AN ECONOMIC ORDER QUANTITY FOR DETERIORATING ITEMS WITH ALLOWABLE REWORK OF DETERIORATED PRODUCTS

MAHDI KARIMI*, SEYED JAFAR SADIADI AND ALIREZA GHASEMI BIJAGHINI

Department of Industrial Engineering, Iran University of Science and Technology
Tehran, Iran

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Abstract. This paper presents an inventory model for deteriorating items with variable demand when shortage is permitted and quantity discount in purchase cost, and rework on deteriorating products are also allowed. The main idea of this research is to study the effects of the discount and the rework on the inventory costs. In this paper, it is assumed that for a certain quantity of purchased items, the seller would offer a discount and the manager would have the choice to either accept the discount or dismiss. On the other hand, there is also a similar decision-making scenario, where the manager makes a decision to reduce the total costs by using the rework and reducing the shortage periods or reducing the total costs by ignoring the rework cost and increasing the shortage periods. The implementation of the mathematical model is illustrated with a numerical example and sensitivity analysis describes the effects of the parameters on the total costs. The results show that the rework will decrease the total costs of the inventory system, significantly.

1. Introduction. The inventory control, as an essential tool for the management of costs required for handling an inventory system, was considered by researchers for almost seven decades. While the key aim of the economic production quantity (EPQ) of the inventory system is to maximize the profit, the economic order quantity (EOQ) seeks the best level of the order to minimize the total cost of the system. The inventory system needs to spend resources for fulfilling of ordering, holding, and shortage costs incurred by the system. The costs related to the holding of deteriorating items is one of the other costs imposed to the system, in addition to the cost items mentioned earlier.

The first contribution for deteriorating items was provided by Ghare and Schrader [15]. Goyal and Giri [19] investigated some special kinds of products which are either deteriorated or obsolescent and there have been numerous contributions devoted to this area. The work of Aggarwal and Jaggi [1] considered lead time and delay in payment for deteriorating items inventory models. Huang [23], developed a model by including a trade credit financing assumption. An inventory model with a variable deteriorating rate under supplier credit was proposed by Chang et al. [8] and

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* Corresponding author.
Liao[31] extended this model. The cost of opportunity was incorporated in the inventory model of Teng et al.[62]. The trade credit was embedded in an EPQ model by Teng and Chang[61]. A two-echelon EPQ for infinity time horizon with fixed demand rate was exhibited by Mahata[32]. The work of Shavandi et al.[52] introduced a multi-item inventory model for deteriorating items. Sarkar[50] added a routing problem to the previous models. Wang et al.[67], proposed a model to find the optimal credit period and the cycle time in a supply chain for deteriorating items with maximum lifetime. Pan and Li[40] presented a probabilistic EOQ model for deteriorating item. In the model proposed by Chowdhury et al.[14], the demand function was assumed to be varied. A three-level EPQ model with the fixed demand over the time horizon was introduced by Ghiami and Williams[16]. Srichandan[54] proposed an EPQ model for two-echelon supply chain by considering variable holding cost. Taleizadeh et al.[57] proposed a model for a two-layer EPQ with the variable demand function. A multi-item EOQ for deteriorating items was proposed by Ghosh et al.[18]. Bounkhel[5] considered the variable demand function in the EPQ model. A one-level model with fixed deteriorating rate along with variable demand rate was seen in the work of Chang[7]. In the EOQ model of Pervin et al.[43] with variable demand rate, the effect of trade credit policy was investigated. A two-warehouses model was proposed by Xu[68]. By considering reference price, Hsieh and Dye[21] exhibited a dynamic pricing for deteriorating items in the circumstance of stimulated demand by inventories. The work of Chan et al.[6] provided a two-layer EPQ with fixed demand for deteriorating items. The uncertainty of demand was imposed to the two-echelon EPQ by Maihami et al.[33].

The main common feature of all contributions mentioned above is that models were developed without considering discount and shortage.

In order to make the inventory models of deteriorating items more realistic, shortage assumption was added to these models. One of the earliest contributions in this area was provided by Jamal et al.[25]. An inventory model with partial exponential typebacklogging was provided by Papachristos and Skouri[41]. Teng et al.[63] proposed an inventory model by allowing partial backlogging and demonstrated the convexity of the cost function of the model and hence the global optimum of that cost function can be provided. After then, the complete backlogging was considered in a model with two levels of supply chain by Taleizadeh[55]. The partial backlogging was proposed under fuzzy assumptions by Jaggi et al.[24]. In the two warehouse model, the lead time was assumed to be positive Bhunia et al.[2]. Tat et al.[58] presented a model with the partial backlogging and fixed rate of demand and deterioration. The effect of variable deterioration rate in the models with partial backlogging was studied by Tayal et al.[59] and Mandal and Islam[34]. The complete backlogging along with variable deterioration rate was investigated by Roy[47] and Choudhury et al.[12]. Kumar and Rajput[28] presented a model by considering complete backlogging with variable deterioration rate and fuzzy demand. While the work of Kumar et al.[29] includes a model with variable deterioration rate and partial backlogging, the complete backlogging with variable deterioration rate was proposed by Mishra[35]. The time-varying demand and partial backlogging are conditions assumed by Bhunia et al.[3]. Samanta[48] incorporated the complete backlogging in the framework of an EPQ. In the work of Mohanty et al.[36] two separate models for fixed and variable demand rate with partial backlogging were presented. While the variable deterioration rate with complete backlogging was proposed by Naik and Patel[38], the backlogging was considered as the lost sale
by Chen and Sarker[11]. In the work of Samanta et al.[49], an EOQ model with complete backlogging and variable deterioration rate in the fuzzy environment was developed. The model of Li et al.[30] with the assumption of allowing partial backlogging is essentially a three-level EPQ. Pervin et al.[44] proposed a novel EOQ model by considering partial backlogging, probabilistic deterioration rate, variable holding cost, and variable demand rate. In brief, in all contributions discussed in this subsection, inflation and discount are neglected.

In recent years, the competitive nature of markets and time-varying of the value of the money, discount and inflation were considered in different studies. A discount on variable inventory cost was discussed by Shinn et al.[53]. In the two-echelon of Yang[70], the discount assumption was taken into account. The model of Zhang et al.[71] includes quantity discount. The future value of money and the maximum available budget was assumed in the model of Jana et al.[26]. The effect of inflation on the two warehouse inventory model was investigated by Bhunia et al.[4]. A model for deteriorating items with the assumption of discount was proposed in the work of Ghoreishi et al.[17] and Chen and Teng[10]. Shah and Cárdenas-Barrón[51] presented EPQ with the assumption of discount in a supply chain with two layers. Inflation rate was incorporated in an EPQ model by Choudri et al.[13]. The inflation assumption along with discount was considered by Chang[9]. While Rajan and Uthayakumar[46] demonstrated an EOQ model with the inflation assumption, in the EOQ model of Mukherjee and Bansal[37], the discount was considered. In sum, while taking into account discount and inflation helps the inventory model to be closer to the real world problems, shortage is not allowed in the contributions of this subsection, hence the model is investigated where the shortage is not permitted.

By considering the inflation, shortage and discount simultaneously, it is possible to build a more realistic inventory model. In the work of Hou[20], Jaggi et al.[23], and Pal et al.[39], the inflation assumption along with shortage was taken into account. The assumptions of the model presented by Yang[69] are partial backlogging, inflation, and variable deterioration rate. The inflation and partial backlogging were assumed in the model of Taleizadeh and Nematollahi[56]. In the work of Teng et al.[60] the discount, partial backlogging, along with variable deterioration rate, were assumed. While a two-warehouse inventory model by considering discount and partial backlogging, was studied by Kumar and Kumar[27], in the work of Tiwari et al.[65] and Tiwari et al.[66], inflation and partial backlogging were presumed. A two-echelon model by allowing partial backlogging shortage, price discount on backorders, and variable holding cost was proposed by Pervin et al. [45].

Another issue with little consideration in the literature of the inventory models is the rework of deteriorating items by making an assumption that there is neither repair nor replacement for deteriorated items. In other words, the inventory models are supposed non-cyclical supply chains and deteriorated products are immediately removed from the models and rework of deteriorated items is not allowed. But considering allowable rework with cyclical supply chain has many applications in real-world for example we can apply the rework of deteriorated items into our works when our study is about information technology companies or green supply chain management.

This assumption was applied in many studies because in the researches pertinent to deteriorating items, the attention is usually focused on items such as fruit, blood, etc. where reworked cannot be defined. But there are other types of items that are classified as deteriorating items and can be reworked when they become
deteriorated. For example, high-tech products such as smart phones and personal computers that are quickly outdated and become useless, even machines that are become worse can be reworked. During these years, many definitions were given for deterioration and one of the best definitions which includes all different cases of deterioration was presented by Pathma Raja[42], according to that work: “the deterioration means to make or become worse or lower in quality, value, character with depreciating value and wear away or disintegrate”, so high-tech items that do not meet the customer’s expected performance are also in this category Goyal and Giri[19]. But when a product with high technology was deteriorated, this deterioration would not occur on all product components, and components that do not have fast-growing technology can be used in new products, and this is what is performed in the successful corporates. Therefore, considering cyclical supply chain and the rework in this field is a novel idea and can be used for high-tech products.

In today’s competitive markets, a retailer can reduce his costs by using the discount provided by a supplier. On the other hand, the benefit of rework of product parts which can be used in the new other products, can reduce the total costs and increase the profit of the retailer. Thus, these assumptions, along with providing a solution and reducing the time which solution found, is the main idea of this paper.

In Table 1 a review of previous works is provided and the contribution of this paper in comparison with the other contributions is shown.

We propose an inventory model for deteriorating items by considering quantity discount and partial backlogging with allowing rework of deteriorated products, based on Teng et al.[63]. First, we introduce the model and its solution along with improvements to the solution as the methodology of our research in section 2. In section 3, a numerical analysis of the model is provided and finally, the results of the research are summarized in the conclusion section.

2. Methodology. In this section, the notations used for modeling and the underlying assumptions are explained.

2.1. Notations. The following notations used for mathematical modeling.

FC: Fixed cost of ordering  
HC: Holding cost per unit, per unit of time  
PPi: Purchase price per unit in ith period  
PPh: Purchase price per unit without discount  
PPl: Purchase price per unit considering discount  
RC: Rework cost per deteriorated unit  
LC: Shortage cost per unit of inventory  
MC: Credit loss cost per unit of lost sale  
Ri: Main inventory held in ith period  
Di: Main inventory deteriorated in ith period  
Qi: Order quantity in ith period  
Si: Shortage amount in ith period  
Bi: Lost sale in ith period  
RRi: Reworked inventory held in ith period  
RWi: Reworked inventory in ith period  
ρ: Deterioration rate of main inventory  
λ: Deterioration rate of reworked inventory  
β: Backlog rate  
ε: A very small positive number
Table 1. A. Review of previous works

| Number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------|---|---|---|---|---|---|---|---|---|----|
| Paper  | 49| 29| 12| 46| 48| 3| 7 | 37| 54|    |
| Model  |   |   |   |   |   |   |   |   |   |    |
| Type   |   |   |   |   |   |   |   |   |   |    |
| Ordering | √ | √ | √ | √ | √ | √ | √ |   |   |    |
| Production |   |   |   |   |   |   |   |   |   |    |
| Levels  |   |   |   |   |   |   |   |   |   |    |
| Three  |   |   |   |   |   |   |   |   |   |    |
| Two    |   |   |   |   |   |   |   |   |   |    |
| one    |   |   |   |   |   |   |   |   |   | √  |
| Multi  |   |   |   |   |   |   |   |   |   |    |
| Product |   |   |   |   |   |   |   |   |   |    |
| No     | √ | √ | √ | √ | √ | √ | √ |   |   |    |
| Yes    |   |   |   |   |   |   |   |   |   |    |
| Inflation and Discount |   |   |   |   |   |   |   |   |   |    |
| Discount |   |   |   |   |   |   |   |   |   |    |
| Inflation |   |   |   |   |   |   |   |   |   |    |
| Not Allowed | √ | √ | √ | √ | √ |   |   |   |   |    |
| Time Horizon |   |   |   |   |   |   |   |   |   |    |
| Infinity | √ | √ | √ | √ | √ | √ | √ |   |   |    |
| Finite   |   |   |   |   |   |   |   |   |   |    |
| Warehouses |   |   |   |   |   |   |   |   |   |    |
| Two    |   |   |   |   |   |   |   |   |   |    |
| One    |   |   |   |   |   |   |   |   |   | √  |
| Supply Chain |   |   |   |   |   |   |   |   |   |    |
| Open   | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Cyclic |   |   |   |   |   |   |   |   |   |    |
| Shortage |   |   |   |   |   |   |   |   |   |    |
| Lost Sales |   |   |   |   |   |   |   |   |   |    |
| Part. Backlog |   |   |   |   |   |   |   |   |   |    |
| Com. Backlog | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Not Allowed |   |   |   |   |   |   |   |   |   |    |
| Lead Time |   |   |   |   |   |   |   |   |   |    |
| Positive | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Zero    |   |   |   |   |   |   |   |   |   |    |
| Deter. rate |   |   |   |   |   |   |   |   |   |    |
| Variable | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Fixed   |   |   |   |   |   |   |   |   |   |    |
| Fuzzy Demand |   |   |   |   |   |   |   |   |   |    |
| Prob. Det. Variable |   |   |   |   |   |   |   |   |   |    |
| Det. Fixed | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Number | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| Paper  | 28| 2 | 6 | 10 | 11| 14| 16| 17| 18| 21 |
| Model  |   |   |   |   |   |   |   |   |   |    |
| Type   |   |   |   |   |   |   |   |   |   |    |
| Ordering | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Production |   |   |   |   |   |   |   |   |   |    |
| Levels  |   |   |   |   |   |   |   |   |   |    |
| Three  |   |   |   |   |   |   |   |   |   |    |
| Two    |   |   |   |   |   |   |   |   |   |    |
| one    |   |   |   |   |   |   |   |   |   | √  |
| Multi  |   |   |   |   |   |   |   |   |   |    |
| Product |   |   |   |   |   |   |   |   |   |    |
| No     | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Yes    |   |   |   |   |   |   |   |   |   |    |
| Inflation and Discount |   |   |   |   |   |   |   |   |   |    |
| Discount |   |   |   |   |   |   |   |   |   |    |
| Inflation |   |   |   |   |   |   |   |   |   |    |
| Not Allowed | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Time Horizon |   |   |   |   |   |   |   |   |   |    |
| Infinity | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Finite   |   |   |   |   |   |   |   |   |   |    |
| Warehouses |   |   |   |   |   |   |   |   |   |    |
| Two    |   |   |   |   |   |   |   |   |   |    |
| One    |   |   |   |   |   |   |   |   |   | √  |
| Supply Chain |   |   |   |   |   |   |   |   |   |    |
| Open   | √ | √ | √ | √ | √ | √ | √ | √ | √ |    |
| Cyclic |   |   |   |   |   |   |   |   |   |    |
| Number | Paper |
|--------|-------|
|       | 11 12 13 14 15 16 17 18 19 20 |
| Loss Sales | [28] [2] [6] [10] [11] [14] [16] [17] [18] [21] |
| Part. Backlog | √ | | |
| Com. Backlog | | | | | | | | | |
| Not Allowed | | | | | | | | | |
| Lead Time | Positive | | | | | | | | |
| Zero | | | | | | | | | |
| Variable | | | | | | | | | |
| Fixed | | | | | | | | | |
| Demand | Fuzzy | | | | | | | | |
| Prob. Det. | | | | | | | | | |
| Variable Det. | | | | | | | | | |
| Fixed | | | | | | | | | |
| Number | 21 22 23 24 25 26 27 28 29 30 |
| Paper | [24] [30] [33] [36] [38] [39] [40] [51] [13] [57] |
| Model Type | Ordering Production | | | | | | | | |
| Levels | Three | | | | | | | | |
| Two | | | | | | | | | |
| one | | | | | | | | | |
| Multi Product | No | | | | | | | | |
| Yes | | | | | | | | | |
| Inflation Discount | Discount Inflation | | | | | | | | |
| Not Allowed | | | | | | | | | |
| Time Horizon | Infinity | | | | | | | | |
| Finite | | | | | | | | | |
| Ware | Two | | | | | | | | |
| -houses | One | | | | | | | | |
| Supply Chain | Open | | | | | | | | |
| Cyclic | | | | | | | | | |
| Shortage | Lost Sales | | | | | | | | |
| Part. Backlog | | | | | | | | | |
| Com. Backlog | | | | | | | | | |
| Not Allowed | | | | | | | | | |
| Lead Time | Positive | | | | | | | | |
| Zero | | | | | | | | | |
| Variable | | | | | | | | | |
| Fixed | | | | | | | | | |
| Demand | Fuzzy | | | | | | | | |
| Prob. Det. | | | | | | | | | |
| Variable Det. | | | | | | | | | |
| Fixed | | | | | | | | | |
| Number | 31 32 33 34 35 36 37 38 39 40 |
| Paper | [58] [60] [65] [66] [59] [27] [68] [3] [20] [23] |
| Model Type | Ordering Production | | | | | | | | |
| Levels | Three | | | | | | | | |
| Two | | | | | | | | | |
| one | | | | | | | | | |
Table 1. C. Review of previous works

| Number | Paper | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|--------|-------|----|----|----|----|----|----|----|----|----|----|
| Multi  | Product | No | √  | √  | √  | √  | √  | √  | √  | √  | √  |
|        | Yes    |    |    |    |    |    |    |    |    |    |    |
| Inflation and Discount | Discount |    |    |    |    |    |    |    |    |    |    |
|        | Inflation |    |    |    |    |    |    |    |    |    |    |
|        | Not Allowed | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Time Horizon | Infinity | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Finitie | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Warehouses | Two | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | One | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Supply Chain | Open | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Cyclic | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Shortage | Lost Sales | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Part. Backlog | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Com. Backlog | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Not Allowed | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Lead Time | Positive | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Zero | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Deter. rate | Variable | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Fixed | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Demand | Fuzzy | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Prob. | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Det. Variable | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Det. Fixed | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Number | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| Paper | 32 | 34 | 35 | 47 | 56 | 63 | 69 | 55 | 31 | 8 |
| Model Type | Ordering | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Production | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Levels | Three | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Two | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | One | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Multi Product | No | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Yes | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Inflation and Discount | Discount | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Inflation | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Not Allowed | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Time Horizon | Infinity | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Finitie | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Warehouses | Two | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | One | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Supply Chain | Open | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Cyclic | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Shortage | Lost Sales | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Part. Backlog | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Com. Backlog | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Not Allowed | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
| Lead Time | Positive | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
|        | Zero | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |
Table 1. D. Review of previous works

| Number | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
|--------|----|----|----|----|----|----|----|----|----|----|
| Paper  | 32 | 34 | 35 | 47 | 56 | 63 | 69 | 55 | 31 | 8  |
| Deter. rate | Variable | Fixed | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Demand | Fuzzy | Prob. | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |

| Number | 51 | 52 | 53 | 54 | Percentage(rounded) | This Paper |
|--------|----|----|----|----|---------------------|------------|
| Paper  | 43 | 45 | 44 | 41 | %                   | ✓          |
| Model Type | Ordering | Production | ✓ | ✓ | ✓ | 73% | ✓ |
| Levels | Three | ✓ | 10% | | | | |
|         | Two   | ✓ | 26% | | | | |
|         | one   | ✓ | 64% | | | | |
| Multi Product | No | ✓ | ✓ | ✓ | 98% | ✓ | |
| Inflation and Discount | Discount | ✓ | 16% | | | | |
|                     | Inflation | ✓ | 24% | | | | |
|                     | Not Allowed | ✓ | ✓ | ✓ | 68% | ✓ | |
| Time Horizon | Infinity | ✓ | 27% | | | | |
|                     | Finite   | ✓ | 73% | | | | |
| Ware houses | Two | ✓ | ✓ | ✓ | 15% | ✓ | |
|                     | One      | ✓ | 85% | | | | |
| Supply Chain | Open | ✓ | ✓ | ✓ | 100% | ✓ | |
|                     | Cyclic   | ✓ | 0% | | | | |
| Shortage | Lost Sales | ✓ | ✓ | ✓ | 2% | ✓ | |
|                     | Part. Backlog | ✓ | ✓ | ✓ | 39% | ✓ | |
|                     | Com. Backlog | ✓ | ✓ | ✓ | 22% | ✓ | |
|                     | Not Allowed | ✓ | ✓ | ✓ | 37% | ✓ | |
| Lead Time | Positive | ✓ | ✓ | ✓ | 8% | ✓ | |
|                     | Zero     | ✓ | ✓ | ✓ | 92% | ✓ | |
| Deter. rate | Variable | ✓ | ✓ | ✓ | 32% | ✓ | |
|                     | Fixed    | ✓ | ✓ | ✓ | 68% | ✓ | |
| Demand | Fuzzy | Prob. | ✓ | ✓ | ✓ | 2% | ✓ | |
|                     | Det. Variable | ✓ | ✓ | ✓ | 72% | ✓ | |
|                     | Det. Fixed | ✓ | ✓ | ✓ | 22% | ✓ | |

M: Discount threshold
H: Planning horizon
η: Percentage of reworking products
\( t_i \): Replenishment time in \( i \)th period
\( s_i \): Shortage start time in \( i \)th period
\( tw_i \): Rework inventory arrival time inventory model in \( i \)th period
\( T_i \): length of \( i \)th period
\( I(t) \): Inventory amount in time \( t \)
c: A constant number
n: Number of periods
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$QH$: The accumulative demand during the planning horizon ($H$)

$TP$: Total purchase costs over the time horizon

$TO$: Total ordering costs over the time horizon

$TH$: Total Holding costs over the time horizon

$TR$: Total reworking costs over the time horizon

$TS$: Total shortage costs over the time horizon

$TB$: Total lost sale costs over the time horizon

$TC$: Total costs of inventory model

$f(t)$: Demand rate in time $t$

2.2. Assumptions.

1. Only one type of product exists and there is no change in deterioration rates.
2. Planning horizon is restricted to parameter $H$.
3. The main issue of this paper is high-tech products so repair and rework of deteriorated products are possible and allowed and a percentage of deteriorated products are reworked. One period is spent on rework so for $i = 1$, as shown in the Figure 2, we do not have any reworked inventory and we have $tw_1 = s_1$ as a result, similarly for $i = n + 1$ we have $tw_{(n+1)} = t_{(n+1)}$. Products are fed into the system as the main inventory is decreasing in the next periods. Percentage of reworked products is constant.
4. Demand rate of $f(t)$ is a continuous, non-negative, monotonic function of time and $(f(t))/(f'(t))$ is non-decreasing.
5. Shortage can occur in all periods and the inventory level at the end of time horizon is zero.
6. While shortage, backlogging rate depends on delay time until replenishment. As the time passes backlogging rate decreases. In other words, the number of customers who accept backlogging at the time $t$ until the next replenishment time is decreasing. In order to implement this assumption, when the inventory is negative, backlogging rate is calculated by $1/(1 + \beta(t_i - t))$, $s_{(i-1)} < t < t_i$ where $\beta$ is a positive constant.
7. There is no lead time.
8. If the order quantity $Q_i$ is less than the threshold $M$ set by supplier, customer has to pay more. However, if the ordering quantity is more than the discount threshold, discount is set on each unit of product by supplier.
9. It is assumed that the cost of credit loss is more than the price per unit ($BC > PPl, PPh$).
10. The rework cost is less than the purchasing price per unit, in other words, rework is beneficial ($RC < PPl, PPh$).

2.3. Mathematical model. In this section, we present a mathematical model for the inventory control of deteriorating items. The proposed model is formulated for a single-product system in a one-level supply chain. The shortage is allowed with partial backlogging, and a discount in purchase cost is considered and the demand rate is assumed to be time-varying. The main idea of this paper is to consider allowable rework of deteriorated products which means that when a product deteriorates, some parts of it can be used in new products. The deterioration rate and rework rate have fixed values over the finite time horizon and there is only one warehouse to store the products. It is necessary to determine the replenishment and shortage times to find the order quantity of each period. We first calculate the inventory
costs. In fact, all of the cost elements used in the model are defined in details in the next subsections.

2.3.1. Total purchase cost of deteriorated items. The lost products due to deterioration in ith period is defined as $D_i$; however, the deteriorated products can be partially reworked, i.e. $\eta D_i$ then the lost products due to deterioration of reworked products is $\lambda \eta D_i$, finally the actual lost products is calculated by $(1-\eta+\eta\lambda)D_i$. So the model becomes as follows,

\[ TP = \sum_{i=1}^{n} (1-\eta+\eta\lambda)D_i(yPPh + (1-y)PPl) \]  

\[ \text{Subject to :} \begin{cases} y = 0 & Q_i \geq M \\ y = 1 & Q_i \leq M \end{cases} \]

However, due to nonlinear behavior of $Q_i$ in the main model, solving the model is significantly complex. In order to solve this problem, the above model is rewritten without constraints and binary variables.

\[ TP = \sum_{i=1}^{n} (1-\eta+\eta\lambda)D_i\{PPh(1 - (\sum_{i=1}^{\lfloor \frac{Q_i}{M} \rfloor} \frac{1}{M})) + PPl((\sum_{i=1}^{\lfloor \frac{Q_i}{M} \rfloor} \frac{1}{M}))\}. \]  

The term $\sum_{i=1}^{\lfloor \frac{Q_i}{M} \rfloor} (1/\lfloor Q_i/M \rfloor)$ functions equally with variable $y$ which omits the role of the binary constraints of $Q_i < M$ and $Q_i \geq M$. For instance if $Q_i < M$ we have $Q_i/M < 1$ then $\lfloor Q_i/M \rfloor = 0$, as a result since the upper limit of $\sum_{i=1}^{\lfloor Q_i/M \rfloor} (1/\lfloor Q_i/M \rfloor)$ is zero, it does not matter what is posed in front of sigma notation and for all values of lower limit we have: $\sum_{i=1}^{\lfloor Q_i/M \rfloor} (1/\lfloor Q_i/M \rfloor) = 0$ and $TP = PPh \sum_{i=1}^{n} (1-\eta+\eta\lambda)D_i$. In contrast if $Q_i \geq M$ we have $Q_i/M \geq 1$ and assume $\lfloor Q_i/M \rfloor = c$, as result $\sum_{i=1}^{c} (1/c) = 1$ and $TP = PPl(\sum_{i=1}^{n} (1-\eta+\eta\lambda)D_i)$.

This transformation fulfills the needs of the model, but it has significant complexity burdened by floor function and nonlinear $Q_i$. In order to solve this problem $Sgn(x)$ function is used:

\[ Sgn(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \]  

Here $x$ is defined as $Q_i - M$ that if $Q_i > M$ the output will be 1 and if $Q_i < M$ the output will be $-1$. As the required values are 0 and 1 instead of $-1$ and $1$, $((Sgn(Q_i - M) + 1)/2)$ is used. But if $Q_i = M$ value of the variable $x$ is equal to zero and derivative is not defined. In order to solve this problem, to ensure that $Q_i \neq M$, $(Sgn(Q_i + \epsilon - M) + 1)/2)$ is used. Hence, $PP_i$ is defined as below:

\[ PP_i = PPh(1 - \frac{Sgn(Q_i + \epsilon - M) + 1}{2}) + PPl(\frac{Sgn(Q_i + \epsilon - M) + 1}{2}), \]  

The function of purchase cost in terms of ordering quantity was drawn in Figure 1.

At last $TP$ is defined as below:

\[ TP = \sum_{i=1}^{n} PP_i(1-\eta+\eta\lambda)D_i. \]
2.3.2. **Total ordering costs.** Fixed cost of ordering in one period is $FC$, so we have:

\[ TO = nFC. \]  

(6)

2.3.3. **Total holding costs.** $f(t)$ is positive and differentiable and since the amount of product which must be ordered in each period, should fulfill the demand of that period and the shortage of the previous period, the ordering quantity in the $i\text{th}$ period is equal to the amount of the inventory in the interval $[s(i-1), tw_i]$, hence we have:

\[
Q_i = \int_{s(i-1)}^{t_i} \frac{f(t)}{1 + \beta(t - t_i)} dt + \int_{t_i}^{tw_i} (e^{\beta(t - t_i)}) f(t) dt.
\]  

(7)

Based on Figure 2 and the assumptions, while the replenishment of $i\text{th}$ period, i.e. $[t_i, tw_i]$, inventory level is decreased because of deterioration and demand, hence
we have:
\[
\frac{dI(t)}{dt} = -\rho I(t) - f(t) \quad \text{for} \quad t_i \leq u \leq s_i, \tag{8}
\]
by solving the equation 8, we have:
\[
I(t) = e^{-\rho t} \int_t^s (e^{\rho u}) f(u) du. \tag{9}
\]
finally, the held and deteriorated inventories in ith period are written as below:
\[
R_i = \int_{t_i}^{tw_i} I(t) dt = \int_{t_i}^{tw_i} \{ e^{-\rho t} \int_t^s (e^{\rho u}) f(u) du \}
= \frac{1}{\rho} \int_{t_i}^{tw_i} (e^{\rho(t-t_i)} - 1)f(t) dt. \tag{10}
\]
\[
D_i = \rho R_i = \int_{t_i}^{tw_i} (e^{\rho(t-t_i)} - 1)f(t) dt. \tag{11}
\]
based on the same analysis but different bounds we have:
\[
RR_i = \frac{1}{\lambda} \int_{tw_i}^{s_i} (e^{\lambda(t-tw)} - 1)f(t) dt. \tag{12}
\]
hence, the holding cost is:
\[
TH = HC(\sum_{i=1}^n R_i + \sum_{i=1}^{n+1} RR_i). \tag{13}
\]
2.3.4. Total costs of rework of deteriorated products. Based on 11, since we have
\[
RW_i = \eta D_i, \quad \text{then:}
RW_i = \eta D_i = \eta \int_{t_i}^{tw_i} (e^{\rho(t-t_i)} - 1)f(t) dt. \tag{14}
\]
furthermore, total reworking cost or TR is defined as:
\[
TR = RC \sum_{i=1}^n RW_i. \tag{15}
\]
2.3.5. Total costs of shortage and lost sales. In the interval of \([s_i, t_{(i+1)}]\) changes in
inventory level which are caused by shortage, is expressed as a differential equation
as follows,
\[
\frac{dI(t)}{dt} = -\frac{f(t)}{1 + \beta(t_{i+1} - t)} \quad \text{for} \quad s_i \leq t \leq t_{i+1}, I(s_{n+1}) = 0, \tag{16}
\]
solving 16 yields,
\[
I(t) = -\int_{s_i}^t \frac{f(u)}{1 + \beta(t_{i+1} - u)} du \quad \text{for} \quad s_i \leq t \leq t_{i+1}, \tag{17}
\]
hence, the shortage and the lost sale in the ith period is calculated as below:
\[
S_i = \int_{s_i}^{(t_{i+1} - t)} \{ \int_{s_i}^t \frac{f(u)}{1 + \beta(t_{i+1} - u)} du \} dt = \int_{s_i}^{(t_{i+1} - t)} \frac{1}{1 + \beta(t_{i+1} - t)} f(t) dt,
B_i = \int_{s_i}^{t_{i+1} - t} (1 - \frac{1}{1 + \beta(t_{i+1} - t)}) f(t) dt
\]
\[
= \beta \int_{s_i}^{t_{i+1} - t} \frac{(t_{i+1} - t)}{1 + \beta(t_{i+1} - t)} f(t) dt = \beta S_i, \tag{19}
\]
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and shortage cost is defined as:

\[ TS = LC \sum_{i=1}^{n} S_i \]

and lost sale cost is:

\[ TB = MC \sum_{i=1}^{n} B_i. \]

2.3.6. Total costs of inventory model. Total cost of the model is defined as below:

\[ TC\{n,t_i,tw_i,s_i\} = TO + TP + TH + TR + TS + TB; \]

\[ TC = nFC \]

\[ + \sum_{i=1}^{n} PP_i(1 - \eta + \eta \lambda)D_i + HC\sum_{i=1}^{n} R_i + \sum_{i=1}^{n+1} RR_i \]

\[ + RC \sum_{i=1}^{n} RW_i + LC \sum_{i=1}^{n} S_i + MC \sum_{i=1}^{n} B_i, \]

\[ TC = nFC + \sum_{i=1}^{n} ((PPh(1 - \frac{Sgn(Q_i + \epsilon - M)}{2}) + \]

\[ + PPl(\frac{Sgn(Q_i + \epsilon - M)}{2})(1 - \eta + \eta \lambda)(\int_{t_i}^{tw_i} (e^{\rho(t-t_i)} - 1)f(t)dt) \]

\[ + HC\{\sum_{i=1}^{n} \frac{1}{\rho} \int_{t_i}^{tw_i} (e^{\rho(t-t_i)} - 1)f(t)dt + \sum_{i=1}^{n+1} \frac{1}{\lambda} \int_{tw_i}^{s_i} (e^{\lambda(t-tw_i)} - 1)f(t)dt \]

\[ + \eta RC \sum_{i=1}^{n} \int_{t_i}^{tw_i} (e^{\rho(t-t_i)} - 1)f(t)dt + LC \sum_{i=1}^{n} \int_{s_i}^{t_i+1} \frac{(t_i+1-t)}{1 + \beta(t_i+1-t)}f(t)dt \]

\[ + \beta MC \sum_{i=1}^{n} \int_{s_i}^{t_i+1} \frac{(t_i+1-t)}{1 + \beta(t_i+1-t)}f(t)dt, \]

2.4. Solution. In order to solve the proposed model, variable \( n \) is assumed to be constant and derivative is calculated and set to zero. Variable \( n \) is increased one by one and \( TC \) decreases until the first time that \( TC \) increases. As a result the previous \( n \) is the optimal solution.

**Theorem 2.1.** Convexity of \( TC\{n,t_i,tw_i,s_i\} \) for all variables is the necessary and sufficient condition for finding the global minimum of \( TC\{n,t_i,tw_i,s_i\} \). (See Appendix)

For a constant \( n \), the necessary conditions for \( TC \) to be minimum are \( \delta TC/\delta t_i = 0, \delta TC/\delta tw_i = 0 \) and \( \delta TC/\delta s_i = 0 \).

Hence we must solve the following equations:

\[ \frac{\delta TC}{\delta t_i} = (LC + \beta MC) \int_{s_i}^{t_i+1} \frac{1}{\{1 + \beta(t_i+1-t)\}^2} f(t)dt \]

\[ - (\rho PP_i(1 - \eta + \eta \lambda) + HC + \eta RC)(\int_{t_i}^{tw_i} (e^{\rho(t-t_i)})f(t)dt) = 0 \]
\[
\delta TC = -(PPh(1 - \eta + \eta \lambda) + \frac{HC}{\rho} + \eta RC)(e^{\rho(t-t_i)} - 1)
\]

\[
-HC \int_{t_{w_i}}^{s_i} (e^{\lambda(t-t_{w_i})}) f(t) dt = 0
\]

\[
\frac{\delta TC}{\delta s_i} = (LC + \beta MC)(\frac{(t_{i+1} - s_i)}{1 + \beta(t_{i+1} - t)}) - \frac{HC}{\lambda}(e^{\lambda(s_i - tw_i)} - 1) = 0,
\]

It is clear that with boundary conditions \((t_1 = 0\) and \(s_n = H\) we can easily find other variables from \(25,26\) and \(27\) and while \(TC\) is convex this answer is unique and yields the global minimum for \(TC\).

2.5. Improvement to the solution. Note that to shorten searching time for optimum \(n\) to a local minimum of \(TC(n)\) we provide a good starting value for \(n\). We need to introduce an estimation for \(n\). So we consider a constant value for purchase cost for all periods and assume that \(PPh = (PPh + PPl)/2\), then holding cost per unit and inventory and deterioration and rework cost will be \(HC + (\rho(1 - \eta + \eta \lambda)(PPh + PPl)/2) + \rho n RC\). The unit cost of lost sales is \(MC - ((PPh + PPl)/2)\) and backlogging rate is \(1/(1 + \beta)\) so the unit cost of stockout is \(LC/(1 + \beta) + (\beta(MC - (PPh + PPl)/2))/(1 + \beta)\). By using Eqn. (15) as in Teng et al.(1997)[64] we estimate the number of replenishments as:

\[
n = \text{rounded integer of } \{ \frac{(HC + [\rho(1 - \eta + \eta \lambda)]((PPh + PPl)/2) + \rho n RC)}{[2FC](HC + [\rho(1 - \eta + \eta \lambda)]((PPh + PPl)/2) + \rho n RC)} + \frac{[LC/(1 + \beta) + (\beta(MC - (PPh + PPl)/2))/(1 + beta)]}{\beta}]\}^{\frac{1}{2}}.
\]

It is clear that starting with \(n\) in \(28\) will simplify and shorten searching time compared with starting with \(n = 1\).

3. Numerical example. To illustrate the mathematical model and solution method we have provided a numerical example as follows:

**Example.** \(f(t) = 100 - 2t, FC = 200, HC = 70, PPh = 35, PPl = 25, \rho = 20, LC = 55, MC = 90, \rho = 0.02, \eta = 0.2, \lambda = 0.015, \beta = 15, H = 2, M = 200, \epsilon = 0.00001\). Substituting the parameters into \(28\) yields \(n = 6\) as starting point for searching for the optimum number, \(n\). According to boundary conditions we have \(t_1 = 0\) and \(s_{n+1} = H\). The solution method was coded in Matlab 2014b, obtained results are shown in Table 2.

| \(n\) | \(TC\) | \(i\) | \(t_i\) | \(tw_i\) | \(s_i\) |
|------|--------|-----|--------|--------|-------|
| 6    | 12785.79 | 1   | 0      | 0.0310 | 0.0310 |
| 7    | 12279.73 | 2   | 0.0310 | 0.0314 | 0.0410 |
| 8    | 7978.37  | 3   | 0.0410 | 0.0510 | 0.0520 |
| 9    | 14197.62 | 4   | 0.0521 | 0.0522 | 0.0814 |
| 10   | 14886.37 | 5   | 0.0816 | 0.0817 | 0.194  |
| 11   | 21056.36 | 6   | 0.1965 | 0.1972 | 0.2075 |
|      |         | 7   | 0.2075 | 0.2086 | 0.2355 |
| \(n^*\)\(=8\) |         | 8   | 0.2345 | 0.2346 | 1.3667 |
| \(TC^*\)=7978.37 |       | 9   | -      | 1.3681 | 2     |

Table 2. Optimal solution of Numerical Example
4. **Sensitivity analysis.** In this part the effect of changes in deterioration rate \((\rho)\), rework percentage \((\eta)\), backlogging rate \((\beta)\) and discount threshold \((M)\), in total costs of inventory model is investigated. A sensitivity analysis is performed with the same parameters used in the numerical example. In order to study the pattern of variations exhibited by parameters, for each parameter, five different levels are considered arbitrarily. Note that changes in \(\lambda\) will be same as \(\rho\).

Tables 3, 4, 5 and 6 and Figures 3, 4, 5 and 6 show the results of sensitivity analysis.

**Table 3.** Sensitivity analysis for backlogging rate

| \(\beta\) | 0 (Complete backlogging) | 10 | 20 | 40 | \(\infty\) (No shortage) |
|---|---|---|---|---|---|
| TC | 7874.34 | 7894.87 | 7978.37 | 8078.46 | 8213.75 |

Based on the results it is found that if lost sales are completely backlogged, inventory costs decrease and when no shortage is allowed total costs increase and

**Table 4.** Sensitivity analysis for deterioration rate

| \(\rho\) | 0.01 | 0.016 | 0.02 | 0.024 | 0.03 |
|---|---|---|---|---|---|
| TC | 7975.35 | 7977.15 | 7978.37 | 7979.56 | 7981.37 |

**Table 5.** Sensitivity analysis for rework percentage

| \(\eta\) | 0.1 | 0.16 | 0.2 | 0.24 | 0.3 |
|---|---|---|---|---|---|
| TC | 7978.63 | 7978.47 | 7978.37 | 7978.25 | 7978.09 |
Figure 4. Sensitivity analysis for deterioration rate

Figure 5. Sensitivity analysis for rework percentage
also it is concluded that the model is most sensitive to backlogging rate. The manager can increase the total profit of company by decreasing the backlog rate which in turn yields lower cost, and for decreasing the backlog rate, the company can enhance the trust of customers who wait for replenishment. In addition, presenting products in monopoly markets which there are few competing companies leads to decrease the backlog rate and increase the profit.

As it was guessed, when the deterioration decreases total costs decrease. Companies can decrease the deterioration rate by improving the holding conditions, for example using refrigerators for fruits and selling products with lower deterioration to increase the profit.

We see when the rework rate increases, total costs of inventory model will also decrease. It is clear that while rework cost is lower than the purchase cost (both high and low), the rework rate has an inverse relationship with total costs which means higher rework rate yields lower total cost so we showed that reworking of the deteriorated product is beneficial. As a managerial insight, using rework instead of ordering (or production) and selling reworkable deteriorating items will decrease the costs of the inventory systems and increase the profit of the business enterprises.

As seen discount threshold has no specific relation with total costs of inventory model and this makes this parameter more important since we should be more careful when we use this parameter to decrease total cost of our inventory model.

Table 6: Sensitivity analysis for discount threshold

| $M$ | 50  | 200 | 400 | 600 | $\infty$ |
|-----|-----|-----|-----|-----|---------|
| $TC$ | 12917 | 7978.37 | 10546 | 13848 | 11180 |

**Figure 6.** Sensitivity analysis for discount threshold
5. Conclusions. In this paper, we have presented an Economic Order Quantity model for deteriorating items with time-varying demand by considering allowed rework, discount in purchase cost and allowed shortage with partial backlogging. In this paper, we have considered discount and rework allowance and these assumptions made the model more realistic for the real world problems because it is more reasonable to consider reusability of outdated products. Considering these assumptions, a development of the binary model and turning it into an unconstrained problem using an innovative method and finally providing a way to reduce the time of finding the solution of the proposed model, are the main advantages of our work over the other researchers’ works. We have shown that by increasing the rework rate total cost would be reduced which is beneficial for remaining in a competitive market. Since it is assumed that the rework cost is marginally less than the purchase cost, therefore an increase in rework rate has an insignificant decrease in total cost. However, in real world little changes in rework rate may cause significant decrease in total cost and also note that even one percent reduction in huge quantity of costs will play a sensational role on the market also it was seen that discount threshold has no specific relation with total cost and we should be careful about it and use it to increase our benefits. As a future study, this model can be developed by adding more realistic assumption such as demand-varying discount, probabilistic demand, inflation, having lead time or salvage values. Developing the model for multi-products is another way to extend the proposed model.

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Appendix. We first will prove that $TC$ is convex for $n$ using dynamic programing. Since main mathematics functions are similar to that developed by Papachristos and Skouri(2000)[41] (notice that on-hand inventory function is exactly same and shortage function is decreasing in time and differentiable so it has similar behavior with their function), this proof is completely similar to that used by Papachristos and Skouri(2000)[41]. Instead of the Eqn. (11) in Papachristos and Skouri(2000)[41] Let:

$$TC\{n, t_i, tw_i, s_i\} = nFC + CI\{n, tw_i, t_i\} + CS\{n, s_i, tw_i\},$$

Where $CI$ is sum of inventory, deterioration, rework and holding costs and $CS$ is sum of shortage and lost sales cost incurred from 0 to $H$.

The rest of the proof is exactly same as the proof that used in Papachristos and Skouri(2000)[41], just instead of one function ($T$ in main study) we have two functions ($CI$ and $CS$), and we can use that proof separately for both of our functions then $CI$ and $CS$ are convex and $nFC$ is linear and convex, therefore $TC$ is strictly convex for $n$.

Next we prove that if $f(t) \geq 0$, then $TC$ is convex for all feasible points using Hessian matrix.

We do this by using Hessian principle which implies that if the value of the determinant of the Hessian matrix (Hessian value) for a function is always positive the function is convex.

We have:

$$TC\{n, t_i, tw_i, s_i\} = TO + TP + TH + TR + TS + TB,$$
It is clear that $TO = nFC$ is linear and convex. We show that $TP$ is convex. We know $TP = \sum_{i=1}^{n} PR_i(1 - \eta + \eta\lambda)D_i$, if $Q_i < M$ we have $TP = PPh\sum_{i=1}^{n} (1 - \eta + \eta\lambda)D_i$.

In contrast if $Q_i \geq M$ then $TP = PPl\sum_{i=1}^{n} (1 - \eta + \eta\lambda)D_i$.

The Hessian value for $D_i$ is: $2f(t)\rho e^{\eta(tw_i - t_i)}$. Then for $f(t) \geq 0$, since $tw_i \geq t_i$ then $2f(t)\rho e^{\eta(tw_i - t_i)}$ is non-negative and $D_i$ is convex also we know that $PPh$ and $PPl$ are parameters, so for all values of the feasible solution ($Q_i < M$ and $Q_i \geq M$), $TP$ is equal to the multiplication of a convex function in a parameter, then $TP$ is strictly convex.

Now we must prove that $TH + TR + TS + TB$ is convex, we have:

$$TH + TR + TS + TB = HC\left(\frac{1}{\rho} \int_{t_i}^{tw_i} (e^{\eta(t-tw_i)} - 1) f(t) dt \right) + \sum_{i=1}^{n+1} \frac{1}{\lambda} \int_{tw_i}^{s_i} (e^{\lambda(t-tw_i)} - 1) f(t) dt + \eta RC \sum_{i=1}^{n} \int_{t_i}^{tw_i} e^{\eta(t-tw_i)} f(t) dt + \beta MC \sum_{i=1}^{n} \int_{s_i}^{tw_i} \frac{(tw_i - t)}{1+\beta (tw_i - t)} f(t) dt,$$

The Hessian value for this equation is:

$$HC \left( \sum_{i=1}^{n} 2f(t)\rho e^{\eta(tw_i - t_i)} + \sum_{i=1}^{n+1} 2f(t)\rho e^{\eta(s_i - tw_i)} \right) + \eta RC \sum_{i=1}^{n} 2f(t)\rho e^{\eta(tw_i - t_i)} + \beta MC \sum_{i=1}^{n} \frac{2f(t)}{\beta^2 s_i^2 + 2\beta^2 s_i t_{i+1} + \beta^2 t_{i+1}^2 - 2\beta s_i + 2\beta t_{i+1} + 1}$$

For $f(t) \geq 0$ this equation is always positive afterwards $TH + TR + TS + TB$ is convex. Therefore, $TC$ is strictly convex and there is a unique and global minimum for the developed model in this paper.

REFERENCES

[1] S. Aggarwal and C. Jaggi, Ordering policies of deteriorating items under permissible delay in payments, *Journal of the operational Research Society*, **46** (1995), 658–662.

[2] A. K. Bhunia, A. A. Shaikh and R. Gupta, A study on two-warehouse partially backlogged deteriorating inventory models under inflation via particle swarm optimisation, *International Journal of Systems Science*, **46** (2015), 1036–1050.

[3] A. K. Bhunia, A. A. Shaikh and L. Sahoo, A two-warehouse inventory model for deteriorating item under permissible delay in payment via particle swarm optimisation, *International Journal of Logistics Systems and Management*, **24** (2016), 45–69.

[4] A. K. Bhunia, A. A. Shaikh, G. Sharma and S. Pareek, A two storage inventory model for deteriorating items with variable demand and partial backlogging, *Journal of Industrial and Production Engineering*, **32** (2015), 263–272.

[5] M. Boukhek, Nonlinear receding horizon control of production inventory systems with deteriorating items, *Yugoslav Journal of Operations Research*, **18** (2018), 37–45.

[6] C. K. Chan, W. H. Wong, A. Langevin and Y. Lee, An integrated production-inventory model for deteriorating items with consideration of optimal production rate and deterioration during delivery, *International Journal of Production Economics*, **189** (2017), 1–13.

[7] C.-T. Chang, Inventory models with stock-and pricependent demand for deteriorating items based on limited shelf space, *Yugoslav Journal of Operations Research*, **20** (2016).
[8] C.-T. Chang, L.-Y. Ouyang and J.-T. Teng, An EOQ model for deteriorating items under supplier credits linked to ordering quantity, *Applied Mathematical Modelling*, 27 (2003), 983–996.

[9] H. J. Chang, A partial backlogging inventory model for non-instantaneous deteriorating items with stock-dependent consumption rate under inflation, *Yugoslav Journal of Operations Research*, 20 (2016).

[10] S.-C. Chen and J.-T. Teng, Inventory and credit decisions for time-varying deteriorating items with upstream and downstream trade credit financing by discounted cash flow analysis, *European Journal of Operational Research*, 243 (2015), 566–575.

[11] Z. Chen and B. R. Sarker, Integrated production-inventory and pricing decisions for a single-manufacturer multi-retailer system of deteriorating items under JIT delivery policy, *The International Journal of Advanced Manufacturing Technology*, 89 (2017), 2099–2117.

[12] K. D. Choudhury, B. Karmakar, M. Das and T. K. Datta, An inventory model for deteriorating items with stock-dependent demand time-varying holding cost and shortages, *Opsearch*, 52 (2015), 55–74.

[13] V. Choudhri, M. Venkatashalam and S. Panayappan, Production inventory model with deteriorating items, two rates of production cost and taking account of time value of money, *Journal of Industrial and Management Optimization*, 12 (2016), 1153–1172.

[14] R. R. Chowdhury, S. Ghosh and K. Chaudhuri, An inventory model for deteriorating items with stock and price sensitive demand, *International Journal of Applied and Computational Mathematics*, 1 (2015), 187–201.

[15] P. Ghare and G. Schrader, An inventory model for deteriorating item for exponentially deteriorating items, *Journal of Industrial Engineering*, 14 (1963), 238–243.

[16] Y. Ghiami and T. Williams, A two-echelon production-inventory model for deteriorating items with multiple buyers, *International Journal of Production Economics*, 159 (2015), 233–240.

[17] M. Ghoreishi, G.-W. Weber and A. Mirmazadeh, An inventory model for non-instantaneous deteriorating items with partial backlogging, permissible delay in payments, inflation-and selling price-dependent demand and customer returns, *Annals of Operations Research*, 226 (2015), 221–238.

[18] S. K. Ghosh, T. Sarkar and K. Chaudhuri, A multi-item inventory model for deteriorating items in limited storage space with stock-dependent demand, *American Journal of Mathematical and Management Sciences*, 34 (2015), 147–161.

[19] S. Goyal and B. C. Giri, Recent trends in modeling of deteriorating inventory, *European Journal of Operational Research*, 134 (2001), 1–16.

[20] K.-L. Hou, An inventory model for deteriorating items with stock-dependent consumption rate and shortages under inflation and time discounting, *European Journal of Operational Research*, 168 (2006), 463–474.

[21] T.-P. Hsieh and C.-Y. Dye, Optimal dynamic pricing for deteriorating items with reference price effects when inventories stimulate demand, *European Journal of Operational Research*, 262 (2017), 136–150.

[22] Y.-F. Huang, Optimal retailer’s ordering policies in the EOQ model under trade credit financing, *Journal of the operational Research Society*, 54 (2003), 1011–1015.

[23] C. K. Jaggi, K. Aggarwal and S. K. Goel, Optimal order policy for deteriorating items with inflation induced demand, *International Journal of Production Economics*, 10 (2006), 707–714.

[24] C. K. Jaggi, S. Pareek, S. K. Goel and Nidhi, An inventory model for deteriorating items with ramp type demand under fuzzy environment, *International Journal of Logistics Systems and Management*, 22 (2015), 436–463.

[25] A. Jamal, B. Sarker and S. Wang, An ordering policy for deteriorating items with allowable shortage and permissible delay in payment, *Journal of the operational Research Society*, 48 (1997), 826–833.

[26] D. K. Jana, B. Das and M. Maiti, Multi-item partial backlogging inventory models over random planning horizon in random fuzzy environment, *Applied Soft Computing*, 21 (2014), 12–27.

[27] N. Kumar and S. Kumar, Effect of learning and salvage worth on an inventory model for deteriorating items with inventory-dependent demand rate and partial backlogging with capability constraints, *Uncertain Supply Chain Management*, 4 (2016), 123–136.

[28] S. Kumar and U. Rajput, Fuzzy inventory model for deteriorating items with time dependent demand and partial backlogging, *Applied Mathematics*, 6 (2015), Article ID 54567, 13 pages.
[29] S. Kumar, A. K. Singh and M. K. Patel, Optimization of Weibull deteriorating items inventory model under the effect of price and time dependent demand with partial backlogging, *Sadhana*, 41 (2016), 977–984.

[30] Y. Li, S. Zhang and J. Han, Dynamic pricing and periodic ordering for a stochastic inventory system with deteriorating items, *Automation*, 76 (2017), 200–213.

[31] J.-J. Liao, A note on an EOQ model for deteriorating items under supplier credit linked to ordering quantity, *Applied Mathematical Modelling*, 31 (2007), 1690–1699.

[32] G. C. Mahata, An EPQ-based inventory model for exponentially deteriorating items under retailer partial trade credit policy in supply chain, *Expert systems with Applications*, 39 (2012), 3537–3550.

[33] R. Maihami, B. Karimi and S. M. Ghomi, Pricing and Inventory Control in a Supply Chain of Deteriorating Items: A Non-cooperative Strategy with Probabilistic Parameters, *International Journal of Applied and Computational Mathematics*, 3 (2017), 2477–2499.

[34] W. A. Mandal and S. Islam, Fuzzy inventory model for weibull deteriorating items, with time depended demand, shortages, and partially backlogging, *Pak. J. Stat. Oper. Res.*, 12 (2016), 101–109.

[35] V. K. Mishra, Inventory Model of Deteriorating Items with Revenue Sharing on Preservation Technology Investment under Price Sensitive Stock Dependent Demand, *International Journal of Mathematical Modelling and Computations*, 21 (2016), 37–48.

[36] D. J. Mohanty, R. S. Kumar and A. Goswami, A two-warehouse inventory model for non-instantaneous deteriorating items over stochastic planning horizon, *Journal of Industrial and Production Engineering*, 33 (2016), 516–532.

[37] B. Mukherjee and A. Bansal, An Approach for Developing an Optimum Quantity Discount Policy of Deteriorating Items Inventory Transportation System, *International Journal of Innovative Technology and Research*, 5 (2017), 5811–5816.

[38] M. Pervin, S. K. Roy and G.-W. Weber, Analysis of inventory control model with shortage under time-dependent demand and time-varying holding cost including stochastic deterioration, *Annals of Operations Research*, 260 (2018), 437–460.

[39] S. Pal, G. Mahapatra and G. Samanta, A production inventory model for deteriorating item with ramp type demand allowing inflation and shortages under fuzziness, *Economic Modelling*, 46 (2015), 334–345.

[40] X. Pan and S. Li, Optimal control of a stochastic production-inventory system under deteriorating items and environmental constraints, *International Journal of Production Research*, 53 (2015), 607–628.

[41] S. Papachristos and K. Skouri, An optimal replenishment policy for deteriorating items with time-varying demand and partial-exponential type-backlogging, *Operations Research Letters*, 27 (2000), 175–184.

[42] P. Pathma Raja, A study on factors affecting quality deterioration in housing construction industry in Klang Valley, in *Open University Malaysia*.

[43] M. Pervin, G. C. Mahata and S. K. Roy, An inventory model with declining demand market for deteriorating items under a trade credit policy, *International Journal of Management Science and Engineering Management*, 11 (2016), 243–251.

[44] M. Pervin, S. K. Roy and G.-W. Weber, Analysis of inventory control model with shortage under time-dependent demand and time-varying holding cost including stochastic deterioration, *Annals of Operations Research*, 260 (2018), 437–460.

[45] M. Pervin, S. K. Roy and G.-W. Weber, A Two-echelon inventory model with stock-dependent demand and variable holding cost for deteriorating items, *Numerical Algebra, Control and Optimization*, 7 (2017), 21–50.

[46] R. S. Rajan and R. Uthayakumar, Optimal pricing and replenishment policies for instantaneous deteriorating items with backlogging and trade credit under inflation, *Journal of Industrial Engineering International*, 13 (2017), 428–443.

[47] A. Roy, Fuzzy inventory model for deteriorating items with price dependent demand, *International Journal of Management Science and Engineering Management*, 10 (2015), 237–241.

[48] G. Samanta, A production inventory model with deteriorating items and shortages, *Yugoslav Journal of Operations Research*, 14 (2016).

[49] P. Samanta, J. Das and S. Indrajitsingha, Fuzzy Inventory Model for Two Parameter Weibull Deteriorating Items, (2017).

[50] B. Sarkar, A production-inventory model with probabilistic deterioration in two-echelon supply chain management, *Applied Mathematical Modelling*, 37 (2017), 3138–3151.
[51] N. H. Shah and L. E. Cárdenas-Barrón, Retailer’s decision for ordering and credit policies for deteriorating items when a supplier offers order-linked credit period or cash discount, Applied Mathematics and Computation, 259 (2015), 569–578.

[52] H. Shavandi, H. Mahlooji and N. E. Nosratian, A constrained multi-product pricing and inventory control problem, Applied Soft Computing, 12 (2012), 2454–2461.

[53] S. W. Shinn, H. Hwang and S. P. Sung, Joint price and lot size determination under conditions of permissible delay in payments and quantity discounts for freight cost, European Journal of Operational Research, 91 (1996), 528–542.

[54] M. Srichandan, Inventory Model of Deteriorating Items for Linear Holding Cost with Time Dependent Demand, (2015).

[55] A. A. Taleizadeh, An economic order quantity model for deteriorating item in a purchasing system with multiple prepayments, Applied Mathematical Modelling, 38 (2014), 5357–5366.

[56] A. A. Taleizadeh and M. Nematollahi, An inventory control problem for deteriorating items with back-ordering and financial considerations, Applied Mathematical Modelling, 38 (2014), 93–109.

[57] A. A. Taleizadeh, M. Noori-daryan and L. E. Cárdenas-Barrón, Joint optimization of price, replenishment frequency, replenishment cycle and production rate in vendor managed inventory system with deteriorating items, International Journal of Production Economics, 159 (2015), 285–295.

[58] R. Tat, A. A. Taleizadeh and M. Esmaeili, Developing economic order quantity model for non-instantaneous deteriorating items in vendor-managed inventory (VMI) system, International Journal of Systems Science, 46 (2015), 1257–1268.

[59] S. Tayal, S. Singh and R. Sharma, An inventory model for deteriorating items with seasonal products and an option of an alternative market, Uncertain Supply Chain Management, 3 (2015), 69–86.

[60] J.-T. Teng, L. E. Cárdenas-Barrón, H.-J. Chang, J. Wu and Y. Hu, Inventory lot-size policies for deteriorating items with expiration dates and advance payments, Applied Mathematical Modelling, 40 (2016), 8605–8616.

[61] J.-T. Teng and C.-T. Chang, Optimal manufacturer’s replenishment policies in the EPQ model under two levels of trade credit policy, European Journal of Operational Research, 195 (2009), 358–363.

[62] J.-T. Teng, L.-Y. Ouyang and L.-H. Chen, A comparison between two pricing and lot-sizing models with partial backlogging and deteriorated items, International Journal of Production Economics, 105 (2007), 190–203.

[63] J. T. Teng, H. Yang and L. Ouyang, On an EOQ model for deteriorating items with time-varying demand and partial backlogging, Journal of the Operational Research Society, 54 (2003), 432–436.

[64] J. T. Teng, M. S. Chern and H. L. Yang, An optimal recursive method for various inventory replenishment models with increasing demand and shortages, Naval Research Logistics, 44 (1997), 791–806.

[65] S. Tiwari, L. E. Cárdenas-Barrón, A. Khanna and C. K. Jaggi, Impact of trade credit and inflation on retailer’s ordering policies for non-instantaneous deteriorating items in a two-warehouse environment, International Journal of Production Economics, 176 (2016), 154–169.

[66] S. Tiwari, C. K. Jaggi, A. K. Bhunia, A. A. Shaikh and M. Goh, Two-warehouse inventory model for non-instantaneous deteriorating items with stock-dependent demand and inflation using particle swarm optimization, Annals of Operations Research, 254 (2017), 401–423.

[67] W.-C. Wang, J.-T. Teng and K.-R. Lou, Seller’s optimal credit period and cycle time in a supply chain for deteriorating items with maximum lifetime, European Journal of Operational Research, 232 (2014), 315–321.

[68] X. Xu, Q. Bai and M. Chen, A comparison of different dispatching policies in two-warehouse inventory systems for deteriorating items over a finite time horizon, Applied Mathematical Modelling, 41 (2017), 359–374.

[69] H.-L. Yang, Two-warehouse partial backlogging inventory models for deteriorating items under inflation, International Journal of Production Economics, 103 (2006), 362–370.

[70] P.-C. Yang, Pricing strategy for deteriorating items using quantity discount when demand is price sensitive, European Journal of Operational Research, 157 (2004), 389–397.
[71] Q. Zhang, H. Dong, J. Luo and A. Segerstedt, Supply chain coordination with trade credit and quantity discount incorporating default risk, *International Journal of Production Economics*, **153** (2014), 352–360.

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E-mail address: Mahdi.Karimi@ind.iust.ac.ir
E-mail address: sjsadjadi@iust.ac.ir
E-mail address: alireza.ghasemi@ind.iust.ac.ir