Vibrations of rigid cantilevered plates in a rectangular tank filled with liquid

Goncharov D.A.¹, Pozhalostin A.A.¹

¹Bauman Moscow State Technical University, 2-ya Baumanskaya str. 5, b.1, Moscow 105005, Russia
E-mail: goncharov@bmstu.ru

Abstract. The article presents the developed analytical (exact) methods for solving problems of calculating the frequencies of natural vibrations of rigid and elastic partitions in a rigid rectangular tank filled with liquid. The results of solving these problems can be used in the design of hydraulic structures, as well as in shipbuilding when analysing the dynamics of structural elements of large oil tankers and gas carriers.

1. Introduction

4 cases were considered. In the first case, the tank is closed at one end by an elastically sealed rigid plate. In the second, the plate is elastically fixed in the middle of the tank. In the third case, two plates from cases 1 and 2 are located in the tank. In the fourth case, the solution of the boundary value problem for an elastic plate-console located in the middle of the tank is considered.

In the latter case, a frequency equation is obtained, the left side of which is a meromorphic function with an infinite number of simple poles.

The article presents an analytical solution to the problem of small oscillations of a liquid in a rigid tank with an elastic baffle. This mechanical system is a model for various oil tankers. The frequency transcendental equation of the problem is obtained in closed form under certain assumptions, the left side of which is a meromorphic function with an infinite number of simple poles. This result is achieved by expanding the homogeneous solution into a series in terms of the eigenfunctions of the Laplace equation in a rectangular coordinate system. One of the first specialists to obtain results in this problem of hydroelasticity was P.K. Ishkov. [4].

![Figure 1. Schematic representation of the tank](image_url)
Later, this technique was successfully used by L.S. Leibenzon. [3], Balabukh L.I. [2] and other researchers.

In the first three cases, the solution is constructed in such a way that the associated moments of inertia from the fluid.

In the latter case, a rigorous solution of the boundary value problem is obtained within the accepted assumptions, namely, for the plane problem and small fluctuations.

2. Boundary value problem

We consider the plane problem of vibrations of a tank closed by an elastically sealed partition is considered. The partition is absolutely rigid, weighing \( m_0 \) performs angular oscillations according to the law \( \varphi(t) \). We put \( w(t) = y\varphi(t) \). There is an elastic connection, torsional stiffness \( C_0 \). The liquid fills a rectangular tank with a length \( L \) to a height \( H \).

This boundary value problem is solved under the following assumptions: the fluid is assumed to be ideal and incompressible, its motion is potential, with the velocity potential \( \Phi \). The dynamic pressure of the fluid through the Lagrange-Cauchy integral is determined by the formula \( P = \rho \frac{\partial\Phi}{\partial t} \).

Here \( \rho \) is the density of the liquid. The velocity of a liquid particle will be determined by the relation:

\[
\nu = -\text{grad} \Phi.
\]

The velocity potentials are solutions of the Laplace equation (Fig. 1)

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0
\]  

The boundary conditions for the velocity potential are:

\[
\frac{\partial \Phi}{\partial x} \Big|_{x=0} = 0, \quad \frac{\partial \Phi}{\partial x} \Big|_{x=L} = \hat{w}.
\]  

We will seek the velocity potential in the form:

\[
\Phi = \sum_{i=1}^{\infty} X_i(x)Y_i(y)\hat{s}(t).
\]

On the free surface, we take the linearized boundary condition \( \Phi \big|_{y=H} = 0 \). Satisfying the boundary conditions (2) and the condition on the free surface, we obtain the velocity potential in the form:

\[
\Phi = \sum_{i=1}^{\infty} C_i \cosh \mu_i x \cos \mu_i y s(t).
\]

With this choice of potential, we focus on plate vibrations with a liquid, while calculating the attached moment of inertia from the liquid. \( \mu_i = \frac{\pi(2i-1)}{2H} \) are the eigenvalues of the considered boundary value problem. The system of eigenfunctions \( Y_i(y) \) is a complete, orthogonal system of functions. Since \( y\varphi = w \), we expand \( y \) in a series in the system of functions \( \cos \mu_i y \),

\[
y = \sum_{i=1}^{\infty} \alpha_i \cos \mu_i y \text{ we get}
\]

\[
\alpha_i = \left( -1 \right)^{i+1} \frac{H}{\mu_i^2} \frac{1}{\mu_i - 1} \mu_i^2
\]

Taking into account (5), the velocity potential takes the form
The differential equation of the rotational motion of the plate under the action of the hydrodynamic pressure of the liquid has the form:
\[ J_0 \ddot{\phi} + C\dot{x} = -\int_0^H \rho \sum_{i=1}^n \mu_i \cosh \mu_i L(x, y) y \ddot{\phi} \mathrm{d}y. \]

Finally, \( J = J_0 + \sum_{i=1}^n \gamma_i \) is a moment of inertia of the plate taking into account the liquid. The expression for the constant \( \gamma \) is not shown, the moment of forces is also not taken into account.

In the second case, an elastically fixed plate is located in the middle of a rectangular tank of length \( L = l_1 + l_2 \). This boundary value problem is solved under the same assumptions that were made in the first case. Velocity potential in the left compartment is \( \Phi_1 \) and in the right is \( \Phi_2 \). They satisfy the Laplace equation and have the form:
\[ \Phi_1 = \sum_{i=1}^n A_i \cos \mu_i y \cosh \mu_i x \hat{s}_1, \quad \mu_i = \frac{\pi (2i-1)}{2H}, \quad (7) \]
\[ \Phi_2 = \sum_{i=1}^n B_i \cos \mu_i y (D_i \cosh \mu_i x + D_2 \sinh \mu_i x) \hat{s}_2. \quad (8) \]

The boundary condition for the potentials is the equality of the fluid velocities at \( x = l_1 \).
\[ \frac{\partial \Phi_1}{\partial x} \bigg|_{x=l_1} = \frac{\partial \Phi_2}{\partial x} \bigg|_{x=l_1}. \quad (9) \]

whence \( \hat{s}_1 = \hat{s}_2 \). Taking into account (9), the velocity potential functions have the form
\[ \Phi_1 = \sum_{i=1}^n D_i \gamma \cos \mu_i y \cosh \mu_i x \hat{s}_1, \]
\[ \Phi_2 = \sum_{i=1}^n D_i \cos \mu_i y (\cosh \mu_i x - \tanh \mu_i L \sinh \mu_i x) \hat{s}_1. \]

In this case, the differential equation of plate vibrations will be:
\[ J \ddot{\phi} + C\dot{x} = 0, \quad \text{where} \quad J = J_0 - \beta_1 + \beta_2. \]

They are determined from the formulas
\[ \int_0^H p_1 y \mathrm{d}y = \beta_1 \ddot{\phi} \quad \text{and} \quad -\int_0^H p_2 y \ddot{\phi} \mathrm{d}y = \beta_2 \ddot{\phi}. \]

In the third case, in a rigid rectangular tank of length \( L = l_1 + l_2 \) are two rigid rectangular plates elastically fixed as in the first case - one plate is located in the middle of the tank, and the second is elastically fixed at the end of the tank at \( x = L \) (Fig. 1). The assumptions under which this boundary value problem is solved are similar to those that were earlier when solving the first two problems. Taking into account the boundary conditions on the free surface, on the bottom and on the walls of the vessel, we obtain the following expressions for the velocity potentials
\[ \Phi_1 = \sum_{i=1}^n A_i \gamma_i \cos \mu_i y \cosh \mu_i x \hat{s}_1, \]
\[ \Phi_i = \sum_{j=1}^{\infty} \cos \mu y \left( A_j \cosh \mu x + A_j \sinh \mu x \right) s_j. \]

Since the velocity of the point on the surface of the middle plate is equal to the velocity of the liquid particle, and the velocity of the point of the side plate 2 is equal to the velocity of the liquid particle of the second potential and the velocities of both potentials are determined in accordance with (9), then after simple transformations and using the expansion \( y \) in a series in the system of functions \( \cos \mu y \), we get
\[ A_1 \dot{s}_1 = \beta_1 \dot{\phi}_1 + \beta_2 \dot{\phi}_2, \]
\[ A_2 \dot{s}_2 = \beta_3 \dot{\phi}_1 + \beta_4 \dot{\phi}_2. \]

Here \( \beta \) are some constants whose form is not given.

The differential equations for the vibrations of the plates are as follows:
\[ J_i \ddot{\phi}_1 + C_i \phi_1 = \gamma_{11} \dot{\phi}_1 + \gamma_{12} \dot{\phi}_2, \]
\[ J_2 \ddot{\phi}_2 + C_2 \phi_2 = \gamma_{21} \dot{\phi}_1 + \gamma_{22} \dot{\phi}_2. \]

Here \( \gamma_{ij} \) are some coefficients that are rapidly converging series. The equations represent a system of two differential equations as in the case of oscillations of a mechanical system with two degrees of freedom with inertial constraint.

3. Conclusion

The compartments of a rectangular rigid tank are separated by a uniform ideally elastic cantilever-sealed beam of linear mass \( \mu_0 \) of height \( H \). The material of the beam obeys Hooke's law, its bending is considered to be straight, and \( EJ_0 \) is its linear bending stiffness [5], \( \Phi(x,y,t) \) is the velocity potential. We will use the Fourier method [4], then the solution to equation (1) has the form similar to examples 1-3, where the constants \( A_j, E_j, B_j, D_j \) are to be determined, \( s(t) \) is a time function. The differential equation of bending vibrations of the beam will have the form
\[ EJ_0 \dot{w}_x^y - \mu_0 \omega^2 w = \beta_1 \Phi_1 - \beta_2 \Phi_2 \quad (10) \]

The velocity potential functions satisfy the boundary conditions on the free surface and the impermeability conditions. Using the property of orthogonality of the vibration modes \( \Phi_i \), we obtain the dependences for arbitrary constants \( A_j, E_j, B_j, D_j \) through the constants of the homogeneous solution of the differential equation (10). Satisfying the conditions for fixing the cantilever beam We obtain the frequency equation of the problem (in closed form) whose left side is a meromorphic function.

References
[1] Kochin N.E. and others. Theoretical hydromechanics. M., FN. GIFML. 1963.
[2] Balabukh L.I., Molchanov A.G. PMT, Vol. 6, 1966, p. 1098-1102.
[3] Leibenzon L.S. Reports of the USSR Academy of Sciences, 1951, Vol. 1, p. 125-133.
[4] Ishkov P.K. PMM. 1937, Vol. 1, pp. 1 - 18
[5] Strelkov S.P. Introduction to the theory of oscillations, Nauka, 1964, 437 p.