Supermanifolds, Rigid Manifolds and Mirror Symmetry

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By providing a general correspondence between Landau-Ginzburg orbifolds and non-linear sigma models, we find that the elusive mirror of a rigid manifold is actually a supermanifold. We also discuss when sigma models with super-target spaces are conformally invariant and describe their chiral rings. Both supermanifolds with and without Kähler moduli are considered. This work leads us to conclude that mirror symmetry should be viewed as a relation among super-varieties rather than bosonic varieties.

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1. Introduction

Mirror symmetry is among the most exciting and intriguing discoveries in string theory. We begin with a brief description: Consider the class of string vacua given by $N = 2$ superconformal non-linear sigma models. These theories are particularly interesting when used to compactify the heterotic string; the resulting theory is then space-time supersymmetric. Mirror symmetry is the statement that strings propagating on inequivalent but mirror target spaces give physically equivalent theories. To understand why a manifold and its mirror give the same field theory, we must examine how geometric rings on the manifold map to rings of operators in the field theory.

In particular, the cohomology ring of the manifold, in the large radius limit, is associated to a ring of primary operators in the field theory. However, there are two possible candidates for the cohomology ring, excluding complex conjugation, in a unitary $N = 2$ theory: Either the ring of left-chiral right-chiral primary operators, known as the $(c, c)$ ring, or the ring of left-antichiral right-chiral primary operators known as the $(a, c)$ ring [1][2]. The operators in these rings are labeled by a left-moving and a right-moving $U(1)$ charge. Chiral primary operators are positively charged while anti-chiral primary operators are negatively charged. Note that the two rings are exchanged by simply changing the sign of the left-moving $U(1)$ charge. If the cohomology ring of a manifold $M$ is identified with the $(c, c)$ ring, then the mirror symmetry conjecture asserts that there is a mirror manifold $\tilde{M}$ whose cohomology ring can be identified with the $(a, c)$ ring of the field theory. For manifolds of complex dimension $d$, the Hodge numbers $h^{p,q}$ of $M$ are related to the Hodge numbers $\tilde{h}^{p,q}$ of $\tilde{M}$ by:

$$\tilde{h}^{p,q} = h^{d-p,q}. \quad (1.1)$$

There are two kinds of moduli for these conformal field theories. Those given by charge $(1, 1)$ primary operators correspond to deformations of the Kähler structure, while those given by charge $(-1, 1)$ primary operators correspond to deformations of the complex structure. The $(-1, 1)$ operators are identified with $(d - 1, 1)$ operators using the asymmetric spectral flow isomorphism of these theories. Under the mirror map, these two moduli spaces are exchanged.

Within the past few years, there has been a great deal of effort devoted to understanding mirror symmetry [3]. A primary motivation has been to understand how string
theory modifies conventional geometry, providing powerful and unexpected relations between a priori unrelated manifolds. However, the existence of rigid manifolds has posed a long-standing obstacle to this effort. A rigid manifold is a Calabi-Yau manifold without any complex structure moduli. The mirror therefore cannot possess any Kähler moduli and so cannot be a Kähler manifold in any conventional sense. In a number of cases, the mirror to a rigid manifold can be described as a Landau-Ginzburg orbifold \cite{4}. These Landau-Ginzburg models fall outside of the class that had previously admitted a sigma model interpretation \cite{5} \cite{6} \cite{7}. Our approach to this problem is to give a sigma model interpretation to these Landau-Ginzburg orbifolds. These theories include a far larger class of models than simply mirrors of rigid manifolds. We then find that strings propagating on a bosonic manifold can be alternatively described by strings propagating on a supermanifold where the fermionic coordinates carry negative dimension!

In fact, the fermionic coordinates perform a second crucial function by ‘cancelling’ out the contribution to the super-first Chern class from the bosonic coordinates; hence, allowing conformal invariance. From this analysis, we conclude that the correct framework for understanding mirror symmetry is not the space of bosonic varieties but the space of super-varieties. This framework would seem a more natural setting for algebraic geometry since the mirror encodes non-trivial information about the original manifold - such as the counting of instantons - in a computable manner.

In the following section, we give a geometric interpretation of Landau-Ginzburg orbifolds as sigma models on supermanifolds. Section three provides a discussion of when such sigma models are conformally invariant. Section four is a study of the chiral primary ring of these theories. This section includes a discussion of the type of supercohomology theory needed to construct physical observables. The final section is devoted to summarizing the findings.

2. Landau-Ginzburg Orbifolds and Non-linear Sigma Models

2.1. A Path-Integral Argument

The action for a two-dimensional $N = 2$ Landau-Ginzburg model is of the form:

\[
\int d^2z d^4\theta K(\Phi_i, \Phi_i) + \left( \int d^2z d^2\theta W(\Phi_i) + \text{c.c.} \right). \tag{2.1}
\]

The chiral superfields $\Phi^i$ satisfy $D^+ \Phi^i = \overline{D}^+ \Phi^i = 0$ where
\[
D^\pm = \frac{\partial}{\partial \theta^\pm} + \theta^\mp \frac{\partial}{\partial z}, \quad \theta^\pm \Rightarrow \overline{\theta}^\mp,
\]
while the anti-chiral fields satisfy conjugate conditions. We take the superpotential to be a quasi-homogeneous polynomial in the chiral superfields with a degenerate critical point at the origin. A simple example is the Fermat type superpotential where the fields \(\Phi_i\) have charge \((\frac{1}{k_i}, \frac{1}{k_i})\) and

\[
W(\Phi_i) = \sum \Phi_i^{k_i}.
\]

Let us denote the charge of the chiral field \(\Phi_i\) by \((\frac{1}{k_i}, \frac{1}{k_i})\) for the general case. The requirement of quasi-homogeneity is needed for conformal invariance. The superpotential then defines a universality class under renormalization group flow. The choice of superpotential determines the chiral primary ring \(\mathcal{R}\) in a simple way \[1\][8],

\[
\mathcal{R} = \mathbb{C}[\Phi_i] \frac{dW(\Phi_i)}{dW(\Phi_i)}.
\]

This is the local ring of \(W\) given by monomials in \(\Phi_i\) modulo the Jacobian ideal. To simplify our discussion, we take the low-energy limit and consider constant superfields which we can treat as complex variables. A computation of \(\hat{c} = \frac{2}{g_s}\), where \(c\) is the central charge, gives the well-known formula \(\hat{c} = \sum 1 - \frac{2}{k_i}\). This is the dimension of the associated sigma model. Our aim in this section is to relate orbifolds of Landau-Ginzburg models with integral \(\hat{c}\) to sigma models. Specifically, the orbifold of the Landau-Ginzburg model by the diagonal sub-group of the phase symmetries for the theory.

\[
\Phi_i \rightarrow e^{\frac{2\pi i}{k_i}} \Phi_i
\]

Let us briefly comment on what we mean by associating a sigma model to a Landau-Ginzburg orbifold. The Landau-Ginzburg theory and the sigma model can be viewed as different ‘phases’ of the same theory. A smooth analytic continuation is believed to exist between the two phases \[3\][9]. For the remainder of this paper, we simply refer to an identification or association of a sigma model to a Landau-Ginzburg orbifold with the above comments in mind.

Before discussing the general case, let us consider a simple but illuminating family of examples with superpotential:
\[ W(\Phi_i) = \sum_{i=1}^{3N} \Phi_i^3. \]  

(2.6)

For the case \( N = 1 \), we can use the results of [5]. Let \( \Lambda = \Phi_1^3 \) and \( z_i = \frac{\Phi_i}{\Phi_1} \) for \( i \neq 1 \). This corresponds to choosing a patch with \( \Phi_1 \neq 0 \). The F-term in the lagrangian (2.1) then becomes:

\[
\int d^2 z d^2 \theta^{-} \Lambda (1 + z_2^3 + z_3^3) + c.c.
\]  

(2.7)

More importantly, the Jacobian for this change of variables is trivial so we can integrate out the superfield \( \Lambda \) as a Lagrange multiplier giving a super-delta function. The change of variables is not one-to-one, so we must further orbifold by the diagonal \( \mathbb{Z}_3 \) sub-group of the phase symmetries. We conclude that the Landau-Ginzburg orbifold is identified with a sigma model on the variety \( W = 0 \) in \( \mathbb{P}^2 \), which is just the maximal \( SU(3) \) torus.

For the more general superpotential with \( n \) fields, this change of variables procedure can be performed when \( \hat{c} = n - 2 \). This condition corresponds to the variety \( W = 0 \) in weighted projective space having vanishing first chern class. There are two other cases that can arise. For \( \hat{c} > n - 2 \), one simply adds quadratic fields to the superpotential \( W \to W + x_1^2 + \ldots \) until \( \hat{c} = n - 2 \). The quadratic fields have no effect on the chiral ring or the conformal fixed point to which the Landau-Ginzburg theory flows.

The more interesting case is \( \hat{c} < n - 2 \) which includes (2.6) for \( N > 1 \). We proceed with the following ansatz: Add bilinears of ghost superfields to the superpotential \( W \to W + \eta_1 \eta_2 + \ldots \). By ghost superfields \( \eta_i \), we mean each component of the superfield has reversed statistics so the lowest component is a spin zero fermion etc. Under a change of variables, the ghost measure transforms inversely to the measure for the bosonic superfields. By adding enough pairs of ghosts, the Jacobian under the change of variables procedure can be made trivial. In addition, there is again no change in the chiral ring for the theory so we expect the conformal Landau-Ginzburg theory to be unchanged. Under the condition that \( \hat{c} \) be integral, the sum of the charges \( \sum \frac{1}{k_i} \) for theories in this class can be integral or half-integral. The first case only requires adding ghost bilinears while the second needs a quadratic bosonic field as well as ghost bilinears. The requirement that the Jacobian be trivial and that the sigma model have dimension \( \hat{c} \) uniquely determines, for \( W \) with degree \( > 2 \) (the nontrivial case), the number of ghost and quadratic bosonic fields needed for this procedure.
Let us return to our examples (2.6) for \( N > 1 \). For these cases, the dimension computed from the Landau-Ginzburg model gives \( \hat{c} = N \). We claim that orbifolding by the diagonal phase group identifies these models with sigma models on target spaces defined by the zero set of the algebraic constraints:

\[
\begin{align*}
N = 2 & \quad \sum_{i=1}^{6} z_i^3 + \eta_1 \eta_2 \\
N = 3 & \quad \sum_{i=1}^{9} z_i^3 + \eta_1 \eta_2 + \eta_3 \eta_4 \\
\vdots & \quad \vdots 
\end{align*}
\]

These constraints define supermanifolds - the subject of the following section - in superprojective space. Note that in the sigma model, the ghost superfields carry negative quantum dimension since they contribute to \( \hat{c} \) with opposite sign to bosonic superfields. In the Landau-Ginzburg theory, however, they have no effect on \( \hat{c} \). That the dimensions of both models agree is an encouraging first sign.

### 2.2. Relation to Rigid Manifolds

Using standard techniques [10], we can compute the Hodge diamonds for the orbifolds of the Landau-Ginzburg models (2.6). These are displayed for the first two cases in figure 2.1.

![Hodge diamonds](image)

*Figure 2.1: Hodge diamonds for the \( N = 2 \) and \( N = 3 \) cases.*

For the case \( N > 2 \), we find that the models have no Kähler deformations. We therefore expect these models to be mirrors of rigid manifolds. Indeed the mirrors for these Landau-Ginzburg orbifolds with \( N > 2 \) are rigid toroidal orbifolds [11]. The mirror for the \( N = 2 \) case is the \( K3 \) surface. The supermanifolds defined by (2.8a,...) should
therefore be the elusive geometric mirrors. Many Landau-Ginzburg orbifolds falling into
this class of theories do, however, possess Kähler forms which are contributed from the
twisted sectors.

The more general superpotential can contain quartic and higher terms in the ghost
superfields. In these cases, the fermionic geometry in the Landau-Ginzburg model is non-
trivial. There can also be mixed terms containing both bosonic and fermionic fields such
as $z_1 \eta_1 \eta_2$. However, since the ghost fields now appear nontrivially in the chiral ring for
these Landau-Ginzburg theories, we believe these models are generally nonunitary.

3. Supermanifolds

3.1. Conditions for Conformal Invariance

Homogeneous coordinates for superprojective space $\text{SP}(n|m)$ are given by

$$(z^1, \ldots, z^{n+1}|\eta^1, \ldots, \eta^m)$$

with $z^\alpha \sim \lambda z^\alpha$ and $\eta^\alpha \sim \lambda \eta^\alpha$. There are $n+1$ coordinate patches with $z^i \neq 0$ in
the $i$-th patch. Inhomogeneous coordinates are then defined in the standard manner
$$(\tilde{z}^1, \ldots, \tilde{z}^i, \ldots, \tilde{z}^{n+1}|\tilde{\eta}^1, \ldots, \tilde{\eta}^m)$$

where $\tilde{z}^j = \frac{z^j}{z^i}$ and $\tilde{\eta}^j = \frac{\eta^j}{z^i}$. Let $\tilde{z}^j_{\{k\}}$ and $\tilde{\eta}^j_{\{k\}}$ be co-
ordinates in the patch where $z_k \neq 0$, then the transition functions are the usual ones:

$\tilde{z}^j_{\{k\}} = \tilde{z}^j_{\{i\}} \frac{z^i}{z^k}$ and $\tilde{\eta}^j_{\{k\}} = \tilde{\eta}^j_{\{i\}} \frac{z^i}{z^k}$. The fermionic coordinates $\tilde{\eta}^j$ are Grassmann-valued
sections of the line bundle $O_{\mathbb{P}^n}(-1)$. This space is a split supermanifold\footnote{Split supermanifolds are also known as DeWitt supermanifolds \cite{12}. The de Rham cohomology for the split case reduces to the usual de Rham cohomology of the body, in this case $\mathbb{P}^n$, so these spaces are quite uninteresting cohomologically.} - a special case
of the super-Grassmannian \cite{13}. The weighted case $\text{WSP}(k_1, \ldots, k_{n+1}|l_1, \ldots, l_m)$ is a
straightforward extension with $z^\alpha \sim \lambda^{k_\alpha} z^\alpha$ and $\eta^\alpha \sim \lambda^{l_\alpha} \eta^\alpha$.

We are interested in $N = 2$ sigma models on subvarieties of these spaces with actions
of the form:

$$\int d^2z d^4\theta K(\Phi^\alpha, \Phi^{\bar{\alpha}}) + \int d^2z d^2\theta^- \Lambda P(\Phi^\alpha) + \text{c.c.}. \quad (3.2)$$
Let the indices $\mu, \nu, \ldots$ refer to bosonic coordinates, $i, j, \ldots$ to ghost coordinates, and $\alpha, \beta, \ldots$ for either case. Let us establish a convention for the component expansion of the superfields:

$$\Phi^\mu = z^\mu - \theta - \psi^\mu + \theta - \bar{\theta} - F^\mu - \ldots \quad (3.3)$$

$$\Phi^i = \eta^i - \theta - \xi^i + \theta - \bar{\theta} - G^i - \ldots$$

More general actions can include twisted-chiral fields [14], gauge symmetries, several polynomial constraints and both fermionic and bosonic Lagrange multipliers. For simplicity, we restrict this discussion to the single polynomial case with a bosonic Lagrange multiplier $\Lambda$ and no twisted-chiral fields. This case includes most of the interesting examples and the extension to the other cases is not difficult.

Rather than consider the action (3.2) with an explicit F-term, let us assume the Kähler potential for the subvariety is known. We can then study the ultraviolet divergence structure for the sigma model defined only by a D-term. All computations can be performed in the $N = 2$ superspace framework, but not while preserving manifest covariance [15]. Nevertheless, the final expressions for the counterterms, which are all corrections to the Kähler potential, are covariant. In the usual bosonic case, the divergent part of the one-loop effective action is proportional to the determinant of the Kähler metric. We choose to expand the Kähler potential in the following way

$$K = K(\Phi_\alpha, \Phi_{\bar{\beta}}) + \tilde{\Phi}_\alpha K_{\alpha\bar{\beta}} \tilde{\Phi}_{\bar{\beta}} + \ldots \quad (3.4)$$

around a classical background $\Phi^\alpha_0$ where $\Phi^\alpha = \Phi^\alpha_0 + \tilde{\Phi}^\alpha$. The order of the terms is important since the fields can anticommute. The supermetric is defined to be $K_{\alpha\bar{\beta}}$. The divergent part of the one-loop counter-term is now proportional to $\ln \text{sdet} K_{\alpha\bar{\beta}}$. The condition for conformal invariance to one-loop is the existence of a super-Ricci flat metric. Standard arguments imply that all higher loop counterterms are cohomologically trivial. In fact, the actual metric for the conformal theory theory will not be the super-Ricci flat metric but a metric corrected for higher loop contributions as in the usual bosonic case [16].

**3.2. Super-Ricci Flat Metrics**

Let us consider the case of $\text{SP}(n|n+1)$ with Kähler potential

$$K = \ln(1 + z^\alpha z_{\alpha}) \quad z_{\alpha} = \delta_{\alpha\bar{\alpha}} z^{\bar{\alpha}}$$

$$K = \ln(1 + z^\mu z_{\mu}) + \sum_p \frac{(-1)^{p+1}(\eta^i \eta_i)^p}{p(1 + z^\mu z_{\mu})^p} \quad (3.5)$$
which is a natural extension of the Fubini-Study Kähler potential to superprojective space. Surprisingly, the metric for this Kähler potential is super-Ricci flat. Unlike the purely bosonic case, this embedding space provides a nonunitary \( \hat{c} = -1 \) conformal field theory! Note that the contribution from the ghosts to (3.5) is in the form of a globally defined section-valued function on \( \mathbb{P}^n \). In this sense, the addition of the ghosts to the potential does not effect the cohomology class of the Kähler form. For the more general case \( \text{SP}(n|m) \), the Kähler potential \( K = \ln(1 + z^\alpha z_\alpha) \) gives a Kähler-Einstein metric. The one-loop counterterm, which gives the potential for the super-Ricci tensor, is in this case:

\[
\ln \text{det} K_{\alpha\bar{\beta}} = -(n - m + 1) K. \tag{3.6}
\]

From this example, we can see that the fermionic coordinates contribute to the super-first Chern class with a negative sign.

Yau’s theorem [17] for bosonic manifolds guarantees that the vanishing of the first Chern class is necessary and sufficient to ensure the existence of a Ricci flat metric. In the super case, there is no such theorem yet, and in fact the situation is more subtle. We will not attempt a full analysis of the necessary conditions for the existence of a super-Ricci flat metric here, but we will present a preliminary investigation.

Let us examine the structure of the counterterm \( \ln \text{det} K_{\alpha\bar{\beta}} \) more closely. Let \( K^\alpha_{\bar{\beta}} \) denote the inverse to the non-degenerate metric \( K_{\alpha\bar{\beta}} \) then

\[
\ln \text{det} K_{\alpha\bar{\beta}} = \ln \text{det}(K_{\mu\nu} - K_{\mu\bar{i}}K^{\bar{i}j}K_{j\nu}) - \ln \text{det}(K_{ij}). \tag{3.7}
\]

For metric non-degeneracy, terms quadratic in the ghost fields must be present in the Kähler potential. These terms ensure that \( K_{ij} \) is invertible, and are also necessary if the bosonic part of this counterterm is to be cohomologically trivial. Physically, these terms ensure that the ghost field propagators are well-defined. The contribution to the fermionic part \( K_{\text{ferm}} \) of the Kähler potential for weighted projective space \( \text{WSP}(1, \ldots, 1|l_1, \ldots, l_m) \) should take the form:

\[
K_{\text{ferm}} = \sum \frac{\eta^i \eta_i}{(1 + z^\mu z_\mu)^{l_i}} + O(\eta^i \eta_i)^2. \tag{3.8}
\]

The bosonic part of the Kähler potential is taken to be the usual Fubini-Study potential. This particular embedding space is important for understanding the examples given in (2.8a,...) and we will infer general features from this case. Note that if the bosonic coordinates scale with nontrivial weight, then a non-degenerate (weighted) extension of
this Fubini-Study potential does not generally exist. To ensure the bosonic part of the counterterm (3.7) is trivial, the super-first Chern class must vanish. Fermions of weight $l_i$ contribute $-l_i$ to the super-first Chern class. Unlike the bosonic case, however, vanishing of the super-first Chern class is not sufficient to ensure the existence of a nontrivial super-Ricci flat metric. Physically, $K_{\text{ferm}}$ can still be renormalized from the one-loop and higher counterterms since the terms quadratic in the ghosts are globally defined. For example, the target space $\mathbf{S}P(1,1|2)$ with bosonic field $z$, ghost field $\eta$ and potential

$$K = \ln(1 + z\bar{z}) + \frac{\eta\bar{\eta}}{(1 + z\bar{z})^2}$$

(3.9)

has a counterterm $\propto \frac{2\eta\bar{\eta}}{(1 + z\bar{z})^2}$. The theory therefore flows to a conformal model with a degenerate metric. If a non-degenerate metric exists for a variety in $\mathbf{W}SP(k_1, \ldots, k_{n+1}|l_1, \ldots, l_m)$ with $\sum k_i - \sum l_j = 0$, and there is no renormalization of Kähler potential terms quadratic in the ghost fields, then the theory flows to a nontrivial super-Ricci flat metric.

For those sigma models admitting a Landau-Ginzburg description, we fully expect from the arguments given in section two that a nontrivial super-Ricci flat metric exists. As a basic check, we can compute the super-first Chern class for Landau-Ginzburg orbifolds corresponding to supermanifolds. Recall that each superfield $\Phi_i$ has charge $(\frac{1}{k_i}, \frac{1}{k_i})$. Each of the $(\sum \frac{1}{k_i} - 1)$ ghost bilinears then subtracts $d$ from the super-first Chern class. The degree $d$ of $W$ is the homogeneity of $W$ as a defining equation in weighted projective space. The bosonic fields contribute $\frac{d}{k_i}$ each while the constraint $W$ further reduces the super-first Chern class by $d$. Just as we expect, the super-first Chern class vanishes. The change of variables procedure is therefore only possible when the super-variety $W = 0$ has vanishing super-first Chern class. A more direct computation from the action (3.2) using the constraint term shows that the induced super-Ricci tensors for the models (2.8) are indeed cohomologically trivial.

Within this framework, we can now interpret the models (2.8) as varieties in $\mathbf{W}SP(1, \ldots, 1|1,2,\ldots, 1,2)$; the $N = 2$ case is a $(4|2)$ supermanifold while the $N = 3$ case is a $(7|4)$ supermanifold. For these models, each ghost bilinear in the defining polynomial scales as $\lambda^3$. The assignment of scaling weight to each ghost field is therefore unique in these cases. For polynomials of higher degree $d$ derived from Landau-Ginzburg theories, there is some freedom in the assignment of scaling weight to each ghost field. However, not all choices admit non-degenerate super-Ricci flat metrics. Further, the chiral rings can differ for different choices of scaling weights. These considerations limit the freedom in choosing ghost scaling weights when identifying Landau-Ginzburg orbifolds with supermanifolds.
4. The Chiral Ring

This section is organized in the following way: We first examine general features of these theories. This discussion is in the context of the topological sigma model. We then proceed to study the structure of differential forms on the body of the supermanifold. Guided by that analysis, we conjecture the form of an analogue to the usual holomorphic $n$-form and study the ‘variation of Hodge structure.’ In this way, we construct the local observables for the sigma model phase. Lastly, we consider the cases with Kähler moduli and explain how these deformations can arise.

4.1. General Features

There are two conserved $U(1)$ charges for an $N = 2$ superconformal model. Let us list the left and right charge decomposition of the operators pertinent to this discussion. The fields $(\psi^\alpha, \psi^\bar{\alpha})$ are assigned charge $(1, 0)$ and $(0, 1)$, $(\psi_\alpha, \psi^\bar{\alpha}_+)$ charge $(-1, 0)$ and $(0, -1)$ respectively. The fields $(z^\alpha, z^{\bar{\alpha}})$ are uncharged. There are also two supercharges $G^\pm$ with charge $(\pm 1, 0)$ together with their conjugates $\bar{G}^\mp$. The topological sigma model (A-model) is obtained by twisting the $N = 2$ theory. This is accomplished by coupling the vector $U(1)$ current to a background gauge field $A_{[18][19], S \rightarrow S + \int \bar{J}A + J\bar{A}}$. (4.1)

The background gauge field is taken to be one-half the spin connection. The spins of the fields in the topological theory are determined by the shifted stress-energy tensor:

$$
T \rightarrow T - \frac{1}{2} \partial J \quad \bar{T} \rightarrow \bar{T} + \frac{1}{2} \partial \bar{J}. \tag{4.2}
$$

In particular, two of the supersymmetry generators, $G^+$ and $\bar{G}^-$, are now spin zero. These nilpotent charges are interpreted as generators of BRST transformations. For this choice of twisting, elements of the $(c, c)$ ring are identified with BRST cohomology classes of the operator $Q = G^+ + \bar{G}^-$. Let us use $\mathcal{M}$ to denote the target supermanifold. The body of $\mathcal{M}$ is a Kähler manifold with $c_1 > 0$ and dimension $\hat{c}_b > \hat{c}$. On a genus $g$ Riemann surface $\Sigma$, the twisted theory has a background charge given by $\hat{c} (1 - g, 1 - g)$. Since we are studying conformal models, this background charge violation is independent of the map $z : \Sigma \rightarrow \mathcal{M}$. This is another way of saying that the axial $U(1)$ charge is conserved. Therefore correlation functions $\langle \vartheta \rangle$ vanish unless the operator $\vartheta$ has charge $(\hat{c}, \hat{c})$ for $\Sigma$ genus 0. The genus zero correlation
functions are of particular interest since they are related to Yukawa couplings for the low energy theory.

In the usual case where $\mathcal{M}$ is a bosonic target space, the local observables for the A-model are identified with elements of the de Rham cohomology for $\mathcal{M}$ \cite{20}. The observables are constructed from the fields $(\psi^\mu_+, \psi^\bar{\mu}_-)$ which we call $(\psi^\mu, \psi^{\bar{\mu}})$ for simplicity. To a differential form $\omega$ on $\mathcal{M}$,

$$\omega = \omega_{\mu_1...\mu_p\bar{\mu}_1...\bar{\mu}_q} dz^{\mu_1} \wedge \cdots \wedge dz^{\bar{\mu}_q},$$  \hspace{1cm} (4.3)

we associate an operator $\vartheta_\omega$ in the field theory by making the substitution:

$$dz^{\mu} \rightarrow \psi^\mu \hspace{1cm} dz^{\bar{\mu}} \rightarrow \psi^{\bar{\mu}}.$$  \hspace{1cm} (4.4)

The action of $Q$ on $\vartheta_\omega$ is just that of the exterior derivative $d$ on $\omega$. When $\mathcal{M}$ is a supermanifold, supersymmetry again provides us with an exterior derivative.

$$d = \partial + \bar{\partial}$$

$$d = \psi^\mu \frac{\partial}{\partial z^\mu} + \xi^i \frac{\partial}{\partial \eta^i} + c.c.$$  \hspace{1cm} (4.5)

The observables are again in the cohomology of this nilpotent operator. However, there are immediate problems if we na"ively try to identify observables with ‘forms’ on $\mathcal{M}$ by substituting the bosonic field $\xi$ for $d\eta$ as above. Let us momentarily restrict to the constant maps $z : \Sigma \rightarrow \mathcal{M}$. Integration over the moduli space of these maps is simply integration over the target space. For an $(n|m)$ supermanifold, there are $n$ zero modes for $\psi$ together with $m$ zero modes for $\xi$. Firstly, when evaluating correlation functions, integration over the tangent vectors $\xi$ to the ghost directions yields infinities. Secondly, the fields $\xi$ are positively charged but adding the ghost coordinates decreases $\hat{c}$. From these considerations, we should expect to identify $d\eta$ with a negatively charged operator. Otherwise, all correlators vanish by charge conservation! Lastly, viewing $d\eta$ as a measure for Berezin integration requires the operator identified with $d\eta$ to transform inversely to $\eta$ under a change of coordinates – unlike $\xi$. We will argue later that distribution-valued forms resolve these issues with the identification:

$$d\eta^i \rightarrow \delta(\xi^i) \hspace{1cm} d\eta^{\bar{i}} \rightarrow \delta(\bar{\xi}^i).$$  \hspace{1cm} (4.6)
At first glance, it appears that the sigma models we are studying must possess a Kähler modulus. After all, there is a natural Kähler form induced from the embedding space:

\[ k = \partial \bar{\partial} K. \]  

(4.7)

Indeed, it was shown long ago that \( N = 2 \) supersymmetry requires a Kähler target space \[21\]. Does this contradict our asserted correspondence between Landau-Ginzburg orbifolds and these sigma models? How can the mirror of a rigid manifold have a Kähler modulus? Let us consider correlation functions involving the Kähler form. The natural volume form for absorbing the fermionic zero modes \((\psi^\mu, \psi^{\bar{\mu}})\) is \( \hat{k} c \bar{b} \), but \( \langle \hat{k} c \bar{b} \rangle \) vanishes by charge conservation. Taking higher powers of \( k \) only worsens the situation. Clearly, the natural volume form for the supermanifold is not constructed from the induced Kähler form. That the \( \mathbb{P}^3 \) Yukawa coupling for the low-energy string theory vanishes is exactly the behavior expected from mirror symmetry. There should be no marginal perturbation associated with the Kähler parameter. However, this is not sufficient. We must further show that the correlator \( \langle k^n \vartheta \rangle \) vanishes for any observable \( \vartheta \) with \( n > 0 \). This condition is the requirement that the Kähler form completely decouple from the chiral ring. We will return to check this condition after studying the properties of ‘good’ observables for \( \mathcal{M} \).

Usually, the ring of observables for the A-model is sensitive to rational curves on the target space. Let us show that when the target is a supermanifold, there is no such sensitivity. Path-integral computations in the A-model therefore reduce to integration over the space of constant maps. For simplicity, let us take maps \( z : \text{SP}(1|0) \to \mathcal{M} \) where \( \mathcal{M} \) is a variety in \( \text{SP}(n|m) \) defined by a degree \( q \) constraint. Note that \( q < n + 1 \) for conformal invariance. As a warmup, let us first consider constant maps. Maps into \( \text{SP}(n|m) \) can be parametrized by \((a^0, \ldots, a^n, \tilde{a}^1, \ldots, \tilde{a}^m)\) with a single scaling relation: \( a^j \sim \lambda a^j \) and \( \tilde{a}^i \sim \lambda \tilde{a}^i \). The \( a^j \) are bosonic while the \( \tilde{a}^i \) are fermionic. This is just a copy of \( \text{SP}(n|m) \) and enforcing the constraint reduces the moduli space to a copy of \( \mathcal{M} \). Let \((x, y)\) denote homogeneous coordinates for \( \text{SP}(1|0) \). Take the coordinates \( z \) for \( \text{SP}(n|m) \) to be sections of \( O(k) \) over \( \text{SP}(1|0) \).

\[
\begin{align*}
  z^j &= \sum a^j_i x^l y^{k-l} & 0 \leq j \leq n \\
  \eta^i &= \sum \tilde{a}^i_l x^l y^{k-l} & 1 \leq i \leq m
\end{align*}
\]  

(4.8)
This space is a copy of \( \text{SP}(\{n+1\}| \{k+1\} - 1|m\{k+1\}) \) where we have implicitly compactified the moduli space. The constraint reduces the bosonic dimension of the moduli space by \( kq + 1 \) to \( k(n + 1 - q) + (n - 1) \) in the generic case. This is to be contrasted with the Calabi-Yau case where the dimension of the moduli space is independent of \( k \). The additional fermionic tangent vectors to the moduli space then annihilate correlation functions for \( k > 0 \). This argument extends straightforwardly to varieties in weighted projective space. The decoupling phenomena described above occurs generally in these models: Topological amplitudes are therefore independent of perturbations of the induced Kähler form.

4.2. The Hodge Structure

In the previous section, we argued that there is no Kähler modulus when \( W = 0 \) is a smooth variety. The interesting moduli are then derived from the \((a,c)\) ring, and are related to deformations of the complex structure. Let us establish some notation. In the Landau-Ginzburg theory, the defining superpotential \( W(z^{1}, \ldots, z^{n+1}) \) of degree \( d \) can be taken to be ghost free for unitary models. Let \( p = W + \Gamma \) denote the superconstraint where

\[
\Gamma = \eta^{1}\eta^{2} + \ldots + \eta^{2m-1}\eta^{2m},
\]

and \( \Gamma^{m+1} = 0 \). Let \( \lambda_{i} = \frac{d}{k_{i}} \) be the weight of \( z^{i} \), and \( \tilde{\lambda}_{i} \) be the weight of \( \eta^{i} \). The supermanifold \( \mathcal{M} \) is then a degree \( d \) variety in \( \text{WSP}(\lambda_{1}, \ldots, \lambda_{n+1}|\tilde{\lambda}_{1}, \ldots, \tilde{\lambda}_{2m}) \). Let us briefly review the structure of the chiral ring expected from spectral flow arguments. This discussion applies generally to \( N = 2 \) superconformal theories with integral \( U(1) \) charges in the Neveu-Schwarz sector. The left-moving \( U(1) \) current can be expressed in bosonized form as

\[
J = i\sqrt{\hat{c}} \partial \phi_{L},
\]

and similarly for the right-mover. The unique states with charge \((\hat{c}, 0)\) and \((0, \hat{c})\) are constructed from the free bosons \( \phi_{L} \) and \( \phi_{R} \):

\[
\Omega = e^{i\sqrt{\hat{c}} \phi_{L}} \quad \bar{\Omega} = e^{-i\sqrt{\hat{c}} \phi_{R}}.
\]

\[2\text{ See } [1]\text{ for an explanation of the normalization conventions, and a more detailed discussion.}\]
As the notation suggests, these operators correspond to the holomorphic and anti-holomorphic forms present on a Calabi-Yau manifold $X$. The operator corresponding to the volume form is just $\Omega \otimes \bar{\Omega}$. States in the $(a, c)$ ring with charge $(-p, q)$ flow to states $(\hat{c} - p, q)$ in the $(c, c)$ ring under spectral flow generated by $\Omega$. The corresponding geometric operation on the Calabi-Yau is the cup product of an element in $H^q(X, \wedge^p TX)$ with the holomorphic $\hat{c}$-form. For theories with integral $U(1)$ charges, spectral flow provides a precise correspondence between the $(a, c)$ and $(c, c)$ rings.

By computing the $(c, c)$ ring for the Landau-Ginzburg orbifold, we know the expected structure of the $(c, c)$ ring for $\mathcal{M}$. Unfortunately, there is no current supercohomology theory that would provide the desired ring, or even the correct Hodge numbers. We must therefore proceed to construct the observables for the sigma model guided by physical considerations.

Let us return momentarily to the family of examples (2.8). It is instructive to examine the Hodge diamonds of the bodies $X$ of these supermanifolds. A straightforward computation reveals that the Hodge numbers $h^{p,q}$ of middle cohomology ($p + q = \hat{c}$) of the Landau-Ginzburg orbifold agree with $h^{p+m,q+m}_o$ of $X$. The subscripted Hodge number $h^{p,q}_o$ refers to the number of primitive forms in the Dolbeault group $H^{p,q}(X)$ [22]. As an illustration, compare figure 4.1 with figure 2.1 showing the $N = 2$ ($m = 1$) case.

$$
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & 1 & 0 & \\
0 & 0 & 0 & 0 \\
0 & 1 & 21 & 1 & 0 \\
0 & 0 & 0 & 0 & \\
0 & 1 & 0 & & \\
0 & 0 & & & \\
1 & & & \\
\end{array}
$$

*Figure 4.1: Hodge diamond for the body of the $N = 2$ case.*

The agreement of these Hodge numbers is a consequence of the Landau-Ginzburg description of these models. To explain this point, let us describe the construction of middle cohomology on the body $X$ defined by $W = 0$. The main theme of this discussion is the relation between pole-order and charge grading of differential forms. Let $Y$ denote the embedding space $\text{WP}(\lambda_1, \ldots, \lambda_{n+1})$. The Poincaré residue of a holomorphic form $\varpi$ on $Y$ with a pole on $X$ is given by [23] [24]:

14
\[
\text{Res}[\varpi] = \int_{\gamma} \varpi.
\] (4.12)

The integration contour is a small one-cycle enclosing the hypersurface \(X\). For a cycle \(\gamma \in H_{k-1}(X)\), the residue satisfies

\[
\int_{\gamma} \text{Res}[\varpi] = \frac{1}{2\pi i} \int_{T(\gamma)} \varpi
\] (4.13)

where \(T(\gamma) \in H_k(Y - X)\) is a tube over \(\gamma\). As an example, let us take \(X\) to be Calabi-Yau. The holomorphic \(n\)-form \(\Omega\) then has the following well-known construction \[25\] \[26\]:

\[
\Omega = \text{Res}[\frac{\varpi}{W}] = \omega = \sum_{i} (-1)^i \lambda_i z^i dz^1 \ldots \wedge dz^i \ldots \wedge dz^{n+1}.
\] (4.14)

This construction is only well-defined in the Calabi-Yau case where the scaling degree of \(W\) equals that of \(\omega\). More generally, the pole-order of the form on \(Y\) determines the Hodge decomposition of the form on \(X\) obtained under the residue map. To construct the middle cohomology of \(X\), take the form

\[
\varpi(P) = \frac{P(z^1, \ldots, z^{n+1}) \omega}{W^k}.
\] (4.15)

with \(\omega\) defined in (1.14). This is only well-defined when the condition

\[
k \deg(W) = \deg(P) + \sum \lambda_i
\] (4.16)

is satisfied. Under the Poincaré residue, this form maps into \(\oplus_{q=1}^{k} H^{n-q,q-1}_o(X)\). Returning to the Landau-Ginzburg theory, recall the definition of the chiral primary ring \(\mathcal{R}\) given in \[2.4\]. After orbifolding, only the ring elements with integral charge survive, so we restrict our discussion to that subring. Let \(P_\alpha\) be a basis for \(\mathcal{R}\), then \(\text{Res}[\varpi(P_\alpha)]\) is a basis for \(H^{n-1}_o(X)\) \[23\] \[27\] \[28\]. More precisely, if \(P \in \mathcal{R}\) satisfies (1.14) for some \(k\), then \(\text{Res}[\varpi(P)] \in H^{n-k,k-1}_o(X)\) is nontrivial. This description of the Hodge structure has recently been extended to hypersurfaces in toric varieties \[29\]. For the concrete case of \[2.6\] with \(N = 2\), the forms

\[
\left( \frac{\omega}{W^2}, \frac{(z^i z^j z^k) \omega}{W^3} (i \neq j \neq k), \frac{(\prod z^i) \omega}{W^4} \right)
\] (4.17)

provide a basis under the residue map for the middle cohomology shown in figure 4.1. In this way, we obtain a map from \(\mathcal{R}\) to the primitive cohomology of the body of \(\mathcal{M}\).
This analysis is quite independent of any sigma model interpretation for the Landau-Ginzburg orbifold. It was shown in \cite{30} that this mapping of Hodge structures is consistent with the real structure and period maps. Computing the chiral ring structure constants on the body of $\mathcal{M}$ produces the same results as computations for the Landau-Ginzburg orbifold including normalization \cite{30,31}. Picard-Fuchs equations can therefore be derived for general Landau-Ginzburg orbifolds with integral $\hat{c}$. Direct computations of period matrices can also be performed for these models \cite{22,33}. Using these techniques, the dependence of the Landau-Ginzburg orbifold on the complex structure moduli can be recovered.\footnote{At least for complex structure moduli that can be parametrized by polynomial deformations \cite{34}. This is not true for more general cases such as embeddings in products of superprojective spaces. Even an embedding in a single weighted projective space can produce moduli not parametrized by polynomial deformations. From the Landau-Ginzburg viewpoint, there are additional moduli contributed from the twisted sectors. See \cite{33} for an attempt to deal with these extra deformations.}

For an orbifold of (2.6) with $N = 3$, explicit computations using these techniques were checked against known results on the mirror manifold in \cite{35}. In the framework of mirror symmetry, the construction proposed in \cite{36} provides a general technique to obtain the body of the desired supermanifold when the mirror is a toric variety.

The identity operator in the ring $\mathcal{R}$ corresponds to the holomorphic $\hat{c}$-form for the Landau-Ginzburg orbifold. Under the mapping described above, the identity maps to a $(\hat{c} + m, m)$ form. This correspondence is only natural in the sense of preserving charge when $X$ is Calabi-Yau. Note that the primitive forms described above are not observables of the sigma model. The expected map from $\mathcal{R}$ to the sigma model should be a morphism of Hodge structures of type $(0, 0)$ i.e. there should be no shift of the charges. However, after integrating out the ghost fields, we expect correlation functions to reduce to intersection theory on the body.

Let us construct the cohomology of $\mathcal{M}$ excluding contributions from any fixed point sets. Those contributions are discussed in the following subsection. The spectral flow arguments are unaffected by the addition of the ghost coordinates. However, the expression for $\Omega$ given in (4.11) is helpful. Take flat superspace with Kähler potential

$$K = z^\mu \bar{z}_\mu + \eta^\dagger \eta$$

(4.18)
as an example. The $U(1)$ current is then a sum of $b - c$ and $\beta - \gamma$ systems. Bosonizing each $b - c$ system provides the identification.
\[ e^{i\phi_i^L} = \psi^\mu. \] (4.19)

However, bosonizing each $\beta - \gamma$ system gives the relation \[ e^{-\phi_i^L} = \delta(\xi^i). \] (4.20)

Combined with the considerations presented in subsection 4.1, this leads to the identification of $d\eta^i$ with $\delta(\xi^i)$. Note that the scaling properties of these operators

\[ \eta^i \to \lambda \eta^i \quad \delta(\xi^i) \to \frac{\delta(\xi^i)}{\lambda} \] (4.21)

imply that $d\eta^1 \ldots d\eta^{2m}$ always scales as $W^{-m}$. This permits a natural conjecture for the holomorphic $\hat{c}$-form

\[ \Omega = \text{Res}\left[ \frac{d\eta^1 \ldots d\eta^{2m} \omega}{p} \right], \] (4.22)

where $p = W + \Gamma$. In clear analogy to the Calabi-Yau case, this construction only makes sense when the super-first Chern class vanishes. Since the hypersurface is defined by the superconstraint $p = 0$, this form is $d$-closed with $d$ given in (4.5). For these models, there are no polynomial deformations in the ghost directions. The construction of the remaining observables is then straightforward. The forms

\[ \varpi(P) = \frac{P(z^1 \ldots z^{n+1}) d\eta^1 \ldots d\eta^{2m} \omega}{p^k} \] (4.23)

under the residue mapping provide a basis for the middle cohomology of $\mathcal{M}$. At least for the single polynomial constraint, these forms are in a one-to-one correspondence with forms on the body described in (4.13). However, the pole-order is shifted ensuring that these operators have the correct charge. By adding the forms $(1, \Omega \wedge \bar{\Omega})$, we recover the Hodge diamond expected from Landau-Ginzburg calculations. Do these forms constitute a complete set for $\mathcal{M}$? Physically, we expect no additional forms. However, we offer no general proof of completeness here. Such a proof requires a suitable mathematical theory of supercohomology. Nevertheless, we can present some heuristic expectations. Since $\{Q, \eta^i\}$ is bosonic, we expect a closed form to depend on the ghosts only through the constraint $p$. These are precisely the forms we have just discussed. Any other closed form $\varpi$ - including the Kähler form - should be cohomologically trivial in an appropriate sense. This would provide a geometric explanation for the decoupling phenomenon.
The path-integral provides a definition for the integral of the volume form over the supermanifold,

$$\int_{\mathcal{M}} \Omega \wedge \bar{\Omega}. \quad (4.24)$$

After integrating out the ghost fields \((\eta, \xi)\), the integral reduces to the body \(X\) of \(\mathcal{M}\). Expanding \(p\) in (4.22) inside the residue,

$$\frac{1}{p} = \frac{1}{W} \left\{ 1 - \frac{\Gamma}{W} + \ldots \right\}, \quad (4.25)$$

and integrating over the \(\eta\) fields selects the term proportional to \(W^{-m}\). Integration over the \(\xi\) fields is trivial. After evaluating the residue on the body, we are left with an integral proportional to the natural volume form on \(X\). This procedure agrees with the identification (4.6) if when integrating over \(\mathcal{M}\) we simply perform the Berezin integrals in (4.22) and then evaluate the residues. This recipe therefore avoids evaluating residues on \(\mathcal{M}\). To show this procedure agrees with first evaluating the residues and then computing the path-integral requires a more detailed investigation of residues than presented here. By comparison with Landau-Ginzburg results, we do expect both procedures to agree, though the proof appears to be quite nontrivial. Without such a proof, the residue construction remains somewhat formal. By noting that the form

$$\varpi = \text{Res}\left[ \frac{q_1 \ldots q_c \eta^1 \ldots \eta^{2m} \omega}{p^{c+1}} \right] \quad (4.26)$$

is proportional to \(\bar{\Omega}\), the computation of Yukawa couplings is straightforward. Each \(q_i(z^1, \ldots, z^{n+1})\) is a degree \(d\) polynomial, and the associated Yukawa coupling is given by:

$$\kappa(q_1, \ldots, q_c) = \frac{\int_{\mathcal{M}} \Omega \wedge \varpi}{\int_{\mathcal{M}} \Omega \wedge \bar{\Omega}}. \quad (4.27)$$

After integrating out the ghosts, this expression reduces to intersection theory on \(X\). For the case of a single polynomial constraint, this agrees with the corresponding computation in the Landau-Ginzburg theory as previously discussed [30]. However, the constructions described here are general, and extend to the case of many constraints, more general embedding spaces etc. These Yukawa couplings then provide a relatively simple way of computing the instanton corrected couplings of the mirror theory.
Let us close this discussion by checking the decoupling condition that the correlator \( < k^n \vartheta > \) vanish. This follows from a counting argument. The operator \( \vartheta \) must have total charge \( 2\hat{c} - 2n \) by charge conservation. Clearly \( \vartheta \) cannot be \( (1, \Omega \wedge \bar{\Omega}) \), so \( \vartheta \) is a product of forms each with charge \( \hat{c} \). Further, each form with charge \( \hat{c} \) absorbs \( \hat{c}_b \) fermionic \( (\psi, \bar{\psi}) \) zero modes. Now \( k^n \) absorbs at most \( 2n (\psi, \bar{\psi}) \) zero modes. There are at least \( 2\hat{c}_b \) such zero modes. To absorb all the zero modes and satisfy charge conservation when \( n \neq 0 \) requires

\[
\frac{\hat{c}_b}{\hat{c}} \leq 1
\]

which is a contradiction. Therefore \( < k^n \vartheta > \) vanishes and the Kähler form decouples for these models.

4.3. Kähler Moduli

From the previous discussion, we found that the only diagonal Hodge numbers that were non-zero corresponded to the the identity and volume forms. How then can Kähler moduli arise? The only possibility is from the resolution of fixed point sets. In the Landau-Ginzburg framework, the only models with Kähler moduli correspond to varieties in weighted superprojective space. Let us take a specific example with superpotential:

\[
W = (y^1)^6 + (y^2)^6 + (x^1)^3 + \ldots + (x^5)^3.
\]

This Landau-Ginzburg model has \( \hat{c} = 3 \). The Hodge diamond obtained after orbifolding by the canonical \( \mathbb{Z}_6 \) is shown in figure 4.2. There are five twisted sectors in the Landau-Ginzburg theory. The untwisted sector provides the forms in the middle cohomology of the first diamond in figure 4.2. The identity and volume form correspond to vacua for the \( (1, 5) \) twisted sectors \([10]\). The existence of these forms for the sigma model follows from our previous discussion. The interesting forms appear in the \( (2, 3, 4) \) twisted sectors and are shown in the second Hodge diamond. These forms should arise from a resolved fixed-point set.
Figure 4.2: Hodge diamond for (4.29) with contributions from the (2, 3, 4) twisted sectors displayed in the second diamond.

The corresponding super-variety $\mathcal{M}$ has defining constraint

$$(y^1)^6 + (y^2)^6 + (x^1)^3 + \ldots + (x^5)^3 + \eta^1 \eta^2 = 0$$

with embedding space $\text{WSP}(1, 1, 2, 2, 2, 2)\tilde{\lambda}_1, \tilde{\lambda}_2)$. The choice of ghost scaling weights must satisfy $\tilde{\lambda}_1 + \tilde{\lambda}_2 = 6$. Ideally, this theory could be identified with an orbifold of homogeneous projective space, and studied using the techniques in [38]. This usually involves the change of coordinates $x^i = (z^i)^{\lambda_i}$ where $x^i$ has degree $\lambda_i$. Obviously, we cannot make such a change of variables for the ghost fields. Nevertheless, orbifold considerations should still be applicable.

The theory should possess a fixed-point set of positive codimension. This requirement uniquely determines the ghost scaling weights to be $(2, 4)$. A purely bosonic fixed-point set would not provide any interesting observables for $\mathcal{M}$. The only fixed point set under the projective identification is given by:

$$(x^1)^3 + \ldots + (x^5)^3 + \eta^1 \eta^2 = 0 \quad y^1 = y^2 = 0.$$  

This $\hat{c} = 1$ ‘supercurve’ is a $\mathbb{Z}_2$-quotient singular set. Forms on $\mathcal{M}$ which arise from a resolution of this fixed point set should correspond to observables in the single twisted sector associated to this $\mathbb{Z}_2$ quotient. This is clearly heuristic reasoning but it will prove useful. Observables in the twisted sector correspond to forms on the fixed-point set [38]. The cohomology for the curve is constructed using the techniques of subsection 4.2; the Hodge numbers are shown in figure 4.3.

In the twisted sector, the fermion vacuum is charged which in this case shifts the charges of the operators in figure 4.3 by $(1, 1)$ [38]. These observables then provide the missing forms on $\mathcal{M}$. Note that the forms $(k, k^2)$ on $\mathcal{M}$ correspond to the identity and volume form respectively on the fixed point set.

1
5 5
1

Figure 4.3: Hodge diamond for the fixed point set.
This line of reasoning, albeit heuristic, implies that the resolved supermanifold should possess a Kähler modulus. To desingularize $\mathcal{M}$, we must smoothly ‘glue’ in an appropriate supermanifold while preserving super-Ricci flatness. The Hodge diamond for the resulting smooth space should coincide with figure 4.2 from the Landau-Ginzburg theory.

5. Conclusions

To provide a general Landau-Ginzburg orbifold with a sigma model phase requires the introduction of supermanifolds. The condition for conformal invariance is the vanishing of the super-Ricci tensor. We described the chiral ring for these theories and argued that Kähler moduli only appear when the target space is singular. Among the supermanifolds considered are the mirrors of rigid manifolds resolving that issue in mirror symmetry. Many interesting questions remain to be solved: What are the necessary and sufficient conditions for the existence of a super-Ricci flat metric? How do index theorems extend to these spaces? What is an appropriate mathematical supercohomology theory? How are singularities resolved for supermanifolds? Clearly, much of algebraic geometry must generalize nontrivially to these spaces.

More physically, our intuitive notion of a string propagating on a target space must be enlarged to accommodate the idea of negative dimensions. Once again, string theory provides unexpected relations - this time between strings on bosonic spaces and strings on supermanifolds. We conclude with a modest conjecture on mirror symmetry: Any Kähler supermanifold giving rise to a nondegenerate conformal field theory has one or more mirror realizations.

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