Quantum Limits to Optical Point-Source Localization

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Many superresolution microscopic techniques rely on the accurate localization of optical point sources from far field. To investigate the fundamental limits to their resolution, here I derive measurement-independent quantum lower bounds on the error of locating point sources in free space, taking full account of the quantum, nonparaxial, and vectoral nature of photons. To arrive at analytic results, I focus mainly on the cases of one and two classical monochromatic sources with an initial vacuum optical state. For one source, a lower bound on the root-mean-square position estimation error is on the order of $\lambda_0/\sqrt{N}$, where $\lambda_0$ is the free-space wavelength and $N$ is the average number of radiated photons. For two sources, owing to a nuisance parameter effect, the error bound diverges when their radiated fields overlap significantly. The use of squeezed light to further enhance the accuracy of locating one classical point source and the localization limits for single-photon sources and partially coherent sources are also discussed.

The resolution limit of optical microscopes is one of the most important problems in science and engineering. The Abbe-Rayleigh criterion with respect to the free-space wavelength $\lambda_0$ has been a widely used resolution limit [1], but it is now well known that the criterion is heuristic and superresolution microscopy is possible. An important class of superresolution microscopy, including stimulated-emission-depletion microscopy [2] and photoactivated-localization microscopy [3], relies on the accurate localization of point sources from far field [4]. The localization accuracy is then limited by the statistics of the optical measurement [5, 6]. Prior analyses of point-source localization accuracy assume classical, scalar, and paraxial optics with statistics specific to the measurement methods [5, 6]. On a more fundamental level, however, optics is governed by the quantum theory of electromagnetic field [7], and the existence of more accurate measurement methods and more fundamental quantum limits remains an open question.

In this Letter, using a quantum Cramér-Rao bound (QCRB) [8, 9] and the full quantum theory of electromagnetic field [7], I derive quantum limits to the accuracy of locating point sources. These quantum resolution limits are more general and fundamental than prior classical analyses in the sense that they apply to any measurement method and take full account of the quantum, nonparaxial, and vectoral nature of photons. To arrive at analytic results, I focus mainly on the cases of one and two monochromatic classical sources and an initial vacuum optical state. The possibility of using squeezed light to further enhance the accuracy of locating one point source will also be discussed, while single-photon sources and partially coherent sources are treated in the Supplemental Material [10].

In quantum optics, there has been a substantial literature on quantum imaging; see, for example, Refs. [8, 11–20], but most of them assume certain quantum optical states without considering how they may be generated by objects relevant to microscopy or consider simply the estimation of mirror displacement. Helstrom’s derivation of the QCRB for one point source [8, 11] is the most relevant prior work, although he used the paraxial approximation, did not consider the use of squeezed light, and studied two sources only in the context of binary hypothesis testing [8]. There have also been intriguing claims of superresolution using the nonclassical photon statistics from single-photon sources [18, 19], but their protocols have not been analyzed using statistical inference, so even though their images appear sharper, the accuracies of their methods in estimating object parameters remain unclear. Localization limits for single-photon sources will be discussed in Ref. [10].

Let the initial quantum state of a system be $|\psi\rangle$. After unitary evolution $U(X, T)$ with respect to Hamiltonian $H(X, t)$ as a function of parameters $X = (X_1, X_2, \ldots)$, the quantum system is measured with outcome $Y$. The probability distribution of $Y$ according to Born’s rule can be expressed as [8, 9, 21]

$$P(Y|X) = \text{tr} \left[ E(Y) U(X, T)|\psi\rangle\langle\psi| U^\dagger(X, T) \right],$$  \hspace{1cm} (1)

where $E(Y)$ is the positive operator-valued measure (POVM) that characterize the quantum measurement and $\text{tr}$ denotes the operator trace.

Denote the estimator of $X$ using $Y$ as $\hat{X}(Y)$. The estimation error matrix is defined as

$$\Sigma_{\mu\nu}(X) \equiv \int dY P(Y|X) \left[ \hat{X}_\mu(Y) - X_\mu \right] \left[ \hat{X}_\nu(Y) - X_\nu \right].$$  \hspace{1cm} (2)

For unbiased estimators, the following QCRB holds for any POVM [8, 9]:

$$\Sigma(X) \geq J^{-1}(X),$$  \hspace{1cm} (3)

which means that $\Sigma - J^{-1}$ is positive-semidefinite. $J$ is the quantum Fisher information (QFI) matrix; it can be
obtained by expressing the fidelity \(|\langle \psi | U^\dagger(X, T)U(X + \delta X, T)|\psi \rangle|^2\) in the interaction picture [22] and expanding it to the second order of \(\delta X\) [23]. The result is

\[ J_{\mu\nu}(X) = 4 \text{Re} \langle \Delta g_\mu(X) \Delta g_\nu(X) \rangle, \]

where \(\text{Re}\) denotes the real part, \(\langle A \rangle \equiv \langle \psi | A |\psi \rangle\), \(\Delta A \equiv A - \langle A \rangle\), and

\[ g_\mu(X) \equiv \frac{1}{\hbar} \int_0^T dt U^\dagger(X, t) \frac{\partial H(X, t)}{\partial X_\mu} U(X, t) \]

is the generator of the parameter shift in the Heisenberg picture. For \(M\) trials, the QFI is simply multiplied by \(M\), and at least one component of the QCRB can be attained in an asymptotic \(M \to \infty\) sense [24]. Various generalizations of the QCRB and alternatives are available [8, 25, 26], but the presented version suffices to illustrate the pertinent physics. In Ref. [10], the QCRB is generalized to a Bayesian version that treats nuisance parameters separately and is used to study partially coherent sources.

Consider first a classical point source, as depicted in Fig. 1. The Hamiltonian is [7]

\[ H(r, t) = H_F + H_I(r, t), \]

\[ H_F = \sum_k \int d^3k \, \hbar \omega(k) a^\dagger(k, s) a(k, s), \]

\[ H_I(r, t) = -p(t) \cdot E(r), \]

\[ E(r) = \sum_k \int d^3k \sqrt{\frac{\hbar \omega(k)}{2(2\pi)^3 \epsilon_0}} (i\epsilon(k, s) a(k, s) e^{ik \cdot r} + \text{H.c.}) \]

where \(k = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}\) is a wavevector, \((\hat{x}, \hat{y}, \hat{z})\) denote unit vectors in the Cartesian coordinate system, \(\int d^3k \equiv \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z\), \(s\) is an index for the two polarizations, \(\epsilon(k, s)\) is a unit polarization vector, \(\omega(k) = c |k|\), \(c\) is the speed of light, \(p(t)\) is the \(c\)-number dipole moment of the source, \(r = x \hat{x} + y \hat{y} + z \hat{z}\) is its position, \(\epsilon_0\) is the free-space permittivity, \(a(k, s)\) is an annihilation operator obeying the commutation relation \([a(k, s), a^\dagger(k', s')] = \delta_{ss'} \delta(\mathbf{k} - \mathbf{k}')\), and H.c. denotes the Hermitian conjugate. Since \(p(t)\) is a \(c\)-number, \(H_F\) implements a field displacement operation [7]. The Heisenberg picture of \(a(k, s)\) is

\[ a(k, s, t) = U^\dagger(X, t) a(k, s) U(X, t) \]

\[ = e^{-i\omega t} \left[ a(k, s) + \alpha(k, s, r, t) \right], \]

where \(\alpha\) is the radiated field. Assuming \(p(t) = p_0 e^{-i\omega_0 t} + \text{c.c.}\), where c.c. denotes the complex conjugate, \(\alpha\) becomes

\[ \alpha(k, s, r, T) = \sqrt{\frac{\omega_0}{2(2\pi)^3 \hbar \epsilon_0}} e^{-i k \cdot r} \epsilon^*(k, s), \]

\[ = \left[ p_0 e^{i(\omega - \omega_0) T/2} \sin(\omega - \omega_0) T/2 \right] \frac{(\omega - \omega_0)/2}{(\omega + \omega_0)/2}, \]

which indicates that only the optical modes with \(\omega(k) = \omega_0\) grow in time, corresponding to the far field, while all the other near-field modes with \(\omega(k) \neq \omega_0\) oscillate. Assuming \(T \gg 2\pi/\omega_0\), such that \(\sin^2[(\omega \pm \omega_0)/2] = 2\pi T \delta(\omega \pm \omega_0)\), using the identity \(\sum_k \epsilon_\mu(k, s) \epsilon^*_\nu(k, s) = \delta_{\mu\nu} - k_\mu k_\nu/|k|^2\) [7], and switching to the spherical coordinate system for \(k\), it can be shown that the average number of radiated photons for an initial vacuum state is

\[ N \equiv \sum_s \int d^3k |a(k, s, r, T)|^2 \approx \frac{|p_0|^2 \omega_0^3 T}{3\pi \hbar \epsilon_0 c^5}. \]

The far-field limit \((\omega_0 T \to \infty)\) will be assumed hereafter. I now focus on two representative cases: a linearly polarized dipole with \(p_0 = p_0 \hat{z}\) and a circularly polarized dipole with \(p_0 = p_0 \sqrt{2} (\hat{x} + i \hat{y})/\sqrt{2}\). Taking the unknown parameters to be \(r\), the generators for \(X = (x, y, z)\) can be expressed, after some algebra, as

\[ \Delta g_\mu(r) = -\sqrt{\frac{2}{W_\mu}} \Delta P_\mu(r), \quad \mu \in \{x, y, z\}, \]

\[ \Delta P_\mu(r) \equiv \frac{1}{\sqrt{2}} \left[ \Delta b_\mu(r) - \Delta b_\mu^*(r) \right], \]

where \(\Delta b_\mu\) is a normalized annihilation operator defined as

\[ \Delta b_\mu(r) \equiv W_\mu \sum_s \int d^3k \left[ -ik_\mu a^*(k, s, r, T) \right] \Delta a(k, s), \]

such that \([\Delta b_\mu(r), \Delta b^*_\nu(r)] = \delta_{\mu\nu}\), and the normalization constants \(W_\mu\) are

\[ W_\mu \equiv \left[ \sum_s \int d^3k k_\mu |a_\mu(k, s, r, T)|^2 \right]^{-1/2}. \]
The $d^3k$ integrals can again be computed with the help of the far-field assumption and spherical coordinates. The results depend on $p_0$; for $p_0 = p_0 \hat{z}$,

$$W_x = W_y = \sqrt{\frac{5}{2} \frac{\lambda_0}{2\pi \sqrt{N}}}, \quad W_z = \sqrt{\frac{5}{2} \frac{\lambda_0}{2\pi \sqrt{N}}}$$

(18)

and for $p_0 = p_0(\hat{x} + i\hat{y})/\sqrt{2}$,

$$W_x = W_y = \sqrt{\frac{10}{3} \frac{\lambda_0}{2\pi \sqrt{N}}}, \quad W_z = \sqrt{\frac{5}{2} \frac{\lambda_0}{2\pi \sqrt{N}}}$$

(19)

but the important point here is that they are all on the order of $\lambda_0/\sqrt{N}$, where $\lambda_0 \equiv 2\pi c/\omega_0$ is the free-space wavelength. The QFI becomes

$$J_{\mu\nu}(r) = \frac{8}{W_\mu} \langle \Delta P_\mu(r) \Delta P_\nu(r) \rangle.$$

(20)

For an initial vacuum state (or any coherent state), $\langle \Delta P_\mu(r) \Delta P_\nu(r) \rangle = \delta_{\mu\nu}/2$, and the QCRB is hence

$$J_{\mu\nu}(r) = \frac{4}{W_\mu} \delta_{\mu\nu}, \quad \Sigma_{\mu\nu}(r) \geq \frac{W_\mu^2}{4},$$

(21)

meaning that the quantum resolution limit in terms of the root-mean-square error is on the order of $\lambda_0/\sqrt{N}$. I call this limit the quantum shot-noise limit. It is on the same order as the limits for optical centroid measurements derived in Refs. [5, 15] when the measured photons have no spatial correlation and the numerical aperture approaches 1. Ref. [10] shows that a single-photon source also obeys this limit with repeated trials. Generalization to lossless media is straightforward and results simply in $\lambda_0$ being replaced by the wavelength in the medium.

Even though the source is classical, quantum enhancement is possible if the initial state $|\psi\rangle$ is nonclassical, as I now show. If $\Delta g_\mu$ were independent of the parameter, the accuracy could be enhanced by squeezing and measuring the conjugate quadrature [27]. Although $\Delta g_\mu(r)$ depends on the unknown $r$ here, the radiated field can be approximated as $\alpha(k, s, r, t) \approx \alpha(k, s, r_0, T)$, resulting in $\Delta g_\mu(r) \approx \Delta g_\mu(r_0)$, provided that

$$|r - r_0| \ll \lambda_0$$

(22)

with respect to a known reference position $r_0$. The acquisition of such prior information will require a fixed amount of overhead resource, but once it is done, one can squeeze the quadrature

$$\Delta Q_\mu(r_0) \equiv \frac{1}{\sqrt{2}} \left[ \Delta b_\mu(r_0) + \Delta b_\mu^\dagger(r_0) \right]$$

(23)

in the initial state and perform a homodyne measurement of $\Delta Q_\mu(r_0)$ to estimate $r$ much more accurately. Since $[\Delta Q_\mu(r_0), \Delta Q_\nu(r_0)] = 0$, all three quadratures can be squeezed and measured simultaneously in principle. The estimation error becomes

$$\Sigma_{\mu\nu}(r) \approx \frac{W_\mu^2}{2} \langle \Delta Q_\mu^2(r_0) \rangle,$$

(24)

and the error reduction below the shot-noise limit is determined by the squeezing factor, which is limited by the average photon number $N_0$ in the initial state (not to be confused with $N$). Using $\langle \Delta Q_\mu^2(r_0) \rangle + \langle \Delta P_\mu^2(r_0) \rangle \leq 2N_0 + 1$ and the uncertainty relation $\langle \Delta Q_\mu^2 \rangle \langle \Delta P_\mu^2 \rangle \geq 1/4$, it can be shown that

$$\langle \Delta Q_\mu^2(r_0) \rangle \geq \frac{f(N_0)}{2}, \quad \langle \Delta P_\mu^2(r_0) \rangle \leq \frac{1}{2f(N_0)}.$$

(25)

$$f(N_0) \equiv (2N_0 + 1) \left[ 1 - \sqrt{1 - (2N_0 + 1)^{-2}} \right].$$

(26)

where $f(0) = 1$ and $f(N_0) \approx 1/(4N_0)$ for $N_0 \gg 1$. With a zero-mean minimum-uncertainty state and all initial photons in the $\Delta b_\mu(r_0)$ mode, the estimation error becomes

$$\Sigma_{\mu\nu}(r) \approx \frac{W_\mu^2}{2} \langle \Delta Q_\mu^2(r_0) \rangle = \frac{W_\mu^2}{4} f(N_0).$$

(27)

The enhancement factor $f(N_0)$ is optimal, as the QCRB can be further bounded by

$$\Sigma_{\mu\nu}(r) \geq J_{\mu\nu}^{-1}(r) = \frac{W_\mu^2}{8\langle \Delta P_\mu^2(r) \rangle} \geq \frac{W_\mu^2}{4} f(N_0).$$

(28)

The optical mode to be squeezed has a profile $ik_\mu \alpha(k, s, r_0, T)$. This means that, in real space, the electric field profile of the mode should be the spatial derivative of the radiated field. This kind of squeezing and measurement has actually been demonstrated experimentally, albeit in the paraxial regime, by Taylor et al. in the context of particle tracking [20], where the weak scatterer under a strong pump can be modeled as a classical source, similar to the implementation of field displacement by a beam splitter [28], and the spatial mode profile of the squeezed light and the local oscillator is a spatial derivative of the scattered field. To realize an enhancement in practice, accurate phase locking of the squeezed light and the local oscillator to the radiated field and a high measurement efficiency are crucial.

Next, consider two classical point sources at $r$ and $r'$, as shown in Fig. 2. The Hamiltonian is now

$$H(r, r', t) = H_F + H_I(r, r', t),$$

(29)

$$H_I(r, r', t) = -\mathbf{p}(t) \cdot \mathbf{E}(r) - \mathbf{p}'(t) \cdot \mathbf{E}(r').$$

(30)

The Heisenberg picture of $\alpha(k, s)$ becomes $a(k, s, t) = e^{-itq}[a(k, s) + \alpha(k, s, r, t) + \alpha'(k, s, r', t)]$, where $\alpha$ and $\alpha'$ are the radiated fields from the two sources, $\alpha$ is the same as before, and $\alpha'$ has the same expression as $\alpha$ except that $\mathbf{p}$ is replaced by $\mathbf{p}'$ and $r$ by $r'$. One can then follow the preceding procedure to obtain the QCRB for estimating...
and $r'$, To highlight the important physics, however, consider here the estimation of just two parameters $X = (x, x')$. The generators $\Delta g_{xx}$ and $\Delta g_{x'x'}$ may not commute, and the QFI matrix for an initial vacuum or any coherent state now has off-diagonal components:

$$J_{xx'}(X) = J_{x'x}(X)$$
$$= 4 \text{Re} \sum_s \int d^3 k k_2^2 \alpha^*(k, s, r, T) \alpha'(k, s, r', T),$$

(31)

while $J_{xx}$ remains the same and $J_{x'x'}$ has a similar expression to $J_{xx}$. $J_{xx}$ and $J_{x'x'}$ still obey a shot-noise scaling with the average photon number, but the nonzero off-diagonal components mean that the parameters act as nuisance parameters to each other, and the QCRB with respect to, say, $x$ is always raised:

$$\Sigma_{xx}(X) \geq \frac{1}{J_{xx}[1 - \kappa(X)]},$$

(32)

where the resolution degradation factor, defined as

$$\kappa(X) = \frac{J_{xx'}^2(X)}{J_{xx} J_{x'x'}} = \frac{(\text{Re} \sum_s \int d^3 k k_2^2 \alpha^* \alpha')^2}{\sum_s \int d^3 k k_2^4 |\alpha|^2 \sum_s \int d^3 k k_2^4 |\alpha'|^2},$$

(33)

is within the range $0 \leq \kappa \leq 1$ and determined by the overlap between the two differential mode profiles. The nuisance-parameter effect generalizes the Rayleigh criterion and other classical results [6] by revealing a fundamental measurement-independent degradation of resolution for two point sources with overlapping radiation.

For example, Fig. 3 plots $\kappa$ against $|x - x'|/\lambda_0$, assuming $p = p' = p_0 e^{-i \omega t} + \text{c.c.}$, $T \gg 2 \pi/\omega_0$, $p_0 = p_0 \hat{x}$, $y = y'$, and $z = z'$. $\kappa \approx 0$ for $|x - x'| \gg \lambda_0$, as expected, but it approaches 1 and leads to a diverging QCRB when $|x - x'| \ll \lambda_0$. Ref. [10] shows that the degradation effect should still exist for single-photon sources and partially coherent sources.

The degradation effect can be avoided by minimizing the overlap before each source is located independently. The overlap can be reduced by making the radiated fields separate in space, time, frequency, quadrature, or polarization; time multiplexing of point sources has especially been the key driver in current superresolution microscopy [2–4]. When the overlap is unavoidable or when the generators do not commute, heterodyne measurements can still be used to measure both quadratures of $a(k, s, T)$ and should have a classical Fisher information within a factor of 1/2 of the QFI. Quantum enhancement may also be possible using entangled squeezed states [29]; the specific experimental design will be left for future studies.

In conclusion, I have derived quantum limits to point-source localization using the QCRB and quantum electromagnetics. These results not only provide general no-go theorems concerning the microscope resolution but should also motivate further progress in microscopy through classical or quantum techniques beyond the current assumptions. For example, the presented theory may be applied to other more exotic optical states interacting with quantum sources, such as multilevel atoms [7, 30], quantum dots [18, 31], diamond defects [19, 32], and second-harmonic nanoparticles [33]. It is also possible to generalize the current formalism for open quantum systems [34] to account for mixed states, decoherence, optical losses, and imperfect measurement efficiency. This work is supported by the Singapore National Research Foundation under NRF Grant No. NRF-NRFF2011-07.
Consider first an initially excited two-level atom in free space. A detailed analysis of atom-photon interaction is formidable [7, 35], but when the initial optical state is vacuum, spontaneous emission can be treated more easily, as the atom must decay to ground state in the long-time limit and the final optical state must contain exactly one photon. Using the continuous Fock space [7], the final optical state in the Schrödinger picture can be written with respect to the vacuum $|0\rangle$ as

$$|\Psi\rangle = c^\dagger|0\rangle,$$

where

$$c^\dagger = \sum_s d^3k \phi(k, s) a^\dagger(k, s),$$

and

$$\phi(k, s) = \langle k, s|\Psi = \langle 0|a(k, s)|\Psi\rangle$$

is the photon configuration-space amplitude. $L_\mu$ can be obtained by considering

$$\frac{\partial}{\partial X_\mu}|\Psi\rangle = \frac{\partial c^\dagger}{\partial X_\mu}|0\rangle = \frac{\partial c^\dagger}{\partial X_\mu} (cc^\dagger)^{-1} cc^\dagger|0\rangle = \frac{\partial c^\dagger}{\partial X_\mu} (cc^\dagger)^{-1} c|\Psi\rangle.$$ 

The QFI is then

$$J_{\mu\nu} = 4 \text{Re}(\langle 0| \frac{\partial c^\dagger}{\partial X_\mu} \frac{\partial c}{\partial X_\nu} |0\rangle) = 4 \text{Re} \sum_s d^3k \frac{\partial \phi^*(k, s)}{\partial X_\mu} \frac{\partial \phi(k, s)}{\partial X_\nu}.$$ 

Following Chap. III.C of Ref. [35], the photon amplitude can be expressed as

$$\phi(k, s) = \frac{\langle k, s| \otimes \langle g| H_1 |e\rangle |0\rangle}{\hbar[\omega - \tilde{\omega}_0 + i/(2T_1)]} e^{-i\omega T},$$

$$H_1 = i\omega_0 (\mu_{12}\sigma - \mu_{12}^*\sigma^\dagger) \cdot A(r),$$

$$A(r) = \sum_s d^3k \frac{\hbar}{2(2\pi)^3\omega_0} \left[a(k, s)e(k, s)e^{ik\cdot r} + \text{h.c.}\right],$$

$$T_1 = \frac{3\pi\hbar\epsilon_0 c^3}{|\mu_{12}|^2\omega_0^2} \frac{1}{i(\omega - \tilde{\omega}_0) + 1/(2T_1)} \sqrt{\frac{\omega_0^2}{2(2\pi)^3\hbar\epsilon_0\mu_{12} \cdot \varepsilon^*(k, s)}} e^{-ik\cdot r - i\omega T}.$$ 

The QFI is now

$$J_{\mu\nu} = 4 \text{Re} \sum_s d^3k \frac{\partial \phi^*(k, s)}{\partial X_\mu} \frac{\partial \phi(k, s)}{\partial X_\nu}.$$ 

Assuming that the decay time is much longer than the optical period ($T_1 \gg 2\pi/\tilde{\omega}_0$) and the Lamb shift is much smaller than the optical $\omega_0$ ($\tilde{\omega}_0 \approx \omega_0$), the QFI becomes

$$J_{\mu\nu} \approx 4 \text{Re} \sum_s d^3k \frac{k_\mu k_\nu}{(\omega - \tilde{\omega}_0)^2 + 1/(4T_1^2)} \frac{\omega_0^2}{2(2\pi)^3\hbar\epsilon_0} \left|\mu_{12} \cdot \varepsilon^*(k, s) \right|^2.$$ 

This turns out to be identical to the QFI derived in the main text for an $N = 1$ classical source:

$$J_{\mu\nu} = \frac{4\delta_{\mu\nu}}{NW_\mu^2} \approx \frac{\delta_{\mu\nu}}{\lambda_0^2}, \quad \Sigma_{\mu\nu} \geq J_{\mu\nu}^{-1} = \frac{NW_\mu^2}{4} \approx \lambda_0^2.$$

Single-photon sources
where $N$ and $W_\mu$ are defined in the main text. Superresolution beyond the classical Abbe-Rayleigh limit can still be obtained, however, if the experiment can be repeated. The QFI is then multiplied by the number of trials $M$, which is also the total number of emitted photons, and the resulting QCRB is identical to that for a classical source with $M$ replacing $N$. The experiments reported by Refs. [18, 19] certainly involved a large number of measurements of many photons in total, which can explain the apparent superresolution, but it remains to be seen whether their methods are accurate or efficient in estimating object parameters.

For atoms with different resonance frequencies, orthogonal dipole moments, or large separation ($|r - r'| \gg \lambda_0$), the one-atom analysis is expected to be applicable to each atom independently. The analysis of two close two-level atoms is much more challenging because of the Dicke superradiance effect [7]. Two initially excited atoms can undergo the superradiant transition

$$|e\rangle \otimes |e\rangle \rightarrow \frac{1}{\sqrt{2}} (|e\rangle \otimes |g\rangle + |g\rangle \otimes |e\rangle) \rightarrow |g\rangle \otimes |g\rangle$$  \hspace{1cm} (46)

or the subradiant transition

$$|e\rangle \otimes |e\rangle \rightarrow \frac{1}{\sqrt{2}} (|e\rangle \otimes |g\rangle - |g\rangle \otimes |e\rangle) \rightarrow |g\rangle \otimes |g\rangle$$  \hspace{1cm} (47)

with different decay rates depending on the separation. Regardless, the number of spontaneously emitted photons from two atoms is at most 2 each time, and it is likely that a large number of trials are needed to obtain significant resolution enhancement, in which case the QCRB must still obey a shot-noise scaling with respect to $M$.

A simplified treatment of the spontaneously emitted photons can be done if $|r - r'| \ll \lambda_0$, such that the superradiant decay rate $1/T_+ \approx 2/T_1$ is much faster than the subradiant decay rate $1/T_- \approx 0$, and the two photons can be assumed to come only from the superradiant transition. A reasonable guess of the superradiant two-photon state is then

$$|\Psi\rangle \approx \frac{1}{\sqrt{2}} c_+^2 |0\rangle,$$  \hspace{1cm} (48)

$$c_+ = \sum_s \int dk \phi_+(k, s) a^\dagger(k, s),$$  \hspace{1cm} (49)

$$\phi_+ \propto \delta(\omega - \omega_0) \mu_{12} \cdot e^*(k, s) \left(e^{-ik\cdot r} + e^{-ik\cdot r'}\right),$$  \hspace{1cm} (50)

where the monochromatic approximation $T_+ \gg 2\pi/\omega_0$ is assumed and $\phi_+$ arises from a constructive interference between the fields from the atoms, as suggested by perturbation theory. The QFI can be determined as follows:

$$\frac{\partial}{\partial X_\mu} |\Psi\rangle = 2 \frac{\partial c_+^\dagger}{\partial X_\mu} \frac{1}{\sqrt{2}} c_+^\dagger |0\rangle = 2 \frac{\partial c_+^\dagger}{\partial X_\mu} (c_+ c_+^\dagger)^{-1} c_+ |\Psi\rangle,$$  \hspace{1cm} (51)

$$J_{\mu \nu} = 8 \text{Re} (0|c_+ \frac{\partial c_+^\dagger}{\partial X_\mu} \frac{\partial c_+^\dagger}{\partial X_\nu} c_+^\dagger |0\rangle).$$  \hspace{1cm} (52)

After some algebra,

$$J_{\mu \nu} = 8 \text{Re} \sum_s \int d^3k \frac{\partial \phi_+^\dagger}{\partial X_\mu} \frac{\partial \phi_+}{\partial X_\nu}.$$  \hspace{1cm} (53)

For the estimation of $x$ and $x'$,

$$J_{xx} = J_{xx'} = \frac{8}{N_2} \sum_s \int d^3k k_2^2 |\alpha(k, s, r, T_+)|^2,$$  \hspace{1cm} (54)

$$J_{xx'} = \frac{8}{N_2} \text{Re} \sum_s \int d^3k k_2^2 \alpha^*(k, s, r, T_+) \alpha(k, s, r', T_+),$$  \hspace{1cm} (55)

$$N_2 \equiv \sum_s \int d^3k |\alpha(k, s, r, T_+) + \alpha(k, s, r', T_+)|^2.$$  \hspace{1cm} (56)

This QFI matrix is the same as that for two identical in-phase classical sources that radiate two photons on average and the QCRB exhibits the same resolution degradation effect.

Beyond the current assumption of spontaneous emission, it will be interesting, though highly nontrivial, to analyze the interaction between two-level atoms and other states of light, such as coherent states or squeezed states, and investigate their quantum localization limits and the possibility of quantum enhancement.
Bayesian quantum Cramér-Rao bound with nuisance parameters

There are many ways to generalize the classical Cramér-Rao bound when nuisance parameters are present [36]. The main text shows one way, which includes the nuisance parameters as part of the wanted parameters $X$. To derive tighter bounds for other nuisance parameters, here I start with a Bayesian QCRB and generalize a classical approach by Miller and Chang [36, 37]. Let $Z$ be a set of nuisance parameters, and suppose first that $Z$ is given. The estimation error matrix is

$$\Sigma_{\mu\nu}(Z) \equiv \int dX dY P(Y|X, Z) P_X(X|Z) \left[ \tilde{X}_\mu(Y, Z) - X_\mu \right] \left[ \tilde{X}_\nu(Y, Z) - X_\nu \right],$$

(57)

where $P_X(X|Z)$ is the prior distribution of $X$ conditioned on $Z$. A Bayesian quantum Cramér-Rao bound valid for any estimator is given by [25]

$$\Sigma(Z) \geq J^{-1}(Z),$$

(58)

$$J(Z) = E_{X|Z} [J(X|Z)] + j(Z),$$

(59)

where $J(X|Z)$ is the same QFI as before, except that it is now conditioned on $Z$, $E_{X|Z}$ denotes expectation over $P_X(X|Z)$, and $j(Z)$ is a prior Fisher information defined as

$$j_{\mu\nu}(Z) \equiv \int dXP_{X|Z}(X|Z) \left[ \frac{\partial}{\partial X_\mu} \ln P_X(X|Z) \right] \left[ \frac{\partial}{\partial X_\nu} \ln P_X(X|Z) \right].$$

(60)

If $Z$ is a random parameter with prior distribution given by $P_Z(Z)$, the estimation error is

$$\Pi_{\mu\nu} \equiv E_Z \left[ \tilde{\Sigma}'(Z) \right],$$

(61)

$$\tilde{\Sigma}'(Z) \equiv \int dX dY P(Y|X, Z) P_X(X|Z) \left[ \tilde{X}_\mu(Y) - X_\mu \right] \left[ \tilde{X}_\nu(Y) - X_\nu \right],$$

(62)

where $E_Z$ denotes expectation over $P_Z$ and the estimator $\tilde{X}_\mu(Y)$ can no longer depend on $Z$. The lower bound in Eq. (58) still holds for $\tilde{\Sigma}'(Z)$, so one can obtain a lower bound on $\Pi$ given by

$$\Pi \geq E_Z \left[ J^{-1}(Z) \right].$$

(63)

The important feature of this bound is that the expectation with respect to the nuisance parameter $Z$ is taken after the inverse of the conditional QFI matrix. This can sometimes lead to a tighter bound than a QCRB that includes $Z$ as part of $X$. Note also that this Bayesian bound is valid for any estimator, not just the unbiased ones, unlike the claim in Ref. [37]. The tightness of the bound should depend on whether the nuisance parameters can be accurately estimated from the measurements.

Classical partially coherent sources

I now use the new bound to study partially coherent sources. First, consider the example of one point source in the main text, but suppose that the complex dipole amplitude $p_0$ is unknown. Assuming $Z = p_0$, the quantum state before measurement is

$$\rho = \int d^2 p_0 P_Z(p_0) U(X, p_0, T) \rho_0 U^\dagger(X, p_0, T).$$

(64)

If $p_0$ is a vacuum state, $U p_0 U^\dagger$ is a coherent state, and $\rho$ is a classical mixed state of light with $P_Z(p_0)$ determining the Sudarshan-Glauber $P$ representation [7]. The random $p_0$ therefore gives rise to a classical partially coherent source model. For an initial vacuum, $J(p_0)$ is given by

$$J_{\mu\nu}(p_0) = \frac{4}{W_\mu^2(p_0)} \delta_{\mu\nu} + j_{\mu\nu} = \frac{N(p_0)}{C_\mu \lambda_0^2} \delta_{\mu\nu} + j_{\mu\nu},$$

(65)
where $W_\mu$ and $N$ now depend on the unknown dipole moment and $C_\mu$ is a constant on the order of 1 (see main text). Assuming that $j$ is diagonal and independent of $p_0$ and taking the inverse and then the expectation according to Eq. (63), I obtain
\begin{equation}
\Pi_{\mu\mu} \geq E_{p_0} \left[ \frac{1}{4/W_\mu^2(p_0) + j_{\mu\mu}} \right] = E_{p_0} \left[ \frac{1}{N(p_0)/(C_\mu \lambda_0^2) + j_{\mu\mu}} \right].
\end{equation}

For example, if $P_Z(p_0)$ corresponds to the $P$ representation of a thermal source, the bound can be written in terms of the average radiated photon number $N$ as
\begin{equation}
E_{p_0} \left[ \frac{1}{N(p_0)/(C_\mu \lambda_0^2) + j_{\mu\mu}} \right] = \frac{C_\mu \lambda_0^2}{N} \int_0^\infty dN \exp \left( -\frac{N}{N} \right) \frac{1}{N + C_\mu \lambda_0^2 j_{\mu\mu}}
\approx \frac{C_\mu \lambda_0^2}{N} \ln \frac{N}{C_\mu \lambda_0^2 j_{\mu\mu}},
\end{equation}
where $\lambda_0$ is a constant on the order of 1 (see main text).

An alternative Bayesian QCRB can be obtained by including $p_0$ as part of $X$. In that case, the off-diagonal QFI elements between $p_0$ and $r$ turn out to be zero, and the expectation with respect to $p_0$ is taken before the inverse, leading to a bound given by $C_\mu \lambda_0^2 / N$. The separate treatment of $p_0$ as nuisance parameter here involves taking the expectation after the inverse, giving rise to an additional factor $\sim \ln N$ and thus a tighter bound for large $N$.

Next, consider the example of two point sources in the main text, and $Z = (p_0, p'_0)$ is now assumed to be unknown to model partially coherent sources. Assuming that $j$ is diagonal and $X = (x, x')$ is independent of $Z$,
\begin{align}
\tilde{J}_{xx}(p_0) &= \frac{4}{W_x^2(p_0)} + j_{xx}, \\
\tilde{J}_{xx'}(p'_0) &= \frac{4}{W_{x'}^2(p'_0)} + j_{xx'}, \\
\tilde{J}_{x'x}(p_0, p'_0) &= E_X [J_{x'x}(p_0, p'_0)].
\end{align}
$\tilde{J}_{xx'}$ is now the expectation of $J_{xx'}$ over $X = (x, x')$, conditioned on the dipole moments. If the two point sources are a priori known to be close relative to $\lambda_0$, $\tilde{J}_{xx'}(p_0, p'_0)$ can still have a significant magnitude for certain $(p_0, p'_0)$. The bound given by Eq. (63) becomes
\begin{equation}
\Pi_{xx} \geq E_{(p_0, p'_0)} \left[ \frac{1}{J_{xx}(p_0)[1 - \bar{\kappa}(p_0, p'_0)]} \right],
\end{equation}
where a new resolution degradation factor is defined as
\begin{equation}
\bar{\kappa}(p_0, p'_0) = \frac{J_{xx'}^2(p_0, p'_0)}{J_{xx}(p_0)J_{x'x}(p_0, p'_0)}.
\end{equation}

The important point here is that $\bar{\kappa}(p_0, p'_0)$ can still be close to 1 for certain values of $(p_0, p'_0)$ if the two sources are a priori known to be close to each other relative to $\lambda_0$, $1/[1 - \bar{\kappa}(p_0, p'_0)] \gg 1$ is possible, and the expectation over $(p_0, p'_0)$ will then be dominated by those large values. This means that the resolution degradation effect derived for coherent sources must still exist for partially coherent sources if their radiated fields may have significant overlap.

An alternative Bayesian QCRB that includes $(p_0, p'_0)$ as part of $X$ can again be computed, but at some stage it involves taking the expectation of $\tilde{J}_{xx'}$ with respect to $(p_0, p'_0)$ before the inverse. For incoherent sources, this can reduce the off-diagonal components significantly; the resulting bound, while still valid, would no longer demonstrate the resolution degradation effect.