Exact results for generalized extremal problems forbidding an even cycle

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Abstract

We determine the maximum number of copies of $K_{s,s}$ in a $C_{2s+2}$-free $n$-vertex graph for all integers $s \geq 2$ and sufficiently large $n$. Moreover, for $s \in \{2,3\}$ and any integer $n$ we obtain the maximum number of cycles of length $2s$ in an $n$-vertex $C_{2s+2}$-free bipartite graph.

1 Introduction

Notation. For a graph $G$, the vertex and edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. Furthermore, we let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. The maximum degree of a vertex in $G$ is denoted by $\Delta(G)$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced on the vertex set $S$. For a vertex $v$, we denote by $G - v$ the graph $G[V(G) \setminus \{v\}]$. Moreover, for a subgraph $H$ of $G$ we write $G - H$ for $G[V(G) \setminus V(H)]$. For $v \in V(G)$, we denote by $N(v)$ the neighborhood of $v$ and we write $N_H(v)$ for $N(v) \cap V(H)$. For a vertex $v$, the set $N(v) \cup \{v\}$ is denoted by $N[v]$. For a set $S$, we denote the set of $k$-element subsets of $S$ by $\binom{S}{k}$. By a copy of $H$ in a graph $G$ we mean a subgraph of $G$ isomorphic to $H$. For graphs $G$ and $H$, we denote by $G + H$ the join of $G$ and $H$, that is the graph obtained by connecting each pair of vertices between a vertex disjoint copy of $G$ and $H$.

The path, cycle and complete graph on $k$ vertices are denoted by $P_k$, $C_k$ and $K_k$, respectively. Let $S_k$ be a $k$-vertex complete bipartite graph with a color class of size 1. We sometimes refer to a $k$-vertex path by a sequence of vertices $v_1v_2 \ldots v_k$, where $v_iv_{i+1}$ is an edge for all $1 \leq i \leq k-1$, and we sometimes refer to a $k$-vertex cycle by a sequence of the form $v_1v_2 \ldots v_kv_1$ where $v_iv_k$ and $v_iv_{i+1}$ are edges for all $1 \leq i \leq k-1$. We denote by $K_{a,b}$ the complete bipartite graph with parts of size $a$ and $b$, and we denote by $K_{a,b}^+$ the join of a clique of size $a$ and an independent set of size $b$. A star is a tree with at most one vertex of degree at least two; this vertex is referred to as the central vertex of the star. A forest in which every connected component is a star is called a star forest. A double star is a tree consisting of two adjacent vertices $u$ and $v$ such that every vertex in $(N(u) \setminus \{v\}) \cup (N(v) \setminus \{u\})$ is a leaf; such vertices $u$ and $v$ are referred to as central vertices of the double star. For any two graphs $H$ and $G$, we denote the number of copies of $H$ in $G$ by $H(G)$. For given graphs $F$ and $G$, we say that $G$ is $F$-free if it does not contain $F$ as a subgraph (not necessarily induced). The maximum of $H(G)$ across $n$-vertex $F$-free graphs $G$ is denoted by $ex(n, H, F)$. The maximum of $H(G)$ across bipartite $n$-vertex $F$-free graphs $G$ is denoted by $ex_{bip}(n, H, F)$.

Background. Generalized extremal problems have a long history dating back to a result of Zykov [16] (and later independently Erdős [3]), who determined for all $s$ and $t$ the value of
ex(n, Ks, K1), extending the classical theorem of Turán [15]. Following this initial result, there has been extensive work determining the value of ex(n, H, F) for various pairs of graphs H and F. An important early result in this direction is a result of Győri, Pach and Simonovits [12] which determined the value of ex(n, H, K3) for all bipartite graphs H containing a matching on all but at most one of its vertices. The generalized extremal function ex(n, H, F) for arbitrary pairs of graphs was introduced by Alon and Shikhelman [1]. In addition to the function ex(n, H, F), we will consider the analogous function under the assumption that the ground graph is bipartite, which we denote by exbip(n, H, F).

In the present paper we will be interested in the setting when the forbidden graph is an even cycle. Determining ex(n, C5, C3) was a long standing open problem of Erdős [4]. A construction is obtained by taking a blow-up of C5 consisting of almost equal classes. This problem was finally settled by Grzesik [9] and independently by Hatami, Hladký, Král, Norine and Razborov [14]. This result has recently been extended Grzesik and Kielak [10] who determined ex(n, C2k+1, C2k−1) asymptotically for all k > 2. The dual problem of determining ex(n, C3, C5) was introduced by Bollobás and Győri [2] who proved an upper bound, and this upper bound was subsequently improved in [1], [6] and [5]. Bounds on ex(n, C3, C2k+1) were obtained by Győri and Li [11]. An exact result for ex(n, C5, C7) was obtained by Górski and Grzesik [13].

In [7] an asymptotic result was obtained for ex(n, C4, C6) and exbip(n, C6, C8). We determine the exact value of exbip(n, C4, C6), exbip(n, C6, C8) for all n, and the exact value of ex(n, C4, C6) for sufficiently large n in the following three theorems. Moreover, we determine the structure of the extremal graphs in each case.

**Theorem 1.** For all positive integers n, we have

\[ \text{exbip}(n, C_4, C_6) = \binom{n-2}{2}, \]

and equality holds only for K2,n−2.

**Theorem 2.** For all positive integers n, we have

\[ \text{exbip}(n, C_6, C_8) = 6 \binom{n-3}{3}, \]

and equality holds only for K3,n−3.

**Theorem 3.** For n > 3(\binom{31}{4}), we have

\[ \text{ex}(n, C_4, C_6) = \binom{n-2}{2} + 2, \]

and equality holds only for a graph obtained from K2,n−2 by adding a single edge in the both independent sets.

Finally, we turn our attention to exact results about the generalized extremal numbers of complete bipartite graphs Ks,s in graphs without a copy of C2s+2. We prove the following exact result.

**Theorem 4.** For integers n and s such that s \geq 3 and n \geq 3s + 1 + \binom{2s+1}{s} \frac{2s+1}{s+1}, we have

\[ \text{ex}(n, K_{s,s}, C_{2s+2}) = \binom{n-s}{s}. \]

Equality holds only for graphs containing Ks,n−s and contained in the graph Ks + H, where H is an (n − s)-vertex graph with exactly one edge.

The proofs of Theorems 1 and 2 are given in Section 2, and the proofs of Theorems 3 and 4 are given in Section 3.
2 Proofs of theorems about bipartite graphs

Proof of Theorem 1. Consider the graph $K_{2,n-2}$. It contains no cycle of length six and $(n-2)$ cycles of length four. Hence we have $\text{ex}_{bip}(n, C_4, C_6) \geq (n-2)$.

Let $G$ be an $n$-vertex $C_6$-free bipartite graph. Let $C$ be a cycle of length four in $G$ such that $C = v_1v_2v_3v_4v_1$. Then since $G$ is a bipartite $C_6$-free graph, either there is no cycle of length four containing the vertices $v_1, v_3$ distinct from $C$ or there is no cycle of length four containing the vertices $v_2, v_4$ distinct from $C$. Hence each $C_4$ in $G$ contains a pair of vertices from the same color class of $G$ which is unique for that cycle.

We define an auxiliary graph $G'$ on the same set of vertices as $G$ by taking the edge $uv$ in $G'$ if and only if there is exactly one $C_4$ in $G$ with opposite vertices $u$ and $v$. Note that since $G$ is bipartite, the vertices $u$ and $v$ belong to the same color class of $G$. The number of copies of $C_4$ in $G'$ is at most the number of edges in $G'$ since each copy of $C_4$ in $G$ contains a pair of opposite vertices which is unique for the cycle. If the color classes of $G$ have sizes $a$ and $n-a$ such that $2 \leq a \leq \frac{n}{2}$, then the number of edges of $G'$ is at most $\binom{a}{2} + \binom{n-a}{2}$. Note that if $a > 2$, then $\binom{a}{2} + \binom{n-a}{2} < \binom{n}{2}$ when $n > 6$. Therefore if $n > 6$ and $G$ contains at least $\binom{n}{2}$ copies of $C_4$, then it is a bipartite graph with color classes of sizes $n-2$ and $2$, and it follows that the number of copies of $C_4$ in $G$ is at most $\binom{n}{2}$ with equality only when $G$ is isomorphic to $K_{2,n-2}$. For $n \leq 6$ it is straightforward to verify that the theorem holds.

Proof of Theorem 2. The graph $K_{3,n-3}$ contains no cycle of length eight and contains $6\binom{n-3}{3}$ cycles of length six. Hence we have $\text{ex}_{bip}(n, C_6, C_8) \geq 6\binom{n-3}{3}$.

Let $G$ be an $n$-vertex bipartite graph with color classes $V_1$ and $V_2$, containing no cycle of length eight and containing the maximum number of cycles of length six. Note that we may assume that $n \geq 8$ since the theorem is simple to verify for $n < 8$. In what follows we are going to find a function $\phi$ from the set of copies of $C_6$ in $G$ to unordered monochromatic triples of the vertices of $G$. Then we are going to show that the inverse image of each triple has size at most six. This implies that

$$C_6(G) \leq 6\left(\frac{|V_1|}{3}\right) + 6\left(\frac{|V_2|}{3}\right).$$

If $|V_1|, |V_2| > 3$, then we have $C_6(G) \leq 6\binom{|V_1|}{3} + 6\binom{|V_2|}{3} < 6\binom{n-3}{3}$ since $n \geq 8$, a contradiction. We may therefore assume $|V_1| = 3$, and it follows that $G$ is isomorphic to $K_{3,n-3}$ since it maximizes the number of cycles of length six. Hence we are done if such a function $\phi$ exists.

We are going to define the function $\phi$ recursively. Let $C$ be a subgraph of $G$ be isomorphic to $C_6$, such that $C = v_1v_2v_3v_4v_5v_1$ with color classes $B = \{v_1, v_3, v_5\}$ and $R = \{v_2, v_4, v_6\}$, and assume that $\phi(C)$ is not yet defined for $C$. If all of the cycles of length six containing $B$ as a color class contain $R$ as the other color class, then we take $\phi(C) := B$; if all of the cycles of length six containing $R$ as a color class contain $B$ as the other color class, then we take $\phi(C) := B$. Otherwise, without loss of generality, we may assume there is a vertex $v'_2$ incident to $v_1$ and $v_3$, and there is a vertex $v'_4$ incident to $v_4$ and $v_6$.

Consider a subgraph $G'$ of $G$ containing all cycles of length six with opposite vertices $v_1$ and $v_4$ such that each edge of $G'$ is in a cycle of length six with opposite vertices $v_1$ and $v_4$. Let the neighborhood of $v_1$ in $G'$ excluding $v_4$ (if they are incident) be denoted by $U$, and let the neighborhood of $v_4$ in $G'$ excluding $v_1$ be denoted by $W$. Note that $|U|, |W| \geq 3$ since $\{v_2, v'_2, v_6\} \subseteq U$ and $\{v_3, v_5, v'_4\} \subseteq W$. Since $G'$ is $C_8$-free, it is straightforward to deduce that the structure of $G'$ is one of the following:

- If $v_3v_6$ is an edge, then $v_1v_4$ is also an edge since $G'$ is $C_8$-free and every edge is in a copy of $C_6$. Then the induced graph $G'[U \cup W]$ is a double star with central vertices $v_3$ and $v_6$. Note that every cycle of length six in $G'$ contains a vertex from $U \setminus \{v_6\}$ and a vertex from $W \setminus \{v_3\}$. Even more, for each pair consisting of a vertex from $U \setminus \{v_6\}$ and a vertex from...
$W \setminus \{v_3\}$, there are exactly two cycles of length six in $G'$. Hence the number of cycles of length six in $G'$ is $2(|U| - 1)(|W| - 1)$.

- Otherwise $v_3v_6$ is not an edge of $G'$, and the graph $G'[U \cup W]$ is a star forest. In particular, it contains at least two stars, one with central vertex $v_3$ and the other with central vertex $v_6$. Note that every $C_6$ in $G'$ contains $v_3$ and $v_6$ and at least two leaves of $G'[U \cup W]$. Each pair of the leaves is contained in at most one cycle of length six in $G'$. Hence the number of cycles of length six in $G'$ is at most $\binom{|U|+|W|-2}{2}$.

Therefore we have the following bound on the number of cycles of length six in $G'$:

$$C_6(G') \leq \max \left( \left( \frac{|U| + |W| - 2}{2} \right), 2(|U| - 1)(|W| - 1) \right).$$

Let us consider the set of triples

$$X_{v_1,v_4} := \{\{v_1, w_1, w_2\} : \{w_1, w_2\} \subseteq W \} \cup \{\{v_4, u_1, u_2\} : \{u_1, u_2\} \subseteq U\}.$$ 

Note that $|X_{v_1,v_4}| = \binom{|U|}{2} + \binom{|W|}{2}$, hence $2|X_{v_1,v_4}| > C_6(G')$.

**Observation 5.** There is no vertex in $V(G - G')$ incident to two vertices of $G'$ since $G$ is $C_8$-free. Hence the only vertices adjacent to more than one vertex from $\{v_1\} \cup W$ are $v_4$ and $v_6$.

There is no path of length three between any two vertices of $G'$ containing a vertex outside of $G'$ since $G$ is $C_8$-free and there is no vertex in $V(G - G')$ incident to two vertices of $G'$.

By the previous two claims, $v_4$ and $w \in W$ do not appear as opposite vertices in any copy of $C_6$ which is not contained in $G'$. Hence if $X_{v_1,v_4} \cap X_u,v \neq \emptyset$ for some opposite pair of vertices $u,v$, then every six-cycle with opposite vertices $u$ and $v$ are six cycles of $G'$. Hence $G'$ doesn't contain any 6-cycle for which $\phi$ is already defined.

Let us denote the set of 6-cycles in $G'$ by $C$. We define $\phi$ on $C$ in the following way: $\phi(\mathcal{C}) \subseteq X_{v_1,v_4}$ and $|\phi^{-1}(x) \cap \mathcal{C}| \leq 2$ for every $x \in X_{v_1,v_4}$.

We repeat the procedure iteratively for all cycles of length six in $G$ for which $\phi$ is not yet defined. By Observation 5, for each triple $\{v_1, w_1, w_2\} \in X_{v_1,v_4}$ we have $|\phi^{-1}(\{v_1, w_1, w_2\})| \leq 2 < 6$. \qed

## 3 Proofs of theorems about general graphs

### 3.1 Maximizing cycles of length four

First we prove an essential lemma which we will need in the proof of Theorem 3.

**Lemma 6.** Let $G$ be a $K_{2,s+1}$-free and $C_6$-free graph for some integer $s \geq 2$, and let $v \in V(G)$. Then the number of copies of $C_4$ incident to $v$ is at most

$$\max \left( 3|N_G(v)|, \frac{(s-1)(s+2)}{2(s+1)} |N_G(v)| \right).$$

If $G$ is also $K_5$-free, then the number of copies of $C_4$ incident to $v$ is at most

$$\max \left( 2|N_G(v)|, \frac{(s-1)(s+2)}{2(s+1)} |N_G(v)| \right).$$

**Proof.** Let $G'$ be the subgraph of $G$ spanned by the edges which are in a copy of $C_4$ incident to $v$. By the linearity of the desired upper bound in Lemma 6, we may assume that $G'$ is 2-connected. Let us denote the degree of $v$ in $G'$ by $x$.

First we consider the case when $v(G') = x + 1$. Observe that $G' - v$ is a connected graph since $G'$ is 2-connected, and moreover we have that $G'[N_G(v)] = G' - v$. Note that the number
of paths of length two in $G' - v$ is equal to the number of copies of $C_4$ incident to $v$ in $G$. The graph $G' - v$ is $P_3$-free since $G'$ is $C_6$-free. Even more, we have $\Delta(G' - v) \leq s$ since $G$ is $K_{2,s+1}$-free. If $G' - v$ is a tree, then it is either a star or a double star, and therefore the number of copies of $C_4$ incident to $v$ in $G$ is at most \( \frac{(s-1)x}{2(s+1)} \). If $G' - v$ contains a cycle of length four, then the number of copies of $C_4$ incident to $v$ is at most $12 \leq 3x$. Note that if $G$ is $K_5$-free (and $G' - v$ contains a $C_4$), then the number of copies of $C_4$ incident to $v$ is at most $8 \leq 2x$. If $G' - v$ contains a cycle and no 4-cycle, then $G' - v$ is a graph obtained from a triangle by adding $x - 3$ pendant edges to one of its vertices. In this case we have at most \( \left( \frac{s-1}{2} \right) + 2 \) cycles of length 4 incident to $v$. Note that $x - 1 \leq s$, and a simple calculation shows that the required upper bound in Lemma 6 holds.

From here we will assume that $v(G') > x + 1$. Hence $V(G') \setminus N_{G'}[v]$ is nonempty. Note that $G'[V(G') \setminus N_{G'}[v]]$ is an independent set, and every vertex from $V(G') \setminus N_{G'}[v]$ has at least two neighbors in $N_{G'}[v]$. Even more, if $|V(G') \setminus N_{G'}[v]| = 1$, then for all vertices $u, u' \in V(G') \setminus N_{G'}[v]$ we have $N_{G'}(u) \neq N_{G'}(u')$ and $|N_{G'}(u)| = 2$ since $G'$ is a 2-connected $C_6$-free graph.

If $G'[N_{G'}(v)]$ is connected, then $G'[N_{G'}(v)]$ is either a star or a triangle. It follows that the number of copies of $C_4$ incident to $v$ in $G'$ is at most

\[
\max \left( 2x, \frac{(s-1)(s+2)}{2(s+1)}x \right).
\]

Note that if $G'[N_{G'}(v)]$ is a star of $s + 1$ vertices and $G'[V(G') \setminus N_{G'}[v]]$ is an independent set of size $s - 1$ incident to the same two vertices in $G'[N_{G'}(v)]$ inducing an edge, then there are exactly \( \frac{(s-1)(s+2)x}{2(s+1)} \) copies of $C_4$ incident to $v$.

If $N_{G'}(v)$ is not connected, then $G'[N_{G'}(v)]$ consists of isolated vertices and a star. If $G'[V(G') \setminus N_{G'}[v]]$ consists of just a vertex, denoted by $u$, then in $G'$ all of the $y$ isolated vertices are incident to $u$, therefore $y < s$ and $x - y - 1 \leq s$. The star $S$ in $G'[N_{G'}(v)]$ is isomorphic to $S_{x-y}$. Since every edge of $G'$ is in a copy of $C_4$ incident to $v$, the star $S$ is not an edge. The only vertex of $S$ incident to $u$ is the central vertex of the star. Then the number of copies of $C_4$ incident to $v$ is \( \left( \frac{y+1}{2} \right) + \left( \frac{y-1}{2} \right) \leq \frac{(s-1)x}{2(s+1)} \leq \max \left( 2x, \frac{(s-1)(s+2)}{2(s+1)}x \right) \) since $x - y - 2 \leq s - 1$ and $y \leq s - 1$. If $G'[V(G') \setminus N_{G'}[v]]$ contains at least two vertices, then each of the vertices has at least two neighbors in $N_{G'}(v)$. Hence $G'[N_{G'}(v)]$ contains two connected components since $G'$ is 2-connected. Therefore $G'[N_{G'}(v)]$ consists of an isolated vertex $w$ and a star isomorphic to $S_{x-1}$ with central vertex $w'$. Note that $S_{x-1}$ is not an edge and every vertex of $G'[V(G') \setminus N_{G'}[v]]$ is incident to $w$ and $w'$. Then the number of copies of $C_4$ is at most $s - 1 + \left( \frac{s-2}{2} \right) \leq \frac{(s-1)x}{2} \leq \max \left( 2x, \frac{(s-1)(s+2)}{2(s+1)}x \right)$. \[\]

**Proof of Theorem 3.** Let $G$ be an $n$-vertex graph with no cycle of length six. Let $K$ be a subgraph of $G$ isomorphic to $K_{2,s}$, where $s$ is the maximum integer for which $G$ contains a copy of $K_{2,s}$. Let the color classes of $K$ of size 2 and $s$ be $A$ and $B$, respectively.

**Observation 7.** For $s > 2$, let $v$ and $u$ be distinct vertices of $V(G - K)$. Then the following properties hold.

1. The vertex $v$ is not adjacent to both vertices in $A$ by the maximality of $s$.
2. Since $G$ is $C_6$-free, $v$ is not adjacent to more than one vertex from $B$.
3. By Items 1 and 2, we have that $v$ and $u$ are adjacent to at most 2 vertices of $K$. If $v$ and $u$ both have two neighbors in $V(K)$, then $N(v) \cap V(K) = N(u) \cap V(K)$ since $G$ is $C_6$-free.
4. For $s > 3$, the graph $G[B]$ contains at least one edge since $G$ is $C_6$-free.
In the following proof we need to classify the copies of \( C_4 \) depending how they meet with \( K \). For simplicity, we introduce the following technical definitions. If there exists a vertex \( v \in V(G - K) \) with two neighbors \( x \) and \( y \) in \( V(K) \), then we call \( xy \) a red edge. Note that there is at most one red edge in \( G[V(K)] \) by Item 3 of Observation 7. We refer to any edge in \( G[A] \) or \( G[B] \) as a blue edge. Note that there are at most two blue edges in \( G[V(K)] \) for \( s > 3 \) by Item 4 of Observation 7.

Let \( C \) be a subgraph of \( G \) isomorphic to \( C_4 \).

- We say \( C \) is of Type 1 if \( |V(K) \cap V(C)| = 1 \).
- We say \( C \) is of Type 2 if \( |V(K) \cap V(C)| = 2 \). We will further divide the class of Type 2 copies of \( C_4 \) into two subclasses. Each copy of \( C_4 \) using a blue edge of \( G[V(K)] \) will be referred to as a blue Type 2 \( C_4 \), and each copy of \( C_4 \) with two opposite vertices belonging to \( V(K) \) (thus, inducing a red edge in \( K \) as its diagonal) will be referred to as a red Type 2 \( C_4 \). Note that there is no \( C_4 \) of Type 2 which is neither blue nor red.
- We say \( C \) is of Type 3 if \( |V(K) \cap V(C)| = 3 \). Note that such a copy of \( C_4 \) is incident to a blue edge, and it induces a red edge of \( K \) as its diagonal.
- We say \( C \) is of Type 4 if \( |V(K) \cap V(C)| = 4 \).

**Observation 8.** It is important to note that if there is a blue Type 2 \( C_4 \), then \( K \) contains exactly one blue edge since \( G \) is \( C_6 \)-free. Thus there are at most \( s - 1 \) blue Type 2 \( C_4 \)'s since \( G \) is \( C_5 \)- and \( K_{2,s+1} \)-free. Even more, if there is a blue edge and a red edge, then the blue edge is in the color class \( A \). All Type 3 \( C_4 \)'s share the same blue edge, hence they share the same three vertices of \( K \). Therefore the number of Type 3 \( C_4 \)'s is at most \( s - 1 \).

In the following we show that for all \( n \geq 31 \) either we have \( C_4(G) \leq \binom{n-2}{2} + 2 \), and equality holds if and only if \( G \) is isomorphic to \( K_{2,n-2} \) or \( C_4(G) = \text{ex}(n-1, C_4, C_6) + n - 3 \). For a vertex \( v \) and for a copy \( K \) of \( K_{2,s} \) with \( v \in V(K) \), the collection of copies of \( C_4 \) incident to \( v \) in \( G \) can be partitioned into five sets: Type 4, Type 3, red Type 2, blue Type 2 and Type 1 \( C_4 \)'s. For convenience in the following we will bound the number of copies of \( C_4 \) incident to a given vertex by the sum of five terms where each term represents an upper bound on the number of \( C_4 \)'s of Type 4, Type 3, red Type 2, blue Type 2 and Type 1 in this given order.

If \( s = 2 \), then for each pair of vertices there is at most one \( C_4 \) containing this pair as opposite vertices. Therefore we have \( C_4(G) \leq \binom{2}{2} = \binom{n-2}{2} + 2 \) for \( n > 5 \).

If \( s = 3 \) and \( G \) contains a \( K_5 \), then without loss of generality we may assume \( G[V(K)] \) is isomorphic to \( K_5 \). Then every \( C_4 \) incident to a vertex of \( K \) is either of Type 4 or Type 1 since \( G \) is \( C_6 \)-free. Note that every vertex of \( G - K \) is adjacent to at most one vertex of \( K \), hence by the pigeonhole principle there is a vertex \( v \) of \( K \) with at most \( \frac{2n}{5} \) neighbors in \( G - K \). Hence by applying Lemma 6 for the vertex \( v \) in \( G[V(G - K) \cup \{v\}] \), we have that the number of copies of \( C_4 \) incident to \( v \) in \( G \) is at most

\[
12 + 0 + 0 + 0 + 3 \cdot \frac{n - 5}{5} < n - 3,
\]

and the last inequality holds for \( n > 30 \).

If \( s = 3 \) and \( G \) is \( K_5 \)-free, then there are at least three vertices of \( K \) not contained in any blue Type 2 \( C_4 \) by Observation 8. Hence by the pigeonhole principle, there is a vertex \( v \) of \( K \) not contained in a blue Type 2 \( C_4 \) with at most \( \frac{2n}{5} \) neighbors in \( G - K \). Thus by applying Lemma 6 and Observation 8 for the vertex \( v \) in \( G[V(G - K) \cup \{v\}] \), we have that the number of copies of \( C_4 \) incident to \( v \) is at most

\[
12 + 2 + 1 + 0 + 2 \cdot \frac{n - 5}{5} < n - 3,
\]
and the last inequality holds for \( n \geq 27 \).

If \( s \geq 4 \) and there is a red edge, then there is no blue edge incident to any of the vertices of \( B \). Let us fix a vertex \( v \) in \( B \) not incident to the red edge and with a neighborhood of the smallest size in \( V(G - K) \). Then by the pigeonhole principle, we have that \(|N_G(v) \cap V(G - K)| \leq \frac{n - s - 2}{s - 1}\) since there are \( n - s - 2 \) vertices in \( G - K \) and at least one is incident to the red edge. Since there are no red or blue edges incident to the vertex \( v \), there are no Type 2 and Type 3 \( C_4 \)'s incident to \( v \). Hence we have that the number of copies of \( C_4 \) incident to \( v \) is at most

\[
(s - 1) + 0 + 0 + 0 + \max \left( 3 \cdot \frac{n - s - 3}{s - 1}, \frac{(s - 1)(s + 2)}{2(s + 1)} \cdot \frac{n - s - 3}{s - 1} \right) < n - 3
\]

for all \( n > 6 \).

Assume \( 4 \leq s < n - 2 \) and there is no red edge. Let us fix a vertex \( v \) in \( B \) with a neighborhood of the smallest size in \( V(G - K) \). Then by the pigeonhole principle, we have \(|N_G(v) \cap V(G - K)| \leq \frac{n - s - 2}{s - 1}\). Thus by applying Lemma 6 and Observation 8 for the vertex \( v \) in \( G[V(G - K) \cup \{v\}] \), we have that the number of copies of \( C_4 \) incident to \( v \) is at most

\[
(s - 1) + 0 + 0 + (s - 1) + \max \left( 3 \cdot \frac{n - s - 2}{s}, \frac{(s - 1)(s + 2)}{2(s + 1)} \cdot \frac{n - s - 2}{s} \right) < n - 3
\]

for all \( n > 18 \).

If \( \frac{n}{3} \leq s < n - 2 \) and there is no red edge in \( K \), then there is a vertex \( v \) in \( B \) not incident to a blue \( C_4 \) such that it has at most one neighbor in \( V(G - K) \). Therefore the number of \( C_4 \)'s incident to \( v \) is at most \((s - 1) + 0 + 0 + 0 + 0 < n - 3\).

If \( s = n - 2 \), then \( G \) contains a copy of \( K_{2,n-2} \). In this case we have \( C_4(G) \leq \binom{n - 2}{2} + 2 \), and equality holds only for a graph obtained from a graph isomorphic to \( K_{2,n-2} \) by adding an edge in each independent set.

For all \( n > 3 \binom{n}{4} \), we have either \( C_4(G) \leq \binom{n - 2}{2} + 2 \) and equality holds only for a graph obtained from \( K_{2,n-2} \) by adding an edge in each independent set, or

\[
C_4(G) \leq 3 \left( \frac{31}{4} \right) + \sum_{i=1}^{n} (i - 4) < \binom{n - 2}{2}.
\]

### 3.2 Maximizing complete bipartite graphs

**Proof of Theorem 4.** Since \( K_{s,s} \) contains \( \binom{n - s}{s} \) copies of \( K_{s,s} \), we have \( \text{ex}(n, K_{s,s}, C_{2s+2}) \geq \binom{n - s}{s} \). In the following we prove a matching upper bound for all integers \( n \) and \( s \) such that \( s \geq 4 \) and \( n \geq 3s + 2 + \left( \frac{2s + 1}{s} \right) \frac{31}{4} \). Let \( G \) be an \( n \)-vertex \( C_{2s+2} \)-free graph with the maximum number of copies of \( K_{s,s} \).

Let \( K \) be a subgraph of \( G \) isomorphic to \( K_{s,s} \) with color classes \( A \) and \( B \). If there is a subgraph \( K' \) of \( G \) isomorphic to \( K_{s,s} \) with color classes \( A' \) and \( B' \) different from \( K \) such that \( A = A' \), then there exists a vertex \( b \in B' \setminus B \) incident to all vertices of \( A \). Similarly if there is a subgraph \( K'' \) of \( G \) isomorphic to \( K_{s,s} \) with color classes \( A'' \) and \( B'' \) different from \( K \) such that \( B = B'' \), then there exists a vertex \( a \in A'' \setminus A \) incident to all vertices of \( B \). Since \( G \) is \( C_{2s+2} \)-free, we have \( a = b \) and \( K \) is a contained in a subgraph of \( G \) isomorphic to \( K_{s,s} + K_1 \). Thus for each subgraph \( K \) of \( G \) isomorphic to \( K_{s,s} \), either there is a color class \( A \) of \( K \) such that the only \( K_{s,s} \) containing \( A \) as a color class is \( K \) or there is a vertex in \( G \) incident to all vertices of \( K \). Observe that the graph \( K_{s,s} + K_1 \) contains cycles of every length \( l \in \{3, 4, \ldots, 2s + 1\} \).

In the following we give a classification of all subgraphs of \( G \) isomorphic to \( K_{s,s} \).

- For each subgraph \( M \) of \( G \) such that \( M \cong K_{s,s} + K_1 \), let us label all copies of \( K_{s,s} \) which are subgraphs of \( G[V(M)] \) by \( G[V(M)] \). We call each such label a *Type 1* label.
• For each maximal subgraph \( M \) of \( G \) such that \( M \cong K_{s,t} \) for some \( t \geq s + 2 \) let us label all copies of \( K_{s,s} \) which are subgraphs of \( G[V(M)] \) by \( G[V(M)] \). We call each such label a Type 2 label.

• Let \( M \) be a maximal subgraph of \( G \) such that \( M \cong K_{s,t} \) for some \( t = s \) or \( t = s + 1 \) and \( K_{s,s} + K_1 \nsubseteq G[V(M)] \). If \( t = s + 1 \), let us label all copies of \( K_{s,s} \) which are subgraphs of \( G[V(M)] \) by \( G[V(M)] \). If \( t = s \), let \( V' \) be the set of vertices from \( V(G - M) \) incident to a vertex from both color classes of \( M \) and incident to at least two vertices from some color class of \( M \). Note that \( |V'| \leq 1 \) since \( G \) is \( C_{2s+2}\)-free. We label each copy of \( K_{s,s} \) which is a subgraph of \( G[V(M) \cup V'] \) by \( G[V(M) \cup V'] \), and we call each such label a Type 3 label.

It is easy to see that each copy of \( K_{s,s} \) is assigned at least one label. Even more, we will now show that each copy of \( K_{s,s} \) in \( G \) receives exactly one label. If a copy \( K \) of \( K_{s,s} \) has a Type 1 label \( G[V(M)] \), then it is immediate from the definition of the labels that it has no Type 3 label. Since \( G \) is \( C_{2s+2}\)-free, there is no vertex from \( V(G - M) \) incident to at least two vertices of \( K \). Therefore \( K \) has no Type 1 label distinct from \( G[V(M)] \) and no Type 2 label either. By the definition of Type 2 and Type 3 labels, it is impossible to have both types of labels for a copy of \( K_{s,s} \). If a copy of \( K_{s,s} \) received a Type 2 label, then it contains a color class \( A \) such that the only \( K_{s,s} \) containing \( A \) as a color class is this \( K_{s,s} \), hence no \( K_{s,s} \) receives two different Type 2 labels. Finally, no copy of \( K_{s,s} \) receives two distinct Type 3 labels, say \( G[V(K) \cup \{v_1\}] \) and \( G[V(K) \cup \{v_2\}] \), for otherwise \( G[V(K) \cup \{v_1, v_2\}] \) contains a cycle of length \( 2s + 2 \), a contradiction.

Claim 9. Let \( K \) and \( K' \) be two copies of \( K_{s,s} \) in \( G \) with color classes \( A, B \) and \( A', B' \), respectively, such that \( K \) and \( K' \) have different labels. Then we have \( A \not\subseteq A' \cup B' \) and \( B \not\subseteq A' \cup B' \).

Proof. Suppose by way of contradiction that \( A \subseteq A' \cup B' \). First note that since the labels are distinct, neither \( A = A' \) nor \( A = B' \). Hence we may assume without loss of generality that \( |A \cap A'| \geq 2 \) and \( |A \cap B'| \geq 1 \). Thus there are at most \( 2s + 1 \) vertices in \( V(K) \cup V(K') \), otherwise there would be at least two vertices in \( B \setminus (A' \cup B') \) incident to two vertices in the class \( A' \) and to a vertex in \( B' \), yielding a cycle of length \( 2s + 2 \). Note that \( V(K) \neq V(K') \) since they have different labels. Hence we have \( |V(K) \cup V(K')| = 2s + 1 \). We denote the only vertex in \( V(K) \setminus V(K') \) by \( v' \). This vertex has at least two neighbors in the color class \( A' \), which we denote by \( a_1, a_2 \), and at least one neighbor in the color class \( B' \), which we denote by \( u \).

Since \( K' \) has a different label than \( K \) and \( v' \) has at least two neighbors in \( A \) and a neighbor in \( B \), there is a vertex \( v'' \) from \( (V(G - K') - K) \) incident to all vertices of \( A' \) or all vertices of \( B' \). Since \( G \) is \( C_{2s+2}\)-free and \( v'' \) has at least two neighbors in \( A' \), the vertex \( v'' \) is incident to all vertices of \( A' \). Even more, \( v'' \) has exactly one neighbor in \( B' \), which was previously denoted by \( u \). Observe that \( B \cap B' \) is nonempty; let us denote an arbitrary vertex from \( B \cap B' \) by \( u' \).

Since \( A' \) and \( (B' \setminus \{u\}) \cup \{v''\} \) induce a complete bipartite graph, there exists a cycle of length \( 2s \) using the edge \( u'a_1 \); let us denote this cycle by \( C \). Note that there is a path \( P = u'uv'a_1 \) of length three. By replacing the edge \( u'a_1 \) of \( C \) with \( P \), we obtain a cycle of length \( 2s + 2 \), a contradiction.

Claim 10. Let \( K \) and \( K' \) be two copies of \( K_{s,s} \) in \( G \) with color classes \( A, B \) and \( A', B' \), respectively, such that \( K \) and \( K' \) have different labels. Then we have either

\[ |V(K) \cap V(K')| \leq 1, \]

or \( |V(K) \cap V(K')| = 2 \) and the vertices of \( V(K) \cap V(K') \) are in the same color class in one of the copies of \( K_{s,s} \) and in different color classes in the other copy of \( K_{s,s} \).

Proof. First we prove \( |A \cap A'| \leq 1 \). Suppose by way of contradiction that \( \{a, a'\} \subseteq A \cap A' \), where \( a \neq a' \). Then by Claim 9, there exist vertices \( a_1 \) and \( b_1 \) such that \( a_1 \in A \setminus V(K) \) and \( b_1 \in B' \setminus V(K) \). Since \( B' \) does not contain the vertices \( a \) and \( a' \), there is a vertex \( b_2 \in B' \setminus (A \cup \{b_1\}) \).
Note that there exists a 2s-cycle $C$ in $K$ such that $a$ and $a'$ are at distance two. Even more, if $b_2 \in B$, then let $C$ be a 2s-cycle in which the vertex $b_2$ is incident to $a$ and $a'$. Then by changing the path of length two in $C$ between the vertices $a$ and $a'$ with the path $ab_1a_1b_2a'$, we obtain a cycle of length $2s + 2$, a contradiction.

In a similar way, we have that if $|A \cap A'| = 1$, then $|B \cap B'| = 0$. This implies the claim. □

Let us fix a maximal copy $M$ of $K_{s,t}$ in $G$ with color classes $D$ of size $s$ and $D'$ of size $t$. If $s = 3$, we may assume that $t > 4$, otherwise for every pair of vertices $u, v$ there are at most 20 copies of $K_{3,3}$ containing $u$ and $v$ in the same color class by Claim 10, since the number of $K_{3,3}$’s in a $K_7$ containing a fixed pair of vertices in the same color class is 20. Then

$$K_{3,3}(G) \leq \frac{20}{6} \binom{n}{2} < \binom{n-3}{3}$$

since $n$ is sufficiently large.

In what follows, we find an injective mapping $\phi$ from the set of copies of $K_{s,s}$ in $G$ to $V(G)^{/D}$. Finding such a mapping implies the desired upper bound.

Let $L_1$ be a copy of $K_{s,s} + K_1$ in $G$, if such a copy exists. Note that $G[V(L_1)]$ is a Type 1 label. If $s > 3$ and $V(L_1) \neq V(M)$, then let $A'$ be a subset of $V(L_1) \setminus M$ of size $s - 1$. Such a subset exists by Claim 10. If $s > 3$ and $V(L_1) = V(M)$, let $A'$ be a subset of $V(L_1) \setminus D$ of size $s - 1$.

Let $X$ be a set of all copies of $K_{s,s}$ in $G[V(L_1)]$. Note that $|X| \leq K_{s,s}(K_{2s+1}) = \frac{1}{2s+1} \binom{2s+1}{s}$. Let $Y$ be a subset of $V(G) \setminus (V(L_1) \cup D)$ of size $|X|$; such a subset exists since $n$ is large. We define $\phi$ on the set of copies of $K_{s,s}$ with label $G[V(L_1)]$ in the following way

$$\phi(X) := \{A' \cup \{v\} : v \in Y\}.$$

If $s = 3$, then $V(L_1) \neq V(M)$ since $t > 4$. Then by Claim 10, we have $|V(L_1) \cap D| \leq 1$ and $|V(L_1) \cap D'| \leq 1$. Let $X$ be a set of all copies of $K_{s,s}$ in $G[V(L_1)]$. Note that $|X| \leq 70$. If $|V(L_1) \cap D| = 0$, then let $Y$ be a set of size $|X|$ containing distinct triples from $\binom{V(L_1)}{3}$ and $\binom{V(L_1)}{2}$ with a vertex from $D \setminus V(L_1)$. Note that $|D \setminus V(L_1)| \geq 2$ by Claim 10. Such a set $Y$ exists since $|D \setminus V(L_1)| \geq 2$ and $|X| \leq 70 < \binom{\binom{\binom{\binom{3}{2}}{2}}{2}}{2}$. If $|V(L_1) \cap D| = 1$, then let $Y$ be a set of size $|X|$ containing distinct triples from $\binom{V(L_1)}{3}$ and $\binom{V(L_1)}{2}$ with vertices from $D \setminus V(L_1)$. Note that $|D \setminus V(L_1)| \geq 4$ by Claim 10 and $70 < \binom{\binom{\binom{3}{2}}{2}}{2}$. We define $\phi$ on the set of copies of $K_{s,s}$ with label $G[V(L_1)]$ by

$$\phi(X) := \{A' \cup \{v\} : v \in Y\}.$$

Let $L_2$ be a maximal subgraph of $G$ isomorphic to $K_{s,t'}$ for some $t' \geq s + 2$ with color classes $A$ of size $s$ and $B$ of size $t'$, if such a subgraph $L_2$ exists. Note that $G[V(L_2)]$ is a Type 2 label. Since $G$ is $C_{2s+2}$-free, each subgraph $K$ of $G[V(L_2)]$ isomorphic to $K_{s,s}$ has color classes $A$ and a subset $B'$ of $B$ of size $s$. Then by Claim 10, we have $|B \cap D| \leq 1$. If $|B \cap D| = 0$, then let $\phi(K) := B'$ and if $|B \cap D| \leq 1$, then let $\phi(K) := (B \cup \{v_{L_2}\}) \setminus D$, where $v_{L_2}$ is any fixed vertex of $D \setminus V(L_2)$.

Let $L_3$ be a subgraph of $G$ such that $G[V(L_3)]$ is a Type 3 label if such a subgraph $L_3$ exists. We define $\phi$ for the copies of $K_{s,s}$ in $G[V(L_3)]$ as we did for all copies of $K_{s,s}$ with Type 1 label. Note that if $s = 3$ and $|V(L_3) \setminus D| \leq 1$ the number of $K_{s,s}$ with such a label is at most 10, and the number of triples $\binom{V(L_3) \setminus D}{3}$ is at least 10. Therefore for the image of $\phi$ we do not need to use any vertex outside of this label. If $s = 3$ and $|V(L_3) \setminus D| = 2$ (by Claim 10 it is at most 2), then $L_3$ induces a $K_{3,3}$ since $G$ is $C_6$-free. Therefore for the image of $\phi$ we do not need to use any vertex outside of this label.

Let $K$ and $K'$ be copies of $K_{s,s}$ in $G$. Note that if $K$ and $K'$ have the same label, then we have

$$\phi(K) \neq \phi(K').$$
If \( K \) and \( K' \) have different labels and \( s > 3 \), then since each triple \( \phi(K) \) and \( \phi(K') \) contains at most one vertex outside of its label and we have \( \phi(K) \neq \phi(K') \) by Claim 10.

If \( K \) and \( K' \) have different labels and \( s = 3 \) suppose by way of contradiction that \( \phi(K) = \phi(K') = \{a, d, d'\} \). By Claim 10 we have that \( a \) is a vertex of both labels, \( d \) is a vertex of label of \( K' \) and \( d, d' \in D' \). There is a path \( dPd' \) of length four going through the vertex \( a \) in \( G[V(K) \cup V(K')] \). There is a path of length 5 in \( G[M] \) from \( d \) to \( d' \) internally disjoint from \( dPd' \) by Claim 10. These two paths constitute a cycle of length 8, a contradiction. Hence \( \phi \) is an injective function and we obtained the extremal number \( ex(n, C_6, C_8) \).

Hence \( \phi \) is an injective function and we obtained the extremal number \( ex(n, C_6, C_8) \).

Note that if there is more than one label, then for each label there is room to choose an extra set from \( \binom{V(G) \setminus D}{s} \) which will not be an image of \( \phi \) at the end of the procedure. Therefore equality holds only for graphs \( G \) containing exactly one label. Hence \( K_{s,n-s} \subseteq G \). It is easy to see that \( G \subseteq K_s + H \), where \( H \) is an \((n-3)\)-vertex graph with exactly one edge since \( G \)-is \( C_8\)-free.

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