EXISTENCE OF COROTATING ASYMMETRIC VORTEX PAIRS FOR EULER EQUATIONS

ZINEB HASSAINIA AND TAOUFIK HMIDI

Abstract. In this paper, we study the existence of co-rotating and counter-rotating unequal-sized pairs of simply connected patches for Euler equations. In particular, we prove the existence of curves of steadily co-rotating and counter-rotating asymmetric vortex pairs passing through a point vortex pairs with unequal circulations.

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1. Introduction

In this paper we shall be concerned with the dynamics of asymmetric vortex pairs for the two-dimensional incompressible Euler equations. These equations describe the motion of an ideal fluid and take the form

\begin{align}
\begin{aligned}
\partial_t \omega + v \cdot \nabla \omega &= 0 \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
v &= -\nabla^\perp (-\Delta)^{-1} \omega, \\
\omega|_{t=0} &= \omega_0,
\end{aligned}
\end{align}

where \( v = (v_1, v_2) \) refers to the velocity field and its vorticity \( \omega \) is given by the scalar \( \omega = \partial_1 v_2 - \partial_2 v_1 \).

The second equation in (1.1) is nothing but the Biot-Savart law which can be written in the complex form:

\[ v(t, z) = \frac{i}{2\pi} \int_{\mathbb{R}^2} \frac{\omega(t, \zeta)}{\zeta - z} dA(\zeta), \quad \forall z \in \mathbb{C}, \]

where we identify \( v = (v_1, v_2) \) with the complex valued-function \( v_1 + iv_2 \) and \( dA \) denotes the planar Lebesgue measure. The global existence and uniqueness of solutions with initial integrable and bounded vorticity was established a long time ago by Yudovich [30]. He proved that the system
(1.1) admits a unique global solution in the weak sense, provided that the initial vorticity $\omega_0$ lies in $L^1 \cap L^\infty$. In this setting, we can rigorously deal with the so-called vortex patches, which are the characteristic function of bounded domains. This specific structure is preserved along the time and the solution $\omega(t)$ is uniformly distributed in the bounded domain $D_t$, which is the image by the flow mapping of the initial domain $D_0$. When $D_0$ is a disc then the vorticity is stationary. However, elliptical vortex patches undergo a perpetual rotation about their centers without changing the shape. This discovery goes back to Kirchhoff [24] and till now, the ellipses are the only explicit examples with such properties in the setting of vortex patches. The existence of general class of implicit rotating patches, called also V-states, was discovered numerically by Deem and Zabusky [7]. Few years later, Burbea [1] gave an analytical proof using complex analytical tools and bifurcation theory to show the existence of a countable family of V-states with $m$-fold symmetry for each integer $m \geq 2$. More precisely, the rotating patches appear as a collection of one dimensional branches bifurcating from Rankine vortices at the discrete angular velocities set $\{ \frac{m-1}{2m}, m \geq 2 \}$. These local branches were extended very recently to global ones in [15], where the minimum value on the patch boundary of the angular fluid velocity becomes arbitrarily small near the end of each branch. The regularity of the V-states boundary has been conducted in a series of papers [5, 6, 15, 21]. The existence of small loops in the bifurcation diagram has been proved recently in [16]. From numerical point of view, Wu, Overman, and Zabusky [29] went further along the same branches and found singular limiting solutions with $90^\circ$ corner. We also refer to the paper [25] where it is proved that corners with right angles is the only plausible scenario for the limiting V-states.

It is worth pointing out that Burbea’s approach has been intensively exploited in the few last years in different directions. For instance, this was implemented to prove the existence of rotating patches close to Kirchhoff’s ellipses [2, 18], multi-connected patches [9, 17], patches in bounded domains [9], non trivial rotating smooth solutions [3, 4] and rotating vortices with non uniform densities [12]. We mention that many of these results apply not only to the Euler equations but also to more singular nonlinear transport equations as the inviscid surface quasi-geostrophic equations or the quasi-geostrophic shallow-water equations, but with much more involved computations; in this context also see [2, 4, 5, 6, 8, 11, 13, 14, 19, 22].

It is important to emphasize that all of the aforementioned analytical results treat connected patches. However, for the disconnected ones the bifurcation arguments discussed above are out of use. The main objective of this paper is to deal with vortex pairs moving without changing the shape. One of the very simplest nontrivial vortex equilibria is given by a pair of point vortices with magnitude $\gamma_1$ and $\gamma_2$ and far away at a distance $d$. It is well known that if the circulations are with opposite signs $\gamma_1 = -\gamma_2$ then the system travels in a rectilinear motion with a constant speed $U_0 = \frac{\gamma_1}{2\pi d}$, otherwise the pair of point vortices rotates steadily with the angular velocity $\Omega_0 = \frac{\gamma_1 + \gamma_2}{2\pi d^2}$.

Coming back to the emergence of steady disconnected vortex patches, translating vortex pairs of symmetric patches were discovered numerically by Deem and Zabusky [7] and Pierrehumbert [26] where they conjectured the existence of a curve of translating symmetric pair of simply connected patches emerging from two point vortices. A similar study was established by Saffman and Szeto in [27] for the co-rotating vortex pairs, where two symmetric patches with the same circulations rotate about the centroid of the system with constant angular velocity. In the same direction Dritschel [10] calculated numerically the V-states of asymmetric vortex pairs and discussed their linear stability.

The analytical study of the corotating vortex pairs of patches was conducted by Turkington [28] using variational principle. However, this approach does not give sufficient information on the topological structure of each vortex patch and the uniqueness problem is left open. In the same direction Keady [24] implemented the same approach to prove the existence part of translating vortex pairs of symmetric patches. Very recently, Hmidi and Mateu [20] gave direct proof confirming the numerical experiments and showing the existence of co-rotating and counter-rotating vortex
pairs using the contour dynamics equations combined with a desingularization of the point vortex pairs and the application of the implicit function theorem.

The main concern of this work is to explore the existence of co-rotating and counter-rotating asymmetric vortex pairs using the contour dynamics equations. Before stating our result we need to make some notation. Let \( \varepsilon \in (0, 1), \gamma_1, \gamma_2 \in \mathbb{R}, b_1, b_2 \in \mathbb{R}_+ \) and \( d > 2(b_1 + b_2) \). Consider two small simply connected domains \( D_1^{\varepsilon} \) and \( D_2^{\varepsilon} \) containing the origin and contained in the open ball \( B(0, 2) \) centered at the origin and with radius 2. Define

\[
\omega_{0,\varepsilon} = \frac{\gamma_1}{\varepsilon^2 b_1^2} \chi_{\tilde{D}_1^{\varepsilon}} + \frac{\gamma_2}{\varepsilon^2 b_2^2} \chi_{\tilde{D}_2^{\varepsilon}},
\]

with

\[
\tilde{D}_1^{\varepsilon} = \varepsilon b_1 D_1^{\varepsilon} \quad \text{and} \quad \tilde{D}_2^{\varepsilon} = -\varepsilon b_2 D_2^{\varepsilon} + d.
\]

Informally stated, our main existence result is the following

**Theorem 1.1.** There exists \( \varepsilon_0 > 0 \) such that the following results hold true.

1. For any \( \gamma_1, \gamma_2 \in \mathbb{R} \) such that \( \gamma_1 + \gamma_2 \neq 0 \) and any \( \varepsilon \in (0, \varepsilon_0] \) there exists two strictly convex domains \( D_1^{\varepsilon} \) and \( D_2^{\varepsilon} \) at least of class \( C^1 \) such that \( \omega_{0,\varepsilon} \) generates a co-rotating vortex pair for (1.1).
2. For any \( \gamma_1 \in \mathbb{R} \) and any \( \varepsilon \in (0, \varepsilon_0] \) there exists \( \gamma_2 = \gamma_2(\varepsilon) \) and two strictly convex domains \( D_1^{\varepsilon}, D_2^{\varepsilon} \) at least of class \( C^1 \) such that \( \omega_{0,\varepsilon} \) generates a counter-rotating vortex pair for (1.1).

**Remark 1.2.** The domains \( D_j^{\varepsilon}, j = 1, 2 \) are small perturbations of the unit disc. Moreover, as a by-product of the proofs the corresponding conformal parametrization \( \phi_j^{\varepsilon} : \mathbb{T} \to \partial D_j^{\varepsilon} \) belongs to \( C^{1+\beta} \) for any \( \beta \in (0, 1) \), and has the Fourier asymptotic expansion

\[
\phi_j^{\varepsilon}(\varepsilon, w) = w + \delta_j \left( \frac{\varepsilon b_j}{d} \right)^2 w + \delta_j \left( \frac{\varepsilon b_j}{d} \right)^3 w^2 + \delta_j \left( \frac{\varepsilon b_j}{d} \right)^4 \left( w^3 + 6(1 + \delta_j) w \right)
\]

\[
+ \frac{\delta_j}{4} \left( \frac{\varepsilon b_j}{d} \right)^5 \left( w^4 + 3(1 + \delta_j) w^3 \right) + o(\varepsilon^5), \quad j = 1, 2,
\]

where \( \delta_j = \frac{\gamma_j - 1}{\gamma_j} \) in the co-rotating case and \( \delta_j = -1 \) in the translating one. In addition, the angular velocity has the expansion

\[
\Omega(\varepsilon) = \frac{\gamma_1 + \gamma_2}{2d^2} + \frac{\varepsilon^4}{2d^6} \left( \gamma_1 b_2^4 + \gamma_2 b_1^4 \right) + o(\varepsilon^4),
\]

and the center of rotation has the expansion

\[
Z(\varepsilon) = \frac{\gamma_2 d}{\gamma_1 + \gamma_2} + \frac{\varepsilon^4}{d^3(\gamma_1 + \gamma_2)^2} \left( \frac{\gamma_2^2 b_1^4}{\gamma_1} - \frac{\gamma_1^2 b_2^4}{\gamma_2} \right) + o(\varepsilon^4).
\]

In the case of translation, the speed has the expansion

\[
U(\varepsilon) = \frac{\gamma_1}{2d} \left( 1 + \frac{\varepsilon^4}{d^4} (2b_1^4 + b_2^4) \right) + o(\varepsilon^4)
\]

and the vorticity \( \gamma_2 \) has the expansion

\[
\gamma_2(\varepsilon) = \gamma_1 \left( 1 + \frac{\varepsilon^4}{d^4} (b_1^4 - b_2^4) \right) + o(\varepsilon^4).
\]

**Remark 1.3.** As we shall see later, the case \( \varepsilon = 0 \) corresponds to the vortex point system. This allows to recover the classical result stating that two point vortices at distance 2d and magnitudes \( 2\pi \gamma_1 \) and \( 2\pi \gamma_2 \) rotate uniformly about their centroid with the angular velocity \( \Omega_0 = \frac{\gamma_1 + \gamma_2}{2d^2} \) provided that \( \gamma_1 + \gamma_2 \neq 0 \). However, they translate uniformly with the speed \( U = \frac{\gamma_1}{2d} \) when \( \gamma_1 \) and \( \gamma_2 \) are opposite.
Remark 1.4. The interval $(0, \varepsilon_0]$ is uniform in $b_1$ and $b_2$ and therefore we may recover the point vortex-vortex patch configuration by letting $b_1$ and $b_2$ go to 0.

The proof of this theorem is done in the spirit of the work [20] using contour dynamics reformulation. It is a known fact that the initial vorticity $\omega_{0, \varepsilon}$ with the velocity $v_{0, \varepsilon}$ generates a rotating solution, with constant angular velocity $\Omega$ around the point $Z$ on the real axis, if and only if

$$
(1.2) \quad \text{Re}\left\{ -i\Omega (\bar{z} - Z) - n(z) \right\} = 0 \quad \forall z \in \partial D_1 \cup \partial D_2,
$$

where $n$ is the exterior unit normal vector to the boundary at the point $z$. According to the Biot-Savart law combined with Green-Stokes formula, we can write

$$
\bar{v}(z) = \frac{i\gamma_1}{2\varepsilon^2 b_1^2} \int_{\partial D_1} \frac{\bar{\xi} - \bar{z}}{\bar{\xi} - z} d\xi + \frac{i\gamma_2}{2\varepsilon^2 b_2^2} \int_{\partial D_2} \frac{\bar{\xi} - \bar{z}}{\bar{\xi} - z} d\xi, \quad \forall z \in \mathbb{C}.
$$

Following the same line of [20] we can remove the singularity in $\varepsilon$ by slightly perturbing the unit disc with a small amplitude of order $\varepsilon$ and taking advantage of its symmetry. Indeed, we seek for conformal parametrization of the boundaries $\phi_j : \mathbb{D}^c \to [D_j^c]^c$, in the form

$$
\phi_j(w) = w + \varepsilon b_j f_j(w), \quad \text{with} \quad f_j(w) = \sum_{n \geq 1} \frac{a_n^j}{w^n}, \quad a_n^j \in \mathbb{R}, j = 1, 2.
$$

Straightforward computations and substitutions we find that the conformal mappings are subject to two coupled nonlinear equations defined as follows: for all $w \in \mathbb{T}$ and $j \in \{1, 2\}$,

$$
(1.3) \quad F_j(\varepsilon, \Omega, Z, f_1, f_2) = \text{Im}\left\{ 2\Omega \left( \varepsilon b_j \frac{\phi_j(w)}{w} + (-1)^j Z - (j - 1)d \right) + J_\varepsilon^{(j)}(w) \right\} w \phi_j(w) = 0,
$$

where $J_\varepsilon$ ranges in the Hölder space $C^\alpha$ and can be extended for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 > 0$. In addition, the functional $F = (F_1, F_2) : (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times \mathbb{R} \times X \to Y$ is well-defined and it is of class $C^1$ where

$$
X \triangleq \left\{ f \in (C^{1+\alpha}(\mathbb{T}))^2, \quad f(w) = \sum_{n \geq 1} A_n w^n, \quad A_n \in \mathbb{R}^2, w \in \mathbb{T} \right\}
$$

and

$$
Y = \left\{ g \in (C^\alpha(\mathbb{T}))^2, \quad g = \sum_{n \geq 1} C_n e_n, \quad C_n \in \mathbb{R}^2, w \in \mathbb{T} \right\}, \quad e_n(w) = \text{Im}\{w^n\}.
$$

In order to apply the Implicit Function Theorem we compute the linearized operator around the point vortex pairs leading to

$$
D_{(f_1, f_2)} F(0, \Omega, Z, 0, 0)(h_1, h_2)(w) = - \begin{pmatrix} \gamma_1 \text{Im}\{h_1(w)\} \\ \gamma_2 \text{Im}\{h_2(w)\} \end{pmatrix},
$$

This operator is not invertible from $X$ to $Y$ but it does from $X$ to $\tilde{Y}$, where

$$
\tilde{Y} \triangleq \left\{ g \in Y : C_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.
$$

The idea to remedy to this defect is to consider the operator: for any $h = (\alpha_1, \alpha_2, h_1, h_2) \in \mathbb{R} \times \mathbb{R} \times X$
Given $\gamma$ advected by the flow, then it is folklore (see, for instance, rotating pair of patches about some point $Z_2(\mathbf{1}) \tau$ where $\bar{n}$ is the exterior unit normal vector to the boundary at the point $z$. From the Biot–Savart law, the velocity can be recovered from the vorticity by

\[
\overline{v(z)} = \frac{i\gamma_1}{2\pi\varepsilon^2 b_1^2} \int_{\partial \tilde{D}_1^\varepsilon} \frac{dA(\zeta)}{\zeta - z} + \frac{i\gamma_2}{2\pi\varepsilon^2 b_2^2} \int_{\partial \tilde{D}_2^\varepsilon} \frac{dA(\zeta)}{\zeta - z}, \quad \forall z \in \mathbb{C}.
\]

Therefore, by using Green-Stokes formula we get

\[
\overline{v(z)} = \frac{i\gamma_1}{2\varepsilon^2 b_1^2} \int_{\partial \tilde{D}_1^\varepsilon} \frac{\bar{\zeta} - \bar{z}}{\zeta - z} d\bar{\xi} + \frac{i\gamma_2}{2\varepsilon^2 b_2^2} \int_{\partial \tilde{D}_2^\varepsilon} \frac{\bar{\zeta} - \bar{z}}{\zeta - z} d\bar{\xi}, \quad \forall z \in \mathbb{C}.
\]
where

\[(2.4) \quad \overline{v}(z) = \frac{i \gamma_1}{2 \varepsilon^2 b_1} \int_{\partial \tilde{D}_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} \, d\xi - \frac{i \gamma_2}{2 \varepsilon^2 b_2} \int_{\partial \tilde{D}_2^\varepsilon} \frac{\overline{\xi} + \overline{z} - d}{\xi + z - d} \, d\xi\]

we have used the notation

\[\tilde{D}^\varepsilon_3 \triangleq - \tilde{D}^\varepsilon_5 + d = \varepsilon b_2 D_2^\varepsilon.\]

Hence by combining the last identity with (2.4) and (2.3) we obtain

\[
\begin{align*}
\text{Re}\left\{\left(2\Omega(\overline{\tau} - Z) + \frac{\gamma_1}{\varepsilon b_1^2} \int_{\partial \tilde{D}_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} \, d\xi - \frac{\gamma_2}{\varepsilon b_2^2} \int_{\partial \tilde{D}_2^\varepsilon} \frac{\overline{\xi} + \overline{z} - d}{\xi + z - d} \, d\xi\right) \tau\right\} &= 0 \quad \forall z \in \partial \tilde{D}_1^\varepsilon, \\
\text{Re}\left\{\left(2\Omega(\overline{\tau} + Z - d) - \frac{\gamma_1}{\varepsilon^2 b_1^2} \int_{\partial \tilde{D}_1^\varepsilon} \frac{\overline{\xi} + \overline{z} - d}{\xi + z - d} \, d\xi + \frac{\gamma_2}{\varepsilon^2 b_2^2} \int_{\partial \tilde{D}_2^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} \, d\xi\right) \tau\right\} &= 0 \quad \forall z \in \partial \tilde{D}_3^\varepsilon
\end{align*}
\]

where \(\tau\) denotes a tangent vector to the boundary of \(\tilde{D}_1^\varepsilon \cup \tilde{D}_3^\varepsilon\) at the point \(z\). Observe that when \(z \in \partial \tilde{D}_1\) then \(-z + d \notin \tilde{D}_3\) and conversely; when \(z \in \partial \tilde{D}_3\), then \(-z + d \notin \tilde{D}_1\). Hence according to residue theorem, we find that for \(j \in \{1, 3\}\),

\[
\int_{\partial \tilde{D}_j^\varepsilon} \frac{\overline{\xi} + \overline{z} - d}{\xi + z - d} \, d\xi = \int_{\partial \tilde{D}_j^\varepsilon} \frac{\overline{\xi}}{\xi + z - d} \, d\xi \quad \forall z \in \partial \tilde{D}_{3-j}^\varepsilon.
\]

Combining this identity with the change of variables \(z \rightarrow \varepsilon b_1 z\) and \(z \rightarrow \varepsilon b_2 z\), which send \(\tilde{D}_1^\varepsilon\) to \(D_1^\varepsilon\) and \(\tilde{D}_3^\varepsilon\) to \(D_2^\varepsilon\) respectively, we get

\[
\begin{align*}
\text{Re}\left\{\left(2\Omega(\varepsilon \overline{\tau} - Z) + \frac{\gamma_1}{\varepsilon b_1} \int_{\partial D_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} \, d\xi - \frac{\gamma_2}{\varepsilon b_2} \int_{\partial D_2^\varepsilon} \frac{\overline{\xi} + \overline{z} - d}{\xi + z - d} \, d\xi\right) \tau\right\} &= 0 \quad \forall z \in \partial D_1^\varepsilon, \\
\text{Re}\left\{\left(2\Omega(\varepsilon \overline{b_3} \overline{\tau} + Z - d) - \frac{\gamma_1}{\varepsilon^2 b_1} \int_{\partial D_1^\varepsilon} \frac{\overline{\xi} + \overline{z} - d}{\xi + z - d} \, d\xi + \frac{\gamma_2}{\varepsilon^2 b_2} \int_{\partial D_2^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} \, d\xi\right) \tau\right\} &= 0 \quad \forall z \in \partial D_2^\varepsilon.
\end{align*}
\]

The structure of the last system may be unified as follows

\[(2.5) \quad \text{Re}\left\{\left(2\Omega(\varepsilon b_j \overline{\tau} + (\varepsilon b_j)^2 \overline{\tau} - (j - 1)\overline{d}) + \frac{\gamma_j}{\varepsilon b_j} \int_{\partial D_j^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} \, d\xi - \gamma_{3-j} \int_{\partial D_{3-j}^\varepsilon} \frac{\overline{\xi}}{\xi + z - d} \, d\xi\right) \tau\right\} = 0, \quad \forall z \in \partial D_j, \quad j = 1, 2.
\]

We shall look for domains \(\{D_j, j = 1, 2\}\) which are slight perturbation of the unit disc of order \(\varepsilon\).

In other words, we shall impose the conformal mapping \(\phi_j : \mathbb{D}^\varepsilon \rightarrow [D_j^\varepsilon]^c\) to satisfy the expansions

\[
\phi_j(w) = w + \varepsilon b_j f_j(w), \quad \text{with} \quad f_j(w) = \sum_{n \geq 1} \frac{a_n^j}{w^n}, \quad a_n^j \in \mathbb{R}, j = 1, 2.
\]

Because a tangent vector to the boundary at \(z = \phi_j(w)\) is determined by

\[
\tau(\phi_j(w)) = i w \phi'_j(w),
\]

then by a change of variables the steady vortex pairs system (2.5) becomes: for all \(w \in \mathbb{T}\),

\[(2.6) \quad \text{Im}\left\{\left(2\Omega(\varepsilon b_j \phi_j(w) + (\varepsilon b_j)^2 Z - (j - 1)\overline{d}) + \gamma_j f_j^2(w) - \frac{\gamma_i}{\varepsilon} \frac{K_{3-j}^\varepsilon(w)}{2} w \phi'_j(w)\right)\right\} = 0, \quad j = 1, 2
\]
where
\[ I_j^\varepsilon(w) = \frac{1}{\varepsilon b_j} \int_\gamma \frac{\phi_j(f_j(w)) - \phi_j(w)}{\phi_j(w) - \phi_j(w)} \phi_j'(\tau)d\tau \]

and
\[ K_j^\varepsilon(w) \int_\gamma \frac{\phi_i(\tau) \phi_i'(\tau)}{\varepsilon b_i \phi_i(\tau) + \varepsilon b_j \phi_j(w) - \tau}d\tau. \]

Now, we shall see how to remove the singularity in \( \varepsilon \) from the full non-linearity \( I_j^\varepsilon(w) \). To alleviate the discussion, we use the notation
\[ A = \tau - w \quad \text{and} \quad B_j = f_j(\tau) - f_j(w). \]

Thus
\[ I_j^\varepsilon(w) = \frac{1}{\varepsilon b_j} \int_\gamma \frac{\overline{A} + \varepsilon b_j B_j}{A + \varepsilon b_j B_j} \left[ 1 + \varepsilon b_j f_j'(\tau) \right] d\tau \]
\[ = \int_\gamma \frac{\overline{A} f_j'(\tau)d\tau + \varepsilon b_j \int_\gamma \frac{\overline{A} \overline{B}_j - \overline{A} B_j}{A(A + \varepsilon b_j B_j)} f_j'(\tau)d\tau + \int_\gamma \frac{\overline{A} \overline{B}_j - \overline{A} B_j}{A^2} d\tau \]
\[ - \varepsilon b_j \int_\gamma \frac{(\overline{A} \overline{B}_j - \overline{A} B_j) B_j}{A^2(A + \varepsilon b_j B_j)} d\tau + \frac{1}{\varepsilon b_j} \int_\gamma \frac{\overline{A}}{A} d\tau. \]

From the identity
\[ \int_\gamma \frac{\overline{A}}{A} d\tau = \int_\gamma \frac{\overline{\varphi} - \overline{\varphi}}{\varphi - \varphi} d\tau = -\overline{\varphi} \]
and by the residue theorem
\[ \int_\gamma \frac{\overline{A} f_j'(\tau)d\tau = 0, \quad \int_\gamma \frac{\overline{A} \overline{B}_j - \overline{A} B_j}{A^2} d\tau = 0. \]

Therefore
\[ (2.7) \quad \text{Im}\left\{ I_j^\varepsilon(w) w \phi_j'(w) \right\} = \varepsilon b_j \text{Im}\left\{ w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_\gamma \frac{\overline{A} \overline{B}_j - \overline{A} B_j}{A(A + \varepsilon b_j B_j)} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \right\} \]
\[ - \text{Im}\left\{ f_j'(w) \right\}. \]

Inserting the last identity in the system (2.6) we get
\[ (2.8) \quad F_j(\varepsilon, g)(w) = 0 \quad \forall w \in \gamma \]
where
\[ g \equiv (\Omega, Z, f_1, f_2) \]
and
\[ (2.9) \quad F_j(\varepsilon, g) \equiv \text{Im}\left\{ 2\Omega \left[ \varepsilon b_j (\overline{w} + \varepsilon b_j \overline{f_j'(w)}) + (-1)^j Z - (j - 1)d \right] w \left( 1 + \varepsilon b_j f_j'(w) \right) - \gamma_j f_j'(w) \right\} \]
\[ + \varepsilon b_j \gamma_j w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_\gamma \frac{\overline{A} \overline{B}_j - \overline{A} B_j}{A(A + \varepsilon b_j B_j)} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \]
\[ - \gamma_{3-j} w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_\gamma \frac{(\overline{\varphi} + \varepsilon b_{3-j} f_{3-j}(\varphi)) \left( 1 + \varepsilon b_{3-j} f_{3-j}(\varphi) \right)}{\varepsilon b_{3-j} f_{3-j}(\varphi) + b_j^2 f_j(w) + \varepsilon^2 (b_{3-j}^2 f_{3-j}(\varphi) + b_j^2 f_j(w)) - d\tau \}
\[ \equiv \text{Im}\left\{ F_{1j}(\varepsilon, g)(w) + F_{2j}(\varepsilon, g)(w) + F_{3j}(\varepsilon, g)(w) \right\} \]
Next we use the following notation

\begin{equation}
(2.10)\quad F(\varepsilon, g)(w) \triangleq \left( F_1(\varepsilon, g)(w), F_2(\varepsilon, g)(w) \right) \quad \forall w \in \mathbb{T}.
\end{equation}

2.2. Counter-rotating vortex pair. We shall consider two bounded simply connected domains \( D_1^\varepsilon \) and \( D_2^\varepsilon \), containing the origin and contained in the ball \( B(0,2) \). For \( b_1, b_2 \in (0, \infty) \) and \( d > b_1 + b_2 \) we set

\begin{equation}
(2.11)\quad \tilde{D}_1^\varepsilon \triangleq \varepsilon b_1 D_1^\varepsilon \quad \text{and} \quad \tilde{D}_2^\varepsilon \triangleq -\varepsilon b_2 D_2^\varepsilon + d.
\end{equation}

Given \( \gamma_1, \gamma_2 \in \mathbb{R} \), we consider the initial vorticity

\begin{equation}
(2.12)\quad \omega_{0,\varepsilon} = \frac{\gamma_1}{\varepsilon^2 b_1^2} \chi_{\tilde{D}_1^\varepsilon} - \frac{\gamma_2}{\varepsilon^2 b_2^2} \chi_{\tilde{D}_2^\varepsilon}.
\end{equation}

We assume that this unequal-sized pair of simply connected patches with vorticity magnitudes \( \gamma_1 \) and \( -\gamma_2 \) travels steadily in \((Oy)\) direction with uniform velocity \( U \). Then in the moving frame the pair of the patches is stationary and consequently, \( \gamma \)

\begin{equation}
(2.13)\quad \text{Re}\left\{ (\overline{v(z)} + iU) \bar{n} \right\} \quad \forall z \in \partial \tilde{D}_1^\varepsilon \cup \partial \tilde{D}_2^\varepsilon,
\end{equation}

where \( \bar{n} \) is the exterior unit normal vector to the boundary of \( \tilde{D}_1^\varepsilon \cup \tilde{D}_2^\varepsilon \) at the point \( z \). From (2.4) one has

\begin{equation}
\text{Re}\left\{ \left( 2U + \frac{\gamma_1}{\varepsilon^2 b_1^2} \int_{\partial \tilde{D}_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi + \frac{\gamma_2}{\varepsilon^2 b_2^2} \int_{\partial \tilde{D}_2^\varepsilon} \frac{\overline{\xi} + \overline{z} - d}{\xi + z - d} d\xi \right) \overline{\tau} \right\} = 0 \quad \forall z \in \partial \tilde{D}_1^\varepsilon,
\end{equation}

\begin{equation}
\text{Re}\left\{ \left( 2U + \frac{\gamma_1}{\varepsilon^2 b_1^2} \int_{\partial \tilde{D}_1^\varepsilon} \frac{\overline{\xi} + \overline{z} - d}{\xi + z - d} d\xi + \frac{\gamma_2}{\varepsilon^2 b_2^2} \int_{\partial \tilde{D}_2^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi \right) \overline{\tau} \right\} = 0 \quad \forall z \in \partial \tilde{D}_3^\varepsilon
\end{equation}

where we have used the notation

\( \tilde{D}_3^\varepsilon \triangleq \varepsilon b_2 D_2^\varepsilon \),

and \( \overline{\tau} \) denote for tangent vector to the boundary of \( \tilde{D}_1^\varepsilon \cup \tilde{D}_3^\varepsilon \) at the point \( z \). We can easily verify that when \( z \in \partial \tilde{D}_1^\varepsilon \), \(-z + d \notin \overline{\tilde{D}_1^\varepsilon} \). Therefore, if \( z \in \partial \tilde{D}_3^\varepsilon \) then \(-z + d \notin \overline{\tilde{D}_1^\varepsilon} \). Thus, by the residue theorem we conclude that for all \( i, j \in \{1, 3\} \) and \( i \neq j \), we have

\begin{equation}
\int_{\partial \tilde{D}_j^\varepsilon} \frac{\overline{\xi} + \overline{z} - d}{\xi + z - d} d\xi = \int_{\partial \tilde{D}_j^\varepsilon} \frac{\overline{\xi}}{\xi + z - d} d\xi \quad \forall z \in \partial \tilde{D}_{3-i}^\varepsilon.
\end{equation}

Inserting the last identity into the system above and changing \( z \to \varepsilon b_1 z \) and \( z \to \varepsilon b_2 z \) we find

\begin{equation}
(2.14)\quad \text{Re}\left\{ \left( 2U + \frac{\gamma_j}{\varepsilon b_j} \int_{\partial D_j^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi + \gamma_{3-j} \int_{\partial D_{3-j}^\varepsilon} \frac{\overline{\xi}}{\xi + z - d} d\xi \right) \overline{\tau} \right\} = 0, \quad \forall z \in \partial D_j
\end{equation}

with \( j = 1, 2 \). We shall now use the conformal parametrization of the boundaries \( \phi_j : \mathbb{D} \to [D_j^\varepsilon]^c \),

\( \phi_j(w) = w + \varepsilon b_j f_j(w) \), with \( f_j(w) = \sum_{n \geq 1} a_n^j w^n \), \( a_n^j \in \mathbb{R}, j = 1, 2 \).

Since the tangent vector to the boundary at \( z = \phi_j(w) \) is given by

\( \overline{\tau}(\phi_j(w)) = i w \phi_j'(w) \),

then by a change of variables the system (2.14) becomes

\begin{equation}
(2.15)\quad \text{Im}\left\{ \left( 2U + \gamma_j I_j^\varepsilon(w) + \gamma_{3-j} K_{3-j}^\varepsilon(w) \right) w \phi_j'(w) \right\} = 0, \quad j = 1, 2
\end{equation}
for all \( w \in \mathbb{T} \), where
\[
I_j^\varepsilon(w) \triangleq \frac{1}{\varepsilon b_j} \int_{\mathbb{T}} \frac{\phi_j(\tau) - \phi_j(w)}{\phi_j(\tau) - \phi_j(w)} \phi_j'(\tau) d\tau,
\]
\[
K_j^\varepsilon(w) \triangleq \int_{\mathbb{T}} \frac{\phi_i(\tau) \phi_j'(\tau)}{\varepsilon b_i(\tau) + \varepsilon b_j \phi_j(w) - d} d\tau.
\]
As in the rotating case, from (2.7), one has
\[
\text{Im}\left\{ I_j^\varepsilon(w) \phi_j'(w) \right\} = \varepsilon b_j \text{Im}\left\{ w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_{\mathbb{T}} \frac{A\bar{B}_j - \bar{A}B_j}{A(A + \varepsilon b_j B_j)} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \right\} - \text{Im}\left\{ f_j'(w) \right\},
\]
where we have used the notation
\[
A = \tau - w \quad \text{and} \quad B_j = f_j(\tau) - f_j(w).
\]
Inserting the last identity into (2.15) we get
\[
F(\varepsilon, g)(w) \triangleq (F_1(\varepsilon, g)(w), F_2(\varepsilon, g)(w)) = 0, \quad \forall w \in \mathbb{T}, \quad j = 1, 2
\]
and for all \( w \in \mathbb{T} \) with \( j = 1, 2 \).

(2.17) \quad F_j(\varepsilon, g) \triangleq \text{Im}\left\{ 2Uw \left( 1 + \varepsilon b_j f_j'(w) \right) - \gamma_j f_j'(w) \right\} + \varepsilon b_j \gamma_j w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_{\mathbb{T}} \frac{A\bar{B}_j - \bar{A}B_j}{A(A + \varepsilon b_j B_j)} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau
\]
\[
+ \gamma_{3-j} w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_{\mathbb{T}} \varepsilon \left( b_{3-j} \tau + b_j w \right) + \varepsilon^2 \left( b_{3-j}^3 f_{3-j}(\tau) + b_j^3 f_j(w) \right) - d \right\}
\]
\[
\triangleq \text{Im}\left\{ F_{1,j}(\varepsilon, g) + F_{2,j}(\varepsilon, g) + F_{3,j}(\varepsilon, g) \right\}.
\]

3. Regularity of the nonlinear functional

This section is devoted to the regularity study of the nonlinear functional \( F \) introduced in (2.10) and (2.17) and which defines the V-states equations. We proceed first with the Banach spaces \( X \) and \( Y \) of Hölder type to which the Implicit Function Theorem will be applied. Recall that for given \( \alpha \in (0, 1) \), we denote by \( C^\alpha \) the space of continuous functions \( f : \mathbb{T} \to \mathbb{C} \) such that
\[
\|f\|_{C^\alpha(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \sup_{\tau \neq w \in \mathbb{T}} \frac{|f(\tau) - f(w)|}{|\tau - w|^\alpha} < \infty.
\]
For any integer \( n \in \mathbb{N} \), the space \( C^{n+\alpha}(\mathbb{T}) \) stands for the set of functions \( f \) of class \( C^n \) whose \( n \)-th order derivative is Hölder continuous with exponent \( \alpha \). This space is equipped with the usual norm
\[
\|f\|_{C^{n+\alpha}(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \left\| \frac{d^m f}{dw^m} \right\|_{C^\alpha(\mathbb{T})}.
\]
Now, consider the spaces
\[
X \triangleq \left\{ f \in \left( C^{1+\alpha}(\mathbb{T}) \right)^2, f(w) = \sum_{n \geq 1} A_n w^n, \quad A_n \in \mathbb{R}^2, \quad w \in \mathbb{T} \right\},
\]
\[
Y = \left\{ g \in \left( C^\alpha(\mathbb{T}) \right)^2, g(w) = \sum_{n \geq 1} C_n e_n, \quad C_n \in \mathbb{R}^2, \quad w \in \mathbb{T} \right\}, \quad e_n = \text{Im}\{ w^n \},
\]
\[
9
\]
(3.1) \[
\tilde{Y} \triangleq \left\{ g \in Y : C_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.
\]

For \( r > 0 \) we denote by \( B_r \) the open ball of \( X \) centered at zero and of radius \( r \),
\[
B_r \triangleq \left\{ f \in X, \ |f|_{C^{\alpha + n}(\mathbb{T})} \leq r \right\}.
\]

It is straightforward that for any \( f \in B_r \) the function \( w \mapsto \phi(w) = w + \varepsilon f(w) \) is conformal on \( \mathbb{C} \setminus \mathbb{D} \) provided that \( r, \varepsilon < 1 \). Moreover according to Kellog-Warshawski result, the boundary of \( \phi(\mathbb{C} \setminus \mathbb{D}) \) is a Jordan curve of class \( C^{\alpha + n} \).

### 3.1. Co-rotating vortex pairs

We propose to prove the following result concerning the regularity of \( F \).

**Proposition 3.1.** The following assertions hold true.

1. The function \( F \) can be extended to \( C^1 \) function from \( (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times \mathbb{R} \times B_1 \to Y \).
2. Two initial point vortex \( \gamma_1, \pi \delta_0 \) and \( \gamma_2, \pi \delta_d \) with \( \gamma_1 \neq -\gamma_2 \) rotate uniformly about the point \( Z_0 \triangleq -\gamma_2 \gamma_1 \) with the angular velocity \( \Omega_0 \triangleq \frac{\gamma_1 + \gamma_2}{2d^2} \).
3. For all \( h = (\alpha_1, \alpha_2, h_1, h_2) \in \mathbb{R} \times \mathbb{R} \times X \) one has
   \[
   D_g F(0, g_0) h(w) = -\frac{2\alpha_1 d}{\gamma_1 + \gamma_2} \left( \frac{\gamma_2}{\gamma_1} \right) \text{Im}\{w\} - \frac{\alpha_2 (\gamma_1 + \gamma_2)}{d^2} \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \text{Im}\{w\} - \left( \gamma_1 \text{Im}\{h_1'(w)\}, \gamma_2 \text{Im}\{h_2'(w)\} \right)
   \]
   with \( g \triangleq (\Omega, Z, f_1, f_2) \) and \( g_0 \triangleq (\Omega_0, Z_0, 0, 0) \).
4. The linear operator \( D_g F(0, g_0) : \mathbb{R} \times \mathbb{R} \times X \to Y \) is an isomorphism.

**Proof.** (1) Notice that the part \( F_{1j} + F_{2j} \) defined in (2.9) appears identically in the boundary equation of co-rotating symmetric pairs and its regularity was discussed in the paper [20]. The only term that one should care about is \( F_{3j} \) describing the interaction between the boundaries of the two patches which are supposed to be disjoint. Therefore the involved kernel is sufficiently smooth and it does not carry significant difficulties in the treatment. and can be dealt with in a very classical way.

(2) According to the formulation developed in Section 2 the two points vortex system is a solution to the equation \( F(0, \Omega, Z, 0, 0) = 0 \). In this case we can easily check that
\[
F_j(0, \Omega, Z, 0, 0)(w) = \left[ 2\Omega \left( (-1)^j Z + (1 - j)d \right) + \frac{\gamma_3 - j}{d} \right] \text{Im}\{w\}
\]
Therefore \( F_j(0, \Omega, Z, 0, 0) = 0 \) if and only if
\[
-2\Omega Z + \frac{\gamma_2}{d} = 0 \quad \text{and} \quad 2\Omega \left( Z - d \right) + \frac{\gamma_1}{d} = 0.
\]
Thus,
\[
\Omega = \Omega_0 \triangleq \frac{\gamma_1 + \gamma_2}{2d^2} \quad \text{and} \quad Z = Z_0 \triangleq \frac{d\gamma_2}{\gamma_1 + \gamma_2}.
\]
(3)-(4) Since \( F = (F_1, F_2) \) then for given \( h = (h_1, h_2) \in X \), we have

\[
D_f F(0, \Omega, Z, 0, 0)h(w) = \left( \frac{\partial F_1(0, \Omega, Z, 0, 0)h_1(w) + \partial F_2(0, \Omega, Z, 0, 0)h_2(w)}{\partial F_2(0, \Omega, Z, 0, 0)h_1(w) + \partial F_2(0, \Omega, Z, 0, 0)h_2(w)} \right).
\]

The Gâteaux derivative of the function \( F_j \) at \((0, \Omega, b, 0, 0)\) in the direction \( h_j \) is given by

\[
\partial_{f_j} F_j(0, \Omega, Z, 0, 0)h_j(w) = \frac{d}{dt} \left[ F_j(0, \Omega, Z, th_j, 0) \right]_{t=0} \quad (w) = -\gamma_j \Im \{ h'_j(w) \}.
\]

On the other hand, straightforward computations allow to get

\[
\partial_{f_{3-j}} F_j(0, \Omega, Z, 0, 0)h_{3-j}(w) = \frac{d}{dt} \left[ F_j(0, \Omega, Z, th_{3-j}) \right]_{t=0} \quad (w) = 0.
\]

From (2.9) one can easily check that

\[
F_j(0, \Omega, Z, f_1, f_2) = \Im \left\{ 2\Omega \left[ (-1)^j Z + (1 - j)d \right] w - \gamma_j f'_j(w) + \frac{\gamma_{3-j}}{d} w \right\}.
\]

Differentiating the last identity with respect to \( \Omega \) at \((\Omega_0, Z_0)\) gives

\[
\partial_{\Omega} F_j(0, \Omega_0, Z_0, f_1, f_2) = 2 \left[ (-1)^j Z_0 + (1 - j)d \right] \Im \{ w \}.
\]

Next, differentiating (3.3) with respect to \( Z \) at \((\Omega_0, Z_0)\) we get

\[
\partial_{Z} F_j(0, \Omega_0, Z_0, f_1, f_2) = 2(-1)^j \Omega_0 \Im \{ w \}.
\]

Therefore, for all \((\alpha_1, \alpha_2) \in \mathbb{R}^2\)

\[
D_{(\Omega, Z)} F(0, \Omega_0, Z_0, f_1, f_2)(\alpha_1, \alpha_2) = \alpha_1 \left( \frac{2\gamma_2 d}{\gamma_1 + \gamma_2} \right) \Im \{ w \} + \alpha_2 \left( \frac{\gamma_1 + \gamma_2}{d^2} \right) \Im \{ \overline{w} \}.
\]

Consequently the restricted linear operator \( D_{(\Omega, Z, f_1, f_2)} F(0, \Omega_0, Z_0, f_1, f_2) \) is invertible if and only if the determinant of the two vectors, given by \(-\frac{2}{d}(\gamma_1 + \gamma_2)\), is non vanishing.

\[\square\]

3.2. Counter-rotating vortex pairs. This section is devoted to the counter-rotating asymmetric pairs and our main result reads as follows.

**Proposition 3.2.** The following assertions hold true.

1. The function \( F \) can be extended to \( C^1 \) function from \((-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times \mathbb{R} \times B_1 \rightarrow Y\).
2. Two initial point vortex \( \gamma_1 \pi \delta_0 \) and \(-\gamma_2 \pi \delta_{(d,0)}\) translate uniformly with the speed \( U_0 \triangleq \frac{\gamma_1}{2d} \).
3. For all \( h = (\alpha_1, \alpha_2, h_1, h_2) \in \mathbb{R} \times \mathbb{R} \times X \) one has

\[
D_g F(0, g_0)h(w) = \alpha_1 \left( \frac{2}{2} \right) \Im \{ w \} - \alpha_2 \left( \frac{1}{0} \right) \Im \{ w \} - 2U_0 \left( \frac{\Im \{ h'_1(w) \}}{\Im \{ h'_2(w) \}} \right)
\]

where

\[
g \triangleq (U, \gamma_2, f_1, f_2) \quad \text{and} \quad g_0 \triangleq (U_0, \gamma_1, 0, 0).
\]
4. The linear operator \( D_g F(U_0, \gamma_1, 0, 0) : \mathbb{R} \times \mathbb{R} \times X \rightarrow Y \) is an isomorphism.
**Proof.** (1) Notice that the function $F_{j2} + F_{j3}$ coincide with the function $F_{j2} + F_{j3}$ appearing in the rotating case and $F_j$ enjoys obviously the required regularity.

(2) The point vortex configuration corresponds to $F(0, U, \gamma_2, 0, 0)$. Thus, for $\varepsilon = 0$ one has

$$F_j(0, \gamma_1, \gamma_2, 0, 0)(w) = \left[2U - \frac{\gamma_3 - j}{d}\right]{\text{Im}}\{w\}.$$ 

Therefore $F(0, U, \gamma_2, 0, 0) = 0$ if and only if

$$\gamma_1 = \gamma_2 = 2Ud.$$ 

(3)-(4) Taking $\varepsilon = 0$ in (2.17) we get

$$F_j(0, U, \gamma_2, f_1, f_2) = \text{Im}\left\{2Uw - \gamma_j f_j'(w) - \frac{\gamma_3 - j}{d}w\right\}.$$ 

Thus, for given $(h_1, h_2) \in X$, we have

$$\partial f_j F_j(0, U, \gamma_2, 0, 0) h_j = -\gamma_j \text{Im}\{h_j'(w)\}$$

and

$$\partial f_{3-j} F_j(0, U, \gamma_2, 0, 0) h_{3-j}(w) = 0.$$ 

On the other hand, differential of $F$ with respect $(U, \gamma_2)$ is given by

$$D_{(U, \gamma_2)}F(0, U, \gamma_2, f_1, f_2)(\alpha_1, \alpha_2) = \alpha_1 \left(\frac{2}{2}\right) \text{Im}\{w\} + \alpha_2 \left(-\frac{1}{0}\right) \text{Im}\{w\}.$$ 

Thus, we conclude that the linear operator $D_g F(0, g_0)$ is invertible for all $\gamma_1 \in \mathbb{R}$. 

\[\square\]

### 4. Existence of asymmetric co-rotating patches

The goal of this section is to provide a full statement of the point (1) of Theorem 1.1. In other words, we shall describe the set of solutions of the equation $F(\varepsilon, g) = 0$ around the point $(\varepsilon, g) = (0, g_0)$ by a one-parameter smooth curve $\varepsilon \mapsto g(\varepsilon)$ using the implicit function theorem.

#### 4.1. Co-rotating vortex pairs.

The main result of this section reads as follows.

**Proposition 4.1.** The following assertions hold true.

1. There exists $\varepsilon_0 > 0$ and a unique $C^1$ function $g \triangleq (\Omega, Z, f_1, f_2) : [-\varepsilon_0, \varepsilon_0] \to \mathbb{R} \times \mathbb{R} \times B_1$ such that

$$F\left(\varepsilon, \Omega(\varepsilon), Z(\varepsilon), f_1(\varepsilon), f_2(\varepsilon)\right) = 0$$

with

$$\left(\Omega(0), Z(0), f_1(0), f_2(0)\right) = (\Omega_0, Z_0, 0, 0).$$

2. For all $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$ one has

$$(f_1(\varepsilon), f_2(\varepsilon)) \neq (0, 0).$$

3. If $(\varepsilon, f_1, f_2)$ is a solution then $(-\varepsilon, \tilde{f}_1, \tilde{f}_2)$ is also a solution, where

$$\forall w \in \mathbb{T}, \quad \tilde{f}_j(w) = f_j(-w), \quad j = 1, 2.$$

4. For all $\varepsilon \in [-\varepsilon_0, \varepsilon] \setminus \{0\}$, the domains $D_1^\varepsilon$ and $D_2^\varepsilon$ are strictly convex.
Proof. (1) From Proposition 3.1 one has \( F : (-\frac{1}{2}, \frac{1}{2}) \mathbb{R} \times \mathbb{R} \times B_1 \rightarrow Y \) is a \( C^1 \) function and \( D_y F(0, g_0) : X \rightarrow Y \) is an isomorphism. Thus the implicit function theorem ensures the existence of \( \varepsilon_0 > 0 \) and a unique \( C^1 \) parametrization \( g = (\Omega, Z, f_1, f_2) : [-\varepsilon_0, \varepsilon_0] \rightarrow B_1 \) verifying the equation \( F(\varepsilon, \Omega(\varepsilon), Z(\varepsilon), f_1(\varepsilon), f_2(\varepsilon)) = 0 \). Moreover, this solution passes through the origin, that is,

\[
(\Omega, Z, f_1, f_2)(0) = (0, 0).
\]

This completes the proof of the desired result.

(3) We shall prove that for any \( \varepsilon \) small enough and any \( \Omega \) we can not get a rotating vortex pair with \( f_1 = 0 \) and \( f_2 = 0 \). In other words, we need to prove that

\[
F(\varepsilon, \Omega, Z, 0, 0) \neq 0.
\]

To this end we write

\[
F_j(\varepsilon, \Omega, Z, 0, 0)(w) = \text{Im} \left\{ 2\Omega \left( (-1)^j Z - (j - 1)d \right) - \gamma_{3-j} \int_{\Omega} \frac{\tau}{\varepsilon(b_{3-j} \tau + b_j w) - d} d\tau \right\} w.
\]

By Taylor expansion, we get

\[
\int_{\Omega} \frac{\tau}{\varepsilon(b_{3-j} \tau + b_j w) - d} d\tau = -\sum_{n \in \mathbb{N}} \frac{\varepsilon^n}{d^{n+1}} \int_{\Omega} \tau(b_{3-j} \tau + b_j w)^n d\tau = -\sum_{n \in \mathbb{N}} \frac{\varepsilon^n}{d^{n+1}} b_j^n w^n = \frac{1}{\varepsilon b_j w - d}.
\]

Consequently,

\[
F_j(\varepsilon, \Omega, Z, 0, 0)(w) = \text{Im} \left\{ 2\Omega \left( (-1)^j Z - (j - 1)d \right) - \frac{\gamma_{3-j}}{\varepsilon b_j w - d} \right\} w,
\]

and this quantity is not zero if \( \varepsilon \neq 0 \) is small enough.

(2) We only need to check that

\[
F_j(\varepsilon, \Omega, Z, f_1, f_2)(-w) = -F_j(-\varepsilon, \Omega, Z, \tilde{f}_1, \tilde{f}_2)(w) \quad \text{for} \quad j = 1, 2.
\]

where \( F_j \) is defined by (2.9). By definition we have

\[
F_{1j}(-\varepsilon, \Omega, Z, \tilde{f}_1, \tilde{f}_2) = 2\Omega \left[ -\varepsilon b_j \left( \overline{\eta} - \varepsilon \eta \tilde{f}_j(\overline{\eta}) \right) + (-1)^j Z - (j - 1)d \right] w \left( 1 - \varepsilon \eta \tilde{f}_j'(w) \right) - \gamma_j \tilde{f}_j'(w).
\]

Since \( \tilde{f}_j'(w) = -f_j'(-w) \) we get

\[
F_{1j}(-\varepsilon, \Omega, Z, \tilde{f}_1, \tilde{f}_2) = 2\Omega \left[ \varepsilon b_j \left( -\overline{\eta} + \varepsilon \eta f_j(-\overline{\eta}) \right) + (-1)^j Z - (j - 1)d \right] w \left( 1 + \varepsilon \eta f_j'(-w) \right) + \gamma_j f_j'(-w) = -F_{ij}(\varepsilon, \Omega, Z, f_1, f_2)(-w).
\]
Straightforward computations will lead to the same properties for the functions $F_{2j}$ and $F_{3j}$. This completes the proof of (4).

(3) Now to prove the convexity of the domains $D_1^\varepsilon$, $D_2^\varepsilon$ we shall reproduce the same arguments of [20]. Recall that the outside conformal mapping associated to the domain $D_j^\varepsilon$ is given by

$$\phi_j = w + \varepsilon b_j f_j(w).$$

and the curvature can be expressed by the formula

$$\kappa(\theta) = \frac{1}{|\phi_j'(w)|} \text{Re}\left(1 + w \frac{\phi_j''(w)}{\phi_j'(w)}\right).$$

We can easily verify that

$$1 + w \frac{\phi_j''(w)}{\phi_j'(w)} = 1 + \varepsilon b_j w \frac{f_j''(w)}{1 + \varepsilon b_j f_j'(w)}$$

and so

$$\text{Re}\left(1 + w \frac{\phi_j''(w)}{\phi_j'(w)}\right) \geq 1 - |\varepsilon| b_j \frac{|f_j''(w)|}{1 - |\varepsilon| b_j |f_j'(w)|} \geq |\varepsilon| b_j.$$ 

The last quantity is non-negative if $|\varepsilon| < 1/2$. Thus the curvature is strictly positive and therefore the domain $D_j^\varepsilon$ is strictly convex. \qed

4.2. Counter-rotating vortex pairs. In this section we shall give the complete statement for the existence of translating unequal-sized pairs of patches.

**Proposition 4.2.** The following holds true.

1. There exists $\varepsilon_0 > 0$ and a unique $C^1$ function $g \triangleq (U, \gamma_2, f_1, f_2) : [-\varepsilon_0, \varepsilon_0] \rightarrow \mathbb{R} \times \mathbb{R} \times B_1$ such that

$$F\left(\varepsilon, U(\varepsilon), \gamma_2(\varepsilon), f_1(\varepsilon), f_2(\varepsilon)\right) = 0,$$

with

$$\left(U(0), \gamma_2(0), f_1(0), f_2(0)\right) = \left(\gamma_1, 0, 0\right).$$

2. If $(\varepsilon, f_1, f_2)$ is a solution then $(-\varepsilon, \tilde{f}_1, \tilde{f}_2)$ is also a solution, where

$$\forall w \in \mathbb{T}, \quad \tilde{f}_j(w) = f_j(-w), \quad j = 1, 2.$$

3. For all $\varepsilon \in [-\varepsilon_0, \varepsilon] \setminus \{0\}$, the domains $D_1^\varepsilon$ and $D_2^\varepsilon$ are strictly convex

**Proof.** Same arguments as in Proposition 4.1. \qed

5. Asymptotic behavior

In this section we study the asymptotic expansion for small $\varepsilon$ of the conformal mappings, the angular velocity, the center of rotation and the circulations.
5.1. Co-rotating pairs. In the case of the co-rotating pairs we get the following expansion at higher order in $\varepsilon$.

**Proposition 5.1.** The conformal mappings of the co-rotating domains, $\phi_j : \mathbb{T} \rightarrow \partial D_{j}^\varepsilon$, have the expansions

$$
\phi_j(\varepsilon, w) = w + \delta_j \left( \frac{\varepsilon b_j}{d} \right)^2 \bar{w} + \frac{\delta_j}{2} \left( \frac{\varepsilon b_j}{d} \right)^3 \bar{w}^2 + \frac{\delta_j}{3} \left( \frac{\varepsilon b_j}{d} \right)^4 \left( \bar{w}^3 + 6(1 + \delta_j)w \right) + \frac{\delta_j}{4} \left( \frac{\varepsilon b_j}{d} \right)^5 \left( \bar{w}^4 + 3(1 + \delta_j)w^2 \right) + o(\varepsilon^5),
$$

for all $w \in \mathbb{T}$ and $j = 1, 2$ where $\delta_j = \frac{\gamma_{j+1} - \gamma_j}{\gamma_j}$. The angular velocity has the expansion

$$
\Omega(\varepsilon) = \frac{\gamma_1 + \gamma_2}{2d^2} + \varepsilon^4 \left( \frac{\gamma_1 b_j^1 + \gamma_2 b_{j+1}^1}{\gamma_1} \right),
$$

and the center of rotation has the expansion

$$
Z(\varepsilon) = \frac{\gamma_2 d}{\gamma_1 + \gamma_2} + \varepsilon^4 \frac{d^2(\gamma_1 + \gamma_2)^2}{d^2(\gamma_1 + \gamma_2)^2} \left( \frac{\gamma_2 b_j^1}{\gamma_1} - \frac{\gamma_2 b_{j+1}^1}{\gamma_2} \right).
$$

**Proof.** Recall that

$$
\phi_j(\varepsilon, w) = w + \varepsilon b_j f_j(\varepsilon, w), \quad j = 1, 2,
$$

and from Proposition 4.1 one has

$$(5.1) \quad F(\varepsilon, g(\varepsilon)) \triangleq F(\varepsilon, \Omega(\varepsilon), Z(\varepsilon), f_1(\varepsilon), f_2(\varepsilon)) = 0.$$

where $F$ is defined in (2.9). Moreover

$$(5.2) \quad g(0) = (\Omega_0, Z_0, 0, 0).$$

By the composition rule we get

$$(5.3) \quad \partial_\varepsilon g(0) = -D_g F(0, g_0)^{-1} \partial_\varepsilon F(0, g_0).$$

In view of (2.9) one has

$$
F_j(\varepsilon, g_0) = -\gamma_{3-j} \text{Im} \left\{ \left( \frac{1}{d} + \int_{\mathbb{T}} \varepsilon \left( b_{3-j} \tau + b_j w \right) - d \right) w \right\}
$$

and from (4.1) we conclude that

$$(5.4) \quad F_j(\varepsilon, g_0) = \gamma_{3-j} \sum_{n=1}^{\infty} \frac{\varepsilon^n b_j^n}{d^{n+1}} \text{Im} \{ w^{n+1} \}.$$

Thus

$$(5.5) \quad \partial_\varepsilon^n F(0, g_0)(w) = \frac{n!}{d^{n+1}} \left( \frac{\gamma_2 b_j^n}{\gamma_1 b_2^n} \right) \text{Im} \{ w^{n+1} \}.$$

Now, we shall write down the explicit expression of $D_g F(0, g_0)^{-1}$. Recall from Proposition 3.1 that for all $h = (\alpha_1, \alpha_2, h_1, h_2) \in \mathbb{R} \times \mathbb{R} \times X$ with

$$
h_j(w) = \sum_{n \geq 1} a_n^j \bar{w}^n, \quad j = 1, 2,
$$

one gets

$$
D_g F(0, g_0)h(w) = -\frac{2\alpha_1 d}{\gamma_1 + \gamma_2} \left( \frac{\gamma_2}{\gamma_1} \right) \text{Im} \{ w \} - \alpha_2 \frac{\gamma_1 + \gamma_2}{d^2} \left( \frac{1}{-1} \right) \text{Im} \{ w \} - \sum_{n \geq 1} n \left( \frac{\gamma_1 a_n^1}{\gamma_2 a_n^2} \right) \text{Im} \{ w^{n+1} \}.
$$
Then, for any \( k \in Y \) with the expansion
\[
k(w) = \sum_{n \geq 0} \left( \frac{A_n}{B_n} \right) \text{Im}\{w^{n+1}\},
\]
one has
\[
D_g F(0, g_0)^{-1} k(w) = \left( -\frac{A_0 + B_0}{2d}, -\frac{(A_0 \gamma_1 - B_0 \gamma_2) d^2}{(\gamma_1 + \gamma_2)^2}, -\sum_{n \geq 1} \frac{A_n}{n \gamma_1} \overline{w}^n, -\sum_{n \geq 1} \frac{B_n}{n \gamma_2} \overline{w}^n \right).
\]
Combining the last identity with (5.5) and (5.3) we get
\[
\partial_\varepsilon g(0, w) = \left( 0, 0, \frac{b_1 \gamma_2}{d^2 \gamma_1}, \frac{b_2 \gamma_1}{d^2 \gamma_2} \right).
\]
Let’s move to the second order derivative in \( \varepsilon \) at 0. By the composition rule we get
\[
\partial^2_{\varepsilon \varepsilon} g(0) = -D_g F(0, g_0)^{-1} \left( \partial^2_\varepsilon F(0, g_0) + 2D^2_{\varepsilon g} F(0, g_0) g'(0) + D^2_{gg} F(0, g_0) [g'(0), g'(0)] \right).
\]
Observe that
\[
F_j(0, g) = \text{Im}\left\{ 2\Omega \left[ (-1)^j Z + (1 - j)d \right] w - \gamma_j f_j'(w) + \frac{\gamma_{3-j}}{d} w \right\}.
\]
Thus, for all \( h = (\alpha_1, \alpha_2, h_1, h_2), k = (\alpha_1, \alpha_2, k_1, k_2) \in \mathbb{R} \times \mathbb{R} \times X \), one has
\[
\partial^2_{gg} F_j(0, g) [h, k] = (-1)^j 2\alpha_1 \beta_2 \text{Im}\{w\}.
\]
From the identity (5.7) we get
\[
\partial^2_{gg} F_j(0, g_0) [g'(0), g'(0)] = 0.
\]
Next, differentiating the identity (2.9) with respect to \( \varepsilon \) we get
\[
\partial_\varepsilon F_j(\varepsilon, g) = \text{Im}\left\{ 2\Omega \right[ \varepsilon b_j \left( \overline{w} + \varepsilon b_j f_j'(w) \right) + \left( (-1)^j Z - (1 - 1)d \right) \right] w b_j f_j'(w)
+ 2b_j \Omega \left( \overline{w} + \varepsilon b_j f_j'(w) \right) w \left( 1 + \varepsilon b_j f_j'(w) \right)
+ b_j \gamma_j w \left( 1 + 2\varepsilon b_j f_j'(w) \right) \int_T \frac{A \overline{B}_j - \overline{A} B_j}{A \left( A + \varepsilon b_j B_j \right)} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau
- \varepsilon b_j^2 \gamma_j w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_T \frac{B_j \left( A \overline{B}_j - \overline{A} B_j \right)}{A \left( A + \varepsilon b_j B_j \right)^2} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau
- \gamma_{3-j} b_j w f_j'(w) \int_T \frac{(\overline{\gamma} + \varepsilon b_{3-j} f_{3-j}(\tau)) (1 + \varepsilon b_{3-j} f_{3-j}(\tau))}{\varepsilon C_j + \varepsilon^2 D_j - d} d\tau
- \gamma_{3-j} b_{3-j} w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_T \frac{f_{3-j}(\tau) (1 + \varepsilon b_{3-j} f_{3-j}(\tau)) + (\overline{\gamma} + \varepsilon b_{3-j} f_{3-j}(\tau)) f_{3-j}(\tau)}{\varepsilon C_j + \varepsilon^2 D_j - d} d\tau
+ \gamma_{3-j} w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_T \frac{(C_j + 2\varepsilon D_j) (\overline{\gamma} + \varepsilon b_{3-j} f_{3-j}(\tau)) (1 + \varepsilon b_{3-j} f_{3-j}(\tau))}{(\varepsilon C_j + \varepsilon^2 D_j - d)^2} d\tau\right\}.
\]
with
\[
A = \tau - w, \quad B_j = f_j(\tau) - f_j(w), \quad C_j = (b_{3-j} \tau + b_j w), \quad D_j = b_{3-j}^2 f_{3-j}(\tau) + b_j^2 f_j(w).
\]
For $\varepsilon = 0$ we have

\[
(5.13) \quad \partial \varepsilon F_j(0, g) = \text{Im} \left\{ 2\Omega \left( (-1)^j Z - (j - 1)d \right) wb_j f'_j(w) + b_j \gamma_j \int_{\mathbb{T}} \frac{A\overline{B_j} - \overline{A}B_j}{A^2} \left[ f'_j(\tau) - \frac{B_j}{A} \right] d\tau \right. \\
+ \frac{\gamma^3 - jb_j}{d} w f'_j(w) + \left. \frac{\gamma^3 - jb_j}{d^2} w^2 \right\}.
\]

Consequently, for all $h = (\alpha_1, \alpha_2, h_1, h_2) \in \mathbb{R} \times \mathbb{R} \times X$, one finds

\[
(5.14) \quad \partial^2_{eg} F_j(0, g) h = \alpha_1 \text{Im} \left\{ 2\Omega \left( (-1)^j Z - (j - 1)d \right) wb_j f'_j(w) \right\} + \alpha_2 \text{Im} \left\{ 2\Omega (-1)^j wb_j f'_j(w) \right\} + \text{Im} \left\{ 2\Omega \left( (-1)^j Z - (j - 1)d \right) wb_j h'_j(w) + \frac{\gamma^3 - jb_j}{d} wh'_j(w) \right\} \\
+ b_j \gamma_j \int_{\mathbb{T}} \frac{2\text{Im} \left\{ A\overline{f_j(\tau)} - \overline{f_j(w)} \right\}}{A^2} \left[ h'_j(\tau) - \frac{h_j(\tau) - h_j(w)}{A} \right] d\tau \\
+ b_j \gamma_j \int_{\mathbb{T}} \frac{2\text{Im} \left\{ A\overline{h_j(\tau)} - \overline{h_j(w)} \right\}}{A^2} \left[ f'_j(\tau) - \frac{f_j(\tau) - f_j(w)}{A} \right] d\tau \right\}.
\]

Then, for $g = g_0 \triangleq \left( \frac{2\gamma + \gamma_2}{2\gamma_1}, \frac{d\gamma_2}{\gamma_1 + \gamma_2}, 0, 0 \right)$ one gets

\[
(5.15) \quad \partial^2_{eg} F_j(0, g_0) h = 0.
\]

Plugging the identities (5.5), (5.6), (5.11) and (5.15) into (5.8) yields

\[
(5.16) \quad g''(0) = \left( 0, 0, \frac{b_1^2 \gamma_2}{d^3 \gamma_1 w^2}, \frac{b_2^2 \gamma_1}{d^3 \gamma_2 w^2} \right).
\]

Next, we move to the third order derivative

\[
(5.17) \quad \partial^3_{ggg} F_j(0, 0) = -D_s F(0, g_0)^{-1} \left( \partial^3_{ggg} F(0, g_0) + 3\partial^2_{eg} F(0, g_0) g''(0) + 3\partial^3_{ggg} F(0, g_0) \left[ g'(0), g'(0) \right] \right) \\
+ 3\partial^2_{eg} F(0, g_0) g''(0) + 3\partial^2_{gg} F(0, g_0) \left[ g'(0), g''(0) \right] + \partial^2_{ggg} F(0, g_0) \left[ g'(0), g'(0), g'(0) \right] \right),
\]

Owing to the identity (5.13), for all $h = (\alpha_1, \alpha_2, h_1, h_2), k = (\beta_1, \beta_2, k_1, k_2) \in \mathbb{R} \times \mathbb{R} \times X$, one has one gets

\[
(5.18) \quad \partial^3_{ggg} F_j(0, g[h, k]) = \beta_1 \text{Im} \left\{ 2\left( (-1)^j Z - (j - 1)d \right) wb_j h'_j(w) \right\} \\
+ \beta_2 \text{Im} \left\{ 2(-1)^j \Omega wb_j h'_j(w) \right\} + \alpha_1 \text{Im} \left\{ 2\left( (-1)^j Z - (j - 1)d \right) wb_j k'_j(w) \right\} \\
+ \alpha_2 \text{Im} \left\{ 2(-1)^j \Omega wb_j k'_j(w) \right\} + (\alpha_1 \beta_2 + \alpha_2 \beta_1) \text{Im} \left\{ 2(-1)^j wb_j f'_j(w) \right\} \\
+ \text{Im} \left\{ b_j \gamma_j \int_{\mathbb{T}} \frac{2\text{Im} \left\{ A\overline{k_j(\tau)} - \overline{k_j(w)} \right\}}{A^2} \left[ h'_j(\tau) - \frac{h_j(\tau) - h_j(w)}{A} \right] d\tau \right\} \\
+ b_j \gamma_j \int_{\mathbb{T}} \frac{2\text{Im} \left\{ A\overline{h_j(\tau)} - \overline{h_j(w)} \right\}}{A^2} \left[ k'_j(\tau) - \frac{k_j(\tau) - k_j(w)}{A} \right] d\tau \right\}.
\]

Thus, for $g = g_0 \triangleq \left( \frac{2\gamma + \gamma_2}{2\gamma_1}, \frac{d\gamma_2}{\gamma_1 + \gamma_2}, 0, 0 \right)$, $h = g'(0)$ and $k = g''(0)$ we find
(5.19) \[ \partial_{\varepsilon}^3 F_j(0, g_0)[\varepsilon'(0), g'(0)] = 2b_j \gamma_j \Im \left\{ -w \frac{b_j^2 \varepsilon_{3-j}}{d^2 \gamma_j^2} \int_{\mathbb{T}} \left( 1 - \frac{(\tau - w)^2}{(\tau - \overline{w})^2} \right) \left( \frac{\tau^2 - \frac{\tau - w}{\tau - \overline{w}}} \right) d\tau \right\} = 2b_j \gamma_j \Im \left\{ -\frac{b_j^2 \varepsilon_{3-j}}{d^4 \gamma_j^2} \right\} = 0. \]

Next, differentiating \((5.12)\) with respect to \(\varepsilon\) gives

(5.20) \[ \partial_{\varepsilon}^2 F_j(\varepsilon, g) = \Im \left\{ 4\Omega \frac{b_j^2 \overline{f}_j(w) f_j'(w)}{A(1 + \varepsilon_b f_j'(w))} + 4b_j^2 \Omega(\overline{w} + 2\varepsilon_d f_j(w)) f_j'(w) \right\} \]

\[ + 2b_j^2 \gamma_j \omega f_j'(w) \int_{\mathbb{T}} \frac{\overline{A B_j} - \overline{A B}_j}{A(A + \varepsilon_b f_j'(w))} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \]

\[ - 2b_j^2 \gamma_j \omega \left( 1 + 2\varepsilon_d f_j'(w) \right) \int_{\mathbb{T}} \frac{B_j(\overline{A B}_j - \overline{A B}_j)}{A(A + \varepsilon_b f_j'(w))^2} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \]

\[ + 2\varepsilon_d b_j \gamma_j \omega \left( 1 + \varepsilon_d f_j(w) \right) \int_{\mathbb{T}} \frac{B_j(\overline{A B}_j - \overline{A B}_j)}{A(A + \varepsilon_b f_j'(w))^3} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \]

\[ - 2\gamma_j \omega b_j b_3 f_j(w) \int_{\mathbb{T}} \left( \frac{C_j + 2\varepsilon D_j}{\varepsilon C_j + \varepsilon^2 D_j - d} \right) \left( 1 + \varepsilon_b \frac{f_3-j(\varepsilon) + 2\varepsilon \frac{f_3-j(\varepsilon)}{f_3-j(\tau)}}{f_3-j(\tau)} \right) d\tau \]

\[ - 2b_j^2 \gamma_j \omega b_3 \omega \int_{\mathbb{T}} \left( 1 + \varepsilon_d f_j'(w) \right) \int_{\mathbb{T}} \frac{f_3-j(\varepsilon) \frac{f_3-j(\varepsilon)}{f_3-j(\tau)}}{\varepsilon C_j + \varepsilon^2 D_j - d} d\tau \]

\[ + 2\gamma_j b_3 \omega b_3 \omega \left( 1 + \varepsilon_d f_j'(w) \right) \int_{\mathbb{T}} \frac{(C_j + 2\varepsilon D_j) \left( 1 + \varepsilon_b \frac{f_3-j(\tau)}{f_3-j(\varepsilon)} \right) \left( \varepsilon C_j + \varepsilon^2 D_j - d \right)^2}{\varepsilon C_j + \varepsilon^2 D_j - d} d\tau \]

\[ + 2\gamma_j b_3 \omega b_3 \omega \left( 1 + \varepsilon_d f_j'(w) \right) \int_{\mathbb{T}} \frac{(C_j + 2\varepsilon D_j) \left( \varepsilon C_j + \varepsilon^2 D_j - d \right)^2}{\varepsilon C_j + \varepsilon^2 D_j - d} d\tau \]

\[ - 2\gamma_j b_3 \omega b_3 \omega \left( 1 + \varepsilon_d f_j'(w) \right) \int_{\mathbb{T}} \frac{(C_j + 2\varepsilon D_j) \left( \varepsilon C_j + \varepsilon^2 D_j - d \right)^2}{\varepsilon C_j + \varepsilon^2 D_j - d} d\tau \]

where we have used the notation

\[ A = \tau - w, \quad B_j = f_j(\tau) - f_j(w), \quad C_j = (b_{3-j} + b_j w), \quad D_j = b_j^2 f_{3-j}(\tau) + b_j^2 f_j(w). \]
At \( \varepsilon = 0 \) one has
\[
\partial_{\varepsilon \varepsilon}^2 F_j(0, g) = 2 \text{Im} \left\{ 2 \Omega b_j^2 \left( w f_j(w) + f_j'(w) \right) + b_j^2 \gamma_j w f_j'(w) \right. \\
- b_j^2 \gamma_j w \left. \int_\mathcal{T} \frac{B_j(\overline{A B_j} - \overline{A B_j})}{A^3} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \right. \\
+ \frac{1}{d^2} b_j^2 \gamma_{3-j} w \left. \int_\mathcal{T} f_{3-j}(\tau) f_{3-j}(\tau) d\tau + \frac{1}{d^2} \gamma_{3-j} b_j^2 w^2 \right\}.
\]

Thus, for all \( h = (\alpha_1, \alpha_2, h_1, h_2) \in \mathbb{R} \times \mathbb{R} \times X \) one finds
\[
\partial_{\varepsilon \varepsilon \varepsilon}^3 F_j(0, g) h = 2 \alpha_1 \text{Im} \left\{ 2 b_j^2 \left( w f_j(w) + f_j'(w) \right) \right. \\
+ 2 \text{Im} \left\{ 2 \Omega b_j^2 \left( w h_j(w) + h_j'(w) \right) + \frac{1}{d^2} \gamma_{3-j} w^2 b_j^2 h_j'(w) + \frac{1}{d^2} \gamma_{3-j} b_j^2 w h_j(w) \right. \\
+ b_j^2 \gamma_j w \left. \int_\mathcal{T} \left[ h_j'(w) - \frac{h_j(\tau) - h_j(w)}{A} \right] \frac{B_j(\overline{A B_j} - \overline{A B_j})}{A^2} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \right. \\
+ b_j^2 \gamma_j w \left. \int_\mathcal{T} \left[ f_j'(w) - \frac{B_j}{A} \right] 2 \text{Im} \left\{ \frac{A(h_j(\tau) - h_j(w))}{A^2} \right\} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \right. \\
+ b_j^2 \gamma_j w \left. \int_\mathcal{T} \left[ f_j'(w) - \frac{B_j}{A} \right] \frac{A B_j(\overline{A B_j} - \overline{A B_j})}{A^2} \left[ h_j'(\tau) - \frac{h(\tau) - h_j(w)}{A} \right] d\tau \right. \\
+ \frac{2}{d} b_j^2 \gamma_{3-j} \text{Im} \left\{ w \int_\mathcal{T} f_{3-j}(\tau) h_{3-j}(\tau) d\tau + w \int_\mathcal{T} h_{3-j}(\tau) f_{3-j}(\tau) d\tau \right\}. 
\]

Therefore, for \( g = g_0 \triangleq \left( \frac{\gamma_1 + \gamma_2}{2d}, \frac{d \gamma_0}{\gamma_1 + \gamma_2}, 0, 0 \right) \), \( h = g'(0) \) we get
\[
\partial_{\varepsilon \varepsilon \varepsilon}^3 F_j(0, g_0) g'(0) = \frac{4 b_j^3 \gamma_{3-j}}{d^2} (\gamma_1 + \gamma_2) \text{Im} \left\{ w^2 \right\}.
\]

Next, in view of (5.10) one has
\[
\partial_{\varepsilon \varepsilon \varepsilon}^3 F_j(0, g_0) [g'(0), g''(0)] = 0
\]
and
\[
\partial_{\varepsilon \varepsilon \varepsilon}^3 F_j(0, g_0) [h, k, l] = 0 \quad \text{for all} \quad h, k, l \in \mathbb{R} \times \mathbb{R} \times X.
\]

Putting together the identities (5.5), (5.19), (5.22), (5.23) (5.24) and (5.17) we conclude that
\[
\partial_{\varepsilon \varepsilon \varepsilon}^3 g(0, w) = \frac{12}{d^4} \left( 0, 0, b_1^3 \left( 1 + \frac{\gamma_2}{\gamma_1} \right) \frac{\gamma_2}{\gamma_1} w + b_1^3 \frac{\gamma_2}{\gamma_1} w^3, b_2^3 \left( 1 + \frac{\gamma_1}{\gamma_2} \right) \frac{\gamma_1}{\gamma_2} w + b_2^3 \frac{\gamma_1}{\gamma_2} w^3 \right).
\]
Now we move to the fourth order derivative in $\varepsilon$ of $g$. By the composition rule we get

\[
\partial^4_{\varepsilon\varepsilon\varepsilon\varepsilon} g(0) = -D_g F(0, g_0)^{-1} \left( \partial^4_{\varepsilon\varepsilon\varepsilon\varepsilon} F(0, g_0) + 4\partial^4_{\varepsilon\varepsilon\varepsilon g} F(0, g_0) g'(0) + 6\partial^3_{\varepsilon\varepsilon g} F(0, g_0) g^{(2)}(0) \\
+ 4\partial_{\varepsilon g} F(0, g_0) g^{(3)}(0) + 6\partial^2_{\varepsilon g} F(0, g_0) [g'(0), g'(0)] + 12\partial^2_{g g} F(0, g_0) [g''(0), g'(0)] \\
+ 4\partial^2_{g g} F(0, g_0) [g'(0), g'(0)] + 3\partial^2_{g g} F(0, g_0) [g''(0), g''(0)] \\
+ 4\partial_{g g} F(0, g_0) [g^{(3)}(0), g'(0)] + 6\partial^3_{g g} F(0, g_0) [g'(0), g''(0), g'(0)] \\
+ \partial^4_{g g g g} F(0, g_0) [g'(0), g'(0), g'(0), g'(0)] \right).
\]

In view of (5.10) one has

\[
\partial^2_{g g} F(0, g_0) [g^{(3)}(0), g'(0)] = \partial^2_{g g} F(0, g_0) [g''(0), g''(0)] = 0
\]

and

\[
\partial^4_{g g g g} F(0, g_0) [g'(0), g'(0), g'(0), g'(0)] = \partial^3_{g g g} F(0, g_0) [g'(0), g''(0), g'(0)] = 0.
\]

Moreover, from (5.15) we find

\[
\partial_{g g} F(0, g_0) g^{(3)}(0) = 0
\]

and by (5.18) we obtain

\[
\partial^3_{g g g} F_j(0, g)[h, k, l] = (\beta_1 \sigma_2 + \beta_2 \sigma_1) \text{Im} \left\{ 2(-1)^j w b_j h_j'(w) \right\} + (\alpha_1 \sigma_2 + \alpha_2 \sigma_1) \text{Im} \left\{ 2(-1)^j w b_j k_j'(w) \right\}
\]

\[
+ (\beta_1 \alpha_2 + \beta_2 \alpha_1) \text{Im} \left\{ 2(-1)^j w b_j t_j'(w) \right\}.
\]

Consequently

\[
\partial^3_{g g g} F_j(0, g_0)[g'(0), g'(0), g'(0)] = 0, \quad \text{for all} \quad h, k, l \in \mathbb{R} \times \mathbb{R} \times X.
\]

Therefore, we conclude that

\[
g^{(4)}(0) = -D_g F(0, g_0)^{-1} \left( \partial^4_{\varepsilon\varepsilon\varepsilon\varepsilon} F(0, g_0) + 4\partial^4_{\varepsilon\varepsilon\varepsilon g} F(0, g_0) g'(0) + 6\partial^3_{\varepsilon\varepsilon g} F(0, g_0) g^{(2)}(0) \\
+ 6\partial^2_{\varepsilon\varepsilon} F(0, g_0) [g'(0), g'(0)] + 12\partial^2_{g g} F(0, g_0) [g''(0), g'(0)] \right).
\]

From (5.20) we get

\[
\partial^4_{\varepsilon\varepsilon\varepsilon g} F_j(0, g_0) h = \frac{6 \gamma_{3-j}}{d^5} \text{Im} \left\{ w^3 h_j' h_j'(w) + b_{3-j} w \int_{\mathbb{T}} (b_j w + b_{3-j} \tau)^2 \left[ \frac{\gamma_{3-j}(\tau)}{\gamma_{3-j}} + \gamma_{3-j}'(\tau) \right] d\tau \right\}
\]

\[
+ 2w \int_{\mathbb{R}} \left( b_j h_j'(w) + b_{3-j} h_{3-j}(\tau) \right) (b_j w + b_{3-j} \tau) d\tau \right\}
\]

for all $h \in \mathbb{R} \times \mathbb{R} \times X$. Replacing $h$ by $g'(0) = 0, 0, \frac{b_{1, \gamma_{3-j}, \gamma_1} g_0}{\gamma_1}, \frac{b_{2, \gamma_{3-j}, \gamma_2} g_0}{\gamma_2}$ gives

\[
\partial^4_{\varepsilon\varepsilon\varepsilon g} F_j(0, g_0) g'(0) = \frac{6 \gamma_{3-j}}{d^5} \text{Im} \left\{ - \frac{b_{1, \gamma_{3-j}, \gamma_1}}{\gamma_j} w + \frac{b_{2, \gamma_{3-j}, \gamma_2}}{\gamma_{3-j}} w \int_{\mathbb{T}} (b_j w + b_{3-j} \tau)^2 [\tau - \tau^3] d\tau \right\}
\]

\[
+ 2w \int_{\mathbb{T}} \left( \frac{b_{1, \gamma_{3-j}, \gamma_1} g_0}{\gamma_j} + \frac{b_{2, \gamma_{3-j}, \gamma_2} g_0}{\gamma_{3-j}} \right) (b_j w + b_{3-j} \tau) d\tau \right\}.
\]
Thus

\[ \partial^3_{\epsilon \epsilon g} F_j(0, g_0)g'(0) = \frac{6}{\gamma_j} \left( \frac{b_3^4}{\gamma_j} \frac{b_3^2}{\gamma_j} \right) \Im \{ w \}. \]  

Replacing \( h \) by \( g''(0) = \left( 0, 0, \frac{b_3^2}{\gamma_j} \frac{w^2}{\gamma_j}, \frac{b_3^2}{\gamma_j} \frac{w^2}{\gamma_j} \right) \) in (5.21) we get

\[ \partial^3_{\epsilon \epsilon g} F_j(0, g_0)g''(0) = \frac{2b_j^4}{\gamma_j} \Im \left( \frac{\gamma_3 - j}{\gamma_j} \right) \left( \frac{\gamma_1 + \gamma_2}{w^3 - 2w^3} - \frac{\gamma_3 - j}{w} \right) \]

\[ = \frac{6b_j^4}{\gamma_j} \frac{\gamma_3 - j}{\gamma_j} (\gamma_1 + \gamma_2) \Im \{ w^3 \} + \frac{2b_j^4}{\gamma_j} \frac{\gamma_3^2 - j}{\gamma_j} \Im \{ w \}. \]

From (5.18) we get

\[ \partial^3_{\epsilon \epsilon g} F_j(0, g_0)[g'(0), g'(0)] \]

\[ = b_j \frac{\gamma_3 - j}{\gamma_j} \Im \left\{ w \int_\tau A^2 \left( \frac{\partial f_j(0, \tau)}{A^2} \right) \frac{\partial^2 f_j(0, \tau)}{A^2} \right\} d\tau 
\]

\[ + w \int_\tau A^2 \left( \frac{\partial f_j(0, \tau)}{A^2} \right) \frac{\partial^2 f_j(0, \tau)}{A^2} \right\} d\tau \]

\[ \begin{align*} 
\Im & \left\{ - b_j \frac{\gamma_3 - j}{\gamma_j} \frac{w^3}{\gamma_j} \int_\tau \left( 1 - \frac{(\tau - w)^2}{(\tau - w)^2} \right) \left( \frac{(\tau - w)^3}{\tau - w} \right) d\tau 
\right. 
\left. - b_j \frac{\gamma_3^2 - j^2}{\gamma_j^2} \frac{w^3}{\gamma_j} \int_\tau \left( w + \tau - \frac{(\tau - w)(\tau^2 - w^2)}{(\tau - w)^2} \right) \left( \frac{\tau^2 + \tau - w}{\tau - w} \right) d\tau \right\}.
\end{align*} \]

Now using the fact that \( \tau - w = -(\tau - w)\overline{w} \) \( \forall w, \tau \in \mathbb{T} \), we get

\[ \partial^3_{\epsilon \epsilon g} F_j(0, g_0)[g''(0), g'(0)] = \frac{b_j^4}{\gamma_j} \frac{\gamma_3^2 - j}{\gamma_j} \Im \{ w \}. \]

From (5.21) we find

\[ \partial^3_{\epsilon \epsilon g} F_j(0, g_0)[h, k] = \alpha_1 \Im \left\{ 4b_j^2 \left( w \overline{k_j}(w) + k_j'(w) \right) \right\} + 2\beta_1 \Im \left\{ 2b_j^2 \left( w \overline{h_j}(w) + h_j'(w) \right) \right\} + \frac{2\gamma_j^2}{\gamma_j} \frac{b_j}{\gamma_j} \Im \left\{ w \int_\tau k_{3-j}(\tau) \overline{h_{3-j}(\tau)} d\tau + w \int_\tau h_{3-j}(\tau) \overline{k_{3-j}(\tau)} d\tau \right\}.
\]

Therefore

\[ \partial^3_{\epsilon \epsilon g} F_j(0, g_0)[g'(0), g'(0)] = \frac{4b_j^4}{\gamma_j} \frac{\gamma_3^2}{\gamma_j^2} \Im \{ w \}. \]

Plugging in the identities (5.5), (5.30), (5.31), (5.32) and (5.33) into (5.63) we get

\[ g^{(4)}(0) = -\frac{24}{d^5} D_g F(0, g_0)^{-1} \left( \frac{\gamma_2 b_4^4}{\gamma_1 b_2^2} \right) \Im \{ w^5 \} + \frac{6}{4} (\gamma_1 + \gamma_2) \left( \frac{\gamma_2 b_4^4}{\gamma_1 b_2^2} \right) \Im \{ w^3 \}
\]

\[ + \left( \gamma_1 b_2^4 - \frac{\gamma_2 b_4^4}{\gamma_1 b_2^2} + \frac{\gamma_2 b_4^4}{\gamma_1 b_2^2} \right) \Im \{ w \}. \]
In view of (5.6) we conclude that
\[
(5.35) \quad g^{(4)}(0) = \frac{24}{d^4} \left( \frac{\gamma_1 b_2^4 + \gamma_2 b_1^4}{2d^3} + \frac{d^2(\gamma_2^3 b_1^4 - \gamma_1^3 b_2^4)}{\gamma_1 \gamma_2 (\gamma_1 + \gamma_2)^2} + \frac{1}{4} \gamma_2 b_1^4 \frac{w^4}{w_1} + \frac{3}{4} \gamma_2 \frac{b_2^4 w^2}{\gamma_1 + 1} \right),
\]

5.2. Counter-rotating pairs. Next we shall prove the following expansion.

**Proposition 5.2.** The conformal mappings of the counter-rotating domains, \( \phi_j : \mathbb{D} \to \partial D_j^\varepsilon \), have the expansions

\[
\phi_j(\varepsilon, w) = w - \left( \frac{\varepsilon b_j}{d} \right)^2 w - \frac{1}{2} \left( \frac{\varepsilon b_j}{d} \right)^3 w^2 - \frac{1}{3} \left( \frac{\varepsilon b_j}{d} \right)^4 w^3 - \frac{1}{4} \left( \frac{\varepsilon b_j}{d} \right)^5 w^4 + o(\varepsilon^5),
\]

for all \( w \in \mathbb{D} \). Moreover,

\[
U(\varepsilon) = \frac{\gamma_1}{2d} \left( 1 + \frac{\varepsilon^4}{d^4} (2b_1^4 + b_2^4) \right) + o(\varepsilon^4)
\]

\[
\gamma_2(\varepsilon) = \gamma_1 \left( 1 + \frac{\varepsilon^4}{d^4} (b_1^4 - b_2^4) \right) + o(\varepsilon^4).
\]

**Proof.** Recall that
\[
\phi_j(\varepsilon, w) = w + \varepsilon b_j f_j(\varepsilon, w), \quad j = 1, 2.
\]

Moreover, in view of Proposition 4.2 one has
\[
(5.36) \quad F(\varepsilon, g(\varepsilon)) = F(\varepsilon, U(\varepsilon), \gamma_2(\varepsilon), f_1(\varepsilon), f_2(\varepsilon)) = 0
\]

and
\[
(5.37) \quad g(0) = \left( U(0), \gamma_2(0), f_1(0), f_2(0) \right) = \left( \frac{\gamma_1}{2d}, \gamma_1, 0, 0 \right).
\]

where \( F \) is defined by (2.17). By the composition rule we get
\[
(5.38) \quad \partial \varepsilon g(0) = -D_g F(0, g_0)^{-1} \partial \varepsilon F(0, g_0).
\]

In view of (2.17) one has
\[
F_j(\varepsilon, g_0) = \gamma_1 \text{Im} \left\{ \left( \frac{1}{d} + \frac{\tau}{d} \right) \left( b_{3-j} \tau + b_j w \right) \right\}
\]

\[
= \gamma_1 \text{Im} \left\{ \left( \frac{1}{d} - \sum_{n=0}^\infty \frac{\varepsilon^n}{d^{n+1}} \int_{\mathbb{D}} \left( b_{3-j} \tau + b_j w \right)^n d\tau \right) w \right\}
\]

\[
= -\gamma_1 \frac{1}{d} \sum_{n=1}^\infty \frac{\varepsilon^n b_j^n}{d^n} \text{Im} \{ w^{n+1} \}.
\]

Thus
\[
(5.39) \quad \partial \varepsilon F(0, g_0)(w) = -\frac{n! \gamma_1}{d^{n+1}} \left( b_j^n \frac{b_j}{b_2} \right) \text{Im} \{ w^{n+1} \}.
\]

Recall from Proposition 3.2 that for all \( h = (\alpha_1, \alpha_2, h_1, h_2) \in \mathbb{R} \times \mathbb{R} \times X \) with
\[
h_j(w) = \sum_{n=1}^\infty a_n \bar{w}^n, \quad j = 1, 2,
\]

where
From the identity \((5.45)\)
where we have used the notation
\((5.41)\)
Thus, for all \((5.47)\) one has
\[
\begin{align*}
k(w) &= \sum_{n \geq 0} \left( \frac{A_n}{B_n} \right) \text{Im} \{ w^{n+1} \},
\end{align*}
\]
one has
\[
\begin{align*}
D_y F(0, g_0)^{-1} k(w) &= - \left( - \frac{B_0}{2}, d(A_0 - B_0), \sum_{n \geq 1} \frac{A_n}{\gamma_1 w^n}, \sum_{n \geq 1} \frac{B_n}{\gamma_1 w^n} \right).
\end{align*}
\]
Combining the last identity with \((5.39), (5.38)\) and \((5.41)\) we get
\[
\begin{align*}
\partial_{\varepsilon} g(0, w) &= - \frac{1}{d^2} \left( 0, 0, b_1 w, b_2 w \right).
\end{align*}
\]
Let’s move to the second order derivative in \(\varepsilon\) at 0. By the composition rule we get
\[
\begin{align*}
\partial^2_{\varepsilon} g(0) &= - D_y F(0, g_0)^{-1} \left( \partial^2_{\varepsilon} F(0, g_0) + 2D^2_{\varepsilon} F(0, g_0) g'(0) + D^2_{gg} F(0, g_0) [g'(0), g'(0)] \right).
\end{align*}
\]
Observe that
\[
\begin{align*}
F_j(0, g) &= \text{Im} \left\{ 2U w - \gamma_j f'_j(w) - \frac{\gamma_3 - j}{d} w \right\}.
\end{align*}
\]
Thus, for all \(h = (\alpha_1, \alpha_2, h_1, h_2), k = (\beta_1, \beta_2, k_1, k_2) \in \mathbb{R} \times \mathbb{R} \times X\), one has
\[
\begin{align*}
\partial^2_{gg} F_1(0, g)[h, k] &= 0, \\
\partial^2_{gg} F_2(0, g)[h, k] &= - \alpha_2 \text{Im} \left\{ k_2'(w) \right\} - \beta_2 \text{Im} \left\{ h_2'(w) \right\}.
\end{align*}
\]
From the identity \((5.42)\) we get
\[
\begin{align*}
\partial^2_{gg} F_j(0, g_0)[g'(0), g'(0)] &= 0.
\end{align*}
\]
Next, differentiating the identity \((2.17)\) with respect to \(\varepsilon\) we get
\[
\begin{align*}
\partial_{\varepsilon} F_j(\varepsilon, g) &= \text{Im} \left\{ 2U w b_j f'_j(w) + b_j \gamma_j w \left( 1 + 2 \varepsilon b_j f'_j(w) \right) \int_{\mathbb{T}} \frac{A B_j - \overline{A} B_j}{A(A + \varepsilon b_j B_j)} \left( f'_j(\tau) - \frac{B_j}{A} \right) d\tau \right\}
\end{align*}
\]
where we have used the notation
\[
\begin{align*}
A &= \tau - w, \\
B_j &= f_j(\tau) - f_j(w), \\
C_j &= (b_{3-j} + b_j w), \\
D_j &= b^2_{3-j} f_{3-j}(\tau) + b_j^2 f_j(w).
\end{align*}
\]
Consequently, for all \( h = (\alpha_1, \alpha_2, h_1, h_2) \in \mathbb{R} \times \mathbb{R} \times X \), one finds

\[
\partial_{\varepsilon g}^2 F_1(0, g) h = \alpha_1 \text{Im} \left\{ 2 w b_j f_j'(w) \right\} - \frac{\alpha_2 b_j \delta_{ij}}{d} \text{Im} \left\{ w f_j'(w) - \frac{1}{d} w^2 \right\} \\
+ \alpha_2 \delta_{2j} \text{Im} \left\{ b_j w \int_\mathbb{T} A B_j - \overline{A B_j} \left[ f_j'(\tau) - B_j \right] d\tau \right\} \\
+ \text{Im} \left\{ 2 U w b_j h_j'(w) - \frac{\gamma_3 - j b_j}{d} w h_j'(w) \right\} \\
+ b_j \gamma_j w \int_\mathbb{T} 2 i \text{Im} \left\{ A (f_j'(\tau) - f_j(w)) \right\} \left[ h_j'(\tau) - \frac{h_j(\tau) - h_j(w)}{A} \right] d\tau \\
+ b_j \gamma_j w \int_\mathbb{T} 2 i \text{Im} \left\{ A (h_j'(\tau) - h_j(w)) \right\} \left[ f_j'(\tau) - \frac{f_j(\tau) - f_j(w)}{A} \right] d\tau,
\]

where \( \delta_{ij} \) is the Kronecker delta. Then, for \( g = g_0 \triangleq \left( \frac{24}{24}, \gamma_1, 0, 0 \right) \) and \( h = g'(0) \) one gets

\[
\partial_{\varepsilon g}^2 F_1(0, g_0) g'(0) = 0.
\]

Plugging the identities (5.39), (5.41), (5.46) and (5.49) into (5.43) yields

\[
g''(0) = -\frac{1}{d^3} \left( 0, 0, b_1^2 w^2, b_2^2 w^2 \right).
\]

Next, we move to the third order derivative

\[
\partial_{\varepsilon e e}^3 g(0) = -D_y F(0, g_0)^{-1} \left( \partial_{\varepsilon e e}^2 F(0, g_0) + 3 \partial_{\varepsilon e g}^2 F(0, g_0) g'(0) + 3 \partial_{e g g}^2 F(0, g_0) [g'(0), g'(0)] \right) \\
+ \partial_{\varepsilon e g}^2 F(0, g_0) g''(0) + 3 \partial_{e g g}^2 F(0, g_0) [g'(0), g''(0)] + \partial_{g g g}^2 F(0, g_0) [g'(0), g'(0), g'(0)].
\]

By virtue of the identity (5.48), for all \( h = (\alpha_1, \alpha_2, h_1, h_2), k = (\beta_1, \beta_2, k_1, k_2) \in \mathbb{R} \times \mathbb{R} \times X \), one has

\[
\partial_{\varepsilon g g}^3 F_j(0, g) [h, k] = \alpha_1 \text{Im} \left\{ 2 w b_j k_j'(w) \right\} + \beta_1 \text{Im} \left\{ 2 w b_j h_j'(w) \right\} \\
- \frac{\alpha_2 b_j \delta_{ij}}{d} \text{Im} \left\{ w k_j'(w) \right\} - \frac{\beta_2 b_j \delta_{ij}}{d} \text{Im} \left\{ w h_j'(w) \right\} \\
+ (\alpha_2 + \beta_2) \delta_{2j} b_j \text{Im} \left\{ w \int_\mathbb{T} 2 i \text{Im} \left\{ A (f_j'(\tau) - f_j(w)) \right\} \left[ h_j'(\tau) - \frac{h_j(\tau) - h_j(w)}{A} \right] d\tau \right\} \\
+ w \int_\mathbb{T} 2 i \text{Im} \left\{ A (h_j'(\tau) - h_j(w)) \right\} \left[ f_j'(\tau) - \frac{f_j(\tau) - f_j(w)}{A} \right] d\tau \\
+ b_j \gamma_j \text{Im} \left\{ \int_\mathbb{T} 2 i \text{Im} \left\{ A (k_j'(\tau) - k_j(w)) \right\} \left[ k_j'(\tau) - \frac{k_j(\tau) - k_j(w)}{A} \right] d\tau \right\} \\
+ w \int_\mathbb{T} 2 i \text{Im} \left\{ A (h_j'(\tau) - h_j(w)) \right\} \left[ k_j'(\tau) - \frac{k_j(\tau) - k_j(w)}{A} \right] d\tau.
\]

Thus, for \( g = g_0 \triangleq \left( \frac{24}{24}, \gamma_1, 0, 0 \right) \), \( h = g'(0) \) and \( k = g'(0) \) we find
Next, differentiating (5.47) with respect to \( \varepsilon \) gives

\[
\partial_{\varepsilon \varepsilon}^2 F_j(\varepsilon, g) = \text{Im} \left\{ 2b_j^2 \gamma_j w f_j'(w) \int_{\mathbb{I}} \frac{A B_j - \overline{A} B_j}{A(A + \varepsilon B_j)} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \right. \\
- b_j^2 \gamma_j w \left( 1 + 2\varepsilon b_j f_j'(w) \right) \int_{\mathbb{I}} \frac{B_j (A B_j - \overline{A} B_j)}{A(A + \varepsilon B_j)^2} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \\
+ 2\varepsilon b_j^3 \gamma_j w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_{\mathbb{I}} \frac{B_j (A B_j - \overline{A} B_j)}{A(A + \varepsilon B_j)^2} \left[ f_j'(\tau) - \frac{B_j}{A} \right] d\tau \\
+ 2\gamma_{3-j} b_j b_{3-j} w f_j'(w) \int_{\mathbb{I}} \left( C_j + 2\varepsilon D_j \right) \left( 1 + \varepsilon b_{3-j} f_{3-j}(\tau) \right) \left( 1 + \varepsilon b_{3-j} f_{3-j}(\tau) \right) d\tau \\
- 2\gamma_{3-j} w b_j f_j'(w) \int_{\mathbb{I}} \frac{f_{3-j}(\tau)}{(C_j + \varepsilon^2 D_j - d)^2} d\tau \\
+ 2b_j^3 \gamma_{3-j} w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_{\mathbb{I}} \frac{f_{3-j}(\tau) f_{3-j}(\tau)}{\varepsilon C_j + \varepsilon^2 D_j - d} d\tau \\
- 2\gamma_{3-j} b_{3-j} w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_{\mathbb{I}} \frac{f_{3-j}(\tau) f_{3-j}(\tau)}{(C_j + \varepsilon^2 D_j - d)^2} d\tau \\
- 2\gamma_{3-j} b_{3-j} w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_{\mathbb{I}} \frac{f_{3-j}(\tau) f_{3-j}(\tau)}{(C_j + \varepsilon^2 D_j - d)^2} d\tau \\
- 2\gamma_{3-j} w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_{\mathbb{I}} \frac{D_j (\tau + \varepsilon b_{3-j} f_{3-j}(\tau)) (1 + \varepsilon b_{3-j} f_{3-j}(\tau))}{(C_j + \varepsilon^2 D_j - d)^2} d\tau \\
+ 2\gamma_{3-j} w \left( 1 + \varepsilon b_j f_j'(w) \right) \int_{\mathbb{I}} \frac{(C_j + 2\varepsilon D_j)^2 (\tau + \varepsilon b_{3-j} f_{3-j}(\tau)) (1 + \varepsilon b_{3-j} f_{3-j}(\tau))}{(C_j + \varepsilon^2 D_j - d)^3} d\tau \right\},
\]

with

\[ A = \tau - w, \quad B_j = f_j(\tau) - f_j(w), \quad C_j = (b_{3-j} \tau + b_j w), \quad D_j = b_{3-j}^2 f_{3-j}(\tau) + b_j^2 f_j'(w). \]

For \( \varepsilon = 0 \) one has

\[
\partial_{\varepsilon \varepsilon}^2 F_j(0, g) = 2\text{Im} \left\{ b_j^2 \gamma_j w \int_{\mathbb{I}} \frac{A B_j - \overline{A} B_j}{A^2} \left[ f_j'(\tau) - \frac{B_j}{A} \right] \left[ f_j'(w) - \frac{B_j}{A} \right] d\tau \right. \\
- \frac{1}{d} \gamma_{3-j} w^2 b_j^2 f_j'(w) - \frac{1}{d} b_{3-j}^2 \gamma_{3-j} w \int_{\mathbb{I}} f_{3-j}(\tau) f_{3-j}(\tau) d\tau - \frac{1}{d^2} \gamma_{3-j} b_j^2 w f_j(w) - \frac{1}{d^3} \gamma_{3-j} b_j^2 w^3 \right\}.
\]
Thus, for all \( h = (\alpha_1, \alpha_2, h_1, h_2) \in \mathbb{R} \times \mathbb{R} \times X \) one finds

\[
(5.55) \quad \partial_{\varepsilon\varepsilon\varepsilon\varepsilon}^2 F_j(0, g) h = 2 \alpha_2 \delta_{j2} \Im \left\{ b_j^2 w \int_{\mathbb{T}} \frac{AB_j - \overline{AB}_j}{A^2} \left[ f'_j(\tau) - \frac{B_j}{A} \right] \left[ f'_j(\tau) - \frac{B_j}{A} \right] d\tau \right\} \\
- 2 \alpha_2 \delta_{j1} \Im \left\{ \frac{1}{d^2} \left[ w^2 b_j^2 f''_j(\tau) w + \frac{1}{d^2} b_j^2 w \int_{\mathbb{T}} f''_{3-j}(\tau) f_{3-j}(\tau) d\tau + \frac{1}{d^2} b_j^2 w f'(\tau) + \frac{1}{d^2} b_j^2 w^3 \right] \right\} \\
+ 2 \Im \left\{ \frac{1}{d^2} \gamma_{3-j} \gamma_{3-j}^2 b_j^2 h_j''(w) - \frac{1}{d^2} \gamma_{3-j} b_j^2 w h_j'(w) \right\} \\
+ b_j^2 \gamma_j w \int_{\mathbb{T}} \left[ h_j''(w) - \frac{h_j(\tau) - h_j(w)}{A} \right] \frac{AB_j - \overline{AB}_j}{A^2} \left[ f'_j(\tau) - \frac{B_j}{A} \right] d\tau \\
+ b_j^2 \gamma_j w \int_{\mathbb{T}} \left[ f''_j(w) - \frac{B_j}{A} \right] \frac{2 \Im \{ A(\alpha_j(\tau) - \alpha_j(w)) \}}{A^2} \left[ f'_j(\tau) - \frac{B_j}{A} \right] d\tau \\
+ b_j^2 \gamma_j w \int_{\mathbb{T}} \left[ f''_j(w) - \frac{B_j}{A} \right] \frac{AB_j - \overline{AB}_j}{A^2} \left[ h_j'(\tau) - \frac{h(\tau) - h_j(w)}{A} \right] d\tau \right\} \\
- 2 \frac{1}{d^2} \gamma_{3-j} \gamma_{3-j} \Im \left\{ \int_{\mathbb{T}} f''_{3-j}(\tau) h_{3-j}(\tau) d\tau + \int_{\mathbb{T}} h''_{3-j}(\tau) f_{3-j}(\tau) d\tau \right\}.
\]

Therefore, for \( g = g_0 = \left( \frac{\alpha_1}{\alpha_2}, \gamma_1, 0, 0 \right) \), \( h = g'(0) \) we get

\[
(5.56) \quad \partial_{\varepsilon\varepsilon\varepsilon\varepsilon}^2 F_j(0, g_0) g'(0) = 0.
\]

Next, in view of (5.45) one has

\[
(5.57) \quad \partial_{g g}^3 F(0, g_0) \left[ g'(0), g''(0) \right] = 0
\]

and

\[
(5.58) \quad \partial_{g g g}^3 F(0, g_0) \left[ h, k, l \right] = 0 \quad \text{for all} \quad h, k, l \in \mathbb{R} \times \mathbb{R} \times X.
\]

Putting together the identities (5.39), (5.53), (5.56), (5.57) (5.58) and (5.51) we conclude that

\[
(5.59) \quad \partial_{\varepsilon\varepsilon\varepsilon\varepsilon}^3 g(0, w) = - \frac{2}{d^4} \left( 0, 0, b_1^3 w^3, b_2^3 w^3 \right).
\]

Now we move to the fourth order derivative in \( \varepsilon \) of \( g \). By the composition rule we get

\[
\partial_{\varepsilon\varepsilon\varepsilon\varepsilon\varepsilon}^4 F(0, g_0) = - D_{g} F(0, g_0)^{-1} \left( \partial_{\varepsilon\varepsilon\varepsilon\varepsilon}^4 F(0, g_0) + 4 \partial_{\varepsilon\varepsilon\varepsilon\varepsilon}^4 F(0, g_0) g'(0) + 6 \partial_{\varepsilon\varepsilon\varepsilon\varepsilon}^4 F(0, g_0) g''(0) \right) + 4 \partial_{\varepsilon\varepsilon}^4 F(0, g_0) g^{(3)}(0) + 6 \partial_{\varepsilon\varepsilon g g}^4 F(0, g_0) \left[ g'(0), g'(0) \right] + 12 \partial_{\varepsilon\varepsilon g g}^4 F(0, g_0) \left[ g''(0), g'(0) \right] + 4 \partial_{g g g g}^4 F(0, g_0) \left[ g'(0), g'(0), g'(0) \right] + 6 \partial_{g g g g}^4 F(0, g_0) \left[ g''(0), g'(0), g'(0) \right]
\]

In view of (5.45) one has

\[
(5.60) \quad \partial_{gg}^2 F(0, g_0) \left[ g^{(3)}(0), g'(0) \right] = \partial_{gg}^2 F(0, g_0) \left[ g''(0), g'(0) \right] = 0
\]

and

\[
(5.61) \quad \partial_{g g g g}^4 F(0, g_0) \left[ g'(0), g'(0), g'(0), g'(0) \right] = \partial_{g g g g}^2 F(0, g_0) \left[ g'(0), g''(0), g'(0) \right] = 0.
\]
Moreover, from (5.48) we find
\begin{equation}
\partial_{eg} F(0, g_0) g^{(3)}(0) = 0
\end{equation}
and by (5.52) we obtain
\begin{align*}
\partial_{egg}^4 F_j(0, g)[h, k, l] &= (\alpha_2 + \beta_2) \delta_{2j} \text{Im} \left\{ b_j w \int \frac{2 \text{Im} \{A(l_j(\tau) - l_j(w))\}}{A^2} \left[ h_j' j - h_j(\tau) - h_j(w) \right] d\tau \right. \\
& \quad \left. + b_j w \int \frac{2 \text{Im} \{A(h_j(\tau) - h_j(w))\}}{A^2} \left[ k_j' j - k_j(\tau) - k_j(w) \right] d\tau \right. \\
& \quad \left. + \sigma_j \text{Im} \left\{ b_j w \int \frac{2 \text{Im} \{A(k_j(\tau) - k_j(w))\}}{A^2} \left[ k_j' j - k_j(\tau) - k_j(w) \right] d\tau \right. \\
& \quad \left. + b_j w \int \frac{2 \text{Im} \{A(h_j(\tau) - h_j(w))\}}{A^2} \left[ k_j' j - k_j(\tau) - k_j(w) \right] d\tau \right. \\
Consequently
\begin{equation}
\partial_{egg}^3 F_j(0, g_0)[g'(0), g'(0), g'(0)] = 0, \text{ for all } h, k, l \in \mathbb{R} \times \mathbb{R} \times X.
\end{equation}
Therefore, we conclude that
\begin{equation}
g^{(4)}(0) = -D_g F(0, g_0)^{-1} \left( \partial_{eeee}^4 F(0, g_0) + 4 \partial_{esee}^2 F(0, g_0) g'(0) + 6 \partial_{egg}^3 F(0, g_0) g^{(2)}(0) \right. \\
& \quad \left. + 6 \partial_{egg}^3 F(0, g_0) [g'(0), g'(0)] + 12 \partial_{egg}^3 F(0, g_0) [g''(0), g'(0)] \right).
\end{equation}
From (5.54) we get
\begin{equation}
\partial_{egg}^4 F_j(0, g_0) h = -\frac{6 \gamma_1}{d^5} \text{Im} \left\{ b_j^3 h_j'(w) + b_3 j w \int \left( b_j w + b_3 j \tau \right)^2 [h_{3-j}(\tau) + \tau h_{3-j}(\tau)] d\tau + 2 w \int \left( b_j^2 h_j(w) + b_3^2 j h_{3-j}(\tau) \right) (b_j w + b_3 j \tau) d\tau \right\},
\end{equation}
for all \( h \in \mathbb{R} \times \mathbb{R} \times X \). Replacing \( h \) by \( g'(0) = \left( 0, 0, -\frac{b_j^2}{d^2} w, -\frac{b_3}{d^2} w \right) \) gives
\begin{equation}
\partial_{egg}^4 F_j(0, g_0) g'(0) = -\frac{6 \gamma_1}{d^5} \text{Im} \left\{ b_j^4 w - b_3^2 j w \int \left( b_j w + b_3 j \tau \right)^2 [\tau - \tau^2] d\tau + 2 w \int \left( b_j^3 w + b_3^3 j \tau \right) (b_j w + b_3 j \tau) d\tau \right\}.
\end{equation}
Thus
\begin{equation}
\partial_{egg}^4 F_j(0, g_0) g'(0) = \frac{6 \gamma_1}{d^5} \left( b_j^4 + b_3^4 \right) \text{Im} \left\{ w \right\}.
\end{equation}
Replacing \( h \) by \( g''(0) = \left( 0, 0, -\frac{b_j^2}{d^2} w^2, -\frac{b_3}{d^2} w^2 \right) \) in (5.55) we get
\begin{equation}
\partial_{egg}^3 F_j(0, g_0) g''(0) = \frac{2 \gamma_1 b_j^4}{d^5} \text{Im} \left\{ w \right\}.
\end{equation}
From (5.52) we get
\[
\partial^3_{\epsilon\eta g} F_j(0, g_0)[g''(0), g'(0)]
= \gamma_1 \text{Im} \left\{ b_j w \int_{\mathbb{T}} \frac{2\text{Im} \{ A(\partial_\epsilon f_j(0, \tau) - \partial_\epsilon f_j(0, w)) \}}{A^2} \left[ \partial^2_{\epsilon\tau} f_j(0, \tau) - \frac{\partial^2_{\epsilon\tau} f_j(0, w)}{A} \right] d\tau + b_j w \int_{\mathbb{T}} \frac{2\text{Im} \{ A(\partial^2_{\epsilon\epsilon} f_j(0, \tau) - \partial^2_{\epsilon\epsilon} f_j(0, w)) \}}{A^2} \left[ \partial^2_{\epsilon\tau} f_j(0, \tau) - \frac{\partial^2_{\epsilon\tau} f_j(0, w)}{A} \right] d\tau \right\}.
\]
\[
= \gamma_1 \text{Im} \left\{ - \frac{b_4^4}{d^3} w \int_{\mathbb{T}} \left( 1 - \frac{(\tau - w)^2}{(\tau - w)^2} \right) \left( 2\tau^2 + \frac{\tau^2 - w^2}{\tau - w} \right) d\tau \right\}.
\]
Now using the fact that
\[
\tau - w = -(\tau - w)\overline{w} \quad \forall \omega, \tau \in \mathbb{T},
\]
we get
\[
(5.66) \quad \partial^3_{\epsilon\eta g} F_j(0, g_0)[g''(0), g'(0)] = -\frac{\gamma_1 b_4^4}{d^3} \text{Im} \{ w \}.
\]
From (5.55) we find
\[
\partial^3_{\epsilon\eta g} F_j(0, g_0)[h, k] = -\frac{2\gamma_1}{d} b^4_{3-j} \text{Im} \left\{ w \int_{\mathbb{T}} k^j_{3-j}(\tau) h_{3-j}(\tau) d\tau + w \int_{\mathbb{T}} h^j_{3-j}(\tau) k_{3-j}(\tau) d\tau \right\}.
\]
Therefore
\[
(5.67) \quad \partial^3_{\epsilon\eta g} F_j(0, g_0)[g'(0), g'(0)] = \frac{4\gamma_1 b^4_{3-j}}{d^3} \text{Im} \{ w \}.
\]
Plugging in the identities (5.33), (5.64), (5.65), (5.66) and (5.67) into (5.63) we get
\[
g^{(4)}(0) = -\frac{24}{d^4} D_g F(0, g_0)^{-1} \left( -\left( \frac{b_4^4}{b_2^4} \right) \text{Im} \{ w^5 \} + \frac{\gamma_1}{d} \left( \frac{b_4^4}{b_2^4 + 2b_1^4} \right) \text{Im} \{ w \} \right).
\]
By virtue of (5.41) we conclude that
\[
(5.68) \quad g^{(4)}(0) = \frac{24}{d^4} \left( \frac{\gamma_1}{2d} (2b_1 + b_3^2), \gamma_1 (b_1^4 - b_2^4), -\frac{1}{4} b_1^4 \overline{w}, -\frac{1}{4} b_2^4 \overline{w} \right),
\]
which achieves the proof of the desired result.

\[\square\]

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NYU ABU DHABI, SAADIYAT MARINA DISTRICT - ABU DHABI, UNITED ARAB EMIRATES.
E-mail address: zh14@nyu.edu

IRMAR, UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, 35 042 RENNES CEDEX, FRANCE
E-mail address: thmidi@univ-rennes1.fr

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