ON REVERSIBLE ASYNCHRONOUS NON-UNIFORM CELLULAR AUTOMATA

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Abstract. We study the class of asynchronous non-uniform cellular automata (ANUCA) over an arbitrary group universe with multiple local transition rules. We introduce the notion of stable injectivity, stable reversibility, stable post-surjectivity and investigate several dynamical properties of such automata. In particular, we establish the equivalence between reversibility, stable reversibility, and stable injectivity for ANUCA. We also prove the invertibility of several classes of injective and stably injective ANUCA. Counter-examples are given to highlight the differences between cellular automata and ANUCA.

1. Introduction

We briefly recall notions of symbolic dynamics. Given a set $A$ equipped with the discrete topology and a group $G$, a configuration $c \in A^G$ is simply a map $x : G \rightarrow A$. Two configurations $x, y \in A^G$ are asymptotic if for some finite subset $E \subset G$, $x|_{G \setminus E} = y|_{G \setminus E}$. The Bernoulli shift action $G \times A^G \rightarrow A^G$ is defined by $(g, x) \mapsto gx$, where $(gx)(h) := x(g^{-1}h)$ for $g, h \in G$ and $x \in A^G$. The space $A^G$ is equipped with the prodiscrete topology, i.e., the smallest topology for which the projection $A^G \rightarrow A^\{g\}$ is continuous for every $g \in G$. When $A$ is finite, the Tychonoff theorem implies that $A^G$ is a compact Hausdorff space. For every $x \in A^G$, we denote

$$\Sigma(x) := \{gx : g \in G\} \subset A^G.$$ 

Following von Neumann [18], a cellular automaton (CA) over a group $G$, the universe, and a set $A$, the alphabet, is a map $\tau : A^G \rightarrow A^G$ admitting a finite memory set $M \subset G$ and a local defining map $\mu : A^M \rightarrow A$ such that

$$(\tau(x))(g) = \mu((g^{-1}c)|_M) \quad \text{for all } x \in A^G \text{ and } g \in G.$$ 

Equivalently, a map $\tau : A^G \rightarrow A^G$ is a CA if and only it is $G$-equivariant and uniformly continuous (cf. [3], [14]). Various physical phenomena in fluid dynamics as well as biological organisms consisting of homogeneous cells whose evolution is described by the same local transition rule can be simulated by CA. In computer science, CA are studied as powerful models
of computation. For instance, the famous two-dimensional CA Game of Life defined by Conway [11] is Turing complete. The mathematical theory of CA also leads to a surprising characterization of amenable groups (see Section [2]) in terms of the Garden of Eden theorem for CA (cf. [15], [16], [6], [1]). Amenable groups were also invented by von Neumann [17].

To generalize the notion of CA, we introduce the class of asynchron ous non-uniform cellular automata (ANUCA) which allows multiple local defining maps. Our definition extends the class of \( \nu \)-CA studied in [7], [8] where the group universe is \( \mathbb{Z} \) or \( \mathbb{Z}^d \) for \( d \geq 2 \).

**Definition 1.1.** Let \( G \) be a group and let \( A \) be a set. Let \( M \subset G \) be a subset and let \( S = A^M \) be the set of all maps \( A^M \to A \). Given \( s \in S^G \), the **asynchronous non-uniform cellular automaton** (ANUCA) \( \sigma_s : A^G \to A^G \) associated with \( s \) is defined for all \( x \in A^G, g \in G \) by the formula:

\[
\sigma_s(x)(g) := s(g)((g^{-1}x)|_M).
\]

The set \( M \) is called the **memory** of \( \sigma_s \). The ANUCA \( \sigma_s \) is said to be **stably injective** if \( \sigma_p \) is injective for every \( p \in \Sigma(s) \). We say that \( \sigma_s \) is **invertible** if it is bijective and the inverse map \( \sigma_s^{-1} \) is an ANUCA with finite memory.

We say that \( \sigma_s \) is **reversible**, or **left-invertible**, if there exists an ANUCA with finite memory \( \tau : A^G \to A^G \) such that \( \tau \circ \sigma = \text{Id} \). Moreover, \( \sigma_s \) is **stably reversible** if there exist \( N \subset G \) finite and \( q \in T^G \) where \( T = A^{AN} \) such that for every \( p \in \Sigma(s) \), we can find \( w \in \Sigma(q) \) such that \( \sigma_w \circ \sigma_p = \text{Id} \).

The configuration \( s \in S^G \) is called the **configuration of local defining maps** of the ANUCA \( \sigma_s \). Note that every CA is an ANUCA with finite memory and constant configuration of local defining maps. Moreover, a CA is injective if and only if it is stably injective since \( \Sigma(c) = \{ c \} \) for all constant configuration \( c \). It is clear from the above definition for ANUCA that

\[(1.1)\]

Stable reversibility \( \Rightarrow \) Reversibility \( \Rightarrow \) Injectivity \( \iff \) Stable injectivity.

The first main result of the paper is the following characterization of reversible ANUCA in terms of stable injectivity (see Section [11]).

**Theorem A.** Let \( M \) be a finite subset of a countable group \( G \). Let \( A \) be a finite set and let \( S = A^M \). Let \( s \in S^G \). Then the following are equivalent:

(i) \( \sigma_s \) is reversible;
(ii) \( \sigma_s \) is stably reversible;
(iii) \( \sigma_s \) is stably injective.

Hence, the first two and the last properties in (1.1) are in fact equivalent and Example 14.1-14.2 show that are strictly stronger than Injectivity.

Our next results concern Gottschalk’s surjunctivity conjecture [13] which asserts that over any universe, any injective CA with finite alphabet must be surjective. Over sofic group universes, the surjunctivity conjecture was famously shown by Gromov-Weiss [12], [23] (see also [3], [9], [21], [22]). The
class of sofic groups was introduced by Gromov [12] as a generalization of residually finite groups and amenable groups (see Section 2). The situation for ANUCA is more complicated. Example 14.3 gives a simple one-dimensional stably injective ANUCA which is not surjective. However, when the configuration of local defining maps of an ANUCA is asymptotic to a constant configuration, we establish the invertibility, i.e. bijectivity plus reversibility, for stably injective ANUCA and even for injective ANUCA over a suitable universe (see Section 2, Theorem 8.2, Theorem 9.1).

**Theorem B.** Let $M$ be a finite subset of a countable group $G$. Let $A$ be a finite set and let $S = A^A^M$. Let $s \in S^G$ be asymptotic to a constant configuration $c \in S^G$. Then $\sigma_s$ is invertible in each of the following cases:

(i) $G$ is amenable, e.g. an abelian group, and $\sigma_s$ is injective;
(ii) $G$ is residually finite, e.g. a free group, and $\sigma_s, \sigma_c$ are injective.

To illustrate the case of higher dimensional ANUCA, i.e., when the universe is $\mathbb{Z}^d$, we can prove the invertibility of stably injective ANUCA where the configuration of local defining maps is no longer required to be asymptotically constant but we only need the mild condition of bounded singularity (see Section 10).

**Theorem C.** Let $M \subset \mathbb{Z}^d$ be a finite subset. Let $A$ be a finite set and let $S = A^A^M$. Suppose that for some $s \in S^{\mathbb{Z}^d}$, the ANUCA $\sigma_s$ is stably injective with bounded singularity. Then $\sigma_s$ is invertible.

For instance, fix $d, n_0 \geq 1$ and two functions $f, g: \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = \lim_{n \to \infty} f(n) - g(n) = \infty.$$

Let $R_n = \{a \in \mathbb{Z}: g(n) \leq |a| \leq f(n)\}^d \subset \mathbb{Z}^d$ for $n \geq n_0$. Let $A$ be a finite set and let $M \subset \mathbb{Z}^d$ be finite. Then for every $s \in S^{\mathbb{Z}^d}$, where $S = A^A^M$, such that $s$ is constant on $R_n$ for every $n \geq n_0$, the ANUCA $\sigma_s$ has bounded singularity. By Theorem C such an ANUCA is automatically invertible whenever it is stably injective.

The paper is organized as follows. We introduce various induced local maps of ANUCA in Section 3. Then we investigate the continuity and closedness properties of ANUCA in Section 4. In Section 5, we formulate an almost equivariance property and several invariant properties of ANUCA under the translations of the configurations of local defining maps (Lemma 5.1). In Section 6, we show that ANUCA over the same universe and alphabet form a monoid under the composition of maps (Theorem 6.2). We shall establish various stable properties of stably injective ANUCA through a series of results: Theorem 7.3, Theorem 9.1, and Theorem 11.1. The implication (iii) $\Rightarrow$ (ii) of Theorem A is proved in Theorem 7.3. We shall prove Theorem B as a direction consequence of Theorem 8.2 and Theorem 9.1 given in Section 8 and Section 9. The proof of Theorem C is given in Section 10. In Section 11, we show that every reversible ANUCA is stably injective (Theorem 11.1) which completes the proof of Theorem A.
In Section 12, we show that over a countable universe, every pre-injective post-surjective ANUCA with finite memory is invertible (Theorem 12.3). Moreover, we prove in Lemma 12.1 the pointwise uniform post-surjectivity property of post-surjective ANUCA. We explore in Section 13 the notion of stably post-surjectivity and establish a uniform post-surjectivity property of the family \( \{ \sigma_p : p \in \Sigma(s) \} \) associated with a stably post-surjective \( \sigma_s \).

Finally, we present several counter-examples in Section 14 to show that our results are in a sense optimal. Examples 14.1 describes an ANUCA which is injective but not stably injective, not surjective, and not reversible. Example 14.2 gives a bijective non reversible (and thus non stably injective) ANUCA. In particular, for ANUCA, there is no implication between bijectivity and reversibility.

2. Amenable and residually finite groups

2.1. Amenable groups. Although we shall not need the precise definition of amenable groups, we include a simple characterization for the sake of completeness. A group \( G \) is amenable if it satisfies the Følner’s condition [10]: for every \( \varepsilon > 0 \) and \( T \subset G \) finite, there exists \( F \subset G \) finite such that \( |TF| \leq (1 + \varepsilon)|F| \). Finitely generated groups of subexponential growth and solvable groups are amenable, while all groups containing a subgroup isomorphic to a free group of rank 2 are not amenable. See e.g. [24] for some more details.

The celebrated Garden of Eden of Moore and Myhill [15], [16] states that a CA with finite alphabet over the universe \( \mathbb{Z} \) is pre-injective if and only if it is surjective. In [6], the Garden of Eden theorem was generalized to hold over amenable group universes (see also [5], [19], [20]).

2.2. Residually finite groups. We recall that a group is residually finite if the intersection of its finite-index subgroups is reduced to the identity element. Equivalently, a group \( G \) is residually finite if for every finite subset \( E \subset G \), there exists a finite group and a surjective group homomorphism \( \varphi : G \rightarrow H \) such that \( \varphi|_E : E \rightarrow H \) is injective.

Notable examples of residually finite groups include finitely generated abelian group, e.g., \( \mathbb{Z}^d \) and cyclic groups, and more generally finitely generated linear groups due to a theorem of Mal’cev.

We formulate below a technical property of the class of residually finite groups which will be useful in Section 8.

**Lemma 2.1.** Let \( G \) be a residually finite group. Then for all finite subsets \( M, E \subset G \), there exist a finite subset \( K \subset G \), a finite group \( H \), and a surjective group homomorphism \( \varphi : G \rightarrow H \) such that:

(a) \( E \cup M \subset K \);
(b) \( \varphi|_K : K \rightarrow H \) is bijective;
(c) \( \varphi(KM \setminus K) \cap \varphi(E) = \emptyset \).
Proof. Up to replacing $E$ and $M$ by the finite set $\{1_G\}\cup E^{-1}\cup E\cup M^{-1}\cup M$, we can suppose without loss of generality that $E = M$, $E = E^{-1}$, and moreover $1_G \in E$. Since $G$ is residually finite, we can find a finite group $H$ and a surjective homomorphism $\varphi : G \to H$ such that the restriction $\varphi|_{E^2} : E^2 \to H$ is injective.

Consider the finite index subgroup $Z = \text{Ker} \varphi \subset G$ and let $K$ be a complete set of representatives of the right cosets of $Z$ in $G$ such that $E^2 \subset K$. Hence, the condition (b) is satisfied. Note that the choice of $K$ also satisfies (a) since $E \subset E^2 \subset K$ and $1_G \in E$.

Let us check the condition (c). Suppose that $k \in K$, $z \in Z$, and $x, y \in E$ verify $kx = zy$. Then $k = zyx^{-1}$ and it follows that $k$ and $yx^{-1}$ belong to the same right coset of $Z$. On the other hand, both $k$ and $yx^{-1}$ are elements of $K$ since $yx^{-1} \in E^2 \subset K$. We deduce from the choice of $K$ that $k = yx^{-1}$.

Consequently, $z = 1_G$ and thus $kx = y \in E \subset K$.

Therefore, we have proved that $KE \cap ZE \subset K$, which is exactly equivalent to the condition (c). The proof is thus complete. \qed

3. Induced local maps of ANUCA

3.1. Local maps for arbitrary group universes. Let $G$ be a group and let $A$ be a set. For every subset $E \subset G$ and $x \in A^E$ we define $gx \in A^{gE}$ by setting $gx(gh) = x(h)$ for all $h \in E$. In particular, we find that $gA^E = A^{gE}$.

Let $M$ be a subset of a group $G$. Let $A$ be a set and let $S = A^M$ be the collection of all maps $A^G \to A$.

For every finite subset $E \subset G$ and $w \in S^E$, we define a map $f_{E,w}^+ : A^{EM} \to A^E$ defined as follows. For every $x \in A^{EM}$ and $g \in E$, we set:

\[
f_{E,w}^+(x)(g) = w(g)((g^{-1}x)|_M).
\]  

(3.1)

In the above formula, note that $g^{-1}x \in A^{g^{-1}EM}$ and $M \subset g^{-1}EM$ since $1_G \in g^{-1}E$ for $g \in E$. Therefore, the map $f_{E,w}^+ : A^{EM} \to A^E$ is well defined.

Consequently, for every $s \in S^G$, we have a well-defined induced local map $f_{E,s|_E}^+ : A^{EM} \to A^E$ for every finite subset $E \subset G$ which satisfies:

\[
\sigma_s(x)(g) = f_{E,s|_E}^+(x|EM)(g)
\]  

(3.2)

for every $x \in A^G$ and $g \in E$. Equivalently, we have for all $x \in A^G$ that:

\[
\sigma_s(x)|_E = f_{E,s|_E}^+(x|EM).
\]  

(3.3)

3.2. Local maps and reversible ANUCA. For the notation, let $M, K$ be finite subsets of a group $G$. Let $A$ be a finite set and let $s, t \in S^G$ where $S = A^M$. Let $\varphi : G \to H$ be a surjective group homomorphism where $H$ is a finite group such that $\varphi|_K : K \to H$ is a bijection. In other words, $K$ forms a complete set of representatives of the right cosets of the subgroup Ker $\varphi$ in $G$.

The configurations $s, t$ induce the maps $\Psi_{K,s}$ and $\Psi_{K,t} : A^K \to A^K$ defined as follows. Given $x \in A^K$, we determine $\hat{x} \in A^G$ by setting $\hat{x}(g) = x(k_g)$
for all \( g \in G \) and the unique \( k_g \in K \) such that \( \varphi(g) = \varphi(k_g) \). Equivalently, 
\( \bar{x}(hk) = x(k) \) for all \( k \in K \) and \( h \in \text{Ker} \varphi \). Then we put 
\[
(3.4) \quad \Psi_{K,s}(x) := \sigma_s(\bar{x})|_K, \quad \Psi_{K,t}(x) := \sigma_t(\bar{x})|_K.
\]

**Lemma 3.1.** With the above notation and hypotheses, suppose in addition that \( \sigma_t \circ \sigma_s = \text{Id} \) and \( s(g) = s(k_g) \) for all \( g \in KM \setminus K \). Then one has: 
\[
\Psi_{K,t} \circ \Psi_{K,s} = \text{Id}.
\]

**Proof.** Let \( x \in A^K \). Consider \( u = f^+_{K,M,s|KM}(\bar{x}|_{KM^2}) \in A^{KM} \) and \( v = u|_K \). Then we deduce from (3.4) and (3.3) that 
\[
(3.5) \quad v = u|_K = \sigma_s(\bar{x})|_K = \Psi_{K,s}(x).
\]

We claim that \( u = \bar{v}|_{KM} \). Indeed, let us fix \( g \in KM \). If \( g \in K \) then there is nothing to prove since \( \bar{v}|_K = v = u|_K \). Suppose now that \( g \in KM \setminus K \).

On the one hand, for the unique \( k_g \in K \) and \( h_g \in \text{Ker} \varphi \) such that 
\( g = h_g k_g \), we have:
\[
(3.6) \quad \bar{v}(g) = v(k_g) = u(k_g) = s(k_g)((k_g^{-1} \bar{x})|_M).
\]

On the other hand, observe that \( (g^{-1} \bar{x})|_{M} = (k_g^{-1} \bar{x})|_{M} \) since for every \( m \in M \), we have by the definition of \( \bar{x} \) that:
\[
\bar{x}(gm) = \bar{x}(h_g k_g m) = \bar{x}(k_g m).
\]

Consequently, as \( s(g) = s(k_g) \) by hypothesis, we deduce that:
\[
(3.7) \quad u(g) = s(g)((g^{-1} \bar{x})|_M) = s(h_g k_g)((k_g^{-1} \bar{x})|_M) = s(k_g)((k_g^{-1} \bar{x})|_M)
\]

Hence, (3.6) and (3.7) imply that \( u = \bar{v}|_{KM} \) and the claim is proved. We can thus compute that:
\[
\Psi_{K,t}(\Psi_{K,s}(x)) = \Psi_{K,t}(v) = \sigma_t(\bar{v})|_K = f^+_{K,t|K}(\bar{v}|_{KM}) = f^+_{K,t|K}(u) = f^+_{K,t|K}(f^+_{KM,s|KM}(\bar{x}|_{KM^2})) = \sigma_t(\sigma_s(\bar{x}))|_K = \bar{x}|_K = x.
\]

Therefore, the proof of the lemma is complete. \( \square \)

### 4. Continuity and closedness property of ANUCA

We first prove that every ANUCA with finite memory is continuous.
Lemma 4.1. Let $M$ be a finite subset of a group $G$. Let $A$ be a set and let $S = A^M$. Then for every $s \in S^G$, the ANUCA $\sigma_s: A^G \to A^G$ is continuous with respect to the prodiscrete topology.

Proof. It suffices to observe that for every finite subset $E \subset G$ and all $x, y \in A^G$ such that $x|_E = y|_E$, we have $\sigma_s(x)|_E = \sigma_s(y)|_E$ and note that $EM \subset G$ is finite.

The following result shows the continuity of ANUCA $\sigma_s$ with respect to the configuration $s \in S^G$. Note that we do not suppose the finiteness of neither the memory nor the alphabet.

Lemma 4.2. Let $M$ be a subset of a group $G$. Let $A$ be a set and let $S = A^M$. Suppose that a sequence $(s_n)_{n \in \mathbb{N}}$ of elements of $S^G$ converges to some $s \in S^G$. Then for every $x \in A^G$, one has $\lim_{n \to \infty} \sigma_{s_n}(x) = \sigma_s(x)$.

Proof. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence in $S^G$ which converges to some $s \in S^G$. Let $x \in A^G$ and let $E \subset G$ be a finite subset. Then there exists $n_0 \in \mathbb{N}$ large enough so that for every $n \geq n_0$, we have $s_n|_E = s|_E$. From (3.1) and (3.3), we can compute:

$$\sigma_{s_n}(x)|_E = f^+_{E,(s_n)|_E} (x|_E)$$
$$= f^+_{E,s|_E} (x|_E)$$
$$= \sigma_s(x)|_E.$$

We conclude that $\lim_{n \to \infty} \sigma_{s_n}(x) = \sigma_s(x)$ and the proof is complete. □

Similarly, we can prove the following continuity of families of ANUCA with the same finite memory.

Lemma 4.3. Let $M$ be a finite subset of a group $G$. Let $A$ be a set and let $S = A^M$. Suppose that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} s_n = s$ for some $x \in A^G$, $s \in S^G$ and sequences $(x_n)_{n \in \mathbb{N}}$, $(s_n)_{n \in \mathbb{N}}$ in $A^G$ and $S^G$ respectively. Then one has $\lim_{n \to \infty} \sigma_{s_n}(x_n) = \sigma_s(x)$.

Proof. Let $s \in S^G$ and let $E \subset G$ be a finite subset. Since $M$ is finite by hypothesis and since $\lim_{n \to \infty} x_n = x$, $\lim_{n \to \infty} s_n = s$ we can find $n_0 \in \mathbb{N}$ large enough so that for every $n \geq n_0$, we have $x_n|_E = x|_E$ and moreover $s_n|_E = s|_E$. We can thus compute again from the formula (3.1) and (3.3):

$$\sigma_{s_n}(x_n)|_E = f^+_{E,(s_n)|_E} (x_n|_E)$$
$$= f^+_{E,s|_E} (x|_E)$$
$$= \sigma_s(x)|_E.$$

We conclude that $\lim_{n \to \infty} \sigma_{s_n}(x) = \sigma_s(x)$ and the proof is complete. □

Note that when $G$ is a countable group, the topological space $A^G$ is metrizable for every set $A$. In fact, we can define in this case the compatible Hamming metric $d$ on $A^G$ associated with any given exhaustion $(E_n)_{n \in \mathbb{N}}$ of finite
subsets of $G$ by setting for all $x, y \in A^G$:

\[(4.1) \quad d(x, y) = 2^{-n(x, y)}, \quad \text{where} \quad n(x, y) = \sup \{k \in \mathbb{N} : x|_{E_k} = y|_{E_k}\}.\]

As for classical CA, we can show that the important closed image property also holds for ANUCA.

**Theorem 4.4.** Let $M$ be a finite subset of a countable group $G$. Let $A$ be a finite set and let $S = A^M$. Then for every $s \in S^G$, the image $\sigma_s(A^G)$ is closed in $A^G$ with respect to the prodiscrete topology.

**Proof.** First observe that $A^G$ is a compact metrizable space for the prodiscrete topology by Tychonoff’s theorem as $A$ is finite. Moreover, as $G$ is countable, the topological space $A^G$ is also metrizable with the standard Hamming metric (cf. (4.1)). On the other hand, we know that $\sigma_s$ is continuous by Lemma 4.1. Consequently, $\sigma_s$ is a closed map and it follows in particular that $\sigma_s(A^G)$ is a closed subset of $A^G$. The proof is thus complete. \(\square\)

As an application, we obtain the following result:

**Corollary 4.5.** Let $M$ be a finite subset of a countable group $G$. Let $A$ be a finite set and let $S = A^M$. Suppose that $s \in S^G$ and $\sigma_s$ is post-surjective. Then $\sigma_s$ is also surjective.

**Proof.** By Theorem 4.4, it suffices to prove that the image $\sigma_s(A^G)$ is dense in $A^G$ with respect to the prodiscrete topology.

Let us fix an arbitrary configuration $x_0 \in A^G$ and let $y_0 = \sigma_s(x_0) \in A^G$. Then by the definition of post-surjectivity, we deduce that every configuration that is asymptotic to $y_0$ must belong to the image of $\sigma_s$. On the other hand, the set of all configurations asymptotic to $y_0$ is clearly a dense subset of $A^G$. We deduce that $\sigma_s(A^G)$ contains a dense subset of $A^G$ and the conclusion follows. \(\square\)

5. **Stably injective ANUCA**

In general, ANUCA do not enjoy the $G$-equivariance property of the smaller class of CA but they satisfy the following fundamental property.

**Lemma 5.1.** Let $M$ be a subset of a group $G$. Let $A$ be a set and let $S = A^M$. Let $s \in S^G$ then for all $g \in G$, we have

\[(5.1) \quad \sigma_{gs}(gx) = g\sigma_s(x), \quad \text{for every} \quad x \in A^G.\]

Moreover, if $\sigma_s$ is injective (resp. surjective, resp. pre-injective, resp. stably injective, resp. post-surjective) then so is $\sigma_{gs}$ for every $g \in G$. 

Indeed, let \( s \in S^G, g \in G, x \in A^G \) and \( h \in G \), then we find from Definition 1.1 that

\[
\sigma_{gs}(gx)(h) = gs(h)((h^{-1}gx)|_M)
\]
\[
= s(g^{-1}h)((g^{-1}h)^{-1}x)|_M
\]
\[
= \sigma_s(x)(g^{-1}h)
\]
\[
= gs(x)(h).
\]

Consequently, we obtain \( \sigma_{gs}(gx) = g\sigma_s(x) \) and the formula (5.1) is proved.

Now fix \( s \in S^G \). Suppose first that \( \sigma_s \) is injective. Let \( g \in G \) and let \( x, y \in A^G \) be such that \( \sigma_{gs}(x) = \sigma_{gs}(y) \). Then it follows from (5.1) that \( gs(g^{-1}x) = g\sigma_s(g^{-1}y) \). We deduce that \( \sigma_s(g^{-1}x) = \sigma_s(g^{-1}y) \). Therefore, as \( \sigma_s \) is injective, we obtain \( g^{-1}x = g^{-1}y \) and consequently \( x = y \). Hence, \( \sigma_{gs} \) is also injective for every \( g \in G \).

Similarly, suppose that \( \sigma_s \) is surjective. Let \( g \in G \) and let \( y \in A^G \). Then we can find \( x \in A^G \) such that \( \sigma_s(x) = g^{-1}y \). It follows from (5.1) that \( \sigma_{gs}(gx) = g\sigma_s(x) = gy^{-1}y = y \). Since \( y \in A^G \) is arbitrary, we conclude that \( \sigma_{gs} \) is surjective as well.

Assume now that \( \sigma_s \) is pre-injective. Let \( g \in G \) and let \( x, y \in A^G \) be two asymptotic configurations such that \( \sigma_{gs}(x) = \sigma_{gs}(y) \). Then we infer from (5.1) that \( gs(g^{-1}x) = g\sigma_s(g^{-1}y) \) and therefore \( \sigma_s(g^{-1}x) = \sigma_s(g^{-1}y) \). Since \( x, y \) are asymptotic, so are \( g^{-1}x \) and \( g^{-1}y \). Thus, the pre-injectivity of \( \sigma_s \) implies that \( g^{-1}x = g^{-1}y \). Consequently, \( x = y \) and we deduce that \( \sigma_{gs} \) is pre-injective for every \( g \in G \).

Next, we suppose that \( \sigma_s \) is post-surjective. Let \( x, y, t \in A^G \) such that \( y = \sigma_{gs}(x) \) and \( y, t \) are asymptotic. Then (5.1) implies that \( y = g\sigma_s(g^{-1}x) \) and thus \( g^{-1}y = \sigma_s(g^{-1}x) \). As \( \sigma_s \) is post-surjective and since \( g^{-1}y, g^{-1}t \) are asymptotic, we deduce that there exists a configuration \( z \in A^G \) which is asymptotic to \( g^{-1}x \) and \( g^{-1}t = \sigma_s(z) \). Again, we infer from (5.1) that \( t = g\sigma_s(z) = \sigma_{gs}(gz) \). Note that \( gz \) is asymptotic to \( gg^{-1}x = x \). Therefore, we find that \( \sigma_{gs} \) is also post-surjective for all \( g \in G \).

Finally, suppose that \( \sigma_s \) is stably injective. Fix \( g \in G \) and observe that:

\[
\Sigma(gs) = \{hgs: h \in G\}
\]
\[
= \{hs: h \in G\} \quad \text{(since } G \text{ is a group)}
\]
\[
= \Sigma(s) \subseteq S^G.
\]

Since \( \sigma_s \) is stably injective, we deduce from the definition that \( \sigma_p \) is injective for every \( p \in \Sigma(s) = \Sigma(gs) \). We can thus conclude that \( \sigma_{gs} \) is stably injective for all \( g \in G \). The proof of the lemma is complete.

Lemma 5.1 implies in particular that an injective ANUCA \( \sigma_s \) is stably injective if and only if \( \sigma_p \) is also injective for every limit point \( p \in \Sigma(s) \), which justifies our choice of the terminology stable injectivity. We shall see
later in Theorem 7.3, Theorem 9.1, Theorem 11.1 more stable properties of stably injective ANUCA.

The next lemma allows us to improve the statement concerning the stable injectivity of Lemma 5.1.

**Lemma 5.2.** Let $M$ be a subset of a group $G$. Let $A$ be a set and let $S = A^M$. Let $s \in S^G$. Then for every $p \in \Sigma(s)$ we have $\Sigma(p) \subset \Sigma(s)$. In particular, $\sigma_s$ is stably injective if and only if so is $\sigma_p$ for every $p \in \Sigma(s)$.

**Proof.** Let $p \in \Sigma(s)$. Then for every $g \in G$, we have $gp \in \Sigma(s)$ as $\Sigma(s)$ is a $G$-invariant subset of $S^G$. Since $\Sigma(s)$ is closed and $\Sigma(p) = \{gs : g \in G\}$ by definition, we deduce that $\Sigma(p) \subset \Sigma(s)$ for all $p \in \Sigma(s)$.

From this, the last statement follows immediately from the definition of stable injectivity and we simply note that $s \in \Sigma(s)$. □

**Remark 5.3.** With the notation as in Lemma 5.2, we remark that the inclusion $\Sigma(s) \subset \Sigma(p)$ may fail for some $p \in \Sigma(s)$.

For example, let $G = \mathbb{Z}$, $A = \{0, 1\}$, and let $M = \{-1, 0, 1\} \subset G$. Let $S = A^M$ and let $u, v : A^M \rightarrow A$ be two distinct maps. Let $s \in S^G$ defined by $s(0) = u$ and $s(n) = v$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $\Sigma(s)$ consists of the translates of $s$ and the configuration constant $p \in S^G$ given by $p(n) = v$ for all $n \in \mathbb{Z}$. It is clear that $\Sigma(p) = \{p\} \subsetneq \Sigma(s)$.

6. The monoid of ANUCA

We begin with a lemma which describes the action of the translations of the configurations of local defining maps in a composition of ANUCA.

**Lemma 6.1.** Let $M$ be a subset of a group $G$. Let $A$ be a set and let $S = A^M$. Let $p, q, s \in S^G$ and suppose that $\sigma_p \circ \sigma_q = \sigma_s$. Then for every $g \in G$, we have $\sigma_{gp} \circ \sigma_{gq} = \sigma_{gs}$. In particular, if $\sigma_p \circ \sigma_q = \text{Id}_{A^G}$, then for every $g \in G$, one has $\sigma_{gp} \circ \sigma_{gq} = \text{Id}_{A^G}$.

**Proof.** Suppose first that $\sigma_p \circ \sigma_q = \sigma_s$ for some $p, q, s \in S^G$. Let $g \in S$ and let $x \in A$. Then we infer from the formula (5.1) the following computation:

\[
\begin{align*}
(\sigma_{gp} \circ \sigma_{gq})(gx) &= \sigma_{gp}(\sigma_{gq}(gx)) \\
&= \sigma_{gp}(g\sigma_q(x)) \quad \text{(by (5.1))} \\
&= g\sigma_p(\sigma_q(x)) \quad \text{(by (5.1))} \\
&= g\sigma_q(x) \quad \text{(as $\sigma_p \circ \sigma_q = \text{Id}_{A^G}$)} \\
&= \sigma_{gs}(gx). \quad \text{(by (5.1))}
\end{align*}
\]

Since $y = g^{-1}x \in A^G$ is arbitrary, we deduce that

\[
(\sigma_{gp} \circ \sigma_{gq})(y) = y
\]

for all $y \in A^G$ and the proof of the first statement is thus complete. The last statement is an obvious consequence. □
The following result shows that the composition of two ANUCA is again an ANUCA. It follows that that set of ANUCA over a given universe and a given alphabet form a monoid with respect to the composition operation.

**Theorem 6.2.** Let \( M, N \subseteq G \) be subsets of a group \( G \). Let \( A \) be a set and let \( s \in S^G \), \( t \in T^G \) where \( S = A^M \), \( T = A^N \). Then there exists \( q \in Q^G \) where \( Q = A^{MN} \) such that \( \sigma_s \circ \sigma_t = \sigma_q \).

**Proof.** Fix \( g \in G \) and \( x \in A^G \). Let us consider the induced local maps \( f_{(g)}: A^M \rightarrow A^g \) and \( f^+_{gMN,tlgMN}: A^{gMN} \rightarrow A^M \) defined in Section \( 3 \). Then we infer from the formula (3.1) and (3.2) the following computation:

\[
\sigma_s(\sigma_t(x))(g) = f_{(g)}(f^+_{gMN,tlgMN}(x|MN)) = s(g)(f^+_{MN,tlMN}((g^{-1}x)|MN))
\]

Therefore, if we define \( q \in Q^G \) by setting \( q(g) \in A^{MN} \) to be the map

\[
q(g) := s(g) \circ f^+_{MN,tlMN}
\]

for every \( g \in G \) then it follows immediately that \( \sigma_s \circ \sigma_t = \sigma_q \). The proof of the theorem is thus complete. \( \square \)

### 7. Reversibility of stably injective ANUCA

Given a group \( G \) and a set \( A \), we say that an ANUCA \( \sigma: A^G \rightarrow A^G \) is **reversible** if it is injective and there exists an ANUCA with finite memory \( \tau: A^G \rightarrow A^G \) such that \( \tau \circ \sigma = \text{Id} \). Note that in our definitions, invertibility implies reversibility for ANUCA. We establish the following reversibility result for stably injective ANUCA.

**Theorem 7.1.** Let \( M \) be a finite subset of a countable group \( G \). Let \( A \) be a finite set and let \( S = A^M \). Suppose that \( \sigma_s \) is stably injective for some \( s \in S^G \). Then \( \sigma_s \) is reversible.

**Proof.** We can suppose without loss of generality that \( 1_G \in M \). Since \( G \) is countable by hypothesis, we can find an increasing sequence of finite subsets \( (E_n)_{n \in \mathbb{N}} \) of \( G \) such that \( G = \bigcup_{n \in \mathbb{N}} E_n \) and \( M \subset E_0 \).

For every \( n \in \mathbb{N} \) and every configuration \( w \in S^{E_n} \), we have an induced local map \( f^+_{E_n,s}: A^{E_n,M} \rightarrow A^{E_n} \) defined as in (3.1).

Let \( \Gamma = \sigma_s(A^G) \). We claim that there exists \( N \subseteq G \) finite and such that \( \sigma_s^{-1}(y)(g) \in A \) depends uniquely on the restriction \( y|_{gN} \) for every configuration \( g \in \Gamma \) and every group element \( g \in G \).

Indeed, suppose on the contrary that the claim is false. Then for every \( n \in \mathbb{N} \), we can find \( g_n \in G \) and \( u_n, v_n \in A^G \) such that \( u_n(g_n) \neq v_n(g_n) \) and \( \sigma_s(u_n)|_{gE_n} = \sigma_s(v_n)|_{gE_n} \). Therefore, \( g^{-1}\sigma_s(u_n)|_{E_n} = g^{-1}\sigma_s(v_n)|_{E_n} \) and we infer from (3.1) that

\[
\sigma_{g^{-1}s}(g_n^{-1}u_n)|_{E_n} = g_n^{-1}\sigma_s(u_n)|_{E_n} = g_n^{-1}\sigma_s(v_n)|_{E_n} = \sigma_{g^{-1}s}(g_n^{-1}v_n)|_{E_n}.
\]
Hence, by setting \( s_n = g_n^{-1}s|_{E_n} \in S^{E_n} \), we can deduce from \((3.3)\) that:

\[
f_{E_{n,s_n}}^+((g_n^{-1}u_n)|_{E_n,M}) = f_{E_{n,s_n}}^+((g_n^{-1}v_n)|_{E_n,M}).
\]

Observe that \( S \) is finite since \( M \) and \( A \) are finite. Hence, the space \( S^G \) is compact with respect to the prodiscrete topology. Thus, the closed subset \( \Sigma(s) \subset S^G \) is also compact.

Consequently, since \( g_n^{-1}s \in \Sigma(s) \) we can find a subsequence \( (g_n^{-1}s)_{k \in \mathbb{N}} \) of \( (g_n^{-1}s)_{n \in \mathbb{N}} \) which converges to a configuration \( p \in \Sigma(s) \). Hence, up to restricting again to another subsequence and reindexing, we can suppose without loss of generality that for every \( k \in \mathbb{N} \), we have

\[
g_n^{-1}s|_{E_k} = p|_{E_k}.
\]

Note that \( E_k M \subset E_n M \) for all \( k \in \mathbb{N} \) since \( n_k \geq k \). Therefore, if we denote \( x_k = g_n^{-1}u_n \in A^G \) and \( y_k = g_n^{-1}v_n \in A^G \), we deduce immediately from \((7.1)\) and \((7.2)\) that for every \( k \in \mathbb{N} \), we have:

\[
f_{E_k,p|_{E_k}}^+(x_k|_{E_k,M}) = f_{E_k,p|_{E_k}}^+(y_k|_{E_k,M}).
\]

Consequently, the combination of the relations \((3.3)\) and \((7.3)\) imply that

\[
\sigma_p(x_k)|_{E_k} = \sigma_p(y_k)|_{E_k}.
\]

Moreover, \( x_k(1_G) \neq y_k(1_G) \) for all \( k \in \mathbb{N} \) since \( u_n(g_n) \neq v_n(g_n) \). As the space \( A^G \times A^G \) is compact with respect to the prodiscrete topology, we can suppose without loss of generality, up to passing to a subsequence, that \( x_k \) converges to some \( x \in A^G \) and \( y_k \) converges to some \( y \in A^G \) as well.

Therefore, since \( \sigma_p \) is continuous by Lemma \((4.1)\) and since \( (E_k)_{k \in \mathbb{N}} \) is an exhaustion of \( G \), we can infer from \((7.4)\) by passing to the limit that

\[
\sigma_p(x) = \sigma_p(y).
\]

On the other hand, we have \( x(1_G) \neq y(1_G) \) since \( x_k(1_G) \neq y_k(1_G) \) for all \( k \in \mathbb{N} \). In particular, \( x \neq y \) and it follows that \( \sigma_p \) is not injective. However, since \( \sigma_s \) is stably injective and \( p \in \Sigma(s) \), we deduce that \( \sigma_p \) is injective, which is a contradiction.

Hence, we have proved the claim that there exists a finite subset \( N \subset G \) such that \( \sigma_s^{-1}(y)(g) \in A \) depends uniquely on the restriction \( y|_{gN} \) for every configuration \( y \in \Gamma \) and every group element \( g \in G \).

To complete the proof of the theorem, let \( T = A^{AN} \). We construct a configuration \( q \in T^G \) as follows. Fix some \( a_0 \in A \). For every \( g \in G \), the property of the set \( N \) shows that we have a well-defined map \( \varphi_g : \Gamma_{gN} \rightarrow A \) given by the formula:

\[
\varphi_g(z) = \sigma_s^{-1}(y)(g)
\]

for every \( z \in \Gamma_{gN} \) and \( y \in \Gamma \) which extends \( z \). We define \( q(g) : A^N \rightarrow A \) by setting \( q(g)(t) = \varphi_g(gt) \) for all \( t \in A^N \) such that \( gt \in \Gamma_{gN} \) and we simply put \( q(g)(t) = a_0 \) whenever \( t \in A^N \) such that \( gt \notin \Gamma_{gN} \).

Hence, we obtain \( q \in T^G \) and it is clear from our construction that \( \sigma_s^{-1}(y) = \sigma_q(y) \) for all \( y \in \Gamma \). The proof is thus complete. \( \square \)
Under the same assumptions of Theorem 7.1, we can actually show that stable injectivity implies stable reversibility which is a stronger property than reversibility.

**Definition 7.2.** Let $M$ be a finite subset of a countable group $G$. Let $A$ be a finite set and let $S = A^M$. Let $s \in S^G$. Then $\sigma_s$ is said to be stably reversible if there exist $N \subset G$ finite and $q \in T^G$ where $T = A^N$ such that for every $p \in \Sigma(s)$, we can find $w \in \Sigma(q)$ such that $\sigma_w \circ \sigma_p = \text{Id}$.

Our result can be stated as follows.

**Theorem 7.3.** Let $M$ be a finite subset of a countable group $G$. Let $A$ be a finite set and let $S = A^M$. Let $s \in S^G$ and suppose that $\sigma_s$ is stably injective. Then $\sigma_s$ is stably reversible.

**Proof.** By Theorem 7.1, there exists a finite subset $N \subset G$ and $q \in T^G$ where $T = A^N$ such that $\sigma_q \circ \sigma_s = \text{Id}$.

Let $p \in \Sigma(s)$. Then there exists a sequence $(g_n)_{n \in \mathbb{N}}$ in $G$ such that the sequence $(g_n s)_{n \in \mathbb{N}}$ converges to $p$ in the space $S^G$ with respect to the prodiscrete topology.

Since $T^G$ is compact with respect to the prodiscrete topology as $T$ is finite, we can suppose, up to passing to a subsequence, that $(g_n q)_{n \in \mathbb{N}}$ converges to some $w \in T^G$. In particular, we have $w \in \Sigma(q)$ by the definition of $\Sigma(q)$.

We are going to prove that $\sigma_w \circ \sigma_p = \text{Id}$. It suffices to show that $\sigma_w(y) = x$ for all $y = \sigma_p(x)$ where $x \in A^G$.

Hence, let us fix $x \in A^G$ and let $y = \sigma_p(x)$. Since $\lim_{n \to \infty} g_n q = w$, Lemma 4.2 implies that $\lim_{n \to \infty} \sigma_{g_n p}(y) = \sigma_w(y)$.

We claim that $\lim_{n \to \infty} \sigma_{g_n q}(\sigma_p(x)) = x$. Indeed, let $F \subset G$ be a finite subset. As $\lim_{n \to \infty} g_n s = p$, there exists $n_0 \in \mathbb{N}$ such that $p|_{F N} = (g_n s)|_{F N}$ for all $n \geq n_0$.

From (3.1) and (3.2), we find that for all $n \geq n_0$:

$$\sigma_{g_n q}(\sigma_p(x))|_F = f_{F, (g_n q)}^+(f_{F, N, p|_{F N}}^+(x|_{F N M})) = f_{F, (g_n q)}^+(f_{F, N, (g_n s)}^+(x|_{F N M})) = \sigma_{g_n q}(\sigma_{g_n s}(x))|_F = x|_F.$$

It follows that $\lim_{n \to \infty} \sigma_{g_n q}(y) = \sigma_{g_n q}(\sigma_p(x)) = x$ and the claim is proved.

Consequently, $x = \sigma_w(y)$ as we also have $\lim_{n \to \infty} \sigma_{g_n p}(y) = \sigma_w(y)$. Therefore, $\sigma_w \circ \sigma_p = \text{Id}$ as $x \in A^G$ is arbitrary. We conclude that $\sigma_s$ is stably reversible and the proof of the theorem is thus complete.

8. DISTURBANCE OF CA OVER RESIDUALLY FINITE GROUP UNIVERSES

Recall that a configuration $s \in S^G$, where $S$ is a set and $G$ is a group, is said to be constant if $s(g) = s(h)$ for all $g, h \in G$. We have the following simple observation.
Lemma 8.1. Let $S$ be a finite set and let $G$ be a group. Suppose that $s \in S^G$ is asymptotic to a constant configuration $c \in S^G$. Then we have:

\begin{equation}
\Sigma(s) = \{gs: g \in G\} \cup \{c\} \subset S^G.
\end{equation}

Proof. For this, let $S$ be a finite symmetric generating set of $G$ and let $B_S(r) \subset G$ be the ball of radius $r$ in the connected Cayley graph $C_S(G)$ of $G$ associated with $S$ and the corresponding metric $d_S: G \times G \to \mathbb{N}$ defined as the length of shortest path in $C_S(G)$ that connects two vertices.

Since $s$ is asymptotic to the constant configuration $c \in S^G$, we can find $r_0 \geq 1$ such that $s(g) = c(1_G)$ for all $g \in G \setminus B_S(r_0)$. Let $(g_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of elements in $G$ such that $d_S(g_n, 1_G) = n + r_0 + 1$ for every $n \in \mathbb{N}$. Then it is clear from the triangle inequality that $g_n s(g) = c(1_G)$ for all $g \in B_S(n)$. Consequently, we deduce that the sequence $(g_n s)_{n \in \mathbb{N}}$ converges to $c \in S^G$. It follows that $c \in \Sigma(s)$.

It is clear from the definition of $g s \in \Sigma(s)$ for all $g \in G$. Hence, we find that $\{gs: g \in G\} \cup \{c\} \subset \Sigma(s)$.

Conversely, let $p \in \Sigma(s)$ and suppose that $p \neq gs$ for all $g \in G$. Then $p$ is the limit of $(g_n s)_{n \in \mathbb{N}}$ for some sequence $(g_n)_{n \in \mathbb{N}}$ of distinct elements of $G$.

Since the ball $B_S(r)$ is finite for every $r \in \mathbb{N}$, we can thus find a subsequence $(g_{n_k})_{k \in \mathbb{N}}$ of $(g_n)_{n \in \mathbb{N}}$ such that $(d_S(g_n, 1_G))_{n \in \mathbb{N}}$ forms a strictly increasing sequence of positive integers. In particular, we deduce that $d_S(g_n, 1_G) \geq n$ for every $n \in \mathbb{N}$.

It is then clear that $(g_{n+k_0+1}s)|_{B_S(n)} = c|_{B_S(n)}$ for all $n \in \mathbb{N}$. It follows that $p|_{B_S(n)} = c|_{B_S(n)}$ for all $n \in \mathbb{N}$. Consequently, we have $p = c$. This shows that $\Sigma(s) \subset \{gs: g \in G\} \cup \{c\}$ and the relation (8.1) is proved. The proof of the lemma is thus complete.

As an application of Theorem 7.3, we obtain the following surjectivity property of locally disturbed injective classical CA.

Theorem 8.2. Let $M$ be a finite subset of a residually finite group $G$. Let $A$ be a finite set and let $S = A^M$. Let $s \in S^G$ be asymptotic to a constant configuration $c \in S^G$. Then $\sigma_s$ and $\sigma_c$ are invertible whenever they are both injective.

Proof. As residually finite groups are sofic, Gromov-Weiss the surjectivity theorem for CA implies that $\sigma_c$ is surjective and thus invertible.

Let $\Gamma = \sigma_s(A^G)$ and suppose on the contrary that $\sigma_s$ is not surjective. As $\sigma_s(A^G)$ is closed in $A^G$ in the prodiscrete topology by Theorem 4.1, there exists a finite subset $\Omega \subset G$ such that $\Gamma_\Omega = \Gamma|_{\Omega^s} \subset \Omega^M \subseteq \Omega^G$.

Since $\sigma_s$ and $\sigma_c$ are injective, we deduce from Lemma 8.1 and Lemma 8.1 that $\sigma_s$ is stably injective. It follows that $\sigma_s$ is reversible by Theorem 7.1. Consequently, up to enlarging $M$ without loss of generality, we can find $t \in S^G$ such that $\sigma_t \circ \sigma_s = \Id$.

Up to enlarging $M$ again, we can suppose without loss of generality that $1_G \in M$ and $M$ is symmetric, i.e., $M = M^{-1}$. Since $s$ and $c$ are asymptotic, we can find a finite subset $E \subset G$ such that $M \cup \Omega \subset E$ and $s|_{G \setminus E} = c|_{G \setminus E}$. 

\[ (8.1) \quad \Sigma(s) = \{gs: g \in G\} \cup \{c\} \subset S^G. \]
Since $G$ is residually finite, Lemma 2.1 provides a finite group $H$ and a surjective group homomorphism $\varphi: G \to H$ such that the restriction map $\varphi|_E: EM^2 \to H$ is injective and there exists a finite subset $K \subset G$ such that $E \subset K$ and $\varphi|_K: K \to H$ is a bijection and $\varphi(KM \setminus K) \cap \varphi(E) = \emptyset$.

The configurations $s, t$ induce the maps $\Psi_{K,s}$ and $\Psi_{K,t}: A^K \to A^K$ defined as in Section 3.2. Every $x \in A^K$ defines $\tilde{x} \in A^G$ by $\tilde{x}(g) = x(k_g)$ for all $g \in G$ and the unique $k_g \in K$ such that $\varphi(g) = \varphi(k_g)$. Then

$$\psi_{K,s}(x) := \sigma_s(\tilde{x})|_K, \quad \psi_{K,t}(x) := \sigma_t(\tilde{x})|_K. \tag{8.2}$$

Since $\sigma_t \circ \sigma_s = \text{Id}$ and $s(g) = s(k_g) = c(0)$ for all $g \in K \setminus K$, we infer from Lemma 3.1 that $\psi_{K,t} \circ \psi_{K,s} = \text{Id}$.

As $A^K$ is finite, we deduce that $\psi_{K,t}$ and $\psi_{K,s}$ are bijections. In particular, since $\Omega \subset K$, it follows that

$$\Gamma_\Omega = \sigma_s(A^G)|_\Omega \supset \{\sigma_s(\tilde{x}): x \in A^K\}|_\Omega = (\text{Im } \psi_{K,s})|_\Omega = (A^K)|_\Omega = A^\Omega.$$

Hence, we obtain a contradiction to the choice of $\Omega$. We conclude that $\sigma_s$ is surjective. Since $\sigma_t \circ \sigma_s = \text{Id}$, it follows at once that $\sigma_s$ and $\sigma_t$ are invertible. The proof of the theorem is thus complete. \qed

Therefore, we see that when an injective ANUCA $\sigma_s$ was obtained by disturbing an injective classical CA, i.e., when the configuration $s$ is asymptotic to a constant configuration, then $\sigma_s$ is in fact invertible if the universe is a residually finite group.

9. Disturbance of CA over amenable group universes

When the universe is an amenable group, Theorem 8.2 can be strengthened as follows. In essence, what happens in this case is that because of the Garden of Eden theorem, one cannot obtain injective ANUCA by disturbing the local transition rules of a finite number of cells of non-injective CA.

Theorem 9.1. Let $G$ be an amenable group and let $M \subset G$ be finite. Let $A$ be a finite set and let $S = A^M$. Suppose that $\sigma_s$ is injective for some $s \in S^G$ asymptotic to a constant configuration $c$. Then $\sigma_c$ and $\sigma_s$ are invertible.

Proof. We can suppose without loss of generality that $G$ is a finitely generated group up to restricting to the subgroup generated by the union of $M$ with the largest finite subset of $G$ over which $c$ is different from $s$. Since $c$ and $s$ are asymptotic, there exists $E \subset F$ finite such that $c|_{G \setminus E} = s|_{G \setminus E}$.

First, we claim that $\sigma_c$ is surjective. Indeed, suppose on the contrary that $\sigma_c$ is not surjective. Then we infer from the Garden of Eden theorem (cf. [4]) that $\sigma_c$ is not pre-injective. Consequently, we can find two distinct asymptotic configurations $u, v \in A^G$ such that $\sigma_c(u) = \sigma_c(v)$. In particular, we can find $F \subset G$ finite such that $u|_{G \setminus F} = v|_{G \setminus F}$. Up to replacing $u, v$ by a suitable translation, we can clearly suppose that $F \cap EM = \emptyset$.

It follows immediately from the formula 3.1 and 3.2 that $\sigma_s(u) = \sigma_s(v)$ and thus $\sigma_s$ is not injective. The obtained contradiction proves the claim that $\sigma_c$ is surjective.
Now fix $z \in A^G$ and a finite subset $N \subset G$ containing $E$. Consider the set $V = \{ x \in A^G : x|_{G \setminus N} = z|_{G \setminus N}\}$. Let $U = \sigma_c^{-1}(V) \subset A^G$. Since $\sigma_c$ is surjective, we have $|U| \geq |V|$.

On the other hand, observe that $\sigma_s(U) \subset V$ as $s|_{G \setminus N} = c|_{G \setminus N}$ so that $|\sigma_s(U)| \leq |V|$. Hence, by combining with the inequality $|U| \geq |V|$ and the injectivity of $\sigma_s$, we find that

$$|U| = |\sigma_s(U)| \leq |V| \leq |U|.$$ 

Therefore, $|\sigma_s(U)| = |V|$ and thus $\sigma_s(U) = V$ as $\sigma_s(U) \subset V$.

Since $N$ is arbitrary, we deduce that the image $\sigma_s(A^G)$ is dense in $A^G$ with respect to the indiscrete topology. We can thus conclude that $\sigma_s(A^G) = A^G$ since $\sigma_s(A^G)$ is closed in $A^G$ by Theorem 4.3.

Therefore, $\sigma_s$ is surjective and thus bijective. Since $\sigma_s$ is injective, we deduce from Lemma 5.1 that $\sigma_{gs}$ is also injective for every $g \in G$.

Note also that $|\sigma_c^{-1}(V)| = |V|$ since they both equal to $|U| = |\sigma_s(U)|$. Hence, as $z \in A^G$ is arbitrary, we deduce that $\sigma_c$ is injective and thus bijective. This proves the first part of the conclusion of the theorem.

As a consequence, $\sigma_s$ is stably injective since we know by Lemma 8.1 that $\Sigma(s) = \{gs : g \in G\} \cup \{c\}$. Therefore, we infer from Theorem 7.1 that $\sigma_c$ and $\sigma_s$ are reversible and thus invertible as they are surjective. Therefore, there exist $R \subset G$ finite and $t, d \in T^G$ where $T = A^{A_R}$ such that $\sigma_s^{-1} = \sigma_t$ and $\sigma_c^{-1} = \sigma_d$. Note that $d$ is constant since $c$ is constant.

The proof of the theorem is thus complete. 

\[ \square \]

10. Generalization to ANUCA of bounded singularity

When the universe is a free abelian group, we can establish the following invertibility result of the large class of stably injective ANUCA of bounded singularity that we describe below.

Given $g, h \in \mathbb{Z}^d$ and a box $K = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{Z}^d$ where $[a_j, b_j] = \{a_j, \ldots, b_j\}$, we say that that $g \equiv h$ (mod $K$) if $g_j \equiv h_j$ (mod $a_j - b_j + 1$) for every $j = 1, \ldots, d$ where $g = (g_1, \ldots, g_d)$ and $h = (h_1, \ldots, h_d)$.

Given subsets $M, K$ of a group $G$. We can define the $M$-interior, the $M$-exterior, and the $M$-boundary of $K$ respectively by

$$\partial_M K := \{ g \in K : gM \subset K \},$$

$$\partial_M^+ K := KM \setminus K,$$

$$\partial_M^- K := \partial_M^+ K \cup (K \setminus \partial_M^- K).$$

**Definition 10.1.** Let $M \subset \mathbb{Z}^d$ ($d \in \mathbb{N}$) be finite. Let $A$ be a finite set and let $S = A^{A^M}$. Given $s \in S^{\mathbb{Z}^d}$, we say that $\sigma_s$ has bounded singularity if for all finite subset $E \subset \mathbb{Z}^d$, there exists a box $K \subset \mathbb{Z}^d$ containing $E$ such that $s(g) = s(k_g)$ for all $g \in \partial_E^+ K$ and the unique $k_g \in K$ with $k_g \equiv g$ (mod $K$).
For example, let $M = [-r, r]^2$ and let $p \geq 2r + 1$. Let $A$ be a finite set and let $S = A^{4M}$. Then $\sigma_s$ has bounded singularity for every $s \in S^{\mathbb{Z}^2}$ which is constant on $\mathbb{Z}^2 \setminus p\mathbb{Z}^2M$. The following example is more general.

**Example 10.2.** For every $r \geq 0$, let $M_r = [-r, r]^2$. Let $(K_n)_{n \geq 0}$ be a nested sequence of boxes such that $\mathbb{Z}^2 = \bigcup_{n \geq 0} K_n$. Let $A$ be a finite set and let $S = A^{4M_0}$. Then the ANUCA $\sigma_s$ has bounded singularity for every $s \in S^{\mathbb{Z}^2}$ such that $s$ is constant on each of $\partial_{M_n} K_n$.

We can now prove the main result Theorem C in the Introduction whose proof is similar to the proof of Theorem 8.2.

**Proof of Theorem C.** Since $\sigma_s$ is stably injective, we infer from Theorem 7.1 that $\sigma_s$ is reversible. Hence, up to enlarging $M$ without loss of generality, we can find $t \in S^G$ such that $\sigma_t \circ \sigma_s = \text{Id}$.

Let $\Gamma = A^{(A^{Z^d})}$ and suppose on the contrary that $\sigma_s$ is not invertible. In particular, $\sigma_s$ is not surjective and we infer Theorem 4.4 that there exists a finite subset $E \subset \mathbb{Z}^d$ such that $M \subset E$ and $\Gamma_E = \bigcup_{s \in E} (A^{EM}) \subseteq A^E$.

Since $\sigma_s$ has bounded singularity, we can find a box $K \subset \mathbb{Z}^d$ which contains $E$ and such that $s(g) = s(k_g)$ for all $g \in \partial_E K$ and the unique $k_g \in K$ with $k_g \equiv g \pmod{K}$.

The configurations $s, t$ induce the maps $\Psi_{K,s}$ and $\Psi_{K,t} : A^K \to A^K$ defined as in Section 3.2. Every $x \in A^K$ defines $\tilde{x} \in A^{Z^d}$ by $\tilde{x}(g) = x(k_g)$ for all $g \in \mathbb{Z}^d$ and the unique $k_g \in K$ such that $k_g \equiv g \pmod{K}$. Then we have:

\begin{equation}
\Psi_{K,s}(x) := \sigma_s(\tilde{x})|_K, \quad \Psi_{K,t}(x) := \sigma_t(\tilde{x})|_K.
\end{equation}

Since $\sigma_t \circ \sigma_s = \text{Id}$ and $s(g) = s(k_g)$ for all $g \in KM \setminus K$ (as $M \subset E$), Lemma 3.1 implies that $\Psi_{K,t} \circ \Psi_{K,s} = \text{Id}$. It follows that $\Psi_{K,t}$ and $\Psi_{K,s}$ are bijective since $A^K$ is finite. As $E \subset K$, we deduce that

$$\Gamma_E = \sigma_s(A^{Z^d})|_E \supseteq \{\sigma_s(\tilde{x}) : x \in A^K\}|_E = (\text{Im } \Psi_{K,s})|_E = (A^K)|_E = A^E,$$

which contradicts the choice of $E$. We conclude that $\sigma_s$ is surjective and thus invertible since $\sigma_t \circ \sigma_s = \text{Id}$. The proof of the theorem is complete. \qed

Using Lemma 2.4 and Lemma 3.1 we see that Definition 10.1 and Theorem C can be easily generalized, *mutatis mutandis*, to finitely generated group universes.

11. Stable reversibility and direct finiteness of ANUCA

**Theorem 11.1.** Let $M$ be a finite subset of a countable group $G$. Let $A$ be a finite set and let $S = A^{4M}$. Let $s, t \in S^G$ and suppose that $\sigma_t \circ \sigma_s = \text{Id}$. Then $\sigma_s$ is stably injective. Moreover, for every $p \in \Sigma(s)$, there exists $q \in \Sigma(t)$ such that $\sigma_q \circ \sigma_p = \text{Id}$. In particular, $\sigma_s$ is stably reversible.

**Proof.** Let $p \in \Sigma(s)$ then there exists a sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $G$ such that $\lim_{n \to \infty} g_n s = p$. Since $M$ and $A$ are finite, $S$ is also finite.
and it follows that $S^G$ is compact by Tychonoff’s theorem. Hence, up to passing to a subsequence, we can suppose without loss of generality that $\lim_{n \to \infty} g_n t = q$ for some $q \in S^G$.

We claim that $\sigma_q \circ \sigma_p = \text{Id}$. Indeed, let $x \in A^G$ and let $E \subset G$ be a finite subset. Since $\lim_{n \to \infty} g_n s = p$ and $\lim_{n \to \infty} g_n t = q$, we can find $n_0 \in \mathbb{N}$ large enough so that for all $n \geq n_0$, we have $(g_n s)|_{EM} = p|_{EM}$ and $(g_n t)|_E = q|_E$.

Note that $\sigma_{g_n t} \circ \sigma_{g_n s} = \text{Id}$ by Lemma 6.1 as $\sigma_1 \circ \sigma_s = \text{Id}$ by hypothesis. Consequently, we infer from the formula (3.1) and (3.2) that:

$$\sigma_q(\sigma_p(x))|_E = f_{E,q|_E}^{-1}\left( f_{E,\{s\}|_E}^{+(G\backslash\{s\})|_E}(x|_{EM^2}) \right)$$

$$= f_{E,(g_n t)|_E}^{+(G\backslash\{g_n t\})|_E}(x|_{EM^2})$$

$$\sigma_{g_n q}(\sigma_{g_n s}(x))|_E = x|_E.$$

Since $E$ is arbitrary, we deduce that $\sigma_q(\sigma_p(x)) = x$ for all $x \in A^G$ and the claim is proved. Since clearly $q \in \Sigma(t)$, the last statement of the theorem is proved. In particular, we find that $\sigma_s$ is stably injective by definition and the proof is therefore complete. □

Combining Theorem 11.1 with Theorem 7.1, we can now give the proof of Theorem A in the Introduction which gives various characterizations of the reversibility of ANUCA.

Proof of Theorem A. It is clear from the definition of stable reversibility that (ii) $\implies$ (i). Theorem 11.1 tells us that (i) $\implies$ (iii). Finally, the implication (iii) $\implies$ (ii) follows from Theorem 7.1. □

12. Pointwise uniform post-surjectivity

Given a map $\tau: A^G \to A^G$ where $G$ is a group and $A$ is a set. Then $\tau$ is pre-injective if $\tau(x) = \tau(y)$ implies $x = y$ whenever $x, y \in A^G$ are asymptotic, and $\tau$ is post-surjective if for all $x, y \in A^G$ with $y$ asymptotic to $\tau(x)$, we can find $z \in A^G$ asymptotic to $x$ such that $\tau(z) = y$. We shall see in Corollary 12.3 that every post-surjective ANUCA is automatically surjective.

The dual-surjectivity version of Gottschalk’s conjecture was introduced recently by Capobianco, Kari, and Taati in [2] and states that if $G$ is a group and $A$ is a finite set, then every post-surjective CA must be pre-injective. Moreover, the authors settled in the same paper [2] the dual-surjectivity conjecture for CA over sofic universes.

We establish the following pointwise uniform post-surjectivity of ANUCA.

Lemma 12.1. Let $M$ be a finite subset of a countable group $G$. Let $A$ be a finite set and let $S = A^{G|_M}$. Suppose that $\sigma_s$ is post-surjective for some $s \in S^G$. Then for each $g \in G$, there exists a finite subset $E \subset G$ such that for all $x, y \in A^G$ with $y|_{G \backslash\{s\}} = \sigma_s(x)|_{G \backslash\{g\}}$, there exists $z \in A^G$ such that $\sigma_s(z) = y$ and $z|_{G \backslash\{g\}} = x|_{G \backslash\{g\}}$. 

Proof. To simplify the notation, we will only treat the case when \( g = 1_G \) since the general case is similar.

Without loss of generality, we can clearly suppose that \( 1_G \in M \) and \( M \) is symmetric, i.e., \( M = M^{-1} \).

Since \( G \) is countable by hypothesis, we can find an increasing sequence \((E_n)_{n \in \mathbb{N}}\) of finite subsets of \( G \) such that \( G = \bigcup_{n \in \mathbb{N}} E_n, \ 1_G \in E_0, \) and moreover, for every \( n \in \mathbb{N} \):

\[
E_n M \subset E_{n+1}.
\]

We suppose on the contrary that there does not exist a finite subset \( E \subset G \) with the property described in the conclusion of the lemma. It follows that there exists for every \( n \in \mathbb{N} \) two configurations \( x_n, y_n \in A^G \) with \( y_n|_{G \setminus \{1_G\}} = \sigma_s(x_n)|_{G \setminus \{1_G\}} \) but for all \( z \in A^G \) satisfying \( z|_{G \setminus E_n} = x_n|_{G \setminus E_n} \), one has \( \sigma_s(z) \neq y_n \).

Note that we have \( G \setminus E_m \subset G \setminus E_n \) for all \( m \geq n \geq 0 \). Hence, by the choice of \( x_n \) and \( y_n \), we find that for every \( m \geq n \geq 0 \), we have \( \sigma_s(z) \neq y_m \) for every \( z \in A^G \) with \( z|_{G \setminus E_n} = x_n|_{G \setminus E_n} \).

On the other hand, the space \( A^G \) is compact by Tychonoff’s theorem. Therefore, we can, up to passing to a subsequence, suppose without loss of generality that \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \) for some \( x, y \in A^G \).

By passing the relation \( y_n|_{G \setminus \{1_G\}} = \sigma_s(x_n)|_{G \setminus \{1_G\}} \) to the limit when \( n \) goes to \( \infty \), we deduce that

\[
y|_{G \setminus \{1_G\}} = \sigma_s(x)|_{G \setminus \{1_G\}}
\]

since \( \sigma_s \) is continuous by Lemma 4.3. Hence, \( y \) is asymptotic to \( \sigma_s(x) \).

It follows from the post-surjectivity of \( \sigma_s \) that there exists \( w \in A^G \) such that \( w \) is asymptotic to \( x \) and \( \sigma_s(w) = y \). In particular, we can find \( k \in \mathbb{N} \) such that \( w|_{G \setminus E_k} = x|_{G \setminus E_k} \).

As \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \), we can choose \( m > k \) such that \( x_m|_{E_k M^2} = x|_{E_k M^2} \) and \( y_m|_{E_k M^2} = y|_{E_k M^2} \).

Consequently, we have \( w|_{E_k M^2 \setminus E_k} = x_m|_{E_k M^2 \setminus E_k} \). Therefore, we obtain a well-defined configuration \( z \in A^G \) by setting

\[
(12.1) \quad z|_{E_k M^2} = w|_{E_k M^2}, \quad z|_{G \setminus E_k} = x_m|_{G \setminus E_k}.
\]

Since \( M \) is a symmetric memory set of \( \sigma_s \) and \( 1_G \in E_k M \), we infer the formula (3.1)-(3.2), and the choice of \( x_m, y_m \) that:

\[
\sigma_s(z)|_{G \setminus E_k M} = \sigma_s(x_m)|_{G \setminus E_k M} = y_m|_{G \setminus E_k M}.
\]

Similarly, we deduce from the relations (3.1)-(3.2), \( y_m|_{E_k M^2} = y|_{E_k M^2} \), and \( \sigma_s(w) = y \) that:

\[
\sigma_s(z)|_{E_k M} = \sigma_s(w)|_{E_k M} = y|_{E_k M} = y_m|_{E_k M}.
\]
Hence, we can conclude that $\sigma_s(z) = y_m$. However, $z|_{G\setminus E_m} = x_m|_{G\setminus E_m}$ by the relation (12.1) as $E_k \subset E_m$. Therefore, we obtain a contradiction to the choice of $x_m, y_m$. The proof is thus complete. \hfill $\Box$

As a consequence of Lemma 12.1 in the case of CA, we obtain the following result proved in [2, Lemma 1].

**Corollary 12.2.** Let $G$ be a countable group and let $A$ be a finite set. Suppose that $\tau: A^G \to A^G$ is a post-surjective CA. Then there exists a finite subset $E \subset G$ such that for all $x, y \in A^G$ with $y|_{G\setminus \{1_G\}} = \sigma_s(x)|_{G\setminus \{1_G\}}$, there exists $z \in A^G$ such that $\sigma_s(z) = y$ and $z|_{G\setminus E} = x|_{G\setminus E}$.

**Proof.** It is a direct consequence of Lemma 12.1 and the fact that $\tau$ is a $G$-equivariant CA. \hfill $\Box$

Using Lemma 12.1 instead of [2, Corollary 2], we see easily that the exact same proof, *mutatis mutandis*, of [2] shows that all pre-injective post-surjective ANUCA with finite memory are invertible.

**Theorem 12.3.** Let $G$ be a countable group and let $A$ be a finite set. Suppose that $\tau: A^G \to A^G$ is a post-surjective pre-injective ANUCA with finite memory. Then $\tau$ is invertible. \hfill $\Box$

13. Stably post-surjective ANUCA

As for stable injectivity, we introduce the following notion of stably post-surjective ANUCA.

**Definition 13.1.** Let $M$ be a subset of a group $G$ and let $S = A^A M$ where $A$ is a set. Given $s \in S^G$, the ANUCA $\sigma_s$ is said to be stably post-surjective if for every $p \in \Sigma(s)$, the ANUCA $\sigma_p$ is post-surjective.

With the above notation, suppose that $\sigma_s$ is stably post-surjective for some $s \in S^G$. It is then immediate that $\sigma_p$ is also stably post-surjective for every $p \in \Sigma(s)$. It suffices to observe that $\Sigma(p) \subset \Sigma(s)$ (see the proof of Lemma 5.1).

Observe also that for every constant configuration $c \in S^G$, the subset $\Sigma(s) \subset S^G$ reduces to the single configuration $c$. Consequently, it follows immediately from the above definition that every post-surjective CA is stably post-surjective.

We now state and prove the fundamental uniform post-surjectivity property of stably post-surjective ANUCA.

**Lemma 13.2 (Uniform post-surjectivity).** Let $M$ be a finite subset of a countable group $G$. Let $A$ be a finite set and let $S = A^A M$. Suppose that $\sigma_s$ is stably post-surjective for some $s \in S^G$. Then there exists a finite subset $E \subset G$ such that for all $g \in G$ and $x, y \in A^G$ with $y|_{G\setminus \{1_G\}} = \sigma_s(x)|_{G\setminus \{1_G\}}$, there exists $z \in A^G$ such that $\sigma_s(z) = y$ and $z|_{G\setminus E} = x|_{G\setminus E}$. 
Proof. By Lemma 12.1, we can find a finite subset $E$ such that for all $x, y \in A^G$ with $y|_{G\setminus\{1_G\}} = \sigma_s(x)|_{G\setminus\{1_G\}}$, there exists $z \in A^G$ such that $\sigma_s(z) = y$ and $z|_{G\setminus E} = x|_{G\setminus E}$.

We can clearly suppose that $1_G \in M$ and $M$ is symmetric, i.e., $M = M^{-1}$. Moreover, since $G$ is countable, we can find an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite subsets of $G$ such that $G = \cup_{n \in \mathbb{N}} E_n$, $1_G \subset E_0$, and for every $n \in \mathbb{N}$:

$$E_n M \subset E_{n+1}.$$  

We suppose on the contrary that for every $n \in \mathbb{N}$, there exists $g_n \in G$ and $u_n, v_n \in A^G$ such that $v_n|_{G\setminus\{g_n\}} = \sigma_s(u_n)|_{G\setminus\{g_n\}}$ but for all $z \in A^G$ with $\sigma_s(z) = v_n$, one must have $z|_{G\setminus\{g_n E_n \}} \neq u_n|_{G\setminus\{g_n E_n \}}$.

For $n \in \mathbb{N}$, let us denote $s_n = g_n^{-1} s$, $x_n = g_n^{-1} u_n$, and $y_n = g_n^{-1} v_n$. Then Lemma 5.1 implies that $y_n|_{G\setminus\{1_G\}} = \sigma_s(x_n)|_{G\setminus\{1_G\}}$ and for $z \in A^G$ with $\sigma_s(z) = y_n$, one has $z|_{G\setminus E_n} \neq x_n|_{G\setminus E_n}$.

Since $A^G$ and $S^G$ are compact by Tychonoff’s theorem, we can pass to a subsequence and suppose without loss of generality that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$ for some $x, y \in A^G$ and $\lim_{n \to \infty} s_n = p$ for some $p \in \Sigma(s)$.

Consequently, by taking the limit of the relation $y_n|_{G\setminus\{1_G\}} = \sigma_s(x_n)|_{G\setminus\{1_G\}}$ when $n$ goes to infinity, we deduce from Lemma 4.3 that

$$(13.1) \quad y|_{G\setminus\{1_G\}} = \sigma_p(x)|_{G\setminus\{1_G\}}.$$  

It follows from the stable post-surjectivity of $\sigma_s$ that $\sigma_p$ is also post-surjective. Hence, we deduce from (13.1) that there exists a configuration $w \in A^G$ asymptotic to $x$ such that $\sigma_p(w) = y$. In particular, we have $w|_{G\setminus E_k} = x|_{G\setminus E_k}$ for some $k \in \mathbb{N}$.

Now choose $m > k$ large enough so that

$$x_m|_{E_k M^2} = x|_{E_k M^2}, \quad y_m|_{E_k M} = y|_{E_k M}, \quad s_m|_{E_k M} = p|_{E_k M}.$$  

It follows that $x_m|_{E_k M^2 \setminus E_k} = w|_{E_k M^2 \setminus E_k}$ and we can define $z \in A^G$ by:

$$z|_{E_k M^2} = w|_{E_k M^2}, \quad z|_{G\setminus E_k} = x_m|_{G\setminus E_k}.$$  

Consequently, we deduce from the formula (3.1)-(3.2) that:

$$\sigma_s_m(z)|_{G\setminus E_k} = \sigma_s_m(x_m)|_{G\setminus E_k}$$  

(as $z|_{G\setminus E_k} = x_m|_{G\setminus E_k}$) (as $1_G \subset E_k$).

On the other hand, we find that:

$$\sigma_s_m(z)|_{E_k M} = \sigma_s_m(w)|_{E_k M}$$  

(as $z|_{E_k M^2} = w|_{E_k M^2}$)  

$$= \sigma_p(w)|_{E_k M}$$  

(as $s_m|_{E_k M} = p|_{E_k M}$)  

$$= y|_{E_k M}$$  

(as $\sigma_p(w) = y$)

$$= y_m|_{E_k M}.$$  

Therefore, we deduce that $\sigma_s_m(z) = y_m$. However, since by construction $z|_{G\setminus E_m} = x_m|_{G\setminus E_m}$ as $E_k \subset E_m$, we obtain a contradiction to the choice of $x_m$ and $y_m$. The proof of the lemma is thus complete. \qed
The next results imply that the above uniform post-surjectivity is a stable property when passing to the limit of the configurations of local defining maps.

**Theorem 13.3.** Let $M$ be a finite subset of a countable group $G$. Let $A$ be a finite set and let $s \in S^G$ where $S = A^M$. Suppose that $\sigma_s$ is stably post-surjective. Then there exists $E \subset G$ finite such that for all $p \in \Sigma(s)$, $g \in G$, and $x, y \in A^G$ with $y|_{G \setminus \{g\}} = \sigma_p(x)|_{G \setminus \{g\}}$, there exists $z \in A^G$ such that $\sigma_p(z) = y$ and $z|_{G \setminus gE} = x|_{G \setminus gE}$.

**Proof.** Let $E \subset G$ be the subset given by Lemma 13.2 and let $p \in \Sigma(s)$. Then we can find a sequence $(g_n)_{n \in \mathbb{N}}$ in $G$ such that $\lim_{n \to \infty} g_ns = p$.

Let $g \in G$ and $x, y \in A^G$ such that $y|_{G \setminus \{g\}} = \sigma_p(x)|_{G \setminus \{g\}}$. For every $n \in \mathbb{N}$, we define $y_n \in A^G$ be setting

$$y_n|_{G \setminus \{g\}} = \sigma_{g_n s}(x)|_{G \setminus \{g\}}, \quad y_n(g) = y(g)$$

Since $\lim_{n \to \infty} g_n s = p$, we deduce from Lemma 4.1 that:

$$\lim_{n \to \infty} y_n|_{G \setminus \{g\}} = \lim_{n \to \infty} \sigma_{g_n s}(x)|_{G \setminus \{g\}} = \sigma_p(x)|_{G \setminus \{g\}} = y|_{G \setminus \{g\}}.$$

In particular, we have $\lim_{n \to \infty} y_n = y$ since $y_n(g) = y(g)$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. We infer from Lemma 5.1 that $(g_n \sigma_s(g_n^{-1} x))|_{G \setminus \{g\}} = y_n|_{G \setminus \{g\}}$. It follows that $\sigma_s(g_n^{-1} x)|_{G \setminus \{g \setminus g_n^{-1} g\}} = (g_n^{-1} y_n)|_{G \setminus \{g \setminus g_n^{-1} g\}}$. Hence, by Lemma 13.2 we can find $t \in A^G$ such that $t|_{G \setminus g_n^{-1} gE} = (g_n^{-1} x)|_{G \setminus g_n^{-1} gE}$ and $\sigma_s(t) = g_n^{-1} y_n$.

Let $z_n = g_n t$ then we deduce again from Lemma 5.1 that

$$z_n|_{G \setminus gE} = x|_{G \setminus gE}, \quad \sigma_{g_n s}(z_n) = y_n.$$

Since $A^G$ is compact, we can pass to a subsequence and suppose without loss of generality that $\lim_{n \to \infty} z_n = z$ for some $z \in A^G$. It follows from Lemma 13.2 that $z|_{G \setminus gE} = x|_{G \setminus gE}$.

Moreover, since $\lim_{n \to \infty} y_n = y$, we have $\sigma_p(z) = y$ by Lemma 4.3. This proves that $E$ satisfies the required condition in the conclusion of the theorem and the proof is thus complete. \hfill $\square$

14. **Counter-examples**

We present a simple example of an ANUCA which is injective but not stably injective, not surjective, and not reversible. Hence, we obtain counterexamples to Theorem A and Theorem 7.1 when we replace the stable injectivity hypothesis by the weaker injectivity hypothesis.

**Example 14.1.** Let $G = \mathbb{Z}$ and let $A = \{0, 1\}$. Let $M = \{-1, 0, 1\}$ and consider the functions $f, g: A^M \to A$ defined for all $(u, v, w) \in A^M$ by the following formula:

$$f(u, v, w) := w, \quad g(u, v, w) := u + v \pmod{2}.$$
Let $S = A^A$ and let $p, q \in S^\mathbb{Z}$ where $p(n) = f$ and $q(n) = g$ for all $n \in \mathbb{Z}$. For every $k \in \mathbb{Z}$, we define $s_k \in S^\mathbb{Z}$ by setting $s_k(n) = f$ if $n \leq k$ and $s(n) = g$ if $n \geq k + 1$.

Denote $s = s_0$ then it is clear that $\Sigma(s) = \{s_k : k \in \mathbb{Z}\} \cup \{p, q\}$. We claim that the ANUCA $\sigma_s : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$ is injective but not stably injective.

Indeed, suppose that $\sigma_s(x) = \sigma_s(y)$ for some $x, y \in A^\mathbb{Z}$. Then we deduce from (14.1) that $x(n) = y(n)$ for all $n \leq 1$ and $x(n) + x(n + 1) = y(n) + y(n + 1)$ (mod 2) for all $n \geq 0$. It follows immediately that $x(n) = y(n)$ for all $n \in \mathbb{Z}$.

We conclude that $x = y$ and thus $\sigma_s$ is injective. On the other hand, observe that $\sigma_q$ is not injective since $\sigma_q(0^\mathbb{Z}) = \sigma_q(1^\mathbb{Z}) = 0^\mathbb{Z}$. As $q \in \Sigma(s)$, we conclude that $\sigma_s$ is not stably injective.

We claim that $\sigma_s$ is not surjective. Indeed, let $c \in A^\mathbb{Z}$ be the configuration given by $c(n) = 0$ for all $n \leq 0$ and $c(n) = 1$ for all $n \geq 1$. Suppose on the contrary that $\sigma_s(x) = c$ for some $x \in A^\mathbb{Z}$. It follows from the definition of $s$ and (14.1) that $x(n) = c(n - 1) = 0$ for all $n \geq 1$ and $x(0) + x(1) = c(1)$. Hence $c(1) = 0$ and we obtain a contradiction.

Finally, let $x \in A^\mathbb{Z}$ and put $y = \sigma_s(x)$. By a direct induction, we infer from the relations $y(n) = x(n + 1)$ for $n \leq 0$ and $y(n) = x(n) + x(n - 1)$ (mod 2) for $n \geq 1$ that for all $n \geq 2$, we have:

$$x(n) = y(n) - y(n - 1) + \cdots + (-1)^{n+1}y(-1) \pmod{2}$$

and that $y|_{n \geq 2}$ can take any value in $A^{n \geq 2}$ (in fact, the only requirement for $y$ is that $y(1) = y(0) + y(-1)$). Hence, $x(n)$ must depend on $y(n), \ldots, y(2)$ for all $n \geq 2$. Consequently, if $\tau : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$ is an ANUCA such that $\tau \circ \sigma_s = \text{Id}$ then $\tau$ cannot have finite memory and we conclude that $\sigma_s$ is not reversible (see Section 7).

The next similar example shows that unlike CA, a bijective ANUCA is not necessarily reversible or stably injective.

**Example 14.2.** Let $G = \mathbb{Z}$ and let $A = \{0, 1\}$. Let $M = \{-1, 0\}$ and consider the functions $f, g : A^M \rightarrow A$ defined for all $(u, v, w) \in A^M$ by the following formula:

$$f(u, v) := v, \quad g(u, v) := u + v \pmod{2}.$$  

Let $S = A^A$ and for $k \in \mathbb{Z} \cup \{\pm \infty\}$, we define $s_k \in S^\mathbb{Z}$ by setting $s_k(n) = f$ if $n \leq k$ and $s(n) = g$ if $n \geq k + 1$. Denote $s = s_0$ then we claim that $\sigma_s : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$ is bijective but it is not reversible and thus not stably injective by Theorem 7.1.

Indeed, let $x \in A^\mathbb{Z}$ and $y = \sigma_s(x)$. As in Example 14.1, we infer from (14.3) that $x(n) = y(n)$ for $n \leq 0$ and we have for all $n \geq 1$:

$$x(n) = y(n) - y(n - 1) + \cdots + (-1)^ny(0) \pmod{2}.$$  

It follows immediately that $\sigma_s$ is bijective but it is not reversible since $x(n)$ depends on $y(0), \ldots, y(n)$ for all $n \geq 1$. The fact that $\sigma_s$ is not stably injective can also be seen directly by checking that $\sigma_{s - \infty}$ is not injective.
Finally, we present an example of a very simple one-dimensional stably injective ANUCA which is not surjective.

**Example 14.3.** Let $G = \mathbb{Z}$ and let $A = \{0, 1\}$. Let $M = \{-1, 0, 1\}$ and consider the functions $f, g, h : A^M \to A$ defined for all $(u, v, w) \in A^M$ by the following formula:

\[
(14.3) \quad f(u, v, w) := w, \quad g(u, v, w) := u, \quad h(u, v, w) = v \pmod{2}.
\]

Let $S = A^M$ and consider $s \in S$ defined by $s(n) = f$ if $n \leq -1$, $s(n) = h$ if $n = 0$, and $s(n) = g$ if $n \geq 1$. Then $\sigma_s$ is injective since $\sigma_s(x) = y$ implies that $x(n) = y(n-1)$ for $n \leq 0$ and $x(n) = y(n+1)$ for $n \geq 0$.

Let $p, q \in S$ where $p(n) = f$ and $q(n) = g$ for all $n \in \mathbb{N}$ then it is clear that $\sigma_p$ and $\sigma_q$ are injective. Since $\Sigma(s) = \{p, q, s\}$, we deduce that $\sigma_s$ is stably injective. On the other hand, $\sigma_s$ is not surjective since we can check directly that

\[
\text{Im } \sigma_s = \{y \in A^\mathbb{Z} : y(-1) = y(0) = y(1)\}.
\]

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