The Singapore Protocol: Incoherent Eavesdropping Attacks

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We thoroughly analyse the novel quantum key distribution protocol introduced recently in [5], which is based on minimal qubit tomography. We examine the efficiency of the protocol for a whole range of noise parameters and present a general analysis of incoherent eavesdropping attacks with arbitrarily many steps in the iterative key generation process. The comparison with the tomographic 6-state protocol shows that our protocol has a higher efficiency (up to 20%) and ensures the security of the established key even for noise parameters far beyond the 6-state protocol’s noise threshold.

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A. Introduction

With the growing interest in quantum information theory, quantum cryptography has become a key research area. One way of ensuring secure communication is to establish a secret key between the two communication partners, Alice and Bob, with which they can later encode and decode secret messages. The distribution of the key can be done using quantum mechanical systems where the laws of physics guarantee the security, in marked contrast to classical schemes which rely on the complexity of mathematical problems. Quantum key distribution could thus replace conventional public key cryptosystems, which can be broken in polynomial time by quantum algorithms as soon as suitable quantum computers are available.

Most quantum key distribution schemes discussed at the current stage of research are based on the BB84 or the E91 protocol. These schemes operate in only a subspace of the whole qubit state space, and so they allow the eavesdropper, Eve, unnecessary freedom to make use of the undetected portion of the Hilbert space. This power is denied to her in fully tomographic key distribution protocols, such as those of Ref. [4,], and in particular by the “minimal qubit tomography protocol” of Ref. [4,] that has become known colloquially as the Singapore protocol. It relies heavily on the minimal qubit tomography (MQT) discussed in Ref. [4,].

In Ref. [4,] it was shown that the maximum theoretical efficiency of a quantum key distribution protocol using the MQT measurement is maximally $\log_2 3 = 0.415$; this is less than the efficiencies of the non-tomographic BB84 and E91 protocol which are $\frac{1}{2}$ and $\frac{2}{3}$, respectively. However, the difference between the mutual information between Alice and Bob (A&B) and the mutual information between Eve and either Alice or Bob (CK-yield) promises to be higher than in the non-tomographic protocols [4,]. Moreover, the MQT measurement has the potential for a significantly higher key yield than the comparable tomographic 6-state protocol introduced in Ref. [3,] and discussed in Refs. [6, 10,], which has an efficiency of only $\frac{1}{3}$.

Using the MQT measurement, one possible way to generate a secret key from the correlated measurement outcomes was proposed in Ref. [4,]. The resulting quantum key distribution protocol recovers 0.4 key-bits per qubit pair, or 96.4% of the potential efficiency in the noise-free case. It is the objective of the present paper to give a detailed account of how the analysis is carried out that yields the thresholds stated in Ref. [4,]. A quantum key-distribution scheme that uses POVMs with tetrahedron structure is also investigated by Renes in Ref. [11,]. This protocol differs from the Singapore protocol by a less efficient key generation procedure that does not fully exploit the potential of the MQT measurement.

In Sec. [B,] we give a brief overview of the MQT measurement and review the key generation in Sec. [C,]. In Sec. [D,] we will discuss the constraints on Eve’s eavesdropping imposed by the tomographic nature of the MQT measurement. We then investigate, in Sec. [E,] the incoherent attacks available to Eve when she exploits the classical information transmitted between the communication partners during the key generation. Finally, in Sec. [F,] we examine the security of the protocol against such attacks and obtain the noise threshold stated in Ref. [4,].

B. Minimal qubit tomography

We suppose Alice and Bob want to establish a secret key and they use a provider that distributes entangled qubit pairs for private communication. As advertised, each will receive one qubit of the pair. Since real communication channels do not usually preserve the signal perfectly, Alice and Bob have to deal with the fact that they will receive a distorted state. Let Alice and Bob agree to accept only a mixed state $\rho_{A&B}$ consisting of the ideal,
perfectly anti-correlated singlet $|s\rangle\langle s| = \frac{1}{4}(\mathbb{1} - \vec{\sigma}_A \cdot \vec{\sigma}_B)$, and white, unbiased noise, weighted with a noise parameter $\epsilon$. The two-qubit state that A&B will share is thus
\[ \rho_{AB}^{\epsilon} = (1 - \epsilon)|s\rangle\langle s| + \frac{\epsilon}{4}\mathbb{1}, \tag{1} \]
where $\epsilon$ ranges from $\epsilon = 0$ (no noise) to $\epsilon = 1$ (nothing but noise). In practical situations, the class of acceptable sources should in fact be chosen in accordance with the experimental setup which produces the singlet and the properties of the transmission line used (fibre, air, ...). But for the sake of simplicity, we impose the above standard criterion which was also used in the tomographic protocols of Ref. \[2\].

The tomographically complete 6-state protocol was analyzed for the above scenario in Ref. \[12\]. In this protocol, Alice and Bob each performs a measurement of a randomly chosen Pauli operator, $\sigma_x, \sigma_y, \sigma_z$, resulting in six outcome probabilities. However, this is not a minimal tomography since only four outcome probabilities are needed to specify the state of a qubit completely. The optimal qubit tomography POVM with the minimum number of four elements is of tetrahedron geometry as shown in Ref. \[2\]: that is the POVM operators can be written in the form
\[ P_k = \frac{1}{4} (\mathbb{1} + \vec{t}_k \cdot \vec{\sigma}) \quad \text{for} \quad k = 1, 2, 3, 4, \tag{2} \]
where the vectors $\vec{t}_k$ point to the corners of a tetrahedron inscribed in the Bloch sphere; see Fig. 1 in Ref. \[2\] for an illustration. The four vectors are linearly dependent,
\[ \sum_{k=1}^{4} \vec{t}_k = 0, \tag{3} \]
with the scalar product
\[ \vec{t}_k \cdot \vec{t}_l = \frac{4}{3} t_{kl}^2 - \frac{1}{3} \quad \text{for} \quad k, l = 1, 2, 3, 4, \tag{4} \]
and fulfill the dyadic completeness relation
\[ \frac{3}{4} \sum_{k=1}^{4} \vec{t}_k \vec{t}_k = \mathbb{1}. \tag{5} \]

Let Alice and Bob each measure the tetrahedron POVM of Eq. \[2\] on many copies of a two-qubit state $\rho$. The resulting joint probabilities of the measurement are then given by $p_{kl} = \text{tr} [\rho P_k Q_l]$, with $P_k$ denoting Alice’s POVM elements and $Q_l$ Bob’s, chosen so that their tetrahedrons are perfectly aligned (if they chose a non-zero angle between their tetrahedrons they would lose the perfect anti-correlations introduced by the singlet). To verify that they indeed received the state $\rho_{AB}^{\epsilon}$ of Eq. \[1\], Alice and Bob sacrifice a fraction of their data and announce them publicly to determine the joint probabilities $p_{kl}$ of Alice measuring $k$ and Bob $l$. They check their results for statistical independence and are able to reconstruct the original state by
\[ \rho = \sum_{k,l=1}^{4} (6P_{kl} - \mathbb{1}) p_{kl} (6Q_{kl} - \mathbb{1}). \tag{6} \]

Naturally, after a finite number of measurements, Alice and Bob cannot infer the values of the $p_{kl}$ exactly, but they can estimate them rather reliably. A discussion of the quality of such estimates was given in Ref. \[2\] for the single qubit case where it was also shown that the measurement of a randomly chosen qubit state with the tetrahedron POVM will on average lead to the best (optimal) estimate of the state’s Pauli vector. Finally A&B compare whether the predicted state of Eq. \[1\] is consistent with Eq. \[1\]. They will only use the provider if this is the case.

Given the shared state is $\rho_{AB}^{\epsilon}$ for some $\epsilon$, their joint probabilities $p_{kl}$ will be
\[ p_{kl} = \frac{4 - \epsilon}{48} (1 - \delta_{kl}) + \frac{\epsilon}{16} \delta_{kl} \quad \text{for} \quad k,l = 1, 2, 3, 4, \tag{7} \]
and the accessible information that A&B can establish between each other is given by
\[ I^{\text{access}}_{AB}(\epsilon) = \left(1 - \frac{\epsilon}{4}\right) \log_2 \frac{4 - \epsilon}{3} + \frac{\epsilon}{4} \log_2 \epsilon, \tag{8} \]
where we used the definition of the classical mutual information for a probability distribution $\{p_{kl}\}_{k,l}$
\[ I = \sum_{kl} p_{kl} \log_2 \frac{p_{kl}}{\sum_{k'} p_{k'l} \sum_{l'} p_{kl'}}. \tag{9} \]
Note that in the noise-free case the accessible information, $I^{\text{access}}_{AB}(0) = 0.415$, is substantially higher (by 24.5%) than the corresponding value of $\frac{1}{4}$ for the tomographically complete 6-state protocol.

C. The Singapore protocol

From now on, we will refer to the possible measurement outcomes as A, B, C, D for $k = 1, 2, 3, 4$, respectively, symbolizing a click in the $k$-th POVM detector. To generate a key from their correlated sequences, Alice and Bob must communicate classically. We use the two-way key generation scheme proposed in Ref. \[2\] which leads to a mutual information of $I^{\text{access}}_{AB}(0) = 0.4$, sufficiently close to the maximally accessible information of $I^{\text{access}}_{AB}(0) = 0.415$. The scheme has a simple structure and can be easily implemented on a computer. We give a brief description of the key generation scheme followed by a more detailed analysis considering the presence of noise which will be relevant for the eavesdropping discussion. We refer the reader to the original paper \[2\] for more details on the key generation scheme.

Let us first consider the noise-free case ($\epsilon = 0$). Alice publicly announces two randomly chosen positions of her
iteration), the probability of Alice and Bob receiving the
other choices of letters. Let us denote the probability of
letters are unequal, i.e. in all off-diagonal cases. Simi-
Table I, where M.P. denotes the marginal probabilities.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
 & A & B & C & D & M.P. \\
\hline
Bob's & \(\frac{1}{4}p_s\) & \(\frac{1}{4}p_s\) & \(\frac{1}{4}p_d\) & \(\frac{1}{4}p_d\) & \(\frac{1}{16}\) \\
1st letter & \(\frac{1}{4}p_s\) & \(\frac{1}{4}p_s\) & \(\frac{1}{4}p_d\) & \(\frac{1}{4}p_d\) & \(\frac{4}{8}\) \\
M.P. & \(\frac{1}{16}\) & \(\frac{1}{8}\) & \(\frac{1}{8}\) & \(\frac{1}{8}\) & \(\frac{1}{4}\) \\
\hline
\end{tabular}
\caption{Bob's two letters given that Alice announced two positions where she got the outcome A.}
\end{table}

sequence for which she has the same letter. With proba-
bility \(\frac{1}{3}\), Bob has different letters at these positions. He
then groups the possible outcomes A, B, C, D in two
groups, one containing the two letters he received and
one with the remaining two letters. He randomly assigns
the values 0 and 1 to the two groups and announces these
groupings. Bob does not reveal which group his letters
belong to, but since A&B's measurement outcomes are
perfectly anti-correlated, they can both write down the
value of the group which contains Alice's letter and thus
generate a key-bit. With probability \(\frac{1}{3}\) Bob has the same
letter in the two positions Alice announced. In this case,
he states this fact and A&B each write their correspond-
ing letter in a new sequence. The above procedure is re-
peate iterateatively with the new sequences thus generated.

In the presence of noise, the key sequence generated
with the above scheme will contain errors with a rate
dependent on \(\epsilon\). For the original letter sequence (first
iteration), the probability of Alice and Bob receiving the
same letter is then non-zero, see Eq. (7),

\[ p_s(\epsilon) = \frac{\epsilon}{4}, \]

and Bob receives one of the other three letters with prob-
ability

\[ p_d(\epsilon) = \frac{4 - \epsilon}{12}. \]

With a priori probability \(\frac{1}{3}\) Alice announces the positions
of two occurrences of the letter A. Then, Bob's corre-
sponding two letters occur with the probabilities given in
Table I where M.P. denotes the marginal probabilities.
The conversion into one key-bit will occur when Bob's
letters are unequal, i.e. in all off-diagonal cases. Simi-
larly, we can construct the probability tables for Alice's
other choices of letters. Let us denote the probability of
successfully generating one key-bit from one letter pair
by \(p_{s\text{ucc}}\). For the first iteration it is then

\[ p_{s\text{ucc}}^{(1)}(\epsilon) = 6p_s p_d (p_s + p_d) = \frac{(4 - \epsilon) (2 + \epsilon)}{12}. \]

However, these successfully generated key-bits will con-
tain errors. The probability that a generated key-bit is
wrong is given by

\[ p_{\text{err}}^{(1)}(\epsilon) = \frac{6p_s p_d}{p_{s\text{ucc}}} = \frac{p_s}{p_s + p_d} = 3\epsilon \frac{4 + 2\epsilon}{4}. \]

accounting for the equally likely cases that Bob writes 0
and Alice 1 and vice versa. The mutual information of
the key itself is thus less than unity and depends on \(p_{\text{err}}^{(1)}\),
which is nonzero for nonzero \(\epsilon\),

\[ I_{\text{key}}(p_{\text{err}}^{(1)}) = 1 + p_{\text{err}}^{(1)} \log_2 p_{\text{err}}^{(1)} \]

\[ + (1 - p_{\text{err}}^{(1)}) \log_2 [1 - p_{\text{err}}^{(1)}]. \]

Let us regard the mutual information as a cryptographic
resource that Alice and Bob can use later to extract a per-
fected correlated key. We are therefore interested in the
expectation value of the mutual information which A&B
share per qubit pair. This expectation value is the prod-
uct of the mutual information of the generated key-bit
and the probability that this key-bit was actually gener-
ated \(p_{\text{succ}}^{(1)}\), divided by the number of qubit pairs needed
(2 in the first iteration) to obtain the key-bit.

\[ I_{\text{A&B}}^{(1)}(\epsilon) = \frac{p_{\text{succ}}^{(1)} I_{\text{key}}(p_{\text{err}}^{(1)})}{2} \]

\[ = \frac{(4 - \epsilon)}{48} \left(4 - \epsilon\right) \log_2 \frac{4 - \epsilon}{2 + \epsilon} + 3\epsilon \log_2 \frac{3\epsilon}{2 + \epsilon}. \]

To deduce similar results for further iterations we first
study the properties of the recycled sequences. The sec-
ond iteration can again be characterized by two probabili-
ties \(p_s'\) and \(p_d'\) defined with analogous meanings as \(p_s\)
and \(p_d\) for the original sequence. The probability \(p_s'\) is
given by the probability of Bob receiving the same letter as
Alice in the original sequence twice \((p_s^2)\), divided by the
total probability of keeping letters in the first iteration,
\(\frac{1}{2}\). The relation between \(\epsilon'\) and \(\epsilon\) is most compactly stated in the form

\[ \frac{3\epsilon'}{4 - \epsilon'} = \left(\frac{3\epsilon}{4 - \epsilon}\right)^2, \]
in the same way, each time replacing the noise \( \epsilon \) of the previous iteration by the new noise parameter \( \epsilon' \).

In particular, the probability of successfully generating a key-bit in the \( n \)-th iteration is, similarly to Eq. (12), given by

\[
p_{succ}^{(n)} = \left( 1 - q^{(n)} \right) \prod_{m=1}^{n-1} \left( 1 - q^{(m)} \right),
\]

where \( q^{(n)} = 4 - \epsilon^{(n)} + \frac{1}{12} \), and

\[
p_{err}^{(n)} = \frac{3\epsilon^{(n)}}{4 + 2\epsilon^{(n)}} = \left[ 1 + \left( \frac{4 - \epsilon}{3\epsilon} \right)^{2^{n-1}} \right]^{-1},
\]

which uses Eq. (19). The contribution to \( I_{total}^{AkB} \) in the \( n \)-th iteration is given by

\[
I_{total}^{AkB}(\epsilon) = 2^{-n} p_{succ}^{(n)} I_{key}(p_{err}^{(n)}),
\]

and the overall expectation value of the mutual information per qubit pair in the limit of infinitely many iterations is thus

\[
I_{total}^{AkB}(\epsilon) = \sum_{n=1}^{\infty} \frac{p_{succ}^{(n)}}{2^n} I_{key}(p_{err}^{(n)}).
\]

This quantity serves as our figure of merit for the comparison with the 6-state protocol. One should however keep in mind that it is an average over the various key-bit sequences of the successive iterations. Each of them has different noise properties which must be taken into account when the data are processed further by a privacy amplification procedure.

In the noiseless case (\( \epsilon = 0 \)), the Singapore protocol yields a mutual information \( I_{total}^{AkB}(0) = \frac{1}{4} \) for the first iteration. This is as much as one can get in the 6-state protocol; but here we can improve the efficiency by continuing the key generation with the left-over sequences, e.g., \( I_{total}^{AkB}(0) + I_{total}^{AkB}(1) = \frac{1}{3} + \frac{1}{15} = 0.389 \) up to the second iteration and so on, with the limiting value of 0.4.

When Alice and Bob share the complete mixture (\( \epsilon = 1 \)) we find the expected result of \( I_{total}^{AkB}(1) \propto \log_2[1] = 0 \).

The numerical plots of the total mutual information for Alice and Bob in the 6-state protocol represent the function \( I_{total}^{6-state}(\epsilon) = \frac{1}{2} \epsilon \log_2(\epsilon) + (2 - \epsilon) \log_2(2 - \epsilon) \). Note, that the total mutual information up to the 3rd, 4th and 5th iteration are so close that they overlap and the 3rd iteration is already a very good approximation of \( I_{total}^{AkB} \). The plot covers the range \( 0 < \epsilon < \frac{1}{4} \). i.e., the \( \epsilon \)-values for which \( \rho_{AkB} \) of Eq. (1) is not separable.

![FIG. 1: Total mutual information between Alice and Bob for the Singapore protocol for the 1st to 5th iteration in comparison with the tomographic 6-state protocol. The plotted average mutual information of Alice and Bob in the 6-state protocol is\( \mathfrak{I} \) the noiseless case and vanishes for \( \rho_{AkB} \) of Eq. (1) is not separable.](image)

D. Constraints on Eve’s eavesdropping

In Sec. B Alice and Bob received a two-qubit state sent by a provider. We must however assume that this provider is not trustworthy and eager to know A&B’s secret. We hence identify the provider as the most dangerous eavesdropper (Eve) possible, and give her full control over the source. In the worst case scenario, Eve is smarter with her technology and can even replace the usually noisy channel between Alice and Bob by a perfect one. She is then in the position to entangle an additional ancilla to each qubit pair she sends, and the disturbances she causes by doing so would imitate noise. But Alice and Bob perform a complete tomography of the shared state and since they agree to use the channel only when the noise has the properties reflected in Eq. (1), they can greatly restrict how Eve can entangle ancillas, and later on deduce the shared key.

Eve will prepare a 3-party pure state because she has
no advantage from creating a mixed state and thus introducing classical noise herself. We decompose the state as

$$|S_e⟩ = \sum_{i=1}^{4} |\bar{l}i⟩|E_i⟩,$$  

(22)

where the (unnormalized) states $|E_i⟩$ represent Eve’s ancilla. The pure states $|\bar{l}i⟩$ belong to Alice and Bob and form a non-orthogonal basis $\{|\bar{l}i⟩\}_{i=1}^{4}$, where the tetrahedron states $|\bar{l}⟩$ and $|l⟩$ are defined as

$$|\bar{l}⟩|l⟩ = \frac{1}{2} (1 + \bar{t}_l \cdot \vec{\sigma}), \quad l = 1, 2, 3, 4,$$

$$|l⟩|\bar{l}⟩ = \frac{1}{2} (1 - \bar{t}_l \cdot \vec{\sigma}),$$

(23)

with the phase-conventions

$$⟨l|k⟩ = ⟨\bar{k}|\bar{l}⟩ \quad \text{and} \quad ⟨l|\bar{l}⟩ = ⟨k|\bar{\bar{l}}⟩,$$  

(24)

the second of which implied by the first. Some useful relations of these states can be found in the Appendix. Note, that the decomposition of Eq. (22) allows for exactly four components on Eve’s side, together forming Eve’s ancilla state. Each ancilla can be thus be represented by a maximally four-dimensional system so that the ancilla can be regarded as another qubit pair.

Eve’s ancilla state is however restricted by the condition that A&B must receive the two-qubit state $\rho_{A&B}$ of Eq. (11) which they check by comparing their outcomes of the tomographic measurement. Assuming that all the noise originates in Eve’s eavesdropping attempt, the 3-party state $|S_e⟩$ must be such that

$$\text{tr}_E [|S_e⟩⟨S_e|] = \rho_{A&B}(\epsilon).$$  

(25)

Let us rewrite $\rho_{A&B}$ using the expansion of the identity matrix in Eq. (A3) and the decomposition of the singlet in Eq. (A3), to get

$$\rho_{A&B}(\epsilon) = \frac{1}{8} \sum_{l,k=1}^{4} |\bar{l}k⟩⟨k\bar{l}| \left( 1 - \frac{3}{2} \epsilon + 3\epsilon \delta_{lk} \right).$$  

(26)

This expression immediately gives the following constraints on Eve’s ancilla state

$$⟨E_k|E_l⟩ = \frac{2 - 3\epsilon}{16} + \frac{3\epsilon}{16} \delta_{kl} \quad \text{for} \quad k, l = 1, ..., 4.$$  

(27)

For $\epsilon = 0$, the $|E_l⟩$ are identical and Eve cannot extract any information. For $\epsilon = \frac{1}{3}$, the scalar products in Eq. (27) show the orthogonality of the $|E_l⟩$ which implies that $\rho_{A&B}$ is separable. Indeed, Alice and Bob share a separable Werner state

$$\rho_{A&B} \left(\frac{2}{3}\right) = \frac{1}{4} \sum_{l=1}^{4} |\bar{l}l⟩⟨\bar{l}l| = \frac{1}{4} \left(1 - \frac{1}{3} \vec{\sigma}_A \cdot \vec{\sigma}_B \right),$$  

(28)

and thus all correlations are classical.

In Ref. 13 it was shown that, given the constraints of Eq. (27), the most general state Eve can construct, up to unitary equivalence, can be written as

$$|S_e⟩ = \alpha|s_{12}⟩|s_{34}⟩ + \beta|s_{13}⟩|s_{24}⟩,$$  

(29)

where the first qubit is held by Alice, the second by Bob and the third and fourth by Eve. The amplitudes $\alpha$ and $\beta$ must now be chosen so that Eve’s ancilla satisfies Eq. (27). By using Eq. (22) and Eq. (27) we find Eve’s states to be

$$|E_k⟩ = \frac{1}{2\sqrt{2}} |s⟩ - \frac{\beta}{2} \left( |k\bar{k}⟩ + \frac{1}{2} |\bar{k}k⟩ \right),$$  

(30)

and evaluating the scalar products $⟨E_k|E_l⟩$ and comparing with Eq. (27), we deduce the constraints on the parameters $\alpha$ and $\beta$,

$$|\beta|^2 = \epsilon \quad \text{and} \quad \alpha + \frac{\beta}{2} = 1 - \frac{3}{4}\epsilon.$$  

(31)

Since we have a freedom of global phase for $|S_e⟩$ we can choose $\beta$ to be real, i.e. $\beta = \sqrt{\epsilon}$. The only free parameter is then the phase $\phi$ in

$$\alpha + \frac{\beta}{2} = e^{i\phi} \sqrt{1 - \frac{3\epsilon}{4}}.$$  

(32)

Eve wants to guess Alice’s key-bit and constructs a state $ρ_E^{(k)}$ for each outcome $k$ Alice could measure regardless of Bob’s result. These conditional ancilla states are

$$ρ_E^{(k)} = \text{tr}_{AB} [P_k [|S_e⟩⟨S_e|]],$$  

(33)

$$= \frac{|\beta|^2}{8} |k\bar{k}⟩⟨k\bar{k}|$$

$$+ \frac{1}{4} \left(\alpha |s⟩ - \frac{\beta}{\sqrt{2}} |k\bar{k}⟩ \right) \left(\alpha^* |s⟩ - \frac{\beta^*}{\sqrt{2}} |\bar{k}k⟩ \right),$$

where $P_k$ is the POVM element for Alice measuring $k$. Note, that all $ρ_E^{(k)}$ are subnormalized to $\frac{1}{4}$, the a priori probability that Alice will measure a particular $k$.

Owing to the symmetry between Alice and Bob, the ancilla states conditional to Bob’s measurement results are unitarily equivalent to these $ρ_E^{(k)}$. Therefore, it does not matter whether Eve tries to learn Alice’s measurement results or Bob’s.

E. Incoherent Eavesdropping attacks

Let us summarize what we have found so far. For each qubit pair that Eve sends to A&B she will keep a qubit pair (the ancilla) for herself. In the first iteration of the key generation scheme Alice and Bob will use the measurement outcomes of two qubit pairs. Eve has thus two corresponding ancillas which she can measure to guess A&B’s generated key-bit. For the second iteration Eve
will have four, for the third she will have eight ancillas and so on.

We suppose Eve has no means of storing her ancillas until classical communication between Alice and Bob is done. She therefore has to measure them individually as she creates them, without being able to include the classical information in her measurement. Her measurement can be optimized such as to maximize her mutual information with Alice. We will call her optimal strategy for doing this an incoherent attack as opposed to a coherent (joint) measurement performed on the bunch of ancillas correlated through the key generation process. The optimal POVM for the $\rho^E_{(k)}$ of Eq. (32) was found using the iterative procedure in Ref. [12] and is analogous to the optimal POVM for the 6-state protocol given there. The 4-member POVM consists of the projectors

$$M_l = |e_l\rangle\langle e_l|, \quad |e_l\rangle = \frac{1 + \sqrt{3}e^{-i\phi}}{2}|s\rangle + \frac{\sqrt{3}}{2}e^{-i\phi}|l\rangle,$$

for $l = 1, 2, 3, 4$, where $\phi$ is the phase of Eq. (32), and the $M_l$ obey $\sum_{l=1}^4 M_l = 1$. Note that the POVM is independent of the noise parameter, $\epsilon$.

Interestingly, in the interval $0 < \epsilon < \bar{\epsilon}$ where $\bar{\epsilon} = 0.1725$ (obtained numerically by solving a transcendental equation), it was found that a 5-member POVM gives a slightly larger mutual information than the 4-member POVM (less than 1% larger). The fifth element has the following expression

$$M_5 = |e_5\rangle\langle e_5| \text{ where } |e_5\rangle = \sqrt{2\mu - 4\mu^2} \sum_{l=1}^4 |e_l\rangle$$

where $\mu$ is a function of $\epsilon$ and $0 \leq \mu \leq 1/2$. All the other $\{|e_l\rangle\}_{l=1}^4$ have to be modified in the following manner

$$|e_j\rangle \rightarrow \left(|e_j\rangle - \mu \sum_{l=1}^4 |e_l\rangle\right)$$

(36)

\[ \text{to ensure the elements sum up to identity. For the purpose of finding the noise threshold, we can just use the simpler 4-member POVM since as we shall see later, the noise threshold is always much larger than } \bar{\epsilon} \text{ for any iteration.} \]

With the 4-member POVM, the joint probabilities $q_{kl}$ of Alice measuring $k$ and Eve measuring $l$ are given by the same expression as Alice and Bob’s joint probabilities in Eq. (17) with $\epsilon$ replaced by a new noise parameter $\eta$, quantifying the noise between Alice and Eve, with

$$\eta(\epsilon) = \left(\sqrt{1 - \frac{3\epsilon}{4}} - \sqrt{\frac{3\epsilon}{4}}\right)^2.$$

(37)

Note that for $\epsilon = 0$ the noise between Alice and Eve reaches a maximum ($\eta = 1$) and when $\epsilon = \frac{3}{4}$ there is no noise between Alice and Eve ($\eta = 0$).

We define the probabilities $q_s$ ($q_d$) for Alice and Eve having the same (a particular different) measurement result in analogy to the $p_s$ ($p_d$) in Eq. (11) by

$$q_s(\eta) = q_{kk} = \frac{\eta}{4} \quad \text{and} \quad q_d(\eta) = q_{k\neq k} = \frac{4 - \eta}{12}.$$  

(38)

When a key-bit is generated between Alice and Bob the joint probabilities between Alice’s and Eve’s results are as given in Table II where we assumed that Bob grouped $AB = 0$ and $CD = 1$ (similarly for other groupings). Note, that the probabilities in Table II are again more anti-correlated than they are correlated since $q_s \leq q_d$ for all $\eta$. We compute the mutual information between Alice and Eve from this table of probabilities. The middle column does not contribute and the mutual information becomes

$$I^{(1)}_{AkE}(\epsilon) = \frac{p_{\text{max}}}{2} \left\{ q_s^2 + q_d^2 \log_2 \left[ \frac{q_s^2 + q_d^2}{2} \right] + 4q_s q_d \log_2 \left[ q_s q_d \right] - (q_s^2 + 3q_d^2) \log_2 \left[ \frac{q_s^2 + 3q_d^2}{4} \right] - 2q_s q_d \right\}.$$  

(39)

where the prefactor is the same as in Eq. (13). This value gives the upper bound to the amount of information Eve can obtain about the key generated in the first iteration. It is valid for incoherent attacks only, by whatever suitable method Eve might employ to extract Alice’s key-bit.

In the $n$-th iteration Eve will have $2^n$ qubit pairs available for measurement. She measures all qubit pairs individually and gets a sequence of $2^n$ letters with A, B, C and D occurring $n_A, n_B, n_C$ and $n_D = 2^n - n_A - n_B - n_C$ times, respectively. In all announced positions which contribute in the key generation of a key-bit, Alice always has the same letter, say A. The probability of Eve measuring a sequence which contains $n_A$ times A is given by, with $q_s$ and $q_d$ from Eq. (32).

$$q_A^n(n_A) = \frac{1}{4} q_s^n q_d^{2^n-n_A}.$$  

(40)
If Bob grouped \([AB] = 0\) and \([CD] = 1\), say, the probability of Eve getting a particular distribution \(\{n_A, n_B, n_C, n_D\}\) and Alice having the key-bit 0 is then
\[
q^n_{n}(n_A, n_B, \ldots) = q^n_A(n_A) + q^n_B(n_B) + q^n_C(n_C) + q^n_D(n_D)
\]
and similarly for group 1. The marginal probabilities for Eve getting a distribution \(\{n_A, n_B, n_C, n_D\}\) is then
\[
q^k_n(n_A, n_B, n_C, n_D) = \frac{q^n_{n}}{2^n} \sum_{j=A}^{D} \binom{2^n}{n_A, n_B, n_C, n_D}
\]
and Alice’s marginals \(q^k_n\) for \(k = 0, 1\) turn out correctly
\[
q^k_n(n_A, n_B, n_C, n_D) = \frac{1}{2^n} \binom{2^n}{n_A, n_B, n_C, n_D}
\]
The contribution to the mutual information that Eve shares with Alice from the \(n\)-th iteration can now be calculated,
\[
I^{(n)}_{A\&E}(\epsilon) = \frac{p_{\text{suc}}^{(n)}}{2^n} \sum_{k=0}^{2^n} \binom{2^n}{n_A, n_B, n_C, n_D = 0} \binom{2^n}{n_A, n_B, n_C, n_D} \log_2 \left[ \frac{q^k_n(n_A, n_B, n_C, n_D)}{q^n_{n}(n_A, n_B, n_C, n_D)} \right],
\]
where \(p_{\text{suc}}^{(n)}\) is again the probability of success in the \(n\)-th iteration (Eq. (43)) and the factor of \(2^{-n}\) gives the mutual information per qubit pair used. The total mutual information that Eve can reach if Alice and Bob perform infinitely many iterations is then
\[
I^{(\text{total})}_{A\&E}(\epsilon) = \sum_{n=1}^{\infty} I^{(n)}_{A\&E}(\epsilon).
\]
In the noise-free case the mutual information of Eve vanishes \(I^{(\text{total})}_{A\&E}(0) \propto \log_2 [1] = 0\). This is clear since the channel between Alice and Bob is perfect and the channel between Alice and Eve is completely noisy (\(\eta = 1\)).

**F. Security**

According to the Csiszár-Körner Theorem in Ref. [14], Alice and Bob are able to share a secret key provided their mutual information \(I^{(\text{total})}_{A\&B}\) exceeds the mutual informations shared between Eve and one of the communication partners \(I^{(\text{total})}_{A\&E}\) and \(I^{(\text{total})}_{B\&E}\). The CK-yield is then
\[
Y_{CK} = I^{(\text{total})}_{A\&B} - I^{(\text{total})}_{A\&E}.
\]
The CK-yield determines the length of the secure key Alice and Bob can obtain from the generated raw key of length \(L\); namely the length of the secure key will maximally be \(Y_{CK} \times L\) (for one-way communication). The intersection between \(I^{(\text{total})}_{A\&B}\) and \(I^{(\text{total})}_{A\&E}\) thus gives the final noise threshold below which a secret key between Alice and Bob can be generated by one-way communication that relies on error correction codes.

In Fig. 2 we plot the CK-yield for the 6-state protocol and different iterations of the Singapore protocol. Observe that the yield of the Singapore protocol is distinctly larger than the yield of the tomographic 6-state protocol from the second iteration onwards. For \(\epsilon = 0\) the gain is already 20\% and it increases significantly for larger noise parameters. Further, the noise threshold for the first iteration of the Singapore protocol is at \(\epsilon = 0.409\) and increases to \(\epsilon = 0.417\) for the 3rd iteration. Additional iterations will raise this value in the 4th decimal place, so that \(\epsilon = 0.417\) can count as the maximum noise Alice and Bob can accept when establishing a secure key with the Singapore protocol. In contrast, the 6-state protocol has its noise threshold at the much smaller value of \(\epsilon = 0.236\). Compared to the 6-state protocol the noise threshold of the Singapore protocol is remarkable 76.7\% higher. This result is our key observation in this paper. Given a number of qubit pairs, measuring the tetrahedron POVM and using the above key-generation scheme thus leads to a raw key substantially longer than the one that could be produced by the 6-state protocol. Admi-
tionally, Alice and Bob are still able to share a secret key when the noise level exceeds the 6-state threshold by a lot.

G. Discussion and conclusions

The analysis of the eavesdropping attacks was carried out for the original protocols, without any error-correction or privacy amplification schemes. It is thus a comparison between the Singapore protocol and the 6-state protocol in their pure form only up to the extraction of a raw key. We have seen the inherent potential of both protocols and analysed in detail how much mutual information the communication partners Alice and Bob can establish between each other when using the Singapore protocol. It turns out that A&B can already stop the key generation after the third iteration without losing much, since the mutual information up to the third iteration is already very close to the limiting value of infinitely many iterations. The comparison with the 6-state protocol showed that the efficiency of the Singapore protocol is up to 20% larger than in the 6-state protocol.

We continued our discussion by constructing incoherent eavesdropping attacks under the following assumptions: 1) A source controlled by the eavesdropper Eve distributes the singlet state, mixed with unbiased, white noise scaled by the noise parameter $\epsilon$. 2) Eve is the cause for all noise; and 3) Eve can not store her ancillas and is thus not able to incorporate knowledge of the classical communication between Alice and Bob when measuring her ancillas. Additionally she is constrained to perform only individual attacks and cannot measure correlated ancillas in a joint measurement. Condition 1) is equivalent to the scenario where Alice sends a qubit in a state orthogonal to one of the tetrahedron states from Eq. (23), each with probability $\frac{1}{4}$. Eve could then intercept the traveling qubit and produce an optimal clone and an anti-clone. Here she also keeps two qubits which she can measure and Bob receives a disturbed state. Together with condition 2) this leads to the worst case scenario for Alice and Bob, where Eve can realize optimal cloning. This is reasonable since we are interested in absolute security statements which rely only on the laws of physics and not on technical abilities of Eve. On the other hand, we assumed condition 3) which is clearly a relaxation to this strictness. But this is still a valid constraint given that modern technology has not developed reliable quantum storage systems and it is not yet feasible to perform joint measurements on demand. However, we have also analysed coherent eavesdropping attacks on the Singapore protocol as discussed in Ref. [4] and will present a detailed and extended report in due time.

Our discussion did assume throughout that Alice and Bob share a state of the form given in Eq. (23). A natural and open question is then how well will the Singapore protocol perform if the state differs from the above, e.g. if the noise is somehow biased, and how does it then compare to the 6-state protocol? Another issue worth addressing is the possible use that Eve can make of the information gained when Alice and Bob perform a privacy amplification or key purification by other means. However, under the conditions 1) - 3), the Singapore protocol provides an efficient alternative for generating a secret key between two communication partners. The measurement of the tetrahedron POVM is from a practical point of view as feasible as the comparable tomographic 6-state measurement (see Ref. [12]) and the efficiency and security under incoherent attacks is significantly higher.

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Appendix

It is expedient to re-express the tetrahedron states of Eq. (23) in terms of the POVM elements $P_l$ of Eq. (23) by

$$|l\rangle\langle l| = 2 P_l, \quad l = 1, 2, 3, 4, \quad (A1)$$

Then the scalar products of the tetrahedron states are given by

$$|\langle l|k\rangle|^2 = 4 \text{tr} [P_l P_k] = \frac{1}{3} (1 + 2 \delta_{lk}), \quad (A2)$$

$$|\langle l|\bar{k}\rangle|^2 = 2 \text{tr} [P_l (P_k - 2 P_l)] = \frac{1}{3} (2 - 2 \delta_{lk}).$$

The singlet $|s\rangle$ can be written in terms of the $|l\rangle$ and $|\bar{l}\rangle$ as

$$|s\rangle = \frac{1}{\sqrt{2}} (|l\rangle - |\bar{l}\rangle) \quad \text{for any } l = 1, 2, 3, 4, \quad (A3)$$

or

$$|s\rangle = \frac{1}{\sqrt{8}} \sum_{l=1}^4 |l\rangle = -\frac{1}{\sqrt{8}} \sum_{l=1}^4 |\bar{l}\rangle. \quad (A4)$$

Furthermore, we expand the identity in terms of $|l\rangle$ and the singlet,

$$1 = \frac{3}{2} \sum_{l=1}^4 |l\rangle\langle l| - 2 |s\rangle\langle s|. \quad (A5)$$
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