Special relativity in terms of Lie groups

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The special theory of relativity is constructed demanding the retention of the rectilinear form of a trajectory and invariance of the wave equation under linear transformations of space and time coordinates. The usual approach to relativity based on manipulations with the impulse of light is shown to be owing to that the symmetry of the particular solution of the wave equation coincides with the symmetry of the very wave equation. Thereof instead of the equation in partial derivatives we may deal with the algebraic form referred to as the interval.

1. INTRODUCTION

The central instrument of the special relativity theory is a pulse of light. The light is usually treated in this theory merely as a signal with no insight into its physical nature. The only property necessary to be specified is a peculiar feature of the speed of delivery of the signal. In the current discourse we construct the theory of relativity taking into account explicitly that the light is a wave. The content of the theory becomes the demand that the wave equation and dynamic equation of mechanics have one and the same symmetry.

2. INERTIAL FRAMES OF REFERENCE: CLASSICAL DEFINITION

Axiom. There exists at least one frame of reference \( x, y, z, t \) where a free material point describes a rectilinear trajectory. For simplicity we will consider only a one-dimensional case

\[
\frac{d^2x}{dt^2} = 0
\]

where \( u \) and \( x_0 \) are constants. We will find other reference frames \( x', t' \) where this trajectory is rectilinear as well. These are obviously all frames of reference which can be obtained by affine transformations of the original reference frame. The translation is

\[
x' = x + \delta, \quad t' = t + \tau,
\]

where \( \delta \) and \( \tau \) are variable parameters. The extension is

\[
x' = x + \lambda t, \quad t' = t + \mu t.
\]

The linear transformation that intermixes the space and time coordinates is

\[
x' = x + \alpha t, \quad t' = t + \beta x.
\]

The particular type of \( (6), (7) \) is given by the Galileo transformation

\[
x' = x - vt, \quad t' = t.
\]

All frames of reference obtained by such transformations are called inertial frames of reference in the classical sense. Otherwise we may define inertial frames of reference as those which do not change the form of the equation

\[
\frac{d^2x}{dt^2} = 0
\]

that specifies the family of straight lines. The form \( (10) \) eliminates transformations \( (7) \) with \( \beta \neq 0 \) and restricts \( (4), (5) \) by a dissimilar extension (see Appendix A).
3. INERTIAL FRAMES OF REFERENCE: RELATIVISTIC DEFINITION

We introduce another restriction in the definition of inertial frames of reference. Consider reference frames obtained by linear transformations that do not change the equation of the electromagnetic wave

\[
\frac{\partial^2 A}{\partial t^2} = c^2 \frac{\partial^2 A}{\partial x^2},
\]

(11)

where \(c\) is the speed of light. The extension (12), (13) complies this requirement when it is a similarity transformation for variables \(x\) and \(ct\). The transformation (6), (7) does not in general leave invariant Eq. (11). It works only provided that \(\beta = \alpha/c^2\)

\[
x' = x + \alpha ct,
\]

(12)

\[
t' = t + \alpha x/c,
\]

(13)

and \(\alpha \to 0\). We may verify this substituting (12), (13) into Eq. (11) and neglecting \(\alpha^2\) terms:

\[
\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'},
\]

(14)

\[
\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x'} \left( \frac{\partial}{\partial x'} + \frac{\alpha \partial}{\partial t'} \right) + \frac{\alpha}{c} \frac{\partial}{\partial t'} \left( \frac{\partial}{\partial x'} + \frac{\alpha \partial}{\partial t'} \right) \approx \frac{\partial^2}{\partial x'^2} + \frac{2\alpha c^2 \partial^2}{\partial t' \partial x'}.
\]

(15)

\[
\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'},
\]

(16)

\[
\frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial t'} \left( \frac{\partial}{\partial t'} + \frac{\alpha \partial}{\partial x'} \right) + \frac{\alpha c}{\partial x'} \left( \frac{\partial}{\partial t'} + \frac{\alpha \partial}{\partial x'} \right) \approx \frac{\partial^2}{\partial t'^2} + \frac{2\alpha c^2}{\partial t' \partial x'}.
\]

(17)

4. LIE GROUPS

We may try to construct a finite transformation by a successive application of infinitesimal steps (12), (13):

\[
x'' = x' + \beta ct' = x + \alpha ct + \beta c(t + \alpha x/c) \approx x + \gamma ct,
\]

(18)

\[
t'' = t' + \beta x'/c = t + \alpha x/c + \beta(x + \alpha ct)/c \approx t + \gamma x/c,
\]

(19)

where

\[
\gamma = \alpha + \beta.
\]

(20)

Relations (12), (13) and (18), (19) with (20) say that these infinitesimal transformations form a one-parameter Lie group.

In general, transformations

\[
x' = \Phi(x, t, \alpha), \quad t' = \Psi(x, t, \alpha)
\]

(21)

form a group with the parameter \(\alpha\) if from (21) and

\[
x'' = \Phi(x', t', \beta), \quad t'' = \Psi(x', t', \beta)
\]

(22)

follows

\[
x'' = \Phi(x, t, \gamma), \quad t'' = \Psi(x, t, \gamma)
\]

(23)

with the group operation

\[
\gamma = \varphi(\alpha, \beta).
\]

(24)
For example, similarity transformations

\[ x' = x + \alpha x, \quad t' = t + \alpha t \]  \hspace{1cm} (25)

form a Lie group with the group operation

\[ \gamma = \alpha + \beta + \alpha \beta. \]  \hspace{1cm} (26)

The group parameter is said to be canonical if it is additive as in (20).

The coordinates transformation group is the way that the transition between certain reference frames can be parameterized.

We can always accommodate the group parameter \( \alpha \) so that the identity transformation will correspond to \( \alpha = 0 \). Then, transformation (21) can be expanded into the Taylor series with the linear part

\[ x' = x + \alpha \zeta(x, t) + ..., \quad t' = t + \alpha \eta(x, t) + ..., \]  \hspace{1cm} (27)

where functions \( \zeta(x, t) \) and \( \eta(x, t) \) are referred to as the kernel of the group. It can be shown (the second Lie theorem) that the kernel \( \zeta, \eta \) of the group having been given we may restore the whole transformation \( \Phi, \Psi \) solving the following set of ordinary differential equations; when the group parameter \( \alpha \) is canonical these equations are

\[ \frac{dx'}{d\alpha} = \zeta(x', t'), \quad \frac{dt'}{d\alpha} = \eta(x', t') \]  \hspace{1cm} (28)

(see Appendix B).

5. THE LORENTZ GROUP

Applying Eqs. (27), (28) to (12), (13) with the account of (20) we can write down the following equations

\[ \frac{dx'}{d\alpha} = ct', \quad \frac{dt'}{d\alpha} = x'/c. \]  \hspace{1cm} (29)

Eqs. (29) have the solution

\[ x' = x \cosh \alpha + ct \sinh \alpha, \quad ct' = x \sinh \alpha + ct \cosh \alpha. \]  \hspace{1cm} (30)

Substituting (30) in

\[ x'' = x' \cosh \beta + ct' \sinh \beta, \quad ct'' = x' \sinh \beta + ct' \cosh \beta, \]  \hspace{1cm} (31)

we may verify that the transformation (30) is a group and the group parameter is canonical:

\[ x'' = (x \cosh \alpha + ct \sinh \alpha) \cosh \beta + (x \sinh \alpha + ct \cosh \alpha) \sinh \beta \]  \hspace{1cm} (32)

\[ = x (\cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta) + ct (\sinh \alpha \cosh \beta + \cosh \alpha \sinh \beta) \]

\[ = x \cosh (\alpha + \beta) + ct \sinh (\alpha + \beta), \]

\[ ct'' = (x \cosh \alpha + ct \sinh \alpha) \sinh \beta + (x \sinh \alpha + ct \cosh \alpha) \cosh \beta \]  \hspace{1cm} (33)

\[ = x (\cosh \alpha \sinh \beta + \sinh \alpha \cosh \beta) + ct (\sinh \alpha \sinh \beta + \cosh \alpha \cosh \beta) \]

\[ = x \sinh (\alpha + \beta) + ct \cosh (\alpha + \beta). \]

Transforming partial derivatives as in (14)-(17) we may be convinced another time that (30) represents a symmetry transformation of the equation (11) (see Appendix C).

Belonging of coordinates transformation to a symmetry group indicates that all reference frames defined by the transformation enjoy equal rights in relation to the property admitting this symmetry group. In other words in the bounds of the Lorentz group, i.e. among inertial frames of reference, there is no a preferable reference frame.
6. AN EXTENSION TO DERIVATIVES

We may attach a physical sense to the parameter of the Lorentz group \((30)\). To this end we will construct the corresponding infinitesimal group transformation for the velocity

\[
\dot{x}' = \frac{dx'}{dt}'.
\]

(34)

Substituting (12) and (13) in (34) and neglecting terms with \(\alpha^2\) we obtain

\[
\dot{x}' = \frac{dx'}{dt}' = \frac{dx + \alpha c dt}{dt + \alpha dx/c} = \frac{\dot{x} + \alpha c}{1 + \alpha \dot{x}/c} \approx \dot{x} + (c - \dot{x}^2/c)\alpha.
\]

(35)

From (35) we may construct the differential equation for the respective group transformation

\[
\frac{d\dot{x}'}{d\alpha} = c - \dot{x}'^2/c.
\]

(36)

The solution to (36) is given by

\[
\ln \left| \frac{1 + \dot{x}'/c}{1 + \ddot{x}/c} \right| - \ln \left| \frac{1 - \dot{x}'/c}{1 - \ddot{x}/c} \right| = 2\alpha.
\]

(37)

Let the frame of reference \(x', t'\) moves with the velocity \(v\) relative to the fixed frame of reference \(x, t\). Then we have for the origin of the reference frame \(x', t'\): \(\dot{x}' = 0\) and \(\ddot{x} = v\). Substituting this to (37) we find

\[
\alpha = \frac{1}{2} \ln \left| \frac{1 - v/c}{1 + v/c} \right|.
\]

(38)

Substituting (38) in (30) gives the Lorentz transformation of space and time coordinates

\[
x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}.
\]

(39)

Substituting (38) in (37) gives the respective group transformation of the velocity \(\dot{x}\)

\[
\dot{x}' = \frac{\dot{x} - v}{1 - \dot{x}^2/c^2}.
\]

(40)

7. RELATIVISTIC MECHANICS

Now we must correct Eq. (10) in order it will be invariant under the Lorentz transformation (39). To this end we will find the infinitesimal group transformation for the acceleration

\[
\ddot{x}' = \frac{d\ddot{x}'}{dt}'.
\]

(41)

Substituting (35) and (13) in (41) and neglecting terms with \(\alpha^2\) we obtain

\[
\ddot{x}' = \frac{d\ddot{x}'}{dt}' = \frac{d\ddot{x} - \alpha^2 \dot{x} \ddot{x}/c}{dt + \alpha dx/c} = \frac{\ddot{x} - \alpha^2 \dot{x} \ddot{x}/c}{1 + \alpha \dot{x}/c} \approx \ddot{x} - 3\dot{x}^2 \alpha/c.
\]

(42)

From (42) we can find the differential equation for the respective group transformation

\[
\frac{d\ddot{x}'}{d\alpha} = -3\dot{x}' \ddot{x}'/c.
\]

(43)

In general, we are searching the form \(G(x', t', \dot{x}', \ddot{x}')\) that does not change under the extended Lorentz transformation, i.e.

\[
\frac{dG}{d\alpha} = 0.
\]

(44)
Using (29), (36) and (43) we find
\[
\frac{dG}{d\alpha} = \frac{\partial G}{\partial t} \frac{dt}{d\alpha} + \frac{\partial G}{\partial x} \frac{dx}{d\alpha} + \frac{\partial G}{\partial \dot{x}} \frac{d\dot{x}}{d\alpha} + \frac{\partial G}{\partial \ddot{x}} \frac{d\ddot{x}}{d\alpha} = \frac{x'}{c} \frac{\partial G}{\partial t} + ct \frac{\partial G}{\partial x} + (c - \frac{\dot{x}'^2}{c}) \frac{\partial G}{\partial \dot{x}} - \frac{3\dot{x}' \ddot{x}'}{c} \frac{\partial G}{\partial \ddot{x}'}.
\]

(45)

Insofar as the form sought for does not depend on \(x\) and \(t\) we must find the solution \(G_3(\dot{x}', \ddot{x}')\) to the following equation in partial derivatives
\[
(c - \frac{\dot{x}'^2}{c}) \frac{\partial G_3}{\partial \dot{x}'} - \frac{3\dot{x}' \ddot{x}'}{c} \frac{\partial G_3}{\partial \ddot{x}'} = 0.
\]

(46)

The differential invariant \(G_3\) can be found as the integral of the respective ordinary differential equation constructed from (46)
\[
\frac{dx'}{c - \dot{x}'^2/c} = \frac{cd\dot{x}'}{3\dot{x}' \ddot{x}'}.
\]

(47)

This integral is
\[
G_3 = \frac{\ddot{x}'}{(1 - \dot{x}'^2/c^2)^{3/2}}.
\]

(48)

The form (48) should replace the left-hand part of Eq. (10):
\[
\ddot{x} \frac{(1 - \dot{x}'^2/c^2)_{3/2}}{0}.
\]

(49)

Equation (49) defines a rectilinear trajectory (1). Eqs. (49) and (11) are invariant under the Lorentz transformation (39) or (39) extended according to Eqs. (36) and (43). Because of the difference in symmetries of equations (49) and (11), extensions are excluded from the consideration. Thus, the class of inertial reference frames is defined by the Lorentz group, and the space and time translations.

Using (48) we may construct the relativistic form of the second law of classical mechanics
\[
\frac{d}{dt} \left[ \frac{m\dot{x}}{(1 - \dot{x}'^2/c^2)^{1/2}} \right] = F.
\]

(50)

8. INTERVAL

Special relativity is usually constructed starting from the notion of the interval
\[
x^2 - c^2t^2.
\]

(51)

Interval (51) is invariant under the Lorentz transformation (39). In this section we will establish a relation between the standard approach and the approach developed in the current report.

First of all we will notice that the form (51) can be obtained as a differential invariant \(G_1\) from the first two terms of (45)
\[
\frac{x}{c} \frac{\partial G}{\partial t} + ct \frac{\partial G}{\partial x} = 0.
\]

(52)

Therefore when studying the symmetry of vacuum we may deal with the algebraic expression (51) instead of the equation (11) in partial derivatives.

Secondly I shall explain why in the phenomenological theory of relativity it is sufficient to use properties of the light signal. The general solution of the d’Alembert equation (11) is
\[
\Phi_1(x - ct) + \Phi_2(x + ct)
\]

(53)

where \(\Phi_1\) and \(\Phi_2\) are arbitrary functions. A differential equation having a symmetry does not imply that a solution of this equation also possesses the same symmetry. However we may find a particular solution of the wave equation
whose symmetry coincides with the symmetry of the very wave equation. Considering a point disturbance emitted from the origin of coordinates we have for \( \delta \)
\[
\delta(x - ct) + \delta(x + ct). \tag{54}
\]
The following relation follows from general properties of the \( \delta \)-function
\[
\delta(x - ct) + \delta(x + ct) = 2a\delta(x^2 - c^2t^2) \tag{55}
\]
where \( a > 0 \) is a constant. Formula (55) can be verified, say, integrating it over \( x \) and taking \( a = ct \) (see Appendix D). The form in the right-hand part of (55) is invariant under the Lorentz transformation (39) since any function of the invariant is the invariant of the group. Physically the form (54) corresponds to a light impulse. Thus, the special theory of relativity can be constructed dealing only with a light signal.

**APPENDIX A: INVARIANCE OF CLASSICAL MECHANICS**

We have for the affine transformation
\[
x' = \Lambda x + \alpha t + \delta, \tag{A1}
\]
\[
t' = \beta x + \Omega t + \tau. \tag{A2}
\]
Taking the differential of (A1) and (A2):
\[
\frac{dx'}{dt'} = \Lambda dx + \alpha dt, \tag{A3}
\]
\[
\frac{dt'}{dt'} = \beta dx + \Omega dt. \tag{A4}
\]
Dividing (A3) by (A4):
\[
\frac{dx'}{dt'} = \frac{\Lambda dx + \alpha dt}{\beta dx + \Omega dt} = \Lambda \frac{dx}{dt} + \frac{\alpha}{\beta \frac{dx}{dt}} + \Omega. \tag{A5}
\]
Rewriting (A5) into the linear form:
\[
\frac{dx'}{dt'}(\beta \frac{dx}{dt} + \Omega) = \Lambda \frac{dx}{dt} + \alpha. \tag{A6}
\]
Differentiating (A6):
\[
(\beta \frac{dx}{dt} + \Omega)\frac{d^2x'}{dt'^2} + \frac{dx'}{dt'} \frac{d^2x}{dt^2} = \Lambda \frac{d^2x}{dt^2}. \tag{A7}
\]
Dividing (A7) by (A4) we get
\[
\frac{d^2x'}{dt'^2} = \frac{\frac{d^2x}{dt^2} - \beta \frac{dx'}{dt'}}{(\Omega + \beta \frac{dx}{dt})^2}. \tag{A8}
\]
The retention of the form (10) requires
\[
\beta = 0, \quad \Lambda = \Omega^2. \tag{A9}
\]
Substituting (A9) into (A1), (A2) we obtain the affine symmetry transformation of the form (10):
\[
x' = \Omega^2 x + \alpha t + \delta, \tag{A10}
\]
\[
t' = \Omega t + \tau. \tag{A11}
\]
APPENDIX B: THE SECOND LIE THEOREM

We consider the transformation of the variable $x$ to $x'$

$$x' = T(x, \alpha) \tag{B1}$$

which depends on the parameter $\alpha$ so that

$$x = T(x, 0). \tag{B2}$$

Further, let $x'$ be transformed to $x''$:

$$x'' = T(x', \beta). \tag{B3}$$

Substituting (B1) into the right-hand part of (B3) we obtain the composition of the two transformations:

$$x'' = T(T(x, \alpha), \beta). \tag{B4}$$

We are interested in the case when (B4) belongs to the same set of transformations as (B1) and (B3):

$$x'' = T(x, \gamma) \tag{B5}$$

where

$$\gamma = \varphi(\alpha, \beta). \tag{B6}$$

If (B1) and (B3) imply (B5), (B6) then we say that (B1), (B3) and (B5) form a Lie group with the group operation (B6).

The function (B1) can be expanded into the Taylor series. We have with the account of (B2)

$$T(x, \alpha) = x + \frac{\alpha}{1!}[\partial_\alpha T(x, \alpha)]_{\alpha=0} + \frac{\alpha^2}{2!}[\partial^2_\alpha T(x, \alpha)]_{\alpha=0} + \frac{\alpha^3}{3!}[\partial^3_\alpha T(x, \alpha)]_{\alpha=0} + .... \tag{B7}$$

So, in order to define a function $T(\alpha)$ we must know all its derivatives for $\alpha = 0$.

**Theorem.** The group $T(x, \alpha)$ is fully defined by its first derivative $\partial_\gamma T(x, \gamma)$ at $\gamma = 0$.

**Proof.** We will make successively the following transitions

$$\alpha^{-1} \to x \to x' \to x'' \tag{B8}$$

where $\alpha^{-1}$ is the parameter of the inverse transformation ($\alpha^{-1} \neq 1/\alpha$). The first transition in (B8) is realized by

$$x = T(x', \alpha^{-1}). \tag{B9}$$

Substituting (B9) in (B1) we obtain similarly to (B4), (B3)

$$x' = T(T(x', \alpha^{-1}), \alpha) = T(x', \gamma) \tag{B10}$$

where by (B9) and with the account of (B2)

$$\gamma = \varphi(\alpha^{-1}, \alpha) = 0. \tag{B11}$$

Differentiating (B10) with respect to $\alpha$ we obtain with the account of (B11)

$$\frac{dx'}{d\alpha} = [\partial_\gamma T(x', \gamma)]_{\gamma=0}\partial_\alpha \varphi(\alpha^{-1}, \alpha). \tag{B12}$$

The autonomous ordinary differential equation (B12) defines unambiguously the group of the transformations (B1).
APPENDIX C: SYMMETRY TRANSFORMATION OF THE WAVE EQUATION

We transform partial derivatives $\partial^2/\partial x^2$ and $\partial^2/\partial t^2$ to $\partial^2/\partial x'^2$ and $\partial^2/\partial t'^2$, respectively, by (30)

\[
\begin{align*}
  x' &= x \cosh \alpha + ct \sinh \alpha, \\
  ct' &= x \sinh \alpha + ct \cosh \alpha.
\end{align*}
\]  

(C1)  

(C2)

Transforming first derivatives with (C1) and (C2):

\[
\begin{align*}
  \frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} = \cosh \alpha \frac{\partial}{\partial x'} + \frac{1}{c} \sinh \alpha \frac{\partial}{\partial t'}, \\
  \frac{\partial}{\partial t} &= \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} = c \sinh \alpha \frac{\partial}{\partial x'} + \cosh \alpha \frac{\partial}{\partial t'}.
\end{align*}
\]  

(C3)  

(C4)

Calculating second derivatives with (C3) and (C4):

\[
\begin{align*}
  \frac{\partial^2}{\partial x^2} &= (\cosh \alpha \frac{\partial}{\partial x'} + \frac{1}{c} \sinh \alpha \frac{\partial}{\partial t'})(\cosh \alpha \frac{\partial}{\partial x'} + \frac{1}{c} \sinh \alpha \frac{\partial}{\partial t'}) \\
  &= \cosh^2 \alpha \frac{\partial^2}{\partial x'^2} + \frac{2}{c} \sinh \alpha \cosh \alpha \frac{\partial^2}{\partial x' \partial t'} + \frac{1}{c^2} \sinh^2 \alpha \frac{\partial^2}{\partial t'^2}, \\
  \frac{\partial^2}{\partial t^2} &= (c \sinh \alpha \frac{\partial}{\partial x'} + \cosh \alpha \frac{\partial}{\partial t'})(c \sinh \alpha \frac{\partial}{\partial x'} + \cosh \alpha \frac{\partial}{\partial t'}) \\
  &= c^2 \sinh^2 \alpha \frac{\partial^2}{\partial x'^2} + 2c \sinh \alpha \cosh \alpha \frac{\partial^2}{\partial x' \partial t'} + \cosh^2 \alpha \frac{\partial^2}{\partial t'^2}.
\end{align*}
\]  

(C5)  

(C6)

Substituting (C5) and (C6) into (11) and using the identity $\cosh^2 \alpha - \sinh^2 \alpha = 1$ we obtain the same form of the wave equation:

\[
\frac{\partial^2 A}{\partial t'^2} = c^2 \frac{\partial^2 A}{\partial x'^2}.
\]  

(C7)

APPENDIX D: THE WAVE FRONT

Integrating the left-hand part of (55):

\[
\int_{-\infty}^{\infty} \delta(x - ct) dx + \int_{-\infty}^{\infty} \delta(x + ct) dx = 2.
\]  

(D1)

Integrating the right-hand part of (55):

\[
2a \int_{-\infty}^{\infty} \delta(x^2 - c^2 t^2) dx = 4a \int_{0}^{\infty} \delta(x^2 - c^2 t^2) dx = 2a \int_{0}^{\infty} \frac{1}{x} \delta(x^2 - c^2 t^2) dx = \frac{2a}{ct}.
\]  

(D2)

Taking in (D2) $a = ct$ we obtain (D1).

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