Malliavin Calculus Techniques for Local Asymptotic Mixed Normality and Their Application to Degenerate Diffusions

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Abstract. We study sufficient conditions for a local asymptotic mixed normality property of statistical models. We develop a scheme with the $L^2$ regularity condition proposed by Jeganathan [Sankhya Ser. A 44 (1982) 173–212] so that it is applicable to high-frequency observations of stochastic processes. Moreover, by combining with Malliavin calculus techniques by Gobet [Bernoulli 7 (2001) 899–912, 2001], we introduce tractable sufficient conditions for smooth observations in the Malliavin sense, which do not require Aronson-type estimates of the transition density function. Our results, unlike those in the literature, can be applied even when the transition density function has zeros. For an application, we show the local asymptotic mixed normality property of degenerate (hypoelliptic) diffusion models under high-frequency observations, in both complete and partial observation frameworks. The former and the latter extend previous results for elliptic diffusions and for integrated diffusions, respectively.

Keywords. degenerate diffusion processes; integrated diffusion processes; local asymptotic mixed normality; $L^2$ regularity condition; Malliavin calculus; partial observations

1 Introduction

In the study of statistical inference for parametric models, asymptotic efficiency plays a key role when we consider the asymptotic optimality of estimators. This notion was first studied for models that satisfy local asymptotic normality (LAN); Hájek [7] showed the convolution theorem, and Hájek [8] showed the minimax theorem under the LAN property. Both theorems give different concepts of asymptotic efficiency. For statistical models with the extended notion of local asymptotic mixed normality (LAMN), Jeganathan [12, 13] showed the convolution theorem and the minimax theorem.

Gobet [5] showed the LAMN property for discretely observed diffusion processes on a fixed interval. In that model, the maximum-likelihood-type estimator proposed by Genon-Catalot and Jacod [2] is asymptotically efficient. For further results related to diffusion processes on a fixed interval, see Gloter and Jacod [4] (LAN for noisy observations of diffusion processes with deterministic diffusion coefficients), Gloter and Gobet [3] (LAMN for integrated diffusion processes), Ogihara [16] (LAMN for nonsynchronously observed diffusion processes), and Ogihara [17] (LAN for noisy, nonsynchronous observations of diffusion processes with deterministic diffusion coefficients).

In the model of discretely observed diffusion processes by Gobet [5], he initiated a scheme based on Malliavin calculus techniques to show the LAMN property. He introduced Malliavin calculus techniques to control the asymptotics of log-likelihood ratios, and his scheme is effective for diffusion processes when the diffusion coefficient matrix is nondegenerate. In his scheme, it is crucial that the transition density functions of diffusion processes are estimated from below and above by Gaussian density functions. Such estimates are known as Aronson’s estimate. Gloter and Gobet [3] gave Aronson’s estimate, and consequently showed the LAMN property for the one-dimensional integrated diffusion processes by using Gobet’s scheme. However, the proof of Aronson’s estimate (Theorem 4) crucially depends on the fact that the latent process is one-dimensional.
For a diffusion model with the degenerate diffusion coefficient and a multi-dimensional integrated diffusion model, it seems difficult to obtain Aronson’s estimate in general, and therefore we cannot apply Gobet’s scheme. On the other hand, Theorem 1 in Jeganathan [12] introduced a scheme by using the so-called \( L^2 \) regularity condition to show the LAMN property. An advantage of this scheme is that we do not need estimates for the transition density functions. However, the results in [12] are not directly applicable to high-frequency observations that require a framework of triangular arrays. Further, for integrated diffusions, following the idea in [3], we need to consider a triangular array of expanding data blocks.

This paper studies four topics. First, we extend Theorem 1 in [12] so that it can be applied to statistical models with triangular array observations appearing in the above diffusion models with high-frequency observations. Second, we show that the new scheme based on the \( L^2 \) regularity condition can be applied under several conditions described via notions of Malliavin calculus. The new scheme is highly compatible with Gobet’s scheme. Indeed, the \( L^2 \) regularity condition is satisfied when observations are smooth in the Malliavin sense, and the inverse of Malliavin matrix and its derivatives have moments (see (B1), (B2), and Theorem 2.2). Moreover, if observations have a Euler–Maruyama approximation, then the sufficient conditions for the LAMN property is simplified (Theorem 2.3). Third, by using these schemes, we show the LAMN property for diffusion processes with the degenerate diffusion coefficient (degenerate diffusion) in which it is difficult to obtain Aronson-type estimates in general. Finally, we deal with the LAMN property for partial observations of degenerate diffusion processes.

Our new schemes can be applied to general statistical models without transition density estimates. In particular, they can be applied even when the transition density function has zero points. The \( L^2 \) regularity condition is related to differences in the roots of transition density functions, and it is not easily applied when the transition density functions have zero points (see (2.3)). However, we will see in Section 2.2 that if the observations are smooth in the Malliavin’s sense and the Malliavin matrix is nondegenerate, then we can apply the \( L^2 \) regularity condition even when the transition density has zero points. Consequently, this scheme enables us to study the LAMN property for several statistical models in Wiener space. First, this scheme allows a simplified proof of the results in Gobet [5]. Moreover, this scheme yields two interesting results. The first one is an extension of the results in Gobet [5] to a wider class including degenerate diffusion processes; we emphasis that this is achieved because we do not rely on Aronson-type estimates. The second one is an extension of the LAMN property for one-dimensional integrated diffusion processes in Gloter and Gobet [3] to the multi-dimensional case. We deal with the integrated diffusion process model in the general framework of partial observations for degenerate diffusion processes. We find that efficient asymptotic variance is the same for an integrated diffusion process model and for a diffusion process model, and that they are exactly twice as large as the statistical model of both observations (see Remark 2.6). Because our scheme does not require transition density estimates, we expect these ideas to be useful also for jump-diffusion process models or Lévy driven stochastic differential equation models. However, we left this for future work.

Our study of integrated diffusion models is motivated by experimental observations of single molecules (see e.g. Li et al. [21]), behind which are Langevin-type molecular dynamics

\[
\dot{Y} = b(\dot{Y}, Y) + a(\dot{Y}) \dot{W}.
\]

Here \( Y \) represents the position of a molecule (or a particle) and \( \dot{W} \) is white noise. When \( a = 0 \) this reduces to the Newtonian equation of classical dynamics. The system can be written as an integrated diffusion

\[
\begin{align*}
dY_t &= X_t dt, \\
dX_t &= b(X_t, Y_t) dt + a(X_t) dW_t.
\end{align*}
\] (1.1)

Our LAMN property enables us to discuss optimality in estimating the coefficient \( a \) based on high-frequency observations of the position \( Y \).

The rest of this paper is organized as follows. In Section 2 we introduce our main results, namely, the extended scheme using the \( L^2 \) regularity condition, the scheme via Malliavin calculus techniques, and the LAMN property of degenerate diffusion processes. Section 3 contains details of Malliavin calculus techniques. We combine the extended scheme of the \( L^2 \) regularity condition with the approaches of Gobet [5] and Gloter and Gobet [3].
2 Main results

2.1 The LAMN property via the \( L^2 \) regularity condition

In this subsection, we extend Theorem 1 in Jeganathan [12] to statistical models of triangular array observations so that it can be applied to high-frequency observations of stochastic processes.

Let \( \{P_{\theta,n}\}_{\theta \in \Theta} \) be a family of probability measures defined on \((\mathcal{R}_n, F_n)\), where \( \Theta \) is an open subset of \( \mathbb{R}^d \). We often regard a \( p \)-dimensional vector \( v \) as a \( p \times 1 \) matrix. \( I_k \) denotes the unit matrix of size \( k \) for \( k \in \mathbb{N} \).

**Condition (L).** The following two conditions are satisfied for \( \{P_{\theta,n}\}_{\theta \in \Theta} \).

1. There exists a sequence \( \{V_n(\theta_0)\} \) of \( F_n \)-measurable \( d \)-dimensional vectors and a sequence \( \{T_n(\theta_0)\} \) of \( F_n \)-measurable \( d \times d \) symmetric matrices such that

\[
P_{\theta,n}[T_n(\theta_0)] \text{ is nonnegative definite} = 1
\]

for any \( n \in \mathbb{N} \), and

\[
\log \frac{dP_{\theta_0 + r_n h_n}}{dP_{\theta,n}} - h^\top V_n(\theta_0) + \frac{1}{2} h^\top T_n(\theta_0) h \to 0
\]

in \( P_{\theta,n} \)-probability for any \( h \in \mathbb{R}^d \), where \( \{r_n\} \) is a sequence of positive definite matrices and \( \top \) denotes the transpose operator for matrices.

2. There exists an almost surely nonnegative definite random matrix \( T(\theta_0) \) such that

\[
\mathcal{L}(V_n(\theta_0), T_n(\theta_0)|P_{\theta,n}) \to \mathcal{L}(T^{1/2}(\theta_0)W, T(\theta_0)),
\]

where \( W \) is a \( d \)-dimensional standard normal random variable independent of \( T(\theta_0) \).

The following definition of the LAMN property is Definition 1 in [12].

**Definition 2.1.** The sequence of the families \( \{P_{\theta,n}\}_{\theta \in \Theta} (n \in \mathbb{N}) \) satisfies the LAMN condition at \( \theta = \theta_0 \in \Theta \) if Condition (L) is satisfied, \( P_{\theta,n}[T_n(\theta_0)] \) is positive definite \( = 1 \) for any \( n \in \mathbb{N} \), and \( T(\theta_0) \) is positive definite almost surely.

For proving the LAMN property for diffusion processes using a localization technique such as Lemma 4.1 in [5], Condition (L) is useful because (L) for the localized model often implies (L) for the original model. See the proofs of Theorems 2.4 and 2.5 for the details.

**Remark 2.1.** When Condition (L) is satisfied and \( T(\theta_0) \) is positive definite almost surely, by setting

\[
\tilde{T}_n(\theta_0) = T_n(\theta_0)1_{\{T_n(\theta_0) \text{ is p.d.}\}} + I_d 1_{\{T_n(\theta_0) \text{ is not p.d.}\}},
\]

the LAMN property holds with \( \tilde{T}_n(\theta_0) \) and \( V_n(\theta_0) \).

Let \( (m_n)_{n=1}^{\infty} \) be a sequence of positive integers. Let \( \{\mathcal{R}_{m_j}\}_{j=1}^{m_n} \) be a sequence of complete, separable metric spaces, and let \( \Theta \) be an open subset of \( \mathbb{R}^d \). Let \( \mathcal{R}_n = \mathcal{R}_{n,1} \times \cdots \times \mathcal{R}_{n,m_n} \). We consider statistical experiments \((\mathcal{R}_n, B(\mathcal{R}_n), \{P_{\theta,n}\}_{\theta \in \Theta})\). Let \( X_j = X_{n,j} : \mathcal{R}_n \to \mathcal{R}_{n,j} \) be the natural projection, \( \tilde{X}_j = \tilde{X}_{n,j} = (X_1, \ldots, X_j) \), and \( F_j = F_{n,j} = \sigma(\tilde{X}_j) \) for \( 0 \leq j \leq m_n \). Suppose that there exists a \( \sigma \)-finite measure \( \mu_j = \mu_{m,j} \) on \( \mathcal{R}_{n,j} \) such that \( P_{\theta,n}(X_j = \cdot) \ll \mu_j \) and \( P_{\theta,n}(X_j \in [\tilde{X}_{j-1}, \tilde{X}_{j-1}]) \ll \mu_j \) for all \( \tilde{X}_{j-1} \in \mathcal{R}_{n,1} \times \cdots \times \mathcal{R}_{n,j-1} \), \( 2 \leq j \leq m_n \). Let \( E_{\theta,n} = E_{\theta,n} \) denote the expectation with respect to \( P_{\theta,n} \), and let \( p_j = p_{n,j} \) be the conditional density functions defined by

\[
p_1(\theta) = \frac{dP_{\theta,n}(X_1 \in \cdot)}{d\mu_1} : \mathcal{R}_{n,1} \to \mathbb{R}, \quad p_j(\theta) = \frac{dP_{\theta,n}(X_j \in [\tilde{X}_{j-1}, \cdot])}{d\mu_j} : \mathcal{R}_{n,j} \to \mathbb{R}
\]

for \( 2 \leq j \leq m_n \). Then we can see that for \( g : \mathcal{R}_{n,1} \times \cdots \mathcal{R}_{n,j} \to \mathbb{R} \),

\[
\int_{\mathcal{R}_{n,j}} p_j(\theta)g(\tilde{X}_{j-1}, \cdot)\mu_j = E_\theta[g(\tilde{X}_{j-1}, X_j)|\tilde{F}_{j-1}].
\]

(2.4)
**Assumption (A1).** There is a $d \times d$ positive definite matrix $r_n$ and measurable functions

$$
\dot{\xi}_{n,j}(\theta_0, \cdot) : \mathcal{G}_{n,1} \times \cdots \times \mathcal{G}_{n,j} \rightarrow \mathbb{R}^d
$$

such that for every $h \in \mathbb{R}^d$,

$$
\sum_{j=1}^{m_n} E_{\theta_0} \left[ \int \left[ \xi_{n,j}(\theta_0, h) - \frac{1}{2} h^\top r_n \dot{\xi}_j(\theta_0) \right]^2 d\mu_j \right] \rightarrow 0
$$

as $n \rightarrow \infty$, where

$$
\xi_{n,j}(\theta_0, h) = \sqrt{p_j(\theta_0 + r_n h)} - \sqrt{p_j(\theta_0)}
$$

and

$$
\dot{\xi}_j(\theta_0) = \dot{\xi}_{n,j}(\theta_0, \bar{X}_{j-1}, \cdot) : \mathcal{G}_{n,j} \rightarrow \mathbb{R}^d.
$$

Condition (A1) is the $L^2$ regularity condition.

For a vector $x = (x_1, \ldots, x_k)$, we denote $\partial_x^k = \left( \frac{\partial}{\partial x_{i_1} \cdots \partial x_{i_k}} \right)_{i_1 \cdots i_k = 1}$. If $p_j$ is smooth with respect to $\theta$ and $p_j \neq 0$, then the log-likelihood ratio is rewritten as

$$
\log \frac{dP_{\theta,n}}{dP_{\theta}} = \sum_{j=1}^{m_n} \log \frac{p_j(\theta')}{p_j(\theta)} = \sum_{j=1}^{m_n} \int_0^1 \frac{\partial p_j}{p_j}(t\theta' + (1-t)\theta)dt(\theta' - \theta).
$$

To show the LAMN property, we must identify the limit distribution of this function under $P_{\theta,n}$. Doing so requires estimates for density ratios with different probability measures, which are not easy to obtain for stochastic processes in general. Gobet [5] dealt with this problem for discretely observed diffusion processes by using estimates from below and above by Gaussian density functions and show the LAMN property of that model.

On the other hand, if $p_j \in C^2(\Theta)$ and $p_j(\theta_t) \neq 0$ for any $t \in [0,1]$ $\mu_j$-a.e. for $\theta_t = \theta_0 + tr_n h$, then by setting

$$
\dot{\xi}_j(\theta_0) = \partial_\theta p_j(\theta_0)p_j(\theta_0)^{-1/2}
$$

we obtain

$$
\int \left[ \xi_{n,j}(\theta_0, h) - \frac{1}{2} h^\top r_n \dot{\xi}_j(\theta_0) \right]^2 d\mu_j
$$

$$
= \int \left[ h^\top r_n \left( \int_0^1 \frac{\partial_\theta p_j(\theta_s)}{2 \sqrt{p_j(\theta_s)}} d\mu_j - \frac{1}{2} h^\top r_n \frac{\partial_\theta p_j(\theta_0)}{\sqrt{p_j(\theta_0)}} \right)^2 d\mu_j
$$

$$
= \int \left[ \frac{h^\top r_n}{2} \int_0^1 \int_0^1 \left( \frac{\partial_\theta^2 p_j(\theta_s)}{2p_j^{3/2}} - \frac{\partial_\theta p_j(\theta_0)^\top(\theta_s)}{2p_j^{3/2}} \right) d\mu_j \right] d\mu_j
$$

$$
\leq \frac{1}{4} \sup_{0 \leq s \leq 1} E_{\theta_0} \left[ \left( h^\top r_n \left( \frac{\partial_\theta^2 p_j(\theta_s)}{p_j(\theta_s)} - \frac{\partial_\theta p_j(\theta_0)^\top(\theta_s)}{2p_j^{3/2}} \right) p_j(\theta_0) \right) r_n h \right]^2 \mathcal{F}_{j-1}.
$$

In the right-hand side of the above inequality, the value $\theta_s$ of the parameter is the same for the probability measure of expectation and $p_j$ in the integrand, and therefore we do not need estimates for the transition density ratios.

Thus, a scheme with the $L^2$ regularity condition does not require estimates for the transition density function. This is a big advantage, and this scheme can be applicable to multi-dimensional integrated diffusion processes and degenerate diffusion processes, where it is difficult to obtain estimates for transition density functions.

Define

$$
\eta_j = \left( \frac{\dot{\xi}_j(\theta_0)}{\sqrt{p_j(\theta_0)}} 1_{\{p_j(\theta_0) \neq 0\}} \right) (X_j).
$$

**Assumption (A2).** $E_{\theta_0}[|\eta_j|^2|\mathcal{F}_{j-1}] < \infty$ and $E_{\theta_0}[\eta_j|\mathcal{F}_{j-1}] = 0$, $P_{\theta_0,n}$-almost surely for every $j \geq 1$. 

4
Assumption (A3). For every $\epsilon > 0$ and $h \in \mathbb{R}^d$,
\[
\sum_{j=1}^{m_n} E_{\theta_0}[|h^\top r_n \eta_j|^2 1_{(|h^\top r_n \eta_j| > \epsilon)}] \to 0.
\]

Assumption (A4). For every $h \in \mathbb{R}^d$, there exists a constant $K > 0$ such that
\[
\sup_{n \geq 1} \sum_{j=1}^{m_n} E_{\theta_0}[|h^\top r_n \eta_j|^2] \leq K.
\]

Let
\[
T_n = r_n \sum_{j=1}^{m_n} E_{\theta_0}[\eta_j \eta_j^\top |\mathcal{F}_{j-1}]r_n \quad \text{and} \quad V_n = r_n \sum_{j=1}^{m_n} \eta_j.
\]

Assumption (A5). There exists a random $d \times d$ symmetric matrix $T$ such that $P[T \text{ is n.d.}] = 1$ and
\[
\mathcal{L}((V_n, T_n)|P_{\theta_0,n}) \to \mathcal{L}(T^{1/2}W, T),
\]
where $W \sim N(0, I_d) \perp T$.

Assumption (P). $T$ in (A5) is positive definite almost surely.

Conditions (A1)–(A4) correspond to (2.A.1), (2.A.2), (2.A.4), and (2.A.5) in [12], respectively. Condition (A5) ensures Point 2 of Condition (L). For the sequential observations in [12], convergence of $r_n \sum_{j=1}^{m_n} \eta_j$ always holds by virtue of Hall [9] (see (2.3) in [12]). However, the results of [9] cannot be applied to the triangular array observations, so we instead assume (A5) for our scheme. To check (A5), the results in Sweeting [20] are useful. Moreover, it is not difficult to check (A5) for statistical models of discretely observed diffusion processes using a martingale central limit theorem. See, for example, Theorems 2.4 and 2.5 and their proofs.

**Theorem 2.1.** Assume (A1)–(A5). Then (L) holds true with $T_n$ and $V_n$ in (2.7) for the family $\{P_{\theta,n}\}_{\theta,n}$ of probability measures. If further (P) is satisfied, then $\{P_{\theta,n}\}_{\theta,n}$ satisfies the LAMN condition at $\theta = \theta_0$ with $T_n$ in Remark 2.2.

A proof is given in the appendix.

**Remark 2.2.** We assumed that $r_n$ is positive definite because this assumption is made in the definition of the LAMN property in Jeganathan [12] (Definition 1). However, we can see that Theorem 2.1 holds even if $r_n$ is a nondegenerate asymmetric matrix. In that case, even though the assumptions of convolution theorem (Corollary 1) in [12] are not satisfied, the convolution theorem in Hájek [7] is satisfied when local asymptotic normality is satisfied (i.e., $T$ in (A5) is non-random) and the operator norm of $r_n r_n^\top$ converges to zero.

### 2.2 The LAMN property via Malliavin calculus techniques

Gobet [5] used Malliavin calculus techniques to show the LAMN property for discretely observed diffusion processes. Gloter and Gobet [8] developed Gobet’s scheme into a more general one and showed the LAMN property for a one-dimensional integrated diffusion process. These approaches require estimates for transition density functions by Gauss density functions, thereby hampers multi-dimensional extension of their results. Our alternative approach introduces tractable sufficient conditions to show the LAMN property for smooth observations in the Malliavin sense, by combining with a scheme with the $L^2$ regularity condition in Section 2.1. In particular, we can show the $L^2$ regularity condition under (B1) and (B2), which are related to the smoothness of observations and estimates for the inverse of the Malliavin matrix (Theorem 2.2). If further, observations have a Gaussian approximation like the Euler–Maruyama approximation, then the sufficient conditions for the LAMN property are simplified as in Theorem 2.5.

We assume that $\Theta$ is convex and that $m_n \to \infty$ as $n \to \infty$. Let $(\epsilon_n)_{n=1}^{\infty}$ be a sequence of positive numbers and $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(k_j)_{j=0}^{m_n}$ be an increasing sequence of nonnegative integers such
that $k_0^n = 0$. Hereinafter, we abbreviate $k^n_j$ as simply $k_j$. Let $N_n = k_{m_n}$ and $X_j^{n,0}$ be an $\mathbb{R}^{k_j-k_{j-1}}$-valued random variable on $(\Omega, \mathcal{F}, P)$ for $1 \leq j \leq m_n$. Let $\mathcal{P}_n$ be the induced probability measure by $\{X_j^{n,0}\}_{j=1}^{m_n}$ on $(\mathbb{R}^{N_n}, \mathcal{B}(\mathbb{R}^{N_n}))$ and $\mathcal{F}_j = \{A \times \mathbb{R}^{N_n-k_j} | A \in \mathcal{B}(\mathbb{R}^{k_j})\} \subset \mathcal{B}(\mathbb{R}^{N_n})$. For each $1 \leq j \leq m_n$, we adopt the notation of Nualart [15]. Specifically, let $H_j$ be a real separable Hilbert space and $W_j = \{W_j(h), h \in H_j\}$ be an isonormal Gaussian process defined on a complete probability space $(\Omega, \mathcal{G}_j, Q_j)$. We assume that $G_j$ is generated by $W_j$. Even though these objects possibly depend on $n$, we omit the dependence in our notation. Let $\delta_i$ be the Hitsuda–Skorokhod integral (the divergence operator), $D_i$ be the Malliavin–Shigekawa derivative, and $S_j = \{ f(W_j(h_1), \ldots, W_j(h_k)) : k \in \mathbb{N}, h_i \in H_j \ (i = 1, \ldots, k), f \in C^{\infty}(\mathbb{R}^k) \}$. For $k \in \mathbb{Z}_+$ and $p \geq 1$, $\|\cdot\|_{k,p}$ denotes the operator on $S_j$ defined by

$$
\|F\|_{k,p} = \left( E_j[|F|^p] + \sum_{l=1}^{k} E_j[\|D_j^lF\|_{H_j,\mathcal{P}}^p]\right)^{1/p},
$$

where $E_j$ denotes the expectation with respect to $Q_j$. Let $\mathbb{D}^{k,p}_j$ be the completion of $S_j$ with respect to the distance $d(F,G) := \|F - G\|_{k,p}$. For general properties of $W_j$, $D_j$, and $\delta_i$, see Nualart [15]. Let $F_{n,\theta,j,x_{j-1}}$ be an $\mathbb{R}^{k_j-k_{j-1}}$-valued random variable on $(\Omega, \mathcal{G}_j)$ such that $Q_j^{-1}[F_{n,\theta,j,x_{j-1}} = P(X_j^{n,\theta} \in \{X_j^{n,\theta} = x_{j-1}\}] - 1$, where $X_j^{n,\theta} = \{X_j^{n,\theta} = x_{j-1}\}$. We assume that $F_{n,\theta,j,x_{j-1}}$ is Fréchet differentiable with respect to $\theta$ on $L^p(\Omega_j)$ for any $p > 1$ and denote its derivative by $\partial_\theta F_{n,\theta,j,x_{j-1}} = (\partial_{\theta_0} F_{n,\theta,j,x_{j-1}}, \ldots, \partial_{\theta_p} F_{n,\theta,j,x_{j-1}})^T$. We often omit the parameter $x_{j-1}$ in $F_{n,\theta,j,x_{j-1}}$ and write $F_{n,\theta,j}$. Let $C_2^\infty$ denote the space of all $C^\infty$ functions with compact support. For a matrix $A$, we denote its element $(i,j)$ by $[A]_{ij}$. Similarly, we denote by $[V]_i$ the $l$-th element of a vector $V$. Let $k_n = \max(j \leq k_j-k_{j-1})$.

We assume the following conditions.

**Assumption (B1).** $\partial_\theta [F_{n,\theta,j}]_i \in \cap_{p>1} \mathbb{D}^{4-1,p}_j$ for any $n, \theta, j, i, 0 \leq l \leq 3$, and supremum over $n, i, j, x_{j-1}, \theta$ of $\|\partial_\theta [F_{n,\theta,j}]_i\|_{4-1,p} < \infty$ for $p > 1$.

**Assumption (B2).** The matrix $K_j^i(\theta) = ([D_j[F_{n,\theta,j}]_k, D_j[F_{n,\theta,j}]_l]_H_j)_{k,l}$ is invertible almost surely for any $j, x_{j-1}$, and $\theta$, and there exists a constant $\alpha_n \geq 1$ such that

$$
\sup_{i,j,x_{j-1},\theta} \|K_j^{-1}(\theta)\|_{i,2,8} \leq \alpha_n, \quad \text{and} \quad c^2_n \mathbb{E}[\|K_j^{-1}(\theta)\|_{i,2,8}]^4 \to 0
$$

as $n \to \infty$.

Let $\theta_0$ be the true value of parameter $\theta$. We will see later in Proposition [15] that $F_{n,\theta,j}$ admits a density $p_{j,x_{j-1}}(x_j, \theta)$ that satisfies $p_{j,x_{j-1}}(x_j, \theta) \in C^2(\theta)$ almost everywhere in $x_j \in \mathbb{R}^{k_j-k_{j-1}}$ under (B1) and (B2). Let $N_j^\times = \{x_j \in \mathbb{R}^{k_j-k_{j-1}} | \sup_{\theta \in \Theta} p_{j,x_{j-1},\theta}(x_j, \theta) > \inf_{\theta \in \Theta} p_{j,x_{j-1},\theta}(x_j, \theta) = 0\}$ and $M_j^\times = \{x_j \in \mathbb{R}^{k_j-k_{j-1}} | \inf_{\theta \in \Theta} p_{j,x_{j-1},\theta}(x_j, \theta) > 0\}$. We further assume the following condition.

**Assumption (N1).** For any $h \in \mathbb{R}^d$,

$$
E_{\theta_0} \left[ \sum_{j=1}^{m_n} \int_{N_j^\times} p_{j,x_{j-1},\theta}(x_j, \theta_0 + r_n h) dx_j \right] \to 0
$$

as $n \to \infty$.

If $\sup_{\theta \in \Theta} p_{j}(x_j, \theta) = 0$ or $\inf_{\theta \in \Theta} p_{j}(x_j, \theta) > 0$, we have $x_j \in N_j^\times$. Condition (N1) says that the probability of other cases is asymptotically negligible. This condition is used to validate an estimate such as [26]. However, if $F_{n,\theta,j}$ is approximated by a Gaussian random variable and satisfies (B3) and (N2) below, then we can check (A1) without (N1) (see Lemma [33]).

With these definitions, the following theorem shows that the $L^2$ regularity condition is automatically satisfied under (B1), (B2), and (N1). Let

$$
\dot{\xi}_j(\theta) = \frac{\partial_\theta p_j}{2\sqrt{p_j}} 1_{M_j}(x_j, \theta), \quad \eta_j = \frac{\partial_\theta p_j}{2p_j} 1_{M_j}(x_j, \theta_0).
$$

(2.8)
Theorem 2.2. Assume (B1), (B2), (N1), (A4), and (A5) with \( \xi_j(\theta) \) and \( \eta_j \) defined in (2.3). Then (L) holds true for \( \{P_{\theta,n}\}_{\theta,n} \) at \( \theta = \theta_0 \) with \( r_n = c_n I_d \). If further (P) is satisfied, then \( \{P_{\theta,n}\}_{\theta,n} \) satisfies the LAMN condition at \( \theta = \theta_0 \).

In the following, we give tractable sufficient conditions for (A4) and (A5) when \( F_{n,\theta,j} \) has a Gaussian approximation \( \tilde{F}_{n,\theta,j} \).

**Assumption (B3).** There exist a matrix \( B_{j,i,\theta} = B_{j,i,\theta,j,x_{j-1},n} \) and \( h_{j,i} = h_{n,\theta,j,i,x_{j-1}} \in H_j \) (1 \( \leq l \leq k_j - k_{j-1} \)) such that \( \tilde{F}_{n,\theta,j,x_{j-1}} = (W_j(h_{j,i}))_{i=1}^{k_j-k_{j-1}} \) is Fréchet differentiable with respect to \( \theta \) on \( L^p \) space, and \( \partial_{\theta} \tilde{F}_{n,\theta,j} = B_{j,i,\theta} \tilde{F}_{n,\theta,j} \). Moreover, \( \partial_{\theta} B_{j,i,\theta} \) exists and is continuous with respect to \( \theta \), and there exists a constant \( C_p \) and a sequence \( (\rho_n)_{n\in \mathbb{N}} \) of positive numbers such that

\[
\sup_{n,i,j,x_{j-1},\theta,l,i_2} \| \partial_{\theta}^l \tilde{F}_{j,i,\theta}[r_1,t_2] \| < \infty,
\]

and

\[
\left\| [F_{n,\theta,j} - \tilde{F}_{n,\theta,j}^\prime] \right\|_{L^p} + \| \partial_{\theta} \left[ F_{n,\theta,j} - \tilde{F}_{n,\theta,j} \right] \|_{L^p} \leq C_p \rho_n
\]

for \( p > 1, 1 \leq j \leq m_n, 1 \leq i \leq d, 1 \leq i' \leq k_j - k_{j-1}, \theta \in \Theta \) and \( x_{j-1} \).

For a statistical model of diffusion processes, \( F_{n,\theta,j} \) corresponds to normalized discrete observations and \( \tilde{F}_{n,\theta,j} \) corresponds their Euler–Maruyama approximations. See (B.1) and (B.3) for an example.

Let \( \tilde{K}_j(\theta) = (h_{j,i}, h_{j,i})_l_{i=1}^{d} \). Then, we will see that for sufficiently large \( n \), \( \tilde{K}_j(\theta) \) is invertible almost surely under (B1)–(B3) and that \( \alpha_n \rho_n K_n^2 \to 0 \) in Lemma 3.1 of Section 3. Let \( \mathcal{L}_{j,i,j,x_{j-1}}(u,\theta) = u^\top B_{j,i,\theta}^\top \tilde{K}_j^{-1}(\theta)u - tr(B_{j,i,\theta}) \).

Let \( \Phi_{j,i} = (B_{j,i,\theta}^\top \tilde{K}_j^{-1}(\theta_0) + \tilde{K}_j^{-1}(\theta_0)B_{j,i,\theta_0})/2 \), and let

\[
\gamma_j(x_{j-1}) = (2tr(\Phi_{j,i} \tilde{K}_j^{-1}(\theta_0)\Phi_{j,i} \tilde{K}_j^{-1}(\theta_0)))_{i',j'=1}^{d}.
\]

**Assumption (B4).** There exist \( \mathbb{R}^d \)-valued random variables \( \{G_{j}^n\}_{1 \leq j \leq m_n, \theta} \) and a filtration \( \{G_j\}_{j=1}^{m_n} \) on \( (\Omega, \mathcal{F}, P) \) such that \( (X_j^{n,\theta_0}, G_j^n) \) is \( G_j \)-measurable, \( E[G_j^n|G_{j-1}] = 0 \), and

\[
Q_j((F_{n,\theta,j}, \mathcal{L}_{j,i,j,x_{j-1}}(\tilde{F}_{n,\theta,j}(\theta_0)))_{i=1}^{d}) \in A) \big| x_{j-1}=X_{j-1}\vphantom{\int} \Big|_{x_{j-1}=X_{j-1}} = P((X_j^{n,\theta_0}, G_j^n) \in A| G_{j-1}) \tag{2.9}
\]

for \( A \in \mathcal{B}(\mathbb{R}^{k_j-k_{j-1}} \times \mathbb{R}^d) \) and sufficiently large \( n \). Moreover,

\[
\sup_n \left( \sum_{j=1}^{m_n} E[\gamma_j(X_j^{n,\theta_0})] \right) < \infty,
\]

\[
\alpha_n \rho_n K_n^2 \to 0 \quad \text{and} \quad \epsilon_n^2 m_n \alpha_n^3 \rho_n K_n^6 \to 0 \tag{2.10}
\]

as \( n \to \infty \), and there exist a random \( d \times d \) matrix \( \Gamma \) and a \( d \)-dimensional standard normal random variable \( \mathcal{N} \) such that

\[
\left( \epsilon_n \sum_{j=1}^{m_n} G_j^n, \epsilon_n^2 \sum_{j=1}^{m_n} \gamma_j(X_j^{n,\theta_0}) \right) \overset{d}{\to} (\Gamma^{1/2} \mathcal{N}, \Gamma) \tag{2.11}
\]

**Assumption (N2).** \( [F_{n,\theta,j}]_i \in \cap_{\theta > 1, \mathcal{R} \subset \mathbb{R}^p} \mathbb{D}^{r,p} \) for \( n, \theta, j, i, x_{j-1} \) and

\[
\sup_{\theta \in \Theta} \| F_{n,\theta,j} \|_{L^p} < \infty
\]

for any \( n, i, j, x_{j-1}, r \in \mathbb{N} \) and \( p > 1 \). Also,

\[
\partial_{\theta} D_j[F_{n,\theta,j}]_k = \sum_{l=1}^{k_j-k_{j-1}} [B_{j,i,\theta}]_{k,i,l} D_j[F_{n,\theta,j}]_l
\]

and \( \sup_{\theta \in \Theta} E_j[|\det K_j^{-1}(\theta)|^p] < \infty \) for any \( p > 1, j, k, x_{j-1} \) and \( 1 \leq i \leq d \).
Assumption (B5). (N1) or (N2) holds true.

Assumption (P'). Γ in (B4) is positive definite almost surely.

Remark 2.3. Because

\[ x^T \gamma_j x = 2\text{tr} \left( \sum_{i=1}^{d} \Phi_{j,i} x_i \right) K_j \left( \sum_{i=1}^{d} \Phi_{j,i'} x_{i'} \right) K_j \]

\[ = 2\text{tr} \left( K_j^{1/2} \sum_{i=1}^{d} \Phi_{j,i} x_i \right) K_j^{1/2} \geq 0 \]

for \( x = (x_1, \cdots, x_d) \in \mathbb{R}^d \), \( \gamma_j \) is symmetric and nonnegative definite. Hence \( \Gamma \) is also symmetric and nonnegative definite almost surely under (B1)--(B3) and the fact that \( \epsilon^2_n \sum_{j=1}^{m_n} \gamma_j (\bar{X}_{n,j-1}) \to \Gamma \) as \( n \to \infty \).

Theorem 2.3. Assume (B1)--(B5). Then \( \{ \theta_n \}_{\theta} \) satisfies (L) with \( T(\theta_0) \) equal to \( \Gamma \) in (B4), \( r_n = \epsilon_n I_d \), \( \xi_j(\theta) \) and \( \eta_j \) are defined by (2.3) if (N1) is satisfied, and

\[ \dot{\xi}_j(\theta) = \frac{\partial \theta p_j}{2 \sqrt{p_j}} 1_{\{p_j \neq 0\}} (x_j, \theta), \quad \eta_j = \frac{\partial \theta p_j}{2 p_j} 1_{\{p_j \neq 0\}} (x_j, \theta) \]

if (N2) is satisfied. If further (P') is satisfied, then \( \{ \theta_n \}_{\theta, \eta} \) satisfies the LAMN property at \( \theta = \theta_0 \).

Note that Theorem 2.3 works without having to identify zero points of the density function \( p_j \), unlike in previous studies. This is a major advantage because it is often not an easy task to show either that \( p_j \) has no zero points or that zero points are common for every \( \theta \). For example, to our knowledge, there are no results related to zero points of transition density functions for the statistical model of multi-dimensional integrated diffusion processes. In the following section, we see that Theorem 2.3 can be applied to this model.

The following lemma is useful when we check (2.12) by using a martingale central limit theorem. The proof is left to Section B in the appendix.

For a matrix \( A \), \( \| A \|_\text{op} \) denotes the operator norm of \( A \).

Lemma 2.1. Assume (B1)--(B3) and that (2.3) is satisfied for any \( A \in \mathcal{B}(\mathbb{R}^{d-k_j+1} \times \mathbb{R}^d) \). Then,

1. \( E[|G_j^n|^4 | G_{j-1}] = \gamma_j (\bar{X}_{n,j-1}) \) for any \( 1 \leq j \leq m_n \), and

2. \( \epsilon^4_n \sum_{j=1}^{m_n} E[|G_j|^4 | G_{j-1}] \to 0 \) as \( n \to \infty \) if

\[ \epsilon^4_n m_n \alpha_n^8 k_n^8 \sup_{j,i,j-1} \| B_{j,i,\theta_0} \bar{K}_j (\theta_0) + \bar{K}_j (\theta_0) B_{j,i,\theta_0}^\top \|_\text{op}^4 \to 0. \] (2.12)

2.3 The LAMN property for degenerate diffusion models

In this section, we show the LAMN property for degenerate diffusion processes by applying the results in Sections 2.2 and 2.3.

Let \( r \in \mathbb{N} \), and let \( (\Omega, \mathcal{F}, P) \) be the canonical probability space associated with an \( r \)-dimensional Wiener process \( W = \{W_t\}_{t \in [0,1]} \), that is, \( \Omega = C([0,1]; \mathbb{R}^r) \), \( P \) is the \( r \)-dimensional Wiener measure, \( W_t(\omega) = \omega(t) \) for \( \omega \in \Omega \), and \( \mathcal{F} \) is the completion of the Borel σ-field of \( \Omega \) with respect to \( P \). Let \( D \) be the Malliavin–Shige kawa derivative related to the underlying Hilbert space \( H = L^2([0,1]; \mathbb{R}^r) \). Let \( \Theta \) be a bounded open convex set in \( \mathbb{R}^d \). We regard \( \partial_{x^i} v = (\partial_{x^i} v)_{j,j} \) as a matrix for vectors \( x \) and \( v \). \( \mathcal{A} \) denotes the closure for a set \( A \).

For \( \theta \in \Theta \), let \( X^\theta = (X^\theta_t)_{t \in [0,1]} \) be an \( m \)-dimensional diffusion process satisfying \( X^\theta_0 = z_{\text{ini}} \), and

\[ dX^\theta_t = b(X^\theta_t, \theta) dt + a(X^\theta_t, \theta) dW_t, \quad t \in [0,1], \] (2.13)

where \( z_{\text{ini}} \in \mathbb{R}^m \) and \( a \) and \( b \) are Borel functions. We consider a statistical model with observations \( \{X^\theta_t\}_{\theta=0} \).

Let \( k = \text{rank}(a(z, \theta)) \). Assume that \( m/2 \leq k < m \) and that \( k \) does not depend on \( (z, \theta) \). Then, by singular value decomposition, we can find an orthogonal matrix \( U_{z,\theta} \) such that \( U_{z,\theta} a(z, \theta) = (\hat{a}(U_{z,\theta} z, \theta)^\top, O_{m-k,l}^\top) \), where \( O_{k,l} \) denotes a \( k \times l \) matrix with each element equal to zero.

First, we assume that \( U_{z,\theta} \) does not depend on \( (z, \theta) \).
Assumption (C1). The derivatives $\partial_i^j\partial_i^0\mu(z,\theta)$ and $\partial_i^j\partial_i^0\nu(z,\theta)$ exist on $\mathbb{R}^{m} \times \Theta$ and can be extended to continuous functions on $\mathbb{R}^{m} \times \Theta$ for $i \in \mathbb{Z}_+$ and $0 \leq j \leq 3$. Moreover, $\sup_{z,\theta}(\partial_i^j\partial_i^0\nu(z,\theta)\vline\partial_i^j\partial_i^0\mu(z,\theta)) < \infty$, and there exist an orthogonal matrix $U$ and $\mathbb{R}^n \otimes \mathbb{R}^r$, $\mathbb{R}^s$, and $\mathbb{R}^{m-r}$-valued Borel functions $\tilde{a}(z,\theta)$, $b(z,\theta)$, and $\tilde{b}(z)$, respectively, such that

\[ Ua(z,\theta) = \begin{pmatrix} \tilde{a}(Uz,\theta) \\ 0_{m-k,r} \end{pmatrix}, \quad Ub(z,\theta) = \begin{pmatrix} \tilde{b}(Uz,\theta) \\ b(Uz) \end{pmatrix} \quad (2.14) \]

for any $z \in \mathbb{R}^m$ and $\theta \in \Theta$. Further, $\tilde{a}\tilde{a}^\top(z,\theta)$ is positive definite for any $(z,\theta) \in \mathbb{R}^m \times \Theta$.

There exists a unique strong solution $(X_{t}^{\theta})_{t \in [0,\mathbb{R}^n]}$ of (B3) under (C1). Let $P_{\theta,n}$ be the distribution of $(X_{k/n}^{\theta})_{n=0}^\infty$, and let $\theta_0$ be the true value of $\theta$. We denote $X_t = X_t^{\theta_0}$.

Under (C1), by setting $Y_t^{\theta} = UX_t^{\theta}$, we obtain

\[ dY_t^{\theta} = \begin{pmatrix} \tilde{b}(Y_t^{\theta},\theta) \\ \tilde{a}(Y_t^{\theta},\theta) \end{pmatrix} dt + \begin{pmatrix} \tilde{a}(Y_t^{\theta},\theta) \\ 0_{m-k,r} \end{pmatrix} dW_t. \quad (2.15) \]

We denote $z = (x,y)$ for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{m-n}$, $\nabla_1 = (\partial_{x_1},\ldots,\partial_{x_n})$, $\nabla_2 = (\partial_{x_{n+1}},\ldots,\partial_{x_m})$, $\text{Ker}(A) = \{x \in \mathbb{R}^l : Ax = 0\}$, and by $A^\top$ the Moore–Penrose inverse for a $k \times l$ matrix $A$.

Assumption (C2). The derivative $\partial_i^j\tilde{b}(z)$ is bounded for $i \in \mathbb{N}$ and

\[ \sup_{z \in \mathbb{R}^m} \|((\nabla_1^{\top}\nabla_1)^{-1}(z))\|_{\text{op}} < \infty. \]

Moreover,

\[ \text{Ker}(\tilde{a}(z,\theta)) \subset \text{Ker}(\partial_{\theta_i}\tilde{a}(z,\theta)), \quad \text{Ker}((\nabla_1^{\top}\nabla_1)^{-1}(z)) \subset \text{Ker}((\nabla_1^{\top}\nabla_1)^{-1}(z)\partial_{\theta_0}\tilde{a}(z',\theta)) \quad (2.16) \]

for any $z, z' \in \mathbb{R}^m$, $1 \leq i \leq d$, and $\theta \in \Theta$. Furthermore, at least one of the following two conditions is satisfied;

1. $\tilde{b}$ is bounded;
2. $\nabla_2\tilde{a}(z,\theta) = 0$ and $\nabla_2\tilde{b}(z,\theta) = 0$ for any $z \in \mathbb{R}^m$ and $\theta \in \Theta$.

We need (C2) to satisfy $\partial_{\theta_i}F_{n,\theta,j} = B_{j,i,\theta}F_{n,\theta,j}$ in (B3). See Section [B] in the appendix for the details.

We can write $\tilde{a}^\top = (\tilde{a}\tilde{a}^\top)^{-1}$ because $\tilde{a}\tilde{a}^\top$ is invertible. If $r = 1$ and (C1) is satisfied, then we can easily check $\text{Ker}(\tilde{a}(z,\theta)) \subset \text{Ker}(\partial_{\theta_0}\tilde{a}(z,\theta))$ because $\tilde{a}(z,\theta)$ is invertible. Similarly, we can easily check $\text{Ker}((\nabla_1^{\top}\nabla_1)^{-1}(z)) \subset \text{Ker}((\nabla_1^{\top}\nabla_1)^{-1}(z)\partial_{\theta_0}\tilde{a}(z',\theta))$ if $m - \kappa = 1$ and $(\nabla_1^{\top}\nabla_1)^{-1}\nabla_1\tilde{b}(z)$ is positive definite.

Let $\Psi_{t,\theta} = (\nabla_1^{\top}\tilde{a}\tilde{a}^{\top}\nabla_1)B(0,X_t,\theta)$, and let

\[ \Gamma = \left( \frac{1}{2} \int_0^1 \text{tr}(\nabla_1^{\top}\tilde{a}\tilde{a}^{\top}\nabla_1^{\top}\partial_{\theta_i}(\tilde{a}\tilde{a}^{\top})\partial_{\theta_j}(\tilde{a}\tilde{a}^{\top})(X_t,\theta_0))dt \right. \]

\[ \left. + \frac{1}{2} \int_0^1 \text{tr}(\nabla_1^{\top}\partial_{\theta_i}\Psi_{t,\theta}\partial_{\theta_j}\Psi_{t,\theta}\Psi_{t,\theta_0}^{-1}\partial_{\theta_i}\Psi_{t,\theta_0})dt \right)_{1 \leq i,j \leq d}. \quad (2.17) \]

Assumption (C3). $\Gamma$ is positive definite almost surely.

Theorem 2.4. Assume (C1)–(C3). Then $\{P_{\theta,n}\}_{\theta \in \Theta}$ satisfies the LAMN property at $\theta = \theta_0$ with $\Gamma$.

Remark 2.4. The proof of Theorem 2.4 in Section [A] shows that we obtain similar results when $\kappa = m$ and $aa^\top$ is positive definite by ignoring $b$ and $\Psi_{t,\theta}$. This approach allows another proof of the LAMN property for nondegenerate diffusion processes by Gobet [5].

Remark 2.5. The first term in the right-hand side of (2.17) is equal to $\Gamma$ in Gobet [5] for nondegenerate diffusion processes. It is also the same as $\Gamma$ for the statistical model with observations $\{Y_{j,n}\}_{0 \leq j \leq n, 1 \leq i \leq n}$. Then, the second term in the right-hand side of (2.17) corresponds to additional information obtained by observation $\{\{Y_{j,n}\}_{0 \leq j \leq n, k+1 \leq i \leq m}$. For the degenerate process.
Example 2.1. Let $\kappa \in \mathbb{N}$. Let $X_t^\theta$ and $\bar{X}_t^\theta$ be a $\kappa$-dimensional diffusion process satisfying

$$
\begin{align*}
dX_t^\theta &= d(X_t^\theta, \bar{X}_t^\theta, \theta)dt + c(X_t^\theta, \theta)dW_t, \\
d\bar{X}_t^\theta &= X_t^\theta dt,
\end{align*}
$$

where $\theta \in \Theta \subset \mathbb{R}^d$ and $W_t$ is a $\kappa$-dimensional standard Wiener process. We assume that $cc^\top(x, \theta)$ is positive definite and $c(x, \theta)$ and $d(z, \theta)$ are smooth functions with bounded derivatives $\partial_x c$ and $\partial_z d$. Then, (C1) and (C2) are satisfied with $U = I_{2\kappa}$. $\Gamma$ is given by

$$
\Gamma = \left( \int_0^1 \text{tr}((cc^\top)^{-1}\partial_\theta(cc^\top)^{-1}\partial_\theta(cc^\top))(X_t^{\theta_0}, \theta_0)dt \right)_{1 \leq i, j \leq \kappa}.
$$

If further $\Gamma$ is positive definite almost surely, then we obtain the LAMN property of this model by Theorem 2.4.

Example 2.2. Let $\kappa' \leq \kappa$. Let $X_i^\theta$ be the same as in Example 2.1, and let $\bar{X}_i^\theta$ be a $\kappa'$-dimensional stochastic process satisfying $[\bar{X}_i^\theta]_{t} = \int_0^t[\bar{X}_i^\theta]_s ds$ for $1 \leq i \leq \kappa'$. Moreover, let $c(x, \theta) = f(x, \theta)A$ for some $\mathbb{R}$-valued function $f$ and matrix $A$ independent of $x$ and $\theta$. We assume that $\Lambda A^\top$ is positive definite, $f$ is positive-valued, and $f(x, \theta)$ and $d(z, \theta)$ are smooth functions with bounded derivatives $\partial_x f$ and $\partial_z d$. Then, (C1) and (C2) are satisfied because $(\nabla^2 f)^\top = (I_{\kappa'}O_{\kappa'-\kappa})$ and $\partial_\theta cc^{-1}(x, \theta) = \partial_\theta f^{-1}(x, \theta)\partial_\theta f$. We have $\Psi_{i, \theta} = f^2(X_i^\theta, \theta)([\Lambda A^\top]_{ij})_{1 \leq i, j \leq \kappa'}$, and hence we have

$$
\begin{align*}
[\Gamma]_{ij} &= \frac{1}{2} \int_0^1 \left\{ \frac{2\partial_\theta f}{f} (X_t, \theta_0) \cdot \kappa + \frac{2\partial_\theta f}{f} (X_t, \theta_0) \cdot \kappa' \right\} dt \\
&= 2(\kappa + \kappa') \int_0^1 \frac{\partial_\theta f}{f_2} \partial_\theta f (X_t, \theta_0) dt.
\end{align*}
$$

If we only observe $(X_{k/n}^\theta)_{k=0}^n$, then $\Gamma$ in Gobet [5] is calculated as

$$
\Gamma = \left( 2\kappa \int_0^1 \frac{\partial_\theta f}{f_2} \partial_\theta f (X_t, \theta_0) dt \right)_{1 \leq i, j \leq \kappa'}.
$$

Therefore, we conclude that $\Gamma$ for observations $X_t^\theta$ and $\bar{X}_t^\theta$ is $(\kappa + \kappa')/\kappa$ times as much as the one for observations $X_t^\theta$.

Example 2.3. Let $X_t^\theta = (X_t^{\theta, 1}, X_t^\theta, \bar{X}_t^\theta, 2)$ be a two-dimensional diffusion process satisfying

$$
\begin{align*}
\begin{cases}
\frac{dX_t^{\theta, 1}}{dt} &= (d(X_t^{\theta, 1}, X_t^{\theta, 2}, \theta) + c(X_t^{\theta, 1}, X_t^{\theta, 2}, \theta))dt + c(X_t^{\theta, 1}, X_t^{\theta, 2}, \theta)dW_t, \\
\frac{dX_t^{\theta, 2}}{dt} &= (d(X_t^{\theta, 1}, X_t^{\theta, 2}, \theta))dt + c(X_t^{\theta, 1}, X_t^{\theta, 2}, \theta)dW_t,
\end{cases}
\end{align*}
$$

where $\theta \in \Theta \subset \mathbb{R}^d$ and $W_t$ is a one-dimensional standard Wiener process. That is, the diffusion coefficients of $X_t^{\theta, 1}$ and $X_t^{\theta, 2}$ are the same. We assume that $c$ is positive-valued, $\sup_{x, y} |\partial_x c(x, y)| < \infty$, and $c(x, y, \theta)$, $d(x, y, \theta)$, and $c(x, y)$ are smooth functions with bounded derivatives $\partial_x c$, $\partial_z d$, and $\partial^2_x c$ for $i \in \mathbb{N}$ $(z=(x,y))$. Moreover, we assume that at least one of the following two conditions holds true:

1. $c$ is bounded;
2. $c(x, y) = \bar{c}(x + y)$ and $c(x, y, \theta) = \bar{c}(x + y, \theta)$ for some functions $\bar{c}$ and $\bar{c}$.

Then (C1) and (C2) are satisfied with

$$
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
$$

10
and $\Gamma$ is given by (2.19). If further $\Gamma$ is positive definite almost surely, then we obtain the LAMN property of this model by Theorem 2.4.

For the statistical model with observations $(X_{j/n}^\theta)^n_{j=0}$, $\Gamma$ is equal to half of the one in (2.19). The above result shows that the efficient asymptotic variance for estimators does not depend on $\epsilon$ and is equal to just half of the one when we observe $(X_{j/n}^\theta)^n_{j=0}$.

Example 2.4. Let $m/2 \leq \kappa < m$. Let $X^\theta_t$ be an $m$-dimensional diffusion process satisfying
\begin{equation}
 dX^\theta_t = (X^\theta_t)dt + f(X^\theta_t, \theta)AdW_t, \tag{2.21}
\end{equation}
where $W$ is a $\kappa$-dimensional standard Wiener process, $f(z, \theta)$ is an $\mathbb{R}$-valued function, and $A$ is an $m \times \kappa$ matrix independent of $z$ and $\theta$. Let $U^T(\Lambda O_{\kappa, \kappa})^TV$ be the singular value decomposition of $A$ for a $\kappa \times \kappa$ diagonal matrix $\Lambda$, and orthogonal matrices $U$ and $V$ of size $m$ and $\kappa$, respectively. Then we have
\begin{equation}
 Uf(z, \theta)A = \left( \begin{array}{c} \tilde{c}(Uz, \theta) \\ O_{m-\kappa, \kappa} \end{array} \right), \quad U\varepsilon(z) = \left( \begin{array}{c} \tilde{\varepsilon}(Uz) \\ \varepsilon(Uz) \end{array} \right),
\end{equation}
where $\tilde{c}(z, \theta) = f(U^Tz, \theta)AV$, and $\tilde{\varepsilon}(z)$ and $\varepsilon(z)$ are suitable functions.

We assume that rank($A$) = $\kappa$ (that is, $\Lambda$ is invertible), $f$ is positive-valued, $f(z, \theta)$ and $\varepsilon(z)$ are smooth functions, and $\partial_z f, \partial^2_z \varepsilon$, and $\|((\nabla_1 \varepsilon)^T \nabla_1 \varepsilon)^{-1}\|_{op}$ are bounded for $i \in \mathbb{Z}_+$. Then we obtain $\partial_\theta \tilde{c}^{-1}(z, \theta) = \partial_\theta f^{-1}(U^Tz, \theta)1_{\kappa}$, and consequently (C1) and (C2) hold. Moreover, we have
\begin{equation}
 \Psi_{t, \theta} = f^2(X^\theta_0, \theta)((\nabla_1 \varepsilon)^T \Lambda^2 \nabla_1 \varepsilon)(UX^\theta_t)
\end{equation}
and hence
\begin{equation}
 \Gamma = \left( 2m \int_0^1 \frac{\partial_\theta f \partial_\theta f}{f^2} - (X^\theta_t, \theta_0) dt \right)_{1 \leq i,j \leq d}.
\end{equation}
If $\Gamma$ is positive definite almost surely, then we have the LAMN property of this model.

We can regard (2.21) as a multi-factor model for stock prices, where each component of $W$ is regarded as a factor that influences the stock prices, $A$ comprises the contributions of each factor to each stock, and $f$ is a scalar that depends on the stock prices. The above results show that we obtain the LAMN property of such a degenerate model if the number $\kappa$ of factors is in $[m/2, m)$.

### 2.4 The LAMN property for partial observations

In this section, we show the LAMN property for degenerate diffusion processes with partial observations. Gloter and Gobet [3] showed the LAMN property for a one-dimensional integrated diffusion process. While this model is similar to the one in Example 2.1, the observations are only the integrated process $\hat{X}^\theta_t$ (partial observations). Their proof depends on Aronson’s estimate, which is difficult to obtain for the multi-dimensional process. We can avoid the Aronson-type estimate by using the scheme with the $L^2$ regularity condition in Sections 2.1 and 2.2 and can consequently extend their results to the multi-dimensional case. We can also generalize the observed components of $X^\theta_t$ and $\hat{X}^\theta_t$, which yields an interesting example of a stock process and integrated volatility observations (Example 2.6).

Let $m \in \mathbb{N}$, and let $(\Omega, \mathcal{F}, P)$, $W$, $D$, $H$, and $\Theta$ be the same as in Section 2.3. We consider a process $Y^\theta_t = (Y^\theta_t, Y^\theta_t)$ that satisfies a slight restricted version of the stochastic differential equation (2.15): $(Y^\theta_0, Y^\theta_0) = (\bar{z}_{ini}, \bar{z}_{ini})$, and
\begin{equation}
 d\tilde{Y}^\theta_t = b(\tilde{Y}^\theta_t, \tilde{Y}^\theta_t, \theta)dt + a(\tilde{Y}^\theta_t, \theta)dW_t, \tag{2.22}
\end{equation}
where $B$ is an $(m - \kappa) \times \kappa$ matrix such that $BB^T$ is positive definite. Let $Q : \mathbb{R}^\kappa \rightarrow \mathbb{R}^\kappa$ be a projection. We assume that $(Q\tilde{Y}^\theta_k)^n_{k=0}$ and $(Y^\theta_k)^n_{k=0}$ are observed. Let $q_1 = \text{rank}(Q)$, $q_2 = m - \kappa$, and let $q = q_1 + q_2$. We assume that $0 \leq q_1 < \kappa$.

For a $k \times l$ matrix $A$, we denote $\text{Im}(A) = \{Ax : x \in \mathbb{R}^l\}$.
There exists an Assumption (C5).

Assumption (C4).

\[ \ker(B) \subset \text{im}(Q), \]  
(2.23)

and

\[ Q\partial_\theta \tilde{a}^\top(x,\theta) = \partial_\theta \tilde{a}^\top(x,\theta)Q \]  
(2.24)

for any \( 1 \leq i \leq d, x \in \mathbb{R}^\kappa, \) and \( \theta \in \Theta. \)

By (2.23), we have

\[ q_1 = \dim \text{im}(Q) \geq \dim \ker(B) = \kappa - \dim \ker(B) = \kappa - q_2, \]  
(2.25)

which implies \( q \geq \kappa. \)

Let \( R_1 : \text{im}(Q) \rightarrow \mathbb{R}^\kappa \) and \( R_2 : \text{im}(I_\kappa - Q) \rightarrow \mathbb{R}^{\kappa - q_1} \) be any isomorphism on vector spaces. We denote \( \tilde{Q}_1 = R_1Q, \tilde{Q}_2 = B, \) and \( \tilde{Q}_3 = R_3(I_\kappa - Q). \) For a \( k \times k \) matrix \( A, \) we denote \( \Upsilon_{i,j}(A) = \tilde{Q}_i A \tilde{Q}_j \) for \( 1 \leq i, j \leq 3, \)

\[ \Xi_1(A) = \begin{pmatrix} \Upsilon_{1,1} & \Upsilon_{1,2}/2 & \Upsilon_{1,3}/2 \\ \Upsilon_{2,1}/2 & \Upsilon_{2,2}/3 & \Upsilon_{2,3}/3 \end{pmatrix} (A), \]

\[ \Xi_3(A) = \begin{pmatrix} \Upsilon_{3,1} & \Upsilon_{3,2}/2 & \Upsilon_{3,3}/2 \\ \Upsilon_{3,2}/2 & \Upsilon_{3,3}/3 & \Upsilon_{3,4}/3 \end{pmatrix} (A), \]

and for \( L \in \mathbb{N}, L \geq 3 \) and \( 1 \leq k, l \leq 2, \) we define an \((Lq + (\kappa - q_1)(k - 1)) \times (Lq + (\kappa - q_1)(l - 1))\) matrix \( \psi_{L}^{k,l}(A) \) by

\[ \psi_{L}^{1,1}(A) = \begin{pmatrix} \Xi_1 & \Xi_2 & O_{q,q} & \cdots & O_{q,q} \\ \Xi_2 & \Xi_3 & \cdots & \cdots & \vdots \\ O_{q,q} & \cdots & \cdots & \cdots & O_{q,q} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ O_{q,q} & \cdots & \cdots & \cdots & \Xi_2 \end{pmatrix} (A), \]

\[ \psi_{L}^{1,2}(A) = \begin{pmatrix} \psi_{L}^{1,1} & O_{(L-1)q,\kappa-q_1} \\ \Upsilon_{1,3}/2 & \Upsilon_{2,3}/6 \end{pmatrix} (A), \]

\[ \psi_{L}^{2,1}(A) = \begin{pmatrix} \psi_{L}^{1,2} & \psi_{L}^{1,1} \\ \Upsilon_{3,1}/2 & \Upsilon_{3,2}/6 \end{pmatrix} (A). \]

Here we ignore \( \Upsilon_{i,j} \) if \( \text{rank}(\tilde{Q}_i) = 0 \) or \( \text{rank}(\tilde{Q}_j) = 0. \) Let

\[ T_{k,l,L}(x) = \left( \text{tr}(\partial_\theta \tilde{a}^{k,k}((\tilde{a}a^\top)^{-1})(x,\theta_0)\psi_{L}^{k,l}((\tilde{a}a^\top)^{-1})(x,\theta_0) \right. \]
\[ \times \partial_\theta \tilde{a}^{k,k}((\tilde{a}a^\top)^{-1})(x,\theta_0)\psi_{L}^{k,l}((\tilde{a}a^\top)^{-1})(x,\theta_0)) \right)_{1 \leq i,j \leq d}. \]

To show the LAMN property of partial observations, we consider an augmented model generated by block observations with some observations of \((I_\kappa - Q)\hat{Y}, \) following the idea of Gloter and Gobet. The matrix \( \psi_{L}^{k,l} \) corresponds to the covariance matrix of the block observations. See Sections 2.2 and 3.3 for the details.

Assumption (C5). There exists an \( \mathbb{R}^d \otimes \mathbb{R}^d \)-valued continuous function \( g(x) \) such that

\[ L^{-1}T_{k,l,L}(x) \rightarrow g(x) \]  
(2.26)

as \( L \rightarrow \infty \) uniformly in \( x \) on compact sets for \( 1 \leq k, l \leq 2. \)

Let

\[ \Gamma' = \frac{1}{2} \int_0^1 g(Y_t)dt. \]  
(2.27)
Assumption (C6). $\Gamma'$ is positive definite almost surely.

Let $P_{\theta,n}$ be the distribution of partial observations $(QY_{k/n}^\theta)^n_{k=0}$ and $(V_{k/n}^\theta)^n_{k=0}$.

Theorem 2.5. Assume (C2) and (C4)–(C6). Then $\{P_{\theta,n}\}_{\theta \in \Gamma'}$ satisfies the LAMN property at $\theta = \theta_0$ with $\Gamma'$.

Example 2.5 (Integral observations). Let $X_\theta^k$ and $X_\theta^l$ be the same as in Example 2.4. We consider a statistical model with observations $(X_{k/n}^\theta)^n_{k=0}$. In this case, we have $m = 2\kappa$, $B = I_{\kappa}$, $Q = O_{\kappa,\kappa}$. We assume that $cc^\top(x,\theta)$ is positive definite and that $c(x,\theta)$ and $d(z,\theta)$ are smooth functions with bounded derivatives $\partial_x c$ and $\partial_x d$. As in Example 2.4, we have (C2′). Moreover, we can check (C4).

We can see that $\psi_{k,l}^{1,2}(A) = V_L \otimes A$, where $\otimes$ denotes the Kronecker product, and $V_L$ is an $(L+1) \times (L+1)$ matrix satisfying

$$[V_L]_{ij} = (2/3)1_{\{i=j\}} + (1/6)1_{\{i=j \}} - (1/3)1_{\{i=j \} \text{ and } i \in \{1,L+1\}}.$$  

(2.28)

Because we obtain similar equations for $\psi_{k,l}^{1,1}(A)$ and $\psi_{k,l}^{1,2}(A)$, together with Lemma F.1, we have (2.26) for

$$g(x) = (\text{tr}((cc^\top)^{-1}\partial_\theta (cc^\top)(cc^\top)^{-1}\partial_\theta (cc^\top))(x,\theta_0))_{i \leq j \leq d}.$$  

(2.29)

Therefore, we have the LAMN property of this model if $\Gamma'$ is positive definite almost surely.

This result is an extension of Gloter and Gobet [3] to multi-dimensional processes. Moreover, the result can be applied to the Langevin-type molecular dynamics in (1.1) with positional observations.

Remark 2.6. If we observe $(X_{k/n}^\theta)^n_{k=0}$ instead of $(X_{k/n}^\theta)^n_{k=0}$, then Gobet [5] shows the LAMN property for this model with $\Gamma'$ the same as in (2.27) and (2.29). On the other hand, if we observe both $(X_{k/n}^\theta)^n_{k=0}$ and $(X_{k/n}^\theta)^n_{k=0}$, then Example 2.4 shows the LAMN property with $\Gamma'$ twice that in (2.27). Therefore, we can say that the efficient asymptotic variance with observations $(X_{k/n}^\theta)^n_{k=0}$ is half of that with observations $(X_{k/n}^\theta)^n_{k=0}$ or half of that with $(X_{k/n}^\theta)^n_{k=0}$.

Example 2.6 (Observations of a stock process and integrated volatility). Let $W$ be a two-dimensional standard Wiener process, and let $c$ be an $\mathbb{R}^2 \otimes \mathbb{R}^2$-valued function with $c^j$ for $j \in \{1,2\}$. Let $X_i = (X_i^j)_{i=1}^3$ be a four-dimensional process satisfying

$$dX_i^1 = d^1(X_i,\theta)dt + c^1(X_i^1, X_i^2, \theta)dW_i,$$

$$dX_i^2 = d^2(X_i,\theta)dt + c^2(X_i^1, X_i^2, \theta)dW_i,$$

$$dX_i^3 = X_i^1 dt.$$  

(2.30)

We assume that we observe $((X_{k/n}^1, X_{k/n}^3))^n_{k=0}$. In this case, we have $m = 3$, $\kappa = 2$, $r = 2$,

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = (0 1).$$  

(2.31)

We assume that $cc^\top(x,\theta)$ is positive definite for each $(x,\theta)$, and $c(x,\theta)$, $d^1(z,\theta)$, and $d^2(z,\theta)$ are smooth functions with bounded derivatives $\partial_x c$, $\partial_x d^1$, and $\partial_x d^2$. We can check (2.23).

We consider the following two cases.

1. The case where $c(x_1, x_2, \theta) = f(x_1, x_2, \theta)A$ for a matrix $A$ and a positive-valued function $f(x_1, x_2, \theta)$:

We have that $AA^\top$ is positive definite and $\partial_\theta (\bar{a}a^\top) = \partial_\theta (cc^\top)^{-1} = \partial_\theta f^{-1}I_2$. Then (C2′) and (C4) are satisfied. Moreover, we obtain

$$\psi_{k,l}^{\kappa,i}(\partial_\theta (\bar{a}a^\top)) = \frac{2\partial_\theta f}{f} \psi_{k,l}^{\kappa,i}(\bar{a}a^\top).$$  

(2.32)

Together with Lemma F.1, we have (C5) with

$$g(x_1, x_2) = \left( \frac{8\partial_\theta f \partial_\theta f}{f^2} (x_1, x_2, \theta_0) \right)_{i,j}.$$  

Therefore, we have the LAMN property if $\Gamma'$ in (2.27) is positive definite almost surely.
2. The case where \( c(x_1, x_2, \theta) \) is a diagonal matrix for any \((x_1, x_2, \theta)\):
Because \( \partial_\theta \tilde{a} \tilde{a}^T \) also becomes a diagonal matrix, (C2') and (C4) are satisfied. Moreover, we have \( Y_{1,2} = 0 \) and \( Y_{1,3} = 0 \). Then, by rearranging the rows and columns of \( \psi_L^{3,2} \) by using an orthogonal matrix \( V_L \) of size \( 2L + 1 \), we have

\[
V_L \psi_L^{3,2}(M)V_L^T = \begin{pmatrix}
|M|_{11}I_L & O_{L,L} \\
O_{L,L} & [M]_{22}V_L
\end{pmatrix}
\]  

(2.33)

for any diagonal matrix \( M \) of size 2, where \( V_L \) is defined in (2.28). Together with Lemma [Fr] and similar equations for \( \psi_L^{1,2} \) and \( \psi_L^{1,1} \), we have (2.29) with

\[
g(x_1, x_2) = \left( \begin{array}{c}
\frac{4\partial_\theta[c]_{11}(\partial_\theta[c]_{11})}{|c|^2} + \frac{4\partial_\theta[c]_{22}(\partial_\theta[c]_{22})}{|c|^2} \\
(x_1, x_2, \theta_0)
\end{array} \right)_{1 \leq i, j \leq d}.
\]  

(2.34)

Then we have the LAMN property if \( \Gamma' \) in (2.27) is positive definite almost surely.

Remark 2.7. If \( c^1(x_1, x_2, \theta) = \sqrt{x_2}v \) for a unit vector \( v \) in Example [24], then we have \( X_3^t = (X^1)_t \), and therefore \( X^t \) represents a stock process for a stochastic volatility model and \( X^3 \) is the integrated volatility process. If we observe daily stock prices and realized volatility calculated from high-frequency data, then we can regard it as an approximation of the integrated volatility process. Even though \( c^1 = \sqrt{x_2} \) does not satisfy our assumptions because \( \partial_\theta c \) is not bounded, we can approximate this function by setting \( c^1 \) as a positive-valued smooth function satisfying \( c^1(x_1, x_2, \theta) = \sqrt{x_2} \) on \( \{|x_2| \geq \epsilon\} \) for small \( \epsilon > 0 \).

3 Malliavin calculus and the \( L^2 \) regularity condition

In this section, we show how to check (A1)–(A5) in Section (2.1) under (B1)–(B5). The equations for density derivatives in Proposition (3.1) are crucial for the proof. From these equations, we obtain Proposition (3.2) and Lemma (3.3) which are necessary for checking (A1).

For a matrix \( A \), \( |A| \) denotes the Frobenius norm, \( |A| = \sqrt{\sum_{i,j} ||A||^2} \). Let

\[
L^\theta(V) = \sum_{k,k'} [K^{-1}(\theta)]_{kk'} D_j[F_{\theta,j}]_{kk'}[V]_{kk'}
\]

for a vector \( V \in \mathbb{R}^{k_j-k_{j-1}} \).

The following proposition is essentially from Proposition 4.1 in [3] and Theorem 5 in [3]. To check (A1), we need an equation for \( \partial_\theta^3 p_{\theta_j} \). The proof is left to Section [D] in the appendix.

Proposition 3.1. Assume (B1) and (B2). Then \( F_{\theta,j} \) admits a density denoted by \( p_{\theta_j}(x_j, \theta) \). Moreover, \( p_{\theta_{j-1}}(x_{j-1}) \in C^2(\theta) \),

\[
\partial_\theta p_{\theta_j}(x_j, \theta) = p_{\theta_j}(x_j, \theta) E_j \left[ \delta_j(L^\theta(\partial_\theta F_{\theta,j}))|F_{\theta,j} = x_j \right],
\]

(3.1)

and

\[
\partial_\theta^3 p_{\theta_j}(x_j, \theta) = p_{\theta_j}(x_j, \theta) E_j \left[ \delta_j(L^\theta(\partial_\theta^3 F_{\theta,j})) + \delta_j(L^\theta(\mathfrak{A}_j))|F_{\theta,j} = x_j \right]
\]

(3.2)

almost everywhere in \( x_j \in \mathbb{R}^{k_j-k_{j-1}} \), where \( \mathfrak{A}_j = (\delta_j(L^\theta(\partial_\theta F_{\theta,j} \partial_\theta F_{\theta,j}^2(k)))) \).

The proof of the following proposition is left to Section [D] in the appendix.

Proposition 3.2. Assume (B1) and (B2). Then

\[
\sup_{i,j,x_{j-1},\theta} E_j \left[ |\partial_\theta p_{\theta_{j-1}}(x_{j-1})|^4 \right] \leq C_0 k_{n}^2,
\]

(3.3)

and

\[
\sup_{i,j,x_{j-1},\theta} E_j \left[ |\partial_\theta^2 p_{\theta_{j-1}}(x_{j-1})|^2 \right] \leq C_0 k_{n}^2.
\]

(3.4)
Let \( \theta_h = \theta_0 + \epsilon_n h \) for \( h \in \mathbb{R}^d \),
\[
\mathcal{E}_j^1(x_j, \theta) = \mathcal{E}_j^1(x_j, \theta, \bar{x}_{j-1}) = (E_j[\delta_j(L^{\theta}(\theta_0, F_{\theta,j,x_{j-1}}))^d]F_{\theta,j,x_{j-1}} = x_j)_{i=1}^d,
\]
\[
\mathcal{E}_j^2(x_j, \theta) = (E_j[\delta_j(L^{\theta}(\theta_0, \bar{F}_{\theta,j}, F_{\theta,j})) + \delta_j(L^{\theta}(\theta_0))]|F_{\theta,j,x_{j-1}} = x_j]_{i=1}^d.
\]
We set the conditional expectations equal to zero when \( p_j(x_j, \theta) = 0 \). Then \( \mathcal{E}_j^1(x_j, \theta) \) and \( \mathcal{E}_j^2(x_j, \theta) \) are measurable with respect to \( \theta \) almost everywhere in \( x_j \) because
\[
[\mathcal{E}_j^1(x_j, \theta)]_i = (\partial_0 p_j/p_j)1_{\{p_j \neq 0\}} \quad \text{and} \quad [\mathcal{E}_j^2(x_j, \theta)]_i = (\partial_0 \partial_0 p_j/p_j)1_{\{p_j \neq 0\}}.
\]

**Proof of Theorem 2.2**
We check \( \text{A1} \)–\( \text{A3} \) in Theorem 2.1 by setting
\[
p_j(\theta) = p_j(x_j, \theta) = p_{j,x_{j-1}}(x_j, \theta).
\]
For sufficiently large \( n \), we have \( \{\theta_{hi}\}_{i=0}^1 \subset \Theta \),
\[
E_{\theta_0} \left[ \sum_{j=1}^{\infty} \int_{N_j} |\sqrt{p_j(x_j, \theta_h)} - \sqrt{p_j(x_j, \theta_0)|^2 dx_j \right]
\leq 2E_{\theta_0} \left[ \sum_{j=1}^{\infty} \int_{N_j} (p_j(x_j, \theta_h) + p_j(x_j, \theta_0))dx_j \right] \to 0
\]
as \( n \to \infty \) by (N1), and
\[
\int_{N_j} \left\{ \sqrt{p_j(x_j, \theta_h)} - \sqrt{p_j(x_j, \theta_0)} - \frac{\epsilon_n h \cdot \hat{\xi}_j(\theta_0)}{2} \right\}^2 dx_j
\]
\[
= \int_{M_j} \left\{ \int_0^1 \epsilon_n \frac{\partial_0 p_j}{2\sqrt{p_j}}(x_j, \theta_h) dt \cdot h - \epsilon_n \frac{\partial_0 p_j}{2\sqrt{p_j}}(x_j, \theta_0) \cdot h \right\}^2 dx_j
\]
\[
= \int_{M_j} \left\{ \int_0^1 \epsilon_n^2 h^\top \partial_0 \left( \frac{\partial_0 p_j}{2\sqrt{p_j}} \right)(x_j, \theta_h) h ds dt \right\}^2 dx_j
\]
\[
\leq \epsilon_n^4 |h|^4 \int_0^1 E\left[ \frac{\partial_0^2 p_j}{2p_j} - \frac{\partial_0 p_j \partial_0 p_j^\top}{4p_j^2} \right] \| \|_\theta^2 \|_L 1_{\{p_j \neq 0\}}(F_{\theta,n,j}, \theta_h) \right] ds.
\]
Together with Proposition 3.2, we have
\[
\int_{N_j} \left\{ \sqrt{p_j(x_j, \theta_h)} - \sqrt{p_j(x_j, \theta_0)} - \frac{\epsilon_n h \cdot \hat{\xi}_j(\theta_0)}{2} \right\}^2 dx_j \to 0, \quad (3.5)
\]
which implies (A1).

Moreover, we have (A2) because
\[
E_{\theta_0} \left[ \frac{\partial_0 p_j}{p_j} 1_{\{p_j \neq 0\}}(x_j, \theta_0) \right]_{\mathcal{F}_{j-1}} = \int \partial_0 p_j(x_j, \theta_0) dx_j = 0,
\]
where \( \mathcal{F}_{j-1} \) is the one in Section 2.1.

Further, Proposition 3.2 yields (A3). \( \Box \)

In the following, we prove Theorem 2.3 To show (A5), we replace \( \mathcal{E}_j^1(F_{\theta,j}, \theta) \) by \( \mathcal{C}_{j,i \geq 1}(F_{\theta,j})_{i=1}^d \), and then we apply (B4). For that purpose, we first estimate the difference between \( K_j \) and \( \tilde{K}_j \).
Lemma 3.1. Assume (B1)–(B3) and that \( \alpha_n \rho_n k_n^2 \to 0 \). Then, for any \( 1 \leq j \leq m_n \) and \( p > 1 \), \( \mathbf{K}_j(\theta) \) is an invertible matrix almost surely and satisfies

\[
\sup_{i,j,x_{j-1},\theta} \| [K_j(\theta) - \mathbf{K}_j(\theta)]_{i\ell} \|_{2,p} \leq C_p \alpha_n, \quad \sup_{j,x_{j-1},\theta} \| \mathbf{K}_j^{-1}(\theta) \|_{\text{op}} \leq C \alpha_n k_n
\]

for sufficiently large \( n \).

The proof is left to Section \([D]\) in the appendix.

Proposition 3.3. Assume (B1)–(B3) and that \( \alpha_n \rho_n k_n^2 \to 0 \) as \( n \to \infty \). Then there exists a positive constant \( C \) such that

\[
\sup_{i,j,x_{j-1},\theta} \mathbb{E}_j \left[ \frac{\partial \alpha_{i,j,x_{j-1}}}{\partial \theta} 1_{\{p_j \neq 0\}}(F_{n,\theta,j},\theta) - L_{j,i,x_{j-1}}(\bar{F}_{n,\theta,j},\theta) \right]^{1/2} \leq C \alpha_n^2 \rho_n k_n^4
\]

for sufficiently large \( n \).

Proof. For \( V \in (\mathbb{D}^1)^{k_j-k_{j-1}} \), we regard \( D_j V = (D_j[V])_j \) as a vector of size \( k_j-k_{j-1} \). Let \( L_{j,i}^\theta = \partial \alpha_{i,j,x_{j-1}}(\bar{F}_{n,\theta,j}) D_j \bar{F}_{n,\theta,j} \). First, we show that

\[
\sup_{i,j,x_{j-1},\theta} \| L_{j,i}^\theta(F_{n,\theta,j}) - L_{j,i}^\theta \|_{\mathbb{D}^{1-p}(H_j)} \leq C \alpha_n^2 \rho_n k_n^4.
\]

Condition (B3) and Lemma 3.1 yield estimates for

\[
(\partial \alpha_{i,j,x_{j-1}}(\bar{F}_{n,\theta,j}) D_j \bar{F}_{n,\theta,j}) - (\partial \alpha_i(\bar{F}_{n,\theta,j}) D_j \bar{F}_{n,\theta,j}).
\]

Because \( K_j^{-1} - \mathbf{K}_j^{-1} = \mathbf{K}_j^{-1}(\mathbf{K}_j - \mathbf{K}_j) \mathbf{K}_j^{-1} \), we also obtain an estimate for \( \partial \alpha_{i,j,x_{j-1}}(\bar{F}_{n,\theta,j}) (K_j^{-1} - \mathbf{K}_j^{-1}) D_j \bar{F}_{n,\theta,j} \). Then we have \( \| \mathbf{K}_j^{-1} - \mathbf{K}_j^{-1} \| \leq C \alpha_n^2 \rho_n k_n^4 \).

Moreover, Proposition 1.3.3 in Nualart \([15]\) and (B3) yield

\[
\| L_{j,i}^\theta(F_{n,\theta,j}) - L_{j,i}^\theta \|_{\mathbb{D}^{1-p}(H_j)} \leq C \alpha_n^2 \rho_n k_n^4.
\]

Together with Proposition 3.2, we have

\[
\| \mathbf{K}_j^{-1} - \mathbf{K}_j^{-1} \| \leq C \alpha_n^2 \rho_n k_n^4
\]

for any \( p \geq 1 \).

Lemma 3.2. Assume (B1), (B2), and (N2). Then, for any \( n \in \mathbb{N} \), \( 1 \leq j \leq m_n \), and \( h \in \mathbb{R}^d \) satisfying \( \{\theta_{1j}\}_{j=1}^m \subset \Theta \), the function \( \sqrt{p_{j-1,x_{j-1}}(x_j,\theta_{1j})} \) is absolutely continuous on \( t \in [0,1] \) almost everywhere in \( x_j \).

The proof is left to Section \([D]\) in the appendix.

Lemma 3.3. Assume (B1)–(B3), (B5), and (2.17). Then (A1) holds true.

Proof. If (N1) is satisfied, then the proof of Theorem 3.2 implies (A1). Thus, we may assume (N2). We fix \( h \in \mathbb{R}^d \) and consider a sufficiently large \( n \) so that \( \{\theta_{1j}\}_{j=1}^m \subset \Theta \). Thanks to Lemma 3.2, \( \partial h \sqrt{p_{j,t}} \) exists almost everywhere in \( t \in [0,1] \) and \( \sqrt{p_{j-1,t}} - \sqrt{p_{j-1,t}} = \int_0^1 \partial h \sqrt{p_{j,t}} \| dt \) almost everywhere in \( x_j \in \mathbb{R}^{k_j-k_{j-1}} \). Moreover, we can see that \( \partial h \sqrt{p_{j,t}} = \partial p_{j,t}/(2\sqrt{p_{j,t}}) \) when \( p_{j,t} \neq 0 \) by Proposition 3.1.
For $t \in (0, 1)$ such that $\partial_t \sqrt{p_{j,t}}$ exists and $p_{j,t} = 0$, we have

$$\liminf_{s \to t} \frac{\sqrt{p_{j,s}}}{s - t} \geq 0 \quad \text{and} \quad \limsup_{s \to t} \frac{\sqrt{p_{j,s}}}{s - t} \leq 0,$$

which imply $\partial_t \sqrt{p_{j,t}} = 0$. Therefore, we obtain

$$\sqrt{p_{j,1}} - \sqrt{p_{j,0}} = \int_0^1 \frac{\partial_t p_{j,t}}{2 \sqrt{p_{j,t}}} 1_{(p_{j,t} \neq 0)} dt.$$

Then we have

$$\sum_{j=1}^m \int \left( \sqrt{p_{j,1}(x_j)} - \sqrt{p_{j,0}(x_j)} - \frac{\partial_t p_{j,t} - \epsilon_n h}{2 \sqrt{p_j}} 1_{(p_{j,t} \neq 0)}(x_j, \theta_0) \right)^2 dx_j$$

$$= \sum_{j=1}^m \int \left( \int_0^1 \frac{\partial_t p_{j,t} - \epsilon_n h}{2 \sqrt{p_j}} 1_{(p_{j,t} \neq 0)}(x_j, \theta_t) dt - \frac{\partial_t p_{j,t} - \epsilon_n h}{2 \sqrt{p_j}} 1_{(p_{j,t} \neq 0)}(x_j, \theta_0) \right)^2 dx_j$$

$$\leq e_n^2 \sum_{j=1}^m \int_0^1 \int \left| \partial_t p_{j,t} - \frac{\partial_t p_{j,t} - \epsilon_n h}{2 \sqrt{p_j}} 1_{(p_{j,t} \neq 0)}(x_j, \theta_t) dt - \frac{\partial_t p_{j,t} - \epsilon_n h}{2 \sqrt{p_j}} 1_{(p_{j,t} \neq 0)}(x_j, \theta_0) \right|^2 dx_j dt. \quad (3.11)$$

Let

$$\Xi_{j,t} = \frac{\partial_t p_{j,t}}{2 \sqrt{p_j}} 1_{(p_{j,t} \neq 0)}(x_j, \theta_t) - \frac{\sqrt{p_j}}{2} (L_{j,i,j-1}(x_j, \theta_t))^2,$$

then (3.10), Proposition 3.13 and (2.10) yield

$$e_n^2 \sum_{j=1}^m \int_0^1 \left| \Xi_{j,t} - \Xi_{j,0} \right|^2 dx_j dt \leq 2e_n^2 \sup_t \left( \sum_{j=1}^m \int \left( |\Xi_{j,t}|^2 + |\Xi_{j,0}|^2 \right) dx_j \right)$$

$$\leq e_n^2 \sup_t \left( \sum_{j=1}^m \left[ \left( \frac{\partial_t p_{j,t} - \epsilon_n h}{2 \sqrt{p_j}} 1_{(p_{j,t} \neq 0)} - (L_{j,i,j-1})^d \right) (F_{n,\theta,t,j}, \theta_t) \right] \right)$$

$$\leq Ce_n^4 m_n (\alpha_n^4 k_n^8 + \alpha_n^2 k_n^4) \to 0$$

for any $x_{j-1}$.

Moreover, because the function $t \mapsto \sqrt{p_{j,t}}(L_{j,i,j-1}(x_j, \theta_t)$ is absolutely continuous, we have

$$e_n^2 \sum_{j=1}^m \int_0^1 \left( \frac{\sqrt{p_j}}{2} L_{j,i}(x_j, \theta_t) - \frac{\sqrt{p_j}}{2} L_{j,i}(x_j, \theta_0) \right)^2 dx_j dt$$

$$= e_n^2 \sum_{j=1}^m \int_0^1 \int \left( \int_0^t \frac{\sqrt{p_j}}{2} \partial_t L_{j,i} + \frac{\partial_t p_{j,t}}{4 \sqrt{p_j}} L_{j,i} \right) (x_j, \theta_t) \cdot \epsilon_n h ds \right)^2 dx_j dt$$

$$\leq Ce_n^4 m_n \sup_t \left[ \left( \frac{1}{2} \partial_t L_{j,i} + \frac{\partial_t p_{j,t}}{4 \sqrt{p_j}} L_{j,i} \right)^2 (F_{n,\theta,t,j}, \theta_t) \right]$$

for any $1 \leq i \leq d$.

Because $\partial_0 \tilde{K}_j^{-1} = -\tilde{K}_j^{-1} \partial_0 \tilde{K}_j \tilde{K}_j^{-1}$ and

$$\partial_0 \tilde{K}_j = (\partial_0 D_j \tilde{F}_{n,\theta,j}, D_j \tilde{F}_{n,\theta,j}) + (D_j \tilde{F}_{n,\theta,j}, \partial_0 D_j \tilde{F}_{n,\theta,j}) = B_{j,i,\theta} \tilde{K}_j + \tilde{K}_j B_{j,i,\theta},$$

Lemma 3.1 yields

$$E_j \left[ |\partial_0 L_{j,i,j-1}(F_{n,\theta,j}, \theta) |^2 \right]$$

$$\leq CE_j \left[ |F_{n,\theta,j}^T \partial_0 B_{j,i,\theta}^{-1} \tilde{K}_j^{-1} F_{n,\theta,j} |^2 \right]$$

$$+ CE_j \left[ |F_{n,\theta,j}^T B_{j,i,\theta}^{-1} \tilde{K}_j^{-1} B_{j,i,\theta} \tilde{K}_j^{-1} F_{n,\theta,j} |^2 \right]$$

$$\leq C\alpha_n^2 k_n^8$$

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for any $\theta \in \Theta$. Moreover, (3.10) yields
\[
E_j[[L_{j,i,x_{j-1}}(F_{n,\theta,j})]^4]\n\leq CE_j[[L_{j,i,x_{j-1}}(\tilde{F}_{n,\theta,j})]^4] + CE_j[[L_{j,i,x_{j-1}}(\tilde{F}_{n,\theta,j})]^4] + C(\alpha_n \rho_n k_n^3 + \alpha_n k_n^4)\]
for any $\theta \in \Theta$, (3.13) and Proposition 3.2 yield
\[
\epsilon_n \sum_{j=1}^{m_n} \int_0^1 \left( \frac{\sqrt{p_j}}{2} L_{j,i,x_{j-1}}(x_j, \theta_0) - \frac{\sqrt{p_j}}{2} L_{j,i,x_{j-1}}(x_j, \theta_0) \right)^2 dx_j dt \leq C\epsilon_n^4 m_n (\alpha_n^2 k_n^8 + \alpha_n^2 \rho_n^2 k_n^6 + \alpha_n^2 k_n^4).
\]
The right-hand side converges to zero by (B2) and (2.10). Together with (3.11) and (3.12), we obtain the conclusion.

**Proof of Theorem 2.3**

Thanks to Remark 2.1, Lemma 3.3 and the proof of Theorem 2.2, it is sufficient to check (A4) and (A5) under (B1)–(B5). Let $X_j = X_j^{n,\theta_0}$, $\bar{X}_{j-1} = (X_1, \ldots, X_{j-1})$, and
\[
\mathcal{H}_j = E[\mathcal{E}_j^1(\mathcal{E}_j^1)^\top (X_j, \theta_0, \bar{X}_{j-1})] \sigma(\bar{X}_{j-1}).
\]
Then it suffices to show that
\[
\sup_m \left( \epsilon_n \sum_{j=1}^{m_n} E[|\mathcal{H}_j|] \right) < \infty
\]
and
\[
\left( \epsilon_n \sum_{j=1}^{m_n} \mathcal{E}_j^1(X_j, \theta_0, \bar{X}_{j-1}), \epsilon_n \sum_{j=1}^{m_n} \mathcal{H}_j \right) \overset{\Delta}{\to} (\Gamma^{1/2} \mathcal{N}, \Gamma).
\]
For sufficiently large $n$, (2.9) and Proposition 3.3 yield
\[
E[|\mathcal{E}_j^1(X_j, \theta_0, \bar{X}_{j-1}) - G_j|^2] \leq C\epsilon_n^4 m_n \alpha_n \rho_n k_n^8.
\]
Together with (2.10) and (B4), we obtain
\[
E\left[ \epsilon_n \sum_{j=1}^{m_n} \mathcal{E}_j^1(X_j, \theta_0, \bar{X}_{j-1}) - \epsilon_n \sum_{j=1}^{m_n} G_j^0 \right] \leq C\epsilon_n^4 m_n \alpha_n \rho_n k_n^8 \to 0
\]
as $n \to \infty$.

Let $\bar{\mathcal{S}}_j = (L_{j,i,x_{j-1}}(\tilde{F}_{n,\theta,j}))_{i=1}^d$, then we have
\[
\gamma_j(\bar{x}_{j-1}) = E_j[[L_{j,i,x_{j-1}}(\tilde{F}_{n,\theta,j})]L_{j,i,x_{j-1}}(\tilde{F}_{n,\theta,j})]u],
\]
and $\sup_j E_j[|\bar{\mathcal{S}}_j|^2]^{1/2} = O(\alpha_n k_n^2)$ by (3.15). Together with (2.10) and Propositions 3.2 and 3.3, we have
\[
\sup_j E_j[|\mathcal{E}_j^1(\mathcal{E}_j^1)^\top (F_{n,\theta,j}, \theta_0, \bar{x}_{j-1})] - \gamma_j(\bar{x}_{j-1})] \leq C\sup_j \left( \epsilon_n \sum_{j=1}^{m_n} \mathcal{E}_j^1(\mathcal{E}_j^1)^\top (F_{n,\theta,j}, \theta_0, \bar{x}_{j-1}) - \bar{\mathcal{S}}_j(\bar{x}_{j-1}) \right) \leq O(\alpha_n k_n^2, \alpha_n^2 \rho_n k_n^4 + \alpha_n^2 \rho_n k_n^3 + \alpha_n \rho_n k_n^4) = o(\epsilon_n^2 m_n^{-1}).
\]
Then, (3.18), (3.20), and (B4) yield
\[
\sup_n \left( \epsilon_n^2 \sum_{j=1}^{m_n} E[|\mathcal{H}_j|] \right) = \sup_n \left( \epsilon_n^2 \sum_{j=1}^{m_n} E[|\gamma_j(\bar{X}_{j-1})|] + O(1) \right) < \infty
\]
and
\[
(\epsilon_n^m \sum_{j=1}^{m_n} E_j^j(X_j, \theta_0, \bar{X}_{j-1}), \epsilon_n^m \sum_{j=1}^{m_n} \mathcal{H}_j) \\
= (\epsilon_n^m \sum_{j=1}^{m_n} G_j^n, \epsilon_n^m \sum_{j=1}^{m_n} \gamma_j(\bar{X}_{j-1})) + o_p(1) \overset{d}{\rightarrow} (\Gamma^{1/2}, \Gamma).
\]

References

[1] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[2] V. Genon-Catalot and J. Jacod. On the estimation of the diffusion coefficient for multi-dimensional diffusion processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 29(1):119–151, 1993.

[3] A. Gloter and E. Gobet. LAMN property for hidden processes: the case of integrated diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.*, 44(1):104–128, 2008.

[4] A. Gloter and J. Jacod. Diffusions with measurement errors. I. Local asymptotic normality. *ESAIM Probab. Statist.*, 5:225–242 (electronic), 2001.

[5] E. Gobet. Local asymptotic mixed normality property for elliptic diffusion: a Malliavin calculus approach. *Bernoulli*, 7(6):899–912, 2001.

[6] E. Gobet. LAN property for ergodic diffusions with discrete observations. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(5):711–737, 2002.

[7] J. Hájek. A characterization of limiting distributions of regular estimates. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 14:323–330, 1969/70.

[8] J. Hájek. Local asymptotic minimax and admissibility in estimation. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. I: Theory of statistics*, pages 175–194, 1972.

[9] P. Hall. Martingale invariance principles. *Ann. Probability*, 5(6):875–887, 1977.

[10] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, second edition, 2013.

[11] J. Jacod. *On continuous conditional Gaussian martingales and stable convergence in law*, volume 1655 of *Lecture Notes in Math*. Springer, Berlin, 1997.

[12] P. Jeganathan. On the asymptotic theory of estimation when the limit of the log-likelihood ratios is mixed normal. *Sankhyā Ser. A.*, 44(2):173–212, 1982.

[13] P. Jeganathan. Some asymptotic properties of risk functions when the limit of the experiment is mixed normal. *Sankhyā Ser. A.*, 45(1):66–87, 1983.

[14] L. Le Cam. *Asymptotic methods in statistical decision theory*. Springer Series in Statistics. Springer-Verlag, New York, 1986.

[15] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[16] T. Ogihara. Local asymptotic mixed normality property for nonsynchronously observed diffusion processes. *Bernoulli*, 21(4):2024–2072, 2015.

[17] T. Ogihara. Parametric inference for nonsynchronously observed diffusion processes in the presence of market microstructure noise. *Bernoulli*, 24(4B):3318–3383, 2018.

[18] P. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1990. A new approach.
A Proof of Theorem 2.1

We use a similar approach to that for Theorem 1 in Jeganathan [12]. First, we show that

$$\sum_{i=1}^{m_n}(h^\top r_n \eta_i \eta_i^\top r_n h - h^\top T_n h) \to 0 \quad \text{(A.1)}$$

in $P_{\theta_0,n}$-probability, which corresponds to Lemma 1 in [12].

Let $X_{n,i} = |h^\top r_n \eta_i|$. We denote $P_{\theta_0,n}$ by $P$, $E_{\theta_0}$ by $E$, and $E_{\theta_0}[\cdot |F_{j-1}]$ by $E_{(j)}$. By (A4), for any $\eta > 0$ there exists $K > 0$ such that

$$\sup_n P\left[\sum_{i=1}^{m_n}E_{(i)}[X_{n,i}^2] > K\right] < \eta/4. \quad \text{(A.2)}$$

Let $\delta > 0$ and then take $\epsilon > 0$ with $16\epsilon^2(K + \epsilon^2)/\delta^2 < \eta$. Let $W_{n,i} = X_{n,i}1_{\{X_{n,i} \leq \epsilon, \sum_{j=1}^{m_n}E_{(j)}[X_{n,j}^2] \leq K\}}$. For sufficiently large $n$, we obtain

$$P[X_{n,i} \neq W_{n,i} \text{ for some } i] \leq \sum_{i=1}^{m_n}P[|X_{n,i}| > \epsilon] + P\left[\sum_{i=1}^{m_n}E_{(i)}[X_{n,i}^2] > K\right] < \eta/2.$$

Moreover, (A3) yields

$$\sum_{i=1}^{m_n}E_{(i)}[X_{n,i}^2 - W_{n,i}^2]$$

$$\leq \sum_{i=1}^{m_n}E_{(i)}[X_{n,i}^2 1_{\{|X_{n,i}| > \epsilon\}}] + \sum_{i=1}^{m_n}E_{(i)}[X_{n,i}^2 1_{\{\sum_{j=1}^{m_n}E_{(j)}[X_{n,j}^2] \leq K\}}]$$

$$\leq \delta/2 + \sum_{i=1}^{m_n}E_{(i)}[X_{n,i}^2 1_{\{\sum_{j=1}^{m_n}E_{(j)}[X_{n,j}^2] > K\}},$$

and so by (A.2) we have

$$P\left[\sum_{i=1}^{m_n}E_{(i)}[X_{n,i}^2 - W_{n,i}^2] > \delta/2\right] < \eta/4.$$

Further,

$$E\left[\sum_{i=1}^{m_n}(W_{n,i}^2 - E_{(i)}[W_{n,i}^2])^2\right] = \sum_{i=1}^{m_n}E[W_{n,i}^4 - E_{(i)}[W_{n,i}^2]^2]$$

$$\leq \epsilon^2 E\left[\sum_{i=1}^{m_n}E_{(i)}[W_{n,i}^2]\right] \leq \epsilon^2(K + \epsilon^2).$$
Therefore, we obtain

\[ P[|\sum_{i=1}^{m_n} (X_{n,i}^2 - E_{(i)}[X_{n,i}^2])| > \delta] \]

\[ \leq P[X_{n,i} \neq W_{n,i} \text{ for some } i] + P[\sum_{i=1}^{m_n} E_{(i)}[X_{n,i}^2 - W_{n,i}^2] > \delta/2] \]

\[ + P[|\sum_{i=1}^{m_n} (W_{n,i} - E_{(i)}[W_{n,i}^2])| > \delta/2] \]

\[ \leq 3\eta/4 + 4/\delta^2 E[c|\sum_{i=1}^{m_n} (W_{n,i}^2 - E_{(i)}[W_{n,i}^2])|^2] < \eta. \]

Because \( \eta, \delta > 0 \) are arbitrary, (A.1) holds true.

(A5) corresponds to Lemma 2 in [12]. Moreover, by setting \( \hat{\eta}_{nj}(\theta_0, h) = (p_j(\theta_0 + r_n h)^{1/2} p_j(\theta_0)^{-1/2} - 1)^1_{\{p_j(\theta_0) \neq 0\}} \), we obtain the following similarly to Lemmas 5–7 in [12]:

\[ \left| \sum_{j=1}^{m_n} \hat{\eta}_{nj}^2(\theta_0, h) - \frac{1}{4} h^T T_n h \right| \rightarrow 0, \]

(A.3)

\[ \max_{1 \leq j \leq m_n} |\hat{\eta}_{nj}(\theta_0, h)| \rightarrow 0, \]

(A.4)

\[ \sum_{j=1}^{m_n} |\hat{\eta}_{nj}(\theta_0, h)|^3 \rightarrow 0, \]

(A.5)

\[ \left| 2 \sum_{j=1}^{m_n} \hat{\eta}_{nj}(\theta_0, h) - \hat{h}^T r_n \sum_{j=1}^{m_n} \eta_j + \frac{1}{4} h^T T_n h \right| \rightarrow 0, \]

(A.6)

in \( P_{\theta, n} \)-probability.

For any \( h \in \mathbb{R}^d \), (A.4) and Taylor’s formula yield

\[ \log \frac{dP_{\theta_0 + r_n h, n}}{dP_{\theta_0, n}} = 2 \sum_{j=1}^{m_n} \log(1 + \hat{\eta}_{nj}(\theta_0, h)) \]

\[ = 2 \sum_{j=1}^{m_n} \hat{\eta}_{nj}(\theta_0, h) - \sum_{j=1}^{m_n} \hat{\eta}_{nj}^2(\theta_0, h) + \sum_{j=1}^{m_n} \alpha_{nj} |\hat{\eta}_{nj}(\theta_0, h)|^3 \]

with probability tending to one, where \( |\alpha_{nj}| \leq 1 \).

Together with (A.5), we have

\[ \left| \log \frac{dP_{\theta_0 + r_n h, n}}{dP_{\theta_0, n}} - 2 \sum_{j=1}^{m_n} \hat{\eta}_{nj}(\theta_0, h) + \sum_{j=1}^{m_n} \hat{\eta}_{nj}^2(\theta_0, h) \right| \rightarrow 0 \]

in \( P_{\theta, n} \)-probability. Therefore, (A.3), (A.6), and (A5) yield Condition (L).

If further (P) is satisfied, then the LAMN property holds by Remark [2.1].

\[ \square \]

B Degenerate diffusion models

In this section, we prove Theorem 2.4.

We set \( X_j^{n, \theta} \) in Section 2.2 as

\[ X_j^{n, \theta} = \left( \begin{array}{cc} \sqrt{n} & 0 \\ 0 & n \sqrt{n} \end{array} \right) \left( Y_j^{n, \theta} - Y_j^{n, \theta}_{(j-1)/n} \right) - \left( \begin{array}{c} b(Y_{(j-1)/n})^0 \\ \dot{b}(Y_{(j-1)/n})^0 \end{array} \right). \]
Let $W = (W_t)_{t \geq 0}$ be an $r$-dimensional standard Wiener process on a canonical probability space, and let $X_t^{n,\theta} = \mathcal{X}_t^{n,\theta,z_0}$ and $Y_t^{n,\theta} = \mathcal{Y}_t^{n,\theta,z_0}$ be $\kappa$- and $(m - \kappa)$-dimensional diffusion processes, respectively, satisfying $(X_0^{n,\theta}, Y_0^{n,\theta}) = 0$, and
\[
\begin{align*}
\frac{dX_t^{n,\theta}}{dt} &= \tilde{b}_n(t, X_t^{n,\theta}, Y_t^{n,\theta}, \theta)dt + \tilde{a}_n(t, X_t^{n,\theta}, Y_t^{n,\theta}, \theta)dW_t, \\
\frac{dY_t^{n,\theta}}{dt} &= \tilde{b}_n(t, X_t^{n,\theta}, Y_t^{n,\theta})dt,
\end{align*}
\]
where $z_0 = (x_0, y_0) \in \mathbb{R}^m$ and
\[
\begin{align*}
\tilde{a}_n(t, x, y, \theta) &= \tilde{a}(x_0 + n^{-1/2}x, y_0 + n^{-3/2}(y + t\sqrt{n}b(z_0)), \theta), \\
\tilde{b}_n(t, x, y, \theta) &= n^{-1/2}\tilde{b}(x_0 + n^{-1/2}x, y_0 + n^{-3/2}(y + t\sqrt{n}b(z_0)), \theta), \\
\tilde{b}_n(t, x, y) &= \sqrt{n}\tilde{b}(x_0 + n^{-1/2}x, y_0 + n^{-3/2}(y + t\sqrt{n}b(z_0))) - \tilde{b}(z_0).
\end{align*}
\]

Then, $F_{n,\theta,j}$ in Section 2.2 is given by
\[
F_{n,\theta} = \left( (\mathcal{X}_t^{n,\theta})^\top, (\mathcal{Y}_t^{n,\theta})^\top \right)^\top
\]
with $m_n = n$ and $k_j = jm$ because
\[
\begin{align*}
\left\{ \left( \begin{array}{c} X_t^{n,\theta} \\ Y_t^{n,\theta} \end{array} \right) \right\}_{t \geq 0} 
&\overset{d}{=} \left\{ \left( \begin{array}{c} \sqrt{n} \theta \\ 0 \end{array} \right) - \left( \begin{array}{c} Y_t^{n,\theta} \\ 0 \end{array} \right) - \left( \begin{array}{c} n \sqrt{n} \theta \\ 0 \end{array} \right) \right\}_{t \geq 0},
\end{align*}
\]
which satisfies the following condition, which is stronger than (C1).

**Assumption (C1').** (C1) is satisfied, $\partial_x^2 \partial_y^2 \tilde{a}$ and $\partial_x^2 \partial_y^2 \tilde{b}$ are bounded for $i \in \mathbb{Z}_+$ and $0 \leq j \leq 3$, and
\[
\sup_{z, \theta} \| (\tilde{a} \tilde{a}^\top)^{-1}(z, \theta) \|_\text{op} < \infty.
\]

First, we show Condition (L) under (C1') and (C2) by using Theorem 2.3, and then we weaken the assumptions to (C1) and (C2) by the localization technique similar to Lemma 4.1 of Gobet 5.

Define
\[
\tilde{F}_{n,\theta} = \left( \begin{array}{c} \tilde{a}(z_0, \theta)W_t \\ (\nabla \tilde{b})^\top(z_0)\tilde{a}(z_0, \theta) \int_0^t W_s dt \end{array} \right),
\]
and
\[
\tilde{X}_j^{n,\theta} = \left( \begin{array}{c} \sqrt{n}\tilde{a}(Y_{(j-1)/n}, \theta)W_{(j-1)/n} \\ n^{3/2}(\nabla \tilde{b})^\top(Y_{(j-1)/n}, \theta)\tilde{a}(Y_{(j-1)/n}, \theta) \int_0^{(j-1)/n} W_t - W_{(j-1)/n} dt \end{array} \right),
\]
then we have
\[
\partial_\theta \tilde{F}_{n,\theta} = \left( \begin{array}{c} \partial_\theta \tilde{a}(z_0, \theta)W_t \\ (\nabla \tilde{b})^\top(z_0)\partial_\theta \tilde{a}(z_0, \theta) \int_0^t W_s dt \end{array} \right),
\]
and $\tilde{K}_j$ in Section 2.2 can be calculated as
\[
\tilde{K}(\theta) = \left( \begin{array}{cc} \tilde{a} \tilde{a}^\top(z_0, \theta) & (1/2)\tilde{a} \tilde{a}^\top \nabla \tilde{b}(z_0, \theta) \\ (1/3)(\nabla \tilde{b})^\top(z_0, \theta) & (1/3)(\nabla \tilde{b})^\top(z_0, \theta) \tilde{a} \tilde{a}^\top \nabla \tilde{b}(z_0, \theta) \end{array} \right).
\]

We denote by $V^\perp$ the orthogonal complement of a subspace $V$ on a vector space. Let $\tilde{B}_{1,\theta} = \partial_\theta \tilde{a}^+(z_0, \theta)$ and $\mathcal{C}_{1,\theta} = (\nabla \tilde{b})^\top \partial_\theta \tilde{a}^+ \nabla \tilde{b}(\nabla \tilde{b})^\top \nabla \tilde{b}^{-1}(z_0, \theta)$. Then because $\tilde{a}^+ \tilde{a}$ is a projection to $(\text{Ker}(\tilde{a}))^\perp$ and $(\text{Ker}(\partial_\theta \tilde{a}))^\perp \subset (\text{Ker}(\tilde{a}))^\perp$ by (C2), we have
\[
\tilde{B}_{1,\theta}(z_0, \theta) = \partial_\theta \tilde{a}(z_0, \theta)
\]
(B.4)
for $1 \leq i \leq d$ and $\theta \in \Theta$. Moreover, because $\nabla 1b((\nabla 1b)^{\top})^{-1}(\nabla 1b)^{\top}$ is a projection to $(\text{Ker}((\nabla 1b)^{\top}))^\perp$, (C2) yields

$$\mathcal{C}_t(\nabla 1b)^\top \tilde{a}(z_0, \theta) = (\nabla 1b)^\top \partial_{\theta} \tilde{a}(z_0, \theta) = (\nabla 1b)^\top \partial_{\theta} \tilde{a}(z_0, \theta)$$

(B.5)

for $1 \leq i \leq d$ and $\theta \in \Theta$. Then, by setting

$$B_{i, \theta} = \begin{pmatrix} \tilde{B}_{i, \theta} & O_{m - n, \kappa} \\ O_{m - n, \kappa} & \mathcal{C}_{i, \theta} \end{pmatrix},$$

(B.6)

and (B.5) yield $\partial_{\theta} \tilde{F}_{n, \theta} = B_{i, \theta} \tilde{F}_{n, \theta}$.

For an $\mathbb{R}^N$-valued random variable $V = (V_i)_{i=1}^N \in (\mathbb{R}^1)^N$, we regard $D_2V$ as an $r \times N$ matrix.

**Lemma B.1.** Assume (C1') and (C2). Then Condition (L) is satisfied for $\{P_{\theta, n}\}_{\theta, n}$.

**Proof.** Thanks to Theorem 2.3 it is sufficient to check (B1)–(B5). Let $\tilde{a}_{n, t} = \tilde{a}_n(t, \mathcal{X}_t^{x, \theta}, \mathcal{Y}_t^{y, \theta}, \theta)$, $\tilde{b}_{n, t} = \tilde{b}_n(t, \mathcal{X}_t^{x, \theta}, \mathcal{Y}_t^{y, \theta}, \theta)$, and $\tilde{b}_{n, t} = \tilde{b}_n(t, \mathcal{X}_t^{x, \theta}, \mathcal{Y}_t^{y, \theta})$.

First, we check (B1). (C1') implies $\sup_{n, \theta, t} E[|\mathcal{X}_t^{x, \theta}|^p]^{1/p} \leq C_p$. Moreover, we obtain

$$\tilde{b}_{n, s} = \int_0^1 \left((\nabla 1b)^\top \mathcal{Y}_s^{y, \theta} + \frac{1}{n} (\nabla 2b)^\top \mathcal{Y}_s^{y, \theta}\right) (x_0 + u \mathcal{Y}_s^{y, \theta} + y_0 + u \mathcal{Y}_s^{y, \theta} n^{1/2}) du,$$

(B.7)

where $\tilde{\mathcal{Y}}_s^{y, \theta} = \mathcal{Y}_s^{y, \theta} + s \sqrt{n} \tilde{b}(z_0)$. Therefore, if $\nabla 2b \equiv 0$, then

$$\sup_{n, \theta} E[|\mathcal{Y}_t^{x, \theta}|^p]^{1/p} \leq \sup_{n, \theta, t} E[|\tilde{b}_{n, t}|^p]^{1/p} \leq C_p.$$

(B.8)

If $\tilde{b}$ is bounded, then

$$E[|\tilde{b}_{n, t}|^p] \leq C_p + C_p E[|\mathcal{Y}_t^{x, \theta}|^p] \leq C_p + C_p \int_0^t E[|\tilde{b}_{n, s}|^p] ds,$$

and hence Gronwall’s inequality yields

$$\sup_{n, \theta} E[|\mathcal{Y}_t^{x, \theta}|^p]^{1/p} \leq \sup_{n, \theta, t} E[|\tilde{b}_{n, t}|^p]^{1/p} < \infty.$$

Then we have $\sup_{n, \theta} ||F_{n, \theta}||_{0, p} < \infty$ under (C1') and (C2).

Let $Z_t^{x, \theta} = (X_t^{x, \theta})^\top, (\mathcal{Y}_t^{y, \theta})^\top$. Theorem 2.2.1 in [15] yields

$$D_1Z_t^{x, \theta} = (\tilde{a}_{n, t}^\top, O_{r, m - n, \kappa}) + \int_t^r (D_1Z_s^{n, \theta} \partial_{\theta} \tilde{b}_{n, s} + D_1Z_s^{n, \theta} \partial_{\theta} \tilde{b}_{n, s}) ds$$

$$+ \left(\begin{array}{c}
\int_t^r \sum_{k,l} [D_2Z_s^{n, \theta} \partial_{\theta} \tilde{a} \tilde{a}]_{ij} dW_s \\
\end{array}\right)_{ij} O_{r, m - n, \kappa}$$

for $r \geq t$. Then (C1') and Gronwall’s inequality yield $\sup_{n, \theta} E[||D_1Z_t^{x, \theta}||_{1/p}]^{1/p} < \infty$. By using Lemma 2.2.2 in [15], we similarly obtain $\sup_{n, \theta} E[||DF_{n, \theta}||_{H}]^{1/p} < \infty$. Thus, by Theorem 39 in Chapter V of Protter [15], we have $\sup_{n, \theta} E[||DF_{n, \theta}||_{4 - l, p}] < \infty$ for any $k \leq n$ and $p \geq 1$.

Therefore, Theorem 39 in Chapter V of Protter [15] yields

$$\begin{pmatrix}
\partial_{\theta} \mathcal{X}_t^{x, \theta} \\
\partial_{\theta} \mathcal{Y}_t^{y, \theta}
\end{pmatrix} = \int_0^t \begin{pmatrix}
\partial_{\theta} \tilde{b}_{n, s} + (\partial_{n} \tilde{b}_{n, s})^\top \partial_{\theta} Z_s^{n, \theta} \\
(\partial_{n} \tilde{b}_{n, s})^\top \partial_{\theta} Z_s^{n, \theta}
\end{pmatrix} ds$$

$$+ \int_0^t \begin{pmatrix}
\partial_{\theta} \tilde{a}_{n, s} + \sum_{k} \partial_{\theta} Z_s^{n, \theta} \partial_{\theta} \tilde{a}_{n, s} \\
0
\end{pmatrix} dW_s.$$
Next, we show (B2). Under (C1) and (C2), we have

\[
\sup_{n,t,\theta} \left( \|\partial_t \tilde{b}_n(t, z, \theta)\|_{\text{op}} \vee \|\partial_t \partial_t \tilde{b}_n(t, z)\|_{\text{op}} \vee \max_{t \leq t \leq m} \|\partial_{\theta} \tilde{a}_n(t, z, \theta)\|_{\text{op}} \right) < \infty
\]

for \(i \in \{0, 1\}, j \in \{1, 2, 3\}, \) and \(l \in \{0, 1\}, \) and

\[
\sup_{n,t,\theta} \left( \|\partial_t \tilde{a}_n, \partial_t \tilde{b}_n)\|_{\text{op}} \vee \|(\tilde{a}_n, \tilde{b}_n^{-1})(t, \theta)\|_{\text{op}} \right) < \infty
\]

Together with Proposition E.1, we obtain (B2), where \(\epsilon_n = 1/\sqrt{n}, \) \(m_n = n, \) \(\tilde{K}_n = m, \) and \(\alpha_n \) is a constant independent of \(n. \) Furthermore, by setting (B.9), we have (N2) and \(\partial_\theta \tilde{F}_{n,\theta} = B_{i,\theta} \tilde{F}_{n,\theta} \).

To verify (B3) with \(\rho_n = 1/\sqrt{n}, \) we only need to check \(\|\partial_\theta \tilde{F}_{n,\theta} - \partial_\theta \tilde{F}_{n,\theta}\|_{3-1,p} \leq C_p/\sqrt{n} \) for \(l \in \{0, 1\} \) and \(p > 1. \) First, we have

\[
F_{n,\theta} - \tilde{F}_{n,\theta} = \int_0^1 \left( b_{n,s} - (\nabla b)_{t}(z_0)\tilde{a}(z_0, \theta) W_s \right) ds + \int_0^1 \left( \tilde{a}_{n,s} - \tilde{a}(z_0, \theta) \right) dW_s.
\]

Because

\[
\tilde{a}_{n,s} - \tilde{a}(z_0, \theta) = \frac{1}{\sqrt{n}} \int_0^1 \left( \nabla_1 \tilde{a}(x, y) + \frac{1}{n} \nabla_2 \tilde{a}(x, y) \right) \left( x_0 + u \frac{X_n, \theta}{\sqrt{n}}, y_0 + u \frac{Y_n, \theta}{\sqrt{n}} \right) du,
\]

we have

\[
\sup_{n,\theta} E \left[ \sup_l |x(n, \theta)| W_l |^p \right]^{1/p} \leq \frac{C_p}{\sqrt{n}}.
\]

Together with (B.7), we have

\[
\sup_{n,\theta} E \left[ \sup_l |\tilde{b}_n_l - (\nabla b)_{n, s}(z_0)\tilde{a}(z_0, \theta) W_l |^p \right]^{1/p} \leq \frac{C_p}{\sqrt{n}}.
\]

(B.9) and (B.11) yield \(\|F_{n,\theta} - \tilde{F}_{n,\theta}\|_{0,p} \leq C_p/\sqrt{n}.\)

Similarly to above, Theorem 39 in Chapter V of Protter \cite{protter2005stochastic} and Theorem 2.2.1 and Lemma 2.2.2 in \cite{protter2010stochastic} yield \(\|\partial_\theta F_{n,\theta} - \partial_\theta \tilde{F}_{n,\theta}\|_{3-1,p} \leq C_p/\sqrt{n} \) for \(p \geq 1 \) and \(l \in \{0, 1\}, \) and consequently (B3) holds.

Finally, we show (B4). We denote \(\hat{\varphi}(A) = (\nabla b)_{t}(z_0)A\nabla b(z_0) \) for a \(\kappa \times \kappa \) matrix \(A. \) Let \(S = (1/12)\hat{\varphi}(\tilde{a} a)_{t}(z_0, \theta). \) Then for \(\epsilon = \inf_{s,\theta}(\|(\tilde{a} a)_{t}(z, \theta)\|_{\text{op}}^{-1}), \) \(\epsilon' = \inf_{s,\theta}(\|(\nabla b)_{t}(z, \theta)\|_{\text{op}}^{-1}), \) and \(y \in \mathbb{R}^{n-\kappa}, \) we obtain

\[
y^\top \hat{\varphi}(\tilde{a} a)_{t}^\top y \geq cy^\top (\nabla b)_{t}^\top \nabla b y \geq \epsilon \epsilon' |y|^2,
\]

which implies that \(S \) is positive definite. Together with (0.8.5.6) in Horn and Johnson \cite{horn1990matrix}, we have

\[
\hat{K}^{-1}(\theta) = \left( \begin{array}{c} \tilde{a} a^\top (\tilde{a} a)_{t}^{-1} + (1/4)\nabla \tilde{b} S^{-1}(\nabla \tilde{b})_{t}^\top - (1/2) (\nabla \tilde{b} S^{-1}(\nabla \tilde{b})_{t})^\top \end{array} \right) (z_0, \theta).
\]

Let \(\tilde{K}_0 = \hat{K}(\theta_0)\) and \(\sigma = \sigma(W_s; s \leq t). \) We can set \(G_{\theta, t}^{n} = (G_{\theta, t}^{n})_{1 \leq i \leq d} \) and \(\gamma_j \) in (B4) by

\[
G_{\theta, t}^{n} = \frac{1}{2} \left( \tilde{a} a_{j}^\top (B_{\theta, t}^{n, \theta} + \tilde{K}_{0}^{-1} B_{\theta, t}^{n, \theta}) - \text{tr}(B_{\theta, t}^{n, \theta}) \right).
\]

and

\[
[\gamma(z_0)]_{\nu} = \frac{1}{2} \text{tr}(\tilde{K}_0^{-1}(B_{\theta, t}^{n, \theta} + \tilde{K}_0 B_{\theta, t}^{n, \theta})) (B_{\theta, t}^{n, \theta} + \tilde{K}_0 B_{\theta, t}^{n, \theta}).
\]

Repeated use of (B.3) and (B.4) yields

\[
B_{\theta, t}^{n, \theta} + \tilde{K}_0 B_{\theta, t}^{n, \theta} = \left( \begin{array}{c} \partial_\theta (\tilde{a} a)_{t}^\top \left( \frac{1}{2}\partial_\theta (\tilde{a} a)_{t}^\top (\nabla \tilde{b})_{t}^\top \partial_\theta (\tilde{a} a)_{t} + (1/3) \hat{\varphi}(\tilde{a} a)_{t}^\top) \right) (z_0, \theta) \end{array} \right).
\]
and hence
\[ \hat{K}_0^{-1}(B_{1,0}, \hat{K}_0 + \tilde{K}_0 B_{1,0}) = \left( \begin{array}{cc} (\tilde{a}a^\top)^{-1} \partial_0 (\tilde{a}a^\top) & (1/2)(\tilde{a}a^\top)^{-1} \partial_0 (\tilde{a}a^\top) \nabla I \dot{b} - \mathcal{S}_t \end{array} \right) (z_0, \theta_0), \]  
(\text{B.13})

where \( \mathcal{S}_t = (1/24) \nabla I \dot{b} S^{-1} \varphi(\partial_0 (\tilde{a}a^\top))(z_0, \theta_0). \) Together with the equations
\[ U \partial_0 (aa^\top) U^\top = \left( \begin{array}{cc} (\tilde{a}a^\top)^{-1} & O_{\kappa,m-\kappa} \\ O_{m-\kappa,\kappa} & O_{m-\kappa,m-\kappa} \end{array} \right) \]
and
\[ U(aa^\top)^\top U^\top = (Uaa^\top U^\top)^\top = \left( \begin{array}{cc} (\tilde{a}a^\top)^{-1} & O_{\kappa,m-\kappa} \\ O_{m-\kappa,\kappa} & O_{m-\kappa,m-\kappa} \end{array} \right), \]
we have
\[ \lceil \gamma(z_0) \rceil_{iv} = \text{tr}((aa^\top)^\top \partial_0 (aa^\top)(aa^\top)^\top \partial_0 (aa^\top))((U^\top z_0, \theta_0))/2 + \text{tr}(\varphi(\tilde{a}a^\top)^{-1} \varphi(\partial_0 (\tilde{a}a^\top))(\tilde{a}a^\top)^{-1} \varphi(\partial_0 (\tilde{a}a^\top)))(z_0, \theta_0)/2. \]

Therefore, we obtain
\[ \frac{1}{n} \sum_{j=1}^{n} \gamma(UX_{(j-1)/n}) \xrightarrow{P} \Gamma \]  
(\text{B.14})

as \( n \to \infty. \) Moreover, Lemma \[\text{B.1}\] yields
\[ \frac{1}{n} \sum_{j=1}^{[nt]} E[G_j^n(G_{j+1}^n\mid \mathcal{G}_{(j-1)/n}]) \xrightarrow{P} \Gamma_t, \quad \frac{1}{n} \sum_{j=1}^{[nt]} E[|G_j^n|^4|\mathcal{G}_{(j-1)/n}] \xrightarrow{P} 0 \]  
(\text{B.15})

for \( t \in [0,1], \) where \( \Gamma_t \) is defined by replacing the interval of integration in the definition of \( \Gamma \) with \([0,t]. \)

Furthermore, it is easy to see that
\[ \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} E[G_j^n(W_{j/n} - W_{(j-1)/n})^\top \mathcal{G}_{(j-1)/n}] = 0, \]
\[ \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} E[G_j^n(N_{j/n} - N_{(j-1)/n})^\top \mathcal{G}_{(j-1)/n}] = 0 \]
for any \( t \in [0,1] \) and any bounded \( (G_t)_{t \in [0,1]} \)-martingale \( N = (N_t)_{t \in [0,1]} \) orthogonal to \( W. \) Together with Theorem 3.2 in Jacod \[\text{[11]}\], we have \( \sum_{j=1}^{n} G_j^n/\sqrt{n} \to \Gamma^{1/2}N \) stably as \( n \to \infty, \) which implies (B4).

\[\square\]

**Proof of Theorem 2.4**

For \( q > 0, \) let \( \phi_q^1 : \mathbb{R}^\kappa \to \mathbb{R}^\kappa \) and \( \phi_q^2 : \mathbb{R}^{m-\kappa} \to \mathbb{R}^{m-\kappa} \) be \( C^\infty \) functions with compact support satisfying \( \phi_q^1(x) = x \) on \( \{ |x| \leq q \} \) and \( \phi_q^2(y) = y \) on \( \{ |y| \leq q \}. \) Let \( \phi_q(z) = (\phi_q^1(x), \phi_q^2(y)), \) and let
\[ a_q(z, \theta) = U^\top \left( \begin{array}{c} \hat{a}(\phi_q(Uz), \theta) \\ 0 \end{array} \right), \quad b_q(z, \theta) = U^\top \left( \begin{array}{c} \hat{b}(\phi_q(Uz), \theta) \\ \hat{b}(Uz) \end{array} \right). \]

Let \( P_{\theta,q,n} \) be the corresponding probability measure, and \( V_{q,n}, T_{q,n}, W_q, T_q \) correspond to \( V_n, T_n, W, T, \) respectively. Then (C1') and (C2) are satisfied for the statistical model of \( P_{\theta,q,n}, \) and hence this model satisfies Condition (L) by Lemma \[\text{B.1}\]. Moreover, we have
\[ \log \frac{dP_{\theta,q+h/\sqrt{n},q,n}}{dP_{\theta,q,n}} = (h^\top V_{q,n} + 1/2 h^\top T_{q,n} h) \to 0 \]
in \( P_{\theta,q,n} \)-probability, and \( \mathcal{L}((T_{q,n}, V_{q,n})|P_{\theta,q,n}) \to \mathcal{L}((T_q, T_q^1/2, W_q)). \) By letting \( q \to \infty, \) Proposition 4.3.2 in Le Cam \[\text{[14]}\] yields Condition (L) for \( (P_{\theta,n})_{\theta,n}. \) Then Remark \[\text{2.1}\] leads to the conclusion.

\[\square\]
C Partial observation models

In this section, we prove Theorem 2.5 by using the results in Section 2.2. It is difficult to apply the scheme in Section 2.2 directly because the conditional distribution \( P(X_{n}^{u} \in \cdot | X_{j-1}^{n} = \tilde{x}_{j-1}) \) is complicated when some components are hidden. Therefore, we follow the idea of [3], that is, we first show Condition (L) of an augmented model obtained by adding some observations of \((I_{\kappa} - \mathcal{Q})\tilde{Y}_{t} \) to the partial observations. Then we show the LAMN property of the original model by approximating the log-likelihood ratios of the augmented model using functionals of the original observations.

C.1 An augmented model

We divide the whole observation interval \([0, 1]\) into several blocks and show Condition (L) for an augmented model that is obtained by adding an observation of \((I_{\kappa} - \mathcal{Q})\tilde{Y}_{t}\) for each block. Let \((e_{n})_{n=1}^{\infty}\) be a sequence of positive integers that diverges to infinity very slowly (the precise diverge rate of \(e_{n}\) is specified in Lemma C.1). Let \(L_{n} = [(n - 1)/e_{n}]\) and \(t_{j,k} = (k + j e_{n})/n\). We consider an augmented model generated by observation blocks

\[
\{\{\mathcal{Q}\tilde{Y}_{t_{j,k}, \tilde{Y}_{t_{j,k}}} e_{n}, (I_{\kappa} - \mathcal{Q})\tilde{Y}_{t_{j+1,k}}\} \quad (C.1)
\]

for \(0 \leq j \leq L_{n} - 1\) and

\[
\{\mathcal{Q}\tilde{Y}_{t_{L_{n}, k}, \tilde{Y}_{t_{L_{n}, k}}} e_{n}, L_{n}\}.
\]

Because \(I_{\kappa} - \mathcal{Q}\) and \(B^{+}B\) are projections to \((\text{Im}(\mathcal{Q}))^{\perp}\) and \((\text{Ker}(B))^{\perp}\), respectively, \((C.4)\) implies

\[
\tilde{Q}_{3}B^{+}B = \tilde{Q}_{3}.
\]

Then, we can approximate

\[
n\tilde{Q}_{3}B^{+}(\tilde{Y}_{k/n} - \tilde{Y}_{(k-1)/n}) = n\tilde{Q}_{3}B^{+}B \int_{(k-1)/n}^{k/n} \tilde{Y}_{t} dt \approx \tilde{Q}_{3}\tilde{Y}_{k/n}.
\]

Therefore, we set \(X_{n}^{u} \) in Section 2.2 as

\[
X_{n}^{u} = \left\{ \begin{array}{l}
\{ \sqrt{n}\tilde{Q}_{1}\Delta_{n,j,1} \tilde{Y}_{n}^{1/2}(\Delta_{n,j,1} \tilde{Y}_{n} - B\tilde{Y}_{t_{j,0}/n}) \} , \\
\{ \sqrt{n}\tilde{Q}_{1}\Delta_{n,j,k} \tilde{Y}_{n}^{1/2}(\Delta_{n,j,k} \tilde{Y}_{n}) \}_{\tilde{k}=2}^{e_{n}} , \\
\sqrt{n}((\tilde{Q}_{3}\tilde{Y}_{t_{j+1,0}} - n\tilde{Q}_{3}B^{+}\Delta_{n,j,0}) \\
\end{array} \right.
\]

for \(0 \leq j \leq L_{n} - 1\), and

\[
X_{n}^{u} = \left\{ \begin{array}{l}
\{ \sqrt{n}\tilde{Q}_{1}\Delta_{n,L_{n},1} \tilde{Y}_{n}^{1/2}(\Delta_{n,L_{n},1} \tilde{Y}_{n} - B\tilde{Y}_{t_{L_{n},0}/n}) \} , \\
\{ \sqrt{n}\tilde{Q}_{1}\Delta_{n,L_{n},k} \tilde{Y}_{n}^{1/2}(\Delta_{n,L_{n},k} \tilde{Y}_{n}) \}_{2 \leq k \leq n-e_{n} L_{n}} , \\
\end{array} \right.
\]

which are obtained as a linear transformation of block observations \((C.1)\) and \((C.2)\). Here we denote \(\Delta_{n,j,l}V = V_{t_{j,l}} - V_{t_{j,l-1}}, \Delta_{n,j,l}V = \Delta_{n,j,l}V - \Delta_{n,j,l-2}V\) for \(l \geq 1, l \geq 2\) and a stochastic process \(V = (V_{t})_{t \in [0, 1]}\).

Thanks to (B.2), the corresponding \(F_{n,\theta,j}\) and \(\tilde{F}_{n,\theta,j}\) are defined by

\[
F_{n,\theta} = \{ \tilde{Q}_{1}\Delta_{k} X_{n}^{u,\theta} - \tilde{Q}_{3}B^{+}(Y_{n}^{u,\theta} - Y_{n}^{u,-1}) \} \quad (C.5)
\]

and

\[
\tilde{F}_{n,\theta} = \left\{ \tilde{Q}_{1}\tilde{a}(x_{0}, \theta) \Delta_{k} W_{t} - \tilde{Q}_{3}\tilde{a}(x_{0}, \theta) \int_{(k-2)/n}^{1} (W_{(t+k-2)/n} - W_{(t+k-2)/n}) dt \right\}^{e_{n}}_{k=1} \\
\cup \left\{ \tilde{Q}_{3}\tilde{a}(x_{0}, \theta) \int_{(k-2)/n}^{1} (W_{n} - W_{t+k-2}) dt \right\}^{e_{n}}_{k=1}
\]

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for $0 \leq j \leq L_n - 1$, where $\chi_t^{n,\theta}$, $\gamma_t^{n,\theta}$ are defined in Section B.

$$F_{n,\theta}^{n} = \left\{ \tilde{Q}_1 \Delta_k \chi_t^{n,\theta}, \Delta_k^2 \gamma_t^{n,\theta} \right\}_{k=1}^{n-e_n L_n},$$

and

$$\tilde{F}_{n,\theta}^{n} = \left\{ \tilde{Q}_1 \tilde{a}(x_0,\theta) \Delta_k W, \tilde{Q}_2 \tilde{a}(x_0,\theta) \int_0^1 (W_{t+k-1} - W_{t+k-2}) dt \right\}_{k=1}^{n-e_n L_n}.$$  

Here, we denote $\Delta_t V = V_t - V_{t-1}$, $\Delta_t^2 V = \Delta_t V - \Delta_{t-1} V$ for $l \geq 1$, $l' \geq 2$, and a stochastic process $V = (V_t)_{t \geq 0}$.

Moreover, we define

$$\tilde{X}_j^{n,\theta} = \left( \sqrt{n} \tilde{Q}_1 \tilde{a}_j \Delta_n,j,k W, n^{3/2} \tilde{Q}_2 \tilde{a}_j \int_{l,j,k-1}^{l,j,k} (W_t - W_{t-1}) dt \right)_{k=1}^{n-e_n L_n},$$

for $0 \leq j \leq L_n - 1$, and $\tilde{X}_{L_n}^{n,\theta}$ similarly, where $\tilde{a}_j = \tilde{a}(Y_{j,\theta},\theta)$.

### C.2 Condition (L) for the augmented model

First, we show Condition (L) for the augmented model under a stronger condition.

**Assumption (C2'').** (C2') is satisfied, $\sup_{x,\theta} ||(\hat{a}a^{-1})^{-1}(x,\theta)||_{op} < \infty$, and $\partial_x^I \partial_x^k \hat{a}(x,\theta)$ and $\partial_x^I \partial_x\partial_x^k \hat{b}(x, y, \theta)$ are bounded for $i, j \in \mathbb{Z}_+$ and $0 \leq k \leq 3$.

**Assumption (C5').** (C5) holds, $g$ is bounded, and the convergence (2.26) holds uniformly in $x$.

Let $F_{\theta,n}^{aug}$ be the probability measure induced by the augmented model.

**Lemma C.1.** Assume (C2''), (C4), and (C5'). Then Condition (L) is satisfied for $\{F_{\theta,n}^{aug}\}_{\theta,n \in \mathbb{N}}$ by setting suitable $(e_n)_{n \in \mathbb{N}}$.

**Proof.** We check (B1)–(B5) in Theorem 2.23 First, the boundedness of $\tilde{a}_n$ and $\tilde{b}_n$ yields

$$\sup_{\theta \in [0,\infty]} E[|\chi_t^{n,\theta} - \chi_{(t-1) \vee 0}^{n,\theta}|^q] < \infty, \sup_{k,\theta} E[|\Delta_k^2 \gamma_t^{n,\theta}|^q] < \infty.$$  

Then by a similar argument to Lemma 3.1 we have

$$\|\partial_x^I \partial_x^k [F_{\theta,n}]_k\|_{l,p} \vee \|\partial_x^I \partial_x^k [F_{\theta,n}]_k\|_{l,p} < \infty$$

for any $l \in \mathbb{N}$, $l' \in \mathbb{N}$, $p \geq 1$, and $1 \leq k \leq e_n$. Let $\tilde{C}_{t,\theta}(x) = B\partial_y \tilde{a}(x,\theta) B\tilde{a}(x,\theta)^{-1},$ (B.5), (C2''), and (C4) yield

$$\tilde{Q}_1 \partial_y \tilde{a} = R_1 Q \partial_y \tilde{a}, \quad \tilde{Q}_2 \partial_y \tilde{a} = \tilde{C}_{t,\theta} B \tilde{a}, \quad \tilde{Q}_3 \partial_y \tilde{a} = R_3 (I - Q) \partial_y \tilde{a},$$

and consequently we obtain

$$\partial_y \tilde{F}_{n,\theta} = B_{t,\theta} \tilde{F}_{n,\theta}$$

for $B_{t,\theta}(x_0) = \text{diag}((R_1 \partial_y \tilde{a}, R_3 \partial_y \tilde{a}^{-1}))_{k=1}^{e_n 3}$, $R_3 \partial_y \tilde{a}^{-1} R_3^{-1}(x_0, \theta)$.

Moreover, similarly to the argument in Section B we have $\|\partial_x^I \partial_x^k [F_{\theta,n}]_k\|_{3-l,p} \leq C_p e_n / \sqrt{n}$ for $1 \leq k \leq q$. For $2 \leq k \leq e_n$, we have

$$([F_{\theta,n} - \tilde{F}_{\theta,n}]_{(k-1)q+1})_{k=1}^{q} = \tilde{Q}_1 \left( \int_{k-1}^{k} \tilde{b}_{n,s} ds + \int_{k-1}^{k} (\tilde{a}_{n,s} - \tilde{a}(x_0,\theta)) dW_s \right).$$
\[(F_n, \theta - \mathcal{F}_n, \theta_{[k-1]q+q_1+t})\]_{t=1}^{\infty} = \overline{Q}_2 \int_{k-1}^{k} (X_t^{n, \theta} - X_{t-1}^{n, \theta} - \tilde{a}(x_0, \theta)(W_t - W_{t-1})) dt,
and
\[(F_n, \theta - \mathcal{F}_n, \theta_{[k-1]q+q_1+t})\]_{t=1}^{\infty} = \overline{Q}_3 \int_{t}^{\infty} (X_t^{n, \theta} - \tilde{a}(x_0, \theta) W_t - X_t^{n, \theta} + \tilde{a}(x_0, \theta) W_t) dt.

Then, by a similar argument to the proof of Lemma B.3, we have \(\|\partial_\theta^k [F_n, \theta] \|_{3-\ell, p} \leq C_{p,n} e_n / \sqrt{n}\) for \(1 \leq k \leq \epsilon_n q + \kappa - q_1\), and a similar estimate for \(\partial_\theta^k [F_n, \theta] \|_{3-\ell, p} \leq C_{p,n} e_n / \sqrt{n}\). Hence we obtain (B3) with \(\rho_n = e_n / \sqrt{n}\).

Let \(K_n(\theta)\) be the Malliavin matrix of \(\{\tilde{Q}_1 X_t^{n, \theta}, \gamma_t^{n, \theta}\}_{t=1}^{\infty}\). By Proposition B.2 (0.8.5.3) in Horn and Johnson [10], and the fact that \(\|DX_t^{n, \theta}\|_{1, p}\) and \(\|DY_t^{n, \theta}\|_{1, p}\) are bounded, we have \(E[\det K_n]^{-\rho} \leq e(p, \epsilon_n)\), where \(e(p, \epsilon_n)\) is a positive constant depending on \(p\) and \(\epsilon_n\).

Because there exists an invertible matrix \(M_n\) depending on \(\epsilon_n\) such that \(\tilde{K}_n(\theta) = (\tilde{D} F_n, \theta, \tilde{D} F_n, \theta) = M_n K_n M_n^T\), there exists a constant \(e(p, \epsilon_n)\) such that \(\sup_p E[\det K_n(\theta)]^{-\rho} \leq e(p, \epsilon_n)\). Because we also have a similar estimate for \(\langle \tilde{D} F_n, \theta, \tilde{D} F_n, \theta \rangle\), we have (B2) and (B.10) by letting \(\epsilon_n\) diverge to infinity sufficiently slowly. Similarly, we have (N2) (the upper bound of \(\sup_p E[\det K_n(\theta)]^{-\rho}\) can depend on \(n\) in (N2)).

Now we only need to show (B4). We define \(\tilde{K}_n(\theta) = \psi_{\epsilon_n}^2(\tilde{a}(x_0, \theta))\) and \(B_{t, \theta} \tilde{K}_n(\theta) + \tilde{K}_n(\theta) B_{t, \theta} = \psi_{\epsilon_n}^2(\partial_{x_0} (\tilde{a} a^T (x_0, \theta)))\) by (C.4).

We define \(G_n^o = (G_{j, n}^o)_{j=1}^{\infty}\) in Section 2.2 by
\[
G_{j, n}^o = \frac{1}{2} (X_t^{n, \theta})^T (B_{t, \theta} \tilde{K}_n^{-1}(\theta_0) + \tilde{K}_n^{-1}(\theta_0) B_{t, \theta})|_{x_0 = \tilde{Y}_{t, 0}^n} X_t^{n, \theta} - \text{tr}(B_{t, \theta})|_{x_0 = \tilde{Y}_{t, 0}^n}.
\]

Then, \(\gamma_j(x_j-1)\) in Section 2.2 is calculated as
\[
|\gamma_j(x_j)|_{w'} = \frac{1}{2} \text{tr}(\psi_{\epsilon_n}^2(\tilde{a} a^T)^{-1} \psi_{\epsilon_n}^2(\tilde{a} a^T) \psi_{\epsilon_n}^2(\tilde{a} a^T)^{-1} \psi_{\epsilon_n}^2(\partial_{x_0} (\tilde{a} a^T))) (x_j, \theta_0)\]
for \(0 \leq j \leq L_n - 1\), and \(\gamma_{L_n}(x_0)\) is similarly calculated with the estimate
\[
\frac{1}{n} ||\gamma_{L_n}(\tilde{Y}_{t_j, 0})||_{w'} \leq \frac{C_{\epsilon_n}^2 \gamma_{\epsilon_n}^2 e^2_n}{n} \rightarrow 0
\]
as \(n \rightarrow \infty\) by Lemma 3.1.

Therefore, (C5) implies that there exists \(n_0 \in \mathbb{N}\) such that
\[
\sup_{n \geq n_0} \left( \frac{1}{n} \sum_{j=0}^{L_n} E[|\gamma_j(\tilde{Y}_{t_j, 0})|]\right) < \infty,
\]
and
\[
\frac{1}{n} \sum_{j=0}^{L_n} |g(\tilde{Y}_{t_j, 0})|_{w'} - \frac{1}{2} \sum_{j=0}^{L_n-1} \int_{t_{j+1, 0}}^{t_{j+1, 0}} [g(\tilde{Y}_{t_j, 0})]_{w'} dt \rightarrow 0.
\]

Lemma 2.4 Theorem 3.2 in Jacod [11], and the inequality
\[
||\psi_{\epsilon_n}^2(\partial_{x_0} (\tilde{a} a^T)) (x_j, \theta)||_{\infty} < C e_n
\]
yield (B4) with \(G_j = \sigma(W_s; s \leq t_j, 0)\).
C.3 Approximation of the log-likelihood ratio

We show that the log-likelihood ratio \(\log(dP^{\text{aug}}_{\theta_0}/dP^\theta_0)\) can be approximated by a random variable that is observable in the original model. Let \(X'_j = ([X^n_{j \theta_0}])_{k=1}^m\) (removing the last element of \(X^n_{j \theta_0}\)), \(\bar{Y}_0 = \bar{z}_{i_{ni}},\bar{Y}_j = Q\bar{Y}_{t,o} + n(\bar{Q}B^+(\bar{Y}_{t,o} - \bar{Y}_{t,-1}))\) for \(j \geq 1\). Let \(U_j = (U_{i,j})_{i=1}^d\), where

\[
U_{i,j} = -\frac{1}{2} \left\{ X_{j}^T \delta_{\theta_i}(\psi_{\epsilon_n}^{-1}(\tilde{a}a^T)^{-1})(\bar{Y}_j, \theta_0)X_{j}^T \right. \\
+ \left. \text{tr}(\delta_{\theta_i}(\psi_{\epsilon_n}^{-1}(\tilde{a}a^T)^{-1})\psi_{\epsilon_n}^{-1}(\tilde{a}a^T))(\bar{Y}_j, \theta_0) \right\},
\]

\[
\mathcal{U}_j = \left( \frac{1}{2} \text{tr}(\psi_{\epsilon_n}^{-1}(\tilde{a}a^T)^{-1}\psi_{\epsilon_n}^{-1}(\tilde{a}a^T)) \psi_{\epsilon_n}^{-1}(\tilde{a}a^T)^{-1}(\delta_{\theta_i'}(\tilde{a}a^T))(\bar{Y}_j, \theta_0) \right)_{1 \leq i, i' \leq d}
\]

for \(0 \leq j \leq L_n - 1\). Then \(U_j\) and \(\mathcal{U}_j\) are functionals of the original observations.

**Proposition C.1.** Assume \((C_{2''}), (C_{4}), \) and \((C_{5'})\). Then

\[
\frac{\log(dP^{\text{aug}}_{\theta_0+h}/dP^\theta_0)}{\sqrt{n}} \xrightarrow{P} 0
\]

as \(n \to \infty\) for any \(h \in \mathbb{R}^d\).

**Proof.** Because the augmented model satisfies Condition (L) and

\[
G^\theta_{L_n}/\sqrt{n} \xrightarrow{P} 0
\]

by Lemma 24 and (C.7), it is sufficient to show that

\[
\frac{1}{\sqrt{n}} \sum_{j=0}^{L_n-1} (U_j - G_{j}^\theta) \xrightarrow{P} 0 \quad \text{and} \quad \frac{h^T}{n} \sum_{j=0}^{L_n-1} (\mathcal{U}_j - \gamma_j(\bar{Y}_{t,o}))h \xrightarrow{P} 0,
\]

where \(G_{j}^\theta = (G_{j}^\theta)^d\).

Let \(\mathcal{J}_i(x) = \partial_{\theta_i}(\psi_{\epsilon_n}^{-1}(\tilde{a}a^T)^{-1})(x, \theta_0)\) and \(\bar{X}'_j = ([\bar{X}^{\theta_0}_{j \theta_0}])_{k=1}^m\). Let

\[
U_{i,j} = -\frac{1}{2} \left\{ X_{j}^T \mathcal{J}_i(\bar{Y}_{t,o})X_{j}^T + \text{tr}(\mathcal{J}_i\psi_{\epsilon_n}^{-1}(\tilde{a}a^T))(\bar{Y}_{t,o}, \theta_0) \right\},
\]

\[
\mathcal{U}_j = \left( \frac{1}{2} \text{tr}(\mathcal{J}_i\psi_{\epsilon_n}^{-1}(\tilde{a}a^T)^{-1}\mathcal{J}_i')\psi_{\epsilon_n}^{-1}(\tilde{a}a^T))\mathcal{J}_i(\bar{Y}_{t,o}, \theta_0) \right)_{1 \leq i, i' \leq d}.
\]

First, we show that \(\sum_{j}(U_{i,j} - \bar{U}_{i,j})/\sqrt{n} \xrightarrow{P} 0\).

Let \(\bar{b}_j = \bar{b}(\bar{Y}_{t,o}, \bar{Y}_{t,o}, \theta_0)\). Because

\[
\Delta_{n,j,k} \bar{Y} = \bar{a}_j \Delta_{n,j,k} W
\]

\[
= \int_{t_{j,k-1}}^{t_{j,k}} (\bar{a}(\bar{Y}_t, \theta_0) - \bar{a}_j)dW_t + \bar{b}_j \Delta_{n,j,k} t + O_p\left( \frac{e_n}{n^{3/2}} \right)
\]

\[
= \int_{t_{j,k-1}}^{t_{j,k}} \sum_{l=1}^{\kappa} [\bar{a}_l(W_l - W_{t,j,o})]d\bar{z}_l \bar{a}_j dW_t + \frac{\bar{b}_j}{n} + O_p\left( \frac{e_n^2}{n^{3/2}} \right)
\]

for \(k \geq 1\), and

\[
\Delta_{n,j,k}^2 \bar{Y} = B\bar{a}_j \int_{t_{j,k-1}}^{t_{j,k}} (W_t - W_{t-1/n})dt
\]

\[
= B \int_{t_{j,k-1}}^{t} \int_{t-1/n}^{t} (\bar{a}(\bar{Y}_s, \theta_0) - \bar{a}_j)dW_s dW_t + \frac{B\bar{b}_j}{n^2} + O_p\left( \frac{e_n}{n^{3/2}} \right)
\]

\[
= B \int_{t_{j,k-1}}^{t} \int_{t-1/n}^{t} \sum_{l=1}^{\kappa} [\bar{a}_l(W_s - W_{t,j,o})]d\bar{z}_l \bar{a}_j dW_s dW_t + \frac{B\bar{b}_j}{n^2} + O_p\left( \frac{e_n^2}{n^{3/2}} \right)
\]

(10)
for $k \geq 2$, together with a similar estimate for

$$\Delta_{n,j,l} \tilde{Y} - B\tilde{y}_{t_{j,l}}/n = B\tilde{a}_j \int_{t_{j,l}}^{t_{j,l+1}} (W_t - W_{t_{j,l}}) dt,$$

we have

$$X_j' - \tilde{X}_j' = X_j'' + O_p(n^{-1/2}e_n^n),$$

where $X_j'' = O_p(n^{-1/2}e_n)$ and $E[\tilde{X}_j'(X_j'')^T | \mathcal{G}_{t_{j,l}}] = 0$.

Moreover, because $\sup_x (\psi^{1,1}_{e_n})_+(\tilde{a}^2(x, \theta_0)) \leq C_\alpha e_n$ by Lemma 3.1 and $\tilde{Y}_j = \tilde{y}_{t_{j,l}}$ is equal to an $n^{-1/2}$-order martingale difference plus an $n^{-1}$-order term, we have

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{L_n-1} \left\{ \text{tr}(\tilde{\mathfrak{3}}_i(\psi^{1,1}_{e_n}(\tilde{a}^2)))(\tilde{Y}_j - \tilde{y}_{t_{j,l}}) \right\} = 0$$

as $n \to \infty$ by rearranging $e_n$ if necessary. Similarly, we have

$$\frac{h^T}{n} \sum_{j=0}^{L_n-1} (\tilde{\mathfrak{3}}_j - \mathfrak{3}_j)h \to 0.$$  \hfill (C.13)

(C.11), (C.12), and a similar estimate yield

$$\frac{1}{\sqrt{n}} \sum_j \tilde{\mathfrak{4}}_{i,j} - \mathfrak{4}_{i,j}$$

and

$$\frac{1}{\sqrt{n}} \sum_j \tilde{\mathfrak{4}}_{i,j} = \frac{1}{2\sqrt{n}} \sum_j \left\{ 2\tilde{X}_j^T \tilde{\mathfrak{3}}_i(\tilde{y}_j')(X_j' - \tilde{X}_j') + (X_j' - \tilde{X}_j')^T \tilde{\mathfrak{3}}_i(\tilde{y}_j')(X_j' - \tilde{X}_j') 

+ \tilde{X}_j^T \tilde{\mathfrak{3}}_i(\tilde{y}_j') - \tilde{\mathfrak{3}}_i(\tilde{y}_{t_{j,l}})X_j' \right\} + o_p(1)$$

(C.14)

Moreover, (C5') yields

$$\frac{h^T}{n} \sum_{j=0}^{L_n-1} (\tilde{\mathfrak{4}}_j - \gamma_j(\tilde{y}_{t_{j,l}}))h \to 0.$$  \hfill (C.15)

Thanks to (C.13), (C.15), it is sufficient to show

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{L_n-1} \tilde{\mathfrak{4}}_j - G_j^n \to 0.$$  \hfill (C.16)
We can easily check $E[\hat{U}_{t,j} - G_{j,t,o}^n|G_{t,o}] = 0$ and (C5') yields
\[
\frac{1}{n} \sum_j E[|\hat{U}_j - G_j^n|^2|G_{t,o}] = \frac{1}{n} \sum_j E[|\hat{U}_j|^2 - 2\hat{U}_j \cdot G_j^n + |G_j^n|^2|G_{t,o}] = \frac{1}{n} \sum_j \text{tr}(T_{1,1,0}(\hat{Y}_{t,o}) - 2T_{1,2,0}(\hat{Y}_{t,o}) + T_{2,2,0}(\hat{Y}_{t,o})) \xrightarrow{P} 0.
\]
(C.17)

Then Lemma 9 in Genon-Catalot and Jacod [2] yields the conclusion.

\[\square\]

### C.4 Proof of Theorem 2.5

In light of Proposition C.1\[C.1\] a similar argument to Proposition 4 in [3] yields
\[
\log \frac{dP_{\theta_0+h/\sqrt{n},n}}{dP_{\theta_0,n}} - h \cdot \sum_{j=0}^{L-1} U_j + \frac{1}{2} h^T \sum_{j=0}^{L-1} \mathfrak{U}_j h \xrightarrow{P} 0.
\]

Therefore, we obtain Condition (L) of the original model under (C2\[C.2\]), (C4), and (C5\[C.5\]).

Let $\phi_q$ and $\phi_q$ be the same as the ones in the proof of Theorem 2.4. Let $P_{\theta,q,n}$ be the probability measure generated by replacing the coefficients $\tilde{a}(x,\theta)$ and $\tilde{b}(z,\theta)$ by $\tilde{a}(\phi_q(x),\theta)$ and $\tilde{b}(\phi_q(x),\phi_q(y),\theta)$ ($z = (x,y)$), respectively. Then (C2\[C.2\]), (C4), and (C5\[C.5\]) are satisfied for $\{P_{\theta,q,n}\}_{\theta,n}$. Therefore, similarly to the proof of Theorem 2.4, we have the conclusion.

\[\square\]

### D Proofs of the results in Section 3

#### D.1 Proof of Lemma 2.1

(3.19) and (2.9) yield Point 1. Moreover, we have
\[
\epsilon_n^4 \sum_{j=1}^{m_n} E[|G_j^n|^4|G_{j-1}] \\
\leq C \epsilon_n \sum_{j=1}^{m_n} E_j \left[ \frac{\hat{F}_{n,\theta,j}^\top B_{j,i,\theta_0} \hat{K}_{j}^{-1}(\theta_0) + \hat{K}_{j}^{-1}(\theta_0) B_{j,i,\theta_0} \hat{F}_{n,\theta,j}}{2} \right]_{j-1} \xrightarrow{x_{j-1} \rightarrow X_{j-1}} 0
\]

as $n \rightarrow \infty$.

\[\square\]

#### D.2 Proof of Proposition 3.1

Theorem 2.1.2 in [13] shows that $F_{n,\theta,j}$ admits a density $p_{j,x_{j-1}}(x_j,\theta)$.
For any $g \in C^1_b(\mathbb{R}^{k_1-k_{i-1}})$ and $h \in C^\infty_o(\Theta)$, we have
\[
- \int \partial_b h(\theta) E_j[g(F_{n,\theta,j})] d\theta = \int h(\theta) \partial_b E_j[g(F_{n,\theta,j})] d\theta = \int h(\theta) E_j \left[ \sum_k \frac{\partial g}{\partial y_k} [F_n,\theta,j]_k \right] d\theta = \int h(\theta) \int g(x_j) E_j[\delta_j(L^\theta(\partial_b F_{n,\theta,j}))] F_{n,\theta,j} = x_j | p_j, x_{j-1}(x_j, \theta) d\theta, d\theta.
\]
Therefore, we obtain
\[
- \int \partial_b h(\theta) p_j, x_{j-1}(x_j, \theta) d\theta = \int h(\theta) E_j[\delta_j(L^\theta(\partial_b F_{n,\theta,j}))] F_{n,\theta,j} = x_j | p_j, x_{j-1}(x_j, \theta) d\theta
\]
almost everywhere in $x_j \in \mathbb{R}^{k_1-k_{i-1}}$ for any $h \in C^\infty_o(\Theta)$. Because $C^\infty_o(\Theta)$ is separable with respect to the Sobolev norm $\| \cdot \|_{H^l}^2$, we have (D.1) for any $h \in C^\infty_o(\Theta)$ almost everywhere in $x_j$. Similarly, we can obtain an equation for $\int \partial_b h(\theta) p_j, x_{j-1} (x_j, \theta) d\theta$ for $l = 2, 3$, then Theorem 5.3 in Shigekawa [19] yields $p_j, x_{j-1}(x_j, \cdot) \in C^2(\Theta)$ almost everywhere in $x_j \in \mathbb{R}^{k_j-k_{i-1}}$. Then a similar argument with $h = \delta_\theta$ (Dirac delta) yields (3.1).

Similarly, we obtain
\[
\int g(x_j) \partial^2_b p_j, x_{j-1}(x_j, \theta) dx_j = E_j \left[ \sum_k \frac{\partial g}{\partial x_k} \partial^2_b [F_{n,\theta,j}]_k + \sum_{k,l} \frac{\partial^2 g}{\partial x_k \partial x_l} \partial_b [F_{n,\theta,j}]_k \partial_b [F_{n,\theta,j}]_l \right]
\]
\[
= E_j \left[ g(F_{n,\theta,j}) \delta_j(L^\theta(\partial^2_b F_{n,\theta,j})) \right] + E_j \left[ \sum_k \frac{\partial g}{\partial x_k} (F_{n,\theta,j})[\partial_j]_k \right]
\]
\[
= E_j \left[ g(F_{n,\theta,j}) \delta_j(L^\theta(\partial^2_b F_{n,\theta,j})) \right] + E_j \left[ |g(F_{n,\theta,j}) \delta_j(L^\theta(\partial_j)) \right]
\]
for any $g \in C^2_b(\mathbb{R}^{k_j-k_{i-1}})$, which implies (3.2).

\[
\Box
\]

D.3 Proof of Proposition [3.2]

The first inequality is obtained because
\[
E[|\delta_j(L^\theta(\partial_b F_{n,\theta,j}))|^4]^{1/4}
\leq C \left\| \sum_{k,l} \partial_b |F_{n,\theta,j}|_k [K_j^{-1}(\theta)]_{k,l} D_j[F_{n,\theta,j}] l \right\|_{1.4}
\leq C \sum_{k,l} ||\partial_b |F_{n,\theta,j}|_k ||_{1.16} [K_j^{-1}(\theta)]_{k,l} ||_{1.8} ||F_{n,\theta,j}||_{2.16} \leq C a_n k_n^2.
\]

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The estimate for $\delta_j(L^\theta(\partial_\theta \partial_\theta F_{n,\theta,j}))$ is similarly obtained. Moreover, we have
\[
\|\delta_j(L^\theta((\delta_j(L^\theta(\partial_\theta F_{n,\theta,j} \partial_\theta |F_{n,\theta,j}|_k)))_k))\|_{0,2} \\
\leq C \left\| \sum_{k,l} \delta_j(L^\theta(\partial_\theta F_{n,\theta,j} \partial_\theta |F_{n,\theta,j}|_k)) |K_j^{-1}(\theta)|_{k,l} D_j[F_n,\theta,j]t \right\|_{1,2} \\
\leq C \alpha n K_n^2 \sum_k \|\delta_j(L^\theta(\partial_\theta F_{n,\theta,j} \partial_\theta |F_{n,\theta,j}|_k))\|_{1,4} \\
\leq C \alpha n K_n^2 \sum_k \|\partial_\theta[F_n,\theta,j]k \sum_{l,t} \partial_\theta[F_n,\theta,j]l |K_j^{-1}(\theta)|_{l,t} D_j[F_n,\theta,j]t \right\|_{2,4} \\
\leq C \alpha^2 n K_n^4.
\]

\[\square\]

d.4 Proof of Lemma 3.1

We denote $F_1 = [F_{n,\theta,j}]_1$ and $\tilde{F}_1 = [\tilde{F}_{n,\theta,j}]_1$. Because
\[
\|K_j - \tilde{K}_j\|_{d,2,p} \leq \|(D_j(F_1 - \tilde{F}_1), D_j F_1)\|_{2,p} + \|(D_j \tilde{F}_1, D_j F_1 - \tilde{F}_1)\|_{2,p}
\]
and
\[
\|\langle D_j(F_1 - \tilde{F}_1), D_j F_1 \rangle\|_{2,p} \leq \|\|D_j(F_1 - \tilde{F}_1)\| H \|D_j F_1\| H\|_{2,p} \\
\leq \|F_1 - \tilde{F}_1\|_{3,2p} \|F_1\|_{3,2p} \leq C_p \rho_n,
\]
we have
\[
\sup_{i,l,j,x,j-1,\theta} |||K_j(\theta) - \tilde{K}_j(\theta)|||_{d,2,p} \leq C_p \rho_n.
\]

Moreover, we have $\tilde{K}_j = K_j + (\tilde{K}_j - K_j) = K_j(I + K_j^{-1}(\tilde{K}_j - K_j))$, $E[||K_j^{-1}||_{op}^{\frac{p}{2}}] \leq C_n \tilde{k}_n$, and
\[
E[||\tilde{K}_j^{-1}(\tilde{K}_j - K_j)||_{op}] \leq C \alpha \tilde{k}_n \cdot \rho_n \tilde{k}_n \to 0
\]
as $n \to \infty$, by (B2) and the fact that $\alpha \rho_n \tilde{k}_n^2 \to 0$. Then, for sufficiently large $n$, we have $||K_j^{-1}(\tilde{K}_j - K_j)||_{op} < 1/2$ with positive probability, and therefore $\tilde{K}_j^{-1}$ exists and
\[
||\tilde{K}_j^{-1}||_{op} \leq ||K_j^{-1}||_{op}(1 - ||K_j^{-1}(\tilde{K}_j - K_j)||_{op}^{-1}) \leq C \alpha \tilde{k}_n
\]
with positive probability. Because $\tilde{K}_j$ is deterministic, we obtain $||\tilde{K}_j^{-1}||_{d} \leq C \alpha \tilde{k}_n$. 

\[\square\]

d.5 Proof of Lemma 3.2

Lemma D.1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and $E \subseteq \mathbb{R}$ be a finite set. Assume that the derivative $f(t)$ exists almost everywhere in $t \in f^{-1}(E^c)$ and $\int_{f^{-1}(E)} |\dot{f}(t)| dt < \infty$. Then, $f$ is absolutely continuous on $[0,1].$

**Proof.** We may assume that $E$ is not empty. We denote $E = \{a_1, \ldots, a_k\}$ for some $k \in \mathbb{N}$ and $a_1 < \cdots < a_k$. It is sufficient to show that
\[
|f(t) - f(s)| \leq \int_{f^{-1}(E) \cap [s,t]} |\dot{f}(u)| du \tag{D.2}
\]
for any $0 \leq s < t \leq 1$.

Fix $s, t \in [0, 1]$ satisfying $s < t$. First, we assume that $f(t), f(s) \notin E$. We only show the case where there exist $k_1, k_2$ such that $1 \leq k_1 \leq k_2 \leq k$, $f(s) < a_{k_1}$, and $a_{k_2} < f(t)$. Other cases are proved in a similar way.

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Let \( K = k_2 - k_1 + 1 \) and \( E_\varepsilon := \{ y \in \mathbb{R} \mid \min_{x \in E} |x - y| \leq \varepsilon \} \). We set a positive number \( \varepsilon \) so that 
\[
\varepsilon < \min_{2 \leq j \leq k} |a_j - a_{j-1}|/2 \text{ and } f(t), f(s) \not\in E_\varepsilon.
\]
We inductively define 
\[
\begin{align*}
  s_0 &= \sup \{ u \in [s, t] \mid f(u) = f(s) \}, \\
  s_j &= \sup \{ u \in (t_{j-1}, t] \mid f(u) = a_{k_1+j-1} + \varepsilon \} \quad (j = 1, \ldots, K), \\
  t_j &= \inf \{ u \in (s_j, t] \mid f(u) = a_{k_1+j} - \varepsilon \} \quad (j = 0, \ldots, K-1), \\
  t_K &= \inf \{ u \in (s_K, t] \mid f(u) = f(t) \}.
\end{align*}
\]
Then, we obtain \( f^{-1}(E) \cap (\cup_{j=0}^K (s_j, t_j)) = \emptyset \) and 
\[
|f(t) - f(s)| \leq \sum_{j=0}^K |f(t_j) - f(s_j)| + \sum_{j=1}^K |f(s_j) - f(t_j-1)| \leq \sum_{j=0}^K \int_{s_j}^{t_j} |f(u)| du + 2K \varepsilon \leq \int_{f^{-1}(E^c) \cap [s, t]} |f(u)| du + 2K \varepsilon.
\]
By letting \( \varepsilon \to 0 \), we obtain (D.2).

In the case where \( f(s) = a_{k_1} \) for some \( k_1 \in \{1, \ldots, k\} \), then by setting \( s_1, t_1, \ldots, s_K, t_K \) similarly, we have 
\[
|f(t) - f(s)| \leq \sum_{j=1}^K |f(t_j) - f(s_j)| + \varepsilon
\]
for sufficiently small \( \varepsilon \), and consequently we have (D.2). We can similarly show (D.2) in the case that \( f(t) = a_{k_2} \) for some \( k_2 \in \{1, \ldots, k\} \).

**Proof of Lemma 3.2.** Let \( p, q > 1 \) with \( 1/p + 1/q = 1 \) and \( 1 - q/2 > 0 \). We abbreviate \( p_j,t(x_j) = p_{j,t-1}(x_j, \theta_{th}) \).

Then, Hölder’s inequality yields 
\[
\begin{align*}
  \int \int_0^1 \sqrt[p]{p_j,t}(x_j)|\mathcal{L}^1_j(x_j, \theta_{th})|dtdx_j \\
  &= \int_0^1 E_j \left[ \frac{1}{\sqrt[p]{p_{j,t}}}|E_j[\delta_j(L^{\theta_{th}}(\theta_{th}, F_{n, \theta_{th}}))]|F_{n, \theta_{th}}|\mathbb{1}_{\{p_j,t \neq 0\}}\right] dt \\
  &\leq \sup_{t \in [0, 1]} \left\{ E_j[|\delta_j(L^{\theta_{th}}(\theta_{th}, F_{n, \theta_{th}}))|^p]^{1/p} \left(\int p_{j,t}^{-1/q}(x_j)dx_j\right)^{1/q}\right\}.
\end{align*}
\]
Proposition 2.1.5 in Nualart [15] and its proof yield \( \sup_{t} \int_0^1 p_{j,t}^{-1/q}(x_j)dx_j < \infty \) under (N2). Together with (B1) and (B2), we obtain 
\[
\int \int_0^1 \sqrt[p]{p_{j,t}}(x_j)|\mathcal{L}^1_j(x_j, \theta_{th})|dtdx_j < \infty,
\]
which implies 
\[
\int_0^1 \frac{[\partial_j p_{j,t}]}{\sqrt[p]{p_{j,t}}}(x_j)\mathbb{1}_{\{p_j,t \neq 0\}} dt = \int_0^1 \sqrt[p]{p_{j,t}}(x_j)|\mathcal{L}^1_j(x_j, \theta_{th})|dt < \infty \tag{D.3}
\]
almost everywhere in \( x_j \).

The function \( \sqrt[p]{p_{j,t}} \) has derivative \( \partial_j \sqrt[p]{p_{j,t}} = \partial_j p_{j,t}/(2\sqrt[p]{p_{j,t}}) \) if \( \sqrt[p]{p_{j,t}} \neq 0 \) by Proposition 3.1. Therefore, Lemma (D.1) and (D.3) yield the conclusion. 

\[ \]
E Nondegeneracy of the Malliavin matrix for degenerate diffusion processes

E.1 The Malliavin matrix on a section

Let $m_1, m_2, r, L \in \mathbb{N}$. Let $(\mathfrak{X}, \mathfrak{F}, \mu)$ be the canonical probability space associated with an $r$-dimensional Wiener process $\mathcal{W} = (W_t)_{t \in [0, L]}$. Let $\mathcal{X}_t$ and $\gamma_t$ be $m_1$- and $m_2$-dimensional diffusion processes, respectively, on a Wiener space satisfying $(\mathcal{X}_0, \gamma_0) = ((x_0, y_0))$ and

\[
\begin{align*}
  d\mathcal{X}_t &= \tilde{B}(t, \mathcal{X}_t, \gamma_t)dt + \tilde{A}(t, \mathcal{X}_t, \gamma_t)dW_t, \\
  d\gamma_t &= \tilde{B}(t, \mathcal{X}_t, \gamma_t)dt,
\end{align*}
\]

where $x_0 \in \mathbb{R}^{m_1}$, $y_0 \in \mathbb{R}^{m_2}$, and $\tilde{B}(t, x, y)$, $\tilde{A}(t, x, y)$, and $\tilde{B}(t, x, y)$ are $\mathbb{R}^{m_1}$-, $\mathbb{R}^{m_1} \otimes \mathbb{R}^r$- and $\mathbb{R}^{m_2}$-valued functions, respectively. We assume that the derivatives $\partial_{(x,y)} \tilde{B}$, $\partial_{(x,y)} \tilde{B}$, and $\partial_{(x,y)} \tilde{A}$ exist and are continuous with respect to $(t, x, y)$ for $i \in \{0, 1\}$, $j \in \{1, 2, 3\}$, and $l \in \{0, 1\}$. We denote $z = (x, y)$ and by $E_\mu$ the expectation with respect to $\mu$. Let $\mathfrak{H} = L^2([0, L]; \mathbb{R}^r)$ and $\mathfrak{D}$ be the Malliavin–Shigekawa derivative operator associated with $\mathcal{W}$. Let $\gamma_{\mathcal{X}, \gamma}$ be the Malliavin matrix of $(\mathcal{X}_1, \gamma_1)$, that is,

\[
\gamma_{\mathcal{X}, \gamma} = \begin{pmatrix}
\langle D\mathcal{X}_1, D\mathcal{X}_1 \rangle & \langle D\mathcal{X}_1, D\gamma_1 \rangle \\
\langle D\gamma_1, D\mathcal{X}_1 \rangle & \langle D\gamma_1, D\gamma_1 \rangle
\end{pmatrix}.
\]

For a multi-index $(i_1, \ldots, i_l)$, we denote $|A_{i_1, \ldots, i_l}|^2 = \sum_{i_1, \ldots, i_l} A_{i_1, \ldots, i_l}^2$.

**Proposition E.1.** Assume that there exist constants $M_1$ and $M_2$ such that

\[
\sup_{t, x, y} \left( |\partial^j_{(x, y)} \tilde{B}(t, x, y)| + |\partial^j_{(x, y)} \tilde{B}(t, x, y)| + |\partial^j_{(x, y)} \tilde{A}(t, x, y)| \right) \leq M_1 \tag{E.1}
\]

for $i \in \{0, 1\}$, $j \in \{1, 2, 3\}$, and $l \in \{0, 1\}$, and

\[
\sup_{t, x, y} (\|\partial^j_{(x, y)} \tilde{B}\|_\infty + \|\partial^j_{(x, y)} \tilde{A}\|_\infty)^{-1} \leq M_2. \tag{E.2}
\]

Then $\gamma_{\mathcal{X}, \gamma}$ is positive definite almost surely, and for any $p \geq 1$, there exists a constant $C_p$ depending only on $x_0$, $y_0$, $p$, $m_1$, $m_2$, $M_1$, and $M_2$ such that $E_\mu[|\det \gamma_{\mathcal{X}, \gamma}|^{-p}] \leq C_p$. If further

\[
\partial_x \tilde{B} \equiv 0 \quad \text{or} \quad |\tilde{B}| \leq M_1, \tag{E.3}
\]

then $C_p$ depends on neither $x_0$ nor $y_0$.

To prove Proposition E.1 first we show the nondegeneracy of the Malliavin matrix $\gamma_{\mathcal{X}} = \langle D\mathcal{X}_1, D\mathcal{X}_1 \rangle \delta_{(x, y)}$ for $\mathcal{X}_1$. Let

\[
\begin{align*}
  B(t, x, y) &= \begin{pmatrix} \tilde{B}(t, x, y) \\ \tilde{B}(t, x, y) \end{pmatrix}, \\
  A(t, x, y) &= \begin{pmatrix} \tilde{A}(t, x, y) \\ \partial_{m_2, r} \end{pmatrix},
\end{align*}
\]

$B_t = B(t, \mathcal{X}_t, \gamma_t)$, and $A_t = A(t, \mathcal{X}_t, \gamma_t)$. We define an $(m_1 + m_2) \times (m_1 + m_2)$ matrix-valued process $(\mathcal{U}_t)_{t \in [0, L]}$ by a stochastic integral equation

\[
\mathcal{U}_t = \delta_{ij} + \sum_{k=1}^{m_1 + m_2} \int_0^t [\nabla B_s]_{kj} \mathcal{U}_s ds + \sum_{k, l=1}^{m_1 + m_2} \int_0^t [\nabla]_{kl} \mathcal{U}_s dW_s,
\]

where $\nabla = \partial_{(x, y)}$. Then by the argument in Section 2.3.1 of Nualart [15], $\mathcal{U}_t$ is invertible and we have

\[
\mathcal{U}_t^{-1} = \delta_{ij} - \sum_{k=1}^{m_1 + m_2} \int_0^t [\mathcal{U}_s^{-1}]_{jk} \left( [\nabla B_s]_{kj} - \sum_{l, \alpha=1}^{m_1 + m_2} [\nabla]_{kl} A_s \right) ds
\]

\[
- \sum_{k, l=1}^{m_1 + m_2} \int_0^t [\mathcal{U}_s^{-1}]_{kij} \mathcal{U}_s dW_s.
\]

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Moreover, we obtain
\[(D_tZ_t)^\top = U_tU_t^{-1}A_t1_{(t \leq t)},\tag{E.4}\]
where \(Z_t = (X_t^\top, Y_t^\top)^\top\).

**Lemma E.1.** Under the assumptions of Proposition [E.1], \(\gamma_X\) is an invertible matrix almost surely and for all \(p \geq 1\), there exists a positive constant \(C_p\) depending only on \(p\), \(m_1\), \(M_1\), and \(M_2\) such that
\[E_p[\|\det(\gamma_X)^{-p}\|] \leq C_p^p.\]

**Proof.** Let \(\mathcal{I} = (I_{m_1}, O_{m_1}, m_2)\). Let \(\tau = 1 - \sup\{t \in [0, 1]: (\|U_t\|_{op} + \|U_t^{-1}\|_{op})^3\|U_t - U_t\|_{op} > (48\sqrt{3}M_1^2M_2)^{-1} \wedge (1/6)\} \lor 0\). Then, because \(AA^\top \geq (1/M_2)I_{m_1}\), we have
\[\gamma_X = \int_1^0 (D_tX_t)^\top D_tX_t\,dt = \int_0^1 \mathcal{U}U_tU_t^{-1}A_tA_t^\top (U_t^\top)^{-1}U_t^\top I^\top dt \geq \frac{1}{M_2} \int_1^{\tau} \mathcal{U}U_tU_t^{-1}I^\top I(U_t^\top)^{-1}U_t^\top I^\top dt.\]

For \(x \in \mathbb{R}^{m_1}\) and \(t \in [1 - \tau, 1]\), simple calculations show that
\[x^\top \mathcal{U}U_tU_t^{-1}I^\top I(U_t^\top)^{-1}U_t^\top I^\top x \geq |x|^2 - |x| |\mathcal{U}U_tU_t^{-1}I^\top I(U_t^\top)^{-1}U_t^\top I^\top|_{op} - |x|^2 |\mathcal{U}U_tU_t^{-1}I^\top I(U_t^\top)^{-1}U_t^\top I^\top|_{op} \geq |x|^2 / 3.\]

Here we used Proposition 2.7 in Chapter II of Conway [1], the equation \(U_t(U_t^\top)^{-1} = (U_t - U_t)U_t^{-1}\), and the fact that
\[\|U_t^{-1}\|_{op} \leq \|U_t^{-1}\|_{op}(1 - \|U_t^{-1}\|_{op}\|U_t - U_t\|_{op})^{-1} \leq 2\|U_t^{-1}\|_{op}.\]

Hence, we have \(\gamma_X \geq \frac{M_2}{3M_2}I_{m_1}\).

Moreover, for any \(q > 0\), there exists a constant \(C''_p\) depending only on \(p\), \(m_1\), \(M_1\), and \(M_2\) such that
\[\mu[\tau < 1/t] \leq \mu \left(\left(\|U_t\|_{op} + \|U_t^{-1}\|_{op}\right)^3 \sup_{0 \leq s \leq 1/t} \|U_{1-s} - U_t\|_{op} \geq \frac{1}{48\sqrt{3}M_1^2M_2} \wedge \frac{1}{6}\right) \leq C''_p (1/t)^{2q}\]
for any \(t > 1\). Together with the equation \(E_p[\tau^{-q}] = \int_0^\infty \mu[\tau < (1/t)^{1/q}]\,dt\), we obtain the conclusion. \(\Box\)

**Proof of Proposition E.5.1** First, we have
\[\langle D\mathcal{Y}_1, D\mathcal{Y}_1 \rangle_\delta = \int_0^1 (D_t\mathcal{Y}_1)^\top D_t\mathcal{Y}_1\,dt, \quad \langle D\mathcal{Y}_1, D\mathcal{X}_1 \rangle_\delta = \int_0^1 (D_t\mathcal{Y}_1)^\top D_t\mathcal{X}_1\,dt.\]

The determinant formula for a partitioned matrix (see (0.8.5.3) in Horn and Johnson [10]) yields \(\det(\gamma_{X, Y}) = \det(\gamma_X)\det F\), where
\[F = \langle D\mathcal{Y}_1, D\mathcal{Y}_1 \rangle_\delta - \langle D\mathcal{Y}_1, D\mathcal{X}_1 \rangle_\delta \gamma_X^{-1} \langle D\mathcal{X}_1, D\mathcal{Y}_1 \rangle_\delta.\]
Therefore, thanks to Lemma E.1, it is sufficient to show that for any \( p \geq 1 \), there exists a positive constant \( C_p^{\infty} \) depending only on \( p, m_1, m_2, M_1 \), and \( M_2 \) such that \( E_p[|\det F|^{-p}] \leq C_p^{\infty} \).

Let \( M = \gamma_n^{-1} \int_0^1 (D_sX_1)^\top D_sY_1 ds \). Then, \( F \) can be rewritten as

\[
F = \int_0^1 \left( D_tY_1 - D_tX_1 M \right)^\top \left( D_tY_1 - D_tX_1 M \right) dt. \tag{E.6}
\]

We also have

\[
D_tY_1 = \int_t^1 D_t(\bar{B}(s, Z_s)) ds = \int_t^1 (D_tX_s \partial_s \bar{B}_s + D_tY_s \partial_s \bar{B}_s) ds \\
= A_t^\top (U_t^{-1})^\top \int_t^1 U_t^\top \partial_s \bar{B}_s ds + \int_t^1 D_tY_s \partial_s \bar{B}_s ds \tag{E.7}
\]

where \( \bar{B}_s = \bar{B}(s, Z_s) \). Let \( Z_{t,0} = (1-t)A_t^\top (U_t^{-1})^\top U_t^\top (I_m, O_{m_1,m_2})^\top \partial_s \bar{B}_1 \) and

\[
\tau' = \tau \land \left( 1 - \sup_{s \in [t,1]} \sup_{u \in [s,1]} \| D_t \bar{B}_s \|^\text{op} > (12 \sqrt{6} M_1 M_2 (1-t)^{-1}) \right) \land \left( 1 - \sup_{t \in (0,1)} \sup_{s \in [t,1]} \| \partial_s \bar{B}_s - \partial_s \bar{B}_1 \|^\text{op} > (32 \sqrt{3} M_1 M_2)^{-1} \right).
\]

Then, because \( \bar{B} \) is linear growth, for any \( q > 0 \), there exists a constant \( \tilde{C}_q \) such that

\[
\mu(\tau' \leq 1/t) \leq \mu(\tau \leq 1/t) + \mu \left( \sup_{s \in [0,1/t]} \sup_{u \in [s,1]} \| D_t \bar{B}_s - \bar{B}_1 - u \|^\text{op} > \frac{t}{12 \sqrt{6} M_1 M_2} \right) \tag{E.8}
\]

\[
+ \mu \left( \sup_{s \in [0,1/t]} \| \partial_s \bar{B}_s - \partial_s \bar{B}_1 \|^\text{op} > \frac{1}{32 \sqrt{3} M_1 M_2} \right)
\]

\[
\leq \tilde{C}_q (1/t)^{2q}
\]

for any \( t \geq 1 \), which implies that \( E_p[\tau'^{-q}] \) is finite.

Let \( \bar{U}_t = \bar{U} \bar{U}_t^{-1} I^\top \) and

\[
F_0 = \int_{1-\tau'}^1 (Z_{t,0} - D_tX_1 M)^\top (Z_{t,0} - D_tX_1 M) dt.
\]

By the matrix inequality

\[
- A^\top A - B^\top B \leq A^\top B + B^\top A \leq A^\top A + B^\top B \tag{E.9}
\]

for matrices \( A \) and \( B \) of the same size, we have

\[
(C - M)^\top (C - M) + (D - M)^\top (D - M) \geq (C - D)^\top (C - D)/2 \tag{E.10}
\]

for matrices \( C \) and \( D \).
Together with the inequalities $\tilde{A}_t \tilde{A}_t^\top \geq (1/M_2)I_{m_1}$ and $(\partial_x \tilde{B}_1)^\top \partial_x \tilde{B}_1 \geq (1/M_2)I_{m_2}$, we obtain
\[
F_0 = \int_{1-\tau'}^1 \mathcal{B}(t)^\top \tilde{U}_t \tilde{A}_t \tilde{A}_t^\top \tilde{U}_t^\top \mathcal{B}(t) dt \\
\geq \frac{1}{M_2} \int_{1-\tau'}^1 \mathcal{B}(t)^\top \tilde{U}_t \tilde{U}_t^\top \mathcal{B}(t) dt \\
\geq \frac{1}{M_2} \inf_{t \in [1-\tau',1]} (\|\tilde{U}_t \tilde{U}_t^\top\|_{op}) \int_{1-\tau'}^1 \mathcal{B}(t)^\top \mathcal{B}(t) dt \\
= \frac{1}{M_2} \inf_{t \in [1-\tau',1], |x|=1} \left| x^\top \tilde{U}_t \tilde{U}_t^\top x \right| \\
\times \int_{1-\tau'}/2 \left\{ \mathcal{B}(t)^\top \mathcal{B}(t) + \mathcal{B}(t - \tau'/2)^\top \mathcal{B}(t - \tau'/2) \right\} dt \\
\geq \frac{1}{2M_2} \frac{4}{9} \int_{1-\tau'}/2 \frac{\tau'^2}{4} (\partial_x \tilde{B}_1)^\top \partial_x \tilde{B}_1 dt \geq \frac{\tau'^3}{36M_2^2} I_{m_2},
\]
where $\mathcal{B}(t) = (1 - t) \partial_x \tilde{B}_1 - M$. Here we used the fact that
\[
|\tilde{U}_t^\top x|^2 = |x + \mathcal{I}(\tilde{U}_t^{-1})^\top (\tilde{U}_t - \tilde{U}_t^\top \mathcal{I}^\top x)|^2 \\
\geq \left( |x| - \|\tilde{U}_t^{-1}\|_{op}\|\tilde{U}_t - \tilde{U}_t^\top\|_{op}\|x\| \right)^2 \geq (1 - 2/6)^2 = 4/9 \tag{E.11}
\]
for $t \in [1 - \tau', 1]$ and $|x| = 1$.

Let $F'$ be similarly defined to $F$ by changing the interval of integration to $[1 - \tau', 1)$. Because (E.9) yields
\[
C^\top C - D^\top D = (C - D)^\top (C - D) + D^\top (C - D) + (C - D)^\top D \\
\geq (C - D)^\top (C - D) - 2(C - D)^\top (C - D) - D^\top D/2 \\
= -(C - D)^\top (C - D) - D^\top D/2 \tag{E.12}
\]
for matrices $C$ and $D$ of the same size, together with (E.6), (E.7), and (E.8), we obtain
\[
F' - F_0 \geq -\frac{1}{2} F_0 - \int_{1-\tau'}^1 (Z_{t,1} + Z_{t,2} - Z_{t,0})^\top (Z_{t,1} + Z_{t,2} - Z_{t,0}) dt \\
\geq -\frac{1}{2} F_0 - 2 \int_{1-\tau'}/2 (Z_{t,1} - Z_{t,0})^\top (Z_{t,1} - Z_{t,0}) dt - 2 \int_{1-\tau'}/2 Z_{t,2} Z_{t,2} dt.
\]
Because
\[
Z_{t,1} - Z_{t,0} = \tilde{A}_t^\top (\tilde{U}_t^{-1})^\top \int_t^1 \left\{ \tilde{U}_s - \tilde{U}_t^\top \mathcal{I}^\top \partial_x \tilde{B}_s + \tilde{U}_t^\top \mathcal{I} \tilde{U}_s^\top (\partial_x \tilde{B}_s - \partial_x \tilde{B}_t) \right\} ds,
\]
we have
\[
F' - F_0/2 \\
\geq -4M_1^4 \tau'^3 \sup_{t \in [1-\tau',1]} (\|\tilde{U}_t^{-1}\|_{op}^2) \sup_{t \in [1-\tau',1]} (\|\tilde{U}_t - \tilde{U}_t^\top\|_{op}^2) I_{m_2} \\
- 4M_2^2 \tau'^3 \sup_{t \in [1-\tau',1]} \|\tilde{U}_t \tilde{U}_t^{-1}\|_{op}^2 \sup_{t \in [1-\tau',1]} \|\partial_x \tilde{B}_t - \partial_x \tilde{B}_1\|_{op}^2 I_{m_2} \\
- 2M_2^2 \tau'^3 \sup_{1-\tau' \leq \tau \leq 1} \|D_t \mathcal{Y}_s\|_{op}^2 I_{m_2} \\
\geq -16M_1^4 \tau'^3 \|\tilde{U}_t^{-1}\|_{op}^2 \sup_{t \in [1-\tau',1]} \|\tilde{U}_t - \tilde{U}_t^\top\|_{op}^2 I_{m_2} \\
- \frac{64}{9} M_1^2 \tau'^3 \sup_{t \in [1-\tau',1]} \|\partial_x \tilde{B}_t - \partial_x \tilde{B}_1\|_{op}^2 I_{m_2} \\
- 2M_2^2 \tau'^3 \sup_{1-\tau' \leq \tau \leq 1} \|D_t \mathcal{Y}_s\|_{op}^2 I_{m_2} \\
\geq -\frac{\tau'^3}{432M_2^2} I_{m_2} - \frac{\tau'^3}{432M_2^2} I_{m_2} - \frac{\tau'^3}{432M_2^2} I_{m_2}.
\]
Here we used (E.5) and the fact that \( \|U_t U_t^{-1}\|_{op} \leq 1 + \|U_1 - U_t\|_{op} \|U_t^{-1}\|_{op} \leq 4/3 \) for \( t \in [1 - \tau^{'}, 1] \). Therefore, we conclude that

\[
\det F \geq \det F' \geq \det \left( \frac{1}{2} F_0 - \frac{\tau^3}{144 M_2^2} I_m \right) \geq \left( \frac{\tau^3}{144 M_2^2} \right)^{m_2} .
\]

If further (E.3) is satisfied, then the upper bound in (E.8) depends on neither \( x_0 \) nor \( y_0 \) because

\[
\partial_x\dot B_t - \partial_x\dot B_s = \int_{s}^{t} \sum_{i} \partial_x\partial_{y_i}\dot B_u[\dot B_u]_i du + \text{(terms with bounded moments)}
\]

by Itô’s formula.

\[\square\]

### E.2 The Malliavin matrix of block observations

Let \( \gamma_l \) be the Malliavin matrix of \( ((X_j, Y_j))_{j=1}^{L} \).

**Proposition E.2.** Assume the conditions of Proposition [E.1]. Let \( C_p \) be the one in Proposition [E.1]. Then, \( \gamma_l \) is positive definite almost surely, and \( E_{\mu}[|\gamma_{l}|^{-p}] \leq C_{pL} \) for any \( p \geq 1 \).

**Proof.** We may assume that \( L \geq 2 \). Let \( 2 \leq l \leq L \). Because (E.4) implies

\[
(D_t \gamma_l)^{T} = U_t U_t^{-1} A_t = U_t U_t^{-1}(D_t \gamma_{l-1})^{T}
\]

for \( t \leq l - 1 \), we have

\[
\langle DZ_j, D\gamma_l \rangle_{\mathcal{B}} = \langle DZ_j, D\gamma_{l-1} \rangle_{\mathcal{B}} (U_{l-1}^{-1})^{T} U_t^{T}
\]

for \( j \leq l - 1 \), and

\[
\langle DZ_l, D\gamma_l \rangle_{\mathcal{B}} = \int_{0}^{t} \langle DZ_l(D_t \gamma_l)^{T} D_t \gamma_l dt
\]

\[
= \langle DZ_l, D\gamma_{l-1} \rangle_{\mathcal{B}} (U_{l-1}^{-1})^{T} U_t^{T} + \int_{l-1}^{t} \langle DZ_l(D_t \gamma_l)^{T} D_t \gamma_l dt.
\]

Then, by setting \( \tilde{\gamma}_{l} = \int_{l-1}^{t} (D_t \gamma_l)^{T} D_t \gamma_l dt \), we have \( \gamma_l = \det \tilde{\gamma}_{l-1} \det \tilde{\gamma}_{l} \) because

\[
\left( \begin{array}{c}
\langle DZ_1, D\gamma_{l-1} \rangle_{\mathcal{B}} (U_{l-1}^{-1})^{T} U_t^{T} \\
\vdots \\
\langle DZ_l, D\gamma_{l-1} \rangle_{\mathcal{B}} (U_{l-1}^{-1})^{T} U_t^{T}
\end{array} \right)
\]

is a linear combination of \( \{[\gamma_l]_{ij} \}_{1 \leq i \leq (m_1 + m_2)l} \}_{(m_1 + m_2)(l-2) < (m_1 + m_2)(l-1)} \). Therefore, Proposition [E.1] implies that

\[
E_{\mu}[|\det \gamma_l|^{-p}] = E_{\mu} \left[ \prod_{l=1}^{L} |\det \tilde{\gamma}_{l}|-p \right] \leq \prod_{l=1}^{L} E_{\mu}[|\det \tilde{\gamma}_{l}|-pL] \leq C_{pL}.
\]

\[\square\]

### F An auxiliary lemma related to partitioned matrices

**Lemma F.1.** Let \( A_1, A_2, B, C \) be matrices of suitable size so that

\[
\begin{pmatrix}
A_1 & B \\
B^{T} & C
\end{pmatrix}
\]

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is a partitioned matrix for \( i = 1, 2 \). Assume that \( A_1 \) and \( C - B^\top A_1^{-1} B \) are invertible. Then we have

\[
A_1^{-1}(A_2 \ B) \left( \begin{array}{cc} A_1 & B \\ B^\top & C \end{array} \right)^{-1} \left( \begin{array}{c} A_2 \\ B^\top \end{array} \right) = (A_1^{-1} A_2)^2 + A_1^{-1}(A_2 A_1^{-1} - I)B(C - B^\top A_1^{-1} B)^{-1}B^\top (A_1^{-1} A_2 - I).
\]  

(F.1)

In particular, the right-hand side of (F.1) is equal to the unit matrix if \( A_1 = A_2 \).

Proof. A simple calculation yields the conclusion by using (0.8.5.6) in Horn and Johnson [10].