FINITE RANK TOEPLITZ OPERATORS: SOME EXTENSIONS OF D.LUECKING’S THEOREM

ALEXEY ALEXANDROV AND GRIGORI ROZENBLUM

ABSTRACT. The recent theorem by D.Luecking about finite rank Bergman-Toeplitz operators is extended to weights being distributions with compact support and to the spaces of harmonic functions.

1. Introduction and the main result

Toeplitz operators play an important role in many branches of analysis. A significant recent development in the theory of such operators is related to the proof, given by D.Luecking [8], of the finite rank conjecture. Let $\mathcal{B}^2$ be the Bergman space of $L^2$-functions analytical in a domain $\Omega \in \mathbb{C}$ and $P$ be the orthogonal projection in $L^2(\Omega)$ onto $\mathcal{B}_2$. For a regular complex Borel measure $\mu$ with compact support, the Toeplitz operator with weight $\mu$,

$$u \mapsto T_\mu u = Pu \mu, u \in \mathcal{B}^2,$$

(1.1)
can be correctly defined. According to the finite rank conjecture, if $T_\mu$ has finite rank then the measure is a finite combination of point masses, exactly as many as the rank is. The nontrivial part of this conjecture is described in [8], [11]. Immediately after the preprint containing the proof appeared, an activity developed in extending and applying this result. On the one hand, the theorem by Luecking was extended to the multi-dimensional case, see [1], [11] (by different methods). On the other hand, interesting application to the theory of Toeplitz operators appeared, see [2], [3], [6], [7], as well as in Function Theory, see [4]. The finite rank result turns out to be useful also in Mathematical Physics, more exactly, to the spectral analysis of the perturbed Landau Hamiltonian, see [10], as well as the discussion and further references in [11].

A number of natural questions arise around Luecking’s theorem. First, it is interesting to find out whether the finite rank property still holds when the analytical Bergman space is replaced by some other, also closed in $L^2$, space of smooth functions. In [11] such a generalization was found for the space of $n$--harmonic functions in a domain in $\mathbb{C}^n$, and in [6] the finite rank property was, in the complex dimension 1, extended to the $L^2$--closed span of certain, not too sparse, sets of

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monomials $z^{n_k}, n_k \in \mathbb{Z}_+$. At the same time, for the problems arising in Mathematical Physics, it is important to generalize the results to the case when the weight measure $\mu$ is replaced by a distribution with compact support.

In the present paper we deal with these questions. First, in the complex dimension 1, for the analytical Bergman space, we describe the procedure of reducing the finite rank problem for a distribution to the same problem for an absolutely continuous measure $\mu$, which is already taken care of. Thus, the finite rank problem finds its solutions also for distributional weights. We note that the reduction above seems to be necessary. The initial proof with measure weight was critically based upon a lemma on the density of symmetric polynomials of a special form in the space of symmetric continuous functions of many variables, proved by an ingenious use of the Stone-Weierstrass theorem. The distributional case requires a similar density result in the space of differentiable functions, where no proper analogy of the Stone-Weierstrass theorem exists. Moreover, the density result itself turns out to be wrong for differentiable functions. We present an example demonstrating this. Therefore our approach seems to be at the moment the only one able to treat the distributional case.

The results on finite rank problem for distributional weights are further extended to the multi-dimensional case. We use a modification of the induction on dimension presented in [11]. It seems that the approach to proving the multi-dimensional Luecking’s theorem, proposed in [1] using Stone-Weierstrass argument would not work for distributions, by the reasons given above.

Finally, we consider the finite rank problem for the Bergman space of harmonic functions. The result follows immediately from the one in the analytical case in an even dimension, since the space of harmonic functions contains the space of $n$-harmonic functions, where the finite rank property is an obvious consequence of the one in the analytical case, see [1]. Quite different is the situation in an odd dimension ($\geq 3$), where no direct coupling of harmonic functions to analytical ones exists. Here we are able to handle only the case of a measure acting as weight, using a sort of dimension-reduction argument and some Harmonic Analysis technique. We give also an example, not disproving the finite rank conjecture directly, but just hinting that the situation here with distributions might be considerably more delicate than the one with measures.

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2. Setting

Let \( \tau \) be a positive measure in a domain \( \Omega \subset \mathbb{C}^d \) such that \( 0 < \int_{\mathbb{C}^d} |P| d\tau < +\infty \) for every polynomial \( P \) of the complex variables \( (z_1, \ldots, z_d) \), \( P \neq 0 \). We consider the space \( L^2(\Omega, \tau) \) and the subspace \( \mathcal{A}(\Omega, \tau) \subset L^2(\Omega, \tau) \) consisting of analytical functions. It is a closed subspace, and we denote by \( P_A(\Omega, \tau) \) the orthogonal projection onto \( \mathcal{A}(\Omega, \tau) \). Further on, as soon as the domain and the measure are fixed, we suppress them in the notations. The typical examples here are the Bergman spaces, for the case of a bounded \( \Omega \) with (say) Lebesgue measure, and the Fock-Bargmann spaces for \( \Omega = \mathbb{C}^d \), \( \tau \) being the Gaussian measure. The projection \( P_A \) is an integral operator with the reproducing kernel \( P(z, \cdot) \), infinitely smooth, analytical in \( z \) and anti-analytical in \( w \) in the domain \( \Omega \).

Let \( F \) be a distribution with compact support in \( \Omega \), \( F \in \mathcal{E}'(\Omega) \). We denote by \( \langle F, \phi \rangle \) the action of the distribution \( F \) on the function \( \phi \in C^\infty(\Omega) \). Then, for \( u \in \mathcal{A}(\Omega, \tau) \), the expression

\[
(T_F u)(z) = \langle F, P(z, \cdot) u(\cdot) \rangle, \quad z \in \Omega,
\]

defines an analytical function \( (T_F u)(z) \in \mathcal{A}(\Omega, \tau) \). The corresponding operator \( u \mapsto T_F u \) is a natural generalization of the Toeplitz operator \( u \mapsto PFu \), \( u \in \mathcal{A}(\Omega, \tau) \) for the case when \( F \) is a bounded measurable function with compact support in \( \Omega \). The operator \( T_F \) is bounded in \( \mathcal{A} \). Its sesquilinear form can be described as

\[
(T_F u, v) = \langle F, uv \rangle, \quad u, v \in \mathcal{A}.
\]

In the special case when the distribution \( F \) is, in fact, a complex Borel measure \( \mu \) with compact support in \( \Omega \), the operator \( T_F \) can be described as

\[
(T_F u)(z) = \int_{\Omega} P(z, w) u(w) d\mu(w),
\]

and the sesquilinear form is given by

\[
(T_F u, v) = \int uv d\mu.
\]

Suppose that the operator \( T_F \) has finite rank, \( \text{rank}(T_F) = m < \infty \). This means, in particular, that for any, finite or infinite, system of functions \( f_\alpha \in \mathcal{A} \), the system of functions \( g_\alpha = T_F f_\alpha \) is linearly dependent and \( \text{rank}\{g_\alpha\} \leq m \). This is correct, in particular, if we take as \( f_\alpha \) the system of polynomials \( f_\alpha = z^\alpha, \alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{Z}_+)^d \). Therefore the infinite matrix

\[
A_F = (a_{\alpha\beta}), \quad a_{\alpha\beta} = (T_F z^\alpha, z^\beta) = \langle F, z^\alpha z^\beta \rangle
\]

has finite rank, \( \text{rank}(A_F) \leq m \). It is important that the matrix \( A_F \) does not depend on the domain \( \Omega \) or the measure \( \tau \), but it depends only on the distribution \( F \). Of course, the rank of \( A_F \) does not change.
if we make a unitary transformation of \(\mathbb{C}^d\) with corresponding change of complex coordinates.

We notice also, following \[11\], that if \(g\) is function analytical and bounded in some polydisk neighborhood of \(\text{supp } F\) and \(Fg\) is the distribution \(|g|^2F\) then \(\text{rank } A_{Fg} \leq \text{rank } A_F\). To show this, we consider first a polynomial \(g_l\) of degree \(l\). The matrix \(A_{Fg_l}\) is obtained by building linear combinations of rows and columns of \(A_F\), therefore the rank does not increase, \(\text{rank } A_{Fg_l} \leq \text{rank } A_F\). We pass to a general analytical function \(g\) using approximations by Taylor polynomials, convergent, together with all derivatives, uniformly on any compact in the polydisk.

In a similar way, we consider Toeplitz operators in spaces of harmonic functions. Denote by \(H(\Omega, \tau)\) the subspace in \(L_2(\Omega, \tau)\), consisting of harmonic functions in a domain \(\Omega \subset \mathbb{R}^d\) and by \(Q\) the orthogonal projection \(Q : L_2(\Omega, \tau) \rightarrow H(\Omega, \tau)\); this projection is an integral operator with kernel \(Q(x, y), x, y \in \Omega\), the kernel being a harmonic function in each variable \(x\) and \(y\). With a distribution \(F\) having compact support in \(\Omega\) we associate, similarly to (2.1) the Toeplitz operator \(T_F^H : u \mapsto T_F^H u, T_F^H u(z) = \langle F, Q(x, \cdot)u(\cdot) \rangle\). The expression for the action of the operator for the case when \(F\) is a Borel measure and the expressions for the sesquilinear form are analogous to (2.3), (2.2), (2.4). Similar to the case of analytical functions, we associate with the distribution \(F\) the matrix \(H_F,\) with entries being \(\langle F, f_\alpha f_\beta \rangle\), where \(f_\alpha\) is some system of harmonic polynomials in \(\mathbb{R}^d\). Again, the rank of the infinite matrix \(H_F\) does not exceed the rank of the operator \(T_F^H\). We, however, may not include, as we have done for analytical functions, the multiplicative functional parameter \(g\), since harmonic functions do not possess a multiplicative structure.

### 3. Finite Rank Operators in Dimension 1

The aim of this section is to give a proof of the following result generalizing the Luecking theorem.

**Theorem 3.1.** Let \(F\) be a distribution with compact support in the domain \(\Omega \subset \mathbb{C}^1\). Suppose that the operator \(T_F\) has finite rank \(m\). Then there exist finitely many points \(z_q \in \Omega\) \(q = 1, \ldots, m_0\), \(m_0 \leq m\), and differential operators \(L_q = L_q(\partial x, \partial y), q = 1, \ldots, m_0\) such that \(F = \sum L_q \delta(z - z_q)\).

We start with some observations about distributions in \(\mathcal{E}'(\mathbb{C})\). For such distribution we denote by \(\text{psupp } F\) the complement of the unbounded component of the complement of \(\text{supp } F\).

**Lemma 3.2.** Let \(F \in \mathcal{E}'(\mathbb{C})\). Then the following two statements are equivalent:

a) there exists a distribution \(G \in \mathcal{E}'(\mathbb{C})\) such that \(\frac{\partial G}{\partial x} = F\), moreover \(\text{supp } G \subset \text{psupp } F\);
b) $F$ is orthogonal to all polynomials of $z$ variable, i.e. $\langle F, z^k \rangle = 0$ for
all $k \in \mathbb{Z}_+$. 

Proof. The implication $a) \implies b)$ follows from the relation

$$\langle F, z^k \rangle = \langle \frac{\partial G}{\partial \bar{z}}, z^k \rangle = \langle G, \frac{\partial z^k}{\partial \bar{z}} \rangle = 0. \quad (3.1)$$

We prove that $b) \implies a)$. Put $G := F * \frac{1}{\pi z} \in \mathcal{S}'(\mathbb{C})$, the convolution being well-defined since $F$ has compact support. Since $\frac{1}{\pi z}$ is the fundamental solution of the Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$, we have $\frac{\partial G}{\partial \bar{z}} = F$ (cf., for example, [5], Theorem 1.2.2). By the ellipticity of the Cauchy-Riemann operator, suppsupp $G \subset$ suppsupp $F$, in particular, this means that $G$ is a smooth function outside psupp $F$, moreover, $G$ is analytic outside psupp $F$ (by suppsupp $F$ we denote the singular support of the distribution $F$, see, e.g., [5], the largest open set where the distribution coincides with a smooth function). Additionally, $G(z) = \langle F, \frac{1}{\pi(z-w)} \rangle = \pi^{-1} \sum_{k=0}^{\infty} z^{-k-1} \langle F, w^k \rangle = 0$ if $|z| > R$ and $R$ is sufficiently large. By analyticity this implies $G(z) = 0$ for all $z$ outside psupp $F$. \hfill $\square$

Proof of Theorem 3.1. The distribution in question $F$, as any distribution with compact support, is of finite order, therefore it belongs to some Sobolev space, $F \in H^s$ for certain $s \in \mathbb{R}^1$. If $s \geq 0$, $F$ is a function and must be zero by Luecking’s theorem. So, suppose that $s < 0$.

Consider the first $m + 1$ columns in the matrix $A_F$, i.e.

$$a_{nk} = \langle T_F z^k, z^l \rangle = \langle F, z^k z^l \rangle, l = 0, \ldots, m; \ k = 0, \ldots. \quad (3.2)$$

Since the rank of the matrix $A_F$ is not greater than $m$, the columns are linearly dependent, in other words, there exist coefficients $c_0, \ldots, c_m$ such that $\sum_{l=0}^{m} a_{kl} c_l = 0$ for any $k \geq 0$. This relation can be written as

$$\langle F, z^k h_1(\bar{z}) \rangle = \langle h_1(\bar{z}) F, z^k \rangle = 0, \ h_1(\bar{z}) = \sum_{k=0}^{m} c_l \bar{z}^l. \quad (3.3)$$

Therefore the distribution $h_1(\bar{z}) F \in H^s$ satisfies the conditions of Lemma 3.2 and hence there exists a compactly supported distribution $F^{(1)}$ such that $\frac{\partial F^{(1)}}{\partial \bar{z}} = F_h$. By the ellipticity of the Cauchy-Riemann operator, the distribution $F^{(1)}$ is less singular than $F$, $F^{(1)} \in H^{s+1}$. At the same time,

$$\langle F^{(1)}, z^k \bar{z}^l \rangle = (l + 1)^{-1} \langle F^{(1)}, \frac{\partial z^k \bar{z}^{l+1}}{\partial \bar{z}} \rangle = (l + 1)^{-1} \langle h(\bar{z}) F, z^k \bar{z}^l \rangle = (l + 1)^{-1} \langle F, z^k \bar{z}^l h(\bar{z}) \rangle, \quad (3.4)$$

and therefore the rank of the matrix $A_{F^{(1)}}$ does not exceed the rank of the matrix $A_F$. 
We repeat this procedure sufficiently many (say, \(N = [-s] + 1\)) times and arrive at the distribution \(F^{(N)}\) in \(L_2\), for which the corresponding matrix \(A_{F^{(N)}}\) has finite rank. By Luecking’s theorem, this may happen only if \(F^{(N)} = 0\).

Now we go back to the initial distribution \(F\). Since, by our construction, \(\frac{\partial F^{(N)}}{\partial z} = h_N(\bar{z}) F^{(N - 1)}\), we have that \(h_N(\bar{z}) F^{(N - 1)} = 0\) and therefore \(\text{supp} \, F^{(N - 1)}\) is a subset of the set of zeroes of the polynomial \(h_N(\bar{z})\). On the next step, since \(\frac{\partial F^{(N - 1)}}{\partial z} = h_{N-1}(\bar{z}) F^{(N - 2)}\), we obtain that \(\text{supp} \, F^{(N - 2)}\) lies in the union of sets of zeroes of polynomials \(h_{N-1}(\bar{z})\) and \(h_N(\bar{z})\). After having gone all the way back to \(F\), we obtain that its support is a finite set of points lying in the union of zero sets of polynomials \(h_j\). A distribution with such support must be a linear combination of \(\delta\)'-distributions in these points and their derivatives, \(F = \sum L_q \delta(z - q)\), where \(L_q\) is some differential operator. Finally, to show that the number of points \(z_q\) does not exceed \(m\), we construct for each of them the interpolating polynomial \(f_q(z)\) such that \(|f_q(z)|^2 \neq 0\) at the point \(z_q\) while at the points \(z_{q'}\), \(q' \neq q\), the polynomial \(f_q\) has zero of sufficiently high order, higher than the order of \(L_{q'}\), so that \(L_{q'}(f_q g)(z_{q'}) = 0\) for any smooth function \(g\). With such choice of polynomials, the matrix with entries \(\langle F, f_q f_{q'} \rangle\) is the diagonal matrix with nonzero entries on the diagonal, and therefore its size (that equals the number of the points \(z_q\)) cannot be greater than the rank of the whole matrix \(A_F\), i.e., cannot be greater than \(m\).

We note here that the attempt to extend the original proof of Luecking’s theorem to the distributional case would probably meet certain complications. Let us recall the crucial place in [8].

The matrix of the type (2.5) is also considered, with a measure \(\mu\) standing on the place of the distribution \(F\). Then, for a given \(N\), the measure \(\mu^N = \otimes^N \mu\) on \(\mathbb{C}^N\) is introduced, and Lemma 5.1 is established, stating that if the Toeplitz operator \(T_\mu\) has rank smaller than \(N\), then for all symmetric polynomials \(H_1(Z), H_2(Z)\) of the multi-dimensional complex variable \(Z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N\),

\[
\int H_1(Z) \overline{H_2(Z)} |V(Z)|^2 d\mu^N = 0, \tag{3.6}
\]

where \(V(Z)\) is the Vandermonde function, \(V(Z) = \prod_{i<j}(z_i - z_j)\). To derive the finite rank result from Lemma 5.1, the following property is needed: the algebra generated by the functions of the form \(H_1(Z) \overline{H_2(Z)}\) is dense (in the sense of the uniform convergence on compacts) in the space of symmetric continuous functions. This latter property is proved in [8] by an ingenious reduction to the Stone-Weierstrass theorem.

Now, if \(\mu = F\) is a distribution that is not a measure, the analogy of reasoning in [8] would require a similar density property, however not in the sense of the uniform convergence on compacts, but in a
Proposition 3.3. The algebra generated by the functions having the form $H_1(Z)\overline{H_2(Z)}$, where $H_1, H_2$ are symmetric polynomials of the variables $Z = (z_1, \ldots, z_N)$ is not dense in the sense of the uniform $C^l$-convergence on compact sets in the space of $C^l$-differentiable symmetric functions, as long as $l \geq N(N - 1)$.

Proof. We introduce the notations: $D_j = \frac{\partial}{\partial z_j}$, $\overline{D_j} = \frac{\partial}{\partial \bar{z}_j}$. Consider the differential operator $V(D) = \prod_{j<k}(D_j - D_k)$. It is easy to check that $V(D)H$ is symmetric for any antisymmetric function $H(Z)$ and $V(D)H$ is antisymmetric for any symmetric function $H(Z)$. Further on, consider any function $H(Z)$ of the form $H(Z) = H_1(z)\overline{H_2(z)}$ where $H_1(Z), H_2(Z)$ are analytic polynomials. If at least one of them is symmetric, we have

$$V(D)V(\bar{D})H(0) = 0. \quad (3.7)$$

In fact, $V(D)V(\bar{D})H_1(Z)\overline{H_2(Z)} = [(V(D)H_1(Z)][V(D)\overline{H_2(Z)}]$. In the last expression, for the symmetric polynomial $H_1$, the corresponding polynomial $V(D)H_1(Z)$ is antisymmetric, and therefore equals zero for $Z = 0$. Now consider the symmetric function $|V(Z)|^2 = V(Z)\overline{V(Z)}$. We have

$$V(D)V(\bar{D})V(Z)\overline{V(Z)} = [V(D)V(Z)][V(\bar{D})\overline{V(Z)}].$$

Now note that $V(Z) = \sum_{\kappa} C_\kappa \prod z_j^{\kappa_j}$ where the summing goes over multi-indices $\kappa = (\kappa_1, \ldots, \kappa_N)$, $|\kappa| = N$ and not all of real coefficients $C_\kappa$ are zeros. Simultaneously, $V(D) = \sum_{\kappa} C_\kappa \prod D_j^{\kappa_j}$ with the same coefficients. We recall now that $\prod D_j^{\kappa_j} \prod z_j^{\kappa_j} = 0$ if $\kappa \neq \kappa'$ and it equals $\kappa!$ if $\kappa = \kappa'$. Therefore, $V(D)V(Z) = \sum_{\kappa} C_\kappa^2 \kappa!$ is a positive constant. In this way we have constructed the differential operator $V(D)V(\bar{D})$ of order $N(N - 1)$, satisfying (3.7) for any function of the form $H(Z) = H_1(z)\overline{H_2(z)}$ with symmetric $H_1, H_2$, and not vanishing on some symmetric differentiable function $|V(Z)|^2$. Therefore the function $|V(Z)|^2$ cannot be approximated by linear combinations of the functions $H(Z) = H_1(z)\overline{H_2(z)}$ in the sense of the uniform $C^{N(N-1)}$ convergence on compacts. □
4. The multi-dimensional case

In this Section we extend our main Theorem 3.1 to the case of Toeplitz operators in Bergman spaces of analytical functions of several variables. For the case of a measure acting as weight, there exist two proofs of this result, in [1] and [11]. The first proof generalizes the approach used in [8], the other one uses the induction on dimension. As it follows from Proposition 3.3, for the case of distribution the approach of [1] is likely to meet some complications. On the other hand, as we are going to show, the approach of [11] can be extended to the distributional case.

**Theorem 4.1.** Let $F$ be a distribution in $E'(C^d)$. Consider the matrix

$$A_F = (a_{\alpha \beta})_{\alpha, \beta \in \mathbb{Z}^d_+}; \quad a_{\alpha \beta} = \langle F, Z^\alpha \overline{Z}^\beta \rangle, \quad Z = (z_1, \ldots, z_d).$$

(4.1)

Suppose that the matrix $A_F$ has finite rank $m$. Then $\text{card supp } F \leq m$ and $F = \sum L_q \delta(Z - Z_q)$, where $L_q$ are differential operators and $Z_q$ are some points in $\mathbb{C}^d$.

We will perform the induction on dimension, proving a statement that is, actually, only formally weaker than Theorem 4.1, since, as it was explained in Sect. 2, the rank of the matrix $A_F$ does not grow if $F$ is replaced by $F_g$.

**Proposition 4.2.** Suppose that for any function $g(Z)$, analytic and bounded in a polydisk neighborhood of the support of the distribution $F$, the conditions of Theorem 4.1 are fulfilled with the distribution $F$ replaced by $\langle g(Z) \rangle^2 F \equiv F_g$. Then $\text{card supp } F \leq m$ and $F = \sum L_q \delta(Z - Z_q)$, where $L_q$ are differential operators.

**Proof.** For $d = 1$ the statement of Proposition 4.2 coincides with the one of Theorem 3.1 that was proved in Sect. 3. We suppose that we have established our statement in dimension $d - 1$ and consider the $d$-dimensional case. We denote the variables as $Z = (z_1, Z')$, $Z' \in \mathbb{C}^{d-1}$.

For a fixed function $g(Z)$ we denote by $G(g) = \pi_* F_g$ the distribution in $E'(\mathbb{C}^{d-1})$ induced from $F_g$ by the projection $\pi : \mathbb{C}^d \mapsto \mathbb{C}^{d-1}$:

$$\langle G(g), u \rangle = \langle F_g, 1_{\mathbb{C}^d} \otimes u \rangle.$$  

(4.2)

Although the function $g$ is defined only in a polydisk, the expression in (4.2) is well defined since this polydisk contains supp $F$.

Consider the submatrix $A'_{F_g}$ in the matrix $A_{F_g}$ consisting only of those $a_{\alpha \beta} = \langle |g|^2 F, Z^\alpha \overline{Z}^\beta \rangle$ for which $\alpha_1 = \beta_1 = 0$. It follows from (4.2), that the matrix $A'_{F_g}$ coincides with the matrix $A_{G(g)}$ constructed for the distribution $G(g)$ in dimension $d - 1$. Thus, the matrix $A_{G(g)}$, being a submatrix of a finite rank matrix, has a finite rank itself, moreover, $\text{rank } A_{G(g)} \leq m$. By the inductive assumption, this implies that the distribution $G(g)$ has finite support consisting of $m(g) \leq m$ points.
\[ \zeta_1(g), \ldots, \zeta_m(g); \, \zeta_q(g) \in \mathbb{C}^{d-1} \] (the notation reflects the fact that both the points and their quantity may depend on the function \( g \)). Among all functions \( g \), we can find the one, \( g = g_0 \), for which \( m(g) \) attains its maximum value \( m_0 \leq m \). Without losing in generality, we can assume that \( g_0 = 1 \).

Fix an \( \epsilon > 0 \), sufficiently small, so that 2\( \epsilon \)-neighborhoods of \( \zeta_q(1) \) are disjoint, and consider the functions \( \varphi_q(z') \in C^\infty(\mathbb{C}^{d-1}), \, q = 1, \ldots, \), such that \( \text{supp} \varphi_q \) lies in the \( \epsilon \)-neighborhood of the point \( \zeta_q(1) \) and \( \varphi(z') = 1 \) in the \( \frac{\epsilon}{2} \)-neighborhood of \( \zeta_q(1) \). We fix an analytic function \( g(z) \) and consider for any \( g \) the distribution \( \Phi_q(t, g) \in \mathcal{E}'(\mathbb{C}^d), \, \Phi_q(t, g) = |1 + tg|^2 \varphi_q(Z')F = \varphi_q(Z')F_{1+tg} \). For \( t = 0 \), \( \Phi_q(t, g) = \varphi_q(Z')F \), the point \( \zeta_q(1) \) belongs to the support of \( \pi_* \Phi_q(0, g) \), and therefore for some function \( u \in C^\infty(\mathbb{C}^{d-1}), \, \langle \pi_* \Phi_q(0, g), u \rangle \neq 0 \). By continuity, for \( |t| \) small enough, we still have \( \langle \pi_* \Phi_q(t, g), u \rangle \neq 0 \), which means that the \( \epsilon \)-neighborhood of the point \( \zeta_q(1) \) contains at least one point in the support of the distribution \( G(1 + tg) \). Altogether, we have not less than \( m_0 \) points of the support of \( G(1 + tg) \) in the union of \( \epsilon \)-neighborhoods of the points \( \zeta_q(1) \). However, recall, the support of \( G(1 + tg) \) can never contain more than \( m_0 \) points, so we deduce that for \( t \) small enough, there are no points of the support of \( G(1 + tg) \) outside the \( \epsilon \)-neighborhoods of the points \( \zeta_q(1) \), so

\[ \text{supp} G(1 + tg) \cap \{ Z' : |Z' - \zeta_q| > \epsilon \} = \emptyset \quad (4.3) \]

for \( |t| \) small enough (depending on \( g \)). Now we introduce a function \( \psi \in C^\infty(\mathbb{C}^{d-1}) \) that equals 1 outside 2\( \epsilon \)-neighborhoods of the points \( \zeta_q(1) \) and vanishes in \( \epsilon \)-neighborhoods of these points. By (4.3), the distribution \( \psi G(1 + tg) \) equals zero for any \( g \), for \( t \) small enough. In particular, applying this distribution to the function \( u = 1 \), we obtain

\[ \langle \psi G(1 + tg), 1 \rangle = \langle \psi F, |1 + tg|^2 \rangle = \langle \psi F, 1 + 2t \text{Re } g + t^2 |g|^2 \rangle = 0. \quad (4.4) \]

By the arbitrariness of \( t \) in a small interval, (4.4) implies that \( \langle \psi F, |g|^2 \rangle = 0 \) for any \( g \). Now we take \( g \) in the form \( g = g_1 + ig_2 \), where \( g_1, g_2 \) are again functions analytical in a polydisk neighborhood of \( \text{supp} F \). Then we have

\[ \langle \psi F, |g_1|^2 + 2 \text{Re } (g_1 \overline{g_2}) + |g_2|^2 \rangle = \langle \psi F, 2 \text{Re } (g_1 \overline{g_2}) \rangle = 0. \]

Replacing here \( g_1 \) by \( ig_1 \), we obtain \( \langle \psi F, 2 \text{Im } (g_1 \overline{g_2}) \rangle = 0 \), and thus

\[ \langle \psi F, g_1 \overline{g_2} \rangle = 0. \quad (4.5) \]

Any polynomial \( p(Z, \bar{Z}) \) can be represented as a linear combination of functions of the form \( g_1 \overline{g_2} \), so, (4.5) gives

\[ \langle \psi F, p(Z, \bar{Z}) \rangle = 0. \quad (4.6) \]

Now we take any function \( f \in C^\infty(\mathbb{C}^d) \) supported in the neighborhood \( V \) of \( \text{supp} F \) such that \( f = 0 \) on the support of \( \psi \). We can approximate \( f \) by polynomials of the form \( p(Z, \bar{Z}) \) uniformly on \( \nabla \) in the sense of
$C^l$, where $l$ is the order of the distribution $F$. Passing to the limit in (4.6), we obtain $\langle \psi F, f \rangle = \langle F, f \rangle = 0$.

The latter relation shows that $\text{supp} F \subset \bigcup_q \{Z : |Z' - \zeta_q(1)| < 2\epsilon\}$. Since $\epsilon > 0$ is arbitrary, this implies that $\text{supp} F$ lies in the union of affine subspaces $Z' = \zeta_j, j = 1, \ldots, m_0$ of complex dimension 1. Now we repeat the same reasoning having chosen instead of $Z = (z_1, Z')$ another decomposition of the complex variable $Z: Z = (Z'', z_d)$. We obtain that for some points $\xi_k \in \mathbb{C}^{d-1}, \leq m_0$ of them, the support of $F$ lies in the union of subspaces $Z'' = \xi_k$. Taken together, this means that, actually, $\text{supp} F$ lies in the intersection of these two systems of subspaces, which consists of no more than $m^2$ points $Z_s$. The number of points is finally reduced to $m_0 \leq m$ in the same way as in Theorem 3.1, by choosing a special system of interpolation functions.

□

5. Harmonic functions

The aim of this section is to establish finite rank results for Toeplitz operators corresponding to the Bergman spaces of harmonic functions. The main difference with the analytical case lies in the circumstance that the space of harmonic functions does not possess the multiplicative structure. Therefore, in the process of dimension reduction, similar to the one we used in the proof of Theorem 4.1, we are not able to introduce the functional parameter (denoted by $g$ there.) As a result of this circumstance, we can prove the finite rank theorem only in the case of $F$ being a measure and not a more singular distribution. In order to justify this shortcoming, we conclude the section by presenting an example of a singular distribution with rather large support (and thus non-discrete), that projects to a discrete measure, whatever the direction of the projection. Thus, a considerable part of $F$ becomes invisible after being projected. This example, although not contradicting directly the finite rank property, indicates that the reduction of dimension might be not sufficient to prove the result.

We start with the even-dimensional case. Here the problem with harmonic spaces reduces easily to the analytical case (in fact, we could have used a reference to [II] instead).

**Theorem 5.1.** Let $d = 2n$ be an even integer. Suppose that for a certain distribution $F \in \mathcal{E}'(\mathbb{R}^n)$ the matrix $H_F$ defined in Section 2 has rank $m < \infty$. Then the distribution $F$ is a sum of $m_0 \leq m$ terms, each supported in one point: $F = \sum L_j \delta(x - x_q), x_q \in \mathbb{R}^n, L_q$ are differential operators in $\mathbb{R}^n$.

**Proof.** We identify the space $\mathbb{R}^d$ with the complex space $\mathbb{C}^n$. Since the functions $z^\alpha, \bar{z}^\beta$ are harmonic, the matrix $A_F$ can be considered as a submatrix of $H_F$, and therefore it has rank not greater than $m$. It
remains to apply Proposition 4.2 to establish that the distribution $F$
has the required form, with no more than $m$ points $x_q$. □

The odd-dimensional case requires considerably more work. We will
use again a kind of dimension reduction, however, unlike the analytic
case, we will need projections of the distribution to one-dimensional
subspaces.

Let $S$ denote the unit sphere in $\mathbb{R}^d$, $S = \{\zeta \in \mathbb{R}^d : |\zeta| = 1\}$ and
let $\sigma$ be the Lebesgue measure on $S$. For $\zeta \in S$, we denote by $\mathcal{L}_\zeta$
the one-dimensional subspace in $\mathbb{R}^d$ passing through $\zeta$, $\mathcal{L}_\zeta = \zeta \mathbb{R}^1$. For
a distribution $F \in \mathcal{E}'(\mathbb{R}^d)$, we define the distribution $F_\zeta \in \mathcal{E}'(\mathbb{R}^1)$ by
setting $\langle F_\zeta, \phi \rangle = \langle F, \phi_\zeta \rangle$, where $\phi_\zeta \in C^\infty(\mathbb{R}^d)$ is
$\phi_\zeta(x) = \phi(x \cdot \zeta)$. The distribution $F_\zeta$ can be understood as result of projecting of $F$
to $\mathcal{L}_\zeta$ with further transplantation of the projection, $\pi_{\mathcal{L}_\zeta} F$, from the line $\mathcal{L}_\zeta$
to the standard line $\mathbb{R}^1$. The Fourier transform $\mathcal{F} F_\zeta$ of $F_\zeta$ is closely
related with $\mathcal{F} F$:

$$\mathcal{F}(F_\zeta)(t) = (\mathcal{F} F)(t\zeta). \quad (5.1)$$

Further on, we will restrict ourselves to the case when the dis-
tribution $F$ is a finite complex Borel measure $\mu$. Here we will use the
notation $\mu_\zeta$ instead of $F_\zeta$.

We need to recall certain facts in harmonic analysis. In the one-
dimensional case, they were proved by N.Wiener as long ago as in
1919; the multi-dimensional version seems to be folklore, however the
formulations we found in the literature, see [9], are slightly weaker than
the ones we need.

Let $\mu$ be a finite complex Borel measure in $\mathbb{R}^d$. We define

$$|\mu| = \left( \sum_{\xi \in \mathbb{R}^d} |\mu(\{\xi\})|^2 \right)^{\frac{1}{2}}.$$

Of course, $|\mu|$ is finite for a finite measure and it vanishes if and only
if $\mu$ has no atoms.

**Lemma 5.2.** Let $\mu$ be a finite Borel measure $\mathbb{R}^d$ and $h$ be a function
in $L_1(\mathbb{R}^1)$. Denote by $\mathcal{F} \mu$ the Fourier transform of $\mu$. Then

$$\lim_{R \to \infty} R^{-d} \int_{\mathbb{R}^d} h(R^{-1} \xi) \mathcal{F} \mu(\xi) d\xi = \mu(\{0\}) \int_{\mathbb{R}^d} h(\xi) d\xi. \quad (5.2)$$

**Proof.** By Plancherel identity, we have

$$\lim_{R \to \infty} R^{-d} \int_{\mathbb{R}^d} h(R^{-1} \xi) \mathcal{F} \mu(\xi) d\xi = \lim_{R \to \infty} \int_{\mathbb{R}^d} (\mathcal{F} h)(Rx) d\mu(x).$$

Now note that $\mathcal{F} h(0) = \int_{\mathbb{R}^d} h(\xi) d\xi$ and $\lim_{R \to 0} (\mathcal{F} h)(Rx) = 0$ for $x \neq 0$
by Riemann-Lebesgue lemma. The proof completes by applying the
Lebesgue dominant convergence theorem. □
The measure \( \mu \) sure on \( W \). We take a function \( \nu \) such that \( \nu = \mu \ast \tilde{\mu} \). Then \( \mathcal{F} \nu = |\mathcal{F} \mu|^2 \) and \( \nu(0) = |\mu|^2 \). It remains to apply Lemma 5.2 to the measure \( \mu \).

We are going to use Corollary 5.3 to relate the properties of the family of measures \( \mu_\zeta, \zeta \in S \), with the properties of \( \mu \).

**Lemma 5.4.** Let \( \mu \) be a finite compactly supported complex Borel measure on \( \mathbb{R}^d \). Then the following two statements are equivalent:

a) The measure \( \mu \) is continuous, i.e., \( \mu(\{x\}) = 0 \) for any \( x \in \mathbb{R}^d \),

b) The measure \( \mu_\zeta \) is continuous for \( \sigma \)-almost all \( \zeta \in S \).

d) The continuous part of \( \mu_\zeta \) is \( \mu \).

**Proof.** We take a function \( h(\xi) \), depending only on \( |\xi| \), \( h(\xi) = H(|\xi|) \) such that \( \int_{\mathbb{R}^d} h(\xi) d\xi = 1 \). So, \( \int_{\mathbb{R}} |r|^{d-1} H(|r|) dr = \frac{2}{\sigma(S)} \). By Corollary 5.3 used in dimension 1 for \( \mu_\zeta \),

\[
|\mu_\zeta|^2 = \lim_{R \to \infty} \frac{\sigma(S)}{2R} \int_{\mathbb{R}} |R^{-1}r|^{d-1} H(R^{-1}|r|)|\mathcal{F} \mu_\zeta(r)|^2 dr.
\]

In what follows we apply the Lebesgue dominant compactness theorem to justify the passing to a limit:

\[
\frac{1}{\sigma(S)} \int_S |\mu_\zeta|^2 d\sigma(\zeta)
\]

\[
= \int_S \lim_{R \to \infty} \frac{1}{2R} \int_{\mathbb{R}} |R^{-1}r|^{d-1} H(R^{-1}|r|)|\mathcal{F} \mu_\zeta(r)|^2 dr d\sigma(\zeta)
\]

\[
= \lim_{R \to \infty} \frac{1}{2R} \int_S \int_{\mathbb{R}} |r|^{d-1} H(R^{-1}|r|)|\mathcal{F} \mu_\zeta(r)|^2 dr d\sigma(\zeta)
\]

\[
= \lim_{R \to \infty} \frac{1}{R^d} \int_S \int_0^\infty r^{d-1} H(R^{-1}r)|\mathcal{F} \mu(r\zeta)|^2 dr d\sigma(\zeta)
\]

\[
= \lim_{R \to \infty} \frac{1}{R^d} \int_{\mathbb{R}^d} h(R^{-1}|\xi|)|\mathcal{F} \mu(\xi)|^2 d\xi = |\mu|^2.
\]

Hence, \( |\mu_\zeta| = 0 \) if and only if \( |\mu_\zeta| = 0 \) for almost all \( \zeta \in S \).

**Corollary 5.5.** For a finite complex Borel measure \( \mu \) with compact support in \( \mathbb{R}^d \) the following three statements are equivalent:

a) \( \mu \) is discrete;

b) \( \mu_\zeta \) is discrete for all \( \zeta \in S \);

c) \( \mu_\zeta \) is discrete for \( \sigma \)-almost all \( \zeta \in S \).

**Proof.** The implications \( a) \implies b) \) and \( b) \implies c) \) are obvious. To establish \( c) \implies a) \), we denote by \( \mu^c \) the continuous part of \( \mu \). Then the statement \( c) \) means that \( (\mu^c)_\zeta \) is discrete for \( \sigma \)-almost all \( \zeta \in S \).

On the other hand, by Lemma 5.4 applied to \( \mu^c \), the measure \( (\mu^c)_\zeta \) is
continuous for \( \sigma \)-almost all \( \zeta \in S \). Being both discrete and continuous, the measure \( (\mu^c)_\zeta \) is zero for \( \sigma \)-almost all \( \zeta \in S \). Passing to the Fourier transform, we obtain \( (\mathcal{F}\mu^c)(r\zeta) = 0 \) for all \( r \) for \( \sigma \)-almost all \( \zeta \in S \). Now, since the Fourier transform \( \mathcal{F}\mu^c \) is smooth, this means that \( \mu^c = 0 \). \( \square \)

Now we return to our finite rank problem.

**Theorem 5.6.** Let \( d \geq 3 \) be an odd integer, \( d = 2n + 1 \). Let \( \mu \) be a finite complex Borel measure in \( \mathbb{R}^d \) with compact support. Suppose that the matrix \( H_\mu \) has finite rank \( m \). Then \( \text{supp} \mu \) consists of no more than \( m \) points.

**Proof.** Fix some \( \zeta \in S \) and choose some \( d - 1 = 2n \)-dimensional linear subspace \( L \subset \mathbb{R}^d \) containing \( L_\zeta \). We choose the co-ordinate system \( x = (x_1, \ldots, x_d) \) in \( \mathbb{R}^d \) so that the subspace \( L \) coincides with \( \{ x : x_d = 0 \} \). The even-dimensional real space \( L \) can be considered as the \( n \)-dimensional complex space \( \mathbb{C}^n \) with co-ordinates \( z = (z_1, \ldots, z_n) \), \( z_j = x_{2j-1} + ix_{2j}, j = 1, \ldots, n \). The functions \( (z, x_d) \mapsto z^\alpha, (z, x_d) \mapsto \bar{z}^\beta \), \( \alpha, \beta \in (\mathbb{Z} + \mathbb{Z})^d \), are harmonic polynomials in \( \mathbb{C}^d \times \mathbb{R}^1 \). Moreover, by definition, \( \langle \mu, z^\alpha \bar{z}^\beta \rangle = \langle \pi_{\mathbb{C}^n} \mu, z^\alpha \bar{z}^\beta \rangle \). Hence, the matrix \( A_{\pi_{\mathbb{C}^n} \mu} \) is a submatrix of the matrix \( H_\mu \), and the former has not greater rank than the latter, \( \text{rank}(A_{\pi_{\mathbb{C}^n} \mu}) \leq m \). So we can apply Theorem 4.1 and obtain that the measure \( \pi_{\mathbb{C}^n} \mu \) is discrete and its support contains not more than \( m \) points. Now we project the measure \( \pi_{\mathbb{C}^n} \mu \) to the real one-dimensional linear subspace \( L_\zeta \) in \( L \). We obtain the same measure as if we had projected \( \mu \) to \( L \zeta \) from the very beginning, and not in two steps i.e., \( \pi_{L_\zeta} \mu \). As a projection of a discrete measure, \( \pi_{L_\zeta} \mu \) is discrete and has no more than \( m \) points in the support. By our definition of the measure \( \mu_\zeta \) as \( \pi_{L_\zeta} \mu \) transplanted to \( \mathbb{R}^1 \), this means that \( \mu_\zeta \) is discrete.

Due to the arbitrariness of the choice of \( \zeta \in S \), we obtain that all measures \( \mu_\zeta \) are discrete. Now we can apply Corollary 5.5 and obtain that the measure \( \mu \) is discrete itself. Finally, in order to show that the number of points in \( \text{supp} \mu \) does not exceed \( m \), we chose \( \zeta \in S \) such that no two points in \( \text{supp} \mu \) project to the same point in \( L_\zeta \). Then the point masses of \( \mu \) cannot cancel each other under the projection, and thus \( \text{card} \text{supp} \mu = \text{card} \text{supp} \mu_\zeta \leq m \).

The number of points in the support of \( \mu \) is estimated in the same way as in Theorems 3.1 and 4.1. \( \square \)

The analysis of the reasoning in the proof shows that the only essential obstacle for extending Theorem 5.6 to the case of distributions is the limitation set by Corollary 5.5. If we were able to prove this Corollary for distributions, all other steps in the proof of Theorem 5.6 would go through without essential changes. However, it turns out that not only the proof of Corollary 5.5 cannot be carried over to the distributional case, but, moreover, the Corollary itself becomes wrong.
The example that we present does not disprove Theorem 5.6 for distributions, however it indicates that the proof, if exists, should involve some other ideas.

**Example 5.7.** Let \( d \geq 2 \). We consider the Schwartz distribution \( F \in \mathcal{S}(\mathbb{R}^d) \) that has \( \cos |\xi| \) as its Fourier transform. By the Paley-Wiener theorem, since \( \mathcal{F}F \) is an entire function of exponential type, \( F \) has compact support, \( F \in \mathcal{E}'(\mathbb{R}^d) \). By (5.1) and spherical symmetry, for any \( \zeta \in S \), \( F\zeta = \mathcal{F}^{-1}(\cos \tau) = \frac{1}{2}(\delta_1 + \delta_{-1}) \). If \( F \) were a measure, then, by Corollary 5.3 it would be discrete. This, however is impossible since \( F \), together with \( \mathcal{F}F \), is rotationally invariant; being both discrete and rotationally invariant, \( F \) must have support in the origin, which contradicts the above expression for \( F\zeta \). The construction also shows that \( F \) is the unique distribution that has \( \frac{1}{2}(\delta_1 + \delta_{-1}) \) as its one-dimensional projections. Of course, we could have directly checked that \( F \) is not a measure, using the fact that \( F \) is, actually, the solution \( u(x,t) \), \( t = 1 \), for the wave equation \( u_{tt} - \Delta u = 0 \) with initial conditions \( u(\cdot,0) = \delta \), \( u_t(\cdot,0) = 0 \). Moreover, from the classical Poisson formulas it follows that \( \text{supp} \ F \) is the sphere \( \{|x| = 1\} \) for odd \( d \) and the ball \( \{|x| \leq 1\} \) for even \( d \). Note, however, that in neither dimension \( F \) generates a finite rank Toeplitz operator.

### 6. Discussion

In the process of exploring the finite rank conjecture, a number of interesting open questions arise. The case of analytical functions is studied completely. However, in the case of harmonic functions the finite rank conjecture is open for weights being distributions that are not measures. The complete solution of this problem would follow from the positive answer to the next question. Let \( d \geq 3 \), \( F \in \mathcal{E}'(\mathbb{R}^d) \). Suppose that \( \pi^H u \) is a distribution with a finite support for every subspace \( H \subset \mathbb{R}^d \) with \( \dim H = d - 1 \). Is it true that the support of \( \mu \) is finite? As Example 5.7 shows, the answer is negative, if we consider subspaces of dimension 1 instead.

Further possible versions of the finite rank conjecture may involve some other elliptic equations playing the part of the Cauchy-Riemann or the Laplace equations in the problem. The first interesting candidate for the study here is the Helmholtz operator \( \mathcal{H}_Eu = \Delta u + Eu, E > 0 \). Let \( \mathcal{P}_H \) be the orthogonal projection from \( L^2(\Omega) \) to the subspace \( \mathcal{H}_E(\Omega) \) consisting of solutions of the Helmholtz equation. With a function (or a compactly supported distribution) \( F \) we associate the Toeplitz operator \( T_F : u \mapsto \mathcal{P}_H uF, u \in \mathcal{H}_E(\Omega) \). Which restrictions on \( F \) are imposed by the the condition that the operator \( T_F \) has a finite rank? The question is of a certain importance for the scattering theory. It is easy to show that if \( T_F \) is zero then \( F \) must be zero. However it is unclear at the moment how to handle the case of a positive...
Finite rank. For the Toeplitz operator corresponding to the projection onto the subspace of solutions of a general elliptic equation, even the case of rank 0 is unresolved.

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(A. Alexandrov) Petersburg Department of Steklov Institute of Mathematics, Russian Academy of Sciences. 27, Fontanka, St. Petersburg, 191023, Russia
E-mail address: alex@pdmi.ras.ru

(G. Rozenblum) Department of Mathematics, Chalmers University of Technology, and Department of Mathematics University of Gothenburg, Chalmers Tvåårgatan, 3, S-412 96 Gothenburg Sweden
E-mail address: grigori@math.chalmers.se