Entanglement dynamics of three-qubit states in local many-sided noisy channels

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Received 9 November 2011, in final form 24 November 2011
Published 10 January 2012
Online at stacks.iop.org/JPhysB/45/035501

Abstract

We study entanglement dynamics of pure three-qubit Greenberger–Horne–Zeilinger-type entangled states when one, two or three qubits are subjected to general local noise. Employing a lower bound for three-qubit concurrence as an entanglement measure, we show that for some many-sided noisy channels the entanglement dynamics can be completely described by the evolution of the entangled states in single-sided channels.

1. Introduction

It is widely accepted nowadays that entangled states of multiparticle systems are the most promising resource for quantum information processing [1, 2]. At the same time, entanglement of complex systems is known to be very fragile with regard to decoherence [3], which may appear, for instance, due to a transmission of the whole quantum system or some of its subsystems through communication channels. In practice, moreover, it is often required to distribute parts of an entangled multiparticle system between several remote recipients [4, 5]. In this case, each subsystem is coupled locally with its environment. Such a coupling of a quantum subsystem to some environmental channel leads to decoherence of the multiparticle system and usually to some loss of entanglement. For the successful practical utilization of entangled states, it is of great importance to quantify entanglement of the complex quantum systems and describe quantitatively their entanglement dynamics under the action of local (noisy) channels.

There are two main approaches in the literature to investigate entanglement dynamics of multiparticle entangled states. First one is to study state evolution of particular (usually maximally entangled) states under the action of some chosen noisy channels and deduce entanglement dynamics from the state evolution [6–9]. This approach, however, is restricted by our choice of the entangled states and the noise models. Recently, a completely different approach for the description of entanglement dynamics, which is based on the evolution equation for entanglement, has been developed [10–13]. This latter approach allows us to obtain a direct relationship between the initial and final entanglement of the system in which just one of its subsystems is subjected to an arbitrary noise. Unfortunately, the suggested evolution equations cannot be straightforwardly generalized to the case when more than one subsystem undergo the action of local noisy channels.

In this work, we investigate entanglement dynamics of initially pure three-qubit states when one, two or three qubits are subjected to general local single-qubit noisy channels. At first, we shall consider the entanglement dynamics of the maximally entangled Greenberger–Horne–Zeilinger (GHZ) state, which can be written, in the computational basis, as

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

(1)

Later, we shall generalize our discussion to all (GHZ-type) pure three-qubit states which can be obtained from the state (1) by local unitary transformations. To describe the influence of the local noisy channels on the entangled states, we shall use quantum operation formalism [1]. Since there is no analytically computable measure of entanglement for multiqubit states [2], a lower bound for multiqubit concurrence [14] will be utilized to access the entanglement dynamics. We shall show that for some noisy channels the complex two- and three-sided entanglement dynamics, i.e. when two or three qubits are affected by local noise, can be completely described by the evolution of the entangled system in single-sided channels (when just one qubit is subjected to local noise).

This work is organized as follows. In the following section, we present the entanglement measure of use, the lower bound for multiqubit concurrence, and recall its properties. In section 3, we analyse cases step-by-step when one, two or three qubits of the three-qubit system are subjected to local noisy channels. A summary is drawn in section 4.
2. The entanglement measure

Concurrence, originally suggested by Wootters [15] to describe the entanglement of an arbitrary state of two qubits, has been recognized as a very powerful measure of entanglement. Although various extensions of the concurrence to the case of bipartite states, if the dimensions of the associated Hilbert (sub-)spaces are larger than two, have been suggested [16–18], a full generalization of the concurrence towards multipartite states still remains challenging [2]. However, a formal extension of the concurrence to a multipartite case can be successfully approximated by an analytically computable function, so-called lower bound, which never exceeds such multipartite concurrence, but nonetheless is close enough to its values. To date, several lower bounds for the multipartite concurrence have been proposed [6, 2, 14]. Let us recall and exploit the lower bound for the multiqubit concurrence as suggested by Li et al [14].

The multiqubit concurrence for a pure three-qubit state \(|\psi\rangle\) is given by

\[
C_3(|\psi\rangle) = \sqrt{1 - \sum_{i=1}^{3} \text{Tr} \rho_i^2},
\]

where \(\rho_i = \text{Tr} |\psi\rangle \langle \psi|\) denote the reduced density matrix of the \(i\)th qubit which is obtained by tracing out the remaining two qubits. The concurrence for an arbitrary mixed three-qubit state can formally be defined by means of the so-called convex roof

\[
C_3(\rho) = \min \sum_i p_i C_3(|\psi_i\rangle),
\]

relying on the fact that any mixed state can be expressed as a convex sum of some pure states \(|\psi_i\rangle\); \(\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|\). However, the minimum in this expression may be found among all possible decompositions of \(\rho\) into pure states \(|\psi_i\rangle\). No solution has been found so far to optimize the concurrence (3) analytically [2].

A simple analytically computable lower bound \(\tau_3(\rho)\) for three-qubit concurrence can be given in terms of the three bipartite concurrences \(C_{abc}\) (\(a, b, c = 1, 2, 3\) and \(a \neq b \neq c \neq a\)) [14] as

\[
\tau_3(\rho) = \frac{1}{3} \sum_{k=1}^{6} (C_{k12}^{13})^2 + (C_{k13}^{12})^2 + (C_{k23}^{12})^2.
\]

Here, each bipartite concurrence \(C_{abc}\) corresponds to a possible (bipartite) cut of the three-qubit system in which just one of the qubits is discriminated from the other two qubits. For a separation \(ab|c\), the bipartite concurrence \(C_{abc}\) is given by a sum of six terms \(C_{k}\) which are expressed as

\[
C_{abc} = \max (0, \lambda_1^m - \lambda_2^m - \lambda_3^m - \lambda_4^m),
\]

and where \(\lambda_m^m\), \(m = 1, \ldots, 4\) are the square roots of the four nonvanishing eigenvalues of the matrix \(\rho \tilde{a}_{k}^{abc}\), if taken in decreasing order. These matrices \(\rho \tilde{a}_{k}^{abc}\) are formed by means of the density matrix \(\rho\) and its complex conjugate \(\rho^*\), and are further transformed by the operators \(S_{k}^{abc} = L_{k}^{abc} \otimes L_{0}\), \(k = 1, \ldots, 6\) as \(\tilde{a}_{k}^{abc} = S_{k}^{abc} \rho \tilde{a}_{k}^{abc}\). In this notation, moreover, \(L_0\) is the (single) generator of the group \(SO(2)\) which is given by the second Pauli matrix \(\sigma_x = -i(0|0\rangle + |1\rangle|1\rangle)\), while \(L_{k}^{abc}\) are the six generators of the group \(SO(4)\) that can be expressed explicitly by means of the totally antisymmetric Lévi–Civita symbol in four dimensions as \((L_{k})_{mn} = -i\varepsilon_{klnm}\). \(k, l, m, n = 1, \ldots, 4\) [19].

Since the lower bound (4) is just an approximation for the convex roof (3), it is of great importance to know how accurate this bound is with respect to the convex roof for multiqubit concurrence. It has been checked [20] by sampling 100 randomly generated density matrices \(\rho\) that the lower bound \(\tau_3(\rho)\) coincides with the numerically simulated convex roof \(C_3(\rho)\) for all density matrices with rank \(r \leq 4\). This result has found its explanation in the fact that the lower bound \(\tau_3(\rho)\), by its construction, relies only on four nonvanishing eigenvalues of the matrix \(\rho \tilde{a}_{k}^{abc}\), while this matrix may have at most eight eigenvalues.

Taking into account the mentioned property of the lower bound \(\tau_3(\rho)\), it is convenient to define a function of at most four variables

\[
f(w, x, y, z) = \min(0, 2\max(w, x, y, z) - w - x - y - z).\]

It is easy to see that \(C_{abc}^{k} = f(\lambda_1^k, \lambda_2^k, \lambda_3^k, \lambda_4^k)\), where \(\lambda_m^k\), \(m = 1, \ldots, 4\) are the square roots of the eigenvalues of the matrix \(\rho \tilde{a}_{k}^{abc}\). Whenever the matrix \(\rho \tilde{a}_{k}^{abc}\) has less than four, e.g. two, eigenvalues, we shall use the function of just two inputs \(f(w, x) = f(w, x, 0, 0)\).

3. Entanglement dynamics in local noisy channels

Quantum operation formalism is a very general and prominent tool to describe how a quantum system has been influenced by its environment. According to this formalism the final state of the quantum system, that is coupled to some environmental channel, can be obtained from its initial state with the help of (Kraus) operators

\[
\rho_{in} = \sum_i K_i \rho_{in} K_i^\dagger,
\]

and the condition \(\sum_i K_i^\dagger K_i = I\) is fulfilled. Note that we consider only such system–environment interactions that can be associated with completely positive trace-preserving maps [1].

If the quantum system of interest consists of just a single qubit, which is subjected to some environmental channel \(A\), then an arbitrary quantum operation can be expressed with the help of at most four operators [1]. Let us define the four operators through the Pauli matrices as

\[
K_1(a_1) = \frac{a_1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2(a_2) = \frac{a_2}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
K_3(a_3) = \frac{a_3}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad K_4(a_4) = \frac{a_4}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \(a_i\) are the real parameters and the condition \(\sum_{i=1}^{4} a_i^2 = 1\) holds.

For the three-qubit system, local interactions of the qubits with the channels \(A, B\) and \(C\) can be described by
operators, which are constructed as tensor products of the single-qubit operators $K_i(a_i), K_i(b_i)$ and $K_i(c_i)$. Therefore, if the three qubits are affected by local noise simultaneously, the final state of the system $\rho_{\text{fin}} \equiv [A \otimes B \otimes C] \rho_{\text{in}}$ can be obtained from its initial state $\rho_{\text{in}}$ with the help of 64 operators $K_i(a_i) \otimes K_j(b_j) \otimes K_l(c_l)$ with $(i, j, l = 1, \ldots, 4)$ through equation (7). If just two qubits undergo the action of local channels, the final state $\rho_{\text{fin}}$ can be obtained in a similar way as given above, but with just 16 Kraus operators in equation (7), e.g., $1 \otimes K_i(a_i) \otimes K_j(b_j)$ and $(i, j = 1, \ldots, 4)$.

### 3.1. Single-sided channels

Suppose, just one qubit of the three-qubit system, which is initially prepared in a pure GHZ state (1), is subjected to a noisy channel $A$. The final state of the system is in general mixed and can be expressed by a rank 4 density matrix $\rho$. This matrix is obtained from equation (7) using the four operators $1 \otimes K_i(a_i)$. Here we assumed, without loss of generality, that the third qubit undergoes the action of the channel. From the matrix $\rho \equiv [1 \otimes 1 \otimes A] |\text{GHZ}\rangle \langle \text{GHZ}|$, we can directly compute the three bipartite concurrences $C^{12|3}, C^{13|2}$ and $C^{23|1}$, and the lower bound (4) following the method which has been given in section 2. The concurrences are given by

$$C^{12|3} = f(a_1^2, a_2^2, a_3^2, a_4^2),$$

$$C^{13|2} = \sqrt{f^2(a_1^2, a_2^2) + f^2(a_3^2, a_4^2)},$$

$$C^{23|1} = \sqrt{f^2(a_1^2, a_3^2) + f^2(a_2^2, a_4^2)},$$

where we dropped the arguments of the concurrences, e.g., $C^{12|3} = C^{12|3}([1 \otimes 1 \otimes A] |\text{GHZ}\rangle \langle \text{GHZ}|)$.

Interestingly, if the channel $A$ is the bit-flip ($a_1 \neq 0, a_2 \neq 0, a_3 = a_4 = 0$) or the bit-phase-flip ($a_1 \neq 0, a_3 \neq 0, a_2 = a_4 = 0$) channel [11], the entanglement of the three-qubit system, described by means of the lower bound $\tau_3(\rho)$, never vanishes.

The description of the entanglement dynamics of the GHZ state (1) in single-sided channels can be generalized to the case of an arbitrary pure three-qubit state $|\psi\rangle$, which can be obtained from the GHZ state by local unitary transformations. Such an extension is possible due to a recently suggested evolution equation for the lower bound $\tau_3$ [13], which manifests that the entanglement dynamics of an arbitrary pure state $|\psi\rangle$ of a three-qubit system, when one of its qubits undergoes the action of an arbitrary noisy channel $A$, is subordinated to the dynamics of the maximally entangled state, i.e.

$$\tau_3([1 \otimes 1 \otimes A]|\psi\rangle \langle \psi|) = \tau_3([1 \otimes 1 \otimes A]|\text{GHZ}\rangle \langle \text{GHZ}|) \tau_3([|\psi\rangle \langle \psi|]).$$

Conclusively, the entanglement dynamics of an arbitrary pure three-qubit state, which is locally equivalent to the GHZ state and is subjected to an arbitrary single-sided channel $A$, is given by the lower bound (4), which is defined through equations (9)–(11), and the evolution equation (12). Particular examples of the entanglement dynamics of multiqubit systems in single-sided channels can be found in [13, 20]. Moreover, based on the numerical simulations in [20], we argue that the description of this particular case of the entanglement dynamics with the lower bound $\tau_3$ is as general as with the convex roof (3) for multipartite concurrence.

### 3.2. Two-sided channels

If the three-qubit system is prepared into the GHZ state (1) and just two (let’s say, the second and third) qubits are affected by the local channels $A$ and $B$ simultaneously, the final state of the system is given by a rank 8 density matrix $\rho \equiv [1 \otimes A \otimes B] |\text{GHZ}\rangle \langle \text{GHZ}|$. In this case, the three bipartite concurrences can be computed in a similar way as in the previous section, following the method from section 2, and can be expressed as

$$C^{12|3} = \sqrt{f_1^2 + f_2^2},$$

$$C^{13|2} = \sqrt{f_3^2 + f_4^2},$$

$$C^{23|1} = \sqrt{f_5^2 + f_6^2},$$

where

$$f_1 = f(a_1 b_2^2 + a_2 b_3^2 + a_3 b_4^2 + a_4 b_5^2 + a_5 b_6^2 + a_6 b_7^2),$$

$$f_2 = f(a_1 b_2^2 + a_2^2 b_3^2 + a_3 b_4^2 + a_4 b_5^2 + a_5 b_6^2 + a_6 b_7^2),$$

$$f_3 = f(a_1 b_4^2 + a_2 b_5^2 + a_3 b_6^2 + a_4^2 b_7^2 + a_5 b_1^2 + a_6 b_2^2) + f(a_1 b_4^2 + a_2 b_5^2 + a_3 b_6^2 + a_4^2 b_7^2 + a_5 b_1^2 + a_6 b_2^2),$$

$$f_4 = f(a_1^2 b_2^2 + a_2 b_3^2 + a_3 b_4^2 + a_4 b_5^2 + a_5 b_6^2 + a_6 b_7^2),$$

$$f_5 = f(a_1 b_4^2 + a_2^2 b_5^2 + a_3 b_6^2 + a_4 b_7^2 + a_5 b_1^2 + a_6 b_2^2),$$

$$f_6 = f(a_1 b_4^2 + a_2^2 b_5^2 + a_3 b_6^2 + a_4 b_7^2 + a_5 b_1^2 + a_6 b_2^2),$$

where $i, j = 2, 3$. Symbolically, equation (16) can be written as

$$\tau_5^2([1 \otimes A \otimes B]|\psi\rangle \langle \psi|) = \frac{1}{4} (A^2 + B^2 + A^2 B^2),$$

where $A \equiv C^{12|3}([1 \otimes 1 \otimes A]|\psi\rangle \langle \psi|)$ and $B \equiv C^{13|2}([1 \otimes 1 \otimes B]|\psi\rangle \langle \psi|)$ and $\rho = |\text{GHZ}\rangle \langle \text{GHZ}|$. In other words, entanglement dynamics of the GHZ state in two-sided noisy channels, when the channels $A$ and $B$ are the bit-flip or the bit-phase-flip channel, is completely defined by the dynamics of the state in the single-sided channels. Although this result holds only for mentioned channels, it can be extended to the case of an arbitrary pure three-qubit state $|\psi\rangle \langle \psi|$ which is locally equivalent to the GHZ state. This extension is based on the evolution equation for bipartite concurrence [11], which has a very similar sense and structure to the evolution equation (12) for the lower bound. For the bipartite concurrence $C^{12|3}$, for example, the evolution equation is given by

$$C^{12|3}([1 \otimes 1 \otimes A]|\psi\rangle \langle \psi|) = C^{12|3}([1 \otimes 1 \otimes A]|\text{GHZ}\rangle \langle \text{GHZ}|) C^{12|3}[|\psi\rangle \langle \psi|].$$
It is important to note that equations (13)–(15) are computed for a rank 8 density matrix and, therefore, the lower bound $\tau_3$ obtained with the help of these equations may differ from the convex roof (3) for the concurrence significantly. However, due to the assumption that the channels $A$ and $B$ are the bit-flip or the bit-phase-flip channels, the state $[1 \otimes A \otimes B] (\text{GHZ})$ (GHZ) has rank 4. Therefore, it can be argued that equation (17), formulated for the lower bound $\tau_3$, remains valid if the convex roof (3) is substituted in its left-hand side (lhs).

### 3.3. Three-sided channels

In the previous subsection, we have analyzed the case when just two qubits of the three-qubit system are subjected to local noisy channels. Similar analysis can be made when all three qubits are affected by local channels $A$, $B$, and $C$ simultaneously. If the system is initially prepared in the GHZ state (1), the final state density matrix, derived from equation (7), has rank 8. The analytically computed concurrences $C^{123}$, $C^{132}$ and $C^{231}$ for this density matrix have a very complex structure and, therefore, we do not display them here. There are significant simplifications in the structure of the concurrences and the lower bound if all $A$, $B$ and $C$ are the bit-flip channels or if two of these channels are bit-phase-flip and the remaining one is a bit-flip. In these two cases, the squared lower bound $\tau_3^2$ can be written as

$$\tau_3^2 ([A \otimes B \otimes C] \rho) = \frac{1}{4} (A^2 B^2 + B^2 C^2 + C^2 A^2), \quad (19)$$

where $A \equiv C^{123} ([1 \otimes 1 \otimes A] |\psi\rangle \langle \psi|)$, $B \equiv C^{132} ([1 \otimes 1 \otimes B] |\psi\rangle \langle \psi|)$, $C \equiv C^{231} ([1 \otimes 1 \otimes C] |\psi\rangle \langle \psi|)$ and $\rho = |\text{GHZ}\rangle \langle \text{GHZ}|$. The entanglement dynamics of the GHZ state in the three-sided noisy channels is given just by the dynamics of the state in the single-sided channels. As in the previous section, this result can be generalized to the case of an arbitrary three-qubit state (which is locally equivalent to the GHZ state) due to the evolution equation for bipartite concurrence [11]. Moreover, the fact that the state $[A \otimes B \otimes C] (\text{GHZ})$ (GHZ) has rank 4, for mentioned channels, suggests to substitute the convex roof (3) in the lhs of equation (19) without loss of any information about the entanglement dynamics.

### 4. Summary

Our analysis in the previous section suggests that, for the first time, the entanglement dynamics of a pure three-qubit GHZ-type entangled state $|\psi\rangle$ in some local many-sided noisy channels can be completely described by evolution of the entangled state in single-sided channels. More specifically, if just two qubits are subjected to the local channels $A$ and $B$, so that the channel $[1 \otimes A \otimes B]$ is a combination of just bit-flip and/or bit-phase-flip channels, the lower bound for three-qubit concurrence $\tau_3 ([1 \otimes A \otimes B] |\psi\rangle \langle \psi|)$ can be expressed without loss of generality through the bipartite concurrences $C^{123} ([1 \otimes 1 \otimes A] |\psi\rangle \langle \psi|)$ and $C^{123} ([1 \otimes 1 \otimes B] |\psi\rangle \langle \psi|)$ by equation (17). If all three qubits undergo the action of the local noisy channels $A$, $B$ and $C$, which are either bit-flip channels or two of these channels are the bit-phase-flip and the remaining one is the bit-flip, the lower bound $\tau_3 ([A \otimes B \otimes C] |\psi\rangle \langle \psi|)$ is again given just by the bipartite concurrences as is displayed in equation (19).

It is important to note that there are only two nonvanishing eigenvalues in expression (9) for the bipartite concurrence $C^{123} ([1 \otimes 1 \otimes A] |\psi\rangle \langle \psi|)$, if the channel $A$ is the bit-flip or the bit-phase-flip channel. Such a bipartite concurrence can be directly measured [21]. Equations (17) and (19) allow one to access the complex many-sided entanglement dynamics just by measuring the bipartite concurrence in the simplest case, when just one qubit is affected by a local noisy channel.

The results of this work may find their application in quantum communication. In the quantum communication protocols by Karlsson and Bourennane [5], for instance, the three-qubit GHZ state (1) is used to establish secure communication between two or three partners. The efficiency of the communication depends on the value of entanglement that is preserved in the three-qubit system after the qubits are distributed between the partners through local (noisy) channels. Equations (17) and (19) provide us with a very simple description of the entanglement dynamics of the GHZ state in local channels. Moreover, these equations simplify significantly testing of the communication lines. Instead of submitting two or three qubits to noisy channels during the test as is required by the protocol, one can always submit just one qubit of the three-qubit system to one of the channels and, nevertheless (by repeating this procedure for each of the channels), obtain full information about the entanglement dynamics in more complex situations, when several qubits are affected by noise simultaneously. In addition, our results may also be suited in investigation of the dynamics of entangled ions in cavity QED systems [22], but if only the ions interact locally with their environments.

### Acknowledgment

A part of this work has been performed during my stay at Heidelberg University. Partial support by the Heidelberg Graduate School for Fundamental Physics is acknowledged.

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