CHERN’S MAGIC FORM AND THE GAUSS-BONNET-CHERN MASS

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Abstract. In this note, we use Chern’s magic form $\Phi_k$ in his famous proof of the Gauss-Bonnet theorem to define a mass for asymptotically flat manifolds. It turns out that the new defined mass is equivalent to the one that we introduced recently by using the Gauss-Bonnet-Chern curvature $L_k$. Moreover, this equivalence implies a simple proof of the equivalence between the ADM mass and the intrinsically defined mass via the Ricci tensor, which was reconsidered by Miao-Tam [15] and Herzlich [12] very recently.

1. Introduction

The ADM mass plays an important role in the Einstein gravity. It is a conserved quantity for asymptotically flat manifolds. A complete manifold $(M^n, g)$ is said to be asymptotically flat (AF) of decay order $\tau$ (with one end) if there exists a compact set $K$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_R(0)$ for some $R > 0$ and in the standard coordinates of $\mathbb{R}^n$, the metric $g$ has the following expansion

$$g_{ij} = \delta_{ij} + \sigma_{ij},$$

with

$$|\sigma_{ij}| + r|\partial\sigma_{ij}| + r^2|\partial^2\sigma_{ij}| = O(r^{-\tau}),$$

where $r$ and $\partial$ denote the Euclidean distance and the standard derivative operator on $\mathbb{R}^n$ with the standard metric $\delta$, respectively. If the scalar curvature $R$ is integrable on $(M^n, g)$ and $\tau > \frac{n-2}{2}$, the ADM mass [1] is defined by

$$m_{\text{ADM}} := \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \to \infty} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu^\delta_j d\sigma^\delta,$$

where $\omega_{n-1}$ is the volume of $(n-1)$-dimensional standard unit sphere and $S_r$ is the Euclidean coordinate sphere, $d\sigma^\delta$ is the volume element on $S_r$ induced by the Euclidean metric, $\nu^\delta$ is the outward unit normal vector to $S_r$ in $\mathbb{R}^n$ and $g_{ij,k} = \partial_k g_{ij}$ are the ordinary partial derivatives. The well-definedness and invariance of $m_{\text{ADM}}$ was proved by Bartnik [3] (see also the work by Chruściel [7]). It is known that there is alternative formulation of the ADM mass via the Einstein tensor and a radial direction Euclidean conformal Killing field (see Ashtekar-Hasen [2] and Chruściel [8]). Precisely,

$$m_I = -\frac{1}{(n-1)(n-2)\omega_{n-1}} \lim_{r \to \infty} \int_{S_r} (\text{Ric} - \frac{1}{2}Rg)(X, \nu)d\sigma^g,$$

where $X$ denotes the Euclidean conformal Killing vector field $r\partial r$, $d\sigma^g$ is the volume element on $S_r$ induced by the metric $g$ and $\nu$ is the outward unit normal vector to $S_r$ in

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\((M^n, g)\). The equivalence of the two masses was proved by applying a density theorem to reduce the general case to the harmonic asymptotics case (see the work of Huang \[13\] for instance). Very recently, there appeared new proofs of the equivalence by Miao-Tam \[18\] by calculating directly in coordinates and by Herzlich \[12\] in a coordinate-free way. One of the aims of this note is to give another, simpler proof of this equivalence.

As a generalization of the ADM mass, recently we introduced a higher order mass by using the Gauss-Bonnet-Chern curvature \(L_k\), which is a natural generalization of the scalar curvature, as follows:

**Definition 1.1.** Let \(n > 2k\). Suppose that \((M^n, g)\) is an asymptotically flat of decay order \(\tau > \frac{n-2k}{k+1}\) and the Gauss-Bonnet curvature \(L_k\) is integrable on \((M^n, g)\). The Gauss-Bonnet-Chern mass (or GBC mass) given by

\[
m_k = m_{GBC} := \frac{(n-2k)!}{2^{k-1}(n-1)! \omega_{n-1}} \lim_{r \to \infty} \int_{S_r} P^{ijls}_{(k)} \partial_s g_{ij} \nu^\delta \, d\sigma^\delta,
\]

exists and does not depend on the choice of coordinates.

For the definition of the Gauss-Bonnet-Chern curvature \(L_k\) and the 4-tensor \(P^{(k)}\), see Section 2 below.

Analog to the work of Ashtekar-Hasen and Chruściel, we can also introduce a higher order intrinsically defined mass, by using a generalized Einstein tensor \(E^{(k)}\), which we will call it the Lovelock curvature, since Lovelock gave a characterization of such tensors in \[17\]. For its definition see (2.7) in Section 2 below.

**Definition 1.2.** Let \(X\) be the radial direction Euclidean conformal Killing vector field \(r \frac{\partial}{\partial r}\), we define

\[
m_I^k := -\frac{(n-2k-1)!}{2^{k-1}(n-1)! \omega_{n-1}} \lim_{r \to \infty} \int_{S_r} E^{(k)}(X, \nu) d\sigma^g,
\]

whenever this limit is convergent.

This is the same as \(m_I\) when \(k = 1\), since \(E^{(1)}\) is exactly the ordinary Einstein tensor \(Ric - \frac{1}{2} Rg\). In this note, we would like to see if it is equivalent to the GBC mass \(m_{GBC}\) for general \(1 \leq k < \frac{n}{2}\).

On the other hand, the Gauss-Bonnet-Chern curvature \(L_k\) looks like the magic form \(\Phi_k\), which was introduced by Chern \[5\] in his famous intrinsic proof of the Gauss-Bonnet theorem. Its definition is recalled in (3.1) below. Therefore it is natural to ask if one can use \(\Phi_k\) alternatively to define a mass.

By a slight modification, we show that one can really do it.

**Definition 1.3.** Under the conditions in Definition 1.1, we define a new mass

\[
m_k^C = \frac{1}{2^k(n-1)! \omega_{n-1}} \lim_{r \to \infty} \int_{S_r} r^{n-2k} \nu^\ast(\Phi_k),
\]

where the \((n-1)\) form \(\Phi_k\) is defined in (3.1) below and the outer unit normal \(\nu\) is viewed as a map from \(S_r\) to the sphere bundle over \(M\).

The well-definedness and the geometric invariance of this quantity can be checked directly and one can refer to Section 4 for more details. We call it Chern mass, or the \(k^{th}\) Chern mass. A more interesting point is that this mass has a simple relation to the GBC mass \(m_{GBC}\) as well as to the intrinsically defined mass \(m_I^k\). To be more precisely, we have
(i) \( d(r^{n-2k} \Phi_k) = (n - 2k)! L_k + O(r^{-(k+1)r-2k}). \) (Lemma 6.2)
(ii) \( r^{n-2k} \nu^*(\Phi_k) = -2 \cdot (n - 2k - 1)! \mathcal{E}^{(k)}(r \frac{\partial}{\partial r}, \frac{\partial}{\partial r}) d\sigma + O(r^{-(k+1)r-2k+1}). \) (Lemma 6.2)

From (ii) it is easy to see that \( m_k^C = m_k^A \). From (i) we can prove \( m_k^C = m_{GBC} \), by applying a trick used recently by Herzlich in [12]. This trick was also used by Schoen in [21]. Therefore, we have

**Theorem 1.4.** If \( n > 2k \) and \((M^n, g)\) is an asymptotically flat of decay order \( \tau > \frac{n-2k}{k+1} \) with integrable Gauss-Bonnet-Chern curvature \( L_k \) on \((M^n, g)\). Then we have the equivalence of the three masses

\[
m_{GBC} = m_k^A = m_k^C.
\]

When \( k = 1 \) this result also provides a simple proof of the equivalence between the ADM mass \( m_{ADM} \) and \( m_1 \), mentioned above.

We remark that from our work, one can see that the mass can be seen as a generalization of Gauss-Bonnet-Chern theorem for asymptotically flat manifolds, or more precisely, as a renormalized Gauss-Bonnet-Chern theorem. See Remark 1.3 below.

In addition to the mass, it is also interesting to define other invariants, such as the center of mass for these higher order masses. Related to the ADM mass \( m_{ADM} \), under some additional parity condition at infinity, the Hamiltonian formulation of center of mass was proposed by Regge-Teitelboim [20] (See also the work of Beig-Ó Murchadha [4]). Similar to (1.2), it is also known that there is an alternative formulation of a center of mass suggested by R. Schoen [13] using the Einstein tensor and Euclidean conformal Killing vector fields. Their equivalence has been first proved by Huang [13] by applying a density theorem and very recently by Miao-Tam [18] and Herzlich [12] via different methods. From our work, one can also define a corresponding center for our masses and show their equivalence. We leave this to the interested reader.

The rest of the paper is organized as follows: In Section 2, we recall some definitions as well as the properties, including the Gauss-Bonnet-Chern curvature, the GBC mass and the generalized Einstein tensor. Chern’s magic forms \( \Phi_k \) are reviewed in Section 3 and we apply it to define the Chern mass in Section 4. Section 5 is devoted to the equivalence of \( m_{GBC} \) and \( m_k^C \). In Section 6, we show that \( m_k^A = m_k^C \).

**2. The Gauss-Bonnet-Chern curvature and mass**

Recall that the Gauss-Bonnet-Chern curvature is given by

\[
L_k := \frac{1}{2k} \delta^{j_1 j_2 \cdots j_{2k-1} j_{2k}}_{11 \cdots j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2 \cdots i_{2k-1} i_{2k}}^{j_1 j_2 \cdots j_{2k-1} j_{2k}}, \tag{2.1}
\]

where \( \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} \) is the generalized Kronecker delta defined by

\[
\delta^{j_1 j_2 \cdots j_r}_{i_1 i_2 \cdots i_r} = \det \begin{pmatrix}
\delta_{i_1}^{j_1} & \delta_{i_2}^{j_2} & \cdots & \delta_{i_r}^{j_r} \\
\delta_{i_1}^{j_2} & \delta_{i_2}^{j_2} & \cdots & \delta_{i_r}^{j_r} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{i_1}^{j_r} & \delta_{i_2}^{j_r} & \cdots & \delta_{i_r}^{j_r}
\end{pmatrix},
\tag{2.2}
\]

and \( R_{ij}^{st} \) is the Riemannian curvature tensor in the local coordinates. One can easily check that \( L_1 \) is just the scalar curvature \( R \). When \( k = 2 \), it is the (second) Gauss-Bonnet curvature

\[
L_2 = \|Rm\|^2 - 4\|Ric\|^2 + R^2,
\]
which appeared at the first time in the paper of Lanczos \[14\] in 1938.

In the definition of the Gauss-Bonnet-Chern mass \[9\], the key observation is that the Gauss-Bonnet curvature has the following decomposition

\[
L_k = P^{ijls}_{(k)} R_{ijls},
\]

with the crucial property that \(P_{(k)}\) is divergence-free, i.e.,

\[
\nabla_i P^{ijls}_{(k)} = 0,
\]

and has the same symmetry as the curvature tensor. Here the 4-tensor \(P_{(k)}\) is defined by

\[
P^{ijls}_{(k)} := \frac{1}{2^k} \delta^{ij}_{i_1 j_1} \cdots \delta^{ls}_{i_2 j_2} \cdots R_{i_1i_2}^{j_1j_2} \cdots R_{i_2k-3i_2k-2}^{j_2k-3j_2k-2} g^{j_2k-1l} g^{j_2k}. \tag{2.4}
\]

For asymptotically flat manifolds, one has the following asymptotic expansion for \(L_k\) \[9\],

\[
L_k = 2 \partial_i (P^{ijls}_{(k)} \partial_s g_{jl}) + O(r^{-(k+1)}),
\]

which suggests the definition of the GBC mass in \[9\] by

\[
m_k = m_{GBC} := c(n,k) \lim_{r \to \infty} \int_{S_r} P^{ijls}_{(k)} \partial_s g_{jl} \nu_i^s \ d\sigma^j,
\]

with

\[
c(n,k) = \frac{(n-2k)!}{2^{k-1} (n-1)! \omega_{n-1}},
\]

provided \(L_k\) is integrable on \((M^n, g)\) and \(\tau > \frac{n-2k}{k+1}\). Here the constant \(c(n,k)\) is determined by calculating the mass of the Schwarzschild-type solution in the Gauss-Bonnet gravity to obtain the expected answer. One can easily see that \(m_1\) is exactly the ADM mass \(m_{ADM}\).

The Einstein-like tensor associated to \(L_k\) is introduced and characterized by Lovelock \[17\] by

\[
\mathcal{E}^{(k)}_{ij} = - \frac{1}{2^{k+1}} \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}} g_{i_1 i_2}^{j_1 j_2} \cdots g_{i_{2k-1} i_{2k}}^{j_{2k-1} j_{2k}} R_{i_1i_2}^{j_1j_2} \cdots R_{i_{2k-1}i_{2k}}^{j_{2k-1}j_{2k}}. \tag{2.7}
\]

As a convention, we set \(\mathcal{E}^{(0)} = g\). When \(k = 1\), then \(\mathcal{E}^{(1)}\) is just the usual Einstein tensor \(E := \text{Ric} - \frac{1}{2} R g\). Like the usual Einstein tensor \(E\), the generalized Einstein-like tensor \(\mathcal{E}^{(k)}\) satisfies a conversation law, namely, \(\text{div} \mathcal{E}^{(k)} = 0\), i.e., \(\nabla_j \mathcal{E}^{(k)}_{ij} = 0\). \(\mathcal{E}^{(k)}\) is called the \(k\)-th Lovelock curvature and one can see easily that \(\text{Trace} \mathcal{E}^{(k)} = -\frac{n-2k}{k+1} L_k\).

As indicated in the Introduction, with this generalized Einstein tensor \(\mathcal{E}^{(k)}\), we can introduce a higher order intrinsically defined mass \(m^k_i\) by

\[
m^k_i := - \frac{(n-2k-1)!}{2^{k-1}(n-1)! \omega_{n-1}} \lim_{r \to \infty} \int_{S_r} \mathcal{E}^{(k)}(X, \nu) d\sigma^g
\]

\[
= - \frac{(n-2k-1)!}{2^{k-1}(n-1)! \omega_{n-1}} \lim_{r \to \infty} \int_{S_r} \mathcal{E}^{(k)}(r \frac{\partial}{\partial r}, \frac{\partial}{\partial r}) d\sigma^g,
\]

since \(\nu = \frac{\partial}{\partial r} + O(r^{-\tau})\) for AF manifolds of decay order \(\tau > \frac{n-2k}{k+1}\).

One can also write \(L_k\) in terms of differential forms as follows,

\[
L_k \ast 1 = \frac{1}{2^k} \delta^{\alpha_1 \alpha_2 \cdots \alpha_{2k}} \beta_1 \beta_2 \cdots \beta_{2k} R_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} \cdots R_{\alpha_{2k-1} \alpha_{2k}}^{\beta_{2k-1} \beta_{2k}} \omega_1 \wedge \cdots \wedge \omega_n
\]

\[
= \frac{1}{(n-2k)!} \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_n} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n},
\]
where \( \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_n} := \delta^{\alpha_1 \alpha_2 \cdots \alpha_n}_{1 \ldots n} \), \{e_\alpha\}_{\alpha=1}^n \) is the dual coframe of a local frame \( \{e_\alpha\}_{\alpha=1}^n \), \(*\) is the volume form and \( \Omega_{\alpha_1 \alpha_2} \) is the curvature two-form given by

\[ \Omega_{\alpha_1 \alpha_2} = \frac{1}{2} R_{\alpha_1 \alpha_2 \beta_1 \beta_2} \omega_{\beta_1} \wedge \omega_{\beta_2}. \]

Define a two form \( Q \) from the \( P(k) \) curvature tensor by

\[ Q^{\alpha \beta} = P^{\alpha \beta \gamma \delta}_{(k)} \omega_{\gamma} \wedge \omega_{\delta}. \]

It is easy to check that (2.3) is equivalent to

\[ d^* Q^{\alpha \beta} = 0, \quad \text{(2.8)} \]

where \( d^* \) is the dual operator of the differential operator \( d \). The Hodge dual operator of \( Q^{\alpha \beta} \) is

\[ (*Q^{\alpha \beta}) = \frac{1}{(n-2k)!} \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_n} \Omega_{\alpha_3 \alpha_4} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n} \]

\[ = \frac{1}{2k-1} \cdot \frac{1}{(n-2k)!} \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_n} R_{\alpha_3 \alpha_4 \beta_3 \beta_4} \cdots R_{\alpha_{2k-1} \alpha_{2k} \beta_{2k-1} \beta_{2k}} \omega_{\beta_3} \cdots \wedge \omega_{\beta_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n}. \quad \text{(2.9)} \]

It is clear from (2.9) that

\[ L_k * 1 = \Omega_{\alpha_1 \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2}). \]

Let \( \omega_{\alpha \beta} \) be the Levi-Civita connection one-form with respect to the local frame \( \{e_\alpha\}_{\alpha=1}^n \). Now we can rewrite (2.5) in terms of differential forms

\[ L_k * 1 = d(\omega_{\alpha_1 \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2})) + O(r^{-(k+1)r-2k}), \quad \text{(2.10)} \]

which is the formula used in [16]. In fact it follows readily from (2.8), if one computes \( d(\omega_{\alpha_1 \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2})) \) directly:

\[ d(\omega_{\alpha_1 \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2})) = d\omega_{\alpha_1 \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2}) = (\Omega_{\alpha_1 \alpha_2} + \omega_{\alpha_1 \beta} \wedge \omega_{\beta \alpha_2}) \wedge (*Q^{\alpha_1 \alpha_2}) = L_k * 1 + \omega_{\alpha_1 \beta} \wedge \omega_{\beta \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2}) = L_k * 1 + O(r^{-(k+1)r-2k}). \quad \text{(2.11)} \]

In the first equality, we have used (2.8). We remark that unlike \( L_k * 1 \), the \( n - 1 \) form \( \omega_{\alpha_1 \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2}) \) does depend on the frame. Or in other words, \( \omega_{\alpha_1 \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2}) \) is an \( n - 1 \) form on the frame bundle over \( M \). Therefore, when we use the expression \( P^{ij \ell}_{(k)} \partial_s g_{j\ell} \) or equivalently using \( \omega_{\alpha_1 \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2}) \) to define a mass, we need to check that the defined mass does not depend on the choice of the frames.

The mass \( m_{GBC} \) defined above is trivially the same if one uses \( \omega_{\alpha_1 \alpha_2} \wedge (*Q^{\alpha_1 \alpha_2}) \) to define a mass, as in [16].

The GBC mass was first studied in [9] and its positivity was proved in [9] for graphic manifolds and in [10] for conformally flat manifolds. The GBC mass for higher codimensional graphs was studied in [15] and [11].
3. Chern’s Magic Forms

To prove the Gauss-Bonnet formula for a general closed Riemannian manifold $M^n$, Chern \cite{5,6} turn to consider the sphere bundle $S(M) := \{(p,v) : p \in M, v \in T_p(M) \text{ and } |v| = 1\}$ of dimension $2n - 1$. More precisely, he introduced the following important forms:

\[
\Phi_k = e^{\alpha_1 \alpha_2 \cdots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1} \alpha_{2k}} \wedge \cdots \wedge \omega_{\alpha_{2n-1} \alpha_{n-1}},
\]

\[
\Psi_k = 2(k+1)e^{\alpha_1 \alpha_2 \cdots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \Omega_{\alpha_{2k+1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+2} \alpha_{2k}} \wedge \cdots \wedge \omega_{\alpha_{2n-1} \alpha_{n-1}},
\]

where $e_n = \nu$ denotes the unit outer normal vector field of $S_r$ and $e^{\alpha_1 \alpha_2 \cdots \alpha_{n-1}} = \delta_{\alpha_1 \alpha_2 \cdots \alpha_{n-1}}$ as before. Note that $\Phi_k$ are $n - 1$ forms, $\Psi_k$ are $n$ forms and if $n$ is even, $\Psi_{\frac{n}{2}-1}$ equals the Paffian $\Omega = L_{\frac{n}{2}} \ast 1$. More importantly, it was proved by Chern \cite{5,6} that $\Omega$ is an exact form (i.e. it is the exterior derivative of a $n - 1$ form) by observing the following iteration relation between these forms:

\[
d\Phi_k = \Psi_{k-1} + \frac{n - 2k - 1}{2(k+1)} \Psi_k, \quad k = 0, 1, \cdots, \left[\frac{n}{2}\right] - 1,
\]

where $\left[\frac{n}{2}\right]$ denotes the largest integer $\leq n/2$. From (3.3) Chern obtained

\[
L_{\frac{n}{2}} \ast 1 = d\Pi,
\]

where

\[
\Pi := (-1)^{\frac{n}{2}-1} \sum_{k=0}^{\frac{n}{2}-1} (-1)^k \frac{(\frac{n}{2} - k - 1)! (\frac{n}{2})!}{(n - 2k - 1)! k! 2^{2k-n+1}} \Phi_k,
\]

when $n$ is even. Therefore, the Euler density $L_{\frac{n}{2}} \ast 1$ is an “exact form” with $\Pi$, a form on the sphere bundle. The intrinsic proof of the Gauss-Bonnet Theorem follows from (3.3) and the Poincaré-Hopf index theorem. See also a nice survey \cite{22}.

If one compares (2.11) with (3.3), one can see that they look quite similar. It should be noticed that $\Phi_k$ ($\Psi_k$ resp.) are $n - 1$ forms ($n$ forms resp.) defined on the sphere bundle, while $L_k \ast 1$ is an $n$ form defined on $M$ and $\omega_{\alpha_1 \alpha_2} \wedge (\ast Q)^{\alpha_1 \alpha_2} (\omega_{\alpha_1 \beta} \wedge \omega_{\beta \alpha_2} \wedge (\ast Q)^{\alpha_1 \alpha_2}$ resp.) are $n - 1$ forms ($n$ forms resp.) on the frame bundle.

Due to this similarity, a natural question arises: can one define an invariant by using $\Phi_k$ for asymptotically flat manifolds?

4. A Mass Defined by Using $\Phi_k$

In this section we modify the Chern form $\Phi_k$ slightly to define a mass for asymptotically flat manifolds.

Let $S_r$ be a coordinate sphere of radius $r$ in the asymptotically flat manifolds $(M^n, g)$, for large $r$. Let $\nu$ be the outerward unit normal vector field along $S_r$. Viewing $\nu$ as a map from $S_r$ to the sphere bundle over $M$, we consider the pull-back form $\nu^* (\Phi_k)$ and define a quantity by

\[
\tilde{m}_k = \lim_{r \to \infty} \int_{S_r} r^{n-2k} \nu^* (\Phi_k),
\]

whenever this limit is convergent.

First, we need the following decay estimates:

**Lemma 4.1.** On an asymptotically flat manifold $(M^n, g)$ with decay order $\tau$, we have $\omega_{jn} = -\frac{1}{r} \omega_j + O(r^{\tau-1})$ for all $j = 1, 2, \cdots, n - 1$. 
Proof. First we choose $e_n = \nu = \frac{\nabla r}{|\nabla r|} = \frac{\partial}{\partial r} + O(r^{-\tau})$, which implies
\[
\omega_n = dr + O(r^{-\tau}).
\]
In view of the first structure equation $d\omega_n = \omega_j \wedge \omega_{jn}$, one can derive
\[
\omega_{jn} = h_{ij} \omega_i + O(r^{-1-\tau}), \quad i = 1, \ldots, n-1.
\]
The conclusion follows by noting that $h_{ij} = -\frac{\delta_{ij}}{r} + O(r^{-1-\tau})$.
\[\Box\]

With the above lemma, it is crucial to observe the following

**Lemma 4.2.**
\[
d(r^{n-2k} \Phi_k) = (n-2k)! L_k \ast 1 + O(r^{-(k+1)\tau-2k}). \tag{4.1}
\]

**Proof.** By using the Chern formula (3.3) together with (3.1) and (3.2), we have
\[
d(r^{n-2k} \Phi_k) = r^{n-2k} d\Phi_k + (n-2k)r^{n-2k-1} dr \wedge \Phi_k
\]
\[
= r^{n-2k} \left( \Psi_{k-1} + \frac{n-2k-1}{2(k+1)} \Psi_k \right) + (n-2k)r^{n-2k-1} dr \wedge \Phi_k
\]
\[
= r^{n-2k} \left( \Psi_{k-1} + \frac{n-2k-1}{r} dr \wedge \Phi_k \right) + O(r^{-(k+1)\tau-2k})
\]
\[
= r^{n-2k} e^{\alpha_1 \cdots \alpha_{n-1}} \left( 2k \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n} \
+ (n-2k) \frac{1}{r} dr \wedge \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n} \right)
\]
\[
+ O(r^{-(k+1)\tau-2k}).
\]

On the other hand, using $\omega_{jn} = -\frac{\omega_i}{r} + O(r^{-1-\tau})$ ($j = 1, \ldots, n-1$) and $\omega_n = dr + O(r^{-\tau})$ (Lemma 4.1), we have
\[
(n-2k)! L_k \ast 1 = e^{\alpha_1 \alpha_2 \cdots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n}
\]
\[
= e^{\alpha_1 \alpha_2 \cdots \alpha_{n-1}} \left( 2k \cdot (-1)^n \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n} \
+ (n-2k) \cdot (-1)^{n-1} dr \wedge \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n} \right)
\]
\[
+ O(r^{-(k+1)\tau-2k})
\]
\[
= r^{n-2k} e^{\alpha_1 \cdots \alpha_{n-1}} \left( 2k \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n} \
+ (n-2k) \frac{1}{r} dr \wedge \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1}} \wedge \cdots \wedge \omega_{\alpha_n} \right)
\]
\[
+ O(r^{-(k+1)\tau-2k}).
\]

Thus we complete the proof. \[\Box\]

**Remark 4.3.** Formula (4.1) means that on an asymptotically flat manifold, up to a higher order term, $L_k \ast 1$ is an “exact form” with $r^{n-2k} \Phi_k$, a form on the sphere bundle. In spirit, it is very similar to (3.3).

From Lemma 4.2 we can define a mass by using the form $\Phi_k$. 

Definition 4.4. Let \((M,g)\) be an asymptotically flat manifold of decay order \(\tau > \frac{n-2k}{k+1}\) and with \(L_k \in L^1(M,g)\). For each given integer \(1 \leq k < \frac{n}{2}\) we define \(k\)-th Chern mass by

\[
m^C_k = \frac{1}{2^k(n-1)!} \lim_{r \to \infty} \int_{S_r} r^{n-2k} \nu^* (\Phi_k).
\] (4.2)

Lemma 4.2 implies that the limit in \(m^C_k\) is convergent. Moreover, one can prove that

Lemma 4.5. \(m^C_k\) is well-defined and a geometric invariant, provided that \(L_k\) is integrable and \(\tau > \frac{n-2k}{2k+1}\).

Proof. In view of Lemma 4.2, one can show this lemma by using methods of Bartnik [3] as in [9, 16]. See also the approach due to Michel [19]. We skip the proof here, since the result also follows from the equivalence proved below. \(\square\)

5. Equivalence of the GBC mass and the Chern mass

In this section, we show that the GBC mass introduced in [9] is the same as the Chern mass introduced in the previous section.

Theorem 5.1. Let \((M,g)\) be an asymptotically flat manifold of decay order \(\tau > \frac{n-2k}{k+1}\) and with \(L_k \in L^1(M,g)\). Then we have

\[
m^C_k = m_{GBC}.
\]

Proof. From the previous section we know that \(L_k\) has two different expansions:

\[
L_k \ast 1 = 2 \partial_i (P_{(k)}^{ij} s^j g_{is}) \ast 1 + O(r^{-(k+1)\tau-2k})
\] (5.1)

\[
= \frac{1}{(n-2k)!} d(r^{n-2k} \Phi_k) + O(r^{-(k+1)\tau-2k}).
\] (5.2)

If we define a one-form \(a\) by \(a = P_{(k)}^{ij} s^j g_{js} dx^i\), then (5.1) is rewritten as

\[
L_k \ast 1 = 2d(\ast a) + O(r^{-(k+1)\tau-2k}).
\] (5.3)

It is clear that \(2 \ast a = (\omega_{\alpha_1 \alpha_2} \wedge (\ast Q^{\alpha_1 \alpha_2}))\). See [211]. The GBC mass \(m_{GBC}\) can certainly be defined by using \(\lim_{r \to \infty} \int_{S_r} 2 \ast a\) as well.

Now we apply a trick used by Herzlich recently in [12]. For any large \(r > 4\), we consider a modified metric \(h\) by gluing the Euclidean metric \(\delta\) inside the ball \(B_{\frac{3}{4}}\) and the original metric \(g\) outside the ball \(B_{r}\) as follows: Let \(\eta : \mathbb{R}^n \to [0, 1]\) be a cut-off function satisfying \(\eta = 0\) in \(B_1\) and \(\eta = 1\) outside \(B_{\frac{3}{4}}\). Set \(\eta_r(x) = \eta(rx)\). It is clear that \(\eta_r\) satisfies

\[
r |\nabla \eta_r| + r^2 |\nabla^2 \eta_r| + r^3 |\nabla^3 \eta_r| \leq c_0,
\]

for some universal constant \(c_0 > 0\), which is independent of \(r \gg 1\). We then define for each \(r \gg 1\) a metric on the annulus \(A = A(\frac{1}{4}, r)\):

\[
h := h_r := \eta_r \cdot g + (1 - \eta) \cdot \delta.
\]

It is clear that \(h\) is also an asymptotically flat metric of decay order \(\tau\) with uniform estimates independent of \(r \gg 1\). Hence, for metric \(h\), (5.1)–(5.3) hold. Note that...
the general }

\[ *a(h) = \Phi_k(h) = 0 \text{ in } B_{\frac{1}{4}r}, \] and \[ *a(h) = *a(g), \Phi_k(h) = \Phi_k(g) \text{ outside } B_{\frac{1}{4}r}. \] Hence we infer from \((5.2)\) and \((5.3)\) that

\[
2 \int_{S_r} *a(g) = 2 \int_{S_r} *a(h) = 2 \int_{A(\frac{1}{4}r, r)} d(*a)(h)
= \int_{A(\frac{1}{4}r, r)} L_k(h) * 1 + \int_{A(\frac{1}{4}r, r)} O(r^{-(k+1)r-2k})
= \int_{A(\frac{1}{4}r, r)} L_k(h) * 1 + o(1),
\]
and

\[
\frac{1}{(n-2k)!} \int_{S_r} r^{n-2k}\Phi_k(g) = \frac{1}{(n-2k)!} \int_{S_r} r^{n-2k}\Phi_k(h) = \frac{1}{(n-2k)!} \int_{A(\frac{1}{4}r, r)} d(r^{n-2k}\Phi_k(h))
= \int_{A(\frac{1}{4}r, r)} L_k(h) * 1 + o(1).
\]
It follows that

\[
2 \int_{S_r} *a(g) = \frac{1}{(n-2k)!} \int_{S_r} r^{n-2k}\Phi_k(g) + o(1),
\]
and hence the conclusion as \( r \to \infty. \)

\[ \square \]

**Remark 5.2.** To prove that \( m_{GBC} = m_k^C \), one may would like to show that \( *a = \frac{1}{2(n-2k)!} \cdot r^{n-2k}\Phi_k + o(r^{-(n-1)}) \) pointwisely. However, this should not be true, since \( r^{n-2k}\Phi_k \) consists of \( k \) times curvature tensors, while \( *a \) has only \( k-1 \) times curvature tensors together with a connection form.

### 6. Equivalence between \( m_k^S \) and \( m_k^C \)

Now we want to see what exactly \( r^{n-2k}\nu^*(\Phi_k) \) is. Notice that the induced area element of \( S_r \) from the volume form of \( g \) is

\[
d\sigma^g = (1)^{n-1}\omega_1 \wedge \cdots \wedge \omega_{n-1}. \tag{6.1}
\]

We first consider the \( k = 1 \) case. By Lemma 4.1

\[
r^{n-2}\nu^*(\Phi_1) = r^{n-2}\epsilon_{\alpha_1\alpha_2\cdots\alpha_{n-1}}\Omega_{\alpha_1\alpha_2} \wedge \omega_{\alpha_3} \wedge \cdots \wedge \omega_{\alpha_{n-1}}\n
= (-1)^{n-1}r^{n-2}\epsilon_{\alpha_1\alpha_2\cdots\alpha_{n-1}}\Omega_{\alpha_1\alpha_2} \wedge \omega_{\alpha_3} \wedge \cdots \wedge \omega_{\alpha_{n-1}} + O(r^{1-2r}).
\]

One can check directly that (or one can refer to the following Lemma 6.2 for a proof of the general \( k \) case)

\[
r^{n-2}\nu^*(\Phi_1) = -2(1)^{n-1} \cdot (n-3)! (\text{Ric} - \frac{R}{2} g)(r \frac{\partial}{\partial r}, \frac{\partial}{\partial r}) \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{n-1} + O(r^{1-2r}),
\]
which implies that

\[
r^{n-2}\nu^*(\Phi_1) = -2 \cdot (n-3)! (\text{Ric} - \frac{R}{2} g)(r \frac{\partial}{\partial r}, \frac{\partial}{\partial r})d\sigma^g + O(r^{1-2r}). \tag{6.2}
\]
This means that the integrands in the definition of \( m_k^C \) and of \( m_f \) are in fact the same, up to a term of order \( O(r^{1-2r}) \) which vanishes after integration at infinity. Therefore, \( m_k^C \) and \( m_f \) are trivially the same. Therefore, Theorem 5.1 implies the equivalence of the ADM mass \( m_{ADM} \) and the intrinsically defined mass \( m_f \). Indeed this fact holds for the general \( k \).
**Theorem 6.1.** Under the conditions in Definition 4.2, we have

\[ m^k = m^G_C = m_{GBC}. \]

As the case \( k = 1 \), the Theorem follows immediately from Theorem 5.1 and the following Lemma.

**Lemma 6.2.**

\[ r^{n-2k} \nu^*(\Phi_k) = -2 \cdot (n-2k-1)! \mathcal{E}^{(k)} \left( r \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) d\sigma_g + O(r^{-(k+1)r-2k+1}). \] (6.3)

**Proof.** First by the definition of \( \Phi_k \), together with Lemma 4.1, we have

\[ r^{n-2k} \nu^*(\Phi_k) = r^{n-2k} \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1} n} \wedge \cdots \wedge \omega_{\alpha_{n-1} n} \]

\[ + O(r^{-(k+1)r-2k+1}) \]

\[ = (n-1) r^{n-1} \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1} n} \wedge \cdots \wedge \omega_{\alpha_{n-1} n} \]

\[ + O(r^{-(k+1)r-2k+1}) \]

\[ = (n-1) r^{n-1} \frac{1}{2^k} \delta^{\alpha_1 \cdots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1} \alpha_{2k}} \wedge \omega_{\alpha_{2k+1} n} \wedge \cdots \wedge \omega_{\alpha_{n-1} n} \]

\[ + O(r^{-(k+1)r-2k+1}), \]

(6.4)

where in the last equality, we have used the simple relation

\[ \delta^{\alpha_1 \cdots \alpha_{n-1}} \delta^{\beta_1 \cdots \beta_{2k+1} \cdots \alpha_{n-1}} = (n-1-2k)! \delta^{\alpha_1 \cdots \alpha_{2k}}. \]

On the other hand, noting \( e_n = \frac{\partial}{\partial r} + O(r^{-\tau}) \), we compute

\[ - \mathcal{E}^{(k)} \left( r \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) = \frac{r}{2^{k+1}} \delta^{\alpha_1 \cdots \alpha_{2k}} \Omega_{\alpha_1 \alpha_2} \beta_1 \beta_2 \cdots \Omega_{\alpha_{2k-1} \alpha_{2k}} \beta_{2k-1} \beta_{2k} + O(r^{-(k+1)r-2k+1}). \] (6.5)

Finally, equalities (6.1), (6.4) and (6.5) imply the conclusion. \( \square \)

**Proof of Theorem 6.1.** Note that \( (k+1)r+2k-1 > n-1 \) for \( \tau > \frac{n-2k}{k+1} \) and \( \nu = \frac{\partial}{\partial r} + O(r^{-\tau}) \), the conclusion follows directly from Theorem 5.1 and Lemma 6.2.

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