EM–GRADED RINGS

TARIQ ALRAQAD, HICHAM SABER, AND RASHID ABU-DAWWAS

ABSTRACT. The main goal of this article is to introduce the concept of EM –$G$–graded rings. This concept is an extension of the notion of EM–rings. Let $G$ be a group and $R$ be a $G$–graded commutative ring. The $G$–gradation of $R$ can be extended to $R[x]$ by taking the components $(R[x])_{σ} = R_{σ}[x]$. We define $R$ to be EM–$G$–graded ring if every homogeneous zero divisor polynomial has an annihilating content. We provide examples of EM–$G$–graded rings that are not EM–rings and we prove some interesting results regarding these rings.

1. Introduction

Throughout this article, all rings are commutative rings with one. The set of all zero divisors of a ring $R$ is $Z(R)$, the set of all regular elements is denoted by $reg(R)$, and the set of all idempotent elements is denoted by $E(R)$. A zero divisor polynomial $f(x) \in R[x]$ is said to have an annihilating content if there are $c_{f} \in Z(R)$ and $f_{1} \in reg(R[x])$ such that $f(x) = c_{f}f_{1}(x)$. Annihilating contents simplify computations of annihilator of polynomials. For instance, in $[3]$ annihilating contents are used to obtain some results related to the zero divisors graph of $R[x]$, where $R$ is a principal ideal rings. As a generalization, Abuosba and Ghanem $[4]$ introduce the concept of EM–rings as follows: A ring $R$ is called EM–ring if every zero divisor polynomial in $R[x]$ has an annihilating content. These rings were studied extensively in $[4]$ and $[10]$. Our goal in this article is to extend the notion of EM–rings by using the concept of graded rings. Let $G$ be a group. A ring $R$ with unity 1, is said to be $G$–graded if there exist additive subgroups $\{R_{σ} \mid σ \in G\}$ such that $R = \oplus_{σ \in G} R_{σ}$ and $R_{σ}R_{τ} \subseteq R_{στ}$ for all $σ, τ \in G$. This gradation is denoted by $(R, G)$. The elements of $R_{σ}$ are called homogeneous of degree $σ$. The set of all homogeneous elements is denoted by $h(R)$, and the set of homogenous zero divisors is denoted by $hZ(R)$. If $x \in R$, then $x$ can be written uniquely as $\sum_{σ \in G} x_{σ}$, where $x_{σ}$ is the component of $x$ in $R_{σ}$.

If $R$ is a $G$–graded ring, then $R[x]$ is $G$–graded with gradation defined by $(R[x])_{σ} = R_{σ}[x]$, $σ \in G$. Clearly, $R[x] = \oplus_{σ \in G} R_{σ}[x]$, and $R_{σ}[x]R_{τ}[x] \subseteq R_{στ}[x]$, for all $σ, τ \in G$. Furthermore, $f(x) = \sum_{i=0}^{n} a_{i}x^{i} \in h(R[x])$ if and only if there exists $σ \in G$ such that $a_{i} \in R_{σ}$ for all $i$, i.e. all coefficients of $f$ are homogenous of the same degree. We will say that $R$ is an EM –$G$–graded ring if every zero divisor homogeneous polynomial in $R[x]$ has an annihilating content. It is evident that every EM–ring (that is $G$–graded) is an EM –$G$–graded ring, however the converse is not true as we will see in Examples $5.2$ and $5.3$.

The next section includes some preliminary results on graded rings and EM–rings that will be needed in the following section. Section $5$ is devoted to the concept of

2010 Mathematics Subject Classification. 13A02, 16W50.

Key words and phrases. Graded rings, EM-rings,, homogeneous, polynomial ring, annihilating content.
EM – G–graded rings and its properties. We show that if \( R_e \) is an EM–ring and the other components satisfy some certain condition then \( R \) is EM – G–graded ring. We deduce that if \((R, G)\) is a cross product and \( R_e \) is an EM–ring then \( R \) is an EM – G–graded ring. Some extensions of EM – G–graded rings are also investigated. Some of these results are similar to results obtained about EM–ring in \([4]\). We show that if \( R \) is EM – G–graded ring then so is \( R[x] \). We also prove that if \( R \) is an EM – G–graded ring and \( S \subseteq h(R) \) is multiplicatively closed then \( S^{-1}R \) is EM – G–graded ring. In this section we also obtain a nice result related to the idealization \( R(+)R \) that generalizes \([10]\, \text{Theorem 5.1}]\). We show that \( R \) is an EM–ring if and only if \( R(+)R \) is an EM – Z_2–graded ring with gradation \( H_0 = R \oplus 0 \) and \( H_1 = 0 \oplus R \).

2. Preliminaries

In this section we list some results on graded rings and EM–rings that will be needed in the sequel. For a general reference on graded rings the reader is referred to \([11]\).

**Definition 2.1.** Let \( G \) be a group. A ring \( R \) with unity 1, is said to be \( G \)-graded if there exist additive subgroups \( \{R_{\sigma} \mid \sigma \in G\} \) such that \( R = \bigoplus_{\sigma \in G} R_{\sigma} \) and \( R_{\sigma}R_{\tau} \subseteq R_{\sigma \tau} \) for all \( \sigma, \tau \in G \).

When \( R \) is \( G \)-graded we denote that by \((R, G)\). The support of \((R, G)\) is defined as \( \text{supp}(R, G) = \{\sigma \in G : R_{\sigma} \neq 0\} \). If \( x \in R \), then \( x \) can be written uniquely as \( \sum_{\sigma \in G} x_{\sigma} \), where \( x_{\sigma} \) is the component of \( x \) in \( R_{\sigma} \). It is well known that \( R_e \) is a subring of \( R \) with \( 1 \in R_e \). An ideal \( A \) of a \( G \)-graded ring \( R \) is called \( G \)-ideal provided that \( A = \bigoplus_{\sigma \in G} (A \cap R_{\sigma}) \). In case \( A \) is a \( G \)-graded ideal of a \( G \)-graded ring \( R \) then the factor ring \( R/A \) is a \( G \)-graded ring with gradation defined by \((R/A)_{\sigma} = R_{\sigma} + A/A \).

For a polynomial \( f(x) \in R[x] \), we denote by \( C(f) \) the ideal generated by the coefficients of \( f \). Abuosba and Ghanem \([3]\) noted that if \( cf \) is an annihilating content of \( f \) then \( \text{Ann}_{R[x]}(f(x)) = \text{Ann}_{R[x]}(cf) \) and \( \text{Ann}_{R}(C(f)) = \text{Ann}_{R}(cf) \). Next we show that if \( f \) is homogenous then \( C(f) \) is a \( G \)-graded ideal.

**Lemma 2.2.** Let \( R \) be a \( G \)-graded ring and consider the \( G \)-grading on \( R[x] \) whose components are \( R([x])_{\sigma} = R_{\sigma}[x] \), \( \sigma \in G \). If \( f(x) = \sum_{i=0}^{n} a_{i}x^{i} \in h(R[x]) \) then \( C(f) \) is a \( G \)-graded ideal of \( R \).

**Proof.** Since \( f \) is homogeneous, there is \( \tau \in G \) such that \( \{a_0, \ldots, a_n\} \subseteq R_{\tau} \). Let \( w = \sum_{\sigma \in G} w_{\sigma} \in C(f) \). Then we can write \( w \) as \( w = \sum_{i=0}^{n} r_{i}a_{i} \). In addition, each \( r_{i} \) can be written as \( r_{i} = \sum_{\sigma \in G} r_{\sigma,i} \). So, we have

\[
\sum_{\sigma \in G} w_{\sigma} = \sum_{i=0}^{n} \left( \sum_{\sigma \in G} r_{\sigma,i} \right) a_{i} = \sum_{\sigma \in G} \left( \sum_{i=0}^{n} r_{\sigma,i}a_{i} \right).
\]

Now, for each \( \sigma \in G \) we have \( r_{\sigma} \) and \( ia_{i} \) belong to \( R_{\sigma}R_{\tau} \subseteq R_{\sigma \tau} \). Therefore \( w_{\sigma} \in C(f) \), for all \( \sigma \in G \). Hence \( C(f) \) is \( G \)-graded ideal of \( R \). \( \square \)

**Lemma 2.3.** Let \( R \) be a \( G \)-graded ring. If for every \( \sigma \in \text{supp}(R, G) \), there exists \( u_{\sigma} \in R_{\sigma} \) such that \( R_{\sigma} = R_{\sigma}u_{\sigma} \) and \( \text{Ann}_{R_{\sigma}}(u_{\sigma}) = \{0\} \), then \( \text{reg}(R_{e}[x]) \subseteq \text{reg}(R[x]) \).

**Proof.** Let \( h(x) = \sum_{i=0}^{n} a_{i}x^{i} \in \text{reg}(R_{e}[x]) \). Then \( \text{Ann}_{R_{e}}(a_0, a_1, \ldots, a_n) = \{0\} \). Let \( t = \sum_{\sigma \in G} t_{\sigma} \in \text{Ann}_{R}(a_0, a_1, \ldots, a_n) \). Then for each \( i \), we have \( 0 = \sum_{\sigma \in G} t_{\sigma,a_i} \). Since \( t_{\sigma}a_i \in R_{\sigma} \), by the unique representation of 0, we get \( t_{\sigma} \in \text{Ann}_{R}(a_0, a_1, \ldots, a_n) \) for
each \( \sigma \in G \). On the other hand, we have that for every \( \sigma \in G \) there exists \( s_{t_{\sigma}} \in R_\lambda \), such that \( t_{\sigma} = u_{\sigma} s_{t_{\sigma}} \). Since \( u_{\sigma} s_{t_{\sigma}} a_i = t_{\sigma} a_i = 0 \) and \( \text{Ann}_{R_\tau} (a_0, \ldots, a_n) = \{0\} \), we obtain that \( s_{t_{\sigma}} a_i = 0 \) for each \( i \). So, \( s_{t_{\sigma}} \in \text{Ann}_{R_\tau} (a_0, \ldots, a_n) = \{0\} \), for all \( \sigma \in G \). This implies that \( t = 0 \). Hence \( h(x) \in \text{reg}(R[x]) \).

**Definition 2.4.** A grading \((R, G)\) is called crossed product over \( \text{supp}(R, G) \) if for every \( \sigma \in \text{supp}(R, G) \), \( R_\sigma \) contains a unit.

**Proposition 2.5.** [2, Proposition 1.7] If \((R, G)\) is crossed product over \( \text{supp}(R, G) \), then for each \( \sigma \in \text{supp}(R, G) \), \( R_\sigma = R_\sigma u \) for some unit \( u \in R_\sigma \).

**Definition 2.6.** Let \( R \) be a ring with two gradations \((R, G)\) and \((R, G')\). Then \((R, G)\) is almost equivalent to \((R, G')\) if there exists an automorphism \( \phi : R \to R \) such that for each \( \tau \in G' \) the exists \( \sigma \in G \) such that \( \phi(R_\sigma) = R_\tau \).

**Proposition 2.7.** [11, Proposition 8.1.2] Let \( R \) be a \( G \)-graded ring. If \( S \subseteq h(R) \) is a multiplicatively closed subset of \( R \), then the ring \( S^{-1}R \) is \( G \)-graded by the gradation \((S^{-1}R)_\lambda = \{ \frac{a}{s} \mid s \in S, a \in R \text{ such that } \lambda = (\text{deg}(s))^{-1}\text{deg}(a) \} \).

The following lemma can be found in [6, Lemma 2.3].

**Lemma 2.8.** If \( f \in R[x] \) with \( C(f) = Ra \) then there exists \( g \in R[x] \) with \( C(g) = R \) and \( f(x) = ag(x) \).

**Definition 2.9.** Let \( H \) be a subset (res. subring, ideal) of a ring \( R \). Then \( H \) is called \( EM \)-subset (res. \( EM \)-subring, \( EM \)-ideal) of \( R \) if for each \( f(x) \in H[x] \cap Z(R[x]) \), there exists \( c_f \in Z(R) \) and \( g(x) \in \text{reg}(R[x]) \) such that \( f(x) = c_f g(x) \).

## 3. \( EM - G \)-Graded Rings

We start this section by introducing the concept of \( EM - G \)-graded ring.

**Definition 3.1.** Let \( R \) be a \( G \)-graded ring and consider the \( G \)-grading on \( R[x] \) whose components are \( R([x])_\sigma = R_\sigma[x] \), \( \sigma \in G \). Then \( R \) is called \( EM - G \)-graded ring if every non-zero polynomial in \( hZ(R[x]) \) has an annihilating content.

We can see that every \( EM \)-ring that is \( G \)-graded is an \( EM - G \)-graded ring. The next example shows that the converse is not true.

**Example 3.2.** The ring \( R = \mathbb{Z}_4[y]/(y^2) = \{ a + bY : a, b \in \mathbb{Z}_4 \text{ and } Y^2 = 0 \} \) is not an \( EM \)-ring because the polynomial \( f(x) = 2 + Y x \in Z(R[x]) \) does not have annihilating content (see [1, Example 2.6]). Now \( R \) is \( \mathbb{Z}_2 \)-graded by \( R_0 = \mathbb{Z}_4 \), \( R_1 = \mathbb{Z}_4Y \). Let \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in hZ(R[x]) \). Then either \( \{ a_0, \ldots, a_n \} \subseteq \{0, 2\} \) or \( \{ a_0, a_1, \ldots, a_n \} \subseteq \{0, 2\} \). If \( \{ a_0, a_1, \ldots, a_n \} \subseteq \{0, 2\} \), then \( a_n = 2 \), and hence 2 is an annihilating content of \( f \). Assume \( \{ a_0, a_1, \ldots, a_n \} \subseteq \{0, 2\} \). Then there is a polynomial \( f_1(x) \) whose one of the coefficients is 1 or 3 such that \( f(x) = Y f_1(x) \) or \( f(x) = 2Y f_1(x) \). Hence either \( Y \) or \( 2Y \) is an annihilating content of \( f \). Therefore \( R \) is an \( EM - \mathbb{Z}_2 \)-graded ring.

**Theorem 3.3.** Let \( R \) be \( G \)-graded ring. Then \( R \) is an \( EM - G \)-graded ring if and only if \( R_\sigma \) is an \( EM - \text{subset} \) of \( R \) for all \( \sigma \in G \).

**Proof.** Straightforward because \( hZ(R[x]) = \bigcup_{y \in G} (R_y[x] \cap Z(R[x])) \).

**Corollary 3.4.** If \( R \) is an \( EM - G \)-graded ring then \( \lambda \) is an \( EM - \text{subring} \) of \( R \).
Theorem 3.5. Let \( R \) be a \( G \)-graded ring such that \( R_e \) is an \( EM \)-ring. If for every \( \sigma \in \text{supp}(R, G) \), there exists \( u_\sigma \in R_e \) such that \( R_\sigma = R_e u_\sigma \) and \( \text{Ann}_{R_e}(u_\sigma) = \{0\} \), then \( R \) is an \( EM - G \)-graded ring.

Proof. Let \( f(x) \in hZ(R[x]) \). Then there exists \( \sigma \in G \) such that \( a_i \in R_\sigma \) for all \( i \). Therefore \( f(x) = u_\sigma h(x) \) for some \( h(x) \in R_e [x] \). If \( h(x) \in \text{reg}(R_e [x]) \), then by Lemma 2.3 \( h(x) \in \text{reg}(R[x]) \), and hence \( u_\sigma \) is an annihilating content of \( f \). Suppose \( h(x) \in Z(R_e [x]) \). Then there exist \( c \in Z(R_e) \) and \( h_1(x) \in \text{reg}(R_e [x]) \) such that \( h(x) = ch_1(x) \). So we have \( f(x) = u_\sigma ch_1(x) \) where \( u_\sigma c \in Z(R) \) and \( h_1(x) \in \text{reg}(R[x]) \). Hence \( u_\sigma c \) is an annihilating content of \( f(x) \). \( \square \)

Example 3.6. The ring \( R = \mathbb{Z}_6[x, y]/(xy) \) is not an \( EM \)-ring (see [4] Example 3.11]). Now \( R \) is \( \mathbb{Z} \times \mathbb{Z} \)-graded by \( R_{(i,0)} = \mathbb{Z}_6, R_{(i,0)} = \mathbb{Z}_6 x^i, R_{(0,i)} = \mathbb{Z}_6 y^i \) \( (i \geq 1) \), and \( R_{(i,j)} = 0 \) otherwise. For each \( i \), \( R_{(i,0)} = R_{(0,0)} x^i, R_{(0,i)} = R_{(0,0)} y^i \), and \( \text{Ann}_{R_{(0,0)}}(x^i) = \text{Ann}_{R_{(0,0)}}(y^i) = \{0\} \). So by Theorem 3.5, we get that \( R \) is an \( EM - \mathbb{Z} \times \mathbb{Z} \)-graded ring.

Let \( R \) be any commutative ring (need not to be \( G \)-graded) and \( n \geq 2 \). Then the ring \( H = R[x]/(x^n) \) is \( \mathbb{Z}_n \)-graded by the gradation \( H_k = Rx^k, k \in \mathbb{Z}_n \). For each \( k \in \mathbb{Z}_n \), we have \( H_k = Rx^k = H_0 x^k \), and \( \text{Ann}_R(x^k) = \{0\} \). So by Theorem 3.5 we obtain following result

Corollary 3.7. Let \( R \) be an \( EM \)-ring and \( n \geq 2 \) be an integer. Then \( H = R[x]/(x^n) \) is \( EM - \mathbb{Z}_n \)-graded by the gradation \( H_k = Rx^k \).

Let \( R \) be a ring and \( G \) be a group. Then the group ring \( R[G] \) is \( EM - G \)-graded ring with gradation \( H_\sigma = R \sigma, \sigma \in G \). The following corollary also follows directly from Theorem 3.5.

Corollary 3.8. Let \( R \) be an \( EM \)-ring and \( G \) be a group. Then \( R[G] \) is \( EM - G \)-graded ring with gradation \( H_\sigma = R \sigma, \sigma \in G \).

Corollary 3.9. Let \( R \) be a \( G \)-graded ring. If \( (R, G) \) is crossed product over \( \text{supp}(R, G) \) and \( R_e \) is an \( EM \)-ring then \( R \) is an \( EM - G \)-graded ring.

Proof. The result follows directly from Proposition 2.5 and Theorem 3.5. \( \square \)

Theorem 3.10. Let \( R \) be a ring with two gradations \( (R, G) \) and \( (R, G') \) such that \( (R, G) \) is almost equivalent to \( (R, G') \). If \( R \) is \( EM - G \)-graded ring then \( R \) is \( EM - G' \)-graded ring.

Proof. Let \( f(x) = \sum_{i=0}^{n} a_i x^i \) be a homogenous zero divisor of \( R[x] \) under the gradation \( (R, G') \). So \( \sum_{i=0}^{n} \phi^{-1}(a_i) x^i \) a homogenous zero divisor of \( R[x] \) under the gradation \( (R, G) \). Hence there exists \( c \in Z(R) \) and \( g(x) \in \text{reg}(R[x]) \) such that \( \sum_{i=0}^{n} \phi^{-1}(a_i) x^i = cg(x) \). This implies that \( \phi(c) \) is an annihilating content of \( f \). \( \square \).

A ring \( R \) is called Bezout ring if every finitely generated ideal is principal. \( R \) is called Armendariz if the product of two polynomials on \( R[x] \) is zero if and only if the product of their coefficients is zero. Abouosba and Ghanem [4] showed that every Bezout ring is \( EM \)-ring and every \( EM \)-ring is Armendariz. We extend these results into the notion of graded rings. We shall call a \( G \)-graded ring \( R \), Bezout-\( G \)-graded if each finitely generated \( G \)-graded ideal is principal. In addition, we will say \( R \) is Armendariz-\( G \)-graded ring if the product of two polynomials in \( hZ(R[x]) \) is zero if and only if the product of their coefficients is zero.
Corollary 3.11. Every Bezout–G–graded ring is an EM – G–graded ring.

Proof. The result follows directly from Lemma 2.2 and Lemma 2.8. □

Theorem 3.12. Every EM – G–graded ring is Arendariz–G–graded.

Proof. Suppose R is an EM – G–graded ring. Let \( f = \sum_{i=0}^{n} a_{i}x^{i} \) and \( g = \sum_{i=0}^{m} b_{i}x^{i} \) be two polynomials in \( h(R[x]) \) such that \( f(x)g(x) = 0 \). If R is an EM – G–graded ring, we get \( f(x) = c_{f}\sum_{i=0}^{n} r_{i}x^{i} \) and \( g(x) = c_{g}\sum_{j=0}^{m} s_{j}x^{j} \) where \( c_{f}c_{g} = 0 \). Hence, for each \( i, j \), we have \( a_{i}b_{j} = c_{f}r_{i}c_{g}s_{j} = 0 \). □

Theorem 3.13. Let R be an EM – G–graded ring and \( S \subseteq h(R) \) be a multiplicatively closed subset of R. Then \( S^{-1}R \) is EM – G–graded ring by the gradation \( (S^{-1}R)_{\lambda} = \{ \frac{a}{s} \mid s \in S, a \in R \text{ such that } \lambda = (\deg(s))^{-1}\deg(a) \} \).

Proof. Assume \( f(x) \in hZ(S^{-1}R[x]) \). Then \( f(x) = \frac{h(x)}{t} \) for some \( t \in S \) and some \( h(x) \in hZ(R[x]) \). Since R is an EM – G–graded ring, there exist \( c \in Z(R) \) and \( g(x) = \sum_{i=0}^{m} b_{i}x^{i} \in reg(R[x]) \) such that \( h(x) = cg(x) \). So \( f(x) = c\frac{h(x)}{t} \). Suppose that there exist \( \frac{d}{k} \in S^{-1}R \) such that \( \frac{d}{k}g(x) = 0 \). Then there exists \( u \in S \) such that \( udg(x) = 0 \). Since \( g(x) \) is regular in \( R[x] \) and \( u \in S \), we get \( d = 0 \). Hence \( \frac{d}{k}g(x) \) is regular in \( S^{-1}R[x] \). Therefore \( S^{-1}R[x] \) is an EM – G–graded ring. □

If R is a commutative ring with unity, then \( reg(R) \) is multiplicatively closed subset of R, and the localization \( T(R) = (\text{reg}(R))^{-1}R \) is called the total quotient ring of R. Abuosba and Ghanem [4] deduced that if R is an EM–ring then so is \( T(R) \). A similar result is obtain for EM – G–graded rings. Let \( h\text{reg}(R) \) be the set of regular homogenous elements of R and define \( hT(R) = (\text{hreg}(R))^{-1}R \).

Corollary 3.14. If R is EM – G–graded ring then so is \( hT(R) \).

Theorem 3.15. Let R be a G–graded ring. If \( hT(R) \) is EM – G–graded ring then for every \( f(x) \in hZ(R[x]) \) there exists \( c \in R \) such that \( \text{Ann}_{R[x]}(f) = \text{Ann}_{R[x]}(c) \).

Proof. Let \( f(x) \in hZ(R[x]) \). Then \( \frac{f(x)}{t} \in hZ(hT(R)[x]) \). So there exist \( \frac{x}{t} \in Z(hT(R)[x]) \), and \( g(x) \in \text{reg}(hT(R)[x]) \) such that \( \frac{f(x)}{t} = \frac{x}{t}g(x) \). So we have \( u(tf(x) - kg(x)) = 0 \) for some \( u \in h\text{reg}(R[x]) \). Since \( u, t, \) and \( g(x) \) are regular, we get \( \text{Ann}_{R[x]}(f) = \text{Ann}_{R[x]}(k) \). □

Let \( R_{\alpha} \), \( \alpha \in I \) be a family of G–graded rings. Then \( R = \prod_{\alpha \in I} R_{\alpha} \) is G–graded by the gradation \( R_{\alpha} = \prod_{\alpha \in I} (R_{\alpha})_{g}, g \in G \). Next we generalize [4] Theorem 3.12.

Theorem 3.16. Let \( R_{\alpha} \), \( \alpha \in I \) be a family of G–graded rings, and \( R = \prod_{\alpha \in I} R_{\alpha} \). Then R is EM – G–graded if and only if \( R_{\alpha} \) is EM – G–graded for each \( \alpha \in I \).

Proof. Suppose R is EM – G–graded. Fix \( j \in I \), and let \( f(x) = \sum_{i=0}^{n} a_{i}x^{i} \in hZ(R_{j}[x]) \). Consider the polynomial \( g(x) = \sum_{i=0}^{m} b_{i,x}x^{i} \in R(x) \), where for each i, \( b_{j,i} = a_{i}, \) and \( b_{\alpha,i} = 0 \), for all \( \alpha \neq j \). Since all \( a_{i} \)'s are in the same component of the G–grading of \( R_{\alpha} \), we get \( g(x) \in hZ(R_{j}[x]) \). Hence there exist \( c = (c_{\alpha}) \in Z(R) \) and \( g_{\alpha}(x) = \sum_{i=0}^{m} (d_{\alpha,i})x^{i} \in \text{reg}(R[x]) \) such that \( g(x) = cg_{\alpha}(x) \), and \( m \geq n \). Then \( f(x) = c_{j}\sum_{i=0}^{m} d_{j,i}x^{i} \). It is clear that \( c_{j} \neq 0 \), because \( 0 \neq a_{i} = c_{j}d_{j,i} \). Assume \( yd_{j,i} = 0 \) for all i. Let \( q = (q_{\alpha}) \) where \( q_{j} = y \) and \( q_{\alpha} = 0 \), when \( \alpha \neq j \). Then \((q_{\alpha})(d_{\alpha,i}) = 0 \) for all i. So \( y = 0 \) and hence \( \sum_{i=0}^{m} d_{j,i}x^{i} \in \text{reg}(R_{\alpha}[x]) \).

For the converse assume \( R_{\alpha} \) is EM – G– graded for all \( \alpha \in I \). Let \( f(x) = \sum_{i=0}^{n} (a_{i})x^{i} \in hZ(R[x]) \). Then there exists \( b_{\alpha} \in R[x] \) such that \( (b_{\alpha})(a_{i}) = (0) \).
for all $i$. For each $\alpha \in I$, let $f_\alpha(x) = \sum_{i=0}^{n} a_{\alpha,i}x^i$ and let $J = \{\alpha \in I \mid f_\alpha \notin \text{reg}(R_\alpha)\}$. So, for each $\alpha \in J$, we have $f_\alpha(x) \in hZ(R_j[x])$, and hence $f_\alpha = c_\alpha g_\alpha(x)$ for some $c_\alpha \in Z(R_\alpha)$ and some $g_\alpha(x) \in \text{reg}(R_\alpha[x])$. Now for $\alpha \notin J$, let $c_\alpha = 1$ and $g_\alpha(x) = f_\alpha(x)$. Hence we have $f(x) = (c_\alpha)(g_\alpha(x))$, where $(c_\alpha) \in Z(R)$ and $(g_\alpha(x)) \in \text{reg}(R[x])$ as desired.

**Theorem 3.17.** If $R$ is an EM $- G$-graded ring then so is $R[x]$.

**Proof.** Let $f(x) = \sum_{i=0}^{n} f_i(x)y^i \in hZ(R[x,y])$. Then there exists $h(x) \in R[x]$ such that $h(x)f_i(x) = 0$ for all $i$. Let

$$g(x) = f_0(x) + f_1(x)x^{\text{deg}(f_0)+1} + f_2(x)x^{\text{deg}(f_0)+2} + \cdots + f_n(x)x^{\sum_{i=0}^{n} \text{deg}(f_i)+n}.$$ 

So $hg = 0$. Moreover, since $f(x,y) \in hZ(R[x,y])$, there exists $\lambda \in G$ such that $R_\lambda$ contains all coefficients of $f_i(x)$, for all $i$. Hence $g(x) \in hZ(R[x])$. Thus there exists $c \in Z(R)$ and $g_i(x) = \sum_{i=0}^{n} b_ix^i \in \text{reg}(R[x])$ such that $g(x) = cg_1(x)$. So $f_0(x) = c \sum_{i=0}^{n} b_ix^i = c\nu_0(x)$, $f_1(x) = c \sum_{i=0}^{n} b_{i+\text{deg}(f_0)+1}x^i = c\nu_1(x)$, and so on. Thus $f(x,y) = c \sum_{i=0}^{n} w_i(x)y^i$, and since $\cap \text{Ann}(b_i) = 0$, we have $\sum_{i=0}^{n} w_i(x)y^i \in \text{reg}(R[x,y])$. Therefore $R[x]$ is EM $- G$-graded.

**Corollary 3.18.** Let $R$ be an EM $- G$-graded ring and $k$ be a positive integer. Then $R[x_1,x_2,\ldots,x_k]$ is an EM $- G$-graded ring.

**Remark 3.19.** If $R$ is a $G$-graded ring then $(x^2)$ is a $G$-graded ideal of $R[x]$. So $R[x]/(x^2)$ is a $G$-graded ring with gradation $(R[x]/(x^2))_g = R_0[y] + (x^2)/(x^2)$ i.e. $a + bX \in h(R[x]/(x^2))$ if and only if there exists $\sigma \in G$ such that $a, b \in R_\sigma$.

**Theorem 3.20.** Let $R$ be $G$-graded such that $h(R) \cap Z(R) = \{0\}$. Then $H = R[x]/(x^2)$ is an EM $- G$-graded ring.

**Proof.** Let $f(y) = \sum_{i=0}^{n} (a_i + b_iX)y^i \in hZ(H[y])$. Then by the argument in Remark 3.19 we get that $a_i, b_i \in h(R)$ for all $i$. Hence $a_i = 0$ for all $i$. So we have $f(y) = X \sum_{i=0}^{n} b_iy^i$, which yields $X$ is an annihilating content for $f$. Thus $H$ is an EM $- G$-graded ring.

**Theorem 3.21.** Let $R$ be a $G$-graded ring. If $H = R[x]/(x^2)$ is an EM $- G$-graded ring, then so is $R$.

**Proof.** Let $f(y) = \sum_{i=0}^{n} r_i y^i \in hZ(H[y])$. Then $f(y) \in hZ(H[y])$. Therefore $f(y) = (c+dX)\sum_{i=0}^{n} (a_i + b_iX)y^i$, with $\cap_{i=0}^{n} \text{Ann}(a_i + b_iX) = \{0\}$. So we have $\cap_{i=0}^{n} \text{Ann}(a_i) = \{0\}$ and $r_i = ca_i$ for each $i$. Hence $f(y) = c \sum_{i=0}^{n} a_i y^i$. Therefore $R$ is an EM $- G$-graded ring.

**Theorem 3.22.** Let $R$ be $G$-graded ring such that for each $a \in h(R)$ there exists $b \in E(R)$ such that $\text{ann}(a) = bR$. Then $R$ is an EM $- G$-graded ring.

**Proof.** Let $f(x) = \sum_{i=0}^{n} a_ix^i \in hZ(R[x])$. Then $a_i \in h(R)$ for all $i$. So for each $i$ there exists $b_i \in E(R)$ and $u_i \in \text{reg}(R)$ such that $a_i = u_ib_i$. Let $b = 1 - \sum_{i=0}^{n} (1-b_i)$. Clearly $1 - b \in E(R)$, and so $b \in E(R)$. Since $b_i(1-b_i) = 0$, we get $b(b_i + 1 - b) = bb_i = b_i - \sum_{j=0}^{n} (1-b_j) = b_i$ and $1 = \sum_{i=0}^{n} (b_i + 1 - b) - \sum_{i=0}^{n} (b_i + 1 - b)$. Hence $b(0+1-b) \in \text{Ann}(b_0 + 1 - b, b_1 + 1 - b, \ldots, b_n + 1 - b) = \{0\}$ and for each $i$, $a_i = bu_i(b_i + 1 - b)$. Thus $f(x) = bg(x)$ where $b \in Z(R)$ and $g(x) = \sum_{i=0}^{n} a_i(b_i + 1 - b)x^i \in \text{reg}(R[x])$. Therefore $R$ is EM $- G$-graded ring.

Let $R$ be a ring and $M$ be an $R$-module. Then the idealization $R(+).M$ is the ring whose elements are those of $R \times M$ equipped with addition and multiplication defined
by \((r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)\) and \((r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)\) respectively. The annihilator of the module \(M\) in \(R\) is the set \(\text{Ann}_RM = \{r \in R \mid rM = \{0\}\}\). It is well known that the idealization \(R(+)M\) is \(\mathbb{Z}_2\)-graded by the gradation \((R(+))\)0 = \(R \oplus 0\) and \((R(+))\)1 = \(0 \oplus M\).

**Theorem 3.23.** Let \(R\) be a ring and \(M\) be an \(R\)-module such that \(\text{Ann}_RM = \{0\}\). If \(R(+)M\) is an \(EM - \mathbb{Z}_2\)-graded by the gradation \(H_0 = R \oplus 0\) and \(H_1 = 0 \oplus M\), then \(R\) is an \(EM\)-ring.

**Proof.** Suppose \(R(+)M\) is an \(EM - \mathbb{Z}_2\)-graded by the gradation \(H_0 = R \oplus 0\) and \(H_1 = 0 \oplus M\). Let \(f(x) = \sum_{i=0}^{n} a_i x^i \in Z(R[x])\). Then \(\text{Ann}_{R(+)M}\{a_i, 0\} \neq \{0\}\). So \(\sum_{i=0}^{n} a_i, 0 = x^i = (c, m) \sum_{i=0}^{k} (b_i, m_i)x^i\) with \(k \geq n\) and \(\text{Ann}_{R(+)M}\{b_i, m_i\} = \{0\}\). Suppose \(0 \neq w \in \text{Ann}_R\{a_i\}\). Since \(\text{Ann}_R(M) = \{0\}\), there exists \(t \in M\) such that \(wt \neq 0\). However, since \((0, wt)(b_i, m_i) = (0, wth_i) = (0, 0)\) for all \(i\), we get \((0, wt) \in \text{Ann}_{R(+)M}\{b_i, m_i\} = \{0\}\), a contradiction. Therefore \(f(x) = c \sum_{i=0}^{n} b_i x^i\) with \(\text{Ann}_R\{b_i\} = \{0\}\) as desired. \(\square\)

Ganam and Abuosba in [10] proved that if \(R(+)R\) (equivalently \(R[x]/(x^2)\)) is an \(EM\)-ring then so is \(R\). However the converse of this result is not true. From Theorem 3.3 and Theorem 3.23 we obtain the following result.

**Corollary 3.24.** A ring \(R\) is an \(EM\)-ring if and only if \(H = R(+)R\) is an \(EM - \mathbb{Z}_2\)-graded ring with gradation \(H_0 = R \oplus 0\) and \(H_1 = 0 \oplus R\).

### 4. Further Questions

In this section we highlight two problems that may be of interest for future research.

Several results on \(EM\)-rings can be extended to \(EM - G\)-graded rings if we can show that every homogenous zero divisor polynomial has at least one homogenous annihilating content. The rings discussed in Theorem 3.3, Corollary 3.7, and Corollary 3.8 have this property, however we don’t know if this always the case. Based on this observation we ask the following question.

**Question 4.1.** In an \(EM - G\)-graded ring \(R\), is it guaranteed that every polynomial in \(hZ(R[x])\) has an annihilating content that belongs to \(hZ(R)\)? If not, then what kinds of graded rings have this property?

The second problem is related to the notion of strongly \(EM\) rings that was defined in [4]. An \(EM\) ring \(R\) is called strongly \(EM\) ring if every zero divisor power series has an annihilating content. If \(R\) is \(G\)-graded ring then \(R[[x]]\) is \(G\)-graded by the gradation \((R[[x]])_g = R_g[[x]], g \in G\). We define an \(EM - G\)-graded ring \(R\) to be strongly \(EM - G\)-graded ring if every nonzero homogeneous power series in \(R[[x]]\) has an annihilating content. It is not guaranteed that if every power series in \(hZ(R[[x]])\) has an annihilating content then every polynomial in \(hZ(R[x])\) has an annihilating content. So similar to a question from [4], we ask the following question.

**Question 4.2.** What types of graded rings have the property that if every power series in \(hZ(R[[x]])\) has an annihilating content then every polynomial in \(hZ(R[x])\) has an annihilating content?

The following theorem describes one of these types.
Theorem 4.3. Let $R$ be a $G$–graded ring such that $Z(R[[x]])$ is an ideal of $R[[x]]$. Then $R$ is strongly EM – $G$–ring if and only if every homogeneous power series in $R[[x]]$ has an annihilating content.

Proof. Suppose that every homogeneous power series in $R[[x]]$ has an annihilating content. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in hZ(R[[x]])$. Then $f(x) = c_f f_1$ for some $c_f \in Z(R)$ and $f_1(x) = \sum_{i=0}^{\infty} b_i x^i \in reg(R[[x]])$. Thus we have, $a_i = c_f b_i$, for $i = 0, 1, \ldots, n$, and $0 = c_f b_i$, for $i \geq n + 1$. So $\sum_{i=n+1}^{\infty} b_i x^i \in Z(R[[x]])$. Since $Z(R[[x]])$ is an ideal of $R[[x]]$ and $f_1(x) \in reg(R[[x]])$, we have $\sum_{i=0}^{n} b_i x^i \in reg(R[[x]])$. Thus $f(x) = c_f \sum_{i=0}^{n} b_i x^i$, where $\sum_{i=0}^{n} b_i x^i \in reg(R[[x]])$. Therefore $R$ is EM – $G$–graded. \qed

References

[1] R. Abu-Dawwas, On graded semi-prime rings, Proceedings of the Jangjeon Mathematical Society, 20 (1), (2017), 19-22.
[2] R. Abu-Dawwas, More on Crossed Product over the Support of Graded Rings, International Mathematical Forum, 5 (63), (2010), 3121 - 3126.
[3] E. Abuosba and O. Alkam, When zero-divisor graphs are divisor graphs, Turk. J. Math. 41 (2017), 797-807.
[4] E. Abuosba and M. Ghanam, Annihilating Content in Polynomial and Power Series Rings, Journal of Korean Mathematical Society, 56(5), (2019), 1403-1418.
[5] K. Al-Zoubi, F. Al-Turman, and E. Celikel, gr-n-ideal in graded commutative rings, Acta Univ. Sapientiae, Mathematica, 11(1), 2019, 18-28.
[6] D. D. Anderson, D. F. Anderson, and R. Markanda, The rings $R(X)$ and $R \langle X \rangle$, J. Algebra, 95, 1985, 96-115.
[7] M. Bataineh and R. Abu-Dawwas, Graded almost 2-absorbing structures, JP Journal of Algebra, Number Theory and Applications, 39 (1), (2017), 63-75.
[8] P. M. Cohn, Reversible Rings, Bulletin of London Mathematical Society, 31 (6), (1999), 641-648.
[9] M. Cohen and L. Rowen, Group graded rings, Comm. in Algebra, 11, (1983), 1253-1270.
[10] M. Ghanam and E. Abuosba, Some Extensions of Generalized Morphic Rings and EM–Rings, An. St. Univ. Ovidius Constanta, 26(1), (2018), 111-123.
[11] C. Nastasescu and F. Van Oystaeyen, Methods of graded rings, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004.
[12] M. Refai and K. Al-Zoubi, On graded primary ideals, Turkish J. Mathematics, 28, (2004), 217-229.
[13] M. Refai, M. Hailat and S. Obeidat, Graded radicals and graded prime spectra, Far East Journal of Mathematical Sciences, Part I, (2000), 59-73.
[14] J. C. Wei, Certain rings whose simple singular modules are nil-injective, Turkish Journal of Mathematics, 32, (2008), 1-16.
[15] H. P. Yu, On quasi-duo rings, Glasgow Mathematical Journal, 37, (1995), 21-31.

Tariq Alraqad, Department of Mathematics, University of Hail, Saudi Arabia.  
E-mail address: t.alraqad@uoh.edu.sa

Hicham Saber, Department of Mathematics, University of Hail, Saudi Arabia.  
E-mail address: hicham.saber7@gmail.com

Rashid Abu-Dawwas, Department of Mathematics, Yarmouk University, Jordan.  
E-mail address: rrashid@yu.edu.jo