GROUPOID EQUIVARIANT PREQUANTIZATION

DEREK KREPSKI

Abstract. In their 2005 paper, C. Laurent-Gengoux and P. Xu define prequantization for pre-Hamiltonian actions of quasi-presymplectic Lie groupoids in terms of $S^1$-central extensions of Lie groupoids. The definition requires that the quasi-presymplectic structure be exact (i.e. the closed 3-form on the unit space of the Lie groupoid must be exact). In the present paper, we define prequantization for pre-Hamiltonian actions of (not necessarily exact) quasi-presymplectic Lie groupoids in terms of Dixmier-Douady bundles. The definition is a natural adaptation of E. Meinrenken’s treatment of prequantization for quasi-Hamiltonian Lie group actions with group-valued moment map. The definition given in this paper is shown to be compatible with the definition of Laurent-Gengoux and Xu when the underlying quasi-presymplectic structure is exact. Properties related to Morita invariance and symplectic reduction are established.

1. Introduction

Quasi-symplectic groupoids were introduced in [32] to provide a unified framework to study several ‘momentum map theories,’ and in [8] (where they are called twisted presymplectic groupoids) as the global objects integrating twisted Dirac structures. Recall that a quasi-presymplectic structure on a Lie groupoid $G_1 \rightarrow G_0$ consists of a pair of differential forms $\eta \in \Omega^3(G_0)$ and $\omega \in \Omega^2(G_1)$ satisfying

$$d\eta = 0, \quad s^*\eta - t^*\eta = d\omega, \quad \text{and} \quad m^*\omega = \text{pr}_1^*\omega + \text{pr}_2^*\omega,$$

where $s, t : G_1 \rightarrow G_0$ are the source and target maps, respectively, $m : G_1 \times_{G_0} G_1 \rightarrow G_1$ is the groupoid multiplication, and $\text{pr}_1, \text{pr}_2$ are the natural projections. Examples include (see Section 2.2) the cotangent bundle of a Lie group, the AMM groupoid associated to the conjugation action on a compact Lie group (as well as its twisted conjugation counterpart), and more generally the dressing action groupoid associated to a group pair integrating a Manin pair.

The resulting momentum map theories consist of Lie groupoid actions of the quasi-symplectic groupoids $G_1 \rightarrow G_0$ on manifolds $X$ equipped with a differential form $\omega_X \in \Omega^2(X)$ satisfying a compatibility condition with the quasi-symplectic structure (see Definition 2.10). These include ordinary Hamiltonian actions of Lie groups on

\begin{itemize}
\item 
\end{itemize}

\textit{Date}: September 28, 2018.
\textit{2010 Mathematics Subject Classification}. Primary: 53D50; Secondary: 53C08, 53D17, 53D20, 58H05.

This work is partially supported by an NSERC Discovery Grant.
symplectic manifolds, quasi-Hamiltonian actions with Lie group-valued moment map \cite{2} and their twisted counterparts \cite{21}, Hamiltonian loop group actions \cite{22}, Poisson-Lie group actions \cite{18,19}, Hamiltonian quasi-Poisson $G$-spaces \cite{2}, and symmetric-space valued moment maps \cite{15}.

In their 2005 paper \cite{14}, C. Laurent-Gengoux and P. Xu define prequantization for pre-Hamiltonian actions of exact quasi-presymplectic groupoids—where the 3-form $\eta$ is exact—using central extensions of Lie groupoids (see also \cite{30,31}). Consequently, their definition is not immediately applicable in certain examples, notably in the case of quasi-Hamiltonian $G$-actions where $G$ is a compact semi-simple Lie group. As noted by the authors, one must first pass to a Morita equivalent quasi-presymplectic groupoid (a Čech groupoid associated to a $G$-equivariant covering of $G$) in order to suitably interpret prequantization in that context.

Following E. Meinrenken’s approach in \cite{20} for quasi-Hamiltonian group actions, this paper presents a definition of prequantization (see Definition 4.1) for Hamiltonian actions of quasi-presymplectic groupoids (with no exactness assumption), using the theory of Dixmier-Douady bundles (DD-bundles), which are bundles of $C^*$-algebras with typical fibre the compact operators on an infinite dimensional separable Hilbert space. An important aspect of this definition relates to the notion of equivariance for DD-bundles used in the present context and investigated further in \cite{12}, which makes use of the ‘higher structure’ inherent in DD-bundles (that is also present for other models of $S^1$-gerbes, e.g. see \cite{24}).

We summarize the main results of the paper. In Theorem 4.7(1), we show that the definition of prequantization presented here (Def. 4.1) is consistent with the definition of Laurent-Gengoux and Xu for pre-Hamiltonian actions of exact quasi-presymplectic Lie groupoids. In this special case, one obtains a prequantum line bundle over the space $X$ acted upon, and Theorem 4.7(2) gives its (equivariant) real Chern class.

Recall that Morita equivalent quasi-presymplectic groupoids give rise to a correspondence between their corresponding Hamiltonian spaces. Theorem 5.5 verifies a Morita invariance property of prequantization, showing that prequantization respects this correspondence. As a corollary (Corollary 5.6), we recover the equivalence of prequantizations for Hamiltonian loop group actions and quasi-Hamiltonian group actions, without the assumption of simple-connectivity on the underlying Lie group in \cite[Theorem A.7]{11}.

Finally, Theorem 6.1 shows that the prequantization of a pre-Hamiltonian space descends to a prequantization of its symplectic quotients (provided they are smooth—see Remark 6.3).

This paper is organized as follows.

In Section 2 we recall elementary simplicial data related to Lie groupoids and establish the notation used throughout the paper. We recall the definition of a
GROUPOID EQUIVARIANT PREQUANTIZATION

quasi-presymplectic groupoid from [32] together with some examples appearing in the literature and review the definition of a pre-Hamiltonian action in this setting.

As stated earlier, the definition of prequantization given in this paper uses the theory of DD bundles. In Section 3, we review some elements of Dixmier-Douady theory (following [1]) and recall their groupoid-equivariant counterparts from [12], which makes use of the fact that DD bundles from a bicategory. We also recall the equivalent (strict) 2-category of differential characters of degree 3 (as well as its groupoid-equivariant counterpart), which we make use of in the proofs of several of our results. We also review the relative versions (i.e. relative equivariant DD bundles and relative equivariant differential characters) associated to a groupoid morphism.

In Section 4, we give the definition of prequantization for pre-Hamiltonian actions in terms of equivariant Dixmier-Douady theory. After recalling the definition of prequantization from [14] for pre-Hamiltonian actions of exact quasi-presymplectic groupoids—phrased in terms of $S^1$-central extensions of Lie groupoids—we state and prove the main result, Theorem 4.7, which verifies that the definitions agree in this case.

In Section 5, we review the definition of Morita equivalence for quasi-presymplectic groupoids and recall the correspondence between related Hamiltonian spaces for Morita equivalent quasi-presymplectic groupoids. The Morita invariance property, stated in Theorem 5.5 is established.

In the final section, we prove Theorem 6.1 on the compatibility of prequantization with symplectic reduction.

Acknowledgements. The author thanks H. Bursztyn, E. Lerman, E. Meinrenken, and J. Watts for helpful conversations and correspondence. This work was partially supported by an NSERC Discovery Grant.

2. Preliminaries and notation

In this section we record some preliminaries on Lie groupoids and recall some definitions surrounding quasi-presymplectic groupoids and their pre-Hamiltonian actions.

2.1. Lie groupoids and double complexes. Mainly to establish notation, we recall aspects related to the simplicial manifold $G_\bullet$ associated to a Lie groupoid $\mathcal{G} = (G_1 \rightrightarrows G_0)$, as well as the resulting double complex arising from a presheaf of chain complexes. Denote the source and target maps by $s, t : G_1 \to G_0$, multiplication (composition) by $m : G_1 \times G_0 G_1 \to G_1$, inversion by $i : G_1 \to G_1$ and the unit by $\epsilon : G_0 \to G_1$. To avoid the possibility of confusion, we may at times decorate the structure maps with a subscript to indicate the underlying groupoid (e.g. $\epsilon_G$ as the unit map of $\mathcal{G}$).
For $k \geq 2$, write

$$C_k = G_1 \times_{G_0} G_1 \times_{G_0} \cdots \times_{G_0} G_1$$

whose elements are $k$-tuples $(g_1, \ldots, g_k)$ of composable arrows (with $s(g_i) = t(g_{i+1})$).

For $0 \leq i \leq k$, let $\partial_i : G_k \to G_{k-1}$ be the face maps given by

$$\partial_i(g_1, \ldots, g_k) =\begin{cases} (g_2, \ldots, g_k) & \text{if } i = 0 \\ (g_1, \ldots, g_{i+1}, \ldots, g_k) & \text{if } 0 < i < k \\ (g_1, \ldots, g_{k-1}) & \text{if } i = k. \end{cases}$$

For convenience, we set $\partial_0 = s$ and $\partial_1 = t$ on $G_1$. It is easily verified that the face maps satisfy the simplicial identities $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$. (We will not require degeneracy maps in this paper.)

A morphism of Lie groupoids $F : G \to H$ yields a map of simplicial manifolds $F : G_• \to H_•$, which for convenience may also be denoted $F$. If $X$ is a $G$-space with anchor map $\Phi : X \to G_0$, let $\Phi : G \ltimes X \to G$ denote the resulting morphism of Lie groupoids.

Let $(C^*, d)$ denote a presheaf of cochain complexes, and consider the double complex $C^*(G_•)$, depicted below.

The horizontal differential is the alternating sum of pullbacks of face maps, $\partial = \sum (-1)^i \partial_i$. Denote the total complex by

$$C(G) := \text{Tot}(C^*(G_•)), \quad \text{with } C^n(G) = \bigoplus_{p+q=n} C^p(G_q),$$

with total differential $\delta = (-1)^q d \oplus \partial$. For example, when $C^* = \Omega^*$ is the de Rham complex, we obtain the Bott-Shulman-Stasheff complex of $G_•$. In this paper, we will use the de Rham complex $\Omega^*$, smooth singular cochains $C^*(-; \mathbb{Z})$ and $C^*(-; \mathbb{R})$, and a cochain complex of Hopkins and Singer [10], denoted $DC^*$ following the notation in [16] (reviewed in Section 3.2). Note that we will abuse notation and use integration of forms to view $\Omega^*(M) \subset C^*(M; \mathbb{R})$ and also view $C^*(M; \mathbb{Z}) \subset C^*(M; \mathbb{R})$. 

---

**Image Diagram**

The diagram shows a double complex with arrows indicating the differential and face maps. The horizontal differential is represented as the alternating sum of pullbacks of face maps.

---
2.2. Quasi-symplectic groupoids and Hamiltonian actions. We recall some definitions from [32] and [8].

Definition 2.1. Let $G = (G_1 \xrightarrow{\pi} G_0)$ be a Lie groupoid. A quasi-presymplectic structure on $G$ is a closed differential form $\omega \oplus \eta \in \Omega^2(G_1) \oplus \Omega^3(G_0)$ of degree 3 in the total complex $\Omega^*(G)$.

In terms of the de Rham differential $d$ and simplicial differential $\partial$, the differential forms $\omega$ and $\eta$ must satisfy

\[ d\eta = 0, \quad d\omega = \partial\eta, \quad \text{and} \quad \partial\omega = 0. \]

Remark 2.2. As noted in [32, Remark 2.2], $\partial\omega = 0$ is equivalent to the condition that $\omega$ be multiplicative, i.e. $m^*\omega = \text{pr}_1^*\omega + \text{pr}_2^*\omega$, where $m$ denotes the groupoid multiplication. Equivalently, the graph of $m : G_2 \to G_1$ is an isotropic submanifold of $G_1 \times G_1 \times \bar{G}_1$ (with 2-form $\omega \oplus \omega \oplus (-\omega)$).

Remark 2.3. Using the terminology in [8], the condition $d\omega = \partial\eta$ is the condition that $\omega$ is relatively $\eta$-closed.

A quasi-presymplectic Lie groupoid $(G, \omega \oplus \eta)$ is quasi-symplectic if the following condition controlling the degeneracy of $\omega$ is satisfied [8, 32]:

\[ \ker(\omega_x) \cap \ker(ds)_x \cap \ker(dt)_x = \{0\} \quad \text{for all} \quad x \in G_0. \]

In this work, we will not make use of the above condition.

Remark 2.4. Quasi-symplectic groupoids $(G, \omega \oplus \eta)$ were also introduced in [8], where they are called called $\eta$-twisted presymplectic groupoids. (In that setting, the prefix $\text{pre}$ alludes to the fact that $\omega$ is possibly degenerate, and the prefix is removed when $\omega$ is non-degenerate.) As noted in [32], symplectic groupoids [29] and twisted symplectic groupoids [9] are important special cases of quasi-symplectic groupoids.

We recall some familiar examples below.

Example 2.5 (Cotangent bundle of a Lie group). Let $G$ be a Lie group with Lie algebra $g$. The symplectic groupoid $T^*G \rightrightarrows g^*$ (with canonical symplectic form on $T^*G$ and zero form on $g^*$) is a quasi-symplectic groupoid.

\[ \square \]

Example 2.6 (AMM groupoid). [3, 8, 32] Let $G$ be a Lie group equipped with a bi-invariant inner product $\langle \cdot, \cdot \rangle$ on $g$. Let $G \times G \rightrightarrows G$ be the action groupoid for the conjugation action of $G$ on itself. Let $\theta^L$ and $\theta^R$ denote, respectively, the left and right invariant Maurer-Cartan forms on $G$. Let $\eta \in \Omega^3(G)$ be the Cartan form,

\[ \eta = \frac{1}{12} \langle [\theta^L, [\theta^L, \theta^L]] \rangle, \]

and $\omega \in \Omega^2(G \times G)$ be given by

\[ \omega_{(g,x)} = -\frac{1}{2} \left( \langle \text{Ad}_x \text{pr}_1^*\theta^L, \text{pr}_1^*\theta^L \rangle + \langle \text{pr}_1^*\theta^L, \text{pr}_2^*(\theta^L + \theta^R) \rangle \right) \]

Then $(G \times G \rightrightarrows G, \omega \oplus \eta)$ is a quasi-symplectic groupoid.

\[ \square \]
Example 2.7 (Twisted AMM groupoid). The AMM groupoid above may be twisted by an automorphism of the Lie group $G$, as in [21]. Let $\kappa : G \to G$ be an automorphism, and consider the $\kappa$-twisted conjugation action of $G$ on itself

$$Ad^\kappa_g(x) = gx\kappa(g^{-1}).$$

Let $G\kappa$ denote the manifold $G$ equipped with the above action. Let $\eta$ be the Cartan form as above, and let $\omega^{(\kappa)} \in \Omega^2(G \times G\kappa)$ be given by

$$\omega^{(\kappa)}_{(g,x)} = -\frac{1}{2} \left( (Ad_x pr^*_1 \kappa^* \theta^L + pr^*_1 \theta^L) \right).$$

As in the untwisted case, it is straightforward to verify that $(G \times G\kappa \rightrightarrows G\kappa, \omega^{(\kappa)} \oplus \eta)$ is a quasi-symplectic groupoid, since the inner product is invariant under automorphisms $\kappa$.

Example 2.8 (Manin pairs). Let $\mathfrak{d}$ be a Lie algebra equipped with an invariant, nondegenerate, symmetric bilinear form of split signature and $\mathfrak{g} \subset \mathfrak{d}$ a Lie subalgebra that is also a maximal isotropic subspace. Suppose that the pair $(\mathfrak{d}, \mathfrak{g})$ (the Manin pair [2]) integrates to a group pair $(D, G)$ consisting of a connected Lie group $D$ with Lie algebra $\mathfrak{d}$, and $G \subset D$ a connected closed Lie subgroup with Lie algebra $\mathfrak{g}$. Let $D/G$ denote the space of right cosets, and consider the dressing action of $G \subset D$ on $D/G$ induced by left translation. As shown in [7], a choice of isotropic connection on the principal $G$-bundle $D \to D/G$ determines a quasi-presymplectic structure on $G \times D/G \rightrightarrows D/G$.

Examples 2.5, 2.6, and 2.7 may all be obtained via Manin pairs. Another important example of this kind is the action groupoid $G \times G^* \rightrightarrows G^*$ of a complete Poisson-Lie group $G$ acting on its Poisson-Lie dual group $G^*$ by the left dressing action [18,19].

Example 2.9 (Coadjoint action groupoid of loop groups). [6,25] Let $G$ be a compact Lie group with bi-invariant inner product on its Lie algebra $\mathfrak{g}$. Fix a real number $s > 1$, and consider the loop group $LG = \text{Map}(S^1, G)$ consisting of maps of Sobolev class $s + 1/2$. The inner product on $\mathfrak{g}$ induces a 2-cocycle on the loop Lie algebra $L\mathfrak{g}$ and hence corresponds to a central extension of Lie algebras $\hat{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbb{R}$. Suppose the central extension integrates to a central extension of Lie groups

$$1 \to S^1 \to \hat{L}G \to LG \to 1.$$

Let $L\mathfrak{g}^* = \Omega^1(S^1; \mathfrak{g})$ be the space of $\mathfrak{g}$-valued 1-forms of Sobolev class $s - 1/2$, with pairing $L\mathfrak{g} \times L\mathfrak{g}^* \to \mathbb{R}$ given by $(\xi, A) \mapsto \int_{S^1} \langle \xi, A \rangle$. Let $\hat{L}\mathfrak{g}^* = L\mathfrak{g}^* \oplus \mathbb{R}$, and observe that the coadjoint action of $\hat{L}G$ on $\hat{L}\mathfrak{g}^*$ factors through $LG$. Identifying $L\mathfrak{g}^* \cong L\hat{\mathfrak{g}}^* \oplus \{1\} \subset \hat{L}\mathfrak{g}^*$ recovers the standard $LG$-action on $L\mathfrak{g}$ by gauge transformations. Let $\omega_{\text{can}}$ be the 2-form on $\hat{L}G \times \hat{L}\mathfrak{g}^*$, defined by the same formula as the canonical symplectic form on the trivialized cotangent bundle $T^*\hat{L}G$. Then the restriction of $\omega_{\text{can}}$ to $\hat{L}G \times L\mathfrak{g}^*$ descends to a 2-form $\nu$ on $LG \times L\mathfrak{g}^*$, and $(LG \times L\mathfrak{g}^* \rightrightarrows L\mathfrak{g}^*, \nu)$ is a quasi-symplectic groupoid.
Definition 2.10. Let $G = (G_1 \rightrightarrows G_0, \omega \oplus \eta)$ be a quasi-presymplectic groupoid. A pre-Hamiltonian $G$-space is a triple $(X, \omega_X, \Phi)$ consisting of a (left) $G$-action on manifold $X$ with anchor $\Phi : X \to G_0$ together with a 2-form $\omega_X$ on $X$ satisfying

$$\Phi^*(\omega \oplus \eta) = -\delta \omega_X \in \Omega^*(G \rtimes X)$$

where $G \rtimes X$ denotes the action groupoid for the $G$-action on $X$.

Remark 2.11. Note that Definition 2.10 employs the opposite sign convention for pre-Hamiltonian spaces appearing in [32].

Hamiltonian actions by quasi-symplectic groupoids were introduced in [32] as pre-Hamiltonian actions satisfying a minimal degeneracy condition, controlling the degeneracy of the 2-form $\omega_X$. This paper does not make use of the minimal degeneracy condition.

Applying Definition 2.10 to the examples of quasi-presymplectic groupoids listed above provides several important examples of (pre)-Hamiltonian actions, such as: ordinary Hamiltonian $G$-actions on symplectic manifolds, quasi-Hamiltonian $G$-actions with group-valued moment map [3] and their twisted counterparts [21], Hamiltonian loop group actions [22], Poisson-Lie group actions [18,19], Hamiltonian quasi-Poisson $G$-spaces [2], and symmetric-space valued moment maps [15].

3. Equivariant Dixmier-Douady bundles and differential characters

In this section, we recall some perspectives on Dixmier-Douady bundles and differential characters. We also briefly review their groupoid equivariant counterparts as in [12]. Dixmier-Douady bundles are geometric models for $S^1$-gerbes. Other such models appearing in the literature are $S^1$-central extensions (see Definition 4.5), bundle gerbes [23], and principal Lie 2-group bundles [4].

3.1. Dixmier-Douady bundles. We provide a brief review of Dixmier-Douady bundles, following [1]. For further background, see also [26].

Let $\mathcal{H}$ denote an infinite dimensional separable Hilbert space, and $\mathbb{K}(\mathcal{H})$ the $C^*$-algebra of compact operators on $\mathcal{H}$. Recall that the automorphism group $\text{Aut}(\mathbb{K}(\mathcal{H})) \cong \text{PU}(\mathcal{H}) = \text{U}(\mathcal{H})/S^1$, where $\text{U}(\mathcal{H})$ (with strong operator topology) acts on $\mathbb{K}(\mathcal{H})$ by conjugation.

Recall that a Dixmier-Douady bundle (DD-bundle) $\mathcal{A} \to M$ is a locally trivial bundle of $C^*$-algebras with typical fibre $\mathbb{K}(\mathcal{H})$ and structure group $\text{PU}(\mathcal{H})$.

A Morita isomorphism of DD-bundles $\mathcal{E} : \mathcal{A}_1 \to \mathcal{A}_2$ is a locally trivial Banach space bundle $\mathcal{E} \to M$ of $\mathcal{A}_2 - \mathcal{A}_1$ bimodules with typical fibre $\mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$, the compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Locally, the bimodule action is given fibrewise by the natural $\mathbb{K}(\mathcal{H}_2) - \mathbb{K}(\mathcal{H}_1)$ bimodule action given by post- and pre-composition of operators, respectively. The composition of two Morita isomorphisms $\mathcal{E}_1 : \mathcal{A}_1 \to \mathcal{A}_2$
and $\mathcal{E}_2 : \mathcal{A}_2 \rightarrow \mathcal{A}_3$ is given by $\mathcal{E}_2 \circ \mathcal{E}_1 = \mathcal{E}_2 \otimes_{\mathcal{A}_2} \mathcal{E}_1$, the fibrewise completion of the (algebraic) tensor product over $\mathcal{A}_2$.

Given two Morita isomorphisms $\mathcal{E}_1, \mathcal{E}_2 : \mathcal{A}_1 \rightarrow \mathcal{A}_2$, a 2-isomorphism $\tau : \mathcal{E}_1 \Rightarrow \mathcal{E}_2$ is a continuous bundle isomorphism $\tau : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ that intertwines the norms and the $(\mathcal{A}_2 - \mathcal{A}_1)$-bimodule structures.

Recall that given a Morita isomorphism $\mathcal{E} : \mathcal{A}_1 \rightarrow \mathcal{A}_2$, the opposite Morita isomorphism $\mathcal{E}^* : \mathcal{A}_2 \rightarrow \mathcal{A}_1$ is given by $\mathcal{E}^* = \mathcal{E}$ as real vector bundles, with opposite (conjugate) scalar multiplication. There are natural 2-isomorphisms $\mathcal{E}^* \otimes_{\mathcal{A}_2} \mathcal{E} \cong \mathcal{A}_1$ and $\mathcal{E} \otimes_{\mathcal{A}_1} \mathcal{E}^* \cong \mathcal{A}_2$.

**Remark 3.1.** Suppose given Morita isomorphisms $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{F}$ as well as a 2-isomorphism $\tau : \mathcal{E}_2 \otimes_{\mathcal{A}_2} \mathcal{E}_1 \rightarrow \mathcal{F}$ as in the diagram below.

At times, we will abuse notation and view $\tau$ instead as the composition

$$
\mathcal{E}_1 \xrightarrow{\cong} \mathcal{A}_3 \otimes_{\mathcal{A}_3} \mathcal{E}_1 \xrightarrow{\cong} \mathcal{E}_2^* \otimes_{\mathcal{A}_2} \mathcal{E}_1 \xrightarrow{id \otimes \tau} \mathcal{E}_2^* \otimes_{\mathcal{A}_3} \mathcal{F}.
$$

The above definitions allow us to view DD-bundles over a fixed manifold $M$ as a bigroupoid (i.e. weak 2-category with coherently invertible 1-arrows and invertible 2-arrows). Hence, for example, we may speak of ‘horizontal’ composition of 2-isomorphisms $\tau \otimes \sigma : \mathcal{F}_1 \otimes_{\mathcal{A}_2} \mathcal{E}_1 \Rightarrow \mathcal{F}_2 \otimes_{\mathcal{A}_2} \mathcal{E}_2$:

as well as ‘vertical’ composition $\tau \circ \sigma$ of 2-isomorphisms, $\mathcal{E} \Rightarrow \mathcal{F} \Rightarrow \mathcal{G}$,

which is the usual composition of the underlying bundle maps.

### 3.1.1. Equivariant Dixmier-Douady bundles

We recall the definition of a $\mathbf{G}$-equivariant DD-bundle below [12].

**Definition 3.2.** Let $\mathbf{G} = (G_1 \rightrightarrows G_0)$ be a Lie groupoid.
Remark 3.3. In Definition 3.2 we implicitly use the simplicial identities—for example, by viewing $\partial_2^* A\rightarrow \partial_1^* A$ and a 2-isomorphism

$$\tau : \partial_2^* \mathcal{E} \otimes_{\partial_0^* \partial_1^* A} \partial_0^* \mathcal{E} \rightarrow \partial_1^* \mathcal{E}$$

satisfying the coherence condition $\partial_2^* \tau \circ (\text{id} \otimes \partial_0^* \tau) = \partial_1^* \tau \circ (\partial_2^* \tau \otimes \text{id})$.

(2) A $G$-equivariant Morita isomorphism $(\mathcal{A}, \mathcal{E}, \tau) \rightarrow (\mathcal{A}', \mathcal{E}', \tau')$ is a pair $G : \mathcal{A} \rightarrow \mathcal{A}'$ and a 2-isomorphism

$$\phi : \mathcal{E}' \otimes_{\partial_2^* \mathcal{A}} \partial_0^* \mathcal{G} \rightarrow \partial_1^* \mathcal{G} \otimes_{\partial_1^* \mathcal{A}} \mathcal{E}$$

satisfying the coherence condition $(\text{id} \otimes \tau)(\partial_2^* \phi \otimes \text{id}) = \partial_1^* \phi (\tau' \otimes \text{id})$.

(3) A $G$-equivariant 2-isomorphism $(\mathcal{F}, \alpha) \rightarrow (\mathcal{G}, \beta)$ is a 2-isomorphism $\sigma : \mathcal{F} \rightarrow \mathcal{G}$ satisfying the coherence condition $\beta \circ \partial_0^* \sigma = \partial_1^* \sigma \circ \alpha$.

Remark 3.4. Similar to Remark 3.3, to make sense of the above definition of pullback, we implicitly use (canonical) bundle isomorphisms so that, for instance, $\Phi^*: \mathcal{E}$ can be viewed as a $\Phi_0^* \mathcal{A}$ bimodule. We continue to abuse notation in this way throughout the paper.

Given a morphism $\Phi : H \rightarrow G$ of Lie groupoids and an $G$-equivariant DD-bundle $(\mathcal{A}, \mathcal{E}, \tau)$ we can pullback along $\Phi$ in the obvious way to get a $H$-equivariant DD-bundle $\Phi^*(\mathcal{A}, \mathcal{E}, \tau)$,

$$\Phi^*(\mathcal{A}, \mathcal{E}, \tau) = (\Phi_0^* \mathcal{A}, \Phi_1^* \mathcal{E}, \Phi_2^* \tau).$$

Remark 3.5. Similar to Remark 3.3, to make sense of the above definition of pullback, we implicitly use (canonical) bundle isomorphisms so that, for instance, $\Phi_1^* \mathcal{E}$ (a $\Phi_1^* \partial_1^* \mathcal{A} - \Phi_1^* \partial_2^* \mathcal{A}$ bimodule) can be viewed as a $\partial_1^* \Phi_0^* \mathcal{A} - \partial_0^* \Phi_1^* \mathcal{A}$ bimodule.

3.1.2. Relative Dixmier-Douady bundles. Recall that given a map $f : M_1 \rightarrow M_2$, a relative DD-bundle for $f$ is a pair $\mathcal{A}$ consisting of a DD-bundle $\mathcal{A} \rightarrow M_2$ together with a Morita isomorphism $\mathcal{E} : M_1 \times \mathbb{C} \rightarrow f^* \mathcal{A}$ (i.e. a Morita trivialization of the pullback of $\mathcal{A}$ along $f$). The equivariant counterpart is defined similarly.

Definition 3.5. Let $\Phi : H \rightarrow G$ be a Lie groupoid morphism.

(1) A relative DD-bundle for $\Phi$ is a pair $(\mathcal{F}, \alpha ; \mathcal{A}, \mathcal{E}, \tau)$ consisting of a $G$-equivariant DD-bundle $(\mathcal{A}, \mathcal{E}, \tau)$ together with a Morita isomorphism

$$(\mathcal{F}, \alpha) : (\mathbb{C}, \mathbb{C}, \text{id}) \rightarrow \Phi^*(\mathcal{A}, \mathcal{E}, \tau).$$

(2) A relative Morita isomorphism $(\mathcal{F}, \alpha) \rightarrow (\mathcal{F}', \alpha')$ is a pair $(\mathcal{G}, \beta; \rho)$ consisting of a $G$-equivariant Morita isomorphism

$$(\mathcal{G}, \beta) : (\mathcal{A}, \mathcal{E}, \tau) \rightarrow (\mathcal{A}', \mathcal{E}', \tau')$$

and an $H$-equivariant 2-isomorphism $\rho : \Phi^*(\mathcal{G}, \beta) \circ (\mathcal{F'}, \alpha') \Rightarrow (\mathcal{F}, \alpha)$. 
3.2. Differential characters. In [12], it was shown that the bicategory of DD-bundles over a manifold is equivalent to a (strict) 2-category $\mathcal{DC}^3_1(M)$, the 2-category of differential characters (of degree 3) on $M$, which we briefly recall below. (See also [16] and [10] for further details on the construction.)

To begin, let $\mathcal{DC}^n_1(M)$ denote the chain complex,

$$\mathcal{DC}^n_1(M) = \{(c, h, \omega) \in \mathcal{C}^n(M; \mathbb{Z}) \times \mathcal{C}^{n-1}(M; \mathbb{Z}) \times \mathcal{W}^n(M) \mid \omega = 0 \text{ if } n = 0\}$$

with differential given by $d(c, h, \omega) = (dc, \omega - c - dh, d\omega)$.

An object of the 2-category $\mathcal{DC}^3_1(M)$, called a differential character of (degree 3) on $M$, is a cocycle $z = (c, h, \omega)$ in $\mathcal{DC}^3_1(M)$. Given differential characters $z_1, z_2$ on $M$, a 1-arrow $z_1 \rightarrow z_2$ is a primitive $y \in \mathcal{DC}^2_1(M)$ of their difference, $z_1 - z_2 = dy$. Given 1-arrows, $y_1, y_2 : z_1 \rightarrow z_2$, a 2-arrow $y_1 \Rightarrow y_2$ is an equivalence class $[x] \in \mathcal{DC}^1_1(M)$, where $x$ is a primitive of their difference, $y_2 - y_1 = dx$—see Remark 3.8 explaining the chosen sign convention. Here we identify $x \sim x + dv$ for all $v \in \mathcal{DC}^0_1(M)$. Composition is given by addition, and hence all 1- and 2-arrows are isomorphisms.

**Remark 3.6.** The subscript in the notation for the above chain complex and the resulting 2-category of differential characters is included for completeness—the above is the $s = 1$ instance of family of chain complexes and 2-categories $\mathcal{DC}_s^*$ and $\mathcal{DC}^*_s$, respectively. The interested reader may consult [16] and [10] for the significance of this parameter.

3.2.1. Equivariant differential characters. Analogous to Definition 3.2, we define the 2-category $\mathcal{DC}^3_1(G)$ of $G$-equivariant differential characters (of degree 3) as follows.

**Definition 3.7.** Let $G = (G_1 \rightrightarrows G_0)$ be a Lie groupoid.

1. A $G$-equivariant differential character (of degree 3) is a triple $(z, y, [x])$ consisting of a differential character $z$ in $\mathcal{DC}^3_1(G_0)$, a 1-isomorphism $y : \partial_0^*_y z \rightarrow \partial_0^*_z z$ in $\mathcal{DC}^2_1(G_1)$ and a 2-isomorphism $[x] : \partial_2^* y + \partial_0^*y \Rightarrow \partial_1^* y$, $x$ in $\mathcal{DC}^1_1(G_2)$, satisfying the coherence condition that $\partial x$ is exact in $\mathcal{DC}^1_1(G_3)$.

2. A 1-isomorphism $(z_1, y_1, [x_1]) \rightarrow (z_2, y_2, [x_2])$ of $G$-equivariant differential characters is a pair $(u, [v])$ where $u : z_1 \rightarrow z_2$ is in $\mathcal{DC}^2_1(G_0)$ and $[v] : y_2 + \partial^*_0 u \Rightarrow \partial^*_1 y_1 + y_1$ is a 2-isomorphism, with $v$ in $\mathcal{DC}^1_1(G_1)$ satisfying the coherence condition that $x_1 - x_2 - \partial v$ is exact in $\mathcal{DC}^1_1(G_2)$.

3. A $G$-equivariant 2-isomorphism $(u, [v]) \Rightarrow (s, [t])$ is a 2-isomorphism $[w] : u \Rightarrow s$ satisfying the coherence condition that $t - v - \partial w$ is exact in $\mathcal{DC}^1_1(G_1)$.

**Remark 3.8.** The sign conventions used here for defining 1- and 2-arrows is chosen mainly to be able to view an object $(z, y, [x])$ in $\mathcal{DC}^3_1(G)$ as a cocycle

$$z \oplus y \oplus x \oplus w \in \bigoplus_{p+q=3} \mathcal{DC}^p_q(G_q)$$

(where $w$ is a primitive of $\partial x$) in the total complex of the double complex $\mathcal{DC}_1(G)$. 
Lemma 3.9. G-equivariant differential characters are in one-to-one correspondence with \( Z^3(\text{DC}_1(G))/\delta \text{DC}_1^0(G_2) \).

Proof. Note that a G-equivariant differential character amounts to a degree 3 cocycle \( \xi \) in the total complex \( \text{DC}_1^*(G) \),

\[
\xi = a \oplus x \oplus y \oplus z \in \text{DC}_1^0(G_3) \oplus \text{DC}_1^1(G_2) \oplus \text{DC}_1^2(G_1) \oplus \text{DC}_1^3(G_0),
\]

where \( da = \partial x \). Changing \( x \) to \( x + df \) for some \( f \in \text{DC}_1^0(G_2) \) and \( a \) to \( a + \partial b \) results in the same equivariant differential character. Therefore, G-equivariant differential characters are in one-to-one correspondence with \( Z^3(\text{DC}_1(G))/\delta \text{DC}_1^0(G_2) \). \( \square \)

Lemma 3.10. The set of 1-isomorphisms between two fixed G-equivariant differential characters is either empty or a torsor for \( Z^2(\text{DC}_1(G))/\delta \text{DC}_1^0(G_1) \).

Proof. Similar to the previous Lemma, (if it exists) a 1-isomorphism of two fixed G-equivariant differential characters, \( (u, [v]): (z_1, y_1, [x_1]) \rightarrow (z_2, y_2, [x_2]) \), amounts to a degree 2 cochain \( \theta = b \oplus v \oplus u \in \text{DC}_1^0(G_2) \oplus \text{DC}_1^1(G_1) \oplus \text{DC}_1^2(G_0) \), where \( db = x_2 - x_1 + dv \). Changing \( v \) to \( v + dg \) for some \( g \in \text{DC}_1^0(G_1) \) and \( b \) to \( b + \partial g \) results in the same 1-isomorphism. Therefore, (if non-empty) the set of 1-isomorphisms of two fixed G-equivariant differential characters is a \( Z^2(\text{DC}_1(G))/\delta \text{DC}_1^0(G_1) \)-torsor. \( \square \)

3.2.2. Relative differential characters. Analogous to Section 3.1.2, we define relative differential characters for a Lie groupoid morphism as follows.

Definition 3.11. Let \( \Phi : H \rightarrow G \) be a Lie groupoid morphism.

1. A relative differential character for \( \Phi \) is a pair \( (u, [v]; z, y, [x]) \) consisting of a G-equivariant differential character \( (z, y, [x]) \) together with a trivialization \( (u, [v]): (0, 0, [0]) \rightarrow \Phi^*(z, y, [x]) \).
2. A relative 1-isomorphism \( (u, [v]; z, y, [x]) \rightarrow (u', [v']; z', y', [x']) \) of relative equivariant differential characters is a pair \( (s, [t]; [w]) \) consisting of a G-equivariant 1-isomorphism \( (s, [t]): (z, y, [x]) \rightarrow (z', y', [x']) \) and an H-equivariant 2 isomorphism \( [w]: (u', [v']) + \Phi^*(s, [t]) \Rightarrow (u, [v]). \)

3.3. The Dixmier-Douady class. Analogous to the classification of complex line bundles by their Chern class, Dixmier-Douady bundles are classified by a degree 3 cohomology class called the Dixmier-Douady class [26]. In [12], it was verified that if \( G \) is a proper Lie groupoid, then Morita isomorphism classes of G-equivariant DD-bundles are classified by \( H^3(G; \mathbb{Z}) \). In particular, an equivariant DD-bundle \( (A, E, \tau) \) is Morita trivial if and only if its DD-class \( DD(A, E, \tau) \in H^3(G; \mathbb{Z}) \) vanishes.

Remark 3.12. Note that the corresponding notion of a DD-class for G-equivariant differential characters is automatic. Indeed, given a differential character \( (z, y, [x]), \)
choose a cocycle \( \xi \) in \( DC^3_1(G) \) that represents it (see Lemma 3.9) and set \( DD(z, y, [x]) = [pr(\xi)] \), where \( pr : DC^3_1(G) \to C^*(G) \) denotes the natural projection. This is indeed the DD-class, since isomorphism classes of differential characters are in bijection with \( H^3(DC_1(G)) \cong H^3(G; \mathbb{Z}) \) (where the isomorphism is induced by \( pr \)—see [12]).

**Remark 3.13.** The equivalence of bicategories [12]

\[
\mathcal{D} : DC^3_1(G) \to B^2S^1(G)
\]
from \( G \)-equivariant differential characters \( DC^3_1(G) \) to \( G \)-equivariant DD bundles \( B^2S^1(G) \) allows for the following description of the DD-class. Given a \( G \)-equivariant DD-bundle \((A, \mathcal{E}, \tau)\), let \((z, y, [x])\) denote a \( G \)-equivariant differential character such that \( D(z, y, [x]) \) is isomorphic to \((A, \mathcal{E}, \tau)\) and set \( DD(A, \mathcal{E}, \tau) = [pr(\xi)] \) as in Remark 3.12.

We briefly describe the relative version of the DD-class. Let \( \Phi : H \to G \) be a morphism of Lie groupoids, and suppose we are given a relative DD-bundle \((\mathcal{F}, \alpha; \mathcal{A}, \mathcal{E}, \tau)\) for \( \Phi \). The relative DD-class \( DD(\mathcal{F}, \alpha; \mathcal{A}, \mathcal{E}, \tau) \) is a cohomology class in \( H^3(\Phi; \mathbb{Z}) \), the cohomology of the algebraic mapping cone of \( \Phi^* : C^*(G; \mathbb{Z}) \to C^*(H; \mathbb{Z}) \) (see [28] for more on relative cohomology and mapping cones). In particular, \( DD(\mathcal{F}, \alpha; \mathcal{A}, \mathcal{E}, \tau) \) is represented by a pair

\[
(b, c) \in C^2(H; \mathbb{Z}) \oplus C^3(G; \mathbb{Z}),
\]

with \( \delta c = 0 \) and \([c] = DD(A, \mathcal{E}, \tau)\) as well as \( \Phi^* c = -\delta b \). Similar to Remark 3.13, we may define the relative DD-class using a corresponding relative \( G \)-equivariant differential character: namely, a differential character \((z, y, [x])\) in \( DC^3_1(G) \) and a 1-isomorphism \((u, [v]) : (0, 0, [0]) \to \Phi^*(z, y, [x])\). The relative DD-class is then obtained by setting \( b = pr(\theta) \) and \( c = pr(\xi) \), where \( \theta \) and \( \xi \) are as in the proofs of Lemmas 3.10 and 3.9, respectively, and \( pr \) is the natural projection from Remark 3.12.

By a *real* DD-class, we mean the image of the DD-class under the coefficient homomorphism induced by \( \mathbb{Z} \hookrightarrow \mathbb{R} \), which we will denote as \( DD_\mathbb{R} \). By the de Rham Theorem, we may also represent such real classes using differential forms, so that \( DD_\mathbb{R}(A, \mathcal{E}, \tau) = [\alpha \oplus \beta \oplus \omega \oplus \eta] \) for some cocycle in the Bott-Shulman-Stasheff complex,

\[
\alpha \oplus \beta \oplus \omega \oplus \eta \in \Omega^0(G_3) \oplus \Omega^1(G_2) \oplus \Omega^2(G_1) \oplus \Omega^3(G_0).
\]
In this case, we may say that the \( G \)-equivariant DD-bundle \((A, \mathcal{E}, \tau) \) represents \( \alpha \oplus \beta \oplus \omega \oplus \eta \).

Similarly, we may use the de Rham Theorem to write a real relative DD-class

\[
DD_\mathbb{R}(\mathcal{F}, \alpha; \mathcal{A}, \mathcal{E}, \tau) = [(\xi \oplus \mu \oplus \nu, \alpha \oplus \beta \oplus \omega \oplus \eta)]
\]
for some cocycle \( \alpha \oplus \beta \oplus \omega \oplus \eta \) as above and total form

\[
\xi \oplus \mu \oplus \nu \in \Omega^0(H_2) \oplus \Omega^1(H_1) \oplus \Omega^2(H_0)
\]
in the Bott-Shulman-Stasheff complex for \( H \), satisfying

\[
\Phi^*(\alpha \oplus \beta \oplus \omega \oplus \eta) = -\delta(\xi \oplus \mu \oplus \nu).
\]  

(3.1)
In this case, we may say that \((F, \alpha; A, E, \tau)\) represents the relation (3.1).

4. Equivariant prequantization

In this section we define prequantization for pre-Hamiltonian actions of quasi-presymplectic groupoids in a manner that is analogous to the definition in the quasi-Hamiltonian case \([20]\).

**Definition 4.1.** Let \((G_1 \rightrightarrows G_0, \omega \oplus \eta)\) be a quasi-presymplectic Lie groupoid, and let \((X, \omega_X, \Phi)\) be a pre-Hamiltonian \(G\)-space. A prequantization of \((X, \omega_X, \Phi)\) consists of a relative DD-bundle for \(\Phi\) whose real DD-class is \([\omega_X, \omega \oplus \eta]) \in H^3(\Phi; \mathbb{R})\).

In other words, a prequantization of \((X, \omega_X, \Phi)\) is a relative DD-bundle \((F, \alpha; A, E, \tau)\) that represents the pre-Hamiltonian condition \(\Phi^*(\omega \oplus \eta) = -\delta \omega_X\).

**Remark 4.2.** In contrast to other definitions of prequantization appearing in the literature, Definition 4.1 does not involve a choice of connective structure.

As with other treatments of prequantization, an obstruction to the existence of a prequantization can be characterized as an integrality condition.

**Definition 4.3.** For a morphism \(\Phi : H \to G\) of Lie groupoids, a relative closed total form \((\alpha, \beta)\) in \(\Omega^3(\Phi)\) is integral whenever \([\alpha, \beta]) \in H^3(\Phi; \mathbb{R})\) is in the image of the coefficient homomorphism \(H^3(\Phi; \mathbb{Z}) \to H^3(\Phi; \mathbb{R})\).

**Proposition 4.4.** Let \((G_1 \rightrightarrows G_0, \omega \oplus \eta)\) be a quasi-presymplectic Lie groupoid, and let \((X, \omega_X, \Phi)\) be a pre-Hamiltonian \(G\)-space. A prequantization of \((X, \omega_X, \Phi)\) exists if and only if \((\omega_X, \omega \oplus \eta)\) is an integral (relative) total form.

**Proof.** It is clear that if a prequantization exists that \((\omega_X, \omega \oplus \eta)\) is an integral relative total form.

Conversely, suppose \((\omega_X, \omega \oplus \eta)\) is integral. Then there exists \((b, c) \in C^2(G \times X; \mathbb{Z}) \oplus C^3(G; \mathbb{Z})\) satisfying \(\delta c = 0\) and \(\Phi^*c = -\delta b\), as well as

\[
\omega_X - b = \delta g + \Phi^*h \tag{4.1}
\]

\[
\omega \oplus \eta - c = -\delta h \tag{4.2}
\]

for some \((g, h) \in C^2(\Phi; \mathbb{R})\). Equation (4.2) says that \(\omega \oplus \eta\) is integral and it is straightforward to verify that \((z, y, [x]) = (c_0, -h_0, \eta; c_1, h_1, \omega; [c_2, -h_2, 0])\) defines a \(G\)-equivariant differential character and hence \(G\)-equivariant DD-bundle \((A, E, \tau)\) with DD-class equal to \([c]\), and real DD-class \([\omega \oplus \eta]\).
Equation (4.1) and the condition that \((b, c)\) is a relative cocycle gives the following system:

\[
\begin{align*}
\Phi^* c_0 &= -db_0 \\
\Phi^* c_1 &= -\partial b_0 + db_1 \\
\Phi^* c_2 &= -\partial b_1 - db_2 \\
\Phi^* c_3 &= -\partial b_2
\end{align*}
\]

Together with the condition \(\delta \omega_X + \Phi^*(\omega \oplus \eta) = 0\), we find that

\[(u, [v]) = (b_0, g_0, \omega_X; [-b_1, g_1, 0])\]

is a trivialization \((0, 0, [0]) \to \Phi^*(z, y, [x])\), which gives a Morita trivialization

\[(\mathcal{F}, \alpha) : (\mathbb{C}, \mathbb{C}, \text{id}) \to \Phi^*(\mathcal{A}, \mathcal{E}, \tau).\]

It follows that the relative DD-bundle \((\mathcal{F}, \alpha; \mathcal{A}, \mathcal{E}, \tau)\) is the desired prequantization. □

4.1. Relation with Laurent-Gengoux & Xu’s definition. The definition of prequantization in Definition 4.1 is readily seen to agree with the definition of prequantization for quasi-Hamiltonian \(G\)-space with \(G\)-valued moment map [20]. In this subsection, we show that Definition 4.1 is consistent with the definition in [14], which employs \(S^1\)-central extensions to model \(S^1\)-gerbes. For convenience, we recall some elementary definitions regarding \(S^1\)-central extensions and prequantization from [14]. (For more further on \(S^1\)-central extensions, the reader may consult [5] or [27].)

**Definition 4.5.** An \(S^1\)-central extension of a Lie groupoid \(G_1 \rightrightarrows G_0\) is a Lie groupoid \(R \rightrightarrows G_0\) together with a Lie groupoid homomorphism

\[
\begin{array}{ccc}
R & \xrightarrow{\pi} & G_1 \\
\downarrow & & \downarrow \\
G_0 & \longrightarrow & G_0
\end{array}
\]

with the property that \(\pi\) is a (left) principal \(S^1\)-bundle and the principal \(S^1\)-action on \(R\) is compatible with the groupoid multiplication on \(R:\ (s \cdot x)(t \cdot y) = st \cdot (xy)\) for all \(s, t \in S^1\) and \((x, y) \in R_2 = R \times_{G_0} R\). Such an \(S^1\)-central extension is denoted by \(R \rightrightarrows G_1 \rightrightarrows G_0\).

Recall from [4] that associated to an \(S^1\)-central extension \(R \rightrightarrows G_1 \rightrightarrows G_0\) is a DD-class \(DD(R)\) in \(H^3(G; \mathbb{Z})\), which by [5, Proposition 4.7] pulls back along \(\epsilon_G\) to zero in \(H^3(G_0; \mathbb{Z})\). In particular, if \((G, \omega \oplus \eta)\) is a quasi-presymplectic groupoid and \(R \rightrightarrows G_1 \rightrightarrows G_0\) is an \(S^1\)-central extension whose real DD-class is \([\omega \oplus \eta]\), then \(\eta\) must be exact.

For a quasi-presymplectic groupoid \((G, \omega \oplus \eta)\) with exact 3-form \(\eta\), the authors in [14] define a prequantization of a Hamiltonian \(G\)-space \((X, \omega_X, \Phi)\) as an \(S^1\)-central extension \(R \rightrightarrows G_1 \rightrightarrows G_0\) of \(G\) together with a principal \(S^1\)-bundle \(p : L \to X\)
equipped with an $R$-action satisfying $(s \cdot r)(t \cdot y) = st \cdot (ry)$ for all $s, t \in S^1$ and $(r, y) \in R \times G_0 L$. Theorem 4.7(1) below shows that this definition is compatible with Definition 4.1.

The proof of Theorem 4.7 requires the following lemma, which describes the coherence condition for 1-isomorphisms of equivariant DD-bundles (see Definition 3.2(2)) in the special case of a trivialization. Note that the statement of the lemma implicitly abuses notation in the spirit of Remark 3.1.

**Lemma 4.6.** Let $G$ be a Lie groupoid, and let $(A, E, \tau)$ be a $G$-equivariant DD-bundle. A trivialization $(\mathcal{G}, \beta) : \epsilon^*(A, E, \tau) \to (C, C, \text{id})$ gives rise to an equality of 2-isomorphisms $\beta = \epsilon^* \tau$.

**Proof.** The coherence condition from Definition 3.2(2) amounts to the following commutative diagram of bundle isomorphisms over $G_0$:

$$
\begin{array}{c}
\begin{array}{ccc}
\epsilon^* E \otimes \epsilon^* E & \to & \epsilon^* E \otimes G \otimes C \\
\text{id} \otimes \beta & \downarrow & \beta \otimes \text{id} \\
\epsilon^* E \otimes \mathcal{G} & \to & \mathcal{G} \otimes C \otimes C \\
\epsilon^* \tau \otimes \text{id} & \downarrow & \\
\epsilon^* E \otimes \mathcal{G}
\end{array}
\end{array}
$$

where the unlabelled arrow is a canonical isomorphism. Hence, the diagram

$$
\begin{array}{c}
\begin{array}{ccc}
\epsilon^* E \otimes \epsilon^* E & \to & \epsilon^* E \otimes G \otimes C \\
\text{id} \otimes \beta & \downarrow & \\
\epsilon^* E \otimes \mathcal{G}
\end{array}
\end{array}
$$

commutes, where the unlabelled arrow is a canonical isomorphism. Therefore, the 2-isomorphism $\beta$ (see Remark 3.1), which may be written as the composition

$$
\epsilon^* E \to \epsilon^* E^* \otimes \epsilon^* E \otimes \epsilon^* E \otimes G \otimes G^* \to \epsilon^* E^* \otimes \epsilon^* E \otimes G \otimes C \otimes G^* \to A
$$

(where unlabelled arrows are canonical isomorphisms) coincides with $\epsilon^* \tau$, which may be written as the composition

$$
\epsilon^* E \to \epsilon^* E^* \otimes \epsilon^* E \otimes \epsilon^* E \otimes G \otimes G^* \to \epsilon^* E^* \otimes \epsilon^* E \otimes G \otimes G^* \to A
$$

(where unlabeled arrows are canonical isomorphisms), as required. □
Theorem 4.7. Let \((G, \omega \oplus \eta)\) be a quasi-presymplectic groupoid, and let \((X, \omega_X, \Phi)\) be a pre-Hamiltonian \(G\)-space. Suppose that \((\mathcal{F}, \alpha; \mathcal{A}, \tau)\) is a prequantization of \((X, \omega_X, \Phi)\) and that \(\epsilon^*_G(\mathcal{A}, \mathcal{E}, \tau)\) admits a trivialization over \(G_0\) (viewed as a trivial groupoid). Then

\begin{enumerate}
  \item there exists an \(S^1\)-central extension \(p : L \to X\) equipped with an \(R\)-action with anchor map \(\Phi \circ p\), satisfying \((s \cdot r)(t \cdot y) = st \cdot (ry)\) for all \(s, t \in S^1\) and \((r, y) \in R \times_{G_0} L\);
  \item the \(R\)-equivariant curvature class of \(L\) is \([\Phi^* \beta_0 - \omega_X] + \Phi^* \theta] \in H^2(R \times X; \mathbb{R})\), where \(\beta_0\) is a primitive of \(\eta\) and \(\theta\) is a primitive of the curvature form of \(\pi : R \to G_1\) pulled back along \(\pi\).
\end{enumerate}

Proof of (1). Let \((G, \beta) : (\mathbb{C}, \mathbb{C}, \text{id}) \to \epsilon^*_G(\mathcal{A}, \mathcal{E}, \tau)\) be a trivialization over \(G_0\). Consider the composition \(C \to \partial_0^* \mathcal{A} \to \partial_1^* \mathcal{A} \to C\) given by the line bundle \(\mathcal{R} = \partial_1^* G^* \otimes_{\partial_1^* \mathcal{A}} \mathcal{E} \otimes_{\partial_0^* \mathcal{A}} \partial_0^* \mathcal{G}\) over \(G_1\). Let \(R \to G_1\) be its associated principal \(S^1\)-bundle. Then \(R \to G_1 \rightrightarrows G_0\) is an \(S^1\)-central extension of \(G\).

Let \((\mathcal{F}, \alpha)\) be the trivialization of \(\Phi^*(\mathcal{A}, \mathcal{E}, \tau)\) given by the prequantization. Consider the composition \(C \to \Phi^* \mathcal{A} \to C\) given by the line bundle \(\mathcal{L} = \mathcal{F}^* \otimes_{\Phi^* \mathcal{A}} \Phi^* \mathcal{G}\) over \(X\). Let \(p : L \to X\) be its associated principal \(S^1\)-bundle.

The 2-isomorphism \(\alpha\) (see Remark 3.1) in the diagram

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\Phi^* \partial_0^* \mathcal{G}} & \partial_0^* \Phi^* \mathcal{A} \xrightarrow{\Phi^* \mathcal{E}} \partial_1^* \Phi^* \mathcal{A} \xrightarrow{\Phi^* \partial_1^* \mathcal{G}^*} C \\
\downarrow{\partial_1^* \mathcal{F}} & & \downarrow{\partial_1^* \mathcal{F}}
\end{array}
\]

gives rise to a bundle isomorphism \(\Phi^* \mathcal{R} \cong \partial \mathcal{L} = \partial_0^* \mathcal{L} \otimes \partial_1^* \mathcal{L}^*\), or equivalently \(\Phi^* \mathcal{R} \cong \partial L\). We verify below that this isomorphism, which will be denoted by \(\psi : \Phi^* \mathcal{R} = R \times_{G_0} X \to \partial L\), defines an action

\[R \times_{G_0} L \to L, \quad (r, y) \mapsto r \cdot y,\]

with anchor map \(\Phi \circ p : L \to G_0\).

It is straightforward to verify that

\[
\partial L = \{(\gamma, x, y_0, y_1) \in G_1 \times X \times L \times L \mid x = p(y_0), \gamma \cdot x = p(y_1)\}/\sim
\]

where \((\gamma, x, y_0, y_1) \sim (\gamma, x, \lambda y_0, \lambda y_1)\) for \(\lambda \in S^1\). (We shall use brackets \([\cdots]\) to denote \(\sim\)-equivalence classes.) Given \((r, y) \in R \times_{G_0} L\), define \(r \cdot y \in L\) by the equality

\[\psi(r, p(y)) = [\pi(r), p(y), y, r \cdot y].\]

This is well-defined since the action of \(S^1\) on \(L\) is free. We check that this formula defines an action of \(R\) on \(L\) with anchor \(\Phi p\). That \(\Phi p(r \cdot y) = \partial_1(r)\) follows immediately from the fact that \(G\) acts on \(X\): since \(p(r \cdot y) = \pi(r) \cdot p(y)\), and hence \(\Phi p(r \cdot y) = \Phi(\pi(r) \cdot p(y)) = \partial_1(\pi(r)) = \partial_1(r)\).
Recall that the unit map \( \epsilon_R : G_0 \to R \) can be viewed as a section of the \( S^1 \)-bundle \( \epsilon^* G R \to G_0 \). We show next that the units of \( R \) act trivially. This will follow from the commutative diagram (4.3) below of \( S^1 \)-bundles over \( X \). (The commutativity of the diagram will be shown subsequently.)

\[
\begin{array}{ccc}
\epsilon^*_{G \times X} \Phi^* \frac{e_{\epsilon_{G \times X}^*}}{\psi} R & \longrightarrow & \epsilon^*_{G \times X} \partial L \\
\Phi^* \epsilon^* G R & \longrightarrow & X \times S^1
\end{array}
\] (4.3)

The vertical maps marked as equalities are canonical isomorphisms. The canonical isomorphism on the right is

\[
[\epsilon_G(\Phi(x)), x, y_0, y_1] \mapsto (x, \lambda(y_0, y_1))
\]

where \( \lambda(y_0, y_1) \in S^1 \) is defined by \( y_1 = \lambda(y_0, y_1)y_0 \). The trivialization \( \Phi^* \epsilon^* G R \to X \times S^1 \) sends \((r, x) \mapsto \lambda(w(\Phi(x)), r)\).

To see that (4.3) guarantees that the units act trivially, let \( a \in G_0 \), and consider \( \epsilon_R(a) \in R \) and \( y \in L \) satisfying \( \Phi(p(y)) = \partial_0(\epsilon_R(a)) = a \). Then \( (\epsilon_R(a), p(y)) \in \epsilon^*_{G \times X} \Phi^* R = \Phi^* \epsilon^* G R \) and the commutativity of the diagram forces

\[
\lambda(y, \epsilon_R(a) \cdot y) = \lambda(w(\Phi(p(y)), \epsilon_R(a))), \quad \lambda(\epsilon_R(a), \epsilon_R(a)) = 1,
\]

as required.

We now verify the commutativity of (4.3), which ultimately relies on the coherence conditions (see Definition 3.2) satisfied by the trivializations \((\mathcal{F}, \alpha)\) and \((\mathcal{G}, \beta)\). Omitting certain canonical isomorphisms, diagram (4.3) may be rewritten as

\[
\begin{array}{ccc}
\Phi^* \mathcal{G}^* \otimes_{\Phi^* \mathcal{A}} \epsilon^*_{G \times X} \Phi^* \mathcal{E} \otimes_{\Phi^* \mathcal{A}} \Phi^* \mathcal{G} & \overset{\text{id} \otimes \epsilon^*_{G \times X} \alpha \otimes \text{id}}{\longrightarrow} & \Phi^* \mathcal{G}^* \otimes_{\Phi^* \mathcal{A}} \mathcal{F} \otimes \mathcal{C} \otimes_{\Phi^* \mathcal{A}} \Phi^* \mathcal{G} \\
\Phi^* \mathcal{G}^* \otimes_{\Phi^* \mathcal{A}} \Phi^* \epsilon^* \mathcal{E} \otimes_{\Phi^* \mathcal{A}} \Phi^* \mathcal{G} & \longrightarrow & \mathcal{C}
\end{array}
\] (4.4)

where the vertical maps marked as equalities are canonical isomorphisms. The lower horizontal map is the composition,

\[
\Phi^* \mathcal{G}^* \otimes_{\Phi^* \mathcal{A}} \Phi^* \epsilon^* \mathcal{E} \otimes_{\Phi^* \mathcal{A}} \Phi^* \mathcal{G} \overset{\text{id} \otimes \Phi^* \beta \otimes \text{id}}{\longrightarrow} \Phi^* \mathcal{G}^* \otimes_{\Phi^* \mathcal{A}} \mathcal{A} \otimes_{\Phi^* \mathcal{A}} \Phi^* \mathcal{G} \cong \mathcal{C}.
\]

Recall from Definition 3.2 that the 2-isomorphisms \( \epsilon^*_{G \times X} \alpha \) and \( \Phi^* \beta \) must satisfy certain coherence conditions. Here—see Remark 3.1—they are being viewed as 2-isomorphisms

\[
\epsilon^*_{G \times X} \alpha : \epsilon^*_{G \times X} \Phi^* \mathcal{E} \to \mathcal{F} \otimes_{\mathcal{C}} \mathcal{F}^*, \quad \text{and} \quad \Phi^* \beta : \Phi^* \epsilon^* \mathcal{E} \to \Phi^* \mathcal{G} \otimes_{\mathcal{C}} \Phi^* \mathcal{G}^* \cong \Phi^* \mathcal{A}.
\]

Writing the 2-isomorphisms \( \Phi^* \epsilon^* \mathcal{T} \) and \( \epsilon^*_{G \times X} \Phi^* \tau \) as

\[
\Phi^* \epsilon^* \mathcal{T} : \Phi^* \epsilon^* \mathcal{E} \to \Phi^* \mathcal{A}, \quad \text{and} \quad \epsilon^*_{G \times X} \Phi^* \tau : \epsilon^*_{G \times X} \Phi^* \mathcal{E} \to \Phi^* \mathcal{A},
\]
the corresponding coherence conditions are
\[ \Phi^* \epsilon^*_G \tau = \Phi^* \beta, \quad \text{and} \quad \epsilon_{G \times X}^* \Phi^* \tau = \epsilon_{G \times X}^* \alpha, \]
by Lemma 4.6. Therefore, the trivializations in diagram (4.4) coincide (i.e. the diagram commutes) because each is a trivialization that is compatible with \( \Phi^* \epsilon^*_G \tau = \epsilon_{G \times X}^* \Phi^* \tau \).

Next, we show that \( r_0 \cdot (r_1 \cdot y) = (r_0 r_1) \cdot y \) for compatible \( r_0, r_1 \in R, y \in L \). As will be verified next, this follows from the commutativity of the diagram below

\[
\begin{array}{ccc}
\partial \Phi^* R & \xrightarrow{\partial \psi} & \partial^2 L \\
\Phi^* \partial R & \xrightarrow{\ } & (G \times X)_2 \times S^1 \\
\end{array}
\]

(where the equalities are canonical isomorphisms). (The commutativity of the diagram follows by an argument similar to the one given above for diagram (4.3), ultimately relying on the coherence condition satisfied by the trivialization \((\mathcal{F}, \alpha)\).)

A straightforward computation gives the identifications
\[
\partial \Phi^* R = \Phi^* \partial R = \{(r_0, r_1, r, x) \mid \pi(r_0) \pi(r_1) = \pi(r), \partial_0(r_1) = \Phi(x)\}/\sim
\]
where \( (r_0, r_1, r, x) \sim (\lambda r_0, r_1, r, x) \sim (r_0, \mu r_1, \mu r, x) \), for \( \lambda, \mu \in S^1 \), and
\[
\partial^2 L = \left\{ (\gamma_0, \gamma_1, x; y_0, y_1, y_0', y_1', y_0'', y_1'') \mid \begin{array}{l}
\partial_0(\gamma_1) = \Phi(x), \partial_0(\gamma_0) = \Phi(\gamma_1 \cdot x) \\
x = p(y_0) = p(y_0') \\
\gamma_1 \cdot x = p(y_1) = p(y_0'') \\
\gamma_0 \gamma_1 \cdot x = p(y_1') = p(y_1'') \\
\end{array} \right\}/\sim
\]
where
\[
(\gamma_0, \gamma_1, x; y_0, y_1, y_0', y_1', y_0'', y_1'') \sim (\gamma_0, \gamma_1, x; \lambda y_0, \lambda \gamma_1, \lambda y_0', \lambda \gamma_1', \mu \gamma_0 y_1', \nu \gamma_1'')
\]
for \( \lambda, \mu, \nu, \alpha, \beta \in S^1 \). For \([r_0, r_1, r_0 r_1, p(y)] \in \partial \Phi^* R\), we compare the two ways of going around the above commutative diagram:

\[
[r_0, r_1, r_0 r_1, p(y)] \quad \xrightarrow{\pi(r_0), \pi(r_1), p(y); y, r_1 \cdot y, y, (r_0 r_1) \cdot y, r_1 \cdot y, r_0 \cdot (r_1 \cdot y)\]} \quad \xrightarrow{\pi(r_0), \pi(r_1), p(y); \lambda((r_0 r_1) \cdot y, r_0 \cdot (r_1 \cdot y))\]} \quad [r_0, r_1, r_0 r_1, p(y)]
\]

(where \( \gamma(\cdot, \cdot) \) is defined analogously as above) and hence \( r_0 \cdot (r_1 \cdot y) = (r_0 r_1) \cdot y \), as required.
Finally, that the $\mathbf{R}$-action is compatible with the $S^1$-action, as stated in the Theorem, follows from the fact that $\psi$ is a bundle map. □

Proof of (2). In addition to making use of the equivalence (of bicategories) between equivariant DD-bundles and equivariant differential characters of degree 3 [12], this proof also makes use of the degree 2 version shown in [16] for principal $S^1$-bundles (the reader is referred there for details). Briefly, we recall that for a Lie groupoid $H_1 \rightrightarrows H_0$, the category of $H$-equivariant $S^1$-bundles is equivalent to the category $\mathcal{DC}^2_1(H)$ of $H$-equivariant differential characters of degree 2. An object in $\mathcal{DC}^2_1(H)$ is a pair $(z, [y])$ consisting of a differential character (i.e. cocycle) $z \in \mathcal{DC}^2_1(H_0)$ and a 1-isomorphism $[y] : \partial^*_0 z \rightarrow \partial^*_1 z$ (i.e. $y \in \mathcal{DC}^1_1(H_1)$ is a primitive of $\partial^*_1 z - \partial^*_0 z$). (In this case, two 1-isomorphisms whose difference is exact are considered equal.) Moreover, $y$ must satisfy a coherence condition that $\partial y$ is exact.

Let $\zeta = (\zeta_0, \zeta_1, [\zeta_2]) = ((c_0, h_0, \eta), (c_1, h_1, \omega), [c_2, h_2, 0])$ be a differential character (of degree 3) corresponding to $(A, E, \tau)$, and let $((b_0, g_0, \beta_0), [b_1, g_1, 0]) : 0 \rightarrow \epsilon_G \zeta$ be a trivialization. Note that one of the defining relations of this trivialization is that $\eta = d\beta_0$.

Let $((B_0, F_0, \omega_X), [B_1, F_1, 0]) : 0 \rightarrow \Phi^* \zeta$ be a prequantization. Then it is straightforward to verify that the principal $S^1$-bundle $\pi : R \rightarrow G_1$ may be represented by the differential character of degree 2 (also denoted by $R$),

$$R = (\partial b_0 + c_1, \partial g_0 + h_1, \partial \beta_0 + \omega),$$

and that the $S^1$-bundle $p : L \rightarrow X$ may be represented by the differential character of degree 2 in $\mathcal{DC}^2_1(X)$ (also denoted by $L$),

$$L = (\Phi^* b_0 - B_0, \Phi^* g_0 - F_0, \Phi^* \beta_0 - \omega_X).$$

This immediately gives the non-equivariant curvature class of $L$ as $[\Phi^* \beta_0 - \omega_X]$.

Next we will find an equivariant extension for $L$ in $\mathcal{DC}^2_1(\mathbf{R} \ltimes X)$. That is, we will find a 1-isomorphism $[y] : \partial^*_0 L \rightarrow \partial^*_1 L$ (so that $\partial L = dy$) satisfying the coherence condition that $\partial y$ is exact. This will exhibit the desired object $(L, [y])$ in $\mathcal{DC}^2_1(\mathbf{R} \ltimes X)$.

To introduce some notation, consider the commutative diagram of $S^1$-central extensions below.

$$
\begin{array}{ccc}
R \times_{G_0} X & \overset{\Phi}{\longrightarrow} & R \\
\downarrow{\pi'} & & \downarrow{\pi} \\
G_1 \times_{G_0} X & \overset{\Phi_1}{\longrightarrow} & G_1 \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
X & \overset{\Phi}{\longrightarrow} & G_0
\end{array}
$$
Since
\[
\partial_{R \ltimes X} L = (\pi')^* \partial_{G \ltimes X} L = (\pi')^*(\Phi_1 R + d(B_1, F_1, 0)) = \hat{\Phi}^*(\pi^* R) + (\pi')^*(d(B_1, F_1, 0)),
\]
it would suffice to find a primitive \( r \) with \( \pi^* R = dr \) such that \( \partial_{R \ltimes X} (\hat{\Phi}^* r + (\pi')^*(B_1, F_1, 0)) \) is exact. (In this case, we set \( y = \hat{\Phi}^* r + (\pi')^*(B_1, F_1, 0) \) to get the data needed for the desired equivariant extension \((L, [y])\).) We verify below that such a primitive exists.

Since the pullback \( S^1 \)-central extension \( \pi^* R \to R \Rightarrow G_0 \) is canonically trivial, it follows that \( \pi^* \zeta \) is canonically trivial as well. That is, there exists a canonical \( 1 \)-isomorphism \((u, [v]) : 0 \to \pi^* \zeta\), where
\[
- \pi^* \zeta_0 = du, \quad -\pi^* \zeta_1 - \partial_R u = dv, \quad \text{and} \quad -\pi^* \zeta_2 - \partial_R v \text{ is exact}. \tag{4.5}
\]
To be more precise, the trivialization \( \sigma = \sigma_R = (u, [v]) \) above satisfies the following property. Given a groupoid morphism \( f : H \to G \), let \( f^* R \twoheadrightarrow H_1 \Rightarrow H_0 \) be the pullback \( S^1 \)-central extension,

\[
\begin{array}{ccc}
f^* R & \xrightarrow{j} & R \\
p & & \pi \\
\downarrow & & \downarrow \\
H_1 & \xrightarrow{f} & G_1 \\
\downarrow & & \downarrow \\
H_0 & \xrightarrow{f_0} & G_0
\end{array}
\]

and consider the pullback central extension \( \hat{f}^*(\pi^* R) \cong p^*(f^* R) \to f^* R \Rightarrow H_0 \),

\[
\begin{array}{ccc}
\hat{f}^*(\pi^* R) & \xrightarrow{j} & \pi^* R \\
\downarrow & & \downarrow \\
f^* R & \xrightarrow{f} & R \\
\downarrow & & \downarrow \\
H_0 & \xrightarrow{f_0} & G_0
\end{array}
\]

Viewing trivializations of \( S^1 \)-central extensions as sections, let \( \sigma : R \to \pi^* R \) be a section of \( \pi \). Then the sections

\[
\sigma_{f^* R}, \hat{f}^* \sigma_R : f^* R \to \hat{f}^*(\pi^* R)
\]
agree (up to natural 2-isomorphism).

Taking \( f = \epsilon_G \), it follows that \( \epsilon^*_G(u, [v]) = (u, [\epsilon^*_G v]) \) and \(((b_0, g_0, \beta_0), [b_1, g_1, 0])\) are 2-isomorphic. In particular, \( u = (b_0, g_0, \beta_0) - dw \), for some \( w \) in \( DC^1_1(G_0) \). Together with \((4.5)\), a straightforward calculation verifies that \( r = \partial_R w - v \) will suffice. Letting \( \theta \)
be the component of \( r \in DC^1_1(R) \) in the \( \Omega^1(R) \) summand, we find that the equivariant curvature class is as stated in the Theorem. \( \square \)

As a special case of Theorem 4.7 above, we briefly consider the case of ordinary Hamiltonian \( G \)-spaces. (See also [14, Example 4.3].)

**Example 4.8.** Let \( (M, \omega) \) be a symplectic manifold equipped with a Hamiltonian \( G \)-action with \( G \)-equivariant moment map \( \Phi : M \to \mathfrak{g}^* \), where \( G \) is a compact Lie group. In other words, \( (M, \omega, \Phi) \) is a Hamiltonian \( T^*G \)-space, for the symplectic groupoid \( (T^*G \rightrightarrows \mathfrak{g}^*, -\alpha) \), where \( \alpha \) is the canonical 1-form on the cotangent bundle (which satisfies \( \partial \alpha = 0 \)). Additionally, suppose \( (M, \omega, \Phi) \) admits a prequantization. Since \( [-d\alpha] = [-\delta\alpha] = 0 \) in \( H^3(T^*G \rightrightarrows \mathfrak{g}^*; \mathbb{R}) \), we may choose the trivial \( S^1 \)-central extension \( R = T^*G \times S^1 \to T^*G \rightrightarrows \mathfrak{g}^* \) in Theorem 4.7. (In the notation of the statement of the theorem, we have \( \beta_0 = 0 \) and \( \theta = \pi^*\alpha \).) Moreover, the \( R \)-equivariant \( S^1 \)-bundle \( p : L \to M \) has (real) curvature class \( [-\omega] \in H^2(M; \mathbb{R}) \)—i.e. \( L \) is a prequantum circle bundle—and its corresponding \( R \)-equivariant extension is \( [-\omega \oplus \Phi^*\pi^*\alpha] \in H^2(R \ltimes M; \mathbb{R}) \).

Since \( R \) is a trivial \( S^1 \)-central extension, we may view \( L \) as a \( G \)-equivariant \( S^1 \)-bundle. Hence, the \( G \)-equivariant curvature is \( [-\omega \oplus \Phi^*\alpha] \) in \( H^2(G \ltimes M; \mathbb{R}) \), which corresponds to \( [-\omega + \Phi] \in H^2_G(M; \mathbb{R}) \) in the Cartan model for \( G \)-equivariant de Rham cohomology. \( \diamond \)

### 5. Morita invariance

This section considers the compatibility of Definition 4.1 with Morita equivalence, and establishes a Morita invariance property for prequantization in Theorem 5.5.

#### 5.1. Morita equivalence of quasi-presymplectic groupoids

Recall that a morphism of Lie groupoids \( F : G \to H \) is a (weak) equivalence if the map

\[
t \circ \text{pr}_1 : H_1 \times_{H_0} G_0 \to H_0
\]

(defined on pairs \((h, x)\) satisfying \( s(h) = F_0(x) \)) is a surjective submersion and if the commutative square below is cartesian:

\[
\begin{array}{ccc}
G_1 & \xrightarrow{(s,t)} & G_0 \times G_0 \\
\downarrow F_1 & & \downarrow F_0 \times F_0 \\
H_1 & \xrightarrow{(s,t)} & H_0 \times H_0
\end{array}
\]

Also, recall that Lie groupoids \( G \) and \( H \) are Morita equivalent if there exists a Lie groupoid \( K \) and a pair of equivalences \( G \xrightarrow{\lambda} K \xrightarrow{\rho} H \). In this case, we shall refer to the pair \((\lambda, \rho)\) as a Morita equivalence.
Definition 5.1. Let \((G, \omega \oplus \eta)\) and \((H, \nu \oplus \chi)\) be quasi-presymplectic groupoids. A Morita equivalence \(G \xleftarrow{\lambda} K \xrightarrow{\rho} H\) is an equivalence of quasi-presymplectic groupoids if the equality \([\lambda^*(\omega \oplus \eta)] = [ho^*(\nu \oplus \chi)]\) holds in \(H^3(K; \mathbb{R})\).

Remark 5.2. If \(G \xleftarrow{\lambda} K \xrightarrow{\rho} H\) is an equivalence of proper quasi-presymplectic groupoids \((G, \omega \oplus \eta)\) and \((H, \nu \oplus \chi)\), then one may choose a primitive \(\alpha \in \Omega^2(K_0)\) with

\[
\rho^*(\nu \oplus \chi) - \lambda^*(\omega \oplus \eta) = \delta\alpha. \tag{5.1}
\]

That is, \(K\) may be viewed as a quasi-presymplectic groupoid in two ways, \((K, \lambda^*(\omega \oplus \eta))\) and \((K, \rho^*(\nu \oplus \chi))\), which differ by a gauge transformation \([32]\).

Recall that two Morita equivalences \(G \xleftarrow{\lambda} K \xrightarrow{\rho} H\) and \(G \xleftarrow{\lambda'} K' \xrightarrow{\rho'} H\) are 2-isomorphic if there exists a Morita equivalence \(K \xleftarrow{\mu} L \xrightarrow{\nu} K'\) such that the following diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\lambda} & K \\
\downarrow{\lambda'} & & \downarrow{\lambda} \\
L & \xleftarrow{\mu} & K' \\
\downarrow{\nu} & & \downarrow{\rho'} \\
H & \xrightarrow{\rho} & H
\end{array}
\]

Hence, if \(G \xleftarrow{\lambda} K \xrightarrow{\rho} H\) is a Morita equivalence of quasi-presymplectic groupoids \((G, \omega \oplus \eta)\) and \((H, \nu \oplus \chi)\), then so is \(G \xleftarrow{\lambda'} K' \xrightarrow{\rho'} H\). In particular, if \((G, \omega \oplus \eta)\) and \((H, \nu \oplus \chi)\) are Morita equivalent quasi-presymplectic groupoids, there exists a Morita equivalence (of quasi-presymplectic groupoids) \(G \xleftarrow{\lambda} K \xrightarrow{\rho} H\), where the object maps \(\lambda_0 : K_0 \to G_0\) and \(\rho_0 : K_0 \to H_0\) are surjective submersions (e.g. see [17, Remark 1.38]).

5.2. Related Hamiltonian spaces. Recall from [32, Theorem 4.19], that Morita equivalent quasi-presymplectic groupoids \((G, \omega \oplus \eta)\) and \((H, \nu \oplus \chi)\) give rise to equivalent theories of Hamiltonian actions. That is, given a Hamiltonian \(G\)-space \((X, \omega_X, \Phi)\) one may construct a corresponding Hamiltonian \(H\)-space \((M, \omega_M, \mu)\) and vice versa. In Theorem 5.5 below, we will show that prequantization respects this correspondence.

To begin, we recall some aspects of the correspondence in [32], in a special case which will suffice in our setting. (We refer the reader to [32, Proposition 4.23] for details.) Let \((G, \omega \oplus \eta)\) and \((H, \nu \oplus \chi)\) be proper quasi-presymplectic groupoids, and let \(F : G \to H\) be an equivalence, viewed as an equivalence of quasi-presymplectic groupoids (i.e. with \(\lambda = \text{id}, \rho = F\)). Suppose in addition that \(F_0 : G_0 \to H_0\) is a
surjective submersion. Let $\alpha \in \Omega^2(G_0)$ satisfy

$$F^*(\nu \oplus \chi) - \omega \oplus \eta = \delta \alpha$$

as in Remark 5.2.

Let $(M, \omega_M, \mu)$ be a pre-Hamiltonian $H$-space. Then the corresponding $G$-space $(X, \omega_X, \Phi)$ is obtained by setting

$$X = G_0 \times_{H_0} M,$$

with natural $G$-action,

$$g \cdot (x, m) = (t(g), F_1(g) \cdot m)$$

for $g \in G_1$ satisfying $s(g) = x$, and moment map $\Phi : X \to G_0$ given by projection to the first factor.

Conversely, $(M, \omega_M, \mu)$ is obtained from $(X, \omega_X, \Phi)$ by setting $M$ to be the quotient

$$M = X/G_*$$

where $G_* \rightrightarrows G_0$ is the subgroupoid $G_* \subset G_1$ consisting of all arrows $g \in G_1$ such that $F_1(g) \in H_1$ is an identity map (i.e. in the image of the unit $e_H$). The moment map $\mu : M \to H_0$ is induced from the moment map $\Phi$, $\mu([p]) = F_0(\Phi(p))$. The $H$-action can be described as follows. For a $G_*$-orbit $[p]$ and $h \in H_1$ with $s(h) = \mu([p])$, choose $y \in G_0$ with $F_0(y) = t(h)$. Then there exists a unique $g \in G_1$ with $F_1(g) = h$, and we set $h \cdot [p] = [g \cdot p]$.

The corresponding 2-forms on $X$ and $M$ are determined by the relation:

$$f_0^* \omega_M = \omega_X - \Phi^* \alpha, \quad (5.2)$$

where $f_0 : X \to M$ denotes the quotient map.

**Lemma 5.3.** Let $G$ and $H$ be proper Lie groupoids, and let $F : G \to H$ be an equivalence with $F_0$ a surjective submersion. Let $X$ and $M$ be $G$ and $H$-spaces with moment maps $\Phi$ and $\mu$, respectively, obtained from either of the above constructions. Then the natural morphism $G \ltimes X \to H \ltimes M$ is an equivalence.

**Proof.** It is straightforward to verify that the above constructions for $X$ and $M$ are inverse to each other (up to diffeomorphism). Therefore, we may assume $X = G_0 \times_{H_0} M$. Abusing notation, let $f : G \ltimes X \to H \ltimes M$ be the induced map of action groupoids:

$$G_1 \times_{G_0} G_0 \times_{H_0} M \xrightarrow{f_1} H_1 \times_{H_0} M$$

Since $F_0$ is a surjective submersion, so is the pullback $f_0$, and hence so is the map

$$t \circ \text{pr}_1 : (H_1 \times_{H_0} M) \times_M (G_0 \times_{H_0} M) \to M.$$

It remains to verify

$$G_1 \times_{G_0} G_0 \times_{H_0} M \cong (G_0 \times_{H_0} M) \times_M (G_0 \times_{H_0} M) \quad (5.3)$$
To that end, let \((x_1, m_1; x_2, m_2; h, m)\) be an element from the right hand side of (5.3), so that \(\phi(x_i) = \mu(m_i)\) \((i = 1, 2)\), \(m_1 = m\), and \(m_2 = h \cdot m\). Since \(F\) is an equivalence, there is a unique \(g \in G_1\) with \(s(g) = x_1\), \(t(g) = x_2\) and \((g, x_1, m_1)\) defines an arrow on the left side of (5.3). This gives the required identification. \(\square\)

Now suppose that \(G \xleftarrow{\lambda} K \xrightarrow{\rho} H\) is a Morita equivalence of the proper quasi-symplectic groupoids \((G, \omega \oplus \eta)\) and \((H, \nu \oplus \chi)\), where \(\lambda_0\) and \(\rho_0\) are surjective submersions. Let \(\alpha \in \Omega^2(K_0)\) satisfy (5.1). We may iterate the construction above to associate a Hamiltonian \(H\)-space to a Hamiltonian \(G\)-space via the corresponding \(K\)-space. Indeed, let \((X, \omega_X, \Phi)\) be a pre-Hamiltonian \(G\)-space. Using the above construction, we may form a Hamiltonian \(K\)-space \((Z, \omega_Z, \Psi)\) for the quasi-presymplectic groupoid \((K, \rho^*(\nu \oplus \chi))\). It is straightforward to verify that \((Z, \omega_Z + \Psi^*\alpha, \Psi)\) is a Hamiltonian \(K\)-space for the quasi-presymplectic groupoid \((K, \lambda^*(\omega \oplus \eta))\). Hence we may form the Hamiltonian \(H\)-space \((M, \omega_M, \mu)\). Using (5.2), it follows that the 2-forms \(\omega_X\), \(\omega_M\), and \(\omega_Z\) satisfy:

\[
\ell^*\omega_X = \omega_Z + \Psi^*\alpha \quad \text{and} \quad r^*\omega_M = \omega_Z,
\]

where \(\ell : Z \to X\) and \(r : Z \to M\) denote the natural quotient maps arising in the construction.

A straightforward application of Lemma 5.3 gives the following Proposition.

**Proposition 5.4.** Let \(G\) and \(H\) be proper Lie groupoids and suppose \(G \xleftarrow{\lambda} K \xrightarrow{\rho} H\) is a Morita equivalence. Let \(X, Z\) and \(M\) be \(G, K\) and \(H\)-spaces with moment maps \(\Phi\), \(\Psi\), and \(\mu\), respectively, as in the above construction. Then the natural morphisms \(G \ltimes X \xleftarrow{} K \ltimes Z \xrightarrow{} H \ltimes M\) provide a Morita equivalence of the corresponding action groupoids.

We are now ready to establish the main result of this section, showing that prequantization respects related Hamiltonian spaces.

**Theorem 5.5.** Let \((G, \omega \oplus \eta)\) and \((H, \nu \oplus \chi)\) be proper quasi-presymplectic groupoids, and let \(G \xleftarrow{\lambda} K \xrightarrow{\rho} H\) be an equivalence of quasi-presymplectic groupoids where \(\lambda_0 : K_0 \to G_0\) and \(\rho_0 : K_0 \to H_0\) are surjective submersions. Let \((X, \omega_X, \Phi)\) and \((M, \omega_M, \mu)\) be Hamiltonian \(G\) and \(H\)-spaces, respectively, as in the discussion above. Then \((X, \omega_X, \Phi)\) admits a prequantization if and only if \((M, \omega_M, \mu)\) does.

**Proof.** Let \(Z\) be the intermediate Hamiltonian \(K\)-space with moment map \(\Psi\), as in the discussion preceding Proposition 5.4. By Proposition 5.4 the following diagram
of Lie groupoids commutes,

\[
\begin{array}{ccc}
H \times Z & \xrightarrow{\mu} & K \\
\downarrow r & & \downarrow \rho \\
K \times Z & \xrightarrow{\Psi} & K \\
\downarrow \ell & & \downarrow \lambda \\
G \times X & \xrightarrow{\Phi} & G
\end{array}
\]

where the vertical maps are equivalences.

Being equivalences, the vertical maps induce isomorphisms on cohomology \(H^*(-)\) (with any coefficients). The two commutative squares in the above diagram induces maps on relative cohomology (with any coefficients) \(H^3(\mu) \to H^3(\Psi) \leftarrow H^3(\Phi)\), which, by easy applications of the 5-Lemma, are isomorphisms. By Proposition 4.4, it suffices to check that \([(\omega_X, \omega \oplus \eta)] \in H^3(\Phi; \mathbb{R})\) is integral if and only if \([(\omega_M, \nu \oplus \chi)] \in H^3(\mu; \mathbb{R})\) is integral. Using the above pair of isomorphisms induced from the commutative diagram, it then suffices to check that \((\ell, \lambda)^*[\omega_X, \omega \oplus \eta] = (r, \rho)^*[\omega_M, \nu \oplus \chi]\).

Indeed, we have

\[
(\ell, \lambda)^*[\omega_X, \omega \oplus \eta] = (r^*\omega_M + \Psi^*\alpha, \rho^*(\nu \oplus \chi) - \delta \alpha) = (r^*\omega_M, \rho^*(\nu \oplus \chi)) + \delta(0, \alpha).
\]

\[\square\]

Hamiltonian loop group actions and quasi-Hamiltonian spaces. For the remainder of this section, let \(G\) be a compact Lie group with bi-invariant inner product \(\langle \cdot, \cdot \rangle\) on \(g\), and \(LG = \text{Map}(S^1, G)\) as in Example 2.9.

As an application of Theorem 5.5 to the equivalence (see [3, Theorem 8.3] and [32, Corollary 4.28]) between Hamiltonian \(LG\)-actions with proper moment map and quasi-Hamiltonian \(G\)-actions with group-valued moment map, we obtain the following corollary (cf. [11, Theorem A.7] which assumes \(G\) is simply connected).

**Corollary 5.6.** Let \((X, \omega_X, \Phi)\) be a pre-Hamiltonian \(LG\)-space, and \((M, \omega_M, \mu)\) its corresponding quasi-Hamiltonian \(G\)-space. Then \((X, \omega_X, \Phi)\) admits a prequantization if and only if \((M, \omega_M, \mu)\) does.

A necessary condition (and ingredient) for the existence of a prequantization is the existence of \(G\)-equivariant DD-bundle representing the quasi-presymplectic structure \(\omega \oplus \eta\). For the case of the symplectic groupoid \(LG \times Lg^* \rightrightarrows Lg^*\) this amounts to a central \(S^1\)-extension (by Theorem 4.7 (1)). And by [3] Theorem 3.3, this central \(S^1\)-extension of groupoids corresponds to a central extension of Lie groups,

\[
1 \to S^1 \to \hat{LG} \to LG \to 1.
\]

For simple \(G\), such central extensions are classified (see [25] and [13]), and a necessary condition for the existence of such a central extension is that the inner product \(\langle \cdot, \cdot \rangle\)
be a multiple of $l_b \cdot B$, where $B$ denotes the basic inner product on $\mathfrak{g}$, and $l_b$ is the basic level. Recall $B$ is the invariant inner product on $\mathfrak{g}$ normalized to make short co-roots have squared length 2, and the integer $l_b$ is the smallest integer $l$ such that $l \cdot B(\lambda_1, \lambda_2) \in \mathbb{Z}$ for all elements $\lambda_1, \lambda_2$ of the integral lattice $\Lambda \subset \mathfrak{t}$ (t is the Lie algebra of a maximal torus $T \subset G$). The integers $l_b$ are computed in [13] for each compact simple Lie group $G$ (e.g. if $G$ is simply connected, $l_b = 1$). Therefore, by Corollary 5.6 we obtain:

**Corollary 5.7.** Let $G$ be a compact simple Lie group and let $l \cdot B$ be a positive integer multiple of the basic inner product on $\mathfrak{g}$. If the quasi-Hamiltonian $G$-space $(M, \omega_M, \mu)$ admits a prequantization, then $l$ is a multiple of the basic level $l_b$.

6. SYMPLECTIC QUOTIENTS

In this section, show how the definition of prequantization is compatible with symplectic quotients.

Let $(G, \omega \oplus \eta)$ be a proper quasi-presymplectic groupoid, and let $(X, \omega_X, \Phi)$ be a pre-Hamiltonian $G$-space. Suppose that $z \in G_0$ is a regular value for $\Phi : X \to G_0$. Then the $G$-action on $X$ restricts to a $G_1(z)$-action on the level set manifold $\Phi^{-1}(z) \subset X$, where $G_1(z) = \{g \in G_1 | s(g) = z = t(g)\}$ denotes the isotropy group of $z$. Below, we show how a prequantization for $(X, \omega_X, \Phi)$ pulls back to a $G_1(z)$-equivariant $S^1$-bundle over $\Phi^{-1}(z)$. When the $G_1(z)$-action on the level set is free, we may view this as a prequantization of the resulting symplectic quotient $\Phi^{-1}(z)/G_1(z)$ (see [32, Theorem 3.18]).

**Theorem 6.1.** Let $(G, \omega \oplus \eta)$ be a proper quasi-presymplectic groupoid, and let $(X, \omega_X, \Phi)$ be a pre-Hamiltonian $G$-space. Suppose that $z \in G_0$ is a regular value for $\Phi : X \to G_0$ and let $j : \Phi^{-1}(z) \hookrightarrow X$ denote the inclusion map. A prequantization of $(X, \omega_X, \Phi)$ gives rise to a $G_1(z)$-equivariant $S^1$-bundle over $\Phi^{-1}(z)$ whose real $G_1(z)$-equivariant curvature class is $[j^*\omega_X] \in H^2(G_1(z) \ltimes \Phi^{-1}(z); \mathbb{R})$.

**Proof.** As in the proof of Theorem 4.17 (2), we shall use the framework of equivariant differential characters. Let $\zeta = (\zeta_0, \zeta_1, [\zeta_2]) = ((c_0, h_0, \eta), (c_1, h_1, \omega), [c_2, h_2, 0])$ be an equivariant differential character (of degree 3) and let $((B_0, F_0, \omega_X), [B_1, F_1, 0]) : 0 \to \Phi^*\zeta$ be a prequantization. Since the composition $\Phi \circ j$ is constant, then $j^*((B_0, F_0, \omega_X), [B_1, F_1, 0])$ defines an equivariant differential character of degree 2. Indeed, a straightforward verification shows $(j^*B_0, j^*F_0, j^*\omega_X)$ is cocycle in $DC_1^2(\Phi^{-1}(z))$ and that $[j^*B_1, j^*F_1, 0]$ defines the required 1-isomorphism $\partial_0((j^*B_0, j^*F_0, j^*\omega_X) \to \partial_1((j^*B_0, j^*F_0, j^*\omega_X)$ to give an equivariant differential character of degree 2. It follows that $[j^*\omega_X]$ in $H^2(G_1(z) \ltimes \Phi^{-1}(z); \mathbb{R})$ is the resulting $G_1(z)$-equivariant curvature class. □

**Remark 6.2.** The proof of Theorem 6.1 shows how one can describe the resulting $G_1(z)$-equivariant $S^1$-bundle with curvature class $j^*\omega$ in bundle-theoretic terms.
On the level set $\Phi^{-1}(z)$, there are two Morita trivializations of the equivariant $DD$-bundle $j^*\Phi^* (\mathcal{A}, \mathcal{E}, \tau)$: one resulting from the pullback of the prequantization $(\mathcal{F}, \alpha) : (\mathbb{C}, \mathbb{C}, \text{id}) \to \Phi^*(\mathcal{A}, \mathcal{E}, \tau)$ along the inclusion $j$, and the canonical trivialization coming from the fact that the composition $\Phi \circ j$ is trivial. Their difference defines a line bundle (or equivalently its associated $S^1$-bundle) over $\Phi^{-1}(z)$.

**Remark 6.3.** Under the conditions of Theorem 6.1, if we assume additionally that $G_1(z)$ acts freely on the level set $\Phi^{-1}(z)$, we see that the $G_1(z)$-equivariant $S^1$-bundle obtained in the Theorem descends to a prequantum $S^1$-bundle on the symplectic quotient $\Phi^{-1}(z)/G_1(z)$.

**References**

[1] A. Alekseev and E. Meinrenken. Dirac structures and dixmier–douady bundles. *International Mathematics Research Notices*, 2012(4):904–956, 2012.

[2] Anton Alekseev and Yvette Kosmann-Schwarzbach. Manin pairs and moment maps. *Journal of Differential Geometry*, 56(1):133–165, 2000.

[3] Anton Alekseev, Anton Malkin, and Eckhard Meinrenken. Lie group valued moment maps. *Journal of Differential Geometry*, 48(3):445–495, 1998.

[4] John C. Baez and Urs Schreiber. Higher gauge theory. *Contemporary Mathematics*, 431:7–30, 2007.

[5] Kai Behrend and Ping Xu. Differentiable stacks and gerbes. *Journal of Symplectic Geometry*, 9(3):285–341, 2011.

[6] Kai Behrend, Ping Xu, and Bin Zhang. Equivariant gerbes over compact simple lie groups. *Comptes rendus. Mathématique*, 336(3):251–256, 2003.

[7] Henrique Bursztyn and Marius Crainic. Dirac geometry, quasi-poisson actions and $d/g$-valued moment maps. *Journal of Differential Geometry*, 82(3):501–566, 2009.

[8] Henrique Bursztyn, Marius Crainic, Alan Weinstein, and Chenchang Zhu. Integration of twisted dirac brackets. *Duke Mathematical Journal*, 123(3):549–607, 2004.

[9] Alberto S. Cattaneo and Ping Xu. Integration of twisted poisson structures. *Journal of Geometry and Physics*, 49(2):187–196, 2004.

[10] M.J. Hopkins and I.M. Singer. Quadratic functions in geometry, topology, and $M$-theory. *Journal of Differential Geometry*, 70(3):329–452, 2005.

[11] Derek Krepski. Pre-quantization of the moduli space of flat $G$-bundles over a surface. *Journal of Geometry and Physics*, 58(11):1624–1637, 2008.

[12] Derek Krepski and Jordan Watts. Differential characters and Dixmier-Douady bundles (in preparation). 2016.

[13] Valerio Toledano Laredo. Positive energy representations of the loop groups of non-simply connected lie groups. *Communications in mathematical physics*, 207(2):307–339, 1999.

[14] Camille Laurent-Gengoux and Ping Xu. Quantization of pre-quasi-symplectic groupoids and their Hamiltonian spaces, pages 423–454. Birkhäuser Boston, Boston, MA, 2005.

[15] Matthew Leingang. Symmetric space valued moment maps. *Pacific journal of mathematics*, 212(1):103–123, 2003.

[16] Eugene Lerman and Anton Malkin. Differential characters as stacks and prequantization. *Journal of Gökova Geometry Topology*, 2:14–39, 2008.

[17] Du Li. Higher groupoid actions, bibundles, and differentiation. arXiv:1512.04209 2015.

[18] Jiang-Hua Lu. Momentum mappings and reduction of poisson actions. In *Symplectic geometry, groupoids, and integrable systems*, pages 209–226. Springer, 1991.

[19] Jiang-Hua Lu and Alan Weinstein. Poisson lie groups, dressing transformations, and bruhat decompositions. *Journal of Differential Geometry*, 31(2):501–526, 1990.
[20] Eckhard Meinrenken. Twisted $K$-homology and group-valued moment maps. *International Mathematics Research Notices*, 2012(20):4563–4618, 2012.

[21] Eckhard Meinrenken. Convexity for twisted conjugation. arXiv:1512.09000, 2015.

[22] Eckhard Meinrenken and Chris Woodward. Hamiltonian loop group actions and verlinde factorization. *Journal of Differential Geometry*, 50(3):417–469, 1998.

[23] Michael K. Murray. Bundle gerbes. *Journal of the London Mathematical Society*, 54(2):403–416, 1996.

[24] Thomas Nikolaus and Christoph Schweigert. Equivariance in higher geometry. *Advances in Mathematics*, 226(4):3367–3408, 2011.

[25] Andrew Presley and Graeme Segal. Loop groups. *Oxford Math. Monogr.*, Calderon Press, Oxford, 1986.

[26] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace C*-algebras*. Number 60. American Mathematical Soc., 1998.

[27] Jean-Louis Tu, Ping Xu, and Camille Laurent-Gengoux. Twisted $K$-theory of differentiable stacks. In *Annales scientifiques de l’Ecole normale supérieure*, volume 37, pages 841–910, 2004.

[28] Charles A. Weibel. *An introduction to homological algebra*. Number 38. Cambridge University Press, 1995.

[29] Alan Weinstein. Symplectic groupoids and poisson manifolds. *Bulletin of the American mathematical Society*, 16(1):101–104, 1987.

[30] Alan Weinstein and Ping Xu. Extensions of symplectic groupoids and quantization. *J. reine angew. Math*, 417:159–189, 1991.

[31] Ping Xu. Classical intertwiner space and quantization. *Communications in mathematical physics*, 164(3):473–488, 1994.

[32] Ping Xu. Momentum maps and morita equivalence. *Journal of Differential Geometry*, 67(2):289–333, 2004.

**Department of Mathematics, University of Manitoba, Winnipeg, MB, Canada**

*E-mail address:* Derek.Krepski@umanitoba.ca

*URL:* http://server.math.umanitoba.ca/~dkrepski/