Law of Large Numbers for Random Dynamical Systems

Katarzyna Horbacz · Maciej Śleszka

Received: 30 July 2015 / Accepted: 24 November 2015 / Published online: 10 December 2015
© The Author(s) 2015. This article is published with open access at Springerlink.com

Abstract We consider random dynamical systems with randomly chosen jumps. The choice of deterministic dynamical system and jumps depends on a position. We prove the existence of an exponentially attractive invariant measure and the strong law of large numbers.

Keywords Dynamical systems · Law of large numbers · Invariant measure

Mathematics Subject Classification 60J25 · 60J75

1 Introduction

In the present paper we are concerned with the problem of proving the law of large numbers (LLN) for random dynamical systems.

The question of establishing the LLN for an additive functional of a Markov process is one of the most fundamental in probability theory and there exists a rich literature on the subject, see e.g. [21] and the citations therein. However, in most of the existing results, it is usually assumed that the process under consideration is stationary and its equilibrium state is stable in some sense, usually in the $L^2$, or total variation norm. Our stability condition is formulated in a weaker metric than the total variation distance.

The law of large numbers we study in this note was also considered in many papers. Our results are based on a version of the law of large numbers due to Shirikyan (see [23, 24]). Recently Komorowski et al. [15] obtained the weak law of large numbers for the passive tracer model in a compressible environment and Walczuk studied Markov processes with the
transfer operator having spectral gap in the Wasserstein metric and proved the LLN in the non-stationary case [30].

Random dynamical systems [6,8,10] take into consideration some very important and widely studied cases, namely dynamical systems generated by learning systems [1,11,13,19], Poisson driven stochastic differential equations [7,18,25,27], iterated function systems with an infinite family of transformations [17,28,29], random evolutions [4,22] and irreducible Markov systems [31], used for the computer modelling of different stochastic processes.

A large class of applications of such models, both in physics and biology, is worth mentioning here: the shot noise, the photo conductive detectors, the growth of the size of structural populations, the motion of relativistic particles, both fermions and bosons (see [3,14,16]), the generalized stochastic process introduced in the recent model of gene expression by Lipniacki et al. [20] see also [2,5,9].

A number of results have been obtained that claim an existence of an asymptotically stable, unique invariant measure for Markov processes generated by random dynamical systems for which the state space need not be locally compact. We consider random dynamical systems with randomly chosen jumps acting on a given Polish space \((Y, \varnothing)\).

The aim of this paper is to study stochastic processes whose paths follow deterministic dynamics between random times, jump times, at which they change their position randomly. Hence, we analyse stochastic processes in which randomness appears at times \(\tau_0 < \tau_1 < \tau_2 < \ldots\). We assume that a point \(x_0 \in Y\) moves according to one of the dynamical systems \(T_i : \mathbb{R}_+ \times Y \to Y\) from some set \(\{T_1, \ldots, T_N\}\). The motion of the process is governed by the equation \(X(t) = T_i(t, x_0)\) until the first jump time \(\tau_1\). Then we choose a transformation \(q_\theta : Y \to Y\) from a family \(\{q_\theta : \theta \in \Theta = \{1, \ldots, K\}\}\) and define \(x_1 = q_\theta(T_1(\tau_1, x_0))\). The process restarts from that new point \(x_1\) and continues as before. This gives the stochastic process \(\{X(t)\}_{t \geq 0}\) with jump times \(\{\tau_1, \tau_2, \ldots\}\) and post jump positions \(\{x_1, x_2, \ldots\}\). The probability determining the frequency with which the dynamical systems \(T_i\) are chosen is described by a matrix of probabilities \(\{p_{ij}\}_{i,j=1}^{N}, p_{ij} : Y \to [0, 1]\). The maps \(q_\theta\) are randomly chosen with place dependent distribution. Given a Lipschitz function \(\psi : X \to \mathbb{R}\) we define

\[ S_n(\psi) = \psi(x_0) + \cdots + \psi(x_n). \]

Our aim is to find conditions under which \(S_n(\psi)\) satisfies law of large numbers. Our results are based on an exponential convergence theorem due to Śłęczka and Kapica (see [12]) and a version of the law of large numbers due to Shirikyan (see [23,24]).

### 2 Notation and Basic Definitions

Let \((X, d)\) be a Polish space, i.e. a complete and separable metric space and denote by \(B_X\) the \(\sigma\)-algebra of Borel subsets of \(X\). By \(B_b(X)\) we denote the space of bounded Borel-measurable functions equipped with the supremum norm, \(C_b(X)\) stands for the subspace of bounded continuous functions. Let \(\mathcal{M}_{\text{fin}}(X)\) and \(\mathcal{M}_1(X)\) be the sets of Borel measures on \(X\) such that \(\mu(X) < \infty\) for \(\mu \in \mathcal{M}_{\text{fin}}(X)\) and \(\mu(X) = 1\) for \(\mu \in \mathcal{M}_1(X)\). The elements of \(\mathcal{M}_1(X)\) are called probability measures. The elements of \(\mathcal{M}_{\text{fin}}(X)\) for which \(\mu(X) \leq 1\) are called subprobability measures. By \(\text{supp } \mu\) we denote the support of the measure \(\mu\). We also define

\[ \mathcal{M}_1^L(X) = \left\{ \mu \in \mathcal{M}_1(X) : \int_X L(x)\mu(dx) < \infty \right\} \]
where $L : X \to [0, \infty)$ is an arbitrary Borel measurable function and

$$\mathcal{M}_1(X) = \left\{ \mu \in \mathcal{M}_1(X) : \int_X d(\bar{x}, x) \mu(dx) < \infty \right\},$$

where $\bar{x} \in X$ is fixed. By the triangle inequality this family is independent of the choice of $\bar{x}$.

The space $\mathcal{M}_1(X)$ is equipped with the Fortet-Mourier metric:

$$\|\mu_1 - \mu_2\|_{FM} = \sup\left\{ \int_X f(x)(\mu_1 - \mu_2)(dx) : f \in \mathcal{F} \right\},$$

where

$$\mathcal{F} = \{ f \in C_b(X) : |f(x) - f(y)| \leq d(x, y) \text{ and } |f(x)| \leq 1 \text{ for } x, y \in X \}.$$

Let $P : B_b(X) \to B_b(X)$ be a Markov operator, i.e. a linear operator satisfying $P1_X = 1_X$ and $P(f(x)) \geq 0$ if $f \geq 0$. Denote by $P^*$ the dual operator, i.e operator $P^* : \mathcal{M}_{fin}(X) \to \mathcal{M}_{fin}(X)$ defined as follows

$$P^* \mu(A) := \int_X P1_A(x) \mu(dx) \quad \text{for } A \in \mathcal{B}_X.$$

We say that a measure $\mu_\ast \in \mathcal{M}_1(X)$ is invariant for $P$ if

$$\int_X Pf(x) \mu_\ast(dx) = \int_X f(x) \mu_\ast(dx) \quad \text{for every } f \in B_b(X)$$
or, alternatively, we have $P^* \mu_\ast = \mu_\ast$. An invariant measure $\mu$ is attractive if

$$\lim_{n \to \infty} \int_X P^n f(x) \mu(dx) = \int_X f(x) \mu(dx) \quad \text{for } f \in C_b(X), \mu \in \mathcal{M}_1(X).$$

By $\{P_x : x \in X\}$ we denote a transition probability function for $P$, i.e. a family of measures $P_x \in \mathcal{M}_1(X)$ for $x \in X$, such that the map $x \mapsto P_x(A)$ is measurable for every $A \in \mathcal{B}_X$ and

$$Pf(x) = \int_X f(y)P_x(dy) \quad \text{for } x \in X \text{ and } f \in B_b(X)$$
or equivalently $P^* \mu(A) = \int_X P_x(A) \mu(dx)$ for $A \in \mathcal{B}_X$ and $\mu \in \mathcal{M}_{fin}(X)$. We say that a vector $(p_1, \ldots, p_N)$ where $p_i : Y \to [0, 1]$ is a probability vector if

$$\sum_{i=1}^N p_i(x) = 1 \quad \text{for } x \in Y.$$

Analogously a matrix $[p_{ij}]_{i,j}$ where $p_{ij} : Y \to [0, 1]$ for $i, j \in \{1, \ldots, N\}$ is a probability matrix if

$$\sum_{j=1}^N p_{ij}(x) = 1 \quad \text{for } x \in Y \text{ and } i \in \{1, \ldots, N\}.$$

**Definition 1** A coupling for $\{P_x : x \in X\}$ is a family $\{B_{x,y} : x, y \in X\}$ of probability measures on $X \times X$ such that for every $B \in \mathcal{B}_X$ the map $X^2 \ni (x, y) \mapsto B_{x,y}(B)$ is measurable and
for every \(x, y \in X\) and \(A \in B_X\).

In the following we assume that there exists a *subcoupling* for \(\{P_x : x \in X\}\), i.e. a family \(\{Q_{x,y} : x, y \in X\}\) of subprobability measures on \(X^2\) such that the map \((x, y) \mapsto Q_{x,y}(B)\) is measurable for every Borel \(B \subset X^2\) and

\[
Q_{x,y}(A \times X) \leq P_x(A) \quad \text{and} \quad Q_{x,y}(X \times A) \leq P_y(A)
\]

for every \(x, y \in X\) and Borel \(A \subset X\).

Measures \(\{Q_{x,y} : x, y \in X\}\) allow us to construct a coupling for \(\{P_x : x \in X\}\). Define on \(X^2\) the family of measures \(\{R_{x,y} : x, y \in X\}\) which on rectangles \(A \times B\) are given by

\[
R_{x,y}(A \times B) = \frac{1}{1 - Q_{x,y}(X^2)}(P_x(A) - Q_{x,y}(A \times X))(P_y(B) - Q_{x,y}(X \times B)),
\]

when \(Q_{x,y}(X^2) < 1\) and \(R_{x,y}(A \times B) = 0\) otherwise. A simple computation shows that the family \(\{B_{x,y} : x, y \in X\}\) of measures on \(X^2\) defined by

\[
B_{x,y} = Q_{x,y} + R_{x,y} \quad \text{for} \quad x, y \in X
\]

is a coupling for \(\{P_x : x \in X\}\).

The following theorem due to M. Śłęczka and R. Kapica (see [12]) will be used in the proof of Theorem 3 in Sect. 4.

**Theorem 1** Assume that a Markov operator \(P\) and transition probabilities \(\{Q_{x,y} : x, y \in X\}\) satisfy

\begin{itemize}
  \item[A0] \(P\) is a Feller operator, i.e. \(P(C_b(X)) \subset C_b(X)\).
  \item[A1] There exists a Lapunov function for \(P\), i.e. continuous function \(L : X \rightarrow [0, \infty)\) such that \(L\) is bounded on bounded sets, \(\lim_{x \to \infty} L(x) = +\infty\) and for some \(\lambda \in (0, 1), c > 0\)

\[
PL(x) \leq \lambda L(x) + c \quad \text{for} \quad x \in X.
\]
  \item[A2] There exist \(F \subset X^2\) and \(\alpha \in (0, 1)\) such that \(\text{supp } Q_{x,y} \subset F\) and

\[
\int_{X^2} d(u,v)Q_{x,y}(du, dv) \leq \alpha d(x, y) \quad \text{for} \quad (x, y) \in F.
\]
  \item[A3] There exist \(\delta > 0, l > 0\) and \(\nu \in (0, 1]\) such that

\[
1 - Q_{x,y}(X^2) \leq ld(x, y)\nu
\]

and

\[
Q_{x,y}((u,v) \in X^2 : d(u,v) < \alpha d(x, y)) \geq \delta
\]

for \((x, y) \in F\)
  \item[A4] There exist \(\beta \in (0, 1), \tilde{C} > 0\) and \(R > 0\) such that for

\[
\kappa((x_n, y_n)_{n \in \mathbb{N}_0}) = \inf\{n \in \mathbb{N}_0 : (x_n, y_n) \in F \quad \text{and} \quad L(x_n) + L(y_n) < R\}
\]

we have

\[
E_{x,y}\beta^{-\kappa} \leq \tilde{C} \quad \text{whenever} \quad L(x) + L(y) < \frac{4c}{1 - \lambda}.
\]
\end{itemize}
where $\mathbb{E}_{x,y}$ denotes here the expectation with respect to the chain starting from $(x,y)$ and with transition function $\{B_{x,y} : x, y \in X\}$. Then operator $P$ possesses a unique invariant measure $\mu_* \in M^1_L(X)$, which is attractive in $M_1(X)$. Moreover, there exist $q \in (0,1)$ and $C > 0$ such that

$$\|P^n \mu - \mu_*\|_{FM} \leq q^n C(1 + \int_X L(x) \mu(dx))$$

for $\mu \in M_1^+ (X)$ and $n \in \mathbb{N}$.

We will also need a version of the strong law of large numbers due to Shirikyan \[23, 24\]. It is originally formulated for Markov chains on a Hilbert space, however analysis of the proof shows that it remains true for Polish spaces.

**Theorem 2** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X$ be a Polish space. Suppose that for a family of Markov chains $((X^n_x)_{n \geq 0}, \mathbb{P}_x)_{x \in X}$ on $X$ with Markov operator $P : B_b(X) \to B_b(X)$ there exists a unique invariant measure $\mu_* \in M_1(X)$, a continuous function $v : X \to \mathbb{R}^+$ and a sequence $(\eta_n)_{n \in \mathbb{N}}$ of positive numbers such that $\eta_n \to 0$ as $n \to \infty$ and

$$\|P^n \delta_x - \mu_*\|_{FM} \leq \eta_n v(x) \quad \text{for} \quad x \in X.$$

If

$$C = \sum_{n=0}^{\infty} \eta_n < \infty$$

and there exists a continuous function $h : X \to \mathbb{R}_+$ such that

$$\mathbb{E}_x(v(X^n_x)) \leq h(x) \quad \text{for} \quad x \in X, n \geq 0,$$

where $\mathbb{E}_x$ is the expectation with respect to $\mathbb{P}_x$, then for any $x \in X$ and any bounded Lipschitz function $f : X \to \mathbb{R}$ we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^n) = \int_X f(y) \mu_*(dy)$$

$\mathbb{P}_x$ almost surely.

### 3 Random Dynamical Systems

Let $(Y, \varrho)$ be a Polish space, $\mathbb{R}_+ = [0, +\infty)$ and $I = \{1, \ldots, N\}$, $\Theta = \{1, \ldots, K\}$, where $N$ and $K$ are given positive integers.

We are given a family of continuous functions $q_\theta : Y \to Y$, $\theta \in \Theta$ and a finite sequence of semidynamical systems $T_i : \mathbb{R}_+ \times Y \to Y$, $i \in I$, i.e.

$$T_i(s + t, x) = T_i(s, (T_i(t, x))) \quad T_i(0, x) = x \quad \text{for} \quad s, t \in \mathbb{R}_+, \ i \in I \ \text{and} \ x \in Y,$$

the transformations $T_i : \mathbb{R}_+ \times Y \to Y$, $i \in I$ are continuous.

Let $p_i : Y \to [0, 1]$, $i \in I$, $\tilde{p}_\theta : Y \to [0, 1]$, $\theta \in \Theta$ be probability vectors and $[p_{ij}]_{i,j \in I}$, $i,j \in I$ be a matrix of probabilities. In the sequel we denote the system by $(T, q, p)$.  

\[ \text{Springer} \]
Finally, let \((\Omega, \Sigma, \mathbb{P})\) be a probability space and \(\{\tau_n\}_{n \geq 0}\) be an increasing sequence of random variables \(\tau_n : \Omega \rightarrow \mathbb{R}_+\) with \(\tau_0 = 0\) and such that the increments \(\Delta \tau_n = \tau_n - \tau_{n-1}\), \(n \in \mathbb{N}\), are independent and have the same density \(g(t) = \lambda e^{-\lambda t}, t \geq 0\).

The intuitive description of random dynamical system corresponding to the system \((T, q, p)\) is the following.

For an initial point \(x_0 \in Y\) we randomly select a transformation \(T_{i_0}\) from the set \(\{T_1, \ldots, T_N\}\) in such a way that the probability of choosing \(T_{i_0}\) is equal to \(p_{i_0}(x_0)\), and we define

\[
X(t) = T_{i_0}(t, x_0) \quad \text{for} \quad 0 \leq t < \tau_1.
\]

Next, at the random moment \(\tau_1\), at the point \(T_{i_0}(\tau_1, x_0)\) we choose a jump \(q_{\theta}\) from the set \(\{q_1, \ldots, q_K\}\) with probability \(\tilde{p}_{\theta}(T_{i_0}(\tau_1, x_0))\) and we define

\[
x_1 = q_{\theta}(T_{i_0}(\tau_1, x_0)).
\]

Finally, given \(x_n, n \geq 1\), we choose \(T_{i_n}\) in such a way that the probability of choosing \(T_{i_n}\) is equal to \(p_{i_{n-1}i_n}(x_n)\) and we define

\[
X(t) = T_{i_n}(t - \tau_n, x_n) \quad \text{for} \quad \tau_n < t < \tau_{n+1}.
\]

At the point \(T_{i_n}(\Delta \tau_{n+1}, x_n)\) we choose \(q_{\theta_n}\) with probability \(\tilde{p}_{\theta_n}(T_{i_n}(\Delta \tau_{n+1}, x_n))\). Then we define

\[
x_{n+1} = q_{\theta_n}(T_{i_n}(\Delta \tau_{n+1}, x_n)).
\]

We obtain a piecewise-deterministic trajectory for \(\{X(t)\}_{t \geq 0}\) with jump times \(\{\tau_1, \tau_2, \ldots\}\) and post jump locations \(\{x_1, x_2, \ldots\}\).

The above considerations may be reformulated as follows. Let \(\{\xi_n\}_{n \geq 0}\) and \(\{\gamma_n\}_{n \geq 1}\) be sequences of random variables, \(\xi_n : \Omega \rightarrow I\) and \(\gamma_n : \Omega \rightarrow \Theta\), such that

\[
\mathbb{P}(\xi_0 = i | x_0 = x) = p_i(x),
\]

\[
\mathbb{P}(\xi_n = k | x_n = x \quad \text{and} \quad \xi_{n-1} = i) = p_{ik}(x),
\]

\[
\mathbb{P}(\gamma_n = \theta | T_{\xi_{n-1}}(\Delta \tau_n, x_{n-1}) = y) = \tilde{p}_\theta(y). \tag{4}
\]

Assume that \(\{\xi_n\}_{n \geq 0}\) and \(\{\gamma_n\}_{n \geq 1}\) are independent of \(\{\tau_n\}_{n \geq 0}\) and that for every \(n \in \mathbb{N}\) the variables \(\gamma_1, \ldots, \gamma_{n-1}, \xi_1, \ldots, \xi_{n-1}\) are also independent.

Given an initial random variable \(\xi_0\) the sequence of the random variables \(\{x_n\}_{n \geq 0}\), \(x_n : \Omega \rightarrow Y\), is given by

\[
x_n = q_{\gamma_n}(T_{\xi_{n-1}}(\Delta \tau_n, x_{n-1})) \quad \text{for} \quad n = 1, 2, \ldots \tag{5}
\]

and the stochastic process \(\{X(t)\}_{t \geq 0}\), \(X(t) : \Omega \rightarrow Y\), is given by

\[
X(t) = T_{\xi_{n-1}}(t - \tau_{n-1}, x_{n-1}) \quad \text{for} \quad \tau_{n-1} \leq t < \tau_n, \quad n = 1, 2, \ldots \tag{6}
\]

It is easy to see that \(\{X(t)\}_{t \geq 0}\) and \(\{x_n\}_{n \geq 0}\) are not Markov processes. In order to use the theory of Markov operators we must redefine the processes \(\{X(t)\}_{t \geq 0}\) and \(\{x_n\}_{n \geq 0}\) in such a way that the redefined processes become Markov.

To this end, consider the space \(Y \times I\) endowed with the metric \(d\) given by

\[
d((x, i), (y, j)) = d_q(x, y) + d_q(i, j) \quad \text{for} \quad x, y \in Y, \quad i, j \in I, \tag{7}
\]

where \(d_q\) is the discrete metric in \(I\). Now define the process \(\{\xi(t)\}_{t \geq 0}, \xi(t) : \Omega \rightarrow I\), by

\[
\xi(t) = \xi_{n-1} \quad \text{for} \quad \tau_{n-1} \leq t < \tau_n, \quad n = 1, 2, \ldots
\]
Then the stochastic process \(\{(X(t), \xi(t))\}_{t \geq 0}, (X(t), \xi(t)) : \Omega \to Y \times I\) has the required Markov property.

We will study the Markov process (post jump locations) \(\{(x_n, \xi_n)\}_{n \geq 0}, (x_n, \xi_n) : \Omega \to Y \times I\).

Define the Markov operator \(P : B_b(Y \times I) \to B_b(Y \times I)\)

\[
P f(x, i) = \sum_{j \in I} \sum_{\theta \in \Theta} \int_0^{+\infty} \lambda e^{-\lambda t} f(q_\theta(T_j(t, x)), j) p_{ij}(x) \tilde{p}_\theta(T_j(t, x)) \, dt.
\]

(8)

Now consider the sequence of distributions

\[
\overline{\mu}_n(A) = \mathbb{P}\left((x_n, \xi_n) \in A\right) \quad \text{for} \quad A \in B(Y \times I), \ n \geq 0.
\]

It is easy to see that

\[
\overline{\mu}_{n+1} = P^* \overline{\mu}_n \quad \text{for} \quad n \geq 0,
\]

where \(P^* \mathcal{M}_1(Y \times I) \to \mathcal{M}_1(Y \times I)\) is the dual operator

\[
P^* \mu(A) = \sum_{j \in I} \sum_{\theta \in \Theta} \int_{Y \times I} \int_0^{+\infty} \lambda e^{-\lambda t} 1_A(q_\theta(T_j(t, x)), j) p_{ij}(x) \tilde{p}_\theta(T_j(t, x)) \, dt \, \mu(dx, di).
\]

(9)

In order to get the existence of an exponentially attractive invariant measure and the strong law of large numbers, we make the following assumptions on the system \((T, q, p)\).

The transformations \(T_i : \mathbb{R}_+ \times Y \to Y, i \in I\) and \(q_\theta : Y \to Y, \theta \in \Theta,\) are continuous and there exists \(x_* \in Y\) such that

\[
\int_{\mathbb{R}_+} e^{-\lambda t} q(q_\theta(T_j(t, x_*)), q_\theta(x_*)) \, dt < \infty \quad \text{for} \quad j \in I, \ \theta \in \Theta.
\]

(10)

For the system \((T, q, p)\) there are three constants \(L \geq 1, \alpha \in \mathbb{R}\) and \(L_q > 0\) such that

\[
\sum_{j \in I} p_{ij}(y) q(T_j(t, x), T_j(t, y)) \leq Le^\alpha t q(x, y) \quad \text{for} \quad x, y \in Y, \ i \in I, \ t \geq 0
\]

(11)

and

\[
\sum_{\theta \in \Theta} \tilde{p}_\theta(x) q(q_\theta(x), q_\theta(y)) \leq L_q q(x, y) \quad \text{for} \quad x, y \in Y.
\]

(12)

We also assume that the functions \(\tilde{p}_\theta, \theta \in \Theta,\) and \(p_{ij}, i, j \in I,\) satisfy the following conditions

\[
\sum_{j \in I} |p_{ij}(x) - p_{ij}(y)| \leq L p q(x, y) \quad \text{for} \quad x, y \in Y, \ i \in I,
\]

\[
\sum_{\theta \in \Theta} |\tilde{p}_\theta(x) - \tilde{p}_\theta(y)| \leq \tilde{L} \tilde{p} q(x, y) \quad \text{for} \quad x, y \in Y,
\]

(13)

where \(L p, \tilde{L} \tilde{p} > 0.\)

For \(x, y \in Y, \ t \geq 0\) we define

\[
I_T(t, x, y) = \{j \in I : q(T_j(t, x), T_j(t, y)) \leq Le^\alpha t q(x, y)\}
\]

\[
I_q(x, y) = \{\theta \in \Theta : q(q_\theta(x), q_\theta(y)) \leq L_q q(x, y)\}
\]

(14)

Assume that there are \(p_0 > 0, q_0 > 0\) such that for every \(i_1, i_2 \in I, x, y \in Y\) and \(t \geq 0\) we have
\[\sum_{j \in I_T(t,x,y)} p_{i_1,j}(x)p_{i_2,j}(y) > p_0,\]
\[\sum_{\theta \in I_q(x,y)} \tilde{p}_\theta(x) \tilde{p}_\theta(y) > q_0.\] (15)

Remark 1 The condition (15) is satisfied if there are \(i_0 \in I, \theta_0 \in \Theta\) such that
\[q(T_{i_0}(t,x), T_{i_0}(t,y)) \leq L \rho(x,y) \text{ for } x, y \in Y, t \geq 0,\]
\[q(q_{\theta_0}(x), q_{\theta_0}(y)) \leq L_q \rho(x,y) \text{ for } x, y \in Y,\] (16)
and
\[\inf_{i \in I} \inf_{x \in X} p_{ii_0}(x) > 0,\]
\[\inf_{x \in X} \tilde{p}_{\theta_0}(x) > 0.\] (17)

4 The Main Theorem

Theorem 3 Assume that system \((T, p, q)\) satisfies conditions (10)–(15). If
\[LL_q + \frac{\alpha}{\lambda} < 1.\] (18)
then
(i) there exists a unique invariant measure \(\mu_* \in \mathcal{M}^1_1(Y \times I)\) for the process \((x_n, \xi_n)_{n \geq 0}\), which is attractive in \(\mathcal{M}^1_1(Y \times I)\).
(ii) there exist \(q \in (0, 1)\) and \(C > 0\) such that for \(\mu \in \mathcal{M}^1_1(Y \times I)\) and \(n \in \mathbb{N}\)
\[\|P^n \mu - \mu_*\|_{FM} \leq q^n C \left(1 + \int_Y q(x, x_*) \mu(dx)\right),\]
where \(x_*\) is given by (10),
(iii) the strong law of large numbers holds for the process \((x_n, \xi_n)_{n \geq 0}\) starting from \((x_0, \xi_0) \in Y \times I\), i.e. for every bounded Lipschitz function \(f : Y \times I \to \mathbb{R}\) and every \(x_0 \in Y\) and \(\xi_0 \in I\) we have
\[\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x_k, \xi_k) = \int_{Y \times I} f(x, \xi) \mu_*(dx, d\xi)\]
\[\mathbb{P}_{x_0, \xi_0} \text{ almost surely.}\]

Proof of Theorem 3 We are going to verify assumptions of Theorem 1. Set \(F = X \times X\) and define
\[Q_{(x_1, i_1)(x_2, i_2)}(A) = \sum_{j \in I} \sum_{\theta \in \Theta} \int_0^{+\infty} \lambda e^{-\lambda t} \left[p_{i_1,j}(x_1) \tilde{p}_\theta(T_j(t, x_1)) \wedge p_{i_2,j}(x_2) \tilde{p}_\theta(T_j(t, x_2))\right] dt \times 1_A((q_{\theta}(T_j(t, x_1)), j), (q_{\theta}(T_j(t, x_2)), j)) dt\]
for \(A \subset X \times X\), where \(a \wedge b\) stands for the minimum of \(a\) and \(b\).
A0. The continuity of functions $p_{ij}, \tilde{p}_\theta, q_\theta$ implies that the operator $P$ defined in (8) is a Feller operator.

A1. Define $L(x, i) = \varrho(x, x_*)$ for $(x, i) \in X$. By (8) we have

$$PL(x, i) \leq \sum_{j \in I} \sum_{\theta \in \Theta} \int_0^{+\infty} \varrho(q_\theta(T_j(t, x)), q_\theta(T_j(t, x_*))) \lambda e^{-\lambda t} p_{ij}(x) \tilde{p}_\theta(T_j(t, x)) \, dt$$

$$+ \sum_{j \in I} \sum_{\theta \in \Theta} \int_0^{+\infty} \varrho(q_\theta(T_j(t, x_*)), q_\theta(x_*)) \lambda e^{-\lambda t} p_{ij}(x) \tilde{p}_\theta(T_j(t, x)) \, dt$$

$$+ \sum_{j \in I} \sum_{\theta \in \Theta} \int_0^{+\infty} \varrho(q_\theta(x_*), x_*) \lambda e^{-\lambda t} p_{ij}(x) \tilde{p}_\theta(T_j(t, x)) \, dt.$$

Further, using (10), (11) and (12) we obtain

$$PL(x, i) \leq aL(x, i) + b,$$

where

$$a = \frac{\lambda LL_q}{\lambda - \alpha},$$

$$b = \sum_{j \in I} \sum_{\theta \in \Theta} \int_0^{+\infty} \lambda e^{-\lambda t} \varrho(q_\theta(T_j(t, x_*)), q_\theta(x_*)) \, dt + \sum_{\theta \in \Theta} \varrho(q_\theta(x_*), x_*),$$

so $L$ is a Lapunov function for $P$.

A2. Observe that by (7), (11) and (12) we have for $(x_1, i_1), (x_2, i_2) \in X$

$$\int_{X^2} d(u, v) Q_{(x_1, i_1)(x_2, i_2)}(du, dv) =$$

$$\sum_{j \in I} \sum_{\theta \in \Theta} \int_0^{+\infty} \lambda e^{-\lambda t} \left\{ p_{i_1, j}(x_1) \tilde{p}_\theta(T_j(t, x_1)) \wedge p_{i_2, j}(x_2) \tilde{p}_\theta(T_j(t, x_2)) \right\} \times$$

$$\times \varrho(q_\theta(T_j(t, x_1)), q_\theta(T_j(t, x_2))) \, dt$$

$$\leq \sum_{j \in I} \sum_{\theta \in \Theta} \int_0^{+\infty} \lambda e^{-\lambda t} p_{i_1, j}(x_1) \tilde{p}_\theta(T_j(t, x_1)) \varrho(q_\theta(T_j(t, x_1)), q_\theta(T_j(t, x_2))) \, dt$$

$$\leq \beta \varrho(x_1, x_2) \leq \beta d((x_1, i_1), (x_2, i_2))$$

with $\beta = \frac{\lambda LL_q}{\lambda - \alpha} < 1$ by (18).

A3. From (13) and (11) it follows that

$$1 - \sum_{j \in I} \sum_{\theta \in \Theta} \left\{ p_{i_1, j}(x_1) \tilde{p}_\theta(T_j(t, x_1)) \wedge p_{i_2, j}(x_2) \tilde{p}_\theta(T_j(t, x_2)) \right\}$$

$$\leq \sum_{j \in I} \sum_{\theta \in \Theta} \left| p_{i_1, j}(x_1) \tilde{p}_\theta(T_j(t, x_1)) - p_{i_2, j}(x_2) \tilde{p}_\theta(T_j(t, x_2)) \right|$$

$$\leq \sum_{j \in I} \sum_{\theta \in \Theta} p_{i_1, j}(x_1) | \tilde{p}_\theta(T_j(t, x_1)) - \tilde{p}_\theta(T_j(t, x_2)) |$$

$$+ \sum_{j \in I} \sum_{\theta \in \Theta} \tilde{p}_\theta(T_j(t, x_2)) | p_{i_1, j}(x_1) - p_{i_2, j}(x_2) |$$

$$\leq LL_{\tilde{p}} e^{\alpha t} \varrho(x_1, x_2) + L_p \varrho(x_1, x_2) + 2N q_d(i_1, i_2)$$
and consequently
\[
1 - Q(x_1,i_1)(x_2,i_2)(X^2) \leq \left( L_p + \frac{\lambda LL\beta}{\lambda - \alpha} \right)\varphi(x_1,x_2) + 2N\varphi_d(i_1,i_2).
\]

Fix \( x_1, x_2 \in Y \) and \( i_1, i_2 \in I \). Define \( B = \{(u, (v, j)): \varphi(v, u) < \beta \varphi(x_1, x_2), j \in I\} \). If \( \alpha \geq 0 \) then there exists \( t_0 > 0 \) such that \( LL_q e^{\alpha t} < \beta \) for \( t < t_0 \). Set \( A = (0, t_0) \). If \( \alpha < 0 \) then there exists \( t_0 > 0 \) such that \( LL_q e^{\alpha t} < \beta \) for \( t > t_0 \). Set \( A = (t_0, \infty) \). In both cases define \( r = \int_A \lambda e^{-\lambda t} \, dt \). For all \( x, y \in Y, t \in A, j \in I_r(t, x, y) \) and \( \theta \in I_q(T_j(t, x), T_j(t, y)) \) we have
\[
\left((q_\theta(T_j(t, x)), j), (q_\theta(T_j(t, y), j)) \right) \in B.
\]

From (15) and (21) we obtain
\[
Q(x_1,i_1)(x_2,i_2)(B) \geq \int_A \lambda e^{-\lambda t} \sum_{j \in I_r(t, x_1, x_2)} \sum_{\theta \in I_q(T_j(t, x), T_j(t, y))} \left\{ p_{i_1j}(x_1)\tilde{\varphi}(T_j(t, x_1)) \right\} \wedge p_{i_2j}(x_2)\tilde{\varphi}(T_j(t, x_2)) \times 1_B((q_\theta(T_j(t, x_1)), j), (q_\theta(T_j(t, x_2), j))) \, dt
\]
\[
= \int_A \lambda e^{-\lambda t} \sum_{j \in I_r(t, x_1, x_2)} \sum_{\theta \in I_q(T_j(t, x), T_j(t, y))} \left\{ p_{i_1j}(x_1)\tilde{\varphi}(T_j(t, x_1)) \right\} \wedge p_{i_2j}(x_2)\tilde{\varphi}(T_j(t, x_2)) \, dt
\]
\[
\geq \int_A \lambda e^{-\lambda t} \sum_{j \in I_r(t, x_1, x_2)} \sum_{\theta \in I_q(T_j(t, x), T_j(t, y))} \left\{ p_{i_1j}(x_1)p_{i_2j}(x_2)\tilde{\varphi}(T_j(t, x_1)) \right\} \times \tilde{\varphi}(T_j(t, x_2)) \, dt
\]
\[
> p_0q_0r > 0,
\]
so A3 is satisfied. Since \( F = X \times X \), assumption A4 is trivially satisfied. From Theorem 1 we obtain (i) and (ii). Set \( v(x, i) = C(\varphi(x, x_n^i) + 1) \) and \( h(x, i) = C(\varphi(x, x_n^i) + 1 + \frac{b}{1-n}) \) for \( x \in X, i \in I, \) with \( a, b \) as in (20). Iterating (19) we obtain
\[
\mathbb{E}_{x_0, \xi_0}(v(x_0, \xi_n^i)) \leq h(x_0, \xi_0) \quad \text{for} \quad x_0 \in X, \xi_0 \in I.
\]

Application of Theorem 2 ends the proof.

The next result describing the asymptotic behavior of the process \( (x_n)_{n \geq 0} \) on \( Y \) is an obvious consequence of Theorem 3. Let \( \tilde{\mu}_0 \) be the distribution of the initial random vector \( x_0 \) and \( \tilde{\mu}_n \) the distribution of \( x_n \), i.e.
\[
\tilde{\mu}_n(A) = \mathbb{P}(x_n \in A) \quad \text{for} \quad A \in \mathcal{B}_Y, n \geq 1.
\]

\[\square\]

**Theorem 4** Under the hypotheses of Theorem 3 the following statements hold:

(i) there exists a measure \( \tilde{\mu}_\ast \in M_1(Y) \) such that for any \( \tilde{\mu}_0 \) the sequence \( (\tilde{\mu}_n)_{n \geq 0} \) converges weakly to \( \tilde{\mu}_\ast \). Moreover, if
\[
\mathbb{P}(x_0 \in A) = \tilde{\mu}_\ast(A) \quad \text{for} \quad A \in \mathcal{B}_Y
\]
then \( \tilde{\mu}_n(A) = \tilde{\mu}_\ast(A) \) for \( A \in \mathcal{B}_Y \) and \( n \geq 1 \).
(ii) there exist \( q \in (0, 1) \) and \( C > 0 \) such that
\[
||\tilde{\mu}_n - \tilde{\mu}_*||_{FM} \leq q^n C(1 + \int_y q(x, x_*) \tilde{\mu}_0(dx))
\]
for any initial distribution \( \tilde{\mu}_0 \in \mathcal{M}_1^1(Y) \) and \( n \geq 1 \).

(iii) for any starting point \( x_0 \in Y \), \( \xi_0 \in I \) and any bounded Lipschitz function \( f \) on \( Y \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x_k) = \int_Y f(x) \tilde{\mu}_*(dx)
\]

\( \mathbb{P}_{x_0, \xi_0} \) almost surely.

The examples below show that our model generalizes some important and widely studied objects, namely dynamical systems generated by iterated function systems \([1,11,13,19]\) and Poisson driven stochastic differential equations \([7,18,25,27]\).

**Example 1** Iterated Function Systems.

Let \( (Y, \| \cdot \|) \) be a separable Banach space. An iterated function system (IFS) consists of a sequence of continuous transformations

\[
q_\theta : Y \to Y, \quad \theta = 1, \ldots, K
\]

and a probability vector

\[
\tilde{p}_\theta : Y \to [0, 1], \quad \theta = 1, \ldots, K.
\]

Such a system is briefly denoted by \((q, \tilde{p})_K = (q_1, \ldots, q_K, \tilde{p}_1, \ldots, \tilde{p}_K)\). The action of an IFS can be roughly described as follows. We choose an initial point \( x_0 \) and we randomly select from the set \( \Theta = \{1, \ldots, K\} \) an integer \( \theta_0 \) in such a way that the probability of choosing \( \theta_0 \) is \( \tilde{p}_{\theta_0}(x_0) \). If a number \( \theta_0 \) is drawn, we define \( x_1 = q_{\theta_0}(x_0) \). Having \( x_1 \) we select \( \theta_1 \) in such a way that the probability of choosing \( \theta_1 \) is \( \tilde{p}_{\theta_1}(x_1) \). Now we define \( x_2 = q_{\theta_1}(x_1) \) and so on.

An IFS is a particular example of a random dynamical system with randomly chosen jumps. Consider a dynamical system of the form \( I = \{1\} \) and \( T_1(t, x) = x \) for \( x \in Y \), \( t \in \mathbb{R}_+ \). Moreover assume that \( p_1(x) = 1 \) and \( p_{11}(x) = 1 \) for \( x \in Y \). Then we obtain an IFS \((q, \tilde{p})_K\).

Denoting by \( \tilde{\mu}_n, n \in \mathbb{N} \), the distribution of \( x_n \), i.e., \( \tilde{\mu}_n(A) = \mathbb{P}(x_n \in A) \) for \( A \in \mathcal{B}(Y) \), we define \( \tilde{P}_* \) as the transition operator such that \( \tilde{\mu}_{n+1} = \tilde{P}_* \tilde{\mu}_n \) for \( n \in \mathbb{N} \). The transition operator corresponding to iterated function system \((q, \tilde{p})_K\) is given by

\[
\tilde{P}_* \mu(A) = \sum_{\theta \in \Theta} \int_Y 1_A(q_\theta(x)) \tilde{p}_\theta(x) \mu(dx) \quad \text{for} \quad A \in \mathcal{B}(Y), \quad \mu \in \mathcal{M}_1(Y).
\]

We assume (13) and (15) and take \( \alpha = 0 \) in (11). If

\[
\sum_{\theta \in \Theta} \tilde{p}_\theta(x)q(x, x_*) \leq L_q q(x, y) \quad \text{for} \quad x, y \in Y.
\]

with \( L_q < 1 \) then from Theorem 4 we obtain existence of an invariant measure \( \mu_* \in \mathcal{M}_1(Y \times I) \) for the process \((x_n, \xi_n)_{n \geq 0} \), which is attractive in \( \mathcal{M}_1(Y \times I) \), exponentially attractive in \( \mathcal{M}_1^1(Y \times I) \) and for which the strong law of large numbers holds (cf. [26]).
Remark 2 The convergence in Theorem 4 is weak and one cannot expect the strong one, i.e. in the total variation norm. Indeed, let \( Y = \mathbb{R} \), \( q_1(x) = x \) and \( q_2(x) = 0 \) for \( x \in \mathbb{R} \). For every probabilistic vector \( (p_1, p_2) \) with \( p_1 < 1 \) condition \( L_q < 1 \) from the above Example is satisfied. Thus for every \( \mu \in \mathcal{M}_1(Y) \) the sequence \( \{\tilde{P}^n\mu\}_{n \geq 1} \) given by (22) converges weakly to \( \mu_0 = \delta_0 \). Obviously, the strong convergence does not hold.

Example 2 Poisson driven stochastic differential equation.

Consider a stochastic differential equation
\[
dX(t) = a(X(t), \xi(t))dt + b(X(t))dp(t) \quad \text{for} \quad t \geq 0
\]
with the initial condition
\[
X(0) = x_0,
\]
where \( a : Y \times I \to Y \), \( b : Y \to Y \) are Lipschitz functions, \( (Y, \| \cdot \|) \) is a separable Banach space, \( \{p(t)\}_{t \geq 1} \) is a Poisson process and \( \{\xi(t)\}_{t \geq 0} \), \( \xi(t) : \Omega \to I \) is a stochastic process describing random switching at random moments \( \tau_n \). Consider a sequence of random variables \( \{x_n\}_{n \geq 0} \), \( x_n : \Omega \to Y \) such that
\[
x_n = q(T_{\xi(\tau_{n-1})}(\tau_n - \tau_{n-1}, x_{n-1})), \quad q(x) = x + b(x)
\]
\[
P\{\xi(0) = k|x_0 = x\} = p_k(x),
\]
\[
P\{\xi(\tau_n) = s|x_n = y, \xi(\tau_{n-1}) = i\} = p_{is}(y), \quad \text{for} \quad n = 1, \ldots
\]
and
\[
\xi(t) = \xi(\tau_{n-1}) \quad \text{for} \quad \tau_{n-1} \leq t < \tau_n, \quad n = 1, 2, \ldots
\]

This is a particular example of continuous random dynamical systems where \( q_\theta(x) = q(x), \theta \in \{1, \ldots, K\} \), and for every \( i \in I \), \( T_i(t, x) = v_i(t) \) are the solutions of the unperturbed Cauchy problems
\[
v'_i(t) = a(v_i(t), i) \quad \text{and} \quad v_i(0) = x, \quad x \in Y.
\]

It is easy to check that \( \mu_\mu = P^{*n}\mu \), where \( P^{*} \) is the transition operator corresponding to the above stochastic equation* and given by
\[
P^{*}\mu(A) = \sum_{j \in I} \int_{Y \times I} \int_{\mathbb{R}_+} \lambda e^{-\lambda t} 1_A(q(T_j(t, x)), j) p_{ij}(x) dt d\mu(x, i)
\]
for \( A \in \mathcal{B}(Y \times I) \) and \( \mu \in \mathcal{M}_1 \).

Assume that there exist positive constants \( L_q, L \) and \( \alpha \) such that
\[
\|q(x) - q(y)\| \leq L_q \|x - y\|
\]
and
\[
\|T_{i_0}(t, x) - T_{i_0}(t, y)\| \leq Le^{\alpha t} \|x - y\|
\]
for some \( i_0 \in I \) such that \( \inf_{i \in I} \inf_{x \in Y} p_{i_0i}(x) > 0 \) and \( x, y \in Y, t \geq 0 \). If
\[
LL_q + \frac{\alpha}{\lambda} < 1.
\]
then there exists a unique invariant measure \( \mu_\star \in \mathcal{M}_1^1(Y \times I) \) for the process \( \{x_n, \xi_n\}_{n \geq 0} \), which is attractive in \( \mathcal{M}_1^1(Y \times I) \), exponentially attractive in \( \mathcal{M}_1^1(Y \times I) \) and the strong law.
of large numbers holds for the process \((x_n, \xi_n)_{n \geq 0}\) starting from \((x_0, \xi_0) \in Y \times I\), i.e. for every bounded Lipschitz function \(f : Y \times I \to \mathbb{R}\) and every \(x_0 \in Y\) and \(\xi_0 \in I\) we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x_k, \xi_k) = \int_{Y \times I} f(x, \xi) \mu_\ast (dx, d\xi)
\]

\(\mathbb{P}_{x_0, \xi_0}\) almost surely.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Barnsley, M.F., Demko, S.G., Elton, J.H., Geronimo, J.S.: Invariant measures arising from iterated function systems with place dependent probabilities. Ann. Inst. H. Poincaré 24, 367–394 (1988)
2. Bobrowski, A.: Degenerate convergence of semigroups related to a model of eukaryotic gene expression. Semigroup Forum 73, 343–366 (2006)
3. Frisch, K.U.: Wave propagation in random media, stability. In: Bharucha-Reid, A.T. (ed.) Probabilistic Methods in Applied Mathematics. Academic Press, New York (1986)
4. Griego, R.J., Hersh, R.: Random evolutions, Markov chains and systems of partial differential equations. Proc. Nat. Acad. Sci. USA 62, 305–308 (1969)
5. Hille, S., Horbacz, K., Szarek, T.: Unique steady-state molecular distribution for a regulatory network with random bursting. Submitted for publication
6. Horbacz, K.: Random dynamical systems with jumps. J. Appl. Probab. 41, 890–910 (2004)
7. Horbacz, K.: Asymptotic stability of a semigroup generated by randomly connected Poisson driven differential equations. Boll. Unione Mat. Ital. 9, 545–566 (2006)
8. Horbacz, K.: Invariant measures for random dynamical systems. Dissertationes Math. 451 (2008)
9. Horbacz, K.: Continuous random dynamical systems. J. Math. Anal. Appl. 408, 623–637 (2013)
10. Horbacz, K., Myjak, J., Szarek, T.: On stability of some general random dynamical system. J. Statist. Phys. 119, 35–60 (2005)
11. Iosifescu, M., Theodorescu, R.: Random Processes and Learning. Springer, New York (1969)
12. Kapica, R., Ślęczka, M.: Random iteration with place dependent probabilities. arXiv:1107.0707
13. Karlin, S.: Some random walks arising in learning models. Pacific J. Math. 3, 725–756 (1953)
14. Keller, J.B.: Stochastic equations and wave propagation in random media. Proc. Sympos. Appl. Math. 16, 1456–1470 (1964)
15. Komorowski, T., Peszat, S., Szarek, T.: On ergodicity of some Markov processes. Ann. Probab. 38(4), 1401–1443 (2010)
16. Kudo, T., Ohba, I.: Derivation of relativistic wave equation from the Poisson process. Pramana 59, 348–367 (2006)
17. Lasota, A., Mackey, M.C.: Cell division and the stability of cellular population. J. Math. Biol. 38, 241–261 (1999)
18. Lasota, A., Traple, J.: Invariant measures related with Poisson driven stochastic differential equation. Stochast. Process. Appl. 106(1), 81–93 (2003)
19. Lasota, A., Yorke, J.A.: Lower bound technique for Markov operators and iterated function systems. Random Comput. Dynam. 2, 41–77 (1994)
20. Lipniacki, T., Paszek, P., Marciniak-Czochra, A., Brasier, A.R., Kimel, M.: Transcriptional stochasticity in gene expression. J. Theor. Biol. 238, 348–367 (2006)
21. Meyn, S.P., Tweedie, R.L.: Computable bounds for geometric convergence rates of Markov chains. Ann. Appl. Probab. 4, 981–1011 (1994)
22. Pinsky, M.A.: Lectures on Random Evolution. World Scientific, Singapore (1991)
23. Shirikyan, A.: A version of the law of large numbers and applications. In: Probabilistic Methods in Fluids. Proceedings of the Swansea Workshop held on 14–19 April 2002, World Scientific, New Jersey, pp. 263–271 (2003)
24. Shirikyan, A.: Law of large numbers and central limit theorem for randomly forced PDE’s. Probab. Theory Relat. Fields 134(2), 215–247 (2006)
25. Szarek, T., Wedrychowicz, S.: Markov semigroups generated by Poisson driven differential equation. Nonlinear Anal. 50, 41–54 (2002)
26. Ślęczka, M.: The rate of convergence for iterated functions systems. Studia Math. 205(3), 201–214 (2011)
27. Traple, J.: Markov semigroup generated by Poisson driven differential equations. Bull. Pol. Acad. Sci. Math. 44, 161–182 (1996)
28. Tyrcha, J.: Asymptotic stability in a generalized probabilistic/deterministic model of the cell cycle. J. Math. Biol. 26, 465–475 (1988)
29. Tyson, J.J., Hamsgen, K.B.: Cell growth and division: a deterministic /probabilistic model of the cell cycle. J. Math. Biol. 23, 231–246 (1986)
30. Walczuk, A.: Central limit theorem for an additive functional of a Markov process, stable in the Wesserstein metric. Ann. Univ. Mariae Curie-Skłodowska Sect. A 62, 149–159 (2008)
31. Werner, I.: Contractive Markov systems. J. London Math. Soc. 2(71), 236–258 (2005)