Time evolution of rapidly rotating stratified neutron stars

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Abstract. We study the time evolution of non-axisymmetric oscillations of rapidly rotating neutron stars. The rotation is assumed uniform and the stellar models have composition gradients that generate gravity restored oscillation modes. We work in linear Newtonian perturbation theory and in Cowling approximation, i.e. we neglect the gravitational potential perturbations. We study for the first time the behaviour of the inertia-gravity modes in fast spinning stars, and carry out a detailed comparison of non-stratified and stratified models. In particular, we show that each g-mode becomes rotation-dominated, i.e. approaches a particular inertial mode, as the rotation rate of the star is increased.

1. Introduction

The spectral properties of stellar oscillations could help us to clarify many aspects of the neutron star physics, as every physical property and configuration of the star could be related to particular features of the spectrum. In particular, asteroseismology of the electromagnetic and gravitational radiation may be very helpful for constraining the equation of state and understand the state of matter at supernuclear densities. In order to facilitate the signal detection and its interpretation, we must still improve the theoretical models and clarify the nature of various stellar pulsation features.

In this contribution, we report our study of the non-axisymmetric oscillations of rapidly rotating stars [1]. Here, we focus on stratified neutron stars and investigate the properties of the inertia-gravity modes, i.e. pulsations that are restored both by buoyancy and Coriolis force. When buoyancy is dominant, as in the non-rotating limit, inertia-gravity modes exhibit the typical features of gravity modes. Instead in the rapidly rotating limit, the Coriolis force overcomes buoyancy and the modes are very similar to the inertial modes of barotropic stars. It is then important to understand better the influence of these two restoring forces on the mode oscillations and in which regions of the parameter space they dominate. For a more detailed analysis of the inertia-gravity modes see [1]. For an analysis of the spectrum of superfluid neutron stars see [2].

We study the non-axisymmetric pulsations in the time domain by adopting the Newtonian linear perturbation theory on an axisymmetric background. The equilibrium configurations are determined by using the self-consistent Hachisu method [3] for polytropic models. We have found very useful to expand the linear perturbations of an axisymmetric background with respect to the azimuthal number $m$. In this ways, the time evolution of non-axisymmetric oscillations becomes a two-dimensional problem.

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Table 1. This table displays the main quantities of our rotating equilibrium configurations. The stellar models are described by a polytropic equation of state with adiabatic index $\Gamma_\beta = 2$. In the first and second columns we show, respectively, the ratio of polar to equatorial axes and the angular velocity of the star. In the third column, the rotation rate is compared to the Kepler velocity $\Omega_K$ that represents the mass shedding limit. The ratio between the rotational kinetic energy and gravitational potential energy $T/|W|$ and the stellar mass are given in the fourth and fifth columns, respectively. All quantities are given in dimensionless units, where $G$ is the gravitational constant, $\rho_c$ represents the central mass density and $R_{eq}$ is the equatorial radius.

| $R_p/R_{eq}$ | $\Omega/\sqrt{G\rho_c}$ | $\Omega/\Omega_K$ | $T/|W| \times 10^{-2}$ | $M/(\rho_c R_{eq}^3)$ |
|--------------|--------------------------|------------------|--------------------------|----------------------|
| 1.000        | 0.000                    | 0.000            | 0.000                    | 1.273                |
| 0.992        | 0.119                    | 0.105            | 0.192                    | 1.261                |
| 0.950        | 0.287                    | 0.260            | 1.170                    | 1.197                |
| 0.900        | 0.403                    | 0.374            | 2.380                    | 1.118                |
| 0.800        | 0.556                    | 0.550            | 4.930                    | 0.956                |
| 0.700        | 0.658                    | 0.710            | 7.570                    | 0.779                |
| 0.600        | 0.717                    | 0.893            | 9.860                    | 0.579                |

2. Neutron star model
We consider Newtonian uniformly rotating stars where matter is composed by three degenerates and non-superfluid components: neutrons $n$, protons $p$ and electrons $e$. The Coulomb interaction is very efficient to lock protons and electrons together [4], forming a neutral charge conglomerate. Furthermore, we assume that the background star is in $\beta$-equilibrium, then the proton fraction $x_p \equiv \rho_p/\rho$ is a function only of the mass density $\rho$. With $\rho_p$ we denote the proton mass density. The equilibrium configuration can be then described by a barotropic equation of state $P = P(\rho)$. For simplicity, we consider a simple polytropic model

$$P = K \rho^{\Gamma_\beta},$$

where $K$ is constant and

$$\Gamma_\beta = \frac{d \log P}{d \log \rho},$$

is the adiabatic index.

The axisymmetric background configurations are determined by using the self-consistent Hachisu method [3] for $\Gamma_\beta = 2$ polytropic models. The properties of the rotating models are given in Table 1.

3. Perturbations
In neutron stars, the mode oscillation period is much shorter than the typical time scales of the weak interaction processes. Therefore, a perturbed fluid element does not have time to reach $\beta$-equilibrium with the surrounding matter within an oscillation period [4]. The composition of the perturbed fluid element can be considered as frozen, i.e. the Lagrangian perturbation of the proton fraction vanishes:

$$\Delta x_p = \delta x_p + \xi \cdot \nabla x_p = 0.$$

The function $\xi$ denotes the Lagrangian displacement of a fluid element, and $\delta x_p$ is the Eulerian perturbation of $x_p$. 

2
The equation of state of the perturbed star is then a function of two variables, and from equation (3) one can write:

\[ \Delta P = \Gamma_f \frac{P}{\rho} \Delta \rho, \tag{4} \]

where

\[ \Gamma_f = \left. \frac{\partial \log P}{\partial \log \rho} \right|_{x_p}, \tag{5} \]

is the adiabatic index for a frozen composition. Equation (4) can be re-written as [1]:

\[ \frac{\delta \rho}{\rho} = \frac{1}{\Gamma_f} \frac{\delta P}{P} - \left( \frac{\Gamma_\beta}{\Gamma_f} - 1 \right) \frac{\delta \chi_p}{\rho}, \tag{6} \]

where we have defined a new perturbation variable,

\[ \delta \chi_p \equiv \frac{\delta x_p}{\rho} \left( \frac{dx_p}{d\rho} \right), \tag{7} \]

and also assumed that \( x_p = x_p(\rho) \). One can show that the difference between \( \Gamma_\beta \) and \( \Gamma_f \) is proportional to the composition gradients [1].

In the Cowling approximation, the non-axisymmetric perturbations of rapidly, and uniformly, rotating stratified neutron stars can be described by five dynamical variables, the perturbations \( \delta P, \delta \chi_p \) and the flux \( f = \rho \mathbf{v} \). In the frame of the rotating background, the perturbed Euler, mass conservation and frozen composition equations can be written as

\[ \partial_t f = -\nabla \delta P - 2(\Omega \times f) + \frac{\nabla \ln P}{\Gamma_f} \delta P + \left( 1 - \frac{\Gamma_\beta}{\Gamma_f} \right) \frac{\nabla P}{\rho} \delta \chi_p, \tag{8} \]

\[ \partial_t \delta P = -\frac{\Gamma_f}{\rho} \nabla \cdot f + \frac{\Gamma_f}{\rho} \left( \frac{\Gamma_f}{\Gamma_\beta} - 1 \right) f \cdot \nabla P, \tag{9} \]

\[ \partial_t \delta \chi_p = -\frac{f \cdot \nabla P}{\Gamma_\beta P}. \tag{10} \]

Equations (8)-(10) reduce to the barotropic perturbation equations by choosing \( \Gamma_\beta = \Gamma_f \).

The details of the numerical code are explained in [1], here we report some main properties. The perturbation equations are evolved in time with a Mac-Cormack algorithm, which is a second order accurate numerical method. A key element of the code is the implementation of the fourth-order Kreiss-Oliger numerical dissipation. This prevents the growth of spurious oscillations.

We impose standard boundary conditions for the perturbation functions. At the stellar centre, we guarantee the regularity of the perturbation equations (8)-(10). On the surface, we set the vanishing of the Lagrangian perturbation of the pressure, i.e. \( \Delta P = 0 \) (see [1] for more details).

4. Results

We study the spectrum of the non-axisymmetric modes by evolving in time the system of equations (8)-(10). We consider generic initial data in order to excite several oscillation modes. The identification of the modes is determined by performing a Fast Fourier Transformation of the perturbation evolution data and using the two-dimensional eigenfunction extraction code developed in [5, 6]. We have been able to properly determine the spectrum of fundamental, pressure modes and inertia-gravity modes in the fast rotating regime. In this context, we discuss the low frequency band of rotating non-barotropic stars, while the high band analysis and further code tests can be found in [1].
Figure 1. This figure displays the variation of the counter rotating $^2 g_1^l$ mode with the adiabatic index $\Gamma_f$ for a sequence of rotating stellar models. The polar-led inertial mode $^3 i_2$ frequencies of the barotropic models with $\Gamma_f = 2$ are shown as circle-solid line. This figure suggests that in rapidly rotating stars the counter rotating $^2 g_1^l$ mode behaves similarly to the barotropic $^3 i_2$ inertial mode. This is expected when the rotation rate is larger than the Brunt-Väisälä frequency $N$, namely $2\Omega > N$, and the Coriolis force starts to dominate over the buoyancy force.

From a local analysis of uniformly rotating stars in the Cowling approximation, we can show that the low frequency modes can be approximately described by the following dispersion relation [7]:

$$\sigma^2 \simeq \frac{N^2 k_\perp^2 + (2\Omega \cdot k)^2}{k^2}, \quad (11)$$

where $N$ is the Brunt-Väisälä frequency that estimates the frequency band of the gravity modes [7], $k$ is the wave vector and $k_\perp$ is its component orthogonal to the apparent gravity. In the barotropic case, $N = 0$, equation (11) describes the inertial waves, while for a non-rotating model it reduces to the usual gravity waves. In stratified and rotating neutron stars, it is thus to be expected that the low-frequency oscillation modes have an hybrid character with gravity dominating at low rotation rates and the Coriolis force taking over at fast stellar spin. It is then natural to call them inertia-gravity waves [7].

In order to study the dependence of the modes on the stellar spin and stratification, we construct a set of equilibrium models by gradually varying the two parameters $\Omega$ and $\Gamma_f$. We can then follow the change of the mode frequency and eigenfunction shape and correctly distinguish it among the other peaks in the spectrum. We consider, for instance, a sequence of rotating models having increasing composition gradients, i.e. increasing values of the parameter $\Gamma_f$. Let us now focus on the counter-rotating $l = m = 2$ g-mode $^2 g_1^l$, and on the $l = 3$ and $m = 2$ barotropic inertial mode $^3 i_2$. As shown in Fig. 1, the g-mode frequency approaches the barotropic inertial mode in the limit of weak stratification and large rotational rate. Generally, when $2\Omega > N$, the Coriolis force is dominating and the $^2 g_1^l$ seems to behave as the barotropic $^3 i_2$ inertial mode. This property is also noted in the mode eigenfunctions [1].

When we extend this study to the other g-modes, we can determine further associations with the barotropic inertial modes. For barotropic models with $\Gamma_\beta = 2$ and stratified stars with $\Gamma_f = 2.05$, we select in Fig. 2 some possible associations for the $m = 2$ gravity and inertial modes. Is it evident that the individual non-barotropic g-modes tend toward specific barotropic inertial modes as the rotation rate of the star is increased.
Figure 2. Frequencies of some selected g-modes (solid lines) for \( \Gamma_f = 2 \) models and polarled inertial modes (dashed lines) of the barotropic \( \Gamma_\beta = \Gamma_f = 2 \) model. All frequencies are measured in the rotating frame. The dominance of the Coriolis force in rapidly rotating stars is evident from this figure. We see that the g-modes are mainly restored by the Coriolis force when \( \Omega / \sqrt{G\rho_c} > 0.3 \), and thus become similar to the barotropic inertial modes.

The inertia-gravity mode frequencies and their associations with the barotropic inertial modes have been tested with the slow rotation approximation results given in [8]. The authors used a frequency domain approach and studied \( \Gamma_\beta = 2 \) polytropic models with weak stratification and slow rotation. In this work, we have considerably extended the parameter space of previous analyses by studying stars with large composition gradients and fast rotation.

5. Conclusions
We have studied the pulsations of rapidly rotating stratified neutron stars with the aim of understanding the dependence of the composition g-modes on the rotation rate of the star. We have used the Cowling approximation where the gravitational potential perturbations are neglected. The low frequency band of the spectrum is characterised by the inertia-gravity modes that have a mixed character. They typically behave as g-modes in the slowly rotating limit, while they assume the properties of the inertial modes when the star rotates rapidly. This property make difficult to determine the presence of composition gradients from the inertia-gravity mode spectrum of fast rotating stars. However, as the neutron star ages and spins down, the dependence of the inertia-gravity modes on the star’s rotation and their deviation from the inertial barotropic modes can, at least in principle, be used to estimate the Brunt-Väisälä frequency and the degree of stratification.

5.1. Acknowledgments
This work was supported by STFC through grant number PP/E001025/1.

References
[1] Passamonti A, Haskell B, Andersson N, Jones D I and Hawke I 2008 (Preprint 0807.3457)
[2] Passamonti A, Haskell B, Andersson N 2008 (Preprint 0812.3569)
[3] Hachisu I 1986 ApJS 62 461–499
[4] Reisenegger A and Goldreich P 1992 ApJ 395 240–249
[5] Stergioulas N, Apostolatos T A and Font J A 2004 MNRAS 352 1089 (Preprint astro-ph/0312648)
[6] Dimmelmeier H, Stergioulas N and Font J A 2006 MNRAS 368 1609–1630 (Preprint astro-ph/0511394)
[7] Unno W, Osaki Y, Ando H, Saio H and Shibahashi H 1989 *Nonradial oscillations of stars* (Tokyo: University of Tokyo Press, 1989, 2nd ed.)

[8] Yoshida S and Lee U 2000 *ApJS* **129** 353–366 (*Preprint* arXiv:astro-ph/0002300)