Quantum Guiding Equation Can Determine Classical Behavior of Earth for Large Quantum Numbers

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For quantum systems, we expect to see classical behavior at the limit of large quantum numbers. Hence, we apply Bohmian approach to describe the Earth evolution around the Sun. We obtain possible trajectories of the Earth system with different initial conditions which converge to a definite stable orbit after a given time, known as the Kepler orbit. The trajectories have resulted from the guiding equation \( p = \nabla S \) in Bohmian mechanics which relates the momentum of the system to the phase part of the wave function. Except for some special situations, Bohmian trajectories are not Newtonian in character. We show that the classical behavior of the Earth can be described as the consequence of the guiding equation at the limit of large quantum numbers.

I. INTRODUCTION

Quantum mechanics works exceedingly well in all practical applications. No examples of conflict between its predictions and experiments are known. The main problem arises when we emphasize on macroscopic systems which behave classically. The transition of quantum to classical mechanics is an unsolved fundamental problem for years. Most physicists believe that macroscopic systems are quantum mechanical in nature. Therefore, the way they behave classically is not clear yet. In the most general sense, the corresponding principle determines the guideline of scientific theories. However, it requires that new theories not only successfully make predictions that previous theories were unable to make, but that they also yield the correct predictions of the previous theories \[1\]. In this issue, Bohmian mechanics which comes up with trajectories for quantum systems can be an appropriate option for a promotion of other theories describing the micro-world.

Bohmian mechanics is a deterministic and distinctly non-Newtonian reformulation of quantum mechanics which the wave function itself is responsible for guiding the motion of the particles \[2\]. However, despite indiscernibility in results with the standard quantum mechanics, it differs in explanations \[3\]. Through self-experiences and the classical description of nature, we comprehend macro-systems with trajectories. Therefore, there is no way to understand the quantum-classical transition, till we reach an explanation for classical trajectories based on the quantum description. Since Bohmian mechanics keeps the quantum results based on a casual description, it can potentially provide a proper connection between the quantum-classical domain. It should be noted that the trajectories which Bohmian mechanics suggests is completely different from the Newtonian trajectories, so that the theory has its exclusive description \[4\]. Interestingly, in this regard, periodic orbits have been, since Kepler, considered as the key concept for describing and understanding classical dynamics which, since Bohr, could be gainful in apprehending the quantum-classical transition \[5\].

The problem of describing Sun-Earth dynamics has been always important in physics \[6\]. The first illustration of Earth’s orbit was computed by Lagrange (1781,1782), and improved by Pontecoulant (1834), Agassiz (1840) and others. Since then, many works have been done to modify the theory and ameliorate the results \[7\]. In recent times, plenty of quantum researches has been engaged to put light on the problem. As an instance, Flöthmann and Welge showed the classical dynamics of electron motion of Hydrogen atom under the gravitational field of the Sun-Earth problem \[5\]. Also, Battista and Esposito studied the effect of quantum corrections to the Newtonian potential for the evaluation of equilibrium points \[5\].

The real issue still has been remained unsolved. However, the dynamics of the famous classical Earth-Sun problem has unanswered questions. Meanwhile, the quantum aspects of the problem has been considered unsolved in general \[9,12\]. Nevertheless, both quantum and classical aspects of the model appear in all subfields of physics \[13,22\]. Regarding the classical Earth-Sun system, many efforts have been made to explain its dynamics in a quantum fashion. David Keeports considered Earth as a quantum object and discussed its quantum properties compared to classical ones \[11\]. Also, studies on quantum correction terms for classical potentials have been reported \[23,25\].

In this paper, we are going to present the quantum aspects responsible for the classical behavior of the Earth. In section II, we present non-relativistic Hydrogen-like
Hamiltonian for the Sun-Earth system to show that the Earth classical energy is dependent upon the principle and the angular momentum quantum numbers \( n \) and \( l \), respectively. In section III, we discuss the role of the Bohmian mechanics and its guiding equation in the appearance of the Earth dynamics. Then, in the next section, we show as one of our main results, that how the large values of the magnetic quantum number \( m \) affect the possible trajectories of the Earth via the Bohmian guiding equation. The wave function is discussed in section V. Finally, at the last section VI, we discuss the results of our work.

II. DEFINITION OF EARTH HAMILTONIAN

Earth is mostly described as a classical object. But there is a belief that classical behavior of macroscopic systems, even Earth in the solar gravitational field, is based on their quantum mechanical nature. Therefore, different forms of Hamiltonians for the description of the Earth dynamics have been proposed in literature \cite{1, 6, 12}. Here, we introduce a Hamiltonian which contains two terms. First, the Hydrogen-like Hamiltonian \( \mathcal{H}_0 \) containing the kinetic and gravitational potential energies as well as a kinetic-type term denoted by \( K \),

\[
H = \mathcal{H}_0 + K
\]

\[
\mathcal{H}_0 = -\frac{\hbar^2}{2m_e} \nabla^2 - \frac{Gm_s m_e}{r},
\]

where \( G \) is the universal gravitational constant, \( m_s \) and \( m_e \) are the Sun and the Earth masses respectively.

The second part \( K \) includes additional terms which should be added to correct the formal kinetic energy \( \mathcal{K}_0 = \langle -\hbar^2 \nabla^2 / 2m_e \rangle \) of the Earth system. In fact, \( \mathcal{H}_0 \) does not give us a real statement of Earth’s energy; therefor we suggest common additional \( r^{-2} \) and \( r^{-1} \) terms as a correction.

Regarding the relation \cite{1}, a similar situation is seen in relativistic energy correction of Hydrogen atom in which the kinetic energy is extended to include other terms due to the high velocity of electron in the inner shell of the atom. This correction contain three different terms with radius orders of \( r^{-2} \), \( r^{-1} \) and \( r^0 \) (a constant term). The difference, however, is that here \( K \) is not supposed to have small effect compared to \( \mathcal{K}_0 \). It should be considered as a real part of the Earth Hamiltonian \( \mathcal{H} \), without which a proper description of the Earth dynamics is not possible, albeit in a classical-like (Bohmian) approach.

Following the Bohmian approach in the description of the problem \cite{29}, we consider \( \psi(x,t) = Re^{-iS/t} \) as an eigenfunction of the Hamiltonian. Thus, by dividing the real and the imaginary parts of the Schrödinger equation, one reaches the following equations

\[
\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m_e} - \frac{\hbar^2}{2m_e} \nabla^2 \frac{R}{r} - \frac{Gm_s m_e}{r} + K = 0
\]

\[
\frac{\partial R^2}{\partial t} + \nabla \cdot \left( \frac{R^2 \nabla S}{m_e} \right) = 0.
\]

The relation \cite{3} is the well-known quantum Hamilton-Jacobi equation. Keepsorts pointed out that how the Hydrogen-like energy of the Earth is related to its classical one \cite{1}. He showed if the Earth-Sun Hamiltonian is assumed similar to the Hydrogen atom, regardless of the additional term \( K \), the energy levels could be calculated as \( E_n = -Gm_s^2 m_e^2 / 2h^2 n^2 \). On the other hand, by considering the continuous limit of the Earth classical energy, he obtained \( E = -Gm_s^2 m_e^2 / 2h^2 (l+1) \), where \( n \) and \( l \) are the principal and the azimuthal quantum numbers. As is clear, he suggested that we should have \( n = l \rightarrow \infty \). Also, Keepsorts showed that the Earth wave function is the same as the Hydrogen atom when the quantum numbers \( n \) and \( l \) tend to large values.

The Hydrogen-like Hamiltonian \( \mathcal{H}_0 \), explains the classical energy of Earth very well, as we mentioned above. So, it is reasonable to assume that the energy remains unchanged in this new framework. This point is crucial in analyzing the role of different terms in the Hamilton-Jacobi equation \cite{6}. In this regard, if we designate the amplitude of the wave function \( R \) similar to what is assumed for a Hydrogen-like system, the following expression in spherical coordinates should be established

\[
\frac{(\nabla S)^2}{2m_e} + K = \frac{m^2 \hbar^2}{2m_e e^2 \sin^2 \theta},
\]

where \( m \) denotes the so-called magnetic quantum number. The equation \cite{3} is obtained assuming that \( (\nabla S)^2 / 2m_e + K \) is similar in its form to \( (\nabla S)^2 / 2m_e \) for the Hydrogen-like system. Yet, the difference is that both \( r \) and \( \theta \) are supposed to be time-dependent. Thereupon our first assumption here is that the energy and the amplitude of the Earth wave function are in a form similar to the same system with Hydrogen-like Hamiltonian. For (Hydrogen-like Earth-Sun system see \cite{1}). This assumption ensures us that we can obtain the proper dynamics along with the classical energy of the Earth as reported before \cite{1}.

Hence, the study of the Earth in Bohmian approach should be in accordance with quantum mechanical results, reaching the energy relation in the same way. For the Hydrogen atom, the gradient of the phase function is \( (\nabla S)^2 / 2m_e = m^2 \hbar^2 / 2m_e e^2 \sin^2 \theta \). So, the equation \cite{3} should have the same form as in the Hydrogen atom. This condition, however, leads to a new phase function and also provides a different dynamics afterward.

III. BOHMIAN DYNAMICS OF THE EARTH

Keeping the energy unchanged as discussed in the previous section, according to the equation \cite{3}, \( K \) plays a
As a suitable choice, we also define the constant momentum, we obtain 

\[ K = \frac{Am_e}{2(r^2 - Z_h^2)} - \mu m_e (\frac{1}{r} - \frac{1}{2a}), \] (6)

where \( a \) is the semi-major axis of Earth and \( \mu \) is defined as \( \mu = G(m_e + m_s) \approx Gm_s \), \( A \) and \( Z_h \) are two constants. Then in (5) we have

\[ (\nabla S)^2 = \frac{m^2 \hbar^2}{2m_e r^2 \sin^2 \theta} - \frac{Am_e}{2(r^2 - Z_h^2)} + \frac{m_e \mu}{r} - \frac{m_e \mu}{2a}. \] (7)

The constant \( Z_h \) somehow represents the \( z \)-axis of the coordinate system. It has been placed in the \( r^{-2} \) term to keep the trajectories on the surface. Let us consider the \( xy \)-plane parallel to the plane of Earth’s orbit. The \( z \)-axis is then perpendicular to this plane. So, the \( z \)-coordinate of the Earth position must be constant, i.e.,

\[ r \cos \theta = Z_h, \] (8)

where one can write

\[ r^2 \sin^2 \theta = r^2 - Z_h^2. \] (9)

As a suitable choice, we also define the constant \( A \) in a way that the first two terms in R.H.S of (7) cancel out each other. So, we adopt

\[ A = \frac{m^2 \hbar^2}{m_e^2}. \] (10)

Consequently, in (7) one can show that the Bohmian guiding equation is equal to

\[ (\nabla S)^2 = m_e^2 (\frac{2 \mu}{r} - \frac{\mu}{a}). \] (11)

Since the guiding equation \( \nabla S = p \), where \( p \) is the linear momentum, we obtain

\[ v^2 = \frac{2 \mu}{r} - \frac{\mu}{a} \] (12)

which is the well-known vis viva relation in Newtonian mechanics. Regarding the equation (12), velocity only depends on the Sun and Earth distance \( r \).

In Bohmian mechanics the dynamics of the wave function is determined by the Schrödinger equation and the dynamics of the system is determined by the guiding equation. There is a wrong general belief that the Bohmian mechanics become Newtonian for the macroscopic systems or the systems with large masses. This misleading concept happens when we ignore the guiding equation \( \nabla S = p \). Dürr and Teufel show that for many systems the Bohmian trajectories obtained from the guiding equation are different from the Newtonian ones [3]. This depends on the circumstances in which the guiding equation is at work. We will show that for the Sun-Earth system, the guiding equation leads to the trajectories which are classical for the large values of quantum numbers.

IV. EARTH TRAJECTORIES

The electron in Hydrogen atom has a circular trajectory in Bohmian mechanics with a constant radius in the \( xy \)-plane. Azimuthal angle \( \phi(t) \) is the only parameter that changes with time, which has linear time-dependency [3] (pp. 148-153). The similarity of the Hydrogen system with the Earth one is a good reason to assume a similar relation for \( \phi(t) \) in the Earth-Sun case. So, our second assumption in this problem is considering the time-dependency of the azimuthal angle \( \phi(t) \) similar to the Hydrogen atom relation in Bohmian mechanics, albeit in its form. However, in this case, the radius \( r \) and the polar angle \( \theta \) are supposed to be time-dependent, contrary to the case of the Hydrogen atom. So we assume that

\[ \phi(t) = \frac{m \hbar t}{m_e r^2 (t) \sin^2 \theta(t)} + \phi_0. \] (13)

The important consequence of this choice is the direct effect of the quantum number \( m \) on the dynamics of the system. This is the main accomplishment of Bohmian approach which appears in this problem. Yet, our second assumption is not a surprise but is completely understandable. It is clear that the idea comes from the circular movement of the Earth around the Sun, and the equation (13) is the simplest way to describe this movement with respect to the theory.

Now, for the angular velocity \( \dot{\phi} \), we have

\[ \dot{\phi} = r \sin \theta \dot{\phi} = \frac{m \hbar t}{m_e r^2 \sin^2 \theta} - \frac{2m \hbar t}{m_e r^2 \sin^2 \theta} \] (14)

where \( v^2 = v_r^2 + v_\theta^2 + v_\phi^2 \). Also using the equation (9), one gets

\[ v_\theta = \dot{\theta} = Z_h \frac{\dot{r}}{r \sin \theta}. \] (15)

Using (14), (15) and (12), the time-dependent equation for \( r \) is obtained as

\[ (1 + \frac{4A r^2}{r^3 \sin^2 \theta} + \frac{Z_h^2}{r^2 \sin^2 \theta} + \frac{4A Z_h^2 r^2}{r^3 \sin^2 \theta} + \frac{8A Z_h^2}{r^3 \sin^2 \theta} r^2) \]

\[ - (\frac{4A t}{r^3 \sin^2 \theta} + \frac{4A Z_h^2 t}{r^3 \sin^2 \theta} \dot{r}) + A \frac{2 \mu}{r^2 \sin^2 \theta} - \frac{\mu}{a} = 0. \] (16)

Here, we need a solution for the radial velocity \( \dot{r} \) to show the trajectories. To do this, we should know the magnitude of each term to make some approximations.

The order of magnitude of the constant \( A \) depends on the Earth mass, Planck constant, and the magnetic quantum number \( m \). Therefore, its order must be determined due to the Earth dynamics. As we know the Earth revolves around the Sun in a year. So, for the azimuthal
angle $\phi(t)$, we have $\phi(\tau + 1\text{year}) = \phi(\tau) + 2\pi$. The radial time-dependency of the Earth motion is negligible and its value remains nearly unchanged. Then, according to the definition of $\phi(t)$ in [13] we can estimate the magnitude of the magnetic quantum number $m \approx 10^7\pi$. As we expect, the Earth dynamics appears in the large values of quantum numbers $l$ and $m$.

Thus, from [10] we have $A \approx 10^{31}m^4s^{-2}$ and due to the equation [5], the constant $Z_h$ should be in the order of the Earth-Sun distance. Now, let us assume that

$$\dot{r} = \xi \sin \theta \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}},$$

(17)

where $\xi$ is a dimensionless constant which will be determined later. Then, for the equation [16], we have

$$\frac{\xi}{r^2} \left( 1 + \frac{4AZ_h^2}{r^6 \sin^4 \theta} \right) \left( 1 + \sin^2 \theta \right) + \frac{4At^2}{r^4} \left( \frac{2\mu}{r} - \frac{\mu}{a} \right) \xi^2 \left( 1 + \sin^2 \theta \right) - \frac{A\xi}{r^2 \sin^2 \theta} \left( \frac{2\mu}{r} - \frac{\mu}{a} \right) + \frac{A}{\beta r^2 \sin^2 \theta} - \left( \frac{2\mu}{r} - \frac{\mu}{a} \right) = 0. \tag{18}$$

With respect to the Earth-Sun distance which is approximately constant and the fact that we are seeking the trajectories for the time domain of $10^7 - 10^8\text{s}$ magnitude, the coefficients $\alpha$ and $\beta$ are almost constant (about $10^3$) during the Earth revolution. Moreover, the fact that $\xi$ is assumed to be constant is due to the limited time domain we supposed here. For long time variations, $\xi$ is time-dependent. But then, the equation [18] cannot be solved rigorously.

Regarding the equation [17], the radial velocity $\dot{r}$ changes with time due to the time-dependency of $r(t)$ and $\dot{\theta}(t)$. There is no analytical solution for this kind of nonlinear differential equation. Yet, by applying some physical and mathematical assumptions, the equation [17] can give us proper trajectories, which are appropriate at the one year time domain. The method used here is to attain the following solution

$$r(t) = \xi \sqrt{F^2(t) - \sqrt{F^2(t)(F^2(t) - 4Z_h^2)}} \tag{19}$$

$$F(t) = -B_2 + C \tan \left( \frac{1}{2} C(t + \tau) \right) \tag{20}$$

In [20] we define

$$B_1 := B - \frac{\mu^2}{8B^3 r_{eq}^2} + \frac{\mu}{2Br_{eq}} \quad B_3 := \frac{\mu^2}{8B^3 r_{eq}^2}$$

$$B_2 := \frac{\mu^2}{4B^3 r_{eq}^2} - \frac{\mu}{2Br_{eq}^2} \quad C := \sqrt{-B_2^2 + 4B_1 B_3}$$

$$B := \sqrt{\frac{2\mu}{r_{eq}} - \frac{\mu}{a}} \tag{21}$$

where $\tau$ is a constant with time dimension. The details of calculating [19] from [17] are given in Appendix.

According to the equations [8], [13] and [19] we know how $\phi(t)$, $\theta(t)$ and $r(t)$ vary with time. So it is now possible to draw the trajectories. However, some points should be made clear first. If we take a closer look at [20], we will find a periodic tangent function with singularity points which means that our trajectories become discrete in these points. So, there is no continuous path prediction for all definite times $0 \leq t < \infty$ in this model. We can only follow and draw the trajectories in limited time domains, e.g., for one year ($10^7 - 10^8\text{s}$). In each time interval (which can be extended even to several years), we can obtain closed cycles, demonstrating nearly the Earth orbit around the Sun.

The equation [19] shows us that $r$ decreases while time goes forward, then it reaches to an equilibrium value. Such a behavior is sketched in Fig. [1]. The origin of coordinates here is not located on Sun but is displaced by $Z_h$. So, the radius $r$ represents $\sqrt{x^2 + y^2 + Z_h^2}$. Nevertheless, when we speak about the Earth-Sun distance, we mean $\sqrt{x^2 + y^2}$. Due to the long distance of the Earth from the Sun $Z_h$ does not affect the value of $r$, because $r$ and the Earth-Sun distance are both in the same order. Therefore, it is only a coordinate displacement. However, the constant $Z_h$ could not be considered as zero, to keep the trajectories on the surface. In fact, $Z_h$ is just a mathematical tool and does not have necessarily any counterpart in the physical world. According to Fig. [1] the Earth-Sun distance varies with time in different forms when the value of $Z_h$ changes. Nonetheless, whatever the initial conditions are supposed, the Earth-Sun distance decreases rapidly and tends to an equilibrium value. In other words, different trajectories produced from different initial conditions converge to a definite equilibrium situation when time passes sufficiently.

As we mentioned before, the trajectory equation [19]
is not the exact answer of the differential equation [18], though by a minute choice for the values of $Z_h$ and $\xi$ which could be made by the classical data, the equation [19] would be practically an appropriate solution for the Earth trajectory in the Bohmian framework. For typical values of $Z_h = 1.496 \times 10^{11} m$, $\phi_0 = 0$ and $\tau = 0$, as the main initial conditions, Fig. 2 shows how the Earth trajectory finally approaches to a stable closed cycle around the sun with an equilibrium distance $r_{eq} = 1.496 \times 10^{11} m$. As an instance, we can assume that $\phi(t) = 2\pi t$. Then, for the $x$ and $y$ components of the velocity we have

$$
\dot{x} = \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + Z_h^2} \times (1 + \frac{Z_h^2}{x^2 + y^2}) \sqrt{\frac{2\mu}{\sqrt{x^2 + y^2 + Z_h^2}} - \frac{\mu}{a} - 2\pi y}
$$

(22)

$$
\dot{y} = \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + Z_h^2} \times (1 + \frac{Z_h^2}{x^2 + y^2}) \sqrt{\frac{2\mu}{\sqrt{x^2 + y^2 + Z_h^2}} - \frac{\mu}{a} + 2\pi x}.
$$

(23)

Fig. 3 shows the stream of vector field $(\dot{x}, \dot{y})$ which is described by the equations (22) and (23). Whatever the initial conditions are supposed, the same equilibrium form is resulted, as mentioned before.

V. THE WAVE FUNCTION

So far we discussed the Earth dynamics and its trajectory. Here, we are going to investigate the Earth wave function. Let us note the time independent amplitude of the wave function, $R(r, \theta)$. Considering the equation (4) we have

$$
R^2 \nabla \cdot \nabla S + \nabla S \cdot \nabla R^2 = 0.
$$

(24)

In spherical coordinates, one can write $\nabla S$ as

$$
\nabla S = m_c \hat{r} \rho_r + m_c \cot \theta \hat{\theta} \rho_{\theta}
$$

$$
+ \frac{\sqrt{A}}{r \sin \theta} (1 - \frac{2\hat{r} t}{r} - \frac{2\hat{\theta} \cos^2 \theta t}{r \sin^2 \theta}) \rho_{\phi},
$$

(25)

where $\hat{r}$ is defined in (17). If we consider $R$ as $R(r, \theta) = \Re(r) \Theta(\theta)$, similar to the Hydrogen-like wave function, with the large values of $l$ and $m$ quantum numbers, one can write $\Theta(\theta) \propto \sin^m \theta$. According to $\Theta(\theta)$ relation, the function $\Theta^2$ as a probability distribution is approximately either 1 for $\theta = \pi/2$ or zero for the other values of $\theta$ due to the large value of $m$. On the one hand, for $\Theta^2 = 0$ the relation (24) is simply valid, but on the other hand, for $\Theta^2 = 1$ we have $R^2(r, \theta) \simeq \Re^2(r)$ and the validity of (24) should be studied. By substituting $\nabla S$ from (25) in (24) and after some algebraic calculations, one can show that

$$
\frac{\partial \Re^2}{\partial r} = \left( \frac{a}{2ar r - \frac{1}{r}} - 1 \right) \Re^2.
$$

(26)

The equation (26) imposes constraints on the amplitude of the wave function $R(r, \theta)$. Accordingly, from (26), one can conclude that

$$
R^2(r, \theta) = \Re^2(r) \Theta^2(\theta) = c \sqrt{\frac{|r - 2a|}{r}} \sin^m \theta,
$$

(27)
where \( c \) is a constant. As is apparent, \( R(r, \theta) \) shows constant probability, when the system reaches an equilibrium situation. Since, after sufficient time, both \( r \) and \( \theta \) become fixed values and only \( \phi(t) \) is responsible for the dynamics of the Earth revolution at equilibrium. This is similar to what Keeports obtain for the Hydrogen-like model of the Earth depiction. Keeports shows that the radial function becomes negligible when \( r \) differs slightly from the equilibrium value. Thus the Earth has the probability of being in the certain radius. This also is in agreement with the conservation of the total probability if all trajectories exist for all times.

Here the wave function is an energy eigenstate as discussed before. However with the large values of \( n \) the differences between the energy levels is negligible and the energy becomes continuous. Therefore although the wave function is an energy eigenstate, the energy is continuous, in agreement with classical results.

\[ \phi(r) \sim \frac{1}{r^2} \sin \left( \frac{\pi}{a} r \right) \]

where \( a \) and \( \frac{\pi}{a} \) are both time-dependent, but have no dependency on each other. For \( \theta \) we have from [8]

\[ \sin \theta = \sqrt{1 - \frac{Z_h^2}{r^2(t)}} \]

During the Earth rotation around the Sun, the time variations of \( r \) are insignificant. With regard to (A.2) and noticing that the constant \( Z_h \) has the same order of magnitude as \( r \), one can neglect the time-dependency of \( \theta \) too. Therefore, to solve the equation (A.1), we can nearly consider the term \( \xi \sin \theta \) as a constant.

Defining the variable \( q \) as

\[ q = r - r_{eq}, \]

where \( r_{eq} \) represents the equilibrium distance, the equation (A.1) yields

\[ \frac{dq}{dt} = \xi \sin \theta \sqrt{\frac{2\mu}{r_{eq}} \left( \frac{2}{r_{eq}} + 1 \right)} - \frac{\mu}{a}. \]

Hence, the variable \( q \) represents the deviation of \( r \) from the equilibrium distance \( r_{eq} \approx a \). The term \( q/r_{eq} \) is small, so that one can expand \( 1 + q/r_{eq} \) to obtain:

\[ \frac{dq}{dt} = \xi \sin \theta \sqrt{\frac{\mu}{r_{eq}} \left( 2 - \frac{q}{r_{eq}} \right)} - \frac{\mu}{a}. \]

By expanding the radical term, one finally gets

\[ \frac{dq}{dt} = \xi \sin \theta \left[ B - \frac{\mu}{2B r_{eq}^2} q - \frac{\mu^2}{8B^3 r_{eq}^4} q^2 \right], \]

where

\[ B := \sqrt{\frac{2\mu}{r_{eq}} - \frac{\mu}{a}}. \]

The equation (A.6) is a linear differential equation which can be solved easily. Consequently the equation (A.1) is obtained as an answer from (A.6), followed by the relations (20)-(21) as definitions in (19).

VI. CONCLUSION

Here, we have considered the Earth as a quantum object to show that the quantum guiding equation makes it behaves classically in the limit of large quantum numbers. Via the quantum formalism, the Earth dynamics is determined by the Schrödinger equation. By describing the Sun-Earth system as a model of Hydrogen atom, we introduced a new Hamiltonian with an additional kinetic energy term \( K \). In order to obtain the predicted quantum trajectory of the Earth, we used the guiding equation \( p = \nabla S \) (see (11)) in the Bohmian regime.

It is amazing to see that the large quantum numbers directly ascertain the Earth dynamics. As Keeports has already shown [1], at large quantum numbers \( n \) and \( l \), the classical energy of the Earth is obtained. Also, we showed here that the Earth dynamics is dependent on the large values of the quantum number \( m \) which affects the Earth trajectory via the guiding equation (11). This makes the well-known Newton’s \( \text{vis \ vita} \) equation (12) as the Earth velocity, which leads to acceptable Bohmian trajectories, after some reasonable approximations.

Interestingly, the main result is independent of the initial conditions. The rotation distance of the Earth-Sun system decreases rapidly and tends to an equilibrium one. All trajectories with different initial conditions approach to a stable closed cycle, illustrating how Earth orbits around the Sun with negligible deviation. As a macrosystem, this is a good achievement to approximately obtain the Earth trajectory in the Bohmian framework, which enables us to see the quantum footprints in a classical domain.

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