D-metric Spaces and Composition Operators Between Hyperbolic Weighted Family of Function Spaces

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ABSTRACT

The aim of this paper is to introduce new hyperbolic classes of functions, which will be called $B_{\alpha, \log}^*$ and $F_{\log}^*(p, q, s)$ classes. Furthermore, we introduce D-metrics space in the hyperbolic type classes $B_{\alpha, \log}^*$ and $F_{\log}^*(p, q, s)$. These classes are shown to be complete metric spaces with respect to the corresponding metrics. Moreover, necessary and sufficient conditions are given for the composition operator $C_\phi$ to be bounded and compact from $B_{\alpha, \log}^*$ to $F_{\log}^*(p, q, s)$ spaces.

RESUMEN

El objetivo de este artículo es introducir nuevas clases hiperbólicas de funciones, que serán llamadas clases $B_{\alpha, \log}^*$ y $F_{\log}^*(p, q, s)$. A continuación, introducimos D-espacios métricos en las clases de tipo hiperbólicas $B_{\alpha, \log}^*$ y $F_{\log}^*(p, q, s)$. Mostramos que estas clases son espacios métricos completos con respecto a las métricas correspondientes. Más aún, damos condiciones necesarias y suficientes para que el operador composición $C_\phi$ sea acotado y compacto desde el espacio $B_{\alpha, \log}^*$ a $F_{\log}^*(p, q, s)$.

Keywords and Phrases: D-metric spaces, Logarithmic hyperbolic classes, Composition operators.

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1 Introduction

Let $\phi$ be an analytic self-map of the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $\partial D$ be its boundary. Let $H(D)$ denote the space of all analytic functions in $D$ and let $B(D)$ be the subset of $H(D)$ consisting of those $f \in H(D)$ for which $|f(z)| < 1$ for all $z \in \mathbb{D}$.

Let the Green’s function of $D$ be defined as $g(z, a) = \log \left| \frac{1}{\varphi_a(z)} \right|$, where $\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}$ is the Möbius transformation related to the point $a \in D$.

A linear composition operator $C_\phi$ is defined by $C_\phi(f) = (f \circ \phi)$ for $f$ in the set $H(D)$ of analytic functions on $D$ (see [9]). A function $f \in B(D)$ belongs to $\alpha$-Bloch space $B_\alpha$, $0 < \alpha < \infty$, if

$$\|f\|_{B_\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f'(z)| < \infty.$$ 

The little $\alpha$-Bloch space $B_{\alpha,0}$ consisting of all $f \in B_\alpha$ so that

$$\lim_{|z| \to 1^-} (1 - |z|^2)|f'(z)| = 0.$$

**Definition 1.** [15] For an analytic function $f$ on $D$ and $0 < \alpha < \infty$, if

$$\|f\|_{B_{\alpha, \log}} = \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f'(z)| \left( \log \frac{2}{1 - |z|^2} \right) < \infty,$$

then, $f$ belongs to the weighted $\alpha$-Bloch spaces $B_{\alpha, \log}$.

If $\alpha = 1$, the weighted Bloch space $B_{\log}$ is the set for all analytic functions $f$ in $D$ for which $\|f\|_{B_{\log}} < \infty$.

The expression $\|f\|_{B_{\log}}$ defines a seminorm while the norm is defined by

$$\|f\|_{B_{\log}} = |f(0)| + \|f\|_{B_{\log}}.$$

**Definition 2.** [14] For $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$, a function $f \in H(D)$ is in $F(p, q, s)$, if

$$\sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \infty.$$

Moreover, if

$$\lim_{|a| \to 1^-} \int_D |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) = 0,$$

then $f \in F_0(p, q, s)$.

El-Sayed and Bakhit [5] gave the following definition:
Definition 3. For $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$, a function $f \in H(D)$ is said to belong to $F_{\log}(p, q, s)$, if

$$
\sup_{I \subset \partial D} \left( \frac{\log \frac{2}{|I|}}{|I|^s} \right)^p \int_{S(I)} |f'(z)|^p (1 - |z|^2)^q \left( \log \frac{1}{|z|} \right)^s dA(z) < \infty.
$$

Where $|I|$ denotes the arc length of $I \subset \partial D$ and $S(I)$ is the Carleson box defined by (see [8, 6])

$$
S(I) = \{ z \in D : 1 - |I| < |z| < 1, \frac{z}{|z|} \in |I| \}.
$$

The interest in the $F_{\log}(p, q, s)$-spaces rises from the fact that they cover some well known function spaces. It is immediate that $F_{\log}(2, 0, 1) = BMOA_{\log}$ and $F_{\log}(2, 0, p) = Q_p^{\log}$, where $0 < p < \infty$.

2 Preliminaries

Definition 4. [11] The hyperbolic Bloch space $B^*_\alpha$ is defined as

$$
B^*_\alpha = \{ f : f \in B(D) \text{ and } \sup_{z \in D} (1 - |z|^2)^\alpha f^*(z) < \infty \}.
$$

Denoting $f^*(z) = \frac{|f(z)|}{1 - |f(z)|}$, the hyperbolic derivative of $f \in B(D)$. [7]

The little hyperbolic Bloch space $B^*_{\alpha, 0}$ is a subspace of $B^*_\alpha$ consisting of all $f \in B^*_\alpha$ so that

$$
\lim_{|z| \to 1^-} (1 - |z|^2)^\alpha f^*(z) = 0.
$$

The space $B^*_\alpha$ is Banach space with the norm defined as

$$
||f||_{B^*_\alpha} = |f(0)| + \sup_{z \in D} (1 - |z|^2)^\alpha |f^*(z)|.
$$

Definition 5. For $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha = \frac{q + 2}{p}$ and $q + s > -1$, a function $f \in H(D)$ is said to belong to $F^*(p, q, s)$, if

$$
\sup_{a \in D} \int_D (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} g^*(z, a) dA(z) < \infty.
$$

Definition 6. For $f \in B(D)$ and $0 < \alpha < \infty$, if

$$
||f||_{B^*_\alpha, \log} = \sup_{z \in D} (1 - |z|^2)^\alpha (f^*(z)) \left( \log \frac{2}{1 - |z|^2} \right) < \infty,
$$

then $f$ belongs to the $B^*_\alpha, \log$. 

We must consider the following lemmas in our study:

Lemma 2.1. Let $0 < r \leq t \leq 1$, then
\[ \log \frac{1}{t} \leq \frac{1}{r}(1 - t^2) \]

Lemma 2.2. Let $0 \leq k_1 < \infty$, $0 \leq k_2 < \infty$, and $k_1 - k_2 > -1$, then
\[ C(k_1, k_2) = \int_{\mathbb{D}} \left( \log \frac{1}{|z|} \right)^{k_1} (1 - |z|^2)^{-k_2} dA(z) < \infty. \]

To study composition operators on $\mathcal{B}^*_\alpha, \log$ and $F^*_{\log}(p, q, s)$ spaces, we need to prove the following result:

Theorem 1. If $0 < p < \infty$, $1 < s < \infty$ and $\alpha = \frac{q + 2}{p}$ with $q + s > -1$. Then the following are equivalent:

(A) $f \in \mathcal{B}^*_\alpha, \log$.

(B) $f \in F^*_{\log}(p, q, s)$.

(C) $\sup_{a \in \mathbb{D}} \left( \frac{\log \frac{2}{|a|^2}}{1 - |a|^2} \right)^p \int_{\mathbb{D}} (|f^*(z)|^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi(z)|^2)^s) dA(z) < \infty$,\n
(D) $\sup_{a \in \mathbb{D}} \left( \frac{\log \frac{2}{|a|^2}}{1 - |a|^2} \right)^p \int_{\mathbb{D}} (|f^*(z)|^p (1 - |z|^2)^{\alpha p - 2} g^*(z, a)) dA(z) < \infty$.

Proof. Let $0 < p < \infty$, $-2 < q < \infty$, $1 < s < \infty$ and $0 < r < 1$. By subharmonicity we have for an analytic function $g \in \mathbb{D}$ that
\[ |g(0)|^p \leq \frac{1}{\pi r^2} \int_{\mathbb{D}(0, r)} |g(w)|^p dA(w). \]

For $a \in \mathbb{D}$, the substitution $z = \varphi_a(z)$ results in Jacobian change in measure given by
\[ dA(w) = |\varphi'_a(z)|^2 dA(z). \]

For a Lebesgue integrable or a non-negative Lebesgue measurable function $f$ on $\mathbb{D}$, we thus have the following change of variable formula:
\[ \int_{\mathbb{D}(a, r)} f(\varphi_a(w)) dA(w) = \int_{\mathbb{D}(0, r)} f(z) |\varphi'_a(z)|^2 dA(z). \]

Let $g = \frac{f' \circ \varphi_a}{1 - |f \circ \varphi_a|^2}$ then we have
\[ \left( \frac{|f(a)|}{1 - |f(a)|^2} \right)^p = (f^*(a))^p \leq \frac{1}{\pi r^2} \int_{\mathbb{D}(0, r)} \left( \frac{|f' \circ \varphi_a(w)|}{1 - |f \circ \varphi_a(w)|^2} \right)^p dA(w) \]
\[ = \frac{1}{\pi r^2} \int_{\mathbb{D}(a, r)} (f^*(z))^p |\varphi'_a(z)|^2 dA(z). \]
Since
\[ |\varphi'_a(z)| = \frac{1 - |\varphi_a(z)|^2}{1 - |a|^2}, \]
and
\[ 1 - |\varphi_a(z)|^2 \leq \frac{4}{1 - |a|^2} \quad a, z \in \mathbb{D}. \]
So we obtain that
\[ (f^*(a))^p \leq \frac{16}{\pi r^2(1 - |a|^2)^2} \int_{D(a,r)} (f^*(z))^p dA(z). \]
Again \( f \in \mathcal{B}^*_\alpha, \log \), and \((1 - |z|^2)^2 \approx (1 - |a|^2)^2 \approx D(a, r)\), for \( z \in D(a, r) \). Thus, we have
\[
\left( \log \frac{2}{1 - |a|^2} \right)^p (f^*(a))^p(1 - |a|^2)^{\alpha p} \leq \frac{16}{\pi r^2(1 - |a|^2)^2} \int_{D(a,r)} (f^*(z))^p dA(z)
\leq \frac{16}{\pi r^2} \times \left( \log \frac{2}{1 - |a|^2} \right)^p \int_{D(a,r)} (f^*(z))^p(1 - |z|^2)^{\alpha p - 2} dA(z)
\leq \frac{16}{\pi r^2} \times \left( \log \frac{2}{1 - |a|^2} \right)^p \int_{D(a,r)} (f^*(z))^p(1 - |z|^2)^{\alpha p - 2} \times \left( 1 - |\varphi_a(z)|^2 \right)^s dA(z)
\leq \frac{16}{\pi r^2(1 - r^2)^s} \times \left( \log \frac{2}{1 - |a|^2} \right)^p \int_{D(a,r)} (f^*(z))^p(1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z).
\]
Where \( M(r) \) is a constant depending on \( r \). Thus, the quantity \((A)\) is less than or equal to constant times the quantity \((C)\).

From the fact
\[ (1 - |\varphi_a(z)|^2) \leq 2 \log \frac{1}{|\varphi_a(z)|} = 2g(z, a) \quad \text{for} \ a, z \in \mathbb{D}, \]
we have
\[
\left( \log \frac{2}{1 - |a|^2} \right)^p \int_{D(a,r)} (f^*(z))^p(1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z)
\leq \left( \log \frac{2}{1 - |a|^2} \right)^p \int_{D(a,r)} (f^*(z))^p(1 - |z|^2)^{\alpha p - 2} g^s(z, a) dA(z).
\]
Hence, the quantity \((C)\) is less than or equal to a constant times \((D)\). By taking \( \alpha = \frac{4 + 2}{p} \), it follows \( f \in F^*_{\log}(p, q, s) \). Thus, the quantity \((C)\) is less than or equal to a constant times the quantity \((B)\).
Finally, from the following inequality, let \( z = \varphi_a(w) \) then \( w = \varphi_a(z) \). Hence,
\[
\left( \log \frac{2}{1 - |a|^2} \right)^P \int_\mathbb{D} (f^*(\varphi_a(w)))^P(1 - |\varphi_a(w)|^2)^{\alpha p - 2} \left( \log \frac{1}{|w|} \right)^s |\varphi_a'(w)|^2 dA(w)
= \left( \log \frac{2}{1 - |a|^2} \right)^P \int_\mathbb{D} (f^*(\varphi_a(w)))^P(1 - |\varphi_a(w)|^2)^{\alpha p} \left( \log \frac{1}{|w|} \right)^s |\varphi_a'(w)|^2 dA(w)
= \left( \log \frac{2}{1 - |a|^2} \right)^P \int_\mathbb{D} (f^*(\varphi_a(w)))^P(1 - |\varphi_a(w)|^2)^{\alpha p} \left( \log \frac{1}{|w|} \right)^s \frac{1}{(1 - |\varphi_a(w)|^2)^2} dA(w)
\leq ||f||_{B^*_\alpha, \log}^p \left( \log \frac{2}{1 - |a|^2} \right)^P \int_\mathbb{D} \left( \log \frac{1}{|w|} \right)^s (1 - |w|^2)^2 dA(w)
= C(s, 2)||f||_{B^*_\alpha, \log}^p.
\]

By lemma 2.2 \( C(s, 2) = \int_\mathbb{D} \left( \log \frac{1}{|w|} \right)^s (1 - |w|^2)^{-2} dA(w) < \infty, \quad \text{for } 1 < s < \infty. \)

Thus, the quantity (D) is less than or equal to a constant times the quantity (A). Hence, it is proved.

Let us give the following equivalent definition for \( F^*_p(p, q, s) \).

**Definition 7.** For \( 0 < p, s < \infty, -2 < q < \infty, \alpha = \frac{q+2}{p} \) and \( q + s > -1 \), a function \( f \in H(\mathbb{D}) \) is said to belong to \( F^*_p(p, q, s) \), if
\[
\sup_{a \in \mathbb{D}} \left( \log \frac{2}{1 - |a|^2} \right)^P \int_\mathbb{D} (f^*(z))^P(1 - |z|^2)^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s dA(z) < \infty.
\]

**Definition 8.** A composition operator \( C_\phi : B^*_\alpha, \log \to F^*_p(p, q, s) \) is said to be bounded if there is a positive constant \( C \) so that \( ||C_\phi f||_{F^*_p(p, q, s)} \leq C||f||_{B^*_\alpha, \log} \) for all \( f \in B^*_\alpha, \alpha \).

**Definition 9.** A composition operator \( C_\phi : B^*_\alpha, \log \to F^*_p(p, q, s) \) is said to be compact if it maps any ball in \( B^*_\alpha, \alpha \) onto a precompact set in \( F^*(p, q, s) \).

The following lemma follows by standard arguments similar to those outline in [13]. Hence, we omit the proof.

**Lemma 2.3.** Assume \( \phi \) is a holomorphic mapping from \( \mathbb{D} \) into itself. Let \( 0 < p, s, \alpha < \infty, -2 < q < \infty \), then \( C_\phi : B^*_\alpha, \log \to F^*_p(p, q, s) \) is compact if and only if for any bounded sequence \( \{f_n\}_{n \in \mathbb{N}} \in B^*_\alpha, \log \) which converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \) we have \( \lim_{n \to \infty} ||C_\phi f_n||_{F^*_p(p, q, s)} = 0. \)

### 3 \( D \)-metric space

Topological properties of generalized metric space called \( D \)-metric space was introduced in [1], see for example, ([2] and [3]). This structure of \( D \)-metric space is quite different from a 2-metric space and natural generalization of an ordinary metric space in some sense.
**Definition 10.** [4] Let $X$ denote a nonempty set and $\mathbb{R}$ the set of real numbers. A function $D : X \times X \times X \to \mathbb{R}$ is said to be a $D$-metric on $X$ if it satisfies the following properties:

(i) $D(x, y, z) \geq 0$ for all $x, y, z \in X$ and equality holds if and only if $x = y = z$ (nonnegativity),

(ii) $D(x, y, z) = D(x, z, y) = \cdots$ (symmetry),

(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$ (tetrahedral inequality).

A nonempty set $X$ together with a $D$-metric $D$ is called a $D$-metric space and is represented by $(X, D)$. The generalization of a $D$-metric space with $D$-metric as a function of $n$ variables is provided in Dhage [2].

**Example 1.1:** [4] Let $(X, d)$ be an ordinary metric space and define a function $D_1$ on $X^3$ by

$$D_1(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

for all $x, y, z \in X$. Then, the function $D_1$ is a $D$-metric on $X$ and $(X, D_1)$ is a $D$-metric space.

**Example 1.2:** [4] Let $(X, d)$ be an ordinary metric space and define a function $D_2$ on $X^3$ by

$$D_2(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for $x, y, z \in X$. Then, $D_2$ is a metric on $X$ and $(X, D_2)$ is a $D$-metric space.

**Remark 1.** Geometrically, the $D$-metric $D_1$ represents the diameter of a set consisting of three points $x, y$ and $z$ in $X$ and the $D$-metric $D_2(x, y, z)$ represents the perimeter of a triangle formed by three points $x, y, z$ in $X$ as its vertices.

**Definition 11.** (Cauchy sequence, completeness) [10] For every $m, n > N$. A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be Cauchy if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that

$$d(x_m, x_n) < \varepsilon.$$

The space $X$ is said to be complete if every Cauchy sequence in $X$ converges (that is, has a limit which is an element of $X$).

The following theorem can be found in [4]:

**Theorem 2.** [4] Let $d$ be an ordinary metric on $X$ and let $D_1$ and $D_2$ be corresponding associated $D$-metrics on $X$. Then, $(X, D_1)$ and $(X, D_2)$ are complete if and only if $(X, d)$ is complete.
4  \( D \)-metrics in \( B^{*}_{\alpha, \log} \) and \( F^{*}_{\log}(p, q, s) \)

In this section, we introduce a \( D \)-metric on \( B^{*}_{\alpha, \log} \) and \( F^{*}_{\log}(p, q, s) \).

Let \( 0 < p, s < \infty, -2 < q < \infty \), and \( 0 < \alpha < 1 \). First, we can find a \( D \)-metric in \( B^{*}_{\alpha, \log} \), for \( f, g, h \in B^{*}_{\alpha, \log} \) by defining

\[
D(f, g, h; B^{*}_{\alpha, \log}) := D_{B^{*}_{\alpha, \log}}(f, g, h) + \|f - g\|_{B^{*}_{\alpha, \log}} + \|g - h\|_{B^{*}_{\alpha, \log}} + \|h - f\|_{B^{*}_{\alpha, \log}} + |f(0) - g(0)| + |g(0) - h(0)| + |h(0) - f(0)|,
\]

where

\[
D_{B^{*}_{\alpha, \log}}(f, g, h) := d_{B^{*}_{\alpha, \log}}(f, g) + d_{B^{*}_{\alpha, \log}}(g, h) + d_{B^{*}_{\alpha, \log}}(h, f)
\]

and

\[
D_{B^{*}_{\alpha, \log}}(f, g, h) := \left( \sup_{z \in D} |f^{*}(z) - g^{*}(z)| + \sup_{z \in D} |g^{*}(z) - h^{*}(z)| + \sup_{z \in D} |h^{*}(z) - f^{*}(z)| \right)
\times (1 - |z|^2)^\alpha \left( \log \frac{2}{1 - |z|^2} \right).
\]

Also, for \( f, g, h \in F^{*}_{\log}(p, q, s) \) we introduce a \( D \)-metric on \( F^{*}_{\log}(p, q, s) \) by defining

\[
D(f, g, h; F^{*}_{\log}(p, q, s)) := D_{F^{*}_{\log}(p, q, s)}(f, g, h) + \|f - g\|_{F^{*}_{\log}(p, q, s)} + \|g - h\|_{F^{*}_{\log}(p, q, s)} + \|h - f\|_{F^{*}_{\log}(p, q, s)} + |f(0) - g(0)| + |g(0) - h(0)| + |h(0) - f(0)|,
\]

where

\[
D_{F^{*}_{\log}(p, q, s)}(f, g, h) := d_{F^{*}_{\log}(p, q, s)}(f, g) + d_{F^{*}_{\log}(p, q, s)}(g, h) + d_{F^{*}_{\log}(p, q, s)}(h, f)
\]

and

\[
d_{F^{*}_{\log}(p, q, s)}(f, g) := \left( \sup_{z \in D} \ell^p(a) \int_{|A|^2} |f^{*}(z) - g^{*}(z)|^p(1 - |z|^2)^q(1 - |\varphi(z)|^2)^s dA(z) \right)^{\frac{1}{p}}.
\]

**Proposition 1.** The class \( B^{*}_{\alpha, \log} \) equipped with the \( D \)-metric \( D(\cdot, \cdot; B^{*}_{\alpha, \log}) \) is a complete metric space. Moreover, \( B^{*}_{\alpha, \log, 0} \) is a closed (and therefore complete) subspace of \( B^{*}_{\alpha, \log} \).

**Proof.** Let \( f, g, h, a \in B^{*}_{\alpha, \log} \). Then, clearly

(i) \( D(f, g, h; B^{*}_{\alpha, \log}) \geq 0 \), for all \( f, g, h \in B^{*}_{\alpha, \log} \).
(ii) \(D(f, g, h; B^*_{\alpha, \log}) = D(f, h, g; B^*_{\alpha, \log}) = D(g, h, f; B^*_{\alpha, \log})\).

(iii) \(D(f, g, h; B^*_{\alpha, \log}) \leq D(f, g, a; B^*_{\alpha, \log}) + D(f, a, h; B^*_{\alpha, \log}) + D(a, g, h; B^*_{\alpha, \log})\)

for all \(f, g, h, a \in B^*_{\alpha, \log}\).

(iv) \(D(f, g, h; B^*_{\alpha, \log}) = 0\) implies \(f = g = h\).

Hence, \(D\) is a \(D\)-metric on \(B^*_{\alpha, \log}\), and \((B^*_{\alpha, \log}, D)\) is \(D\)-metric space.

To prove the completeness, we use Theorem 2 let \((f_n)_{n=1}^\infty\) be a Cauchy sequence in the metric space \((B^*_{\alpha, \log}, d), \) that is, for any \(\varepsilon > 0\) there is an \(N = N(\varepsilon) \in \mathbb{N}\) such that \(d(f_n, f_m; B^*_{\alpha, \log}) < \varepsilon\), for all \(n, m > N\). Since \((f_n) \subset B(\mathbb{D})\), the family \((f_n)\) is uniformly bounded and hence normal in \(\mathbb{D}\). Therefore, there exists \(f \in B(\mathbb{D})\) and a subsequence \((f_{n_j})_{j=1}^\infty\) such that \(f_{n_j}\) converges to \(f\) uniformly on compact subsets of \(\mathbb{D}\). It follows that \(f_n\) also converges to \(f\) uniformly on compact subsets, and by the Cauchy formula, the same also holds for the derivatives. Now let \(m > N\). Then, the uniform convergence yields

\[
\left| f^*(z) - f^*_m(z) \right| (1 - |z|^2)^{\alpha} \left( \log \frac{2}{1 - |z|^2} \right) \\
= \lim_{n \to \infty} \left| f^*_n(z) - f^*_m(z) \right| (1 - |z|^2)^{\alpha} \left( \log \frac{2}{1 - |z|^2} \right) \\
\leq \lim_{n \to \infty} d(f_n, f_m; B^*_{\alpha, \log}) \leq \varepsilon
\]

for all \(z \in \mathbb{D}\), and it follows that \(||f||_{B^*_{\alpha, \log}} \leq ||f_m||_{B^*_{\alpha, \log}} + \varepsilon\). Thus \(f \in B^*_{\alpha, \log}\) as desired. Moreover, the above inequality and the compactness of the usual \(B^*_{\alpha, \log}\) space imply that \((f_n)_{n=1}^\infty\) converges to \(f\) with respect to the metric \(d\), and \((B^*_{\alpha, \log}, D)\) is complete \(D\)-metric space.

Since \(\lim_{n \to \infty} d(f_n, f_m; B^*_{\alpha, \log}) \leq \varepsilon\), the second part of the assertion follows.

Next we give characterization of the complete \(D\)-metric space \(D(\cdot, \cdot; F^*_{\log}(p, q, s))\).

**Proposition 2.** The class \(F^*_{\log}(p, q, s)\) equipped with the \(D\)-metric \(D(\cdot, \cdot; F^*_{\log}(p, q, s))\) is a complete metric space. Moreover, \(F^*_{\log, 0}(p, q, s)\) is a closed (and therefore complete) subspace of \(F^*_{\log}(p, q, s)\).

**Proof.** Let \(f, g, h, a \in F^*_{\log}(p, q, s)\). Then clearly

(i) \(D(f, g, h; F^*_{\log}(p, q, s)) \geq 0\), for all \(f, g, h \in F^*_{\log}(p, q, s)\).

(ii) \(D(f, g, h; F^*_{\log}(p, q, s)) = D(f, h, g; F^*_{\log}(p, q, s)) = D(g, h, f; F^*_{\log}(p, q, s))\).
(iii) $D(f, g, h; F^*_\log (p, q, s)) \leq D(f, g, a; F^*_\log (p, q, s)) + D(f, a, h; F^*_\log (p, q, s))$

$$+ D(a, g, h; F^*_\log (p, q, s))$$

for all $f, g, h, a \in F^*_\log (p, q, s)$.

(iv) $D(f, g, h; F^*_\log (p, q, s)) = 0$ implies $f = g = h$.

Hence, $D$ is a $D$-metric on $F^*_\log (p, q, s)$, and $(F^*_\log (p, q, s), D)$ is $D$-metric space.

For the complete proof, by using Theorem 2 let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $(F^*_\log (p, q, s), d)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ so that $d(f_n, f_m; F^*_\log (p, q, s)) < \varepsilon$, for all $n, m > N$. Since $(f_n) \subset B(\mathbb{D})$, such that $f_{n_j}$ converges to $f$ uniformly on compact subsets of $\mathbb{D}$. It follows that $f_n$ also converges to $f$ uniformly on compact subsets, now let $m > N$, and $0 < r < 1$. Then, the Fatou’s yields

$$\int_{\mathbb{D}(0, r)} \left| f^*_n(z) - f^*_m(z) \right|^p (1 - |z|^2)^q (1 - |\varphi_n(z)|^2)^s dA(z)$$

$$= \int_{\mathbb{D}(0, r)} \lim_{n \to \infty} \left| f^*_n(z) - f^*_m(z) \right|^p (1 - |z|^2)^q (1 - |\varphi_n(z)|^2)^s dA(z)$$

$$\leq \lim_{n \to \infty} \int_{\mathbb{D}(0, r)} \left| f^*_n(z) - f^*_m(z) \right|^p (1 - |z|^2)^q (1 - |\varphi_n(z)|^2)^s dA(z) \leq \varepsilon^p,$$

and by taking $r \to 1^-$, it follows that,

$$\int_{\mathbb{D}} (f^*_n(z))^p (1 - |z|^2)^q (1 - |\varphi_n(z)|^2)^s dA(z)$$

$$\leq 2^p \varepsilon^p + 2^p \int_{\mathbb{D}} (f^*_m(z))^p (1 - |z|^2)^q (1 - |\varphi_m(z)|^2)^s dA(z).$$

This yields

$$\|f\|_{F^*_\log (p, q, s)}^p \leq 2^p \|f_m\|_{F^*_\log (p, q, s)}^p + 2^p \varepsilon^p.$$ 

And thus $f \in F^*_\log (p, q, s).$ We also find that $f_n \to f$ with respect to the metric of $(F^*_\log (p, q, s), D)$ and $(F^*_\log (p, q, s), D)$ is complete $D$-metric space. The second part of the assertion follows.

5 Composition operators of $C_\phi : B^*_\alpha, \log \to F^*_\log (p, q, s)$

In this section, we study boundedness and compactness of composition operators on $B^*_\alpha, \log$ and $F^*_\log (p, q, s)$ spaces. We need the following notation:

$$\Phi_\phi(\alpha, p, s; a) = \ell^p(a) \int_{\mathbb{D}} |\phi'(z)|^p \frac{(1 - |z|^2)^{\alpha p - 2}(1 - |\varphi_n(z)|^2)^s}{(1 - |\phi(z)|^2)^{\alpha p \left( \log \frac{2}{1 - |\phi(z)|^2} \right)}} dA(z),$$
where \( \ell^p(a) = \left( \log \frac{2}{1-|a|^p} \right)^p \).

For \( 0 < \alpha < 1 \), we suppose there exist two functions \( f, g \in \mathcal{B}^*_\alpha, \log \) such that for some constant \( C \),

\[
(|f^*(z)| + |g^*(z)|) \geq \frac{C}{(1 - |z|^2)^\alpha \left( \log \frac{2}{1-|a|^p} \right)^p} > 0, \quad \text{for each } z \in \mathbb{D}.
\]

Now, we provide the following theorem:

**Theorem 3.** Assume \( \phi \) is a holomorphic mapping from \( \mathbb{D} \) into itself and let \( 0 < p, 1 < s < \infty, 0 < \alpha \leq 1 \). Then the induced composition operator \( C_\phi \) maps \( \mathcal{B}^*_\alpha, \log \) into \( F^*_\log (p, \alpha p - 2, s) \) is bounded if and only if,

\[
\sup_{z \in \mathbb{D}} \Phi_\phi (\alpha, p, s; a) < \infty. \tag{5.1}
\]

**Proof.** First assume that \( \sup_{z \in \mathbb{D}} \Phi_\phi (\alpha, p, s; a) < \infty \) is held, and \( f \in \mathcal{B}^*_\alpha, \log \) with \( ||f||_{\mathcal{B}^*_\alpha, \log} \leq 1 \), we can see that

\[
||C_\phi f||^p_{\mathcal{F}^*_\log (p, \alpha p - 2, s)} = \sup_{a \in \mathbb{D}} \ell^p(a) \int_{\mathbb{D}} ((f \circ \phi)^*(z))^{p(1 - |z|^2)^{\alpha p - 2}} \left( 1 - |\varphi_a(z)|^2 \right)^s dA(z)
\]

\[
= \sup_{a \in \mathbb{D}} \ell^p(a) \int_{\mathbb{D}} (f^*(\phi(z)))^{p|\phi'(z)|^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s} dA(z)
\]

\[
\leq ||f||^p_{\mathcal{B}^*_\alpha, \log} \sup_{a \in \mathbb{D}} \ell^p(a) \int_{\mathbb{D}} \left| \frac{\phi'(z)}{|\phi(z)|^{\alpha p - 2}(1 - |\varphi_a(z)|^2)^s} \right| \frac{(1 - |\phi(z)|^2)}{(1 - |z|^2)^{\alpha p (\log \frac{2}{1-|a|^p})}} dA(z)
\]

\[
= ||f||^p_{\mathcal{B}^*_\alpha, \log} \Phi_\phi (\alpha, p, s; a) < \infty.
\]

For the other direction, we use the fact that for each function \( f \in \mathcal{B}^*_\alpha, \log \), the analytic function
$C_\phi(f) \in F_{\log}^*(p, \alpha p - 2, s)$. Then, using the functions of lemma 1.2

$$
2^p \left\{ ||C_\phi f_1||_{F_{\log}^*(p, \alpha p - 2, s)}^p + ||C_\phi f_2||_{F_{\log}^*(p, \alpha p - 2, s)}^p \right\}
= 2^p \left\{ \sup_{a \in \mathbb{D}} \ell^p(a) \int_{\mathbb{D}} \left[ ((f_1 \circ \phi)^*(z))^p + ((f_2 \circ \phi)^*(z))^p \right] \right.
\times (1 - |z|^2)^{\alpha p - 2} (1 - |\phi_a(z)|^2)^s dA(z) \right\}
\geq \left\{ \sup_{a \in \mathbb{D}} \ell^p(a) \int_{\mathbb{D}} \left[ (f_1 \circ \phi)^*(z) + (f_2 \circ \phi)^*(z) \right] \right.
\times (1 - |z|^2)^{\alpha p - 2} (1 - |\phi_a(z)|^2)^s dA(z) \right\}
\geq C \left\{ \sup_{a \in \mathbb{D}} \ell^p(a) \int_{\mathbb{D}} |\phi'(z)|^p (1 - |z|^2)^{\alpha p - 2} (1 - |\phi_a(z)|^2)^s dA(z) \right\}
\geq C \sup_{a \in \mathbb{D}} \Phi_\phi(\alpha, p, s; a).
$$

Hence $C_\phi$ is bounded, the proof is completed.

The composition operator $C_\phi : B_{\alpha, \log}^* \to F_{\log}^*(p, \alpha p - 2, s)$ is compact if and only if for every sequence $f_n \in \mathbb{N} \subset F_{\log}^*(p, \alpha p - 2, s)$ is bounded in $F_{\log}^*(p, \alpha p - 2, s)$ norm and $f_n \to 0, n \to \infty$, uniformly on compact subset of the unit disk (where $\mathbb{N}$ be the set of all natural numbers), hence,

$$
||C_\phi(f_n)||_{F_{\log}^*(p, \alpha p - 2, s)} \to 0, n \to \infty.
$$

Now, we describe compactness in the following result:

**Theorem 4.** Let $0 < p, 1 < s < \infty, \alpha < \infty$. If $\phi$ is an analytic self-map of the unit disk, then the induced composition operator $C_\phi : B_{\alpha, \log}^* \to F_{\log}^*(p, \alpha p - 2, s)$ is compact if and only if $\phi \in F_{\log}^*(p, \alpha p - 2, s)$, and

$$
\lim_{r \to 1} \sup_{a \in \mathbb{D}} \Phi_\phi(\alpha, p, s; a) \to 0. \quad (5.2)
$$

**Proof.** Let $C_\phi : B_{\alpha, \log}^* \to F_{\log}^*(p, \alpha p - 2, s)$ be compact. This means that $\phi \in F_{\log}^*(p, \alpha p - 2, s)$.

Let

$$
U_r^1 = \{ z : |\phi(z)| > r, r \in (0, 1) \},
$$
and

\[ U_r^2 = \{ z : |\phi(z)| \leq r, r \in (0, 1) \}. \]

Let \( f_n(z) = \frac{e^{2\pi i n}}{n} \) if \( \alpha \in [0, \infty) \) or \( f_n(z) = \frac{e^{2\pi i n}}{n} \) if \( \alpha \in (0, 1) \). Without loss of generality, we only consider \( \alpha \in (0, 1) \). Since \( \|f_n\|_{B_{ \alpha, \log}^*} \leq M \) and \( f_n(z) \to 0 \) as \( n \to \infty \), locally uniformly on the unit disk, then \( \|C_\phi(f_n)\|_{F_{ \log}^*(p, (\alpha p - 2, s)), n} \to \infty \). This means that for each \( r \in (0, 1) \) and for all \( \varepsilon > 0 \), there exist \( N \in N \) so that if \( n \geq N \), then

\[
\frac{N^{\alpha p}}{r^{p(1-\alpha p)}} \sup_{a \in \mathbb{B}} \|\phi(z)\|^{(1-|z|^2)^{\alpha p - 2}}(1 - |\varphi_a(z)|^2)^s dA(z) < \varepsilon.
\]

If we choose \( r \) so that \( \frac{N^{\alpha p}}{r^{p(1-\alpha p)}} = 1 \), then

\[
\sup_{a \in \mathbb{B}} \|\phi(z)\|^{(1-|z|^2)^{\alpha p - 2}}(1 - |\varphi_a(z)|^2)^s dA(z) < \varepsilon. \tag{5.3}
\]

Let now \( f \) be with \( \|f\|_{B_{ \alpha, \log}^*} \leq 1 \). We consider the functions \( f_t(z) = f(tz), t \in (0, 1) \). \( f_t \to f \) uniformly on compact subset of the unit disk as \( t \to 1 \) and the family \( (f_t) \) is bounded on \( B_{ \alpha, \log}^* \), thus

\[ \|f_t \circ \phi - f \circ \phi\| \to 0. \]

Due to compactness of \( C_\phi \), we get that for \( \varepsilon > 0 \) there is \( t \in (0, 1) \) so that

\[
\sup_{a \in \mathbb{B}} \|\phi(z)\|^{(1-|z|^2)^{\alpha p - 2}}(1 - |\varphi_a(z)|^2)^s dA(z) < \varepsilon,
\]

where

\[ F_t(\phi(z)) = [(f \circ \phi)^* - (f_t \circ \phi)^*]. \]

Thus, if we fix \( t \), then

\[
\sup_{a \in \mathbb{B}} \|\phi(z)\|^{(1-|z|^2)^{\alpha p - 2}}(1 - |\varphi_a(z)|^2)^s dA(z)
\leq \quad 2^p \sup_{a \in \mathbb{B}} \|\phi(z)\|^{(1-|z|^2)^{\alpha p - 2}}(1 - |\varphi_a(z)|^2)^s dA(z)
\leq \quad 2^p \varepsilon + \|f_t^*\|_{H_\infty}^{p} \sup_{a \in \mathbb{B}} \|\phi(z)\|^{(1-|z|^2)^{\alpha p - 2}}(1 - |\varphi_a(z)|^2)^s dA(z)
\leq \quad 2^p \varepsilon + 2^p \varepsilon. \]
i.e,
\[
\sup_{a \in D} \ell^p(a) \int_{U^*_1} ((f \circ \phi)^*(z))^{p}(1 - |z|^2)^{\alpha p - 2}(1 - \rho_a(z))^2 dA(z) \\
\leq 2^p \varepsilon (1 + ||f^*_1||_{H^\infty}),
\] (5.4)
where we have used (4). On the other hand, for each \( ||f||_{g^*_n, \log} \leq 1 \) and \( \varepsilon > 0 \), there exists a \( \delta \) depending on \( f \) and \( \varepsilon \), so that for \( r \in [\delta, 1) \),
\[
\sup_{a \in D} \ell^p(a) \int_{U^*_1} ((f \circ \phi)^*(z))^{p}(1 - |z|^2)^{\alpha p - 2}(1 - \rho_a(z))^2 dA(z) < \varepsilon.
\] (5.5)

Since \( C_\phi \) is compact, then it maps the unit ball of \( B^*_\alpha, \log \) to a relatively compact subset of \( F^*_\log(p, q, s) \). Thus, for each \( \varepsilon > 0 \), there exists a finite collection of functions \( f_1, f_2, ..., f_n \) in the unit ball of \( B^*_\alpha, \log \) so that for each \( ||f||_{g^*_n, \log} \), there is \( k \in \{1, 2, 3, ..., n\} \) so that
\[
\sup_{a \in D} \ell^p(a) \int_{U^*_1} |F_k(\phi(z))|^{p}(1 - |z|^2)^{\alpha p - 2}(1 - \rho_a(z))^2 dA(z) < \varepsilon,
\]
where
\[
F_k(\phi(z)) = [(f \circ \phi)^* - (f_k \circ \phi)^*].
\]

Also, using (5), we get for \( \delta = \max_{1 \leq k \leq n} \delta(f_k, \varepsilon) \) and \( r \in [\delta, 1) \), that
\[
\sup_{a \in D} \ell^p(a) \int_{U^*_1} ((f_k \circ \phi)^*(z))^{p}(1 - |z|^2)^{\alpha p - 2}(1 - \rho_a(z))^2 dA(z) < \varepsilon.
\]

Hence, for any \( f, ||f||_{g^*_n, \log} \leq 1 \), combining the two relations as above, we get the following
\[
\sup_{a \in D} \ell^p(a) \int_{U^*_1} ((f \circ \phi)^*(z))^{p}(1 - |z|^2)^{\alpha p - 2}(1 - \rho_a(z))^2 dA(z) \leq 2^p \varepsilon.
\]

Therefore, we get that (2) holds. For the sufficiency, we use that \( \phi \in F^*_{\log}(p, \alpha p - 2, s) \) and (2) holds.

Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of functions in the unit ball of \( B^*_\alpha, \log \) so that \( f_n \to 0 \) as \( n \to \infty \), uniformly on the compact subsets of the unit disk. Let also \( r \in (0, 1) \). Then,
\[
||f_n \circ \phi||_{F^*_{\log}(p, \alpha p - 2, s)}^p \leq 2^p |f_n(\phi(0))|
\] 
\[+ 2^p \sup_{a \in D} \ell^p(a) \int_{U^*_1} ((f_n \circ \phi)^*(z))^{p}(1 - |z|^2)^{\alpha p - 2}(1 - \rho_a(z))^2 dA(z)
\] 
\[+ 2^p \sup_{a \in D} \ell^p(a) \int_{U^*_1} ((f_n \circ \phi)^*(z))^{p}(1 - |z|^2)^{\alpha p - 2}(1 - \rho_a(z))^2 dA(z)
\] 
\[= 2^p (I_1 + I_2 + I_3).
\]
Since \( f_n \to 0 \) as \( n \to \infty \), locally uniformly on the unit disk, then \( I_1 = |f_n(\phi(0))| \) goes to zero as \( n \to \infty \) and for each \( \varepsilon > 0 \), there is \( N \in \mathbb{N} \) so that for each \( n > N \),

\[
I_1 = |f_n(\phi(0))| \to 0 \quad \text{as} \quad n \to \infty.
\]

We also observe that

\[
I_2 = \sup_{a \in D} \| f_n \|_{\ell^p(a)} \int_{U^2_1} |(f_n \circ \phi)(z)|^p (1 - |z|^2)^{\alpha p - 2} (1 - |\varphi_a(z)|^2)^s dA(z)
\]

\[
\leq \varepsilon \| \phi \|_{B^{p,\alpha p - 2, s}}^p.
\]

Under the assumption that (2) holds, then for every \( n > N \) and for every \( \varepsilon > 0 \), there exists \( r_1 \) so that for every \( r > r_1 \), \( I_3 < \varepsilon \).

Thus, if \( \phi(z) \in F^{p, \alpha p - 2, s} \), we get

\[
\| f_n \circ \phi \|_{\ell^p(a)} \leq 2^p \left\{ 0 + \varepsilon \| \phi \|_{F^{p, \alpha p - 2, s}} + \varepsilon \right\} \leq C \varepsilon.
\]

Combining the above, we get \( \| C_\phi(f_n) \|_{F^{p, \alpha p - 2, s}} \to 0 \) as \( n \to \infty \) which proves compactness.

Thus, the theorem we presented is proved.

### 6 Conclusions

We have obtained some essential and important \( D \)-metric spaces. Moreover, the important properties for \( D \)-metric on \( B^{p, \alpha p - 2, s} \) and \( F^{p, \alpha p - 2, s} \) are investigated in Section 4. Finally, we introduced composition operators in hyperbolic weighted family of function spaces.

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