CONVERGENCE OF SEQUENCES OF SCHRÖDINGER MEANS

PER SJÖLIN AND JAN-OLOV STRÖMBERG

Abstract. We study convergence almost everywhere of sequences of Schrödinger means. We also replace sequences by uncountable sets.

1. Introduction

For $f \in L^2(\mathbb{R}^n), n \geq 1$ and $a > 0$ we set
$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx, \xi \in \mathbb{R}^n,$$
and
$$S_t f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} e^{i t |\xi|^a} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, t \geq 0.$$

For $a = 2$ and $f$ belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ we set $u(x,t) = S_t f(x)$. It then follows that $u(x,0) = f(x)$ and $u$ satisfies the Schrödinger equation $i \partial u / \partial t = \Delta u$.

We introduce Sobolev spaces $H^s = H^s(\mathbb{R}^n)$ by setting
$$H^s = \{ f \in \mathcal{S}'; ||f||_{H^s} < \infty \}, s \in \mathbb{R},$$
where
$$||f||_{H^s} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}.$$ 

In the case $a = 2$ and $n = 1$ it is well-known (see Carleson [3] and Dahlberg and Kenig [5]) that
$$\lim_{t \to 0} S_t f(x) = f(x)$$
almost everywhere if $f \in H^{1/2}$. Also it is known that $H^{1/4}$ cannot be replaced by $H^s$ if $s < 1/4$.

In the case $a = 2$ and $n > 2$ Sjölin [11] and Vega [13] proved independently that [11] holds almost everywhere if $f \in H^s(\mathbb{R}^n), s > 1/2$. This result was improved by Bourgain [1] who proved that $f \in H^s(\mathbb{R}^n), s > 1/2 - 1/4n$, is sufficient for convergence almost everywhere. On the other hand Bourgain [2] has proved that $s \geq n/2(n+1)$ is necessary for convergence for $a = 2$ and $n \geq 2$.

In the case $n = 2$ and $a = 2$, Du, Guth and Li [6] proved that the condition $s > 1/3$ is sufficient. Recently Du and Zhang [7] proved that the condition $s > n/2(n+1)$ is sufficient for $a = 2$ and $n \geq 3$.

In the case $a > 1, n = 1$, [11] holds almost everywhere if $f \in H^{1/4}$ and $H^{1/4}$ cannot be replaced by $H^s$ if $s < 1/4$. In the case $a > 1, n = 2$, it is known that [11] holds almost everywhere if $f \in H^{1/2}$ and in the case $a > 1, n \geq 3$ convergence has been proved for

Mathematics Subject Classification (2010): 42B99.

Key Words and phrases: Schrödinger equation, convergence, Sobolev spaces
$f \in H_s$ with $s > 1/2$. For the results in the case $a > 1$ see Sjölin [11, 12] and Vega [16, 17].

If $f \in L^2(\mathbb{R}^n)$ then $S_tf \to f$ in $L^2$ as $t \to 0$. It follows that there exists a sequence $(t_k)_1^\infty$ satisfying

$$1 > t_1 > t_2 > t_3 > \cdots > 0 \quad \text{and} \quad \lim_{k \to \infty} t_k = 0. \tag{2}$$

such that

$$\lim_{k \to \infty} S_{t_k}f(x) = f(x) \tag{3}$$

almost everywhere.

In Sjölin [13] we studied the problem of deciding for which sequences $(t_k)_1^\infty$ one has (3) almost everywhere if $f \in H_s$ for $s > 0$. The following result was obtained in [13].

**Theorem A** Assume that $n \geq 1$ and $a > 1$ and $s > 0$. We assume that (2) holds and that

$$\sum_{k=1}^\infty t_k^{\frac{2s}{a}} < \infty \quad \text{and} \quad f \in H_s(\mathbb{R}^n).$$

Then

$$\lim_{k \to \infty} S_{t_k}f(x) = f(x)$$

for almost every $x \in \mathbb{R}^n$.

We shall here continue the study of conditions on sequences $(t_k)_1^\infty$ which imply that (3) holds almost everywhere. We shall also replace the set $\{t_k; k = 1, 2, 3, \ldots \}$ with sets $E$ which are not countable, for instance the Cantor set. Our first theorem is an extension of Theorem A in which we replace the spaces $H_s$ with Bessel potential spaces $L_p^s$.

We need some more notations. Let $1 < p \leq 2$ and $s > 0$. Set $k_s(\xi) = (1 + |\xi|^{-s})^{-1/2}$ for $\xi \in \mathbb{R}^n$.

Let the operator $\mathcal{J}_s$ be defined by

$$\mathcal{J}_sf = 2^{-s/2}f$$

where $\mathcal{F}$ denotes the Fourier transformation, i.e. $\mathcal{F}f = \hat{f}$. Then $\mathcal{J}_s$ can be extended to a bounded operator on $L_p$, that is $k_s \in M_p$, where $M_p$ denotes the space of Fourier multipliers on $L_p$ (see Stein [14], p.132).

We introduce the Bessel potential space $L^p_s$ by setting $L^p_s = \{\mathcal{J}_sg; g \in L_p\}$, $s > 0$.

We let $I$ denote an interval defined in the following way. In the case $n = 1, s < a/2$, and in the case $n \geq 2$, we have $I = [p_0, 2[$, where $p_0 = 2/(1 + 2s/na)$.

In the remaining case $n = 1, s \geq a/2$, we have $I = (1, 2]$.

For $f \in L^p_s, p \in I$, and $a > 1$, and $0 < s < a$, we shall define $S_tf$ so that

$$(S_tf)(\xi) = e^{it|\xi|^a} \hat{f}(\xi)$$

and then have the following theorem.

**Theorem 1.** Assume $a > 1$, $0 < s < a$, and $f \in L^p_s$. Let the sequence $(t_k)_1^\infty$ satisfy (4), and assume also that $\sum_{t=1}^\infty t_k^{\frac{ps}{a}} < \infty$. Then

$$\lim_{k \to \infty} S_{t_k}f(x) = f(x)$$

almost everywhere.
In the proof of Theorem 1 we shall use the following theorem on Fourier multipliers.

**Theorem 2.** Let \( a > 1, 0 < s < a, \) and assume also that \( 0 < \delta < 1. \) Set

\[
m(\xi) = \frac{e^{i\delta|\xi|^a} - 1}{(1 + |\xi|^2)^{s/2}}, \quad \xi \in \mathbb{R}^n.
\]

Then \( m \in M_p \) and

\[
\|m\|_{M_p} \leq C_p \delta^{s/a} \quad \text{for} \quad p \in I,
\]

where \( C_p \) does not depend on \( \delta. \)

We remark that in Sjölin [13] we used Theorem 2 in the special case \( p = 2. \)

Now let the sequence \((t_k)\) satisfy \((2)\) and set

\[
A_j = \{t_k; 2^{-j-1} < t_k \leq 2^{-j}\} \quad \text{for} \quad j = 1, 2, 3, \ldots.
\]

Let \#\( A \) denote the number of elements in a set \( A. \) We have the following theorem.

**Theorem 3.** Assume that \( n \geq 1, a > 1, \) and \( 0 < s \leq 1/2 \) and \( b \leq 2s/(a - s). \) Assume also that

\[
\#A_j \leq C 2^{bj} \quad \text{for} \quad j = 1, 2, 3, \ldots \quad (4)
\]

and that \( f \in H_s. \) Then

\[
\lim_{k \to \infty} S_{t_k} f(x) = f(x)
\]

almost everywhere.

Theorem 3 has the following two corollaries.

**Corollary 1.** Assume that \((t_k)\) satisfies \((2)\) and that \( n \geq 1, a > 1, 0 < s \leq 1/2, \) and that \( \sum_{i=1}^{\infty} t_i^\gamma < \infty, \) where \( \gamma = 2s/(a - s). \) If also \( f \in H_s \) then \((3)\) holds almost everywhere.

We remark that Corollary 1 gives an improvement of Theorem A.

**Corollary 2.** Assume that \((t_k)\) satisfies \((3)\), and that \( n \geq 1, a > 1, 1 < p < 2, r > 0, \) and

\[
s = \frac{n}{2} + r - \frac{n}{p}.
\]

If \( f \in L^p_\gamma \) and \( s > 1/2 \) then \((5)\), holds almost everywhere.

If \( 0 < s \leq 1/2 \) set \( \gamma = 2s/(a - s). \) If also \( \sum_{i=1}^{\infty} t_i^\gamma < \infty, \) and \( f \in L^p_\gamma \) then \((5)\) holds almost everywhere.

Now let \( E \) denote a bounded set in \( \mathbb{R}. \) For \( r > 0 \) we let \( N_E(r) \) denote the minimal number \( N \) of intervals \( I_l, l = 1, 2, \ldots, N, \) of length \( r, \) such that \( E \subset \bigcup_1^N I_l. \)

For \( f \in \mathcal{S} \) we introduce the maximal function

\[
S^* f(x) = \sup_{t \in E} |S_t f(x)|, \quad x \in \mathbb{R}^n.
\]

We shall prove the following estimate.
Theorem 4. Assume $n \geq 1, a > 0$, and $s > 0$. If $f \in \mathcal{S}$ then one has
\[
\int |S^* f(x)|^2 dx \leq C \left( \sum_{m=0}^\infty N_E(2^{-m}) 2^{-2ms/a} \right) \|f\|_{H^s}^2.
\]

The following corollary follows directly

Corollary 3. Assume that $n \geq 1, a > 0, s > 0, f \in \mathcal{S}$, and
\[
\sum_{m=0}^\infty N_E(2^{-m}) 2^{-2ms/a} < \infty.
\] (5)

Then one has
\[
\left( \int |S^* f(x)|^2 dx \right)^{1/2} \leq C \|f\|_{H^s}.
\]

Now let $E = \{t_k, k = 1, 2, 3, \ldots \}$ where the sequence $(t_k)\infty$ satisfies (2). We define $S^* f$ as above so that
\[
S^* f(x) = \sup_k |S_{t_k} f(x)|, \quad f \in \mathcal{S}.
\]

We then have the following corollary.

Corollary 4. We let $n \geq 1, a > 0, s > 0$, and assume that
\[
\sum_{m=0}^\infty N_E(2^{-m}) 2^{-2ms/a} < \infty,
\]
and $f \in H^s$. Then (3) holds almost everywhere.

Now assume $0 < \kappa < 1$ and that let $m_\kappa$ denote $\kappa$-dimensional Hausdorff measure on $\mathbb{R}$ (see Mattila \[8\], p.55). Let $E \subset \mathbb{R}$ be a Borel set with Hausdorff dimension $\kappa$ and $0 < m_\kappa(E) < \infty$. Assume also that $0 \in E$.

We shall use a precise definition of $S_t f(x)$ for $f \in L^2(\mathbb{R}^n)$ and $(x,t) \in \mathbb{R}^n \times E$. Let $Q$ denote the unit cube $[-\frac{1}{2}, \frac{1}{2}]^n$ in $\mathbb{R}^n$. Set
\[
f_N(x,t) = (2\pi)^{-n} \int_{NQ} e^{i\xi \cdot x} e^{it|\xi|^a} \hat{f}(\xi) d\xi, \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times E
\]
and $N = 1, 2, 3, \ldots$. It follows from well-known estimates (See Sjölin \[10\]) that there exists a set $F \subset \mathbb{R}^n \times E$ with $m \times m_\kappa((\mathbb{R}^n \times E) \setminus F) = 0$ such that
\[
\lim_{N \to \infty} f_N(x,t)
\]
exists for every $(x,t) \in F$. Here $m$ denotes Lebesque measure. We set $S_t f(x)$ equal to this limit for $(x,t) \in F$ and $S_t f(x)$ will then be a measurable function on $\mathbb{R}^n \times E$ with respect to the measure $m \times m_\kappa$.

Then one has the following convergence result

Theorem 5. Let $n \geq 1, a > 0$, and assume that $s > 0$ and
\[
\sum_{m=0}^\infty N_E(2^{-m}) 2^{-2ms/a} < \infty
\] (6)
and $f \in H_s$. Then for almost every $x \in \mathbb{R}^n$ we can modify $S_t f(x)$ on a $m_\kappa$-nullset so that

$$\lim_{t \to 0} S_t f(x) = f(x).$$

Note that if $0 < a < 2s$ then (6) holds when $E$ is the interval $[0, 1]$. Thus one of the consequences of the above results is the following well-known fact (see Cowling [4]).

**Corollary 5.** If $0 < a < 2s$ and $f \in H_s$ then (7) holds.

We also have

**Corollary 6.** In Theorem 3 the conditions $a > 1$ and $b \leq 2s/(a - s)$ can be replaced by the conditions $a \geq 2s$ and $1/b > (a - 2s)/2s$.

and

**Corollary 7.** Assume that $(t_k)\infty_{k=1}$ satisfies (3), and that $n \geq 1, a \geq 2s, 0 < s \leq 1/2$, and that $\sum_{k=1}^{\infty} t_k^{\gamma} < \infty$, where $1/\gamma > (a - 2s)/2s$. If also $f \in H_s$ then (3) holds almost everywhere.

We remark that Corollary 7 gives an improvement of Theorem A and Corollary 1.

We shall now study the case where $E$ is a Cantor set. Assume $0 < \lambda < 1/2$. We set $I_{0,1} = [0, 1], I_{1,1} = [0, \lambda]$ and $I_{1,2} = [1 - \lambda, 1]$. Having defined $I_{k-1,1}, \ldots, I_{k-1,2^k-1}$, we define $I_{k,1}, \ldots, I_{k,2^k}$ by taking away from the middle of each interval $I_{k-1,j}$ an interval of length $(1 - 2\lambda)l(I_{k-1,j}) = (1 - 2\lambda)\lambda^{k-1}$, where $l(I)$ denotes the length of an interval $I$. We then define Cantor sets by setting

$$C(\lambda) = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} I_{k,j}.$$ 

It can be proved that $C(\lambda)$ has Hausdorff dimension

$$\kappa = \log 2 / \log(1/\lambda)$$

and that $m_\kappa(C(\lambda)) = 1$ (See [3], p. 60-62). We have the following result, where $S_t f(x)$ is defined as in Theorem 5 with $E = C(\lambda)$.

**Theorem 6.** Assume $n \geq 1, a > 0$, and $0 < \lambda < 1/2$. Also assume $s > a\kappa/2$ and $f \in H_s$. Then we can for almost every $x$ modify $S_t f(x)$ on $m_\kappa$-nullset so that

$$\lim_{t \to 0} S_t f(x) = f(x).$$

**Remark.** In the proofs of Corollary 4 and Theorem 5 we first in the main part of the proof obtain a maximal estimate for smooth functions and then prove a convergence result for functions in $H_s$. In the passage from the maximal estimate for smooth functions to the convergence result we use an approach which was mentioned to one of the authors by P. Sjögren in a conversation, 2009.
In Section 2 we shall prove Theorems 1 and 2, and Section 3 contains the proof of Theorem 3. In section 4 we prove Theorem 4, and in Section 5 the proofs of Theorems 5 and 6 are given.

We shall finally construct a counter-example which gives the following theorem.

**Theorem 7.** Assume \( t_k = 1/(\log k) \) for \( k = 2, 3, 4, \ldots \), and set
\[
S^* f(x) = \sup_k |S_{t_k} f(x)|, \ x \in \mathbb{R}^n,
\]
for \( f \in L^2(\mathbb{R}^n) \). Then \( S^* \) is not a bounded operator on \( L^2(\mathbb{R}^n) \) in the case \( n = 1, a > 1 \), and also in the case \( n \geq 2, a = 2 \).

2. PROOFS OF THEOREMS 1 AND 2

For \( m \in L^\infty(\mathbb{R}^n) \) and \( 1 < p < \infty \) we set
\[
T_m f = \mathcal{F}^{-1}(m\hat{f}), \quad f \in L^p \cup L^2.
\]
We say that \( m \) is a Fourier multiplier for \( L^p \) if \( T_m \) can be extended to a bounded operator on \( L^p \), and we let \( M_p \) denote the class of multipliers on \( L^p \). We set \( \|m\|_{M_p} \) equal to the norm of \( T_m \) as an operator on \( L^p \).

Now let \( 1 < p \leq 2 \) and \( 0 < s < a \). For \( f \in \mathcal{S} \) and with \( \hat{f}(\xi) = (1 + |\xi|^2)^{-s/2}\hat{g}(\xi) \) one obtains
\[
S_{tf}(x) = (\mathcal{F}^{-1}(\mu(\xi)\hat{g}(\xi))) (x) = T_\mu g(x),
\]
where
\[
\mu(\xi) = e^{i\xi |a|^s} \frac{1}{(1 + |\xi|^2)^{s/2}}.
\]
We shall prove that \( \mu \in M_p \) for \( p \in I \), where \( I \) is an interval defined in the introduction. We need some well-known results.

**Lemma 1.** Assume that \( m \in M_p \) for some \( p \) which \( 1 < p < \infty \). Let \( b \) be a positive number and let \( k(\xi) = m(b\xi) \) for \( \xi \in \mathbb{R}^n \). Then \( k \in M_p \) and \( \|k\|_{M_p} = \|m\|_{M_p} \).

We shall also use the following multiplier theorem (see Stein ([14], p. 96).

**Theorem B:** Assume that \( m \) is a bounded function on \( \mathbb{R}^n \setminus \{0\} \) and that
\[
|D^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}
\]
for \( \xi \neq 0 \) and \( |\alpha| \leq k \), where \( k \) is an integer and \( k > n/2 \). Then \( m \in M_p \) for \( 1 < p < \infty \).

We shall also need the following result (see Miyachi [9], p 283)

**Theorem C:** Assume \( \psi \in C^\infty(\mathbb{R}^n) \) and that \( \psi \) vanishes in a neighbourhood of the origin and is equal to 1 outside a compact set. Set
\[
m_{a,s}(\xi) = \psi(\xi)|\xi|^{-s}e^{i|\xi|^a}, \quad \xi \in \mathbb{R}^n,
\]
where \( a > 1 \) and \( 0 < s < a \). Then \( m_{a,s} \in M_p \) if \( 1 < p < \infty \) and \( |1/p - 1/2| \leq s/na \).

**Remark.** In Miyachi's formulation of this result the function \( \psi \) is replaced by a function \( \psi_1 \) with the properties that \( \psi_1 \in C^\infty, 0 \leq \psi_1 \leq 1, \psi_1(\xi) = 0 \) for \( |\xi| \leq 1 \), and \( \psi_1(\xi) = 1 \) for \( |\xi| \geq 2 \). However, the two formulations are equivalent since the function
$(\psi - \psi_1)|\xi|^{-s} e^{i|\xi|a}$ belongs to $C_0^\infty$.

It follows from Theorem C that $m_{a,s} \in M_p$ if $p \in I$.

We shall then give the proof of the above statement about the function $\mu$.

**Lemma 2.** Assume $a > 1$ and $0 < s < a$ and also $t > 0$. Set

$$\mu(\xi) = e^{it|\xi|^a} (1 + |\xi|^2)^{-s/2}, \quad \xi \in \mathbb{R}^n.$$  

Then $\mu \in M_p$ for $p \in I$.

Proof of Lemma 2. We first take $\psi$ as in Theorem C and also set $\varphi = 1 - \psi$. One then has

$$\mu(\xi) = \varphi(\xi) e^{it|\xi|^a} (1 + |\xi|^2)^{-s/2} + \psi(\xi) e^{it|\xi|^a} (1 + |\xi|^2)^{-s/2} = \mu_1(\xi) + \mu_2(\xi).$$

We write $\mu_2 = \mu_3\mu_4$, where

$$\mu_3(\xi) = \psi(\xi) \frac{e^{it|\xi|^a}}{|\xi|^s}$$

and

$$\mu_4(\xi) = \frac{|\xi|^s}{(1 + |\xi|^2)^{s/2}}.$$  

We have

$$\mu_3(t^{-1/a} \eta) = \psi(t^{-1/a} \eta) \frac{e^{i|\eta|^a}}{|t^{-1/a} \eta|^s} = \psi(t^{-1/a} \eta) e^{i|\eta|^a} \frac{e^{i|\eta|^a}}{|\eta|^s}.$$  

We let $p \in I$ and it then follows from the Remark after Theorem C that $\mu_3 \in M_p$. Also $\mu_4 \in M_p$ since $I \subset (1, \infty)$ (see Stein [14], p. 133). Finally

$$\mu_1(\xi) = \psi(\xi) \frac{e^{it|\xi|^a}}{(1 + |\xi|^2)^{s/2}}$$

and it is easy to see that $\mu_1$ satisfies the conditions in Theorem B. We conclude that $\mu_1 \in M_p$ and thus also $\mu \in M_p$.

□

For $f \in L_p, p \in I$, and $a > 1$, and $0 < s < a$, we define $S_{t}f$ by setting $S_{t}f = T_{\mu}g$. It is then easy to see that

$$(S_{t}f)^{*}(\xi) = e^{it|\xi|^a} \hat{f}(\xi).$$

Observe that according to the Hausdorff-Young theorem $\hat{f} \in L^q$ where $1/p + 1/q = 1$.

We shall then give the proof of Theorem 2. We shall write $A \lesssim B$ if there is a constant $K$ such that $A \leq KB$.

**Proof of Theorem 2.** We set $C = \delta^{-1/a}$ and then have $C^{-s} = \delta^{s/a}$. It follows that

$$m(C\xi) = \frac{e^{i|\xi|^a} - 1}{(1 + C^2|\xi|^2)^{s/2}} = m_1(\xi) + m_2(\xi) - m_3(\xi),$$

where

$$m_1(\xi) = \varphi(\xi) \frac{e^{i|\xi|^a} - 1}{(1 + C^2|\xi|^2)^{s/2}}.$$
\[ m_2(\xi) = \psi(\xi) \frac{e^{i|\xi|^a}}{(1 + C^2|\xi|^2)^{s/2}} \]

and

\[ m_3(\xi) = \psi(\xi) \frac{1}{(1 + C^2|\xi|^2)^{s/2}}. \]

Here \( \varphi \) and \( \psi \) are defined as in the proof of Lemma 2, and we may assume that \( \varphi \) and \( \psi \) are radial functions.

We have

\[ m_2(\xi) = m_4(\xi) m_5(\xi), \]

where

\[ m_4(\xi) = \psi(\xi) \frac{e^{i|\xi|^a}}{(C^2|\xi|^2)^{s/2}} = \delta^{s/a} \psi(\xi) \frac{e^{i|\xi|^a}}{|\xi|^s} \]

and

\[ m_5(\xi) = \frac{(C^2|\xi|^2)^{s/2}}{(1 + C^2|\xi|^2)^{s/2}}. \]

It follows from Theorem C that \( m_4 \in M_p \) and \( \|m\|_{M_p} \lesssim \delta^{s/a} \) for \( p \in I \). Also \( m_5 \) has the same multiplier norm as the function \( |\xi|^s(1 + |\xi|^2)^{-s/2} \). We conclude that \( \|m_2\|_{M_p} \lesssim \delta^{s/a} \) for \( p \in I \).

We want to show that

\[ |D^\alpha m_1(\xi)| \lesssim C^{-s} |\xi|^{-|\alpha|} \text{ for } \xi \in \mathbb{R}^n \setminus \{0\} \]

for all multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \), where \( \alpha_i \) are non-negative integers. Invoking Theorem B we conclude that

\[ \|m_1\|_{M_p} \lesssim C^{-s} = \delta^{s/a} \]

for \( 1 < p < \infty \).

First we set

\[ m_{10}(x) = \varphi_0(x) \frac{e^{ix^a/2} - 1}{(1 + C^2x)^{s/2}}, \]

where we define \( \varphi_0 \) by taking \( \varphi_0(x) = \varphi(\xi) \) if \( x = |\xi|^2 \) and we then have \( m_1(\xi) = m_{10}(|\xi|^2) \).

We get for \( x > 0 \)

\[ D^j \frac{1}{(1 + C^2x)^{s/2}} = \frac{C_j C^{2j}}{(1 + C^2x)^{s/2+2j}}. \]

Hence we have

\[ |D^j \frac{1}{(1 + C^2x)^{s/2}}| \lesssim x^{-j} C^{-s} x^{-s/2}. \] (7)

on support of \( \varphi_0 \). One also has \( |e^{ix^a/2} - 1| \lesssim x^{a/2} \) and \( D^j(e^{ix^a/2} - 1) \) are linear combinations of functions \( e^{ix^a/2} x^{ka/2-j} \) for \( j \geq 1 \), where \( k \) is an integer \( 1 \leq k \leq j \). Hence

\[ |D^j(e^{ix^a/2} - 1)| \lesssim x^{a/2-j}, \quad j = 0, 1, 2, \ldots, \] (8)
for \( x \in \text{supp} \varphi \).

A combination of (7) and (8) then gives

\[
|D_j m_{10}(x)| \lesssim x^{-j} C^{-s} x^{a/2 - s/2}
\]

Let \( \alpha \) and \( \beta \) denote \( n \)-dimensional multi-index. By induction over \( j = 0, 1, 2, \ldots \), and \( |\alpha| = j \) we can write \( D^\alpha m_1(\xi) \) as a finite linear combination of functions of the form

\[
D^k m_{10}(|\xi|^2) \xi^\beta
\]

with \( j/2 \leq k \leq j \) and \( |\beta| = 2k - j \). We conclude that

\[
|D^\alpha m_1(\xi)| \lesssim \max_{|\alpha|/2 \leq k \leq |\alpha|} |\xi|^{-2k} C^{-s} |\xi|^{a-s} |\xi|^{2k-j} = C^{-s} |\xi|^{-|\alpha|} \lesssim \delta^{s/a} |\xi|^{-|\alpha|}.
\]

It remains to study \( m_3 \). Define \( m_{30}(x) \) analogously to the definition of \( m_{10}(x) \) on \( \text{supp} \varphi_0 \), such that

\[
m_{30}(x) = \psi_0(x) \frac{1}{(1 + C^2 x)^{s/2}}
\]

and invoking (7)

\[
|D^j (1 + C^2 x)^{-s/2}| \lesssim C^{-s} x^{-j}
\]
on \( \text{supp} \psi_0 \). Also \( |D^j \psi_0(x)| \lesssim x^{-j} \) on \( \text{supp} \psi_0 \).

We conclude that

\[
|D^j m_{30}(x)| \lesssim C^{-s} x^{-j}
\]

and arguing as above we obtain

\[
|D^\alpha m_3(\xi)| \lesssim \max_{|\alpha|/2 \leq k \leq |\alpha|} |\xi|^{-2k} C^{-s} |\xi|^{2k-j} = C^{-s} |\xi|^{-|\alpha|} \lesssim \delta^{s/a} |\xi|^{-|\alpha|}
\]

for \( \xi \in \text{supp} m_3 \) and \( j = 0, 1, 2, \ldots \). Invoking Theorem B we conclude that \( \|m_3\|_{M_p} \lesssim \delta^{s/a} \) for \( 1 < p < \infty \). This completes the proof of Theorem 2 \( \Box \).

We shall finally give the proof of Theorem 1.

**Proof of Theorem 1.** We set

\[
\mu_0(\xi) = \frac{e^{it_\xi |\xi|^a}}{(1 + |\xi|^2)^{s/2}}
\]

\[
m(\xi) = \frac{e^{it_\xi |\xi|^a} - 1}{(1 + |\xi|^2)^{s/2}}
\]

and also have

\[
k_s(\xi) = (1 + |\xi|^2)^{-s/2}.
\]

It follows that

\[
T_{\mu_0} g - \mathcal{J}_s g = T_m g
\]

for \( g \in \mathcal{S} \).

We have \( f \in L^p \) where \( p \in I \) and it follows that \( f = \mathcal{J}_s g \) for some \( g \in L^p \). We choose a sequence \((g_j)\) such that \( g_j \in \mathcal{S} \) and \( g_j \to g \) in \( L^p \) as \( j \to \infty \).

One then has

\[
T_{\mu_0} g_j - \mathcal{J}_s g_j = T_m g_j
\]
for every \( j \). Letting \( j \) tend to \( \infty \) we obtain
\[
T_{\mu_0} g - \mathcal{J} s g = T_m g
\]
since the three operators \( T_{\mu_0}, \mathcal{J} s \) and \( T_m \) are all bounded on \( L^p \). It follows that
\[
S_{t_k} f - f = T_m g.
\]
Here we have used Lemma 2 and Theorem 2.

We now set \( h_k = S_{t_k} f - f \) and hence \( h_k = T_m g \). It follows from Theorem 2 that
\[
\|h_k\|_p \lesssim t_k^{s/a} \|g\|_p
\]
and we conclude that
\[
\infty \sum_{k=1} \int |h_k|^p dx \leq \left( \sum_{k=1}^\infty t_k^{ps/a} \right) \int |g|^p dx < \infty.
\]
Applying the theorem on monotone convergence on then obtain
\[
\int \left( \sum_{1}^\infty |h_k|^p \right) dx < \infty
\]
and hence \( \sum_1^\infty |h_k|^p \) is convergent almost everywhere. It follows that \( \lim_{k \to \infty} |h_k| = 0 \) almost everywhere and we conclude that
\[
\lim_{k \to \infty} S_{t_k} f(x) = f(x)
\]
almost everywhere. This completes the proof of Theorem 1. \( \square \)

3. Proof of Theorem 3 and its corollaries

We first give the proof of Theorem 3.

Proof of Theorem 3. We may assume \( b = 2s/(a-s) \). Fix \( j \). By adding points to \( A_j \) we can get an increasing sequence \( (v_k)_{k=0}^N \) and \( \tilde{A}_j = \{v_k; k = 0, \ldots, N\} \) such that \( v_0 = 0, v_N = 2^{-j}, \# \tilde{A}_j \leq C 2^b j \), and \( v_k - v_{k-1} \leq C 2^{-j} 2^{-bj} \).

We split the operator \( S_{v_k} \) into a low frequency part and a high frequency part
\[
S_{v_k} f(x) = S_{v_k, \text{low}, j} f(x) + S_{v_k, \text{high}, j} f(x)
\]
where
\[
S_{k, \text{low}, j} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi x} e^{iv_k |\xi|^a} \chi_{E_j} \hat{f}(\xi) d\xi,
\]
and
\[
S_{k, \text{high}, j} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi x} e^{iv_k |\xi|^a} \chi_{E_j} \hat{f}(\xi) d\xi,
\]
with \( E_j = \{\xi \in \mathbb{R}^n; |\xi| \leq 2^{b_1 j}\} \) and \( b_1 = b/2s \).

We shall prove that
\[
\sum_{j} 2^{bj} \sum_{v_k \in \tilde{A}_j \atop k > 0} \|S_{k, \text{low}, j} f - S_{k-1, \text{low}, j} f\|_2^2 \leq C \|f\|_{H^s}^2 \quad (9)
\]
and
\[ \sum_{j} \sum_{v_k \in A_j} \|S_{k, \text{high}, j} f\|_2^2 \leq C \|f\|_{H_s}^2. \] (10)

We first assume that (9) and (10) hold. Using the Schwarz inequality we then have
\[ \sup_{v_k \in A_j} |S_{k, \text{low}, j} f(x) - f(x)|^2 \leq \left( |S_{0, \text{high}, j} f(x)| + \sum_{v_k \in A_j} \sum_{k > 0} |S_{k, \text{low}, j} f(x) - S_{k-1, \text{low}, j} f(x)| \right)^2 \]
\[ \leq 2 |S_{0, \text{high}, j} f(x)|^2 + C_{\text{low}}^2 \sum_{v_k \in A_j} \sum_{k > 0} |S_{k, \text{low}, j} f(x) - S_{k-1, \text{low}, j} f(x)|^2 \]

and invoking (9) and (10)
\[ \sum_{j} \sup_{v_k \in A_j} |S_{k, \text{low}, j} f(x) - f(x)|^2 \leq 2 \sum_{j} |S_{0, \text{high}, j} f(x)|^2 + C \sum_{j} 2^{bj} \sum_{v_k \in A_j} \sum_{k > 0} |S_{k, \text{low}, j} f(x) - S_{k-1, \text{low}, j} f(x)|^2 \]

and
\[ \int \sum_{j} \sup_{v_k \in A_j} |S_{k, \text{low}, j} f(x) - f(x)|^2 \, dx \leq C \|f\|_{H_s}^2. \] (11)

Using (10) we also obtain
\[ \int \sum_{j} \sup_{v_k \in A_j} |S_{k, \text{high}, j} f(x)|^2 \, dx \]
\[ \leq \int \sum_{j} \sup_{v_k \in A_j} |S_{k, \text{low}, j} f(x)|^2 \, dx \leq C \|f\|_{H_s}^2. \] (12)

The theorem follows from (11) and (12).

We shall now prove (9) an first observe that
\[ S_{k, \text{low}, j} f(x) - S_{k-1, \text{low}, j} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left( e^{i\xi \cdot x} - e^{i\xi \cdot (x-\hat{f}(\xi))} \right) \chi_{E_j} \hat{f}(\xi) \, d\xi, \]

Applying Plancherel's theorem we obtain
\[ \|S_{k, \text{low}, j} f - S_{k-1, \text{low}, j} f\|_2^2 = C \int_{E_j} \left| e^{i\xi \cdot \hat{f}(\xi)} - e^{i\xi \cdot \hat{f}(\xi)} \right|^2 \, d\xi \]
\[ \leq C \int_{E_j} \left| \sum_{v_k \in A_j} |e^{i\xi \cdot \hat{f}(\xi)} - e^{i\xi \cdot \hat{f}(\xi)}|^2 \, d\xi \leq C 2^{-2j} 2^{-2bj} \int_{E_j} |\xi|^{2a} |\hat{f}(\xi)|^2 \, d\xi. \]

and
\[ \sum_{j} 2^{bj} \sum_{v_k \in A_j} \|S_{k, \text{low}, j} f - S_{k-1, \text{low}, j} f\|_2^2 \]
\[ \leq C \sum_{j} 2^{-2j} \left( 2^{-bj} \sum_{v_k \in A_j} 1 \int_{E_j} |\xi|^{2a} |\hat{f}(\xi)|^2 \, d\xi \right) \]
\[ \leq C \int \left( \sum_{2^{b_1 j} \geq |\xi|} 2^{-2j} \right) |\xi|^{2a} |\hat{f}(\xi)|^2 \, d\xi. \]

The inequality \( 2^{b_1 j} \geq |\xi| \) implies \( 2^j \geq |\xi|^{1/b_1} \) and thus we get
\[ \sum_{2^{b_1 j} \geq |\xi|} 2^{-2j} \leq C |\xi|^{-2/b_1}. \]
Hence the left hand side of (9) is majorized by
\[ C \int |\xi|^{2n-2|b|} |\hat{f}(\xi)|^2 \, d\xi. \]

We have \( b = 2s/(a-s) \) and \( b_1 = 1/(a-s) \) and \( 2a - 2/b_1 = 2a - 2(a-s) = 2s \) and the inequality (9) follows.

To prove (10) we first observe that Plancherel’s theorem implies
\[ \|S_{k, \text{high}} f\|_2^2 \leq C \int_{|\xi| \geq 2^{b_1}j} |\hat{f}(\xi)|^2 \, d\xi, \]
and hence
\[ \sum_j \sum_{k \in \mathcal{A}_j} \|S_{k, \text{high}} f\|_2^2 \leq \sum_j 2^{b_1} 2^{j-|b_1|} \int_{|\xi| > 2^{j-1}} |\hat{f}(\xi)|^2 \, d\xi = C \int_{|\xi| > 2^{j-1}} |\hat{f}(\xi)|^2 \, d\xi. \]

Since \( b = 2s/(a-s) \) and \( b_1 = 1/(a-s) \) we obtain \( b/b_1 = 2s \) and (10) follows. Thus the proof of Theorem 3 is complete. \( \square \)

We shall then prove the two corollaries to Theorem 3.

**Proof of Corollary 1.** Since \( \sum_j t_k^{1} \) is convergent we obtain
\[ \left( \# \{ k; t_k > 2^{-j-1} \} \right)^2 2^{(-j-1)\gamma} \leq \sum_{t_k > 2^{-j-1}} t_k^\gamma \leq C \]
an \( \# A_j \leq C 2^{j\gamma} \) for \( j = 1, 2, 3, \ldots \). Since \( \gamma = 2s/(a-s) \) the corollary follows from Theorem 3. \( \square \)

**Proof of Corollary 2.** Assume that \( f \in L^p_r \), where \( 1 < p < 2 \), and \( r > 0 \). Also let \( s = n/2 + r - n/p \). Then there exists \( g \in L^p \) such that \( f = \mathcal{F}_r(g) = \mathcal{F}_s(\mathcal{F}_{r-s} g) \) and we have
\[ \frac{1}{2} = \frac{1}{p} - \frac{r-s}{n}. \]
It follows from the Hardy-Littlewood-Sobolev theorem that \( \mathcal{F}_{r-s} g \in L^2 \) and hence \( f \in H_s \) (see Stein [14], p. 119). The corollary then follows from Theorem 3. \( \square \)

### 4. Proofs of Theorem 4 and its corollaries

In Sections 4 and 5 we assume \( n \geq 1 \) and \( a > 0 \). We remark that (11) holds almost everywhere if \( f \in H_s \) and \( n = 1, 0 < a < 1 \), and \( s > a/4 \) or \( n \geq 1, a = 1 \) and \( s > 1/2 \) (see Walther [18],[19]).

Before proving Theorem 4 we need some preliminary estimates. We set \( B(x_0; r) = \{ x; |x-x_0| \leq r \} \). Using the estimate
\[ |e^{it|\xi|^a} - e^{iu|\xi|^a}| \leq |t-u| |\xi|^a \]
and with \( A \geq 1 \) and supp \( \hat{f} \subset B(0; A) \) we obtain by Schwarz inequality
\[ \|S_uf - S_u f\|_\infty \leq \int_{|\xi| \leq A} |t-u| |\xi|^a |\hat{f}(\xi)| \, d\xi \leq |t-u| \left( \int_{|\xi| \leq A} |\xi|^{2a} \, d\xi \right)^{1/2} \left( \int |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \leq C |t-u| \left( \int_0^A r^{2a+n-1} \, dr \right)^{1/2} \|f\|_2 \leq C |t-u| A^{a+n/2} \|f\|_2 \]
(13)
Now assume \( T = \{ t_j; j = 0, 1, 2, \ldots, N \} \) where \( t_j \in \mathbb{R} \) and \( t_{j-1} < t_j \). We shall prove that if \( \text{supp} \hat{f} \subset B(0; A) \) then

\[
\int \max_{t, u \in T} |S_t f(x) - S_u f(x)|^2 \, dx \leq C \max_{t, u \in T} |t - u|^2 A^{2a} ||f||^2_2. \tag{14}
\]

Using the Schwarz inequality we obtain

\[
\max_{t, u \in T} |S_t f(x) - S_u f(x)| \leq \sum_{i=1}^N |S_{t_i} f(x) - S_{t_{i-1}} f(x)|
\]

\[
\leq \sum_{i=1}^N |t_i - t_{i-1}|^{-1/2} |S_{t_i} f(x) - S_{t_{i-1}} f(x)| |t_i - t_{i-1}|^{1/2}
\]

\[
\leq \left( \sum_{i=1}^N |t_i - t_{i-1}|^{-1} |S_{t_i} f(x) - S_{t_{i-1}} f(x)|^2 \right)^{1/2} \left( \sum_{i=1}^N |t_i - t_{i-1}| \right)^{1/2}
\]

where the last sum equals \( \max_{t, u \in T} |t - u| \), and the Plancherel theorem gives

\[
\int \max_{t, u \in T} |S_t f(x) - S_u f(x)|^2 \, dx \leq (\max_{t, u \in T} |t - u|) \sum_{i=1}^N |t_i - t_{i-1}|^{-1} \int |S_{t_i} f(x) - S_{t_{i-1}} f(x)|^2 \, dx
\]

\[
\leq (\max_{t, u \in T} |t - u|)^2 \sum_{i=1}^N |t_i - t_{i-1}| |t_i - t_{i-1}|^2 |\xi|^{2a} |f(\xi)|^2 \, d\xi
\]

\[
\leq (\max_{t, u \in T} |t - u|)^2 \int |\xi|^{2a} |f(\xi)|^2 \, d\xi \leq C \max_{t, u \in T} |t - u|^2 A^{2a} ||f||^2_2.
\]

Hence (14) is proved.

We shall then prove the following lemma

**Lemma 3.** Let \( I \) denote an interval of length \( r \). Then

\[
\int \sup_{t, u \in I} |S_t f(x) - S_u f(x)|^2 \, dx \leq Cr^2 A^{2a} ||f||^2_2 \tag{15}
\]

if \( f \in L^2(\mathbb{R}^n) \) and \( \text{supp} \hat{f} \subset B(0; A) \).

**Proof of Lemma 3.** Assume \( I = [b, b + r] \) and let \( N \) be a positive integer. Set \( t_i = b + ir/N, i = 0, 1, 2, \ldots, N \), and \( T = \{ t_i; i = 0, 1, 2, \ldots, N \} \). We have

\[
S_t f(x) - S_u f(x) = S_{t_i} f(x) - S_{t_j} f(x) + S_{t_j} f(x) - S_u f(x) - (S_u f(x) - S_{t_j} f(x)),
\]

where we choose \( t_i \) close to \( t \) and \( t_j \) close to \( u \). Invoking (14) we obtain

\[
|S_t f(x) - S_{t_i} f(x)| \leq C |t - t_i| A^{a+n/2} ||f||_2 \leq C r \frac{r}{N} A^{a+n/2} ||f||_2 = C_f \frac{r}{N},
\]

and

\[
|S_u f(x) - S_{t_j} f(x)| \leq C |u - t_j| A^{a+n/2} ||f||_2 \leq C r \frac{r}{N} A^{a+n/2} ||f||_2 = C_f \frac{r}{N},
\]

where \( C_f \) depends on \( f \). It follows that

\[
|S_t f(x) - S_u f(x)| \leq \max_{i, j} |S_{t_i} f(x) - S_{t_j} f(x)| + C_f \frac{r}{N}.
\]

Setting \( F_N(x) = \max_{i, j} |S_{t_i} f(x) - S_{t_j} f(x)| \) we obtain

\[
|S_t f(x) - S_u f(x)| \leq F_N(x) + C_f \frac{r}{N}.
\]

Letting \( N \to \infty \) we obtain

\[
|S_t f(x) - S_u f(x)| \leq \lim_{N \to \infty} F_N(x).
\]
An application of Fatou’s lemma and the inequality (14) then gives
\[
\int \sup_{t,u \in I} |S_t f(x) - S_u f(x)|^2 \, dx \leq \int \lim_{N \to \infty} F_N(x)^2 \, dx \\
\leq \lim_{N \to \infty} \int F_N(x)^2 \, dx \leq C r^2 A^{2a} \|f\|_2^2
\]
and the lemma follows. □

Let \( I \) and \( f \) have the properties in the above lemma. Then
\[
\int \sup_{t \in I} |S_t f(x) - f(x)|^2 \, dx \leq C (r^2 A^{2a} + 1) \|f\|_2^2. \tag{16}
\]
To prove (16) we take \( u_0 \in I \) and observe that
\[
\sup_{t \in I} |S_t f(x) - f(x)| \leq \sup_{t \in I} |S_t f(x) - S_{u_0} f(x)| + |S_{u_0} f(x)| + |f(x)|
\]
and (16) follows from Lemma 3 and Plancherel theorem.

We shall then prove the following lemma

Lemma 4. Let \( f \) have the same properties as in Lemma 3. Assume \( r > 0 \) and set \( I_l = [t_l - r/2, t_l + r/2], l = 1, 2, \ldots, N \). Assume that \( E \) is a set and \( E \subseteq \bigcup_{l=1}^{N} I_l \). Then
\[
\int \sup_{t \in E} |S_t f(x) - f(x)|^2 \, dx \leq C N (r^2 A^{2a} + 1) \|f\|_2^2. \tag{17}
\]

Proof of Lemma 4. The lemma follows from the inequality (16) and the inequality
\[
\sup_{t \in E} |S_t f(x) - f(x)|^2 \leq \sum_{l=1}^{N} \sup_{t \in I_l} |S_t f(x) - f(x)|^2
\]
Now assume \( f \in \mathcal{F} \) and write
\[
f = \sum_{k=0}^{\infty} \hat{f}_k,
\]
where \( \hat{f}_0 \) is supported in \( B(0; 1) \) and \( \hat{f}_k \) has support in \( \{\xi; 2^{k-1} \leq |\xi| \leq 2^k\} \) for \( k = 1, 2, 3, \ldots \). We shall prove the following lemma

Lemma 5. Let \( f \in \mathcal{F} \) and \( s > 0 \) and and let \( E \) be a bounded set in \( \mathbb{R} \). Then
\[
\int \sup_{t \in E} |S_t f(x) - f(x)|^2 \, dx \leq C \|f\|_{H^s}^2 \left( \sum_{k=0}^{\infty} N_E(2^{-ks}) 2^{-2ks} \right),
\]
where \( N_E(r) \) for \( r > 0 \) denotes the minimal number \( N \) of intervals \( I_l, l = 1, 2, \ldots, N, \) of length \( r \) such that \( E \subseteq \bigcup_{l=1}^{N} I_l \).

Proof of Lemma 5. With real numbers \( g_k > 0, k = 0, 1, 2, \ldots, \) we have
\[
\sup_{t \in E} |S_t f(x) - f(x)| \leq \sum_{k=0}^{\infty} \sup_{t \in E} |S_t f_k(x) - f_k(x)|
\]
\[
= \sum_{k=0}^{\infty} g_k^{-1/2} \sup_{t \in E} |S_t f_k(x) - f_k(x)| g_k^{1/2}
\]
\[
\leq (\sum_{k=0}^{\infty} g_k)^{1/2} \sup_{t \in E} |S_t f_k(x) - f_k(x)|^{1/2} (\sum_{k=0}^{\infty} g_k)^{1/2}
\]
\[
\leq C \|f\|_{H^s} \left( \sum_{k=0}^{\infty} N_E(2^{-ks}) 2^{-2ks} \right),
\]
and invoking Lemma 4 with \( r = 2^{-ka} \) and \( A = 2^k \) we obtain
\[
\int \sup_{t \in E} |S_t f(x) - f(x)|^2 \, dx \leq \left( \sum_{k=0}^{\infty} g_k \right) \left( \sum_{k=0}^{\infty} g_k^{-1} C N_E(2^{-ka})(2^{-2ak}2^{2ak} + 1) \|f_k\|_2^2 \right)
\]
Choosing \( g_k = N_E(2^{-ka})2^{-2ks} \) one obtains
\[
\int \sup_{t \in E} |S_t f(x) - f(x)|^2 \, dx \leq C \left( \sum_{k=0}^{\infty} g_k \right) \left( \sum_{k=0}^{\infty} 2^{2ks} \|f_k\|_2^2 \right)
\]
and the proof of the lemma is complete. \( \square \)

We shall prove Theorem 4.

**Proof of Theorem 4.** Let \( m \) take the values 0, 1, 2, \ldots. If
\[
2^{-m-1} < 2^{-ka} \leq 2^{-m}
\]
for some integer \( k \geq 0 \) then
\[
N_E(2^{-ka}) \leq C N_E(2^{-m})
\]
and since \( a > 0 \) there is for any fixed \( m \) only a bounded number of values of \( k \) for which (18) holds. It follows that
\[
N_E(2^{-ka})2^{-2ks} \leq C N_E(2^{-m})2^{-2ms/a}.
\]
Combining this inequality with the estimate
\[
\sup_{t \in E} |S_t f(x)| \leq \sup_{t \in E} |S_t f(x) - f(x)| + |f(x)|
\]
one obtains the theorem from Lemma 5. \( \square \)

Corollary 3 follows directly from Theorem 4 and we shall then prove Corollary 4.

**Proof of Corollary 4.** Set \( E_0 = E \cup \{0\} \) and
\[
S_0^* f(x) = \sup_{E_0} |S_t f(x)|, \; x \in \mathbb{R}^n.
\]
It then follows from Corollary 3 that for \( f \in \mathcal{S} \) one has
\[
\|S_0^* f\|_2 \leq C \|f\|_{H_s}.
\]
It follows that for every cube \( I \) in \( \mathbb{R}^n \) one has
\[
\int_I S_0^* f(x) \, dx \leq C \|f\|_{H_s}, \; f \in \mathcal{S}.
\]
Now fix \( f \in H_s \) and a cube \( I \). Then there exists a sequence \( (f_j) \) such that \( f_j \in C_0^\infty \) and
\[
\|f_j - f\|_{H_s} < 2^{-j}, \; j = 1, 2, 3, \ldots.
\]
One then has \( \|f_j - f_{j+1}\|_{H_s} < 2 \cdot 2^{-j} \) and
\[
\int_I \sup_{t \in E_0} |S_t f_j(x) - S_t f_{j+1}(x)| \, dx \leq C 2^{-j}.
\]
Hence

\[ \sum_{i} \sup_{t \in E_0} |S_t f_j(x) - S_t f_{j+1}(x)| < \infty \]  \hspace{1cm} (19)

for almost every \( x \in I \).

Then choose \( x \) so that (19) holds. It follow that \( S_t f_j(x) \to u_x(t) \), as \( j \to \infty \), uniformly in \( t \in E_0 \), where \( u_x \) is a continuous function on \( E_0 \).

It is also clear that \( S_t f_j \to S_t f \) in \( L^2 \) as \( j \to \infty \), for every \( t \in E_0 \). Since \( E_0 \) is countable we can find a subsequence \( (f_{j_i})_i^\infty \) such that for almost every \( x \in E \) \( S_t f_{j_i} \to S_t f(x) \) for all \( t \in E_0 \).

It follows that for almost every \( x \in I \) one has \( S_t f(x) = u_x(t) \) for all \( t \in E_0 \). Since

\[ \lim_{t \to 0 \atop t \in E} u_x(t) = u_x(0) \]

almost everywhere one also has

\[ \lim_{t \to 0 \atop t \in E} S_t f(x) = f(x) \]

for almost every \( x \in I \). Since \( I \) is arbitrary it follows that (3) holds almost everywhere in \( \mathbb{R}^n \). \( \square \)

5. Proofs of Theorems 5 and 6 and Corollaries 6 and 7

We shall first give the proof of Theorem 5.

Proof of Theorem 5. It follows from Corollary 3 that

\[ \| S^* f \|_2 \leq C \| f \|_{H_\kappa}, \quad f \in J, \]

where

\[ S^* f(x) = \sup_{t \in E} |S_t f(x)|, \quad x \in \mathbb{R}^n, f \in J. \]

Now take \( f \in H_\kappa \).

Let \( I \) denote a cube in \( \mathbb{R}^n \). It follows that

\[ \int_I S^* f(x) \, dx \leq C_I \| f \|_{H_\kappa} \quad \text{for} \quad f \in C^\infty_0. \]

We choose a sequence \( (f_j)_j^\infty \) such that \( f_j \in C^\infty_0 \) and

\[ \| f_j - f \|_{H_\kappa} < 2^{-j}, \quad j = 1, 2, 3, \ldots. \]

One then has \( \| f_j - f_{j+1} \|_{H_\kappa} < C 2^{-j} \) and

\[ \int_I \sup_{t \in E} |S_t f_j(x) - S_t f_{j+1}(x)| \, dx \leq C 2^{-j}. \]

It follows that

\[ \sum_{j=1}^\infty \sup_{t \in E} |S_t f_j(x) - S_t f_{j+1}(x)| < \infty \]

for almost every \( x \in I \). Now choose \( x \) such that the above inequality holds. We conclude that \( S_t f_j(x) \to u_x(t) \), as \( j \to \infty \), uniformly in \( t \in E \), where \( u_x \) is a continuous function on \( E \).

On the other hand \( S_t f_j \to S_t f \) in \( L^2(\mathbb{R}^n \times E; m \times m_x) \) as \( j \to \infty \). Hence there is a subsequence \( (f_{j_i})_i^\infty \) such that \( S_t f_{j_i}(x) \to S_t f(x) \) almost everywhere in \( \mathbb{R}^n \times E \) with
CONVERGENCE OF SEQUENCES OF SCHRÖDINGER MEANS

It follows that for almost every \( x \in I \) one has \( S_t f(x) = u_x(t) \) for almost all \( t \in E \) with respect to \( m \times \kappa \). We have

\[
\lim_{t \to 0} u_x(t) = f(x)
\]

for almost every \( x \in I \) and it follows that for almost every \( x \in I \) we can modify \( S_t f(x) \) on a \( m \times \kappa \)-nullset so that

\[
\lim_{t \to 0} S_t f(x) = f(x).
\]

This completes the proof of Theorem 5.

For the proof of Corollary 6 we need the following lemma

**Lemma 6.** Let \( A_j \) be defined as in Theorem 3 satisfying

\[
\# A_j \leq C 2^{bj} \quad \text{for } j = 0, 1, 2, \ldots
\]

for some \( b > 0 \). Let \( E = \bigcup_{j=1}^{\infty} A_j \) and \( N_E \) be as above then

\[
N_E(2^{-m}) \leq C 2^{bm/(b+1)}
\]

**Proof of Lemma 6.** Fix a \( k \). We have

\[
\# \left( \bigcup_{j=1}^{k} A_j \right) \leq C \sum_{j=1}^{k} 2^{bj} \leq C 2^{bk}
\]

and \( \bigcup_{j=k+1}^{\infty} A_j \subset \{ t; 0 \leq t \leq 2^{-(k-1)} \} \), which can be covered by \( 2^{m-k+1} \) intervals of length \( 2^{-m} \). Thus

\[
N_E(2^{-m}) \leq 2^{m-k+1} + C 2^{bk}
\]

Choose \( k \) such that \( k \leq (m+1)/(b+1) < k + 1 \) We get \( 2^{b+1} \cdot 2^{(b+1)k} > 2^{m+1} \) and \( 2^{bk} \leq C 2^{mb/(b+1)} \). We conclude that

\[
N_E(2^{-m}) \leq C 2^{bk} \leq C 2^{bm/(b+1)}.
\]

This ends the proof of the Lemma 6.

We can now prove Corollary 6 by using Lemma 6 and Corollary 4

**Proof of Corollary 6.** With \( 1/b > (a-2s)/2s \) as in Corollary 6 we get

\[
b/(b+1) = \frac{1}{1 + 1/b} < 1 \left( 1 + \frac{a-2s}{2s} \right) = 2s/a,
\]

and we get

\[
\sum_{l} N_E(2^{-m}) 2^{-2ms/a} \leq C \sum_{l} 2^{bm/(b+1)} 2^{-2ms/a} \leq C \sum_{l} 2^{m(b/(b+1)-2s/a)} < \infty
\]

since \( b/(b+1) - 2s/a < 0 \).

By Corollary 4 the Corollary 6 will follow.

The Corollary 7 will now follow by similar arguments as in the proof Corollary 1. Finally we shall give the proof of Theorem 6.
Proof of Theorem 6. We shall use Theorem 5 with
\[ \kappa = \log 2/(\log 1/\lambda). \]
For \( k = 0, 1, 2, 3, \ldots, C(\lambda) \) can be covered by \( 2^k \) intervals of length \( \lambda^k \).
Let \( m \) be a positive integer. Choose \( k \) such that \( \lambda^{k+1} < 2^{-m} \leq \lambda^k \). It follows that \( N_2(2^{-m}) \leq 2^{k+1} \) and that
\[ (1/\lambda)^k \leq 2^m \]
and
\[ k \leq m \frac{\log 2}{\log(1/\lambda)} = \kappa m. \]
Hence
\[ \sum_{m=1}^{\infty} N_2(2^{-m})2^{-2sm/a} \leq C \sum_{m=1}^{\infty} 2^{sm}2^{-2sm/a} < \infty, \]
if \( \kappa - 2s/a < 0 \), i.e. \( s > \kappa/2 \). Theorem 6 follows from an application of Theorem 5. \( \square \)

6. PROOF OF THEOREM 7

We first assume \( n = 1 \) and \( a > 1 \). We choose a function \( \varphi \in C_0^\infty(\mathbb{R}) \) with the property that \( \varphi(\xi) = 1 \) for \( |\xi| = a^{-1/(a-1)} \) and also \( \varphi \geq 0 \). We also assume that there exists a constant \( A > 1 \) such that \( \text{supp} \varphi \subset \{ \xi \in \mathbb{R}; 1/A \leq |\xi| \leq A \} \). We then define a function \( f_\nu \) by setting \( \hat{f}_\nu(\xi) = \varphi(2^{-\nu} \xi) \) where \( \nu = 1, 2, 3, \ldots \). One then has
\[ \| f_\nu \|_2 = c\| \hat{f}_\nu \|_2 = c \left( \int |\varphi(2^{-\nu} \xi)|^2 \, d\xi \right)^{1/2} = c \left( \int |\varphi(\eta)|^2 \, d\eta \right)^{1/2} = c 2^{\nu/2}, \]
where \( c \) denotes positive constants. Setting \( \eta = 2^{-\nu} \xi \) we also obtain
\[ S_t f_\nu(x) = c \int e^{ix\eta} e^{it|\xi|^a} \varphi(2^{-\nu} \xi) \, d\xi = c 2^{\nu} \int e^{i2^{\nu} \eta \xi} e^{it2^{\nu}|\eta|^a} \varphi(\eta) \, d\eta = c 2^{\nu} \int e^{iF(\xi)} \varphi(\xi) \, d\xi, \]
where \( F(\xi) = t2^{\nu}|\xi|^a + 2^\nu x\xi \).
We then assume \( C 2^{-\nu} \leq x \leq 1 \) where \( C \) denotes a large positive constant. It is clear that \( F = G + H \), where
\[ G(\xi) = 2^\nu x|\xi|^a + 2^\nu x\xi \]
and
\[ H(\xi) = t2^{\nu}|\xi|^a - 2^\nu x|\xi|^a. \]
We shall first study the integral
\[ \int e^{iG(\xi)} \varphi(\xi) \, d\xi = \int e^{i2^{\nu} xK(\xi)} \varphi(\xi) \, d\xi, \]
where \( K(\xi) = |\xi|^a + \xi \) for \( \xi \in \mathbb{R} \).
For \( \xi > 0 \) we have \( K'(\xi) = a|\xi|^{a-1} + 1 \) and for \( \xi < 0 \) one has \( K'(\xi) = 1 - a|\xi|^{a-1} \). It follows that \( K'(\xi) = 0 \) for \( \xi = -a^{-1/(a-1)} \). Also \( K''(\xi) \neq 0 \) for \( \xi \in \text{supp} \varphi \). We now apply the method of stationary phase (see Stein [15], p. 334). One obtains
\[ \left| \int e^{iG(\xi)} \varphi(\xi) \, d\xi \right| \lesssim (2^\nu x)^{-1/2} = 2^{-\nu/2} x^{-1/2}. \]
Hence
\[
\left| \int e^{iF} \varphi \, d\xi \right| = \left| \int e^{i(G+H)} \varphi \, d\xi \right| = \left| \int e^{iG} \varphi \, d\xi + \int (e^{iG+H} - e^{iG}) \varphi \, d\xi \right|
\geq 2^{-\nu/2} x^{-1/2} - O \left( \int |e^{iH} - 1| \varphi \, d\xi \right) \geq 2^{-\nu/2} x^{-1/2} - O \left( \int |H| \varphi \, d\xi \right),
\]
and we need an estimate of \( H \). One obtains
\[
\|H(\xi)\| = \|t^{2\alpha} - 2^\nu x\| \|\xi\|^\alpha \lesssim |t^{2\alpha} - 2^\nu x|
\]
on supp \( \varphi \). We then choose \( k \) such that
\[
t_{k+1} < \frac{2^\nu x}{2\alpha} \leq t_k
\]
where we assume that \( \nu \) is large. It follows that
\[
t_k \leq \frac{2^{2\nu} x}{2\alpha} \leq \frac{2^\nu}{2\alpha} = 2 \cdot 2^{\nu(1-\alpha)}
\]
and hence
\[
\log k \geq \frac{1}{2} \cdot 2^{\nu(a-1)} \geq 2^{\nu \epsilon}
\]
where \( \epsilon > 0 \). It is then easy to see that
\[
k \geq e^{2^{\nu \epsilon}}
\]
and
\[
t_k - t_{k+1} \leq \frac{1}{k} \leq e^{-2^{\nu \epsilon}}
\]
which implies that
\[
\left| t_k - \frac{2^\nu x}{2\alpha} \right| \leq t_k - t_{k+1} \leq e^{-2^{\nu \epsilon}}
\]
We conclude that
\[
|t_k 2^{\nu a} - 2^\nu x| \leq 2^{\nu a} e^{-2^{\nu \epsilon}} e^{-100 \nu}
\]
for \( \nu \) large.
Setting \( t = t_k \), invoking the inequality (20), and using the fact that \( x \leq 1 \), one obtains
\[
\left| \int e^{iF} \varphi \, d\xi \right| \geq 2^{-v/2} x^{-1/2} - O \left( e^{-100 \nu} \right) \geq 2^{-v/2} x^{-1/2}.
\]
It follows that
\[
\int |S^* f(x)|^2 \, dx \geq \int_{C_{2^{-v}}}^{1} \frac{2^\nu 1}{x} \, dx \geq 2^\nu \nu
\]
for \( \nu \) large.
We have \( \|f_{\nu}\|_2 = C^{2^{\nu/2}} \) and we have proved that \( \|S^* f_{\nu}\|_2 \gtrsim 2^{\nu/2} \nu^{1/2} \) and it follows that \( S^* \) is not a bounded operator on \( L^2(\mathbb{R}) \).

We shall then study the case \( n \geq 2 \) and \( a = 2 \). We let \( \varphi \in C_0^\infty(\mathbb{R}) \) be the same function as in the case \( n = 1 \). Also let \( \psi \in C_0^\infty(\mathbb{R}^{n-1}) \) and assume that \( \|\psi\|_2 > 0 \).
For \( x \in \mathbb{R}^n \) we write \( x = (x_1, x') \), where \( x' = (x_2, x_2, \ldots, x_n) \). We define \( f_{\nu} \) by setting \( \hat{f}_{\nu}(\xi) = \varphi(2^{-\nu} \xi_1) \psi(\xi') \) for \( v = 1, 2, 3, \ldots \).
It is then easy to see that \( \|f_\nu\|_2 = c 2^{\nu/2} \) for some constant \( c \).
We also have
\[
S_t f_\nu(x) = c \int_{\mathbb{R}^{n-1}} e^{i(\xi x_1 + \xi' x')} e^{i(t/2) |\xi|} |\varphi(2^{-\nu} \xi_1)| \psi(\xi') \, d\xi_1 d\xi',
\]
where \( c \) denotes a constant. Setting \( \eta_1 = 2^{-\nu} \xi_1 \) we obtain
\[
S_t f_\nu(x) = c 2^\nu \left( \int_{\mathbb{R}} e^{i(t/2) \eta_1^2 + 2^\nu \eta_1 x_1} \varphi(\eta_1) \, d\eta_1 \right) \left( \int_{\mathbb{R}^{n-1}} e^{i(\xi' x' + t/2) |\xi'|} \psi(\xi') \, d\xi' \right).
\]

We then choose \( t_k \) as an approximation for \( 2^\nu \) as in the one-dimensional case and set \( t(x_1) = t_k \). It follows that
\[
S_t f_\nu(x) = c 2^\nu I(x_1) J(x_1, x')
\]
where
\[
I(x_1) = \int_{\mathbb{R}} e^{i(t/2) \eta_1^2 + 2^\nu \eta_1 x_1} \varphi(\eta_1) \, d\eta_1
\]
and
\[
J(x_1, x') = \int_{\mathbb{R}^{n-1}} e^{i(\xi' x' + t/2) |\xi'|} \psi(\xi') \, d\xi'.
\]

Above we proved that \( |I(x_1)| \gtrsim 2^{-\nu/2} x_1^{-1/2} \) for \( C 2^{-\nu} \leq x_1 \leq 1 \). We also have
\[
S^* f_\nu(x) \gtrsim 2^\nu \|I(x_1)\| |J(x_1, x')|.
\]
It follows that
\[
\int_{\mathbb{R}^{n-1}} (S^* f_\nu(x))^2 \, dx' \gtrsim 2^{2\nu} \|I(x_1)\|^2 \int_{\mathbb{R}^{n-1}} |J(x_1, x')|^2 \, dx',
\]
and invoking Plancherel’s theorem we obtain
\[
\int_{\mathbb{R}^{n-1}} (S^* f_\nu(x))^2 \, dx' \gtrsim 2^{2\nu} \|I(x_1)\|^2 \int_{\mathbb{R}^{n-1}} |\psi(\xi')|^2 \, d\xi' = c 2^{2\nu} \|I(x_1)\|^2 \gtrsim 2^{2\nu} 2^{-\nu} x_1^{-1} = 2^\nu x_1^{-1}
\]
for \( C 2^{-\nu} \leq x_1 \leq 1 \).

We conclude that
\[
\int \int_{\mathbb{R}^{n-1}} (S^* f_\nu(x))^2 \, dx_1 dx' \gtrsim 2^\nu \int_{C 2^{-\nu}} 1/x_1 \, dx_1 \gtrsim 2^\nu
\]
and
\[
\|S^* f_\nu\|_2 \gtrsim 2^{\nu/2} \nu^{1/2}.
\]
Since \( \|f_\nu\|_2 = c 2^{\nu/2} \) it follows that \( S^* \) is not a bounded operator on \( L^2(\mathbb{R}^n) \).
REFERENCES

[1] Bourgain, J., *On the Schrödinger maximal function in higher dimensions*. Proc. Steklov Inst. Math., 280, 46-60 (2013).

[2] Bourgain, J., *A note on the Schrödinger maximal function*. J. Anal. Math. 130, 393-396 (2016).

[3] Carleson, L., *Some analytical problems related to statistical mechanics*. Euclidean Harmonic Analysis, Lecture Notes in Mathematics, vol 779, pp. 5-45. Springer, Berlin (1979).

[4] Cowling, M.G., *Pointwise behavior of solutions to Schrödinger equations*. Harmonic analysis, Lecture Notes in Mathematics vol 992, pp. 83-90, Springer Berlin (1983).

[5] Dahlberg, B.E.J., and Kenig, C.E., *A note on the almost everywhere behaviour of solutions to the Schrödinger equation*. Harmonic Analysis, Lecture Notes in Mathematics, vol 908, pp. 205-209, Springer, Berlin (1981).

[6] Du, X., Guth, L., and Li, X., *A sharp Schrödinger maximal estimate in \( \mathbb{R}^2 \)*. Ann. Math. 186, 607-640 (2017).

[7] Du, X. and Zhang, R., *Sharp \( L^2 \) estimate of Schrödinger maximal function in higher dimensions*. arXiv: 1805.02775v2.

[8] Mattila, P., *Geometry of Sets and Measures in Euclidean Spaces*. Fractals and rectifiability, Cambridge Univ. Press, 1995.

[9] Miyachi A., *On some singular Fourier multipliers*. J. of the Faculty of Science, University of Tokyo sec. I.A., Vol 28 No.2, 267-315 (1981).

[10] Sjölin, P., *Convergence almost everywhere of certain singular integrals and multiple Fourier series*. Arkiv för matematik 9, 66-90 (1971).

[11] Sjölin, P., *Regularity of solutions to the Schrödinger equation*. Duke Math. J. 55, 699-715 (1987).

[12] Sjölin, P., *Nonlocalization of operators of Schrödinger type*. Ann. Acad. Sci. Fenn. Math. 38, 141-147 (2013).

[13] Sjölin, P., *Two theorems on convergence of Schrödinger means*. To appear in J. of Fourier Analysis and Applications.

[14] Stein, E., *Singular integrals and differentiability properties of functions*. Princeton University Press, 1970.

[15] Stein, E., *Harmonic analysis. Real-variable methods, orthogonality, and oscillatory integrals*. Princeton Univ. Press, 1993.

[16] Vega, L., *Schrödinger equations: pointwise convergence to the initial data*. Proc. Amer. Math. Soc. 102, 874-878 (1988).

[17] Vega, L., *El multiplicador de Schrödinger, la Funcion Maximal y los Operadores des Restricciones*. Departamento de Matematicas, Univ. Auto´noma de Madrid, Madrid 1988.

[18] Walter, B., *Maximal estimates for oscillatory integrals with concave phase*. Harmonic analysis and Operator theory(Caracas 1994), 485-495, Contemp. Math., 189, Amer. Math. Soc., Providence, R.I., 1995.

[19] Walter, B., *Some \( L^p(\mathbb{L}^\infty) \) - and \( L^2(\mathbb{L}^2) \) - estimates for oscillatory Fourier transforms*. Analysis of Divergence (Orono, ME, 1997), 213-231, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998.

Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden
E-mail addresses: persj@kth.se, jostromb@kth.se