HOW TO REALIZE LIE ALGEBRAS BY VECTOR FIELDS

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Abstract. An algorithm for embedding finite dimensional Lie algebras into Lie algebras of vector fields (and Lie superalgebras into Lie superalgebras of vector fields) is offered in a way applicable over ground fields of any characteristic. The algorithm is illustrated by reproducing Cartan’s interpretations of the Lie algebra of G(2) as the Lie algebra that preserves certain non-integrable distributions. Similar algorithm and interpretation are applicable to other exceptional simple Lie algebras, as well as to all non-exceptional simple ones and many non-simple ones, and to many Lie superalgebras.

1. Introduction

In memory of Felix Aleksandrovich Berezin

Here I offer an algorithm which explicitly describes how to embed any \( \mathbb{Z} \)-graded Lie algebra (or Lie superalgebra) \( \mathfrak{n} := \bigoplus_{k \geq -d} \mathfrak{n}_k \) such that \( \mathfrak{n}_{-1} \) generates \( \mathfrak{n} := \bigoplus_{k < 0} \mathfrak{n}_k \) and \( \dim \mathfrak{n}_- < \infty \)

\begin{equation}
\mathfrak{n}_{-1} \text{ generates } \mathfrak{n} := \bigoplus_{k < 0} \mathfrak{n}_k \text{ and dim } \mathfrak{n}_- < \infty
\end{equation}

into a Lie algebra (resp., Lie superalgebra) of polynomial vector fields over \( \mathbb{R} \) or \( \mathbb{C} \) or over a field \( K \) of characteristic \( p > 0 \).

For almost a decade, whenever asked, I described the algorithm I propose here but was reluctant to publish it as a research paper: the algorithm is straightforward and was, actually, used more than a century ago by Cartan \( C \), and recently by Yamaguchi \( Y \). For the same reason, the draft of \( ShE \) with some examples of embeddings based on this algorithm was also being put aside and will appear as a sequel to this paper; in the meantime \( La1 \) and \( La2 \) appeared with some more examples\(^2\).

Grozman and Leites convinced me, however, that the algorithm, and its usefulness, were never expressed explicitly. Most convincingly, they used the algorithm not only for interpreting known, but mysterious, simple Lie algebras, and Lie superalgebras, especially in characteristic \( p > 0 \), but in order to get new examples in the absence of classification \( GL \).

So here it is. Grozman already implemented it in his \texttt{SuperLie} package \( Gr \).

Having started to write, I added something new as compared with \( C \): a description by means of differential equations of partial prolongs — subalgebras of the Lie algebras of polynomial vector fields embedded “projective-like”. Such description is particularly important if \( p > 0 \), and for some Lie superalgebras.

\medskip

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\(^1\)Although \( p \) denotes the characteristic of the ground field, parity, and is used as an index, the context is always clear.

\(^2\)I was unable to follow the details of both \( La1 \) and \( La2 \) and hopefully this paper will help to elucidate important realizations of \( La1 \) and \( La2 \).
At the last moment, I learned that, for $p > 0$, Fei and Shen [FSH] proved existence of embeddings I consider and illustrated it with a description of the simple Lie algebras of contact vector fields for $p = 2$. They also formulate questions this paper answers.

For reviews of related to our result realizations of Lie (super)algebras by differential operators (not necessarily first order homogeneous ones), see [BGLS] [M] and [Sh2].

**Problem formulation, facts known, and our reasons.** Let $\mathfrak{n} := \bigoplus_{k = -d}^{-1} \mathfrak{n}_k$, be an $n$-dimensional $\mathbb{Z}$-graded Lie algebra of depth $d > 1$ satisfying \( (\Pi) \). Let $f : \mathfrak{n} \rightarrow \text{vect}(n) = \text{der}K[x_1, \ldots, x_n]$ be an embedding. The image $f(\mathfrak{n})$ is a subspace in the space of vector fields; every vector field can be evaluated at any point; let $f(\mathfrak{n}(0))$ be the span of these evaluations at 0.

**Problem 1.** Embed $\mathfrak{n}$ into $\text{vect}(n)$ so that the $\dim f(\mathfrak{n})(0) = n$.

**Comment.** Roughly speaking, we wish the image of $\mathfrak{n}$ be spanned by all partial derivatives modulo vector fields that vanish at the origin.

Such an embedding determines a non-standard\(^3\) grading of depth $d$ on $\text{vect}(n)$. We will denote $\text{vect}(n)$ considered with this non-standard grading by $\mathfrak{v} = \bigoplus_{k = -d}^{\infty} \mathfrak{v}_k$. Let $\mathfrak{g}_-$ be the image of $\mathfrak{n}$ in $\mathfrak{v}$, i.e., $\mathfrak{g}_- \subset \mathfrak{v}_- := \bigoplus_{k < 0} \mathfrak{v}_k$.

**Problem 2.** Compute the complete algebraic prolong of $\mathfrak{g}_-$, i.e., the maximal subalgebra $(\mathfrak{g}_-)_* = \bigoplus_{k \geq -d} \mathfrak{g}_k \subset \mathfrak{v}$ with the given negative part.

**Problem 3.** Single out partial prolongs of $\mathfrak{g}_-$ in $(\mathfrak{g}_-)_*$. In particular, given not only $\mathfrak{n}$, but $\mathfrak{n}_0 \subset \text{der}_0 \mathfrak{n}$, where the subscript 0 singles out derivations that preserve the $\mathbb{Z}$-grading, we should automatically have an embedding $\mathfrak{n}_0 \subset \mathfrak{g}_0$.

If the inclusion $\mathfrak{n}_0 \subset \mathfrak{g}_0$ is a strict one, we wish to be able to single out $\mathfrak{n}_0$ in $\mathfrak{g}_0$ as well as to single out the algebraic prolong $(\mathfrak{g}_-, \mathfrak{n}_0)_* - \text{the maximal subalgebra of } \mathfrak{v}$ with a given non-positive part — in $(\mathfrak{g}_-)_*$.

If $\mathfrak{n}_0 = \mathfrak{g}_0$ but the component $\mathfrak{g}_1$ forms a reducible $\mathfrak{g}_0$-module with a submodule $\mathfrak{h}_1$, how to single out the maximal subalgebra $(\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{h}_1)_* \subset \mathfrak{v}$ with a given "beginning part" (components of grading $\leq 1$)?

In utmost generality, single out in $\mathfrak{v}$ the maximal subalgebra $\mathfrak{h}_* = \bigoplus_{k \geq -d} \mathfrak{h}_k$ with a given beginning part $\mathfrak{h} = \mathfrak{g}_- \oplus \bigoplus_{0 \leq k \leq K} \mathfrak{h}_k$. Naturally, the beginning part $\mathfrak{h}$ should be compatible with the bracket, i.e., $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{i+j}$ for all $i, j$ such that $i + j \leq K$.

The components $\mathfrak{h}_k$ with $k > K$ are defined recurrently:

$$h_k = \{X \in \mathfrak{g}_k \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{k-1}\}. \quad (2)$$

We prove the inclusion $[\mathfrak{h}_k, \mathfrak{h}_l] \subset \mathfrak{h}_{k+l}$ for all indices by induction on $k + l$ with an appeal to (1); it guarantees that $\mathfrak{h}_* := \mathfrak{g}_- \oplus \bigoplus_{0 \leq k \leq K} \mathfrak{h}_k$ is a subalgebra of $\mathfrak{v}$. The Lie algebra $\mathfrak{h}_* = \bigoplus_{k \geq -d} \mathfrak{h}_k$ is a generalization of Cartan prolong.

**Remark 1.1.** *De facto*, for simple Lie algebras over $\mathbb{R}$ and $\mathbb{C}$, the number $K$ is always $\leq 1$, but if $\text{char} \mathbb{K} > 0$, and for superalgebras, then $K > 1$ is possible.

**Discussion.** If a Lie group $N$ with a Lie algebra $\mathfrak{n}$ is given explicitly, i.e., if we know explicit expressions for the product of the group elements in some coordinates, then there is

\(^3\)The grading deg $x_i = 1$ for all $i$ associated with the $(x)$-adic filtration is said to be *standard*; any other grading is *non-standard*. 
no problem to describe an embedding $n \subset \text{vect}(n)$: the Lie algebras of left- and right-invariant vector fields on $N$ are isomorphic to $n$. (This is, actually, one of the definitions of the Lie algebra of $N$.) If the group $N$ is not explicitly given, then to describe an embedding $n \subset \text{vect}(n)$ is a part of the problem of recovering the Lie group from its Lie algebra (in the cases where one can speak about Lie groups). Of course, the Campbell-Hausdorff formula gives a solution to this problem. Unfortunately, despite its importance in theoretical discussions, the Campbell-Hausdorff formula is not convenient in actual calculations.

For $K = \mathbb{R}$ and $\mathbb{C}$, another method of constructing an embedding $n \subset \text{vect}(n)$ and recovering a Lie group from its Lie algebra is integration of the Maurer-Cartan equations, cf. [DFN]. Although the algorithm I offer does not use a Lie group of $n$ and is applicable even for the cases where no analog of a Lie groups can be offered, it is viewing a given Lie algebra as the Lie algebra of left-invariant vector fields on a Lie group that gives us a key lead.

For other algorithms for embedding $n \subset \text{vect}(n)$, based on explicit descriptions of the $n$-action in $U(n)$, see [BGLS, M]. Now, let me list reasons that lead to the algorithm.

Reason 1. Let $X_1, \ldots, X_n$ be vector fields linearly independent at each point of an $n$-dimensional (super)domain, and

$$\left[ X_i, X_j \right] = \sum_k c^k_{ij} X_k, \quad c^k_{ij} \in K.$$  

Let $\omega^1, \ldots, \omega^n$ be the dual basis of differential 1-forms ($\omega^i(X_j) = \delta^i_j$). Then (a standard exercise)

$$d\omega^k = -\frac{1}{2} \sum_{ij} c^k_{ij} \omega^i \wedge \omega^j = -\sum_{i<j} c^k_{ij} \omega^i \wedge \omega^j,$$

and vise versa: if the 1-forms $\omega^1, \ldots, \omega^n$ satisfy (4) then the dual vector fields $X_1, \ldots, X_n$ satisfy (3).

Observe that, although in the super setting the expression for $d\omega$, i.e.,

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]),$$

acquires some signs, eq. (4) is valid for superalgebras as well: the extra signs in eq. (5) appearing due to super nature of its constituents do not affect (4).

Recall that if the fields $X_i$ form a basis of left-invariant vector fields on the group $N$, eqs. (4) are called the Maurer-Cartan equations; in this case, $c^k_{ij}$ are the structure constants of the Lie algebra $n$.

If $\omega^i = \sum_k V^i_k(x) dx^k$, then eqs. (4) can be expressed as equations for the functions $V^i_k$:

$$\partial_j V^k_i - \partial_i V^k_j = \sum_{p,q} c^k_{ij} V^p_i V^q_j.$$  

Reason 2. In the real or complex situation, eqs. (3) are easy to integrate in “nice” coordinates for any Lie algebra $n$ (not only nilpotent). Namely, introduce functions

$$W^i_j(t, x) = t V^i_j(\exp(tx)), \quad \text{where} \ t \in \mathbb{R}, \ x \in n^*.$$  

(In other words, we should integrate eqs. (3) along one-parameter subgroups.) As is easy to check, the functions $W$ satisfy ODE

$$\frac{dW^i_j}{dt} = \delta^i_j + \sum_{p,q} c^i_{pq} W^p_j x^q$$

with the initial condition $W^i_j(0, x) = 0$.  

Actually, since \( n \) is \( \mathbb{Z} \)-graded nilpotent, the system \((6)\) is so simple that one can integrate it directly, without appealing to auxiliary functions \( W \), and over any ground field. This direct solution of \((6)\) allows us to construct an embedding \( n \to \text{vect}(n) \) most suitable for our purposes\(^4\), and find all possible embeddings.

Namely, select a basis \( B = \{e_1, \ldots, e_n\} \) of \( n \) compatible with the grading. This means that its first \( n_1 \) elements form a basis of \( n_{-1} \), the next \( n_2 \) elements form a basis of \( n_{-2} \), and so on. Let \( I_s \) be the set of indices corresponding to \( n_{-s} \), and \( I = \bigcup I_s \). Let \( c^k_{ij} \) be the structure constants in this basis:

\[
[e_i, e_j] = \sum_k c^k_{ij} e_k, 
\]

and \( x_1, \ldots, x_n \) be the determined by \( B \) coordinates of \( n^* \), the dual space to \( n \). The nonstandard \( \mathbb{Z} \)-grading \( v = \bigoplus_{k \geq -d} v_k \) of \( \text{vect}(n) \) compatible with the \( \mathbb{Z} \)-grading of \( n \) is determined by setting

\[
\deg x_i = s \quad \text{for any} \quad i \in I_s. 
\]

Let \( X_i \in v \) be the image of \( e_i \) under our embedding. Then the value of \( X_i \) at 0 is equal to \( \partial_{x_i} := \partial_i \) and the value of the dual form \( \omega^i \) at 0 is equal to \( dx^i \). If \( i \in I_s \), then the field \( X_i \) and the form \( \omega^i \) are homogeneous of degree \(-s\) and \( s \) respectively. We have

\[
\begin{align*}
\omega^i &= dx^i & \text{for} \quad i \in I_1; \\
\omega^i &= dx^i + \sum_{j,k \in I_1} a^i_{jk} x^j dx^k & \text{for} \quad i \in I_2; \\
\omega^i &= dx^i + \sum_{j \in I_1, k \in I_2} a^i_{jk} x^j dx^k + \\
\sum_{k \in I_3} \left( \sum_{s,t \in I_1} a^i_{stk} x^s x^t + \sum_{s \in I_2} a^i_{sk} x^s \right) dx^k & \text{for} \quad i \in I_3; \\
\end{align*}
\]

The grading guarantees automatic fulfillment of a part of conditions \((6)\): For example, for \( k \in I_1 \), all the functions \( V^k_i \) are known: \( V^k_i = \delta^k_i \); for \( k \in I_2 \), the rhs of \((6)\) only contains the known functions \( (V^k_i \text{ with } k \in I_1) \), and so on.

The system for \( a^i_{jk} \) is highly undetermined but if we are interested in getting some embedding only, we do not need all the solutions; any solution (the simpler looking, the better) will do. Then we proceed in the same way with \( V^k_i \) for \( k \in I_3 \), and so on. The Jacobi identity guarantees the compatibility of the system.

**Reason 3. How the complete prolongations are singled out.** Over \( \mathbb{R} \), the connected simply connected Lie group \( \mathcal{N} \) with Lie algebra \( n \), left-invariant forms \( \omega^i \), where \( i \in I \), and the structure constants \( c^k_{ij} \) given by \((4)\) possess a universal property \((\text{SH})\):

Let \( M \) be a smooth manifold with a collection of linearly independent at each point differential 1-forms \( \alpha^i \), satisfying \((4)\) with the same constants \( c^k_{ij} \). Then, for every point \( x \in M \), there exists its neighborhood \( U \) and a diffeomorphism \( f: U \to \mathcal{N} \) such that

\[
\alpha^i = f^*(\omega^i). 
\]

\(^4\)For example, why even the authors of \([\text{BGLS}]\) were reluctant to use any of the three algorithms presented in \([\text{BGLS}]\)? I tested all of their three algorithms: they work, although some clarifications (see \([\text{Sh2}]\)) are needed. The only explanation I can deduce from the questions Grozman and Leites asked me, is the fact that the formulas in \([\text{BGLS}]\) are fixed, and some of them involve divisions. And what to do if, say, one wants to avoid division (by 2 or 3) in coefficients?! Whereas I give the customer a possibility to select the embedding to taste.
Any two such diffeomorphisms differ by a translation.

Hence, as soon as we have found forms $\omega^i$ satisfying $[\Pi]$, we can think of them as of left-invariant forms of the group $\mathcal{N}$ and of the dual vector fields $X_i$ as of left-invariant vector fields, $\mathfrak{g}_- = \text{Span}\{X_1, \ldots, X_n\} \subset \text{vect}(\mathcal{N})$.

Let $Y_1, \ldots, Y_n$ be the right-invariant vector fields, such that

$$X_i(e) = Y_i(e),$$

and $\theta^1, \ldots, \theta^n$ be the dual right-invariant 1-forms.

Clearly, both $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ span Lie subalgebras of $\mathfrak{v}_-$.

Let us define a right-invariant distribution $\mathcal{D}$ on $\mathcal{N}$ such that $D(e) = \mathfrak{n}_{-1}$. Clearly, $\mathcal{D}$ is singled out by the system of equations for $X \in \text{vect}(n)$:

$$\theta^i(X) = 0 \text{ for any } i \in I_2 \cup I_3 \cup \cdots \cup I_d.$$

Since left- and right-invariant vector fields on a Lie group always commute with each other, each $X_j$ preserves $\mathcal{D}$ and hence the Lie algebra $\mathfrak{g}_-$ preserves $\mathcal{D}$. Moreover, since $\mathfrak{n}$ is $\mathbb{Z}$-graded, it follows that the fact “$X \in \mathfrak{n}_-$ preserves $\mathcal{D}$" is equivalent to the fact “$X$ commutes with all $Y_i$, where $i \in I_1$”, and hence with all $Y_i$, where $i \in I$, since $\mathfrak{n}_{-1}$ generates $\mathfrak{n}$.

Thus, $\mathfrak{g}_-$ is characterized as the maximal subalgebra of $\mathfrak{v}_-$ preserving $\mathcal{D}$. But then the complete prolongation of $\mathfrak{g}_-$ is the maximal subalgebra of $\text{vect}(n)$ preserving $\mathcal{D}$.

Of course we can reformulate all this without appealing to $\mathcal{N}$. All we need is $\text{cent}_{\mathfrak{v}_-}(\mathfrak{g}_-)$, the centralizer of $\mathfrak{g}_-$ in $\mathfrak{v}_-$. It is also clear that, having represented $Y \in \mathfrak{v}_{-s}$ as a sum of homogeneous components in the standard grading ($\deg x^i = 1$):

$$Y = \sum_{p=-1}^{d-s-1} Y_{(p)},$$

we see that for the fields that vanish at the origin (for them, $Y_{(-1)} = 0$), the lowest component of $[X_i, Y]$ coincides with the bracket of the lowest component of $Y$ with $\partial_i$, and therefore is nonzero.

The other way round, for any $Y$ such that $Y_{(-1)} \neq 0$ the equations $[X_i, Y] = 0$, where $i = 1, \ldots, n$, enable us to uniquely recover, consecutively, all the components $Y_{(p)}$ for $p \geq 0$ starting with $Y_{(-1)}$ using the recurrence:

$$[\partial_i, Y_{(p)}] = -\sum_{s=-1}^{p-1} [\partial_i, Y_{(p-1-s)}], Y_{(s)}] \quad \text{for } i = 1, \ldots, n.$$  

Let $Y_i \in \text{cent}_{\mathfrak{v}_-}(\mathfrak{g}_-) \subset \mathfrak{v}_-$ be such that $(Y_i)_{(-1)} = \partial_i$. Then

$$[Y_i, Y_j]_{(-1)} = [\partial_i, (Y_j)_{(0)}] + [(Y_i)_{(0)}, \partial_j] = -[(X_i)_{(0)}, \partial_j] - [\partial_i, (X_j)_{(0)}] =$$

$$-[X_i, X_j]_{(-1)} = -$ \sum_k c^k_{ij}(X_k)_{(-1)} = -$ \sum_k c^k_{ij}(Y_k)_{(-1)},$$

and, since the fields from $\text{cent}_{\mathfrak{v}_-}(\mathfrak{g}_-)$ are uniquely determined by their $(-1)$st components, we get:

$$[Y_i, Y_j] = -$ \sum_k c^k_{ij} Y_k,$$

i.e., $\text{cent}_{\mathfrak{v}_-}(\mathfrak{g}_-)$ is isomorphic to $\mathfrak{n}$.

Let the $\theta^i$ constitute a basis of 1-forms dual to the $\{Y_i\}_{i \in I}$ (i.e., $\theta^i(Y_j) = \delta^i_j$). Then any vector field $X \in \text{vect}(n)$ is of the form

$$X = \sum_i \theta^i(X) Y_i,$$
Since \([X_i, Y_j] = 0\) for any \(i, j = 1, \ldots, n\), we have
\[
\theta^i([X_j, X]) = X_j(\theta^i(X)).
\]

Now let us consider the distribution \(D\) defined by (10). As we have already observed, \(g_-\) is characterized as the maximal subalgebra of \(v_-\) preserving \(D\).

Observe, first of all, that although “any \(X_i\) preserves any form \(\theta^j\)”, the condition in quotation marks does not survive the operation (2) of complete prolongation whereas the condition “preserve \(D\)” is not so strong and survives it.

Indeed, a field \(X \in v\) preserves \(D\) if and only if
\[
\theta^k([X, Y_i]) = 0 \quad \text{for any} \quad i = 1, \ldots, n_1, \text{and any} \quad k > n_1.
\]

Let (14) be valid for any \(X \in g_{s-1}\). Then, due to (2), \(X \in g_s\) if and only if
\[
\theta^k([[X_j, X], Y_i]) = X_j \theta^k([X, Y_i]) = 0 \quad \text{for any} \quad i, j = 1, \ldots, n_1, \text{and} \quad k > n_1.
\]

(We have taken (13) into account.)

Finally, since \(n_{s-1}\) generates the algebra \(n\), (15) is equivalent to
\[
\partial_j(\theta^k([X, Y_i])) = 0 \quad \text{for all} \quad j = 1, \ldots, n.
\]

But if \(k \in I_l\) \((l \geq 2)\), then \(\theta^k([X, Y_i])\) is a homogeneous (in our nonstandard grading) polynomial of degree \(s - 1 + l \geq s + 1 \geq 1\), and hence (16) is equivalent to
\[
\theta^k([X, Y_i]) = 0 \quad \text{for any} \quad i = 1, \ldots, n_1, \text{and} \quad k > n_1,
\]
and hence \(X\) preserves \(D\).

Let us rewrite the system (14) for coordinates of \(X\) more explicitly:
\[
Y_i(\theta^k(X)) - \sum_j (-1)^{p(Y_i)p(\theta^j(X))} c_{ij}^k \theta^j(X) = 0
\]
for any \(i = 1, \ldots, n_1, \text{and} \quad k = n_1 + 1, \ldots, n\).

Since \(g_-\) is \(\mathbb{Z}\)-graded, eqs. (18) are of a particular form. Let
\[
F_i = \theta^{n-nd+i}(X), \quad \text{where} \quad i = 1, \ldots, n_d,
\]
be the coordinates of a vector field \(X\) lying in the component \(g_{-d}\) of maximal depth.

If the functions \(F_i\) are given, then eqs. (18), where \(k \in I_{d-1}\), constitute a system of linear (not differential) equations for the coordinates \(\theta^j(X)\) corresponding to the component \(g_{-d+1}\), and if this component does not contain central elements of the whole algebra \(g_-\), then all the coordinates of the level \(-d + 1\) enter the system. After all these coordinates are determined, eqs. (18), where now \(k \in I_{d-2}\), become a system of linear equations for coordinates on the next level, \(-d + 2\), and so on.

Therefore, the \(F_i\) are generating functions for \(X\). In the general case, one should take for generating functions the functions corresponding to all central basis elements of \(g_-\).

Now, we are able to formulate the algorithm for the first two of our problems.

2. The algorithm: Solving Problems 1 and 2

- In \(n\), take a basis \(B\) compatible with the grading and compute the corresponding structure constants \(c_{ij}^k\).
- Seek the basis of 1-forms \(\{\omega^j\}_{i \in I}\) satisfying (11), i.e., solve system (9) upwards, i.e., starting with degree 1 and proceeding up to degree \(d\).
- Seek the dual basis of vector fields \(\{X_i\}_{i \in I}\) upwards, i.e., starting with degree \(-d\) and proceeding up to degree \(-1\). The fields \(\{X_i\}_{i \in I}\) determine an embedding of \(n\) into \(\text{vect}(n)\).

Problem 1 is solved.
• Seek a basis \( \{Y_i\}_{i \in I} \) of \( \text{cent}_{\mathfrak{g}_-}(\mathfrak{g}_-) \) in \( \mathfrak{v}_- \) by means of \( \{\theta_i\}_{i \in I} \) and the dual basis of 1-forms \( \{\theta^i\}_{i \in I} \).

• To find the component \( \mathfrak{g}_s \) of the complete prolongation of \( \mathfrak{g}_- \), we seek the field \( X \in \mathfrak{g}_s \) in the form \( X = \sum \theta^i(X)Y_i \). For this, we express each of the \( n_d \) generating functions \( F^i = \theta^{n - n_d + i}(X) \) as a sum of monomials of degree \( d + s \) (in the nonstandard grading) with undetermined coefficients and solve the system \( \{\mathbf{I}\} \) of linear homogeneous equations for these coefficients. For debugging, we compare, for \( s < 0 \), the fields thus obtained with the \( X_i \).

Problem 2 is solved.

**Example.** Consider the exceptional Lie algebra \( \mathfrak{g}(2)^5 \) in its \( \mathbb{Z} \)-grading of depth 3, as in \( [\mathbb{C}, \mathfrak{Y}] \). In what follows,

\[
x^{(k)} \text{ denotes } \begin{cases} 
\frac{x^k}{k!} & \text{over } \mathbb{R} \text{ or } \mathbb{C} \\
u^k & \text{in characteristic } p > 0.
\end{cases}
\]

Then (recall that \( \mathfrak{n} \) is the given abstract algebra whose image in the Lie algebra of vector fields is designated by \( \mathfrak{g} \))

\[
\mathfrak{n} = \mathfrak{n}_{-3} \oplus \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}, \quad \text{where } \dim \mathfrak{n}_{-1} = 2, \dim \mathfrak{n}_{-2} = 1, \dim \mathfrak{n}_{-3} = 2.
\]

Let us see how the algorithm works for the embedding \( f(\mathfrak{n}) = \mathfrak{g}_- \subset \mathfrak{vect}(5) \).

1. A basis compatible with the \( \mathbb{Z} \)-grading and structure constants are of the form:

\[
\begin{align*}
\mathfrak{n}_{-1} &= \text{Span}(e_1, e_2), \quad \mathfrak{n}_{-2} = \text{Span}(e_3), \quad \mathfrak{n}_{-3} = \text{Span}(e_4, e_5); \\
[e_1, e_2] &= e_3, \quad [e_1, e_3] = e_4, \quad [e_2, e_3] = e_5; \\
c_{ij}^k &= 0 \text{ for } k = 1, 2; \quad c_{12}^3 = -c_{21}^3 = 1, \quad c_{ij}^k = 0 \text{ otherwise } ; \\
c_{13}^4 &= -c_{31}^4 = 1, \quad c_{ij}^4 = 0 \text{ otherwise}; \\
c_{23}^5 &= -c_{32}^5 = 1, \quad c_{ij}^5 = 0 \text{ otherwise}.
\end{align*}
\]

2. We have

\[
\begin{align*}
\omega^1 &= dx^1, \quad \omega^2 = dx^2 \implies V_i^k = \delta_i^k \text{ for } k = 1, 2 \\
\omega^3 &= dx^3 + \sum_{i,j=1}^2 a_{ij}^3 x^i dx^j \implies V_1^3 = V_2^3 = 0; \quad V_3^3 = 1; \\
V_1^3 &= a_{11}^3 x^1 + a_{21}^3 x^2; \quad V_2^3 = a_{12}^3 x^1 + a_{22}^3 x^2.
\end{align*}
\]

Eqs. \( \{\mathbf{6}\} \) give one non-trivial relation on the \( V_1^3 \):

\[
\partial_2 V_1^3 - \partial_1 V_2^3 = V_1^1 V_2^2 - V_1^2 V_2^1 = 1,
\]

or, equivalently,

\[
(19) \quad a_{21}^3 - a_{12}^3 = 1.
\]

Select a solution which seems to be a simplest one:

\[
a_{21}^3 = a_{12}^3 = 0, \quad a_{21}^3 = 1.
\]

(In canonical coordinates of first kind, \( a_{11}^3 = a_{22}^3 = 0, \ a_{12}^3 = a_{21}^3 = \frac{1}{2}. \) These are most symmetric coordinates. We wish, however, to evade division if possible.) Thus, \( V_1^3 = x^2, \ V_2^3 = 0 \), and hence

\[
\omega^3 = dx^3 + x^2 dx^1.
\]

\(^5\)We denote the exceptional Lie algebras in the same way as the serial ones, like \( \mathfrak{s}(n) \); we thus avoid confusing \( \mathfrak{g}(2) \) with the second component \( \mathfrak{g}_2 \) of a \( \mathbb{Z} \)-graded Lie algebra \( \mathfrak{g} \).
Further, for $k = 4, 5$,

$$\omega^k = dx^k + \sum_{j=1}^{3} V_j^k dx^j,$$

where

$$V_3^k = a_1^k x^1 + a_2^k x^2,$$

$$V_j^k = \alpha_j^k (x^1)^{(2)} + \beta_j^k x^1 x^2 + \gamma_j^k (x^2)^{(2)} + \varepsilon_j^k x^3, \quad j = 1, 2.$$

Eqs. (10) give three nontrivial relation for each function $V_j^k$, where $k = 4, 5, j = 1, 2, 3$:

$$\partial_2 V_1^4 - \partial_1 V_2^4 = 0; \quad \partial_3 V_1^4 - \partial_1 V_3^4 = 1; \quad \partial_3 V_2^4 - \partial_2 V_3^4 = 0;$$

$$\partial_2 V_1^5 - \partial_1 V_2^5 = -x^2; \quad \partial_3 V_1^5 - \partial_1 V_3^5 = 0; \quad \partial_3 V_2^5 - \partial_2 V_3^5 = 1,$$

or, in terms of coefficients:

$$\beta_1^4 x^1 + \gamma_1^4 x^2 = \alpha_2^4 x^1 + \beta_2^4 x^2; \quad \varepsilon_1^4 - a_1^4 = 1; \quad \varepsilon_2^4 - a_2^4 = 0;$$

$$\beta_1^5 x^1 - \gamma_5^5 x^2 - \alpha_2^5 x^1 - \beta_2^5 x^2 = -x^2; \quad \varepsilon_5^5 - a_1^5 = 1; \quad \varepsilon_2^5 - a_2^5 = 1.$$

Select a simpler looking solution:

$$\omega^4 = dx^4 - x^1 dx^3, \quad \omega^5 = dx^5 - x^2 dx^3 - (x^2)^{(2)} dx^1.$$

Finally,

$$\omega^1 = dx^1, \quad \omega^2 = dx^2,$$

$$\omega^3 = dx^3 + x^2 dx^1,$$

$$\omega^4 = dx^4 - x^1 dx^3,$$

$$\omega^5 = dx^5 - x^2 dx^3 - (x^2)^{(2)} dx^1.$$

3. Now seek the dual fields $X_i$:

$$X_5 = \partial_5, \quad X_4 = \partial_4,$$

$$X_3 = \partial_3 + x^1 \partial_4 + x^2 \partial_5,$$

$$X_2 = \partial_2, \quad X_1 = \partial_1 - x^2 \partial_3 - x^1 x^2 \partial_4 - (x^2)^{(2)} \partial_5.$$

We get $\mathfrak{g}_- = \text{Span}\{X_1, \ldots, X_5\}$.

4. Now we seek homogeneous fields $Y_i = \partial_1 + \ldots$, commuting with all the $X_j$. Since the brackets with $X_2, X_4, X_5$ vanish, the coordinates of the $Y_i$ can only depend on $x^1$ and $x^3$. Therefore

$$Y_4 = \partial_4, \quad Y_5 = \partial_5;$$

$$Y_3 = \partial_3 + ax^1 \partial_4 + bx^1 \partial_5,$$

and $[X_1, Y_3] = 0$ implies that $Y_3 = \partial_3$.

Finally, for $i = 1, 2$, we have

$$Y_i = \partial_i + \alpha_i x^1 \partial_3 + \sum_{j=4}^{5} (\beta_i^j (x^1)^{(2)} + \gamma_i^j x^3) \partial_j.$$

Bracketing $Y_i$ with $X_1$ and $X_3$, we get

$$(20)$$

$$Y_1 = \partial_1 + x^3 \partial_4,$$

$$Y_2 = \partial_2 - x^1 \partial_3 - (x^1)^{(2)} \partial_4 + x^3 \partial_5.$$
It only remains to find the forms $\theta^i$ left-dual to $Y_i$. The routine computations yield: $\theta^i = dx^i$ for $i = 1, 2$ and

\begin{align*}
\theta^3 &= dx^3 + x^1 dx^2; \\
\theta^4 &= dx^4 - x^3 dx^1 + (x^1)^{(2)} dx^2; \\
\theta^5 &= dx^5 - x^3 dx^2.
\end{align*}

(21)

5. Now we seek all the vector fields $X$ preserving $\mathcal{D} = \text{Span}\{Y_1, Y_2\}$, or, which is the same, all the fields that belong to the complete prolongation of $\mathfrak{g}_-$. Let $X = \sum f^i Y_i$, where $f^i = \theta^i(X)$. To find the $f^i$, we solve eqs. (18). In our case they are:

\begin{align*}
Y_1(f^4) &= f^3, \quad Y_1(f^5) = 0, \quad Y_2(f^4) = 0, \quad Y_2(f^5) = f^3, \quad Y_1(f^3) = f^2, \quad Y_2(f^3) = -f^1.
\end{align*}

(22)

We see that $X$ is completely determined by the functions $f^4$ and $f^5$ which must satisfy the three relations:

\begin{align*}
Y_1(f^5) &= 0, \quad Y_2(f^4) = 0, \quad Y_1(f^4) = Y_2(f^5).
\end{align*}

For control, let us look what are the corresponding fields in the component $\mathfrak{v}_{-2}$. In this case, both $f^4$ and $f^5$ should be of degree 1 in our grading, i.e., must be of the form $f^i = a^i x^1 + b^i x^2$ for $i = 4, 5$. Then $Y_1(f^i) = a^i, Y_2(f^i) = b^i$, and hence eqs. (22) mean that

\begin{align*}
f^4 &= ax^1, \quad f^5 = ax^2 \implies f^3 = a, f^1 = f^2 = 0.
\end{align*}

Therefore, any field preserving $\mathcal{D}$ and lying in $\mathfrak{v}_{-2}$ is proportional to

\begin{align*}
X = Y_3 + x^1 Y_4 + x^2 Y_5 = \partial_3 + x^1 \partial_4 + x^2 \partial_5 = X_3,
\end{align*}

as should be.

We similarly check that, in $\mathfrak{v}_{-1}$, our equations single out precisely the subspace spanned by $X_1$ and $X_2$.

Now, let us compute $\mathfrak{g}_0$. Its generating functions must be of degree 3 in our grading, i.e., of the form (for $i = 4, 5$)

\begin{align*}
f^i &= a_1^i (x^1)^{(3)} + a_2^i (x^1)^{(2)} x^2 + a_3^i x^1 (x^2)^{(2)} + a_4^i (x^2)^{(3)} + b_1^i x^1 x^3 + b_2^i x^2 x^3 + c_1^i x^4 + c_2^i x^5.
\end{align*}

Then

\begin{align*}
Y_1(f^i) &= a_1^i (x^1)^{(2)} + a_2^i x^1 x^2 + a_3^i (x^2)^{(2)} + (b_1^i + c_1^i) x^3; \\
Y_2(f^i) &= (a_2^i - 2b_1^i - c_1^i) (x^1)^{(2)} + (a_3^i - b_2^i) x^1 x^2 + a_4^i (x^2)^{(2)} + (b_2^i + c_2^i) x^3.
\end{align*}
In this case, eqs. (22) take the following form:

\[
\begin{align*}
    a_1^5 &= a_2^5 = a_3^5 = 0, \\
    b_1^5 + c_1^5 &= 0, \\
    a_4^1 &= 0, \\
    b_2^4 + c_2^4 &= 0, \\
    a_3^4 - b_4^5 &= 0, \\
    a_2^4 - 2b_1^4 + c_1^4 &= 0, \\
    a_1^4 &= a_2^5 - 2b_1^5 - c_1^5, \\
    a_2^5 &= a_3^2 - b_3^2, \\
    a_3^5 &= a_3^5, \\
    b_1^4 + c_1^4 &= b_2^2 + c_2^5.
\end{align*}
\]

The solution to this system is:

\[
\begin{align*}
    a_1^4 &= -b_5^5 = c_1^5 = \alpha, \\
    a_2^4 &= -b_5^5 = \beta, \\
    a_4^4 &= a_4^5 = b_2^1 = -c_2^4 = \gamma, \\
    a_4^5 &= a_1^5 = a_2^5 = a_3^5 = 0, \\
    b_1^4 &= \delta, \\
    c_1^4 &= \beta - 2\delta, \\
    c_2^5 &= 2\beta - \delta.
\end{align*}
\]

Hence

\[
\begin{align*}
    f^4 &= \alpha(x^1)^{(3)} + \beta(x^1)^{(2)}x^2 + \gamma(x^1)^{(2)}x^2 + \delta x^1 x^3 + \gamma x^2 x^3 + (\beta - 2\delta)x^4 - \gamma x^5, \\
    f^5 &= \gamma(x^2)^3 - \alpha x^1 x^3 - \beta x^2 x^3 + \alpha x^4 + (2\beta - \delta)x^5, \\
    f^3 &= \alpha(x^1)^{(2)} + \beta x^1 x^2 + \gamma(x^2)^{(2)} + (\beta - \delta)x^3, \\
    f^2 &= Y_1(f^3) = \alpha x^1 + \beta x^2, \\
    f^1 &= -Y_2(f^3) = -\delta x^1 - \gamma x^2.
\end{align*}
\]

For a basis of \(g_0\), we take the vectors \(X_\alpha, X_\beta, X_\gamma, X_\delta\) corresponding to the only one non-zero parameter (for example, \(X_\alpha\) corresponds to \(\alpha = 1, \beta = \gamma = \delta = 0\) and so on):

\[
\begin{align*}
    X_\alpha &= x^1 Y_2 + (x^1)^{(2)} Y_3 + (x^1)^{(3)} Y_4 + (-x^1 x^3 + x^4) Y_5 = x^1 \partial_2 + x^4 \partial_5 - (x^1)^{(2)} \partial_3 - 2(x^1)^{(2)} \partial_4, \\
    X_\beta &= x^2 \partial_2 + x^3 \partial_3 + x^4 \partial_4 + 2x^5 \partial_5, \\
    X_\gamma &= -x^2 \partial_1 - x^3 \partial_4 + (x^2)^{(2)} \partial_3 + x^1 (x^2)^{(2)} \partial_4 + (x^2)^{(3)} \partial_5, \\
    X_\delta &= -x^1 \partial_1 - x^3 \partial_3 - 2x^4 \partial_4 - x^5 \partial_5.
\end{align*}
\]

For \(\alpha = \gamma = 0, \delta = -\beta = 1\), we get the grading operator

\[
    X = -x^1 \partial_1 - x^2 \partial_2 - 2x^3 \partial_3 - 3x^4 \partial_4 - 3x^5 \partial_5.
\]

The higher components can be calculated in a similar way.
**Interpretation.** There are three realizations of \( \mathfrak{g} = \mathfrak{g}(2) \) as a Lie algebra that preserves a non-integrable distribution on \( \mathfrak{g}_- \) related with the three (incompressible) \( \mathbb{Z} \)-gradings of \( \mathfrak{g} \): with one or both coroots of degree 1. Above we considered the grading \((1,0)\); Cartan used it to give the first interpretation of \( \mathfrak{g}(2) \), then recently discovered by Killing, see [C].

In this realization (by fields \( X_i \)) \( \mathfrak{g} = \mathfrak{g}(2) \) preserves the distribution in the tangent bundle on \( \mathfrak{g}_- \) given by the system of Pfaff equations for vector fields \( X 
\]
\[
\theta^3(X) = 0; \quad \theta^4(X) = 0; \quad \theta^5(X) = 0.
\]
Equivalently, but a bit more economically, we can describe \( \mathfrak{g} = \mathfrak{g}(2) \) as preserving the codistribution in the cotangent bundle on \( \mathfrak{g}_- \) given by the vectors \((20)\), i.e., as the following system of equations for 1-forms \( \alpha \):
\[
\alpha(Y_1) = 0; \quad \alpha(Y_2) = 0.
\]

Obviously, description in terms of codistributions is sometimes shorter: any distribution of codimension \( r \) requires for its description \( r \) Pfaff equations, whereas the dual codistribution requires \( n - r \) equations.

One can similarly describe the remaining realizations of \( \mathfrak{g}(2) \) corresponding to the other \( \mathbb{Z} \)-gradings, various realizations of \( \mathfrak{f}(4) \) and \( \mathfrak{e}(6) - \mathfrak{e}(8) \) and of exceptional Lie superalgebras, as well as Lie algebras over fields of characteristic \( p \). There seemed to be no need to consider nonintegrable distributions associated with various \( \mathbb{Z} \)-gradings of non-exceptional Lie algebras (their usual description as preserving volume or a nondegenerate form seems to be sufficiently clear); Cartan himself, though understood importance of description of Lie algebras in terms of distributions, only considered one or two \( \mathbb{Z} \)-gradings and related distributions of exceptional Lie algebras and none for non-exceptional. If, however, we apply the algorithm presented here to \( \mathfrak{g}(2) \), \( \mathfrak{o}(7) \), \( \mathfrak{sp}(4) \) and \( \mathfrak{sp}(10) \) in characteristic \( p = 2, 3 \) or 5, we elucidate the meaning of some of the simple Lie algebras specific to \( p = 2, 3, 5 \) and, with luck and in the absence of classification, distinguish new examples, as in [GL].

Other gradings of other algebras are now being under consideration.

3. **How to single out partial prolongs: Solving Problem 3**

Thus, we have described the complete prolong of the Lie (super)algebra \( \mathfrak{g}_- \), i.e., as we have already observed, the maximal subalgebra \( \mathfrak{g} = (\mathfrak{g}_-)_* \subset \mathfrak{v} \) with a given negative part.

Let us consider now a subspace \( \mathfrak{h} = \mathfrak{g}_- \oplus \bigoplus_{0 \leq k \leq K} \mathfrak{h}_k \subset \mathfrak{g} \), closed with respect to the bracket within limits of its degrees, i.e., such that \([\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{i+j} \) whenever \( i + j \leq K \). Let us describe the partial prolong \( \mathfrak{h}_* = \bigoplus_{k \geq -d} \mathfrak{h}_k \subset \mathfrak{g} \) of the subspace \( \mathfrak{h} \), i.e., the maximal subalgebra of \( \mathfrak{g} \) with the given beginning part \( \mathfrak{h} \). The components \( \mathfrak{h}_k \) with \( k > K \) are singled out by condition \((2)\). Here by description we mean a way to single out \( \mathfrak{h} \) in \( \mathfrak{g} \) by a system of differential equations.

**Remark 3.1.** Observe that the Cartan prolong \((\mathfrak{g}_-, \mathfrak{g}_0)_* \) (where \( \mathfrak{g}_- \) is commutative, the depth is \( d = 1 \), and \( \mathfrak{g}_0 \subset \mathfrak{gl}(n) \)) is a particular case of the above construction with \( \mathfrak{g} = \text{Vect}(n) \), and \( \mathfrak{h} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \). For examples of descriptions of Cartan prolongations by means of differential equations, see [Sh14], [ShFA], and [ShP].

The homogeneous component \( \mathfrak{h}_m \) of \( \mathfrak{h} \) is said to be defining, if \( \mathfrak{h}_k = \mathfrak{g}_k \) for all \( k < m \) but \( \mathfrak{h}_m \neq \mathfrak{g}_m \). Let us consider an algorithm of description of \( \mathfrak{h}_* \) in the case where the defining component is of the maximal degree — \( \mathfrak{h}_K \). The case of defining component of smaller
degree \( m < K \) can be reduced to our case; indeed, we first describe the partial prolong \( h^*_m = \bigoplus_{0 \leq k \leq m} h_k \), compare the components \( h_k \), where \( m < k \leq K \) with the corresponding components of this prolong \( h^*_m \), find out the new defining component, if any, and so on.

Thus, let \( Z_1, \ldots, Z_{\dim h_K} \) be a basis of the defining component \( h_K \subset g_K \).

The first thing to do is to single out the subspace \( h_K \) in \( g_K \) by means of a system of linear (algebraic) equations (i.e., find out a basis of the annihilator of \( h_K \) in \( (g_K)^* \), or, equivalently, find out the fundamental system of solutions \( \alpha^1, \ldots, \alpha^r \) of the system of equations for an unknown 1-form \( \alpha \in (g_K)^* \):

\[
(23) \quad \alpha(Z_i) = 0 \text{ for all } i = 1, \ldots, \dim h_K.
\]

The subspace \( h_K \) is then singled out by a system of homogeneous linear equations for an unknown vector field \( X \in g_K \):

\[
(24) \quad \alpha^l(X) = 0 \text{ for all } i = 1, \ldots, r.
\]

Observe now that in \( g_K \) there is a convenient for us basis consisting of the fields of the form \( fY_j \), where \( f \) is a monomial of degree \( K + s \) if \( j \in I_s \). Accordingly, the dual basis consists of the elements of degree \( -K \) and of the form

\[
A^i_{j_1, \ldots, j_t} = S(Y_{i_1} \ldots Y_{i_t}) \theta^j,
\]

and the forms \( \alpha^l \) can be expressed in this basis as

\[
(25) \quad \alpha^l = \sum a^{l i_1, \ldots, i_t} A^i_{j_1, \ldots, j_t}.
\]

Substituting (23) into (24) we get a system of homogeneous linear differential equations with constant coefficients for the coordinates of the vector field \( X = \theta^i(X)Y_i \in h_K \):

\[
(26) \quad \sum a^{l i_1, \ldots, i_t} S(Y_{i_1} \ldots Y_{i_t}) \theta^j(X) \text{ for all } l = 1, \ldots, r.
\]

Observe now that, for Lie algebras, equation (26) survive prolongation procedure (2). Indeed, for \( k > K \), by the induction hypothesis \( X = \theta^i(X)Y_i \in h_k \) if and only if the brackets \([X_i, X]\) satisfy (26) for any \( i = 1, \ldots, n_1 \). Set

\[
f^l = \sum a^{l i_1, \ldots, i_t} S(Y_{i_1} \ldots Y_{i_t}) \theta^j(X).
\]

Since all the \( X_i \) commute with all the \( Y_j \), the system (26) for the brackets \([X_i, X]\) is equivalent to the system

\[
(27) \quad X_i(f^l) = 0 \text{ for all } i = 1, \ldots, n_1 \text{ and } l = 1, \ldots, r,
\]

which thanks to (1) is, in its turn, equivalent to the system

\[
\partial_i(f^l) = 0 \text{ for all } i = 1, \ldots, n \text{ and } l = 1, \ldots, r.
\]

This implies that \( f^l = \text{const} \) for all \( l = 1, \ldots, r \). Since the functions \( f^l \) are homogeneous polynomials of degree \( k - K > 0 \), it follows that \( f^l = 0 \). Hence, \( X \in h_k \) if and only if \( X \) satisfies system (26).

In super case the fields \( X_i \) and \( Y_j \) supercommute, not commute, and this does not allow us, generally speaking, break out the \( X_i \) and pass from system (26) for the brackets to the system (27). There is, however, a simple and well-know consideration that saves us. Recall that \( p \) denotes the parity function and \( Pty \) is the parity operator, i.e.,

\[
Pty(x) = (-1)^{p(x)} x.
\]

**Lemma 3.2.** Let \( X, Y \in \text{End} V \) supercommute and \( p(Y) = \overline{1} \). Then \( X \) and \( \hat{Y} = Y Pty \) commute (in the usual sense), i.e., \( X\hat{Y} = \hat{Y}X \).
Indeed,

\[ XY' (v) = XYPty(v) = (-1)^{p(v)} XY(v) = \]
\[ (-1)^{p(v)} (-1)^{p(X)p(Y)} YX(v) = (-1)^{p(v)+p(X)} YX(v) = \hat{Y}X(v). \]

Therefore, in the super case, the system (26) should be written with operators \( \hat{Y}_i \) instead of \( Y_i \) if \( p(Y_i) = 0 \), we set \( \hat{Y}_i = Y_i \).

Finally, if \( d > 1 \), then any field \( X \in \mathfrak{g} \) is completely determined by its generating functions \( F^i \). Therefore, it suffices to write equations (26) for the generating functions only.

**Examples: Depth 1.** Let \( \mathfrak{g}_- = \mathfrak{g}_{-1} \) be commutative, hence \( \mathfrak{g} = (\mathfrak{g}_-)_* = \mathfrak{vect}(n) \) in the standard \( \mathbb{Z} \)-grading (the degree of each indeterminate is equal to 1). Let \( \mathfrak{g}_0 = \mathfrak{gl}(n) = \mathfrak{vect}(n)_0 \). The degree 1 component \( \mathfrak{vect}(n)_1 \) consists, as is well-known, of the two irreducible \( \mathfrak{gl}(n) \)-modules. Let

\[
(28) \quad X = \sum_i a_{ij}^k x^i x^j \partial_k := \sum f^k(x) \partial_k.
\]

Then these submodules are:

\[
(29) \quad \mathfrak{h}_{1(1)} := \text{Span} \{ x^i \partial_j | i = 1, \ldots, n \}
\]

and

\[
(30) \quad \mathfrak{h}_{1(2)} := \text{Span} \{ X = \sum_{j=1}^n d_{ij}^k x^i x^j \partial_k | d_{ii}^i + \sum d_{ij}^i = 0 \text{ for } i = 1, \ldots, n. \}.
\]

Let us single out the partial prolongs \( \mathfrak{h}_{i(i)} = (\mathfrak{g}_{-1} \oplus \mathfrak{gl}(n) \oplus \mathfrak{h}_{1(i)})_* \), where \( i = 1, 2 \) in \( \mathfrak{vect}(n) \) by means of differential equations on the functions \( f^k(x) \), see (28). In this case \( X_i = Y_i = \partial_i \).

The conditions on \( \mathfrak{h}_{1(2)} \) can be immediately expressed as

\[
\sum_{j} \partial_i \partial_j (f^j) = 0 \quad \text{for all } i = 1, \ldots, n
\]

or, equivalently, as

\[
(31) \quad \partial_i \left( \sum_{j} \frac{\partial f^j}{\partial x^j} \right) = 0 \quad \text{for all } i = 1, \ldots, n.
\]

This is exactly the system (26) for \( \mathfrak{h}_{s(2)} \) which can be rewritten in a well-known way:

\[
\sum_{j} \frac{\partial f^j}{\partial x^j} = \text{div} X = \text{const}.
\]

Hence, as is well-known, \( \mathfrak{h}_{s(2)} = \mathfrak{dsvect}(n) := \mathfrak{vect}(n) \in \mathbb{K}E \), where \( E = \sum x^i \partial_i \).

Now let us consider \( \mathfrak{h}_{1(1)} \) (which is, of course, \( \mathfrak{sl}(n+1) \) embedded into \( \mathfrak{vect}(n) \)). Having expressed \( X \in \mathfrak{h}_{1(1)} \) as

\[
(c_1 x^1)^2 + c_2 x^1 x^2 + \cdots + c_n x^1 x^n)\partial_1 + \cdots + (c_1 x^1 x^n + c_2 x^2 x^n + \cdots + c_n x^n)^2)\partial_n
\]

we immediately see that \( d_{ij}^k = 0 \) if \( i \neq k \) and \( j \neq k \), and \( d_{kk}^k = d_{ki}^k \) for any \( i \neq k \). The corresponding system of differential equations is

\[
(32) \quad \frac{\partial^2 f^k}{\partial x^i \partial x^j} = 0 \quad \text{for } i, j \neq k;
\]

\[
\frac{1}{2} \frac{\partial^2 f^i}{\partial x^k} = \frac{\partial^2 f^i}{\partial x^i \partial x^k} \quad \text{for } i \neq k.
\]
Superization. For superalgebras, as we have seen, one should take compositions of $Y_i = \partial_t$ with the parity operators, i.e., instead of the $\partial_i$ we should take operators

$$\nabla_i(f) := (-1)^{p(f)p(\partial_i)}\partial_i(f).$$

These $\nabla_i$ commute (not supercommute) with any operator $X_j = \partial_j$ from $\mathfrak{g}_{-1}$. The system \ref{31} will take form

$$\nabla_i \left( \sum_j \nabla_j (f^j) \right) = 0 \text{ for all } i = 1, \ldots, n,$$

which yields, nevertheless, the same condition $\text{div} X = \text{const}$. (This is one more way to see why the coordinate expression of divergence in the super case must contain some signs: eqs. \ref{31} do not survive the prolongation procedure \ref{2}.)

Having in mind that

$$d^k_{ij} = -(-1)^{p(f^k)}\frac{\partial^2 f^k}{\partial x^i \partial x^j}; \quad \text{so } d^k_{kk} = 0 \text{ for } x^k \text{ odd}$$

we deduce that the second line in \ref{32} takes the following form

$$\frac{1}{2} \frac{\partial^2 f^k}{(\partial x^k)^2} = (-1)^{p(x^i)p(f^j)+1} \frac{\partial^2 f^i}{\partial x^i \partial x^k} \text{ for } p(x^i) = 0 \text{ and } i \neq k;$$

$$( -1)^{p(x^i)p(f^j)} \frac{\partial^2 f^i}{\partial x^i \partial x^k} = (-1)^{p(x^i)p(f^j)} \frac{\partial^2 f^i}{\partial x^i \partial x^k} \text{ for } p(x^i) = \overline{1} \text{ and } i, j \neq k.$$

Depth $> 1$ We consider several more or less well-known examples and a new one (as).

Let $n = n_{-1} \oplus n_{-2}$ be the Heisenberg Lie algebra: $\dim n_{-1} = 2n$, $\dim n_{-2} = 1$. The complete prolong of $n$ is the Lie algebra $\mathfrak{e}(2n+1)$ of contact vector fields. Having embedded $n$ into $\text{vect}(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$ with the grading $\text{deg} p_i = \text{deg} q_i = 1$ for all $i$ and $\text{deg} t = 2$, we can take for the $X$-vectors, for example,

$$X_{q_i} = \partial_{q_i}, \quad X_{p_i} = \partial_{p_i}, \quad X_t = \partial_t.$$

Hence $\mathfrak{g}_{-} = \text{Span}\{X_{p_1}, \ldots, X_{p_n}, X_{q_1}, \ldots, X_{q_n}, X_t\}$ and the contact vector fields in consideration preserve the distribution $\mathcal{D}$ given by the Pfaff equation $\alpha(X) = 0$ for vector fields $X$, where $\alpha = dt + \sum_i (p_i dq_i - q_i dp_i)$.

The $Y$-vectors in this case are of the form

$$Y_{q_i} = \partial_{q_i}, \quad Y_{p_i} = \partial_{p_i}, \quad Y_t = \partial_t.$$

In this particular example, a contact vector field $K$ is determined by only one generating function $F$ which is exactly the coefficient of $Y_t$ in the decomposition of $K$ with respect to the $Y$-basis and there are no restrictions on the function $F$. Denoting $F = 2f$ and solving eqs. \ref{18}, we get the formula for any contact vector field $K_f$:

$$K_f = 2f Y_t + \sum_i (-Y_{q_i}(f)Y_{p_i} + Y_{p_i}(f)Y_{q_i}) = (2 - E)(f)\partial_t + \frac{\partial f}{\partial t}E + \sum_i \left( \frac{\partial f}{\partial p_i} \partial_{q_i} - \frac{\partial f}{\partial q_i} \partial_{p_i} \right),$$

where $E = \sum_i (p_i \partial_{p_i} + q_i \partial_{q_i})$. Of course, this is exactly the standard formula of the contact vector field with the generating function $f$. Further we use the realization of $\mathfrak{e}(2n+1)$ in generation functions $f$.

If $\mathfrak{h}_0 \neq \mathfrak{e}_0$, then $\mathfrak{h}_0$ is the defining component. The component $\mathfrak{e}_0$ is generated by 2nd order homogeneous polynomials in $p, q, t$. Thus, for a basis $Z$ in $\mathfrak{e}_0$ we can take monomials $p_ip_j$,
The Poisson algebra. This equation singles out in \( k \) and, respectively, spanned by \( Y \) while eqs. (36) imply deg \( W \) for eqs. (36) imply deg \( W \) and \( h \) and \( q \).

Analogously, \( Y(f) = 0 \), and hence \( Y(f) = 0 \), and hence \( Y(f) = 0 \), which means \( f = ct + f_0 \) while eqs. (36) imply deg \( f_0 \leq 1 \). Hence the prolong of \( g_\ast \oplus h_0 \) coincides with \( g_\ast \oplus h_0 \).

Let \( h_0 = t_0 \) and \( h_1 \subset t_1 \). As a \( t_0 \)-module, \( t_1 \) decomposes into the direct sum of two (over \( \mathbb{C} \); for char \( \mathbb{K} = 3 \) and in super setting, even over \( \mathbb{C} \), the situation is more involved) irreducible submodules, \( W_1 \) spanned by cubic monomials in \( p \) and \( q \), and \( W_2 \) spanned by \( tp_i \) and \( tq_i \).

The dual bases of \( W_1 \) and \( W_2 \) are given by order 3 symmetric polynomials in the \( Y_{p_i}, Y_{q_i} \), and, respectively, spanned by \( Y_{p_i}, Y_{t} \) and \( Y_{q_i} Y_{t} \).

Hence the subspace \( W_1 \) is singled out by conditions

\[
Y_{p_i} Y_{t}(f) = Y_{q_i} Y_{t}(f) = 0 \implies Y_{t}(f) = \text{const.}
\]

This equation singles out in \( t(2n + 1) \) the derivation algebra \( \mathfrak{der}(\mathfrak{so}(2n)) = \mathfrak{so}(2n) \oplus \mathbb{C} K_t \) of the Poisson algebra.

To single out \( W_2 \), we have the system

\[
\begin{align*}
Y_{p_i} Y_{p_j}(f) = 0,
Y_{p_i} Y_{q_j}(f) = 0, \\
Y_{p_i} Y_{p_j} Y_{q_k}(f) = 0,
Y_{p_i} Y_{q_j} Y_{q_k}(f) = 0, \\
Y_{p_i} Y_{p_j} Y_{q_j}(f) = 0, \\
Y_{p_i} Y_{p_j} Y_{q_k}(f) = 0,
\end{align*}
\]

which implies that \( Y_{t}(f) \) satisfies eqs. (36), and hence

\[
\frac{\partial f}{\partial t} \in \text{Span}\{1; p_1, \ldots, p_n, q_1, \ldots, q_n; t\},
\]

whereas

\[
f \in \text{Span}\{1; p_1, \ldots, p_n, q_1, \ldots, q_n; t, \; tp_1, \ldots, tp_n, \; tq_1, \ldots, tq_n; \; t^2\}.
\]

Hence the complete prolongation \( \mathfrak{f}(\mathfrak{e}_1(1|6))_p \) is isomorphic to \( \mathfrak{sp}(2n + 2) \).

Let \( \mathfrak{n} = \mathfrak{n}_{-1} \oplus \mathfrak{n}_{-2} \) be the Heisenberg Lie superalgebra \( \mathfrak{hei}(1|6) \): \( \dim \mathfrak{n}_{-1} = 0|6, \dim \mathfrak{n}_{-2} = 1|0 \). The complete prolong of \( \mathfrak{n} \) is \( \mathfrak{g} = \mathfrak{f}(1|6) \) with \( \mathfrak{g}_0 = \mathfrak{co}(6) \). The component \( \mathfrak{g}_1 \) consists of three irreducible \( \mathfrak{g}_0 \)-modules.
If we consider $\mathfrak{f}(1|6)$ in realization by generating functions in $t, \theta_1, \ldots, \theta_6$, i.e., when
\[
K_f = (2 - E)(f)\partial_t + \frac{\partial f}{\partial t} E - (-1)^p(f) \sum_i \frac{\partial f}{\partial \theta_i} \partial_{\theta_i}, \quad f \in \mathbb{C}[t, \theta_1, \ldots, \theta_6],
\]
where $E = \sum_i \theta_i \partial_{\theta_i}$ and $\{\theta_i, \theta_j\}_{k,b} = \delta_{ij}$, then $\mathfrak{g}_1 \simeq t\Lambda(\theta) \oplus \Lambda^3(\theta) = t\Lambda(\theta) \oplus \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$ with $\mathfrak{g}_1^\pm \subset \Lambda^3(\theta)$ singled out with the help of the Hodge star $^*$:
\[
(38) \quad \mathfrak{g}_1^\pm = \{f \in \Lambda^3(\theta) \mid f^* = \pm \sqrt{-1}f\}.
\]
Recall that the Hodge star $^*$ is just the Fourier transformation in odd indeterminates whereas $t$ is considered a parameter:
\[
\ast : f(\xi,t) \mapsto f^*(\eta,t) = \int \exp(\sum \eta_i \xi_i) f(\xi,t) \text{vol}(\xi).
\]

The exceptional simple Lie superalgebra $\mathfrak{h}$ is defined as a partial prolong of $\mathfrak{h} = \bigoplus_{k=-2}^1 \mathfrak{h}_k$, where $\mathfrak{h}_k = \mathfrak{f}(1|6)_k$ for $-2 \leq k \leq 0$ and where $\mathfrak{h}_1 = t\Lambda(\theta) \oplus \mathfrak{g}_1^+$. Hence $\mathfrak{h}_1$ is the defining component.

Then
\[
X_i = \partial_{\theta_i} + \theta_i \partial_t \text{ for } i = 1, \ldots, 6, \quad X_7 = \partial_t,
\]
and
\[
Y_i = \partial_{\theta_i} - \theta_i \partial_t \text{ for } i = 1, \ldots, 6, \quad Y_7 = X_7.
\]

Let $I = \{i_1, i_2, i_3\} \subset \{1, \ldots, 6\}$ be an ordered subset of indices, and $I^* = \{j_1, j_2, j_3\}$ the dual subset of indices (i.e., $\{I, I^*\}$ is an even permutation of $\{1, \ldots, 6\}$). Set:
\[
Y_I = Y_{i_1}Y_{i_2}Y_{i_3}, \quad Y_{I^*} = Y_{j_1}Y_{j_2}Y_{j_3},
\]
and define $\Delta_{Y_I} : \mathbb{C}[t, \theta] \longrightarrow \mathbb{C}[t, \theta]$ by the eqs.
\[
\Delta_{Y_I}(f) = (-1)^p(f)Y_I(f).
\]

Observe that $\Delta_{Y_I}(t\theta_s) = 0$ for any $s = 1, \ldots, 6$. Therefore $\mathfrak{h}_1$ can be singled out in $\mathfrak{f}(1|6)$ by the following 10 equations parameterized by partitions $(I, I^*)$ of $\{1, \ldots, 6\}$ constituting even permutations:
\[
(39) \quad (\Delta_{Y_I} - \sqrt{-1}\Delta_{Y_{I^*}})(f) = 0.
\]
Clearly, (39) is equivalent to
\[
(Y_I - \sqrt{-1}Y_{I^*})(f) = 0.
\]
The solutions of this system span the following subspace of the space of generating functions:
\[
\begin{align*}
& f(t) - \sqrt{-1}f''(t)1^*, \\
& f_j(t)\theta_j - \sqrt{-1}f_j'(t)\theta_j^* , \\
& f_{jk}(t)\theta_j\theta_k - \sqrt{-1}f_{jk}'(t)(\theta_j\theta_k)^*, \\
& f_{jkl}(t)\left(\theta_j\theta_k\theta_l - \sqrt{-1}(\theta_j\theta_k\theta_l)^*\right).
\end{align*}
\]
In (40), $j, k, l$ are distinct indices 1 to 6.
The above description is in agreement with equations from [CK6].
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