Relativistic Spherical Functions on the Lorentz Group

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Abstract

Matrix elements of irreducible representations of the Lorentz group are calculated on the basis of complex angular momentum. It is shown that Laplace-Beltrami operators, defined in this basis, give rise to Fuchsian differential equations. An explicit form of the matrix elements of the Lorentz group has been found via the addition theorem for generalized spherical functions. Different expressions of the matrix elements are given in terms of hypergeometric functions both for finite-dimensional and unitary representations of the principal and supplementary series of the Lorentz group.

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1 Introduction

As is known, an expansion problem of relativistic amplitudes requires the most simple form for the matrix elements of irreducible representations of the Lorentz group. Matrix elements of this group are studied for a long time by many authors. So, in 1956, Dolginov [1] (see also [2, 3, 4]) considered an analytic continuation of the Fock four-dimensional spherical functions (four-dimensional spherical functions of an euclidean space was introduced by Fock [5] for the solution of the hydrogen atom problem in momentum representation). Basis functions, called in the works [1, 2, 3] as relativistic spherical functions, depend on angles of the radius-vector in the four-dimensional spacetime. It should be noted that Dolginov-Toptygin relativistic spherical functions present the most degenerate form of the matrix elements of the Lorentz group. Different realizations of these elements were studied in the works [6, 7, 8, 9, 10, 11, 12, 13]. The most complete form of the matrix elements of the Lorentz group was given in the works [8, 9] within the Gel’fand-Naimark basis [14, 15]. However, matrix elements in the Ström form, and also in the Sciarrino-Toller form [11], are very complicate and cumbersome. Smorodinsky and Huszar [16, 17, 18] found more simple and direct method for definition of the matrix elements of the Lorentz group by means of a complexification of the three-dimensional rotation group and solution of the equation on eigenvalues of the Casimir operators (see also [19]).

In the present work matrix elements of irreducible representations of the Lorentz group are found on the basis of complex angular momentum (SU(2) ⊗ SU(2)-basis). It is shown that Laplace-Beltrami operators, defined in this basis, lead to Fuchsian differential equations which can be reduced to hypergeometric equations. An explicit form of the matrix elements has been found via the addition theorem for generalized spherical functions, where the functions \( P_{mn}^l \) and \( \Phi_{mn}^l \) are components. As is known [20], the matrix elements of SU(2)
are defined by the functions $P_{mn}^l$, and matrix elements of the group QU(2) of quasiunitary matrices of the second order, which is isomorphic to the group SL(2, $\mathbb{R}$)\(^1\), are expressed via the functions $\mathfrak{P}_{mn}^l$. The groups SU(2) and SU(1, 1) are real forms of the group SL(2, $\mathbb{C}$). The factorization of the matrix elements of SL(2, $\mathbb{C}$) with respect to the subgroups SU(2) and SU(1, 1) allows us to express these elements via the product of two hypergeometric functions both for finite-dimensional and unitary representations of the principal and supplementary series of the Lorentz group (it should be noted that matrix elements in Ström form are expressed via the product of three hypergeometric functions). On the other hand, matrix elements of the Lorentz group play an essential role in quantum field theory on the Poincaré group \([23, 24, 25, 26, 27, 28]\), where the field operators are expressed via generalized Fourier integrals (it leads to harmonic analysis on the homogeneous spaces). Solutions of relativistic wave equations are reduced also to expansions in relativistic spherical functions \([29, 30]\). Moreover, the Biedenharn type relativistic wavefunctions \([31]\) are defined completely in this framework \([27, 28]\).

2 Relativistic Spherical Functions

As is known, the group $\text{Spin}_+(1, 3) \simeq \text{SL}(2, \mathbb{C})$ is an universal covering of the proper orthochronous Lorentz group $\text{SO}_0(1, 3)$. The group $\text{SL}(2, \mathbb{C})$ of all complex matrices

$$ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} $$

of 2-nd order with the determinant $\alpha\delta - \gamma\beta = 1$, is a complexification of the group SU(2). The group SU(2) is one of the real forms of SL(2, $\mathbb{C}$). The transition from SU(2) to SL(2, $\mathbb{C}$) is realized via the complexification of three real parameters $\varphi$, $\theta$, $\psi$ (Euler angles) of SU(2).

Let $\theta^c = \theta - i\tau$, $\varphi^c = \varphi - i\epsilon$, $\psi^c = \psi - i\varepsilon$ be complex Euler angles, where

$$
\begin{align*}
0 & \leq \text{Re} \theta^c = \theta & \leq \pi, & -\infty < \text{Im} \theta^c = \tau < +\infty, \\
0 & \leq \text{Re} \varphi^c = \varphi & < 2\pi, & -\infty < \text{Im} \varphi^c = \epsilon < +\infty, \\
-2\pi & \leq \text{Re} \psi^c = \psi & < 2\pi, & -\infty < \text{Im} \psi^c = \varepsilon < +\infty.
\end{align*}
$$

(1)

Infinitesimal operators $A_i$ and $B_i$ of the group $\text{SL}(2, \mathbb{C})$ form a basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and satisfy the relations

$$
\begin{align*}
[A_1, A_2] &= A_3, & [A_2, A_3] &= A_1, & [A_3, A_1] &= A_2, \\
[B_1, B_2] &= -A_3, & [B_2, B_3] &= -A_1, & [B_3, B_1] &= -A_2, \\
[A_1, B_1] &= 0, & [A_2, B_2] &= 0, & [A_3, B_3] &= 0, \\
[A_1, B_2] &= B_3, & [A_1, B_3] &= -B_2, \\
[A_2, B_3] &= B_1, & [A_2, B_1] &= -B_3, \\
[A_3, B_1] &= B_2, & [A_3, B_2] &= -B_1.
\end{align*}
$$

(2)

Let us consider the operators

$$
X_l = \frac{1}{2} i (A_l + i B_l), \quad Y_l = \frac{1}{2} i (A_l - i B_l),
$$

(l = 1, 2, 3).

\(^1\)Other designation of this group is SU(1, 1) known also as three-dimensional Lorentz group, representations of which was studied by Bargmann, \([22]\).
At this point, we see that operators (7) contain the well known Casimir operators. Further, introducing generators of the form

\[ L = X_1 + iX_2, \quad X = X_1 - iX_2, \]

\[ Y = Y_1 + iY_2, \quad Y = Y_1 - iY_2, \]

we see that in virtue of commutativity of the relations (4) a space of an irreducible finite-dimensional representation of the group \( SL(2, \mathbb{C}) \) can be spanned on the totality of \( (2l + 1)(2\hat{m} + 1) \) basis vectors \( |l, m; \hat{l}, \hat{m}\rangle \), where \( l, m, \hat{l}, \hat{m} \) are integer or half-integer numbers, \(-l \leq m \leq l, -\hat{l} \leq \hat{m} \leq \hat{l} \). Therefore,

\[
X_+ |l, m; \hat{l}, \hat{m}\rangle = \sqrt{(l + m)(l - m + 1)} |l, m - 1, \hat{l}, \hat{m}\rangle \quad (m > -l), \\
X_+ |l, m; \hat{l}, \hat{m}\rangle = \sqrt{(l - m)(l + m + 1)} |l, m + 1, \hat{l}, \hat{m}\rangle \quad (m < l), \\
X_3 |l, m; \hat{l}, \hat{m}\rangle = m |l, m; \hat{l}, \hat{m}\rangle, \\
Y_+ |l, m; \hat{l}, \hat{m}\rangle = \sqrt{(\hat{l} + \hat{m})(\hat{l} - \hat{m} + 1)} |l, m; \hat{l} - 1, \hat{m}\rangle \quad (\hat{m} > -\hat{l}), \\
Y_+ |l, m; \hat{l}, \hat{m}\rangle = \sqrt{(\hat{l} - \hat{m})(\hat{l} + \hat{m} + 1)} |l, m; \hat{l} + 1, \hat{m}\rangle \quad (\hat{m} < \hat{l}), \\
Y_3 |l, m; \hat{l}, \hat{m}\rangle = \hat{m} |l, m; \hat{l}, \hat{m}\rangle. 
\]

From relations (4), it follows that each of the sets of infinitesimal operators \( X \) and \( Y \) generates the group \( SU(2) \) and these two groups commute with each other. Thus, from the relations (4) and (6) it follows that the group \( SL(2, \mathbb{C}) \), in essence, is equivalent locally to the group \( SU(2) \otimes SU(2) \). The basis (5) was first introduced by Van der Waerden in [32].

On the group \( SL(2, \mathbb{C}) \) there exist the following Laplace-Beltrami operators:

\[
X^2 = X_1^2 + X_2^2 + X_3^2 = \frac{1}{4}(A^2 - B^2 + 2iAB), \\
Y^2 = Y_1^2 + Y_2^2 + Y_3^2 = \frac{1}{4}(\tilde{A}^2 - \tilde{B}^2 - 2i\tilde{A}\tilde{B}).
\]

At this point, we see that operators (7) contain the well known Casimir operators \( A^2 - B^2, AB \) of the Lorentz group. Using expressions (11), we obtain a Euler parametrization of the Laplace-Beltrami operators,

\[
X^2 = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left[ \frac{\partial^2}{\partial \varphi^2} - 2 \cos \theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \psi} + \frac{\partial^2}{\partial \psi^2} \right], \\
Y^2 = \frac{\partial^2}{\partial \hat{\theta}^2} + \cot \hat{\theta} \frac{\partial}{\partial \hat{\theta}} + \frac{1}{\sin^2 \hat{\theta}} \left[ \frac{\partial^2}{\partial \hat{\varphi}^2} - 2 \cos \hat{\theta} \frac{\partial}{\partial \hat{\varphi}} \frac{\partial}{\partial \hat{\psi}} + \frac{\partial^2}{\partial \hat{\psi}^2} \right].
\]

Here \( \hat{\theta} = \theta + i\tau, \hat{\varphi} = \varphi + i\epsilon, \hat{\psi} = \psi + i\varepsilon \) are complex conjugate Euler angles.

Matrix elements \( t_{mn}(\theta) = \mathcal{M}^l_{mn}(\varphi, \theta, \psi) \) of irreducible representations of the group \( SL(2, \mathbb{C}) \) are eigenfunctions of the operators (8),

\[
[X^2 + l(l + 1)] \mathcal{M}^l_{mn}(\varphi, \theta, \psi) = 0, \\
[Y^2 + \hat{l}(\hat{l} + 1)] \mathcal{M}^l_{nm}(\varphi, \hat{\theta}, \hat{\psi}) = 0.
\]
where

\[
M_{mn}(g) = e^{-i(m\varphi + n\psi)} Z^l_{mn}(\cos \theta),
\]
\[
\dot{M}_{mn}(g) = e^{i(\dot{m}\varphi + \dot{n}\psi)} \dot{Z}^l_{mn}(\cos \theta). \tag{10}
\]

Here \(M_{mn}(g)\) are general matrix elements of the representations of \(SO_0(1,3)\), and \(Z^l_{mn}(\cos \theta)\) are hyperspherical functions. Substituting the functions (10) into (9) and taking into account the operators (8) and substitutions \(z = \cos \theta\), \(\dot{z} = \cos \dot{\theta}\), we arrive at the following differential equations:

\[
\left[ (1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{m^2 + n^2 - 2mnz}{1 - z^2} + l(l + 1) \right] Z^l_{mn}(z) = 0, \tag{11}
\]
\[
\left[ (1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{\dot{m}^2 + \dot{n}^2 - 2\dot{m}\dot{n}z}{1 - z^2} + i(l + 1) \right] \dot{Z}^l_{mn}(\dot{z}) = 0. \tag{12}
\]

The latter equations have three singular points \(-1, +1, \infty\). The equations (11), (12) are Fuchsian equations. Indeed, denoting \(w(z) = Z^l_{mn}(z)\), we write the equation (11) in the form

\[
\frac{d^2 w(z)}{dz^2} - p(z) \frac{dw(z)}{dz} + q(z) w(z) = 0, \tag{13}
\]

where

\[
p(z) = \frac{2z}{(1 - z)(1 + z)}, \quad q(z) = \frac{l(l + 1)(1 - z^2) - m^2 - n^2 + 2mnz}{(1 - z)^2(1 + z)^2}.
\]

Let us find solutions of (11). Applying the substitution

\[ t = \frac{1 - z}{2}, \quad w(z) = t^{\frac{|m-n|}{2}} (1 - t)^{\frac{|m+n|}{2}} v(t), \]

we arrive at hypergeometric equation

\[
t(1 - t) \frac{d^2 v}{dt^2} + \left[ c - (a + b + 1)t \right] \frac{dv}{dt} - abv(t) = 0, \tag{14}
\]

where

\[
a = l + 1 + \frac{1}{2}(|m-n| + |m+n|),
\]
\[
b = -l + \frac{1}{2}(|m-n| + |m+n|),
\]
\[
c = |m-n| + 1.
\]

Therefore, a solution of (14) is

\[
v(t) = C_{12} F_1 \begin{pmatrix} a, b \\ c \end{pmatrix} \left( t \right) + C_{21} t^{1-c} \frac{1}{2} F_1 \begin{pmatrix} b - c + 1, a - c + 1 \\ 2 - c \end{pmatrix} \left( t \right).\]
Coming back to initial variable, we obtain

\[ w(z) = C_1 \left( \frac{1 - z}{2} \right)^{\frac{\left| m - n \right|}{2}} \left( \frac{1 + z}{2} \right)^{\frac{\left| m + n \right|}{2}} \times \]
\[ \times {}_2F_1 \left( l + 1 + \frac{1}{2}(|m - n| + |m + n|), -l + \frac{1}{2}(|m - n| + |m + n|) \middle| \frac{1 - z}{2} \right) + \]
\[ + C_2 \left( \frac{1 - z}{2} \right)^{-\frac{\left| m - n \right|}{2}} \left( \frac{1 + z}{2} \right)^{\frac{\left| m + n \right|}{2}} \times \]
\[ \times {}_2F_1 \left( -l + \frac{1}{2}(|m + n| - |m - n|), l + 1 + \frac{1}{2}(|m + n| - |m - n|) \middle| 1 - \frac{z}{2} \right). \] (15)

Carrying out the analogous calculations for the equation (12), we find that

\[ w(z^*) = C_1 \left( \frac{1 - z^*}{2} \right)^{\frac{\left| \bar{m} - \bar{n} \right|}{2}} \left( \frac{1 + z^*}{2} \right)^{\frac{\left| \bar{m} + \bar{n} \right|}{2}} \times \]
\[ \times {}_2F_1 \left( l + 1 + \frac{1}{2}(|\bar{m} - \bar{n}| + |\bar{m} + \bar{n}|), -l + \frac{1}{2}(|\bar{m} - \bar{n}| + |\bar{m} + \bar{n}|) \middle| \frac{1 - z^*}{2} \right) + \]
\[ + C_2 \left( \frac{1 - z^*}{2} \right)^{-\frac{\left| \bar{m} - \bar{n} \right|}{2}} \left( \frac{1 + z^*}{2} \right)^{\frac{\left| \bar{m} + \bar{n} \right|}{2}} \times \]
\[ \times {}_2F_1 \left( -l + \frac{1}{2}(|\bar{m} + \bar{n}| - |\bar{m} - \bar{n}|), l + 1 + \frac{1}{2}(|\bar{m} + \bar{n}| - |\bar{m} - \bar{n}|) \middle| 1 - \frac{z^*}{2} \right). \] (16)

As follows from (15) and (16), the functions \( Z^g_{mn} \) and \( Z^g_{\bar{m}\bar{n}} \) are expressed via the hypergeometric function. In virtue of the full development of the theory of hypergeometric functions, the representations (15) and (16) are the most useful. Indeed, from (15) it follows that the function \( Z^g_{mn} \) can be represented by the following particular solution:

\[ Z^g_{mn}(\cos \theta^c) = C_1 \sin^{\left| m-n \right|} \frac{\theta^c}{2} \cos^{\left| m+n \right|} \frac{\theta^c}{2} \times \]
\[ \times {}_2F_1 \left( l + 1 + \frac{1}{2}(|m - n| + |m + n|), -l + \frac{1}{2}(|m - n| + |m + n|) \middle| \frac{1 - \sin^2 \theta^c}{2} \right). \] (17)

Let us give now a general definition for spherical functions on the group \( G \). Let \( T(g) \) be an irreducible representation of the group \( G \) in the space \( L \) and let \( H \) be a subgroup of \( G \). The vector \( \xi \) in the space \( L \) is called an invariant with respect to the subgroup \( H \) if for all \( h \in H \) the equality \( T(h)\xi = \xi \) holds. The representation \( T(g) \) is called a representation of the class one with respect to the subgroup \( H \) if in its space there are non-null vectors which are invariant with respect to \( H \). At this point, a contraction of \( T(g) \) onto its subgroup \( H \) is unitary:

\[ (T(h)\xi_1, T(h)\xi_2) = (\xi_1, \xi_2). \]

Hence it follows that a function

\[ f(g) = (T(g)\eta, \xi) \]
corresponds the each vector $\eta \in L$. $f(g)$ are called spherical functions of the representation $T(g)$ with respect to $H$.

Spherical functions can be considered as functions on homogeneous spaces $\mathcal{M} = G/H$. In its turn, a homogeneous space $\mathcal{M}$ of the group $G$ has the following properties:

a) It is a topological space on which the group $G$ acts continuously, that is, let $y$ be a point in $\mathcal{M}$, then $gy$ is defined and is again a point in $\mathcal{M}$ ($g \in G$).

b) This action is transitive, that is, for any two points $y_1$ and $y_2$ in $\mathcal{M}$ it is always possible to find a group element $g \in G$ such that $y_2 = gy_1$.

There is a one-to-one correspondence between the homogeneous spaces of $G$ and the coset spaces of $G$. Let $H_0$ be a maximal subgroup of $G$ which leaves the point $y_0$ invariant, $h_0 y_0 = y_0$, $h \in H_0$, then $H_0$ is called the stabilizer of $y_0$. Representing now any group element of $G$ in the form $g = gh$, where $h \in H_0$ and $g \in G/H_0$, we see that, by virtue of the transitivity property, any point $y \in \mathcal{M}$ can be given by $y = gh_0y_0 = gcy$. Hence it follows that the elements $g_\psi$ of the coset space give a parametrization of $\mathcal{M}$. The mapping $\mathcal{M} \leftrightarrow G/H_0$ is continuous since the group multiplication is continuous and the action on $\mathcal{M}$ is continuous by definition. The stabilizers $H$ and $H_0$ of two different points $y$ and $y_0$ are conjugate, since from $H_0y_0 = g_0$, $y_0 = g^{-1}y$, it follows that $Hg_0y^{-1}y = y$, that is, $H = gH_0g^{-1}$.

Coming back to the Lorentz group $G = SO_0(1,3)$, we see that there are the following homogeneous spaces of $SO_0(1,3)$ depending on the stabilizer $H$. First of all, when $H = 0$ the homogeneous space $\mathcal{M}_6$ coincides with a group manifold $\mathcal{L}_6$ of $SO_0(1,3)$. Therefore, $\mathcal{L}_6$ is a maximal homogeneous space of the Lorentz group. Further, when $H = \Omega^c_\psi$, where $\Omega^c_\psi$ is a group of diagonal matrices $\left( \begin{array}{cc} e^{i \varphi} & 0 \\ 0 & e^{-i \varphi} \end{array} \right)$, the homogeneous space $\mathcal{M}_4$ coincides with a two-dimensional complex sphere $S_2^\ast$, $\mathcal{M}_4 = S_2^\ast \sim \text{SL}(2, \mathbb{C})/\Omega^c_\psi$. The sphere $S_2^\ast$ can be constructed from the quantities $z_k = x_k + iy_k$, $z_k^* = x_k - iy_k$ ($k = 1, 2, 3$) as follows:

$$S_2^\ast: \quad z_1^* + z_2^* + z_3^* = x^2 - y^2 + 2ixy = r^2. \quad (18)$$

The complex conjugate (dual) sphere $\hat{S}_2^\ast$ is

$$\hat{S}_2^\ast: \quad \hat{z}_1^* + \hat{z}_2^* + \hat{z}_3^* = x^2 - y^2 - 2ixy = \hat{r}^2. \quad (19)$$

The following homogeneous space $\mathcal{M}_3$ we obtain when the stabilizer $H$ coincides with a maximal compact subgroup $K = \SO_0(3)$ of $SO_0(1,3)$. In this case we have a three-dimensional two-sheeted hyperboloid $\mathcal{M}_3 = H_3 \sim SO_0(1,3)/SO_0(3) \simeq \text{SL}(2, \mathbb{C})/\text{SU}(2)$, defined by the equation

$$H_3 = \{ x \in \mathbb{R}^{1,3} | [x, x] = 1 \}.$$ 

In the case $[x, x] = 0$ we arrive at a cone $C_3$ which can be considered also as a homogeneous space of $SO(1,3)$. Usually, only the upper sheets $H_3^+$ and $C_3^+$ are considered in applications.

Finally, a minimal homogeneous space $\mathcal{M}_2$ of $SO_0(1,3)$ is a two-dimensional real sphere $S_2 \sim SO_0(3)/SO(2)$. In contrast to the previous homogeneous spaces, the sphere $S_2$ coincides with a quotient space $SO_0(1,3)/P$, where $P$ is a minimal parabolic subgroup of $SO_0(1,3)$. From the Iwasawa decompositions $SO_0(1,3) = KNA$ and $P = MAN$, where $M = SO(2)$, $N$ and $A$ are nilpotent and commutative subgroups of $SO_0(1,3)$, it follows that $SO_0(1,3)/P = KNA/MNA \sim K/M \sim SO_0(3)/SO(2)$.

Taking into account the list of homogeneous spaces of $SO_0(1,3)$, we introduce now the following types of spherical functions $f(g)$ on the Lorentz group:
• \( f(\mathbf{g}) = \mathcal{M}_{mn}^{t}(\mathbf{g}) \). This function is defined on the group manifold \( \mathcal{L}_6 \) of SO\(_0(1,3)\). It is the most general spherical function on the group SO\(_0(1,3)\). In this case \( f(\mathbf{g}) \) depends on all the six parameters of SO\(_0(1,3)\) and for that reason it should be called as a function on the Lorentz group. An explicit form of \( \mathcal{M}_{mn}^{t}(\mathbf{g}) \) (respectively \( \mathcal{M}_{mn}^{t}(\mathbf{g}) \)) for finite-dimensional representations and of \( \mathcal{M}_{mn}^{t}(\mathbf{g}) \) (resp. \( \mathcal{M}_{mn}^{t}(\mathbf{g}) \)) for infinite-dimensional representations of SO\(_0(1,3)\) will be given in the sections 3 and 4, respectively.

• \( f(\varphi^c, \theta^c) = \mathcal{M}^{m}_{l}(\varphi^c, \theta^c, 0) \). This function is defined on the homogeneous space \( \mathcal{M}_4 = S^2_2 \sim SO(1,3)/\Omega_2^c \), that is, on the surface of the two-dimensional complex sphere \( S^c_2 \). The function \( \mathcal{M}^{m}_{l}(\varphi^c, \theta^c, 0) \) is a relativistic analogue of the usual spherical function \( Y^m_l(\varphi, \theta) \) defined on the surface of the real two-sphere \( S^2 \). In its turn, the function \( f(\varphi^c, \theta^c) = \mathcal{M}^{m}_{l}(\varphi^c, \theta^c, 0) \) is defined on the surface of the dual sphere \( \hat{S}^c_2 \). General solutions of relativistic wave equations have been found via an expansion in spherical functions \( f(\varphi^c, \theta^c) \) [30]. An explicit form of the functions \( \mathcal{M}^{m}_{l}(\varphi^c, \theta^c, 0) \) (\( \mathcal{M}^{m}_{l}(\varphi^c, \theta^c, 0) \)) and \( \mathcal{M}^{m}_{\frac{1}{2} + ip}(\varphi^c, \theta^c, 0) \) (\( \mathcal{M}^{m}_{\frac{1}{2} - ip}(\varphi^c, \theta^c, 0) \)) will be given in the section 3 and 4.

• \( f(\epsilon, \tau, \varepsilon) = e^{-im\varphi} \mathcal{P}^{t}_{mn}(\cosh \tau)e^{-in\varepsilon} \). This function is defined on the homogeneous space \( \mathcal{M}_3 = H^+_3 \sim SO(0,1,3)/SO(3) \), that is, on the upper sheet of the hyperboloid \( x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1 \). In essence, we come here to representations of SO\(_0(1,3)\) restricted to the subgroup SU\((1,1)\) [11] [13].

• \( f(\varphi, \theta, \psi) = e^{-im\varphi} \mathcal{P}^{t}_{mn}(\cos \theta)e^{-in\psi} \). This function is defined on the homogeneous space \( \mathcal{M}_2 = S_2 \sim SO(3)/SO(2) \), that is, on the surface of the two-dimensional real sphere \( S_2 \). We come here to the most degenerate representations of SO\(_0(1,3)\) restricted to the subgroup SU\((2)\).

We see that only first two functions \( f(\mathbf{g}) \) and \( f(\varphi^c, \theta^c) \) can be considered as functions on the Lorentz group SO\(_0(1,3)\); other two functions \( f(\epsilon, \tau, \varepsilon) \) and \( f(\varphi, \theta, \psi) \) present degenerate cases corresponding to the subgroups SU\((1,1)\) and SU\((2)\). For that reason the functions \( f(\mathbf{g}) \) and \( f(\varphi^c, \theta^c) \) should be called relativistic spherical functions on the Lorentz group.

### 3 Hyperspherical Functions and Addition Theorem for Generalized Spherical Functions

In this section we will find expressions for the matrix elements (relativistic spherical functions) containing explicitly all six parameters of the Lorentz group. Moreover, such a form of the matrix elements to be the most suitable for forthcoming tasks of harmonic analysis on the Lorentz and Poincaré groups.

As is known, the groups SU\((2)\) and SU\((1,1)\) \(\cong\) SL\((2,\mathbb{R})\) are real forms of SL\((2,\mathbb{C})\). As a direct consequence of this, a structure of the matrix elements of these groups is very similar with a corresponding structure of matrix elements for the group SL\((2,\mathbb{C})\). Indeed, matrix elements of irreducible representations of SU\((2)\) have the form [21] [33]

\[
t^l_{mn}(u) = e^{-im\varphi} P^t_{mn}(\cos \theta)e^{-in\psi},
\]
where

\[ P_{mn}^{l}(\cos \theta) = i^{m-n} \sqrt{\frac{\Gamma(l-m+1)\Gamma(l-n+1)}{\Gamma(l+m+1)\Gamma(l+n+1)}} \times \]
\[ \times \cos^{m+n} \frac{\theta}{2} \sin^{m-n} \frac{\theta}{2} \sum_{t=0}^{l-m} \frac{(-1)^t \Gamma(l+m+t+1)}{\Gamma(t+1)\Gamma(m-n+t+1)\Gamma(l-m-t+1)} \sin^{2t} \frac{\theta}{2}. \]  

(20)

Here $\varphi, \theta, \psi$ are real Euler parameters for SU(2) (the first column from the relations (1)).

At $m \geq n$ the function $P_{mn}^{l}(\cos \theta)$ is expressed via the hypergeometric function as

\[ P_{mn}^{l}(\cos \theta) = i^{m-n} \sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(l-n+1)}} \times \]
\[ \times \cos^{m+n} \frac{\theta}{2} \sin^{m-n} \frac{\theta}{2} 2F1 \left( \frac{l+m+1, m-l}{m-n+1} \left| \sin^2 \frac{\theta}{2} \right. \right). \]  

(21)

Analogously, at $n \geq m$

\[ P_{mn}^{l}(\cos \theta) = i^{n-m} \sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(l-n+1)}} \times \]
\[ \times \cos^{m+n} \frac{\theta}{2} \sin^{m-n} \frac{\theta}{2} 2F1 \left( \frac{l+n+1, n-l}{l-m+1} \left| \sin^2 \frac{\theta}{2} \right. \right). \]  

(22)

It is easy to see that the functions (21) and (22) with an accuracy of the constant coincide with the function (15) (correspondingly (17)) if to open the modules and suppose $z = \cos \theta$.

Other expression for the function $P_{mn}^{l}(\cos \theta)$, related with (20), is defined by the transformation $u = k z$, where $k = \left( \begin{array} {cc} \bar{\alpha}^{-1} & \beta \\ 0 & \bar{\alpha} \end{array} \right)$ and $z = \left( \begin{array} {cc} 1 & 0 \\ -\beta/\bar{\alpha} & 1 \end{array} \right)$. This expression has the form

\[ P_{mn}^{l}(\cos \theta) = i^{m-n} \sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(l-n+1)}} \times \]
\[ \cos^{2l} \frac{\theta}{2} \tan^{m-n} \frac{\theta}{2} \times \]
\[ \sum_{j=\max(0,n-m)}^{\min(l-m,l+n)} \frac{i^{2j} \tan^{2j} \frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+n-j+1)\Gamma(m-n+j+1)}. \]  

(23)

Correspondingly, the functions (23) are expressed via the hypergeometric function as follows:

\[ P_{mn}^{l}(\cos \theta) = i^{m-n} \sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}} \times \]
\[ \times \cos^{2l} \frac{\theta}{2} \tan^{m-n} \frac{\theta}{2} 2F1 \left( \frac{m-l, -n-l}{m-n+1} \left| -\tan^2 \frac{\theta}{2} \right. \right), \ m \geq n; \]  

(24)
\[ P_{mn}^l(\cos \theta) = i^{n-m} \sqrt{\frac{\Gamma(l+n+1)\Gamma(l-m+1)}{\Gamma(l-n+1)\Gamma(l+m+1)}} \times \]
\[ \times \cos^2 \frac{\theta}{2} \tan^{n-m} \frac{\theta}{2} F_1 \left( \begin{array}{c} n-l, -m-l \\ n-m+1 \end{array} \right) \left( -\tan^2 \frac{\theta}{2} \right), \quad n \geq m. \quad (25) \]

In turn, matrix elements of irreducible representations of the group SU(1, 1) have the form \[ t_{mn}^l(g) = e^{-im\epsilon} \Psi_{mn}^l(\cosh \tau) e^{-i\epsilon}, \]
where in the case of finite-dimensional representations
\[ \Psi_{mn}^l(\cosh \tau) = \sqrt{\frac{\Gamma(l-m+1)\Gamma(l-n+1)}{\Gamma(l+m+1)\Gamma(l+n+1)}} \times \]
\[ \times \cosh^{m+n} \frac{\theta}{2} \sinh^{m-n} \frac{\theta}{2} \sum_{s=0}^{l-m} \frac{(-1)^s \Gamma(l+m+s+1)}{\Gamma(s+1)\Gamma(m-n+s+1)\Gamma(l-m-s+1)} \sinh^{2s} \frac{\theta}{2}. \quad (26) \]

or
\[ \Psi_{mn}^l(\cosh \tau) = \sqrt{\frac{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-n+1)\Gamma(l+n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}} \times \]
\[ \cosh^{m+n+l+n} \frac{\tau}{2} \tan^{m-n} \frac{\tau}{2} \times \]
\[ \sum_{s=\max(0,n-m)}^{\min(l-m,l+n)} \frac{\cosh^{2s} \frac{\tau}{2}}{\Gamma(s+1)\Gamma(l-m-s+1)\Gamma(l+n-s+1)\Gamma(m-n+s+1)}. \quad (27) \]

Here \( \epsilon, \tau, \varepsilon \) are real Euler parameters for the group SU(1, 1) (the second column from [11] at the restriction of the parameters \( \epsilon \) and \( \varepsilon \) within the limits \( 0 \leq \epsilon \leq 2\pi \) and \( -2\pi \leq \varepsilon < 2\pi \)). The functions \( \Psi_{mn}^l(\cosh \tau) \) can be reduced also to hypergeometric functions. So, at \( m \geq n \) we have
\[ \Psi_{mn}^l(\cosh \tau) = \frac{1}{\Gamma(m-n+1)} \sqrt{\frac{\Gamma(l-n+1)\Gamma(l+m+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}} \times \]
\[ \times \cosh^{m+n} \frac{\tau}{2} \sinh^{m-n} \frac{\tau}{2} F_1 \left( \begin{array}{c} l+m+1, m-l \\ m-n+1 \end{array} \right) \left( -\sinh^2 \frac{\tau}{2} \right) = \]
\[ = \sqrt{\frac{\Gamma(l+n+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}} \cosh^{2l} \frac{\tau}{2} \tan^{m-n} \frac{\tau}{2} F_1 \left( \begin{array}{c} m-l, -n-l \\ m-n+1 \end{array} \right) \left( \tanh^2 \frac{\tau}{2} \right). \quad (28) \]

Correspondingly, at \( n \geq m \)
\[ \Psi_{mn}^l(\cosh \tau) = \frac{1}{\Gamma(n-m+1)} \sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l-n+1)\Gamma(l+m+1)}} \times \]
\[ \times \cosh^{m+n} \frac{\tau}{2} \sinh^{n-m} \frac{\tau}{2} F_1 \left( \begin{array}{c} l+n+1, n-l \\ n-m+1 \end{array} \right) \left( -\sinh^2 \frac{\tau}{2} \right) = \]
\[ = \sqrt{\frac{\Gamma(l+n+1)\Gamma(l-m+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}} \cosh^{2l} \frac{\tau}{2} \tan^{n-m} \frac{\tau}{2} F_1 \left( \begin{array}{c} n-l, -m-l \\ n-m+1 \end{array} \right) \left( \tanh^2 \frac{\tau}{2} \right). \quad (29) \]
In the case of principal series of unitary representations, matrix elements are (see, for example, [21]

\[ \mathcal{P}_{mn}^{\pm1+i\rho} (\cosh \tau) = \sqrt{\frac{\Gamma(i\rho - n + \frac{1}{2})\Gamma(i\rho + n + \frac{1}{2})\Gamma(i\rho - m + \frac{1}{2})\Gamma(i\rho + m + \frac{1}{2})}{\Gamma(i\rho - n - s + \frac{1}{2})\Gamma(i\rho + n + \frac{1}{2})}} \times \]

\[ \sum_{s = \max(0,m-n)}^{\infty} \frac{\cosh 2i\rho - \frac{1}{2} \tanh \frac{n-m}{2} \times}{\tanh 2s \frac{1}{2}} \Gamma(s + 1) \Gamma(i\rho - n - s + \frac{1}{2}) \Gamma(n - m + s + 1) \Gamma(i\rho + m - s + \frac{1}{2}) \times \]

or

\[ \mathcal{P}_{mn}^{\mp1+i\rho} (\cosh \tau) = \sqrt{\frac{\Gamma(i\rho + n + \frac{1}{2})\Gamma(i\rho - m + \frac{1}{2})}{\Gamma(i\rho - n + \frac{1}{2})\Gamma(i\rho + m + \frac{1}{2})}} \times \]

\[ \cosh 2i\rho - \frac{1}{2} \tanh \frac{n-m}{2} F_1 \left( m - i\rho + \frac{1}{2}, -n - i\rho + \frac{1}{2} \left| \tanh^2 \frac{1}{2} \right. \right). \]

at \( m \geq n \) and

\[ \mathcal{P}_{mn}^{\pm1+i\rho} (\cosh \tau) = \sqrt{\frac{\Gamma(i\rho + m + \frac{1}{2})\Gamma(i\rho - m + \frac{1}{2})}{\Gamma(i\rho - m + \frac{1}{2})\Gamma(i\rho + m + \frac{1}{2})}} \times \]

\[ \cosh 2i\rho - \frac{1}{2} \tanh \frac{n-m}{2} F_1 \left( n - i\rho + \frac{1}{2}, -m - i\rho + \frac{1}{2} \left| \tanh^2 \frac{1}{2} \right. \right). \]

at \( n \geq m \).

As is known [21], generalized spherical functions \( P_{mn}^{l} (\cos \theta) \) satisfy the following addition theorem:

\[ e^{-i(m\varphi + n\psi)} P_{mn}^{l} (\cos \theta) = \sum_{k = -l}^{l} e^{-ik\varphi} P_{mk}^{l} (\cos \theta_1) P_{kn}^{l} (\cos \theta_2), \]

where the angles \( \varphi, \psi, \theta, \theta_1, \varphi_2, \theta_2 \) are related by the formulae

\[ \cos \theta = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \varphi_2, \]

\[ e^{i\varphi} = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \cos \varphi_2 + i \sin \theta_2 \sin \varphi_2 \]

\[ e^{-i\varphi} = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\varphi} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi} \]

\[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\varphi} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi} \]

Let \( \cos(\theta - i\tau) \) and \( \varphi_2 = 0 \), then the formulae (31) - (36) take the form

\[ \cos \theta^c = \cos \theta \cosh \tau + i \sin \theta \sinh \tau, \]

\[ e^{i\varphi} = \frac{\sin \theta \cosh \tau - i \cos \theta \sinh \tau}{\sin \theta^c} = 1, \]

\[ e^{-i\varphi} = \frac{\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\varphi} - \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i\varphi} \cos \frac{\theta^c}{2}}{2} = 1. \]

Hence it follows that \( \varphi = \psi = 0 \) and formula (33) can be written as

\[ Z_{mn}^{l} (\cos \theta^c) = \sum_{k = -l}^{l} P_{mk}^{l} (\cos \theta) \mathcal{P}_{kn}^{l} (\cosh \tau). \]
Therefore, using the addition theorem, we derived a new representation for the hyperspherical function. Further, taking into account (23) and (27), we obtain an explicit expression for $Z_{lm}^i(\cos \theta^e)$,

$$Z_{mn}^i(\cos \theta^e) = \sum_{k=-l}^{l} i^{m-k} \sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)} \times$$

$$\cos^2 \frac{\theta}{2} \tan^{m-k} \frac{\theta}{2} \times$$

$$\max(l-m,l+k) \sum_{j=\max(0,k-m)}^{\min(l-m,l+k)} \frac{i^{2j} \tan^{2j} \frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)} \times$$

$$\sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-k+1)\Gamma(l+k+1)} \cosh^{2l} \frac{\tau}{2} \tanh^{n-k} \frac{\tau}{2} \times$$

$$\min(l-n,l+k) \sum_{s=\max(0,k-n)}^{\min(l-n,l+k)} \tan^{2s} \frac{\tau}{2} \Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1).$$

(37)

By way of example let us calculate matrix elements $\mathcal{M}_{mn}^l(\varphi^e) = e^{-i\varphi^e} Z_{mn}^l(\cos \theta^e)e^{-i\psi^e}$ at $l = 0, 1/2, 1$, where $Z_{mn}^l(\cos \theta^e)$ is defined via (37). The matrices of finite-dimensional representations at $l = 0, 1/2, 1$ have the following form:

$$T_0(\varphi^e, \theta^e, \psi^e) = 1,$$

(38)

$$T_2(\varphi^e, \theta^e, \psi^e) = \begin{pmatrix}
\mathcal{M}_{-1-1}^2 & \mathcal{M}_{-1-1/2}^2 & \mathcal{M}_{-11}^2 \\
\mathcal{M}_{1-1/2}^1 & \mathcal{M}_{1-1}^1 & \mathcal{M}_{11}^1 \\
\mathcal{M}_{11}^1 & \mathcal{M}_{1-1/2}^1 & \mathcal{M}_{1-1}^1 \\
\end{pmatrix} = \begin{pmatrix}
e^{i\varphi^e} Z_{-1-1}^2 & e^{i\varphi^e} Z_{-1-1/2}^2 & e^{i\varphi^e} Z_{-11}^2 \\
e^{-i\varphi^e} Z_{-1-1/2}^2 & e^{-i\varphi^e} Z_{-11}^2 & e^{-i\varphi^e} Z_{-11/2}^2 \\
e^{-i\varphi^e} Z_{-11}^2 & e^{-i\varphi^e} Z_{-11/2}^2 & e^{-i\varphi^e} Z_{-1-1/2}^2 \\
\end{pmatrix} =$$

$$\begin{pmatrix}
e^{i\varphi^e} \cos \frac{\vartheta}{2} & e^{i\varphi^e} \sin \frac{\vartheta}{2} & i e^{i\varphi^e} \sin \frac{\vartheta}{2} \\
e^{-i\varphi^e} \sin \frac{\vartheta}{2} & e^{-i\varphi^e} \cos \frac{\vartheta}{2} & -e^{-i\varphi^e} \sin \frac{\vartheta}{2} \\
e^{-i\varphi^e} \sin \frac{\vartheta}{2} & e^{-i\varphi^e} \cos \frac{\vartheta}{2} & -e^{-i\varphi^e} \sin \frac{\vartheta}{2} \\
\end{pmatrix}.$$
\[
Z^{l}_{mn} = \left( \left[ \cos^2 \frac{\theta}{2} \sinh^2 \frac{\tau}{2} + \sin^2 \frac{\theta}{2} \sinh^2 \frac{\tau}{2} \right] e^{\epsilon+i\psi} \left[ \frac{1}{\sqrt{2}} (\cos \theta \mp i \sin \theta \cosh \tau) \right] e^{\epsilon+i\psi} \cos \theta \cosh \tau \right) . \quad (40)
\]

Let us express now \( Z^{l}_{mn} (\cos \theta) \) via the hypergeometric functions. So, at \( m \geq n \) from (24) and (28) it follows that

\[
Z^{l}_{mn} (\cos \theta) = \sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}} \cos^{2l} \frac{\theta}{2} \cosh^{2l} \frac{\tau}{2} \times \sum_{k=-l}^{l} i^{m-k} \tan^{m-k} \frac{\theta}{2} \sinh^{k-n} \frac{\tau}{2} \times 2F_{1} \left( \frac{m-l, -k-l, m-k+1}{k-l, n-l, k-n+1} \right) \tan^{2} \frac{\tau}{2} . \quad (41)
\]

Analogously, at \( n \geq m \) from the formulae (24) and (28) we have

\[
Z^{l}_{mn} (\cos \theta) = \sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(l-n+1)}} \cos^{2l} \frac{\theta}{2} \cosh^{2l} \frac{\tau}{2} \times \sum_{k=-l}^{l} i^{k-m} \tan^{k-m} \frac{\theta}{2} \sinh^{n-k} \frac{\tau}{2} \times 2F_{1} \left( \frac{k-l, -m-l, k-m+1}{n-l, -k-l, k-n+1} \right) \tan^{2} \frac{\tau}{2} . \quad (42)
\]

Other representation for \( Z^{l}_{mn} (\cos \theta) \) in the form of hypergeometric function we obtain from the formulae (21), (22) and (29), (30). Namely,

\[
Z^{l}_{mn} (\cos \theta) = \sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}} \sum_{k=-l}^{l} \frac{i^{m-k}}{\Gamma(m-k+1)\Gamma(n-k+1)} \times \cos^{m+k} \frac{\theta}{2} \sin^{m-k} \frac{\theta}{2} \sinh^{k-n} \frac{\tau}{2} \cosh^{k+n} \frac{\tau}{2} \times 2F_{1} \left( \frac{l+m+1, m-l}{m-k+1} \right) 2F_{1} \left( \frac{k+l+1, k-l}{k-n+1} \right) \sin^{2} \frac{\theta}{2} , \quad m \geq n ;
\]

\[
Z^{l}_{mn} (\cos \theta) = \sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(l-n+1)}} \sum_{k=-l}^{l} \frac{i^{k-m}}{\Gamma(k-m+1)\Gamma(n-k+1)} \times \cos^{m+k} \frac{\theta}{2} \sin^{m-k} \frac{\theta}{2} \sinh^{n-k} \frac{\tau}{2} \cosh^{k+n} \frac{\tau}{2} \times 2F_{1} \left( \frac{l+k+1, k-l}{k-m+1} \right) 2F_{1} \left( \frac{n+l+1, n-l}{n-k+1} \right) \sin^{2} \frac{\theta}{2} , \quad n \geq m .
\]
Analogous expressions take place for the functions $Z_{m\nu}^l(\cos \theta)$. Relativistic spherical functions of the second type $f(\varphi^e, \theta^e) = \mathfrak{M}_m^l(\varphi^e, \theta^e, 0) = e^{-im\varphi^e} Z_{l}^{m}(\cos \theta^e)$, where

$$Z_{l}^{m}(\cos \theta^e) = \sum_{k=-l}^{l} P_{mk}^l(\cos \theta) \mathfrak{P}_l^k(\cosh \tau),$$

are defined on the surface of complex two-sphere [18]. In its turn, the functions $f(\varphi^e, \hat{\theta}^e) = e^{im\varphi^e} Z_{l}^{m}(\cos \hat{\theta}^e)$ are defined on the surface of dual sphere [19]. Explicit expressions and hypergeometric type formulae for $f(\varphi^e, \theta^e)$ and $f(\varphi^e, \hat{\theta}^e)$ follow directly from the previous expressions for $f(g)$ at $n = 0$.

4 Relativistic spherical functions of unitary representations of $SO_0(1, 3)$

Relativistic spherical functions $\mathfrak{M}_m^l(\varphi^e, \theta^e, \psi^e)$, considered in the previous sections, define matrix elements of non-unitary finite-dimensional representations of the group $SO_0(1, 3)$. As is known [14], finite-dimensional (spinor) representations of $SO_0(1, 3)$ in the space of symmetric polynomials $\text{Sym}_{(k,r)}$ have the following form:

$$T_\theta q(\xi, \bar{\xi}) = (\gamma \xi + \delta)^{l_0+l_1-1}(\gamma \bar{\xi} + \bar{\delta})^{l_0-l_1+1} q \left( \frac{\alpha \xi + \beta}{\gamma \xi + \delta} : \frac{\alpha \bar{\xi} + \beta}{\gamma \bar{\xi} + \bar{\delta}} \right), \quad (43)$$

where $k = l_0 + l_1 - 1$, $r = l_0 - l_1 + 1$, and the pair $(l_0, l_1)$ defines some representation of $SO_0(1, 3)$ in the Gel’fand-Naimark basis:

$$H_3 \xi_{k\nu} = m \xi_{k\nu},$$

$$H_+ \xi_{k\nu} = \sqrt{(k + \nu + 1)(k - \nu)} \xi_{k,\nu+1},$$

$$H_- \xi_{k\nu} = \sqrt{(k + \nu)(k - \nu + 1)} \xi_{k,\nu-1},$$

$$F_3 \xi_{k\nu} = C_1 \sqrt{k^2 - \nu^2} \xi_{k-1,\nu} - A_1 \nu \xi_{k,\nu} - C_{k+1} \sqrt{(k + 1)^2 - \nu^2} \xi_{k+1,\nu};$$

$$F_+ \xi_{k\nu} = C_k \sqrt{(k - \nu)(k - \nu - 1)} \xi_{k-1,\nu+1} - A_k \sqrt{(k - \nu)(k + \nu + 1)} \xi_{k,\nu+1} + C_{k+1} \sqrt{(k + \nu + 1)(k + \nu + 2)} \xi_{k+1,\nu+1};$$

$$F_- \xi_{k\nu} = -C_k \sqrt{(k + \nu)(k + \nu - 1)} \xi_{k-1,\nu-1} - A_k \sqrt{(k + \nu)(k - \nu + 1)} \xi_{k,\nu-1} - C_{k+1} \sqrt{(k - \nu + 1)(k - \nu + 2)} \xi_{k+1,\nu-1},$$

$$A_k = \frac{i l_0 l_1}{k(k + 1)}, \quad C_k = \frac{i}{k} \sqrt{(k^2 - l_0^2)(k^2 - l_1^2) / 4k^2 - 1}, \quad (44)$$

where $l_0$ is a positive integer or half-integer number, $l_1$ is an arbitrary complex number. These formulae define a finite-dimensional representation of the group $SO_0(1, 3)$ when $l_1^2 = (l_0 + \ldots$
\( (l_0, l_1) = (l, l + 1) \), whence it immediately follows that

\[
l = \frac{l_0 + l_1 - 1}{2}.
\]  

(45)

As is known \([14]\), if an irreducible representation of the proper Lorentz group \( \text{SO}_0(1, 3) \) is defined by the pair \( (l_0, l_1) \), then a conjugated representation is also irreducible and is defined by a pair \( \pm (l_0, -l_1) \). Therefore,

\[
(l_0, l_1) = (-\hat{l}, \hat{l} + 1).
\]

The relation between the numbers \( l_0 \), \( l_1 \) and the number \( l \) (the weight of representation in the basis \([3]\)) is given by a following formula:

\[
(l_0, l_1) = (l, l + 1),
\]

whence it immediately follows that

\[
l = \frac{l_0 + l_1 - 1}{2}.
\]

(45)

As is known \([14]\), if an irreducible representation of the proper Lorentz group \( \text{SO}_0(1, 3) \) is defined by the pair \( (l_0, l_1) \), then a conjugated representation is also irreducible and is defined by a pair \( \pm (l_0, -l_1) \). Therefore,

\[
(l_0, l_1) = (-\hat{l}, \hat{l} + 1).
\]

Hence it follows that

\[
\hat{l} = \frac{l_0 - l_1 + 1}{2}.
\]

(46)

For the unitary representations, that is, in the case of principal series representations of the group \( \text{SO}_0(1, 3) \), there exists an analogue of the formula \([13]\)

\[
V_\alpha f (z) = (a_{12} z + a_{22})^{\frac{1}{2} + i \frac{\hat{\alpha}}{2} - 1} (a_{12} z + a_{22})^{\frac{1}{2} + i \frac{\alpha}{2} - 1} f \left( \frac{a_{11} z + a_{21}}{a_{12} z + a_{22}} \right),
\]

(47)

where \( f(z) \) is a measurable function of the Hilbert space \( L_2 (Z) \), satisfying the condition \( \int |f(z)|^2 dz < \infty \), \( z = x + iy \). A totality of all representations \( \alpha \to T^\alpha \), corresponding to all possible pairs \( \lambda, \rho \), is called a principal series of representations of the group \( \text{SO}_0(1, 3) \) and denoted as \( \mathcal{S}_{\lambda, \rho} \). At this point, a comparison of \([17]\) with the formula \([13]\) for the spinor representation \( \mathcal{S}_I \) shows that the both formulas have the same structure; only the exponents at the factors \( (a_{12} z + a_{22}), (a_{12} z + a_{22}) \) and the functions \( f(z) \) are different. In the case of spinor representations the functions \( f(z) \) are polynomials \( p(z, \bar{z}) \) in the spaces \( \text{Sym}_{(k, r)} \), and in the case of a representation \( \mathcal{S}_{\lambda, \rho} \) of the principal series \( f(z) \) are functions from the Hilbert space \( L_2 (Z) \).

We know that a representation \( S_I \) of the group \( \text{SU}(2) \) is realized in terms of the functions

\[
t^l_{mn} (u) = e^{-im \psi} P^l_{mn} (\cos \theta) e^{-im \psi}.
\]

(48)

We use below the following
The comparison of these formulae with (50) and (51) shows that representations $S_k$ of SU(2) is contained in $\mathfrak{g}_{\lambda,\rho}$ not more than one time. At this point, $S_k$ is contained in $\mathfrak{g}_{\lambda,\rho}$, when $\frac{1}{2}$ is one from the numbers $-k, -k + 1, \ldots, k$.

Let us find a relation between the parameters $l_0, l_1$ in the formulae (44) and the parameters $\lambda, \rho$ of the representation $\mathfrak{g}_{\lambda,\rho}$. The number $l_0$ is the lowest from the weights $k$ of representations $S_k$ contained in $\mathfrak{g}_{\lambda,\rho}$. Hence it follows that $k \geq \left| \frac{1}{2} \right|$ and $l_0 = \left| \frac{1}{2} \right|$. Let us consider the operators

$$\Delta = F_+ F_- + F_- F_+ + 2F_3^2 - (H_+ H_- + H_- H_+ + 2H_3^2) = A^2 - B^2,$$  \hspace{1cm} (48)

$$\Delta' = H_+ F_- + H_- F_+ + F_+ H_- + F_- H_+ + 4H_3F_3 = AB.$$  \hspace{1cm} (49)

Applying the formulae (44), we obtain

$$\Delta \xi_{k\nu} = -2(l_0^2 + l_1^2 - 1)\xi_{k\nu},$$  \hspace{1cm} (50)

$$\Delta' \xi_{k\nu} = -4il_0l_1\xi_{k\nu}.$$  \hspace{1cm} (51)

On the other hand, calculating infinitesimal operators of the representation $\mathfrak{g}_{\lambda,\rho}$ in the space $L_2(Z)$, we have

$$A_1 f = \frac{i}{2}(1 - z^2)\frac{\partial f}{\partial z} - \frac{i}{2}(1 - z^2)\frac{\partial f}{\partial \bar{z}} + \frac{i}{2} \left[ \left( \frac{\lambda}{2} + i\frac{\rho}{2} - 1 \right) z - \left( -\frac{\lambda}{2} + i\frac{\rho}{2} - 1 \right) \bar{z} \right] f,$$  \hspace{1cm} (52)

$$A_2 f = \frac{1}{2}(1 + z^2)\frac{\partial f}{\partial z} + \frac{1}{2}(1 + z^2)\frac{\partial f}{\partial \bar{z}} - \frac{1}{2} \left[ \left( \frac{\lambda}{2} + i\frac{\rho}{2} - 1 \right) z + \left( -\frac{\lambda}{2} + i\frac{\rho}{2} - 1 \right) \bar{z} \right] f,$$  \hspace{1cm} (53)

$$A_3 f = iz\frac{\partial f}{\partial z} - i\bar{z}\frac{\partial f}{\partial \bar{z}} - \frac{i}{2}\lambda f,$$  \hspace{1cm} (54)

$$B_1 f = \frac{1}{2}(1 - z^2)\frac{\partial f}{\partial z} + \frac{1}{2}(1 - z^2)\frac{\partial f}{\partial \bar{z}} + \frac{1}{2} \left[ \left( \frac{\lambda}{2} + i\frac{\rho}{2} - 1 \right) z + \left( -\frac{\lambda}{2} + i\frac{\rho}{2} - 1 \right) \bar{z} \right] f,$$  \hspace{1cm} (55)

$$B_2 f = -\frac{i}{2}(1 + z^2)\frac{\partial f}{\partial z} + \frac{i}{2}(1 + z^2)\frac{\partial f}{\partial \bar{z}} + \frac{i}{2} \left[ \left( \frac{\lambda}{2} + i\frac{\rho}{2} - 1 \right) z - \left( -\frac{\lambda}{2} + i\frac{\rho}{2} - 1 \right) \bar{z} \right] f,$$  \hspace{1cm} (56)

$$B_3 f = \bar{z}\frac{\partial f}{\partial z} + \bar{z}\frac{\partial f}{\partial \bar{z}} + \left( 1 - i\frac{\rho}{2} \right) f,$$  \hspace{1cm} (57)

Substituting the latter expressions into (48) and (49), we find that

$$\Delta f(z) = -2 \left[ \left( \frac{\lambda}{2} \right)^2 - \left( \frac{\rho}{2} \right)^2 - 1 \right] f(z),$$

$$\Delta' f(z) = -\lambda\rho f(z).$$

The comparison of these formulae with (50) and (51) shows that

$$\left( \frac{\lambda}{2} \right)^2 - \left( \frac{\rho}{2} \right)^2 = l_0^2 + l_1^2,$$  \hspace{1cm} (58)

$$\lambda\rho = 4il_0l_1.$$  \hspace{1cm} (59)
Let \( l_0 \neq 0 \); since \( l_0 = \left| \frac{\lambda}{2} \right| \), then from (59) it follows that
\[
l_1 = -i (\text{sign} \lambda) \frac{\rho}{2}.
\]
If \( l_0 = 0 \) and, therefore, \( \lambda = 0 \), from (58) we obtain
\[
l_1 = \pm i \frac{\rho}{2}.
\]
Thus, the numbers \( l_0, l_1, \lambda, \rho \) are related by the formulas
\[
\begin{align*}
  l_0 &= \left| \frac{\lambda}{2} \right|, & l_1 &= -i (\text{sign} \lambda) \frac{\rho}{2} \quad \text{if} \quad \lambda \neq 0, \\
  l_0 &= 0, & l_1 &= \pm i \frac{\rho}{2} \quad \text{if} \quad \lambda = 0.
\end{align*}
\]

On the other hand, we see from (7) that Laplace-Beltrami operators
\[
X^2 = -l(l + 1) \quad \text{and} \quad Y^2 = -\dot{l}(\dot{l} + 1)
\]
contain Casimir operators \( \Delta = A^2 - B^2 \) and \( \Delta' = A \cdot B \) as real and imaginary parts:
\[
\begin{align*}
  X^2 &= \frac{1}{4} \Delta + \frac{i}{2} \Delta' = -l(l + 1), \\
  Y^2 &= \frac{1}{4} \Delta - \frac{i}{2} \Delta' = -\dot{l}(\dot{l} + 1).
\end{align*}
\]
Taking into account in the latter operators the formulae (50) and (51), we arrive at relations (45) and (46). The relations (45) and (46) define a relation between parameters \( l_0, l_1 \) of the Gelfand-Naimark basis (44) and parameters \( l, \dot{l} \) of the Van der Waerden basis (6).

As is known, all the unitary representations of the group \( \text{SO}_0(1, 3) \) are infinite-dimensional. The group \( \text{SO}_0(1, 3) \) is non-compact and one from its real forms, the group \( \text{SU}(1, 1) \), is also non-compact group involving unitary infinite-dimensional representations. In the previous section it has been shown that the matrix elements of \( \text{SO}_0(1, 3) \) are defined via the addition theorem for the matrix elements of the subgroups \( \text{SU}(2) \) and \( \text{SU}(1, 1) \). This factorization allows us to separate explicitly in the matrix element all the parameters changing in infinite limits.

In such a way, using Theorem 1, formulae (30), (37) and (45), we find that matrix elements of the principal series representations of the group \( \text{SO}_0(1, 3) \) have the form
\[
\begin{align*}
\mathcal{M}^{\frac{1}{2} + i \varrho, l_0}_{mn}(g) &= e^{-m(\epsilon + i\varphi) - n(\epsilon + i\psi)} Z^{\frac{1}{2} + i \varrho, l_0}_{mn} = e^{-m(\epsilon + i\varphi) - n(\epsilon + i\psi)} \times \\
&\sum_{t=-l_0}^{l_0} i^{m-t} \sqrt{\Gamma(l_0 - m + 1)\Gamma(l_0 + m + 1)\Gamma(l_0 - t + 1)\Gamma(l_0 + t + 1)} \times \\
&\cos^{2l_0} \frac{\theta}{2} \tan^{m-t} \frac{\theta}{2} \times \\
&\sum_{j=\max(0, t-m)}^{\min(l_0 - m, l_0 + t)} i^{2j} \tan^{2j} \frac{\theta}{2} \frac{\Gamma(j + 1)\Gamma(l_0 - m - j + 1)\Gamma(l_0 + t - j + 1)\Gamma(m - t + j + 1)}{\Gamma(\frac{1}{2} + i\varrho - n) \Gamma(\frac{1}{2} + i\varrho + n) \Gamma(\frac{1}{2} + i\varrho - t) \Gamma(\frac{1}{2} + i\varrho + t) \cosh^{-1+2i\varrho} \frac{\tau}{2} \tanh^{n-t} \frac{\tau}{2} \times \\
&\sum_{s=\max(0, t-n)}^{\infty} \frac{\tan^{2s} \frac{\tau}{2} \Gamma(s + 1)\Gamma(\frac{1}{2} + i\varrho - n - s)\Gamma(\frac{1}{2} + i\varrho + t - s)\Gamma(n - t + s + 1)}{\Gamma(s + 1)\Gamma(\frac{1}{2} + i\varrho - n - s)\Gamma(\frac{1}{2} + i\varrho + t - s)\Gamma(n - t + s + 1)}. \quad (60)
\end{align*}
\]
where \( l_0 = \left| \frac{k}{2} \right| \) and \( \frac{k}{2} \) is one from the numbers \(-k, -k + 1, \ldots, k\). It is obvious that \( \mathcal{M}_{m\bar{n}}^{-\frac{1}{2} + i\rho, l_0}(g) \) cannot be attributed as matrix elements to single irreducible representation. From the latter expression it follows that relativistic spherical functions \( f(g) \) of the principal series can be defined by means of the function

\[
\mathcal{M}_{m\bar{n}}^{-\frac{1}{2} + i\rho, l_0}(g) = e^{-m(\xi + i\rho)} Z_{m\bar{n}}^{-\frac{1}{2} + i\rho, l_0}(\cos \theta^c) e^{-n(\xi + i\rho)},
\]

(61)

where

\[
Z_{m\bar{n}}^{-\frac{1}{2} + i\rho, l_0}(\cos \theta^c) = \sum_{t=-l_0}^{l_0} P_{mt}^{l_0}(\cos \theta) \mathcal{P}_{-\frac{1}{2} + i\rho}^t(\cosh \tau).
\]

In the case of relativistic spherical functions \( f(\varphi^c, \theta^c) \) we have

\[
Z_{m\bar{n}}^{-\frac{1}{2} + i\rho, l_0}(\cos \theta^c) = \sum_{t=-l_0}^{l_0} P_{mt}^{l_0}(\cos \theta) \mathcal{P}_{-\frac{1}{2} + i\rho}^t(\cosh \tau),
\]

(62)

where \( \mathcal{P}_{-\frac{1}{2} + i\rho}^t(\cosh \tau) \) are conical functions (see [37]). In this case our result agrees with the paper [38], where matrix elements (eigenfunctions of Casimir operators) of non-compact rotation groups are expressed in terms of conical and spherical functions (see also [21]).

When \( \rho \) is a cleanly imaginary number, \( \rho = i\sigma \), we have

\[
T^\alpha f(z) = |a_{12}z + a_{22}|^{-2-\sigma} f \left( \frac{a_{11}z + a_{21}}{a_{12}z + a_{22}} \right).
\]

This formula defines an unitary representation \( a \to T^\alpha \) of supplementary series \( \mathfrak{D}_\sigma \) of the group \( \text{SO}_0(1, 3) \). In its turn, for the supplementary series \( \mathfrak{D}_\sigma \) the following theorem holds.

**Theorem 2** (Naimark [13]). *The representation \( S_k \) of SU(2) is contained in \( \mathfrak{S}_{\lambda, \rho} \) when \( k \) is an integer number. In this case, \( S_k \) is contained in \( \mathfrak{D}_\sigma \) exactly one time.*

We see that \( \frac{k}{2} = 0 \) should be one from the numbers \(-k, -k + 1, \ldots, k\), therefore, when \( k \) is integer.

Let us find a relation between the parameters \( l_0, l_1 \) in (44) and the parameter \( \sigma \) of \( \mathfrak{D}_\sigma \). First, the lowest wight \( l_0 \) from the weights \( k \) of the representations \( S_k \), contained in \( \mathfrak{D}_\sigma \), is equal to zero, that is, \( l_0 = 0 \). With the aim to define the parameter \( l_1 \) let us consider again the Casimir operator \( \Delta = A^2 - B^2 \). Calculating this operator with the help of formulae (44) and (47)–(51), where \( \frac{k}{2} + i\frac{\rho}{2} - 1 \) and \(-\frac{k}{2} + i\frac{\rho}{2} - 1 \) should be replaced by \(-\frac{\sigma}{2} - 1 \) and \(-\frac{\sigma}{2} - 1 \), we obtain

\[
\Delta \xi_{k\nu} = -2(l^2_1 - 1) \xi_{k\nu}, \quad \Delta f(z) = -2 \left[ \left( \frac{\sigma}{2} \right)^2 - 1 \right] f(z).
\]

Hence it follows that \( l^2_1 = \left( \frac{\sigma}{2} \right)^2 \) and \( l_1 = \pm \frac{\sigma}{2} \) (the choice of the sign is not important). Thus, for the supplementary series the relations

\[
l_0 = 0, \quad l_1 = \pm \frac{\sigma}{2}
\]

hold. In the case of \( \mathfrak{D}_\sigma \), Laplace-Beltrami operators \( X^2 \) and \( Y^2 \) are coincide with each other, \( X^2 = Y^2 = A^2 - B^2 \). This means that we come here to representations of \( \text{SO}_0(1, 3) \) restricted to the subgroup SU(1, 1).
Thus, matrix elements of supplementary series appear as a particular case of the matrix elements of the principal series at $l_0 = 0$ and $\rho = i\sigma$:

$$M_{mn}^{-\frac{1}{2}-\sigma}(g) = e^{-m(\epsilon+i\varphi)-n(\epsilon+i\psi)} Z_{mn}^{-\frac{1}{2}-\sigma} = e^{-m(\epsilon+i\varphi)-n(\epsilon+i\psi)} \times \sqrt{\Gamma\left(\frac{1}{2} - \sigma - n\right) \Gamma\left(\frac{1}{2} - \sigma + n\right) \Gamma\left(\frac{1}{2} - \sigma - m\right) \Gamma\left(\frac{1}{2} - \sigma + m\right) \cosh^{-1-2\sigma} \frac{\tau}{2} \tanh^{n-m} \frac{\tau}{2} \times \sum_{s=\max(0,m-n)}^\infty \frac{\tanh^{2s} \frac{\tau}{2}}{\Gamma(s+1) \Gamma\left(\frac{1}{2} - \sigma - n - s\right) \Gamma\left(\frac{1}{2} - \sigma + m - s\right) \Gamma(n - m + s + 1)}}, \quad (63)$$

Or

$$M_{mn}^{-\frac{1}{2}-\sigma}(g) = e^{-m(\epsilon+i\varphi)} P_{mn}^{-\frac{1}{2}-\sigma}_\rho(\cosh \tau) e^{-n(\epsilon+i\psi)},$$

that is, the hyperspherical function $Z_{mn}^{-\frac{1}{2}+i\rho,l_0}(\cos \theta^c)$ in the case of supplementary series is reduced to the Jacobi function $P_{mn}^{-\frac{1}{2}+\rho}(\cosh \tau)$. For the relativistic spherical functions $f(\varphi^c, \theta^c) \sim f(\varphi^c, \tau)$ of supplementary series we obtain

$$M_{mn}^{-\frac{1}{2}-\sigma}(g) = e^{-m(\epsilon+i\varphi)} P_{mn}^{-\frac{1}{2}-\sigma}(\cosh \tau).$$

Let us express now the relativistic spherical function $M_{mn}^{-\frac{1}{2}+\rho,l_0}(g)$ of the principal series representations of $SO_0(1,3)$ via the hypergeometric function. Using the formulae (60), (61), (62) and (61), (62), we find

$$M_{mn}^{-\frac{1}{2}+i\rho,l_0}(g) = e^{-m(\epsilon+i\varphi)-n(\epsilon+i\psi)} \sqrt{\Gamma\left(l_0 + m + 1\right) \Gamma\left(i\rho - n + \frac{1}{2}\right)} \times \cos^{2l_0} \frac{\theta}{2} \cosh^{-1+2i\rho} \frac{\tau}{2} \times \sum_{t=-l_0}^{l_0} t^{m-t} \tan^{m-t} \frac{\theta}{2} \tanh^{n-t} \frac{\tau}{2} \times 2F_1\left(\begin{array}{c} m - l_0, -t - l_0 \\ m - t + 1 \end{array} \right) - \tan^2 \frac{\theta}{2} 2F_1\left(\begin{array}{c} t - i\rho + \frac{1}{2}, -n - i\rho + \frac{1}{2} \\ t - n + 1 \end{array} \right) \tanh^2 \frac{\tau}{2}\right), \quad m \geq n;$$

$$M_{mn}^{-\frac{1}{2}+i\rho,l_0}(g) = e^{-m(\epsilon+i\varphi)-n(\epsilon+i\psi)} \sqrt{\Gamma\left(l_0 + m + 1\right) \Gamma\left(i\rho - n + \frac{1}{2}\right)} \times \cos^{2l_0} \frac{\theta}{2} \cosh^{-1+2i\rho} \frac{\tau}{2} \times \sum_{t=-l_0}^{l_0} t^{m-t} \tan^{m-t} \frac{\theta}{2} \tanh^{n-t} \frac{\tau}{2} \times 2F_1\left(\begin{array}{c} t - l_0, -m - l_0 \\ t - m + 1 \end{array} \right) - \tan^2 \frac{\theta}{2} 2F_1\left(\begin{array}{c} n - i\rho + \frac{1}{2}, -t - i\rho + \frac{1}{2} \\ n - t + 1 \end{array} \right) \tanh^2 \frac{\tau}{2}\right), \quad n \geq m.$$

For the supplementary series we have

$$M_{mn}^{-\frac{1}{2}-\sigma}(g) = e^{-m(\epsilon+i\varphi)-n(\epsilon+i\psi)} \sqrt{\Gamma\left(m - \sigma + \frac{1}{2}\right) \Gamma\left(-n - \sigma + \frac{1}{2}\right)} \times \cosh^{-1-2\sigma} \frac{\tau}{2} \tanh^{m-n} \frac{\tau}{2} \times 2F_1\left(\begin{array}{c} m + \sigma + \frac{1}{2}, -n + \sigma + \frac{1}{2} \\ m - n + 1 \end{array} \right) \tanh^2 \frac{\tau}{2}\right), \quad m \geq n;$$
\[ M_{mn}^{\frac{1}{2}-\sigma}(g) = e^{-m(\epsilon+i\phi)-n(\epsilon+i\psi)} \sqrt{\frac{\Gamma(n-\sigma+\frac{1}{2})\Gamma(-m-\sigma+\frac{1}{2})}{\Gamma(-n-\sigma+\frac{1}{2})\Gamma(m-\sigma+\frac{1}{2})}} \times \cosh^{-1-2\sigma}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2} F_{1}\left(\begin{array}{c} n + \sigma + \frac{1}{2}, -m + \sigma + \frac{1}{2} \\ n - m + 1 \end{array}\right) \left| \frac{\tanh^{2}\frac{\tau}{2}}{2} \right), \quad n \geq m.\]

In like manner we can define conjugated spherical functions \( f(g) = M_{mn}^{\frac{1}{2}-i\rho,l_0}(g) \) and \( f(\dot{\phi}^c, \dot{\theta}^c) = M_{mn}^{\frac{1}{2}-i\rho,l_0}(\dot{\phi}^c, \dot{\theta}^c, 0) \) of the principal series \( \mathfrak{S}_{\lambda,\rho} \), since a conjugated representation of \( SO_{0}(1,3) \) is defined by the pair \( \pm(l_0, -l_1) \). It is obvious that in the case of supplementary series \( \mathfrak{D}_{\sigma} \) we arrive at the same functions \( M_{mn}^{\frac{1}{2}-\sigma}(g) \).

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