Solutions to the Quasi-flatness and Quasi-lambda Problems

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Abstract

Big Bang models of the Universe predict rapid domination by curvature, a paradox known as the flatness problem. Solutions to this problem usually leave the Universe exactly flat for every practical purpose. Explaining a nearly but not exactly flat current Universe is a new problem, which we label the quasi-flatness problem. We show how theories incorporating time-varying coupling constants could drive the Universe to a late-time near-flat attractor. A similar problem may be posed with regards to the cosmological constant $\Lambda$, the quasi-lambda problem, and we exhibit a solution to this problem as well.

1 Introduction

Recent investigations of the cosmological effects of introducing a time-variation of the speed of light, $c$, into the gravitational field equations have revealed a number of tantalising possibilities. If the speed of light falls sufficiently rapidly over an interval of time then it is possible to solve the standard horizon and flatness problems in a way that differs from the inflationary universe, \cite{3,1,2}. Moreover, unlike in inflationary models, \cite{5}, or pre-big-bang scenarios, \cite{6}, a sufficiently fast fall-off in $c$ can also solve the cosmological constant problem \cite{1,2,4}.

In this letter we display another appealing feature of these varying-$c$ models: they permit a solution of the quasi-flatness and quasi-lambda problems. Whereas a solution of the flatness problem must provide for the flat $\Omega = 1$ cosmology being a late-time attractor for ever-expanding universes, a solution to the quasi-flatness problem must offer an explanation for a late-time attractor with $\Omega$ taking a value or order unity, between zero and one, for example $\Omega_0 \sim 0.2$. An inflationary universe model could only provide a resolution of
this problem by appealing to a particular (small) amount of inflation that left the expansion short of asymptoting very close to flatness. The motivation for seeking a model in which such an attractor with \( \Omega < 1 \) arises naturally is the persistent trend of the observational evidence [7]. It almost all points towards such a value in our Universe.

Similarly, whereas a solution of the lambda problem would require that the contribution of the cosmological constant term, \( \Lambda \), to the Friedmann equation be negligible at late times in ever-expanding universes, [8], a solution of the quasi-lambda problem would explain why its effect is of the same order as the density term in the Friedmann equation. The motivation for this type of model is the recent evidence from supernovae suggesting that the cosmological dynamics are best fitted by a model with \( \Lambda \neq 0 \), [9].

This type of concern has also stimulated new investigations in quantum cosmology, [10], following earlier investigations of open inflationary universes, [11]. It is possible for inflationary universe models to provide a solution to the quasi-lambda problem if there is a residual cosmological constant remaining at the end of inflation: the dynamics then approach a model in which the curvature is negligibly small and \( \Omega_0 = 1 - \Lambda / 3H_0^2 \), where \( H_0 \) is the Hubble expansion rate.

2 The Quasi-flatness Problem

The cosmological evolution of an isotropic and homogeneous universe incorporating varying speed of light can be described by the standard evolution equations for the Friedmann universe with \( c = c(t) \):

\[
\begin{align*}
\frac{\dot{a}^2}{a^2} &= \frac{8\pi G(t)\rho}{3} - \frac{Kc^2(t)}{a^2}, \\
\ddot{a} &= -\frac{4\pi G(t)}{3}(\rho + \frac{3p}{c^2(t)})a,
\end{align*}
\]

where \( p \) and \( \rho \) are the density and pressure of the matter, respectively, and \( K \) is the metric curvature parameter. By differentiating (1) with respect to time and substituting in (2), we find the generalised conservation equation [1] incorporating possible time variations in \( c(t) \),

\[
\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \frac{p}{c^2}) = \frac{3Kc\dot{c}}{4\pi Ga^2}.
\]
We shall assume that the matter obeys an equation of state of the form

\[ p = (\gamma - 1) \rho c^2(t), \]  

(4)

where \( \gamma \) is a constant. We shall assume that the universe undergoes a sufficiently long period of evolution during which \( c \) changes at a rate proportional to the expansion of the universe:

\[ c = c_0 a^n, \]  

(5)

where \( c_0 > 0 \) and \( n \) are constants, [2].

Elsewhere, [1], [14], we have discussed the lagrangian formulation of such a theory and its relationship, via transformations of units, to a theory in which \( c \) is constant (set equal to unity for example) and the electron charge varies. This is just a manifestation of the fact that an invariant operational meaning can only be attributed to the variation of dimensionless constants. In this case, varying \( e \) or \( c \) would be different representations of a theory displaying the observational consequences of varying fine structure constant, in which there has been much recent observational interest [12]. An example of a theory with varying \( e \) but constant \( c \) has been given by Bekenstein, [13], but as shown in [14] the theory leading to (1)-(3) differs from Bekenstein’s because he postulates \textit{ab initio} that there be no effects on the gravitational field equations.

The cosmological density parameter, \( \Omega \), is defined in the usual way as the density of the universe in units of the critical density, \( \rho_c \), which defines the \( K = 0 \) solution of eq. (1). Thus,

\[ \Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G \rho}{3H^2} \]  

(6)

and so, again using (1), we have

\[ \frac{\Omega}{\Omega - 1} = \frac{8\pi G \rho}{3Kc^2(t)a^{-2}}. \]  

(7)

We can solve eq. (3), using (4) and (5), to obtain [2]

\[ \rho = \frac{B}{a^{3\gamma}} + \frac{3Kc_0^2 \rho a^{2(n-1)}}{4\pi G(2n - 2 + 3\gamma)}, \]  

(8)

where \( B \geq 0 \) is constant if \( 2n - 2 + 3\gamma \neq 0 \). When the speed of light is constant \( (n = 0) \) this reduces to the usual adiabatic density evolution law \( \rho \propto a^{-3\gamma} \).
Substituting in (1) we have

\[ \frac{\dot{a}^2}{a^2} = \frac{8\pi GB}{3a^{3\gamma}} + \frac{Kc_0^2a^{2(n-1)}(2 - 3\gamma)}{(2n - 2 + 3\gamma)} \]  

(9)

and hence (7) reduces to

\[ \frac{\Omega}{\Omega - 1} = \frac{8\pi G}{3Kc_0^2} \left[ Ba^{2-2n-3\gamma} + \frac{3Kc_0^2n}{4\pi G(2n - 2 + 3\gamma)} \right]. \]  

(10)

These equations allow us to determine the possible cosmological behaviours that can arise as \( n \), the rate of variation of the speed of light, varies. Thus, we see that if \( 2 - 2n - 3\gamma > 0 \) then, as the universe expands, \( a \to \infty \), the right-hand side of (10) diverges and we have \( \Omega \to 1 \) and the expansion asymptotes to \( a(t) \propto t^{2/3\gamma} \) for \( \gamma > 0 \); that is, we provide an explanation for flatness at large \( a \) or time, \( t \), whenever, \([1,2]\),

\[ n < \frac{1}{2}(2 - 3\gamma). \]  

(11)

By contrast, if \( 2 - 2n - 3\gamma < 0 \), then as \( a \to \infty \), we have

\[ \frac{\Omega}{\Omega - 1} \to \frac{2n}{2n - 2 + 3\gamma} \]  

(12)

and hence we have a solution of the quasi-flatness problem. We predict that at large \( a \)

\[ \Omega \to \frac{-2n}{3\gamma - 2} \]  

(13)

whenever

\[ 0 > n > \frac{1}{2}(2 - 3\gamma) \]  

(14)

where \( n < 0 \) is required to ensure \( \Omega > 0 \) when the equation of state satisfies \( 3\gamma > 2 \). From (9) we see that at late times the scale factor asymptotes to

\[ a(t) \propto t^{\frac{1}{1-n}}. \]  

(15)

For example, if the universe is radiation (\( \gamma = 4/3 \)) or dust (\( \gamma = 1 \)) dominated then a solution of the quasi-flatness problem would require a period of
evolution during which $0 > n(\text{rad}) > -1$ or $0 > n(\text{dust}) > -\frac{1}{2}$, leading to asymptotic expansion with $\Omega(\text{rad}) = -n$ and $\Omega(\text{dust}) = -2n$, respectively. Thus, asymptotic expansion with $\Omega_0$ less than but of order unity is possible in these models. If the fluid which dominates the expansion dynamics during the period when $c$ varies violates the strong energy condition (as is required for inflation to occur), so $0 < 3\gamma < 2$, then a quasi-flat asymptote cannot arise because we also require $n > 0$ for $\Omega > 0$.

The conditions required for the solution of the horizon and monopole problems are identical to those for the flatness problem [1,2].

3 The Quasi-lambda Problem

Let us now consider the more stringent requirements on $c$ variation that are required to resolve the problems associated with the possible existence of a non-negligible cosmological constant term in the cosmological equations. If we wish to incorporate a positive cosmological constant term, $\Lambda$, (which we shall assume to be constant) into a theory with varying speed of light then we can define a vacuum stress obeying an equation of state

$$p_\Lambda = -\rho_\Lambda c^2,$$

where

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \geq 0.$$ (17)

Then, replacing $\rho$ by $\rho + \rho_\Lambda$ in (3), we have the generalisation [1]

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \frac{p}{c^2}) + \dot{\rho}_\Lambda = \frac{3K\dot{c}\ddot{c}}{4\pi Ga^2}.$$ (18)

The Friedmann equation is now

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{Kc^2(t)}{a^2} + \frac{\Lambda c^2(t)}{3}.$$ (19)

We define

$$\Omega \equiv \Omega_m + \Omega_\Lambda = \frac{8\pi G(\rho + \rho_\Lambda)a^2}{3Kc^2}.$$/}\end{equation} (20)
where we distinguish the contributions from the matter \((m)\) and the lambda term \((\Lambda)\) by the ratio

\[
\frac{\Omega_m}{\Omega_\Lambda} \equiv \frac{\rho}{\rho_\Lambda}.
\]  \hfill (21)

Again, we assume that \(c\) varies according to (5), so eq. (18) integrates to give [2]

\[
\rho = \frac{B}{a^{3\gamma}} + \frac{3Kc_0^2na^{2(n-1)}}{4\pi G(2n - 2 + 3\gamma)} - \frac{\Lambda nc_0^2a^{2n}}{4\pi G(2n + 3\gamma)}.
\]  \hfill (22)

Substituting in (19) we have

\[
\frac{\dot{a}^2}{a^2} = \frac{8\pi GB}{3a^{3\gamma}} + \frac{Kc_0^2a^{2(n-1)}(2 - 3\gamma)}{(2n - 2 + 3\gamma)} + \frac{\Lambda c_0^2 a^{2n}}{(3\gamma + 2n)}. \tag{23}
\]

and

\[
\frac{\Omega_m}{\Omega_\Lambda} = \frac{8\pi GB a^{-3\gamma - 2n}}{\Lambda c_0^2} + \frac{6Kn a^{-2}}{\Lambda (2n - 2 + 3\gamma)} - \frac{2n}{(2n + 3\gamma)}. \tag{24}
\]

Eq. (23) allows us to determine what happens at large \(a\). We note that the curvature term is always dominated by the \(\Lambda\) term at sufficiently large \(a\). Eq. (24) allows us to infer whether a solution of the quasi-lambda problem is possible.

There are two cases to consider:

3.1 Subcase 1: \(n < -3\gamma/2\)

If \(-3\gamma > 2n\) then we see that the flatness and the lambda problems are both solved as before since the \(B\) term dominates the right-hand side of eq. (23) at large \(a\), with \(a(t) \propto t^{2/3\gamma}\). In this case there is no possible resolution of the quasi-flatness problem since, from (24), we see that \(\Omega_m/\Omega_\Lambda \to \infty\).

3.2 Subcase 2: \(0 > n > -3\gamma/2\)

In this case the \(\Lambda\) term dominates the dynamics of eq. (23) yielding

\[
\frac{\dot{a}^2}{a^2} \to \frac{\Lambda c_0^2 a^{2n}}{(3\gamma + 2n)}. \tag{25}
\]
So, at large $t$, we have

$$a \propto t^{-\frac{1}{n}}. \quad (26)$$

However, for negative $n$, now there is a solution to the quasi-lambda problem (i.e., why $\Omega_m$ and $\Omega_\Lambda$ are of similar order at large $a$) since the ratio of the densities contributed by the matter and lambda stresses approaches a constant positive value determined by $n$:

$$\frac{\Omega_m}{\Omega_\Lambda} \to -\frac{2n}{2n + 3\gamma} > 0. \quad (27)$$

We can also express the asymptotic form of the scale factor as $t \to \infty$ in the form

$$a(t) \propto t^\lambda \quad (28)$$

where

$$\lambda \equiv -\frac{3\gamma \Omega_m}{2(\Omega_\Lambda + \Omega_m)}. \quad (29)$$

An interesting prediction of our model is that if $\Omega_\Lambda \neq 0$ then one must have $\Omega_m + \Omega_\Lambda = 1$. Therefore if we are to solve the quasi-lambda problem, we must have exact flatness. This is in agreement with recent observational indications [9].

### 4 Discussion

We have shown that a cosmological theory in which the velocity of light experiences a period of change can have a number of appealing consequences. Elsewhere, we have shown that a suitable fall-off in the value of $c$ can provide solutions to either or both of the flatness and cosmological constant problems. The flatness problem is solved in a different way to its resolution in inflationary universes: the curvature term is made to decrease faster than the matter term in the Friedmann equation whereas in general relativistic inflationary cosmologies the matter terms are made to fall off more slowly than the curvature term by appeal to matter fields with $\rho + 3p < 0$. Inflation does not resolve the lambda problem at all. Here we extend those results by showing that solutions of the more difficult quasi-flatness and quasi-lambda problems can also be found in such a scenario. For a simple power-law change of $c = c_0 a^n$
we have determined the range of values of \( n \) which provide solutions of the flatness, quasi-flatness, lambda, and quasi-lambda problems in the presence of matter with a perfect fluid equation of state.

A number of open problems remain, to be investigated in a longer publication. Standard varying-\( c \) scenarios solve the flatness problem due to violations of energy conservation, as encoded in (3) and (18). These violations are only non-negligible when curvature “attempts” to dominate. Then, matter is produced if the Universe is open (and therefore sub-critical), and destroyed if the Universe is closed (and therefore super-critical). A changing \( c \) introduces a self-regulating mechanism which stabilizes the flat or critical model. The implications of this process for the entropy of the Universe were also discussed in [1]. In the scenarios we discuss in this Letter, on the other hand, the energy source term is present even after the attractor is achieved. We have therefore a scenario of “permanent reheating”, which we will detail in a future publication.

We will return the study of the general class of solutions to this model elsewhere. In particular we shall consider the effects of the radiation to matter transition, and also the possibility of a changing \( G \). The latter results from the theories discussed in [2] which are a generalization of the usual Brans-Dicke theories.

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References

[1] A. Albrecht and J. Magueijo, A time-varying speed of light as a solution to cosmological puzzles, Phys. Rev. D, in press, (1998).

[2] J.D. Barrow, Cosmologies with varying light speed, Phys. Rev. D, in press, (1998).

[3] J. Moffat, International Journal of Physics D, Vol. 2, No. 3 (1993) 351-365; Foundations of Physics, Vol. 23 (1993) 411.

[4] J. Moffat, astro-ph/9811390.

[5] A.H. Guth, Phys. Rev. D 23, 347 (1981); A. Linde, Phys. Lett. B 108, 389 (1982); S.W. Hawking and I. Moss, Phys. Lett. B 110 35 (1982); A. Albrecht and P. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982)
[6] G. Veneziano, Phys. Lett. B 406, 297 (1997); M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993).

[7] P. Coles and G.F.R. Ellis, Is the Universe Open or Closed?, Cambridge UP, Cambridge, (1997).

[8] S.W. Hawking, Phil. Trans. Roy. Soc. A 310, 303 (1983); S.W. Hawking, Phys. Rev. D 37, 904; S. Coleman, Nucl. Phys. B 310, 643 (1988); S.W. Weinberg, Rev. Mod. Phys. 61, 1 (1989).

[9] S. Perlmutter et al, Ap. J. 483, 565 (1997); S. Perlmutter et al (The Supernova Cosmology project), Nature 391 51 (1998); Garnavich, P.M et al 1998 Ap.J. Letters 493:L53-57 Schmidt, B.P. 1998 Ap.J. 507:46-63 Riess, A.G. et al 1998 AJ 116:1009

[10] J.R. Gott III, Nature 295, 304 (1982); A. Vilenkin, astro-ph/9805252; N. Turok and S. Hawking, hep-th/9803156; A. Linde, gr-qc/9802038.

[11] M. Bucher, A.S. Goldhaber, and N. Turok, Phys. Rev D52, 3314 (1995); K. Yamamoto, M. Sasaki, and T. Tanaka, Astrophys. J. 455, 412 (1995).

[12] M.J. Drinkwater, J.K. Webb, J.D. Barrow & V.V. Flambaum, Mon. Not. Roy. astron. Soc. 295, 457 (1998); T. Damour and F. Dyson, Nucl. Phys. B 480, 37 (1996); A.I. Shylakhter, Nature 264, 340(1976); J.D. Barrow, Phys. Rev. D 35, 1805 (1987). J.D. Barrow and F.J. Tipler, The Anthropic Cosmological Principle, Oxford UP, Oxford (1986). J.K. Webb, V.V. Flambaum, C.W. Churchill, M.J. Drinkwater and J.D. Barrow, et al., Search for Time Variation of the Fine Structure Constant, Phys. Rev. Lett. , in press, astro-ph/9803165.

[13] J.D. Bekenstein, Phys. Rev. D 25, 1527 (1982)

[14] J.D. Barrow and J. Magueijo, Varying-α theories and solutions to the cosmological problems, Phys. Lett. B (in press 1998)