RATIONAL POINTS AND PRIME VALUES OF POLYNOMIALS IN MODERATELY MANY VARIABLES

KEVIN DESTAGNOL AND EFTHYMIOS SOFOS

Abstract. We derive the Hasse principle and weak approximation for pencils of certain varieties in the spirit of work by Colliot-Thélène–Sansuc and Harpaz–Skorobogatov–Wittenberg. Our varieties are defined through polynomials in many variables and part of our work is devoted to establishing Schinzel’s hypothesis for polynomials of this kind. This last part is achieved by using arguments behind Birch’s well-known result regarding the Hasse principle for complete intersections with the notable difference that we prove our result in 50% fewer variables than in the classical Birch setting. We also study the problem of square-free values of an integer polynomial with 66.6% fewer variables than in the Birch setting.

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1. Introduction

1.1. Prime values of polynomials and rational points. Let \( n \geq 1 \) be an integer and assume that \( f \in \mathbb{Q}[t_1, \ldots, t_n] \). Let \( K_1, \ldots, K_r \) be cyclic extensions of \( \mathbb{Q} \), denote the degree \( [K_i : \mathbb{Q}] \) by \( d_i \) and fix a basis \( \{\omega_{1,i}, \ldots, \omega_{d_i,i}\} \) for \( K_i \) as a vector space over \( \mathbb{Q} \). We will denote

\[
N_{K_i/\mathbb{Q}}(x_i) = N_{K_i/\mathbb{Q}}(x_1,i\omega_{1,i} + \cdots + x_{d,i,i}\omega_{d,i,i}), \quad (1 \leq i \leq r)
\]

where \( N_{K_i/\mathbb{Q}} \) denotes the field norm. Let now the quasi-affine variety \( X \subset \mathbb{A}^n \times \mathbb{A}^{d_1} \times \cdots \times \mathbb{A}^{d_r} \) be defined via

\[
X : (0 \neq f(t_1, \ldots, t_n) = N_{K_1/\mathbb{Q}}(x_1) = \cdots = N_{K_r/\mathbb{Q}}(x_r))
\]  

(1.1)

and \( V \) be a smooth proper model of the affine variety of \( \mathbb{A}^n \times \mathbb{A}^{d_1} \times \cdots \times \mathbb{A}^{d_r} \) defined via

\[
f(t_1, \ldots, t_n) = N_{K_1/\mathbb{Q}}(x_1) = \cdots = N_{K_r/\mathbb{Q}}(x_r).
\]

(1.2)

The Hasse principle and weak approximation for varieties of this kind have been the object of intensive study. There are cases where the Hasse principle and weak approximation hold

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and there are examples for which they fail. However, it has been conjectured by Colliot-Thélène that all such failures are accounted for by the Brauer–Manin obstruction.

The main objective of this paper is to study the Hasse principle and weak approximation for the class of varieties defined by and under the restriction that the polynomial is an irreducible form and has many variables, but only moderately so as it will appear in due course. To this end, information on prime values assumed by integer polynomials can be exploited. The prototypical example is due to Hasse, whose proof of the Hasse principle for smooth quadratic forms in four variables relies on Dirichlet’s theorem on primes in arithmetic progressions combined with the global reciprocity law and the Hasse principle for non-singular quadratic forms in three variables. This fibration argument was later generalized in an important work by Colliot-Thélène and Sansuc to establish that, conditionally under Schinzel’s hypothesis, various pencils of varieties over satisfy the Hasse principle and weak approximation. Their result was then extended by many authors, see the introduction of for a list of relevant references. Theorem below will allow us to replace Schinzel’s hypothesis in order to prove unconditionally the Hasse principle and weak approximation for the varieties defined by and in the case of an irreducible form with moderately many variables. Let us conclude by mentioning that unconditional proofs in this subject exist in cases where the underlying polynomials have small degree (see for instance Theorem 9.3) or special factorisation over . For example, polynomials that are completely split over are treated in . Our main result provides an example where the polynomial needs to have moderately many variables compared to its degree but has no restriction on its shape.

For a homogeneous polynomial that is irreducible over we let be the affine dimension of the singular locus of . Observe that and that if and only if is non-singular.

**Theorem 1.1.** Let , and be as in with an irreducible form and assume that
\[ n - \sigma_f \geq \max\{4, 1 + 2^{\deg(f) - 1}(\deg(f) - 1)\}. \]
Then is Zariski dense as soon as it is non-empty. In addition, satisfies the Hasse principle and weak approximation.

Note that, thanks to the fact that our Theorem below (which is the main ingredient of the proof of Theorem holds in half as many variables as in the work of Birch, a direct application of §7, Th.1] would not prove Theorem . Our strategy will be to establish an analogue of Prop.1.2] and then to adapt the argument in the proof of Th.1.3]. Due to the introductory remarks in the proof of Theorem , one can deduce the following corollary from Theorem since by birational invariance it is enough to establish the result for the smooth model of provided by .

**Corollary 1.2.** Keep the assumptions of Theorem and let be like in . Then is Zariski dense as soon as . In addition, satisfies the Hasse principle and weak approximation.

**1.2. Primes represented by polynomials of general shape.** As mentioned above, a key tool in our proof of Theorem is a generalization of Schinzel’s hypothesis for polynomials in moderately many variables. Let be a, not necessarily homogeneous, polynomial that is irreducible over and denote by the top degree part of .
We define $\sigma_f := \sigma_{f_0}$. For a non-empty compact box $B \subset \mathbb{R}^n$ with the property that $f_0(B) \subset (1, \infty)$, we define

$$
\pi_f(B) := \# \{ x \in \mathbb{Z}^n \cap B : f(x) \text{ is a prime} \} \quad \text{and} \quad \operatorname{Li}_f(B) := \int_B \frac{dx}{\log f_0(x)}.
$$

(1.3)

Our main result in this section is the following theorem.

**Theorem 1.3.** Assume that $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is any integer polynomial which is irreducible over $\mathbb{Q}[x_1, \ldots, x_n]$ and let $B \subset \mathbb{R}^n$ be any non-empty compact box with $f_0(B) \subset (0, \infty)$. If

$$
n - \sigma_f \geq \max\{4, (\deg f - 1)2^{\deg f - 1} + 1\},
$$

(1.4)

then for every fixed $A > 0$ the following holds for all sufficiently large $P$,

$$
\pi_f(PB) = \left( \prod_{p \text{ prime}} \frac{(1 - p^{-n} \# \{ x \in \mathbb{F}_p^n : f(x) = 0 \})}{(1 - 1/p)} \right) \operatorname{Li}_f(PB) + O_{A,B,f}(\frac{P^n}{(\log P)^A})
$$

where the implied constant depends at most on $A, B$ and $f$.

Note that the assumption $f_0(B) \subset (0, \infty)$ shows that for all sufficiently large $P$ every $x \in PB$ satisfies $f_0(x) > P^{\deg(f)} > 1$, thus $\operatorname{Li}_f(PB)$ is well-defined. We shall see in Lemma 3.21 that $\operatorname{Li}_f(PB) - \operatorname{vol}(B)P^n/(\log(P^{\deg(f)})) \ll_{f,B} P^n/(\log P)^2$, hence

$$
\pi_f(PB) = \frac{\operatorname{vol}(B)}{\deg(f)} \left( \prod_{p \text{ prime}} \frac{(1 - p^{-n} \# \{ x \in \mathbb{F}_p^n : f(x) = 0 \})}{(1 - 1/p)} \right) \frac{P^n}{\log P} + O_{f,B}(\frac{P^n}{(\log P)^2}).
$$

(1.5)

Bateman and Horn [11] provided heuristics that led to a conjecture regarding the density of primes among the values of an integer polynomial in a single variable. One can modify their heuristics in the case that the polynomial has arbitrarily many variables, thus resulting in an analogous conjecture regarding the prime values of an integer polynomial in many variables. We refer the reader to Appendix A for a quick overview of Schinzel’s hypothesis, the Bateman-Horn conjecture and their generalisations. Theorem 1.3 then establishes the analogous conjecture provided that the polynomial has sufficiently many variables compared to its degree.

There are currently no available techniques capable of settling any case of the Bateman–Horn conjecture in one variable apart from the case of one linear polynomial, which is Dirichlet’s theorem for primes in arithmetic progressions. Efforts have therefore focused on settling such problems for polynomials in more variables. Notable examples in cases with $n = 2$ are Iwaniec’s work [25] for quadratic polynomials, Fouvry–Iwaniec’s work [15] for $x_1^2 + x_2^2$ with $x_2$ prime, Friedlander–Iwaniec’s work [16] for $x_1^2 + x_2^4$, Heath-Brown’s work [22] for $x_3^3 + 2x_2^3$, Heath-Brown–Moroz’s work [24] for binary cubic forms and the recent work of Heath-Brown–Li [23] on $x_1^3 + x_2^3$ with $x_2$ prime. The special shape of these polynomials plays a central role in the proofs of these results; they are all related to norms of a number field. In cases with $n > 2$ it should be noted that Green–Tao–Ziegler [19] studied simultaneous prime values of certain linear polynomials by a variety of methods, Friedlander and Iwaniec [17] studied the prime values of $x_1^2 + x_2^2 + x_3^2$ via the class number formula of Gauss, while Maynard’s work [28] employs geometry of numbers to cover the case of incomplete norm forms.

It is therefore a natural question whether the problem of representing primes by polynomials can be studied for polynomials with no special shape. Let us recall here that one of the important theorems in the frontiers between analytic number theory and Diophantine
geometry concerns the Hasse principle for systems of polynomials in many variables and with no special shape by Birch [2]. To prove Theorem 1.3 we shall employ the Hardy–Littlewood circle method in the form used by Birch and use several of his estimates.

While Birch’s work applies to every non-singular homogeneous polynomial $f$ having at least $n \geq (\deg(f) - 1)2^{\deg(f)} + 1$ variables (which was recently improved by Browning and Prendiville [6] to $n \geq (\deg(f) - \sqrt{\deg(f)}/2)2^{\deg(f)}$), the assumption (1.4) of our Theorem 1.3 is less restrictive, as it allows for half as many variables. The improved range is due to the use of $L_2$-norm inequalities in the minor arcs, as well as bounds for exponential sums due to Browning–Heath-Brown [4] and Deligne [13] to show that the singular series in Birch’s work converges absolutely in the range (1.4).

Let us finally give a direct consequence of Theorem 1.3.

**Corollary 1.4.** Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be an integer homogeneous polynomial which is irreducible over $\mathbb{Q}[x_1, \ldots, x_n]$ and assume that $n - \sigma_f \geq \max\{4, (\deg(f) - 1)2^{\deg(f)} + 1\}$. Then $f(x)$ takes infinitely many distinct prime values as $x$ ranges over $\mathbb{Z}^n$ if and only if $f(\mathbb{R}^n)$ is not included in $(-\infty, 0]$ and for every prime $p$ the set $f(\mathbb{Z}^n)$ is not included in $p\mathbb{Z}$.

**Proof.** We clearly need to focus only on the sufficiency. If $f(\mathbb{R}^n) \subset (-\infty, 0]$ holds, then we can obviously find a non-empty box $B \subset \mathbb{R}^n$ with $f(B) \subset (0, +\infty)$, so that $\text{vol}(B) = 0$. If $f(\mathbb{Z}^n) \subset p\mathbb{Z}$ holds, then the $p$-adic factor in (1.5) is strictly positive and we shall see in Lemma 3.16 that the product over $p$ is absolutely convergent. Hence by (1.5) we deduce that $\pi_f(P) = f_{\mathbb{Q}} P^n / \log P$. If $f(x) = q$ was soluble only for finitely many primes $q$, say $q_1, \ldots, q_r$, then the standard estimate $\#\{x \in \mathbb{Z}^n \cap P B : f(x) = q\} \ll_q \#(B)^{1 - \delta}$ would lead to

$$\frac{P^n}{\log P} = f_{\mathbb{Q}} P^n / \log P \gg \sum_{i=1}^r \#\{x \in \mathbb{Z}^n \cap P B : f(x) = q\} \ll_q \#(B)^{1 - \delta} P^n,$$

which is a contradiction. \hfill \Box

1.3. **Square-free integers represented by polynomials of general shape.** An integer $m$ is called square-free if for every prime $p$ we have $p^2 \nmid m$. In particular, 0 is not square-free and $m$ is square-free if and only if $-m$ is.

Assume that we are given a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ that is separable as an element of $\mathbb{Q}[x_1, \ldots, x_n]$ and let $f_0$ and $\sigma_f$ be as in (1.2). A similar approach to the one for Theorem 1.3 allows us to study the set $S_f := \{x \in \mathbb{Z}^n : f(x) \text{ is square-free}\}$.

**Theorem 1.5.** Assume that $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is any integer polynomial which is separable as an element of $\mathbb{Q}[x_1, \ldots, x_n]$ and let $B \subset \mathbb{R}^n$ be any non-empty closed box. If

$$n - \sigma_f > \max\left\{1, \frac{1}{3}(\deg(f) - 1)2^{\deg(f)}\right\},$$

then there exists $\beta = \beta(f) > 0$ such that for all $P \geq 2$ the equality

$$\frac{\#\{S_f \cap P B\}}{\#(\mathbb{Z}^n \cap P B)} = \prod_{p \text{ prime}} \left(1 - p^{-2n} \#\{x \in (\mathbb{Z}/p^2\mathbb{Z})^n : f(x) \equiv 0 \mod{p^2}\}\right) + O_{f, B}(P^{-\beta})$$

holds with an implied constant that depends at most on $f$ and $B$.

The problem of square-free values of integer polynomials has a very long history, see [3] for a list of references. Many cases are still open, for example, there is no irreducible quartic
integer polynomial in one variable for which we know that it takes infinitely many square-free. One of the most general results, conditional on the \(abc\) conjecture, is due to Poonen \[29\] where polynomials of arbitrary shape are treated. Our Theorem 1.5 covers unconditionally general polynomials of fixed degree and number of variables with the proviso that the number of variables is suitably large compared to the degree. Theorem 1.5 features a saving of two thirds of the variables compared to the Birch setting \[2\]. This saving comes from the fact that exponential sums whose terms are restricted to square-free integers can be bounded in a satisfactory manner, this was done in the work of Brüdern, Granville, Perelli, Vaughan and Wooley \[7\] and Keil \[27\].

**Notation.** We shall use the notation \(x\) to refer to vectors \(x = (x_1, \ldots, x_n)\) for \(n\)-tuples. We will also make use of the classical von Mangoldt function denoted \(\Lambda\) and of the classical Möbius function denoted \(\mu\). The letter \(d\) will refer exclusively to the degree of the polynomial \(f\) in Theorem 1.3. Finally, throughout the paper, we shall make use of the notation

\[
e(z) := \exp(2\pi iz), z \in \mathbb{C}.
\]

(1.7)

The polynomial \(f\) and the box \(\mathcal{B}\) will be considered constant throughout. This is taken to mean that, although each implied constant in the big \(O\) notation will depend on several quantities related to \(f\), we shall avoid recording these dependencies. The list of the said quantities consists of

\[f_0, n, d, \sigma_f, \theta_0, \delta, \eta, \lambda_1, A, \lambda,\]

whose meaning will become evident in due course. The symbol \(\varepsilon\) will be used for a small positive parameter whose value may vary, allowing, for example, inequalities of the form

\[x^\varepsilon \ll x^{\varepsilon/4}.\]

Further dependency of the implied constants on other quantities will be recorded explicitly via an appropriate use of subscript.

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2. The proof of Theorem 1.1

Denote by \(\mathbb{Q}_v\) the completion of \(\mathbb{Q}\) with respect to the place \(v\), let \(|\cdot|_p\) be the \(p\)-adic norm defined by \(|x|_p = p^{-\nu_p(x)}\) for \(x \in \mathbb{Q}_p\) if \(v = p\) is finite and define \(|\cdot|_\infty\) as the classical absolute value for the real place. We will use the notation \(\mathbb{Z}_S = \mathbb{Z}[S^{-1}]\) for any finite set of finite places \(S\).

2.1. Preliminary lemmas. We begin by establishing the following analogue of \[9\] Lem. 2].

**Lemma 2.1.** Let \(f \in \mathbb{Z}[x_1, \ldots, x_n]\) be a non-zero polynomial with content equal to 1. If \(p\) is such that \(f(\mathbb{Z}^n) \subseteq p\mathbb{Z}\), then \(p \leq \deg(f)\).

**Proof.** Define \(d := \deg(f)\) and let \(p\) be a prime such that \(f(\mathbb{Z}^n) \subseteq p\mathbb{Z}\). On one hand, we have by assumption that

\[
\# \{x \in (\mathbb{Z}/p\mathbb{Z})^n : f(x) \equiv 0 \pmod{p} \} = p^{n+1}.
\]

On the other hand, since \(f\) has content one, \[33\] Lemma 2.7 implies that

\[
\# \{x \in (\mathbb{Z}/p\mathbb{Z})^n : f(x) \equiv 0 \pmod{p} \} \leq dp^n
\]

and hence \(p \leq d\), thus concluding the proof of the lemma. \(\square\)
We now use Lemma 2.1 to verify the following analogue of [20, Prop.1.2] and of the hypothesis (H₁) of [11] over \( \mathbb{Q} \).

**Proposition 2.2.** Let \( f \in \mathbb{Q}[x₁, \ldots, xₙ] \) be an irreducible homogeneous polynomial satisfying the assumptions \( \{1.3\} \) and \( f(1,0, \ldots, 0) > 0 \) and suppose we are given \((λ₁, p), \ldots, (λₙ, p)\) \( \in \mathbb{Q}_p^n \) for \( p \) in a finite set of finite places \( S \) containing all primes \( p \leq \deg(f) \) and such that \( f \) does not have \( p \)-integral coefficients as well as all primes \( p \) such that \( ν_p(f(1,0, \ldots, 0)) > 0 \), \( C \) a positive real constant and \( ε > 0 \). Then there exists infinitely many \((λ₁, \ldots, λₙ) \in \mathbb{Z}_S^n \) such that

\[
λ_1 > Cλ_i > 0 \quad \text{for all } i \in \{2, \ldots, n\}, \quad |λ_i - λ_{i,p}|_p < ε \quad \text{for all } i \in \{1, \ldots, n\} \quad \text{and } p \in S \quad \text{and} \quad f(λ₁, \ldots, λₙ) = \ell u \quad \text{for a prime } \ell \notin S \quad \text{and } u \in \mathbb{Z}_S^*, \quad u > 0.
\]

**Proof.** Up to multiplication of \((λ₁, \ldots, λₙ)\) and \((λ₁, p), \ldots, (λₙ, p)\) by a product of powers of primes in \( S \), we can assume without loss of generality that \((λ₁, p), \ldots, (λₙ, p) \in \mathbb{Z}_p^n \) for \( p \in S \). The assumption that \( f(1,0, \ldots, 0) > 0 \) provides with \( a_i \in \mathbb{Q} \) and \( a > 0 \) such that

\[
f(x₁, \ldots, xₙ) = ax₁^d + \sum_{\substack{i₁+\cdots+iₙ=d \\atop 0 \leq i₁ \cdots iₙ \leq d \\atop i₁ \neq 0}} a_i x₁^{i₁} \cdots xₙ^{iₙ}, \tag{2.1}
\]

Let \( N \) be the number of \( i \) with \( a_i < 0 \). We can assume that \( C > 1 \) and that \( C > \frac{Na}{\sum a_i} \) whenever \( a_i < 0 \). As in the proof of [20, Prop.1.2], we can now find \((λ₀,1, \ldots, λ₀, n) \in \mathbb{Z}^n \) such that \(|λ₀,i - λ₀,p|_p < ε \) for all \( p \in S \). We can choose them such that \( λ₀,1 > Cλ₀,i > 0 \) for all \( i \in \{2, \ldots, n\} \). We can now see that \( f(λ₀,1, \ldots, λ₀, n) > 0 \) by alluding to

\[
f(λ₀,1, \ldots, λ₀, n) ≥ aλ₀,d - \sum_{a_i < 0} a_i λ₀,i \cdots λ₀,n,
\]

and the inequalities

\[
aλ₀,d = \frac{a}{N} \sum a_i λ₀,i = \frac{a}{N} \sum λ₀,i \cdots λ₀,n > \sum a_i C^{d-i} λ₀,i \cdots λ₀,n > \sum a_i λ₀,i \cdots λ₀,n.
\]

Let \( A = \prod_{p \in S} p \). For \( i \in \{1, \ldots, n\} \) we let \( λ_i = λ₀,i + x_i A^N \) for a fixed integer \( N \) big enough so that \(|λ_i - λ₀,i|_p < ε \) for all \( p \in S \) and

\[
g(x₁, \ldots, xₙ) = f(λ₁, \ldots, λₙ) = f(λ₀,1 + x₁ A^N, \ldots, λ₀,n + xₙ A^N).
\]

The polynomial \( g \) can be expressed as \( g = t\tilde{g} \) for \( t \in \mathbb{Z}_S^\times \) and \( \tilde{g} \) a polynomial with integer coefficients which is irreducible over \( \mathbb{Q} \). Let us denote by \( c \) the product of all fixed prime factors of \( \tilde{g} \). We will now establish that if \( p \) is a prime factor of \( c \) then \( p \in S \). Let \( p \mid c \). Either \( p \) divides the content of \( \tilde{g} \) and in particular, with the notation \( \{2.1\} \), \( ν_p(a) \neq 0 \) which immediately implies that \( p \in S \) either, denoting by \( \tilde{c} \) the content of \( \tilde{g} \), \( p \) is a fixed prime factor of the polynomial \( \tilde{g}/\tilde{c} \) which has integral coefficients and content equal to one. By Lemma 2.1 this implies that \( p \leq \deg(f) \) and hence that \( p \in S \). Moreover, \( \tilde{g}_0 = A^{dN} f \) and the conditions \( x₁ > Cx_i > 0 \) define an open cone in \( \mathbb{R}^n \). In addition, when \( f \) is evaluated at \((λ₀,1, \ldots, λ₀, n) \in C \) it produces a strictly positive value, therefore we can find a box \( \mathcal{B} \subseteq C \) such that \( f(\mathcal{B}) \subset (0, \infty) \). Since for all \( P \) we have \( P \mathcal{B} \subset C \), we obtain from Theorem 1.3 that there exist infinitely many \( x \in \mathbb{Z}^n \cap C \) such that \( \tilde{g}(x)/c \) is prime. This yields the result because \( λ₀,1 > Cλ₀,i > 0 \) and \( x₁ > Cx_i > 0 \), which implies \( λ₁ = (λ₀,1 + x₁ A^N) > Cλ_i = C(λ₀,i + x_i A^N) \). \( \square \)
2.2. Conclusion of the proof of Theorem 1.1

Proof. We proceed by adapting the proof of [20, Th.1.3]. We are given $1 > \varepsilon > 0$, a finite set of places $S$ and a solution $(t_v, x_{1,v}, \ldots, x_{r,v}) \in X(\mathbb{Q}_v)$ for every place $v \in S$ and we want to find $(t, x_1, \ldots, x_r) \in X(\mathbb{Q})$ such that for all $v \in S$,

\[
\begin{cases}
|t_i - t_{i,p}|_v < \varepsilon & (i \in \{1, \ldots, n\}), \\
|x_{j,i} - x_{j,i,v}|_v < \varepsilon & (i \in \{1, \ldots, r\}, \ j_i \in \{1, \ldots, d_i\}).
\end{cases}
\]

2.2.1. First step. By density and continuity, we can assume that $t_\infty \in \mathbb{Q}^n$ and by a linear change of variables, we can assume that $t_\infty = (1, 0, \ldots, 0)$. Note that the solubility over $\mathbb{R}$ implies that $f(1, 0, \ldots, 0) > 0$ in the case where there is a totally imaginary $K_i$. In addition, it implies that $f(1, 0, \ldots, 0)$ can be strictly positive or strictly negative when all $K_i$ are totally real. We denote by $s \in \{-1, +1\}$ the sign of $f(1, 0, \ldots, 0)$. We can enlarge $S$ so that the field $K_i$ is unramified outside $S$ for all $i \in \{1, \ldots, r\}$, $S$ contains all primes $p \leq \deg(f)$ and such that $f$ does not have $p$-integral coefficients as well as primes $p$ such that $\nu_p(f(1, 0, \ldots, 0)) > 0$.

2.2.2. Second step. Let $L = [d_1, \ldots, d_r]$ and $M = \max_{v \in S} \max_{1 \leq i \leq r} |x_{i,v}|_v$. By [14, Prop.6.1], we know that the map $N_{K_{i,p}/\mathbb{Q}_p} : K_{i,p}^* \to \mathbb{Q}_p^*$ is open and that a polynomial function is continuous for the $p$-adic topology hence there exists $\varepsilon' > 0$ such that $|\lambda_i - t_{i,p}|_p < \varepsilon'$ implies that $f(\lambda_1, \ldots, \lambda_n)$ is a local norm for $K_i/\mathbb{Q}$ for the place $p$ and there exists $\varepsilon'' > 0$ such that $|\lambda_i - t_{i,p}|_p < \varepsilon''$ implies that

\[
|f(\lambda_1, \ldots, \lambda_n) - f(t_{1,p}, \ldots, t_{n,p})|_p < \frac{\varepsilon}{2M}|f(t_{1,p}, \ldots, t_{n,p})|_p|L|_p.
\]

(2.2)

Then applying Proposition 2.2 yields $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_S^n$ such that $\lambda_1 > C\lambda_i > 0$ for all $i \in \{2, \ldots, n\}$, $|\lambda_i - t_{i,p}|_p < \min\{\varepsilon/2, \varepsilon', \varepsilon''/2\}$ for all $i \in \{1, \ldots, n\}$ and $p \in S$ and $f(\lambda_1, \ldots, \lambda_n) = s\ell u$ for a prime $\ell \notin S$ and $u \in \mathbb{Z}_S^*$ with $u > 0$. We thus obtain that $f(\lambda_1, \ldots, \lambda_n)$ is a local norm for $K_i/\mathbb{Q}$ for all places of $S$. This is also the case for the real place because $f(\lambda_1, \ldots, \lambda_n) > 0$ in the case that there is a totally imaginary $K_i$.

2.2.3. Third step. Now, $f(\lambda_1, \ldots, \lambda_n) = \ell s u$ is a unit for every $\mathbb{Q}_p$ and $p \notin S \cup \{\ell\}$ and we know by [20, Prop.3.11] that this implies that $f(\lambda_1, \ldots, \lambda_n)$ is a local norm for $K_i/\mathbb{Q}$ for all $p \notin S \cup \{\ell\}$. By the global reciprocity law and the fact that $K_i/\mathbb{Q}$ is unramified outside $S$ we see that $f(\lambda_1, \ldots, \lambda_n)$ is also a local norm for $K_i/\mathbb{Q}$ at the place $\ell$. The conclusion is that $f(\lambda_1, \ldots, \lambda_n)$ is a local norm for $K_i/\mathbb{Q}$ at every place of $\mathbb{Q}$ and then by the Hasse norm principle [26, Th.12.9], one gets that there exists $(x_1, \ldots, x_r) \in \mathbb{Q}^{d_1} \times \cdots \times \mathbb{Q}^{d_r}$ such that

\[0 \neq f(\lambda_1, \ldots, \lambda_n) = N_{K_i/\mathbb{Q}}(x_1) = \cdots = N_{K_i/\mathbb{Q}}(x_r).
\]

2.2.4. Fourth step. By continuity, there exists $\varepsilon_1 > 0$ such that for all $q_1 \in \mathbb{Q}^\times$ such that $|q_1 - \lambda_1|_\infty < \varepsilon_1$, then $\left|\frac{1}{q_1} - \frac{1}{\lambda_1}\right|_\infty < \frac{\varepsilon}{2\max_{1 \leq i \leq n} |\lambda_i|_\infty}$. Writing $m = [L, \deg(f)]$, by weak approximation in $\mathbb{Q}$, since $\lambda_1 > 0$ one can find $\rho \in \mathbb{Q}$ such that $|\rho - 1|_p < \min\{\frac{\varepsilon}{2}, \varepsilon''/2\}$ min\{1, |\lambda_1|_p\} for all $p \in S$ and $|\rho^{m/\deg(f)} - \lambda_1|_\infty < \min\{\frac{\varepsilon}{2}, \min\{1, \lambda_1 - \frac{\varepsilon}{2}\}, \varepsilon_1\}$ where we can assume that $\varepsilon < 2\lambda_1$. In particular, this implies that $|\rho|_p = 1$ for all $p \in S$. We now make the following change of variables,

\[
\lambda_i = \rho^{m/\deg(f)} \lambda_i', \quad i \in \{1, \ldots, n\}, \quad x_i = \rho^{m/d_i} x_i', \quad i \in \{1, \ldots, r\},
\]

such that $(\lambda_1', \ldots, \lambda_n') \in \mathbb{Z}_S^n$.
so that for all finite place \( p \in S \) we have

\[
|\lambda'_i - \lambda_i|_p = |\lambda'_i|_p \rho^{n/\deg(f)} - 1|_p \leq |\lambda_i|_p \rho - 1|_p < \frac{\varepsilon}{2}
\]

and therefore \( |\lambda'_i - t_i|_p < \varepsilon \) for all \( i \in \{1, \ldots, n\} \). Moreover, we have

\[
0 \neq f(\lambda'_1, \ldots, \lambda'_n) = N_{K_i/Q}(\lambda'_1) = \cdots = N_{K_i/Q}(\lambda'_n).
\]  

(2.3)

As for the real place, we have \( |\lambda'_i - 1|_\infty < \varepsilon \) and \( |\lambda'_i - \lambda_i|_\infty < \varepsilon/2 \). The treatment of the archimedian place is now concluded similarly as in [20], by alluding to 0 < \( \lambda_i/\lambda_1 < C^{-1} \) and by taking \( C \) big enough, namely \( C > \frac{2}{\varepsilon} \).

2.2.5. Fifth step. To conclude the proof, it remains to find \( (x''_1, \ldots, x''_r) \in \mathbb{Q}^{d_1} \times \cdots \times \mathbb{Q}^{d_r} \) \( v \)-adically close to \( (x_{1,v}, \ldots, x_{r,v}) \) for all \( v \in S \) and such that \( N_{K_i/Q}(x''_1) = N_{K_i/Q}(x'_1) \) for all \( i \in \{1, \ldots, r\} \). By (2.2) and the choice of \( \lambda' \), one can write \( f(\lambda'_1, \ldots, \lambda'_n) = f(t_{1,p}, \ldots, t_{n,p})\beta_p \) with \( \beta_p \in \mathbb{Q}_p \) satisfying \( |\beta_p - 1|_p < \frac{1}{2M}|L|_p \) for all finite place \( p \in S \). In particular, \( |\beta_p|_p = 1 \) and \( \beta_p \in \mathbb{Z}_p^* \) for all finite place \( p \in S \). Now, Hensel’s lemma implies that there exists \( \alpha_p \in \mathbb{Z}_p \) such that \( \beta_p = \alpha_p^L \) and

\[
|\alpha_p - 1|_p = \left| \frac{\beta_p - 1}{L} \right|_p < \frac{\varepsilon}{2M}.
\]

Of course, there exists \( \alpha_\infty \) such that \( \beta_\infty = \alpha_{\infty}^L \) and \( |\alpha_\infty - 1|_\infty < \varepsilon/(2M) \) since one can always ensure that \( f(1,0,\ldots,0) \) and \( f(\lambda'_1, \ldots, \lambda'_n) \) have the same sign. Now alluding to the facts that \( (t_{1,v}, x_{1,v}, \ldots, x_{r,v}) \in X(\mathbb{Q}_v) \) and to (2.3), we obtain that for all \( v \in S \)

\[
0 \neq f(\lambda'_1, \ldots, \lambda'_n) = N_{K_i/Q}(\alpha_v^{L/d_i}x_{1,v}) = \cdots = N_{K_i/Q}(\alpha_v^{L/d_r}x_{r,v}).
\]

In other words, for every \( i \in \{1, \ldots, r\} \), we have \( N_{K_i/Q}(\alpha_v^{L/d_i}x_{i,v}) = N_{K_i/Q}(x'_i) \) for all \( v \in S \). Thanks to the fact that weak approximation holds for the norm tori \( N_{K_i/Q}(z) = 1 \), one gets the existence of \( x''_i \in \mathbb{Q}^{d_i} \) such that \( |x''_{j,i} - \alpha_v^{L/d_i}x_{j,i}\!\!_v|_v < \varepsilon/2 \) for all \( v \in S \) and \( j \in \{1, \ldots, d_i\} \) and \( N_{K_i/Q}(x''_i) = N_{K_i/Q}(x'_i) \). Therefore, we have

\[
0 \neq f(\lambda'_1, \ldots, \lambda'_n) = N_{K_i/Q}(x''_1) = \cdots = N_{K_i/Q}(x''_r)
\]

along with

\[
|x''_{j,i} - x_{j,i}\!\!_v|_v \leq |x''_{j,i} - \alpha_v^{L/d_i}x_{j,i}\!\!_v|_v + |x_{j,i}\!\!_v|_v|\alpha_v - 1|_v < \varepsilon,
\]

thus concluding the proof of Theorem 1.1.

\[\square\]

3. The proof of Theorem 1.3

3.1. First steps and auxiliary estimates. The proof of Theorem 1.3 is initiated by using the following exponential sums for real \( \alpha \),

\[
S(\alpha) := \sum_{x \in \mathbb{Z}^n \cap P\mathcal{B}} e(\alpha f(x)) \quad \text{and} \quad W(\alpha) := \sum_{\frac{1}{d} \min\{f_0(\mathcal{B})\} \leq \alpha \leq 2 \max\{f_0(\mathcal{B})\}} e(\alpha p),
\]

(3.1)

where we used that, in the setting of Theorem 1.3, the succeeding quantities are positive

\[
\min\{f_0(\mathcal{B})\} = \min\{f_0(x) : x \in \mathcal{B}\} \quad \text{and} \quad \max\{f_0(\mathcal{B})\} := \max\{f_0(x) : x \in \mathcal{B}\}.
\]
The fact that \( \int_{0}^{1} e(\alpha \{ f(x) - p \}) d\alpha \) is 1 when \( f(x) = p \) and is otherwise 0, shows that for all \( P \gg f, B \) 1, we have the equality

\[
\pi_f(P\mathcal{B}) = \int_{0}^{1} S(\alpha)\overline{W(\alpha)} d\alpha. \tag{3.2}
\]

This identity has the useful feature that it completely separates the problem of evaluating \( \pi_f \) into two problems, one regarding the evaluation of the sum \( S \) (that is only related to the values of the polynomial \( f \)) and one regarding the evaluation of the sum \( W \) (that is only related to the distribution of primes). Birch [2] has a similar identity, save for the factor \( W_p\alpha \). The main idea is that the presence of this extra factor can be turned to our advantage, as it attains small values for certain \( \alpha \) for which \( |S(\alpha)| \) is large. Let us comment that we could have defined \( W \) in an alternative way by replacing the range for the primes \( p \) by the condition \( \min \{ f_0(\mathcal{B}) \} P^d \leq p \leq \max \{ f_0(\mathcal{B}) \} P^d \), however, our choice will make more transparent the proof of Lemma 3.1.

Before proceeding let us recall here the estimates from the work of Birch [2] that we shall need later. First, following [2, pg.251,Eq.(5)], we let for \( \theta \in (0, 1) \) and \( a \in \mathbb{Z} \cap [0, q) \) with \( \gcd(a, q) = 1 \),

\[
\mathcal{M}_{a,q}(\theta) := \{ \alpha \in (0, 1] : 2|q\alpha - a| \leq P^{-d+(d-1)\theta} \} \tag{3.3}
\]

and

\[
\mathcal{M}(\theta) = \bigcup_{1 \leq q \leq P^{(d-1)\theta}} \bigcup_{\substack{a \in \mathbb{Z} \cap [0, q) \\\gcd(a, q) = 1}} \mathcal{M}_{a,q}(\theta). \tag{3.4}
\]

Birch then gives the following upper bound for the volume of \( \mathcal{M}(\theta) \).

Lemma 3.1 (Birch [2, Lemma 4.2]). \( \mathcal{M}(\theta) \) has volume at most \( P^{-d+2(d-1)\theta} \).

Next, we choose any positive \( \delta, \theta_0 \) satisfying

\[
1 > \delta + 6d\theta_0 \quad \text{and} \quad \frac{n - \sigma_f}{2d-1} - (d - 1) > \delta\theta_0^{-1}. \tag{3.5}
\]

As in [2, pg.252,Eq.(13)-(14)] it is easy to see that there exists \( T \in \mathbb{N} \) and positive real numbers \( \theta_1, \ldots, \theta_T \) with the properties

\[
\begin{cases}
T \ll P^\delta, \\
\theta_T > \theta_{T-1} > \ldots > \theta_1 > \theta_0 > 0, \\
d = 2(d - 1)\theta_T \\
\frac{1}{2} \delta > 2(d - 1)(\theta_{t+1} - \theta_t) \quad \text{for} \ 0 \leq t \leq T - 1.
\end{cases} \tag{3.6}
\]

We next recall [2, Lem.4.3]. Note that it was proved for homogeneous \( f \), but, as noted by Schmidt [31, §9] a similar argument works for inhomogeneous \( f \), because the Weyl differencing process is not affected by lower order terms.

Lemma 3.2 (Birch [2 Lemma 4.3]). Let \( 0 < \theta \leq 1 \) and \( \varepsilon > 0 \). Then if \( \alpha \) is not in \( \mathcal{M}(\theta) \) modulo 1,

\[
|S(\alpha)| \ll P^{n - \frac{n - \sigma_f}{2d-1} + \varepsilon}.
\]


Following the notation in [2, pg.253] and for $\theta, a, q$ as above we also let
\[ M_{a,q}^1(\theta) := \left\{ \alpha \in (0, 1] : |qa - a| \leq qP^{-d+(d-1)\theta} \right\} \]  
and
\[ M'(\theta) = \bigcup_{1 \leq q \leq P^{(d-1)\theta}} \bigcup_{ae \mathbb{Z} \cap [0,q)} a \gcd(a,q) = 1 M_{a,q}'(\theta). \]  

With $\theta_0$ as in (3.5), we let
\[ \eta := (d-1)\theta_0 \]  
and for $a \in \mathbb{Z}, q \in \mathbb{N}$ we define
\[ S_{a,q} := \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^n} e\left( \frac{f(x)}{q} \right). \]  

Finally, for any $\gamma \in \mathbb{R}$ and any measurable $C \subset [-1,1]^n$ we define
\[ I(C; \gamma) := \int_{x \in C} e(\gamma f_0(x)) \, dx. \]

The following result, due to Birch, gives an upper bound for the quantity $I(C; \gamma)$.

**Lemma 3.3** (Birch [2, Lemma 5.2]). Let $C$ be a box contained in $[-1,1]^n$ with sidelength at most $\sigma < 1$. Then
\[ I(C; \gamma) \ll \sigma^n \min \left[ 1, (\sigma^d \max_{1 \leq i \leq n} |\gamma_i|)^{-\frac{n-d}{2d-1(d-1)+\varepsilon}} \right]. \]

The next result was proved by Birch with $f$ instead of $f_0$ in the definition of $I(C; \gamma)$, however the following result holds in light of the remarks concerning $\mu(x, B)$ in Schmidt’s work [31, §9].

**Lemma 3.4** (Birch [2, Lemma 5.1]). Assume that we are given coprime integers $q \in \mathbb{N}$ and $a \in \mathbb{Z} \cap [0,q)$ and let $\alpha \in M_{a,q}(\theta_0)$ with the notations (3.7) and (3.7). Denoting $\beta := \alpha - \frac{a}{q}$, we have
\[ S(\alpha) = q^{-n} S_{a,q} I(\mathcal{B}; P^d \beta) + O(P^{n-1+2\eta}). \]

Let us now turn to the quantity $S_{a,q}$ defined in (3.10).

**Lemma 3.5** (Birch [2, Lemma 5.4]). For every $\varepsilon > 0$ and for $a \in \mathbb{Z}, q \in \mathbb{N}$ with $\gcd(a, q) = 1$ we have
\[ S_{a,q} \ll q^{n-\frac{n-\sigma_f}{2d-1(d-1)+\varepsilon}}. \]

The next result is of key importance in the proof of Theorem 1.3. It is concerned with the convergence of the singular series and its proof relies on recent results on the estimation of exponential sums due to Browning–Heath-Brown [4] and Browning–Prendiville [6].

**Proposition 3.6.** Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be an irreducible polynomial and define
\[ T_f(q) := q^{-n} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^n} |S_{a,q}|, \quad q \in \mathbb{N}. \]

(1) If $n - \sigma_f \geq \max\{5, (\deg(f) - 1)^2 \deg(f) - 1 + 2\}$ then the abscissa of convergence of the Dirichlet series of $T_f$ is strictly negative.
(2) If \( n - \sigma_f \geq \max\{4, (\deg(f) - 1)2^{\deg(f) - 1} + 1\} \) then there exists a constant \( C' = C'(f) > 0 \) such that \( \sum_{\substack{q \leq x}} T_f(q) \ll (\log x)^{C'} \).

**Proof.** Part (1). It is sufficient to prove that there exists \( \lambda_1 > 0 \) such that \( \sum_q q^{\lambda_1}T_f(q) < \infty \). By [2, §7], the function \( T_f \) is multiplicative, hence the series over \( q \) converges absolutely if the analogous Euler product converges absolutely, i.e.

\[
\sum_{p \text{ prime } k \in \mathbb{N}} p^{k\lambda_1}T_f(p^k) = \sum_{p \text{ prime } k \in \mathbb{N}} p^{k(\lambda_1-n)} \sum_{\sigma \in \mathbb{Z}/p^k\mathbb{Z}^*} |S_{a,p}| < \infty. \tag{3.12}
\]

By Lemma [5.5] the terms with \( k > 2^{d-1}(d-1) \) contribute

\[
\ll \sum_p \sum_{k \geq 1 + 2^{d-1}(d-1)} p^{k \left(1 - \frac{n-\sigma_f}{2^{d-1}(d-1)} + \varepsilon + \lambda_1\right)}.
\tag{3.13}
\]

By the assumption \( n - \sigma_f \geq (d-1)2^{d-1} + 2 \)

\[
\frac{n - \sigma_f}{(d-1)2^{d-1}} \geq 1 + \frac{2}{(d-1)2^{d-1}}
\]

we have

\[
p^{\left(1 - \frac{n-\sigma_f}{2^{d-1}(d-1)} + \varepsilon + \lambda_1\right)} \leq p^{\left(- \frac{2}{(d-1)2^{d-1}} + \varepsilon + \lambda_1\right)},
\]

thus taking \( \varepsilon, \lambda_1 \) sufficiently small we can ensure that this is at most \( p^{-\frac{1}{2^{d-1}(d-1)}} \leq 2^{-\frac{1}{2^{d-1}(d-1)}} \), which is of the form \( 1 - \delta \) for some \( 0 < \delta < 1 \). Note that if \( \delta \in (0,1) \) then for all \( z \in \mathbb{R} \) with \( 0 \leq z \leq 1 - \delta \) and all \( k_0 \in \mathbb{N} \) we have

\[
\sum_{k \geq k_0} z^m = \frac{z^{k_0}}{1-z} \leq \frac{z^{k_0}}{\delta}.
\]

Therefore the sum in (3.13) is

\[
\ll \sum_p p^{(1+2^{d-1}(d-1))\left(1 - \frac{n-\sigma_f}{2^{d-1}(d-1)} + \varepsilon + \lambda_1\right)} \ll \sum_p p^{-2 + (\varepsilon + \lambda_1)(1 + 2^{d-1}(d-1))} \ll \sum_p p^{-3/2} < \infty,
\]

where we have taken \( \varepsilon, \lambda_1 \) sufficiently small to ensure \( (\varepsilon + \lambda_1)(1 + 2^{d-1}(d-1)) \leq 1/2 \). Next, we study the contribution towards (3.12) of any \( k \in [2, 2^{d-1}(d-1)] \). By [4, Lemma 25] we infer that the said contribution is

\[
\ll \sum_p p^{k(\lambda_1-n)+k(k-1)n+\sigma_f} = \sum_p p^{(1+\lambda_1)k-n+\sigma_f} \ll \sum_p p^{(1+\lambda_1)(2^{d-1}(d-1)) - n + \sigma_f}.
\]

The assumption \( n - \sigma_f \geq (d-1)2^{d-1} + 2 \) shows that the exponent is \( \leq \lambda_1(2^{d-1}(d-1)) - 2 \) and for small \( \lambda_1 \) the sum converges. To conclude the proof of (3.12) it only remains to bound the contribution of terms with \( k = 1 \). As noted in [6, §5], one can prove

\[
T_f(p) = p^{-n} \sum_{\sigma \in \mathbb{Z}/p^k\mathbb{Z}^*} |S_{a,p}| \ll p^{1 - \frac{n-\sigma_f}{2^2}} \tag{3.14}
\]

by Deligne’s estimate and induction on \( \sigma_f \). Taking small \( \lambda_1 < 1/4 \) and using the assumption \( n - \sigma_f \geq 5 \) shows that the terms with \( k = 1 \) in (3.12) form a convergent series. This completes our proof.

Part (2). If \( k \in [2, 2^{d-1}(d-1)] \) then [4, Lemma 25] and \( n - \sigma_f \geq 1 + 2^{d-1}(d-1) \) imply that \( T_f(p^k) \ll p^{-1} \). Furthermore, using Lemma [5.5] and \( n - \sigma_f \geq 1 + 2^{d-1}(d-1) \) we have
that if $k \geq 1 + 2^{d-1}(d-1)$ then $T_f(p^k) \ll p^{-1-2^{d-1}d^{-1}}$. Finally, $n - \sigma_f \geq 4$, thus (3.14) ensures that $T_f(p) \ll p^{-1}$. Putting everything together yields $\sum_{k \geq 1} T_f(p^k) \leq C'p^{-1}$ for some $C' = C'(f) > 0$ and the proof is concluded by using $\sum_{q \leq x} T_f(q) \leq \prod_{q \leq x}(1+\sum_{k \geq 1} T_f(p^k))$. □

3.2. The minor arcs. For $\theta \in (0, 1]$ and $a \in \mathbb{Z} \cap [0, q]$ with $\gcd(a, q) = 1$ we use the sets $\mathcal{M}(\theta)$ and $\mathcal{M}_{a,q}(\theta)$ defined by (3.4) and (3.3). Next, we choose any positive $\delta, \theta_0$ satisfying (3.5).

**Lemma 3.7.** For any $0 < \theta \leq 1$ we have
\[
\left| \int_{\alpha \in \mathcal{M}\setminus(\theta)} S(\alpha)\overline{W(\alpha)}d\alpha \right| \ll \left( \int_{\alpha \in \mathcal{M}\setminus(\theta)} |S(\alpha)|^2 d\alpha \right)^{1/2} P^{d/2}(\log P)^{-1/2}.
\]

**Proof.** By Schwarz’s inequality the integral on the left side is bounded by
\[
\left( \int_{\alpha \in \mathcal{M}\setminus(\theta)} |S(\alpha)|^2 d\alpha \right)^{1/2} \left( \int_0^1 |W(\alpha)|^2 d\alpha \right)^{1/2}.
\]

The proof is concluded by noting that
\[
\int_0^1 |W(\alpha)|^2 d\alpha = \sum_{\frac{1}{2} \min\{f_0(\mathcal{A})\} P^{d} \leq P^{d}} 1 \ll P^d/\log P.
\]

**Lemma 3.8.** Keep the assumptions of Theorem 1.3 and (3.5). Then we have,
\[
\left( \int_{\alpha \in \mathcal{M}\setminus(\theta_0)} |S(\alpha)|^2 d\alpha \right)^{1/2} = O(P^{n-d/2-\delta/2}).
\]

**Proof.** Using the entities $(\theta_t)_{t=0}^T$, given in (3.6), we have for sufficiently small $\varepsilon > 0$,
\[
\int_{\alpha \in \mathcal{M}\setminus(\theta_0)} |S(\alpha)|^2 d\alpha \ll P^{2(n-(\frac{n-\sigma_f}{2d-f})+\varepsilon) + \varepsilon} \ll P^{2n-d-\delta},
\]
due to Lemma 3.2, the third equation of (3.6) and (1.4). For $t < T$ and $\varepsilon > 0$ we get
\[
\int_{\mathcal{M}\setminus(\theta_t+1)\setminus\mathcal{M}\setminus(\theta_t)} |S(\alpha)|^2 d\alpha \ll P^{-d+2(d-1)\theta_t+2(n-(\frac{n-\sigma_f}{2d-f})+\varepsilon)
\]
by Lemmas 3.1 and 3.2. The proof can now be completed easily by using the last equation of (3.6), (3.5) and $T \ll P^\delta$, as in the last stage of the proof of [2, Lem.4.4]. □

**Lemma 3.9.** Keep the assumptions of Theorem 1.3 and (3.5). Then we have,
\[
\left| \int_{\alpha \in \mathcal{M}\setminus(\theta_0)} S(\alpha)\overline{W(\alpha)}d\alpha \right| = O(P^{n-\delta/2}).
\]

**Proof.** The proof follows immediately by tying together Lemmas 3.7 and 3.8 □

Recall the definition of $\mathcal{M}'(\theta_0)$ and $\mathcal{M}_{a,q}'(\theta_0)$ given in (3.8) and (3.7). The next lemma is analogous to [2, Lem.4.5].

**Lemma 3.10.** Keep the assumptions of Theorem 1.3 and (3.5). Then we have
\[
\pi_f(P\mathcal{B}) = \sum_{q \leq P^{(d-1)/2}} \sum_{a \in \mathbb{Z}\cap [0, q]} \gcd(a,q)=1 \int_{\mathcal{M}_{a,q}'(\theta_0)} S(\alpha)\overline{W(\alpha)}d\alpha + O(P^{n-\delta/2}).
\]
Before proceeding we note that one can take an arbitrarily small positive value for \( \theta_0 \) in Lemma 3.10 because the system of inequalities (3.5) can be solved for any \( \theta_0 > 0 \) small enough. This will come at the cost of a worse error term in Lemma 3.10, however, it will still exhibit a power saving and it will thus be acceptable for the purpose of verifying Theorem 1.3.  

3.3. The intermediate range. Under the assumptions of Theorem 1.3 and (3.5) we can use Lemmas 3.1, 3.3 and the trivial bound \( W(\alpha) \ll P^d \) to evaluate the quantity \( S(\alpha) \) in Lemma 3.10. This yields

\[
\frac{\pi_f(P B)}{P^n} - \sum_{q \leq P^{(d-1)\theta_0}} q^{-n} \sum_{a \in \mathbb{Z} \cap [0,q)} \sum_{\gcd(a,q)=1} S_{a,q} \int_{|\gamma| \leq P^n} I(\mathcal{B}; \gamma) W(a/q + \gamma P^{-d}) \frac{1}{P^d} \ d\gamma \ll (\log P)^{-A}, \tag{3.15}
\]

valid for all \( A > 0 \), where \( \eta, S_{a,q} \) and \( I(\mathcal{B}; \gamma) \) are defined respectively in (3.9), (3.10) and (3.11).

For \( A, q \in \mathbb{N} \) and \( a \in \mathbb{Z} \cap [0,q) \) with \( \gcd(a,q) = 1 \) we let

\[
\mathcal{M}_{a,q}(A) := \{ \alpha \in \mathbb{R} \mod 1 : |\alpha - a/q| \leq P^{-d}(\log P)^A \},
\]

\[
\mathcal{M}(A) := \bigcup_{1 \leq q \leq (\log P)^A} \bigcup_{a \in \mathbb{Z} \cap [0,q)} \mathcal{M}_{a,q}(A), \tag{3.17}
\]

and we observe that \( \mathcal{M}(A) \subset \mathcal{M}'(\theta_0) \) for all \( P \gg 1 \). We denote the difference by

\[
t(A) := \mathcal{M}'(\theta_0) \setminus \mathcal{M}(A). \tag{3.18}
\]

The set \( t(A) \) is therefore to be thought of as lying ‘between’ the major arcs \( \mathcal{M}'(\theta_0) \) and the minor arcs \( [0,1) \setminus \mathcal{M}'(\theta_0) \). We shall see in 3.10 that \( \mathcal{M}(A) \) gives rise to the main term in Theorem 1.3.

Next, we observe that Lemma 3.3 and our assumption \( n - \sigma_f \geq 1 + 2d-1(d-1) \) yield

\[
\int_{|\gamma| \geq Q} |I(\mathcal{B}; \gamma)| \ d\gamma \ll Q^{-\frac{1}{2(d-1)}}, \quad (Q \gg 1), \tag{3.19}
\]

in particular showing that \( \int_{\mathbb{R}} |I(\mathcal{B}; \gamma)| \ d\gamma \) converges under assumption (1.4).

**Lemma 3.11.** If (1.4) holds then

\[
\sum_{(\log P)^A < q < P^n} q^{-n} \sum_{a \in \mathbb{Z} \cap [0,q)} \sum_{\gcd(a,q)=1} |S_{a,q}| \int_{|\gamma| \leq P^n} |I(\mathcal{B}; \gamma)| \frac{|W(a/q + \gamma P^{-d})|}{P^d} \ d\gamma \ll (\log P)^{-A/2+3+C'}. \]

**Proof.** If \( \alpha \) is not in the union of the sets \( \{ \alpha \mod 1 : |\alpha - a/q| \leq P^{-d+(d-1)\theta_0} \} \) taken over all \( q \in \mathbb{N} \cap [1, (\log P)^A] \) and \( a \in \mathbb{Z} \cap [0,q) \) with \( \gcd(a,q) = 1 \), then by Dirichlet’s approximation theorem there are coprime integers \( 1 \leq a' \leq q' \) with \( q' \leq P^{d-(d-1)\theta_0} \) and \( |\alpha - a'/q'| \leq P^{-d+(d-1)\theta_0}/q' \). Thus we must have \( q' > (\log P)^A \). Alluding to Vaughan’s estimate [12] §25 and using partial summation we obtain

\[
|W(\alpha)| \ll (P^d q'^{-1/2} + P^{4d/5} + (P^d q')^{1/2})(\log P)^3 \leq (P^d(\log P)^{-A/2} + P^{4d/5} + P^{d-n/2})(\log P)^3,
\]

which is \( \ll P^d(\log P)^{-A/2+3} \). For each \( a \) and \( q \) as in our lemma we get by (3.19) that

\[
\int_{|\gamma| \leq P^n} |I(\mathcal{B}; \gamma)| \frac{|W(a/q + \gamma P^{-d})|}{P^d} \ d\gamma \ll (\log P)^{-A/2+3},
\]
hence by the second part of Proposition 3.6 we see that the sum over \( q \) in the lemma is

\[
\ll \sum_{(\log P)^A < q \leq P^d} \sum_{\substack{a \in \mathbb{Z} \cap [0, q) \atop \gcd(a, q) = 1}} \frac{|S_{a, q}|}{q^n} (\log P)^{-A/2 + 3} \ll (\log P)^{-A/2 + 3} + C'.
\]

\[\square\]

**Lemma 3.12.** Assume \( \{14\} \). Then we have

\[
\sum_{q \leq (\log P)^A} q^{-n} \sum_{\substack{a \in \mathbb{Z} \cap [0, q) \atop \gcd(a, q) = 1}} |S_{a, q}| \int_{(\log P)^A < |\gamma| \leq P^n} \frac{|I(\mathcal{B}; \gamma)| |W(a/q + \gamma P^{-d})|}{P^d} d\gamma \ll \frac{\log \log P}{(\log P)^{2(d-1)}}.
\]

**Proof.** The proof follows immediately by combining the bound \( W(\alpha) \ll P^{d}, \) the inequality (3.19) for \( Q = (\log P)^A \) and the second part of Proposition 3.6

Tying Lemmas 3.11 and 3.12 proves the following lemma.

**Lemma 3.13.** Keep the assumptions of Theorem 1.3. Then there exists a strictly positive constant \( \lambda = \lambda(f) \) such that for every fixed sufficiently large \( A > 0 \) we have

\[
\left| \int_{\alpha \in (A)} S(\alpha) \overline{W(\alpha)} d\alpha \right| \ll \frac{P^n}{(\log P)^{A\lambda}}.
\]

3.4. **The major arcs.** Bringing together (3.16), (3.18), and Lemma 3.13 we see that under the assumptions of Lemma 3.13 there exists \( \lambda > 0 \) such that for all large \( A > 0 \) we have

\[
\frac{\pi_f(P^{\mathcal{B}})}{P^n} - \sum_{q \leq (\log P)^A} q^{-n} \sum_{\substack{a \in \mathbb{Z} \cap [0, q) \atop \gcd(a, q) = 1}} S_{a, q} \int_{|\gamma| \leq (\log P)^A} I(\mathcal{B}; \gamma) \frac{|W(a/q + \gamma P^{-d})|}{P^d} d\gamma \ll (\log P)^{-A\lambda}. \quad (3.20)
\]

Using the Siegel–Walfisz theorem as in [12] pg.147 we can show that there exists \( c = c(A) > 0 \) such that if \( |\beta| \ll P^{-d}(\log P)^A, \) \( q \leq (\log P)^A, \) \( a \) coprime to \( q \) and \( x \in [P^{d/2}, P^{2d}] \) then

\[
\sum_{m \leq x} \Lambda(m) e(m(a/q + \beta)) = \frac{\mu(q)}{\varphi(q)} \left( \int_2^x e(\beta t) dt \right) + O \left( (1 + |\beta|x) \exp \left(-c\sqrt{\log P}\right) \right),
\]

where \( \mu, \varphi \) and \( \Lambda \) denote the Möbius, Euler and von Mangoldt functions. We now see that

\[
\sum_{p \leq x} (\log p) e(p(a/q + \beta)) = \frac{\mu(q)}{\varphi(q)} \left( \int_2^x e(\beta t) dt \right) + O \left( (1 + |\beta|x) \exp \left(-c\sqrt{\log P}\right) \right)
\]

due to the estimate \( \sum_{m \leq x \atop m \neq p} \Lambda(m) \ll x^{1/2} \). Partial summation shows that \( W(a/q + \beta) \) equals

\[
\frac{\mu(q)}{\varphi(q)} \left( \int_2^{\max\{f_0(\mathcal{B})\} P^d} e(\beta t) dt \right) - \frac{1}{2} \min\{f_0(\mathcal{B})\} P^d \int_2^{\max\{f_0(\mathcal{B})\} P^d} e(\beta t) dt - \int_2^{\max\{f_0(\mathcal{B})\} P^d} \left( \int_2^u e(\beta t) dt \right) \left( \frac{1}{\log u} \right) du \]

up to an error of size \( \ll (1 + |\beta| P^d) P^d \exp \left(-c\sqrt{\log P}\right) \). Partial integration now yields

\[
W(a/q + \gamma P^{-d}) = \frac{\mu(q)}{\varphi(q)} \left( \int_2^{\max\{f_0(\mathcal{B})\} P^d} \frac{e(\gamma P^{-d} t)}{\log t} dt \right) + O \left( (1 + |\gamma| P^d \exp \left(-c\sqrt{\log P}\right) \right).
\]
The error term makes the following contribution towards (3.20),
\[
\ll \exp \left( -c \sqrt{\log P} \right) \sum_{q \leq (\log P)^A} q^{-n} \sum_{\substack{a \in \mathbb{Z} \cap [0,q) \atop \gcd(a,q)=1}} |S_{a,q}| \int_{|\gamma| \leq (\log P)^A} |I(\mathcal{B}; \gamma)| (1 + (\log P)^4) \, d\gamma
\]
and, by the second part of Proposition 3.6 this is \( \ll \exp \left( -c \sqrt{\log P} \right) (\log P)^{A+1} \), which is obviously \( \ll \exp \left( -c/2 \sqrt{\log P} \right) \). Hence, letting
\[
\Xi_A(P) := \sum_{q \leq (\log P)^A} \frac{\mu(q)}{\varphi(q)q^n} \sum_{\substack{a \in \mathbb{Z} \cap [0,q) \atop \gcd(a,q)=1}} S_{a,q}
\]
and
\[
\Psi_A(P) := \int_{|\gamma| \leq (\log P)^A} I(\mathcal{B}; \gamma) \left( \int_{\frac{1}{2} \min\{f_0(\mathcal{B})\}^{Pd}}^{\frac{1}{2} \max\{f_0(\mathcal{B})\}^{Pd}} \frac{e(-\gamma P^{-d}t)}{\log t} \, dt \right) \, d\gamma,
\]
we obtain the following result via (3.20).

Lemma 3.14. Under the assumptions of Theorem 1.3 there exists \( \lambda = \lambda(f) > 0 \) such that for every \( A > 0 \) we have \( \pi_f(P \mathcal{B}) = \Xi_A(P) \Psi_A(P) P^{n-d} + O(P^n(\log P)^{-A\lambda}) \) for all sufficiently large \( P \).

3.5. The non-archimedean densities. If \( n - \sigma_f \geq 3 \) then (3.14) along with the multiplicativity of \( T_j \) \([2 \S 7]\) gives
\[
\sum_{q > x} \frac{\left| \mu(q) \right|}{\varphi(q)q^n} \sum_{a \in \mathbb{Z}/q\mathbb{Z}^*} |S_{a,q}| \leq \sum_{q > x} \frac{\left| \mu(q) \right|}{\varphi(q)q^n} q^{1 - \frac{(n-\sigma_f)}{2} + \varepsilon}.
\]
Hence, for \( q \in \mathbb{N} \), the estimate \( q/\varphi(q) \ll \log(4q) \) that can be found for example in [36 Th.5.6] implies
\[
\sum_{q > x} \frac{\left| \mu(q) \right|}{\varphi(q)q^n} \sum_{a \in \mathbb{Z}/q\mathbb{Z}^*} |S_{a,q}| \ll x^{-1/2 + \varepsilon}.
\]
Therefore, we have
\[
\Xi_A(P) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)q^n} \sum_{a \in \mathbb{Z}/q\mathbb{Z}^*} S_{a,q} + O((\log P)^{-A/4}).
\]
The multiplicativity of the last sum over \( a \) shows that the above sum over \( q \) is \( \prod_p \beta_p \), where
\[
\beta_p := 1 - \frac{1}{(p-1)p^n} \sum_{a \in \mathbb{Z}/p\mathbb{Z}^*} S_{a,p}.
\]
Finally, the succeeding lemma is obtained by observing that
\[
\sum_{a \in \mathbb{Z}/p\mathbb{Z}^*} S_{a,p} = \sum_{\bar{x} \in \mathbb{F}_p^n} \left( 1 + \sum_{a \in \mathbb{Z}/p\mathbb{Z}} e(a f(\bar{x})/p) \right) = -p^n + p \# \{ \bar{x} \in \mathbb{F}_p^n : f(\bar{x}) = 0 \}. \tag{3.21}
\]

Lemma 3.15. If \( n - \sigma_f \geq 3 \) then
\[
\Xi_A(P) = \prod_p \left( \left( 1 - \frac{\# \{ \bar{x} \in \mathbb{F}_p^n : f(\bar{x}) = 0 \} \bigg) \right) \left( 1 - \frac{1}{p} \right)^{-1} \right) + O((\log P)^{-A/4}).
\]
Combining (3.14) and (3.21) yields \( p^{-n} \# \{ x \in \mathbb{F}_p^n : f(x) = 0 \} = 1/p + O(p^{(n-\sigma_f)/2}) \), thus verifying the following lemma.

**Lemma 3.16.** If \( n - \sigma_f \geq 3 \) then the product in Theorem 1.3 converges absolutely.

### 3.6. The archimedean densities.

Letting for \( P^d > \frac{1}{2} \min \{ f_0(\mathcal{B}) \} \)

\[
\Psi(P) := \int_{\gamma \in \mathbb{R}} I(\mathcal{B}; \gamma) \left( \int_{\frac{1}{2} \min \{ f_0(\mathcal{B}) \} }^{2 \max \{ f_0(\mathcal{B}) \} } e(-\gamma \mu) \frac{\log(\mu^{P^d})}{\mu^{P^d}} \, d\gamma \right) \, d\gamma,
\]

we see by (3.19) and our assumption (1.4) that there exists \( \lambda_2 = \lambda_2(f) > 0 \) such that

\[
\Psi_A(P)P^{-d} = \Psi(P) + O_A((\log P)^{-\lambda_2 A}).
\]  

(3.22)

Now we observe that for all reals \( z, \mu \) with \( z > \mu > 0 \) and \( z \notin \{1/\mu, 1\} \) we have

\[
\frac{1}{\log(\mu z)} = \frac{1}{\log z} \left( 1 + \frac{1}{\log \mu} \right) = \frac{1}{\log z} \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{\log z} \right)^k (\log \mu)^k,
\]

(3.23)

therefore, letting for \( k \in \mathbb{Z}_{\geq 0} \),

\[
J(k) := \int_{\gamma \in \mathbb{R}} I(\mathcal{B}; \gamma) \left( \int_{\frac{1}{2} \min \{ f_0(\mathcal{B}) \} }^{2 \max \{ f_0(\mathcal{B}) \} } e(-\gamma \mu) (\log \mu)^k \, d\mu \right) \, d\gamma,
\]  

(3.24)

we infer that for all sufficiently large \( P \) we have

\[
\Psi(P) = \frac{1}{\log(P^d)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\log(P^d))^k} J(k).
\]  

(3.25)

Let us furthermore introduce the succeeding entity for all \( n \in \mathbb{N} \) and \( k \in \mathbb{Z}_{\geq 0} \),

\[
J_n(k) := \int_{\gamma \in \mathbb{R}} e^{-\frac{\pi^2 \gamma^2}{n^2}} I(\mathcal{B}; \gamma) \left( \int_{\frac{1}{2} \min \{ f_0(\mathcal{B}) \} }^{2 \max \{ f_0(\mathcal{B}) \} } e(-\gamma \mu) (\log \mu)^k \, d\mu \right) \, d\gamma.
\]  

(3.26)

**Lemma 3.17.** Under the assumption (1.4) we have \( \lim_{n \to +\infty} J_n(k) = J(k) \) for every \( k \in \mathbb{Z}_{\geq 0} \).

**Proof.** The difference \( J(k) - J_n(k) \) has modulus at most

\[
\ll_k \left( \int_{|\gamma| \leq \log n} + \int_{|\gamma| > \log n} \right) \left( 1 - e^{-\frac{\pi^2 \gamma^2}{n^2}} \right) |I(\mathcal{B}; \gamma)| \, d\gamma.
\]

We have \( I(\mathcal{B}; \gamma) \ll 1 \) due to (3.19), hence, the first integral is

\[
\ll (\log n) \left( 1 - e^{-\frac{\pi^2 (\log n)^2}{n^2}} \right) = o(1).
\]

Again, by (3.19), the second integral is \( \ll (\log n)^{-\lambda_1} = o(1) \) for some positive \( \lambda_1 = \lambda_1(f) \). \( \square \)

**Lemma 3.18.** Under the assumption (1.4) we have the following for every \( k \in \mathbb{Z}_{\geq 0} \),

\[
\lim_{n \to +\infty} J_n(k) = \int_{t \in \mathcal{B}} (\log f_0(t))^k \, dt.
\]
It is standard to see that the Fourier transform of the function \( \varphi_n : \mathbb{R} \to \mathbb{R} \) defined through \( \varphi_n(x) := \pi^{-1/2}n \exp(-n^2 x^2) \) satisfies \( \hat{\varphi}_n(\gamma) = \exp(-\pi n^{-2} \gamma^2) \). Therefore, the Fourier inverse formula yields \( \varphi_n(x) = \int_{\mathbb{R}} e(x \gamma) \hat{\varphi}_n(\gamma) d\gamma \). Using this for \( x = f_0(t) - y \) and rewriting (3.26) as

\[
\int_{t \in \mathcal{B}} \int_{\gamma \in \mathbb{R}} \text{e}^{2\max\{f_0(\mathcal{B})\}^{2\gamma^2}} \text{e}((f(t) - \mu)\gamma) d\gamma d\mu dt,
\]

we infer that \( J_n(k) = \int_{\mathcal{B}} g_n(t) dt \), where

\[
g_n(t) := \int_{\gamma \in \mathbb{R}} \text{e}^{2\max\{f_0(\mathcal{B})\}^{2\gamma^2}} \text{e}((f(t) - \mu)\gamma) d\mu.
\]

It is obvious that for any reals \( a < c < b \) and any continuous function \( h : [a, b] \to \mathbb{R} \) one has

\[
\lim_{n \to +\infty} \int_{a}^{b} h(\mu) \varphi_n(c - \mu) d\mu = h(c).
\]

Recalling that \( f_0(\mathcal{B}) \subset (0, \infty) \) we infer that whenever \( t \in \mathcal{B} \) then the following inequality holds, \( \frac{1}{2} \min\{f_0(\mathcal{B})\} < f_0(t) < 2 \max\{f(\mathcal{B})\} \). This shows that \( \lim_n g_n(t) = (\log f_0(t))^k \) and an use of the dominated convergence theorem concludes the proof of the lemma.

**Lemma 3.19.** Under the assumption (1.4) we have, for all sufficiently large \( P \), \( \Psi(P) = P^{-n}\text{Li}_f(P, \mathcal{B}) \).

**Proof.** Combining Lemmas 3.17 and 3.18 we get \( J(k) = \int_{\mathcal{B}} (\log f(t))^k dt \). Injecting this into (3.25) and interchanging the sum over \( k \) and the integral over \( t \) yields

\[
\Psi(P) = \int_{\mathcal{B}} \left( \frac{1}{\log(P^d)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\log(P^d))^k} (\log f_0(t))^k \right) dt.
\]

The proof is concluded by alluding to (3.23) and making the change of variables \( x = Pt \).

Combining Lemma 3.19 with (3.22) provides us with the following result.

**Lemma 3.20.** Under the assumptions of Theorem 1.3 there exists \( \lambda_2 = \lambda_2(f) > 0 \) such that for every \( A > 0 \) and every sufficiently large \( P \) we have

\[
\Psi_A(P) = \text{Li}_f(P, \mathcal{B}) P^{n-d} + O_A(P^n(\log P)^{-A\lambda_1}).
\]

Our final result offers an asymptotic expansion of \( \text{Li}_f(P, \mathcal{B}) \) in terms of \( (\log P)^{-1} \).

**Lemma 3.21.** For \( f \) and \( \mathcal{B} \) as in Theorem 1.3 and \( P \) large enough we have

\[
\text{Li}_f(P, \mathcal{B}) = \frac{\text{vol}(\mathcal{B})}{d} \frac{P^n}{\log(P^d)} + P^n \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{d^k} \left( \int_{\mathcal{B}} (\log f_0(t))^{k-1} dt \right) \frac{1}{(\log P)^k}.
\]

In particular, we have

\[
\text{Li}_f(P, \mathcal{B}) = \frac{\text{vol}(\mathcal{B})}{d} \frac{P^n}{\log P} + O_f,\mathcal{B} \left( \frac{P^n}{(\log P)^2} \right).
\]
Proof. We have \( J(k) = \int \log f_0(t)^k dt \) as in the proof of Lemma 3.19. Substituting this into (3.25) and alluding to Lemma 3.19 concludes the proof of the first equality in the lemma. To prove the second, note that if \( \log P > 2 \) then
\[
\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{d^k} \left( \int \log f_0(t)^{k-1} dt \right) \frac{1}{(\log P)^k} \lesssim \sum_{k=2}^{\infty} \frac{1}{(\log P)^k} < \frac{1}{(\log P)^2} \sum_{k=2}^{\infty} \frac{1}{2^{k-2}},
\]
thus concluding the proof. \(\Box\)

3.7. The proof of Theorem 1.3. It follows by merging Lemmas 3.14, 3.15 and 3.20. \(\Box\)

4. The proof of Theorem 1.5

4.1. First steps and auxiliary estimates. Similarly as in §3.1 we may write
\[
\#\{S_f \cap P_B\} = \int_0^1 S(\alpha)Q(\alpha) d\alpha,
\]
where \( S(\alpha) \) is defined in (3.1) and
\[
Q(\alpha) := \sum_{m \text{ square-free}, m \neq 0} e(\alpha m).
\]
As in the case of Theorem 1.3 this particular choice allows us to deal with cases where \([\min\{f_0(\mathcal{B})\}, \max\{f_0(\mathcal{B})\}] \cap (-\infty, 0) \neq \emptyset\) and will make more transparent the end of the proof of Theorem 1.5.

We shall later need certain estimates concerning exponential sums taking values over square-free integers that we record here. For \( \alpha \in \mathbb{R} \) and \( N \in \mathbb{R}_{\geq 1} \) define
\[
f_2(\alpha, N) := \sum_{1 \leq n \leq N} \mu(n)^2 e(\alpha n).
\]
The following result is the very special case corresponding to the choices \( k = 2, p = 3/2 \) and \( N = P^d \) in the work of Keil [27].

Lemma 4.1 (Keil [27] Theorem 1.2). We have
\[
\int_0^1 |f_2(\alpha, N)|^{3/2} d\alpha \ll N^{1/2}(\log N)^2.
\]

For \( p \) prime and \( \ell, m \) non-negative integers, now define the function \( g(p^\ell, p^m) \) by
\[
p^\ell(1 - p^{-2})g(p^\ell, p^m) = \begin{cases} 0, & \text{if } \ell \geq m \geq 2, \\ 1, & \text{if } m < \min\{2, \ell\}, \\ 1 - p^{\ell-2}, & \text{if } \ell = m \leq 1. \end{cases}
\]
We extend this definition by defining the following whenever \( d, q \in \mathbb{N} \) are such that \( d | q \),
\[
g(q, d) := \prod_{p | q} g(p^{\nu_p(q)}, \gcd(d, p^{\nu_p(q)})).
\]
We can now introduce the following entity for \( q \in \mathbb{N} \),
\[
G(q) := \sum_{b=1}^{q} e(b/q)g(q, \gcd(b, q)).
\]
Brüdern, Granville, Perelli, Vaughan and Wooley studied $Q(\alpha)$ in [7]. A very special case of their results in [7, §3] is the following result.

**Lemma 4.2** (Brüdern, Granville, Perelli, Vaughan and Wooley, [7]). There exist absolute positive constants $\delta_1, \delta_2$ such that for all $q \in \mathbb{N}$ with $q \leq P^{\delta_1}$, all $a \in \mathbb{Z} \cap [1, q)$, $d \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and all $c_1 < c_2 \in \mathbb{R}$ we have

$$
\sum_{\substack{m \text{ square-free, } m \neq 0 \\ c_1 \leq m \leq c_2}} e(m(a/q + \gamma P^{-d})) = \frac{G(q)}{\zeta(2)} \left( \int_{c_1 P^d}^{c_2 P^d} e(\gamma P^{-d} t) \, dt \right) + O_{c_1, c_2} \left( (1 + |\gamma|) P^{d - \delta_2} \right),
$$

where $\zeta$ denotes the Riemann zeta function and the implied constant depends at most on $c_1$ and $c_2$.

Finally, the next result is shown in the proof of [7, Lemma 3.1].

**Lemma 4.3** (Brüdern, Granville, Perelli, Vaughan and Wooley, [7, Lemma 3.1]). The function $G$ is multiplicative, supported in cube-free integers and satisfies for all prime $p$ the identity

$$
G(p) = G(p^2) = -p^{-2}(1 - p^{-2})^{-1}.
$$

### 4.2. Continuation of the proof.

Recalling the meaning of $\mathcal{M}(\theta)$ and $\mathcal{M}_{a,q}(\theta)$ in (3.4) and (3.3), we allude to Hölder’s inequality and Lemma 4.1 to obtain

$$
\left| \int_{\alpha \notin \mathcal{M}(\theta)} S(\alpha) \overline{Q(\alpha)} \, d\alpha \right| \leq \left( \int_{\alpha \notin \mathcal{M}(\theta)} |S(\alpha)|^3 \, d\alpha \right)^{1/3} \left( \int_0^1 |Q(\alpha)|^{3/2} \, d\alpha \right)^{2/3} \leq \left( \int_{\alpha \notin \mathcal{M}(\theta)} |S(\alpha)|^3 \, d\alpha \right)^{1/3} P^{d/3}(\log P)^{4/3}.
$$

The proof of Lemma 3.8 can be adapted straightforwardly to show that if

$$
1 > \delta + 6d\theta_0 \quad \text{and} \quad \frac{n - \sigma_f}{2d - 1} - \frac{2}{3}(d - 1) > \delta \theta_0^{-1}
$$

then

$$
\left( \int_{\alpha \notin \mathcal{M}(\theta)} |S(\alpha)|^3 \, d\alpha \right)^{1/3} \ll P^{n - \frac{4}{3} - \frac{4}{3d}}.
$$

Let $\eta := (d - 1)\theta_0$. Under the assumptions of Theorem 1.5 and for $\theta_0$ as in (1.1), one obtains the following inequality that is in analogy with Lemma 3.10,

$$
\#\{S_f \cap P\mathcal{B}\} = \sum_{q \leq P^n} \sum_{a \in \mathbb{Z} \cap [0, q)} \sum_{\gcd(a,q) = 1} \int_{\mathcal{M}_{a,q}(\theta_0)} S(\alpha) \overline{Q(\alpha)} \, d\alpha + O \left( P^{n - \frac{d}{m}} \right).
$$

Similarly as in the proof of (3.15), one may now acquire some $\delta_1 = \delta_1(f) > 0$ such that

$$
\frac{\#\{S_f \cap P\mathcal{B}\}}{P^n} - \sum_{q \leq P^n} q^{-n} \sum_{a \in \mathbb{Z} \cap [0, q)} \sum_{\gcd(a,q) = 1} \int_{\mathcal{M}_{a,q}(\theta_0)} S(\alpha) \overline{Q(\alpha)} \, d\alpha \ll P^{-\delta_1}.
$$

(4.2)

By Lemma 4.2 we see that for suitably small $\eta$ and all $a, q$ as in (4.2) one has

$$
Q(a/q + \gamma P^{-d}) = \frac{G(q)}{\zeta(2)} \left( \int_{R_{\max(f_0)}}^{(\max(f_0) + 1)P^{d}} e(\gamma P^{-d} t) \, dt \right) + O \left( (1 + |\gamma|) P^{d - \delta_2} \right).
$$
Therefore, as in the proof of Lemma \[3.14\] we may infer that there exists a positive constant \(\delta_3 = \delta_3(f)\) such that the quantity \(\#\{S_f \cap \mathcal{B}\} = \sum_{q \leq P^n} \frac{G(q)}{q^n} \sum_{a \in \mathbb{Z}_C(0,q), \text{gcd}(a,q) = 1} S_{a,q} \left( \prod_p \left( 1 - p^{-2} \right) \sum_{a \in \mathbb{Z}_C(0,p), \text{gcd}(a,p) = 1} S_{a,p} + \frac{1}{p^{2n}} \sum_{a \in \mathbb{Z}_C(0,p^2), \text{gcd}(a,p) = 1} S_{a,p^2} \right) \)

(4.3)

up to an error term which is \(O(P^{n-\delta_3})\). We shall now use Lemma \[3.3\] to show that the sum over \(q\) forms an absolutely convergent series. Bringing into play (3.14) and [4, Lemma 25] we obtain the bounds

\[ |G(p)T_f(p)| \ll p^{-(n-\sigma_f)/2} \quad \text{and} \quad |G(p^2)T_f(p^2)| \ll p^{-n+\sigma_f}. \]

Hence, assuming \(n - \sigma_f \geq 2\), these two estimates allow to modify easily the proof of Proposition \[3.6\], thereby showing that the abscissa of convergence of the Dirichlet series \(\zeta(p)\) has

\[ \text{abs}(\zeta) \leq \min \{\frac{1}{4}, \frac{1}{\delta_4} \} \]

such that the quantity \(\#\{S_f \cap \mathcal{B}\}\) is strictly negative. This provides \(\delta_4 = \delta_4(f) > 0\) such that for all \(x \geq 2\), one has \(\sum_{q \geq x} |G(q)| |T_f(q)| < x^{-\delta_4}\), hence the sum over \(q\) in \[4.3\] is \(\Pi' + O(P^{-\eta_5})\), where \(\Pi'\) is

\[ \sum_{q=1}^{\infty} \frac{G(q)}{q^n} \sum_{a \in \mathbb{Z}_C(0,q), \text{gcd}(a,q) = 1} S_{a,q} = \prod_p \left( 1 - p^{-2} \right) \left( 1 + \frac{1}{p^{2n}} \sum_{a \in \mathbb{Z}_C(0,p), \text{gcd}(a,p) = 1} S_{a,p} + \frac{1}{p^{2n}} \sum_{a \in \mathbb{Z}_C(0,p^2), \text{gcd}(a,p) = 1} S_{a,p^2} \right) \]

from which we can show that \(\Pi'/\zeta(2)\) is

\[ \prod_p \left( 1 - \frac{|\{x \in (\mathbb{Z}/p^2\mathbb{Z})^n : f(x) \equiv 0 \pmod{p^2}\}|}{p^{2n}} \right). \]

This is in agreement with the infinite product in Theorem \[1.5\]

To deal with the integral in \[4.3\] we observe that the transformation \(t = P^d\mu\) gives

\[ P^d \int_{\text{max}\{f_0(\mathcal{B})\}}^{\text{min}\{f_0(\mathcal{B})\} + 1} e(-\gamma P^d t) dt = \int_{\text{min}\{f_0(\mathcal{B})\} - 1}^{\text{max}\{f_0(\mathcal{B})\} + 1} e(-\gamma \mu) d\mu \ll \min\{1, |\gamma|^{-1}\}, \]

hence Lemma \[3.3\] shows that the integral in \[4.3\] converges absolutely and equals

\[ \int_{\gamma \in \mathbb{R}} I(\mathcal{B}; \gamma) \left( \int_{\text{min}\{f_0(\mathcal{B})\} - 1}^{\text{max}\{f_0(\mathcal{B})\} + 1} e(-\gamma \mu) d\mu \right) d\gamma + O(P^{-\eta_5}) \]

(4.4)

for some \(\delta_5 = \delta_5(f) > 0\). Let us now observe that in the special case \(k = 0\) the proof of Lemmas \[3.17\] and \[3.18\] only uses the absolute convergence of the integral over \(\gamma\) in \[4.4\], thus the assumption \[1.4\] is not required for their validity in our situation. Subsequently, a completely analogous argument as in \$3.6\) yields that the integral over \(\gamma\) in \[4.4\] is equal to \(\text{vol}(\mathcal{B})\). Thereby alluding to the well-known estimate

\[ \#\{\mathbb{Z}^n \cap \mathcal{B}\} = \text{vol}(\mathcal{B}) \cdot P^n + O(\mathcal{B}(P^{n-1})) \]

allows us to conclude the proof of Theorem \[1.5\].
APPENDIX A. THE BATEMAN–HORN HEURISTICS IN MANY VARIABLES

In this section we extend the Bateman–Horn heuristics from the setting of univariate polynomials to that of polynomials with arbitrarily many variables; we do so because we were unable to find a reference for this extension in the literature.

In 1958, Schinzel [30] formulated the following conjecture concerning prime values of univariate polynomials.

Conjecture A.1 (Schinzel’s hypothesis H, [30]). Let \( f_1, \ldots, f_r \in \mathbb{Z}[x] \) be univariate irreducible polynomials with positive leading coefficient. If \( \prod_{i=1}^r f_i \) has no repeated polynomial factors and, for every prime \( p \), there exists \( x_p \in \mathbb{Z} \) such that \( p \nmid f_1(x_p) \cdots f_r(x_p) \), then there exist infinitely many integers \( m \) such that \( f_1(m), \ldots, f_r(m) \) are all primes.

This conjecture was later refined by Bateman and Horn [1] who, based on the Cramér model and the heuristics behind the Hardy–Littlewood conjecture (see [34, pg.6-8]), gave a quantitative version of Schinzel’s conjecture.

Conjecture A.2 (Bateman–Horn’s conjecture, [1]). Keep the assumptions of Conjecture A.1. Then the number of integers \( m \in [1, P] \) such that every \( f_1(m), \ldots, f_r(m) \) is prime is asymptotically equivalent to the succeeding quantity as \( P \to +\infty \),

\[
\left( \prod_{p \text{ prime}} \frac{1 - p^{-1} \# \{ x \in \mathbb{F}_p : f_1(x) \cdots f_r(x) = 0 \} }{(1 - 1/p)^r} \right) \frac{1}{\deg(f_1) \cdots \deg(f_r)} \int_2^P \frac{dx}{(\log x)^r}.
\]

The convergence of the infinite product is established in [1] using the prime ideal theorem. These two conjectures lie very deep and imply a number of notoriously difficult conjectures as immediate corollaries (the twin primes conjecture among others; see [30] for a non-exhaustive list of implications). There are applications to the arithmetic of algebraic varieties, see [9, 35] or [20], where Schinzel’s hypothesis is assumed in order to prove that the Hasse principle and weak approximation holds.

Let us now record the multivariable version of the Bateman–Horn conjecture.

Conjecture A.3 (Extension of the Bateman–Horn conjecture). Assume that we are given irreducible polynomials \( f_1, \ldots, f_r \in \mathbb{Z}[x_1, \ldots, x_n] \) such that \( \prod_{i=1}^r f_i \) has no repeated polynomial factors. Moreover, we assume that \( \mathcal{B} \subset \mathbb{R}^n \) is a non-empty box such that \( f_{i0}(\mathcal{B}) \subset (1, \infty) \) for all \( i \). Then the number of integer vectors \( \mathbf{x} \in \mathbb{Z}^n \cap P\mathcal{B} \) for which every \( f_1(\mathbf{x}), \ldots, f_r(\mathbf{x}) \) is a prime number is asymptotic to the following quantity as \( P \to +\infty \),

\[
\left( \prod_{p \text{ prime}} \frac{1 - p^{-n} \# \{ \mathbf{x} \in \mathbb{F}_p^n : f_1(\mathbf{x}) \cdots f_r(\mathbf{x}) = 0 \} }{(1 - 1/p)^r} \right) \int_{P\mathcal{B}} \frac{d\mathbf{x}}{\prod_{i=1}^r \log f_{i0}(\mathbf{x})}.
\]

Remark A.4. Before providing the heuristics behind Conjecture A.3 let us note that one can prove that the product over \( p \) converges via partial summation from

\[
\sum_{p \leq x} \# \{ x \in \mathbb{F}_p^n : f_1(x) \cdots f_r(x) = 0 \} = r \left( \int_2^x \frac{dt}{\log t} \right) + O(x^n e^{-c\sqrt{\log x}}),
\]

which holds for some \( c = c(f_1, \ldots, f_r) > 0 \). The former asymptotic is a version of the prime number theorem for schemes over \( \mathbb{Z} \) found in the work of Serre [32, Cor. 7.13].
We end this section by adopting the heuristics behind Conjecture A.2 to the multivariate case. Recall that the Cramér model asserts that a random positive integer \( m \) of size \( X \) has probability \( 1/\log X \) of being a prime. An analogous statement can be made if the extra condition that \( m \) lies in a primitive arithmetic progression modulo \( q \) for some positive integer \( q \) is added, in this case the probability is \( 1/(\varphi(q) \log X) \) owing to Dirichlet’s theorem on primes in arithmetic progressions. This implies that for coprime \( a, q \), the conditional probability that a positive integer \( m \) of size \( X \) is prime provided that \( m \equiv a \pmod{q} \) equals

\[
\text{Prob}[m \sim X \text{ is a prime } | \ m \equiv a \pmod{q}] \approx \frac{1/(\varphi(q) \log X)}{1/q} = \frac{q}{\varphi(q) \log X}.
\]  

(A.1)

In the setting of Conjecture A.2 observe that for typical \( x \in \mathbb{Z}^n \) the integer \( f_i(x) \) can be prime only if \( f_i(x) \) is coprime to all small primes. Therefore, letting \( z = z(P) \) be a function that slowly tends to infinity with \( f \), \( P \) and by (A.1) one now gets

\[
\frac{\pi_{f_1,\ldots,f_r}(P \mathcal{B})}{\# \{(\mathbb{Z}^n \cap P \mathcal{B}) \}} \approx \sum_{a \in (\mathbb{Z}/\mathcal{P}\mathbb{Z})^n} \text{Prob}[x_i \equiv a_i \pmod{\mathcal{P}}] \quad \text{for all } 1 \leq i \leq n \cdot \mathcal{P}_a, \quad (A.2)
\]

where \( \mathcal{P}_a \) denotes the joint probability defined through

\[
\mathcal{P}_a := \text{Prob}[m_i \sim P^{\deg(f_i)} \text{ is a prime for all } 1 \leq i \leq r \ | \ m_i \equiv f_i(a) \pmod{\mathcal{P}}].
\]

This is because the integer \( f_i(x) \) is typically of size \( P^{\deg(f_i)} \) when \( x \in P \mathcal{B} \) and the values \( f_i(x) \) are thought to behave like a random integer \( m_i \) lying in the arithmetic progression \( f_i(a) \pmod{\mathcal{P}} \), provided that \( x \equiv a \pmod{\mathcal{P}} \). Note that for \( i \neq j \) the polynomials \( f_i \) and \( f_j \) are coprime due to the assumption that \( \prod_i f_i \) has no repeated factors, therefore it is reasonable to expect that for \( i \neq j \) the integer values \( f_i(x) \) and \( f_j(x) \) behave independently. This suggests that

\[
\mathcal{P}_a, = \prod_{i=1}^{r} \text{Prob}[m_i \sim P^{\deg(f_i)} \text{ is a prime } | \ m_i \equiv f_i(a) \pmod{\mathcal{P}}]
\]

and by (A.1) one now gets \( \mathcal{P}_a = \mathcal{P} \varphi(\mathcal{P})^{-r}(\log P)^{-r} \prod_{i=1}^{r} (\deg(f_i))^{-1} \). Injecting this into (A.2) and noting that \( \text{Prob}[x_i \equiv a_i \pmod{\mathcal{P}}] = 1/\mathcal{P} \) yields

\[
\frac{\pi_{f_1,\ldots,f_r}(P \mathcal{B})}{\text{vol}(\mathcal{B})P^n} \approx \left( \frac{\mathcal{P}}{\varphi(\mathcal{P}) \log P} \right)^r \frac{1}{\prod_{i=1}^{r} \deg(f_i)} \frac{1}{\mathcal{P}^n} \sum_{a \in (\mathbb{Z}/\mathcal{P}\mathbb{Z})^n} 1, \quad 1 \leq i \leq r \Rightarrow f_i(a) \in (\mathbb{Z}/\mathcal{P}\mathbb{Z})^n.
\]

The sum over \( a \) forms a multiplicative function of \( \mathcal{P} \) that can be evaluated as

\[
\prod_{p \leq z} \left(p^n - \# \{x \in \mathbb{F}_p^n : f_1(x) \cdots f_r(x) = 0\}\right).
\]

Putting everything together shows that we expect \( \pi_{f_1,\ldots,f_r}(P \mathcal{B}) \) to be approximated by

\[
\frac{\text{vol}(\mathcal{B})P^n}{(\log P)^r \prod_{i=1}^{r} \deg(f_i)} \prod_{p \leq z} \left( \frac{p}{p-1} \right)^r \left( \frac{p^n - \# \{x \in \mathbb{F}_p^n : f_1(x) \cdots f_r(x) = 0\}}{p^n} \right).
\]
In view of Remark A.4 the product over $p \leq z(P)$ converges to the product in Conjecture A.3 as $P \to +\infty$. For $x \in P \mathcal{B}$ we have $f_i^p(x) = P^{\deg(f_i)}$ and using $\deg(f_i) = \deg(f_i^p)$ we get
\[
\frac{\text{vol}(\mathcal{B}) P^n}{(\log P)^r \prod_{i=1}^r \deg(f_i)} = \int_{P \mathcal{B}} \prod_{i=1}^r \log(P^{\deg(f_i)}) \, dx = \int_{P \mathcal{B}} \prod_{i=1}^r \log f_0(x),
\]
thereby concluding our explanation of the asymptotic in Conjecture A.3.

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Max Planck Institute for Mathematics, Vivatsgasse 7, Bonn, 53111, Germany  
E-mail address: kdestagnol@mpim-bonn.mpg.de, sofos@mpim-bonn.mpg.de