ON FINITE GROUPS ACTING ON ACYCLIC COMPLEXES OF DIMENSION TWO

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Abstract

We conjecture that every finite group $G$ acting on a contractible $CW$-complex $X$ of dimension 2 has at least one fixed point. We prove this in the case where $G$ is solvable, and under this additional hypothesis, the result holds for $X$ acyclic.

Dedicat a la memòria d’en Pere Menal

0. Introduction

Let $G$ be a group and $A$ an abelian group. Dicks and Dunwoody ([4, Chapter IV]) proved that for each element $\zeta$ of $H^1(G; AG)$ there exists a $G$-tree $T$ with finite edge stabilizers, with the property that for each subgroup $H$ of $G$, the restriction of $\zeta$ to $H$ is zero if and only if $H$ fixes a point of $T$. It is natural to look for analogous geometric explanations of elements of higher cohomology groups; thus, for example, one can ask if for each element $\zeta$ of $H^2(G; AG)$ there exists a contractible 2-dimensional $CW$-complex $X$ admitting an action of $G$ with finite stabilizers for 2-cells, with the property that for each subgroup $H$ of $G$, the restriction of $\zeta$ to $H$ is zero if and only if $H$ acts trivially on $X$ in some sense, perhaps leaving invariant a subtree of the 1-skeleton of $X$. The restriction of $\zeta$ to any finite subgroup of $G$ is zero, but if a finite group leaves a subtree invariant then it fixes a point. With this motivation, we optimistically conjecture that every finite group $G$ acting on a contractible 2-dimensional $CW$-complex $X$ has at least one fixed point.

In this note we prove this conjecture in the case where $G$ is solvable. Our argument is based on a classical result of P.A. Smith ([8], [9]), stating that every action of a finite $p$-group on a finite dimensional $\mathbb{Z}/p$-acyclic $CW$-complex has a $\mathbb{Z}/p$-acyclic fixed-point set (see [2, Chapter III] and further developments e.g. in [1], [3], [7]).
In our context, the hypothesis that $X$ has no cells above dimension 2 is essential. It is known that any finite nilpotent group whose order is not a prime power acts on some contractible 3-dimensional CW-complex without fixed points ([1]).

On the other hand, we shall prove that for a finite solvable group $G$ acting on a 2-dimensional CW-complex $X$, in order to ensure the existence of a fixed point it suffices to assume that $X$ is acyclic. For $X$ acyclic, however, the condition that $G$ be solvable cannot be removed, because the alternating group $A_5$ acts on the 2-skeleton of the Poincaré sphere—which is acyclic—without fixed points ([6]). Recall that the 1-skeleton of the Poincaré sphere is the complete graph on 5 vertices, and the 2-skeleton is obtained by adding 6 pentagonal faces so as to extend the natural action of $A_5$ on the set of vertices. The fundamental domain of the action is a triangle with angles $\pi/2$, $\pi/5$, $3\pi/10$, and the 60 copies of this fundamental domain triangulate the 2-skeleton, from which it follows that there are no fixed points. The fundamental group of this space is isomorphic to $SL_2(\mathbb{F}_5)$.

Since $X$ being contractible is equivalent to $X$ being simply-connected and acyclic, the question that remains open is: If we add the condition that $X$ be simply-connected, can we delete the condition that $G$ be solvable?

1. Statement and proof of the result

Let $G$ be a finite group acting on a CW-complex $X$ of dimension 2, and denote by $X^G$ the set of fixed points under the action of $G$. We shall assume that the action is cellular ([5]); that is, each translation of an open cell is an open cell, and, if a cell is invariant, then it is pointwise fixed. Thus $X^G$ is a subcomplex of $X$. For a subcomplex $Y \subseteq X$, we denote by $C_n(X,Y)$ the group of relative cellular $n$-chains of the pair $(X,Y)$.

Given a nonzero abelian group $A$, a space $X$ is said to be $A$-acyclic if $\tilde{H}_k(X;A) = 0$ for all $k$, where $\tilde{H}$ denotes reduced homology. Recall that the condition $\tilde{H}_{-1}(X;A) = 0$ is equivalent to the augmentation homomorphism $C_0(X) \otimes A \rightarrow A$ being surjective, and hence equivalent to $X$ being nonempty.

We prove

**Theorem 1.1.** Let $G$ be a finite solvable group acting on a CW-complex $X$ of dimension 2. If $\tilde{H}_*(X;\mathbb{Z})$ is finite, and the orders of $G$, $H_1(X;\mathbb{Z})$ are coprime, then the natural map $\tilde{H}_*(X^G;\mathbb{Z}) \rightarrow \tilde{H}_*(X;\mathbb{Z})$ is injective.
Proof: Under our assumptions, the graded group $\widetilde{H}_s(X; \mathbb{Z})$ is necessarily concentrated in degree 1, since it is free abelian in all other degrees. Moreover, $H_1(X; \mathbb{Z}/p) = 0$ (and hence $X$ is $\mathbb{Z}/p$-acyclic) for every prime $p$ dividing the order of $G$.

Since $G$ is solvable, we can find a series of subgroups

\begin{equation}
\{1\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_k = G
\end{equation}

in which each $G_{i-1}$ is normal in $G_i$ and $G_i/G_{i-1} \cong \mathbb{Z}/p$, where $p_i$ is a prime. Then the action of $G$ on $X$ induces an action of $G_i/G_{i-1}$ on $X^{G_{i-1}}$ and

\begin{equation}
X^{G_i} = \left(X^{G_{i-1}}\right)^{G_i/G_{i-1}}.
\end{equation}

We prove inductively that the map $\widetilde{H}_s(X^{G_i}; \mathbb{Z}) \to \widetilde{H}_s(X; \mathbb{Z})$ is a monomorphism for all $i = 0, \ldots, k$. This is trivial for $G_0$. Thus suppose that it has been established for $G_{i-1}$. Then $X^{G_{i-1}}$ is $\mathbb{Z}/p$-acyclic for every prime $p$ dividing the order of $G$. Since the order of $G_i/G_{i-1}$ is a prime $p_i$, applying Smith’s Theorem ([9]) to the action of $G_i/G_{i-1}$ on $X^{G_{i-1}}$ we obtain, by (1.2), that $X^{G_i}$ is $\mathbb{Z}/p_i$-acyclic. This tells us in particular that $X^{G_i}$ is nonempty and connected. Further, for every abelian group $A$ we have an exact sequence

\begin{equation}
0 \to H_2(X^{G_i}; A) \to H_2(X; A) \to H_2(X, X^{G_i}; A) \to H_1(X^{G_i}; A) \to H_1(X; A) \to H_1(X, X^{G_i}; A) \to 0,
\end{equation}

from which we infer that $H_2(X, X^{G_i}; \mathbb{Z}/p_i) = 0$. But, since $X$ has no cells above dimension 2, the group $H_2(X, X^{G_i}; \mathbb{Z})$ embeds in the free abelian group $C_2(X, X^{G_i})$ and hence it is free abelian itself. This forces $H_2(X, X^{G_i}; \mathbb{Z}) = 0$, showing that $H_1(X^{G_i}; \mathbb{Z})$ embeds in $H_1(X; \mathbb{Z})$. ■

Corollary 1.2. Every action of a finite solvable group $G$ on a $\mathbb{Z}$-acyclic CW-complex $X$ of dimension 2 has at least one fixed point.

Proof: It follows from Theorem 1.1 that the fixed-point set $X^G$ is $\mathbb{Z}$-acyclic, so in particular it is nonempty. ■

Note. Robert Oliver has kindly informed us that an, as yet unpublished, paper by Yoav Segev contains a different proof of Corollary 1.2, with the additional assumption that $X$ be finite.

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