Interaction signatures and non-Gaussian photon states from a strongly driven atomic ensemble coupled to a nanophotonic waveguide

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We study theoretically a laser-driven one-dimensional chain of atoms interfaced with the guided optical modes of a nanophotonic waveguide. The period of the chain and the orientation of the laser field can be chosen such that emission occurs predominantly into a single guided mode. We find that the fluorescence excitation line shape changes as the number of atoms is increased, eventually undergoing a splitting that provides evidence for the waveguide-mediated all-to-all interactions. Remarkably, in the regime of strong driving the light emitted into the waveguide is non-classical, with a significant negativity of the associated Wigner function. We show that both the emission properties and the non-Gaussian character of the light are robust against voids in the atom chain, enabling the experimental study of these effects with present-day technology. Our results offer a route towards novel types of fiber-coupled quantum light sources and an interesting perspective for probing the physics of interacting atomic ensembles through light.

Introduction. Investigating the interaction of light and matter at the level of single photons and atoms is central to quantum optics. Nanophotonic systems enable strong coupling of emitters to a well-defined optical mode, thereby providing a powerful platform for studying light-matter interaction experimentally [1–4]. In particular, the strong transversal confinement of light in optical nanostructures naturally enables the implementation of chiral light-matter coupling, where the light propagation direction symmetry is broken [5–11]. This recently led to the demonstration of, e.g., optical isolators [12] and circulators [13]. Moreover, this directional coupling is ideally suited for the implementation of so-called cascaded quantum systems [14, 15] which can show surprising effects such as the formation of entangled spin dimers by driving the system into a steady-state [16–18].

Nanophotonic waveguides provide not only strong but also homogeneous coupling, even for large ensembles of emitters, e.g., for optically trapped laser-cooled atoms. The propagation losses of light in nanofiber-based waveguides are, in addition, small [19, 20]. For these reasons, the light-mediated atom-atom interactions in this system are of extremely long-range character. This enables the theoretical and experimental investigation of many-body-physics with all-to-all interactions [21–26]. Here, the physics depends strongly on the absolute number of constituents which is also the case, for example, for gravitating systems [27]. Moreover, since these interactions are mediated by the light in the waveguide, the atom-waveguide coupling can be drastically enhanced when a constructive interference condition is met, as demonstrated, e.g., by the observation of large Bragg reflections using “mirrors” that consist of only a few hundred atoms [28, 29].

In this work, we shed light on the signatures of all-to-all interactions in an atomic system that is driven by a laser field and coupled to a nanophotonic waveguide. We operate in the regime of large coupling, where the majority of scattered photons is unidirectionally emitted into the waveguide [30]. All-to-all interactions manifest in the fluorescence excitation line shape, whose form is strongly particle number dependent. We find a characteristic splitting into a double-peak as the atom number increases. We, moreover, investigate the so far little explored regime of strong driving, where the laser Rabi frequency becomes comparable with the single atom decay rate. Here, the light scattered into the guided mode can become non-classical, which is analyzed through the negativity of the Wigner function of the output light field. This negativity exhibits, as well, a strong particle number dependence, but also shows a high degree of robustness.

![FIG. 1. System](image-url) Periodic array of two-level atoms with nearest neighbor separation $a$, placed at a distance $h$ from a nanofiber. The atomic $|g\rangle \rightarrow |e\rangle$ transition with dipole moment $\mathbf{d} = d/\sqrt{2}(1, 0, -i)$ is driven by a laser field with Rabi frequency $\Omega$ and detuning $\Delta$. Photons emitted in the right-propagating mode are detected.
against voids in the atomic chain due to the peculiar nature of the all-to-all interactions. Our study shows that strong driving may enable the controlled realization of non-Gaussian quantum states of light with a potential use as quantum resources [31–33].

System and model. We consider a periodic array of $N$ atoms with nearest neighbor separation $a$ held at a distance $h$ from the surface of a cylindrical nanofiber (see Fig. 1). We model the internal structure of each atom as a two-level system with states $|g\rangle$ and $|e\rangle$. The wavelength of the $|g\rangle \rightarrow |e\rangle$ transition, $\lambda$, and the radius and dielectric constant of the nanofiber are chosen such that it can only support two pairs of counter-propagating guided modes [34, 35]. The dipole moment $d = d/\sqrt{2(1,0,-i)}$ and the position of the atoms relative to the nanofiber ensure that the atoms emit only into one of the pairs of modes. The emission is chiral, i.e., each atom has a larger probability to emit, e.g., into the right- than the left-propagating guided mode (accessible with current experimental setups [39]). Moreover, the atoms are coupled to the radiation (unguided) modes of the free electromagnetic field. An external laser field drives the $|g\rangle \rightarrow |e\rangle$ transition with Rabi frequency $\Omega$ and detuning $\Delta$, and the wave vector of the laser, $k$, with $k = |k| = 2\pi/\lambda$, forms an angle $\varphi$ with respect to the atomic chain (as depicted in Fig. 1).

The dynamics of the system is described by the quantum master equation (within the Born-Markov and secular approximations)

$$\dot{\rho}(t) = -\frac{i}{\hbar} [H, \rho] + \sum_{ij} \Gamma_{ij} \left( \sigma_i \rho a_j^\dagger - \frac{1}{2} \left\{ a_j^\dagger \sigma_i, \rho \right\} \right)$$

(1)

with $\sigma_i = |g_i\rangle \langle e_i|$, where $|g_i\rangle$ and $|e_i\rangle$ are the ground and excited state of the i-th atom in the chain, respectively [37, 38]. The Hamiltonian can be split as $H = H_1 + H_{\text{int}}$, where the first part accounts for the action of the external laser field

$$H_1 = \hbar \sum_{j=1}^N \left( \Omega \left[ e^{ika(j-1)\cos \varphi} \sigma_j + \text{h.c.} \right] - \Delta \sigma_j^\dagger \sigma_j \right),$$

(2)

and the second term describes the interatomic interactions due to the exchange of virtual photons among the atoms,

$$H_{\text{int}} = \hbar \sum_{i\neq j} V_{ij} \sigma_i^\dagger \sigma_j.$$  

(3)

The coherent interaction matrix coefficients $V_{ij}$ can be decomposed into the contributions of the unguided (u) and the right- and left-propagating guided modes (R,L) as $V_{ij} = V_{ij}^u + V_{ij}^R + V_{ij}^L$. While $V_{ij}^u$ decays with the distance between the atoms in the chain, the guided modes give rise to all-to-all coherent interactions. Similarly, the incoherent emission, which is determined by the matrix coefficients $\Gamma_{ij}$, can be decomposed as $\Gamma_{ij} = \Gamma_{ij}^u + \Gamma_{ij}^R + \Gamma_{ij}^L$. Again here, the guided terms $\Gamma_{ij}^R$ and $\Gamma_{ij}^L$ have all-to-all connectivity, which gives rise to collective emission properties largely independent of the nearest neighbor distance $a$ [20, 38, 39]. On the other hand, $\Gamma_{ij}^u$ decays with the distance between the atoms, such that the atoms emit independently into the unguided modes as the ratio $a/\lambda$ is increased [37].

Enhanced atom-nanofiber coupling. In Ref. [30] it was shown that enhanced emission takes place predominantly into a single guided mode when the distance between neighboring atoms $a$ and the driving angle $\varphi$ satisfy the relation

$$\cos \varphi = \frac{m\lambda}{a} - \frac{\lambda}{\lambda_i}, \quad m \in \mathbb{N}$$

(4)

with $\lambda_i$ being the wavelength of the light guided by the nanofiber. Under this matching condition, not only is the efficiency of the coupling between the atomic array and the nanofiber collectively enhanced [20, 30], but the photon emission also becomes unidirectional. This phase-matching mechanism can be nicely illustrated in the weak driving limit, where one can assume that the collective state of the atoms which is excited by the laser takes the form

$$|\psi\rangle = \sum_{j=1}^N \psi_j |e_j\rangle,$$

(5)

with $\psi_j = e^{ika(j-1)\cos \varphi}/\sqrt{N}$. The effective decay rate of this state can be calculated as $\Gamma_{|\psi\rangle} = \sum_{n,j=1}^N \gamma_n D_{nj} \psi_j$, where $\gamma_n$ and $D_{nj}$ are the collective decay rates and modes, respectively, obtained from the diagonalization of the matrix $\Gamma_{ij}$. As shown in Fig. 2 this decay rate indeed matches the maximally achievable collective rate of the right-propagating mode when the ratio $a/\lambda$ satisfies Eq. (4) (red dashed lines) [41].
**Interaction signatures at weak driving.** To analyze the properties of the light emitted into the right-propagating mode, we calculate the photon emission rate

$$\Gamma_R = \sum_{ij} \Gamma_{ij}^R \left\langle \sigma_i^+ \sigma_j^- \right\rangle_{ss},$$

where $$\left\langle \ldots \right\rangle_{ss}$$ denotes the expectation value in the stationary state. In the weak driving limit, where at most one excitation is present at all times, this quantity can be calculated for large atom numbers $$N$$ [30]. In Fig. 3a we show that, indeed, as one approaches the matching condition (4) (red dashed lines), $$\Gamma_R$$ quickly becomes maximal (density plot), as almost all light is emitted into the fiber. In fact, here $$\Gamma_R \approx N \Omega^2 / \gamma$$.

Remarkably, the fluorescence excitation line shape changes as the number of atoms increases. Under the phase-matching condition (4), we observe a splitting of the line into two peaks (see Fig. 3b), whose distance increases proportionally to $$N$$ (for $$N$$ sufficiently large) as shown in Fig. 3c. This splitting is consequence of the fiber-mediated coherent interactions. This can be better understood by inspecting the eigenvalues $$v_n$$ of the interaction matrix $$V_{ij}$$ (Fig. 3b), which represent the energy shift of the corresponding collective eigenstate, $$C_{nj}$$, with respect to single-atom resonance. At the phase-matching point, the state (3) can be approximately written as an equal weight superposition of the two (guided) extremal eigenmodes $$C_{1j}$$ and $$C_{Nj}$$. Their eigenvalues, $$v_1$$ and $$v_N$$, increase linearly with $$N$$, but with opposite sign, as shown in Fig. 3a. This $$N$$-dependence is directly reflected in the observed line splitting.

**Strong driving.** For strong driving, an expansion of the atomic state in form of Eq. (3) is no longer possible, since more than one atom can be excited at a time. Here, we need to solve the exact master equation and the number of atoms that we can simulate is constrained to a maximum of $$N = 7$$.

In Fig. 4, we show the normalized emission rate $$\Gamma_R$$ at $$\Delta = 0$$ as a function of the Rabi frequency $$\Omega$$. If the atoms were not-interacting, all curves would fall into the $$N = 1$$ one. We observe a clear departure from this behavior as $$N$$ is increased. Deviations are particularly visible when the Rabi frequency is comparable to the single atom decay rate $$\gamma$$. Here, the normalized emission rate exhibits a local maximum whose value increases with the number of atoms. Similarly, the collective beta factor, $$\beta = (\Gamma_R + \Gamma_L)/(\Gamma_R + \Gamma_L + \Gamma_a)$$, which quantifies the fraction of the photons that are emitted into the guided modes, as well as the chirality of the photon emission $$\chi = (\Gamma_R - \Gamma_L)/(\Gamma_R + \Gamma_L)$$ (see Figs. 4b and c) increase with the number of atoms. Both quantities are maximal at weak driving. For strong driving they decay to zero, as the atoms become uncorrelated and all collective effects cease.

**Non-Gaussian state of the guided light.** Signatures of non-classicality can be observed at intermediate driving strength, where the laser excites interacting many-body states and at the same time the beta factor and the chirality remain sufficiently large such that most of the light is still emitted into a single mode. To characterise those we use the Wigner distribution (WD) $$W(X,P)$$ (with $$X$$ and $$P$$ being the two light field quadratures in the stationary state), which is one of the most widely used tools for detecting, e.g. non-Gaussianity of photon states. The WD is of particular interest in experimental physics as it can be directly measured via homodyne detection [12]. We will quantify the non-Gaussianity of the state of light emitted by the nanofiber through the negativity of the WD [43, 44], which can be calculated as the volume of its negative part, i.e.

$$\delta W = \int_{g^2} \left[ |W(X,P)| - W(X,P) \right] dX dP. \quad (7)$$

A non-zero $$\delta W$$ is symptomatic of a non-Gaussian state of light, while it is zero for coherent and squeezed states [45, 46].

We follow the approach presented in Ref. [47] for obtaining the steady state WD from a deformed quantum master equation (details given in Appendix A). The results for the negativity as well as the full WD for selected values of $$\Omega/\gamma$$ are shown in Fig. 4d and e, respectively. The negativity increases with the number of atoms and tends to zero for small and large values of the driving strength, reaching a maximum for an intermediate value of $$\Omega$$ that increases with $$N$$. The direction of the WD displacement is given by the driving field angle [48].
FIG. 4. Strong driving and Wigner distribution. a, b and c: Emission rate into the right-propagating guided mode, collective beta factor and chirality, respectively (on resonance) for \( N = 1 - 7 \) atoms. d: Negativity of the WD for \( N = 1 - 6 \) atoms. e: WD for \( \Omega/\gamma = 0.05, 0.5 \) and 1.25. For low values of the driving field, the WD is always positive, regardless of the number of atoms. For \( \Omega/\gamma = 0.5 \) the negative area of the distribution grows with \( N \). As the driving field is increased further, the size of the negative areas of the WD decreases.

To understand the non-monotonic behavior of \( \delta W \) with \( \Omega \), let us first note that the negativity of the light emitted into the right-propagating mode is reduced when emission takes place into any other modes (guided or un-guided). The losses render the observed state a statistical mixture of the measured guided light and vacuum noise [48, 49]. For very weak driving, the collective beta factor and chirality are large (see Fig. 4b and c) and, hence, the losses into other modes are minimized. However, here the driving is not strong enough to produce a sufficiently large photon rate that displaces the light state away from the vacuum (first row in Fig. 4e). Conversely, when \( \Omega \gg \gamma \) the displacement from the vacuum state is large due to the strong driving, but both the beta factor and the chirality decrease (see Figs. 4b and c), increasing the effect of the noise coming from the undetected photons, ultimately leading to positive Wigner distributions (third row in Fig. 4b). Hence, the maximum negativity arises at intermediate values of \( \Omega/\gamma \), when both the emission rate as well as the beta factor and chirality are high enough to allow for an increase of the emission probability into the right-propagating mode. Since the beta factor and the chirality increase with \( N \), so does the value of \( \Omega/\gamma \) at which the maximum of the negativity occurs.

Robustness due to all-to-all interactions. In a realistic experimental apparatus involving atoms interacting with a nanofiber, often the filling of the traps, created either by the evanescent field of the nanofiber or an external lattice, is imperfect, with gaps appearing in the chain. Here, we demonstrate that the effects described in this work are robust against this experimental imperfection.

In the limit of weak driving, we show in Figs. 5a and b the beta factor and chirality as a function of the number of atoms in the chain \( N \), for perfect filling (joined by the solid line) and for different configurations of the same number of atoms leaving a single gap within the chain (see inset in Fig. 5a for all configurations considered for \( N = 4 \)). The impact on both \( \beta \) and \( \chi \) is very small, sometimes even improving on the regularly spaced array. Also the non-Gaussianity of the photon state is robust against imperfect filling. E.g., in Fig. 5c we show the WD, virtually unchanged for several gapped configurations of the same number of atoms \( N = 5 \) in the chain, with differences in the negativity of the order of \( 10^{-3} \) between different configurations. We attribute this robustness to the periodicity of the all-to-all interactions between the atoms mediated by the nanofiber: the origin of the collective enhancement of the beta factor and chirality is geometrical and hence any displacement of the atoms will result in a relative phase of \( 2\pi \), which does not affect the emission properties significantly.

Conclusions. We have shown that the light scattered unidirectionally from a laser-driven array of atoms into a nanofiber contains signatures of the all-to-all interactions induced by the atom-fiber coupling. Moreover, in the
strong driving regime, a situation can be achieved where a non-Gaussian photon state is emitted unidirectionally into the fiber. These findings are particularly relevant for optical quantum computing and simulation. In fact, it has been shown that a quantum circuit containing operations that generate states represented by a negative Wigner function cannot be efficiently simulated with classical algorithms [31,32]. Hence, the atom-nanofiber setup considered in this work is an excellent candidate for generating quantum computational resources by making use of state-of-the-art light-matter technologies.

The authors acknowledge fruitful discussions with Christian Liedl, Yijian Meng and Sebastian Pucher. The research leading to these results has received funding from the European Union’s H2020 research and innovation programme [Grant Agreement No. 800942 (ErBeStA)] and EPSRC [Grant No. EP/R04340X/1]. IL acknowledges support from the "Wissenschaftler-Rückkehrprogramm GSO/CZS of the Carl-Zeiss-Stiftung and the German Scholars Organization e.V.. BO was supported by the Royal Society and EPSRC [Grant No. DH130145].

APPENDIX A: WIGNER DISTRIBUTION

Here we give a detailed explanation on the procedure for evaluating the Wigner distribution of the photon emission. We analyze the quantum state of emitted light by studying the statistics of the quadrature trajectories of the light emitted into the reservoir (here the right-propagating guided mode). The generalized quadratures of the light emitted into this mode are defined as:

\[ X_\alpha(t) = X \cos \alpha + P \sin \alpha, \quad (8) \]

where \( \alpha \) defines an angle in phase-space with respect to the axes \( X \) and \( P \), which represent the real and imaginary part of the complex amplitude of the electromagnetic field. We now consider projections of the density matrix \( \rho(t) \) on to the subspace for which the time-integrated quadrature for light coming out of the system up to time \( t \) has a specific value \( X_\alpha \). We denote a corresponding projected reduced density matrix by \( \rho_\alpha^{(X_\alpha)} \). This quantity is related to an \( s \)-biased reduced density matrix \( \rho^s(t) \), via the Laplace transform [50,51],

\[ \rho^s(t) = \int \rho^{(X_\alpha)}(t) e^{-sX_\alpha} dX_\alpha. \quad (9) \]

We obtain then the moment generating function \( Z_\alpha(s) = \text{Tr} \rho^s(t) \), which at long times takes the large deviation form \( e^{t \theta_{X_\alpha}(s)} \). The full statistics of each \( X_\alpha \) at long times is contained within the scaled cumulant generating functions (SCG) \( \theta_{X_\alpha}(s) \), which are also identified as the largest real eigenvalue of the deformed master equation [17]:

\[
\dot{\rho}^s(t) = -\frac{i}{\hbar} [H, \rho^s] + \sum_{ij} \Gamma_{ij} \left( \sigma_i \rho^s \sigma_j^\dagger - \frac{1}{2} \left( \sigma_j^\dagger \sigma_i, \rho^s \right) \right) - s \left( e^{-i\alpha J_R \rho^s} + e^{i\alpha J_R \rho^s} \right) + \frac{s^2}{8} \rho^s \quad (10)
\]

where \( J_R = \sum_j j_{jj} e^{-i\frac{\pi}{2}\tilde{n}(j-1)} \sigma_j \), is the jump operator associated with the right-propagating guided mode in the nanofiber. When \( s \to 0 \), Eq. (10) collapses to the standard trace-preserving master equation [1]. Away from \( s = 0 \) the \( s \)-field biases the dynamics towards rare events in the measured light field. From the projected reduced density matrix \( \rho_\alpha^{(X_\alpha)} \) it is possible to evaluate the marginal probability to observe a particular value of \( X_\alpha \),

\[ P_\alpha(X_\alpha) = \text{Tr} [\rho^{(X_\alpha)}(t)] \approx e^{-t \phi(X_\alpha)} \quad (11) \]

where the large deviation function \( \phi(X_\alpha) \) contains the information about the statistical properties of the underlying probability distribution at long times. The SCG and the LD functions are related via the Legendre transform:

\[ \phi(X_\alpha) = -\min_s (\theta_{X_\alpha}(s) + X_\alpha s). \quad (12) \]

The determination of the whole set of distributions (12) allows to reconstruct the Wigner distribution of the emission at a given long time, \( t \). In fact, first Vogel and Risken [52] demonstrated that \( P_\alpha(X_\alpha) \) is given by the following integral over the Wigner distribution:

\[ e^{-t \phi(X_\alpha)} = \int \mathcal{W}(X \cos \alpha - P \sin \alpha, X \sin \alpha + P \cos \alpha) dP. \]

This can be inverted, by employing the inverse Radon transform, to obtain the WD in terms of the distributions \( \phi(X_\alpha) \) of the homodyne measurements. In a quantum optics experiment not all \( \phi(X_\alpha) \) are measured, but a discrete set of them. Hence, the WD can be estimated numerically using tomographic imaging software.

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