SOME REMARKS ABOUT INTRINSIC PARITY IN RYDER’S DERIVATION OF
THE DIRAC EQUATION*

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Ryder in one of the most nowadays widely used textbooks on quantum field theory [5, 6], has recently proposed an alternative derivation of the Dirac equation. His proposal contains, in essence, the steps followed to derive the plane wave solutions of the free Dirac equation in an arbitrary frame from the plane wave solutions in the rest frame, but in the opposite direction [3]. We consider that the underlying idea in his derivation has a significant pedagogical value, but it omits important conceptual features, as e.g., the physical meaning of the negative energies [3] and the relative intrinsic parity of the elementary particles [4]. In this comment we repeat Ryder’s derivation amending such omissions, which becomes a more instructive derivation since it reveals the physical content of the Dirac equation. Moreover, we explicitly show the few physical hypotheses which are enough to uniquely determine it.

Let us outline Ryder’s argument which corresponds to \( \epsilon = 1 \) in the following formulas. We are searching for a wave equation for spin 1/2 particles, so let us consider Weyl chiral spinors. It is well known that right and left handed Weyl spinors are two spin 1/2 irreducible representations of the Lorentz group interchanged under parity transformation, which transform under boosts as [3, 4]

\[
\varphi_R(\vec{v}) = \exp[\pm \frac{1}{2} \vec{\sigma} \vec{n} \phi] \varphi_R(0)
\]

where \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices, \( \vec{v} = v \vec{n} \) is the velocity and \( \phi \) the additive parameter of the boost (\( \tanh \phi = v; \ c = 1 \)). Supposing now that \( \varphi_R(0) \) are the spinors corresponding to the frame where a particle (or an antiparticle) is at rest. Using trigonometric algebra we can rewrite (1) as

\[
\varphi_R(\epsilon E_p, \epsilon \vec{p}) = \left[ \sqrt{\frac{\gamma + 1}{2}} \pm \frac{\vec{\sigma} \cdot \vec{p}}{P} \sqrt{\frac{\gamma - 1}{2}} \right] \varphi_R(\epsilon m, 0),
\]

where \( \vec{p} = \epsilon m \gamma \vec{v}, \ m > 0, \ \gamma = (1 - v^2)^{-\frac{1}{2}} = E_p/m, \) and \( \epsilon = +1 \) for particles and \( \epsilon = -1 \) for antiparticles.

The well known relativistic energy-momentum relation for particles and antiparticles,

\[
E = \epsilon E_p = \epsilon \sqrt{\vec{p}^2 + m^2},
\]

where \( E \) represents the energy of both particles and antiparticles [we assume that the antiparticles has negative energy; see the comment (a) below], allows to put Eq. (2) in the form,

\[
\varphi_R(\epsilon E_p, \epsilon \vec{p}) = \frac{E_p + m \pm \vec{\sigma} \cdot \vec{p}}{\sqrt{2m(E_p + m)}} \varphi_R(\epsilon m, 0).
\]

Let us assume for a moment that, in the rest frame,

\[
\varphi_R(\epsilon m, 0) = \epsilon \varphi_L(\epsilon m, 0).
\]

(We will later discuss the physical meaning of this statement.) Then, using the algebra of Pauli matrices we can immediately verify that

\[
\varphi_R(\epsilon E_p, \epsilon \vec{p}) = \frac{E_p \pm \vec{\sigma} \cdot \vec{p}}{m} \varphi_R(\epsilon E_p, \epsilon \vec{p}).
\]

The system of equations (6) is nothing else than another way of writing the “Dirac equation,” as was originally proposed by Weyl [7],

\[
(\epsilon E_p + \vec{\sigma} \cdot \vec{p}) \varphi_L = m \varphi_R,
\]

(7)

\[
(\epsilon E_p - \vec{\sigma} \cdot \vec{p}) \varphi_R = m \varphi_L.
\]

This can be easily verified, defining the \( \gamma \) matrices and the four-spinor \( \psi \) by

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & -i \vec{\sigma} \\ i \vec{\sigma} & 0 \end{pmatrix}, \quad \psi = (\varphi_R, \varphi_L),
\]

and rewriting Eqs. (6) and (7) in a more standard way,

\[
(\gamma^\mu p_\mu - m) \psi(p_\mu) = 0,
\]

(9)

where \( p_\mu = (E, -\vec{p}) \) [8]. However, strictly speaking, Eq. (10) is not the Dirac equation yet, but it could be obtained, in a heuristic way, replacing \( p_\mu \) by the “operator” \( \hat{p}_\mu = i \partial_\mu \) (\( \hbar = 1 \)) (see Ref. [3]),

\[
(i \gamma^\mu \partial_\mu - m) \psi_D(x^\mu) = 0,
\]

(11)

where \( \psi_D(x^\mu) = \psi(p_\mu) e^{-ip_\mu x^\mu} \). Nevertheless, this result cannot be rigorously reached, unless one considers the inhomogeneous Lorentz group from the very beginning.

Let us now examine the hypotheses which lead to Eq. (10) in detail.

(a) We have assumed the standard formulas of classical relativistic dynamics [e.g., Eq. (4)], associating particles and antiparticles with positive and negative energy values, according to St"uckelberg-Feynman interpretation [10].

(b) We have also assumed that the desired equation must describe a spin 1/2 system, so we have chosen spin 1/2 representations of the homogeneous Lorentz group. In this way we have derived the Weyl equations (6) and (7). At this point from these equations, making the replacement \( p_\mu \) by \( \hat{p}_\mu \), we can build a first order equation [Eq. (11)] for a four-components wave function \( \psi_D \) following Dirac [11], which is the direct sum of the two Weyl representations [Eq. (6)]. Alternatively, following Feynman and Gell-Mann [12], we can combine Eqs. (6) and (7) to form a second order equation, in which is necessary to consider only one of the chiral two-components wave functions.

(c) We want finally to remark that Eq. (3) is equivalent to postulating that the relative intrinsic parity [4] of
particles and antiparticles is opposite. This fact means that, for instance, considering that $\varphi_\mu (e, \vec{p})$ represents the state of the spin $1/2$ system, we can choose

$$P\varphi_\mu (e, 0) = e\varphi_\mu (e, 0),$$

where $P$ is the intrinsic parity operator. However, we know that

$$P\varphi_\mu (e, 0) = \varphi_\mu (e, 0).$$

Then, comparing the two sets of equations we obtain Eq. (12). (It is easy to check that in each set of equations there is a redundant one.)

To conclude, we have shown that the derivation exposed above highlights the essential content of the Dirac equation [hypotheses (a), (b), and (c)]. This fact contrasts with the standard derivation introduced in classical textbooks [2], which follows Dirac’s original idea [3] of linearizing the Klein-Gordon equation, i.e., to obtain a Schrödinger equation putting on an equal footing both the temporal and the spatial derivatives by relativistic invariance considerations [3]. However, as it was later admitted [2], the linearization condition does not uniquely lead to spin $1/2$ systems. In Dirac’s original scheme, the appearance of the spin $1/2$, the negative energies (which are common to all spins), and the relative intrinsic parity, i.e., conditions (a), (b), and (c), came as unexpected consequences [2]. (This is what brought about the broad impact of Dirac’s work.) But now, a retrospective view, based on the experimental facts, allows us to uniquely obtain the desired relativistic equation for a massive, spin $1/2$ system. Here mainly lies the already mentioned pedagogical value of Ryder’s modified derivation of the Dirac equation that we have revised in this comment.

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[1] L. Ryder, Quantum Field Theory (Cambridge University, Cambridge, 1988), Chap. 2.
[2] See, e.g., A. Messiah, Mécanique Quantique (Dunod, Paris, 1964), Chap. 20.
[3] Ryder’s argument is based on positive energy states. Negative energies are put in ad hoc at the end of his derivation.
[4] This fact was first noticed by Berestetskii in 1948. See, V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, Relativistic Quantum Theory (Pergamon, New York, 1971), Sec. 27.
[5] An extensive treatment of the representations of the Lorentz group can be found in, I.M. Gel’fand, R.A. Minlos, and Z.Ya. Shapiro, Representation of the Rotation and Lorentz Groups and Their Applications (Pergamon, New York, 1963). See also, N.A. Doughty, Electrodynamics and Relativistic Wave Equations: An Introduction to Tensors, Spinors and Lagrangian Fields (Addison-Wesley, Reading, 1988), Chaps. 17 and 18. The traditional derivation of relativistic wave equations from representations of the Lorentz group can be seen in, V. Bargmann and E.P. Wigner, “Group Theoretical Discussion of Relativistic Wave Equations,” Proc. Natl. Acad. Sci. (USA) 34, 211-223 (1948). The readers should be aware that the Bargmann-Wigner formalism is restricted to positive energy states. A refined mathematical treatment can be found in, A.O. Barut and R. Rączka, Theory of Group Representations and Applications (World Scientific, Singapore, 1986), Chap. 21.
[6] The reader who desires to avoid group theoretical considerations can find an adequate introduction to spinors in, W.L. Bade and H. Jehle, “An Introduction to Spinors,” Rev. Mod. Phys. 25, 714-728 (1953), and in, C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation (Freeman, San Francisco, 1973), Chap. 41.
[7] H. Weyl, The Theory of Groups and Quantum Mechanics (Dover, New York, 1931), Chap. 4.
[8] As we have just pointed out in Ref. [3], in Ryder’s original derivation $p_0 = E_\mu$, so Eq. (13) would only describe a system with positive energy in this case.
[9] We would like to remark that in Dirac’s work the symmetry between space and time is incomplete because time is a $c$ number and the spatial position an operator. Moreover, as a consequence of this fact, the identification of $p_\mu$ with $(i\partial_\tau, i\partial_x)$ is not trivial, since the differential operator $-i\partial_\tau$ corresponds to the coordinate representation of the momentum operator $\vec{p}$, while the differential operator $i\partial_x$ is not associated with a particular representation of any operator. This fact has also been overlooked in Ref. [2]. A complete framework for re-establishing this symmetry can be found in, J.P. Aparicio, F.H. Gaioli, and E.T. Garcia Alvarez, “Formulación de la Mecánica Cuántica Relativista parametrizada con un tiempo propio,” Anales AFA 4 (1993) (in press); “Proper Time Derivatives in Quantum Mechanics,” Phys. Rev. A 51, 96-103 (1995), and references therein.
[10] E.C.G. Stückelberg, “La signification du temps propre en mécanique ondulatoire,” Helv. Phys. Acta 14, 322-323 (1941); “La mécanique du point matériel en théorie de relativité et en théorie des quanta,” 15, 23-37 (1942). See also, R.P. Feynman, “The Theory of Positrons,” Phys. Rev. 76, 749-759 (1949).
[11] P.A.M. Dirac, “The Quantum Theory of electron,” Proc. R. Soc. London, Ser. A 117, 610-624 (1928). For a historical review of Dirac’s work see also, D.F. Moyer, “Origins of Dirac’s Electron, 1925-1928,” Am. J. Phys. 49, 944-949 (1981).
[12] R.P. Feynman and M. Gell-Mann, “Theory of Fermi interaction,” Phys. Rev. 109, 193-198 (1958).
[13] See, for example, J.D. Björken and S.D. Drell, Relativistic Quantum Mechanics (Mc Graw-Hill, New York, 1964), Chap. 1.
[14] R.J. Duffin, “On the Characteristic Matrices of Covariant Systems,” Phys. Rev. 54, 1114-1114 (1938). N. Kemmer, “The Particle Aspect of Meson Theory,” Proc. R. Soc. London, Ser. A 173, 91-116 (1939); H. Fesbach, “Relativistic Wave Equations,” Phys. Rev. 98, 801-802 (1955).
[15] P.A.M. Dirac, Recollections of an Exciting Era, in Proceedings of the International School of Physics “Enrico Fermi,” Course 17, History of Twentieth Century Physics, ed. C. Weiner (Academic, New York, 1977), pp. 109-146. P.A.M. Dirac, “Pretty Mathematics,” Int. J. Theor. Phys. 21, 603-605 (1982).