Light propagation in the averaged universe

Samae Bagheri and Dominik J. Schwarz

Fakultät für Physik, Universität Bielefeld, Universitätsstr. 25, Bielefeld, Germany

E-mail: s_bagheri@physik.uni-bielefeld.de, dschwarz@physik.uni-bielefeld.de

Received April 10, 2014
Revised August 28, 2014
Accepted October 1, 2014
Published October 29, 2014

Abstract. Cosmic structures determine how light propagates through the Universe and consequently must be taken into account in the interpretation of observations. In the standard cosmological model at the largest scales, such structures are either ignored or treated as small perturbations to an isotropic and homogeneous Universe. This isotropic and homogeneous model is commonly assumed to emerge from some averaging process at the largest scales. We assume that there exists an averaging procedure that preserves the causal structure of space-time. Based on that assumption, we study the effects of averaging the geometry of space-time and derive an averaged version of the null geodesic equation of motion. For the averaged geometry we then assume a flat Friedmann-Lemaître (FL) model and find that light propagation in this averaged FL model is not given by null geodesics of that model, but rather by a modified light propagation equation that contains an effective Hubble expansion rate, which differs from the Hubble rate of the averaged space-time.

Keywords: redshift surveys, cosmic web

ArXiv ePrint: 1404.2185
1 Introduction

The standard model of modern cosmology describes large-scale structures as perturbations of an isotropic and homogeneous Universe, the Friedmann-Lemaître (FL) model. Measurements of the cosmic microwave background (CMB) and many other observations confirm that the standard model is a good description of the Universe. Despite of its success, the FL model is only a large-scale approximation to highly nonlinear structures at small scales. Consequently, one can ask how to justify this high degree of symmetry at the largest scales and how to connect the smallest scales to the largest ones. Eventually, we must not ignore the effects of local inhomogeneities from which an averaged space-time with certain symmetries seems to emerge. By local we refer to scales on which gravitationally bound structures exist, i.e. from $\sim 100$ Mpc down to the Planck scale. Above the 100 Mpc, the Universe appears to be statistically homogeneous and isotropic, but on smaller scales, unlike the FL model, it is inhomogeneous. For testing large-scale homogeneity, several tests have been applied to the data from the Sloan Digital Sky Survey [1] and the WiggleZ Dark Energy Survey [2]. In both cases a transition to homogeneity at scales of about 100 Mpc is found. In this work, we are specifically interested on the effects of the local inhomogeneities at and below the 100 Mpc scale on the propagation of light.

The averaging problem was introduced in general relativity by Shirkov and Fisher in 1963 [3]. They proposed a space-time averaging procedure, but it was not covariant such that a tensor did not remain to be a tensor after applying averaging. The issue was not very well known until 1984 when Ellis gave a description of the concept of backreaction from small to large structures [4]. The question was further considered by Futamase [5, 6] who studied the gravitational correlation by employing the metric perturbations and by Zotov and Stoeger [7], whose procedure was equivalent to the one by Shirkov and Fisher, hence not covariant.

Two breakthroughs in the study of the averaging problem and backreaction were achieved by Zalaletdinov [8, 9] in a covariant and exact way and by Buchert [10, 11], who restricted the problem to scalar quantities only. A connection to dark energy has been proposed in [12–15], and [16], who attempted to explain dark energy by means of a backreaction of small scale structures on the large scale evolution of the Universe, while many others like [17, 18] were completely against that idea. The question is still open. So far nobody could present a proof...
that would exclude this idea and nobody could prove that the backreaction effects are large enough to explain dark energy. However, it seems to be generally accepted that backreaction effects cannot be neglected if one is interested in precision cosmology. The idea has been later on discussed in [19–24] and many others.

The aim of this paper is to investigate the effects of averaging on light propagation in the Universe, or how to derive the equation of motion of light in an averaged description of the Universe from the null geodesic equation in the inhomogeneous universe. Therefore we ask if some effects can be seen in observations in the lumpy universe. For example how do averaged inhomogeneities affect the redshift of photons.

The motion of photons in an averaged geometry has already been studied in [25] and in a more precise way in [26, 27]. Yet in a different approach using a gauge invariant formalism, the averaged geometry on the past null cone has been introduced [28, 29]. This allows to average the luminosity-redshift relation [30–32]. The study of light propagation in inhomogeneous Swiss cheese models by simulating Hubble diagrams has been probed recently in [33–35].

Here we follow a new approach. We make the plausible assumption that an averaging procedure that respects the causal structure of space-time exists. Based on this and a second assumption specified in section 3, we derive an effective equation for light propagation in an averaged Universe.

The central finding of our work is that photons in an averaged Universe follow a FL geodesic equation of motion, but with the Hubble rate replaced by an effective Hubble rate that does not coincide with the Hubble rate that one would infer from the averaging of the space-time itself. In contrast to many previous studies, this result is not based on a perturbative approach and does not make use of a toy model.

The work is organized as follows. In the next section we introduce the concept of a covariant averaging of a tensor and briefly discuss what has been achieved in the works by Zalaletdinov and Buchert. It is essential for our work that a covariant procedure to average a space-time metric exists. As we show below, it is irrelevant for our purpose how this is defined in detail. In the third section we derive our central result — an effective equation for the propagation of light in an averaged space-time and in section four we evaluate that equation for a Universe that can be described by a FL model after averaging. The last two sections contain a discussion and a conclusion.

2 Averaging procedures of space-time

Averaging involves the integration of tensors over a (space-time or spatial or null) volume $V$, and is not easily well defined, because the result can change by changing the coordinates and is typically not unique. For treating this problem one can define the covariant averaging of tensors via bilocal operators, as proposed by Zalaletdinov [8, 9, 38], or one can simplify the problem and consider only scalar quantities and average them, as has been first proposed by Buchert [10, 11, 39].

Besides the question of how to average a tensor, general relativity is non-linear in the components of the metric tensor $g_{\mu \nu}$, and thus in general for the Einstein tensor $G_{\alpha \beta}$

\[
\langle G_{\alpha \beta}(g_{\mu \nu}) \rangle \neq G_{\alpha \beta}(\langle g_{\mu \nu} \rangle),
\]

where the brackets denote some averaging procedure.

In the procedure provided by Zalaletdinov one considers the bilocal extension of a tensor, e.g. for a vector $P^\alpha(x)$ one defines its extension as $\hat{P}^\alpha(x', x) = A^\alpha_{\alpha'}(x, x')P^{\alpha'}(x')$. The bilocal
operator \( A \) parallel transports an object at point \( x' \) to an object at a reference point \( x \). One possibility is to define the operator as the product of a basis of vector fields (the tetrad fields) at two different points \( x \) and \( x' \)

\[
A_a^\alpha(x, x') = e_a^\alpha(x) e_a'^\alpha(x'),
\]

(2.2)

where the latin index \( a \) labels the basis vectors. Then the average of the vector \( P^\alpha \) is defined as

\[
\langle P^\alpha \rangle = \frac{1}{V_\Sigma} \int_{\Sigma} d^4x' \sqrt{-g} \tilde{P}^\alpha(x', x),
\]

(2.3)

where \( V_\Sigma \) is the volume of the region \( \Sigma \),

\[
V_\Sigma = \int_{\Sigma} d^4x' \sqrt{-g'}.
\]

(2.4)

For higher rank tensors the bilocal extension works on each space-time index and the average is defined analogously. Especially this allows us to define the bilocal extension of the metric tensor and the definition of an averaged metric.

Zalaletdinov defines a line element for the macroscopic space-time

\[
ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu,
\]

(2.5)

where \( \bar{g}_{\mu\nu} \) is the macroscopic metric tensor, which is used to calculate the macroscopic Christoffel symbols, denoted by \( \langle \Gamma^\mu_{\nu\alpha} \rangle \). The so defined macroscopic Christoffel symbols guarantee that the macroscopic space-time is a Riemannian manifold itself and that there are no metric correlations \([38]\). Thus the averaged metric tensor \( \langle g_{\mu\nu} \rangle \) and the averaged inverse metric tensor \( \langle g^{\mu\nu} \rangle \) can be identified as

\[
\langle g_{\mu\nu} \rangle = \bar{g}_{\mu\nu}, \quad \langle g^{\mu\nu} \rangle = \bar{g}^{\mu\nu}.
\]

(2.6)

Now Zalaletdinov defines a macroscopic Riemann tensor

\[
M^\mu_{\nu\alpha\beta} = \partial_\alpha \langle \Gamma^\nu_{\beta\gamma} \rangle - \partial_\beta \langle \Gamma^\mu_{\gamma\alpha} \rangle + \langle \Gamma^\mu_{\sigma\beta} \rangle \langle \Gamma^\nu_{\gamma\sigma} \rangle - \langle \Gamma^\mu_{\sigma\gamma} \rangle \langle \Gamma^\nu_{\sigma\alpha} \rangle,
\]

(2.7)

which is different from the average of the microscopic Riemann tensor \( \langle R^\mu_{\nu\alpha\beta} \rangle \). This non-perturbative approach is also called macroscopic gravity. One considers a macroscopic description of gravity based on a continuous matter model, instead of a microscopic description in which matter would be described by a discrete model.

Following this structure the obtained macroscopic field equations are

\[
(g^{\beta\rho}) M_{\mu\beta} - \frac{1}{2} \delta^\rho_\gamma (g^{\mu\nu}) M_{\mu\nu} = 8\pi G \langle T^\rho_\gamma \rangle + \langle g^{\mu\nu} \rangle \left( Z^\alpha_{\mu\nu\gamma} - \frac{1}{2} \delta^\rho_\gamma Z^\alpha_{\mu\nu\alpha} \right),
\]

(2.8)

where

\[
Z^\alpha_{\beta[\gamma\rho]} = \langle \Gamma^\mu_{\beta[\gamma} \Gamma^\mu_{\rho]} \rangle - \langle \Gamma^\mu_{\beta[\gamma} \rangle \langle \Gamma^\mu_{\rho]} \rangle.
\]

(2.9)

underlined indices are not included in anti-symmetrization and \( Z^\alpha_{\mu\nu\beta} = 2Z^\alpha_{\mu[\rho| \nu\beta]} \). Some solutions of macroscopic field equations have been studied in \([40-42]\).

A technically simpler approach has been proposed by Buchert. He decomposed Einstein equations into a set of dynamical equations for scalar quantities. The disadvantage of this approach is that the set of scalar equations is not closed and an assumption like an effective
equation of state has to be introduced. Despite of its limitation for using only scalars, Buchert’s formalism is the only formalism apart from Zalaletdinov’s macroscopic gravity, that treats the inhomogeneities in an exact way, and gives new insight.

The metric can be written in the synchronous gauge
\[ ds^2 = -dt^2 + (3)g_{\mu\nu}dx^\mu dx^\nu, \] (2.10)
where \((3)g_{\mu\nu}\) is the metric on hypersurface of constant \(t\). The spatial average of a scalar quantity \(f\) is defined on these hypersurfaces as
\[ \langle f \rangle (t, \mathbf{x}) = \frac{\int d^3\mathbf{x} \sqrt{g(t, \mathbf{x})} f(t, \mathbf{x})}{\int d^3\mathbf{x} \sqrt{g(t, \mathbf{x})}}. \] (2.11)

The averaged scale factor is defined via the comoving volume on spatial hypersurfaces
\[ a_D(t) = \left( \frac{\int d^3\mathbf{x} \sqrt{g(t, \mathbf{x})} \int d^3\mathbf{x} \sqrt{g(t_0, \mathbf{x})}}{\int d^3\mathbf{x} \sqrt{g(t, \mathbf{x})}} \right)^{\frac{1}{3}}. \] (2.12)

A key point here is that the time evolution and spatial averaging do not commute
\[ \partial_t \langle f \rangle - \langle \partial_t f \rangle = \langle f \theta \rangle - \langle f \rangle \langle \theta \rangle, \] (2.13)
where \(\theta = (\sqrt{\det g})^{-1}\partial_t (\sqrt{\det g})\) is the expansion rate. By considering the expansion shear tensor \(\sigma_{\mu\nu}\), a kinematic backreaction term is defined as \(Q_D \equiv \frac{2}{3}(\langle \theta^2 \rangle - \langle \theta \rangle^2) - 2\langle \sigma^2 \rangle\). For a vanishing cosmological constant and irrotational dust (with mass density \(\rho\)) this leads to Buchert’s equations (see e.g. \[43\])
\[ 3\frac{\dot{a}_D^2}{a_D^2} = 8\pi G\langle \rho \rangle_D - \frac{1}{2}\langle R \rangle_D - \frac{1}{2}Q_D, \] (2.14)
\[ 3\frac{\dot{a}_D}{a_D} = -4\pi G\langle \rho \rangle_D + Q_D, \] (2.15)
where \(R_D\) denotes the 3-Ricci scalar and \(Q_D\) and \(\langle R \rangle_D\) have to obey to
\[ a_D^{-2}\left(a_D^2\langle R \rangle_D \right) + a_D^{-6}(a_D^6 Q_D) = 0. \] (2.16)

The most important result of Buchert’s approach is that it is possible to cast a spatially volume averaged irrotational dust model in the form of a FL model with an effective mass density and pressure.

We notice that Buchert’s formalism has been considered in FL space-time where the Weyl tensor vanishes and thus bundles of light rays are subject to Ricci focussing (i.e. associated with a smooth distribution of matter). However, in a clumpy universe, light rays propagate in underdense regions and are sensitive to Weyl focussing (i.e. induced by the gradient of the gravitational potential). The issue of relating Weyl focussing of point like sources to Ricci focussing of smooth matter sources has been considered recently in \[44\].

Recently, Skarke \[45\] realized that one could avoid the non-commutativity of averaging and time evolution of scalars by utilizing a mass weighted averaging scheme. Other averaging schemes that invoke the past light cone will be discussed in section 6 of this work.

A problematic aspect of spatial averaging is that we do not observe spatial volumes, but rather null volumes and that Buchert’s and Zalaletdinov’s approaches neglect possible effects on the propagation of photons.
3 Propagation of light in an averaged space-time

In standard cosmology the background geometry is used for observing the large scale of universe. Speaking of background means the homogeneous isotropic flat FL universe with neglecting the details of small scales and local inhomogeneities.

Observational cosmology is based on light trajectories and the paths of light are on null geodesics. One of the significant effects of inhomogeneity is on the light trajectories. Therefore we should see these effects on observations in the lumpy universe.

Some aspects of this are very well understood and studied in great depth, e.g. CMB photons are related to density fluctuations by the Sachs-Wolfe effect. The integrated Sachs-Wolfe effect [36], which is caused by gravitational redshift, and the gravitational lensing [37] play important roles in interpreting the effects of light propagation in the universe.

The crucial issue is how we justify the smaller scales to the background geometry and transform from lumpy universe to the smoothed one.

The key point here will be to use an averaged metric that describes the smoothed manifold. This allows us to consider the paths of light propagation in the averaged space-time. But the metric tensor cannot be averaged easily, and many current approaches of averaging cannot be used to construct such an averaged metric. The exception is the averaging procedure defined by Zalaletdinov and we view it as a proof of existence of such an average. Thus we are going to assume that the average of a metric is a metric.

As a second critical assumption, we are proposing that the averaged space-time agrees perfectly with the causal structure of the microscopic space-time. We think that this is a plausible assumption. At least this assumption is implicitly made in modern cosmology when it is assumed that the light rays in the Universe that is assumed to be isotropic and homogeneous on large scales are null in a FL model. The example of the bilocal extension (2.2) has this property. Unfortunately, it is not useful for our purpose as it gives \( \langle g_{\mu\nu} \rangle = g_{\mu\nu} \). Nevertheless, this proves that at least one bilocal extension that satisfies both required properties exists.

Let \( k^\mu \) denote a null vector field. Its geodesic equation reads

\[
k^\mu_{\nu,\nu} + \Gamma^\mu_{\nu\rho} k^\rho = 0,
\]

where the Christoffel symbol can be calculated from the metric,

\[
\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (g_{\sigma\rho,\nu} + g_{\sigma\nu,\rho} - g_{\nu\rho,\sigma}).
\]

We cancel out the inverse metric in (3.1) by a multiplication with \( g_{\lambda\mu} \) and using

\[
(k^\mu g_{\mu\nu} k^\nu)_{,\lambda} = 0.
\]

We arrive at a more convenient form of the geodesic equation for null vector fields,

\[
(k^\mu_{,\nu} g_{\lambda\mu} + k^\mu_{,\lambda} g_{\mu\nu} + k^\mu g_{\mu\lambda\nu}) k^\nu = 0.
\]

The big advantage for our purpose is that this form is linear in the metric.

As a next step, we average this equation in the following sense: we consider a particular light ray, so \( k^\mu \) is not subject to the averaging, but the metric and it’s derivative are. Thus we assume that averaging and contractions with \( k^\mu \) commute. This assumption might not hold for all possible averaging schemes, but we think that this is a sensible assumption to make.
Finally, we also assume that derivatives of the wave vector are not subject to averages. The reason is again that we take to point of view that we only average over the metric of space-time, but consider the same light ray in the averaged and microscopic space-time. Thus the coordinate derivatives of the wave vector should not be affected by this averaging procedure. In other words, the same coordinate values correspond to the same physical event in the averaged and microscopic space-times. Distances, angles and time intervals between physical events are different however.

A consequence of the two assumptions mentioned above is that we preserve the null condition,

\[ (k^\mu g_{\mu \nu} k^\nu) = k^\mu (g_{\mu \nu} k^\nu) = 0. \tag{3.5} \]

Note that the derivative of an averaged metric is different from the average of the derivative of the metric, i.e. \( (g_{\mu \lambda ; \nu}) \neq (g_{\mu \lambda})_{;\nu} \).

Therefore we arrive at an averaged version of (3.4),

\[ \left( k^\mu (g_{\mu \lambda}) + k^\mu (g_{\mu \lambda} ; \nu) + k^\mu (g_{\mu \lambda})_{;\nu} \right) k^\nu = k^\mu T_{\mu \lambda \nu} k^\nu \equiv I_\lambda, \tag{3.6} \]

where \( T_{\mu \lambda \nu} \equiv (g_{\mu \lambda})_{;\nu} - (g_{\mu \lambda})_{;\nu} \). The left hand side represents the equation of a null geodesic of an averaged metric and the right hand side represents the modification due to averaging.

Let us now prove two important properties. Firstly, it turns out that the object \( T_{\mu \lambda \nu} \) is a tensor. This is a non-trivial and non-obvious statement. Secondly, we can show that \( I_\lambda k^\lambda = 0 \).

The tensor property follows from a brute force argument, based on the well known transformation properties of vectors and tensors (the averaged metric has been assumed to be a tensor itself)

\[ k^\mu (x') = \frac{\partial x'^\mu}{\partial x^\alpha} k^\alpha (x), \tag{3.7} \]

and

\[ (g'_{\mu \nu}) (x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} (g_{\alpha \beta}) (x), \tag{3.8} \]

and their derivatives

\[ \frac{\partial k^\mu}{\partial x'^\nu} = \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial k^\alpha}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial^2 x^\mu}{\partial x^\alpha \partial x^\beta} k^\alpha, \tag{3.9} \]

and

\[ \frac{\partial (g'_{\mu \nu})}{\partial x'^\lambda} = \frac{\partial x^\sigma}{\partial x'^\lambda} \frac{\partial^2 x^\mu}{\partial x^\sigma \partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} (g_{\alpha \beta}) + \frac{\partial x^\sigma}{\partial x'^\lambda} \frac{\partial x^\alpha}{\partial x^\sigma} \frac{\partial^2 x^\beta}{\partial x^\nu \partial x^\alpha} (g_{\alpha \beta}) + \frac{\partial x^\sigma}{\partial x'^\lambda} \frac{\partial x^\alpha}{\partial x^\sigma} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial (g_{\alpha \beta})}{\partial x^\sigma}. \tag{3.10} \]

We then check explicitly that the left hand side is a vector and conclude that \( k^\mu k^\nu T_{\mu \lambda \nu} \) transforms as a vector. For doing so, one has to use the relation

\[ \frac{\partial^2 x^\mu}{\partial x^\gamma \partial x^\alpha} \frac{\partial x^\beta}{\partial x^\mu} = \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial^2 x^\beta}{\partial x^\gamma \partial x^\mu}, \tag{3.11} \]

besides the null condition. Finally we can argue that as \( I_\lambda = k^\mu T_{\mu \lambda \nu} k^\nu \) is a vector, the symmetric part, \( T_{\mu \lambda \nu}^{\text{sym}} = (T_{\mu \lambda \nu} + T_{\nu \lambda \mu})/2 \), must also be a tensor.

Notice that by construction \( T_{\mu \lambda \nu} \) is a symmetric object under the exchange of \( \mu \) and \( \lambda \), \( T_{\mu \lambda \nu} = T_{\lambda \mu \nu} \), but not necessarily a tensor. Also note that \( T_{\mu \lambda \nu}^{\text{sym}} \neq T_{\lambda \mu \nu}^{\text{sym}} \). In fact only \( T_{\mu \lambda \nu}^{\text{sym}} \) is of relevance to the averaged null geodesic equation.
The second property, $I_\lambda k^\lambda = 0$, follows from contracting the left hand side of (3.6) with $k^\lambda$ and using the fact that the null property of the wave vector is preserved. A straightforward calculation shows that the left hand side vanishes identically and thus the second statement holds.

4 Propagation of light through an averaged Universe

Let us denote the metric of a FL model by $\bar{g}_{\mu\nu}$ and the four-velocity of a comoving observer (the one that sees the light ray under consideration) by $\bar{u}_\mu$. The observed photon frequency is then given by $\omega = -\bar{u}_\mu k^\mu$.

We assume that $\langle g_{\mu\nu} \rangle = \bar{g}_{\mu\nu}$, as the cosmological principle tells us that we should be able to describe the averaged Universe by an isotropic and homogeneous model. We still do not define how this average works in detail, but we assume that it exists and argue that observations confirm that such an approach must be possible.

Applying the cosmological principle to the tensor $T^\text{sym}_{\mu\lambda\nu}$, we can write down the most general algebraic structure compatible with isotropy and homogeneity. It is

$$T^\text{sym}_{\mu\lambda\nu} = f_1 \frac{1}{2} (\bar{g}_{\mu\lambda} \bar{u}_\nu + \bar{g}_{\nu\lambda} \bar{u}_\mu) + f_2 \bar{u}_\mu \bar{u}_\lambda \bar{u}_\nu + f_3 \bar{g}_{\mu\nu} \bar{u}_\lambda,$$

as $\bar{u}_\mu$ and $\bar{g}_{\mu\nu}$ are the only non-trivial tensors of first and second rank that can be used to construct a third rank tensor that is symmetric in two of its indices. $f_1$, $f_2$ and $f_3$ are three functions of cosmic time $t$ only, which cannot be fixed by pure symmetry considerations. However, only the combination

$$I_\lambda = k^\mu T^\text{sym}_{\mu\lambda\nu} k^\nu = f_1 (\omega) \bar{g}_{\mu\lambda} k^\mu + f_2 \omega^2 \bar{u}_\lambda,$$

enters the averaged light geodesic equation. We further consider the contraction

$$I_\lambda k^\lambda = -f_2 \omega^3,$$

which must vanish as shown in the previous section and thus $f_2 \equiv 0$.

Thus the inhomogeneity of the light propagation equation is given by

$$I_\lambda = -f_1 \omega \bar{g}_{\mu\lambda} k^\mu,$$

and all effects of averaging on the light propagation must be encoded in a single function $f_1(t)$. Without any further knowledge, this generic structure of the inhomogeneity of the null geodesic equation allows us to make some non-trivial and generic statements about light propagation in an averaged Universe.

The Hubble rate $H = \dot{a}/a$, where $a(t)$ denotes the scale factor of the averaged FL metric and the dot denotes a derivative with respect to cosmic time. For a comoving observer with $\bar{u}_\mu = (-1,0)$ we find $k^0 = \omega$ and $k^i = \omega e^i/a$, where $e^i$ is a spatial unit vector, indicating the spatial direction the light ray is pointing at.

Let us first look at the time component of the averaged null geodesic equation. The left hand side is well known from the equation of null geodesic motion in the FL model,

$$(-\omega) \left( \dot{\omega} + \frac{\omega}{a} \dot{\omega} + H \omega \right) = I_0 = f_1 \omega^2.$$
The equation might be more familiar in terms of the affine parameter
\[
\frac{D \omega}{d \lambda} = \frac{d \omega}{d \lambda} = k^i \frac{\partial \omega}{\partial x^i} = \omega \left( \dot{\omega} + \frac{e_i}{a} \omega_i \right),
\]
(4.6)
and thus
\[
\frac{d \omega}{d \lambda} + H_{\text{eff}} \omega^2 = 0, \quad H_{\text{eff}} \equiv H + f_1.
\]
(4.7)
For \( f_1 = 0 \) this reduces to the famous result \( \omega \propto 1/a \), the redshift of photons (note that \( dt = \omega d\lambda \)). Thus we conclude that any \( f_1 \neq 0 \) leads to a modification of the redshift of photons, so we would expect that the actual redshift of a photon in an averaged description of an inhomogeneous universe must differ from the redshift that the same photon would have in the corresponding homogeneous and isotropic universe.

The spatial components of the modified light propagation equation becomes
\[
a \gamma_{ij} e^j \omega \left( \dot{\omega} + \frac{e_k}{a} \omega_k + H \omega \right) + a \omega^2 \gamma_{ij} \left( \dot{e}^j + e^j_k e^k \right) = I_i = - \omega^2 a \gamma_{ij} e^j f_1,
\]
(4.8)
where \( \tilde{g}_{ij} = a^2 \gamma_{ij} \) and \( \gamma \) denotes a covariant derivative with respect to the 3-metric \( \gamma_{ij} \). By means of (4.5), equation (4.8) can be further simplified to yield
\[
\dot{e}^j + e^j_k e^k = 0,
\]
(4.9)
or in terms of the affine parameter
\[
\frac{D e_i}{d \lambda} = 0,
\]
(4.10)
i.e. light rays propagate along straight lines. This result holds for an exact FL models and for equation of motion for light in the averaged Universe.

To sum up, based on the principles of statistical isotropy and homogeneity, there is one global effect on the propagation of light, which is a modification of the redshift of a photon, which can be described by an effective Hubble expansion rate.

5 Discussion

In order to estimate the function \( f_1(t) \) we consider an irrotational model without gravitational waves. Then an ansatz for the metric that allows for density perturbations and can easily be compared with the zero shear gauge (or longitudinal Newtonian gauge) of linear perturbation theory is
\[
\begin{align*}
    ds^2 &= -e^{2\phi} dt^2 + a^2(t) e^{-2\psi} \delta_{ij} dx^i dx^j. 
\end{align*}
\]
(5.1)
We split the exact metric \( g_{\mu\nu} = \tilde{g}_{\mu\nu} + \delta g_{\mu\nu} \), where we do not make the assumption that \( \delta g_{\mu\nu} \) is small. By construction \( \langle \delta g_{\mu\nu} \rangle = 0 \).

We now evaluate
\[
T^{\text{sym}}_{00} = -\langle \delta g_{00,0} \rangle = 2 \langle e^{2\phi} \dot{\phi} \rangle.
\]
(5.2)
Note that \( \tilde{g}_{00} = -1 \) and thus its derivative vanishes before and after averaging. By comparing this result with our ansatz for \( T^{\text{sym}}_{00} = f_1 \) we have
\[
f_1 = 2 \langle e^{2\phi} \dot{\phi} \rangle.
\]
(5.3)
Alternatively $\delta g_{ij} = a^2 (e^{-2\psi} - 1) \gamma_{ij} \psi$ allows us to estimate $f_1$ from $T^{\text{sym}}_{ij0}$, where now $f_1 = 2\langle e^{2\psi} \dot{\psi} \rangle$. Without anisotropic pressure, the off-diagonal components of the Einstein tensor must vanish, which implies $\phi = \psi$ (exact!) and thus both estimates are consistent with each other.

As already stated above, $\langle e^{2\phi} \rangle \equiv 1$ (by construction). However, since averaging and time derivative do not commute in general, $f_1$ is in general non-zero. In linear perturbation theory, $\phi = 0$ in the Einstein-de Sitter model (EdS), but this is not the case for the $\Lambda$CDM model. For higher orders in perturbation, both the EdS and the $\Lambda$CDM model have $\phi \neq 0$ (these are the integrated Sachs-Wolfe effect [36] and the Rees-Sciama effect [46]). Consequently, this implies that for the fully nonlinear theory we have $\langle e^{2\phi} \dot{\phi} \rangle \neq 0$ in general.

Let us now estimate qualitatively what are the effects of the effective Hubble expansion rate $H_{\text{eff}}$. By means of (4.7), $H_{\text{eff}} = H + 2\langle e^{\phi} \dot{\phi} \rangle$. In the following, we define the density contrast w.r.t. the averaged matter density $\bar{\rho}(t)$, i.e.

$$\delta(r, t) \equiv \frac{\rho(r, t) - \bar{\rho}(t)}{\bar{\rho}(t)}. \quad (5.4)$$

$\rho \geq 0$ implies $\delta \geq -1$. For an over-dense, collapsing region ($\delta > 0$), we expect from Newtonian reasoning that $\dot{\phi} < 0$. Similarly, for an underdense, expanding region ($\delta < 0$), $\dot{\phi} > 0$. However, if the overdense region is virialized, its gravitational potential does not change any more and we expect no effect. Thus it is impossible to predict the sign of $f_1$ without a detailed investigation. Another important aspect is that most of the volume of the Universe is under-dense. An arbitrary light-ray will typically pass through a dominantly under-dense universe, and thus we expect that $H_{\text{eff}} > H$ at times long after the formation of cosmic structure started. On the other hand, observed light is typically emitted in an over-dense and observed in an over-dense region. Thus for objects at not too far distances we expect that over-densities dominate the trajectory of the light ray. For a quantitative discussion, which is beyond the scope of this work, some numerical simulations are necessary.

We can nevertheless conclude that the one-to-one association of redshift with the scale factor and thus with cosmic time that we know from the standard model of cosmology is not possible if the effect from the averaged description is taken into account.

Let us finally put our work in the context of a previous result. In the work of Räsänien [27, 47] the propagation of a bundle of light has been studied. The redshift $z \equiv (\omega_s - \omega_o)/\omega_o$, where the suffixes denote source and observer, is found to be

$$1 + z = \exp \left( \int_{\lambda_s}^{\lambda_o} d\lambda \omega \left[ \frac{1}{3} \theta + \sigma_{\mu\nu} e^\mu e^\nu \right] \right), \quad (5.5)$$

where $\sigma_{\mu\nu}$ denotes the shear and $\theta$ the expansion rate and $e^\mu$ denotes as above the spatial direction of light propagation. This result agrees very well with our result in equation (4.7), which after integration can be written as (using $dt = \omega d\lambda$)

$$1 + z = \exp \left[ \int_{t_o}^{t} H_{\text{eff}} dt \right]. \quad (5.6)$$

Räsänien argued that the shear is negligible for the averaged geometry, and that the only important contribution would come from the averaged expansion rate. Therefore the distance redshift relation is in terms of the averaged expansion rate. In [48] it has been discussed that if the metric remains close to a FL model, the change in redshift respect to its background value is small.
6 Conclusion

In this work, we have considered the propagation of light rays in an averaged space-time. Our central result is a modification of the equation of null geodesic motion, see (3.6). This new equation of motion is a fully covariant vector equation for the wave-vector $k^\mu$. Rays describing the propagation of light in an averaged space-time are generated by this wave vector, which is null w.r.t. the averaged space-time. In order to prove those points we assume that the averaged space-time (pseudo-)metric is a tensor and that it respects the causal structure of the microscopic space-time. That such averaging procedures exist has been shown by Zalaletdinov \cite{38}. As we consider a fixed light ray (source and observer are fixed events on the manifold) we think that it is justified not to average the wave vector and its derivative, but to just average the metric and its derivatives.

We then apply this light propagation equation (recall, it is not the geodesic equation of the averaged space-time) to a cosmological model. We assume that the averaged metric is a flat, spatially isotropic and homogeneous (as suggested by the success of the standard model of cosmology). We have shown that the relation between photon frequency and affine parameter is modified. This modification can be expressed as an effective Hubble rate, as shown in (4.7). Our result is in perfect agreement with previous non-perturbative investigations \cite{48} and with the results of the study of toy models, like the Swiss cheese model \cite{35}. Also perturbative studies are in line with our findings \cite{32}.

So far we restricted our attention to the study of a single light ray. The next logical step is to study the equation of geodesic deviation in order to ask if an analogous modification occurs, which would allow us to find a modification to the luminosity and angular diameter distances. In this context it will be interesting to ask if it is true that a microscopic Weyl focusing leads to an effective Ricci focusing after averaging.

We thus have shown that the Hubble rate associated with the averaged space-time metric does not necessarily coincide with the effective Hubble rate that should be considered for photon propagation. A quantitative study of the order of magnitude of the effect is beyond the scope of this work. The most important result of this work is that the averaging effects on light propagation can be absorbed into an effective Hubble rate. This might be one of the more fundamental reasons for the great success of the Friedmann-Lemaître models.

Acknowledgments

We thank Thomas Buchert, Domenico Giulini, Seshadri Nadathur and Harald Skarke for valuable discussions and comments. We acknowledge support by Deutsche Forschungsgemeinschaft (DFG) within the Research Training Group 1620 Models of Gravity.

References

[1] D.W. Hogg, D.J. Eisenstein, M.R. Blanton, N.A. Bahcall, J. Brinkmann et al., Cosmic homogeneity demonstrated with luminous red galaxies, Astrophys. J. 624 (2005) 54 [astro-ph/0411197] [insPIRE].

[2] M. Scrimgeour, T. Davis, C. Blake, J.B. James, G. Poole et al., The WiggleZ Dark Energy Survey: the transition to large-scale cosmic homogeneity, Mon. Not. Roy. Astron. Soc. 425 (2012) 116 [arXiv:1205.6812] [insPIRE].
[3] M.F. Shirokov and I.Z. Fisher, Isotropic space with discrete gravitational field sources. On the theory of non homogeneous isotropic universe, Sov. Astron. J. 6 (1963) 699, reprinted in: Gen. Rel. Grav. 30 (1998) 1411.

[4] G.F.R. Ellis, Relativistic cosmology: its nature, aims and problems, in General Relativity and Gravitation, B. Bertotti et al. eds., Reidel, Dordrecht (1984) pp. 215.

[5] T. Futamase, Approximation scheme for constructing a clumpy universe in General Relativity, Phys. Rev. Lett. 61 (1988) 2175.

[6] T. Futamase, Averaging of a locally inhomogeneous realistic universe, Phys. Rev. D 53 (1996) 681 [inSPIRE].

[7] N.V. Zotov and W.R. Stoeger, Averaging Einsteins equations, Class. Quant. Grav. 9 (1992) 1023.

[8] R.M. Zalaletdinov, Averaging out the Einstein equations and macroscopic space-time geometry, Gen. Rel. Grav. 24 (1992) 1015 [inSPIRE].

[9] R. Zalaletdinov, Towards a theory of macroscopic gravity, Gen. Rel. Grav. 25 (1993) 673 [inSPIRE].

[10] T. Buchert, On average properties of inhomogeneous fluids in general relativity. 1. Dust cosmologies, Gen. Rel. Grav. 32 (2000) 105 [gr-qc/9906016] [inSPIRE].

[11] T. Buchert, On average properties of inhomogeneous fluids in general relativity: Perfect fluid cosmologies, Gen. Rel. Grav. 33 (2001) 1381 [gr-qc/0102049] [inSPIRE].

[12] D.J. Schwarz, Accelerated expansion without dark energy, astro-ph/0209584 [inSPIRE].

[13] S. Räsänen, Dark energy from backreaction, JCAP 02 (2004) 003 [astro-ph/0311257] [inSPIRE].

[14] D.L. Wiltshire, Viable inhomogeneous model universe without dark energy from primordial inflation, (2005) [gr-qc/0503099].

[15] D.L. Wiltshire, Dark energy without dark energy, in Dark Matter in Astroparticle and Particle Physics, H.V. Klapdor-Kleingrothaus and G.F. Lewis eds., (2008) pp. 565 arXiv:0712.3984 [inSPIRE].

[16] E.W. Kolb, S. Matarrese and A. Riotto, On cosmic acceleration without dark energy, New J. Phys. 8 (2006) 322 [astro-ph/0506534] [inSPIRE].

[17] A. Ishibashi and R.M. Wald, Can the acceleration of our universe be explained by the effects of inhomogeneities?, Class. Quant. Grav. 23 (2006) 235 [gr-qc/0509108] [inSPIRE].

[18] S.R. Green and R.M. Wald, A new framework for analyzing the effects of small scale inhomogeneities in cosmology, Phys. Rev. D 83 (2011) 084020 [arXiv:1011.4920] [inSPIRE].

[19] A.A. Coley, Averaging in cosmological models, arXiv:1001.0791 [inSPIRE].

[20] E.W. Kolb, V. Marra and S. Matarrese, Cosmological background solutions and cosmological backreactions, Gen. Rel. Grav. 42 (2010) 1399 [arXiv:0901.4566] [inSPIRE].

[21] S. Räsänen, Accelerated expansion from structure formation, JCAP 11 (2006) 003 [astro-ph/0607626] [inSPIRE].

[22] N. Li and D.J. Schwarz, On the onset of cosmological backreaction, Phys. Rev. D 76 (2007) 083011 [gr-qc/0702043] [inSPIRE].

[23] N. Li and D.J. Schwarz, Scale dependence of cosmological backreaction, Phys. Rev. D 78 (2008) 083531 [arXiv:0710.5073] [inSPIRE].

[24] C. Clarkson, G. Ellis, J. Larena and O. Umeh, Does the growth of structure affect our dynamical models of the universe? The averaging, backreaction and fitting problems in cosmology, Rept. Prog. Phys. 74 (2011) 112901 [arXiv:1109.2314] [inSPIRE].
[25] A.A. Coley, Null geodesics and observational cosmology, arXiv:0812.4565 [inSPIRE].
[26] S. Räsänen, Evaluating backreaction with the peak model of structure formation, JCAP 04 (2008) 026 [arXiv:0801.2692] [inSPIRE].
[27] S. Räsänen, Light propagation in statistically homogeneous and isotropic dust universes, JCAP 02 (2009) 011 [arXiv:0812.2872] [inSPIRE].
[28] M. Gasperini, G. Marozzi and G. Veneziano, Gauge invariant averages for the cosmological backreaction, JCAP 03 (2009) 011 [arXiv:0901.1303] [inSPIRE].
[29] M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, Light-cone averaging in cosmology: Formalism and applications, JCAP 07 (2011) 008 [arXiv:1104.1167] [inSPIRE].
[30] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, Backreaction on the luminosity-redshift relation from gauge invariant light-cone averaging, JCAP 04 (2012) 036 [arXiv:1202.1247] [inSPIRE].
[31] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, Do stochastic inhomogeneities affect dark-energy precision measurements?, Phys. Rev. Lett. 110 (2013) 021301 [arXiv:1207.1286] [inSPIRE].
[32] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, Average and dispersion of the luminosity-redshift relation in the concordance model, JCAP 06 (2013) 002 [arXiv:1302.0740] [inSPIRE].
[33] P. Fleury, H. Dupuy and J.-P. Uzan, Interpretation of the Hubble diagram in a nonhomogeneous universe, Phys. Rev. D 87 (2013) 123526 [arXiv:1302.5308] [inSPIRE].
[34] P. Fleury, H. Dupuy and J.-P. Uzan, Can all cosmological observations be accurately interpreted with a unique geometry?, Phys. Rev. Lett. 111 (2013) 091302 [arXiv:1304.7791] [inSPIRE].
[35] P. Fleury, Swiss-cheese models and the Dyer-Roeder approximation, JCAP 06 (2014) 054 [arXiv:1402.3123] [inSPIRE].
[36] R.K. Sachs and A.M. Wolfe, Perturbations of a cosmological model and angular variations of the microwave background, Astrophys. J. 147 (1967) 73 [Gen. Rel. Grav. 39 (2007) 1929] [inSPIRE].
[37] A. Blanchard and J. Schneider, Gravitational lensing effect on the fluctuations of the cosmic background radiation, Astron. Astrophys. 184 (1987) 1.
[38] R. Zalaletdinov, The Averaging Problem in Cosmology and Macroscopic Gravity, Int. J. Mod. Phys. A 23 (2008) 1173 [arXiv:0801.3256] [inSPIRE].
[39] T. Buchert, Dark Energy from Structure: A Status Report, Gen. Rel. Grav. 40 (2008) 467 [arXiv:0707.2153] [inSPIRE].
[40] A.A. Coley, N. Pelavas and R.M. Zalaletdinov, Cosmological solutions in macroscopic gravity, Phys. Rev. Lett. 95 (2005) 151102 [gr-qc/0504115] [inSPIRE].
[41] A.A. Coley and N. Pelavas, Averaging in Spherically Symmetric Cosmology, Phys. Rev. D 75 (2007) 043506 [gr-qc/0607079] [inSPIRE].
[42] R.J. van den Hoogen, A Complete Cosmological Solution to the Averaged Einstein Field Equations as found in Macroscopic Gravity, J. Math. Phys. 50 (2009) 082503 [arXiv:0909.0070] [inSPIRE].
[43] T. Buchert, Toward physical cosmology: focus on inhomogeneous geometry and its non-perturbative effects, Class. Quant. Grav. 28 (2011) 164007 [arXiv:1103.2016] [inSPIRE].
[44] C. Clarkson, G.F.R. Ellis, A. Faltenbacher, R. Maartens, O. Umeh et al., (Mis-)Interpreting supernovae observations in a lumpy universe, Mon. Not. Roy. Astron. Soc. 426 (2012) 1121 [arXiv:1109.2484] [inSPIRE].
[45] H. Skarke, *Inhomogeneity implies Accelerated Expansion*, *Phys. Rev. D* \textbf{89} (2014) 043506 [arXiv:1310.1028] [inSPIRE].

[46] M.J. Rees and D.W. Sciama, *Large scale Density Inhomogeneities in the Universe*, *Nature* \textbf{217} (1968) 511 [inSPIRE].

[47] S. Räsänen, *Light propagation in statistically homogeneous and isotropic universes with general matter content*, *JCAP* \textbf{03} (2010) 018 [arXiv:0912.3370] [inSPIRE].

[48] S. Räsänen, *Light propagation and the average expansion rate in near-FRW universes*, *Phys. Rev. D* \textbf{85} (2012) 083528 [arXiv:1107.1176] [inSPIRE].