Abstract

In 1933 Kolmogorov constructed a general theory that defines the modern concept of conditional probability. In 1955 Rényi formulated a new axiomatic theory for probability motivated by the need to include unbounded measures. We introduce a general concept of conditional probability in Rényi spaces. In this theory improper priors are allowed, and the resulting posteriors can also be improper.

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An often voiced criticism of the use of improper priors in Bayesian inference is that such priors sometimes don’t lead to a proper posterior distribution. This can happen when the marginal law of the data $X$ is not $\sigma$-finite (Taralsden and Lindqvist, 2010), as sometimes encountered in applied settings with sparse data (Bord et al., 2018; Tufto et al., 2012).

The dangers of improper posteriors in Markov Chain Monte Carlo based methods of inference are well recognized (e.g. Hobert and Casella, 1996). Within the theory to be presented here, improper posteriors as such are well-defined, however, and in practical applied statistics it will be of interest to develop numerical methods for computing such posterior densities. One possible method is indicated by Tufto et al. (2012, Appendix S4) for data on tropical butterflies, and is illustrated in Fig 1. The key idea is to consider the family of posteriors obtained from restriction to intervals, and then glue the resulting posteriors together in a postprocessing step. This simple idea is also the key for the general definition of the posterior we introduce in Section 3. The definition is based on the family of conditional probabilities appearing in the axioms of a conditional probability space as introduced by Renyi (1955).

As a simpler motivating example, suppose you observe a homogeneous Poisson process with a scale invariant prior density (Jeffreys, 1939, p.122)

$$\pi(\lambda) = \frac{c}{\lambda}$$

on the Poisson intensity $\lambda$. The constant $c$ is arbitrary, carries no information, and $c = 1$ is used below. Similar arbitrary constants will, however, play an important role in the theory in later parts of this paper. The marginal law of the number $X$ of events in the interval $(0, t]$ is then not $\sigma$-finite since

$$P(X = 0) = \int_0^\infty P(X = 0|\lambda)\pi(\lambda)\,d\lambda = \int_0^\infty \frac{(\lambda t)^0}{0!} e^{-\lambda t} \frac{d\lambda}{\lambda} = \infty$$
If you observe $X = 0$ and formally multiply the prior by the likelihood you obtain an improper posterior

$$
\pi(\lambda|X = 0) = \frac{e^{-\lambda t}}{\lambda}
$$

This posterior law for $\lambda$ is different from the initial prior law, and we claim that this is a correct way of incorporating the information given by $X = 0$. High values for $\lambda$ are less probable given the observation $X = 0$. Further updating can be done with this posterior as a prior, and this is consistent with only one updating based on the initial prior.

A related example is the Beta posterior density for the success probability $p$ given by

$$
\pi(p|x) = p^{x-1}(1-p)^{n-x-1}
$$

for a Bernoulli sequence with $x$ successes out of $n$ trials. This corresponds to the improper Haldane (1932) prior (Jeffreys, 1939, p.123)

$$
\pi(p) = p^{-1}(1-p)^{-1}
$$

The posterior is improper if $x$ is zero as in the previous example. In all cases, however, the observation of the number of successes $x$ results in a corresponding updating of the uncertainty associated with $p$. The posterior is in this case improper for $x = 0$ and for $x = n$. In all cases, however, the posterior in equation (4) contains the information given by the observation $x$ and the prior in equation (5).

The Haldane prior is the prior that corresponds to the formal Bayes estimator $\hat{p} = x/n$ (Robert, 2007, p.29). This is the optimal frequentist estimator in the sense of being the unique uniformly minimum variance unbiased estimator. Similar optimality phenomena motivates the use of improper priors more generally (Berger, 1985, p.409), and is also linked to fiducial inference (Taraldsen and Lindqvist, 2013).

Unfortunately, even people accepting the use of improper priors reject the above form of inference, on the ground that the posterior is not a probability distribution, and a mathematical theory is lacking for this (Robert et al., 2009). This is understandable, and we agree initially with this point of view. We will demonstrate, however, that the above forms of, so far, formal inference can be made consistent with the axiomatic system of Rényi which allows improper laws. In Section 3 we develop his mathematical theory further to include general conditioning on a $\sigma$-field. This gives a rigorous mathematical foundation for inference with unbounded laws - including the previous three examples. The aim of this paper is to present key elements in a mathematical and philosophical theory of statistics that allows improper laws both as priors and posteriors based on the concept of a Rényi space $(\Omega, \mathcal{E}, P)$ as defined in the next section.

### 2 Statistics in Rényi space

The mathematical theory of statistics is thoroughly presented by Schervish (1995). We will next present the initial ingredients in this theory. The purpose is to have a platform for a generalization that effectively replaces the probability space of Kolmogorov (1933) with the conditional probability space of Rényi (1970). The reader may feel that we include too many elementary standard definitions, and we apologise for this. The reason is that there are small differences in most books on foundations, and the easiest way to be precise is to be explicit. For measure theory we follow mostly the conventions in the elegant treatment by Rudin (1987). A particular interpretation is indicated together with the mathematical theory, but the reader should recognise that many other interpretations are possible. We do not claim that there is one
single “correct” interpretation, but we do claim that the indicated philosophical interpretation is useful in many applied concrete problems.

The initial ingredient is an abstract underlying space \( \Omega \). This space \( \Omega \) is a non-empty set equipped with a law \( P \) which assigns a weight \( P(A) \) to all measurable sets \( A \subset \Omega \). The family \( \mathcal{E} \) of measurable sets is assumed to be a \( \sigma \)-field: (i) \( A \in \mathcal{E} \), (ii) \( A \in \mathcal{E} \) implies \( A^c \in \mathcal{E} \), and (iii) \( A_1, A_2, \ldots \in \mathcal{E} \) implies \( \bigcup_i A_i \in \mathcal{E} \). The set \( \Omega \) equipped with the family \( \mathcal{E} \) of measurable sets is then a measurable space (Rudin, 1987, p.8). A measurable set \( A \) is also referred to as an event with a corresponding philosophical abstraction (Renyi, 1970, p.1-37).

The law \( P \) is a positive measure defined on \( \mathcal{E} \): \( P(\emptyset) = 0 \), (ii) \( A \in \mathcal{E} \) implies \( 0 \leq P(A) \leq \infty \), (iii) If \( A_1, A_2, \ldots \in \mathcal{E} \) are disjoint, then \( P(\bigcup_i A_i) = \sum_i P(A_i) \). Property (iii) is referred to as countable additivity. It is the distinctive feature that separates the theory here from the alternative approach of including improper laws by allowing finitely additive measures (Heath and Sudderth, 1989) (Schervish, 1995, p.21). Additionally, in the theory of Kolmogorov, \( P(\Omega) = 1 \) is assumed. The set \( \Omega \) equipped with \( \mathcal{E} \) and \( P \) is then a probability space. This is assumed in the following paragraphs until the concept of a Rényi space is introduced.

A random quantity \( Z \) is a measurable function \( Z : \Omega \to \Omega_Z \) (Schervish, 1995, p.583, p.606). A function \( Z \) is measurable if \( (Z \in A) = Z^{-1}(A) = \{ \omega \mid Z(\omega) \in A \} \) is measurable for any measurable \( A \subset \Omega_Z \). Using this notation, the law \( P_Z \) of \( Z \) is well defined by

\[
P_Z(A) = P(Z \in A)
\]

Another random quantity \( W = \phi(Z) \) is defined by \( W(\omega) = \phi(Z(\omega)) \) when \( \phi : \Omega_Z \to \Omega_W \) is measurable. This implies \( P_W = P_Z \circ \phi^{-1} \) so the law of \( W = \phi(Z) \) is determined by the law of \( Z \). The general change-of-variables theorem \( E(\phi(Z)) = E_Z(\phi) \) is also a consequence (Schervish, 1995, Thm B.12). The notation \( E(W) = \int W(\omega) \, P(d\omega) \) is here used for the expectation of \( W \). These observations explain partly why the abstract space \( \Omega \) can be left unspecified in applications.

The previous paragraph defines the law \( P_T \) of a random quantity \( T \). The notation \( P^t(A) = P(A \mid T = t) \) is here used for the conditional law \( P^t \) on \( \Omega \). It is defined by the equation

\[
P((T \in C) \cdot A) = \int_C P^t(A) \, P_T(dt)
\]

The left-hand-side defines for each event \( A \) a measure on \( \Omega_T \) which is absolutely continuous with respect to \( P_T \), and \( h(t) = P^t(A) \) is the unique density \( h \in L^1(P_T) \) obtained from the Radon-Nikodym theorem (Rudin, 1987, p.121). The conditional law \( P^t_Z \) of \( Z \) given \( T = t \) is defined by

\[
P^t_Z(A) = P^t(Z \in A) = P(Z \in A \mid T = t)
\]

The general change-of-variables theorem implies that \( P^t_Z(A) = (P_{Z,T})^t(A \times \Omega_T) \) so the conditional law \( P^t_Z \) is determined by the joint law \( P_{Z,T} \).

The previous gives some basic ingredients from probability theory needed in a mathematical theory of statistics. For the theory of statistics Schervish (1995, p.82) assumes, as we also do, that there is a single space \( \Omega \) underlying also the statistical analysis for a particular model \( \theta \) with observed data \( x \). The data \( X \) and the model \( \Theta \) are random quantities. This means that \( X : \Omega \to \Omega_X \) and \( \Theta : \Omega \to \Omega_\theta \) are measurable functions. The data space \( \Omega_X \) is the set corresponding to the possible observations \( x \). It is commonly referred to as the sample space. The model space \( \Omega_\theta \) is the set of possible model parameters \( \theta \). It is sometimes referred to as the model parameter space. A parameter \( \Gamma = \psi(\Theta) \) is by definition a function of the model \( \Theta \). It is hence also consistent to refer to \( \Theta \) as the model parameter. A statistic \( Y = \phi(X) \) is by definition a function of the data \( X \). The function \( \phi : \Omega_X \to \Omega_Y \) is sometimes referred to as an action, and the set \( \Omega_Y \) is then the action space. The parameter \( \Gamma \) is sometimes referred to as the focus.
parameter, and the set $\Omega_\Gamma$ is then the focus space. These concepts are as explained in much more detail by Schervish (1995), but with some differences in notation and naming conventions. The involved concepts are illustrated in the commutative diagram in Figure 2.

A statistical model for observed data $x$ is conventionally specified by a family $\{P_{\theta}^X \mid \theta \in \Omega_\Theta\}$ of probability measures $P_{\theta}^X$ indexed by the unknown model parameter $\theta$ (Lehmann and Casella, 1998, p.1). We assume additionally, as does Schervish (1995, p.83), that the statistical model is given by the conditional law $P_{\theta}^X(A) = P(X \in A \mid \Theta = \theta)$ (9) of the data $X$ given $\Theta = \theta$. This requires also a specification of the data space $\Omega_X$ and the model space $\Omega_\Theta$. The task for the statistician is to infer something about a chosen focus parameter $\gamma = \psi(\theta) = \Gamma(\omega)$ from the observed data $x = X(\omega)$. This is done by reporting a statistic $y = \phi(x)$. The problem is then to choose or characterise a suitable action $\phi$, and to implement and perform associated calculations.

In Bayesian inference the prior $P_\Theta$ is also specified, and together with the conditional law $P_{\theta}^X$ this determines the joint law of $X$ and $\Theta$. The joint law of the data $X$ and the model $\Theta$ determines the posterior law $P_{\theta}^X$. The simplicity and generality of this transformation of prior knowledge $P_\Theta$ into the posterior knowledge $P_{\theta}^X$ given the data $x$ is one major argument in favour of the Bayesian paradigm. Additionally, it can be observed that the Bayes posterior expectation $\phi(x) = E^\theta(\Gamma)$ exemplify that the Bayes posterior can be used to define many possible actions $\phi$ in addition to the distribution estimator given by the posterior law $\phi_{D}\big(\theta\big) = P_{\theta}^X$ itself.

The previous paragraphs give a condensed presentation of some of the initial ingredients in the well established mathematical theory of statistics as presented in considerable more detail by Schervish (1995). We now turn to the more general case where $\Omega$ is a Rényi space as will be defined in the next few paragraphs. Assume first that $(\Omega, P, E)$ is a $\sigma$-finite measure space. Let $B \subset E$ denote the family of elementary conditions $B$ defined by the requirement $0 < P(B) < \infty$. A family $\{P(\cdot \mid B) \mid B \in B\}$ of conditional probability measures is then defined by

$$P(A \mid B) = \frac{P(AB)}{P(B)}, \quad \forall A \in E, \forall B \in B$$

(10)

It can be verified that $B$ is a bunch (Rényi, 1970, Def.2.2.1): (i) $\emptyset \notin B$, (ii) $B_1, B_2 \in B$ implies $B_1 \cup B_2 \in B$, (iii) There exists a sequence $B_1, B_2, \ldots \in B$ with $\Omega = \cup_i B_i$. Condition (iii) follows since $P$ is $\sigma$-finite. Furthermore, $B_1, B_2 \in B$ and $B_1 \subset B_2$ imply $P(B_1 \mid B_2) > 0$, and imply also
the consistency requirement

\[ P(A \mid B_1) = \frac{P(AB_1 \mid B_2)}{P(B_1 \mid B_2)} \]  

This shows that a σ-finite measure \((Ω, E, P)\) generates a conditional probability space \((Ω, E, B, P)\). A conditional probability space is a measurable space equipped with a consistent family of conditional probabilities indexed by a bunch (Rényi, 1970, Def.2.2.2). The Rényi structure theorem shows that every conditional probability space is generated by a corresponding σ-finite measure (Rényi, 1970; Taraldsen and Lindqvist, 2016). It should be noted, however, that the above construction given by equation (10) gives a maximal bunch and then a maximal family of conditional probabilities. Consequently, every conditional probability space can be extended to a maximal conditional probability space.

It can be noted that the family of conditional probabilities \(P(A \mid B)\) and the family \(B\) of elementary conditions are unchanged if \(P\) is replaced by \(cP\) where \(c\) is a positive constant. The Rényi state defined by \(P\) is the equivalence class \([P] = \{cP \mid c > 0\}\). The measures \(P\) and \(cP\) are equivalent when interpreted as Rényi states, and the conditional probabilities \(P(A \mid B)\) give the philosophical interpretation in statistical models. A Rényi space is here defined to be a measurable space equipped with a Rényi state. It corresponds to a conditional probability space where the bunch is maximal. Our definition here of a Rényi space is equivalent with the definition of a full conditional probability space as used by Rényi (1970, p.43). We will follow conventional abuse of notation and use the same symbol \(P\) for the equivalence class, a representative σ-finite measure, and the family of conditional measures.

Consider now again the commutative diagram in Figure 2 corresponding to a general statistical inference problem. It can be interpreted as before also when \(Ω\) is assumed to be a Rényi space. A random quantity \(Z\) is a measurable function. It is said to be σ-finite if the law \(P_Z\) is σ-finite. The σ-finite functions define the natural arrows in the category of Rényi spaces. In the case \(Ω_Z = \mathbb{R}\) our definition of \(Z\) being σ-finite is equivalent with \(Z\) being a regular random variable as defined by Rényi (1970, p.73).

The prior \(P_Θ(A) = P(Θ ∈ A)\) defines a Rényi state if \(Θ\) is σ-finite. The interpretation is in terms of the conditional probabilities \(P_Θ(A \mid B)\) for \(B ∈ B_Θ = \{B \mid 0 < P_Θ(B) < ∞\}\). If the variable \(Z = (X, Θ)\) and the data \(X\) are σ-finite, then the posterior \(P_Θ^x\) is well defined with \(P_Θ^x(Π_Θ) = 1\). This is discussed and exemplified by Taraldsen and Lindqvist (2010) and Lindqvist and Taraldsen (2018). In the next section this theory will be generalised so that the posterior \(P_Θ^x\) is also allowed to be a conditional Rényi state as needed for the butterfly, Poisson process, and Bernoulli examples in Section 1.

3 Improper posteriors as conditional Rényi states

Taraldsen and Lindqvist (2010, 2016) define the posterior law \(P^x(A) = P(A \mid X = x)\) for the case where the data \(X\) is σ-finite. The aim now is to prove existence and uniqueness of a posterior law without assuming that \(X\) is σ-finite. The simple idea in the following is to define \(P^x(A)\) from a family \(P^x(A \mid B)\) indexed by the elementary conditions \(B ∈ B\). The later is defined by the family of conditional probabilities \(P(A \mid B)\) defined by the Rényi state \(P\). It is assumed throughout this Section that \((Ω, E, P)\) is a Rényi space, and that all random quantities are defined on this space. The bunch \(B\) associated to \(Ω\) is the family of events \(B ⊂ Ω\) defined by the requirement \(0 < P(B) < ∞\).

Assume that \(T\) is a random quantity. If \(B ∈ B\), then

\[ P((T ∈ C)A \mid B) = \int_C P^x(A \mid B) P_T(dt \mid B) \]  

6
defines $P^t(A \mid B)$ similarly to how $P^t(A)$ was defined by equation (7). The left-hand-side defines for each event $A$ a measure on $\Omega_T$ which is absolutely continuous with respect to $P_T(dt \mid B)$, and $h(t) = P^t(A \mid B)$ is the unique density $h \in L^1(P_T(dt \mid B))$ obtained from the Radon-Nikodym theorem (Rudin, 1987, p.121).

If $X$ is a random quantity, then the previous defines a family of posterior laws $P^x(\cdot \mid B)$ indexed by $B \in \mathcal{B}$. This is the necessary ingredient for the interpretation of a posterior law. This family is taken as the definition of the posterior law $P^x$. The construction holds also more generally for a conditional probability space with an arbitrary bunch. In the following we restrict attention to Rényi spaces. The posterior law defines then a conditional Rényi state.

The next aim is to prove existence of a posterior law $P^x$, and show that

$$P^x(AB) = P^x(A \mid B) P^x(B), \quad \forall A \in \mathcal{E}, \forall B \in \mathcal{B}, \forall x \in \Omega_X$$

This generalization of the structure theorem of Rényi is the main result given below in Theorem 1. Its precise statement requires some more definitions.

A $\sigma$-finite measure $Q_T$ is by definition a pseudo-law of a random quantity $T$ if $\{C \mid Q_T(C) = 0\} = \{C \mid P_T(C) = 0\}$. If $Q_T$ is another pseudo-law, then the Radon-Nikodym theorem gives existence of a unique $c > 0$ in $L^1_{lo}(Q_T)$ such that $Q_T(dt) = c(t)Q_T(dt)$. Existence of a pseudo-law follows by defining $Q_T(C) = Q(T \in C)$ where $Q$ is a probability measure such that $P(d\omega) = W(\omega)Q(d\omega)$ with $W > 0$ (Rudin, 1987, 6.9 Lemma). Given a pseudo-law $Q_T$, or more generally a law $Q_T$ that dominates $P_T$, we define the conditional law $P^t$ by the relation

$$P((T \in C)A) = \int_C P^t(A) Q_T(dt)$$

The left-hand-side defines for each event $A$ a measure on $\Omega_T$ which is absolutely continuous with respect to $Q_T$, and $h(t) = P^t(A)$ is the unique density obtained from the Radon-Nikodym theorem (Rudin, 1987, p.121, p.123). If $c(t) > 0$, then the previous shows that $c(t) P^t(d\omega)$ is the conditional law corresponding to the pseudo-law $Q_T(dt)/c(t)$. This defines an equivalence between conditional laws, and defines the unique conditional Rényi state $P^t$ as an equivalence class. The main mathematical result can now be stated.

**Theorem 1.** A random quantity $T$ determines a unique conditional Rényi state $P^t$, and a unique family of conditional Rényi states $P^t(\cdot \mid B)$ for $B \in \mathcal{B}$ such that $\forall A \in \mathcal{E}$

$$P^t(AB) = P^t(A \mid B) P^t(B), \quad \forall t \in \Omega_T$$

Proof. All that remains to prove is equation (15). Observe first that $P_T(C \mid B) = P((T \in C)B)/P(B)$ and $P((T \in C)B) = \int_C P^t(B)Q_T(dt)$ give

$$P_T(dt \mid B) = \frac{P^t(B)}{P(B)}Q_T(dt)$$

Using this gives

$$\int_C P^t(AB)Q_T(dt)/P(B) = P((T \in C)A \mid B) = \int_C P^t(A \mid B)P_T(dt \mid B)$$

and equation (15) is proved. \qed
All of the previous can be repeated with a replacement of the measurable set \( A \) with a positive measurable function \( A : \Omega \rightarrow [0, \infty] \) and \( E^t(A) = P^t(A) \). Conditional expectation of complex valued functions can be defined by decomposition in positive and negative parts and then in real and complex parts. Consideration of the dual space gives conditional expectation of a separable Banach space valued \( A : \Omega \rightarrow \mathbf{B} \). The conditional expectation \( E(A \mid T = t) \) is in particular well defined when \( A \) takes values in a separable Hilbert space. Separability is assumed to ensure almost everywhere definition on \( \Omega_T \).

Conditional expectation with respect to a \( \sigma \)-field \( \mathcal{T} \subset \mathcal{E} \) is defined by \( E(W \mid T) = E(W \mid T) \) where \( T(\omega) = \omega \) and \( (\Omega_T, \mathcal{E}_T) = (\Omega, \mathcal{T}) \). It can be noted that we define \( P^t(A) \) directly following Kolmogorov instead of more indirectly by first defining \( P(A \mid T) \) as is more common. This has the advantage of allowing a completely general measurable space \( \Omega_T \), whereas the common approach requires separability properties according to Schervish (1995, p.616, Prop.B.24).

4 Examples

4.1 The uniform Rényi state on \( \mathbb{R} \)

The most familiar example of an improper prior is given by Lebesgue measure \( m \) on the real line \( \Omega_\Theta = \mathbb{R} \) equipped with the family \( \mathcal{E}_\Theta \) of Borel sets \( A \). The corresponding Rényi state is the equivalence class \( P_\Theta = [m] = \{cm \mid c > 0\} \). The Rényi state is equivalently given by the bunch \( \mathcal{B}_\Theta = \{B \in \mathcal{E}_\Theta \mid 0 < m(B) < \infty\} \), and the family of conditional probabilities \( P_\Theta(A \mid B) = m(AB)/m(B) \) for \( B \in \mathcal{B}_\Theta \) and \( A \in \mathcal{E}_\Theta \). This defines a full conditional probability space in the sense of Renyi (1970, p.43), or equivalently, in our terminology, a Rényi space.

In the context of statistical modeling it is furthermore assumed, as in Figure 2, that \( \Theta \) is a random variable defined on the underlying Rényi space \((\Omega, \mathcal{E}, P)\) with \( P_\Theta(A \mid B) = P(\Theta \in AB)/P(\Theta \in B) \). The latter can also be written as \( P_\Theta = P \circ \Theta^{-1} \) or as \( P_\Theta(A) = P(\Theta \in A) \) where, as always, \( (\Theta \in A) = \{\omega \mid \Theta(\omega) \in A\} \). These equations are interpreted as given for representative measures in the equivalence classes.

The family \( \mathcal{B}_0 = \{-n, n \mid n \in \mathbb{N}\} \) does not contain the empty set, it is closed under finite unions, and there is a sequence \( B_j \in \mathcal{B}_0 \) with \( B_1 \cup B_2 \cdots = \Omega_\Theta \). The family \( \mathcal{B}_0 \subset \mathcal{B}_\Theta \) is therefore a bunch. The family of probability measures defined by \( m_n(A) = m(AB_n)/m(B_n) \) for \( B_n = [-n, n] \) defines a conditional probability space \((\mathbb{R}, \mathcal{E}_\Theta, \mathcal{B}_0, \{m_n\})\) in the sense of Rényi (1970, Definition 2.2.2, p.38). The Rényi structure theorem ensures that this space is generated by a \( \sigma \)-finite measure, or equivalently, that this conditional probability space can be extended to a unique Rényi space. This Rényi space is given by the Rényi state \( P_\Theta \) described in the previous paragraphs.

In applications the uniform law on the real line is often described as the limit of the probability measures \( m_n \) as \( n \rightarrow \infty \). The previous paragraph identifies the uniform law not as a limit, but as given by the collection \( \{m_n \mid n \in \mathbb{N}\} \) of probability measures itself. The uniform Rényi state \( m = P_\Theta \) can, however, also be obtained as \( \lim_n m_n = m \). Each \( m_n \) and \( m \) are interpreted as Rényi states. The limit can be defined as in the convergence of conditional probability spaces defined by (Rényi, 1970, p.57), but also in the sense of convergence of Rényi states given by equivalence classes (Taraldsen and Lindqvist, 2016, p.5015) (Bioche and Druilhet, 2016).

The interpretation of \( P_\Theta \) comes from the definition of a conditional probability space as discussed in more detail by Rényi (1970, p.34-38). Given that \( \Theta \in B \) the law is the probability distribution \( P_\Theta(\cdot \mid B) \) concentrated on \( B \). The interpretation of all these conditional probabilities can be, depending on the situation at hand, in a frequentist sense or in a subjective Bayesian sense. This generalizes to other unbounded laws including the priors and posteriors for the butterfly, Poisson process, and Bernoulli examples in the Introduction. It is most important
since it gives the needed interpretation of the mathematical theory in the context of statistical inference. The same interpretation is in particular used for both the prior and the posterior. They are on an equal footing, and this is how uncertainty is represented in the statistical model.

4.2 Conditional Rényi state densities

Assume that \((X, \Theta) \sim f(x, \theta)\mu(dx)\nu(d\theta)\) for \(\sigma\)-finite measures \(\mu\) and \(\nu\). It follows that \((\Theta | X = x) \sim f(x, \theta)\nu(d\theta)\) by choosing \(Q_X = \mu\). This can be verified directly by the defining equation (14).

It follows in particular that this is consistent with the definition of an improper posterior as used by Bioche and Druilhet (2016, p.1716). The previous can also be reformulated simply as

\[
 f(\theta | x) = f(x, \theta)
\]

There is no need for a normalization constant since two proportional densities are equivalent when considered as conditional Rényi states. The symbol \(f\) is used here, and in the following, as a generic symbol for a density and also for conditional densities. The arguments \(x \in \Omega_X\) and \(\theta \in \Omega_\Theta\) give the interpretation as different functions.

Let \(c\) be a function with \(c(\theta) > 0\). In the context here this statement is interpreted as stating that \(c : \Omega_\Theta \rightarrow \mathbb{R}\) is measurable and that \(P(c(\Theta) \leq 0) = 0\). Similar context dependent interpretations are also used elsewhere, but then without further explanation. It follows then that \((X | \Theta = \theta) \sim c(\theta)f(x | \theta)\mu(dx)\), and so

\[
 f(\cdot | \theta) = c(\theta)f(\cdot | \theta)
\]

when interpreted as a conditional Rényi densities. We will then also write \(f(x | \theta) = c(\theta)f(x | \theta)\) with this interpretation. The resulting equivalence class of conditional densities is the conditional Rényi state density. These observations are special cases of the discussion before Theorem 1 leading to the definition of a conditional Rényi state as an equivalence class.

A formal prior density \(f(\theta)\) gives the joint density \(c(\theta)f(x | \theta)f(\theta)\), and this shows that the interpretation of \(f(\theta)\) as prior information is dubious in this case. It is only when the density \(f(x | \theta)\) is normalized that the common procedure of combining a prior density \(f(\theta)\) with the model density \(f(x | \theta)\) into a resulting joint density \(f(x, \theta) = f(x | \theta)f(\theta)\) and a posterior density \(f(\theta | x)\) is well defined. In all cases, however, the posterior density \(f(\theta | x)\) is well defined as a conditional Rényi state density from the joint density \(f(x, \theta)\) as in equation (16). In general, the problem with the prior arises when the statistical model \(P_X^\theta\) itself is allowed to be a conditional Rényi state. The likelihood \(L(\theta | x) = f(x | \theta) = c(\theta)f(x | \theta)\) is not well defined in this case.

A concrete example with an undefined likelihood is discussed by Lavine and Hodges (2012, p.43) and Lindqvist and Taraldsen (2018, p.102). They consider a Gaussian density

\[
 f(x | \theta) = c(\theta)\exp(-\theta x^TQx/2)
\]

with a known \(n \times n\) precision matrix \(Q \succeq 0\). This is an improper density if \(Q\) has at least one eigenvalue equal to zero, and then the likelihood is undefined due to the ambiguity introduced by \(c\). The normalization constant \(c\) is undefined. A seemingly natural candidate, motivated by the proper model case \(Q > 0\), is given by \(c(\theta) = \theta^{n/2}\), and this was used initially in the computer software WinBUGS (Lindqvist and Taraldsen, 2018, p.102). This choice in WinBUGS was later changed into \(c(\theta) = \theta^{(n-1)/2}\). It is clear that, in this situation, a prior information in the form of a prior density \(f(\theta)\) can not be combined with the given improper model to give a well defined posterior density \(f(\theta | x)\).
A possible solution is given by restricting \( x \) to the orthogonal complement of the null space of \( Q \). The model density is then proper, and \( c(\theta) = \theta^{(n-k)/2} \) is the correct normalization when \( k \) is the dimension of the null space of \( Q \). In the \( k = 1 \) case considered by Lindqvist and Taraldsen (2018, p.103) this corresponds to a change from a uniform to a point mass distribution at 0 for \((x_1 + \cdots + x_n)/n\). More generally, the model in equation (18) can be further specified as a Gaussian distribution for \( x \) with point masses at \( k \) components. Anyhow, a well defined posterior requires that the joint density \( f(x, \theta) \), or more generally as just exemplified, a well defined joint distribution of the data \( X \) and the model \( \Theta \). Lindqvist and Taraldsen (2018, p.103) obtain a unique normalized posterior \( P^\alpha_\Theta \) only in the case where the data \( X \) is \( \sigma \)-finite. Theorem 1 ensures, however, that a unique posterior Rényi state is defined also without requiring a \( \sigma \)-finite \( X \).

A more transparent example is given by letting \( P_{X,\Theta}(dx, d\theta) = dx d\theta \) correspond then both to Lebesgue measure on the plane. The law of \( X \) given \( \Theta = \theta \) and the posterior law of \( \Theta \) given \( X = x \) correspond both to Lebesgue measure on the line. The factorization \( f(x, \theta) = 1 = c(\theta) \pi(\theta) \) with \( \pi(\theta) = 1/c(\theta) \) is completely arbitrary. This can be interpreted according to Hartigan (1983, p.26) as saying that the marginal law is not determined by the joint law. The choice of a pseudo-law \( Q_\Theta \) plays a role similar to the role of choosing a marginal law in the theory of Hartigan. The interpretation of Hartigan is discussed in more detail by Taraldsen and Lindqvist (2010), but it differs from the interpretation here. We insist that the marginal law of \( \Theta \) is uniquely determined from the joint law of \( X \) and \( \Theta \). In the case here it is given by the measure \( P_\Theta(d\theta) = \infty \cdot d\theta \) which is not \( \sigma \)-finite. It follows in particular that the decomposition \( P_{X,\Theta}(dx, d\theta) = P_X^\pi(dx) P_\Theta(d\theta) \) fails in this case. However, regardless of the choice of a pseudo-law \( Q_\Theta \), the decomposition \( P_{X,\Theta}(dx, d\theta) = P_X^\pi(dx) Q_\Theta(d\theta) \) defines \( P_X^\pi(dx) = dx \) uniquely as a conditional Rényi state.

### 4.3 Elementary conditional Rényi states

Let \( P_\Theta(d\theta) = d\theta_1 d\theta_2 \) be Lebesgue measure in the plane, and consider the indicator function of the upper half plane: \( \Gamma = \psi(\Theta) = (\Theta_2 > 0) \). It follows that \( P_\Gamma = \infty \delta_0 + \infty \delta_1 \) so \( \Gamma \) is not \( \sigma \)-finite. The conditional law \( P^\alpha_\Theta(d\theta) = [(\gamma = 1)(\theta_2 > 0) + (\gamma = 0)(\theta_2 \leq 0)]d\theta_1 d\theta_2 \) is, however, a well defined unique conditional Rényi state. It corresponds to the dominating measure \( Q_\Gamma = \delta_0 + \delta_1 \). The conditional law \( P^\alpha_\Theta \) is Lebesgue measure restricted to the upper half-plane and \( P^\alpha_\Theta \) is Lebesgue measure restricted to the lower half-plane. This demonstrates directly that the conditional law is also defined when \( \Gamma \) is not \( \sigma \)-finite.

Consider more generally a random natural number \( T : \Omega \to \mathbb{N} \). A dominating measure for \( T \) is the counting measure \( Q_T \) on \( \mathbb{N} \). This gives \( P(A \mid T = t) = P^\pi(A) = P(A(T = t)) \). Let \( B = (T = 1) \). The previous gives then the elementary definition of the law

\[
P(A \mid B) = P(AB), \quad P(B) > 0 \quad (19)
\]

The conditional \( P(A \mid B) \) is not defined from this argument when \( P(B) = 0 \) since \( P^\pi(A) \) can be arbitrarily specified in this case. The previous is consistent with the familiar \( P(A \mid B) = P(AB)/P(B) \) for the case where \( 0 < P(B) < \infty \). A Rényi state is arbitrary up to multiplication by a positive constant. It is an equivalence class of \( \sigma \)-finite measures. Theorem 1 gives the existence of conditional expectations in full generality - including this elementary case. The restriction \( 0 < P(B) < \infty \) for defining \( P(A \mid B) \) has here been relaxed to the condition \( P(B) > 0 \) by Theorem 1.

### 4.4 The marginalization paradox

Stone and Dawid (1972, p.370) consider inference for the ratio \( \theta \) of two exponential means. They assume that \( X \) and \( Y \) are independent exponentially distributed with hazard rates \( \theta \phi \) and \( \phi \)
respectively, so $Z = Y/X$ will have a distribution that only depends on $\theta$. In fact, $Z = \theta F$, where $F$ has a Fisher distribution with 2 and 2 degrees of freedom since a standard exponential variable is distributed like a $\chi^2_2$ variable. Stone and Dawid (1972) conclude that the density is

$$f(z | \theta) = \frac{1}{(\theta + z)^2} \theta^{-1}(1+z/\theta)^{-2} = \theta(\theta + z)^{-2}$$

and that the posterior density corresponding to a prior density $\pi(\theta)$ is

$$\pi(\theta | z) \propto \frac{\theta \pi(\theta)}{(\theta + z)^2}$$

A second argument considers a joint density for $(x,y,\theta,\phi)$ from a joint prior $\pi(\theta)d\theta d\phi$. This gives

$$\pi(\theta, \phi | x,y) \propto \theta \phi \exp(-\phi(x+y))$$

Integration over $\phi$ gives

$$f(x,z,\theta) = \frac{\theta \pi(\theta)}{(\theta x + y)^3} \theta \phi \exp(-\phi(x+y))$$

which implies

$$\pi(\theta | x,z) = \frac{\theta \pi(\theta)}{(\theta z + x)^3} \theta \phi \exp(-\phi(x+y))$$

The second equality holds since it is equality in the sense given by an equivalence class as in Theorem 1. The right hand side can be multiplied by an arbitrary positive function $c(x,z)$ without changing the equality sign. Equation (25) is equivalent with equation (22) since there is a one-one correspondence between $(x,y)$ and $(x,z)$.

An alternative is to integrate equation (23) over $x$ to obtain

$$f(z,\theta,\phi) = \pi(\theta)\theta \phi \exp(-\phi(x+y))$$

which implies

$$\pi(\theta | z, \phi) = \pi(\theta)\theta \phi \exp(-\phi(x+y))$$

This is similar to equation (21), but the conditioning differs.

Reconsider now the argument leading to equation (21). The first observation was that $Z = Y/X$ has a distribution that only depends on $\theta$. This is true, but it is still conditionally given both $\theta$ and $\phi$ as assumed initially in the model. Equation (21) and equation (20) are wrong as stated, interpreted as conditional Rényi states, but can be corrected by a replacement of $f(z | \theta)$ by $f(z | \theta, \phi)$ and $\pi(\theta | z)$ by $\pi(\theta | z, \phi)$. The error in the original argument, as interpreted in the theory presented here, is that it can not be concluded that $\pi(\theta | z) = \pi(\theta | z, \phi)$ even though the
later does not depend on \( \phi \). Equation (27) is not in conflict with equation (25) for the same reason.

More generally, it can be noted that even if a conditional law \( P^{x,z} \) does not depend on \( x \) it can not be concluded that it equals \( P^z \). This is demonstrated by equation (25) and equation (27). The rule \( P^{x,z} = P^z \) holds for probability distributions, and also more generally if \( Z \) and \( (X,Z) \) are \( \sigma \)-finite given that \( P^{x,z} \) does not depend on \( x \). Stone and Dawid (1972) calculated formally as if the rule where generally valid. This resulted in two conflicting results. This example, and the other examples constructed by Stone and Dawid (1972) pointed out that purely formal manipulations with improper distributions, treated as if they obeyed all the rules of proper distributions, could lead to paradoxical inconsistencies — which by reductio ad absurdum — is an argument against doing such formal computations.

### 4.5 The Jeffreys-Lindley paradox

Observations can give rejection of a simple hypothesis at the 5% level, but a Bayesian analysis can give the hypothesis a posterior probability larger than 95%. Lindley (1957) discussed this seemingly paradoxical phenomena with reference to previous work by Jeffreys (1939). Both Berger (1985, p.148-156) and Robert (2007, p.230-236) give thorough discussions of the problem of testing a point null hypothesis, and explain that the use of improper priors is a delicate issue in this case. This has also been emphasized in several discussion papers (Shafer, 1982; Berger and Sellke, 1987; Berger and Delampady, 1987; Robert et al., 2009; Robert, 2014). A full discussion of this problem in the context of the theory of Rényi will not be given here, but we will indicate some consequences and observations.

The most important is to note that any prior, improper or not, contains information. We agree with Robert (2007, p.29) that it is a mistake to think of improper priors as representing ignorance. This is particularly important when testing a point null hypothesis, which in most situations implies a non-symmetric treatment of the hypothesis and the alternative hypothesis. The relevance of the information is specific to each particular case with its own interpretation. Rényi explained that improper laws can be interpreted in terms of the associated family of conditional probabilities. This holds for both prior and posterior laws, and also so in a hypothesis testing problem.

Assume that \( x \sim N(\theta, \sigma^2) \) with unknown mean \( \theta \in \Omega = \mathbb{R} \) and known variance \( \sigma^2 \) so \( f(x|\theta) = (\sqrt{2\pi}\sigma)^{-1} \exp\left(-\frac{1}{2}(x-\theta)^2/\sigma^2\right) \). Consider the hypothesis \( H_0 = \{0\} \subset \Omega = \mathbb{R} \) versus the alternative \( H_1 = H^c_0 = \{\theta \in \Omega \mid \theta \not\in H_0\} \). This basic hypothesis testing problem is often the first example of hypothesis testing presented to statistics students using the notation \( H_0 : \theta = 0 \) versus \( H_1 : \theta \neq 0 \). Our notation identifies the hypothesis and its alternative more explicitly with a partition \( H_0 + H_1 \) of the model parameter space \( \Omega \).

The uniformly most powerful unbiased level \( \alpha \) test rejects \( H_0 \) if (Casella and Berger, 1990, p.374)

\[
t = \phi(x) = 2\Phi(-|x/\sigma|) \leq \alpha
\]

where \( \Phi(z) = P(Z \leq z) \) with \( Z \sim N(0,1) \). The test statistic \( T = \phi(X) \) is the p-value. It is a probability, but it must not be confused with the posterior probability of \( H_0 \) given the data. The posterior probability is undetermined in this classical analysis.

Consider next a Bayesian analysis with a prior density \( \pi \) with respect to the measure \( \nu(d\theta) = \delta_0(d\theta) + d\theta \). The Dirac measure \( \delta_0 \) is dimensionless, and it is hence assumed that \( \theta, x, \) and \( \sigma \) are dimensionless in the following. A Bayesian test with minimal posterior risk rejects \( H_0 \) if the
Figure 3: Two test statistics for testing $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$ based on $x \sim \mathcal{N}(\theta, \sigma^2)$.

Posterior probability of $H_0$ given the data is small (Berger, 1985, p.164)

$$s = \phi_\pi(x) = \pi(0 \mid x) = \left[ 1 + \int \frac{f(x \mid \theta)\pi(\theta)d\theta}{f(x \mid 0)\pi(0)} \right]^{-1} \leq \frac{L_{II}}{L_{II} + L_I}$$

$L_I$ is the loss corresponding to a type I error, $L_{II}$ is the loss corresponding to a type II error, and the loss is zero otherwise. The classical and Bayesian tests are similar in form, but the Bayesian test statistic $S = \phi_\pi(X)$ depends on the prior density $\pi$. We have here restricted attention to the case where the posterior is proper, and the above integral is then finite.

Consider first the constant prior $\pi_\infty(\theta) = c$. This gives

$$\phi_\infty(x) = [1 + f(x \mid 0)^{-1}]^{-1} = [1 + \sqrt{2\pi\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right)]^{-1}$$

Figure 3 shows the remarkable similarity between the p-value and the posterior probability for the case $\sigma = 1$. Robert (2007, p.234) also notes this similarity in his Table 5.2.5, and discusses this phenomena. This similarity, and more generally the close resemblance in practice between Bayesian and classical methods for many common statistical problems, is a common theme in the early fundamental texts on Bayesian statistics (Jeffreys, 1939; Lindley, 1965; Savage, 1954).

A common way to justify usage of improper priors is to consider limits of proper priors. Consider the sequence $\pi_1, \pi_2, \ldots$ of proper prior densities defined by $\pi_n(0) = \pi(0)$ and $\pi_n(\theta) = (1 - \pi(0))g_n(\theta)$ for $\theta \neq 0$. Let $g_n(0) = 0$ and $g_n(\theta) = [\sqrt{2\pi\tau_n}]^{-1} \exp\left(-\frac{1}{2}\theta^2/\tau_n^2\right)$ for $\theta \neq 0$. Define also $g_\infty(0) = 0$ and $g_\infty(\theta) = 1$ for $\theta \neq 0$. If the variance $\tau_n^2 \to \infty$, then $g_n \to g_\infty$ intuitively as densities since a normally distributed variable with infinite variance should correspond to a variable with a constant density. This convergence is in fact true if interpreted as densities with respect to $d\theta$ in the sense of $q$-vague convergence (Bioche and Druilhet, 2016). It seems
hence reasonable to take the sequence \( \pi_1, \pi_2, \ldots \) of proper densities as an approximation of the improper density \( \pi_\infty(\theta) = c = \pi(0) \). The point mass \( \pi(0) \) at \( \theta = 0 \) is then fixed, and the densities for \( \theta \neq 0 \) approximate the flat density. Equation (29) gives, however, that \( \pi_n(0 \mid x) \to 1 \), since \( \int f(x \mid \theta) \pi_n(\theta) \, d\theta \to 0 \). This is in conflict with \( \pi_\infty(0 \mid x) \leq [1 + \frac{\sqrt{2\pi} \sigma}{\sigma}]^{-1} \). The source of the problem is that the generalized density \( \pi_n = \pi(0) \delta_0 + (1 - \pi(0)) g_n \) with respect to \( d\theta \) does not converge to the density \( \pi_\infty = \pi(0) \delta_0 + (1 - \pi(0)) g_\infty \) as intuition would suggest, but instead \( \pi_n \to \delta_0 \) as explained by Bioche and Druilhet (2016).

The previous can be used to illustrate the Jeﬀreys-Lindley paradox. Assume that \( x \) is statistically signiﬁcantly different from 0 at the \( \alpha \) level of signiﬁcance in the sense of \( \phi(x) \leq \alpha \) as deﬁned by equation (28). The convergence \( \pi_n(0 \mid x) \to 1 \) ensures, however, that the posterior probability \( \pi_N(0 \mid x) > 1 - \alpha \) for a prior \( \pi_N \) with a suﬃciently large \( N \). This can, of course, only be considered to be paradoxical in a situation where the prior \( \pi_N \) is reasonable. Consider instead a symmetric proper prior on the form \( \pi(0) = 1/2 \) and \( \pi(\theta) = g(\theta)/2 \) for \( \theta \neq 0 \) where \( g(\theta) \) is non-increasing in \( |\theta| \). The critical values \( 1.645, 1.960 \) and \( 2.576 \) for \( x/\sigma \) corresponds to the \( p \)-values 10%, 5% and 1% as also indicated in Figure 3. The corresponding posterior values \( \pi(0 \mid x) \) are, however, bounded from below by 39%, 29%, and 10% for \( \alpha \) any prior on the given form (Berger, 1985, p.154, Table 4.4). The conclusion is that a large class of reasonable symmetric proper priors gives a posterior probability much larger than the classical \( p \)-value. This is an even more striking illustration of the Jeﬀreys-Lindley paradox.

Consider again the improper prior density \( \pi_\infty(\theta) = c \) with respect to \( \nu(d\theta) = \delta_0(d\theta) + d\theta \). The value of the constant \( c \) is of no concern, as we know from the general theory, and also explicitly from equation (30). Equation (30) has, however, a dependency on \( \sigma \) that is a concern. It follows that the prior \( \pi_\infty \) corresponds effectively to two diﬀerent priors if a concrete problem is formulated first in terms of one measurement scale and then alternatively in terms of a diﬀerent measurement scale. The \( p \)-value does not share this defect, since it only depends on the scaled variable \( x^* = x/\sigma \) as given in equation (28). An alternative prior is obtained by reformulating the original problem in scaled variables and using the prior \( \pi_\infty \) for this problem. Transforming back gives the improper density \( \pi^*(0) = \sigma \) and \( \pi^*(\theta) = 1 \) for \( \theta \neq 0 \). The result is a posterior probability \( \pi^*(0 \mid x) \) that only depends on \( x^* = x/\sigma \), and it is remarkably close to the \( p \)-value as shown in Figure 3 for all values of \( \sigma \).

It was seen above that a seemingly reasonable approximation by proper priors failed. An alternative sequence of proper priors is obtained by using the interpretation R´enyi gives for the prior corresponding to the density \( \pi^* \). Let \( B_n = [-n \sigma, n \sigma] \) for \( n = 1, 2, \ldots \). The conditional probability \( P_B(4 \mid B_n) \) is then given by the proper density \( \pi^*_n \) deﬁned by \( \pi^*_n(0) = 1/(1 + 2n) \) and \( \pi^*_n(\theta) = \frac{1}{n \leq \theta/\sigma \leq n}/(\sigma^2 + 2n\sigma) \) for \( \theta \neq 0 \). In this case \( \pi^*_n(0 \mid x) \to \pi^*(0 \mid x) \), and this gives in particular a proper prior with a posterior that approximates the \( p \)-value as shown in Figure 3. The appropriateness of a prior on this form can not be decided in general, but must be decided in each concrete case.

Consider ﬁnally a concrete problem where it is assumed that the prior density \( \pi^* \) gives a reasonable prior for \( \theta \). Assume that a measurement is done and \( x \sim N(\theta, \sigma^2) \) is observed. In this case, the classical and Bayesian procedures are very similar if \( \alpha = L_I / (L_I + L_{II}) = 5\% \). Assume next that the experimenter chooses to repeat the measurement \( N - 1 \) more times. The prior information is, of course, not changed by this decision, so the prior is still given by \( \pi^* \). A suﬃcient statistic is given by the empirical mean \( \bar{x} \sim N(\theta, \sigma^2/N) \). The classical \( p \)-value and the posterior probability \( \pi^*(0 \mid \bar{x}) \) are in this case very diﬀerent if \( N \) is large. It follows in particular that \( \pi^*(0 \mid \bar{x}) \to 1 \) as \( N \to \infty \) for all ﬁxed \( \bar{x} \), and the Jeﬀreys-Lindley paradox reappears. We see this, in fact, as no paradox, but as a most important and striking demonstration of an important diﬀerence between Bayesian and classical inference.
4.6 Hypothesis testing with improper posteriors

The possibility of improper posteriors was not considered in the previous discussion of the Jeffrey-Lindley paradox. It was, in fact, demonstrated that there exist a proper prior so that the classical and the Bayesian decision rules essentially coincides as shown in Figure 3. This proper prior appears naturally from the Rényi interpretation of a corresponding improper prior in terms of a family of conditional probabilities. It was also noted that this improper prior can be approximated arbitrary well by a sequence of proper priors in a natural topology for Rényi states given by q-vague convergence (Bioche and Druilhet, 2016).

Another observation is that, in general, a classical matching prior is typically improper. DeGroot (1973) demonstrates, by an elegant argument, how a matching prior can be determined for a different problem. A classical matching prior is here defined to be a prior such that the posterior coincides with the p-value. The prior for a different problem. A classical matching prior is here defined to be a prior such that the posterior coincides with the p-value. The prior $\pi^*$ is only approximately matching as shown in Figure 3. A matching prior - if it exists - is determined by the integral equation that follows by equating $\alpha$ and $L_{II}/(L_{I} + L_{II})$ in equation (28) and equation (29). We will not discuss this further here, but observe that a solution is given explicitly by an inverse Fourier transformation.

The butterfly, Poisson process, and Bernoulli examples in the Introduction can be used to exemplify a hypothesis testing problem with an improper posterior. Consider, instead, testing of $H_0 : \gamma \leq 0$ versus $H_1 : \gamma > 0$ based on observing $x \sim N(\gamma, \sigma^2)$ with unknown $\theta = (\gamma, \sigma) \in \Omega_\theta = \mathbb{R} \times \mathbb{R}_+$. This problem, but with known variance $\sigma^2$, is considered by Berger (1985, p.147-148). He notes that a constant prior corresponds to an infinite mass to both hypothesis, but argues that this can be tackled by consideration of increasingly larger intervals. The essence of the following argument is that this argument should in principle then be equally possible for the posterior. This is given by the general interpretation of any Rényi state by its corresponding family of conditional probabilities.

Assume that the prior density is $\pi(\theta) = 1/\sigma$ with respect to $\nu(d\theta) = d\gamma d\sigma$. The posterior is then improper, and given by the density $\pi(\theta | x) = \sigma^{-2} \exp(-\frac{1}{2}(\theta - x)^2/\sigma^2)$. It follows that $P(\Theta \in H_0 | X = x) = P(\Theta \in H_1 | X = x) = \infty$, and it is not obvious how to formulate a decision rule. The interpretation of Rényi leads to consideration of the posterior probability $P_{\phi}(H_0 | B)$ for all $B$ with $0 < P(\Theta \in B) < \infty$. In an application it can, possibly, be argued that it is sufficient to consider elementary events on the form $B(m,n) = (-m, m) \times (1/n, n)$. This determines corresponding posterior probabilities $\phi_{m,n}(x)$ for $H_0$ which should be considered when deciding to reject $H_0$ or not. We leave the further discussion of this for the future.

Another possible approach is presented next. The posterior gives the two improper marginal densities

$$\pi(\sigma | x) = \sigma^{-1}$$

$$\pi(\gamma | x) = |\gamma - x|^{-1} \quad (31)$$

The posterior for $\sigma$ has no dependence on $x$ as intuition would suggest without any further argument. The posterior for $\gamma$ is symmetric around $x$ - again in harmony with intuition. The posterior for $\gamma$ is improper, but it clearly represents an updating of the state of knowledge regarding $\gamma$. As a technical aside it can be observed that the posterior for $\gamma$ is in fact not $\sigma$-finite, but the posterior for $\sigma$ is $\sigma$-finite. It is only the full posterior that is guaranteed to be represented by a conditional Rényi state by Theorem 1.

For the given hypothesis testing problem it can seem natural to replace the model space $\Omega_{\theta}$ with the focus space $\Omega_\gamma = \mathbb{R}$. The hypothesis are then represented by $H_0^* = \{\gamma | \gamma \leq 0\}$ and $H_1^* = \{\gamma | \gamma > 0\}$. Motivated by the interpretation from Rényi it seems natural to consider the elementary events $B(m,n) = (-m, m) \setminus (x - 1/n, x + 1/n)$ with $m > |x|$. The singularity of the posterior at $\gamma = x$ can then be tackled by taking the limit $n \to \infty$ for the resulting posterior.
probabilities $\phi_{m,n}(x)$ for $H_0$. This leads to $\phi_m(x) = \phi_{m,\infty}(x) = (x \leq 0)$. The conclusion from this argument is to reject $H_0 : \gamma \leq 0$ if $x > 0$.

5 Final remarks

Lindley (1965, p.xi) wrote in 1964 in the preface of his classic book on Bayesian statistics:

The axiomatic structure used here is not the usual one associated with the name of Kolmogorov. Instead one based on the ideas of Rényi has been used.

It can be concluded that Lindley initially supported the use of conditional probability spaces as introduced by Rényi. We have argued that Lindley’s initial intuition is correct. The theory of Rényi gives a natural approach to Bayesian statistics including commonly used objective priors.

The marginalization paradoxes seem to have been the main reason for Lindley’s change in opinion on this. Tony O’Hagan interviewed Lindley for the Royal Statistical Society’s Bayes 250 Conference held in June 2013. Lindley explains very nicely that all probabilities are conditional probabilities, but also recalls his reaction to the marginalization paradoxes presented by Stone and Dawid (1972): Good heavens, the world is collapsing about me. In the interview, Lindley continuous to argue that Bayesian statistics is a sound theory, and that the focus should be on how to quantify the prior uncertainty of the unknown parameters. The parameters should be viewed as real physical quantities regardless of which experiment is later used for decreasing their uncertainty. This clearly disqualifies the choice of data dependent priors, and even the choice of priors depending on the particular statistical model used. We wholeheartedly agree with Lindley on this, but we claim that this can be done also within the more general theory introduced by Rényi and continued here.

Historically, the most influential initial work on Bayesian inference is possibly given by the book by Jeffreys (1939). Jeffreys (1939, p.21) argues in particular that the normalization of probabilities is a rule generally adopted, but that the value $\infty$ is needed in certain cases. This is in line with current usage of Bayesian arguments (Berger, 1985; Robert, 2007). It is well established that inference based on the posterior gives, indeed, a most rewarding path for obtaining useful inference procedures from both a frequentist and a Bayesian perspective (Berger, 1985; Schervish, 1995; Lehmann and Romano, 2005). Parts of Jeffreys arguments were mainly intuitive, and there is a lack of mathematical rigor as also observed by Robert et al. (2009). We suggest that a rigorous reformulation of some of the original and most important ideas of Jeffreys (1939) can be done within the mathematical theory presented here.

Within this framework we reach the view that improper posteriors, just as improper priors, are not ‘improper’ but reflect the updated state of knowledge about a parameter after conditioning on the data. Returning to the introductory Poisson-process example, at time $t$, we have clearly learned something about $\lambda$ in that our belief in large values of the Poisson intensity $\lambda$ has decreased while our relative degree of belief in small values of $\lambda$ has remained approximately unchanged. An improper posterior does not imply that our prior was wrong, but only that more data perhaps needs to be collected if possible. Proceeding by using the improper posterior at time $t$ as prior in subsequent inference, say based on the number of occurrences observed in a sufficiently long subsequent interval $[t,t_2]$, we indeed eventually reach the same proper final posterior as the one reached by combining the initial scale prior and the likelihood for the data on $[0,t_2]$. We hope that the reader can appreciate that this argument indicates also the potential philosophical importance of unbounded laws more generally.

An unbounded law can, according to Rényi, be interpreted by the corresponding family of conditional probabilities given by conditioning on the events in the bunch. These elementary
conditional probabilities are probabilities in the sense of Kolmogorov, and the interpretation depends on the application. They can, as Lindley (2006) advocates convincingly, be interpreted as personal probabilities corresponding to a range of real life events. They can also, as needed in for instance quantum physics, be interpreted as objectively true probabilities representing a law for how a system behaves when observed repeatedly under idealized conditions.

Assume now that you accept a theory where the prior uncertainty is given by a possibly unbounded law. It is then natural, we claim, that you accept that a resulting posterior uncertainty can also be represented by a possibly unbounded law. Both the prior and the posterior represent uncertainty of the same kind. Hopefully, many can agree on this on an intuitive level. The main mathematical result presented here is Theorem 1 which provides a key ingredient in a mathematical model for statistics in which this can be done consistently without paradoxical results. This key ingredient is a well defined extension of the concept of conditional expectation as introduced originally by Kolmogorov (1933) to also include the case of Rényi spaces.

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