BRST ANALYSIS OF PHYSICAL STATES IN TWO-DIMENSIONAL BLACK HOLE

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Abstract

We study the BRST cohomology for $SL(2,R)/U(1)$ coset model, which describes an exact string black hole solution. It is shown that the physical spectrum could contain not only the extra discrete states corresponding to those in $c = 1$ two-dimensional gravity but also many additional new states with ghost number $N_{FP} = -1 \sim 2$. We also discuss characters for nonunitary reps. and the relation of our results to other approaches.
1 Introduction

Two-dimensional gravity or string theory has been widely studied as the only viable candidate for consistent theory of quantum gravity coupled to matter. In this approach, the study of curved space-time as string backgrounds should be useful to investigate how gravity behaves at the Planck scale.

 Particularly relevant in this analysis is the background of black holes. Indeed, many of the most interesting and important questions in quantum gravity concern the physics of black holes. Some insight into these problems has already been obtained from two-dimensional models of black holes \cite{1, 2}. However, most of the works employed semiclassical approximations and have been unable to give a consistent picture of quantum theory. Clearly it is necessary to understand full quantum effects in order to get such a picture.

 Recently there has been substantial progress in developing nonperturbative techniques to handle strings in nontrivial backgrounds \cite{3}, and this has stimulated the remarkable advance in our understanding of the continuum theory of two-dimensional gravity. In particular, it has been found that there are an infinite number of extra states at discrete values of momenta \cite{4, 3} which represent the \( W_\infty \) symmetry of the theory \cite{5, 6, 7, 8, 9}. It is expected that these states play very important roles in our understanding of the structure of the theory.

 String backgrounds may be most effectively studied with conformal field theory. However most of the existing works have been restricted to compact curved space. In order to study string theory in general curved backgrounds, it will be useful to examine noncompact coset models \cite{10, 11, 12}. In particular, it has been shown that a gauged \( SL(2, R) \) Wess-Zumino-Witten model is a conformal field theory for two-dimensional string in a black hole background \cite{13, 14}. It is thus quite important to understand the structure of this coset model. To this aim, it is first necessary to clarify the physical spectrum in the theory. This problem has been examined by several groups by adopting the BRST approach \cite{15, 16, 17}. These analyses indicate that there are also an infinite number of
discrete states in the theory. However, it seems that the precise structure of the physical spectrum in $SL(2, R)/U(1)$ has not been clarified yet.

The main purpose of this paper is to compute the BRST cohomology and derive the spectrum for the $SL(2, R)/U(1)$ coset model. We study the BRST cohomology in two steps. We first compute the cohomology of the $SL(2, R)/U(1)$ parafermion (PF) modules with irreducible base reps. The results agree with those obtained by Distler and Nelson [15]. These PF modules, however, are in general reducible and contains null states. To get irreps. of the coset, we must subtract submodules generated from the null states. We next identify the null states and examine how the cohomology is changed if one takes them into account. As a result, we find that there are more discrete physical states than those given in ref. [15]. We also derive the character formulae for these reps. based on our knowledge of the null states.

It would be appropriate to explain why we find more discrete states here than $c = 1$ matter coupled to gravity. The situation is best understood in the free boson realization. After the division by $U(1)$, our coset theory may be described by two bosons, which look quite similar to the $c = 1$ matter and Liouville fields. Thus, if we study the spectrum in Fock spaces for these bosons and the ghosts, we are bound to get the same spectrum as $c = 1$ matter coupled to gravity. All the other states are in a BRST doublet, $Q_B \psi = \chi$.

In our discussion, we divide Fock spaces first to the PF modules with irreducible base reps. and next to PF irreps. In the first step there are cases that $\psi$ and $\chi$ are in different modules on different irreducible base reps. Thus each of the doublet becomes a physical state in each module. This gives us new discrete states obtained in sect. 3. In the second step $\chi (\psi)$ may be null states and this makes $\psi (\chi)$ physical. The real physical states for string in the black hole background must be the subset of those in sect. 5. There is also a possibility to find more than one string theory with the same black hole geometry.

In sect. 2, we discuss $SL(2, R)$ algebra, the coset $SL(2, R)/U(1)$ and their free field realization [18]. In this paper, we will consider the Euclidean black hole: namely, we divide out the negative metric field in the coset construction. Another formulation to gauge away the $U(1)$ with the BRST formalism is explained in appendix A. In sect. 3, we
compute the BRST cohomology for the $SL(2, R)/U(1)$ PF Verma modules (this is the case considered in [13]). In computing the cohomology of the coset, we use the cohomology analysis for $2D$ gravity [4]. A brief discussion of exact sequences necessary to our analysis is relegated to appendix B. In sect. 4, we construct null states in the $SL(2, R)$ algebra and $SL(2, R)/U(1)$ coset by using the free field realization. The result is consistent with the determinant formula of Kac and Kazhdan [11, 19]. This result is then used in sect. 5 to derive the physical spectrum for the $SL(2, R)/U(1)$ coset model. Sect. 6 is devoted to discussions of the character formulae for various reps. [20]. The relation of our results to other approaches [10] is also illustrated by taking simple examples of the discrete physical states.

2 $SL(2, R)$ and free field realization

In this section, we briefly summarize the reps. and free field realizations for $SL(2, R)$ current and PF algebras.

The $SL(2, R)$ algebra has three generators with operator product expansions (OPEs)

$$J^i(z)J^j(w) \sim \frac{K g^{ij}/2}{(z-w)^2} + \frac{i \epsilon^{ijk} J^k(w)}{z-w},$$

(2.1)

where the metric $g_{ij} = \text{diag.} (1, 1, -1)$ and $K$ is the level of this algebra. If we set

$$J^\pm (z) = J^1 \pm i J^2,$$

(2.2)

the OPEs take the form

$$J^+(z)J^-(w) \sim \frac{K}{(z-w)^2} - \frac{2}{z-w} J^3(w),$$

$$J^3(z)J^\pm (w) \sim \pm \frac{z-w}{z-w} J^\pm (w),$$

(2.3)

$$J^3(z)J^3(w) \sim \frac{-K/2}{(z-w)^2},$$

and we have the hermiticity

$$(J^+(z))^\dagger = J^-(\frac{1}{\bar{z}})(\bar{z})^{-2},$$

$$(J^3(z))^\dagger = J^3(\frac{1}{\bar{z}})(\bar{z})^{-2}.$$
The stress tensor is given by

\[ T^{SL(2)}(z) = \frac{1}{2(K-2)} \left[ J^+(z)J^-(z) + J^-(z)J^+(z) - 2(J^3(z))^2 \right], \tag{2.5} \]

which satisfies the conformal OPE with central charge

\[ c = \frac{3K}{K-2}. \tag{2.6} \]

In order to construct reps. of the \( SL(2, R) \) algebra, it is useful to introduce the mode operator \( J_n^i \) by

\[ J^i(z) = \sum_n J^i_n z^{-n-1}. \tag{2.7} \]

The hermiticity (2.4) implies

\[ (J^+_n)^\dagger = J^-_{-n}, \quad (J^3_n)^\dagger = J^3_{-n}. \tag{2.8} \]

By the standard procedure, we find the commutation relations from the OPEs (2.3)

\[ [J^+_n, J^-_m] = Kn\delta_{n+m,0} - 2J^3_{n+m}, \]
\[ [J^3_n, J^\pm_m] = \pm J^\pm_{n+m}, \tag{2.9} \]
\[ [J^3_n, J^3_m] = -\frac{K}{2}n\delta_{n+m,0}, \]

as well as the Virasoro operator

\[ L_0^{SL(2)} = \frac{1}{K-2} \left[ J^2 + \sum_{n=1}^{\infty} (J^+_nJ^-_n + J^-_nJ^+_n - 2J^3_nJ^3_n) \right], \]
\[ J^2 \equiv \frac{1}{2} (J_0^+J_0^- + J_0^-J_0^+) - (J^3_0)^2. \tag{2.10} \]

The reps. of the \( SL(2, R) \) current algebra are built over irreps. of the global \( SL(2, R) \) \( (generated by J_0^\pm \text{ and } J_0^3) \) by applying the \( J^\pm_{-n}, n > 0 \), in all possible ways.

Since we consider a target space with the time direction, we do not restrict ourselves to unitary reps. of the \( SL(2, R)/U(1) \) coset. Instead, we take hermitian reps., where the currents are realized as hermitian operators and the inner product is non-degenerate. The unitarity should be discussed in the physical subspace as in the critical string theory.
The reps. of the global $SL(2, R)$ are constructed in exactly the same way as those of $SU(2)$. These are characterized by spin $J^2 = -j(j - 1)$. It follows that the reps. with $j$ and $1 - j$ are equivalent. To avoid the double counting, we restrict $j \geq 1/2$.

The $SL(2, R)$ reps. are obtained by determining the eigenvalues of $J_0^3$. We first consider a state $|m\rangle$ with $J_0^3 = m$ and then act on this state by the ladder operators $J_0^\pm$.

After some algebra, one finds that the hermitian irreps. of the global $SL(2, R)$ fall into the following four classes:

1. Lowest weight (LW) reps.: $\hat{\Delta}^+ \equiv \{|j\rangle, |j+1\rangle, |j+2\rangle, \cdots\}$ or $\Delta^+ \equiv \{|-j+1\rangle, |-j+2\rangle, \cdots\}$, $j \geq 1/2$ with $J_0^-|j\rangle = 0$ or $J_0^-|-j+1\rangle = 0$;

2. Highest weight (HW) reps.: $\hat{\Delta}^- \equiv \{|-j\rangle, |-j-1\rangle, |-j-2\rangle, \cdots\}$ or $\Delta^- \equiv \{|j-1\rangle, |j-2\rangle, \cdots\}$, $j \geq 1/2$ with $J_0^+|j\rangle = 0$ or $J_0^+|j-1\rangle = 0$;

3. Double-sided reps.: $U \equiv \{|-j+1\rangle, |-j+2\rangle, \cdots, |j-2\rangle, |j-1\rangle\}$, $j = 1/2, 1, 3/2, \cdots$, with $J_0^-|-j+1\rangle = 0$, $J_0^+|j-1\rangle = 0$;

4. Continuous reps.: $C \equiv \{|k + \phi_0\rangle\}$ with integer $k$ and $1 > \phi_0 \geq 0$ and $j \geq 1/2$.

where we indicate only $J_0^3$ eigenvalues. These reps. provide the bases to build up reps. of the $SL(2, R)$ current algebra and will be called base reps. in what follows.

In order to make a connection with the two-dimensional gravity coupled to matter, it is useful to bosonize the currents as

\[ J^\pm = \left(i\sqrt{\frac{K}{2}}\partial\phi^M \pm \sqrt{\frac{K-2}{2}}\partial\phi^L\right) \exp\left(\pm\sqrt{\frac{2}{K}}(i\phi^M + \phi^3)\right), \]

\[ J^3 = -\sqrt{\frac{K}{2}}\partial\phi^3, \quad (2.11) \]

where $\phi^i(z)\phi^j(w) \sim -\delta^{ij}\ln(z-w)$. It is easy to see that the OPEs in (2.3) are satisfied by the currents in (2.11).

\[ ^1{} \]For $j = \frac{1}{2}$, $J_0^\pm$ are interchanged in these conditions.
\[ ^2{} \]For the continuous reps., $j$ is allowed to take the complex value $j = 1/2 + i\lambda$ with $\lambda > 0$. In this paper this possibility will not be studied since it cannot satisfy the on-shell condition.
The base reps. may be constructed in terms of vertex operators,

\[ U_{jm}(z) = \exp \left[ \sqrt{\frac{2}{K-2}} j\phi^L + \sqrt{\frac{2}{K}} m(i\phi^M + \phi^3) \right], \tag{2.12} \]

as

\[ |j, m\rangle = U_{j,m}(0)|0\rangle, \tag{2.13} \]

which has the dimension \(-j(j-1)/(K-2)\) and \(J^3_0 = m\). The LW rep. \(\tilde{\Delta}^+\) corresponds to the set of values \(\{m = j, j + 1, \ldots\}\), while the HW \(\tilde{\Delta}^-\) to \(\{m = -j, -j - 1, \ldots\}\) with \(j\) replaced by \(-j\). Indeed, we find the OPEs

\[
J^3(z) U_{jm}(w) \sim \frac{m}{z-w} U_{jm}(w),
\]

\[
J^\pm(z) U_{jm}(w) \sim \frac{m \pm j}{z-w} U_{j,m \pm 1}(w), \tag{2.14}
\]

which ensure that the reps. split precisely at \(m = \pm j\).

A comment is in order. In the free boson realization, it would be natural to start with a generally reducible rep. for global \(SL(2,R)\) rep., \(F \equiv \{|k + m_0\}\) with integer \(k\) and some \(m_0\), and decompose it into irreps. For example, when \(m_0 = j\), one sees from eq. (2.14) that there is a LW state \(|j\rangle\) in \(F\). We thus get a LW rep. \(\tilde{\Delta}^+\) generated from \(|j\rangle\) by acting \(J^3_0^+\). On the other hand, we obtain a HW rep. \(\Delta^-\) by setting the above LW rep. equal to zero in \(F\). Namely, \(\Delta^-\) is defined as the coset

\[ \Delta^- = F/\tilde{\Delta}^+. \tag{2.15} \]

This shows that the LW state \(|j\rangle\) can be interpreted as a null state in \(F\) when we consider the HW rep. \(\Delta^-\). In similar manners, one finds all the irreps. from \(F\) depending on whether \(m \pm j\) is integer or not.

The \(SL(2,R)/U(1)\) coset model is obtained by requiring \(J^3 \sim 0\). This eliminates the negative metric field and gives a model corresponding to a Euclidean black hole \cite{13}. The stress tensor for this coset is given by

\[
T(z) = T^{SL(2)}(z) + \frac{1}{K} (J^3(z))^2, \\
= -\frac{1}{2} (\partial \phi^M)^2 - \frac{1}{2} (\partial \phi^L)^2 + \frac{1}{\sqrt{2(K-2)}} \partial^2 \phi^L. \tag{2.16}
\]
The central charge of the coset model is given by
\[ c = \frac{3K}{K-2} - 1. \] (2.17)

An alternative formulation to give the coset model is discussed in appendix A.

This coset may be regarded as the standard parafermionic model defined with PF fields \( \psi^\pm(z) \),
\[ J^\pm(z) = \sqrt{K} \psi^\pm(z) \exp \left( \pm \sqrt{\frac{2}{K}} \phi^3(z) \right). \] (2.18)

The base for PF modules is also given by (2.13), which has the dimension
\[ h_{jm} = \frac{-j(j-1)}{K-2} + \frac{1}{K} m^2 \]
\[ = h_{j,-m} = h_{1-j,m}. \] (2.19)

The PF modules may be obtained by restricting \( SL(2,R) \) modules by the conditions, \( J^3_n = 0, n \geq 1 \). We will denote, following ref. \[15\], the PF reps. generated on the lowest (highest) reps. \( \Delta^+(\Delta^-) \) by \( \hat{\mathcal{D}}^+(\hat{\mathcal{D}}^-) \) and those on the complementary reps. \( \Delta^-(\Delta^+) \) by \( \mathcal{D}^-(\mathcal{D}^+) \). The reps. generated on double-sided \( U \) and continuous reps. \( C \) are denoted by \( \mathcal{U} \) and \( \mathcal{C} \), respectively.

In our discussions of the physical spectrum, an important role is played by the nilpotent operators
\[ S^\pm = \oint \frac{dz}{2\pi i} \exp \left( \sqrt{\frac{K-2}{2}} \phi^L \pm i \sqrt{\frac{K}{2}} \phi^M \right), \]
\[ (S^+)^2 = (S^-)^2 = 0, \] (2.20)
which commute with the currents and hence satisfy
\[ [S^\pm, \psi^\pm(z)] = [S^\pm, \psi^\mp(z)] = 0. \] (2.21)
It follows that these operators provide an isomorphism of the PF modules \[15\].

Similarly to the discussion after (2.14) for global \( SL(2,R) \) rep., PF Verma modules can be obtained from free boson Fock modules. We define the vector spaces \( \mathcal{F}_j^\pm \) as
\[ \mathcal{F}_j^\pm \equiv \bigoplus_{m=\pm j+Z} \mathcal{F}_{jm}, \] (2.22)
with $\mathcal{F}_{jm}$ being the $\phi^L$ and $\phi^M$ Fock modules over $U_{jm}$. The PF modules $\tilde{\mathcal{D}}_j^\pm$ and $\mathcal{D}_j^\pm$ may be defined by

$$\tilde{\mathcal{D}}_j^\pm = \mathcal{F}_j^\pm \cap \text{Ker} S^\pm, \quad \mathcal{D}_j^\pm = \mathcal{F}_j^\mp / \tilde{\mathcal{D}}_j^\mp,$$  \hspace{1cm} (2.23)

where we have used (2.21) and the fact that the base rep. $\tilde{\Delta}^\pm$ is characterized as $\text{Ker} S^\pm$. The PF modules $\mathcal{D}_j^\pm$ are obtained from vector spaces $\mathcal{F}_j^\mp$ by setting the states in the complementary modules $\tilde{\mathcal{D}}_j^\mp$ equal to zero.

In order to consider dual spaces to these modules $\mathcal{F}_j^\pm$, $\tilde{\mathcal{D}}_j^\pm$ and $\mathcal{D}_j^\pm$, we note that $\phi^L$ has a background charge in eq. (2.16). This leads to the inner product in base reps. as

$$\langle j', m'| j, m \rangle = \delta_{j',1-j} \delta_{m', m},$$ \hspace{1cm} (2.24)

which implies that the dual space to $\mathcal{F}_{jm}$ is isomorphic to $\mathcal{F}_{1-j,m}$:

$$(\mathcal{F}_{jm})^* \simeq \mathcal{F}_{1-j,m}.$$ \hspace{1cm} (2.25)

By means of (2.25) and the hermiticity of $\psi^\pm$ implied by (2.4) and (2.18), we can show that the following isomorphisms hold for the dual spaces:

$$\mathcal{F}_j^\pm)^* \simeq \mathcal{F}_{1-j}^\mp,$$

$$\tilde{\mathcal{D}}_j^\pm)^* \simeq \mathcal{D}_{1-j}^\mp,$$ \hspace{1cm} (2.26)

where we have used the hermiticity relation $(S^+)^\dagger = S^-$ which follows from (2.20).

The total Fock space consists of the tensor product of $\mathcal{F}_{jm}$ and the ghost Fock space. We introduce the BRST operator

$$Q_B = \oint \frac{dz}{2\pi i} (cT + bc \partial c).$$ \hspace{1cm} (2.27)

Requiring that the BRST charge be nilpotent or the total central charge be zero, we get

$$K = \frac{9}{4}.$$ \hspace{1cm} (2.28)

In the rest of this paper, we restrict our discussions to this value of $K$ except in sect. 4. Our physical state condition is then

$$Q_B |\text{phys} \rangle = 0.$$ \hspace{1cm} (2.29)
We decompose the BRST charge as usual with respect to ghost zero modes

\[ Q_B = c_0 L_0 + b_0 M + d. \]  

(2.30)

We note that any physical states are BRST-exact unless they satisfy the on-shell condition \( L_0 = 0 \). It is convenient to reduce the physical subspace by restricting to states annihilated by \( b_0 \) (known as relative cohomology). Our physical state condition is then reduced to

\[ L_0|\text{phys}\rangle = b_0|\text{phys}\rangle = d|\text{phys}\rangle = 0. \]  

(2.31)

We note that \( d^2 = 0 \) in this subspace.

3 BRST cohomology over PF Verma modules

As the first step to derive the complete relative cohomology for the irreducible \( SL(2, R)/U(1) \) PF reps., let us discuss the cohomology over PF Verma modules.

By using the parameters \( k \) and \( l \) defined by

\[ j = \frac{1}{4}(k + l + 2), \quad m = \frac{3}{4}(k - l), \]  

(3.1)

the on-shell condition

\[ h_{j,m} + N \equiv -4j(j - 1) + \frac{4}{9} m^2 + N = 1, \]  

(3.2)

is cast into

\[ N = kl. \]  

(3.3)

This implies that the on-shell condition is satisfied only if \( k, l \geq 0 \) or \( k, l \leq 0 \). Here we will study the case \( k, l \geq 0 \), which corresponds to the restriction \( j \geq 1/2 \) as mentioned in sect. 2. The region \( k, l \leq 0 \) is obtained by the replacement \((j, m) \rightarrow (1 - j, m)\) and is isomorphic to dual spaces as explained in sect. 2.

For \( N = 0 \), we find from the known cohomology \( H^1(\mathcal{F}) = \mathbb{C} \) \([4]\) that we have a physical state called tachyon with continuous spectrum:

\[ H^1(\mathcal{C}) = \mathbb{C}, \quad \text{for } k = 0 \text{ or } l = 0, \quad (m = \pm(3j - \frac{3}{2})). \]  

(3.4)
This tachyon state is in the continuous rep. $C$ for a generic momentum. For special momenta such that $j \pm m \in \mathbb{Z}$, however, this is in other reps. as discussed below.

For $N > 0$, we will show that there are discrete physical states. For this purpose we use the mathematical tools of exact sequences which are recapitulated in appendix B. First note that we have the short exact sequence

$$0 \to \tilde{D}^\pm_{jm} \to F_{jm} \to D^\pm_{jm} \to 0. \quad (3.5)$$

Here the first map is the embedding of $\tilde{D}^\pm_{jm} = F^\pm_{jm} \cap \text{Ker} S^\pm$ [see (2.23)] into the free boson module.

In the following study of cohomology, we use isomorphism among PF modules provided by $S^\pm$. $S^\pm$ cannot act on all states with arbitrary $j$ and $m$. Indeed, the action of $S^+$ is well defined only for states with

$$m - j = \text{integer} \quad \text{or} \quad l = \frac{1}{2}(k - 1) + (\text{integer}), \quad (3.6)$$

and it maps a state at $(k,l)$ to that at $(k + 1, l - \frac{1}{2})$. Similarly $S^-$ is well defined for

$$m + j = \text{integer} \quad \text{or} \quad l = 2k - (\text{odd integer}), \quad (3.7)$$

and it changes $(k,l)$ to $(k - \frac{1}{2}, l + 1)$. On these lines (3.6,7), we are interested in points $(k,l)$, $(k + 1, l - \frac{1}{2})$ and $(k - \frac{1}{2}, l + 1)$ with $k,l \in \mathbb{Z}$. The points $(k + 1, l - \frac{1}{2})$ and $(k - \frac{1}{2}, l + 1)$ are related by the maps $S^\pm$ to points $(k,l)$, on which the cohomology of free boson modules is nontrivial [4]. Consequently the cohomology for PF modules can be nontrivial on these points. We depict these points and the directions of the actions of $S^\pm$ in Fig. 1.

From the definition (2.23) of PF modules and the nilpotency of $S^\pm$, it is easy to see that maps $S^\pm$ give an isomorphism among PF modules

$$D^\pm_{j,m} \cong \tilde{D}^\pm_{j + \frac{1}{2}, m + \frac{1}{2}}, \quad (3.8)$$

This implies an isomorphism of the BRST cohomology classes

$$H^n(D^\pm_{jm}) = H^n(\tilde{D}^\pm_{j + \frac{1}{2}, m + \frac{1}{2}}), \quad (3.9)$$

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since \( S^\pm \) commute with \( Q_B \).

From the long exact sequence associated with the short exact sequence (3.5) and the facts that \( H^n(\mathcal{F}) = 0 \) for \( n \geq 3 \) and \( n \leq 0 \) in the region \( k, l \geq 0 \), we find

\[
H^{n+1}(\tilde{D}_{jm}^\pm) = H^n(D_{jm}^\mp),
\]

for \( n \geq 3 \) and \( n \leq -1 \). Using eq. (3.9), one finds

\[
\begin{aligned}
H^{n+1}(\tilde{D}_{jm}^\pm) &= H^n(\tilde{D}_{j+\frac{1}{2},m\pm\frac{q}{8}}^\pm), \\
H^{n+1}(D_{j-\frac{1}{2},m\mp\frac{q}{8}}^\mp) &= H^n(D_{jm}^\mp),
\end{aligned}
\]

for \( n \geq 3 \) and \( n \leq -1 \). Iterating (3.11) for \( n \geq 3 \), we get

\[
H^n(\tilde{D}_{jm}^\pm) = H^{n+q}(\tilde{D}_{j-\frac{1}{2},m\mp\frac{q}{8}}^\pm),
\]

for any positive integer \( q \). On the other hand, a state with \( j, m \) and level \( N(>0) \) can be nontrivial only if it satisfies the on-shell condition. Due to eq. (3.2), any state representing the RHS in (3.12) must be at the level

\[
N = \left(2j - \frac{q}{4} - 1\right)^2 - \frac{4}{9} \left(m \mp \frac{9}{8}q\right)^2
\rightarrow -\frac{8}{16}q^2 < 0 \quad \text{for} \quad q \rightarrow \infty.
\]

(3.13)

Obviously any state in the module cannot be on-shell for sufficiently large \( q \). We thus conclude the cohomology is trivial for \( n \geq 3 \):

\[
H^n(\tilde{D}_{jm}^\pm) = 0 \quad \text{for} \quad n \geq 3.
\]

(3.14)

For \( n \leq -1 \), we may repeat the same argument into the large negative \( n \) and show

\[
H^n(\tilde{D}_{jm}^\pm) = 0 \quad \text{for} \quad n \leq 0.
\]

(3.15)

The same conclusion is valid for the modules \( D_{jm}^\pm \). So the nontrivial states can exist only for \( n = 1 \) and 2.
To determine $H^{1,2}$, we will start from points on the line $l = 0$ where $S^\pm$ are well defined, then move up to larger $l$. Consider states at $(k,l) = (2p - 1,0)$ for positive integer $p$ and write the long exact sequence associated with (3.5)

\[
\begin{align*}
0 &\xrightarrow{0} H^2(\bar{\mathcal{D}}^+) \xrightarrow{\text{bijection}} H^2(\mathcal{F}) \xrightarrow{0} H^2(\mathcal{D}^-) \\
&\xleftarrow{0} H^1(\bar{\mathcal{D}}^+) \xrightarrow{\text{bijection}} H^1(\mathcal{F}) \xrightarrow{0} H^1(\mathcal{D}^-) \\
&\xleftarrow{0}.
\end{align*}
\]

(3.16)

We know $H^2(\mathcal{F}) = 0$ and $H^1(\mathcal{F}) = \mathbb{C}$ on the line $l = 0$. Also from eq. (3.9), we have $H^{1,2}(\mathcal{D}^-_{\pm}) = H^{1,2}(\bar{\mathcal{D}}^+_{\pm,m+\frac{q}{R}})$. Using this relation to the cohomology classes on the RHS of (3.16), we find

\[
H^2(\mathcal{D}^-) = H^1(\mathcal{D}^-) = 0,
\]

(3.17)

since the states in $\bar{\mathcal{D}}^+_{\pm,\frac{q}{R}+m}$ have $k > 0$ but $l < 0$ and thus cannot satisfy the on-shell condition (3.3). Substituting these into (3.16), we find

\[
H^2(\bar{\mathcal{D}}^+) = 0, \quad H^1(\bar{\mathcal{D}}^+) = \mathbb{C}.
\]

(3.18)

Similarly at $(k,l) = (p - \frac{1}{2},0)$, we have

\[
\begin{align*}
0 &\xrightarrow{0} H^2(\bar{\mathcal{D}}^-) \xrightarrow{\text{bijection}} H^2(\mathcal{F}) \xrightarrow{0} H^2(\mathcal{D}^+) \\
&\xleftarrow{0} H^1(\bar{\mathcal{D}}^-) \xrightarrow{\text{bijection}} \mathbb{C} \xrightarrow{0} H^1(\mathcal{D}^+) \\
&\xleftarrow{0}.
\end{align*}
\]

(3.19)

In this case, we can relate the cohomology on the left to that of $\mathcal{D}^+$ with $l < 0$ by using the action of $S^-$. We find from (3.9)

\[
H^2(\bar{\mathcal{D}}^-) = H^1(\bar{\mathcal{D}}^-) = 0,
\]

(3.20)

which, when substituted into (3.19), tells us

\[
H^2(\mathcal{D}^+) = 0, \quad H^1(\mathcal{D}^+) = \mathbb{C}.
\]

(3.21)
By repeating this procedure, moving up to larger $l$ and using the fact that $H^2(F) = H^1(F) = \mathbb{C}$ for $k, l > 0$, we can determine all the cohomology $H^{1,2}(D^\pm)$ and $H^{1,2}(\tilde{D}^\pm)$. On the points where $S^+$ ($S^-$) can act, the cohomology $H^{1,2}(D^\pm)$ ($H^{1,2}(\tilde{D}^\pm)$) on the right (left) of the long exact sequence may be obtained from $H^{1,2}(\tilde{D}^\pm)$ ($H^{1,2}(D^\pm)$) with smaller $l$ by (3.9). Then the cohomology on the left (right) $H^{1,2}(\tilde{D}^\pm)$ ($H^{1,2}(D^\pm)$) can be found from the long exact sequence.

For the points with both $k$ and $l$ positive odd integers, further study is necessary since the modules $D^\pm$ split into $\tilde{D}^\pm$ and $U$. This splitting yields the short exact sequence

$$0 \to \tilde{D}^\pm \to D^\pm \to U \to 0. \tag{3.22}$$

The long exact sequences associated with (3.22) tell us nontrivial cohomology classes $H^n(U)$.

For continuous reps. $\mathcal{C}$ which appear on the points $k, l \in 2\mathbb{Z}_+$, there is no difference between Fock and PF modules. The cohomology coincides with that of the free boson modules $[4]$.

Proceeding this way, we may determine all the nontrivial cohomology in the region $k, l > 0$:

$$\begin{align*}
\tilde{D}^+ : & \quad H^2 = \mathbb{C} \quad \text{at} \quad (k, l) = (2p, q - \frac{1}{2}), \\
& \quad H^1 = \mathbb{C} \quad \text{at} \quad (k, l) = (2p - 1, q), \\
D^- : & \quad H^2 = \mathbb{C} \quad \text{at} \quad (k, l) = (2p - 1, 2q), \\
& \quad H^1 = \mathbb{C} \quad \text{at} \quad (k, l) = (2p, q - \frac{1}{2}), \\
\tilde{D}^- : & \quad H^2 = \mathbb{C} \quad \text{at} \quad (k, l) = (p - \frac{1}{2}, 2q), \\
& \quad H^1 = \mathbb{C} \quad \text{at} \quad (k, l) = (p, 2q - 1), \\
D^+ : & \quad H^2 = \mathbb{C} \quad \text{at} \quad (k, l) = (2p, 2q - 1), \\
& \quad H^1 = \mathbb{C} \quad \text{at} \quad (k, l) = (p - \frac{1}{2}, 2q), \\
U : & \quad H^2 = H^0 = \mathbb{C} \quad \text{at} \quad (k, l) = (2p - 1, 2q - 1), \\
\mathcal{C} : & \quad H^2 = H^1 = \mathbb{C} \quad \text{at} \quad (k, l) = (2p, 2q), \tag{3.23}
\end{align*}$$
for all positive integers \( p \) and \( q \). Others are trivial. The results agree with those in ref. [15] and are depicted in Fig. 2.

In the above analysis, we neglected the presence of null states in the PF modules: they could be reducible. We will see how the results are modified when null states are taken into account.

4 Null states in PF modules

We would like to identify null states in PF modules. A PF module may be obtained from an \( SL(2,R) \) current algebra module by the condition \( J^3 \sim 0 \). Thus null states in \( SL(2,R) \) current algebra modules are always in PF modules; there is actually a one-to-one correspondence between null states in two modules. So it is sufficient to find null states in \( SL(2,R) \) modules.

In this section, we first describe some general properties of modules by using currents rather than the free field realization. Next we construct null states with charge screening operators by the procedure to be explained shortly. The arguments are generic and the level of the algebra may take arbitrary value. Some special features for \( K = 9/4 \) will be used in the second part of this section. These will be explained in some examples.

For concreteness let us take a LW rep. as a base rep., whose elements are written as \(|j,m\rangle\) with LW state as \(|j,j\rangle\). The LW state is characterized by the primary conditions: \( J^-_0|j,j\rangle = 0 \) and \( J^a_n|j,j\rangle = 0 \), \( (a = \pm, 3 \text{ and } n > 0) \). A rep. of the algebra is generated by the action of \( J^+_0 \) and \( J^a_{\pm n}, (n > 0) \) on \(|j,j\rangle\). Among the states in the generated module, we may have states which satisfy the same condition as the LW state, \( J^-_0|\chi\rangle = 0 \) and \( J^a_n|\chi\rangle = 0 \), \( (a = \pm, 3 \text{ and } n > 0) \). These are called null states and are to be removed to have an irrep.

It is well known that the algebra (2.9) is invariant under the Weyl transformation

\[ J^+_n \rightarrow \tilde{J}^+_n, \quad J^-_n \rightarrow -\tilde{J}^-_n + \frac{K}{2} \delta_{n,0}. \]  

Under this transformation, the rep. is rearranged into another LW rep. From the follow-
ing relations
\[ \mathbf{J}_0^3 |j, m\rangle = (\frac{K}{2} - m) |j, m\rangle, \quad \mathbf{J}_3^2 |j, j\rangle = (\frac{K}{2} - j)(1 - \frac{K}{2} + j) |j, j\rangle, \] (4.2)
we read changes in quantum numbers
\[ m \to \tilde{m} = \frac{K}{2} - m, \quad -j(j - 1) \to (\frac{K}{2} - j)(1 - \frac{K}{2} + j). \] (4.3)

The original rep. may contain a null state \( |\chi\rangle \equiv \mathcal{O}(J_0^+, J_{-n}) |j, j\rangle \). Since the algebra does not change under the Weyl transformation, the state obtained from \( |\chi\rangle \) by the replacement (4.1) is a null state in the transformed rep. In other words, the null states in two reps. are related by the Weyl transformation.

Turning to the free field realization, a general formula for null states is written in terms of charge screening operators. For the \( SL(2, R) \) currents in eq. (2.11), the screening operator is given by
\[ W(z) = \left( \sqrt{\frac{K - 2}{2}} \partial \phi^L + i \sqrt{\frac{K}{2}} \partial \phi^M \right) \exp \left( \sqrt{\frac{2}{K - 2}} \phi^L \right). \] (4.4)
It is easy to see that \( W \) obeys expected OPEs
\[ J^{\pm, 3}(z) W(w) \sim \partial_w (\text{something}). \] (4.5)

We define singular vertex operators \( \Phi_{jm}^s \) corresponding to null fields by
\[ \Phi_{jm}^s(z) = \oint \frac{dz_s}{2\pi i} \int_z^{z_s} \frac{dz_{s-1}}{2\pi i} \cdots \int_z^{z_2} \frac{dz_1}{2\pi i} W(z_s) W(z_{s-1}) \cdots W(z_1) U_{jm}(z) \]
\[ \equiv Q^s U_{jm}(z). \] (4.6)
Since the screening charge (4.4) does not depend on \( J^3 \) or \( \phi^3 \), these null states in current algebra modules are also the null states in PF Verma modules, which are of our interest.

Let us find conditions that \( \Phi_{jm}^s \) is well-defined and non-vanishing. If we normal order the integrand in (4.6), we find that the most singular part has the factor
\[ \prod_{i>j} (z_i - z_j)^{-2/(K-2)} \prod_{i=1}^{s} (z_i - z)^{-2j/(K-2)-1}. \] (4.7)

3The integrations are defined by the analytic continuation in the level \( K \).
Thus the contour integral over $z_s$ is dragging the integrations over $z_i, (i = 1, 2, \cdots, s-1)$. By changing the integration variables as $z_i \rightarrow z_i z_s$, we see that the $z_s$ integration is well-defined and non-vanishing when

$$(s - 1) - \frac{2}{K - 2} \frac{s(s - 1)}{2} - \frac{2js}{K - 2} - s = -N - 1,$$

(4.8)

where $N$ is a non-negative integer, which will be found to be the level of the null state. Solving (4.8) for $j$ yields

$$j = \frac{K - 2 N}{2} s + \frac{1 - s}{2}.$$  

(4.9)

This is only a necessary condition for the presence of singular vertex operators (4.6). As in the Virasoro null states [21, 22], the necessary and sufficient condition is actually that

$$\frac{N}{s} = r$$

is a non-negative integer. Thus there exists a null state for $U_{jm}$ with $j = j(r, -s)$, where

$$j(r,s) = \frac{K - 2}{2} r + \frac{1 + s}{2}.$$  

(4.10)

Since $W$ carries $j$ quantum number, the Casimirs for $\Phi^s_{jm}$ and $U_{jm}$ are given by $J^2 = -j(j - 1)$ with $j = j(r,s)$ and $j = j(r,-s)$, respectively; the map does not change $m$. Therefore the level of a null state (cf. (3.2)) is

$$\frac{1}{K - 2}\left[j(r,s)(j(r,s) - 1) - j(r,-s)(j(r,-s) - 1)\right] = rs = N.$$  

(4.11)

The primary conditions on singular vertex operators are reduced to the same conditions on $U_{jm}$ due to (4.5). Thus $U_{jm}$ itself must correspond to a LW (HW) state for $\Phi^s_{jm}$ to be null vectors. This gives us a condition on $m$ as $m \mp j = 0$.

Proceeding as in the Virasoro case [22], we are led to the Kac-Kazhdan formula [19, 11]

$$D_N = (-1)^{r_3(N)} C_N (K - 2)^{r_3(N)} \prod_{rs = 1}^{N} \left[-j(j - 1) + j(r,s)(j(r,s) - 1)\right]^{p_3(N-rs)},$$

(4.12)

for the determinant of inner products of all states with fixed $m$ at level $N$. Here $C_N$ is a numerical constant. The exponent $r_3(n)$ is the number of factors of $J^3_{-p}$ in all states at level $n$ in a system with three generators; $p_3(n)$ is the number of states at level $n$ in a theory of three free bosons.
The complete degenerate reps. of $SL(2, R)$ current algebra are clarified in refs. [23, 24].

The case of our interest corresponds to subcase F in the classifications [24] and primary states are on the boundary of a conformal grid. Indeed in our model of $SL(2, R)_{9/4}$, the conformal grid has the size $1 \times 4$ and all the states are on its boundary.

A characteristic feature of these boundary modules is that the embedding structure of null states is quite simple; null states at the higher levels belong to submodules generated from null states at lower levels. Therefore in order to get an irrep., it is enough to subtract the submodule generated from the lowest level null state.

The embedding structure of null states is a little complicated when there is a $\mathcal{U}$ rep. since we have two null states at the same level in this case. Let us explain this for $K = \frac{9}{4}$, in a simple example, by using the above consideration on the Weyl transformation and the construction of null states by free fields. Consider a current algebra module on the base with $m = 17/8, 17/8 + 1, \cdots$ in Fig. 3(a). There are three null states in this module at levels 3, 9 and 10 as dictated by the determinant formula (4.12). This may be understood in the free field realization as well: since the Casimir corresponds to $j = 17/8$, we may choose $(r, s) = (1, 3), (5, 2)$ and $(9, 1)$. Therefore we may construct singular vertex operators associated with $(r, -s) = (1, -3), (5, -2)$ and $(9, -1)$ at levels 3, 10 and 9. Null states at higher levels are contained in modules generated from those at lower levels [24].

After Weyl transformation, this module becomes that over the state indicated as $\otimes$ in Fig. 3(b). We have one null state at level 0 and other two null states at the same level 8. Each pair of null states with the same symbols in the figures are related by the transformation.

This new module is $\mathcal{D}^+$ with $j = 2$ or $-j(j - 1) = -2$, for which we may construct three null states associated with $(r, -s) = (0, -3), (4, -2)$ and $(8, -1)$. The null state mapped from $(0, -3)$ has level $N = 0$ and it decomposes this LW module to $\mathcal{U}$ and $\tilde{\mathcal{D}}^+$ modules. The null states with $\tilde{m} = 0$ and 1 is mapped from $(4, -2)$ and $(8, -1)$, respectively.

Since the embedding structure does not change under the Weyl transformation, we can conclude that $\mathcal{U}$ module has no null state. In addition, the null state with $\tilde{m} = 1$ is
obtained from that with $\tilde{m} = 0$ by the action of $\tilde{J}_0^+$. We only need to remove the module generated from the latter in order to obtain a $\tilde{D}^+$ irrep.

Some explanations on $N = 0$ null state are in order. The parent $j_{(0, -3)}$ module related to this null state has a LW state with $\tilde{m} = -1$ rather than $\tilde{m} = 2$. It thus appears that it gives a null at $\tilde{m} = -1$. In this $N = 0$ case, however, the map of screening charge (4.6) is special because only the most singular part contributes to the integral and its coefficient is proportional to $\tilde{m}(\tilde{m}^2 - 1)$. Thus the state $\tilde{m} = 2$ in $j_{(0, -3)}$ module consistently creates the null state with $\tilde{m} = 2$ in $j = 2$ module.

These embedding structures are common in any $D^+ (= U \oplus \tilde{D}^+)$ with $m \pm j \in \mathbb{Z}$.

In $D^+$ there is a null state at level zero which separates the module into $U$ and $\tilde{D}^+$ modules. All the higher level null states are in $\tilde{D}^+$ module. Thus there is no null state in $U$ module. There are two degenerate lowest null states in $\tilde{D}^+$. The module $D^+$ over the null with smaller $\tilde{m}$ is separated into $U$ and $\tilde{D}^+$ due to another null with larger $\tilde{m}$. All the higher null states generate submodules in this $D^+$. Hence one can obtain the irrep. by subtracting this module $D^+ = U \oplus \tilde{D}^+$.

One can easily find a similar structure of null states in HW modules.

Finally we consider the embedding structure of null states in $C$ module.\(^4\) Using the map of the screening charge (4.6), we can formally construct two null states at the same level. However, one of these maps cannot be defined by simple analytic continuation. It is not clear for us when this map is well-defined. This leaves us with two possibilities:\(^5\)

1. There are two lowest null states in a module.

The embedding structure of this case is obtained by counting the number of null states

\(^4\) A null state in $C$ is defined by the condition $J_n^{\pm 3} \Phi_{jm}^s = 0$ for $n > 0$.

\(^5\) Similar indeterminancy of the number of null states appears whenever $K - 2 = \frac{1}{n}$ with positive integer $n$. We have explicitly checked that there is only one null state at the level $N = 2$ in the case $n = 1 (K = 3)$. We thus expect the second structure is the correct one.
in all the submodules. This is illustrated by the following diagrams

\[ j = \text{half integer} : \quad \bullet \quad \underset{\rightarrow}{\square} \quad \underset{\rightarrow}{\square} \quad \cdots \quad \underset{\rightarrow}{\square} \quad \bullet \]

\[ j = \text{integer} : \quad \bullet \quad \underset{\rightarrow}{\square} \quad \underset{\rightarrow}{\square} \quad \cdots \quad \underset{\rightarrow}{\square} \quad \bullet \]

where \( \bullet \) represent null states. The null state at the end of an arrow is in the module over that at the origin of the arrow.

(2) There is only one lowest null state in a module.

The embedding structure of this case is the same as the LW or HW modules. Null states at higher levels are in the module over the lower null states.

### 5 Physical states for \( SL(2, R)/U(1) \)

As explained in the previous section, once we identify the lowest null state, we may subtract the submodule generated over it to get an irrep.

We have shown that there is a null state for

\[ j = \frac{1}{8}(r + 4s + 4); \quad r, s \in \mathbb{Z}_+ \]

at the level \( N = rs \) for \( K = \frac{9}{4} \). Since \( j \) is invariant under the following change of \( r \) and \( s \):

\[ r \rightarrow r' \equiv r + 4p, \quad s \rightarrow s' \equiv s - p, \quad p \in \mathbb{Z} \]

the module has several null states at levels

\[ N = r's' = (r + 4p)(s - p). \]

Each null state is generated from the module at \((r', -s')\) with screening operators \( Q^{s'} \).

It is easy to count the number of null states in a module. Since the level (5.3) must be
positive, we find
\[ s > p > -\frac{r}{4}. \]  
(5.4)

Using the freedom (5.2), we may always choose \( r \) in the range \( 4 \geq r \geq 1 \). It then follows from (5.4) that \( p \) can take values \( p = 0, 1, \ldots, s - 1 \), which implies that there are \( s \) null states. From (5.1) and (3.1), \((k, l)\) and \((r, s)\) are related as \( k + l = r/2 + 2s \). This defines lines on the \((k, l)\) plane and multiplicities of null states on them are \( s \). On the \((k, l)\) plane in Fig. 4, we show the number of null states at points studied in sect. 3.

Now let us compute the cohomology of irreps. Consider the short exact sequence
\[ 0 \to D_{j(r,s),m}^{(n-1)} \to D_{j(r,s),m}^{(n)} \to \hat{D}_{j(r,s),m} \to 0, \]  
(5.5)

where \( D_{jm}^{(n)} \) is the PF Verma module with \( n \) null states and \( \hat{D}_{jm} \) is an irrep., whose cohomology is of our interest. The first map with \( Q^s \) is the embedding of \( D_{jm}^{(n)} \) onto the submodule over the lowest level null state in \( D_{jm}^{(n)} \). The cohomology for the modules \( D^{(n-1)} \) and \( D^{(n)} \) will be obtained from our earlier results in sect. 3. The cohomology for the irrep. \( \hat{D} \) is then calculated from the long exact sequences associated with (5.5).

Let us explain our procedure in more detail. Take a point in the first quadrant on the \((k, l)\) plane studied in sect. 3. From (3.1) and (5.1), we find \( 2j(r,s) - 1 = r/4 + s = (k+l)/2 \) and \( 2m/3 = (k - l)/2 \). We know the cohomology for modules with \( j(r,s) \) and \( m \).

In order to construct an irrep. \( \hat{D}_{j(r,s),m} \), we would like to identify \( D_{j(r,-s),m}^{(n-1)} \) with less null states by one for a given \( D_{j(r,s),m}^{(n)} \). Note that the two modules have different \( j \) but the same \( m \). From this observation, we may find \((k, l)\) coordinates for \( D_{j(r,-s),m}^{(n-1)} \), \((k', l')\), as follows: \( k' = k - 2s \) and \( l' = l - 2s \). When \( 4 \geq r \geq 1 \), \( Q^s \) realizes the desired map from \( D_{j(r,-s),m}^{(n-1)} \) to \( D_{j(r,s),m}^{(n)} \) so that we have an irrep. \( \hat{D} \). The changes in \((k, l)\) coordinates by the map are

\[
\begin{align*}
(k', l') &= (-2q, -2p + i/2) \to (2p, 2q + i/2) = (k, l), \\
(-2q + i/2, -2p) &\to (2p + i/2, 2q), \\
(-2q, -2p) &\to (2p + 2, 2q + 2), \\
(1 - 2q, 1 - 2p) &\to (2p + 1, 2q + 1),
\end{align*}
\]  
(5.6)

20
where \( p, q \) are positive integers and \( i = 1, 2, 3 \). Since both \( k' \) and \( l' \) are negative for \( D_{j(r-s),m}^{(n-1)} \), we realize that it is dual to some module in the first quadrant listed in Fig. 4. When a module has quantum numbers \((j, m)\), the dual module carries \((1 - j, m)\). This corresponds to the change \((k, l) \leftrightarrow (-l, -k)\) on the \((k, l)\) plane. With the isomorphism in (2.26), the cohomology at \((k', l')\) is obtained by the replacement

\[
\tilde{D}^- \leftrightarrow D^-, \tilde{D}^+ \leftrightarrow D^+, H^2 \leftrightarrow H^0.
\]  
(5.7)

from the results for \((-l', -k')\) in sect. 3. Here the third relation is due to the dual structure of ghost Fock space. It is useful to see relations between \((-l', -k')\) and \((k, l)\) on Fig. 4. Note that the points \((-l', -k')\) and \((k, l)\) are on the same line \(k - l = \frac{4}{3}m\) and the change of \(k\) coordinate is \(k \rightarrow -l' = 2s - l\). We see that (1) when either \(k\) or \(l\) is a half-integer, \((-l', -k')\) is the closest point to the left on the line; (2) when both \(k\) and \(l\) are integers, \((-l', -k')\) is the next closest point to the left on the line. The change of \(s\), the number of null states, is one for (1) and two for (2).

As \( D_{j(r-s),m}^{(n)} \), we study four different cases with \((k, l) = (i) (2p, q-1/2), (ii)(2p-1, 2q), (iii)(2p - 1, 2q - 1) \) and (iv)(2p, 2q). Other possibilities are realized by exchanging \(k\) and \(l\). The possible patterns for long exact sequences to be studied later are given below:

\[
\begin{array}{c}
0 \\
\Downarrow \\
a \rightarrow C \rightarrow H^2 \\
\Downarrow \\
C \rightarrow b \rightarrow H^1 \\
\Downarrow \\
0 \rightarrow 0 \rightarrow H^0 \\
\Downarrow \\
0 \rightarrow H^{-1} \\
\end{array} \quad \begin{array}{c}
0 \\
\Downarrow \\
0 \rightarrow 0 \rightarrow H^2 \\
\Downarrow \\
a \rightarrow C \rightarrow H^1 \\
\Downarrow \\
b \rightarrow 0 \rightarrow H^0 \\
\Downarrow \\
0 \rightarrow H^{-1} \\
\end{array}
\]
(5.8)

(A) \quad (B)

We will indicate below the long exact sequence for each case, from which the desired cohomology can be obtained. The cohomology is summarized in eq. (5.10).
Case (i) \((k, l) = (2p, q - 1/2)\)

From Fig. 2, we find that \(H^2(\tilde{D}^+) = H^1(\tilde{D}^-) = C\) for \(\mathcal{D}^{(n)}\). When \(q = 1, 2, l' = 0\) and the only nontrivial cohomology is \(H^1(\tilde{D}^+) = C\), corresponding to the tachyon. Applying the rule (5.7), we find \(H^1(\tilde{D}^+) = C\) for \(\mathcal{D}^{(n-1)}\). The long exact sequence for \(\tilde{\mathcal{D}}^+\) is in the pattern (A) with \(a = b = 0\). It is easy to find that \(H^{0,2}(\tilde{D}^+) = C\). For \(\tilde{\mathcal{D}}^-\), the sequence is in the pattern (B) with \(a = b = 0\) and the result is \(H^1(\tilde{D}^-) = C\). When \(q > 2\), \(H^2(\tilde{D}^-) = H^1(\tilde{D}^+) = C\) for \((-l', -k')\), so \(H^0(\tilde{D}^-) = H^1(\tilde{D}^+) = C\) for \(\mathcal{D}^{(n-1)}\) at \((k', l')\). The sequences are (A) with \(a = b = 0\) for \(\tilde{\mathcal{D}}\) and (B) with \(a = 0\) and \(b = C\) for \(\tilde{\mathcal{D}}^-\).

Case (ii) \((k, l) = (2p - 1, 2q)\)

We have \(H^1(\tilde{D}^+) = H^2(\tilde{D}^-) = C\) for \(\mathcal{D}^{(n)}\). We find \(H^2(\tilde{D}^+) = H^1(\tilde{D}^-) = C\) at \((-l', -k')\), so \(H^0(\tilde{D}^+) = H^1(\tilde{D}^-) = C\) for \(\mathcal{D}^{(n-1)}\). The sequences are (A) with \(a = b = 0\) for \(\tilde{\mathcal{D}}^-\) and (B) with \(a = 0\) and \(b = C\) for \(\tilde{\mathcal{D}}\). When \(p = 1\), we have to modify the above argument slightly: \(H^0(\tilde{D}^+) = 0\) for \(\mathcal{D}^{(n-1)}\) and the pattern is (B) with \(a = b = 0\) for \(\tilde{\mathcal{D}}^+\).

For cases (iii) and (iv), there are two lowest null states associated with two possible values for \((r, -s)\) (with \(r = 4\) or \(s = 1\)). So other than the mapping given in (5.6), there could be another map, for which \((k, l)\) coordinates change as

\[
(k', l') = (p - 2, q - 2) \rightarrow (p, q) = (k, l)
\]

Hence we need further investigation to obtain the cohomology of irreps.

Case (iii) \((k, l) = (2p - 1, 2q - 1)\)

For this case, we have already clarified the embedding structure of the null states in the previous section. We should use (5.6) to obtain the irreps.

Nontrivial cohomology classes for \(\mathcal{D}^{(n)}\) are \(H^{2,0}(\mathcal{U}) = H^1(\tilde{D}^\pm) = C\). As we have shown in sect. 4, there is no null state in \(\mathcal{U}\) reps. so that the results in sect. 3 are valid for them. We only have to study \(\tilde{\mathcal{D}}^\pm\) reps. When \(p = q = 1\), there is no null state and the module is irreducible. When either \(p\) or \(q\) is 1, say \((k, l) = (2p - 1, 1), (k', l') = (1, 3 - 2p)\) and the cohomology for \(\mathcal{D}^{(n-1)}\) is trivial. The long exact sequence is (B) with \(a = b = 0\) for \(\tilde{\mathcal{D}}^\pm\). For a generic \((p, q)\), we find \((k', l') = (3 - 2q, 3 - 2p)\). The sequence is (B) with
\(a = C\) and \(b = 0\) for \(\tilde{D}^{\pm}\). Due to the pattern, \(0 \to H^0 \to C \to C \to H^1 \to 0\), there are two possible solutions; \(H^{1,0}(\tilde{D}^{\pm}) = 0\) or \(C\). Accordingly we find two possibilities for the cohomology for \(\tilde{D}^{\pm}\). The first possibility is shown in Fig. 5.

Case (iv) \((k, l) = (2p, 2q)\)

Here we have only \(C\) modules. It is crucial to know the embedding structure of null states since two possibilities of \(D^{(n-1)}\) in (5.6) and (5.9) give different cohomologies. We have the following two possibilities for the structure.

1. There are two null states in a module: the above two maps give different null states.
2. There is only one null state in a module: the map \((2p - 2, 2q - 2) \to (2p, 2q)\) give the null state and the map \((2 - 2q, 2 - 2p) \to (2p, 2q)\) is ill-defined.\(^6\)

Let us compute the cohomology of irreps. for each case.

For the first possibility, the embedding structure of null states is not so simple, as explained in the previous section. It is not enough to subtract modules over the lowest level null states; we have to subtract irreps. over all the null states in order to get the right cohomology of an irrep. Note, however, that if we know the cohomology of irreps. with less null states, we can find the cohomology of a module by subtracting the irreps. Since submodules on the points with less \(k + l\) (on the \((k, l)\) plane) have less null states, we can compute the cohomology of irrep. inductively, starting from modules at smaller \(k + l\) to those at larger \(k + l\). In this way, we can compute all the cohomology of irreps. in principle. However, the long exact sequence does not give a unique cohomology as in the case (iii). Since the cohomology of an irrep. is obtained by using that with less null states, we have more possibilities for the former cohomology than the latter.

For the second possibility, we find \(H^{2,1}(C) = C\) for \(D^{(n)}\). When \(p = 1\) or \(q = 1\), we have \(H^1 = C\) for \(D^{(n-1)}\). The pattern is (A) with \(a = 0\) and \(b = C\) and \(H^2(C) = C\). As for \(H^{1,0}(\hat{C})\), we encounter the same indeterminancy mentioned for Case (iii), obtaining \(H^{1,0}(\hat{C}) = 0\) or \(C\). Fig. 5 shows the latter possibility. When \(p, q > 1\), \(H^1(C) = H^2(C) = C\)

\(^6\) The map \((2p - 2, 2q - 2) \to (2p, 2q)\) is always well-defined since it has only one integration with a closed contour.
at \((k', l') = (2p-2, 2q-2)\) for \(D^{(n-1)}\). The long exact sequence is (A) with \(a = b = C\). We have four possibilities for the cohomology: \((H^0, H^1, H^2) = (0, 0, 0), (0, C, C), (C, C, 0)\) or \((C, C \oplus C, C)\). The second possibility is shown in Fig. 5.

Let us summarize the nontrivial cohomology obtained in our study:

\[
\hat{\mathcal{D}}^+: \quad H^2 = H^0 = C \quad \text{at} \quad (2p, q - \frac{1}{2}),
\]
\[
H^1 = C \quad \text{at} \quad (2p - 1, 2q), (2p - 1, 1), (1, 2q - 1),
\]
\[
H^1 = H^0 = 0 \text{ or } C \quad \text{at} \quad (2p + 1, 2q + 1)
\]
\[
H^{-1} = C \quad \text{at} \quad (2p + 1, 2q),
\]
\[
\hat{\mathcal{D}}^-: \quad H^2 = H^0 = C \quad \text{at} \quad (2p - 1, 2q),
\]
\[
H^1 = C \quad \text{at} \quad (2p, q - \frac{1}{2}),
\]
\[
H^{-1} = C \quad \text{at} \quad (2p, q + \frac{3}{2}),
\]
\[
\hat{\mathcal{D}}^-: \quad H^2 = H^0 = C \quad \text{at} \quad (p - \frac{1}{2}, 2q),
\]
\[
H^1 = C \quad \text{at} \quad (2p, 2q - 1), (2p - 1, 1), (1, 2q - 1),
\]
\[
H^1 = H^0 = 0 \text{ or } C \quad \text{at} \quad (2p + 1, 2q + 1)
\]
\[
H^{-1} = C \quad \text{at} \quad (2p, 2q + 1),
\]
\[
\hat{\mathcal{D}}^+: \quad H^2 = H^0 = C \quad \text{at} \quad (2p, 2q - 1),
\]
\[
H^1 = C \quad \text{at} \quad (p - \frac{1}{2}, 2q),
\]
\[
H^{-1} = C \quad \text{at} \quad (p + \frac{3}{2}, 2q),
\]
\[
\hat{U}: \quad H^2 = H^0 = C \quad \text{at} \quad (2p - 1, 2q - 1),
\]
\[
\hat{C}: \quad (H^2, H^1, H^0) = (0, 0, 0) \text{ or } (0, C, C) \text{ or } (C, C, 0) \text{ or } (C, C \oplus C, C)
\]
\[
\text{at} \quad (2p + 2, 2q + 2),
\]
\[
H^2 = C, H^1 = H^0 = 0 \text{ or } C \quad \text{at} \quad (2, 2q), (2p, 2).
\]

(5.10)

for all positive integers \(p\) and \(q\). Here we only list the results of case (2) for \(\mathcal{C}\) rep. There is some indeterminancy in our results at the points \((2p, 2q)\) and \((2p + 1, 2q + 1)\). We have shown in Fig. 5 one possibility of these results.
Now let us compare this spectrum with that of $c = 1$ gravity. The discrete states of $c = 1$ gravity correspond to states with $k, l \in \mathbb{Z}_+$ and ghost numbers $N_{FP} = 1, 2$. Our maximum spectrum includes these states. In addition to these states, there are an infinite number of discrete states in the spectrum. We give a simple example to show how these physical states appear in the next section.

6 Discussions

Using the cohomological terms, we have identified the nontrivial cohomology in the $SL(2, R)/U(1)$ coset model. The physical states in the theory exist for the ghost number $N_{FP} = -1 \sim 2$.

Let us now discuss the character formulae for reps. of the coset model and their relation to our results. As discussed by Distler and Nelson [15], if we define the character-valued index by

$$
\text{Ind}(q) = q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^2 \chi_{jm}(q),
$$

(6.1)

where $\chi_{jm}(q)$ is the character $\text{Tr}(q^{L_0})$ of the $SL(2, R)/U(1)$ PF module, the $q^0$ term in $\text{Ind}(q)$ gives the index of the BRST operator

$$
\text{Index}(Q_B) = \sum_n (-1)^{n+1} \text{dim}(H^n),
$$

(6.2)

Using the character formulae, Distler and Nelson computed the index, the alternating sum of the dimensions of the cohomology. Let us see how this works.

The characters for hermitian reps. are known as [20, 15]

$$
\chi_{jm}(\mathcal{C}) = q^{hjm} \prod_{n=1}^{\infty} (1 - q^n)^{-2},
$$

$$
\chi_{jm}(\mathcal{D}^-) = q^{hjm} \prod_{n=1}^{\infty} (1 - q^n)^{-2} \sum_{s=0}^{\infty} (-1)^s q^{s(s-2m-2j+1)/2},
$$

$$
\chi_{jm}(\mathcal{D}^+) = \chi_{j,-m}(\mathcal{D}^-),
$$

$$
\chi_{jm}(\mathcal{D}^-) = \chi_{1-j,m}(\mathcal{D}^-),
$$

$$
\chi_{jm}(\mathcal{D}^+) = \chi_{1-j,-m}(\mathcal{D}^-),
$$

$$
\chi_{jm}(\mathcal{D}^-) = \chi_{1-j,-m}(\mathcal{D}^-),
$$

25
\[ \chi_{jm}(\mathcal{U}) = \prod_{n=1}^{\infty} (1 - q^n)^{-2} \left[ 1 - \sum_{s=0}^{\infty} (-1)^s q^{s(s-2j-1)/2} (q^{ms} + q^{-ms}) \right]. \quad (6.3) \]

Actually these characters must be modified since there exist null states in general. If we use them without modifications, they give the indices corresponding to the cohomology over the PF Verma modules. Let us show that our results obtained in sect. 3 are consistent with these indices.

For \( \tilde{D}^- \) reps., using (3.2) and (3.3), we see that the condition that the index of \( Q_B \) is not zero gives

\[- kl + \frac{1}{2} s(s - 2k + l) = 0, \quad (6.4)\]

for non-negative integer \( s \). This has the solution

\[ k = \frac{s}{2}, \quad (6.5) \]

Combined with the condition \( m - j = \frac{1}{2} (k - 2l - 1) = \text{integer} \), this gives the following solutions. If \( s \) is even \( s = 2\bar{s} \), the index is +1 for

\[ k = \bar{s}, \quad l = 2r + 1; \quad r, \bar{s} \geq 0. \quad (6.6) \]

If \( s \) is odd \( s = 2\bar{s} + 1 \), the index is -1 for

\[ k = \bar{s} + \frac{1}{2}, \quad l = 2r; \quad r, \bar{s} \geq 0. \quad (6.7) \]

This result is consistent with that in sect. 3.

It is easy to check that for all other reps. this analysis gives results consistent with those in sect. 3, although it does not determine precisely where the physical states are. Moreover, it cannot determine whether there are physical states when the index is zero.

The characters of the irreps. are obtained by subtracting the contributions of null states. Let us discuss how the above analysis is modified when this point is taken into account. In order to derive the correct characters, note that it is enough to subtract submodule generated on the null states which are indicated in Fig. 4. For example, consider the \( \tilde{D}^+ \) rep. with \( j = \frac{1}{4}(2p+2q-\frac{1}{2}+2) \) for positive integers \( p, q \). The contributions
of the null states are subtracted if we subtract the character at \( j' = -j + \frac{7}{4} \). We have the character

\[
\hat{\chi}_{jm}(\hat{D}^\pm) = \chi_{jm}(\hat{D}^\pm) - \chi_{-j+7/4,m}(\hat{D}^\pm)
\]  

(6.8)

The same analysis as above then tells us that the nonvanishing index is

\[
\text{Index}(Q_B) = -2,
\]  

(6.9)

in agreement with the results in sect. 5. At more general points where the cohomology is nontrivial, we get the characters of irreps.

\[
\hat{\chi}_{jm} = \chi_{jm} - \chi_{-j+(8-i)/4,m},
\]  

(6.10)

for \((k,l) = (2p,2q-i/2) \) or \((2p-i/2,2q)\) with \(i = 1, 2, 3\). For the points \((k,l) = (2p+1,2q+1)\), the null state structure is a bit complicated as shown in sect. 4. There is no null state in \(U\) reps. and the complementary \(\tilde{D}^\pm\) reps. have two null states at the lowest level. Thus we get for these reps.

\[
\hat{\chi}_{jm}(\hat{U}) = \chi_{jm}(U),
\]  

(6.11)

\[
\hat{\chi}_{jm}(\hat{\tilde{D}}^\pm) = \chi_{jm}(\tilde{D}^\pm) - \chi_{2-j,m(U)} - \chi_{j-1,m(\tilde{D}^\pm)}.
\]  

(6.12)

For \(C\) reps., we obtain two possible characters corresponding to the cases given in the previous section. For case (1), the character is given by an alternative sum as usual complete degenerate reps. Here we give it as a recursion relation

\[
\hat{\chi}_{jm}(\hat{C}) = \chi_{jm}(C) - \chi_{2-j,m}(C) - \hat{\chi}_{j-1,m}(\hat{C}),
\]  

(6.13)

where the initial characters are

\[
\hat{\chi}_{\frac{1}{2},m} = \chi_{\frac{3}{2},m} - \chi_{\frac{1}{2},m},
\]

\[
\hat{\chi}_{2,m} = \chi_{2,m} - \chi_{0,m} - \chi_{1,m},
\]  

(6.14)

for \(j = \) half-integer and integer, respectively. For case (2), the character has the similar form to other reps. It is simply obtained by single subtraction:

\[
\hat{\chi}_{jm}(\hat{C}) = \chi_{jm}(C) - \chi_{j-1,m}(C).
\]  

(6.15)
One can see that these characters give indices consistent with our results in sect. 5.

In this paper, we have considered only holomorphic sector of the theory. For constructing a complete theory of the string, we must take tensor products of holomorphic and antiholomorphic sectors so as to give modular invariant partition function. The complete classification of the modular invariant combinations of the characters is left to future investigation.

We have proved that there are an infinite number of discrete states in the string theory obtained from the \( SL(2, R)/U(1) \) coset model. These discrete states include additional states which are not present in the \( c = 1 \) gravity. In ref. [16], however, it has been claimed that the spectrum is isomorphic to that of \( c = 1 \) gravity and there is no additional physical state in contrast to our results. Let us discuss why there is such a discrepancy and how our new physical states appear by a simple example.

Our example is a set of the states with \( j = m = 9/8 \) at level 1, which correspond to the additional discrete states absent from the spectrum of the \( c = 1 \) gravity (Fig. 6).

We first examine the BRST quartet structure in the vector space \( \mathcal{F}^+_j \otimes \mathcal{F}_{gh} \), where \( \mathcal{F}^+_j \) is defined in sect. 2 and \( \mathcal{F}_{gh} \) is the ghost Fock space. The vector space \( \mathcal{F}^+_{9/8} \) has two on-shell states at level \( N = 1 \):

\[
|\psi_1\rangle = J^+_{-1}\left|\frac{9}{8}, \frac{9}{8}\right\rangle + \frac{5}{4} J^-_{-1}\left|\frac{9}{8}, \frac{17}{8}\right\rangle,
\]

\[
|\psi_2\rangle = J^-_{-1}\left|\frac{9}{8}, \frac{17}{8}\right\rangle - \frac{8}{9} J^3_{-1}\left|\frac{9}{8}, \frac{9}{8}\right\rangle,
\]

which satisfy the condition

\[
J^3_n|\psi_{1,2}\rangle = 0, \quad n \geq 1.
\]

We find the following structure under the action of the BRST charge on the states:

\[
Q_B^b_{-1}\left|\frac{9}{8}, \frac{9}{8}\right\rangle = 9\left|\psi_2\right\rangle,
\]

\[
Q_B\left|\psi_1\right\rangle = \frac{5}{2} c_{-1}\left|\frac{9}{8}, \frac{9}{8}\right\rangle,
\]

where we denote the tensor product state with ghost physical vacuum as \( ||*\rangle\rangle = |*\rangle\otimes c_1|0\rangle\).

Thus all the states in (6.18) and (6.19) are in the BRST quartet and decouple from the physical subspace in the vector space \( \mathcal{F}^+_j \otimes \mathcal{F}_{gh} \).
In the PF irrep. (with the ghost Fock space), however, the BRST quartet structure is different. First of all, it is true that there is a state which has nonzero inner product with \( |\psi_2\rangle \) in the vector space \( \mathcal{F}^+_j \), but \( |\psi_2\rangle \) is a null state in the PF module and must be set equal to zero to obtain the PF irrep.; \( b_{-1} |\begin{pmatrix} 9 \\ 8 \\ 9 \\ 8 \end{pmatrix}\rangle \) is a singlet in the PF rep.

To discuss the second doublet relation (6.19), we must first explain how the space of base states \( \mathcal{F}^\pm_j = \bigoplus_{m=\pm j} \mathbb{Z} \mathcal{F}_{j,m} \) (see (2.22)) splits into the base reps. As explained in sect. 2, the space \( \mathcal{F}^\pm_j \) splits into two irreps. as \( \mathcal{F}^\pm_j = \tilde{\Delta}^\pm \oplus \Delta^\mp \) when \( j \mp m \in \mathbb{Z} \). However, the transformations by currents do not respect this property in the whole space \( \mathcal{F}^\pm_j \). From the OPE (2.14), we obtain

\[
J_0^+ |j,m\rangle = (m \pm j) |j,m \pm 1\rangle, \\
J_0^- |j,m\rangle = m |j,m\rangle,
\]

(6.20)

which shows that \( J_0^- \) correctly annihilates the lowest state \( |j,j\rangle \) in \( \tilde{\Delta}^+ \):

\[
J_0^- |j,j\rangle = 0,
\]

(6.21)

but \( J_0^+ \) does not the highest \( |j,j-1\rangle \) in \( \Delta^- \):

\[
J_0^+ |j,j-1\rangle = (2j - 1) |j,j\rangle \neq 0,
\]

(6.22)

which must be put to zero. Thus, for the whole PF modules, the states in \( \tilde{\mathcal{D}}^\pm \) are null if they appear upon applying the PF currents on \( \mathcal{D}^\pm \), as defined in eq. (2.23).

In the above example, we immediately notice that the state \( |\psi_1\rangle \) does not make sense as a PF rep. since it consists of two states \( J_{-1}^+ |\begin{pmatrix} 9 \\ 8 \\ 1/8 \end{pmatrix}\rangle \) and \( J_{-1}^- |\begin{pmatrix} 9 \\ 8 \\ 17/8 \end{pmatrix}\rangle \) in different reps. \( \mathcal{D}^- \) and \( \tilde{\mathcal{D}}^+ \), respectively (see Fig. 6). We must reinterpret these states as the PF reps.

For the state in \( \mathcal{D}^- \), we take \( |\psi'_1\rangle = J_{-1}^- |\begin{pmatrix} 9 \\ 8 \\ 1/8 \end{pmatrix}\rangle \). At first sight, this state does not appear to be in the PF module since

\[
J_1^3 |\psi'_1\rangle = J_0^- |\begin{pmatrix} 9 \\ 8 \\ 1/8 \end{pmatrix}\rangle = \frac{5}{4} |\begin{pmatrix} 9 \\ 8 \\ 9 \\ 8 \end{pmatrix}\rangle,
\]

(6.23)

which is responsible for the doublet structure in (6.19). However, as explained above, the RHS must be set equal to zero to obtain the irrep. \( \mathcal{D}^- \). The doublet relation in eq.
(6.19) should then be replaced by

$$Q_B ||\psi'_1|| = 0. \tag{6.24}$$

Thus $$||\psi'_1||$$ is a singlet in $$\mathcal{D}^-$$. The ghost excited state $$c_{-1}||\frac{q}{8}, \frac{q}{8}||$$ is then a singlet in $$\tilde{\mathcal{D}}^+$$ since its would-be BRST partner is in the different rep. $$\mathcal{D}^-$$.

To summarize, in the above example there is no physical state in the vector space $$\mathcal{F}^\pm$$. All the states fall into a BRST quartet rep. and decouple from the physical subspace. In the PF irreps., however, there are three physical discrete states $$b_{-1}||\frac{q}{8}, \frac{q}{8}||$$, $$||\psi'_1||$$ and $$c_{-1}||\frac{q}{8}, \frac{q}{8}||$$ with ghost number $$N_{FP} = 0, 1$$ and 2, respectively. This conclusion agrees with the general results obtained in sect. 5.

In this way, the physical spectrum may be isomorphic to that of $$c = 1$$ gravity if we take the vector space $$\mathcal{F}_j^\pm$$ as a starting point of the string theory on the black hole background. On the other hand, if we start from the irreps. of the $$SL(2,R)/U(1)$$ coset, the spectrum does contain additional infinite number of discrete states, which we have proved in this paper.

We hope that our discovery of the new physical states provides some clues to uncover the consistent picture of the quantum theory of gravity.

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Appendix

A An alternative formulation of $SL(2, R)/U(1)$

In this appendix, we discuss an alternative formalism to realize reps. of the coset $SL(2, R)/U(1)$. This is useful for the following reason. The states in PF modules as defined by $J^3_n = 0$, $n \geq 1$ have the same momenta (apart from the factor $i$) for $\phi^M$ and $\phi_3$ but the operators $S^\pm$ carry only $\phi^M$ momentum. One may wonder that the mapping of $S^\pm$ is not closed in the PF modules. This is not a problem because two modules with different $\phi^3$ momenta give the equivalent reps. of the same PF module. We can take the modules with vanishing $\phi^3$ momentum as the representative of PF modules in which the $S^\pm$ mapping is closed. Still it may be better to have a formulation which avoids this complication. This is what we describe below.

The idea is the following. We introduce an additional free field $X$ as well as two “fermionic ghosts” $\eta, \xi$, and impose the constraint that they make a BRST quartet and all decouple from the theory. Let us define the $U(1)$ BRST charge by

$$Q^{U(1)} = \oint \frac{dz}{2\pi i} \xi \partial (\phi^3 - iX).$$

(A.1)

By imposing the constraint

$$Q^{U(1)} |\psi\rangle = 0,$$

(A.2)

on the reps. $|\psi\rangle$ of the current algebra $SL(2, R)$, we get those of $SL(2, R)/U(1)$.

We define

$$\tilde{\psi}^\pm = \frac{1}{\sqrt{K}} J^+_\pm \exp \left( \mp i \sqrt{\frac{2}{K}} X \right),$$

$$= \left( \frac{i}{\sqrt{2}} \partial \phi^M \mp \sqrt{\frac{K-2}{2K}} \partial \phi^L \right) \exp \left( \pm \sqrt{\frac{2}{K}} (i\phi^M + \phi^3 - iX) \right),$$

$$\tilde{S}^\pm = \exp \left( \sqrt{\frac{K-2}{2}} \phi^L \pm \sqrt{\frac{K}{2}} (i\phi^M + \phi^3 - iX) \right).$$

(A.3)

It is easy to see that

$$[Q^{U(1)}, \tilde{\psi}^\pm(z)] = 0.$$

(A.4)
Due to the constraint (A.2), these operators are equivalent to the corresponding ones defined in the text. This can be seen by computing the OPE among the operators.

\[
\tilde{\psi}^\pm(z)\tilde{\psi}^\pm(w) \sim O((z - w)^{2/K}) \\
\sim \psi^\pm(z)\psi^\pm(w), \\
\tilde{\psi}^+(z)\tilde{\psi}^-(w) \sim (z - w)^{-2/K}\left[\frac{1}{(z - w)^2} + \frac{\sqrt{2}}{K(K - 2)}(\partial\phi^3 - i\partial X) + \frac{1}{\sqrt{2}K}\partial^2\phi^L\right] \\
\sim \psi^+(z)\psi^-(w) + \left\{Q^U(1), (z - w)^{-1-2/K}\sqrt{\frac{2}{K}}\eta(w) + (z - w)^{-2/K}\left[\frac{1}{K}\eta(w)(\partial\phi^3 - i\partial X) + \frac{1}{\sqrt{2}K}\partial\eta(w)\right]\right\}. \quad (A.5)
\]

where \(\psi^\pm\) are the PF’s defined in the text. These relations show that the operators in (A.3) are PF’s up to \(U(1)\) BRST-exact form. We also have

\[
\tilde{S}^+(z)\tilde{S}^+(w) \sim \text{regular}, \\
\tilde{S}^+(z)\tilde{S}^-(w) \sim \partial\left[-\frac{1}{\sqrt{K(z - w)}}\exp\left(\sqrt{\frac{K - 2}{2}}(\phi^L - \frac{K - 2}{\sqrt{2K}}(i\phi^M + \phi^3 - iX)\right)\right] \\
- \frac{1}{\sqrt{2(z - w)}} \times \left\{Q^U(1), \eta\exp\left(\sqrt{\frac{K - 2}{2}}(\phi^L - \frac{K - 2}{\sqrt{2K}}(i\phi^M + \phi^3 - iX)\right)\right\}. \quad (A.6)
\]

which shows that \(\tilde{S}^\pm\) commute with \(\tilde{\psi}^\pm\) up to \(U(1)\) BRST-exact form. An important difference from the formulation in the text is that the reps. in the \(SL(2, R)/U(1)\) are defined without spoiling the dependence on \(\phi^M\) and \(\phi^3\), and that \(\tilde{S}^\pm\) acts in PF modules without the complication noted above.
B Exact sequences

In this appendix, we summarize mathematical tools necessary to the discussions in the text. This is already well known, but we include it for completeness. We refer the reader to ref. [25] for details.

Our purpose is to examine the structure of the differential complex $\mathcal{C}$ which is a direct sum of the vector space $\mathcal{C}^q$, ($q$: integer) on which the nilpotent differential operator $d$ acts as

$$
\ldots \xrightarrow{d} \mathcal{C}^{q-1} \xrightarrow{d} \mathcal{C}^q \xrightarrow{d} \mathcal{C}^{q+1} \xrightarrow{d} \ldots .
$$

(B.1)

The object of our interest is the cohomology of $\mathcal{C}$:

$$
H(\mathcal{C}) = \bigoplus_{q \in \mathbb{Z}} H^q(\mathcal{C}),
$$

(B.2)

where $H^q(\mathcal{C})$ is defined by

$$
H^q(\mathcal{C}) = \text{Ker}(d) \cap \mathcal{C}^q / \text{Im}(d) \cap \mathcal{C}^q.
$$

(B.3)

A map $f: \mathcal{A} \rightarrow \mathcal{B}$ between two differential complexes is a chain map if it commutes with the differential operator $d$.

A sequence of vector spaces $V_i$

$$
\ldots \xrightarrow{f_{i-2}} V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \xrightarrow{f_{i+1}} \ldots
$$

(B.4)

is said to be exact if for all $i$

$$
\text{Ker}(f_i) = \text{Im}(f_{i-1}).
$$

(B.5)

What is most important to us is the short exact sequence

$$
0 \rightarrow \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \rightarrow 0.
$$

(B.6)

The first part of this sequence means that the exact map $f$ is one-to-one since it has only 0 as its kernel. Similarly the latter part implies that the exact map $g$ is onto. Given exact maps $f$ and $g$, $\mathcal{A}$ is embedded into the space $\mathcal{B}$ and the rest of the space $\mathcal{B}$ is mapped to $\mathcal{C}$. 

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Given differential complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and chain maps $f : \mathcal{A} \to \mathcal{B}, g : \mathcal{B} \to \mathcal{C}$ which form a short exact sequence, we can get the long exact sequence of cohomology groups

\[
\begin{align*}
H^{q+1}(\mathcal{A}) &\xrightarrow{f^*} \cdots \\
\uparrow d^* & \text{ } \\
H^q(\mathcal{A}) &\xrightarrow{f^*} H^q(\mathcal{B}) \xrightarrow{g^*} H^q(\mathcal{C}) \\
\uparrow d^* & \text{ } \\
H^{q-1}(\mathcal{A}) &\xrightarrow{f^*} H^{q-1}(\mathcal{B}) \xrightarrow{g^*} H^{q-1}(\mathcal{C}) \\
\uparrow d^* & \text{ } \\
\cdots & \text{ } 
\end{align*}
\]  

(B.7)

The map $f^*$ is naturally induced as follows. Suppose $f$ maps an element $a$ of $\mathcal{A}$ to $b$ in $\mathcal{B}$. If $da = 0$, one has $db = df(a) = f(da) = f(0) = 0$. Also for $a = d\alpha$, $b = f(d\alpha) = df(\alpha)$. Hence this gives a well-defined map between the elements of the cohomology groups. Denote their representatives by $[a], [b]$, respectively. The map $f^*$ is then defined as

\[
f^* : [a] \to [b].
\]  

(B.8)

The map $g^*$ is similarly induced from $g$.

The map $d^*$ is defined as follows. First let us recall the structure of the map

\[
\begin{align*}
0 &\to \mathcal{A}^{q+1} \xrightarrow{f} \mathcal{B}^{q+1} \xrightarrow{g} \mathcal{C}^{q+1} \to 0 \\
0 &\to \mathcal{A}^q \xrightarrow{f} \mathcal{B}^q \xrightarrow{g} \mathcal{C}^q \to 0
\end{align*}
\]  

(B.9)

For an element $c \in \mathcal{C}^q, dc = 0$, there exist a $b \in \mathcal{B}^q$ such that $g(b) = c$ since $g$ is onto. Moreover $db \in Ker(g)$ because $0 = dc = dg(b) = g(db)$. From the short exact sequence, it follows that there is an element $a \in \mathcal{A}^{q+1}$ such that $db = f(a)$. Since $0 = df(a) = f(da)$, we find $da = 0$. In this way, we can make a map between the closed forms in $\mathcal{A}^{q+1}$ and $\mathcal{C}^q$. 

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This map is not one-to-one. However, the naturally induced map $d^* : [c] \to [a]$ gives a homomorphism. Suppose there are two elements $b_1, b_2 \in B^q$ which give the same $c \in C^q$: $g(b_1) = g(b_2) = c$. We have $db_1 = f(a_1), db_2 = f(a_2)$. Since $0 = g(b_1) - g(b_2) = g(b_1 - b_2)$ and hence $b_1 - b_2 \in \text{Ker}(g) = \text{Im}(f)$, we find $f(a_{12}) = b_1 - b_2$ for some $a_{12} \in A^{q+1}$. These relations lead to $db_1 - db_2 = f(da_{12}) = f(a_1 - a_2)$, from which $a_1 - a_2 = da_{12}$ follows.

Similarly it can be shown that a cohomologically trivial element is mapped to a trivial element. This completes the proof that $d^*$ is a homomorphism.
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**Figure captions**

Fig. 1. The points related by $S^\pm$ to those with $k = l = \text{integers}$. The directions of the map are shown by arrows.

Fig. 2. BRST nontrivial states in PF modules. (a) for $N_{FP} = 2$, (b) for $N_{FP} = 1$, (c) for $N_{FP} = 0$. Here · stands for $\hat{D}^+$, × for $D^-$, ○ for $\hat{D}^-$, • for $D^+$, Δ for $\hat{U}$, ⊙ for $\hat{C}$. ⊙ implies both $\hat{D}^+$ and $\hat{D}^-$.

Fig. 3(a). The current algebra module with the Casimir $-j(j-1) = -\frac{17}{8} \cdot \frac{3}{8}$ generated over the state with $m = 17/8$ ($\otimes$). Null states are indicated by Δ, ○ and •. This module is related to that in Fig. 3(b) by the Weyl transformation. States with the same symbols are transformed into each other.

Fig. 3(b). The module generated over the state with $\tilde{m} = -1$ ($\otimes$). Null states are indicated by Δ, ○ and •. Due to the null indicated by Δ, the module is separated into $\hat{U}$ and $\hat{D}^+$.

Fig. 4. The number of null states on points studied in sect. 3. In a region separated by two lines, the line to the right included, the number is as indicated. At points shown with * the embedding structure and the number of null states are not completely clear.

Fig. 5. Possible physical states in irreps. (a) for $N_{FP} = 2$, (b) for $N_{FP} = 1$, (c) for $N_{FP} = 0$, (d) for $N_{FP} = -1$. Here · stands for $\hat{D}^+$, × for $D^-$, ○ for $\hat{D}^-$, • for $D^+$, Δ for $\hat{U}$, ⊙ for $\hat{C}$. ⊙ implies both $\hat{D}^+$ and $\hat{D}^-$.

Fig. 6. Examples of physical states.
Fig. 1

Fig. 2(a)
Fig. 2(b)

Fig. 2(c)
Fig. 3(a)

Fig. 3(b)
Fig. 4

Fig. 5(a)
Fig. 5(b)

Fig. 5(c)
Fig. 5(d)

Fig. 6