Twenty Questions for Localizing Multiple Objects by Counting: Bayes Optimal Policies for Entropy Loss

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Abstract

We consider the problem of twenty questions with noiseless answers, in which we aim to locate multiple objects by querying the number of objects in each of a sequence of chosen sets. We assume a joint Bayesian prior density on the locations of the objects and seek to choose the sets queried to minimize the expected entropy of the Bayesian posterior distribution after a fixed number of questions. An optimal policy for accomplishing this task is characterized by the dynamic programming equations, but the curse of dimensionality prevents its tractable computation. We first derive a lower bound on the performance achievable by an optimal policy. We then provide explicit performance bounds relative to optimal for two computationally tractable policies: greedy, which maximizes the one-step expected reduction in entropy; and dyadic, which splits the search domain in successively finer partitions. We also show that greedy performs at least as well as the dyadic policy. This can help when choosing the policy most appropriate for a given application: the dyadic policy is easier to compute and nonadaptive, allowing its use in parallel settings or when questions are inexpensive relative to computation; while the greedy policy is more computationally intensive but also uses questions more efficiently, making it the better choice when robust sequential computation is possible. Numerical experiments demonstrate that both procedures outperform a divide-and-conquer benchmark policy from the literature, called sequential bifurcation. Finally, we further characterize performance under the dyadic policy by showing that the entropy of the posterior distribution is asymptotically normal.

1 Introduction

We consider the following set-guessing problem. Let $\Omega = \mathbb{R}$ be the real line and $\theta = (\theta_1, \ldots, \theta_k) \in \Omega^k$ be a vector containing the unknown locations of $k$ objects, where $k \geq 1$ is known. One can sequentially choose subsets $A_1, A_2, \ldots$ of $\Omega$, query the number of objects in each set, and obtain a series of noiseless answers $X_1, X_2, \ldots$. In studying this problem, our goal is to devise a method for choosing the questions that allows us to find $\theta$ as accurately as possible, given a finite budget of questions. We work in a Bayesian setting, and use the entropy of the posterior distribution on $\theta$ to measure accuracy.

While the adaptive method with minimal expected posterior entropy is described by the dynamic programming principle, and could in principle be computed using dynamic programming, current
computational techniques do not allow doing this in a tractable way. In this paper, we provide a lower bound on this minimal expected entropy; and analyze two specific methods, providing in one case an explicit expression for the expected entropy, and in the other case a tractable upper bound.

The previous literature on similar problems can be classified into two groups: those that consider a single object \((k = 1)\); and those that consider multiple objects \((k \geq 1)\).

Among single-object versions of this problem, the earliest is the Rényi-Ulam game \([1, 2]\). In this game, one person (the responder) thinks of a number between one and one million and another person (the questioner) chooses a sequence of subsets to query in order to find this number. The responder can answer either YES or NO and is allowed to lie a given number of times.

Variations of the Rényi-Ulam game have been considered in \([3]\). Among these variations, the following continuous probabilistic version, first studied in \([4]\), is similar to the problem we consider: The responder thinks of a number \(\theta \in [0, 1]\) and the questioner aims to find a set \(A \subset [0, 1]\) with measure less than \(\epsilon\) such that \(\theta \in A\) with probability at least \(q\). In addition, the responder lies with probability no more than \(p\). Whether the questioner can win this game based on the error probability \(p\) is analyzed and searching algorithms using \(O(\log \frac{1}{\epsilon})\) queries are provided.

Among previous work on the single-object problem, perhaps the closest to the current work is \([5]\), which considered a Bayesian setting and used the entropy of the posterior distribution to measure of accuracy, as we do here. It considered two policies, a greedy policy called probabilistic bisection, which was originally proposed in \([6]\) and further studied in \([7, 8]\), and the dyadic policy. \([9]\) generalized the probabilistic bisection policy to multiple questioners. Here, we generalize both policies to multiple objects.

Our work contrasts with this previous work on the single-object problem by considering multiple objects.

The previous literature includes work on three multiple-object problems: the Group Testing problem \([10, 11, 12, 13, 14]\); the subset-guessing game associated with the Random Chemistry algorithm \([15, 16]\); and the Guessing Secret game \([17]\). In the Group Testing problem, questions are of the form: Is \(A \cap S \neq \emptyset\)?

In the subset-guessing game associated with the Random Chemistry algorithm, questions are of the form Is \(S \subset A\)?. In the Guessing Secret game, when queried with a set \(A\), the responder chooses an element from \(S\) according to any rule that he likes, and tells the questioner whether this chosen element is in \(A\). The chosen element itself is not revealed, and may change after each question. Thus, the answer is 1 when \(S \subset A\), 0 when \(A \cap S = \emptyset\), and can be either 0 or 1 otherwise.

Our work contrasts with this previous work by considering a problem where the answer provided by the responser is not binary but instead counts the number of objects in the queried set.

These multiple-object-localization games find application in constructions of block codes \([18, 19]\), searching for auto-catalytic sets of molecules \([15]\), searching for collections of multiple contingencies leading to cascading power failures in models of electrical networks \([20]\), computer vision \([21, 22, 23]\), and screening for stochastic simulation \([24, 25]\).

Now, in Section 2, we state the problem more formally, and summarize our main results.

2 Problem Formulation and Summary of Main Results

Let \(\theta = (\theta_1, \ldots, \theta_k)\) be a random vector taking values in \(\mathbb{R}^k\). \(\theta_i\) represents the location of the \(i\)th object of interest, \(i = 1, \ldots, k\). We assume that \(\theta_1, \ldots, \theta_k\) are i.i.d. with density \(f_0\), and joint density \(p_0\). We refer to \(p_0\) as the Bayesian prior probability distribution on \(\theta\). We will ask a
series of $N > 0$ questions to locate $\theta_1, \ldots, \theta_k$, where each question takes the form of a subset of $\mathbb{R}$, and the answer to this question is the number of objects in this subset. More precisely, for each $n \in \{1, 2, \ldots, N\}$, the $n^{th}$ question is $A_n \subset \mathbb{R}$ and its answer is

$$X_n = 1_{A_n}(\theta_1) + \cdots + 1_{A_n}(\theta_k),$$

(1)

where $1_A$ is the indicator function of the set $A$. Our choice of the set $A_n$ may depend upon the answers to all previous questions, and upon some initial randomization through a uniform random variable $Z$ on $[0, 1]$ chosen independently of $\theta$. Thus, the set $A_n$ is random, through its dependence on $Z$, and the answers to previous questions.

We call a rule for choosing the questions $A_n$ a policy. Formally, we define a policy $\pi$ to be a sequence $\pi = (\pi_1, \ldots, \pi_N)$, where $\pi_n$ is a Borel-measurable subset of $[0, 1] \times \{0, 1, \ldots, k\}^{n-1} \times \mathbb{R}$. With a policy $\pi$ specified, the choice of $A_n$ is then $A_n = \{t \in \mathbb{R} : (Z, X_{1:n-1}, t) \in \pi_n\}$, so that specifying $\pi_n$ implicitly specifies a rule for choosing $A_n$ based on the random seed $Z$ and the history $X_{1:n-1}$. Here, we have used the notation $X_{a:b}$ for any natural numbers $a$ and $b$ to indicate the sequence $(X_a, \ldots, X_b)$ if $a \geq b$, and the empty sequence if $a < b$. We define $\theta_{a:b}$ and $A_{a:b}$ similarly.

The distribution of $A_n$ thus implicitly depends on $\pi$. When we wish to highlight this dependence, we will use the notation $P_\pi$ and $E_\pi$ to indicate probability and expectation respectively. However, when the policy being studied is clear, we will simply use $P$ and $E$.

We refer to the posterior probability distribution on $\theta$ after $n$ questions as $p_n$, so $p_n$ is the conditional distribution of $\theta$ given $X_{1:n}$ and $A_{1:n}$. Equivalently, under any fixed policy $\pi$, $p_n$ is the conditional distribution of $\theta$ given $Z$ and $X_{1:n}$. This posterior $p_n$ can be computed using Bayes rule: $p_n(u)$ is proportional to $p_0(u)$ over the set $\{u \in \mathbb{R}^k : X_m = \sum_{i=1}^k 1_{A_m}(u_i), 1 \leq m \leq n\}$, and $0$ outside. The dependence on $Z$ arises because $A_n$ may depend on $Z$, in addition to $X_{1:n-1}$.

After we exhaust our budget of $N$ questions, we will measure the quality of what we have learned from them via the differential entropy $H(p_N)$ of the posterior distribution $p_N$ on $\theta$ at this final time,

$$H(p_N) = -E[\log p_N] = -\int_{\mathbb{R}^k} p_N(u_{1:k}) \log(p_N(u_{1:k})) \, du_{1:k}. \quad (2)$$

Throughout this paper, we use “$\log$” to denote the logarithm to base 2. We let $H_0 = H(p_0)$, and we assume $-\infty < H(p_0) < +\infty$. The posterior distribution $p_N$, as well as its entropy $H(p_N)$, are random for $N > 0$, as they depend on $X_{1:N}$ and $Z$. Thus, we measure the quality of a policy $\pi \in \Pi$ when given $N$ questions using

$$R(\pi, N) = E_\pi[H(p_N)]. \quad (3)$$

Our goal in this paper is to characterize the solution to the optimization problem

$$\inf_{\pi \in \Pi} R(\pi, N). \quad (4)$$

Any policy that attains this infimum is called optimal.

While (4) can be formulated as a partially observable Markov decision process [26], and can be solved, in principle, via dynamic programming, the state space of this dynamic program is the space of posterior distributions over $\theta$, and the extreme size of this space prevents solving this dynamic program through brute-force computation.
Thus, in this paper, rather than attempting to compute the optimal policy, we provide an easily computed lower bound on (4), and then study two classes of policies relative to this lower bound: greedy policies, and dyadic policies.

By a greedy policy, we mean any policy that chooses each of its questions to minimize the expected entropy of the posterior distribution one step forward in time,

\[ A_n \in \arg\min_A E[H(p_n)|p_{n-1}, A_n = A], \text{for all } n = 1, 2, \ldots, N, \]

where the argmin is taken over all Borel-measurable subsets of \( \mathbb{R} \). We show in Section 6 that this argmin exists.

To define the dyadic policy, let us recall that the quantile function of \( \theta_1 \) is

\[ Q(p) = \inf \{ u \in \mathbb{R} : p \leq F_0(u) \}, \]

where \( F_0 \) is the cumulative distribution function of \( \theta_1 \), corresponding to its density \( f_0 \). The dyadic policy consists in choosing at step \( n \geq 1 \) the set

\[ A_n = \left( \bigcup_{j=0}^{2^{n-1}-1} \left( Q\left( \frac{2j + 1}{2^n} \right), Q\left( \frac{2j + 2}{2^n} \right) \right) \right) \cap \text{supp}(f_0), \]

where \( \text{supp}(f_0) \) is the support of \( f_0 \), i.e., the set of values \( u \in \mathbb{R} \) for which \( f_0(u) > 0 \). For example, when \( f_0 \) is uniform over \((0, 1]\), the dyadic policy is the one in which the first question is \( A_1 = \left( \frac{1}{2}, 1 \right] \), the second question is \( A_2 = \left( \frac{1}{4}, \frac{1}{2} \right] \cup \left( \frac{3}{4}, 1 \right] \), and each subsequent question is obtained by subdividing \((0, 1]\) into \( 2^n \) equally sized subsets, and including every second subset.

A further illustration of the dyadic question sets \( A_n \) is provided in Figure 3 in Section 5. This definition of the dyadic policy generalizes a definition provided in [5] for single objects.

We are now ready to present our main results:

\[ H_0 - \log(k+1)N \leq \inf_{\pi \in \Pi} R(\pi, N) \leq R(\pi_G, N) \leq R(\pi_D, N) = H_0 - H\left( \text{Bin}\left( k, \frac{1}{2} \right) \right) N, \]

where \( \pi_G \) is any greedy policy, \( \pi_D \) is the dyadic policy, and \( \text{Bin} \) indicates the binomial distribution.

The first inequality in (8) is an information theoretic inequality (proved in Section 3). The second inequality is trivial since an optimal policy is at least as good as any other policy. The third inequality comes from a detailed computation of the posterior distribution \( p_N \) of \( \theta \) after observing \( N \) answers for any possible sequence of \( N \) questions (see Section 6.2). Additionally, we show that this inequality cannot be reversed, by presenting a special case in which there is a greedy policy whose performance is strictly better than that of the dyadic policy (see Section 6.3). The last equality comes from the characterization of the posterior distribution \( p_N \) in the special case of the dyadic policy (see Section 5.2).

The power of these results is illustrated by Figure 1, which shows, as a function of the number of objects \( k \), the number of questions required to reduce the expected entropy of the posterior on their locations by 20 bits per object. The figure shows the number of questions needed under the dyadic policy (solid line, and right-most expression in [8]); under two benchmark policies described below, Benchmark 1 and Benchmark 2 (dotted, and dash-dotted lines); and a lower bound on the number needed under the optimal policy (dashed line, and left-most expression in [8]). By (8), we
know that the number of extra questions required by using either the dyadic or the greedy, instead of the optimal policy, is bounded above by the distance between the solid and dashed lines.

\[ \text{Figure 1: Number of questions needed to reduce the entropy by 20 bits per object under two benchmark policies and the dyadic policy, and a lower bound on the number under the optimal policy. The dyadic policy significantly outperforms both benchmarks and its performance is relatively close to the lower bound on the optimal possible from (8). The performance of the greedy policy is between that of the dyadic and optimal policies.} \]

Benchmark 1 identifies each object individually, using an optimal single-object strategy. It first asks questions to localize the first object \( \theta_1 \), reducing the entropy of our posterior distribution on that object’s location by 20 bits. This requires 20 questions, and can be achieved, for example, by a bisection policy. It then uses the same strategy to localize each subsequent second object, requiring 20 questions per object. The total number of questions required under this policy to achieve 20 bits of entropy reduction per object is \( 20k \).

Benchmark 2 is adapted from the sequential bifurcation policy of [25]. While [25] considered an application setting somewhat different from the problem that we consider here (screening for discrete event simulation), we were able to modify their policy to allow it to be used in our setting. A detailed description of the modified policy is provided in Appendix A. It makes full use of the ability to ask questions about multiple objects simultaneously, and improves slightly over Benchmark 1. We view this policy as the best previously proposed policy from the literature for solving the problem that we consider.

The figure shows that a substantial saving over both benchmarks is possible through the dyadic or greedy policy. For example, for \( k = 2^4 = 16 \) objects, Benchmark 1 and Benchmark 2 require 320 and 304 questions respectively. In contrast, the dyadic policy requires 106 questions, which is nearly 3 times smaller than required by the benchmarks. Furthermore, (8) shows that the greedy policy performs at least as well as the dyadic policy. Thus, localizing objects’ locations jointly can

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1Implementing Benchmark 1 would require the ability to ask questions about whether or not a single specified object (e.g., object \( \theta_i \)) resides in a queried set, rather than the number of objects in that set. While this ability is not included in our formal model, Benchmark 1 is nevertheless a useful comparator.
be much more efficient than localizing them one-at-a-time, and the dyadic and greedy policies are implementable policies that can achieve much of the potential efficiency gains.

The figure also shows, again at \( k = 2^4 = 16 \) objects, that the optimal policy requires at least 80 questions, while the dyadic and greedy require no more than 106 questions, and so are within a factor of 1.325 of optimal. This is remarkable, when we compare how little is lost when going from the hard-to-compute optimal policy to the easily computed dyadic policy, with how much is gained by going to the dyadic from one of the two benchmark policies considered.

The dyadic policy can be computed extremely quickly, and can even be pre-computed, as the questions asked do not depend on the answers to previous questions. This makes it convenient in settings where multiple questions can be asked simultaneously, e.g., in a parallel or distributed computing environment. The greedy policy requires more computational effort than the dyadic policy, but is still substantially easier to compute than the optimal policy, and provides performance at least as good as that of the dyadic policy, as shown by (8), and sometimes strictly better, as will be shown in Section 5.3.

We see in the figure that the dyadic policy’s value and the value of the optimal policy come together at \( k = 1 \). This can also be seen directly from our theoretical results. When \( k = 1 \), the left-hand and right-hand sides of (8) are equal, since Bin\( (k, \frac{1}{2}) \) becomes a Bernoulli\( (\frac{1}{2}) \) random variable, whose entropy is \( \log(2) = 1 \). This shows, when \( k = 1 \), that the expected entropy reduction under the dyadic is the same as the lower bound on this reduction under the optimal policy, which in turn shows that both dyadic and greedy policies are optimal, and the lower bound is tight. This result can also seen through results obtained in [5]. When \( k = 1 \), the well-known bisection policy is a greedy policy, and the dyadic is also greedy, i.e., satisfies [5].

We begin our analysis in Section 3, by justifying the left-most inequality in (8). We then provide an explicit expression for the posterior distribution in Section 4, which is used in later analysis. We analyze the dyadic policy in Section 5 and the greedy policy in Section 6. Finally, we offer concluding remarks in Section 7.

### 3 A Lower Bound on the Expected Entropy after a Fixed Number of Questions and Answers

In this section, below in Theorem 1, we prove the first inequality in (8), which is a lower bound on the expected entropy after a fixed number of questions and answers.

We first introduce some notation, used here, and throughout the paper. For any pair of random variables \( W, V \), we define \( H(W \parallel V) \) to be the random variable taking the value

\[
- \int_{-\infty}^{\infty} f(w|V = v) \log f(w|V = v) \, dw
\]

for each \( V = v \), assuming the conditional density function \( f(w|V = v) \) exists. The “usual” conditional entropy is related to it by

\[
H(W|V) = E[H(W \parallel V)].
\]

We now provide here, in Lemma 1, an expression for the expected entropy after additional questions. This lemma is based on the idea that each additional question reduces the entropy of \( \theta_{1:k} \) by an amount that can be expressed in terms of the conditional entropy of the answer to that
question. The total entropy reduction can then be computed as a sum of the contributions from each question, which we use later to study the expected total entropy reduction under specific policies.

**Lemma 1.** Under any policy $\pi$,

$$E[H(p_{n+1})|B_n] = H(p_n) - H(X_{n+1}|B_n), \text{ for all } n = 0, 1, \ldots, N - 1, \quad (11)$$

where $B_n = (Z, X_{1:n})$ denotes the random vector in the history observable before asking the question $A_{n+1}$, which is deterministic once $B_n = b_n$ is fixed. Moreover,

$$E[H(p_N)] = H_0 - \sum_{n=0}^{N-1} H(X_{n+1}|B_n). \quad (12)$$

**Proof.** First of all, we prove the recursive relation (11). $H(p_n)$ is the entropy of the posterior distribution of $\theta$, which is random through its dependence on the past history $B_n$, hence we can rewrite it as $H(p_n) = H(\theta||B_n)$. Similarly, $H(p_{n+1}) = H(\theta||B_{n+1}) = H(\theta||B_n, X_{n+1})$. Since all three terms in (11) are $\sigma(B_n)$-measurable random variables, it suffices to prove (11) holds for any fixed history $B_n = b_n$, i.e.

$$E[H(\theta||B_n, X_{n+1})|B_n = b_n] = H(\theta|B_n = b_n) - H(X_{n+1}|B_n = b_n). \quad (13)$$

Using information theoretic arguments, we have

$$E[H(\theta||B_n, X_{n+1})|B_n = b_n] = \sum_{x_{n+1}=0}^{k} H(\theta|B_n = b_n, X_{n+1} = x_{n+1})P(X_{n+1} = x_{n+1}|B_n = b_n) \quad (14a)$$

$$= H(\theta|X_{n+1}, B_n = b_n) \quad (14b)$$

$$= H(\theta, X_{n+1}|B_n = b_n) - H(X_{n+1}|B_n = b_n) \quad (14c)$$

$$= H(X_{n+1}|\theta, B_n = b_n) + H(\theta|B_n = b_n) - H(X_{n+1}|B_n = b_n) \quad (14d)$$

$$= H(\theta|B_n = b_n) - H(X_{n+1}|B_n = b_n) \quad (14e)$$

where (14b) comes from the definition of conditional entropy and (14c), (14d) come from the chain rule for conditional entropy. (14e) holds as the first term in (14d) vanishes because the information of $\theta$ completely determines the answer $X_{n+1}$. This proves (13).

Now, in order to prove (12), let us first obtain a recursive relation in unconditional expected entropy of posterior distributions. Taking the expectation over $B_n$ on both sides of (11),

$$E[E[H(p_{n+1})|B_n]] = E[H(p_n)] - E[H(X_{n+1}|B_n)] \quad (15)$$

Note that $E[E[H(p_{n+1})|B_n]] = E[H(p_{n+1})]$ by the iterated conditioning property of conditional expectation. Moreover, $E[H(X_{n+1}|B_n)] = H(X_{n+1}|B_n)$ according to the definition of conditional entropy in (10). Hence, (15) is equivalent to

$$E[H(p_{n+1})] = E[H(p_n)] - H(X_{n+1}|B_n). \quad (16)$$

Applying (16) iteratively for $n = N - 1, \ldots, 0$, we obtain (12), which concludes the proof. $\Box$
Now, applying (12) in Lemma and using an information theoretic argument, we are able to show the first inequality in our main result (8).

**Theorem 1.**

\[
\inf_{\pi \in \Pi} R(\pi, N) \geq H_0 - \log(k + 1)N.
\]  

(17)

Moreover, when \( k > 1 \), this inequality is strict.

**Proof.** Since conditioning always reduces entropy, we have

\[
H(X_{n+1}|B_n) \leq H(X_{n+1}), \text{ for all } n = 0, 1, \ldots, N - 1.
\]  

(18)

Combining (12) with (18), the expected entropy must satisfy

\[
E[H(p_N)] \geq H_0 - \sum_{n=1}^{N} H(X_n).
\]  

(19)

Recall that for all \( n = 1, 2, \ldots, N \), \( X_n \) is a discrete random variable with \( k + 1 \) possible outcomes, namely 0, 1, \ldots, \( k \). The maximum possible value for the entropy \( H(X_n) \) is \( \log(k + 1) \), obtained when each outcome of \( X_n \) has the same probability \( \frac{1}{k+1} \), i.e. \( H(X_n) \leq \log(k + 1) \). Thus, by (19),

\[
E[H(p_N)] \geq H_0 - \log(k + 1)N.
\]  

(20)

Since (20) is true for any policy \( \pi \), and indicating the dependence of \( E[H(p_N)] \) on the policy \( \pi \) in our notation, we have

\[
\inf_{\pi \in \Pi} R(\pi, N) = \inf_{\pi \in \Pi} E^\pi[H(p_N)] \geq H_0 - \log(k + 1)N.
\]  

(21)

This proves our claim (17).

We now prove that the inequality (17) is strict when \( k > 1 \), i.e. when there is more than one object. Consider any fixed \( B_0 = Z = z \), which specifies the questions set \( A_1 \). Recall from (1) that \( X_1 = 1_{A_1}(\theta_1) + \cdots + 1_{A_1}(\theta_k) \) and that \( \theta_1, \ldots, \theta_k \) are independent. As a consequence, \( X_1 | Z = z \sim \text{Bin}(k, p) \), where \( p = \int_{A_1} f_0(u) du \). Therefore, \( H(X_1|Z = z) = H(\text{Bin}(k, p)) < \log(k + 1) \) when \( k > 1 \), implying \( H(X_1|B_0) < \log(k+1) \), so that there is no policy that can achieve the lower bound. 

4 Explicit Characterization of the Posterior Distribution

In this section, we first derive in Section 4.1 an explicit formula for the posterior distribution on the locations of the objects, and introduce some additional notation. We then provide in Section 4.2 an example illustrating this notation and the posterior distribution. This example also will be used later, in Section 6.3, to show that greedy is sometimes strictly better than dyadic. Finally, in Section 4.3, we compute the conditional distribution of the next answer \( X_n \) given previous answers \( X_{1:n-1} \), which we will use later to analyze the value of a policy.
4.1 The Posterior Distribution of the Objects

Consider a fixed \( n \), where \( 1 \leq n \leq N \). For each binary sequence of length \( n \), \( s = \{s_1, \ldots, s_n\} \), let

\[
C_s = \left( \bigcap_{1 \leq j \leq n; s_j = 1} A_j \right) \cap \left( \bigcap_{1 \leq j \leq n; s_j = 0} A_j^c \right) \cap \text{supp}(f_0). \tag{22}
\]

The collection \( \{C_s : C_s \neq \emptyset, s \in \{0, 1\}^n\} \) is a partition of the support of \( f_0 \). A history of \( n \) questions provides information on which sets \( C_s \) contain which objects among \( \theta_{1:k} \).

We will think of a sequence of binary sequences \( s^{(1)}, \ldots, s^{(k)} \) as a sequence of codewords indicating the sets in which each of the objects \( \theta_{1:k} \) reside, i.e., indicating that \( \theta_1 \) is in \( C_{s^{(1)}} \), \( \theta_2 \) is in \( C_{s^{(2)}} \), etc. We may consider each binary sequence \( s^{(1)}, \ldots, s^{(k)} \) to be a column vector, and place them into an \( n \times k \) binary matrix, \( S \). This binary matrix then codes the location of all \( k \) objects, and is a codeword for their joint location.

Moreover, to characterize the location of the random vector \( \theta = (\theta_{1:k}) \) in terms of its codeword \( S \), define \( C_S \subset \mathbb{R}^k \) to be the Cartesian product

\[
C_S = C_{s^{(1)}} \times \cdots \times C_{s^{(k)}}. \tag{23}
\]

To be consistent with an answer \( X_j \), we must have exactly \( X_j \) objects located in the question set \( A_j \) for each \( 1 \leq j \leq n \). This can be described in terms of a constraint on the matrix \( S \) as \( s_j^{(1)} + \cdots + s_j^{(k)} = X_j \), i.e., that the sum of the \( j \)th row in the matrix \( S \) is \( X_j \). Thus, after observing the answers to the questions \( X_{1:n} = x_{1:n} \), the set of all possible joint codewords describing \( \theta_{1:k} \) is

\[
E_n = \{S| s^{(1)}, \ldots, s^{(k)} \in \{0, 1\}^n, C_{s^{(1)}}, \ldots, C_{s^{(k)}} \neq \emptyset, s_j^{(1)} + \cdots + s_j^{(k)} = x_j, \text{for all } 1 \leq j \leq n\}. \tag{24}
\]

An example will be provided in Section 4.2 to illustrate this construction.

Given this notation, we observe the following lemma:

**Lemma 2.** Let the random seed \( Z = z \) be fixed. Then, for each \( x_{1:n} \), the event \( \{X_{1:n} = x_{1:n}\} \) can be rewritten

\[
\{X_{1:n} = x_{1:n}\} = \left\{ \theta \in \bigcup_{S \in E_n} C_S \right\}, \tag{25}
\]

where we recall that \( E_n \) depends on \( x_{1:n} \) and \( z \). Moreover for any \( S, T \in E_n \) with \( S \neq T \), the two sets \( C_S \) and \( C_T \) are disjoint.

**Proof.** Clearly, according to the definition of \( E_n \) in (24), when \( \theta \in \bigcup_{S \in E_n} C_S \), the answers that we observe must satisfy \( X_{1:n} = x_{1:n} \). On the other hand, suppose \( \theta_{1:k} \not\in \bigcup_{S \in E_n} C_S \). Then \( \theta_{1:k} \) belongs to some nonempty set \( C_S \) where \( S \not\in E_n \). Hence, there exists \( j, 1 \leq j \leq n \), such that \( s_j^{(1)} + \cdots + s_j^{(k)} \neq x_j \), which implies that the answer to the question \( A_j \) is \( X_j = s_j^{(1)} + \cdots + s_j^{(k)} \neq x_j \). This proves (25).

Now, for any \( S \neq T \), there exists \( i \) with \( 1 \leq i \leq k \) such that \( s^{(i)} \neq t^{(i)} \). This implies that \( C_{s^{(i)}} \) and \( C_{t^{(i)}} \) are disjoint and the last assertion follows.

At this point, the explicit characterization of the posterior distribution is immediate and we have the following lemma.
Lemma 3.
\[ p_n(u_{1:k}) = \frac{p_0(u_{1:k})}{p_0\left(\bigcup_{S \in E_n} C_S\right)}, \text{for } u_{1:k} \in \bigcup_{S \in E_n} C_S, \]  
(26)

and \( p_n(u_{1:k}) = 0 \) for \( u_{1:k} \notin \bigcup_{S \in E_n} C_S \). Here, for any measurable set \( A \), \( p_0(A) \) denotes the integral \( \int_A p_0(u_{1:k}) du_{1:k} \). Moreover,
\[ p_0\left(\bigcup_{S \in E_n} C_S\right) = \sum_{S \in E_n} p_0(C_S) = \sum_{S \in E_n} f_0(C^{(1)}_S) \cdots f_0(C^{(k)}_S), \]  
(27)

where \( f_0(C^{(i)}_S) \) denotes the integral \( \int_{C^{(i)}_S} f_0(u) du \).

4.2 Examples Illustrating the Posterior Distribution

To illustrate the previous construction, and also to provide the foundation for a later analysis in Section 6.3 showing the greedy policy is strictly better than the dyadic policy in some settings, we provide two examples of the posterior distribution, arising from two different responses to the same sequence of questions.

Suppose \( \theta_1, \theta_2 \) are two objects located in \((0,1]\) with a uniform prior distribution \( f_0 \). Let \( A_1 \) and \( A_2 \) be the first two questions of the dyadic policy, so \( A_1 = (\frac{1}{4}, \frac{1}{2}] \) and \( A_2 = (\frac{1}{4}, \frac{1}{2}] \cup (\frac{3}{4}, 1] \). Then consider two possibilities for the answers to these questions:

Example 1: Suppose \( X_1 = 0 \) and \( X_2 = 2 \). According to (24), there is only one matrix \( S \) in the collection \( E_2 \), which has \( s^{(1)} = s^{(2)} = (0,1)^T \). Thus \( E_2 = \{S_1\} \) where
\[ S_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]  
(28)

By (26) in Lemma 3, we have that \( p_2(u_{1:2}) = 16 \) when \( u_{1:2} \) is in \((\frac{1}{4}, \frac{1}{2}] \times (\frac{1}{4}, \frac{1}{2}]\), and 0 otherwise.

Example 2: Suppose \( X_1 = 1 \) and \( X_2 = 1 \). According to (24), there are four matrices in the collection \( E_2 = \{S_1, S_2, S_3, S_4\} \),
\[ S_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S_4 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \]  
(29)

By (26) in Lemma 3, the posterior distribution has density \( p_2(u_{1:2}) = 16 \) when \( u_{1:2} \) is in \((0, \frac{1}{4}] \times (\frac{3}{4}, 1] \) or \((\frac{1}{4}, \frac{3}{4}] \times (\frac{1}{2}, \frac{3}{4}] \) or \((\frac{3}{4}, \frac{1}{2}] \times (\frac{1}{4}, \frac{1}{2}] \) or \((\frac{3}{4}, 1] \times (0, \frac{3}{4}] \), and is 0 otherwise.

All possible joint locations of \( \theta_1, \theta_2 \) in the two examples above are shown in Figure 2.
**Figure 2:** Illustration of the locations of the two objects $\theta_1, \theta_2$ specified by each matrix given in (28) and (29). The dark subsets mark the location of the objects $\theta_1, \theta_2$.

### 4.3 The Posterior Predictive Distribution of $X_{n+1}$

We now provide an explicit form for the posterior predictive distribution of $X_{n+1}$, i.e., its conditional distribution given the history $X_{1:n}$ and the external source of randomness in the policy $Z$. This is useful because Lemma 1 shows that the expected entropy $E[H(p_N)]$ can be computed using the conditional entropy of $X_{n+1}$ given $B_n = (Z, X_{1:n})$. We use this in Sections 5.2 and 6.2 to compute the expected entropy for the dyadic and greedy policies respectively.

For $n = 0$, we have demonstrated in the proof of Theorem 1 that $X_1$ follows the binomial distribution Bin($k, f_0(A_1)$) given $Z$.

Now, consider any $n \in \{1, 2, \ldots, N - 1\}$, and any fixed history $b_n = (z, x_{1:n})$. Using the equality (25) presented in Lemma 2 we have,

$$P(X_{n+1} = x | B_n = b_n) = \sum_{S \in E_n} P(X_{n+1} = x, \theta \in C_S | B_n = b_n)$$

$$= \sum_{S \in E_n} P(X_{n+1} = x | \theta \in C_S, B_n = b_n)P(\theta \in C_S | B_n = b_n).$$

Now, since for any $S \in E_n$, $\{\theta \in C_S, Z = z\} \subset \{B_n = b_n\}$ according to Lemma 2 we can simplify:

$$P(X_{n+1} = x | \theta \in C_S, B_n = b_n) = P(X_{n+1} = x | \theta \in C_S, Z = z).$$

Also, using Lemma 3 we obtain

$$P(\theta \in C_S | B_n = b_n) = \frac{f_0(C_{\theta(1)}) \cdot \ldots \cdot f_0(C_{\theta(k)})}{\sum_{S \in E_n} f_0(C_{\theta(1)}) \cdot \ldots \cdot f_0(C_{\theta(k)})}.$$
Finally, according to (1), $X_{n+1}$ is the sum of $k$ Bernoulli random variables $\mathbb{1}_{A_{n+1}}(\theta_1), \ldots, \mathbb{1}_{A_{n+1}}(\theta_k)$. Given the event \{\(\theta \in C_S, Z = z\), these $k$ Bernoulli r.v.’s are conditionally independent with respective parameters \(q_1 = \frac{f_0(A_{n+1} \cap C_{s(1)})}{f_0(C_{s(1)})}, \ldots, q_k = \frac{f_0(A_{n+1} \cap C_{s(k)})}{f_0(C_{s(k)})}\). This conditional independence can be verified as follows. Consider any fixed binary vector \(w \in \{0,1\}^k\). For each $i = 1, \ldots, k$, let $D_i$ be equal to $A_{n+1}$ if $w_i = 1$ and its complement $A_{n+1}^c$ if $w_i = 0$. Then,

$$P(\mathbb{1}_{A_{n+1}}(\theta_i) = w_i, i = 1, \ldots, k|\theta \in C_S, Z = z) = P(\theta_i \in D_i, i = 1, \ldots, k|\theta \in C_S, Z = z)$$

$$= \prod_{i=1}^k \frac{p_0(D_i \cap C_{s(i)})}{p_0(C_{s(i)})} = \prod_{i=1}^k \frac{p_0(D_i \cap C_{s(i)})}{p_0(C_{s(i)})}$$

Using the fact that $X_{n+1}$ is the sum of $k$ conditionally independent Bernoulli random variables given $\theta \in C_S$ and $Z = z$, we may provide an explicit probability mass function. When \(q_1 = \ldots = q_k\), $X_{n+1}$ is conditionally Bin$(k,q_1)$ given $\theta \in C_S$ and $Z = z$. In general, let $W_1, \ldots, W_n$ be $n$ independent discrete random variables with $W_i \sim$ Bernoulli$(q_i)$, where $q_1, \ldots, q_n$ are any real numbers in [0,1]. The distribution of $Y = W_1 + \ldots + W_n$ is called Poisson Binomial distribution, which was first studied by S. D. Poisson in [27]. We denote the distribution of $Y$ by $PB(q_1, \ldots, q_n)$ and its probability mass function $P(Y = y) = f_{PB}(y; q_1, \ldots, q_n)$ is given by

$$f_{PB}(y; q_1, \ldots, q_n) = \sum_{w=(w_1, \ldots, w_n) \in \{0,1\}^n \text{ with } \sum_{j=1}^n w_j = y} \prod_{j=1}^n q_j^{w_j}(1-q_j)^{1-w_j},$$

and has mean and variance given by

$$E[Y] = q_1 + \ldots + q_n,$$

$$Var[Y] = q_1(1-q_1) + \ldots + q_n(1-q_n).$$

Using this definition of the Poisson Binomial distribution, the conditional distribution of $X_{n+1}$ given $\theta \in C_S$ and $Z = z$ is $PB(q_1, \ldots, q_n)$. Finally, putting together equations (30), (32), and the fact that $X_{n+1}$ is conditionally $PB(q_1, \ldots, q_n)$ given $\theta \in C_S$ and $Z = z$ provides the following characterization of the conditional probability mass function of $X_{n+1}$ given $B_n = (Z, X_{1:n}) = b_n$.

**Theorem 2.** For $n = 0$, given \(\{B_0 = b_0\} = \{Z = z\}, X_1 \sim$ Bin$(k,f_0(A_1))$. For $n = 1, 2, \ldots, N-1$, given $B_n = (Z, X_{1:n}) = b_n$, $X_{n+1}$ is a mixture of Poisson Binomial distributions with probability mass function:

$$P(X_{n+1} = x|B_n = b_n) = \sum_{S \subseteq E_n} \sum_{T \subseteq E_n} \frac{f_0(C_{s(1)} \cap \cdots \cap C_{s(k)})}{f_0(C_{s(1)}) \cdots f_0(C_{s(k)})} f_{PB} \left( x, q_1 = \frac{f_0(A_{n+1} \cap C_{s(1)})}{f_0(C_{s(1)})}, \ldots, q_k = \frac{f_0(A_{n+1} \cap C_{s(k)})}{f_0(C_{s(k)})} \right).$$

5 The Dyadic Policy for Localizing Multiple Objects

We now present the first policy of interest: the dyadic policy. This policy is easy to implement, and is non-adaptive, allowing its use in parallel computing environments. The description of the dyadic
policy will be given in Section 5.1. In Section 5.2, we will prove the theorem concerning the value of this policy and derive the last equality in our main results (8). Finally, asymptotic normality of $H(p_N)$ under the dyadic policy will be provided in Section 5.3.

5.1 Description of the dyadic policy

The definition of the dyadic policy is given in (7). In this section, we provide an iterative construction of this policy, introducing notation which will be useful later on.

First, we partition the support of $f_0$ into two subsets, $A_{1,0}$ and $A_{1,1}$:

\begin{align}
A_{1,0} &= \left( Q\left(0\right), Q\left(\frac{1}{2}\right) \right] \cap \text{supp}(f_0), \quad (37a) \\
A_{1,1} &= \left( Q\left(\frac{1}{2}\right), Q\left(1\right) \right] \cap \text{supp}(f_0), \quad (37b)
\end{align}

where $Q$, as defined in (6), denotes the quantile function. With this partition, the question asked at time 1 is

$$A_1 = A_{1,1}. \quad (38)$$

Then we adopt a similar procedure recursively for each $n = 1, \ldots, N - 1$ to partition $A_{n,j}$ into two subsets, $A_{n+1,2j}$ and $A_{n+1,2j+1}$ and then construct the question from these partitions. For $j = 0, \ldots, 2^n - 1$, define

\begin{align}
A_{n+1,2j} &= \left( Q\left(\frac{2j}{2^{n+1}}\right), Q\left(\frac{2j + 1}{2^{n+1}}\right) \right] \cap \text{supp}(f_0), \quad (39a) \\
A_{n+1,2j+1} &= \left( Q\left(\frac{2j + 1}{2^{n+1}}\right), Q\left(\frac{2j + 2}{2^{n+1}}\right) \right] \cap \text{supp}(f_0), \quad (39b)
\end{align}

Then the question asked at time $n + 1$ is

$$A_{n+1} = \bigcup_{j=0}^{2^n-1} A_{n+1,2j+1}. \quad (40)$$

An illustration of these sets $A_n$ is provided below in Figure 3.

Note that the dyadic policy is non-adaptive, as only the prior distribution is used to construct the next set and not the answer to previous questions.
5.2 The value of the dyadic policy

The value of the dyadic policy is stated as follows:

**Theorem 3.** Under the dyadic policy $\pi_D$, 

$$ R(\pi_D, N) = H_0 - H\left(\text{Bin}\left(k, \frac{1}{2}\right)\right)N. \quad (41) $$

**Proof.** In this proof, we will first simplify the equation (36) in Theorem 2 to obtain the posterior distribution of $X_{n+1}$ under the dyadic policy. Then we will calculate the entropy $H^{\pi_D}(X_{n+1}|B_n)$ and employ Lemma 1 to compute the value of the dyadic policy.

At time $n$, where $1 \leq n \leq N$, the support of $f_0$ is partitioned into pairwise disjoint subsets $\{A_{n,0}, \ldots, A_{n,2^n-1}\}$. Recall the definition of $C_s$ in (22). The sets $C_s$ provide a bijection which maps a binary sequence $s \in \{0,1\}^n$ to a subset $A_{n,j(s)}$ for some $j(s) \in \{0,1,\ldots,2^n-1\}$. Hence, $C_{s(i)}$ in (36) can be rewritten as 

$$ C_{s(i)} = A_{n,j(s(i))}, \text{ for some index } j(s(i)) \in \{0,1,\ldots,2^n-1\}. \quad (42) $$

According to the construction of dyadic questions in Section 5.1, $A_{n+1} = \bigcup_{j=0}^{2^n-1} A_{n+1,2j+1}$. Moreover, $A_{n+1,2j(s(i))+1} \subset A_{n,j(s(i))}$ and $A_{n+1,2j+1} \cap A_{n,j(s(i))} = \emptyset$, for all $j \neq j(s(i))$. Thus, by (42) we have 

$$ A_{n+1} \cap C_{s(i)} = A_{n+1,2j(s(i))+1}. \quad (43) $$
Combining the above result with the fact that \( f_0(A_{n+1,2j(s^{(i)})+1}) = \frac{1}{2} f_0(A_{n,j(s^{(i)})}) \) yields

\[
\frac{f_0(A_{n+1} \cap C_{s^{(i)}})}{f_0(C_{s^{(i)}})} = \frac{1}{2},
\]

and this is true for all \( i = 1, 2, \ldots, k \).

Thus, for \( n \geq 1 \), we can simplify \( \text{(36)} \) in Theorem 2 as

\[
p_n(X_{n+1} = x | B_n = b_n) = \sum_{S \in E_n} \sum_{T \in E_{n+1}} \frac{f_0(C_{s^{(1)}}) \ldots f_0(C_{s^{(i)}})}{f_0(C_{s^{(i)}}) \ldots f_0(C_{s^{(k)}})} f_{PB}(x, q_1 = \frac{1}{2}, \ldots, q_k = \frac{1}{2})
\]

\[
= f_{PB}(x, q_1 = \frac{1}{2}, \ldots, q_k = \frac{1}{2}).
\]

The density above is just the density of the binomial distribution \( \text{Bin} \left( k, \frac{1}{2} \right) \). We proved that given \( \{ B_n = b_n \} \), \( X_{n+1} \) is distributed as \( \text{Bin} \left( k, \frac{1}{2} \right) \) and \( H^{\pi_D}(X_{n+1}|B_n = b_n) = H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) \) for all \( n = 1, \ldots, N - 1 \). Thus, taking the expectation over all possible realizations of \( B_n \), we obtain

\[
H^{\pi_D}(X_{n+1}|B_n) = H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right).
\]

Since \( f_0(A_1) = \frac{1}{2} \) under the dyadic policy, according to Theorem 2, \( X_1|Z = z \) is distributed as \( \text{Bin} \left( k, \frac{1}{2} \right) \) for any fixed \( z \) and \( H^{\pi_D}(X_1|B_0) = H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) \) as well.

Therefore, according to \( \text{(12)} \) in Lemma 1

\[
R(\pi_D, N) = E^{\pi_D}[H(p_N)] = H_0 - \sum_{n=0}^{N-1} H^{\pi_D}(X_{n+1}|B_n) = H_0 - H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) N.
\]

Note that this is the last equality in our main result \( \text{(8)} \).

### 5.3 Convergence in entropy under the dyadic policy

In real applications, however, we are concerned not only about the expected entropy \( E^{\pi_D}[H(p_N)|p_0] \) but also about the actual entropy \( H(p_N) \) that we obtain in a specific trial. It would be beneficial if the actual entropy did not deviate too much from its expected value. It turns out to be the case for the dyadic policy under the assumptions that the prior density \( f_0 \) is bounded from above. Lemma 4 provides a decomposition formula for the actual entropy \( H(p_n) \) into a sum of two terms. The first term is a sum of i.i.d. random variables. The second term is a converging martingale as will be shown in Lemma 5. Finally, Theorem 4 provides almost sure convergence and asymptotic normality for \( H(p_n) \) as a direct consequence of Lemma 4 and 5. Note that the dyadic policy is deterministic, i.e., it does not make use of the random seed \( Z \). As a consequence, in this section, we use \( X_{1:n} \) to denote the history up to time \( n \) without including \( Z \).
Lemma 4. Under the dyadic policy, for all $n = 1, 2, \ldots, N$,

$$H(p_n) = - \sum_{j=1}^{n} Z_j + I_2(n), \quad (48)$$

where $I_2(n)$ is a random variable and $Z_j = k - \log \left( \binom{k}{x_j} \right)$ with $X_j$ following i.i.d binomial distribution $\text{Bin}(k, \frac{1}{2})$.

Proof. Let $X_{1:n} = x_{1:n}$ be fixed. According to Lemma 3

\begin{equation}
\begin{aligned}
p_n(u_{1:k}) &= \frac{p_0(u_{1:k})}{p_0 \left( \bigcup_{S \in E_n} C_S \right)} = \frac{f_0(u_1) \ldots f_0(u_k)}{\sum_{S \in E_n} f_0(C_{s(1)}) \ldots f_0(C_{s(k)})},
\end{aligned}
\end{equation}

where $(u_{1:k}) \in C := \bigcup_{S \in E_n} C_S$.

Under the dyadic policy, the support of $f_0$ is partitioned into $2^n$ subsets with identical probability masses after the final step and each $C_{s(i)}$ is one such subset, for $i = 1, 2, \ldots, k$. Thus, we have

$$f_0(C_{s(i)}) = 2^{-n}, \text{ for } i = 1, 2, \ldots, k \text{ and } S \in E_n. \quad (50)$$

Let $|E_n|$ be the cardinality of $E_n$. Note that under the dyadic policy, every binary sequence $s$ of length $N$ corresponds to a nonempty set $C_s$. Furthermore, in step $j$, there are \( \binom{k}{x_j} \) ways to choose the $j^{th}$ row in the matrix satisfying the definition in (54), for $j = 1, 2, \ldots, n$. Thus, by the product rule,

$$|E_n| = \prod_{j=1}^{n} \binom{k}{x_j}. \quad (51)$$

By (50) and (51),

$$p_0(C) = \sum_{S \in E_n} f_0(C_{s(1)}) \ldots f_0(C_{s(k)}) = 2^{-nk} \prod_{j=1}^{n} \binom{k}{x_j}. \quad (52)$$

Combining the result above and the definition of the differential entropy, we have

\begin{equation}
\begin{aligned}
H(p_n) &= - \int_C p_n(u_{1:k}) \log(p_n(u_{1:k})) \, du_{1:k} \\
&= - \int_C \frac{p_0(u_{1:k})}{p_0(C)} \log \left( \frac{p_0(u_{1:k})}{p_0(C)} \right) \, du_{1:k} \\
&= \left[ \log \left( \frac{p_0(C)}{p_0(C)} \right) \int_C p_0(u_{1:k}) \, du_{1:k} \right] + \left[ - \frac{1}{p_0(C)} \int_C p_0(u_{1:k}) \log(p_0(u_{1:k})) \, du_{1:k} \right] \\
&= I_1(n) + I_2(n),
\end{aligned}
\end{equation}

where $I_1(n)$ and $I_2(n)$ denote the first term and the second term in the last equation above. $I_1(n)$ can be easily computed as

\begin{equation}
\begin{aligned}
I_1(n) &= \frac{\log(p_0(C))}{p_0(C)} \int_C p_0(u_{1:k}) \, du_{1:k} = \log(p_0(C)) = - \left( nk - \sum_{j=1}^{n} \log \left( \binom{k}{x_j} \right) \right). \quad (54)
\end{aligned}
\end{equation}

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Now consider $X_{1:n}$ as random variables. By Theorem 2, we see that under the dyadic policy, $X_{1:n}$ is a sequence of i.i.d. random variables $\text{Bin}(k, \frac{1}{2})$. Moreover, $I_2(n)$ is random through its dependence on the random support $C$. Therefore, combining (53) and (54), we prove the claim in Lemma 4 by setting $Z_j = k - \log \left( \frac{k}{X_j} \right)$.

Define $I_2(0) = H(p_0) = H_0$ so that (48) is also satisfied for $n = 0$. Applying the result above, we can furthermore analyze the term $I_2(n)$ and derive the following lemma.

**Lemma 5.** Assume there exists $M > 0$ such that $f_0(u) \leq M$ for all $u \in \mathbb{R}$. Then the random variable $I_2(n)$ in (48) converges to a random variable $I_2(\infty)$ almost surely as $n \to \infty$, where $I_2(\infty)$ is a random variable and $E[|I_2(\infty)|] < \infty$.

**Proof.** We prove almost sure convergence using the martingale convergence theorem (see Theorem 35.5 in [28]). First, let us calculate the expected value of $Z_j$ as follows.

\[
E(Z_j) = \sum_{j=0}^{k} \left( k - \log \left( \frac{k}{j} \right) \right) \left( \frac{k}{j} \right)^2 2^{-k}. \tag{55}
\]

Therefore, $E(Z_j) = H\left( \text{Bin}(k, \frac{1}{2}) \right)$ since

\[
H\left( \text{Bin}(k, \frac{1}{2}) \right) = -\sum_{j=0}^{k} \left( \frac{k}{j} \right) 2^{-k} \log \left( \left( \frac{k}{j} \right)^2 \right) = \sum_{j=0}^{k} \left( k - \log \left( \frac{k}{j} \right) \right) \left( \frac{k}{j} \right)^2 2^{-k}. \tag{56}
\]

Now, let us verify that $I_2(n)$ is a martingale. According to (48),

\[
E[I_2(n+1)|X_{1:n}] = E \left[ H(p_{n+1}) + \sum_{j=1}^{n+1} Z_j | X_{1:n} \right] \tag{57a}
\]

\[
= H(p_n) - H(X_{n+1}|X_{1:n}) + \sum_{j=1}^{n} Z_j + E[Z_{n+1}|X_{1:n}] \tag{57b}
\]

\[
= I_2(n) - H\left( \text{Bin} \left( k, \frac{1}{2} \right) \right) + E[Z_{n+1}] \tag{57c}
\]

\[
= I_2(n), \tag{57d}
\]

where (57b) is true by (11) in Lemma 1 and the fact that $Z_{1:n}$ is $\sigma(X_{1:n})$-measurable. (57d) holds because we have proved under the dyadic policy, $X_{n+1}|X_{1:n} \sim \text{Bin} \left( k, \frac{1}{2} \right)$, which is independent of $X_{1:n}$, and $Z_{n+1}$ is also independent of $X_{1:n}$. (57c) holds because we have proved $E[Z_{n+1}] = H\left( \text{Bin} \left( k, \frac{1}{2} \right) \right)$. 

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Next, we want to show that $E[|I_2(n)|] < \infty$. Let us fix $X_{1:n} = x_{1:n}$ and expand $I_2(n)$ in as

$$I_2(n) = -\frac{1}{p_0(C)} \sum_{S \in E_n^{C_i}} \int f_0(u_1) \ldots f_0(u_k) \log (f_0(u_1) \ldots f_0(u_k)) \, du_{1:k}$$

$$= -\frac{1}{p_0(C)} \sum_{S \in E_n} \sum_{i=1}^k \left( \int f_0(u_i) \log(f_0(u_i)) \, du_i \prod_{j \neq i} \int f_0(u_j) \, du_j \right)$$

$$= -\frac{1}{p_0(C)} \sum_{S \in E_n} \sum_{i=1}^k 2^{-n(k-1)} \int f_0(u_i) \log(f_0(u_i)) \, du_i. \tag{58}$$

Now consider the integral $\int_{C_s(i)} f_0(u_i) \log(f_0(u_i)) \, du_i$. Since $f_0(u_i) \leq M$, we can obtain an upper bound for $\int_{C_s(i)} f_0(u_i) \log(f_0(u_i)) \, du_i$ as

$$\int_{C_s(i)} f_0(u_i) \log(f_0(u_i)) \, du_i \leq \log M \int_{C_s(i)} f_0(u_i) \, du_i = 2^{-n} \log M. \tag{59}$$

Substituting (52) and (59) into (58), we have

$$I_2(n) \geq -k \log M. \tag{60}$$

Furthermore, define $I_2^+(n) = \max(I_2(n), 0), I_2^-(n) = \max(-I_2(n), 0)$ and we have

$$E[|I_2(n)|] = E[I_2^+(n)] + E[I_2^-(n)] = E[I_2(n)] + 2E[I_2^-(n)] \leq H_0 + 2k \log M, \tag{61}$$

where the last equation follows from the fact that $E[I_2(n)] = I_2(0) = H_0$ since $I_2$ is a martingale and $I_2^-(n) \leq k \log M$ by (60). Therefore, using the martingale convergence theorem, $I_2(n)$ converges to a random variable $I_2(\infty)$ almost surely with $E[|I_2(\infty)|] \leq H_0 + 2k \log M$.

From the proof above we can see that if $f_0$ is uniform over $(0, 1]$, $f_0(u_i) = 1$ for all $u_i \in (0, 1]$ and thus the term $I_2$ is 0. Therefore, in this case, $H(p_n) = -\left( nk - \sum_{j=1}^n \log \left( \frac{k}{X_j} \right) \right)$.

Now the following theorem is a direct consequence of the preceding lemmas.

**Theorem 4.** Assume there exists $M > 0$ such that $f_0(u) \leq M$ for all $u \in \mathbb{R}$. Then under the dyadic policy,

$$\lim_{N \to \infty} \frac{H(p_N)}{N} = -H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) \text{ almost surely,} \tag{62}$$

and

$$\lim_{N \to \infty} \frac{H(p_N) + NH \left( \text{Bin} \left( k, \frac{1}{2} \right) \right)}{\sqrt{N}} \overset{d}{=} N(0, \sigma^2), \tag{63}$$

where $\sigma^2$ is the variance of the random variable $\log \left( \frac{k}{X} \right)$ with $X \sim \text{Bin} \left( k, \frac{1}{2} \right)$. 

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Proof. According to Lemma 5, \( \lim_{N \to \infty} \frac{I_2(N)}{N} = \lim_{N \to \infty} \frac{I_2(\infty)}{N} = 0 \) almost surely. Hence, by (48) in Lemma 4

\[
\lim_{N \to \infty} \frac{H(p_N)}{N} = \lim_{N \to \infty} \frac{I_2(N)}{N} - \frac{1}{N} \sum_{j=1}^{N} Z_j = 0 - E[Z_1] = -H\left(\text{Bin}\left(k, \frac{1}{2}\right)\right) \tag{64}
\]

almost surely.

To prove (63), note that

\[
\lim_{N \to \infty} \frac{H(p_N) + NH\left(\text{Bin}\left(k, \frac{1}{2}\right)\right)}{\sqrt{N}} = \frac{I_2(N) - \sum_{i=1}^{N} Z_j + NH\left(\text{Bin}\left(k, \frac{1}{2}\right)\right)}{\sqrt{N}}.
\tag{65}
\]

Furthermore, since by Lemma 5, \( I_2(N) \) converges to \( I_2(\infty) \) almost surely and \( E|I_2(\infty)| < \infty \), \( \frac{I_2(N)}{\sqrt{N}} \to 0 \) almost surely, which implies \( \frac{I_2(N)}{\sqrt{N}} \sim \mathcal{L} \to 0 \). On the other hand, \( E(Z_j) = H\left(\text{Bin}\left(k, \frac{1}{2}\right)\right) \) and \( Var(Z_j) = Var\left(\log\left(\frac{k}{X}\right)\right) = \sigma^2 \), where \( X \sim \text{Bin}\left(k, \frac{1}{2}\right) \). Hence, by the central limit theorem, we have \( \frac{-\sum_{i=1}^{N} Z_j + NH\left(\text{Bin}\left(k, \frac{1}{2}\right)\right)}{\sqrt{N}} \sim \mathcal{N}(0, \sigma^2) \). Therefore, by Slutsky’s Theorem (Theorem 25.4 in [28]),

\[
\lim_{N \to \infty} \frac{H(p_N) + NH\left(\text{Bin}\left(k, \frac{1}{2}\right)\right)}{\sqrt{N}} = \frac{I_2(N)}{\sqrt{N}} + \frac{-\sum_{i=1}^{N} Z_j + NH\left(\text{Bin}\left(k, \frac{1}{2}\right)\right)}{\sqrt{N}} \mathcal{L} \to \mathcal{N}(0, \sigma^2). \tag{66}
\]

Figure 4 below shows the simulation results for localizing one object, two objects, and three objects under the dyadic policy, respectively. We assume the prior density \( f_0 \) is uniform over \((0, 1]\) and ask 100 questions to locate the objects. The top line corresponds to locating a single object. In this case, the dyadic policy is actually optimal and identical to the greedy policy as was proved in [5]. Moreover, the entropy process \( H(p_n) \) is in this case deterministic. The middle and bottom lines show the results for respectively \( k = 2 \) and \( k = 3 \) objects. In this case, the entropy process \( H(p_n) \) is not deterministic anymore. The entropy reduction per question which is visualized in the second column is asymptotically equal to \( H\left(\text{Bin}\left(k, \frac{1}{2}\right)\right) \) according to the law of large numbers. The third column illustrates the asymptotic normality of the entropy process for the dyadic policy.
Figure 4: Simulation results for localizing one, two, three objects under the dyadic policy. $N = 100$ and $f_0$ is uniform over $(0, 1]$. The horizontal graphs above show the actual trajectories of entropy $H(p_n)$, average reduction in entropy $\frac{H(p_n)}{n}$, and normality of $\frac{H(p_n) + N H(Bin(k, \frac{1}{2}))}{\sqrt{N}}$, respectively.

6 The Greedy Policy for Localizing Multiple Objects

In this section, we will present the second policy of interest—the greedy policy. The greedy policy is a family of policies (not unique) which pursue a maximal one-step expected reduction in entropy. Despite having a better performance than the dyadic policy, the greedy policy is difficult for us to parametrize and implement. A description of the greedy policy will be given in Section 6.1 and an upper bound of its value is shown in Section 6.2, which verifies our claim of the third inequality in the main results (8). Furthermore, we will provide an example in which the greedy policy outperforms the dyadic policy in Section 6.3 and thus this inequality cannot be reversed.

6.1 Description of the greedy policy

Unlike the dyadic policy, the greedy policy is adaptive, that is, the actual policy depends on the previous answers that we already observed, and at each step the question set $A_n \subset \mathbb{R}$ is defined in [5] to maximize the one-step expected reduction in entropy.
We prove that this argmin exists below in Theorem 5. The computation of the greedy policy might be complicated in some cases, however, the greedy policy is strictly better than the dyadic policy and we will demonstrate this point in Section 6.3.

6.2 The value of the greedy policy

Although deriving the value of the greedy policy seems impossible, we are able to employ Lemma 1 to derive an upper bound of it as the following.

**Theorem 5.** The argmin (5) defining the class of greedy policies exists. Under any greedy policy \( \pi_G \),

\[
R(\pi_G, N) \leq H_0 - H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) N. \tag{67}
\]

**Proof.** Fix some history \( B_n = (Z, X_{1:n}) = b_n \). We first show existence of the argmin from [5], restated here as

\[
\arg \min_A E[H(p_{n+1}|p_n, A_{n+1} = A)], \tag{68}
\]

where we recall that the minimum is taken over all Borel-measurable subsets of \( \mathbb{R} \).

Since conditioning on the posterior distribution \( p_n \) under any fixed policy is equivalent to conditioning on \( \{B_n = (Z, X_{1:n}) = b_n\} \), using (11) in Lemma 1, we have

\[
E[H(p_{n+1}|p_n, A_{n+1} = A)] = E[H(p_{n+1}|B_n = b_n, A_{n+1} = A)]
= H(p_n|B_n = b_n, A_{n+1} = A) - H(X_{n+1}|B_n = b_n, A_{n+1} = A). \tag{69}
\]

Since the first term \( H(p_n|B_n = b_n, A_{n+1} = A) \) does not depend on \( \{A_{n+1} = A\} \), (68) can be rewritten as

\[
\arg \min_A H(p_n|B_n = b_n, A_{n+1} = A) - H(X_{n+1}|B_n = b_n, A_{n+1} = A)
= \arg \max_A H(X_{n+1}|B_n = b_n, A_{n+1} = A). \tag{70}
\]

When \( n = 0 \), according to Theorem 2, we can rewrite the above argmax as

\[
\arg \max_A \left( \text{Bin} \left( k, f_0(A) \right) \right). \tag{71}
\]

The maximum is achieved by any questions set \( A \) such that \( f_0(A) = \frac{1}{2} \). For example, the first dyadic question \( (Q \left( \frac{1}{2} \right), Q(1)] \cap \text{supp} f_0 \) is one of such sets. This also proves \( H_{\pi_G}(X_1|B_0) = H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) \).

When \( n \geq 1 \), using (36) in Theorem 2, we can rewrite the argmax in (70) as

\[
\arg \max_A \left( \sum_{S \in E_n} \alpha(S) \text{PB} \left( \frac{f_0(A \cap C_s)}{f_0(C_s)} \right) \right), \tag{72}
\]

where \( \alpha(S) = \frac{f_0(C_{s(1)}) \cdots f_0(C_{s(k)})}{\tau_{s(1)} f_0(C_{s(1)}) \cdots f_0(C_{s(k)})} \) and \( \sum_{S \in E_n} \alpha(S) = 1 \).

Let \( S = \{ s \in \{0, 1\}^n : C_s \neq \emptyset \} \), and fix some arbitrary order of these elements so that \( S \) becomes a sequence rather than a set. For each \( s \in S \), let \( r_s(A) = f_0(A \cap C_s)/f_0(C_s) \) so that (72) can be rewritten as

\[
\arg \max_A \left( \sum_{S \in E_n} \alpha(S) \text{PB} \left( r_s(A) \right) \right). \tag{73}
\]
For each Borel-measurable subset \( A \) of \( \mathbb{R} \), \( r(A) = (r_s(A) : s \in S) \) is an element of \([0,1]^{|S|}\). Moreover, for each \( r \in [0,1]^{|S|} \), there is a Borel-measurable \( A \subset \mathbb{R} \) such that \( r(A) = r \). This is because the continuity of the prior cumulative density function allows us to construct the desired subset \( A \) as a union of sets, one for each element of \( S \). In this construction, the subset of \( A \) corresponding to \( s \in S \) is a subset of \( C_s \) containing a fraction \( r_s \) of the prior mass of \( C_s \). This shows that the argmax \((\ref{argmax})\) exists iff the following argmax exists:

\[
\arg \max_{r \in [0,1]^{|S|}} H \left( \sum_{S \in E_n} \alpha(S) \text{PB} \left( r_s^{(1)}, \ldots, r_s^{(k)} \right) \right). \tag{74}
\]

The function \( r \mapsto H \left( \sum_{S \in E_n} \alpha(S) \text{PB} \left( r_s^{(1)}, \ldots, r_s^{(k)} \right) \right) \) is continuous, and the set \([0,1]^{|S|}\) is compact, so this argmax is attained. This shows that the argmax \((\ref{argmax})\) defining the class of greedy policies is well-defined.

We now show an upper bound on the value of any greedy policy \( \pi_G \) by showing a lower bound on this quantity. The argument above also shows that under any greedy policy \( \pi_G \), for \( n \geq 1 \),

\[
H^\pi_G(X_{n+1}|B_n = b_n) = \max_A H(X_{n+1}|B_n = b_n, A_{n+1} = A) \tag{75a}
\]

\[
= \max_{r \in [0,1]^{|S|}} H \left( \sum_{S \in E_n} \alpha(S) \text{PB} \left( r_s^{(1)}, \ldots, r_s^{(k)} \right) \right) \tag{75b}
\]

\[
\geq \max_{r \in [0,1]^{|S|}} \sum_{S \in E_n} \alpha(S) H \left( \text{PB} \left( r_s^{(1)}, \ldots, r_s^{(k)} \right) \right) \tag{75c}
\]

\[
\geq \sum_{S \in E_n} \alpha(S) H \left( \text{PB} \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \right) \tag{75d}
\]

\[
= H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) \tag{75e}
\]

Above, we use the concavity of the entropy function to obtain the inequality \((\ref{argmax})\), and that \( \text{PB}(\frac{1}{2}, \ldots, \frac{1}{2}) \) a special case of a Poisson Binomial Distribution to obtain \((\ref{argmax})\). The last line, \((\ref{argmax})\), follows from \( \sum_{S \in E_n} \alpha(S) = 1 \) and the fact that \( \text{PB}(\frac{1}{2}, \ldots, \frac{1}{2}) \) is the Bin \( \left( k, \frac{1}{2} \right) \) distribution.

Furthermore, taking the expectation over all possible realizations of \( B_n \), we obtain for \( n \geq 1 \),

\[
H^\pi_G(X_{n+1}|B_n) \geq H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) \tag{76}
\]

Recall that we already have \( H^\pi_G(X_1|B_0) = H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) \) from previous arguments.

Finally, \((\ref{argmax})\) in Lemma \( \ref{lemma} \) shows

\[
R(\pi_G, N) = E^\pi_G [H(p_N)] = H_0 - \sum_{j=0}^{N-1} H^\pi_G(X_{n+1}|B_n) \leq H_0 - H \left( \text{Bin} \left( k, \frac{1}{2} \right) \right) N. \tag{77}
\]

### 6.3 A setting in which the greedy policy is strictly better than the dyadic policy

In this section, we show that the greedy policy is strictly better than the dyadic policy under some circumstances.
Example 3: Suppose $\theta_1, \theta_2$ are two objects located in $(0,1]$ with the prior $f_0$ being uniform over $(0,1]$, and $A_1$ and $A_2$ the first two questions of the dyadic policy, $A_1 = \left[\frac{1}{4}, \frac{1}{2}\right]$ and $A_2 = \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left(\frac{3}{4}, 1\right]$. Now consider the following family of questions $A_3$ indexed by $0 \leq \alpha, \beta \leq 1$:

$$A_3 = \left(\frac{1 - \alpha}{4}, \frac{1}{4}\right] \cup \left(\frac{2 - \beta}{4}, \frac{1}{2}\right] \cup \left(\frac{3 - \beta}{4}, \frac{3}{4}\right] \cup \left(\frac{4 - \alpha}{4}, 1\right].$$

(78)

According to (36), given $X_1 = 2$ and $X_2 = 0$, the point mass function of $X_3$ is

$$P(X_3 = x) = f_{PB}(x; q_1 = \beta, q_2 = \beta),$$

(79)

which is a Binomial distribution with parameter $\beta$. The maximum entropy is then achieved when $\beta = 0.5$. Note that the dyadic question, corresponding to $\alpha = \beta = 0.5$, verifies this condition and as a consequence is also a valid question for the greedy policy.

Now, more interestingly, assume that $X_1 = X_2 = 1$, then, according to (36), the point mass function of $X_3$ is

$$p_2(X_3 = x) = \frac{1}{4} f_{PB}(x; q_1 = \alpha, q_2 = \alpha) + \frac{1}{4} f_{PB}(x; q_1 = \beta, q_2 = \beta) + \frac{1}{4} f_{PB}(x; q_1 = \beta, q_2 = \beta) + \frac{1}{4} f_{PB}(x; q_1 = \alpha, q_2 = \alpha),$$

(80)

which simplifies to

| $x$ | $p_2(X_3 = x)$ |
|-----|----------------|
| 0   | $\frac{1}{4} (1 - \alpha)^2 + \frac{1}{4} (1 - \beta)^2$ |
| 1   | $\alpha (1 - \alpha) + \beta (1 - \beta)$ |
| 2   | $\frac{1}{4} \alpha^2 + \frac{1}{4} \beta^2$ |

Now, one can choose values for $\alpha$ and $\beta$ such that $p_2(X_3 = x) = \frac{1}{3}$, $x = 0, 1, 2$. Specifically,

$$\alpha = \frac{1 + \frac{\sqrt{3}}{2}}{2} \text{ and } \beta = \frac{1 - \frac{\sqrt{3}}{2}}{2}.$$  

(81)

In this case $H(p_2(X_3 = \cdot)) = \log(3) > 1.5$ which shows that the greedy policy is in this case strictly better than the dyadic policy.

7 Conclusion

We have considered the problem of twenty questions with noiseless answers, in which we aimed at locating multiple objects simultaneously. There are a variety of applications associated with this problem, such as group testing, computer vision, stochastic simulation and bioinformatics. By adopting the approach of minimizing the expected entropy of the posterior distribution, we derived a lower bound on the expected entropy and studied two classes of policies, the dyadic policy and the greedy policy. Although the greedy policy, as we have shown, outperforms the dyadic policy in reducing the expected entropy, the latter employs a series of pre-determined question sets and thus is easy to implement. In addition, the dyadic policy beats traditional policies such as the sequential bifurcation policy and is relatively stable in the sense that the average reduction in entropy converges under certain assumptions (Section 5.3).
Also, there are several questions calling for future works. First, in real applications, noisy answers provide a more natural and accurate approximation but we only considered noiseless answers in this paper. Second, we assumed the number of the objects is known, but in a more general setting, this assumption should be released. Third, another objective function such as the mean-squared error can replace the expected entropy, which measures the performance of a specific policy differently. We feel that researches in these and other questions will be prosperous and fruitful.

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A Definition of the Sequential Bifurcation Policy

In this appendix, we define the sequential bifurcation policy used as a benchmark in Figure 1. This policy is based on the sequential bifurcation policy of [25], but adapted slightly to the setting considered in this paper.

We define the sequential bifurcation (SB) policy as follows. At each point in time $n$, SB maintains a disjoint collection of intervals $D_n = \{D_{n,1}, \ldots, D_{n,m_n}\}$. At time 0, $D_0 = \{\mathbb{R}\}$, and for each $n$, SB obtains $D_{n+1}$ and $A_{n+1}$ recursively as follows. First, SB chooses the interval $D_{n}^*$ in $D_n$ with the largest mass under the prior, i.e.,

$$D_{n}^* \in \arg \max_{D \in D_n} \int_D f_0(u) du. \quad (82)$$

Then, SB obtains $A_{n+1}$ by splitting $D_{n}^*$ at its conditional median under the posterior, and taking the left-hand portion. SB then creates $D_{n+1}$ by adding to $D_n \setminus D_{n}^*$ those intervals $A_{n+1}$ and $D_{n}^* \setminus A_{n+1}$ shown by $X_{n+1}$ to have at least one object.

This version of the sequential bifurcation policy differs slightly from the policy presented in [25] in that (1) it is designed for the continuum rather for a discrete domain; (2) it is designed for the case with known $k$, while running it for unknown $k$ (as does [25]) would require an additional query of the number of objects in $\mathbb{R}$ at the start; (3) it is generalized for the case of a non-uniform prior distribution.

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