Preparation and measurement: two independent sources of uncertainty in quantum mechanics

Willem M. de Muynck
Department of Theoretical Physics, Eindhoven University of Technology, Eindhoven, the Netherlands

In the Copenhagen interpretation the Heisenberg uncertainty relation is interpreted as the mathematical expression of the concept of complementarity, quantifying the mutual disturbance necessarily taking place in a simultaneous or joint measurement of incompatible observables. This interpretation has already been criticized by Ballentine a long time ago, and has recently been challenged in an experimental way. These criticisms can be substantiated by using the generalized formalism of positive operator-valued measures, from which a new inequality can be derived, precisely illustrating the Copenhagen concept of complementarity. The different roles of preparation and measurement in creating uncertainty in quantum mechanics are discussed.

I. INTRODUCTION

The Copenhagen view on the meaning of quantum mechanics largely originated from the consideration of so-called “thought experiments”, like the double-slit experiment and the γ microscope. These experiments demonstrate that there is a mutual disturbance of the measurement results in a joint measurement of two incompatible observables \( A \) and \( B \) (like position \( Q \) and momentum \( P \)). The Heisenberg-Kennard-Robertson uncertainty relation

\[
\Delta A \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|,
\]

in which \( \Delta A \) and \( \Delta B \) are standard deviations, has often been interpreted as the mathematical expression of this disturbance (in Heisenberg’s paper\(^1\) only position \( Q \) and momentum \( P \) are considered). However, as noted by Ballentine\(^2\), this uncertainty relation does not seem to have any bearing on the issue of joint measurement, because it can be experimentally tested by measuring each of the observables separately, subsequently multiplying the standard deviations thus obtained. Moreover, such an interpretation is at variance with the standard formalism developed by Dirac and von Neumann, which only allows the joint measurement of compatible observables. According to Ballentine\(^2\) quantum mechanics is silent about the joint measurement of incompatible observables. If this were true, however, what would this mean for the disturbance idea originating from the “thought experiments”? How could these experiments be useful in clarifying the meaning of a mathematical formalism that is not capable of yielding a description of such experiments?

Nowadays measurements like the double-slit experiment no longer are “thought” experiments\(^3\)\(^−\)\(^9\), and complementarity, in the sense of mutual disturbance, has been experimentally demonstrated in an unequivocal way. However, in agreement with Ballentine’s observation the relation of these experiments with the Heisenberg-Kennard-Robertson inequality \( [\bullet] \) has proved controversial\(^10\)\(^,\)\(^11\). Whereas Storey et al.\(^10\) conclude that “the principle of complementarity is a consequence of the Heisenberg uncertainty relation,” Scully et al.\(^11\) observe that “The principle of complementarity is manifest although the position-momentum uncertainty relation plays no role.” Duerr et al.\(^9\) stress that quantum correlations due to the interaction of
object and detector, rather than “classical” momentum transfer, enforces the loss of interference in a which-way measurement. In their experiment momentum disturbance is not large enough to account for the loss of interference if the measurement arrangement is changed so as to yield ‘which-way’ information.

Actually, two different questions are at stake here. First, the question might be posed whether the Heisenberg inequality of position and momentum is the relevant one for interference experiments. Second, there is the problem observed by Ballentine, which is the more fundamental question whether the Heisenberg inequality is applicable at all. Contrary to the latter question, the former might be thought to have a relatively simple answer. In general, interference experiments like the one of Ref. 9 are not joint measurements of position and momentum but of a different pair of observables $A$ and $B$ (see section IV for an example). Hence, rather than the inequality $\Delta Q \Delta P \geq \hbar/2$ relation (1) for observables $A$ and $B$ seems to be relevant to the experiment. Although position and momentum may also be disturbed by the interaction with the detector, this need not be related to complementarity because $A$ and $B$ rather than $Q$ and $P$ are involved in the correlations between object and detector. Hence, the controversy could be resolved by pointing out which (incompatible) observables are measured jointly in the experiment. However, we would then have to deal with quantum mechanics’ alleged silence with respect to such experiments.

It seems that Ballentine’s problem with respect to the applicability of (1) to the joint measurement of incompatible observables $A$ and $B$ has more far-reaching consequences because it points to a fundamental confusion regarding complementarity within the Copenhagen interpretation. This is due to the poor distinction made between the different aspects of preparation and measurement involved in physical experimentation. As a matter of fact, in the Copenhagen interpretation a measurement is not perceived as a means of obtaining information about the initial (pre-measurement) state of the object, but as a way of preparing the object in some final (post-measurement) state. Due to this view on the meaning of “measurement” there is insufficient awareness that both the preparation of the initial state as well as the measurement may contribute to the dispersion of an observable. The Copenhagen issue of complementarity actually has two different aspects, viz. the aspects of preparation and measurement, which are not distinguished clearly enough. If such a distinction is duly made, it is not difficult to realize that the notion of “measurement disturbance” should apply to the latter aspect, whereas the Heisenberg-Kennard-Robertson uncertainty relation refers to the former. With no proper distinction between preparation and measurement the Copenhagen interpretation was bound to amalgamate the two forms of complementarity, thus interpreting the Heisenberg-Kennard-Robertson uncertainty relation as a property of (joint) measurement. Unfortunately, remnants of this view are still abundant in the quantum mechanical literature.

The purpose of the present paper is to demonstrate that the Copenhagen confusion of preparation and measurement largely is a consequence of the inadequateness of the standard formalism for the purpose of yielding a description of certain quantum mechanical experiments, and joint measurements of incompatible observables in particular. To describe such measurements it is necessary to generalize the quantum mechanical formalism so as to encompass positive operator-valued measures (POVMs); the standard formalism is restricted to the projection-valued measures corresponding to the spectral representations of selfadjoint operators. The gen-
eralized formalism will briefly be discussed in sect III. In sect. IV the generalized formalism will be applied to neutron interference experiments that can be seen as realizations of the double-slit experiment. By employing the generalized formalism of POVMs it is possible to interpret such experiments as joint non-ideal measurements of incompatible observables like the ones considered in the “thought experiments”. An inequality, derived from the generalized theory by Martens, yields an adequate expression of the mutual disturbance of the information obtained on the initial probability distributions of two incompatible observables in a joint measurement of these observables. How both contributions to complementarity can be distinguished in the measurement results obtained in such experiments will be discussed in sect. V. A proof of the Martens inequality is given in Appendix B.

II. CONFUSION OF PREPARATION AND MEASUREMENT

The confusion of preparation and measurement is already present in the Copenhagen thesis that quantum mechanics is a complete theory. As a consequence of this thesis a physical quantity cannot have a well-defined value preceding the measurement (because this would correspond to an “element of physical reality” as employed by Einstein, Podolsky and Rosen to demonstrate the incompleteness of quantum mechanics). For this reason a quantum mechanical measurement cannot serve to ascertain this value in the way customary in classical mechanics. Heisenberg proposed an alternative for quantum mechanics, to the effect that the value of an observable is well-defined immediately after the measurement, and, hence, is more or less created by the measurement. For Heisenberg his uncertainty relation did not refer to the past (i.e. to the initial state), but to the future (i.e. the final state): it was seen as a consequence of the disturbing influence of the measurement on observables that are incompatible with the measured one. Hence, for Heisenberg a quantum mechanical measurement was a preparation (of the final state of the object), rather than a determination of certain properties of the initial state. As emphasized by Ballentine, the interpretation of the Heisenberg-Kennard-Robertson uncertainty relation usually found in quantum mechanics textbooks, is in disagreement with Heisenberg’s views, because in the textbook view this relation is not considered a property of the measurement process but, rather, of the initial object state.

Also Bohr did not draw a clear distinction between preparation and measurement. He always referred to the complete experimental arrangement (often indicated as “the measuring instrument”) serving to define the measured observable. For Bohr the uncertainty relation was an expression of our limitations in jointly defining complementary quantities (like position and momentum) within the context of a measurement. He did not distinguish different phases of the measurement. More particularly he did not distinguish different contributions to complementarity from the preparation of the initial state and from the disturbance by the measurement. According to Bohr the uncertainty relation refers to the “latitudes” of the definition of incompatible observables within the context of a well-defined measurement arrangement, deemed valid for the measurement as a whole. Incidentally, we see a manifest difference here with Heisenberg’s views, a difference that may have confused anyone trying to understand the Copenhagen interpretation as a consistent way of looking at quantum mechanics. Moreover, the discrepancy between the Copenhagen interpretations of the uncertainty relation (viz. as a property of the measurement, either during this measurement (Bohr), or afterwards (Heisenberg)) and the textbook interpretation (viz. as a property of the preparation preceding the
Figure 1: Double-slit experiment

measurement may have caused some uneasiness in many students.

Obviously, two completely different issues are at stake here, corresponding to different forms of complementarity. As stressed by Ballentine, the Heisenberg-Kennard-Robertson uncertainty relation (1), in which $\Delta A$ and $\Delta B$ are standard deviations in separately performed measurements, should be taken, in agreement with textbook interpretation, as referring to the preparation of the initial state. On the other hand, the Copenhagen idea of complementarity in the sense of mutual disturbance in a joint measurement of incompatible observables, is certainly not without a physical basis. Thus, in the double-slit experiment (cf. figure 1) Bohr demonstrated that, if the quantum mechanical character of screen $S$ is taken into account, our possibility to define the position and momentum of a particle passing the slits is limited by the Heisenberg uncertainty relation

$$\Delta z_S \Delta p_{zS} \geq \hbar / 2$$

of the screen observables $z_S$ and $p_{zS}$. As a matter of fact, the lower bounds with which the latitudes $\delta z$ and $\delta p_z$ of particle position and momentum are defined, are equal to the standard deviations $\Delta z_S$ and $\Delta p_{zS}$, respectively. Hence, these latitudes must satisfy the inequality

$$\delta z \delta p_z \geq \hbar / 2.$$  (3)

In Heisenberg’s terminology this inequality can be interpreted as expressing a lower bound for the disturbing influence exerted by the measuring instrument on the particle, thus causing the post-measurement state of the object to satisfy an uncertainty relation.

Inequality (3) should be distinguished from the uncertainty relation

$$\Delta z \Delta p_z \geq \hbar / 2$$

satisfied by the standard deviations $\Delta z$ and $\Delta p_z$ of position and momentum of the particle in its initial state. Whereas inequality (4), being an instance of inequality (2), does not refer in any way to joint measurement of position and momentum, but can be interpreted as a property of the preparation of the object preceding the measurement, inequality (3) does refer to the measurement process, since it is derived from a relation (viz. (2)) satisfied by a part of the measurement arrangement (screen $S$).

Unfortunately, in discussions of the double-slit experiment such a distinction usually is not made. On the contrary, equating the quantities $\delta z$ and $\delta p_z$ from (3) with the standard deviations $\Delta z$ and $\Delta p_z$, the derivation of (3) is generally interpreted as
an illustration of the relation (I). As a consequence it is not sufficiently realized that preparation of the initial state and (joint) measurement are two distinct physical sources of uncertainty, yielding similar but physically distinct uncertainty relations that express different forms of complementarity. Only the former one is represented by a relation (viz. (I)), which can straightforwardly be derived from the standard formalism. Bohr’s analysis of the double-slit experiment demonstrates that there is a second form of complementarity, which is not a property of the preparation of the initial state as represented by the Heisenberg-Kennard-Robertson relation, but which is due to mutual disturbance in a joint measurement of position and momentum.

One important cause of the mixing up of the two forms of complementarity is the fact, as stressed by Ballentine, that the quantum mechanical formalism as axiomatized by von Neumann and Dirac defies a description of joint measurement of incompatible observables. In particular, such a measurement would have to yield joint probability distributions of the incompatible observables. However, within the standard formalism no mathematical quantities can be found that are able to play such a role. Thus, according to Wigner’s theorem\(^{20}\) no positive phase space distribution functions \(f(q, p)\) exist that are linear functionals of the density operator \(\rho\) such that \(\int dp \ f(q, p) = \langle q | \rho | q \rangle\) and \(\int dq \ f(q, p) = \langle p | \rho | p \rangle\). Also von Neumann’s projection postulate is often interpreted as prohibiting the joint measurement of incompatible observables, since there is no unambiguous eigenstate that can serve as the final state of such a measurement. For this reason only measurements of one single observable, for which the Heisenberg-Kennard-Robertson relation has an unambiguous significance, are usually considered in axiomatic treatments.

On the other hand, Ballentine’s judgment with respect to the inability of the quantum mechanical formalism to deal with the second kind of complementarity seems to be too pessimistic. Thus, for specific measurement procedures generalized Heisenberg uncertainty relations have been derived\(^{7,8,21,22}\), different from the Heisenberg-Kennard-Robertson relation, in which the uncertainties seem to contain contributions from both sources. Moreover, in the following it will be demonstrated that the generalized quantum mechanical formalism is able to deal with the two forms of complementarity separately, thus distinguishing the contributions due to preparation and (joint) measurement.

III. GENERALIZED MEASUREMENTS

In the generalized quantum mechanical formalism the notion of a quantum mechanical measurement is generalized so as to encompass measurement procedures that can be interpreted as joint measurements of incompatible observables of the type considered in the “thought experiments”. A possibility to do so is offered by the so-called operational approach\(^{12}\), in which the interaction between object and measuring instrument is treated quantum-mechanically, and measurement results are associated with pointer positions of the latter. If \(\rho_o\) and \(\rho_a\) are the initial density operators of object and measuring instrument, respectively, then the probability of a measurement result is obtained as the expectation value of the spectral representation \(\{E_m^{(a)}\}\) of some observable of the measuring instrument in the final state \(\rho_f = U \rho \rho_o U^\dagger\), \(U = \exp(-iHT/\hbar)\) of the measurement. Thus, \(p_m = Tr_{oa} \rho_f E_m^{(a)}\). This quantity can be interpreted as a property of the initial object state, \(p_m = Tr_o \rho M_m\), with \(M_m = Tr_a \rho_a U^\dagger E_m^{(a)} U\).
The quantum mechanical formalism is generalized to a certain extent by the operational approach. Whereas in the standard formalism quantum mechanical probabilities \( p_m \) are represented by the expectation values of mutually commuting projection operators (\( p_m = \langle E_m \rangle \), \( E_m^2 = E_m \), \( [E_m, E_{m'}]_\pm = O \)), the generalized formalism allows these probabilities to be represented by expectation values of operators \( M_m \) that are not necessarily projection operators, and need not commute (\( M_m^2 \neq M_m \), \( [M_m, M_{m'}]_\pm \neq O \) in general). The operators \( M_m \), \( O \leq M_m \leq I \), \( \sum_m M_m = I \) generate a so-called positive operator-valued measure (POVM); the observables of the standard Dirac-von Neumann formalism are restricted to those POVMs of which the elements are mutually commuting projection operators (so-called projection-valued measures).

After having generalized the notion of a quantum mechanical observable it is possible to define a relation of partial ordering between observables, expressing that the measurement represented by one POVM can be interpreted as a non-ideal measurement of another. Thus, we say that a POVM \( \{R_m\} \) represents a non-ideal measurement of the (generalized or standard) observable \( \{M_{m'}\} \) if the following relation holds between the elements of the POVMs:

\[
R_m = \sum_{m'} \lambda_{mm'} M_{m'}, \quad \lambda_{mm'} \geq 0, \quad \sum_m \lambda_{mm'} = 1.
\] (5)

The matrix \( (\lambda_{mm'}) \) is the non-ideality matrix. It is a so-called stochastic matrix. Its elements \( \lambda_{mm'} \) can be interpreted as conditional probabilities of finding measurement result \( a_m \) if an ideal measurement had yielded measurement result \( a_{m'} \). In the case of an ideal measurement the non-ideality matrix \( (\lambda_{mm'}) \) reduces to the unit matrix \( (\delta_{mm'}) \). As an example we mention photon counting using an inefficient photon detector (quantum efficiency \( \eta < 1 \)), for which the probability of detecting \( m \) photons during a time interval \( T \) can be found (cf. Kelley and Kleiner) as:

\[
p_m(T) = Tr\rho N \left( \frac{(\eta a^\dagger a)^m}{m!} \exp(-\eta a^\dagger a) \right)
\] (6)

(in which \( a^\dagger \) and \( a \) are photon creation and annihilation operators, and \( N \) is the normal ordering operator). Defining the POVM \( \{R_m\} \) of the inefficient measurement by means of the equality \( p_m(T) = Tr\rho R_m \), it is not difficult to prove that \( R_m \) can be written in the form

\[
R_m = \sum_{n=0}^{\infty} \lambda_{mn} |n\rangle\langle n|,
\] (7)

with \( |n\rangle\langle n| \) the projection operator projecting on number state \( |n\rangle \), and

\[
\lambda_{mn} = \begin{cases} 
0, & m > n, \\
\binom{n}{m} \eta^m (1-\eta)^{n-m}, & m \leq n.
\end{cases}
\] (8)

For \( \eta = 1 \) the non-ideality matrix is seen to reduce to the unit matrix, and the POVM \( \{\[\]\} \) to coincide with the projection-valued measure corresponding to the spectral representation of the photon number observable \( N = \sum_{n=0}^{\infty} n|n\rangle\langle n| \).

Non-ideality relations of the type (5) are well-known from the theory of transmission channels in the classical theory of stochastic processes, where the non-ideality matrix describes the crossing of signals between subchannels. It should be noted,
however, that, notwithstanding the classical origin of the latter subject, the non-ideality relation (8) may be of a quantum mechanical nature. Thus, the interaction of the electromagnetic field with the inefficient detector is a quantum mechanical process just like the interaction with an ideal photon detector is. Relations of the type (5) are abundant in the quantum theory of measurement. They can be employed to characterize the quantum mechanical idea of mutual disturbance in a joint measurement of incompatible observables.

Generalizing the notion of quantum mechanical measurement to the joint measurement of two (generalized) observables, it seems reasonable to require that such a measurement should yield a bivariate joint probability distribution $p_{mn}$, satisfying $p_{mn} \geq 0, \sum_{mn} p_{mn} = 1$. Here $m$ and $n$ label the possible values of the two observables measured jointly, corresponding to pointer positions of two different pointers (one for each observable) being jointly read for each individual preparation of an object. It is assumed that, analogous to the case of single measurement, the probabilities $p_{mn}$ of finding the pair $(m,n)$ are represented in the formalism by the expectation values $\langle R_{mn} \rangle$ of a bivariate POVM $\{R_{mn}\}$, $R_{mn} \geq 0, \sum_{mn} R_{mn} = I$ in the initial state of the object. Then the marginal probabilities $\{\sum_n p_{mn}\}$ and $\{\sum_m p_{mn}\}$ are expectation values of POVMs $\{M_m = \sum_n R_{mn}\}$ and $\{N_n = \sum_m R_{mn}\}$, respectively, which correspond to the (generalized) observables jointly measured.

In Appendix A it is proven that, if the observables corresponding to the POVMs $\{M_m\}$ and $\{N_n\}$ are standard observables (i.e. if the operators $M_m$ and $N_n$ are projection operators), then joint measurement is only possible if these observables commute$^{26}$. This result, derived here from the generalized formalism, corroborates the standard formalism for those measurements to which the latter is applicable. Note, however, that in general commutativity of the operators $M_m$ and $N_n$ is not a necessary condition for joint measurability of generalized observables (see section IV for an example).

The notion of joint measurement can be extended in the following way. We say that a measurement, represented by a bivariate POVM $\{R_{mn}\}$, can be interpreted as a joint non-ideal measurement of the observables $\{M_m\}$ and $\{N_n\}$ if the marginals $\{\sum_n R_{mn}\}$ and $\{\sum_m R_{mn}\}$ of the bivariate POVM $\{R_{mn}\}$ describing the joint measurement represent non-ideal measurements of observables $\{M_m\}$ and $\{N_n\}$. Then, in accordance with (8) two non-ideality matrices $(\lambda_{mn'})$ and $(\mu_{mn'})$ should exist, such that

$$\sum_n R_{mn} = \sum_{n'} \lambda_{mn'} M_{m'}, \lambda_{mn'} \geq 0, \sum_{n} \lambda_{mn'} = 1,$$
$$\sum_m R_{mn} = \sum_{m'} \mu_{mn'} N_{n'}, \mu_{mn'} \geq 0, \sum_{m} \mu_{mn'} = 1. \quad (9)$$

It is possible that $\{M_m\}$ and $\{N_n\}$ are standard observables. To demonstrate that the joint measurement scheme, given above, is a useful one, neutron interference experiments will be discussed in the next section as an example, satisfying the definition of a joint non-ideal measurement of two standard observables. It should be noted that this example is not an exceptional one, but can be supplemented by many others$^{27,28,29,30}$. For instance, in analogy “eight-port optical homodyning”$^7$ can be interpreted as a joint non-ideal measurement of the observables $Q = (a + a^\dagger)/\sqrt{2}$ and $P = (a - a^\dagger)/i\sqrt{2}$ of a monochromatic mode of the electromagnetic field.

If $\{M_m\}$ and $\{N_n\}$ are standard observables the non-idealities expressed by the non-ideality matrices $(\lambda_{mn'})$ and $(\mu_{mn'})$ can be proven$^{13}$ to satisfy the characteristic traits of the type of complementarity that is due to mutual disturbance in a joint measurement of incompatible observables as dealt with in the “thought experiment”.
A measure of the departure of a non-ideality matrix from the unit matrix is required for this. A well-known quantity serving this purpose is Shannon’s channel capacity \(25\).

Here we consider a closely related quantity, viz. the average row entropy of the non-ideality matrix \(\lambda_{mm'}\),

\[
J(\lambda) = -\frac{1}{N} \sum_{mm'} \lambda_{mm'} \ln \frac{\lambda_{mm'}}{\sum_{m''} \lambda_{mm''}},
\]

that (restricting to square \(N \times N\) matrices) satisfies the following properties:

\[
0 \leq J(\lambda) \leq \ln N, \\
J(\lambda) = 0 \text{ if } \lambda_{mm'} = \delta_{mm'}, \\
J(\lambda) = \ln N \text{ if } \lambda_{mm'} = \frac{1}{N}.
\]

Hence, the quantity \(J(\lambda)\) vanishes in the case of an ideal measurement of observable \(\{M_m\}\), and obtains its maximal value if the measurement is uninformative (i.e. does not yield any information on the observable measured non-ideally) due to maximal disturbance of the measurement results. For a joint non-ideal measurement as defined by \(9\), the non-idealities of both non-ideality matrices \(\lambda_{mm'}\) and \(\mu_{nn'}\) can be quantified in a similar way.

In Appendix B it is demonstrated that for a joint non-ideal measurement of two standard observables \(A = \sum_m a_m M_m\) and \(B = \sum_n b_n N_n\), with eigenvectors \(|a_m\rangle\) and \(|b_n\rangle\), respectively, the non-ideality measures \(J(\lambda)\) and \(J(\mu)\) obey the following inequality:

\[
J(\lambda) + J(\mu) \geq -2 \ln \{max_{mn} |\langle a_m | b_n \rangle|\}.
\]

It is evident that \(11\) is a nontrivial inequality (the right-hand side unequal to zero) if the two observables \(A\) and \(B\) are incompatible in the sense that the operators do not commute. I shall refer to inequality \(11\) as the Martens inequality. It is important to note that this inequality is derived from relation \(9\), and, hence, must be satisfied in any measurement procedure that can be interpreted as a joint non-ideal measurement of two incompatible standard observables\(31\). In relation \(4\) only the observables (i.e. the measurement procedures) are involved. Contrary to the Heisenberg-Kennard-Robertson inequality \(1\), the Martens inequality is completely independent of the initial state of the object. Hence, the Martens inequality does not refer to the preparation of the initial state, but to the measurement process.

The Martens inequality should be clearly distinguished from the entropic uncertainty relation\(32,33\) for the standard observables \(A = \sum_m a_m M_m\) and \(B = \sum_n b_n N_n\),

\[
H_{\{M_m\}}(\rho) + H_{\{N_n\}}(\rho) \geq -2 \ln \{max_{mn} |\langle a_m | b_n \rangle|\},
\]

in which \(H_{\{M_m\}}(\rho) = -\sum_m p_m \ln p_m\), \(p_m = Tr \rho M_m\) (and analogously for \(B\)). The inequality \(12\), although quite similar to the Martens inequality, should be compared with the Heisenberg-Kennard-Robertson inequality \(1\), expressing a property of the initial state \(\rho\), to be tested by means of separate measurements of observables \(\{M_m\}\) and \(\{N_n\}\).

IV. NEUTRON INTERFERENCE EXPERIMENTS

Instead of the classical double-slit experiment we shall consider an interference experiment performed with neutrons\(34,35,36\). Due to the simplicity of its mathematical description this experiment yields a better illustration of the problem of
complementarity due to mutual disturbance in a joint measurement of incompatible observables, than is provided by the “thought experiment”. The interferometer consists of a silicon crystal with three parallel slabs (cf. figure 2) in which the neutron can undergo Bragg reflection. A neutron impinging in A at the Bragg angle is then either transmitted in the same direction or Bragg reflected. Hence, the neutron may take one of two possible paths. After reflection in the middle slab (B resp. C) the partial waves of the two paths are brought into interference again in the third slab (D). After that the neutron may be found in one of the two out-going beams by detector $D_1$ or $D_2$. Since it is possible to achieve a separation of the paths by several centimeters in the interferometer, it is possible to influence each of the partial beams separately (cf. figure 3). For instance, we can insert an aluminum plate into one of the paths, causing a phase shift $\chi$ of the partial wave, depending on the plate’s thickness. By varying the thickness an interference pattern is obtained when registering the number of neutrons detected by detector $D_1$ (or $D_2$). Summhammer, Rauch and Tuppinger\textsuperscript{35} performed experiments in which, apart from a phase shifter, an absorbing medium was also inserted into one of the paths (indicated in figure 3 by its transmission coefficient $a$), consisting of gold or indium plates. Then the interference pattern also depends on the value of $a$. The visibility of the interference is maximal if $a = 1$. In such a case we have a pure interference experiment. If the absorbing plate is very thick (such that $a = 0$) every neutron taking that path will be absorbed. In that case it is certain that a neutron that is registered by one of the detectors has taken the other path. Then we have a pure “which path” measurement, in which the visibility of the interference pattern completely vanishes. For $0 < a < 1$ the situation is an intermediate one. This situation will be considered in the following. Whereas the experiments corresponding to the limiting values $a = 1$ and $a = 0$ can be dealt with using the standard formalism, this is not the case for the intermediate values of $a$.

Let $|k_1\rangle$ and $|k_2\rangle$ correspond to the plane waves that impinge at the Bragg angle (cf. figure 3). It is assumed\textsuperscript{36} that each Bragg reflection induces a phase shift of $\pi/2$ in the wave vector. The phase shifter changes the phase of the wave passing it by $\chi$; the absorber alters its amplitude by a factor of $\sqrt{a}$. Thus for an arbitrary incoming state $|in\rangle = \alpha|k_1\rangle + \beta|k_2\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, we find\textsuperscript{27} the following out-going state:

$$
|out\rangle = \frac{1}{2} \{ \alpha (-1 - \sqrt{ae^{i\chi}}) + \beta (i + \sqrt{ae^{i\chi}}) \}|k_1\rangle + \{ \alpha (i - i\sqrt{ae^{i\chi}}) + \beta (-1 - \sqrt{ae^{i\chi}}) \}|k_2\rangle + \sqrt{1 - a^2} (i\alpha - \beta)|abs\rangle.
$$

(13)

Here $|abs\rangle$ denotes the state of the absorbed neutron, assumed to be orthogonal to
Figure 3: Neutron interference experiment by Summhammer, Rauch and Tuppinger

$|k_1\rangle$ and $|k_2\rangle$. The detection probabilities $p_1$ and $p_2$ of the two detectors, and the absorption probability $p_3$ are found from this as

$$p_1 = |\langle k_1 | \text{out} \rangle|^2, \quad p_2 = |\langle k_2 | \text{out} \rangle|^2, \quad p_3 = |\langle \text{abs} | \text{out} \rangle|^2.$$ 

With

$$p_1 = \langle \text{in} | M_1 | \text{in} \rangle, \quad p_2 = \langle \text{in} | M_2 | \text{in} \rangle, \quad p_3 = \langle \text{in} | M_3 | \text{in} \rangle$$

the measured detection probabilities are related to the incoming state, thus yielding an operational definition of the POVM \{\$M_1, M_2, M_3\$\} representing the generalized observable measured in the Summhammer-Rauch-Tuppinger experiment.

We first consider the limits $a = 1$ and $a = 0$. From (13) and (14) for $a = 1$ we find that $p_3 = 0$ and $M_1 = Q_1$, $M_2 = Q_2$, with $Q_1$ and $Q_2$ projection operators, in the two-dimensional representation of the vectors $|k_1\rangle$ and $|k_2\rangle$ being represented by the matrices

$$Q_1 = \begin{pmatrix} \cos^2 \frac{1}{2} \chi & -\frac{1}{2} \sin \chi \\ -\frac{1}{2} \sin \chi & \sin^2 \frac{1}{2} \chi \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \sin^2 \frac{1}{2} \chi & \frac{1}{2} \sin \chi \\ \frac{1}{2} \sin \chi & \cos^2 \frac{1}{2} \chi \end{pmatrix}. \quad (15)$$

The standard observable having these operators as its spectral representation will be referred to as the interference observable.

For $a = 0$ we analogously find $M_1 = M_2 = P_+ / 2$, $M_3 = P_-$, with $P_+$ and $P_-$ projection operators represented by the matrices

$$P_+ = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad P_- = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \quad (16)$$

Also the operators $P_+$ and $P_-$ constitute a spectral representation of a standard observable, the path observable, being incompatible with the interference observable.

For $0 < a < 1$ the operators $M_m$, $m = 1, 2, 3$ are found in an analogous way, according to

$$M_1 = \frac{1}{2} [P_+ + aP_- + \sqrt{a} (Q_1 - Q_2)],$$
$$M_2 = \frac{1}{2} [P_+ + aP_- - \sqrt{a} (Q_1 - Q_2)], \quad (17)$$
$$M_3 = (1 - a)P_-.$$

It is important to note that in this case the operators $M_1$, $M_2$ and $M_3$ are not projection operators. Only in the limits $a = 1$ and $a = 0$ is there a direct link
with a standard observable. For the majority of experiments \((0 < a < 1)\) the detection probabilities are not described by the expectation values of the spectral representation of one single selfadjoint operator (as would be the case within the standard formalism).

It is possible, using definition (9), to interpret the neutron interference experiment as a joint non-ideal measurement of the interference and path observables defined by (14) and (15). In order to do so the operators \(M_m\) of the experiment are ordered in a bivariate form according to
\[
R_{mn} = \begin{pmatrix} M_1 & M_2 \\ \frac{1}{2}M_3 & \frac{1}{2}M_3 \end{pmatrix}, \quad m = +, -, n = 1, 2.
\]

Then the marginals \(\{\sum_m R_{m1}, \sum_m R_{m2}\}\) and \(\{\sum_n R_{+n}, \sum_n R_{-n}\}\) can easily be verified to satisfy the conditions (9) for non-ideal measurements of the path and interference observables, respectively, with non-ideality matrices
\[
(\lambda_{mm'}) = \begin{pmatrix} 1 & a \\ 0 & 1 - a \end{pmatrix}, \quad (\mu_{nn'}) = \frac{1}{2} \begin{pmatrix} 1 + \sqrt{a} & 1 - \sqrt{a} \\ 1 - \sqrt{a} & 1 + \sqrt{a} \end{pmatrix}.
\]

It is interesting to consider the \(a\) dependence of the non-ideality matrices (19). For \(a = 0\) we have \(\lambda_{mm'} = \delta_{mm'}, \mu_{nn'} = 1/2\). In this case the path measurement is ideal, whereas the non-ideality of the interference measurement is maximal (the corresponding POVM is given by \(\{I/2, I/2\}\), implying that the POVM's expectation values do not provide information about the incoming state of the neutron). For \(a = 1\) the situation is just the opposite. Then \(\lambda_{+m'} = 1, \lambda_{-m'} = 0, \mu_{nn'} = \delta_{nn'}\). Now the interference measurement is ideal, and the path measurement is uninformative. For \(0 < a < 1\), in which the standard formalism is not applicable, both measurements are non-ideal. In going from \(a = 0\) to \(a = 1\) the non-ideality of the path measurement increases; that of the interference measurement decreases.

For the non-ideality measures \(J(\lambda)\) and \(J(\mu)\) defined by (10) we obtain
\[
J(\lambda) = \frac{1}{2} [(1 + a) \ln(1 + a) - a \ln a],
J(\mu) = \frac{1}{2} [2 \ln 2 - (1 + \sqrt{a}) \ln(1 + \sqrt{a}) - (1 - \sqrt{a}) \ln(1 - \sqrt{a})].
\]

From the parametric plot in figure 4 it can be seen that the Martens inequality (11) is satisfied. This illustrates the impossibility that both non-ideality measures \(J(\lambda)\) and \(J(\mu)\) jointly have a small value. Figure 4 clearly illustrates the idea of complementarity as this arises in the “thought experiments”. If \(a\) is varied, then the measurement arrangement is altered. For Bohr this would signify a different definition of the path and interference observables for each different value of \(a\); the “latitudes” of the definition of the observables depending on \(a\). For Heisenberg the path observable is disturbed more by the measurement process as \(a\) increases, whereas the interference observable is disturbed less. Both would interpret this as an expression of the complementarity of the path and interference observables, due to the fact that the operators \(P_+\) and \(P_-\) do not commute with \(Q_1\) and \(Q_2\). Evidently, the \(a\) dependence of the non-ideality matrices \((\lambda_{mm'})\) and \((\mu_{nn'})\) precisely expresses the complementarity that is connected with the mutual disturbance in a joint non-ideal measurement of the incompatible interference and path observables.

It should be noted, however, that there also is a difference with Heisenberg’s disturbance ideas. In the neutron interference experiment the non-idealities do not
Figure 4: Parametric plot of $J_\lambda$ versus $J_\mu$, for $0 \leq a \leq 1$. The shaded area is the region that is forbidden by the Martens inequality.

refer to the final object state, but to the information obtained on the initial state. Hence, these quantities do not refer to the preparative aspect of measurement (as is the case in Heisenberg’s interpretation of his uncertainty relation), but to the determinative one. Contrary to the standard formalism the generalized formalism as embodied by (9) is capable of referring to the past (rather than to the future), even if measurements are involved in which measurement disturbance plays an important role. The generalized formalism enables us to consider quantum mechanical measurements in the usual determinative sense, and allows us to distinguish this determinative aspect from the question in which (post-measurement) state the object is prepared by the measurement. Incidentally it is noted that the marginals \( \{ \sum_m R_{m1}, \sum_m R_{m2} \} \) and \( \{ \sum_n R_{n1}, \sum_n R_{n2} \} \) of the bivariate POVM \( \{\mathcal{R}\} \) constitute a non-commuting pair of generalized observables jointly measured.

V. DISCUSSION

For unbiased non-ideal measurements, i.e. measurements for which the non-ideal and the ideal versions in (5) yield the same expectation values for operators \( \sum_m a_m M_m \) and \( \sum_m a_m R_m \), the non-ideality matrix \( (\lambda_{mm'}) \) should satisfy the equality \( a_{m'} = \sum_m a_m \lambda_{mm'} \). If we restrict to unbiased non-ideal measurements it also is possible to demonstrate that there are two sources of uncertainty by using standard deviations. Thus, using the notation \( r_m = \text{Tr} \rho R_m, p_m = \text{Tr} \rho M_m \), the relation \( r_m = \sum_{m'} \lambda_{mm'} p_{m'} \) between the probability distributions \( \{p_m\} \) (of the ideal measurement) and \( \{r_m\} \) (obtained in the non-ideal one) is found from (5). For unbiased measurements the standard deviation of the measurement results \( a_{m'} \) of observable \( A = \sum_m a_m M_m \), obtained in the non-ideal measurement, can easily be seen to satisfy the equality

\[
\Delta(\{r_m\})^2 = \Delta(\{p_m\})^2 + \sum_{m'} \Delta_{m'}^2 p_{m'},
\]

with

\[
\Delta_{m'}^2 := \sum_m a_m^2 \lambda_{mm'} - (\sum_m a_m \lambda_{mm'})^2.
\]

The quantity \( \Delta(\{p_m\})^2 \) obtained in an ideal measurement, which is independent of the parameters of the
measurement arrangement, and, for this reason interpretable in the usual way as a property of the initial state of the object, and ii) a contribution $\sum_{m'} \Delta_{m'^2}^2 \rho_{m'}$ due to the non-ideality of the measurement procedure. Also it is not difficult to see that

$$\Delta(\{r_m\}) \geq \Delta(\{p_m\}).$$

If in a joint non-ideal measurement of two incompatible observables $A$ and $B$ both non-ideal measurements are unbiased, then for the joint non-ideal measurement the generalized Heisenberg uncertainty relation

$$\Delta(\{r_m\}) \Delta(\{s_n\}) \geq \Delta A \Delta B \geq \frac{1}{2} | \langle [A, B] \rangle |$$

($\{s_n\}$ the non-ideally measured probability distribution of observable $B$) immediately follows from the Heisenberg-Kennard-Robertson relation (1).

A disadvantage of (21) is that not all non-ideal measurements are unbiased. For instance, as easily follows from (8), detector inefficiency will cause the average measured photon number to be smaller than the ideal one. For this reason (21) is not universally valid. Moreover, in the expressions for $\Delta(\{r_m\})$ and $\Delta(\{s_n\})$ the two contributions to uncertainty are merged into one single quantity. An inequality, analogous to (21), that is valid for biased measurements too might be obtained by combining the entropic uncertainty relation (12) with the Martens inequality (11), thus yielding

$$(H_{\{M_m\}} + J_{(\lambda)}) + (H_{\{N_n\}} + J_{(\mu)}) \geq -4 \ln \max_{mn} |\langle a_m | b_n \rangle|.$$  (22)

However, it is evident that it is not very meaningful to do this because in (22) the two different contributions are once again merged, thus veiling their different origins. As follows from (12) and (11) both sources satisfy their own inequality. The opportunity entropic quantities offer for exhibiting this seems to be an important advantage of these quantities over the widely used standard deviations. It has occasionally been noted$^{22}$ that for specific measurement procedures an uncertainty relation for the joint measurement of incompatible observables can be formulated in terms of standard deviations. It is not at all clear, however, whether a relation exists, that is comparable to the Martens inequality, and valid for all quantum mechanical measurements interpretable as joint measurements of incompatible standard observables.

Failure to distinguish the different contributions to uncertainty represented by the different terms in (21) and (22) is at the basis of the Copenhagen confusion with respect to the uncertainty relations originating with the discussion of the double-slit experiment. Because no clear distinction was drawn between preparation and measurement, these could not be properly distinguished as different sources of “uncertainty”, both contributing in their own way. Since inequality (3) refers to the measurement process rather than to the preparation of the initial state, it should be compared to the Martens inequality rather than to the Heisenberg-Kennard-Robertson one. The fact that (3) has the same mathematical form as (4) is caused by the more or less accidental circumstance that the uncertainty induced by the measurement process in the double-slit example is a consequence of the preparation uncertainty of a part of the measurement arrangement (viz. screen $S$) described by (2). However, as demonstrated by the neutron interference example, the measurement disturbance seems to more generally originate from the quantum mechanical
character of the whole interaction process of object and measuring instrument. Fluc-
tuations of the latter may be a part of this, but need not always play an essential
role in the complementarity issue.

It is important to stress that the Martens inequality is obtained from the gen-
eralized formalism, being capable of describing measurements represented by POVMs.
The founders of the Copenhagen interpretation did not dispose of this formalism.
Indeed, in the “thought experiments” a measurement is always thought to be rep-
resented by a selfadjoint operator (i.e. a projection-valued measure). In the example
of neutron interferometry this implies a restriction to the extreme values \( a = 0 \) and
\( a = 1 \). The restriction to these extreme values was responsible for the view in which
interference is completely disturbed in a ‘which-way’ measurement (and vice versa).
This, indeed, is confirmed by the limiting values of the non-ideality matrices \( (19) \),
yielding an uninformative marginal for path if interference is measured ideally (and
vice versa). In the intermediate region \( 0 < a < 1 \) information on both observables
is obtained, be it that this information is disturbed in the way described by the
Martens inequality.

From the generalized formalism it is clear that in the neutron interference experi-
ment complementarity of the interference and path observables \((15) \) and \((16) \) is at
stake. Nevertheless, as is evident from the recent discussion referred to above\(^9,10,11\),
this effect is still sometimes associated with the Heisenberg inequality for position
and momentum. It seems that in this discussion the confusion between complemen-
tarity of preparation and measurement still exists. Of course, since a measurement
may also be a preparation procedure for a post-measurement state of the object,
the Heisenberg inequality \( \Delta Q \Delta P \geq \hbar / 2 \) (as well as inequality \((1) \) for any choice of
observables \( A \) and \( B \) ) should also hold in the post-measurement object state. This,
however, is independent of this procedure being a measurement. As a matter of
fact, \( Q \) and \( P \) must satisfy the Heisenberg inequality in the post-measurement state
independently of which observables \( A \) and \( B \) have been measured jointly.

Complementarity in the sense of mutual disturbance in a joint measurement of
incompatible observables, as characterized by the Martens inequality, does not refer
to the preparation of the post-measurement state, but to a restriction with respect
to obtaining information on the initial object state. Apart from this difference,
the Martens inequality nevertheless seems to be the mathematical expression of the
Copenhagen concept of complementarity, viz. mutual disturbance in a joint (or si-
multaneous) measurement of incompatible observables. It seems that the physical
intuition that was expressed by the “thought experiments” was perfect in this re-
spect. However, confusion had to arise because of the impossibility of dealing with
joint measurements of incompatible observables using the standard formalism. Bohr
and Heisenberg were led astray by the availability of the uncertainty relation \((4) \) (or,
more generally, \((1) \)) following from this latter formalism, unjustifiedly thinking that
this relation provided a materialization of their physical intuition.

APPENDIX A

In this Appendix it is proven that standard observables \( A \) and \( B \) can be measured
jointly if and only if they commute. Thus, let \( M_m \) and \( N_n \) be projection operators of
the spectral representations of \( A \) and \( B \), and \( M_m = \sum_n R_{mn}, N_n = \sum_m R_{mn} \), \( \{ R_{mn} \} \)
a POVM. Then \( [M_m, N_n]_\pm = 0 \), and \( R_{mn} = M_m N_n \).

The proof makes use of a well-known property of positive operators, stating that
if $B$ is a positive operator and $P$ a projection operator satisfying $B \leq P$, then $B = PBP$.

Since $R_{mn} \leq M_m$, if $M_m$ is a projection operator we have $R_{mn} = M_m R_{mn} M_m$. Since $R_{mn} \leq N_n$, also $R_{mn} = N_n R_{mn} N_n$. Hence, $M_m = \sum_n R_{mn} = \sum_n N_n R_{mn} N_n$. Because of $N_n N_n' = \delta_{nn'} N_n$, multiplying this expression from both sides by $N_n$ yields:

$$M_m N_n = N_n M_m = N_n R_{mn} N_n.$$ 

Hence, $[M_m, N_n] = 0$, and $R_{mn} = M_m N_n$.

Conversely, if $[M_m, N_n] = 0$, then $\{R_{mn} = M_m N_n\}$ is a POVM satisfying $\sum_n R_{mn} = M_m$, $\sum_m R_{mn} = N_n$.

**APPENDIX B**

In this appendix a derivation is given\(^{13}\) of the Martens inequality \(\text{(III)}\). We shall restrict ourselves to maximal standard observables for which the operators $M_m$ and $N_n$ are one-dimensional projection operators. From

$$M_m |a_m\rangle = \delta_{mm'} |a_{m'}\rangle, \quad N_n |b_n\rangle = \delta_{nn'} |b_{n'}\rangle$$

and

$$\sum_n R_{mn} = \sum_m \lambda_{mm'} M_{m'}, \quad \sum_m R_{mn} = \sum_n \mu_{nn'} N_{n'}$$

it follows that

$$\lambda_{mm'} = \langle a_{m'} | \sum_n R_{mn} | a_{m'} \rangle, \quad \mu_{nn'} = \langle b_{n'} | \sum_m R_{mn} | b_{n'} \rangle.$$ 

It is not difficult to see that $J(\lambda)$ can be written as

$$J(\lambda) = \frac{1}{N} \sum_m (\text{Tr} \sum_n R_{mn}) H_{\{M_m\}} \left( \frac{\sum_{n'} R_{mn'}}{\text{Tr} \sum_{n''} R_{mn''}} \right),$$

and, analogously,

$$J(\mu) = \frac{1}{N} \sum_n (\text{Tr} \sum_m R_{mn}) H_{\{N_n\}} \left( \frac{\sum_{m'} R_{m'n}}{\text{Tr} \sum_{m''} R_{m'n'p'n}} \right).$$

In these expressions the arguments of the functions $H_{\{M_m\}}$ and $H_{\{N_n\}}$ are positive operators with trace equal to 1. Therefore it is possible to use the well-known inequality\(^{37}\)

$$H_{\{M_m\}}(\sum_n p_n \rho_n) \geq \sum_n p_n H_{\{M_m\}}(\rho_n).$$

$O < \rho_n < 1, \text{Tr} \rho_n = 1, 0 \leq p_n \leq 1, \sum_n p_n = 1$\(^{(23)}\)

to find a lower bound to $J(\lambda)$ (and analogously for $J(\mu)$). Taking in \(\text{(23)}\):

$$p_n = \frac{\text{Tr} R_{mn}}{\text{Tr} \sum_{m'} R_{mn'}}, \quad \rho_n = \frac{R_{mn}}{\text{Tr} R_{mn}}$$

we obtain the inequality

$$J(\lambda) \leq \frac{1}{N} \sum_n (\text{Tr} \sum_{n'} R_{mn'}) H_{\{M_m\}} \left( \sum_n p_n \rho_n \right) \geq \frac{1}{N} \sum_{mn} (\text{Tr} R_{mn}) H_{\{M_m\}} \left( \frac{R_{mn}}{\text{Tr} R_{mn}} \right).$$

\(\text{(24)}\)
Analogously we find

\[ J(\mu) \geq \frac{1}{N} \sum_{mn} (TrR_{mn})H\{N_n\} \left( \frac{R_{mn}}{TrR_{mn}} \right). \]  

(25)

From (24) and (25) it then follows that

\[ J(\lambda) + J(\mu) \geq \frac{1}{N} \sum_{mn} (TrR_{mn}) \left( H\{M_m\} \left( \frac{R_{mn}}{TrR_{mn}} \right) + H\{N_n\} \left( \frac{R_{mn}}{TrR_{mn}} \right) \right). \]

Since also \( R_{mn}/TrR_{mn} \) is a positive operator with trace 1, we can use inequality (12), with \( \sum_{mn} R_{mn} = I \), \( TrI = N \) and \( TrM_nN_n = |\langle a_m|b_n \rangle|^2 \), to arrive at the Martens inequality (11).

\[ \square \]

NOTES AND REFERENCES

1) W.M.d.Muynck@phys.tue.nl
2) W. Heisenberg, “Über den anschaulichen Inhalt der quantentheoretischen Kinetik und Mechanik,” Zeitschr. f. Phys. 43, 172–198 (1927) (English translation: “The physical content of quantum kinematics and mechanics,” in: J.A. Wheeler and W.H. Zurek, Quantum theory and measurement (Princeton University Press, 1983) pp. 62–84).
3) L. E. Ballentine, “The statistical interpretation of quantum mechanics,” Rev. Mod. Phys. 42, 358–380 (1970).
4) E.g. G. Möllenstedt and C. Jönsson, “Elektronen-Mehrfachinterferenzen an regelmässig hergestellten Feinspalten,” Zeitschr. f. Phys. 155, 472–474 (1959).
5) J. H. Shapiro and S. S. Wagner, “Phase and amplitude uncertainties in heterodyne detection,” Journ. Quant. Electr. QE 20, 803–813 (1984).
6) N. G. Walker and J. E. Caroll, “Simultaneous phase and amplitude measurements on optical signals using a multiport junction,” Électr. Lett. 20, 981–983 (1984).
7) H. Rauch, “Tests of quantum mechanics by neutron interferometry,” in: G. Gruber et al. eds, Les fondements de la mécanique quantique, 25e Cours de perfectionnement de l’Association Vaudoise des Chercheurs en Physique (1983), pp. 330–350; G. Badurek, H. Rauch and D. Tuppinger, “Neutron-interferometric double-resonance experiment,” Phys. Rev. A34, 2600–2608 (1986).
8) E. Arthurs and J. L. Kelly Jr., “On the simultaneous measurement of a pair of conjugate observables,” Bell Syst. Techn. Journ. 44, 765–779 (1965).
9) Y. Yamamoto, S. Machida, S. Saito, N. Imoto, T. Yanagawa, M. Kitagawa, and G. Björk, “Quantum mechanical limit in optical precision measurement and communication,” in: E. Wolf ed., Progress in optics (Elsevier Science Publishers B.V., 1990), Vol. XXVIII, pp. 87–179.
10) S. Duerr, T. Nonn and G. Rempe, “Origin of quantum-mechanical complementarity probed by a ‘which-way’ experiment in an atom interferometer,” Nature 395, 33-37 (1998).
11) E.P. Storey, S.M. Tan, M.J. Collett and D.F. Walls, Nature 375, 368 (1995).
12) M.O. Scully, B.-G. Englert and H. Walther, “Quantum optical tests of complementarity,” Nature 351, 111-116 (1991); B.-G. Englert, M.O. Scully and H. Walther, Nature 375, 367-368 (1995).
12E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, London, 1976); A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982); G. Ludwig, *Foundations of Quantum Mechanics* (Springer, Berlin, 1983, Vols. I and II); P. Busch, M. Grabowski and P. J. Lahti, *Operational quantum mechanics* (Springer-Verlag, Berlin, Heidelberg, 1995).

13H. Martens and W. de Muynck, “Nonideal quantum measurements,” Found. of Phys. 20, 255–281 (1990); H. Martens and W. de Muynck, “The inaccuracy principle,” Found. of Phys. 20, 357–380 (1990).

14A. Einstein, B. Podolsky, and N. Rosen, “Can quantum-mechanical description of physical reality be considered complete?” Phys. Rev. 47, 777–780 (1935).

15W. Heisenberg, *The physical principles of quantum theory* (Dover Publications, Inc., 1930), pp. 20–46.

16See also P. Jordan, “Quantenphysikalische Bemerkungen zur Biologie und Psychologie,” Erkenntnis 4, 215–252 (1934). It should be noted that this does certainly not agree with Bohr’s views.

17N. Bohr, Como Lecture, “The quantum postulate and the recent development of atomic theory,” in: J. Kalckar ed., *N. Bohr, Collected Works* (North-Holland, Amsterdam, 1985), Vol. 6, pp. 113–136.

18N. Bohr, “Discussion with Einstein on epistemological problems in atomic physics,” in: P. A. Schilpp ed., *Albert Einstein: Philosopher–Scientist* (Open Court, La Salle, Ill. 1982), 3rd ed., pp. 199-241.

19See, for instance, E. Scheibe, *The logical analysis of quantum mechanics* (Pergamon Press, Oxford, etc., 1973), pp. 42–49.

20E.P. Wigner, “Quantum-mechanical distribution functions revisited,” in: W. Yourgrau, A. van der Merwe eds., *Perspectives in quantum theory* (The MIT Press, Cambridge, Mass., 1971), pp. 25–36.

21E. Arthurs and M. S. Goodman, “Quantum correlations: A generalized Heisenberg uncertainty relation,” Phys. Rev. Lett. 60, 2447–2449 (1988).

22M. G. Raymer, “Uncertainty principle for joint measurement of noncommuting variables,” Am. J. Phys. 62, 986–993 (1994).

23e.g. F.R. Gantmacher, *Application of the theory of matrices* (Interscience Publishers Inc., New York, 1959).

24P.L. Kelley and W.H. Kleiner, “Theory of electromagnetic field measurement and photoelectron counting,” Phys. Rev. A136, 316–334 (1964).

25e.g. R. McEliece, *The theory of information and coding* (Addison–Wesley, London, 1977).

26A different proof can be found in A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982), p. 71. Using the method employed in Appendix A it is possible to prove that commutativity is necessary and sufficient for joint measurement already if only one of the marginals is a standard observable (W.M. de Muynck and J.M.V.A. Koelman, “On the joint measurement of incompatible observables in quantum mechanics,” Phys. Lett. 98A, 1–4 (1983)).

27W. M. de Muynck and H. Martens, “Neutron interferometry and the joint measurement of incompatible observables,” Phys. Rev. 42A, 5079–5086 (1990).

28W. M. de Muynck, W. W. Stoffels, and H. Martens, “Joint measurement of interference and path observables in optics and neutron interferometry,” Physica B 175, 127–132 (1991).

29H. Martens and W. M. de Muynck, “Single and joint spin measurements with a Stern-Gerlach device,” Journ. Phys. A: Math. Gen. 26, 2001–2010 (1993).
W. M. de Muynck and H. Martens, “Quantum mechanical observables and positive operator valued measures,” in: W. Florek, D. Lipinski and T. Lulek eds., Symmetry and structural properties of condensed matter (World Scientific, Singapore, 1993), pp. 101–120.

Note that the inequality can easily be generalized to nonmaximal standard observables having a discrete spectrum. From the example of “eight-port optical homodyning” it is seen that also in case of continuous spectra a similar behaviour of mutual disturbance can be observed. For this case the problem of finding an inequality like the Martens one has not been completely solved however (cf. V. Dorofeev and J. de Graaf, “Some maximality results for effect-valued measures,” Indag. Mathem. N.S. 8, 349–369 (1997)).

D. Deutsch, “Uncertainty in quantum measurements,” Phys. Rev. Lett. 50, 631-633 (1983); M. H. Partovi, “Entropic formulation of uncertainty for quantum mechanics,” Phys. Rev. Lett. 50, 1883-1885 (1983); K. Kraus, “Complementary observables and uncertainty relations,” Phys. Rev. D35, 3070–3075 (1987).

H. Maassen and J. B. M. Uffink, “Generalized entropic uncertainty relations,” Phys. Rev. Lett. 60, 1103–1106 (1988).

S. A. Werner and A. G. Klein, “Neutron optics,” in: K. Sköld and D. L. Price eds, Methods of Experimental Physics (Academic Press, Orlando, 1985), Vol. 23, Part A, pp. 259–337.

J. Summhammer, H. Rauch, and D. Tuppinger, “Stochastic and deterministic absorption in neutron-interferometric experiments,” Phys. Rev. A36, 4447–4455 (1987).

A. Zeilinger, “Complementarity in neutron interferometry,” Physica 137B, 235–244 (1986).

R. Balian, From microphysics to macrophysics (Springer-Verlag, Berlin, etc., 1991), Vol. I, pp. 111–122.