Active Growth and Pattern Formation in Membrane-Protein Systems

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Inspired by recent experimental observations of patterning at the membrane of a living cell, we propose a generic model for the dynamics of a fluctuating interface driven by particle-like inclusions which stimulate its growth. We find that the coupling between interfacial and inclusions dynamics yields microphase separation and the self-organisation of travelling waves. These patterns are strikingly similar to those detected in experiments on biological membranes. Our results further show that the active growth kinetics do not fall into the Kardar-Parisi-Zhang universality class for growing interfaces, displaying instead a novel superposition of scaling and sustained oscillations.

Active membranes and interfaces have revealed fascinating complex pattern formation and nontrivial dynamical features \(^1\). An interface is termed active when its dynamics violate detailed balance due to the presence of local non-thermal forces. A paradigmatic example is that of the plasma membrane of an eukaryotic cell, which is driven far from equilibrium by its constant interaction with ion channels, membrane proteins, and the actin cytoskeletal network \(^2\), all of which are intimately coupled to the membrane fluctuations \(^2\).

Recent experiments have unveiled a wide variety of organised dynamical structures formed within the plasma membrane of crawling cells. Membrane-binding proteins such as GTPases of the Rho and Ras families, for instance, form dynamic nanoclusters \(^3\), \(^4\), \(^5\), while ripples develop on the membrane itself and surf as a travelling wave \(^6\), \(^7\), \(^8\). A generic picture accounting for the emergence of all these structures is still lacking. Could the fact that the proteins activate growth be the underlying cause of such a complex scenario?

In this work we explore this possibility by introducing a minimal non-equilibrium model for pattern formation in a system of active inclusions embedded in an active interface. The feedback between such particle-like inclusions and the interfacial dynamics, rooted in experimental observations, assumes the membrane motion to be regulated by transmembrane proteins \(^9\), which, in turn, are coupled to the membrane local shape \(^10\). The mechanism we identify relies on activity alone, and dispenses with the need for nonlinear biochemistry as invoked previously in models assuming an underlying activator-inhibitor dynamics \(^1\), \(^12\). We also stress that the mechanism requires no assumption on the polar patterns which may be formed by the underlying actin cortex \(^10\); all that is required is polymerisation normal to the surface. Besides being relevant to pattern formation on eukaryotic membranes, our model extends the problem of semi-autonomous systems, such as randomly advected passive scalar fields \(^13\) or passive sliders on fluctuating interfaces \(^14\), \(^15\), into the active matter realm.

First, we show that the inclusion-interface coupling provides a generic route to patterning along with microphase separation and waves. This is a general result, which does not depend on fine tuning of model parameters. We further provide a simple theory, based on the analysis of shock and rarefaction waves, which, on the one hand, correctly predicts the numerically observed scaling of cluster size and wave velocity with the model parameters, and, on the other hand, reveals the intimate connection between clustering, waves and the underlying motion of the interface. Importantly, the feedback requires noise to be effective, as only damped waves survive in a mean-field deterministic framework (see e.g. \(^19\)). Second, our work suggests that an actively growing interface cannot be described by the Kardar-Parisi-Zhang (KPZ) equation \(^20\), which successfully represents the universal features of the passive case. Instead, we find non-trivial sustained oscillations in the roughening dynamics, which could be the key signature to look for in future experiments with active membranes.

As the leading edge of a crawling cell is an essentially 1D object, we model the fluctuating interface as a directed random walk in (1+1)-D (Fig. 1). The dynamics, as in standard models of stochastic growth, entail only local single-step moves \(^21\). Pictorially, the interface comprises positive and negative slopes / and \ join-

![FIG. 1. Schematics of our active interface (black solid line) - inclusions (red circles) model. Dashed lines denotes the moves defining our interface updating rule. The detailed-balance breaking action of the inclusions enhances the growth rate (thicker upward arrow) and hampers the reverse move (thinner downward arrow) proportionally to the local number of inclusions.](image-url)
ing \( L \) sites. Each downward kink \( \vee \) transforms into an upward one \( \wedge \) (and vice versa), at rate \( p_+ (p_-) \) (see Fig. 1). The inclusions in the interface break detailed balance by stimulating interface growth (i.e., biasing its motion towards the top in Fig. 1). This is inspired by the upregulation of actin polymerisation due to Rho GTPases such as Rac1 and Cdc42 \[13\]. Note, however, that we are not considering any specific function (except growth stimulation) or shape for the inclusions, as done, for instance, in \[6\] with asymmetric pumps. We set

\[
p_{\pm} = p(1 \pm \lambda n_i)
\]

where \( n_i \) is the number of inclusions at the \( i \)-th site. With this choice growth is favoured on occupied sites, as \( p_+ - p_- \propto \lambda n_i \), and we can control its strength by varying \( \lambda \geq 0 \). Setting \( \lambda \neq 0 \) is the key ingredient that makes our interface active, and our problem different from the semiautonomous systems cited above.

Additionally, \( N \) inclusions diffuse and are advected by the interfacial slope, mimicking the coupling of protein transport to local surface curvature observed for several membrane-binding proteins \[14, 19\]—we consider a “cuvophobic” coupling, where proteins tend to drift towards regions of negative curvature. Each inclusion jumps independently left or right with rates \( q_+ \) and \( q_- \),

\[
q_{\pm} = q \left(1 \pm \frac{\gamma}{2} \nabla h_i\right),
\]

where \( \nabla h_i = (h_{i+1} - h_{i-1})/a \), so that \( \gamma \) measures the strength of the slope-mediated advection. We highlight here that the feedback between inclusions and interface dynamics is realised only when both \( \lambda \) and \( \gamma \) are greater than 0, making \( \gamma \) a key ingredient of our model. We set the unbiased rates \( p \) and \( q \) (obtained when \( \gamma = \lambda = 0 \)) to 1, implying comparable timescales of inclusion and interface dynamics, and the global particle concentration to 1 (i.e. \( N = L \)). As explained in the SM, the specific values of such parameters do not alter the physics of the system, unless pushed to extreme values. This set of update rules, augmented with periodic boundary conditions, leads to stochastic dynamics for the active interface-inclusions system.

Fig. 2 shows typical snapshots of the interface profile and inclusion distribution as a function of time, when \( \gamma \) and \( \lambda \) are both strictly positive (for more details of simulation methods see the supplemental material (SM) \[22\]). Initially, the surface is flat and inclusions are uniformly distributed (bottom snapshot). Later on, the interface roughens and inclusions accumulate in valleys (centre and top snapshots). They do so since \( \gamma > 0 \) favours advection towards regions with negative curvature (valleys). Interestingly, clusters are also strongly affected by \( \lambda \). In the \( \lambda \rightarrow 0 \) limit, our model reduces to the passive problem considered in \[22\], where particles slide on an equilibrium fluctuating interface. In this limit the density fluctuations grow in time so as to reach a steady state scaling with the system size \( L \) as \( L^{0.6} \) (Fig. 2 inset), consistent with numerical predictions \[23\]. Notably, as soon as active growth is turned on (\( \lambda > 0 \)), we find a completely different scenario, in which the steady-state density fluctuations no longer scale with \( L \) (Fig. 2 inset).

It is also useful to compare our results to those obtained in \[17, 18, 24\], where particles slide either on a KPZ or equilibrium interface. The former case corresponds to the limit \( \lambda n_i \rightarrow \lambda \) of Eq. (1), which removes the local concentration dependence in the interface dynamics. In both these passive cases, inclusions aggregate in interface valleys in a fluctuating fashion due to the noise-induced flipping of valleys. As a result of this phase separation, the steady-state density fluctuations scale as a power of the system size \( L \). Conversely, the absence of scaling we observe means that the clusters reach a self-limiting size. In other words, the active growth term \( \lambda n_i \) leads to noisy microphase (rather than macrophase) separation. The mechanism underlying cluster formation is that advection promotes particle congregation in valleys. The clustering cannot proceed indefinitely, however, as inclusions stimulate the growth of a local “bump” in the interface, which eventually drives them away, arresting coarsening. The higher \( \lambda \) is, the sooner we expect the cluster to disperse, and the smaller its size: this is what we find numerically. Intriguingly, clustering requires thermal fluctuations; a mean-field deterministic description of our model leads to an advancing flat inter-
of shallow water waves. Using a standard procedure (see SM), one may derive the following stochastic partial differential equations for the coarse-grained inclusion density field \( n(x,t) \) and interface height \( h(x,t) \)

\[
\partial_t n = \gamma \partial_x (n \partial_x h) + a \partial_x^2 n + \xi_c,
\]

\[
\partial_t h = \lambda n \left[ 1 - (\partial_x h)^2 \right] + a \partial_x^2 h + \eta,
\]

where \( a \) is the lattice spacing of the microscopic model. Both \( \xi_c \) and \( \eta \) in Eq. are Gaussian, whereas \( \xi_c \) is the divergence of a random current, so as to ensure conservation of the number of inclusions. Note that these equations can be deduced on general symmetry grounds, at the price of losing the relation between the coefficients and the microscopic model parameters \( \lambda \). The active terms in the height equation are those controlled by the inclusion density \( n(x,t) \), at variance with early models such as \( \xi \) where activity enters as coloured noise.

Clusters and travelling waves emerge as shock solutions in the inviscid limit \( (a \to 0) \) of the deterministic version of equations \( \Xi \). By introducing the slope variable \( u = \partial_x h \), Eq. \( \Xi \) acquires the structure of a hyperbolic set of conservation laws \( \Xi \).

\[
\partial_t \left( \frac{n}{u} \right) + \partial_x \left( \frac{-\gamma n u}{-\lambda n} \right) = \partial_t v + \partial_x f(v) = 0,
\]

where we introduced a vectorial notation and further neglected the KPZ non-linearity, so as to highlight that our patterns are generated by activity alone. We will show that neglecting the KPZ term gives reasonable results, although its relevance for other aspects of the model is an open question. We call \( F \) the matrix with elements \( F_{\mu \nu} = \partial f_\mu / \partial v_\nu \), and \( \zeta_\mu \), \( \mathbf{r}_\mu \) its \( v \)-dependent eigenvalues and corresponding right eigenvectors \( (\mu, \nu = 1, 2) \). For each positive value of \( \gamma \) and \( \lambda \), \( F \) obeys the genuine non-linearity condition \( \partial x \zeta_\mu \mathbf{r}_\mu > 0 \) \( \Xi \). As a consequence, Eq. \( \Xi \) admits rarefaction fan and shockwave solutions in the whole \( \lambda, \gamma > 0 \) range of parameters. Such solutions can be explicitly obtained by studying the corresponding Riemann problem, i.e. Eq. \( \Xi \) on an infinite domain with a Heaviside-function initial condition \( \mathbf{v} = \mathbf{v}_l \) for \( x < 0 \), \( \mathbf{v}_r \) for \( x > 0 \), then using the outcomes as building blocks for the full problem. In a shockwave, for instance, the initial discontinuity travels ballistically with a fixed speed \( \sigma \) depending on initial state, as well as \( \gamma \) and \( \lambda \).

Two conditions are required for a shockwave to develop. One is the Rankine-Hugoniot condition relating the wave speed to the currents across the shock front,

\[
\sigma[|\mathbf{v}|] = ([f(\mathbf{v})]),
\]

where \([\mathbf{v}]\) denotes the size of the discontinuity across the shock. The other is the requirement that the interfacial slope on the right of the shockwave is higher than that on the left, as shockwaves arise within valleys. A representative case is the evolution of a valley uniformly filled
The system size. If \( \lambda = 0 \), the interface dynamics decouples from the inclusions and its width grows as in the Edwards-Wilkinson (EW) model, with \( \beta = 1/4 \) and \( \alpha = 1/2 \). If \( \lambda \neq 0 \), and the protein density is uniform, the dynamics is described by the KPZ scaling, \( \beta = 1/3 \) and \( \alpha = 1/2 \). Intriguingly, the growth of our active in-
cluster which, in turn, produces a kinematic wave due to feedback between inclusion and interface dynamics. Furthermore, we found the active interface roughening to consist of a scale-invariant component, well described in 1D by Edward-Wilkinson exponents, but with superposed oscillations, whose features are determined by the collective behaviour of the system components. Our theory predicts scaling relations for the features of the kinematic waves and microphase separation which can be experimentally checked, provided an estimate of $\lambda$ and $\gamma$ can be made. To what extent such features are retained in 2D is a question to be addressed in future work. Preliminary results of an extension of our stochastic dynamics to 2D (see SM for details) do suggest the occurrence of microphase separation. However, 2D affords a variety of extensions of the model due to the tensorial nature of curvature and it would be of interest to explore further the different possibilities.

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