Effects of the Coulomb interaction on parameters of resonance states in mirror three-cluster nuclei

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Abstract

We investigate how the Coulomb interaction affects the energy $E$ and width $\Gamma$ of resonance states in mirror nuclei. We employ a three-cluster microscopic model to determine position of resonance states in two- and three-body continua. Two parameters are introduced to quantify effects of the Coulomb interactions. As the energy and width of the corresponding resonance states of mirror nuclei are displayed on an $E-\Gamma$ plane, these parameters determine a rotation and a dilatation. With the help of these parameters we found resonance states with strong, small and medium effects of the Coulomb interaction. We also found two different scenarios of the motion of resonance states due to the Coulomb interaction. The first standard (major) scenario represent resonance states with the larger energy and larger width than their counterparts have. The second rear scenario includes resonance states with the larger energy but smaller width.

Keywords:
Cluster model, Resonating Group Method, Coulomb interaction, three-cluster microscopic model, mirror nuclei, resonance states

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1. Introduction

The main aim of this paper is to study effects of the Coulomb forces on the energy and width of resonance states residing in two- and three-cluster continua. We believe that the ideal objects for these studies are mirror nuclei. If we formulate our many-cluster model in such a way that inter-cluster interactions, originated only from a nucleon-nucleon interaction, are the same in both mirror nuclei, then the relative position of bound and resonance states and their widths are totally determined by the Coulomb interaction of protons. Consider, for example, the mirror nuclei $^8\text{Li}$ and $^8\text{B}$. It is naturally to present them as three-cluster configurations $\alpha + t + n$ and $\alpha + ^3\text{He} + p$, respectively. Cluster models with such three-cluster configurations are shown repeatedly [1, 2, 3, 4, 5, 6] to provide the correct description of many observed properties of these nuclei. In the nucleus $^8\text{Li}$, the Coulomb interaction affects the interaction between an alpha particle and a triton only. In the mirror $^8\text{B}$ nucleus, the Coulomb interaction reduces the effective attraction in the all pairs of interacting clusters: $\alpha + ^3\text{He}$, $\alpha + p$ and $^3\text{He} + p$. Moreover, the Pauli principle generates the three-body Coulomb interaction, provided that the full antisymmetrization of a compound system is taken into account correctly.

This problem has been repeatedly studied in literature [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. However, in many of these publications, the main attention was devoted to bound states. Meanwhile the most intriguing is the impact of the Coulomb interaction on resonance states. There are some new publications [28, 29] dealing with this problem. In Ref. [28], the structure of mirror nuclei $^{11}\text{Li}$ and $^{11}\text{O}$ has been studied experimentally and within the Gamow coupled-channel approach [30]. Within this approach, the mirror nuclei $^{11}\text{Li}$ and $^{11}\text{O}$ are considered as three-body system with an inert core and two valence neutrons and protons, respectively. The calculated density distributions explicitly demonstrate effects of the Coulomb interaction on structure of the $J^\pi = 3/2^-$, $3/2^-$, $5/2^+$ and $5/2^+$ resonance states. In Ref. [29] similar analysis was performed to study mirror nuclei $^{11}\text{Li}-^{11}\text{O}$ and $^{12}\text{Be}-^{12}\text{O}$.

Effects of the Coulomb interaction on mirror or isobaric nuclei have been repeatedly investigated by many authors in different many-particle models. Very often the influence of the Coulomb potential on the spectrum of such
nuclei is associated with the Thomas-Erhman effect or shift (see original papers Ref. [31] and Ref. [32] and recent discussion of the effect, for example, in Ref. [21]), which is connected with the shift of energy of single particle levels in mirror nuclei due to the Coulomb interaction. Recently this effect is also discussed in context of a cluster model.

By analyzing the spectra of the mirror nuclei $^{13}$C and $^{13}$N, Thomas in Ref. [31] and Erhman in Ref. [32] independently discovered that the almost degenerated single-particle s- and d-orbitals give the different contribution to the spectrum of a compound nucleus due to the Coulomb interaction. The more compact orbital yields the larger Coulomb shift of the single-particle energy than the more dispersed orbital. Such a difference in contribution of the Coulomb forces with compact and dispersed single-particle orbitals is called the Thomas-Erhman effect. Since these publications, the Thomas-Erhman effect has been numerously examined in different mirror nuclei. Last decades, this effect is intensively studied within many-cluster models (see, for example, Refs. [14, 15, 23, 24, 26, 27]). It was shown that the different cluster orbitals utilizing for a description of mirror nuclei give also different contribution of the Coulomb energy. Some of these orbitals describes relatively compact many-cluster configurations, and other orbitals suggest the loosely many-cluster configurations. For example, in Ref. [27] the mirror nuclei $^{14}$C and $^{14}$O have been studied with the antisymmetric molecular dynamics (AMD) and it was demonstrated that the Coulomb potential had a different contribution to the triangular, and linear $\sigma$- and $\pi$-bond configurations.

We will not discuss the Thomas-Erhman effect as this is out of the scope of the present paper. Such a discussion requires a decomposing of complicated wave functions of the three-clusters systems into simple orbitals. And it leads to very bulky calculations. Below, we will study the properties wave functions of bound and resonance states and employ other way of their decomposition which is used in many-cluster models. The main aim of the present paper is to suggest an adequate way (manner) of analysis of resonance state behavior in mirror nuclei and to apply it for reveal general features of motion of resonance states in real three-cluster systems caused by the Coulomb forces.

In Ref. [33] the impact of the Coulomb interaction on energy and width of resonance states in three-cluster continua $\alpha + \alpha + n$ and $\alpha + \alpha + p$ of the mirror nuclei $^9$Be and $^9$B have been studied. As resonance states being poles of the $S$ matrix in the complex plane, it was introduced a Coulomb rotating angle to determine and to quantify how strong are the effects of Coulomb interactions. With this parameter, it was discovered three groups of
resonance states with the weak, medium and strong impact of the interaction on the position of resonance states. However, we feel that this analysis was not complete. To make this analysis more complete we introduce a new parameter which determines the relative shift of the energy and width of the resonance state in a mirror nucleus with a large number of protons due to the stronger Coulomb interaction.

We are going to perform such an analysis for different couples of mirror nuclei, namely, $^7\text{Li}$ and $^7\text{Be}$, $^8\text{Li}$ and $^8\text{B}$, $^{11}\text{B}$ and $^{11}\text{C}$. All these nuclei are considered within a three-cluster microscopic model. For all these nuclei we selected dominant three-cluster configurations. In Table 1 we show partners of mirror nuclei and their dominant three-cluster channels. The partners are marked by the letters $L$ and $R$. In Table 1 we also show a microscopic model applied and a source of calculations, and the charge difference $\Delta Z = Z_R - Z_L$ as well.

Table 1: List of nuclei to be investigated, dominant three-cluster configurations, a microscopic model applied and references.

| $L$-nucleus       | $R$-nucleus       | $\Delta Z$ | Source | Model   |
|-------------------|-------------------|------------|--------|---------|
| $^7\text{Li}=\alpha + d + n$ | $^7\text{Be}=\alpha + d + p$ | 1          | [34, 35] | AMGOB   |
| $^8\text{Li}=\alpha + t + n$ | $^8\text{B}=\alpha + ^3\text{He} + p$ | 2          | [6]     | AMGOB   |
| $^9\text{Be}=\alpha + \alpha + n$ | $^9\text{B}=\alpha + \alpha + p$ | 1          | [33, 36] | AMHHB   |
| $^{11}\text{B}=\alpha + \alpha + t$ | $^{11}\text{C}=\alpha + \alpha + ^3\text{He}$ | 1          | [37]    | AMHHB   |

To study effects of the Coulomb interaction on resonance states in three-cluster systems, we employ two microscopical models as shown in Table 1. They are a modification of the resonating group method. These methods are designed to study a three-cluster structure of light atomic nuclei. Both of these methods employ the square-integrable bases to describe dynamics of inter-cluster motion. The first model was formulated in Ref. [35] and will be referred as AMGOB, it utilizes of the Gaussian basis to describe bound and pseudo-bound states in a two-cluster subsystem, while Oscillator basis describes relative motion of the third cluster with two-cluster subsystem. The main merit of the AMGOB is that it allows us to study influence of a cluster polarization, i.e. to study how the shape and size of a nucleus comprised of two clusters are changed when another nucleus (the third cluster) is moving closer. Thus this model provides more exact description of nuclei with prominent two-cluster structure. By exploiting a three-cluster configuration, the method involves up to three different binary channels with the
lowest energy of two-body decay of a compound three-cluster nucleus. It was shown in Refs. [35, 34, 36, 38] that the cluster polarization has a large impact on the spectrum of bound and resonance states of light nuclei, and especially on the astrophysical S-factor of the capture reactions. The second model (AMHHB) is used the hyperspherical harmonics to investigate relative motions of clusters and was designed in Ref. [39] to study processes in the three-cluster continuum.

The layout of our paper is following. In Sec. 2, we formulate in more detail our main aims and consider possible scenarios of motion of resonance states in mirror nuclei stimulated by the Coulomb forces. In Sec. 2 we also introduce parameters which allow us to study thoroughly effects of the Coulomb interaction on bound and resonance states. A short explanation of the main idea of the microscopic method for description of bound and scattering states of a three-cluster system are presented in Sec. 3. In what follows, we will concentrate our main attention on resonance states in three-cluster continuum of mirror nuclei. That is why in Sec. 3 we present only one microscopic model (AMHHB) of two mentioned above. How strong are the effects of the Coulomb interaction are demonstrated in Sec. 4. We start by considering resonance states in the three-cluster continuum of the mirror nuclei $^9$Be and $^9$B, $^{11}$B and $^{11}$C, and then we proceed with bound and resonance states in the two-body continuum of $^7$Li and $^7$Be, $^8$Li and $^8$B. Analysis of wave functions of resonance states in the mirror nuclei is presented in Sec. 5. We close the paper by summarizing the obtained results in Sec. 6.

2. The Coulomb interaction in mirror nuclei

To formulate more clearly our aim, let us consider a schematic picture which demonstrates effects of the Coulomb interaction in mirror nuclei. By letters $L$ and $R$ we denote two mirror nuclei assuming that the charges of these nuclei obey the relation $Z_L < Z_R$. In Fig. 1 we show effective potentials of two mirror nuclei and the position of two bound states and one resonance state.

The next figure (Fig. 2) demonstrates the effective potentials above the decay threshold. This picture shows that the Coulomb interaction increases the height and width of the effective barrier. This figure also suggests the two possible scenarios representing effects of the Coulomb interaction on parameters of resonance states. The first scenario assumes that the Coulomb
Figure 1: Effective potentials $V(r)$ of two mirror nuclei which have the charges $Z_L < Z_R$ as a function of distance $r$. 
interaction increases energy of the resonance state in the $R$ nucleus in such a way that the width of the Coulomb barrier at this energy becomes rather small, which results in increasing of the resonance width. In the second scenario the energy of a resonance state in the $R$ nucleus is also increased but not so high as in the first scenario. At this energy the width of the Coulomb barrier is large and yields a smaller width of the resonance state.

![Figure 2: Effective potential barriers in mirror nuclei and position of resonance states.](image)

By considering the mirror nuclei, one can suggest four main scenarios for changing of the parameters of resonance states due to the Coulomb forces. Increasing of the Coulomb interaction leads to decreasing of the attractive effective interaction in each channel of a many-channel system. That may
shift up the energy of resonance states and may also increase the width of resonance states. That is the first scenario. The second scenario, the Coulomb interaction makes an effective barrier more thicker, that may increase the energy but decrease the width of a resonance state. We add the third and fourth scenarios, when the more wider effective barrier may decrease the energy of resonance states and increase or decrease the width of resonance states, respectively. In the present paper we will investigate what scenario dominates in three-cluster mirror nuclei.

Resonance states are characterized by two parameters: the energy $E$ and width $\Gamma$. Being a pole of the $S$-matrix, the resonance state is usually determined by a complex value $E - i\Gamma/2$. Thus, it is natural to consider the parameters of the resonance state in a two-dimensional space. We select the plane $E$ and $\Gamma$. In this plane the impact of the Coulomb forces on the resonance state can be reduced to two operations: a rotation and a shift. That is why we introduce the Coulomb rotational angle $\theta_C$

$$\theta_C = \arctan \left( \frac{\Gamma (R) - \Gamma (L)}{E (R) - E (L)} \right)$$

and the Coulomb shift $R_C$

$$R_C = \sqrt{[E (R) - E (L)]^2 + [\Gamma (R) - \Gamma (L)]^2}.$$ 

These relations connect resonance states of mirror nuclei with the same total angular momentum $J$ and the same parity $\pi$. They can be also applied for a bound state in the $L$ nucleus and a resonance state in the $R$ nucleus, and for two bound states in these nuclei. In the latter case, these relations give the trivial result: $\theta_C = 0$ and $R_C = E (R) - E (L)$.

The Coulomb rotational angle $\theta_C$ was introduced in Ref. [33] and used to study effects of the Coulomb forces in $^9\text{Be}$ and $^9\text{B}$.

Formulae (1) and (2) suggest that it is more expedient to use differences

$$\Delta E = E (R) - E (L) = R_C \cos \theta_C,$$

$$\Delta \Gamma = \Gamma (R) - \Gamma (L) = R_C \sin \theta_C$$

as the axis $x$ and axis $y$, respectively. Both differences can be positive or negative. In term of the parameters $\Delta E$ and $\Delta \Gamma$ we can consider the four following hypothetical scenarios:
Both parameters are positive, thus the Coulomb interaction increases the energy and width of resonance state in the $R$ nucleus.

The parameter $\Delta E$ is positive and $\Delta \Gamma$ is negative, the energy of resonance state in the $R$ nucleus is larger however its width is smaller than in the $L$ nucleus.

The parameter $\Delta E$ is negative and $\Delta \Gamma$ is positive, the Coulomb interaction reduces the energy of the resonance state in the $R$ nucleus but increases its width.

Both parameters are negative, it means the Coulomb interaction makes smaller the energy and width of the resonance state in the $R$ nucleus.

Note that in the first and second scenarios the Coulomb interaction increases the energy of resonance state in $R$ nucleus, meanwhile the third and fourth scenarios assume that the energy is diminished by the Coulomb interactions.

3. Model formulation

In this section we shortly present the main ideas of a microscopic three-cluster model used to study spectrum of bound and resonance states in mirror nuclei. The principal restrictions and approximations in any model are imposed on a Hamiltonian and wave functions. In the present model we use a microscopic Hamiltonian which includes the kinetic energy in the system of center mass of $A$ nucleons, central and spin-orbital components of a semirealistic nucleon-nucleon interaction and the Coulomb interaction of protons. Exact wave function of the Hamiltonian is approximated by the wave function for the system consisting of three $s$-clusters

$$
\Psi_{JMJ}^F = \sum_{L,S,\lambda,l} \hat{A} \{ [\Phi_1 (A_1, b, s_1) \Phi_2 (A_2, b, s_2) \Phi_3 (A_3, b, s_3)]_S \psi_{E,J}^{E,J} (x, y) \{ Y_{\lambda} (\vec{x}) Y_{l} (\vec{y}) \}_L \}_{JMJ}.
$$

The internal structure of clusters ($\alpha = 1, 2, 3$) is described by the antisymmetric and translationally invariant wave functions $\Phi_\alpha (A_\alpha, b, s_\alpha)$. In this function we indicated the main parameters which determines this function: $A_\alpha$ is the number of nucleons, $b$ is an oscillator length and the spin $s_\alpha$ of
the cluster. The function \( \Phi_\alpha (A_\alpha, b_\alpha, s_\alpha) \) is a wave function of the many-particle shell model with the most compact configuration of nucleons. The antisymmetrization operator \( \hat{A} \) makes antisymmetric the wave function of the compound three-cluster system. Within the standard approximation of the resonating group method all functions \( \Phi_\alpha (A_\alpha, b_\alpha, s_\alpha) \) are fixed, and thus to calculate a spectrum and wave functions of the compound system one has to determine a wave function of inter-cluster motion \( \psi_{E,J, LS}^{E,J} (x,y) \). This function is also translationally invariant and depends on two Jacobi vectors \( x \) and \( y \), locating relative position of clusters in the space. The vector \( x \) determines the distance between selected pair of clusters and the vector \( y \) is proportional to the displacement of the third cluster with respect to the center of mass of two-cluster subsystem. In Eq. (4) \( \hat{x} \) and \( \hat{y} \) are unit vectors, and \( \lambda \) and \( l \) are the partial angular momenta associated with the vectors \( x \) and \( y \) respectively.

Wave functions of inter-cluster motion \( \psi_{E,J, LS}^{E,J} (x,y) \) obey an infinite set of the two-dimensional (in terms of variables \( x \) and \( y \)) integro-differential equations. Two-dimensional equations can be reduced to one-dimensional integro-differential equations by employing the hyperspherical coordinates (one hyper radius \( \rho \) and five hyperspherical angles \( \Omega_5 \)) and hyperspherical harmonics.

Instead of six variables \( x \) and \( y \) or \( x, y, \) and two unit vectors \( \hat{x} \) and \( \hat{y} \) we introduce a hyperspherical radius \( \rho \) and a hyperspherical angle \( \theta \)

\[
\rho = \sqrt{x^2 + y^2}, \quad \theta = \arctan \left( \frac{x}{y} \right). \quad (5)
\]

For a fixed value of \( \rho \), the angle \( \theta \) determines relative length of the vectors \( x \) and \( y \)

\[
x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad \theta \in [0, \pi/2]. \quad (6)
\]

Thus the set of hyperspherical angles \( \Omega_5 \) is \( \Omega_5 = \{ \theta, \hat{x}, \hat{y} \} = \{ \theta, \theta_x, \phi_x, \theta_y, \phi_y \} \). With such a definition of the hyperspherical angles, we can employ the hyperspherical harmonics in the form suggested by Zernike and Brinkman in Ref. [40]. They have simple form and the quantum numbers numerating them have clear physical meaning.

The represented set of the hyperspherical angles is very popular scheme of the hyperspherical coordinates for investigating three-body [41, 42, 43], and three-cluster systems [39, 41, 45, 46].
In new coordinates

\[\Psi_{J,M_J}^E = \sum_{L,S,\lambda,l} \hat{A} \{ [\Phi_1 (A_1, b_1, s_1) \Phi_2 (A_2, b_2, s_2) \Phi_3 (A_3, b_3, s_3)]_S \times \phi_{c,J}^{E,J}(\rho) Y_c (\Omega_5) \} _{J,M_J},\]

where \(Y_c (\Omega_5)\) stands for the product

\[Y_c (\Omega_5) = \chi^{(\lambda,l)}_K (\theta) \{ Y_{\lambda} (\tilde{x}) Y_l (\tilde{y}) \} _{L,M_L},\]

and represents a hyperspherical harmonic for a three-cluster channel

\(c = \{ K, \lambda, l, L \} .\)

Definition of all components of the hyperspherical harmonic \(Y_c (\Omega_5)\) can be found, for instance, in Ref. [39]. Being a complete basis, the hyperspherical harmonics account for any shape of the three-cluster triangle and its orientation and thus they allow one to describe all possible modes of relative motion of three interacting clusters.

To determine the hyperradial wave functions \(\phi_{c,J}^{E,J}(\rho)\) one has to solve a system of integro–differential equations with nonlocal effective potentials for three clusters. This system of equations can be represented as

\[\sum_{\tilde{c}} \left[ \delta_{c,\tilde{c}} \hat{T}_K \phi_{c,J}^{E,J} + \int d\tilde{\rho} \tilde{\rho}^5 V_{c,\tilde{c}} (\rho, \tilde{\rho}) \phi_{\tilde{c},J}^{E,J} (\tilde{\rho}) \right] = E \sum_{\tilde{c}} \int d\tilde{\rho} \tilde{\rho}^5 N_{c,\tilde{c}} (\rho, \tilde{\rho}) \phi_{\tilde{c},J}^{E,J} (\tilde{\rho}) ,\]

where

\[\hat{T}_K = \frac{-\hbar^2}{2m} \left[ \frac{\partial^2}{\partial \rho^2} + \frac{5}{\rho} \frac{\partial}{\partial \rho} - \frac{K(K + 4)}{\rho^2} \right].\]

The potential energy \(V_{c,\tilde{c}} (\rho, \tilde{\rho})\) and the norm kernel \(N_{c,\tilde{c}} (\rho, \tilde{\rho})\) can be obtained with the help of the projection operator \(\hat{P}_c\) which is presented in Ref. [47].

It is well-known that the Pauli principle leads to a nonlocal form of potential energy operator and to appearance of the energy dependent part in the effective potential (the right-hand side of equations \(10\)). To simplify solving a set of equations \(10\), we employ a full set of cluster oscillator functions to expand the total wave function

\[\Psi_{J,M_J}^E = \sum_{n_\rho,c} C_{n_\rho,c}^{E,J} | n_\rho, c, J \rangle .\]
This reduces a set of integro-differential equations (10) to an algebraic form, i.e. to the system of linear algebraic equations

\[ \sum_{n,\xi} \left[ \langle n, c, J | \hat{H} | n, \xi, J \rangle - E \langle n, c, J | n, \xi, J \rangle \right] C_{n,\xi}^{E,J} = 0. \] (12)

Cluster oscillator functions for a three-cluster configuration \( A = A_1 + A_2 + A_3 \) are determined as

\[ |n, c, J \rangle = |n, K; \lambda, l; L; J \rangle \]

\[ = \hat{A} \{ [\Phi_1 (A_1, b_1, s_1) \Phi_2 (A_2, b_2, s_2) \Phi_3 (A_3, b_3, s_3)]_S \times R_{n,K} (\rho, b) Y_c (\Omega_5) \}_J L_J, \]

where \( R_{n,K} (\rho, b) \) is an oscillator function

\[ R_{n,K} (\rho, b) = (-1)^n n! k^n r^K \exp \left\{ -\frac{1}{2} r^2 \right\} L_{n_K}^{K+3} (r^2), \] (14)

\[ r = \rho/b, \quad N_{n,K} = b^{-3} \sqrt{\frac{2\Gamma (n + 1)}{\Gamma (n + K + 3)}}, \]

\( L_{n}^{\alpha} (x) \) is the Laguerre polynomial and \( b \) is an oscillator length.

System of equations (12) can be solved numerically by imposing restrictions on the number of hyperradial excitations \( n_\rho \) and on the number of hyperspherical channels \( c_1, c_2, \ldots, c_{N_{ch}} \). The diagonalization procedure may be used to determine energies and wave functions of the bound states. However, the proper boundary conditions have to be implemented to calculate elements of the scattering \( S \)-matrix and corresponding wave functions of continuous spectrum. Such boundary conditions and their implementation in the proposed discreet scheme has been discussed in Ref. [39]. By solving the system of equations (12), we obtain \( N_{ch} \) wave functions \( \{C_{n_\rho,c}^{E,J}\} \) and \( N_{ch}^2 \) elements of the unitary \( S \) matrix \( S_{c,\xi} \). By employing the Breit-Wigner formula for a resonance state, we deduce the energy, the total and partial widths of such state in three-cluster continuum.

Having obtained the expansion coefficients for any state of the three-cluster system, we can easily construct its wave function in the coordinate space. It can be done, the first of all, for the hyperradial wave function

\[ \phi_c^{E,J} (\rho) = \sum_{n_\rho} C_{n_\rho,c}^{E,J} R_{n_\rho,K} (\rho, b). \] (15)
It can be also done for the total inter-cluster wave function

\[ \psi_{E,J}(x, y) = \sum_{n_{\rho,c}} C_{n_{\rho,c}}^{E,J} R_{n_{\rho,K}}(\rho, b) \mathcal{Y}_c(\Omega_5). \] (16)

To get more information about the state under consideration we will study different quantities which can be obtained with the wave function in discrete or coordinate spaces. With wave functions in the discrete oscillator quantum number representation we can determine a weight \( W_{sh} \) of the oscillator function belonging to the oscillator shell \( N_{sh} \) in this wave function:

\[ W_{sh}(N_{sh}) = \sum_{n_{\rho,c} \in N_{sh}} \left| C_{n_{\rho,c}}^{E,J} \right|^2, \] (17)

where the summation is performed over all quantum numbers of hyperspherical harmonics and hyperradial excitations obeying the following condition:

\[ N_{os} = 2n_{\rho} + K. \]

Here the number of oscillator quanta \( N_{os} \) is fixed.

Basis wave functions (14) belongs to the oscillator shell with the number of oscillator quanta \( N_{os} = 2n_{\rho} + K \), then it is convenient to numerate the oscillator shells by \( N_{sh} (= 0, 1, 2, \ldots) \), which we determine as

\[ N_{os} = 2n_{\rho} + K = 2N_{sh} + K_{min}, \]

where \( K_{min} = L \) for normal parity states \( \pi = (-1)^L \) and \( K_{min} = L + 1 \) for abnormal parity states \( \pi = (-1)^{L+1} \). Thus we account oscillator shells starting from a "vacuum" shell \( (N_{sh} = 0) \) with minimal value of the hypermomentum \( K_{min} \) compatible with a given total orbital momentum \( L \) and the parity \( \pi \).

The weights \( W_{sh} \) can be calculated both for bound and resonance states. For a bound state, the wave function \( \Psi_{E,J} \) is normalized by the condition

\[ \langle \Psi_{E,J} | \Psi_{E,J} \rangle = \sum_{n_{\rho,c}} \left| C_{n_{\rho,c}}^{E,J} \right|^2 = 1, \] (18)

and this quantity \( W_{sh} \) determines the probability. For the continuous spectrum state, when the wave function \( \Psi_{E,J} \) is normalized by the condition

\[ \langle \Psi_{E,J} | \Psi_{E,J} \rangle = \sum_{n_{\rho,c}} C_{n_{\rho,c}}^{E,J} C_{n_{\rho,c}}^{E,J} = \delta (k - \tilde{k}), \] (19)
this quantity has a different meaning. It determines the relative contribution of the different oscillator shells and also the shape of the resonance wave function in the oscillator representation.

It is worthwhile to notice that oscillator functions have some important features. For instance, oscillator functions belonging to an oscillator shell $N_{sh}$ allow one to describe a many-particle system in a finite range of hyperradius $0 < \rho \leq b\sqrt{4N_{sh} + 2K_{\text{min}} + 3}$. Outside this region, these oscillator functions give a negligible small contribution to the many-particle wave function. This statement is, for example, demonstrated in Ref. [48]. Besides, oscillator wave functions belonging to the oscillator shell $N_{sh}$ yield the mass root-mean-square radius which equals $b\sqrt{2N_{sh} + K_{\text{min}} + 3(A - 1)/2}/A$. Thus, oscillator functions with a small value of $N_{sh}$ describe very compact configurations of a three-cluster system with small distances between clusters. When $N_{sh}$ is large, the oscillator functions represent dispersed three-cluster configurations. There are two principal regimes in these configurations. The first regime is associated with a two-body type of asymptotic when two clusters are at a small distance and the third cluster is moved far away. The second regime accounts for the case when all three clusters are well separated. Taking these into account, we will deduce from analysis of shell weights $W_{sh}$ whether a wave function of a bound or resonance state describes a compact or dispersed three-cluster configuration.

The weights oscillator shells $W_{sh}$ have been used by different authors to study wave functions obtained in different microscopic models. For example, they were employed in Refs. [49, 50] to study the Hoyle states in $^{12}\text{C}$ within the fermionic molecular dynamics. Note that our definition of the weights $W_{sh}$ is consistent with more general definition suggested by Y. Suzuki et al in Ref. [51].

Important ingredients of the present microscopic model are a nucleon-nucleon potential and the Coulomb interaction. As we pointed out above, we formulated our model in such a way that the cluster-cluster interaction originated from the nucleon-nucleon potential are the same in mirror nuclei. Thus difference in the position and width of resonance states is due to the Coulomb interaction. Let us consider in more detail the Coulomb potential energy in mirror nuclei. For the $L$ and $R$ nuclei of $Z_L < Z_R$ it has the form

$$V_C^{(L)} = \sum_{i<j}^{Z_L} \frac{e^2}{|\vec{r}_i - \vec{r}_j|}, \quad V_C^{(R)} = \sum_{i<j}^{Z_R} \frac{e^2}{|\vec{r}_i - \vec{r}_j|}. \quad (20)$$
It is easy to see that the potential energy for the $R$ nucleus, where $Z_R > Z_L$, can be represented as a combination of three components

$$V_C^{(R)} = \sum_{i<j\in Z_R} \frac{e^2}{|r_i - r_j|} + \sum_{i<j\in Z_L} \frac{e^2}{|r_i - r_j|} + \sum_{i\in Z_L, j\in (Z_R-Z_L)} \frac{e^2}{|r_i - r_j|}.$$  \hspace{2cm} (21)

In the right hand side of equation (21), the first component is the Coulomb potential energy of the $L$ nucleus, the third component is the potential energy of extra protons (with respect to protons of the $L$ nucleus), and the second component represents the potential energy of the interaction of extra protons with protons of the $L$ nucleus. The last two components, which we denote as

$$\Delta V_C = V_C^{(R)} - V_C^{(L)}$$  \hspace{2cm} (22)

$$= \sum_{i\in Z_L, j\in (Z_R-Z_L)} \frac{e^2}{|r_i - r_j|} + \sum_{j>i\in (Z_L-Z_R)} \frac{e^2}{|r_i - r_j|},$$

determine the shift of bound and resonance states in the $R$ nucleus with respect to their position in the $L$ nucleus. In all but one nuclei we have only one extra proton, thus for a pair of nuclei $^7\text{Li}$ and $^7\text{Be}$, $^9\text{Be}$ and $^9\text{B}$, $^{11}\text{B}$ and $^{11}\text{C}$ only the second component in Eq. (21) determines the Coulomb shift. While for nuclei $^8\text{Li}$ and $^8\text{B}$, where there are two extra protons, the second and the third components take part in shifting of parameters of bound and resonance states.

Taking into account a three-cluster structure of mirror nuclei, we obtain an alternative way to present the differences of the Coulomb interaction. The Coulomb potential energy for both the $L$ and $R$ nuclei can be represented as

$$V_C = \sum_{c=1}^{3} \sum_{i<j\in Z_c} \frac{e^2}{|r_i - r_j|} + \sum_{i\in Z_1, j\in Z_2} \frac{e^2}{|r_i - r_j|} + \sum_{i\in Z_1, j\in Z_3} \frac{e^2}{|r_i - r_j|} + \sum_{i\in Z_2, j\in Z_3} \frac{e^2}{|r_i - r_j|} + \sum_{i\in Z_1, j\in Z_3} \frac{e^2}{|r_i - r_j|}.$$  \hspace{2cm} (23)

The first three terms of eq. (23) represent the internal Coulomb potential energy of a cluster $c$ ($c=1, 2, 3$) and the last three-terms are the Coulomb
interactions of different clusters. It is obvious, that the internal Coulomb energy is nonzero for a cluster containing 2 and more protons, and the Coulomb interactions between different clusters are nonzero when both interacting clusters contain one and more protons. Note that in such representation of the Coulomb potential energy, the Coulomb potential energy difference $\Delta V_C$ may originates from the difference of the internal energy of a cluster and from the interaction between clusters. It can be represented as

$$
\Delta V_C = \sum_{c=1}^{3} \sum_{i<j \in Z_{c,R} - Z_{c,L}} \frac{e^2}{|r_i - r_j|} + \sum_{i \in Z_{2,R} - Z_{1,L}} \sum_{j \in Z_{2,R} - Z_{3,L}} \frac{e^2}{|r_i - r_j|} + \sum_{i \in Z_{3,R} - Z_{2,L}} \sum_{j \in Z_{3,R} - Z_{3,L}} \frac{e^2}{|r_i - r_j|}.
$$

Let us consider the nuclei in our hands. The nuclei $^7$Li and $^7$Be, as was shown in Table 1, differ in one proton. Thus, in $^7$Be we have additional terms caused by the Coulomb interaction, namely, the interaction of a valence proton with alpha-particle and the interaction of that proton with a deuteron. The similar picture is observed in a pair $^9$Be and $^9$B, where $\Delta V_C$ consists of the interaction of the valence proton with the first and second alpha-particles. In nuclei $^8$Li and $^8$B and $^{11}$B and $^{11}$C we have got a rather different situation. In these nuclei, the Coulomb interaction contribute to the internal energy of the cluster $^3$He and makes a stronger interaction (repulsion) between $^3$He and alpha-particle(s) in $^8$B ($^{11}$C) with respect to the $t - \alpha$ interaction in $^8$Li ($^{11}$B).

The Coulomb potential energy difference $\Delta V_C$ can be treated as a perturbation and thus the Coulomb shift can be evaluated by using wave functions of the $L$ nucleus, or, by introducing a factor $\lambda_C$ and considering the interaction $\lambda_C \Delta V_C$, one can study the trajectory of bound and resonance states when the parameter $\lambda_C$ is changed from zero to one. However, in our calculations the energies and widths and wave functions of resonance states in both $L$ and $R$ nuclei are obtained in the same way with the corresponding boundary conditions.
In the next Section we will study how the Coulomb potential energy difference $\Delta V_C$ changes the energy and width of resonance states in the $R$ nucleus with respect to its position in the $L$ nucleus.

4. Effects of Coulomb forces

We will not discuss details of calculations as they were discussed in papers mentioned in Table 1. We just outline some general steps of these calculations. In our calculations we use a common oscillator length $b$ for all interacting clusters. The oscillator length was chosen to minimize the energy of the three-cluster threshold. This optimizes a description of the internal structure of clusters. The Minnesota potential (MP) [52] and the modified Hasegawa-Nagata potential (MHNP) [53, 54] were involved in all calculations. The exchange parameter $u$ of the MP and the Majorana parameter $m$ of the MHNP was slightly adjusted to reproduce the ground state energy of the $L$ nucleus. The same values of $m$ or $u$ were used for the $R$ nucleus. In this case, the interactions between clusters, generated by the nucleon-nucleon interaction, are the same in the $L$ and $R$ nuclei.

In this paper we do not compare our results with the available experimental data, as it was done in the references mentioned in Table 1. However, we will compare our results with the results of other theoretical approaches.

4.1. $^9$Be and $^9$B

Spectrum of resonance states in $^9$Be and $^9$B has been obtained in Refs. [33, 36] within the AMHHB. This method was selected to study parameters and nature of resonance states in $^9$Be and $^9$B because all resonance states of these nuclei are embedded in three-cluster continuum and this method implements proper boundary conditions for the three-cluster continuous spectrum states.

In Table 2 we demonstrate energies and widths of resonance states in $^9$Be and $^9$B. They were obtained in Ref. [33] with the MHNP within the AMHHB. Table 2 also displays the Coulomb shift $R_C$ and rotational angle $\theta_C$.

Let us look closely what possible scenarios are realized in nuclei $^9$Be and $^9$B and how it depends on the total angular momentum $J$.

The first effects of the Coulomb forces in the mirror nuclei $^9$Be and $^9$B can be seen in Fig. 3 where spectra of these nuclei are shown. Five dashed lines, connecting levels with the same total angular momentum $J$ and parity
Figure 3: Spectra of bound and resonance states in $^9$Be and $^9$B.
Table 2: Spectrum of bound and resonance states in $^{9}$Be and $^{9}$B calculated with the MHNP. Energies $E$ and widths $\Gamma$ are in MeV.

| $J^\pi$ | $^{9}$Be $E$ | $^{9}$Be $\Gamma$ | $^{9}$B $E$ | $^{9}$B $\Gamma$ | $R_C$ | $\theta_C$ |
|--------|-------------|-----------------|-------------|-----------------|-------|-------------|
| $3/2^-_1$ | -1.574 | 0.00 | 1.1 $\times 10^{-6}$ | 1.953 | 3.23 $\times 10^{-6}$ |
| $1/2^+_1$ | 0.338 | 0.168 | 0.477 | 0.429 | 46.04 |
| $5/2^-_1$ | 0.897 | 2.4 $\times 10^{-5}$ | 2.805 | 0.018 | 1.908 | 0.54 |
| $1/2^-_1$ | 2.866 | 1.597 | 3.398 | 3.428 | 1.907 | 73.80 |
| $5/2^+_1$ | 2.086 | 0.112 | 3.670 | 0.415 | 1.613 | 10.83 |
| $3/2^+_1$ | 4.062 | 1.224 | 4.367 | 3.876 | 2.669 | 83.44 |
| $3/2^-_2$ | 2.704 | 2.534 | 3.420 | 3.361 | 1.094 | 49.12 |
| $7/2^-_1$ | 4.766 | 0.404 | 6.779 | 0.896 | 2.072 | 13.74 |
| $9/2^-_1$ | 4.913 | 1.272 | 6.503 | 2.012 | 1.754 | 24.96 |
| $5/2^-_2$ | 5.365 | 4.384 | 5.697 | 5.146 | 0.831 | 66.46 |
| $7/2^+_1$ | 5.791 | 3.479 | 7.100 | 4.462 | 1.637 | 36.90 |

$\pi$ in $^{9}$Be and $^{9}$B, show that Coulomb forces significantly shift up levels ($J^\pi=$ $3/2^-, 5/2^-, 5/2^+, 7/2^-$ and $9/2^-$) and four dashed lines indicate a moderate shift up of energy of resonance states ($J^\pi=$ $1/2^+, 3/2^-, 1/2^-$ and $3/2^+$) in $^{9}$B comparing with correspondent states in $^{9}$Be.

To see effects of the Coulomb interaction more vividly we present Fig. 4. In this figure and other figures below, the arcs (grey dashed curves) mark the Coulomb shift $R_C=1$, 2 and 3 MeV, and a set of rays (grey solid lines) indicate the Coulomb rotational angles $\theta_C$ every 15 degrees. As we can see the largest group of resonance states are concentrated around $R_C=2$ and almost all states of this group except for one lie below $\theta_C=45^\circ$.

Thus, the Coulomb interaction has week (the first group of $\Delta E = 0.25 \sim 0.75$ MeV), moderate (the second group of $\Delta E = 1.25 \sim 1.8$ MeV) or strong (the third group of $\Delta E > 1.9$ MeV) influence on parameters of resonance states in mirror nuclei $^{9}$Be and $^{9}$B. And these three groups are observed in terms of the Coulomb shift $R_C$ and the Coulomb angle $\theta_C$. We also observed that the first scenario is realized in these nuclei as the Coulomb interaction increases both energy and width of resonance states in $^{9}$B with respect to their values in $^{9}$Be.
Figure 4: The shift and rotation of resonance states in $^9$B caused by the Coulomb interactions. Counterparts of these resonance states in $^9$Be are put in the origin of coordinates.
4.1.1. CSM

The complex scaling method (CSM) has been used in Ref. [55] to determine energies and widths of resonance states in mirror nuclei $^9$Be and $^9$B. Parameters of resonance states were obtained with the MP. The detail comparison of results of the CSM with the AMHHB was carried out in Ref. [36]. Here we wish to present the results of CSM on the $E - \Gamma$ plain in order to see explicitly effects of the Coulomb interactions detected within this method. We display these results in Fig. 5. As we see, all resonance states lie between $R_C = 1.25$ and $R_C = 2.0$ MeV and this results is consistent with results of the AMHHB displayed in Fig. 4. However, contrary to the first group of the results in the AMHHB, there are no resonance states with the weak effects of the Coulomb interactions ($R_C \approx 1$ MeV) in the CSM. The Coulomb shift angles $\theta_C$ in the CSM do not exceed 45° which is smaller than the values of $\theta_C$ in the AMHHB. The difference between results of the CSM and the AMHHB may be ascribed to the different methods of location of resonance states and partially to different nucleon-nucleon potentials used in each approach.

4.2. $^{11}$B and $^{11}$C

In this section we consider bound and resonance states in $^{11}$B and $^{11}$C. Parameters of these states were determined in Ref. [37] within the AMHHB by employing the MHNP. Some additional information on the structure of wave functions of resonance states in $^{11}$B and $^{11}$C can be found in Ref. [47].

Some remarks should be made about an application of the AMHHB method to the nuclei $^{11}$B and $^{11}$C. As it was pointed out above, this method was designed and applied to study light nuclei with a prominent three-cluster structure. These nuclei can be easily split on three clusters and thus the three-cluster threshold has a lowest energy among other two- and three-cluster decay channels. However, it is not the case for nuclei $^{11}$B and $^{11}$C. For these nuclei, the lowest decay channel is the binary channel $^7$Li+$\alpha$ and $^7$Be+$\alpha$, respectively. At the present time and with the present version of the AMHHB model, it is difficult to incorporate simultaneously both two- and three-body channels. Such a work is in the progress. The first step in this direction was made in Ref. [48] where the ability of the oscillator three-cluster functions (13) to describe binary and ternary channels has been investigated. In the present paper, as in Refs. [37, 47] we disregard the binary channels in $^{11}$B and $^{11}$C and consider the three-cluster continuum only. In Refs. [37, 47], such an approximation was aimed to study resonance states in the cluster continuum of $^{11}$B and $^{11}$C and to search for the Hoyle-analog states. It was
Figure 5: The Coulomb shift of resonance states in $^9$B with respect to those in $^9$Be, obtained with the complex scaling method. The data are taken from Ref. [55].
assumed that a binary channel does not change dramatically the spectrum and structure of three-cluster resonance states. Here we use the same approximation and the same motivation to study effects of the Coulomb interaction on parameters of three-cluster resonance states in the mirror nuclei $^{11}$B and $^{11}$C.

The richest spectra of resonance states were obtained for $^{11}$B and $^{11}$C. We detected 20 resonance states in $^{11}$B and 18 resonance states in $^{11}$C, at least three resonance states for a fixed values of the total angular momentum $J$ and parity $\pi$. Such a large number of resonance states is stipulated by huge centrifugal and Coulomb barriers.

However, we start with the spectra of bound states in $^{11}$B and $^{11}$C, which are also very rich. Energies of bound states are shown in Table 3. As we can see, the nucleus $^{11}$B has ten bound states, while the nucleus $^{11}$C has only seven bound states. Thus the Coulomb interaction moves three bound states to the continuous spectrum and transforms them into resonance states. The Coulomb shift for bound states starts from 1.98 MeV for the ground states and slowly decreases to 1.45 MeV. As one could expect the Coulomb shift is reduced for weakly bound states, which have a dispersed three-cluster configuration.

In Fig. 6 we display how the Coulomb interaction shifts the bound state of $^{11}$C with respect to those in $^{11}$B. This figure demonstrates the Coulomb interaction shifts bound states of $^{11}$C approximately on the almost same values (1.4~2.0 MeV) for all bound states.

| $J^\pi$ | $^{11}$B $E$ (MeV) | $^{11}$C $E$ (MeV) | $R_C$(MeV) |
|--------|-----------------|-----------------|------------|
| 3/2$^-$ | -11.0520        | -9.0710         | 1.981      |
| 1/2$^-$ | -9.6440         | -7.7210         | 1.923      |
| 5/2$^-$ | -7.3770         | -5.4420         | 1.935      |
| 3/2$^+$ | -5.6630         | -3.8320         | 1.831      |
| 1/2$^+$ | -2.7590         | -1.2700         | 1.489      |
| 5/2$^+$ | -2.7360         | -1.1680         | 1.568      |
| 3/2$^+$ | -1.5320         | -0.0850         | 1.447      |
| 5/2$^-$ | -1.0173         | 1.45            |            |
| 1/2$^-$ | -0.1895         |                 |            |
| 5/2$^+$ | -0.0414         |                 |            |
Figure 6: Spectra of bound states in $^{11}$B and $^{11}$C.
In Table 4, we display parameters of negative parity resonance states in the three-cluster continuum of $^{11}$B and $^{11}$C. In Table 4, we also included two bound states in $^{11}$B which are transformed into resonance states in $^{11}$C due to the repulsive Coulomb interaction. In Table 4, we also show the rotational $\theta_C$ and shift $R_C$ parameters caused by the Coulomb interaction.

Resonance states of the positive parity are displayed in Table 5. This table contains also one bound $5/2^+_1$ state in $^{11}$B, which is transformed into the narrow resonance state due to the Coulomb interaction.

Effects of the Coulomb interaction on resonance states of the positive and negative parity are demonstrated in Figs. 7 and 8, respectively. These figures demonstrate that for the main part of resonance states in $^{11}$C have approximately the same Coulomb shift $R_C$ with respect to their counterparts in $^{11}$B. More detail information about effects of the Coulomb interaction on resonance states in the three-cluster continuum of $^{11}$B and $^{11}$C is presented in Figs. 9 and 10.

In Fig. 9, we show effects of the Coulomb interaction on the negative parity resonance states in $^{11}$B and $^{11}$C nuclei. As it can be seen, a large group of resonance states are concentrated around $R_C=1$ MeV. There are two resonance states with the Coulomb shift $R_C \approx 2$ MeV. One of these resonance states

| $J^\pi$   | $^{11}$B $E$ | $^{11}$B $\Gamma$ | $^{11}$C $E$ | $^{11}$C $\Gamma$ | $R_C$ | $\theta_C$ |
|-----------|-------------|------------------|-------------|------------------|-------|---------|
| $3/2^-_1$ | 0.755       | 5.8x10^{-4}      | 0.805       | 9.9x10^{-6}      | 0.050 | -0.65   |
| $3/2^-_2$ | 1.402       | 0.185            | 1.920       | 0.105            | 0.524 | -8.78   |
| $3/2^-_3$ | 1.756       | 0.143            | 2.324       | 0.619            | 0.741 | 39.96   |
| $1/2^-_4$ | -0.190      | 0.0              | 1.142       | 7.1x10^{-4}      | 1.332 | 0.03    |
| $1/2^-_5$ | 1.436       | 0.374            | 2.266       | 0.790            | 0.928 | 26.62   |
| $1/2^-_6$ | 1.895       | 0.101            | 3.014       | 0.366            | 1.150 | 13.32   |
| $5/2^-_3$ | -1.017      | 0.0              | 0.783       | 9.6x10^{-8}      | 1.800 | 3.05x10^{-6} |
| $5/2^-_4$ | 0.583       | 5.1x10^{-7}      | 1.897       | 0.006            | 1.314 | 0.26    |
| $5/2^-_5$ | 1.990       | 0.032            | 3.026       | 0.183            | 1.047 | 8.29    |
| $5/2^-_6$ | 2.251       | 0.138            | 3.491       | 0.393            | 1.266 | 11.62   |
| $7/2^-_1$ | 1.591       | 0.004            | 2.700       | 0.067            | 1.111 | 3.25    |
| $7/2^-_2$ | 1.778       | 0.003            | 3.538       | 0.021            | 1.760 | 0.59    |
Figure 7: Spectra of the positive parity resonance states in $^{11}$B and $^{11}$C.
Table 5: Energies and widths of positive parity resonance states in $^{11}$B and $^{11}$C, and the Coulomb shift parameters as well. Energies $E$ are in MeV and widths $\Gamma$ are in keV.

| $J^\pi$ | $E$ | $\Gamma$ | $E$ | $\Gamma$ | $R_C$ | $\theta_C$ |
|---------|-----|---------|-----|---------|-------|----------|
| 1/2$^+$ | 0.437 | 0.015 | 0.906 | 0.162 | 0.492 | 17.40 |
| 1/2$^+$ | 0.702 | 0.012 | 1.930 | 0.059 | 1.229 | 2.19 |
| 1/2$^+$ | 1.597 | 0.016 | 2.679 | 0.086 | 1.084 | 3.70 |
| 3/2$^+$ | 1.147 | 1.49×10$^{-3}$ | 2.268 | 0.034 | 1.121 | 1.671 |
| 3/2$^+$ | 1.367 | 8.58×10$^{-3}$ | 2.478 | 0.159 | 1.121 | 7.73 |
| 3/2$^+$ | 1.715 | 4.12×10$^{-2}$ | 2.850 | 0.115 | 1.137 | 3.73 |
| 5/2$^+$ | -0.041 | 0.0 | 1.460 | 9.00×10$^{-4}$ | 1.502 | 0.03 |
| 5/2$^+$ | 1.047 | 1.54×10$^{-3}$ | 2.346 | 8.27×10$^{-2}$ | 1.302 | 3.58 |
| 5/2$^+$ | 1.951 | 4.02×10$^{-2}$ | 3.179 | 0.123 | 1.231 | 3.85 |

Figure 8: Spectra of negative parity resonance states in the mirror nuclei $^{11}$B and $^{11}$C.
states, namely the $5/2^-$ resonance state, has a very small width and was regarded in Ref. [47] as the Hoyle-analog states. The majority of the negative parity resonance states have small rotational angle $0 \leq \theta_C \leq 18^\circ$. Only two resonance states of $3/2^-$ and $1/2^-$ have the relatively large Coulomb rotational angles $\theta_C \approx 37^\circ$ and $\theta_C \approx 40^\circ$. This is due to the fact that resonance states in $^{11}$B and $^{11}$C nuclei have very small widths comparing to the resonance states in $^9$Be and $^9$B. Thus one may conclude that the Coulomb interaction has moderate effects on the energies and widths of the negative parity resonance states in $^{11}$B and $^{11}$C.

What is new for resonance states in $^{11}$B and $^{11}$C? There are three resonance states in $^{11}$C with the width smaller than their counterparts in $^{11}$B. They are the $3/2^+_1$, $3/2^-_2$ and $1/2^-_6$ resonance states. They have very small ($R_C \approx 0.05$ MeV and $R_C \approx 0.5$ MeV) or moderate ($R_C \approx 0.9$ MeV) values of the Coulomb shift. Thus for the resonance states of the negative parity in mirror nuclei $^{11}$B and $^{11}$C, we observe the second scenario of motion of resonance states caused by the Coulomb interaction.

For the positive parity states, the Coulomb parameters $R_C$ and $\theta_C$ are presented in Fig. [10]. As we can see, all these resonance states belong to the first scenario, as both $\Delta E > 0$ and $\Delta \Gamma > 0$. The main part of these resonance state are more tightly concentrated around $R_C \approx 1$ MeV then the negative parity states. Besides, the Coulomb rotational angles $\theta_C$ for the positive parity states are also smaller then for the negative parity states. They do not exceed $8^\circ$. There is one exception from this rule: the $1/2^+_1$ resonance state has a small value of the Coulomb shift $R_C \approx 0.5$ MeV and relatively large value of the Coulomb rotational angles $\theta_C \approx 17^\circ$. As for the negative parity states, the largest Coulomb shifts $R_C \approx 1.3$ MeV and $R_C \approx 1.5$ MeV are obtained for very narrow resonance states $5/2^+_2$ and $5/2^+_1$, respectively. The latter resonance state as was shown in Ref. [47] is the Hoyle-analog state.

Results presented in this section indicate that the Coulomb interaction increases energy, and in many cases increases and in a few cases decreases width of resonance states in $^{11}$C. Thus resonance states in the three-cluster continuum of the mirror nuclei $^{11}$B and $^{11}$C realize the first and the second scenarios of motion of resonance states in the $E - \Gamma$ plane.

4.3. $^7$Li and $^7$Be

Spectra of bound and resonance states in $^7$Li and $^7$Be have been calculated within AMGOB. In this model, three-cluster configurations, specified
Figure 9: The shift and rotation of negative parity resonance states in \(^{11}\text{C}\) with respect to the position of their counterparts in \(^{11}\text{B}\).
Figure 10: The Coulomb shift parameters for positive parity resonance states in $^{11}\text{B}$ and $^{11}\text{C}$ nuclei.
Table 6: Spectra of bound and resonance states in $^7\text{Li}$ and $^7\text{Be}$. Energies and widths are in MeV.

| $J^\pi$ | $^7\text{Li}$ | $^7\text{Be}$ | Coulomb shifts |
|--------|---------------|---------------|----------------|
|        | $E$ | $\Gamma$ | $E$ | $\Gamma$ | $R_C$ | $\theta_C$ |
| $3/2^-$ | -2.721 | 0.0 | -1.702 | 0.0 | 1.019 | 0 |
| $1/2^-$ | -2.469 | 0.0 | -1.412 | 0.0 | 1.057 | 0 |
| $7/2^-$ | 2.052 | 0.073 | 2.820 | 0.130 | 0.770 | 4.24 |
| $5/2^-$ | 4.270 | 1.104 | 5.040 | 1.343 | 0.806 | 17.24 |

in Table 1 were projected onto a set of the two-body channels $^4\text{He}+^3\text{H}$ and $^6\text{Li}+n$ in $^7\text{Li}$ and $^4\text{He}+^3\text{He}+^6\text{Li}+p$ in $^7\text{Be}$. These are the dominant two-body channels of $^7\text{Li}$ and $^7\text{Be}$. The the AMGOB model also allowed us to study polarizability of interacting clusters when they approach each other. It was shown in Refs. [34, 35, 6] that the polarization of interacting clusters substantially decreases energy and width of resonance states in a compound nucleus.

In Table 6 we collect the energy of bound states and the energy and width of resonance states in the mirror nuclei $^7\text{Li}$ and $^7\text{Be}$. These quantities were calculated with the MP in Refs. [34, 35]. Note that the $7/2^-$ resonance states presented in Table 6 reside in the $^4\text{He}+^3\text{H}$ and $^4\text{He}+^3\text{He}$ two-body continuum, while the $5/2^-$ resonance states belong to the energy region where there are two open channels $^4\text{He}+^3\text{H}$ and $^6\text{Li}+n$ in $^7\text{Li}$, and $^4\text{He}+^3\text{He}$ and $^6\text{Li}+p$ in $^7\text{Be}$.

In Fig. 11 we show the relative position of bound and resonance states in $^7\text{Li}$ and $^7\text{Be}$, and in Fig. 12 we display their Coulomb shift and rotation.

These figures and Table 6 show that the stronger Coulomb interaction in $^7\text{Be}$ shifts all bound and resonance states (with respect to their position in $^7\text{Li}$) approximately on the same value and rotate the broad $5/2^-$ resonance state on 17 degrees while it rotates the narrow $7/2^-$ state on 4 degrees.

4.4. $^8\text{Li}$ and $^8\text{B}$

Let us now consider how the Coulomb interaction affects the spectra of bound and resonance states in mirror nuclei $^8\text{Li}$ and $^8\text{B}$. These nuclei similar to nuclei $^7\text{Li}$ and $^7\text{Be}$ were studied within the AMGOB in Ref. [6]. The three-cluster configurations $\alpha+t+n$ and $\alpha+^3\text{He}+p$ were projected on the dominant the two-cluster channels $^7\text{Li}+n$ and $^7\text{Be}+p$. We restricted ourselves with a single-channel approximation in an asymptotic region of $^8\text{Li}$ and $^8\text{B}$, as bound
Figure 11: Spectrum of bound and resonance states in $^7$Li and $^7$Be.
Figure 12: The Coulomb shift of bound and resonance states in $^7$Be with respect to corresponding states in $^7$Li.
Table 7: Spectra of bound and resonance states in $^8$Li and $^8$B and the Coulomb shift parameters. Results are obtained with the MHNP. Energies and widths are in MeV.

| $J^\pi$ | $^8$Li  | $^8$B  | Coulomb shift |
|--------|--------|--------|---------------|
|        | $E$    | $\Gamma$ | $E$    | $\Gamma$ | $R_C$ | $\theta_C$ |
| $2^+_1$ | 1.908  | 0.0    | -0.1393 | 0.0 | 1.769 | 0.00 |
| $1^+_1$ | -0.977 | 0.0    | 0.615   | 0.044 | 1.592 | 1.57 |
| $3^+_1$ | 0.610  | 0.165  | 2.560   | 0.572 | 1.992 | 11.79 |
| $1^+_2$ | 0.014  | 0.002  | 1.305   | 0.600 | 1.423 | 24.86 |
| $1^+_3$ | 1.002  | 1.433  | 3.218   | 2.089 | 2.311 | 16.49 |
| $1^+_4$ | 2.129  | 0.913  | 3.916   | 0.272 | 1.898 | -19.71 |
| $2^+_2$ | 1.436  | 0.658  | 3.321   | 1.139 | 1.945 | 14.31 |
| $2^+_3$ | 3.175  | 0.976  | 3.889   | 0.346 | 0.953 | -41.45 |
| $4^+_1$ | 3.190  | 0.002  | 4.226   | 0.775 | 1.293 | 36.74 |
| $2^-_1$ | 3.494  | 0.365  | 3.747   | 0.712 | 0.430 | 53.92 |
| $1^-_1$ | 0.681  | 0.6245 | 1.132   | 1.828 | 1.285 | 69.45 |
| $3^-_1$ | 3.756  | 0.883  | 3.957   | 1.495 | 0.644 | 71.83 |

States exist only in two-cluster subsystems $\alpha + t$ and $\alpha + ^3$He. Thus resonance states in $^8$Li and $^8$B, which are displayed in Table 7, together with bound states, belong to the two-body continua $^7$Li+$n$ and $^7$Be+$p$, respectively.

As we see in Table 7, the Coulomb interaction diminished number of bound states in $^8$B with respect to $^8$Li. Thus, the effective interaction between clusters is reduced by the Coulomb interaction, and it results in decreasing (diminishing) energy of the $2^+$ ground state and moving up the $1^+$ excited state to continuous spectrum (i.e. transforming the $1^+$ bound state into a resonance state).

More interesting and intriguing is the influence of the Coulomb forces on energy and width of resonance states. As was shown in Ref. 56 effects of the Coulomb forces on resonance states even in two-cluster systems are not trivial. Here we deal with three-cluster system projected onto a set of two-cluster channels. In Fig. 13 we compare spectrum of bound and resonance states in $^8$Li and $^8$B calculated with the MHNP. Dot-dashed lines connect states with the same value of the total angular momentum $J$ and parity $\pi$. We can see that the Coulomb interaction shifted up energy of all bound and resonance states. Effects of the Coulomb interaction are the same for all states except for $3^+$ and $2^-$ resonance states. As we can see the $2^-$ state has the smallest impact of the Coulomb interaction on energy of this state, while
the largest impact is observed for the $3^+$ resonance state. The main result of our consideration is that the Coulomb forces substantially increase width of resonance states in $^8$B with respect to the corresponding resonance states in $^8$Li.

![Image of Figure 13: Effects of the Coulomb forces on position of resonance states in $^8$Li and $^8$B.]

More detail information about effects of the Coulomb interaction in mirror nuclei $^8$Li and $^8$B are displayed in Fig. 14. In this case we can also distinguish resonance states with the strong effects, and they are located around $R_C \approx 2$ MeV, with the medium effects they are close to $R_C \approx 2$ MeV, and resonance states with the small effects which have $R_C \approx 0.5$ MeV. The relative position of resonance states in $^8$B with respect to their counterparts in $^8$Li shows that there are three resonance states ($1^+_3$, $1^+_4$ and $2^+_3$) with negative values of $\Delta \Gamma$, thus in this pair of nuclei we observe the second scenario. The Coulomb
interaction decreases the width of two resonance states in $^8$B, but increases their energy. It is worthwhile to recall that resonance states $^8$Li and $^8$B are determined in the two-body continuum.

5. Analysis of resonance wave functions

In this section we consider how the Coulomb forces affect wave functions of resonance states in the mirror nuclei. These effects are considered for the nuclei $^9$Be and $^9$B, $^{11}$B and $^{11}$C as they are common for all nuclei under consideration. Besides, two scenarios are realized in the $^{11}$B and $^{11}$C nuclei. It is then interesting to study wave functions of resonance states when (i) the Coulomb forces increase the energy and width of resonance state, and (ii) when they increase the energy but decrease the width of resonance state. A wave function of many-channel system is a very complicated many-dimension object which is difficult to analyze. Thus we present wave functions of bound and resonance states of three-cluster systems in a compressed form through the weights $W_{sh}$ of oscillator shell $N_{sh}$ ($N_{sh}=0, 1, 2, \ldots$) in the corresponding wave function. The weights suggest a spectral decomposition of a three-cluster wave function in terms of the probability (for bound states) or contribution (for scattering states) of many-body oscillator functions belonging to the oscillator shell $N_{sh}$. It is important to recall that the oscillator functions with small values of $N_{sh}$ describe a compact configuration of three-cluster system, while the oscillator functions with large values of $N_{sh}$ reproduce a dispersed three-cluster configuration. It is also important to recall that wave functions of discrete and continuous spectrum states are properly normalized (see Eqs. (23) and (24) in Ref. [47]). As in Refs. [33, 47] we will display only the internal part of a wave function of resonance states.

We start with the case of resonance states for which we observed the smallest impact of the Coulomb forces. These are the $1/2^+$ resonance states in $^{11}$B and $^{11}$C. In Fig. [15] we compare the shell weights for the first $1/2^+$ resonance state in $^{11}$B and $^{11}$C. We can see they have approximately the same shape, however the amplitude of $W_{sh}$ for $^{11}$B is 15 times larger than these weights for $^{11}$C. This is the effect of the Coulomb forces. Fig. [15] helps us to explain why the effects of the Coulomb forces on the energy and width of the $1/2^+$ resonance state is small. We see that the weights of compact three-cluster configuration are negligible small for both nuclei, and only dispersed configurations dominate. For them the Coulomb interaction is not so strong as for compact configurations. These conclusions are valid
Figure 14: The shift and rotation of parameters of resonance states in mirror nuclei $^8$Li and $^8$B caused by the Coulomb interaction.
for the $1/2^+$ resonance states in the mirror nuclei $^9$Be and $^9$B (see Fig. 5 in Ref. [17]). In this case, the effects of the Coulomb interaction are not so strong as in $^{11}$B and $^{11}$C, however the shapes of the resonance wave functions are very similar.

Let us now turn our attention to the $3/2^-$, $3/2^-$, and $3/2^-$ resonance states in $^{11}$B and $^{11}$C. As we can see from Table 4, the Coulomb interaction realizes two scenarios for these resonance states. The second scenario is observed for the $3/2^-$ and $3/2^-$ resonance states, and the first scenario is realized for the $3/2^-$ resonance state. Thus we consider wave functions of these three-resonance states. In Figs. 16, 17 and 18 we display the weights of the oscillator shells for the $3/2^-, 3/2^-$ and $3/2^-$ resonance states, respectively. The resonance states $3/2^-, 3/2^-$ and $3/2^-$ are denotes as $R_1$, $R_2$ and $R_3$, respectively. The $R_1$ states are very narrow resonance states in $^{11}$B and $^{11}$C. The Coulomb interaction makes this resonance state in $^{11}$C more narrow than in $^{11}$B, as the result the amplitude of the $^{11}$C wave function is approximately 25 larger than the amplitude of the $^{11}$B wave function. Thus the stronger Coulomb interaction in $^{11}$C does not decrease the amplitude of the wave function, as was observed in other cases demonstrated above or in Refs. [37, 33, 47]. On the contrary, it substantially increases the amplitude of the wave function of such a narrow resonance state. There is an interplay of two factors which affects the wave function of the resonance state. First, the stronger Coulomb interaction suppresses stronger a wave functions of resonance states at small distance between interacting clusters, or, in our representation, oscillator shells with small values of $N_{sh}$. Second, the narrower is the resonance state, the larger is the amplitude of the resonance wave function. Thus, in the $R_1$ resonance state the second factor dominates over the first factor and as a results increases the amplitude of the wave function of $^{11}$C in the internal region.

The $R_2$ resonance states are not so narrow as the $R_1$ resonance states. These resonance states realize the second scenario when the Coulomb interaction decreases the width but increases the energy of the resonance state. As in previous case, the Coulomb interaction increases the amplitude of the $R_2$ wave function in $^{11}$C comparatively to the wave function of $^{11}$B, however only two times. The $R_3$ resonance states realizes the first scenario. In this case, the Coulomb interaction yields the moderate shift $R_C=0.741$ MeV but the large rotational angle $\theta_C=39.96^\circ$. The amplitude of the $R_3$ wave function in $^{11}$B is slightly larger than the amplitude of the $R_3$ wave function in $^{11}$C.

To gain more information about structure of resonance wave functions
Figure 15: Weights of different oscillator shells $W_{sh}$ as a function of $N_{sh}$ in the wave functions of the $1/2^+_1$ resonance state in $^{11}$B and $^{11}$C.
Figure 16: Weights $W_{sh}$ of different oscillator shells in the wave functions of the $3/2^-$ resonance states in $^{11}$B and $^{11}$C.
Figure 17: Weights $W_{sh}$ of different oscillator shells in the wave functions of the $3/2^-_2$ resonance states in $^{11}$B and $^{11}$C.
Figure 18: Weights $W_{sh}$ of different oscillator shells in the wave functions of the $3/2^{-}$ resonance states in $^{11}$B and $^{11}$C.
which are related by the first and second scenarios, we consider the $1/2_3^-$, $1/2_4^-$ and $1/2_5^-$ resonance states in $^{11}$B and $^{11}$C. We do not consider the $1/2_3^-$ states as being a weakly bound state in $^{11}$B it transformed into a resonance state in $^{11}$C.

Structure of wave functions of two $1/2^-$ resonance states in $^{11}$B and $^{11}$C are presented in Figs. 19 and 20 in terms of the weights $W_{sh}$ of different oscillator shells. We selected wave functions of the $1/2_4^-$ and $1/2_6^-$ resonance states. The $1/2_4^-$ resonance states are related by the first scenario and we see that the Coulomb interaction suppress the wave function in $^{11}$C with respect to the one in $^{11}$B. The same results are also obtained for the $1/2_5^-$ resonance states which are united the first scenario. The $1/2_6^-$ resonance states represents the second scenario and in this case the wave function of the resonance state in $^{11}$C at small distance (or at small values of $N_{sh}$) is slightly larger than in $^{11}$B. Comparing Fig. 20 with Figs. 16, 17 we came to the conclusion that for resonance states related by the second scenario the smaller is the width of resonance state in the $^{11}$C nucleus the larger is the resonance wave function in the internal region for the $^{11}$C nucleus with respect to one in the $^{11}$B nucleus.

It is interesting to consider the case when the Coulomb interaction has the largest impact on the parameters of resonance states. For this aim we selected the $7/2^-$ and $3/2^+$ resonance states in mirror nuclei $^9$Be and $^9$B. First, we consider the $7/2^-$ resonance states. The Coulomb interaction shifts the $7/2^-$ resonance state on $R_C=2.072$ MeV and rotates on $\theta_C=13.74^\circ$. Fig. 21 displays wave functions of the $7/2^-$ resonance states. As we see, wave functions of these resonance states describe compact three-cluster configurations since the oscillator shells with small values of $N_{sh}$ ($0 \leq N_{sh} < 10$) give the main contribution to these wave functions. The Coulomb interaction reduces the amplitude of $W_{sh}$ of the $^9$B wave function approximately two times with respect to the wave function of $^9$Be.

The $3/2^+$ resonance states in $^9$Be and $^9$B are related by the largest Coulomb shift $R_C=2.669$ MeV and the largest Coulomb rotational angle $\theta_C=83.44^\circ$. The weights of the oscillator shells displayed in Fig. 22 demonstrate that in this case the Coulomb interaction dramatically changes the structure of the resonance wave function in $^9$B. It strongly diminishes the weights of the oscillator shells with small values of $N_{sh}$ ($0 \leq N_{sh} \leq 15$) and substantially suppress the weights in the region $15 < N_{sh} \leq 24$.

In Figs. 13, 16, 17, 18, 21, 22 we display behavior of the resonance wave functions which are typical for (or similar to) all other resonance states and
Figure 19: The weights of different oscillator shells in the wave functions of the $1/2^-_4$ resonance states in mirror nuclei $^{11}$B and $^{11}$C.
Figure 20: The structure of the wave functions of the $1/2_-$ resonance states in $^{11}$B and $^{11}$C.
Figure 21: The weights of the oscillator shells in the wave functions of the $7/2^-$ resonance states in $^{9}\text{Be}$ and $^{9}\text{B}$.
Figure 22: The weights of oscillator shells in wave functions of the $3/2^+$ resonance states in $^9$Be and $^9$B.
nuclei considered.

6. Conclusions

We have considered effects of the Coulomb interaction on energies and widths of resonance states in mirror nuclei $^7$Li and $^7$Be, $^8$Li and $^8$B, $^9$Be and $^9$B, $^{11}$B and $^{11}$C. We have analyzed resonance states embedded in two- and three-cluster continua of these nuclei. Resonance states were obtained in the framework of microscopic three-cluster models. For the proper investigation of effects of the Coulomb interaction we introduced two parameters which determine a rotation and shift (displacement) of the relative position of resonance parameters on the energy and width $E - \Gamma$ plane. It was shown that the Coulomb shift for bound states is larger than for resonance states, since bound states are more compact than resonance states. However, for very narrow resonance states the Coulomb shift is close to the shift of the bound states. This indicates that narrow resonance states can be treated as compact objects as it has been demonstrated, for example, in Refs. [37, 33, 47]. Such narrow resonance states in the three-cluster continuum of $^9$Be and $^9$B, $^{11}$B and $^{11}$C are the Hoyle-analog states as was shown in Ref. [47].

It was also found that the smallest Coulomb shift is observed for broad resonance states. They are, for example, the $1/2^{+}$ resonance states in the three-cluster continuum of $^9$Be and $^9$B, $^{11}$B and $^{11}$C. As it was shown in Refs. [33, 37, 47], these resonance states have very a dispersed three-cluster structure. Therefore, the Coulomb shifts are equal $R_C = 0.429$ and $R_C = 0.492$ MeV, respectively, for these pairs of the mirror nuclei. There is one pair of resonance states in the two-cluster continuum of the mirror nuclei $^8$Li and $^8$B, which also has the smallest Coulomb shift $R_C = 0.304$ MeV. This pair of resonance states is the $1_3^{+}$ resonance states.

We have investigated different scenarios of motion of resonance states due to the Coulomb forces. The dominated scenario for three-cluster systems is when both energy and width of the $R$ nucleus are increased with respect to the position of corresponding resonance state in the $L$ nucleus. This scenario is observed for resonance states residing in two- and three-cluster continua. We also observed a few cases of the second scenario, when the Coulomb interaction increases energy of resonance state but decreases its width. We have not observed the third and fourth scenarios when the Coulomb interaction decreases the energy of resonance states.
And final remark. We have suggested a method how to analyze effects of a specific interaction (the Coulomb interaction) on resonance states in two- and three-cluster continuum. It is obvious that this method can be applied to study effects of other types of forces or different factors on the energies and widths of resonance states in many-channel and/or many-cluster systems.

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