Noise as a Boolean algebra of $\sigma$-fields. II.
Classicality, blackness, spectrum

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Abstract

Similarly to noises, Boolean algebras of $\sigma$-fields [1] can be black. A noise may be treated as a homomorphism from a Boolean algebra of regular open sets to a Boolean algebra of $\sigma$-fields. Spectral sets are useful also in this framework.

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Introduction

A noise is called black, if its classical part is trivial (but the whole noise is not) [2] Def. 7a1]. The same definition applies to noise-type Boolean algebras
(of $\sigma$-fields) introduced in [1]. Triviality of the classical part is treated here as absence of non-zero square-integrable random variables $\psi$ satisfying the following additivity condition:
\[
\psi = \mathbb{E}(\psi \mid \mathcal{E}) + \mathbb{E}(\psi \mid \mathcal{E}')
\]
for all $\sigma$-fields $\mathcal{E}$ of the given Boolean algebra. The set of all such $\psi$ is the so-called first chaos space (generalizing the first Wiener chaos space). Surprisingly, it is sufficient to check the additivity condition only for $\sigma$-fields $\mathcal{E}$ of a Boolean subalgebra, provided that the corresponding measure $\mathcal{E} \mapsto \mathbb{E}|\mathbb{E}(\psi \mid \mathcal{E})|^2$ on the subalgebra is atomless (Theorem [1i]). This result is useful when dealing with a noise over $\mathbb{R}^2$ that is not rotation-invariant. (Probably, the Arratia flow leads to such noise.) Its projections to different axes may behave quite differently, but anyway, if one of them is black then others must be black.

The spectral theory of noises [2, Sect. 9] is reformulated here (Sect. 2) for a noise-type Boolean algebra. Instead of spectral measures on the space of closed sets we get spectral measure spaces. Sect. 3 relates the new framework to the old one. If $\mathbb{R}^2$ is divided in two domains by a curve, a noise over $\mathbb{R}^2$ is thus divided in two independent components if and only if almost all spectral sets avoid the curve (Prop. 3b).

1 Classicality and blackness

1a Definitions; preservation under completion

Let $B$ be a noise-type Boolean algebra [1, Def. 2a1] of $\sigma$-fields on a probability space $(\Omega, \mathcal{F}, P)$. The corresponding projections \( Q_x \ [1, \text{Sect. 1d}] \) for $x \in B$, acting on $H = L^2(\Omega, \mathcal{F}, P)$, satisfy [1, Lemma 2a2]
\[
(1a1) \quad Q_x Q_y = Q_{x \wedge y}.
\]

1a2 Definition. (a) The first chaos space $H^{(1)}$ is a (closed linear) subspace of $H$ consisting of all $\psi \in H$ such that for all $x, y \in B$
\[
(1a3) \quad x \wedge y = 0 \implies Q_{x \vee y} \psi = Q_x \psi + Q_y \psi.
\]

(b) $B$ is called classical if the first chaos space generates the whole $\sigma$-field $\mathcal{F}$.
(c) $B$ is called black if the first chaos space contains only $0$.

\[\text{1Throughout, “projection” means “orthogonal projection”}.\]
Taking $x = y = 0$ in (1a3) we see that

\[(1a4) \quad Q_0 \psi = 0 \quad \text{for all } \psi \in H^{(1)}.\]

Note that $H^{(1)}$ is the set of all $\psi \in H$ such that $Q_0 \psi = 0$ and for all $x, y \in B$

\[(1a5) \quad Q_{x \lor y} \psi + Q_{x \land y} \psi = Q_x \psi + Q_y \psi;\]

for the proof, apply (1a3) twice: to $x, y \land x'$ and also to $y \land x, y \land x'$.

Recall the noise-type completion $C$ of $B$ [1].

1a6 Proposition. If $\psi$ satisfies (1a3) for all $x, y \in B$ then $\psi$ satisfies (1a3) for all $x, y \in C$, where $C$ is the noise-type completion of $B$. 

Proof. We use (1a5) instead of (1a3). For every $x \in C$ the maps $y \mapsto x \land y$ and $y \mapsto x \lor y$ are continuous on $C$ [1, (2a6) and 2b6] in the “strong operator” topology [1, Sect. 1d]: $x_n \to x$ in this metrizable topology if and only if $\forall \psi \in H \quad \|Q_{x_n} \psi - Q_x \psi\| \to 0$.

Thus, (1a5) extends by continuity from the case $x, y \in B$ to the more general case $x \in B$, $y \in C$. And then it extends further to $x, y \in C$. 

We see that $H^{(1)}$ (as well as classicality and blackness) is uniquely determined by the completion $C$ of $B$. Recall also that $C$ is uniquely determined by the closure of $B$ [1, Intro].

1b Beyond the completion

A partition of unity in $B$ consists, by definition, of $x_1, \ldots, x_n \in B$ such that $x_1 \lor \cdots \lor x_n = 1$, $x_i \neq 0$ for all $i$, and $x_i \land x_j = 0$ whenever $i \neq j$.

We say that a vector $\psi \in H^{(1)}$ is atomless, if for every $\varepsilon > 0$ there exists a partition of unity $x_1, \ldots, x_n$ such that $\|Q_{x_i} \psi\| \leq \varepsilon$ for all $i = 1, \ldots, n$.

Assume that $b \subset B$ is a Boolean subalgebra, and a vector $\psi \in H$ satisfies (1a3) for all $x, y \in b$. The notion “$b$-atomless” is defined as before (using partitions of unity in $b$ rather than $B$).

1b1 Theorem. If $b$ is a Boolean subalgebra of $B$, $\psi \in H$ satisfies (1a3) for all $x, y \in b$ and is $b$-atomless, then $\psi \in H^{(1)}$.

The proof is given after some lemmas.

Note that

$$
\langle Q_x \psi, Q_y \psi \rangle = 0 \quad \text{whenever } \psi \in H^{(1)} \text{ and } x \land y = 0,
$$
since $\langle Q_x \psi, Q_y \psi \rangle = \langle Q_y Q_x \psi, \psi \rangle = \langle Q_0 \psi, \psi \rangle = 0$ by (1a1) and (1a4). It follows that

$$x \mapsto \|Q_x \psi\|^2$$

is an additive function $B \to [0, \infty)$ for $\psi \in H^{(1)}$.

1b2 Lemma. $Q_x + Q_y \leq Q_{x \lor y} + Q_{x \land y}$ for all $x, y \in B$.

Proof. By (1a1), $Q_x$ and $Q_y$ are commuting projections, which implies $Q_x + Q_y = Q_x \lor Q_y \lor Q_x \land Q_y$, where $Q_x \lor Q_y$ and $Q_x \land Q_y$ are projections onto $Q_x H + Q_y H$ and $Q_x H \cap Q_y H$ respectively. Using (1a1) again, $Q_x \land Q_y = Q_x Q_y$, $Q_{x \lor y}$ follows since $\langle \cdot, \cdot \rangle$ is the complement of $\langle \cdot, \cdot \rangle$. It remains to note that $Q_x \lor Q_y \leq Q_{x \lor y}$ just because $Q_x \leq Q_{x \lor y}$ and $Q_y \leq Q_{x \lor y}$.

Taking into account that $\|Q_x \psi\|^2 = \langle Q_x \psi, \psi \rangle$ we get the following.

1b3 Corollary. For every $\psi \in H$ such that $Q_0 \psi = 0$ we have

$$x \mapsto \|Q_x \psi\|^2$$

is a superadditive function $B \to [0, \infty)$, that is, $\|Q_x \psi\|^2 + \|Q_y \psi\|^2 \leq \|Q_{x \lor y} \psi\|^2$ whenever $x \land y = 0$.

1b4 Lemma. If $\psi = Q_x \psi + Q_{x'} \psi$ for all $x \in B$, then $\psi \in H^{(1)}$.

(Here $x'$ is the complement of $x$ in $B$, of course.)

Proof. If $x \land y = 0$ then $Q_{x \lor y} \psi = Q_{x \lor y} (Q_x \psi + Q_{x'} \psi) = Q_{x \lor y} Q_x \psi + Q_{x \lor y} Q_{x'} \psi = Q_{x \lor y} \psi + Q_{x \lor y} \psi = Q_x \psi + Q_y \psi$.

Recall the sub-$\sigma$-fields $F_x \subset F$ and subspaces $H_x = L_2(F_x) \subset H$ for $x \in B$ [I Sect. 1a]; $H_x = Q_x H$.

For every $x \in B$ the $\sigma$-fields $F_x, F_{x'}$ are independent [I Sect. 2a], therefore the pointwise product $\xi \eta$ belongs to $H$ for all $\xi \in H_x$, $\eta \in H_{x'}$.

1b5 Lemma. The following two conditions on $x \in B$ and $\psi \in H$ are equivalent:

(a) $\psi = Q_x \psi + Q_{x'} \psi$;
(b) $E \psi = 0$, and $E (\psi \xi \eta) = 0$ for all $\xi \in H_x$, $\eta \in H_{x'}$ satisfying $E \xi = 0$, $E \eta = 0$.

(Here $E \psi = \int \! \psi \mathrm{d}P = \langle \psi, \mathbb{1} \rangle$.)

The proof uses a construction important to [I] (see first of all [I proof of Prop. 1d13]). Let $x \in B$. Up to the natural unitary equivalence we have $H = H_x \otimes H_{x'}$ and $Q_{u \lor v} = Q_u^{(x)} \otimes Q_v^{(x'}$ for all $u, v \in B$ such that $u \leq x$ and $v \leq x'$. Here $Q_u^{(x)} : H_x \to H_x$ is the projection onto $H_u \subset H_x$;
similarly, $Q_v^{(x')}: H_{x'} \rightarrow H_{x'}$ is the projection onto $H_v \subset H_{x'}$. In particular, $Q_x = Q_x^{(x)} \otimes Q_0^{(x')} = 1 \otimes Q_0^{(x')}$ and $Q_y = Q_y^{(x)} \otimes 1$.

It may be puzzling that $H_x$ is both a subspace of $H$ and a tensor factor of $H$ (which never happens in the general theory of Hilbert spaces). Here is an explanation. All spaces $H_x$ contain the one-dimensional space $H_0$ of constant functions (on $\Omega$). Multiplying an $\mathcal{F}_Q$ of $(x)$ becomes $1 \otimes H$ similarly, $\mathcal{F}_Q$ is an explanation. All spaces $H_x$ contain the one-dimensional space $H_0$ of constant functions (on $\Omega$). Multiplying an $\mathcal{F}_x$-measurable function $\psi \in H_x$ by the constant function $\xi \in H_{x'}$, $\xi(\cdot) = 1$, we get the (puzzling) equality $\psi \otimes \xi = \psi$.

Proof of Lemma 1b5. Treating $H$ as $H_x \otimes H_{x'}$ we have $H = ((H_x \oplus H_0) \otimes (H_{x'} \oplus H_0) = (H_x \oplus H_0) \otimes (H_{x'} \oplus H_0) \otimes H_0 \oplus H_0 \otimes (H_x \oplus H_0) \oplus (H_{x'} \oplus H_0)$; here $H_x \oplus H_0$ is the orthogonal complement of $H_0$ in $H_x$ (it consists of all zero-mean functions of $H_x$). In this notation $Q_x + Q_{x'}$ becomes $1 \otimes Q_0^{(x')} + Q_x \otimes 1 = ((1 - Q_0^{(x')} + Q_0^{(x)}) \otimes Q_0^{(x')} + Q_0^{(x)} \otimes ((1 - Q_0^{(x)} + Q_0^{(x)})) = (1 - Q_0^{(x)}) \otimes Q_0^{(x')} + Q_0^{(x)} \otimes (1 - Q_0^{(x)}) + 2Q_0^{(x)} \otimes Q_0^{(x')}$, the projection onto $(H_x \ominus H_0) \oplus H_0 \ominus H_0 \ominus (H_{x'} \ominus H_0)$ plus twice the projection onto $H_0 \ominus H_0 = H_0$. Thus, the equality $\psi = (Q_x + Q_{x'})(\psi) (\text{Item (a)})$ becomes $\psi \in (H_x \ominus H_0) \otimes H_0 \ominus H_0 \ominus (H_{x'} \ominus H_0)$, or equivalently, orthogonality of $\psi$ to $H_0$ and $(H_x \ominus H_0) \ominus (H_{x'} \ominus H_0)$, which is Item (b).

1b6 Remark. The proof given above shows also that

$$\{ \psi : \psi = Q_x \psi + Q_{x'} \psi \} = (H_x \ominus H_0) \oplus (H_{x'} \ominus H_0)$$

for all $x \in B$.

Proof of Theorem 1b1. Let $x \in B$; we have to prove that $\psi = Q_x \psi + Q_{x'} \psi$. Let $\xi \in H_x \ominus H_0$, $\eta \in H_{x'} \ominus H_0$; by Lemma 1b5 it is sufficient to prove that $\mathbb{E}(\psi \xi \eta) = 0$.

Given $\varepsilon > 0$, we take a partition of unity $y_1, \ldots, y_n$ in $b$ such that $\|Q_{y_i} \psi\| \leq \varepsilon$ for all $i$. We have $\psi = \sum_i Q_{y_i} \psi$ (by 1a3 for $b$), thus, $\mathbb{E}(\psi \xi \eta) = \sum_i \mathbb{E}(Q_{y_i} \psi \xi \eta)$, where $\mathbb{E}(Q_{y_i} \psi \xi \eta) = \langle Q_{y_i} \psi, \xi \xi \eta \rangle = \langle Q_{y_i} \psi, Q_{y_i}(\xi \otimes \eta) \rangle = \langle Q_{y_i} \psi, (Q_{u_i}^{(x)} \otimes Q_{v_i}^{(x)}) (\xi \otimes \eta) \rangle = \langle Q_{y_i} \psi, (Q_{v_i}^{(x)} \otimes (Q_{u_i}^{(x)}) \eta) \rangle$, where $u_i = y_i \wedge x$ and $v_i = y_i \wedge x'$; it follows that $\mathbb{E}(\psi \xi \eta) \leq \sum_i \|Q_{y_i} \psi\| \cdot \|Q_{u_i}^{(x)} \xi\| \cdot \|Q_{v_i}^{(x)} \eta\|$. By additivity, $\sum_i \|Q_{y_i} \psi\|^2 = \|\psi\|^2$. By superadditivity (Corollary 1b3), $\sum_i \|Q_{u_i}^{(x)} \xi\|^2 \leq \|\xi\|^2$ and $\sum_i \|Q_{v_i}^{(x)} \eta\|^2 \leq \|\eta\|^2$. We get $\mathbb{E}(\psi \xi \eta) \leq (\max_i \|Q_{y_i} \psi\|) \sum_i \|Q_{u_i}^{(x)} \xi\| \cdot \|Q_{v_i}^{(x)} \eta\|) \leq \varepsilon \|\xi\| \|\eta\|$ for all $\varepsilon$. 


2 Spectrum

2a Preliminaries: commutative von Neumann algebras and measure class spaces

Every commutative von Neumann algebra \( \mathcal{A} \) of operators on a separable Hilbert space \( H \) is isomorphic to the algebra \( L_\infty(S, \Sigma, \mu) \) on some measure space \((S, \Sigma, \mu)\) ([3 Sect. 1.7.3], [4 Th. 1.22]). Here and henceforth all measures are positive, finite and such that the corresponding \( L_2 \) spaces are separable. The measure \( \mu \) may be replaced with any equivalent (that is, mutually absolutely continuous) measure \( \mu_1 \). Thus we may turn to a measure class space (see [5 Sect. 14.4]) \((S, \Sigma, \mathcal{M})\) where \( \mathcal{M} \) is an equivalence class of measures, and write \( L_\infty(S, \Sigma, \mathcal{M}) \). We have an isomorphism \( \alpha : \mathcal{A} \to L_\infty(S, \Sigma, \mathcal{M}) \) of von Neumann algebras. (See [5, 14.4] for the Hilbert space \( L_2(S, \Sigma, \mathcal{M}) \) on which \( L_\infty(S, \Sigma, \mathcal{M}) \) acts by multiplication.)

Let \( \Sigma_1 \subset \Sigma \) be a sub-\( \sigma \)-field. Restrictions \( \mu|_{\Sigma_1} \) of measures \( \mu \in \mathcal{M} \) are mutually equivalent; denoting their equivalence class by \( \mathcal{M}|_{\Sigma_1} \) we get a measure class space \((S, \Sigma_1, \mathcal{M}|_{\Sigma_1})\). Clearly, \( L_\infty(S, \Sigma_1, \mathcal{M}|_{\Sigma_1}) \subset L_\infty(S, \Sigma, \mu) \) or, in shorter notation, \( L_\infty(\Sigma_1) \subset L_\infty(\Sigma) \). We have \( L_\infty(\Sigma_1) = \alpha(\mathcal{A}_1) \) where \( \mathcal{A}_1 = \alpha^{-1}(L_\infty(\Sigma_1)) \subset \mathcal{A} \) is a von Neumann algebra. And conversely, if \( \mathcal{A}_1 \subset \mathcal{A} \) is a von Neumann algebra then \( \alpha(\mathcal{A}_1) = L_\infty(\Sigma_1) \) for some sub-\( \sigma \)-field \( \Sigma_1 \subset \Sigma \) (which follows easily from [6]).

Given two von Neumann algebras \( \mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A} \), we denote by \( \mathcal{A}_1 \vee \mathcal{A}_2 \) the von Neumann algebra generated by \( \mathcal{A}_1, \mathcal{A}_2 \). Similarly, for two \( \sigma \)-fields \( \Sigma_1, \Sigma_2 \subset \Sigma \) we denote by \( \Sigma_1 \vee \Sigma_2 \) the \( \sigma \)-field generated by \( \Sigma_1, \Sigma_2 \).

2a1 Lemma. \( L_\infty(\Sigma_1) \vee L_\infty(\Sigma_2) = L_\infty(\Sigma_1 \vee \Sigma_2) \).

Proof. “\( \subset \)” is trivial; we prove “\( \supset \)”.

Let \( A \in L_\infty(\Sigma_1 \vee \Sigma_2) \); we have to prove that \( A \in L_\infty(\Sigma_1) \vee L_\infty(\Sigma_2) \). Without loss of generality we assume the following. First, that \( A \) is an indicator, \( A = 1_{X_1} \), \( X \in \Sigma_1 \vee \Sigma_2 \). Second, that \( X \) belongs to the algebra generated by \( \Sigma_1, \Sigma_2 \). Third, that \( X = X_1 \cap X_2 \) for some \( X_1 \in \Sigma_1, X_2 \in \Sigma_2 \). Now, \( 1_{X_1} = 1_{X_1} 1_{X_2} \in L_\infty(\Sigma_1) \vee L_\infty(\Sigma_2) \).

2a2 Corollary. If \( \alpha(\mathcal{A}_1) = L_\infty(\Sigma_1) \) and \( \alpha(\mathcal{A}_2) = L_\infty(\Sigma_2) \) then \( \alpha(\mathcal{A}_1 \vee \mathcal{A}_2) = L_\infty(\Sigma_1 \vee \Sigma_2) \).

Proof. \( \alpha(\mathcal{A}_1 \vee \mathcal{A}_2) = \alpha(\mathcal{A}_1) \vee \alpha(\mathcal{A}_2) \), since \( \alpha \) is an isomorphism; use 2a1.

The product \((S, \Sigma, \mu) = (S_1, \Sigma_1, \mu_1) \times (S_2, \Sigma_2, \mu_2)\) of two measure spaces leads to the tensor product of commutative von Neumann algebras, \( L_\infty(S, \Sigma, \mu) = L_\infty(S_1, \Sigma_1, \mu_1) \otimes L_\infty(S_2, \Sigma_2, \mu_2) \). The same situation appears whenever two sub-\( \sigma \)-fields \( \Sigma_1, \Sigma_2 \subset \Sigma \) are independent (that is, \( \mu(X \cap Y) = \mu(X)\mu(Y) \) for all \( X \in \Sigma_1, Y \in \Sigma_2 \)), similarly to [1 Sect. 1c].
2a3 Definition. Let \((S, \Sigma, \mathcal{M})\) be a measure class space. Two sub-\(\sigma\)-fields \(\Sigma_1, \Sigma_2 \subset \Sigma\) are \(\mathcal{M}\)-independent, if they are \(\mu\)-independent for some \(\mu \in \mathcal{M}\).

If \(\Sigma_1, \Sigma_2\) are \(\mathcal{M}\)-independent then (up to a natural unitary equivalence) 
\[
L_\infty(\Sigma_1 \vee \Sigma_2) = L_\infty(\Sigma_1) \otimes L_\infty(\Sigma_2) \quad \text{(as before, } L_\infty(\Sigma_1) = L_\infty(S, \Sigma_1, \mathcal{M}|_{\Sigma_1}) \text{ etc.).}
\]

The product \((S, \Sigma, \mathcal{M}) = (S_1, \Sigma_1, \mathcal{M}_1) \times (S_2, \Sigma_2, \mathcal{M}_2)\) of two measure class spaces is a measure class space \([5, 14.4]\); namely, \((S, \Sigma) = (S_1, \Sigma_1) \times (S_2, \Sigma_2)\), and \(\mathcal{M}\) is the equivalence class containing \(\mu_1 \times \mu_2\) for some (therefore all) \(\mu_1 \in \mathcal{M}_1, \mu_2 \in \mathcal{M}_2\). In this case 
\[
L_\infty(S, \Sigma, \mathcal{M}) = L_\infty(S_1, \Sigma_1, \mathcal{M}_1) \otimes L_\infty(S_2, \Sigma_2, \mathcal{M}_2).
\]

Given two commutative von Neumann algebras \(\mathcal{A}_1\) on \(H_1\) and \(\mathcal{A}_2\) on \(H_2\), their tensor product \(\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2\) is a von Neumann algebra on \(H = H_1 \otimes H_2\). Given isomorphisms \(\alpha_1: \mathcal{A}_1 \to L_\infty(S_1, \Sigma_1, \mathcal{M}_1)\) and \(\alpha_2: \mathcal{A}_2 \to L_\infty(S_2, \Sigma_2, \mathcal{M}_2)\), we get an isomorphism \(\alpha = \alpha_1 \otimes \alpha_2: \mathcal{A} \to L_\infty(S, \Sigma, \mathcal{M})\), where \((S, \Sigma, \mathcal{M}) = (S_1, \Sigma_1, \mathcal{M}_1) \times (S_2, \Sigma_2, \mathcal{M}_2);\) namely, \(\alpha(\mathcal{A}_1 \otimes \mathcal{A}_2) = \alpha_1(\mathcal{A}_1) \otimes \alpha_2(\mathcal{A}_2)\) for \(\mathcal{A}_1 \in \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{A}_2\). Note that \(\alpha(\mathcal{A}_1 \otimes \mathcal{1}) = L_\infty(\Sigma_1)\) and \(\alpha(\mathcal{1} \otimes \mathcal{A}_1) = L_\infty(\Sigma_2)\), where \(\Sigma_1 = \{S_1 \times \mathcal{A}_1 : A_1 \in \Sigma_1\}\) and \(\Sigma_2 = \{S_2 \times \mathcal{A}_2 : A_2 \in \Sigma_2\}\) are \(\mathcal{M}\)-independent sub-\(\sigma\)-fields of \(\Sigma\), and \(\Sigma_1 \vee \Sigma_2 = \Sigma\).

2a4 Corollary. For every isomorphism \(\alpha: \mathcal{A}_1 \otimes \mathcal{A}_2 \to L_\infty(S, \Sigma, \mathcal{M})\) there exist \(\mathcal{M}\)-independent \(\Sigma_1, \Sigma_2 \subset \Sigma\) such that \(\alpha(\mathcal{A}_1 \otimes \mathcal{1}) = L_\infty(\Sigma_1), \alpha(\mathcal{1} \otimes \mathcal{A}_2) = L_\infty(\Sigma_2),\) and \(\Sigma_1 \vee \Sigma_2 = \Sigma\).

2b Spectrum as a measure class factorization

As before, \(B\) is a noise-type Boolean algebra. The corresponding projections \(Q_x\) commute (by \([1a1]\)), and generate a commutative von Neumann algebra \(\mathcal{A}\). Sect. \(2a\) gives us a measure class space \((S, \Sigma, \mathcal{M})\) and an isomorphism 
\[
\alpha: \mathcal{A} \to L_\infty(S, \Sigma, \mathcal{M}).
\]

Projections \(Q_x\) turn into indicators:
\[
\alpha(Q_x) = \mathbb{1}_{S_x}, \quad S_x \in \Sigma
\]
(of course, \(S_x\) is an equivalence class rather than a set); \([1a1]\) gives
\[
(2b1) \quad S_x \cap S_y = S_{x \wedge y}.
\]
(In contrast, the evident inclusion \(S_x \cup S_y \subset S_{x \vee y}\) is generally strict.) Every \(\Sigma\)-measurable set \(E \subset S\) leads to a subspace \(H(E) \subset H\) such that 
\[
\alpha(\text{Pr}_{H(E)}) = \mathbb{1}_E,
\]

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where $\text{Pr}_{H(E)}$ is the projection onto $H(E)$. Note that
\[
H(E_1 \cap E_2) = H(E_1) \cap H(E_2),
H(E_1 \cup E_2) = H(E_1) \oplus H(E_2),
H(E_1 \cup E_2) = H(E_1) + H(E_2),
H(S_x) = H_x
\]
(the second line differs from the third line by assuming that $E_1, E_2$ are disjoint and concluding that $H(E_1), H(E_2)$ are orthogonal); $E \mapsto H(E)$ is a projection measure.

Every subset of $B$ leads to a subalgebra of $\mathcal{A}$, thus, to a sub-$\sigma$-field of $\Sigma$. In particular, for every $x \in B$ we introduce the von Neumann algebra
\[
\mathcal{A}_x \subset \mathcal{A}
\]
generated by \( \{Q_y : y \in B, x \lor y = 1\} \)
and the $\sigma$-field $\Sigma_x \subset \Sigma$ such that
\[
\alpha(\mathcal{A}_x) = L_\infty(\Sigma_x).
\]
Note that
\[
(2b2) \quad x \leq y \quad \text{implies} \quad \mathcal{A}_x \subset \mathcal{A}_y \quad \text{and} \quad \Sigma_x \subset \Sigma_y.
\]

The Boolean algebra $B$ contains the least element 0; the corresponding operator $Q_0$, — the projection onto the one-dimensional space $H_0$ (of constant functions on $(\Omega, \mathcal{F}, P)$), — is a \textit{minimal} projection in $\mathcal{A}$. Combined with the relation $\alpha(Q_0) = \mathbb{1}_{S_0}$ it shows that $S_0$ is an atom of $\Sigma$. Similarly, $Q_{x'}$ is a minimal projection in $\mathcal{A}_x$, and therefore $S_{x'}$ is an atom of $\Sigma_x$.

2b3 Proposition. $\Sigma_x \lor \Sigma_y = \Sigma_{x \lor y}$ for all $x, y \in B$.

\textbf{Proof.} By (2b2), $\Sigma_x \lor \Sigma_y \subset \Sigma_{x \lor y}$. By 2a2 it is sufficient to prove that $\mathcal{A}_{x \lor y} \subset \mathcal{A}_x \lor \mathcal{A}_y$, that is, $Q_z \in \mathcal{A}_x \lor \mathcal{A}_y$ whenever $x \lor y \lor z = 1$. We have $z = (z \lor x') \land (z \lor y')$. By (1a1), $Q_z = Q_{z \lor x'}Q_{z \lor y'} \in \mathcal{A}_x \lor \mathcal{A}_y$, since $Q_{z \lor x'} \in \mathcal{A}_x$ and $Q_{z \lor y'} \in \mathcal{A}_y$.

2b4 Proposition. If $x \land y = 0$ then $\Sigma_x, \Sigma_y$ are $\mathcal{M}$-independent.

\textbf{Proof.} It is sufficient to prove that $\Sigma_x, \Sigma_{x'}$ are $\mathcal{M}$-independent (since $\Sigma_y \subset \Sigma_{x'}$ by (2b2)).
As was noted before the proof of 1b5, we have (up to the natural unitary equivalence) \( H = H_x \otimes H_{x'} \) and \( Q_{u \lor v} = Q_u^{(x)} \otimes Q_v^{(x')} \) for all \( u, v \in B \) such that \( u \leq x \) and \( v \leq x' \).

By 2a4 it is sufficient to prove that all operators of \( A_x \) are of the form \( A \otimes 1 \) (for \( A : H_x \to H_x \)), and all operators of \( A_{x'} \) are of the form \( 1 \otimes B \). We prove the former; the latter is similar. If \( z \) satisfies \( x \lor z = 1 \) then \( z = (z \land x) \lor x' \) and therefore \( Q_z = Q_{z \land x}^{(x)} \otimes Q_{x'}^{(x')} = Q_x^{(x)} \otimes 1 \), as needed. 

2b5 Remark. The completion of \( B \) (thus, also the closure of \( B \)) determines uniquely the algebra \( \mathcal{A} \) (since \( \mathcal{A} \) is closed in the strong operator topology) and therefore also the spectral space.

2c Spectral filters, spectral sets

Taking into account that every noise-type Boolean algebra contains a dense countable Boolean subalgebra and both algebras lead to the same spectral space, we assume here (in Sect. 2c) that \( B \) is a countable noise-type Boolean algebra.

Having only countably many equivalence classes \( S_x \) we may, and will, treat them as sets (rather than equivalence classes), satisfying (2b1) exactly (rather than almost everywhere). Then sets \( \Phi_s = \{ x \in B : s \in S_x \} \) for \( s \in S \) satisfy \( (x, y \in \Phi_s) \iff (x \land y \in \Phi_s) \), which shows that \( \Phi_s \) is either a filter in \( B \) (if \( s \notin S_0 \)) or the improper filter, the whole \( B \) (if \( s \in S_0 \)). This way, points of the spectral space may be interpreted as filters on \( B \) (“spectral filters”).

Every countable Boolean algebra \( B \) is isomorphic to the Boolean algebra of all clopen (that is, both closed and open) subsets of a totally disconnected compact metrizable space, so-called Stone space of \( B \) (homeomorphic to the Cantor set, if \( B \) is atomless). Filters on \( B \) (maybe improper) correspond bijectively to closed subsets (maybe empty) of the Stone space. This way, points of the spectral space may be interpreted as closed subsets of the Stone space (“spectral sets”), and the relation \( s \in S_x \) holds if and only if the closed set corresponding to \( s \) is contained in the clopen set corresponding to \( x \).

3 Digression: planar spectral sets, etc.

As noted in [1, Intro], in the framework of a “noise as a Boolean algebra of \( \sigma \)-fields” we consider the \( \sigma \)-fields irrespective of the corresponding domains (in \( \mathbb{R}^n \) or another parameter space). In contrast, spectral sets defined before
for a noise over $\mathbb{R}$ are compact subsets of $\mathbb{R}$ (rather than a Stone space). In this section we return to a parameter space and its spectral subsets. The parameter space is usually $\mathbb{R}^n$, but an arbitrary topological space can be used equally well.

3a Preliminaries: regular open sets

Let $X$ be a topological space. We introduce the set

$$\text{Reg}(X) = \{(G, F) : G = \text{Int}(F), F = \text{Cl}(G)\}$$

of all pairs $(G, F)$ of subsets of $X$ such that $G$ is the interior of $F$ and at the same time $F$ is the closure of $G$. For $r \in \text{Reg}(X)$ we denote

$$G = \text{Int}(r), \quad F = \text{Cl}(r),$$

somewhat abusing the symbols “Int” and “Cl”, since $r$ is not a subset of $X$. We introduce on $\text{Reg}(X)$ a partial order

$$r \leq s \iff \text{Int}(r) \subset \text{Int}(s) \iff \text{Cl}(r) \subset \text{Cl}(s)$$

(the second and third relations being evidently equivalent). It appears that $\text{Reg}(X)$ is a Boolean algebra, and

$$\text{Int}(r \land s) = \text{Int}(r) \cap \text{Int}(s),$$
$$\text{Cl}(r \lor s) = \text{Cl}(r) \cup \text{Cl}(s),$$
$$\text{Int}(r^\prime) = X \setminus \text{Cl}(r), \quad \text{Cl}(r^\prime) = X \setminus \text{Int}(r).$$

Also,

$$\text{Cl}(r \land s) = \text{Cl}(\text{Int}(r) \cap \text{Int}(s)) \subset \text{Cl}(r) \cap \text{Cl}(s),$$
$$\text{Int}(r \lor s) = \text{Int}(\text{Cl}(r) \cup \text{Cl}(s)) \supset \text{Int}(r) \cup \text{Int}(s).$$

For every $r \in \text{Reg}(X)$ the set $G = \text{Int}(r)$ is equal to the interior of its closure; such sets are called regular open. Every regular open set $G$ is $\text{Int}(r)$ for some $r \in \text{Reg}(X)$, namely, $r = (G, \text{Cl}(G))$. Thus, the Boolean algebra $\text{Reg}(X)$ is naturally isomorphic to the Boolean algebra of all regular open sets. The same holds for regular closed sets.

See [7, Sect. 4].
3b Back to a topological base

Let $X$ be a topological space, $A \subseteq \text{Reg}(X)$ a Boolean subalgebra, $B$ a noise-type Boolean algebra, and $h : A \to B$ a homomorphism. We are interested in a map $F$ from $S$ to the set of closed subsets of $X$ such that for every $a \in A$, 

$$(3b1) \quad S_{h(a)} = \{ s \in S : F(s) \subset \text{Cl}(a) \} \quad (\text{mod } 0).$$

Here are two relevant assumptions.

3b2 Assumption. There exists a countable subset $A_0 \subset A$ such that $\{\text{Int}(a) : a \in A_0\}$ is a (topological) base of $X$.

3b3 Assumption. For every $a \in A$ there exist $a_1, a_2, \ldots \in A$ such that $a_n \leq a_{n+1}$, $\text{Cl}(a_n)$ is compact, $\text{Cl}(a_n) \subset \text{Int}(a)$ for all $n$, and $h(a_n) \downarrow h(a)$.

3b4 Lemma. Assumption $3b2$ ensures uniqueness of $F$ satisfying $(3b1)$.

Proof. Every open set is the union of some sets of the base. In particular,

$$(3b5) \quad X \setminus F(s) = \bigcup_{a \in A_0, s \in S_{h(a')}} \text{Int}(a),$$

since $\text{Int}(a) \subset X \setminus F(s) \iff F(s) \subset X \setminus \text{Int}(a) \iff F(s) \subset \text{Cl}(a) \iff s \in S_{h(a')}$. 

3b6 Theorem. Assumptions $3b2, 3b3$ ensure existence of $F$ satisfying $(3b1)$.

Proof. Assumption $3b2$ gives us $A_0$. We define $F(\cdot)$ by $(3b5)$ and prove $(3b1)$.

Let $a \in A$ and $s \in S_{h(a)}$; we’ll prove that $F(s) \subset \text{Cl}(a)$. To this end it is sufficient to prove that $X \setminus F(s) \supset \text{Int}(a_0)$ for every $a_0 \in A_0$ satisfying $\text{Int}(a_0) \subset \text{Int}(a')$. We note that $a_0 \leq a'$, $a'_0 \geq a$, $S_{h(a'_0)} \supset S_{h(a)}$, thus $s \in S_{h(a'_0)}$. By $(3b5)$, $X \setminus F(s) \supset \text{Int}(a_0)$.

Let $a \in A$ and $F(s) \subset \text{Cl}(a')$; we’ll prove that $s \in S_{h(a')}$. Assumption $3b3$ gives us $a_1, a_2, \ldots$. It is sufficient to prove that $s \in S_{h(a'_n)}$ for every $n$, since $S_{h(a'_n)} \uparrow S_{h(a')}$. We have $X \setminus F(s) \supset X \setminus \text{Cl}(a') = \text{Int}(a) \supset \text{Cl}(a_n)$. Using $(3b5)$ and compactness of $\text{Cl}(a_n)$ we find $b_1, \ldots, b_k \in A_0$ (dependent on $n$, of course) such that $\text{Int}(b_1) \cup \cdots \cup \text{Int}(b_k) \supset \text{Cl}(a_n)$ and $s \in S_{h(b'_1)} \cap \cdots \cap S_{h(b'_k)}$. Introducing $b = b_1 \lor \cdots \lor b_k$ we have $\text{Int}(b) \supset \text{Cl}(a_n)$ by $(3a2)$, and $s \in S_{h(b')}$ by $(2b1)$. Finally, $S_{h(b')} \subset S_{h(a'_n)}$ since $b \geq a_n$, and we get $s \in S_{h(a'_n)}$. 


From now on we assume \(3b2\) and \(3b3\) and consider \(F\) satisfying \(3b1\).

In particular, if \(B\) is countable, \(X\) is the Stone space of \(B\), \(A\) consists of all clopen sets, and \(h\) is the natural isomorphism \(A \to B\), then \(F(s)\) is the spectral set in the sense of Sect. 2c.

In general, every monotone sequence in \(B\) converges in \(\text{Cl}(B)\) (the closure of \(B\) in \(\Lambda\), see [1, Sect. 2a]). Thus, \(\lim_n h(a_n)\) exists in \(\text{Cl}(B)\) for every monotone sequence \((a_n)\) in \(A\).

3b7 Proposition. The following two conditions on an increasing sequence \((a_n)\) in \(A\) are equivalent:

(a) \(\lim_n h(a_n) = 1\);

(b) for almost every \(s\) there exists \(n\) such that \(F(s) \subseteq \text{Cl}(a_n)\).

Proof. \(h(a_n) \uparrow 1 \iff S_{h(a_n)} \uparrow S \iff \forall s \in S_{h(a_n)} \exists s \in F(s) \subseteq \text{Cl}(a_n)\), where "\(\forall\)" means "for almost all".

3b8 Corollary. If there exist \(a_1 \leq a_2 \leq \ldots\) such that \(\lim_n h(a_n) = 1\) and \(\text{Cl}(a_n)\) is compact for every \(n\), then \(F(s)\) is compact for almost all \(s\).

Given \(r \in \text{Reg}(X), r \notin A\), we may try to extend \(h\) to \(r\) by approximation from the inside:

\[h_-(r) = \sup\{h(a) : a \in A, \text{Cl}(a) \subseteq \text{Int}(r)\}\,.

Then \(h_-(r) \land h_-(r') = 0\), but the question is, whether \(h_-(r) \lor h_-(r') = 1\) or not.

Denote \(\text{Bd}(r) = \text{Cl}(r) \setminus \text{Int}(r)\) (the boundary).

3b9 Proposition. (a) If \(h_-(r) \lor h_-(r') = 1\) then \(F(s) \cap \text{Bd}(r) = \emptyset\) for almost all \(s\).

(b) If \(F(s) \cap \text{Bd}(r) = \emptyset\) for almost all \(s\), and \(F(s)\) is compact for almost all \(s\), and \(X\) is a regular topological space, then \(h_-(r) \lor h_-(r') = 1\) (and therefore \(h_-(r)\) belongs to the noise-type completion of \(B\), and \((h_-(r))' = h_-(r')\)).

Proof. We have \(h_-(r) \lor h_-(r') = \sup\{h(a) : a \in A, \text{Cl}(a) \cap \text{Bd}(r) = \emptyset\} = \sup h(a_n)\) for some \(a_n \in A\) satisfying \(\text{Cl}(a_n) \cap \text{Bd}(r) = \emptyset\) and \(a_1 \leq a_2 \leq \ldots\)

Thus, Item (a) follows from 3b7. We turn to Item (b). Using Assumption \(3b2\) and regularity of \(X\) we take \(a_1, a_2, \ldots \in A\) such that \(\text{Cl}(a_n) \cap \text{Bd}(r) = \emptyset\) for all \(n\), and \(\cup_n \text{Int}(a_n) = X \setminus \text{Bd}(r)\). Introducing \(b_n = a_1 \lor \cdots \lor a_n\) we have \(\text{Cl}(b_n) \cap \text{Bd}(r) = \emptyset\) and by \(3a2\), \(\cup_n \text{Int}(b_n) = X \setminus \text{Bd}(r)\). Compactness of \(F(s)\) implies \(F(s) \subseteq \text{Int}(b_n)\) for some \(n\) (dependent on \(s\)). By \(3b7\), \(h(b_n) \uparrow 1\). On the other hand, \(h(b_n) \leq \sup_k h(a_k) = h_-(r) \lor h_-(r')\).
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