Casimir effect across a layered medium

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Using nonstandard recursion relations for Fresnel coefficients involving successive stacks of layers, we extend the Lifshitz formula to configurations with an inhomogeneous, n-layered, medium separating two planar objects. The force on each object is the sum of a Lifshitz like force and a force arising from the inhomogeneity of the medium. The theory correctly reproduces very recently obtained results for the Casimir force/energy in some simple systems of this kind. As a by product, we obtain a formula for the force on an (unspecified) stack of layers between two planar objects which generalizes our previous result for the force on a slab in a planar cavity.

Keywords: Casimir force; Fresnel coefficients; nonstandard recursion.

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1. Introduction

Very recently, several papers appeared dealing with the theory of the Casimir effect in systems consisting of two perfectly reflecting plates separated by a layered medium. Among the other results, these works provided formulas for the Casimir force and/or energy for a few simple systems of this sort (with up to five layers medium between the plates). Using the theory of the Casimir effect in multilayers and nonstandard recursion relations for Fresnel coefficients, in this Note we derive formulas for the Casimir force and energy for systems with arbitrary plates separated by arbitrary inhomogeneous, generally n-layered, media.

2. Casimir effect across a layered medium

Consider the system consisting of two planar objects (plates) separated by a layered medium, as depicted in Fig. 1. According to the theory of the Casimir effect in multilayers, the Casimir forces on the left ($L$) and the right ($R$) plate are given by

$$ F_L = F_{1-} = T_{zz}^{(1)} \quad \text{and} \quad F_R = F_{n+} = -T_{zz}^{(n)}, $$

respectively, where (unless necessary, we omit the polarization index $q = p, s$ when writing Fresnel coefficients)

$$ T_{zz}^{(j)} = \frac{\hbar}{2\pi^2} \int_0^\infty d\xi \int_0^\infty dk k \kappa_j \sum_q \frac{r_j - r_{j+} e^{-2\kappa_j d_j}}{1 - r_j - r_{j+} e^{-2\kappa_j d_j}} $$

(2)
is the relevant component of the vacuum-field (Minkowski) stress tensor in the layer \( j \). Here \( \kappa_j = \sqrt{n_j^2(i\xi)\xi^2/c^2 + k^2} \) is the perpendicular wave vector at the imaginary frequency \( (\omega = i\xi) \) in the layer, \( k = \sqrt{k_x^2 + k_y^2} \) is the magnitude of the wave vector parallel to the system surfaces and \( r_{j\pm}(i\xi, k) \) are the reflection coefficients of the right and left stack of layers bounding the layer \( j \). These reflection coefficients obey generalized recursion relations \( 4, 5, 6, 7 \)

\[
r_{j\pm} = r_{j/l} + \frac{t_{j/l}t_{l\pm}e^{-2\kappa_j d_l}}{1 - r_{j/l}t_{l\pm}e^{-2\kappa_j d_l}}, \quad t_{j\pm} = \frac{t_{j/l}t_{l\pm}}{1 - r_{j/l}t_{l\pm}e^{-2\kappa_j d_l}},
\]

where \( l \) denotes an intermediate layer and where the symbol \( a/b \equiv a . . . b \) is used to denote the stack of layers between layers \( a \) and \( b \). As seen, these recurrence relations look the same as the standard ones \( 8, 9 \) (to which they reduce in case that layers \( j \) and \( l \) are neighbor layers), however, this time they generally involve Fresnel coefficients \( r_{j/l}, r_{l/j}, t_{j/l}, t_{l/j} \) of the stack between the layers \( j \) and \( l \).

Using the above recursion relations, reflection coefficients \( r_{1+} \) and \( r_{n-} \) can be expressed as

\[
\begin{align*}
    r_{1+} &= \frac{r_{1/n} + a_{1/n} R_{RE}^{-2\kappa_n d_n}}{1 - r_{1/n} R_{RE}^{-2\kappa_n d_n}}, \\
    r_{n-} &= \frac{r_{n/1} + a_{n/1} R_{LE}^{-2\kappa_1 d_1}}{1 - r_{n/1} R_{LE}^{-2\kappa_1 d_1}},
\end{align*}
\]

where we have introduced the quantity

\[
a_{1/n} = t_{1/n} t_{n/1} - r_{1/n} r_{n/1} = a_{n/1}
\]

and identified \( r_{n+} \) and \( r_{1-} \) as the reflection coefficients \( R_R \) and \( R_L \), respectively, of the plates. Therefore, from \( 1 \) and \( 2 \) we obtain for the forces on the plates

\[
\begin{align*}
    F_L &= \frac{\hbar}{2\pi^2} \int_0^\infty d\xi \int_0^\infty d\kappa K_1 \sum_q \frac{1}{N_n} (r_{1/n} + a_{1/n} R_{RE}^{-2\kappa_n d_n}) R_{RE}^{-2\kappa_1 d_1}, \\
    F_R &= -\frac{\hbar}{2\pi^2} \int_0^\infty d\xi \int_0^\infty d\kappa K_1 \sum_q \frac{1}{N_n} (r_{n/1} + a_{1/n} R_{LE}^{-2\kappa_1 d_1}) R_{LE}^{-2\kappa_n d_n},
\end{align*}
\]
where

$$N_n = 1 - (r_{1/n} R_L e^{-2 \kappa_1 d_1} + r_{n/1} R_e e^{-2 \kappa_n d_n}) - a_{1/n} R_L R_e e^{-2 \kappa_1 d_1 - 2 \kappa_n d_n}. \quad (7)$$

As seen, since $r_{1/n} \neq r_{n/1}$ unless the medium is not symmetric across the gap, these forces are not generally equal in magnitude and each of them consists of a Lifshitz-like force (given by the second terms in (6)) and a force due to the inhomogeneity of the medium. We also note that owing to the medium inhomogeneity there is a force for $F_S = T_z^{(n)} - T_z^{(1)} = -F_R - F_L$ on the central stack of the medium given explicitly by

$$F_S = \frac{h}{2 \pi^2} \int_0^\infty d\xi \int_0^\infty dk k \sum_q \frac{1}{N_n} (\kappa_n r_{n/1} R_L e^{-2 \kappa_n d_n} - \kappa_1 r_{1/n} R_L e^{-2 \kappa_1 d_1})$$

$$+ (\kappa_n - \kappa_1) a_{1/n} R_L R_e e^{-2 \kappa_1 d_1 - 2 \kappa_n d_n}. \quad (8)$$

When $n_1 = n_n$, this generalizes previously obtained result for the force on a slab in a planar cavity to configurations with the slab replaced by an unspecified multilayered stack.

Having determined forces on the plates, we can calculate the Casimir energy of the system from $F_L = \partial E / \partial d_1$ or $F_R = - \partial E / \partial d_n$. We obtain

$$E = \frac{\hbar}{(2\pi)^2} \int_0^\infty d\xi \int_0^\infty dk k \sum_q \ln N_n. \quad (9)$$

In the following, we illustrate this formula by discussing its implications for several simple systems and comparing them with the results obtained in Refs. [13]

### 3. Discussion

It is easy to see that (7) and (9) give correctly the Casimir energy in case of a homogeneous medium between the plates. Indeed, assuming that all $n$ medium layers are made of the same matter ($n_a = n$, $\kappa_a = \kappa$), we have $r_{1/n} = r_{n/1} = 0$ and $t_{1/n} = t_{n/1} = \exp[-\kappa(d - d_1 - d_n)]$ so that $a_{1/n} = \exp[-2\kappa(d - d_1 - d_n)]$ and

$$N_n = 1 - R_L R_e e^{-2\kappa d}, \quad (10)$$

where $d$ is the distance between the plates. This leads to the standard Lifshitz-type formula for the Casimir energy. For perfectly reflecting plates, we must let here $R_L R_e = 1$.

In the $n = 2$ case (two media between the plates), we have $r_{1/2} = r_{12} = -r_{21}$ and $t_{1/2} = t_{12} = (\mu_2 \kappa_1 / \mu_1 \kappa_2) t_{21} = (\mu_2 \kappa_1 / \mu_1 \kappa_2) t_{2/1}$, where $r_{12}$ and $t_{12}$ are the single-interface Fresnel coefficients,

$$r_{12} = \frac{\kappa_1 - \gamma_{12} \kappa_2}{\kappa_1 + \gamma_{12} \kappa_2} = -r_{21}, \quad t_{12} = \sqrt{\frac{\gamma_{12}}{\gamma_{12}} (1 + r_{12})} = \frac{\mu_2 \kappa_1}{\mu_1 \kappa_2} t_{21}, \quad (11)$$

with $\gamma_{12} = \varepsilon_1 / \varepsilon_2$ and $\gamma_{12} = \mu_1 / \mu_2$. Noting that $a_{1/2} = 1$, we have

$$N_2 = 1 - r_{12} (R_L e^{-2 \kappa_1 d_1} - R_e e^{-2 \kappa_2 d_2}) - R_L R_e e^{-2 (\kappa_1 d_1 + \kappa_2 d_2)}, \quad (12)$$
which, in conjunction with (9), gives the Casimir energy for the present system. This result coincides with the corresponding result obtained in Ref. [2] providing that we let for perfectly reflecting plates $R_L^p = -1$. We note, however, that perfect reflectors are standardly simulated by media with infinitely large permit-tivities (conductivities) in which case (11) implies ($\varepsilon_2 \to \infty$) that $R_L^p = -1$ but $R_L^p = 1$. Therefore, with this convention, our result disagrees with that of Ref. [2] regarding the $p$ contribution to the Casimir force/energy.

Using recursion relations (3), $E$ for more complex ($n \geq 3$) systems can be written in terms of lower-layered stacks and, owing to the number of the medium layers, this can be done in a number of ways. Clearly, to obtain the effective Casimir energy, we can drop from these results the terms not involving $d_1$ or $d_n$. Thus, for example, from (3) we have

$$ r_{1/n} = \frac{r_{1/l} + a_{1/l} r_{l/n} e^{-2\kappa_1 d_l}}{D_l}, \quad r_{n/1} = \frac{r_{n/l} + a_{n/l} r_{l/n} e^{-2\kappa_1 d_l}}{D_l}, \quad (13a) $$

$$ a_{1/n} = \frac{a_{1/l} a_{n/l} e^{-2\kappa_1 d_l} - r_{1/l} r_{n/l}}{D_l}, \quad D_l = 1 - r_{1/l} r_{l/n} e^{-2\kappa_1 d_l}. \quad (13b) $$

Using this in (7) and rearranging, we find

$$ N_n = N_n^{(l)} \frac{D_l}{l}, \quad (14) $$

where

$$ N_n^{(l)} = (1 - r_{1/l} R_L e^{-2\kappa_1 d_l})(1 - r_{n,l} R_R e^{-2\kappa_n d_n} - e^{-2\kappa_1 d_l}(a_{1/l} R_L e^{-2\kappa_1 d_l} + r_{1/l})(a_{n/l} R_R e^{-2\kappa_n d_n} + r_{n/l}). \quad (15) $$

Finally, inserting this $N_n$ in (7) and dropping the (ineffective) term involving $D_l$, we find for the effective Casimir energy $E_l$ of the system (with respect to the layer $l$)

$$ E_l = \frac{\hbar}{(2\pi)^2} \int_0^\infty d\xi \int_0^\infty d\kappa \sum_q \ln N_n^{(l)}. \quad (16) $$

This generalizes the ($T = 0$) result for the Casimir interaction energy between two slabs obtained in Ref [1] using a realistic Casimir piston approach and a five layer model for the medium to arbitrary multilayered slabs and plates. Note that, when removing the plates by letting $d_1(n) \to \infty$, we have $N_n^{(l)} \to D_l$ and (13) and (16) give the Casimir interaction energy of the two stacks of layers separated by a layer of medium $l$, as derived in Ref. [4].

We illustrate the above result by considering the $n=3$ system. In this case, there is only one intermediate layer and the effective Casimir energy (16) is entirely expressed in terms of the single-interface reflection coefficients $r_{12} = -r_{21}$ and $r_{32} = -r_{23}$. From (13), we have ($a_{1/2} = a_{3/2} = 1$)

$$ N_3^{(2)} = (1 - r_{12} R_L e^{-2\kappa_1 d_1})(1 - r_{32} R_R e^{-2\kappa_3 d_3}) - e^{-2\kappa_2 d_2}(R_L e^{-2\kappa_1 d_1} - r_{12})(R_R e^{-2\kappa_3 d_3} - r_{32}). \quad (17) $$
As mentioned, for perfectly reflecting plates we must let here $R^q_{L(R)} = \delta_{qp} - \delta_{qs}$. This result then coincides with the corresponding result derived in Ref. 1 whereas the results obtained in Refs. 2 and 3 correspond to plates with $R^q_{L(R)} = -1$ and $R^q_{L(R)} = 1$, respectively.

4. Summary
Effectively, in this work we have extended the Lifshitz formula to configurations with an inhomogeneous, n-layered, medium separating two planar objects. The force on each object is the sum of a Lifshitz-like force and a force arising from the inhomogeneity of the medium. Owing to this inhomogeneity, there is also a force acting on the medium. When the first and the last medium layer are made of the same matter, this result generalizes previously obtained one for the force on a slab in a planar cavity to arbitrary multilayered slabs.

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