I. INTRODUCTION

In this paper we discuss the problem of isometric embedding of the surface of a rapidly rotating black hole in a flat space.

It is well known that intrinsically defined Riemannian manifolds can be isometrically embedded in a flat space. According to the Cartan-Janet theorem, every analytic Riemannian manifold of dimension $n$ can be locally real analytically isometrically embedded into $\mathbb{E}^N$ with $N = n(n + 1)/2$. The so called Fundamental Theorem of Riemannian geometry (Nash, 1956) states that every smooth Riemannian manifold of dimension $n$ can be globally isometrically embedded in a Euclidean space $\mathbb{E}^N$ with $N = (n + 2)(n + 3)/2$.

The problem of isometric embedding of 2D manifolds in $\mathbb{E}^3$ is well studied. It is known that any compact surface embedded isometrically in $\mathbb{E}^3$ has at least one point of positive Gauss curvature. Any 2D compact surface with positive Gauss curvature is always isometrically embeddable in $\mathbb{E}^3$, and this embedding is unique up to rigid rotations. (For general discussion of these results and for further references, see e.g. [4]). It is possible to construct examples when a smooth geometry on a 2D ball with negative Gauss curvature cannot be isometrically embedded in $\mathbb{E}^3$ (see e.g. [3]). On the other hand, it is easy to construct an example of a global smooth isometric embedding for a surface of the topology $S^2$ which has both, positive and negative Gauss curvature ball-regions, separated by a closed loop where the Gauss curvature vanishes. An example of such an embedding is shown in Figure 1.

The surface geometry of a charged rotating black hole and its isometric embedding in $\mathbb{E}^3$ was studied long time ago by Smarr [7]. He showed that when the dimensionless rotation parameter $\alpha = J/M^2$ is sufficiently large, there are two regions near poles of the horizon surface where the Gauss curvature becomes negative. Smarr proved that these regions cannot be isometrically embedded (even locally) in $\mathbb{E}^3$ as a revolution surface, but such local embedding is possible in a 3D Minkowsky space. More recently different aspects of the embedding of a surface of a rotating black hole and its ergosphere in $\mathbb{E}^3$ were discussed in [8, 9]. A numerical scheme for construction of the isometric embedding for surfaces with spherical topology was proposed in [9]. The surface geometry of a rotating black hole in an external magnetic field and its embedding in $\mathbb{E}^3$ was studied in [12, 13, 14].

The purpose of this paper is to obtain the global isometric embedding of a surface of a rapidly rotating black hole in $\mathbb{E}^3$. In Section 2 we discuss general properties of 2D axisymmetric metrics and prove that if the Gauss curvature is negative at the fixed points of the rotation group it is impossible to isometrically embed a region containing such a fixed point in $\mathbb{E}^3$. In Section 3 we demonstrate that such surfaces can be globally embedded in $\mathbb{E}^4$. We obtain the embedding of surfaces of rapidly rotating black holes in $\mathbb{E}^4$ in an explicit form in Section 4. Section 5 contains a brief summary and discussions.

FIG. 1: This picture shows the "croissant" surface. A solid line separates two regions with opposite signs of the Gauss curvature. Each of these regions has the topology of a 2D ball. The Gauss curvature is negative in the upper ball-region.

II. GEOMETRY OF 2D AXISYMMETRIC DISTORTED SPHERES

Let us consider an axisymmetric deformation $S$ of a unit sphere $S^2$. Its metric can be written in the form

$$dl^2 = h(x) dx^2 + f(x) d\phi^2.$$  \hfill (1)

Here $\xi = \partial_\phi$ is a Killing vector field with closed trajectories. Introducing a new coordinate $\mu = \int dx \sqrt{h}$ one can rewrite (1) in the form

$$dl^2 = f(\mu)^{-1} d\mu^2 + f(\mu) d\phi^2.$$  \hfill (2)

We assume that the function $f$ is positive inside the interval $(\mu_0, \mu_1)$ and vanishes at its ends. We choose $\mu_0 = -1$. The surface area of $S$ is $2\pi(\mu_1 + 1)$. By multiplying the

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metric (1) by a constant scale factor \((\mu_1 + 1)^{-1/2}\) one can always put \(\mu_1 = 1\). We shall use this choice, for which the surface area of \(S\) is \(4\pi\) and the fixed points of \(\xi\) are located at \(\mu = \pm 1\).

The metric (2) is regular (no conical singularities) at the points \(\mu = \pm 1\) (where \(r = 0\)) if \(f'(\pm 1) = \mp 2\). (Here and later \((\ldots)' = d(\ldots)/dz\).) The Gaussian curvature for the metric (2) is

\[
K = -\frac{1}{2}f''.
\] (3)

Let us introduce a new coordinate \(r = \sqrt{f(\mu)}\). The metric (2) in these coordinates is

\[
dl^2 = V(r)dr^2 + r^2d\phi^2, \quad V = \frac{4}{f'}. \] (4)

We denote by \(K_0\) the value of \(K\) at a fixed point of rotations. Then in the vicinity of this point one has

\[
V \approx 1 + \frac{1}{2}K_0r^2 + \ldots . \] (5)

For \(K_0 < 0\) the region in the vicinity of the fixed point cannot be embedded as a revolution surface in a Euclidean space \(\mathbb{E}^n\) for any \(n \geq 3\). Indeed, consider a space \(\mathbb{E}^n\) with the metric

\[
dS^2 = dX^2 + dY^2 + \sum_{i=3}^n dZ_i^2. \] (6)

For the surface of revolution \(X = r\cos \phi, Y = r\sin \phi, Z_i = Z_i(r)\) the induced metric is

\[
dl^2 = V(r)dr^2 + r^2d\phi^2, \quad V(r) = 1 + \sum_{i=3}^n (dZ_i/dr)^2. \] (7)

For a regular surface \(V(0) = 1\) and \(V(r) \geq 1\) in the vicinity of \(r = 0\). According to (5) this is impossible when \(K_0 < 0\).

We show now that if \(K_0 < 0\) then a ball-region near the fixed point \(p_0\) of axisymmetric 2D geometry cannot be isometrically embedded in \(\mathbb{E}^3\). Let us assume that such an embedding (not necessarily as a revolution surface) exists. One can choose coordinates \((X^1, X^2, Z)\) in \(\mathbb{E}^3\) so that \(X^1 = X^2 = 0\) at \(p_0\), and in its vicinity

\[
Z = \frac{1}{2}(k_1X^1 + k_2X^2) + \ldots , \] (8)

where \(k_a\) (\(a = 1, 2\)) are principal curvatures at \(p_0\). Here and later ‘dots’ denote omitted higher order terms. The metric on this surface induced by its embedding is

\[
dl^2 = (1 + k_1^2X^1)dx^1dX^1 + (1 + k_2^2X^2)dx^2dX^2
\nonumber
+ 2k_1k_2X^1X^2dX^1dX^2 + \ldots . \] (9)

In the vicinity of \(p_0\) the Killing vector \(\xi\) generating rotations has the form

\[
\xi = p^a\partial_a, \] (10)

where \(p^a\) (\(a = 1, 2\)) are regular functions of \((X^1, X^2)\) vanishing at \((0, 0)\). Their expansion near \(p_0\) has the form

\[
p^a = P^a_{bc}X^bX^c + P^a_{bcd}X^bX^cX^d + \ldots . \] (11)

Consider the Taylor expansion near \(p_0\) of the Killing equation

\[
\xi_{ab} = \xi_{ab} - \Gamma_{ab}^c\xi_c = 0 \] (12)

in the metric (2). Since the expansion of both \(\Gamma_{ab}^c\) and \(\xi_c\) starts with a linear in \(X^a\) terms, the equation (12) can be used to obtain restrictions on the coefficients \(P^a_{bc}, P^a_{bcd}\) and \(P^a_{bcd}\) in (11). Simple calculations give

\[
P^1_1 = P^2_2 = 0, \quad P^1_2 = -P^2_1 = q, \quad P^a_{bc} = P^a_{bcd} = 0, \] (13)

\[
qk_1(k_1 - k_2) = qk_2(k_1 - k_2) = 0. \] (14)

If the Killing vector does not vanish identically then \(q \neq 0\) and the equations (13) imply that \(k_1 = k_2\). This contradicts to the assumption of the existence of the embedding with \(K_0 = k_1k_2 < 0\).

### III. Embedding of a 2D Surface with \(K_0 < 0\) in \(\mathbb{E}^4\)

Increasing the number of dimensions of the flat space from 3 to 4 makes it possible to find an isometric embedding of 2D manifolds with \(K_0 < 0\). Denote by \((X, Y, Z, R)\) Cartesian coordinates in \(\mathbb{E}^4\) and determine the embedding by equations

\[
X = \frac{r}{\Phi_0}\xi(\psi), \quad Y = \frac{r}{\Phi_0}\eta(\psi), \quad Z = \frac{r}{\Phi_0}\zeta(\psi), \] (15)

\[
R = R(r), \] (16)

where \(0 \leq \psi \leq 2\pi, \) and functions \(\xi, \eta, \zeta\) obey the condition

\[
\xi^2(\psi) + \eta^2(\psi) + \zeta^2(\psi) = 1. \] (17)

In other words, \(n = (\xi, \eta, \zeta)\) as a function of \(\psi\) is a line on a unit sphere \(S^2\). We require that this line is a smooth closed loop \((n(0) = n(2\pi))\) without self-intersections. Since a loop on a unit sphere allows continuous deformations preserving its length, there is an ambiguity in the choice of functions \((\xi, \eta, \zeta)\). We denote \(\Phi = (\xi, \eta, \zeta)^{1/2}\) then

\[
2\pi\Phi_0 = \int_0^{2\pi} d\psi\Phi(\psi). \] (18)
is the length of the loop. Instead of the coordinate $\psi$ it is convenient to use a new angle coordinate $\phi$ which is proportional to the proper length of a curve $r = \text{const}$

$$\phi = \Phi_0^{-1} \int_0^\psi d\psi' \Phi(\psi').$$

(19)

The coordinate $\phi$ is a monotonic function of $\psi$ and for $\psi = 0$ and $\psi = 2\pi$ it takes values 0 and $2\pi$, respectively.

Equations (15) give the embedding in $\mathbb{E}^3$ of a linear surface formed by straight lines passing through $r = 0$. This surface has $K = 0$ outside the point $r = 0$ where, in a general case, it has a cone-like singularity with the angle deficit $2\pi(1 - \Phi_0)$.

We shall use the embedding (15)–(16) for the case when the angle deficit is negative. In this case one can use, for example, the following set of functions

$$\xi = \cos \psi / F, \quad \eta = \sin \psi / F, \quad \zeta = a \sin(2\psi) / F,$$

(20)

$$F = \sqrt{1 + a^2 x^2}, \quad x = \sin^2(2\psi).$$

(21)

This embedding for the functions $(\xi, \eta, \zeta)$ defined by (20) is shown in Figure 2.

For this choice

$$\Phi = (1 + 4a^2 - 3a^2 x^2)^{1/2}(a^2 x + 1)^{-1},$$

(22)

$$\Phi_0 = \frac{1}{\pi} \int_0^1 dx (1 + 4a^2 - 3a^2 x^2)^{1/2} / \sqrt{x(1-x)(a^2 x + 1)}.$$  

(23)

Calculations give

$$\Phi_0 = \frac{8}{\pi \sqrt{1 + 4a^2}} [1 + a^2 \Pi(-a^2, k) - 3/4K(k)],$$

(24)

$$k = \sqrt{3a/(1 + 4a^2)}.$$  

(25)

Here $K(k)$ and $\Pi(\nu, k)$ are complete elliptic integrals of the first and third kind, respectively. The function $\Phi_0$ monotonically increases from 1 (at $a = 0$) to 2 (at $a \to \infty$) (see Fig. 3).

The induced metric for the embedded 2D surface defined by (15)–(16) is

$$dl^2 = \Phi_0^{-2} + (dR/dr)^2 dr^2 + r^2 d\phi^2.$$  

(26)

If the angle deficit is positive ($\Phi_0 < 1$), the pole point $r = 0$ in the metric (26) remains a cone singular point for any $R(r)$. For $\Phi > 1$ (the negative angle deficit), the polepoint $r = 0$ in the metric (20) is regular if $(dR/dr)_0^2 = 1 - \Phi_0^2$. By comparing (16) and (20) one obtains

$$r = f^{1/2}, \quad (dR/dr)_0^2 = (V - \Phi_0^2).$$

(27)

This relation gives the following equation relating $R(\mu)$ with $f(\mu)$

$$R' = (1 - f'^2 / (4\Phi_0^2))^{1/2} f^{-1/2}.$$  

(28)

It is easy to check that $R'' = 0$ at points where $f' = 0$. In order $R'$ to be real, the following condition must be valid $\Phi_0 \geq \frac{1}{2} \max_{\mu \in (-1,1)} |f'(\mu)|$. At a point where $|f'|$ reaches its maximum the quantity $f'' = -2K$ vanishes. Thus it is sufficient to require that $\Phi_0$ is greater or equal to the values of $|f'|$ calculated at the points separating regions with the positive and negative Gauss curvature.

IV. EMBEDDING OF THE SURFACE OF THE KERR-NEWMAN HORIZON IN $\mathbb{E}^4$

The surface geometry of the Kerr-Newman black hole is described by the metric $ds^2 = N^2 dl^2$, where

$$dl^2 = (1 - \beta^2 \sin^2 \theta) d\theta^2 + \sin^2 \theta [1 - \beta^2 \sin^2 \theta]^{-1} d\phi^2,$$

(29)

$$N = (r_+^2 + a^2)^{1/2}, \quad \beta = a(r_+^2 + a^2)^{-1/2}.$$  

(30)

Here $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The metric $dl^2$ is normalized so that the area of the surface with this metric is $4\pi$. In the coordinates $\mu = \cos \theta$ the metric (29) takes the form (24) with

$$f(\mu) = (1 - \mu^2)[1 - \beta^2(1 - \mu^2)]^{-1}.$$  

(31)
For the black hole with mass $M$, charge $Q$ and the angular momentum $J = Ma$

$$r_+ = M - (M^2 - a^2 - Q^2)^{1/2}. \quad (32)$$

The rotation parameter $a$ and mass $M$ can be written in terms of the distortion parameter $\beta$ as follows

$$a = \beta N, \quad M = \frac{1}{2} N (1 - \beta^2)^{-1/2} (1 + Q^2/N^2). \quad (33)$$

The condition $M^2 \geq a^2 + Q^2$ for given parameters $N$ and $\beta$ requires that

$$0 \leq Q \leq N (1 - \beta^2)^{1/2}, \quad (34)$$

$$\frac{1}{2} N (1 - \beta^2)^{-1/2} \leq M \leq N (1 - \beta^2)^{1/2}. \quad (35)$$

The distortion parameter has its maximal value $\beta_{\text{max}} = 1/\sqrt{2}$ for $Q = 0$. The Gauss curvature of the surface with the metric (34) is

$$K = [1 - \beta^2 (1 + 3 \mu^2) (1 - \beta^2 (1 - \mu^2))]^{-3}. \quad (36)$$

For $\frac{1}{2} < \beta \leq \frac{1}{\sqrt{2}}$, the Gauss curvature is negative in the vicinity of poles in the region $\mu_c \leq |\mu| \leq 1$

$$\mu_c = (1 - \beta^2)^{1/2} (\sqrt{3} \beta)^{-1}. \quad (37)$$

At $|\mu| = \mu_c$ the Gauss curvature vanishes. As it was shown earlier, at this point $|f'|$ has its maximum

$$|f'|_{\text{max}} = |f'|_{\mu_c} = \frac{3\sqrt{3}}{8\beta(1 - \beta^2)^{3/2}}. \quad (38)$$

and one must choose the parameter $\Phi_0$ so that $\Phi_0 \geq \frac{1}{2} |f'|_{\text{max}}$. Simplest possible choice is

$$\Phi_0 = \frac{1}{2} |f'|_{\mu_c}. \quad (39)$$

Using (39) and integrating the equation (28) one determines $R$ as a function of $\mu$. A plot of this function for $\beta = 0.7$ is shown in Figure 4. Plot 1 at Figure 5 shows $R$ as a function of $r$ for the same values of $\beta$.

The metric (20) can also be written in the form

$$d\ell^2 = [1 + (dR/d\rho)^2]d\rho^2 + \Phi_0^2 \rho^2 d\phi^2, \quad (40)$$

where $\rho = r/\Phi_0$. Plot 2 at Figure 5 shows $R$ as a function of $\rho$ for $\beta = 0.7$. The metric (40) coincides locally with the metric on the revolution surface determined by the equation $R = R(\rho)$ in $E^3$. This does not give a global isometric embedding since the period of the angle coordinate is $2\pi\Phi_0$. This surface can be obtained by gluing two figures shown in Figure 6 along their edges. For the left figure $\phi$ changes from $0$ to $\pi$, while for the right one it changes from $\pi$ to $2\pi\Phi_0$.

V. CONCLUDING REMARKS

We demonstrated that a surface of a rapidly rotating black hole, which cannot be isometrically embedded in $E^3$, allows such a global embedding in $E^4$. To construct this embedding one considers first a $2D$ surface in $E^3$ formed by straight lines passing through one point ($r=0$) which has a cone singularity at $r = 0$ with negative angle deficit. Its Gauss curvature outside $r = 0$ vanishes. Next element of the construction is finding a function $R(r)$. The revolution surface for this function in $E^3$ has a positive angle deficit at $r = 0$. By combining these two maps in such a way that positive and negative angle deficits cancel one another, one obtains a regular global embedding in $E^4$. This construction can easily be used.
to find the embedding in $E^4$ of surfaces of rapidly rotating stationary black holes distorted by an action of external forces or fields, provided the axial symmetry of the spacetime is preserved. An interesting example is a case of a rotating black hole in a homogeneous at infinity magnetic field directed along the axis of the rotation (see e.g. \cite{12, 13, 14}).

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