as follows

\begin{align}
||v||^2_T &\equiv \int_{u+r=T} v(C^u + C^r)v \\
||q^0||^2 &\equiv \int_{u=0} \sum(q^0)^2 \\
||w^0||^2 &\equiv \int_{r=r_0} \sum(w^0)^2
\end{align}

As argued in Section II this in turn ensures that the solutions are stable with respect to small perturbations of the data.

Notice that the surface $u + r = T$ for fixed value of $T$ is a spacelike surface with respect to the hyperbolic operator because $C^u + C^r$ is positive definite. Additionally, one can verify that the foliation by constant values of $u + r$ is also spacelike with respect to the Schwarzschild metric, and thus it corresponds to some timelike coordinate for Schwarzschild space other than the standard Schwarzschild time. This is, of course, immaterial for the purpose of defining a positive-definite norm of the solution $v$.

III. CONCLUDING REMARKS

The main result is a formulation of the linearized characteristic problem of the Einstein equations that is a counterpart to a symmetric hyperbolic formulation of the 3+1 Einstein equations with respect to the stability of the solutions under small variations of the data. The equations are (11b), (15a), (15b), (15d), (15e), (15f) and (15g), which are thought to be solved as a characteristic problem of partial differential equations, from normal data $P^{ab}_u(r, x^a)$ on the surface $u = 0$ and null data $(Q^a_{00}(u, x^a), T^a_{00}(u, x^a), J^{a}_{n}(u, x^a), h_{ab}(u, x^a), \beta^0(u, x^a), U^a_{0}(u, x^a))$ on the worldtube $r = r_0$ outside of the event horizon of the background Schwarzschild black hole. The solution to this problem is used as a known source for the remaining metric variable $V$ which is obtained by integrating an ordinary differential equation in $r$, Eq. (15l), from data $\tilde{V}^a(0, x^a)$ given on the worldtube. The problem has 14 variables in all, a significant reduction with respect to the standard hyperbolic formulations of the Einstein equations which generically require 30 dependent variables.

With respect to the reduction in the number of variables, it is perhaps appropriate to point out that the reduction has not so much to do with the use of characteristic coordinates, but much to do with the fact that the original equations are of second-order. In this respect, consider the original second-order formulation of the Einstein equations in terms of one timelike and three spacelike coordinates: they are six second-order equations for six variables where all second-order derivatives of all variables occur generically. The first-order representation of such a system generically requires that all first-derivatives of the six variables be promoted to new variables, that is: an addition of $6 \times 4$ new variables. Hence the generic number of variables of a first-order formulation of the Einstein equations: 30. Any one of such formulations is hyperbolic in the sense that it admits characteristic surfaces: surfaces that are not appropriate as initial-data surfaces for the Cauchy problem. One can write the characteristic problem of such a hyperbolic formulation, but this problem will have no less than 30 variables – the characteristic problem of any first-order hyperbolic system of equations has as many variables (and data) as the initial-value problem.

But now consider writing the second-order form of the Einstein equations in terms of a coordinate that labels characteristic surfaces of the second-order system. By construction, the second-order derivatives with respect to the characteristic coordinate do not occur in the equations (that’s why the coordinate is characteristic). Therefore, the first-order representation of the characteristic problem of the Einstein equations must have at least six fewer variables than the generic first-order hyperbolic formulation, that is, at most 24 variables. A simpler instance of the non-commutativity of the operations of casting a system into characteristic form and of reducing to first-order is that of the two characteristic problems of the (scalar) wave equation. The problem of reducing to first-order and then going to a characteristic formulation can be found in (12), whereas the problem of first transforming to characteristic coordinates and then going to first-order appears, for instance, in (12).

The characteristic problem that we have examined here does not return a solution to the Einstein equations as it stands, because we have neglected four equations: the “trivial” equation and the “supplementary conditions” (13), which do not seem to appear in explicit form in the literature. Irrespective of their explicit form, these equations are essential if one is to obtain a metric that solves the Einstein equations via the characteristic approach. Nevertheless, considering them jointly with the main equations for a global statement of well-posedness is outside of the scope of the current work.

A number of issues remain open to further study. One question is whether estimates can be established for the derivatives of the solution of the characteristic problem of the Einstein equations. As stated in (10), manifest well-posedness of a linear homogeneous characteristic problem in general does not guarantee the existence of a priori estimates for the derivatives (in stark contrast to the case of initial-value problems (24)). Another open issue is how the well-posedness of the characteristic problem relates to well-posedness of a hyperbolic formulation of the Einstein equations.

Lastly, one may want to consider other first-order reductions of Sachs’ characteristic problem in order to establish estimates that are of greater relevance to the nonlinear case, in particular the 18-variable first-order reduction developed in (24). On the other hand, whether or not the argument used in the linearized case is rele-
vant to the full non-linear case is not really an issue at this time, since a concept of well-posedness for non-linear characteristic problems is not available yet.

Acknowledgments

I am deeply indebted to Roberto Gómez for numerous conversations. This work was supported by the NSF under grants No. PHY-9803301, No. PHY-0070624 and No. PHY-0244752 to Duquesne University.

[1] B. Gustafsson, H.-O. Kreiss, and J. Oliger, *Time-dependent problems and difference methods* (Wiley, New York, 1995).
[2] Y. Choquet-Bruhat and T. Ruggeri, Commun. Math. Phys. 89, 269 (1983).
[3] O. A. Reula, Living Reviews in Relativity 1, 3 (1998), http://www.livingreviews.org/.
[4] R. Gómez, L. Lehner, R. Marsa, and J. Winicour, Phys. Rev. Lett. 80, 3915 (1998), gr-qc/9801069.
[5] R. Gómez, L. Lehner, R. Marsa, and J. Winicour, Phys. Rev. D 57, 4778 (1998), gr-qc/9710138.
[6] R. Gómez, Phys. Rev. D 64, 024007 (2001), gr-qc/0103011.
[7] R. M. Balean, Ph.D. thesis, University of New England, Armidale, NSW Australia, 1996.
[8] R. Balean, Commun. PDE 22, 1325 (1997).
[9] R. Balean and R. Bartnik, P. Roy. Soc. Lond. A 454, 2041 (1998).
[10] S. Frittelli, Estimates for first-order homogeneous linear characteristic problems, math-ph/0408010.
[11] G. F. D. Duff, Can. J. Math. 10, 127 (1958).
[12] S. Frittelli, Estimates for the characteristic problem of the first-order reduction of the wave equation, to appear in J. Phys. A, math-ph/0408007.
[13] R. K. Sachs, Proc. R. Soc. Lond. A 270, 103 (1962).
[14] R. A. Isaacson, J. S. Welling, and J. Winicour, J. Math. Phys. 24, 1824 (1983).
[15] D. Goldwirth and T. Piran, Phys. Rev. D 36, 3575 (1987).
[16] N. Bishop, C. Clarke, and R. A. d’Inverno, Class. Quantum Grav. 7, L23 (1990).
[17] J. M. Stewart, in *Approaches to Numerical Relativity*, edited by R. d’Inverno (Cambridge University, Cambridge, 1992).
[18] C. Clarke and R. A. d’Inverno, Class. Quantum Grav. 11, 1463 (1994).
[19] R. Gómez, P. Papadopoulos, and J. Winicour, J. Math. Phys. 35, 4184 (1994).
[20] N. T. Bishop, R. Gómez, L. Lehner, and J. Winicour, Phys. Rev. D 54, 6153 (1996).
[21] J. Winicour, Living Reviews in Relativity 4, 3 (2001), http://www.livingreviews.org/.
[22] J. Winicour, J. Math. Phys. 24, 1193 (1983).
[23] S. Frittelli and L. Lehner, Phys. Rev. D 59, 084012 (1999).
[24] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, New York-London, 1962), Vol. II.
[25] R. Gómez and S. Frittelli, Phys. Rev. D 68, 084013 (2003), gr-qc/0303104.