Complex FIOs and composition of Toeplitz operators

Lewis Coburn¹ · Michael Hitrik² · Johannes Sjöstrand³

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Abstract
We study Toeplitz operators on the Bargmann space, with Toeplitz symbols that are exponentials of complex quadratic forms, from the point of view of Fourier integral operators in the complex domain. Sufficient conditions are established for the composition of two such operators to be a Toeplitz operator.

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1 Introduction and statement of results

Various algebras of pseudodifferential and Fourier integral operators have long played a fundamental role in PDE and applications [13, 19, 22, 27]. Parallel developments in the setting of Toeplitz operators on exponentially weighted spaces of holomorphic

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Michael Hitrik
hitrik@math.ucla.edu

Lewis Coburn
lcoburn@buffalo.edu

Johannes Sjöstrand
johannes.sjostrand@u-bourgogne.fr

¹ Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260, USA
² Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, USA
³ IMB (UMR 5584 CNRS), Université de Bourgogne, 9 Av. A. Savary, BP 47870, 21078 Dijon, France
functions on \( C^n \) (Bargmann spaces), motivated in part by the general ideas of quantization \([3, 7]\), are comparatively more recent, and may present some surprises \([4, 8]\). In the recent series of papers \([10–12]\), certain links have been established between the theory of Toeplitz operators on the Bargmann space and Fourier integral operators (FIOs) in the complex domain. In particular, the point of view of complex FIOs has been used in those works to establish the validity of the Berger-Coburn conjecture \([4]\), stating that a Toeplitz operator is bounded precisely when its Weyl symbol is bounded, for Toeplitz symbols that are exponentials of complex quadratic forms.

A basic, still unresolved issue concerns the problem of composing Toeplitz operators \([9]\), and the work \([8]\) gives an explicit example of a bounded Toeplitz operator on the Bargmann space, whose square cannot be approximated by bounded Toeplitz operators in the operator norm. Now the Toeplitz symbol of the operator in the example of \([8]\) is given by an exponential of a complex quadratic form, which makes it natural therefore to apply the machinery developed in \([10–12]\), to a systematic study of the composition of such metaplectic Toeplitz operators acting on the Bargmann space.

The purpose of this paper is precisely to carry out such a study and to derive sufficient conditions for the composition of two metaplectic Toeplitz operators to be an operator of the same class. In addition to placing the example of \([8]\) into a more conceptual framework, it seems that by doing so, one gains some additional insight into the analytic structure of the space of metaplectic Toeplitz operators, while also clarifying the link to the Weyl quantization. Let us now proceed to describe the assumptions and state the main results of this work.

Let \( \Phi_0 \) be a strictly plurisubharmonic quadratic form on \( C^n \) and let us introduce the Bargmann space

\[
H_{\Phi_0}(C^n) = L^2(C^n, e^{-2\Phi_0} L(dx)) \cap \text{Hol}(C^n),
\]

(1.1)

with \( L(dx) \) being the Lebesgue measure on \( C^n \). We have the orthogonal projection

\[
\Pi_{\Phi_0} : L^2(C^n, e^{-2\Phi_0} L(dx)) \to H_{\Phi_0}(C^n).
\]

(1.2)

Let \( q \) be a complex valued quadratic form on \( C^n \) and assume that

\[
\text{Re } q(x) < \Phi_{\text{herm}}(x) := (1/2) (\Phi_0(x) + \Phi_0(ix)) , \quad x \neq 0.
\]

(1.3)

In this work, we shall be concerned with (bounded) Toeplitz operators of the form

\[
\text{Top}(e^q) = \Pi_{\Phi_0} \circ e^q \circ \Pi_{\Phi_0} : H_{\Phi_0}(C^n) \to H_{\Phi_0}(C^n).
\]

(1.4)

Any such operator can be represented as the Weyl quantization,

\[
\text{Top}(e^q) = a^w(x, D_x),
\]

(1.5)

see \([23, 27]\), where the Weyl symbol \( a \) is given by

\[
a(x, \xi) = \left( \exp \left( \frac{1}{4} (\Phi''_{0, i\xi})^{-1} \frac{\partial_x \cdot \partial_{\xi}}{i} \right) e^q \right)(x), \quad (x, \xi) \in \Lambda_{\Phi_0}.
\]

(1.6)
Here we have introduced the real linear subspace

$$\Lambda_{\Phi_0} = \left\{ \left( x, 2 \frac{\partial \Phi_0}{i \partial x} (x) \right), x \in \mathbb{C}^n \right\} \subset \mathbb{C}_x^n \times \mathbb{C}_\xi^n = \mathbb{C}^{2n},$$

(1.7)

which is I-Lagrangian and R-symplectic, in the sense that the restriction of the complex symplectic form on $\mathbb{C}^{2n}$ to $\Lambda_{\Phi_0}$ is real and non-degenerate. In particular, $\Lambda_{\Phi_0}$ is maximally totally real, so that its complexification is given by $\mathbb{C}^{2n}$.

As observed in [10], an application of the method of quadratic stationary phase to (1.6) allows us to write,

$$a(x, \xi) = C \exp (i F(x, \xi)), \quad (x, \xi) \in \Lambda_{\Phi_0},$$

(1.8)

for some constant $C \neq 0$, where $F$ is a holomorphic quadratic form on $\mathbb{C}^{2n}$. We let

$$\mathcal{F} = \frac{1}{2} \begin{pmatrix} F''_{\xi x} & F''_{\xi \xi} \\ -F''_{xx} & F''_{x \xi} \end{pmatrix}$$

(1.9)

be the fundamental matrix of $F$ and assume, following [10], that $\pm 1 \notin \text{Spec}(\mathcal{F})$. Let $\tilde{q}$ be a second complex valued quadratic form on $\mathbb{C}^n$ satisfying, similarly to (1.3),

$$\text{Re} \tilde{q}(x) < \Phi_{\text{herm}}(x), \quad x \neq 0. \quad (1.10)$$

Letting $\tilde{a} \in \mathcal{C}^\infty(\Lambda_{\Phi_0})$ be the Weyl symbol of the Toeplitz operator $\text{Top}(e^{\tilde{q}})$, let us write as in (1.8),

$$\tilde{a}(x, \xi) = \tilde{C} \exp (i \tilde{F}(x, \xi)), \quad (x, \xi) \in \Lambda_{\Phi_0},$$

(1.11)

for a holomorphic quadratic form $\tilde{F}$ on $\mathbb{C}^{2n}$ and a constant $\tilde{C} \neq 0$. Let $\tilde{\mathcal{F}}$ be the fundamental matrix of $\tilde{F}$ and assume that $\pm 1 \notin \text{Spec}(\tilde{\mathcal{F}})$.

We shall assume that the Weyl symbols of $\text{Top}(e^{\tilde{q}}), \text{Top}(e^{\tilde{q}})$ satisfy

$$a \in \mathcal{L}^\infty(\Lambda_{\Phi_0}), \quad \tilde{a} \in \mathcal{L}^\infty(\Lambda_{\Phi_0}).$$

(1.12)

As established in [10, 12], each of the assumptions in (1.12) is equivalent to the boundedness of the corresponding operator,

$$\text{Top}(e^{\tilde{q}}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n), \quad \text{Top}(e^{\tilde{q}}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n).$$

(1.13)

The following is the first main result of this work, with $\text{Op}^w$ denoting the Weyl quantization in the complex domain, see [23], [27, Chapter 13], [16].

**Theorem 1.1** Let $\Phi_0$ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, and let $q$, $\tilde{q}$ be complex valued quadratic forms on $\mathbb{C}^n$ such that (1.3), (1.10) hold. Let $a, \tilde{a} \in \mathcal{C}^\infty(\Lambda_{\Phi_0})$ be the Weyl symbols of the Toeplitz operators $\text{Top}(e^q), \text{Top}(e^{\tilde{q}})$, respectively, which are of the form (1.8), (1.11), and let us assume that $\pm 1 \notin \text{Spec}(\mathcal{F}), \pm 1 \notin \text{Spec}(\tilde{\mathcal{F}})$. 


Spec($\tilde{F}$). Here $F$, $\tilde{F}$ are the fundamental matrices of the quadratic forms $F$, $\tilde{F}$, respectively. Assume that (1.12) holds and that $-1 \notin \text{Spec}(\tilde{F}F)$. We have
\[
\text{Top}(e^{i\theta}) \circ \text{Top}(e^{i\theta}) = C \text{Op}^w(e^{i\tilde{F}}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n),
\]
for some $0 \neq C \in \mathbb{C}$. Here $\tilde{F}$ is a holomorphic quadratic form on $\mathbb{C}^{2n}$ such that
\[
\text{Im} \tilde{F}|_{\Lambda_{\Phi_0}} \geq 0.
\]
The fundamental matrix $\hat{F}$ of $\tilde{F}$ given by
\[
\hat{F} = (1 + F)(1 + \tilde{F}F)^{-1}(1 + F) - 1.
\]
Before stating the second main result, we shall recall the notion of polarization of a quadratic form on $\mathbb{C}^n$. Given a complex valued quadratic form $f(x)$ on $\mathbb{C}^n$, the polarization $f^\pi(x, y)$ of $f$ is the unique holomorphic quadratic form on $\mathbb{C}^{2n}$ such that $f^\pi(x, x) = f(x), x \in \mathbb{C}^n$. We shall use this system of notation below, with the following two exceptions: the polarization of the positive definite Hermitian form $\Phi_{\text{herm}}$ in (1.3) will be denoted by $\Psi_{\text{herm}}$ and the polarization of the strictly plurisubharmonic quadratic form $\Phi_0$ will be denoted by $\Psi_0$.

Let $G$ be a holomorphic quadratic form on $\mathbb{C}^{2n}$ and let us observe that the polarization of the quadratic form
\[
f(x) = G\left(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x)\right), \quad x \in \mathbb{C}^n,
\]
is given by
\[
f^\pi(x, y) = G\left(x, \frac{2}{i} \frac{\partial \Psi_0}{\partial x}(x, y)\right), \quad x, y \in \mathbb{C}^n.
\]

**Theorem 1.2** Let $\Phi_0$ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, and let $G$ be a holomorphic quadratic form on $\mathbb{C}^{2n}$ such that
\[
\text{Im} G|_{\Lambda_{\Phi_0}} \geq 0.
\]
Assume that the fundamental matrix of $G$ does not have the eigenvalues $\pm 1$. Assume also that the holomorphic quadratic form
\[
iG\left(x, \frac{2}{i} \frac{\partial \Psi_0}{\partial x}(x, z)\right) + 4\Psi_{\text{herm}}(x, z), \quad (x, z) \in \mathbb{C}^{2n}
\]
is non-degenerate, and let us set
\[
Q^\pi(y, \theta) = \text{vc}_{x,z}\left(4\Psi_{\text{herm}}(x - y, z - \theta) + iG\left(x, \frac{2}{i} \frac{\partial \Psi_0}{\partial x}(x, z)\right)\right), \quad (y, \theta) \in \mathbb{C}^{2n},
\]
where \( v_{c,z} \) indicates that we take the critical value with respect to \( x,z \). Here the critical value is attained at a unique critical point which is non-degenerate. Assume furthermore that the restriction \( Q(y) = \overline{Q^n}(y, \overline{y}) \) of the holomorphic quadratic form \( Q^n \) on \( C^{2^n} \) to the anti-diagonal satisfies
\[
\text{Re } Q(y) < \Phi_{\text{herm}}(y), \quad 0 \neq y \in C^n.
\] (1.22)

Then the Weyl quantization \( \text{Op}^w(e^{iG}) \) is a Toeplitz operator, bounded on \( H_{\Phi_0}(C^n) \), with the Toeplitz symbol of the form \( Ce^Q \), for some constant \( C \neq 0 \).

**Remark** Applying Theorem 1.2 in the case when \( G = \hat{F} \) in Theorem 1.1, we obtain a general criterion for when the composition of two metaplectic Toeplitz operators in (1.14) is a Toeplitz operator.

**Remark** As we shall recall in Sect. 5 below, if the condition (1.22) fails, then the bounded operator \( \text{Op}^w(e^{iG}) \) in Theorem 1.2 does not need to be a Toeplitz operator.

The composition problem for Toeplitz operators on the Bargmann space \( H_{\Phi_0}(C^n) \) has been studied in [1, 8], see also [17, 26], as well as [23, 27, Chapter 13] for the semiclassical case. In fact, the situation is particularly satisfying in the latter case, where the composition calculus often works with \( O(h^\infty) \) – errors. To recall a rough statement of it, following [23, 27, Theorem 13.11], [9, 15], let \( p_1, p_2 \in C^\infty(C^n) \) be such that \( \partial^\alpha_x \partial^\beta_x p_j \in L^\infty(C^n) \), \( j = 1, 2 \), for all \( \alpha, \beta \in \mathbb{N}^n \), and let us consider the semiclassical Toeplitz quantizations
\[
\text{Top}_h(p_j) = \Pi_{\Phi_0,h} \circ p_j \circ \Pi_{\Phi_0,h} = \mathcal{O}(1) : H_{\Phi_0,h}(C^n) \to H_{\Phi_0,h}(C^n).
\] (1.23)

Here, similarly to (1.1), we set \( H_{\Phi_0,h}(C^n) = L^2(C^n, e^{-2\Phi_0/h}L(dx)) \cap \text{Hol}(C^n) \), and
\[
\Pi_{\Phi_0,h} : L^2(C^n, e^{-2\Phi_0/h}L(dx)) \to H_{\Phi_0,h}(C^n)
\]
is the orthogonal projection. We then have
\[
\text{Top}_h(p_1)\text{Top}_h(p_2) - \text{Top}_h(p) = \mathcal{O}(h^\infty) : H_{\Phi_0,h}(C^n) \to H_{\Phi_0,h}(C^n).
\] (1.24)

Here \( p \in C^\infty(C^n) \) admits a complete asymptotic expansion in integer powers of \( h \), as \( h \to 0^+ \), that we shall only recall in the case when \( \Phi_0(x) = \frac{|x|^2}{4} \), see [27, Theorem 13.11] for the case of a general quadratic weight,
\[
p(x) \sim \sum_{|\alpha| \geq 0} \frac{(-2h)^{|\alpha|}}{\alpha!} \partial_x^\alpha p_1(x) \partial_x^\alpha p_2(x), \quad x \in C^n.
\] (1.25)

Accurate remainder bounds in the semiclassical expansion (1.25) have been established in [6]. In the non-semiclassical case, i.e. for \( h = 1 \), the composition formula (1.25) is still valid and becomes exact when the Toeplitz symbols \( p_1, p_2 \) are polynomials in \( x, \overline{x} \), see [8]. Furthermore, it also holds, with absolute convergence, for \( p_1, p_2 \)
which are Fourier-Stieltjes transforms of compactly supported measures on $\mathbb{C}^n$, [8].

The Toeplitz symbols considered in this work may be unbounded, exhibiting some super-exponential growth at infinity, and when understanding the composition of the corresponding operators we shall rely crucially on the complex FIO point of view, developed in [10–12].

Remark As we also observed in [12], while very special, the Toeplitz symbols considered here, given by exponentials of complex quadratic forms, may still be of some interest since the class of the associated Toeplitz operators includes those that are “at the edge” of boundedness, with the unboundedness of the symbols attenuated by their fast oscillations at infinity. As such, it has also been exploited as a source of various counter-examples, see [2, 8].

The plan of the paper is as follows. In Sect. 2, we review some essentially well known results concerning the composition of metaplectic Fourier integral operators in the complex domain associated to complex linear canonical transformations, that are positive relative to the maximally totally real subspace $\Lambda_{q_0}$ in (1.7). This discussion is specialized in Sect. 3 to FIOs given as Weyl quantizations of symbols of the form $e^{iF(x, \xi)}$, where $F$ is a holomorphic quadratic form on $\mathbb{C}^{2n}$, and we show that under mild additional assumptions, the composition of two such operators is again an operator of this form. The proofs of Theorem 1.1 and 1.2 are then completed in Sect. 4, by passing from the Toeplitz symbols to the Weyl ones, along the lines of [10, 11], and then back, by means of a well known critical value inversion formula, somewhat in the spirit of the inversion formula for the Legendre transformation. In Sect. 5 we discuss composition properties of some explicit families of metaplectic Toeplitz operators on a model Bargmann space, closely related to the example of [8], and in particular we illustrate Theorems 1.1 and 1.2 in this case. “Appendix A”, finally, is devoted to some remarks concerning adjoints of complex metaplectic FIOs quantizing positive complex linear canonical transformations, realized as linear continuous maps between spaces of entire holomorphic functions with quadratic exponential weights. We compute the canonical transformation associated to the complex adjoint of such an operator.

2 Composition of metaplectic FIOs in the complex domain

The discussion in this section is essentially well known, see [21, Chapter 4], [5, 10], and serves as a convenient starting point for us. Let

$$\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \quad \text{(2.1)}$$

be a complex linear canonical transformation, and let $\varphi(x, y, \theta)$ be a holomorphic quadratic form on $\mathbb{C}^n_x \times \mathbb{C}^n_y \times \mathbb{C}^N_\theta$, which is a non-degenerate phase function in the sense of Hörmander [18],

$$\operatorname{rank} \left( \varphi''_{\partial x} \varphi''_{\partial y} \varphi''_{\partial \theta} \right) = N, \quad \text{(2.2)}$$
generating the graph of $\kappa$, so that

$$\kappa : \mathbb{C}^{2n} \ni (y, -\varphi'_y(x, y, \theta)) \mapsto (x, \varphi'_x(x, y, \theta)) \in \mathbb{C}^{2n}, \quad \varphi'_\theta(x, y, \theta) = 0. \quad (2.3)$$

For future reference, let us recall from [5] that the fact that the canonical relation

$$\{ (x, \varphi'_x(x, y, \theta); y, -\varphi'_y(x, y, \theta)), \varphi'_\theta(x, y, \theta) = 0 \} \subset \mathbb{C}^{2n} \times \mathbb{C}^{2n} \quad (2.4)$$

is the graph of a linear canonical transformation is equivalent to the following condition,

$$\det \begin{pmatrix} \varphi''_{xy} & \varphi''_{x\theta} \\ \varphi''_{y\theta} & \varphi''_{\theta\theta} \end{pmatrix} \neq 0. \quad (2.5)$$

Assume that

$$\kappa(\Lambda\Phi_0) = \Lambda\Phi_1, \quad (2.6)$$

where $\Phi_0, \Phi_1$ are strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$. Here we have set as in (1.7),

$$\Lambda_{\Phi_j} = \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi_i}{\partial x}(x) \right), \ x \in \mathbb{C}^n \right\} \subset \mathbb{C}^{2n} = C^n_x \times C^n_\zeta, \quad (2.7)$$

for $j = 0, 1$. It follows from [21], [5, Appendix B] that the plurisubharmonic quadratic form

$$\mathbb{C}^n \times \mathbb{C}^N \ni (y, \theta) \mapsto -\text{Im} \varphi(0, y, \theta) + \Phi_0(y) \quad (2.8)$$

is non-degenerate of signature $(n + N, n + N)$. Let $(y_c(x), \theta_c(x)) \in \mathbb{C}^n \times \mathbb{C}^N$ be the unique critical point of

$$\mathbb{C}^n \times \mathbb{C}^N \ni (y, \theta) \mapsto -\text{Im} \varphi(x, y, \theta) + \Phi_0(y), \quad (2.9)$$

for each $x \in \mathbb{C}^n$, and let us recall from [5, Appendix B] that

$$\Phi_1(x) = \text{vc}_{y, \theta} (-\text{Im} \varphi(x, y, \theta) + \Phi_0(y)). \quad (2.10)$$

It follows that there exists an affine subspace $\Gamma(x) \subset \mathbb{C}^{n+N}_{y,\theta}$ of real dimension $n + N$, passing through the critical point $(y_c(x), \theta_c(x))$ such that

$$-\text{Im} \varphi(x, y, \theta) + \Phi_0(y) \leq \Phi(x) - \frac{1}{C} \text{dist} ((y, \theta), (y_c(x), \theta_c(x)))^2,$$

along $\Gamma(x)$. In such a situation, here and below, we say that $\Gamma(x) \subset \mathbb{C}^{n+N}_{y,\theta}$ is a good contour for the plurisubharmonic function

$$\mathbb{C}^n \times \mathbb{C}^N \ni (y, \theta) \mapsto -\text{Im} \varphi(x, y, \theta) + \Phi_0(y).$$
We conclude, following [21], [5, Appendix B] that the corresponding realization of a Fourier integral operator \( A \) quantizing \( \kappa \),

\[
A_\Gamma u(x) = \int \int_{\Gamma(x)} e^{i\varphi(x,y,\theta)} au(y) \, dy \, d\theta, \quad a \in \mathbf{C},
\]

defines a bounded linear map,

\[
A_\Gamma = A : H_{\Phi_0}(\mathbf{C}^n) \to H_{\Phi_1}(\mathbf{C}^n).
\]

Here the Bargmann space \( H_{\Phi_0}(\mathbf{C}^n) \) is defined in (1.1), with the space \( H_{\Phi_1}(\mathbf{C}^n) \) having an analogous definition.

Let next \( \tilde{\kappa} : \mathbf{C}^{2n} \to \mathbf{C}^{2n} \) be a second complex linear canonical transformation, and let \( \psi(x, y, w) \) be a holomorphic quadratic form on \( \mathbf{C}^n_x \times \mathbf{C}^n_y \times \mathbf{C}^M_w \), which is a non-degenerate phase function in the sense of Hörmander, such that

\[
\tilde{\kappa} : \mathbf{C}^{2n} \ni (y, -\psi'(x, y, w)) \mapsto (x, \psi'(x, y, w)) \in \mathbf{C}^{2n}, \quad \psi'(w, x, y) = 0.
\]

Similarly to (2.6), assume that

\[
\tilde{\kappa}(\Lambda_{\Phi_1}) = \Lambda_{\Phi_2},
\]

where \( \Phi_2 \) is a strictly plurisubharmonic quadratic form on \( \mathbf{C}^n \). Letting \( \tilde{\Gamma}(x) \subset \mathbf{C}^{n+M}_{y,w} \) be a good contour for the plurisubharmonic function

\[
\mathbf{C}^n_x \times \mathbf{C}^M_w \ni (y, w) \mapsto -\text{Im} \psi(x, y, w) + \Phi_1(y),
\]

we consider the corresponding realization of a Fourier integral operator \( B \) quantizing \( \tilde{\kappa} \),

\[
B_{\tilde{\Gamma}} u(x) = \int \int_{\tilde{\Gamma}(x)} e^{i\psi(x,y,w)} b u(y) \, dy \, dw, \quad b \in \mathbf{C},
\]

defining a bounded linear map,

\[
B_{\tilde{\Gamma}} = B : H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_2}(\mathbf{C}^n).
\]

The composition \( B_{\tilde{\Gamma}} \circ A_\Gamma \) takes the form

\[
(B_{\tilde{\Gamma}} \circ A_\Gamma u)(x) = \int \int \int_{\tilde{\Gamma}(x)} e^{i(\psi(x,z,w)+\varphi(z,y,\theta))} ab u(y) \, dy \, d\theta \, dz \, dw,
\]

where \( \tilde{\Gamma}(x) \subset \mathbf{C}^{n}_{z} \times \mathbf{C}^{M}_{w} \times \mathbf{C}^{n}_{y} \times \mathbf{C}^{N}_{\theta} \) is the composed contour of real dimension \( 2n + N + M \) given by

\[
\tilde{\Gamma}(x) = \{(z, w, y, \theta); (z, w) \in \tilde{\Gamma}(x), (y, \theta) \in \Gamma(z)\}.
\]
Let us set
\[ \Phi(x, y; z, w, \theta) = \psi(x, z, w) + \varphi(z, y, \theta), \] (2.19)
with \( \chi = (z, w, \theta) \in \mathbb{C}^n_z \times \mathbb{C}^M_w \times \mathbb{C}^N \) viewed as the fiber variables. We claim that the holomorphic quadratic form \( \Phi(x, y; \chi) \) is a non-degenerate phase function in the sense of Hörmander, and when verifying the claim we proceed similarly to [24, Chapter 6]. We need to show that the \((n + M + N) \times (n + n + M + N)\) matrix
\[
\begin{pmatrix}
\Phi''_{xx} & \Phi''_{xy} & \Phi''_{xz} & \Phi''_{xw} & \Phi''_{x\theta} \\
\Phi''_{wx} & \Phi''_{wy} & \Phi''_{wz} & \Phi''_{ww} & \Phi''_{w\theta} \\
\Phi''_{\theta x} & \Phi''_{\theta y} & \Phi''_{\theta z} & \Phi''_{\theta w} & \Phi''_{\theta \theta}
\end{pmatrix}
\] (2.20)
has full rank, and using (2.19) we see that the matrix in (2.20) is of the form
\[
\begin{pmatrix}
\psi''_{zx} & \psi''_{zy} & \psi''_{zz} + \varphi''_{zz} & \psi''_{zw} & \psi''_{z\theta} \\
\psi''_{wx} & 0_{M \times n} & \psi''_{wz} & \psi''_{ww} & 0_{M \times N} \\
0_{N \times n} & \varphi''_{\theta y} & \varphi''_{\theta z} & 0_{N \times M} & \varphi''_{\theta \theta}
\end{pmatrix}.
\] (2.21)
Here in view (2.5), the \((n + N) \times (n + N)\) matrix
\[
\begin{pmatrix}
\varphi''_{zy} & \varphi''_{z\theta} \\
\varphi''_{\theta y} & \varphi''_{\theta \theta}
\end{pmatrix}
\] (2.22)
is non-degenerate, and we have
\[
\text{rank } \begin{pmatrix}
\psi''_{wx} & \psi''_{wz} & \psi''_{ww}
\end{pmatrix} = M.
\] (2.23)
Let \( B \) be an invertible \( M \times M \) matrix, whose columns are among the columns of the matrix \( \begin{pmatrix} \psi''_{wx} & \psi''_{wz} & \psi''_{ww} \end{pmatrix} \). Observing that an \((n + M + N) \times (n + M + N)\) matrix of the form
\[
\begin{pmatrix}
A & \varphi''_{zy} & \varphi''_{z\theta} \\
B & 0_{M \times n} & 0_{M \times N} \\
C & \varphi''_{\theta y} & \varphi''_{\theta \theta}
\end{pmatrix}
\] (2.24)
is non-degenerate, independently of matrices \( A, C \) of size \( n \times M \) and \( N \times M \), respectively, we conclude that the matrix in (2.20) is of full rank, giving the claim.

It follows that the associated canonical relation
\[
\mathbb{C}^{2n} \ni (y, -\Phi'_x(x, y; \chi)) \mapsto (x, \Phi'_x(x, y; \chi)) \in \mathbb{C}^{2n}, \quad \Phi'_z = 0, \quad \Phi'_w = 0, \quad \Phi'_\theta = 0,
\] (2.25)
is of dimension \( 2n \) and is given by
\[
\mathbb{C}^{2n} \ni (y, -\phi'(z, y, \theta)) \mapsto (x, \psi'(x, z, w)) \in \mathbb{C}^{2n}, \quad \psi'_z(x, z, w) + \phi'_z(z, y, \theta) = 0, \quad \psi'_w(x, z, w) = 0, \quad \phi'_\theta(z, y, \theta) = 0.
\] (2.26)
(2.27)
It is therefore clear that the canonical relation (2.25) is the graph of the canonical transformation \( \tilde{\kappa} \circ \kappa \).
We see furthermore, directly from the definitions, that the plurisubharmonic quadratic form
\[ C^n \times C^M \times C^n \times C^N \ni (z, w, y, \theta) \mapsto -\text{Im} \psi(0, z, w) - \text{Im} \varphi(z, y, \theta) + \Phi_0(y) \]
(2.28)
is negative definite along the contour \( \tilde{\Gamma}(0) \) of real dimension \( 2n + N + M \), and therefore, the quadratic form (2.28) is non-degenerate of signature \((2n + N + M, 2n + N + M)\). It follows that the composed contour \( \tilde{\Gamma}(x) \) is good for the function
\[ C^n \times C^M \times C^n \times C^N \ni (z, w, y, \theta) \mapsto -\text{Im} \psi(x, z, w) - \text{Im} \varphi(z, y, \theta) + \Phi_1(y), \]
(2.29)
and we conclude therefore that the composition \( B_{\tilde{\Gamma}} \circ A_{\Gamma} \) is a realization of a Fourier integral operator quantizing the canonical transformation \( \tilde{\kappa} \circ \kappa \).

Assume next that the complex linear canonical transformations \( \kappa, \tilde{\kappa} \) are positive relative to \( \Lambda_{\Phi_0} \), so that
\[ \frac{1}{i} \left( \sigma(\kappa(\rho), \iota \Phi_0 \kappa(\rho)) - \sigma(\rho, \iota \Phi_0(\rho)) \right) \geq 0, \quad \rho \in C^{2n}, \]
(2.30)
and similarly for \( \tilde{\kappa} \), see [10]. Here \( \iota : C^{2n} \to C^{2n} \) is the unique anti-linear involution such that \( \iota|_{\Lambda_{\Phi_0}} = 1 \), and
\[ \sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j \]
(2.31)
is the complex symplectic form on \( C^{2n} = C^n \times C^n \). It follows from [10, Theorem 1.1] that (2.6) holds, with the strictly plurisubharmonic quadratic form \( \Phi_1 \) satisfying \( \Phi_1 \leq \Phi_0 \). Let us next check that (2.14) holds as well, with \( \Phi_2 \) quadratic strictly plurisubharmonic on \( C^n \) satisfying \( \Phi_2 \leq \Phi_0 \). When doing so, we observe that since \( \tilde{\kappa} \) is positive relative to \( \Lambda_{\Phi_0} \), we have that the plurisubharmonic quadratic form
\[ C^n \times C^M \ni (y, w) \mapsto -\text{Im} \psi(0, y, w) + \Phi_0(y) \]
(2.32)
is non-degenerate of signature \((n + M, n + M)\). Using that
\[ -\text{Im} \psi(0, y, w) + \Phi_1(y) \leq -\text{Im} \psi(0, y, w) + \Phi_0(y), \]
(2.33)
we conclude that since the left hand side in (2.33) is a plurisubharmonic quadratic form, it is also non-degenerate of signature \((n + M, n + M)\). It follows that (2.14) holds with
\[ \Phi_2(x) = v_{\xi, w} (-\text{Im} \psi(x, y, w) + \Phi_1(y)). \]
(2.34)
An application of the fundamental lemma of [21] allows us to conclude that \( \Phi_2 \) is plurisubharmonic, and since the real linear subspace \( \Lambda_{\Phi_2} = \tilde{\kappa}(\Lambda_{\Phi_1}) \) is R-symplectic,
the plurisubharmonicity of $\Phi_2$ is necessarily strict. We also have

$$
\Phi_2(x) = vc_{y,w} \left( -\operatorname{Im} \psi(x, y, w) + \Phi_1(y) \right) \leq vc_{y,w} \left( -\operatorname{Im} \psi(x, y, w) + \Phi_0(y) \right),
$$

(2.35)

and the strictly plurisubharmonic quadratic form in the right hand side is $\leq \Phi_0$, in view of the positivity of $\tilde{\kappa}$ relative to $\Lambda_{\Phi_0}$.

We may summarize the discussion in this section in the following essentially well known result, see also \[5, Proposition B.4\].

**Theorem 2.1** Let $\Phi_0$ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, and let $\kappa, \tilde{\kappa} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be complex linear canonical transformations that are positive relative to $\Lambda_{\Phi_0}$. Let $A, B$ be metaplectic Fourier integral operators quantizing $\kappa, \tilde{\kappa}$, respectively, realized with the help of good contours. Then the composition $B \circ A$ is a Fourier integral operator associated to the canonical transformation $\tilde{\kappa} \circ \kappa$, which is also positive relative to $\Lambda_{\Phi_0}$. The operator $B \circ A$ can be realized with the help of a good contour and we have that

$$
B \circ A : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)
$$

is bounded.

### 3 Composing complex Weyl quantizations

Let $\Phi_0$ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let $F$ be a holomorphic quadratic form on $\mathbb{C}_x^2 \times \mathbb{C}_\xi^2$ such that

$$
\operatorname{Im} F \geq 0 \quad \text{along} \quad \Lambda_{\Phi_0}.
$$

(3.1)

Here the real $2n$-dimensional linear subspace $\Lambda_{\Phi_0} \subset \mathbb{C}^{2n}$ has been introduced in (1.7). Assume that the fundamental matrix of $F$,

$$
\mathcal{F} = \frac{1}{2} \begin{pmatrix} F''_{\xi x} & F''_{\xi \xi} \\ -F''_{xx} & -F''_{x \xi} \end{pmatrix}
$$

(3.2)

satisfies

$$
\pm 1 \notin \operatorname{Spec}(\mathcal{F}).
$$

(3.3)

Then, as explained in \[10, 11\], the Weyl quantization $\operatorname{Op}^w(e^{iF})$ can be regarded as a Fourier integral operator in the complex domain associated to the complex linear canonical transformation

$$
\kappa = (1 - \mathcal{F}) (1 + \mathcal{F})^{-1}.
$$

(3.4)

Here we may notice that the map

$$
\kappa + 1 = (1 - \mathcal{F} + 1 + \mathcal{F}) (1 + \mathcal{F})^{-1} = 2 (1 + \mathcal{F})^{-1}
$$

(3.5)
is bijective, and we have the inverse relation
\[ \mathcal{F} = (1 + \kappa)^{-1}(1 - \kappa). \] (3.6)

Furthermore, as we have seen in [10, Proposition B.1], the assumption (3.1) implies that \( \kappa \) is positive relative to \( \Lambda_{\Phi_0} \), i.e., that (2.30) holds, and that the operator
\[ \text{Op}^w(e^{i\tilde{F}}) : H_{\Phi_0}(C^n) \to H_{\Phi_0}(C^n) \] (3.7)
is bounded.

Let \( \tilde{F} \) be a second holomorphic quadratic form on \( C_{x,\xi}^{2n} \) such that
\[ \text{Im} \tilde{F} \geq 0 \quad \text{along} \quad \Lambda_{\Phi_0}. \] (3.8)

Assume also that \( \pm 1 \notin \text{Spec} (\tilde{F}) \), where \( \tilde{F} \) is the fundamental matrix of \( \tilde{F} \). It follows therefore from Theorem 2.1 that the composition \( \text{Op}^w(e^{i\tilde{F}}) \circ \text{Op}^w(e^{iF}) \) is a Fourier integral operator associated to the complex linear canonical transformation
\[ \hat{\kappa} := \tilde{\kappa} \circ \kappa : C^{2n} \to C^{2n}, \] (3.9)
which is positive relative to \( \Lambda_{\Phi_0} \). Here \( \tilde{\kappa} = (1 - \tilde{F})(1 + \tilde{F})^{-1} \). Assume that \( -1 \notin \text{Spec}(\hat{\kappa}) \) and let us set, similarly to (3.6),
\[ \hat{\mathcal{F}} = (1 + \hat{\kappa})^{-1}(1 - \hat{\kappa}). \] (3.10)

Using the fact that \( \hat{\kappa} \) is canonical, we see that the complex linear map \( \hat{\mathcal{F}} \) is skew-symmetric with respect to \( \sigma \),
\[ \hat{\mathcal{F}} + \hat{\mathcal{F}}^\sigma = 0, \] (3.11)
where \( \hat{\mathcal{F}}^\sigma \) is the symplectic transpose of \( \hat{\mathcal{F}} \), given by
\[ \sigma(\hat{\mathcal{F}}\mu, \nu) = \sigma(\mu, \hat{\mathcal{F}}^\sigma\nu), \quad \mu, \nu \in C^{2n}. \]

Writing
\[ \sigma(\mu, \nu) = J\mu \cdot \nu, \quad \mu, \nu \in C^{2n}, \] (3.12)
where
\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J^t = -J, \quad J^2 = -1, \] (3.13)
we see that (3.11) is equivalent to the statement that \( J \hat{\mathcal{F}} \) is symmetric. It follows that the holomorphic quadratic form
\[ \hat{F}(\rho) = \sigma(\rho, \hat{\mathcal{F}}\rho) = -J\hat{\mathcal{F}}\rho \cdot \rho, \quad \rho \in C^{2n}, \] (3.14)
satisfies
\[ \sigma(t, H_{\hat{F}}(\rho)) = Jt \cdot H_{\hat{F}}(\rho) = d\hat{F}(\rho) \cdot t = -2J\hat{\mathcal{F}}\rho \cdot t = 2\hat{\mathcal{F}}\rho \cdot Jt, \quad t \in C^{2n}. \] (3.15)
We get $H_{\hat{F}}(\rho) = 2\hat{F}\rho$, where $H_{\hat{F}}$ is the Hamilton vector field of $\hat{F}$, and therefore, in view of (3.2), we conclude that $\hat{F}$ is the fundamental matrix of the quadratic form $\hat{F}$. Using (3.10), we observe also that the linear map

$$\hat{F} + 1 = (1 + \hat{\kappa})^{-1}(1 - \hat{\kappa} + 1 + \hat{\kappa}) = 2(1 + \hat{\kappa})^{-1}$$  \hspace{1cm} (3.16)

is bijective, and therefore $1 - \hat{F}$ is bijective as well. It follows furthermore from (3.10) that the canonical transformation $\hat{\kappa}$ takes the form

$$(1 + \hat{F}) \rho \mapsto (1 - \hat{F}) \rho,$$  \hspace{1cm} (3.17)

and recalling the positivity of $\hat{\kappa}$ relative to $\Lambda_{\Phi_0}$, we conclude, following [10, Proposition B.1], that the holomorphic quadratic form in (3.14) satisfies

$$\im \hat{F} \geq 0 \text{ along } \Lambda_{\Phi_0}.$$  \hspace{1cm} (3.18)

It follows that the holomorphic quadratic form

$$C^n_x \times C^n_y \times C^n_{\theta} \ni (x, y, \theta) \mapsto (x - y) \cdot \theta + \hat{F} \left( \frac{x + y}{2}, \theta \right)$$

is a non-degenerate phase function in the sense of Hörmander, which generates the positive complex linear canonical transformation $\hat{\kappa} = \hat{\kappa} \circ \kappa$ in (3.9).

The discussion above can be summarized in the following result.

**Proposition 3.1** Let $\Phi_0$ be a strictly plurisubharmonic quadratic form on $C^n$, and let $F, \tilde{F}$ be holomorphic quadratic forms on $C^{2n}$ such that

$$\im F \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) \geq 0, \quad \im \tilde{F} \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) \geq 0, \quad x \in C^n.$$

Assume that the fundamental matrices $\mathcal{F}, \tilde{\mathcal{F}}$ of the quadratic forms $F, \tilde{F}$, respectively, satisfy $\pm 1 \notin \text{Spec}(\mathcal{F}), \pm 1 \notin \text{Spec}(\tilde{\mathcal{F}})$. Let

$$\kappa = (1 - \mathcal{F})(1 + \mathcal{F})^{-1}, \quad \tilde{\kappa} = (1 - \tilde{\mathcal{F}})(1 + \tilde{\mathcal{F}})^{-1},$$  \hspace{1cm} (3.19)

and assume that $-1 \notin \text{Spec}(\tilde{\kappa} \circ \kappa)$. We have then

$$\text{Op}^w(e^{i\tilde{F}}) \circ \text{Op}^w(e^{iF}) = C \text{ Op}^w(e^{i\hat{F}}),$$  \hspace{1cm} (3.20)

for some constant $0 \neq C \in C$, where $\hat{F}$ is a holomorphic quadratic form on $C^{2n}$ satisfying

$$\im \hat{F} \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) \geq 0, \quad x \in C^n,$$
and such that \( \pm 1 \not\in \text{Spec}(\hat{F}) \), where \( \hat{F} \) is the fundamental matrix of \( F \). The operators in (3.20) are bounded: \( H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n) \).

**Remark** The constant \( C \neq 0 \) in (3.20) can be computed, see [20].

**Remark** Assume that one of the canonical transformations \( \kappa, \tilde{\kappa} \) in (3.19) is strictly positive relative to \( \Lambda_{\Phi_0} \), in the sense that the inequality in (2.30) is strict, for all \( \rho \neq 0 \). It follows that \( \hat{\kappa} = \tilde{\kappa} \circ \kappa \) is also strictly positive relative to \( \Lambda_{\Phi_0} \),

\[
\frac{1}{i} \left( \sigma(\hat{\kappa}(\rho), t_{\Phi_0} \hat{\kappa}(\rho)) - \sigma(\rho, t_{\Phi_0}(\rho)) \right) > 0, \quad 0 \neq \rho \in \mathbb{C}^{2n},
\]  

(3.21)

and in particular \( -1 \not\in \text{Spec}(\hat{\kappa}) \). More generally, we may observe that the spectrum of a strictly positive complex linear canonical transformation avoids the set \( \{ \lambda \in \mathbb{C}; |\lambda| = 1 \} \). We also recall from [10] that the strict positivity of \( \kappa \) in (3.19) relative to \( \Lambda_{\Phi_0} \) is equivalent to the ellipticity property,

\[
\text{Im} F \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) \asymp |x|^2, \quad x \in \mathbb{C}^n.
\]  

(3.22)

We shall finish this section by deriving an explicit expression for the fundamental matrix \( \hat{F} \) of the quadratic form \( F \) in (3.20) in terms of the fundamental matrices \( F, \tilde{F} \). See also [25] for closely related computations.

Let us use (3.6) and the corresponding expression for \( \tilde{F} \) in terms of \( \tilde{\kappa} \) to write

\[
F = (1 + \kappa)^{-1}(1 - \kappa), \quad \hat{F} = (1 + \tilde{\kappa})^{-1}(1 - \tilde{\kappa}).
\]  

(3.23)

It follows that

\[
1 + \hat{F}F = 1 + (1 + \tilde{\kappa})^{-1}(1 - \tilde{\kappa})(1 + \kappa)^{-1}(1 - \kappa)
= (1 + \tilde{\kappa})^{-1}\left( (1 + \tilde{\kappa})(1 + \kappa) + (1 - \tilde{\kappa})(1 - \kappa) \right)(1 + \kappa)^{-1}
= 2(1 + \tilde{\kappa})^{-1}(1 + \kappa)(1 + \kappa)^{-1} = \frac{1}{2}(1 + \hat{F})(1 + \tilde{\kappa} \kappa)(1 + F).
\]  

(3.24)

Here in the last equality we have used (cf. (3.5)) that

\[
F + 1 = 2(1 + \kappa)^{-1}, \quad \hat{F} + 1 = 2(1 + \tilde{\kappa})^{-1}.
\]  

(3.25)

Using (3.24), we get that

\[
2(1 + \hat{F})^{-1}(1 + \hat{F}F)(1 + F)^{-1} = 1 + \tilde{\kappa} \kappa.
\]  

(3.26)

In Proposition 3.1 we have assumed that the linear map \( 1 + \tilde{\kappa} \kappa = 1 + \hat{\kappa} \) is bijective, and using (3.26) we conclude that this assumption is equivalent to the bijectivity of \( 1 + \hat{F}F \). We get

\[
2(1 + \tilde{\kappa})^{-1} = (1 + F)(1 + \tilde{F})^{-1}(1 + \hat{F}),
\]  

(3.27)
and recalling (3.16) we conclude that
\[ \hat{\mathcal{F}} = (1 + \mathcal{F})(1 + \hat{\mathcal{F}} \mathcal{F})^{-1}(1 + \hat{\mathcal{F}}) - 1. \] (3.28)

Combining Proposition 3.1 with (3.28), we shall be able to complete the proof of Theorem 1.1 in Sect. 4 below, once we have recalled how to express a Toeplitz operator of the form (1.4) as a Weyl quantization.

4 From Toeplitz quantization to Weyl quantization and back

Let \( \Phi_0 \) be a strictly plurisubharmonic quadratic form on \( \mathbb{C}^n \) and let \( q \) be a complex valued quadratic form on \( \mathbb{C}^n \) such that (1.3) holds. From [10, 11] we recall that when equipped with the maximal domain
\[ D(\text{Top}(e^q)) = \left\{ u \in H_{\Phi_0}(\mathbb{C}^n); e^q u \in L^2(\mathbb{C}^n, e^{-2\Phi_0 L}(dx)) \right\}, \] (4.1)
the Toeplitz operator
\[ \text{Top}(e^q) = \Pi_{\Phi_0} \circ e^q \circ \Pi_{\Phi_0} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n) \] (4.2)
becomes densely defined. Here the orthogonal projection \( \Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0 L}(dx)) \to H_{\Phi_0}(\mathbb{C}^n) \) has been introduced in (1.2).

**Remark** Let \( \Psi_0 \) be the polarization of \( \Phi_0 \). Using the well known property
\[ 2\text{Re} \Psi_0(x, \overline{y}) - \Phi_0(x) - \Phi_0(y) = -\Phi_{\text{herm}}(x - y) \asymp -|x - y|^2, \] (4.3)
see [23], together with (1.3), we obtain that
\[ e^{2\Psi_0(\cdot, \overline{\cdot})} \in D(\text{Top}(e^q)), \quad y \in \mathbb{C}^n. \] (4.4)
We may also observe that the linear span of \( \{ e^{2\Psi_0(\cdot, \overline{\cdot})}; y \in \mathbb{C}^n \} \) is dense in \( H_{\Phi_0}(\mathbb{C}^n) \).

Let us write, following [10, 23],
\[ \text{Top}(e^q) = a^w(x, D_x), \] (4.5)
where \( a \in C^\infty(\Lambda_{\Phi_0}) \) is the Weyl symbol of the Toeplitz operator \( \text{Top}(e^q) \), given by
\[ a(x, \xi) = \exp \left( \frac{1}{4} \left( \Phi''_{0,xx} \right)^{-1} \partial_x \cdot \partial_\xi \right) e^q \right) \) \( x, (x, \xi) \in \Lambda_{\Phi_0}. \) (4.6)

An application of the method of exact stationary phase allows us to conclude that
\[ a(x, \xi) = C \exp \left( i(F(x, \xi)) \right), \quad (x, \xi) \in \Lambda_{\Phi_0}, \] (4.7)
for some constant $C \neq 0$, where $F$ is a holomorphic quadratic form on $C^{2n}$. See also (4.16) and the computations below. We may write therefore

$$\text{Top}(e^q) = C \text{Op}^w(e^{iF}).$$

(4.8)

In what follows, we shall assume that the fundamental matrix $F$ of $F$ satisfies $\pm 1 \notin \text{Spec}(F)$.

Let $\tilde{q}$ be a second complex valued quadratic form on $C^n$ satisfying (1.10), and let us write similarly to (4.8),

$$\text{Top}(e^{\tilde{q}}) = \tilde{C} \text{Op}^w(e^{i\tilde{F}}), \quad \tilde{C} \neq 0.$$

(4.9)

Here $\tilde{F}$ is a holomorphic quadratic form on $C^{2n}$. Assume also that the fundamental matrix $\tilde{F}$ of $\tilde{F}$ is such that $\pm 1 \notin \text{Spec}(\tilde{F})$.

Assume that the Weyl symbols satisfy

$$e^{iF} \in L^\infty(\Lambda\Phi_0), \quad e^{i\tilde{F}} \in L^\infty(\Lambda\Phi_0),$$

(4.10)

and that $1 + \tilde{F}F : C^{2n} \to C^{2n}$ is bijective. The discussion in Sect. 3 applies therefore to the composition $\text{Top}(e^{\tilde{q}}) \circ \text{Top}(e^q)$, in view of (4.8), (4.9), as it stands, implying Theorem 1.1.

We shall now proceed to give a proof of Theorem 1.2, and to this end, it will first be convenient to take a closer look at the formula for the Weyl symbol (4.6), and to give a more explicit description of the Fourier multiplier in (4.6). When doing so, we observe that the symbol of the second order constant coefficient differential operator on $C^n$,

$$-\frac{1}{4} \left( \Phi''_{0,x\overline{x}} \right)^{-1} \partial_x \cdot \partial_{\overline{x}}$$

(4.11)

is given by the positive definite quadratic form

$$\frac{1}{2} \left( 8\Phi''_{0,x\overline{x}} \right)^{-1} \overline{\xi} \cdot \xi, \quad \xi \in C^n.$$

(4.12)

Here it will be convenient to recall from [14] that the dual of a real valued non-degenerate quadratic form $R^N \ni x \mapsto \frac{1}{2}Ax \cdot x$ is by definition the quadratic form $R^N \ni \xi \mapsto \frac{1}{2}A^{-1}\overline{\xi} \cdot \overline{\xi}$. Assuming that the quadratic form is positive definite, we can express its dual as the Legendre transform,

$$\frac{1}{2}A^{-1}\overline{\xi} \cdot \overline{\xi} = \sup_x \left( x \cdot \xi - \frac{1}{2}Ax \cdot x \right).$$

(4.13)
It follows therefore that the dual of the quadratic form in (4.12) is given by the positive definite quadratic form

\[
\sup_{\xi} \left( \text{Re} \left( \xi \cdot \bar{x} \right) - \frac{1}{2} (8 \Phi''_{0,x\bar{x}})^{-1} \bar{\xi} \cdot \bar{\xi} \right) = 4 \Phi''_{0,x\bar{x}} \bar{x} \cdot x = 4 \Phi_{\text{herm}}(x). \tag{4.14}
\]

Combining this observation with the standard formula,

\[
e^{-\frac{AD\cdot D}{2}} u(x) = \frac{1}{(2\pi)^{N/2}} \frac{1}{(\det A)^{1/2}} \int e^{-\frac{A^{-1}x\cdot x}{2}} u(x) \, dy,
\tag{4.15}
\]

where \( A \) is an \( N \times N \) real symmetric positive definite matrix and \( u \in \mathcal{S}(\mathbb{R}^N) \), we conclude, in view of (4.6), that we have

\[
a(x, \xi) = C_{\Phi_0} \int_{\mathbb{C}^n} \exp \left( -4 \Phi_{\text{herm}}(x - y) e^{q(y)} \right) L(dy), \quad (x, \xi) \in \Lambda_{\Phi_0}.
\tag{4.16}
\]

Here \( C_{\Phi_0} \neq 0 \) and the integral converges thanks to (1.3).

We shall next evaluate a general Gaussian integral of the form (4.16). To this end, let \( Q \) be a complex valued quadratic form on \( \mathbb{C}^n \) such that

\[
\text{Re} \ Q(x) < \Phi_{\text{herm}}(x), \quad 0 \neq x \in \mathbb{C}^n.
\tag{4.17}
\]

Introducing the polarizations \( Q^\pi \) of \( Q \) and \( \Psi_{\text{herm}} \) of \( \Phi_{\text{herm}} \), we may write in view of (4.16), for some \( C \neq 0 \),

\[
\left( \exp \left( \frac{1}{4} \left( \Phi''_{0,x\bar{x}} \right)^{-1} \partial_x \cdot \partial_{\bar{x}} \right) e^Q \right)(x) = C \int_\Gamma \exp \left( -4 \Psi_{\text{herm}}(x - y, \bar{x} - \theta) + Q^\pi(y, \theta) \right) \, dy \, d\theta.
\tag{4.18}
\]

Here \( \Gamma \subset \mathbb{C}^{2n}_{y,\theta} \) is the contour given by \( \theta = \bar{y} \) (the anti-diagonal). An application of [12, Proposition 2.1] together with (4.17) allows us to conclude that the holomorphic quadratic form

\[
\mathbb{C}^{2n}_{y,\theta} \ni (y, \theta) \mapsto -4 \Psi_{\text{herm}}(y, \theta) + Q^\pi(y, \theta)
\tag{4.19}
\]

is non-degenerate, and therefore the holomorphic function

\[
\mathbb{C}^{2n}_{y,\theta} \ni (y, \theta) \mapsto -4 \Psi_{\text{herm}}(x - y, z - \theta) + Q^\pi(y, \theta)
\tag{4.20}
\]

has a unique critical point which is non-degenerate, for each \( (x, z) \in \mathbb{C}^n \times \mathbb{C}^n \). In view of the method of exact (quadratic) stationary phase and (4.18), it is clear therefore that

\[
\left( \exp \left( \frac{1}{4} \left( \Phi''_{0,x\bar{x}} \right)^{-1} \partial_x \cdot \partial_{\bar{x}} \right) e^Q \right)(x) = C \exp \left( \psi_{x,\theta} \left( -4 \Psi_{\text{herm}}(x - y, \bar{x} - \theta) + Q^\pi(y, \theta) \right) \right),
\tag{4.21}
\]
for some $C \neq 0$.

Let $G$ be a holomorphic quadratic form on $\mathbb{C}^{2n}$, such that

$$\text{Im} \ G \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} (x) \right) \geq 0, \quad x \in \mathbb{C}^n,$$

and such that the fundamental matrix of $G$ does not have the eigenvalues $\pm 1$. When proving Theorem 1.2, we would like to give a general criterion for when an operator of the form $\text{Op}^w (e^{iG})$ is a non-vanishing multiple of an operator the form $\text{Top} (e^{Q})$, where $Q$ is quadratic. In view of (4.21), this holds precisely when the polarization

$$iG \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} (x, z) \right), \quad (x, z) \in \mathbb{C}^{2n} \tag{4.22}$$

of the quadratic form $iG \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} (x) \right)$ is of the form

$$iG \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} (x, z) \right) = \text{vc}_{y, \theta} (-4 \Psi_{\text{herm}} (x - y, z - \theta) + Q^n (y, \theta)). \tag{4.23}$$

We need to invert the critical value expression (4.23), and to this end we shall make use of the following well known result, see [21, Chapter 4], [5].

**Proposition 4.1** Let $f(X, Y)$ be a holomorphic quadratic form on $\mathbb{C}^N_X \times \mathbb{C}^N_Y$ such that $\det f''_{XY} (X, Y) \neq 0$ and let $g(Y)$ be a holomorphic quadratic form on $\mathbb{C}^N_Y$ such that $f(0, Y) + g(Y)$ is non-degenerate. Set

$$h(X) = \text{vc}_Y (f(X, Y) + g(Y)), \quad X \in \mathbb{C}^N, \tag{4.24}$$

where the critical value is attained at a unique critical point which is non-degenerate. Then $-f(X, 0) + h(X)$ is non-degenerate, and we have the inversion formula,

$$g(Y) = \text{vc}_X (-f(X, Y) + h(X)). \tag{4.25}$$

**Proof** We recall the main ideas of the proof for completeness. Let us introduce the complex linear canonical transformation

$$\kappa : \mathbb{C}^{2N} \ni (Y, -f'_Y (X, Y)) \mapsto (X, f'_X (X, Y)) \in \mathbb{C}^{2N} \tag{4.26}$$

and the $\mathbb{C}$-Lagrangian plane

$$\Lambda_g = \left\{ (Y, g'(Y)), \ Y \in \mathbb{C}^N \right\} \subset \mathbb{C}^{2N}. \tag{4.27}$$

The holomorphic quadratic form $f(0, Y) + g(Y)$ is non-degenerate on $\mathbb{C}^N$ precisely when $\Lambda_g$ and $\kappa^{-1} \left( T_0^* \mathbb{C}^N \right)$ are transversal, and it follows that the $\mathbb{C}$-Lagrangian plane
κ(Λg) is of the form

\[ Λ_h = \left\{ (Y, h'(Y)) \mid Y \in \mathbb{C}^N \right\} \subset \mathbb{C}^{2N}, \]

where the holomorphic quadratic form h is given by (4.24). Next, the C-Lagrangian planes Λ_h and κ(T^*_0\mathbb{C}^N) are transversal, so that \(-f(X, 0) + h(X)\) is non-degenerate. Writing Λ_g = κ^{-1}(Λ_h) and using that κ^{-1} is of the form

\[ κ^{-1} : \mathbb{C}^{2N} \ni (X, f'_X(X, Y)) \mapsto (Y, -f'_Y(X, Y)) \in \mathbb{C}^{2N}, \quad (4.28) \]

we infer therefore the inversion formula (4.25).

When applying Proposition 4.1 to (4.23), we let \( N = 2n \), \( Y = (x, z) \in \mathbb{C}^{2n} \), \( X = (y, \theta) \in \mathbb{C}^{2n} \),

\[ g(Y) = iG \left( x, \frac{2}{i} \frac{∂Ψ_0}{∂x}(x, z) \right), \]

and

\[ f(X, Y) = 4Ψ_{\text{herm}}(x - y, z - \theta) = 4Φ''_{0,xx}(y - x) \cdot (\theta - z). \quad (4.29) \]

We have

\[ f''_{XY} = f''_{(y, \theta), (x, z)} = \begin{pmatrix} f''_{\bar{y}x} & f''_{\bar{y}z} \\ f''_{\bar{\theta}x} & f''_{\bar{\theta}z} \end{pmatrix} = \begin{pmatrix} 0 & -4Φ''_{0,xx} \bar{x} \\ -4Φ''_{0,xx} \bar{x} & 0 \end{pmatrix} \quad (4.30) \]

is invertible, and therefore Proposition 4.1 applies. The proof of Theorem 1.2 is complete.

Remark In the discussion above, we have considered links between the Weyl and Toeplitz quantizations in the complex domain. The purpose of this remark is to observe that such links become more direct when considering the anti-classical, rather than Weyl, quantization. Indeed, let us assume that for simplicity that the pluriharmonic part of Φ_0 vanishes, so that Φ_0(x) = Φ''_{0,xx} \bar{x} \cdot x. Given \( p \in L^∞(\Lambda_Φ_0) \), let us consider the anti-classical quantization of \( p \),

\[ Ω_0(p)u(x) = \frac{1}{(2π)^n} \int \int_{\Gamma_0} e^{i(x-y)·θ} p(y, \theta)u(y) \, dy \wedge dθ, \quad (4.31) \]

where the contour of integration \( \Gamma_0 \subset \mathbb{C}^{2n} \) is given by

\[ θ = \frac{2}{i} \frac{∂Φ_0}{∂\bar{y}}(y) = \frac{2}{i} Φ''_{0,yy} \bar{y}. \quad (4.32) \]
Along $\Gamma_0$, we have $dy \wedge d\theta = 2^n \det(\Phi_{0,y\bar{y}}''') L(dy)$, provided that the orientation has been chosen properly. It follows that

$$\text{Op}_0(p)u(x) = \frac{2^n \det(\Phi_{0,y\bar{y}}''')}{\pi^n} \int e^{2(x-y) \cdot \Phi_{0,y\bar{y}}'''} p \left( y, \frac{2}{i} \Phi_{0,y\bar{y}}''' \right) u(y) L(dy)$$

$$= \frac{2^n \det(\Phi_{0,y\bar{y}}''')}{\pi^n} \int e^{2\Psi_0(x,y)} p \left( y, \frac{2}{i} \Phi_{0,y\bar{y}}''' \right) u(y) e^{-2\Phi_0(y)} L(dy)$$

$$= \text{Top}(p|_{\Lambda_{\Phi_0}})u(x).$$

(4.33)

5 Example: composing special metaplectic Toeplitz operators

The purpose of this section is to illustrate Theorems 1.1 and 1.2, by applying them to an explicit class of metaplectic Toeplitz operators on a model Bargmann space $H_{\Phi_0}(C^n)$. It will be assumed throughout this section that $\Phi_0(x) = \frac{|x|^2}{4}$, $x \in C^n$. (5.1)

Let

$$q(x) = \lambda |x|^2, \quad \tilde{q}(x) = \tilde{\lambda} |x|^2, \quad x \in C^n,$$

where $\lambda, \tilde{\lambda} \in C$ satisfy $\text{Re}\lambda < \frac{1}{4}, \text{Re}\tilde{\lambda} < \frac{1}{4}$, so that the assumptions (1.3), (1.10) hold. The Weyl symbol $a$ of the operator $\text{Top}(e^q)$ has been computed in [11, Section 4] by evaluating the Gaussian integral (4.16), and we recall from that work that it is given by

$$a \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} (x) \right) = C \exp \left( \frac{\lambda}{1-\lambda} |x|^2 \right), \quad x \in C^n, \quad C \neq 0.$$  

(5.3)

Here we notice that

$$\text{Re} \left( \frac{\lambda}{1-\lambda} \right) = \frac{1 - |1 - 2\lambda|^2}{4 |1 - \lambda|^2}.$$  

(5.4)

Similarly, the Weyl symbol $\tilde{a}$ of the operator $\text{Top}(e^{\tilde{q}})$ has the form

$$\tilde{a} \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} (x) \right) = \tilde{C} \exp \left( \frac{\tilde{\lambda}}{1-\tilde{\lambda}} |x|^2 \right), \quad x \in C^n, \quad \tilde{C} \neq 0.$$  

(5.5)

Following (1.12), we shall assume that

$$a \in L^\infty(\Lambda_{\Phi_0}), \quad \tilde{a} \in L^\infty(\Lambda_{\Phi_0}),$$  

(5.6)

which, in view of (5.4), is equivalent to the conditions

$$|1 - 2\lambda| \geq 1, \quad |1 - 2\tilde{\lambda}| \geq 1.$$  

(5.7)
respectively. Using (5.1), (5.3), and (5.5), we get next

\[ a(x, \xi) = C \exp(i F(x, \xi)), \quad F(x, \xi) = \frac{2\lambda}{1 - \lambda} x \cdot \xi, \quad (x, \xi) \in \mathbb{C}^{2n}, \quad (5.8) \]

\[ \tilde{a}(x, \xi) = \tilde{C} \exp(i \tilde{F}(x, \xi)), \quad \tilde{F}(x, \xi) = \frac{2\tilde{\lambda}}{1 - \lambda} x \cdot \xi, \quad (x, \xi) \in \mathbb{C}^{2n}. \quad (5.9) \]

and recalling (1.9), we see that the fundamental matrices \( F, \tilde{F} \) of the quadratic forms \( F, \tilde{F} \), respectively, are given by

\[ F = \frac{\lambda}{1 - \lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{F} = \frac{\tilde{\lambda}}{1 - \tilde{\lambda}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.10) \]

In particular, \( \pm 1 \notin \text{Spec}(F), \, \pm 1 \notin \text{Spec}(\tilde{F}) \). In order to apply Theorem 1.1, we should also check that \( -1 \notin \text{Spec}(\tilde{F}F) \), where the product \( \tilde{F}F \) is of the form

\[ \tilde{F}F = \frac{\tilde{\lambda}}{1 - \tilde{\lambda}} \frac{\lambda}{1 - \lambda} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.11) \]

When verifying that \( -1 \) is not an eigenvalue of \( \tilde{F}F \), we observe that this is the case provided that at least one of the inequalities in (5.7) is strict. Indeed, it follows from the remark following Proposition 3.1 that the strict inequality \( |1 - 2\lambda| > 1 \), say, implies that the complex linear canonical transformation associated to the Fourier integral operator \( \text{Top}(e^q) \) is strictly positive relative to \( \Lambda_{\Phi_0} \), and the general arguments of Sect. 3 imply then that \( -1 \notin \text{Spec}(\tilde{F}F) \). To discuss the remaining case, it suffices to make the following elementary observation.

**Lemma 5.1** Let \( \lambda, \tilde{\lambda} \in \mathbb{C} \) be such that \( \text{Re} \lambda < \frac{1}{4}, \, \text{Re} \tilde{\lambda} < \frac{1}{4} \), and assume that

\[ \frac{1}{2} (|1 - 2\lambda| = 1, \quad |1 - 2\tilde{\lambda}| = 1. \quad (5.12) \]

Then we have

\[ \frac{\tilde{\lambda}}{1 - \tilde{\lambda}} \frac{\lambda}{1 - \lambda} \neq -1. \quad (5.13) \]

**Proof** It follows from (5.12) that we have

\[ \text{Re} \lambda = |\lambda|^2, \quad \text{Re} \tilde{\lambda} = |\tilde{\lambda}|^2, \quad (5.14) \]

and therefore,

\[ |1 - \lambda|^2 = 1 - \text{Re} \lambda, \quad |1 - \tilde{\lambda}|^2 = 1 - \text{Re} \tilde{\lambda}, \quad (5.15) \]

implying that

\[ \frac{\lambda}{1 - \lambda} = \frac{i \text{Im} \lambda}{1 - \text{Re} \lambda}, \quad \frac{\tilde{\lambda}}{1 - \tilde{\lambda}} = \frac{i \text{Im} \tilde{\lambda}}{1 - \text{Re} \tilde{\lambda}}. \quad (5.16) \]
It suffices to check that
\[
\left( \frac{\tilde{\lambda}}{1 - \lambda} - \frac{\lambda}{1 - \tilde{\lambda}} \right)^2 = \frac{(\operatorname{Im} \lambda)^2(\operatorname{Im} \tilde{\lambda})^2}{(1 - \operatorname{Re} \lambda)^2(1 - \operatorname{Re} \tilde{\lambda})^2} \neq 1,
\]
and to this end, we observe that (5.14) gives
\[
\frac{(\operatorname{Im} \lambda)^2(\operatorname{Im} \tilde{\lambda})^2}{(1 - \operatorname{Re} \lambda)^2(1 - \operatorname{Re} \tilde{\lambda})^2} = \frac{\operatorname{Re} \lambda \operatorname{Re} \tilde{\lambda}}{(1 - \operatorname{Re} \lambda)(1 - \operatorname{Re} \tilde{\lambda})} \neq 1,
\] (5.17)
since \( \operatorname{Re} \lambda + \operatorname{Re} \tilde{\lambda} < 1 \). The proof is complete. \( \square \)

An application of Theorem 1.1 gives therefore that
\[
\operatorname{Top}(e^{\tilde{q}}) \circ \operatorname{Top}(e^q) = C \operatorname{Op}^w(e^{i\hat{F}}), \quad C \neq 0,
\] (5.18)
where the fundamental matrix \( \hat{F} \) of the quadratic form \( \hat{F} \) is given by
\[
\hat{F} = (1 + F)(1 + \tilde{F}F)^{-1}(1 + \tilde{F}) - 1
= (1 + \tilde{F}F)^{-1}(1 + \tilde{F})(1 + F) - 1 = (1 + \tilde{F}F)^{-1}(F + \tilde{F}). \quad (5.19)
\]
Here we have used (1.16) as well as the fact that the matrices \( F, \tilde{F} \) commute, in view of (5.10). A simple computation using (5.10), (5.11), and (5.19) gives that
\[
\hat{F} = \frac{\lambda + \tilde{\lambda} - 2\lambda \tilde{\lambda}}{1 - \lambda - \tilde{\lambda} + 2\lambda \tilde{\lambda}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (5.20)

The computations above may be summarized in the following result.

**Proposition 5.2** Let \( \Phi_0(x) = \frac{|x|^2}{4} \), and let \( q(x) = \lambda |x|^2, \tilde{q}(x) = \tilde{\lambda} |x|^2 \), with \( \lambda, \tilde{\lambda} \in C \) such that \( \operatorname{Re} \lambda < \frac{1}{4}, \operatorname{Re} \tilde{\lambda} < \frac{1}{4} \). Assume that
\[
|1 - 2\lambda| \geq 1, \quad |1 - 2\tilde{\lambda}| \geq 1.
\] (5.21)

We have
\[
\operatorname{Top}(e^{\tilde{q}}) \circ \operatorname{Top}(e^q) = C \operatorname{Op}^w(e^{i\hat{F}}) : H_{\Phi_0}(C^n) \to H_{\Phi_0}(C^n),
\] (5.22)
for some \( C \neq 0 \). Here the holomorphic quadratic form \( \hat{F} \) is given by
\[
\hat{F}(x, \xi) = \frac{2(\lambda + \tilde{\lambda} - 2\lambda \tilde{\lambda})}{1 - \lambda - \tilde{\lambda} + 2\lambda \tilde{\lambda}} x \cdot \xi, \quad (x, \xi) \in C^{2n}.
\] (5.23)

We have
\[
\operatorname{Im} \hat{F} \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) \geq 0, \quad x \in C^n,
\]
and the fundamental matrix of $\widehat{F}$ does not have the eigenvalues $\pm 1$.

We shall next apply Theorem 1.2 to the Weyl quantization in (5.22). To this end, we observe that the holomorphic quadratic form

$$i \widehat{F} \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} (x, z) \right) + 4 \Psi_{\text{herm}}(x, z) = i \frac{2(\lambda + \tilde{\lambda} - 2\lambda \tilde{\lambda})}{1 - \lambda - \tilde{\lambda} + 2\lambda \tilde{\lambda}} x \cdot \frac{2}{i} \frac{z}{4} + x \cdot z$$

is non-degenerate on $\mathbb{C}^{2n}_{x,z}$, and following (1.21), let us set

$$Q^\pi(y, \theta) = \nu_{x,z} \left( (x - y) \cdot (z - \theta) + \frac{\lambda + \tilde{\lambda} - 2\lambda \tilde{\lambda}}{1 - \lambda - \tilde{\lambda} + 2\lambda \tilde{\lambda}} x \cdot z \right).$$

We obtain after a straightforward computation that

$$Q^\pi(y, \theta) = (\lambda + \tilde{\lambda} - 2\lambda \tilde{\lambda}) y \cdot \theta,$$

and an application of Theorem 1.2 gives us therefore the following result.

**Proposition 5.3** Let us make the same assumptions as in Proposition 5.2 and assume furthermore that

$$\text{Re} \left( \lambda + \tilde{\lambda} - 2\lambda \tilde{\lambda} \right) < \frac{1}{4}.$$  

Then we have

$$\text{Top}(e^{\tilde{q}}) \circ \text{Top}(e^q) = C \text{Top}(\hat{q}) : H_{\Phi_0}(\mathbb{C}^n) \rightarrow H_{\Phi_0}(\mathbb{C}^n),$$

for some constant $C \neq 0$, where $\hat{q}(x) = (\lambda + \tilde{\lambda} - 2\lambda \tilde{\lambda}) |x|^2$.

**Remark** In the case when $\lambda = i, \tilde{\lambda} = -i$, the result of Proposition 5.3 has been observed in [2].

**Remark** Let us set, following [8],

$$\lambda = \tilde{\lambda} = \frac{1 + 2i}{5},$$

so that (5.21) holds, with the equality sign. Proposition 5.2 applies in this case, and we find that the quadratic form $\widehat{F}$ given in (5.23) satisfies

$$\text{Im} \left( \frac{2}{i} \frac{\partial \Phi_0}{\partial x} (x) \right) = 0, \quad x \in \mathbb{C}^n.$$  

It has been established in [8] that the composition in (5.22) satisfies

$$\| \left( \text{Top}(e^q) \right)^2 - \text{Top}(p) \|_{L(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n))} \geq 1,$$
for all $p : \mathbb{C}^n \to \mathbb{C}$ measurable such that

$$e^{2\Psi_0(\cdot, y)} \in D(\text{Top}(p)), \quad y \in \mathbb{C}^n. \quad (5.31)$$

Notice that in this case we have

$$\text{Re} \left( 2\lambda - 2\lambda^2 \right) = \frac{16}{25} > \frac{1}{4}, \quad (5.32)$$

and therefore the assumption (1.22) in Theorem 1.2 cannot be removed entirely.

**Remark** The purpose of this remark is to discuss the composition of more general metaplectic Toeplitz operators of the form $\text{Top}(e^{\theta})$, considered in [12, Theorem 4.1]. Thus, let

$$q(x) = \lambda |x|^2 + A \bar{x} \cdot x, \quad \tilde{q}(x) = \tilde{\lambda} |x|^2 + \tilde{A} \bar{x} \cdot x, \quad x \in \mathbb{C}^n, \quad (5.33)$$

where $\lambda, \tilde{\lambda} \in \mathbb{C}$ and $A, \tilde{A}$ are $n \times n$ complex symmetric matrices, such that

$$\text{Re} \lambda + ||A|| < \frac{1}{4}, \quad \text{Re} \tilde{\lambda} + ||\tilde{A}|| < \frac{1}{4}, \quad (5.34)$$

and

$$4||A|| \leq \frac{1 - |\gamma|^2}{|\gamma|^2}, \quad 4||\tilde{A}|| \leq \frac{1 - |\tilde{\gamma}|^2}{|\tilde{\gamma}|^2}. \quad (5.35)$$

Here the norm is Euclidean and

$$\gamma = \frac{1}{1 - 2\lambda}, \quad \tilde{\gamma} = \frac{1}{1 - 2\tilde{\lambda}}. \quad (5.36)$$

It has been established in [12, Theorem 4.1] that the conditions (5.35) are equivalent to the boundedness of the operators $\text{Top}(e^{\theta})$, $\text{Top}(e^{\tilde{\theta}})$, respectively, on the Bargmann space $H_{\Phi_0}(\mathbb{C}^n)$.

When computing the bounded operator $\text{Top}(e^{\tilde{\theta}}) \circ \text{Top}(e^{\theta})$, rather than applying Theorem 1.1, following [11, 12], we shall consider the composition acting directly on the space of coherent states given by

$$k_w(x) = C_{\Phi_0} e^{2\Psi_0(x, \bar{y}) - \Phi_0(w)}, \quad w \in \mathbb{C}^n. \quad (5.37)$$

Here the constant $C_{\Phi_0} > 0$ is chosen suitably so that $||k_w||_{H_{\Phi_0}(\mathbb{C}^n)} = 1, \quad w \in \mathbb{C}^n$. A straightforward computation making use of [12, equations (2.32), (2.33)], or alternatively, of [11, equations (4.13), (4.14), (4.15)] shows that

$$\left( \text{Top}(e^{\tilde{\theta}}) \circ \text{Top}(e^{\theta}) e^{2\Psi_0(\cdot, \bar{w})} \right)(x) = (\tilde{\gamma} \gamma)^n e^{2\Psi_0(x, \tilde{\gamma} y \bar{w})} \exp \left( \tilde{A} \tilde{\gamma} y \bar{w} \cdot \tilde{\gamma} y \bar{w} + A y \bar{w} \cdot y \bar{w} \right). \quad (5.38)$$
Here we have also used the following more precise version of [11, equation (4.13)],

\[
\left( \text{Top}(e^{k|x|^2})e^{2\Psi_0(x,\overline{w})} \right)(x) = \gamma^n e^{2\Psi_0(x,\gamma\overline{w})},
\]

which follows by the exact stationary phase. Setting

\[
\hat{\gamma} = \tilde{\gamma} \gamma, \quad \hat{A} = \tilde{A} \gamma + \frac{1}{\gamma^2} A,
\]

we obtain from (5.38) that

\[
\left( \text{Top}(e^{\tilde{q}}) \circ \text{Top}(e^q) k_w \right)(x) = C_{\Phi_0} \hat{\gamma}^n e^{2\Psi_0(x,\overline{\hat{\gamma}w})-\Phi_0(w)} \exp \left( \hat{A} \hat{\gamma} \overline{w} \cdot \hat{\gamma} w \right).
\]

Let us now set

\[
\hat{q}(x) = \hat{\lambda} |x|^2 + \hat{A} \overline{x} \cdot x, \quad \hat{\lambda} = \lambda + \tilde{\lambda} - 2\lambda \tilde{\lambda},
\]

so that

\[
\hat{\gamma} = \frac{1}{1 - 2\lambda}.
\]

Assuming that

\[
\text{Re} \hat{\lambda} + || \hat{A} || < \frac{1}{4},
\]

so that the Toeplitz operator \( \text{Top}(e^{\tilde{q}}) \) is densely defined, we conclude, in view of (5.41), that the following identity holds on the common dense domain given by the linear span of the coherent states \( k_w, w \in \mathbb{C}^n \).

\[
\text{Top}(e^{\tilde{q}}) \circ \text{Top}(e^q) = \text{Top}(e^{\tilde{q}}).
\]  

Here we observe that in view of (5.35), we have

\[
4|| \hat{A} || \leq 4|| \tilde{\gamma} || + \frac{4}{|\tilde{\gamma}|^2} || A || \leq \frac{1 - |\tilde{\gamma}|^2}{|\tilde{\gamma}|^2} + \frac{1 - |\gamma|^2}{|\gamma|^2 |\gamma|^2} = \frac{1 - |\tilde{\gamma}|^2}{|\tilde{\gamma}|^2},
\]

and hence the operator \( \text{Top}(e^{\tilde{q}}) \) is bounded on \( H_{\Phi_0}(\mathbb{C}^n) \), in view of [12, Theorem 4.1]. We obtain the composition result,

\[
\text{Top}(e^{\tilde{q}}) \circ \text{Top}(e^q) = \text{Top}(e^{\tilde{q}}) : H_{\Phi_0}(\mathbb{C}^n) \rightarrow H_{\Phi_0}(\mathbb{C}^n),
\]

provided that (5.34), (5.35), and (5.43) hold.

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**Declarations**

**Competing interests** The authors declare no competing interests.
A Adjointsof complex FIOs

The purpose of this appendix is to continue the discussion started in [10, Appendix A] and to review some of the basic facts concerning adjoints of metaplectic Fourier integral operators in the complex domain. Let $\Phi_0$ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, and let us recall from [16, 23] that the orthogonal projection

$$\Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)) \rightarrow H_{\Phi_0}(\mathbb{C}^n)$$  \hspace{1cm} (A.1)

is given by

$$\Pi_{\Phi_0} u(x) = a_0 \int e^{2(\Psi_0(x,\overline{y})-\Phi_0(y))} u(y) \, dy \, d\overline{y}, \quad a_0 \neq 0.$$  \hspace{1cm} (A.2)

Here $\Psi_0$ is the polarization of $\Phi_0$. In polarized form we may write for $u$ holomorphic,

$$\Pi_{\Phi_0} u(x) = \int_{\Gamma} e^{2(\Psi(x,\theta)-\Psi(y,\theta))} a_0 u(y) \, dy \, d\theta.$$  \hspace{1cm} (A.3)

Here the contour $\Gamma \subset \mathbb{C}^{2n}$ is given by $\theta = \overline{y}$. Omitting the contour of integration, we obtain a formal factorization

$$\Pi_{\Phi_0} = A \circ B,$$  \hspace{1cm} (A.4)

where

$$Av(x) = \int e^{2\Psi_0(x,\theta)} a_0 v(\theta) \, d\theta, \quad Bu(\theta) = \int e^{-2\Psi_0(y,\theta)} u(y) \, dy.$$  \hspace{1cm} (A.5)

We notice that from this point of view, the construction of $\Pi_{\Phi_0}$ is reduced to the problem of inverting the operator $B$, i.e. finding a suitable constant amplitude $a_0 \neq 0$ such that $AB = 1$ in suitable $H_{\Phi}$-spaces. The canonical transformations associated to $A, B$ are given by

$$\kappa_A = \kappa_{2\Psi_0/i} : \left( \theta, -\frac{2}{i} \partial_\theta \Psi_0(x, \theta) \right) \mapsto \left( x, \frac{2}{i} \partial_x \Psi_0(x, \theta) \right),$$  \hspace{1cm} (A.6)

and

$$\kappa_B = \kappa_{-1\Psi_0/i} : \left( \theta, \frac{2}{i} \partial_\theta \Psi_0(y, \theta) \right) \mapsto \left( \theta, -\frac{2}{i} \partial_\theta \Psi_0(y, \theta) \right),$$  \hspace{1cm} (A.7)

respectively. Since $\kappa_B$ is equal to the inverse of $\kappa_A$ it is clear that $AB$ is a multiple of the identity operator, and that we can choose the constant amplitude $a_0 \neq 0$ in (A.5) so that $AB = 1$.

Putting $\theta = \overline{y}$ in (A.7), we get

$$\kappa_B : \left( y, \frac{2}{i} \partial_y \Phi_0(y) \right) \mapsto \left( \overline{y}, -\frac{2}{i} \partial_{\overline{y}} \Phi_0(y) \right) = \left( \overline{y}, -\frac{2}{i} \partial(\Phi_0 \circ \dagger)(\overline{y}) \right) \in \Lambda_{-\Phi_0 \circ \dagger} = \dagger(\Lambda_{\Phi_0}),$$
where \( \dagger \) is the operator of complex conjugation of complex numbers or elements in \( \mathbb{C}^N \), \( \dagger(z) = \overline{z} \). Thus we have the following map between two maximally totally real subspaces of \( \mathbb{C}^{2n} \),

\[
\kappa_B : \Lambda \Phi_0 \to \Lambda - \Phi_0 \circ \dagger,
\]

and \( \kappa_B \) in (A.7) is the holomorphic extension of this map. Here we observe that 

\[
- \Phi_0 \circ \dagger \text{ is strictly pluri-super-harmonic so it would not be meaningful to say that} \quad \text{“} B : H_{\Phi_0} \to H_{- \Phi_0 \circ \dagger} \text{”, or that “} A : H_{- \Phi_0 \circ \dagger} \to H_{\Phi_0} \text{”}.
\]

Let \( \Phi_j, j = 1, 2 \), be strictly plurisubharmonic quadratic forms on \( \mathbb{C}^n \), and let \( \kappa : C^{2n} \to C^{2n} \) be a complex linear canonical transformation which is positive relative to \( (\Lambda \Phi_2, \Lambda \Phi_1) \), in the sense that

\[
\frac{1}{i} \left( \sigma(\kappa(\rho), \iota \Phi_2 \kappa(\rho)) - \sigma(\rho, \iota \Phi_1(\rho)) \right) \geq 0, \quad \rho \in C^{2n}.
\] (A.8)

Here, as above, \( \iota \Phi_j : C^{2n} \to C^{2n} \) is the unique anti-linear involution which is equal to the identity on the maximally totally real subspace \( \Lambda \Phi_j \subset C^{2n}, j = 1, 2 \).

Let \( A \) be a metaplectic Fourier integral operator quantizing \( \kappa \), and let us recall from [10] that we can realize \( A \) as a linear continuous map

\[
A : H_{\Phi_1}(C^n) \to H_{\Phi_2}(C^n).
\] (A.9)

We shall then also write \( \kappa = \kappa_A \). With \( \Psi_1 = \Phi_1^\pi \) being the polarization of \( \Phi_1 \), let

\[
\Pi_1 u(x) = a_1 \int \int e^{2\Phi_1(x, \overline{y})} u(y) e^{-2\Phi_1(y)} \frac{dy \, d\overline{y}}{(2i)^n}, \quad a_1 \neq 0,
\] (A.10)

be the Bergman projection: \( L^2(\mathbb{C}^n, e^{-2\Phi_1} L(dx)) \to H_{\Phi_1}(\mathbb{C}^n) \). Here we observe that the \( (n, n) \)-form

\[
\frac{dy \, d\overline{y}}{(2i)^n} = \frac{1}{(2i)^n} dy_1 \wedge \ldots \wedge dy_n \wedge d\overline{y}_1 \wedge \ldots \wedge d\overline{y}_n
\]

can be naturally identified with the Lebesgue volume form \( L(dy) \) on \( \mathbb{C}^n \). An application of [10, Theorem A.1] allows us to write for \( u \in H_{\Phi_1}(\mathbb{C}^n) \),

\[
Au(x) = \int \int K_A(x, \overline{y}) u(y) e^{-2\Phi_1(y)} \frac{dy \, d\overline{y}}{(2i)^n}.
\] (A.11)

Here the kernel \( K_A(x, \theta) \) is holomorphic on \( \mathbb{C}^n \times \mathbb{C}^n \), with \( y \mapsto K_A(x, \overline{y}) \in H_{\Phi_1}(\mathbb{C}^n) \), uniquely determined by (A.11). Arguing as in [10] we see, using the reproducing property of \( \Pi_1 \) on \( H_{\Phi_1}(\mathbb{C}^n) \), that

\[
K_A(x, \theta) = A \left( a_1 e^{2\Psi_1(\cdot, \theta)} \right)(x), \quad \text{(A.12)}
\]

where \( \hat{z} \) is the operator of complex conjugation of complex numbers or elements in \( \mathbb{C}^N \), \( \hat{z}(z) = \overline{z} \). Thus we have the following map between two maximally totally real subspaces of \( \mathbb{C}^{2n} \),

\[
\kappa_B : \Lambda \Phi_0 \to \Lambda - \Phi_0 \circ \dagger,
\]
see also [10, equation (3.9)]. We infer furthermore from the discussion in [10, Section 3] that the kernel $K_A$ in (A.12) is of the form

$$K_A(x, \theta) = \hat{a} e^{2\Psi(x, \theta)}, \quad (A.13)$$

for some $\hat{a} \in \mathbb{C}$, where $\Psi(x, \theta)$ is a holomorphic quadratic form on $\mathbb{C}_x^n \times \mathbb{C}_\theta^n$, such that $\det \Psi_{x\theta}'' \neq 0$. Combining (A.11) and (A.13), we get therefore,

$$Au(x) = \hat{a} \int\int e^{2(\Psi(x, \theta)-\Psi_1(y, \theta))}u(y) \frac{dy \, d\theta}{(2i)^n}. \quad (A.14)$$

Introducing the formal Fourier integral operator

$$\widetilde{A}v(x) = \int e^{2\Psi(x, \theta)}\hat{a} \, v(\theta) \, d\theta, \quad (A.15)$$

with the associated canonical transformation

$$\kappa_{\widetilde{A}} : \left( \theta, -\frac{2}{i} \partial_{\theta} \Psi(x, \theta) \right) \mapsto \left( x, \frac{2}{i} \partial_x \Psi(x, \theta) \right), \quad (A.16)$$

we obtain the factorization

$$\kappa_A = \kappa_{\widetilde{A}} \circ \kappa_{2\Psi_1/i}. \quad (A.17)$$

Here we have set, similarly to (A.6),

$$\kappa_{2\Psi_1/i} : \left( \theta, -\frac{2}{i} \partial_{\theta} \Psi_1(y, \theta) \right) \mapsto \left( y, \frac{2}{i} \partial_y \Psi_1(y, \theta) \right). \quad (A.18)$$

The Hilbert space adjoint $A^* : H_{\Phi_1}(\mathbb{C}^n) \to H_{\Phi_2}(\mathbb{C}^n)$ of $A$ in (A.9) satisfies

$$A^*v(y) = \int\int K_{A^*}(y, \overline{x})v(x)e^{-2\Phi_2(x)} \frac{dx \, d\overline{x}}{(2i)^n}, \quad (A.19)$$

where $K_{A^*}(y, \overline{x}) = \overline{K_A(x, y)}$, so that

$$K_{A^*}(y, x) = \overline{K_A(x, y)}. \quad (A.20)$$

We shall now compute the canonical transformation $\kappa_{A^*}$ associated to the Fourier integral operator $A^*$. When doing so, recalling the notation $\dagger(z) = \overline{z}$, $z \in \mathbb{C}^N$, let us put $f^\dagger = \dagger \circ f \circ \dagger$, if $f$ is a continuous function on $\mathbb{C}^N$. When $f$ is of class $C^1$, we have $\partial(\dagger \circ f) = \dagger \circ \overline{\partial f}$, $\overline{\partial}(\dagger \circ f) = \dagger \circ \partial f$. We have the analogous relations for the composition with $\dagger$ to the right and it follows that

$$\partial(f^\dagger) = (\partial f)^\dagger, \quad \overline{\partial}(f^\dagger) = (\overline{\partial} f)^\dagger. \quad (A.21)$$
From (A.13) and (A.20), we get $K_{A^*}(y, x) = \tilde{a} e^{2\Psi^\dagger(x, y)}$, for some $\tilde{a} \in \mathbb{C}$, and combining this with (A.19), we get that

$$A^* v(y) = \tilde{a} \int_\Gamma e^{2(\Psi^\dagger(\theta, y) - \Psi_2(x, \theta))} v(x) \frac{dx \, d\theta}{(2i)^n}. \quad (A.22)$$

Here the contour $\Gamma$ given by $\theta = \overline{x}$ and $\Psi_2 = \Phi^\pi_2$ is the polarization of $\Phi_2$. Associated to the formal Fourier integral operator

$$\tilde{A}^* v(y) = \tilde{a} \int e^{2\Psi^\dagger(\theta, y)} v(\theta) \, d\theta \quad (A.23)$$

is the canonical transformation

$$\kappa^{\tilde{A}^*} : (\theta, -\frac{2}{i} \partial_\theta \Psi^\dagger(\theta, y)) \mapsto (y, \frac{2}{i} \partial_y \Psi^\dagger(\theta, y)), \quad (A.24)$$

and it follows from (A.22), similarly to (A.17), that the following factorization holds,

$$\kappa_A^* = \kappa_{A^*}^{\tilde{A}^*} \circ \kappa_{2\Psi_2/i}. \quad (A.25)$$

Here $\kappa_{2\Psi_2/i}$ is defined similarly to (A.18). We shall now simplify (A.24). In view of (A.21) we have

$$\kappa^{\tilde{A}^*} : \dagger (\dagger \theta, \frac{2}{i} \partial_\theta \Psi(\dagger \theta, \dagger y)) \mapsto \dagger \left(\dagger y, -\frac{2}{i} \partial_y \Psi(\dagger \theta, \dagger y)\right), \quad (A.26)$$

and replacing $\dagger \theta, \dagger y$ by $\theta, y$, we get

$$\kappa^{\tilde{A}^*} : \dagger (\theta, \frac{2}{i} \partial_\theta \Psi(\theta, y)) \mapsto \dagger \left(y, -\frac{2}{i} \partial_y \Psi(\theta, y)\right). \quad (A.27)$$

Using the fact that $\dagger^{-1} = \dagger$ together with (A.16), (A.27) we obtain that

$$\dagger \circ \kappa^{\tilde{A}^*} \circ \dagger = \kappa^{-1}_A. \quad (A.28)$$

We combine (A.28) with (A.17), (A.25), and get

$$\dagger \circ \kappa_A^* \circ \kappa_{2\Psi_2/i} \circ \dagger = \kappa_{2\Psi_1/i}^{-1} \circ \kappa_A^{-1},$$

$$(\kappa_{2\Psi_1/i} \circ \dagger) \circ \kappa_A^* \circ (\kappa_{2\Psi_2/i} \circ \dagger) = \kappa_A^{-1}. \quad (A.29)$$

Here (A.18) gives that

$$\kappa_{2\Psi_1/i} \circ \dagger : \left(\overline{y}, \frac{2}{i} \partial_y \Psi_1(x, y)\right) \mapsto \left(x, \frac{2}{i} \partial_x \Psi_1(x, y)\right),$$
or after the substitution \( y \mapsto \bar{y} \),

\[
\kappa_{2\psi_1/i} \circ \dagger : \left( y, \frac{2}{i} \partial_y \psi_1(x, \bar{y}) \right) \mapsto \left( x, \frac{2}{i} \partial_x \psi_1(x, \bar{y}) \right). \tag{A.30}
\]

Recalling [10, equation (2.4)], we conclude that \( \kappa_{2\psi_1/i} \circ \dagger = \iota_{\Phi_1} \) is the unique antilinear involution

\[
\iota_{\Phi_1} : \mathbb{C}^{2n} \to \mathbb{C}^{2n},
\]

which is equal to the identity on the maximally totally real subspace \( \Lambda_{\Phi_1} \subset \mathbb{C}^{2n} \). Similarly, \( \kappa_{2\psi_2/i} \circ \dagger = \iota_{\Phi_2} \) is the unique antilinear involution on \( \mathbb{C}^{2n} \) which agrees with the identity on \( \Lambda_{\Phi_2} \subset \mathbb{C}^{2n} \). We get from (A.29) that

\[
\iota_{\Phi_1} \circ \kappa_A^* \circ \iota_{\Phi_2} = \kappa_A^{-1},
\]

or in other words,

\[
\kappa_A^* = \iota_{\Phi_1} \circ \kappa_A^{-1} \circ \iota_{\Phi_2}. \tag{A.31}
\]

The discussion in the appendix can be summarized in the following result.

**Theorem A.1** Let \( \Phi_j \) be strictly plurisubharmonic quadratic forms on \( \mathbb{C}^n \), \( j = 1, 2 \), and \( \kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \) be a complex linear canonical transformation which is positive relative to \( (\Lambda_{\Phi_2}, \Lambda_{\Phi_1}) \), in the sense that (A.8) holds. Let \( A : H_{\Phi_1}(\mathbb{C}^n) \to H_{\Phi_2}(\mathbb{C}^n) \) be a realization of a Fourier integral operator associated to the canonical transformation \( \kappa = \kappa_A \). The canonical transformation \( \kappa_A^* \) associated to the complex adjoint \( A^* : H_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n) \) of \( A \) is given by

\[
\kappa_A^* = \iota_{\Phi_1} \circ \kappa_A^{-1} \circ \iota_{\Phi_2}. \tag{A.32}
\]

Here \( \iota_{\Phi_j} \) is the anti-holomorphic reflection in \( \Lambda_{\Phi_j} \), \( j = 1, 2 \).

**Remark** It follows from (A.32) and the positivity of \( \kappa_A \) relative to \( (\Lambda_{\Phi_2}, \Lambda_{\Phi_1}) \) that \( \kappa_A^* \) is positive relative to \( (\Lambda_{\Phi_1}, \Lambda_{\Phi_2}) \),

\[
\frac{1}{i} \left( \sigma(\kappa_A^*(\rho), \iota_{\Phi_1}\kappa_A^*(\rho)) - \sigma(\rho, \iota_{\Phi_2}(\rho)) \right) \geq 0, \quad \rho \in \mathbb{C}^{2n}. \tag{A.33}
\]

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