CONSTRAINT ALGEBRAS
IN GAUGE INVARIANT SYSTEMS

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Abstract
The Hamiltonian description for a wide class of mechanical systems, having local symmetry transformations depending on time derivatives of the gauge parameters of arbitrary order, is constructed. The Poisson brackets of the Hamiltonian and constraints with each other and with arbitrary function are explicitly obtained. The constraint algebra is proved to be the first class.

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1 Introduction

Gauge invariance gives rise to constraints of both Lagrangian and Hamiltonian formalisms (see [1, 2, 3, 4, 5, 6, 7] and references therein). Lagrangian constraints appear as a consequence of a functional dependence of the Lagrange equations. In general, there exist gauge invariant systems having no Lagrangian constraints. These systems correspond to the gauge transformations of the form

$$\delta \varepsilon q^r = \varepsilon^\alpha \psi^r_\alpha(q, \dot{q}).$$

(1.1)

But for all known systems of such type it is possible to find an equivalent gauge invariant system with Lagrangian constraints. Such a situation is observed e.g. for relativistic particles and strings.

To construct the Hamiltonian description of the system, it is necessary to use the Legendre transformation, but for the case of gauge invariant systems it turns out to be a singular mapping. This fact leads to the constraints in Hamiltonian formalism. Thus, since the origins of Lagrangian and Hamiltonian constraints seem to be different, the questions of the correspondence between Lagrangian and Hamiltonian descriptions of gauge invariant systems, the connection between Lagrangian and Hamiltonian constraints, are risen (see [8, 9] and refs. therein).

The structure of the theory (i.e. the Noether identities, gauge algebras, hierarchy of constraints, etc.) is determined by the type of gauge transformations. In particular, the systems of Yang–Mills type [10] are characterized by the gauge symmetry transformations of trajectory

$$\delta \varepsilon q^r(t) = \varepsilon^\alpha(t)\xi^r_\alpha(q(t)) + \varepsilon^\alpha(t)\psi^r_\alpha(q(t)),$$

(1.2)

whereas the systems describing theories of gravity, strings, relativistic particles have the local symmetry under transformations of the form [4, 6, 11, 17]

$$\delta \varepsilon q^r(t) = \varepsilon^\alpha(t)\left(\xi(q(t)) + \dot{q}^s(t)\xi^s_{\alpha s}(q(t))\right) + \varepsilon^\alpha(t)\psi^r_\alpha(q(t)).$$

(1.3)

We see that various dependences on the velocity phase space coordinates in the gauge transformations lead to essentially different physical theories.

All of the most interesting from the physical point of view systems, as we know, correspond to the gauge transformations depending only on (up to) first order time derivative of infinitesimal gauge parameters $\varepsilon^\alpha(t)$. Note also that the most general form of the gauge symmetry transformations for the quadratic systems are given by [11]. Now we shall generalize consideration of Refs.[4, 6] to the class of the gauge symmetry transformations with higher (arbitrary) order time derivatives of infinitesimal gauge parameters.

The paper is organized as follows. In Section 1 the Lagrangian formalism for systems, invariant under gauge transformations of a general form, but depending only on the velocity phase space coordinates, is presented. Section 2 is devoted to the Hamiltonian description of such systems; using the Noether identities and the gauge algebra relations obtained in Section 1, we get the Poisson brackets of the Hamiltonian and constraints on the primary constraint surface. The correspondence between Lagrangian and Hamiltonian formalisms is clarified. In Section 3, using the notion of the standard extension [4, 6, 7], we find the explicit form of the constraint algebras in the total phase space. Besides, we calculate the
Poisson brackets of the Hamiltonian and constraints with an arbitrary function on the phase space; these expressions may be useful for some applications.

Here we consider only bosonic mechanical systems. Note that one can easily generalize the results of the paper to the case of mechanical systems, described by both even and odd variables [12, 13].

The summation over repeated indexes is assumed.

2 Gauge invariance in Lagrangian mechanics

Let us consider mechanical system given by Lagrangian \( L(q, \dot{q}) \) in the \( 2R \)-dimensional velocity phase space [14] with coordinates \( q^r, \dot{q}^r, \ r = 1, \ldots, R \). Hereafter, \( q^r \) and \( \dot{q}^r \) are the generalized coordinates and the generalized velocities of the system, respectively. It is convenient to present the Lagrange equations as follows:

\[
L_r(q, \dot{q}, \ddot{q}) \equiv W_{rs}(q, \dot{q}) \ddot{q}^s - R_r(q, \dot{q}) = 0,
\]

where

\[
R_r(q, \dot{q}) = \frac{\partial L(q, \dot{q})}{\partial q^r} - \dot{q}^s \frac{\partial^2 L(q, \dot{q})}{\partial q^s \partial \dot{q}^r},
\]

\[
W_{rs}(q, \dot{q}) = \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^r \dot{q}^s}.
\]

The matrix \( W_{rs} \) is called the Hessian of the system.

Assume that the system has a gauge symmetry under infinitesimal trajectory transformations of the form

\[
\delta_\varepsilon q^r = \sum_{k=0}^{N} (\varepsilon^k) \psi^{[N-k]}_\alpha(q, \dot{q}), \quad \alpha = 1, \ldots, A,
\]

where \( \varepsilon^\alpha \) are arbitrary infinitesimal functions of time:

\[
\delta_\varepsilon L = \frac{d}{dt} \Sigma_\varepsilon.
\]

In this paper we use the notations: integers within parentheses over characters display an order of time derivative of corresponding functions, and all the integers within square brackets (both subscripts and superscripts of characters) just mark the functions, simply giving them numbering. Hence, the integer \( N \) is the maximal order of time derivatives of the gauge parameters \( \varepsilon^\alpha(t) \) for the gauge invariant system we consider. The case of \( N = 1 \), which is the most interesting from the physical point of view, was considered in Refs. [4, 6]. Now we shall treat the case of arbitrary \( N > 1 \).

From the symmetry equations (2.4), (2.5) we get the Noether identities

\[
\sum_{k=0}^{N} (-1)^k \frac{d^k}{dt^k} \left( \psi^{[N-k]}_\alpha L_r \right) = 0.
\]
Besides, for $\Sigma_\varepsilon$ we have

$$\Sigma_\varepsilon = \sum_{k=0}^{N} (k)^{[N-k]} \sigma_{\alpha}^\varepsilon,$$  \hspace{1cm} (2.7)

where

$$\sigma_{\alpha} = \phi_{\alpha}^r \frac{\partial L}{\partial \dot{q}^r} - \sum_{l=0}^{k} (-1)^l \psi_{\alpha}^{[k-l]} \frac{\partial L}{\partial q^r} \frac{d^l}{dt^l} \left( \psi_{\alpha}^{[k-1-l]} L_r \right).$$  \hspace{1cm} (2.8)

One can rewrite the Noether identities (2.6) in the following equivalent form

$$[k+1] \Lambda_{\alpha} = [k] \psi_{\alpha}^r R_r - \dot{q}^s \frac{\partial [k] \Lambda_{\alpha}}{\partial q^s},$$  \hspace{1cm} (2.9)

$$[k] \psi_{\alpha}^r W_{rs} = - \frac{\partial [k] \Lambda_{\alpha}}{\partial q^s},$$  \hspace{1cm} (2.10)

where $k = 0, 1, \ldots, N$ and $[0] \Lambda_{\alpha} = [N+1] \Lambda_{\alpha} \equiv 0$. Using the Noether identities in the form of (2.9)–(2.10), we get more convenient form of $\Sigma_\varepsilon$, namely

$$\Sigma_\varepsilon = \delta_{\varepsilon} q^r \frac{\partial L}{\partial \dot{q}^r} + \sum_{k=0}^{N-1} (k)^{[N-k]} \sigma_{\alpha}^\varepsilon \Lambda_{\alpha}. \hspace{1cm} (2.11)$$

It follows from Eq. (2.11) that the Hessian of the system has $A$ null vectors $\psi_{\alpha}^r$, $\alpha = 1, \ldots, A$. Suppose that the vectors $\psi_{\alpha}^r$ are linearly independent, and any null vector of the matrix $W_{rs}$ is a linear combination of the vectors $\psi_{\alpha}^r$. Hence, we have

$$\text{rank } W_{rs}(q, \dot{q}) = R - A, \quad \text{rank } \psi_{\alpha}^r(q, \dot{q}) = A \hspace{1cm} (2.12)$$

for any values of $q^r$ and $\dot{q}^r$. We consider the systems for which any choice of arbitrary functions $\varepsilon_{\alpha}(t)$ and any trajectory gives the gauge transformations (2.4) to be nontrivial. One can show that this condition is equivalent to the linear independence of the set formed by the vectors $\psi_{\alpha}^r$, $k = 0, 1, \ldots, N$.

The rank of the Hessian is less than dimension of the configuration space of the system, hence, the Cauchy problem for the Lagrange equations (2.1) has no unique solution and there are intersecting trajectories in the system \{4\}. In other words, using the Lagrange equations, we can express only $R - A$ accelerations $\ddot{q}^r$ through the generalized coordinates and the generalized velocities. The remaining equations have the form

$$\psi_{\alpha}^r R_r = 0, \hspace{1cm} (2.13)$$

that follows directly from Eqs. (2.1),(2.10). Such relations restrict the possible values of $q^r$ and $\dot{q}^r$ and are called the primary Lagrangian constraints.

Using the Noether identities we get from the stability condition for the primary Lagrangian constraints $\Lambda_{\alpha} = \psi_{\alpha}^r R_r$ other Lagrangian constraints of the system, which are the
relations $\Lambda_{\alpha} = 0$, $k = 2, \ldots, N$. The Lagrangian constraints of $(k + 1)$-th stage $\Lambda_{\alpha}^{[k+1]}$ appear as a consequence of stability (with respect to time evolution) of the preceding Lagrangian constraints of $k$-th stage $\Lambda_{\alpha}^{[k]}$.

Suppose now that gauge transformations (2.4) form a closed gauge algebra. So, for any two sets of infinitesimal functions $\varepsilon_{1}^{\alpha}(t)$ and $\varepsilon_{2}^{\alpha}(t)$ we have the commutator of corresponding gauge transformations of type (2.4) to be of the same type

$$[\delta_{\varepsilon_{1}}, \delta_{\varepsilon_{2}}] \ = \delta_{\varepsilon_{1}} \varepsilon_{2}^{\alpha}. \quad (2.14)$$

In this, $\varepsilon^{\alpha}$ are, in general, some functions of $\varepsilon_{1}^{\alpha}$, $\varepsilon_{2}^{\alpha}$ and the trajectory of the system.

Using (2.4) and taking into account the linear independence of the vectors $\psi_{\alpha}^{r}$, $k = 0, 1, \ldots, N$, we obtain from Eq. (2.14) the relations

$$\left[ \partial \psi_{\alpha}^{r}, \partial q_{s} \right] = \left( \partial \psi_{\alpha}^{r}, \partial q_{s} \right) + \left( \partial \psi_{\beta}^{r}, \partial q_{s} \right) \frac{\partial \psi_{\alpha \beta}}{\partial q^{s}}, \quad (2.15)$$

where $n = 0, 1, \ldots, N + 1$; $m = 0, 1, \ldots, 2N + 1$. Here $\bar{A}_{\alpha \beta}^{[k]}$ are some functions of the generalized coordinates $q_{r}$, called the structure functions of the gauge algebras (remember that for the case of $N = 1$ [4, 3] the structure functions of the corresponding gauge algebras, in general, depend on both generalized coordinates and generalized velocities).

These functions satisfy the symmetry equations

$$\bar{A}_{\alpha \beta}^{[l]} = - \bar{A}_{\alpha \beta}^{[k-l]}, \quad l \leq k \leq l + 1, \quad (2.16)$$

and relate the infinitesimal parameters of gauge transformations in (2.14) as follows

$$\varepsilon^{\gamma} = \sum_{k=0}^{N+1} \sum_{l=0}^{k} \varepsilon_{1}^{(k-l)\alpha} \varepsilon_{2}^{(l)\beta} \bar{A}_{\alpha \beta}^{[k]}, \quad (2.17)$$

However, we get from (2.14) that only the structure functions $\bar{A}_{\alpha \beta}^{[0]}$, $\bar{A}_{\alpha \beta}^{[0]}$, $\bar{A}_{\alpha \beta}^{[1]}$, $\bar{A}_{\alpha \beta}^{[1]}$, $\bar{A}_{\alpha \beta}^{[2]}$ are nonzero. Besides, as follows from Eq. (2.15), for the cases of $N > 2$ all the structure functions are turned out to be constant, thus the terms within the square brackets in the r. h. s. of (2.13) are equal to zero. These properties of the gauge algebras essentially simplify the further analysis.

Note also that Eq. (2.15) contains the relations

$$\psi_{\alpha}^{r} \frac{\partial \psi_{\beta}^{s}}{\partial q^{t}} = \bar{A}_{\alpha \beta}^{[N-n]} \psi_{r}^{[0]}, \quad n = 0, 1, \ldots, N. \quad (2.18)$$
We see that the r. h. s. of Eq. (2.18) for values of $n < N - 1$ is zero because of the above properties of the structure functions.

Thus, we have obtained in this Section the Noether identities, the gauge algebra relations and the hierarchy of the Lagrangian constraints

$$[k] \Lambda^\alpha_\alpha = 0, \quad k = 1, \ldots, N.$$  \hspace{1cm} (2.19)

These expressions will be used in the next Section to construct Hamiltonian description for the systems under consideration.

### 3 Hamiltonian description of gauge invariant systems

To write down Hamiltonian description of the systems under consideration, one needs the algebras of constraints and the Hamiltonian. It forces us to find out the correspondence between Lagrangian and Hamiltonian constraints in the spirit of Refs. [4, 6].

To realize a Hamiltonian formalism, let us introduce $2R$-dimensional phase space described by canonical pairs of the generalized coordinates $q^r$ and generalized momenta $p_r$, and define a mapping of the velocity phase space to the (canonical) phase space with the help of usual relation

$$p_r(q, \dot{q}) = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^r}. \hspace{1cm} (3.1)$$

As it was established in the previous Section, the Hessian of the system $W_{rs}(q, \dot{q}) = \partial p_r(q, \dot{q})/\partial \dot{q}^s$ is singular, hence the mapping given by Eq. (3.1) has no inverse. One can show [8] that under this mapping an inverse image of a point of the phase space is either empty or consists of one or several $A$-dimensional surfaces having the parametric representation of the form

$$q^r(\tau) = q^r, \hspace{1cm} (3.2)$$

$$\dot{q}^r(\tau) = \dot{q}^r + \tau^a \psi^r_\alpha(q, \dot{q}). \hspace{1cm} (3.3)$$

Taking this fact into account and disregarding the degenerative cases, we see that the image of the velocity phase space under the mapping (3.1) is a $(2R - A)$-dimensional surface in the phase space, which may be given with the help of $A$ functionally independent functions as follows

$$[0] \Phi_\alpha(q, p) = 0, \quad \alpha = 1, \ldots, A. \hspace{1cm} (3.4)$$

Hence, we have introduced irreducible set of the so-called primary constraints $[0] \Phi_\alpha$ and defined the primary constraint surface by Eq. (3.4).

For any function $F(q, p)$, defined on the phase space, we can introduce the corresponding function $f(q, \dot{q})$ on the velocity phase space by the relation

$$f(q, \dot{q}) = F(q, p(q, \dot{q})). \hspace{1cm} (3.5)$$

It follows from the relation

$$p_r(q(\tau), \dot{q}(\tau)) = p_r(q, \dot{q}), \hspace{1cm} (3.6)$$

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where $\tau^\alpha$ parametrize the surfaces (3.2), (3.3), that the function $f(q, \dot{q})$ is constant on such surfaces. This fact implies differential expressions of the form
\[
\psi^r_\alpha \frac{\partial f}{\partial \dot{q}^r} = 0, \quad \alpha = 1, \ldots, A. \tag{3.7}
\]

However, given a function $f(q, \dot{q})$ on the velocity phase space, it is not always possible to define a function $F(q, p)$ on the phase space, connected with $f$ by the relation
\[
F(q, p(q, \dot{q})) = f(q, \dot{q}). \tag{3.8}
\]

The necessary conditions for the existence of such a function are (3.7). These conditions become sufficient if each point of the primary constraint surface is the image of only one surface of form (3.2), (3.3). We suppose that it is really so.

Thus, in the case under consideration for any function $f(q, \dot{q})$, satisfying the conditions (3.7), we can find a function $F(q, p)$, such that Eq.(3.8) is fulfilled. When the functions $F$ and $f$ are connected by the relation (3.8), we shall use the notations of Refs.[4, 6] and write
\[
F \doteq f. \tag{3.9}
\]

We shall also call such a function $f$ projectable to the primary constraint surface, or simply projectable.

Note that
\[
\Phi^0_\alpha \doteq 0, \tag{3.10}
\]
thus, if a function $F_0$ satisfies (3.4), then any function $F$ of the form
\[
F = F_0 + F^\alpha_0 \Phi_\alpha, \tag{3.11}
\]
where $F^\alpha$ are arbitrary functions, satisfies this equality as well. Treating (3.9) as an equation for $F$, and $F_0$ as a partial solution of this equation, one can show that (3.11) gives the general solution of this equation. In other words, the relation (3.9) defines the function $F$ only on the primary constraint surface, and it may be extended from this surface to the total phase space arbitrarily, but according to (3.11) various extensions will differ from each other in a linear combination of the primary constraints [3].

Now let us define the Hamiltonian of the system. To this end, introduce the energy function $E(q, \dot{q})$ on the velocity phase space by the relation
\[
E = \dot{q}^r \frac{\partial L}{\partial \dot{q}^r} - L. \tag{3.12}
\]

The energy function is projectable to the primary constraint surface, hence we can find the corresponding function $H(q, p)$ on the phase space, such that
\[
H \doteq E, \tag{3.13}
\]
and treat this function as the Hamiltonian of the system. According to the above reasonings, Eq.(3.13) determines the Hamiltonian only at the points of the primary constraint surface.
The function \( H(q, p) \) as function on the total phase space may be obtained by arbitrary extensions from this surface.

Recall now that the Lagrangian constraints \( \lambda_k(q, \dot{q}) \), \( k = 1, \ldots, N \), take constant values on the surfaces of the form (3.2), (3.3), i.e. they satisfy the conditions (3.7). It follows directly from the Noether identities (2.11). Hence, one can find on the phase space the functions \( \Phi_\alpha(q, p) \), such that

\[
\Phi_\alpha \equiv \lambda_\alpha, \quad k = 1, \ldots, N.
\]  

(3.14)

Let us compute the Poisson brackets of the Hamiltonian \( H \), primary constraints \( \Phi_\alpha \) and functions \( \Phi_\alpha, k = 1, \ldots, N \), corresponding to the Lagrangian constraints. To this end, we shall obtain the partial derivatives of all these functions over the canonical coordinates and momenta.

One can write the explicit form of Eq.(3.11)

\[
[0] \Phi_\alpha(q, p(q, \dot{q})) = 0.
\]  

(3.15)

Differentiating these relations over \( \dot{q}^r \) and \( q^r \), respectively, we get the following expressions

\[
\frac{\partial[0] \Phi_\alpha}{\partial p_r} \equiv -[0] u_\beta^\alpha [0] \psi_\beta^r,
\]  

(3.16)

\[
\frac{\partial[0] \Phi_\alpha}{\partial q_r} \equiv [0] u^s_\alpha [0] \psi_s^\gamma \frac{\partial^2 L}{\partial \dot{q}^s \partial q^r},
\]  

(3.17)

where the matrix \([0] u_\alpha^\beta(q, \dot{q})\) has to be invertible since the primary constraints are functionally independent.

Taking into account (2.18) we obtain that the matrix elements of \([0] u_\alpha^\beta(q, \dot{q})\) are projectable to the primary constraint surface:

\[
[0] \psi_\gamma^s \frac{\partial[0] u_\alpha^\gamma}{\partial q^r} = 0.
\]  

(3.18)

For any choice of the primary constraints, which define one and the same primary constraint surface, we shall get different matrices \([0] u_\alpha^\beta(q, \dot{q})\), satisfying Eq.(3.18).

Differentiating the energy function \( E(q, \dot{q}) \) over \( q^r \) and \( \dot{q}^r \) we get

\[
\frac{\partial E}{\partial q^r} = \dot{q}^s W_{sr},
\]  

(3.19)

\[
\frac{\partial E}{\partial \dot{q}^r} = \dot{q}^s \frac{\partial^2 L}{\partial \dot{q}^s \partial q^r} - \frac{\partial L}{\partial q^r}.
\]  

(3.20)

Taking into account (3.13), we obtain for the partial derivatives of the Hamiltonian the expressions

\[
\frac{\partial H}{\partial p_r} \equiv \dot{q}^r - \mu_\alpha [0] \psi_\alpha^r,
\]  

(3.21)

\[
\frac{\partial H}{\partial q_r} \equiv -\frac{\partial L}{\partial q^r} + \mu_\alpha [0] \psi_\alpha^s \frac{\partial^2 L}{\partial \dot{q}^s \partial q^r},
\]  

(3.22)
where the functions $\mu^\alpha(q, \dot{q})$ satisfy the equalities

$$ \psi^r_\alpha \frac{\partial \mu^\beta}{\partial \dot{q}^r} = \delta^\beta_\alpha. \quad (3.23) $$

Using the Noether identities one can obtain the partial derivatives of the functions $\Phi^k_\alpha$ from Eq.(3.14). We have for any $k = 1, \ldots, N$

$$ \frac{\partial \Phi^k_\alpha}{\partial p_r} = -\psi^r_\alpha + \psi_{\alpha}^{[k]} u^{\beta}_{\alpha} \psi^r_{\beta}, \quad (3.24) $$

$$ \frac{\partial \Phi^k_\alpha}{\partial \dot{q}^r} = \frac{\partial \Lambda^k_\alpha}{\partial \dot{q}^r} + (\psi^{[k]}_\alpha - \psi_{\alpha}^{[k]} u^{\beta}_{\alpha} \psi^s_{\beta}) \frac{\partial^2 L}{\partial \dot{q}^s \partial \dot{q}^r}, \quad (3.25) $$

where the functions $u^{k\beta}_{\alpha}(q, \dot{q})$ fulfil the relations

$$ \psi^r_\alpha \frac{\partial u^{k\gamma}_{\beta}}{\partial \dot{q}^r} = \frac{[N-k]}{[N-k+1]} \Lambda^{\beta}_{\alpha \gamma}, \quad (3.26) $$

following from Eq.(2.18).

Note that the arbitrariness in the choice of functions $\mu^\alpha$ and $u^{k\beta}_{\alpha}$ is a consequence of the ambiguity of the extension of the Hamiltonian and functions, corresponding to the Lagrangian constraints, from the primary constraint surface. But for all the possible extensions these functions must satisfy Eqs.(3.23) and (3.26), respectively.

Using the relations of the gauge algebra (2.15)–(2.18) and the Noether identities (2.9), (2.10), we get from Eqs.(3.16), (3.17), (3.21), (3.22), (3.24), (3.25) the following expressions for the Poisson brackets

$$ \{H, \Phi^0_\alpha\} = u^{[k]}_{\alpha} \Lambda^{[1]}_{\beta}, \quad (3.27) $$

$$ \{H, \Phi^k_\alpha\} = \frac{[k+1]}{[N-k+1]} \Lambda^{[1]}_{\alpha \beta} - \psi^{[k]}_{\alpha} + \psi_{\alpha}^{[k]} u^{[1]}_{\alpha} \Lambda^{[1]}_{\beta}, \quad (3.28) $$

We see that the Poisson brackets of the Hamiltonian with the functions $\Phi^k_\alpha$, $k = 0, 1, \ldots, N$, give rise to the Lagrangian constraints of the next order $\Lambda^{[k+1]}_{\alpha}$. This fact is in accordance with the Lagrangian approach, given in the previous Section.

It is not also difficult to obtain the expressions for the Poisson brackets of the primary constraints $\Phi^0_\alpha$ with each other and with the functions $\Phi^k_\alpha$. We have for any $k = 1, \ldots, N$ that

$$ \{\Phi^0_\alpha, \Phi^0_\beta\} = 0, \quad (3.29) $$

$$ \{\Phi^k_\alpha, \Phi^0_\beta\} = u^{[k]}_{\beta} \frac{[1]}{[N-k+1]} \Lambda^{[1]}_{\alpha \gamma}, \quad (3.30) $$

where we again made use of the Noether identities, the gauge algebra relations and the expressions for the partial derivatives (3.16), (3.17) and (3.24), (3.25).
Taking into account Eqs. (3.27)–(3.30) we see that the functions \( \Phi_\alpha \), introduced by Eq. (3.14), are nothing but the secondary Hamiltonian constraints of \( k \)-th stage [13, 14].

The most difficult calculational problem here is to obtain the Poisson brackets of the functions \( \Phi_\alpha, k = 1, \ldots, N \), with each other. Using the above results we get from Eqs. (3.24), (3.25), that

\[
\{ \Phi_\alpha, \Phi_\beta \} \equiv \left( u_\alpha^\delta \left[ \frac{A}{[N-l+1]_{\beta\delta}} - u_\beta^\delta \left[ \frac{A}{[N-k+1]_{\alpha\delta}} \right] \right] \right) \Lambda_\gamma + \left[ \frac{X}{[l]} \right]_\alpha\beta, \quad (3.31)
\]

for any \( k, l = 1, \ldots, N \), where we have introduced the notation

\[
\left[ \frac{X}{[l]} \right]_\alpha\beta = \psi_\alpha^{[k]} \frac{\partial \Lambda_\beta}{\partial q^r} - \psi_\beta^{[k]} \frac{\partial \Lambda_\alpha}{\partial q^r} - \psi_\alpha^{[k]} \psi_\beta^{[k]} \left( \frac{\partial R_r}{\partial q^s} + \dot{q}^s \frac{\partial W_{rs}}{\partial q^t} \right). \quad (3.32)
\]

Recall that for \( N > 2 \) from the gauge algebra it follows that the structure functions are constant on the velocity phase space. Now, considering the case of \( N > 2 \) we get, after tiresome calculations, that the functions \( \left[ \frac{X}{[l]} \right]_\alpha\beta, k, l = 1, \ldots, N \), satisfy the following recursive relations

\[
\left[ \frac{X}{[l]} \right]_\alpha\beta = \frac{1}{2} \left[ \frac{X}{[l-1]} \right]_\alpha\beta + \frac{1}{2} \left[ \frac{X}{[l+1]} \right]_\alpha\beta + \frac{1}{2} \frac{\partial \Lambda_\alpha}{\partial q^r} X_\beta^{[l]} + \frac{1}{2} \frac{\partial \Lambda_\beta}{\partial q^r} X_\alpha^{[l]} \quad (3.33)
\]

Using the relations (2.9) and the properties (2.16) of the structure functions, we get the recursive relations (3.33) to be equivalent to the recursive equations of a more simple form

\[
\left[ \frac{X}{[l]} \right]_\alpha\beta + \frac{1}{2} \left[ \frac{X}{[l-1]} \right]_\alpha\beta + \frac{1}{2} \left[ \frac{X}{[l+1]} \right]_\alpha\beta = \left[ \frac{Z}{[l]} \right]_\alpha\beta, \quad (3.34)
\]

where

\[
\left[ \frac{Z}{[l]} \right]_\alpha\beta = \frac{1}{2} \sum_{i=0}^{2} \sum_{j=0}^{i} \left( \frac{2N-2-k-l-i}{N+1-l-j} \right) \left[ \frac{A}{[l]} \right]_\gamma^{[k+1-N+i]} \Lambda_\gamma \quad (3.35)
\]

We obtain that the solution to the recursive relations (3.34) is given by the expression

\[
(k+l) \left[ \frac{X}{[l]} \right]_\alpha\beta = (-1)^k l \left[ \frac{X}{[k]} \right]_\alpha\beta + (-1)^l k \left[ \frac{X}{[l]} \right]_\alpha\beta + 2kl \left[ \frac{Z}{[l]} \right]_\alpha\beta + 2l \sum_{m=1}^{k-1} (-1)^m (k-m) \left[ \frac{Z}{[l]} \right]_\alpha\beta + 2k \sum_{m=1}^{l-1} (-1)^m (l-m) \left[ \frac{Z}{[l]} \right]_\alpha\beta, \quad (3.36)
\]

where we have used the notations

\[
\left[ \frac{X}{[k]} \right]_\alpha\beta = \frac{A}{[N-k-l+1]} \alpha_\beta \Lambda_\gamma, \quad \left[ \frac{X}{[k]} \right]_\alpha\beta = \frac{1}{[N-k-l+1]} \alpha_\beta \Lambda_\gamma, \quad (3.37)
\]

\[
\left[ \frac{X}{[l]} \right]_\alpha\beta = \frac{A}{[N-l+1]} \alpha_\beta \Lambda_\gamma, \quad \left[ \frac{X}{[l]} \right]_\alpha\beta = \frac{1}{[N-l+1]} \alpha_\beta \Lambda_\gamma.
\]
Using the explicit form of the functions \( \frac{[k]}{[l]} \), given by Eq. (3.35), we finally obtain

\[
\frac{[k]}{[l]} X_{\alpha\beta} = \sum_{i=0}^{2} \sum_{j=0}^{1} \left( 2N - k - l - i \right) \frac{[j]}{[i]} A_{\alpha\beta}^{[k+l-N+i]} \Lambda_{\gamma}.
\] (3.38)

To get the last expression we also made use of the properties of the binomial coefficients \([16]\).

Note that

\[
\frac{[k]}{[l]} X_{\alpha\beta} \equiv 0 \quad \text{for} \quad k + l < N - 1.
\] (3.39)

Consider now the case of \( N = 2 \), which is distinguished from the others by the fact that the structure functions depend on the generalized coordinates \( q^r \). Performing the corresponding calculations, we see that this dependence gives rise to the additional term to the expression (3.38) for \( \frac{[k]}{[l]} X_{\alpha\beta} \). This term is equal to \( \dot{q}^r \partial / \partial q^r \left( \frac{[N-l]}{[2N-k-l]} A_{\alpha\beta}^{[k+l-N+i]} \Lambda_{\gamma} \right) \).

Thus, we have the following expression for the Poisson brackets of the \( k \)-th and \( l \)-ary stage secondary constraints \( (k, l = 1, \ldots, N > 1) \)

\[
\{ \frac{[k]}{[l]} \Phi_{\alpha}, \frac{[l]}{[k]} \Phi_{\beta} \} \equiv \left( \frac{[k]}{[l]} u^{\delta}_{\alpha} \frac{[l]}{[k]} A_{\beta\delta}^{[N-l]} \gamma - \frac{[k]}{[l]} \dot{u}^{\delta}_{\beta} \frac{[l]}{[k]} A_{\alpha\delta}^{[N-k+1]} \gamma + \dot{q}^r \partial / \partial q^r \frac{[N-l]}{[2N-k-l]} A_{\alpha\beta}^{[k+l-N+i]} \Lambda_{\gamma} \right)
+ \sum_{i=0}^{2} \sum_{j=0}^{1} \left( 2N - k - l - i \right) \frac{[j]}{[i]} A_{\alpha\beta}^{[k+l-N+i]} \Lambda_{\gamma}.
\] (3.40)

Remember that we have obtained all the expressions for the Poisson brackets only on the primary constraint surface. Hence, these formulae determine the relations of the constraint algebra up to linear combinations of the primary constraints. To have the constraint algebra in the total phase space, it is necessary to define a way of extension of functions from the primary constraint surface to the whole phase space. One of such ways, called the standard extension \([4]\), will be described in the next Section.

Summarizing the above consideration, we see that the gauge invariance of the form (2.4), (2.5) gives rise to the singular system with the set of \( N \times A \) projectable Lagrangian constraints. Note that we have also proved that there appear in Hamiltonian description of such systems \( (N+1) \times A \) constraints being in involution, at least on the primary constraint surface.

### 4 The Poisson brackets within the standard extension

Recall now main results and definitions dealing with the notion of the standard extension. Following Ref.\([4]\), introduce the set of functions \( \chi^\alpha(q, \dot{q}), \alpha = 1, \ldots, A \), such that

\[
\frac{[0]}{[0]} u_{\alpha}^{\delta} \frac{[0]}{[0]} \dot{\psi}_{\beta}^{r} \partial \chi_{\beta}^{\alpha} / \partial \dot{q}^r = \delta_{\alpha}^{\beta},
\] (4.1)

i.e. the vectors \( \partial \chi_{\alpha}^{\alpha} / \partial \dot{q}^r \) are dual to the vectors \( \frac{[0]}{[0]} u_{\alpha}^{\delta} \frac{[0]}{[0]} \dot{\psi}_{\beta}^{r} \).
The quite nontrivial aspects of existence of the functions $\chi^\alpha$, satisfying (4.1) were discussed in Ref. [4] (see also [6, 11, 17]) for the case of $N = 1$. Following to that discussion, we assume the conditions of the existence of the functions $\chi^\alpha$ to be valid.

Choose the functions $\chi^\alpha(q, \dot{q})$ as follows:

$$\chi^\alpha(q, \dot{q}) = \dot{q}^r \chi^\alpha_r(q) + \nu^\alpha(q). \quad (4.2)$$

Hence, the vectors $\chi^\alpha_r(q)$ have to satisfy the duality relations

$$^{[0]}\tilde{u}_\alpha^\beta \psi^\beta_r \chi^\alpha_r = \delta^\beta_\alpha, \quad (4.3)$$

where $\nu^\alpha$ are some (arbitrary) functions of the generalized coordinates $q^r$. We call a function $F(q, p)$ standard if $[4, 6, 10, 11, 17]$

$$\chi^\alpha_r \frac{\partial F}{\partial p_r} = 0. \quad (4.4)$$

One can show that for any function, defined on the primary constraint surface, there exists a unique extension to the total phase space, which is a standard function. This extension is called the standard extension. The standard function coinciding with a function $F(q, p)$ on the primary constraint surface is denoted by $F^0$. It is clear that for any projectable function $f(q, \dot{q})$ one can find an unique standard function $F(q, p)$, such that $F = f$. We denote this standard function by $f^0$.

Using the properties of the standard extension [4], one can obtain the expression for the Poisson brackets of two standard functions $F^0$ and $G^0$:

$$\{F^0, G^0\} = \{F, G\}^0 + \frac{\partial F^0}{\partial p_r} \chi^\alpha_r \frac{\partial G^0}{\partial p_s} \Phi^\alpha_{rs}, \quad (4.5)$$

where

$$\chi^\alpha_r = \frac{\partial \chi^\alpha_s}{\partial q^s} - \frac{\partial \chi^\alpha_s}{\partial q^r}. \quad (4.6)$$

Note that we have introduced the so-called standard primary constraints in Eq.(4.5), defined by the relations

$$^{[0]}\Phi^\alpha = 0, \quad \frac{\partial^{[0]} \Phi^\alpha}{\partial p_r} = -(^{[0]}\tilde{u}_\alpha^\beta \psi^\beta_r)^0. \quad (4.7)$$

Let the Hamiltonian $H$ and all the constraints of the system $^{[0]}\Phi^\alpha, \, ^{[k]}\Phi^\alpha, \, k = 1, \ldots, N$, be the standard functions. Then, using Eqs.(3.27)–(3.30), (3.40), we get from (4.3) that the constraint algebra of the system under consideration is given by the expressions

$$\{H, ^{[0]}\Phi^\alpha\} = (^{[0]}\tilde{u}_\alpha^\beta)^0 \Phi^\beta + \frac{\partial H}{\partial p_r} \chi^\alpha_r \frac{\partial^{[0]} \Phi^\alpha}{\partial p_s} \Phi^\beta, \quad (4.8)$$

$$\{H, ^{[k]}\Phi^\alpha\} = ^{[k+1]}\Phi^\alpha - (^{[k]}\tilde{u}_\alpha^\beta + \mu_k A^\beta_{N-k+1} \alpha^\delta)^0 \Phi^\beta + \frac{\partial H}{\partial p_r} \chi^\alpha_r \frac{\partial^{[k]} \Phi^\alpha}{\partial p_s} \Phi^\beta, \quad (4.9)$$

11
\[
\{ \Phi_\alpha, \Phi_\beta \} = \frac{\partial \Phi_\alpha}{\partial q^r} \frac{\partial \Phi_\beta}{\partial p^s} \Phi_{\gamma},
\]
(4.10)

\[
\{ \Phi_\alpha, \Phi_\beta \} = \left( \frac{\partial}{\partial q^r} \frac{1}{A[k \times [N-k+1]} \gamma_\delta \right) 0 [1] \Phi_{\gamma} + \frac{\partial \Phi_\alpha}{\partial q^r} \frac{\partial \Phi_\beta}{\partial p^s} \Phi_{\gamma},
\]
(4.11)

\[
\{ \Phi_\alpha, \Phi_\beta \} = \left( \frac{\partial}{\partial q^r} \frac{1}{A[k \times [N-k+1]} \gamma_\delta - \frac{\partial}{\partial q^r} \frac{1}{A[k \times [N-k+1]} \gamma_\delta + \frac{\partial}{\partial q^r} \frac{1}{A[k \times [N-k+1]} \gamma_\delta \right) 0 [1] \Phi_{\gamma}
\]

\[
+ \sum_{i=0}^{2} \sum_{j=0}^{1} \left( \frac{2N - k - l - i}{N - l - j} \right) \frac{\partial \Phi_\alpha}{\partial q^r} \frac{\partial \Phi_\beta}{\partial p^s} \Phi_{\gamma},
\]
(4.12)

for any \( k, l = 1, \ldots, N > 1 \), and from (3.39) we have that \( i > N - k - l \). Besides, the functions \( \mu^\alpha(q, \dot{q}) \) and \( W^\alpha(q, \dot{q}) \) are of the form

\[
\mu^\alpha = \dot{q}^r \Gamma_k^\alpha u^\beta_r,
\]
(4.13)

\[
W^\alpha = \frac{\partial}{\partial q^r} \frac{1}{A[k \times [N-k+1]} \gamma_\delta \frac{\partial}{\partial p^s} \Phi_{\gamma},
\]
(4.14)

It is seen from Eq.(4.10), that the primary constraints form a subalgebra of the constraint algebra.

Now let us compute the Poisson brackets of the standard Hamiltonian and constraints with arbitrary standard function \( F \). It is useful to have the corresponding formulas from the point of view of possible applications (see e.g. [11, 17, 18]). To obtain these expressions, define the projector

\[
\Pi^r_s = \delta^r_s - \chi^\alpha_s u^\beta_r \psi^\alpha_r, \quad \Pi^r_s = \Pi^r_s,
\]
(4.15)

and introduce the so-called pseudo-inverse matrix \( W^{rs}(q, \dot{q}) \) [19, 20] for the singular Hessian \( W^{rs}(q, \dot{q}) \). It can be shown [20] that for any singular matrix \( W^{rs} \) there exists a pseudo-inverse matrix \( W^{rs} \), defined uniquely by the relations

\[
W^{rt} W^{rs} = \Pi^r_s, \quad W^{rs} \chi^\alpha_s = 0.
\]
(4.16)

Consider a standard function \( F(q, p) \) on the phase space. It is clear that there exists a function \( f(q, \dot{q}) \) on the velocity phase space, connected with \( F \) by the relation (3.3). Using the definitions of a standard function and the pseudo-inverse matrix (4.4), (4.15)−(4.17), we obtain the following expressions for the partial derivatives of the standard function \( F \) [13, 17]

\[
\frac{\partial F}{\partial p^r} = W^{rs} \frac{\partial f}{\partial q^s},
\]
(4.18)

\[
\frac{\partial F}{\partial q^r} = \frac{\partial f}{\partial q^r} - \frac{\partial f}{\partial q^r} W^{st} \frac{\partial^2 L}{\partial q^i \partial q^i},
\]
(4.19)

Now, making use of the expressions for the partial derivatives of the standard functions \( F \), \( H \), \( \Phi_\alpha \), \( \Phi_\alpha \), \( k = 1, \ldots, N > 1 \), we get the Poisson brackets of the form

\[
\{ H, F \} \equiv -T(f) + \mu^\alpha \left( \psi^\alpha_s \frac{\partial f}{\partial q^s} + \left( \psi^\alpha_r + T(\psi^\alpha_r) \right) \frac{\partial f}{\partial q^r} \right)
\]

\[\{ H, F \} \equiv -T(f) + \mu^\alpha \left( \psi^\alpha_s \frac{\partial f}{\partial q^s} + \left( \psi^\alpha_r + T(\psi^\alpha_r) \right) \frac{\partial f}{\partial q^r} \right)\]
\[-\mu^\alpha v_\alpha^\beta \left( \frac{\partial f}{\partial \dot{q}^r} W^{rs} \frac{\partial [u]^r_\beta}{\partial \dot{q}^s} \right) [1] \Lambda_\gamma, \tag{4.20}\]

\[\{ \Phi_\alpha \, F \} = \left[ u_\alpha^\beta \left( \psi^r_\beta \frac{\partial f}{\partial q^r} + (\psi^r_\beta + T(\psi^r_\beta)) \frac{\partial f}{\partial \dot{q}^r} \right) \right] - \left( \frac{\partial f}{\partial q^r} W^{rs} \frac{\partial [u]^r_\beta}{\partial \dot{q}^s} \right) [1] \Lambda_\beta, \tag{4.21}\]

where \( v_\alpha^\beta \) is the inverse matrix for the matrix \([u]^\alpha_\beta\), and

\[\{ \Phi_\alpha \, F \} = \left[ u_\alpha^\beta \left( \psi^r_\beta \frac{\partial f}{\partial q^r} + (\psi^r_\beta + T(\psi^r_\beta)) \frac{\partial f}{\partial \dot{q}^r} \right) \right] - \left( \frac{\partial f}{\partial q^r} W^{rs} \frac{\partial [u]^r_\beta}{\partial \dot{q}^s} \right) [1] \Lambda_\beta. \tag{4.22}\]

Here we have introduced the notation

\[T = \dot{q}^t \frac{\partial}{\partial q^t} + R^t W^{st} \frac{\partial}{\partial \dot{q}^s}. \tag{4.23}\]

Note that on the Lagrange equations \( L_r = 0 \) we have

\[T(f) = \frac{d}{dt}(f), \tag{4.24}\]

so the differential operator \( T \) is nothing but the evolution operator of gauge invariant systems.

Note again that the Eqs.\((4.20)\)–\((4.22)\) determine the Poisson brackets only on the primary constraint surface. To obtain the corresponding formulae in the total phase space, it is sufficient to apply Eq.\((4.3)\) for standardly extended Poisson brackets to these expressions.
5 Conclusion

In this paper we have considered the mechanical systems, which are invariant under gauge transformations of the form (2.4) and established the correspondence between Lagrangian and Hamiltonian descriptions of such systems. On the base of the notion of the standard extension we have obtained the explicit form of the constraint algebra, the latter turned out to be the first class.

The results of this paper and Refs.[4, 6] complete the consistent analysis of gauge invariant systems of general form, where only projectable Lagrangian constraints appear.

Note finally that the gauge transformations (2.4) are mapped to the phase space as follows (cf. Refs.[21, 4, 22, 7])

\[
\delta q^r = \{q^r, G_\varepsilon\}, \quad (5.1) \\
\delta p_r = \{p_r, G_\varepsilon\} + \frac{\partial \delta q^s}{\partial q^r} L_s, \quad (5.2)
\]

Where \(G_\varepsilon\) is the linear combination of the constraints

\[
G_\varepsilon = \sum_{k=0}^{N} \frac{[k]}{g^k_\alpha(\varepsilon) \Phi_\alpha}, \quad (5.3)
\]

\[
\frac{[0]}{g^0_\alpha(\varepsilon)} = - \left( (N)_{\beta} + \sum_{k=0}^{N-1} (k)_{\gamma} \frac{[N-k]}{u^\gamma_{\beta}} \right) v_{\alpha}^\beta, \quad (5.4)
\]

\[
\frac{[k]}{g^k_\alpha(\varepsilon)} = - \frac{(N-k)}{\varepsilon}, \quad k = 1, \ldots, N. \quad (5.5)
\]

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