High-energy scalarons in $R^2$ gravity as a model for Dark Matter in galaxies

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January 20, 2013

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Abstract

We show that in the framework of $R^2$ gravity and in the linearized approach it is possible to obtain spherically symmetric stationary states that can be used as a model for galaxies. Such approach could represent a solution to the Dark Matter Problem. In fact, in the model, the Ricci

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$^1$CC is partially supported by a Research Grant of The R. M. Santilli Foundation Number RMS-TH-5735A2310

$^2$HJMC is fellow of the Ceará State Foundation for the Development of Science and Technology (FUNCAP), Fortaleza, CE, Brazil
curvature generates a high energy term that can in principle be identified as the dark matter field making up the galaxy. The model can also help to have a better understanding on the theoretical basis of Einstein-Vlasov systems. Specifically, we discuss, in the linearized $R^2$ gravity, the solutions of a Klein-Gordon equation for the spacetime curvature. Such solutions describe high energy scalarons, a field that in the context of galactic dynamics can be interpreted like the no-light-emitting galactic component. That is, these particles can be figured out like wave-packets showing stationary solutions in the Einstein-Vlasov system. In such approximation, the energy of the particles can be thought of as the galactic dark matter component that guarantees the galaxy equilibrium. Thus, because of the high energy of such particles the coupling constant of the $R^2$-term in the gravitational action comes to be very small with respect to the linear term $R$. In this way, the deviation from standard General Relativity is very weak, and in principle the theory could pass the Solar System tests. As pertinent to the issue under analysis in this paper, we present an analysis on the gravitational lensing phenomena within this framework.

Although the main goal of this paper is to give a potential solution to the Dark Matter Problem within galaxies, we add a Section where we show that an important property of the Bullet Cluster can in principle be explained in the scenario introduced in this work.

To the end, we discuss the generic prospective to give rise to the Dark Matter component of most galaxies within extended gravity.

PACS numbers: 95.35.+d, 04.50.Kd.

Keywords: galactic high energy scalarons; Einstein-Vlasov system; Dark Matter.

1 Introduction

The accelerated expansion of the universe, that is today observed, suggests that cosmological dynamics is dominated by the so-called Dark Energy field which provides a large negative pressure. This is the standard picture, in which such new ingredient is considered as a source of the right hand side of the field equations. It should be some form of non-clustered non-zero vacuum energy which, together with the clustered Dark Matter, drives the global dynamics. This is the so-called “concordance model” (LCDM) which gives, in agreement with the Cosmic Microwave Background Radiation (CMBR), dim Lyman Limit Systems (LLS) and type Ia supernovae (SNeIa) data, a good framework to understand the today observed Universe. However, it presents several shortcomings as the well known ”coincidence” and ”cosmological constant” problems [1]. An alternative approach is to change the left hand side of the field equations, and check if observed cosmic dynamics can be achieved by extending general relativity [2][3][4][5][6]. In this different context, it is not required to search candidates for Dark Energy and Dark Matter, which till now have not been found. Rather, one
can only stand on the “observed” ingredients: curvature and baryon matter, to account for the observations. Considering this point of view, one can think of that gravity is not scale-invariant. Such an assumption opens a room for alternative theories to be introduced \cite{7, 8, 9}. In principle, the most popular Dark Energy and Dark Matter models can be achieved by considering $f(R)$ theories of gravity \cite{2}-\cite{9}, where $R$ is the Ricci curvature scalar. 

In this picture, even the sensitive detectors for gravitational waves like bars and interferometers (i.e. those which are currently in operation and the ones which are in a phase of planning and proposal stages) \cite{10} could, in principle, be important to confirm or rule out the physical consistency of general relativity or of any other theory of gravitation. This is because, in the context of Extended Theories of Gravity, some differences between General Relativity and the alternative theories can be pointed out as far as the linearized theory of gravity is concerned \cite{11, 12}.

In the general picture of high order theories of gravity, recently the $R^2$ theory, which was originally proposed by Starobinski \cite{13}, has been analysed in various interesting frameworks, see \cite{14, 15, 16, 17} for example. Specifically, the non-singular behaviour of this class of models is discussed in \cite{14}. In \cite{15} $R^2$ inflation is combined with the Dark Energy stage and in \cite{16} an oscillating Universe, which is well tuned with some cosmological observations is discussed. Finally, in \cite{17} the production and potential detection of gravity-waves from this particular theory has been shown.

It is also quite important to emphasize that the $R^2$ is the simplest one among the class of viable models with $R^m$ terms in addition to the Einstein-Hilbert theory. In Ref. \cite{5}, it has been shown that such models may lead to the (cosmological constant or quintessence) acceleration of the universe as well as an early time era of inflation. Moreover, they seem to pass the Solar System tests, i.e. they have the acceptable newtonian limit, no instabilities and no Brans-Dicke problem (decoupling of the scalar) in the scalar-tensor version.

In this paper it is shown that in the framework of $R^2$ gravity, in the linearized approach, it is possible to obtain spherically symmetric and stationary galaxy states which can be interpreted like an approximated solution to the Dark Matter Problem. In fact, in the proposed model the Ricci curvature scalar generates an energy term that can in principle be identified as the Dark Matter content in the galaxy. The model can also help to have a better understanding of the physics of Einstein-Vlasov systems \cite{18}.

As the observed gravitational lensing phenomena are well related to the S-type galaxies in the Hubble sequence, here we briefly discuss in Section 4 the implications of this theory for observations of bending of light by foreground galaxies.

The main goal of this paper is to give a potential solution to the Dark Matter Problem within galaxies. In other words, we do not ban the presence of a Dark Matter component in our Universe which can explain properties of clusters of galaxies which cannot be achieved by alternative gravity theories. In any case, we add a Section where we show that an important property of the bullet cluster can be, in principle, explained in the scenario given in this paper \cite{44}.
At the end of the paper, we discuss the general possibility to give rise to a Dark Matter within extended gravity.

2 The field equations and the linearized approach

Let us consider the high order action \[15, 16, 17\]

\[ S = \int d^4x \sqrt{-g} (R + bR^2 + \mathcal{L}_m). \] (1)

In order to avoid confusion, in all this paper \(b\) represents the coupling constant of the \(R^2\) term.

The action (1) is a particular choice with respect to the well known canonical action characterizing General Relativity (the Einstein - Hilbert action \[19\]), which reads

\[ S = \int d^4x \sqrt{-g} (R + \mathcal{L}_m). \] (2)

If the gravitational Lagrangian is not linear in the curvature invariants, the Einstein field equations has an order higher than second. For this reason, such theories are often called higher-order gravitational theories \[2\]-\[9\]. This is exactly the case of the action (1). Note that in this paper we work with \(8\pi G = 1\), \(c = 1\) and \(\hbar = 1\), while the sign conventions for the line element, which generate the sign conventions for the Riemann/Ricci tensors, are \((-,,+,,+\)).

By varying the action (1) with respect to \(g_{\mu\nu}\) (see refs. \[11, 16, 17\] for a parallel computation) the field equations are obtained:

\[ G_{\mu\nu} + b\{2R[R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R] + \]

\[ -2R_{\rho\mu\nu} + 2g_{\mu\nu}\Box R\} = T_{\mu\nu}^{(m)}, \] (3)

with the associated Klein - Gordon equation for the Ricci curvature scalar

\[ \Box R = E^2(R + T), \] (4)

which is obtained by taking the trace of equation (3), where \(\Box\) is the d’Alembertian operator and the energy term, \(E\), has been introduced for dimensional motivations:

\[ E^2 \equiv \frac{1}{6b}, \] (5)

thus, \(b\) has to be positive \[16\].

In the above equations \(T_{\mu\nu}^{(m)}\) is the standard stress-energy tensor of the matter. Note that General Relativity is obtained for \(b = 0\) in Eq. (3).

Because we want to study interactions between stars at galactic scales, the linearized theory in vacuum \((T_{\mu\nu}^{(m)} = 0)\), which gives a better approximation
than Newtonian theory, can be analysed, by considering a small perturbation of
the background, which is assumed to be given by a Minkowskian background.

In the linearization procedure we will perform a computation very similar to
the ones in [16, 17], but with some difference that will emphasize the importance
of the Ricci scalar for the purpose of this paper: to provide a curvature-inspired
model for galaxies.

Putting
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]  
(6)

to first order in \( h_{\mu\nu} \), calling \( \tilde{R}_{\mu\nu\rho\sigma}, \tilde{R}_{\mu\nu} \) and \( \tilde{R} \), respectively, the linearized
quantity which correspond to \( R_{\mu\nu\rho\sigma}, R_{\mu\nu} \) and \( R \), the linearized field equations
are obtained as [16, 17]:
\[ \tilde{R}_{\mu\nu} - \frac{\tilde{R}}{2}\eta_{\mu\nu} = -\partial_\mu \partial_\nu b\tilde{R} + \eta_{\mu\nu} \Box b\tilde{R} \]  
\[ \Box \tilde{R} = E^2 \tilde{R}. \]  
(7)

Since \( \tilde{R}_{\mu\nu\rho\sigma} \) and Eqs. (7) are invariants for gauge transformations of the
kind [16, 17]
\[ h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} - \partial(\eta_{\mu\nu}^b) \]  
\[ b\tilde{R} \rightarrow b\tilde{R}' = b\tilde{R}; \]  
then the gauge transformation
\[ \tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}^b + \eta_{\mu\nu} b\tilde{R} \]  
(9)
can be defined. Thus, by considering the transformation rule for the parameter \( \epsilon^\mu \)
\[ \Box \epsilon_\nu = \partial^\mu \tilde{h}_{\mu\nu}, \]  
(10)
a gauge analogous to the Lorenz one for electromagnetic waves can be chosen, which then reads
\[ \partial^\mu \tilde{h}_{\mu\nu} = 0. \]  
(11)
In this way, the field equations then read like
\[ \Box \tilde{h}_{\mu\nu} = 0, \]  
(12)
\[ \Box b\tilde{R} = E^2 b\tilde{R}. \]  
(13)

Solutions of eqs. (12) and (13) are plan waves:
\[ \tilde{h}_{\mu\nu} = A_{\mu\nu}(\vec{p}) \exp[ip^\beta x_\beta) + c.c. \]  
(14)
\[ b\tilde{R} = a(\vec{p}) \exp(iq^\beta x_\beta) + c.c. \]  

(15)

where

\[ p^\beta \equiv (\omega, \vec{p}), \quad \omega = p \equiv |\vec{p}| \]  

(16)

\[ q^\beta \equiv (\omega_E, \vec{p}), \quad \omega_E = \sqrt{E^2 + p^2}. \]

In Eqs. (12) and (14) the equation and the solution for the tensor waves, exactly like in general relativity, have been obtained \[16, 17\], while Eqs. (13) and (15) are, respectively, the equation and the solution for the third mode stemming from the curvature (see also \[16, 17\]).

The fact that the dispersion law for the modes of the field \( b\tilde{R} \) is not linear has to be emphasized. The velocity of every “ordinary” (i.e. which arises from General Relativity) mode \( \bar{h}_{\mu\nu} \) is the light speed \( c \), but the dispersion law (the second of Eqs. (16)) for the modes of \( b\tilde{R} \) is that of a wave-packet \[16, 17\]. Also, the group-velocity of a wave-packet of \( b\tilde{R} \) centred in \( \vec{p} \) is

\[ \bar{v}_G = \frac{\vec{p}}{\omega}. \]  

(17)

From the second of Eqs. (16) and Eq. (17) it is simple to obtain:

\[ v_G = \frac{\sqrt{\omega^2 - E^2}}{\omega}. \]  

(18)

Then, it is also written like \[16, 17\]

\[ E = \sqrt{(1 - v_G^2)\omega}. \]  

(19)

Now, the analysis can remain in the Lorenz gauge with transformations of the type \( \square \epsilon_\nu = 0 \). This gauge gives a condition of transverse effect for the ordinary part of the field, i.e. \( k^\mu A_{\mu\nu} = 0 \), but does not give the transverse effect for the total field \( h_{\mu\nu} \).

From Eq. (9) it reads

\[ h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} + \eta_{\mu\nu} b\tilde{R}. \]  

(20)

At this point, if one were still to remain within general relativity one would impose \[23\],

\[ \square \epsilon^\mu = 0 \]  

(21)

\[ \partial_\mu \epsilon^\mu = -\frac{\hbar}{\tau} + b\tilde{R}, \]

which would give the total transverse effect of the field. However, for the present case this is impossible. In fact, applying the d’Alembertian operator to
the second of Eqs. (21) and using the field equations (12) and (13), it comes out that

\[ \partial_\mu \Box e^\mu = E^2 b \tilde{R}, \]  

(22)

which is in contrast with the first of Eqs. (21). In the same way, it is possible to show that no linear relation exists at all between the tensor field \( \tilde{h}_{\mu\nu} \) and the linearized term \( b \tilde{R} \) stemming from the curvature scalar. Thus, a gauge in which \( h_{\mu\nu} \) is purely spatial cannot be chosen (i.e. one cannot put \( h_{\mu0} = 0 \), see Eq. (20)). Nonetheless, the traceless condition to the field \( \tilde{h}_{\mu\nu} \) can be written in the form:

\[ \Box e^\mu = 0 \]

(23)

\[ \partial_\mu e^\mu = -\frac{\tilde{h}}{2}. \]

These equations imply

\[ \partial^\mu \tilde{h}_{\mu\nu} = 0. \]  

(24)

To save the conditions \( \partial_\mu \tilde{h}_{\mu\nu} = 0 \) and \( \tilde{h} = 0 \), transformations like

\[ \partial_\mu \Box e^\mu = 0 \]

(25)

\[ \partial_\mu e^\mu = 0 \]

can be used and, by taking \( \vec{p} \) in the \( z \) direction, a gauge in which only \( A_{11}, A_{22}, \) and \( A_{12} = A_{21} \) are different from zero can be chosen. The condition \( \tilde{h} = 0 \) gives \( A_{11} = -A_{22} \).

Now, by substituting these equations in Eq. (20), one gets

\[ h_{\mu\nu}(t, z) = A^+(t - z)e_{\mu\nu}^{(+)} + A^x(t - z)e_{\mu\nu}^{(x)} + b \tilde{R}(t, z) \eta_{\mu\nu}. \]  

(26)

The term \( A^+(t - z)e_{\mu\nu}^{(+)} + A^x(t - z)e_{\mu\nu}^{(x)} \) describes the two standard polarizations of gravitational waves obtained from general relativity, while the term \( b \tilde{R}(t, z) \eta_{\mu\nu} \) represents a mode arising from the curvature term associate to the \( R^2 \) high order theory [16, 17].

In other words, in the \( R^2 \) theory of gravity, the linearized Ricci scalar is a source of a third polarization mode for gravitational waves which is not present in standard general relativity. This third mode is associated to a “curvature” energy \( E \) (see Eq. (4)).

We also recall that the idea of considering the Ricci scalar as an effective scalar field (scalaron) arose from Starobinski [13].
3 Application to the Einstein-Vlasov system: stationary galaxies

Now, we are going to discuss a model of stationary, spherically symmetric galaxy, assuming that the dynamics of the matter, i.e. of the stars making up the galaxy, is described by the Einstein-Vlasov system \[18\]. In this way, the gravitational forces between the particles of the system, i.e., a galaxy, will be mediated by the third mode of Eq. \[26\], i.e. by the spacetime curvature. Thus, the key assumption is that in a cosmological context such a mode, which is given by the (linearized) spacetime curvature, becomes dominant at galactic and cosmological scales (i.e. \(A^+ , A^- \ll b\tilde{R}\)). In this way the "curvature" energy \(E\) can be identified as the dark matter content of a galaxy of typical mass-energy: \(E \simeq 10^{45} g\) \[25\], in ordinary c.g.s. units. These two assumptions constitute an adaptation to modelling a galaxy of the assumptions related to the model describing an oscillating Universe in \[16\].

Now, let us emphasize an important point. Assuming \(E \simeq 10^{45} g\), from Eq. \(5\) we get \(b \simeq 10^{-34} cm^4\) in natural units. Thus, in our assumption, the constant coupling of the \(R^2\) term in the gravitational action results infinitesimal with respect to the linear term \(R\). In this way, the variation from standard General Relativity is very weak, thus the theory can pass the Solar System tests. Regarding this important issue, it is important to provide citations of precedent work illustrating this and explicitly show that the bounds there agree with the preferred values of the number \(E\). The key point is that as the effective scalar field arising from curvature is very energetic, then the constant coupling of the the \(R^2\) nonlinear term \(\to 0\) \[31\]. In this case, the Ricci curvature, which is an extra dynamical quantity in the metric formalism, must have a range longer than the size of the Solar System. An important work is ref. \[32\], where it is shown that this is correct if the effective length of the scalar field \(l\) is much shorter than the value of 0.2 mm. In such a case, the presence of this effective scalar is hidden from Solar System and terrestrial experiments. The value of \(E\) that we are assuming here guarantees the condition \(l \ll 0.2 mm\).

Another important test concerns the deflection of light by the Sun. This effect was studied in \(R^2\) gravity by calculating the Feynman amplitudes for photon scattering, and it was found that, to linearized order, this deflection is the same as in standard General Relativity \[33\].

On the other hand, Eq. \(15\) guarantees that spacetime curvature does not remain too small and thus it can, in principle, become the Dark Matter component in the galaxy \[24\].

The model that we shall discuss is similar to the one introduced by Nordstrom in \[21\]. The relevance of the Einstein-Vlasov model was emphasized in the 1992 in a famous paper by Rein and Rendall \[18\]. And the following results will be obtained adapting the ideas introduced in \[18, 22, 24, 26, 28\].

In the hypothesis \(A^+, A^- \ll b\tilde{R}\), the spacetime of our model will be given by the conformally flat metric
$$ds^2 = [1 + b\tilde{R}(t, z)](dx^2 + dy^2 + dz^2 - dt^2).$$ \hspace{1cm} (27)$$

Note: in general, conformal transformations are performed by rescaling the line-element like [23]
$$\tilde{g}_{\alpha\beta} = e^\Phi g_{\alpha\beta}. \hspace{1cm} (28)$$
Here we choose the scalar field as being
$$\Phi \equiv b\tilde{R}. \hspace{1cm} (29)$$
which also implies
$$e^\Phi = 1 + b\tilde{R}, \hspace{1cm} (30)$$
in our linearized approach. Thus, the conformal transformation which translates the analysis into the conformal frame (Einstein frame, see [5]) is performed by the spacetime curvature. The potential of using such conformally flat line-element in the perspective of describing an oscillating Universe has been discussed in [16].

The condition that the particles in the spacetime make up an ensemble with no collisions [27] is satisfied if the particle density is a solution of the Vlasov equation [18, 22, 26, 28]
$$\partial_t f + \frac{p_\alpha}{p^0} \partial_{x^\alpha} f - \Gamma^\alpha_{\mu\nu} \frac{p^\mu p^\nu}{p^0} \partial_{p^\alpha} f = 0, \hspace{1cm} (31)$$
where $\Gamma^\alpha_{\mu\nu}$ are the Christoffel coefficients, $f$ is the particle density and $p^0$ is given by $p^a(a = 1, 2, 3)$ according to the relation [18, 22, 26, 28]
$$g_{\mu\nu} p^\mu p^\nu = -1. \hspace{1cm} (32)$$
Eq. (32) implies that the four momentum $p^\mu$ lies on the mass-shell of the spacetime (greek indices run from 0 to 3) [18 22 26 28].

We recall that, in general, the Vlasov-Poisson system is given by [18 22 26 28]
$$\partial_t f + v \cdot \nabla_x f - \nabla U \cdot \nabla_v f = 0$$
$$\nabla \cdot U = 4\pi \rho$$
$$\rho(t, x) = \int dv f(t, x, v),$$
where $t$ denotes the time and $x$ and $v$ the position and the velocity of the stars. The function $U = U(t, x)$ is the average Newtonian potential generated by the stars. This system represents the non-relativistic kinetic model for an ensemble of particles with no collisions interacting through gravitational forces which they generate collectively [18 22 26 28].
Thus, such a system can be used for a description of the motion of the stars within a galaxy, if stars are considered as pointlike particles, and the relativistic effects are negligible [18, 22, 26, 28]. In this approach, the function $f(t, x, v)$ in the Vlasov-Poisson system (33) is non-negative and gives the density on phase space of the stars within the galaxy.

In other words, we are going to discuss the solutions of a Klein-Gordon equation for the spacetime curvature like galactic high energy scalarons, i.e. particles that can be figured out like wave-packets, and stationary solutions in terms of an Einstein-Vlasov system will be shown.

In such approximation, the energy of the particle will be seen like the Dark Matter component that guarantees the galaxy’s equilibrium. It is stressed that such approximation is not as precise as we would, but it could be a starting point for further robust analysis.

The Vlasov equation (31) implies that the function $f$ is constant on the geodesic flow over the spacetime (27). The Christoffel coefficients of such a spacetime are obtained from (note that as we are working in the linearized approach, in the following computations only terms up to first order in the linearized curvature $\tilde{R}$ will be considered while high-order terms will be assumed equal to zero)

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \left( \delta^\alpha_{\nu} \partial_{\mu} \tilde{R} + \delta^\alpha_{\mu} \partial_{\nu} \tilde{R} - \frac{1}{1 + 2b\tilde{R}} g_{\mu\nu} \partial^\alpha \tilde{R} \right).$$

(34)

In this way, the Vlasov equation in the spacetime defined by the line-element (27) becomes

$$\partial_t f + \frac{p^a}{p^0} \partial_{x^a} f - \frac{1}{2} \left[ 2(p^a \partial_{\mu} \tilde{R}) \frac{p^\mu}{p^0} + \frac{\partial^a \tilde{R}}{1 + 2b\tilde{R})p^0} \right] \partial_{p^a} f = 0.$$  

(35)

Now, let us recall that two quantities are important for the Vlasov equation in a curved spacetime [18, 22, 26, 28]. The first is the current density

$$N^\mu = - \int \frac{dp}{p^0} \sqrt{g} p^\mu f$$

(36)

and the second is the stress-energy tensor

$$T^{\mu\nu} = - \int \frac{dp}{p^0} \sqrt{g} p^\mu p^\nu f.$$  

(37)

Here $g$ is the usual determinant of the metric tensor, which in the case of the line-element (27) is given by

$$g = 1 + 4b\tilde{R}.$$  

(38)

Keep in mind that both $N^\mu$ and $T^{\mu\nu}$ are divergence free (conservation of energy):
\[ \nabla_\mu N^\mu = 0, \]  
\[ \nabla_\mu T^{\mu \nu} = 0. \]  (39)

The mass shell conditions \(^{(32)}\) can be rewritten as
\[ p^0 = \sqrt{(1 + b\tilde{R})^{-1} + \delta_{ab}p^a p^b}. \]  (40)

From the Christoffel connections \(^{(34)}\), computing the Riemann tensor, Ricci tensor and Ricci scalar, the “effective” Einstein field equations
\[ G_{\mu \nu} = T_{\mu \nu}, \]  (41)
can be obtained together with the “effective” Klein - Gordon equation
\[ \Box b\tilde{R} = -T, \]  (42)
where \( T = T_\mu^\mu \) is the trace of the stress-energy tensor.

To simplify the computations the analysis can be performed in a conformal frame. Thus, rescaling the stress-energy tensor in the form
\[ T_{\mu \nu}^* = (1 + 3b\tilde{R})T_{\mu \nu}, \]  (43)
one obtains
\[ T_* = (1 + 3b\tilde{R})T. \]  (44)

Then, equation \(^{(42)}\) becomes
\[ \Box b\tilde{R} = -T_. \]  (45)

It should be noticed that the particle density is still defined on the mass shell of the starting line-element \( g_{\alpha \beta} \). In order to remove even this last connection with the starting frame (Jordan Frame) we can rescale the momentum as
\[ p_*^\mu = (1 + \frac{b\tilde{R}}{2})p^\mu \]  (46)
and we can define the particle density in the Einstein frame as
\[ f_*(t, x, p_*) = f \left( t, x, (1 - \frac{b\tilde{R}}{2})p_* \right). \]  (47)

Hence, we can write our adaptation of the Vlasov system in the following form
\[ \Box b\tilde{R} = (1 + 2b\tilde{R}) \int \frac{dp_*}{p_0} f_*(t, x, p_*), \]  (48)
\[ p_0^* = \sqrt{1 + \delta_{ab}p_v^a p_v^b}. \]  (49)
\[ \partial_t f_\ast + \frac{p^a}{p^\mu} \partial_{p^\mu} f_\ast - \frac{1}{p^\mu} p_c^c \partial_p b \tilde{R} f_\ast + \partial^\rho b \tilde{R} \partial_{p^\rho} f_\ast = 0. \]  

(50)

Because we want to restrict ourselves to spherical symmetry in the present approximation, the line-element \( (27) \) can be rewritten as

\[ ds^2 = [1 + b \tilde{R}(t, r)](dr^2 - dt^2). \]  

(51)

In this equation \( r \) is the radial coordinate. Thus, in spherical coordinates, equations \( (48), (49) \) and \( (50) \) can be written as

\[ - \frac{d^2 b \tilde{R}}{dt^2} + \frac{1}{r^2} \frac{d}{dr} \left( \frac{d}{dr} b \tilde{R} r^2 \right) = (1 + 2 b \tilde{R}) \mu(t, r), \]  

(52)

\[ \mu(t, r) = \int \frac{dp}{\sqrt{1 + p^2}} f(t, x, p), \]  

(53)

\[ \partial_t f + \frac{p}{\sqrt{1 + p^2}} \partial_x f - \left( \frac{d}{dt} b \tilde{R} + \frac{x \cdot p}{\sqrt{1 + p^2}} \frac{1}{r} \frac{d}{dr} b \tilde{R} \right) p + \frac{x}{\sqrt{1 + p^2}} \frac{1}{r} \frac{d}{dr} b \tilde{R} \cdot \partial_p f = 0, \]  

(54)

where the suffix \( \ast \) has been removed for the sake of simplicity, and we have denoted by \( p \) the vector \( p = (p_1, p_2, p_3) \) with \( p^2 = |p|^2 \), and also defined \( x \) for the vector \( x_i = (x_1, x_2, x_3) \).

Then, in searching for stationary states, and following [16], we can call \( \lambda \) the wavelength of the “galactic” gravitational wave \( (26) \), i.e. the characteristic length of our gravitational perturbation, and assume that \( \lambda \gg d \), where \( d \) is the galactic scale, i.e. \( d \sim 10^5 \) light-years [25]. In this way, the gravitational wave is “frozen-in” with respect to the galactic scale.

Thus, the system of equations which defines the stationary solutions of eqs. \( (52), (53) \) and \( (54) \), for our galaxy model is

\[ \frac{1}{r^2} \frac{d}{dr} \left( \frac{d}{dr} b \tilde{R} r^2 \right) = (1 + 2 b \tilde{R}) \mu(r), \]  

(55)

\[ \mu(r) = \int \frac{dp}{\sqrt{1 + p^2}} f(x, p), \]  

(56)

\[ p \cdot \partial_x f - \frac{1}{r} \frac{d}{dr} b \tilde{R}[\cdot f - \partial_{p^\mu} (p \cdot x)] \cdot \partial_p f = 0, \]  

(57)

In other words, the idea is that the spin-zero degree of freedom introduced via the addition of an \( R^2 \) term in the gravitational Lagrangian may be a candidate for the dark matter. This new degree of freedom is termed the scalaron [13].

Thus, we assume that within a galaxy the dominant contribution to the curvature comes from the new degree of freedom (as it is also the case to first approximation in the general relativity+cold dark matter case). We deduce this contribution via the scalaron field equation which itself has a baryonic
source term. The baryons themselves are then taken to evolve according to a collisionless Boltzmann equation, propagating on a background perturbed by the scalaron. The collective set of equations can in principle be solved for given initial data.

4 The gravitational lensing

The attentive reader notices that, in principle, as the metric is conformally flat, this could mean that the scalaron Dark Matter in galaxies will, in and of itself, result in no gravitational lensing of light rays. This could be a serious problem for the model. This is because there exists substantial evidence for gravitational lensing by the large scale structure of a galaxy beyond that due to the constituent baryons alone. For instance, the observed absence of lensing by the effective dark matter in a class of metric formulations of Modified Newtonian Dynamics formed the basis for a tentative no-go theorem for such theories [30].

Actually, we show that in the proposed model the gravitational lensing can be, in principle, obtained like an effect of spacetime curvature, i.e. due to the Ricci curvature scalar.

In our linearized approach, gravitational lensing can be described in a local Lorentz frame perturbed by the first order post-Newtonian potential [27]. Calling $V$ such a potential one can define a refractive index [27]

$$n \equiv 1 + 2|V|.$$  \hspace{1cm} (58)

Some clarifications are needed concerning this point. In the usual Geometrical Optics, the condition $n > 1$ implies that the light in a medium is slower than in vacuum [36]. Then, the effective speed of light in a gravitational field is expressed by [36]

$$v = \frac{1}{n} \approx 1 - 2|V|.$$  \hspace{1cm} (59)

Thus, one can obtain the Shapiro delay [37] by integrating over the optical path between the source and the observer:

$$\int_{\text{source}}^{\text{observer}} 2|V|dl.$$  \hspace{1cm} (60)

The situation is analogous to the prism [36].

Now, we recall that, in the weak field approximation, the connection between the post-Newtonian potential and the linearized theory is given by the $g_{00}$ component of the line-element [19]:

$$g_{00} \approx 1 + 2V.$$  \hspace{1cm} (61)

Thus, from eq. [27] one obtains

$$V \approx \frac{b\dddot{R}(t,z) - 1}{2},$$  \hspace{1cm} (62)
and

\[ n \approx 1 + 2|b\tilde{R}(t, z) - 1| \]  

(63)

is the equation which can, in principle, be used to discuss the gravitational lensing in our model. Then, in this case, the gravitational lensing is performed directly by spacetime curvature, i.e. by the Ricci scalar.

For a sake of completeness, we discuss the gravitational lensing from another point of view.

The condition \( \lambda \gg d \), on the characteristic length of the gravitational perturbation, guarantees that the Ricci curvature remains “frozen, i.e. constant,” with respect to the galactic characteristic distance scale. Thus, we put

\[ b\tilde{R} = K \text{(a constant!)}. \]  

(64)

We can search deviations from standard General Relativity within the galaxy by considering a spherically symmetric Schwarzschild-like metric generated by the ordinary (barion) galaxy mass \( M \) with the corrections that are generated by curvature [38]:

\[ ds^2 = -\exp(-\lambda r)dt^2 + \exp(\lambda r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \]  

(65)

Then, following [38], the Ricci scalar is given by:

\[ R = \exp(-\lambda r)(\lambda'' - \frac{4\lambda'}{r} + \frac{2}{r^2}) + \frac{2}{r^2}, \]  

(66)

where \( ' \) stands for the derivative with respect to \( r \).

Putting the condition (64) in Eq. (66) one gets [38]:

\[ \lambda(r) = -\ln \left( \frac{\alpha}{r} - \frac{\beta}{r^2} - \frac{K}{12b}r^2 \right), \]  

(67)

and, by choosing \( \alpha = -2M, \beta = 0 \) in analogy with the standard Schwarzschild metric, one gets [19, 38]

\[ ds^2 = -(1 - \frac{2M}{r} - \frac{K}{12b}r^2)dt^2 + (1 - \frac{2M}{r} + \frac{K}{12b}r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \]  

(68)

The physical interpretation implies that within some radius \( r_{\text{min}} \) deviation from General Relativity are highly suppressed and we say that the object is screened [39, 40, 41]. This is because we are assuming that, within the radius \( r_{\text{min}} \), the spacetime curvature which is due by the ordinary, i.e. barion, mass of the galaxy dominates with respect to the intrinsic constant curvature \( R = K \).

On the other hand, for values \( r > r_{\text{min}} \), the spacetime curvature which is due by the intrinsic curvature dominates with respect to the spacetime curvature due by the barion mass of the galaxy.

In order to clarify the issue of the bending of light in this \( R^2 \) scenario, in what follows, and as matter of illustration, we will reproduce a piece of the discussion
presented by Smith in Ref. [41], which, in comparison with our discussion at the beginning of Section II, appears as a generalization of that digression (see the Appendix).

Thus, for a general action $S = \int d^4x \sqrt{-g} (R + f(R) + L_m)$ screening occurs within a radius $r_{\text{min}}$ implicitly given by [39, 40, 41]

$$|f'(R_{\text{max}})| < \rho(r_{\text{min}}) r_{\text{min}}^2,$$

where $\rho$ is the local value of the density.

In our case it is $f'(R) = 2bR$ and $R_{\text{max}} = \frac{K}{b}$. Then, Eq. (69) becomes

$$K < \frac{1}{2} \rho(r_{\text{min}}) r_{\text{min}}^2.$$

Now, following [41], we can use Eq. (70) to further discuss the gravitational lensing.

Let us consider a galactic Navarro, Frank and White (NFW) halo of the form [40, 41]

$$\rho(r) = \rho_c \delta_c \frac{r_s^3}{r(r + r_s)^2},$$

where $r_s$ is the scale radius and $\rho_c$ the critical density of the universe [40, 41]. By assuming the mass-concentration relation [41, 42, 43]

$$c = \frac{9}{1 + z} \left( \frac{M}{8.12 \times 10^{12} h^{-1} M_\odot} \right),$$

where $z$ is the redshift, $M_\odot$ the mass of the Sun and $h$ is the Hubble parameter in units of 100 km/(sMpc), the amplitude $\delta_c$, that relates the concentration to the virial radius with an overdensity $\Delta = 119$ is given by [41, 42, 43]

$$\delta_c = \frac{\Delta c^3}{3[\ln(1 + c) - (\frac{c}{1 + c})]}.$$

Eq. (72) reduces the NFW halo profile to a one-parameter family which is taken to be dependent on the virial mass, $M$ [41]. For $r < r_s \ \rho_{\text{NFW}}$ scales like $r^{-1}$, while for $r > r_s \ \rho_{\text{NFW}}$ scales like $r^{-4}$ [41]. Thus, the innermost point at which deviations from general relativity are suppressed in will occur at the scale radius $r_s$. Considering Eq. (70) together with the NFW density profile on sees that halos with masses which satisfy

$$K > \rho_c \delta_c (M)$$

As $\delta_c (M)$ decreases with decreasing $M$ [41], an upper limit is setted to the mass of halos which can be screened given $K$, i.e. given the value of the Ricci curvature, see Eq. (63). For masses below this threshold the theory strongly deviates from general relativity and strong lensing around these halos is permitted. We plan to further develop in this direction of research.
5 Can $R^2$ gravity explain the Bullet Cluster dynamics?

The scalaron has two aspects as a field (like classical electric or magnetic field) and a particle (like quantized photon). We treated the scalaron as a field in the Vlasov equation (see [44] for a recent cosmological application) whose scenario is in some sense similar to that of the MOND [45]. In such a scenario, it seems to be difficult to explain the Bullet Cluster [46] although the Bullet Cluster could be naturally explained if the scalaron could be a heavy particle.

Aside from this, we emphasize that the main goal of this paper is to give a potential solution to the Dark Matter Problem within galaxies. Therefore, we do not ban the presence of a Dark Matter component in our Universe which can explain properties of clusters of galaxies which cannot be achieved by current alternative gravity theories [46]. In any case, in this Section we show that an important property of the Bullet Cluster can in principle be explained in the scenario proposed in this paper.

5.1 A note on the Bullet Cluster 1E0657-56

Since 2006, the argument "But the Bullet Cluster ..." has come into scene in most discussions about the Dark Matter Problem in astrophysics. It relates to the observations of the post-collision clusters of galaxies in the source 1E0657-56 (known as the Bullet Cluster) [46]. It is seen that the X-ray emission is mid the couple of colliding clusters, and thus is shifted with respect to the bulk of matter making up each group of galaxies, which resides in the outskirts. In most studies of X-ray emission from galaxy clusters the gas usually sits at its center. This offset can be understood by recalling that as the gas pass each other it interacts electromagnetically, heats up, ionize to emit X-rays, and hence is slowed-down during the collision. This may explain why the gas appears a bit behind in the heading direction of each of the group of galaxies involved. Meanwhile, non interacting Dark Matter components do only interact through gravity and thus pass each other unhindered without being slowed-down like the gas. In the image, the field purported to represent the vast of mass in each cluster, as compared to the X-ray emitting counterpart, is inferred from gravitational lensing analysis of images of background galaxies, under the assumption that general relativity is the theory of gravity (but be aware that most modified gravity theories describe correctly the phenomenon of gravitational lensing. See the Appendix). It follows the distribution of galaxies, not the gas. As the most massive component of the galaxy cluster is not centered at the gas, it is argued, upon that lens analysis, that the large part of matter in the cluster is close to, and around each galaxy group. And as the visible mass in the galaxies is not enough to account for the velocity dispersion of a galaxy cluster, it is suggested that there is Dark Matter, too. As noninteracting component, the distribution of the Dark Matter can pass through each other just like the galaxies, and that is why the bulk of mass should be found close to the galaxies in a cluster collision.
The analysis of the overimposed optical and X-ray images of the cluster can be interpreted as an endorsement on the existence of Dark Matter. This issue is based on a number of assumptions which are not totally accepted by the astrophysics community because the large set of problems for the Dark Matter hypothesis are, in any case, independent from the observations of the Bullet Cluster. In a different context, the collision velocity of the Bullet Cluster seems to be in disagreement with the standard concordance model of cosmology (the relative velocity of the clusters in the collision is an issue being addressed next in the framework of this gravity theory). At the same time, alternative theories of gravitation, while often said to fail in explaining dynamics of galaxy clusters, can account for them rather naturally.

On the other hand, even if the Bullet Cluster can only be explained by invoking Dark Matter, the problems on small scales persist. The Bullet Cluster does not improve our understanding of the Local Group of Galaxies, nor on the dynamics of individual galaxies, (as is on focus in the present paper). The two arguments are absolutely independent. A conclusion could be that the lensing mass and the hot gas have a spatial shift, and hence the visible, hot gas cannot make up the bulk of mass in the system. In any case, whether the cause of this phenomenon is a nonstandard law of gravitation or missing mass is not that easy to unveil.

5.2 Scalaron and the Bullet Cluster dynamics

As pointed out above, it is a common opinion that alternative gravity theories cannot explain all the properties of the bullet cluster, even if scalar-vector-tensor theory could, in principle, take into account the apparent discrepancy between the gravitational lensing and X-ray maps of the colliding clusters, without requiring collisionless Dark Matter.

Meanwhile, hydrodynamic simulations of the colliding galaxy clusters 1E0657-06 show that at a distance of 4.6 Mpc between the two clusters, an infall velocity of 3000 km/s is required in order to explain the observed X-ray brightness and morphology of the cluster. It has also been argued that such a high infall velocity is incompatible with predictions of the standard cosmology (the concordance ΛCDM model).

Next, we extend our scalaron model in order to show that the infall velocity of the two clusters can in principle be explained in the framework introduced in the present paper. To rigorously show that the scalaron gravity can cope with the Bullet Cluster dynamics we need first to modify our assumptions. In particular, we will assume that the characteristic length of the scalaron gravitational perturbation is much longer not only than the galactic scale, but than the distance between the colliding clusters too. In other words, the gravitational wave is “frozen-in” with respect to the Bullet Cluster’s scale. In this way, we can consider the colliding clusters (the main cluster and the “bullet”) like test particles in the gravitational field of the scalaron.

In the linearized approach of this paper, the coordinate system in which the
space-time is locally flat has to be used and the distance between any two points (the colliding clusters) is given simply by the difference in their coordinates in the sense of Newtonian physics [19]. This frame is the proper reference frame of a local observer, which we assume to be located within the main cluster. In this frame gravitational signals manifest them-self by exerting tidal forces on the test masses. In other words, we assume that the space-time within the main cluster is locally flat with respect to the global curvature generated by the scalaron which is described by the line element (27). By using the proper reference frame of a local observer the time coordinate $x_0$ is the proper time of the observer $O$ and the spatial axes are centred in $O$. In the special case of zero acceleration and zero rotation the spatial coordinates $x_j$ are the proper distances along the axes and the frame of the local observer reduces to a local Lorentz frame [19]. As the origin of the coordinates is located within the main cluster, the proper distances between the two clusters are the coordinates of the “bullet”. The line element is [19]

$$ds^2 = +(dx^0)^2 - \delta x^j dx^j - O(|x_j|^2)dx^\alpha dx^\beta.$$  \hfill (75)

The effect of the gravitational force on test masses is described by the equation

$$\ddot{x}^i = -\tilde{R}_{ik0}^i x^k,$$  \hfill (76)

which is the equation for geodesic deviation in this frame [19]. $\tilde{R}_{ik0}^i$ is the linearized Riemann tensor and $x^i$ are the coordinates of the “bullet” which represent the separation vector between the two test masses [19].

To study the effect of the scalaron on the two clusters, $\tilde{R}_{ik0}^i$ has to be computed in the proper reference frame of the main cluster. But, as the linearized Riemann tensor $\tilde{R}_{\mu\nu\rho\sigma}$ is invariant under gauge transformations [19], it can be directly computed from Eq. (27).

From [19] we get

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\{\partial_\mu \partial_\nu h_{\rho\sigma} + \partial_\rho \partial_\sigma h_{\mu\nu} - \partial_\nu \partial_\sigma h_{\mu\rho} - \partial_\mu \partial_\rho h_{\sigma\nu}\},$$  \hfill (77)

that, in the case Eq. (27), gives

$$\tilde{R}_{\alpha\gamma0}^0 = \frac{b}{2}\{\partial_\alpha \partial_0 \tilde{R}_{\eta\eta\gamma} + \partial_0 \partial_\gamma \tilde{R}_{\delta\delta0} - \partial_\alpha \partial_\gamma \tilde{R}_{\eta\eta0} - \partial_0 \partial_\gamma \tilde{R}_{\delta\delta0}\}.$$  \hfill (78)

The different elements are (only the non zero ones will be written down explicitly)

$$\partial^\alpha \partial_0 \tilde{R}_{\eta\eta\gamma} = \begin{cases} \frac{\partial^2 \tilde{R}}{\partial t^2} & \text{for } \alpha = \gamma = 0, \\ -\partial_\gamma \partial_0 \tilde{R} & \text{for } \alpha = 3; \gamma = 0. \end{cases}$$  \hfill (79)

$$\partial_0 \partial_\gamma \tilde{R}_{\delta\delta0} = \begin{cases} \frac{\partial^2 \tilde{R}}{\partial x^2} & \text{for } \alpha = \gamma = 0, \\ \partial_\gamma \partial_\alpha \tilde{R} & \text{for } \alpha = 0; \gamma = 3. \end{cases}$$  \hfill (80)
\[ -\partial^{\alpha}\partial_{\gamma} \tilde{R}_{00} = \partial^{\alpha}\partial_{\gamma} \tilde{R} = \begin{cases} 
-\partial_{t}^{2} \tilde{R} & \text{for } \alpha = \gamma = 0 \\
\partial_{z}^{2} \tilde{R} & \text{for } \alpha = \gamma = 3 \\
-\partial_{t}\partial_{z} \tilde{R} & \text{for } \alpha = 0; \gamma = 3 \\
\partial_{z}\partial_{t} \tilde{R} & \text{for } \alpha = 3; \gamma = 0 
\end{cases} \] (81)

\[ -\partial_{t}\partial_{t} \tilde{R} \delta_{i}^{\alpha} = -\partial_{t}^{2} \tilde{R} & \text{for } \alpha = \gamma. \] (82)

By inserting these results in Eq. (78) we obtain

\[ \tilde{R}_{010}^{1} = -\frac{b}{2} \tilde{R}, \]
\[ \tilde{R}_{010}^{2} = -\frac{b}{2} \tilde{R}, \]
\[ \tilde{R}_{030}^{3} = \frac{b}{2}(\partial_{t}^{2} \tilde{R} - \partial_{z}^{2} \tilde{R}). \] (83)

At the Bullet Cluster’s scale we have homogeneity and isotropy, which imply \( \partial_{t}^{2} \tilde{R} = 0 \). Therefore, Eqs. (83) become

\[ \tilde{R}_{010}^{1} = -\frac{b}{2} \tilde{R}, \]
\[ \tilde{R}_{010}^{2} = -\frac{b}{2} \tilde{R}, \]
\[ \tilde{R}_{030}^{3} = -\frac{b}{2} \tilde{R}. \] (84)

By using Eq. (76), we obtain

\[ \ddot{x} = \frac{b}{2} \tilde{R}x, \] (85)
\[ \ddot{y} = \frac{b}{2} \tilde{R}y \] (86)

and

\[ \ddot{z} = \frac{b}{2} \tilde{R}z. \] (87)

These are three perfectly symmetric oscillations of the scalaron (wave-packet). If one re-introduces the radial distance \( r \), Eqs. (85), (86) and (87) are summarized by

\[ \ddot{r} = \frac{b}{2} \tilde{R}r. \] (88)

Eq. (88) can be solved with the perturbation method [19]
\[ r(t) \simeq r(0)[1 + \frac{b}{2}\ddot{R}(t)]. \]  

(89)

Assuming that the wave-packet is in the contraction phase we get the relative infall velocity as

\[ v_{\text{infall}} = \frac{\text{d}}{\text{d}t} r(t) \simeq r(0)\frac{b}{2}\ddot{R}(t). \]  

(90)

Therefore, in order to have consistency with the observations we need

\[ r(0)\frac{b}{2}\ddot{R}(t) \simeq 3000 \text{ km/s}. \]  

(91)

Being \( r(0) \simeq 4.6 \text{ Mpc} \), the model will be consistent with observations if

\[ \frac{b}{2}\ddot{R}(t) \simeq \frac{3000 \text{ km}}{4.6 \text{ Mpc}} s^{-1} \simeq 2 \times 10^{-17} s^{-1}. \]  

(92)

The assumption that the characteristic length of the scalaron gravitational perturbation is much longer than the distance between the colliding clusters implies that the frequency of the wave-packet has to be \( \omega_E \ll 10^{-14} \text{s}^{-1} \). For example, for a value \( \omega_E \sim 10^{-16} \text{s}^{-1} \) Eq. (92) guarantees theoretical consistence if \( \frac{b}{2}\ddot{R} \sim 10^{-1} \).

6 Final discussion and closing remarks

It has been shown that in the framework of \( R^2 \) gravity and in the linearized approach, it is possible to obtain spherically symmetric and stationary galaxy states which can be interpreted like an approximated solution of the Dark Matter problem. In fact, in the model the Ricci curvature generates a high energy term that can in principle be identified as the dark matter content of most the galaxies. The model can also help to gain a better understanding of the basics of the Einstein-Vlasov system.

In other words, we have discussed, in the linearized \( R^2 \) gravity, the solutions of a Klein-Gordon equation for the spacetime curvature like galactic high energy scalarons, i.e. particles that has been analysed like wave-packets, and we have shown stationary solutions in terms of the Einstein-Vlasov system.

In this approximation, the energy of the galaxy-particle has been seen like the dark matter component which guarantees the equilibrium of galaxies.

An important point is that, because of the high energy of such particle-galaxies, the constant coupling of the \( R^2 \) in the gravitational action comes out to be infinitesimal with respect to the linear term \( R \). In this way, the deviation from standard general relativity is very weak, and, in principle, the theory could pass the Solar System tests. An analysis on the gravitational lensing has been also performed in Section 4.

We recall that the main goal of this paper is to give a potential solution to the Dark Matter Problem within galaxies. Thus, we do not ban the presence
of a Dark Matter component in our Universe which can explain properties of clusters of galaxies which cannot be achieved by alternative gravity theories. In any case, in Section 5 we showed that an important property of the bullet cluster, i.e., an infall velocity of 3000 km/s between the two clusters, can be, in principle, explained in the scenario given in this paper.

Acknowledgements

The authors thank an unknown referee for precious advices and comments which permitted to improve this paper. Christian Corda thanks the R. M. Santilli Foundation for partially supporting this research (Research Grant Number RMS-TH-5735A2310). Herman J. Mosquera Cuesta thanks Fundação Cearense para o Desenvolvimento Científico e Tecnológico (FUNCAP), Ceará, Brazil for financial support.

Appendix: dynamics in \( f(R) \) theories of gravity and gravitational lensing

We emphasize that hereafter we closely follow the paper by Smith [41], and the set of references quoted there. A generic branch of scalar-tensor theories in which the Brans-Dicke parameter \( \omega \equiv 0! \) bring in unique predictions: they feature a peculiar effect known as the chameleon mechanism, by which the modifications to general relativity get quickly suppressed nearby sufficiently massive objects, such as galaxies. This rapid decay in the theory’s gravitational effects present a unique size-dependent lensing footprint which is discussed next.

The action for \( f(R) \)-theories (functional of the Ricci scalar \( R \))

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [R + f(R)] + S_m, \tag{93}
\]

where \( S_m \) action for the matter fields. The field equations turn out to be

\[
[1 + f'(R)] G_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left[ R f'(R) - f + 2 \Box f'(R) \right] - \nabla_\mu \nabla_\nu f'(R) = \kappa T_{\mu\nu}. \tag{94}
\]

In this class of theories the Ricci scalar becomes a dynamical quantity whose equation of motion is determined by the trace of the field equation,

\[
\Box f'(R) = \frac{1}{3} \left( \kappa T + R [1 - f'(R)] + 2f \right). \tag{95}
\]

In the limit where \( f \to 0 \), Eq. (95) reduces to the standard relation \( R = -\kappa T \) of general relativity. Here \( T \) is the trace of the stress-energy tensor.

Using the trace equation to rewrite the gravitational equation of motion one obtains
\[ T_{\text{eff}} = \frac{1}{3\kappa} \left[ \kappa T + R \right] + \frac{1}{3\kappa} \left[ 2Rf'(R) - f \right]. \] (96)

Solutions to the Eq. (95) determine the lensing predictions for the theory, and can be understood by rewriting the trace of the field equation in the form

\[ \Box f'(R) = -\frac{dV}{df'(R)}, \] (97)

where

\[ \frac{dV}{df'(R)} = \frac{1}{3} \left( \kappa T + R \left[ 1 - f'(R) \right] + 2f \right). \] (98)

Notice that for functions \( f(R) \) which reproduce the observed expansion history of the universe, the minimum of this potential yields the general relativistic relationship between \( R \) and \( T \), i.e, \( R = -\kappa T \).

Hence, within a given object, a typical galaxy; for instance, far away from the center the scalar curvature starts off nearly at its asymptotic value, \( R_{\text{max}} \), and evolves with radial distance from the center. If the object is too small compared to the wavelength of the gravitational perturbation, then \( R \sim R_{\text{max}} \) throughout the object, and deviations from general relativity will be of order unity. Meanwhile, if the object is large enough then the scalar curvature is forced to the minimum of its potential and \( R = -\kappa T \) within some radius \( r_{\text{min}} \). Within that radius deviations from general relativity are highly suppressed, and thus one says that the object is “screened”. Finally, on the generic prospective for it to give rise to dark matter within extended gravity, we recall that a further argument in favor of the potential of modifying the gravity theory at the galactic scale so as to explain dark matter has been provided by Ellis [24].

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