SHORT COMMUNICATIONS

On the Number of Singular Points of Terminal Factorial Fano Threefolds

Yu. G. Prokhorov*

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia

Received December 6, 2016

DOI: 10.1134/S0001434617050364

Keywords: Fano threefold, terminal singularity.

1. INTRODUCTION

Throughout this paper, by $X$ we denote a Fano threefold with terminal $\mathbb{Q}$-factorial singularities and $\text{rk} \, \text{Pic}(X) = 1$. Let $\iota = \iota(X)$ be its Fano index and $g := (-K_X)^3/2 + 1$ be its genus. When $\iota(X) \geq 2$, a more convenient invariant of a Fano threefold $X$ is its degree $d := (-K_X)^3/\iota^3$. Recall that any $X$ as above has a one-parameter smoothing $\mathfrak{X}$ [1]. Let $h^{1,2}(\mathfrak{X})_{\text{s}}$ be the Hodge number of the general fiber of $\mathfrak{X}$. Denote by $s(V)$ be the number of singular points of $V$.

The upper bounds of $s(X)$ are interesting for classification problems, in particular, for the classification of finite subgroups of the space Cremona group (see, e.g., [2], [3], [4], [5]). Y. Namikawa in [1] proved the inequality

$$s(X) \leq 20 - \text{rk} \, \text{Pic}(X) + h^{1,2}(\mathfrak{X})_{\text{s}}$$

which holds for any Fano threefold $X$ with terminal Gorenstein singularities. However, this estimate is quite rough. For Fano threefolds with nondegenerate singularities; the inequality $s(X) \leq h^{1,2}(\mathfrak{X})_{\text{s}}$ is known (see, e.g., [6, Sec. 10]).

Theorem 1 (cf. [7, Theorem 4.1]). Suppose that either

1) $\iota = 1$ and $g \geq 7$ or 2) $\iota = 2$ and $d \geq 3$.

Then $s(X) \leq h^{1,2}(\mathfrak{X})_{\text{s}}$ and this bound is attained for some $X$ having only ordinary double points.

Recall (see [8], see also the references in [9]) that $h^{1,2}(\mathfrak{X})_{\text{s}}$ takes the following values:

| $\iota$ | $g$ | $h^{1,2}$ |
|--------|-----|-----------|
| 1      | 12  | 0         |
|        | 10  | 2         |
|        | 9   | 3         |
|        | 8   | 5         |
| 1      | 7   | 6         |
|        | 6   | 5         |
|        | 5   | 4         |
|        | 4   | 3         |
|        | 3   | 2         |

| $\iota$ | $d$ | $h^{1,2}$ |
|--------|-----|-----------|
| 2      | 5   | 0         |
|        | 4   | 2         |
|        | 3   | 1         |
|        | 2   | 10        |
|        | 1   | 21        |

Note that, in our theorem, we do not assume that the singularities of $X$ are nondegenerate. Moreover, our construction allows us to obtain better bounds in degenerate cases. If $\iota = 1$, $g = 6$ or $\iota = d = 2$, then under the additional assumption that the variety $X$ has a $cA_1$-point, our computations allow us to obtain the estimate $s(X) \leq 15$ (cf. [7, proof of 1.3]). However, we do not claim that this bound is sharp.

*E-mail: prokhoro@mi.ras.ru
A Sarkisov link is the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\chi} & Y' \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xrightarrow{\chi} & Z
\end{array}
\]

where \(f\) and \(f'\) are extremal Mori contractions and \(\chi\) is a flop. In our situation, the morphism \(f\) is birational. Let \(E\) be the exceptional divisor and \(E' \subset Y'\) be its proper transform. If \(f'\) is also birational, then we denote by \(D'\) its exceptional divisor and \(\Gamma := f'(D')\). If, moreover, \(\Gamma\) is a curve, then it is irreducible, contained in the nonsingular locus of \(Z\), \(f'\) is the blow-up of \(\Gamma\), and the singularities of \(\Gamma\) are planar (if there are any; see [10]).

2. BIRATIONAL TRANSFORMATIONS IN SINGULAR POINTS

Recall that a three-dimensional singularity \(P \in X\) is of type \(cA_1^n\) if it is analytically equivalent to the hypersurface singularity \(x_1x_2 + x_3^n + x_4^n\). The blow-up \(f : Y \to X\) of the maximal ideal \(m_{P,X}\) of such a point is a Mori contraction (see [10]). If \(m = 2\), then \(Y\) is smooth along \(f^{-1}(P)\). If \(m > 2\), then \(Y\) has exactly one singular point on \(f^{-1}(P)\) and this point is of type \(cA_1^{m-2}\).

Sarkisov links with centers at singular points \(P \in X\) of type \(cA_1\) appeared earlier in different papers (see, e.g., [11], [12], [14, Sec. 3], [13]), but usually under some additional restrictions. We systematize and generalize this information as follows.

**Theorem 2.** Suppose that \(X\) has a \(cA_1\)-point \(P \in X\). Let \(f : Y \to X\) be the blow-up of \(m_{P,X}\). If \(\iota = 2, d \geq 2\), then \(f\) is included in the Sarkisov link (1), where \(\chi\) is an isomorphism and \(f'\) is defined by the linear system \(|\frac{1}{2}(-K_Y, -E')|\). Except for case 3\(^o\), the morphism \(f'\) contracts a divisor \(D'\) to a curve \(\Gamma\):

| no. | \(d\)    | \(Z\)                        | \(\deg\Gamma\) | \(p_4(\Gamma)\) |
|-----|---------|-----------------------------|-----------------|-----------------|
| 1\(^o\) | 4      | a smooth quadric in \(\mathbb{P}^4\) | 4               | 1               |
| 2\(^o\) | 3      | \(\mathbb{P}^3\)           | 6               | 4               |
| 3\(^o\) | 2      | \(\mathbb{P}^2\), \(f'\) is a conic bundle with discriminant curve \(\Delta \subset \mathbb{P}^2\), \(\deg\Delta = 6\) | \text{---}      | \text{---}      |

If \(\iota = 1, g \geq 4\), then \(f\) is included in the Sarkisov link (1), where, except for case 10\(^o\), the morphism \(f'\) is defined by the linear system \(|-K_Y, -E'|\), and, except for cases 8\(^o\) and 9\(^o\), \(f'\) contracts a divisor \(D'\) to a curve \(\Gamma\):

| no. | \(g\)  | \(Z\)                        | \(\deg\Gamma\) | \(p_4(\Gamma)\) |
|-----|-------|-----------------------------|-----------------|-----------------|
| 4\(^o\) | 10    | a nonsingular Fano threefold with \(\iota = 2\) and \(d = 5\) | 6               | 1               |
| 5\(^o\) | 9     | a \(\mathbb{Q}\)-factorial Fano threefold with \(\iota = 2\) and \(d = 4\) | 4               | 0               |
| 6\(^o\) | 8     | a smooth quadric in \(\mathbb{P}^4\) | 8               | 4               |
| 7\(^o\) | 7     | \(\mathbb{P}^3\)           | 8               | 6               |
| 8\(^o\) | 6     | \(\mathbb{P}^2\), \(f'\) is a conic bundle with discriminant curve \(\Delta \subset \mathbb{P}^2\), \(\deg\Delta = 6\) | \text{---}      | \text{---}      |
| 9\(^o\) | 5     | \(\mathbb{P}^3\), \(f'\) is a del Pezzo fibration of degree 4 | \text{---}      | \text{---}      |
| 10\(^o\) | 4    | a \(\mathbb{Q}\)-factorial Fano threefold with \(\iota = 1\) and \(g = 4\) | 4               | 0               |

In case 10\(^o\), \(D' \in |-K_Y, -E'|\).
Proof. In all our cases, the linear system $|−K_X|$ is very ample and defines an embedding $X \subset \mathbb{P}^{g+1}$ as a variety of degree $2g−2$. Consider the projection $\psi : X → Y_0 \subset \mathbb{P}^2$ from $P$. The blow-up $f : Y → X$ of the maximal ideal $m_{P,X}$ resolves the indeterminacy points of $\psi$ and gives a morphism $Y → Y_0$. Note that the exceptional divisor $E \subset Y$ is isomorphic to an irreducible quadric $Q \subset \mathbb{P}^3$ and $\mathcal{O}_E(E) ≃ \mathcal{O}_Q(-1)$ (see [10]). It is easy to see that $K_Y = f^*K_X + E$. This shows that the morphism $Y → Y_0$ is given by the anticanonical linear system and $−K_Y^2 = −K_X^2 − 2 > 0$. Therefore, $\dim Y_0 = 3$ and, in the Stein factorization $Y → Y → Y_0$, the morphism $f : Y → Y$ is birational and crepant. Since there are at most a finite number of lines passing through $P$, the exceptional locus of the projection $\psi$ is at most one-dimensional. Thus, $f : Y → Y$ is a small crepant contraction. Hence there exists a flop $\chi : Y → Y'$ and an extremal Mori contraction $f' : Y' → Z$. Using the restriction exact sequences for $E ≃ Q \subset \mathbb{P}^3$, it is easy to compute that $\dim |−K_{Y'} − E'| ≥ \dim |−K_X| = g − 4$ always and $|\frac{1}{2}(−K_{Y'} − E')| ≥ d$ for $\iota = 2$. Using this and solving the corresponding Diophantine equations, as in [8, Chap. 4] and [7, Sec. 4], we obtain the possibilities $1^0\text{-}10^0$. The smoothness of $Z$ in cases $1^0$, $4^0$ and $6^0$ follows from $Q$-factoriality (see Corollary 2 below). If $\iota = 2$, then the threefold $X$ does not contain any lines (for the anticanonical embedding). Therefore, the divisor $−K_Y$ is ample and $Y ≃ Y'$. This proves Theorem 2.

Remark 1. Let the pair $(Z, \Gamma)$ be such as in cases $1^0$, $2^0$, $4^0\text{-}7^0$. Suppose that

(i) the curve $\Gamma$ has planar singularities and is contained in the nonsingular locus of $Z$;

(ii) $\Gamma$ is the scheme-theoretic intersection of elements of the linear system $|\iota(Z)L|$, where $L$ is the ample generator of $\text{Pic}(Z) ≃ \mathbb{Z}$;

(iii) the number of $\iota(Z)$-secant lines of $\Gamma$ is at most finite.

Then the blow-up of $\Gamma$ can be completed to a Sarkisov link with the corresponding Fano threefold $X$ (i.e., the construction (1) can be reversed). Indeed, it follows from condition (i) that $Y'$ has only terminal factorial singularities, condition (ii) ensures that $|−K_{Y'}|$ has no base points, and condition (iii) guarantees that the corresponding morphism defines a crepant small contraction. The rest is similar to the proof of Theorem 2.

Corollary 1. Suppose that, under the conditions of Theorem 1, the threefold $X$ has a point of type $cA_1$. Then $s(X) ≤ h^{1,2}(\mathbb{X}_a)$ and this bound is attained for some $X$ having only ordinary double points.

Proof. Note that three-dimensional terminal flops preserve types of singularities. Therefore, in the notation of Theorem 2 and diagram (1), we have

$$s(X) ≤ s(Y) + 1 = s(Y') + 1 = s(Z) + s(\Gamma) + 1 ≤ s(Z) + p_a(\Gamma) + 1$$

(see [7, 5.1(ii)]). In all cases $1^0\text{-}7^0$, we can choose $Z$ and $\Gamma$ so that equalities are attained. This proves Corollary 1.

3. PROOF OF THEOREM 1: THE CASE $g ≠ 7$

For $\iota = 1$, $g ≥ 9$, Theorem 1 was proved in [7, 1.3] and, for $\iota = 1$, $g = 8$, and $\iota = 2$, $d = 3$, the assertion follows directly from Corollary 1 and the following lemma.

Lemma 1. Let one of the following conditions hold:

1) $\iota = 1$ and $g = 8$ or 2) $\iota = 2$ and $d = 3$.

If $s(X) > 2$, then $X$ has a singularity of type $cA_1$. 

MATHEMATICAL NOTES Vol. 101 No. 6 2017
Proof. In the case \( g = 8 \), according to [7, 4.1], the double projection from a line contained in the nonsingular locus determines a Sarkisov link, where \( f' \) is a conic bundle over \( \mathbb{P}^2 \) with discriminant curve \( \Delta \) of degree 5. Similarly, for a cubic hypersurface with \( \mathbb{Q} \)-factorial terminal singularities, the usual projection from a line defines a Sarkisov link, where \( f' \) is a conic bundle over \( \mathbb{P}^2 \) with discriminant curve \( \Delta \) of degree 5 (in this case, \( \chi \) is isomorphism). Moreover, \( Y' \) has the same collection of singularities as \( X \). Assume that \( s(X) > 2 \) and all the singularities of \( X \) are worse than \( cA_1 \). Let \( R \in Y' \) be a singular point, and let \( o := f'(R) \).

We claim that \( R \) is the only singular point of \( Y' \) lying on the fiber \( f := f'^{-1}(o) \) and the singularity of the curve \( \Delta \) at \( o \) is of multiplicity \( \geq 3 \). Let \( U \subset Z = \mathbb{P}^2 \) be a small analytic neighborhood of \( o \) and \( Y'_U := f'^{-1}(U) \). Then \( Y'_U \) can be embedded to \( \mathbb{P}^2_{x_0,x_1,x_2} \times U_{u_1,u_2} \) and defined there by a homogeneous equation of degree 2 with respect to the variables \( x_0, x_1, x_2 \). We may assume that \( R \) has coordinates \((0 : 0 : 1; 0, 0)\).

If \( F \) is a reducible conic, then the equation of \( Y'_U \) can be written in the form

\[
x_0x_1 + \alpha(u_1, u_2)x_0^2 = 0, \quad \text{where} \quad \alpha \in \mathfrak{m}^2_{u,U}.
\]

In this case, \( R \) is the only singular point of \( Y'_U \) and the discriminant curve \( \Delta = \{ \alpha = 0 \} \) has a singularity of multiplicity \( \geq 3 \) at \( o \).

If \( F \) is a double line, then the equation of \( Y'_U \) can be written in the form

\[
x_0^2 + \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 = 0,
\]

where \( \alpha \in \mathfrak{m}_{u,U}^3, \beta, \gamma \in \mathfrak{m}_{u,U}^2 \). Since singularities of \( Y'_U \) are isolated, we have \( \alpha \notin \mathfrak{m}_{u,U}^2 \). This means that \( R \) is the only singular point of \( Y'_U \). The discriminant curve \( \Delta \) is given by \( \beta^2 = 4\alpha \gamma \) and again has a singularity of multiplicity \( \geq 3 \) at \( o \). This proves our assertion.

Thus, \( s(Y') \) is less than or equal to the number of points of multiplicity \( \geq 3 \) of the curve \( \Delta \). On the other hand, a plane (possibly, reducible) curve of degree 5 has at most two triple points. Lemma 1 is proved. \( \square \)

It remains to consider the case \( \nu = 2, d \geq 4 \). The following lemma is well known in the nonsingular case (see, e.g., [8]). The singular case is similar.

Lemma 2. Let \( X = X_d \subset \mathbb{P}^{d+1} \) be a Fano threefold with \( \nu = 2, d \geq 4 \) and let \( l \) be a line contained in the nonsingular locus of \( X \). Then the projection from \( l \) determines a Sarkisov link, where \( \chi \) is an isomorphism. Moreover,

(i) if \( d = 4 \), then \( Z \simeq \mathbb{P}^3 \), and \( f' \) is the blow-up of an (irreducible) curve of degree 5 and arithmetic genus 2 lying on a quadric;

(ii) if \( d = 5 \), then \( Z \) is smooth quadric in \( \mathbb{P}^4 \) and \( f' \) is the blow-up of a smooth rational cubic curve lying on a hyperplane section.

Corollary 2. If \( \nu = 2, d = 5 \), then \( X \) is smooth. Let \( \nu = 2, d = 4 \). In that case, \( s(X) \leq 2 \) and if \( s(X) = 2 \), then both singularities of \( X \) are of type \( cA_1 \).

4. THE CASE \( g = 7 \)

Theorem 3. Let \( \nu = 1, g = 7 \) and let \( C \subset X \) be a sufficiently general conic. Then there exists a Sarkisov link (1) with center \( C \) and there are two possibilities:

(i) \( Z \subset \mathbb{P}^4 \) is a smooth quadric, \( f' \) is the blow-up of a curve \( \Gamma \subset Z \) with \( \deg \Gamma = 10, p_a(\Gamma) = 7 \);

(ii) \( Z \subset \mathbb{P}^4 \) is a cubic and \( f' \) is the blow-up of a smooth rational curve \( \Gamma \subset Z \) of degree 4.
The proof is substantially similar to [8, Sec. 4.4], but, in the singular case, some of the arguments must be modified. According to [1] and [8, 4.2.5], the family \( \mathcal{C} \) parametrizing nondegenerate conics on \( X \) is not empty and two-dimensional and, according to [15, A.1.2], conics from the family \( \mathcal{C} \) cover an open subset in \( X \). Further, we need the following.

**Lemma 3.**
(i) For each line \( l \subset X \), there are only a finite number of lines on \( X \) meeting \( l \).

(ii) Let \( l \) be a line contained in the nonsingular locus of \( X \). Then it meets a one-dimensional family of conics.

(iii) There is a nondegenerate conic contained in the nonsingular locus of \( X \).

**Proof.** (i) The assertion follows from the fact that \( \dim |H - 2l| \geq 1 \) and \( \text{Pic}(X) = \mathbb{Z} \cdot [H] \).

(ii) Consider the double projection from \( l \) and the corresponding Sarkisov link [7]. In the notation of (1), the morphism \( f \) is the blow-up of \( l \), \( Z \cong \mathbb{P}^1 \) and \( f' \) is a del Pezzo fibration of degree 5. The proper transform \( C' \subset Y' \) of a conic \( C \subset X \) meeting \( l \) is a line in a fiber of \( f' \) (because \( -K_{Y'} \cdot C' = 1 \) and \( E' \cdot C' \geq 1 \)). There is at most a one-dimensional family of such lines. This proves (ii).

(iii) Assume that all of the conics from the (two-dimensional) family \( \mathcal{C} \) pass through a point \( P \in X \). Take a line \( l \subset X \) that does not pass through \( P \) and such that all the lines \( l_i \) meeting \( l \) also do not pass through \( P \). Then the map \( \chi \circ f^{-1} : X \rightarrow Y' \) is an isomorphism near \( P \). The line \( l \) intersects a one-dimensional family of conics from the family \( \mathcal{C} \) and the proper transforms; these conics on \( Y' \) are lines in the fiber of \( f' \) passing through the point \( P' := \chi(f^{-1}(P)) \). Hence this fiber is a cone in \( \mathbb{P}^5 \) with vertex \( P' \). But since the singularities \( Y' \) are hypersurface, it follows that the dimension of the tangent space to a fiber of \( f' \) is at most 4. The contradiction proves our lemma.

Thus, there exists a conic \( C \) contained in the nonsingular locus of \( X \). Take such a conic so that it is not contained in the surface swept out by the lines. Consider the projection of \( \psi : X \rightarrow Y_0 \subset \mathbb{P}^5 \) from the linear span of \( C \). Since \( X \cap (\mathcal{C}) = C \), the blow-up \( f : Y \rightarrow X \) of the conic \( C \) resolves the indeterminacy points of \( \psi \) and defines a morphism \( f_0 : Y \rightarrow Y_0 \). Here, in the Stein factorization \( Y \rightarrow \tilde{Y} \rightarrow Y_0 \), the morphism \( \tilde{f} : Y \rightarrow \tilde{Y} \) is birational and is given by a multiple of the linear system \( |-K_Y| \), i.e., \( \tilde{f} \) is crepant.

**Lemma 4.** The exceptional locus of \( \tilde{f} \) is one-dimensional.

**Proof.** Suppose that \( \tilde{f} \) contracts a divisor \( D \). It is easy to compute (see [8, 4.1.2]) that
\[
D \sim \alpha(2(-K_Y) - 3E) \quad \text{for some } \alpha \in \mathbb{Z}_{>0} \quad \text{and} \quad D \cdot E \cdot (-K_Y) = 14\alpha > 0.
\]
In particular, \( \Lambda := f_0(D \cap E) \) is a curve. Since the conic \( C \) meets only a finite number of lines, the general fiber of \( D \) must be a conic meeting \( C \) at two points [8, 4.4.1]. If the restriction \( f_0|_E : E \rightarrow f_0(E) \) is a birational morphism, \( f_0(E) \) is a ruled surface of degree 4 singular along \( \Lambda \), where
\[
\deg \Lambda = \frac{1}{2} D \cdot E \cdot (-K_Y) = 7\alpha \geq 7.
\]
This is impossible; see, e.g., [8, 4.4.8]. Therefore, \( f_0 \) is a morphism of degree 2, \( f_0(E) \) is a quadric, and \( X_0 \subset \mathbb{P}^5 \) is a subvariety of degree 3. Since \( \text{rk Cl}(\tilde{Y}) = \text{rk Cl}(Y_0) = 1 \), it follows that the variety \( Y_0 \) is a cone over a rational twisted cubic curve with a vertex on a line. However, this cone contains no quadrics. The contradiction proves Lemma 4.

Further, as in [8, Sec. 4.1], there exists a flop \( \chi : Y \rightarrow Y' \) and an extremal Mori contraction \( f' : Y' \rightarrow Z \). By solving the corresponding Diophantine equations, as in [8, Chap. 4] or [7, Sec. 4], we obtain the possibilities (i)–(ii). Theorem 3 is proved. For the proof of Theorem 1 in the case \( g = 7 \), similar to (2), we write
\[
s(X) = s(Y) = s(Y') = s(Z) + s(\Gamma) \leq s(Z) + p_a(\Gamma).
\]
If the possibility (i) of Theorem 3 occurs, then the variety \( Z \) is smooth and therefore \( s(X) \leq p_a(\Gamma) = 7 \). Otherwise, the curve \( \Gamma \) is smooth and therefore \( s(X) = s(Z) \). Then \( s(Z) \leq 5 \) according to the already proved case of Theorem 1.
ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation under grant 14-50-00005.

REFERENCES

1. Y. Namikawa, J. Algebraic Geom. 6 (2), 307 (1997).
2. Yu. Prokhorov, J. Algebraic Geom. 21 (3), 563 (2012).
3. Yu. Prokhorov and C. Shramov, Jordan Constant for Cremona Group of Rank 3, arXiv: 1608.00709 (2016).
4. Yu. Prokhorov and C. Shramov, p-Subgroups in the Space Cremona Group, arXiv: 1610.02990 (2016).
5. V. V. Przhiyalkovskii and K. A. Shramov, in Trudy Mat. Inst. Steklov., Vol. 294: Contemporary Problems of Mathematics, Mechanics, and Mathematical Physics. II (MAIK, Moscow, 2016), pp. 167–190 [Proc. Steklov Inst. Math. 294, 154 (2016)].
6. Yu. Prokhorov, Adv. Geom. 13 (3), 389 (2013).
7. Yu. G. Prokhorov, Izv. Ross. Akad. Nauk, Ser. Mat. 79 (4), 159 (2015) [Izv. Math. 79 (4), 795 (2015)].
8. V. A. Iskovskikh and Yu. G. Prokhorov, in Encyclopaedia Math. Sci., Vol. 47: Algebraic Geometry. V (Springer, Berlin, 1999), pp. 1–245.
9. V. Przyjalkowski and C. Shramov, Int. Math. Res. Not. 2015 (21), 11302 (2015).
10. S. Cutkosky, Math. Ann. 280 (3), 521 (1988).
11. J. W. Cutrone and N. A. Marshburn, Cent. Eur. J. Math. 11 (9), 1552 (2013).
12. P. Jahnke, Th. Peternell, and I. Radloff, Cent. Eur. J. Math. 9 (3), 449 (2011).
13. K. Takeuchi, Weak Fano Threefolds with Del Pezzo Fibration, arXiv: 0910.2188 (2009).
14. I. Cheltsov, V. Przyjalkowski, and C. Shramov, Which Quartic Double Solids are Rational? arXiv: 1508.07277 (2015).
15. A. Kuznetsov, Yu. Prokhorov, and C. Shramov, Hilbert Schemes of Lines and Conics and Automorphism Groups of Fano Threefolds, arXiv: 1605.02010 (2016).