Ratio of Symmetries Relating Graphs Differing by an Edge Set

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Given an undirected graph $G = (V, E)$ and a subset of its edges $E' \subseteq E$, one can consider the subgraph $G' = (V, E \setminus E')$. We relate the total number of automorphisms of $G$ and of $G'$ via the automorphism orbit of $E'$. That is, we consider the automorphism orbit of $E'$ in $G$ (i.e. $AO_G(E')$) as an orbit of a set of edges and the automorphism orbit of $E'$ in $G'$ (i.e. $AO_{G'}(E')$) as an orbit of a set of non-edges. We then prove that the following ratio holds for any $G$ and any $E' \subseteq E$:

$$\frac{|\text{Aut}(G)|}{|AO_G(E')|} = \frac{|\text{Aut}(G')|}{|AO_{G'}(E')|}.$$  

The proof is relatively brief and makes use of the Erdős-Rényi graph generation model.

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I. INTRODUCTION

The automorphism groups of a graph $G$ and of the same graph with some subset of its edges $E'$ removed can be arbitrarily different. For instance, Hartke, Kolb, Nishikawa, and Stolee show this when they proved that given any two finite groups $A$ and $B$, which can be arbitrarily different, they can construct a graph $G$ which has some subgraph $G - E'$, whose automorphism groups are isomorphic to $A$ and $B$ respectively [13]. This holds for all pairs of groups. In short, this result means that merely knowing the group structure of a graph’s automorphism group (the “$A$”) tells you almost nothing about the automorphism group if an edge (or set of edges) is deleted (the “$B$”).

If instead of looking solely at the automorphism group of the graph $G$ we also take into account the way the deleted edges $E'$ relate to the rest of the graph, what then can we conclude about the automorphism group of $G - E'$?

We show that the two graphs’ automorphism groups are related by the size of the automorphism orbit of the deleted edge set $E'$. Our result applies to all finite undirected graphs and the removal of any set of edges.

In addition to the fact that this result is fascinating in its own right, we also believe it may have implications for the longstanding open problem of graph reconstruction (and the simpler but still open problem of edge reconstruction) [2–4, 8, 15, 17]. The graph reconstruction conjecture was originally formulated in 1941 by Kelly and Unam, and postulates that for any graph $G = (V, E)$, $G$ is uniquely identified by the multiset of $|V|$ subgraphs obtained by deleting a single vertex from $G$ [4]. The multiset of subgraphs is called the “deck” (as in a deck of cards). The conjecture has been described as a “disease” due to the way it infects mathematicians with interest in the problem [10].

In the reconstruction conjecture, removing a vertex involves removing both the vertex itself and the edges that were incident to the vertex. However, if one constructs an augmented deck by removing only the edges of a vertex but leaving the vertex, one gets a deck which is identical to the original deck except that each graph in the augmented deck has an extra isolated vertex. Thus one effectively has the same problem to which our theoretical result directly applies. We spell out this connection a bit more fully in our conclusion in Sec. V.

Beyond the graph reconstruction conjecture itself, there has been a fairly large amount of work on vertex-deleted subgraphs [3, 7, 14, 16, 18], and less on edge-deleted (or edge-set-deleted) subgraphs, though there is some [1]. The results of the present work are likely to be relevant to many questions of symmetry in these areas. For instance, our results provide a bound on the change in the size of the automorphism group by a factor of at most max $(1 + m, 1 + (\binom{n}{2} - m))$.

This paper begins with a preliminary introduction to the key concepts in Sec. II for a more thorough review of the formal concepts see the work of Gross, Yellen, and Anderson [9] as well as the sources cited throughout our preliminaries. Our main result is described in Sec. III and the full proof is offered in only a few pages in Sec. IV. Finally, the concluding section provides a brief discussion on possible application and extensions of the result with special attention paid to the Graph Reconstruction Conjecture.

II. PRELIMINARIES

A. Notation

Let $G = (V, E)$ be an undirected graph where $V$ is the set of vertices and $E \subseteq V \times V \setminus \{(x, x) \mid x \in V\}$ is the set of edges. We consider $(a, b) \equiv (b, a)$ due to the undirected nature of the graph.

Let $S$ be a set and let $f : S \rightarrow S$ be a bijection. Given two elements $a, b \in S$, we use $f((a, b))$ to denote $(f(a), f(b))$. Likewise, given a set $S' \subseteq S$ we use $f(S')$ to denote $\{f(x) \mid x \in S\}$. Lastly, given a set of pairs $X \subseteq S \times S$ we use $f(X)$ to denote $\{f((c, d)) \mid (c, d) \in X\}$.

Given a graph $G = (V, E)$ and an edge $e \in E$, we use “$G - e$” to denote $(V, E \setminus \{e\})$. Likewise, given $E' \subseteq E$, we use “$G - E'$” to denote $(V, E \setminus E')$.

B. Iso- and Auto-morphisms

Given an undirected graph $G = (V, E)$ and a graph $G' = (V', E')$, an isomorphism between $G$ and $G'$ is a bijection $f : V \rightarrow V'$ such that $(a, b) \in E \iff f((a, b)) \in E'$.

An automorphism of $G$ is simply an isomorphism from $G$ to itself.

Let $\text{Aut}(G)$ denote the set of all automorphisms of $G$.

Isomorphic graphs $G$ and $G'$ are denoted to be isomorphic by writing “$G \cong G'$”.

Likewise, given a set $S$ and a function $f : S \rightarrow S$, we use $f(S)$ to denote $\{f(x) \mid x \in S\}$. Given a set of pairs $X \subseteq S \times S$, we use $f(X)$ to denote $\{f((c, d)) \mid (c, d) \in X\}$.

In the reconstruction conjecture, removing a vertex involves removing both the vertex itself and the edges that were incident to the vertex. However, if one constructs an augmented deck by removing only the edges of a vertex but leaving the vertex, one gets a deck which is identical to the original deck except that each graph in the augmented deck has an extra isolated vertex. Thus one effectively has the same problem to which our theoretical result directly applies. We spell out this connection a bit more fully in our conclusion in Sec. V.

This paper begins with a preliminary introduction to the key concepts in Sec. II for a more thorough review of the formal concepts see the work of Gross, Yellen, and Anderson [9] as well as the sources cited throughout our preliminaries. Our main result is described in Sec. III and the full proof is offered in only a few pages in Sec. IV. Finally, the concluding section provides a brief discussion on possible application and extensions of the result with special attention paid to the Graph Reconstruction Conjecture.
Example Graph $G$

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
|   |   |   |   |   |   |   |   |

Automorphism Orbit of Edge Set $\{(1,5),(5,6)\}$ in $G$

\[
AO_G(\{(1,5),(5,6)\}) = \{\{(1,5),(5,6)\}, \{(6,7),(7,3)\}, \{(2,5),(5,6)\}, \{(6,7),(7,4)\}\}
\]

**FIG. 1. Example of an Automorphism Orbit of a Set of Edges:** On the right, each distinct set of edges in the automorphism orbit is shown in a unique color and labeled with a letter. The multi-edges on the right are shown solely to indicate that the single edges on the left participate in multiple edge sets in the orbit. Note that sets such as $\{(1,5),(5,6)\}$ are not part of the example orbit.

### C. Automorphism Orbits

Let $x$ be a graph “entity” where $x$ could be a node ($x \in V$), a node pair ($x \in V \times V$), or a set of node pairs ($x \subseteq V \times V$); $x$ does not need to be an induced subgraph. The automorphism orbit of $x$ is the set of all entities that play the same structural role in $G$ that $x$ does:

\[
AO_G(x) = \{f(x) \mid f \in Aut(G)\}
\]

Note that if $x = (a,b)$, $x$ could be an edge in $E$, or $x$ could just as easily be a non-edge. Likewise, if $x$ is a set of node pairs, $x$ could be a set of edges, a set of non-edges, or a mixture. Lastly, note that the automorphism orbit of an entity $x$ contains the same kind of entities that $x$ is, per our notation in Section II A; the orbit of a node is a set of nodes, the orbit of an edge is a set of edges, etc. To see an example of an automorphism orbit of a set of edges, see Figure 1.

### D. Erdős-Rényi Graph Generation

The Erdős-Rényi graph generation model \[6\] takes two parameters $n$ and $m$ where $n \geq 1$ and $0 \leq m \leq \binom{n}{2}$. It generates a graph with $n$ nodes (labeled 1 through $n$) and $m$ edges by randomly sampling $m$ node-pairs, each with equal probability. That is, we sample $m$ node-pairs uniformly without replacement from $(V \times V) \setminus \{(a,a) \mid a \in V\}$ where $V = \{1,2,\ldots,n\}$.

Given these generation parameters $n$ and $m$, the probability of generating a graph $H$ isomorphic to $G = (V,E)$ where $|V| = n$ and $|E| = m$ is found via the equation:

\[
P(H \cong G) = \frac{1}{\binom{n}{m}} \cdot \frac{n!}{|Aut(G)|}
\]

where the first fraction is the probability of generating a specific set of $m$ edges on $n$ nodes. The second fraction is the number of distinct sets of edges that form a graph isomorphic to $G$ \[11\] \[12\].

### III. MAIN RESULT

For any undirected graph $G = (V,E)$ and any set of undirected node pairs $E' \subseteq E$ where $|E'| > 0$, we prove that:

\[
\frac{|Aut(G)|}{|AO_G(E')|} = \frac{|Aut(G - E')|}{|AO_G - E'(E')|}
\]
FIG. 2. **Examples of Main Result:** The two rows show two different edge sets being removed from the same graph. Column 3 shows the different elements in the orbit of $E'$. That is, column 3 shows the edge sets that play the same structural role in $G$ that $E'$ plays; each different edge set in the orbit is labeled with a different letter. As in Figure 1, multi-edges are drawn solely to show that the edge in the original graph participates in multiple orbit edge sets. Column 5 shows the same information as column 3 but for the graph $G - E'$; in graph $G - E'$, the set $E'$ is a set of non-edges. Observe that in both rows the ratio equality holds: $\frac{12}{3} = \frac{8}{2}$ and $\frac{12}{6} = \frac{4}{2}$.

Remember that $E'$ is simply a set of node pairs and therefore $E'$ can both be thought of as a set of edges in graph $G$ and as a set of non-edges in graph $G - E'$. The above formula states that the ratio of automorphisms of $G$ to the number of edge sets that $E'$ is equivalent to in $G$ is equal to the ratio of automorphisms of $G - E'$ to the number of non-edge sets that $E'$ is equivalent to in $G - E'$.

### IV. PROOF

#### A. Erdős-Rényi Generation and Edge Sampling Order

Our proof revolves around analyzing the probability that a graph $H$ generated by the Erdős-Rényi model is isomorphic to some graph $G$ and the probability that when $H$ was generated “the edges in $H$ corresponding to the edges $E'$ in $G$” were the last edges sampled. However, given that there is not necessarily just one isomorphism between $G$ and $H$, there is not necessarily a single set of edges in $H$ which uniquely corresponds to an edge set $E'$ in $G$.

Consequently, we consider the set of all possible correspondences. If the generated graph $H$ is isomorphic to $G$, we can represent this as $H$ being drawn uniformly from $\{G'' = (V, E'') \mid (V, E'') \cong G\}$, and then we consider a random correspondence (isomorphism) $f : V \to V$ sampled uniformly from the isomorphisms between $G$ and $H$.

We can consider the edges sampled during the Erdős-Rényi generation process as having been sampled (without replacement) from the set of all possible edges one at a time. Given any particular subset $S$ of the $m$ sampled edges, we can observe that the probability it contains the last $|S|$ edges sampled is $\frac{1}{\binom{m}{|S|}}$. This is independent of the structure of the generated graph $H$, because the identity of the other sampled edges has nothing to do with the conceptual order in which they were sampled.

Given a set of edges $S$ in a randomly generated graph $H$, let $\ell(S)$ denote the event that the edges in $S$ were the last $|S|$ edges sampled.

Likewise, if we are given a set of edge sets $\mathcal{T}$, we can talk about the event that one of the edge sets in $\mathcal{T}$ was the collection of the last edges sampled. We denote this as $\ell(\mathcal{T})$, where $\ell(\mathcal{T}) = \bigvee_{S \in \mathcal{T}} \ell(S)$. 

| Example Graph $G$ | Edge Set $E'$ to Delete | $\text{AO}_{G}(E')$ | Resultant Graph $G - E'$ | $\text{AO}_{G - E'}(E')$ |
|-------------------|------------------------|-------------------|--------------------------|------------------------|
| ![Graph](image1.png) | ![Graph](image2.png) | $|\text{Aut}(G)| = 12$ | $|\text{AO}_{G}(E')| = 3$ | $|\text{Aut}(G - E')| = 8$ | $|\text{AO}_{G - E'}(E')| = 2$ |
| ![Graph](image3.png) | ![Graph](image4.png) | $|\text{Aut}(G)| = 12$ | $|\text{AO}_{G}(E')| = 6$ | $|\text{Aut}(G - E')| = 4$ | $|\text{AO}_{G - E'}(E')| = 2$ |
B. The Derivation

Note that if \(|E| = |E'|\), then the result follows trivially because \(\text{Aut}(G - E')\) would be the symmetric group on \(n\) with size \(n!\), \(|\text{AO}_G(E')|\) would be 1, and \(|\text{AO}_{G - E'}(E')|\) would be the number of distinct edge sets that produce the graph \(G\), which as discussed in Sec. [V.D] is \(\frac{n!}{|\text{Aut}(G)|}\). Thus from here on out we assume that \(|E'| > |E'|\).

As in Section [V.A], we use \(H\) to refer to our randomly generated graph, and, in the event that \(H\) and \(G\) are isomorphic, we use \(f\) to denote a random isomorphism from \(G\) to \(H\).

We can now ask the question: “Given a set of edges \(E' \subset E\), what is the probability that \(H\) is isomorphic to \(G\) and \(E'\) was isomorphic to \(G - E'\)?” This event can be expressed as the event that \(H \cong G\) and the event that \(\ell(f(E'))\) given \(H \cong G\). Let \(k = |E'|\). Because the edge set that \(f\) maps \(E'\) to had a probability of \(\frac{1}{k!}\) of being sampled last, this quantity is:

\[
\mathbb{P}(\ell(f(E')) \mid H \cong G) \cdot \mathbb{P}(H \cong G) = \frac{1}{|E'|!} \mathbb{P}(H \cong G)
\]  (2)

However, we can also break the quantity down in a different way. Let \(H'\) denote the graph generated by including all but the last \(k\) sampled edges. Note that the following holds:

\[H \cong G \land \ell(f(E')) \rightarrow H' \cong G - E'
\]

That is, if \(H\) is isomorphic to \(G\) and \(E'\) were the last \(k\) edges added, then it follows by the definition of \(H'\) that \(H'\) was isomorphic to \(G - E'\). In the same vein, note that the following conditions also hold:

\[\ell(f(E')) \rightarrow \ell(f(\text{AO}_G(E')))
\]

and

\[H \cong G \land \ell(f(\text{AO}_G(E')))) \rightarrow H' \cong G - E'
\]

These observations allow us to break down the event “\(H \cong G \land \ell(f(E'))\)” into the equivalent event “\(H \cong G \land \ell(f(E')) \land \ell(f(\text{AO}_G(E'))) \land H' \cong G - E''\) so as to make use of standard probability conditioning:

\[
\mathbb{P}(\ell(f(E')) \mid H \cong G) \cdot \mathbb{P}(H \cong G) = \mathbb{P}(H \cong G \land \ell(f(\text{AO}_G(E')))) \land H' \cong G - E')
\]

\[
\cdot \mathbb{P}(H \cong G \land \ell(f(\text{AO}_G(E')))) \land H' \cong G - E')
\]

\[= \frac{1}{|\text{AO}_G(E')|} \cdot \frac{|\text{AO}_{G - E'}(E')|}{\binom{n}{k} - (m-k)} \cdot \mathbb{P}(H' \cong G - E')
\]  (3)

The three final quantities in the last line of the equation correspond to the three probabilities in the middle. To explain the first of these three quantities, observe that:

\[
\mathbb{P}(\ell(f(E')) \mid H \cong G \land \ell(f(\text{AO}_G(E')))) \land H' \cong G - E')
\]

\[= \mathbb{P}(\ell(f(E')) \mid H \cong G \land \ell(f(\text{AO}_G(E'))))
\]

\[= \frac{1}{|\text{AO}_G(E')|}
\]

In other words, this is the probability that if the final graph is isomorphic to \(G\) and if the last \(k\) edges added play the same structural role as \(E'\) does in \(G\) (i.e. if the last set of the \(k\) edges are in \(\text{AO}_H(f(E'))\)), then those \(k\) edges in \(H\) correspond (via \(f\)) to \(E'\) in particular. Note that \(\text{AO}_H(f(E')) = f(\text{AO}_G(E'))\). The reason we can let the “\(H' \cong G - E''\)” drop out from the equation is because, as we discussed earlier, \(H \cong G \land \ell(f(\text{AO}_G(E')))) \rightarrow H' \cong G - E'.

Further, \(|\text{AO}_{G - E'}(E')| = \mathbb{P}(H \cong G \land \ell(f(\text{AO}_G(E')))) \land H' \cong G - E')\). The numerator \(|\text{AO}_{G - E'}(E')|\) is the number of non-edge sets in \(G - E'\) which are equivalent to the missing edge set \(E'\) (i.e. the number of potential edge sets which, if added to a graph isomorphic to \(G - E'\), will yield a graph \(G'\) isomorphic to \(G\) where that added edge set plays
the same role in $G'$ that $E'$ plays in $G$). Because $H'$ has $m - k$ edges, then \( \binom{m}{k} \) is the total number of $k$-size non-edge sets in $H'$ from which a $k$-size edge set will be uniformly sampled by the ER generation process. Thus the ratio of $|AO_{G-E}(E')|$ to $\binom{n}{k}$ is the probability that if $H'$ is isomorphic to $G - E'$ and if $k$ edges are added to $H'$, then the resulting graph will be isomorphic to $G$ and the $k$ added edges will have the same role in $H$ as $E'$ has in $G$.

Putting equations 2 and 3 together, we get that:

\[
\frac{1}{\binom{m}{n}} \cdot \mathbb{P}(H \cong G) = \frac{1}{|AO_G(E')|} \cdot \frac{|AO_{G-E'}(E')|}{\binom{n}{k} - (m-k)} \cdot \mathbb{P}(H' \cong G - E')
\]

(4)

Multiplying both sides by $\binom{m}{k}$ gives us:

\[
\mathbb{P}(H \cong G) = \frac{\binom{m}{k}}{|AO_G(E')|} \cdot \frac{|AO_{G-E'}(E')|}{\binom{n}{k} - (m-k)} \cdot \mathbb{P}(H' \cong G - E')
\]

(5)

If we substitute in the formula for the probability of generating a type of graph according to the Erdős-Rényi model, this equation becomes:

\[
\frac{1}{\binom{m}{n}} \cdot \frac{n!}{|Aut(G)|} = \frac{\binom{m}{k}}{|AO_G(E')|} \cdot \frac{|AO_{G-E'}(E')|}{\binom{n}{k} - (m-k)} \cdot \frac{1}{\binom{m}{k}} \cdot \frac{n!}{|Aut(G - E')|}
\]

(6)

Re-arranging and expanding gives us:

\[
\frac{|AO_G(E')|}{|Aut(G)|} = \frac{\binom{m}{k}}{\binom{m}{k-1}} \cdot \frac{\binom{n}{k}}{\binom{n}{k-1}} \cdot \frac{|AO_{G-E'}(E')|}{|Aut(G - E')|}
\]

\[
= \frac{\binom{m}{k} \cdot (\binom{m}{k} - 1) \ldots (\binom{m}{k} + 1 - m)}{\binom{m}{k} \cdot (\binom{m}{k} - 1) \ldots (\binom{m}{k} + 1 - m) - \binom{m}{m-k} \cdot \frac{m(m-1) \ldots (m+1-k)}{k!} \cdot \frac{|AO_{G-E'}(E')|}{|Aut(G - E')|}
\]

(7)

To simplify this expression, we define the following shorthand:

- $a = \binom{m}{k} \cdot (\binom{m}{k} - 1) \ldots (\binom{m}{k} + 1 - m)$
- $b = m!$
- $c = \binom{m}{k} \cdot (\binom{m}{k} - 1) \ldots (\binom{m}{k} + 1 - (m-k))$
- $d = (m-k)!$
- $e = m \cdot (m-1) \ldots (m+1-k)$
- $f = k!$
- $g = (\binom{m}{k} - (m-k)) \cdot (\binom{m}{k} - ((m-k) + 1)) \ldots (\binom{m}{k} + 1 - m)$
- $h = k!$

Using this shorthand, Equation 7 becomes:

\[
\frac{|AO_G(E')|}{|Aut(G)|} = \frac{a}{b} \cdot \frac{e}{f} \cdot \frac{|AO_{G-E'}(E')|}{|Aut(G - E')|} = \frac{a d e h}{b c f g} \cdot \frac{|AO_{G-E'}(E')|}{|Aut(G - E')|}
\]

(8)

Observe that $a$, $b$, $c$, $d$, $e$, and $g$ all take the form of $x \cdot (x-1) \ldots (x-y)$ for different values of $x$ and $y$. Further, observe that $c$ and $g$ combine to form $a$: That is, $c$ and $a$ both begin with $\binom{m}{k}$, but $c$ stops before $a$ does; then $g$ picks up where $c$ left off, and $g$ and $a$ end with the same value. Thus $a = cg$. In the same manner, $b = ed$. Given that $f = h$, these observations give us that $\frac{a d e h}{b c f g} = 1$, and thus our result has been derived. □
V. CONCLUSIONS

A. Extensions

We expect that this result can be extended to other kinds of graphs, including directed graphs and graphs with node colors. The formula for the probability of generating a directed graph with the Erdős-Rényi model would be very similar to the one for undirected graphs shown in Section [13] because the same logic about symmetry and edge sets applies.

The main limitation of our result is that it only relates the symmetries of graphs where one graph’s edges is a superset of the other graph’s edges; it would be a significant addition if this kind of result could be obtained for any two graphs. Such a modification might simultaneously consider a set of edge additions and a set of edge deletions.

B. Application to the Graph Reconstruction Conjecture

As we began to discuss in the introduction, we believe this result may have a bearing on the graph reconstruction conjecture. We begin to explore that connection here.

Given a graph \( G = (V, E) \) and a vertex \( v \in V \), define the set of edges incident to \( v \) to be \( e_G(v) = \{(a, b) : (a, b) \in E \land (a = v \lor b = v)\} \) and define \( G - v \) to be the graph \( (V \setminus \{v\}, E \setminus e_G(v)) \).

In the graph reconstruction problem, the deck \( D \) is defined to be the multiset \( \{G - v \mid v \in V\} \). We define the augmented deck \( D_A \) to be the multiset \( \{G - e_G(v) \mid v \in V\} \). Note that the two multisets are equivalent in the sense that if you are given \( D \) you can obtain \( D_A \) by adding an isolated vertex to each graph, and given \( D_A \) you can obtain \( D \) by removing an isolated vertex from each graph. Thus the graph reconstruction conjecture can be framed as the question, “Given the augmented deck \( D_A \), can you uniquely determine \( G \)?”

The reconstruction conjecture only considers graphs with 3 or more nodes because the two two-node graphs are not reconstructable (their decks are indistinguishable from each other), and we know that a graph can be reconstructed if it is disconnected [4]. Thus the conjecture is only an open question for graphs which are connected and have 3 or more nodes, which in turn entails that each vertex either has a degree greater than 1 or the neighbor of each vertex of degree of 1 has a degree greater than 1. From here on out we assume that \( G \) meets these basic conditions. These conditions give us that no two vertices have the same edge set. Formally, \( u \neq v \rightarrow e_G(u) \neq e_G(v) \).

We also note that for any vertex \( v \in V \) and any automorphism \( f \) of \( G \), \( f(e_G(v)) = e_G(f(v)) \) and thus \( |AO_G(e_G(v))| = |AO_G(v)| \). Someone with the full deck knows these quantities because they are the number of times a graph equivalent to \( G - v \) appears in the deck. Given a graph \( G' = G - e_G(v) \) in the augmented deck, let \( M(G') = |AO_G(e_G(v))| \) denote this quantity (i.e. the multiplicity of \( G' \) in \( D_A \)).

Given a graph \( G' \) in the augmented deck, we let \( v(G') \) denote a (arbitrary) vertex for which \( G - e_G(v(G')) \cong G' \). Applying our result from this paper gives us that for any graph \( G' \in D_A \):

\[
\frac{|Aut(G)|}{|AO_G(e_G(v(G')))|} = \frac{|Aut(G')|}{|AO_G(e_G(v(G')))|}
\]

Re arranging gives us:

\[
|Aut(G)| = \frac{|Aut(G')|}{|AO_G(e_G(v(G')))|} \cdot |AO_G(e_G(v(G')))|
\]

As we said above, for relevant graphs \( |AO_G(e_G(v(G')))| = |AO_G(v(G'))| = M(G') \), which can be determined from the (augmented) deck. Thus, our equation becomes:

\[
|Aut(G)| = \frac{|Aut(G')|}{|AO_G(e_G(v(G')))|} \cdot M(G')
\]

This applies to the reconstruction conjecture as follows: We know that the number of edges missing from each graph in the deck is reconstructible [2]. Given a graph \( G' \) in the (augmented) deck, let \( \mathcal{E}(G') \) denote the set of all edge sets that represent adding the appropriate number of edges to an isolated vertex in \( G' \). Further, let \( AOP(\mathcal{E}(G')) \) denote the partitioning of \( \mathcal{E}(G') \) into automorphism orbits. That is, \( AOP(\mathcal{E}(G')) = \{AO_{\mathcal{G'}}(S) \mid S \in \mathcal{E}(G')\} \). If for a subset of the augmented deck \( \{G_1, G_2, \ldots, G_k \} \subseteq D_A \) we get that there is a unique collection of edge set orbits \( AO_1 \in AOP(\mathcal{E}(G_1)), AO_2 \in AOP(\mathcal{E}(G_2)), \ldots, AO_k \in AOP(\mathcal{E}(G_k)) \) where:
∀ 1 ≤ i, j ≤ k. \[
\frac{M(G_i) \cdot |\text{Aut}(G_i)|}{|\text{AO}_i|} = \frac{M(G_j) \cdot |\text{Aut}(G_j)|}{|\text{AO}_j|}
\]
then we know there is only one way to add the appropriate numbers of edges to the deck graphs which yields the same automorphism group size in all the resulting graphs, which means \( G \) can be uniquely reconstructed from the deck.

C. Closing

We have proven an elegant combinatorial result on graphs relating the symmetries of any two graphs differing by an edge set. Our proof utilizes the Erdős-Rényi graph generation model, but the combinatorial result itself makes no explicit reference to the model. We expect that due to its fundamental nature and simplicity, this result will have much to say about various mathematical questions, perhaps including the graph reconstruction conjecture.

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