Supersonic flow of a Chaplygin gas past a delta wing

Bingsong Long\textsuperscript{1,*} & Chao Yi\textsuperscript{2}

\textsuperscript{1}School of Mathematics and Statistics, Huanggang Normal University, Huanggang 438000, China; \textsuperscript{2}Center for Mathematical Sciences, Huazhong University of Science and Technology, Wuhan 430074, China

Email: bslong20@163.com, chaoyi@hust.edu.cn

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Abstract We consider the problem of supersonic flow of a Chaplygin gas past a delta wing with a shock or a rarefaction wave attached to the leading edges. The flow under study is described by the three-dimensional steady Euler system. In conical coordinates, this problem can be reformulated as a boundary value problem for a nonlinear equation of mixed type. The type of this equation depends fully on the solutions of the problem itself, and thus it cannot be determined in advance. We overcome the difficulty by establishing a crucial Lipschitz estimate, and finally prove the unique existence of the solution via the method of continuity.

Keywords supersonic flow, delta wing, Chaplygin gas, boundary value problem, equation of mixed type

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1 Introduction

The problem of supersonic flow over delta wings is of great importance in aeronautics, because most supersonic aircraft, like hypersonic planes or missiles, are designed as a triangle or a body having a delta wing (see [1]). When a sharp-edged delta wing is placed at a small angle of attack in the supersonic flow, there arise a shock front on its compression side and a rarefaction wave on its expansion side (see [13]). The shock or the rarefaction wave may be attached to or detached from the leading edges, depending on the Mach number in the flow, the angle of attack and the sweep angle of the wing. During the past decades, many experimental and computational efforts have been made to investigate this problem (see [2,3,15,16,19,24,28] and the references therein). However, there has been no rigorous mathematical theory for the global existence of solutions even until now. For the case of a three-dimensional wedge, which can be regarded as the most special delta wing, some related results were announced in [6,8,22]. Under the assumption that the sweep angle is nearly close to zero, the global existence of conical solutions was obtained by Chen and Yi [12]. Also, see Chen [9] for a linear approximate solution under the same assumption. In practice, the sweep angle of supersonic aircraft is not that small, so it is still necessary to develop a more general theory.

*Corresponding author
In this paper, we mainly focus on the study of the above problem for the flow of a Chaplygin gas (see [5]). The Chaplygin gas is a perfect fluid obeying the following equation of state:

\[ p(\rho) = A \left( \frac{1}{\rho} - \frac{1}{\rho^*} \right), \tag{1.1} \]

where \( p, \rho > 0 \) are pressure and density, respectively, and \( \rho^* \) and \( A \) are positive constants. As a model of cosmology, the Chaplygin gas can be used to describe the expansion of the universe (see, for example, [20,25]). It follows from (1.1) that

\[ \rho_c = \sqrt{A}, \]

where \( c = c(\rho) \) is the speed of sound. This implies that any shock is reversible and characteristic (see [26,27] for more details). In other words, any rarefaction wave can be treated as a shock but with negative strength. This allows us to discuss the case of rarefaction waves in the same way as we discuss the case of shocks. Based on the special properties of the Chaplygin gas, some multidimensional Riemann problems have been well studied (see, for example, [10,11,21,26,27]).

In what follows, we will first investigate the problem for the special case of a triangular plate, and then in Section 4 turn our attention to some thin delta wings with specific shapes. Hereafter, both shocks and rarefaction waves are called pressure waves, and only the attached case is considered.

Now we describe our problem in more detail. Let \( W_\sigma \) be a flat, infinite-span delta wing in which the angle of apex is \( \pi - 2\sigma \) with \( \sigma \in (0, \pi/2) \). In the rectangular coordinates \((x_1, x_2, x_3)\), it is placed symmetrically on the \( x_2Ox_3 \)-plane with the apex at the origin and the root chord along the positive \( x_3 \)-axis, i.e.,

\[ W_\sigma = \{(x_1, x_2, x_3) : x_1 = 0, |x_2| < x_3 \cot \sigma, x_3 > 0 \}. \tag{1.2} \]

Thus, the sweep angle of \( W_\sigma \) at the leading edges is just \( \sigma \) (see Figure 1). Throughout the paper, the oncoming flow of the uniform state \((\rho_\infty, q_\infty)\) is assumed to be supersonic by passing the wing \( W_\sigma \) at an angle of attack \( \alpha \), where \( \alpha \in (0, \pi/2) \). Then the velocity \( v_\infty = (v_{1\infty}, v_{2\infty}, v_{3\infty}) \) of the oncoming flow is given by \( v_{1\infty} = q_\infty \sin \alpha \) and \( v_{3\infty} = q_\infty \cos \alpha \). Let \( x_1 = s(x_2, x_3) \) and \( x_1 = r(x_2, x_3) \) be the equations for the shock and the rarefaction wave attached to the leading edges, respectively. Clearly, both \( s \) and \( r \) are homogeneous functions of degree one. Write

\[
\mathcal{R}_\sigma := \{s(x_2, x_3) < x_1 < 0, x_2 > 0, x_3 > 0 \},
\]

\[
\mathcal{R}_\sigma' := \{0 < x_1 < r(x_2, x_3), x_2 > 0, x_3 > 0 \}.
\]

By the symmetry of \( W_\sigma \), it suffices to discuss the problem in the regions \( \mathcal{R}_\sigma \) and \( \mathcal{R}_\sigma' \). For the attached pressure waves, since the flow fields in the compression region and the expansion region are independent, we are allowed to consider the case of shocks and the case of rarefaction waves separately.

It is shown in [26,27] that if the piecewise smooth steady flow of a Chaplygin gas is isentropic and irrotational initially, then it remains so forever. Recall that the oncoming flow has been assumed to be uniform. Hence, the flow under consideration is exactly the potential flow. Let us introduce a velocity potential \( \Phi \) by \( v = \nabla_x \Phi \), where \( x := (x_1, x_2, x_3) \). Then the flow is governed by the conservation of mass

\[
\text{div}_x (\rho \nabla_x \Phi) = 0 \tag{1.3}
\]
and the Bernoulli equation
\[ \frac{1}{2} |\nabla_\sigma \Phi|^2 + h(\rho) = \frac{1}{2} B_\infty, \]
where \( \rho \) and \( h(\rho) \) are the density and specific enthalpy, respectively; \( B_\infty/2 \) is the Bernoulli constant determined by the oncoming flow, i.e.,
\[ B_\infty = q_\infty^2 - c_\infty^2. \]
Combining (1.1) and (1.4), we can express \( \rho \) as a function of \( \Phi \), i.e.,
\[ \rho = \sqrt{\frac{4}{|\nabla_\sigma \Phi|^2 - B_\infty}}. \]

Then we obtain a quasilinear equation for \( \Phi \), by substituting (1.6) into (1.3).

Next, we denote by \( S_\sigma \) and \( S'_\sigma \) the shock and the rarefaction wave, respectively. Then the Rankine-Hugoniot condition yields
\[ [\rho \nabla_\sigma \Phi] \cdot n_s = 0 \quad \text{on} \quad S_\sigma, \]
where \([\cdot]\) denotes the jump of quantities across the shock, and \( n_s \) is the exterior normal to \( S_\sigma \). Note that any shock is a characteristic, so the boundary condition (1.7) is naturally satisfied. Likewise, we have the same conclusion for the rarefaction wave.

Consequently, our problem for the flow of a Chaplygin gas can be formulated mathematically as follows.

**Problem 1.1.** For the wing \( W_\sigma \) and the oncoming flow given above, we wish to seek a solution \( \Phi \) of the system (1.3)–(1.4) in the region \( \mathcal{R}_\sigma \) (resp., \( \mathcal{R}'_\sigma \)) with the Dirichlet boundary condition
\[ \Phi = \Phi_\infty \quad \text{on} \quad S_\sigma \quad \text{(resp.,} \quad S'_\sigma \quad \text{)} \quad \text{(1.8)} \]
and the slip boundary conditions
\[ \nabla_\sigma \Phi \cdot n_w = 0 \quad \text{on} \quad \{x_1 = 0\}, \quad \text{(1.9)} \]
\[ \nabla_\sigma \Phi \cdot n_{sw} = 0 \quad \text{on} \quad \{x_2 = 0\}, \quad \text{(1.10)} \]
where \( \Phi_\infty = v_{1\infty}x_1 + v_{3\infty}x_3 \) is the potential of the oncoming flow, \( n_w = (1, 0, 0) \) is the exterior normal to \( \{x_1 = 0\} \), and \( n_{sw} = (0, -1, 0) \) is the exterior normal to \( \{x_2 = 0\} \).

The following theorem is the main result of this paper.

**Theorem 1.2** (The main theorem). Assume that the state \((\rho_\infty, q_\infty)\) of the oncoming flow is uniform and supersonic, and the wing \( W_\sigma \) is a triangular plate given by (1.2). Then for the case of shocks, we can find a critical angle \( \alpha_0 = \alpha_0(\rho_\infty, q_\infty) \in (0, \pi/2) \) so that for any fixed \( \alpha \in (0, \alpha_0) \), there exists \( \sigma_0 = \sigma_0(\rho_\infty, q_\infty, \alpha) \in (0, \pi/2) \) such that when \( \sigma \in [0, \sigma_0] \), Problem 1.1 admits a piecewise smooth solution.

Similarly, by replacing the angle \( \alpha_0 \) with \( \alpha'_0 \in (0, \pi/2) \), the angle \( \sigma_0 \) with \( \sigma'_0 \in (0, \sigma_0) \), and the condition \( \sigma \in [0, \sigma_0] \) with \( \sigma \in [0, \sigma'_0] \), we obtain the same result for the case of rarefaction waves.

Although Problem 1.1 is three-dimensional in nature, it can be treated mathematically as a two-dimensional one, due to the features of the wing \( W_\sigma \) and the resulting conical flow. In fact, we can view the scaled variables \( \xi_1 = x_1/x_3 \) and \( \xi_2 = x_2/x_3 \) as new coordinates, and then restate Problem 1.1 in the \((\xi_1, \xi_2)\) coordinates, as demonstrated in Subsection 2.3. This finally leads to a boundary value problem for a nonlinear mixed-type equation (see Problem 2.5). Unlike those discussed in [7, 12], the type of our equation is far from being known, due to the fact that the sweep angle of \( W_\sigma \) is no longer sufficiently small. Moreover, from (1.1) and the equation of state for a polytropic gas, namely, \( p(\rho) = A \rho^\gamma \) with constants \( A, \gamma > 0 \), we obtain \( \gamma = -1 \) for the Chaplygin gas. This means that the ellipticity principle proved by Elling and Liu [14] for self-similar potential flow cannot be applied to our problem either, because the approach adopted there is valid only for \( \gamma > -1 \). Motivated by Serre [26, 27], we find that the type of our equation can be determined completely by a priori estimates for the solutions of Problem 2.5. It should be pointed out that our estimates can be established under a weaker condition that may allow the nonlinear equation to degenerate inside the domain. By contrast, the corresponding equation in [26, 27]...
is assumed to be elliptic over the whole domain. We also note that in [26,27] only the Dirichlet problem has been studied. In Section 4, with some new ingredients added into the strategy mentioned above, we are able to treat a more general class of boundary value problems, such as those with mixed boundary conditions or in a Lipschitz domain.

The rest of this paper is arranged as follows. Section 2 mainly presents some preliminaries to the investigation of Problem 1.1. We first determine the undisturbed states of the downstream flow near the leading edges of the wing, and then analyze the global structures of the pressure waves in conical coordinates. After that, we reduce Problem 1.1 to a boundary value problem for a nonlinear mixed-type equation, i.e., Problem 2.5, and meanwhile we rewrite the main theorem as Theorem 2.6. Section 3 is devoted to the proof of Theorem 2.6. With the use of two key auxiliary functions, we establish a priori estimates for the solutions of Problem 2.5, which ensure the ellipticity of the equation under consideration, and therefore allow us to obtain the existence and the uniqueness for Problem 2.5. Section 4 considers the problem of supersonic flow over a thin delta wing of diamond cross-sections. We achieve a similar result by solving a Neumann boundary value problem in a Lipschitz domain.

2 Preliminary analysis of Problem 1.1

When the wing \( W_\sigma \) becomes a half-plane, i.e., \( \sigma = 0 \), Problem 1.1 is essentially a two-dimensional problem, so it can be solved by the analysis of shock polars, as shown in Appendix A. From now on, we only consider the case \( \sigma > 0 \).

2.1 The uniform downstream flow near the leading edges

Let us begin with the potential equation (1.3). Expanding (1.3), together with (1.6), we obtain a quasi-linear equation of second-order

\[
(c^2 - \Phi^2)\Phi_{x_1x_1} + (c^2 - \Phi^2)\Phi_{x_2x_2} + (c^2 - \Phi^2)\Phi_{x_3x_3}
- 2\Phi_{x_1}\Phi_{x_2}\Phi_{x_1x_2} - 2\Phi_{x_2}\Phi_{x_3}\Phi_{x_1x_3} - 2\Phi_{x_2}\Phi_{x_3}\Phi_{x_2x_3} = 0.
\]

(2.1)

The characteristic equation of (2.1) is

\[
Q(\zeta) = c^2 - |\nabla_x \Phi \cdot \zeta|^2 \quad \text{for any } \zeta \in \mathbb{R}^3, \quad |\zeta| = 1,
\]

(2.2)

so the equation (2.1) is elliptic in a subsonic domain and hyperbolic in a supersonic domain. Also, since any pressure wave is a characteristic, the normal component of the flow velocity across the pressure waves is sonic. Then if a stream of the flow is initially supersonic, it stays so forever. This means that the downstream flow is supersonic, under the assumption that the oncoming flow is supersonic. Accordingly, the equation (2.1) is hyperbolic and there exists a Mach cone of the apex of the wing. Thanks to the property of finite propagation for hyperbolic equations, the solution of (2.1) outside the Mach cone is undisturbed, so it can be analyzed in the same way as that for the supersonic flow past a wedge. In particular, for a uniform state of the downstream flow, the pressure wave is flat and attached to the leading edges of the wing. For simplicity of notation, we use \( S_{ob} \) and \( S'_{ob} \), respectively, to denote the flat shock and the flat rarefaction wave.

We then calculate the solution of (2.1) outside the Mach cone. Let us first consider the case of shocks. Since the oncoming flow is not perpendicular to the leading ledges, we introduce

\[
e_1 := (1, 0, 0), \quad e_i := (0, \cos \sigma, \sin \sigma), \quad e_j := (0, -\sin \sigma, \cos \sigma).
\]

Obviously, \( \{e_1, e_i, e_j\} \) is an orthogonal basis for a system of coordinates \((x_1, x_i, x_j)\). Then the velocity \( v_\infty \) is decomposed as

\[
v_\infty = v_{1\infty}e_1 + v_{3\infty} \sin \sigma e_i + v_{3\infty} \cos \sigma e_j.
\]

(2.3)

For notational convenience, we write

\[
\hat{v}_\infty = v_{1\infty}e_1 + v_{3\infty} \cos \sigma e_j,
\]
and thus \( \tilde{q}_\infty = \sqrt{v_{1\infty}^2 + v_{3\infty}^2 \cos^2 \sigma} \) and \( \alpha_n = \arctan (\tan \alpha / \cos \sigma) \), where \( \alpha_n \) is the angle between \( \tilde{v}_\infty \) and the \( x_2Ox_3 \)-plane. Let \( \psi_\sigma = (0, v_{2\sigma}, v_{3\sigma}) \) be the velocity of the uniform flow behind the shock, and \( q_{j\sigma} \) be the speed of the flow along \( e_j \). Since the velocity of the flow along \( e_i \) is unchanged across the flat shock \( S_{ob} \), we have

\[
v_\sigma = v_{1\infty} \sin \sigma e_i + q_{j\sigma} e_j,
\]

which implies

\[
v_{2\sigma} = v_{3\infty} \sin \sigma \cos \sigma - q_{j\sigma} \sin \sigma, \quad v_{3\sigma} = v_{3\infty} \sin^2 \sigma + q_{j\sigma} \cos \sigma.
\]  

(2.4)

Note that \( q_{j\sigma} \) can be derived by (A.4) with the choices \( u_0 = \tilde{q}_\infty \), \( c_0 = c_\infty \) and \( \alpha = \alpha_n \). From (2.4), we obtain the explicit expression of \( v_\sigma \). So the solution of the equation (2.1) outside the Mach cone is \( \Phi_\sigma = v_{2\sigma} x_2 + v_{3\sigma} x_3 \). In addition, we get the corresponding sound speed

\[
c_\sigma = \sqrt{\| \nabla x \Phi_\sigma \|^2 - B_\infty},
\]

where \( B_\infty \) is given by (1.5). Analogously, we can obtain a uniform state \((c_\sigma', (0, v_{2\sigma}', v_{3\sigma}'))\) and a potential function \( \Phi'_\sigma = v_{2\sigma}' x_2 + v_{3\sigma}' x_3 \) for the case of rarefaction waves.

Now we turn to the role of the angles \( \alpha \) and \( \sigma \). From Appendix A, we see that for the Chaplygin gas, there may be a phenomenon of concentration or cavitation, if the angle of the wedge changes excessively. To avoid this, we restrict the ranges of \( \alpha \) and \( \sigma \). Let \( \beta_n \) be the angle between the flat shock \( S_{ob} \) and the \( x_2Ox_3 \)-plane, and \( \beta'_n \) be the angle between the flat rarefaction wave \( S'_{ob} \) and the \( x_2Ox_3 \)-plane. Then from the shock polar given in Appendix A, we know that the circle with the center \( O_\infty (\tilde{q}_\infty \cos \alpha_n, \tilde{q}_\infty \sin \alpha_n) \) and the radius \( c_\infty \) is tangent to \( S_{ob} \) at point \( P \), and also tangent to \( S'_{ob} \) at point \( P' \) (see Figure 2). Note that in the \( x_1Ox_j \)-plane, both \( \tilde{q}_\infty \) and \( c_\infty \) are invariant under any rotation transformation. We here apply the conclusion in Appendix A directly.

To avoid concentration, we deduce from (A.5) that

\[
c_\infty < \tilde{q}_\infty < \frac{c_\infty}{\sin \alpha_n}.
\]  

(2.5)

Owing to the relation \( \tan \alpha = v_{1\infty} / v_{3\infty} \), the speed \( \tilde{q}_\infty \) equals

\[
\tilde{q}_\infty = v_{1\infty} \sqrt{1 + \frac{\cos^2 \sigma}{\tan^2 \alpha}}.
\]  

(2.6)

Substituting (2.6) into the right-hand side of the inequality (2.5), we get

\[
q_\infty \sin \alpha = v_{1\infty} < c_\infty,
\]

which leads to

\[
\alpha_0 := \arcsin \left( \frac{c_\infty}{q_\infty} \right) > \alpha.
\]  

(2.7)

Figure 2: The determination of flat pressure waves
In addition, the left-hand side of the inequality (2.5) can be reduced to
\[ \sigma_0 := \arcsin \left( \frac{\sqrt{q_\infty^2 - c_\infty^2}}{v_{3\infty}} \right) > \sigma. \]  

(2.8)

Then to avoid cavitation, it follows from (A.6) that
\[ \tilde{q}_\infty > \frac{c_\infty}{\cos \alpha_n}. \]  

(2.9)

Since the speed \( \tilde{q}_\infty \) also takes the form
\[ \tilde{q}_\infty = v_{3\infty} \cos \sigma \sqrt{1 + \tan^2 \alpha \cos^2 \sigma}, \]  

(2.10)

combining (2.9) and (2.10), we have
\[ v_{3\infty} > \frac{c_\infty}{\cos \sigma}. \]  

(2.11)

Then inserting \( \sigma = 0 \) into (2.11) gives
\[ \alpha_0' := \arccos \left( \frac{c_\infty}{\tilde{q}_\infty} \right) > \alpha. \]  

(2.12)

Moreover, we infer from (2.11) that
\[ \sigma_0' := \arccos \left( \frac{c_\infty}{v_{3\infty}} \right) > \sigma. \]  

(2.13)

Obviously, \( \sigma_0' < \sigma_0 \) by (2.8) and (2.13).

**Remark 2.1.** For the case of shocks, if we fix \( \alpha \in (0, \alpha_0) \) and let \( \sigma \) vary in \( (0, \sigma_0) \), then from
\[ \tilde{q}_\infty \sin \alpha_n = q_\infty \sin \alpha \quad \text{and} \quad \tilde{q}_\infty \cos \alpha_n = q_\infty \cos \alpha \cos \sigma, \]
we see that the \( v_1 \)-coordinate of \( O_\infty \) remains unchanged. Note that the radius of the circle in Figure 2 is always \( c_\infty \). This implies that the angle \( \beta_n \) is a monotonically increasing function of \( \sigma \). Since the phenomenon of concentration occurs only when \( \beta_n = 0 \), it would never occur unless it happened for \( \sigma = 0 \).

However, for the case of rarefaction waves, the phenomenon of cavitation could really occur as long as we fix \( \alpha \in (0, \alpha_0') \) and let \( \sigma \) approach \( \sigma_0' \). It can be observed from Figure 2 that since the center \( O_\infty \) moves left as \( \sigma \) increases, the circle finally touches the \( v_1 \)-axis when \( \sigma = \sigma_0' \).}

### 2.2 Pressure wave patterns in conical coordinates

As mentioned in Section 1, we can treat the case of shocks and the case of rarefaction waves separately. In what follows, we fix \( \alpha \in (0, \alpha_0) \) for the case of shocks, and \( \alpha \in (0, \alpha_0') \) for the case of rarefaction waves, where \( \alpha_0 \) and \( \alpha_0' \) are given by (2.7) and (2.12), respectively. We will show that the attached shock appears for \( \sigma \in (0, \sigma_0) \), while the attached rarefaction wave appears only for \( \sigma \in (0, \sigma_0') \). Moreover, the patterns of these waves will be demonstrated explicitly in a rectangular system of conical coordinates, as defined below.

Notice that the boundary value problem (1.3)–(1.4) with (1.8)–(1.10) is invariant under the scaling
\[ x \rightarrow \zeta x, \quad (\rho, \Phi) \rightarrow \left( \frac{\rho}{\zeta}, \frac{\Phi}{\zeta^2} \right) \text{ for } \zeta \neq 0. \]

Thus, we seek a solution with the following form:
\[ \rho(x) = \rho(\xi_1, \xi_2), \quad \Phi(x) = x_3 \phi(\xi_1, \xi_2), \]  

(2.14)
where \((\xi_1, \xi_2) := (x_1/x_3, x_2/x_3)\) are called conical coordinates. Hereafter, our discussion is carried out in these coordinates, along with the notations
\[
\psi := \frac{\phi}{\sqrt{B_\infty}}, \quad a := \frac{c}{\sqrt{B_\infty}},
\]
where the positive constant \(B_\infty\) is defined by (1.5).

Before proceeding, we introduce some notations. Let \(C_\infty, C_\sigma\) and \(C'_\sigma\) be the Mach cones of the apex of the wing, determined by the oncoming flow, the flow behind the shock, and the flow behind the rarefaction wave, respectively. By abuse of notation but without misunderstanding, we continue to write \(C_\infty, C_\sigma\) and \(C'_\sigma\) for the corresponding curves of the Mach cones, \(S_{ob}\) and \(S'_{ob}\) for the corresponding oblique shock and oblique rarefaction wave, respectively, in the conical coordinates.

Let us first derive the equations for \(S_{ob}\) and \(S'_{ob}\). By the continuity of \(\Phi\) on the flat pressure waves, together with (2.14), we have \(\psi_\infty = \psi_\sigma\) on \(S_{ob}\) and \(\psi_\infty = \psi'_\sigma\) on \(S'_{ob}\). Moreover, using (2.14)–(2.15) and the explicit expressions of \(\Phi_\infty, \Phi_\sigma\) and \(\Phi'_\sigma\), we get
\[
\psi_\infty = \frac{v_{1\infty}\xi_1 + v_{3\infty}}{\sqrt{B_\infty}}, \quad \psi_\sigma = \frac{v_{2\sigma}\xi_2 + v_{3\sigma}}{\sqrt{B_\infty}}, \quad \psi'_\sigma = \frac{v'_{2\sigma}\xi_2 + v'_{3\sigma}}{\sqrt{B_\infty}}.
\]
Then from (2.16), it follows that the equations for \(S_{ob}\) and \(S'_{ob}\) are given by
\[
S_{ob} : v_{1\infty}\xi_1 + v_{3\infty} = v_{2\sigma}\xi_2 + v_{3\sigma},
\]
\[
S'_{ob} : v_{1\infty}\xi_1 + v_{3\infty} = v'_{2\sigma}\xi_2 + v'_{3\sigma}.
\]

We then turn to the equations for \(C_\infty, C_\sigma\) and \(C'_\sigma\). It follows from (B.4) and (2.14)–(2.15) that in the conical coordinates, the equation for a Mach cone of the apex of the wing takes the form
\[
|D\psi|^2 + |\psi - D\psi \cdot \xi|^2 - \frac{\psi^2}{1 + |\xi|^2} = a^2,
\]
where \(\xi := (\xi_1, \xi_2)\). By (1.6) and (2.14)–(2.15), and noting \(pc = \sqrt{A}\), we have
\[
a^2 = |D\psi|^2 + |\psi - D\psi \cdot \xi|^2 - 1.
\]
Substituting (2.20) into (2.19) yields
\[
\psi^2 = 1 + |\xi|^2.
\]
Plugging (2.16) into (2.21), we obtain the following equations:
\[
C_\infty : (v_{1\infty}\xi_1 + v_{3\infty})^2 = B_\infty(1 + |\xi|^2),
\]
\[
C_\sigma : (v_{2\sigma}\xi_2 + v_{3\sigma})^2 = B_\infty(1 + |\xi|^2),
\]
\[
C'_\sigma : (v'_{2\sigma}\xi_2 + v'_{3\sigma})^2 = B_\infty(1 + |\xi|^2).
\]

Now we are ready to analyze the global structures of pressure waves. Since any shock is a characteristic, the oblique shock \(S_{ob}\) must be tangent to the curve \(C_\infty\) at a point, denoted by \(P_1\). This is also true for the case of rarefaction waves, except for the tangent point, denoted by \(P'_1\). Also, we denote by \(P_0\) the intersection point of \(C_\infty\) and the \(\xi_2\)-axis, by \(P_2\) the intersection point of \(C_\infty\) and the negative \(\xi_1\)-axis, by \(P_4\) (resp. \(P'_4\)) the intersection point of \(C_\sigma\) (resp. \(C'_\sigma\)) and the \(\xi_2\)-axis, and by \(P_{5\alpha}\) the intersection point of the oblique shocks and the \(\xi_2\)-axis (see Figure 3(a)). In addition, by using (2.17) and (2.22), we have \(P_{5\alpha}(0, 0, \sec \alpha)\) and \(P_{5\alpha}(0, \sqrt{c_{\infty}^2 - v_{1\infty}^2}/\sqrt{B_\infty})\). It follows from (2.8) that when \(\sigma \in (0, \sigma_0)\), the point \(P_{3\alpha}\) is above \(P_0\) all the time. Then we establish the relationship between the curve \(C_\infty\) and the curve \(C_\sigma\) (resp. \(C'_\sigma\)) as follows.

**Lemma 2.2.** Let \(C_\infty, C_\sigma\) and \(C'_\sigma\) be defined as in (2.22)–(2.24). For any fixed \(\alpha \in (0, \alpha_0)\), if \(\sigma \in (0, \sigma_0)\), then the curve \(C'_\sigma\) is tangent to \(C_\infty\) only at the point \(P_1\), and the point \(P_1'\) is always below \(P_0\), where \(\alpha_0\) and \(\sigma_0\) are given by (2.7) and (2.8), respectively.

With \(\alpha_0, \sigma_0, P_1\) and \(P_4\) replaced, respectively, by \(\alpha'_0, \sigma'_0, P'_1\) and \(P'_4\), we obtain the same result for \(C_\infty\) and \(C'_\sigma\) except that the point \(P'_{1\alpha}\) is located between \(P_0\) and \(P_{5\alpha}\), where \(\alpha'_0\) and \(\sigma'_0\) are given by (2.12) and (2.13), respectively.
In fact, (2.26) is the equation for the oblique shock $\xi$ is perpendicular to the $\xi$-axis. Also, we know from Appendix A that if the angle between the flat rarefaction wave and the velocity of the uniform flow behind the point $P_0$ is below $45^\circ$. Therefore, we infer from (2.11) and (2.13) that $\sigma = \sigma_0$. Consequently, the vertex angle of the wing $W_\sigma$ is greater than the apex angle of the Mach cone $C_\sigma$ until $\sigma = \sigma_0$. This implies that the attached shock occurs

\begin{proof}
Let us first prove that $C_\infty$ and $C_\sigma$ are tangent at the point $P_1$. It follows from (2.22) and (2.23) that the intersection points of $C_\infty$ and $C_\sigma$ satisfy

$$|v_{1\infty}\xi_1 + v_{3\infty}| = |v_{2\sigma}\xi_2 + v_{3\sigma}|.$$  \hfill (2.25)

We claim that the function $\psi_\infty$ is positive on the curve $C_\infty$. The explicit expression of $\psi_\infty$ implies $\psi_\infty > 0$ on the arc $C_\infty \cap \{\xi_2 \geq 0, \xi_1 \geq 0\}$. Then it suffices to consider the function $\psi_\infty$ on the remaining part of $C_\infty \cap \{\xi_2 \geq 0, \xi_1 < 0\}$. Denoted by $P_6$ the intersection point of the extension line of $S_{ob}$ and the $\xi_1$-axis. Since the oblique shock $S_{ob}$ is tangent to the curve $C_\infty$, the point $P_6$ must lie on the left-hand side of $P_2$ (see Figure 3(a)). From (2.16)–(2.17) and the explicit expressions of $\psi_\infty$ and $S_{ob}$, it follows that in the triangle $\Delta P_3P_5P_6$, the following inequality holds:

$$\psi_\infty(\xi) \geq \frac{v_{1\infty}\xi_1 + v_{3\infty}}{\sqrt{B_\infty}} = \frac{v_{3\sigma}}{\sqrt{B_\infty}} > 0,$$

where $\xi_{P_6}$ denotes the $\xi_1$-coordinate of $P_6$. In addition, it is clear that $C_\infty \cap \{\xi_2 \geq 0, \xi_1 < 0\} \subset \Delta P_3P_5P_6$. Thus we have shown the positivity of $\psi_\infty$. Moreover, in the conical coordinates, we know $\psi_\sigma > 0$ on the curve $C_\sigma$. Therefore, the equality (2.25) is equivalent to

$$v_{1\infty}\xi_1 + v_{3\infty} = v_{2\sigma}\xi_2 + v_{3\sigma}.$$  \hfill (2.26)

In fact, (2.26) is the equation for the oblique shock $S_{ob}$. Also, since the curve $C_\infty$ is tangent to $S_{ob}$ at the point $P_1$, it follows that there is only one intersection point of $C_\infty$ and $C_\sigma$, and moreover they are tangent at the point $P_1$.

Then we prove that for any fixed $\alpha \in (0, \alpha_0)$, the point $P_4$ is always below $P_0$ when $\sigma \in (0, \sigma_0)$. Note that the point $P_3$ is above $P_0$ when $\sigma \in (0, \sigma_0)$, and then $P_1$ is the only intersection point of $C_\infty$ and $C_\sigma$ from the above discussion. We also know that the point $P_4$ is below $P_0$ when $\sigma = 0$. Then for $\sigma \in (0, \sigma_0)$, the point $P_4$ is below $P_0$, as the coordinate of $P_4$ is a continuous function of $\sigma$.

As for the point $P_4'$, since the curve $C_\sigma'$ is tangent to the oblique rarefaction wave $S_{ob}'$, it is always below $P_5$. Also, noting that the point $P_4'$ is above $P_0$ for $\sigma = 0$, we deduce that the point $P_4'$ must be located between $P_0$ and $P_5$. The proof is completed. \hfill $\square$

**Remark 2.3.** We explain here that the oblique rarefaction wave $S_{ob}'$ is perpendicular to the $\xi_2$-axis when $\sigma = \sigma_0'$. Note that the location of $C_\infty$ is known (see the curve $P_2P_0P_2'$ in Figure 3(b)). Then for any fixed $\alpha \in (0, \alpha_0)$, as $\sigma$ varies from zero to $\sigma_0$, there must exist a critical angle such that $S_{ob}'$ is perpendicular to the $\xi_2$-axis. Also, we know from Appendix A that if the angle between the flat rarefaction wave and the velocity of the uniform flow behind $S_{ob}'$ approaches $\pi/2$, then the phenomenon of cavitation will occur. Therefore, we infer from (2.11) and (2.13) that $\sigma = \sigma_0'$ is the critical angle.

With the above analysis, we are able to draw the patterns of pressure waves as in Figure 3(a). Note that the points $P_1$ and $P_3$ meet at $P_0$ when $\sigma = \sigma_0$. Consequently, the vertex angle of the wing $W_\sigma$ is greater than the apex angle of the Mach cone $C_\sigma$ until $\sigma = \sigma_0$. This implies that the attached shock occurs

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Patterns of pressure waves in the ($\xi_1, \xi_2$)-plane}
\end{figure}
only for $\sigma \in (0, \sigma_0]$. Furthermore, we conclude from Lemma 2.2 and Remark 2.3 that there is a rarefaction wave attached to the leading edge for $\sigma \in (0, \sigma_0')$. Since the discussion of the case of rarefaction waves is similar to that of the case of shocks, we mainly consider the case of shocks afterwards.

Let us denote by $\Gamma_{\sigma \text{ cone}}$ and $\Gamma_{\text{sym}}$ the arcs $P_1P_2$ and $P_1P_4$, respectively; denote by $\Gamma_{\text{sym}}$ and $\Gamma_{\text{wing}}$ the lines $P_2P_3$ and $P_3P_5$, respectively. In addition, let $U$ be the domain $P_1P_2P_3P_5$, and $\Omega$ be the domain $P_1P_2P_3P_4$.

Finally, we conclude Subsections 2.1 and 2.2 by the following proposition.

**Proposition 2.4.** Assume that the state $(\rho_\infty, q_\infty)$ of the oncoming flow is uniform and supersonic, and the wing $W_\sigma$ is a triangular plate given by (1.2). Then we can find a critical angle $\alpha_0 = \alpha_0(\rho_\infty, q_\infty)$ so that for any fixed $\alpha \in (0, \alpha_0)$, there exists $\sigma_0 = \sigma_0(\rho_\infty, q_\infty, \alpha) \in (0, \pi/2)$ such that

(i) for $\sigma \in (0, \sigma_0)$, there is uniform flow in the domain $U \setminus \Omega$ (see Figure 4(a)), and the corresponding potential function satisfies

\[
\psi = \frac{v_2\alpha_2 + v_3\alpha_3}{\sqrt{B_\infty}};
\]

(ii) for $\sigma = \sigma_0$, the boundary $\Gamma_{\sigma \text{ cone}}$ degenerates into the point $P_0$, and the domain $\Omega$ coincides with $U$ (see Figure 4(b)). In particular, there is no uniform flow behind the shock.

Here, $\alpha_0$ and $\sigma_0$ are defined by (2.7) and (2.8), respectively.

### 2.3 The boundary value problem (BVP) for a nonlinear mixed-type equation

In this subsection, we will reformulate Problem 1.1 in the conical coordinates. Before proceeding further, we introduce a useful notation

\[
D^2f[a, b] := \sum_{i,j=1}^2 a_ib_j \partial_{ij}f \quad \text{for } f \in C^2 \quad \text{and} \quad a, b \in \mathbb{R}^2.
\]

Let us first derive the potential equation for $\psi$. By (1.3) and (2.14)–(2.15), we obtain

\[
\text{div}(\rho(D\psi - (\psi - D\psi \cdot \xi)) + 2\rho(\psi - D\psi \cdot \xi)) = 0,
\]

or equivalently,

\[
a^2(\Delta \psi + D^2\psi [\xi, \xi]) - D^2\psi [D\psi - \chi, D\psi - \chi] = 0, \quad (2.27)
\]

where $\Delta$ and $D$ denote the Laplacian and the gradient operator with respect to $\xi$, respectively, and $\chi$ is given by

\[
\chi = \psi - D\psi \cdot \xi. \quad (2.28)
\]
Define
\[ L^2 := \frac{|D\psi|^2 + |\psi - D\psi \cdot \xi|^2 - \frac{\psi^2}{1 + |\xi|^2}}{a^2}. \] (2.29)

The type of the equation (2.27) is determined by \( L \). To be specific, the equation (2.27) is hyperbolic if \( L > 1 \), elliptic if \( L < 1 \), and parabolic degenerate if \( L = 1 \). It follows from (2.19) and (2.29) that the equation (2.27) is hyperbolic in the domain \( U \setminus \Omega \) and parabolic degenerate on the arc \( \Gamma_{cone}^\infty \cup \Gamma_{cone}^\sigma \).

Then we simplify the form of the boundary conditions. Since we have shown in Lemma 2.2 that both the function \( \psi_\infty \) on \( C_\infty \) and the function \( \psi_\sigma \) on \( C_\sigma \) are positive, it follows from (2.20) and (2.29) that \( L = 1 \) is equivalent to
\[ \psi = \sqrt{1 + |\xi|^2}. \] (2.30)

In other words, Problem 1.1 can be rewritten as follows.

**Problem 2.5.** Let \( \alpha_0, \alpha \) and \( \sigma_0 \) be as in Proposition 2.4. Then for any \( \sigma \in (0, \sigma_0] \), we expect to seek a solution \( \psi \) of the following boundary value problem:
\[
\begin{aligned}
\psi &= \sqrt{1 + |\xi|^2} \quad \text{in} \quad \Omega, \\
\psi &= \sqrt{1 + |\xi|^2} \quad \text{on} \quad \Gamma_{cone}^\infty \cup \Gamma_{cone}^\sigma, \\
D\psi \cdot \nu_w &= 0 \quad \text{on} \quad \Gamma_{wing}, \\
D\psi \cdot \nu_{sy} &= 0 \quad \text{on} \quad \Gamma_{sym},
\end{aligned}
\] (2.31)

where \( \nu_w = (1, 0) \) and \( \nu_{sy} = (0, -1) \) are the exterior normals to \( \Gamma_{wing} \) and \( \Gamma_{sym} \), respectively. We emphasize here that when \( \sigma = \sigma_0 \), the domain \( \Omega \) coincides with \( U \), and the boundary \( \Gamma_{cone}^\infty \) degenerates into the point \( P_0 \).

Correspondingly, to prove Theorem 1.2, we only need to show the following theorem.

**Theorem 2.6.** Let \( \alpha_0, \alpha \) and \( \sigma_0 \) be as in Proposition 2.4. Then for any \( \sigma \in (0, \sigma_0] \), Problem 2.5 admits a unique solution \( \psi \) satisfying
\[ \psi \in C^\infty(\bar{\Omega} \setminus \Gamma_{cone}^\infty \cup \Gamma_{cone}^\sigma) \cap \text{Lip}(\bar{\Omega}) \]
and
\[ \psi > \sqrt{1 + |\xi|^2} \quad \text{in} \quad \bar{\Omega} \setminus \Gamma_{cone}^\infty \cup \Gamma_{cone}^\sigma. \]

Before proving Theorem 2.6, we briefly make some comments on the equation (2.27). This equation is hyperbolic in the domain \( U \setminus \Omega \) and degenerate on the boundary \( \Gamma_{cone}^\infty \cup \Gamma_{cone}^\sigma \). Then it is natural to find a solution such that the equation (2.27) is elliptic in \( \bar{\Omega} \setminus \Gamma_{cone}^\infty \cup \Gamma_{cone}^\sigma \). We also expect to know whether there are some parabolic bubbles inside the domain. For this purpose, we only assume \( \psi \gg \sqrt{1 + |\xi|^2} \) in \( \bar{\Omega} \setminus \Gamma_{cone}^\infty \cup \Gamma_{cone}^\sigma \) afterwards. Fortunately, with the help of the auxiliary function given below, we can prove that no such parabolic bubbles exist in this domain. The detailed proof of Theorem 2.6 will be given in Section 3.

### 3 Unique solvability of Problem 1.1

Notice that the equation (2.27) is of reflection symmetry with respect to the axes. In addition, the boundaries \( \Gamma_{cone}^\infty \) and \( \Gamma_{cone}^\sigma \) are perpendicular to the \( \xi_1 \)-axis and the \( \xi_2 \)-axis, respectively. Therefore, we can reflect the domain \( \Omega \) with respect to the axes to obtain a domain \( \Omega_{ext} \), and accordingly extend the function \( \psi \) to the domain \( \Omega_{ext} \) by an even reflection. For simplicity of notation, we still write \( \psi \) for the function after extension. It should be noted that when \( \sigma = \sigma_0 \) (see Figure 4(b)), the shock \( \Gamma_{cone}^\infty \) is not perpendicular to the \( \xi_2 \)-axis, and thus the corner point \( P_0 \) cannot be removed by reflecting the domain \( U \) about the \( \xi_2 \)-axis. In what follows, we mainly focus on the case \( \sigma \in (0, \sigma_0) \), and only give a brief explanation for the case \( \sigma = \sigma_0 \), if necessary.
3.1 The comparison principle

After the extension above, the mixed boundary value problem (2.31) is reduced to a Dirichlet problem as follows:

\[
\begin{aligned}
\text{Equation (2.27) in } \Omega_{\text{ext}}, \\
\psi = \sqrt{1 + |\xi|^2} \quad \text{on } \partial\Omega_{\text{ext}},
\end{aligned}
\] (3.1)

Now, our purpose is to seek a solution to the problem (3.1) with \(\psi \geq \sqrt{1 + |\xi|^2}\) in \(\Omega_{\text{ext}}\). The main difficulty in solving this problem is that the type of the equation (2.27) in \(\Omega_{\text{ext}}\) cannot be determined in advance. Thus, we expect to find a sufficient condition to ensure the uniform ellipticity of (2.27). It can be verified that the equation (2.27) is uniformly elliptic if and only if there exists a positive number \(\varepsilon_0 > 0\) such that

\[
L^2 < 1 - \varepsilon_0,
\]

where \(L^2\) is defined by (2.29).

Set

\[
w := \frac{\psi}{\sqrt{1 + |\xi|^2}}.
\] (3.2)

A simple computation gives

\[
D\psi = \frac{1}{\sqrt{1 + |\xi|^2}} (w\xi + (1 + |\xi|^2)Dw),
\] (3.3)

\[
a^2 = w^2 - 1 + (1 + |\xi|^2)(|Dw|^2 + |Dw \cdot \xi|^2).
\] (3.4)

Since the domain \(\Omega_{\text{ext}}\) is bounded, we substitute (2.29) and (3.3)–(3.4) into the relation \(L^2 < 1 - \varepsilon_0\) to obtain

\[
\frac{|Dw|^2 + |Dw \cdot \xi|^2}{w^2 - 1} < C(\varepsilon_0),
\] (3.5)

where the constant \(C(\varepsilon_0)\) depends only on \(\varepsilon_0\). It follows from (3.2) and (3.5) that if there exist a bounded constant \(C\) and a positive number \(\varepsilon_0 > 0\) such that \(\psi\) satisfies

\[
\sqrt{1 + |\xi|^2} + \varepsilon_0 \leq \psi < C, \\
|D\psi| < C,
\] (3.6) (3.7)

then the equation (2.27) is uniformly elliptic.

To establish the estimates (3.6)–(3.7), as well as the uniqueness of the solutions of the problem (3.1), we may expect that the equation (2.27) satisfies a comparison principle, but this is not easy to verify directly. Fortunately, we can proceed by employing the auxiliary function \(w\). By (2.27) and (3.2), the equation for \(w\) has the form

\[
\begin{aligned}
a^2(\Delta w + D^2w[\xi, \xi]) - (1 + |\xi|^2)D^2w[\xi]Dw + (Dw \cdot \xi)\xi, Dw + (Dw \cdot \xi)\xi \\
+ 2(w^2 - 1)Dw \cdot \xi + \frac{w(a^2 + w^2 - 1)}{1 + |\xi|^2} = 0.
\end{aligned}
\] (3.8)

For notational simplicity, we denote by \(N_1w\) the left-hand side of (3.8). Now, we prove that the equation (3.8) satisfies the following comparison principle.

**Lemma 3.1.** Let \(\Omega_D \subset \mathbb{R}^2\) be an open bounded domain. Also, let \(w_+ \in C^0(\Omega_D) \cap C^2(\Omega_D)\) satisfy \(w_+ > 1\) in \(\Omega_D\). Assume that the operator \(N_1\) is locally uniformly elliptic with respect to either \(w_+\) or \(w_-\), and it holds that

\[
N_1w_- \geq 0, \quad N_1w_+ \leq 0 \quad \text{in } \Omega_D
\]

with \(w_- \leq w_+\) on \(\partial\Omega_D\). Then it follows that \(w_- \leq w_+\) in \(\Omega_D\).

**Proof.** Owing to \(w_+ > 1\) in \(\Omega_D\), we introduce a function \(z > 0\) defined implicitly by

\[
w(\xi) = \cosh z(\xi).
\] (3.9)
Since
\[ Dw = \sinh z Dz, \quad Dw \cdot \xi = \sinh z Dz \cdot \xi, \]
\[ a^2 = \sinh^2 z (1 + (1 + |\xi|^2)(|Dz|^2 + |Dz \cdot \xi|^2)), \]
(3.10)
it follows from (3.8) that
\[
\frac{a^2}{\sinh^2 z} (\Delta z + D^2 z[\xi, \xi]) - (1 + |\xi|^2)D^2 z[Dz + (Dz \cdot \xi)\xi, Dz + (Dz \cdot \xi)\xi] 
+ 2 \left( Dz \cdot \xi + \frac{a^2}{1 + |\xi|^2 \sinh^2 z \tanh z} \right) = 0.
\]
(3.12)

We may as well assume that the operator \( N_1 \) is locally uniformly elliptic with respect to \( w_+ \). Then we see from (3.5) and (3.9) that the operator defined by the left-hand side of (3.12) is also locally uniformly elliptic with respect to the corresponding \( z_+ \). In addition, we notice from (3.11) that the function \( a^2/\sinh^2 z \) is independent of \( z \), which means that the principal coefficients of (3.12) depend only on \( Dz \) and \( \xi \). Also, the lower-order term of (3.12) is non-increasing in \( z \) for each \((\xi, Dz) \in \Omega_D \times \mathbb{R}^2\).

By [18, Theorem 10.1], such an equation satisfies the comparison principle in the sense that if \( z_- \) is a super-solution and \( z_+ \) is a sub-solution of the equation (3.12) with \( z_- \leq z_+ \) on the boundary \( \partial \Omega_D \), then it holds that \( z_- \leq z_+ \) everywhere in the domain \( \Omega_D \). Since the function \( \cosh z \) is increasing in \((0, +\infty)\), the equation (3.8) also satisfies the comparison principle. This completes the proof.

**Remark 3.2.** From (3.2) and the boundedness of \( \Omega_{\text{ext}}, \) we know that the equation (3.8) has the same type as the equation (2.27), and the solutions of these two problems have the same regularity. So, under the assumption that \( \psi \geq \sqrt{1 + |\xi|^2} \) in \( \Omega_{\text{ext}} \), if we can prove \( \psi > \sqrt{1 + |\xi|^2} \) in \( \Omega_{\text{ext}} \) and the equation (2.27) is locally uniformly elliptic in \( \Omega_{\text{ext}} \), then the uniqueness of the solution in \( C^0(\Omega_{\text{ext}}) \cap C^2(\Omega_{\text{ext}}) \) to the problem (3.1) can be obtained from Lemma 3.1 immediately.

### 3.2 The strategy of the proof

In this subsection, we will give the strategy of the proof for the existence of the solution to the problem (3.1). Inspired by the work in [26,27] and the estimates (3.6)–(3.7), we consider the following Dirichlet problem:

\[
\mathcal{F}(\mu, \psi) := a^2(\Delta \psi + D^2 \psi[\xi, \xi]) - \mu D^2 \psi[D\psi - \chi \xi, D\psi - \chi \xi] = 0 \quad \text{in} \ \Omega_{\text{ext}}
\]
(3.13)
with
\[
\psi = \sqrt{1 + |\xi|^2} + \varepsilon \quad \text{on} \ \partial \Omega_{\text{ext}},
\]
(3.14)
where \( \mu \in [0, 1] \) and \( \varepsilon > 0 \) are parameters, and \( \chi \) is given by (2.28). Obviously, the equation (3.13) can be rearranged in the form

\[
\mathcal{A}(\mu; \xi, \psi, D\psi) : D^2 \psi := \sum_{i,j=1}^{2} A_{ij} \partial_{ij} \psi = 0,
\]
(3.15)
where \( \mathcal{A}(\mu; \xi, \psi, D\psi) \) has its expression as shown in (3.13).

We find that when \( \mu = 0 \), the equation (3.13) is reduced to a linear elliptic equation. Such a property motivates us to solve the problem (3.13)–(3.14) by the method of continuity. To apply this approach, we analyze the equation (3.13) in the same way as we analyze the equation (2.27) in Subsection 3.1.

Denote by \( \psi_{\mu, \varepsilon} \) a solution to the problem (3.13)–(3.14), and define

\[
L_{\mu}^2 := \mu(D\psi)^2 + |\psi - D\psi \cdot \xi|^2 - \frac{\psi^2}{1 + |\xi|^2}.
\]
It can be verified that the equation (3.13) is elliptic if \( L_{\mu}^2 < 1 \), i.e.,

\[
\psi_{\mu, \varepsilon} > \sqrt{(1 + |\xi|^2)(1 + \frac{\mu - 1}{\mu} a^2)} \quad \text{for} \ \mu \in (0, 1].
\]
We deduce from this relation that if \( \psi_{\mu, \varepsilon} > \sqrt{1 + |\xi|^2} \), then the equation (3.13) is always elliptic for any \( \mu \in [0, 1] \). Moreover, using the analysis as in Subsection 3.1, we know that for any \( \mu \in [0, 1] \), the equation (3.13) is uniformly elliptic if we can find a positive number \( \varepsilon_0 > 0 \) and a bounded constant \( C \) so that

\[
\sqrt{1 + |\xi|^2} + \varepsilon_0 \leq \psi_{\mu, \varepsilon} < C, \tag{3.16}
\]

\[
|D\psi_{\mu, \varepsilon}| < C. \tag{3.17}
\]

Next, let us verify that the corresponding equation for \( w_{\mu, \varepsilon} \) also satisfies the comparison principle (i.e., Lemma 3.1), which will enable us to obtain the estimate (3.16) later. By (3.2) and (3.13), the equation for \( w_{\mu, \varepsilon} \) is

\[
a^2(\Delta w + D^2w[\xi, \xi]) - \mu(1 + |\xi|^2)D^2w[Dw + (Dw \cdot \xi)\xi, Dw + (Dw \cdot \xi)\xi] \\
+ 2((1 - \mu)a^2 + \mu(w^2 - 1))Dw \cdot \xi + (2 - \mu)a^2 + \mu(w^2 - 1)) \frac{w}{1 + |\xi|^2} = 0. \tag{3.18}
\]

For simplicity, denote by \( N_{\mu, w} \) the left-hand side of the equation (3.18). As before, we also utilize the auxiliary function \( z_{\mu, \varepsilon} \). Using (3.9), we have

\[
(1 + m(\xi, Dz))(\Delta z + D^2z[\xi, \xi]) - \mu(1 + |\xi|^2)D^2z[Dz + (Dz \cdot \xi)\xi, Dz + (Dz \cdot \xi)\xi] \\
+ 2(1 + (1 - \mu)m(\xi, Dz))Dz \cdot \xi + (2 + (1 - \mu)m(\xi, Dz)) \frac{1 + m(\xi, Dz)}{(1 + |\xi|^2) \tanh z} = 0, \tag{3.19}
\]

where \( m(\xi, Dz) \) is defined as

\[
m(\xi, Dz) := (1 + |\xi|^2)(|Dz|^2 + |Dz \cdot \xi|^2).
\]

Let us write the equation (3.19) in the form

\[
\sum_{i,j=1}^{2} a_{ij}(\mu; \xi, Dz)\partial_{ij}z + H(\mu; \xi, z, Dz) = 0. \tag{3.20}
\]

It follows from (3.19) that for any \( \mu \in [0, 1] \), the coefficients \( a_{ij} \) are independent of \( z \), and the lower-order term \( H(\mu; \xi, z, Dz) \) is non-increasing in \( z \) for each \((\xi, Dz) \in \Omega_{\text{ext}} \times \mathbb{R}^2 \). Then we see that the equation (3.19) has the same form as the equation (3.12). Consequently, the conclusion in Lemma 3.1 is still valid for the equation (3.18).

Define

\[
J_\varepsilon := \{ \mu \in [0, 1] : \text{such that } \psi_{\mu, \varepsilon} \in C^0(\Omega_{\text{ext}}) \cap C^2(\Omega_{\text{ext}}) \text{ satisfies } (3.13) - (3.14) \text{ with } \psi_{\mu, \varepsilon} \geq \sqrt{1 + |\xi|^2 + \varepsilon} \text{ in } \Omega_{\text{ext}} \}. \tag{3.21}
\]

According to the above analysis, the strategy of our proof can be described as follows. For any fixed \( \varepsilon > 0 \), we first prove \( J_\varepsilon = [0, 1] \). It is required to verify that \( J_\varepsilon \) is open, closed and not empty. To this end, we will establish a priori estimates which are independent of \( \mu \) and \( \varepsilon \) in Subsection 3.3. Next, we show that the limit of \( \psi_{1, \varepsilon} \) as \( \varepsilon \to 0+ \) is exactly the solution to the problem (3.1).

### 3.3 The Lipschitz estimate

For simplicity of notation, the subscripts of \( \psi_{\mu, \varepsilon} \), \( w_{\mu, \varepsilon} \) and \( \Phi_{\mu, \varepsilon} \) will be omitted in this subsection.

Let us first derive the estimate (3.16). Since the equation (3.18) satisfies the comparison principle, it is more convenient to derive the \( L^\infty \)-estimate for \( w \), which in turn leads to the corresponding estimate for \( \psi \) directly.
Lemma 3.3. Let \( w \in C^0(\Omega_{\text{ext}}) \cap C^2(\Omega_{\text{ext}}) \) satisfy \( \mathcal{N}_\mu w = 0 \), \( w > 1 \) in \( \Omega_{\text{ext}} \) and \( w = 1 + \varepsilon \) on \( \partial \Omega_{\text{ext}} \). Then there exists a constant \( C > 0 \), independent of \( \mu \) and \( \varepsilon \) such that

\[
1 + \varepsilon < w \leq C \quad \text{in} \quad \Omega_{\text{ext}},
\]

where \( \mathcal{N}_\mu w = 0 \) denotes the equation (3.18).

Proof. The main work of the proof is to find proper sub- and super-solutions to the equation \( \mathcal{N}_\mu w = 0 \) with the boundary condition \( w|_{\partial \Omega_{\text{ext}}} = 1 + \varepsilon \). For this purpose, we first seek an exact solution to the equation \( \mathcal{N}_\mu w = 0 \). By (2.14)–(2.15) and (3.13), the corresponding \( \Phi \) is a solution of

\[
(|\nabla_x \Phi|^2 - B_\infty) \Delta_x \Phi - \mu D_x^2 \Phi\nabla_x \Phi, \nabla_x \Phi = 0.
\]

(3.23)

Notice that the equation (3.23) is independent of \( \Phi \) itself. Then for any given constant vector \( \eta \in \mathbb{R}^3 \), the linear function \( \Phi^n = \sqrt{B_\infty} \eta \cdot x \) is a solution to (3.23). From (2.14)–(2.15) and (3.2), we know that

\[
w^n(\xi) = \frac{\eta \cdot (\xi, 1)}{\sqrt{1 + |\xi|^2}}
\]

is an exact solution to the equation \( \mathcal{N}_\mu w = 0 \). A simple computation gives

\[
\partial_i w^n(\xi) = \frac{\sqrt{1 + |\xi|^2} \eta_i - w^n \xi_i}{1 + |\xi|^2} \quad \text{for} \quad i = 1, 2,
\]

(3.24)

where \( \partial_i \) stands for \( \partial_{\xi_i} \).

For any fixed \( \varepsilon > 0 \), let us define the sets

\[
\Sigma_+ := \{\eta \in \mathbb{R}^3 : w^n > 1 + \varepsilon \text{ on } \partial \Omega_{\text{ext}}\},
\]

(3.26)

\[
\Sigma_- := \{\eta \in \mathbb{R}^3 : w^n < 1 + \varepsilon \text{ on } \partial \Omega_{\text{ext}}\}.
\]

(3.27)

It follows from (3.24) that for any fixed constant vector \( \eta \in \mathbb{R}^3 \), \( w^n(\xi) \) is a decreasing function of the angle between \( \frac{(\xi_1, \xi_2, 1)}{\sqrt{1 + |\xi|^2}} \) and \( \eta \), which implies that when \( \eta \in \Sigma_+ \), the function \( w^n \) is larger than \( 1 + \varepsilon \) everywhere in \( \Omega_{\text{ext}} \). Furthermore, for any compact subdomain \( \Omega_{\text{ext}}^C \subset \Omega_{\text{ext}} \), we have

\[
w^n > 1 + \varepsilon + \delta \quad \text{in} \quad \Omega_{\text{ext}}^C,
\]

(3.28)

where the constant \( \delta > 0 \) depends only on \( \Omega_{\text{ext}}^C \).

We deduce from (3.24)–(3.25) and (3.28) that when \( \eta \in \Sigma_+ \), the operator \( \mathcal{N}_\mu \) is locally uniformly elliptic with respect to \( w^n \). Then from Lemma 3.1, it follows that \( w \leq w^n \) in \( \Omega_{\text{ext}} \). Taking \( w^+ \) as the infimum of all these \( w^n \), namely,

\[
w^+ = \inf\{w^n : \eta \in \Sigma_+\},
\]

(3.29)

we obtain \( w \leq w^+ \) in \( \Omega_{\text{ext}} \) and \( w^+ \geq 1 + \varepsilon \) on \( \partial \Omega_{\text{ext}} \). In particular, the convexity of the domain \( \Omega_{\text{ext}} \) yields

\[
w^+ = 1 + \varepsilon \quad \text{on} \quad \partial \Omega_{\text{ext}}.
\]

(3.30)

The situation is analogous for sub-solutions. Define

\[
w^- := \sup\{w^n : \eta \in \Sigma_-\}.
\]

(3.31)

Then we have \( \mathcal{N}_\mu w^- \geq 0 \) in \( \Omega_{\text{ext}} \).

We claim here that for any compact subdomain \( \Omega_{\text{ext}}^C \subset \Omega_{\text{ext}} \), there exists a positive number \( \delta > 0 \) depending only on \( \Omega_{\text{ext}}^C \), such that

\[
w^- \geq 1 + \varepsilon + \delta \quad \text{in} \quad \Omega_{\text{ext}}^C.
\]

(3.32)

For any fixed point \( \xi_0 = (\xi_{10}, \xi_{20}) \in \Omega_{\text{ext}}^C \), we can choose a suitably small constant \( 0 < \delta \ll 1 \) and a constant vector \( \eta_0 = (1 + \varepsilon + \delta) \frac{(\xi_{10}, \xi_{20}, 1)}{\sqrt{1 + |\xi|^2}} \) such that \( \eta_0 \) belongs to \( \Sigma_- \). Therefore, by (3.31) and the definition of \( w^- \), we obtain the relation (3.32).
By the continuity of the function $w^-$, (3.32) implies $w^- > 1 + \varepsilon$ in $\Omega_{\text{ext}}$ and $w^- \geq 1 + \varepsilon$ on $\partial\Omega_{\text{ext}}$. In addition, it is clear that $w^- \leq 1 + \varepsilon$ on $\partial\Omega_{\text{ext}}$. Consequently, we have
\begin{equation}
\label{3.33}
w^- = 1 + \varepsilon \quad \text{on } \partial\Omega_{\text{ext}}.
\end{equation}

It follows from (3.24)–(3.25) and (3.32) that the operator $\mathcal{N}_\mu$ is locally uniformly elliptic with respect to $w^-$. According to Lemma 3.1, we obtain $w \geq w^-$ in $\Omega_{\text{ext}}$.

We conclude from the discussion above that
\begin{equation}
\label{3.34}
1 + \varepsilon < w^- \leq w^+ \text{ in } \Omega_{\text{ext}},
\end{equation}
where $w^\pm$ satisfy the conditions (3.30) and (3.33) on $\partial\Omega_{\text{ext}}$. This completes the proof. \hfill \Box

**Remark 3.4.** It is shown from the above proof that only the equality (3.30) depends on the convexity of $\Omega_{\text{ext}}$, while (3.33) holds even for the domain $\Omega_{\text{ext}}$ being non-convex.

On account of the boundedness of $\Omega_{\text{ext}}$, we do not distinguish the small numbers $\varepsilon$ and $\sqrt{1 + |\xi|^2} \varepsilon$, and write them as $\varepsilon$ all the time. Then from (3.2) and (3.34), we obtain
\begin{equation}
\label{3.35}
\sqrt{1 + |\xi|^2} + \varepsilon < \psi^- \leq \psi \leq \psi^+ \leq C \quad \text{in } \Omega_{\text{ext}},
\end{equation}
where $\psi^\pm = \sqrt{1 + |\xi|^2}w^\pm$, and the constant $C$ is independent of $\mu$ and $\varepsilon$. Moreover, it still holds that
\begin{equation}
\label{3.36}
\psi = \psi^+ \quad \text{on } \partial\Omega_{\text{ext}}.
\end{equation}

Next, we establish the Lipschitz estimate (3.17).

**Lemma 3.5.** Let $\psi \in C^0(\overline{\Omega_{\text{ext}}}) \cap C^2(\Omega_{\text{ext}})$ satisfy the problem (3.13)–(3.14) and $\psi > \sqrt{1 + |\xi|^2}$ in $\Omega_{\text{ext}}$. Then there exists a constant $C > 0$, independent of $\mu$ and $\varepsilon$ such that
\begin{equation}
\label{3.37}
\|D\psi\|_{L^\infty(\Omega_{\text{ext}})} \leq C.
\end{equation}

**Proof.** First, we consider the estimate on the boundary. It follows from (3.35)–(3.36) that
\begin{equation}
\label{3.38}
\|D\psi\|_{L^\infty(\partial\Omega_{\text{ext}})} \leq \|D\psi^+\|_{L^\infty(\partial\Omega_{\text{ext}})}.
\end{equation}
Also, by (3.25) and (3.29), and noting $\psi^+ = \sqrt{1 + |\xi|^2}w^+$, we know that the right-hand side above is bounded by a constant $C$ independent of $\mu$ and $\varepsilon$.

Then we turn to the interior Lipschitz estimate. For any function $\psi \in C^2(\Omega_{\text{ext}})$, since we can choose a sequence $\{\psi_l\} \subset C^3(\Omega_{\text{ext}})$ such that $\psi_l \to \psi$ in $C^2(\Omega_{\text{ext}})$ as $l \to \infty$, we may as well assume that $\psi \in C^3(\Omega_{\text{ext}})$.

For the equation (3.15), a standard calculation shows that the function $\frac{1}{2}|D\psi|^2$ satisfies
\begin{equation}
\label{3.39}
A : D^2 \left(\frac{1}{2}|D\psi|^2\right) = \sum_{k=1}^{2} (D\partial_k \psi)^T \cdot A \cdot (D\partial_k \psi) + \sum_{i,j,k=1}^{2} A_{ij} \partial_k \psi \partial_{ijk} \psi.
\end{equation}

By differentiating $A : D^2 \psi = 0$ with respect to $\xi_k$ and multiplying on its both sides by $\partial_k \psi$, we have
\begin{equation}
\label{3.40}
\sum_{i,j=1}^{2} \partial_k (A_{ij}) \partial_{ij} \psi \partial_k \psi + \sum_{i,j=1}^{2} A_{ij} \partial_k \psi \partial_{ijk} \psi = 0.
\end{equation}

Plugging (3.40) into (3.39) gives
\begin{equation}
\label{3.41}
A : D^2 \left(\frac{1}{2}|D\psi|^2\right) = \sum_{k=1}^{2} (D\partial_k \psi)^T \cdot A \cdot (D\partial_k \psi) + \sum_{k=1}^{2} h_k(\xi, \psi, D\psi) \partial_k \left(\frac{1}{2}|D\psi|^2\right),
\end{equation}
where $h_k$ is given by
\begin{equation}
\label{3.42}
h_k(\xi, \psi, D\psi) = -\sum_{i,j=1}^{2} \partial_i (A_{kj}) \frac{\partial_j \psi}{\partial \psi}.
\end{equation}
Also, it follows from (3.35) that the equation (3.15) is elliptic in $\Omega_{\text{ext}}$. Then the matrix $A$ is positive definite, i.e.,

$$
\sum_{k=1}^{2} (D\partial_k \psi)^T \cdot A \cdot (D\partial_k \psi) > 0.
$$

(3.43)

We assert here that the function $|D\psi|^2$ cannot achieve its maximum at any interior point unless it is constant. In fact, for any interior point $\xi_0 \in \Omega_{\text{ext}}$, by (3.39), (3.41) and (3.43), this conclusion is established when $D\psi(\xi_0) = 0$, or both $\partial_1 \psi(\xi_0)$ and $\partial_2 \psi(\xi_0)$ are not equal to zero. Then it suffices to discuss the case where either $\partial_1 \psi(\xi_0)$ or $\partial_2 \psi(\xi_0)$ equals zero. In either case, we can choose a rotation transformation in the $(\xi_1, \xi_2)$-plane such that $\partial_1 \psi \neq 0$ and $\partial_2 \psi \neq 0$ at the corresponding point of $\xi_0$ in the new coordinates. Also note that, since the equation (3.15) and the function $|D\psi|^2$ are rotation invariant, the form of the equation (3.41) still holds under the rotation transformation. In the new coordinates, using (3.41) and (3.43) again, we can verify this assertion. Therefore, its maximum is achieved on the boundary, and we obtain

$$
\|D\psi\|_{L^\infty(\Omega_{\text{ext}})} \lesssim \|D\psi\|_{L^\infty(\partial\Omega_{\text{ext}})}.
$$

(3.44)

Combining this estimate and (3.38), we complete the proof of the lemma. \qed

3.4 The continuation procedure

Now we are ready to solve the problem (3.1) by using the strategy mentioned in Subsection 3.2. For a fixed $\varepsilon > 0$, we first prove $J_\varepsilon = [0, 1]$, which will be divided into three steps.

Step 1. $J_\varepsilon$ is not empty. To this end, we consider

$$
\mathcal{F}(0, \psi) := \Delta \psi + D^2 \psi [\xi, \xi] = 0 \quad \text{in } \Omega_{\text{ext}}
$$

(3.45)

with the boundary condition (3.14). This is a Dirichlet problem for a linear elliptic equation, which is uniquely solvable by the Fredholm alternative. Since $\Gamma_{\text{cone}}^\infty$ is tangent to $\Gamma_{\text{cone}}^\infty$ at the point $P_1$ and both $\Gamma_{\text{cone}}^\infty$ and $\Gamma_{\text{cone}}^\infty$ are smooth, the boundary $\partial\Omega_{\text{ext}} \in C^1$. Thanks to [18, Theorems 6.13 and 6.17], we have $\psi_{0,\varepsilon} \in C^0(\Omega_{\text{ext}}) \cap C^\infty(\Omega_{\text{ext}})$. In addition, we obtain $\psi_{0,\varepsilon} \geq \sqrt{1 + |\xi|^2 + \varepsilon}$ in the domain $\Omega_{\text{ext}}$. Thus, there exists a subsequence $\mu_{k\varepsilon}$ such that the corresponding $\psi_{m_{k\varepsilon},\varepsilon}$ are convergent in $C^0(\Omega_{\text{ext}})$.

Let us denote by $\psi_{m_{k\varepsilon},\varepsilon} \in C^0(\Omega_{\text{ext}})$ the limit of $\psi_{m_{k\varepsilon},\varepsilon}$. Then we improve the regularity of $\psi_{m_{k\varepsilon},\varepsilon}$ in the domain $\Omega_{\text{ext}}$. Since $\psi_{m_{k\varepsilon},\varepsilon}$ satisfies the estimates (3.35) and (3.37), i.e., $\psi_{m_{k\varepsilon},\varepsilon}$ is uniformly bounded in $\text{Lip}(\Omega_{\text{ext}})$ with respect to $\mu \in [0, 1]$. Thus, there exist a subsequence $\mu_{m_{k\varepsilon}}$ such that the corresponding $\psi_{m_{k\varepsilon},\varepsilon}$ are convergent in $C^0(\Omega_{\text{ext}})$.

Moreover, the linearized equation

$$
A(\mu_{m_{k\varepsilon}}; \xi, \psi_{m_{k\varepsilon},\varepsilon}, D\psi_{m_{k\varepsilon},\varepsilon}) : D^2 \psi
$$

$$
= \lambda^2 (\Delta \psi + D^2 \psi [\xi, \xi]) - \mu_{m_{k\varepsilon}} D^2 \psi [D\psi_{m_{k\varepsilon},\varepsilon} - \chi \xi, D\psi_{m_{k\varepsilon},\varepsilon} - \chi \xi]
$$

(3.46)

is uniformly elliptic. For any compact subdomain $\Omega_{\text{sub}} \subset \Omega_{\text{ext}}$, using [18, Theorem 6.17], we see that $\psi_{m_{k\varepsilon},\varepsilon}$ is uniformly bounded in $C^\infty(\Omega_{\text{sub}}^\infty)$ with respect to $\mu \in [0, 1]$. Hence we can find a subsequence $\mu_{m_{k\varepsilon}}$ of the subsequence $\mu_{m_{k\varepsilon}}$ such that the corresponding $\psi_{m_{k\varepsilon},\varepsilon}$ are convergent in $C^2(\Omega_{\text{sub}}^\infty)$, which implies $\psi_{m_{k\varepsilon},\varepsilon} \in C^0(\Omega_{\text{sub}}^\infty) \cap C^2(\Omega_{\text{ext}})$.

Obviously, the limit $\psi_{m_{k\varepsilon},\varepsilon}$ is a solution to the problem (3.13)–(3.14) and $\psi_{m_{k\varepsilon},\varepsilon} \geq \sqrt{1 + |\xi|^2 + \varepsilon}$ in $\Omega_{\text{ext}}$. Then $\mu_{m_{k\varepsilon},\varepsilon} \in J_\varepsilon$, which means that $J_\varepsilon$ is closed.

Step 2. $J_\varepsilon$ is closed. Let $(\mu_{m_{k\varepsilon}}, \psi_{m_{k\varepsilon},\varepsilon})$ be a sequence such that $\mu_{m_{k\varepsilon}} \in J_\varepsilon$ and $\psi_{m_{k\varepsilon},\varepsilon}$ is the solution of the corresponding problem. From (3.21) and Lemmas 3.3 and 3.5, we know that $\psi_{m_{k\varepsilon},\varepsilon}$ satisfies the estimates (3.35) and (3.37), i.e., $\psi_{m_{k\varepsilon},\varepsilon}$ is uniformly bounded in $\text{Lip}(\Omega_{\text{ext}})$ with respect to $\mu \in [0, 1]$. Thus, there exists a subsequence $\mu_{m_{k\varepsilon}}$ such that the corresponding $\psi_{m_{k\varepsilon},\varepsilon}$ are convergent in $C^0(\Omega_{\text{ext}})$.

Let us denote by $\psi_{m_{k\varepsilon},\varepsilon} \in C^0(\Omega_{\text{ext}})$ the limit of $\psi_{m_{k\varepsilon},\varepsilon}$. Then we improve the regularity of $\psi_{m_{k\varepsilon},\varepsilon}$ in the domain $\Omega_{\text{ext}}$. Since $\psi_{m_{k\varepsilon},\varepsilon}$ satisfies the estimates (3.35) and (3.37), we have $\psi_{m_{k\varepsilon},\varepsilon} \in \text{Lip}(\Omega_{\text{ext}}) \cap C^2(\Omega_{\text{ext}})$. Moreover, the linearized equation

$$
\sum_{i,j=1}^{2} a_{ij} (\mu_0; \xi, D\psi_{\mu_0,\varepsilon}) \partial_i \partial_j \varepsilon + b_i (\mu_0; \xi, D\psi_{\mu_0,\varepsilon}) \partial_i \varepsilon + d (\mu_0; \xi, D\psi_{\mu_0,\varepsilon}) \varepsilon = f,
$$

(3.47)
sections. Let $W^2$ denote such a wing, in which the angle of the apex is $\pi - 2\sigma$ and the angle between the upper and lower root chords is $|2\theta|$, where $\sigma \in (0, \pi/2)$ and $\theta \in (-\pi/2, 0)$. As in Section 1, we place $W^\theta$ in the rectangular coordinates $(x_1, x_2, x_3)$ such that it is symmetric about the $x_1 Ox_2$-plane and the $x_2 Ox_3$-plane with the apex at the origin (see Figure 5), namely,

$$W^\theta = \{(x_1, x_2, x_3) : |x_2| < x_3 \cot \sigma, |x_1| < (x_2 \tan \sigma - x_3) \tan \theta, x_3 > 0\}. \quad (4.1)$$

Obviously, when $\theta = 0$ it becomes a triangular plate, which is the case discussed in the previous sections. Moreover, we assume the oncoming flow as in Section 1, and still use the notations as defined in Sections 1 and 2 except the boundary $\Gamma_{wing}$, which denotes the upper surface of $W^\theta$ in this section.
Without loss of generality, we only discuss the case of attached shocks. Write

\[ R_\sigma := \{ s(x_2, x_3) < x_1 < (x_3 - x_2 \tan \sigma) \tan \theta, x_2 > 0 \}, \]

where \( x_1 = s(x_2, x_3) \) is the equation for the shock attached to the leading edges. In view of the symmetry of the wing \( W_\theta \sigma \), it suffices to consider the problem in the region \( R_\sigma \). Clearly, the potential function \( \Phi \) satisfies (1.3)–(1.4) in \( R_\sigma \) with the boundary condition (1.10) and

\[ \nabla x \Phi \cdot n'_w = 0 \text{ on } \{ x_1 = (x_3 - x_2 \tan \sigma) \tan \theta, x_2 > 0 \}, \]

where \( n'_w = (1, \tan \theta \tan \sigma, -\tan \theta) \) is the exterior normal to the upper surface of \( W_\theta \sigma \). By the continuity of \( \Phi \), we also have (1.8) on \( S_\sigma \). We point out here that the equation (2.1) is still hyperbolic in \( R_\sigma \), and thus there exists a Mach cone of the apex of the wing.

Proceeding as in the analysis of Section 2, we can determine the location of the shock and the uniform flow state outside the Mach cone. The only difference is that at this point the oncoming flow should satisfy the condition

\[ c_\infty < \tilde{q}_\infty < \frac{c_\infty}{\sin(\alpha_n - \theta_n)} \]

instead of (2.5), where \( \theta_n = \arctan(\tan \theta / \cos \sigma) \) denotes the angle between the upper surface of \( W_\theta \sigma \) and the \( x_2Ox_3 \)-plane. With the relations

\[ \tilde{q}_\infty \sin \alpha_n = q_\infty \sin \alpha \quad \text{and} \quad \tilde{q}_\infty \cos \alpha_n = q_\infty \cos \alpha \cos \sigma, \]

the right-hand side of (4.3) can be reduced to

\[ q_\infty \sin(\alpha - \theta_n) + q_\infty \cos \alpha \sin \theta_n (1 - \cos \sigma) < c_\infty. \]

Substituting \( \sigma = 0 \) into (4.4), we have

\[ \alpha < \arcsin \left( \frac{c_\infty}{q_\infty} \right) + \theta = \alpha_0 + \theta. \]

This implies that our model is valid only for \( \theta > -\alpha_0 \).

Next, we focus on the flow state inside the Mach cone. Unlike that for the triangular plate, if the scaling transformation (2.14) is carried out directly, then there is an oblique derivative condition on \( \Gamma_{\text{wing}} \). Since the type of the equation (2.27) is \textit{a priori} unknown, it is quite involved to establish the Lipschitz estimate for the solution in this case. To overcome the difficulty, we will first make a rotation of coordinates before performing a scaling transformation, as shown in (4.5)–(4.6). Accordingly, the oblique derivative condition is reduced to a Neumann condition on \( \Gamma_{\text{wing}} \). Later, we will develop our method to treat the problem (3.13) with a Neumann condition on \( \Gamma_{\text{wing}} \cup \Gamma_{\text{sym}} \) and a Dirichlet condition on \( \Gamma_{\text{cone}} \cup \Gamma_{\sigma, \text{cone}} \) in a Lipschitz domain. It is worth pointing out that the root chord of the wing \( W_\theta \sigma \) corresponds to a corner point in the \( (\xi_1, \xi_3) \)-plane (see Figure 6), which needs more careful examination.

We consider a rotation transformation as follows:

\[ (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (x_1 \cos \theta - x_3 \sin \theta, x_2, x_1 \sin \theta + x_3 \cos \theta). \]
Then the corresponding scaling is given by
\[
(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \rightarrow (\tilde{\xi}_1, \tilde{\xi}_2) := \left( \frac{\tilde{x}_1}{\tilde{x}_3}, \frac{\tilde{x}_2}{\tilde{x}_3} \right), \quad (\rho, \Phi) \rightarrow (\rho, \tilde{\phi}) := \left( \rho, \frac{\Phi}{\tilde{x}_3} \right),
\]
where \((\tilde{\xi}_1, \tilde{\xi}_2)\) are the new conical coordinates. Similarly, we introduce \(\tilde{\psi} := \tilde{\phi} / \sqrt{B_{\infty}}\), where \(B_{\infty}\) is defined by (1.5). Since the equation (1.3) does not change under the rotation transformation (4.5), this equation can be expressed in terms of \(\tilde{\psi}\) as
\[
\text{div} \left( \rho D_{\tilde{\xi}} \tilde{\psi} - (\tilde{\psi} - D_{\tilde{\xi}} \tilde{\psi} \cdot \tilde{\xi}) \tilde{\xi} \right) + 2\rho(\tilde{\psi} - D_{\tilde{\xi}} \tilde{\psi} \cdot \tilde{\xi}) = 0,
\]
where \(\tilde{\xi} := (\tilde{\xi}_1, \tilde{\xi}_2)\), and \(\text{div}\tilde{\xi}\) and \(D_{\tilde{\xi}}\) stand for the divergence and gradient operators with respect to \(\tilde{\xi}\), respectively. Then the problem for \(\tilde{\psi}\) is
\[
\begin{cases}
\text{Equation (4.7)} & \text{in } \Omega, \\
\tilde{\psi} = \sqrt{1 + |\tilde{\xi}|^2} & \text{on } \Gamma_{\text{cone}} \cup \Gamma_{\text{cone}}, \\
D_{\tilde{\xi}} \tilde{\psi} \cdot \tilde{\nu}_w = 0 & \text{on } \Gamma_{\text{wing}}, \\
D_{\tilde{\xi}} \tilde{\psi} \cdot \tilde{\nu}_{sy} = 0 & \text{on } \Gamma_{\text{sym}},
\end{cases}
\]
where \(\tilde{\nu}_w := (\sin \theta \tan \sigma)\) and \(\tilde{\nu}_{sy} := (0, -1)\) are the exterior normals to \(\Gamma_{\text{wing}}\) and \(\Gamma_{\text{sym}}\), respectively.

Hereafter, we will work in these new coordinates and drop “\(\sim\)” for simplification.

We also use the strategy mentioned in Subsection 3.2 to prove the existence of the solution to the problem (4.8). Let us consider the following boundary value problem:
\[
\begin{cases}
\text{Equation (3.13)} & \text{in } \Omega, \\
\psi = \sqrt{1 + |\xi|^2} + \varepsilon & \text{on } \Gamma_{\text{cone}} \cup \Gamma_{\text{cone}}, \\
D\psi \cdot \nu_w = 0 & \text{on } \Gamma_{\text{wing}}, \\
D\psi \cdot \nu_{sy} = 0 & \text{on } \Gamma_{\text{sym}},
\end{cases}
\]
Still, denote by \(\psi_{\mu, \varepsilon}\) a solution to the problem (4.9). In addition, for any open bounded domain \(\hat{\Omega} \subset \mathbb{R}^2\), we introduce the operator
\[
Mw := Dw \cdot \nu + bw \quad \text{in } \partial \hat{\Omega},
\]
where \(\nu\) is the exterior normal to \(\partial \hat{\Omega}\), and \(b \geq 0\) is a function of \(\xi\). As before, we first show a comparison principle.
Lemma 4.1. Let $\Omega_D \subset \mathbb{R}^2$ be an open bounded domain with its boundary composed of $\partial^1 \Omega_D$ and $\partial^2 \Omega_D$, and let $\nu$ be the exterior normal to $\partial^2 \Omega_D$. Also, let $w_\pm \in C^0(\Omega_D) \cap C^1(\Omega_D \setminus \partial^2 \Omega_D) \cap C^2(\Omega_D)$ satisfy $w_\pm > 1$ in $\Omega_D$. Assume that for any $\mu \in [0, 1]$, the operator $N_\mu$ is locally uniformly elliptic with respect to either $w_+$ or $w_-$, and it holds that

$$N_\mu w_- \geq 0, \quad N_\mu w_+ \leq 0 \text{ in } \Omega_D$$

with $w_- \leq w_+$ on $\partial^1 \Omega_D$ and $Mw_- < Mw_+$ on $\partial^2 \Omega_D$, where $N_\mu w = 0$ denotes the equation (3.18). Then it follows that $w_- \leq w_+$ in $\Omega_D$.

Proof. Setting

$$\bar{w} := w_- - w_+,$$

by Lemma 3.1 and the analysis in Subsection 3.2, we have

$$\sup_{\Omega_D} \bar{w} \leq \sup_{\partial \Omega_D} \bar{w}.$$

It remains to prove that $\bar{w}$ achieves its maximum on $\partial^2 \Omega_D$. If this conclusion was false, then $\bar{w}$ could achieve its maximum at a point $P$ on $\partial^2 \Omega_D$; namely, there is $D\bar{w} \cdot \nu|_P > 0$. Since $M\bar{w} < 0$ and $b > 0$ on $\partial^2 \Omega_D$, we have $\bar{w}|_P < 0$. This leads to a contradiction. Hence we have $\bar{w} \leq 0$ in $\Omega_D$. The proof is completed.

Next, we construct the sub- and super-solutions to the problem (4.10). For the super-solution, let us define two domains $\Omega_{\sigma}$ and $\Omega_{\nu}$ as follows:

$$\Omega_{\sigma} = \{ \arctan(\cot \theta \tan \theta) < \theta < \pi \},$$

$$\Omega_{\nu} = \{ \arctan(\cot \theta \tan \theta) + \pi < \theta < 2\pi \},$$

where $\bar{w} = \arctan(\xi_2/\xi_1) \in [0, 2\pi)$. For simplicity, the subscripts of $\psi_{\mu, \varepsilon} w_{\mu, \varepsilon}$ will be omitted in the proofs of Lemmas 4.2 and 4.3 below.

Lemma 4.2. Let $w \in C^0(\Omega) \cap C^1(\Omega \setminus \Gamma_{\cap \Gamma_{\sigma}}) \cap C^2(\Omega \setminus \Gamma_{\sigma})$ be a solution of the problem (4.10) with $w > 1$ in $\Omega$. Then there exists a constant $C$ independent of $\mu$ and $\varepsilon$ such that

$$1 + \varepsilon < w \leq C \text{ in } \Omega \setminus \Gamma_{\cap \Gamma_{\sigma}}.$$

Proof. We first analyze the function $w_0$, which is an exact solution to the equation $N_\mu w = 0$. From (3.25), we have

$$Dw_0(\xi) \cdot w_0 = \frac{\hat{\eta} \cdot w_0}{\sqrt{1 + |\xi|^2}}, \quad Dw_0(\xi) \cdot w_\sigma = \frac{\hat{\eta} \cdot w_\sigma}{\sqrt{1 + |\xi|^2}},$$

where $\hat{\eta} = (\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3)$. Let $\hat{O} = (\xi_1, \xi_3)$ denote the corresponding coordinates of $\hat{\eta}$ with $\eta_3 = 1$. Note that $\arctan(\cot \theta \tan \theta)$ is the included angle of the line $P_1 P_2$ and the $\xi_3$-axis (see Figure 6). Then we infer from (4.13) that when $O \hat{\in} \Omega_1$, both $Dw_0 \cdot w_0$ on $\Gamma_{\sigma}$ and $Dw_0 \cdot w_\sigma$ on $\Gamma_{\sigma}$ are negative; when $O \hat{\in} \Omega_2$, however, both of them are positive.

Next, we construct the sub- and super-solutions to the problem (4.10). For the super-solution, let us consider the set with a fixed $\varepsilon > 0$:

$$\Sigma^\varepsilon := \{ \eta \in \mathbb{R}^3 : O_\eta \in \Omega_2 \text{ and } w_\eta > 1 + \varepsilon \text{ on } \Gamma_{\cap \Gamma_{\sigma}} \}. $$

(4.14)
When \( \eta \in \Sigma^\theta_+ \), the function \( w^\eta \) satisfies

\[
\begin{cases}
N_\mu w^\eta = 0 & \text{in } \Omega, \\
w^\eta > 1 + \varepsilon & \text{on } \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{cone}}^\sigma, \\
Dw^\eta \cdot \nu_w > 0 & \text{on } \Gamma_{\text{wing}}, \\
Dw^\eta \cdot \nu_{\text{sym}} > 0 & \text{on } \Gamma_{\text{sym}}.
\end{cases}
\]

Recall that for any fixed constant vector \( \eta \in \mathbb{R}^3 \), the function \( w^\eta \) decreases with respect to the angle between \( \frac{(\xi_1, \xi_2, 1)}{\sqrt{1 + |\xi|^2}} \) and \( \eta \). Then from (3.24)–(3.25) and (4.14), it follows that the operator \( N_\mu \) is locally uniformly elliptic with respect to \( w^\eta \). Thanks to Lemma 4.1, we have \( w \leq w^\eta \) in \( \Omega \). Let \( w^+ \) be the infimum of all these \( w^\eta \)'s. Then \( w \leq w^+ \) in \( \Omega \). Moreover, owing to the convexity of \( \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{cone}}^\sigma \), we know that \( w^+ = 1 + \varepsilon \) on this boundary.

For the sub-solution, we consider the set with a fixed \( \varepsilon > 0 \):

\[
\Sigma^\theta_- := \{ \eta \in \mathbb{R}^3 : w^\eta < 1 + \varepsilon \text{ on } \partial \Omega \},
\]

and define

\[
w^- := \sup \{ w^\eta : \eta \in \Sigma^\theta_- \},
\]

where \( \partial \Omega := \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{cone}}^\sigma \cup \Gamma_{\text{wing}} \cup \Gamma_{\text{sym}} \). Obviously, we have \( N_\mu w^- \geq 0 \) in \( \Omega \).

From (3.27) and (4.15), we find that the set \( \Sigma^\theta_- \) is the same as the set \( \Sigma_- \) by replacing the boundary \( \partial \Omega \) by \( \partial \Omega_{\text{ext}} \). Then proceeding as in the argument of Lemma 3.3, we have \( w^- > 1 + \varepsilon \) in \( \Omega \) and \( w^- = 1 + \varepsilon \) on \( \partial \Omega \). Moreover, for any compact subdomain \( \Omega_{\text{sub}} \subset \Omega \), there is

\[
w^- > 1 + \varepsilon + \delta \text{ in } \Omega_{\text{sub}},
\]

with the constant \( \delta \) only depending on \( \Omega_{\text{sub}} \). This means that the operator \( N_\mu \) is locally uniformly elliptic with respect to \( w^- \). Notice that the function \( w^- \) is less than or equal to \( 1 + \varepsilon \) outside \( \Omega \). Then it holds that \( Dw^- \cdot \nu_w < 0 \) on \( \Gamma_{\text{wing}} \) and \( Dw^- \cdot \nu_{\text{sym}} < 0 \) on \( \Gamma_{\text{sym}} \). From Lemma 4.1, we know

\[
w \geq w^- > 1 + \varepsilon \text{ in } \Omega.
\]

For the point \( P_3 \) and the interior points of \( \Gamma_{\text{wing}} \) and \( \Gamma_{\text{sym}} \), we only have the estimate \( w \geq w^- = 1 + \varepsilon \).

When \( \varepsilon = 0 \), this estimate cannot deduce that the equation (2.27) is uniformly elliptic in the corresponding domains. Hence we need to improve the estimate for these points. To this end, let us consider the set with a fixed \( \varepsilon > 0 \):

\[
\tilde{\Sigma}^\theta_- := \{ \eta \in \mathbb{R}^3 : \Omega^\eta \in \Lambda_1 \text{ and } w^\eta < 1 + \varepsilon \text{ on } \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{cone}}^\sigma \}.
\]

Obviously, when \( \eta \in \tilde{\Sigma}^\theta_- \), the function \( w^\eta \) satisfies

\[
\begin{cases}
N_\mu w^\eta = 0 & \text{in } \Omega, \\
w^\eta < 1 + \varepsilon & \text{on } \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{cone}}^\sigma, \\
Dw^\eta \cdot \nu_w < 0 & \text{on } \Gamma_{\text{wing}}, \\
Dw^\eta \cdot \nu_{\text{sym}} < 0 & \text{on } \Gamma_{\text{sym}}.
\end{cases}
\]

It follows from (4.16) and (4.17) that the operator \( N_\mu \) is locally uniformly elliptic with respect to \( w^\eta \). Therefore, from Lemma 3.1 or Lemma 4.1, it follows that if \( \eta \in \tilde{\Sigma}^\theta_- \), then \( w^\eta \) is a sub-solution to the problem (4.10) in the subdomain \( \Omega_{\text{sub}} \), where \( w^\eta \) is larger than 1. Define \( \tilde{w}^- \) as the supremum of all these \( w^\eta \). Then \( w \geq \tilde{w}^- \) in \( \Omega \). Using the argument as in the proof of (3.32), we can find a constant \( \delta > 0 \) independent of \( \mu \) and \( \varepsilon \) such that \( \tilde{w}^- \geq 1 + \varepsilon + \delta \) at the point \( P_3 \) and the interior points of \( \Gamma_{\text{wing}} \) and \( \Gamma_{\text{sym}} \). Thus, from the discussion above, we have

\[
\begin{align*}
1 + \varepsilon < \tilde{w}^- & \leq w^+ \quad \text{in } \Omega \setminus \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{cone}}^\sigma, \\
w = w^+ = 1 + \varepsilon & \quad \text{on } \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{cone}}^\sigma.
\end{align*}
\]

This completes the proof. \( \square \)
From the estimates (4.18) and (4.19), it follows that
\[
\sqrt{1 + \|\xi\|^2} + \epsilon < \psi \leq C \quad \text{in} \; \Omega \setminus \Gamma_{\text{cone}} \cup \Gamma_{\text{sym}}^\sigma,
\]
\[
\psi = \psi_{\pm} \quad \text{on} \; \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{sym}}^\sigma,
\]
where \(\psi_{\pm} = \sqrt{1 + \|\xi\|^2} \pm \epsilon\), and \(C\) is a constant independent of \(\mu\) and \(\epsilon\).

**Lemma 4.3.** Let \(\psi \in C^0(\Omega) \cap C^1(\Omega \setminus \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{sym}}^\sigma) \cap C^2(\Omega \cup \Gamma_{\text{sym}} \cup \Gamma_{\text{wing}})\) satisfy the problem (4.9) with \(\psi > \sqrt{1 + \|\xi\|^2}\) in \(\Omega\). Then there exists a constant \(C > 0\) independent of \(\mu\) and \(\epsilon\) such that
\[
\|D\psi\|_{L^\infty(\Omega)} \leq C.
\]

**Proof.** From the discussion of interior Lipschitz estimates in Lemma 3.5, we have
\[
\|D\psi\|_{L^\infty(\Omega)} \leq \|D\psi\|_{L^\infty(\partial \Omega)}.
\]
Since the problem (4.9) is invariant under a rotation transformation and is also of reflection symmetry with respect to the straight boundaries \(\Gamma_{\text{wing}}\) and \(\Gamma_{\text{sym}}\), it follows that any point on \(\Gamma_{\text{wing}}\) and \(\Gamma_{\text{sym}}\) can be treated as an interior point of the domain by an even extension. In addition, we find that the normal vectors of \(\Gamma_{\text{wing}}\) and \(\Gamma_{\text{sym}}\) are different. Then \(Dw = 0\) at the point \(P_3\) follows from the boundary conditions \(Dw \cdot \nu_{x_y} = 0\) on \(\Gamma_{\text{sym}}\) and \(Dw \cdot \nu_{w} = 0\) on \(\Gamma_{\text{wing}}\). Consequently, the estimate (4.23) is reduced to
\[
\|D\psi\|_{L^\infty(\Omega)} \leq \|D\psi\|_{L^\infty(\Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{sym}}^\sigma)}.
\]
Using the argument as in Lemma 3.5, along with (4.20) and (4.21), we obtain the boundedness of \(\|D\psi\|_{L^\infty(\Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{sym}}^\sigma)}\). Therefore, this lemma is proved.

According to Lemmas 4.2 and 4.3, we now define
\[
J_{\epsilon}^0 := \{\mu \in [0, 1] : \text{such that} \; \psi_{\mu, \epsilon} \text{ satisfies (4.9) with} \; \psi_{\mu, \epsilon} \geq \sqrt{1 + \|\xi\|^2} + \epsilon \text{ in} \; \Omega \; \text{and}
\psi_{\mu, \epsilon} \in C^0(\Omega) \cap C^1(\Omega \setminus \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{sym}}^\sigma) \cap C^2(\Omega \cup \Gamma_{\text{sym}} \cup \Gamma_{\text{wing}})\}.
\]
For any fixed \(\epsilon > 0\), we prove that the set \(J_{\epsilon}^0\) is closed as follows. Let \((\mu_n, \psi_{m, \epsilon})\) be a sequence such that \(\mu_n \in J_{\epsilon}^0\) and \(\psi_{m, \epsilon}\) is the solution of the corresponding problem. By Lemmas 4.2 and 4.3, the function \(\psi_{m, \epsilon}\) satisfies the estimates (4.20) and (4.22). Hence the corresponding linearized equation (3.46) is uniformly elliptic and \(\psi_{m, \epsilon} \in \text{Lip}(\Omega) \cap C^1(\Omega \setminus \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{sym}}^\sigma) \cap C^2(\Omega \cup \Gamma_{\text{sym}} \cup \Gamma_{\text{wing}})\). Obviously, there exists a subsequence \(\mu_{n_k}\) such that \(\psi_{m_{n_k, \epsilon}} \to \psi_{m, \epsilon}\) in \(C^0(\Omega_{\text{ext}})\). The crucial point here is to improve the regularity of \(\psi_{m, \epsilon}\) at the corner point \(P_3\). To this end, we apply [23, Lemma 1.3] to the neighborhood of the point \(P_3\). Then there exists a positive constant \(\kappa = \kappa(\theta, \sigma) \in (0, 1)\) such that \(|\psi_{m_{n_k, \epsilon}}|^{1-\kappa} \leq C\), where the constant \(C\) only depends on \(\Omega\). For the norm \(|\cdot|_{2-1-\kappa}\), we refer the reader to [17, 23]. Owing to \(|\psi_{m_{n_k, \epsilon}}|_{2-1-\kappa} \leq C\), the regularity of \(\psi_{m_{n_k, \epsilon}}\) at the point \(P_3\) is \(C^{1, \kappa}\) for \(\kappa \in (0, 1)\). Then by the Arzela-Ascoli theorem, there is \(\psi_{m, \epsilon} \in C^0(\Omega) \cap C^1(\Omega \setminus \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{sym}}^\sigma)\).

It remains to verify that [23, Lemma 1.3] is valid for the problem (3.46) with the boundary conditions as in (4.9). When \(\mu_{n_k} \in J_{\epsilon}^0\), the coefficients of the equation (3.46) belong to \(C^0(\Omega \setminus \Gamma_{\text{cone}}^\infty \cup \Gamma_{\text{sym}}^\sigma) \cap C^1(\Omega \cup \Gamma_{\text{sym}} \cup \Gamma_{\text{wing}})\). In addition, this equation does not have lower-order terms, so [23, (1.4a)–(1.5d)] are naturally satisfied. Moreover, since the equation (3.46) is uniformly elliptic and the angle at \(P_3\) is equal to \(\pi - \arctan(\cot \sigma / \tan \theta)\), the required conditions in [23, (1.6a)–(1.7)] also hold.

Furthermore, as mentioned above, any point on \(\Gamma_{\text{wing}} \cup \Gamma_{\text{sym}}\) can be treated as an interior point of the domain. Then proceeding as in the analysis of Step 2 in Subsection 3.4, we can prove that \(J_{\epsilon}^0\) is closed.

The remaining part of the proof can be completed in much the same way as in Subsection 3.4, so we just omit the details.

We summarize this section by stating the following theorem.

**Theorem 4.4.** Assume that the state \((\rho_{\infty}, q_{\infty})\) of the oncoming flow is uniform and supersonic, and the thin wing \(W_{\sigma}^0\) is defined by (4.1). Then we can find a critical angle \(\alpha_0 = \alpha_0(\rho_{\infty}, q_{\infty}) \in (0, \pi/2)\) so
that if $\theta \in (-\alpha_0, 0)$, then for any $\alpha \in (0, \alpha_0 + \theta)$, there exists $\sigma_0 = \sigma_0(\rho_\infty, q_\infty, \alpha) \in (0, \pi/2)$ such that when $\sigma \in [0, \sigma_0]$, there exists a constant $\kappa = \kappa(\theta, \sigma) \in (0, 1)$, and the problem (1.3)–(1.4) with (1.8), (1.10) and (4.2) admits a piecewise smooth solution

$$\Phi(\tilde{x}) = \sqrt{B_\infty} \tilde{x}_3 \psi\left(\frac{\tilde{x}}{\tilde{x}_3}\right)$$

in the domain $\mathcal{R}_\alpha^\sigma$ satisfying

$$\psi \in \text{Lip}(\tilde{U}) \cap C^{1, \kappa}((\tilde{U} \setminus \Gamma_{\text{cone}} \cup \Gamma_{\text{cone}}^\sigma) \cap C^\infty((\tilde{U} \setminus (\Gamma_{\text{cone}} \cup \Gamma_{\text{cone}}^\sigma \cup \{P_3\})))$$

and

$$\psi > \sqrt{1 + |\xi|^2} \quad \text{in} \quad \tilde{\Omega} \setminus \Gamma_{\text{cone}} \cup \Gamma_{\text{cone}}^\sigma.$$

Here, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ and $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)$ are given by (4.5) and (4.6), respectively, $\alpha_0$ and $\sigma_0$ are given by (2.7) and (2.8), respectively, and the constant $B_\infty$ is defined by (1.5).

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26. Popov V A. Dark energy and dark matter unification via superfluid Chaplygin gas. Phys Lett B, 2010, 686: 211–215
Appendix A  The shock polar for a Chaplygin gas

It is shown in [27] that a pressure wave between two constant states must be tangent to the sonic circles, whose centers are the velocities and radii are the sound speeds. With this property, we now characterize the shock polar for a Chaplygin gas, by studying the model of the supersonic flow around a corner.

Let us first discuss the case of shocks. We assume that a shock line \( S \) in the \((x_1, x_2)\)-plane is straight and passes through the origin \( O \), and moreover the oncoming flow state \((\rho_0, (u_0, 0))\) and the outgoing flow state \((\rho_1, (u_1, v_1))\) on the both sides are constant (see Figure 7(a)). Given \((\rho_0, (u_0, 0))\) with \( u_0 > c_0 \) and \( c_0 = \sqrt{\frac{A}{\rho_0}} > 0 \), we need to derive \((\rho_1, (u_1, v_1))\) under the condition
\[
\tan \alpha = \frac{v_1}{u_1}, \tag{A.1}
\]
where \( \beta > 0 \) is the angle between the shock line \( S \) and the velocity of the oncoming flow. To get the state \((\rho_1, (u_1, v_1))\), we first determine the position of \( O_1 \). Note that the circle \( C_1 \) is also tangent to the shock line \( S \) at the point \( P \). This can be done immediately by using (A.1) as shown in Figure 8. Then from (A.1) and (A.2), we obtain the following explicit expressions:
\[
c_1 = \frac{c_0 - \tan \alpha \sqrt{u_0^2 - c_0^2}}{c_0 \tan \alpha + \sqrt{u_0^2 - c_0^2}} \sqrt{u_0^2 - c_0^2}, \tag{A.3}
\]
\[
u_1 = \frac{u_0 \sqrt{u_0^2 - c_0^2}}{\sqrt{u_0^2 - c_0^2} + c_0 \tan \alpha}, \quad v_1 = \frac{u_0 \tan \alpha \sqrt{u_0^2 - c_0^2}}{\sqrt{u_0^2 - c_0^2} + c_0 \tan \alpha}. \tag{A.4}
\]
These equations show that for a given state \((\rho_0, (u_0, 0))\), the angle \( \alpha \) determines the state \((\rho_1, (u_1, v_1))\). Moreover, the trajectory of the point \( O_1 \) in the \((u, v)\)-plane describes the shock polar \( O_0P \) as \( \alpha \) varies.

Similarly, we assume that a rarefaction wave \( S \) in the \((x_1, x_2)\)-plane is straight and passes through the origin \( O \), and moreover the oncoming flow state \((\rho_0, (u_0, 0))\) and the outgoing flow state \((\rho_1', (u_1', v_1'))\) on the both sides are constant (see Figure 7(b)). For a given state \((\rho_0, (u_0, 0))\) with \( u_0 > c_0 \), using the argument as above, we have
\[
c_1' = \frac{c_0 - \tan \alpha' \sqrt{u_0^2 - c_0^2}}{c_0 \tan \alpha' + \sqrt{u_0^2 - c_0^2}} \sqrt{u_0^2 - c_0^2}, \tag{A.3'}
\]
\[
u_1' = \frac{u_0 \sqrt{u_0^2 - c_0^2}}{\sqrt{u_0^2 - c_0^2} + c_0 \tan \alpha'}, \quad v_1' = \frac{u_0 \tan \alpha' \sqrt{u_0^2 - c_0^2}}{\sqrt{u_0^2 - c_0^2} + c_0 \tan \alpha'},
\]
where \( \alpha' < 0 \) is the angle between the oncoming flow and the outgoing flow.

In conclusion, the shock polar for a Chaplygin gas is a half-line, extending infinitely from the tangent point \( P \) and always perpendicular to the pressure wave \( S \) (see Figure 8).

It is well known that there may occur a phenomenon of concentration or cavitation for a Chaplygin gas. Now, using the shock polar discussed above, we impose some restriction on the oncoming flow to avoid these phenomena.
From Figure 8, we see that as $\alpha \to \beta$, the sound speed $c_1 \to 0$. Then the angle $\beta$ must be greater than $\alpha$, i.e., $\sin \beta > \sin \alpha$. Using this inequality and (A.2), we have $u_0 < c_0/\sin \alpha$. This condition means that for a fixed $\alpha$, if the oncoming flow passes the wing too quickly, i.e.,

$$u_0 \geq \frac{c_0}{\sin \alpha}$$

then the flow between the shock and the wedge will concentrate at once. Such a phenomenon is called concentration (see [4]). To avoid this phenomenon, the oncoming flow should satisfy the condition

$$c_0 < u_0 < \frac{c_0}{\sin \alpha}.$$  \hspace{1cm} (A.5)

Also, Figure 8 shows that the sound speed

$$c'_1 = \sqrt{\Lambda/\rho'_1} \to +\infty$$

as $\alpha' \to \beta - \pi/2$. In other words, if the angle between the rarefaction wave and the velocity of the outgoing flow approaches $\pi/2$, then a phenomenon of cavitation occurs. Hence the angle $\alpha'$ should be greater than $\beta - \pi/2$, i.e., $\sin \beta < \cos \alpha'$. Then it follows from this condition and (A.2) that

$$u_0 > \frac{c_0}{\cos \alpha'}.$$  \hspace{1cm} (A.6)
Appendix B  Mach cones in the 3-D potential flow

For the reader’s convenience, we calculate the explicit expression of Mach cones for the three-dimensional potential equation (see also [12, Lemma 1.1]). From (2.2), we have

\[
\begin{align*}
\begin{cases}
|\nabla_x \Phi \cdot \zeta(\tau)| = c, \\
x \cdot \zeta(\tau) = 0, \\
x \cdot \zeta'(\tau) = 0,
\end{cases}
\end{align*}
\]

(B.1)

where $|\zeta(\tau)| = 1$ and $\tau \in [0, 2\pi)$. By eliminating the parameter $\tau$, we reduce (B.1) to the form

\[
\left( (q^2 - v_1^2)x_1 - v_1v_2x_2 - v_1v_3x_3 \right)^2 + \frac{q^2(v_2x_2 - v_3x_3)^2}{q^2 - c^2} = \frac{c^2(q^2 - v_1^2)}{q^2 - c^2} (v_1x_1 + v_2x_2 + v_3x_3)^2
\]

(B.2)

with $\nabla_x \Phi = (v_1, v_2, v_3)$ and $q^2 = v_1^2 + v_2^2 + v_3^2$. A tedious computation shows that the left-hand side of (B.2) can be rewritten as

\[
(q^2 - v_1^2)(q^2(x_1^2 + x_2^2 + x_3^2) - (v_1x_1 + v_2x_2 + v_3x_3)^2).
\]

This further reduces (B.2) to the form

\[
(q^2 - c^2)(x_1^2 + x_2^2 + x_3^2) = (v_1x_1 + v_2x_2 + v_3x_3)^2,
\]

or equivalently,

\[
|\nabla_x \Phi \cdot x|^2 - (|\nabla_x \Phi|^2 - c^2)|x|^2 = 0 \tag{B.3}
\]

by the definition of $\Phi$. Hence,

\[
|\nabla_x \Phi|^2 - \left| \nabla_x \Phi \cdot \frac{x}{|x|} \right|^2 = c^2 \quad \text{for} \ x \in \mathbb{R}^3 \setminus \{0\}. \tag{B.4}
\]