A q-deformation of the parastatistics and an alternative to the Chevalley description of $U_q[osp(2n+1/2m)]$

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Abstract. The paper contains essentially two new results. Physically, a deformation of the parastatistics in a sense of quantum groups is carried out. Mathematically, an alternative to the Chevalley description of the quantum orthosymplectic superalgebra $U_q[osp(2n + 1/2m)]$ in terms of $m$ pairs of deformed parabosons and $n$ pairs of deformed parafermions is outlined.

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1. Introduction

In this paper we give an alternative to the Chevalley definition of the quantum superalgebra \( U_q[osp(2n+1/2m)] \) in terms of generators and relations (see Eqs. (30)). We generalize to the quantum case a result we have recently obtained, namely that the universal enveloping algebra \( U[osp(2n+1/2m)] \) of the orthosymplectic Lie superalgebra \( osp(2n+1/2m) \) is an associative unital algebra with generators, called Green generators (operators),

\[
a_1^+, a_2^+, \ldots, a_{m-1}^+, a_m^+, a_{m+1}^+, \ldots, a_{m+n}^+ \equiv a_N^+, \tag{1}
\]

and relations

\[
[a_{N-1}^+, a_N^+], a_N^+] = 0, \quad [[a_i^+, a_j^-], a_k^+] = 2\eta^{(k)} \delta_{jk} a_i^+, \quad \forall |i-j| \leq 1, \quad \eta = \pm, \tag{2}
\]

where

\[
\deg(a_i^+) \equiv \langle i \rangle = \begin{cases} 1, & \text{for } i \leq m \\ 0, & \text{for } i > m. \end{cases} \tag{3}
\]

Here and throughout \([a, b] = ab - (-1)^{\deg(a)\deg(b)}ba\), \([a, b] = ab - ba\), \(\{a, b\} = ab + ba\).

The motivation for the present work stems from the observation that the Green operators provide a description of \( osp(2n+1/2m) \) via generators, which, contrary to the Chevalley elements, have a direct physical significance. As it was shown in Ref. 1 \( a_1^+, a_2^+, \ldots, a_{m-1}^+, a_m^+, a_{m+1}^+, \ldots, a_{m+n}^+ \) (resp. \( a_m^+, a_{m+1}^+, \ldots, a_{m+n}^+ \equiv a_N^+ \)) are para-Bose (pB) (resp. para-Fermi (pF)) operators. These operators were introduced in the quantum field theory as a possible generalization of the statistics of the tensor (resp. spinor) fields.\(^2\) Therefore what we are actually doing here is a simultaneous deformation of the pB and the pF operators (2) in the sense of quantum groups.\(^3\),\(^4\)

The fact that \( n \) pairs of pF creation and annihilation operators \( \mathrm{CAOs} \) generate the Lie algebra \( so(2n+1) \) was first observed in Refs. 5 and 6. It took some time to incorporate the para-Bose statistics into an algebraic structure: \( a_1^+, \ldots, a_m^+ \) are odd elements, generating a Lie superalgebra \(^7\) isomorphic to \( osp(1/2m) \).\(^8\) It is usually assumed that the pB operators commute with pF operators. Other possibilities were also investigated.\(^9\) In Ref. 10 it was indicated that the relations between the pB and pF operators can be selected in such a way that they generate \( osp(2n+1/2m) \).

The identification of the parastatistics with a well known algebraic structure has far going consequences. Firstly, it indicates that the representation theory of \( n \) pairs of pF operators (or of \( m \) pairs of pB operators) is completely equivalent to the representation theory of \( so(2n+1) \) (resp. \( osp(1/2m) \)). On this way one may enlarge considerably \(^11\),\(^12\) the class of the known representations, those corresponding to a fixed order of the statistics. In particular, since the (complex) Lie algebra \( so(2n+1) \) has infinite-dimensional representations, so do the pF operators. Similarly \( osp(1/2m) \) has finite-dimensional representations (for instance the defining one) and therefore the pB operators have also such representations. Secondly, it provides a natural background for further generalizations of the quantum statistics. In order to give a hint of where this possibility comes from consider in the frame of the quantum field theory a field \( \Psi(x) \).

In the momentum space the translation invariance of the field is expressed as a commutator between the energy-momentum \( P^m, m = 0, 1, 2, 3 \) and the CAOs \( a_i^+ \) of the field:

\[
[P^m, a_i^+] = \pm k_i^m a_i^+, \tag{4}
\]
where the index $i$ replaces all (continuous and discrete) indices of the field and

$$P^m = \sum_i k_i^m H_i. \quad (5)$$

In the case of the pF statistics $H_i = \frac{1}{2}[a_i^+, a_i^-]$, whereas for pB fields $H_i = \frac{1}{2}\{a_i^+, a_i^-\}$. In a unified form $H_i = \frac{1}{2}[a_i^+, a_i^-]$ with the pF considered as even elements and the pB - as odd. To quantize the field means, loosely speaking, to find solutions of Eqs. (4) and (5), where the unknowns are the CAOs $a_i^\pm$.

The first opportunity for further generalizations is based on the observation that both $so(2n + 1)$ and $osp(1/2m)$ belong to the class $B$ superalgebras in the classification of Kac.\textsuperscript{13} Therefore it is natural to try to satisfy the quantization equations (4) and (5) with CAOs, generating superalgebras from the classes $A$, $C$ or $D$\textsuperscript{13} or generating other superalgebras from the class $B$. It turns out this is possible indeed. Examples of this kind, notably the $A$--statistics, related to the completion and the central extension of $sl_\infty$ were studied in Refs. 14 and 15 (see also Example 2 in Ref. 16 and the the other references in this paper). The Wigner quantum systems (WQSs), introduced in Refs. 17 and 18, are also examples of this kind, however in the frame of a noncanonical quantum mechanics. Some of these systems possess quite unconventional physical features, properties which cannot be achieved in the frame of the quantum mechanics. The $(n + 1)$--particle WQS, based on $sl(1/3n) \in A$,\textsuperscript{19} exhibits a quark like structure: the composite system occupies a small volume around the centre of mass and within it the geometry is noncommutative. The underlying statistics is a Haldane exclusion statistics,\textsuperscript{20} a subject of considerable interest in condensed matter physics. The $osp(3/2) \in B$ WQS, studied in Ref. 21, leads to a picture where two spinless point particles, curling around each other, produce an orbital (internal angular) momentum $1/2$.

The second opportunity for generalization of the statistics is based on deformations of the relations (2). Assume only for simplicity that in (5) $i = 1, \ldots, n$ (the consideration remain valid for $n = \infty$). Then both in the pF and the pB case $H_i$ are elements from the Cartan subalgebra $H$ of $so(2n + 1)$ and $osp(1/2n)$, respectively. The CAOs are root vectors of these (super)algebras (see Ref. 15 for more detailed discussions). The important point now comes from the observation that the commutation relations between the Cartan elements and the root vectors, in particular the quantization equations (4) and (5), remain unaltered upon $q$-deformations. Therefore one can satisfy the quantization equations (4) and (5) also with deformed pF (resp. pB) operators. Certainly in this case the relations $H_i = \frac{1}{2}[a_i^+, a_i^-]$ cannot be preserved anymore. One has to postulate the expression (5), introducing the additional (Cartan) generators $H_i$ similarly as in the case of a deformed harmonic oscillator, where one is forced to introduce also number operators.\textsuperscript{22,23,24}

The conclusion is that the deformed pF operators $\{a_i^\pm, H_i|i = 1, \ldots, n\}$, being solutions of the quantization equations (4) and (5), enlarge the class of the possible statistics. It turns out these are the operators, which provide an alternative to the Chevalley description of $U_q[so(2n + 1)]$. This was shown in Ref. 25. A similar problem for $U_q[osp(1/2m)]$, corresponding to a $q$--deformation of the pB operators, was first carried out for $m = 1$\textsuperscript{26} and then for any $m$.\textsuperscript{27,28,29} Here we generalize the results for any $U_q[osp(2n + 1/2m)]$, $n, m > 1$, namely when both para-Bose and para-Fermi operators are involved. This amounts to simultaneous deformation of the parabosons and the parafermions as one single supermultiplet.

In Sect. 2, after recalling the definition of $U_q[osp(2n + 1/2m)]$, we introduce the deformed Green generators (13) and derive the relations (30) they satisfy. In Sect 3 we solve the inverse problem. We express the Chevalley elements via the Green generators and show that the relations among the Chevalley generators
follow from the properties of the Green operators. This leads to the conclusion that the deformed Green operators provide an alternative description of $U_q(osp(2n + 1/2m))$.

Throughout the paper we use the notation (some of them standard):

- $C$ - all complex numbers;
- $C[[h]]$ - the ring of all formal power series in $h$ over $C$;
- $q = e^h \in C[[h]]$, $\bar{q} = q^{-1}$;
- $[a, b] = ab - (-1)^{deg(a)deg(b)}ba$, $[a, b] = ab - ba$, $\{a, b\} = ab + ba$;
- $[a, b]_x = ab - (-1)^{deg(a)deg(b)}xba$, $[a, b]_x = ab - xba$, $\{a, b\}_x = ab + xba$;

$$\deg(a_i^+) \equiv \langle i \rangle = \begin{cases} 1, & \text{for } i \leq m \\ 0, & \text{for } i > m. \end{cases}$$

$q_i = q^{(-1)^{(i+1)}}$, i.e. $q_i = \bar{q}$, $i < m$, $q_i = q$, $i \geq m$.

For a convenience of further references we list here some deformed identities, which will be often used.

(Id(3) follows from Id(2)).

$Id(1)$: If $[a, c] = 0$, then $(x + x^{-1})[h, [a, [b, c]]] = [a, [b, [c, c]]] - [[h, [b, a]], c]]$ - all complex numbers;

$Id(2)$: If $B$ or $C$ is an even element, then for any values of $x, y, z, t, r, s$ subject to the relations

$$x = zs, \ y = zr, \ t = zs;$$

$$[A, [B, C]] = [[A, B], C] = (1)^{deg(A)deg(B)}z[A, [B, C]]_s;$$

$Id(3)$: If $C$ is an even element and $[A, C] = 0$, then

$$[A, [B, C]] = [[A, B], C] = 0.$$

2. Deformed Green Generators and their relations

The $q$-deformed superalgebra $U_q(osp(2n + 1/2m))$, a Hopf superalgebra, is by now a classical concept. See, for instance, Refs. 30-33, where all Hopf algebra operations are explicitly given. Here, following Ref. 33, we write only the algebra operations.

Let $(\alpha_{ij})$, $i, j = 1, \ldots, m + n = N$ be an $N \times N$ symmetric Cartan matrix chosen as:

$$(\alpha_{ij}) = (-1)^{(i)}\delta_{i+1,j} + (-1)^{(i)}\delta_{i,j+1} - (-1)^{(i+1)} + (-1)^{(j)}\delta_{ij} + \delta_{i,m+n}\delta_{j,m+n}. \quad (6)$$

For instance the Cartan matrix, corresponding to $m = n = 4$ is $8 \times 8$ dimensional matrix:

$$(\alpha_{ij}) = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}. \quad (7)$$
\textbf{Definition.} $U_q(osp(2n+1/2m))$ is a Hopf algebra, which is a topologically free module over $\mathbb{C}[[\hbar]]$ (complete in $\hbar$-adic topology), with (Chevalley) generators $h_i$, $e_i$, $f_i$, $i = 1, \ldots, N$ and $N$

1. Cartan-Kac relations:

\begin{align*}
[h_i, h_j] &= 0, \quad \text{(8a)} \\
[h_i, e_j] &= a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad \text{(8b)} \\
[e_i, f_j] &= \delta_{ij} \frac{k_i - k_i^-}{q - q^-}, \quad k_i = q^{h_i}, \quad k_i^- = q^{-h_i}, \quad \text{(8c)}
\end{align*}

2. e-Serre relations

\begin{align*}
[e_i, e_j] &= 0, \quad |i - j| \neq 1, \quad \text{(9a)} \\
[e_i, [e_i, e_{i \pm 1}]_q] &= 0, \quad i \neq m, \quad i \neq N \quad \text{(9b)} \\
\{e_m, e_{m-1}\}_q &\equiv [e_m, e_{m-1}]_q = 0, \quad \{e_m, e_{m+1}\}_q = 0, \quad \text{(9c)} \\
[e_N, [e_N, e_{N-1}]_q] &= 0 \quad \text{(9d)}
\end{align*}

3. f-Serre relations

\begin{align*}
[f_i, f_j] &= 0, \quad |i - j| \neq 1, \quad \text{(10a)} \\
[f_i, [f_i, f_{i \pm 1}]_q] &= 0, \quad i \neq m, \quad i \neq N \quad \text{(10b)} \\
\{f_m, f_{m-1}\}_q &\equiv [f_m, f_{m-1}]_q = 0, \quad \{f_m, f_{m+1}\}_q = 0, \quad \text{(10c)} \\
[f_N, [f_N, f_{N-1}]_q] &= 0 \quad \text{(10d)}
\end{align*}

The grading on $U_q(osp(2n+1/2m))$ is induced from:

\begin{align*}
\deg(h_j) &= 0, \forall j, \quad \deg(e_m) = \deg(f_m) = 1, \quad \deg(e_i) = \deg(f_i) = 0 \quad \text{for} \quad i \neq m. \quad \text{(11)}
\end{align*}

The (9c) and (10c) relations are the additional Serre relations,\textsuperscript{33,34,35} which were initially omitted. We do not write the other Hopf algebra maps ($\Delta$, $\varepsilon$, $S$)\textsuperscript{33} since we will not use them. They are certainly also a part of the definition of $U_q(osp(2n+1/2m))$.

From (8a,b) and the definition of $k_i$ one derives:

\begin{align*}
k_i k_i^{-1} &= k_i^- k_i = 1, \quad k_i k_j = k_j k_i, \quad \text{(12a)} \\
k_i e_j &= q^{\alpha_{ij}} e_j k_i, \quad k_i f_j = q^{-\alpha_{ij}} f_j k_i. \quad \text{(12b)}
\end{align*}

Introduce the following $3N$ elements in $U_q(osp(2n+1/2m))$ ($i = 1, \ldots, N - 1$):

\begin{align*}
a_i^- &= (-1)^{(m-i)(i)} \sqrt{2} [e_i, [e_{i+1}, [\ldots, [e_{N-2}, [e_{N-1}, e_N]_q]_{q_{N-1} = 1} \ldots]_{q_{i+2} = 1}]_{q_i} = -\sqrt{2} e_N, \quad \text{(13a)} \\
a_i^+ &= (-1)^{N-i+1} \sqrt{2} [[\ldots [f_N, f_{N-1}]]_{q_{N-1}} = -\sqrt{2} f_N, \quad \text{(13b)} \\
H_i &= h_i + h_{i+1} + \ldots + h_N \quad \text{(including} \ i = N)), \quad \text{(13c)}
\end{align*}

We refer to the above operators as to (deformed) Green generators since in the nondeformed case they coincide with the Green generators of $U_q(osp(2n+1/2m))$ (see Eq. (36) in Ref. 1). Therefore $a_1^+, a_2^+, \ldots, a_{m-1}^+, a_m^+$.
Proposition 1. The following "mixed" relations between the Chevalley and the Green generators take place:

\[ a_i^- = (\pm 1)^{(m-i)+(m-j)} [e_i, [e_{i+1}, [\ldots [e_{j-2}, [e_{j-1}, a_{j-1}^+]_{q_{j-2}}\ldots]_{q_{j+1}}]_{q_i}], \text{ if } i < j < N, \]  

\[ a_i^+ = (\pm 1)^{(m-i)+(m-j)} [e_i, [e_{i+1}, [\ldots [a_{j+1}^+, [f_{j-1}, f_{j-2}]_{q_{j-2}}\ldots]_{q_{j+1}}, f_i]_{q_i}], \text{ if } i < j < N. \]  

Taking into account (9a), (10a) and applying repeatedly \( \text{Id}(3) \) one rewrites the Green generators also as \((i < j < N)\)

\[ a_i^- = (\pm 1)^{(m-1)(i)+(m-j)} [e_i, [e_{i+1}, [\ldots [e_{j-2}, [e_{j-1}, a_{j-1}^+]_{q_{j-2}}\ldots]_{q_{j+1}}]_{q_i}], a_{j}^+]_{q_{j-1}}, \]  

\[ a_i^+ = (\pm 1)^{(m-j)(i)+(m-j)} [a_{j+1}^+, [\ldots [f_{j-1}, f_{j-2}]_{q_{j-2}}\ldots]_{q_{j+1}}, f_i]_{q_i} a_{j-1}^- \]  

The next proposition plays an important role in several intermediate calculations.

**Proposition 1.** The following "mixed" relations between the Chevalley and the Green generators take place:

\[ [e_i, a_j^+] = -\delta_{ij}(\pm 1)^{(i+1)} k_i a_{i+1}^+, \text{ if } i \neq N, \]  

\[ [a_j^-, f_i] = \delta_{ij} a_{i+1}^- k_i, \text{ if } i \neq N, \]  

\[ [e_i, a_j^-] = 0, \text{ if } i < j - 1 \text{ or } i > j, i \neq N, \]  

\[ [e_i, a_{i+1}^-]_{q_i} = (\pm 1)^{(i+1)} a_i^-, i \neq N, \]  

\[ [e_i, a_i^-]_{q_{i-1}} = 0, \text{ if } i \neq N, \]  

\[ [a_j^+, f_i] = 0, \text{ if } i < j - 1 \text{ or } i > j, i \neq N, \]  

\[ [a_{i+1}^+, f_i]_{q_i} = -a_i^+, i \neq N. \]  

\[ [a_i^+, f_i]_{q_{i-1}} = 0, i \neq N. \]  

**Proof.** We stress on some of the intermediate steps in the proof.

1. Begin with (16).

(i) Let \( i < j \). Then from (13b) and (8c) one immediately has \([e_i, a_j^+] = 0.\)

(ii) Let \( i > j \). From (14b) \([e_i, a_j^+] \sim [e_i, [\ldots [[[a_{i+1}^+, f_i], f_{i-1}], f_{i-2}]_{q_{i-2}}\ldots]_{q_{i-1}}, f_j]_{q_i}] \) (applying repeatedly \( \text{Id}(3) \) and (8c)) \( = [\ldots [A, f_{i-2}]_{q_{i-2}}\ldots]_{q_{i-1}}, f_j]_{q_i} \), where \( A = [e_i, [a_{i+1}^+, f_i]_{q_i}, f_{i-1}]_{q_{i-1}} \) (from (i) and \( \text{Id}(3) \)) \( = [a_{i+1}^+, [e_i, f_i]]_{q_i, f_{i-1}}_{q_{i-1}} \sim [a_{i+1}^+, k_i, f_{i-1}]_{q_{i-1}} = q_i^{-1}[a_{i+1}^+, f_{i-1}]_{q_i} k_i = 0, \) since evidently \( f_{i-1} \) commutes with \( a_{i+1}^+ \) (see (13b)). Hence

\[ [e_i, a_j^+] = 0 \text{ for } i > j. \]  

(iii) Let \( i = j \). \([e_i, a_j^+] = -[e_i, [a_{i+1}^+, f_i]_{q_i}] \) (from (i) and \( \text{Id}(3) \)) \( = -[a_{i+1}^+, [e_i, f_i]]_{q_i} = -[a_{i+1}^+, \frac{k_i-k_j}{q-q_i}]_{q_i} = -(\pm 1)^{(i+1)} q_i a_{i+1}^+ k_i = -(\pm 1)^{(i+1)} k_i a_{i+1}^+ \). The unification of (i)-(iii) yields (16).

2. Eq. (17) is proved in a similar way.
3. We pass to prove (18a).

(i) The case \(i < j - 1\) is evident.

(ii) Take \(i = m > j\). Note first that according to (9) and Id(3) \([e_m, [e_{m-1}, [e_m, e_{m+1}]]_q]_q\) = \([e_m, [e_{m-1}, e_m]_q, e_{m+1}]_q\) (using Id(2)) = \([e_m, e_{m-1}]_q, [e_m, e_{m+1}]_q\) = \([e_m, [e_{m-1}, e_m]]_q\) = 0, according to (9c) and (9a), i.e.,
\[
B = [e_m, [e_{m-1}, [e_m, e_{m+1}]]_q]_q = 0. \tag{21}
\]

If \(m = N - 1\), then \([e_m, a_{m-1}^-] = B = 0\). Let \(m < N - 1\). From (15a) the and Id(3) \([e_m, a_{m-1}^-] \sim [B, a_{m+2}^-]_q = 0\).

(iii) Let \(i \neq m > j\). From (15b) \(a_{i-1}^- \sim [e_{i-1}, [e_i, e_{i+1}]_q]_{q-1}, a_{i+2}^-_{q+1}\). Then Id(3) yields \([e_i, a_{i-1}^-] \sim [z, a_{i+2}]_{q+1}\) with \(z = [e_i, [e_i, e_{i+1}]_q]_q\). If \(i > m\), then using Id(1), \(z = [e_i, [e_i-1, [e_i, e_{i+1}]_q]]_q\) \(\sim [e_{i-1}, [e_i, e_{i+1}]_q]_{q+2} - [e_i, [e_i, e_{i-1}]_q]_{q+1} - e_i_{i+1}q^2 = 0\) from (9b). If \(i < m\) again from Id(1) \(z = 0\). Hence \([e_i, a_{i-1}^-] = 0\), if \(m \neq i\).

So far we have from (ii) and (iii) that
\[
[e_i, a_{i-1}^-] = 0. \tag{22}
\]

The rest of the proof is by induction. Assume \([e_i, a_j^-] = 0\) for a certain \(i > j, i \neq N\). Then from (9a), (14a) and Id(3) \([e_i, a_{j-1}^-] \sim [e_{j-1}, [e_i, e_{i+1}]_q]_{q-1}, a_{j+2}^-_{q+1}\) = \([e_{j-1}, [e_i, e_{i+1}]_q]_{q} = 0\). Therefore \([e_i, a_j^-] = 0\) for any \(i > j, i \neq N\). Combining the last with (i), one obtains (18a).

4. Eq. (18b) follows from the definition of \(a_i^-\) and the observation that \((-1)^{(m-i-1)(i+1)-(m-i)} = (-1)^{(i+1)}\).

5. It remains to verify (18c). Since \(a_i^- \sim [e_i, e_{i+1}]_q, a_{i+2}^-_{q+1}, [e_i, a_i^-]_{q+1} \sim [e_i, [e_i, e_{i+1}]_q, a_{i+2}^-_{q+1}]_{q-1}\) (from (18a) and Id(3)) \([z, a_{i+2}]_{q+1}\), where \(z = [e_i, [e_i, e_{i+1}]_q]_{q-1}\). If \(i > m\), \(z = [e_i, [e_i, e_{i+1}]_q] = 0\) (see (9b)); if \(i < m\), \(z = [e_i, [e_i, e_{i+1}]_q]_{q} = 0\) again from (9b); if \(i = m\), \(z = [e_m, [e_m, e_{m+1}]_q]_q = e_m^2e_m = 0\), since according to (9a) \(e_m^2 = 0\). Hence \([e_i, a_i^-]_{q-1} = 0\).

6. Eqs. (19) are proved in a similar way as Eqs. (18). This completes the proof of the proposition.

**Proposition 2.** The deformed Green operators (13) generate \(U_q[osp(2n + 1/2m)]\).

**Proof.** The proof is an immediate consequence of the relations:
\[
[a_i^-, a_{i+1}^+] = 2L_{i+1}e_i, \quad i = 1, 2, \ldots, N - 1, \tag{23a}
\]
\[
[a_{i+1}^-, a_i^+] = -2(-1)^{(i+1)}f_iL_{i+1}, \quad i = 1, 2, \ldots, N - 1, \tag{23b}
\]
\[
[a_i^-, a_i^+] = -\frac{L_i - \bar{L}_i}{q - \bar{q}}, \quad L_i = q^{H_i}, \quad \bar{L}_i = q^{-H_i}, \quad i = 1, \ldots, N. \tag{23c}
\]

These equations are proved by induction on \(i\).

1. The Serre relation (8c) together with the definitions of \(a_N^\pm, L_N\) immediately yield \([a_i^-, a_i^+] = -\frac{2L_i - \bar{L}_i}{q - \bar{q}}\).

From (18b) \(a_{N-1}^- = [e_{N-1}, a_N^+]_{qN-1} = [e_{N-1}, a_N^+]_{qN-1} = 0\) and Id(3), one has \([a_{N-1}^-, a_N^+] = [a_{N-1}^-, a_N^+] = [e_{N-1}, a_N^+]_{qN-1} = 2L_Ne_{N-1}, i.e.,
\[
[a_{N-1}^-, a_N^+] = 2L_Ne_{N-1}. \tag{24a}
\]

In a similar way one shows that
\[
[a_N^-, a_{N-1}^+] = -2f_{N-1}L_N. \tag{24b}
\]
In order to compute $[a_{N-1}^-, a_{N-1}^+]$ set from (13b) $a_{N-1}^+ = -[a_N^-, f_{N-1}]_{\bar{q}_{N-1}}$. Since $n \geq 1$, $\bar{q}_{N-1} = \bar{q}$. Therefore $[a_{N-1}^-, a_{N-1}^+] = -[a_{N-1}^-, [a_N^+, f_{N-1}]]_{\bar{q}}$. Apply to the last supercommutator the identity $Id(2)$ with $y = y = r = 1$ and $x = s = t = \bar{q}$, namely

$$[A, [B, C]]_{\bar{q}} = [[[A, B], C]]_{\bar{q}} + (-1)^{deg(A)deg(B)}[B, [A, C]]_{\bar{q}}, \tag{25}$$

where $A = a_{N-1}^-$, $B = a_N^+$, $C = f_{N-1}$. Then

$$[a_{N-1}^-, a_{N-1}^+] = -[[a_{N-1}^-, a_N^+], f_{N-1}]_{\bar{q}} - a_{N-1}^+ [a_{N-1}^-, f_{N-1}]_{\bar{q}} = (from (17) and (24a)) = -[2kNe_{N-1}, f_{N-1}]_{\bar{q}} - [a_N^+, a_{N-1}^- k_{N-1}]_{\bar{q}} = -2kN[e_{N-1}, f_{N-1}]_{\bar{q}} + [a_N^+, a_{N-1}^- k_{N-1}]_{\bar{q}} = -2kN - \bar{k}_{N-1} = -2kN\bar{k}_{N-1} - \bar{k}_{N-1} = -2kN\bar{k}_{N-1} - \bar{k}_{N-1}, \quad i.e.,$$

$$[a_{N-1}^-, a_{N-1}^+] = -2L_{N-1} - \bar{L}_{N-1}. \tag{26}$$

From (24) and (26) we conclude that Eqs. (23) are fulfilled for $i = N - 1$.

2. Assume Eqs. (23) hold for $i$ replaced by $i + 1$:

$$[a_{i+1}^-, a_{i+2}^+] = 2L_{i+2}e_{i+1}, \tag{27a}$$
$$[a_{i+1}^-, a_{i+1}^+] = -2(-1)^{(i+1)} f_{i+1}L_{i+2}, \tag{27b}$$
$$[a_{i+1}^-, a_{i+1}^+] = -2L_{i+1} - \bar{L}_{i+1}. \tag{27c}$$

We proceed to show that then Eqs. (23) hold too. Set from (13b) $a_i^- = (-1)^{(i+1)}[e_i, a_{i+1}^-]_{\bar{q}}$. Take into account that according to (16) $[e_i, a_{i+1}^-] = 0$ and $Id(3)$. Then $[a_{i}^-, a_{i+1}^+] = (-1)^{(i+1)}[[e_i, a_{i+1}^-]_{\bar{q}}, a_{i+1}^+] = (-1)^{(i+1)}[e_i, [a_{i+1}^-, a_{i+1}^+]_{\bar{q}}]_{\bar{q}}$ (from (27a)) $2(ar{q} - q)^{-1}(-1)^{(i+1)}[e_i, L_{i+1} - \bar{L}_{i+1}]_{\bar{q}}$, which after some rearrangement of the multiples finally yields (23a). The verification of (23b) is similar. So far we have derived from (27) that

$$[a_{i}^-, a_{i+1}^+] = 2L_{i+1}e_i, \quad i = 1, 2, \ldots, N - 1, \tag{28a}$$
$$[a_{i+1}^-, a_{i}^+] = -2(-1)^{(i+1)} f_iL_{i+1}, \quad i = 1, 2, \ldots, N - 1. \tag{28b}$$

Set in $[a_{i}^-, a_{i+1}^+] a_i^- = -[a_i^+, f_i]_{\bar{q}}$. Either $a_i^+$ or $f_i$ is an even element. Therefore applying again the identity (25), one has $[a_{i}^-, a_{i+1}^+] = -[[a_{i}^-, [a_{i+1}^+, f_i]]_{\bar{q}}, f_i]_{\bar{q}} = -[a_{i}^-, [a_{i+1}^+, f_i]]_{\bar{q}} - (-1)^{(i+1)}[a_{i+1}^+, [a_{i}^+, f_i]]_{\bar{q}}$ (from (28a) and (17)) $-2[2L_{i+1}e_i, f_i]_{\bar{q}} - (-1)^{(i+1)}[a_{i+1}^+, a_{i+1}^- k_i]_{\bar{q}} = -2L_{i+1}[e_i, f_i] + [a_{i+1}^-, a_{i+1}^+] k_i = -2L_{i+1}[e_i, f_i] - \bar{L}_{i+1} - \bar{L}_{i+1}$. Hence (23c) holds too. From here and (28) we conclude that, if Eqs. (27) hold, then also Eqs. (23) are fulfilled too.

This completed the proof of the validity of Eqs. (23).

From (13) and (23a,b) one obtains $(i = 1, \ldots, N - 1)$:

$$h_i = H_i - H_{i+1}, \quad H_N = h_N, \tag{29a}$$
$$e_i = \frac{1}{2}L_{i+1}[a_i^-, a_{i+1}^+], \quad e_N = \frac{1}{\sqrt{2}}a_N^+, \tag{29b}$$
$$f_i = -\frac{1}{2}(-1)^{(i+1)}[a_{i+1}^-, a_i^+] L_{i+1} = \frac{1}{2}[a_i^+, a_{i+1}^+] L_{i+1}, \quad f_N = -\frac{1}{\sqrt{2}}a_N^+. \tag{29c}$$

Since the Chevalley elements generate $U_q[osp(2n + 1/2m)]$, so do the Green operators. This completes the proof.
Proposition 3. The Green generators $H_i, a_i^\pm, i = 1, \ldots, N$ satisfy the following relations 
$(i, j = 1, \ldots, N, \xi, \eta = \pm 1)$:

$$
[H_i, H_j] = 0, \tag{30a}
$$

$$
[H_i, a_j^\pm] = \pm \delta_{ij}(-1)^{(i)}a_j^\pm, \tag{30b}
$$

$$
[a_i^-, a_i^+] = -2\frac{L_i - \bar{L}_i}{q - \bar{q}}, \tag{30c}
$$

$$
[[a_{N-1}^\xi, a_N^\xi], a_N^\xi]_q = 0, \tag{30d}
$$

$$
[[a_i^\eta, a_{i+1}^-], a_j^\eta]_{q^{-\xi(i)}s_{ij}} = 2(\eta)^{(j)}\delta_{j,i+\xi}L_j - \xi\eta a_i^\eta. \tag{30e}
$$

Proof. The commutation relations (30a) are evident. (30b) follows from the definitions of the Green generators, the Cartan relations (8a,b) and the observation that

$$
\sum_{s=1}^{N} \sum_{r=j}^{N} a_{sr} = -(1)^{(i)}\delta_{ij}. \tag{31}
$$

The Eq. (30c) was derived in Proposition 2. The Eq. $[[a_{N-1}^\xi, a_N^\xi], a_N^\xi]_q = 0$ is the same as the Serre relation (9b), if one takes into account that $[e_{N-1}, e_N]_q = 0$ and $e_N \sim a_N$. Similarly one shows that (30d) with $\xi = +$ is the same as (10d).

The proof of (30c) is based on a case by case considerations ($\xi, \eta = \pm$). To this end one has to replace $e_i$ and $f_i$ in Eqs. (16)-(19) with their expressions through the CAOs from (29). Using the relations (which follow from (30b))

$$
L_i a_j^\pm = a_j^\pm L_i, \quad \bar{L}_i a_j^\pm = a_j^\pm \bar{L}_i, \quad i \neq j = 1, \ldots, N, \tag{32a}
$$

$$
L_i a_i^\pm = q^{\pm(-1)^{(i)}} a^\pm L_i, \quad \bar{L}_i a_i^\pm = q^{\mp(-1)^{(i)}} a^\pm \bar{L}_i, \quad i = 1, \ldots, N, \tag{32b}
$$

after long, but simple calculations one verifies (30c).

3. Description of $osp_q(2n+1/2m)$ via Deformed Green Generators

So far we have established that the Green generators (13) satisfy the relations (30). Here we solve the inverse problem: we show that the operators $H_i, a_i^\pm, i = 1, \ldots, N$ subject to the relations (30) provide an alternative description of $U_q[osp(2n + 1/2m)]$.

In Sect. 2 we have derived the relations (16)-(19) from the definition (13) of the Green generators and the Cartan-Kac and the Serre relations, satisfied by the Chevalley generators. Now as a first step we derive (16)-(19) on the ground of Eqs. (30).

Proposition 4. The ”mixed” relations (16)-(19) follow from (29) and (30).

Proof. Consider Eq. (16). Since $i \neq N$, from (29b)

$$
[e_i, a_j^+] = \frac{1}{2} L_{i+1} [a_i^-, a_{i+1}^+] a_j^+ - \frac{1}{2} a_j^+ \bar{L}_{i+1} [a_i^-, a_{i+1}^+]. \tag{33}
$$

(i) If $i + 1 < j$ or $i > j$, then $L_{i+1}$ and $a_j^+$ commute (see (32)). Therefore, using (30e),

$$
[e_i, a_j^+] = \frac{1}{2} L_{i+1} [a_i^-, a_{i+1}^+] a_j^+ = -\frac{1}{2} (-1)^{(i+1)} L_{i+1} [a_{i+1}^+, a_i^-] a_j^+ = 0.
$$
(ii) If \( i = j \), again from (30c) \([e_i, a^+_i] = -\frac{1}{2}(-1)^{(i+1)}L_{i+1}[a^+_{i+1}, a^-_i], a^+_i = (-1)^{(i+1)}L_{i+1}L_ia^+_i = -(1)^{(i+1)}k_i a^+_i\).

(iii) If \( i + 1 = j \), from (33) and taking into account (32b)
\[
[e_i, a^+_{i+1}] = \frac{1}{2}L_{i+1}[[a^-_i, a^+_i], a^+_i] - q^{-1}(1)^{(i+1)}a^+_{i+1}[a^-_i, a^+_i]] = \frac{1}{2}L_{i+1}[[a^-_i, a^+_i], a^+_i]]_{q^{-(i+1)}} = \frac{1}{2}L_{i+1}(-1)^{(i+1)}[[a^+_{i+1}, a^-_i], a^+_i]]_{q^{-(i+1)}} = 0. The unification of (i)-(iii) yields (16).

The remaining equalities (17)-(19) are proved in a similar way.

**Proposition 5.** The Cartan-Kac relations are a consequence of the relations (30).

**Proof.** The first two equations (8a) and (8b) are easily verified. We proceed to prove (8c).

1. The case \( i = j \). If \( i = N \), then (8c) is the same as (30c). Let \( i < N \). From (29c) and the graded Leibnitz rule
\[
[e_i, f_j] = \frac{1}{2}[[e_i, a^+_i], [a^-_i, L_{i+1}]] = \frac{1}{2}[[e_i, a^+_i], a^-_i + 1 + 1] + (-1)^{(i+1)(i)}[a^+_i, [e_i, a^-_i + 1 + 1]].
\]
Insert above
\[
[e_i, a^+_i] = -(-1)^{(i+1)}k_i a^+_i \quad \text{and} \quad [e_i, a^-_i + 1 + 1] = [e_i, a^-_i + 1 + 1].
\]
After some rearrangement of the multiples one obtains:
\[
[e_i, f_j] = \frac{1}{2}[a^-_i + 1 + 1, a^+_i + 1] + \frac{1}{2}[a^-_i, a^+_i + 1]L_{i+1} \quad \text{from (23c) = } \frac{k_i - k_j}{q - q}.
\]
Hence
\[
[e_i, f_j] = \frac{k_i - k_j}{q - q}, \quad i = 1, \ldots, N. \quad (34)
\]

2. The case \( i \neq j \). Eq. (8c) is easily verified for \( i = N \) or \( j = N \). We consider \( i \neq j \neq N \). From (29c)
\[
[e_i, f_j] = \frac{1}{2}[[e_i, a^+_j], a^-_j + 1 + 1] = \frac{1}{2}[[e_i, a^+_j], a^-_j + 1 + 1] + (-1)^{(i+1)(j)}[a^+_j, [e_i, a^-_j + 1 + 1]].
\]
The first term in the r.h.s. cancels out, since \([e_i, a^+_j] = 0 \text{ according to (16). From the second term evaluate only the internal supercommutator } A = [e_i, a^-_j + 1 + 1].

(i) If \( i < j \) or \( i > j + 1 \), \( A = [e_i, a^-_j + 1 + 1]L_{j+1} = 0 \text{ according to (18a);}

(ii) If \( i = j + 1 \), then \( L_i e_i = \tilde{q}_{i-1} e_i L_i \). Therefore \( A = [e_i, a^-_j] = \tilde{q}_{i-1} e_i L_i = 0 \text{ according to (18c).}

Hence \([e_i, f_j] = 0, \text{ if } i \neq j = 1, \ldots, N. \text{ The latter together with (34) shows that also the last Cartan-Kac relation (8c) is fulfilled. This completes the proof.}

**Proposition 6.** The Serre relations (9) and (10) are a consequence of the relations (30).

**Proof.**

1. First we prove that \([e_i, e_j] = 0 \text{ if } |i - j| > 1 \text{. Assume for definiteness that } i + 1 < j\).

(i) Let \( i + 1 < j = N \). From (29b) and the observation that \( L_{i+1} \text{ commutes with } a^-_N \text{ one has}
\[
[e_i, e_N] = [L_{i+1}[a^-_i, a^+_i], a^-_N] = L_{i+1}[a^-_i, a^+_i], a^-_N] \text{ from (30c)=0}.
\]

(ii) Let \( i + 1 < j < N \). From (29b) \([e_i, e_j] = [L_{i+1}[a^-_i, a^+_i], L_{j+1}[a^-_j, a^+_j]] \text{ (} L_{j+1} \text{ commutes with } a^-_j \text{ and}
\]
\[
a^+_j, L_{j+1} \text{ commutes with } a^-_j \text{ and } a^+_j, L_{j+1} \text{ commutes with } a^-_j \text{ and } a^+_j, [a^-_j, a^+_j]] = L_{j+1}[a^-_j, a^+_j], a^-_j + 1 + 1] \text{ and } [[a^-_i, a^+_i], a^-_j + 1 + 1] \text{ and }[[a^-_i, a^+_i], a^-_j + 1 + 1] = 0 \text{ and }[[a^-_i, a^+_i], a^-_j + 1 + 1] = 0 \text{ according to (30c).}
\]

2. The proof of \([f_i, f_j] = 0, \text{ if } |i - j| > 1 \text{ is similar.}

3. Proof of \([e_i, [e_i, e_i + 1]] = 0, \text{ if } i \neq m, \text{ i} \neq N \text{ and } q' = q \text{ or } q' = \tilde{q} \text{.} \)
We choose \( q' = q_{t-1} = q^{(-1)^i} \). Therefore the relation to be proved is
\[
[e_i, [e_i, e_{i+1}]_{q_{t-1}}]_{q_{t-1}} = 0, \quad i \neq m, \quad i \neq N.
\]
(35)
As a preliminary step compute \([e_i, e_{i+1}]_{q_{t-1}}\) (see (29b)) = \( \frac{1}{4} [L_{i+1} [a_i, a_{i+1}^+] + L_{i+2} [a_{i+1}, a_{i+2}^+]]_{q_{t-1}} \).
From (32) one has
\[
[a_i, a_{i+1}^+] L_{i+2} = L_{i+2} [a_i, a_{i+1}^+] \quad \text{and} \quad [a_{i-1}^+, a_{i+2}^-] L_{i+1} = q^{(-1)^i} L_{i+1} [a_{i-1}^+, a_{i+2}^-] = \tilde{q}_i L_{i+1} [a_{i-1}^+, a_{i+2}^-].
\]
Therefore, \([e_i, e_{i+1}]_{q_{t-1}} = \frac{1}{4} L_{i+1} L_{i+2} ([a_i, a_{i+1}^+] [a_{i-1}^+, a_{i+2}^-] - q_{t-1} \tilde{q}_i [a_{i-1}^+, a_{i+2}^-] [a_{i-1}^+, a_{i+2}^-]) \).

Since \( i \neq m, \quad q_{t-1} \tilde{q}_i \) = 1. Thus, \([e_i, e_{i+1}]_{q_{t-1}} = \frac{1}{4} L_{i+1} L_{i+2} ([a_i, a_{i+1}^+] [a_{i-1}^+, a_{i+2}^-]) \)
\[
= \frac{1}{4} L_{i+1} L_{i+2} ([a_i, a_{i+1}^+] [a_{i-1}^+, a_{i+2}^-] + \frac{1}{4} L_{i+1} L_{i+2} (-1)^{(i+1)(i+1)} [a_{i-1}^+, [a_i, a_{i+1}^+] a_{i+2}^-]).
\]
The second term in the r.h.s. is zero, since from (30c) \([a_{i-1}^+, a_i] [a_{i+1}^+, a_{i+2}^-] = 0 \). Again from (30c)
\[
[a_i^-, a_{i+1}^+], a_{i+1}^+] = 2(-1)^{i+1} L_{i+1} a_i^-. \quad \text{Therefore}
\]
\([e_i, e_{i+1}]_{q_{t-1}} = \frac{1}{4} (-1)^{i+1} L_{i+2} [a_i^-, a_{i+1}^+]. \quad \text{(36)}
\]
Insert \( e_i \) from (29b) and \([e_i, e_{i+1}]_{q_{t-1}} \) from (36) in the l.h.s. of (35):
\[
[e_i, [e_i, e_{i+1}]_{q_{t-1}}]_{q_{t-1}} = \frac{1}{4} L_{i+1} [a_i^-, a_{i+1}^+], \quad (-1)^{i+1} = \frac{1}{4} L_{i+1} L_{i+2} [a_i^-, a_{i+1}^+]_{q_{t-1}}
\]
\[
= \frac{1}{4} (-1)^{i+1} L_{i+1} L_{i+2} ([a_i^-, a_{i+1}^+] [a_{i-1}^+, a_{i+2}^-])_{q_{t-1}} \quad \text{(use the circumstance that} \quad [a_i^-, a_{i+1}^+] [a_{i-1}^+, a_{i+2}^-] = 0 \quad \text{and} \quad Id(3))
\]
\[
= \frac{1}{4} (-1)^{i+1} L_{i+1} L_{i+2} [a_i^-, a_{i+1}^+] [a_{i-1}^+, a_{i+2}^-]_{q_{t-1}} = 0, \quad \text{since} \quad [a_i^-, a_{i+1}^+], a_{i+1}^-]_{q_{t-1}} = [a_i^-, a_{i+1}^+], a_{i+1}^-]_{q_{t-1}} = 0 \quad \text{according to (30e). \quad \text{Hence the Serre relation (35) holds.}}
\]

4. The proof of \([e_i, [e_i, e_{i-1}]_{q}] = 0, \quad i \neq 1, \quad i \neq m, \quad i \neq N \) and \( q' = q \) or \( q' = \tilde{q} \) is similar. For \( q' \) one has to take \( q' = \tilde{q}_{t-1} = q^{(-1)^i} \).

5. The proof of the Serre relations (10b) is similar as for (9b).

6. Proof of \( e_m^2 = 0 \) (i.e., of (9a) for \( i = j = m \)).
\[
e_m^2 \sim [e_m, e_m] q^2 \quad \text{(use (29b))} \sim [L_{m+1} [a_m, a_{m+1}^+], L_{m+1} [a_m, a_{m+1}^+]] q^2 = q E_m^2 [a_m, a_{m+1}^+] [a_m, a_{m+1}^+] q^2.
\]
In order to evaluate \([a_m, a_{m+1}^+] [a_m, a_{m+1}^+] q^2 \) set \( A = [a_m, a_{m+1}^+], \quad B = a_m, \quad C = a_{m+1}^- \). Note that \( deg(A) = deg(B) = 1, \quad deg(C) = 0 \) and use the identity \( Id(2) \) with \( x = 1, \quad y = q^2, \quad z = r = t = q, \quad s = \tilde{q} \). It yields
\[
[a_m, a_{m+1}^-] [a_m, a_{m+1}^+] q^2 = [a_m, a_{m+1}^-] [a_m, a_{m+1}^+] q - q [a_m, [a_m, a_{m+1}^+] a_{m+1}^- q] = 0,
\]
since, as it follows from (30c), \([a_m, a_{m+1}^-], a_m q = 0 \) and \([a_m, a_{m+1}^-], a_{m+1}^- q = 0 \).

7. The proof of \( f_m^2 = 0 \) is similar.

8. Proof of \([e_m, e_{m-1} q, e_m, e_{m+1} q] = 0 \).
So far we have proved the validity of the Cartan-Kac relations (8) and of the Serre relations (9a,b) and (10a,b). Therefore we can refer to them. In particular Eqs. (15) hold. Using (15a), write \( a_{m-1} = -[e_{m-1}, e_m, e_{m+1}] q, a_{m+2} = [e_{m-1}, e_m, e_{m+1}] q \). According to (18a), which was proved in Proposition 4, \([e_m, a_{m-1}] = 0 \). Therefore, \( 0 = [e_m, a_{m-1}] = -[e_m, [e_{m-1}, e_m, e_{m+1}] q, a_{m+2} q] \) and since, again from (18a), \([e_m, a_{m+2}] = 0 \), applying \( Id(3) \), we have \([e_m, a_{m-1}] = -[y, a_{m+2} q], \quad \text{where} \quad y = [e_m, e_{m-1}, e_m, e_{m+1}] q] \), which can be written also as
\[
y = [e_m, [e_{m-1}, e_m] q, e_{m+1}] q \quad \text{(37)}
\]
and
\[
[y, a_{m+2} q] = 0.
\]
(38)
From (13b), (37) and (8c) one immediately concludes that \([y, a_{m+2}^{-}] = 0\). Therefore, applying \(Id(3)\), one has

\[0 = [y, a_{m+2}^{-}]_{q}, a_{m+2}^{+} = [y, [a_{m+2}^{-}, a_{m+2}^{+}]]_{q} (\text{use (30c)}) = -2(q - \bar{q})^{-1}[y, L_{m+2} - \bar{L}_{m+2}]_{q},\]

which after, pushing \(L_{m+2} \text{ and } \bar{L}_{m+2} \) to the right, yields \((1 - q^{2}) y L_{m+2} = 0\). Hence

\[y = [e_{m}, [e_{m-1}, e_{m}]_{\bar{q}}, e_{m+1}]_{q} = 0. \tag{39}\]

Set in (39) \(A = e_{m}, \ C = [e_{m-1}, e_{m}]_{\bar{q}}, \ B = e_{m+1} \) \text{ and use the following identity, which follows from } \text{Id}(2): \text{If } B \text{ is an even element, then}

\[[A, [C, B]_{q}] = -q[[A, B]_{\bar{q}}, C] - [B, [A, C]_{q}]. \tag{40}\]

This yields \(y = -q[[e_{m}, e_{m+1}]_{q}, [e_{m-1}, e_{m}]_{q}] - [e_{m+1}, [e_{m-1}, e_{m}]_{q}]_{q} = 0\). The second term in the r.h.s. above is zero, since \([e_{m}, [e_{m-1}, e_{m}]_{q}]_{q} = q e_{m-1}^{-1/2} - \bar{q} e_{m-1}^{2} \text{ and } e_{m}^{2} = 0\). Therefore,

\[[e_{m}, e_{m-1}]_{q}, [e_{m}, e_{m+1}]_{\bar{q}} = 0, \]

which proves (9c).

9. The proof of (10c) is similar.

10. Proof of \([e_{N}, [e_{N}, e_{N-1}]_{q}]_{q} = 0\).

From (29b) \(e_{N-1} = \frac{1}{2\sqrt{2}} L_{N}[a_{N-1}, a_{N}] \) \text{ and } \(e_{N} = \frac{1}{\sqrt{2}} a_{N}^{-}. \) Therefore, \([e_{N}, e_{N-1}]_{\bar{q}} = \frac{1}{\sqrt{2}} a_{N}^{-} \frac{1}{2\sqrt{2}} L_{N}[a_{N-1}, a_{N}]_{\bar{q}} = \frac{1}{4\sqrt{2}} q L_{N}[a_{N}, [a_{N-1}, a_{N}]_{\bar{q}}],\) \text{ which, applying (30e), yields:}

\[[e_{N}, e_{N-1}]_{\bar{q}} = -\frac{q}{2\sqrt{2}} a_{N-1}^{-}. \tag{41}\]

Therefore, \([e_{N}, [e_{N}, e_{N-1}]_{q}]_{q} = -\frac{q}{2\sqrt{2}} [a_{N}^{-}, [a_{N}, a_{N-1}]_{\bar{q}}] = -\frac{1}{4\sqrt{2}} [[a_{N-1}, a_{N}], a_{N}]_{\bar{q}} = 0, \text{ according to } (30d). \) \text{Hence, (9d) holds.}

11. The proof of (10d) is similar.

This completes the proof of Proposition 6.

The relations (29), written in the form \((i = 1, \ldots, N - 1)\)

\[h_{i} = H_{i} - H_{i+1}, \quad H_{N} = h_{N}, \quad \tag{42a}\]

\[e_{i} = \frac{1}{2} q^{-H_{i+1}}[a_{i}^{-}, a_{i+1}^{+}], \quad e_{N} = \frac{1}{2\sqrt{2}} a_{N}^{-}, \quad \tag{42b}\]

\[f_{i} = \frac{1}{2}[a_{i}^{+}, a_{i+1}^{-}] q^{H_{i+1}}, \quad f_{N} = -\frac{1}{2\sqrt{2}} a_{N}^{+}, \quad \tag{42c}\]

indicate that the Chevalley elements are functions of the Green generators. More precisely, \(h_{i}, e_{i}, f_{i}\) \text{ are in the closure of the subalgebra of all polynomials of the Green operators over } \mathbb{C}\{[h]\}.

So far we were considering \(U_{q}[osp(2n + 1/2m)]\) as a topologically free module over the ring \(\mathbb{C}\{[h]\}\) of the formal power series over an indeterminate \(h\). Due to this, for instance, \(q^{H_{i}}\) \text{ is a well defined element from } \(U[osp(2n + 1/2m)]_{q}\). It is important to note however that all our considerations remain true, if one goes to the factor algebra \(U_{h_{e}}[osp(2n + 1/2m)]\), replacing \(h\) by a complex number \(h_{e}\), such that \(h_{e} \notin i\pi \mathbb{Q} \) (\(\mathbb{Q}\) - all
rational numbers), namely considering \( q \) to be a number, which is not a root of 1. Then in the limit \( h_c \to 0 \) the deformed Green generators become ordinary parabosons and parafermions. This is the justification to call the operators (13) deformed Green generators, and the corresponding to them statistics - quantum deformation of the parastatistics. We conclude the paper, formulating our main result as a theorem.

**Theorem.** \( U_q[osp(2n + 1/2m)] \) is a topologically free \( \mathbb{C}[[h]] \) module and a unital algebra with generators \( H_i, a_i^\pm, \ i = 1, \ldots, N \) and relations (30). The generators consist of \( m \) pairs of deformed parabosons and \( n \) pairs of deformed parafermions.

This theorem established a link between the quantum groups in the sense of Drinfeld-Jimbo\(^3,4\) and the quantum statistics in the sense of Green.\(^2\)

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