Decomposition of the Height Function of
Scherk’s First Surface
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Abstract We show that Scherk’s first surface, a one-parameter family of solutions to the
minimal surface equation, may be written as a linear superposition of other solutions with
specific parametric values.

KEYWORDS: minimal surface, soliton, nonlinear partial differential equation

A surface in the Monge representation, represented by its height function \( h[x, y] \), is a
minimal surface whenever
\[
(1 + h_y^2)h_{xx} - 2h_xh_yh_{xy} + (1 + h_x^2)h_{yy} = 0. \tag{1}
\]

A well-known solution is Scherk’s first surface \([1]\), shown in Fig. 1:
\[
z = h[x, y; \alpha] \equiv -\sec(\frac{1}{2}\alpha)\tan^{-1}\left\{\frac{\tanh\left[\frac{1}{2}x\sin(\alpha)\right]}{\tan\left[y\sin(\frac{1}{2}\alpha)\right]}\right\} \tag{2}
\]
As \( x \to \pm\infty \), \( z = \pm y\tan(\frac{1}{2}\alpha) + (n + \frac{1}{2})\pi\sec(\frac{1}{2}\alpha) \), respectively for integers \( n \). Thus Scherk’s
first surface connects two infinite sets of parallel planes at \( x = \pm\infty \), equally spaced by \( \pi \)
and rotated by an angle \( \alpha \) with respect to each other. It has been used as a model for grain
boundaries in diblock copolymers and smectic liquid crystals \([2,3]\) where the multi-valued
height function represents the peak of the one-dimensional density modulation of these
materials. In the limit that \( \alpha \to 0 \), the solution \( z = h[x, y; \alpha] \) in (2) is the height function
for another well-known minimal surface, the helicoid:
\[
\lim_{\alpha \to 0} h[x, y; \alpha] = -\tan^{-1}\left\{\frac{\alpha x}{\alpha y}\right\} = \tan^{-1}\left(\frac{y}{x}\right) - \frac{\pi}{2}. \tag{3}
\]
The helicoid corresponds to a topological defect (specifically a screw dislocation) in a quies-
cent layered structure \([4]\). As we shall see, Scherk’s first surface is an infinite superposition
of these topological defects.

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It has long been known that the arctangent function satisfies a number of remarkable identities \([5]\). Employing these we have two results:

**Theorem 1:** The solution \(h[x, y; \alpha]\) of (2) may be decomposed into an infinite superposition of dilated helicoids:

\[
h[x, y; \alpha] = \sec\left(\frac{1}{2}\alpha\right) \sum_{n=-\infty}^{\infty} \left\{ \tan^{-1}\left(\frac{y - n\ell}{x \cos\left(\frac{1}{2}\alpha\right)}\right) - \frac{\pi}{2} \right\}
\]  

(4)

where \(\sin\left(\frac{1}{2}\alpha\right) = \frac{\pi}{\ell}\) defines \(\ell\).

**Proof:** This follows from a result of Ramanujan \([5]\):

\[
\tan^{-1}[\tanh a \cot b] = \sum_{k=-\infty}^{\infty} \tan^{-1}\left(\frac{a}{b + k\pi}\right)
\]  

(5)

by taking \(a = \frac{1}{2}x \sin(\alpha)\) and \(b = y \sin\left(\frac{1}{2}\alpha\right)\). The result \((4)\) may also be derived via the Poisson summation formula applied to derivatives of \((2)\). That derivation only proves the equality of \((4)\) and \((5)\) up to an additive constant. In \((3)\) an infinite additive constant was neglected in comparison with \((1)\). However since the height function may be arbitrarily shifted along the \(z\)-axis, an additive constant is irrelevant. It is important to note that the dilation preserves the topology, but not the geometry of the helicoids and so the dilated helicoids are neither true helicoids nor are they minimal surfaces. However, it is surprising that a pure dilation along \(x\) and not a function of \(x\) makes a sum of helicoids into a minimal surface.

A class of finite decompositions of \((2)\) are also possible:
Theorem 2: The solution \( h[x, y; \alpha] \) of (4) may be decomposed into a finite superposition of dilated Scherk’s first surfaces:

\[
h[x \sec \beta, y; 2\beta] = \frac{\cos \tilde{\beta}}{\cos \beta} \sum_{m=0}^{n-1} h[x \sec \tilde{\beta}, y + \frac{m}{n} \pi \csc \tilde{\beta}; 2\tilde{\beta}]
\]

with \( \sin \beta = n \sin \tilde{\beta} \).

Proof: This follows by noting that

\[
\tan^{-1} \left[ \frac{\tanh x}{\tan y} \right] = \Im \ln \sin(y + ix)
\]

and the identity (6):

\[
\sin(nz) = 2^{n-1} \prod_{m=0}^{n-1} \sin \left( z + \frac{m\pi}{n} \right).
\]

An alternative proof is based on (5). Note that for an arbitrary positive integer \( n \), the sum can be decomposed into a “sum of sums”:

\[
\sum_{k=-\infty}^{\infty} \tan^{-1} \left( \frac{a}{b + k\pi} \right) = \sum_{m=0}^{n-1} \sum_{p=-\infty}^{\infty} \tan^{-1} \left( \frac{a}{b + (np + m)\pi} \right)
\]

\[
= \sum_{m=0}^{n-1} \tan^{-1} \left[ \tanh \left( \frac{a}{n} \right) \cot \left( \frac{b + m\pi}{n} \right) \right]
\]

Again, the dilation of the solutions by \( \sec \tilde{\beta} \) preserves the topology of the “subboundaries”. Note, however, that for small \( x \), \( h[x, y; \alpha] \) is linear in \( x \) and so the factors of \( \sec \beta \) multiplying \( x \) will cancel the factors of \( \cos \beta \) multiplying \( h \). Thus, in some sense, (5) extends the small \( x \) limit, where true, undistorted linear superposition holds, to the entire \( x \) axis.

Note that as the subboundaries are moved apart \( \ell \) necessarily grows, \( \alpha \) tends toward 0, the factors of cosine and secant tend to unity, and true linear superposition prevails. This is reminiscent of the multi-soliton solutions of the KdV or sine-Gordon equations – as the superposed solutions are moved further apart, undistorted linear superposition prevails [7]. Unlike these integrable systems, however, in moving the solutions apart we must change the angle of rotation \( \alpha \) which also alters the form of the subboundaries. Nonetheless, these decompositions suggest that \( h[x, y; \alpha] \) has greater symmetry than arbitrary solutions of (4).

The decompositions provided by the two theorems presented here should prove useful in studying the stability of grain boundaries in layered systems. Scherk’s first surface
has been used to model such boundaries [2]. However it is not clear that Scherk’s first surface minimizes nonlinear elastic energy functionals such as those considered in [3]. By using these decompositions it is possible to study long wavelength perturbations to the boundaries. For instance, while a typical approach might consider the stability of the location of a single topological defect, by utilizing (6) for \( n = 2 \), we can shift half of the topological defects with respect to the other half and thus probe long-wavelength deformations of the grain boundary. This work is in progress [8].

A variational approach to minimizing the energy functional would also be possible. For instance, one could deform the functional form of \( h[x, y; \alpha] \) via

\[
\tilde{h}[x, y; \alpha, \gamma] = h[\text{sgn}(x)|x|\gamma, y; \alpha],
\]

(10)
calculate the energy as a function of \( \gamma \), and then minimize over \( \gamma \). The decompositions here would allow a much broader class of variational ansätze that would still be computationally manageable.

Though the remarkable linear superposition properties of this solution might suggest a connection to the linear Weierstrass-Enneper representation of minimal surfaces [9], this is, unfortunately, not the case. The Weierstrass-Enneper representation allows for linear superposition of a minimal surface in parametric form, whereas (6) is in nonparametric form: we are unaware of any connection between that representation and our result.

Whether the hidden symmetry that gives rise to these decompositions can lead to similar decompositions of other height functions of minimal surfaces is an open but interesting question. It is amusing to note that the Born-Infeld equation:

\[
(1 - \phi_1^2)\phi_{xx} + 2\phi_x\phi_y\phi_{xy} - (1 + \phi_x^2)\phi_{yy} = 0
\]

(11)
is related to the minimal surface equation through the Wick rotation \( y = it \). It is worth noting that \( f(x-t) \) and \( f(x+t) \) are each solutions to (11) for any \( f(\cdot) \), and that soliton-like scattering properties exist for this equation [10]. Unfortunately, upon analytic continuation these solutions become complex and are inadmissible as height functions. Moreover, the hodograph transformation, on which the results in [10] rely, cannot be performed when branch-cut singularities are present as in (3).

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