Material space and dual canonical wave formulation: Application to nonlinear elastic solids

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Abstract. It is shown that the kinematic wave theory develops in parallel with the "materials mechanics" but in terms of frequency and material wave vector instead of time and material coordinates. A conservation law of material wave momentum involving a material wave Eshelby stress is thus deduced together with the conservation of wave action. This formalism and the Whitham-Newell averaging method are then used in an illustration to the case of nonlinear dispersive bulk waves in elastic crystals. A nonlinear "dispersive" dispersion relation is thus constructed by means of asymptotics, which allows one to obtain slowly varying small amplitude modulated dynamical solutions such as envelope solitons.

1. Introduction

Since the space-time parametrization ($X$, $t$) - where $X$ are material coordinates and $t$ is Newtonian time - provides the natural background for the canonical thermomechanics of continua and the accompanying theory of motion of defects of various types [1]-[6], one may naturally wonder what happens in the dual space-time manifold spanned by a material wave vector $K$ and an angular frequency $\omega$ since a space-time invariant phase is defined by $\phi = K \cdot X - \omega t$, exploiting in the like manner consequences of Noether’s theorem [7]-[8]. The observation of the invariance of this scalar quantity was an essential tool in the intellectual construct of Louis de Broglie that yielded quantum wave mechanics with further works by L. de Broglie, E.Schrödinger and others in the 1920s [9]. This strictly applies to a monochromatic linear wave process. Dispersion may be envisaged but no nonlinearity. However, the kinematic wave theory due essentially to Lighthill [10]-[11], Whitham [12]-[14], and Hayes [15]-[17] allows one to introduce a more inclusive definition of wave vector and angular frequency from a general space-time dependent phase function. The cited authors also noticed the interest of variational formulations and of the exploitation of basic theorems of field theory (the theory of the averaged Lagrangian) in the solution of nonlinear wave problems with
almost monochromatic features but including some degree of dispersion and nonlinearity, with refinements by Benney [18] and Newell [19]. Very few applications have followed in fluid mechanics where this started, probably due to the difficult grasp of some of the basic ideas. This is even more true in wave-like phenomena in deformable solids where the only available problem solution seems to be the one given by the author and Hadouaj [20] in a complicated nonlinear surface wave problem.

Here below, exploiting the noticed duality between the parametrization \((X,t)\) and the wave-like entity \((K,\omega)\) and having recalled the basic structure of variational continuum mechanics and elasticity in finite strains (the ideal paragon case) -Sections 2 and 3-, we examine the related wave-kinematics in the theory of the averaged Lagrangian theory of Whitham and Hayes and show how, via the application of Noether’s theorem, such notions as those of wave action, material wave momentum (quite similar to the wave momentum of de Broglie), and an Eshelby “wave” material tensor emerge naturally with corresponding continuum conservation laws of wave action, material wave momentum, and wave energy -Section 4. By way of application (Section 5) the case of one-dimensional dispersive weakly nonlocal elasticity is considered with the possibility of existence of modulated localized nonlinear solutions akin to bright solitons. This work has been preceded by reflections on the relative nature of waves (phonons) in the Eulerian and material descriptions (pp.34-38 in Ref. [1] -work in cooperation with C.Trimarco-, further elaboration in [21]) and first considerations on wave kinematics in [22]. Recent works by Kienzler and Herrmann [23]-[24] go in the same direction. As a rule we use the standard intrinsic notation of nonlinear continuum mechanics [25].

2. Elements of field theory

We are concerned with simple general features of field theories in a continuum with space-time parametrization \(\{X,t\}\), where \(X\) stands for material coordinates of classical continuum mechanics (e.g., in Truesdell and Toupin [25]), and \(t\) for a timelike scalar variable (Newton’s absolute time). We consider first Hamiltonian actions of the type

\[
A(\phi;V) = \int_{V \times I} \left[ L(\phi^\alpha, \partial_\mu \phi^\alpha; X^{\mu}) \right] d^4 X ,
\]

where \(\phi^\alpha, \alpha = 1,2,\ldots,N\), denotes the ordered array of fields, say the independent components of a certain geometric object, and \(d^4 X = dV dt\). This is a Cartesian-Newtonian notation, with

\[
\left\{ \partial_\mu = \partial / \partial X^{\mu}; \mu = 1,2,3,4 \right\} = \left\{ \partial / \partial X^K, K = 1,2,3; \partial / \partial X^4 \equiv \partial / \partial t \right\} .
\]

The summation over dummy indices (Einstein convention) is enforced. We do not pay attention to boundary and initial conditions and impose no external sources. From expression (2.1) two types of equations can be derived: there are those relating to each one of the fields \(\phi^\alpha, \alpha\) chosen, and those which express a general conservation law of the system governing all fields simultaneously (involving a summation over \(\alpha\)). The first group is none other than the set of Euler-Lagrange variational equations:

\[
E_\alpha \equiv \frac{\delta L}{\delta \phi^\alpha} = \frac{\partial L}{\partial \phi^\alpha} - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial (\partial_\mu \phi^\alpha)} \right) - \frac{\partial}{\partial X^K} \left( \frac{\partial L}{\partial (\partial_\mu \phi^\alpha / \partial X^K)} \right) = 0 ,
\]

(2.3)
for each $\alpha = 1, 2, \ldots, N$, at any regular material point $X$ and for any time $t$.

The second group of equations are the result of the variation of the parametrization, and these results, on account of the former group, express the invariance or lack of invariance of the whole system under changes of this parametrization. [This invariance can be written as]

$$\int_{X} L(\phi^\alpha, \partial \phi^\alpha / \partial X^\mu; X^\mu) d^4 X = \int_{X} L(\phi^\alpha, \partial \phi^\alpha / \partial X'^\mu; X'^\mu) d^4 X'$$

for any reparametrization $X'^\mu = \overline{X}'^{\mu}(X^\nu)$.

The resulting equations are none other than the expression of the celebrated Noether's theorem (Noether [7]; Soper [26]). The latter states that to any symmetry of the system there corresponds the conservation (or lack of strict conservation) of a current (cf. Maugin [1], pp.99-103). Here, with respect to time and space-like parametrizations we have the following two equations, a scalar conservation law in the explicit form

$$\frac{\partial H}{\partial t} |_X - \nabla_k Q = h,$$  \hspace{1cm} (2.4)

and a co-vectorial conservation (material) law:

$$\frac{\partial P}{\partial t} |_X - \text{div}_k b = f^{inh}.$$

where we defined the following canonical quantities:

- **energy (Hamiltonian density):**

  $$H := \sum_{\alpha} \dot{\phi}^\alpha \left( \frac{\partial L}{\partial \dot{\phi}^\alpha} \right) - L, \quad \dot{\phi}^\alpha \equiv \partial \phi^\alpha / \partial t;$$  \hspace{1cm} (2.6)

- **energy flux vector:**

  $$Q = \left\{ Q^k := -\sum_{\alpha} \dot{\phi}^\alpha \frac{\partial L}{\partial \left( \partial_k \phi^\alpha \right)} \right\};$$  \hspace{1cm} (2.7)

- **canonical (here material) momentum:**

  $$P = \left\{ P_k := -\sum_{\alpha} \frac{\partial \phi^\alpha}{\partial X^k} \frac{\partial L}{\partial \left( \partial \phi^\alpha / \partial t \right)} \right\};$$  \hspace{1cm} (2.8)

- **canonical stress tensor:**
\[ b = \left\{ b^L = - \left( L \delta^L - \sum_a \frac{\partial \phi^a}{\partial X^L} \frac{\partial L}{\partial (\partial \phi^a / \partial X^k)} \right) \right\}. \hspace{1cm} (2.9) \]

and the source terms

\[ h^L = - \left. \frac{\partial L}{\partial t} \right|_{\text{expl}}, \quad f^{\text{inh}} = \left. \frac{\partial L}{\partial X^L} \right|_{\text{expl}}. \hspace{1cm} (2.10) \]

where the notation « expl » means that the « explicit » derivatives are taken at fixed fields \( \phi^a \). The symbol \( \nabla \) means the material gradient and \( \text{div} \) is used to represent the material divergence taken at the left of rank-two tensors.

Physical systems in which \( h \) does not vanish are said to be rheonomic (after Boltzmann, cf. Lanczos [27], p.32). Systems in which the « material »force \( f^{\text{inh}} \) does not vanish are said to be materially inhomogeneous. A clear meaning of this is best illustrated by the case of hyperelasticity (see below).

Had we considered a Lagrangian function \( L \) depending on higher-order space-time gradients, e.g., second-order one,

\[ L = L(\phi^a, \partial_t \phi^a, \partial_{\mu} \phi^a; X, t), \hspace{1cm} (2.11) \]

we would have obtained longer expressions for \( E_{\alpha} \), \( Q \) and \( b \); e.g., in place of eqn.(2.3),

\[ E_{\alpha} = \frac{\delta L}{\delta \phi^a} = \frac{\partial L}{\partial \phi^a} - \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} \phi^a)} \right) \]

\[ + \partial_{\mu} \partial_{\nu} \left( \frac{\partial L}{\partial (\partial_{\mu} \partial_{\nu} \phi^a)} \right) = 0. \hspace{1cm} (2.12) \]

But apart for some paradoxical theories with strange inertial terms (see, e.g., Maugin and Christov [28]), in classical theories \( L \) depends at most on the first-order time derivatives and the second-order gradients in (2.11) are purely spatial. When this is the case the formal definitions of both \( H \) and \( P \) are left unchanged while \( Q \) and \( b \) take on the following more general component forms including second-order space-like (here material) gradients:

\[ Q^k = - \sum_\alpha \left\{ \dot{\phi}^a \left( \frac{\partial L}{\partial (\partial_k \phi^a)} - \partial_L \left( \frac{\partial L}{\partial (\partial_k \partial_L \phi^a)} \right) \right) \right\}, \hspace{1cm} (2.13) \]

and
\[
\begin{aligned}
&b_{L}^{K} = - \left\{ L \delta_{L}^{K} - \sum_{\alpha} \frac{\partial \phi^{\alpha}}{\partial X^{L}} \frac{\partial L}{\partial (\partial \phi^{\alpha} / \partial X^{K})} \right. \\
&\quad - \sum_{\alpha} 2 \frac{\partial^{2} \phi^{\alpha}}{\partial X^{L} \partial X^{M}} \frac{\partial L}{\partial (\partial^{2} \phi^{\alpha} / \partial X^{M} \partial X^{K})} \\
&\left. + \sum_{\alpha} \frac{\partial}{\partial X^{L}} \left( \frac{\partial \phi^{\alpha}}{\partial X^{M}} \frac{\partial L}{\partial (\partial^{2} \phi^{\alpha} / \partial X^{M} \partial X^{K})} \right) \right\} \\
&= \frac{\partial^{2} \phi^{\alpha}}{\partial X^{L} \partial X^{M}} \frac{\partial L}{\partial (\partial^{2} \phi^{\alpha} / \partial X^{M} \partial X^{K})} .
\end{aligned}
\]

(2.14)

Other symmetries, and thus other consequences of Noether’s theorem can be applied - e.g., rotations ; cf. Maugin [1], Sec.5.2-5.5 - but these are not reminded here as they will not be exploited. More is to be found on Noether’s theorem and its relationship to the theory of Lie groups in Olver’s book [29]. The above is quite sufficient for our purpose.

3. The case of hyperelasticity
This corresponds to a trivial and isomorphic application of the above given formulae with a special choice of the fields \( \phi^{\alpha} = \bar{x}^{\alpha} \), with \( \alpha \) reduced to \( i = 1,2,3 \). They are the the physical components (i.e., components in physical space at time \( t \)) of the placement of the material « particle » \( X \); that is

\[
x = \bar{x}(X,t).
\]

(3.1)

The physical velocity of \( X \) and the deformation gradient are defined by

\[
v_{i} := \frac{\partial \bar{x}}{\partial t} \bigg|_{x}, \quad F_{i} := \frac{\partial \bar{x}}{\partial X_{i}} \equiv \nabla_{k} \bar{x} .
\]

(3.2)

For a first-order gradient theory of elasticity, the Lagrangian density \( L \) per unit volume in the reference configuration of the continuous body reads

\[
L = \bar{E}(v,F;X,t) = K(v;X,t) - W(F;X,t) ,
\]

(3.3)

where \( W \) is the elasticity potential and \( K \) is the kinetic energy given in a general manner by

\[
K = \frac{1}{2} \rho(X,t) v^{2} .
\]

(3.4)

Accordingly, equations (2.3) through (2.5) render the balance of linear physical momentum (three components of the field \( x \) in physical space),

\[
\frac{\partial p}{\partial t} \bigg|_{x} - \text{div}_{k} T = 0 .
\]

(3.5)
and the balance of energy (a scalar equation), and the balance of material momentum (a co-vectorial equation on the material manifold) with the following expressions for the various involved quantities:

\[
\mathbf{p} := \rho_0 \mathbf{v}, \quad \mathbf{T} := \frac{\partial W}{\partial \mathbf{F}}, \quad H = K + W, \quad \mathbf{Q} = \mathbf{T} \cdot \mathbf{v}, \quad \mathbf{P} := -\rho_0 \mathbf{v} \cdot \mathbf{F}, \quad \mathbf{b} = -\left( \mathbf{L} \mathbf{1} + \mathbf{T} \cdot \mathbf{F} \right),
\]

\[
P := -\rho_0 \mathbf{v} \cdot \mathbf{F}, \quad \mathbf{b} = -\left( \mathbf{L} \mathbf{1} + \mathbf{T} \cdot \mathbf{F} \right), \quad H = K + W, \quad \mathbf{Q} = \mathbf{T} \cdot \mathbf{v}, \quad \mathbf{P} := -\rho_0 \mathbf{v} \cdot \mathbf{F},
\]

\[
\mathbf{b} = -\left( \mathbf{L} \mathbf{1} + \mathbf{T} \cdot \mathbf{F} \right), \quad \mathbf{h} := -\left( \frac{K}{\rho_0} \right) \frac{\partial \rho_0}{\partial t} + \frac{\partial W}{\partial t} + \frac{\partial W}{\partial t} \left|_\text{expl} \right.
\]

\[
f^{inh} := \left( \frac{K}{\rho_0} \right) \frac{\partial \rho_0}{\partial \mathbf{X}} - \frac{\partial W}{\partial \mathbf{X}} \left|_\text{expl} \right.
\]

These are, respectively, the physical linear momentum \( \mathbf{p} \), the first Piola-Kirchhoff stress \( \mathbf{T} \), the Hamiltonian (energy) density \( H \), the energy flux or Poynting vector \( \mathbf{Q} \), the material momentum \( \mathbf{P} \), the Eshelby material stress \( \mathbf{b} \) (cf. Maugin and Trimarco [30], who gave this name to what Eshelby himself used to refer to as the energy-momentum tensor or Maxwell stress), the energy source \( h \) in a rheonomic system, and the material inhomogeneity force \( f^{inh} \) (cf. Maugin [1]). Rheonomic systems are rarely studied but in such systems the energy is not conserved. Equation (2.4) becomes a strict conservation law (no source), only when the system considered is scleronomic (i.e., with no explicit time dependence of the Lagrangian density, according to the classification of Boltzmann and Lanczos). The possibility that the reference material density be an explicit function of time is nonetheless the basic working hypothesis in the theory of material growth (such as in biological tissues; cf. Epstein and Maugin [31]). The possibility that the elasticity potential \( W \) might depend explicitly on time would refer to the phenomenon of aging (i.e., evolution in time of material elasticity coefficients), a domain of research that remains practically unexplored probably due to the difficulty to measure experimentally the time evolution of interest. Accordingly, the large majority of elastic materials considered correspond to scleronomic systems. However, the explicit dependency of both \( \rho_0 \) and \( W \) on the material point \( \mathbf{X} \) in materially inhomogeneous solids is an evidence, and its manifestation as a source term in eqn.(2.5) provides the basis for the theory of all types of « material » forces in solids. As a conclusion of this point, we note that hyperelasticity clearly is a paradigmatic theory of fields.

Now, the dynamical elasticity encapsulated in a Lagrangian such as (3.3) may be fully non linear as regards deformation processes, but it does not contain any characteristic length, except for inhomogeneous materials where the material spatial variation of the material properties (density, elasticity coefficients) may provide such a length. Homogeneous elasticity exhibiting a characteristic intrinsic length of necessity involves gradients of higher order than the first, e.g., the second material gradient of the placement or, just the same, the material gradient of the deformation gradient. That is, we may have to consider a generalization of (3.3) of the type

\[
L = \mathcal{L}(\mathbf{v}, \mathbf{F}, \nabla_k \mathbf{F}; \mathbf{X}, t) = K(\mathbf{v}; \mathbf{X}, t) - W(\mathbf{F}, \nabla_k \mathbf{F}; \mathbf{X}, t),
\]

where the inertial contribution is left unchanged. In this case the representations (2.13) and (2.14) apply other quantities being formally unchanged. The presence of an intrinsic length in the system is
paramount to the existence of dispersion in the case of wave motion. Together with the nonlinearity potentially present in (3.1), this provides the most interesting framework for wave-mechanics studies in elastic crystals (cf. Maugin [32] for a general background).

4. Kinematic wave theory

First, it is reminded that the phase of a plane linear wave in a continuum is defined in the material description by

$$\varphi(X, t) = \hat{\varphi}(K, \omega) = K \cdot X - \omega t , \quad (4.1)$$

where $K$ is the material wave vector and $\omega$ is the associated circular frequency. But in the kinematic wave theory due to the mastery of a small group of scientists (Lighthill [10]-[11], Whitham [12]-[14], Hayes [15]-[17]; elements of this are also given in Ostrovski and Potapov [33], Maugin [32]) a general phase function

$$\varphi = \varphi(X, t) . \quad (4.2)$$

is introduced from which the material wave vector $K$ and the frequency $\omega$ are defined by

$$K = \nabla \varphi = \nabla_k \varphi , \quad \omega = -\frac{\partial \varphi}{\partial t} , \quad (4.3)$$

Whence there follows at once the two equations (curl-free nature of $K$, and conservation of wave vector)

$$\nabla \times K = 0 \quad (4.4)$$

$$\frac{\partial K}{\partial t} + \nabla \times \omega = 0 . \quad (4.5)$$

In particular, eqns.(4.3) are trivially satisfied for plane wave solutions for which the last of (4.1) holds true. For an inhomogeneous rheonomic linear behavior with dispersion we have the dispersion relation

$$\omega = \Omega(K, X, t) . \quad (4.6)$$

Accordingly, the conservation of wave vector (4.6) becomes

$$\frac{DK}{Dt} = -\frac{\partial \Omega}{\partial X} \bigg|_{\text{expl}} , \quad (4.7)$$

where we have set
\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla_g , \quad \mathbf{V}_g = \frac{\partial \Omega}{\partial \mathbf{K}}.
\] (4.8)

Simultaneously we have the Hamilton-Jacobi equation (compare the second of eqn.(4.3))

\[
\frac{\partial \phi}{\partial t} + \Omega(\mathbf{X}, t; \mathbf{K} = \frac{\partial \phi}{\partial \mathbf{X}}) = 0.
\] (4.9)

If we now consider a wave in an inhomogeneous rheonomic dispersive nonlinear material, the frequency will also depend on the amplitude. Let \( \mathbf{a} \) the \( n \)-vector of \( \mathbb{R}^n \) that characterizes this small slowly varying amplitude of a complex system (in general with several degrees of freedom). Thus, now,

\[
\omega = \Omega(\mathbf{K}, \mathbf{X}, t, \mathbf{a}),
\] (4.10)

where dependency on derivatives of \( \mathbf{a} \) is discarded by virtue of the slow-variation hypothesis. Accordingly, eqn.(4.7) will now read [22]

\[
\frac{D \mathbf{K}}{Dt} = -\frac{\partial \Omega}{\partial \mathbf{X}} + \mathbf{A}(\nabla_g \mathbf{a})^T, \quad \mathbf{A} = -\frac{\partial \Omega}{\partial \mathbf{a}}.
\] (4.11)

The relationship with the developments in Section 2 above follows from the remarkable considerations of Whitham on a so-called averaged Lagrangian. For a wave motion depending on the phase (3.2) and with all characteristic quantities varying slowly over space-time (derivatives of \( \mathbf{a} \), \( \omega \), and \( \mathbf{K} \) are small in an appropriate mathematical sense and can thus be neglected), Whitham proposes to replace the initial variational problem (3.1) by one pertaining to the averaged Lagrangian, i.e.,

\[
\delta \int \tilde{L} d\mathbf{X} dt = 0 , \quad \tilde{L} = \frac{1}{2\pi} \int_{0}^{2\pi} L d\phi ,
\] (4.12)

with

\[
\tilde{L} = \tilde{\mathcal{L}} \left( \frac{\partial \bar{\phi}}{\partial \mathbf{X}} = \mathbf{K}, \frac{\partial \bar{\phi}}{\partial t} = -\omega, \mathbf{a}, \mathbf{X}, t \right),
\] (4.13)

where the fields are the amplitude \( \mathbf{a} \) and the phase \( \phi \), the latter being not involved per se. Accordingly, the associated equations (2.3) through (2.5) read

\[
\frac{\partial \tilde{L}}{\partial \mathbf{a}} = 0 ,
\] (4.14)

\[
\frac{\partial \tilde{S}}{\partial t} - \nabla_g \cdot \mathbf{W} = 0 ,
\] (4.15)
for the variational (i.e., field) equations, and
\[
\frac{\partial \tilde{H}}{\partial t} - \nabla_{R} \cdot \tilde{Q} = \tilde{h} \quad \text{and} \quad \frac{\partial \tilde{P}}{\partial t} - \text{div}_{R} \tilde{b} = \tilde{f}_{\text{inh}},
\]
(4.16)

for the equations that follow from Noether’s theorem for the translational invariance under \( t \) and \( X \), where we have set
\[
\tilde{S} := \frac{\partial \tilde{L}}{\partial \omega} , \quad W := \frac{\partial \tilde{L}}{\partial K} ,
\]
(4.17)
\[
\tilde{H} = \omega \tilde{S} - \tilde{L} , \quad \tilde{Q} = \omega W , \quad \tilde{h} = -\frac{\partial \tilde{L}}{\partial t}_{\text{expl}} ,
\]
(4.18)
\[
\tilde{P} = \tilde{S} K , \quad \tilde{b} = -\left( \tilde{L} R - W \otimes K \right) , \quad \tilde{f}_{\text{inh}} = \frac{\partial \tilde{L}}{\partial X}_{\text{expl}} .
\]
(4.19)

Dimensionally, \( \tilde{S} \) is an action and may be called the wave action, while eqn.(4.15) may be referred to as a strict conservation law for the wave action in which \( W \) is the action flux. Note that in continuum mechanics the action density \( S \) is seldom considered but it would be given in terms of canonical momentum \( P \) and Hamiltonian \( H \) by \( S = P.X - H t ; \) cf. the discussion in Maugin [22]). For a rheonomic system \( \left[ \left( \partial L / \partial t \right)_{\text{expl}} \neq 0 \right] \), this requires special attention. But returning to the above results, the material co-vector \( \tilde{P} \) may be called the material wave momentum (notice that its formula reminds us of the quantum wave-mechanics relationship due to de Broglie: \( \tilde{P} = hK \) where \( h \) is the reduced elementary quantum of action (Planck’s constant)), and \( \tilde{b} \), the associated flux, may be called the material wave Eshelby stress. This tensor is not symmetric unless \( W \) is proportional to \( K \).

The canonical conservation equations of energy and material wave momentum (4.16) are a consequence of Noether’s identity. But the wave action conservation equation (4.15) plays here the central role (equivalent to the balance of linear physical momentum (3.5) for hyperelasticity). Indeed, in the same way as eqns.(2.4) and (2.5) for elasticity can be deduced from (3.5) by right scalar multiplication by \( v \) and \( F \), respectively, and some further manipulations on account of the expression for \( L \) [34], eqns.(4.17) can be deduced from (4.16) by scalar and tensorial multiplication, respectively, by \( \omega \) and \( K \) on account of the functional dependency assumed for \( \tilde{L} \) and eqn.(4.14)- see Appendix A.

5. Application to nonlinear dispersive waves in elastic crystals
For the sake of example we chose to consider a particular one-dimensional nonlinear elastic bulk motion in a dispersive crystal. With \( u_t \) and \( u_{\xi} \) replacing \( v \) and \( F \), we have the following scleronomic Lagrangian density
\[
L = \frac{1}{2} \rho_{0} u_t^2 - \frac{1}{2} E \left( u_{\xi}^2 - \frac{1}{6} \beta u_{\xi}^4 \right) + \frac{1}{2} E \delta^2 \left( u_{xx} \right)^2 ,
\]
(5.1)
where \( \rho_0 \) is a fixed density, \( E \) is an elasticity coefficient, \( \beta \) is a coefficient of nonlinearity for elasticity (it may be of any sign) and \( \delta \) is a characteristic (intrinsic) length. The model (5.1) belongs in the class of models described by eqn.(2.1) for which (2.12)-(2.14) apply. The elastic component is not necessarily longitudinal (i.e., along \( x \)). The variational field equation (2.12) reads

\[
u_u - c_E^2 u_{xx} \left(1 - \beta u_X^2 \right) - c_E^2 \delta^2 u_{xxxx} = 0 , \tag{5.2}\]

in which we recognize a modified Boussinesq (crystal) equation (cf. Ref. [28] and [32], for the so-called « Boussinesq paradigm »). Here \( c_E = (E / \rho_0)^{1/2} \) is the linear elastic speed. The crystal is cut in such a way that there appears no cubic term in the expression (4.1) so that the first non-quadratic term is quartic. In appropriate nondimensional form eqn.(5.2) is rewritten as

\[
u_u - u_{xx} \left(1 - \beta u_X^2 \right) - \delta^2 u_{xxxx} = 0 , \tag{5.3}\]

Obviously, it is known that this admits soliton solutions of the kink type [34]. In our notation such a solution is given by

\[
u(X,t) = \pm u_0 \tan^{-1} \left[ \exp \left( \frac{X - ct}{L} + X_1 \right) \right] , \tag{5.4}\]

where \( X_1 \) is an initial value and we have defined the amplitude \( u_0 \) and the width \( L \) by

\[
u_0 = \left[- 6 \delta^2 / \beta \right]^{1/2} , \quad L = \left[\delta^2 / (c^2 - 1) \right]^{1/2} . \tag{5.5}\]

Accordingly, with the sign chosen in front of the \( \delta^2 \) term in (5.1), the kink has to be supersonic with respect to the elastic speed, and the nonlinearity coefficient \( \beta \) has to be negative to guarantee the existence of solutions (5.4). For a positive sign in front of \( \delta^2 \), the kink is subsonic and exists provided \( \beta \) is positive. The derivative function of \( \nu \) will provide a pulse or hump soliton in the same conditions. Flytzanis et al [35] offer a rather exhaustive study of the possibilities of nonlinear localized waves that can be obtained on the basis of eqn.(5.3). However, envelope solitons such as the ones exhibited in the present work are better obtained by working directly on the discrete (lattice) equation from which (5.3) can be deduced in the continuum long wave-length approximation, that is, by considering an ansatz such as \( \nu_n (t) = F_n (t) \exp (i \varphi_n) + \text{cc} \) for the displacement \( \nu \) at discrete point \( n \) in a one-dimensional nonlinear chain. Here the phase \( \varphi_n \) would be defined by

\[
\varphi_n = \overline{\varphi}_n (K, \omega) = n K D - \omega t , \quad \text{where } n \text{ is an integer and } D \text{ is the lattice spacing.}
\]

Here below we present a fully continuum approach by considering the wave mechanics associated with eqn.(5.32). Accordingly, first in the linear case, trying a solution \( \nu = a \exp (i \varphi) \), with \( \varphi = K X - \omega t \), we deduce from (5.3) the following « linear » dispersion relation :

\[
D_L (\omega, K) : = \omega^2 - K^2 \left(1 - \delta^2 K^2 \right) = 0 . \tag{5.6}\]

It is readily checked that the averaged Lagrangian is given by
\[
\tilde{L} = \frac{1}{2\pi} \int_0^{\pi} L d\varphi = D_L(\omega, K)a^2.
\] (5.7)

Then the associated Euler-Lagrange variational equations (4.14) and (4.15) for the amplitude \(a\) and the phase \(\varphi\) read (for \(a \neq 0\))
\[
D_L(\omega, K) = 0,
\] (5.8)

and
\[
\frac{\partial(\omega a^2)}{\partial t} + \frac{\partial(Ka^2)}{\partial X} = 0,
\] (5.9)

which are the linear dispersion relation and the conservation of wave action, respectively.

Now we look for the influence of the elastic nonlinearity. First, following Newell [19], also Ref. [20], we construct an amplitude dependent dispersion relation (remember eqn.(4.10)). For this purpose we consider the following asymptotic solution with slow space \(x\) and time \(T\) variables:
\[
u(X,t) = a \exp(i\nu \varphi) + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots\quad x = \varepsilon X, \quad T = \varepsilon t,
\] (5.10)

where \(\varepsilon\) is an order parameter and the phase is defined in a general way by eqns.(4.2)-(4.3). We immediately have
\[
\frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial \varphi} + \varepsilon \frac{\partial}{\partial T}, \quad \frac{\partial}{\partial X} = K \frac{\partial}{\partial \varphi} + \varepsilon \frac{\partial}{\partial x}.
\] (5.11)

On substituting (5.10) and (5.11) in eqn.(5.3), at the order zero in \(\varepsilon\) we obtain \(D_L \neq 0\)
\[
\frac{\partial^2 u_0}{\partial \varphi^2} + v^2 u_0 = 0,
\] (5.12)

where
\[
u_0 = a \exp(i\nu \varphi), \quad v^2 = -\frac{\beta a^2}{D_L(\omega, K)}.
\] (5.13)

The zeroth-order solution is only \(2\pi\) periodic with no smaller period when \(v^2 = 1\), i.e., when there holds the following amplitude-dependent dispersion relation:
\[
D_{NL}(\omega, K; a) = D_L(\omega, K) + \beta a^2 = 0.
\] (5.14)

Alternatively, this equation may be considered as providing the slowly varying amplitude in terms of \(\omega, K\), and the material parameters. However, following Newell [19], also Maugin, [32] (Appendix A6) and in anticipation of more generality, it is worthwhile to view (5.14) as a first (zeroth) approximation and write the general dispersion relation in the perturbed form of a «dispersive» nonlinear dispersion relation:
where the ellipsis in the arguments of $D_{n\ell}$ on the left-hand side stands for higher-order space and time derivatives in the slow variables. The first-order term in the expansion of eqn.(5.15) indeed yields $u_0 = a \exp(i \phi)$. Same as in Maugin [32] p.279, the next order yields $g_1 = 0$, $u_1 = 0$, and none other than the wave-action conservation (5.9). The fact that $u_1$ vanishes is not surprising since the quartic nonlinearity of the initial potential is a third-harmonic generator. At order two, we will obtain that $g_2$ must be given by

$$g_2 = -a^{-1} \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial x^2} \right) a.$$  

(5.16)

Gathering these results in (5.15) we finally have the following «wave-like» equation for the dispersive nonlinear dispersion relation:

$$\omega^2 - K^2(1 - \delta^2 K^2) + \beta a^2 - \varepsilon^2 a^{-1} \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial x^2} \right) a = 0.$$  

(5.17)

We have constructed thus a set of partial differential equations for the unknown $(\omega, K, a)$, a system that consists of eqns. (4.5), (5.9) and (5.17). The general solution of this is impracticable. That is why an interesting approximation consists in considering an almost monochromatic regime $(\omega_0, K_0)$ whose point belongs to the linear dispersion relation (5.6); (i.e., when $\beta$ and $\varepsilon$ vanish). But in the nonlinear case we have to consider couples $(\omega, K)$ in the neighborhood of this working point. On account of the representation (4.3) we shall therefore write

$$K = K_0 + \varepsilon \phi_t, \quad \omega = \omega_0 - \varepsilon \phi_t,$$  

(5.18)

where $\phi$ is the perturbation phase. Moreover, we introduce a moving coordinate $\xi$ and a new scaling by (cf. [18])

$$\xi = x - \omega_0' T, \quad \tau = \varepsilon T, \quad a \rightarrow \varepsilon a,$$  

(5.19)

where a prime denotes the derivative with respect to $K$ at $K_0$. Then eqns.(5.9) and (5.17) first yield the following two equations:

$$\frac{\partial}{\partial T} \left( (\omega_0 - \varepsilon \phi_t) a^2 \right) + \frac{\partial}{\partial x} \left( (K_0 + \varepsilon \phi_t) a^2 \right) = 0,$$  

(5.20)

and

$$a^{-1} \left( \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial x^2} \right) a - \left( \phi_t^2 - \phi_x^2 + \beta a^2 + \delta \phi_t^4 \right) + 2 \left( \omega_0 \phi_t + K_0 \phi_t \right) = 0.$$  

(5.21)
Then using (5.19) these two equations provide two first-order time differential equations:

\[ 2\omega_0 a a \tau + \left( 1 - \omega_0^2 \right) a\left( 2a_\tau \phi_\tau + a \phi_{\tau \tau} \right) + h.o.t = 0, \]  

(5.22) 

and

\[ \left( 1 - \omega_0^2 \right) a^{-1} \left( a_{s\tau \tau} \phi_\tau - \phi_{\tau \tau} \right) - \left( \beta a^2 + \beta \phi_{\tau \tau} \right) + 2\omega_0 \phi_\tau + h.o.t = 0, \]  

(5.23) 

where \( h.o.t \) stands for higher-order terms. These two equations therefore yield \( \left( \omega_0 \neq 0 \right) \)

\[ \alpha_\tau = \frac{\omega_0 ^2 - 1}{2\omega_0} \left( a_{\tau} \phi_\tau + a \phi_{\tau \tau} \right), \]  

(5.24) 

and

\[ \phi_\tau = \frac{1 - \omega_0 ^2}{2\omega_0} \left( a^{-1} a_{s\tau \tau} \phi_\tau - \phi_{\tau \tau} \right) + \frac{\beta a^2}{2\omega_0}. \]  

(5.25) 

Finally, introducing a complex amplitude \( A = a \exp(i\phi) \), eqns.(5.24) and (5.25) combine to give a single **nonlinear Schrödinger equation** (NLS equation; the asterisk denotes complex conjugacy):

\[ iA_\tau + p A_{s\tau \tau} + q|A|^2 A = 0, \]  

(5.26) 

with the definitions

\[ p = \frac{1}{2} \omega_0 ^{''}, \quad q = \frac{\beta}{2\omega_0}. \]  

(5.27) 

Here \( \omega_0 ^{''} \) is the curvature of the linear dispersion relation at the working point \( (\omega_0, K_0) \).

Equation (5.26) is a canonical equation of the theory of envelope waves, i.e., it always obtains but with different values of \( p \) and \( q \). According to a celebrated result of Zakharov and Shabat [34], eqn.(5.26) admits stable « bright » envelope solitons as solutions whenever the product \( pq \) is positive, that is, when the product of the curvature of the linear dispersion relation and of the quartic nonlinearity parameter is positive. If the curvature is negative then the material must be selected such as to have a negative parameter \( \beta \) to allow for the propagation of these « bright » solitons, the qualification « bright » originating from the « optical » heritage of the naming. Thus the choice of the working point and that of the specific material are instrumental for the existence of this wave phenomenon. Now solutions of this type should not come here as a surprise - except for the exploited methodology which is seldom used in solid mechanics - because the original modified Boussinesq equation has associated with it an evolution equation (one-directional equation deduced by means of the so-called reductive perturbation method), which is a modification of the traditional Korteweg-deVries equation, and the latter itself will provide modulated envelope waves in the appropriate conditions of slow variation of the modulation of a small amplitude almost monochromatic signal (Benney and Newell, [18]; Ostrovsky and Potapov [33]).

**6. Concluding remarks**

By way of closure, we briefly return to the notion of Eshelby stress. Model (5.1) fits into the second-gradient scheme of eqns.(3.11), and (2.12) through (2.14). According to (2.14) and after a short computation, the material Eshelby stress (here reduced to a scalar) is
and the balance of material (or field) momentum (here a strict conservation law) reads

\[
\frac{\partial (-u_t u_x)}{\partial t} + \frac{\partial}{\partial X} \left[ \frac{u_t^2 + u_x^2}{2} + \frac{3}{2} \delta^2 u_{xx}^2 - \frac{\beta}{4} u_x^4 - \delta^2 u_x u_{xxx} \right] = 0 . \tag{6.2}
\]

Here some of the groupings of terms are misleading and due to the specific one-dimensional nature of the problem (see remark in p. 211, Ref. [1]). But now if we try to think in terms of the averaged Lagrangian, in the linear (but dispersive) case the latter will be provided by (4.13) in which \(D_L\) is given by the left-hand side of (5.6). On using the definitions in (4.18)-(4.20) we shall find that the general balance of wave momentum is given by

\[
\frac{\partial (\omega Ka^2)}{\partial t} + \frac{\partial}{\partial X} \left[ \frac{\omega^2 + K^2}{2} - \frac{3}{2} \delta^2 K^4 \right] a^2 + \ldots = 0 , \tag{6.3}
\]

where the ellipsis in the left-hand side stands for unknown terms due to the nonlinear nature of the full problem. Noting then that \(\tilde{P} = -u_t u_x = \omega Ka^2\) in the zeroth approximation (i.e., up to a factor of modulus one), we obviously identify the analogies between eqns. (6.2) and (6.3). To go further in the exploitation of (6.3) we would have to find the missing nonlinear terms and enforce the already indicated perturbation of the dispersion relation, introduce an averaged Lagrangian that depends also on the time derivative and space derivative of the amplitude, and apply the almost monochromatic approximation (5.18).

We should also note that eqn. (5.26) itself partakes of the wave-particle dualism in the sense that, via a new application of the canonical formalism [37], a quasi-particle having Newtonian mechanics as its « point particle » mechanics (cf. [28],[32]) can be associated with eqn. (5.26). The « point mass » involved in this mechanics is none other than the « wave action » or « number of phonons » defined by

\[
M = \int_R A^2 dX . \tag{6.4}
\]

on the real line \(R\). Thus, successive application of the wave-mechanics dualism, averaged Lagrangian method, and perturbation techniques led us again to some kind of particle-wave dualism. Although quantization in the linear wave case with the concept of phonons applies directly, here the path has been long and indirect.

\textbf{Appendix A : Noether’s identity for wave mechanics}

Multiplying both sides of eqn. (4.16) by co-vector \(K\), we have
\[ 0 = \left\{ \frac{\partial \tilde{S}}{\partial t} - (\nabla_r \cdot W) \right\} \hat{K} = \frac{\partial (\tilde{S} K)}{\partial t} - \tilde{S} \frac{\partial K}{\partial t} - \text{div}_r \left( W \otimes K \right) + (W \cdot \nabla_r) K. \] (A.1)

But
\[ \tilde{S} \frac{\partial K}{\partial t} = -\tilde{S} \nabla_r \omega, \] (A.2)

and
\[ (W \cdot \nabla_r) K = \frac{\partial \tilde{L}}{\partial K} \nabla_r K = \frac{\partial \tilde{L}}{\partial K} (\nabla_r K)^T. \] (A.3)

where we used the definitions of \( \omega \) and \( K \) in terms of the phase. Furthermore, from the total material gradient of \( \tilde{L} \), we obtain that
\[
\frac{\partial \tilde{L}}{\partial K} (\nabla_r K)^T = \nabla_r \tilde{L} - \frac{\partial \tilde{L}}{\partial X} \bigg|_{\text{exp}} \nabla_r \omega - \frac{\partial \tilde{L}}{\partial a} (\nabla_r a)^T
\]
\[
= \text{div}_r \left( \tilde{L} 1_K \right) - \frac{\partial \tilde{L}}{\partial X} \bigg|_{\text{exp}} \tilde{S} \nabla_r \omega.
\] (A.4)

Accordingly, combining (A.2) through (A.4) in (A.1) yields
\[ 0 = \left\{ \frac{\partial \tilde{S}}{\partial t} - (\nabla_r \cdot W) \right\} \hat{K} = \frac{\partial \tilde{P}}{\partial t} - \text{div}_r \tilde{b} - \frac{\partial \tilde{L}}{\partial X} \bigg|_{\text{exp}} \] (A.5)

with the definitions (4.20). As a matter of fact, here one of Noether’s identities is none other than
\[ \left\{ \frac{\partial \tilde{S}}{\partial t} - (\nabla_r \cdot W) \right\} \hat{K} - \left\{ \frac{\partial \tilde{P}}{\partial t} - \text{div}_r \tilde{b} - \frac{\partial \tilde{L}}{\partial X} \bigg|_{\text{exp}} \right\} = 0. \] (A.6)

References
[1] Maugin G A 1993 Material Inhomogeneities in Elasticity (London: Chapman and Hall)
[2] Kienzler K and Herrmann G 1999 Mechanics in Material Space (Berlin: Springer)
[3] Gurtin M E 1999 Configurational forces as basic concepts of continuum physics (New York: Springer)
[4] Maugin G A 1997 ARI (Springer-Verlag) 50 41
[5] Maugin G A 1998 ARI (Springer-Verlag) 50 141
[6] Maugin G A 2006 Mechanics Research Communications 33 705
[7] Noether E 1918 Klg.-Ges.Wiss.Nach.Göttingen, Math-Physik K1.2 235.
