ORDINAL DEFINABILITY AND COMBINATORICS OF EQUIVALENCE RELATIONS

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Abstract. Assume $ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $E$ be a $\Sigma^1_1$ equivalence relation coded in HOD. $E$ has an ordinal definable equivalence class without any ordinal definable elements if and only if HOD $\models E$ is unpinned.

$ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ proves $E$-class section uniformization when $E$ is a $\Sigma^1_1$ equivalence relation on $\mathbb{R}$ which is pinned in every transitive model of ZFC containing the real which codes $E$: Suppose $R$ is a relation on $\mathbb{R}$ such that each section $R_x = \{y : (x,y) \in R\}$ is an $E$-class, then there is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $R(x, f(x))$.

$ZF + AD$ proves that $\mathbb{R} \times \kappa$ is Jónsson whenever $\kappa$ is an ordinal: For every function $f : [\mathbb{R} \times \kappa]^{<\omega} \to \mathbb{R} \times \kappa$, there is an $A \subseteq \mathbb{R} \times \kappa$ with $A$ in bijection with $\mathbb{R} \times \kappa$ and $f([A]^{<\omega}) \neq \mathbb{R} \times \kappa$.

1. Introduction

The questions of concern here are problems of independent interests that appeared during the study of the Jónsson property for nonwellorderable sets under the axiom of determinacy.

Let $N \in \omega \cup \{\omega\}$ and $X$ be some set. Define $[X]^N = \{x \in X : (i,j < N)(i \neq j \Rightarrow x(i) \neq x(j))\}$ and $[X]^{<\omega} = \bigcup_{n \in \omega} [X]^n$. Let $\approx$ denote the relation of being in bijection. Define $\mathcal{P}_N(X) = \{Y \subseteq X : Y \approx N\}$ and $\mathcal{P}_{<\omega}(X) = \bigcup_{n \in \omega} \mathcal{P}_n(X)$.

An $N$-Jónsson function for $X$ is a function $f : [X]^N \to X$ so that for all $Y \subseteq X$ with $Y \approx X$, $f([Y]^N) = X$. A function $f : [X]^{<\omega} \to X$ is a Jónsson function if and only if for all $Y \subseteq X$ with $Y \approx X$, $f([Y]^{<\omega}) = X$. A set $X$ has the Jónsson property if and only if there are no Jónsson functions for $X$.

The classical study of the Jónsson property involved wellordered sets. For wellordered sets $X$, Jónsson functions for $X$ are formulated using $\mathcal{P}_N(X)$ rather than $[X]^N$. Under AC, the following results are known: [4] showed that every infinite set has an $\omega$-Jónsson function. The existence of such a function is also where Kunen’s proof of the Kunen’s inconsistency uses AC. The existence of a cardinal with the Jónsson property implies $\theta^+ \not\in \theta$. Results of Erdős and Hajnal (see [3] and [4]) imply that under CH, $2^{\aleph_0}$ is not Jónsson. Hence $\mathbb{R}$ is not Jónsson under CH. On the other hand, real valued measurable cardinals are Jónsson (see [3] Corollary 11.1). Solovay showed it is consistent relative to a measurable cardinal that $2^{\aleph_0}$ is real valued measurable. Hence it is consistent relative to a measurable cardinal that $\mathbb{R}$ is Jónsson.

Using the axiom of determinacy AD, [15] showed that $\aleph_n$ is Jónsson for each $n \in \omega$. [7] showed that every cardinal $\kappa < \Theta$ is Jónsson under $ZF + AD + V = L(\mathbb{R})$. In fact, Woodin showed that $ZF + AD^+$ can prove every cardinal $\kappa < \Theta$ is Jónsson.

Under AD, there are some sets which cannot be wellordered. Some important examples are quotients of $\Delta^1_1$ equivalence relations such as $=, E_0, E_1, E_2,$ and $E_3$ (see Definition 2.15). Holshouser and Jackson (see [6] and [7]) showed that $\mathbb{R}$ has the Jónsson property and there are no 2-Jónsson functions for $\mathbb{R}/E_0$ under AD. [2] showed that under AD, there is a 3-Jónsson function for $\mathbb{R}/E_0$. Results from [2] seem to suggest that $\mathbb{R}/E_1, \mathbb{R}/E_2,$ and $\mathbb{R}/E_3$ do not have that Jónsson property, but no Jónsson functions for these quotients have yet to be constructed.

For the $\Delta^1_1$ equivalence relations mentioned above, various dichotomy theorems assert the significance of these equivalence relations in the degree structure of $\Delta^1_1$ equivalence relations under $\Delta^1_1$ reducibility. The proofs of these dichotomy results give specific combinatorial structures to sets $A$ such that $E \leq_{\Delta^1_1} E \mid A$, when $E$ is one of the $\Delta^1_1$ equivalence relations above. For example, if $A \subseteq \mathbb{R}$ is $\Sigma^1_1$ and $E_0 \leq_{\Delta^1_1} E_0 \mid A$, then $A$ contains an $E_0$-tree (a perfect tree with very specific symmetry conditions; see [2] Definition 5.2).

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Similarly, if \( A \subseteq \mathbb{R} \) is \( \Sigma^1_1 \) and \( E_2 \leq \Delta^1_1 \) \( E_2 \upharpoonright A \), then \( A \) contains an \( E_2 \)-tree (a perfect tree with certain summability conditions; see \cite{2} Fact 14.14).

The following describes the techniques from \cite{2} for investigating the Jónsson property for \( \mathbb{R}/E_0 \): To study functions \( f : [\mathbb{R}/E_0]^2 \rightarrow \mathbb{R}/E_0 \), one would like to lift \( f \) to a function \( F : \mathbb{R}^2 \rightarrow \mathbb{R} \) with the property that for all \( (x_1, x_2) \in \mathbb{R}^2 \), \( [F(x_1, x_2)]_{E_0} = f([x_1]_{E_0}, [x_2]_{E_0}) \). Such a function \( F \) is called a lift of \( f \). Then one tries to produce an \( E_0 \)-tree on which the collapse of \( F \) misses elements of \( \mathbb{R}/E_0 \). On the other hand, using the specific combinatorial structure of \( E_0 \)-trees, one can define a map \( F : \mathbb{R}^3 \rightarrow \mathbb{R} \) which is \( E_0 \)-invariant and given any real \( x \), there is a triple \( (x_1, x_2, x_3) \) of \( E_0 \)-unrelated reals so that \( F(x_1, x_2, x_3) \in E_0 \). The collapse of \( F \) would then be a 3-Jónsson map.

As described in the above example, the existence of lifts of functions from \( \mathbb{R}/E \rightarrow \mathbb{R}/E' \), where \( E \) and \( E' \) are equivalence relations on \( \mathbb{R} \), seems to be useful in the study of functions on quotients. The existence of a lift is an immediate consequence of uniformization. \( \text{AD}_\mathbb{R} \) has full uniformization. Moreover, a lift of a function \( f : \mathbb{R}/E \rightarrow \mathbb{R}/E' \) requires only uniformization for relations whose sections are \( E' \)-classes. Woodin showed that countable section uniformization holds in \( \text{AD}^+ \). Thus lifts exist for functions into \( \mathbb{R}/E_0 \) under \( \text{AD}^+ \). Moreover for the purpose of showing that there are no 2-Jónsson functions for \( \mathbb{R}/E_0 \), \( \text{AD} \) alone has a sufficient uniformization: Let \( f : \mathbb{R}/E_0]^2 \rightarrow \mathbb{R}/E_0 \). One can apply comeager uniformization (which holds in just \( \text{AD} \)) to find a function \( F : C \rightarrow \mathbb{R} \), where \( C \subseteq \mathbb{R}^2 \) is comeager, which lifts \( f \) on \( C \). Then the 2-Mycielski property for \( E_0 \) shows that there is a set \( A \) such that \( E_0 \leq \Delta_1^1 \) \( E_0 \upharpoonright A \) and \( \{ (x_1, x_2) \in A^2 : \lnot \langle x_1, E_0, x_2 \rangle \} \subseteq C \).

Motivated by this question of \( E \)-class section uniformization, Zapletal asked a related question: Does every ordinal definable \( E_2 \) equivalence class contain an ordinal definable real, under \( \text{ZF} + \text{AD} + \forall V = L(\mathbb{R}) \)? He informed the author that the equivalence relation \( =^+ \), defined on \( \omega^\omega \) as equality of range, has ordinal definable classes with no ordinal definable elements assuming \( \text{AD} + \forall V = L(\mathbb{R}) \), and that this phenomenon can be viewed as a consequence of the unpinnedness of \( =^+ \). He asked then whether pinnedness can be used to characterize those \( \Delta^1_1 \) equivalence relations with ordinal definable equivalence classes without any ordinal definable elements.

For countable equivalence relations, Zapletal’s question has a positive answer under \( \text{AD}^+ \): Under \( \text{AD}^+ \), every ordinal definable countable set of reals contains only ordinal definable elements. The proof of this can be found within the proof of Woodin’s countable section enumeration under \( \text{AD}^+ \), which states that for every relation \( R \) with countable sections there is a function that takes \( x \) to a wellordering of the section \( R_x \). The main idea is to consider the canonical wellordering of \( R_z \) in \( \text{HOD}^{(S,z)}_{\omega} \) as \( z \) ranges over a Turing cone of reals and \( S \) is some set of ordinals from an \( \omega^\omega \)-Borel code for \( R \). (See \cite{14} for the proof.) This implies that under \( \text{AD}^+ \), every ordinal definable \( E \) class contains only ordinal definable elements if \( E \) is an equivalence relation with all countable classes defined using only ordinal parameters.

The determinacy assumptions are important for these questions since \cite{11} showed that in a forcing extension of the constructible universe \( L \), there is an ordinal definable \( E_0 \) equivalence class with no ordinal definable elements. Similar examples are given in \cite{12} which showed that in a forcing extension of \( L \), there are definable relations with each section an \( E_0 \)-class but have no uniformizations which are ordinal definable in a real.

Section \cite{2} will show roughly that in \( L(\mathbb{R}) \models \text{AD} \), if a \( \Sigma^1_1 \) equivalence relation \( E \) has an OD equivalence class without any OD elements, then \( \text{HOD} \) must think that \( E \) is unpinned:

\textbf{Theorem 2.12} Assume \( \text{ZF} + \text{AD}^+ + \forall V = L(\text{\mathcal{P}(\mathbb{R})}) \). Let \( T \) be a set of ordinals. Let \( E \) be an equivalence relation which is \( \Sigma^1_1(s) \) for some \( s \in \text{HOD}_T \) and let \( A \) be an OD_\mathbb{R} E\text{-class}. If \( A \) has no OD_\mathbb{T} elements, then \( \text{HOD}_T \models E \) is unpinned.

Models of \( \text{ZF} + \text{AD}^+ + \forall V = L(\text{\mathcal{P}(\mathbb{R})}) \) are considered natural models of \( \text{AD}^+ \). If \( L(\mathbb{R}) \models \text{AD} \), then \( L(\mathbb{R}) \) satisfies this theory. Woodin, \cite{14} Corollary 3.2, has shown that if \( \text{ZF} + \text{AD}^+ + \forall V = L(\text{\mathcal{P}(\mathbb{R})}) \) holds, then either there is a set of ordinals \( J \) so that \( V = L(J, \mathbb{R}) \) or else \( V \models \text{AD}_\mathbb{R} \).
The proof of this theorem uses the idea of taking ultraproducts of $\text{HOD}^{L_{S,z}}$ (where the Turing degree of $z$ serves as the index and $S$ is a set of ordinals) using Martin’s Turing cone measure. This technique appears in Woodin’s proof that sets of reals have $\infty$-Borel codes in $L(\mathbb{R})$ when $L(\mathbb{R}) \models AD$ as exposited in [10] Claim 1.6.

**Theorem 2.13** (ZF + AD) Let $E$ be a $\Sigma^1_1$ equivalence relation defined in $\text{HOD}_R$, where $R$ is some set. Suppose $\text{HOD}_R \models E$ is unpinned. Then there is an OD $E$-class with no OD elements.

These two results together give a very succinct answer to Zapletal’s question in natural models of AD$^+$:

**Corollary 2.14** Assume ZF + AD$^+$ + $V = L(\mathcal{P}(\mathbb{R}))$. Let $E$ be a $\Sigma^1_1$ equivalence relation coded in HOD. $E$ has an OD $E$-class with no OD elements if and only if $\text{HOD} \models E$ is unpinned.

Many important examples of pinned $\Delta^1_1$ equivalence relations include $=$, $E_0$, $E_1$, $E_2$, smooth, hyperfinite, and hypersmooth equivalence relations.

Using the previous theorem, one obtains $E$-class section uniformization for equivalence relations satisfying some definable pinnedness condition. This is particular useful when the equivalence relations are provably pinned:

**Theorem 3.1** Assume ZF + AD$^+$ + $V = L(\mathcal{P}(\mathbb{R}))$. If $E$ is a $\Sigma^1_1$ equivalence relation which is pinned in every transitive model of ZFC containing the real that codes $E$, then every relation $R$ whose sections are all $E$-classes can be uniformized.

As a consequence, every function $f : \mathbb{R}/E \to \mathbb{R}/F$ has a lift under AD$^+$ + $V = L(\mathcal{P}(\mathbb{R}))$ when $F$ is $=$, $E_0$, $E_1$, $E_2$, smooth, hyperfinite, essentially countable, or hypersmooth.

Section 4.1 will study the Jónsson property of some nonwellorderable sets. Holshouser and Jackson have shown that $\mathbb{R} \times \kappa$ for any $\kappa < \Theta$ has the Jónsson property. They use that $\mathbb{R}$ and all ordinals $\kappa < \Theta$ have the Jónsson property. A natural question would be whether $\mathbb{R} \times \kappa$ is Jónsson for all ordinals $\kappa$. The proof that $\mathbb{R}$ is Jónsson has a clear flavor of classical descriptive set theory since it uses comeagerness, continuity, the Mycielski property, and fusions of perfect trees. The proof that ordinals $\kappa < \Theta$ is Jónsson have a somewhat different flavor. A related question would be whether the Jónsson property for $\kappa$ is relevant to showing $\mathbb{R} \times \kappa$ is Jónsson. Does there exists a more classical proof that $\mathbb{R} \times \kappa$ is Jónsson? It will be shown that:

**Theorem 4.15** (ZF + AD) For any ordinal $\kappa$, $\mathbb{R} \times \kappa$ has the Jónsson property.

Whether or not $\kappa$ is Jónsson does not appear in the proof of the above theorem. This result is proved while investigating the Jónsson property for wellordered disjoint unions $\bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ where each $E_\alpha$ is an equivalence relation with all classes countable and $\mathbb{R}/E_\alpha \simeq \mathbb{R}$. The techniques have a very classical flavor using results about lengths of wellordered sequences of reals, additivity of the meager ideal, comeager uniformization, and fusions of perfect trees. There are also some discussions about the cardinality of $\bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. However, it remains open whether $\bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ has the Jónsson property.

This section concludes by producing a 6-Jónsson function for $(\mathbb{R}/E_0) \times \kappa$ for any $\kappa < \Theta$ under AD. This shows that $(\mathbb{R}/E_0) \times \kappa$ for $\kappa < \Theta$ is not Jónsson under AD.

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### 2. Ordinal Definable Equivalence Classes

$V$ will denote the universe of set theory in consideration. If $M$ is a model of set theory and $A$ is some concept given by some formula, then $A^M$ will denote the relativization of that formula inside $M$. If a concept $A$ is unrelativized, then it is assumed to mean $A^V$, although it may be written $A^V$ for emphasis. $\mathbb{R}$ will denote $\omega_1^\omega$, the Baire space, consisting of functions from $\omega$ to $\omega$ with its usual metric. (Although it may sometimes denote $\omega_2$, the Cantor space.) The elements of $\mathbb{R}$ will be called reals.
If $X$ is a set, then $\text{OD}_X$ denotes the class of sets which are ordinal definable using $X$ as a parameter. $\text{HOD}_X$ is the collection of sets which are hereditarily ordinal definable from $X$. $\text{HOD}_X \models \text{ZFC}$ and has a canonical wellordering definable using $X$.

**Fact 2.1. (Vopěnka)** Suppose $S$ is a set of ordinals. Let $x \in \mathbb{R}$.

In $L[S,x]$, let $\mathbb{P}$ denote the forcing of nonempty $\text{OD}_S$ subsets of $\mathbb{R}$ ordered by $\subseteq$. Using the canonical $S$-definable bijection of $\text{OD}_S$ subsets onto $\text{ON}$, let $\mathbb{Q}_S \in \text{HOD}_S$ be the forcing that results by transferring $\mathbb{P}$ onto $\text{ON}$ using this map.

Then there is a $G \in L[S,x]$, which is $\mathbb{Q}_S$-generic over $\text{HOD}_S$, so that $L[S,x] = \text{HOD}_S[G] = \text{HOD}_S[x]$. 

**Proof.** See [8] Theorem 15.46. □

**Definition 2.2.** Let $X \subseteq \mathbb{R}$, $S$ be a set of ordinals, and $\phi$ be a formula in the language of set theory. $(S,\phi)$ is an $\infty$-Borel code for $X$ if and only if for all $x \in \mathbb{R}$, $x \in X \Leftrightarrow L[S,x] \models \phi(S,x)$.

**Definition 2.3.** ([18] Section 9.1) $\text{AD}^+$ consists of the following:

1. $\text{DC}_\mathbb{R}$.
2. Every $A \subseteq \mathbb{R}$ has an $\infty$-Borel code.
3. For all $\lambda < \Theta$, $A \subseteq \mathbb{R}$, and continuous function $\pi : \omega \lambda \rightarrow \mathbb{R}$, $\pi^{-1}[A]$ is determined.

(\lambda is given the discrete topology. $\Theta$ is the supremum of the ordinals which are surjective images of $\mathbb{R}$. Games with moves from $\lambda$ are defined the same way as the more familiar games on $\omega$.)

**Definition 2.4.** ([19]) Let $E$ be an equivalence relation on $\mathbb{R}$. Let $\mathbb{P}$ be a forcing. Let $\tau$ be a $\mathbb{P}$-name.

Let $\tau_{\text{left}}, \tau_{\text{right}}$ be the canonical $\mathbb{P} \times \mathbb{P}$-names with the property that $\tau_{\text{left}}$ and $\tau_{\text{right}}$ are evaluated according to $\tau$ using the left and right $\mathbb{P}$-generic filters, respectively, coming from a $\mathbb{P} \times \mathbb{P}$-generic filter.

$\tau$ is an $E$-pinned name if and only if $1_{\mathbb{P} \times \mathbb{P}} \Vdash \mathbb{P} \times \mathbb{P} \tau_{\text{left}} E \tau_{\text{right}}$.

An $E$-pinned name $\tau$ is an $E$-trivial name if and only if there is some $x \in \mathbb{R}$ so that $1_{\mathbb{P}} \Vdash \mathbb{P} \tau \in x \mapsto x$.

$E$ is a pinned equivalence relation if and only if all forcings $\mathbb{P}$, every $E$-pinned $\mathbb{P}$-names is $E$-trivial.

Pinnedness is more accurately a property of a fixed definition for the equivalence relation $E$ (which is to be used to interpret $E$ in generic extensions). This paper is concern only with $\Sigma_1$ equivalence relations and such equivalence relations are always defined as the projection of certain trees on $\omega \times \omega \times \omega$.

**Definition 2.5.** Let $\leq_T$ denote the Turing reducibility relation on $\omega \omega$. For $x, y \in \omega \omega$, let $x \equiv_T y$ if and only if $x \leq_T y$ and $y \leq_T x$. A Turing degree is a $\equiv_T$ equivalence class. If $x, y \in \omega \omega$, then define $[x]_{\equiv_T} \leq_T [y]_{\equiv_T}$ if and only if $x \leq_T y$.

Let $D$ denote the set of Turing degrees. A Turing cone with base $C \in D$ is the set $\{D \in D : C \leq_T D\}$. Define Martin’s measure $U$ by: for $A \in \mathcal{P}(D)$, $A \in U$ if and only if $A$ contains a Turing cone.

Under $\text{AD}$, the Martin’s measure is a countably complete ultrafilter on $\mathcal{D}$.

**Definition 2.6.** (ZF + AD) Let $T$ be some set. Let $\mathcal{H}$ be a (usually proper class) function on $\mathcal{D}$ which is definable using only $T$ and ordinals as parameters and takes each $X$ to some transitive class. Assume that there is some (usually proper class) function $\mathcal{R}$ definable using only $T$ and ordinals as parameters so that for each $X \in \mathcal{D}$, $\mathcal{R}(X)$ is a wellordering of $\mathcal{H}(X)$.

Let $M^T_{\mathcal{H},\mathcal{R}}$ denote the collection of $\text{OD}_T$ functions on $D$ taking each $X \in D$ to an element in $\mathcal{H}(X)$. For $F,G \in M^T_{\mathcal{H},\mathcal{R}}$, let $F \sim G$ if and only if $\{X \in D : F(X) = G(X)\} \in U$.

Let $M^\sim_{\mathcal{H},\mathcal{R}}$ denote the collection of equivalence classes of $M^T_{\mathcal{H},\mathcal{R}}$ under $\sim$. Define $[F]_\sim \in [G]_\sim$ if and only if $\{X \in D : F(X) \in G(X)\} \in U$.

**Fact 2.7.** (ZF + AD) $M^T_{\mathcal{H},\mathcal{R}}$ is a $T$-definable class consisting of $\text{OD}_T$ elements. Using the $T$-definable bijection of $\text{OD}_T$ and $\text{ON}$, $M^T_{\mathcal{H},\mathcal{R}}$ is isomorphic to a class inside $\text{HOD}_T$. $M^T_{\mathcal{H},\mathcal{R}}$ is well-founded; hence, it can be considered as a transitive structure inside $\text{HOD}_T$.

The Los’s theorem holds for $M^T_{\mathcal{H},\mathcal{R}}$. Suppose $F_0,\ldots,F_{k-1} \in M^T_{\mathcal{H},\mathcal{R}}$ and $\phi$ is a formula of $\{\dot{\epsilon}\}$, then $M^T_{\mathcal{H},\mathcal{R}} \models \phi([F_0]_\sim,\ldots,[F_{k-1}]_\sim)$ if and only $\{X \in D : \mathcal{H}(X) \models \phi(F_0(X),\ldots,F_{k-1}(X))\} \in U$.

For each $\alpha < \omega_1$, let $c_\alpha : D \rightarrow \{\emptyset, \alpha\}$ be the constant function taking value $\alpha$. The class $[c_\alpha]_\sim$ represents the ordinal $\alpha$ in $M^T_{\mathcal{H},\mathcal{R}}$.

For each $r \in \mathbb{R}$ which is $\text{OD}_T$ and belongs to $\mathcal{H}(X)$ for a cone of $X \in D$, define the function $c_r : D \rightarrow \{0, r\}$ by $c_r(X) = r$ if $r \in \mathcal{H}(X)$ and $c_r(X) = \emptyset$ if otherwise. Then $[c_r]_\sim$ represents $r$ in $M^T_{\mathcal{H},\mathcal{R}}$. 

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Proof. $\mathcal{M}^T_{H,R}$ is a structure in $\text{OD}_T$ since $\mathcal{M}^T_{H,R} \subseteq \text{OD}_T$. Note that $\in$ relation of $\mathcal{M}^T_{H,R}$ is definable from $T$. Using the the canonical bijection of $\text{OD}_T$ and ON, one can transfer $\mathcal{M}^T_{H,R}$ and its $\in$-relation onto ON. This new isomorphic structure consists entirely of ordinals and hence elements of $\text{HOD}_T$.

Let $F \in \mathcal{M}^T_{H,R}$. Suppose $[F]_\sim$ is not wellfounded. There is some set $X \subseteq \{[G]_\sim : [G]_\sim \in [F]_\sim\}$ without an $\in$-$\mathcal{M}^T_{H,R}$-minimal element. Let $L(0)$ be the $\text{OD}_T$-least function $G$ so that $[G]_\sim \in X$. Suppose $L(n)$ has been defined. Let $L(n+1)$ be the $\text{OD}_T$-least function $G$ so that $[G]_\sim \in [L(n)]_\sim$. Let $A_n = \{ x \in D : L(n+1)(x) \in L(n)(x) \}$ Each $A_n \in U$. Since $U$ is countably complete, $\bigcap_{n \in \omega} A_n \neq \emptyset$. Let $x \in \bigcap_{n \in \omega} A_n$. Then $L(n)(x) : n \in \omega$ is an $\in$-decreasing sequence in $V$. Contradiction. $\mathcal{M}^T_{H,R}$ is well-founded. Using the Mostowski collapse, one may consider $\mathcal{M}^T_{H,R}$ as a transitive structure inside of $\text{HOD}_T$.

The proof of $\text{Lo}^\prime$'s theorem is by induction on formula complexity: The result holds for the atomic formulas by definition. Assume the result holds for $\varphi$ and $\psi$, then the result holds for $\neg \varphi$ and $\varphi \land \psi$ by the usual arguments. (Note the case involving $\neg$ requires that $U$ is an ultrafilter.) Suppose the result has been shown for $\varphi$. If $\mathcal{M}^T_{H,R} \models (\exists x)\varphi(x, [F_0]_\sim, \ldots, [F_{k-1}]_\sim)$, then there exists some $G \in \mathcal{M}^T_{H,R}$ so that $\mathcal{M}^T_{H,R} \models \varphi([G]_\sim, [F_0]_\sim, \ldots, [F_{k-1}]_\sim)$. Using the induction hypothesis, $\{ X \in D : H(X) \models (\exists x)\varphi(x, F_0(X), \ldots, F_{k-1}(X)) \} \in U$. Define $G$ on $D$ by letting $G(X)$ be the $\mathcal{R}(X)$-least element $z$ of $H(X)$ such that $H(X) \models \varphi(z, F_0(X), \ldots, F_{k-1}(X))$ if such an element exists and $\emptyset$ otherwise. $G$ is $\text{OD}_T$ and so belongs to $\mathcal{M}^T_{H,R}$. By the induction hypothesis, $\mathcal{M}^T_{H,R} \models \varphi([G]_\sim, [F_0]_\sim, \ldots, [F_{k-1}]_\sim)$. Therefore, $\mathcal{M}^T_{H,R} \models (\exists x)\varphi(x, [F_0]_\sim, \ldots, [F_{k-1}]_\sim)$. This completes the sketch of $\text{Lo}^\prime$'s theorem.

Suppose $[F]_\sim \in [c]_\sim$. Let $A = \{ X \in D : F(X) \in \alpha \}$. $A \in U$. Let $A_\beta = \{ X \in D : F(X) = \beta \}$. $A = \bigcup_{\beta<\alpha} A_\beta$. Since $U$ is countably complete and $\alpha$ is countable, there is some $\beta<\alpha$ so that $A_\beta \in U$. Then $c_\beta \sim F$. This shows that $[c]_\sim$ represents $\alpha$ in $\mathcal{M}^T_{H,R}$ when $\alpha<\omega_1$. □

Fact 2.8. (Woodin, [1] Theorem 3.4) Assume $ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $T$ be a set of ordinals. A set $X \subseteq \mathbb{R}$ which is $\text{OD}_T$ has an $\infty$-Borel code $(S, \varphi)$ which is $\text{OD}_T$.

Fact 2.9. (Woodin, [1] Theorem 2.18) Assume $ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $T$ be a set of ordinals. There is some set of ordinals $\mathbb{X}$ so that $\text{HOD}_T = L[\mathbb{X}]$. (Note that $\mathbb{X}$ is $\text{OD}_T$.)

In the case of $L(\mathbb{R})$ and $T = \emptyset$, the set $\mathbb{X}$ can be taken to be $\mathbb{P}^\omega$ which is the direct limit indexed by $n \in \omega$ of Vopěnka forcing on $\mathbb{R}^n$. This follows from Woodin’s result that $L(\mathbb{R})$ is a symmetric collapse extension of its HOD. One can find an exposition of this result in [10].

Fact 2.10. (Woodin, [1] Section 2.2) Assuming $ZF + AD^+$, $\Pi_{\mathcal{X} \in T} \text{ON}/U$ is wellfounded.

Assume $AD^+$, the wellfoundedness of $\mathcal{M}^T_{H,R}$ can also be proved from Fact 2.10. For the question of Zapletal, one will need to form an ultraproduct of the form $\mathcal{M}^T_{H,R}$ so that all the reals of $\text{HOD}$ belong to this ultraproduct.

Fact 2.11. Assume $ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $T$ be a set of ordinals. Let $\mathbb{X}$ be a set of ordinals as given by Fact 2.9, so that $\text{HOD}_T = L[\mathbb{X}]$. For each $X \in D$, let $\mathcal{H}(X) = \text{HOD}^{L[X,X]}_\mathbb{X}$ and $R(X)$ be the canonical wellordering of $\text{HOD}^{L[X,X]}_\mathbb{X}$. Then $\mathcal{M}^T_{H,R}$ is wellfounded, $\mathcal{M}^T_{H,R} \subseteq \text{HOD}_T$, and $\mathcal{R}^{\text{HOD}_T} \subseteq \mathcal{M}^T_{H,R}$.

Proof. Note that $\mathbb{X}$ is $\text{OD}_T$. Observe that for all $X \in D$, $\text{HOD}_T = L[\mathbb{X}] \subseteq \text{HOD}^{L[X,X]}_\mathbb{X}$. So if $r \in \text{HOD}_T$, then $r \in \text{HOD}^{L[X,X]}_\mathbb{X}$. The function $c_r$ is $\text{OD}_X$ and belongs to $\mathcal{M}^T_{H,R}$. This result now follows from Fact 2.7. □

Theorem 2.12. Assume $ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $T$ be a set of ordinals. Let $E$ be an equivalence relation which is $\Sigma^1_1(s)$ for some $s \in \text{HOD}_T$ and let $A$ be an $\text{OD}_T$ $E$-class. If $A$ has no $\text{OD}_T$ elements, then $\text{HOD}_T \models E$ is unpinned.

Proof. For simplicity, let $T = \emptyset$. By Fact 2.9 let $\mathbb{X}$ be a set of ordinals so that $\text{HOD} = L[\mathbb{X}]$. By Fact 2.8 $A$ has an $\infty$-Borel code in $\text{HOD} = L[\mathbb{X}]$. Modifying $\mathbb{X}$ by including an ordinal if necessary, one may as well assume that there is some formula $\varphi$ so that $(\mathbb{X}, \varphi)$ forms an $\infty$-Borel code for $A$.

Recall that $E$ is $\Sigma^1_1(s)$ means there is some $s$-recursive tree $T$ on $\omega \times \omega$ such that $x \in E y$ if and only if $L[s, x, y] \models T^x y$ is illfounded, where $T^x y = \{ u : (x \upharpoonright |u|, y \upharpoonright |u|, u) \in T \}$. In this way, $E$ is $\infty$-Borel with a code that is a subset of $\omega$. 5
Suppose $y \geq_T x$ for some $x \in A$. By Fact 2.1 there is some $\mathcal{O}_X^{[X,y]}$-name $\tau \in \text{HOD}^{[X,y]}_X$ and some $\mathcal{O}_X^{[X,y]}$-generic over $\text{HOD}^{[X,y]}_X$ filter $G \in L[X,y]$ so that $\tau[G] = x$ and $L[X,y] = \text{HOD}^{[X,y]}_X[G]$. Since $V \models L[X,y] = \varphi(X,x)$, $L[X,y] = \varphi(X,x)$. Since $L[X,y] = \text{HOD}^{[X,y]}_X[G]$, one has $\text{HOD}^{[X,y]}_X[G] \models L[X,y] = \varphi(X,x)$. There is some $q \in \mathcal{O}_X^{[X,y]}$ so that $\text{HOD}_X^{[X,y]} = q \forces L[X,\tau] = \varphi(X,\tau)$. Let $q_x$ and $\tau_y$ be the $\text{HOD}^{[X,y]}_X$-least such $q$ and $\tau$ with the above properties. In order to satisfy the technical requirement of using the largest condition of the forcing in the definition of pinnings, let $U_y = \{ p \in \mathcal{O}_X^{[X,y]} : p \leq q_x, q_y \}$, $\leq_{U_y} = \leq_{\mathcal{O}_X^{[X,y]}}|_{U_y}$, and $1_{U_y} = q_y$. If $y$ does not Turing compute any element of $A$, then one can just let $U_y$ and $\tau_y$ be $\emptyset$.

If $x \equiv_T y$, $\text{HOD}_X^{[X,x]} = \text{HOD}_X^{[X,y]}$ and their canonical global wellorderings are the same. This shows that $U_x = U_y$ and $\tau_x = \tau_y$. If $X \in D$ and $x \in X$, let $\text{HOD}_X^{[X,x]} = \text{HOD}_X^{[X,y]}$, $U_x = U_y$, and $\tau_x = \tau_y$. For $X \in D$, let $\mathcal{H}(X) = \text{HOD}_X^{[X,x]}$ and $R(X)$ be the canonical global wellordering of $\text{HOD}_X^{[X,x]}$. For $X \in D$, define $\Phi_U(X) = U_x$ and $\Phi_r(X) = \tau_x$. Let $M = M_{\mathcal{H}, R}^X$. Note that $\Phi_U, \Phi_r \in M_{\mathcal{H}, R}^X$. Let $U = [\Phi_U]_\sim$ and $\tau = [\Phi_r]_\sim$. Let $c_x$ be the constant function taking value $X$. Note that $c_x \in M_{\mathcal{H}, R}^X$. By Fact 2.7 $M$ will be identified as a transitive class in $\text{HOD}^V$. Thus $U$, $\tau$, and $X^\infty$ belong to $\text{HOD}^V$.

By Los’s theorem, $M$ is a model of ZFC, $U$ is some forcing, $\tau$ is some $U$-name adding a real, $X^\infty$ is a set of ordinals, and $M \models 1_U \forces L[X^\infty, \tau] = \varphi(X^\infty, \tau)$.

Claim 1:

$M \models 1_{U \times U} \forces \forall x \forall y((L[X,x,x] = \varphi(X^\infty, x) \land L[X,y,y] = \varphi(X^\infty, y))) \Rightarrow x \equiv E y$

(Note that the ultrapower moves $X$ to $X^\infty$. However, $E$ as a $\Sigma_1^1(s)$ equivalence relation has the real $s$ as its $\omega$-Borel code. The constant function $c_s$ taking value $s$ belongs to $M_{\mathcal{H}, R}^X$. In $M$, $[c_s]_\sim$ represents $s$. That is, $s$ is not moved by the ultrapower. Hence it is appropriate to continue to denote $E$ by $E$ in $M$ as it is still the same $\Sigma_1^1$ equivalence relation.)

To see the claim: Fix some $z \in A$. By Los’s theorem, it suffices to prove that for all $r \geq_T z$:

$\text{HOD}_X^{[X,r]} = 1_{U \times U} \forces \forall x \forall y((L[X,x,x] = \varphi(X^\infty, x) \land L[X,y,y] = \varphi(X^\infty, y))) \Rightarrow x \equiv E y$

Fix some $(p, q) \in U_r \times U_r$. Since $L[X,r] = AC$ and $V = AD$, $\omega^V$ is inaccessible in $\text{HOD}_X^{[X,r]}$. Hence $U_r \times U_r$ and its power set in $\text{HOD}_X^{[X,r]}$ are countable in $V$. There exists $G \times H \in V$ containing $(p, q)$ which is $U_r \times U_r$-generic over $\text{HOD}_X^{[X,r]}$. Since $G \times H \in V$, all sets of $\text{HOD}_X^{[X,r]}[G \times H]$ belong to $V$. Let $x$ and $y$ be reals of $\text{HOD}_X^{[X,r]}[G \times H]$ so that $\text{HOD}_X^{[X,r]}[G \times H] \models L[X,x] = \varphi(X,x) \land L[X,y] = \varphi(X,y)$. Then $V \models L[X,x] = \varphi(X,x) \land L[X,y] = \varphi(X,y)$. Since $(X, \varphi)$ is an $\omega$-Borel code for $A$ in $V$, $x \in A$ and $y \in A$. Since $A$ is an $E$-class, $x \equiv E y$. By Mostowski absoluteness, $\text{HOD}_X^{[X,r]}[G \times H] \models x \equiv E y$. This shows that $\text{HOD}_X^{[X,r]}[G \times H]$ satisfies the formula behind the above forcing relation. Since $G \times H$ is generic, there is some $(p', q') \leq_{U \times U} (p, q)$ so that in $\text{HOD}_X^{[X,r]}$, $(p', q')$ forces that formula. Since $(p, q)$ was arbitrary, this establishes the claim.

Claim 2:

$M \models 1_{U \times U} \forces \forall x \forall y((L[X,x,x] = \varphi(X^\infty, x) \land x \equiv E y) \Rightarrow L[X,y,y] = \varphi(X^\infty, y))$

The proof essentially uses the same idea as Claim 1.

Now to show that $U$ and $\tau$ witness that $E$ is unpinning in $\text{HOD}^V$:

First to show that $\tau$ is an $E$-pinned name in $\text{HOD}^V$: Let $G \times H$ be any $U \times U$-generic filter over $\text{HOD}^V$. Since $M \subseteq \text{HOD}^V$, if $G$ and $H$ are generic over $\text{HOD}^V$, then $G$ and $H$ are generic over $M$. By the forcing theorem, $M[G] \models L[X^\infty, \tau(G)] = \varphi(X^\infty, \tau(G))$ and $M[H] \models L[X^\infty, \tau(H)] = \varphi(X^\infty, \tau(H))$. By Claim 1, $M[G \times H] = \tau(G) \land \tau(H)$. By Mostowski absoluteness, $\text{HOD}^V[G \times H] = \tau(G) \land \tau(H)$. Since $G \times H$ was arbitrary, $\text{HOD}^V \models 1_{U \times U} \forces \tau(G) \land \tau(H)$. This shows that $\tau$ is an $E$-pinned $\mathcal{U}$-name in $\text{HOD}^V$.

Finally, to show that $\tau$ is not $E$-trivial: Suppose there is some $x \in \text{HOD}^V$ so that $\text{HOD}^V \models 1_U \forces \tau(x)$. Let $G \subseteq U$ be a $U$-generic over $\text{HOD}^V$ filter. Then $\text{HOD}^V[G] = \tau(G) \times E x$. By Mostowski absoluteness, $M[G] = \tau(G) \land \tau(E \times x)$. $G$ is also generic over $M$. By the forcing theorem, $M[G] \models L[X^\infty, \tau(G)] = \varphi(X^\infty, \tau(G))$. Since $x \in \text{HOD}^V$, Fact 2.11 and Fact 2.7 imply that $[c_x]_\sim$ represents $x$ in $M$. By Claim 2 applied in $M[G \times H]$ where $H$ is any $U$-generic filter over $M[G]$, $[c_x]_\sim$ represents $x$. Thus $M \models L[c_x]_\sim, [c_x]_\sim = \varphi(X^\infty, x)$. This completes the proof.
\[
\varphi([x]_\sim,[y]_\sim). \text{ By Lö"{s}' theorem, for a Turing cone of } X\text{'s (such that } x \in \text{HOD}_X^{L[X,X]}, \text{HOD}_X^{L[X,X]} \models L[X,x] = \varphi(X,x). \text{ This implies } V \models L[X,x] = \varphi(X,x). V \models x \in A \text{ since } (X,\varphi) \text{ is the } \infty\text{-Borel code for } A \text{ in } V. \text{ This contradicts the assumption that } A \text{ has no OD elements.}
\]

This completes the proof. \hfill \qed

**Theorem 2.13.** (ZF + AD) Let \( E \) be a \( \Sigma^1_1 \) equivalence relation defined in HOD\(_R\), where \( R \) is some set. Suppose \( \text{HOD}_R \models E \) is unpinned. Then there is an OD\(_R\) \( E \)-class with no OD\(_R\) elements.

**Proof.** Since HOD\(_R \models E \) is unpinned, there exists some forcing \( \mathbb{P} \in \text{HOD}_R \) and \( \mathbb{P}\)-name \( \sigma \in \text{HOD}_R \) so that within HOD\(_R\), \( \mathbb{P} \) and \( \sigma \) witness that \( E \) is not pinned.

Inside \( \text{HOD}_R \) (which models AC), let \( N \) be an elementary substructure of some large enough rank initial segment of HOD\(_R\) with the property that (1) \( N \) contains the code for \( E \), (2) \( \mathbb{R} \subseteq N \), (3) \( \mathbb{P},\sigma \subseteq N \), and (4) \( N \) has cardinality \(|\mathbb{R}|\). Let \( M \) be the Mostowski collapse of \( N \). Let \( Q \) and \( \tau \) be the image of \( \mathbb{P} \) and \( \sigma \) under the Mostowski collapse map. As \( E \) is \( \Sigma^1_1 \), the code for \( E \) is a tree on \( \omega \times \omega \times \omega \) whose projection is \( E \). So a code for \( E \) is merely a subset of \( \omega \). Hence the Mostowski collapse map does not move the code for \( E \). Note that \(|M|^V = |\text{HOD}_R[\mathbb{P}]|^V = \aleph_0 \) since AD holds. Hence there are generics for \( \mathbb{Q} \) over \( M \) that lie in \( V \).

Suppose \( G \) and \( H \) are two generic filters for \( \mathbb{Q} \) over \( M \) which belong to \( V \). Since \( M[G] \) and \( M[H] \) are countable in \( V \), one can construct a generic filter \( J \in V \) so that \( G \times J \) and \( H \times J \) are generic filters for \( \mathbb{Q} \times \mathbb{Q} \). By elementarity, \( M \models \tau \text{ is } E\text{-pinned.} \) Thus \( M[G \times J] \models \tau[G] E \tau[J] \) and \( M[H \times J] \models \tau[H] E \tau[J] \). By Mostowski absoluteness, \( \tau[G] E \tau[J] \) holds in \( V \). Since \( E \) is an equivalence relation, \( \tau[G] E \tau[H] \). This shows that whenever \( G \) and \( H \) are \( \mathbb{Q}\)-generic filters over \( M \) that belong to \( V \) (but may not be mutually generic), \( \tau[G] E \tau[H] \).

\( M \models \tau \) is not \( E\)-trivial by elementarity. Since \( \text{HOD}_R \subseteq M \), for any \( G \subseteq \mathbb{Q} \) which is \( \mathbb{Q}\)-generic over \( M \) and any \( x \in \text{HOD}_R \), \( M[G] \models \neg(\tau[G] E x) \). By absoluteness, if \( G \in V \), then \( \neg(\tau[G] E x) \).

In \( V \), let \( A \) be the set of \( x \in \mathbb{R} \) so that there exists some \( G \subseteq \mathbb{Q} \) which is \( \mathbb{Q}\)-generic over \( M \) and \( x \in \mathbb{Q} \). Since \( \mathbb{Q},\tau \in M \) and \( M \in \text{HOD}_R \), \( A \) is OD\(_R\). By the discussion of the above two paragraphs, \( A \) is a single \( E\)-class and has no elements of OD\(_R\).

Note that the only consequence of AD that is used is that there is no uncountable wellordered set of reals. \hfill \qed

The following answers the question of Zapletal.

**Corollary 2.14.** Assume \( \text{ZF + AD}^+ + V = L(\mathcal{P}(\mathbb{R})) \). Let \( E \) be a \( \Sigma^1_1 \) equivalence relation coded in HOD. \( E \) has an OD \( E \)-class with no OD elements if and only if \( \text{HOD} \models E \) is unpinned.

The rest of this section will give some examples.

**Definition 2.15.** The following are some important \( \Delta^1_1 \) equivalence relations.

Let \( =^+ \) denote the identity equivalence relation on \( \mathbb{R} \).

Let \( =^+ \) denote the Friedman-Stanley jump of \( = \) which is defined on \( \omega^\omega \text{by } x =^+ y \text{ if and only if } \{x(n) : n \in \omega\} = \{y(n) : n \in \omega\} \). \( =^+ \) is equality of range.

Let \( E_0 \) be the equivalence relation on \( \mathbb{R} \) \((\text{or } \text{2})\) defined by \( x E_0 y \text{ if and only if } (\exists k)(\forall n \geq k)(x(n) = y(n)) \).

Let \( E_1 \) be the equivalence relation on \( \omega^\omega \text{ defined by } x E_1 y \text{ if and only if } (\exists k)(\forall n > k)(x(n) = y(n)) \).

Let \( E_2 \) be the equivalence relation on \( \omega^\omega \text{ defined by } x E_2 y \text{ if and only if } \sum \{1 : n \in x \Delta y \} < \infty \text{, where } \Delta \text{ denotes the symmetric difference operation.} \)

**Fact 2.16.** The equivalence relations \( =, E_0, E_1, \text{ and } E_2 \) are pinned \( \Delta^1_1 \) equivalence relations. Every \( \Delta^1_1 \) equivalence relation with countable classes is pinned. Every smooth, hyperfinite, essentially countable, or hypersmooth equivalence relation is pinned.

The equivalence relation \( =^+ \) is unpinned.

**Proof.** See [10] Chapter 11.

The Solovay product lemma states: Let \( \mathbb{P} \) and \( \mathbb{Q} \) be two forcings. Suppose \( G \times H \) is \( \mathbb{P} \times \mathbb{Q}\)-generic over \( V \). Then \( V[G] \cap V[H] = V \).

From the Solovay product lemma, it follows that \( =, E_0, \) and \( E_1 \) are pinned equivalence relations.

If \( E \leq^{\Delta^1_1} \) \( F \) and \( F \) is pinned, then \( E \) is also pinned. This implies that smooth, hyperfinite, and hypersmooth equivalence relations are pinned.
Theorem 17.1.3 (iii) states that $\Delta^1_1$ equivalence relations with all classes $\Sigma^0_2$ are pinned. This implies that $E_2$ and every $\Delta^1_1$ equivalence relation with countable classes are pinned. Therefore, essentially countable equivalence relations are pinned.

Let $Q = \text{Coll}(\omega, \mathbb{R})$. Let $\tau$ be the name for the generic surjection of $\omega$ onto $\mathbb{R}$. $Q$ and $\tau$ witness that $=^+$ is unpinned since if $\tau$ was forced to be $=^+$ related to a ground model element, then $\mathbb{R}$ would be countable in the ground model.

Example 2.17. The proof above that $=^+$ is unpinned can be used to produce an OD $=^+$-class with no OD elements assuming $(\mathcal{P}(\mathbb{R}))^{\text{HOD}}$ is countable.

Let $Q = \text{Coll}(\omega, \mathbb{R})$ and $\tau$ be the generic surjection of $\omega$ onto $\mathbb{R}$ as defined inside of HOD. (Note that $\tau$ is an $=^+$-pinned name.) By the assumption, there exists $Q$-generics over HOD in $V$. Let $A$ be the collection of $x \in \mathbb{R}$ such that there exist some $G \subseteq Q$ which is $Q$-generic over HOD and $x =^+ \tau[G]$. $A$ is an OD $=^+$ equivalence class. $A$ cannot contain any OD elements for otherwise HOD would think $\mathbb{R}^{\text{HOD}}$ is countable.

3. Equivalence Class Section Uniformization and Lifting

Theorem 3.1. Assume $\text{ZF} + \text{AD}^+ + \mathbb{V} = L(\mathcal{P}(\mathbb{R}))$. Let $T$ be a set of ordinals. Let $E$ be a $\Sigma^1_1$ equivalence relation coded in $\text{HOD}_T$. Suppose $E$ is pinned in $\text{HOD}_{T,x}$ for all $x \in \mathbb{R}$. Let $R \subseteq \mathbb{R} \times \mathbb{R}$ be $\text{OD}_T$ and have the property that for all $x \in \mathbb{R}$, $R_x = \{ y : R(y,x) \}$ is an $E$-class. Then there is a function $F : \mathbb{R} \to \mathbb{R}$ which is $\text{OD}_T$ and uniformizes $R$: that is, for all $x \in \mathbb{R}$, $R(x, F(x))$.

If $E$ is a $\Sigma^1_1$ equivalence relation which is pinned in every transitive model of $\text{ZFC}$ containing the real that codes $E$, then every relation $R$ whose sections are $E$-classes can be uniformized. (For example, $E$ could be any of the pinned equivalence relations from Fact [2.16].)

Proof. Under these assumptions, for each $x \in \mathbb{R}$, $R_x$ is an $\text{OD}_{T,x}$-class. Since $\text{HOD}_{T,x} \models E$ is unpinned, Theorem 2.12 implies that $R_x$ must have an $\text{OD}_{T,x}$ element. For each $x \in \mathbb{R}$, let $F(x)$ be the least element of $\text{HOD}_{T,x}$ under the canonical global wellordering of $\text{HOD}_{T,x}$ which belongs to $R_x$. $F$ is an $\text{OD}_T$ uniformization of $R$.

For the second statement, under $\text{AD}^+$, any such relation $R$ has an $\infty$-Borel code $(S, \varphi)$. By modifying $S$ if necessary, one may assume that $\text{HOD}_S$ contains a code for $E$ as a $\Sigma^1_1$ set. By the hypothesis, $E$ is pinned in every $\text{HOD}_{S,x}$, where $x \in \mathbb{R}$. The second statement follows from the first statement. $\square$

[19] has shown that if $E$ is a $\Delta^1_1$ equivalence relation coded in some transitive model $M$ and $N$ is some transitive model with $M \subseteq N$, then $E$ is pinned in $M$ if and only if $E$ is pinned in $N$. Therefore, in the first statement of Theorem 3.1, it suffices just to have $\text{HOD}_S \models E$ is pinned, when $E$ is a $\Delta^1_1$ equivalence relation.

However [19] also shows that, in general, pinnedness for $\Sigma^1_1$ equivalence relation is not absolute by producing a pinned $\Sigma^1_1$ equivalence relation in $L$ which is unpinned in a forcing extension of $L$. However, in the present situation, one is concerned with models of the form $\text{HOD}^T_M$ and $\text{HOD}^V_T$, where $V$ is a model of determinacy. Possible more can be said in such settings. This suggests the following question.

Question 3.2. In the first statement of Theorem 3.1, can the condition that $E$ is pinned in $\text{HOD}_{T,x}$ for all $x \in \mathbb{R}$ be replace by just $E$ is pinned in $\text{HOD}_T$ when $E$ is a $\Sigma^1_1$ equivalence relation coded in $\text{HOD}_T$?

Regardless, most natural examples are $\Delta^1_1$. Moreover, for most of the natural examples, pinnedness is provable in $\text{ZFC}$.

Definition 3.3. Let $E$ be an equivalence relation on some set $X$. Let $F$ be an equivalence relation on some set $Y$. Let $n \in \omega$. Let $f : (X/E)^n \to (Y/F)$ be some function. A function $F : X^n \to Y$ is a lift of $f$ if and only if for all $x_0, \ldots, x_{n-1} \in X$, $[F(x_0, \ldots, x_{n-1})]_F = f([x_0]_E, \ldots, [x_{n-1}]_E)$.

Corollary 3.4. Assume $\text{ZF} + \text{AD}^+ + \mathbb{V} = L(\mathcal{P}(\mathbb{R}))$. Suppose $E$ is an equivalence relation on $\mathbb{R}$. Suppose $F$ is a $\Sigma^1_1$ equivalence relation on $\mathbb{R}$ which is pinned in every transitive models of $\text{ZFC}$ containing the real that codes $F$. For all $n \in \omega$, every function $f : (\mathbb{R}/E)^n \to (\mathbb{R}/F)$ has a lift.

In particular, this lifting property holds when $F = E_0, E_1, E_2$, smooth, hyperfinite, essentially countable, or hypersmooth.
Proof. Define the relation \( R(x_0, \ldots, x_{n-1}, y) \) if and only if \( y \in f([x_0]_E, \ldots, [x_{n-1}]_E) \). For each \( (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \), \( R(x_0, \ldots, x_{n-1}) = f([x_0]_E, \ldots, [x_{n-1}]_E) \), which is an \( F \)-class. By assumption, \( F \) is pinned in every model of ZFC containing the real that codes \( F \). Theorem 3.1 implies that \( R \) has a uniformizing function \( G \). \( G \) is a lift of \( f \).

Example 3.5. Under \( \text{ZF} + \text{AD} \), every relation can be uniformized. Hence, \( E \)-class section uniformization and lifting for \( E \) holds for every equivalence relation \( E \) on \( \mathbb{R} \). However \( \text{ZF} + \text{AD}^+ \) is not able to prove \( E \)-class section uniformization when \( E \) is an unpinned equivalence relation. The following is an example.

Assume \( \text{ZF} + \text{AD} + V = \mathbb{L}(\mathbb{R}) \).

Define \( R(x, y) \) if and only if \( y \) is not OD. \( R \) has no uniformizing function: Suppose \( f : \mathbb{R} \rightarrow \mathbb{R} \) uniformized \( R \). Since \( V = \mathbb{L}(\mathbb{R}) \), every set of reals is ordinal definable from some real. Thus \( f \) is OD for some \( z \in \mathbb{R} \). Hence \( f(z) \) is OD. However, \( R(z, f(z)) \) implies that \( f(z) \) is not OD. Contradiction.

Define \( S(x, y) \) if and only if \( \{y_n : n \in \omega\} = HOD_x \), where \( y_n \in \mathbb{R} \) denotes the \( n \)-th section of \( y \) under some coding of pairs of integers by integers. If \( S(x, y) \), then \( y \notin OD \) for otherwise \( \mathbb{R}^{HOD} \) would be countable in \( HOD_\omega \). Since \( S \subseteq R \) and \( R \) has no uniformization, \( S \) also has no uniformization.

Every instance of \( F \)-class section uniformization gives a lift of a function from \( f : \mathbb{R} \rightarrow (\mathbb{R} / F) \). Therefore, failure of \( F \)-class section uniformization is a failure of lifting for \( F \). However, the more interesting instance of the lifting property involving function of the form \( f : (\mathbb{R} / F) \rightarrow (\mathbb{R} / F) \). Zapletal informed the author of an example:

Example 3.6. (Zapletal) Assume \( \text{ZF} + \text{AD} + V = \mathbb{L}(\mathbb{R}) \). There is a \( f : (\omega^\omega / =^+) \rightarrow (\omega^\omega / =^+) \) which does not have a lift.

Define \( f \) as follows: Let \( b \in \omega^\omega \) be such that for all \( n \in \omega \), \( b(n) \) is the constant 0 function. Let \( C \in (\omega^\omega / =^+) \). If there is some \( x \in \mathbb{R} \) so that \( C \) is the \( =^+ \) equivalence class of enumerations of \( [x]_T \) (the Turing degree of \( x \)), then \( f(C) \) is the \( =^+ \) equivalence class of enumerations of \( \mathbb{R}^{HOD} \). Note that \( f(C) \) does not depend on \( x \). If \( C \) is the not the \( =^+ \) equivalence class of enumerations of any Turing degree, then let \( f(C) = [h]_{=^+} = \{b\} \).

Now suppose that \( f \) has a lift \( F : \omega^\omega \rightarrow \omega^\omega \). Since \( V = \mathbb{L}(\mathbb{R}) \), \( F \) is OD for some \( z \in \mathbb{R} \). Since \( [z]_T \subseteq HOD_\omega \) and \( HOD_\omega \) thinks \( [z]_T \) is countable, there is a \( c \in (\omega^\omega)^{HOD} \) such that \( c \) enumerates \( [z]_T \). Thus \( F(c) \in HOD_\omega \). Since \( F \) is a lift of \( f \), \( F(c) \in f([c]_{=^+}) \). By definition, \( F(c) \in \omega^\omega \) is an enumeration of \( \mathbb{R}^{HOD} \). Then \( HOD_\omega \) would think its own set of reals are countable. Contradiction.

4. Jónsson Property

Definition 4.1. Let \( X \) be a set and \( n \in \omega \). Let \( E \) be an equivalence relation on \( X \). Let \( [X]^E_n = \{(x_0, \ldots, x_{n-1}) \in \mathcal{P}^n X : (\forall i < n)(\forall j < n)(i \neq j \Rightarrow \neg(x_i, x_j))\} \). Let \( [X]_E^\omega = \bigcup_{n \in \omega} [X]^E_n \).

A set \( X \) has the Jónsson property if and only if for all functions \( f : [X]_E^\omega \rightarrow X \), there is some \( Y \subseteq X \) with \( Y = X \) and \( f|[Y]^E_n \neq X \). (The symbol \( = \) is the relation of being in bijection.)

For \( n < \omega \), an \( n \)-Jónsson function for \( X \) is a map \( f : [X]^n \rightarrow X \) so that for all \( Y \subseteq X \) with \( Y \approx X \), \( f|[X]^n \approx X \).

Fact 4.2. Under \( \text{ZF} + \text{AD} \),

(5) \( R \) has the Jónsson property.
(2) There is a 3-Jónsson function for \( \mathbb{R}/E_0 \). Hence \( \mathbb{R}/E_0 \) does not have the Jónsson property.

For the rest of this section, \( \mathbb{R} \) will refer to \( \omega^2 \), the set of infinite binary sequences.

Definition 4.3. A nonempty subset \( p \) of \( \omega^2 \) is a tree if and only if for all \( s \in p \) and \( t \subseteq s \), \( t \in p \). A tree \( p \) is a perfect tree if and only if for all \( s \in p \), there is a \( t \supseteq s \) so that \( t^0, t^1 \in p \).

Let \( S \) be the set of all perfect trees. Let \( \leq_S = \subseteq \).

Let \( p \in S \). A node \( s \in p \) is a split node if and only if \( s^0, s^1 \in p \). A node \( s \in p \) is a split of \( p \) if and only if \( s \mid ([s] - 1) \) is a split node of \( p \). For \( n \in \omega \), \( s \) is an \( n \)-split of \( p \) if and only if \( s \) is a \( \leq \)-minimal element of \( p \) with exactly \( n \)-many proper initial segments which are split nodes of \( p \).

Let \( \text{split}^n(p) \) denote the set of \( n \)-splits of \( p \). Note that \( |\text{split}^n(p)| = 2^n \) and \( \text{split}^0(p) = \{\emptyset\} \).

If \( p, q \in S \), define \( p \leq_S q \) if and only if \( p \leq_S q \) and \( \text{split}^n(p) = \text{split}^n(q) \).

If \( p \in S \) and \( s \in p \), then define \( p_s = \{t \in p : t \subseteq s \subseteq s \subseteq t\} \).

Let \( p \in S \). Let \( A \) be defined as follows:
(i) \( \Lambda(p, \emptyset) = \emptyset \).
(ii) Suppose \( \Lambda(p, s) \) has been defined for all \( s \in \omega \). Fix an \( s \in \omega \) and \( i \in 2 \). Let \( t \supseteq \Lambda(p, s) \) be the minimal split node of \( p \) extending \( \Lambda(p, s) \). Let \( \Lambda(p, s' i) = t' i \).

Let \( \Xi(p, s) = p_{\Lambda(p, s)} \).

**Fact 4.4.** A fusion sequence is a sequence \( \langle p_n : n \in \omega \rangle \) in \( S \) so that for all \( n \in \omega \), \( p_{n+1} \leq_{\Xi} p_n \). Let \( p_\omega = \bigcap_{n \in \omega} p_n \). Then \( p_\omega \in S \) and is called the fusion of the above fusion sequence.

**Fact 4.5.** Suppose \( p \in \mathbb{S} \). Let \( \langle r_n : n \in \omega \rangle \) be a sequence of positive integers. Let \( \langle f_n : n \in \omega \rangle \) be a sequence such that for all \( n \in \omega \), \( f_n : [\|p\|]_{\Xi}^n \to \mathbb{N} \) is a continuous function. Then there is some \( q \leq \mathbb{S} \leq \mathbb{P} \) and \( z \in \mathbb{R} \) so that for all \( m, n \in \omega \) and \( y \in f_n([\|q\|]_{\Xi}^m) \), \( z \neq y(m) \).

**Proof.** Let \( B : \omega \to \omega \times \omega \) be a surjection with the property that the inverse image of any \( (e, g) \) is infinite.

Objects \( \langle z_n : n \in \omega \rangle \) and \( \langle q_n : n \in \omega \rangle \) will be built with the following properties.

(I) For each \( n \in \omega \), \( z_n \in \omega \) and \( z_n \notin z_{n+1} \). For each \( n \in \omega \), \( q_n \in \mathbb{S} \), \( q_n \leq p \), and \( q_{n+1} \leq q_n \).

(II) For each \( n \in \omega \), suppose \( B(n) = (e, g) \). Then for each sequence \( \langle \sigma_1, ..., \sigma_{r_n} \rangle \) of pairwise distinct strings in \( \omega^2 \), there is some \( \tau \in \omega^2 \) so that for all \( y \) with

\[
y \in f_{e}([\Xi(q_{n+1}, \sigma_1)] \times ... \times [\Xi(q_{n+1}, \sigma_{r_n})])
\]

\( y(g) \in N_\tau \) and \( z_n \) and \( \tau \) are incompatible.

Suppose these objects can be constructed. Then \( \langle q_n : n \in \omega \rangle \) forms a fusion sequence. By Fact 4.4, \( q = \bigcap_{n \in \omega} q_n \) is a perfect tree. Let \( z = \bigcap_{n \in \omega} z_n \). Let \( e, g \in \omega \). Suppose \( (x_1, ..., x_{r_n}) \in [\|q\|]_{\Xi}^n \). By the assumption on \( B \), there is some \( n \) large enough so that \( B(n) = (e, g) \) and there are pairwise distinct strings \( \sigma_1, ..., \sigma_{r_n} \in \omega^2 \) with \( \Lambda(q, \sigma_1) \subset x_1, ..., \Lambda(q, \sigma_{r_n}) \subset x_r \). Then by (II), \( z_n+1 \) is not an initial segment of \( y(g) \).

Hence \( y(g) \neq z \).

It remains to construct these objects.

Let \( z_0 = \emptyset \) and \( q_0 = p \).

Suppose \( q_n \) and \( z_n \) have been constructed. Suppose that \( B(n) = (e, g) \). Enumerate all the \( r_n \)-tuples of distinct strings in \( \omega^2 \) as \( (\sigma_1^n, ..., \sigma_{r_n}^n), ..., (\sigma_1^M, ..., \sigma_{r_n}^M) \) for some \( M \in \omega \).

Let \( s_0 = q_n \). Let \( \ell_0 = z_n \). Suppose \( s_k \) and \( \ell_k \) have been defined for some fixed \( k \leq M \). For each \( 1 \leq i \leq r_n \), let \( c_i = \sigma_i^k \emptyset \). Let \( d_i = \bigcap_{n < \omega} \Lambda(r_k, c_i \upharpoonright n) \). By the continuity of \( f_e \) on \( [\|p\|]_{\Xi}^e \), there is some \( N > n \) so that for all

\[
y \in f_{e}([\Xi(s_k, c_1 \upharpoonright N)] \times ... \times [\Xi(s_k, c_{r_n} \upharpoonright N)]],
\]

\( f_{e}(d_1, ..., d_{r_n})(g) \upharpoonright |\ell_k+1| \subseteq y(g) \). Define \( \ell_{k+1} = \ell_k(1 - f_{e}(d_1, ..., d_{r_n})(g)(|\ell_k|)) \), that is \( \ell_{k+1} \) extends \( \ell_k \) by one using the opposite of the value of \( f_{e}(d_1, ..., d_{r_n})(g)(|\ell_k|) \). Let \( s_{k+1} \leq s_k \) be such that for all \( \sigma \in \omega^2 \), if \( \sigma = \sigma_k^i \) for some \( 1 \leq i \leq r_n \), then \( \Xi(s_{k+1}, \sigma) = \Xi(s_k, c_i \upharpoonright N) \) and if \( \sigma \) is otherwise, then \( \Xi(s_{k+1}, \sigma) = \Xi(s_k, \sigma) \).

Finally, let \( q_{n+1} = s_{M+1} \) and \( z_{n+1} = \ell_{M+1} \). This completes the construction.

**Fact 4.6.** Let \( \delta \) be an ordinal. Let \( \langle A_\alpha : \alpha < \delta \rangle \) be a sequence of meager subsets of \( \mathbb{R} \). Define a prewellordering on \( \bigcup_{\alpha < \delta} A_\alpha \) by \( x \preceq y \) if and only if the least ordinal \( \xi \) such that \( x \in A_\xi \) is less or equal to the least ordinal \( \xi \) such that \( y \in A_\xi \). Assume that \( z \) as a subset of \( \mathbb{R} \times \mathbb{R} \) has the Baire property. Then \( \bigcup_{\alpha < \delta} A_\alpha \) is meager.

(\( \text{ZF + AD} \)) Every wellordered union of meager sets is meager.

**Proof.** See [12]. The second statement follows from the fact that every subset of \( \mathbb{R} \times \mathbb{R} \) has the Baire property under AD.

**Fact 4.7.** (Mycielski) Suppose \( \langle C_n : n \in \omega \rangle \) is a sequence so that each \( C_n \) is a comeager subset of \( \mathbb{R}^n \). Then there is a perfect tree \( p \) so that for all \( n \in \omega \), \( [\|p\|]_{\Xi} \subseteq C_n \).

**Fact 4.8.** (\( \text{ZF + AD} \)) (Comeager uniformization) Let \( R \subseteq \mathbb{R} \times \mathbb{R} \) be a relation. Then there is a comeager set \( C \subseteq \mathbb{R} \) and a function \( f : C \to \mathbb{R} \) so that for all \( x \in C \), \( R(x, f(x)) \).

**Fact 4.9.** (\( \text{ZF + AD} \)) Let \( E \) be an equivalence relation on \( \mathbb{R} \) with all classes countable and \( \mathbb{R} / E \approx \mathbb{R} \). Let \( p \) be perfect tree. Then \( [\|p\|] / E \approx \mathbb{R} \).

**Proof.** Note that \( [\|p\|] / E \) injects into \( \mathbb{R} / E \) by inclusion. Composing with the bijection then shows that \( [\|p\|] / E \) injects into \( \mathbb{R} \). Let \( \Phi : \mathbb{R} / E \to \mathbb{R} \) be a bijection. Since \( E \) has only countable classes and countable unions of countable sets are countable under AD, \( [\|p\|] / E \) is an uncountable set. Hence \( \Phi([\|p\|] / E) \) is an uncountable
Using the notation from Definition 4.3, define \( \Phi \). Hence \( R/E \) injects into \( p/E \). By Cantor-Schröder-Bernstein, \( p/E \approx R \).

**Fact 4.10.** (ZF + AD) Let \( A \subseteq R \). Let \( E \) be a prewordering on \( A \) with all classes countable so that \( R/E \approx R \). Does the disjoint union \( \bigcup_{\alpha<\kappa} R/E_\alpha \) have the Jónsson property? Note that one is not given a sequence of bijections \( \Phi_\alpha : \alpha < \kappa \) witnessing \( R/E_\alpha \approx R \).

**Question 4.11.** (Holshouser-Jackson) (ZF + AD) Let \( \kappa \) be an ordinal. Let \( \langle E_\alpha : \alpha < \kappa \rangle \) be a sequence of equivalence relations on \( R \) with all classes countable so that \( R/E_\alpha \approx R \). Does the disjoint union \( \bigcup_{\alpha<\kappa} R/E_\alpha \) have the Jónsson property?

Let \( \kappa \) be an ordinal. Let \( \langle E_\alpha : \alpha < \kappa \rangle \) be a sequence of equivalence relations on \( R \) with all classes countable so that \( R/E_\alpha \approx R \). Does the disjoint union \( \bigcup_{\alpha<\kappa} R/E_\alpha \) have the Jónsson property?

The following theorem gives some information concerning the Jónsson property.

**Theorem 4.13.** (ZF + AD) Let \( \kappa \) be an ordinal. Let \( \langle E_\alpha : \alpha < \kappa \rangle \) be a sequence of equivalence relations on \( R \) with all classes countable. Then \( f \) is a bijection witnessing \( \bigcup_{\alpha<\kappa} R/E_\alpha \approx R \). In this case, Theorem 4.15 below would imply \( \bigcup_{\alpha<\kappa} R/E_\alpha \) has the Jónsson property. The following is an interesting question.

**Question 4.12.** (Holshouser-Jackson) (ZF + AD) Let \( \kappa \) be an ordinal. Let \( \langle E_\alpha : \alpha < \kappa \rangle \) be a sequence of equivalence relations on \( R \) with all classes countable so that \( R/E_\alpha \approx R \). Is \( \bigcup_{\alpha<\kappa} R/E_\alpha \approx R \times \kappa \)?
$H^\sigma(x) \in \mathbb{R}$ enumerates $A^\sigma_\alpha$. By Fact 4.5 there is some $q \leq_\mathbb{B} p$ and some $z \in \mathbb{R}$ so that for all $\sigma \in X$, $j \in \omega$, and $(x_1, \ldots, x_{m(\sigma)}) \in [q]_{m(\sigma)}$, $H^\sigma(x)(j) \neq z$.

Now suppose $(r_1, \alpha_1), \ldots, (r_n, \alpha_n) \in \{q\} \times \kappa$ are such that $([([r_1, r_2], \alpha_1, \alpha_2)])_{E} \in \bigsqcup_{\alpha<\kappa} [q]_{E} \approx$. There is some $m \leq n$, $(x_1, \ldots, x_m) \in [q]_{m}$ and surjection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ so that $(r_1, \ldots, r_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then $z \notin A^\sigma_{(x_1, \ldots, x_m)}$ implies that $(z, \beta) \notin f([([r_1, r_2], \alpha_1, \alpha_2)])_{E}$ for all $\beta < \kappa$.

This shows that $f([\bigsqcup_{\alpha<\kappa} [q]_{E}]^\approx) \neq \bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. □

Let $p$ be the perfect tree given by Theorem 4.13. Assume that each $\mathbb{R}/E_\alpha \approx \mathbb{R}$. By Fact 4.9 each $[p]/E_\alpha \cong \mathbb{R}$. If $\bigcup_{\alpha<\kappa} [p]/E_\alpha \cong \bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$, then Theorem 4.13 would imply $\bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ has the Jónsson property. This suggests the following natural question.

**Question 4.14.** (ZF + AD) Let $\kappa$ be an ordinal. Let $\langle E_\alpha : \alpha < \kappa \rangle$ be a sequence of equivalence relations on $\mathbb{R}$ with all classes countable and $\mathbb{R}/E_\alpha \approx \mathbb{R}$ for each $\alpha < \kappa$. Let $p$ be a perfect tree. Is $\bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha \approx \bigcup_{\alpha<\kappa} [p]/E_\alpha$?

When all the $E_\alpha$’s are the identity equivalence relation, $=$, then one can exhibit the desired bijection. This gives the following result.

**Theorem 4.15.** (ZF + AD) For any ordinal $\kappa$, $\mathbb{R} \times \kappa$ has the Jónsson property.

**Proof.** Let $\langle E_\alpha : \alpha < \omega \rangle$ be a sequence where each $E_\alpha$ is the identity equivalence relation, $=$, on $\mathbb{R}$. Note that $\bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha \approx \mathbb{R} \times \kappa$. Apply Theorem 4.13 to this sequence. For any perfect tree $p$, $\bigcup_{\alpha<\kappa} [p]/E_\alpha \approx \mathbb{R} \times \kappa$. □

Many of the results above are trivial if the sequence $\langle E_\alpha : \alpha < \kappa \rangle$ is accompanied by a sequence $\langle \Phi_\alpha : \alpha < \kappa \rangle$ where each $\Phi_\alpha : \mathbb{R}/E_\alpha \to \mathbb{R}$ is a bijection. A natural question would be to construct an example $\langle E_\alpha : \alpha < \kappa \rangle$ such that for each $\alpha < \kappa$, $\mathbb{R}/E_\alpha \approx \mathbb{R}$ but there does not exist a sequence $\langle \Phi_\alpha : \alpha < \kappa \rangle$ which uniformly witnesses these bijections exist. Also, is the condition that each $E_\alpha$ be an equivalence relation with all classes countable necessary in Question 4.12 and 4.14? The following example of Holshouser-Jackson answers these questions.

**Example 4.16.** Fix some recursive coding of binary relations on $\omega$ by reals. Let WO denote the collection of reals that code wellorderings on $\omega$. For $\alpha < \omega_1$, let WO$\alpha$ denote the reals coding wellorderings of ordertype $\alpha$. For $\alpha < \omega_1$, let $E_\alpha$ be the equivalence relation on $\mathbb{R}$ defined by $x E_\alpha y$ if and only if $(x = y) \lor (x \notin \text{WO}_\alpha \land y \notin \text{WO}_\alpha)$. For each $\alpha < \omega_1$, $E_\alpha$ is $\Delta^1_1$ bireducible to $\approx$. Hence $\mathbb{R} \approx \mathbb{R}/E_\alpha$.

For each $\alpha < \omega_1$, if $x \in \text{WO}_\alpha$, identify $[x]_{E_\alpha} = \{x\}$ with $x$. For each $\alpha < \omega_1$, $\mathbb{R} \setminus \text{WO}_\alpha$ is a single $E_\alpha$ equivalence class. Identify it with $\alpha$. Under this identification, one has a bijection of $\bigcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ with WO $\cup \omega_1 \approx \mathbb{R} \cup \omega_1$.

As mentioned above, if $\langle E_\alpha : \alpha < \omega_1 \rangle$ was accompanied by a sequence of bijections $\langle \Phi_\alpha : \alpha < \omega_1 \rangle$, then one can construction a bijection between $\bigcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ and $\mathbb{R} \times \omega_1$. Thus, there cannot be such a sequence of bijections under AD.

Note that $E_\alpha$ has exactly one uncountable class. This example shows Question 4.12 has a negative answer without the condition that each $E_\alpha$ has all countable classes.

Let $p$ be a perfect tree such that $[p] \subseteq \mathbb{R} \setminus \text{WO}$. Then $\bigcup_{\alpha<\omega_1} [p]/E_\alpha \approx \omega_1$. $\omega_1$ is not in bijection with $\bigcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha \approx \mathbb{R} \cup \omega_1$. Hence Question 4.14 has a negative answer if all the equivalence relations do not have all classes countable.

If all the equivalence relations in $\langle E_\alpha : \alpha < \kappa \rangle$ have all classes countable and $\mathbb{R} \approx \mathbb{R}/E_\alpha$, then $\bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$ contains a subset which is in bijection with $\mathbb{R} \cup \omega_1$ but itself is not in bijection with $\mathbb{R} \cup \omega_1$.

**Fact 4.17.** (ZF + AD) Let $\kappa$ be an uncountable ordinal. Let $\langle E_\alpha : \alpha < \kappa \rangle$ be a sequence of equivalence relations on $\mathbb{R}$ so that for each $\alpha < \kappa$, $E_\alpha$ has all classes countable and $\mathbb{R} \approx \mathbb{R}/E_\alpha$. Then $\mathbb{R} \cup \kappa$ injects into $\bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$, but $\mathbb{R} \cup \kappa$ is not in bijection with $\bigcup_{\alpha<\kappa} \mathbb{R}/E_\alpha$. 12
Proof. Let \( \bar{0} : \omega \to \{0\} \), be the constant 0 function. For each \( \alpha < \kappa \), identify \( [\bar{0}]_{E_{\alpha}} \) with \( \alpha \). Let \( \Phi : \mathbb{R} \to (\mathbb{R}/E_0) \setminus \{\bar{0}\}_{E_0} \) be a bijection. Identify \( \Phi(r) \) with \( r \). Using this identification, there is a subset of \( \bigcup_{\alpha < \kappa} \mathbb{R}/E_\alpha \) which is in bijection with \( \mathbb{R} \times \omega \).

Suppose there is a bijection \( \Phi : \mathbb{R} \times \omega \to \bigcup_{\alpha < \kappa} \mathbb{R}/E_\alpha \). \( \bigcup_{\alpha < \kappa} \Phi(\alpha) \) can be prewellordered by \( x \subseteq y \) if and only if the least \( \alpha \) such that \( x \in \Phi(\alpha) \) is less than or equal to the least \( \alpha \) such that \( y \in \Phi(\alpha) \). Each \( \subseteq \)-class is countable. Fact 4.10 implies that \( \bigcup_{\alpha < \kappa} \Phi(\alpha) \) is countable. Let \( r \in \mathbb{R} \) with \( r \notin \bigcup_{\alpha < \kappa} \Phi(\alpha) \). Let \( X = \{[r]_{E_\alpha} : \alpha < \kappa\} \). \( \Phi^{-1}(X) \) is an uncountable sequence of distinct reals in \( \mathbb{R} \). Contradiction. \( \square \)

Theorem 2 shows that under ZF + DC + AD\( \mathbb{R} \), the only uncountable cardinals below \( \mathbb{R} \times \omega_1 \) are \( \omega_1 \), \( \mathbb{R} \times \omega_1 \), and \( \mathbb{R} \times \omega_1 \). Thus under these assumptions, if \( \bigcup_{\alpha < \omega_1} \mathbb{R}/E_\alpha \) is not in bijection with \( \mathbb{R} \times \omega_1 \), then \( \bigcup_{\alpha < \omega_1} \mathbb{R}/E_\alpha \) cannot inject into \( \mathbb{R} \times \omega_1 \). Moreover, Theorem 2 shows that under ZF + DC + AD\( \mathbb{R} \), the only uncountable cardinals below \( [\omega_1]^\omega \) are \( \omega_1 \), \( \mathbb{R} \), \( \mathbb{R} \times \omega_1 \), and \( [\omega_1]^\omega \). An interesting question would be to compare the cardinality of \( [\omega_1]^\omega \) and \( \bigcup_{\alpha < \omega_1} \mathbb{R}/E_\alpha \) when each \( E_\alpha \) is an equivalence relation with all classes countable.

Fact 4.18. (ZF + AD) Let \( \langle E_\alpha : \alpha < \omega_1 \rangle \) be a sequence of equivalence relations on \( \mathbb{R} \) such that each \( E_\alpha \) has all classes \( \Pi^1_3 \). There is no injection of \( [\omega_1]^\omega \) into \( \bigcup_{\alpha < \omega_1} \mathbb{R}/E_\alpha \).

Proof. Recall that \( \mathcal{U} \) is Martin’s cone measure on \( \mathcal{D} \), the set of Turing degrees. For each \( x \in \mathcal{D} \), let \( \Lambda(x) \) denote the collection of countable \( x \)-admissible ordinals. For each \( x \in \mathcal{D} \), let \( \Gamma(x) \in [\omega_1]^\omega \) be the increasing sequence of the first \( \omega \)-many \( x \)-admissible ordinals.

Suppose \( \Phi : [\omega_1]^\omega \to \bigcup_{\alpha < \omega_1} \mathbb{R}/E_\alpha \) is an injection. A sequence of Turing degrees \( \{x_n : n \in \omega\} \) and a sequence \( \{\sigma_n : n \in \omega\} \) in \( <^\omega 2 \) will be constructed by recursion with the property that for all \( n \in \omega \), \( |\sigma_n| = n \), \( \sigma_n \subseteq \sigma_{n+1} \), and whenever \( f \in [\Lambda(x_n)]^\omega \), there is some \( r \in \Phi(f) \) so that \( \sigma_n \subseteq r \).

Let \( x_0 = \bar{0} \), where \( 0 \) is the constant 0 function. Let \( \sigma_0 = \emptyset \).

Suppose \( x_n \) and \( \sigma_n \) have been defined with the desired properties. Let \( E_n^{n+1} = \{x \in \mathcal{D} : (\exists r \in \Phi(\Gamma(x))) (|\sigma_n \setminus \varnothing| \leq r) \} \) and \( E_1^{n+1} = \{x \in \mathcal{D} : (\exists r \in \Phi(\Gamma(x))) (|\sigma_n \setminus \varnothing| \leq r) \} \). Note that the cone above \( x_n \) is contained in \( E_n^{n+1} \cup E_1^{n+1} \) since \( \mathcal{U} \) is an ultrafilter. It contains the cone above \( x_{n+1} \). Let \( \sigma_{n+1} = \sigma_n \cup \{i \} \) for some \( i \in 2 \) and \( x_{n+1} \geq_T x_n \) so that \( E_1^{n+1} \) contains the cone above \( x_{n+1} \).

Let \( f \in [\Lambda(x_{n+1})]^\omega \). A result of Jensen (9) shows that for every increasing \( \omega \)-sequence of \( x_{n+1} \)-admissible ordinals \( f \), there is some \( y \geq_T x_{n+1} \) so that \( \Gamma(y) = f \). Then \( y \in E_1^{n+1} \). Hence there is some \( r \in \Phi(\Gamma(y)) \) that is \( \sigma_{n+1} \subseteq r \).

Let \( r = \bigcup_{n \in \omega} \sigma_n \). Let \( z \) be the join \( \bigoplus_{n \in \omega} x_n \). Suppose \( f \in [\Lambda(z)]^\omega \). For all \( n \in \omega \), there is some \( r_n^f \in \Phi(f) \) so that \( \sigma_n \subseteq r_n^f \). \( \Phi(f) \) is an \( E_\alpha \) class for some \( \alpha < \omega \) so \( \Phi(f) \) is \( \Pi^1_3 \). Since \( r \) is the limit of \( \{r_n^f : n \in \omega\} \subseteq \Phi(f) \), \( r \in \Phi(f) \). It has been shown that for all \( f \in [\Lambda(z)]^\omega \), \( \Phi(f) \subseteq \{[r]_{E_n} : \alpha < \omega_1 \} \approx \omega_1 \). Then \( \Phi \) induces an injection of \( [\Lambda(z)]^\omega \) into \( \omega_1 \). This is impossible since such an injection would yield a wellordering of \( \mathbb{R} \) since \( \mathbb{R} \) injects into \( [\Lambda(z)]^\omega \). \( \square \)

The above argument incorporates Martin’s proof of the partition relation \( \omega_1 \to (\omega_1)^2 \). The following result captures the essential idea of the above argument.

Fact 4.19. (ZF + AD) Let \( \kappa \in \text{ON} \). Let \( \langle E_\alpha : \alpha < \kappa \rangle \) be a sequence of equivalence relations on \( \mathbb{R} \). Let \( \Phi : [\omega_1]^\omega \to \bigcup_{\alpha < \kappa} \mathbb{R}/E_\alpha \). Let \( R \subseteq [\omega_1]^\omega \times \mathbb{R} \) be defined by \( R(f, x) \iff x \in \Phi(f) \). If \( R \) has a uniformizing function then \( \Phi \) is not an injection.

Proof. Let \( \Psi \) be a uniformizing function for \( R \).

For each \( n \in \omega \), let \( E_n^\omega = \{x \in \mathcal{D} : \Psi(\Gamma(x))(n) = i\} \). Since \( \mathcal{U} \) is an ultrafilter, there is some \( a_n \in 2 \) such that \( E_n^\omega \in \mathcal{U} \). Let \( x_n \in \mathcal{D} \) be such that the cone above \( x_n \) lies inside of \( E_n^\omega \). Now suppose that \( f \in [\Lambda(x_n)]^\omega \). A result of Jensen (9) states that for any such \( f \), there is some \( y \geq_T x_n \) so that \( \Gamma(y) = f \). As \( y \in E_n^\omega \), \( \Psi(\Gamma(y))(n) = \Phi(f)(n) = a_n \).

Let \( \mathcal{U} \) be such that for all \( n \), \( r(n) = a_n \). Let \( x = \bigoplus x_n \). If \( f \in [\Lambda(x)]^\omega \), then \( \Phi(f) = r \).

It has been shown that there is an uncountable set \( X \subseteq \omega_1 \) and some real \( r \) so that \( \Psi[X]^\omega \cup \{r\} \). By definition of \( R \), \( \Phi[X]^\omega \subseteq \{[r]_{E_n} : \alpha < \kappa \} \). The latter set is in bijection with \( \kappa \). \( |X|^\omega \approx [\omega_1]^\omega \). Therefore, \( \Phi \) induces an injection of \( [\omega_1]^\omega \) into the ordinal \( \kappa \). As \( \mathbb{R} \) injects into \( [\omega_1]^\omega \), this would imply that one could wellorder \( \mathbb{R} \). \( \square \)
Note that in Fact 4.19, \( R \) only needs to be uniformized on a set of cardinality \( [\omega_1]^\omega \). To see this, suppose \( R \) is uniformized on \( Z \subseteq [\omega_1]^\omega \) of cardinality \( [\omega_1]^\omega \). Let \( L : [\omega_1]^\omega \to Z \) be a bijection. Let \( \Phi' = \Phi \circ L \). The relation \( R' \) associated to \( \Phi' \) can be uniformized. Hence \( \Phi' \) is not injective by Fact 4.19. This implies \( \Phi \) is not injective.

The class of equivalence relations with \( \Pi^0_1 \) classes is very restrictive. However, it does include equivalence relations with all finite classes. However, in such cases, there is a more natural argument: Fix some linear ordering \( < \) of \( \mathbb{R} \). For \( f \in [\omega_1]^\omega \), let \( L(x) \) denote the \( < \)-least element of \( \Phi(x) \) (which exists since \( \Phi(x) \) is finite). Now apply Fact 4.19.

**Fact 4.20.** (With Jackson.) Assume \( \text{ZF} + \text{AD}^+ \). Let \( \kappa \in \text{ON} \) and \( \langle E_\alpha : \alpha < \kappa \rangle \) be a sequence of equivalence relations on \( \mathbb{R} \) such that each \( E_\alpha \) has all classes countable. Then there is no injection \( \Phi : [\omega_1]^\omega \to \bigcup_{\alpha < \kappa} \mathbb{R}/E_\alpha \).

**Proof.** This is proved by verifying the uniformization condition of Fact 4.19. Note that if \( \langle E_\alpha : \alpha < \kappa \rangle \) is a sequence so that each \( E_\alpha \) is an equivalence relation with all classes countable, then for any \( \Phi \), the associated relation has all countable sections.

Woodin’s countable section uniformization states that every relation on \( \mathbb{R} \) with countable section can be uniformized under \( \text{AD}^+ \). In the present situation, the relations are on \( [\omega_1]^\omega \times \mathbb{R} \). Some modification of Woodin’s ideas can be used to show countable section uniformization holds for such relations under \( \text{AD}^+ \).

The main ideas of Woodin’s countable section uniformization on \( \mathbb{R} \) can be found in [1] and [13].

Originally, Theorem 4.13 was proved under \( \text{AD}^+ \) using Woodin’s countable section uniformization. However, it was observed that for the purpose of the Jónsson property, one did not need total uniformization provided by Woodin’s countable section uniformization but rather partial uniformization on a set of cardinality \( \mathbb{R} \) (as provided by comeager uniformization) was adequate. As mentioned above, partial uniformization on a set of cardinality \( [\omega_1]^\omega \) is adequate for the conclusion of Fact 4.19. This suggests the following:

**Question 4.21.** Using just \( \text{AD} \), is it provable that for all relations \( R \subseteq [\omega_1]^\omega \times \mathbb{R} \) with countable sections, there is some \( Z \subseteq [\omega_1]^\omega \) and \( \Phi : Z \to \mathbb{R} \) such that \( |Z| = |[\omega_1]^\omega| \) and for all \( z \in Z, R(z, \Phi(z)) \)?

The rest of this section will show the failure of the Jónsson property for \( (\mathbb{R}/E_0) \times \kappa \) where \( E_0 \) is the equivalence relation from Definition 2.15 and \( \kappa < \Theta \).

**Fact 4.22.** (\( \text{ZF} + \text{AD} \)) Suppose \( A \subseteq (\mathbb{R}/E_0) \times \kappa \) and \( A \approx \mathbb{R}/E_0 \), where \( \kappa \) is an ordinal. Let \( \pi_1 : (\mathbb{R}/E_0) \times \kappa \to \mathbb{R}/E_0 \) be the projection onto the first coordinate. Then \( \pi_1[A] \approx \mathbb{R}/E_0 \).

**Proof.** Note that \( A \) injects into \( \pi_1[A] \times \kappa \). Hence \( \mathbb{R}/E_0 \) injects into \( \pi_1[A] \times \kappa \). Let \( f : \mathbb{R}/E_0 \to \pi_1[A] \times \kappa \) denote this injection. For each \( \alpha < \kappa \), let \( A_\alpha = \{ x \in \mathbb{R} : \pi_2(f([x]_{E_0})) = \alpha \} \), where \( \pi_2 : (\mathbb{R}/E_0) \times \kappa \to \kappa \) is the projection onto the second coordinate. Then \( \bigcup_{\alpha < \kappa} A_\alpha = \mathbb{R} \). By Fact 4.6, there must be some \( \alpha < \kappa \) so that \( A_\alpha \) is nonmeager. Using the Baire property, \( A_\alpha \) is comeager in some basic open set \( O \). (Actually since \( A_\alpha \) is \( E_0 \)-invariant, it can be shown that \( A_\alpha \) is comeager.) Hence \( A_\alpha \supseteq \bigcap_{n \in \omega} D_n \), where \( \langle D_n : n \in \omega \rangle \) is a sequence of topologically dense open sets relative to \( O \). One can build an \( E_0 \)-tree inside of \( A_\alpha \). (See [2] Definition 5.2.) This implies that there is a continuous reduction of \( E_0 \) into \( E_0[ A_\alpha ] \). Hence \( \mathbb{R}/E_0 \) injects into \( A_\alpha/E_0 \). Using \( f, A_\alpha/E_0 \) injects into \( \pi_1[A] \times \{ \alpha \} \approx \pi_1[A] \). It has been shown that \( \mathbb{R}/E_0 \) injects into \( \pi_1[A] \approx \mathbb{R}/E_0 \).

**Fact 4.23.** Let \( \kappa < \Theta \). There is a 6-Jónsson function for \( (\mathbb{R}/E_0) \times \kappa \).

(\( \mathbb{R}/E_0 \times \kappa \) is not Jónsson.

**Proof.** By Fact 4.2, let \( \Phi : [\mathbb{R}/E_0]^3 \to \mathbb{R}/E_0 \) be a 3-Jónsson map for \( \mathbb{R}/E_0 \). Let \( \Psi : \mathbb{R} \to \kappa \) be a surjection. Since \( = \) reduces into \( E_0 \), there is an injection \( \Gamma : \mathbb{R} \to \mathbb{R}/E_0 \). Let \( \Lambda : [\mathbb{R}/E_0]^3 \to \kappa \) be defined by

\[
\Lambda(x) = \begin{cases} 
0 & (\forall r \in \mathbb{R})(\Phi(x) \neq \Gamma(r)) \\
\Psi(r) & \Phi(x) = \Gamma(r)
\end{cases}
\]

Finally, let \( \Upsilon : ([\mathbb{R}/E_0] \times \kappa]_{E_0}^3 \to (\mathbb{R}/E_0) \times \kappa \) be defined by

\[
((x_1, \alpha_1), (x_2, \alpha_2), (x_3, \alpha_3), (x_4, \alpha_4), (x_5, \alpha_5), (x_6, \alpha_6)) \mapsto (\Phi(x_1, x_2, x_3), \Lambda(x_4, x_5, x_6))
\]
Suppose $B \subseteq (\mathbb{R}/E_0) \times \kappa$ is in bijection with $(\mathbb{R}/E_0) \times \kappa$. Let $f : (\mathbb{R}/E_0) \times \kappa \to B$ be a bijection. Let $A = f(\mathbb{R}/E_0) \times \{0\}$. Then $A \approx \mathbb{R}/E_0$. By Fact 4.22 $\pi_1[A] \approx \mathbb{R}/E_0$.

Suppose that $(y, \beta) \in (\mathbb{R}/E_0) \times \kappa$. Suppose $\Psi(y) = \beta$. Since $\Phi$ is a 3-\( \lambda \)-map and $\pi_1[A] \approx \mathbb{R}/E_0$, one can find $((x_1, \alpha_1), (x_2, \alpha_2), (x_3, \alpha_3), (x_4, \alpha_4), (x_5, \alpha_5), (x_6, \alpha_6)) \in [A]^{\leq} \subseteq [B]^{\leq}$ so that $\Phi(x_1, x_2, x_3) = y$ and $\Phi(x_4, x_5, x_6) = \Gamma(r)$. Then $\Upsilon((x_1, \alpha_1), (x_2, \alpha_2), (x_3, \alpha_3), (x_4, \alpha_4), (x_5, \alpha_5), (x_6, \alpha_6)) = (y, \beta)$. $\Upsilon$ is a 3-\( \lambda \)-map for $(\mathbb{R}/E_0) \times \kappa$.

\[\Box\]

\begin{question}
[2] showed that $(\mathbb{R}/E_0)$ has no 2-\( \lambda \)-map but has a 3-\( \lambda \)-map. What is the least $n$ so that $(\mathbb{R}/E_0) \times \kappa$ has a $n$-\( \lambda \)-map, where $\kappa < \Theta$?

If $\kappa$ is any ordinal, is $(\mathbb{R}/E_0) \times \kappa$ also not \( \lambda \)-? 

\end{question}

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