Frobenius and Separable Functors for the Category of Entwined Modules over Cowreaths, I: General Theory

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Abstract
Entwined modules over cowreaths in a monoidal category are introduced. They can be identified to coalgebras in an appropriate monoidal category. It is investigated when such coalgebras are Frobenius (resp. separable), and when the forgetful functor from entwined modules to representations of the underlying algebra is Frobenius (resp. separable). These properties are equivalent when the unit object of the category is a \(\otimes\)-generator.

Keywords
Module category · Cowreath · Entwined module · Frobenius functor · Separable functor · Frobenius coalgebra · Coseparable coalgebra

1 Introduction
This paper is part of a series that has as the final aim the study of Frobenius and separable properties for forgetful functors defined on categories of entwined modules over cowreaths.

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obtained from certain quasi-Hopf actions and coactions. In this paper we present a general
theory that allows us not only to achieve the mentioned goal but also to unify similar results
obtained so far for various generalizations of Hopf algebras. It can be seen as a sequel of [6,
9] and as the theoretical support for [7].

Central elements in the enveloping algebra \( A \otimes A^{\text{op}} \) of an algebra \( A \) are often called
Casimir elements, and they play a crucial role in the theory of Frobenius and of separable
algebras. The fact that they appear in both theories is well understood, and has a categorical
explanation related to the properties that an algebra is Frobenius if the restriction of scalars
functor \( G \) is Frobenius, that is, its right adjoint is also a left adjoint, and that it is separable
if and only if \( G \) is separable in the sense of [22]. This can be exploited in order to study
Frobenius and separable functors simultaneously. This idea originated in the study of separ-
ability and Frobenius properties for Doi-Hopf modules in [12–14], and was later refined
and applied to entwined modules, see [4].

Entwined modules over entwining structures were introduced by Brzeziński in [3] in
order to extend the Hopf-Galois theory to coalgebras. One of the attractive aspects is that
many structures that appear in Hopf algebra theory, such as relative Hopf modules, Doi-
Hopf and Yetter-Drinfeld modules, turn out to be special cases. An entwining structure is a
kind of local braiding between an algebra and a coalgebra. In fact an entwining struc-
ture with underlying algebra \( A \) can be viewed as a coalgebra in the monoidal category
\( \mathcal{T}_A \) of transfer morphisms through \( A \) as introduced by Tambara in [29]. Tambara’s con-
struction can be obtained from Street’s formal theory of monads, see [26]. Monads in a
2-category \( \mathcal{C} \) can be organized into a new 2-category \( \text{Mnd}(\mathcal{C}) \). For an algebra (or monad) in
a strict monoidal category \( \mathcal{C} \) (a 2-category with single 0-cell), Tambara’s category \( \mathcal{T}_A \) is the
category \( \text{Mnd}(\mathcal{C})(A, A) \) of endomorphisms of \( A \) in \( \text{Mnd}(\mathcal{C}) \).

There is a second way to organize monads into a 2-category, see [19]; the second 2-
category is the Eilenberg-Moore 2-category \( \text{EM}(\mathcal{C}) \). It coincides with \( \text{Mnd}(\mathcal{C}) \) at the level
of 0-cells and 1-cells, but has different 2-cells. A cowreath in \( \mathcal{C} \) is a comonad in \( \text{EM}(\mathcal{C}) \), and
consists of an algebra in \( \mathcal{C} \) together with a coalgebra in \( \mathcal{T}_{\mathcal{A}}^\# = \text{EM}(\mathcal{C})(A, A) \), the category
of endomorphisms of \( A \) in the Eilenberg-Moore 2-category. Note that, in the case where \( \mathcal{K} \)
is a 2-category, a comonad in \( \text{EM}(\mathcal{K}) \) was called by Street a mixed wreath, see [28]. So the
cowreaths we are dealing with are nothing but mixed wreaths (or comonads) in \( \text{EM}(\mathcal{K}) \)
in the sense of Street, in the case where \( \mathcal{K} \) is a 2-category with a single 0-cell. If this is the
case, we can introduce entwined modules over a cowreath. The main aim of this paper
is to study when the forgetful functor from entwined modules to \( A \)-modules is Frobenius
or separable. This is related to the question when a coalgebra in \( \mathcal{T}_{\mathcal{A}}^\# \) is a Frobenius or a
coseparable coalgebra.

Compared to the classical situation, we have a two-fold generalization: first of all, the
category of vector spaces is replaced by an arbitrary (strict) monoidal category \( \mathcal{C} \). The
best results are obtained in the situation where the unit object \( 1 \) is a \( \otimes \)-generator of the
monoidal category \( \mathcal{C} \), as introduced in [9]. The following monoidal categories satisfy this
condition: the category of vector spaces, the category of bimodules \( R \mathcal{M} R \) over an Azumaya
\( k \)-algebra \( R \), the category of finite dimensional Hilbert complex vector spaces \( \text{FdHilb} \), and
the category \( \mathcal{Z}_k \) as introduced in [10]. We refer to [9, Examples 3.2].

Secondly, we work over cowreaths which can be viewed as generalized entwining struc-
tures. Our motivation to investigate such cowreaths comes from the applications that we
have in mind, namely the study of categories of Doi-Hopf modules, two-sided Hopf modules
and Yetter-Drinfeld modules over a quasi-Hopf algebra, which can be defined as entwined
modules over certain cowreaths that are not ordinary entwining structures. This study will be done in the forthcoming paper [7].

The paper is organized as follows. In Section 2, we present preliminary results on monoidal categories and bimodules. In Section 3, we introduce cowreaths in monoidal categories, and entwined modules over them. In Section 4, we introduce generalized factorization structures; these are algebras in $T_A^\#$, or, equivalently, wreaths in $C$. Given a generalized factorization structure, we can define an algebra in $C$, called the wreath product algebra or the generalized smash product. Duality arguments turn cowreaths into generalized factorization structures, and the category of entwined modules is isomorphic to the category of modules over the generalized smash product, see Theorem 4.4. In Section 5, we discuss when the forgetful functor $F$ is Frobenius. $F$ always has a right adjoint $G$; in order to investigate when $G$ is also a left adjoint, we need to investigate natural transformations from the identity functor to $FG$, and from $GF$ to the identity functor. Propositions 5.6 and 5.7 tell us that the necessary and sufficient information that is needed to produce such natural transformations is encoded in the so-called Frobenius elements and Casimir morphisms, at least in the case where $1$ is a $\otimes$-generator. Using these results, it is straightforward to prove the main Theorem 5.8, stating that $F$ is a Frobenius functor if and only if the coalgebra corresponding to the given cowreath is Frobenius. In Section 6 it is shown that there is a strong monoidal functor from the category of generalized transfer morphisms $T_A^\#$ to the category of $A$-bimodules, as introduced in the preliminary Section 2.2. Consequently, a cowreath produces an $A$-coring, that is a coalgebra in the category of $A$-bimodules. The main result is that this $A$-coring is Frobenius if and only if the corresponding coalgebra $(A, X)$ in $T_A^\#$ is Frobenius, see Theorem 6.2. Under the assumption that $X$ has a right adjoint $Y$, we have additional results, see Theorem 6.6. Separability is investigated in Section 7. The main result is Theorem 7.5 stating that a coalgebra $(X, \psi)$ in $T_A^\#$ is coseparable if and only if the forgetful functor is coseparable. Again, additional results can be stated if $X$ has a right adjoint.

Our theory can be applied to various cowreaths coming from (co)actions of Hopf algebras and their generalizations, see Section 5 of the paper [6]. But perhaps the most interesting are those cowreaths $(A, X)$ with $X$ regarded as an object in $T_A^\#$ rather than $T_A$. Such examples occur in the quasi-Hopf case and, as we already mentioned, when they are Frobenius or separable cowreaths is the topic of the forthcoming paper [7].

2 Preliminaries

2.1 Monoidal Categories

Monoidal Categories A monoidal category is a category $C$ together with a functor $\otimes : C \times C \to C$, called the tensor product, an object $1 \in C$, called the unit object, and natural isomorphisms $\alpha : \otimes \circ (\otimes \times \text{Id}) \to \otimes \circ (\text{Id} \times \otimes)$ (the associativity constraint), $l : \otimes \circ (1 \times \text{Id}) \to \text{Id}$ (the left unit constraint) and $r : \otimes \circ (\text{Id} \times 1) \to \text{Id}$ (the right unit constraint) satisfying appropriate coherence conditions, see for example [18, XI.2] for a detailed discussion.

To a monoidal category $(C, \otimes, 1, a, l, r)$ we can associate a new one $C^{\text{op}} := (C^{\text{op}}, \otimes, 1, a^{-1}, l^{-1}, r^{-1})$, called the opposite category associated to $C$; the terminology is
based on the fact that this new monoidal structure is build on $C^{\text{op}}$, the opposite category of $C$. Apart from $C^{\text{op}}$ we can also introduce the so called reverse monoidal category of $C$, $C^{\text{rev}} := (\overline{\otimes}, \otimes := \otimes \circ \tau, 1, \alpha, r, l)$; here $\tau : C \times C \to C \times C$ is the switch functor and $\overline{a}_{X,Y,Z} = a_{Z,Y,X}^{-1}$, for all $X, Y, Z \in C$. By mixing the two construction we also get a third monoidal category $C^{\text{oprev}} := (C^{\text{op}})^{\text{rev}} = (C^{\text{rev}})^{\text{op}}$ that one can associate to $C$.

A monoidal category $C$ is called strict if $\alpha, l$ and $r$ are the identity natural transformations. It is well-known that every monoidal category is monoidally equivalent to a strict monoidal category, and this enables us to assume without loss of generality that $C$ is strict. We will often delete the tensor symbol $\otimes$, and write $X \otimes Y = XY$. We write $X^n$ for the tensor product of $n$ copies of $X$. The identity morphism of an object $X \in C$ will be denoted by $\text{Id}_X$ or simply $X$. For morphisms $\text{Id}_X = X : X \to X$, $f : X \to Y$, $g : XY \to Z$ and $h : X \to YZ$ in $C$, we adopt the following graphical notation

$$
\text{Id}_X = X = \begin{array}{c} X \\ X \end{array}, \quad f = \begin{array}{c} X \\ \otimes \\ Y \end{array}, \quad g = \begin{array}{c} X \\ \otimes \\ Y \end{array}, \quad \text{and } h = \begin{array}{c} X \\ \otimes \\ Y \end{array}.
$$

**Algebras and Coalgebras** An algebra in $C$ is a triple $(A, m, \eta)$, where $A$ is an object in $C$ and $m : AA \to A$ (the multiplication) and $\eta : 1 \to A$ (the unit) are morphisms in $C$ satisfying the associativity and unit conditions $m \circ m A = m \circ Am$ and $m \circ \eta A = m \circ A\eta = A$.

The graphical notation for $m$ and $\eta$ is the following:

$$
m = \begin{array}{c} A \\ A \\ A \end{array} \quad \text{and } \eta = \begin{array}{c} 1 \\ A \end{array}.
$$

We use $A$ as a shorter notation for the algebra $(A, m, \eta)$; the multiplication on an algebra $A$ is typically denoted by $m$, and the unit by $\eta$; we put subscripts whenever convenient, so that we can write $A = (A, mA, \eta A)$. Similar conventions are used for other structures, such as coalgebras, modules over an algebra, adjunctions, entwining structures etc.

A coalgebra in $C$ is a triple $C = (C, \Delta : C \to CC, \varepsilon : C \to 1)$, satisfying the appropriate coassociativity and counit conditions. The graphical notation takes the form

$$
\Delta = \begin{array}{c} C \\ C \\ C \end{array} \quad \text{and } \varepsilon = \begin{array}{c} C \\ 1 \end{array}.
$$

**Adjunctions** An adjunction $X \dashv Y$ in $C$ is a quadruple $(X, Y, b, d)$, with $X, Y$ objects in $C$ and morphisms $b : 1 \to YX$ and $d : XY \to 1$ satisfying

$$
Yd \circ bY = Y \quad \text{and } \quad dX \circ Xb = X. \quad (2.1)
$$

With the graphical notation

$$
d = \begin{array}{c} X \\ \otimes \\ Y \\ 1 \end{array}, \quad b = \begin{array}{c} 1 \\ \otimes \\ Y \end{array}.
$$
Equation 2.1 can be rewritten as

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cc}
\downarrow Y & \downarrow X
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{cc}
\downarrow Y & \downarrow X
\end{array}
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{cc}
\downarrow Y & \downarrow X
\end{array}
\end{array}
\end{array} .
\]

Y is called a right adjoint of X, and X a left adjoint of Y. Right adjoints are unique in the following sense. If \((X, Y', b', d')\) is another adjunction, then \(\lambda = Y'd \circ b'Y : Y \to Y'\) is an isomorphism with inverse \(\lambda^{-1} = Yd' \circ bY'\). It is easy to show that \(\lambda\) is designed in such a way that

\[
b' = \lambda X \circ b \text{ and } d = d' \circ X \lambda.
\]

(2.2)

For two adjunctions \((X, Y, b, d)\) and \((X', Y', b', d')\), we have a new adjunction \((XX', Y'Y, b \cdot b' = Y'bX' \circ b', d \cdot d' = d \circ Xd'Y)\).

In particular, we have an adjunction \((X^2, Y^2, b^2 = b \cdot b, d^2 = d \cdot d)\). Given a morphism \(f : X \to X'\), we have

\[
g = Yd' \circ YfY' \circ bY' : Y' \to X',
\]

which reproduces \(f = dX' \circ XgX' \circ Xb'\).

If every object in \(C\) has a right (resp. left) adjoint, then we say that \(C\) has right (resp. left) duality; \(C\) is called rigid if it has left and right duality. Assume that \(C\) has right duality, and choose a right dual \(*X\) for every object \(X\). For every morphism \(f : X \to X'\) in \(C\), we have \(*f : *X' \to *X\), and this defines a functor \(*(-) : C \to C^{\text{op}}\). If \(C\) is rigid, then \((*(-), (-)*\) is a pair of inverse equivalences between \(C\) and \(C^{\text{op}}\).

For \(X, X' \in C\), \(*XX'\) and \(*X'*X\) are right duals of \(XX'\), so we have an isomorphism \(\varphi_2(X, X') : *XX' \to *X'*X\). \((1_1, 1_1, 1_1)\) is an adjunction, so we can put \(*1_1 = 1_1\), and define \(\varphi_0 = 1_1 : 1_1 \to 1_1\). \((*(-), \varphi_0, \varphi_2) : C \to C^{\text{op prev}}\) is a strong monoidal functor.

Let \(C\) be a coalgebra in \(C\), and assume that we have an adjunction \(C \dashv A\). Then \(A\) is an algebra, with structure maps

\[
m = Ad^2 \circ A \Delta AA \circ bAA : AA \to A \text{ and } \eta = A \varepsilon \circ b : 1 \to A.
\]

(2.3)

In this situation \(C\) is a right \(A\)-module (the definition of an \(A\)-module is given below), with structure map

\[
\mu = Cd \circ \Delta A.
\]

(2.4)

In a similar way, if a coalgebra \(C\) has a left adjoint \(A\), then \(A\) is an algebra, and \(C\) is a left \(A\)-module.

**Module Categories** Let \(C\) be a monoidal category. A right \(C\)-category is a quadruple \((\mathcal{D}, \diamond, \Psi, r)\), where \(\mathcal{D}\) is a category, \(\diamond : \mathcal{D} \times C \to \mathcal{D}\) is a functor, and \(\Psi : \diamond \circ (\circ \times \text{Id}) \to \diamond \circ (\text{Id} \times \circ)\) and \(r : \diamond \circ (\text{Id} \times 1) \to \text{Id}\) are natural isomorphisms such that the diagrams

\[
((M)(YZ))Z \xrightarrow{\varphi_{M,Y,Z}^-} (M)(Y(YZ)) \cong M((X)(YZ)) \quad \text{and} \quad (M)(X) \xrightarrow{\varphi_{M,X,1}} M((M)X) \cong M(1X) \xrightarrow{\varphi_{M,1,1}^+}
\]

commute, for all \(M \in \mathcal{D}\) and \(X, Y, Z \in C\). Obviously \(C\) itself is a right \(C\)-category. As before, we deleted the diamond product symbols, and wrote \(MX = M \circ X\), for \(M \in \mathcal{D}\) and \(X \in C\). The above mentioned coherence theorem for monoidal categories can be extended to \(C\)-categories, enabling us to assume throughout that \(\Psi\) and \(r\) are natural identities. In the literature, \(C\)-categories are also named module categories.
Let $\mathcal{D}$ be a right $C$-category, and consider an algebra $A$ in $C$. A right $A$-module in $\mathcal{D}$ is a pair $M = (M, \mu)$, with $M \in \mathcal{D}$ and $\mu : MA \to M$ satisfying $\mu \circ M\eta = M$ and $\mu \circ \mu A = \mu \circ Ma$. A morphism $f : M \to N$ between two right $A$-modules $M$ and $N$ in $\mathcal{D}$ is called right $A$-linear if $f \circ \mu = \mu \circ f A$. $\mathcal{D}_A$ will be the category of right $A$-modules and right $A$-linear morphisms in $\mathcal{D}$. In a similar way, we can define left $A$-modules $N = (N, \nu)$ in a left $C$-category $\mathcal{E}$ and the category $\mathcal{A}_E$. We will typically use the notation $\mu$ for a right action and $\nu$ for a left action. The next step is to introduce two-sided $C$-categories, and two-sided $A$-bimodules in a two-sided $C$-category. We leave it to the reader to formulate the precise definitions.

We can also define the notions of a right $C$-comodule $(M, \rho)$ in a right $C$-category $\mathcal{D}$, and right $C$-colinearity of a morphism between two right $C$-comodules in $\mathcal{D}$. The category of right comodules and right $C$-colinear morphisms in $\mathcal{D}$ will be denoted as $\mathcal{D}^C$. We will use the following diagrammatic notation for actions and coactions:

$$
\mu = \begin{tabular}{c}
\begin{array}{c}
M \\
\hline
A
\end{array}
\end{tabular}, \quad \nu = \begin{tabular}{c}
\begin{array}{c}
A \\
\hline
N
\end{array}
\end{tabular}, \quad \rho = \begin{tabular}{c}
\begin{array}{c}
M \\
\hline
C
\end{array}
\end{tabular}.
$$

2.2 The Category of Bimodules

The results in this Subsection will be needed in Sections 6 and 7. The results are well-known, see for example [23, 25] or [5]. What follows is an original reformulation, which is why we decided to keep the details. Let $\mathcal{C}$ be a (strict) monoidal category with coequalizers. Recall that $X \in \mathcal{C}$ is called left coflat if the functor $-X : \mathcal{C} \to \mathcal{C}$ preserves coequalizers. Let $A$ be an algebra in $\mathcal{C}$. For $X \in \mathcal{C}_A$ and $Y \in \mathcal{A}_C$, $(X \otimes_A Y, q)$ is the coequalizer of the parallel morphisms $\mu Y, X \nu$:

$$
XAY \xrightarrow{\mu Y} X = XAY \xrightarrow{q} \otimes_A Y.
$$

We compactify our notation by writing $X \otimes_A Y = X \cdot Y$. Now let $f : X \to X'$ in $\mathcal{C}_A$ and $g : Y \to Y'$ in $\mathcal{A}_C$. The universal property of coequalizers tells us that there is a unique $f \otimes_A g = f \cdot g$ in $\mathcal{C}$ such that (2.5) commutes.

$$
\begin{array}{c}
\begin{tabular}{c}
XAY \\
\hline
\otimes_A
\end{tabular}
\end{array} \xrightarrow{\mu Y} \begin{tabular}{c}
\begin{array}{c}
X \\
\hline
\otimes_A
\end{array}
\end{tabular} \xrightarrow{q} \begin{tabular}{c}
\begin{array}{c}
X' \otimes_A Y' \\
\hline
\otimes_A
\end{array}
\end{tabular}
\end{array}
$$

Proposition 2.1 Let $X \in \mathcal{C}_A$ and $M \in \mathcal{C}$. Then $(AM, mM) \in \mathcal{A}_C$, and

$$
\begin{array}{c}
\begin{tabular}{c}
XAA \\
\hline
A
\end{tabular}
\end{array} \xrightarrow{\mu AM} \begin{tabular}{c}
\begin{array}{c}
XAM \\
\hline
A
\end{array}
\end{tabular} \xrightarrow{\mu M} \begin{tabular}{c}
\begin{array}{c}
XM \\
\hline
A
\end{array}
\end{tabular}
\end{array}
$$

is a coequalizer in $\mathcal{C}$. If $M \in \mathcal{C}_A$ (resp. $X \in \mathcal{A}_C$), then this is also a coequalizer in $\mathcal{C}_A$ (resp. $\mathcal{A}_C$).
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Proof \( XAA \xrightarrow{Xm} XA \xrightarrow{\mu A} X \) is a split coequalizer through the morphisms \( X\eta : X \rightarrow XA \) and \( XAN : XA \rightarrow XAA \). Thus it is an absolute coequalizer, see [21, VI.6], and therefore it is preserved by the functor \( -M : \mathcal{C} \rightarrow \mathcal{C}, Y \mapsto YM \). Actually, if \( f : XAM \rightarrow P \) is such that \( f \circ \mu AM = f \circ XmM \) then
\[
g = f \circ X\eta M : XM \rightarrow P
\]
is unique with the property that \( f = g \circ \mu M \).

Finally, let \( M \in \mathcal{C}_A \), and assume that \( f \) is right \( A \)-linear. The morphism \( g \) defined by Eq. 2.6 is also right \( A \)-linear, and this shows that \( (XM, \mu M) \) is also a coequalizer in \( \mathcal{C}_A \). Similar arguments hold in the case where \( X \in A\mathcal{C}_A \).

It follows from Proposition 2.1 that we have a unique isomorphism \( \Upsilon : X \bullet (AM) \rightarrow XM \) such that \( \Upsilon \circ q = \mu M \), and \( \Upsilon^{-1} = q \circ X\eta M \). Otherwise stated, there is a unique isomorphism of coequalizers \( (X \bullet (AM), \eta) \cong (XM, \mu M) \). Now coequalizers are defined only up to isomorphisms, so we can go one step further, and declare \( (X \bullet (AM), \eta) = (XM, \mu M) \). This identification will also be useful at the level of morphisms. Before we explain this, we state the following Lemma, which is a well-known and basic fact.

Lemma 2.2 For \( M \in \mathcal{C} \) and \( Y \in A\mathcal{C} \), we have an isomorphism \( \alpha : A\mathcal{C}(AM, Y) \rightarrow \mathcal{C}(M, Y) \), given by the formulas
\[
\alpha(f) = f \circ \eta N ; \quad \alpha^{-1}(f) = \nu Y \circ Af.
\]

Now take \( f : X \rightarrow X' \) in \( \mathcal{C}_A \), \( M, M' \in \mathcal{C} \) and \( g : AM \rightarrow AM' \) in \( A\mathcal{C} \). Let \( g = \alpha(g) : M \rightarrow AM' \). Making the identification \( X \bullet (AM) \cong XM \) and \( X' \bullet (AM') \cong X'M' \), we have \( f \bullet g : XM \rightarrow X'M' \). According to Eq. 2.5, \( f \bullet g \) is determined by the commutativity of the diagram
\[
\begin{array}{ccc}
XAM & \xrightarrow{\mu M} & XM \\
\downarrow f \circ g & & \downarrow f \circ g \\
X'AM' & \xrightarrow{\mu M'} & X'M'
\end{array}
\]
It follows from Eq. 2.6 that
\[
f \bullet g = \mu M' \circ f \circ g \circ X\eta M = \mu M' \circ f g.
\]

Definition 2.3 Let \( A \) be an algebra in \( \mathcal{C} \), \( Y \in A\mathcal{C} \) is called robust as a left \( A \)-module if
\[
MXAY \xrightarrow{MX\nu} MXY \xrightarrow{Mq} M(X \bullet Y)
\]
is a coequalizer in \( \mathcal{C} \), for all \( M \in \mathcal{C} \) and \( X \in \mathcal{C}_A \).

This definition can be restated as follows: the universal property of coequalizers implies the existence of a unique \( \theta : (MX) \bullet Y \rightarrow M(X \bullet Y) \) such that \( \theta \circ q = Mq \). \( Y \) is robust if and only if \( \theta \) is an isomorphism for all \( X \) and \( M \).

Proposition 2.4 For all \( N \in \mathcal{C} \), \( AN \in A\mathcal{C} \) is robust.
Proposition 2.5

(1) The forgetful functor $U : C_A \to C$ creates the coequalizers which are preserved by $-A$ and $-AA$. Consequently, if $A$ is left coflat, $X \in C_A$ and $Y \in A C_A$, then $X \bullet Y \in C_A$ and $(X \bullet Y, q)$ is also a coequalizer in $A C_A$.

(2) Let $X \in A C_A$ and $Y \in A C$. If $Y$ is robust as a left $A$-module, then $X \bullet Y \in A C_A$ and $(X \bullet Y, q)$ is also a coequalizer in $A C_A$.

(3) If both $X$ and $Y$ are $A$-bimodules, $A$ is left coflat and $Y$ is left $A$-robust, then $X \bullet Y \in A C_A$ and $(X \bullet Y, q)$ is also a coequalizer in $A C_A$.

Proof

(1) Follows from [1, Proposition 4.3.2] applied to the monad $(-A, -m, -\eta)$. Note only that the right $A$-module structure on $X \bullet Y$ is determined by the unique morphism $\mu$ that makes the diagram below commutative.

\[
\begin{array}{ccc}
XAY & \xrightarrow{X\nu A} & XAY \\
\downarrow XA\mu & & \downarrow X\mu \\
Y & \xrightarrow{\mu Y} & Y
\end{array} \quad \begin{array}{ccc}
XY & \xrightarrow{q A} & (X \bullet Y)A \\
\downarrow X\nu & & \downarrow X\mu \\
X \bullet Y & \xrightarrow{q} & X \bullet Y
\end{array}
\] (2.8)

(2) If $Y$ is left $A$-robust, then the top row in the diagram

\[
\begin{array}{ccc}
AXAY & \xrightarrow{AX\nu} & AXY \\
\downarrow vAY & & \downarrow vY \\
XAY & \xrightarrow{\mu Y} & XY \xrightarrow{q} X \bullet Y
\end{array} \quad \begin{array}{ccc}
A XY & \xrightarrow{Aq} & A(X \bullet Y) \\
\downarrow A\mu Y & & \downarrow A\mu \\
AXY & \xrightarrow{AX\nu} & AXY
\end{array}
\] (2.9)

is a coequalizer, and the universal property brings the left action $v$ on $X \bullet Y$. The rest of the proof of part (2) is left to the reader.

(3) Now we assume that both $X$ and $Y$ are bimodules. We show that the actions $\mu$ and $v$ on $X \bullet Y$ are compatible. To this end, consider the cubic diagram

\[
\begin{array}{ccc}
AXYA \xrightarrow{Aq A} & A(X \bullet Y)A \\
\downarrow AX\mu & & \downarrow A\mu \\
AXY & \xrightarrow{Aq} & A(X \bullet Y) \\
\downarrow vAY & & \downarrow vA \\
XY & \xrightarrow{q A} & (X \bullet Y)A \\
\downarrow X\mu & & \downarrow \mu \\
XY & \xrightarrow{q} & X \bullet Y
\end{array}
\]
Commutativity of the top and bottom faces follows from the definition of $\mu$, and commutativity of the front and back faces follows from the definition of $\nu$. It is obvious that the left face commutes. From this we deduce that

$$\mu \circ \nu A = \nu \circ A \circ \mu = A \circ \nu A \circ \mu = A \circ \mu \circ \nu A.$$ 

From the robustness of $Y$, we know that $(A(X \bullet Y), AQ)$ is a coequalizer, and from the left coflatness of $A$ that $(A(X \bullet Y)_A, AQ)$ is a coequalizer. It then follows that $\mu \circ \nu A = \nu \circ A \circ \mu$, which is the compatibility that we need.

**Remark 2.6** By using the left handed version of Proposition 2.5 (1) one can see that $X \bullet Y \in \mathcal{C}$, and $(X \bullet Y, q)$ is also a coequalizer in $\mathcal{C}$, provided that $A$ is right coflat, $X \in \mathcal{C}_A$ and $Y \in \mathcal{C}_A$. Furthermore, the morphism $\nu$ that turns $X \bullet Y$ into a left $A$-module is uniquely determined by the equality $\nu \circ AQ = q \circ \nu Y$, so it coincides to the one defined by Eq. 2.9 in the case when $Y$ is robust as a left $A$-module.

By combining Proposition 2.5 (1) with its left handed version we get that the forgetful functor from $\mathcal{C}$ to $\mathcal{C}$ creates the coequalizers which are preserved by the functors $A \circ$ and $AA \circ$. Therefore, if $A$ is coflat (i.e. simultaneously left and right coflat) and $X, Y \in \mathcal{C}$ then $X \bullet Y \in \mathcal{C}_A$ (with the same structure morphisms as in the case when $Y$ is left $A$-robust) such that $(X \bullet Y, q)$ is a coequalizer in $\mathcal{C}_A$, too.

Let $A$ be a left coflat algebra in $\mathcal{C}$, and let $\mathcal{C}_A$ be the full subcategory of $\mathcal{C}$ consisting of bimodules that are left coflat as objects in $\mathcal{C}$, and robust as left $A$-modules. Our aim is to show that $\mathcal{C}_A$ is a monoidal category, with tensor product $\otimes_A$ and unit object $A$.

**Lemma 2.7** Let $A$ be left coflat. If $X, Y \in \mathcal{C}_A$, then $X \bullet Y$ is left coflat.

**Proof** It is easy to show that the tensor product (in $\mathcal{C}$) of two left coflat objects is left coflat. Let $(P, h)$ be the coequalizer of two parallel morphisms $f, g : M \to N$ in $\mathcal{C}$. We have to show that $(P(X \bullet Y), h(X \bullet Y))$ is the coequalizer of $f(X \bullet Y), g(X \bullet Y)$. Towards this end, consider the diagram

$$
\begin{array}{cccccc}
M XAY & \xrightarrow{hXAY} & N XAY & \xrightarrow{PXAY} & P XAY \\
MXY & \xrightarrow{hXY} & NXY & \xrightarrow{PY} & PXY \\
M(X \bullet Y) & \xrightarrow{h(X \bullet Y)} & N(X \bullet Y) & \xrightarrow{PQ} & P(X \bullet Y)
\end{array}
$$

The first two rows are coequalizers since $XY$ and $XAY$ are left coflat, and the three columns are coequalizers since $Y$ is left $A$-robust. The rectangles in the diagram commute, and everything follows now from [30, Lemma 2.10].

Recall that coequalizers are colimits, see for example [21, III.3]; in particular, the tensor product $X \bullet Y$ of $X \in \mathcal{C}_A$ and $Y \in \mathcal{C}_A$ is a colimit: consider the category $J$ with two objects
labelled \( x ay \) and \( xy \), and two non-identity arrows \( my \), \( xn \) : \( x ay \rightarrow xy \), and let \( F : J \rightarrow C \) be the following functor:

\[
F(xay) = XAY, \quad F(xy) = XY, \quad F(my) = \mu Y, \quad F(xn) = Xv.
\]

Cones from \( F \) to the vertex \( P \) in \( C \) correspond to morphisms \( f : XY \rightarrow P \) such that \( f \circ \mu Y = f \circ Xv \), and the colimit \( \text{Colim} F = (X \bullet Y, q) \) consists of an object \( X \bullet Y \in C \) and a universal cone \( q \) from \( F \) to \( X \bullet Y \).

We now generalize this construction. Let \( J_2 \) be the category with four objects \( xayaz \), \( xyaz \), \( xayz \) and \( xyz \), and morphisms

\[
\begin{align*}
& xayaz \\
\downarrow & xamz \\
& xayz \\
\end{align*}
\]

and their compositions, subject to the relations

\[
\begin{align*}
xmz \circ myaz &= myz \circ xamz = mmz; \\
xyn \circ xmaz &= myz \circ xayn = myn; \\
xmz \circ xnaz &= xnz \circ xamz = xmnz; \\
xyn \circ xnaz &= xnz \circ xayn = xnn.
\end{align*}
\]

Consider \( X \in \mathcal{A}_A \), \( Y \in \mathcal{A}_C \) and \( Z \in \mathcal{A}_C \). \( F_2 : J_2 \rightarrow C \) is defined in the following way: \( F_2(xayaz) = XAYAZ \), \( F_2(xyaz) = XYAZ \), \( \cdots \), \( F_2(xmaz) = X\mu AZ \) etc. We can also consider the full subcategory \( J'_2 \) of \( J_2 \), with objects \( xayz \), \( xayz \) and \( xyz \), and the restriction \( F'_2 \) of \( F_2 \) to \( J'_2 \). It is easy to establish that cones from \( F_2 \) to \( P \in C \) correspond bijectively to cones from \( F'_2 \) to \( P \), so that \( F_2 \) and \( F'_2 \) have the same colimit. We now define

\[
\text{Colim} F'_2 = (X \bullet Y \bullet Z, q_2).
\]

**Proposition 2.8** Let \( A \) be a left coflat algebra in \( C \), and consider \( X \in \mathcal{A}_A \) and \( Y, Z \in \mathcal{A}_C \). Then we have isomorphisms of cones

\[
(X \bullet Y \bullet Z, q_2) \cong (X \bullet (Y \bullet Z), q \circ Xq) \cong ((X \bullet Y) \bullet Z, q \circ q Z).
\]

If \( X \) is an \( A \)-bimodule, then \( F_2 \) and \( F'_2 \) corestrict to functors with values in \( \mathcal{A}_C \), and the above cones are also the colimits of these corestrictions.

**Proof** Consider the diagram

\[
\begin{align*}
XAYAZ & \xrightarrow{XAYv} XAYZ & \xrightarrow{XAQ} XA(Y \bullet Z) \\
\downarrow & \downarrow & \downarrow \\
\mu YAZ & \xrightarrow{X\mu AZ} \xrightarrow{XvZ} \xrightarrow{\mu YZ} \xrightarrow{\mu (Y \bullet Z)} Xv \\
\downarrow & \downarrow & \downarrow & \downarrow \\
XYAZ & \xrightarrow{XYv} \xrightarrow{XY} \xrightarrow{Xq} X(Y \bullet Z) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
qAZ & \xrightarrow{qZ} \xrightarrow{q} \xrightarrow{q} \xrightarrow{(X \bullet Y)v} (X \bullet Y)Z \xrightarrow{\exists f} X \bullet (Y \bullet Z)
\end{align*}
\]
The two top rows are coequalizers since $Z$ is robust as a left $A$-module, and the two left columns are coequalizers since $AZ$ and $Z$ are left coflat. Clearly the third column is a coequalizer, too.

The two rectangles in the top right corner of the diagram commute, mostly because of the definition of the left action $\nu$ on $Y \cdot Z$; see Eq. 2.9. Similarly, the rectangles in the left side of the diagram commute. Thus, by using the fact that the central column is a coequalizer, we get a unique morphism $f : (X \cdot Y)Z \to X \cdot (Y \cdot Z)$ such that $f \circ qZ = q \circ Xq$.

By applying again [30, Lemma 2.10], we deduce that the third row of the diagram is a coequalizer as well. So $(X \cdot (Y \cdot Z), f)$ is a coequalizer for the parallel morphisms $\mu Z$ and $(X \cdot Y)\nu$. But the same is $((X \cdot Y) \cdot Z, q)$, and therefore there exists an isomorphism $\alpha : X \cdot (Y \cdot Z) \to (X \cdot Y) \cdot Z$ satisfying $\alpha \circ f = q$.

According to [16, Lemma 0.17], the diagonal of the diagram is a coequalizer and it can be easily seen that this coequalizer coincides to $(X \cdot Y \cdot Z, q)$. From the above it follows that $(X \cdot (Y \cdot Z), q \circ Xq)$ and $((X \cdot Y) \cdot Z, q \circ qZ)$ are also coequalizers of the diagonal of the diagram, and this gives us the desired isomorphisms.

Finally, it follows from Proposition 2.5 that $(X \cdot (Y \cdot Z), q \circ Xq)$ is the colimit of the corestriction of $F'_2$ to $A^C_A$. Similar arguments show that $((X \cdot Y) \cdot Z, q \circ qZ)$ is a colimit of $F'_2$, so we are done.

Proposition 2.9 If $A$ is left coflat, then $A^C_A$ is closed under the tensor product over $A$. 

Take $X, Y, Z \in A^C_A$. It follows from the universal property of colimits that there exists a unique isomorphism

$$\alpha = \alpha_{X,Y,Z} : X \cdot (Y \cdot Z) \to (X \cdot Y) \cdot Z$$

in $A^C_A$ such that the diagram

$$
\begin{array}{c}
XYZ \\
\downarrow{Xq} \\
X(Y \cdot Z) \\
\downarrow{X(\nu)} \\
X(Y \cdot Z) \\
\downarrow{XqZ} \\
(X \cdot Y)Z \\
\downarrow{\alpha} \\
(X \cdot Y) \cdot Z \\
\end{array}
$$

commutes. The diagram

$$
\begin{array}{c}
XYZ \\
\downarrow{Xq} \\
X(Y \cdot Z) \\
\downarrow{Xq} \\
X(Y \cdot Z) \\
\downarrow{XqZ} \\
(X \cdot Y) \cdot Z \\
\downarrow{\alpha} \\
(X \cdot Y) \cdot Z \\
\end{array}
$$

commutes. Indeed, the commutativity of the pentangle follows from Eq. 2.10 combined with Eq. 2.5; the triangle commutes: this is the definition of $\theta$. Then we compute that

$$\alpha \circ q \circ \theta \circ q = \alpha \circ q \circ Xq = q \circ Z \circ q,$$

hence $\alpha \circ q \circ \theta = Z \circ q$, so the rectangle commutes.

Proposition 2.9 If $A$ is left coflat, then $A^C_A$ is closed under the tensor product over $A$. 

\[ \text{Springer} \]
Proof. Take $Y,Z \in \overset{1}{\mathbf{C}_A}$. We know from Lemma 2.7 that $Y \bullet Z$ is left coflat. We are left to show that $Y \bullet Z$ is robust as a left $A$-module. Take $M \in \mathcal{C}$ and $X \in \mathcal{C}_A$ and consider the diagram

\begin{equation}
\begin{array}{ccc}
MX(Y \bullet Z) & \xrightarrow{q} & (MX) \bullet (Y \bullet Z) \\
\downarrow \theta^{-1} & & \downarrow \alpha \\
MXY \bullet Z & \xrightarrow{(Mq) \bullet Z} & (MX) \bullet (Y \bullet Z) \\
\downarrow q & & \downarrow \theta \\
M((XY) \bullet Z) & \xrightarrow{Mq} & M((X \bullet Y) \bullet Z) \\
\downarrow M\theta & & \downarrow M\alpha^{-1} \\
MX(Y \bullet Z) & \xrightarrow{Mq} & M(X \bullet (Y \bullet Z)) \\
\end{array}
\end{equation}

(2.12)

Commutativity of the top and bottom triangles and rectangles follows from Eq. 2.11. The commutativity of the two remaining triangles follows from the definition of $\theta$, and the commutativity of the two remaining quadrangles follows from Eq. 2.5. We conclude that the whole diagram commutes. Now let

$$\Theta = M\alpha^{-1} \circ \theta \circ \theta \bullet Z \circ \alpha : (MX) \bullet (Y \bullet Z) \to M(X \bullet (Y \bullet Z)).$$

$\Theta$ is an isomorphism, and $\Theta \circ q \circ MXq = Mq \circ MXq$, so that $\Theta \circ q = Mq$. It follows from the (reformulation of) Definition 2.3 that $Y \bullet Z$ is robust as a left $A$-module.

Theorem 2.10 Let $A$ be a left coflat algebra in $\mathcal{C}$. Then we have a monoidal category $(\overset{1}{\mathbf{C}_A}, \otimes_A = \bullet, A, \alpha, \lambda, \rho)$. The category $\mathcal{C}_A$ is a right $\overset{1}{\mathbf{C}_A}$-category.

Proof. We know from Proposition 2.4 that $A$ is robust as a left $A$-module, so $A \in \overset{1}{\mathbf{C}_A}$.

We have shown in Proposition 2.9 that the tensor product over $A$ of two objects in $\overset{1}{\mathbf{C}_A}$ is again in $\overset{1}{\mathbf{C}_A}$. The associativity constraint $\alpha$ was defined as an application of Proposition 2.8.

The unit constraint follows as an application of Proposition 2.1. $(X, \mu)$ and $(X \bullet A, q)$ are both coequalizers in $\mathcal{C}$ (and in $\mathbf{C}_A$) of $Xm, \mu A : XAA \to XA$, so there exists a unique isomorphism $\rho_X : X \bullet A \to X$ in $\mathbf{C}_A$ such that $\rho_X \circ q = \mu$, with inverse $\rho_X^{-1} = q \circ \eta_X$. In a similar way, we have a unique isomorphism $\lambda_X : A \bullet X \to X$ in $\mathbf{C}_A$ such that $\lambda_X \circ q = \nu$, with inverse $\lambda_X^{-1} = q \circ \eta X$. We are left to show that the coherence conditions are satisfied.

Take $X, Y, Z, T \in \overset{1}{\mathbf{C}_A}$. We have to show that the following diagrams commute.

\begin{equation}
\begin{array}{ccc}
X \bullet (A \bullet Z) & \xrightarrow{\alpha_{X,A,Z}} & (X \bullet A) \bullet Z \\
\downarrow X \bullet \rho & & \downarrow X \bullet \rho \\
X \bullet Z & & \\
\end{array}
\end{equation}

(2.13)
\[ \alpha_{X,A,Z} \text{ is the unique morphism that makes the diagram (2.10) commutative. If we can show that } (\rho_X \bullet Z)^{-1} \circ X \bullet \lambda_Z \text{ has the same property, then it follows that Eq. 2.13 commutes. This means that we have to show that the diagram} \]

\[ XAZ \xrightarrow{Xq} X(A \bullet Z) \xrightarrow{q} X \bullet (A \bullet Z) \]

\[ (X \bullet A)Z \xrightarrow{q} (X \bullet A) \bullet Z \xrightarrow{\rho_X \bullet Z} X \bullet Z \]

\[ \text{commutes. This is an easy computation:} \]

\[ X \bullet \lambda_Z \circ q \circ Xq = q \circ X \lambda_Z \circ Xq = q \circ Xv = q \circ \mu Z = q \circ \rho_X \circ q \circ Z = \rho_X \bullet Z \circ q \circ qZ. \]

Now consider the category \( J^\prime_3 \), consisting of four objects and six morphisms that are not identities:

\[ \text{We define } F^\prime_3 : J^\prime_3 \rightarrow C \text{ in the obvious way: } F^\prime_3(xyazt) = XyAZT, F^\prime_3(xmzt) = X\mu ZT, \text{ etc. The fourfold tensor product is defined as the colimit of } F^\prime_3: \text{Colim}_{F^\prime_3} = (X \bullet Y \bullet Z \bullet T, q_3). \]

Proceeding as in Proposition 1.8, we can show that \((X \bullet (Y \bullet (Z \bullet T))), q \circ Xq \circ XYq\), \(((X \bullet Y) \bullet (Z \bullet T), q \circ qT \circ qZT), (X \bullet ((Y \bullet Z) \bullet T), q \circ Xq \circ XqT)\) and \((X \bullet ((Y \bullet Z) \bullet T), q \circ qT \circ XqT)\) are all colimits of \( F^\prime_3 \) (and of the corestriction of \( F^\prime_3 \) to \( A \mathcal{C}_A \)). This means that these five cones are isomorphic. For example, the isomorphism between the first two cones is the unique morphism that makes the diagram

\[ \begin{array}{c}
XZT \xrightarrow{Xq} X(Y \bullet (Z \bullet T)) \xrightarrow{q} X \bullet (Y \bullet (Z \bullet T)) \\
\end{array} \]

\[ \text{commutative. Here we used the equality } qq = q(Z \bullet T) \circ XYq. \]

\[ \text{In view of Eq. 2.10, this morphism is } \alpha_{X,Y,Zt}. \text{ In a similar way, we can prove that the maps in the diagram (2.14) establish isomorphisms between the 5 coequalizers above. Therefore the two compositions in the diagram also establish isomorphisms between coequalizers, hence they are equal, since these isomorphisms are unique. This tells us that Eq. 2.14 commutes.} \]

A coalgebra \( C \) in \( \mathcal{C}_A \) is called an \( A \)-coring. The category of right \( C \)-comodules and right \( C \)-colinear morphisms in \( \mathcal{C}_A \) is denoted by \( \mathcal{C}^C \).
3 Entwined Modules over Cowreaths

A (strict) monoidal category \( \mathcal{C} \) can be viewed as a 2-category with a single 0-cell, hence we can consider the 2-categories \( \text{Mnd}(\mathcal{C}) \) [26] and \( \text{EM}(\mathcal{C}) \) [19]. These have the same 0-cells and 1-cells, but are different at the level of 2-cells. The 0-cells are algebras (or monads) in \( \mathcal{C} \). Fix an algebra \( A \) in \( \mathcal{C} \) and consider the endomorphism categories

\[
\mathcal{T}(\mathcal{C})_A = \mathcal{T}_A = \text{Mnd}(\mathcal{C})(A, A) \quad \text{and} \quad \mathcal{T}(\mathcal{C})^\#_A = \mathcal{T}_A^\# = \text{EM}(\mathcal{C})(A, A),
\]

where \( A \) is regarded as a 0-cell in \( \text{EM}(\mathcal{C}) \) as in [6, Proposition 4.1].

The notation \( \mathcal{T}(\mathcal{C})_A \) is taken from [29], where \( \mathcal{T}(\mathcal{C})_A \) appears in a different context, and where it is called the category of right transfer morphisms through \( A \). \( \mathcal{T}_A \) and \( \mathcal{T}_A^\# \) are (strict) monoidal categories. A monad in \( \text{Mnd}(\mathcal{C}) \) is called a distributive law [26, Sec. 6] or a factorization structure. A comonad in \( \text{Mnd}(\mathcal{C}) \) is called a mixed distributive law or an entwining structure [2]. A monad in \( \text{EM}(\mathcal{C}) \) is called a wreath in \( \mathcal{C} \) [19], an alternative name suggested in [19] is generalized distributive law. A comonad in \( \text{EM}(\mathcal{C}) \) was called in [6] a cowreath in \( \mathcal{C} \), or a mixed wreath in [28]; we can also refer to it as a generalized entwining structure. Note also that the notion of cowreath appears, with a different meaning, in [17], too. Namely, to a bicategory \( \mathcal{B} \) it is associated the so called Eilenberg-Moore bicategory of comonads \( \text{REM}(\mathcal{B}) \), a dual version of \( \text{EM}(\mathcal{B}) \): the objects (0-cells) of \( \text{REM}(\mathcal{B}) \) are comonads in \( \mathcal{B} \) (for \( \text{EM}(\mathcal{B}) \) the objects are the monads in \( \mathcal{B} \)), and so on. Then a cowreath in \( \mathcal{B} \) in the sense of [17] is nothing but a comonad in \( \text{REM}(\mathcal{B}) \).

With our definition, a cowreath in \( \mathcal{C} \) consists of an algebra \( A \) in \( \mathcal{C} \) and a coalgebra in \( \mathcal{T}_A^\# \). In a similar way, a wreath in \( \mathcal{C} \) consists of an algebra \( A \) in \( \mathcal{C} \) and an algebra in \( \mathcal{T}_A^\# \). For later use, we spell out the explicit definition of a cowreath.

3.1 The Monoidal Categories \( \mathcal{T}_A \) and \( \mathcal{T}_A^\# \)

Let \( A \) be an algebra in \( \mathcal{C} \). A (right) transfer morphism through \( A \) is a pair \( X = (X, \psi) \), with \( X \in \mathcal{C} \) and \( \psi : XA \to AX \) in \( \mathcal{C} \) such that \( \psi \circ Xm = Xm \circ A\psi \circ \psi A \) and \( \psi \circ X\eta = \eta X \); in diagrammatic notation

\[
\psi = \begin{array}{c|c}
X & A \\
\hline
A & X \\
\end{array}
\]

satisfies (a)

\[
\begin{array}{c|c}
X & A \\
\hline
A & X \\
\end{array}
\]

and (b)

\[
\begin{array}{c|c}
X & A \\
\hline
A & X \\
\end{array}
\]

\[
\text{(3.1)}
\]

The categories \( \mathcal{T}_A \) and \( \mathcal{T}_A^\# \) coincide at the level of objects; their objects are right transfer morphisms through \( A \). A morphism \( X \to Y \) in \( \mathcal{T}_A \) is a morphism \( f : X \to Y \) in \( \mathcal{C} \) such that \( \psi \circ f A = Af \circ \psi \). A morphism \( X \to Y \) in \( \mathcal{T}_A^\# \) is a morphism \( f : X \to AY \) in \( \mathcal{C} \) such that

\[
mY \circ Af \circ \psi = mY \circ A\psi \circ f A.
\]

(3.2)

The composition of two morphisms \( f : X \to Y \) and \( g : Y \to Z \) in \( \mathcal{T}_A \) is \( g \circ f = mZ \circ Ag \circ f \). The identity on \( (X, \psi) \) is \( \eta X \). The tensor product of \( X \) and \( Y \) is \( XY = (XY, \psi_X \cdot \psi_Y = \psi_X Y \circ \psi_Y) \). The tensor product of \( f : X \to X' \) and \( g : Y \to Y' \) in \( \mathcal{T}_A^\# \) is given by the composition \( mXY \circ A\psi Y \circ fg \). The unit object is \( (1, A) \). \( \mathcal{T}_A \) and \( \mathcal{T}_A^\# \) are strict monoidal categories, and we have a strong monoidal functor \( F : \mathcal{T}_A \to \mathcal{T}_A^\# \), which is the identity on objects, and \( F(f) = \eta f \), for \( f : X \to Y \) in \( \mathcal{T}_A \). If a morphism in \( \mathcal{T}_A^\# \) is of the form \( \eta f \), with \( f : X \to Y \) in \( \mathcal{C} \), then \( f \) is a morphism in \( \mathcal{T}_A \).
In a similar way we introduce left transfer morphisms through \( A \), consisting of pairs \( X = (X, \varphi) \), with \( \varphi = \frac{A X}{X A} : AX \toXA \). We leave it to the reader to write down the precise definition of the categories \( _A^A \mathcal{T} \) and \( ^A_A \mathcal{T} \). In fact \( _A^A \mathcal{T} = \text{Mnd}(\mathcal{C}^{\text{op}})(A, A) \) and \( ^A_A \mathcal{T} = \text{EM}(\mathcal{C}^{\text{op}})(A, A) \).

The tensor product in \( _A^A \mathcal{T} \) and \( ^A_A \mathcal{T} \) is given by the formula \( XY = (XY, \varphi_X \cdot \varphi_Y = X\varphi_Y \circ \varphi_X Y) \).

### 3.2 Cowreaths

A cowreath (mixed wreath or generalized entwining structure) in \( \mathcal{C} \) is a triple \( (A, X, \psi) \), where \( A \) is an algebra in \( \mathcal{C} \), and \( (X, \psi) \) is a coalgebra in \( \mathcal{T}_{A}^{\#} \), which is an object \( (X, \psi) \in \mathcal{T}_{A}^{\#} \) together with morphisms

\[
\delta = \begin{array}{c}
\xymatrix{ X \\
A \ar[u]^X \ar[r]_(0.45)X & AX \\
A \ar[u]_X \ar[r]_X & X \\
A 
}
\end{array} : X \to AX X,
\quad
\epsilon = \begin{array}{c}
\xymatrix{ X \\
A \ar[u]^X \\
A 
}
\end{array} : X \to A
\]

in \( \mathcal{C} \) such that the following relations hold:

\[
\begin{align*}
(a) & \quad \delta : X \to AX X, \\
(b) & \quad \epsilon : X \to A
\end{align*}
\]

Conditions (a) and (c) mean that \( \delta \) and \( \epsilon \) define morphisms \( X \to AX \) and \( X \to 1 \) in \( \mathcal{T}_{A}^{\#} \). (b) is the coassociativity of the comultiplication \( \delta \) and (d) and (e) are the left and right counit property.

### 3.3 Entwined Modules over Cowreaths

Let \( \mathcal{D} \) be a right \( \mathcal{C} \)-category, and let \( A \) be an algebra in \( \mathcal{C} \). Then \( \mathcal{D}_A \) is a right \( \mathcal{T}_A \)-category, see [8, Prop. 4.3]. We will now show that it is also a right \( \mathcal{T}_A^{\#} \)-category.
Proposition 3.1 Let $A$ be an algebra in a (strict) monoidal category $C$, and let $D$ be a (strict) right $C$-category. Then $D_A$ is a right $\mathcal{T}_A^\#$-category. The tensor product of $N \in D_A$ and $X \in \mathcal{T}_A^\#$ is given by the formula $N \diamond X = (NX, \mu_{NX} = \mu_X \circ N \psi)$. The tensor product of $f : N \to M$ in $D_A$ and $g : X \to Y$ in $\mathcal{T}_A^\#$ is given by the formula $f \diamond g = \mu_Y \circ fg$.

Proof It is an easy exercise left to the reader. A more conceptual proof is the following. As $C$ acts on the right on $D$, we can view both as forming a bicategory $B$ with two objects, say 0 and 1, with $B(1,1) = C, B(1,0) = D$ and $B(0,0) = 1$. For $A$ an algebra in $C$, $(1,A)$ can be regarded as an object of $EM(B)$, and we have that $EM(C)(A,A) = EM(B)((1,A),(1,A))$ and $D_A = EM(B)((0,1),(1,A))$. Thus the above action of $\mathcal{T}_A^\#$ on $D_A$ is just the composition functor in $EM(B)$.

Proposition 3.1 justifies the following definition.

Definition 3.2 Let $(A, X, \psi)$ be a cowreath in $C$. An entwined module in $D$ over $(A, X, \psi)$ is a right $(X, \psi)$-comodule in $D_A$.

An entwined module over $(A, X, \psi)$ consists of an object $M \in D_A$ and a morphism $\rho : M \to MX$ in $D_A$ satisfying
\[
\rho X \circ \rho = \mu XX \circ M \delta \circ \rho; \tag{3.4}
\]
\[
\mu \circ M \epsilon \circ \rho = M. \tag{3.5}
\]

Equation 3.4 is the coassociativity of the coaction, and Eq. 3.5 is the counit property. The fact that $\rho$ is right $A$-linear is expressed by the formula
\[
\rho \circ \mu = \mu X \circ M \psi \circ \rho A. \tag{3.6}
\]

A mixed distributive law (or entwining structure) $(A, X, \psi)$ can be considered as a cowreath (or generalized entwining structure): take $\delta = \eta \Delta$ and $\epsilon = \eta \epsilon$. It is easy to see that entwined modules over $(A, X, \psi)$ considered as a mixed distributive law coincide with entwined modules over $(A, X, \psi)$ considered as a monoidal cowreath.

A morphism between two entwined modules $M$ and $N$ is a right $A$-linear morphism $f : M \to N$ such that $fX \circ \rho = \rho \circ f$. The category of entwined modules in $D_A$ over $(X, \psi, \delta, \epsilon)$ will be denoted as $D(\psi)^X_A$.

4 Wreaths, Wreath Product Algebras and Duality

4.1 Duality Between Left and Right Transfer Morphisms

Theorem 4.1 Let $A$ be an algebra in a (strict) monoidal category $C$. Take $X \in \mathcal{T}_A$, and assume that $X \dashv Y$ in $C$. Consider
\[
\varphi = \begin{array}{c}
A \\
\downarrow Y
\end{array}^X \quad , \quad \varphi = Y Ad \circ Y \psi Y \circ bAY = \begin{array}{c}
A \\
\downarrow Y \end{array}^X : AY \to YA. \tag{4.1}
\]

Then $(Y, \varphi) \in \mathcal{T}_A^\#$. If $C$ has right duality, then we have strong monoidal functors $^*(-) : \mathcal{T}_A^\# \to ^\# \mathcal{T}_A^{oprev}$ and $^*(-) : \mathcal{T}_A \to ^\# \mathcal{T}_A^{oprev}$.

\( \square \) Springer
Proof We first compute that

\[
\begin{align*}
A^\ast(XX') & \xrightarrow{\psi \cdot \psi'} * (XX') A \\
A \varphi_2(X,X') & \xrightarrow{\psi' \cdot \varphi} * (X'X) A \\
A^{*\ast X} & \xrightarrow{\psi' \cdot \varphi} * (X'X) A
\end{align*}
\]

It follows immediately from Eqs. 2.1, 3.1 that \( \varphi \circ \eta Y = Y \eta \), hence \((Y, \varphi) \in A T\).

\( \varphi \) is independent of the choice of the right adjoint \( Y \) of \( X \) in the following sense. If \((X, Y, b', d')\) is another adjunction, leading to \( \varphi' : AY' \to Y'A \), then it follows from Eq. 2.2. that

\[
\lambda A \circ \varphi = \varphi' \circ A \lambda,
\]

where \( \lambda = Y'd \circ b'Y : Y \to Y' \).

Let \((X, \psi), (X', \psi') \in T_A, X \dashv Y \) and \( X' \dashv Y' \). For \( f : X \to X' \) in \( T_A^\# \),

\[
g = YAd' \circ YfY' \circ bY' : Y' \to YA
\]

is a morphism \( g : Y' \to Y \) in \( T^\# \).

Assuming that \( C \) has right duality, and fixing a right dual \( *X \) for every \( X \in C \), we obtain a functor \( *(-) : T_A^\# \to \# T^\text{op} \), putting \( *X, \psi = (\ast X, \varphi) \) and \( *f = g \). We leave it to the reader to verify that \( *(f' \bullet f) = *f \circ *f' \) and \( *\text{Id}(X, \psi) = \text{Id}(\ast X, \varphi) \).

Let us finally show that \( *(-) \) is strong monoidal as a functor from \( T_A^\# \) to \( \# T^\text{op}. \) It suffices to show that, for \( X, X' \in T_A, \varphi_2(X, X') : *X X' \to *X' X \) defines an isomorphism \( *(XX'), \psi \cdot \psi' \to (\ast XX', \varphi' \cdot \varphi) \) in \( A T \), and, a fortiori, in \( A T \). We have \( X \dashv Y = \ast X, X' \dashv Y' = \ast X' \) and \( XX' \dashv YY' \), and we claim that \( \varphi' \cdot \varphi = \psi' \cdot \psi. \) To this end it suffices to observe that the following diagram commutes.
commutes, which is precisely what we need.

\[ \square \]

### 4.2 Factorization Structures

**Definition 4.2** Let \( C \) be a (strict) monoidal category. A left wreath (or left generalized factorization structure) in \( C \) is a triple \((A, X, \psi)\), where \( A \) is an algebra in \( C \), and \((X, \psi)\) is an algebra in \( \mathcal{T}_A^\# \). A right wreath is a triple \((A, Y, \varphi)\), where \( A \) is an algebra in \( C \) and \((Y, \varphi)\) is an algebra in \( \mathcal{T}_A^\# \).

Explicitly, a right wreath is a triple \((A, Y, \varphi)\), where \( A \) is an algebra in \( C \), and \((Y, \varphi) \in \mathcal{T}_A^\#\), together with morphisms

\[
\begin{align*}
\eta_Y & : 1 \to YA \\
m_Y & : YY \to YA
\end{align*}
\]

in \( C \) such that

\[
\begin{align*}
(a) & \quad AYA = AYA \\
(b) & \quad YYY = YYA \\
(c) & \quad AYA = AYA \\
(d) & \quad YeYA = YrYA \\
(e) & \quad YYA = YrYA
\end{align*}
\]

\[
(4.4)
\]

\( m_A \) and \( \eta_A \) are the multiplication and unit of the algebra \((Y, \varphi)\). (a) and (c) express the fact that \( m_Y : YY \to Y \) and \( \eta_Y : 1 \to Y \) are morphisms in \( \mathcal{T}_A^\# \); (b) is the associativity and (d) and (e) are the unit conditions.

If \((A, Y, \varphi)\) is a right wreath, then \( YA \) is an algebra in \( C \) with multiplication

\[
\begin{align*}
\eta_\#: & \quad 1 \to YA \\
m_\#: & \quad YY \to YA
\end{align*}
\]

and unit \( \eta_\# = \eta_Y : 1 \to YA \), see for example [6]. In the literature, this algebra is called the wreath product or generalized smash product, and is denoted as \( Y\#_\varphi A \).
If \( F : \mathcal{C} \to \mathcal{D} \) is strong monoidal, and \( C \) is a coalgebra in \( \mathcal{C} \), then \( F(C) \) is a coalgebra in \( \mathcal{D} \) with comultiplication and counit given by the formulas

\[
\Delta_{F(C)} = \varphi_2^{-1}(C, C) \circ F(\Delta); \quad \varepsilon_{F(C)} = \varphi_0^{-1} \circ F(\varepsilon).
\] (4.5)

Let \((A, X, \psi)\) be a cowreath, and assume that \( X \vdash Y = \ast X \) in \( \mathcal{C} \). Then \((X, \psi)\) is a coalgebra in \( \mathcal{T}_A^\# \), and \((Y, \varphi)\) is coalgebra in \( \# \mathcal{T}^{oprev} \), by Theorem 4.1, and therefore an algebra in \( \# \mathcal{T} \), so that \((A, Y, \varphi)\) is a right wreath. We compute the multiplication and unit using Eq. 4.3, with \( f : X \to X' \) in \( \mathcal{T}_A^\# \) replaced by \( \delta : X \to XX \) and \( \epsilon : X \to 1 \). We find that

\[
m_Y = YAd^2 \circ Y\delta YY \circ bYY : YY \to YA \quad \text{and} \quad \eta_Y = Y\epsilon \circ b.
\]

This proves the first part of Proposition 4.3. The proof of the second part is similar and is left to the reader. Note also that a different proof can be given by using the techniques used in [28].

**Proposition 4.3** Let \( \mathcal{C} \) be a (strict) monoidal category.

(i) If \((A, X, \psi)\) is a cowreath and \( X \vdash Y \) in \( \mathcal{C} \), then \((A, Y, \varphi)\), with \( \varphi \) given by Eq. 4.1, is a right wreath, with multiplication \( m_Y \) and unit \( \eta_Y \) given by the formulas

\[
m_Y = \begin{array}{c}
Y \\
\end{array} and \eta_Y = \begin{array}{c}
1 \\
Y \ A
\end{array}.
\]

The wreath product \( YA \) is an algebra in \( \mathcal{C} \), with structure maps

\[
m_\# = \begin{array}{c}
Y \ A \ Y \ A
\end{array} and \eta_\# = \begin{array}{c}
1 \\
Y \ A
\end{array}.
\] (4.6)

(ii) If \((A, X, \psi)\) is a left wreath then \((A, Y, \varphi)\) is a left cowreath (a coalgebra in \( \#_A^T \)), with comultiplication and counit given by the formulas

\[
\delta = \begin{array}{c}
Y
\end{array} and \bar{\epsilon} = \begin{array}{c}
Y
\end{array}.
\]
4.3 Modules Versus Entwined Modules

Theorem 4.4 is the main result of this Subsection. It is a generalization of [15, Cor. 6.3] and its proof follows from Proposition 4.3 and an old result of Eilenberg-Moore recalled in Section 1 of the paper [26]. This is why we only define the functors that provide the desired isomorphism of categories, leaving the details to the reader.

Take \( M \in \mathcal{D}(\psi)_X^A \). The coaction \( \rho : M \to MX \) is right \( A \)-linear, and satisfies (3.4) and (3.5). In diagrammatic notation, these conditions take the form

\[
\begin{align*}
M & \xrightarrow{\rho} MX \\
M_C & \xrightarrow{\mu} M
\end{align*}
\]

and

\[
\begin{align*}
M & \xrightarrow{\rho} MX \\
M & \xrightarrow{\rho} M
\end{align*}
\]

(4.7)

Theorem 4.4 Let \( A \) be an algebra in a (strict) monoidal category \( C \), let \( D \) be a right \( C \)-category and let \( (A, X, \psi) \) be a cowreath in \( C \). If \( X \dashv Y \) in \( C \) then the categories \( \mathcal{D}(\psi)_X^A \) and \( \mathcal{D}_YA \) are isomorphic.

Proof We have a functor \( F : \mathcal{D}(\psi)_X^A \to \mathcal{D}_YA \). For \( M \in \mathcal{D}(\psi)_X^A \), \( F(M) = M \in \mathcal{C}_YA \) via

\[
\mu = \begin{array}{c}
M \\
Y \xrightarrow{\mu} A
\end{array}
\]

(4.8)

We can also define a functor \( G : \mathcal{D}_YA \to \mathcal{D}(\psi)_X^A \). \( G(M) = M \in \mathcal{D}(\psi)_X^A \) via

\[
\begin{align*}
\mu & : M \xrightarrow{\mu} A \\
\rho & : M_X \xrightarrow{\rho} M
\end{align*}
\]

(4.9)

It can be seen easily that the functors \( F \) and \( G \) are inverses, and this completes the proof.

5 Frobenius Functors Versus Frobenius Coalgebras

5.1 Frobenius Functors

Throughout this Section \( (A, X, \psi) \) is a cowreath in a (strict) monoidal category \( C \). Recall that a Frobenius functor is a functor having left and right adjoints which are naturally equivalent. The aim of this Section is to investigate when the forgetful functor \( F : \)
\( \mathcal{C}(\psi)^X_A \rightarrow C_A \) is Frobenius. Lemma 5.1 tells us that \( F \) always has a right adjoint \( G \), so that our problem reduces to examining whether \( G \) is a left adjoint of \( F \).

**Lemma 5.1** Let \( \mathcal{D} \) be a right \( \mathcal{C} \)-category. The forgetful functor \( F : \mathcal{D}_A \rightarrow \mathcal{D}(\psi)^X_A \), defined as follows: \( G(N) = NX \) is an object of \( \mathcal{D}(\psi)^X_A \) via

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix@R=2pc@C=2pc{N \ar[r] & X \\
\ar@{.>}[r] & A}
\end{array}
\end{array}
\]

Lemma 5.1 tells us that \( F \) always has a right adjoint \( G \), so that our problem reduces to examining whether \( G \) is a left adjoint of \( F \).

**Proof** The unit and the counit of the adjunction are given by the formulas, \( \eta_M = \rho : M \rightarrow GF(M) = MX \) and \( \varepsilon_N = \mu \circ N \varepsilon : FG(N) = NX \rightarrow N \), for all \( M \in \mathcal{D}(\psi)^X_A \) and \( N \in \mathcal{D}_A \).

\[ \square \]

### 5.2 Frobenius Coalgebras

The notion of Frobenius algebra in a monoidal category (as introduced in [27], see also [9, Def. 4.1]) can be dualized: a coalgebra in a monoidal category is Frobenius if and only if the corresponding algebra in the opposite category is Frobenius. This leads to the following definition.

**Definition 5.2** A coalgebra \( C \) in \( \mathcal{C} \) is called Frobenius if there exists a Frobenius system \((t, B)\) consisting of morphisms \( t : 1 \rightarrow C \) (the Frobenius element) and \( B : CC \rightarrow 1 \) (the Casimir morphism) in \( \mathcal{C} \) such that

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix@R=2pc@C=2pc{C \ar[r] & C \\
\ar@{.>}[r] & B}
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\xymatrix@R=2pc@C=2pc{C \ar[r] & C \\
\ar@{.>}[r] & B}
\end{array}
\end{array}
\]

(5.1)

**Remark 5.3** Several equivalent characterizations of a Frobenius algebra are known, see for example [9, Theorem 5.1]. Now a Frobenius coalgebra is a Frobenius algebra in the opposite category, and this leads to the following equivalent characterizations of a Frobenius coalgebra.

(i) There is an adjunction \( C \dashv A \), and \( C \) is isomorphic to \( A \) as a right \( A \)-module. Recall from Eqs. 2.3–2.4 that \( A \) is an algebra, and that \( C \) is a right \( A \)-module.

(ii) There is an adjunction \( A' \dashv C \) and \( C \) is isomorphic to \( A' \) as a left \( A' \)-module.

(iii) \( C \) is an algebra in \( \mathcal{C} \) with \( C \)-bicolinear multiplication.

(iv) There is an adjunction \( A' \dashv C \) and there exists a balanced right non-degenerate morphism \( B_r : CC \rightarrow 1 \) in \( \mathcal{C} \). This means that \( CB_r \circ \Delta_C = B_r \Delta_C \circ C \Delta_C \) (balanced) and that \( \Phi_{B_r} = B_r A' \circ Cb : C \rightarrow A' \) is an isomorphism (non-degenerate).

(v) There is an adjunction \( C \dashv A \) and there exists a balanced left non-degenerate morphism \( B_l : CC \rightarrow 1 \) in \( \mathcal{C} \). The fact that \( B_l \) is left non-degenerate means that \( \Psi_{B_l} = AB_l \circ b'C : C \rightarrow A \) is an isomorphism.

(vi) There is an adjunction \((b, d) : C \dashv C \) such that \( C \Delta \circ b = \Delta_C \circ b \).

(vii) There is an adjunction \((b, d) : C \dashv C \) such that \( b = \Delta \circ t \), for some \( t : 1 \rightarrow C \) in \( \mathcal{C} \).
A Frobenius coalgebra $C$ is also a Frobenius algebra (and vice versa), with multiplication $m = CB \circ \Delta C = BC \circ C \Delta$, unit $\eta = t$, and Frobenius system $(\varepsilon, \Delta \circ \eta)$.

Specializing Definition 5.2 to coalgebras in $T^A$, we obtain the following result.

**Lemma 5.4** A coalgebra $(X, \psi)$ in $T_A^#$ is Frobenius if and only if there exist morphisms $t : 1 \to AX$ and $B : XX \to A$ in $C$ such that

\begin{align*}
(a) & \quad A t = t A X; \\
(b) & \quad X X A B = X X A e A; \\
(c) & \quad X X \delta B = X X \delta e A; \\
(d) & \quad X t B = \epsilon = X t e A.
\end{align*}

**Proof** This is basically a reformulation of Definition 5.2 in the special case where $C = T^A$. (a) and (b) express the fact that $t$ and $B$ are morphisms in $T^A$, and (c) and (d) are a reformulation of (a) and (b) in Eq. 5.1. We leave the verification of the details to the reader. \qed

### 5.3 Natural Transformations, Frobenius Elements and Casimir Morphisms

**Definition 5.5** [9, Def. 3.1] An object $P$ in a monoidal category $C$ is called a left $\otimes$-generator if the following condition is satisfied: if $f, g : YZ \to W$ are morphisms in $C$ such that $f \circ hZ = g \circ hZ$ for all $h : P \to Y$ in $C$, then $f = g$.

It is easy to see that a left $\otimes$-generator is a generator in the classical sense. If $1$ is a left $\otimes$-generator for $C$, then $C$ is a Frobenius coalgebra if and only if the forgetful functor $C^C \to C$ is Frobenius. In Theorem 5.8, we will prove the following result: under the hypothesis that $1$ is a left $\otimes$-generator in $C$, the forgetful functor $F : C(\psi)_A \to C_A$ is Frobenius if and only if $(X, \psi)$ is a Frobenius coalgebra in $T^A$. We have to determine when $G$ is a left adjoint of $F$, and to this end we have to investigate natural transformations $\theta : \text{Id}_{C_A} \to FG$ and $\vartheta : GF \to \text{Id}_{C(\psi)_A}$. We show that these natural transformations correspond to Frobenius elements (Proposition 5.6) and Casimir morphisms (Proposition 5.7). We recall from Lemma 2.2 that, for $N \in C_A$, we have an isomorphism $\alpha : C_A(A, N) \to C(1, N)$, $\alpha(h) = h$, with $h = h \circ \eta$ and $h = \mu \circ h A$. 

\[\square\] Springer
Proposition 5.6 Let \((A, X, \psi)\) be a cowreath in \(\mathcal{C}\). If \(1\) is a left \(\otimes\)-generator for \(\mathcal{C}\), then we have an isomorphism \(\text{Nat}(\text{Id}_{\mathcal{C}^A}, FG) \cong \mathcal{T}_A^\#(1, X)\).

Proof Consider a natural transformation \(\theta : \text{Id}_{\mathcal{C}^A} \to FG\). We claim that \(t = \alpha(\theta_A) = \theta_A \circ \eta : 1 \to AX\) is a morphism \(1 \to X\) in \(\mathcal{T}_A^\#\). Take \(h \in \mathcal{C}(1, A)\). From the naturality of \(\theta\), it follows that \(\theta_A \circ h = hX \circ \theta_A\), hence

\[
mX \circ At \circ h = mX \circ hAX \circ t = hX \circ \theta_A \circ \eta = \theta_A \circ h \circ \eta = \theta_A \circ m \circ A\eta \circ h = \theta_A \circ h,\]

so that

\[
mX \circ At = \theta_A, \tag{5.3}\]

since \(1\) is a left \(\otimes\)-generator. Using the right \(A\)-linearity of \(\theta_A\), we find that

\[
\theta_A = \theta_A \circ m \circ \eta A = mX \circ A\psi \circ \theta_AA \circ \eta A = mX \circ A\psi \circ tA.
\]

It follows that \(\theta_A = mX \circ At = mX \circ A\psi \circ tA\), which is precisely (5.2.a), expressing that \(t \in \mathcal{T}_A^\#(1, X)\).

Our next aim is to show that \(\theta\) is completely determined by \(t\). \(\theta_A\) is given by Eq. 5.3. Take \(N \in \mathcal{C}_A\) and \(h : 1 \to N\) in \(\mathcal{C}\). Then

\[
\theta_N \circ \mu \circ hA = \theta_N \circ h \circ \eta X \circ \theta_A = \mu X \circ hAX \circ \theta_A = \mu X \circ N\theta_A \circ hA.
\]

At \((\ast)\), we used the naturality of \(\theta\). From the fact that \(1\) is a left \(\otimes\)-generator, it follows that \(\theta_N \circ \mu = \mu X \circ N\theta_A\) (5.2.a) and \(\theta_N = \theta_N \circ \mu \circ N\eta = \mu X \circ N\theta_A \circ N\eta = \mu X \circ N\eta\). We conclude that

\[
\theta_N = \mu X \circ Nt \tag{5.4}
\]

is completely determined by \(t\).

Finally, for \(t \in \mathcal{T}_A^\#(1, X)\), we define \(\theta\) using Eq. 5.4. We show that \(\theta_N\) is right \(A\)-linear, for all \(N \in \mathcal{C}_A\).

\[
\mu_{NX} \circ \theta_N A \overset{(\ast)}{=} \mu X \circ N\psi \circ \mu X A \circ Nt A = \mu X \circ \mu AX \circ N\psi \circ Nt A \\
\overset{(x)}{=} \mu X \circ NmX \circ N\psi \circ Nt A \overset{(\gamma)}{=} \mu X \circ NmX \circ NAt \\
\overset{(x)}{=} \mu X \circ \mu AX \circ NAt = \mu X \circ Nt \circ \mu \overset{(5.4)}{=} \theta_N \circ \mu.
\]

At \((\ast)\) we used the associativity of \(\mu\), and at \((\gamma)\), we used the fact that \(t \in \mathcal{T}_A^\#(1, X)\). \(\square\)

Proposition 5.7 Let \((A, X, \psi)\) be a cowreath in \(\mathcal{C}\). If \(1\) is a left \(\otimes\)-generator for \(\mathcal{C}\), then we have a bijective correspondence between \(\text{Nat}(GF, \text{Id}_{\mathcal{C}(\psi)A})\) and the set of Casimir morphisms for \((X, \psi)\), that is the subset of \(\mathcal{T}_A^\#(XX, 1)\) consisting of morphisms \(B : XX \to A\) satisfying (5.2.c).

Proof Consider a natural transformation \(\theta : GF \to \text{Id}_{\mathcal{C}(\psi)A}\), and take \(N \in \mathcal{C}_A\). Then \(G(N) = NX\) is an entwined module, and for all \(h \in \mathcal{C}(1, N)\), we have that

\[
\theta_{NX} \circ hXX = \theta_{AX} \circ hXX \circ \eta XX \overset{(a)}{=} hX \circ \theta_{AX} \circ \eta XX \\
= \mu X \circ hAX \circ \theta_{AX} \circ \eta XX = \mu X \circ N\theta_{AX} \circ hAXX \circ \eta XX \\
= \mu X \circ N\theta_{AX} \circ N\eta XX \circ hXX.
\]
At (a), we used the naturality of $\vartheta$. Let $\zeta = \vartheta_{AX} \circ \eta XX$. From the fact that $1$ is a left $\otimes$-generator for $C$, it follows that, for all $N \in C_A$,

$$\vartheta_{NX} = \mu X \circ N \zeta.$$  \hspace{1cm} (5.5)

For an entwined module $M$, the coaction $\rho : M \to MX$ is a morphism of entwined modules, and it follows from the naturality of $\vartheta$ that

$$\rho \circ \vartheta_M = \vartheta_{M\rho X}.$$ \hspace{1cm} (5.6)

This enables us to compute that

$$\vartheta_M(3.5) = \mu \circ M \epsilon \circ \rho \circ \vartheta_M(5.5,5.6) = \mu \circ m \circ A \epsilon \circ M \zeta \circ \rho X = \mu \circ m \circ A \epsilon \circ M \zeta \circ \rho X,$$ \hspace{1cm} (5.7)

with

$$B = m \circ A \epsilon \circ \zeta = m \circ A \epsilon \circ \vartheta_{AX} \circ \eta XX : XX \to A.$$ \hspace{1cm} (5.8)

This shows that $\vartheta$ is completely determined by $B$.

We claim that $B \in T^\#(XX, 1)$. To this end, we need to show that Eq. 5.2.b holds, that is,

$$m^2 \circ A \epsilon A \circ \vartheta_{AX} \circ \eta XX A = m \circ A \epsilon \circ m X \circ A \vartheta_{AX} \circ A \eta XX \circ \psi^2.$$ \hspace{1cm} (5.9)

Observe that (5.5) specialized for $N = A$ gives

$$m X \circ A \vartheta_{AX} \circ A \eta XX = \vartheta_{AX} = \vartheta_{AX} \circ m XX \circ A \eta XX,$$

so that the right hand side of Eq. 5.9 equals $m \circ A \epsilon \circ m X \circ A \vartheta_{AX} \circ A \eta XX \circ \psi^2$. Equation 5.9 then follows from the commutativity of the diagram

The commutativity of the two squares is obvious, and the commutativity of the rectangle in the middle follows from the right $A$-linearity of $\vartheta_{AX}$. It follows from Eq. 3.3 that $A A \epsilon \circ A \psi = A A \epsilon$.

Our next step is to show that $B$ as defined in Eq. 5.8 satisfies (5.2.c), or

$$m X \circ A \psi \circ A X m \circ A X A \epsilon \circ A X \zeta \circ \delta X = m^2 X \circ A A \epsilon X \circ A \zeta X \circ \psi XX \circ X \delta.$$ \hspace{1cm} (5.10)
We will show that the two sides of Eq. 5.10 are equal to \( \zeta \). First

\[
\zeta = \vartheta_{AX} \circ \eta XX
\]

\[(5.7)\]

\[
= mX \circ A\psi \circ AXm \circ AX\epsilon \circ AX\zeta \circ mXXX \circ A\delta X \circ \eta XX
\]

\[
= mX \circ A\psi \circ AXm \circ AX\epsilon \circ AX\zeta \circ mXXX \circ \eta AXX \circ \delta X,
\]

the left hand side of Eq. 5.10. The diagram below is commutative.

The septangle in the middle commutes because \( \theta_{AX} \) preserves the right \( X \)-coaction. The commutativity of all the other parts of the diagram is obvious. It follows from the commutativity of the diagram that \( \zeta = mX \circ A\eta X \circ \vartheta_{AX} \circ \eta XX \) is equal to the right hand side of Eq. 5.10.

Finally, for \( B \in \mathcal{T}_{A}^{\#}(XX, 1) \) satisfying (5.2.c), we define \( \vartheta \) using the formula

\[
\vartheta_{M} = \mu \circ MB \circ \rho X : MX \to M,
\]

(5.11)

for any entwined module \( M \). It is left to the reader to show that \( \vartheta_{M} \) is a morphism of entwined modules, and that \( \vartheta \) is natural in \( M \).

\[\blacksquare\]

### 5.4 Frobenius Functors and Frobenius Systems

**Theorem 5.8** Let \( (A, X, \psi) \) be a cowreath in \( \mathcal{C} \). If \( 1 \) is a left \( \otimes \)-generator for \( \mathcal{C} \), then the forgetful functor \( F : \mathcal{C}(\psi)^{X}_{A} \to \mathcal{C}_{A} \) is Frobenius if and only if \( (X, \psi) \) is a Frobenius coalgebra in \( \mathcal{T}_{A}^{\#} \).

**Proof** Let \( F \dashv G \) be the adjunction described in Lemma 5.1. The functor \( F \) is Frobenius if and only if \( G \dashv F \), and this is equivalent to the existence of \( \theta \in \text{Nat}(\text{Id}_{C_{A}}, FG) \) and \( \vartheta \in \text{Nat}(GF, \text{Id}_{C(\psi)^{X}_{A}}) \) such that \( F(\theta_{M}) \circ \theta_{F(M)} = F(M) \) and \( \vartheta_{G(N)} \circ G(\vartheta_{N}) = G(N) \), for all \( M \in C(\psi)^{X}_{A} \) and \( N \in C_{A} \).

Fix \( \theta \in \text{Nat}(\text{Id}_{C_{A}}, FG) \) and \( \vartheta \in \text{Nat}(GF, \text{Id}_{C(\psi)^{X}_{A}}) \), and let \( t \in \mathcal{T}_{A}^{\#}(1, X) \) and \( B \in \mathcal{T}_{A}^{\#}(XX, 1) \) be the Frobenius element and the Casimir morphism corresponding to \( \theta \) and \( \vartheta \), see Propositions 5.6 and 5.7. \((t, B)\) is a Frobenius system if and only if Eq. 5.2.d holds, which comes down to the following: \( f = g = \epsilon \), where \( f = m \circ AB \circ \psi X \circ Xt \) and

\[\blacksquare\]
\[ g = m \circ AB \circ tX. \] It is easy to verify that \( f_M = F(\theta_M) \circ \theta_F(M) = \mu \circ MB \circ \rho X \circ \mu X \circ Mt, \)
the composition in the top row of the diagram
\[
\begin{array}{ccccccc}
M & \xrightarrow{f_M} & MAX & \xrightarrow{\mu X} & MX & \xrightarrow{\rho X} & MXX & \xrightarrow{MB} & MA & \xrightarrow{\mu} & M.
\end{array}
\]

The pentangle in the diagram commutes by Eq. 3.6. The right square commutes by the associativity of \( \mu, \) and the commutativity of the two other squares is obvious. We conclude that the diagram commutes.

If \( f = \epsilon, \) then it follows that \( F(\theta_M) \circ \theta_F(M) = f_M = \mu \circ ME \circ \rho \overset{(3.5)}{=} M. \)

Conversely, if \( f_M = M, \) for every entwined module \( M, \) in particular, \( f_{AX} = AX. \) From the commutativity of the above diagram, it follows that \( \mu_{AX} \circ AXf \circ \rho_{AX} = AX. \) Using the unit property of \( A, \) we find that
\[
m \circ A \epsilon \circ \eta X = m \circ \eta A \circ \epsilon = \epsilon. \tag{5.12}
\]

Then we compute that
\[
\begin{align*}
\epsilon & = m \circ A \epsilon \circ \mu_{AX} \circ AXf \circ \rho_{AX} \circ \eta X \\
& = m \circ A \epsilon \circ m X \circ A \psi \circ AXf \circ mXX \circ A \delta \circ \eta X \\
& = m \circ m A \circ A A \epsilon \circ A \psi \circ AXf \circ \delta \\
& \overset{(3.3.c)}{=} m \circ m A \circ A \epsilon A \circ AXf \circ \delta \\
& = m \circ Af \circ mX \circ A \epsilon X \circ \delta \\
& \overset{(3.3.d)}{=} M \circ Af \circ \eta X = m \circ \eta A \circ f = f.
\end{align*}
\]
We conclude that \( f = \epsilon \) if and only if \( F(\theta_M) \circ \theta_F(M) = F(M), \) for all \( M \in C(\psi)_A^X. \) In a similar way, we show that \( g = \epsilon \) if and only if \( g_N = \theta_{G(N)} \circ G_{\theta_N} = G(N), \) for all \( N \in C_A. \)

\[
g_N = \mu X \circ N \psi \circ NXB \circ \mu XX \circ N \delta X \circ \mu XX \circ NtN,
\]
and some (complicated) diagram chasing arguments using Eq. 5.2.a–c show that
\[
g_N = \mu X \circ N \mu X \circ NAgX \circ N \delta. \tag{5.13}
\]
If \( g = \epsilon, \) then it follows that \( g_N = \mu X \circ N \mu X \circ N A \epsilon X \circ N \delta \overset{(4.3.d)}{=} \mu X \circ N \eta X = NX. \)

The converse implication is more subtle. The tensor product in \( T_A^\# \) of \( t \in T_A^\#(1, X) \)
and the identity \( \eta X \in T_A^\#(X, X) \) is \( tX \in T_A^\#(X, XX) \). The composition in \( T_A^\# \) of \( tX \in T_A^\#(X, XX) \) and \( B \in T_A^\#(XX, 1) \) is then precisely \( g \in T_A^\#(X, 1) \), see Section 3. We conclude that \( g \) is a morphism in \( T_A^\#(X, 1) \), so that it follows from Eq. 3.2 that
\[
m \circ Ag \circ \psi = m \circ gA. \tag{5.14}
\]
If \( g_A = AX, \) then it follows that
\[
\begin{align*}
\epsilon & \overset{(5.12)}{=} m \circ A \epsilon \circ \eta X = m \circ A \epsilon \circ AX \circ \eta X = \mu \circ A \epsilon \circ m^2 X \circ AAgX \circ A \delta \circ \eta X \\
& = m \circ Am \circ AAgA \circ AX \circ \delta \overset{(5.14)}{=} m \circ Am \circ AAg \circ A \psi \circ AX \circ \delta \\
& = m \circ m A \circ AAg \circ A \psi \circ AX \circ \delta \overset{(3.3.e)}{=} m \circ Ag \circ mX \circ A \psi \circ AX \circ \delta \\
& = m \circ Ag \circ \eta X = m \circ \eta A \circ g = g.
\end{align*}
\]
as desired.  □

6 Frobenius Coalgebras Versus Frobenius Corings

Throughout this Section, $\mathcal{C}$ is a (strict) monoidal category with coequalizers and $A$ is a left coflat algebra in $\mathcal{C}$. $\mathcal{C}_A^\#$ is the full subcategory of $\mathcal{T}_A^\#$ consisting of objects $(X, \psi)$ with $X$ left coflat in $\mathcal{C}$.

**Lemma 6.1** We have a fully faithful strong monoidal functor $H : \mathcal{T}_A^\# \to \mathcal{A}_A\mathcal{C}_A$.

**Proof** Take $(X, \psi) \in \mathcal{T}_A^\#$. It follows from Proposition 2.4 that $AX$ is robust as a left $A$-module; $AX$ is left coflat since $A$ and $X$ are left coflat. $AX$ is an $A$-bimodule, with left $A$-action $v = mX$ and right $A$-action $\mu = mX \circ A\psi$. We conclude that $AX \in \mathcal{A}_A\mathcal{C}_A$, and we define $H(X, \psi) = AX$.

Let $f : X \to Y$ in $\mathcal{T}_A^\#$, and define $H(f) = mY \circ Af$. It is clear that $Hf$ is left $A$-linear, and the right $A$-linearity follows from Eq. 3.2. $H : \mathcal{T}_A^\#(X, Y) \to \mathcal{A}_A\mathcal{C}_A(AX, AY)$ is bijective. The inverse of $H$ is the unique isomorphism of coequalizers, see Proposition 2.1.

In fact, if we make the identification of coequalizers $(AX \bullet AY, q) = (AXY, \mu_{AXY})$, then $H$ becomes strictly monoidal. It follows from Lemma 6.1 that there exists a bijective correspondence between coalgebra structures on $(X, \psi) \in \mathcal{T}_A^\#$ and $A$-coring structures on $AX$. Moreover, Frobenius systems on $(A, \psi)$ correspond bijectively to Frobenius systems on $AX$, and the following result follows.

**Theorem 6.2** With notation and assumptions as above, $(X, \psi)$ is a Frobenius coalgebra in $\mathcal{T}_A^\#$ if and only if $\mathcal{C} = AX$ is a Frobenius $A$-coring, i.e. a Frobenius coalgebra in $\mathcal{A}_A\mathcal{C}_A$.

Combining Theorems 5.8 and 6.2, we obtain the following result.

**Corollary 6.3** Assume that $1$ is a left $\otimes$-generator for $\mathcal{C}$. Let $A$ be a left coflat algebra in $\mathcal{C}$, and take $(X, \psi) \in \mathcal{T}_A^\#$. Then the following assertions are equivalent:

(i) The forgetful functor $U : \mathcal{C}^X \to \mathcal{C}_A$ is Frobenius;

(ii) $\mathcal{C} := AX$ is a Frobenius $A$-coring.

**Proof** From [6, Theorem 4.8] we know that the categories $\mathcal{C}^X$ and $\mathcal{C}(\psi)^X_A$ are isomorphic. Thus $U$ is a Frobenius functor if and only if the forgetful functor $F : \mathcal{C}(\psi)^X_A \to \mathcal{C}_A$ is Frobenius. Since $1$ is a left $\otimes$-generator for $\mathcal{C}$, it follows from Theorem 5.8 that $F$ is Frobenius if and only if $(X, \psi)$ is a Frobenius coalgebra in $\mathcal{T}_A^\#$, and this is equivalent to $AX$ being a Frobenius $A$-coring, by Theorem 6.2. □

Another consequence of Lemma 6.1 is the following. If $(X, \psi)$ has a right adjoint in $\mathcal{T}_A^\#$, then $AX$ has a right adjoint in $\mathcal{A}_A\mathcal{C}_A$. Thus, if $(X, \psi)$ is a Frobenius coalgebra in $\mathcal{T}_A^\#$ then it
is selfadjoint and $AX$ is selfadjoint in $\mathcal{A}C_A$. Proposition 6.4 is a result of the same type, but with a more complicated proof. Take $(X, \psi) \in \mathcal{T}_A^\#$. If $X$ has a right adjoint $Y$ in $\mathcal{C}$, then $(Y, \varphi) \in \mathcal{T}$ and $YA \in \mathcal{A}C_A$.

**Proposition 6.4** With notation as above, we assume that $YA \in \mathcal{A}C_A$. Then $AX \dashv YA$ in $\mathcal{A}C_A$.

**Proof** We have an adjunction $(X, Y, b, d)$ in $\mathcal{C}$. Recall that $b : 1 \to YX$, $d : XY \to 1$.

Let $B = Y\psi \circ bA : A \to YAX = YA \bullet AX$ and $m \circ AdA : AXYA \to A$ are morphisms of $A$-bimodules. Let $D$ be the unique morphism in $\mathcal{A}C_A$ that makes the diagram

$$
\begin{array}{c}
\begin{array}{ccc}
AXAYA & \xrightarrow{AXv} & AXYA \\
\mu YA & & q \\
\downarrow AdA & & \downarrow \exists D \\
A & & A
\end{array}
\end{array}
$$

commutative. By applying Propositions 2.1 and 2.8, we have isomorphisms of colimits

$$
(AX \bullet YA \bullet AX, q_2) \cong (AX \bullet YAX, q \circ AXYmX) \cong ((AX \bullet YA)X, \mu_{AX} \bullet YA \circ qAX).
$$

Consider the diagram ($\mu X$ is a shorter notation for $\mu_{AX} \bullet YA$)

$$
\begin{array}{ccc}
AXA & \xrightarrow{q} & AX \bullet A \\
\downarrow & & \downarrow \\
AXYAX & \xrightarrow{AXmX} & AXYAX
\end{array}
$$

The commutativity of top and bottom right squares follows from Eq. 2.5; the commutativity of the rectangle in the middle follows from Eq. 6.2; the commutativity of the bottom left square follows from the definition of $D$. We conclude that the diagram commutes. Let $\delta$ be the composition of $D \bullet AX$ and the isomorphism $AX \bullet YAX \cong (AX \bullet YA)X$, as indicated in the diagram, and consider

$$
\begin{array}{ccc}
AXYAX & \xrightarrow{AXmX} & AXYAX & \xrightarrow{q} & AX \bullet YAX \\
\downarrow AdAX & & \downarrow \delta & & \\
AAAX & \xrightarrow{mX} & AX
\end{array}
$$

We used the associativity of $m$ and the fact that the left square in Eq. 6.4 commutes. It follows that $\delta \circ q = mX \circ AdAX$, since $(AXYAX, AXmX)$ is a coequalizer, see
Proposition 2.1. We conclude that the diagram (6.4) commutes. We therefore have commutative diagrams

\[
\begin{array}{ccc}
AXA & q & \rightarrow & AX \bullet A \\
\downarrow & & \downarrow & \\
AXB & & AX \bullet B \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
AXA & q & \rightarrow & AX \bullet A \\
\downarrow & \mu_{AX} & \rightarrow & \cong \\
AX & & AX \\
\end{array}
\]

and

\[
\begin{array}{ccc}
AXYAX & q & \rightarrow & AX \bullet YAX \\
\downarrow & & \downarrow & \\
\quad & & \\
AX & & AX \\
\end{array}
\]

so

\[
mX \circ \text{AdAX} \circ AXB = mX \circ \text{AdAX} \circ AXY \psi \circ AXbA = mX \circ A \psi \circ \text{AdXA} \circ AXbA = \mu_{AX}.
\]

Since \((AX \bullet A, q)\) is a coequalizer, we conclude that \(\delta \circ (AX \bullet B)\) is the canonical isomorphism \(AX \bullet A \cong AX\), as needed.

In a similar way, the diagram commutes

\[
\begin{array}{ccc}
AYA & q & \rightarrow & A \bullet YA \\
\downarrow & & \downarrow & \\
BYA & & B \bullet YA \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
AYA & q & \rightarrow & A \bullet YA \\
\downarrow & & \downarrow & \\
AYA & & YA \\
\end{array}
\]

Then we consider the diagram

\[
\begin{array}{ccc}
YAAXYA & YmXYA & \rightarrow & YAXYA \\
\downarrow & & \downarrow & \\
YAAXYA & & YAXYA \\
\downarrow & & \downarrow & \\
AYA & & YA \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
YAAXYA & YmXYA & \rightarrow & YAXYA \\
\downarrow & & \downarrow & \\
YAAXYA & & YAXYA \\
\downarrow & & \downarrow & \\
AYA & & YA \\
\end{array}
\]

and compute that

\[
\delta \circ q \circ YmXYA = Ym \circ YA \circ YAA \text{Ad}A = Ym \circ Ym \circ YA \text{Ad}A = Ym \circ Y \text{Ad}A \circ YmXYA.
\]

Therefore \(\delta \circ q = Ym \circ Y \text{Ad}A\), so we have the commutative diagrams

\[
\begin{array}{ccc}
AYA & q & \rightarrow & A \bullet YA \\
\downarrow & & \downarrow & \\
BYA & & B \bullet YA \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
AYA & q & \rightarrow & A \bullet YA \\
\downarrow & \nu_YA & \rightarrow & \cong \\
AYA & & YA \\
\end{array}
\]

We see now that

\[
Ym \circ Y \text{Ad}A \circ BYA = Ym \circ Y \text{Ad}A \circ Y \psi \circ YA \circ bAYA = \nu_YA,
\]

from where we conclude that \(\delta' \circ B \bullet YA\) is the canonical isomorphism \(A \bullet YA \cong YA\), finishing the proof.
In the setting of Proposition 6.4, assume moreover that \((X, \psi)\) is a coalgebra in \(\mathcal{T}_A^\#\). It follows from Lemma 6.1 that \(AX\) is an \(A\)-coring; we know from Proposition 6.4 that \(AX \rhd YA\) in \(\mathcal{C}_A\), hence \(YA\) is an \(A\)-ring, with structure given by Eq. 2.3. The structure maps are denoted as \(m_{YA} : YA \rhd YA = YYA \rightarrow YA\) and \(\eta_{YA} : A \rightarrow YA\).

We have seen in Proposition 4.3 that \((A,Y,\varphi)\) is a right wreath, and we can consider the wreath product \(YA = Y \varphi A\), which is an algebra in \(\mathcal{C}_A\). As in Proposition 4.3, the multiplication and the unit are denoted as \(m_{\varphi} : YAYA \rightarrow YA\) and \(\eta_{\varphi} : 1 \rightarrow YA\).

In Proposition 6.5 we investigate the relation between these two sets of structures.

**Proposition 6.5** With notation as above, \(m_{\#} = m_{YA} \circ Y_{\psi}YA, \eta_{\#} = \eta_{YA} \circ \eta\) and \(\overline{\mu} = \overline{\mu} \circ q\).

**Proof** According to Eq. 2.3, \(m_{YA} = YA \circ D^2 \circ Y\Delta \circ YYA \circ B \circ YYA\), where \(\Delta = mX \circ A\delta : AX \rightarrow AX \bullet AX = AXX\) is the comultiplication on \(AX\). Consider the commutative diagram

\[
\begin{array}{ccc}
AYYA & \xrightarrow{AY\psi} & AYYA \\
BYYA \downarrow & & \downarrow B \bullet YYA \\
YAXYYA & \xrightarrow{q} & YAX \bullet YYA \\
Y\Delta YYYA \downarrow & & \downarrow Y\Delta \bullet YYA \\
YAXXYYA & \xrightarrow{q} & YAXX \bullet YYA \\
YAd^2A \downarrow & & \downarrow Y\Delta A \bullet D^2 \\
YAA & \xrightarrow{Ym} & YA \bullet A = YA \\
\end{array}
\]

\(m_{YA}\) is the unique morphism such that \(m_{YA} \circ Y_{\psi}YA = f\), where \(f = Ym \circ YAd^2A \circ Y\Delta YYYA \circ BYYA\). We have to show that the triangle in the diagram below commutes.

\[
\begin{array}{ccc}
AYAYA & \xrightarrow{AY\psi} & AYYA \\
YAYA & \xrightarrow{m_{\#}} & YYA \\
\end{array}
\]

\((YAYA, Y_{\psi}YA)\) is a coequalizer and the rectangle in the diagram commutes, so it suffices to check

\[
m_{\#} \circ Y_{\psi}YA = f \circ AY_{\psi}YA. \tag{6.5}
\]

The fact that \(\delta : X \rightarrow AXX\) defines a morphism in \(\mathcal{T}_A^\#\) is expressed by the formula

\[
mXX \circ A\delta \circ \psi = mXX \circ A\psi X \circ AX\psi \circ \delta A = \mu_{AXX} \circ \delta A.
\]

Using this formula and the definition of \(B\), we can show that \(f \circ AY_{\psi}YA = g_1 \circ g\), where \(g_1 = Yf_1\),

\[
f_1 = m \circ Ad^2A \circ \mu_{AXX}Y_{\psi}YA : AXXXYAYA \rightarrow A.
\]
\[ g = Y \delta A Y A A Y A \circ b A Y A Y A : AYAY \to Y A X A Y A Y A. \]

Using Eq. 4.6, we compute that \( m_# \circ \nu_Y A Y = g_2 \circ g \), with \( g_2 = Yf_2 \), where

\[ f_2 = m^2 \circ A A d A \circ A \psi Y A \circ A X d A Y A \circ A X \nu Y A Y A : A X X A Y A Y A \to A. \]

Therefore it suffices to show that \( f_1 = f_2 \). To this end, we consider the diagram in Eq. 6.6. We slightly simplified the notation for the morphisms in the diagram, deleting identity morphisms on tensor factors; for example, \( \psi \) in the top left corner is a shorter notation for \( A X \psi Y A Y A \). It follows from Eq. 6.7 that the top left square in the diagram commutes. We deduce from Eq. 6.7 that \( dA \circ X \nu Y A = m \circ A d A = \psi Y A \), so the lower pentagon in the diagram commutes. The associativity of \( m \) entails that the lower right square commutes. The commutativity of the three remaining squares, the remaining pentangle, and the octangle at the lower left is an obvious consequence of the property that \( Cg \circ fB = fD \circ Ag \) for \( f : A \to C \) and \( g : B \to D \). Summing up, the diagram commutes. The composition taking \( A X X A Y A Y A \) in the top left corner to \( A \) via the southwestern route is \( f_1 \), and the composition via the northeastern route is \( f_2 \). Hence \( g_1 = g_2 \) and Eq. 6.5 holds.

From the definition of \( \varphi \), see Eq. 4.1, it easily follows that

\[ Ad \circ \psi Y = dA \circ X \varphi. \]

Let us prove the second formula. As an application of Eq. 2.7, we find that \( YA \cdot \varepsilon_{AX} = Y \varepsilon_{AX} : Y A \cdot AX = YAX \to YA \cdot A = YA. \)

\[ \eta_{Y A} \circ \eta \overset{(2.3)}{=} Y A \cdot AX \circ B \circ \eta = Y \varepsilon_{AX} \circ Y \psi \circ b A \circ \eta = Y m \circ Y A \varepsilon \circ Y \psi \circ b A \circ \eta \]

\[ Y m \circ Y \varepsilon A \circ b A \circ \eta = Y m \circ Y A \eta \circ Y \varepsilon \circ b \overset{(4.6)}{=} \eta_. \]
It follows from Eq. 2.4 that \( \tilde{\mu} = AX \bullet D \circ \Delta \bullet YA \). This fits into the diagram

\[
\begin{array}{ccc}
AXYA & \xrightarrow{\eta} & AX \bullet YA \\
\Delta YA & \downarrow & \Delta \bullet YA \\
AXXYA & \xrightarrow{\eta} & AX \bullet YA \cong AX \bullet AX \bullet YA \\
AXdA & \downarrow & AX \bullet D \\
AXA & \xrightarrow{\mu AX} & AX \\
\end{array}
\]

It follows that \( \tilde{\mu} \circ q = \mu AX \circ AXdA \circ \Delta YA = \overline{\mu} \). \( \square \)

From now on \( C \) is a (strict) monoidal category with coequalizers for which any object is left coflat; \( A \) is an algebra in \( C \) such that every left \( A \)-module in \( C \) is robust. Thus \( 1^A_A \) and \( AC_A \) coincide.

An algebra morphism \( A \to S \) is also called an algebra extension; algebra extensions correspond to \( A \)-rings, these are algebras in the category \( AC_A \). \( A \to S \) is called a Frobenius algebra extension if \( S \) is a Frobenius algebra in \( AC_A \). Now we can state the main result of this Section.

**Theorem 6.6** Let \((A, X, \psi)\) be a cowreath, and assume that \( X \cong Y \) in \( C \). Then the following assertions are equivalent:

(i) \((X, \psi)\) is a Frobenius coalgebra in \( T_A^\#; \)

(ii) \(Y \) is a Frobenius \( A \)-ring;

(iii) \((Y, \varphi)\) is a Frobenius algebra in \( A^\#; \)

(iv) the algebra extension \( Ym \circ \eta YA : A \to YA \) is Frobenius;

(v) \(AX \) and \( YA \) are isomorphic as left \( A \), right \( YA \)-modules in \( C \).

(vi) \(AX \) and \( YA \) are isomorphic as left \( A \)-modules and as entwined modules;

(vii) there exists \( t : 1 \to X \) in \( T_A^\# \) (that is a Frobenius element for \((X, \psi)\) in \( T_A^\# \)) such that

\[
\Phi = m^2 X \circ AA \psi A \circ AAXdA \circ A\delta YA \circ tYA : YA \to AX
\]

is an isomorphism in \( C \);

(viii) there exists \( B : X \otimes X \to 1 \) in \( T_A^\# \) satisfying 5.2.c (that is a Casimir morphism for \((X, \psi)\) in \( T_A^\# \)) such that

\[
\Psi = Ym \circ YAB \circ Y\psi X \circ bAX : AX \to YA
\]

is an isomorphism in \( C \).

If \( 1 \) is a left \( \otimes \)-generator of \( C \), then these statements are equivalent to

(ix) The functor \( F : C(\psi)X_A \to CA \) is a Frobenius functor.

**Proof** (i) \( \Leftrightarrow \) (ii). By Theorem 6.2, \((X, \psi)\) is a Frobenius coalgebra in \( T_A^\# \) if and only if \(AX \) is a Frobenius coalgebra in \( AC_A \). Since \(AX \cong YA \) in \( AC_A \), this is equivalent to \(YA \) is a Frobenius algebra in \( AC_A \), which is a Frobenius \( A \)-ring.

(ii) \( \Leftrightarrow \) (iii) \( \Leftrightarrow \) (iv) follows from [9, Corollary 8.8] applied to the wreath \((A, Y, \varphi)\) in \( CA^{rev} \). A direct proof for (ii) \( \Leftrightarrow \) (iii) can be done by using arguments similar to the ones in the proof of Theorem 6.2.
(i) ⇔ (v). We use the characterization (i) in Remark 5.3, applied to the coalgebra $AX$ in $A^{C_A}$. We have that $(X, \psi)$ in $T^#_A$ is a Frobenius coalgebra if and only if $AX$ is a Frobenius $A$-coring. Since $AX \to YA$ in $A^{C_A}$, this is equivalent to $AX$ and $YA$ being isomorphic as right $YA$-modules in $A^{C_A}$. By Proposition 6.5 it then follows that $(X, \psi)$ is a Frobenius coalgebra if and only if $AX$ and $YA$ are isomorphic as $A$-bimodules and right $YA$-modules in $C$. But the right $A$-module structure on $AX$ and $YA$ is inherited from the right action of $YA$ on them via the restriction of scalars functor defined by the algebra extension $Ym \circ \eta_{YA} : A \to YA$. So we conclude that $(X, \psi)$ is a Frobenius coalgebra in $T^#_A$ if and only if $AX$ and $YA$ are isomorphic as left $A$, right $YA$-modules.

(v) ⇔ (vi) is an immediate consequence of Theorem 4.4.

(v) ⇔ (vii) is based on the elementary observation that a right $YA$-linear morphism $\Phi : YA \to AX$ is completely determined by the morphism $t = \Phi \circ \eta^A : 1 \to AX$ in $C$. Looking at the right $A$-action $\tilde{t} : AXYA \to AX$ (see the proof of Theorem 4.4 and Proposition 6.5), we easily find that $\Phi$ is given by Eq. 6.8. Since

we obtain that $\Phi$ is left $A$-linear if and only if $t$ is a morphism in $T^#_A$.

(v) ⇔ (viii). If $\Psi : AX \to YA$ is left $A$-linear, then

This shows that any left $A$-linear morphism $\Psi : AX \to YA$ is of the form (6.9), for some $B : X \otimes X \to A$ in $C$. A long but straightforward computation shows that $\Psi$ is right $YA$-linear if and only if $B$ is a morphism in $T^#_A$ satisfying (5.2.c).

(i) ⇔ (ix) follows from Theorem 5.8.

\[ \square \]
7 Separability Properties for Entwined Modules

The aim of this Section is to study the separability of the forgetful functor \( F : \mathcal{C}(\psi)_A^X \to \mathcal{C}_A \) and its right adjoint \( G \). Separable functors were introduced in [22]. Consider a pair of adjoint functors \( F \dashv G \) between two categories \( \mathcal{D} \) and \( \mathcal{E} \), with unit \( \eta : \text{Id}_\mathcal{D} \to GF \) and counit \( \varepsilon : FG \to \text{Id}_\mathcal{E} \). The following result is due to Rafael [24].

- \( F \) is separable if and only if the unit \( \eta \) of the adjunction splits: there is a natural transformation \( \vartheta : GF \to \text{Id}_\mathcal{D} \) such that \( \vartheta \circ \eta = \text{Id}_\mathcal{D} \);
- \( G \) is separable if and only if the counit \( \varepsilon \) of the adjunction cosplits: there is a natural transformation \( \theta : \text{Id}_\mathcal{E} \to FG \) such that \( \varepsilon \circ \theta = \text{Id}_\mathcal{E} \).

We will apply Rafael’s Theorem to \( F : \mathcal{C}(\psi)_A^X \to \mathcal{C}_A \) and its right adjoint \( G \). The natural transformations \( \vartheta \) and \( \theta \) will be obtained as an application of Propositions 5.6 and 5.7.

**Proposition 7.1** Assume that \( 1 \) is a left \( \otimes \)-generator of the (strict) monoidal category \( \mathcal{C} \), and let \((A, X, \psi)\) be a cowreath.

\[
\begin{align*}
(1) & \quad \text{The forgetful functor } F : \mathcal{C}(\psi)_A^X \to \mathcal{C}_A \text{ is separable if and only if there exists a Casimir morphism } B : XX \to A \text{ for the coalgebra } (X, \psi) \text{ in } \mathcal{T}_A^\# \text{ such that } m \circ AB \circ \delta = \varepsilon. \\
(2) & \quad G : \mathcal{C}_A \to \mathcal{C}(\psi)_A^X \text{ is separable if and only if there exists a morphism } t : 1 \to X \text{ in } \mathcal{T}_A^\# \text{ such that } m \circ Ae \circ t = \eta.
\end{align*}
\]

**Proof** By Rafael’s Theorem, \( F \) is separable if and only if there exists \( \vartheta : GF \to \text{Id}_{\mathcal{C}(\psi)_A^X} \) such that \( \vartheta \circ \eta \) is the identity natural transformation. Let \( B \) be the Casimir morphism corresponding to \( \vartheta \), see Proposition 5.7. Then we can easily show that \( \vartheta_M \circ \eta_M = \mu \circ MB \circ \rhoX \circ \rho = \mu \circ Mh \circ \rho, \) where \( h = m \circ AB \circ \delta = \varepsilon. \)

If \( h = \varepsilon \), then it follows that \( \vartheta_M \circ \eta_M = \mu \circ M\varepsilon \circ \rho \) \((\text{5.5})\).

Conversely, if \( \theta \circ \eta \) is the identity natural transformation, then

\[
AX = \thetaAX \circ \etaAX = \muAX \circ AXh \circ \rhoAX = mX \circ A\psi \circ AXh \circ mXX \circ \Delta,
\]

and

\[
\varepsilon \overset{(5.12)}{=} m \circ A\varepsilon \circ \etaX = m \circ A\varepsilon \circ AX \circ \etaX = m^2 \circ A\varepsilon \circ A\psi \circ AXh \circ mXX \circ \Delta \circ \etaX \overset{(3.3.c)}{=} m^2 \circ A\varepsilon \circ AXh \circ mXX \circ \Delta \circ \etaX = m \circ Ah \circ mX \circ AmX \circ A\varepsilonX \circ \Delta \circ \etaX \overset{(3.3.d)}{=} m \circ Ah \circ mX \circ A\etaX \circ \etaX = m \circ Ah \circ \etaX = m \circ \etaA \circ h = h.
\]

The proof of the second statement is similar. \( G \) is separable if and only if there exists \( \theta : \text{Id}_{\mathcal{C}_A} \to FG \) such that \( \varepsilon \circ \theta \) is the identity natural transformation, that is, \( \varepsilon_N \circ \theta_N = N \), for all \( N \in \mathcal{C}_A \). Fix \( \theta \), and let \( t \in \mathcal{T}_A^\#(1, X) \) be the Frobenius element corresponding to \( \theta \), see Proposition 5.6. Then \( \varepsilon_N \circ \theta_N = \mu \circ N\varepsilon \circ \muX \circ Nt \), fitting into the commutative diagram

\[
\begin{array}{ccccccccc}
N & N \otimes t & NAX & N \otimes N & N \otimes N & NAX & N \otimes N & N \otimes N & N \otimes N \\
& \downarrow \quad \etaM & \downarrow \quad \muX & \downarrow \quad \mu & \downarrow \quad \muX & \downarrow \quad \mu & \downarrow \quad \muX & \downarrow \quad \mu & \downarrow \quad \muX \\
& NAA & NA & NA & NA & NA & NA & NA & NA
\end{array}
\]
If \( m \circ A \varepsilon \circ t = \eta \), then \( \varepsilon_N \circ \theta_N = \mu \circ N \eta = N \).
Conversely, if \( \varepsilon \circ \theta = \text{Id}_{C_A} \), then \( \varepsilon_A \circ \theta_A = A \), and we find from the commutativity of the diagram that \( \eta = m \circ Am \circ AA \varepsilon \circ At \circ \eta = m \circ mA \circ \eta AA \circ A \varepsilon \circ t = m \circ A \varepsilon \circ t \). 

Coseparable coalgebras were introduced by Larson in [20]. This notion can be generalized to coalgebras in (strict) monoidal categories. Remark that a coalgebra \( C \) is a \( C \)-bicomodule, with left and right \( C \)-coaction induced by comultiplication.

**Definition 7.2** A coalgebra \( C \) is coseparable if it is a relative injective \( C \)-bicomodule in \( C \), which comes down to the following property. If \( i : M \rightarrow N \) in \( C^C \) has a left inverse \( p : M \rightarrow N \) in \( C \), then every \( f : M \rightarrow C \) in \( C^C \) factors through \( i \) in \( C\cdot C^C \): there exists a \( C \)-bilinear morphism \( g : N \rightarrow C \) such that \( g \circ i = f \).

The proof of the next result is similar to the one of [20, Lemma 1], so we will skip it.

**Proposition 7.3** For a coalgebra \( C \) in a (strict) monoidal category \( C \), the following assertions are equivalent.

(i) \( C \) is coseparable;
(ii) the comultiplication \( \Delta \) has a \( C \)-bilinear left inverse \( \gamma : CC \rightarrow C \);
(iii) there exists a morphism \( B : CC \rightarrow 1 \) in \( C \) such that

\[
B \circ \Delta = \varepsilon \quad \text{and} \quad CB \circ \Delta C = BC \circ C \Delta. \quad (7.1)
\]

A morphism \( B : CC \rightarrow 1 \) satisfying the second condition in Eq. 7.1 is a Casimir morphism for \( C \), see Definition 5.2. A Casimir morphism is called normalized if it also satisfies the first condition in Eq. 7.1. A coseparable coalgebra is a coalgebra together with a normalized Casimir morphism.

**Proposition 7.4** For a cowreath \( (A, X, \psi) \) in \( C \), the following assertions are equivalent.

(i) \( (X, \psi) \) is a coseparable coalgebra in \( T_A^\# \);
(ii) there exists a morphism \( \gamma : XX \rightarrow AX \) in \( C \) such that

\[
\begin{align*}
(a) & \quad \begin{array}{c}
\xymatrix{
XX & XX \ar[rr]^-{\gamma} & & XX \\
A & A & X & X \ar[rr]^-{\gamma} & & A & X & X
}
\end{array} \\
(b) & \quad \begin{array}{c}
\xymatrix{
XX & XX \ar[rr]^-{\gamma} & & XX \\
A & A & X & X \ar[rr]^-{\gamma} & & A & X & X
}
\end{array} \\
(c) & \quad \begin{array}{c}
\xymatrix{
XX & XX \ar[rr]^-{\gamma} & & XX \\
A & A & X & X \ar[rr]^-{\gamma} & & A & X & X
}
\end{array} \\
\end{align*}
\]
(iii) there exists a Casimir morphism $B$ for the coalgebra $(X, \psi)$ in $T^\#_A$ such that $m \circ AB \circ \delta = \epsilon$.

If $X \vdash Y$ in $C$, then these conditions are equivalent to

(iv) there exists a left $A$-linear $\Psi : AX \to YA$ in $C(\psi)_A^X$ such that

$$m \circ AdA \circ AX\Psi \circ AX\eta X \circ \delta = \epsilon;$$

(7.3)

(v) there exists a morphism $\overline{\Psi} : X \to YA$ in $C$ satisfying the equations:

$$\gamma \mapsto (a), \quad (b), \quad (c) \quad (\gamma \mapsto \epsilon).$$

Proof (i) $\iff$ (ii) $\iff$ (iii). (7.2.a) says that $\gamma$ is a morphism in $T^\#_A$, (7.2.b) that $\gamma$ is an $(X, \psi)$-bicolinear morphism in $T^\#_A$ and (7.2.c) that $\gamma$ is a left inverse of the comultiplication $\delta$ of the coalgebra $(X, \psi)$ in $T^\#_A$, so condition (ii) is condition (ii) from Proposition 7.3 in the special case where $C = T^\#_A$. A similar observation holds for condition (iii), and the equivalence of (i), (ii) and (iii) follows.

(iii) $\iff$ (iv). We have seen in the proof of the equivalence (v) $\iff$ (viii) in Theorem 6.6 that there is a bijective correspondence between left $A$-linear morphisms $\Psi : AX \to YA$ in $C(\psi)_A^X$ and Casimir morphisms $B$ for the coalgebra $(X, \psi)$ in $T^\#_A$. Moreover, it is easy to show that $\Psi$ satisfies (7.3) if and only if the corresponding $B$ has the property that $m \circ AB \circ \delta = \epsilon$.

(iv) $\iff$ (v). It follows from Lemma 2.2 that we have an isomorphism $\alpha : A\hat{C}(AX,AY) \to \hat{C}(X,AY)$, given by $\alpha(\Psi) = \Psi \circ \eta A$ and $\alpha^{-1}((\Psi)) = \nu Y A \circ A\overline{\Psi} = Ym \circ YAdA \circ Y\psi AYA \circ bAYA \circ A\overline{\Psi}$. It is left to the reader to check that $\Psi$ is a left $A$-linear morphism in $C(\psi)_A^X$ if and only if $\alpha(\Psi) = \overline{\Psi}$ satisfies (7.4.a, b). Finally, Eq. 7.3 is equivalent to Eq. 7.4.c.

Our next result is a generalization of [11, Theorem 2.3].

Theorem 7.5 Assume that $\underline{1}$ is a left $\otimes$-generator for $C$. For a cowreath $(A, X, \psi)$ in $C$, the following statements are equivalent:

(i) The forgetful functor $F : C(\psi)_A^X \to C_A$ is separable;

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(ii) \((X, \psi)\) is a coseparable coalgebra in \(T^\#_A\).

If \(C\) has coequalizers and \(A\) and \(X\) are left coflat in \(C\), these statements are also equivalent to

(iii) \(AX\) is a coseparable \(A\)-coring in \(C\), that is a coseparable coalgebra in the monoidal category \(\mathcal{C}_A\);

(iv) the forgetful functor \(U : \mathcal{C}^AX \to \mathcal{C}_A\) is separable.

**Proof**

(i) \iff (ii). Follows from Proposition 7.1 and the equivalence (i) \iff (iii) in Proposition 7.4.

(ii) \iff (iii). We proceed as in the proof of Theorem 6.2. As before we identify \(AX \otimes AX = AXX\). Applying Lemma 2.2, we obtain an isomorphism \(\alpha : \mathcal{C}(AXX, AX) \to \mathcal{C}(XX, AX)\). A direct verification shows that \(\Omega \in \mathcal{C}(AXX, AX)\) is right \(A\)-linear if and only if \(\alpha(\Omega) = \gamma\) satisfies (7.2.a). In this situation, \(\Omega\) is left and right \(AX\)-colinear if and only if \(\gamma\) satisfies (7.2.b). Finally \(\Delta \circ \Omega = AX\) if and only if (7.2.c) holds.

(i) \iff (iv). The categories \(\mathcal{C}^AX\) and \(\mathcal{C}(\psi)X\) are isomorphic, see [6, Theorem 4.8], and this implies immediately that the separability of \(F\) and \(U\) is equivalent.

More equivalent conditions for the coseparability of a coalgebra \((X, \psi)\) in \(T^\#_A\) can be given under the assumption that \(X \rightharpoonup Y\). For the definition of a separable algebra extension in a monoidal category, we refer to [9, Def. 4.5 (ii)].

**Proposition 7.6** Let \(C\) be a monoidal category with coequalizers, and assume that every object of \(C\) is flat. Let \((A, X, \psi)\) be a cowreath in \(C\). If \(X \rightharpoonup Y\) in \(C\) and every left \(A\)-module is robust, then the following assertions are equivalent:

(i) \((X, \psi)\) is a coseparable coalgebra in \(T^\#_A\);

(ii) \((Y, \varphi)\) is a separable algebra in \(#_A^\mathcal{T}\), where \(\varphi\) is defined in Eq. 4.1;

(iii) The smash product \(YA\) is a separable algebra extension of \(A\) in \(\mathcal{C}\);

(iv) \(YA\) is a separable \(A\)-ring, that is a separable algebra in \(#_A^\mathcal{C}\).

If \(1\) is a left \(\otimes\)-generator in \(C\) then (i)-(iv) are also equivalent to

(v) The restriction of scalars functor \(F' : \mathcal{C}_Y \to \mathcal{C}_A\) is separable.

**Proof** This is an immediate consequence of Theorems 4.4 and 7.5, and [9, Cor. 8.9] applied to the wreath \((Y, A, \varphi)\).

Our final result is a Maschke type Theorem for entwined modules. It generalizes [11, Theorem 2.7] and [4, Theorem 4.2].

**Theorem 7.7** Let \((X, \psi)\) be a coseparable coalgebra in \(T^\#_A\).

(i) If a morphism in \(\mathcal{C}(\psi)_A^X\) has a section (resp. a retraction) in \(\mathcal{C}_A\) then it has a section (resp. a retraction) in \(\mathcal{C}(\psi)_A^X\);

(ii) If an object in \(\mathcal{C}(\psi)_A^X\) is semisimple (resp. projective, injective) as a right \(A\)-module then it is semisimple (resp. projective, injective) as an entwined module over \((A, X, \psi)\).
Every $M \in C(\psi)_{A}^{X}$ is relative injective (see Definition 7.2 for the definition of relative injectivity).

Proof The forgetful functor $F : C(\psi)_{A}^{X} \to C_{A}$ is separable since $(X, \psi)$ is a coseparable coalgebra in $T_{A}^{\#}$. The three assertions then follow immediately from [12, Prop. 47 and 48, Cor.7].

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