Quantum critical fluctuations in disordered $d$-wave superconductors

Julia S. Meyer$^a$, Igor V. Gornyi$^{b,\ast}$, and Alexander Altland$^a$

$^a$Institut für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Köln, Germany
$^b$Institut für Nanotechnologie, Forschungszentrum Karlsruhe, 76021 Karlsruhe, Germany;
Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, 76128 Karlsruhe, Germany
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Quasiparticles in the cuprates appear to be subject to anomalously strong inelastic damping mechanisms. To explain the phenomenon, Sachdev and collaborators recently proposed to couple the system to a critically fluctuating order parameter mode of either $id_{xy}$- or $is$-symmetry. Motivated by the observation that the energies relevant for the dynamics of this mode are comparable to the scattering rate induced by even moderate impurity concentrations, we here generalize the approach to the presence of static disorder. In the $id$-case, we find that the coupling to disorder renders the order parameter dynamics diffusive but otherwise leaves much of the phenomenology observed in the clean case intact. In contrast, the interplay of impurity scattering and order parameter fluctuations of $is$-symmetry entails the formation of a secondary superconductor transition, with a critical temperature exponentially sensitive to the disorder concentration.

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Recent experimental work shows that essential aspects of the low-temperature behaviour of cuprate superconductors lie beyond the scope of orthodox BCS-type formulations in terms of a non-interacting quasiparticle (QP) system. Notably, high resolution angle-resolved photoemission (ARPES) experiments indicate a QP relaxation rate $\kappa \sim |\epsilon|$ proportional to the excitation energy (for temperatures $T < |\epsilon|$), while the standard BCS picture predicts $\kappa \sim |\epsilon|^3$. The much weaker $\sim |\epsilon|$ scaling most likely represents a correlation effect.

Arguing that a qualitative modification of the $|\epsilon|^3$-law required coupling of the QP system to some kind of collective mode, Sachdev and collaborators recently proposed a dynamical generalization of the BCS order parameter field, viz.

$$\hat{\Delta}(x, t) = \hat{\Delta}_0(x) + i\hat{\Delta}_1(x, t),$$

where $\hat{\Delta}_0$ is the background field of $d_{x^2-y^2}$ angular momentum dependence and $\hat{\Delta}_1 \equiv i\hat{\Delta}^s/d_{xy}$ a time-dependent component of either $s$- or $d_{xy}$-wave symmetry.

The basic idea behind the proposal is that somewhere in close vicinity to the phase diagram of the system there might be a quantum critical point towards formation of a stable $i\hat{\Delta}_1$ order parameter amplitude (a $d + is/d_{xy}$ state.) Although the actual transition point may lie outside the accessible parameter space, its proximity will manifest itself through dynamical order parameter fluctuations, as described by $\hat{\Delta}_1$.

The coupling of the collective mode $\hat{\Delta}_1(t)$ to the relativistic QPs strongly alters the low-energy phenomenology of the system. Specifically, (i) scattering off the time-dependent order parameter fluctuations enhances the QP relaxation to a linear rate $\kappa \sim |\epsilon|$. This amplification mechanism appears to be insensitive to microscopic details such as the ultraviolet structure of the QP dispersion relation. In particular, (ii) no qualitative differences between $s$- and $d_{xy}$-scattering exist. The coupling of the QPs to $\hat{\Delta}_1$ leads (iii) to the formation of long-ranged collective fluctuations which should be observable by neutron or Raman scattering. However, (iv) if sufficiently weak, the attractive interaction mediated by $\hat{\Delta}_1$ does not open a QP gap, i.e., it does not drive the system into a secondary superconductor instability.

It is the purpose of this Letter to explore whether elements of the phenomenology outlined above survive generalization to the presence of static disorder. In view of the fact that all features (i)-(iv) rely on a conspiracy of order parameter fluctuations and the Dirac spectrum of the QPs – the latter being thoroughly corrupted by impurity scattering – one might expect the answer to be largely negative. However, as we are going to show below this presumption is overly conservative: owing to a decoupling of the order parameter from the impurity scattering vertex ($id$-case) and/or the renormalization of the polarization operator by infrared-singular diffusion modes ($is$-case), the disordered system, too, turns out to be strongly affected by fluctuations of $\hat{\Delta}_1(t)$.

To summarize our main findings, (i) notwithstanding the presence of disorder, the coupling of QPs to a dynamical order parameter of $id$-symmetry continues to induce an inelastic relaxation rate $\kappa_{\text{in}} \sim |\epsilon|$. However, this contribution sits now superimposed on a structure-less elastic background $\kappa_{\text{el}} \sim \tau^{-1}$, where $\tau$ denotes the impurity scattering time. Further, at low frequencies, $|\epsilon| < \tau^{-1}$, a logarithmically singular third contribution $\kappa_{\text{sing}} \sim \ln(\tau|\epsilon|)$ to the relaxation rate $\kappa = \kappa_{\text{in}} + \kappa_{\text{el}} + \kappa_{\text{sing}}$ signals the formation of long-ranged diffusive excitations which eventually (for asymptotically low excitation energies/temperatures) fully absorb the spectral weight carried by individual QP states. (ii) Unlike in the clean case, the system is no longer indifferent to the symmetry of $\hat{\Delta}_1(t)$. While in the $id$-case, QPs remain (marginally) stable objects and (iii) the order parameter continues...
to exhibit fluctuation behaviour similar to that found in Refs. \[3, 4\], a conspiracy of impurity scattering and dynamical fluctuations of \textit{is}-symmetry (iv') drives the system into a secondary superconductor transition. More specifically, at the critical temperature

\[ \tilde{T}_c \sim \omega_0 \exp(-\tilde{r}/\nu_0), \]

the system enters a $d + is$ superconducting state and the QPs become gapped. Here $\omega_0 = \min[\omega_D, 1/\tau]$, where $\omega_D$ is a phenomenological parameter setting the maximum (Debye) frequency of the order parameter fluctuations, $1/\tau$ denotes the amplitude of the order parameter fluctuations, and $\nu_0$ the residual low-energy density of states (DoS) induced by impurity scattering. Notice that through the dependence on the DoS on the impurity concentration, $\nu_0 \propto 1/\tau$, the transition temperature varies exponentially with the disorder strength. Since the latter can be deliberately changed by doping, the observability of a disorder-dependent BCS superconductor transition at low temperatures appears to impose a relatively straightforward criterion testing the presence of dynamical \textit{is}-order parameter fluctuations in the system.

To prepare our discussion of the interplay order parameter fluctuations/disorder, let us briefly recapitulate the physics of the clean system. Denoting the two-component QP field by $\psi^a$, where $a = 1, 1, 2, 2$ enumerates the low-energy nodes in the Brillouin zone, and the time-dependent amplitude of the order parameter by $\phi(x)$, the low-energy physics of the system is described by a composite action $S[\phi, \psi] = S[\psi] + S[\phi] + S_c[\phi, \psi]$. Here, $x = (\tau, x)^T$ is a three-component space-time argument and $S[\psi] = \int d^3x \psi^a \![\partial_\tau \psi^a + i\nu_F (s^2_{\tau} \!-\! 1) \sigma_1 \partial_1 + s^2_{\tau} \sigma_2 \partial_2] \!\psi^a$ the Dirac action of the unperturbed QPs where $\nu_F$ is the Fermi velocity, $\gamma \equiv t/|\Delta_0|$, $t$ the tight binding energy, $s^a_j$ are sign factors depending on the node index, and $\sigma_j$ Pauli matrices in particle/hole space \[3\]. Further,

\[ S[\phi] = \int d^3x \!\left[ \frac{1}{2} (\partial_\tau \phi)^2 + c^2 (\partial_\xi \phi)^2 + r \phi^2 \right] + \frac{u}{4!} \phi^4 \]

is a $\phi^4$-type action controlling the fluctuation behaviour of the order parameter field. ($c, r,$ and $u$ are phenomenological constants, where $r = 0$ marks the position of the quantum critical point into the $d_x^2 - y^2 + is/d_{xy}$ phase.) Finally, the coupling between the QP fields and the collective mode $\phi$ is mediated by the scattering vertex

\[ S_c[\phi, \psi] = \lambda \int d^3x \!\psi^a \Gamma^a \sigma_3 \psi^a \phi, \]

where $\lambda$ is a coupling constant, and $\Gamma^a$ a symmetry factor discriminating between the two different types of order parameters: $\Gamma^a = 1$ (\textit{Γ} = $(-\Gamma)^a$) for the case of \textit{is} (\textit{id}) symmetry. Perhaps the most direct way to understand the consequences of this coupling mechanism is to integrate over the QP fields. This leads to an effective $\phi$-action $S_{eff}[\phi] = S[\phi] + \delta S[\phi]$, where $\delta S[\phi] = -\ln\exp(-S[\phi, \psi])$ measures the tendency of the system to create/annihilate a Cooper pair in response to fluctuations of the order parameter and $\langle \ldots \rangle = \int \mathcal{D}(\psi, \phi) \exp(-S[\psi, \phi]) \langle \ldots \rangle$ is the functional average over the QP action. For sufficiently weak coupling, $\delta S[\phi]$ can be approximated by its second order expansion in $\phi$ (an RPA type approximation), i.e., $\delta S[\phi] = \frac{\lambda^2}{2} \int d^3x d^3x' \phi(x) \chi(x, x') \phi(x')$, where

\[ \chi(x, x') \equiv \text{tr} \left[ \Gamma^a \sigma_3 G^{a\alpha}(x, x') \Gamma^b \sigma_3 G^{b\alpha}(x', x) \right] \]

is the Cooper pair propagator, ‘tr’ stands for a trace over particle/hole indices, and $G^{a\alpha}(x, x') = \langle \psi^a(x) \psi^a(\tau) \rangle$ is the Gorkov Green function.

Evaluating $\chi$ for the clean case (in which $\mathcal{G}$ becomes diagonal in momentum and nodal space), one finds

\[ \delta S[\phi] = -\int d^3k |\phi_k|^2 \left[ C_1 - C_2 (\omega_m^2 + k^2)^{1/2} \right], \]

where $\int d^3k \equiv T \sum_{\omega_m} \int d^2k$ denotes a summation over the three-momentum $k \equiv (\omega_m, k_T^2)$, and $\omega_m$ is a bosonic Matsubara frequency. While the constant contribution $C_1$ can be absorbed into a shifted value of the parameter $r$, the non-analytic operator $\sim (k^2 + \omega_m^2)^{1/2} - \omega_0^2$ arising from the infrared singularity of the Dirac operator – dominantly influences the scaling behaviour of the model, and holds responsible for the anomalously strong QP relaxation rate.

As a corollary we remark that (ii') the structures above are insensitive to the symmetry of $\Delta_\pm(t)$. The difference between an $s$- and a $d$-scattering amplitude, respectively, is that the former is constant while the latter changes sign from node to node. However, being \textit{quadratic} in the scattering vertex, the pair susceptibility operator (and therefore the induced action $\delta S$) is oblivious to this difference. Further, (iv') the rough estimate, $C_1 \sim \lambda^2 \int d\nu(\nu) \exp(\nu)/(\nu - |\epsilon|) \phi$ implies that the same linearity of the DoS, $\nu(\epsilon) \sim |\epsilon|$, that led to the non-analytic structure of $\delta S$ excludes the formation of a Cooper instability in the $\phi$-action. In contrast, coupling $\phi$ to a fictitious QP species with an extended Fermi surface, $\nu(\epsilon) = \nu_0$, would lead to $C_1(T) \sim \lambda^2 \nu_0 \ln(\omega_0/T)$, i.e., the formation of a superconductor transition at a critical temperature $\tilde{T}_c \sim \omega_0 \exp(-\tilde{r}/\nu_0)$, where $\tilde{r} = r/\lambda^2$.

How then do these structures change in the presence of disorder? Before addressing this question in quantitative detail, let us briefly summarize a number of key elements entering the theory of the weakly disordered \[3\] system: first, the picture developed in Refs. \[3, 4\] essentially relies on the linear singularity of the low-energy DoS. In contrast, disorder scattering leads to a randomization of spectral structures over scales $\sim \tau^{-1}$ and, therefore, to a smearing of the low-energy cusp of the Dirac spectrum. Relatedly, the QP dynamics crosses over from ballistic
at $|c| > \tau^{-1}$ to diffusive at $|c| < \tau^{-1}$. As relativistic QP dynamics is an essential input to the theory \[1,2\] one may expect $|c| \sim \tau^{-1}$ to be a lower bound for its applicability. Finally, and in a way to be discussed in more detail below, the relaxation time (as well as DoS, spin- and thermal conductance) are affected by mechanisms of quantum interference, similar to the weak localization corrections known from the physics of normal metals.

To quantitatively explore the ramification of these mechanisms in the present context \[3,4,10\], we consider the configurational average of the susceptibility operator $\langle \chi(x, x') \rangle_{\text{dis}}$. The most basic effect of impurity scattering will be that the two Green functions appearing in \[3\] acquire a self energy, $G^{-1}(i\epsilon_n) \to G^{-1}(i\epsilon_n + \epsilon_c \text{sgn} (\epsilon_n))$, where the real constants $\epsilon_c = (2\tau)^{-1}$ and $\zeta$ are obtained for $|c| < \tau^{-1}$ from (depending on the microscopic distribution of the disorder) self-consistent Born or T-matrix perturbation theory \[4\].

We next observe that in the is-case the inclusion of a Green function self energy is partially compensated for by vertex corrections. Evaluating the quasiparticle/order parameter vertex within a ladder approximation shown in Fig. \[4\] (the latter stabilized by the parameter $\gamma \gg 1$), one finds that the vertex is coupled to a mode $D_{nn'}(q) = (\delta q^2 + \epsilon_n + \epsilon_{n'})^{-1}$ similar to the singular ‘diffusion’ and ‘Cooperon’ modes of disordered metals. Here, $D = (\gamma + \gamma^{-1})/(\pi^2 \nu_0)$ is the diffusion constant of the d-wave superconductor and $\epsilon_{n,n'}$ are fermionic Matsubara frequencies. Including this mode in the calculation of the polarization operator, we find that

$$\delta S^i_s[\phi] = -\int d^4k \ |\phi_k|^2 \left[ C_1 (T) + \ldots \right], \quad (7)$$

where, apart from a $T$-independent shift $\delta r (\ll r)$, the constant term $C_1(T) = \lambda^2 \nu_0 \ln(\omega_r/T) + \delta r$ now displays the characteristic low-temperature profile of s-wave superconductors, and the ellipses stand for operators involving derivatives. The low-$T$ singularity of $C_1$ implies that at the critical temperature \[4\] the $\phi$-action becomes unstable, and a BCS transition into a phase with finite expectation value of the order parameter $\langle \phi \rangle \neq 0$ and gapped QP states occurs. The only significant difference to the BCS transition in conventional disordered metals is that presently the low-energy DoS $\nu_0$ — an essential ingredient to the buildup of a BCS instability — is generated by impurity scattering (rather than by the existence of an extended Fermi surface). I.e., unlike in normal metals, the critical temperature is exponentially sensitive to the disorder concentration.

The key phenomenon prescribing much of the physics of the complementary id-case is that here the susceptibility operator is not affected by vertex corrections. Formally, this follows from the fact that the mode $D$ obtained by summing an impurity ladder (cf. Fig. \[4\]) is isotropic in node space. As a consequence, the two nodal summations over the symmetry factors $\Gamma^a = (-)^a$ at the vertices decouple and annihilate all contributions to the susceptibility operator, safe for the ‘bare bubble’ without vertex corrections. Heuristically, this mechanism can be understood by noting that multiple impurity scattering occurring in transit between two consecutive interactions with the order parameter field distributes the QP amplitude homogeneously over all nodes. The vanishing of the nodal average of $\Delta^{d_{xy}}$ then implies the absence of multiple scattering contributions to the pair propagator.

Evaluating Eq. \[3\] for the self-energy decorated Green functions, we find

$$\delta S^{id}[\phi] = -\int d^4k \ |\phi_k|^2 \left[ C_1 - C_2 \left( |\omega_m| + D k^2 \right) \right], \quad (8)$$

with the (now again finite at $T \to 0$) constant $C_1 \sim \lambda^2 \omega_D f(\omega_D \tau \zeta)$, where $f(x)$ decreases from 0 to $1$ at $x \to 1$ to $x \to 0$, and $C_2 \sim \lambda^2$. Importantly, a non-analytic contribution $\sim |\omega_m| \delta^2$ survives the blurring of the low-energy spectrum by the imaginary part of the self-energy. The continued presence of an operator of engineering dimension 1 indicates that the collective mode displays fluctuation behaviour similar to that in the clean case. Relatedly, a straightforward calculation along the lines of Ref. \[4\] shows that in the vicinity of the instability $(r \ll C_2 |\epsilon|)$ $\kappa_{\text{in}} \sim |\epsilon|$, as in the absence of disorder, while for $r \gg C_2 |\epsilon|$ we obtain Fermi liquid behaviour $\kappa_{\text{in}} \sim \epsilon^2$.

Having identified the two terms $\kappa_{\text{in}}$ and $\kappa_{\text{el}}$, we proceed to discuss the contribution of disorder-induced quantum interference to the QP self energy. To leading order in an expansion in the small parameter $g^{-1}$, where $g = D \nu_0$, the soft modes $D$ couple to the self-energy operator through the diagram shown in Fig. \[4\]. Physically (cf. Fig. \[4\]), this diagram describes the self-interference of a QP traversing a scattering path twice. In close analogy to the weak localization corrections to the conductance of normal metals, these processes lead to a negative logarithmic correction $\kappa_{\text{sing}}(\epsilon) \sim g^{-1} \ln(\tau |\epsilon|)$ to the self energy. In passing we note that the singularity of the interference correction to the DoS entails the existence of a crossover scale $\omega^* \sim \tau^{-1} \exp(-g)$, below which a proliferation of diffusion modes holds responsible for both, Anderson localization of the QP states and a linear vanishing of their DoS, $\nu(\epsilon) \sim \nu_0 |\epsilon|/\omega^*$.\[3,4,12\]

While the logarithmic singularity of the self energy

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Top: structure of the polarization operator in the clean, disordered id, and disordered is case, respectively. Bottom: effective order parameter propagator.}
\end{figure}
is primarily caused by disorder scattering, the presence of a dynamical id order parameter necessitates to replace the bare diffusion mode by the ‘interacting’ mode shown in Fig. 2 (bottom). However, the quantitative summation over these self-interaction processes merely leads to an inessential renormalization of the prefactor, \( g^{-1} \rightarrow g^{-1}(1 + F) \), where \( F \ll 1 \) is an interaction-dependent constant, playing a role similar to the Fermi-liquid constants in normal metals.

Similar logarithmic corrections, weakly renormalized by QP interactions, are found for the (spin-) conductance and the DoS \([3, 4]\). These findings conform with the RG analysis of Ref. \([1]\) which showed (albeit for the case of globally broken time-reversal invariance – symmetry class C in the notation of Ref. \([1]\), while the present problem falls into the more complex symmetry class CI) that interactions only weakly affect the low-energy disorder-generated interference phenomena \([12]\).

Summarizing, we have explored the interplay of disorder scattering and critical fluctuations of an is- or id-order parameter component in two-dimensional cuprate superconductors. In the is-case, the most prominent effect caused by disorder is a secondary BCS transition whose critical temperature sensitively depends on the impurity concentration. In the complementary id-case, the addition of disorder to the system leaves much of the phenomenology derived in Refs. \([3, 4]\) intact. Notably, for temperatures \( T < |\epsilon| \), the characteristic singularity \( \kappa_{\text{in}} \sim |\epsilon| \) pertains to the disordered case. Heuristically, these phenomena can be understood by noting that in the presence of disorder (and for mass parameters fine tuned to criticality) the collective mode corresponding to the order parameter amplitude diffusion and, therefore, continues to act as a strong agent of QP scattering. Finally, we included the (essentially disorder-generated) formation of logarithmic singularities at low energies into our discussion of the relaxation rate. Given that even moderate impurity concentrations \([17]\) give rise to scattering times comparable to the energies \( O(20K) \) estimated \([3]\) as characteristic for the fluctuations of \( \Delta_1 \), we believe that, for sufficiently low temperatures, our findings may be made visible experimentally.

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\( ^* \) Also at A.F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia.

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