UNSTEADY NON-NEWTONIAN FLUID FLOWS WITH BOUNDARY CONDITIONS OF FRICTION TYPE: THE CASE OF SHEAR THINNING FLUIDS.

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ABSTRACT. Following the previous part of our study on unsteady non-Newtonian fluid flows with boundary conditions of friction type we consider in this paper the case of pseudo-plastic (shear thinning) fluids. The problem is described by a $p$-Laplacian non-stationary Stokes system with $p < 2$ and we assume that the fluid is subjected to mixed boundary conditions, namely non-homogeneous Dirichlet boundary conditions on a part of the boundary and a slip fluid-solid interface law of friction type on another part of the boundary. Hence the fluid velocity should belong to a subspace of $L^p(0,T; (W^{1,p} (\Omega))^3)$, where $\Omega$ is the flow domain and $T > 0$, and satisfy a non-linear parabolic variational inequality. In order to solve this problem we introduce first a vanishing viscosity technique which allows us to consider an auxiliary problem formulated in $L^{p'}(0,T; (W^{1,p'}(\Omega))^3)$ with $p'$ the conjugate number of $p$ and to use the existence results already established in $[4]$. Then we apply both compactness arguments and a fixed point method to prove the existence of a solution to our original fluid flow problem.

1. DESCRIPTION OF THE PROBLEM

Following the previous part of our study on unsteady general incompressible fluid flows with boundary conditions of friction type ($[4]$), we focus in this paper on the case of pseudo-plastic (or shear thinning) fluids like molten polymers. More precisely we consider non-Newtonian fluids satisfying the following power law

$$\sigma = 2\mu(\theta, v, |D(v)|)|D(v)|^{p-2}D(v) - \pi \text{Id}_{\mathbb{R}^3}$$

with $p \in (1, 2)$, where $\mu$ is a given mapping, $v$ is the fluid velocity, $\pi$ is the pressure, $\theta$ is the temperature, $\sigma$ is the stress tensor and $D(v) = \left(d_{ij}(v)\right)_{1 \leq i,j \leq 3}$ is the strain rate tensor given by

$$d_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad 1 \leq i,j \leq 3.$$

The problem is then described by the conservation of mass and momentum i.e.

$$\begin{cases}
\frac{\partial v}{\partial t} - 2\text{div}(\mu(\theta, v, |D(v)|) |D(v)|^{p-2}D(v)) + \nabla \pi = f \quad \text{in } (0, T) \times \Omega \\
\text{div}(v) = 0 \quad \text{in } (0, T) \times \Omega
\end{cases}$$

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where $f$ represents the vector of external forces, $(0, T)$ is the time interval $(T > 0)$ and $\Omega \subset \mathbb{R}^3$ is the fluid flow domain. The evolution of the temperature $\theta$ is described by the heat equation which is fully decoupled from the flow problem when the convection term is neglected. Thus $\theta$ appears as a data in (1.2).

Since the 90’s several experimental studies (see for instance [2, 8, 21, 16, 20, 24]) have shown that shear thinning fluids may exhibit complex behaviour at the boundary like non-linear threshold slip phenomena of friction type, which can be described by

$$
\nu_n = 0, \quad \nu_\tau - s \in -\partial \psi_{B_R^3(0,k)}(\sigma_\tau)
$$

where $k$ is a given positive threshold, $\partial \psi_{B_R^3(0,k)}$ is the subdifferential of the indicator function of the closed ball $B_R^3(0,k)$, $s$ is the sliding velocity of the wall, $\nu_n, \nu_\tau$ and $\sigma_\tau$ are the normal component of the velocity, the tangential component of the velocity and the shear stress respectively. This kind of boundary condition, reminiscent of Tresca’s friction for solids ([7]), had been introduced by H.Fujita in [9], leading to an abundant literature in the case of steady or unsteady Newtonian fluid flows (see for instance [10, 11, 12, 22, 13, 14, 17, 23, 18, 19, 25, 5, 6]). The case of non-Newtonian fluids satisfying a power law with $p \neq 2$ yields a very different mathematical framework since both the velocity and the pressure are expected to belong to Banach spaces depending on $p$ and a first result is given in [3] for steady flows.

Motivated by lubrication and extrusion / injection applications, we consider

$$
\Omega = \{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega, 0 < x_3 < h(x')\}
$$

where $\omega$ is a non-empty bounded domain of $\mathbb{R}^2$ with a Lipschitz continuous boundary and $h$ is a Lipschitz continuous function which is bounded from above and from below by some positive real numbers.

We denote by $u \cdot v$ the Euclidean inner product of two vectors $u$ and $v$ in $\mathbb{R}^3$. We decompose the boundary of $\Omega$ as $\partial \Omega = \Gamma_0 \cup \Gamma_L \cup \Gamma_1$ with

$$
\Gamma_0 = \{(x', x_3) \in \Omega : x_3 = 0\}, \quad \Gamma_1 = \{(x', x_3) \in \Omega : x_3 = h(x')\}
$$

and $\Gamma_L$ the lateral part of $\partial \Omega$. Let $n = (n_1, n_2, n_3)$ be the unit outward normal vector to $\partial \Omega$ and $g : \partial \Omega \to \mathbb{R}^3$ such that

$$
\int_{\partial \Omega} g \cdot n \, dY = 0, \quad g = 0 \text{ on } \Gamma_1, \quad g \neq 0 \text{ on } \Gamma_L, \quad g \cdot n = 0 \text{ on } \Gamma_0.
$$

We define the normal and tangential velocities on $\partial \Omega$ as $\nu_n = v \cdot n$ and $\nu_\tau = v - \nu_n n$ and the normal and tangential components of the stress vector are given by

$$
\sigma_n = \sum_{i,j=1}^{3} \sigma_{ij} n_i n_j, \quad \sigma_\tau = \left( \sum_{j=1}^{3} \sigma_{ij} n_j - \sigma_n n_i \right)_{1 \leq i \leq 3}.
$$

We assume that the fluid is subjected to non-homogeneous Dirichlet boundary conditions on $\Gamma_1 \cup \Gamma_L$ and friction boundary conditions on $\Gamma_0$, i.e.

$$
(1.3) \quad v = 0 \text{ on } (0, T) \times \Gamma_1, \quad v = g \xi \text{ on } (0, T) \times \Gamma_L
$$

where $\xi$ is a function depending only on the time variable such that

$$
(1.4) \quad \xi(0) = 1
$$
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and

\[ \nu_n = 0 \text{ on } (0, T) \times \Gamma_0 \quad \text{(slip condition)} \]

while \( \nu_\tau \) is unknown on \( \Gamma_0 \) and satisfies Tresca’s friction law, i.e.

\[
\begin{align*}
|\sigma_\tau| = k & \Rightarrow \exists \lambda \geq 0 \quad \nu_\tau = s - \lambda \sigma_\tau \\
|\sigma_\tau| < k & \Rightarrow \nu_\tau = s
\end{align*}
\]

on \((0, T) \times \Gamma_0\)

where \( s \) is the sliding velocity of the lower part of the boundary and \( k \) is the positive friction threshold.

We complete the description of the problem with the initial condition

\[ \nu(0) = \nu_0 \text{ in } \Omega \]

and we assume that \( \nu_0 \in \left( W^{1,p}(\Omega) \right)^3 \) such that

\[ \text{div}(\nu_0) = 0 \text{ in } \Omega, \quad \nu_0 = 0 \text{ on } \Gamma_1, \quad \nu_0 = g \text{ on } \Gamma_L, \quad \nu_0 \cdot n = 0 \text{ on } \Gamma_0. \]

At a first glance this problem seems very similar to the case \( p \geq 2 \) already treated in [4]. Unfortunately, when \( p \in (1, 2) \), we have to deal with additional mathematical difficulties. Indeed we expect the fluid velocity to take its values in \( \left( W^{1,p}(\Omega) \right)^3 \). Since we consider an evolution problem we need a Gelfand triplet with some pivot space between \( \left( W^{1,p}(\Omega) \right)^3 \) and its dual. Moreover the proof strategy in [4] relies on a finite difference approximation of the evolution term and monotonicity methods to solve the corresponding elliptic inequalities in a functional framework of the form \( K^p \subset H = H^\prime \subset (K^p)^\prime \) where \( K^p \) and \( H \) are two Hilbert spaces such that the unknown velocity \( \nu = \nu_0 \xi \) belongs to \( K^p \) and \( K^p \) is a subset of \( L^p \left( 0, T; \left( W^{1,p}(\Omega) \right)^3 \right) \).

Of course when \( p \in (1, 2) \) we can not obtain such embeddings. In order to overcome this difficulty we will apply a vanishing viscosity technique, namely we will consider a sequence of approximate problems where the stress tensor is now given by

\[ \sigma^\varepsilon = \sigma + 2\varepsilon |D(\nu)|^{p'-2}D(\nu) \]

where \( 0 < \varepsilon << 1 \) and \( p' = \frac{p}{p-1} > 2 \) is the conjugate number of \( p \).

The paper is organized as follows. In Section 2 we describe the mathematical framework and we derive the variational formulation of the problem. In Section 3 we introduce an auxiliary flow problem where the two first arguments of the mapping \( \mu \) are given data and we consider the modified constitutive law given by (1.9). For any \( \varepsilon > 0 \) the existence results obtained in [4] can be applied leading to a sequence of approximate solutions and we establish a priori estimates. Then we pass to the limit as \( \varepsilon \) tends to zero by using monotonicity properties and we obtain the existence and uniqueness of a solution to the auxiliary flow problem. Finally in Section 4 we apply a fixed point technique to prove the existence of a solution to our original problem.
2. Mathematical framework

We adopt the same notations as in [4]. Throughout this paper we will denote by \( X \) the functional space \( X^3 \). For all \( p > 1 \) we introduce the following subspaces of \( W^{1,p}(\Omega) \):

\[
V_0^p = \{ \varphi \in W^{1,p}(\Omega); \ \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L \text{ and } \varphi \cdot n = 0 \text{ on } \Gamma_0 \}
\]

and

\[
V_{0, \text{div}}^p = \{ \varphi \in V_0^p; \ \text{div}(\varphi) = 0 \text{ in } \Omega \}, \quad V_{\Gamma_1}^p = \{ \varphi \in W^{1,p}(\Omega); \ \varphi = 0 \text{ on } \Gamma_1 \}
\]

endowed with the norm

\[
\| v \|_{1,p} = \left( \int_{\Omega} |\nabla v|^p \, dx \right)^{\frac{1}{p}}.
\]

We observe that the mapping \( z \mapsto z^p \) is convex on \( \mathbb{R}_+^3 \), thus

\[
(2.1) \quad \left( \int_{\Omega} |D(u)|^p \, dx \right)^{1/p} = \| D(u) \|_{(L^p(\Omega))^{3\times 3}} \leq \| u \|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega).
\]

and, with Korn’s inequality ([26]), there exists \( C_{\text{Korn},p} > 0 \) such that

\[
(2.2) \quad \left( \int_{\Omega} |D(u)|^p \, dx \right)^{1/p} = \| D(u) \|_{(L^p(\Omega))^{3\times 3}} \geq C_{\text{Korn},p} \| u \|_{1,p} \quad \forall u \in V_{\Gamma_1}^p.
\]

Let \( \mathcal{Y} = \{ \psi \in L^2(\Omega); \ \text{div}(\psi) \in L^2(\Omega) \} \) endowed with its canonical norm

\[
\| \psi \|_{\mathcal{Y}} = \left( \| \psi \|^2_{L^2(\Omega)} + \| \text{div}(\psi) \|^2_{L^2(\Omega)} \right)^{1/2} \quad \forall \psi \in \mathcal{Y}
\]

and let \( H \) its subspace given by \( H = \{ \psi \in L^2(\Omega); \ \text{div}(\psi) = 0 \text{ in } \Omega, \ \psi \cdot n = 0 \text{ on } \partial \Omega \} \).

Starting from the constitutive power law (1.1) we introduce the mapping \( \mathcal{F} : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3} \) given by

\[
\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) = \begin{cases} 
2\mu(\lambda_0, \lambda_1, |\lambda_2|)|\lambda_2|^{p-2}\lambda_2 & \text{if } \lambda_2 \neq 0_{\mathbb{R}^{3\times 3}}, \\
0_{\mathbb{R}^{3\times 3}} & \text{otherwise}
\end{cases}
\]

and we assume that

\[
\begin{align*}
(2.3) \quad & (o, e, d) \mapsto \mu(o, e, d) \quad \text{is continuous on } \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+, \\
(2.4) \quad & d \mapsto \mu(\ldots, d) \quad \text{is monotone increasing on } \mathbb{R}_+, \\
(2.5) \quad & \text{there exists } (\mu_0, \mu_1) \in \mathbb{R}^2 \text{ such that } \\
& \quad \quad 0 < \mu_0 \leq \mu(o, e, d) \leq \mu_1 \quad \text{for all } (o, e, d) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+.
\end{align*}
\]

With (2.5) we have immediately

\[
(2.6) \quad |\mathcal{F}(\lambda_0, \lambda_1, \lambda_2)| \leq 2\mu_1|\lambda_2|^{p-1} \quad \forall (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}.
\]

Let \( q > 1 \) such that \( q - p + 1 > 0 \). Then for any \( \theta \in L^{\tilde{q}}(0, T; L^{\tilde{p}}(\Omega)) \) with \( \tilde{q} \geq 1 \) and \( \tilde{p} \geq 1 \), and for any \( u \in L^q(0, T; W^{1,p}(\Omega)) \) we have

\[
\mathcal{F}(\theta, u + v_0 \xi, D(u + v_0 \xi)) \in \mathcal{L}_{\tilde{p}'}(0, T; (L^{p'}(\Omega))^{3\times 3})
\]

where \( p' = \frac{p}{p-1} \) is the conjugate number of \( p \) and we may define the integral term

\[
\int_0^T \int_{\Omega} \mathcal{F}(\theta, u + v_0 \xi, D(u + v_0 \xi)) : D(\overline{\varphi}) \, dx \, dt \quad \text{for all } \overline{\varphi} \in L^{\frac{q}{q-1}}(0, T; W^{1,p}(\Omega)).
\]
Since \( L^q(0, T; V_{0,p}^\theta) \subset L^{\frac{q}{r}}(0, T; V_{0,p}^\theta) \) if and only if \( q \geq p \), we consider the operator \( A : L^q(0, T; V_{0,p}^\theta) \to (L^q(0, T; V_{0,p}^\theta))' \) defined by

\[
[A(u), \varphi] = \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0 \xi, D(u + v_0 \xi)) : D(\varphi) \, dx \, dt \quad \forall \varphi \in L^q(0, T; V_{0,p}^\theta)
\]

with \( q \geq p \), where \([\ldots]\) denotes the duality product between \( L^q(0, T; V_{0,p}^\theta) \) and its dual \((L^q(0, T; V_{0,p}^\theta))'\).

With (2.6) we have

\[
(2.7) \quad \|A(u)(L^q(0,T;V_{0,p}^\theta))'\| \leq 2\mu_1 \|u + v_0 \xi\|_{L^q(0,T;V_{1,p}^\theta)}^{p-1} \quad \forall u \in L^q(0, T; V_{0,p}^\theta)
\]

and thus \( A \) is a bounded operator. Furthermore

\[
[A(u), u - \tilde{u}] \geq 2(C_{\text{Korn,p}})^p \mu_0 \|u\|_{L^p(0,T;V_{0,p}^\theta)} - \|v_0 \xi\|_{L^p(0,T;V_{1,p}^\theta)} \bigg\| \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0 \xi, D(u + v_0 \xi)) : D(\varphi) \, dx \, dt \quad \forall \varphi \in L^q(0, T; V_{0,p}^\theta)
\]

for any \((u, \tilde{u}) \in (L^q(0, T; V_{0,p}^\theta))^2\) and we can not infer that \( A \) is coercive unless \( p = q \).

In such a case (i.e. \( p = q \)) we may expect \( \frac{\partial \varphi}{\partial t} \in L^p(0, T; (V_{0,\text{div}}^p)') \). Moreover the embedding of \( V_{0,\text{div}}^p \) into \( H \) is continuous and dense if and only if \( p \geq \frac{6}{5} \) and the functional space

\[
\{ \varphi \in L^p(0, T; V_{0,\text{div}}^p) : \frac{\partial \varphi}{\partial t} \in L^p(0, T; (V_{0,\text{div}}^p)') \}
\]

is continuously embedded into \( C^0([0,T]; H) \). Hence we will consider from now on \( q = p \geq \frac{6}{5} \).

In order to deal with homogeneous boundary conditions on \((0, T) \times (\Gamma_1 \cup \Gamma_L)\), we set \( \varphi = v - v_0 \xi \). The variational formulation of the problem (1.1), (1.2) with the boundary conditions (1.3), (1.6) and the initial condition (1.7) is given by

**Problem (P)** Let \( f \in L^p(0, T; L^2(\Omega)) \), \( k \in L^p(0, T; L^2(\Gamma_0)) \), \( \mu \) satisfying (2.3)-(2.5), \( \theta \in L^2(0, T; L^2(\Omega)) \) with \( \tilde{q} \geq 1 \) and \( \tilde{p} \geq 1 \), \( s \in L^p(0, T; L^p(\Gamma_0)) \), \( \xi \in W^{1,p}(0, T) \) satisfying (1.4) and \( v_0 \in W^{1,p}(\Omega) \) satisfying (1.8). Find \( \varphi \in C([0,T]; L^2(\Omega)) \cap L^p(0, T; V_{0,\text{div}}^p) \) with \( \frac{\partial \varphi}{\partial t} \in L^p(0, T; (V_{0,\text{div}}^p)') \) and \( \pi \in H^{-1}(0, T; L_{\text{div}}^2(\Omega)) \) satisfying the following parabolic variational inequality

\[
\langle \frac{\partial}{\partial t} \varphi, \tilde{\varphi} \rangle_{L^2(\Omega)} + \int_0^T \int_\Omega \mathcal{F}(\theta, \varphi + v_0 \xi, D(\varphi + v_0 \xi)) : D(\tilde{\varphi}) \, dx \, dt \quad - \langle \int_\Omega \pi \, \text{div}(\tilde{\varphi}) \, dx, \zeta \rangle_{D'(0,T),D(0,T)} + J(\varphi + \tilde{\varphi} - J(\varphi))
\]

\[
\geq \int_0^T \int_\Omega \left( f + \frac{\partial \xi}{\partial t} v_0, \tilde{\varphi} \right) \zeta \, dt \quad \forall \tilde{\varphi} \in V_{0,p}, \forall \zeta \in D(0,T)
\]

and the initial condition

\[
\varphi(0) = v_0 - v_0 \xi(0) = 0 \quad \text{in } \Omega
\]
where

\[
J : \begin{cases} 
L^p(0,T;V_0^p) \to \mathbb{R} \\
\nabla \to \int_0^T \int_{\Gamma_0} k|\nabla - \bar{s}| \, dx \, dt,
\end{cases}
\]

and \( \langle . \rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \) and \( \langle . \rangle_{L^2(\Omega)} \) denote respectively the duality product between \( \mathcal{D}(0,T) \) and \( \mathcal{D}'(0,T) \) and the inner product in \( L^2(\Omega) \).

3. Auxiliary flow problem

We consider in this section an auxiliary problem where the mapping \( \mu \) depends on the modulus of the strain tensor while its two other arguments are given data. More precisely let \( u \in L^p(0,T;L^p(\Omega)) \) be given. We consider the following auxiliary flow problem

**Problem (P\(_u\))** Let \( f \in L^{q'}(0,T;L^2(\Omega)) \), \( k \in L^p(0,T;L^p(\Gamma_0)) \), \( \mu \) satisfying (2.3)-(2.5), \( \theta \in L^q(0,T;L^q(\Omega)) \) with \( q \geq 1 \) and \( \tilde{p} \geq 1 \), \( s \in L^p(0,T;L^p(\Gamma_0)) \), \( \xi \in W^{1,q'}(0,T) \) satisfying (1.3) and \( v_0 \in W^{1,q}(\Omega) \) satisfying (1.8). Find \( \nabla \in C([0,T];L^2(\Omega)) \cap L^p(0,T;V_0^p) \) with \( \frac{\partial \nabla}{\partial t} \in L^{q'}(0,T;\{V_0^p,\text{div}\}) \) and \( \pi \in H^{-1}(0,T;L^p(\Omega)) \) satisfying the following parabolic variational inequality

\[
\begin{align*}
\left\langle \frac{\partial \nabla}{\partial t}, \partial \theta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} + & \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0 \xi, D(\nabla + v_0 \xi)) : D(\partial \theta) \, \xi \, dx \, dt \\
- \left\langle \int_\Omega \pi \text{div}(\partial \theta), \xi \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} + & \int_\Omega \pi \text{div}(\partial \theta) \, \xi \, dx + J(\nabla) + J(\partial \theta) - J(\nabla) \\
\geq & \int_0^T \left( f + \frac{\partial \xi}{\partial t} v_0, \partial \theta \right)_{L^2(\Omega)} \, \xi \, dt \quad \forall \theta \in V_0^p, \forall \xi \in \mathcal{D}(0,T)
\end{align*}
\]

and the initial condition

\[
\nabla(0) = v_0 - v_0 \xi(0) = 0 \quad \text{in } \Omega.
\]

By Lemma 1 in [3] we know that the mapping \( \lambda_2 \mapsto \mathcal{F}_p(\lambda_1,\lambda_2) \) is monotone in \( \mathbb{R}^3 \times \mathbb{R}^3 \) for any \( p > 1 \). So the mathematical framework seems the same as in Section 3 in [4] where we consider a similar problem in the case \( p \geq 2 \). Unfortunately the space \( L^p(0,T;V_0^p) \) is not embedded into its dual when \( p \in [6/5,2) \) and some key properties of the semi-group of contractions \( (S(h))_{h \geq 0} \) used in [4] (see Proposition 3.1 in [4]), are not any more satisfied. So we can not reproduce the same proof strategy as in [4].

In order to overcome this difficulty we will apply a vanishing viscosity technique and we introduce a perturbed constitutive law given by

\[
\sigma^\varepsilon = 2\mu(\theta, u + v_0 \xi, D(\nabla + v_0 \xi)) |D(\nabla + v_0 \xi)|^{p-2} D(\nabla + v_0 \xi) + 2\varepsilon |D(\nabla)|^{p-2} D(\nabla) - \pi \text{Id}_{\mathbb{R}^3}
\]
where we recall that \( p' = \frac{p}{p - 1} > 2 \) is the conjugate number of \( p \). Hence we consider the following approximate variational inequality

\[
\left\langle \frac{\partial}{\partial t}(\tau_\varepsilon, \tilde{\phi})_{L^2(\Omega)}, \zeta \right\rangle_{D'(0,T), D(0,T)} + \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0, D(\tau_\varepsilon + v_0 \xi)) : D(\tilde{\phi}) \zeta \, dx \, dt + 2\varepsilon \int_0^T \int_\Omega \left| D(\tau_\varepsilon) \right|^{p'} - 2 D(\tau_\varepsilon) : D(\tilde{\phi}) \zeta \, dx \, dt - \left\langle \int_\Omega \pi_\varepsilon \text{div}(\tilde{\phi}) \, dx, \zeta \right\rangle_{D'(0,T), D(0,T)} + J(\tau_\varepsilon + \tilde{\phi}) - J(\tau_\varepsilon) \geq \int_0^T \left( f + \frac{\partial \xi}{\partial t} v_0, \tilde{\phi} \right)_{L^2(\Omega)} \zeta \, dt \quad \forall \tilde{\phi} \in V_0', \, \forall \zeta \in D(0,T).
\]

Let \( u \in L^{p'}(0,T; V_{0,\text{div}}^{p'}) \). For the sake of notational simplicity let us define \( A_\varepsilon^u(u) \) by

\[
\left[ [A_\varepsilon^u(u), \varphi] \right] = \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0, D(u + v_0 \xi)) : D(\varphi) \, dx \, dt + 2\varepsilon \int_0^T \int_\Omega \left| D(u) \right|^{p'-2} D(u) : D(\varphi) \, dx \, dt \quad \forall \varphi \in L^{p'}(0,T; V_{0,\text{div}}^{p'})
\]

where \([\cdot, \cdot]\) denotes the duality product between \( L^{p'}(0,T; V_{0,\text{div}}^{p'}) \) and its dual \( (L^p(0,T; V_{0,\text{div}}^p))' \). By observing that \( L^{p'}(0,T; V_{0,\text{div}}^{p'}) \subset L^p(0,T; V_{0,\text{div}}^p) \) we obtain immediately with (2.7) and (2.8) that \( A_\varepsilon^u \) is bounded and coercive on \( L^{p'}(0,T; V_{0,\text{div}}^{p'}) \).

Moreover (2.3) implies that \( A_\varepsilon^u \) is hemicontinuous and using (2.3)-(2.4) and Lemma 1 in [3] we obtain that \( A_\varepsilon^u \) is monotone. Furthermore the mapping \( \tilde{s} \mapsto \int_0^T \int_{\Gamma_0} k|\tilde{s} - \hat{s}| \, dx \, dt \) is convex and Lipschitz continuous on \( L^{p'}(0,T; V_{0,\text{div}}^{p'}) \) and fits that we can apply the existence result obtained in [4] and have the

**Proposition 3.1.** Let \( f \in L^{p'}(0,T; L^2(\Omega)), k \in L^{p'}(0,T; L^p_+(\Gamma_0)), \mu \) satisfying (2.3)-(2.5), \( \theta \in L^q(0,T; L^\tilde{q}(\Gamma_0)) \) with \( \tilde{q} \geq 1 \) and \( \bar{p} \geq 1, s \in L^p(0,T; L^p(\Gamma_0)), \xi \in W^{1,p'}(0,T) \) satisfying (1.4) and \( v_0 \in W^{1,p'}(0,T) \) satisfying (1.8). For any \( \varepsilon > 0 \) there exists \( \tau_\varepsilon \in C([0,T]; L^2(\Omega)) \cap L^{p'}(0,T; V_{0,\text{div}}^{p'}) \) with \( \frac{\partial \tau_\varepsilon}{\partial t} \in L^p(0,T; (V_{0,\text{div}}^{p'})) \) and \( \pi_\varepsilon \in H^{-1}(0,T; L^p_+(\Omega)) \) satisfying the following parabolic variational inequality

\[
\left\langle \frac{\partial}{\partial t}(\tau_\varepsilon, \tilde{\phi})_{L^2(\Omega)}, \zeta \right\rangle_{D'(0,T), D(0,T)} + \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0, D(\tau_\varepsilon + v_0 \xi)) : D(\tilde{\phi}) \zeta \, dx \, dt + 2\varepsilon \int_0^T \int_\Omega \left| D(\tau_\varepsilon) \right|^{p'} - 2 D(\tau_\varepsilon) : D(\tilde{\phi}) \zeta \, dx \, dt - \left\langle \int_\Omega \pi_\varepsilon \text{div}(\tilde{\phi}) \, dx, \zeta \right\rangle_{D'(0,T), D(0,T)} + J(\tau_\varepsilon + \tilde{\phi}) - J(\tau_\varepsilon) \geq \int_0^T \left( f + \frac{\partial \xi}{\partial t} v_0, \tilde{\phi} \right)_{L^2(\Omega)} \zeta \, dt \quad \forall \tilde{\phi} \in V_0', \, \forall \zeta \in D(0,T)
\]

and the initial condition

\[
\tau_\varepsilon(0) = v_0 - v_0 \xi(0) = 0 \quad \text{in } \Omega.
\]
Let us observe that for any \( \varphi = \tilde{\varphi} \) with \( \tilde{\varphi} \in V_{0,\text{div}}^{p'} \) and \( \zeta \in \mathcal{D}(0,T) \) we have

\[
\left\langle \frac{\partial}{\partial t}(\varphi_{\varepsilon}, \tilde{\varphi})_{L^2(\Omega)}, \zeta \right\rangle_{\mathcal{D}'(0,T), \mathcal{D}(0,T)} + \left[ A^\varepsilon_{\mu}(\varphi_{\varepsilon}), \varphi_{\varepsilon} \right]_{\mathcal{D}(0,T)}
\]

(3.3)

\[
= \int_0^T \left\langle \frac{\partial \varphi_{\varepsilon}}{\partial t}, \varphi_{\varepsilon} \right\rangle_{(V_{0,\text{div}}^{p'}, V_{0,\text{div}}^{p'})} dt
+ J(\varphi_{\varepsilon} + \varphi) - J(\varphi_{\varepsilon}) \geq \int_0^T \left( f + \frac{\partial \xi}{\partial t} \right)_{L^2(\Omega)} \right. dt
\]

By density of \( \mathcal{D}(0,T) \otimes V_{0,\text{div}}^{p'} \) into \( L^p(0,T; V_{0,\text{div}}^{p'}) \) the same inequality is true for any \( \varphi_{\varepsilon} \in L^p(0,T; V_{0,\text{div}}^{p'}) \).

3.1. A priori estimates. Let us establish now some a priori estimates for the sequence \( (\varphi_{\varepsilon}, \pi_{\varepsilon})_{A.0} \).

**Proposition 3.2.** Let \( u \in L^p(0,T; L^p(\Omega)) \). Let \( f \in L^p(0,T; L^q(\Omega)) \), \( k \in L^p(0,T; L^p(\Gamma_0)) \), \( \mu \) satisfying \( (2.6) \) and \( (2.7) \), \( \theta \in L^q(0,T; L^q(\Omega)) \) with \( q \geq 1 \) and \( \bar{p} \geq 1 \), \( s \in L^p(0,T; L^p(\Gamma_0)) \), \( \xi \in W^{1,p}(0,T) \) satisfying \( (1.4) \) and \( v_0 \in W^{1,p}(\Omega) \) satisfying \( (1.3) \). Then there exists a constant \( C \), independent of \( \varepsilon \) such that, for all \( \varepsilon \in (0,1] \), we have

(3.4)

\[ \| \varphi_{\varepsilon} \|_{L^p(0,T; V_{0,\text{div}}^{p'})} \leq C \]

(3.5)

\[ \varepsilon^{1/p'} \| \pi_{\varepsilon} \|_{L^p(0,T; V_{0,\text{div}}^{p'})} \leq C \]

and

(3.6)

\[ \| \pi_{\varepsilon} \|_{L^\infty(0,T; L^2(\Omega))} \leq C. \]

**Proof.** Let \( t \in (0,T) \) and \( \varphi_{\varepsilon} = -\varphi_{\varepsilon} 1_{[0,t]} \in L^p(0,T; V_{0,\text{div}}^{p'}) \) where \( 1_{[0,t]} \) is the indicator function of the time interval \([0,t]\). With \( \text{(3.3)} \) we obtain

\[
\int_0^t \left\langle \frac{\partial \varphi_{\varepsilon}}{\partial t}, \varphi_{\varepsilon} \right\rangle_{(V_{0,\text{div}}^{p'}, V_{0,\text{div}}^{p'})} d\tilde{\alpha} + \int_0^t \int_{\Omega} \mathcal{F} (\theta, u + v_0 \xi, D(\varphi_{\varepsilon} + v_0 \xi)) : D(\varphi_{\varepsilon}) \, dx \, d\tilde{\alpha}
+ 2\varepsilon \int_0^t \int_{\Omega} \| D(\varphi_{\varepsilon}) \|_{p'-2}^2 D(\varphi_{\varepsilon}) : D(\varphi_{\varepsilon}) \, dx \, d\tilde{\alpha}
\leq \int_0^t \left( f + \frac{\partial \xi}{\partial t} \right)_{L^2(\Omega)} dx + \int_0^t k \| \tilde{\alpha} \|_{L^2(\Omega)} \, dx. \]

From \( \text{(2.5)} \) and \( \text{(2.1)} - \text{(2.2)} \) we have

\[
\int_0^t \int_{\Omega} \mathcal{F} (\theta, u + v_0 \xi, D(\varphi_{\varepsilon} + v_0 \xi)) : D(\varphi_{\varepsilon}) \, dx \, d\tilde{\alpha}
\geq 2(C_{\text{Korn},p})^p \mu_0 \int_0^t \| \varphi_{\varepsilon} + v_0 \xi \|_{1,p}^p d\tilde{\alpha} - 2\mu_1 \int_0^t \| v_0 \xi \|_{1,p} \| \varphi_{\varepsilon} + v_0 \xi \|_{p-1}^p d\tilde{\alpha}.
\]
and with Young’s inequality

\[
\int_0^t \int_\Omega F(\theta, u + v_0, D(\varpi_\varepsilon + v_0 \varepsilon), D(\varpi_\varepsilon)) \, dx \, dt \geq 2(C_{\text{Korn},p}^p) \mu_0 \left\| \varpi_\varepsilon \right\|_{L^p(0,t; V^p_{0,\text{div}})}^p - \left\| v_0 \varepsilon \right\|_{L^p(0,t; V^p_{1,1})}^p \\
-2\mu_1 \left( \left\| \varpi_\varepsilon \right\|_{L^p(0,t; V^p_{0,\text{div}})} + \left\| v_0 \varepsilon \right\|_{L^p(0,t; V^p_{1,1})} \right)^{p-1} \left\| v_0 \varepsilon \right\|_{L^p(0,t; V^p_{1,1})}.
\]

Similarly

\[
2\varepsilon \int_0^t \int_\Omega |D(\varpi_\varepsilon)|^{p-2} D(\varpi_\varepsilon) \, dx \, dt = 2\varepsilon \int_0^t \int_\Omega |D(\varpi_\varepsilon)|^{p'} \, dx \, dt \\
\geq 2\varepsilon(C_{\text{Korn},p})^{p'} \left\| \varpi_\varepsilon \right\|_{L^{p'}(0,t; V^p_{0,\text{div}})}^{p'}.
\]

For the sake of notational simplicity let us define \( \mathcal{F} = f + v_0 \partial \varepsilon \partial_t \in L^p(0,T; L^2(\Omega)). \)

Then we obtain

\[
\frac{1}{2} \left\| \varpi_\varepsilon(t) \right\|_{L^2(\Omega)}^2 + 2(C_{\text{Korn},p}^p) \mu_0 \left\| \varpi_\varepsilon \right\|_{L^p(0,t; V^p_{0,\text{div}})}^p - \left\| v_0 \varepsilon \right\|_{L^p(0,t; V^p_{1,1})}^p \\
+2\mu_1 \left( \left\| \varpi_\varepsilon \right\|_{L^p(0,t; V^p_{0,\text{div}})} + \left\| v_0 \varepsilon \right\|_{L^p(0,t; V^p_{1,1})} \right)^{p-1} \left\| v_0 \varepsilon \right\|_{L^p(0,t; V^p_{1,1})}.
\]

where \( \mathcal{C} \) denotes the norm of the continuous injection of \( V^p_0 \) into \( L^2(\Omega). \)

Let us consider first \( t = T \) and assume that \( \varepsilon \in (0,1]. \) We get

\[
2(C_{\text{Korn},p}^p) \mu_0 \left\| \varpi_\varepsilon \right\|_{L^p(0,T; V^p_{0,\text{div}})}^p - \left\| v_0 \varepsilon \right\|_{L^p(0,T; V^p_{1,1})}^p \\
\leq \mathcal{C}|\mathcal{F}|_{L^p(0,T; L^2(\Omega))} \left\| \varpi_\varepsilon \right\|_{L^p(0,T; V^p_{0,\text{div}})} + J(0) \\
+2\mu_1 \left( \left\| \varpi_\varepsilon \right\|_{L^p(0,T; V^p_{0,\text{div}})} + \left\| v_0 \varepsilon \right\|_{L^p(0,T; V^p_{1,1})} \right)^{p-1} \left\| v_0 \varepsilon \right\|_{L^p(0,T; V^p_{1,1})}.
\]

If \( \left\| \varpi_\varepsilon \right\|_{L^p(0,T; V^p_{0,\text{div}})} \neq 0 \) it follows that

\[
2(C_{\text{Korn},p}^p) \mu_0 \left( 1 - \frac{\left\| v_0 \varepsilon \right\|_{L^p(0,T; V^p_{1,1})}}{\left\| \varpi_\varepsilon \right\|_{L^p(0,T; V^p_{0,\text{div}})}} \right)^p \\
\leq \mathcal{C}|\mathcal{F}|_{L^p(0,T; L^2(\Omega))} \left\| \varpi_\varepsilon \right\|_{L^p(0,T; V^p_{0,\text{div}})}^{1-p} + \frac{J(0)}{\left\| \varpi_\varepsilon \right\|_{L^p(0,T; V^p_{0,\text{div}})}^p} \\
+2\mu_1 \left( 1 + \frac{\left\| v_0 \varepsilon \right\|_{L^p(0,T; V^p_{1,1})}}{\left\| \varpi_\varepsilon \right\|_{L^p(0,T; V^p_{0,\text{div}})}} \right)^{p-1} \frac{\left\| v_0 \varepsilon \right\|_{L^p(0,T; V^p_{1,1})}}{\left\| \varpi_\varepsilon \right\|_{L^p(0,T; V^p_{0,\text{div}})}}.
\]

By observing that the mapping

\[
z \mapsto 2(C_{\text{Korn},p}^p) \mu_0 \left( 1 - \frac{\left\| v_0 \varepsilon \right\|_{L^p(0,T; V^p_{1,1})}}{z} \right)^p - \mathcal{C}|\mathcal{F}|_{L^p(0,T; L^2(\Omega))} z^{p-1} - \frac{J(0)}{z^p} \\
-2\mu_1 \left( 1 + \frac{\left\| v_0 \varepsilon \right\|_{L^p(0,T; V^p_{1,1})}}{z} \right)^{p-1} \frac{\left\| v_0 \varepsilon \right\|_{L^p(0,T; V^p_{1,1})}}{z}
\]
admits $2(C_{\text{Kern},p})^p\mu_0 > 0$ as limit when $z$ tends to $+\infty$, we infer that there exists a real number $C > 0$, independent of $\varepsilon$, such that

$$|\mathcal{F}_z|_{L^p(0,T;V^p_{\text{div}})} \leq C \quad \forall \varepsilon \in (0,1]$$

which yields (3.3). Going back to (3.7) we obtain immediately (3.5) and (3.6). □

**Remark 3.3.** Let us emphasize that the constant $C > 0$ is independent of $u$.

**Proposition 3.4.** Under the same assumptions as in Proposition 3.2 there exists a constant $C' > 0$, independent of $\varepsilon$ such that, for all $\varepsilon \in (0,1]$, we have

$$(3.8) \quad \left\| \frac{\partial \mathcal{F}_z}{\partial t} \right\|_{L^p(0,T;V^p_{\text{div}}')} \leq C'$$

and

$$(3.9) \quad \| \pi_{\varepsilon} \|_{H^{-1}(0,T;L^p(\Omega))} \leq C'.$$

**Proof.** Let us choose $\bar{\mathcal{F}} = \pm \hat{\mathcal{F}}$ with $\hat{\mathcal{F}} \in V^p_{\text{div}}$ and $\zeta \in D(0,T)$ in (3.3) we obtain

$$\int_0^T \left\langle \frac{\partial \mathcal{F}_z}{\partial t}, \pm \hat{\mathcal{F}} \right\rangle_{(V^p_{\text{div}})^\prime, V^p_{\text{div}}} dt = \left[ [A^p_u(\mathcal{F}_z), \pm \hat{\mathcal{F}}] \right]$$

$$+ J(\mathcal{F}_z) - J(\mathcal{F}_z) \geq \int_0^T (\bar{f}, \pm \hat{\mathcal{F}})_{L^2(\Omega)} dt.$$

But

$$\left| J(\mathcal{F}_z) - J(\mathcal{F}_z) \right| \leq \int_0^T \int_{\Gamma_0} k |\hat{\mathcal{F}}| dx' dt$$

and recalling that $k \in L^p(0,T; L^p(\Gamma_0)) \subset L^p(0,T; L^2(\Gamma_0))$ we get

$$\left| J(\mathcal{F}_z) - J(\mathcal{F}_z) \right| \leq \| \gamma_{\mathcal{F}} \|_{L(W^{1,p'}(\Omega), L^p(\partial \Omega))} \| k \|_{L^p(0,T; L^p(\Gamma_0))} \| \hat{\mathcal{F}} \|_{L^p(0,T; V^p_{\text{div}})}$$

where $\gamma_{\mathcal{F}}$ denotes the trace operator from $W^{1,p'}(\Omega)$ into $L^p(\partial \Omega)$. Since $\mathcal{F} \in L^p(0,T; L^2(\Omega)) \subset L^p(0,T; L^2(\Omega))$ we obtain

$$(3.10) \quad \int_0^T \left\langle \frac{\partial \mathcal{F}_z}{\partial t}, \hat{\mathcal{F}} \right\rangle_{(V^p_{\text{div}})^\prime, V^p_{\text{div}}} dt$$

$$\leq \| \gamma_{\mathcal{F}} \|_{L(W^{1,p'}(\Omega), L^p(\partial \Omega))} \| k \|_{L^p(0,T; L^p(\Gamma_0))} \| \hat{\mathcal{F}} \|_{L^p(0,T; V^p_{\text{div}})}$$

$$+ C' \| f \|_{L^p(0,T; L^2(\Omega))} \| \hat{\mathcal{F}} \|_{L^p(0,T; V^p_{\text{div}})} + \left[ [A^p_u(\mathcal{F}_z), \hat{\mathcal{F}}] \right]$$

where $C'$ denotes the norm of the continuous injection of $V^p_{\text{div}}$ into $L^2(\Omega)$.

On the other hand

$$\left[ [A^p_u(\mathcal{F}_z), \hat{\mathcal{F}}] \right] \leq 2 \mu_1 |D(\mathcal{F}_z + \varepsilon_0 \xi)|_{L^p(0,T; L^p(\Omega))} \| \hat{\mathcal{F}} \|_{L^p(0,T; L^p(\Omega))}$$

$$+ 2 |D(\mathcal{F}_z)|_{L^p(0,T; L^p(\Omega))} \| \hat{\mathcal{F}} \|_{L^p(0,T; L^p(\Omega))}$$

$$\leq 2 \mu_1 (\varepsilon_0 T)^{\frac{p}{2}} \| D(\mathcal{F}_z + \varepsilon_0 \xi) \|_{L^{p-1}(0,T; L^{p}(\Omega))} \| \hat{\mathcal{F}} \|_{L^p(0,T; L^p(\Omega))}$$

$$+ 2 |D(\mathcal{F}_z)|_{L^p(0,T; L^p(\Omega))} \| \hat{\mathcal{F}} \|_{L^p(0,T; L^p(\Omega))}.$$
Thus
\[
\left\| \left[ [A_0^\varepsilon(\tau_\varepsilon), \tilde{\vartheta}] \right] \right\|_{L^p(0,T;\mathbb{V}_0',\mathbb{V}_0')} \\
\leq 2\mu_1 (\text{meas}(\Omega)T)^{\frac{2-p}{p}} \left( \|\nabla \tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} + \|v_0\xi\|_{L^p(0,T;\mathbb{V}_0')} \right)^{p-1} \|\tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} \\
+ 2\varepsilon \|\nabla \tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')}^{p-1} \|\tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} \\
\leq 2\mu_1 (\text{meas}(\Omega)T)^{\frac{2-p}{p}} \left( \|\nabla \tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} + \|v_0\xi\|_{L^p(0,T;\mathbb{V}_0')} \right)^{p-1} \|\tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} \\
+ \varepsilon^{1/p'} \left( \|\nabla \tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} \right)^{p-1} \|\tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')}.'
\]

By using (3.4) and (3.5) we obtain
\[
\left\| \left[ [A_0^\varepsilon(\tau_\varepsilon), \tilde{\vartheta}] \right] \right\| \leq \left( 2\mu_1 (C + \|v_0\xi\|_{L^p(0,T;\mathbb{V}_0')} \right)^{2-p} (\text{meas}(\Omega)T)^{\frac{2-p}{p}} \\
+ \varepsilon^{1/p'} \left( \|\nabla \tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} \right)^{p-1} \|\tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')}.'
\]

Going back to (3.10) we obtain (3.8).

Let us prove now (3.9). We choose \( \varphi = \pm \tilde{\vartheta} \) with \( \tilde{\vartheta} \in W^{1,p'}_0(\Omega) \) and \( \zeta \in D(0,T) \) in (3.11). We obtain
\[
\left\langle \int_{\Omega} \pi_\varepsilon \text{div}(\tilde{\vartheta}) \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} = - \int_0^T (\tau_\varepsilon, \tilde{\vartheta})_{L^2(\Omega)} \zeta' \, dt \\
+ \int_0^T \int_{\Omega} \mathcal{F}(\theta, u + v_0\xi, D(\tau_\varepsilon + v_0\xi)) : D(\tilde{\vartheta}) \zeta \, dx \, dt \\
+ 2\varepsilon \int_0^T \int_{\Omega} \|D(\tau_\varepsilon)\|^{p-2} D(\tau_\varepsilon) : D(\tilde{\vartheta}) \zeta \, dx \, dt - \int_0^T (\mathcal{F}, \tilde{\vartheta})_{L^2(\Omega)} \zeta \, dt.
\]

We estimate the right-hand side with the same kind of computations as previously, i.e.
\[
\left\langle \int_{\Omega} \pi_\varepsilon \text{div}(\tilde{\vartheta}) \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} \leq \sqrt{T} \|\nabla \tilde{\vartheta}\|_{L^p(0,T;L^2(\Omega))} \|\tilde{\vartheta}'\|_{L^2(0,T;L^2(\Omega))} \\
+ 2\mu_1 (C + \|v_0\xi\|_{L^p(0,T;\mathbb{V}_0')} \right)^{p-1} (\text{meas}(\Omega)T)^{\frac{2-p}{p}} \|\tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} + \varepsilon^{1/p'} \left( \|\nabla \tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} \right)^{p-1} \|\tilde{\vartheta}\|_{L^p(0,T;\mathbb{V}_0')} \\
+ \|\mathcal{F}\|_{L^p(0,T;L^2(\Omega))} \|\tilde{\vartheta}\|_{L^p(0,T;L^2(\Omega))}
\]

and with (3.6) we get
\[
\left\langle \int_{\Omega} \pi_\varepsilon \text{div}(\tilde{\vartheta}) \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} \leq \sqrt{T} C' + 2\mu_1 C_{\infty} T^{1/p'} (C + \|v_0\xi\|_{L^p(0,T;\mathbb{V}_0')})^{p-1} (\text{meas}(\Omega)T)^{\frac{2-p}{p}} \\
+ \varepsilon^{1/p'} C T^{1/p'} C'_{\infty} + \left( \mathcal{F}, \tilde{\vartheta} \right)_{H^{1}_0(0,T;\mathbb{V}_0')} \|\tilde{\vartheta}\|_{H^{1}_0(0,T;\mathbb{V}_0')}.
\]

where \( C_{\infty} \) is the norm of the continuous injection of \( H^1(0,T;\mathbb{R}) \) into \( L^\infty(0,T;\mathbb{R}). \)

Moreover, for any \( p' > 1 \), there exists a linear and continuous operator \( P_{p'} : L^p_0(\Omega) \rightarrow W^{1,p'}_0(\Omega) \) such that
\[
\text{div} \left( P_{p'}(\varphi) \right) = \varphi \quad \forall \varphi \in L^p_0(\Omega)
\]
(see Corollary 3.1 in [1]). It follows that for any \( \varpi \in L^p_0(\Omega) \) and \( \zeta \in D(0,T) \) we have
\[
\left| \left\langle \int_{\Omega} \pi_\varepsilon \varpi \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} \right| \\
\leq \left\| P_\varepsilon \right\|_{L(L^p_0(\Omega),W^{1,p'}(\Omega))} \left( \sqrt{2C_1\varepsilon + 2\mu_1\varepsilon T^{1/p'}(C + \|\varpi_0\|_{L^p(0,T;V^p_0)})} \right)^{p-1} \left( \operatorname{meas}(\Omega)T \right)^{\frac{2-p}{p}} \\
+ \varepsilon^{1/p'} C_\infty^{1/p'} C^{p'-1} + C_\infty^{1/p} \left( \|\varpi_0\|_{L^p(0,T;L^2(\Omega))} \right) \left\| \varpi \zeta \right\|_{H^1_0(0,T;L^{p'}(\Omega))}.
\]
Hence there exists a real number \( C' > 0 \), independent of \( \varepsilon \), such that for all \( \varepsilon \in (0,1] \) we have
\[
\left| \left\langle \int_{\Omega} \pi_\varepsilon \varpi \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} \right| \\
\leq C' \|\varpi\|_{H^1_0(0,T;L^{p'}(\Omega))} \forall \varpi \in L^p_0(\Omega), \forall \zeta \in D(0,T).
\]
Furthermore, for any \( \varpi^* \in L^p(\Omega) \), we may define \( \varpi \in L^p_0(\Omega) \) by
\[
\varpi = \varpi^* - \frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} \varpi^* \, dx.
\]
We have \( \|\varpi\|_{L^{p'}(\Omega)} \leq 2 \|\varpi^*\|_{L^{p'}(\Omega)} \) and since \( \pi_\varepsilon \in H^{-1}(0,T;L^p_0(\Omega)) \) we have
\[
\left| \left\langle \int_{\Omega} \pi_\varepsilon \left( \varpi^* - \frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} \varpi^* \, dx \right) \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} \right| \\
= \left| \left\langle \int_{\Omega} \pi_\varepsilon \varpi^* \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} \right| - \frac{1}{\operatorname{meas}(\Omega)} \left( \int_{\Omega} \varpi^* \, dx \right) \left\langle \int_{\Omega} \pi_\varepsilon \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} \\
= \left| \left\langle \int_{\Omega} \pi_\varepsilon \varpi^* \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} \right|.
\]
It follows that
\[
\left| \left\langle \int_{\Omega} \pi_\varepsilon \varpi^* \, dx, \zeta \right\rangle_{D'(0,T),D(0,T)} \right| \\
\leq 2C' \|\varpi^*\|_{H^1_0(0,T;L^{p'}(\Omega))} \forall \varpi^* \in L^p(\Omega), \forall \zeta \in D(0,T).
\]
Finally we may conclude by using the density of \( D(0,T) \odot L^p(\Omega) \) into \( H^1_0(0,T;L^p(\Omega)) \).

\[ \square \]

**Remark 3.5.** Let us emphasize once again that these estimates are independent of \( u \).

### 3.2. Existence and uniqueness result for \((P_u)\).

With the previous estimates we infer that, by possibly extracting a subsequence still denoted \((\pi_\varepsilon, \pi_\varepsilon)_{\varepsilon > 0}\), there exists \( \pi \in L^p(0,T;V^p_0) \cap L^\infty(0,T;L^2(\Omega)) \) such that \( \frac{\partial \pi}{\partial t} \in L^p(0,T;\left(V^p_0\right)'\)\), and \( \pi \in H^{-1}(0,T;L^0_0) \) such that

\[(3.12) \quad \pi_\varepsilon \rightharpoonup \pi \quad \text{weakly in } L^p(0,T;V^p_0)\]

\[(3.13) \quad \pi_\varepsilon \rightharpoonup \pi \quad \text{weakly in } L^r(0,T;L^2(\Omega)) \text{ for any } r > 1 \text{ and weakly* in } L^\infty(0,T;L^2(\Omega))\]

\[(3.14) \quad \frac{\partial \pi_\varepsilon}{\partial t} \rightharpoonup \frac{\partial \pi}{\partial t} \quad \text{weakly in } L^p(0,T;\left(V^p_0\right)')\]
and
\begin{equation}
\pi_\varepsilon \rightharpoonup \pi \quad \text{weakly}^* \text{ in } H^{-1}(0,T;L^p_0(\Omega)).
\end{equation}

Owing that \(V_{0,\text{div}}^r \subset L^2(\Omega) \subset (V_{0,\text{div}}^r)'\) for any \(r \geq 6/5\) with a compact injection of \(V_{0,\text{div}}^r\) into \(L^2(\Omega)\) whenever \(r > 6/5\), we infer that \(V_{0,\text{div}}^p\) is compactly embedded into \((V_{0,\text{div}}^p)'\). Thus we may apply Aubin’s lemma and, by possibly extracting another subsequence still denoted \((\pi_\varepsilon, \pi_\varepsilon)_{\varepsilon > 0}\), we have
\begin{equation}
\pi_\varepsilon \rightharpoonup \pi \quad \text{strongly in } L^p(0,T;(V_{0,\text{div}}^p)').
\end{equation}

With Simon’s lemma we have also
\begin{equation}
\pi_\varepsilon(t = 0, \cdot) \rightharpoonup \pi(t = 0, \cdot) = 0.
\end{equation}

Moreover the sequence \((\pi_\varepsilon(T))_{\varepsilon > 0}\) is bounded in \(L^2(\Omega)\). The compact embedding of \(L^2(\Omega)\) into \(\tilde{H}\) combined with \(5.17\) implies that, by possibly extracting another subsequence still denoted \((\pi_\varepsilon, \pi_\varepsilon)_{\varepsilon > 0}\), we have
\begin{equation}
\pi_\varepsilon(T) \rightharpoonup \pi(T) \quad \text{weakly in } L^2(\Omega) \text{ and strongly in } \tilde{H}.
\end{equation}

Furthermore

\begin{lemma}
We have
\begin{equation}
\gamma_p(\pi_\varepsilon) \rightharpoonup \gamma_p(\pi) \quad \text{strongly in } L^p(0,T;L^p(\partial\Omega))
\end{equation}
where \(\gamma_p\) is the trace operator form \(W^{1,p}(\Omega)\) into \(L^p(\partial\Omega)\).
\end{lemma}

\begin{proof}
Let us prove that for any \(\eta > 0\), there exists \(c_\eta > 0\) such that
\begin{equation}
\|\gamma_p(v)\|_{L^p(\partial\Omega)} \leq \eta\|v\|_{V_{0,\text{div}}^p} + c_\eta\|v\|_{(V_{0,\text{div}}^p)'} \quad \forall v \in V_{0,\text{div}}^p.
\end{equation}

Indeed if \(3.20\) is not true, there exists \(\eta > 0\) and a sequence \((v_m)_{m \geq 1}\) of \(V_{0,\text{div}}^p\) such that
\begin{equation}
\|\gamma_p(v_m)\|_{L^p(\partial\Omega)} > \eta\|v_m\|_{V_{0,\text{div}}^p} + m\|v_m\|_{(V_{0,\text{div}}^p)'} \quad \forall m \geq 1.
\end{equation}

It follows that \(\|\gamma_p(v_m)\|_{L^p(\partial\Omega)} > 0\) for all \(m \geq 1\). Since \(\gamma_p \in \mathcal{L}_e(W^{1,p}(\Omega),L^p(\partial\Omega))\) and \(V_{0,\text{div}}^p\) is a subspace of \(W^{1,p}(\Omega)\) we infer that \(\|v_m\|_{V_{0,\text{div}}^p} \neq 0\) for all \(m \geq 1\) and we may define \(w_m = \frac{1}{\|v_m\|_{V_{0,\text{div}}^p}}v_m\) for all \(m \geq 1\).

Thus \(\|w_m\|_{V_{0,\text{div}}^p} = 1\) for all \(m \geq 1\) and the sequence \(\{(\gamma_p(w_m))_{m \geq 1}\}\) is bounded. Owing that
\begin{equation}
\|\gamma_p(w_m)\|_{L^p(\partial\Omega)} > \eta + m\|w_m\|_{(V_{0,\text{div}}^p)'} \geq \eta > 0 \quad \forall m \geq 1
\end{equation}
we infer that
\begin{equation}
w_m \rightharpoonup 0 \quad \text{strongly in } (V_{0,\text{div}}^p)'.
\end{equation}

Moreover by possibly extracting a subsequence still denoted \((w_m)_{m \geq 1}\), we obtain that there exists \(w_* \in V_{0,\text{div}}^p\) such that
\begin{equation}
w_m \rightharpoonup w_* \quad \text{weakly in } V_{0,\text{div}}^p
\end{equation}
and with (3.22) we get \( w_\ast = 0 \). Recalling that the trace operator \( \gamma_p \) is compact from \( W^{1,p}(\Omega) \) into \( L^p(\partial \Omega) \) it follows that
\[
\gamma_p(w_\ast) \to \gamma_p(w_\ast) = 0 \quad \text{strongly in } L^p(\partial \Omega),
\]
which gives a contradiction with (3.21).

By using (3.20) and (3.4) we obtain that, for any \( \eta > 0 \) there exists \( c_\eta > 0 \) such that
\[
\begin{align*}
||\gamma_p(\nabla_\varepsilon) - \gamma_p(\nabla_\varepsilon')||_{L^p(0,T;L^p(\partial \Omega))} &= ||\gamma_p(\nabla_\varepsilon - \nabla_\varepsilon')||_{L^p(0,T;L^p(\partial \Omega))} \\
&\leq \eta ||\nabla_\varepsilon - \nabla_\varepsilon'||_{L^p(0,T;V^p_{div})} + c_\eta ||\nabla_\varepsilon - \nabla_\varepsilon'||_{L^p(0,T;V^p_{div})')}, \\
&\leq 2\eta C + c_\eta ||\nabla_\varepsilon - \nabla_\varepsilon'||_{L^p(0,T;V^p_{div})'} \quad \forall \varepsilon > 0
\end{align*}
\]
and (3.16) allows us to conclude. \( \square \)

For the sake of notational simplicity we will identify the functions and their traces on \( \partial \Omega \) in the rest of the paper.

Next we can prove that the limit pressure \( \pi \) and \( \frac{\partial \pi}{\partial t} \) satisfy better regularity properties.

**Proposition 3.7.** Let \( u \in L^p(0,T;L^2(\Omega)) \). Let \( f \in L^{p'}(0,T;L^2(\Omega)) \), \( k \in L^{p'}(0,T;L^p(\Gamma_0)) \), \( \mu \) satisfying (2.3)–(2.5), \( \theta \in L^{\tilde{q}}(0,T;L^{\tilde{p}}(\Omega)) \) with \( \tilde{q} \geq 1 \) and \( \tilde{p} \geq 1 \), \( s \in L^p(0,T;L^p(\Gamma_0)) \), \( \xi \in W^{1,p'}(0,T) \) satisfying (1.4) and \( v_0 \in W^{1,p}(\Omega) \) satisfying (1.3). Then \( \pi \in H^{-1}(0,T;L^p(\partial \Omega)) \) and \( \frac{\partial \pi}{\partial t} \in L^{p'}(0,T;V^p_{div})' \).

**Proof.** Let us prove that \( \pi \in H^{-1}(0,T;L^p(\partial \Omega)) \). Indeed, for all \( \varepsilon \in (0,1] \), for all \( \tilde{\theta} \in W^{1,p'}_0(\Omega) \) and for all \( \zeta \in D(0,T) \) we have (3.11) and thus
\[
\begin{align*}
\left| \left\langle \int_\Omega \varepsilon \text{div}(\tilde{\theta}) \, dx, \zeta \right\rangle \right|_{D'(0,T),D'(0,T)} &\leq \left( \sqrt{T C \bar{C}} + 2\mu_1 C_\infty T^{1/p'} \left( C + ||v_0 \xi||_{L^p(0,T;V^p_{div})'} \right)^p \right)^{-1} \\
&+ \bar{C} \left\| L^{p'}(0,T;L^2(\Omega)) \right\| C_\infty T^{1/p'} ||\tilde{\theta}||_{H^1(0,T;V^p_{div})'} + \varepsilon^{1/p'} ||C_\infty T^{1/p'} C^{-1} ||\tilde{\theta}||_{H^1(0,T;V^p_{div})'}
\end{align*}
\]
where \( C > 0 \) is the constant introduced in Proposition 3.2 and \( \bar{C} \) and \( C_\infty \) are the norms of the continuous injections of \( V^p_{div} \) into \( L^2(\Omega) \) and \( H^1(0,T;\mathbb{R}) \) into \( L^\infty(0,T;\mathbb{R}) \) respectively. We may pass to the limit as \( \varepsilon \) tends to zero in the previous inequality and we get
\[
\begin{align*}
\left| \left\langle \int_\Omega \varepsilon \text{div}(\tilde{\theta}) \, dx, \zeta \right\rangle \right|_{D'(0,T),D'(0,T)} &\leq \left( \sqrt{T C \bar{C}} + 2\mu_1 C_\infty T^{1/p'} \left( C + ||v_0 \xi||_{L^p(0,T;V^p_{div})'} \right)^p \right)^{-1} \\
&+ \bar{C} \left\| L^{p'}(0,T;L^2(\Omega)) \right\| C_\infty T^{1/p'} ||\tilde{\theta}||_{H^1(0,T;V^p_{div})'}
\end{align*}
\]
By using Green’s formula we infer that
\[
\begin{align*}
\left| \left\langle \int_\Omega (\nabla \pi, \tilde{\theta}) \, dx, \zeta \right\rangle \right|_{D'(0,T),D'(0,T)} &\leq \left( \sqrt{T C \bar{C}} + 2\mu_1 C_\infty T^{1/p'} \left( C + ||v_0 \xi||_{L^p(0,T;V^p_{div})'} \right)^p \right)^{-1} \\
&+ \bar{C} \left\| L^{p'}(0,T;L^2(\Omega)) \right\| C_\infty T^{1/p'} ||\tilde{\theta}||_{H^1(0,T;W^{1,p}_0(\Omega))}
\end{align*}
\]
for all \( \tilde{\theta} \in (D(\Omega))^3 \) and for all \( \zeta \in D(0,T) \), where \((\cdot,\cdot)\) denotes the Euclidean inner product of \( \mathbb{R}^3 \).
With the density of $D(0, T) \otimes W^{1,p}_0(\Omega)$ into $H^1_0(0, T; W^{-1,p}_0(\Omega))$, we conclude that $\pi \in H^{-1}(0, T; W^{-1,p}_0(\Omega))$. Then the properties of the gradient operator (I) imply that $\pi \in H^{-1}(0, T; L^p(\Omega))$. Owing that $\pi \in H^{-1}(0, T; L^p(\Omega))$ we obtain finally that $\pi \in H^{-1}(0, T; L^p(\Omega))$.

Let us prove now that $\frac{\partial \tilde{\pi}}{\partial t} \in L^p'(0, T; (V^p_0, d\omega'))$. We choose $\varphi = \pm \tilde{\theta} \zeta$ with $\tilde{\theta} \in V^p_0$ and $\zeta \in D(0, T)$ in (3.1) and we obtain

$$\left\langle \frac{\partial}{\partial t} (\varphi, \tilde{\theta} \zeta)_{L^2(\Omega)}, \varphi \right\rangle_{D'(0, T), D(0, T)} + \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0 \xi, D(\varphi, \varepsilon) + v_0 \xi)) : D(\tilde{\theta}) \zeta dx dt + 2\varepsilon \left\langle \int_0^T \int_\Omega D(\varphi) \zeta dx dt \right\rangle_{D'(0, T), D(0, T)}$$

By recalling that $\mathbf{T} \in L^p'(0, T; L^2(\Omega))$ and $k \in L^p'(0, T; L^p(\Omega))$, we perform the same kind of computations as in Proposition 3.4 and we get

$$\left\langle \int_0^T (\varphi, \tilde{\theta} \zeta)_{L^2(\Omega)}, \varphi \right\rangle_{D'(0, T), D(0, T)} \leq \|\gamma_p\|_L(\mathbf{W}^{1,\infty}(\Omega) L^p(\partial\Omega)) \|k\|_{L^p(0, T; L^p(\Omega))} + \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0 \xi, D(\varphi, \varepsilon) + v_0 \xi)) : D(\tilde{\theta}) \zeta dx dt$$

where $\mathbf{C}$ denotes the norm of the continuous injection of $V^p_0$ into $L^2(\Omega)$ and

$$\left\langle \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0 \xi, D(\varphi, \varepsilon) + v_0 \xi)) : D(\tilde{\theta}) \zeta dx dt \right\rangle_{D'(0, T), D(0, T)} \leq 2\mu_1 \left( \|\varphi\|_{L^p(0, T; V^p_0, d\omega')} + \|v_0 \xi\|_{L^p(0, T; V^p_0)} \right)^{p-1} \left\|\tilde{\theta} \zeta\right\|_{L^p(0, T; V^p_0)}$$

We pass to the limit as $\varepsilon$ tends to zero. We obtain

$$\left\langle \int_0^T (\varphi, \tilde{\theta} \zeta)_{L^2(\Omega)}, \varphi \right\rangle_{D'(0, T), D(0, T)} \leq \|\gamma_p\|_L(\mathbf{W}^{1,\infty}(\Omega) L^p(\partial\Omega)) \|k\|_{L^p(0, T; L^p(\Omega))} + \int_0^T \int_\Omega \mathcal{F}(\theta, u + v_0 \xi, D(\varphi, \varepsilon) + v_0 \xi)) : D(\tilde{\theta}) \zeta dx dt$$

Moreover we know that $\pi \in H^{-1}(0, T; L^p(\Omega))$. It follows that

$$\left\langle \int_\Omega \pi \text{div} (\tilde{\theta}) dx, \zeta \right\rangle_{D'(0, T), D(0, T)} \leq \|\pi\|_{H^{-1}(0, T; L^p(\Omega))} \|\text{div} (\tilde{\theta})\|_{H^1_0(0, T; L^p(\Omega))}$$
We infer that

\[ u \equiv \int_t^T (\mathbf{\pi}, \tilde{v})_{L^2(\Omega)} \, dt \leq C'' \left| \int_t^T (\mathbf{\pi}, \tilde{v})_{L^2(\Omega)} \, dt \right| + \| \pi \|_{L^p(0,T;V_0^p)} \| \text{div}(\tilde{\phi}) \zeta \|_{L^p_0(0,T;L^p(\Omega))} \]

for all \( \tilde{\phi} \in V_0^p \) and \( \zeta \in \mathcal{D}(0,T) \).

Furthermore \( V_0^p \) is dense into \( V_0^p \). Indeed, by definition of \( V_0^p \) we have \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \in V_0^p \) if and only if \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \in W_1^{1,p}(\Omega) \times W_1^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \) where

\[ W_1^{1,p}(\Omega) = \{ w \in W_1^{1,p}(\Omega); w = 0 \text{ on } \Gamma_1 \cup \Gamma_L \}. \]

Let us define

\[ W = \{ w \in \mathcal{D}(\Omega); w = 0 \text{ on } \Gamma_1 \cup \Gamma_L \} \]

and let \( Q_+ \) and \( Q \) be given as

\[ Q_+ = \Omega = \{ (x', x_3) \in \mathbb{R}^3; x' = (x_1, x_2) \in \omega \text{ and } 0 < x_3 < h(x') \}, \]

\[ Q = \{ (x', x_3) \in \mathbb{R}^3; x' = (x_1, x_2) \in \omega \text{ and } |x_3| < h(x') \}. \]

For any \( u \in W_1^{1,p}(\Omega) \) we build the extension \( u^* \) of \( u \) to \( Q \) by reflexion, i.e.

\[ u^*(x', x_3) = \begin{cases} u(x', x_3) & \text{if } x_3 > 0, \\ u(x', -x_3) & \text{if } x_3 < 0. \end{cases} \]

With classical results we know that \( u^* \in W_1^{1,p}(Q) \). Hence for all \( u \in W_1^{1,p}(\Omega) \) we have \( u^* \in W_0^{1,p}(Q) \). Since \( \mathcal{D}(Q) \) is dense into \( W_0^{1,p}(Q) \) there exists a sequence \( (v_n)_{n \geq 1} \) such that \( v_n \in \mathcal{D}(Q) \) for all \( n \geq 1 \) and

\[ v_n \to u^* \text{ strongly in } W_0^{1,p}(Q). \]

It follows that \( u = u^*|_{Q_+} \in W \) for all \( n \geq 1 \) and

\[ u_n \to u \text{ strongly in } W_1^{1,p}(\Omega). \]

We infer that \( W \) is dense into \( W_1^{1,p}(\Omega) \) and \( W \times W \times \mathcal{D}(\Omega) \) is dense into \( V_0^p \).

Since \( W \times W \times \mathcal{D}(\Omega) \subset V_0^p \subset V_0^p \) we obtain the announced density result.

Going back to \( (3.23) \) we obtain

\[ \left| \int_0^T (\mathbf{\pi}, \tilde{v})_{L^2(\Omega)} \, dt \right| \leq C'' \left| \int_0^T (\mathbf{\pi}, \tilde{v})_{L^2(\Omega)} \, dt \right| + \| \pi \|_{L^p(0,T;L^p(\Omega))} \| \text{div}(\tilde{\phi}) \zeta \|_{L^p_0(0,T;L^p(\Omega))} \]

for all \( \tilde{\phi} \in V_0^p \) and for all \( \zeta \in \mathcal{D}(0,T) \). Moreover if \( \tilde{\phi} \in V_{0,\text{div}}^p \) we get

\[ \left| \int_0^T (\mathbf{\pi}, \tilde{v})_{L^2(\Omega)} \, dt \right| \leq C'' \left| \int_0^T (\mathbf{\pi}, \tilde{v})_{L^2(\Omega)} \, dt \right| \forall \tilde{\phi} \in V_{0,\text{div}}^p, \forall \zeta \in \mathcal{D}(0,T). \]

With the density of \( \mathcal{D}(0,T) \otimes V_{0,\text{div}}^p \) in \( L^p(0,T;V_{0,\text{div}}^p) \) we may conclude that \( \frac{\partial \pi}{\partial t} \in L^{p'}(0,T;V_{0,\text{div}}^p) \).

\[ \square \]
Let us recall the definition of the space $\mathcal{Y}$ already introduced in Section [2]

\[ \mathcal{Y} = \{ \psi \in L^2(\Omega); \; \text{div}(\psi) \in L^2(\Omega) \} \]

endowed with its canonical norm

\[ \| \psi \|_{\mathcal{Y}} = \left( \| \psi \|_{L^2(\Omega)}^2 + \| \text{div}(\psi) \|_{L^2(\Omega)}^2 \right)^{1/2} \quad \forall \psi \in \mathcal{Y} \]

and its subspace $H$ given by $H = \{ \psi \in L^2(\Omega); \; \text{div}(\psi) = 0 \text{ in } \Omega; \; \psi \cdot n = 0 \text{ on } \partial \Omega \}$.

For any $p \geq 6/5$ the embedding of $V_{0,\text{div}}^p$ into $H$ is continuous and dense ([15]). Hence we may consider the Gelfand triplet $V_{0,\text{div}}^p \subset H \equiv H' \subset (V_{0,\text{div}}^p)'$ and we obtain that $\overline{v} \in C([0,T]; H)$.

**Theorem 3.8.** Let $u \in L^p(0,T; L^p(\Omega))$. Let $f \in L^{p'}(0,T; L^2(\Omega))$, $\mu \in L^q(0,T; L^q(\Gamma_0))$, $\nu \in L^r(0,T; L^r(\Gamma_1))$ satisfying (2.13) and (2.12), $\theta \in L^q(0,T; L^q(\Omega))$ with $q \geq 1$ and $\tilde{p} \geq 1$, $s \in L^p(0,T; L^p(\Gamma_0))$, $\xi \in W^{1,p'}(0,T)$ satisfying (4.4) and $\psi_0 \in W^{1,p'}(\Omega)$ satisfying (4.5). Then problem $(P_u)$ admits an unique solution.

**Proof.** Let us introduce the operator $A_u : L^p(0,T; V_{0,\text{div}}^p) \rightarrow L^{p'}(0,T; (V_{0,\text{div}}^p)')$ defined by

\[ [A_u(u), \overline{\psi}] = \int_0^T \int_\Omega F(\theta, u + \nu_0 \xi, D(u + \nu_0 \xi)) : D(\overline{\psi}) \ dx \ dt \quad \forall \overline{\psi} \in L^p(0,T; V_{0,\text{div}}^p) \]

where $[,]$ denotes the duality product between $L^p(0,T; V_{0,\text{div}}^p)$ and its dual $(L^p(0,T; V_{0,\text{div}}^p))'$.

Now let $\varepsilon > 0$ and $\overline{\psi} = \overline{\partial \xi}$ with $\overline{\partial} \in V_{0,\text{div}}^{p'} \subset V_{0,\text{div}}^p$ and $\xi \in D(0,T)$. With (3.3)

\[ \int_0^T \langle \frac{\partial \overline{\sigma}_x}{\partial t} + \overline{\sigma}_x - \overline{\psi}_x \rangle_{(V_{0,\text{div}}^p)'} \langle V_{0,\text{div}}^p \rangle' \ dt + [A_u(\overline{\sigma}_x), \overline{\psi}_x - \overline{\psi}] 
+ 2\varepsilon \int_0^T \int_\Omega |D(\overline{\psi}_x)|^{p'-2} D(\overline{\psi}_x) : D(\overline{\psi}_x) \ dx \ dt + J(\overline{\psi}) - J(\overline{\psi}) \geq \int_0^T \langle \overline{\psi}, \overline{\psi} - \overline{\psi}_x \rangle_{L^2(\Omega)} \]

Hence

\[ [A_u(\overline{\psi}_x), \overline{\psi}_x - \overline{\psi}] \leq \int_0^T \langle \frac{\partial \overline{\sigma}_x}{\partial t} + \overline{\sigma}_x - \overline{\psi}_x \rangle_{(V_{0,\text{div}}^p)'} \langle V_{0,\text{div}}^p \rangle' \ dt + [A_u(\overline{\psi}_x), \overline{\psi}_x - \overline{\psi}] 
- 2\varepsilon \int_0^T \int_\Omega |D(\overline{\psi}_x)|^{p'} \ dx \ dt + 2\varepsilon \int_0^T \int_\Omega \left| D(\overline{\psi}_x) \right|^{p'-2} \ dx \ dt 
+ J(\overline{\psi}) - J(\overline{\psi}) - \int_0^T \langle \overline{\psi}, \overline{\psi} - \overline{\psi}_x \rangle_{L^2(\Omega)} \ dt 
+ \langle \overline{\psi}, \overline{\psi} - \overline{\psi}_x \rangle_{L^2(\Omega)} < 0 \]

With (3.13) we have immediately

\[ \int_0^T \langle \overline{\psi}, \overline{\psi} - \overline{\psi}_x \rangle_{L^2(\Omega)} \ dt \rightarrow \int_0^T \langle \overline{\psi}, \overline{\psi} - \overline{\psi}_x \rangle_{L^2(\Omega)} \ dt, \]

\[ \int_0^T \langle \overline{\psi}, \overline{\psi} \rangle_{L^2(\Omega)} \ dt \rightarrow \int_0^T \langle \overline{\psi}, \overline{\psi} \rangle_{L^2(\Omega)} \ dt. \]
Moreover, if we denote \( A_u \) by the bounded operator from \( L^p(0, T; V_0^p) \) to \( L^p(0, T; (V_0^p)') \) and \( (\pi_{\epsilon})_{\epsilon > 0} \) is bounded in \( L^p(0, T; V_0^p) \). So, by possibly extracting another subsequence still denoted \( (\pi_{\epsilon}, \pi_{\epsilon'})_{\epsilon > 0} \), there exists \( \chi \in L^p(0, T; (V_0^p)') \) such that
\[
A_u(\pi_{\epsilon}) \rightarrow \chi \quad \text{weakly in } L^p(0, T; (V_0^p)').
\]

By using estimate \((3.15)\) we infer from \((3.24)\)
\[
\limsup_{\epsilon \to 0} [A_u(\pi_{\epsilon}), \pi_{\epsilon} - \pi] \leq -\int_0^T (\pi', \hat{\theta})_{L^2(\Omega)} \zeta' dt - \frac{1}{2} \liminf_{\epsilon \to 0} \|\pi_{\epsilon}(T)\|_{L^2(\Omega)}^2
\]
and with \((3.18)\) we get
\[
\limsup_{\epsilon \to 0} [A_u(\pi_{\epsilon}), \pi_{\epsilon} - \pi] \leq -\int_0^T (\pi', \hat{\theta})_{L^2(\Omega)} \zeta' dt - \frac{1}{2} \liminf_{\epsilon \to 0} \|\pi_{\epsilon}(T)\|_{L^2(\Omega)}^2
\]

With Proposition\((3.7)\) we know that \( \frac{\partial \pi}{\partial t} \in L^p(0, T; (V_0^p)'_{\text{div}}) \) and \( \hat{\theta} \in L^p(0, T; V_0^p) \).
So
\[
-\int_0^T (\pi', \hat{\theta})_{L^2(\Omega)} \zeta' dt = \int_0^T \left\langle \frac{\partial \pi}{\partial t}, \hat{\theta} \right\rangle_{D'(0, T), D(0, T)} = \int_0^T \left\langle \frac{\partial \pi}{\partial t}, \hat{\theta} \right\rangle_{(V_0^p)_{\text{div}}'}_{(V_0^p)_{\text{div}}} dt
\]
and by density of \( D(0, T) \otimes V_0^p \) into \( L^p(0, T; V_0^p) \) we have
\[
\limsup_{\epsilon \to 0} [A_u(\pi_{\epsilon}), \pi_{\epsilon} - \pi] \leq \int_0^T \left\langle \frac{\partial \pi}{\partial t}, \pi \right\rangle_{(V_0^p)_{\text{div}}'}_{(V_0^p)_{\text{div}}} dt - \frac{1}{2} \|\pi(T)\|_{L^2(\Omega)}^2
\]
\[
+ \left\langle \chi, \pi - \pi \right\rangle_{(V_0^p)_{\text{div}}'}_{(V_0^p)_{\text{div}}} + \left\langle J(\pi) - J(\pi) \right\rangle_{L^2(\Omega)} dt \quad \forall \varphi \in L^p(0, T; V_0^p). \]

By choosing \( \varphi = \pi_{\epsilon'}. \) and taking the limit as \( \epsilon' \) tends to zero in the right-hand side of the previous inequality we obtain
\[
\limsup_{\epsilon \to 0} [A_u(\pi_{\epsilon}), \pi_{\epsilon} - \pi] \leq \int_0^T \left\langle \frac{\partial \pi}{\partial t}, \pi \right\rangle_{(V_0^p)_{\text{div}}'}_{(V_0^p)_{\text{div}}} \frac{1}{2} \|\pi(T)\|_{L^2(\Omega)}^2 = 0.
\]

By recalling that the mapping \( \lambda_2 \mapsto \mathcal{F}(\cdot, \cdot, \lambda_2) \) is monotone in \( \mathbb{R}^{3 \times 3} \) for any \( p > 1 \) (see Lemma 1 in \[3\]) we infer that the operator \( A_u \) is monotone. Moreover with \((2.3)\) and \((2.7)\) we obtain that \( A_u \) is bounded and hemi-continuous. Hence \( A_u \) is pseudo-monotone. We obtain
\[
\liminf_{\epsilon \to 0} [A_u(\pi_{\epsilon}), \pi_{\epsilon} - \pi] \geq [A_u(\pi), \pi - \pi] \quad \forall \varphi \in L^p(0, T; V_0^p)
\]
and with (3.25) we get

\[ (3.27) \lim \inf [A_u(\varphi), \varphi] - [\varphi, \varphi] \geq [A_u(\bar{\varphi}), \bar{\varphi}] \quad \forall \varphi \in L^p(0, T; \phi).
\]

By choosing \( \bar{\varphi} = \varphi \) we infer that

\[ \lim \inf [A_u(\varphi), \varphi] \geq [\varphi, \varphi]. \]

On the other hand, with (3.26) we have

\[ \lim \sup [A_u(\varphi), \varphi - \varphi] = \lim \sup [A_u(\varphi), \varphi] - [\varphi, \varphi] \leq 0. \]

It follows that

\[ [\varphi, \varphi] = \lim \inf [A_u(\varphi), \varphi] = \lim \sup [A_u(\varphi), \varphi]. \]

Going back to (3.27) we obtain

\[ [\varphi, \varphi - \varphi] \geq [A_u(\varphi), \varphi - \varphi] \quad \forall \varphi \in L^p(0, T; \phi) \]

and with \( \bar{\varphi} = \varphi + \psi \) with \( \psi \in L^p(0, T; \phi) \) we may conclude that

\[ [\varphi - A_u(\varphi), \bar{\varphi}] = 0 \quad \forall \bar{\varphi} \in L^p(0, T; \phi) \]

which yields \( \chi = A_u(\varphi) \) in \( L^p(0, T; (V_0^p)' \), i.e.

\[ A_u(\varphi) \rightarrow A_u(\varphi) \quad \text{weakly in } L^p(0, T; (V_0^p)'). \]

Now let \( \tilde{\vartheta} \in V_0^p \) and \( \zeta \in D(0, T) \). Owing that \( V_0^p' \subset V_0^p \) we obtain

\[ [A_u(\varphi), \tilde{\vartheta}] \rightarrow [A_u(\varphi), \tilde{\vartheta}] . \]

We already know that

\[ |J(\varphi) - J(\bar{\varphi})| \leq \|k\|_{L^p(0, T; L^p'(\Gamma_0)))} \|\gamma_p(\varphi) - \gamma_p(\bar{\varphi})\|_{L^p(0, T; L^p(\partial\Omega))} \rightarrow 0 \]

and similarly

\[ |J(\varphi + \tilde{\vartheta}) - J(\varphi + \tilde{\vartheta})| \leq \|k\|_{L^p(0, T; L^p'(\Gamma_0)))} \|\gamma_p(\varphi + \tilde{\vartheta}) - \gamma_p(\varphi)\|_{L^p(0, T; L^p(\partial\Omega))} \rightarrow 0 . \]

Moreover

\[ 2\varepsilon \int_0^T \int_\Omega \left| D(\varphi) \right|^{\rho' - 2} D(\varphi) \cdot D(\tilde{\vartheta}) \, dx \, dt \]

\[ \leq 2\varepsilon^{1/\rho'} \left( \varepsilon^{1/\rho'} \|\varphi\|_{L^p(0, T; V_0^p)} \right)^{\rho' - 1} \|D(\tilde{\vartheta})\|_{L^p(0, T; (L^p')^3)} \rightarrow 0 \]

and

\[ \left\langle \frac{\partial}{\partial t}(\varphi, \tilde{\vartheta}) \right\rangle_{D'(0, T), D(0, T)} = - \int_0^T \langle \varphi, \tilde{\vartheta} \rangle_{D'^2(\Omega)} \, dt 

\rightarrow - \int_0^T \langle \varphi, \tilde{\vartheta} \rangle_{D'^2(\Omega)} \, dt = \left\langle \frac{\partial}{\partial t}(\varphi, \tilde{\vartheta}) \right\rangle_{D'(0, T), D(0, T)} . \]

Finally with (3.15) we get

\[ \left\langle \int_\Omega \pi \text{div}(\tilde{\vartheta}), \zeta \right\rangle_{D'(0, T), D(0, T)} \rightarrow \left\langle \int_\Omega \pi \text{div}(\tilde{\vartheta}), \zeta \right\rangle_{D'(0, T), D(0, T)} . \]
Thus we can pass to the limit in (3.1) as \( \varepsilon \) tends to zero and we get

\[
\left\langle \frac{\partial}{\partial t}(\overline{\pi}, \vartheta)_{L^2(\Omega)}, \zeta \right\rangle_{D'(0,T), D(0,T)} + \int_0^T \int_{\Omega} F(\theta, u + v_0 \xi, D(\overline{\pi} + v_0 \xi)) : D(\vartheta) \zeta \, dx \, dt
- \int_{\Omega_T} \pi \, \text{div}(\vartheta) \, dx, \zeta \rangle_{D'(0,T), D(0,T)} + J(\overline{\pi} + \vartheta \zeta) - J(\overline{\pi}) \\
\geq \int_0^T \left( f + \frac{\partial \xi}{\partial t} v_0, \vartheta \right)_{L^2(\Omega)} \zeta \, dt \quad \forall \vartheta \in V^p_0, \forall \zeta \in D(0,T).
\]

Finally, by using the density of \( V^p_0 \) into \( V^p_0 \) and reminding that \( \pi \in H^{-1}(0,T; L^2(\Omega)) \), the same inequality holds for any \( \vartheta \in V^p_0 \) and \( \zeta \in D(0,T) \) and we may conclude that \((\overline{\pi}, \pi)\) is a solution of problem \((P_u)\).

Let \( \overline{\pi} = \vartheta \zeta \) with \( \vartheta \in V^p_{0, \text{div}} \) and \( \zeta \in D(0,T) \). Owing that \( \frac{\partial \pi}{\partial t} \in L^p(0,T; (V^p_{0, \text{div}})'') \)
we have

\[
\left\langle \frac{\partial}{\partial t}(\overline{\pi}, \vartheta)_{L^2(\Omega)}, \zeta \right\rangle_{D'(0,T), D(0,T)} + \left[ A_u(\overline{\pi}), \vartheta \right]_{V^p_0, \text{div}} + J(\overline{\pi} + \vartheta \zeta) - J(\overline{\pi}) \geq \int_0^T \left( f + \frac{\partial \xi}{\partial t} v_0, \vartheta \right)_{L^2(\Omega)} \zeta \, dt
\]

and by density of \( D(0,T) \otimes V^p_{0, \text{div}} \) into \( L^p(0,T; V^p_{0, \text{div}}) \) the same inequality is true for any \( \vartheta \in L^p(0,T; V^p_{0, \text{div}}) \).

Let us prove now that \((P_u)\) admits an unique solution. Let us argue by contradiction and let us assume that this problem admits two solutions \((\overline{\pi}_1, \pi_1)\) and \((\overline{\pi}_2, \pi_2)\). So, we have the following two inequalities

\[
\int_0^T \left\langle \frac{\partial \overline{\pi}_1}{\partial t}, \vartheta - \overline{\pi}_1 \right\rangle_{V^p_{0, \text{div}}', V^p_{0, \text{div}}} dt + \left[ A_u(\overline{\pi}_1), \vartheta - \overline{\pi}_1 \right]_\Omega + J(\vartheta) - J(\overline{\pi}_1)
\geq \int_0^T (\overline{\pi}_1 - \vartheta)_{L^2(\Omega)} dt
\]

\[
\int_0^T \left\langle \frac{\partial \overline{\pi}_2}{\partial t}, \vartheta - \overline{\pi}_2 \right\rangle_{V^p_{0, \text{div}}', V^p_{0, \text{div}}} dt + \left[ A_u(\overline{\pi}_2), \vartheta - \overline{\pi}_2 \right]_\Omega + J(\vartheta) - J(\overline{\pi}_2)
\geq \int_0^T (\overline{\pi}_2 - \vartheta)_{L^2(\Omega)} dt
\]

for all \( \vartheta \in L^p(0,T; V^p_{0, \text{div}}) \).

Let \( t \in (0, T) \). We may choose \( \vartheta = \overline{\pi}_2 1_{[0,t]} + \overline{\pi}_1 (1 - 1_{[0,t]}) \) in the first inequality
and \( \vartheta = \overline{\pi}_1 1_{[0,t]} + \overline{\pi}_2 (1 - 1_{[0,t]}) \) in the second one, where \( 1_{[0,t]} \) denotes the indicatrix function of the interval \([0,t] \). By adding the two inequalities, we get

\[
\frac{1}{2} \| \overline{\pi}_1(t) - \overline{\pi}_2(t) \|_{L^2(\Omega)} + \left[ A_u(\overline{\pi}_1) - A_u(\overline{\pi}_2), (\overline{\pi}_1 - \overline{\pi}_2) 1_{[0,t]} \right]_\Omega \leq \frac{1}{2} \| \overline{\pi}_1(0) - \overline{\pi}_2(0) \|_{L^2(\Omega)} = 0.
\]

On the other hand

\[
\left[ A_u(\overline{\pi}_1) - A_u(\overline{\pi}_2), (\overline{\pi}_1 - \overline{\pi}_2) 1_{[0,t]} \right]_\Omega
= \int_0^t \int_{\Omega} (F(\theta, u + v_0 \xi, D(v_1 + v_0 \xi)) - F(\theta, u + v_0 \xi, D(v_2 + v_0 \xi))) : D(v_1 - v_2) \, dx \, dt.
\]
With Lemma 1 in [3] we know that the mapping $\Lambda : L^p(0,T;L^p(\Omega)) \to L^p(0,T;L^p(\Omega))$ defined by $\Lambda(u) = \overline{\sigma}$ where $(\overline{\sigma}, \overline{\pi})$ is the unique solution of problem (P) for all $u \in L^p(0,T;L^p(\Omega))$ i.e. $\Lambda(u) = \overline{\sigma}$ satisfies

$$\int_0^T \left( \frac{\partial \overline{\sigma}}{\partial t}, \varphi \right)_{L^2(\Omega)} + \int_0^T [A_u(\overline{\sigma}), \overline{\sigma}]_{L^2(\Omega)} \, dt + J(\overline{\sigma} + \overline{\pi}) - J(\overline{\sigma}) \geq \int_0^T \left( f + \frac{\partial \xi}{\partial t} v_0, \overline{\sigma} \right)_{L^2(\Omega)} \, dt \ \forall \overline{\sigma} \in L^p(0,T;V_{0,div}^p).$$

By using Schauder’s fixed point theorem we will prove that $\Lambda$ admits a fixed point and we will use De Rham’s theorem to construct the pressure term.

**Theorem 4.1.** Let $f \in L^{\hat{p}'}(0,T;L^2(\Omega))$, $k \in L^{\hat{p}'}(0,T;L^p(\Gamma_0))$, $\mu$ satisfying (2.4), $\theta \in L^{\hat{q}'}(0,T;L^p(\Omega))$ with $\hat{q} \geq 1$ and $\hat{p} \geq 1$, $s \in L^p(0,T;L^p(\Omega))$, $\xi \in W^{1,\hat{p}'}(0,T)$ satisfying (1.4) and $v_0 \in W^{1,\hat{p}'}(\Omega)$ satisfying (1.6). Then problem (P) admits a solution i.e. there exist $\overline{\sigma} \in C([0,T];L^2(\Omega)) \cap L^p(0,T;V_{0,div}^p)$ with $\frac{\partial \overline{\sigma}}{\partial t} \in L^{\hat{p}'}(0,T;V_{0,div}^{\hat{p}'})$ and $\pi \in H^{-1}(0,T;L^2(\Omega))$ such that

$$\int_0^T \left( \frac{\partial \overline{\sigma}}{\partial t}, \varphi \right)_{L^2(\Omega)} + \int_0^T \int_\Omega \bar{F}(\theta, \overline{\sigma} + v_0, \xi, D(\overline{\sigma} + v_0, \xi)) \cdot D(\overline{\sigma}) \, dx \, dt$$

$$- \int_0^T \int_\Omega \overline{\pi} \text{div}(\overline{\sigma}) \, dx \, dt + J(\overline{\sigma} + \overline{\pi}) - J(\overline{\sigma}) \geq \int_0^T \left( f + \frac{\partial \xi}{\partial t} v_0, \overline{\sigma} \right)_{L^2(\Omega)} \, dt \ \forall \overline{\sigma} \in V_{0,div}^p, \ \forall \overline{\pi} \in D(0,T)$$. 
and the initial condition
\[ \varphi(0) = \nu_0 - \nu_0 \xi(0) = 0 \quad \text{in } \Omega. \]

**Proof.** As a first step we prove that \( \Lambda \) satisfies the assumptions of Schauder’s fixed point theorem.

Let \( u \in L^p(0, T; L^p(\Omega)) \) and \( \Lambda(u) = \varphi \). With \( \varphi = -\varphi \) in (4.1) we obtain
\[
\int_0^T \left( \frac{\partial \varphi}{\partial t} \right) (\nu_0^p, \nu_0^p) dt + [A_u(\varphi, \varphi)] + J(\varphi) \leq \int_0^T (\overline{f}, \varphi)_{L^2(\Omega)} dt + J(0).
\]
Hence
\[
[A_u(\varphi, \varphi)] \leq \frac{1}{2} \| \varphi(T) \|^2_{L^2(\Omega)} + [A_u(\varphi, \varphi)] + J(\varphi) \leq \int_0^T (\overline{f}, \varphi)_{L^2(\Omega)} dt + J(0).
\]

With the same kind of computations as in Proposition 3.2 we get
\[
2(C_{\text{Korn},p})^p \mu_0 \left[ 1 - \frac{\| v_0 \xi \|_{L^p(0, T; V_0^p)}^p}{\| \varphi \|_{L^p(0, T; V_0^p)}^p} \right] \leq \tilde{C} \| \varphi \|_{L^p(0, T; L^2(\Omega))} \| \varphi \|_{L^p(0, T; V_0^p)}^{1-p}
\]
\[
+ \frac{J(0)}{\| \varphi \|_{L^p(0, T; V_0^p)}^p} + 2 \mu_1 \left( 1 + \frac{\| v_0 \xi \|_{L^p(0, T; V_0^p)}^p}{\| \varphi \|_{L^p(0, T; V_0^p)}^p} \right)^{-1} \frac{\| v_0 \xi \|_{L^p(0, T; V_0^p)}^p}{\| \varphi \|_{L^p(0, T; V_0^p)}^p}
\]
if \( \| \varphi \|_{L^p(0, T; V_0^p)}^p \neq 0 \), where we recall that \( \tilde{C} \) denotes the norm of the continuous injection of \( V_0^p \) into \( L^2(\Omega) \). Since the mapping
\[
z \mapsto 2(C_{\text{Korn},p})^p \mu_0 \left[ 1 - \frac{\| v_0 \xi \|_{L^p(0, T; V_0^p)}^p}{z} \right] \leq \tilde{C} \| \varphi \|_{L^p(0, T; L^2(\Omega))} \| \varphi \|_{L^p(0, T; V_0^p)}^{1-p} \frac{J(0)}{z^{p-1}} - \frac{J(0)}{z^p}
\]
\[
-2 \mu_1 \left( 1 + \frac{\| v_0 \xi \|_{L^p(0, T; V_0^p)}^p}{z} \right)^{-1} \| v_0 \xi \|_{L^p(0, T; V_0^p)}^p \frac{J(0)}{z}
\]
admits \( 2(C_{\text{Korn},p})^p \mu_0 > 0 \) as limit when \( z \) tends to \( +\infty \), there exists a real number \( C > 0 \), independent of \( u \), such that
\[
(4.2) \| \Lambda(u) \|_{L^p(0, T; V_0^p)} = \| \varphi \|_{L^p(0, T; V_0^p)} \leq C \quad \forall u \in L^p(0, T; L^p(\Omega)).
\]

Hence
\[
\| \Lambda(u) \|_{L^p(0, T; L^p(\Omega))} \leq C \quad \forall u \in L^p(0, T; L^p(\Omega))
\]
where \( C_p \) is the norm of the identity mapping from \( L^p(0, T; V_0^p) \) into \( L^p(0, T; L^p(\Omega)) \).

Let us consider now \( \varphi = \pm \partial \zeta \) with \( \partial \in V_0^p \) and \( \zeta \in D(0, T) \). With the same computations as in Proposition 3.2 we obtain
\[
\left| \int_0^T \left( \frac{\partial \varphi}{\partial t}, \partial \zeta \right)_{(V_0^p)^*, (V_0^p)^*} dt \right| \leq \left( \| \gamma_p \|_{L^1(\mathbb{R}^n \times \Omega)} \| k \|_{L^p(0, T; L^p(\Omega))} + \tilde{C} \| \varphi \|_{L^p(0, T; L^2(\Omega))} \right) \| \partial \zeta \|_{L^p(0, T; V_0^p)} + \| [A_u(\varphi), \partial \zeta] \|
\]
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Let us choose

\[ \text{(4.3)} \]

and

\[ \gamma \int_\Omega \partial_t \mathbf{u} \cdot \mathbf{v} \, \text{d}x = 0 \]

It follows that

\[ \frac{\partial \sigma}{\partial t} \in L^2(0,T;V^p_{0,div}) \]

for all \( \mathbf{u} \in L^p(0,T;L^p(\Omega)) \).

By using Aubin’s lemma we infer that the closed ball \( B = B_{L^p(0,T;L^p(\Omega))}(0,C) \) satisfies \( \Lambda(B) \subset B \) and that \( \Lambda(B) \) is relatively compact in \( L^p(0,T;L^p(\Omega)) \).

It remains to prove that the mapping \( \Lambda \) is continuous. Indeed, let \( (\mathbf{u}_n)_{n \geq 0} \) be a sequence of \( L^p(0,T;L^p(\Omega)) \) which converges strongly to \( \mathbf{u} \). Let us prove that

\[ \Lambda(\mathbf{u}_n) = \mathbf{v}_n \to \mathbf{v} = \Lambda(\mathbf{u}) \text{ strongly in } L^p(0,T;L^p(\Omega)). \]

The sequence \( (\mathbf{v}_n)_{n \geq 0} \) satisfies (4.2) and (4.3) so it admits strongly converging subsequences in \( L^p(0,T;L^p(\Omega)) \). We consider such a subsequence, still denoted \( (\mathbf{v}_n)_{n \geq 0} \). For all \( \mathbf{v} \in L^p(0,T;V^p_{0,div}) \) we have

\[ \int_0^T \left< \frac{\partial \mathbf{v}_n}{\partial t}, \mathbf{v} - \mathbf{v}_n \right>_{(V^p_{0,div})',V^p_{0,div}} \, dt + \left[ A_{u_0}(\mathbf{v}_n), \mathbf{v} - \mathbf{v}_n \right] + J(\mathbf{v}) - J(\mathbf{v}_n) \geq \int_0^T \left< \mathbf{T}, \mathbf{v} - \mathbf{v}_n \right>_{L^2(\Omega)} \, dt \]

and

\[ \int_0^T \left< \frac{\partial \mathbf{v}_n}{\partial t}, \mathbf{v} - \mathbf{v}_n \right>_{(V^p_{0,div})',V^p_{0,div}} \, dt + \left[ A_{u_0}(\mathbf{v}), \mathbf{v} - \mathbf{v} \right] + J(\mathbf{v}) - J(\mathbf{v}) \geq \int_0^T \left< \mathbf{T}, \mathbf{v} - \mathbf{v} \right>_{L^2(\Omega)} \, dt. \]

Let us choose \( \mathbf{v} = \mathbf{n} \) as test-function in the first inequality and \( \mathbf{v} = \mathbf{v}_n \) as test-function in second one. By adding the two inequalities we obtain

\[ \int_0^T \left< \frac{\partial \mathbf{v}_n}{\partial t} - \frac{\partial \mathbf{v}}{\partial t}, \mathbf{v}_n - \mathbf{v} \right>_{(V^p_{0,div})',V^p_{0,div}} \, dt + \left[ A_{u_0}(\mathbf{v}_n) - A_{u_0}(\mathbf{v}), \mathbf{v}_n - \mathbf{v} \right] \leq 0. \]

Thus

\[ \frac{1}{2} \left\| \mathbf{v}_n(T) - \mathbf{v}(T) \right\|_{L^2(\Omega)}^2 + \left[ A_{u_0}(\mathbf{v}_n) - A_{u_0}(\mathbf{v}), \mathbf{v}_n - \mathbf{v} \right] \leq \left[ A_{u_0}(\mathbf{v}) - A_{u_0}(\mathbf{v}), \mathbf{v} - \mathbf{v} \right]. \]

The second term of the left hand side may be rewritten as

\[ \left[ A_{u_0}(\mathbf{v}_n) - A_{u_0}(\mathbf{v}), \mathbf{v}_n - \mathbf{v} \right] \]

\[ = \int_0^T \int_\Omega \left( \mathcal{F}(\theta, \mathbf{u}_n + v_0 \xi, D(\mathbf{n}_n + v_0 \xi)) - \mathcal{F}(\theta, \mathbf{u}_n + v_0 \xi, D(\mathbf{n} + v_0 \xi)) \right) : D(\mathbf{n}_n - \mathbf{n}) \, dxdt \]

\[ = \int_0^T \int_\Omega \left( \mathcal{F}_1(D(\mathbf{n}_n + v_0 \xi)) - \mathcal{F}_1(D(\mathbf{n} + v_0 \xi)) \right) : D(\mathbf{n}_n - \mathbf{n}) \, dxdt \]

\[ + \int_0^T \int_\Omega \left( \mathcal{F}_2(\theta, \mathbf{u}_n + v_0 \xi, D(\mathbf{n}_n + v_0 \xi)) - \mathcal{F}_2(\theta, \mathbf{u}_n + v_0 \xi, D(\mathbf{n} + v_0 \xi)) \right) : D(\mathbf{n}_n - \mathbf{n}) \, dxdt \]
where 
\[ \mathcal{F}_1(\lambda_2) = \mu_0 |\lambda_2|^{p-2} \lambda_2 \quad \text{if } \lambda_2 \neq 0_{\mathbb{R}^{3 \times 3}}, \quad \mathcal{F}_1(\lambda_2) = 0_{\mathbb{R}^{3 \times 3}} \quad \text{otherwise} \]

\[
\begin{align*}
\mathcal{F}_2(\lambda_0, \lambda_1, \lambda_2) &= 2\pi(\lambda_0, \lambda_1, |\lambda_2|) |\lambda_2|^{p-2} \lambda_2 \quad \text{if } \lambda_2 \neq 0_{\mathbb{R}^{3 \times 3}}, \\
\mathcal{F}_2(\lambda_0, \lambda_1, \lambda_2) &= 0_{\mathbb{R}^{3 \times 3}} \quad \text{otherwise}
\end{align*}
\]

and \[ \overline{\pi} = \mu - \frac{\mu_0}{2}. \] Since \( \overline{\pi} \) satisfies
\[ d \mapsto \overline{\pi}(\cdot, d) \] is monotone increasing on \( \mathbb{R}_+ \),

\[ 0 < \frac{\mu_0}{2} \leq \overline{\pi}(o, e, d) \leq \mu_1 - \frac{\mu_0}{2} \] for all \((o, e, d) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+ \),

we infer with Lemma 1 in [3] that \( \lambda_2 \mapsto \mathcal{F}_2(\cdot, \cdot, \lambda_2) \) is monotone in \( \mathbb{R}^{3 \times 3} \). Hence

\[
\int_0^T \int_{\Omega} \left( \mathcal{F}_2(\theta, u_n + v_0 \xi, D(\overline{\pi}_n + v_0 \xi)) - \mathcal{F}_2(\theta, u_n + v_0 \xi, D(\overline{\pi} + v_0 \xi)) \right) : D(\overline{\pi}_n - \overline{\pi}) \, dx \, dt \geq 0
\]

and

\[
\left[ \mathcal{A}_{u_n}(\overline{\pi}_n) - \mathcal{A}_{u_n}(\overline{\pi}, \overline{\pi}_n - \overline{\pi}) \right] \geq \int_0^T \int_{\Omega} \left( \mathcal{F}_1(D(\overline{\pi}_n + v_0 \xi)) - \mathcal{F}_1(D(\overline{\pi} + v_0 \xi)) \right) : D(\overline{\pi}_n - \overline{\pi}) \, dx \, dt.
\]

But for all \((\lambda, \lambda') \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \) we have

\[
(\lambda + |\lambda'|)^{2-p} (\mathcal{F}_1(\lambda) - \mathcal{F}_1(\lambda')) : (\lambda - \lambda') \geq \mu_0 (p-1)|\lambda - \lambda'|^2.
\]

Indeed if \( \lambda = 0_{\mathbb{R}^{3 \times 3}} \) and/or \( \lambda' = 0_{\mathbb{R}^{3 \times 3}} \) the result is obvious. If \( |\lambda| = |\lambda'| \neq 0 \) we have

\[
(\lambda + |\lambda'|)^{2-p} (\mathcal{F}_1(\lambda) - \mathcal{F}_1(\lambda')) : (\lambda - \lambda') = \mu_0 2^{2-p}|\lambda - \lambda'|^2
\]

and the conclusion follows from the inequality \( 2^{2-p} > p - 1 \) for all \( p \in (1, 2) \). Otherwise, if \( \lambda \neq 0_{\mathbb{R}^{3 \times 3}}, \lambda' \neq 0_{\mathbb{R}^{3 \times 3}} \) and \( |\lambda| \neq |\lambda'| \), without loss of generality we may assume that \( |\lambda| > |\lambda'| \) and we let

\[
G(\lambda, \lambda') = \frac{(\lambda + |\lambda'|)^{2-p} (\mathcal{F}_1(\lambda) - \mathcal{F}_1(\lambda')) : (\lambda - \lambda')}{|\lambda - \lambda'|^2}.
\]

With the elementary algebraic computations we get

\[
G(\lambda, \lambda') = \frac{\mu_0}{2} \left( (\lambda + |\lambda'|)^{2-p} \right) \frac{2^{2-p}}{|\lambda - \lambda'|^2} \left( (|\lambda|^{p-2} + |\lambda'|^{p-2}) |\lambda - \lambda'|^2 + (|\lambda|^2 - |\lambda'|^2) (|\lambda|^{p-2} - |\lambda'|^{p-2}) \right)
\]

\[
\geq \frac{\mu_0}{2} \left( (|\lambda| + |\lambda'|)^{2-p} \right) \frac{2^{2-p}}{|\lambda - \lambda'|^2} \left( (|\lambda|^{p-2} + |\lambda'|^{p-2}) + \frac{(|\lambda| + |\lambda'|) (|\lambda|^{p-2} - |\lambda'|^{p-2})}{|\lambda| - |\lambda'|} \right).
\]

Let \( t = \frac{|\lambda|}{|\lambda'|} > 1 \). Thus

\[
G(\lambda, \lambda') \geq \frac{\mu_0}{2} \left( (1 + t)^{2-p} \right) \left( (1 + t^{p-2}) + \frac{(1 + t)(t^{p-2} - 1)}{t - 1} \right)
\]

\[
= \frac{\mu_0}{2} \left( (1 + t^{p-2}) + (t^{p-2} - 1) \right) \geq \frac{\mu_0}{t} (1 - t^{1-p}) = \mu_0 \left( 1 - \frac{t^{2-p} - 1}{t - 1} \right).
\]

But, for all \( t > 1 \) we have \( \frac{t^{2-p} - 1}{t - 1} < 2 - p \) and (4.4) is satisfied.
Hence
\[
\left( (|\lambda| + |\lambda'|)^{p-\frac{1}{2}} \left( \mathcal{F}_1(\lambda) - \mathcal{F}_1(\lambda') \right) : (\lambda - \lambda') \right)^{\frac{2}{p-1}} \geq \left( \mu_0(p-1) \right)^{\frac{2}{p-1}} |\lambda - \lambda'|^p.
\]

Since \( p > 1 \) we have also
\[
(|\lambda| + |\lambda'|)^p \leq 2^{p-1} (|\lambda|^p + |\lambda'|^p)
\]
which yields
\[
(|\lambda| + |\lambda'|)^{p-\frac{1}{2}} \left( \mathcal{F}_1(\lambda) - \mathcal{F}_1(\lambda') \right) : (\lambda - \lambda') \geq \frac{\left( \mu_0(p-1) \right)^{\frac{2}{p-1}}}{2^{\frac{p-1}{2}}} |\lambda - \lambda'|^p 
\]
for all \((\lambda, \lambda') \in \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \).

By replacing \( \lambda = D(\nabla_n + v_0\xi) \), \( \lambda' = D(\nabla + v_0\xi) \) and using Hölder’s inequality we obtain
\[
\frac{\left( \mu_0(p-1) \right)^{\frac{2}{p-1}}}{2^{\frac{p-1}{2}}} \int_0^T \int_\Omega |D(\nabla_n - \nabla)|^p \, dx \, dt
\leq \left( \int_0^T \int_\Omega \left( \mathcal{F}_1(D(\nabla_n + v_0\xi)) - \mathcal{F}_1(D(\nabla + v_0\xi)) \right) : D(\nabla_n - \nabla) \, dx \, dt \right)^{\frac{2}{p-1}}
\]
\[
\times \left( \int_0^T \int_\Omega \left( |D(\nabla_n + v_0\xi)|^p + |D(\nabla + v_0\xi)|^p \right) \, dx \, dt \right)^{\frac{2}{p-1}}
\leq \left( \int_0^T \int_\Omega \left( \mathcal{F}_1(D(\nabla_n + v_0\xi)) - \mathcal{F}_1(D(\nabla + v_0\xi)) \right) : D(\nabla_n - \nabla) \, dx \, dt \right)^{\frac{2}{p-1}}
\]
\[
\times \left( \|\nabla_n + v_0\xi\|_{L^p(0,T;V_{\xi,T}^p)}^p \right)^{\frac{2}{p-1}}.
\]

and with (4.2)
\[
\frac{\left( \mu_0(p-1) \right)^{\frac{2}{p-1}}}{2^{\frac{p-1}{2}}} \|D(\nabla_n - \nabla)\|_{L^p(0,T;(L^p(\Omega))^{3\times3})}^p
\leq \left( \int_0^T \int_\Omega \left( \mathcal{F}_1(D(\nabla_n + v_0\xi)) - \mathcal{F}_1(D(\nabla + v_0\xi)) \right) : D(\nabla_n - \nabla) \, dx \, dt \right)^{\frac{2}{p-1}}
\]
\[
\times \left( 2(C + \|v_0\xi\|_{L^p(0,T;V_{\xi,T}^p)})^p \right)^{\frac{2}{p-1}}.
\]

Hence
\[
[A_{u_n}(\nabla_n) - A_{u_n}(\nabla), \nabla_n - \nabla] \geq \frac{\mu_0(p-1)}{2(2-p) \left( C + \|v_0\xi\|_{L^p(0,T;V_{\xi,T}^p)} \right)^{2-p}} \|D(\nabla_n - \nabla)\|_{L^p(0,T;(L^p(\Omega))^{3\times3})}^2.
\]

On the other hand,
\[
[A_{u}(\nabla) - A_{u_n}(\nabla), \nabla_n - \nabla]
= \int_0^T \int_\Omega \left( \mathcal{F}_2(\theta, \nabla, v_0\xi, D(\nabla + v_0\xi)) - \mathcal{F}_2(\theta, \nabla_n + v_0\xi, D(\nabla + v_0\xi)) \right) : D(\nabla_n - \nabla) \, dx \, dt
\leq \left( \int_0^T \int_\Omega \left( \mathcal{F}_2(\theta, \nabla, v_0\xi, D(\nabla + v_0\xi)) - \mathcal{F}_2(\theta, \nabla_n + v_0\xi, D(\nabla + v_0\xi)) \right) : D(\nabla_n - \nabla) \, dx \, dt \right)^{\frac{2}{p-1}}
\]
\[
\times \|D(\nabla_n - \nabla)\|_{L^p(0,T;(L^p(\Omega))^{3\times3})}^p.
\]
and we get
\[
\left\| \mathcal{F}_2(\theta, u + v_0 \xi, D(\nabla + v_0 \xi)) - \mathcal{F}_2(\theta, u_n + v_0 \xi, D(\nabla + v_0 \xi)) \right\|_{L^{p'}(0,T;(L^{p'}(\Omega))^3 \times 3)} \\
\times \left\| D(\nabla_n - \nabla) \right\|_{L^p(0,T;(L^p(\Omega))^3 \times 3)} \\
\leq \frac{1}{4c} \left\| \mathcal{F}_2(\theta, u + v_0 \xi, D(\nabla + v_0 \xi)) - \mathcal{F}_2(\theta, u_n + v_0 \xi, D(\nabla + v_0 \xi)) \right\|^2_{L^{p'}(0,T;(L^{p'}(\Omega))^3 \times 3)} \\
+ \epsilon \left\| D(\nabla_n - \nabla) \right\|^2_{L^p(0,T;(L^p(\Omega))^3 \times 3)}
\]

where we choose \( \epsilon = \frac{1}{2} \frac{\mu_0(p - 1)}{2^{2(p-1)}} (C + \|v_0\xi\|_{L^p(0,T;V^p_{div}})^{(2-p)} \). We obtain

\[
\frac{1}{2} \left\| \nabla_n(T) - \nabla(T) \right\|^2_{L^2(\Omega)} + \epsilon (C_{Korn,p})^2 \left\| \nabla_n - \nabla \right\|^2_{L^p(0,T;L^p(\Omega))} \\
\leq \frac{1}{4c} \left\| \mathcal{F}_2(\theta, u + v_0 \xi, D(\nabla + v_0 \xi)) - \mathcal{F}_2(\theta, u_n + v_0 \xi, D(\nabla + v_0 \xi)) \right\|^2_{L^{p'}(0,T;(L^{p'}(\Omega))^3 \times 3)}.
\]

The sequence \((u_n)_{n \geq 0}\) converges to \(u\) strongly in \(L^p(0,T;L^p(\Omega))\). So, by possibly extracting a subsequence still denoted \((u_n)_{n \geq 0}\), we have

\[
u_n(t, x) \longrightarrow u(t, x) \quad \forall \text{ a.a. } (x, t) \in (0, T) \times \Omega.
\]

By using the continuity and the uniform boundedness of the mapping \(\nabla\), we infer from Lebesgue's theorem that

\[
\mathcal{F}_2(\theta, u_n + v_0 \xi, D(\nabla + v_0 \xi)) \longrightarrow \mathcal{F}_2(\theta, u + v_0 \xi, D(\nabla + v_0 \xi)) \quad \text{strongly in} \quad L^{p'}(0,T;(L^{p'}(\Omega))^3 \times 3).
\]

It follows that

\[
\lim_{n \to +\infty} \left\| \nabla_n - \nabla \right\|_{L^p(0,T;L^p(\Omega))} = 0.
\]

So, any subsequence \((\nabla_n)_{n \geq 0}\) which is strongly convergent in \(L^p(0,T;L^p(\Omega))\) converges to \(\nabla = \Lambda(u)\). Recalling that the whole sequence \((\nabla_n)_{n \geq 0}\) is bounded in \(L^p(0,T;L^p(\Omega))\), we infer that the whole sequence \((\nabla_n)_{n \geq 0}\) converges to \(\nabla = \Lambda(u)\) which proves the continuity of the mapping \(\Lambda\).

With Schauder's fixed point theorem, we may conclude that \(\Lambda\) admits a fixed point \(u \in L^p(0,T;L^p(\Omega))\). Let us denote \(\Lambda(u) = u = \nabla\). We obtain that \(\nabla \in C(0,T;L^2(\Omega)) \cap L^p(0,T;V^p_{div})\) with \(\frac{\partial \nabla}{\partial t} \in L^{p'}(0,T; (V^p_{div})')\) and satisfies

\[
\int_0^T \left( \frac{\partial \nabla}{\partial t}, V_{div}^p, V^p_{div} \right) dt + \left[ A_p(\nabla), \nabla \right] \\
+ J(\nabla + \nabla) - J(\nabla) \geq \int_0^T \left( f + \frac{\partial \xi}{\partial t} v_0, \nabla \right)_{L^2(\Omega)} dt \
\forall \nabla \in L^p(0,T;V^p_{div})
\]

with \(\nabla(0) = 0\).

In order to conclude the study of problem (P) it remains now to construct the pressure term. As usual the key tool is De Rham's theorem.
Reminding that $\nabla \in C([0, T]; L^2(\Omega))$ we may define $F(t) \in (V^p_0)'$ for all $t \in [0, T]$ by

$$F(t)(\hat{\phi}) = \left( \int_0^t \nabla \cdot d\tilde{\tau}, \hat{\phi} \right)_{L^2(\Omega)} - (\nabla(t), \hat{\phi})_{L^2(\Omega)}$$

$$- \int_0^t \int_{\Omega} f(\theta, \nabla + \nabla, D(\nabla + \nabla)) : D(\hat{\phi}) \, dx \, dt$$

which implies that $F(t)(\hat{\phi}) = 0$. We infer that, for all $t \in [0, T]$, there exists a unique distribution $\tilde{\pi}(t) \in L^p_0(\Omega)$ such that

$$F(t) = \nabla \tilde{\pi}(t)$$

(see for instance Lemma 2.7 in [1]). Since the gradient operator is an endomorphism from $L^p_0(\Omega)$ into $W^{-1,p'}(\Omega)$ (see Corollary 2.5 in [1]), we obtain that $\nabla \tilde{\pi} \in C([0, T]; W^{-1,p'}(\Omega))$ and $\tilde{\pi} \in C([0, T]; L^p_0(\Omega))$. Then, for all $t \in [0, T]$ and for all $\hat{\phi} \in D(\Omega)$, we have

$$F(t)(\hat{\phi}) = \left( \int_0^t \nabla \cdot d\tilde{\tau}, \hat{\phi} \right)_{L^2(\Omega)} - (\nabla(t), \hat{\phi})_{L^2(\Omega)}$$

$$- \int_0^t \int_{\Omega} f(\theta, \nabla + \nabla, D(\nabla + \nabla)) : D(\hat{\phi}) \, dx \, dt$$

and with Green’s formula

$$\left( \int_0^t \nabla \cdot d\tilde{\tau}, \hat{\phi} \right)_{L^2(\Omega)} - (\nabla(t), \hat{\phi})_{L^2(\Omega)} - \int_0^t \int_{\Omega} f(\theta, \nabla + \nabla, D(\nabla + \nabla)) : D(\hat{\phi}) \, dx \, dt$$

$$= - (\nabla \tilde{\pi}(t), \hat{\phi})_{D'(\Omega), D(\Omega)} = - \int_{\Omega} \tilde{\pi}(t) \, div(\hat{\phi}) \, dx.$$
Hence, for all $\zeta \in D(0, T)$ and $\tilde{\vartheta} \in D(\Omega) = (D(\Omega))^3$, we have
\[
\int_0^T \frac{\partial}{\partial t} (\varphi, \tilde{\vartheta})_{L^2(\Omega)} \zeta \, dt + \int_0^T \int_\Omega F(\varphi, \varpi + \nu_0 \xi, D(\varpi + \nu_0 \xi)) : D(\tilde{\vartheta} \zeta) \, dx dt \\
- \left\langle \int_\Omega \pi \text{div}(\tilde{\vartheta}) \, dx, \zeta \right\rangle_{D'(0, T), D(0, T)} = \int_0^T (\mathcal{F}, \tilde{\vartheta})_{L^2(\Omega)} \zeta \, dt.
\]

But, for all $\omega \in L^p(\Omega)$ we know that there exists $\tilde{\vartheta} \in W^{1, p}(\Omega)$ such that
\[
\text{div}(\tilde{\vartheta}) = \omega
\]
and the mapping $P_p : L^p(\Omega) \to W^{1, p}(\Omega)$ given by $\tilde{\vartheta} = P_p(\omega)$ is linear and continuous (see Corollary 3.1 in [4]). So for all $\omega \in L^p(\Omega)$ we have
\[
\left\langle \int_\Omega \pi \omega \, dx, \zeta \right\rangle_{D'(0, T), D(0, T)} = - \int_0^T (\varpi, P_p(\omega))_{L^2(\Omega)} \zeta' \, dt \
+ \int_0^T \int_\Omega F(\varphi, \varpi + \nu_0 \xi, D(\varpi + \nu_0 \xi)) : D(\omega'(\varpi) \zeta) \, dx dt - \int_0^T (\mathcal{F}, P_p(\omega))_{L^2(\Omega)} \zeta \, dt
\]
and
\[
\left\langle \int_\Omega \pi \omega \, dx, \zeta \right\rangle_{D'(0, T), D(0, T)} \leq \|\varpi\|_{L^2(0, T; L^2(\Omega))} \|P_p(\omega)\|_{W^{1, p}(\Omega)} + 2 \mu T \|\varpi\|_{L^p(0, T; V^p_{d, a})} + \|D(\nu_0 \xi)\|_{L^p(0, T; (L^p(\Omega)^{3 \times 3}))}^{L^{-1}} \|P_p(\omega)\|_{W^{1, p}(\Omega)} + T \|\mathcal{F}\|_{L^p(0, T; L^2(\Omega))} \|P_p(\omega)\|_{W^{1, p}(\Omega)}.
\]

By using the continuous embedding of $W^{1, p}(\Omega)$ into $L^2(\Omega)$ and $H^1_0(0, T; \mathbb{R})$ into $L^\infty(0, T; \mathbb{R})$ we infer from (4.6) that there exists a positive real number $C^*$, independent of $\omega$ and $\zeta$, such that
\[
\left\langle \int_\Omega \pi \omega \, dx, \zeta \right\rangle_{D'(0, T), D(0, T)} \leq C^* \|\omega\|_{H^1(0, T; L^p(\Omega))} \quad \forall \omega \in L^p(\Omega), \forall \zeta \in D(0, T).
\]

Then, for all $\omega^* \in L^p(\Omega)$, we may apply the previous inequality with
\[
\omega = \omega^* - \frac{1}{\text{meas}(\Omega)} \int_\Omega \omega^* \, dx \quad \in L^p(\Omega).
\]

Since $\pi \in D'(0, T; L^p_0(\Omega))$ we have
\[
\left\langle \int_\Omega \pi \omega \, dx, \zeta \right\rangle_{D'(0, T), D(0, T)} = \left\langle \int_\Omega \pi \left(\omega^* - \frac{1}{\text{meas}(\Omega)} \int_\Omega \omega^* \, dx\right) \, dx, \zeta \right\rangle_{D'(0, T), D(0, T)}
= \left\langle \int_\Omega \pi \omega^* \, dx, \zeta \right\rangle_{D'(0, T), D(0, T)}
\]
and
\[
\|\omega\|_{L^p(V)} \leq 2 \|\omega^*\|_{L^p(\Omega)}.
\]

So, for all $\omega^* \in L^p(\Omega)$ and $\zeta \in D(0, T)$ we obtain
\[
\left| \left\langle \int_\Omega \pi \omega^* \, dx, \zeta \right\rangle_{D'(0, T), D(0, T)} \right| \leq 2C^* \|\omega^*\|_{H^1(0, T; L^p(\Omega))}.
\]

Finally the density of $D(0, T) \otimes L^p(\Omega)$ into $H^1_0(0, T; L^p(\Omega))$ allows us to conclude that $\pi \in H^{-1}(0, T; L^p_0(\Omega))$. 

Now let \( \hat{\vartheta} \in D(\Omega) = \{D(\Omega)\}^3 \) and \( \zeta \in D(0,T) \). We have
\[
\int_0^T (\overline{\tau}, \hat{\vartheta})_{L^2(\Omega)} dt = \int_0^T \frac{\partial}{\partial t} (\overline{\tau}, \hat{\vartheta})_{L^2(\Omega)} dt
\]
\[
= \int_\Omega \int_0^T \mathcal{F}(\theta, \overline{\vartheta} + v_0 \xi, D(\overline{\vartheta} + v_0 \xi)) : D(\hat{\vartheta} \zeta) dxdt + \int_0^T \langle \nabla \varpi, \hat{\vartheta} \rangle_{D'(\Omega), D(\Omega)} dt
\]
and with Green’s formula
\[
\int_0^T (\overline{\tau}, \hat{\vartheta})_{L^2(\Omega)} dt = \int_0^T \frac{\partial}{\partial t} (\overline{\tau}, \hat{\vartheta})_{L^2(\Omega)} dt = - \int_0^T \langle \text{div}(\sigma), \hat{\vartheta} \rangle_{D'(\Omega), D(\Omega)} dt
\]
where \( \sigma = \mathcal{F}(\theta, \overline{\vartheta} + v_0 \xi, D(\overline{\vartheta} + v_0 \xi)) - \pi \text{Id} \). It follows that
\[
\left| \int_0^T \langle \text{div}(\sigma), \hat{\vartheta} \rangle_{D'(\Omega), D(\Omega)} dt \right| 
\leq (C_\infty T)^\frac{1}{2} \| \|_{L^p(0,T; L^2(\Omega))} + \| \overline{\tau} \|_{L^2(0,T; L^2(\Omega))} \| \hat{\vartheta} \|_{H^1(0,T; L^2(\Omega))}
\]
where we recall that \( C_\infty \) denotes the norm of the continuous injection of \( H^1(0,T; \mathbb{R}) \) into \( L^\infty(0,T; \mathbb{R}) \). By density of \( D(0,T) \otimes D(\Omega) \) into \( H^1_0(0,T; L^2(\Omega)) \) we may conclude that \( \text{div}(\sigma) \in H^{-1}(0,T; L^2(\Omega)) \) and we have
\[
\int_0^T (\overline{\tau}, \hat{\vartheta})_{L^2(\Omega)} dt - \int_0^T \frac{\partial}{\partial t} (\overline{\tau}, \hat{\vartheta})_{L^2(\Omega)} dt
\]
\[
= - \int_0^T \int_\Omega \text{div}(\sigma) \hat{\vartheta} \zeta dxdt \quad \forall \hat{\vartheta} \in L^2(\Omega), \forall \zeta \in D(0,T).
\]
Next we observe that \( \sigma \in H^{-1}(0,T; \tilde{\mathcal{Y}}) \) where
\[
\tilde{\mathcal{Y}} = \{ \hat{\vartheta} \in (L^2(\Omega))^{3 \times 3}; \text{div}(\hat{\vartheta}) \in L^2(\Omega) \}.
\]
It follows that we may use Green’s formula and we get
\[
- \int_0^T \int_\Omega \text{div}(\sigma) \hat{\vartheta} \zeta dxdt = \int_0^T \int_\Omega \sigma \cdot \nabla \hat{\vartheta} \zeta dxdt - \int_0^T \int_{\partial \Omega} \sum_{i,j=1}^3 \sigma_{ij} n_j \hat{\vartheta}_i \zeta dY dt
\]
for all \( \hat{\vartheta} \in W^{1,2}(\Omega) \) and for all \( \zeta \in D(0,T) \).

Therefore, for all \( \hat{\vartheta} \in V^0_\Omega \subset V^0_\Omega \) and for all \( \zeta \in D(0,T) \) we get
\[
\int_0^T \frac{\partial}{\partial t} (\pi, \hat{\vartheta})_{L^2(\Omega)} dt + \int_\Omega \int_0^T \mathcal{F}(\theta, \overline{\vartheta} + v_0 \xi, D(\overline{\vartheta} + v_0 \xi)) : D(\hat{\vartheta} \zeta) dxdt
\]
\[
- \int_0^T \int_\Omega \pi \text{div}(\hat{\vartheta}) \zeta dxdt + J(\overline{\vartheta} + \hat{\vartheta} \zeta) - J(\overline{\vartheta}) = \int_0^T (\overline{\tau}, \hat{\vartheta})_{L^2(\Omega)} dt + A(\overline{\vartheta}, \hat{\vartheta})
\]
where
\[
A(\overline{\vartheta}, \hat{\vartheta}) = \int_0^T \int_{\Gamma_0} (\hat{\vartheta}_\tau \cdot \hat{\vartheta}) \zeta dx'dt + J(\overline{\vartheta} + \hat{\vartheta} \zeta) - J(\overline{\vartheta}).
\]
But
\[
\int_{\partial \Omega} \hat{\vartheta} \cdot n dY = 0 \quad \forall \hat{\vartheta} \in V^0_\Omega.
\]
We infer that there exists \( \hat{\vartheta} \in W^{1,2}(\Omega) \) satisfying
\[
\text{div}(\hat{\vartheta}) = 0 \quad \text{in} \ \Omega, \quad \hat{\vartheta} = \hat{\vartheta} \quad \text{on} \ \partial \Omega.
\]
(see for instance Lemma 3.3 in [1] or Chapter 1, Lemma 2.2 in [15]). Hence \( \hat{\vartheta} \in V_0^{2, \text{div}} \subset V_0^p \). \( \text{div} \) and with \( \varphi = \hat{\vartheta} \zeta \) in (4.5) we obtain
\[
\int_0^T \frac{\partial}{\partial t} (\vartheta, \hat{\vartheta})_{L^2(\Omega)} \, dt + \int_0^T \int_{\Omega} F(\theta, \vartheta + v_0 \xi, D(\vartheta + v_0 \xi)) : D(\hat{\vartheta} \zeta) \, dx \, dt
\]
and thus \( A(\vartheta, \hat{\vartheta}) \geq 0 \). By observing that \( A(\vartheta, \hat{\vartheta}) = A(\vartheta, \hat{\vartheta}) \) since \( \hat{\vartheta} = \hat{\vartheta} \) on \( \partial \Omega \) we get \( A(\vartheta, \hat{\vartheta}) \geq 0 \) and (4.7) becomes
\[
\int_0^T \frac{\partial}{\partial t} (\vartheta, \hat{\vartheta})_{L^2(\Omega)} \, dt + \int_0^T \int_{\Omega} F(\theta, \vartheta + v_0 \xi, D(\vartheta + v_0 \xi)) : D(\hat{\vartheta} \zeta) \, dx \, dt
\]
\[
- \int_0^T \int_{\Omega} \pi \text{div}(\vartheta) \zeta \, dx \, dt + J(\vartheta + \hat{\vartheta} \zeta) - J(\vartheta) \geq \int_0^T (f, \hat{\vartheta})_{L^2(\Omega)} \, dt \quad \forall \hat{\vartheta} \in V_0^p, \forall \zeta \in \mathcal{D}(0, T).
\]
By density of \( V_0^p \) into \( V_0^p \) the same inequality holds for all \( \hat{\vartheta} \in V_0^p \) and \( \zeta \in \mathcal{D}(0, T) \) and we may conclude that \((\vartheta, \pi)\) is a solution of problem (P).

\[\square\]

**Remark 4.2.** If the viscosity \( \mu \) does not depend on the velocity, then Theorem 3.8 yields directly the existence and uniqueness of a solution to problem (P).

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