A Spectral Construction of a Treed Domain that is not Going-Down

David E. Dobbs
Marco Fontana
Gabriel Picavet

Abstract

It is proved that if $2 \leq d \leq \infty$, then there exist a treed domain $R$ of Krull dimension $d$ and an integral domain $T$ containing $R$ as a subring such that the extension $R \subseteq T$ does not satisfy the going-down property. Rather than proceeding ring-theoretically, we construct a suitable spectral map $\varphi$ connecting spectral (po)sets, then use a realization theorem of Hochster to infer that $\varphi$ is essentially $\text{Spec}(f)$ for a suitable ring homomorphism $f$, and finally replace $f$ with an inclusion map $R \hookrightarrow T$ having the asserted properties.

1 Introduction and Summary

The purpose of this note is to construct a ring-theoretic example by using some order-theoretic machinery and relatively little calculation. In the next paragraph, we review the relevant ring-theoretic background and state the main result. In the following paragraph, we review the relevant order-theoretic machinery and outline our approach. Full details are given in Section 2.

All rings considered below are commutative with identity; all ring extensions and all ring homomorphisms are unital. A ring homomorphism $f : A \rightarrow B$ is said to satisfy going-down if, whenever $P_2 \subseteq P_1$ are prime ideals of $A$ and $Q_1$ is a prime ideal of $B$ such that $f^{-1}(Q_1) = P_1$, there exists a prime ideal $Q_2$ of $B$ such that $Q_2 \subseteq Q_1$ and $f^{-1}(Q_2) = P_2$. A ring extension $A \subseteq B$ is said to satisfy going-down if the inclusion map $i : A \hookrightarrow B$ satisfies going-down. Following [2] and [7], we say that an integral domain $R$ is a going-down domain in case $R \subseteq T$ satisfies going-down for all integral domains $T$ containing $R$ as a subring (equivalently, for all integral domains $T$ contained between $R$ and its quotient field). The most natural examples
of going-down domains are arbitrary Prüfer domains and integral domains of Krull dimension at most 1. The fundamental order-theoretic fact about such rings is [2, Theorem 2.2]: any going-down domain is a treed domain. (For each integral domain $A$, the set $\text{Spec}(A)$ of all prime ideals of $A$ is a poset via inclusion; $A$ is said to be a \textit{treed domain} in case $\text{Spec}(A)$, as a poset, is a tree, that is, in case no prime ideal of $A$ contains incomparable prime ideals of $A$.) Remarkably, the converse is false, as [8, Example 4.4] presents a construction, due to W. J. Lewis, of an extension $R \subseteq T$ of two-dimensional domains such that $R$ is a treed domain and $R \subseteq T$ does not satisfy going-down. Like the construction of Lewis, the only other known example of this phenomenon [4, Example 2.3] depends on a specific type of ring-theoretic construction ($k + J(A)$, as in [13, (E2.1), p. 204]) whose analysis involves a considerable amount of calculation. It seems natural to ask if one can use order-theoretic methods to produce a treed domain $R$ that is not a going-down domain without having to appeal to the details of a specific ring-theoretic construction. We do so here for all possible Krull dimensions $d$ of $R$, namely, $2 \leq d \leq \infty$.

A key concept in our approach is that of an $L$-spectral set. Recall from [11, p. 53] that the underlying set of any $T_0$-topological space $Z$ can be given the structure of a poset as follows: for $x, y \in Z$, $x \leq y \iff y \in \{x\}$. A $T_0$-topology $T$ on a poset $(W, \leq)$ is said to be \textit{compatible with} $\leq$ in case $\leq$ coincides with the partial order induced by $T$ on $W$. Recall from [1, Exercice 2, p. 89] that the finest topology on $W$ that is compatible with the given partial order $\leq$ is the \textit{left topology} on $W$, namely, the topology having an open basis consisting of the sets $w^{\downarrow} := \{v \in W \mid v \leq w\}$ as $w$ runs through the elements of $W$. Let $W^L$ denote $W$ equipped with the left topology. As in [6], a poset $W$ is called an $L$-\textit{spectral set} if $W^L$ is a \textit{spectral space}, i.e., is homeomorphic to $\text{Spec}(A)$ (with the Zariski topology) for some ring $A$. (As usual, the Zariski topology on $\text{Spec}(A)$ is defined to be the topology that has an open basis consisting of the sets $\{P \in \text{Spec}(A) \mid a \notin P\}$ as $a$ runs through the elements of $A$.) In Section 2, we construct $L$-spectral sets $Y, X$ and a spectral map $\varphi : Y^L \rightarrow X^L$, in the sense of [11, p. 43], namely, a continuous map of spectral spaces for which the inverse image of any quasi-compact open set is quasi-compact. $Y$ and $X$ are chosen as small as possible for $\varphi$ to fail to satisfy the order-theoretic analogue of the going-down property. Verification of the above-stated topological properties of $X, Y$ and $\varphi$ proceeds order-theoretically, by appealing to some results in [6]. Then, since the spectral map $\varphi$ is surjective, we can apply [11, Theorem 6 (a)]. This result allows us to
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avoid introducing — or analyzing — a specific ring theoretic construction, for it essentially permits the identifications $Y = \text{Spec}(B), X = \text{Spec}(A)$ and $\varphi = \text{Spec}(f)$, for a suitable ring homomorphism $f : A \to B$. (As usual, $\text{Spec}(f) : \text{Spec}(B) \to \text{Spec}(A)$ is defined by $Q \mapsto f^{-1}(Q)$.) The proof concludes by using standard ring-theoretic tools to replace $f : A \to B$ with an inclusion map $i : R \hookrightarrow T$ having the desired properties.

2 The construction

We begin by defining the three-element poset $Y := \{y_0, y_1, y_2\}$ by imposing the requirements that $y_0 \leq y_1$ and $y_0 \leq y_2$ (with $y_1$ and $y_2$ unrelated). Before analyzing $Y$ order-theoretically with essentially no calculations, we indicate how detailed a ring-theoretical approach to the properties of $Y$ would be. It can be seen ring-theoretically that $Y$ is a spectral set: consider, for instance, the poset structure imposed by the Zariski topology on $\text{Spec}(D)$, where $D$ is the localization of $\mathbb{Z}$ at the multiplicatively closed set $\mathbb{Z} \setminus (2\mathbb{Z} \cup 3\mathbb{Z})$. From this point of view, the Prime Avoidance Lemma (cf. [10, Proposition 4.9]) allows the identifications $y_0 = \{0\}, y_1 = 2D$ and $y_2 = 3D$. Using the definition of the Zariski topology, one can then show after some case analysis that the open sets of $Y = \text{Spec}(D)$ are $\emptyset, Y, \{y_0\}, \{y_0, y_1\}$ and $\{y_0, y_2\}$.

Fortunately, $Y$ can be studied directly by order-theoretic means, without recourse to the above ring $D$. In the process, one learns even more: $Y$ is an $L$–spectral set. To see this, one need only verify the four order-theoretic conditions $(\alpha) - (\delta)$ in the characterization of $L$–spectral sets in [6, Theorem 2.4]. Since $Y$ is finite, it is evident that the following three conditions

$(\alpha)$ each nonempty linearly ordered subset of $Y$ has a least upper bound,
$(\beta)$ $Y$ has only finitely many maximal elements, and
$(\gamma)$ for each pair of distinct elements $x, y \in Y$, there exist only finitely many elements of $Y$ which are maximal in the set of common lower bounds of $x$ and $y$

all hold in $Y$. Moreover, checking $(\beta)$ amounts to the easy verification that each nonempty lower-directed subset $Z$ of $Y$ has a greatest lower bound $z$ such that $\{y \in Y \mid z \leq y\} = \{y \in Y \mid w \leq y \text{ for some } w \in Z\}$. By using the definition of the left topology on $Y$, we obtain the same list of open sets as in the above ring-theoretic approach. This is not a coincidence, since an application of either the Main Theorem (whose order-theoretic criteria evidently hold in any finite poset) or Corollary 2.6 of [3] reveals that any
finite poset has only one order-compatible topology.

We next introduce the three-element linearly ordered poset \( X := \{x_0, x_1, x_2\} \) by imposing the requirements that \( x_0 \leq x_1 \leq x_2 \). (Since \( X \) has a unique maximal element, the eventual treed domain \( A \) will be automatically quasilocal, that is, will have a unique maximal ideal.) One could verify ring-theoretically that \( X \) is a spectral set (arising from, for instance, a valuation domain of Krull dimension 2) and then, by considering the Zariski topology, identify the open sets of \( X \) as \( \emptyset, X, \{x_0\} \) and \( \{x_0, x_1\} \). We leave these details to the reader, as the above-cited results from [3] ensure that a “left topology” approach would produce the same list of open sets in \( X \). Of course, such an approach is appropriate, for by considering conditions (\( \alpha \))-(\( \delta \)) in [6, Theorem 2.4], one shows easily that any finite linearly ordered set is an \( L \)-spectral set.

The function \( \varphi : Y \to X \) is defined by \( \varphi(y_i) = x_i \) for \( i = 1, 2, 3 \). Observe that \( \varphi \) is surjective and order-preserving. Of course, \( \varphi \) is not an order-isomorphism. Indeed, we have constructed \( \varphi \) so as to fail to have the order-theoretic analogue of the going-down property, for no \( y_i \) satisfies both \( y_i \leq y_2 \) and \( \varphi(y_i) = x_1 \). One could use the above lists of open sets to check that when viewed as a map \( Y^\mathcal{L} \to X^\mathcal{L} \), \( \varphi \) is continuous, since \( \varphi^{-1}(\{x_0\}) = \{y_0\} \) and \( \varphi^{-1}(\{x_0, x_1\}) = \{y_0, y_1\} \). However, this detail can be avoided by appealing to [6, Lemma 2.6 (a)], which states that any order-preserving map of posets is continuous when these posets are each equipped with the left topology. Being a continuous function between finite spectral spaces, \( \varphi \) is also a spectral map (as the quasi-compact open subsets are the same as the open subsets). In short, \( \varphi : Y^\mathcal{L} \to X^\mathcal{L} \) is spectral and surjective.

The above data are made to order for the realization assertion in [11, Theorem 6 (b)]. This result states that when \( \text{Spec} \) is viewed as a contravariant functor from the category of commutative rings (and ring homomorphisms) to the category of spectral spaces (and spectral maps), then \( \text{Spec} \) is invertible on the (nonfull) subcategory of all spectral spaces and surjective spectral maps. In particular, one infers the existence of a ring homomorphism \( f : A \to B \) and homeomorphisms \( \alpha : \text{Spec}(A) \to X, \beta : \text{Spec}(B) \to Y \) (where \( \text{Spec}(A) \) and \( \text{Spec}(B) \) are each endowed with the Zariski topology) such that \( \alpha \circ \text{Spec}(f) = \varphi \circ \beta \). It follows that \( \text{Spec}(f) \) is surjective. Moreover, since the homeomorphisms \( \alpha, \beta \) are necessarily order-isomorphisms, it also follows that \( \text{Spec}(f) \) has all the order-theoretic properties of \( \varphi \). In particular, \( f \) does not satisfy going-down.

We next reduce to the case of injective \( f \). Indeed, the First Isomorphism
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Theorem gives the factorization \( f = j \circ \pi \), where \( \pi : A \to A/\ker(f) \) is the canonical projection and \( j : A/\ker(f) \to B \) is the canonical injection. Note that \( \text{Spec}(\pi) \) is a homeomorphism (hence, an order-isomorphism), the key point being that \( P \supseteq \ker(f) \) for each prime ideal \( P \) of \( A \). (To see this, take a prime ideal \( Q \) of \( B \) such that \( P = \text{Spec}(f)(Q) = f^{-1}(Q) \) and observe that \( \ker(f) = f^{-1}(\{0\}) \subseteq f^{-1}(Q) \).) As \( \text{Spec}(j) = (\text{Spec}(\pi))^{-1} \circ \text{Spec}(f) \), we see that \( j \) does not satisfy going-down. By abus de langage, we henceforth replace \( f \) with \( j \), viewed as an inclusion (and thus replace \( A \) with \( A/\ker(f) \)). Notice also that (either the “old” or the “new”) \( A \) is a quasilocal ring of Krull dimension 2, thanks to the order-isomorphism \( \alpha \) and the construction of \( X \).

Since \( f \) does not satisfy going-down, we see via [5, Lemma 3.2 (a)] that the injection \( f_{\text{red}} : A_{\text{red}} \to B_{\text{red}} \) of associated reduced rings also does not satisfy going-down. (Recall that if \( E \) is any ring, then \( E_{\text{red}} := E/\sqrt{E} \), where \( \sqrt{E} \) denotes the set of all nilpotent elements of \( E \). It is well known that applying the \( \text{Spec} \) functor to the canonical projection \( E \to E_{\text{red}} \) produces a homeomorphism. Of course, \( f_{\text{red}} \) is defined by \( a + \sqrt{A} \mapsto f(a) + \sqrt{B} \).) By more abus de langage, we replace \( f \) with \( f_{\text{red}} \) (which is now viewed as an inclusion). Observe that (the “new”) \( A \) is quasilocal and of Krull dimension 2. Moreover, we have now reduced to the case in which both \( A \) and \( B \) are reduced rings (that is, rings with no nonzero nilpotents) each having a unique minimal prime ideal, that is, integral domains.

For \( d = 2 \), putting \((R, T, i) := (A, B, f)\) produces, as asserted, an inclusion map \( i : R \to T \) of integral domains such that \( R \) is (quasilocal and) of Krull dimension \( d \) and \( i \) does not satisfy going-down. To produce such an example in which \( T \) is contained between \( R \) and its quotient field, one need only invoke the characterization of going-down domains in [7, Theorem 1].

Suppose next that \( 3 \leq d \leq \infty \). Take \( R \) and (either) \( T \) as above, and let \( F \) denote the quotient field of \( T \). Using, for instance, the proof of [10, Corollary 18.5], we can construct a valuation domain of the form \( V = F + M \) such that \( V \) has Krull dimension \( d - 2 \) and \( M \) is the maximal ideal of \( V \). (As usual, we take \( \infty \pm r := \infty \) for each real number \( r \).) Observe that the integral domains \( R + M \subseteq T + M \) have the same quotient field, since they share \( M \) as a common nonzero ideal. The standard lore of the classical \((D + M)\)-construction, as in [10, Exercise 12, p. 202], yields that \( \text{Spec}(R + M) = \text{Spec}(V) \cup \{P + M \mid P \in \text{Spec}(R)\} \); of course, one also has a similar description of \( \text{Spec}(T + M) \). (The same conclusions are available via [9, Theorem 1.4] since, for instance, \( R + M \) is the pullback of the canonical projection \( V \to V/M \cong F \) and
the inclusion $R \rightarrow F$.) As valuation domains are quasilocal treed domains, it follows that $R + M$ inherits from $R$ the property of being a (quasilocal) treed domain. Moreover, with “dim” denoting Krull dimension, we have that 
\[ \dim(R + M) = \dim(R) + \dim(V) = 2 + (d - 2) = d. \] Finally, as in the proof of [7, Corollary], the above description of prime spectra implies that the extension $R + M \subseteq T + M$ inherits from $R \subseteq T$ the failure of the going-down property. Consequently, $R + M \hookrightarrow T + M$ has the asserted properties, to complete the proof.

In closing, we contrast the above role of pullbacks (which we used only in the case $d \geq 3$) with their role in Lewis’s two-dimensional example. That example had been only sketched in [8]. A fuller explanation of it, as in [4, Remark 2.1 (a), second paragraph], involves the use of either the “maximal quotient map” machinery of [12] or the fundamental gluing result on the prime spectra of pullbacks [9, Theorem 1.4] to analyze a pullback of the form $k + J(A)$. On the other hand, our approach needed such gluing information only for (the arguably more computation ally tractable) pullbacks of classical $D + M$ type. In sum, our approach has used the order-theoretic characterization of spectral spaces when the ambient topology on a poset is the left topology and an order-theoretic verification that the function $\varphi$ is a spectral map, Hochster’s fundamental result on invertibility of the Spec functor for surjective spectral maps, and relatively straightforward ring theory consisting of isomorphism theorems and a description of the prime spectrum of the classical $(D + M)$-construction.

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DAVID E. DOBBS
UNIVERSITY OF TENNESSEE
DEPARTMENT OF MATHEMATICS
KNOXVILLE
TENNESSEE 37996-1300
USA
dobbs@math.utk.edu

MARCO FONTANA
UNIVERSITÀ DEGLI STUDI, ROMA TRE
DEPARTMENT OF MATHEMATICS
LAGRO SAN LEONARDO MURIALDO, 1
00146 ROMA
ITALY
fontana@matrm3.mat.uniroma3.it

GABRIEL PICAVET
UNIVERSITÉ BLAISE PASCAL
LABORATOIRE DE MATHÉMATIQUES PURES
LES CÉZEAXS
63177 AUBIERE CEDEX
FRANCE
Gabriel.Picavet@math.univ-bpclermont.fr