SYNCHRONIZATION BY NOISE

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Abstract. We provide sufficient conditions for synchronization by noise, i.e., under these conditions we prove that weak random attractors for random dynamical systems consist of single random points. In the case of SDE with additive noise, these conditions are also essentially necessary. In addition, we provide sufficient conditions for the existence of a minimal weak point random attractor consisting of a single random point. As a result, synchronization by noise is proven for a large class of SDE with additive noise. In particular, we prove that the random attractor for an SDE with drift given by a (multidimensional) double-well potential and additive noise consists of a single random point. All examples treated in [41] are also included.

1. Introduction

In this paper we introduce new, checkable conditions for synchronization by noise for general white noise, random dynamical systems (RDS) $\varphi$ on complete, separable metric spaces $E$. Here, synchronization by noise means that the (weak) random attractor $A$ for $\varphi$ consists of a single random point, i.e. $A(\omega) = \{a(\omega)\}$ a.s. and thus the long-time dynamics are asymptotically globally stable. In particular, for each $x, y \in E$ it follows that

$$\lim_{t \to \infty} d(\varphi_t(\omega, x), \varphi_t(\omega, y)) = 0$$

in probability.

We are especially interested in SDE with additive noise

$$dX_t = b(X_t)dt + \sigma dW_t \quad \text{on } \mathbb{R}^d$$  \hspace{1cm} (1.1)$$

with $\sigma > 0$, for choices of $b$ such that the deterministic dynamics corresponding to $\sigma = 0$ are not asymptotically globally stable. We provide general conditions on the...
coefficients $b, \sigma$ that lead to synchronization by noise. Hence, in these cases the inclusion of additive noise in (1.1) stabilizes the long-time dynamics.

As a model example, one may consider the multidimensional double-well potential with additive noise, that is

$$dX_t = (X_t - |X_t|^2 X_t) \, dt + \sigma dW_t \quad \text{on } \mathbb{R}^d. \tag{1.2}$$

In this case, for $\sigma = 0$ the long-time dynamics are not asymptotically globally stable, but the attractor is given by the closed unit ball $\bar{B}(0,1)$. We shall also analyze the associated point attractor, which consists of all invariant points, i.e. $S^{d-1} \cup \{0\}$, where $S^{d-1}$ is the $(d-1)$-dimensional unit sphere. It follows from the general conditions developed in this paper, that for $\sigma > 0$ synchronization occurs, that is, the random attractor collapses into a single (random) point.

In the first part of this paper (Section 2.1), we identify general and new sufficient conditions for synchronization by noise. In the case of SDE driven by additive noise these conditions are essentially sharp, i.e. sufficient and necessary. More precisely, we show that asymptotic stability (a local stability condition), swift transitivity (an irreducibility condition) and contraction on large sets imply synchronization by noise. If $E$ is locally compact, then asymptotic stability and contraction on large sets are also necessary conditions. Moreover, swift transitivity is satisfied by SDE of the type (1.1) with locally Lipschitz drift satisfying a one-sided Lipschitz condition. In particular, this proves synchronization for (1.2).

Although contraction on large sets is a necessary condition for synchronization, it is not always easy to check for SDE. In Section 4 for (1.1) we prove that $b$ being monotone on large sets (cf. Proposition 4.9) implies contraction on large sets. However, monotonicity on large sets is not necessary for synchronization. Therefore, in the second part (Section 2.2), we concentrate on a weaker concept of synchronization, so-called weak synchronization. Weak synchronization means that there is a minimal weak point attractor $A$ consisting of a single random point. The main improvement is that we are able to prove weak synchronization without assuming contraction on large sets, which in turn allows us to consider drifts $b$ not necessarily monotone on large sets. More precisely, for strongly mixing, white noise RDS we prove that weak asymptotic stability (a pointwise local stability condition), pointwise strong swift transitivity and a pointwise stability condition imply weak synchronization. Again, weak asymptotic stability and the pointwise stability condition are also necessary for weak synchronization, while pointwise strong swift transitivity is easily checked for (1.1) under mild conditions as above. The proof of weak synchronization is based on an analysis of the support properties of the statistical equilibrium, which leads us to (partial) generalizations of results developed in [26,27].
Our results on weak synchronization are particularly complete in the case of
gradient-type SDE, i.e. for
\[ dX_t = -\nabla V(X_t)dt + \sigma dW_t \quad \text{on} \quad \mathbb{R}^d, \quad (1.3) \]
with \( V \in C^2(\mathbb{R}^d, \mathbb{R}) \), \( \sigma > 0 \) and \( b := -\nabla V \) satisfying a one-sided Lipschitz condition.
Assuming weak asymptotic stability and \( \rho(x) := e^{-\frac{\sigma^2}{2}V(x)} \in L^1(\mathbb{R}^d) \) we prove weak
synchronization for (1.3). Note that no contraction on large sets, or monotonicity
on large sets has to be assumed.

In the final Section 4, we consider SDE of the type (1.1) and provide sufficient
conditions in terms of the coefficients \( b, \sigma \) for asymptotic stability, swift transitivity
and contraction on large sets and, thus, synchronization by noise. In particular, we
establish a discrete-in-time local stable manifold theorem and prove that a negative
top Lyapunov exponent implies asymptotic stability. We then provide conditions
on \( b, \sigma \) leading to a negative top Lyapunov exponent.

Let us now comment on the existing literature. There are several distinct ap-
proaches to synchronization by noise to be found in the literature. We distinguish
three main types of arguments (without aiming for completeness here): Order-
preserving RDS, Local stability and transitivity of the two point motion, pertur-
bation techniques based on large deviation results.

Synchronization by noise for order-preserving, strongly mixing RDS \( \varphi \) has been
analyzed, for example, in [2, 5, 9, 10, 19] and rather general results on (weak) syn-
chronization have been obtained. However, assuming \( \varphi \) to be order-preserving is
a significant restriction, leading to stringend assumptions on the drift \( b \) for (1.1)
in dimensions larger than one (cf. [9]). In particular, our model example (1.2) is
covered for \( d = 1 \) only.

In [4], Baxendale proves synchronization for SDE on manifolds, assuming er-
godicity, local stability, in the sense that the top Lyapunov exponent is supposed
to be negative, and assuming transitivity of the two point motion. Transi-
tivity of the two-point motion implies, in particular, that the two-point motion
\( t \mapsto (\varphi_t(\omega, x), \varphi_t(\omega, y)) \) gets arbitrarily close to the diagonal \( \Delta \subset E \times E \), a.s. for
all \( x, y \in E \). In the case of SDE with additive noise (1.1), transitivity of the two
point motion is not easy to check. Indeed, note that additive noise just shifts the
two-point motion parallel to the diagonal. In particular, it remains unclear how
to make use of this technique in our model case of the double-well potential (1.2).
It is thus one of the aims of this paper to replace the assumption of transitivity
of the two point motion by alternative conditions that are checkable for SDE with
additive noise.

Another approach, based on large deviation techniques, has been introduced
in [29, 30, 11]. Besides several technical assumptions, assuming for (1.1) that \( b \)
has only finitely many fixed points and that \( \sigma \) is small enough, these works prove
synchronization by noise. Again, we note that the model example (1.2) is covered
for \( d = 1 \) only. In contrast, all examples treated in [11] are easily seen to be included in our results.

Synchronization by linear multiplicative noise has been investigated in [30]. For the related effect of synchronization in master-slave systems we refer to [11] and the references therein. Synchronization for discrete time random dynamical systems (iterated function systems) has also been investigated and the recent results are deep and advanced, see [21, 23, 32] and references therein.

Synchronization has been advocated as a relevant property for certain applications. From the theoretical physics literature let us mention [23, 34, 35, 37]. In climate dynamics it has been mentioned as an indication of the possibility to reduce variability of predictions, see [8, 17, 20]. In neurophysiology, synchronous firing of neurons subject to the same input, which may be seen as a dynamical system driven by the same noise path but different initial conditions, is a phenomenon of interest, see [40] and the references therein. Finally, synchronization plays a role in Richardson-Romberg extrapolation numerical method, see [28].

Outline of the paper: In Section 1.1 we introduce some notation and recall some of the fundamentals of the theory of RDS and random attractors. In Section 2 we present the main results in detail, divided in two subsections corresponding to results on synchronization and weak synchronization respectively. The proofs are given in Section 3. In Section 4 we present applications of our main results to SDE.

1.1. Preliminaries and notation. Let \((E, d)\) be a complete separable metric space with Borel \(\sigma\)-algebra \(\mathcal{E}\) and \((\Omega, \mathcal{F}, P, \theta)\) be an ergodic metric dynamical system, i.e. \((\Omega, \mathcal{F}, P)\) is a (not necessarily complete) probability space and \(\theta := (\theta_t)_{t \in \mathbb{R}}\) is a group of jointly measurable maps on \((\Omega, \mathcal{F}, P)\) with ergodic invariant measure \(P\).

Further, let \(\varphi : \mathbb{R}_+ \times \Omega \times E \to E\) be a perfect cocycle: i.e. \(\varphi\) is measurable, \(\varphi_0(\omega, x) = x\) and \(\varphi_{t+s}(\omega, x) = \varphi_t(\theta_s \omega, \varphi_s(\omega, x))\) for all \(x \in E, t, s \geq 0\), \(\omega \in \Omega\). We will assume that \(\varphi_s(\omega, \cdot)\) is continuous for each \(s \geq 0\) and \(\omega \in \Omega\). The collection \((\Omega, \mathcal{F}, P, \theta, \varphi)\) is then called a random dynamical system (in short: RDS), see [11] for a comprehensive treatment.

By definition, \((\Omega, \mathcal{F}, P, \theta, \varphi)\) is a local RDS if \((\Omega, \mathcal{F}, P, \theta)\) is as above and \(\varphi : \mathbb{R}_+ \times \Omega \times \bar{E} \to \bar{E}\) is measurable, where \(\bar{E} := E \cup \{\partial\}\) and \(\partial\) is some adjoined state with the following properties: \(D := \varphi^{-1}(E) \subseteq \mathbb{R}_+ \times \Omega \times E\) and for each \(\omega \in \Omega\) the set \(D(\omega) := \{(t, x) \in \mathbb{R}_+ \times E : (t, \omega, x) \in D\}\) is open, \((t, x) \in D(\omega)\) and \(0 \leq s \leq t\) imply \((s, x) \in D(\omega), x \mapsto \varphi_t(\omega, x)\) is continuous at \(x_0 \in E\) whenever \(\varphi_t(\omega, x_0) \in E\), \(\varphi_0(\omega, \cdot) = \text{Id}\) and \(\varphi\) has the perfect cocycle property (as above). Note that a local RDS is an RDS iff \(D = \mathbb{R}_+ \times \Omega \times E\). Given a (local) RDS \((\Omega, \mathcal{F}, P, \theta, \varphi)\) we may define the skew-product flow \(\Theta\) on \(\Omega \times \bar{E}\) by \(\Theta_t(\omega, x) = (\theta_t \omega, \varphi_t(\omega, x))\). In the following we will often omit the qualifier local. We say that a local RDS is weakly complete if \(\varphi_t(\cdot, x) \in E, P\text{-a.s. for all } t \geq 0, x \in E\).
Since our main applications are RDS generated by SDE driven by Brownian motion, we will assume that the RDS $\varphi$ is suitably adapted to a filtration and is of white noise type. More precisely, we will assume that we have a family $\mathbb{F} = (\mathcal{F}_{s,t})_{-\infty < s \leq t < \infty}$ of sub-$\sigma$ algebras of $\mathcal{F}$ such that $\mathcal{F}_{t,u} \subseteq \mathcal{F}_{s,v}$ whenever $s \leq t \leq u \leq v$, $\theta^{-1}_r(\mathcal{F}_{s,t}) = \mathcal{F}_{s+r,t+r}$ for all $r, s, t$ and $\mathcal{F}_{s,t}$ and $\mathcal{F}_{u,v}$ are independent whenever $s \leq t \leq u \leq v$. For each $t \in \mathbb{R}$, let us denote the smallest $\sigma$-algebra containing all $\mathcal{F}_{s,t}$, $s \leq t$ by $\mathcal{F}_t$ and the smallest $\sigma$-algebra containing all $\mathcal{F}_{t,u}$, $t \leq u$ by $\mathcal{F}_{t,\infty}$. Note that for each $t \in \mathbb{R}$, the $\sigma$-algebras $\mathcal{F}_t$ and $\mathcal{F}_{t,\infty}$ are independent. We will further assume that $\varphi_t(\cdot, x)$ is $\mathcal{F}_{0,s}$-measurable for each $s \geq 0$. The collection $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta, \varphi)$ is then called a white noise (filtered) random dynamical system.

An invariant measure for an RDS $\varphi$ is a probability measure on $\Omega \times E$ with marginal $\mathbb{P}$ on $\Omega$ that is invariant under $\varTheta_t$ for $t \geq 0$. For each probability measure $\mu$ on $\Omega \times E$ with marginal $\mathbb{P}$ on $\Omega$ there is a unique disintegration $\omega \mapsto \mu_\omega$ and $\mu$ is an invariant measure for $\varphi$ iff $\varphi_t(\omega) \mu_\omega = \mu_\vartheta \omega$ for all $t \geq 0$, almost all $\omega \in \Omega$. Here $\varphi_t(\omega) \mu_\omega$ denotes the push-forward of $\mu_\omega$ under $\varphi_t(\omega)$. An invariant measure $\mu_\omega$ is said to be a Markov measure, if $\omega \mapsto \mu_\omega$ is measurable with respect to the past $\mathcal{F}_0$. In case of a weakly complete, white noise RDS $\varphi$ we may define the associated Markovian semigroup by

$$P_t f(x) := \mathbb{E} f(\varphi_t(\cdot, x)),$$

for $f$ being measurable, bounded. There is a one-to-one correspondence between invariant measures for $P_t$ and Markov invariant measures for $\varphi$: If $\mu$ is $P_t$-invariant, then

$$\mu_\omega := \lim_{t \to \infty} \varphi_t(\theta^{-t} \omega) \mu \quad (1.4)$$

exists $\mathbb{P}$-a.s. and defines a Markov invariant measure for $\varphi$. Vice versa, $\mu := \mathbb{E} \mu_\omega$ defines an invariant measure for $P_t$. Note that the proof of these facts given in [12] applies without change to local RDS.

We say that a Markovian semigroup $P_t$ with invariant measure $\mu$ is strongly mixing if

$$P_t f(x) \to \int_E f(y) d\mu(y) \quad \text{for } t \to \infty$$

for each continuous, bounded $f$ and all $x \in E$. Similarly, we say that an RDS $\varphi$ is strongly mixing if the law of $\varphi_t(\cdot, x)$ converges to $\mu$ for $t \to \infty$ for all $x \in E$.

As a notational convention, we let

$$B(x, r) := \{ y \in E : d(x, y) < r \}$$

be the open ball of radius $r$ centered at $x$ and $\overline{B}(x, r)$ the respective closed ball. For a set $A \subseteq E$ we let

$$A^\varepsilon := \{ y \in E : d(y, A) = \inf_{a \in A} d(y, a) < \varepsilon \}$$

and

$$\text{diam}(A) := \sup_{a, b \in A} d(a, b).$$
Definition 1.1. A family \( \{D(\omega)\}_{\omega \in \Omega} \) of non-empty subsets of \( E \) is said to be

1. a random closed (resp. compact) set if it is \( \mathbb{P} \)-a.s. closed (resp. compact)
   and \( \omega \mapsto d(x, D(\omega)) \) is \( \mathcal{F} \)-measurable for each \( x \in E \). In this case we also
   call \( D \), \( \mathcal{F} \)-measurable.

2. \( \varphi \)-invariant, if for all \( t \geq 0 \)
   \[
   \varphi_t(\omega, D(\omega)) = D(\theta_t \omega),
   \]
   for almost all \( \omega \in \Omega \).

Next, we recall the definition of a pullback attractor and a weak (random) attractor (cf. [16, 33]).

Definition 1.2. Let \((\Omega, \mathcal{F}, \mathbb{P}, \theta, \varphi)\) be an RDS. A random, compact set \( A \) is called a pullback attractor, if

1. \( A \) is \( \varphi \)-invariant, and
2. for every compact set \( B \) in \( E \), we have
   \[
   \lim_{t \to \infty} \sup_{x \in B} d(\varphi_t(\theta_t \omega, x), A(\omega)) = 0, \text{ almost surely.}
   \]

The map \( A \) is called a weak attractor, if it satisfies the properties above with almost sure convergence replaced by convergence in probability in (2). It is called a \( \text{(weak)} \) point attractor, if it satisfies the properties above with compact sets \( B \) replaced by single points in (2).

A (weak) point attractor is said to be minimal if it is contained in each (weak) point attractor.

Clearly, every pullback attractor is a weak attractor but the converse is not true (see e.g. [39] for examples).

Lemma 1.3. Weak attractors (and hence pullback attractors) are unique in the sense that if an RDS has two weak attractors, then they agree almost surely.

Proof. Let \( A, \tilde{A} \) be two weak random attractors. Since \( \tilde{A} \) is a random compact set, by [14] Proposition 3.15] for each \( \varepsilon > 0 \) there is a compact, deterministic set \( K_\varepsilon \) and such that

\[
\mathbb{P}[\tilde{A} \subseteq K_\varepsilon] \geq 1 - \varepsilon.
\]

Since \( A \) weakly attracts compact sets, for all \( \delta, \varepsilon > 0 \) there is a \( t_0(\delta, \varepsilon) \) such that

\[
\mathbb{P}[d(\varphi_t(\omega, K_\varepsilon), A(\theta_t \omega)) > \delta] \leq \varepsilon, \quad \forall t \geq t_0.
\]

Hence, also

\[
\mathbb{P}[d(\varphi_t(\omega, \tilde{A}(\omega)), A(\theta_t \omega)) > \delta] \leq 2\varepsilon, \quad \forall t \geq t_0.
\]

By invariance \( \varphi_t(\omega, \tilde{A}(\omega)) = \tilde{A}(\theta_t \omega), \mathbb{P}\text{-a.s.} \). Thus,

\[
\mathbb{P}[d(\tilde{A}(\omega), A(\omega)) > \delta] = \mathbb{P}[d(\tilde{A}(\theta_t \omega), A(\theta_t \omega)) > \delta] \leq 2\varepsilon, \quad \forall t \geq t_0.
\]
Since $\varepsilon$ is arbitrary we conclude
\[ \mathbb{P}[d(\bar{A}(\omega), A(\omega)) > \delta] = 0 \quad \forall \delta > 0, \]
which implies the claim. \qed

If an RDS has a weak attractor $A$, then $A$ can be chosen to be $\mathcal{F}_0$-measurable by Lemma 1.3 and [15, Corollary 4.5].

When discussing (weak or pullback) attractors we will always assume that the underlying RDS is global. In contrast, we allow the RDS to be local when we discuss invariant measures and (weak) point attractors. The existence of an invariant measure does not guarantee that the RDS is global but it does impose some obvious constraints on the set $D$ in the definition of a local RDS.

2. Main results

2.1. Synchronization. We can now define formally what we mean by synchronization for a given RDS $\varphi$ which has a weak attractor $A$.

Definition 2.1. We say that synchronization occurs if $A(\omega)$ is a singleton, for $\mathbb{P}$-a.e. $\omega \in \Omega$.

We will now formulate sufficient conditions for synchronization to occur.

Definition 2.2. Let $U \subset E$ be a (deterministic) non-empty open set. We say that $\varphi$ is asymptotically stable on $U$ if there exists a (deterministic) sequence $t_n \uparrow \infty$ such that
\[ \mathbb{P}\left(\lim_{n \to \infty} \text{diam}(\varphi_{t_n}(\cdot, U)) = 0\right) > 0. \] (2.1)

Remark 2.3 (Necessity of asymptotic stability). Assume that synchronization holds and that there is at least one non-empty, open set $U \subset E$ that is attracted by $A(\omega) = \{a(\omega)\}$ (this is always true if $E$ is locally compact), i.e.
\[ d(\varphi_{t}(\omega, U), A(\theta_{t}\omega)) \to 0 \quad \text{for } t \to \infty, \]
in probability. Then, $\varphi$ is asymptotically stable on $U$. Indeed:
\[ \text{diam}(\varphi_{t}(\omega, U)) = \sup_{x,y \in U} d(\varphi_{t}(\omega, x), \varphi_{t}(\omega, y)) \]
\[ \leq \sup_{x,y \in U} d(\varphi_{t}(\omega, x), a(\omega)) + d(a(\omega), \varphi_{t}(\omega, y)) \]
\[ \to 0 \quad \text{for } t \to \infty \]
in probability.

Clearly, property (2.1) follows from the stronger assumption
\[ \mathbb{P}\left(\lim_{t \to +\infty} \text{diam}(\varphi_{t}(\cdot, U)) = 0\right) > 0, \] (2.2)
but there are a number of interesting cases in which (2.1) holds but (2.2) does not (cf. also Remark 4.2 below):
Example 2.4. We provide an example of an RDS \( \psi \) satisfying asymptotic stability, i.e. (2.1), but not satisfying (2.2) regardless of the choice of \( U \). Consider the one-dimensional SDE
\[
dX_t = -X_t dt + dW_t
\]
with associated RDS \( \varphi \). Obviously, \( \varphi_t(\omega, x) - \varphi_t(\omega, y) = (x - y)e^{-t} \). Let now \( t_n, x_n \uparrow \infty \) such that
\[
P\left( \sup_{t \in [t_{n-1}, t_n]} \varphi_t(\cdot, x) \geq x_n \right) \to 1 \quad \text{for} \quad n \to \infty,
\]
for all \( x \in \mathbb{R} \). We choose \( f : \mathbb{R} \to \mathbb{R} \) smooth, strictly increasing with range \( (f) = \mathbb{R} \) such that
\[
f'(x) \geq ne^{t_n} \quad \forall x \in [x_n, x_{n+1}]
\]
and set \( \psi_t(\omega, x) := f(\varphi_t(\omega, f^{-1}(x))) \). Let \( y > x \). Then
\[
\psi_t(\omega, y) - \psi_t(\omega, x) \geq n(f^{-1}(y) - f^{-1}(x))
\]
if \( \varphi_t(\omega, f^{-1}(x)) \geq x_n \) and \( t \in [t_{n-1}, t_n] \). Due to (2.3) this happens i.o. \( \mathbb{P} \)-a.s.. Hence, for all \( y > x \) we have
\[
\limsup_{t \to \infty} |\psi_t(\cdot, x) - \psi_t(\cdot, y)| = \infty \quad \mathbb{P}\text{-a.s.}
\]
and thus (2.2) does not hold. In contrast, (2.1) is easily verified for \( \psi \).

Let us first state Lemma 2.5, a very general and almost obvious criterion for synchronization.

Lemma 2.5. Let \( \varphi \) be asymptotically stable on \( U \) and \( A \) be an \( \mathcal{F}_0 \)-measurable, \( \varphi \)-invariant, random closed set with
\[
P( A \subset U ) > 0.
\]
Then \( A \) is a singleton \( \mathbb{P} \)-a.s..

The proof, as for the other claims of this section, is given in Section 3. Let us now discuss the two assumptions (2.1) and (2.4).

In applications to SDE, assumption (2.1) will be a consequence of the property that the top Lyapunov exponent \( \lambda_{\text{top}} \) is negative, although being more general (cf. Section 4.1 below). Example 2.3 provides an RDS satisfying (2.1), but the top Lyapunov exponent does not exist.

Let us come to the second assumption of Lemma 2.5. We can view it as an obvious consequence of the following condition.

Definition 2.6. We say that a random closed set \( A \) has full support if
\[
P( A \subset U ) > 0
\]
for every non-empty (deterministic) open set \( U \subset E \).
Let us give a sufficient condition for full support.

**Definition 2.7.** We say that \( \varphi \) is *swift transitive* if, for every (starting) ball \( B(x, r) \) and every (arrival) point \( y \), there is a time \( t > 0 \) such that
\[
P(\varphi_t(\cdot, B(x, r)) \subset B(y, 2r)) > 0.
\]

**Lemma 2.8.** If \( \varphi \) is swift transitive, \( A \) is an \( \mathcal{F}_0 \) measurable, \( \varphi \)-invariant random closed set and
\[
\text{ess inf} \{\text{diam}(A(\omega)); \omega \in \Omega\} = 0 \tag{2.6}
\]
then \( A \) has full support.

Condition (2.6) means that
\[
P(\text{diam}(A) < \varepsilon) > 0
\]
for every \( \varepsilon > 0 \) and is equivalent to the statement that for every \( \varepsilon > 0 \) there is an \( x_0 \in E \) such that
\[
P(A \subset B(x_0, \varepsilon)) > 0,
\]
see Lemma 3.1 below for a proof.

Let us state as the main abstract result of this section the following combination of the previous facts.

**Theorem 2.9.** Assume that \( \varphi \) is asymptotically stable on some non-empty open set \( U \subset E \) and is swift transitive. Let \( A \) be an \( \mathcal{F}_0 \) measurable, \( \varphi \)-invariant random closed set satisfying property (2.6). Then \( A \) is a singleton.

In particular, if \( A \) is a weak attractor, then synchronization occurs.

The property of swift transitivity is generally true for SDE with additive noise and drift satisfying a local one-sided Lipschitz condition, see Section 4. Concerning property (2.6), it looks also very general; we proceed to provide a sufficient condition.

The examples we have in mind which fulfill property (2.6) have the following features. With some (presumably very small) probability, their attractors are driven to regions of strong contraction, where the size of the attractor strictly decreases (cf. Section 4.2 below for examples). Possibly this procedure has to be iterated, until we reach a specified small value of the diameter. Let us formalize one of these steps in a definition.

**Definition 2.10.** We say that \( \varphi \) is *contracting on large sets* if for every \( R > 0 \), there is a ball \( B(y, R) \) and a time \( t > 0 \) such that
\[
P\left(\text{diam}\left(\varphi_t(\cdot, B(y, R))\right) \leq \frac{R}{4}\right) > 0.
\]
Remark 2.11 (Necessity of contraction on large sets). Assume that synchronization holds and that $A(\omega) = \{a(\omega)\}$ weakly attracts all closed, bounded sets (which is always true if $E$ is locally compact), then $\varphi$ is contracting on large sets. This follows as in Remark 2.3.

Lemma 2.12. Assume that $\varphi$ is contracting on large sets and is swift transitive. Then property (2.6) holds for all $\mathcal{F}_0$-measurable, $\varphi$-invariant random compact sets $A$.

We finish this section with a simple example which illustrates the concepts introduced above.

Example 2.13. Consider the one-dimensional SDE

$$dX_t = X_t \, dW_t, \ X_0 = x.$$ 

where $W$ is standard Brownian motion. The RDS generated by the solution is given by

$$\varphi_t(\omega, x) = xe^{-\frac{1}{2}t W_t(\omega)}.$$ 

$A(\omega) = \{0\}$ is the weak attractor of $\varphi$, so synchronization occurs. The RDS $\varphi$ is asymptotically stable on any bounded open set $U \subset \mathbb{R}$ and is contracting on large sets but $\varphi$ is not swift transitive. Lemma 2.5 can be applied but Lemma 2.8 and Theorem 2.9 cannot.

2.2. Weak synchronization. We now investigate a weaker form of synchronization. We will assume throughout this subsection that $\varphi$ is a local, white noise RDS.

Definition 2.14. We say that weak synchronization occurs if there is a minimal weak point attractor $A(\omega)$ being a singleton, for $\mathbb{P}$-a.e. $\omega \in \Omega$.

If there is a weak attractor $A$, then $A$ contains each minimal weak point attractor. In particular, synchronization implies weak synchronization.

We now introduce a weaker concept of asymptotic stability. The point is, that asymptotic stability in the sense of Definition 2.2 is not necessary for weak synchronization, while the following concept of weak asymptotic stability obviously is:

Definition 2.15. Let $U \subset E$ be a (deterministic) non-empty open set. We say that $\varphi$ is weakly asymptotically stable on $U$ if there exists a (deterministic) sequence $t_n \uparrow \infty$ and a set $\mathcal{M} \subset \Omega$ of positive $\mathbb{P}$-measure, such that, for all $x, y \in U$

$$1_{\mathcal{M}}(\cdot) d(\varphi_t(., x), \varphi_t(., y)) \to 0 \quad \text{for } n \to \infty,$$ 

in probability.
Remark 2.16. If weak synchronization occurs, then weak asymptotic stability is satisfied with $U = E$, $M = \Omega$ and every sequence $t_n \to \infty$ since, for all $x, y \in E$ we have
\[ d(\varphi_t(., x), \varphi_t(., y)) \to 0 \quad \text{for} \ t \to \infty, \]
in probability.

Assume that the Markov semigroup corresponding to $\varphi$ has an ergodic invariant measure $\mu$ with disintegration $\mu_\omega$, the statistical equilibrium (cf. (1.4)).

Local stability in terms of weak asymptotic stability can be nicely captured in terms of the support of the statistical equilibrium $\mu_\omega$, i.e. if $\varphi$ is weakly asymptotically stable then the support has to consist of finitely many random points. For RDS with negative top Lyapunov exponent and on compact manifolds this goes back to [27].

Lemma 2.17. (1) The statistical equilibrium $\mu_\omega$ is either discrete or diffuse. More precisely, either $\mu_\omega$ consists of finitely many atoms of the same mass \(\mathbb{P}\)-a.s., i.e. there is an $N \in \mathbb{N}$ and $\mathcal{F}_0$-measurable random variables $a_1, \ldots, a_N$ such that
\[ \mu_\omega = \left\{ \frac{1}{N} \delta_{a_i(\omega)} : i = 1, \ldots, N \right\} \]
or $\mu_\omega$ does not have point masses $\mathbb{P}$-a.s..

(2) Assume that $\varphi$ is weakly asymptotically stable on $U$ with $\mu(U) > 0$. Then $\mu_\omega$ is discrete.

Let
\[ E_0 := \{ x \in E : \lim_{t \to \infty} P_t(x,.) = \mu \}, \]
where $P_t(x,.)$ denotes the transition probability and convergence is to be understood in the weak sense. Note that if the support of $\mu_\omega$ is compact with strictly positive probability then it is compact with probability one.

Proposition 2.18. (1) Assume that $A(\omega) := \text{supp}(\mu_\omega)$ is (almost surely) compact. Then $A$ is a weak point attractor of the set $E_0$. In particular, if $\varphi$ is strongly mixing then $A$ is a minimal weak point attractor.

(2) If $\varphi$ is strongly mixing and weakly asymptotically stable on $U$ with $\mu(U) > 0$, then there is an $N \in \mathbb{N}$ and $\mathcal{F}_0$-measurable random variables $a_1, \ldots, a_N$ such that
\[ A(\omega) = \text{supp}(\mu_\omega) = \{ a_i(\omega) : i = 1, \ldots, N \} \]
is a minimal weak point attractor.

Under a compact absorption assumption, Proposition 2.18 (1) corresponds to [26 Theorem 2.4]. The more general setting treated here, however, requires a quite different proof.
By Proposition 2.18 without any assumption on $A$ having full support, asymptotic stability of $\varphi$ implies that the minimal weak point attractor consists of finitely many points. We note that if $E$ is connected, then this is true for a weak attractor iff synchronization occurs. Indeed, if $E$ is connected then so are weak attractors (which follows from the same proof as for [16, Proposition 3.13]). The following example shows that Proposition 2.18 is not true for weak attractors.

**Example 2.19.** Consider
\[ \alpha_t = \cos(2\alpha_t) \circ dW^1_t + \sin(2\alpha_t) \circ dW^2_t \]
on the one-dimensional sphere $S^1$. Then the weak attractor is the whole sphere $S^1$ while the minimal weak point attractor consists of two (antipodal) random points $P$-a.s. (cf. [4, Remark 4.11]).

**Definition 2.20.** We say that $\varphi$ is pointwise strongly swift transitive if there is a time $t > 0$ such that for every $x^1, x^2 \in E$ and every (arrival) point $y$,
\[ \mathbb{P} \left( \varphi_t \left( \cdot, \{x^1, x^2\} \right) \subset B \left( y, 2d(x^1, x^2) \right) \right) > 0. \]

We obtain

**Proposition 2.21.** Assume that $\varphi$ has right-continuous trajectories, is strongly mixing, weakly asymptotically stable on $U$ with $\mu(U) > 0$, pointwise strongly swift transitive and
\[ \liminf_{t \to \infty} d(\varphi_t(\omega, x), \varphi_t(\omega, y)) = 0 \quad \text{a.s., } \forall x, y \in E. \tag{2.8} \]
Then, there is a minimal weak point attractor $A$ consisting of a single random point $a(\omega)$ and
\[ A(\omega) = \text{supp}(\mu_\omega) = \{a(\omega)\} \quad \mathbb{P}\text{-a.s..} \]

In the case of gradient-type SDE we can use the results above, in order to deduce weak synchronization without assuming contraction of large balls (as compared to Section 2.1). In fact, we will prove that (2.8) is always satisfied as soon as there is an invariant measure.

More precisely, consider the SDE
\[ dX_t = -\nabla V(X_t)dt + \sigma dW_t \quad \text{on } \mathbb{R}^d, \tag{2.9} \]
with $V \in C^2(\mathbb{R}^d, \mathbb{R})$, $\sigma > 0$ and $b := -\nabla V$ satisfying a one-sided Lipschitz condition of the type
\[ (b(x) - b(y), x - y) \leq \lambda |x - y|^2, \]
for all $x, y \in \mathbb{R}^d$ and some $\lambda > 0$. By [18] there is an associated white noise RDS $\varphi$ to (2.9). Further assume $\rho(x) := e^{-\frac{1}{2\sigma^2}V(x)} \in L^1(\mathbb{R}^d)$. Then, by [42] Theorem,
The Markovian semigroup defined by \( P_t f(x) := \mathbb{E} f(\varphi_t(\cdot, x)) \) has as an invariant probability measure, where \( Z := \int_{\mathbb{R}^d} e^{-\frac{2}{\sigma^2} V(x)} dx \). By [38, Theorem 3], \( P_t \) is strongly mixing with ergodic measure \( \mu \).

In Section 4, we prove that (weak) asymptotic stability is satisfied for (2.9) if \( V \) is radially symmetric or \( \sigma \) is small, under some weak growth conditions on \( V \).

**Theorem 2.22.** Assume that \( \rho(x) := e^{-\frac{2}{\sigma^2} V(x)} \in L^1(\mathbb{R}^d) \) and that \( \varphi \) is weakly asymptotically stable on \( U \) with \( \mu(U) > 0 \). Then, there is a minimal weak point attractor \( A \) consisting of a single random point \( a(\omega) \) and
\[
A(\omega) = \text{supp}(\mu_\omega) = \{a(\omega)\} \quad \mathbb{P}\text{-a.s.}
\]

3. Proofs

3.1. Synchronization.

**Proof of Lemma 2.5.** By property (2.1) there exists a sequence \( t_n \to \infty \) such that
\[
\mathbb{P}\left( \lim_{n \to \infty} \text{diam}(\varphi_{t_n}(\cdot, U)) = 0 \right) > 0.
\]
Since \( \{A \subset U\} \) is \( \mathcal{F}_0 \)-measurable, \( \{\lim_{n \to \infty} \text{diam}(\varphi_{t_n}(\cdot, U)) = 0\} \) is \( \mathcal{F}_{0,\infty} \)-measurable and \( \mathcal{F}_0 \) and \( \mathcal{F}_{0,\infty} \) are independent, we obtain
\[
\mathbb{P}\left( \lim_{n \to \infty} \text{diam}(\varphi_{t_n}(\cdot, A)) = 0 \right) > 0.
\]
In particular, since \( \text{diam}(\varphi_{t_n}(\cdot, A)) \) has the same law as \( \text{diam}(A) \), we get
\[
\mathbb{P}(\text{diam}(A) = 0) > 0.
\]
We need to show that this probability is in fact 1. We observe that for each \( t \geq 0 \) we have
\[
\{\text{diam}(A(\theta_t \omega)) = 0\} \subseteq \{\text{diam}(A(\omega)) = 0\}.
\]
Since \( \theta_t \) is \( \mathbb{P} \)-invariant these events have the same \( \mathbb{P} \)-mass and thus coincide almost surely. Note that \( \{\text{diam}(A(\theta_{t} \omega)) = 0\} \) is \( \mathcal{F}_{t} \)-measurable. Hence, \( \{\text{diam}(A(\omega)) = 0\} \) is measurable with respect to \( \cap_{t<0} \mathcal{F}_{t} \) which is trivial by Kolmogorov’s 0-1 law. Here, \( \mathcal{F}_{t} \) is the \( \mathbb{P} \)-completion of \( \mathcal{F}_{t} \). Therefore, we get \( \text{diam}(A(\omega)) = 0 \) almost surely and the proof of the lemma is complete.

**Lemma 3.1.** Let \( A \) be a closed random set. Then (2.6) is satisfied iff for each \( \varepsilon > 0 \) there is an \( x_0 \in E \) such that
\[
\mathbb{P}(A \subset B(x_0, \varepsilon)) > 0.
\]

\(^2\)In fact, [42, Theorem, p.243] assumes \( b \) to be smooth. However, it is an easy exercise to see that only \( b \in C^\delta \) for some \( \delta > 0 \) is required for the proof (cf. also [25, Theorem 10.4.1] for an according regularity result for linear, non-degenerate second order PDE with Hölder coefficients).
**Proof.** Assume (2.6) and consider a countable family of balls of the form $B(x_n, \varepsilon)$ where $\{x_n, n \in \mathbb{N}\}$ is a dense countable set in $E$. We know that
\[ \mathbb{P}(\text{diam}(A) < \varepsilon) > 0. \]
We have
\[ \{\text{diam}(A) < \varepsilon\} \subset \{A \subset B(x_n, \varepsilon) \text{ for some } n \in \mathbb{N}\} \]
hence
\[ 0 < \mathbb{P}(A \subset B(x_n, \varepsilon) \text{ for some } n \in \mathbb{N}) = \mathbb{P} \left( \bigcup_{n \in \mathbb{N}} \{A \subset B(x_n, \varepsilon)\} \right) \leq \sum_{n \in \mathbb{N}} \mathbb{P}(A \subset B(x_n, \varepsilon)) \]
and thus $\mathbb{P}(A \subset B(x_n, \varepsilon)) > 0$ for some $n \in \mathbb{N}$, proving the claim. \(\square\)

**Proof of Lemma 2.8.** Let $U$ be a non-empty open set and $B(y, R) \subset U$. By Lemma 3.1 there is an $x_0 \in E$ such that $\mathbb{P}(A \subset B(x_0, R/2)) > 0$. By Definition 2.7 with the starting ball $B(x_0, R/2)$ and the arrival point $y$, there is a time $t > 0$ such that
\[ \mathbb{P}(\varphi_t(\cdot, B(x_0, R/2)) \subset B(y, R)) > 0. \]
By the independence of $\mathcal{F}_{0,t}$ and $\mathcal{F}_0$ and the fact that $\varphi_t$ is $\mathcal{F}_{0,t}$-measurable and $A$ is $\mathcal{F}_0$-measurable, it follows that
\[ \mathbb{P}(A \subset B(x_0, R/2), \varphi_t(\cdot, B(x_0, R/2)) \subset B(y, R)) > 0 \]
and thus
\[ \mathbb{P}(\varphi_t(\cdot, A) \subset B(y, R)) > 0. \]
We have $\varphi_t(\omega, A(\omega)) = A(\theta_t \omega)$, hence $\mathbb{P}(A(\theta_t) \subset B(y, R)) > 0$. By $\theta_t$ invariance of $\mathbb{P}$ we get
\[ \mathbb{P}(A \subset B(y, R)) > 0 \]
hence $\mathbb{P}(A \subset U) > 0$. The proof is complete. \(\square\)

**Lemma 3.2.** Under the assumptions of Lemma 2.12 the following is true: if $\mathbb{P}(A \subset B(x_0, r_0)) > 0$ for some $r_0 > 0$, $x_0 \in E$ then
\[ \mathbb{P}(A \subset B(x_1, \frac{2}{3}r_0)) > 0 \]
for some $x_1 \in E$. 

Proof. Since $A$ is a random compact set we can choose $r_0 > 0$, $x_0 \in E$ such that
\[ P(A \subset B(x_0, r_0)) > 0. \]

Apply Definition 2.10 with $R = 2r_0$: there is $y_1 \in E$, $t_1 > 0$ such that
\[ P\left( \text{diam} \left( \varphi_{t_1} \left( \cdot, B(y_1, 2r_0) \right) \right) \leq \frac{r_0}{2} \right) > 0. \]

For every $t_0 > 0$, since $P$ is invariant under $\theta_{t_0}$, we also have
\[ P\left( \text{diam} \left( \varphi_{t_1} \left( \theta_{t_0} \cdot, B(y_1, 2r_0) \right) \right) \leq \frac{r_0}{2} \right) > 0. \]

Apply Definition 2.7 with the starting ball equal to $B(x_0, r_0)$ and the arrival point equal to $y_1$: there is a time $t_0 > 0$ such that
\[ P\left( \varphi_{t_0} \left( \cdot, B(x_0, r_0) \right) \subset B(y_1, 2r_0) \right) > 0. \]

We have \( \{ \varphi_{t_0} (\cdot, B(x_0, r_0)) \subset B(y_1, 2r_0) \} \in \mathcal{F}_{0,t_0} \) and
\[ \left\{ \text{diam} \left( \varphi_{t_1} \left( \theta_{t_0} \cdot, B(y_1, 2r_0) \right) \right) \leq \frac{r_0}{2} \right\} \in \mathcal{F}_{t_0,t_0+t_1}, \]

since \( \{ \text{diam} \left( \varphi_{t_1} \left( B(y_1, 2r_0) \right) \right) \leq \frac{r_0}{2} \} \in \mathcal{F}_{0,t_1} \) and \( \theta_{t_0}^{-1} \mathcal{F}_{0,t_1} = \mathcal{F}_{t_0,t_0+t_1}. \) Since \( \mathcal{F}_{0,t_0} \) and \( \mathcal{F}_{t_0,t_0+t_1} \) are independent, and
\[ \varphi_{t_1} \left( \theta_{t_0} \cdot, \varphi_{t_0} \left( \omega, B(x_0, r_0) \right) \right) = \varphi_{t_1+t_0} \left( \omega, B(x_0, r_0) \right) \]

we deduce
\[ P\left( \text{diam} \left( \varphi_{t_1+t_0} \left( \cdot, B(x_0, r_0) \right) \right) \leq \frac{r_0}{2} \right) > 0. \]

Hence there is $x_1 \in E$ such that
\[ P\left( \varphi_{t_1+t_0} \left( \cdot, B(x_0, r_0) \right) \subset B\left( x_1, \frac{2}{3} r_0 \right) \right) > 0. \]

By the independence of $\mathcal{F}_{0,t_1+t_0}$ and $\mathcal{F}_0$ and the fact that $\varphi_{t_1+t_0}$ is $\mathcal{F}_{0,t_1+t_0}$-measurable and $A$ is $\mathcal{F}_0$-measurable, it follows that
\[ P\left( A \subset B \left( x_0, r_0 \right), \varphi_{t_1+t_0} \left( \cdot, B(x_0, r_0) \right) \subset B \left( x_1, \frac{2}{3} r_0 \right) \right) > 0. \]

Hence
\[ P\left( \varphi_{t_1+t_0} \left( \cdot, A \right) \subset B \left( x_1, \frac{2}{3} r_0 \right) \right) > 0. \]

We have $\varphi_{t_1+t_0} \left( \omega, A(\omega) \right) = A \left( \theta_{t_1+t_0} \omega \right)$, hence $P \left( A \left( \theta_{t_1+t_0} \cdot \right) \subset B \left( x_1, \frac{2}{3} r_0 \right) \right) > 0$. By $\theta_{t_1+t_0}$ invariance of $P$ we get
\[ P\left( A \subset B \left( x_1, \frac{2}{3} r_0 \right) \right) > 0. \]

\[ \square \]
Proof of Lemma 2.12. The proof is now obvious. Since $A$ is a random compact set we can choose $r_0 > 0$, $x_0 \in E$ such that

$$\mathbb{P}(A \subset B(x_0, r_0)) > 0.$$ 

Given any $\varepsilon > 0$, we may apply Lemma 3.2 iteratively until we get

$$\mathbb{P}(A \subset B(x, \varepsilon)) > 0$$

for some $x \in E$ and the proof is complete. \hfill \Box

3.2. Weak synchronization.

Proof of Lemma 2.17: The proof uses modified arguments from [27].

(1): Step 1: Let $(a_i(\omega))_{i=1}^N$ with $N \in \mathbb{N} \cup \{\infty\}$ be the weights of point masses of $\mu_\omega$, counted with multiplicity and ordered in non-increasing manner. Now let

$$F^M(\omega) := \sum_{i=1}^M a_i(\omega)$$

for all $M \in \mathbb{N}$. Since $t \mapsto \mu_{\theta_t \omega}$ is stationary, so is $t \mapsto F^M(\theta_t \omega)$ for all $M \in \mathbb{N}$. Since $\phi_t(\omega)\mu_\omega = \mu_{\theta_t \omega}$, weights of point masses of $\mu_\omega$ can only increase or merge under $t \mapsto \mu_{\theta_t \omega}$. Thus, $t \mapsto F^M(\theta_t \omega)$ is non-decreasing for all $M \in \mathbb{N}$. Since also $t \mapsto F^M(\theta_t \omega)$ is stationary, this implies that $F^M$ is constant $\mathbb{P}$-a.s. for all $M \in \mathbb{N}$. Hence, $a_i(\cdot)$ is constant $\mathbb{P}$-a.s.. In particular, the number of point masses with mass $m$ is constant $\mathbb{P}$-a.s. for each $m \in \mathbb{R}^+$. 

Step 2: Let now $F_m = \{(\omega, x) : \mu_\omega(\{x\}) = m\}$. We let $Q$ be the probability measure on $\Omega \times E$ given by $Q = \mathbb{E}_{\mu_\omega}$. Then, for $t \geq 0$

$$\Theta_t F_m = \{(\phi_t(\omega, x), \theta_t \omega) : \mu_\omega(\{x\}) = m\}
= \{(\omega, x) : \mu_{\theta_{-t} \omega}(\phi_t^{-1}(\theta_{-t} \omega)(\{x\})) = m\}
= \{(\omega, x) : \phi_t(\theta_{-t} \omega)\mu_{\theta_{-t} \omega}(\{x\}) = m\}
= \{(\omega, x) : \mu_\omega(\{x\}) = m\}
= F_m,$$

i.e. $F_m$ is $\Theta_t$ invariant. Since $\Theta_t$ is $Q$ ergodic (cf. [7]) this implies $Q(F_m) \in \{0, 1\}$.

Using step one we observe

$$Q(F_m) = \int_{\Omega} 1_{F_m}(\omega, x) d\mu_\omega(x) d\mathbb{P}(\omega)
= m \int_{\Omega} \#\{\text{atoms of } \mu_\omega \text{ with mass } m\} d\mathbb{P}(\omega)
= m \#\{\text{atoms of } \mu_\omega \text{ with mass } m\}.$$

Hence, if $Q(F_m) = 1$ for some $m \in \mathbb{R}$, then $\mu_\omega$ consists of $1/m$ point masses each having mass $m$. 

(2): Due to (1) we only have to show that \( \mu_\omega \) has a point mass with positive probability.

Let \( \psi : (E \times E) \setminus \Delta \to [0, \infty) \) be measurable such that \( \psi(x, y) \to \infty \) for \( d(x, y) \to 0 \) and

\[
\mathbb{E} \int_{(E \times E) \setminus \Delta} \psi(x, y) d\mu_\omega(x) d\mu_\omega(y) < \infty.
\]

Further let \( U \) be as in the assumption of weak asymptotic stability. By invariance of \( \mu_\omega \) we have

\[
\mathbb{E} \int_{(E \times E) \setminus \Delta} \psi(x, y) d\mu_\omega(x) d\mu_\omega(y)
\geq \mathbb{E} \int_{(E \times E) \setminus \Delta} \psi(\varphi_t(\omega, x), \varphi_t(\omega, y)) d\mu_\omega(x) d\mu_\omega(y)
\geq \mathbb{E} \int_{(E \times E) \setminus \Delta} 1_U(x) 1_U(y) \psi(\varphi_t(\omega, x), \varphi_t(\omega, y)) d\mu_\omega(x) d\mu_\omega(y).
\]

By weak asymptotic stability there is a set \( \mathcal{M} \subseteq \Omega \) with positive \( \mathbb{P} \)-measure and a sequence \( t_n \to \infty \) such that, for all \( x, y \in U \)

\[
1_{\mathcal{M}}(\cdot) d(\varphi_{t_n}(\cdot, x), \varphi_{t_n}(\cdot, y)) \to 0 \quad \text{for } n \to \infty,
\]

in probability. We define \( C(n, x, y, R) := \{ \omega \in \Omega : \psi(\varphi_{t_n}(\omega, x), \varphi_{t_n}(\omega, y)) \geq R \} \) and observe

\[
\liminf_{n \to \infty} \mathbb{P}(C(n, x, y, R)) \geq \mathbb{P}(\mathcal{M}).
\]

From (3.1) we obtain

\[
\mathbb{E} \int_{(E \times E) \setminus \Delta} \psi(x, y) d\mu_\omega(x) d\mu_\omega(y)
\geq R \mathbb{E} \int_{(E \times E) \setminus \Delta} 1_U(x) 1_U(y) 1_{C(n, x, y, R)}(\omega) d\mu_\omega(x) d\mu_\omega(y).
\]

Since \( \mu_\omega \) is \( \mathcal{F}_0 \)-measurable, \( C(n, x, y, R) \) is \( \mathcal{F}_{0, \infty} \)-measurable and \( \mathcal{F}_0, \mathcal{F}_{0, \infty} \) are independent, we conclude

\[
\mathbb{E} \int_{(E \times E) \setminus \Delta} 1_U(x) 1_U(y) 1_{C(n, x, y, R)}(\omega) d\mu_\omega(x) d\mu_\omega(y)
= \mathbb{E} \mathbb{E} \left[ \int_{(E \times E) \setminus \Delta} 1_U(x) 1_U(y) 1_{C(n, x, y, R)}(\omega) d\mu_\omega(x) d\mu_\omega(y) \right| \mathcal{F}_0
= \mathbb{E} \mathbb{E} \left[ \int_{(E \times E) \setminus \Delta} 1_U(x) 1_U(y) 1_{C(n, x, y, R)}(\omega) d\mu_\omega(x) d\mu_\omega(y) \right| \mathcal{F}_0
= \mathbb{E} \int_{(E \times E) \setminus \Delta} 1_U(x) 1_U(y) \mathbb{P}[C(n, x, y, R)] d\mu_\omega(x) d\mu_\omega(y).
\]
Using this above, taking $\liminf_{n \to \infty}$ and using Fatou’s Lemma yields

$$
\mathbb{E} \int_{(E \times E) \setminus \Delta} \psi(x, y) d\mu_\omega(x) d\mu_\omega(y) \\
\geq R \mathbb{E} \int_{(E \times E) \setminus \Delta} 1_U(x) 1_U(y) \liminf_{n \to \infty} \mathbb{P}[C(n, x, y, R)] d\mu_\omega(x) d\mu_\omega(y) \\
\geq \mathbb{P}[\mathcal{M}] R \mathbb{E} \int_{(E \times E) \setminus \Delta} 1_U(x) 1_U(y) d\mu_\omega(x) d\mu_\omega(y)
$$

If $\mu_\omega$ has no point masses, then for $\mu_\omega$ a.a. $x$ and all $\delta > 0$ we have $\mu_\omega(B(x, \delta) \setminus \{x\}) > 0$. In particular,

$$
\mathbb{E} \int_{(E \times E) \setminus \Delta} 1_U(x) 1_U(y) d\mu_\omega(x) d\mu_\omega(x) > 0.
$$

Since $R > 0$ is arbitrary we obtain a contradiction. This concludes the proof. □

**Proof of Proposition 2.18 (1):** We show that $A$ attracts each $x \in E_0$ in probability.

Fix $\varepsilon > 0$. There exists some measurable function $\beta(\omega) > 0$ such that there exists a finite (random) number of open balls of radius $\varepsilon/3$ which cover $A(\omega)$ and which each have $\mu_\omega$-measure at least $\beta(\omega)$. Let $\tilde{A}(\omega)$ be the union of these balls and note that $\tilde{A}(\omega)$ is a random bounded open set.

For $b > 0$ and $t > 0$, we define

$$
A_t^{(b)}(\omega) = \bigcup_{x \in D_b(t, \omega)} B(x, \frac{\varepsilon}{3}),
$$

where $D_b(t, \omega)$ is the set of all $x \in E$ for which $(\varphi_t(\theta_{-t}\omega)\mu)(B(x, \frac{\varepsilon}{3})) \geq b$. Since, $\mathbb{P}$-a.s. $(\varphi_t(\theta_{-t}\omega)\mu) \rightarrow \mu_\omega$ weakly$^*$ and $\mu_\omega(A(\omega)\tilde{\omega}) = 1$ we have $(\varphi_t(\theta_{-t}\omega)\mu)(A(\omega)) \geq 1 - \frac{b}{2}$ for $t$ large enough. Hence, for $x \in D_b(t, \omega)$ we have $B(x, \frac{\varepsilon}{3}) \cap A(\omega)\tilde{\omega} \neq \emptyset$, for $t$ large enough. Thus,

$$
\lim_{t \to \infty} \mathbb{P}(A_t^{(b)}(\omega) \subseteq A^c(\omega)) = 1.
$$

Further, we have

$$
\liminf_{t \to \infty} \mathbb{P}(\tilde{A} \subseteq A_t^{(b)}) \geq \mathbb{P}(\beta(\omega) > b).
$$

For given $\delta > 0$, we find $b$ so small and $t_0$ so large that

$$
\mathbb{P}(\tilde{A} \subseteq A_t^{(b)} \subseteq A^c) \geq 1 - \frac{\delta}{4},
$$

for all $t \geq t_0$.

Observe that there exists some $t_1 \geq t_0$ such that

$$
\mathbb{P}\left((\varphi_t(\cdot)\mu)(A_t^{(b)}) \geq 1 - \frac{\delta}{3}\right) \geq 1 - \frac{\delta}{3}
$$

for all $t \geq t_1$. 
Fix $x \in E_0$. Then, for $t \geq t_1$,
\[
\liminf_{s \to \infty} \mathbb{P}(\varphi_{t+s}(\theta_{-(t+s)} \omega), x) \in A^\varepsilon
\geq \liminf_{s \to \infty} \mathbb{P}(\varphi_{t+s}(\theta_{-(t+s)} \omega), x) \in A^{(b)}_t) - \frac{\delta}{4}
= \liminf_{s \to \infty} \mathbb{P}(\varphi_s(\theta_{-(t+s)} \omega), x) \in \varphi_s^{-1}((\theta_{-t} \omega)(A^{(b)}_t)) - \frac{\delta}{4}
\geq \liminf \mathbb{E}((\varphi_t(\theta_{-t} \omega) \mu)(A^{(b)}_t)) - \frac{\delta}{4}
\geq (1 - \frac{\delta}{3}) - \frac{\delta}{4} 
\]
where we used the independence of $\mathcal{F}_{-t,0}$ and $\mathcal{F}_{-\infty,-t}$ and the fact that $x \in E_0$ in the step from the third to the fourth line. Since $\delta > 0$ and $\varepsilon > 0$ are arbitrary, the claim follows.

Let now $\varphi$ be strongly mixing, i.e. $E_0 = E$. Minimality of $A$ follows from the fact that every $\varphi$-invariant Markov measure is supported by every weak point attractor $A'$ (cf. [13]), i.e. $\mu_\omega(A'(\omega)) = 1$ a.s.. Hence, $A' \subseteq A$ a.s..

(2): Follows from Lemma 2.17 and (1).

\[\square\]

Proof of Proposition 2.21. Since $\varphi$ is strongly mixing and weakly asymptotically stable on some non-empty open set with positive $\mu$-measure, by Proposition 2.18 there are $\mathcal{F}_0$-measurable random variables $a_i(\omega)$, $i = 1, \ldots, N$ such that

$A(\omega) := \{a_i(\omega) : i = 1, \ldots, N\}$

is a minimal weak point attractor.

Step 1: By weak asymptotic stability there is an open set $U$, a sequence $t_n \to \infty$ and a $\delta > 0$ such that for all $x, y \in U$ and all $\eta > 0$

\[
\liminf_{n \to \infty} \mathbb{P}(d(\varphi_{t_n}(\cdot), x), \varphi_{t_n}(\cdot, y)) \leq \eta) \geq \delta > 0.
\]  

(3.2)

Without loss of generality we may assume $U = B(\varepsilon, x_0)$ for some $x_0 \in E$, $\varepsilon > 0$. Let $x, y \in E$. By assumption, the stopping time

$\tau^{x,y}_\varepsilon(\omega) := \inf\{t \geq 0 : d(\varphi_t(\omega, x), \varphi_t(\omega, y)) \leq \frac{\varepsilon}{4}\}$

is finite $\mathbb{P}$-almost surely.

Let now $a(\omega), b(\omega) \in A(\omega)$ be two $\mathcal{F}_0$-measurable selections and let $\tau^{x}(\omega) := \tau^{a(\omega),b(\omega)}$, where $\tau^{x,y}_\varepsilon$ is defined as above. Due to independence of $\mathcal{F}_0$ and $\mathcal{F}_{0,\infty}$, $\tau^{x}_\varepsilon$ is finite a.s.. Right-continuity of the trajectories implies that there is a $t : \Omega \to \mathbb{R}_+ \setminus \{0\}$ such that

\[
d(\varphi_{\tau^{x}(\omega)+t}(\omega, a(\omega)), \varphi_{\tau^{x}(\omega)+t}(\omega, b(\omega))) \leq \frac{\varepsilon}{3}
\]  

(3.3)
for all \( t \in [0, \iota(\omega)] \), \( \mathbb{P} \)-a.s.. Hence, there is a \( \bar{t}_0 \geq 0 \) such that
\[
\mathbb{P} \left( d(\varphi_{\bar{t}_0}(\cdot, a(\cdot)), \varphi_{\bar{t}_0}(\cdot, b(\cdot))) \leq \frac{\varepsilon}{3} \right) > 0.
\]
Indeed: Assume not. Then
\[
\mathbb{P} \left( d(\varphi_t(\cdot, a(\cdot)), \varphi_t(\cdot, b(\cdot))) > \frac{\varepsilon}{3} \right) > 0,
\]
in contradiction to (3.3).

**Step 2**: By pointwise strong swift transitivity and using that \( \varphi \) is a white noise RDS there is a time \( \bar{t}_1 \geq 0 \) such that
\[
\mathbb{P}(\varphi_{\bar{t}_0 + \bar{t}_1}(\cdot, \{a(\cdot), b(\cdot)\}) \subseteq U) > 0.
\]
Again using that \( \varphi \) is a white noise RDS we conclude
\[
\lim \inf_{n \to \infty} \mathbb{P}(d(\varphi_{\bar{t}_0 + \bar{t}_1 + t_n}(\cdot, a(\cdot)), \varphi_{\bar{t}_0 + \bar{t}_1 + t_n}(\cdot, b(\cdot))) \leq \eta) \geq \frac{\delta}{2} > 0.
\]

**Step 3**: Assume \( A(\omega) \) is not a singleton \( \mathbb{P} \)-a.s.. Then
\[
F(\omega) := \min_{i,j=1, \ldots, N, i \neq j} d(a_i(\omega), a_j(\omega)) > 0,
\]
\( \mathbb{P} \)-a.s.. Moreover, since \( \varphi_t(\omega, A(\omega)) = A(\theta_t \omega) \) we have
\[
F(\theta_t \omega) = \min_{i,j=1, \ldots, N, i \neq j} d(a_i(\theta_t \omega), a_j(\theta_t \omega))
\]
\[
= \min_{i,j=1, \ldots, N, i \neq j} d(\varphi_t(\omega, a_i(\omega)), \varphi_t(\omega, a_j(\omega)))
\]
\[
\leq d(\varphi_t(\omega, a_1(\omega)), \varphi_t(\omega, a_2(\omega))).
\]
Hence, for all \( \eta > 0 \)
\[
\mathbb{P}(F(\cdot) \leq \eta) = \mathbb{P}(F(\theta_{\bar{t}_0 + \bar{t}_1 + t_n}) \leq \eta)
\]
\[
\geq \mathbb{P}(d(\varphi_{\bar{t}_0 + \bar{t}_1 + t_n}(\cdot, a_1(\cdot)), \varphi_{\bar{t}_0 + \bar{t}_1 + t_n}(\cdot, a_2(\cdot))) \leq \eta).
\]
Taking \( \lim \inf_{n \to \infty} \) and using (3.4) we conclude:
\[
\mathbb{P}(F(\cdot) \leq \eta) \geq \frac{\delta}{2} > 0,
\]
for all \( \eta \) in contradiction to (3.3). \( \square \)

We now proceed to the proof of Theorem 2.22. Hence, let \( b, V, \varphi \) as for (2.9). Note that by Proposition 4.9 below, \( \varphi \) is strongly (pointwise) swift transitive.

**Lemma 3.3.** Assume that \( \rho(x) := e^{-\frac{\sigma^2}{2}V(x)} \in L^1(\mathbb{R}^d) \). Then, for each pair \( x, y \in \mathbb{R}^d \), we have \( \lim \inf_{t \to \infty} |\varphi_t(x) - \varphi_t(y)| = 0 \), almost surely.
Proof. Step 1: We claim that for each \( \delta > 0 \) and \( s \in S^{d-1} \) there exists some \( z \in \mathbb{R}^d \), such that

\[
V(z) < \frac{1}{2} \left( V(z + \delta s) + V(z - \delta s) \right).
\]

Assume this is wrong for some particular \( \delta > 0 \) and \( s \in S^{d-1} \), then for every \( z \in \mathbb{R}^d \) we have

\[
V(z) \geq \frac{1}{2} \left( V(z + \delta s) + V(z - \delta s) \right).
\]

Therefore, one the functions \( n \mapsto V(z + n\delta s), V(z - n\delta s) \) is non-increasing for \( n \in \mathbb{N} \). Without loss of generality let \( n \mapsto V(z + n\delta s) \) be non-increasing. Let \( n \in \mathbb{N} \). Due to the one-sided Lipschitz condition on \( b \) the function \( g(h) := V(z + n\delta s + h\delta s) + \frac{1}{2} h^2 \) is convex on \( h \in [0,1] \). Moreover, \( g(0) \leq V(z) \) and \( g(1) \leq V(z) + \frac{1}{2} \). Since \( g \) is convex this implies \( \sup_{h \in [0,1]} g(h) \leq V(z) + \frac{1}{2} \). Hence, \( \sup_{h \in [0,1]} V(z + n\delta s + h\delta s) \leq V(z) + \frac{1}{2} \) for all \( n \in \mathbb{N} \) which implies that

\[
h \mapsto V(z + hs)
\]

where \( h \in [0,\infty) \), is bounded from above. In particular, \( \int_{\mathbb{R}} \rho(z + hs) \, dh = \infty \) holds for each \( z \in \mathbb{R}^d \), and therefore \( \rho \) cannot be integrable.

Step 2: For each \( \delta > 0 \) and \( s \in S^{d-1} \) there exists some \( z \in \mathbb{R}^d, \alpha \in [0, \delta] \), such that

\[
(b(z + \alpha s) - b(z + \alpha s - \delta s), s) < 0.
\]

To see this, let \( f(h) := V(z + hs) - V(z) \). Due to (3.7) we have

\[
f(0) = 0 < \frac{1}{2} (f(\delta) + f(-\delta)).
\]

We aim to show that there is an \( \alpha \in [0, \delta] \) such that \( f'(\alpha) > f'(\alpha - \delta) \). Assume the contrary, i.e. \( f'(\alpha) \leq f'(\alpha - \delta) \) for all \( \alpha \in [0, \delta] \). Integrating over \([0, \delta]\) yields \( f(\delta) \leq -f(-\delta) \) in contradiction to (3.7), which proves the claim.

Step 3: Assume that the claim of the lemma is wrong. Then there exist \( x, y \in \mathbb{R}^d \) such that for

\[
S := \{ \omega \in \Omega : \liminf_{t \to \infty} |\varphi_t(\omega, x) - \varphi_t(\omega, y)| > 0 \}
\]

we have \( \mathbb{P}(S) > 0 \). Fix such \( x \) and \( y \) and let \( A(\omega) \) be the set of accumulation points of \((\varphi_t(x), \varphi_t(y))\) as \( t \to \infty \). Note that \( A(\omega) \) is closed. By the strong mixing property there exists a set \( \Omega_0 \) of full measure such that \( A(\omega) \) is non-empty for \( \omega \in \Omega_0 \).

Claim: There is a set \( \Omega_1 \subseteq \Omega_0 \) of full measure such that, for all \( \omega \in \Omega_1 \)

\[
(a, b) \in A(\omega), \ c \in \mathbb{R}^d \Rightarrow (a + c, b + c) \in A(\omega).
\]

(3.10)
To see this, observe that the fact that (2.10) is driven by additive noise implies that for each fixed \( q \in \mathbb{R}^d \), each closed ball \( \overline{B} \subset \mathbb{R}^d \) with positive radius and each \( \varepsilon > 0 \) we have

\[
\{ \omega : A(\omega) \cap \overline{B} \neq \emptyset \} \subseteq \{ \omega : A(\omega) \cap (\overline{B} + (q, q))^\varepsilon \neq \emptyset \} \tag{3.11}
\]

almost surely, which follows from swift transitivity (cf. Proposition 4.9 below) and Borel-Cantelli. The exceptional set may depend on \( q, \overline{B} \) and \( \varepsilon \) but if we consider rational \( \varepsilon, q \) with rational coordinates and \( \overline{B} \) with a rational radius and a center with rational coordinates, then there exists a set \( \Omega_1 \subseteq \Omega_0 \) of full measure such that (3.11) holds for all \( \omega \in \Omega_1 \) for all such \( \varepsilon, q \) and \( \overline{B} \). Now assume that \( \omega \in \Omega_1 \), \( v \in A(\omega) \) and \( c \in \mathbb{R}^d \). Let \( \overline{B}_n \) be a sequence of (closed) balls with rational center and radius \( 1/n \) containing \( v \) and let \( q_n \) be a sequence of points in \( \mathbb{R}^d \) with rational coordinates converging to \( c \). Then

\[
\{ \omega \in \Omega_1 : v \in A(\omega) \} \subseteq \bigcap_{n=1}^{\infty} \{ \omega \in \Omega_1 : A(\omega) \cap \overline{B}_n \neq \emptyset \}
\]

\[
\subseteq \bigcap_{n=1}^{\infty} \{ \omega \in \Omega_1 : A(\omega) \cap (\overline{B}_n + (q_n, q_n))^1/n \neq \emptyset \}.
\]

Since \( A(\omega) \) is closed we obtain (3.10), which proves the claim.

Let \( \delta(\omega) := \inf \{ |a - b| : (a, b) \in A(\omega) \} \) and note that \( \delta(\omega) \in (0, \infty) \) on the set \( S \cap \Omega_1 \). Choose \( \delta > 0 \) such that for each \( \varepsilon > 0 \) we have

\[
\mathbb{P}(\delta(\omega) \in (\delta - \varepsilon, \delta + \varepsilon)) > 0.
\]

**Claim:** There exist (deterministic) \( a, b \in \mathbb{R}^d \) such that \( |a - b| = \delta \) and

\[
\mathbb{P}\left( \{ \omega : d\left( A(\omega), \left(\begin{array}{c} a \\ b \end{array}\right) \right) < \varepsilon \mid \delta(\omega) \in (\delta - \varepsilon, \delta + \varepsilon) \} \right) > 0
\]

for every \( \varepsilon > 0 \), where \( d(A, b) := \inf_{a \in A} |a - b| \).

To see this, we first note that using (3.10) with \( c(\omega) = -\frac{a(\omega) + b(\omega)}{2} \) implies that for \( \omega \in \Omega_1 \) there are \( (a, b) = (a(\omega), b(\omega)) \in A(\omega) \) such that \( |b - a| = \delta(\omega) \) and \( |a| = |b| = \delta(\omega)/2 \). Given \( \varepsilon > 0 \), for \( \omega \in \{ \delta(\cdot) \in (\delta - \varepsilon, \delta + \varepsilon) \} \), in particular, we have \( a(\omega), b(\omega) \in \overline{B}(0, \frac{\delta + \varepsilon}{2}) \setminus B(0, \frac{\delta - \varepsilon}{2}) \). Hence, we can choose (deterministic) \( a_\varepsilon, b_\varepsilon \in \overline{B}(0, \frac{\delta + \varepsilon}{2}) \setminus B(0, \frac{\delta - \varepsilon}{2}) \) such that

\[
\mathbb{P}\left( \{ \omega : d\left( A(\omega), \left(\begin{array}{c} a_\varepsilon \\ b_\varepsilon \end{array}\right) \right) < \varepsilon \mid \delta(\omega) \in (\delta - \varepsilon, \delta + \varepsilon) \} \right) > 0
\]

and \( |a_\varepsilon - b_\varepsilon| \in (\delta - \varepsilon, \delta + \varepsilon) \). We may now choose a convergent subsequence of \( (a_\varepsilon, b_\varepsilon) \) for \( \varepsilon \to 0 \), which proves the claim.

Define \( s := \frac{b - a}{|b - a|} \). By Step 2, there exist some \( z \in \mathbb{R}^d \), \( \alpha \in [0, \delta] \) for which (3.8) holds. Continuity of \( b \) guarantees that there exist two (small) closed cylinders \( C_1 \) and \( C_2 \) in \( \mathbb{R}^d \) with respective centers \( z + \alpha s \) and \( z + \alpha s - \delta s \) and with axes parallel to \( s \) such that \( (b(z_1) - b(z_2), s) < 0 \) whenever \( z_1 \in C_1 \) and \( z_2 \in C_2 \). If \( C_1 \)
and $\bar{C}_2$ are cylinders with the same centers but only half the height and radius as $C_1$ respectively $C_2$, then there exists some $\kappa > 0$ such that the probability that the distance of trajectories starting from $z_1 \in \bar{C}_1$ and $z_2 \in \bar{C}_2$ will have a distance smaller than $\delta - \kappa$ before exiting $C_1 \times C_2$ is bounded away from 0 uniformly for all $z_1 \in \bar{C}_1$ and $z_2 \in \bar{C}_2$. Choose $\varepsilon < \kappa$ so small that $\bar{C}_1 \times \bar{C}_2$ contains the $\varepsilon$-neighborhood of $(z + \alpha s, z + \alpha s - \delta s)$.

By (3.10) with $c = z + \alpha s - a$ we have

$$\left\{ \omega : d\left( A(\omega), \left(\begin{array}{c} a \\ b \end{array}\right)\right) < \varepsilon \right\} \subseteq \left\{ \omega : d\left( A(\omega), \left(\begin{array}{c} z + \alpha s \\ z + \alpha s - \delta s \end{array}\right)\right) < \varepsilon \right\} \text{ P-a.s..}$$

Hence, on $\left\{ \omega : d\left( A(\omega), \left(\begin{array}{c} a \\ b \end{array}\right)\right) < \varepsilon \right\} \cap \{ \delta(\omega) \in (\delta - \varepsilon, \delta + \varepsilon) \}$ we have $\delta(\omega) \leq \delta - \kappa$ almost surely which is a contradiction and therefore the proof of the lemma is complete. □

**Proof of Theorem 2.22.** This is now an immediate consequence of Proposition 2.21 and Lemma 3.3. □

### 4. Examples

In this section we provide examples of SDE satisfying asymptotic stability, swift transitivity and contraction on large sets. The section is divided into two parts. In the first part we will focus on asymptotic stability. We first develop a local stable manifold theorem for general, differentiable RDS and prove that a negative top Lyapunov exponent leads to asymptotic stability. We then provide sufficient conditions for SDE to have a negative top Lyapunov exponent. In the second part we will prove swift transitivity and contraction on large sets for SDE with additive noise.

All the (concrete) examples in this section deal with finite dimensional SDE driven by $d$-dimensional Brownian motion, i.e.

$$dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t,$$

(4.1)

with $\sigma$ being Lipschitz continuous, $b$ being locally Lipschitz continuous and $b$ satisfying the following one-sided Lipschitz condition

$$(b(x) - b(y)) \cdot (x - y) \leq \lambda |x - y|^2,$$

(4.2)

for all $x, y \in \mathbb{R}^d$ and some $\lambda > 0$.

By [18] there is a white noise RDS $\varphi$ associated to (4.1), with respect to the canonical setup: The space $\Omega$ is $C\left(\mathbb{R}; \mathbb{R}^d\right)$, $\mathcal{F}$ is the (not completed) Borel $\sigma$-field, $\mathbb{P}$ is the two-sided Wiener measure, $\mathcal{F}_{s,t}$ is the $\sigma$-algebra generated by $\xi_u - \xi_v$ for $s \leq u \leq v \leq t$, where $\xi_s : \Omega \to \mathbb{R}^d$ is defined as $\xi_s(\omega) = \omega(s)$, and $\theta_t$ is the shift

$$(\theta_t \omega)(s) = \omega(s + t) - \omega(t)$$

which is ergodic.
4.1. Asymptotic stability and top Lyapunov exponent. In this section we provide sufficient conditions for asymptotic stability for certain diffusions. We start by considering general RDS and proving that a negative top Lyapunov exponent implies asymptotic stability. Then we provide sufficient conditions for SDE to have negative top Lyapunov exponents.

4.1.1. A time-discrete, local stable manifold theorem and asymptotic stability. Let \( \varphi \) be a white-noise RDS on \( \mathbb{R}^d \) with respect to an ergodic metric dynamical system \((\Omega, \mathbb{P}, \theta)\) and let \( P_t \) be the associated Markovian semigroup. In this section we will introduce the associated Lyapunov spectrum under appropriate assumptions on \( \varphi \) and provide a local stable manifold theorem for discrete time and negative top Lyapunov exponent. We then prove that this implies asymptotic stability.

**Lemma 4.1.** Assume that \( \varphi_t(\omega, \cdot) \in C^{1,\delta} \) for some \( \delta \in (0, 1) \) and all \( t \geq 0 \). Further assume that \( P_t \) has an ergodic invariant measure \( \mu \) such that

\[
\mathbb{E} \int_{\mathbb{R}^d} \log^+ \|D\varphi_1(\omega, x)\|d\mu(x) < \infty. \tag{4.3}
\]

Then

1. There are constants \( \lambda_N < \cdots < \lambda_1 \) (the Lyapunov spectrum) such that

   \[
   \lim_{n \to \infty} \frac{1}{n} \log |D\varphi_n(\omega, x)v| \in \{\lambda_i\}_{i=1}^N,
   \]

   for all \( v \in \mathbb{R}^d \setminus \{0\} \) and \( \mathbb{P} \otimes \mu \)-a.a. \( (\omega, x) \in \Omega \times \mathbb{R}^d \).

2. For every \( \varepsilon \in (\lambda_{\text{top}}, 0) \) there is a measurable map \( \beta : \Omega \times \mathbb{R}^d \to \mathbb{R}^+ \setminus \{0\} \) such that for \( \mu \)-a.a. \( x \in \mathbb{R}^d \)

   \[
   \mathcal{S}(\omega) := \{y \in \mathbb{R}^d : |\varphi_n(\omega, y) - \varphi_n(\omega, x)| \leq \beta(\omega, x)e^{\varepsilon n}, \forall n \in \mathbb{N}\}
   \]

   is an open neighborhood of \( x \), \( \mathbb{P} \)-a.s..

**Proof.** (1): The introduction of the Lyapunov spectrum and the time-discrete stable manifold theorem will be based on [36]. In order to do so, we need to rewrite the dynamics in an appropriate form. This essentially follows the setup put forward in [31].

We define the following extension of the probability space (cf. e.g. [31] p. 626 and Corollary 3.1.1, Remark (iii)]: Let \( M := \Omega \times \mathbb{R}^d \), \( \mathcal{F} := \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d) \), \( \rho = \mathbb{P} \otimes \mu \) and \( \tau : M \to M \) be defined by

\[
\tau(m) := (\theta_1 \omega, \varphi_1(\omega, x)) \quad \text{for } m = (\omega, x) \in M.
\]

By [7] \( \tau \) is ergodic. We then obtain a perfect (time-discrete) cocycle on \( (M, \rho, \tau) \) by

\[
Z_n(m, y) := \varphi_n(\omega, y + x) - \varphi_n(\omega, x) \quad \text{for } m = (\omega, x) \in M.
\]
Note that \( Z_n(m, 0) = 0 \). We further set \( F_m(y) := Z_1(m, y) \),
\[
F^n_m := F_{n-1,m} \circ \cdots \circ F_m
\]
and observe \( F^n_m = Z_n(m, \cdot) \). Obviously, \( F_m \in C^{1,\delta} \) and
\[
T_m = T(m) := DF_m(0) = D\varphi_1(\omega, x) \quad \text{for } m = (\omega, x) \in M.
\]
Setting \( T^n_m := T_{n-1,m} \circ \cdots \circ T_m \) we have \( T^n_m = DF^n_m(0) = D\varphi_n(\omega, x) \) for \( m = (\omega, x) \in M \).
By assumption
\[
\int \log^+ \|T(m)\| d\rho(m) = \mathbb{E} \int_{\mathbb{R}^d} \log^+ \|D\varphi_1(\omega, x)\| d\mu(x) < \infty.
\]
Since \( \rho \) is ergodic, by the multiplicative ergodic theorem \[36\] Theorem 1.6], there are constants \( \{\lambda_i\}_{i=1}^N \) (the Lyapunov spectrum) such that
\[
\lim_{n \to \infty} \frac{1}{n} \log |T^n_m(v)| \in \{\lambda_i\}_{i=1}^N
\]
and the limit exists for \( \rho \)-a.a. \( m \in M \). Since \( T^n_m v = D\varphi_n(\omega, x)v \) this finishes the proof.

(2): By \[36\] Theorem 5.1] (a), there are measurable maps \( \beta > \alpha > 0 \) such that, for a.a. \( m = (\omega, x) \in M \),
\[
\{y \in B(0, \alpha(m)) : |F^n_{m,y}| \leq \beta(m)e^{n\varepsilon}, \forall n \in \mathbb{N}\}
= \{y \in B(0, \alpha(\omega,x)) : |\varphi_n(\omega, y + x) - \varphi_n(\omega, x)| \leq \beta(\omega,x)e^{n\varepsilon}, \forall n \in \mathbb{N}\}
\]
is an open neighborhood of \( 0 \in \mathbb{R}^d \), which implies (2).

\[\square\]

Remark 4.2. In contrast to the time-continuous local stable manifold theorem developed in \[31\], Lemma 4.1 (2) only yields local stability along the natural numbers \( n \in \mathbb{N} \). On the other hand, the assumptions of \[31\] do not cover our model example of a double-well potential. At this point we make use of the weaker form of asymptotic stability introduced in Definition 2.2. In fact, as we will see below, Lemma 4.1 (2) will be sufficient to deduce asymptotic stability, which significantly simplifies the proof of asymptotic stability in cases for which no local stable manifold theorem has (yet) been established.

From Lemma 4.1 (2) we obtain the existence of random neighborhoods of points, that are contracted under the (time-discrete) flow. The following Lemma clarifies the relation to asymptotic stability in the sense of Definition 2.2.

Lemma 4.3. Let \( U_1 \) be a random, non-empty, open set and assume that there is a sequence \( t_n \to \infty \) such that
\[
\mathbb{P} \left( \lim_{n \to \infty} \text{diam} (\varphi_{t_n} (\cdot, U_1(\cdot))) = 0 \right) > 0. \tag{4.4}
\]
Then there is a (deterministic) non-empty, open set \( U \) such that
\[
\mathbb{P} \left( \lim_{n \to \infty} \text{diam} (\varphi_{t_n} (\cdot, U)) = 0 \right) > 0.
\]
In particular, $\varphi$ is asymptotically stable in the sense of Definition 2.2.

**Proof.** Consider the countable family of balls of the form $B(x_m, r_m)$ where $(x_m, r_m)$ is an enumeration of pairs of points $x_m$ of $\mathbb{R}^d$ with rational coordinates and positive rational radii $r_m$. We have

$$\lim_{n \to \infty} \text{diam}(\varphi_{t_n}(\cdot, U_1(\cdot))) = 0 \quad \text{and} \quad \lim_{n \to \infty} \text{diam}(\varphi_{t_n}(\cdot, B(x_m, r_m))) = 0,$$

for some $m \in \mathbb{N}$. Hence, there exists $m \in \mathbb{N}$ such that

$$\mathbb{P}\left(\lim_{n \to \infty} \text{diam}(\varphi_{t_n}(\cdot, B(x_m, r_m))) = 0\right) > 0.$$

The ball $B(x_m, r_m)$ is the set $U$ of the definition of asymptotic stability. The proof is complete.

As immediate consequence we obtain

**Corollary 4.4.** Let $\varphi$ be as in Lemma 4.1 and assume $\lambda_{\text{top}} < 0$. Then $\varphi$ is asymptotically stable in the sense of Definition 2.2.

### 4.1.2. Examples with negative top Lyapunov exponent.

In this section we provide two main classes of SDE for which we prove the top Lyapunov exponent to be negative. The first class of examples will be SDE with eventually monotone drifts and large noise. The second class consists of SDE with gradient structure and small noise.

Consider the following SDE with additive noise

$$dX_t = b(X_t) \, dt + \sigma dW_t \quad \text{on} \quad \mathbb{R}^d,$$

where $\sigma > 0$, $b \in C^{1,\delta}_{\text{loc}}(\mathbb{R}^d)$ for some $\delta \in (0,1)$ and $b$ satisfies (1.2). Hence, there is a corresponding white noise RDS $\varphi$ with $\varphi_t(\omega, \cdot) \in C^{1,\delta}$ and $D\varphi_t(\omega, x)$ satisfies the equation

$$\frac{d}{dt} D\varphi_t(\omega, x) = Db(\varphi_t(\omega, x)) \, D\varphi_t(\omega, x), \quad D\varphi_0(\omega, x) = \text{Id}.$$ 

In particular, given any $v \in \mathbb{R}^d \setminus \{0\}$,

$$\frac{d}{dt} |D\varphi_t(\omega, x) v|^2 = 2 (Db(\varphi_t(\omega, x)) \, D\varphi_t(\omega, x) v, D\varphi_t(\omega, x) v)$$

and we obtain the bound

$$\log^+ \|D\varphi_t(\omega, x)\|^2 \leq 2 \int_0^t \| Db(\varphi_r(\omega, x)) \| \, dr.$$ 

Since $\mathbb{P} \otimes \mu$ is invariant with respect to the skew product flow $\Theta_t(\omega, x) := (\theta_t \omega, \varphi_t(\omega, x))$ we obtain

$$\mathbb{E} \int_{\mathbb{R}^d} \log^+ \|D\varphi_1(\omega, x)\|^2 d\mu(x) \leq 2 \int_{\mathbb{R}^d} \| Db(x) \| d\mu(x).$$

It thus remains to establish $Db \in L^1_\mu$ under appropriate conditions. We assume
(1) Subexponential growth of $Db$: For all $c > 0$
\[ \|Db(x)e^{-c|x|}\| = 0, \quad \text{for } |x| \to \infty. \] (4.8)

(2) Coercivity:
\[ (b(x), x) \leq -c|x|^2 + C \quad \text{for all } x \in \mathbb{R}^d, \] (4.9)

for some $c > 0$, $C \geq 0$.

By [24, Theorem 4.3] we know that $\varphi$ is strongly mixing with invariant probability measure $\mu$. Due to (4.9) it is not difficult to show, by Itô’s formula applied to $e^{\gamma |X_t|^2}$ that we have
\[ \int_{\mathbb{R}^d} e^{\gamma |x|^2} d\mu(x) < \infty, \] (4.10)

for $\gamma > 0$ small enough. In combination with (4.8) this yields $Db \in L^1_\mu$ and thus (4.3).

Hence, an application of Lemma 4.1 implies the existence of a corresponding (deterministic) Lyapunov spectrum with top Lyapunov exponent $\lambda_{\text{top}}$. Due to Corollary 4.4 it only remains to establish $\lambda_{\text{top}} < 0$ in order to prove asymptotic stability for (4.5). We define

**Definition 4.5.** A vector field $b: \mathbb{R}^d \to \mathbb{R}^d$ is said to be eventually strictly monotone if there exists an $R > 0$ such that
\[ (b(x) - b(y), x - y) \leq -\lambda_1 |x - y|^2 \quad \text{for all } |x|, |y| > R \]

for some $\lambda_1 > 0$.

**Example 4.6.** Let $b \in C^{1,\delta}_{\text{loc}}$ for some $\delta \in (0, 1)$ satisfy (1.2), (4.8), (4.9) and consider the SDE
\[ dX_t = b(X_t) dt + \sigma dW_t \quad \text{on } \mathbb{R}^d \]

with $\sigma > 0$. If $b$ is eventually strictly monotone and $\sigma$ is large enough, then $\lambda_{\text{top}} < 0$.

**Proof. Step 1:** By (4.6), for any $v \in \mathbb{R}^d \setminus \{0\}$,
\[ \frac{d}{dt} |D\varphi_t(\omega, x) v|^2 = 2(Db(\varphi_t(\omega, x)) D\varphi_t(\omega, x) v, D\varphi_t(\omega, x) v) \]
\[ = 2(Db(\varphi_t(\omega, x)) r_t(\omega, x, v), r_t(\omega, x, v)) |D\varphi_t(\omega, x) v|^2 \]

where $r_t(\omega, x, v) = \frac{Db(\varphi_t(\omega, x))}{|D\varphi_t(\omega, x)|}$. Hence,
\[ |D\varphi_t(\omega, x) v|^2 = |v|^2 e^{2\int_0^t (Db(\varphi_s(\omega, x)) r_s(\omega, x, v), r_s(\omega, x, v)) ds}. \]

Recall that there exists a $v \in \mathbb{R}^d \setminus \{0\}$ such that
\[ \lambda_{\text{top}} = \lim_{n \to \infty} \frac{1}{n} \log |D\varphi_n(\omega, x) v|. \]
Hence,  
\[ \lambda_{\text{top}} = \lim_{n \to \infty} \frac{1}{n} \int_0^n (Db(\phi_s(\omega, x)) r_s(\omega, x, v), r_s(\omega, x, v)) ds. \]

With  
\[ \lambda^+(x) := \max_{|r|=1} (Db(x), r) \]
\[ \lambda^-(x) := \min_{|r|=1} (Db(x), r) \]
we thus have  
\[ \limsup_{n \to \infty} \frac{1}{n} \int_0^n \lambda^-(\phi_s(\omega, x)) ds \leq \lambda_{\text{top}} \leq \liminf_{n \to \infty} \frac{1}{n} \int_0^n \lambda^+(\phi_s(\omega, x)) ds. \]

Since \( Db \in L^1_\mu \) we have \( \lambda^\pm \in L^1_\mu \) and ergodicity yields
\[ \int_{\mathbb{R}^d} \lambda^-(x) d\mu(x) \leq \lambda_{\text{top}} \leq \int_{\mathbb{R}^d} \lambda^+(x) d\mu(x). \]  
(4.11)

We aim to estimate the right hand side.

By eventual strict monotonicity of \( b \) we have
\[ \int_{\mathbb{R}^d} \lambda^+(x) d\mu(x) = \int_{B_R} \lambda^+(x) d\mu(x) + \int_{B_R^c} \lambda^+(x) d\mu(x) \]
\[ \leq \| Db \|_{C^0(B_R)} \mu(B_R) - \lambda_1 \mu(B_R^c). \]  
(4.12)

Next, we will prove that for \( \sigma \gg 1 \) the invariant measure \( \mu \) “flattens”, i.e. for each \( R \geq 0, \mu(B_R) \to 0 \) for \( \sigma \to \infty \). Thus, the right hand side in (4.12) becomes negative for \( \sigma \) large enough, which finishes the proof.

**Step 2:** For each \( R \geq 0, \mu(B_R) \to 0 \) for \( \sigma \to \infty \).

Indeed: Given \( \sigma > 0 \) let \( \mu^\sigma \) be the corresponding invariant measure, thus solving the Fokker-Planck equation
\[ \frac{\sigma^2}{2} \Delta \mu^\sigma - \text{div}(b \mu^\sigma) = 0 \]
in distributional sense. Consequently, for all \( \varphi \in C^\infty_c \) we have  
\[ \int (\Delta \varphi + \frac{2}{\sigma^2} b \cdot \nabla \varphi) d\mu^\sigma = 0. \]
Since \( \mu^\sigma(\mathbb{R}^d) = 1 \), there is a weakly* convergent subsequence \( \mu^{\sigma_n} \rightharpoonup^* \mu \) in the space of all signed measures of total variation on \( \mathbb{R}^d \). Clearly, \( \mu(\mathbb{R}^d) \leq 1 \). Since,
\[ - \int (\frac{2}{\sigma^2} b \cdot \nabla \varphi) d\mu^\sigma \leq \frac{2}{\sigma^2} \| b \cdot \nabla \varphi \|_{C^0} \mu^\sigma(\mathbb{R}^d) = \frac{2}{\sigma^2} \| b \cdot \nabla \varphi \|_{C^0} \]
we have  
\[ \int \Delta \varphi d\mu^{\sigma_n} \leq \frac{2}{\sigma^2} \| b \cdot \nabla \varphi \|_{C^0}. \]
Taking the limit yields \( \int \Delta \varphi \, d\mu \leq 0 \), for all \( \varphi \in C_\infty^c \). Thus, also \( \int \Delta \varphi \, d\mu = 0 \), for all \( \varphi \in C_\infty^c \). We next show that this implies \( \mu = 0 \). Let \( \varphi_\lambda (x) = e^{-\lambda |x|^2} \) and note
\[
\Delta \varphi_\lambda (x) = 2\lambda \varphi_\lambda (x) (2\lambda |x|^2 - d).
\]
Given \( R > 0 \) we can choose \( \lambda \) small enough such that \( -\Delta \varphi_\lambda (x) \geq \lambda \) for all \( x \in B_R \).

Let \( R(\lambda) = \sqrt{\frac{d}{2\lambda}} \) and note \( R(\lambda) \geq R \) for all \( \lambda \) small enough. Then
\[
\mu (B_R) \leq \frac{1}{\lambda} \int_{B_{R(\lambda)}} -\Delta \varphi_\lambda \, d\mu
\]
\[
\leq \frac{1}{\lambda} \int_{B_{(\lambda)}} -\Delta \varphi_\lambda \, d\mu
\]
\[
= \frac{1}{\lambda} \int_{B_{(\lambda)}} \Delta \varphi_\lambda \, d\mu.
\]
Note that \( \frac{1}{\lambda} \Delta \varphi_\lambda (x) \leq 4e^{-\lambda |x|^2} |x|^2 \leq C \) for some constant \( C \) independent of \( \lambda \geq 0 \). Since \( R(\lambda) \to \infty \) for \( \lambda \to 0 \) and \( \mu (\mathbb{R}^d) \leq 1 \) we obtain
\[
\mu (B_R) \leq C \mu (B_{R(\lambda)}) \to 0
\]
for \( \lambda \to 0 \). Hence, \( \mu = 0 \) and thus \( \mu^{\sigma_n} \rightharpoonup^* 0 \), which finishes the proof. \( \square \)

We next consider the case of SDE with gradient structure, i.e.
\[
dX_t = -\nabla V (X_t) \, dt + \sigma dW_t \quad \text{on} \quad \mathbb{R}^d,
\]
with \( \sigma > 0 \) and \( V \in C^{2,\delta}_{\text{loc}} (\mathbb{R}^d) \) for some \( \delta > 0 \) satisfying
\[
V (x) \geq C_0 \log |x|
\]
\[
\|D^2V (x)\| \leq C_0 |x|^N \quad \forall \ |x| \geq R_0,
\]
for some \( R_0 > 1, N \geq 0 \). Further assume that \( b := -\nabla V \) satisfies \ref{ass:sigma}. By \ref{ass:V} we have \( \rho (x) := e^{-\frac{1}{\sigma^2} V (x)} \in L^1 (\mathbb{R}^d) \) for \( \sigma \) small enough.

We have seen in Section 2.2 that there is a corresponding white noise RDS \( \varphi \) with strongly mixing invariant probability measure \( \mu \). Using \ref{ass:V}, it is easy to see that \ref{ass:det} is satisfied for \( \sigma \) small enough. Thus, the top Lyapunov exponent \( \lambda_{\text{top}} \) is well-defined and it only remains to show \( \lambda_{\text{top}} < 0 \).

**Example 4.7.** Consider the SDE in \( \mathbb{R}^d \)
\[
dX_t = -\nabla V (X_t) \, dt + \sigma dW_t,
\]
with \( \sigma > 0 \), \( V \in C^{2,\delta}_{\text{loc}} (\mathbb{R}^d) \) for some \( \delta > 0 \), \( V \) satisfying \ref{ass:V} and \( b := -\nabla V \) satisfying \ref{ass:sigma}. Further assume
\[
\inf \left\{ \min_{|r|=1} (D^2V (x) r, r) : x \text{ global minimum of } V \right\} > 0.
\]
Then \( \lambda_{\text{top}} < 0 \) for \( \sigma \) small enough.
Proof. Recall that
\[ d\mu(x) = \frac{1}{Z_\sigma} e^{-\frac{2}{\sigma^2} V(x)} dx, \]
where \( Z_\sigma = \int e^{-\frac{2}{\sigma^2} V(x)} dx \). This integral is finite for \( \sigma \) small enough, because \( V(x) \geq C_0 \log |x| \) for large \( x \). Let \( \mathcal{M} \) denote the set of global minima of \( V \). Without loss of generality, we may assume \( V = 0 \) on \( \mathcal{M} \) (hence \( V \geq 0 \) on \( \mathbb{R}^d \)) and \( 0 \in \mathcal{M} \). We also have \( DV = 0 \) on \( \mathcal{M} \).

**Step 1:** We prove that, for some constant \( C > 0 \), we have
\[ Z_\sigma \geq C \sigma^d \quad \forall \sigma \in (0, 1]. \]
Let \( C_1 := \sup_{B(0,1)} \|D^2 V\| \). For \( x \in B(0,1) \) we have
\[ V(x) = \frac{1}{2} \left( D^2 V \left( \theta_x x, x \right) \right) \leq C_1 |x|^2 \]
for some \( \theta_x \in (0,1) \). Hence, for \( x \in B(0,\sigma) \), we have
\[ V(x) \leq C_1 \sigma^2 \]
and therefore
\[ e^{-\frac{2}{\sigma^2} V(x)} \geq e^{-2C_1} \]
for a suitable constant \( C > 0 \).

**Step 2:** We prove that, for every \( R \geq R_0 \),
\[ \lim_{\sigma \to 0} \frac{1}{Z_\sigma} \int_{B(0,R) \setminus U} \left( 1 + \|D^2 V(x)\| \right) e^{-\frac{2}{\sigma^2} V(x)} dx = 0. \]
We have (using Step 1)
\[ \frac{1}{Z_\sigma} \int_{B(0,R) \setminus U} \left( 1 + \|D^2 V(x)\| \right) e^{-\frac{2}{\sigma^2} V(x)} dx \leq \frac{1}{C \sigma^d} \int_{B(0,R)} \left( 1 + C_0 |x|^N \right) |x|^{-\frac{2C_0}{\sigma^2}} dx \]
and the result follows by dominated convergence.

**Step 3:** Let \( U \) be an open neighborhood of \( \mathcal{M} \). We prove that
\[ \lim_{\sigma \to 0} \frac{1}{Z_\sigma} \int_{\mathcal{U}} e^{-\frac{2}{\sigma^2} V(x)} dx = 1 \]
and that, for every \( R \geq R_0 \) such that \( U \subset B(0,R) \),
\[ \lim_{\sigma \to 0} \frac{1}{Z_\sigma} \int_{B(0,R) \setminus U} e^{-\frac{2}{\sigma^2} V(x)} dx = 0. \]
We have \( k = \inf_{U} V > 0 \). Let \( U' \subset U \) be such that \( V(x) \leq \frac{k}{2} \) for all \( x \in U' \). Then
\[
\int_{U'} e^{-\frac{x^2}{\sigma^2}} dx \geq |U'| e^{-\frac{k}{\sigma^2}}
\]
\[
\int_{B(0,R) \setminus U} e^{-\frac{x^2}{\sigma^2}} dx \leq |B(0,R)| e^{-\frac{2k}{\sigma^2}}.
\]
Hence
\[
\int_{B(0,R) \setminus U} e^{-\frac{x^2}{\sigma^2}} dx \leq g(\sigma) \int_{U} e^{-\frac{x^2}{\sigma^2}} dx
\]
where \( g(\sigma) := \frac{|B(0,R)| e^{-\frac{2k}{\sigma^2}}}{|x'| e^{-\frac{1}{\sigma^2}}} \to 0 \) as \( \sigma \to 0 \). Moreover, we have seen in Step 2 that
\[
\lim_{\sigma \to 0} \frac{1}{Z_\sigma} \int_{B(0,R)^c} e^{-\frac{x^2}{\sigma^2}} dx = 0.
\]
Therefore
\[
\frac{1}{Z_\sigma} \int_{U} e^{-\frac{x^2}{\sigma^2}} dx
\]
\[
= 1 - \frac{1}{Z_\sigma} \int_{B(0,R) \setminus U} e^{-\frac{x^2}{\sigma^2}} dx - \frac{1}{Z_\sigma} \int_{B(0,R)^c} e^{-\frac{x^2}{\sigma^2}} dx
\]
\[
\geq 1 - g(\sigma) \frac{1}{Z_\sigma} \int_{U} e^{-\frac{x^2}{\sigma^2}} dx - \frac{1}{Z_\sigma} \int_{B(0,R)^c} e^{-\frac{x^2}{\sigma^2}} dx
\]
and the result follows by dominated convergence. The proof of the second claim is similar.

**Step 4:** We may now complete the proof. Under our assumptions, there exists an open neighborhood \( U \) of \( M \) such that \( c := \inf \{ \min_{|r| = 1} (D^2 V(x) r, r) : x \in U \} > 0 \). Since \( \lambda^+ (x) = -\min_{|r| = 1} (D^2 V(x) r, r) \), we have
\[
\lambda^+ (x) \leq \| D^2 V(x) \| \quad \text{for all } x \in \mathbb{R}^d
\]
\[
\lambda^+ (x) \leq -c \quad \text{for all } x \in U.
\]
Hence, for \( R \geq R_0 \) such that \( U \subset B(0,R) \), we have
\[
\int \lambda^+ (x) d\mu(x) \leq \frac{1}{Z_\sigma} \int_{B(0,R)^c} \| D^2 V(x) \| e^{-\frac{x^2}{\sigma^2}} dx
\]
\[
+ \left( \sup_{B(0,R)} \| D^2 V(x) \| \right) \frac{1}{Z_\sigma} \int_{B(0,R) \setminus U} e^{-\frac{x^2}{\sigma^2}} dx
\]
\[
- \frac{c}{Z_\sigma} \int_{U} e^{-\frac{x^2}{\sigma^2}} dx.
\]
Form the results of the previous steps we get
\[
\int \lambda^+ (x) d\mu(x) < 0
\]
for \( \sigma \) small enough, hence \( \lambda_{\text{top}} < 0 \) by the same arguments as used in the proof of Example 4.6. \[ \square \]

We next consider SDE of the type (4.13) with radially symmetric potential. Note that we neither need to assume \( \sigma \) small nor an assumption of the type (4.14) here.

**Example 4.8.** Consider the SDE

\[
dX_t = -\nabla V(X_t) \, dt + \sigma \, dW_t \quad \text{on } \mathbb{R}^d,
\]

with \( \sigma > 0 \) and \( b = -\nabla V \) satisfying (4.2). Further assume that \( V \) is radially symmetric with \( V(x) = g(|x|^2) \), \( g \in C_{\text{loc}}^{2,\delta} \) being a convex function and \( \rho(x) = e^{-\frac{2}{\sigma^2}V(x)} \in L^1(\mathbb{R}^d) \). Then \( \lambda_{\text{top}} < 0 \).

**Proof.** We first note that, since \( \rho \in L^1 \), by the same arguments as in Section 2.2, Lemma 4.1 applies.

Since \( V(x) = g(|x|^2) \) we compute

\[
\nabla V(x) = 2g'(|x|^2)x \\
D^2 V(x) = 2g''(|x|^2)x \otimes x + 2g'(|x|^2)Id.
\]

Thus (using \( g'' \geq 0 \))

\[
\min_{|r|=1} (D^2 V(x)r, r) = 2g'(|x|^2).
\]

We note that

\[
\int e^{-\frac{2}{\sigma^2}V(x)} \, dx = \int e^{-\frac{2}{\sigma^2}g(|x|^2)} \, dx = \int_{\mathbb{R}^d} r^{d-1} e^{-\frac{2}{\sigma^2}g(r^2)} \, dr < \infty.
\]

Hence, there is a sequence \( t_n \uparrow \infty \) such that \( t_n^{d-1} e^{-\frac{2}{\sigma^2}g(t_n^2)} \rightarrow 0 \) for \( n \rightarrow \infty \). We conclude, (assume \( d \geq 2 \), for \( d = 1 \) one may proceed similarly)

\[
\int \lambda^+(x) d\mu(x) = -\frac{2}{Z} \int g'(|x|^2) e^{-\frac{2}{\sigma^2}g(|x|^2)} \, dx \\
= -c \int_{\mathbb{R}^d} r^{d-1} g'(r^2) e^{-\frac{2}{\sigma^2}g(r^2)} \, dr \\
= c \int_{\mathbb{R}^d} r^{d-2} \frac{d}{dr} e^{-\frac{2}{\sigma^2}g(r^2)} \, dr \\
\leq \lim_{n \rightarrow \infty} t_n^{d-2} e^{-\frac{2}{\sigma^2}g(t_n^2)} - c \int_{\mathbb{R}^d} r^{d-3} e^{-\frac{2}{\sigma^2}g(r)} \, dr < 0,
\]

which implies \( \lambda_{\text{top}} < 0 \) by the same arguments as used in the proof of Example 4.6. \[ \square \]
4.2. Properties of dissipativity, contraction and swift transitivity. This section is devoted to the proof of contraction on large sets and swift transitivity. We consider SDE with additive noise

\[ dX_t = b(X_t)dt + \sigma dW_t, \quad (4.15) \]

where \( b \) is locally Lipschitz and satisfies (4.2).

**Proposition 4.9.** Let \( \varphi \) be the RDS associated to (4.15). Then, for all balls \( B(x, r) \) and \( \tau > 0, y \in \mathbb{R}^d \) one has

\[ \mathbb{P}(\varphi_t (\cdot, B(x, r)) \subset B(y, r(1 + \tau))) > 0 \quad \forall t > 0. \]

In particular, the swift transitivity property holds.

Assume, in addition, that \( b \) is monotone on large sets, i.e. for each \( r > 0 \) there exists some \( z \in \mathbb{R}^d \) such that

\[ (b(x) - b(y), x - y) < 0 \quad \text{for all } x \neq y, x, y \in B(z, r). \]

Then the property of contraction on large sets holds.

**Proof. Part 1** (swift transitivity). Let \( B(x, r) \) be the starting ball and \( y \) the arrival point in the property of swift transitivity. Let \( \varepsilon > 0 \) be given. Let us first construct a function \( \omega^\varepsilon(t), t \in [0, \varepsilon] \), such that \( \varphi_\varepsilon (\omega^\varepsilon, x) = y \). Let

\[ \psi^\varepsilon(t) := x + \frac{t}{\varepsilon} (y - x), \quad t \in [0, \varepsilon]. \]

Define the function

\[ \omega^\varepsilon(t) := \psi^\varepsilon(t) - x - \int_0^t b(\psi^\varepsilon(s)) \, ds, \quad t \in [0, \varepsilon]. \]

We have \( \varphi_t (\omega^\varepsilon, x) = \psi^\varepsilon(t), t \in [0, \varepsilon] \), and in particular \( \varphi_\varepsilon (\omega^\varepsilon, x) = y \).
Let \( \phi_t(\omega, x) := \varphi_t(\omega, x) - \omega(t) \). Then, using that \( b \) is one-sided Lipschitz and locally Lipschitz we observe

\[
\frac{1}{2}|\phi_t(\omega, x) - \phi_t(\omega^\varepsilon, x)|^2 \\
= \int_0^t (b(\varphi_s(\omega, x)) - b(\varphi_s(\omega^\varepsilon, x)), \phi_s(\omega, x) - \phi_s(\omega^\varepsilon, x))ds \\
= \int_0^t (b(\phi_s(\omega, x) + \omega(s)) - b(\phi_s(\omega^\varepsilon, x) + \omega(s)), \phi_s(\omega, x) - \phi_s(\omega^\varepsilon, x))ds \\
= \int_0^t (b(\phi_s(\omega, x) + \omega(s)) - b(\phi_s(\omega^\varepsilon, x) + \omega(s)), \phi_s(\omega, x) - \phi_s(\omega^\varepsilon, x))ds \\
\leq \int_0^t (\lambda + 1)|\phi_s(\omega, x) - \phi_s(\omega^\varepsilon, x)|^2ds \\
+ \int_0^t |b(\phi_s(\omega^\varepsilon, x) + \omega(s)) - b(\phi_s(\omega^\varepsilon, x) + \omega(s))|^2ds.
\]

Let \( \mathcal{V}_1 \) be the set of all \( \omega \in \Omega \) such that \( \sup_{s \in [0,\varepsilon]} |\omega(s) - \omega^\varepsilon(s)| \leq 1 \). By local Lipschitz continuity of \( b \) we obtain the existence of some \( L \geq 0 \) such that

\[
\frac{1}{2}|\phi_t(\omega, x) - \phi_t(\omega^\varepsilon, x)|^2 \\
\leq \int_0^t (\lambda + 1)|\phi_s(\omega, x) - \phi_s(\omega^\varepsilon, x)|^2ds + L^2 \int_0^t |\omega(s) - \omega^\varepsilon(s)|^2ds,
\]

for all \( \omega \in \mathcal{V}_1 \). Gronwall’s Lemma thus implies

\[
|\phi_t(\omega, x) - \phi_t(\omega^\varepsilon, x)| \leq 2e^{(\lambda+1)t}L \sup_{s \in [0,t]} |\omega(s) - \omega^\varepsilon(s)|.
\]

Now let \( \mathcal{V}_2 \) be the set of all \( \omega \in \mathcal{V}_1 \) such that \( \sup_{s \in [0,\varepsilon]} |\omega(s) - \omega^\varepsilon(s)| \leq \frac{\pi}{4(\lambda+1)L} r \). Then

\[
|\varphi_t(\omega, x') - \varphi_t(\omega^\varepsilon, x)| \leq \left(1 + \frac{\pi}{2}\right) r,
\]

for all \( t \in [0,\varepsilon] \) on \( \mathcal{V}_2 \). Choosing \( t = \varepsilon \) yields \( |\varphi_\varepsilon(\omega, x') - y| \leq \left(1 + \frac{\pi}{2}\right) r \). Moreover, \( P(\mathcal{V}_2) > 0 \). The proof of swift transitivity is complete.

**Part 2** (contraction on large sets).

Let \( R > 0 \). By assumption there is a \( z \in \mathbb{R}^d \) such that

\[
(b(x) - b(y), x - y) < 0 \quad \text{for all } x \neq y, x, y \in B(z, 2R).
\]  

(4.16)

**Step 1:** We first prove that there is an \( \eta \in (0, 1) \) and a time \( T_0 > 0 \) such that for all \( r \in (0, R) \):

\[
P \left( \text{diam} \left( \varphi_{T_0}(\omega, B(z, r)) \right) \leq r(1 - 2\eta) \right) > 0.
\]
Indeed: By (4.16), there is a $c > 0$ such that
\[(b(x) - b(y), x - y) < -c|x - y|^2 \quad \forall x \neq y, x, y \in B(z, 2R), |x - y| \geq \frac{R}{8}. \quad (4.17)\]

Since $b$ is Lipschitz continuous on $B(z, 2R)$ there is a $T_0 > 0$ (independent of $r$) and a set $\Omega_0 \subseteq \Omega$ of positive $\mathbb{P}$ measure such that
\[\varphi_t(\omega, B(z, r)) \subseteq B(z, 2R) \quad \forall t \in [0, T_0], \; \omega \in \Omega_0.\]

For $x, y \in B(z, r)$ let
\[\tau(\omega) := \inf \{t \in \mathbb{R}_+ : |\varphi_t(\omega, x) - \varphi_t(\omega, y)| \leq \frac{R}{8}\}.\]

Due to (4.17), on $\{\tau \geq T_0\} \cap \Omega_0$ we have
\[|\varphi_t(\omega, x) - \varphi_t(\omega, y)|^2 \leq r - c \int_0^t |\varphi_r(\omega, x) - \varphi_r(\omega, y)|^2 dr \quad \forall t \in [0, T_0].\]

By Gronwall’s Lemma this implies
\[|\varphi_{T_0}(\omega, x) - \varphi_{T_0}(\omega, y)|^2 \leq re^{-cT_0} \quad \forall \omega \in \{\tau \geq T_0\} \cap \Omega_0.\]

Moreover, due to (4.16), $t \mapsto |\varphi_t(\omega, x) - \varphi_t(\omega, y)|^2$ is non-increasing for a.a. $\omega \in \Omega$. Thus,
\[|\varphi_{T_0}(\omega, x) - \varphi_{T_0}(\omega, y)|^2 \leq re^{-cT_0} \vee \frac{R}{8} \quad \forall \omega \in \Omega_0.\]

In particular,
\[\text{diam}(\varphi_{T_0}(\omega, B(z, r))) \leq re^{-cT_0} \vee \frac{R}{8} \quad \forall \omega \in \Omega_0.\]

Setting $\eta = \frac{1}{2}(1 - e^{-cT_0})$ proves the claim.

Step 2: By step one there is an $x \in \mathbb{R}^d$ such that
\[\varphi_{T_0}(\cdot, B(z, R)) \subseteq B(x, R(1 - 2\eta) \vee \frac{R}{8}),\]

with positive probability. By part one we have
\[\varphi_1(\cdot, B(x, R(1 - 2\eta) \vee \frac{R}{4})) \subseteq B(z, R(1 - \eta) \vee \frac{R}{4}),\]

with positive probability. Since $\varphi$ is a white noise RDS this implies
\[\varphi_{T_0 + 1}(\cdot, B(z, R)) \subseteq B(z, R(1 - \eta) \vee \frac{R}{4}),\]

with positive probability.

Iterating this argument $n$ times yields
\[\varphi_{n(T_0 + 1)}(\cdot, B(z, R)) \subseteq B(z, R(1 - \eta)^n \vee \frac{R}{4}),\]

with positive probability. Choosing $n$ large enough finishes the proof. \[\square\]
4.3. **Summary, explicit examples and open problems.** We close the paper by giving a short summary of general conditions on SDE for synchronization by noise and providing some explicit examples of our general results.

**Theorem 4.10** (Synchronization for general drift). For SDE of the form (1.1), with drift \( b \in C_{loc}^{1,\delta}(\mathbb{R}^d) \) satisfying (4.2), (4.8) and assuming eventual strict monotonicity (cf. Definition 4.5), we have synchronization by noise for sufficiently large noise intensity \( \sigma \).

**Proof.** First, notice that eventual strict monotonicity implies monotonicity on large sets (cf. Proposition 4.9). Moreover, it also implies (4.9), because it gives us

\[
(b(x) - b(0), x) \leq -\lambda_1 |x|^2 \text{ for all } |x| > R, \text{ hence } (b(x), x) \leq -\lambda_1 |x|^2 + |b(0)||x| \text{ for all } |x| > R,
\]

which easily yields (4.9) using local boundedness of \( b \).

Condition (4.2) and the local Lipschitz property, implied by \( b \in C_{loc}^{1,\delta}(\mathbb{R}^d) \), yield the existence of an RDS. Assumptions (4.8), (4.9) and \( b \in C_{loc}^{1,\delta}(\mathbb{R}^d) \) imply the assumptions of Lemma 4.1 and thus the existence of the Lyapunov spectrum, in particular \( \lambda_{top} \). By Example 4.6 and the property of eventual strict monotonicity we deduce \( \lambda_{top} < 0 \) for large noise intensity \( \sigma \). By Corollary 4.4 this implies asymptotic stability.

The local Lipschitz property and the monotonicity on large sets guarantee swift transitivity and contraction on large sets, by Proposition 4.9. Then, by Theorem 2.9 and Lemma 2.12, we deduce synchronization. \( \square \)

For gradient systems, \( b = -\nabla V \), condition (4.2) on \( b \) can be expressed more naturally as

\[
(D^2 V(x) \xi, \xi) \leq \lambda |\xi|^2
\]

for all \( x, \xi \in \mathbb{R}^d \) and some \( \lambda \geq 0 \).

**Theorem 4.11** (Synchronization for gradient systems). Consider an SDE of the form (1.3) with gradient drift \( b = -\nabla V \). Assume weak asymptotic stability for (1), asymptotic stability for (2) and consult the next theorem for a list of sufficient conditions. Then:

1. If \( V \in C^2(\mathbb{R}^d) \) satisfies condition (4.18) and \( e^{-\frac{t}{\sigma^2}}V \in L^1(\mathbb{R}^d) \), then weak synchronization holds.
2. If \( V \in C_{loc}^{2,\delta}(\mathbb{R}^d) \) and that there exists an \( R > 0 \) such that \( (D^2 V(x) \xi, \xi) < 0 \) for all \( |x| > R \) and \( \xi \in \mathbb{R}^d \), then synchronization holds.

**Proof.** (1): Follows by Theorem 2.22. (2): Condition (4.19) plus local boundedness of \( D^2 V(x) \) imply condition (4.18). Condition (4.18), which implies (4.2) for \( b = -\nabla V \), and the local Lipschitz property of \( b \), implied by \( V \in C_{loc}^{2,\delta}(\mathbb{R}^d) \), yield

\( ^3\)Note that this property is implied by (4.20).
the existence of an RDS. Then Proposition 4.9 applies because \( b = -\nabla V \) is locally Lipschitz and (4.19) implies monotonicity on large sets. By Theorem 2.9 and Lemma 2.12, we deduce synchronization. \( \square \)

**Theorem 4.12** (Sufficient conditions for asymptotic stability of gradient systems). Consider an SDE of the form (1.3) with gradient drift \( b = -\nabla V \). Assume that \( V \in C^{2,\delta}_{\text{loc}}(\mathbb{R}^d) \) satisfies conditions (4.14) and (4.18).

1. If the positivity assumption on the infimum in Example 4.7 is satisfied, then asymptotic stability holds for small noise intensity \( \sigma > 0 \).
2. If there exists an \( R > 0 \) such that
   \[
   \left(D^2V(x)\xi,\xi\right) \leq -\lambda_1 |\xi|^2 \tag{4.20}
   \]
   for all \( |x| > R \), \( \xi \in \mathbb{R}^d \) and some \( \lambda_1 > 0 \), then asymptotic stability holds for large noise intensity \( \sigma > 0 \).
3. If \( V(x) = g(|x|^2) \), \( g \in C^{2,\delta}_{\text{loc}} \) is convex, and we have \( e^{-\frac{2}{\sigma^2}}V \in L^1(\mathbb{R}^d) \) for some \( \sigma > 0 \), then asymptotic stability holds. Note that (4.14) is not needed here.

**Proof.** Conditions (4.18) and \( V \in C^{2,\delta}_{\text{loc}}(\mathbb{R}^d,\mathbb{R}) \) give us the existence of an RDS, as for (2) of the previous theorem. Assumptions (4.14) implies that the assumptions of Lemma 4.1 hold, hence \( \lambda_{\text{top}} \) exists. In case (3), \( V \in C^{2,\delta}_{\text{loc}}(\mathbb{R}^d,\mathbb{R}) \) follows from \( g \in C^{2,\delta}_{\text{loc}} \) and see the proof of Example 4.8 to realize that Lemma 4.1 applies without assumption (4.14). Asymptotic stability follows by Corollary 4.4 as soon as we prove \( \lambda_{\text{top}} < 0 \). Let us recall the proof in three cases. For (1) see Example 4.7 and for (3) see Example 4.8. Finally, for (2), condition (4.20) implies eventual strict monotonicity of \( b \), hence \( \lambda_{\text{top}} < 0 \) for large \( \sigma \) by Example 4.6, where the assumptions (4.2), (4.8) and (4.9) of Example 4.6 hold by (4.18), (4.14) and the fact that eventual strict monotonicity implies (4.9). \( \square \)

As in (11) our results apply to

\[
V_E(x) := (0.5 - 10e(-|x-p_1|^2) - 10e(-|x-p_2|^2))|x|^2,
\]

\[
V_S(x) := (2 - 5e(-|x-p_3|^2) - 6e(-|x-p_4|^2) - 7e(-|x-p_5|^2))|x|^2,
\]

where \( p_1 = (0,1), p_2 = (0,-1), p_3 = (0,2), p_4 = (2,-2), p_5 = (-2,-2) \). In contrast to (11), where only small noise \( \sigma \) can be treated, our results also yield synchronization for large noise \( \sigma \).

As pointed out in the introduction, the model example of a double-well potential

\[
V_D(x) = \frac{1}{4}|x|^4 - \frac{1}{2}|x|^2,
\]

is not covered by the techniques in (11) for \( d \geq 2 \). In contrast, our results imply synchronization in this case for all \( \sigma > 0 \). In particular, no restriction to small or large noise \( \sigma \) is required here.
We close the paper by pointing out some open problems: In Theorem 2.22 we assumed that weak asymptotic stability holds. We leave it as an open problem whether this condition is always satisfied for gradient type SDE with additive noise (1.3). Our general results may also be applied to infinite dimensional examples. In particular, synchronization for SPDE could be investigated. This will be subject of subsequent work. Numerical evidence suggests that the top Lyapunov exponent for the Lorentz system perturbed by strong noise (i.e. for $\sigma$ large) is negative and (weak) synchronization occurs. The Lorentz system, however, is not covered by the techniques put forward in Section 4. We leave this as an open problem. We prove swift transitivity for a large class of SDE with (non-degenerate) additive noise in Section 4. It is left as an open problem to establish swift transitivity in other situations, such as degenerate additive or multiplicative noise.

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