Asymptotic behavior of the $W^{1/q,q}$-norm of mollified $BV$ functions and applications to singular perturbation problems

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Abstract

Motivated by results of Figalli and Jerison [8] and Hernández [7], we prove the following formula:

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \|\eta_\varepsilon \ast u\|_{W^{1/q,q}(\Omega)}^q = C_0 \int_{J_u} |u^+(x) - u^-(x)|^q \, d\mathcal{H}^{N-1}(x),$$

where $\Omega \subset \mathbb{R}^N$ is a regular domain, $u \in BV(\Omega) \cap L^\infty$, $q > 1$ and $\eta_\varepsilon(z) = \varepsilon^{-N} \eta(z/\varepsilon)$ is a smooth mollifier. In addition, we apply the above formula to the study of certain singular perturbation problems.

1 Introduction

Figalli and Jerison found in [8] a relationship between the perimeter of a set and a fractional Sobolev norm of its characteristic function. More precisely, for the mollifying kernel $\eta_\varepsilon(z) = \varepsilon^{-N} \eta(z/\varepsilon)$, where $\eta(z)$ denotes the standard Gaussian in $\mathbb{R}^N$, they showed that there exist constants $C_1 > 0$ and $C_2 > 0$ such that for every set $A \subset \mathbb{R}^N$ of finite perimeter $P(A)$ we have

$$C_1 P(A) \leq \liminf_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \|\eta_\varepsilon \ast \chi_A\|_{H^{1/2}(\mathbb{R}^N)}^2 \leq \limsup_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \|\eta_\varepsilon \ast \chi_A\|_{H^{1/2}(\mathbb{R}^N)}^2 \leq C_2 P(A), \quad (1.1)$$

where $\chi_A$ is the characteristic function of $A$. More recently, Hernández improved this result in [7] as follows. For $\eta_\varepsilon$ as above he showed that there exist a constant $C_0 > 0$ such that for every $u \in BV(\mathbb{R}^N) \cap L^\infty$ we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\ln \varepsilon|} \|\eta_\varepsilon \ast u\|_{H^{1/2}(\mathbb{R}^N)}^2 = C_0 \int_{J_u} |u^+(x) - u^-(x)|^2 \, d\mathcal{H}^{N-1}(x). \quad (1.2)$$

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A related result in which the same R.H.S. as in (1.2) appears, was obtained in [13]. More precisely, we showed in [13] that for every radial \( \eta \in C_c^\infty(\mathbb{R}^N, \mathbb{R}) \) there exists a constant \( C = C_\eta > 0 \) such that for every \( u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty \) we have

\[
\lim_{\varepsilon \to 0^+} \varepsilon \| \eta_\varepsilon \ast u \|_{H^1(\Omega)}^2 = C_\eta \int_{J_u} |u^+(x) - u^-(x)|^2 \, d\mathcal{H}^{N-1}(x). \tag{1.3}
\]

More recently, we showed in [14] yet another related result:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set with bounded Lipschitz boundary and let \( u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d) \). Then, for any \( q > 1 \) we have

\[
\lim_{\varepsilon \to 0^+} \int_{B_\varepsilon(x) \cap \Omega} \frac{1}{\varepsilon^N} \left| \frac{u(y) - u(x)}{|y - x|} \right|^q dy \, dx = C_N \int_{J_u} |u^+(x) - u^-(x)|^q \, d\mathcal{H}^{N-1}(x), \tag{1.4}
\]

with the dimensional constant \( C_N > 0 \) defined by

\[
C_N := \frac{1}{N} \int_{S^{N-1}} |z_1| \, d\mathcal{H}^{N-1}(z), \tag{1.5}
\]

where we denote \( z := (z_1, \ldots, z_N) \in \mathbb{R}^N \).

In the present paper we generalize the formula (1.2) in several aspects:

- We allow a general mollifying kernel \( \eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R}) \) (not only the Gaussian as before),
- We allow a general domain \( \Omega \subset \mathbb{R}^N \), of certain regularity, while previous results required \( \Omega = \mathbb{R}^N \),
- We treat the \( W^{1/q, q}(\Omega) \)-norm for any \( q > 1 \), while previous results were restricted to the case \( q = 2 \).

Recall that the Gagliardo seminorm \( \| u \|_{W^{1/q, q}(\Omega, \mathbb{R}^d)} \) is given by

\[
\| u \|_{W^{1/q, q}(\Omega, \mathbb{R}^d)} := \left( \int_{\Omega} \left( \int_{\Omega} \frac{|u(x) - u(y)|^q}{|x - y|^{N+1}} \, dy \right)^{\frac{1}{q}} dx \right)^{\frac{1}{q}}. \tag{1.6}
\]

Our first main result is

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^N \) be an open set and let \( u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d) \) be such that \( \| Du \| (\partial \Omega) = 0 \). For \( \eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R}) \), every \( x \in \mathbb{R}^N \) and every \( \varepsilon > 0 \) define

\[
u_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta \left( \frac{y - x}{\varepsilon} \right) u(y) \, dy = (\eta_\varepsilon \ast u)(x). \tag{1.7}
\]

Then, for any \( q > 1 \) we have

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\ln \varepsilon} \| \nu_\varepsilon \|^q_{W^{1/q, q}(\Omega, \mathbb{R}^d)} = 2 \int_{\mathbb{R}^N} \eta(z) \, dz \left( \int_{\mathbb{R}^{N-1}} \frac{dv}{(\sqrt{1 + |v|^2})^{N+1}} \right) \int_{J_u \cap \Omega} |u^+(x) - u^-(x)|^q \, d\mathcal{H}^{N-1}(x). \tag{1.8}
\]
Theorem 1.2 enables us to prove an upper bound, in the limit $\varepsilon \to 0^+$, for the following singular perturbation functionals with differential constraints:

(i) $$E_\varepsilon^{(1)}(v) := \begin{cases} \frac{1}{|\ln \varepsilon|} \|v\|_{W^{1/q,q}(\Omega,\mathbb{R}^d)}^q + \frac{1}{\varepsilon} \int_\Omega W(v,x) \, dx & \text{if } A \cdot \nabla v = 0 \varepsilon \to 0^+ \\ +\infty & \text{otherwise,} \end{cases}$$ for $v : \Omega \to \mathbb{R}^d$; 

(ii) $$E_\varepsilon^{(2)}(v) := \begin{cases} \frac{1}{|\ln \varepsilon|} \left( \|v\|_{W^{1/q,q}(\mathbb{R}^N,\mathbb{R}^d)}^q - \|v\|_{W^{1/q,q}(\mathbb{R}^N \setminus \Omega,\mathbb{R}^d)}^q \right) + \frac{1}{\varepsilon} \int_\Omega W(v,x) \, dx & \text{if } A \cdot \nabla v = 0 \varepsilon \to 0^+ \\ +\infty & \text{otherwise,} \end{cases}$$ for $v : \mathbb{R}^N \to \mathbb{R}^d$.

In both cases $A : \mathbb{R}^{d \times N} \to \mathbb{R}^l$ is a linear operator (possibly trivial). The most important particular cases are the following:

(a) $A \equiv 0$ (i.e., without any prescribed differential constraint),

(b) $d = N$, $l = N^2$ and $A \cdot \nabla v \equiv \text{curl } v := \{\partial_k v_j - \partial_j v_k\}_{1 \leq k,j \leq N}$,

(c) $l = d$ and $A \cdot \nabla v \equiv \text{div } v$.

The $\Gamma$-limit of the functional (1.9) in the $L^p$-topology when $A \equiv 0$, $q = 2$, $N = 1$ and $W$ is a double-well potential was found by Alberti, Bouchitté and Seppecher [1]. The result was generalized to any dimension $N \geq 1$, for the functional (1.10), by Savin and Valdinoci [15].

Note that the functional (1.9) resembles the energy functional in the following singular perturbation problem:

$$\hat{E}_\varepsilon(v) := \begin{cases} \varepsilon^{q-1} \|v\|_{W^{1/q,q}(\Omega,\mathbb{R}^d)}^q + \frac{1}{\varepsilon} \int_\Omega W(v,x) \, dx & \text{if } A \cdot \nabla v = 0 \varepsilon \to 0^+ \\ +\infty & \text{otherwise,} \end{cases}$$

that attracted a lot of attention by many authors, starting from Modica and Mortola [10], Modica [9], Sternberg [16] and others, who studied the basic special case of (1.11) with $A \equiv 0$, $q = 2$ and $W$ being a double-well potential. The $\Gamma$ limit of (1.11) with $A \equiv 0$, $q = 2$ and a general $W \in C^0$ that does not depend on $x$, was found by Ambrosio in [2]. As an example with a nontrivial differential constraint we mention the Aviles-Giga functional, that appear in various applications. It is defined for scalar functions $\psi$ by

$$\tilde{E}_\varepsilon(\psi) := \int_\Omega \left\{ \varepsilon |\nabla^2 \psi|^2 + \frac{1}{\varepsilon} \left( 1 - |\nabla \psi|^2 \right)^2 \right\} \, dx$$

(see [3, 5, 6]),

and the objective is to study the $\Gamma$-limit, as $\varepsilon \to 0^+$. This can be seen as a special case of (1.11) if we set $v := \nabla \psi$ and let $A \cdot \nabla v \equiv \text{curl } v$, $q = 2$ and $W(v,x) = (1 - |v|^2)^2$.

Our second result provides an upper bound for the energies (1.9)-(1.10):
**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^N$ be an open set and let $W : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}$ be a Borel measurable nonnegative function, continuous and continuously differentiable w.r.t. the first argument, such that $W(0, \cdot) \in L^1(\Omega, \mathbb{R})$. Assume further that for every $D > 0$ there exists $C := C_D > 0$ such that

$$
|\nabla_b W(b, x)| \leq C_D \quad \forall x \in \mathbb{R}^N, \ \forall b \in B_D(0).
$$

(1.13)

Let $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d)$ be such that $W(u(x), x) = 0$ a.e. in $\Omega$, $\|Du\|_\partial = 0$, and $A \cdot Du = 0$ in $\mathbb{R}^N$, where $A : \mathbb{R}^{d \times N} \to \mathbb{R}^d$ is a prescribed linear operator (possibly trivial). Then, for any $q > 1$ there exists a sequence of functions $\{\psi_\varepsilon\} \subset C^\infty(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^d)$ such that $A \cdot Du_\varepsilon = 0$ in $\mathbb{R}^N$, $\psi_\varepsilon(x) \to u(x)$ strongly in $L^p(\mathbb{R}^N, \mathbb{R}^d)$ for every $p \geq 1$, and

$$
\limsup_{\varepsilon \to 0^+} \left( \frac{1}{|\ln \varepsilon|} \left( \|\psi_\varepsilon\|_{W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^d)} - \|\psi_\varepsilon\|_{W^{1,q}(\mathbb{R}^N, \mathbb{R}^d)} \right) + \frac{1}{\varepsilon} \int_{\Omega} W(\psi_\varepsilon(x), x) \, dx \right) =
$$

$$
\limsup_{\varepsilon \to 0^+} \left( \frac{1}{|\ln \varepsilon|} \|\psi_\varepsilon\|_{W^{1,q}(\mathbb{R}^N, \mathbb{R}^d)} + \frac{1}{\varepsilon} \int_{\Omega} W(\psi_\varepsilon(x), x) \, dx \right) =
$$

$$
\left( \int_{\mathbb{R}^{N-1}} \frac{2}{\sqrt{1 + |v|^2}^{N+1}} \, dv \right) \int_{J_u \cap \Omega} |u^+(y) - u^-(y)|^q \, d\mathcal{H}^{N-1}(y).
$$

(1.14)

Moreover, in the case $A \equiv 0$ we can choose $\psi_\varepsilon$ to satisfy also

$$
\int_{\Omega} \psi_\varepsilon(x) \, dx = \int_{\Omega} u(x) \, dx \quad \forall \varepsilon > 0.
$$

(1.15)

Unfortunately, the upper bound found in Theorem 1.3 is not sharp in the most general case with a nontrivial prescribed differential constraint. For example, in the particular case of (1.9) with $N = 2$, $A \cdot \nabla v \equiv \text{curl} v$, $q > 3$ and $W(v, x) = (1 - |v|^2)^2$, the functional on the R.H.S. of (1.14) is not lower semicontinuous, hence cannot be the $\Gamma$-limit (see [3]). However, we still hope that the result of the above theorem could provide the sharp upper bound in some cases with $A = 0$. Indeed, the $\Gamma$-limit, computed in [1] for the special case of (1.9) with $A \equiv 0$, $q = 2$, $N = 1$ and $W$ being a double well potential, coincides with the upper bound found in Theorem 1.3. Moreover, since the functional in (1.10) is superior to the functional in (1.9), the $\Gamma$-limit, found in [15] (see also [12]) for the energy (1.10) in any dimension $N \geq 1$ with $A \equiv 0$, $q = 2$ and $W$ being a double well potential, coincides again with our upper bound.

The paper is organized as follows. In section 2 we prove our two main results. For the convenience of the reader, in the Appendix we recall some known results on $BV$ functions, needed for the proofs.

# 2 Proof of the main results

**Proposition 2.1.** Let $q > 1$, $\Omega \subset \mathbb{R}^N$ be an open set and $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d)$ be such that $\|Du\|_\partial = 0$. Let $\eta \in C_c(\mathbb{R}^N, \mathbb{R})$ and for every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ define

$$
u_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta \left( \frac{y - x}{\varepsilon} \right) u(y) \, dy = (\eta * u)(x).
$$

(2.1)
Then,

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\ln \varepsilon} \left\| u_\varepsilon \right\|_{W^{1, q/(q-1)}(\Omega, \mathbb{R}^d)}^q = 2 \left( \int_{\mathbb{R}^N} |\eta(z)|^q dz \right) \left( \int_{\mathbb{R}^{N-1}} \frac{1}{\sqrt{1 + |v|^2}} d\mathcal{H}^{N-1} \right) \int_{J_0} \left| u^+(x) - u^-(x) \right|^q d\mathcal{H}^{N-1}(x). \quad (2.2)
\]

**Proof.** We start with some notations. For every \( \nu \in S^{N-1} \) and \( x \in \mathbb{R}^N \) set

\[
H_+(x, \nu) = \{ \xi \in \mathbb{R}^N : (\xi - x) \cdot \nu > 0 \},
\]

\[
H_-(x, \nu) = \{ \xi \in \mathbb{R}^N : (\xi - x) \cdot \nu < 0 \}
\]

and

\[
H_0(\nu) = \{ \xi \in \mathbb{R}^N : \xi \cdot \nu = 0 \}.
\]

Let \( R > 0 \) be such that \( \text{supp} \eta \subset B_R(0) \). For every \( x \in \mathbb{R}^N \) and every \( \varepsilon > 0 \) we rewrite \((2.1)\) as:

\[
u_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta \left( \frac{y - x}{\varepsilon} \right) u(y) dy = \int_{\mathbb{R}^N} \eta(z) u(x + \varepsilon z) dz = \int_{B_R(0)} \eta(z) u(x + \varepsilon z) dz. \quad (2.6)
\]

By \((2.6)\) we have

\[
\frac{d}{d\varepsilon} u_\varepsilon(x) := -\frac{N}{\varepsilon^{N+1}} \int_{\mathbb{R}^N} \eta \left( \frac{y - x}{\varepsilon} \right) u(y) dy - \int_{\mathbb{R}^N} \frac{y - x}{\varepsilon^2} \cdot \nabla \eta \left( \frac{y - x}{\varepsilon} \right) u(y) dy = -\frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \text{div} \left\{ \eta \left( \frac{y - x}{\varepsilon} \right) \frac{y - x}{\varepsilon} \right\} u(y) dy = \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta \left( \frac{y - x}{\varepsilon} \right) \frac{y - x}{\varepsilon} \cdot d[Du(y)]. \quad (2.7)
\]

Moreover, by \((1.6)\) we have

\[
\left\| u_\varepsilon \right\|_{W^{1, q/(q-1)}(\Omega, \mathbb{R}^d)}^q = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{\left| u_\varepsilon(x) - u_\varepsilon(y) \right|^q}{|x - y|^{N+1}} \chi_\Omega(y) dy \right) \chi_\Omega(x) dx = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{\left| u_\varepsilon(x + z) - u_\varepsilon(x) \right|^q}{|z|^{N+1}} \chi_\Omega(x + z) \chi_\Omega(x) dz \right) dx, \quad (2.8)
\]

where

\[
\chi_\Omega(x) := \begin{cases} 1 & \forall x \in \Omega \\ 0 & \forall x \in \mathbb{R}^N \setminus \Omega \end{cases}. \quad (2.9)
\]

Thus,

\[
\frac{1}{-\ln \varepsilon} \left\| u_\varepsilon \right\|_{W^{1, q/(q-1)}}^q = -\frac{1}{-\ln \varepsilon} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{\left| u_\varepsilon(x + z) - u_\varepsilon(x) \right|^q}{|z|^{N+1}} \chi_\Omega(x + z) \chi_\Omega(x) dz \right) dx. \quad (2.10)
\]

Since \(-\ln \varepsilon \to +\infty \) as \( \varepsilon \to 0^+ \), applying L'Hôpital's rule to the expression in \((2.10)\) yields

\[
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \left\| u_\varepsilon \right\|_{W^{1, q/(q-1)}}^q = -\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{\varepsilon}{|z|^{N+1}} \left( \frac{d}{d\varepsilon} \left( u_\varepsilon(x + z) - u_\varepsilon(x) \right) \right) \cdot \nabla F_q \left( u_\varepsilon(x + z) - u_\varepsilon(x) \right) \chi_\Omega(x + z) \chi_\Omega(x) dz \right) dx,
\]

\[
(2.11)
\]
where $F_q \in C^1(\mathbb{R}^d, \mathbb{R})$ is defined by

$$F_q(h) := |h|^q \quad \forall h \in \mathbb{R}^d.$$  \hspace{1cm} (2.12)

Thus, by (2.11), (2.6) and (2.7) we get

$$
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \|u_\varepsilon\|^q_{L^1(\mathbb{R}^d)} = \\
- \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varepsilon}{|z|^{N+1}} \left\{ \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \left( \eta \left( \frac{y - (x + z)}{\varepsilon} \right) \frac{y - (x + z)}{\varepsilon} - \eta \left( \frac{y - x}{\varepsilon} \right) \frac{y - x}{\varepsilon} \right) \cdot [Du(y)] \right\} \times \\
\times \nabla F_q \left( \int_{\mathbb{R}^N} \eta(\xi) \left( u(x + z + \varepsilon \xi) - u(x + \varepsilon \xi) \right) d\xi \right) \chi_{\Omega}(x + z) \chi_{\Omega}(x) dz dx = \\
- \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varepsilon}{|z|^{N+1}} \left( \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \left( \eta \left( \frac{y - (x + z)}{\varepsilon} \right) \frac{y - (x + z)}{\varepsilon} - \eta \left( \frac{y - x}{\varepsilon} \right) \frac{y - x}{\varepsilon} \right) \times \\
\times \nabla F_q \left( \int_{\mathbb{R}^N} \eta(\xi) \left( u(x + z + \varepsilon \xi) - u(x + \varepsilon \xi) \right) d\xi \right) \chi_{\Omega}(x + z) \chi_{\Omega}(x) dz dx \cdot d[Du(y)] \right) = \\
- \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|z|^{N+1}} \left( \eta(x - z)(x - z) - \eta(x)x \right) \times \\
\times \nabla F_q \left( \int_{\mathbb{R}^N} \eta(\xi - z) - \eta(\xi) \left( u(y + \varepsilon x + \varepsilon \xi - \varepsilon x) - u(y + \varepsilon x + \varepsilon \xi) \right) d\xi \right) \chi_{\Omega}(y - \varepsilon x + \varepsilon z) \chi_{\Omega}(y - \varepsilon x) dz dx \cdot d[Du(y)] \right) \right).  \hspace{1cm} (2.13)
$$

Changing variable, $z/\varepsilon \rightarrow z$, in the integration on the R.H.S. of (2.13) gives

$$
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \|u_\varepsilon\|^q_{L^1(\mathbb{R}^d)} = \\
- \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} \frac{1}{\varepsilon^N} \left( \eta \left( \frac{y - x}{\varepsilon} - z \right) \left( \frac{y - x}{\varepsilon} - z \right) - \eta \left( \frac{y - x}{\varepsilon} \right) \frac{y - x}{\varepsilon} \right) \times \\
\times \nabla F_q \left( \int_{\mathbb{R}^N} \eta(\xi) \left( u(x + z + \varepsilon \xi) - u(x + \varepsilon \xi) \right) d\xi \right) \chi_{\Omega}(x + \varepsilon z) \chi_{\Omega}(x) dz dx \cdot d[Du(y)] = \\
- \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|z|^{N+1}} \left( \eta(x - z)(x - z) - \eta(x)x \right) \times \\
\times \nabla F_q \left( \int_{\mathbb{R}^N} \eta(\xi - z) - \eta(\xi) \left( u(y + \varepsilon x + \varepsilon \xi - \varepsilon x) - u(y + \varepsilon x + \varepsilon \xi) \right) d\xi \right) \chi_{\Omega}(y - \varepsilon x + \varepsilon z) \chi_{\Omega}(y - \varepsilon x) dz dx \cdot d[Du(y)] \right).  \hspace{1cm} (2.14)
$$

Therefore,

$$
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \|u_\varepsilon\|^q_{L^1(\mathbb{R}^d)} = \\
- \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|z|^{N+1}} \left( \eta(x - z)(x - z) - \eta(x)x \right) \times \\
\times \nabla F_q \left( \int_{\mathbb{R}^N} \left( \eta(\xi - z) - \eta(\xi) \right) \left( u(y + \varepsilon x + \varepsilon \xi) d\xi \right) \chi_{\Omega}(y - \varepsilon x + \varepsilon z) \chi_{\Omega}(y - \varepsilon x) dz dx \cdot d[Du(y)] \right) = \\
- \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|z|^{N+1}} \left( \eta(x - z)(x - z) - \eta(x)x \right) \times \\
\times \nabla F_q \left( \int_{\mathbb{R}^N} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) u(y + \varepsilon x) d\xi \right) \chi_{\Omega}(y - \varepsilon x + \varepsilon z) \chi_{\Omega}(y - \varepsilon x) dz dx \cdot d[Du(y)].  \hspace{1cm} (2.15)
$$
On the other hand, by (3.1) in the Appendix, for every \( x, z \in \mathbb{R}^N \) and \( \mathcal{H}^{N-1}\) a.e. \( y \in \mathbb{R}^N \) we have

\[
\lim_{\varepsilon \to 0^+} \left\{ \int_{\mathbb{R}^N} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) u(y + \varepsilon \xi) \, d\xi \right\} = 
\]

\[
u^+(y) \int_{H_+(0, \nu(y))} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi + \nu^-(y) \int_{H_-(0, \nu(y))} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi.
\]

(2.16)

with \( H_+(x, \nu) \) as defined in (2.3) and (2.4). Thus, since \( \|D\nu\|_{\partial \Omega} = 0 \), by (2.16) and the Dominated Convergence Theorem we obtain:

\[
\lim_{\varepsilon \to 0^+} \frac{1}{1 - \ln \varepsilon} \left\| u_\varepsilon \right\|_{W^{1/q, q}}^q = 
\]

\[
- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|z|^{N+1}} \left( \eta(x - z) (x - z) - \eta(x) x \right) \nabla F_q \left[ u^+(y) \right] \int_{H_+(0, \nu(y))} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi 
\]

\[
+ u^-(y) \int_{H_-(0, \nu(y))} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi \chi_\Omega^2(y) \, dzdx \cdot d[Du(y)] = 
\]

\[
- \int_{\Omega} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|z|^{N+1}} \left( \eta(x - z) (x - z) - \eta(x) x \right) \nabla F_q \left[ u^+(y) \right] \int_{H_+(0, \nu(y))} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi 
\]

\[
+ u^-(y) \int_{H_-(0, \nu(y))} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi \right] \, dzdx \cdot d[Du(y)].
\]

(2.17)

It follows that

\[
\lim_{\varepsilon \to 0^+} \frac{1}{1 - \ln \varepsilon} \left\| u_\varepsilon \right\|_{W^{1/q, q}}^q = - \int_{\Omega} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|z|^{N+1}} \left( \eta(x - z) (x - z) - \eta(x) x \right) \times 
\]

\[
\nabla F_q \left( u^+(y) - u^-(y) \right) \int_{H_+(0, \nu(y))} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi 
\]

\[
+ u^-(y) \int_{\mathbb{R}^N} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi \right] \, dzdx \cdot d[Du(y)] 
\]

\[
- \int_{\Omega} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|z|^{N+1}} \left( \eta(x - z) (x - z) - \eta(x) x \right) \times 
\]

\[
\nabla F_q \left( u^+(y) - u^-(y) \right) \int_{H_+(0, \nu(y))} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi 
\]

\[
+ u^-(y) \int_{\mathbb{R}^N} \left( \eta(\xi + x - z) - \eta(\xi + x) \right) \, d\xi \right] \, dzdx \cdot d[Du(y)],
\]

(2.18)

where we used in the last step the fact that \( \int_{\mathbb{R}^N} \eta(\xi + x - z) \, d\xi = \int_{\mathbb{R}^N} \eta(\xi + x) \, d\xi \). Next, by
where

\[ (2.18) \text{ and } (2.12) \text{ we infer that} \]

\[
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \| u_\varepsilon \|_{W^{1/q,q}}^q = -\int_\Omega \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1 \left( \eta(x - z)(x - z) - \eta(x) x \right) \times
\]

\[
\nabla F_q \left( u^+(y) - u^-(y) \right) \left( \int_{H_+(x,z,\nu(y))} \eta(\xi)d\xi - \int_{H_+(x,\nu(y))} \eta(\xi)d\xi \right) dzdx \cdot d[Du(y)]
\]

\[
= \int_{J_\nu \cap \Omega} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 1 \left( \eta(x) x \cdot \nu(y) - \eta(x - z)(x - z) \cdot \nu(y) \right) \times
\]

\[
\frac{dG_q}{d\rho} \left( \int_{(x-z,\nu(y))} H_0(\nu(y)) \eta(t\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi)dt \right) dx dz |u^+(y) - u^-(y)|^q d\mathcal{H}^{N-1}(y), \tag{2.19}
\]

where \( G_q(\rho) \in C^1(\mathbb{R}, \mathbb{R}) \) is defined by

\[
G_q(\rho) := |\rho|^q \quad \forall \rho \in \mathbb{R}, \tag{2.20}
\]

and \( H_0(\nu) \) is defined in \( (2.15) \). Therefore,

\[
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \| u_\varepsilon \|_{W^{1/q,q}}^q =
\]

\[
\int_{J_\nu \cap \Omega} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{H_0(\nu(y))} 1 \left( \eta(s\nu(y) + \zeta) s - \eta \left( (s - z \cdot \nu(y)) \nu(y) + \zeta \right) (s - z \cdot \nu(y)) \right) \times
\]

\[
\frac{dG_q}{d\rho} \left( \int_{s-z,\nu(y)} H_0(\nu(y)) \eta(t\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi)dt \right) d\mathcal{H}^{N-1}(\xi)ds dz |u^+(y) - u^-(y)|^q d\mathcal{H}^{N-1}(y)
\]

\[
= \int_{J_\nu \cap \Omega} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \frac{1}{\sqrt{\tau^2 + |w|^2}} \right)^{N+1} \right. \times
\]

\[
\left. \left( \int_{H_0(\nu(y))} \eta(s\nu(y) + \zeta) s - \eta \left( (s - \tau \cdot \nu(y)) \nu(y) + \zeta \right) (s - \tau) \right) d\mathcal{H}^{N-1}(\xi) \right) \times
\]

\[
\frac{dG_q}{d\rho} \left( \int_{s-\tau,\nu(y)} H_0(\nu(y)) \eta(t\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi)dt \right) d\mathcal{H}^{N-1}(\xi) ds dw |u^+(y) - u^-(y)|^q d\mathcal{H}^{N-1}(y). \tag{2.21}
\]

Introducing the notation

\[
\Lambda(y, a, b) = \int_a^b \int_{H_0(\nu(y))} \eta(t\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi) dt
\]

allows us to rewrite \( (2.21) \) as

\[
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \| u_\varepsilon \|_{W^{1/q,q}}^q =
\]

\[
\int_{J_\nu \cap \Omega} \left\{ \int_{\mathbb{R}^N} \int_{\mathbb{R}} \left( \frac{1}{\tau^2} \right) \left( \frac{1}{\tau} \right)^{N-1} \left( \frac{1}{\sqrt{1 + |w|^2}} \right)^{N+1} \times
\]

\[
\left( \int_{H_0(\nu(y))} \left( \eta \left( s\nu(y) + \zeta \right) s - \eta \left( (s - \tau) \nu(y) + \zeta \right) (s - \tau) \right) d\mathcal{H}^{N-1}(\xi) \right) \times
\]

\[
\times \frac{dG_q}{d\rho} \left( \Lambda(y, s - \tau, s) \right) d\mathcal{H}^{N-1}(\xi) ds dw \right\} |u^+(y) - u^-(y)|^q d\mathcal{H}^{N-1}(y). \tag{2.23}
\]

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The change of variables $w/|\tau| \to v$ in the R.H.S. of (2.23) gives

$$\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \|u_\varepsilon\|_{W^{1,q}}^q = D_N \int_{J_0 \cap \Omega} \left( \int_{\mathbb{R}} \int_{J_{H_0(\nu(y))}} \frac{1}{\tau^2} \left( \int_{H_0(\nu(y))} \left( \eta(s\nu(y) + \zeta) s - \eta((s - \tau)\nu(y) + \zeta)(s - \tau) \right) d\mathcal{H}^{N-1}(\zeta) \right) \times \frac{dG_q}{d\rho}(\Lambda(y, s - \tau, s)) d\tau ds \right) \left|u^+(y) - u^-(y)\right|^q d\mathcal{H}^{N-1}(y), \tag{2.24}$$

where $D_N$ is the dimensional constant given by

$$D_N := \int_{\mathbb{R}^{N-1}} \frac{1}{(\sqrt{1 + |v|^2})^{N+1}} dv. \tag{2.25}$$

Then we rewrite (2.24) as

$$\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \|u_\varepsilon\|_{W^{1,q}}^q = \lim_{M \to +\infty} \left( D_N \int_{J_0 \cap \Omega} \left( \int_{\mathbb{R}} \int_{-M}^{M} \frac{1}{\tau^2} \left( \int_{H_0(\nu(y))} \eta((s - \tau)\nu(y) + \zeta) d\mathcal{H}^{N-1}(\zeta) \right) \times \frac{dG_q}{d\rho}(\Lambda(y, s - \tau, s)) d\tau ds \right) \left|u^+(y) - u^-(y)\right|^q d\mathcal{H}^{N-1}(y) \right) \times \left( \int_{\mathbb{R}} \int_{-M}^{M} \frac{1}{\tau^2} \left( \int_{H_0(\nu(y))} \left( \eta(s\nu(y) + \zeta) s - \eta((s - \tau)\nu(y) + \zeta)(s - \tau) \right) d\mathcal{H}^{N-1}(\zeta) \right) \times \frac{dG_q}{d\rho}(\Lambda(y, s - \tau, s)) d\tau ds \right) \left|u^+(y) - u^-(y)\right|^q d\mathcal{H}^{N-1}(y) \right). \tag{2.26}$$

Integration by parts of (2.26) and using (2.20) give

$$\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \|u_\varepsilon\|_{W^{1,q}}^q = - \lim_{M \to +\infty} D_N \int_{J_0 \cap \Omega} \left|u^+(y) - u^-(y)\right|^q \left( \int_{\mathbb{R}} \int_{-M}^{M} \frac{1}{\tau^2} \left|\Lambda(y, s - \tau, s)\right|^q d\tau ds \right) d\mathcal{H}^{N-1}(y) + \lim_{M \to +\infty} D_N \int_{J_0 \cap \Omega} \left( \int_{\mathbb{R}} \left|\Lambda(y, s - M, s)\right|^q ds + \int_{\mathbb{R}} \left|\Lambda(y, s, s + M)\right|^q ds \right) \left|u^+(y) - u^-(y)\right|^q d\mathcal{H}^{N-1}(y) = \lim_{M \to +\infty} \frac{D_N}{M} \int_{J_0 \cap \Omega} \left( \int_{\mathbb{R}} \left|\Lambda(y, s - M, s)\right|^q ds + \int_{\mathbb{R}} \left|\Lambda(y, s, s + M)\right|^q ds \right) \left|u^+(y) - u^-(y)\right|^q d\mathcal{H}^{N-1}(y). \tag{2.27}$$
Corollary 2.1. Therefore, applying L'Hôpital's rule in (2.27), using (2.20), we deduce that

\[
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \| u_\varepsilon \|^q_{W^{1/q,q}} = 
\lim_{M \to +\infty} D_N \int_{J_a \cap \Omega} \left( \int_{\mathbb{R}} \frac{dG_q}{d\rho} \left( \Lambda(y, s - M, s) \right) \left( \int_{H_0(\nu(y))} \eta((s - M)\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi) \right) ds \right.
\]

\[+ \left. \int_{\mathbb{R}} \frac{dG_q}{d\rho} \left( \Lambda(y, s + M) \right) \left( \int_{H_0(\nu(y))} \eta((s + M)\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi) \right) ds \right) \times |u^+(y) - u^-(y)|^q d\mathcal{H}^{N-1}(y). \tag{2.28} \]

Changing variables of integration we rewrite (2.28) as

\[
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \| u_\varepsilon \|^q_{W^{1/q,q}} = 
\lim_{M \to +\infty} D_N \int_{J_a \cap \Omega} \left( \int_{\mathbb{R}} \frac{dG_q}{d\rho} \left( \Lambda(y, s + M) \right) \left( \int_{H_0(\nu(y))} \eta(s\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi) \right) ds \right.
\]

\[+ \left. \int_{\mathbb{R}} \frac{dG_q}{d\rho} \left( \Lambda(y, s - M, s) \right) \left( \int_{H_0(\nu(y))} \eta(s\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi) \right) ds \right) \times |u^+(y) - u^-(y)|^q d\mathcal{H}^{N-1}(y) \]

\[= D_N \int_{J_a \cap \Omega} \left( \int_{\mathbb{R}} \frac{dG_q}{d\rho} \left( \Lambda(y, s, \infty) \right) \left( \int_{H_0(\nu(y))} \eta(s\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi) \right) ds \right.
\]

\[+ \left. \int_{\mathbb{R}} \frac{dG_q}{d\rho} \left( \Lambda(y, -\infty, s) \right) \left( \int_{H_0(\nu(y))} \eta(s\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi) \right) ds \right) \times |u^+(y) - u^-(y)|^q d\mathcal{H}^{N-1}(y). \tag{2.29} \]

Applying Newton-Leibniz formula in (2.29) and using (2.20) we obtain that

\[
\lim_{\varepsilon \to 0^+} \frac{1}{-\ln \varepsilon} \| u_\varepsilon \|^q_{W^{1/q,q}} = 
2D_N \int_{J_a \cap \Omega} \left| \int_{-\infty}^{\infty} \int_{H_0(\nu(y))} \eta(t\nu(y) + \xi) d\mathcal{H}^{N-1}(\xi) dt \right|^q |u^+(y) - u^-(y)|^q d\mathcal{H}^{N-1}(y)
\]

\[= 2D_N \left| \int_{\mathbb{R}^N} \eta(z) dz \int_{J_a \cap \Omega} |u^+(y) - u^-(y)|^q d\mathcal{H}^{N-1}(y) , \tag{2.30} \right. \]

and (2.22) follows.

\[\square\]

**Corollary 2.1.** Let \( q > 1 \) and let \( \Omega \subset \mathbb{R}^N \) be an open set. Assume \( W : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R} \) is a Borel measurable function such that, \( W(0, \cdot) \in L^1(\Omega, \mathbb{R}) \) and for every \( D > 0 \) there exists \( C := C_D > 0 \) such that

\[
|W(b, x) - W(a, x)| \leq C_D |b - a| \quad \forall x \in \mathbb{R}^N, \; \forall a, b \in B_D(0). \tag{2.31} \]

Let \( u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d) \) be such that \( \|Du\|((\partial \Omega)) = 0 \) and \( W(u(x), x) = 0 \) a.e. in \( \Omega \). Let \( \eta \in C_c^\infty(\mathbb{R}^N, \mathbb{R}) \) be such that \( \int_{\mathbb{R}^N} \eta(z) dz = 1 \) and \( \text{supp} \; \eta \subset B_R(0) \). For every \( \rho > 0 \) set

\[
\eta_\rho(z) := \frac{1}{\rho^N} \eta \left( \frac{z}{\rho} \right) \quad \forall z \in \mathbb{R}^N. \tag{2.32} \]
Finally, for every $x \in \mathbb{R}^N$ and every $\varepsilon > 0$ define
\[
u_{\rho, \varepsilon}(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta \left( \frac{y - x}{\varepsilon} \right) u(y) dy = \int_{\mathbb{R}^N} \eta(z) u(x + \varepsilon \rho z) dz = \int_{B_R(0)} \eta(z) u(x + \varepsilon \rho z) dz. \tag{2.33}
\]
Then,
\[
\lim_{\rho \to 0^+} \left\{ \limsup_{\varepsilon \to 0^+} \left( \frac{1}{-\ln(\varepsilon)} \left( \|u_{\rho, \varepsilon}\|_{W^{1,q}(\Omega)}^q - \|u_{\rho, \varepsilon}\|_{W^{1,q}(\Omega \setminus \Omega)}^q \right) \right) \right\}
= \lim_{\rho \to 0^+} \left\{ \limsup_{\varepsilon \to 0^+} \left( \frac{1}{-\ln(\varepsilon)} \left( \|u_{\rho, \varepsilon}\|_{W^{1,q}(\Omega)}^q \right) \right) \right\}
= \left( \int_{\mathbb{R}^{N-1}} \frac{2}{(1 + |v|^2)^{N+1}} dv \right) \int_{J_u \cap \Omega} |u^+(y) - u^-(y)|^{q} d\mathcal{H}^{N-1}(y). \tag{2.34}
\]
\[\text{Proof.} \text{ Since } \int_{\mathbb{R}^N} \eta(z) dz = 1, \text{ applying Proposition 2.1 first for } \mathbb{R}^N, \text{ then for } \mathbb{R}^N \setminus \overline{\Omega}, \text{ and finally for } \Omega, \text{ yields, for every } \rho > 0,
\]
\[
\lim_{\varepsilon \to 0^+} \left( \frac{1}{-\ln(\varepsilon)} \left( \|u_{\rho, \varepsilon}\|_{W^{1,q}(\Omega)}^q - \|u_{\rho, \varepsilon}\|_{W^{1,q}(\Omega \setminus \Omega)}^q \right) \right) \right\}
= 2D_N \int_{J_u} |u^+(y) - u^-(y)|^{q} d\mathcal{H}^{N-1}(y) - \int_{J_u \cap \Omega} |u^+(y) - u^-(y)|^{q} d\mathcal{H}^{N-1}(y)
= 2D_N \int_{J_u \cap \Omega} |u^+(y) - u^-(y)|^{q} d\mathcal{H}^{N-1}(y) = \lim_{\varepsilon \to 0^+} \left( \frac{1}{-\ln(\varepsilon)} \left( \|u_{\rho, \varepsilon}\|_{W^{1,q}(\Omega)}^q \right) \right), \tag{2.35}
\]
where $D_N$ is the constant defined in (2.25). On the other hand, since $W(u(x), x) = 0$ a.e. in $\Omega$ and $u \in L^\infty$, by (2.31) we get that
\[
\frac{1}{\varepsilon} \int_{\Omega} W(u_{\rho, \varepsilon}(x), x) dx = \frac{1}{\varepsilon} \int_{\Omega} (W(u_{\rho, \varepsilon}(x), x) - W(u(x), x)) dx \leq C \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |u_{\rho, \varepsilon}(x) - u(x)| dx
\leq C \int_{B_R(0)} |\eta(z)| \left( \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |u(x + \varepsilon \rho z) - u(x)| dx \right) dz
= C \rho \int_{B_R(0)} |z| |\eta(z)| \left( \int_{\mathbb{R}^N} \frac{1}{\varepsilon \rho |z|} |u(x + \varepsilon \rho z) - u(x)| dx \right) dz, \tag{2.36}
\]
for some constant $C > 0$, independent of $\varepsilon$ and $\rho$. Thus, taking into account the following well known uniform bound from the theory of $BV$ functions,
\[
\int_{\mathbb{R}^N} \frac{1}{\rho \varepsilon |z|} |u(x + \rho z) - u(x)| dx \leq C_0 \|Du\|_1(\mathbb{R}^N) \quad \forall z \in \mathbb{R}^N, \forall \rho, \varepsilon > 0, \tag{2.37}
\]
we obtain that
\[
\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\Omega} W(u_{\rho, \varepsilon}(x), x) dx \leq CC_0 \|Du\|_1(\mathbb{R}^N) \rho \int_{B_R(0)} |z| |\eta(z)| dz = O(\rho). \tag{2.38}
\]
By (2.38) and (2.35) we finally derive (2.34). \qed
Proof of Theorem 1.3. Let \( \eta, \eta_\rho \) and \( u_{\rho, \varepsilon} \) be defined as in Corollary 2.1. Then \( u_{\rho, \varepsilon} \in C^\infty(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^N, \mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^d) \) and by Corollary 2.1 we have

\[
\lim_{\rho \to 0^+} \left\{ \limsup_{\varepsilon \to 0^+} \left( \frac{1}{-\ln(\varepsilon)} \left( \| u_{\rho, \varepsilon} \|_{W^{1,q}(\mathbb{R}^N, \mathbb{R}^d)}^q - \| u_{\rho, \varepsilon} \|_{W^{1,q}(\mathbb{R}^N, \mathbb{R}^d)}^q \right) + \frac{1}{\varepsilon} \int_{\Omega} W(u_{\rho, \varepsilon}(x), x) \, dx \right) \right\}
= \lim_{\rho \to 0^+} \left\{ \limsup_{\varepsilon \to 0^+} \left( \frac{1}{-\ln(\varepsilon)} \left( \| u_{\rho, \varepsilon} \|_{W^{1,q}(\Omega, \mathbb{R}^d)}^q \right) + \frac{1}{\varepsilon} \int_{\Omega} W(u_{\rho, \varepsilon}(x), x) \, dx \right) \right\}
= \left( \int_{\mathbb{R}^N} \frac{2}{(1 + |x|^2)^{N+1}} \, dv \right) \int_{\Omega \cap \Omega} |u^+(y) - u^-(y)|^q \, dH^{N-1}(y). \quad (2.39)
\]

Clearly, for every \( x \in \mathbb{R}^N \) we have \( A \cdot \nabla u_{\rho, \varepsilon}(x) = 0 \) and \( u_{\rho, \varepsilon}(x) \to u(x) \) strongly in \( L^p(\mathbb{R}^N, \mathbb{R}^d) \) as \( \varepsilon \to 0^+ \) for every fixed \( \rho \) and \( p \). Therefore, by the above and by (2.39) we can complete the proof of the first assertion of the theorem using a standard diagonal argument.

It remains to show the second assertion of the theorem, namely, that in the case \( A \equiv 0 \) we can construct \( \psi_\varepsilon \) satisfying the additional condition (1.15). Let \( \varphi \in C^\infty_c(\mathbb{R}^N, \mathbb{R}) \) be such that \( \int_{\Omega} \varphi(x) \, dx = 1 \). Define

\[
\tilde{u}_{\rho, \varepsilon}(x) := u_{\rho, \varepsilon}(x) - \varphi(x)c_{\varepsilon, \rho}, \quad (2.40)
\]

where

\[
c_{\varepsilon, \rho} := \int_{\Omega} u_{\rho, \varepsilon}(y) \, dy - \int_{\Omega} u(y) \, dy. \quad (2.41)
\]

In particular,

\[
\int_{\Omega} \tilde{u}_{\rho, \varepsilon}(x) \, dx = \int_{\Omega} u(x) \, dx, \quad (2.42)
\]

and \( \lim_{\varepsilon \to 0^+} c_{\varepsilon, \rho} = 0 \). On the other hand, since \( W(u(x), x) = 0 \) a.e. in \( \Omega \), \( W(b, x) \) is nonnegative and \( W(b, x) \) is differentiable with respect to the \( b \) variable, we have

\[
\nabla_b W(u(x), x) = 0 \quad \text{a.e. in } \Omega. \quad (2.43)
\]
Thus, since \( u \in L^\infty \), by (2.40) we get that

\[
\frac{1}{\varepsilon} \int_\Omega \left| \frac{1}{\varepsilon} \int_\Omega \nabla_b W\left( u_{\rho,\varepsilon}(x) - s \varphi(x) c_{\varepsilon,\rho}, x \right) \right| dx = \frac{c_{\varepsilon,\rho}}{\varepsilon} \int_0^1 \int_\Omega \nabla_b W\left( u_{\rho,\varepsilon}(x) - s \varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds
\]

\[
\leq C \left( \int_{\mathbb{R}^N} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^N} u_{\rho,\varepsilon}(x) - u(x) \right| dx \right) \left( \int_0^1 \int_\Omega \nabla_b W\left( u_{\rho,\varepsilon}(x) - s \varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds \right)
\]

\[
\leq C \left( \int_{\mathcal{B}_{R}(0)} |\eta(z)| \left( \int_{\mathbb{R}^N} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^N} u(x + \varepsilon \rho z) - u(x) \right| dx \right) dz \right) \times
\]

\[
\times \int_0^1 \int_\Omega \nabla_b W\left( u_{\rho,\varepsilon}(x) - s \varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds
\]

\[
= C \rho \left( \int_{\mathcal{B}_{R}(0)} |z| |\eta(z)| \left( \int_{\mathbb{R}^N} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^N} u(x + \varepsilon \rho z) - u(x) \right| dx \right) dz \right) \times
\]

\[
\times \int_0^1 \int_\Omega \nabla_b W\left( u_{\rho,\varepsilon}(x) - s \varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds
\]

\[
= C \rho \left( \int_{\mathcal{B}_{R}(0)} |z| |\eta(z)| \left( \int_{\mathbb{R}^N} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^N} u(x + \varepsilon \rho z) - u(x) \right| dx \right) dz \right) \times
\]

\[
\times \int_0^1 \int_\Omega \nabla_b W\left( u_{\rho,\varepsilon}(x) - s \varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds
\]

\[
= C \rho \left( \int_{\mathcal{B}_{R}(0)} |z| |\eta(z)| \left( \int_{\mathbb{R}^N} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^N} u(x + \varepsilon \rho z) - u(x) \right| dx \right) dz \right) \times
\]

\[
\times \int_0^1 \int_\Omega \nabla_b W\left( u_{\rho,\varepsilon}(x) - s \varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds = 0. \tag{2.44}
\]

On the other hand, taking into account (2.37) and using the Dominated Convergence Theorem and (2.43), we obtain that

\[
\limsup_{\varepsilon \to 0^+} \left( \int_{\mathcal{B}_{R}(0)} |z| |\eta(z)| \left( \int_{\mathbb{R}^N} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^N} u(x + \varepsilon \rho z) - u(x) \right| dx \right) dz \right) \times
\]

\[
\times \int_0^1 \int_\Omega \nabla_b W\left( u_{\rho,\varepsilon}(x) - s \varphi(x) c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds \leq C_0 \left( \| Du \|_{(\mathbb{R}^n)} \right) \left( \int_{\mathcal{B}_{R}(0)} |z| |\eta(z)| dz \right) \times
\]

\[
\times \int_0^1 \int_\Omega \nabla_b W\left( \lim_{\varepsilon \to 0^+} u_{\rho,\varepsilon}(x) - s \varphi(x) \lim_{\varepsilon \to 0^+} c_{\varepsilon,\rho}, x \right) \varphi(x) dx ds
\]

\[
= C_0 \left( \| Du \|_{(\mathbb{R}^n)} \right) \left( \int_{\mathcal{B}_{R}(0)} |z| |\eta(z)| dz \right) \int_\Omega \nabla_b W\left( u(x), x \right) \varphi(x) dx = 0. \tag{2.45}
\]

Using (2.45) in (2.44) yields

\[
\limsup_{\varepsilon \to 0^+} \left| \frac{1}{\varepsilon} \int_\Omega \left( W\left( u_{\rho,\varepsilon}(x), x \right) - W\left( u_{\rho,\varepsilon}(x), x \right) \right) dx \right| = 0. \tag{2.46}
\]

Plugging (2.46) into (2.39) we get that

\[
\lim_{\rho \to 0^+} \left\{ \limsup_{\varepsilon \to 0^+} \left( \frac{1}{1 - \ln(\varepsilon)} \left( \| u_{\rho,\varepsilon} \|^q_{W^{1,q}(\mathbb{R}^N\cap \Omega,\mathbb{R}^d)} \right) - \| u_{\rho,\varepsilon} \|^q_{W^{1,q}(\mathbb{R}^N\cap \Omega,\mathbb{R}^d)} \right) + \frac{1}{\varepsilon} \int_\Omega W\left( u_{\rho,\varepsilon}(x), x \right) dx \right\}
\]

\[
\leq \lim_{\rho \to 0^+} \left\{ \limsup_{\varepsilon \to 0^+} \left( \frac{2}{1 + |y|_1} \int_{\mathbb{R}^N \cap \Omega} W\left( u_{\rho,\varepsilon}(x), x \right) dx \right) \right\}
\]

\[
= \left( \int_{\mathbb{R}^N} \left( \sqrt{1 + |y|_1^2} \right)^{N+1} \right) \int_\Omega \left| u^+(y) - u^-(y) \right|^q d\mathcal{H}^{N-1}(y). \tag{2.47}
\]
Moreover, \( \tilde{u}_{\rho, \varepsilon} \to u \) strongly in \( L^p(\mathbb{R}^N, \mathbb{R}^d) \) as \( \varepsilon \to 0^+ \) for every fixed \( \rho \) and \( p \). Therefore, by the above and (2.47) we complete again the proof by a standard diagonal argument.

The next lemma is needed for the proof of Theorem 1.2 (in the general case \( \eta \in W^{1,1} \)).

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set and let \( u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d) \). For \( \eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R}) \), every \( x \in \mathbb{R}^N \) and every \( \varepsilon > 0 \) define

\[
u_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{y - x}{\varepsilon}\right) u(y) dy = \int_{\mathbb{R}^N} \eta(z) u(x + \varepsilon z) dz.
\]

(2.48)

Then, for every \( q > 1 \) and for every \( \varepsilon \in (0, 1) \) we have

\[
\frac{1}{\omega_{N-1} \ln |\varepsilon|} \int_\Omega \left( \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^q}{|x - y|^{N+1}} dy \right) dx \leq \frac{2^q \|u\|_{L^1(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|^q_{L^1(\mathbb{R}^N, \mathbb{R})}}{\ln |\varepsilon|} + \left( 3 \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^q \|D\eta\|_{L^1(\mathbb{R}^N, \mathbb{R})} (q - 1) \ln |\varepsilon| + \left( 3 \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^q \|D\eta\|_{L^1(\mathbb{R}^N, \mathbb{R})},
\]

(2.49)

where \( \omega_{N-1} \) denotes the surface area of the unit ball in \( \mathbb{R}^N \).

**Proof.** Assume first that \( \eta(z) \in C_c^\infty(\mathbb{R}^N, \mathbb{R}) \). Then, by (2.48) we have

\[
\varepsilon \nabla u_\varepsilon(x) = - \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \nabla \eta\left(\frac{y - x}{\varepsilon}\right) u(y) dy = - \int_{\mathbb{R}^N} \nabla \eta(z) u(x + \varepsilon z) dz.
\]

(2.50)

By (2.48) and (2.50) we get that

\[
\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} + \|\varepsilon \nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \leq \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \quad \text{and}
\]

\[
\|u_\varepsilon\|^q_{L^q(\mathbb{R}^N, \mathbb{R}^d)} \leq \|u\|_{L^1(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|^q_{L^1(\mathbb{R}^N, \mathbb{R})} \|\eta\|^q_{L^1(\mathbb{R}^N, \mathbb{R})} \quad \forall \varepsilon > 0, \forall q \in [1, +\infty).
\]

(2.51)

Next, for every \( \varepsilon \in (0, 1) \) we have

\[
\int_{\Omega} \left( \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^q}{|x - y|^{N+1}} dy \right) dx \leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^q}{|x - y|^{N+1}} dy \right) dx = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x + y) - u_\varepsilon(x)|^q}{|y|^{N+1}} dy \right) dx
\]

\[
+ \int_{\mathbb{R}^N} \left( \int_{B_1(0) \setminus B_{1/2}(0)} \frac{|u_\varepsilon(x + y) - u_\varepsilon(x)|^q}{|y|^{N+1}} dy \right) dx + \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N \setminus B_1(0)} \frac{|u_\varepsilon(x + y) - u_\varepsilon(x)|^q}{|y|^{N+1}} dy \right) dx
\]

\[
= \int_{B_{1/2}(0) \setminus B_{1/4}(0)} \frac{1}{|y|^{N+1-q}} \left( \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x + y) - u_\varepsilon(x)|^q}{|y|^q} dx \right) dy
\]

\[
+ \int_{B_{1/2}(0) \setminus B_{1/4}(0)} \frac{1}{|y|^N} \left( \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x + y) - u_\varepsilon(x)|^q}{|y|^q} dx \right) dy
\]

\[
+ \int_{\mathbb{R}^N \setminus B_1(0)} \frac{1}{|y|^{N+1}} \left( \int_{\mathbb{R}^N} |u_\varepsilon(x + y) - u_\varepsilon(x)|^q dx \right) dy.
\]

(2.52)
On the other hand, (2.51) yields
\[|u_\varepsilon(x+y) - u_\varepsilon(x)| + \frac{\varepsilon |u_\varepsilon(x+y) - u_\varepsilon(x)|}{|x-y|} \leq 3\|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \quad \forall \varepsilon > 0, \forall x, y \in \mathbb{R}^N. \tag{2.53}\]

Thus, inserting (2.53) into (2.52) we deduce that
\[
\int_{\Omega} \left( \int_{\Omega} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^q}{|x-y|^{N+1}} dy \right) dx \leq 2^q \|u_\varepsilon\|_{L^q(\mathbb{R}^N, \mathbb{R}^d)}^q \int_{\mathbb{R}^N \setminus B_1(0)} \frac{dy}{|y|^{N+1}}
\]
\[+ \left( 3\|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^{q-1} \int_{\mathbb{R}^N} \frac{1}{|y|^{N+1-q}} \left( \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x+y) - u_\varepsilon(x)|}{|y|} dx \right) dy \]
\[+ \left( 3\|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^{q-1} \int_{B_1(0) \setminus B_1(0)} \frac{1}{|y|^N} \left( \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x+y) - u_\varepsilon(x)|}{|y|} dx \right) dy. \tag{2.54}\]

Inserting (2.48) into (2.54) and using the second inequality in (2.51) we infer,
\[
\int_{\Omega} \left( \int_{\Omega} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^q}{|x-y|^{N+1}} dy \right) dx \leq 2^q \|u\|_{L^1(\mathbb{R}^N, \mathbb{R}^d)} \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{L^1(\mathbb{R}^N, \mathbb{R})} \int_{\mathbb{R}^N \setminus B_1(0)} \frac{dy}{|y|^{N+1}}
\]
\[+ \left( 3\|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^{q-1} \times
\]
\[\times \int_{B_1(0)} \frac{1}{|y|^{N+1-q}} \left( \int_{\mathbb{R}^N} |\eta(z)| \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x+\varepsilon z+y) - u_\varepsilon(x+\varepsilon z)|}{|y|} dx \right) dy \]
\[+ \left( 3\|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^{q-1} \times
\]
\[\times \int_{B_1(0) \setminus B_1(0)} \frac{1}{|y|^N} \left( \int_{\mathbb{R}^N} |\eta(z)| \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x+\varepsilon z+y) - u_\varepsilon(x+\varepsilon z)|}{|y|} dx \right) dy. \tag{2.55}\]

Taking into account the following well known uniform bound from the theory of BV functions:
\[
\int_{\mathbb{R}^N} \frac{|u(x+\varepsilon z+y) - u(x+\varepsilon z)|}{|y|} dx = \int_{\mathbb{R}^N} \frac{|u(x+y) - u(x)|}{|y|} dx \leq \|Du\|(\mathbb{R}^N) \quad \forall y \in \mathbb{R}^N, \tag{2.56}\]

we rewrite (2.53) as
\[
\int_{\Omega} \left( \int_{\Omega} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^q}{|x-y|^{N+1}} dy \right) dx \leq 2^q \|u\|_{L^1(\mathbb{R}^N, \mathbb{R}^d)} \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{L^1(\mathbb{R}^N, \mathbb{R})} \int_{\mathbb{R}^N \setminus B_1(0)} \frac{dy}{|y|^{N+1}}
\]
\[+ \left( 3\|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^{q-1} \|\eta\|_{L^1(\mathbb{R}^N, \mathbb{R})} \int_{\mathbb{R}^N} \frac{dy}{|y|^{N+1-q}}
\]
\[+ \left( 3\|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^d)} \|\eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^{q-1} \|\eta\|_{L^1(\mathbb{R}^N, \mathbb{R})} \int_{B_1(0) \setminus B_1(0)} \frac{dy}{|y|^N}. \tag{2.57}\]

Computing the integrals on the R.H.S. of (2.57) yields (2.49) in the case \(\eta \in C_c^\infty(\mathbb{R}^N, \mathbb{R})\).

Next consider the general case \(\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})\). Thanks to the density of \(C_c^\infty(\mathbb{R}^N, \mathbb{R})\) in \(W^{1,1}(\mathbb{R}^N, \mathbb{R})\), there exists a sequence \(\{\eta_n\}_{n=1}^\infty \subset C_c^\infty(\mathbb{R}^N, \mathbb{R})\) such that
\[
\lim_{n \to +\infty} \|\eta_n - \eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} = 0. \tag{2.58}\]
Thus if we define
\[
    u_{n,\varepsilon}(x) := \frac{1}{\varepsilon^{N}} \int_{\mathbb{R}^N} \eta_n\left(\frac{y-x}{\varepsilon}\right) u(y) \, dy = \int_{\mathbb{R}^N} \eta_n(z) u(x + \varepsilon z) \, dz,
\]  
then
\[
    \lim_{n \to +\infty} u_{n,\varepsilon}(x) = u_{\varepsilon}(x) \quad \forall x \in \mathbb{R}^N, \forall \varepsilon > 0.
\]  
On the other hand, since we proved (2.49) for the case \(\eta_n \in C_c^\infty(\mathbb{R}^N, \mathbb{R})\), for every \(q > 1\), for every \(n = 1, 2, \ldots\) and for every \(\varepsilon \in (0, 1)\) we have:
\[
    \frac{1}{\omega_{N-1} \ln \varepsilon} \int_\Omega \left( \int_\Omega \left| u_{n,\varepsilon}(x) - u_{n,\varepsilon}(y) \right|^q \frac{1}{|x-y|^{N+1}} \, dy \right) \, dx \leq \frac{2^q \|u\|_{L^1(\mathbb{R}^N, \mathbb{R}^q)} \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^q)}^{q-1} \|\eta_n\|_{L^1(\mathbb{R}^N, \mathbb{R})}^{q-1} \|\eta_n\|_{L^1(\mathbb{R}^N, \mathbb{R})} \|Du\|_{(\mathbb{R}^N)} \right)
\]
\[
    + \left(3 \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^q)} \|\eta_n\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^{q-1} \|\eta_n\|_{L^1(\mathbb{R}^N, \mathbb{R})} \|Du\|_{(\mathbb{R}^N)} \right)
\]
\[
    + \left(3 \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^q)} \|\eta_n\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^{q-1} \|\eta_n\|_{L^1(\mathbb{R}^N, \mathbb{R})} \|Du\|_{(\mathbb{R}^N)} := H_n.
\]  
Letting \(n\) go to infinity in (2.61), using (2.58) in the R.H.S. and (2.60) together with Fatou’s Lemma in the L.H.S., we obtain (2.49) in the general case \(\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})\). \(\square\)

**Proof of Theorem 2.2.** In the case \(\eta \in C_c^\infty(\mathbb{R}^N, \mathbb{R})\) the result follows by Proposition 2.1. Next consider the general case \(\eta \in W^{1,1}(\mathbb{R}^N, \mathbb{R})\). As before, by the density of \(C_c^\infty(\mathbb{R}^N, \mathbb{R})\) in \(W^{1,1}(\mathbb{R}^N, \mathbb{R})\), there exists a sequence \(\{\eta_n\}_{n=1}^{\infty} \subset C_c^\infty(\mathbb{R}^N, \mathbb{R})\) such that
\[
    \lim_{n \to +\infty} \|\eta_n - \eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} = 0.
\]  
Next, as before, define
\[
    u_{n,\varepsilon}(x) := \frac{1}{\varepsilon^{N}} \int_{\mathbb{R}^N} \eta_n\left(\frac{y-x}{\varepsilon}\right) u(y) \, dy = \int_{\mathbb{R}^N} \eta_n(z) u(x + \varepsilon z) \, dz.
\]  
Defining \(u_{n,\varepsilon}\) as in (2.59) we get by Proposition 2.1 for all \(n \geq 1\) (see (2.25)),
\[
    \lim_{\varepsilon \to 0^+} \frac{1}{\ln \varepsilon} \|u_{n,\varepsilon}\|_{W^{1/r,q}(u,\Gamma, \mathbb{R}^q)} = 2D_N \left( \int_{\mathbb{R}^N} \eta_n(z) \, dz \right)^q \int_{J_{\varepsilon} \cap \Omega} \left| u^+(x) - u^-(x) \right|^q \, d\mathcal{H}^{N-1}(x) := L_n
\]
and then
\[
    \lim_{n \to +\infty} L_n = \bar{L} := 2D_N \left( \int_{\mathbb{R}^N} \eta(z) \, dz \right)^q \int_{J_{\varepsilon} \cap \Omega} \left| u^+(x) - u^-(x) \right|^q \, d\mathcal{H}^{N-1}(x).
\]  
On the other hand, by Lemma 2.1 for all \(n \geq 1\) and every \(\varepsilon \in (0, 1/\varepsilon)\) we have
\[
    \frac{1}{\omega_{N-1} \ln \varepsilon} \int_\Omega \left( \int_\Omega \frac{1}{|x-y|^{N+1}} \left| u_{n,\varepsilon}(x) - u_{n,\varepsilon}(y) - \left( u_{\varepsilon}(x) - u_{\varepsilon}(y) \right) \right|^q \, dy \right) \, dx
\]
\[
    \leq \frac{1}{\omega_{N-1} \ln \varepsilon} \int_\Omega \left( \int_\Omega \frac{1}{|x-y|^{N+1}} \left| u_{n,\varepsilon}(x) - u_{n,\varepsilon}(y) \right|^q \, dy \right) \, dx
\]
\[
    \leq 2^q \|u\|_{L^1(\mathbb{R}^N, \mathbb{R}^q)} \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^q)}^{q-1} \|\eta_n - \eta\|_{L^1(\mathbb{R}^N, \mathbb{R})}^{q-1} \|\eta_n - \eta\|_{L^1(\mathbb{R}^N, \mathbb{R})} \|Du\|_{(\mathbb{R}^N)}
\]
\[
    + \left(3 \|u\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^q)} \|\eta_n - \eta\|_{W^{1,1}(\mathbb{R}^N, \mathbb{R})} \right)^{q-1} \|\eta_n - \eta\|_{L^1(\mathbb{R}^N, \mathbb{R})} \|Du\|_{(\mathbb{R}^N)} := H_n.
\]  
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Thus, by the triangle inequality we get, for every \( n \geq 1 \) and every \( \varepsilon \in (0, 1/e) \),
\[
\frac{1}{\ln |\varepsilon|^{1/q}} \left| \| u_{n, \varepsilon} \|_{W^{1/q, q}} - \| u_{\varepsilon} \|_{W^{1/q, q}} \right| \leq \frac{\| u_{n, \varepsilon} - u_{\varepsilon} \|_{W^{1/q, q}}}{\ln |\varepsilon|^{1/q}} \leq (\omega_{N-1} H_n)^{1/q}.
\tag{2.67}
\]

Then, by (2.67) and (2.64), for all \( n \geq 1 \) we obtain:
\[
\limsup_{\varepsilon \to 0^+} \frac{\| u_{\varepsilon} \|_{W^{1/q, q}} - \bar{L}^{1/q}}{\ln |\varepsilon|^{1/q}} \leq \limsup_{\varepsilon \to 0^+} \frac{1}{\ln |\varepsilon|^{1/q}} \left| \| u_{n, \varepsilon} \|_{W^{1/q, q}} - \| u_{\varepsilon} \|_{W^{1/q, q}} \right| + \limsup_{\varepsilon \to 0^+} \frac{\| u_{n, \varepsilon} \|_{W^{1/q, q}} - L_n^{1/q}}{\ln |\varepsilon|^{1/q}} + |L_n^{1/q} - \bar{L}^{1/q}| \leq (\omega_{N-1} H_n)^{1/q} + 0 + |L_n^{1/q} - \bar{L}^{1/q}|.
\tag{2.68}
\]

Letting \( n \) go to infinity in (2.68), using (2.65), the definition of \( \bar{L} \) in (2.65) and the fact that \( \lim_{n \to +\infty} H_n = 0 \), we finally deduce (1.8). \( \square \)

### 3 Appendix: Some known results on BV-spaces

In what follows we present some known definitions and results on BV-spaces; some of them were used in the previous sections. We rely mainly on the book [4] by Ambrosio, Fusco and Pallara.

**Definition 3.1.** Let \( \Omega \) be a domain in \( \mathbb{R}^N \) and let \( f \in L^1(\Omega, \mathbb{R}^m) \). We say that \( f \in BV(\Omega, \mathbb{R}^m) \) if the following quantity is finite:
\[
\int_{\Omega} |Df| := \sup \left\{ \int_{\Omega} f \cdot \text{div} \varphi \, dx : \varphi \in C^1_c(\Omega, \mathbb{R}^{m \times N}), |\varphi(x)| \leq 1 \, \forall x \right\}.
\]

**Definition 3.2.** Let \( \Omega \) be a domain in \( \mathbb{R}^N \). Consider a function \( f \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m) \) and a point \( x \in \Omega \).

i) We say that \( x \) is an **approximate continuity point** of \( f \) if there exists \( z \in \mathbb{R}^m \) such that
\[
\lim_{\rho \to 0^+} \frac{\int_{B_{\rho}(x)} |f(y) - z| \, dy}{\rho^N} = 0.
\]
In this case we denote \( z \) by \( \tilde{f}(x) \). The set of approximate continuity points of \( f \) is denoted by \( G_f \).

ii) We say that \( x \) is an **approximate jump point** of \( f \) if there exist \( a, b \in \mathbb{R}^m \) and \( \nu \in S^{N-1} \) such that \( a \neq b \) and
\[
\lim_{\rho \to 0^+} \frac{\int_{B_{\rho}(x)} |f(y) - \chi(a, b, \nu)(y)| \, dy}{\rho^N} = 0,
\tag{3.1}
\]
where \( \chi(a, b, \nu) \) is defined by
\[
\chi(a, b, \nu)(y) := \begin{cases} 
b & \text{if } \nu \cdot y < 0, 
a & \text{if } \nu \cdot y > 0. 
\end{cases}
\]

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The triple \((a, b, \nu)\), uniquely determined, up to a permutation of \((a, b)\) and a change of sign of \(\nu\), is denoted by \((f^+(x), f^-(x), \nu_f(x))\). We shall call \(\nu_f(x)\) the approximate jump vector and we shall sometimes write simply \(\nu(x)\) if the reference to the function \(f\) is clear. The set of approximate jump points is denoted by \(J_f\). A choice of \(\nu(x)\) for every \(x \in J_f\) determines an orientation of \(J_f\). At an approximate continuity point \(x\), we shall use the convention \(f^+(x) = f^-(x) = \tilde{f}(x)\).

**Theorem 3.1** (Theorems 3.69 and 3.78 from [4]). Consider an open set \(\Omega \subset \mathbb{R}^N\) and \(f \in BV(\Omega, \mathbb{R}^m)\). Then:

i) \(\mathcal{H}^{N-1}\)-a.e. point in \(\Omega \setminus J_f\) is a point of approximate continuity of \(f\).

ii) The set \(J_f\) is \(\sigma\mathcal{H}^{N-1}\)-rectifiable Borel set, oriented by \(\nu(x)\). I.e., the set \(J_f\) is \(\mathcal{H}^{N-1}\) \(\sigma\)-finite, there exist countably many \(C^1\) hypersurfaces \(\{S_k\}_{k=1}^\infty\) such that \(\mathcal{H}^{N-1}\left(J_f \setminus \bigcup_{k=1}^\infty S_k\right) = 0\), and for \(\mathcal{H}^{N-1}\)-a.e. \(x \in J_f \cap S_k\), the approximate jump vector \(\nu(x)\) is normal to \(S_k\) at the point \(x\).

iii) \([\left((f^+ - f^-) \otimes \nu_f\right)](x) \in L^1(J_f, d\mathcal{H}^{N-1})\).

**Theorem 3.2** (Theorems 3.92 and 3.78 from [4]). Consider an open set \(\Omega \subset \mathbb{R}^N\) and \(f \in BV(\Omega, \mathbb{R}^m)\). Then, the distributional gradient \(Df\) can be decomposed as a sum of two Borel regular finite matrix-valued measures \(\mu_f\) and \(D^j f\) on \(\Omega\),

\[
Df = \mu_f + D^j f,
\]

where

\[
D^j f = (f^+ - f^-) \otimes \nu_f \mathcal{H}^{N-1}\llcorner J_f
\]

is called the jump part of \(Df\) and

\[
\mu_f = (D^a f + D^c f)
\]

is a sum of the absolutely continuous and the Cantor parts of \(Df\). The two parts \(\mu_f\) and \(D^j f\) are mutually singular to each other. Moreover, \(\mu_f(B) = 0\) for any Borel set \(B \subset \Omega\) which is \(\mathcal{H}^{N-1}\) \(\sigma\)-finite.

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