TRIANGULATED CORES OF PUNCTURED-TORUS GROUPS

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Abstract. We show that the interior of the convex core of a quasifuchsian punctured-torus group admits an ideal decomposition (usually an infinite triangulation) which is canonical in two different senses: in a combinatorial sense via the pleating invariants, and in a geometric sense via an Epstein-Penner convex hull construction in Minkowski space. The result extends to certain non-quasifuchsian punctured-torus groups, and in fact to all of them if a strong version of the Pleating Lamination Conjecture is true.

0. Introduction

Among Kleinian groups with infinite covolume, quasifuchsian groups, which are deformations of Fuchsian surface groups, are fundamental examples. It has been noted [Jø] that such groups can be analyzed much more explicitly when the base surface is just a once-punctured torus — the curve complex is then dual to a (locally finite) tree, which simplifies many issues. Punctured-torus groups, which still retain many features of general quasifuchsian groups, have thus become a favorite “training ground”: Minsky’s work [Min] on end invariants is an example. Similarly, Caroline Series [Se] was able to prove the Pleating Lamination Conjecture for quasifuchsian punctured-torus groups (general pleating laminations, in contrast, seem to pose tremendous technical challenges). In this paper, we prove that the convex core $V$ of any quasifuchsian punctured-torus group admits an ideal triangulation (or slightly more general decomposition) which relates “as nicely as one could hope” both to the intrinsic geometry of $V$, and to the combinatorics of the boundary pleatings of $V$. This answers several conjectures made in [ASWY1]. As a byproduct, we get enough control to re-prove the result of [Se].

0.1. Objects of study. Let $S$ be the once-punctured torus endowed with its differential structure. Let $C$ be the set of homotopy classes of simple closed curves in $S$. Choose in $C$ a meridian $m$ and longitude $l$ whose intersection number is 1. Fix an identification $s$, called the slope, between $C$ and $\mathbb{P}^1 \mathbb{Q}$, so that $s(m) = \infty$ and $s(l) = 0$. Let $(\alpha^+, \beta^+)$ and $(\alpha^-, \beta^-)$ be elements of $\mathbb{R}^+$ such that $\beta^+/\alpha^+$ and $\beta^-/\alpha^-$ are distinct irrationals. Define the pleatings $\lambda^\pm : C \to \mathbb{R}^\pm$ by

$$\lambda^+(\eta/\xi) = \left| \begin{array}{cc} \beta^+ & \eta \\ \alpha^+ & \xi \end{array} \right| \quad \text{and} \quad \lambda^-(\eta/\xi) = - \left| \begin{array}{cc} \beta^- & \eta \\ \alpha^- & \xi \end{array} \right|,$$

where the double bars denote the absolute value of the determinant (and $\xi$, $\eta$ are coprime integers). Replacing a pair $(\alpha^\pm, \beta^\pm)$ by its negative does not change $\lambda^\pm$. Notice that $\lambda^- < 0 < \lambda^+$ (as functions on $C$).
Theorem 1. There exists a cusped, non-complete hyperbolic 3-manifold $V$ homeomorphic to $S \times \mathbb{R}$ whose metric completion $\overline{V}$ is homeomorphic to $S \times [0, 1] = V \sqcup S_{-\infty} \sqcup S_{+\infty}$, where $S_{-\infty}, S_{+\infty}$ are pleated surfaces whose pleating measures restrict to $\lambda^-$ and $\lambda^+$ (respectively) on $\mathcal{C}$.

The bulk of this paper is devoted to producing such a $V$ (with some adjustments, the method also applies to rational pleatings $\lambda^\pm$). The puncture of $S$ is required to correspond to a cusp of $V$, so $\overline{V}$ will be the convex core of a quasifuchsian punctured-torus group $\Gamma \subset \text{Isom}^+(\mathbb{H}^3)$, i.e. the convex core of the manifold $\mathbb{H}^3/\Gamma$ (a punctured-torus group is a group freely generated by two elements with parabolic commutator). The interior $V$ of $\overline{V}$ is called the open convex core.

Specifically, $V$ is constructed as an infinite union of ideal hyperbolic tetrahedra $(\Delta_i)_{i \in \mathbb{Z}}$ glued along their faces, and the $\lambda^\pm$ encode the gluing rule between $\Delta_i$ and $\Delta_{i+1}$ (see Section 1): so this ideal decomposition $\mathcal{D}$ of $V$ is canonical in a combinatorial sense, with respect to the data $\lambda^\pm$. By $\text{Se}$, the group $\Gamma$ is determined uniquely up to conjugacy in $\text{Isom}(\mathbb{H}^3)$ by the $\lambda^\pm$. Our construction therefore provides a decomposition $\mathcal{D}$ of the open convex core $V$ of an arbitrary quasifuchsian, non-Fuchsian punctured-torus group $\Gamma$. A good drawing of $V$ is Figure 3 of $\text{Th}$.

0.2. Geometric canonicity. Akiyoshi and Sakuma $\text{AS}$ generalized the Epstein-Penner convex hull construction in Minkowski space $\mathbb{R}^{3+1}$ (see $\text{EP}$) to show that $V$ also admits a decomposition $\mathcal{D}^G$, canonical in a purely geometric sense, and related to the Ford-Voronoi domain of $\mathbb{H}^3/\Gamma$. Roughly speaking, $\mathcal{D}^G$ is defined by considering the $\Gamma$-orbit $\mathcal{O} \subset \mathbb{R}^{3+1}$ of an isotropic vector representing the cusp, and projecting the cell decomposition of the boundary of the convex hull of $\mathcal{O}$ back to $\mathbb{H}^3/\Gamma$ (see Section 9 for more detail). Our main theorem is

Theorem 2. If $V$ is the open convex core of a quasifuchsian once-punctured torus group, the decompositions $\mathcal{D}$ and $\mathcal{D}^G$ of $V$ are the same.

It is reasonable to understand ending laminations of geometrically infinite surface groups as infinitely strong pleatings, and to conjecture that the group is determined by its ending and/or pleating laminations. Indeed, our method allows to construct (conjecturally unique) punctured-torus groups with arbitrary admissible ending and/or pleating laminations: the precise statement, with a full description of $\mathcal{D}^G$ (especially in the case of rational laminations), is Theorem 35 in Section 10.

0.3. Context. The identity $\mathcal{D} = \mathcal{D}^G$ of Theorem 2 (and the existence of $\mathcal{D}$, as realized by positively oriented cells) was Conjecture 8.2 in $\text{ASWY1}$, also called the Elliptic-Parabolic-Hyperbolic (EPH) Conjecture. Akiyoshi subsequently established the identity in the case of two infinite ends, in $\text{Ak}$. Near a finite end however, the ideal tetrahedra $\{\Delta_i\}_{i \in \mathbb{Z}}$ of $\mathcal{D}$ flatten at a very quick rate: the smallest angle of $\Delta_i$ typically goes to 0 faster than any geometric sequence, as $i$ goes to $\pm \infty$. In a sense, the difficulty is to show that these angles nevertheless stay positive for the hyperbolic metric.

Under the hypotheses of Theorem 2, for finitely many indices $i \in \mathbb{Z}$, the tetrahedron $\Delta_i$ of $\mathcal{D}^G$ comes from a spacelike face in Minkowski space, and is therefore dual to a singular point (a vertex) of the Ford-Voronoi domain of $\mathbb{H}^3/\Gamma$. The latter domain is described in great detail in $\text{Jø, ASWY1, ASWY2}$ and $\text{ASWY3}$, relying on a geometric continuity argument in the space of quasifuchsian groups (which is known to be connected). To study all tetrahedra $\Delta_i$ at once, including those
not seen in the Ford-Voronoï domain, the present paper takes a somewhat opposite approach: first describe geometric shapes for the tetrahedra of the candidate triangulation \( \mathcal{D} \), then establish that the gluing of these tetrahedra defines (the open convex core of) a quasifuchsian group.

The construction of \( \mathcal{D} \) (namely, of angles for the tetrahedra) will be fairly explicit: the solution will arise as the maximum of an explicit concave “volume” functional \( \mathcal{V} \) over an explicit convex domain. The domain has infinite dimension, but there are explicit bounds on the contributions to \( \mathcal{V} \) of the “tail” coordinates. This should allow for numerically efficient implementations.

The geometrically canonical decomposition \( \mathcal{D}^G \) can be defined for arbitrary cusped manifolds, but is quite mysterious and hard to study in general. It seems to be unknown, for instance, whether \( \mathcal{D}^G \) is always locally finite (see [AS]). For quasifuchsian punctured-torus groups however, Theorem \( 4 \) can be said to completely describe the combinatorics of \( \mathcal{D}^G \). In fact, aside from coarsenings of \( \mathcal{D} \), the author does not know of any ideal cell decomposition that is invariant under the hyperelliptic involution (a property which \( \mathcal{D}^G \) must a priori enjoy).

An identity of the form \( \mathcal{D} = \mathcal{D}^G \) can be established by the same methods in several closely related contexts: punctured-torus bundles (where the result is due to Lackenby [La]); complements of two-bridge links (see the announcement in [ASWY3]) or of certain arborescent links — see [G2] for a synthesis, but the present paper contains the key ideas. We will use several results (and notation) from [GF].

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1. Strategy

1.1. Setup. We return to the irrational pleating data \( \lambda^\pm \). In the hyperbolic plane \( \mathbb{H}^2 \) with boundary \( \partial \mathbb{H}^2 = \mathbb{P}^1 \mathbb{R} \), consider the Farey triangulation (the ideal triangle \( 01\infty \) iteratedly reflected in its sides, see e.g. Section 2 of [Min]). The irrational ratios \( \beta^+ / \alpha^+ \) and \( \beta^- / \alpha^- \) belong to \( \partial \mathbb{H}^2 \), and the oriented line \( \Lambda \) from \( \beta^- / \alpha^- \) to \( \beta^+ / \alpha^+ \) crosses infinitely many Farey edges \( (e_i)_{i \in \mathbb{Z}} \) (the choice of \( e_0 \) is arbitrary). To every pair of consecutive integers \( i, i+1 \) is associated a letter, \( R \) or \( L \), according to whether \( e_i \) and \( e_{i+1} \) share their Right or Left end, with respect to the orientation of \( \Lambda \) (we say that \( \Lambda \) makes a Right or makes a Left across the Farey triangle between \( e_i \) and \( e_{i+1} \)). We thus get a bi-infinite word \( \ldots RLLLRR \ldots \) with infinitely many \( R \)'s (resp. \( L \)'s) near either end.

Two rationals of \( \mathbb{P}^1 \mathbb{Q} \) are Farey neighbors exactly when the corresponding elements (simple closed curves) of \( \mathcal{C} \) have number of intersection 1. Therefore, each Farey triangle \( \tau \) defines an ideal triangulation of the punctured torus \( S \) in the following way. The vertices of \( \tau \) are the slopes of three curves of \( \mathcal{C} \), each parallel to a properly embedded line running from the puncture to itself (so we may speak of the slopes of such properly embedded lines). These three lines are embedded disjointly and separate \( S \) into two ideal triangles. Moreover, two triangulations corresponding to Farey triangles which share an edge differ by a diagonal move (see Figure 1). Such a diagonal move must be seen as a (topological) ideal tetrahedron in \( S \times \mathbb{R} \) filling the space between two (topological) pleated surfaces, pleated along the two ideal triangulations.
Our strategy will be to consider triangulated surfaces (pleated punctured tori) corresponding to the Farey triangles living between \( e_i \) and \( e_{i+1} \) for \( i \) ranging over \( \mathbb{Z} \), interpolate these surfaces with tetrahedra \( \Delta_i \) corresponding to the diagonal moves, and provide geometric parameters (dihedral angles) for all these objects, using Rivin’s Maximum Volume Principle (see [Ri]).

Throughout the paper, we will deal with an infinite family of tetrahedra \((\Delta_i)_{i \in \mathbb{Z}}\), separated by pleated once-punctured tori \( S_i \). By an arbitrary choice, we resolve that \( S_i \) is the surface between \( \Delta_i \) and \( \Delta_{i-1} \) — or equivalently, that \( \Delta_i \) is bounded by the surfaces \( S_i \) and \( S_{i+1} \). However, the numbering of the tetrahedra should be seen as the more essential one (see especially Definition 3 below).

1.2. Plan of the paper. In Section 2 we describe the space of possible dihedral angle assignments \( x_i, y_i, z_i \) for the \( \Delta_i \). In Section 3 we encode \( \lambda^\pm \) into constraints on the \( x_i, y_i, z_i \). In Section 4 we carry out (constrained) volume maximization. Important asymptotic features of the solution are analyzed in Section 5. In Section 6 we describe the Euclidean triangulation of the cusp. In Sections 7 and 8 we show that the pleated surfaces \( S_i \) converge in a strong enough sense, so that their limit as \( i \) goes to \( \pm \infty \) describes the (pleated) boundary of the metric completion of \( V = \bigcup_{i \in \mathbb{Z}} \Delta_i \). At that point, we have constructed (the convex core of) a quasifuchsian group. The corresponding instance of Theorem 2 then follows from a computation, carried out in Section 9. In Section 10 we provide a similar construction of punctured-torus groups with rational pleating slopes \( \beta^\pm/\alpha^\pm \) and/or with infinite ends, and re-prove that \((\lambda^+, \lambda^-)\) are continuous coordinates for the space of quasifuchsian groups (see [Se]).

2. Dihedral angles

In this section we find positive dihedral angles for the ideal tetrahedra \( \Delta_i \), following Section 5 of [GF]. More precisely, we describe the convex space \( \Sigma \) of positive angle configurations for the \( \Delta_i \) such that:

- the three dihedral angles near each ideal vertex of \( \Delta_i \) add up to \( \pi \) (this is true in any ideal tetrahedron of \( \mathbb{H}^3 \));
- the dihedral angles around any edge of \( V = \bigcup_{i \in \mathbb{Z}} \Delta_i \) add up to \( 2\pi \) (this is necessary, though not sufficient, for a hyperbolic structure at the edge);
• the three pleating angles of each pleated punctured torus \( S_i \) add up to 0 (this is necessary, though not sufficient, to make the puncture \( p \) of \( S \) a cusp of \( V \), i.e. make the loop around \( p \) lift to a parabolic isometry of \( \mathbb{H}^3 \)).

(The first condition implies that opposite edges in \( \Delta_i \) have the same dihedral angle.)

Later on we shall apply Rivin’s Maximum Volume Principle on a certain convex subset of (the closure of) \( \Sigma_i \).

If the tetrahedron \( \Delta_i \) realizes a diagonal exchange that kills an edge \( \epsilon' \) and replaces it with \( \epsilon \), denote by \( \pi - w_i \) the interior dihedral angle of \( \Delta_i \) at \( \epsilon \) and \( \epsilon' \). Observe that the slope of \( \epsilon \) (resp. \( \epsilon' \)) is the rational located opposite the Farey edge \( e_i \) in the Farey diagram, on the side of \( \beta^+ / \alpha^+ \) (resp \( \beta^- / \alpha^- \)).

Thus, the pleating angles of the surface \( S_i \) living between \( \Delta_{i-1} \) and \( \Delta_i \) are

\[
\begin{align*}
    w_{i-1}, & \quad -w_i \quad \text{and} \quad w_i - w_{i-1}.
\end{align*}
\]

Observe the sign convention: the pleated punctured torus embedded in \( S \times \mathbb{R} \) receives an upward transverse orientation from \( \mathbb{R} \), and the angles we consider are the dihedral angles above the surface, minus \( \pi \). Thus, the “new” edge of \( \Delta_{i-1} \), pointing upward, accounts for a positive pleating \( w_{i-1} \), while the “old” edge of \( \Delta_i \), pointing downward, accounts for a negative pleating \( -w_i \). This is in accordance with the convention \( \lambda^- < 0 < \lambda^+ \) of the Section 0.1. One may write the three numbers \( \mathbf{1} \) in the corners of the corresponding Farey triangles (Figure 2 top).

In the tetrahedron \( \Delta_i \), let \( x_i \) (resp. \( y_i \)) be the interior dihedral angle at the edge whose slope is given by the right (resp. left) end of the Farey edge \( e_i \). Let

\[
    z_i = \pi - w_i
\]

be the third dihedral angle of \( \Delta_i \). For instance, \( 2x_i \) (resp. \( 2y_i \)) is the difference between the numbers written just below and just above the right (resp. left) end of \( e_i \) in Figure 2 (the factor 2 comes from the fact that the two edges of \( \Delta_i \), with angle \( x_i \) [resp. \( y_i \)] are identified). For notational convenience, write \((w_{i-1}, w_i, w_{i+1}) = (a, b, c)\). By computing differences between the pleating angles given in Figure 2 (bottom), we find the following formulae for \( x_i, y_i, z_i \) (depending on the letters, \( R \) or \( L \), living just before and just after the index \( i \)):

\[
\begin{array}{c|c|c|c|c}
    & LL & RR & LR & RL \\
    x_i & \frac{1}{2}(a + c) & \frac{1}{2}(-a + 2b - c) & \frac{1}{2}(a + b - c) & \frac{1}{2}(-a + b + c) \\
    y_i & \frac{1}{2}(-a + 2b - c) & \frac{1}{2}(a + c) & \frac{1}{2}(-a + b + c) & \frac{1}{2}(a + b - c) \\
    z_i & \pi - b & \frac{1}{2}(a + c) & \pi - b & \frac{1}{2}(a + b - c) \\
\end{array}
\]

The first of the three conditions defining \( \Sigma \) can be checked immediately; the other two are true by construction. From (2), the condition for all angles to be positive is that:

\[
\begin{align*}
    & \text{For all } i \text{ one has } 0 < w_i < \pi. \\
    & \text{If } i \text{ separates identical letters (first two cases), } 2w_i > w_{i+1} + w_{i-1}. \\
    & \text{If } i \text{ separates different letters (last two cases), } |w_{i+1} - w_{i-1}| < w_i.
\end{align*}
\]

Denote by \( \Sigma \) the non-empty, convex solution set of (3).

3. Bounding the Bending

3.1. A natural constraint on the pleating of \( S_i \). Next, we describe a certain convex subset of \( \Sigma \). It is obtained in the following way. Consider the pleated punctured torus, \( S_i \), lying between the tetrahedra \( \Delta_i \) and \( \Delta_{i-1} \). Let \( \epsilon_1, \epsilon_2, \epsilon_3 \) be
the edges of $S_i$ and $\delta_1, \delta_2, \delta_3$ the corresponding pleating angles (exterior dihedral angles, counted positively for salient edges as in figure above). Then, define the pleating measure $\lambda_i : C \to \mathbb{R}^+$ of $S_i$ by
\[
\lambda_i(\gamma) = \nu_1^+ \delta_1 + \nu_2^+ \delta_2 + \nu_3^+ \delta_3,
\]
where $\nu_i^+ \in \mathbb{N}$ is the intersection number of $\gamma$ with the simple closed curve $\varepsilon_s$ parallel to $\varepsilon_s$. We shall require that
\[
\lambda^-(\varepsilon_s) < \lambda_i(\varepsilon_s) < \lambda^+(\varepsilon_s)
\]
for each $s \in \{1, 2, 3\}$, and the same thing for every pleated punctured torus $S_i$ in the $\mathbb{Z}$-family (1).

In other words, denote by $\eta/\xi$ the (rational) slope of $\varepsilon_s$. Observe that $\lambda_i(\varepsilon_s) = \delta_{s'} + \delta_{s''} - \delta_s$ where $\{s, s', s''\} = \{1, 2, 3\}$, because the slopes of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are Farey neighbors. Therefore the requirement is that, for every $\varepsilon_s$ as above,
\[
-\left| \begin{array}{c} \beta^+ \\ \alpha^+ \end{array} \right| \frac{\eta}{\xi} < \delta_s < \left| \begin{array}{c} \beta^- \\ \alpha^- \end{array} \right| \frac{\eta}{\xi}.
\]

To express (5) in terms of the $w_i$, we need some notation (the $\delta_s$ are determined by the $w_i$ via (1) above). For each Farey edge $e_i$, let $q_i^+$ (resp. $q_i^-$) be the rational located opposite $e_i$, on the same side of $e_i$ as $\beta^+/\alpha^+$ (resp. $\beta^-/\alpha^-)$.

For an arbitrary rational $p = \eta/\xi$ (reduced form), introduce the slightly abusive (2) notation
\[
\frac{\beta}{\alpha} \wedge p := \left| \begin{array}{c} \beta \\ \alpha \end{array} \right| \frac{\eta}{\xi} \quad \text{(absolute value of the determinant)}.
\]

---

1. In fact a similar inequality will follow for all simple closed curves $\gamma$ of $C$ (see Lemma (4) below), namely, $\lambda^- < \lambda_i < \lambda^+$ as functions on $C$. Forcing this “natural” inequality is the whole point of our constraint.

2. Abusive in that it depends on the ordered pair $(\alpha, \beta)$ rather than just on the real $\frac{\beta}{\alpha}$.
Then, if $l, r$ are the rationals living at the left and right ends of the Farey edge $e_i$, one has

\begin{equation}
(\hat{p}^+ \land l) + (\hat{p}^+ \land r) = \hat{p}^+ \land q_i^-; \quad (\hat{p}^- \land l) + (\hat{p}^- \land r) = \hat{p}^- \land q_i^+.
\end{equation}

Indeed, the $|| \cdot ||$-notation is invariant under $PSL_2(\mathbb{Z})$, acting on $(\mathbb{H}^2, \partial \mathbb{H}^2)$ by isometries and on ordered pairs $(\alpha, \beta)$ as a matrix group. Since $PSL_2(\mathbb{Z})$ acts transitively on oriented Farey edges, we are reduced to the case $(l, r, q_i^+, q_i^-) = (\infty, 0, 1, -1)$ where $\beta^-/\alpha^- < 0 < \beta^+/\alpha^+$, which is straightforward.

Let us now translate Equation (8) in terms of the $w_i$. Let $e_{i-1}, e_i$ be two consecutive Farey edges; $p$ and $p'$ are the ends of $e_{i-1}$; $p$ and $p''$ are the ends of $e_i$. One has $q_i^- = p''$ and $q_i^+ = p'$. In view of (11) and Figure 2 (top), Equation (5) translates to

\begin{align*}
-(\frac{\beta^+}{\alpha^+} \land p''') &< w_{i-1} < \frac{\beta^-}{\alpha^-} \land p'' \\
-(\frac{\beta^+}{\alpha^+} \land p') &< -w_i < \frac{\beta^-}{\alpha^-} \land p' \\
-(\frac{\beta^+}{\alpha^+} \land p) &< w_i - w_{i-1} < \frac{\beta^-}{\alpha^-} \land p
\end{align*}

Using the fact that the $w_i$ are positive, this simplifies to (respectively)

\begin{align*}
-(\frac{\beta^+}{\alpha^+} \land q^-_i) &< w_{i-1} < \frac{\beta^-}{\alpha^-} \land q^+_i \\
-(\frac{\beta^+}{\alpha^+} \land q^-_i) &< -w_i < \frac{\beta^-}{\alpha^-} \land q^+_i \\
-(\frac{\beta^+}{\alpha^+} \land p) &< w_i - w_{i-1} < \frac{\beta^-}{\alpha^-} \land p
\end{align*}

Finally, observe that $\frac{\beta^+}{\alpha^+} \land p = (\frac{\beta^+}{\alpha^+} \land q^-_{i-1}) - (\frac{\beta^+}{\alpha^+} \land q^-_i)$ while $\frac{\beta^-}{\alpha^-} \land p = (\frac{\beta^-}{\alpha^-} \land q^+_i) - (\frac{\beta^-}{\alpha^-} \land q^+_1)$, by Equation (7). Therefore, if we introduce

\begin{equation}
(8) \quad \phi^+_i := \frac{\beta^+}{\alpha^+} \land q^-_i \quad \text{and} \quad \phi^-_i := \frac{\beta^-}{\alpha^-} \land q^+_i
\end{equation}

then Equation (11) reduces to

\begin{equation}
(9) \quad \phi^-_{i-1} - \phi^-_i < w_{i-1} - w_i < \phi^+_i - \phi^+_i \quad \forall i \in \mathbb{Z}.
\end{equation}

3.2. Study of $\phi^+$ and $\phi^-$.  

Definition 3. In (11) we associated to each $i \in \mathbb{Z}$ a Farey edge $e_i$ living between two letters of $\{R, L\}$. We call $i$ a hinge index, and $\Delta_i$ a hinge tetrahedron, if the two letters are distinct ($RL$ or $LR$). Non-hinges correspond to $RR$ or $LL$.

Lemma 4. The following holds concerning the sequences $\phi^+, \phi^- : \mathbb{Z} \to \mathbb{R}_+^*$.

i. First, $\phi^-$ is strictly increasing and $\phi^+$ is strictly decreasing.

ii. For all $i$ one has $1 < \phi^+_{i+1}/\phi^-_i < 2$ and $1 < \phi^-_{i-1}/\phi^+_i < 2$.

iii. If $i \in \mathbb{Z}$ is non-hinge then $\phi^+_{i-1} + \phi^+_{i+1} = 2\phi^+_i$ and $\phi^-_{i-1} + \phi^-_{i+1} = 2\phi^-_i$.

iv. If $i \in \mathbb{Z}$ is hinge then $\phi^+_{i-1} = \phi^+_i + \phi^-_{i-1}$ and $\phi^-_{i+1} = \phi^+_i + \phi^+_{i+1}$.

v. The sequences $\phi^+$ are (weakly) convex, i.e. $2\phi^+_i \leq \phi^+_{i-1} + \phi^+_{i+1}$.

vi. If $i < j$ are consecutive hinge indices, then $1 + \frac{j-i}{2} \leq \left\{ \frac{\phi^+_{i+1}}{\phi^-_j}, \frac{\phi^-_{i-1}}{\phi^+_j} \right\} \leq 1 + j-i$.

vii. One has $\lim_{+\infty} \phi^+ = \lim_{-\infty} \phi^- = 0$ and $\lim_{-\infty} \phi^+ = \lim_{+\infty} \phi^- = +\infty$.

viii. One has $\lim_{+\infty} (\phi^+_{i-1} - \phi^+_i) = \lim_{+\infty} (\phi^-_{i+1} - \phi^-_i) = +\infty$. 
Remark 5. Points iii and iv are just equality cases of the inequalities (9), which encode positivity of the angles \(x_i, y_i, z_i\).

Proof. We will deal only with \(\phi\): the arguments for \(\phi^{-}\) are analogous. Let \(e_{i-1} = pr_{i-1}\) and \(e_i = pr_i\) be consecutive Farey edges. We have \(\phi_i^+ = \frac{\alpha_i\pm}{\alpha_i} \cap r_{i-1}\) while \(\phi_{i-1}^+ = \frac{\alpha_i\pm}{\alpha_i} \cap r_{i-1} + \frac{\alpha_i\pm}{\alpha_i} \cap p\) by Equation (7), hence i.

For ii, we need only about the upper bound. Just observe that \(\phi_{i-1}^+ = \left(\frac{\alpha_i\pm}{\alpha_i} \cap r_i + \frac{\alpha_i\pm}{\alpha_i} \cap p\right) + \frac{\alpha_i\pm}{\alpha_i} \cap p\) while \(\phi_i^+ = \frac{\alpha_i\pm}{\alpha_i} \cap r_i + \frac{\alpha_i\pm}{\alpha_i} \cap p\).

For iii, assume \(e_{i+1} = pr_{i+1}\) so that \(\phi_{i+1}^+ = \frac{\alpha_i\pm}{\alpha_i} \cap r_{i+1} + \frac{\alpha_i\pm}{\alpha_i} \cap p\) for \(|i-k| \leq 1\). For \(k \in \{i, i+1\}\) the right hand side is \(\frac{\alpha_i\pm}{\alpha_i} \cap r_{i+1}\), so \(\phi_{i+1}^+ - \phi_i^+ = \frac{\alpha_i\pm}{\alpha_i} \cap p\), which is sufficient.

For iv, assume \(e_{i+1} = pr_{i+1}\). In the notations of Formula (8), we have \(q_{i+1} = p\) and \(q_i = r_{i-1}\), the ends of \(e_{i-1}\). This together with Equation (7) yields the result. Point v follows from iv and ii at hinges indices, and from iii at other indices. For Point vi, observe that \(\phi_i^+ = \phi_j^+ + (j-i)\phi_{j+1}\) by iii-iv, and conclude using ii. Point vii follows from vii, and the presence of infinitely many hinge indices near either end. Point viii follows from vii, v, and iv. □

3.3. Behavior of the pleatings \(\lambda_i : C \to \mathbb{R}\). For any real sequence \(u \in \mathbb{R}^Z\), define the real sequence \(\nabla u\) by \((\nabla u)_i = u_{i+1} - u_i\). Let us summarize the conditions imposed on \(w : Z \to \mathbb{R}_+\) (Eq. (9) and (9) above):

\[
\begin{cases}
0 < w_i < \min\{\phi_i^+, \phi_i^-\} < \nabla w_i \\
|w_{i+1} - w_i| < w_i \\
w_{i+1} + w_i < 2w_i
\end{cases}
\]

It is a simple exercise to check that \(w_i = \tanh \phi_i^+ \tanh \phi_i^-\), for instance, satisfies this system (tanh may be replaced by any strictly concave nonmonotous function from \(\mathbb{R}_+\) to \([0, 1]\) with the same 1-jet at 0).

Definition 6. If (10) denotes the system (10) in which all strong inequalities have been turned into weak ones, let \(W \subset \mathbb{R}^Z\) be the solution space of (10).

Suppose \((w_i)_{i \in \mathbb{Z}} \in W\) and consider the corresponding pleating measures \(\lambda_i : C \to \mathbb{R}\) of \(S_i\), the pleated punctured torus lying between the tetrahedra \(\Delta_i\) and \(\Delta_{i-1}\).

Lemma 7. For any curve \(\gamma \in C\), the sequence \((\lambda_i(\gamma))_{i \in \mathbb{Z}}\) is nondecreasing, with

\[
\lambda^-(\gamma) \leq \lim_{i \to -\infty} \lambda_i(\gamma) \leq \lim_{i \to +\infty} \lambda_i(\gamma) \leq \lambda^+(\gamma).
\]

Proof. First, observe that the pleating angles of \(S_{i-1}, S_i\) are always of the form given in Figure 8 where \(x\) (resp. \(y\)) is positive, equal to the interior dihedral angle of the tetrahedron at the horizontal (resp. vertical) edges of the square. The closed curve \(\gamma\) in the surface traverses the given square a number of times, either vertically, or horizontally, or diagonally (cutting off one of the four corners). The pleating along \(\gamma\) increases by \(2y\) per vertical passage, \(2x\) per horizontal passage, and \(0\) per diagonal passage, hence the monotonicity statement. (This argument, or a variant of it, is also valid for higher-genus surfaces and non-simple closed curves, as long as the tetrahedron has positive angles.)

For the bounding, we will focus only on the positive side. Consider the slope \(s = \frac{\eta}{\xi} \in \mathbb{P}^1\mathbb{Q}\) of \(\gamma\) (a reduced fraction); recall the definition \(\lambda^+(s) = ||\frac{\beta^+}{\alpha^+} \eta||\) from
the Introduction. Consider a large enough index $i$, such that the Farey edge $e_{i-1}$ separates $s$ from $\beta^+/\alpha^+$. Consider the points $p, p', p'' \in \mathbb{P}^1\mathbb{Q}$ such that $(e_{i-1}, e_i) = (pp', pp'')$: the points $(s, p, \frac{\beta^+}{\alpha^+}, p', s)$ are cyclically arranged in $\mathbb{P}^1\mathbb{R}$. Observe that the $\wedge$-notation (6) applied at rationals is just the (geometric) intersection number of the corresponding curves. Therefore, using the angle information given in Figure 2 (top), one has

$$\lambda_i(\gamma) = (p \wedge s)(w_i - w_{i-1}) - (p' \wedge s)w_i + (p'' \wedge s)w_{i-1}$$

$$= w_i(p \wedge s) + \nabla w_i(p' \wedge s) \quad \text{since } p'' \wedge s = p \wedge s + p' \wedge s$$

$$\leq \phi^+_i(p \wedge s) + \nabla \phi^+_i(p' \wedge s)$$

$$= \left(\frac{\beta^+}{\alpha^+} \wedge p'\right)(p \wedge s) + \left(\frac{\beta^+}{\alpha^+} \wedge p\right)(p' \wedge s) \quad (*)$$

by definition $\Phi$ of $\phi^+$. The last quantity is $\frac{\beta^+}{\alpha^+} \wedge s$ (hence the upper bound): by $SL_2$-invariance of the $\wedge$-notation, it is enough to check this when $p = \infty$ and $p' = 0$ — in that case, $\frac{\beta^+}{\alpha^+}$ and $s = \frac{\eta}{\xi}$ have opposite signs, and $(*)$ does indeed become $|\beta^+\xi| + |\alpha^+\eta| = \|\beta^+ \eta\|_{\alpha^+}$. \qed

4. Hyperbolic volume

The product topology on $\mathbb{R}^\mathbb{Z}$ induces a natural topology on the space $W$ of Definition 4, clearly, $W$ is nonempty, convex, and compact.

If $(x, y, z)$ is a nonnegative triple such that $x + y + z = \pi$, let $\mathcal{V}(x, y, z)$ be the hyperbolic volume of an ideal tetrahedron whose interior dihedral angles are $x, y, z$. We wish to compute the total hyperbolic volume of all tetrahedra when $w \in W$, i.e.

$$\mathcal{V}(w) := \sum_{i \in \mathbb{Z}} \mathcal{V}(x_i, y_i, z_i)$$
where \(x_i, y_i, z_i\) are defined from the \(w_i\) via Table (2). This poses the problem of well-definedness — the sum of the volumes might diverge. Let us estimate \(\mathcal{V}\): a well-known explicit formula \([\text{Mil}]\) gives

\[
\mathcal{V}(x, y, z) = -\int_0^x \log 2 \sin \tau \, d\tau - \int_0^y \log 2 \sin \tau \, d\tau - \int_0^z \log 2 \sin \tau \, d\tau
\]

\(= \int_0^x \log \frac{\sin(\tau + y)}{\sin \tau} \, d\tau \quad \text{(as } \int_0^\pi \log 2 \sin \tau = 0)\)

\(\leq \int_0^x \log \frac{\tau + y}{\tau} \, d\tau \quad \text{(by concavity of } \sin)\)

\(= x \log \frac{x + y}{x} + y \log \frac{x + y}{y} \quad \text{(concavity of } \log)\)

\[
(12)
\]

**Lemma 8.** There exists a universal constant \(K > 0\) such that the sum of the volumes of the tetrahedra \(\Delta_i\) for \(j \geq i\) (resp. \(j \leq i\)) is at most \(K \phi_i^+\) (resp. \(\phi_i^-\)).

**Proof.** We will focus only on the \(\phi_i^+\)-statement. First, by the computation above, the volume of the tetrahedron \(\Delta_i\) is at most \(w_i \log 2 \leq \phi_i^+ \log 2\) (see Table (2)). In view of Lemma 4-vi, this implies that the total volume of all hinge tetrahedra beyond the index \(i\) is at most \(3\phi_i^+ \log 2\). For the same reason, it is sufficient to prove

**Sublemma 9.** There exists a universal constant \(L > 0\) such that if \(0\) and \(N \in \mathbb{N}\) are two consecutive hinge indices, then the sum of the volumes of the tetrahedra \(\Delta_1, \Delta_2, \ldots, \Delta_{N-1}\) is at most \(L \phi_0^+\).

**Proof.** In view of homogeneity in the estimation \([\text{Mil}]\), it is sufficient to assume \(\phi_0^+ = 1\) and replace the volume with its estimate. Therefore, let \((w_i)_{0 \leq i \leq N}\) be a concave sequence in \([0, 1]\); following Table (2), we want to find a universal upper bound \(L\) (not depending on \(N\)) for

\[
\sum_{i=1}^{N-1} \frac{w_{i+1} + w_{i-1}}{2} \log \left(\frac{2w_i}{w_{i+1} + w_{i-1}}\right) + \frac{2w_i - w_{i+1} - w_{i-1}}{2} \log \left(\frac{2w_i}{2w_i - w_{i+1} - w_{i-1}}\right).
\]

If \(A < B\) are positive integers, denote by \(\Sigma_A^B\) the restriction of the above sum to indices \(A < i < B\). Observe that the general term of \(\Sigma_A^B\) is bounded by \(2\epsilon^{-1}\), because \(\frac{1}{2} \log 2 \leq e^{-1}\) for all positive \(\tau\). To be more efficient, we bound the first half of the general term by \((\frac{w_{i+1} + w_{i-1}}{2w_i})(\frac{2w_i}{w_{i+1} + w_{i-1}} - 1)\), and the second half, by concavity of \(\log\). This produces

\[
\Sigma_A^B \leq \sum_{i=A}^{B-1} \left(\frac{2w_i - w_{i+1} - w_{i-1}}{2} \times \frac{w_{i+1} + w_{i-1}}{w_{i+1} + w_{i-1}}\right)
\]

\[\quad + \left(\frac{2w_i - w_{i+1} - w_{i-1}}{2} \right) \log \left(\frac{\sum_{i=A}^{B-1} w_i}{\sum_{i=A}^{B-1} 2w_i - w_{i+1} - w_{i-1}}\right)\]

\[= \sigma \log \left(\frac{\sum_{i=A}^{B-1} w_i}{\sigma}\right), \quad \text{where } \sigma = \frac{w_A - w_{A-1} + w_B - w_B}{2}\]

Denote by \(M \in [0, N]\) a value of the index \(i\) for which \(w_i\) is maximal. If \(A < B \leq M\), we have

\[
0 \leq \sigma \leq \frac{w_A - w_{A-1}}{2} \leq \frac{1}{2A} \leq 1
\]
After a similar argument for the indices \( M < i < N \) the volume functional 
\[ \text{Corollary 10.} \]

The latter numbers (for \( k \) ranging over \( \mathbb{N}^* \)) add up to some universal \( L' < +\infty \). After a similar argument for the indices \( M < i < N \), we can take \( L = 2e^{-1} + 2L' \). \( \square \)

Finally, we can take \( K = 3L + 3\log 2 \). Lemma 8 is proved. \( \square \)

**Corollary 10.** The volume functional \( \mathcal{V} : W \to \mathbb{R}^+ \) is well-defined, continuous, and concave.

**Proof.** Well-definedness is the point of Lemma 8. Given \( \varepsilon > 0 \), only finitely many indices \( i \) satisfy \( \min\{\phi_i^+, \phi_i^-\} > \varepsilon / K \), and the others contribute at most \( 2\varepsilon \) to the volume: hence continuity in the product topology. Concavity follows from the concavity of the volume of one tetrahedron (parametrized by its angles): see e.g. Proposition 8 of \([GF]\). \( \square \)

Therefore, by compactness, there exists a sequence \( (w_i)_{i \in \mathbb{Z}} \in W \) which maximizes the hyperbolic volume \( \mathcal{V} \). From this point on, \( w \) will denote that maximizer.

**Proposition 11.** For each \( j \in \mathbb{Z} \), if \( x_j y_j z_j = 0 \) then \( \max\{x_j, y_j, z_j\} = \pi \).

**Proof.** Assume the tetrahedron \( \Delta_j \) has exactly one vanishing angle, and aim for a contradiction. If \( \mathcal{V}(\Delta^l) \) is the volume of a tetrahedron \( \Delta^l \) having angles \( x^l, y^l, (\pi - x^l - y^l) \) with \( (x^l, y^l) \geq 0 \) smooth, \( x^0 = 0 < y^0 < \pi \) and \( dx^l/dt|_{t=0} > 0 \), then \( d\mathcal{V}(\Delta^l)/dt|_{t=0} = +\infty \) (by Formula (12) above).

Let \( (w_i')_{i \in \mathbb{Z}} \) be a sequence satisfying all (strict) inequalities of \([10]\) and define \( w^t := w + t(w' - w) \) for \( 0 \leq t \leq 1 \). Denote by \( \Delta^t_j \) the \( i \)-th tetrahedron determined via (2) by \( (w_i')_{i \in \mathbb{Z}} \): then the angles of \( \Delta^t_j \) satisfy the hypotheses above, so

\[
\mathcal{V}(w^t) = \mathcal{V}(\Delta^t_j) + \left( \sum_{i \neq j} \mathcal{V}(\Delta^t_i) \right)
\]

has right derivative \(+\infty\) at \( t = 0 \) (the second summand is concave and continuous, so it has a well-defined right derivative in \( \mathbb{R} \cup \{+\infty\} \) at \( 0 \). Therefore, \( \mathcal{V} \) was not maximal at \( w \). \( \square \)

In Section 10 we will need the following consequence of Proposition 11.

**Proposition 12.** If \( j \in \mathbb{Z} \) and \( x_j y_j z_j = 0 \), then \( j \) is a hinge index and \( w_j = 0 \). \( \square \)

This is Proposition 13 in \([GF]\). But in fact much more is true:

**Proposition 13.** All the (strict) inequalities of \([10]\) are true at \( w \).

**Proof.** If some inequality of \([10]\) involving \( \phi^+ \) fails to be strict, it is easy to see by induction that \( w = \phi^+ \) near \(+\infty\), so all tetrahedra \( \Delta_i \) (for \( i \) large enough) have exactly one vanishing angle: \( w \) was not maximal, by Proposition 11. Therefore all inequalities of \([10]\) involving \( \phi^\pm \) are strict. The arguments in \([GF]\) (especially Lemma 16 and the argument of Section 9 there) can then be used to show that all inequalities \([10]\) are strict at \( w \), so \( w \) is a critical point of the volume \( \mathcal{V} \). \( \square \)
Proposition 13 implies that the holonomy representation is trivial, i.e. the gluing of any finite number of consecutive tetrahedra $\Delta_i$ defines a complete hyperbolic metric with (polyhedral) boundary (the shapes of the $\Delta_i$ “fit together correctly” around the edges): see [Ri], [CH] or [GF]. Therefore, the links of the vertices of the ideal tetrahedra (Euclidean triangles) form a triangulation of the link of the puncture: the latter is naturally endowed with a Euclidean structure and its universal cover can be drawn in the plane (Figure 4 — more on the combinatorics of this triangulation in Section 6; see also [GF]). Denote by $\Gamma$ the image of the induced holonomy representation $\pi_1(S) \to \text{Isom}^+(\mathbb{H}^3)$.

**Figure 4.** The cusp triangulation is shown, in anticipation, against the limit set $\mathcal{L} \subset \mathbb{P}^1\mathbb{C}$ of the quasifuchsian group (a complicated Jordan curve). The picture extends periodically to the right and left. Each broken line is the puncture link of a pleated surface $S_i$; the vertices (whose design artificially sets the $S_i$ apart from each other) are all parabolic fixed points at which the limit set $\mathcal{L}$ becomes pinched. Infinitely many very flat triangles accumulate along the top and bottom horizontal lines. This picture was generated with Masaaki Wada’s computer program Opti [Wa].
Lemma 14. Recall that \( \nabla \phi^- < 0 < \nabla \phi^+ \) (Lemma 3). One has

\[
\max \left\{ \lim_{z \to \infty} \frac{w}{\phi^+}, \lim_{z \to \infty} \frac{-w}{\phi^+} \right\} = \max \left\{ \lim_{z \to \infty} \frac{w}{\phi^-}, \lim_{z \to \infty} \frac{-w}{\phi^-} \right\} = 1.
\]

Proof. We focus on the \( \phi^+ \)-statement; the \( \phi^- \)-part is analogous. Since \( \frac{w}{\phi^+} < 1 \), assuming \( \lim_{z \to \infty} \frac{w}{\phi^+} < 1 \) implies \( \sup_{\phi^+} \frac{w}{\phi^+} < 1 \), and the same holds true for \( \frac{-w}{\phi^+} \) (see Equation 10). Therefore, suppose \( \sup_{\phi^+} \frac{w}{\phi^+} \leq 1 - \varepsilon \) and \( \sup_{\phi^+} \frac{-w}{\phi^+} \leq 1 - \varepsilon \) for some \( \varepsilon > 0 \), and aim at a contradiction.

Recall the ordered pair \((\alpha^+, \beta^+)\) that helped define \( \phi^+ \). For each \( \mu > 0 \), define \((\alpha^\mu, \beta^\mu) := (\mu \alpha^+, \mu \beta^+)\). This defines a new pleating function \( \nu^\mu = \mu \lambda^+ : C \to \mathbb{R}^+ \), a new \( \phi^\mu = \mu \phi^+ \) and a new domain \( W^\mu \) by (10) (the numbers \( \alpha^- \) and \( \beta^- \) are left unchanged). By definition, \( W = W^1 \) and \( W^\mu \subset W^{\mu'} \) if and only if \( \mu \leq \mu' \). Let \((w^\mu_i)_{i \in \mathbb{Z}}\) be the maximizer of the volume functional \( \mathcal{V} \) on \( W^\mu \). By assumption, we have \( w^1 \in W^{1-\varepsilon} \subset W \), so \( w^1 = w^{1-\varepsilon} \).

Write \( V(\mu) = \mathcal{V}(w^\mu) \), so that \( V(1) = V(1-\varepsilon) \). It is straightforward to check that for any \( t \in [0, 1] \), one has \( (t \cdot w^\mu + (1 - t) \cdot w^{\mu'}) \in W^{t \mu + (1 - t) \mu'} \). Since the volume of any tetrahedron is a concave function of its angles, this is enough to imply that \( V : \mathbb{R}^+ \to \mathbb{R}^+ \) is (weakly) concave. By inclusion, \( V \) is also nondecreasing. In fact, \( V \) is strictly increasing (which will finish the proof by contradiction): to prove this, since \( V \) is concave, we just need to produce arbitrarily large values of the volume \( \mathcal{V} \) at points \( \nu^\mu \in W^\mu \), for large enough \( \mu \). We may assume (up to a translation of indices, see Lemma 3 vii-viii) that \( \phi_0^+ > \pi \) and \( -\nabla \phi_0^- > \pi \). Then, start by defining \( v^\mu_i = \min \{ \phi_i^+, \phi_i^- \} \), so that \( \nu^\mu = \pi \) on \([0, N + 1]\) for arbitrarily large \( N \). Without loss of generality, by just taking \( \mu \) large enough, we may further assume \( |\nabla \phi^\mu| > \pi \) on \([0, N + 1]\). Then, each time \( i \in [1, N] \) is a hinge index, replace \( v^\mu_i \) by \( 2\pi/3 \); this is allowed in \( W^\mu \), by our assumptions on \( \nabla \phi^- \), \( \nabla \phi^+ \). The angles of the tetrahedron \( \Delta_i \) are then all in \( \{ \pi/2, \pi/3, \pi/6 \} \) (see Table 3). Since there are infinitely many hinge indices near \( +\infty \), the volume can become arbitrarily large: QED.

Corollary 15. In fact, \( \lim_{i \to +\infty} \min \left\{ \frac{w_i}{\phi_i^+}, \frac{\nabla w_i}{\nabla \phi_i^+} \right\} = \lim_{i \to +\infty} \min \left\{ \frac{w_i}{\phi_i^-}, \frac{-\nabla w_i}{-\nabla \phi_i^-} \right\} = 1 \).

Proof. Again, we focus only on \( \phi^+ \). By Lemma 14 there exists a subsequence \((w_{\nu(i)})_{i \in \mathbb{N}}\) such that \( w_{\nu(i)} \sim \phi_{\nu(i)}^+ \) or \( \nabla w_{\nu(i)} \sim \nabla \phi_{\nu(i)}^+ \). Suppose the latter is the case. For an arbitrary integer \( i \), let \( n \) be the smallest hinge index larger than or equal to \( \nu(i) \): observe that \( \phi_{n+1}^+ = \nabla \phi_n^+ = \nabla \phi_{\nu(i)}^+ \) by Lemma 4 iii-iv, while \( w_{n+1} \geq \nabla w_n \geq \nabla w_{\nu(i)} \) by the positivity conditions 10. Therefore, \( \frac{w_{n+1}}{\phi_{n+1}^+} \geq \frac{\nabla w_{\nu(i)}}{\nabla \phi_{\nu(i)}^+} \) so up to redefining \( \nu \) we may assume simply \( w_{\nu(i)} \sim \phi_{\nu(i)}^+ \).
Pick $\varepsilon > 0$. Take $i$ such that

$$(13) \quad w_{\nu(i)} \geq (1 - \varepsilon)\phi^+_{\nu(i)}.$$  

Let $n$ be the smallest hinge index strictly larger than $\nu(i)$. If $n = \nu(i) + 1$ then

$$w_n \geq w_{n-1} - \nabla \phi^+_n \geq (1-\varepsilon)\phi^+_n - \phi^+_{n+1} = \phi^+_n - \varepsilon\phi^+_{n-1} \geq (1-2\varepsilon)\phi^+_n;$$

$$\nabla w_n = w_{n-1} - w_n \geq (1-\varepsilon)\phi^+_{n-1} - \phi^+_n = \phi^+_{n+1} - \varepsilon\phi^+_{n-1} \geq (1-3\varepsilon)\phi^+_{n+1} = (1-3\varepsilon)\nabla \phi^+_n$$

where Lemma 4 has been used several times. Therefore, $\min\{\nabla w_n, \nabla w_k\} \geq 1 - 3\varepsilon$.

If $n \geq \nu(i) + 2$, we can find an index $k$ such that $\frac{k-\nu(i)}{n-\nu(i)} \in [\frac{1}{2}, \frac{3}{4}]$. We will show that

$$\min\{\frac{n_k}{\phi_k}, \frac{n_w}{\phi_w}\} \geq 1 - 8\varepsilon,$$

which will finish the proof.

• By positivity of $w$ and concavity of $w$ between the points $(\nu(i), k, n)$ one has $w_k \geq \frac{1}{k}w_{\nu(i)}$. Therefore,

$$\phi^+_k - w_k \leq \phi^+_{\nu(i)} - w_{\nu(i)} \leq \left(\frac{1}{1-\varepsilon} - 1\right)w_{\nu(i)} \leq \frac{3\varepsilon}{1-\varepsilon}w_k$$

(here the first inequality holds because $(\phi^+ - w)$ is decreasing by Condition (10), and the second follows from the assumption above). Hence,

$$\frac{w_k}{\phi_k} \geq \frac{1-\varepsilon}{1+2\varepsilon} \geq 1 - 3\varepsilon.$$

• Observe that

$$(1-\varepsilon)\phi^+_{\nu(i)} \leq w_{\nu(i)} \leq w_k + (k-\nu(i))\nabla w_k \leq \phi^+_k + (k-\nu(i))\nabla w_k$$

where the second inequality follows from concavity of $w$ between the points $(\nu(i), k-1, k)$. It follows that

$$(k-\nu(i))\nabla w_k \geq \phi^+_{\nu(i)} - \phi^+_k - \varepsilon\phi^+_{\nu(i)} = (k-\nu(i))\nabla \phi^+_k - \varepsilon\phi^+_{\nu(i)}$$

hence

$$\nabla w_k \geq \nabla \phi^+_k - \frac{\varepsilon}{k-\nu(i)}\phi^+_{\nu(i)} \geq \nabla \phi^+_k - \frac{2\varepsilon}{n-\nu(i)}\phi^+_{\nu(i)}.$$  

Since Lemma 4 vi gives us $\phi^+_{\nu(i)} \leq 2(n-\nu(i))\phi^+_n$, this yields

$$\nabla w_k \geq \nabla \phi^+_k - 4\varepsilon\phi^+_n \geq \nabla \phi^+_k - 8\varepsilon\phi^+_n = (1-8\varepsilon)\nabla \phi^+_n.$$  

\[ \square \]

**Corollary 16.** Recall the pleating $\lambda_i$ of the pleated surface $S_i$. For any simple closed curve $\gamma \in C$ we have $\lim_{i \to +\infty} \lambda_i(\gamma) = \lambda^+(\gamma)$ and $\lim_{i \to -\infty} \lambda_i(\gamma) = \lambda^-(\gamma)$.

**Proof.** By Corollary 15, the member ratio in the inequality 14 can be made arbitrarily close to 1. \[ \square \]
6. The cusp link

We now aim to investigate the behavior of the pleated surfaces $S_i$ as $i$ goes to $\pm \infty$ — or, more precisely, to find two limiting pleated surfaces $S_{\pm \infty}$ with pleatings $\lambda^\pm$ such that $V = \bigcup_{i \in \mathbb{Z}} \Delta_i \sqcup S_{+\infty} \sqcup S_{-\infty}$ is metrically complete with locally convex boundary. The difficult part is to prove that the intrinsic moduli of the $S_i$ converge in Teichmüller space. This question will be addressed in the next section. In this section, we just describe the cusp link, introduce notation and prove a few inequalities.

As always, we will mainly work near $+\infty$. Define $V = \bigcup_{i \in \mathbb{Z}} \Delta_i$. We begin by orienting all the edges of $V$ in a way that will be consistent with the pleating data $(\alpha^\pm, \beta^\pm)$. Namely, denote by $E \subset \mathbb{P}^1 \mathbb{Q}$ the collection of all the endpoints of all the Farey edges $(e_i)_{i \in \mathbb{Z}}$. For $s \in E$, let $Q_s$ be the edge of $V$ of slope $s$. Recall that the punctured torus is defined as $(\mathbb{R}^2 \setminus \mathbb{Z}^2) / \mathbb{Z}^2$. Orienting $Q_s$ is therefore equivalent to orienting the line $F_s$ of slope $s$ in $\mathbb{R}^2$. We decide that the positive half of $F_s$ should be on the same side of the line $\mathbb{R}(\alpha^-, \beta^-)$ as $(\alpha^+, \beta^+)$, and orient $Q_s$ accordingly.

Let us now describe the link of the puncture, or cusp triangulation. Each tetrahedron $\Delta_i$ contributes four similar Euclidean triangles at infinity to the link of the puncture, corresponding to the four ideal vertices of $\Delta_i$. The bases of these four triangles form a closed curve which is a broken line of four segments, and the triangles point alternatively up and down from this broken line (see Figure 5). The two upward (resp. downward) pointing triangles have the same Euclidean size, an effect of the hyperelliptic involution (rotation of $180^\circ$ around the puncture) which acts isometrically on $V$ (reversing all edge orientations) and as a horizontal translation on the cusp link.

**Figure 5.** In the left panel, $\star$ marks the cusp. In the right panel, the cusp is at infinity.

**Definition 17.** If the loop around the cusp has Euclidean length 4, let $b_i$ (resp. $b'_i$) be the length of the base of a downward-pointing (resp. upward-pointing) triangle contributed by the tetrahedron $\Delta_i$, and define $\sigma_i = b'_i / b_i$ (see e.g. Figure 7).
Figure 5 also shows some additional information, assuming that $e_i = 0\infty$ and $\alpha^+, \beta^+ > 0$ (hence $\beta^-/\alpha^- < 0$). Namely, the pleated surfaces $S_{i+1}$ and $S_i$ (above and below $\Delta_i$) are pleated along $Q_0, Q_{\infty}, Q_1$ and $Q_0, Q_{\infty}, Q_{-1}$ respectively, and the orientations of the $Q_a$ are as shown in the left panel of Figure 5 (edge $Q_{-1}$ has no determined orientation). The orientations of the lines to/from the puncture are also shown in the right panel, at the vertices, with the help of a color code. Moreover, each segment $\epsilon$ of the upper broken line in the right panel corresponds to an arc about a vertex $v$ of a face $f$ of $S_{i+1}$ in the left panel, so $\epsilon$ receives the orientation of the edge of $f$ opposite $v$. If $\tau$ is one of the upward-pointing triangles drawn in the plane $\mathbb{C}$ (right panel), consider the tetrahedron $\Delta$ whose vertices are $\infty$ and those of $\tau$: all edges of $\Delta$, except one, receive orientations from the construction above, and $\Delta$ is isometric (respecting these orientations) to $\Delta_i$. Finally, notice the labels in the 3 corners of each triangle in the right panel: the corner of the free vertex is labeled $z$ (the angle there being $z_i$); the other two corners are labeled $x$ and $y$ accordingly. The labels $x$-$y$-$z$ appear clockwise in each triangle.

The contribution of the tetrahedron $\Delta_{i-1}$ to the link at infinity is also a union of four triangles bounded by two broken lines. Moreover, the upper broken line from $\Delta_{i-1}$ is the lower broken line from $\Delta_i$, and the orientations of the lines to/from infinity must agree. Inspection shows that there are only two possibilities, corresponding to whether the letter living on the surface $S_i$ is $R$ or $L$: for $R$, the $x$-corners of the two 4-triangle families live near a common vertex $C$; for $L$, the same is true of the $y$-corners (Figure 6).

**Definition 18.** In a downward-pointing triangle defined by $\Delta_i$, the edge lengths are $b_i$ (the basis from Def. 17), $b_{i-1}$, and a third number which we call $c_i$.

**Property 19.** For $i$ large enough, $(b_i)$ is increasing and $(b'_i)$ is decreasing.

**Proof.** Let $\tau_i$ (resp. $\tau'_i$) be a downward-pointing (resp. upward-pointing) triangle defined by the tetrahedron $\Delta_i$. For large enough $i$ we have $z_i = \pi - w_i > \pi - \phi_i^+ > \pi/2$, so $b_i$ is the longest edge of $\tau_i$, and $b'_i$ the longest edge of $\tau'_i$. Since $b_{i-1}$ is an edge of $\tau_i$ and $b'_{i+1}$ is an edge of $\tau'_i$ (Figure 4 or 6), the conclusion follows. 

**Property 20.** We have $\lim_{i \to +\infty} \sigma_i = 0$.

**Proof.** We already know that $(\sigma_i) = (b'_i/b_i)$ is ultimately decreasing. It is therefore enough to show that $\sigma_{i+1}/\sigma_{i-1} \leq 1/2$ for large enough hinge indices $i$. Consider Figure 6 where angles labeled $z$ are obtuse (as a rule, we shade a cusp triangle.
contributed by $\Delta_i$ whenever $i$ is a hinge index). Check that

$$\frac{\sigma_{i+1}}{\sigma_i} = \frac{b'_{i+1} b_{i-1}}{b'_i b_{i+1}} \leq \frac{\sin x_i \sin y_i}{\sin^2 z_i} \leq \frac{\sin^2(w_i/2)}{\sin^2 w_i} \leq \frac{1}{2}$$

(the last two inequalities follow from an easy study of $\sin$, using $x_i + y_i = \pi - z_i = w_i < \pi/2$). As an immediate consequence, we find $\lim_{+\infty} b'_i = 0$ and $\lim_{+\infty} b_i = 2$. Similarly, $\lim_{-\infty} b'_i = 2$ and $\lim_{-\infty} b_i = 0$. (Compare with Figure 4).

**Definition 21.** Let $J \subset \mathbb{Z}$ be the set of all integers $j$ such that $j-1$ is a hinge index.

**Proposition 22.** If $j < l$ are two large enough consecutive elements of $J$, and $k$ is not in $J$, then

$$\frac{c_k}{c_{k-1}} = \sigma_{k-1} \quad \text{and} \quad \frac{c_l}{c_j} \leq \sigma_{j-2}.$$  

**Proof.** Since $k-1$ is not a hinge index, $b_{k-1}$ shares the same end with $b_k$ and with $b_{k-2}$ (see Figure 8 left) so $c_k/c_{k-1} = b_{k-1}/b_{k-2} = \sigma_{k-1}$.

Since $c_l$ is always an edge of $\tau'_l-1$, we have $c_l \leq b'_l-1$, hence

$$\frac{c_l}{c_j} \leq \frac{b'_{l-1}}{c_j} \leq \frac{b'_j}{c_j} \leq \frac{b'_j-2}{b_{j-2}} = \sigma_{j-2}$$

where the equality in the middle just translates the similarity property of the "hinge" triangles $\tau_{j-1}, \tau'_{j-1}$ (shaded in Figure 8 right).

**Figure 7.** The index $i$ is a hinge; the nature of $i \pm 1$ is undetermined.

**Figure 8.** Left: $k-1$ is not a hinge. Right: $j-1$ is a hinge.
7. Intrinsic convergence of the surfaces \( S_i \)

7.1. **Thickness of the tetrahedra.** Consider a tetrahedron \( \Delta_i \) bounded by the pleated surfaces \( S_i \) and \( S_{i+1} \). Let \( Q \) (resp. \( Q' \)) be the pleating edge of \( S_i \) (resp. \( S_{i+1} \)) not lying in \( S_i \cap S_{i+1} \). Let \( s_i \) be the shortest segment between \( Q \) and \( Q' \), across \( \Delta_i \).

**Definition 23.** Recall the orientations on the edges of \( V = \bigcup_{i \in \mathbb{Z}} \Delta_i \). Let \( \ell_i \) be the complex length of the hyperbolic loxodromy along \( s_i \) sending \( Q \) to \( Q' \), respecting the orientations of \( Q, Q' \) (with \( -\pi < \Im \ell_i \leq \pi \)).

**Proposition 24.** The series \((\ell_i)_{i \in \mathbb{Z}}\) is absolutely convergent.

**Proof.** Consider a downward-pointing triangle \( \tau \) contributed by \( \Delta_i \). Label the vertices of \( \tau \) by \( A, B, C \) in such a way that \( AC = b_i, BC = b_{i-1} \) and \( AB = c_i \) (see Figure \( \text{A} \)). Let \( \gamma_i \) be the hyperbolic loxodromy of complex length \( \ell_i \) along the common perpendicular to \( B\infty \) and \( AC \), sending \( B, \infty \) to \( A, C \) (in that order).

**Sublemma 25.** Let \( \ell = \ell_i = \rho + \theta \sqrt{-1} \) be the complex length of \( \gamma_i \), with \( \theta \in [-\pi, \pi] \). Then \( \max \{\rho, |\theta|\} \leq \pi \sqrt{c_i/b_i} \).

**Proof.** Up to a plane similarity, we may assume \( A = 1 \) and \( C = -1 \). Also, for convenience, relabel the edges of \( ABC \) by \( a, b, c \). Let \( L \) be the fixed line of \( \gamma_i \). The hyperbolic isometry defined by \( z \mapsto f(z) = \frac{2B+1-z}{z+1} \) exchanges the oriented lines \( AC \) and \( B\infty \), so it reverses the orientation of \( L \) around the center of the tetrahedron \( ABC\infty \). Therefore, \( \gamma_i \) is given by \( \gamma_i(z) = 2B - f(z) \). If \( M \) is a matrix of \( \gamma_i \), one has

\[
\frac{\text{tr}^2 M}{4 \det M} = \frac{\cosh \ell + 1}{2}.
\]

Use \( M = \begin{pmatrix} 2B + 1 & -1 \\ 1 & 1 \end{pmatrix} \) to find \( B = \cosh \ell \). Compute

\[
\frac{a + c}{b} = \frac{|B + 1| \pm |B - 1|}{2} = \left| \cosh \frac{\ell}{2} \right| \pm \left| \sinh \frac{\ell}{2} \right| = \cosh \frac{\ell}{2} \pm \sinh \frac{\ell}{2} = \cosh \frac{\ell}{2}.
\]

Use \( \cosh(i\theta) = \cos \theta \) to get \( \cosh(\rho) = \frac{a + c}{b} \) and \( \cos(\theta) = \frac{a - c}{b} \). The estimates \( \text{Argcosh}(y) \leq 2 \sqrt{\frac{y-1}{2}} \) and \( \text{Arccos}(y) \leq \pi \sqrt{\frac{1-y}{2}} \), since \( \frac{1}{2} \left| \frac{a + c}{b} \right| - 1 \leq \frac{\ell}{2} \), finally yield \( \rho \leq 2 \sqrt{c/b} \) and \( |\theta| \leq \pi \sqrt{c/b} \), hence Sublemma 25.

Since \((b_i)\) goes to 2, Proposition 24 will follow if the \( c_i \) go to 0 fast enough near \(+\infty\) (with a similar argument near \(-\infty\)). Such fast decay is given by Proposition 29 using the fact that \((\sigma_i)\) goes to 0, we can bound the \( c_i \) by decreasing geometric sequences on intervals of the form \([j, l-1]\) where \( j < l \) are consecutive elements of the set \( J \).

7.2. **Teichmüller charts.** Let \( T_S \) be the Teichmüller space of the punctured torus: for each \( i \in \mathbb{Z} \) we denote by \( \mu(S_i) \in T_S \) the intrinsic modulus of the marked surface \( S_i \). In order to prove that \( \mu(S_i) \) converge in \( T_S \), let us first introduce appropriate charts for \( T_S \).

Consider a topological ideal triangulation \( E \) of the punctured torus \( S \), with labeled edges \( e_1, e_2, e_3 \). Then \( E \) defines an isomorphism

\[
h_E : \mathbb{P}^2 \mathbb{R}_+^* = ([\epsilon_1, \epsilon_2, \epsilon_3]) / \mathbb{R}_+^* \rightarrow T_S.
\]

Namely, given a hyperbolic metric \( g \) on \( S \), in order to compute \( h_E^{-1}(g) \), straighten \( E \) to an ideal triangulation for \( g \) and return the positive projective triple of Euclidean
lengths defined (in the link of the puncture) by the sectors opposite $\epsilon_1, \epsilon_2, \epsilon_3$. We consider the $h_E$ as charts of $T_S$. We endow $\mathbb{P}^2 \mathbb{R}_+^*$ with the distance $d$ given by

$$d([a : b : c], [a' : b' : c']) := \min_{\lambda > 0} \max \left\{ \left| \log \frac{\lambda a}{a'} \right|, \left| \log \frac{\lambda b}{b'} \right|, \left| \log \frac{\lambda c}{c'} \right| \right\}.$$ 

Also define $\overline{h}_E : \mathbb{P}^2 \mathbb{C}^* \to T_S$ by $\overline{h}_E([a : b : c]) := h_E([|a| : |b| : |c|])$.

In particular, if the pleated punctured torus $S_{i+1}$, pleated along the ideal triangulation $E_{i+1}$, gives rise in the cusp link to a broken (oriented) line whose segments have complex coordinates $(a, b, c)$, then

$$\mu(S_{i+1}) = \overline{h}_{E_{i+1}}([a : b : c]).$$

Moreover, Figure 9 shows two broken lines corresponding respectively to $S_{i+1}$ and the previous pleated surface $S_i$. Since the triangles (links of vertices of $\Delta_i$) are similar, the complex coordinates of the segments forming the lower broken line are functions of $(a, b, c)$, as shown. Therefore, if we define the substitution formula

$$\Psi([a : b : c]) := \left( \left[ a + b : \frac{b}{a + b} c : \frac{a}{a + b} c \right] \right)$$

(which is a birational isomorphism from $\mathbb{P}^2 \mathbb{C}$ to itself), then

$$\mu(S_i) = \overline{h}_{E_i}(\Psi([a : b : c])).$$

Finally, by viewing the diagonal exchange between $E_i$ and $E_{i+1}$ as a flat tetrahedron (as in Figure 1), we see that

$$\mu(S_{i+1}) = h_{E_i}(\Psi([|a| : |b| : |c|])).$$

In other words, the restriction $\psi$ of $\Psi$ to $\mathbb{P}^2 \mathbb{R}_+^*$ is the chart map $h_{E_{i+1}} \to h_{E_i}$, i.e.

$$\begin{array}{cccc}
\mathbb{P}^2 \mathbb{R}_+^* & \psi \leftarrow & \mathbb{P}^2 \mathbb{R}_+^* \\
\overline{h}_E & \searrow & \overline{h}_{E_{i+1}} \\
T_S & \nearrow & h_{E_{i+1}}
\end{array}$$

commutes.

**Property 26.** We have $d(h_{E_i}^{-1}\mu(S_i), h_{E_i}^{-1}\mu(S_{i+1})) = \log \frac{|a| + |b|}{|a + b|}$.
Proof. The right member is
\[
d \left( \left[ \frac{|a + b|}{|b|} : \frac{b}{a + b} \right] ; \left[ \frac{|a + b|}{|a + b|} \right] \right).
\]

It is easy to see that \(\psi : \mathbb{P}\mathbb{P}^2 \rightarrow \mathbb{P}^2\) is 3-Lipschitz for \(d\). In fact,

**Proposition 27.** There exists \(K > 0\) such that the \(n\)-th iterate \(\psi^n\) is \(Kn\)-bilipschitz for all \(n > 0\).

Proof. Since \(\psi([a : b : c]) = \left[ (a+b)^2 \right] : [a : b] \) and \(\psi^{-1}([a : b : c]) = \left[ c : b : \frac{(c+b)^2}{a} \right]\), it is enough to check the Lipschitz statement. Set \(A := \sqrt{a} \); \(C := \sqrt{c} \);
\[
P_n := \frac{A^{n+1}}{C^n} + \sum_{i,j \in \mathbb{Z}} \left( \frac{j}{i-1} \right) \left( \frac{n-i}{j-i} \right) A^{2i-n-1} C^{-n-2i} \quad \text{for all } n \geq -1
\]
(the sum is really on \(0 < i \leq j \leq n\), so that \(P_0 = A \); \(P_{-1} = C \); \(P_1 = \frac{A^2+1}{C^2}\)). We claim that \(\psi^n[a : 1 : c] = [P_2^n : 1 : P_{-2}^n]\). The claim is seen by induction on \(n\): the only difficult thing is the induction step \(P_{n+1}^2 = \frac{(P_{n+1}^2+1)^2}{P_{n-1}^2}\). First, it is straightforward to check that \(P_{n+1} + P_{n-1} = (\frac{A}{C} + \frac{C}{A} + \frac{1}{AC})P_n\) (using the Pascal relation twice). Hence, for all \(n \geq 1\), one has
\[
(P_{n+1}P_{n-1} - P_n^2) - (P_nP_{n-2} - P_{n-1}^2) = P_{n-1}(P_{n+1} + P_{n-1}) - P_n(P_n + P_{n-2}) = (P_{n-1}P_n - P_nP_{n-1})(\frac{A}{C} + \frac{C}{A} + \frac{1}{AC}) = 0.
\]
Therefore \(P_{n+1}P_{n-1} - P_n^2 = P_1P_{-1} - P_0^2 = 1\), which proves the induction step. Since \(P_n\) is a Laurent polynomial in \(A, C\) with partial degrees of order \(n\) and positive coefficients, we see that \(\log P_n\) is \(Ln\)-bilipschitz in \(\log a, \log c\) for some universal \(L\). The Proposition follows. \(\square\)

The proof of Proposition 27 may seem extremely \textit{ad hoc} and unsatisfactory. However, Proposition 27 is a special case of a more general phenomenon for \textit{Markoff maps} (in the sense of Boalch), which we describe in 7.4.

### 7.3. Convergence of the moduli.

**Proposition 28.** The moduli \((\mu(S_i))_{i \rightarrow +\infty}\) converge in Teichmüller space \(T_S\).

Proof. To prove the \(+\infty\)-statement, we will fix a large enough index \(i\), and prove that the series
\[
\eta_j := d \left( h_{E_i}^{-1}(\mu(S_j), \mu(S_{j+1})) \right),
\]
defined for \(j > i\), has finite sum.

For \(j > i\), consider a downward-pointing triangle \(\tau\) contributed by \(\Delta_j\), with its edge lengths \(b_j, b_{j-1}\) and \(c_j\). The angles of \(\tau\) at the ends of \(b_j\) are \(x_j\) and \(y_j\). By Property 26, we have
\[
d \left( h_{E_i}^{-1}(\mu(S_j), \mu(S_{j+1})) \right) = \log \frac{c_jb_{j-1} + b_j}{b_j}.
\]
Compute
\[
\log \frac{c_j + b_{j-1}}{b_j} \leq \frac{c_j + b_{j-1} - b_j}{b_j} \leq \frac{\sin x_j + \sin y_j - \sin z_j}{\sin z_j} \leq \frac{2\sin \frac{x_j}{2} \sin \frac{y_j}{2}}{\cos \frac{x_j + y_j}{2}} \leq \frac{\sin x_j \sin y_j}{\sin^2 z_j} \sin^2 z_j = \frac{c_jb_{j-1}}{b_j} \sin^2 z_j \leq \frac{c_jb_{j-1}}{b_j} + \frac{b_j}{\theta_j^2}
\]
(the inequality at the start of the second line holds for large enough $j$ because $x_j + y_j = 2w_j \to 0$). Define $\delta_j = \frac{1}{b_j} \phi_j^+$, and let $M_j$ denote the best bilipschitz constant for the chart map $h_{E_j} \to h_{E_i}$. Observe that $\eta_j \leq M_j \delta_j$.

Let $j < l$ be two consecutive elements of $J$ (see Definition 21), and $k \notin J$ an integer. We shall bound the $M_n \delta_n$ by geometric sequences on intervals of the form $[j, l - 1]$, using Proposition 22 as in the proof of Proposition 24. Since $\psi$ is 3-bilipschitz, one clearly has $M_k \leq 3M_{k-1}$. By Proposition 22

$$\frac{M_k \delta_k}{M_{k-1} \delta_{k-1}} \leq \frac{3 \delta_k}{\delta_{k-1}} \leq \frac{c_k}{c_{k-1}} = 3\sigma_{k-1}$$

because $(\phi_j^+)$ and $(1/b_k)$ are decreasing. The right member goes to 0 for large $k$.

By Sublemma 27, $M_i \leq (l - j) L \cdot M_j$ for some universal $L$, and by Lemma 3 (i-iii-iv), $\phi_j^+ = \phi_{l-1}^+ + (l - j - 1) \phi_l^+ \geq (l - j) \phi_l^+$. Using Proposition 24 it follows that

$$\frac{M_l \delta_l}{M_j \delta_j} \leq (l - j) L \cdot \frac{c_l}{c_j} \left( \frac{\phi_l^+}{\phi_j^+} \right)^2 \leq \frac{L \sigma_{l-2}}{l - j} \leq L \sigma_{j-2}.$$

The right member goes to 0 as $j$ goes to infinity, hence Proposition 28. \hfill \Box

8. EXTRINSIC CONVERGENCE OF THE SURFACES $S_i$

8.1. Pleated surfaces. Propositions 24 and 28 together with Lemma 7 are the key ingredients to prove that the metric completion of $V = \bigcup_{i \in \mathbb{Z}} \Delta_i$ has two boundary components which are pleated punctured tori with pleating measure $\lambda^\pm$. A pleated surface is by definition (up to taking a universal cover) a map $\varphi : \mathbb{H}^2 \to \mathbb{H}^3$ which sends rectifiable arcs to rectifiable arcs of the same length, such that through each point $p$ of $\mathbb{H}^2$ runs an open segment $s_p$ on which $\varphi$ is totally geodesic. It is known (see CEC, 5.1.4) that the direction of $s_p$ is unique if and only if $p$ belongs to a certain geodesic lamination $\Lambda$ (closed union of disjoint geodesics), and that $\varphi$ is totally geodesic away from $\Lambda$.

To wrestle with pleated surfaces, we will use the fact that if $\varphi$ is a locally convex immersion, and $\Lambda$ has zero Lebesgue measure, then $\Lambda$ comes with a transverse (pleating) measure $\nu_\Lambda$. More precisely, $\nu_\Lambda$ can be defined on any segment $s$ transverse to $\Lambda$ in the following way (see Sections 7 to 9 of [Lon]). Immers $\varphi(\mathbb{H}^2)$ into the Poincaré upper half-space model. Each component of $\mathbb{H}^2 \setminus \Lambda$ crossed by $s$ can be extended to a subset $A$ of $\mathbb{H}^2$ bounded by only one or two lines of $\Lambda$ crossed by $s$. Endow $s$ with a transverse orientation. The boundary component of $A$ in $\partial \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ on the positive side of the transverse orientation defines a circle arc $c_A$, of angle $\theta_A \in (-2\pi, 2\pi)$ (we may assume $\infty \notin c_A$). The closure of the union of all the $c_A$ forms a rectifiable curve $c$, of length $\sum A \text{length}(c_A)$. Then $c$ has a well-defined regular curvature $RC(c)$, defined as the absolutely convergent sum $\sum A \theta_A$. But $c$ also has a total curvature $TC(c)$, defined (up to an appropriate multiple of $2\pi$) as the difference between the arguments of the initial and final tangent vectors to $c$. (The appropriate multiple of $2\pi$ can be determined by closing off the embedded arc $c$ with a broken line, and requiring that the resulting Jordan curve have total curvature $2\pi$). Then, $\nu_\Lambda(s)$ is defined as the singular curvature $SC(c) = TC(c) - RC(c)$. 


Conversely, if \( \varphi : \mathbb{H}^2 \to \mathbb{H}^3 \) is a pleated immersion and \( SC(c) \) is well-defined and non-negative for all transverse segments \( s \), then \( \varphi \) is locally convex with pleating measure \( \nu_A \) as above. We refer to [Bon] for greater detail.

8.2. Setup. Consider the marked once-punctured torus \( S_{+\infty} \), endowed with the hyperbolic metric \( \lim_{\mathbb{R}} \mu(S_i) \). There exists a unique compact geodesic lamination \( \Lambda^+_c \) of slope \( \beta^+/\alpha^+ \) on \( S_{+\infty} \). Exactly two lines \( \ell, \ell' \) issued from the puncture of \( S_{+\infty} \) fail to meet \( \Lambda^+_c \): define \( \Lambda^+ = \Lambda^+_c \cup \ell \cup \ell' \). Then \( S_{+\infty} \) is the disjoint union of \( \Lambda^+ \) and the interiors of two ideal triangles \( A, A' \). The union \( A \cup A' \cup \ell \cup \ell' \) is a punctured ideal bigon.

Recall the ideal tetrahedra \( \Delta_i \), the space \( V = \bigcup_{i \in \mathbb{Z}} \Delta_i \) and the hyperelliptic involution \( h : V \to V \) which reverses all edge orientations. The slopes of the (oriented) pleating lines of the surface \( S_i \) (between \( \Delta_i-1 \) and \( \Delta_i \)) are elements of \( \mathbb{P}^1 \mathbb{Q} \) projecting to 0, 1, \( \infty \) in \( \mathbb{P}^1(\mathbb{Z}/2) \): accordingly, we call these pleating lines \( \ell^\circ, \ell^1, \ell^\infty \). For \( \ast \in \{0, 1, \infty \} \), denote by \( \omega^\ast \) the unique point of \( \ell^\ast \) fixed under the hyperbolic involution (the \( \omega^\ast \) are called Weierstrass points). Let \( s^\ast \) be the segment from \( \omega^\ast \) to \( \omega^\ast_{i+1} \) (although the tetrahedron \( \Delta_i \) is \( \omega^\ast \neq \omega^\ast_{i+1} \)): each \( s^\ast \) is contained in a (pointwise) fixed line \( \Omega^\ast \) of the hyperelliptic involution.

Fix the value of the superscript \( \ast \) (soon we shall omit it). Fix a point \( \omega \in \mathbb{H}^2 \) and an oriented line \( l \) through \( \omega \). For each (oriented) surface \( S_i \), consider the oriented, marked universal covering \( \pi_i : (\mathbb{H}^2, \omega, l) \to (S_i, \omega^\ast, \ell^\ast) = (S_i, \omega_i, l_i) \). Endow a universal covering \( \tilde{V} \) of \( V \) with lifts \( \tilde{\omega}_i \) of the \( \omega_i \) connected by lifts of the segments \( s_i \), and fix a developing map \( \Phi : \tilde{V} \to \mathbb{H}^3 \). There is a unique map \( h_i \) such that

\[
\begin{align*}
(\mathbb{H}^2, \omega, l) & \xrightarrow{\pi_i} (S_i, \omega_i, l_i) \\
\tilde{V} & \xrightarrow{\Phi} \mathbb{H}^3
\end{align*}
\]

(15)

commutes. We will prove that the (developing) pleated immersions

\[
\varphi_i = \Phi \circ h_i : \mathbb{H}^2 \to \mathbb{H}^3
\]

(16)

converge as pleated maps.

8.3. Convergence of the \( \varphi_i \). By Proposition 22 it is already clear that the restriction of \( \varphi_i \) to \( l \) converges to a totally geodesic embedding of the line \( l \) into \( \mathbb{H}^3 \) (the convergence is uniform on all compacts of \( l \)). By Ascoli’s theorem, since the \( \varphi_i \) are 1-Lipschitz, there exists an increasing sequence \( \nu \) such that the \( \varphi_{\nu(i)} \) converge to a certain map \( \varphi_{+\infty} \), uniformly on all compact sets of \( \mathbb{H}^2 \). In Section 3.4 below, \( \varphi_{+\infty} \) is shown to be independent of the subsequence \( \nu \); in anticipation, we now abusively write \( \varphi_i \) instead of \( \varphi_{\nu(i)} \).

We shall work in the projective tangent bundle \( \mathcal{E} = \mathbb{PT} \mathbb{H}^2 \), a circle bundle over \( \mathbb{H}^2 \) in which geodesic laminations naturally live as closed sets. Length and angle measurements define a (canonical) complete Riemannian metric on \( \mathcal{E} \). For \( \mathcal{K} \subset \mathbb{H}^2 \) compact and \( A, B \subset \mathcal{E} \) closed, let \( K^\mathcal{E} \subset \mathcal{E} \) be the preimage of \( K \) under the natural projection \( \mathcal{E} \to \mathbb{H}^2 \), and define

\[ d_K(A, B) := \inf \{ \delta > 0 \mid A \cap K^\mathcal{E} \subset B + \delta \, , \, B \cap K^\mathcal{E} \subset A + \delta \} \]
where $X + \delta$ denotes the set of points within $\delta$ of $X$. Then \( \inf(1, d_{K}) \) is a pseudo-metric, and the set of closed subsets of $E$ is compact for the Hausdorff metric

\[
d_H = \sum_{n>0} 2^{-n} \inf(1, d_{K_n})
\]

where the $K_n$ are concentric balls of radius $n$.

Observe that $S_{+\infty}$ also has Weierstrass points $\omega_{+\infty}$, belonging to leaves $l_{+\infty}$ of the lamination $\Lambda^+$. Fixing the value of $\ast$ as before, denote by $\pi_{+\infty} : (\mathbb{H}^2, \omega, I) \to (S_{+\infty}, \omega_{+\infty}, l_{+\infty})$ an oriented universal cover.

A consequence of Proposition 28 is that the lifts to $E$ of the $\pi_i^{-1}(l_{0}^i \cup l_{1}^i \cup l_{+\infty}^i)$ converge for $d_H$ to the lift of $\pi_{+\infty}^{-1}(\Lambda^+)$. We know that $U = \pi_{+\infty}^{-1}(S_{+\infty} \setminus \Lambda^+) \subset \mathbb{H}^2$ is a disjoint union of (open) ideal triangles, of full Lebesgue measure in $\mathbb{H}^2$. For any connected compact set $K \subset U$, we have $K \cap \pi_i^{-1}(l_{0}^i \cup l_{1}^i \cup l_{+\infty}^i) = \emptyset$ for all $i$ large enough, so $\varphi_{+\infty}$ is totally geodesic on $K$. Therefore, $\varphi_{+\infty}$ is totally geodesic on each component of $U$. Since $\varphi_{+\infty}$ is clearly 1-Lipschitz, we can approximate any segment in $\mathbb{H}^2 \setminus U$ by segments in $U$ to show that $\varphi_{+\infty}$ is totally geodesic on each leaf of $\mathbb{H}^2 \setminus U$. By Lemma 5.2.8 in [CEG], $\varphi_{+\infty}$ sends rectifiable segments to rectifiable segments of the same length, and is a pleated map.

8.4. **The map $\varphi_{+\infty}$ is a topological immersion.** Define $\mathcal{P} := \pi_{+\infty}^{-1}(\Lambda^+)$, which contains the pleating locus of $\varphi_{+\infty}$. To prove $\varphi_{+\infty}$ is an immersion, it is enough to find a short geodesic segment $m$ of $\mathbb{H}^2$, through the base point $\omega$, transverse to $\mathcal{P}$, and prove that $\varphi_{+\infty}$ is an immersion on the union $\Upsilon$ of all strata (lines and complementary ideal triangles) of $\mathcal{P}$ crossed by $m$ (indeed, $\pi_{+\infty}(\Upsilon) = S_{+\infty}$).

Clearly, $\varphi_{+\infty}$ is already an immersion near any point of $\mathbb{H}^2 \setminus \mathcal{P}$. At other points, the key fact will be an “equidistribution” property of the 3 pleating lines of the surface $S_i$, as $i$ goes to $+\infty$.

Choose a small $\mu_1 > 0$, and pick $k \in \mathbb{Z}$ large enough so that $\phi_k^\ast \leq \mu_1$. Let $q$ be the rational opposite the Farey edge $e_k$, on the same side as $\beta^\ast/\alpha^\ast$, so that $\lambda^\ast(q) = \phi_k^\ast$. Let $m_k^C$ be the simple closed geodesic of slope $q$ in $S_i$ (equipped with the intrinsic metric): for some superscript $\ast$ independent of $i$, the Weierstrass point $\omega_i = \omega_k^i$ belongs to $m_k^C$. By Proposition 28 and Hausdorff convergence, there exists $\mu_2 > 0$ such that for all $i > k$, the angle between $m_k^C$ and the pleating line $l_i$ of $S_i$ at $\omega_i$ is at least $\mu_2$, and there exists $\mu_3 > 0$ such that the segment $m_i$ of length $2\mu_3$, centered at $\omega_i$, is embedded in $S_i$. The ends of $m_i$ are at distance at least $\frac{1}{2} \mu_3 \mu_3$ from the pleating line $l_i$ in $S_i$.

Observe that the simple closed curve $m_i^C$ meets the pleating edges of $S_i$ in a perfectly equidistributed (Sturmian) order (as would a straight line on a Euclidean grid): therefore, the algebraic sum of the pleating angles crossed by any given subsegment of $m_i$ does not exceed $2\mu_1$. Finally, let $\kappa_i$ be a subsegment of the pleating line $l_i$, centered at $\omega_i$, of length $2\mu_3$: if $\mu_3$ is small enough, any pleating line $L$ met by $m_i$ makes an angle at least $\frac{1}{2} \mu_2$ with $m_i$, and comes within $3\mu_3$ of both ends of $\kappa_i$ for the intrinsic metric of $S_i$.

Arrange the developing map $\Phi : \tilde{V} \to \mathbb{H}^3$ in the upper half-space model so that the $\varphi_i(\omega)$ lie on the line $0\infty$ at heights less than 1, and $\varphi_{+\infty}(l)$ is the oriented line from $-1$ to 1. Consider lifts $\tilde{m}_i$ of the arcs $m_i$ through the $\varphi_i(\omega)$. By the above (considering lifts of the $\kappa_i$), if the $\mu$’s are small enough, any pleating line of $\varphi_i(\mathbb{H}^2)$ met by $\tilde{m}_i$ has its endpoints within distance $1/2$ from 1 and $-1$ in $\mathbb{C}$ (recall $\varphi_i$
is 1-Lipschitz). Following Subsection 8.3 let \( c_{i}^{\pm 1} \) (resp. \( c_{i}^{-1} \)) denote the piecewise smooth curve defined by the transverse segment \( m_{i} \) of \( \varphi_{i}(\mathbb{H}^{2}) \) near 1 (resp. \(-1\)).

Let \( \tau \) be a subsegment of \( m_{i} \) across an ideal triangle of \( \varphi_{i}(\mathbb{H}^{2}) \). Let \( \tau' \) be the circle arc contributed by \( \tau \) to \( c_{i}^{\pm 1} \). By the above, for some universal \( K_{1} > 0 \),

\[
\frac{\mu_{2}}{K_{1}} \leq \frac{\text{euclidean length of } \tau'}{\text{hyperbolic length of } \tau} \leq K_{1}.
\]

In particular, the \( c_{i}^{\pm 1} \) have length at most \( 2K_{1}\mu_{3} \). But the regular curvature radii of \( c_{i}^{\pm 1} \) are at least \( \frac{1}{2} \) (the corresponding circles come near 1 and \(-1\)), so the total regular curvature of \( c_{i}^{\pm 1} \) is at most \( 4K_{1}\mu_{3} \). By the above, the total singular curvature on any subinterval of \( c_{i}^{\pm 1} \) is at most \( 2\mu_{1} \). If the \( \mu \)'s are small enough, it follows that all tangent vectors of \( c_{i}^{\pm 1} \) have complex arguments within \([\pi/4, 3\pi/4]\).

As a consequence, if \( \tau \) ranges over the ideal triangles of \( \varphi_{i}(\mathbb{H}^{2}) \) crossed by \( m_{i} \), and \( p : \mathbb{H}^{3} \to \mathbb{C} \) is the vertical projection, the different \( p(\tau) \) intersect only along their edges: so \( \varphi_{i}(\bigcup_{\tau} \tau) \) is an embedded surface (which we can see, say, as the graph of a function from an open set of \( \mathbb{C} \) to \( \mathbb{R}^{+} \)). Moreover, define the breadth of \( p(\tau) \) as the length of the segment \( p(\tau) \cap \sqrt{-1}\mathbb{R} \) (of the imaginary axis). Then, for some universal \( K_{2} > 0 \),

\[
\frac{\text{breadth of } p(\tau)}{\text{hyperbolic length of } \tau \cap m_{i}} \geq \frac{\mu_{2}}{K_{2}}.
\]

To conclude concerning \( \varphi_{+\infty} \), define a lift \( m^{c} \) through \( \omega \in \mathbb{H}^{2} \) of the simple closed geodesic of slope \( q \) in \( S_{+\infty} \), and a subsegment \( m \) of \( m^{c} \), of length \( 2\mu_{3} \), centered at \( \omega \). The angle between \( m \) and any pleating line of \( \Lambda^{+} \) it encounters is at least \( \frac{\pi}{4} \mu_{2} \). Let \( I \) be the (infinite) collection of ideal triangles of \( \tau_{+\infty}^{-1}(S_{+\infty} \setminus \Lambda^{+}) \) crossed by \( m \). For \( \tau \in I \), denote by \( |\tau| \) the length of \( m \cap \tau \). By convergence in the Hausdorff metric, \( \varphi_{+\infty}(\tau) \) is approached by triangles of the \( \varphi_{i}(\mathbb{H}^{2}) \), of breadth at least \( \frac{\pi \mu_{2}}{2K_{2}} \), so \( p(\varphi_{+\infty}(\tau)) \) has nonzero breadth. Injectivity follows: if \( x, x' \in \bigcup_{\tau \in I} \tau \) do not belong to the same stratum of the lamination \( \Lambda^{+} \), find \( \tau \) separating \( x \) from \( x' \) to prove that \( \varphi_{+\infty}(x) \neq \varphi_{+\infty}(x') \). By vertical projection to \( \mathbb{C} \), we see that \( \varphi_{+\infty}(\mathbb{H}^{2}) \) is topologically immersed in \( \mathbb{H}^{3} \).

8.5. Pleating measure of \( \varphi_{+\infty} \). These arguments can be extended to prove that the pleating measure of \( \varphi_{+\infty} \), as defined in 8.1, is the limit of the pleating measures of the \( \varphi_{i} \): the rectifiable curves \( c_{i}^{\pm 1} \) defined near \( \pm 1 \) by \( \varphi_{+\infty} \) have lengths \( \ell \pm 1 \), and for any \( \varepsilon > 0 \), there exists a finite disjoint union of circle arcs \( \gamma_{s} \) in \( \tau_{i}^{\pm 1} \) whose lengths add up to at least \( \ell s \mp 1 - \varepsilon \) (moreover the direction of \( c_{i}^{\pm 1} \), like that of \( \tau^{\pm 1} \), is everywhere within \( \pi/4 \) of the vertical axis). The \( \gamma_{s} \) can be approached by (unions of) arcs of the \( c_{i}^{-1} \), and the regular curvature not contributed by the \( \gamma_{s} \) is bounded by \( 3\varepsilon \). It follows that the pleating of \( \varphi_{+\infty} \) is the limit of the pleatings of the \( \varphi_{i} \) (on any transverse arc, and therefore, on any simple closed curve): that pleating is simply \( \lambda^{+} \). In particular, \( \varphi_{+\infty} = \lim_{i \to \infty} \varphi_{i} \) is independent of the original subsequence \( \nu \), and the \( \varphi_{i} \) converge to a pleated map whose pleating is given by \( \lambda^{+} \), as \( i \) goes to \( \pm \infty \).

8.6. Completeness. The construction of 8.3 further allows us to embed the universal cover \( \tilde{V} \) of \( V = \bigcup_{i \in \mathbb{Z}} \Delta_{i} \) into a (topological) manifold with boundary \( \tilde{V}_{\partial} \), as follows. For each \( x \in \mathbb{H}^{2} \), consider a neighborhood \( U_{x} \) of \( x \) such that \( \varphi_{+\infty} \) is
an embedding on \( U_x \). Then \( \varphi_+(U_x) \), which has a well-defined transverse ("outward") orientation, splits a small ball \( B_x \) centered at \( \varphi_+(x) \) into two (topological) hemispheres, which we can call "inner" and "outer", referring to the transverse orientation. The inner hemispheres \( H_x \), for \( x \) ranging over \( \mathbb{H}^2 \), can be patched together to obtain a manifold with boundary \( H \). Without loss of generality, the balls \( B_x \) can be chosen small enough so that, by the construction of Section 5.4 each \( H_x \setminus \varphi_+(U_x) \) is identified with a subset of \( \bar{V} \), embedded in \( \mathbb{H}^3 \). Then, \( H \) can further be patched to \( \bar{V} \). Since each \( H_x \) is homeomorphic to \( \mathbb{R}^2 \times \mathbb{R}^+ \), the space \( \bar{V} = V \cup H \) is a (possibly non-complete) topological manifold with boundary.

**Proposition 29.** The action of the fundamental group \( \Gamma \) of the punctured torus \( S \) on \( \bar{V} \) extends to a properly discontinuous action on \( \bar{V}_\partial \).

**Proof.** Consider the representation \( \rho : \Gamma \to \text{Isom}^+(\mathbb{H}^3) \) given by the developing map \( \Phi \) from \( 15 \), and the representations \( \rho_n : \Gamma \to \text{Isom}(\mathbb{H}^2) \) which satisfy \( \varphi_n \circ \rho_n(g) = \rho(g) \circ \varphi_n \) for all \( g \in \Gamma \). By Lemma 28 the \( \rho_n \) converge to some \( \rho_+\infty \). Convergence of the \( \varphi_n \) immediately implies \( \varphi_+\infty \circ \rho_+\infty(g) = \rho(g) \circ \varphi_+\infty \). Therefore, the hemispheres \( H_x \) can be chosen in an equivariant fashion, and the action of \( \Gamma \) on \( \bar{V}_\partial \) is well-defined. The action is already properly discontinuous at every point \( x \) of \( \bar{V} \) (namely, as \( g \) ranges over \( \Gamma \), the \( gx \) do not accumulate at \( x \)). But if \( \Gamma \) acts without fixed points and by isometries on a locally compact metric space \( X \), the set of \( x \in X \) such that the action is properly discontinuous at \( x \) is open (obviously) and closed: if \( g_n x \to x \) for some sequence \( (g_n) \) of \( \Gamma \setminus \{1\} \) and \( U \) is a compact neighborhood of \( x \), then \( U \) contains a ball of radius \( \varepsilon \) centered at \( x \), and whenever \( d(x, x') \leq \varepsilon/2 \), the \( g_n x' \) accumulate at some point of \( U \), so the action is not totally discontinuous at \( x' \). The Proposition follows by connectedness of \( \bar{V}_\partial \). \( \square \)

As a consequence, the space \( V \) := \( \bar{V}_\partial / \Gamma \) is a (topological) manifold with boundary, containing \( V = \bigcup_{i \in \mathbb{Z}} \Delta_i \); and \( \partial V \) consists of two pleated punctured tori (intrinsic) isometric to \( S_+\infty \) and another surface \( S_-\infty \), with pleatings \( \lambda^+ \) and \( \lambda^- \).

**Proposition 30.** The manifold with boundary \( V \) is complete.

**Proof.** Consider the metric completion \( \nabla : V \to V \) and assume the inclusion is strict. Define a continuous function \( f : V_\partial \to \mathbb{R}^+ \) by \( f(x) = d(x, \nabla \setminus V_\partial) \). By assumption, \( \inf(f) = 0 \).

Consider the immersion of \( V_\partial \) into the upper half-space model of \( \mathbb{H}^3 \) obtained by sending a lift of the cusp to infinity (Figures 4 and 5). The image of \( \partial V_\partial \) contains, in particular, vertical half-planes (interrupted at some height above \( \mathbb{C} \)). Therefore, any geodesic of \( V_\partial \) starting high enough above \( \mathbb{C} \) is defined for all times \( t \leq 1 \) (unless it hits \( \partial V_\partial \)). As a result, if \( H \subset V_\partial \) denotes a small enough open horoball neighborhood of the cusp, we have \( f \geq 1 \) on \( H \).

For \( i \in \mathbb{Z} \), consider the compact set \( K_i := S_i \cap H \) in \( V_\partial \). There exists a ball \( B \) of \( \mathbb{H}^2 \) centered at the base point \( \omega \), with radius independent of \( i \), such that \( K_i \subset \pi_i(B) \): by convergence of \( (\varphi_i) \), the \( \pi_i(B) \) converge metrically to a compact subset \( K' \) of \( \partial V_\partial \), on which \( f \) is positive. Therefore \( f \) is bounded away from 0 on \( \partial K' \) in \( V_\partial \), and \( K_i \subset K' \) for large enough \( i \). So \( f \) is bounded away from 0 on \( \bigcup_{i \in \mathbb{Z}} K_i \), and therefore on \( \bigcup_{i \in \mathbb{Z}} S_i \).

However, assume \( \gamma(t) \) is a rectifiable 1-Lipschitz arc of \( V_\partial \), defined for \( t < M \), with no limit at \( M \). For any \( \varepsilon > 0 \), the restriction \( \gamma_{|[M-\varepsilon, M)} \) meets \( V_\partial \setminus \partial V_\partial \) (because
\[ \partial V_\partial = S_{+\infty} \sqcup S_{-\infty} \text{ is complete}, \text{ but then } \gamma_{[M-n,M]} \text{ must meet } \Delta_i \text{ for an unbounded set of indices } i \text{ (any finite union of tetrahedra is complete). Therefore, we can find a sequence } t_n \to M \text{ such that } \gamma(t_n) \in \bigcup_{i \in \mathbb{Z}} S_i. \text{ Clearly, } f(\gamma(t_n)) \leq M - t_n \text{ which goes to } 0: \text{ a contradiction. So } V_\partial \text{ is complete.} \]

8.7. A quasifuchsian punctured-torus group. The end of the argument is now quite standard: recall the complete manifold with locally convex boundary \( \tilde{V}_\partial \), which is a universal cover of \( V = V_\partial \). Given distinct points \( x, x' \in \tilde{V}_\partial \), consider a shortest possible path \( \gamma \) from \( x \) to \( x' \). If \( \gamma \) has an interior point in \( \partial \tilde{V}_\partial \), by local convexity, we must have \( \gamma \subset \partial \tilde{V}_\partial \), and \( \gamma \) is a geodesic segment of \( \partial \tilde{V}_\partial \). If not, \( \gamma \) is (the closure of) a geodesic segment of \( \tilde{V} \). At any rate, the extended developing map \( \Phi : \tilde{V}_\partial \to \mathbb{H}^3 \) is an embedding (it sends \( \gamma \) to a segment with distinct endpoints) and has a closed convex image \( C \), endowed with a properly discontinuous action of the fundamental group \( \Gamma \) of \( S \) (Proposition 29). The action extends properly discontinuously to \( \mathbb{H}^3 \) (this can be seen by projecting any point of \( \mathbb{H}^3 \) to \( C \)). The manifold \( \mathbb{H}^3/\Gamma \) contains \( \nabla \simeq C/\Gamma \), which has the desired boundary pleatings \( \lambda^\pm \). Clearly, \( C \) is the smallest closed, convex set containing all parabolic fixed points of \( \Gamma \); therefore \( \nabla \) is the convex core, and \( \Gamma \) is quasifuchsian, with pleating data \( \lambda^\pm \). Theorem 4 is proved.

9. Application: the EPH theorem

Let us quickly recall the correspondence between the horoballs of \( \mathbb{H}^3 \) and the vectors in the positive light cone of Minkowski space. Endow \( \mathbb{R}^4 \) with the Lorentzian product \( \langle (x, y, z, t)(x', y', z', t') \rangle := xx' + yy' + zz' - tt' \). Define

\[ X := \{ v = (x, y, z, t) \in \mathbb{R}^4 \mid t > 0 \text{ and } \langle v | v \rangle = -1 \}. \]

Then \( \langle . | . \rangle \) restricts to a Riemannian metric on \( X \) and there is an isometry \( X \simeq \mathbb{H}^3 \), with \( \text{Isom}^+(X) \) a component of \( \text{SO}_{3,1}(\mathbb{R}) \). We will identify the point \( (x, y, z, t) \) of \( X \) with the point at Euclidean height \( \frac{r}{t} \) above the complex number \( \frac{y + iz}{\sqrt{r}} \) in the Poincaré upper half-space model. Under this convention, the closed horoball \( H_{d,\xi} \) of Euclidean diameter \( d \) centered at \( \xi = \xi + in \in \mathbb{C} \) in the half-space model corresponds to \( \{ v \in X \mid \langle v | v_{d,\xi} \rangle \geq -1 \} \), where \( v_{d,\xi} = \frac{1}{d}(2\xi, 2n, 1 - |\xi|^2, 1 + |\xi|^2) \).

We therefore identify \( H_{d,\xi} \) with the point \( v_{d,\xi} \) of the isotropic cone. Similarly, the closed horoball \( H_{h,\infty} \) of points at Euclidean height no less than \( h \) in the half-space model corresponds to \( \{ v \in X \mid \langle v | v_{h,\infty} \rangle \geq -1 \} \) where \( v_{h,\infty} = (0, 0, -h, h) \), so we identify \( H_{h,\infty} \) with \( v_{h,\infty} \).

Consider the following objects: a complete oriented hyperbolic 3-manifold \( M \) with one cusp, a horoball neighborhood \( H \) of the cusp, the universal covering \( \pi : \mathbb{H}^3 \to M \), and the group \( \Gamma \subset \text{Isom}^+(\mathbb{H}^3) \) of deck transformations of \( \pi \). Then \( H \) lifts to a family of horoballs \( \{ H_i \}_{i \in \mathbb{I}} \) in \( \mathbb{H}^3 \), corresponding to a family of isotropic vectors \( \{ v_i \}_{i \in \mathbb{I}} \) in Minkowski space. The closed convex hull \( C \) of \( \{ v_i \}_{i \in \mathbb{I}} \) is \( \Gamma \)-invariant, and its boundary \( \partial C \) comes with a natural decomposition \( \tilde{K} \) into polyhedral facets. Let \( U \) be the convex core of \( M \), minus the pleating locus. In [AS], Akiyoshi and Sakuma extended the Epstein-Penner convex hull construction to prove that \( \tilde{K} \) defines an \( H \)-independent decomposition \( K \) of \( U \) into ideal hyperbolic polyhedra, typically tetrahedra, allowing for a few clearly defined types of degeneracies. In [ASWY1], with Wada and Yamashita, they also conjectured
Theorem 31. If $M$ is quasifuchsian, homeomorphic to the product of the punctured torus with the real line, and has irrational pleating laminations of slopes $\beta^+ / \alpha^+$ and $\beta^- / \alpha^-$, then the restriction of $K$ to the interior of the convex core of $M$ is combinatorially the triangulation $(\Delta_i)_{i \in \mathbb{Z}}$ defined in Section 8.

Proof. It is known (see [Se] or Corollary 42 below) that quasifuchsian punctured-torus groups are fully determined by their measured pleating laminations. It follows that the vertices $A, B, C$ together with its iterated images under $\Gamma$ is locally convex, with $\mathcal{V}$ pointing inward). Define $\partial J := \bigcup \partial \Delta_i$. The central projection to the hyperboloid $X$ with respect to the origin sends $\tau_\Delta$ homeomorphically to the lift of $\Delta$ in $X$, so the interiors of the $\tau_\Delta$ are pairwise disjoint, and each $\tau_\Delta$ comes with a transverse orientation $\mathcal{U}$ (given by any ray through $\Delta$ issuing from the origin). The theorem claims exactly that $D \subset \partial C$ (the inclusion is expected to be strict, since $\partial C$ also contains faces projecting, say, to the boundary of the convex core — they are analyzed in detail in [AS]).

Suppose that $D$ is locally convex, with $\mathcal{U}$ pointing inward. Define $I := [1, +\infty)$ and, for any subset $Y$ of a vector space, $I Y := \{ ty \mid t \in I, y \in Y \}$. Since the interior of the convex core is convex, $ID$ is a convex set. Moreover, $ID$ contains all the $v_i$, so its closure $\overline{ID}$ contains their convex hull $C$. Conversely, $D$ is clearly included in $C$ and it is easy to check that $IC \subset C$ (because $I \{ v_i \} \subset C$ for all $i$, see [AS]). So $ID \subset C$ and, by closedness, $\overline{ID} = C$. Clearly, $D \subset \partial \overline{ID} = \partial C$, as wished. So we only need to prove

Lemma 32. The codimension-1 simplicial complex $D \subset \mathbb{R}^4$ is locally convex ($\mathcal{U}$ pointing inward).

Proof — Consider adjacent ideal tetrahedra $\Delta, \Delta'$ in $\mathbb{H}^3$ which are lifts from tetrahedra $\Delta_{i-1}, \Delta_i$ of the manifold. We must prove that the dihedral angle in $\mathbb{R}^4$ between $\tau_\Delta$ and $\tau_{\Delta'}$ points “downward”. We will assume that the letter between $i - 1$ and $i$ is an $L$ belonging to a subword $RL^m R$ of $\Omega$. In the link of the cusp, the pleated surface $S_i$ between $\Delta_{i-1}$ and $\Delta_i$ contributes a broken line $(-1, \zeta, \zeta', 1)$ in $\mathbb{C}$ together with its iterated images under $u \mapsto u \pm 2$, as in Figure 10 (we assume that the vertices $-1, 1$ both belong to the base segments of the Euclidean triangles just below and just above the broken line, in the sense of Figure 8). We use the notation

$$
\zeta + 1 = \overrightarrow{a} = a e^{iA} \\
\zeta' - \zeta = \overrightarrow{b} = b e^{iB} \\
1 - \zeta' = \overrightarrow{c} = c e^{iC}
$$

(so far $A, B, C$ are only defined modulo $2\pi$). Above this broken line lives a lift of $S_i$ which admits as a deck transformation

$$
f : u \mapsto 1 + \frac{(\zeta + 1)(\zeta' - 1)}{u + 1},
$$
because \( f(-1) = \infty \); \( f(\infty) = 1 \); \( f(\zeta) = \zeta' \). Therefore, \( f(H_{1,\infty}) = H_{[\zeta+1][\zeta'-1],1} = H_{ac,1} \). Similarly, the following horoballs all belong to the same orbit:

\[
H_{1,\infty}; H_{ac,-1}; H_{ab,z}; H_{bc,z'}; H_{ac,1}.
\]

If \( \zeta = \xi + \eta \sqrt{-1} \) and \( \zeta' = \xi' + \eta' \sqrt{-1} \), the corresponding isotropic vectors in Minkowski space are respectively

\[
\begin{align*}
v_{\infty} &= (0, 0, -1, 1) \\
v_{-1} &= \frac{1}{\zeta} (2\xi, 2\eta, 1 - |\zeta|^2, 1 + |\zeta|^2) \\
v_{\zeta} &= \frac{1}{\zeta'} (2\xi', 2\eta', 1 - |\zeta'|^2, 1 + |\zeta'|^2) \\
v_{1} &= \frac{1}{ac} (2, 0, 0, 2).
\end{align*}
\]

(17)

\[
\begin{array}{c}
\text{Figure 10. Adjacent tetrahedra } \Delta_{i-1}, \Delta_i \text{ (cusp view).}
\end{array}
\]

To prove that the dihedral angle at the codimension-2 face projecting to \((\zeta'\zeta')\infty\) is convex, it is enough to show that if \(Pv_\zeta + Qv_{\zeta'} + Rv_\infty = \lambda v_1 + (1 - \lambda) v_{-1}\) then \(P + Q + R > 1\) (moreover, this will in fact take care of all codimension-2 faces of the simplicial complex \(D\)). One easily finds the unique solution

\[
P = \frac{-bn}{c(\eta - \eta')}; Q = \frac{bn}{a(\eta - \eta')}; R = \frac{\eta(1 - |\zeta'|^2) - \eta'(1 - |\zeta|^2)}{ac(\eta - \eta')}
\]

hence

\[
P + Q + R = 1 + \frac{Z}{ac(\eta - \eta')} \quad \text{where } Z = bc\eta - abn' + \eta(1 - |\zeta'|^2) - \eta'(1 - |\zeta|^2) + ac(\eta' - \eta).
\]

Observe that \(\eta > \eta'\) because the triangles \(-1\zeta'\zeta'\) and \(1\zeta'\zeta\) are counterclockwise oriented. So it is enough to prove that \(Z > 0\). Endow \(\mathbb{C} \simeq \mathbb{R}^2\) with the usual scalar product \(\cdot\) and observe that \(1 - |\zeta|^2 = \vec{a} \cdot (\vec{b} + \vec{c})\) and \(1 - |\zeta'|^2 = (\vec{a} + \vec{b}) \cdot \vec{c}^\prime\). Hence

\[
Z = \eta(bc + \frac{\vec{b}}{a} \cdot \vec{c}) - \eta'(ab + \frac{\vec{b}}{a} \cdot \vec{c}) + (\eta' - \eta)(ac - \frac{\vec{a}}{c} \cdot \vec{c})
\]

\[
= abc \left[\frac{\eta}{a} (1 + \cos(B - C)) - \frac{\eta'}{c} (1 + \cos(A - B)) + \frac{\eta' - \eta}{b} (1 - \cos(A - C))\right]
\]

\[
= abc \left[\sin A (1 + \cos(B - C)) + \sin C (1 + \cos(A - B)) + \sin B (1 - \cos(A - C))\right]
\]

\[
= 4abc \sin \frac{A + C}{2} \cos \frac{A - B}{2} \cos \frac{B - C}{2}.
\]
by standard trigonometric formulae. Observe that the last expression is a well-defined function of $A, B, C \in \mathbb{R}/2\pi\mathbb{Z}$ (although each factor is defined only up to sign). Next, however, we shall carefully pick representatives of $A, B, C$ in $\mathbb{R}$.

**Observation 33.** There exists a unique triple of representatives $(A, B, C) \in \mathbb{R}^3$ such that $\{A, B, C\} \subset J$ where $J$ is an open interval of length less than $\pi$, containing 0. (It is easy to see that the broken line $(\ldots, -1, \zeta, \zeta', 1, \ldots)$ has no self-intersection if and only if there exists an open half-plane $H$ containing the vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$: the observation follows since $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = 2$ belongs to $H$).

We pick the representatives $A, B, C \in \mathbb{R}$ given by Observation 33, and do the same for the broken line contributed by each surface $S_i$ for $i \in \mathbb{Z}$ (each broken line is oriented from $-\infty$ to $+\infty$, so each edge $s$ of the cusp link inherits an orientation, as in Figure 10: we write its complex coordinate $\overrightarrow{d}_s = d_s e^{D_s}$). By Observation 33 all the complex arguments $D_s$ are in $(\pi, \pi)$. But since $0 \leq w \leq \pi$, the existence of the interval $J$ also implies (in the notation of Figure 10):

$$A = B + w_{i-1} \quad \text{and} \quad C = B + w_i$$

hence $B = \inf J$ and $B \in (\pi, 0)$. It follows that $B + x_i \in (\pi, \pi)$, hence $D_s = B + x_i$ and similarly $D_{s'} = B + x_{i-1}$.

In other words (by transitivity), for any two edges $s_1, s_2$ of the cusp link, the difference of arguments $D_{s_1} - D_{s_2} \in \mathbb{R}$ can be read off “naively” as a linear combination of the $\{w_i\}_{i \in \mathbb{Z}}$, with no multiple of $2\pi$ added.

Now that all arguments are fixed in the real interval $(\pi, \pi)$, we can see that $\cos \frac{A}{2} = \cos \frac{B}{2}$ and $\cos \frac{B-C}{2} = \cos \frac{w_i}{2}$ are positive. So to prove $Z > 0$ it remains to see that $0 < A + C < 2\pi$. This in turn follows from a small variant of Lemma 16 in [GF], which we prove now (it implies an empirical observation of Jørgensen which was Conjecture 8.6 in [ASWY1]).

**Proposition 34.** With the above notation, one has $0 < A + C < 2\pi$.

![Figure 11. Cusp view: a subword $RL^nR$, bounded by hinge indices 0 and $n$.](image-url)
Consider the maximal subword $RL^nR$, with the broken line $(-1, \zeta, \zeta', 1)$ corresponding to some $L$. There is in fact a sequence of points $(\zeta_0, \zeta_1, \ldots, \zeta_n)$ in $C$ such that the broken line corresponding to the $j$-th letter $L$ is $(-1, \zeta_j, \zeta_j, -1, 1)$ for all $1 \leq j \leq n$. The broken line corresponding to the initial (resp. final) $R$ is $(-1, \zeta_0, \zeta_1, 1)$ for some $\zeta_1$ (resp. $(-1, \zeta_{n+1}, \zeta_n, 1)$ for some $\zeta_{n+1}$). There exists $1 \leq k \leq n$ such that $([\zeta, \zeta']) = ([\zeta_k, \zeta_{k-1}])$.

Set $\xi_j := \zeta_j + 2$ for all $j$. By construction, the rays issued from 1 through $\zeta_1, \zeta_0, \ldots, \zeta_n, \zeta_{n+1}, \zeta_n, \ldots, \zeta_1, \zeta_0, \zeta_1$ (in that cyclic order) divide $C$ clockwise into salient angular sectors of sum $2\pi$ (these rays realize the link of the vertex 1 in the cusp triangulation, see Figure 11). Comparing the angles at 1 and at $-1$, we see immediately that

$$A + C = D_{-1\xi_k} + D_{\xi_{k-1}} = D_{-1\xi_{k+1}} + D_{\xi_k} = \cdots = D_{-1\xi_0} + D_{\xi_1}$$

(for all $0 \leq j \leq n + 1$). Since the triangles $\xi_0\zeta_{-1}$ and $\xi_1\zeta_{-1}$ are counterclockwise oriented, we have $\text{Im}(\zeta_{-1}) < \text{Im}(1)$ i.e. $\text{Im}(\zeta_{-1}) < 0$. Similarly, $\xi_n\zeta_{n+1}$ and $\xi_{n+1}\zeta_{1}$ are counterclockwise oriented, so $\text{Im}(\zeta_{n+1}) > 0$. So there exists $0 \leq j \leq n + 1$ such that $\text{Im}(\zeta_{j-1}) < 0$ and $\text{Im}(\zeta_j) \geq 0$. This implies $D_{-1\xi_j} \in (0, \pi)$ and $D_{\xi_{j-1}} \in [0, \pi)$. By (18), this implies $A + C \in (0, 2\pi)$. Theorem 31 is proved. □

In [G], we studied punctured-torus bundles over the circle by indexing the tetrahedra $\Delta_i$ in $Z/mZ$ (instead of $Z$): note that the proof of Theorem 2 applies without alteration to that context.

10. Generalizations

In this section, we extend all the previous results to punctured-torus groups with rational pleatings and/or infinite ends. The general theorem is as follows.

**Theorem 35.** Let $\lambda^+ \neq \lambda^-$ be nonzero, possibly projective (if irrational), measured laminations on the punctured torus $S$, and let $s(\lambda^\pm) \in \mathbb{P}^1 \mathbb{R}$ be the slope of $\lambda^\pm$. If $\lambda^\pm$ is a closed leaf, denote its weight by $|\lambda^\pm|$ and assume $|\lambda^\pm| \leq \pi$. There exists a punctured-torus group $\Gamma$ with ending and/or pleating laminations $\lambda^\pm$, and the open convex core $V$ of $\mathbb{H}^3/\Gamma$ has an ideal decomposition $D$ into polyhedral cells (of positive volume) whose combinatorics are given by $\lambda^\pm$ in the following sense: if $\Lambda$ is the line from $s(\lambda^-)$ to $s(\lambda^+)$ across the Farey diagram in $\mathbb{H}^2$, then

i - If $s(\lambda^+)$ and $s(\lambda^-)$ are irrational, $D$ consists of ideal tetrahedra $(\Delta_i)_{i \in \mathbb{Z}}$ in natural bijection with the Farey edges crossed by $\Lambda$, as in Section 4

ii - If only $s(\lambda^+)$ is rational and $|\lambda^+| < \pi$, then $D$ has one ideal tetrahedron per Farey edge crossed by $D$, and one cell $T$ homeomorphic to a solid torus: $\partial_1 = \partial T \cap \partial V$ is a punctured torus pleated along a simple closed curve of slope $s(\lambda^+)$, and $\partial_2 = \partial T \setminus \partial V$ is a punctured torus pleated along the ideal triangulation associated to the Farey triangle with vertex $s(\lambda^+)$ crossed by $\Lambda$. Finally, $\partial_1 \cap \partial_2$ is a line from the puncture to itself of slope $s(\lambda^+)$. See the left panel of Figure 15.

iii - If only $s(\lambda^+) = \pi$, all the statements of the previous case apply, except that $\partial_1 = \partial T \cap \partial V$ becomes a thrice-punctured sphere (the simple closed curve of slope $s(\lambda^+)$ has been “pinched” to become a cusp); see the right panel of Figure 15.

iv - If only $s(\lambda^-)$ is rational, the situation is similar to the two previous cases, exchanging $\lambda^-$ and $\lambda^+$. 
v – If \( s(\lambda^+), s(\lambda^-) \) are rationals but not Farey neighbors, the situation is again similar, with two solid torus cells \( T^+ \) and \( T^- \).

vi – If \( s(\lambda^+), s(\lambda^-) \) are Farey neighbors, \( D \) only consists of two solid tori \( T^+ \) and \( T^- \) as above, glued along a punctured torus \( S \) pleated along only two lines, of slopes \( s(\lambda^+) \) and \( s(\lambda^-) \).

Moreover, \( D \) agrees with the geometrically canonical decomposition \( D^G \) of the open convex core given by the Epstein-Penner convex hull construction.

Note that the combinatorics of \( D^G \) do not depend on the nature (finite or infinite) of the ends of \( \mathbb{H}^3/\Gamma \). At this point, we have treated the case of two irrational pleatings (finite ends). We proceed to prove the remaining cases of Theorem 35.

Figure 12. Toric cells: the exterior dihedral angle \( \theta \) is the weight \( |\lambda^\pm| \). Shaded faces are identified.

10.1. One rational pleating. We focus on the case of two finite ends, with only \( \beta^+/\alpha^+ \) rational. We can choose to end the word \( \Omega \in \{R, L\}^\mathbb{Z} \) with an infinite suffix \( LRR...R... \) (or of \( RLL...L... \): that is an arbitrary choice) and proceed from Section 11 onward. We shall assume that \( i = 0 \) is the greatest hinge index. Sections 12 through 13 are unchanged: the sequence \( (w_i) \) is just concave (thus convergent and non-decreasing) on \( \mathbb{N} \). In Subsection 13.2 we find that the sequence \( (\phi_i^+) \) is constant on \( \mathbb{N} \), equal to some \( \theta > 0 \). By (8), the number \( \theta \) is the weight of the rational lamination \( \lambda^+ \), so we assume \( \theta < \pi \). Section 14 goes through essentially unchanged: by the computation of Sublemma 9 (and with the same notation), the sum of the volumes of all tetrahedra \( \Delta_i \) for \( i \geq 2 \) is at most

\[
\sum_{k>n} \Sigma_{2k-1}^k \leq \sum_{k>n} 2^{-k} \left[ 1 + (2k - 1) \log 2 \right] = O(2^{-n/2}).
\]

Therefore the volume functional \( V \) is bounded, continuous for the product topology, and concave. We can find a maximizer \( w \) of \( V \), and it still satisfies Propositions 11 and 12, in particular, all tetrahedra \( \Delta_i \) for \( i > 0 \) have positive angles.

Something new is required in Section 13: we must prove that \( \lim_{i \to +\infty} w_i = \theta \).

The \( \{w_i\}_{i \geq 2} \) contribute only to the angles of the \( \{\Delta_i\}_{i \geq 1} \), which are positive: so the volume \( V \) is critical with respect to each \( w_i \) for \( i \geq 2 \). By Sublemma 6 of CGH, it follows that the cusp triangles of the \( \{\Delta_i\}_{i \geq 1} \) fit together correctly and can be drawn in the Euclidean plane \( \mathbb{C} \). More precisely, there exists a sequence of complex numbers \( (\zeta_i)_{i \geq 0} \) such that the triangles contributed by \( \Delta_i \) have vertices
at \((-1, \zeta_i, \zeta_{i-1})\) and \((1, \zeta_i, \zeta_{i+1})\) (Figure 13 left). These triangles being similar, we have \((\zeta_i + 1)(\zeta_{i+1} - 1) = (\zeta_{i-1} + 1)(\zeta_i - 1) = \ldots = (\zeta_0 + 1)(\zeta_1 - 1)\), hence
\[
\zeta_{i+1} = \frac{\zeta_i + \kappa}{\zeta_i + 1} =: \varphi(\zeta_i)
\]
for some complex number \(\kappa \neq 1\) independent of \(i\). Observe that the complex length \(\ell\) of the hyperbolic isometry (extending) \(\varphi\) satisfies \(\cosh \ell = \frac{1+\kappa}{1-\kappa}\). In the end, \(\varphi\) will be a lift of the loop along the rational pleating line of the convex core.

**Proposition 36.** The number \(\kappa\) lies in the real interval \((0, 1)\).

**Proof.** Denote by \(Z_i\) the periodic broken line \((\ldots, -1, \zeta_{i-1}, \zeta_i, 1, 2+\zeta_{i-1}, 2+\zeta_i, \ldots)\) contributed by the pleated surface \(S_i\). First, \(\kappa\) is real: if not, the \(\zeta_i\) have a limit in \(\mathbb{C} \setminus \mathbb{R}\) (a square root of \(\kappa\)), so \(w_i - w_{i-1}\) (the angle of \(Z_i\) at 1) cannot go to 0. If \(\kappa < 0\), then \(\varphi\) is a pure rotation: the \(\zeta_i\) all belong to a circle of the lower half-plane, which also contradicts \(w_i - w_{i-1} \to 0\). If \(\kappa > 1\), then \(\varphi\) is a gliding axial symmetry: the \(\zeta_i\) go to \(\pm \sqrt{\kappa}\) and belong alternatively to the upper and lower half-plane, so \(Z_i\) has self-intersection for large \(i\). If \(\kappa = 0\), then \(\varphi\) is a parabolic transformation fixing 0: the \(\zeta_i\) go to 0 along a circle tangent to \(\mathbb{R}\), and one can see that \(w_i = \hat{w}_{i+1}\) goes to \(\pi > \lim_{i \to +\infty} \phi^+ = \theta\). The only remaining possibility is \(\kappa \in (0, 1)\), where \(\varphi\) is a pure translation. 

Thus, the \(\zeta_i\) lie on a circle \(C\) of the lower half-plane which meets the real line at \(\pm \sqrt{\kappa}\), and \(\lim_{i \to +\infty} \zeta_i = \sqrt{\kappa}\). We denote by \(\theta^*\) the angle between \(C\) and the segment \([-\sqrt{\kappa}, \sqrt{\kappa}]\) (more precisely, the angle between their half-tangents at \(\sqrt{\kappa}\)). It is easy to see that
\[
\theta^* = \lim_{i \to +\infty} w_i
\]
(see Figure 13). Hence, \(\theta^* \leq \theta\).

**Figure 13.** All marked angles (grey) are \(\pi - \theta^*\), and \(\zeta_{j+1} = \varphi(\zeta_j) = \frac{\zeta_j + \kappa}{\zeta_j + 1}\).

**Proposition 37.** One has \(\theta^* = \theta\).

**Proof.** First, it is easy to check that any data \(0 \leq w_0 < w_1 < \theta^* < \pi\) smoothly determines a unique pair of complex numbers \(\zeta_0, \zeta_1\) such that:

i – the broken line \((-1, \zeta_0, \zeta_1, 1, 2+\zeta_0, 2+\zeta_1, \ldots)\) has angles \((w_0, -w_1, w_1 - w_0)\) as in Figure 13 above;

ii – the number \(\kappa\) such that \(\zeta_1 = \frac{\zeta_0 + \kappa}{\zeta_0 + 1}\) lies in \((0, 1)\);
iii – the circle through \( \zeta_0 \) and \( \zeta_1 \) centered on the imaginary axis intersects the real axis at an angle \( \theta^* \).

These \( \zeta_0, \zeta_1 \) in turn define all \( \{\zeta_j\}_{j \geq 2} \) via \( \zeta_{j+1} = \varphi(\zeta_j) \), and we can read off the angle \( w_j = 1_{\zeta_{j+1}} \zeta_j \leq \theta^* \) and construct the associated ideal tetrahedron \( \Delta_j \). In what follows, we investigate the shape of the space \( U := \bigcup_{j \geq 1} \Delta_j \), whose boundary (the punctured torus \( S_1 \)) has pleating angles \((w_0, w_1, w_1 - w_0)\) (see (II) above).

Define \( f(\zeta) := \frac{\zeta + \sqrt{\zeta^2 + 4}}{2} \), so that \( f(\varphi(\zeta)) = \rho f(\zeta) \) where \( \rho := \frac{1 + \sqrt{\zeta}}{1 - \sqrt{\zeta}} \). The convex hull of \((\infty, 1, \zeta_j, \zeta_{j+1})\) is isometric to the tetrahedron \( \Delta_j \): pushing forward by \( f \), we obtain a tetrahedron \( \Delta_j' \), isometric to \( \Delta_j \), with vertices \((1, \rho \sqrt{f(\zeta)}), \rho \sqrt{f(\zeta)} \) (Figure 13 right). Moreover, all the \( f(\zeta_j) = \rho^j \cdot f(\zeta_0) \) lie on the half-line \( e^{(\pi - \theta^*)} \mathbb{R}^+ \). By reasoning in a fundamental domain of the loxodromy \( \Phi : \pi \mapsto \rho \pi \), it is then easy to see that \( U \) has the same volume as \( D := D_1 \cup D_2 \), where \( D_1, D_2 \) are ideal tetrahedra of vertices \((\infty, f(\zeta_1), 1, \rho)\) and \((\infty, f(\zeta_1), 1, f(\zeta_0))\) respectively. Moreover, \( \Phi \) identifies the faces \((\infty, 1, f(\zeta_0))\) and \((\infty, \rho, f(\zeta_1))\) of \( D \), so that \( D/\Phi \) is a manifold with polyhedral boundary, homeomorphic to a solid torus, with interior dihedral angles

\[
(\pi - w_1, \pi + w_0, \frac{w_1 - w_0}{2}, \frac{w_1 - w_0}{2}, \pi - \theta^*).
\]

The edge of \( D/\Phi \) with dihedral angle \( \pi - \theta^* \) is a simple closed curve of length \( \log \rho \), toward which \( D_1, D_2 \) spiral. A picture of \( D/\Phi \) is obtained by replacing \( \theta \) with \( \theta^* \) in the left panel of Figure 13.

Using the smooth dependence on \( \theta^* \), the Schl"{a}fl"{i} volume formula then gives

\[
dV(D/\Phi)/d\theta^* = \frac{1}{2} \log \rho > 0; \quad \text{so } D/\Phi \text{ (and therefore } U) \text{ has largest volume when } \theta^* \text{ is largest.}
\]

Regard the \( \{w_j\}_{j \leq 1} \) as fixed, and the \( \{w_j\}_{j \geq 2} \) as unknowns: then \((w_2, w_3, \ldots)\) is clearly the solution to the maximization problem for the volume of \( U \) (with fixed pleating angles on \( S_1 \)). Therefore, the \( w_i \) will choose the largest possible limit \( \theta^* \) at \( +\infty \), namely \( \theta^* = \theta \).

The above proof does more than determining \( \lim_{t \to \infty} w_j \): as in Sections 5, 6 it gives a full description of \( U := \bigcup_{j \geq 1} \Delta_j \) and of its boundary (whose pleating turns out to be \( \lambda^+ \)). Here is, however, an important observation:

**Observation 38.** Proposition 34 above, “\( A + C > 0 \)” does not hold for the family of pleated surfaces \((S_i)_{i \geq 0} \). Instead, we have \( A + C = 0 \). This simply means that \(-1, \zeta_i \) and \( \zeta_{i-1}, \zeta_i \) make opposite angles with the real line. Indeed, \((\zeta_i + 1)(1 - \zeta_i) = 1 - \kappa^2 \) is a positive real.

We can now establish

**Proposition 39.** All the (strict) inequalities of (III) are true.

**Proof.** This is Proposition 34 (in the new context where \( \beta^+ / \alpha^+ \) is rational). The proof is the same, with the following caveat: in ruling out \( w_j = 0 \) for \( j \) hinge, we call upon (especially Lemma 16 and the argument of Section 9 there). The strategy is to assume \( w_j = 0 \), and then perturb \( w \) to a well-chosen \( w^\tau \) so as to make the volume increase: \( \partial V/\partial \varepsilon > 0 \). The latter inequality holds essentially because the inequality of Proposition 34 (“\( A + C > 0 \)” is true, both in the \( R^\infty \)-word preceding \( j \) and in the \( L^m \)-word following \( j \) (9)). More exactly, \( \partial V/\partial \varepsilon \) will be positive when

\[\text{in Section 9 of [5], the inequality “}A + C > 0\text{” is formulated in terms of lengths and takes the guise “}Q < P + T\text{”}\]
at least one of the two instances of \(A + C \geq 0\) is strict. But it is always strict, except in a single case (the infinite suffix \(LRR...R...\)): so we can conclude. \(\square\)

As a result, all the tetrahedra \(\Delta_i\) have positive angles and fit together correctly: Sections 5-8 carry through for the \(\lambda^-\)-end, and \(\bigcup_{i \in \mathbb{Z}} \Delta_i\) is the open convex core of a quasifuchsian punctured-torus group, with the prescribed pleatings \(\lambda^\pm\).

The results of Section 9 extend readily: the only modification is that tetrahedra \((\Delta_i)_{i \geq 0}\) lift to a family of coplanar cells in Minkowski space, because the key inequality of Proposition 34 has become an equality. Therefore the geometrically canonical decomposition of the convex core contains the non-contractible cell \(D/\Phi\) (Figure 12, left).

10.2. Two rational pleatings. When both \(\beta^+ / \alpha^+\) and \(\beta^- / \alpha^-\) are rational, the encoding word \(\Omega \in \{R, L\}^\mathbb{Z}\) can be chosen with an infinite prefix \(...R...RRL\) and an infinite suffix \(LRR...R...\). We apply the same argument as above to both ends simultaneously. Again, Proposition 39 holds because no hinge index \(j\) belongs both to the prefix and to the suffix.

If the rational pleating slopes \(s(\lambda^\pm)\) are not Farey neighbors, we can convert the prefix to \(...LLR\) and/or the suffix to \(RLL...\), obtaining different triangulations of the convex core of the same quasifuchsian group.

If \(s(\lambda^+), s(\lambda^-)\) are Farey neighbors, then \(\Omega = ...RRLRR...\) (observe that prefix and suffix overlap, so the two word conversions do not commute). If the indices before and after the \(L\) are 0 and 1, we obtain \(w_0 = w_1\) by applying Observation 38 to prefix and suffix. In other words, the points \(a, b, c\) in Figure 14 (left) are collinear. It is therefore possible to triangulate the same convex core according to a word \(\Omega = ...LLLRRR...\), provided that we allow the hinge tetrahedron \(\Delta_0\) to become flat (black in Figure 14, right). In any case, the sequence \((w_i)_{i \in \mathbb{Z}}\) maximizes the total volume.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure14.png}
\caption{Two triangulations seen against the same limit set.}
\end{figure}

10.3. Pinching. The case where one (or both) of the pleatings \(\lambda^\pm\) has weight \(\pi\) is a straightforward limit case of Subsections 10.1 and 10.2 (the term “pinching” refers to the fact that the pleating curve becomes shorter and shorter, and eventually turns into a cusp as the pleating angle reaches \(\pi\)).

Suppose \(|\lambda^+| = \pi\) (note that the conditions of 10.1 involving \(\phi^+\) become vacuous, because \(\phi^+ \geq \pi\)). Subsection 10.1 carries through with \(\theta = \pi\) and \(\kappa = 0\). The circle arc \(C\) of Figure 13 becomes a full circle, tangent to \(R\) at 0. The analysis of Proposition 34 (existence and uniqueness of \(\zeta_0, \zeta_1 \in \mathbb{C}\)) extends smoothly to \(\theta^+ = \pi\). One then finds that the tetrahedron with vertices \((\infty, 1, \zeta, \zeta + 1)\) is sent by \(f : u \mapsto 1/u\) to a tetrahedron \(\Delta_j\) of \(\mathbb{H}^3\), isometric to \(\Delta_j\), with vertices \((0, 1, \tau + j, \tau + j + 1)\) for a certain \(\tau \in \mathbb{C}\) independent of \(j\) (in the right panel of Figure 13, the grey
bounding rays are replaced by parallel lines). It is straightforward to check that \( \bigcup_{j \geq 1} \Delta_j \) is the solid torus pictured in Figure 12 (right), and the Schl"afli formula again implies that \( \theta^* = \pi \) realizes the maximum volume.

After Proposition 39, the argument is unchanged.

10.4. **Infinite ends.** By [Min], quasifuchsian groups are dense in the set of all discrete, faithful, type-preserving representations of \( \pi_1(S) \to \text{Isom}^+(\mathbb{H}^3) \). In fact, a geometrically infinite end of such a representation comes with an *ending lamination*, namely an irrational projective measured lamination which should be thought of as an “infinitely strong pleating”. In this section we will assume \( \beta^+ / \alpha^+ \notin \mathbb{P}^1 \mathbb{Q} \) and consider \( w^T \), the solution of the volume maximization problem for \((\phi^-, T\phi^+)\), where \( T > 0 \) (namely, \( w \) is subject to conditions (H), where \( \phi^+ \) is replaced by \( T\phi^+ \); and Table 2 still expresses \( x_i, y_i, z_i \)).

In a discrete, type-preserving representation of a surface group, each measured lamination \( \lambda \) on the surface receives a *length*, which can be computed by measuring the (weighted) lengths of weighted curves converging to \( \lambda \). It is known [Min] that the lengths of the pleating laminations of a quasifuchsian group are bounded by a constant depending only on the underlying surface (here the punctured torus). By Thurston’s double limit theorem (see Theorems 4.1 and 6.3 of [Th]), the space of discrete, type-preserving representations in which two fixed measured laminations have length bounded by a given constant is *compact*. Therefore, up to taking a subsequence, the groups \( \Gamma^T \) corresponding to \( w^T \) converge algebraically to a certain \( \Gamma \). By [Min], \( \mathbb{H}^3 / \Gamma \) is homeomorphic to \( S \times \mathbb{R} \) and must have an infinite end (otherwise, \( \Gamma \) would be quasifuchsian and the volumes would stay bounded).

**Proposition 40.** The \( \lambda^- \)-end of \( \mathbb{H}^3 / \Gamma \) is finite, with pleating lamination \( \lambda^- \). The \( \lambda^+ \)-end is infinite, with projective ending lamination [\( \lambda^+ \)].

**Proof.** By [Bon] (Theorem D), the space of type-preserving representations of the abstract group \( \Gamma \) is smoothly (in fact, holomorphically) parametrized by the data \((\tau, \omega)\) of a point \( \tau \) of Teichmüller space \( T \), and a *transverse* \( \mathbb{R} / 2\pi \mathbb{Z} \)-valued cocycle relative to a fixed topological lamination \( \mu \) (such cocycles include pleating measures as special cases). Taking for \( \mu \) the support of \( \lambda^- \), we see that the moduli of the \( \lambda^- \)-boundaries of the convex cores of the \( \Gamma^T \) must converge in \( T \). Therefore, \( \mathbb{H}^3 / \Gamma \) contains a locally convex pleated surface \( H \) with pleating \( \lambda^- \), which must be a boundary of the convex core (\( \partial H \) contains all parabolic fixed points).

The \( \lambda^+ \)-end of \( \mathbb{H}^3 / \Gamma \) must therefore be infinite. The parabolic fixed points of the limit group \( \Gamma \) determine a version of Figure 1 (a Euclidean cusp link), and therefore a family of *non-negative* angle assignments for the tetrahedra \( \Delta_i \). By algebraic convergence, the \( w^T \) converge to \( w \) in the product topology. For any \( i \leq j \), the total volume of \( \Delta_{i-1}, \Delta_i, \ldots, \Delta_j, \Delta_{j+1} \) is maximal with respect to \( w_i, \ldots, w_j \); in particular, Propositions 11 and 12 are still true. The techniques of [GF] (Lemma 16 and Section 9 there) show that all \( \Delta_i \) have positive angles, and Proposition 54 hence also Lemma 22 still hold: \( \{\Delta_i\}_{i \in \mathbb{Z}} \) is the geometrically canonical decomposition of \( \mathbb{H}^3 / \Gamma \). In particular, the family of all edges of all tetrahedra \( \{\Delta_i\}_{i \geq 0} \) forms a sequence of laminations which exits \( \mathbb{H}^3 / \Gamma \): therefore \( [\lambda^+] \) is the end invariant. \( \Box \)

The case of two infinite ends is already treated in [AK], Theorem 35, is proved.

10.5. **The Pleating Lamination Conjecture for punctured-torus groups.**
Proposition 41. The group $\Gamma$ constructed at the end of Section 4 is continuously parametrized by $(\lambda^+, \lambda^-)$.

Proof. Our first observation is that if $\beta^+/\alpha^+$ is rational, the initial choice of infinite prefix/suffix in Subsection 10.1 does not change the resulting group $\Gamma$: it just induces different triangulations of the toric piece of Figure 12 whose deformation space is still the same.

Define the open set
\[ U := \mathbb{R}^2 \setminus \bigcup_{m,n \in \mathbb{Z}} [\pi, +\infty) \cdot \{(m, n)\} \]
(note that $0 \notin U$). An admissible pleating lamination $\lambda$ can be identified with an element $\pm(\alpha, \beta)$ of $U/\pm$. Suppose $(\alpha_n^+, \beta_n^+) \to (\alpha^+, \beta^+)$ and $(\alpha_n^-, \beta_n^-) \to (\alpha^-, \beta^-)$ in $U$, and define the oriented line $\Lambda_n$ from $\beta_n^-/\alpha_n^-$ to $\beta_n^+/\alpha_n^+$ across the Farey diagram. Also define the associated functions $\varphi^{\pm,n}$ as in (9), domains $W^n$ as in Definition (9) and solutions $(w^n_i)_{i \in \mathbb{Z}}$ to the volume maximization problem over $W^n$. If $\beta^+/\alpha^+$ is rational, we may assume (up to restricting to two subsequences) that the $(\alpha_n^+, \beta_n^+)$ converge to $(\alpha^+, \beta^+)$ in the clockwise direction for the natural orientation of $\mathbb{P}^1 \mathbb{R}$. We also make a similar assumption for $\beta^-/\alpha^-$. A priori, the sequences $(w^n_i)_{i \in \mathbb{Z}}$ are defined only up to a shift of the index $i$.

However, we can choose these shifts in a consistent way: there exists a Farey edge $e$ which is crossed by all the lines $\Lambda_n$ for $n$ large enough, so we decide that $w^n_{0}$ always lives on $e$ (namely, $e^n_0 = e$). By compactness of $[0, \pi]^2$, some subsequence of $(w^n_i)_{i \in \mathbb{N}}$ converges to some $w^*$. It is enough to show that $w^* = w$: indeed, the group $\Gamma$ is completely determined by the shapes of a finite number of tetrahedra $\Delta_i$.

The main observation is that the words $\Omega^n \in \{R, L\}^\mathbb{Z}$ converge pointwise to $\Omega$, and $\varphi^{\pm,n} \to \varphi^{\pm}$ (pointwise in $\mathbb{R}^2$), by definition (9). Therefore, $w^*$ belongs to the space $W$, hence the volume inequality $\mathcal{V}(w^*) \leq \mathcal{V}(w)$. Since $\max W \mathcal{V}$ is achieved at a unique point (the volume is a strictly concave function), it is enough to prove the reverse inequality. We procede by contradiction.

Suppose $\mathcal{V}(w^*) < \mathcal{V}(w)$. Pick $\varepsilon > 0$: there exist integers $m < 0 < M$ such that the tetrahedra $\{\Delta_i\}_{m \leq i < M}$ defined by $w$ have total volume at least $\mathcal{V}(w) - \varepsilon$. If we can extend $(w_i)_{i \leq M}$ to a sequence $(v_i)_{i \in \mathbb{Z}}$ of $W^n$ for some large $n$, we will obtain a contradiction (assuming $\varepsilon$ small enough).

By Corollary 15 we can assume $\frac{\varphi_m}{\varphi_m^+} > \frac{w_{m+1}}{\varphi_m}$ and $\frac{\varphi_M}{\varphi_{M-1}} > \frac{w_M}{\varphi_{M-1}}$. Since $w$ satisfies (10) (strong inequalities), for $n$ large enough, the restricted sequence $(w_m, \ldots, w_M)$ satisfies the corresponding inequalities defining $W^n$, by convergence of the $\varphi^{\pm,n}$. Pick such a large $n$ and define
\[ v_i := \begin{cases} \phi_i^{-n} \frac{w_{m}}{\varphi_m^{+}} & \text{if } i \leq m; \\ w_i & \text{if } m \leq i \leq M; \\ \phi_i^{+n} \frac{w_M}{\varphi_M^{+}} & \text{if } M \leq i. \end{cases} \]

It is a straightforward exercise to check that $(v_i)_{i \in \mathbb{Z}}$ belongs to $W^n$. \hfill \Box

Corollary 42. (C. Series) A quasifuchsian punctured-torus group $\Gamma$ is determined up to conjugacy in $\text{Isom}(\mathbb{H}^3)$ by its pleating measures $\lambda^\pm$.

Proof. We will use the well-known fact that the space $\mathcal{QF}$ of quasifuchsian, non-fuchsian (punctured-torus) groups is a connected real manifold of dimension 4.
Recall the open set $\mathcal{U}$ from the proof of Proposition 31 and consider the map 
$$f : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{QF}$$
defined by the construction of the group $\Gamma$ (end of Section 4). We know that $f$ is well-defined and injective (Theorem 1), and continuous (Proposition 41). Since $\mathcal{U}^2$ has dimension 4, the theorem of domain invariance states that the image $\text{Im}(f)$ is open. It remains to show that $\text{Im}(f)$ is closed.

Consider pairs $(\lambda_+^n, \lambda_-^n)$ such that the corresponding groups $\Gamma_n = f(\lambda_+^n, \lambda_-^n)$ converge to some $\Gamma$ in $\mathbb{QF}$. The function which to a group associates its pleatings is continuous (see [KS]), so the $\lambda_+^n$ converge to some $\lambda^+$ in $\mathcal{U}$. Proposition 41 then implies that $\Gamma = f(\lambda^+, \lambda^-)$. □

Theorem 2 now follows from Theorem 35.

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