RANDOM DIOPHANTINE EQUATIONS, I

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Abstract. We consider additive diophantine equations of degree $k$ in $s$ variables and establish that whenever $s \geq 3k + 2$ then almost all such equations satisfy the Hasse principle. The equations that are soluble form a set of positive density, and among the soluble ones almost all equations admit a small solution. Our bound for the smallest solution is nearly best possible.

I. Introduction.

In this memoir, we investigate the solubility of diagonal diophantine equations

$$a_1 x_1^k + a_2 x_2^k + \ldots + a_s x_s^k = 0,$$

and the distribution of their solutions. This is a theme that has received much interest in the past (see Vaughan [22], Vaughan and Wooley [23], Heath-Brown [10], Swinnerton-Dyer [19] and the extensive bibliographies in [22, 23]). Our main concern is with the validity of the Hasse principle, and with a bound for the smallest non-zero solution in integers whenever such a solution exists. The approach is of a statistical nature. Very roughly speaking, we shall show that whenever $s \geq 3k + 2$ and the vector $a = (a_1, \ldots, a_s) \in \mathbb{Z}^s$ is chosen at random, then almost surely the Hasse principle holds for (1.1), and if there are solutions in integers, not all zero, then there is one with $|x|$ not much larger than $|a|^{1/(s-k)}$. Here and later, we write $|x| = \max |x_j|$. The bound on the smallest solutions turns out to be nearly best possible.

We now set the scene to describe our results in precise form. Throughout this memoir, suppose that $k \in \mathbb{N}$, $k \geq 2$. We reserve the letter $a_j$ with and without subscripts, to denote non-zero integers only. This applies also when $a_j$ appears in a summation condition. Similarly, $\mathbf{a} = (a_1, \ldots, a_s)$ denotes a vector with non-zero integral coordinates. Furthermore, since (1.1) has the trivial solution $x = 0$, it will be convenient to describe the equation (1.1) as soluble over a given field if there exists a solution in that field other than the trivial one. If (1.1) is soluble over $\mathbb{R}$ and over $\mathbb{Q}_p$ for all primes $p$, then (1.1) is called locally soluble. We denote by $C = C(k, s)$ the set of all $\mathbf{a}$ with $a_j \in \mathbb{Z}\{0\}$ for which (1.1) is locally soluble. Note that whenever (1.1) is soluble over $\mathbb{Q}$, then $\mathbf{a} \in C$. The reverse implication is known as the Hasse principle for the equation (1.1). Recall that when $k = 2$, then the Hasse principle holds for any $s$, as a special case of the Hasse-Minkowski theorem.

Whenever $s > 2k$, a formal use of the Hardy-Littlewood method leads one to expect an asymptotic formula for the number $\varrho_\mathbf{a}(B)$ of solutions of (1.1) in integers.
\( x_j \) satisfying \( 1 \leq |x_j| \leq B \) for \( 1 \leq j \leq s \). This takes the shape

\[
q_a(B) = B^{s-k} J_a \prod_p \chi_p(a) + o(B^{s-k}) \quad (B \to \infty)
\]

where

\[
\chi_p(a) = \lim_{h \to \infty} p^{h(1-s)} \# \{ 1 \leq x_j \leq p^h : a_1 x_1^k + \ldots + a_s x_s^k \equiv 0 \mod p^h \}
\]

is a measure for the density of the solutions of (1.1) in \( \mathbb{Q}_p \), and similarly, \( J_a \) is related to the surface area of the real solutions of (1.1) within the box \([-1,1]^s\). A precise definition of \( J_a \) is given in (3.7) below.

As we shall see later, a condition milder than the current hypothesis \( s > 2k \) suffices to confirm that the limits (1.3) exist for all primes \( p \), and that the Euler product

\[
\mathcal{E}_a = \prod_p \chi_p(a)
\]

is absolutely convergent. Moreover, an application of Hensel’s lemma shows that \( \chi_p(a) \) is positive if and only if (1.1) is soluble in \( \mathbb{Q}_p \). Likewise, one finds that \( J_a \) is positive if and only if (1.1) is soluble over \( \mathbb{R} \). It follows that (1.1) is locally soluble if and only if

\[
J_a \mathcal{E}_a > 0.
\]

Consequently, if (1.2) holds, then the equation (1.1) obeys the Hasse principle.

The validity of (1.2), and hence of the Hasse principle for the underlying diophantine equations, is regarded to be a save conjecture in the range \( s > 2k \), and in the special case \( k = 2 \), \( s > 4 \) rigorous proofs of (1.2) are available by various methods (see chapter 2 of [22] for one approach). When \( k = 3 \), the formula (1.2) is known to hold whenever \( s \geq 8 \) (implicit in Vaughan [21]), and the Hasse principle holds for \( s \geq 7 \) (Baker [1]). For larger \( k \), much less is known. As a consequence of very important work of Wooley [23, 24], the asymptotic formula (1.2) can now be established when \( s \) is slightly smaller than \( 2k^2 \), and the Hasse principle may be verified when \( s \geq k \log k (1+o(1)) \), see Wooley [21]. It seems difficult to establish (1.2) on average over \( a \) when \( s \) is significantly smaller than in the aforementioned work of Wooley. However, one may choose \( B \) as a suitable function of \(|a|\), say \( B = |a|^\theta \), and then investigate whether (1.5) holds for almost all \( a \). This approach is successful whenever \( s > 3k \) and \( \theta \) is only slightly larger than \( 1/(s-k) \), thus confirming the conclusions alluded to in the introductory section. The principal step is contained in the following mean value theorem. Recall the convention concerning the use of \( \tilde{s} \) which is applied within the summation below. Also, when \( s \) is a natural number, let \( \hat{s} \) denote the largest even integer strictly smaller than \( s \).

**Theorem 1.1.** Let \( k \geq 3 \) and \( \tilde{s} \geq 3k \). Then there is a positive number \( \delta \) such that whenever \( A, B \) are real numbers satisfying

\[
1 \leq B^{2k} \leq A \leq B^{\tilde{s}-k},
\]

one has

\[
\sum_{|a| \leq A} |q_a(B) - J_a \mathcal{E}_a B^{s-k}|^2 \ll A^{s-2-\delta} B^{2s-2k}.
\]
This theorem actually remains valid when \( k = 2 \), but the proof we give below needs some adjustment. We have excluded \( k = 2 \) from the discussion mainly because in that particular case one can say much more, by different methods. Hence, from now on, we assume throughout that \( k \geq 3 \).

As a simple corollary, we note that subject to the conditions in Theorem 1.1, the number of \( a \) with \(|a| \leq A\) for which the inequality
\[
|\varrho_n(B) - J_n \sigma a B^{s-k}| > |a|^{-1} B^{s-k-\delta}
\]
holds, does not exceed \( O(A^{s-\delta}) \). To deduce the Hasse principle for those \( a \) where (1.7) fails, one needs a lower bound for \( J_n \sigma a \) whenever this number is non-zero. When \( k \) is odd, (1.1) is soluble over \( \mathbb{R} \), and one may show that
\[
J_n \gg |a|^{-1}
\]
holds for all \( a \). When \( k \) is even, (1.1) is soluble over \( \mathbb{R} \) if and only if the \( a_j \) are not all of the same sign, and if this is the case, then again (1.3) holds. These facts will be demonstrated in §3. For the ‘singular product’ we have the following result.

**Theorem 1.2.** Let \( s \geq k + 3 \), and let \( \eta \) be a positive number. Then there exists a positive number \( \gamma \) such that
\[
\#\{|a| \leq A : 0 < \sigma a < A^{-\eta}\} \ll A^{s-\gamma}.
\]

We are ready to derive the main result. Let \( \hat{s} \geq 3k \), and let \( \delta \) be the positive number supplied by Theorem 1.1. Suppose that \( a \in \mathcal{C}(k, s) \) satisfies \( \frac{1}{2} A < |a| \leq A \), and choose \( B = A^{1/(\hat{s}-k)} \) in accordance with (1.6). In Theorem 1.2, we take \( 4\eta = \delta/(\hat{s} - k) \) so that \( A^\eta = B^{\delta/4} \). If \( a \) is not counted in Theorem 1.2, then \( \sigma a \geq A^{-\eta} \), and if \( a \) also violates (1.7), then by (1.8) one has
\[
\varrho_n(B) \geq J_n \sigma a B^{s-k} - |a|^{-1} B^{s-k-\delta} \gg B^{s-k} A^{-1-\eta}.
\]
It follows that (1.1) has an integral solution with \( 0 < |x| \leq B \ll |a|^{1/(\hat{s}-k)} \), for these choices of \( a \). The remaining \( a \in \mathcal{C}(k, s) \) with \( \frac{1}{2} A < |a| \leq A \) are counted in (1.7) or in Theorem 1.2. Therefore, there are at most \( O(A^{s-\min(\delta, \gamma)}) \) such \( a \). We now sum for \( A \) over powers of 2 to conclude as follows.

**Theorem 1.3.** Let \( \hat{s} \geq 3k \). Then, there is a positive number \( \theta \) such that the number of \( a \in \mathcal{C}(k, s) \) for which the equation (1.1) has no integral solution in the range \( 0 < |x| \leq |a|^{1/(\hat{s}-k)} \), does not exceed \( O(A^{s-\theta}) \).

Browning and Dietmann [4] have recently shown that whenever \( s \geq 4 \), then
\[
\#\{|a| \in \mathcal{C}(k, s) : |a| \leq A\} \gg A^s,
\]
so that the estimate in Theorem 1.3 is indeed a non-trivial one. In particular, it follows that when \( \hat{s} \geq 3k \), then for almost all \( a \in \mathcal{C}(k, s) \), the equation (1.1) is soluble over \( \mathbb{Q} \). Since the Hasse principle may fail for \( a \in \mathcal{C}(k, s) \) only, this implies that whenever \( \hat{s} \geq 3k \), then the Hasse principle holds for almost all \( a \in \mathcal{C}(k, s) \), but also for almost all \( a \in \mathbb{Z}^s \). Finally, in the same range for \( s \), Theorem 1.3 implies that for almost all \( a \) for which (1.1) has non-trivial integral solutions, there exists a solution with \( 0 < |x| \leq |a|^{1/(\hat{s}-k)} \). This last corollary is rather remarkable, and very close to the best such upper bound possible, as the following result shows.

**Theorem 1.4.** Let \( s \geq 2k \), and let \( \eta > 0 \). Then, there exists a number \( c = \)}
c(k, s, η) > 0 such that the number of \( a \) with \(|a| \leq A\) for which (1.1) admits an integral solution in the range \( 0 < |x| \leq c|a|^{1/(s-k)}\), does not exceed \( ηA^s\).

One should compare this with the lower bound (1.9): even among the locally solvable equations (1.1), those that have an integral solution with \( 0 < |x| \leq c|a|^{1/(s-k)}\) form a thin set, at least when \( c \) is small. It follows that the exponent \( 1/(\hat{s} - k) \) that occurs in Theorem 1.3 cannot be replaced by a number smaller than \( 1/(s-k)\).

An estimate for the smallest non-trivial solution of an additive diophantine equation is of considerable importance in diophantine analysis, also for applications in diophantine approximation; see Schmidt [17] for a prominent example and Birch [2] for further comments. There are some bounds of this type available in the literature (eg. Pitman [12]), most notably by Schmidt [15, 16]. In this context, it is worth recalling that when \( s > k^2 \) then the equation (1.1) is soluble over \( \mathbb{Q}_p \), for all primes \( p \) (Davenport and Lewis [7]). When \( k \) is odd, we then expect that (1.1) is soluble over \( \mathbb{Q} \), and Schmidt [16] has shown that for any \( ε > 0 \) there exists \( s_0(k, ε) \) such that whenever \( s \geq s_0 \) then any equation (1.1) has an integer solution with \( 0 < |x| \ll |a|^ε\). The number \( s_0(k, ε) \) is effectively computable, but Schmidt’s method only yields poor bounds (see Hwang [11] for a discussion of this matter).

When \( k \) is even and \( s > k^2 \), then (1.1) is locally soluble provided only that the \( a_j \) are not all of the same sign. In this situation Schmidt [15] demonstrated that there still is some \( s_0(k, ε) \) such that whenever at least \( s_0(k, ε) \) of the \( a_j \) are positive, and at least \( s_0(k, ε) \) are negative, then the equation (1.1) is soluble in integers with

\[
0 < |x| \leq |a|^{1/k+ε};
\]

see also Schlickewei [13] when \( k = 2 \). Schmidt’s result is essentially best possible: if \( a \leq b \) are coprime natural numbers, and \( k \) is even, then any nontrivial solution of (1.10)

\[
a(x_1^k + \ldots + x_t^k) = b(x_{t+1}^k + \ldots + x_s^k)
\]

must have \( b|x_1^k + \ldots + x_t^k \), whence \(|x| \geq (b/s)^{1/k}\). Thus, there are equations (1.1) where the smallest solution is as large as \(|a|^{1/k}\), even when \( s \) is very large. Moreover, for general, not necessarily even \( k \) the following example shows that the smallest non-trivial solution of (1.1) can be as large as \(|a|^{2/(s-1)}\) for arbitrarily large \( s \): Let \( p \) be a prime, and let \( q \) be a \( k \)-th power non-residue modulo \( p \). It is well known that such \( q \) exist where \( 1 < q < p^{1/2} \). Then the congruence \( x_1^k - qx_2^k \equiv 0 \mod p \) has only the trivial solution \( x_1 = x_2 = 0 \mod p \). Therefore, the equation (1.11)

\[
x_1^k - qx_2^k + p(x_3^k - qx_4^k) + \ldots + p^{k-1}(x_{2t-1}^k - qx_{2t}^k) = 0
\]

has no non-trivial zero \( x \in \mathbb{Z}^{2t} \) with \(|x| < p \) (if \((x_1, x_2) \neq (0, 0)\), then (1.11) forces \( x_1^k - qx_2^k \equiv 0 \mod p \), thus \( x_1 \equiv x_2 \mod p \), whence \( \max\{|x_1|, |x_2|\} \geq p \) by \((x_1, x_2) \neq (0, 0)\); if \((x_1, x_2) = (0, 0)\), then we can take out one factor \( p \) from (1.11), and iterate the argument). Moreover, in the notation (1.1) this equation (1.11) has \(|a| = p^{k-1}q \ll p^{k-1/2} \). Noting that \( s = 2t \), we conclude that the smallest non-trivial integer solution \( x \) of (1.11) satisfies \(|x| \gg p \gg |a|^{2/(s-1)} \). However, in Theorem 1.3 the exponent \( 1/(\hat{s} - k) \) is smaller than \( 1/k \) and \( 2/(s-1) \), respectively. It follows that the exceptional set for \( a \) that is estimated in Theorem 1.3, is non-empty. On the other hand, Theorem 1.3 tells us that examples such as (1.10) or (1.11) where the smallest integer solution is large, must be sparse.

We are not aware of any previous attempts to examine additive diophantine equations on average, save for the dissertation of Breyer [3]. There, a first moment
similar to the quadratic moment in Theorem 1.1 is estimated, in which one takes \( B \asymp A^{1/k} \), and where the sum over \( a \) is restricted to a rather unnaturally defined, but reasonably dense subset of \( \mathbb{Z}^s \). Breyer’s estimates give non-trivial results for \( s > 4k \) only, and are not of strength sufficient to derive the Hasse principle for almost all equations (1.1) with \( a \in C(k, s) \), even when \( s \) is much larger than \( 4k \).

Some mention should be made of the work of Poonen and Voloch [13]. They discuss local solubility on average for general, not only additive, homogeneous equations and formulate a conjecture what to expect for the global problem. Our results, in some sense, confirm their expectations for the thinner average over additive equations in the ranges for \( s \) indicated in the theorems.

A natural approach to establish Theorem 1.1 would be a dispersion argument. This would entail opening the square, and sum the individual terms. Then, one may try to exchange the roles of coefficients and variables in (1.1), and apply the geometry of numbers to handle the lattice point counting problems that then arise. This line of attack is a pure counting device, we cannot describe the exceptional sets beyond bounds on their cardinality. We shall follow this idea in spirit, but apply the indicated tactics after taking Fourier transforms. This brings the circle method into play, and largely facilitates the overall treatment. We postpone a detailed description of our methods until they are needed in the course of the argument, but remark that the ideas developed herein can be refined further, and may be applied to related problems as well. With more work and a different use of the geometry of numbers, we may advance into the range \( 2k < s \leq 3k \). Perhaps more importantly, one may derive results similar to those announced as Theorem 1.3 for the class of general forms of a given degree. Details must be deferred to sequels of this paper.

**Notation.** Beyond the notational conventions already introduced, our notation is standard, or is otherwise explained within the text. Vectors are typeset in bold, and have dimension \( s \) unless indicated otherwise. We use \((x_1; \ldots; x_s)\) and sometimes \( \gcd x_j \) to denote the greatest common divisor of the integers \( x_j \). The exponential \( \exp(2\pi i\alpha) \) is abbreviated to \( e(\alpha) \). Whenever \( \varepsilon \) occurs in a statement, it is asserted that the statement is valid for any positive real number \( \varepsilon \). Implicit constants in Landau’s or Vinogradov’s symbols are allowed to depend on \( \varepsilon \) in such circumstances.

## II. Applications of the geometry of numbers

### 2.1 Lattices and their dual

We recall some elementary facts about lattices. Concepts and exposition are modelled on the work of Heath-Brown [8], p. 336, but we have to develop the theory further.

Let \( n \) be a natural number. We work in \( \mathbb{R}^n \), equipped with the standard inner product \( x \cdot y = x_1y_1 + \ldots + x_ny_n \). Consider a submodule \( \Lambda \) of \( \mathbb{Z}^n \) of rank \( r \). Then there are \( b_1, \ldots, b_r \) in \( \mathbb{Z}^n \) that are linearly independent over \( \mathbb{R} \), and such that \( \Lambda = \mathbb{Z}b_1 + \ldots + \mathbb{Z}b_r \). Hereafter, we refer to \( \Lambda \) as a sublattice of \( \mathbb{Z}^n \) of rank \( r \), and to \( b_1, \ldots, b_r \) as its base.

Choose orthonormal vectors \( e_{r+1}, \ldots, e_n \) such that \( b_i \) and \( e_j \) are perpendicular for all \( i, j \). Then, the number

\[
(2.1) \quad d(\Lambda) = |\det(b_1, \ldots, b_r, e_{r+1}, \ldots, e_n)|
\]

is the volume of the \( r \)-dimensional parallelepiped spanned by \( b_1, \ldots, b_r \). It is readily checked that \( d(\Lambda) \) is independent of the particular base chosen, and we refer to \( d(\Lambda) \) as the discriminant of \( \Lambda \). On computing the determinant of the product of
the matrix \((b_1, \ldots, b_r, e_{r+1}, \ldots, e_n)\) with its transpose one finds that
\begin{equation}
(2.2) \quad d(\Lambda)^2 = \det \left( (b_i \cdot b_j)_{1 \leq i, j \leq r} \right).
\end{equation}
One may apply Laplace’s identity (Schmidt [18], Lemma IV.6D) to the determinant on the right hand side of (2.2). It follows that \(d(\Lambda)\) is the euclidean length of the exterior product \(b_1 \wedge b_2 \wedge \ldots \wedge b_r\) in the Grassmann algebra of \(\mathbb{R}^n\). This length may be computed using Lemma IV.6A of Schmidt [18]. One then finds that
\begin{equation}
(2.3) \quad d(\Lambda)^2 = \sum_I (\det B_I)^2
\end{equation}
where \(I\) runs over \(r\)-element subsets of \(\{1, 2, \ldots, n\}\), and \(B_I\) denotes the \(r \times r\)-minor with rows indexed in \(I\) of the matrix \(B = (b_1, \ldots, b_r)\) formed with columns \(b_j\).

The dual lattice \(\Lambda^*\) of \(\Lambda\) is defined by
\begin{equation}
(2.4) \quad \Lambda^* = \{ x \in \mathbb{Z}^n : b_j \cdot x = 0 (1 \leq j \leq r) \}.
\end{equation}
This is indeed a sublattice of \(\mathbb{Z}^n\) of rank \(n - r\) (see [3], p. 336). We proceed to compute its discriminant. The formula to be announced features the number
\begin{equation}
(2.5) \quad G(\Lambda) = \gcd \det B_I.
\end{equation}
Here again, \(I\) runs over \(r\)-element subsets of \(\{1, 2, \ldots, n\}\). The next lemma shows that \(G(\Lambda)\) is independent of the choice of the base for \(\Lambda\).

**Lemma 2.1.** \(d(\Lambda^*) = d(\Lambda)/G(\Lambda)\).

This identity is probably familiar to workers in the geometry of numbers, but is apparently a lacuna in the literature. We therefore provide a proof.

The lattice \(\Lambda\) is called primitive if \(b_1, \ldots, b_r\) is part of a base of \(\mathbb{Z}^n\). According to Lemma 9.2.1 of Cassels [5], the lattice \(\Lambda\) is primitive if and only if \(G(\Lambda) = 1\). For primitive \(\Lambda\), one has \(d(\Lambda) = d(\Lambda^*)\) (Heath-Brown [3], Lemma 1). This proves Lemma 2.1 when \(G(\Lambda) = 1\).

We now proceed by induction on the number of prime factors of \(G(\Lambda)\). Let \(p\) be a prime with \(p \mid G(\Lambda)\). Reduce the matrix \(B = (b_1, \ldots, b_r)\) modulo \(p\). By (2.5), the reduced matrix has rank at most \(r - 1\) over \(\mathbb{F}_p\) so that by elementary column operations one may generate a column with entries divisible by \(p\). More precisely, there exists \(T \in \mathbb{Z}^{r \times r}\) with \(\det T = 1\) such that \(BT\) has its last column divisible by \(p\). We may write \(BT = (c_1, \ldots, c_r)\) with \(c_j \in \mathbb{Z}^n\) for \(1 \leq j \leq r\), and \(c_r \in p\mathbb{Z}^n\). Then \(c_1, \ldots, c_r\) is a base of \(\Lambda\) so that
\[\Lambda^* = \{ x \in \mathbb{Z}^n : c_j \cdot x = 0 (1 \leq j \leq r) \}\].

Now consider
\[M = \mathbb{Z}c_1 + \ldots + \mathbb{Z}c_{r-1} + \frac{1}{p}\mathbb{Z}c_r\]
which is again a sublattice of \(\mathbb{Z}^n\). Note that \(M^* = \Lambda^*\). Since \(\det T = 1\), it follows that \(G(\Lambda)\) may be computed from \((c_1, \ldots, c_r)\) in place of \((b_1, \ldots, b_r)\), and it is then immediate that \(G(\Lambda) = pG(M)\). Computing the discriminant from the base \(c_1, \ldots, c_r\) of \(\Lambda\) it follows that \(d(\Lambda) = pd(M)\). Hence, \(d(\Lambda)/G(\Lambda) = d(M)/G(M)\). Since \(G(M)\) has fewer prime factors than \(G(\Lambda)\), one may apply the induction hypothesis to \(M\). One then obtains
\[d(\Lambda^*) = d(M^*) = d(M)/G(M) = d(\Lambda)/G(\Lambda)\].
This completes the proof of Lemma 2.1.
Lemma 2.2. Let \( \Lambda \) be a sublattice of rank \( r \) in \( \mathbb{Z}^n \). Let \( A \geq d(\Lambda) \). Then, the box \( |a| \leq A \) contains \( O(A^r) \) elements of \( \Lambda \).

Proof. See Heath-Brown \[9\], Lemma 1 (v), for example.

2.2. Proof of Theorem 1.4. Let \( \Xi(A, B) \) denote the number of all \( a, x \in \mathbb{Z}^s \) that satisfy the equation (1.1), and that lie in the range \( |a| \leq A, 0 < |x| \leq B \). We consider (1.1) as a linear equation defining a lattice, and compute its discriminant from Lemma 2.1 and (2.3). Then, for \( B^k \leq A \), Lemma 2.2 supplies the estimate

\[
\Xi(A, B) \ll \sum_{0 <|x| \leq B} A^{s-1} \frac{(x_1^k; \ldots; x_s^k)}{|x|^k}.
\]

By symmetry, it suffices to sum over all \( x \) with \( x_1 = |x| \). We sort the remaining sum according to \( d = (x_1; x_2; \ldots; x_s) \). Then \( d|x_j \) for all \( j \), and we infer that

\[
\Xi(A, B) \ll A^{s-1} \sum_{1 \leq x_1 \leq B} \sum_{d|x_1} \left( \frac{d}{x_1} \right)^k \left( \sum_{y \leq x_1 \atop d|y} 1 \right)^{s-1}.
\]

Since \( x_1/d \geq 1 \) holds for all \( d|x_1 \), it follows that

\[
\Xi(A, B) \ll A^{s-1} \sum_{1 \leq x \leq B} \sum_{d|x} \left( \frac{x}{d} \right)^{s-1}.
\]

Here, one exchanges the order of summation, and then concludes as follows.

Lemma 2.3. Let \( s \geq k + 2 \), and suppose that \( B^k \leq A \). Then

\[
\Xi(A, B) \ll A^{s-1} B^{s-k}.
\]

This simple estimate is of strength sufficient to establish Theorem 1.4. Let \( P(A, B) \) denote the number of all \( a \) with \( |a| \leq A \) for which the equation (1.1) has an integral solution with \( 0 < |x| \leq B \). Then, on exchanging the order of summation,

\[
P(A, B) \leq \sum_{|a| \leq A} \# \{ 0 < |x| \leq B : a_1 x_1^k + \ldots + a_s x_s^k = 0 \} = \sum_{0 < |x| \leq B} \# \{ |a| \leq A : a_1 x_1^k + \ldots + a_s x_s^k = 0 \} = \Xi(A, B).
\]

When \( s \geq 2k \) and \( 0 < C \leq 1 \), then the choice \( B = C A^{1/(s-k)} \) is admissible in Lemma 2.3. Let \( \eta > 0 \). Thus, if \( C \) is sufficiently small, Lemma 2.3 supplies the inequality \( P(A, C A^{1/(s-k)}) \leq \Xi(A, C A^{1/(s-k)}) < \eta A^s \). If \( a \) is a vector such that \( |a| \leq A \) and (1.1) has an integral solution with \( 0 < |x| < C |a|^{1/(s-k)} \), then \( a \) is also counted by \( P(A, C A^{1/(s-k)}) \), and Theorem 1.4 follows.

2.3. An auxiliary mean value estimate. Our next task is the derivation of an estimate for the number of solutions of a certain symmetric diophantine equation. The result will be one of the cornerstones in the proof of Theorem 1.1. We begin with an examination of a congruence related to \( k \)-th powers.

Lemma 2.4. The number of pairs \((u, v) \in \mathbb{Z}^2 \) with \(|u| \leq B, |v| \leq B \) and \( u^k \equiv v^k \mod d \) does not exceed \( O(B^{1+\varepsilon} + B^{2+\varepsilon} d^{-2/k}) \).
Proof. We sort the pairs \((u, v)\) according to the value of \(e = (u; v)\), and write \(u = eu_0, v = ev_0\) and \(d_0 = d/(d; e^k)\). The congruence then reduces to \(u_0^k \equiv v_0^k \mod d_0\) with \(1 \leq |u_0| \leq B/e, 1 \leq |v_0| \leq B/e\) and \((u_0; v_0) = 1\), and the latter condition implies that \((u_0; d_0) = 1\). There are \(1 + 2B/e\) choices for \(v_0\), and the theory of \(k\)-th power residues coupled with \((u_0; d_0) = 1\) and a divisor function estimate yields the bound \(O(d_0^2(1 + B/(ed_0)))\) for the number of choices for \(v_0\), for any admissible choice of \(v_0\). It follows that the number in question does not exceed

\[
\ll B e \sum_{e \leq B} \frac{B}{e} \left(1 + \frac{B}{ed_0}\right) \ll B^{1+2\varepsilon} + B^{2+\varepsilon} \sum_{e \leq B} \frac{(d; e^k)}{e^{2d}},
\]

from which the desired estimate is routinely deduced.

Now let \(t\) be a natural number, and let \(\Upsilon_t(A, B)\) denote the number of solutions of the equations

\[
\sum_{j=1}^{2t} a_j x_j^k = \sum_{j=1}^{2t} a_j y_j^k = 0
\]

in integers \(a_j, x_j, y_j\) constrained to

\[
0 < |a_j| \leq A, \quad 0 < |x_j| \leq B, \quad 0 < |y_j| \leq B.
\]

Lemma 2.5. Let \(2t \geq k + 2\), and suppose that \(A \geq B^{2k} \geq 1\). Then

\[
\Upsilon_t(A, B) \ll A^{2t-2} B^{4t-2k+\varepsilon} + A^{2t-1} B^{2t-k+\varepsilon}.
\]

Proof. We begin with a localisation process for the variables \(x_j, y_j\) in (2.8). Let

\[
F(\alpha, \beta) = \sum_{1 \leq |a| \leq A} \sum_{1 \leq |x| \leq B} \sum_{1 \leq |y| \leq B} e(a(\alpha x^k + \beta y^k)).
\]

Then, by orthogonality,

\[
\Upsilon(A, B) = \int_0^1 \int_0^1 F(\alpha, \beta)^{2t} \, d\alpha \, d\beta.
\]

Let \(F_{ij}(\alpha, \beta)\) be the portion of the sum (2.9) where \(2^{-j} B < |x| \leq 2^{1-j} B\) and \(2^{-i} B < |y| \leq 2^{1-i} B\). Then

\[
F(\alpha, \beta) = \sum_{i=1}^L \sum_{j=1}^L F_{ij}(\alpha, \beta)
\]

where \(L \ll \log B\). Hence, by (2.10) and Hölder’s inequality,

\[
\Upsilon(A, B) \ll B^{2t} \sum_{i=1}^L \sum_{j=1}^L \int_0^1 \int_0^1 F_{ij}(\alpha, \beta)^{2t} \, d\alpha \, d\beta.
\]

By orthogonality again, and on considering the symmetry of the underlying diophantine equations, it follows that

\[
\Upsilon(A, B) \ll B^t \max_{1 \leq 2X \leq B} \Psi(A, X, Y)
\]

where \(\Psi(A, X, Y)\) denotes the number of solutions of the system (2.7) in the ranges

\[
0 < |a_j| \leq A, \quad X < |x_j| \leq 2X, \quad Y < |y_j| \leq 2Y.
\]
We are reduced to estimating Ψ(A, X, Y). Let Ψ'(A, X, Y) denote the number of solutions counted by Ψ(A, X, Y) where \((x_1^k, x_2^k, \ldots, x_{2t}^k)\) and \((y_1^k, y_2^k, \ldots, y_{2t}^k)\) are linearly independent over \(\mathbb{R}\), and let Ψ''(A, X, Y) denote the number of those solutions where these vectors are parallel. Then

\[(2.13) \quad Ψ(A, X, Y) = Ψ'(A, X, Y) + Ψ''(A, X, Y).\]

We estimate Ψ''(A, X, Y) by an argument similar to the deduction of Lemma 2.3. When \(x\) and \(y\) is a pair that contributes to Ψ'', then the two equations in (2.7) for \(a\) are equivalent. Hence, for given \(x\), we may determine \(a\) through the first equation in (2.7), and then count how many \(y\) may occur for the given value of \(x\). For \(2X \leq B\), the argument that produced (2.6) now delivers the bound

\[
Ψ''(A, X, Y) \ll A^{2t-1} X^{-k} \sum_{X < |x| \leq 2X} (x_1^k; x_2^k; \ldots; x_{2t}^k) Θ(x, Y)
\]

where \(Θ(x, Y)\) denotes the number of \(y \in \mathbb{Z}^k\) with (2.12) such that \((y_1^k, y_2^k, \ldots, y_{2t}^k)\) and \((x_1^k, x_2^k, \ldots, x_{2t}^k)\) are parallel.

To bound Θ(x, Y), let \(d = (x_1; x_2; \ldots; x_{2t})\) and \(z = \frac{1}{d}x\). Then, by unique factorisation, a vector \(y \in \mathbb{Z}^k\) with (2.12) is counted by Θ(x, Y) if and only if there is a non-zero integer \(a\) such that

\[(y_1^k, y_2^k, \ldots, y_{2t}^k) = \pm a^k(z_1^k, z_2^k, \ldots, z_{2t}^k).\]

In particular, one has \(|y_j| = |az_j| = |ax_j|/d\) for all \(j\), and it follows that \(Θ(x, Y) \ll Yd/|x|\). Consequently,

\[
Ψ''(A, X, Y) \ll A^{2t-1} Y X^{-k-1} \sum_{X < |x_j| \leq 2X} (x_1; x_2; \ldots; x_{2t})^{k+1}.
\]

One rearranges the sum according to the value of \(d\). For \(X \leq B\), \(Y \leq B\) and \(2t \geq k + 2\) one then finds that

\[(2.14) \quad Ψ''(A, X, Y) \ll A^{2t-1} Y X^{-k-1} \sum_{d \leq 2X} d^{k+1}(X/d)^{2t} \ll A^{2t-1} B^{2t-k+ε}.
\]

This bound will enter the final bound for \(Υ(A, B)\) through (2.13) and (2.11), and is responsible for the second term on the right hand side in the inequality claimed in Lemma 2.5.

It remains to estimate Ψ'(A, X, Y). For fixed \(x\) and \(y\) that contribute to Ψ'(A, X, Y), put

\[Δ_{ij} = x_i y_j - x_j y_i, \quad D = \gcd_{1 \leq i < j \leq 2t} Δ_{ij}.
\]

Then, by Lemma 2.1 and (2.3), the solutions \(a \in \mathbb{Z}^k\) of (2.7) form a lattice of rank \(2t - 2\) and discriminant

\[D^{-1}\left(\sum_{1 \leq i < j \leq 2t} |Δ_{ij}|^2\right)^\frac{1}{2}.
\]

Subject to the constraints in (2.11), one has \(|Δ_{ij}| \leq 2(4XY)^k \leq 2B^{2k}\). Hence, the discriminant is \(O(A)\), and Lemma 2.2 supplies the bound

\[(2.15) \quad Ψ'(A, X, Y) \ll A^{2t-2} \sum_{x, y} \frac{D}{\max |Δ_{ij}|},
\]
where the sum runs over pairs $x, y$ that meet the linear independence condition typical for $\Psi'$. By symmetry, it suffices to consider the portion of the sum in (2.15) where $\Delta_{12} = \max |\Delta_{ij}|$, and the linear independence condition is then equivalent to $\Delta_{12} \geq 1$. It follows that

$$\Psi'(A, X, Y) \ll A^{2t-2} \sum_{x_1, x_2, y_1, y_2} \sum_{D|\Delta_{12}} \frac{D}{\Delta_{12}} \Omega(x_1, x_2, y_1, y_2, D)$$

where $\Omega(x_1, x_2, y_1, y_2, D)$ is the number of choices for $x_3, x_4, \ldots, x_{2t}$ and $y_3, y_4, \ldots, y_{2t}$ satisfying (2.12) and the conditions

$$|\Delta_{ij}| \leq \Delta_{12}, \quad D | \Delta_{ij} \quad (3 \leq i < j \leq 2t).$$

In order to obtain an upper bound we ignore most of the constraints from the preceding list and only keep the conditions $D 
 \Delta_{2l-1,2l}$ for $2 \leq l \leq t$. The conditions on $x_j, y_j$ then factorise into blocks of indices $2l - 1, 2l$, with $2 \leq l \leq t$. In particular, one finds that

$$\Omega(x_1, x_2, y_1, y_2, D) \leq H(X, Y, D)^t-1$$

where $H(X, Y, D)$ denotes the number of solutions of $D | (x_3 y_3)^k - (x_4 y_3)^k$ with $x_3, x_4, y_3, y_4$ satisfying (2.12). Lemma 2.4 coupled with a divisor function estimate yields

$$H(X, Y, D) \ll (XY)^{1+\varepsilon} + (XY)^{2+\varepsilon} D^{\varepsilon-2/k}.$$  

Now recall that $2t \geq k + 2$. Then, on collecting together, it follows that

$$(2.16) \Psi'(A, X, Y) \ll A^{2t-2} \sum_{x_1, x_2, y_1, y_2} \sum_{D|\Delta_{12}} \frac{D}{\Delta_{12}} ((XY)^{t-1+\varepsilon} + (XY)^{2t-2+\varepsilon} D^{\varepsilon-1})$$

$$\ll A^{2t-2}(XY)^{t+1+\varepsilon} + A^{2t-2}(XY)^{2t-2+\varepsilon} \Phi(X, Y)$$

where

$$\Phi(X, Y) = \sum_{x_1, x_2, y_1, y_2} \Delta_{12}^{-1}.$$  

We write $u = x_1 y_2, v = x_2 y_1$ and apply a divisor function estimate to infer that

$$(2.17) \Phi(X, Y) \ll (XY)^{\varepsilon} \sum_{u^k > v^k, u^k - v^k} (u^k - v^k)^{-1}.$$  

When $k$ is even, in the sum above it suffices to consider summands with $u > v > 0$. Binomial expansion then gives $u^k - v^k \gg (u - v)(XY)^{k-1}$, and it follows that

$$(2.18) \Phi(X, Y) \ll (XY)^{2-2-k+\varepsilon}.$$  

When $k$ is odd, the signs of $u$ and $v$ affect the estimation. The argument that we used in the case where $k$ is even still applies to those terms where $u$ and $v$ have the same sign, and this portion still contributes $O((XY)^{2-k})$ to the sum on the right hand side of (2.16). When $u$ and $v$ have opposite signs, one has $u^k - v^k = |u|^k + |v|^k \gg (XY)^k$, and it is immediate that this portion also contributes at most $O((XY)^{2-k})$ to the sum on the right hand side of (2.16). Hence, (2.18)
holds irrespective the parity of \( k \), so that for \( X \leq B, Y \leq B \), one finds from (2.10) that \( \Psi'(A, X, Y) \ll A^{2t-2}B^{4t-2k+\varepsilon} \). In view of (2.13) and (2.11), this completes the proof of Lemma 2.5.

III. Local solubility

3.1. The singular integral. Local solubility of additive equations has been investigated by Davenport and Lewis [7], and by Davenport [6]. The analytic condition (1.5) for local solubility is implicit in [7]. Unfortunately, these prominent references are insufficient for our purposes. A lower bound for \( J_a \) in terms of \(|a|\) is needed whenever this product in non-zero, at least for almost all \( a \). An estimate of this type is supplied in this section.

We begin with the singular integral. Most of our work is routine, so we shall be brief. When \( \beta \in \mathbb{R}, B > 0 \), let

\[
(3.1) \quad v(\beta, B) = \int_{-B}^{B} e(\beta \xi^k) \, d\xi.
\]

A partial integration readily confirms the bound

\[
(3.2) \quad v(\beta, B) \ll B(1 + Bk|\beta|)^{-1/k}
\]

whence whenever \( s > k \) one has

\[
(3.3) \quad \int_{-\infty}^{\infty} |v(\beta, B)|^s \, d\beta \ll B^{s-k}.
\]

We also see that for \( s > k \) and \( a \in (\mathbb{Z} \setminus \{0\})^s \), the integral

\[
(3.4) \quad J_a(B) = \int_{-\infty}^{\infty} v(a_1 \beta, B) \ldots v(a_s \beta, B) \, d\beta
\]

converges absolutely. By Hölder’s inequality and (3.3),

\[
\int_{-\infty}^{\infty} |v(a_1 \beta, B) \ldots v(a_s \beta, B)| \, d\beta
\leq \prod_{j=1}^{s} \left( \int_{-\infty}^{\infty} |v(a_j \beta, B)|^s \, d\beta \right)^{1/s} \ll |a_1 \ldots a_s|^{-1/s} B^{s-k}.
\]

In particular, it follows that

\[
(3.5) \quad J_a(B) \ll |a_1 \ldots a_s|^{-1/s} B^{s-k}.
\]

The integral \( J_a(B) \) arises naturally as the singular integral in our application of the circle method in section 4. The dependence on \( B \) can be made more explicit. By (3.1), one has \( v(\beta, B) = B v(\beta B^k, 1) \). Now substitute \( \beta \) for \( \beta B^k \) in (3.4) to infer that

\[
(3.6) \quad J_a(B) = B^{s-k} J_a
\]

where \( J_a = J_a(1) \) is the number that occurs in (1.2), and in Theorem 1.1.

It remains to establish a lower bound for \( J_a \). The argument depends on the parity of \( k \), and we shall begin with the case when \( k \) is even. Throughout, we suppose that

\[
(3.7) \quad |a_s| \geq |a_j| \quad (1 \leq j < s).
\]
Define \( \sigma_j = a_j/|a_j| \in \{1,-1\} \). Then, by \((3.1)\),
\[
v(a_j \beta, 1) = 2 \int_0^1 e(a_j \beta \xi^k) \, d\xi = \frac{2}{k} |a_j|^{-1/k} \int_0^{|a_j|} \eta^{(1-k)/k} e(\sigma_j \beta \eta) \, d\eta.
\]
Let \( \mathfrak{A} = [0, |a_1|] \times \ldots \times [0, |a_s|] \), and define the linear form \( \tau \) through the equation
\[
\sigma_s \tau = \sigma_1 \eta_1 + \ldots + \sigma_s \eta_s.
\]
Then, we may rewrite \((3.4)\) as
\[
J_\mathfrak{A} = \left(\frac{2}{k}\right)^s |a_1 \ldots a_s|^{-1/k} \int_{-\infty}^{\infty} \int_{\mathfrak{A}} (\eta_1 \ldots \eta_s)^{(1-k)/k} e(\sigma_s \tau \beta) \, d\eta \, d\beta.
\]
Now substitute \( \tau \) for \( \eta_s \) in the innermost integral. Then, by Fubini’s theorem and \((3.9)\), one has
\[
\int_{\mathfrak{A}} (\eta_1 \ldots \eta_s)^{(1-k)/k} e(\sigma_s \tau \beta) \, d\eta = \int_{-\infty}^{\infty} E(\tau) e(\sigma_s \tau \beta) \, d\tau
\]
where
\[
E(\tau) = \int_{\mathfrak{E}(\tau)} (\eta_1 \ldots \eta_{s-1} \eta_s(\tau, \eta_1, \ldots, \eta_{s-1}))^{(1-k)/k} \, d\eta_1 \ldots \eta_{s-1},
\]
in which \( \eta_s \) is the linear form defined implicitly by \((3.9)\), and \( \mathfrak{E}(\tau) \) is the set of all \( (\eta_1, \ldots, \eta_{s-1}) \) satisfying the inequalities
\[
0 \leq \eta_j \leq |a_j| \quad (1 \leq j < s),
\]
\[
0 \leq \tau - \sigma_s \eta_1 - \sigma_s \eta_2 - \ldots - \sigma_s \eta_{s-1} - 1 |a_s|.
\]
It transpires that \( E \) is a non-negative continuous function with compact support, and that for \( \tau \) near 0, this function is of bounded variation. Therefore, by Fourier’s integral theorem,
\[
\lim_{N \to \infty} \int_{-N}^{N} \int_{-\infty}^{\infty} E(\tau) e(\sigma_s \tau \beta) \, d\tau \, d\beta = E(0),
\]
and we infer that
\[
J_\mathfrak{A} = \left(\frac{2}{k}\right)^s |a_1 \ldots a_s|^{-1/k} E(0).
\]
In particular, it follows that \( J_\mathfrak{A} \geq 0 \). Also, when all \( a_j \) have the same sign, then \( \mathfrak{E}(0) = \{0\} \), and \((3.11)\) yields \( J_\mathfrak{A} = 0 \).

Now suppose that not all the \( a_j \) are of the same sign. First, consider the situation where \( \sigma_1 = \ldots = \sigma_{s-1} \). Then we have \( \sigma_s \sigma_j = -1 \) \( (1 \leq j < s) \). By \((3.8)\), we see that the set of \( (\eta_1, \ldots, \eta_{s-1}) \) defined by
\[
\frac{|a_j|}{2s} \leq \eta_j \leq \frac{|a_j|}{s} \quad (1 \leq j < s)
\]
is contained in \( \mathfrak{E}(0) \), and its measure is bounded below by \( (2s)^{-s} |a_1 a_2 \ldots a_{s-1}| \). By \((3.10)\), we now deduce that
\[
E(0) \gg |a_1 \ldots a_{s-1}|^{1/k} |a_s|^{(1-k)/k},
\]
and \((3.11)\) then implies the bound \( J_\mathfrak{A} \gg |a_s|^{-1} = |a|^{-1} \).

In the remaining cases, both signs occur among \( \sigma_1, \ldots, \sigma_{s-1} \). We may therefore suppose that for some \( r \) with \( 2 \leq r < s \) we have
\[
\sigma_r \sigma_j = -1 \quad (1 \leq j < r), \quad \sigma_s \sigma_j = 1 \quad (r \leq j < s).
\]
Take $r = 0$ in (3.9). Then $\eta_s$ is the linear form
\begin{equation}
\eta_s = \eta_1 + \ldots + \eta_1 - \eta_r - \ldots - \eta_{s-1}.
\end{equation}

By symmetry, we may suppose that
\[ |a_1| \leq |a_2| \leq \ldots \leq |a_{r-1}|, \quad |a_r| \leq |a_{r+1}| \leq \ldots \leq |a_{s-1}|. \]

We define $t$ by $t = r - 1$ when $|a_{r-1}| \leq |a_r|$, and otherwise as the largest $t$ among $r, r+1, \ldots, s-1$ where $|a_t| \leq |a_{r-1}|$. Now consider the set of $(\eta_1, \ldots, \eta_{s-1})$ defined by the inequalities
\[ \frac{|a_j|}{2s} \leq \eta_j \leq \frac{|a_j|}{s}, \quad (1 \leq j \leq r-1), \]
\[ \frac{|a_j|}{8s^2} \leq \eta_j \leq \frac{|a_j|}{4s^2}, \quad (r \leq j \leq t), \]
\[ \frac{|a_{r-1}|}{8s^2} \leq \eta_j \leq \frac{|a_{r-1}|}{4s^2}, \quad (t < j < s). \]

It is readily checked that on this set, the number $\eta_s$ defined in (3.12) satisfies the inequalities $\frac{|a_s|}{4s^2} \leq \eta_s \leq |a_{r-1}|$. Moreover, the measure of this set is $\gg |a_1 \ldots a_t|^{s-t+2}$. By (3.10), it follows that
\[ E(0) \gg |a_1 \ldots a_t|^{1/k} |a_{r-1}|^{(s-t+2)/k} |a_{r-1}|^{(1-k)/k}, \]
and again one then deduces from (3.11) the bound $J_a \gg |a|^{-1}$.

Finally, we discuss the case where $k$ is odd. The main differences in the treatment occur in the initial steps. When $k$ is odd, one may transform (3.11) into
\[ v(a_j, 1) = \frac{1}{k} |a_j|^{-1/k} \int_0^{a_j} \eta^{(1-k)/k} e(\beta \eta + e(-\beta \eta)) \, d\eta. \]

Let $\sigma = (\sigma_1, \ldots, \sigma_s)$ with $\sigma_j \in \{1, -1\}$. For any such $\sigma$, define $\tau$ through (3.9). Then, following through the argument used in the even case, we first arrive at the identity
\[ J_a = k^{-s} |a_1 \ldots a_s|^{-1/k} \sum_{\sigma} \int_{-\infty}^{\infty} \int_{\mathbb{R}} (\eta_1 \ldots \eta_s)^{(1-k)/k} e(\sigma_1 \tau \beta) \, d\eta \, d\beta. \]

Here the sum is over all $2^s$ choices of $\sigma$. Again as before, we see that each individual summand is non-negative, and when not all of $\sigma_1, \ldots, \sigma_s$ have the same sign, then one finds the lower bound $\gg |a|^{-1}$ for this summand. Thus, we now see that $J_a \gg |a|^{-1}$ again holds, this time for any choice of $a$.

For easy reference, we summarise the above results as a lemma.

**Lemma 3.1.** Suppose that $s > k$. Then the singular integral $J_a$ converges absolutely, and one has $0 \leq J_a \ll |a_1 a_2 \ldots a_s|^{1/s}$. Furthermore, when $k$ is odd, or when $k$ is even and $a_1, \ldots, a_s$ are not all of the same sign, then $J_a \gg |a|^{-1}$. Otherwise $J_a = 0$.

### 3.2. The singular series

In the introduction, we defined the classical singular series as a product of local densities. We briefly recall its representation as a series. Though this is standard in principle, our exposition makes the dependence on the coefficients $a$ in (1.1) as explicit as is necessary for the proof of Theorem 1.2 in the next section. Recall that $k \geq 3$. 
For \( q \in \mathbb{N}, r \in \mathbb{Z} \) define the Gaussian sum
\[
S(q,r) = \sum_{x=1}^{q} e(rx^k/q).
\]

Let \( \kappa(q) \) be the multiplicative function that, on prime powers \( q = p^l \), is given by
\[
\kappa(p^{uk+v}) = p^{-u-1} \quad (u \geq 0, 2 \leq v \leq k), \quad \kappa(p^{uk+1}) = kp^{u-1/2}.
\]

Then, as a corollary to Lemmas 4.3 and 4.4 of Vaughan [22], one has \( S(q,r) \ll q\kappa(q) \)
whenever \((q;r) = 1\), and one concludes that
\[
S(q,r) \ll q^{-1}\kappa(q/(q;r))
\]
holds for all \( q \in \mathbb{N}, r \in \mathbb{Z} \). Now let
\[
T_a(q) = q^{-s} \sum_{(r;q)=1}^{q} S(q, a_1r) \ldots S(q, a_sq).
\]

Then, by (3.14),
\[
T_a(q) \ll q\kappa(q/(q;a_1)) \ldots \kappa(q/(q;a_s)).
\]

Moreover, by working along the proof of Lemma 2.11 of Vaughan [22], one finds
\[
T_a(p) \ll q^{-s} \sum_{(r;p)=1}^{p} S(p, a_1r) \ldots S(p, a_sq).
\]

Then, by (3.14),
\[
T_a(p) \ll q\kappa(p/(p;a_1)) \ldots \kappa(p/(p;a_s)).
\]

We may take the limit for \( l \to \infty \) in (3.18) because all sums in (3.17) are convergent.
This shows that the limit \( \chi_p \), as defined in (1.3), exists. In view of (3.17) and (1.4),
we may summarize our results as follows.

**Lemma 3.2.** Let \( s \geq k+2 \). Then, for any \( a \in (\mathbb{Z}\{0\})^s \), the singular product (1.4)
converges, and has the alternative representation
\[
S_a = \sum_{q=1}^{\infty} T_a(q).
\]

A slight variant of the preceding argument also supplies an estimate for \( \chi_p(a) \) when \( p \) is large.

**Lemma 3.3.** Let \( s \geq k+2 \). Then there is a real number \( c = c(k,s) \) such that for any choice of \( a_1, \ldots, a_s \in \mathbb{Z}\{0\} \) for which at least \( k+2 \) of the \( a_j \) are not divisible by \( p \), one has \(|\chi_p(a) - 1| \leq cp^{-2}\).
Proof. We begin with (3.18), and note that \( T(a)(1) = 1 \). Then

\[
p^{l(1-s)}M_a(p^j) - 1 = \sum_{h=1}^{l} T_a(p^h).
\]

One has \( \kappa(q) \leq k \) for any prime power \( q \). Hence, by (3.16), and since \( k + 2 \) of the \( a_j \) are coprime to \( p \), one finds that \( |T_a(p^h)| \leq k^s \kappa(p^h)^{k+2}p^h \). Consequently, a short calculation based on the definition of \( \kappa \) reveals that

\[
|p^{l(1-s)}M_a(p^j) - 1| \leq k^s \sum_{h=1}^{l} \kappa(p^h)^{k+2}p^h \leq k^{s+k+2}p^{-2}.
\]

The lemma follows on considering the limit \( l \to \infty \).

3.3. Proof of Theorem 1.2. Throughout, we suppose that \( s \geq k + 3 \). For \( a \in (\mathbb{Z}\setminus\{0\})^s \), let \( S(a) \) denote the set of all primes that divide at least two of the integers \( a_j \). Lemma 3.3 may then be applied to all primes \( p \notin S(a) \), and we deduce that there exists a number \( C = C(k, s) > 0 \) such that the inequalities

\[
(3.19) \quad \frac{1}{2} \leq \prod_{p \notin S(a) \atop p > C} \chi_p(a) \leq 2
\]

hold for all \( a \). It will be convenient to write

\[
\mathcal{P}(a) = S(a) \cup \{ p : p \leq C \};
\]

this set contains all primes not covered by (3.19). For a prime \( p \in \mathcal{P}(a) \), let

\[
l(p) = \max\{l : p^l \mid a_j \text{ for some } j\},
\]

and then define the numbers

\[
P(a) = \prod_{p \in \mathcal{P}(a)} p, \quad P_0(a) = \prod_{p \in S(a) \atop p > C} p, \quad P^l(a) = \prod_{p \in \mathcal{P}(a)} p^{l(p)}, \quad H = \prod_{p \leq C} p.
\]

For later use, we note that \( P(a) = HP_0(a) \).

Now fix a number \( \delta > 0 \), to be determined later, and consider the sets

\[
(3.20) \quad \mathcal{A}_1 = \{ |a| \leq A : P(a) > A^\delta \},
\]

\[
(3.21) \quad \mathcal{A}_2 = \{ |a| \leq A : P(a) \leq A^\delta, P^l(a) > A^{2\delta} \}.
\]

It transpires that the set \( \mathcal{A}_1 \cup \mathcal{A}_2 \) contains all \( a \) where the singular series is likely to be smallish. Fortunately, \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are defined by divisibility constraints that are related to convergent sieves, so one expects \( \mathcal{A}_1, \mathcal{A}_2 \) to be thin sets. This is indeed the case, as we shall now show.

We begin by counting elements of \( \mathcal{A}_1 \). For a natural number \( d \), let \( \mathcal{A}_1(d) = \{ a \in \mathcal{A}_1 : P_0(a) = d \} \). If there is some \( a \in \mathcal{A}_1(d) \), then by the definition of \( S(a) \), we have \( \delta^2|a_1a_2\ldots a_s \), whence \( d \leq A^{s/2} \). On the other hand, \( A^\delta < P(a) \leq HP_0(a) \leq Hd \). This shows that

\[
\#\mathcal{A}_1 = \sum_{A^\delta/H < d \leq A^{s/2}} \#\mathcal{A}_1(d) \leq \sum_{A^\delta/H < d \leq A^{s/2}} \#\{ |a| \leq A : \delta^2|a_1\ldots a_s \}.
\]

By a standard divisor argument, we may conclude that

\[
(3.22) \quad \#\mathcal{A}_1 \ll A^{s+\varepsilon} \sum_{A^\delta/H < d \leq A^{s/2}} d^{-2} \ll A^{s-\delta+\varepsilon}.
\]
The estimation of $\#A_2$ proceeds along the same lines, but we will have to bound the number of integers with small square-free kernel. When $n$ is a natural number, let

$$n^* = \prod_{p \mid n} p$$

denote its squarefree kernel. One then has the following simple bound (Tenenbaum [20], Theorem II.1.12).

**Lemma 3.4.** Let $\nu \geq 1$ be a real number. Then,

$$\# \{ n \leq X^\nu : n^* \leq X \} \ll X^{1+\varepsilon}.$$ 

For $d \in \mathbb{N}$, let $A_2(d) = \{ a \in A_2 : P^1(a) = d \}$. Since we have $P^1(a)|a_1a_2\ldots a_s$, we must have

$$A^{2\delta} < d \leq A^s$$

whenever $A_2(d)$ is non-empty. Moreover, $P(a)$ is the square-free kernel of $P^1(a)$, so that $d^* \leq P^h$. This yields the bound

$$\#A_2 = \sum_{A^{2\delta} < d \leq A^s} \#A_2(d) \leq \sum_{A^{2\delta} < d \leq A^s} \#\{ a \leq A : d|a_1a_2\ldots a_s \},$$

The divisor argument used within the estimation of $\#A_1$ also applies here, and gives

$$\#A_2 \ll A^{s+\varepsilon} \sum_{A^{2\delta} < d \leq A^s} \frac{1}{d} \ll A^{s-2\delta+\varepsilon} \sum_{d \leq A^s} 1.$$

By Lemma 3.4, it follows that

$$\#A_2 \ll A^{s-\delta+\varepsilon}.$$ 

We are ready to establish Theorem 1.2. It will suffice to find a lower bound for $\mathcal{S}_n$ for those $|a| \leq A$ where $\mathcal{S}_n > 0$ and $a \notin A_1 \cup A_2$. Let $p \in P(a)$. We have $\chi_p(a) > 0$, whence (1.1) is soluble in $\mathbb{Q}_p$. By homogeneity, there is then a solution $x \in \mathbb{Z}_p$ of (1.1) with $p \mid x$. In particular, for any $h \in \mathbb{N}$, we can find integers

$$y_1, \ldots, y_s$$

that are not all divisible by $p$, and satisfy the congruence

$$a_1y_1^k + \ldots + a_s y_s^k \equiv 0 \mod p^h. \tag{3.24}$$

It will be convenient to rearrange indices to assure that $p \mid y_1$. Let $\nu(p)$ be defined by $p^{\nu(p)}|k$, and recall that a $k$-th power residue mod $p^{\nu(p)+2}$ is also a $k$-th power residue modulo $p^\nu$, for any $\nu \geq \nu(p)+2$. We choose $h = \nu(p) + \nu(p)+2$ in (3.24), and define $e$ by $p^e|a_1$. For $l > h$, choose numbers $x_j$, for $2 \leq j \leq s$, with $1 \leq x_j \leq p^l$ and $x_j \equiv y_j \mod p^h$. Then, by (3.24),

$$-a_1 y_1^k + \frac{a_2 x_2^k + \ldots + a_s x_s^k}{p^{e}} \equiv \mod p^{h-e},$$

and we have $e \leq \nu(p)$, whence $h - e \geq \nu(p)+2$. Thus, for any choice of $x_2, \ldots, x_s$ as above, there is a number $x_1$ with

$$a_1 x_1^k + \ldots + a_s x_s^k \equiv 0 \mod p^l.$$ 

Counting the number of possibilities for $x_2, \ldots, x_s$ yields $M_\nu(p^l) \geq p^{(s-1)(l-h)}$, and consequently,

$$\chi_p(a) \geq p^{(1-s)h}.$$
We may combine this with (3.19) to infer that
\begin{equation}
\mathcal{E}_a \geq \frac{1}{2} \prod_{p \in P(a) \atop l(p)=0} p^{(1-s)h}.
\end{equation}

In this product, we first consider primes $p \in P(a)$ where $l(p) = 0$. Then $p \nmid a_1 a_2 \ldots a_s$, and the definition of $P(a)$ implies that $p \leq C$. Also, since $\nu(p) \leq k$, we have $h \leq k + 2$ so that
\begin{equation}
\prod_{p \in P(a) \atop l(p)=0} p^{(1-s)h} \geq \prod_{p \leq C} p^{(1-s)(k+2)} \geq H^{(1-s)(k+2)}.
\end{equation}

Next, consider $p \in P(a)$ with $l(p) \geq 1$. Then, much as before, $h \leq k + 2 + l(p) \leq l(p)(k + 3)$. Hence,
\begin{equation}
\prod_{p \in P(a) \atop l(p) \geq 1} p^{(1-s)h} \geq P^1(a)^{(1-s)(k+3)}.
\end{equation}

However, since $a \not\in A_1 \cup A_2$, we have $P^1(a) \leq A^{2\delta}$, so that we now deduce from (3.26) that
\begin{equation}
\mathcal{E}_a \gg A^{2\delta (1-s)(k+3)}.
\end{equation}

The synthesis is straightforward. Let $\gamma > 0$. Then choose $\delta = \gamma/(8(s-1)(k+3))$, and suppose that $A$ is large. Then (3.26) implies that $\mathcal{E}_a \geq A^{-\gamma}$. If that fails, then $\mathcal{E}_a = 0$, or else $a \in A_1 \cup A_2$. The estimates (3.22) and (3.23) imply Theorem 1.2.

IV. The circle method

4.1. Preparatory steps. In this section, we establish Theorem 1.1. The argument is largely standard, save for the ingredients to be imported from the previous sections of this memoir.

We employ the following notational convention throughout this section: if $h : \mathbb{R} \rightarrow \mathbb{C}$ is a function, and $a \in \mathbb{Z}^s$, then we define
\begin{equation}
h_a(\alpha) = h(a_1 \alpha)h(a_2 \alpha)\ldots h(a_s \alpha).
\end{equation}

As is common in problems of an additive nature, the Weyl sum
\begin{equation}
f(\alpha) = \sum_{1 \leq |x| \leq B} e(\alpha x^k)
\end{equation}
is prominently featured in the argument to follow, because by orthogonality, one has
\begin{equation}
g_a(B) = \int_0^1 f_a(\alpha) \, d\alpha.
\end{equation}

The circle method will be applied to the integral in (4.3). With applications in mind that go well beyond those in the current communication, we shall treat the ‘major arcs’ under very mild conditions on $A, B$, and for the range $s > 2k$.

Let $A \geq 1$, $B \geq 1$, and fix a real number $\eta$ with $0 < \eta \leq 1/3$. Then put $Q = B^\eta$. Let $\mathfrak{M}$ denote the union of the intervals
\begin{equation}
|\alpha - \frac{r}{q}| \leq \frac{Q}{AB^k}
\end{equation}
with \(1 \leq r \leq q < Q\), and \((r, q) = 1\). These intervals are pairwise disjoint, and we write \(m = \lfloor Q/(AB^k) \rfloor (1 + Q/(AB^k))\backslash \mathfrak{M}\). When \(\mathfrak{A}\) is one of \(\mathfrak{M}\) or \(m\), let
\[
\varrho_a(B, \mathfrak{A}) = \int_\mathfrak{A} f_a(\alpha) \, d\alpha
\]
and note that
\[
\varrho_a(B) = \varrho_a(B, \mathfrak{M}) + \varrho_a(B, m)
\]

4.2. The major arc analysis. In this section we make heavy use of the results in Vaughan’s book \([22]\) on the subject. He works with the Weyl sum
\[
g(\alpha) = \sum_{1 \leq x \leq B} e(\alpha x^k)
\]
that is related with our \(f\) through the formulae
\[
f(\alpha) = 2g(\alpha) \quad (k \text{ even}), \quad f(\alpha) = g(\alpha) + g(-\alpha) \quad (k \text{ odd}).
\]
Thus, in particular, Theorem 4.1 of \([22]\) yields the following.

Lemma 4.1. Let \(\alpha \in \mathbb{R}, r \in \mathbb{Z}, q \in \mathbb{N}\) and \(a \in \mathbb{Z}\) with \(a \neq 0\). Then
\[
f(\alpha a) = q^{-s}S(q, ar)S(q, a) + O(q^{1/2+\varepsilon}(1 + |a|B^k|\alpha - r/q|)^{1/2}).
\]
Here, and throughout the rest of this section, we define \(v(\beta) = v(\beta, B)\) through (3.1). When \(|\alpha| \leq A\) and \(\alpha \in \mathfrak{M}\) is in the interval (4.4), we find that
\[
f(\alpha a) = q^{-s}S(q, ar)v(\alpha - r/q) + O(Q^2).
\]
This we use with \(a = a_j\) and multiply together. Then
\[
f_a(\alpha) = q^{-s}S(q, ar)S(q, a) + O(Q^2 B^{s-1}).
\]
Now integrate over \(\mathfrak{M}\), and recall the definition of the latter. By (4.3) and (4.15), we then arrive at
\[
\varrho_a(B, \mathfrak{M}) = \sum_{q < Q} T_a(q) \int_{-Q/(AB^k)}^{Q/(AB^k)} v_a(\beta) \, d\beta + O(Q^2 B^{s-1-k} A^{-1}).
\]
Here, we complete the sum over \(q\) to the singular series, and the integral over \(\beta\) to the singular integral. Some notation is required to make this precise. When \(R \geq 1\), define the tail of \(\mathfrak{S}_a\) as
\[
\mathfrak{S}_a(R) = \sum_{q \geq R} T_a(q)
\]
which is certainly convergent for \(s \geq k + 2\); compare Lemma 3.2. Also, note that \(\mathfrak{S}_a = \mathfrak{S}_a(1)\). Moreover, on recalling (4.3) we write
\[
\int_{-Q/(AB^k)}^{Q/(AB^k)} v_a(\beta) \, d\beta = J_a(B) + E_a,
\]
and then infer that
\[
\varrho_a(B, \mathfrak{M}) = (\mathfrak{S}_a - \mathfrak{S}_a(Q))J_a(B) + E_a + O(Q^5 B^{s-1-k} A^{-1}).
\]
Consequently,
\[
\varrho_a(B, \mathfrak{M}) - \mathfrak{S}_a J_a(B) \ll |\mathfrak{S}_a(Q)||J_a(B)| + E_a + |\mathfrak{S}_a|E_a| + Q^5 B^{s-1-k} A^{-1}.
\]
Lemma 4.2. Let

\[ \sum_{|a| \leq A} |g_a(B, \mathfrak{M}) - \mathcal{S}_a J_a B^{s-k}|^2 \ll V_1 + V_2 + A^{s-2} B^{2s-2k-2} Q^{10} \]

in which

\[ V_1 = \sum_{|a| \leq A} |\mathcal{S}_a(Q)|^2 |J_a(B) + E_a|^2, \quad V_2 = \sum_{|a| \leq A} |\mathcal{S}_a E_a|^2. \]

By \[ \text{(4.8)} \] and \[ \text{(3.5)} \], one has \[ J_a(B) + E_a \ll |a_1 \ldots a_s|^{-1/s} B^{s-k}. \] It follows that

\[ V_1 \ll B^{2s-2k} \sum_{|a| \leq A} |\mathcal{S}_a(Q)|^2 |a_1 \ldots a_s|^{-2/s}. \]

Before we proceed with the estimation of \( V_1 \) we prepare \( V_2 \) in a similar vein. By \[ \text{(4.8)} \] followed by an application of Hölder's inequality,

\[ |E_a| \leq \left( \int_{|\beta| \geq Q/(AB^k)} |v_a(\beta)|^s d\beta \right)^{1/s} \leq \prod_{j=1}^s \left( \int_{|\beta| \geq Q/(AB^k)} |v(\beta)|^s d\beta \right)^{1/s}. \]

However, whenever \( 0 < |a| \leq A \) and \( s > 2k \), then by \[ \text{(3.2)} \], one has

\[ \int_{Q/(AB^k)} |v(\beta)|^s d\beta \ll \frac{B^{s-k}}{|a|} \int_{Q/|a|/(AB^k)} (1 + \gamma)^{-2} d\gamma \ll \frac{AB^{s-k}}{|a|^2 Q}, \]

and therefore,

\[ |E_a| \ll AB^{s-k} Q^{-1} |a_1 a_2 \ldots a_s|^{-2/s}. \]

This shows that

\[ V_2 \ll A^2 B^{2s-2k} Q^{-2} \sum_{|a| \leq A} |a_1 a_2 \ldots a_s|^{-4/s} \mathcal{S}_a^2. \]

Further progress now depends on a mean square estimate related to the singular series. As we shall prove momentarily, when \( s > 2k, A \geq R \geq 1 \) and \( 0 \leq \tau \leq 2/k \), one has

\[ \sum_{|a| \leq A} |a_1 a_2 \ldots a_s|^{-\tau} \mathcal{S}_a(R)^2 \ll A^{s(1-\tau)} R^{s-2/k}. \]

Equipped with this bound, one finds from \[ \text{(4.10)} \] and \[ \text{(4.11)} \] that

\[ V_1 + V_2 \ll A^{s-2} B^{2s-2k+\frac{s}{k}} Q^{-2/k}, \]

We finally choose \( \eta = \frac{1}{6} \), and then by \[ \text{(4.9)} \], conclude as follows.

Lemma 4.2. Let \( A \geq 1, B \geq 1 \) and \( Q = B^{1/6} \). Then, whenever \( s > 2k \), one has

\[ \sum_{|a| \leq A} |g_a(B, \mathfrak{M}) - \mathcal{S}_a J_a B^{s-k}|^2 \ll A^{s-2} B^{2s-2k-1/(3k)}. \]

It remains to confirm \[ \text{(4.12)} \]. First observe that \( \kappa(q) \ll q^{-1/k} \). This is immediate from the definition of \( \kappa \). Then, by \[ \text{(4.7)} \] and \[ \text{(3.10)} \],

\[ \mathcal{S}_a(R) \ll \sum_{q > R} q^{1-s/k+\varepsilon} (q, a_1)^{1/k} \ldots (q, a_s)^{1/k}. \]

Hence, since \( s \geq 2k + 1 \), one deduces that

\[ \mathcal{S}_a(R)^2 \ll \sum_{q > R} \sum_{r > R} (qr)^{1-s/k+\varepsilon} (qr, a_1)^{2/k} \ldots (qr, a_s)^{2/k}. \]
One multiplies with \((a_1 a_2 \ldots a_s)^{-r}\) and then sums over \(a\) first. The desired bound then follows immediately.

### 4.3. The minor arcs

We begin the endgame with a variant of Weyl’s inequality.

**Lemma 4.3.** Let \(A \geq 1, B \geq 1\), and suppose that \(r \in \mathbb{Z}\) and \(q \in \mathbb{N}\) are coprime with \(|\alpha - (r/q)| \leq q^{-2}\). Then

\[
\sum_{0 < |a| \leq A} |f(a\alpha)|^2 \ll AB^{2k-1} \left( \frac{1}{q} + \frac{1}{B} + \frac{q}{AB^k} \right) (ABq)^2.
\]

This is well known, but we give a brief sketch for completeness. Write \(K = 2^{k-1}\).

Then, as an intermediate step towards the ordinary form of Weyl’s inequality, one has

\[
|f(\beta)|^K \ll B^{K-1} + B^{K-k+\varepsilon} \sum_{1 \leq h \leq 2B^k} \min(B, \|h\beta\|^{-1})
\]

where \(\|\beta\|\) denotes the distance of \(\beta\) to the nearest integer; compare the arguments underpinning Lemma 2.4 of Vaughan [22]. Now choose \(\beta = a\alpha\) and sum over \(a\). A divisor function argument then yields

\[
\sum_{0 < |a| \leq A} |f(a\alpha)|^K \ll AB^{K-1} + B^{K-k} (AB)^2 \sum_{h \in A B^k} \min(B, \|h\beta\|^{-1}),
\]

and Lemma 4.3 follows from Lemma 2.2 of Vaughan [22].

Now let \(\alpha \in \mathfrak{m}\). By Dirichlet’s theorem on diophantine approximations, there are coprime \(r \in \mathbb{Z}\), \(q \in \mathbb{N}\) with \(q \leq Q^{-1} AB^k\) and

\[
|q\alpha - r| \leq Q(AB^k)^{-1}.
\]

But \(\alpha \notin \mathfrak{m}\), whence \(q > Q\). Lemma 4.3 in conjunction with Hölder’s inequality now yields

\[
(4.13) \quad \sup_{\alpha \in \mathfrak{m}} \sum_{0 < |a| \leq A} |f(a\alpha)|^2 \ll A^{1+\varepsilon} B^{2+\varepsilon} Q^{-2^{2-k}}.
\]

We now apply this estimate to establish the following.

**Lemma 4.4.** Let \(s \in \mathbb{N}\), \(s = 2t + u\) with \(t \in \mathbb{N}\), \(u = 1\) or \(2\) and \(\delta = \frac{1}{4} 2^{1-k}\). Then, whenever \(1 \leq B^{2k} \leq A \leq B^{2t-k}\) holds, one has

\[
\sum_{|a| \leq A} |\varrho_{\alpha}(B, m)|^2 \ll A^{s-2} B^{2s-2k-\delta+\varepsilon}.
\]

**Proof.** By (4.9) and (2.20), one has

\[
\sum_{|a| \leq A} |\varrho_{\alpha}(B, m)|^2 = \sum_{|a| \leq A} \int_{\mathfrak{m}} f_{\alpha}(\alpha) \, d\alpha \int_{\mathfrak{m}} f_{\alpha}(-\beta) \, d\beta = \int_{\mathfrak{m}} \int_{\mathfrak{m}} F(\alpha, -\beta)^r \, d\alpha \, d\beta.
\]

By orthogonality and Lemma 2.5, it follows that

\[
\int_0^1 \int_0^1 |F(\alpha, -\beta)|^{2t} \, d\alpha \, d\beta \ll \Upsilon(A, B) \ll A^{2t-2} B^{4t-2k+\varepsilon}.
\]

Also, by (2.9), (4.13), and Cauchy’s inequality,

\[
\sup_{\alpha, \beta \in \mathfrak{m}} |F(\alpha, -\beta)| \ll AB^{2-\delta+\varepsilon}.
\]
Lemma 4.4 now follows on combining the information encoded in the last three displays.

Theorem 1.1 is also available: one has $\hat{s} = 2t$, and the theorem follows on combining (4.6) with Lemma 4.2 and Lemma 4.4.

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