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GENERIC BERNSTEIN-SATO POLYNOMIAL ON AN IRREDUCIBLE AFFINE SCHEME

ROUCHDI BAHLOUL

Abstract. Given \( p \) polynomials with coefficients in a commutative unitary integral ring \( C \) containing \( \mathbb{Q} \), we define the notion of a generic Bernstein-Sato polynomial on an irreducible affine scheme \( V \subset \text{Spec}(C) \). We prove the existence of such a non zero rational polynomial which covers and generalizes previous existing results by H. Biosca. When \( C \) is the ring of an algebraic or analytic space, we deduce a stratification of the space of the parameters such that on each stratum, there is a non zero rational polynomial which is a Bernstein-Sato polynomial for any point of the stratum. This generalizes a result of A. Leykin obtained in the case \( p = 1 \).

Introduction and Main Results

Fix \( n \geq 1 \) and \( p \geq 1 \) two integers and \( v \in \mathbb{N}^p \). Let \( x = (x_1, \ldots, x_n) \) and \( s = (s_1, \ldots, s_p) \) be two systems of variables. Let \( k \) be a field of characteristic \( 0 \). Let \( A_n(k) \) be the ring of differential operators with coefficients in \( k[x] = k[x_1, \ldots, x_n] \) and \( D \) (resp. \( \mathcal{O} \)) be the sheaf of rings of differential operators (resp. analytic functions) on \( C^n \) for which we denote by \( D_{x_0} \) (resp. \( \mathcal{O}_{x_0} \)) the fiber in \( x_0 \).

Let \( f = (f_1, \ldots, f_p) \) be in \( k[x]^p \) (resp. \( \mathcal{O}_{x_0}^p \)) and consider the following functional identity:

\[
b(s)f^s \in A_n(k)[s] \cdot f^{s+v},
\]

(resp. \( D_{x_0}[s] \) instead of \( A_n(k)[s] \)) where \( f^{s+v} = f_1^{s_1+v_1} \cdots f_p^{s_p+v_p} \). This identity takes place in the free module generated by \( f^s \) over \( k[x, \frac{1}{f_1}, \cdots, \frac{1}{f_p}, s] \) (resp. \( \mathcal{O}_{x_0}[\frac{1}{f_1}, \cdots, \frac{1}{f_p}, s] \)).

The set of such \( b(s) \) is an ideal of \( k[s] \) (resp. \( C[s] \)). This ideal is called the (global) Bernstein-Sato ideal of \( f \) (resp. local Bernstein-Sato ideal in \( x_0 \)) and we denote it by \( \mathcal{B}(f) \) (resp. \( \mathcal{B}_{x_0}(f) \)). When \( p = 1 \), this ideal is principal and its monic generator is called the Bernstein polynomial associated with \( f \). Historically, I.N. Bernstein introduced the (global) Bernstein polynomial and proved its existence (i.e. the fact that it is not zero). J.E. Björk has given the proof in the analytic case. Let us cite also M. Kashiwara who proved, moreover, the rationality of the roots of the local Bernstein polynomial. For \( p \geq 2 \), the algebraic case can be easily treated in the same way as for \( p = 1 \). For the analytic case, the proof of the non nullity of \( \mathcal{B}_{x_0}(f) \) is due to C. Sabbah. Let us also cite A. Gyoja who proved that \( \mathcal{B}_{x_0}(f) \) contains a non zero rational polynomial. The absolute Bernstein-Sato polynomial naturally leads to the notion of a generic Bernstein-Sato polynomial which we shall explain in what follows.

Let \( C \) be a unitary commutative integral ring with the following condition:

For any prime ideal \( P \subset C \) and for any \( n \in \mathbb{N} \setminus \{0\} \), we have:

\[ n \in P \Rightarrow 1 \in P. \]

1 all the fields considered in this paper are of characteristic 0
This condition is equivalent to the fact that for any $\mathcal{P} \subset \mathcal{C}$, the fraction field of $\mathcal{C}/\mathcal{P}$ is of characteristic 0. Note that this condition is satisfied if and only if there exists an injective ring morphism $\mathbb{Q} \hookrightarrow \mathcal{C}$.

We shall see $\mathcal{C}$ as the ring of coefficients or parameters. Indeed, let $f = (f_1, \ldots, f_p)$ in $\mathcal{C}[x]^p = \mathcal{C}[x_1, \ldots, x_n]^p$.

Let us denote by $A_n(\mathcal{C})$ the ring of differential operators with coefficients in $\mathcal{C}[x]$, that is the $\mathcal{C}$-algebra generated by $x_i$ and $\partial_{x_i}$ ($i = 1, \ldots, n$) where the only non trivial commutation relations are $[\partial_{x_i}, x_j] = 1$ for $i = 1, \ldots, n$ (hence $\mathcal{C}$ is in the center of $A_n(\mathcal{C})$).

We denote by Spec($\mathcal{C}$) (resp. Specm($\mathcal{C}$)) the set of prime (resp. maximal) ideals of $\mathcal{C}$ which is the spectrum of $\mathcal{C}$ (resp. the maximal spectrum). For an ideal $\mathcal{I} \subset \mathcal{C}$, we denote by $V(\mathcal{I}) = \{ \mathcal{P} \in \text{Spec}(\mathcal{C}) ; \mathcal{P} \supset \mathcal{I} \}$ the affine scheme defined by $\mathcal{I}$ and $V_m(\mathcal{I}) = V(\mathcal{I}) \cap \text{Specm}(\mathcal{C})$. Remark that we shall only work with the closed subsets of Spec($\mathcal{C}$) and forget the sheaf structure of a scheme.

We are going to introduce the notion of a generic Bernstein-Sato polynomial on an irreducible affine scheme $V = V(\mathbb{Q}) \subset \text{Spec}(\mathcal{C})$ (that is when $\mathbb{Q}$ is prime).

Fix a positive integer $d$ and a field $k$.

For each $j = 1, \ldots, p$, take $f_j = \sum_{|\alpha| \leq d} a_{\alpha,j} x^\alpha$ with $\alpha \in \mathbb{N}^n$ and $a_{\alpha,j}$ an indeterminate.

By the troubles with the real numbers, a rational generic Bernstein-Sato polynomial of $V$ is the spectrum of $\mathcal{C}$ (that is when $\mathcal{Q}$ is prime). So let $\mathcal{Q}$ be a prime ideal of $\mathcal{C}$ and suppose that none of the $f_j$’s is in $\mathcal{Q}[x]$.

The main result of this article is the following.

Theorem 1. There exists $h \in \mathcal{C} \setminus \mathcal{Q}$ and $b(s) \in \mathbb{Q}[s_1, \ldots, s_p] \setminus 0$ such that

$$h b(s)f^s \in A_n(\mathcal{C})[s]f^{s+v} + \left(\mathcal{Q}[x, \frac{1}{f_1 \cdots f_p}, s]\right)f^s.$$ 

Such a $b(s)$ is called a (rational) generic Bernstein-Sato polynomial of $f$ on $V = V(\mathbb{Q})$ (see the notation and the remark below).

In the case where $p = 1$, the generic and relative (not introduced here) Bernstein polynomial has been studied by F. Geandier in [1] and by J. Brianchon, F. Geandier and P. Maisonobe in [Br-Ge-M] in an analytic context (where $f$ is an analytic function of $x$).

In [B] (see also [B12]), H. Biosca studied these notions with $p \geq 1$ in the analytic and the algebraic context (that which we are concerned with) and proved that when

- $\mathcal{C} = \mathbb{C}[a_1, \ldots, a_m]$ or
- $\mathcal{C} = \mathbb{C}[a_1, \ldots, a_m]$ and $\mathcal{Q} = (0)$ so that $V$ is smooth and equal to $\mathbb{C}^m$ or $(\mathbb{C}^m, 0)$, we have a generic Bernstein-Sato polynomial. It does not seem straightforward to adapt her proof to the case where $\mathcal{Q} \neq (0)$ (i.e. when $V$ is singular). Let us also say that she did not mention the fact that the polynomial she constructed is rational even though a detailed study of her proof shows that it is. As it appears, our main result covers and generalizes the previous existing results in this affine situation.

Notation. Let $\mathcal{P}$ be a prime ideal of $\mathcal{C}$. For $c$ in $\mathcal{C}$, denote by $[c]_\mathcal{P}$ the class of $c$ in the quotient $\mathcal{C}/\mathcal{P}$ and $(c)_\mathcal{P} = \frac{[c]_\mathcal{P}}{1}$ this class viewed in the fraction field of $\mathcal{C}/\mathcal{P}$. We naturally extend these notations to $\mathcal{C}[x], A_n(\mathcal{C})$ and $\mathcal{C}[x, \frac{1}{f_1 \cdots f_p}, s]$.

Remark. Using these notations, we can see that the polynomial $b(s)$ of theorem [1] is a Bernstein-Sato polynomial of $(f)_\mathcal{P}$ for any $\mathcal{P} \in V(\mathbb{Q}) \setminus V(h)$. This justifies the name of a generic Bernstein-Sato polynomial on $V(\mathbb{Q})$.

As an application of theorem [1], we obtain some consequences:

Corollary 2. Fix a positive integer $d$ and a field $k$.

For each $j = 1, \ldots, p$, take $f_j = \sum_{|\alpha| \leq d} a_{\alpha,j} x^\alpha$ with $\alpha \in \mathbb{N}^n$ and $a_{\alpha,j}$ an indeterminate.
Take $a = (a_{\alpha,j})$ for $|\alpha| \leq d$ and $j = 1, \ldots, p$ such that we see $f = (f_1, \ldots, f_p)$ in $k[a][x]^p$. Denote by $m$ the number of the $a_{\alpha,j}$’s. Then there exists a finite partition of $k^m = \bigcup W$ where each $W$ is a locally closed subset of $k^m$ (i.e. $W$ is a difference of two Zariski closed sets) such that for any $W$, there exists a polynomial $b_W(s) \in k[s_1, \ldots, s_p] \setminus 0$ such that for each $a_0 \in W$, $b_W(s)$ is in $B^v(f(a_0, x))$.

Remark.

- This corollary generalizes to the case $p \geq 2$ the main result of A. Leykin and J. Briançon and Ph. Maisonobe in the case $p = 1$.
- There is another way to generalize these results: Given a well ordering $<_N$ on $\mathbb{N}$ compatible with sums, it is possible to prove the existence of a partition $k^m = \bigcup W$ into locally closed subsets with the following property: For any $W$, there exists a finite subset $G_W \subset k[a][x]$ such that for any $a_0 \in W$, the set $G_W(a_0)$ is a $<_N$-Gröbner basis of the Bernstein-Sato ideal $B^v(f(a_0, x))$, see [Ba] and [Br-Mai].

Proof of Corollary 2. We remark that we can give the same statement as in corollary 2 for any algebraic subset $Y \subset k^m$ as a space of parameters. The statement of corollary 2 will then follow from the proof of this more general statement, that we shall give by an induction on the dimension of $Y$. If $\dim Y = 0$, the result is trivial. Suppose $\dim Y \geq 1$.

Write $Y = \bigcup_{i=1}^r V_m(Q_i)$, where the $Q_i$’s are prime ideals in $k^m$ (we identify the maximal ideals of $k[a]$ and the points of $k^m$). For each $i$, let $h_i \in k[a] \cap Q_i$ and $b_i(s) \in k[s] \setminus 0$ be the $h$ and $b(s)$ of theorem 2 applied to $Q_i$. Now, write

$$Y = \left( \bigcup_{i=1}^r V_m(Q_i) \setminus V_m(h_i) \right) \bigcup Y',$$

with $Y' = \bigcup \left( (V_m(Q_i) \cap V_m(h_i)) \right)$ for which $\dim Y' < \dim Y$. Apply the induction hypothesis to $Y'$. We obtain that $Y$ is a union (not necessarily disjoint) of locally closed subsets $V$ such that for each $V$ there exists $b_V(s) \in k[s] \setminus 0$ which is in $B^v(f(a_0, x))$ for any $a_0 \in V$.

Let us show now how to obtain the announced partition. Let $B$ be the set of the obtained polynomials $b_V$’s. Set $B = \{b_1, \ldots, b_e\}$. For any $i = 1, \ldots, e$, let $E_i$ be the set of the $V$’s for which $b_i = b_V$. Put

- $W_1 = \bigcup_{V \in E_1} V$,
- $W_2 = \left( \bigcup_{V \in E_2} V \right) \setminus \left( \bigcup_{V \in E_1} V \right)$,
- $\vdots$
- $W_e = \left( \bigcup_{V \in E_e} V \right) \setminus \left( \bigcup_{V \in E_{e-1} \cup \cdots \cup E_1} V \right)$.

Note that some of the $W_i$’s may be empty. The set $\{(b_1, W_1), \ldots, (b_e, W_e)\}$ gives a partition $Y = \bigcup W_i$ in a way that $b_i \in B^v(f(a_0, x))$ for any $a_0 \in W_i$.

Corollary 3. Take $f_1(a, x), \ldots, f_p(a, x) \in \mathcal{O}(U)[x]$ where $\mathcal{O}(U)$ denotes the ring of holomorphic functions on an open subset $U$ of $\mathbb{C}^n$.

Then there exists a finite partition of $U = \bigcup W$ where each $W$ is an (analytic) locally closed subset of $U$ (i.e. each $W$ is a difference of two analytic subsets of $U$) such that for any $W$, there exists a rational non zero polynomial $b(s)$ which belongs to $B^v(f(a_0, x))$ for any $a_0 \in W$. 

\[ \square \]
Remark. As it will appear in the proof, we have the same result if we replace \( O(U) \) by \( \mathbb{C}\{a_1, \ldots, a_m\} \) or \( k[[a_1, \ldots, a_m]] \) (\( k \) being an arbitrary field).

Proof. Let us write \( f_j(a, x) = \sum g_{a,j}(a)x^a \) where \( g_{a,j} \in O(U) \). Let \( m \) be the number of the \( g_{a,j} \)'s and let us introduce \( m \) new variables \( b_{a,j} \). Consider the (analytic) map \( \phi : U \ni a \mapsto (b_{a,j} = g_{a,j}(a))_{a,j} \in \mathbb{C}^m \) where \( \mathbb{C} \) is a fixed arbitrary field. Now apply corollary 2 to this situation. Let \( k^m = \bigcup \mathcal{W} \) be the obtained partition and for any \( W \), let \( b_W \in \mathbb{Q}[s] \) be the polynomial given in 2. Now apply \( \phi^{-1} \). This gives a partition \( U = \bigcup \phi^{-1}(W) \). Since \( \phi \) is analytic, the sets \( \phi^{-1}(W) \) are locally closed analytic subsets of \( U \). It is then clear that for any \( W \) and \( a_0 \in \phi^{-1}(W) \), we have \( b_W \in \mathcal{B}^o(f(a_0, x)) \). \(\square\)

**Proof of the main theorem**

In order to prove theorem 4, we shall first prove the following.

**Theorem 4.** Let \( k \) be a field and \( f \in k[x]^p \). Then \( \mathcal{B}^o(f) \cap \mathbb{Q}[s] \) is not zero.

Note that in [Br], the author proved (for \( p = 1 \)) that the global Bernstein polynomial has rational roots for any field \( k \) of characteristic zero. The proof of 4 will use the following propositions.

**Proposition 5.** Let \( K \) be a subfield of a field \( L \). Suppose that \( f \in K[x]^p \). Let \( b(s) \in K[s] \) be such that \( b(s)f^s \in A_n(L)[s]f^{s+v} \). Then 
\[
 b(s)f^s = A_n(K)[s]f^{s+v}.
\]

**Proof.** The proof is inspired by [Br] in which the case \( p = 1 \) is treated. As \( L \) is a \( K \)-vector space, let us take \( \{1\} \cup \{l_\gamma; \gamma \in \Gamma\} \) as a basis so that \( L[x, s; \sum_{1 \leq j \leq p} \frac{1}{f_j}]f^s \) is a free \( K[x, s; \sum_{1 \leq j \leq p} \frac{1}{f_j}] \)-module with \( f^s \cup \{l_\gamma f^s; \gamma \in \Gamma\} \) as a basis. Now let \( P \) be in \( A_n(L)[s] \) such that \( b(s)f^s = P f^{s+v} \). We decompose \( P = P_0 + P' \) where \( P_0 \in A_n(K)[s] \) and \( P' \) has its coefficients in \( \bigoplus_{\gamma \in \Gamma} K \cdot l_\gamma \). Now, we have:

\[
 b(s)f^s = P_0 f^{s+v} + P' f^{s+v},
\]

with \( b(s)f^s \) and \( P_0 f^{s+v} \) in \( K[x, s; \sum_{1 \leq j \leq p} \frac{1}{f_j}]f^s \) and \( P' f^{s+v} \) in \( \bigoplus_{\gamma \in \Gamma} K[x, s; \sum_{1 \leq j \leq p} \frac{1}{f_j}]l_\gamma f^s \). By identification, we obtain:

\[
 b(s)f^s = P_0 f^{s+v}.
\]

\(\square\)

**Proposition 6.** (Br and Br-Mai) Given \( f \in \mathbb{C}[x]^p \), we have:

1. The set \( \{\mathcal{B}^o_v(f); x_0 \in \mathbb{C}^n\} \) is finite.
2. \( \mathcal{B}^o_v(f) \) is the intersection of all the \( \mathcal{B}^o_v(f) \) where \( x_0 \in \mathbb{C}^n \).

**Proof of theorem 4.** We shall divide the proof into two steps:

(a) First, suppose that \( k = \mathbb{C} \). By [S1], [S2] and [Gy], as mentioned in the introduction, each \( \mathcal{B}^o_v(f) \) contains a non zero rational polynomial. By the previous proposition, we can take a finite product of these polynomials and obtain a rational polynomial in \( \mathcal{B}^o_v(f) \).

(b) Now suppose that \( k \) is arbitrary. Let \( c_1, \ldots, c_N \) be all the coefficients that appear in the writing of the \( f_j \)'s and consider the field \( K = \mathbb{Q}(c_1, \ldots, c_N) \). There exist \( e_1, \ldots, e_N \in \mathbb{C} \) and an injective morphism of fields \( \phi : K \rightarrow \mathbb{C} \) such that \( \phi(e_i) = e_i \) for any \( i \). We
denote by the same symbol $\phi$ the natural extension of $\phi$ from $K[x]$ to $\mathbb{C}[x]$ and from $A_n(K)[s]$ to $A_n(\mathbb{C})[s]$. Now, consider in $\mathbb{C}[s]$ the Bernstein-Sato ideal $B^s(\phi(f))$ (where $\phi(f) = (\phi(f_1), \ldots, \phi(f_p))$). Using the result of case (a), there exists $b(s) \in \mathbb{Q}[s] \setminus 0$ that belongs to $B^s(\phi(f))$. So we have a functional equation:

$$b(s)\phi(f)^s = P \cdot \phi(f)^{s+v},$$

where $P \in A_n(\mathbb{C})[s]$. By proposition 3, we can suppose $P \in A_n(\phi(K))[s]$. Apply $\phi^{-1}$ to this equation. Since $b(s) \in \mathbb{Q}[s]$, $\phi^{-1}(b(s)) = b(s)$, thus we obtain:

$$b(s)f^s = \phi^{-1}(P) \cdot f^{s+v}.$$  

In conclusion $b(s)$ is in $B^v(f)$.

Now we dispose of a sufficient material to give the

**Proof of theorem 3.** By theorem 3, there exists a non zero rational polynomial $b(s)$ in $B^s((f)\mathbb{Q})$. Hence, we have the following equation:

$$b(s)\left(\frac{[f]\mathbb{Q}}{1}\right)^s = \frac{[U(s)]\mathbb{Q}}{[h]\mathbb{Q}} \cdot \left(\frac{[f]\mathbb{Q}}{1}\right)^{s+v},$$

where $U(s) \in A_n(\mathbb{C})[s]$ and $h \in \mathcal{C} \setminus \mathbb{Q}$. It follows that:

$$h b(s)f^s - U(s) \cdot f^{s+v} \equiv 0 \mod \mathbb{Q}$$

in $\mathcal{C}[x, \frac{1}{f_1 \cdots f_p}, s]f^s$. Since $f_1 \cdots f_p \notin \mathbb{Q}[x]$ and $\mathbb{Q}$ is prime, we obtain:

$$h b(s)f^s - U(s) \cdot f^{s+v} \in \mathbb{Q}[x, \frac{1}{f_1 \cdots f_p}, s]f^s.$$

This article is a more general and simplified version of some results of my thesis [Ba].

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