Nonlinear realizations and the orbit method

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Abstract

Given a symmetry group one can construct the invariant dynamics using the technique of nonlinear realizations or the orbit method. The relationship between these methods is discussed. Few examples are presented.

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I Introduction

It is very common in physics that the symmetry principles provide the starting point for constructing the relevant dynamics, both on classical and quantum levels. Among numerous examples we quote only a few: Poincare \cite{1} and Galilean \cite{2} symmetries, Newton-Hooke group \cite{3}, relativistic \cite{4} and nonrelativistic \cite{5} conformal groups etc.

The natural question arises whether there exist methods which allow to find and, if possible, to classify the dynamical systems exhibiting a given in advance symmetry. At least in the case of Hamiltonian systems with finite number of degrees of freedom the corresponding algorithm does exist and is called the orbit method \cite{6}-\cite{11}. It allows, in particular, to classify all symplectic manifolds on which a given group $G$ acts transitively and preserves symplectic structure.

An alternative method relies on the theory of nonlinear realizations of groups \cite{12}. It allows to construct various invariant dynamical systems in terms of geometry of group manifolds. It was successfully applied to the case of conformal $SL(2,\mathbb{R})$ symmetry \cite{13} in the interesting paper by Ivanov et al. \cite{14} (for supersymmetric version see \cite{15}). Following their method Fedoruk at al. \cite{16} considered dynamical realizations of $N=2$ conformal Galilei algebras; this was further generalized \cite{17} to arbitrary $N$.

The method of nonlinear realizations has many advantages. It allows to write out invariant equations of first order once a subgroup of symmetry group is selected and the relevant Cartan-Maurer forms computed. On the other hand, it cannot be taken for granted that the resulting equations can be put in Hamiltonian form; in fact, it is the case only provided the subgroup selected is the stability subgroup of same point on coadjoint orbit.

In the present paper we analyze in some detail the relation between the method based on nonlinear realizations and the orbit method. In Sec.II we show that the orbit method can be reformulated in terms of nonlinear action of group on coset space. In particular, the Hamiltonian equations of motion are formulated with the help of Cartan-Maurer forms and the group action in both formalisms is described. In Sec.III few examples are considered. First, the results of Ref. \cite{14} are discussed within the general framework presented in Sec.II. Then, two examples related to $N=2$ conformal Galilei group are described. While for the (centrally extended) $N$ odd conformal Galilean groups the situation seems to be clear \cite{18}-\cite{20}, the case of $N$ even is more complicated. The algebra does not allow central extension (except in two dimensions) and the classification of coadjoint orbits is much more involved. Therefore, we restrict ourselves to two special cases. The first onereduces to the $SL(2,\mathbb{R})$-invariant dynamics; the second describes nontrivial yet simple dynamics. Sec.IV is devoted to short conclusions.
The orbit method and nonlinear realizations

The orbit method [6]-[11] provides a powerful tool for constructing the Hamiltonian systems with dynamical symmetry. In fact, it allows to classify all phase manifolds (i.e. symplectic ones) on which a given Lie group $G$ acts transitively as a group of canonical transformations. To specify the dynamics the Hamiltonian is assumed to be an element of the Lie algebra of $G$ (or, more generally, its universal enveloping algebra). The power of the method relies on the fact that the dynamical systems can be defined and classified by making only general assumptions concerning the symmetry structure; no further a priori assumptions have to be made.

Let us discuss the main ingredients of the orbit method. Let $G$ be a Lie group with Lie algebra $\mathcal{G}$ spanned by the generators $X_i$ obeying the commutation rules

$$[X_i, X_j] = i c_{i j}^k X_k$$

Any element of $\mathcal{G}$ can be written as

$$\xi = \xi^i X_i$$

The adjoint action of $G$ is defined by

$$Ad_g(X_i) = gXg^{-1} \equiv D^j_i(g) X_j$$

In particular, for one-parameter subgroups $g = \exp \{it\xi\}$ one obtains

$$Ad_g(X_i) = e^{it\xi} X_i e^{-it\xi} = X_i - t\xi^k c_{ki}^j X_j + O(t^2)$$

or, equivalently

$$D^j_i(e^{it\xi}) = \delta^j_i - t\xi^k c_{ki}^j + O(t^2) = \delta^j_i + t\xi^k c_{ik}^j + O(t^2)$$

For a general element $\xi \in \mathcal{G}$ we get

$$Ad_g\xi = \xi^i X_i, \quad \xi^i = D^i_j(g)\xi^j$$

The basis $\{X^i\}$ in the dual space $\mathcal{G}^*$ is defined by pairing

$$\langle X^i, X_j \rangle = \delta^i_j$$

If $\zeta \equiv \zeta^i X^i$ is a general element of dual space, then

$$\langle \zeta, \xi \rangle = \zeta^i \xi^i$$
The coadjoint action of $G$ in the dual space obeys

$$
\langle Ad^*_g \zeta, Ad_g \xi \rangle = \langle \zeta, \xi \rangle
$$

which yields

$$
Ad^*_g \zeta = \zeta' X^i, \quad \zeta'_i = D^j_i (g^{-1}) \zeta_j
$$

On $G^*$ one defines $Ad^*_g$-invariant Poisson bracket

$$
\{ \zeta_i, \zeta_j \} = c_{ij}^k \zeta_k
$$

or, generally

$$
\{ f_1(\zeta), f_2(\zeta) \} = \frac{\partial f_1}{\partial \zeta_i} \frac{\partial f_2}{\partial \zeta_j} c_{ij}^k \zeta_k
$$

The above Poisson structure is degenerate; this is easily seen by considering the function $f(\zeta)$ corresponding to any Casimir operator of $G$.

Let us now introduce the dynamics by selecting a particular element of $G$ as the Hamiltonian

$$
H = \alpha^i X_i
$$

Let, further,

$$
\xi(t) = Ad_{e^{-itH}} \xi = \xi^i e^{-itH} X_i e^{itH} \equiv \xi^i(t) X_i;
$$

then one easily finds

$$
\dot{\xi}^k(t) = c^k_{ij} \xi^j(t) \alpha^i
$$

Defining the dynamics on dual space by

$$
\langle \zeta(t), \xi(t) \rangle = \langle \zeta(0), \xi(0) \rangle
$$

we arrive at the following equations of motion

$$
\dot{\zeta}_j(t) = c_{ji}^k \alpha^i \zeta_k(t)
$$
with the solution (cf. eqs. (10) and (14))

\[ \zeta_j(t) = D^i{}_j (e^{itH}) \zeta_i(0) \]  

(18)

Eqs. (17) can be put in Hamiltonian form. Indeed, by defining

\[ \tilde{H} = \alpha^i \zeta_i \]  

(19)

we rewrite eq. (17) as the canonical one

\[ \dot{\zeta}_i(t) = \{ \zeta_i(t), \tilde{H}(\zeta(t)) \} \]  

(20)

Having defined the Hamiltonian dynamics in \( G^* \) let us now analyze the action of \( G \) as the symmetry group. In doing that one should keep in mind that \( H \), being an element of Lie algebra \( G \), does not commute in general with symmetry generators. Therefore, they should depend explicitly on time.

Define the explicitly time dependent generators

\[ \tilde{X}_i(\zeta, t) \equiv D^l{}_j (e^{-itH}) \zeta_j \]  

(21)

They obey the (Poisson) commutation rules of \( G \)

\[ \{ \tilde{X}_i(\zeta, t), \tilde{X}_j(\zeta, t) \} = c_{ij}^k \tilde{X}_k(\zeta, t) \]  

(22)

Moreover, by virtue of eq. (18) they are constants of motion

\[ \tilde{X}_i(\zeta(t), t) = \zeta_i(0) \]  

(23)

or

\[ \{ \tilde{X}_i(\zeta, t), \tilde{H}(\zeta) \} + \frac{\partial \tilde{X}_i(\zeta, t)}{\partial t} = 0 \]  

(24)

The last equation can be also viewed as stating that \( \tilde{X}_i(\zeta, t) \) generate symmetry transformation. Alternatively, this is easily verified by considering infinitesimal transformations

\[ \delta \zeta_i(t) = \{ \lambda^k \tilde{X}_k(\zeta(t), t), \zeta_i(t) \} = \lambda^k D^j{}_k (e^{-itH}) c_{ji}^l \zeta_l(t) = \]

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The global counterpart of eq. (25) defines the action of finite symmetry transformations at time $t$:

$$
\zeta'_i = D^m_i \left( e^{itH} \right) \left\{ \delta \zeta_m(0) \right\} = D^m_i \left( e^{itH} \right) \delta \zeta_m(0) \tag{26}
$$

We use above the exponential parametrization of global symmetry transformations; however, one has to keep in mind that in most cases the exponential parametrization does not provide the global map for group manifold and the global topological properties must be carefully taken into account.

In the Hamiltonian formalism the description of the symmetries refers to a fixed time. On the other hand, in Lagrangian (i.e. "explicitly covariant") approach one considers also the action of symmetry group on time variable. The necessity of taking into account the nontrivial action of symmetry group on time variable appears often in the following context. Let us consider the symmetry transformations acting in the phase space and pose the question whether they can be viewed as canonical counterpart of some point transformations. In general, the answer is no because, for example, new coordinates are not only the function of old ones but also depend on initial momenta. However, it can happen that the momentum dependence can be accounted for by admitting the time variation. Then our initial canonical transformation is equivalent to the point one, provided the latter includes time variation.

Let us assume that there exists a (generally nonlinear) realization of our symmetry group $G$ on one-dimensional manifold parametrized by time. This is possible if $G$ contains a subalgebra which after adjoining $H$ spans the whole $G$. This means that $G$ contains the subgroup $K$ such that $G/K$ is one-dimensional coset space generated by $H$. The action of $G$ on $t$ variable is given by

$$
t' = f(t; \lambda), \quad f(f(t, \lambda'); \lambda) = f(t; \varphi(\lambda, \lambda')) \tag{27}
$$

where $\varphi(\lambda, \lambda')$ gives the composition law in some (in general - local) coordinates on $G$. The full action of $G$ reads

$$
\zeta'_i(t') = D^m_i \left( e^{it'H} \right) D^k_m \left( g^{-1}(\lambda) \right) D^j_k \left( e^{-itH} \right) \zeta_j(t) \tag{28}
$$

It is easy to verify that the action of $G$, defined above, obeys the appropriate composition rule; moreover, $\zeta'_i(t')$ obeys the equation of motion (17) (with respect to the
new time variable $t'$.

Note that the eqs. (28) may be rewritten as follows. First, the second formula (28) in a more compact form reads

$$
\zeta'_i(t') = D^i_j \left( \left( e^{-it'H} ge^{itH} \right)^{-1} \right) \zeta_j(t)
$$

(29)

Moreover, the action of $g$ on time variable is given, according to the remark above, by

$$
ge^{itH} = e^{it'k(g, t)}, \quad k(g, t) \in K
$$

(30)

which yields the first eq.(28).

The dynamics considered up to now is rather trivial. In fact, eqs.(17) are linear and their solution (18) is immediately known. However, as mentioned above, the Poisson bracket (11) is degenerate so there exist in $G^*$ submanifolds which are invariant under the dynamics (17). The main point of the orbit method is that these submanifolds, under the assumption that $G$ acts transitively on them (i.e. the orbits of coadjoint action of $G$ on $G^*$) provide all $G$-invariant phase manifolds with transitive action of $G$. The Poisson bracket can be consistently restricted to the orbits and becomes then nondegenerate [6]-[11].

The geometry of orbits may be described in terms of the geometry of $G$. Let us select a particular orbit $O$ and let $\zeta^{(0)} \in O$ be any point on it. Let $H \subset G$ be the stability subgroup of $\zeta^{(0)}$:

$$
\zeta^{(0)} = D^j_i (h^{-1}) \zeta_j^{(0)} \iff h \in H
$$

(31)

Then $O$ is isomorphic to the left coset space $W = G/H$. Using this isomorphism it is not difficult to give an explicit expression for the symplectic form on $O$ defining the Poisson bracket (Kirillov form). Namely, let

$$
\omega(w) \equiv w^{-1} dw = i\omega^i(w)X_i, \quad w \in W
$$

(32)

be the left-invariant Cartan form on $W$. Define

$$
\tilde{\omega}(w) = \omega^k(w) \zeta^{(0)}_k ;
$$

(33)

then $d\tilde{\omega}(w)$ is the Kirillov form on $O$. Due to the isomorphism of $W$ and $O$ the coordinates $w$ on $W$ provide simultaneously the coordinates on $O$. Therefore, we would like to rewrite the dynamical equations (17) in terms of $w$. To this end we put

$$
\zeta_i(t) = D^k_i \left( w^{-1}(t) \right) \zeta^{(0)}_k = D^{-1}(w(t))^k i \zeta^{(0)}_k
$$

(34)
Inserting eq. (34) into eq. (17) one obtains

\[
\left( \Omega^l_m(w, \frac{dw}{dt}) D^{-1}(w)^m_i + c_{ij}^k \alpha^j D^{-1}(w)^l_k \right) \zeta_l^{(0)} = 0, \tag{35}
\]

or

\[
\left( \Omega^l_m(w, \frac{dw}{dt}) + c_{ij}^k \alpha^j D(w)^i m D^{-1}(w)^l_k \right) \zeta_l^{(0)} = 0 \tag{36}
\]

here \( \Omega^l_m \) is the Cartan form \( w^{-1} dw \) in adjoint representation

\[
\Omega^l_m(w, \frac{dw}{dt}) = (D^{-1})^l_k (w) dD^k_m(w) \tag{37}
\]

Note that due to the invariance of the tensors \( c^i_{\ jk} \) under the adjoint representation eq. (36) is equivalent to

\[
\left( \Omega^l_m \left( w, \frac{dw}{dt} \right) + c^l_{\ \mk} (D^{-1})^k_j (w) \alpha^j \right) \zeta_l^{(0)} = 0 \tag{38}
\]

To solve the eq. (36) note that the Hamiltonian in adjoint representation reads

\[
(iH)^i_j = \alpha^k c^i_{\ jk} \tag{39}
\]

This allows to rewrite eq. (38) symbolically as

\[
(iw^{-1} H w dt + w^{-1} dw) \zeta^{(0)} = 0 \tag{40}
\]

or

\[
(e^{itH} w(t))^{-1} d \left( e^{itH} w(t) \right) \zeta^{(0)} = 0 \tag{41}
\]

which implies that \( (e^{itH} w(t))^{-1} d \left( e^{itH} w(t) \right) \) belongs to the Lie algebra \( \eta \) of \( \mathcal{H} \). So we arrive at the following formula for equations of motion

\[
(e^{itH} w(t))^{-1} d \left( e^{itH} w(t) \right) \in \eta \tag{42}
\]

which, in fact, yields the constraints on Cartan forms. The general solution to (42)
reads

\[ e^{itH} w(t) = w_0 h(t), \quad w_0 \in W, \quad h(t) \in \mathcal{H} \tag{43} \]

or

\[ e^{-itH} w_0 = w(t) h^{-1}(t) \tag{44} \]

We conclude that the dynamics is given by nonlinear action of \( \exp(-itH) \) on coset space \( G/\mathcal{H} \) (which is rather natural conclusion).

Consider now the action of the symmetry group \( G \). By virtue of eqs.\((29)\) and \((34)\) one gets

\[ \zeta_j'(t') = D^k_j \left( \left( e^{-it'H} g e^{itH} w \right)^{-1} \right) \zeta_k^{(0)} \tag{45} \]

In terms of geometry of coset space eq.\((45)\) yields

\[ e^{-it'H} g e^{itH} w = w' h(g, w, t), \quad h(g, w, t) \in \mathcal{H} \tag{46} \]

or

\[ g e^{itH} w = e^{it'H} w' h(g, w, t) \tag{47} \]

We see that the action of symmetry group, as defined within the framework of the orbit method, coincides with the standard nonlinear realization of the symmetry group a la Coleman et al. \[12\]. In fact, let us compare eqs.\((43)\) and \((47)\). We see that, starting from the actual state of the system \( w(t) \) we first travel back to the initial moment, \( t = 0 \), then act with the element \( g \) of symmetry group and finally come back to the state at the transformed time \( t' \)

\[ g e^{itH} w(t) = gw_0 h(t) = w_0' h(g, w_0) h(t) = e^{it'H} w'(t') h(t') \tag{48} \]

i.e. time-dependent action of \( g \) consists in relating two states at \( t \) and \( t' \) which correspond to the two initial states related by the standard action of \( G \) (which, again, is not surprising).

9
III Examples

III.1 Conformal quantum mechanics

The conformal group in 1 + 0-dimensions coincides with $SL(2, \mathbb{R})$. It is generated by time translations ($H$), dilatations ($D$) and special conformal generator ($K$). The corresponding Lie algebra reads

$$[D, H] = -iH, \quad [D, K] = iK, \quad [K, H] = -2iD$$

(49)

Its basic representation is given by

$$H = i\sigma_+, \quad K = -i\sigma_-, \quad D = -\frac{i}{2}\sigma_3$$

(50)

$SL(2, \mathbb{R})$ is locally isomorphic to $SO(2, 1)$. Indeed, defining

$$M_0 = \frac{1}{2}(H + K)$$
$$M_1 = \frac{1}{2}(-H + K)$$
$$M_2 = D$$

(51)

one obtains

$$[M_\alpha, M_\beta] = -i\epsilon_{\alpha\beta\gamma}M_\gamma, \quad \epsilon_{012} = 1, \quad g_{\alpha\beta} = diag(+ --)$$

(52)

The counterpart of eq (50) is

$$M_0 = -\frac{1}{2}\sigma_2$$
$$M_1 = -\frac{i}{2}\sigma_1$$
$$M_2 = -\frac{i}{2}\sigma_3$$

(53)

Due to the local isomorphism of $SL(2, \mathbb{R})$ and $SO(2, 1)$ the adjoint (and, due to the semisimplicity, coadjoint) action of $SL(2, \mathbb{R})$ is the same as that of Lorentz group in 1 + 2-dimensions. This allows for simple classification of orbits. Let us take the orbit

$$(\zeta_0)^2 - (\zeta_1)^2 - (\zeta_2)^2 = \lambda^2, \quad \zeta_0 > 0$$

(54)

As the standard vector we take $\zeta^{(0)} = (\lambda, 0, 0)$. Its stability subgroup is one-dimensional
group generated by \( M_0 = \frac{1}{2}(H + K) \). The convenient coset parametrization reads

\[
w = e^{iw_1 K} e^{iw_2 D}
\] (55)

Let

\[
w_0 = e^{ic_1 K} e^{ic_2 D}
\] (56)

then eq. (44) takes the form

\[
e^{itH} e^{iw_1 K} e^{iw_2 D} = e^{ic_1 K} e^{ic_2 D} e^{-\frac{ic}{2}(H+K)}
\] (57)

which, up to notation and the choice of constants, coincides with eqs. (3.1) and (3.4) of Ref. [14]. Now, eq. (57) implies \( i(\omega_H H + \omega_K K + \omega_D D) = -\frac{1}{2} \frac{d}{dt}(H + K) \) i.e. \( \omega_D = 0 \), \( \omega_H = \omega_K \) on trajectories.

Eq. (57) yields (after some redefinition of \( \tau \) ):

\[
w_1 = \left( \frac{e^{-c_2} + c_1^2 e^{c_2}}{2} \right) \sin \tau
\]

\[
w_2 = -2 \ln \cos \frac{\tau}{2} - \ln (e^{-c_2} + c_1^2 e^{c_2})
\] (58)

\[
t = \frac{e^{c_2}}{1 + c_1^2 e^{2c_2}} \tan \left( \frac{\tau}{2} \right) + \frac{c_1 e^{2c_2}}{1 + c_1^2 e^{2c_2}}
\]

Using eq. (34) together with (55) we find the parametrization of our orbit in terms of variables \( w_1, w_2 \):

\[
\zeta_0 = \lambda \left( \text{ch} w_2 + \frac{1}{2} w_1^2 e^{w_2} \right)
\]

\[
\zeta_1 = \lambda \left( \text{sh} w_2 - \frac{1}{2} w_1^2 e^{w_2} \right)
\]

\[
\zeta_2 = -\lambda w_1^2 e^{w_2}
\] (59)

The Poisson brackets

\[
\{\zeta_\alpha, \zeta_\beta\} = -\epsilon_{\alpha\beta} \gamma \zeta_\gamma
\] (60)

yield

\[
\{w_1, w_2\} = -\frac{1}{\lambda} e^{-w_2}
\] (61)
The same result is obtained by computing the Cartan forms (32)

\[ \omega^0 = e^{u_2} dw_1, \quad \omega^1 = e^{u_2} dw_1, \quad \omega^2 = dw_2 \]  

(62)

According to the eq. (33) we find

\[ \tilde{\omega} = \lambda e^{u_2} dw_1 \]  

(63)

and

\[ d\tilde{\omega} = -\lambda e^{u_2} dw_1 \wedge dw_2 \]  

(64)

in agreement with eq.(61).

The Hamiltonian reads

\[ \tilde{H} = \lambda (w_1^2 e^{u_2} + e^{-u_2}) \]  

(65)

and implies the following equations of motion

\[ \begin{align*} 
\dot{w}_1 &= -w_1^2 + e^{-2u_2} \\
\dot{w}_2 &= 2w_1 
\end{align*} \]  

(66)

again in agreement with eqs.(2.14) of Ref.[14]

Let us define the canonical variables

\[ x = \sqrt{2\lambda} e^{\frac{u_2}{2}}, \quad p = \sqrt{2\lambda} w_1 e^{\frac{u_2}{2}} \]  

(67)

Then \( x > 0, p \in \mathbb{R}, \{x, p\} = 1 \) and

\[ \tilde{H}(x, p) = \frac{p^2}{2} + \frac{\lambda^2}{x^2} \]  

(68)

The action functional reads

\[ S = \int \left( -\tilde{\omega} \omega - \tilde{H}(w) dt \right) = \lambda \int \left( -e^{u_2} dw_1 - \tilde{H}(w) dt \right) \]  

(69)

Up to an exact form \( S \) can be rewritten as

\[ S = \int \left( w_1 e^{u_2} dw_2 - \tilde{H}(w) dt \right) = \int \left( p dx - \tilde{H}(x, p) dt \right) \]  

(70)
which is the standard form of action for conformal mechanics.

It is also easy to find the action of $SL(2, \mathbb{R})$. To this end we consider eq. (46) (or (47)). Taking into account the definition of coset space (55) we see that $h(g, w, t) = 1$ and the action of $SL(2, \mathbb{R})$ is simply given by group multiplication, again with perfect accordance with eq.(2.6) of Ref. [14].

### III.2 Galilean conformal mechanics

S. Fedoruk at al. [16] constructed, using the method of nonlinear realizations, the dynamical systems invariant under the action of $N$-Galilean conformal symmetry with $N = 2$. It is well known that the $N$-Galilean conformal algebras with $N$ even do not admit (except in two dimension) the central extension. This makes the classification of coadjoint orbits more complicated than in the case of centrally extended algebras with $N$-odd.

Fedoruk at al. construction is based on the following ingredients: (i) the choice of the subgroup $\mathcal{H} \subset G$ which defines the coset space $G/\mathcal{H}$; (ii) the construction of Cartan forms on $G/\mathcal{H}$; (iii) the application of the so-called inverse Higgs mechanism.

The advantage of this scheme is that one gets an algorithm which allows to produce various invariant dynamical equations. The difficulty is that one cannot take for granted that the resulting equation have automatically Hamiltonian form. In fact, this depends on whether the selected subgroup $\mathcal{H}$ is or is not a stability group of some point on coadjoint orbit.

Let us remind the form of conformal algebra for $N = 2$. It is spanned by $so(d)$ generators $J_{ij}$, $sl(2, \mathbb{R})$ ones $K, D, H$ and additional generators $P_i, B_i$ and $F_i$ which span $D^{(1,1)}$ representation of $so(d) \oplus sl(2, \mathbb{R})$:

\begin{align*}
[H, P_k] &= 0 & [H, F_k] &= 2iB_k & [H, B_k] &= iP_k \\
[K, P_k] &= -2iB_k & [K, F_k] &= 0 & [K, B_k] &= -iF_k \\
[D, P_k] &= -iP_k & [D, F_k] &= iF_k & [D, B_k] &= 0
\end{align*}

(71)

$P_k, B_k$ and $F_k$ themselves span abelian subalgebra.

We shall discuss two examples of coadjoint orbits and the corresponding dynamical systems; for simplicity we assume $d = 3$. Defining the tensor $X_{\alpha i}$ by

\begin{align*}
X_{0i} &\equiv \frac{1}{2} (P_i + F_i) \\
X_{1i} &\equiv \frac{1}{2} (P_i - F_i) \\
X_{2i} &\equiv B_i
\end{align*}

(72)

one finds
\[
\begin{align*}
[J_i, J_j] &= i \epsilon_{ijk} J_k \\
[M_\alpha, M_\beta] &= -i \epsilon_{\alpha \beta}^\gamma M_\gamma \\
[J_i, X_{\alpha j}] &= i \epsilon_{ijk} X_{\alpha k} \\
[M_\alpha, X_{\beta i}] &= -i \epsilon_{\alpha \beta}^\gamma X_{\gamma i}
\end{align*}
\]

(73)

the remaining commutators being zero. With \(J^i, M^\alpha, X^{\alpha i}\) being the dual basis, the general element of dual space reads

\[
j_i J^i + m_\alpha M^\alpha + x_{\alpha i} X^{\alpha i}
\]

(74)

Assume first that the stability group contains \(SO(3)\). The general element (74) depends on four coefficients, \(j_i, X_{\alpha i}\), belonging to spin-1 irreducible representations of \(so(3)\) and the scalars. Therefore, for \(SO(3)\) to be contained in stability subgroup of (74) it is necessary that \(j_i = 0, x_{\alpha i} = 0\). The stability group of such a point is generated by \(J_i\) and \(X_{\alpha i}\). We are left with \(SL(2, \mathbb{R})\) group and the dynamics considered in previous subsection.

As a second example we choose initial point on the coadjoint orbit which breaks explicitly \(SO(d)\) symmetry. To this end let us first write out explicitly the coadjoint action using eqs. (3) and (10). We put

\[
g = e^{i \sum \omega^\alpha M_\alpha} e^{i \sum \eta_i J_i} e^{i \sum y^{\alpha i} X_{\alpha i}}
\]

(75)

which yields

\[
\begin{align*}
g^{-1} X_{\beta j} g &= (R^{-1})_{kj} (\Lambda^{-1})^\alpha_\beta X_{\alpha k} \\
g^{-1} J_j g &= (R^{-1})_{kj} (J_k - \epsilon_{kil} y^{\alpha} X_{\alpha l}) \\
g^{-1} M_\alpha g &= (\Lambda^{-1})^\beta_\alpha (M_\beta + \epsilon_{\beta \gamma \rho} y^{\gamma} X_{\rho i})
\end{align*}
\]

(76)

where \(R \in SO(3), \Lambda \in SO(2, 1)\). The orbit under consideration is selected by taking the initial point in the form:

\[
\zeta = m_\alpha M^\alpha + j_k J^k + x_{\alpha i} X^{\alpha i}
\]

(77)

with \(j_k = 0, m_\alpha = 0, x_{\alpha i} = \zeta_\alpha \cdot s_\alpha, \zeta_\alpha \equiv (\lambda, 0, 0), \lambda > 0, s_\alpha \equiv (0, 0, 1)\). It is now easy to see that the stability subgroup is generated by \(M_0, J_3, X_{03}, X_{11}, X_{12}, X_{21}\) and \(X_{22}\). We are then left with eight generators \(M_1, M_2, J_1, J_2, X_{01}, X_{02}, X_{13}\) and \(X_{23}\). Therefore the coset element can be written as

\[
w = e^{i \sum \omega^\alpha M_\alpha} e^{i \sum \eta_i J_i} e^{i \sum y^{\alpha i} X_{\alpha i}}
\]

(78)
where \( \sum' \omega^\alpha M_\alpha \equiv \omega^1 M_1 + \omega^2 M_2 \), \( \sum' \eta_i J_i \equiv \eta_1 J_1 + \eta_2 J_2 \), \( \sum' y^\alpha i X_{\alpha i} \equiv y^0_1 X_{01} + y^0_2 X_{02} + y^1_3 X_{13} + y^2_3 X_{23} \equiv y^\alpha_i X_{\alpha i} \).

The resulting orbit reads finally

\[
\begin{align*}
  m_\alpha &= (\Lambda^{-1})^\beta_\alpha \left( \epsilon_{\beta\gamma} \phi y^\gamma_i \xi^\beta_j \right) \\
  j_i &= - (R^{-1})_{kj} \epsilon_{kij} y^\alpha_i \xi^\beta_j \\
  x_{\beta j} &= (R^{-1})_{kj} (\Lambda^{-1})^\alpha_{\beta} \xi^\beta_j
\end{align*}
\] (79)

where \( \Lambda \) (respectively \( R \)) is generated by the elements \( M_1, M_2 \) (respectively \( J_1, J_2 \)). The orbit (75) may be rewritten in more elegant form by defining

\[
\begin{align*}
  \zeta_\alpha &\equiv (\Lambda^{-1})^\beta_\alpha \xi^\beta_j \\
  s_k &\equiv (R^{-1})_{jk} \xi^\beta_j \\
  y^\alpha_i &\equiv (\Lambda^{-1})^\beta_\alpha (R^{-1})_{ki} \xi^\beta_k
\end{align*}
\] (80)

Then eqs (79) take the form

\[
\begin{align*}
  m_\alpha &= \epsilon_{\alpha\beta\gamma} y^\beta_k \xi^\gamma_i \xi^\beta_j s_k \\
  j_i &= - \epsilon_{ijk} y^\alpha_j \xi^\alpha_k s_k \\
  x_{\alpha i} &= \zeta_\alpha s_i
\end{align*}
\] (81)

The price one has to pay for the new form of orbit are constraints. First, from eqs. (80) one gets

\[
s_i s_i = 1, \quad \zeta_\alpha \xi^\alpha = \lambda^2
\] (82)

Moreover, one has to reduce the number of independent components of \( y^\alpha_i \). The relevant constraints read

\[
\begin{align*}
  \zeta_\alpha s_i y^\alpha_i &= 0 \\
  \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} s_j \xi^\beta_k y^\gamma_i &= 0
\end{align*}
\] (83)

The constraints (83) express the fact that there are only four independent variables.
They can be replaced by new ones

\[ t_i \equiv y^\alpha_i \zeta_\alpha \]
\[ \eta^\alpha \equiv y^\alpha_i s_i \]  \hspace{1cm} \text{(84)}

obeying

\[ s_i t_i = 0 \]
\[ \zeta_\alpha \eta^\alpha = 0 \]  \hspace{1cm} \text{(85)}

The orbit (81) is now parametrized as

\[ m_\alpha = \epsilon_{\alpha\beta} \eta^\beta \zeta_\gamma \]
\[ j_i = \epsilon_{ijk} s_j t_k \]
\[ x_{\alpha i} = \zeta_\alpha s_i \]  \hspace{1cm} \text{(86)}

with parameters being constarined by eqs. (82) and (85). It is not difficult to find the relevant Poisson brackets for new basic variables:

\[ \{ s_i, s_j \} = 0 \]
\[ \{ s_i, t_k \} = \delta_{ik} s_i s_k \]
\[ \{ t_i, t_k \} = t_i s_k - t_k s_i \]
\[ \{ \zeta_\alpha, \zeta_\beta \} = 0 \]
\[ \{ \zeta_\alpha, \eta^\beta \} = \delta_\alpha^\beta - \frac{1}{\lambda^2} \zeta_\alpha \zeta_\beta \]
\[ \{ \eta^\alpha, \eta^\beta \} = \frac{1}{\lambda^2} \left( \eta^\alpha \zeta_\beta - \eta^\beta \zeta_\alpha \right) \]  \hspace{1cm} \text{(87)}

with all remaining Poisson brackets vanishing.

Finally, the Hamiltonian reads

\[ H = m_0 - m_1 = (\eta^2 \zeta_1 - \eta^1 \zeta_2 + \eta^0 \zeta_2 - \eta^2 \zeta_0) \]  \hspace{1cm} \text{(88)}

Let us describe in some detail our model. The phase space is the product of cotangent bundles to the unit sphere and to the upper sheet of hyperboloid \( \zeta^\alpha \zeta_\alpha = \lambda^2 \). The \( SO(3) \times SL(2, \mathbb{R}) \) group acts naturally. On the other hand the action of the abelian subgroup \( \exp (iz^\alpha \iota X_\alpha) \) reads
\( s'_i = s_i \)
\( \zeta'^\alpha = \zeta^\alpha \)
\( t'_i = t_i + (\delta_{ik} - s_i s_k)\zeta^\alpha z^\alpha_k \) \hspace{1cm} (89)
\( \eta'^\alpha = \eta^\alpha + \left( \delta^\beta_\alpha - \frac{1}{\lambda^2} \zeta^\alpha \zeta^\beta \right) z^\beta_i s_i \)

One can easily check that the Poisson structure (87) is invariant under the above action.

The dynamics of \( s_i \) and \( t_i \) variables is trivial. On the other hand the remaining equations of motion read

\[
\begin{align*}
\dot{\zeta}_0 &= \zeta_2 - \frac{2}{\lambda^2} (\zeta_0)^2 \zeta_2 \\
\dot{\zeta}_1 &= -\zeta_2 - \frac{2}{\lambda^2} \zeta_0 \zeta_1 \zeta_2 \\
\dot{\zeta}_2 &= -\zeta_0 + \zeta_1 - \frac{2}{\lambda^2} \zeta_0 (\zeta_2)^2 \\
\dot{\eta}_0 &= \eta^2 + \frac{2}{\lambda^2} \eta^0 \zeta_0 \zeta_2 \\
\dot{\eta}_1 &= -\eta^2 + \frac{2}{\lambda^2} \eta^1 \zeta_0 \zeta_2 \\
\dot{\eta}_2 &= -\eta^0 + \eta^1 + \frac{2}{\lambda^2} \eta^2 \zeta_0 \zeta_2 
\end{align*}
\] \hspace{1cm} (90)

They are consistent with the constraints (82), (85).

One can easily construct the explicitly time dependent symmetry generators and verify that they obey the correct Poisson algebra of our conformal Galilei group.

It is also straightforward but slightly tedious to show that the orbit construction can be reformulated in terms of the geometry of nonlinear realizations defined by the appropriate choice of stability subgroup (generated by \( J_3, M_0, X_{03}, X_{11}, X_{12}, X_{21} \) and \( X_{22} \)). The relevant constraints on Cartan forms are read off from eqs. (42) and (43). We omit the details here.

As it has been mentioned above the full classification of coadjoint orbits for \( N \)-conformal Galilei algebras with \( N \) even (i.e. not admitting the central extension) is rather involved even for \( N = 2 \). For the special choice of the orbit, eq. (77), the relevant dynamical system is described by eqs. (81) - (90). However, the particular form of dynamics seems to depend strongly on the choice of the orbit. This is also the case for \( N \) odd if we assume the central charge is vanishing. On the contrary, for nonvanishing central charge the dynamics is essentially unique [18], [19].
IV Conclusions

There are two basic methods of explicit construction of dynamical systems on which a
given group $G$ acts transitively as the symmetry group. One is based on the technique
of nonlinear realizations [12]. It allows for elegant and algorithmic construction of in-
variant dynamical equations by selecting an appropriate coset space and computing the
relevant Cartan-Maurer forms (and applying the so-called inverse Higgs phenomenon
[10]). The main problem with this approach is that it does not always lead to the
dynamics which admits Hamiltonian form.

The second method is based on the idea of coadjoint orbits. They are equipped with
invariant symplecting form (Kirillov form) and exhaust the list of all symplectic mani-
folds on which a given group acts transitively as a group of canonical transformations.
It appears to be the dynamical symmetry group provided the Hamiltonian belongs to
its Lie algebra (or universal enveloping algebra). A coadjoint orbit can be identified
with an appropriate coset space once the stability subgroup of an arbitrarily selected
point of the orbit is determined. This identification allows to establish the relation be-
tween both methods; in particular, the Kirillov form is expressible in terms of Cartan
forms [11]. Hamiltonian equations can be also rewritten with the help of these forms.
These general considerations were illustrated by few examples. First, we showed that
the elegant result of Ivanov at al. concerning the conformal (i.e. $SL(2, \mathbb{R})$) invariant
mechanics fits perfectly into the framework of orbit method. The second example is
related to the $N = 2$ conformal Galilei group. We find that if the stability subgroup
contains $SO(3)$ then the Hamiltonian dynamics reduces to that of $SL(2, \mathbb{R})$ confor-
mal mechanics. More complicated example has been also constructed which describes
the system with four degrees of freedom running over nontrivial configuration space
$S^2 \times \mathcal{H}^2$, where $\mathcal{H}^2$ is the upper sheet of the hyperboloid $\zeta_0^2 - \zeta_1^2 - \zeta_2^2 = \lambda^2$.

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