A TWO-PHASE FLOW MODEL WITH DELAYS

THEODORE TACHIM MEDJO

Department of Mathematics, Florida International University, DM413B
University Park
Miami, Florida 33199, USA

(Communicated by José A. Langa)

Abstract. In this article, we study a coupled Allen-Cahn-Navier-Stokes model with delays in a two-dimensional domain. The model consists of the Navier-Stokes equations for the velocity, coupled with an Allen-Cahn model for the order (phase) parameter. We prove the existence and uniqueness of the weak and strong solution when the external force contains some delays. We also discuss the asymptotic behavior of the weak solutions and the stability of the stationary solutions.

1. Introduction. It is well accepted that the incompressible Navier-Stokes (NS) equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [11]. For instance, this approach is used in [1] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with the phase-field system, [2, 11, 10, 12]. In the isothermal compressible case, the existence of a global weak solution is proved in [9]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity $v$ and the order parameter $\phi$. This system can be written as a NS equation coupled with a convective Allen-Cahn equation, [11]. The associated initial and boundary value problem was studied in [11] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor. They also established the existence of an exponential attractor. This entails that the global attractor has a finite fractal dimension, which is estimated in [11] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated although some important issues remain unresolved, [21]. In the case of binary fluids, the analysis is even more complicate and the mathematical studied is still at it infancy as noted in [11]. As noted in [10], the mathematical analysis of binary fluid flows is far from being well understood. For instance, the spinodal decomposition under shear consists of a two-stage evolution of a homogeneous initial mixture: a phase separation stage in which some macroscopic patterns appear, then a shear stage in which these patterns organize themselves into parallel layers (see, e.g. [18] for experimental snapshots).

2010 Mathematics Subject Classification. 35Q30, 35Q35, 35Q72.
Key words and phrases. Allen-Cahn, Navier-Stokes, delays, strong solutions, stability.
This model has to take into account the chemical interactions between the two phases at the interface, achieved using a Cahn-Hilliard approach, as well as the hydrodynamic properties of the mixture (e.g., in the shear case), for which a Navier-Stokes equations with surface tension terms acting at the interface are needed. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but non-zero thickness, a well-known model is the so-called “Model H” (cf. [13]). This is a system of equations where an incompressible Navier-Stokes equation for the (mean) velocity $v$ is coupled with a convective Cahn-Hilliard equation for the order parameter $\phi$, which represents the relative concentration of one of the fluids.

Time delays are omnipresent and virtually unavoidable in everyday life. They appear in applications such as physics, biology, epidemics, fluid control, transport and population models. Because of their importance, differential equations with delays have received considerable attention, [15, 3, 5, 6, 7, 16, 8, 17, 20, 22]. In real world applications, to control a system by applying some type of external forces, it is natural that these forces take into account not only the present state of the system, but also its history, [3]. In fact, in many cases it was shown that the presence of a delay term in a differential equations can be a source of instability and even an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay, [16, 8]. The presence of a delay term in a differential equations can drastically change the mathematical analysis of the model. For instance, when a delay term, general enough is added in equations such as the 2D Navier-Stokes system or the 2D Allen-Cahn-Navier-Stokes system, special attention is needed to analyse the global asymptotic behavior of the system since one needs to consider a semigroup in a different phase space, [3, 19]. In the stochastic setting, the behaviors of a stochastic differential equation with delays and without delays can be very different. More precisely, if a delay term is added to a stochastic differential equation, its solution is no longer Markov and losing the Markovian properties makes the analysis of the system more difficult and complex, [15].

In [5, 6, 7], the authors studied the NS equations in which the forcing term contains some hereditary features. The model can be used for instance to control a system by applying a force which takes into account not only the present state of the system, but also the history of the solutions. The existence and uniqueness of solutions to the 2D NS equations with delays was investigated in [5] and the asymptotic behavior of the solutions is studied in [6]. The existence of attractors for the 2D NS equations with delays is proved in [7]. In [4], the authors studied the existence of an attractor for the 3D Lagrangian averaged Navier-Stokes–$\alpha$ (3D LAN-$\alpha$) model with delays. Instead of working directly with the 3D LAN-$\alpha$ model, they proved the existence of attractors for an abstract delay model and then applied the result to the 3D LAN-$\alpha$ model.

Motivated by the above works, we study in this article an AC-NS model with delays. We prove the existence and uniqueness of a weak and a strong solutions when the external force contains some delays. Let us note that the coupling between the Navier-Stokes and the Allen-Cahn systems makes the analysis more involved. In [19], we proved the existence of an attractor for the model using the theory of pullback attractors.

The article is divided as follows. In the next section, we introduce the AC-NS model with delays and its mathematical setting. The third section studies the
existence of solutions when the delay term satisfies some hypothesis similar to that of [5, 6, 7]. In the fourth section, we study the asymptotic behavior of the weak solutions when the delay term satisfies some hypothesis used in [20]. The stability of the stationary solutions is analyzed in the fifth section.

2. The AC-NS model and its mathematical setting.

2.1. Governing equations. In this article, we consider a model of homogeneous incompressible two-phase flow with delays. More precisely, we assume that the domain $M$ of the fluid is a bounded domain in $\mathbb{R}^2$. Then, we consider the system

$$\begin{cases}
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + \nabla p - K\mu \nabla \phi = Q(t - \tau(t), (v, \phi)(t - \tau(t))), \\
\text{div} v = 0, \\
\frac{\partial \phi}{\partial t} + v \cdot \nabla \phi + \mu = 0, \mu = -\epsilon \Delta \phi + \alpha f(\phi),
\end{cases}$$

(1)
in $M \times (0, +\infty)$.

In (1), the unknown functions are the velocity $v = (v_1, v_2)$ of the fluid, its pressure $p$ and the order (phase) parameter $\phi$. The quantity $\mu$ is the variational derivative of the following free energy functional

$$F(\phi) = \int_M \left( \frac{\epsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) ds,$$

(2)

where, e.g., $F(r) = \int_0^r f(\zeta)d\zeta$. Here, the constants $\nu > 0$ and $K > 0$ correspond to the kinematic viscosity of the fluid and the capillarity (stress) coefficient respectively, $\epsilon, \alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, $\epsilon$ is related with the thickness of the interface separating the two fluid. Hereafter, as in [11] we assume that $\epsilon \leq \alpha$.

We endow (1) with the boundary condition

$$v = 0, \frac{\partial \phi}{\partial \eta} = 0 \text{ on } \partial M \times (0, +\infty),$$

(3)

where $\partial M$ is the boundary of $M$ and $\eta$ is its outward normal.

The initial condition is given by

$$(v, \phi)(s) = \vartheta(s) = (\vartheta_1, \vartheta_2)(s) \in [-r, 0],$$

(4)

The external forcing $Q$ takes into account not only the present state of the system, but also the history of the solutions.

2.2. Mathematical setting. We first recall from [11] a weak formulation of (1)-(4). Hereafter, we assume that the domain $M$ is bounded with a smooth boundary $\partial M$ (e.g., of class $C^2$). We also assume that $f \in C^1(\mathbb{R})$ satisfies

$$\begin{cases}
\lim_{|r| \to +\infty} f'(r) > 0, \\
|f'(r)| \leq c_f(1 + |r|^k), \forall r \in \mathbb{R},
\end{cases}$$

(5)

where $c_f$ is some positive constant and $k \in [1, +\infty)$ is fixed. It follows from (5) that

$$|f(r)| \leq c_f(1 + |r|^{k+1}), \forall r \in \mathbb{R}.$$
If $X$ is a real Hilbert space with inner product $(\cdot, \cdot)_X$, we will denote the induced norm by $\| \cdot \|_X$, while $X'$ will indicate its dual. We set 

$$V_1 = \{ v \in C_c^\infty(\mathcal{M}) : \text{div} v = 0 \text{ in } \mathcal{M} \}.$$ 

We denote by $H_1$ and $V_1$ the closure of $V_1$ in $(L^2(\mathcal{M}))^2$ and $(H_0^1(\mathcal{M}))^2$ respectively. The scalar product in $H_1$ is denoted by $(\cdot, \cdot)_{L^2}$ and the associated norm by $\| \cdot \|_{L^2}$. Moreover, the space $V_1$ is endowed with the scalar product 

$$(u, v) = \sum_{i=1}^2 (\partial_x u, \partial_x v)_{L^2}, \quad \| u \| = ((u, u))^{1/2}. $$

We now define the operator $A_0$ by

$$A_0 u = \mathcal{P}\Delta u, \quad \forall u \in D(A_0) = H^2(\mathcal{M}) \cap V_1,$$

where $\mathcal{P}$ is the Leray-Helmholtz projector in $L^2(\mathcal{M})$ onto $H_1$. Then, $A_0$ is a self-adjoint positive unbounded operator in $H_1$ which is associated with the scalar product defined above. Furthermore, $A_0^{-1}$ is a compact linear operator on $H_1$ and $|A_0 \cdot |_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the $H^2-$norm. 

Note that from (5), we can find $\gamma > 0$ such that

$$\lim_{|r| \to +\infty} f'(r) > 2\gamma > 0. \quad (7)$$

We define the linear positive unbounded operator $A_\gamma$ on $L^2(\mathcal{M})$ by:

$$A_\gamma \phi = -\Delta \phi + \gamma \phi, \quad \forall \phi \in D(A_\gamma), \quad (8)$$

where

$$D(A_\gamma) = \left\{ \rho \in H^2(\mathcal{M}) ; \frac{\partial \rho}{\partial \eta} = 0 \text{ on } \partial \mathcal{M} \right\}.$$ 

Note that $A_\gamma^{-1}$ is a compact linear operator on $L^2(\mathcal{M})$ and $|A_\gamma \cdot |_{L^2}$ is a norm on $D(A_\gamma)$ that is equivalent to the $H^2-$norm.

We introduce the bilinear operators $B_0$, $B_1$ (and their associated trilinear forms $b_0$, $b_1$) as well as the coupling mapping $R_0$, which are defined from $D(A_0) \times D(A_0)$ into $H_1$, $D(A_0) \times D(A_\gamma)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M}) \times D(A_\gamma^{3/2})$ into $H_1$, respectively. More precisely, we set

$$(B_0(u, v), w) = \int_{\mathcal{M}} [(u \cdot \nabla)v] \cdot w dx = b_0(u, v, w), \quad \forall u, v, w \in D(A_0),$$

$$(B_1(u, \phi), \rho) = \int_{\mathcal{M}} [(u \cdot \nabla)\phi] \rho dx = b_1(u, \phi, \rho), \quad \forall u \in D(A_0), \quad \phi, \rho \in D(A_\gamma), \quad (9)$$

$$(R_0(\mu, \phi), w) = \int_{\mathcal{M}} \mu [\nabla \phi \cdot w] dx = b_1(w, \phi, \mu), \quad \forall w \in D(A_0),$$

$$(\mu, \phi) \in L^2(\mathcal{M}) \times D(A_\gamma^{3/2}).$$

Note that

$$R_0(\mu, \phi) = \mathcal{P}_1 \mu \nabla \phi.$$ 

We recall that $B_0$, $B_1$ and $R_0$ satisfy the following estimates

$$|B_0(u, v)|_{V_1^2} \leq c |u|_{L^2}^{1/2} |v|^{1/2} |\nabla|_{L^2}^{1/2} |v|^{1/2}, \quad \forall u, v \in V_1, \quad (10)$$

$$|B_1(u, \phi)|_{V_1^2} \leq c |u|_{L^2}^{1/2} |u|^{1/2} |\phi|^{1/2} |\phi|^{1/2}, \quad \forall u \in V_1, \phi \in V_2, \quad (11)$$
Now we define the Hilbert spaces $Y$ and $V$ by

$$Y = H_1 \times H^1(M), \quad V = V_1 \times D(A_{\gamma})$$

endowed with the scalar products whose associated norms are

$$\langle (v, \phi), (f, \varphi) \rangle_Y = \kappa^{-1} |v|^2 + \epsilon (|\nabla \phi|^2 + |\phi|^2) + \kappa^{-1} |v|^2 + \epsilon |A_{\gamma} \phi|^2$$

and observe that $f_{\gamma}$ still satisfies (7) with $\gamma$ in place of $2\gamma$ since $\epsilon \leq \alpha$. Also its primitive $F_{\gamma}(r) = \int_0^r f_{\gamma}(\zeta)\zeta$ is bounded from below.

Hereafter, we denote by $\lambda_1 > 0$ a positive constant such that

$$\lambda_1|w|_2^2 \leq |w|^2, \quad \forall w \in V_1, \quad \lambda_1|\psi|^2_2 \leq |A_{\gamma} \psi|^2_2, \quad \forall \psi \in H^2(M).$$

Using the notations above, we rewrite (11)-(13) in the form

$$d\phi dt + \nu A_0 v + B_0(v, v) = \kappa R_0(\epsilon A_{\gamma} \phi, \phi) = Q(t - \tau(t), (v, \phi)(t - \tau(t))),$$

$$d\phi dt + \mu + B_1(v, \phi) = 0, \quad \mu = \epsilon A_{\gamma} \phi + \alpha f_{\gamma}(\phi),$$

$$v, \phi \in \mathcal{C}([0, T]; \gamma) \cap L^2([0, T]; \gamma), \quad d\phi dt \in L^1([0, T]; V^*), \quad \mu \in L^1([0, T]; V^*)$$

and $(v, \phi)$ satisfies (14), and (14) in $V_1^*$ and $V^*$ respectively.

For $(v_0, \phi_0) \in \gamma$, a weak solution $(v, \phi)$ is called a strong solution on the time interval $[\tau, T]$ if in addition to (12), it satisfies

$$v \in C([0, T]; V_1) \cap L^2(0, T; D(A_0)), \quad \phi \in C([0, T); D(A_{\gamma})) \cap L^2(0, T; D(A_{\gamma}^{1/2})).$$

(16)

In the case when the delay $r$ is zero, the weak formulation of (14) was proposed and studied in [11, 10], where the existence and uniqueness of solution was proved. Hereafter, to simplify the notation, we set $K = 1$.

We assume that the function $\tau(t)$ is differentiable and there exists a constants $\tau^*$ and $r > 0$ such that

$$\tau : [0, \infty) \to [0, r], \quad \frac{d\tau(t)}{dt} \leq \tau^* < 1.$$ (17)

Throughout this article we also suppose that the forcing $Q$ satisfies the local Lipschitz condition.

$$Q$$ satisfies the local Lipschitz condition. (18)
3. Existence of solution. In this section we discuss the global existence of the weak solution and the strong solution for the AC-NS system with delay (14). Since the injection of $Y \subset V$ is compact, let $\{(w_i, \psi_i), i = 1, 2, 3, \cdots \} \subset V$ be an orthonormal basis of $Y$, where $\{w_i, i = 1, 2, \cdots \}$, $\{\psi_i, i = 1, 2, \cdots \}$ are eigenvectors of $A_0$ and $A$, respectively. We set $\mathbb{V}_m = \mathbb{Y}_m = \text{span}\{(w_1, \psi_1), \cdots (w_m, \psi_m)\}$.

We look for $(u_m, \phi_m) \in \mathbb{V}_m$ solution to the ordinary differential equations
\[
\frac{d u_m}{d t} + \mathcal{P}_m^1 (\nu A_0 u_m + B_0(u_m, u_m) - R_0 (\epsilon A_\gamma \phi_m, \phi_m)) = \mathcal{P}_m^1 Q(t - \tau(t), (u_m, \phi_m)(t - \tau(t))),
\]
\[
\frac{d \phi_m}{d t} + \mathcal{P}_m^2 (\mu_m + B_1(u_m, \phi_m)) = 0, \quad \mu_m = \epsilon A_\gamma \phi_m + f_\gamma(\phi_m),
\]
\[
(w_m, \psi_m)(s) = \mathcal{P}_m \vartheta(s), \ s \in [-r, 0],
\]
where $\mathcal{P}_m = (\mathcal{P}_m^1, \mathcal{P}_m^2) : H_1 \times L^2(M) \to \mathbb{V}_m$ is the orthogonal projection. Since $\mathcal{P}_m(0, Q(t, (v, \phi)))$ is a local Lipschitz function in $(v, \phi)$, it follows from the theory of ordinary differential equation that this equation has a solution $(v_m, \phi_m)$, (see also Theorem A1 of [5]). Hereafter $C$ denotes a constant independent of $m$ and depending only on data such as $M$ and whose value may be different in each inequality. Finally, $c$ will denote a generic constant.

Hereafter, for any $(w, \psi) \in Y$, we set
\[
\mathcal{E}(w, \psi) = |(w, \psi)|^2_2 + 2\langle F_\gamma(\psi), 1 \rangle + \alpha_0,
\]
where $\alpha_0 > 0$ is a constant large enough and independent on $(w, \psi)$ such that $\mathcal{E}(w, \psi)$ is nonnegative (note that $F_\gamma$ is bounded from below).

We can check that (see [10] for details) there exists a monotone non-decreasing function $C_0$ (independent on time and the initial condition) such that
\[
|(w, \psi)|^2_2 \leq \mathcal{E}(w, \psi) \leq C_0(|(w, \psi)|^2_2), \ \forall (w, \psi) \in Y.
\]

Let $g$ be a continuous nonnegative scalar function defined on $[-r, +\infty)$ and let $R$ be a continuous positive monotone nondecreasing function defined on $[0, +\infty)$. As in [20], we set
\[
G = \int \frac{1}{R(x)} \, dx
\]
and we assume that the inverse $G^{-1}$ of $G$ is well-defined on $[0, +\infty)$. Then we have the following result.

**Theorem 3.1.** We suppose that [17]-[18] are satisfied. We also assume that there exists a constant $b_f \geq 0$ such that
\[
|Q(t, (v, \phi))|^2_2 \leq g(t) R((v, \phi)|^2_2) + b_f, \ \forall v \in H_1, \ t \geq -r.
\]
Then [13] has a unique weak solution $(v, \phi)$ for any initial value $(v_0, \phi_0) \in Y, \ \vartheta \in L^2(-r, 0; Y)$.

**Proof.** By taking the scalar product in $H_1$ of [19] with $v_m$, then taking the scalar product in $L^2(M)$ of [19] with $\mu_m$, we derive that (see [11] for the details)
\[
\frac{dE}{dt} + 2\nu \|v_m\|^2_2 + 2|\mu_m|^2_2 = 2\langle Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t))), v_m \rangle,
\]
where $E = E(t) = \mathcal{E}(v_m(t), \phi_m(t))$. 


Thus we have by the compactness theorem (see [21]) that
\[\frac{dE}{dt} + \nu\|v_m\|^2 + 2\mu_m \|z\|_{L^2}^2 \leq c_2|Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t)))|_{L^2}^2\]
\[\leq c_2g(t - \tau(t))R((v_m, \phi_m)(t - \tau(t))) + c_2b_f\]
\[\leq c_2g(t - \tau(t))R(E(t - \tau(t))) + c_2b_f.\] (25)

Let
\[K = E(0) + \frac{c_2}{(1 - \tau^*)} \int_0^0 g(s)R(\|\varphi\|_2^2)ds.\] (26)

Then for \(T \geq t \geq 0\), we have
\[E(t) \leq K + \frac{c_2}{(1 - \tau^*)} \int_0^0 g(s)R(E(t))ds + c_2b_fT.\] (27)

Therefore we have by the Bihari inequality (see [20])
\[E(t) \leq G^{-1}\left\{G(K + c_2Tb_f) + \frac{c_2}{1 - \tau^*} \int_0^t g(s)ds\right\} \equiv M_1, \ 0 \leq t \leq T.\] (28)

This proves that \((v_m, \phi_m)\) is uniformly bounded in \(L^\infty(0, T; \mathcal{Y}) \cap L^2(0, T; \mathcal{V})\).

Note that from
\[\mu_m = \epsilon \lambda_\gamma, \phi_m + \alpha f_\gamma(\phi_m),\]
we derive that
\[|A, \phi_m|_{L^2} \leq c|\mu_m|_{L^2} + Q_1(\|\phi_m\|),\] (29)
where \(Q_1\) is a monotone non-decreasing function independent on time, the initial condition and \(m\). It follows from [20] and (28) that \(A, \phi_m\) is bounded in \(L^2(0, T; H_2)\).

Using (27), (29) and (10)-(12), we can check that
\[\frac{d}{dt}(v_m, \phi_m)\] is bounded in \(L^{4/3}(0, T; V_1^*) \times L^2(0, T; H_2).\) (30)

Therefore, we can take a subsequence (still denoted \((v_m, \phi_m)\) such that
\[\begin{align*}
(v_m, \phi_m) &\to (v, \phi) \text{ weakly-star in } L^\infty(0, T; \mathcal{Y}), \\
(v_m, \phi_m) &\to (v, \phi) \text{ weakly in } L^2(0, T; \mathcal{V}), \quad (31)
\end{align*}\]

\[\frac{d}{dt}(v_m, \phi_m) \to \frac{d}{dt}(v, \phi) \text{ weakly in } L^{4/3}(0, T; V_1^*) \times L^2(0, T; H_2).\]

Thus we have by the compactness theorem (see [21]) that
\[\begin{align*}
(v_m, \phi_m) &\to \text{ strongly in } L^2(0, T; \mathcal{V}).
\end{align*}\] (32)

Using [31]-[32] and standard methods as in [21], we can pass to the limit in (19) as \(m \to \infty\), and derive that \((v, \phi)\) is a weak solution to (14). Let us recall that the passage to the limit in the delay force is obtained as in [20, 5].

For the uniqueness of weak solutions and their continuous dependence (from \(\mathcal{Y} \times L^2(-r, 0; \mathcal{Y})\) into \(\mathcal{Y}\)) with respect to the initial data, we proceed as in [20]. Let \(u_1 = (v_1, \phi_1)\) and \(u_2 = (v_2, \phi_2)\) be two weak solutions to (14). Let \(w = v_1 - v_2, \psi = \phi_1 - \phi_2\) and
\[t_1 = \sup\{\rho > 0, |u_1(s) - u_2(s)|_\mathcal{Y} = 0, \forall s \in [0, \rho]\}.\] (33)
We recall that \((w, \psi)\) satisfies
\[
\begin{cases}
\frac{dw}{dt} + \nu A_0 w + B_0(v_1, w) + B_0(w, v_2) - R_0(\epsilon A_1, \psi, \phi_1) - R_0(\epsilon A_1, \phi_2, \psi) \\
= Q(t - \tau(t), (v_1, \phi_1)(t - \tau(t))) - Q(t - \tau(t), (v_2, \phi_2)(t - \tau(t))),(34)
\end{cases}
\]

\[
\frac{d\psi}{dt} + \epsilon A_1 \psi + \alpha f_1(\phi_1) - \alpha f_1(\phi_2) + B_1 (w, \phi_1) + B_1 (v_2, \psi) = 0.
\]

We set \(w_1 = (v_1, \phi_1)(t_1)\). Since \(Q(t, \psi, \phi)\) satisfies a local Lipschitz condition, for any positive constant \(\epsilon_0 > 0\), there exists \(L(\epsilon_0) > 0\) such that
\[
|Q(t, \zeta) - Q(t, w)|_{L^2} \leq L(\epsilon) |\zeta - w|_Y, \forall (t, \zeta), (t, w) \in \Omega(t_1, \epsilon_0),
\]
where
\[
\Omega(t_1, \epsilon_0) = \{(t, \zeta) : |t - t_1| < \epsilon, |\zeta - w|_Y < \epsilon_0\}.
\]

We can assume without loss of generality that \((t - \tau(t), (v_1, \phi_1)(t - \tau(t))), (t - \tau(t), (v_2, \phi_2)(t - \tau(t))) \in \Omega(t_1, \epsilon_0)\).

Now let
\[
y(t) = |(w, \psi)|_Y^2,
\]
\[
\Upsilon(t) = c(||v_1||^2 + ||\phi_1||^2 |A_1\psi_1|^2 ||v_2||^2 + ||\phi_2||^2 |A_1\psi_2|^2 + ||v_2||^2 ||v_2||^2)
\]
\[
+ Q_1(|\phi_1|_{H^1}, |\phi_2|_{H^1}).
\]

Then as in [11] [14], we can check that
\[
\frac{dy}{dt} \leq \Upsilon(t)y(t) + c|Q(t - \tau(t), (v_1, \phi_1)(t - \tau(t))) - Q(t - \tau(t), (v_2, \phi_2)(t - \tau(t))|_{L^2}^2
\]
\[
\leq \Upsilon(t)y(t) + cL(\epsilon_0)^2 |(w, \psi)(t - \tau(t))|_Y^2
\]
\[
\leq \Upsilon(t)y(t) + cL(\epsilon_0)^2 y(t - \tau(t))
\]
which gives
\[
y(t) \leq y(0) + \int_0^t \left(\Upsilon(s) + \frac{cL(\epsilon_0)^2}{1 - \tau^*}\right) y(s)ds.
\]

By the Gronwall lemma, we obtain that
\[
y(t) \leq y(0)\exp \left[\int_0^t \left(\Upsilon(s) + \frac{cL(\epsilon_0)^2}{1 - \tau^*}\right) ds\right].
\]

This proves the uniqueness of weak solutions and the continuous dependence (in the \(Y\)-norm) on the initial data follow.

As a corollary, we have

**Corollary 1.** We suppose that (7)-(18) are satisfied. We also assume that there exist constants \(a_f > 0\) and \(b_f \geq 0\) such that
\[
|Q(t, (v, \phi))|_{L^2}^2 \leq a_f |(v, \phi)|_Y^2 + b_f, \forall (v, \phi) \in Y, t \geq -r.
\]

Then (14) has a unique weak solution \((v, \phi) \in L^\infty(0, T; Y) \cap L^2(0, T; V)\) for any initial value \((v_0, \phi_0) \in Y, \theta \in L^2(-r, 0; Y)\).
3.1. Global strong solution. In this part we discuss the existence and uniqueness of the strong solution to (14).

**Theorem 3.2.** The assumptions are the same as in Theorem 3.1. Let \((v_0, \phi_0) \in \mathbb{V}\). Then, there exists a unique strong solution \((v, \phi)\) to (14) such that

\[
(v, \phi) \in L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; D(A_0) \times D(A_0^{3/2})).
\] (42)

More precisely, \(\forall t \in [0, T]\), we have

\[
\|(v, \phi)(t)\|_\mathbb{V}^2 \leq M_0, \quad \int_0^t \left(\|A_0 v(s)\|_{L^2}^2 + \|A_0^{3/2} \phi(s)\|_{L^2}^2\right) ds \leq M_0,
\] (43)

where \(M_0 > 0\) depends on \(T\) as well as the initial condition \((v_0, \phi_0)\).

**Proof.** Let \((v_m, \phi_m)\) be any fixed approximation to the solution to (14). It follows from the proof of Theorem 3.1 that there exists a constant \(M_1 = M_1(T) > 0\) such that

\[
|(v_m, \phi_m)|^2_\mathbb{V} \leq M_1,
\]

uniformly in \(m\).

Now taking the inner product in \(H_1\) of (14) with \(2A_0v_m\), the inner product in \(L^2(\mathcal{M})\) of (14) and (14) with \(2A_0^{3/2} \phi_m\) and adding the resulting equalities gives (see [11] for the details)

\[
\frac{d\mathcal{Y}}{dt} + 2\nu \|A_0 v_m\|_{L^2}^2 + 2\epsilon \|A_0^{3/2} \phi_m\|_{L^2}^2 = 2\epsilon (R_0(A_0^{1/2} f_m, A_0 v_m) + 2(Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t))), A_0 v_m) \tag{44}
\]

\[
-2\epsilon (A_0^{1/2} f_\gamma(\phi_m), A_0^{3/2} \phi_m)_{L^2} - 2\epsilon A_0^{1/2} B_1(v_m, \phi_m, A_0^{3/2} \phi),
\]

where

\[
\mathcal{Y}(t) = \|(v_m, \phi_m)|^2_\mathbb{V} = \|v_m(t)|^2 + |A_0 \phi_m(t)|^2_{L^2}.
\]

If we set

\[
\mathcal{G}(t) = \epsilon \|A_0 v_m\|_{L^2}^2 + \|v_m\|_{L^2}^2 \|v_m\|^2 + \|v_m\|^2.
\] (45)

Then as in [11], we can check that (setting \(2\tilde{\nu} = \nu\))

\[
\frac{d\mathcal{Y}}{dt} + 2\tilde{\nu} \|A_0 v_m\|_{L^2}^2 + \epsilon \|A_0^{3/2} \phi_m\|_{L^2}^2
\]

\[
\leq \mathcal{G}(t) \mathcal{Y}(t) + \epsilon (f_\gamma(\phi_m) \nabla \phi_m, \phi_m)_{L^2}^2 + 2(Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t))), A_0 v_m).
\] (46)

Let us set

\[
Z(t) = \mathcal{Y}(t) + \frac{1}{(1 - \tau^*)} \int_{t - \tau(t)}^t Q(s, (v_m, \phi_m)(s))ds.
\]

Then

\[
\int_0^T Z(s)ds < \infty
\] (47)
and

$$\frac{dZ}{dt} \leq -2\bar{\nu}|A_0v_m|^2_{L^2} - \epsilon|A^{3/2}_\gamma \phi_m|^2_{L^2} + \frac{1}{(1 - \tau^\gamma)\bar{\nu}}|Q(t, (v_m, \phi_m(t)))|^2_{L^2}$$

$$- \frac{1}{\bar{\nu}}|Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t)))|^2_{L^2} + G(t)\mathcal{Y}(t) + c|f_\gamma(\phi_m)\nabla \phi_m|^2_{L^2}$$

$$+ 2\langle Q(t - \tau(t), (v_m, \phi_m)(t - \tau(t))), A_0v_m \rangle$$

$$\leq -\bar{\nu}|A_0v_m|^2_{L^2} - \epsilon|A^{3/2}_\gamma \phi_m|^2_{L^2} + \frac{1}{(1 - \tau^\gamma)\bar{\nu}}|Q(t, (v_m, \phi_m(t)))|^2_{L^2}$$

$$+ G(t)\mathcal{Y}(t) + c|f_\gamma(\phi_m)\nabla \phi_m|^2_{L^2}$$

$$\leq -\bar{\nu}|A_0v_m|^2_{L^2} - \epsilon|A^{3/2}_\gamma \phi_m|^2_{L^2} + \frac{1}{(1 - \tau^\gamma)\bar{\nu}}(g_TR(M_1) + b_f) + G(t)\mathcal{Y}(t)$$

$$+ c|f_\gamma(\phi_m)\nabla \phi_m|^2_{L^2}. \quad (48)$$

It follows from the Gronwall lemma that

$$\|(v_m, \phi_m)\|_\mathcal{Y}^2 \leq Z(t) \leq C,$$

$$\int_0^T (|A_0v_m|^2_{L^2} + |A^{3/2}_\gamma \phi_m|^2_{L^2}) ds \leq C. \quad (49)$$

Using (49) and (10)-(12), we can check that

$$\frac{d}{dt}(v_m, \phi_m) \text{ is bounded in } L^2(0, T; \mathbb{Y}). \quad (50)$$

Therefore, we can take a subsequence (still) denoted \((v_m, \phi_m)\) such that

\((v_m, \phi_m) \rightharpoonup (v, \phi) \text{ weakly-star in } L^\infty(0, T; \mathbb{V}),\)

\((v_m, \phi_m) \rightharpoonup (v, \phi) \text{ weakly in } L^2(0, T; D(A_0) \times D(A^{3/2}_\gamma)), \quad (51)\)

$$\frac{d}{dt}(v_m, \phi_m) \rightharpoonup \frac{d}{dt}(v, \phi) \text{ weakly in } L^2(0, T; \mathbb{Y}).$$

Using (49)-(51), we derive from the compactness theorem (see [21]) that

\((v_m, \phi_m) \rightarrow \text{ strongly in } L^2(0, T; \mathbb{V}). \quad (52)\)

Therefore, by passing to the limit in (19), we derive the desired conclusion. It is clear that (43) follows from (49). The uniqueness of strong solution follows from Theorem 3.3 below.

3.1.1. Continuity in \(\mathbb{V}\) with respect to the initial data.

**Theorem 3.3.** The assumptions are the same as in Theorem 3.1. Let \((v_i, \phi_i), i = 1, 2\) be two strong solutions corresponding to the initial conditions \((v^0_i, \phi^0_i) \in \mathbb{V}\). Let the initial data on the time interval \((-r, 0)\) be denoted \(\Phi_1, \Phi_2\) respectively. Then
Let \( \epsilon > 0 \) be any positive constant. We can check that

\[
\frac{\partial (\epsilon A, \psi, \epsilon^2)}{\partial t} \in \Psi \leq \frac{\epsilon L(\epsilon_0)}{1 - \tau^*} \int_{-\tau}^0 |\Phi_1 - \Phi_2|^2 \, ds \times \exp \left[ c \int_0^t \left( \Psi(s) + \frac{cL(\epsilon_0)^2}{1 - \tau^*} \right) \, ds \right],
\]

where \( L(\epsilon_0) \) and \( \Psi \) are defined below.

**Proof.** Let

\[
t_2 = \sup \{ \rho > 0, \| (v_1, \phi_1)(s) - (v_2, \phi_2)(s) \|_V = 0, \forall s \in [0, \rho] \}.
\]

We set \( w_1 = (v_1, \phi_1)(t_2) \). Since \( Q(t, v, \phi) \) satisfies a local Lipschitz condition, for any positive constant \( \epsilon > 0 \), there exists \( L(\epsilon_0) > 0 \) such that

\[
|Q(t, \zeta) - Q(t, w)| \leq L(\epsilon)|\zeta - w|_V, \quad \forall (t, \zeta), (t, w) \in \Omega_1(t_2, \epsilon_0),
\]

where \( \Omega_1(t_2, \epsilon_0) \) is defined in \((36)\). We can assume without loss of generality that \((t - \tau(t), (v_1, \phi_1)(t - \tau(t))), (t - \tau(t), (v_2, \phi_2)(t - \tau(t))) \in \Omega_1(t_2, \epsilon_0)\).

Let \((w, \psi) = (v_1, \phi_1) - (v_2, \phi_2)\). Then \((w, \psi)\) satisfies \((34)\). We multiply \((34)\) by \( A_0 w \) and \((129)\) by \( A_0^2 \psi \) to derive that

\[
\frac{dy}{dt} + 2\nu|A_0 w|_{L^2}^2 + 2\epsilon|A_0^{3/2} \psi|_{L^2}^2 = -2b_0(v_1, w, A_0 w) - 2b_1(v_2, w, A_0 w) + 2(R_0(\epsilon A, \psi, \phi, A_0 w) - 2b_1(w, \phi_1, A_0^2 \psi) - 2b_1(v_2, \psi, A_0^2 \psi) - 2\alpha(f_\gamma(\phi_1) - f_\gamma(\phi_2), A_0^2 \psi) + 2(Q(t - \tau(t), (v_1, \phi_1)(t - \tau(t))) - Q(t - \tau(t), (v_2, \phi_2)(t - \tau(t))), A_0 w),
\]

where

\[
y(t) = \| (w, \psi)(t) \|^2_V.
\]

We can check that

\[
|2b_0(v_1, w, A_0 w)| \leq \frac{L}{8} |A_0 w|_{L^2}^2 + c|v_1|^2 \| v_1 \|^2 \| w \|^2,
\]

\[
|2b_0(w, v_2, A_0 w)| \leq \frac{L}{8} |A_0 w|_{L^2}^2 + c|x_2|^2 \| x_2 \|^2 \| w \|^2,
\]

\[
|\langle R_0(\epsilon A, \psi, \phi, A_0 w) \rangle| = |b_1(A_0 w, \phi_1, \epsilon A, \psi)|
\]

\[
\leq \frac{L}{8} (\nu |A_0 w|_{L^2}^2 + \epsilon |A_0^{3/2} \psi|_{L^2}^2) + c|\phi_1|^2 \| A_0 \phi_1 \|^2 |x_2| |x_2| |A_0 \psi|_{L^2}^2,
\]

\[
|\langle R_0(\epsilon A, \psi, \phi, A_0 w) \rangle| = |b_1(A_0 w, \psi, \epsilon A, \phi_2)|
\]

\[
\leq \frac{L}{8} (\nu |A_0 w|_{L^2}^2 + \epsilon |A_0^{3/2} \psi|_{L^2}^2)
\]

\[
+ c|A_0^{3/2} \phi_2 |x_2|^2 |A_0 \phi_2|^2 |A_0 \psi|_{L^2}^2.
\]

\[(58)\]
\[ |2b_1(w, \phi_1, A^2_\gamma \psi)| = \langle A^{1/2}_\gamma B_1(w, \phi_1), A^{3/2}_\gamma \psi \rangle \]
\[ \leq \frac{\epsilon}{8} |A^{3/2}_\gamma \psi|_{L^2}^2 + c(\|\phi_1\|_{A_\gamma \phi_1 L^2} + |A_\gamma \phi_1|_{L^2} |A^{3/2}_\gamma \phi_1|_{L^2}) \|w\|^2 \]
\[ + c\|\phi_1\|_{A_\gamma \phi_1 L^2} |A_\gamma \psi|_{L^2}^2, \]
\[ |2b_1(v_2, \psi, A^2_\gamma \psi)| = \langle A^{1/2}_\gamma B_1(v_2, \psi), A^{3/2}_\gamma \psi \rangle \]
\[ \leq \frac{\epsilon}{8} |A^{3/2}_\gamma \psi|_{L^2}^2 + c\|v_2\|_{A_0 v_2 L^2} + |v_2|_{L^2}^2 \|v_2\|^2 |A_\gamma \psi|_{L^2}^2, \]
\[ 2\alpha (f_\gamma (\phi_1) - f_\gamma (\phi_2), A^2_\gamma \psi) = 2\alpha (A^{1/2}_\gamma (f_\gamma (\phi_1) - f_\gamma (\phi_2)), A^{3/2}_\gamma \psi) \]
\[ \leq \frac{\epsilon}{8} |A^{3/2}_\gamma \psi|_{L^2}^2 + Q_2(|A_\gamma \phi_1|_{L^2}, |A_\gamma \phi_1|_{L^2}), A_\gamma \psi|_{L^2}^2, \]
for a suitable monotone non-decreasing function independent of time \(Q_2\), (see [11] for details).

Let us set
\[ \Psi(t) = c(\|v_1\|_{L^2}^2 \|v_1\|^2 + \|v_2\|_{A_0 v_2 L^2} + |\phi_1|_{L^2}^2 |A_\gamma \phi_1|_{L^2} + |A^{3/2}_\gamma \phi_1|_{L^2} |A_\gamma \phi_2|_{L^2}) \]
\[ + c(\|\phi_1\|_{A_\gamma \phi_1 L^2} + |A_\gamma \phi_1|_{L^2} |A^{3/2}_\gamma \phi_1|_{L^2} + |\phi_1|_{L^2} |A_\gamma \phi_1|_{L^2} \]
\[ + \|v_2\|_{A_0 v_2 L^2} + \|v_2|_{L^2}^2 \|v_2\|^2 + Q_2(|A_\gamma \phi_1|_{L^2}, |A_\gamma \phi_1|_{L^2}). \]
\[ (62) \]

Since \((v_i, \phi_i), i = 1, 2\) are strong solutions, then \(\int_0^t \Psi(s)ds < \infty, \forall t > 0.\)

From (56)-(62), we have (with \(2\nu = \nu\))
\[ \frac{d\nu}{dt} + 2\nu |A_0 w|_{L^2}^2 + \epsilon |A^{3/2}_\gamma \psi|_{L^2}^2 \leq \Psi(t)\nu(t) \]
\[ + 2(\nu (t - \tau(t)), (v_1, \phi_1)(t - \tau(t))) - Q(t - \tau(t), (v_2, \phi_2)(t - \tau(t))), A_0 w). \]
\[ (63) \]

Let
\[ Z(t) = \nu(t) + \frac{1}{(1 - \tau^*)^\nu} \int_{t - \tau(t)}^t |Q(s, (v_1, \phi_1)(s)) - Q(s, (v_2, \phi_2)(s))|_{L^2}^2. \]
\[ (64) \]

As in (46)-(48), it follows from (56)-(64) that
\[ \frac{dZ}{dt} \leq \Psi(t)\nu(t) + \frac{1}{(1 - \tau^*)^\nu} |Q(t, (v_1, \phi_1)(t)) - Q(t, (v_2, \phi_2)(t))|_{L^2}^2. \]
\[ (65) \]

Note that
\[ \int_0^T Z(s)ds < \infty, \]
\[ |Q(t, (v_1, \phi_1)(t)) - Q(t, (v_2, \phi_2)(t))|_{L^2}^2 \leq |L(\epsilon_0)|^2 |(w, \psi)(t)|_{L^2}^2 \leq L(\epsilon)^2 Z(t), \]
\[ y(t) \leq Z(t), \]
\[ (66) \]
\[ Z(0) \leq y(0) + \frac{cL(\epsilon_0)^2}{1 - \tau^*} \int_{-\tau}^0 |\Phi_1 - \Phi_2|_{L^2}^2 ds. \]
It follows from (65)–(66) that
\[ Z(t) \leq Z(0) \exp \left[ c \int_0^t \left( \Psi(s) + \frac{L(e_0)^2}{1 - \tau^*} \right) ds \right] \] (67)
and
\[ y(t) \leq \left( y(0) + \frac{cL(e_0)^2}{1 - \tau^*} \int_{-r}^0 |\Phi_1 - \Phi_2|^2 ds \right) \exp \left[ \int_0^t \left( \Psi(s) + \frac{L(e_0)^2}{1 - \tau^*} \right) ds \right]. \] (68)

Then for a fixed time \( t > 0 \), the Lipschitz continuous dependence (in the \( V^- \) norm) on the initial data follow.

As a corollary, we have:

**Corollary 2.** We assume that (17)–(18) are satisfied. We also suppose that there exist constants \( a_f > 0 \) and \( b_f \geq 0 \) such that
\[ |Q(t, (v, \phi))|_{L^2}^2 \leq a_f |(v, \phi)|_{L^2}^2 + b_f, \ \forall (v, \phi) \in V, \ t \geq -r. \] (69)
Then for every \((v_0, \phi_0) \in V \) and \( \vartheta \in L^2(-r, 0; V) \), there exists a unique strong solution of the system (93), which depends continuously (from \( V \times L^2(-r, 0; V) \) into \( V \)) on the initial data.

4. **Exponential behavior of weak solutions.** In this part we discuss the exponential behavior of weak solutions to the AC-NS with delay (14).

**Theorem 4.1.** The assumptions are the same as in Corollary 2. Let \( \kappa, \alpha_1, \lambda_1, c_1 \) be given respectively by (80), (81), (13) and (79) below. We also assume that
\[ -\kappa + \frac{a_f}{(1 - \tau_1)\alpha_1\lambda_1} < 0. \] (70)

Then we have the following asymptotic behavior of weak solutions.
\[ |(v, \phi)(t)|_{V^-}^2 \leq (M_0(|(v_0, \phi_0)|_{V^-}^2) + K)e^{-\rho t} + \frac{b_f e^{\rho t}}{(1 - \tau^*)\rho} + \frac{c_1}{\rho}, \] (71)
where
\[ K = \frac{1}{(1 - \tau^*)\alpha_1\lambda_1} \int_{-r}^0 e^{\rho s} e^{\rho t}(a_f |\vartheta(s)|_{V^-}^2 + b_f) ds, \] (72)
\( \rho > 0 \) is a positive number such that
\[ \rho - \kappa + \frac{a_f e^{\rho t}}{(1 - \tau_1)\alpha_1\lambda_1} = 0, \] (73)
and hereafter \( M_0 \) denotes a suitable monotone non-decreasing function independent of time.

**Proof.** As in [11], we can check that
\[ \frac{dE}{dt} + \kappa E(t) = \wedge_1(t), \] (74)
where
\[ E(t) = |(v, \phi)(t)|_{V^-}^2 + 2\alpha(F_\gamma(\phi)(t), 1)_{L^2} + C_\epsilon, \] (75)
and
\[ \wedge_1(t) = -2\nu |v|_{L^2}^2 + \kappa |v|_{L^2}^2 - 2|\mu|_{L^2}^2 - (2 - \kappa)\epsilon(|\nabla \phi|_{L^2}^2 + \gamma|\phi(t)|_{L^2}^2) \\
+2\alpha |F_\gamma(\phi - f_\gamma(\phi)\phi, 1)_{L^2} - (1 - \kappa)(f_\gamma(\phi)\phi, 1)_{L^2} | \\
+2(v, Q(t - \tau(t), (v, \phi)(t - \tau(t)))) + \kappa|\phi(t)|_{L^2}^2 + 2\kappa\alpha|\mathcal{M}|. \] (76)
From (70), we have
\[ c_s |f_\gamma(y)|(1 + |y|) \leq 2f_\gamma(y)y + c_f(1 + \alpha^{-1}\epsilon), \]
(77)
\[ F_\gamma(y) - f_\gamma(y)y \leq c_f^*(1 + \alpha^{-1}\epsilon)|y|^2 + c_f^{**}, \]
for any \( y \in \mathbb{R}, \) where \( c_f, c_s, c_f^* \) and \( c_f^{**} \) are positive, sufficiently large constants that depend only on \( f. \)

From (11), we also note that
\[ \Lambda(t) \leq -(\nu - \kappa C_m |\mathcal{M}|) \|v(t)\|^2 - 2\mu(t)\|\phi(t)\|^2 - (2 - \kappa)\epsilon |\nabla \phi(t)|^2_{L^2} \]
- \( (2 - \kappa)(1 + 2c_f'(\alpha + \epsilon))(\epsilon \gamma)^{-1}\epsilon \gamma |\phi(t)|^2_{L^2} \]
- \( c_s \alpha (1 - \kappa)(|f_\gamma(\phi(t))|, 1 + |\phi(t)|)_{L^2} + 2\langle v, Q(t - \tau(t), (v, \phi)(t - \tau(t)) \rangle + c_1, \)
(78)
where \( C_m \) depends on the shape of \( \mathcal{M}, \) but not its size and \( c_1 \) is given by
\[ c_1 = 2\kappa \alpha C_{F\gamma} |\mathcal{M}| + 2\alpha c_f^*|\mathcal{M}| + c_f(\alpha + \epsilon)(1 - \kappa)|\mathcal{M}|. \]
(79)
Let us choose \( \kappa \in (0, 1) \) as
\[ \kappa = \min \left\{ \nu(2C_m |\mathcal{M}|)^{-1}, (1 + 2c_f'(\alpha + \epsilon))(\epsilon \gamma)^{-1}\right\}. \]
(80)
From now on, \( c_i \) will denote a positive constant independent on the initial data and on time. Let us set
\[ 2\alpha_1 = \nu - \kappa C_m |\mathcal{M}|, \quad 2\alpha_2 = \min(2 - \kappa, (2 - \kappa)(1 + 2c_f'(\alpha + \epsilon))(\epsilon \gamma)^{-1}). \]
(81)
It follows from (75)-(80) that
\[ \frac{dE}{dt} + \kappa E(t) + 2\alpha_1 \|v(t)\|^2 + \epsilon |\nabla \phi(t)|^2_{L^2} + \epsilon \gamma |\phi(t)|^2_{L^2} + 2\mu(t)\|\phi(t)\|^2_{L^2} \]
\[ + c_3 (|f_\gamma(\phi(t))|, 1 + |\phi(t)|)_{L^2} \leq 2\langle v, Q(t - \tau(t), (v, \phi)(t - \tau(t)) \rangle + c_1, \]
(82)
which gives
\[ \frac{dE}{dt} + \kappa E(t) + 2\alpha_1 \|v(t)\|^2 + 2\alpha_2 \|\phi(t)\|^2 + 2\mu(t)\|\phi(t)\|^2_{L^2} \]
\[ + c_3 (|f_\gamma(\phi(t))|, 1 + |\phi(t)|)_{L^2} \leq 2\langle v, Q(t - \tau(t), (v, \phi)(t - \tau(t)) \rangle + c_1. \]
(83)
Let
\[ Z(t) = e^{\theta t} E(t) + \frac{1}{(1 - \tau^*)\alpha_1 \lambda_1} \int_{t-\tau(t)}^t e^{\theta s} e^{\theta r} |Q(s, (v, \phi)(s))|^2_{L^2} ds, \]
(84)
where \( \theta > 0 \) is a positive number such that
\[ \theta - \kappa + \frac{a_f e^{\theta r}}{(1 - \tau^*)\alpha_1 \lambda_1} < 0. \]
(85)
Note that if (70) is satisfied, we can find \( \theta > 0 \) small enough such that (85) holds.
Then from \([82, 84]\), we have
\[
\frac{dZ}{dt} = \theta e^{\theta t} E + e^{\theta t} \frac{dE}{dt} + \frac{1}{(1 - \tau^*)\alpha_1 \lambda_1} e^{\theta t} e^{\theta r} |Q(t, (v, \phi)(t))|^2_{L^2} \\
- \frac{1}{\alpha_1 \lambda_1} e^{\theta t} |Q(t - \tau(t), (v, \phi)(t - \tau(t)))|_{L^2}^2 \\
\leq \theta e^{\theta t} E(t) - \kappa e^{\theta t} E + e^{\theta t} (-2\alpha_2 \|\phi\|^2 - 2\alpha_1 \|v\|^2) \\
+ 2(v, Q(t - \tau(t), (v, \phi)(t - \tau(t)))) + c_1 \\
+ \frac{1}{(1 - \tau^*)\alpha_1 \lambda_1} e^{\theta t} e^{\theta r} |Q(t, (v, \phi)(t))|^2_{L^2} \\
- \frac{1}{\alpha_1 \lambda_1} e^{\theta t} |Q(t - \tau(t), (v, \phi)(t - \tau(t)))|_{L^2}^2 \\
\leq (\theta - \kappa) e^{\theta t} E(t) + e^{\theta t} (-\alpha_1 \|v\|^2 - 2\alpha_2 \|\phi\|^2) \\
+ \frac{1}{(1 - \tau^*)\alpha_1 \lambda_1} e^{\theta t} e^{\theta r} |Q(t, (v, \phi)(t))|^2_{L^2} + e^{\theta t} c_1,
\]
which gives (see \([85]\))
\[
\frac{dZ}{dt} \leq (\theta - \kappa) e^{\theta t} E + \frac{1}{(1 - \tau^*)\alpha_1 \lambda_1} e^{\theta t} e^{\theta r} |Q(t, (v, \phi)(t))|^2_{L^2} + c_1 e^{\theta t} \\
\leq (\theta - \kappa) e^{\theta t} E + \frac{1}{(1 - \tau^*)\alpha_1 \lambda_1} e^{\theta t} e^{\theta r} (a_{\theta f}(v, \phi))^2_{L^2} + b_f + c_1 e^{\theta t} \\
\leq \left(\theta - \kappa + \frac{a_{\theta f} e^{\theta r}}{(1 - \tau^*)\alpha_1 \lambda_1}\right) e^{\theta t} E + \frac{b_{\theta f} e^{\theta t} e^{\theta r}}{(1 - \tau^*)\alpha_1 \lambda_1} + c_1 e^{\theta t} \\
\leq \frac{b_{\theta f} e^{\theta t} e^{\theta r}}{(1 - \tau^*)\alpha_1 \lambda_1} + c_1 e^{\theta t}.
\]
It follows that
\[
e^{\theta t} E(t) \leq E(0) + \int_{-\tau}^{0} e^{\theta s} e^{\theta r} |Q(s, (v, \phi)(s))|^2_{L^2} ds \\
+ c_1 \int_{0}^{t} e^{\theta s} ds + \frac{b_f}{(1 - \tau^*)\alpha_1 \lambda_1} \int_{0}^{t} e^{\theta s} e^{\theta r} ds
\]
\[
\leq E(0) + K + \frac{c_1}{\theta} e^{\theta t} + \frac{b_{\theta f} e^{\theta r}}{(1 - \tau^*)\alpha_1 \lambda_1 \theta},
\]
which yields (letting \(\theta \to \rho\))
\[
E(t) \leq (E(0) + K) e^{-\rho t} + \frac{b_{\theta f} e^{\theta r}}{(1 - \tau^*)\alpha_1 \lambda_1 \theta} + \frac{c_1}{\rho}
\]
and \([71]\) follows. Recall that \(E(0) \leq M_0((v_0, \phi_0))^2_{L^2}\). \(\square\)

Note that if \([70]\) is satisfied, there exists \(\rho > 0\) such that \([73]\) holds true. Moreover, \([85]\) is satisfied for \(\theta > 0\) and small enough provided that \([70]\) holds.
5. Stability of stationary solutions. Hereafter, we study the stability of stationary solutions of (14). We first prove the existence of stationary solutions to (14) when the delay has a special form, provided that viscosity $\nu$ and the physical parameter $\epsilon$ are large enough. Then, we prove that all weak solutions to (14) converge exponentially to this unique stationary solution. We assume that the delay term is given by

$$Q(t,(v_1,\phi_1)) = Q_0((v,\phi)(t-\tau(t)),$$

where

$$Q_0: \mathcal{V} \rightarrow V^*_1$$

satisfies

$$\|Q_0(v_1,\phi_1) - Q_0(v_2,\phi_2)\|_{V^*_1} \leq L_1\|v_1,\phi_1 - v_2,\phi_2\|_\mathcal{V}, \forall (v_1,\phi_1), (v_2,\phi_2) \in \mathcal{V},$$

for some fixed constant $L_1 > 0$.

A stationary solution to (14) is a $(v^*,\phi^*)$ such that

$$\begin{cases}
\nu A_0 v^* + B_0(v^*,v^*) - R_0(\epsilon A_\gamma \phi^*,\phi^*) = Q_0(v^*,\phi^*), \\
\epsilon A_\gamma \phi^* + \alpha f_{\gamma}(\phi^*) + B_1(v^*,\phi^*) = 0.
\end{cases}$$

(93)

5.1. Existence and uniqueness of stationary solution. Let $\{(w_i,\psi_i), i = 1,2,3,\cdots\} \subset \mathcal{V}$ be an orthonormal basis of $\mathcal{V}$, where $\{w_i, i = 1,2,\cdots\}$, $\{\psi_i, i = 1,2,\cdots\}$ are eigenvectors of $A_0$ and $A_\gamma$, respectively. We set $\mathcal{V}_m = \text{span}\{(w_1,\psi_1),\cdots,(w_m,\psi_m)\}$. For fixed $(U,\Phi) \in \mathcal{V}_m$, we consider the following approximating problems: find $(v_m,\phi_m) \in \mathcal{V}_m$ such that

$$\begin{cases}
\nu A_0 v_m + B_0(U,v_m) - R_0(\epsilon A_\gamma \phi_m,\Phi) = Q_0(U,\Phi), \\
\epsilon A_\gamma \phi_m + \alpha f_{\gamma}(\Phi) + B_1(v_m,\Phi) = 0.
\end{cases}$$

(94)

It is clear (using the Lax-Milgram theorem) that for $(U,\Phi) \in \mathcal{V}_m$, there exists a unique solution $(v_m,\phi_m)$ to (94). Define $T_m: \mathcal{V}_m \rightarrow \mathcal{V}_m$ the linear operator given by $T_m(U,\Phi) = (v_m,\phi_m)$.

We will see that for each $m$, we may apply a fixed point theorem to the map $T_m$ (restricted to a suitable subset $\wedge_m \subset \mathcal{V}_m$) to ensure that we can obtain the existence of a solution $(v_m,\phi_m)$ to (94).

Lemma 5.1. We assume that $Q_0$ satisfies (94)-(92). Then any solution $(v_m,\phi_m)$ to (94) satisfies the estimate

$$\|(v_m,\phi_m)\|_\mathcal{V} \leq \kappa_1^{-1}\left(\|Q_0(0,0)\|_{V^*_1} + L_1\|(U,\Phi)\|_\mathcal{V} + M_1(\|\Phi\|)\right),$$

where $\kappa_1$ is given by (99) below and $M_1(\cdot)$ is a suitable monotone non-decreasing function independent of $m$.

Proof. If $(v_m,\phi_m) \in \mathcal{V}$ is a solution to (94), we can easily check that

$$\nu\|v_m\|^2 + \epsilon^2|A_\gamma \phi_m|^2_{L^2} + \alpha|f_{\gamma}(\Phi), A_\gamma \phi_m| = \langle Q_0(U,\Phi), v_m \rangle,$$

which gives

$$\nu\|v_m\|^2 + \epsilon^2|A_\gamma \phi_m|^2_{L^2} \leq L_1\|v_m\|\|(U,\Phi)\|_\mathcal{V} + \alpha|f_{\gamma}(\Phi)A_\gamma \phi_m|_{L^2} + \|Q_0(0,0)\|_{V^*_1}\|v_m\|,$$

and

$$\nu\|v_m\|^2 + \epsilon^2|A_\gamma \phi_m|^2_{L^2} \leq (\|Q_0(0,0)\|_{V^*_1} + L_1\|(U,\Phi)\|_\mathcal{V} + \alpha|f_{\gamma}(\Phi)|_{L^2})\|(v_m,\phi_m)\|_\mathcal{V}.$$
Let 
\[ \kappa_1 \equiv \min(\nu, \epsilon^2). \] (99)

It follows that
\[ \kappa_1 \| (v_m, \phi_m) \|_V^2 \leq (\| Q_0(0,0) \|_{V^*} + L_1 \| (U, \Phi) \|_V + \alpha \| f_\gamma(\Phi) \|_{L^2}) \| (v_m, \phi_m) \|_V, \] (100)

which gives
\[ \| (v_m, \phi_m) \|_V \leq \kappa_1^{-1} (\| Q_0(0,0) \|_{V^*} + L_1 \| (U, \Phi) \|_V + M_1(\| \Phi \|)), \] (101)

for a suitable monotone non-decreasing function independent of \( m \). Note that from [6], \( |f_\gamma(\Phi)|_{L^2} \leq M_1(\| \Phi \|). \)

**Theorem 5.2.** Suppose that \( Q_0 \) satisfies [90]-[92]. We also assume that
\[ \kappa_1 - L_1 > 0. \] (102)

Then there exists at least one solution to \([94]\).

**Proof.** Recall that \((v_m, \phi_m)\) satisfies the a priori estimates [95]. Let \( K_0 > 0 \) such that \( K_0(\kappa_1 - L_1) \geq \| Q_0(0,0) \|_{V^*} + M_1(K_0) \). Then from (101) we can check that \( \| (v_m, \phi_m) \|_V \leq \kappa_0 \) provided that \( \| (U, \Phi) \|_V \leq \kappa_0 \).

Now let 
\[ \land_m = \{ (U, \Phi) \in \mathbb{V}_m, \| (U, \Phi) \|_V \leq K_0 \}. \] (103)

Then \( \land_m \) is a compact and convex subset of \( \mathbb{V}_m \). It is also clear that \( T_m \) maps \( \land_m \) into itself. To prove the existence of solution, we apply the Brouwer fixed point theorem to the restriction of \( T_m \) to \( \land_m \). Therefore it remains to check that \( T_m \) is continuous.

For the continuity of \( T_m \), we proceed as follows. Let \((v_1, \phi_1) = T_m(U_1, \Phi_1) \) and \((v_2, \phi_2) = T_m(U_2, \Phi_2) \), where \((U_1, \Phi_1), (U_2, \Phi_2) \in \mathbb{V}_m \). Let \((w, \psi) = (v_1, \phi_1) - (v_2, \phi_2), (U, \Phi) = (U_1, \Phi_1) - (U_2, \Phi_2) \). Then from [94] can check that we \((w, \psi)\) satisfies
\[ \begin{align*}
\nu A_0 w + B_0(U, v_1) + B_0(U, w) - R_0(\epsilon A_\gamma \phi_1, \Phi) - R_0(\epsilon A_\gamma \psi, \Phi_2) \\
= Q_0(U_1, \Phi_1) - Q_0(U_2, \Phi_2), \\
\epsilon A_\gamma \psi + \alpha f_\gamma(\Phi_1) - \alpha f_\gamma(\Phi_2) + B_1(v_1, \Phi) + B_1(w, \Phi_2) = 0.
\end{align*} \] (104)

Note that
\[ \langle R_0(\epsilon A_\gamma \psi, \Phi_2), w \rangle = \langle B_1(w, \Phi_2), \epsilon A_\gamma \psi \rangle, \]
\[ |\langle Q_0(U_1, \Phi_1) - Q_0(U_2, \Phi_2), w \rangle| \leq L_1 \| (U, \Phi) \|_V \|w\|, \]
\[ |\langle B_0(U, v_1), w \rangle| \leq c \| U \| \| v_1 \| \|w\|, \]
\[ |\langle R_0(\epsilon A_\gamma \phi_1, \Phi), w \rangle| \leq c \| A_\gamma \Phi \|_{L^2} \| A_\gamma \phi_1 \|_{L^2} \|w\|, \]
\[ |\langle B_1(v_1, \Phi), \epsilon A_\gamma \psi \rangle| \leq c \| A_\gamma \Phi \|_{L^2} \| A_\gamma \psi \|_{L^2} \| v_1 \|, \]
\[ |\langle \alpha f_\gamma(\Phi_1) - \alpha f_\gamma(\Phi_2), \epsilon A_\gamma \psi \rangle| \leq M_2(\| \Phi_1 \|, \| \Phi_2 \|) \| \Phi \| \| A_\gamma \psi \|_{L^2}. \] (105)
Multiplying (104)1 and (104)2 by \( w \) and \( \epsilon A, \psi \) respectively and using (105), we derive that
\[
\nu \| w \|^2 + \epsilon^2 |A, \psi|_{L^2}^2 \leq c |U| \| \psi \| \| w \| + \epsilon c |A, \Phi|_{L^2} |A, \phi|_{L^2} \| w \|
\]
\[
+ \epsilon c |A, \Phi|_{L^2} |A, \psi|_{L^2} \| \psi \| + M_2(\| \Phi \|, | \Phi |) \| A, \psi \|_{L^2} + L_1 \| U, \Phi \| \| w \|,
\]
which gives
\[
\kappa_1 \|(w, \psi)\|_V \leq (\| \psi \| + \epsilon |A| \phi|_{L^2} + M_2(\| \Phi \|, | \Phi |) + L_1) \| U, \Phi \|_V,
\]
and the continuity of the mapping \( T_m \) follows. Note that \( M_2(\cdot, \cdot) \) denotes a suitable monotone non-decreasing function independent of \( m \).

It follows that there exists a fixed point \((v_m, \phi_m)\) of \( T_m \) in \( \Lambda_m \). Therefore we can extract a subsequence (still) denoted \((v_m, \phi_m)\) that converges to \((v^*, \phi^*)\) strongly in \( \mathcal{V} \). Using the same argument as in [11], we can prove that \((v^*, \phi^*)\) is a weak solution to (93). \(\square\)

5.2. **Some a priori estimates on** \((v^*, \phi^*)\). Hereafter, we assume that \( f, \gamma \) satisfies the additional condition
\[
\alpha f_\gamma'(\psi) \geq -\kappa_0, \quad \forall \psi \in H_2,
\]
where \( \kappa_0 > 0 \) is a fixed constant. We will derive some explicit a priori estimates in the \( \mathcal{V} \)–norm and under some additional assumptions, we prove the uniqueness of solutions. In particular, we assume that \( \epsilon > 0 \) is larger enough such that
\[
\epsilon > \kappa_0.
\]

**Theorem 5.3.** Suppose that \( Q_0 \) satisfies (90)-(92). We also assume that (108)-(109) are satisfied. Then any solution \((v^*, \phi^*)\) to (93) satisfies the following estimate:
\[
\|(v^*, \phi^*)\|_V \leq c \|Q_0(0, 0)\|_{\mathcal{V}_1^*} \equiv K_1.
\]

Moreover if
\[
\kappa_1 - c(2K_1 + M_2(K_1, K_1)) > 0,
\]
then the solution to (93) is unique.

**Proof.** To prove (110), by multiplying (93)1 by \( v^* \) and (93)2 by \( \epsilon A, \phi^* \), to derive that
\[
\nu \| v^* \|^2 + \epsilon^2 |A, \gamma|_{L^2}^2 + (\alpha f_\gamma(\phi^*), \epsilon A, \phi^*) = \langle Q_0(v^*, \phi^*), v^* \rangle
\]
\[
\leq \|Q_0(0, 0)\|_{\mathcal{V}_1^*} \| v^* \| + L_1 \| (v^*, \phi^*) \|_V \| v^* \|
\]
\[
\leq \|Q_0(0, 0)\|_{\mathcal{V}_1^*} \| (v^*, \phi^*) \|_V + L_1 \| (v^*, \phi^*) \|_V^2
\]
which gives (assuming (109))
\[
(\alpha_1 - L_1) \| (v^*, \phi^*) \|_V^2 \leq \|Q_0(0, 0)\|_{\mathcal{V}_1^*} \| (v^*, \phi^*) \|_V,
\]
where
\[
\alpha_1 = \min(\nu, \epsilon^2 - \epsilon \kappa_0) > 0.
\]
We derive that
\[
\| (v^*, \phi^*) \|_V \leq (\alpha_1 - L_1)^{-1} \|Q_0(0, 0)\|_{\mathcal{V}_1^*},
\]
and (110) is proved. \(\square\)
For the uniqueness, let \((v_1^*, \phi_1^*), (v_2^*, \phi_2^*)\) be two solutions. We set \((w, \psi) = (v_1^*, \phi_1^*) - (v_2^*, \phi_2^*)\). Then \((w, \psi)\) satisfies

\[
\begin{align*}
\nu A_0 w + B_0(w, v_1^*) + B_0(v_2^*, w) - R_0(\epsilon A_\gamma \phi_2^*, \psi) - R_0(\epsilon A_\gamma \psi, \phi_1^*) \\
= Q_0(v_1^*, \phi_1^*) - Q_0(v_2^*, \phi_2^*),
\end{align*}
\]

\(115\)

\(\epsilon A_\gamma \psi + \alpha f_\gamma(\phi_1^*) - \alpha f_\gamma(\phi_2^*) + B_1(v_2^*, \psi) + B_1(w, \phi_1^*) = 0.\)

Note that

\[
\langle R_0(\epsilon A_\gamma \phi_2^*), w \rangle = \langle B_1(w, \phi_1^*), \epsilon A_\gamma \psi \rangle,
\]

\[
|\langle B_0(w, v_1^*), w \rangle| \leq \epsilon \|v_1^*\| \|w\|^2,
\]

\[
|\langle R_0(\epsilon A_\gamma \phi_2^*), w \rangle| \leq c \epsilon \|A_\gamma \psi\|_{L^2} \|A_\gamma \phi_2^*\|_{L^2} \|w\|,
\]

\[
|\langle B_1(v_2^*, \Phi), \epsilon A_\gamma \psi \rangle| \leq c \epsilon A_\gamma \psi^2 \|v_2^*\|,
\]

\[
|\langle \alpha f_\gamma(\phi_1^*) - \alpha f_\gamma(\phi_2^*), \epsilon A_\gamma \psi \rangle| \leq M_2(\|\phi_1^*\|, \|\phi_2^*\|) \|A_\gamma \psi\|_{L^2} \|w\| \leq M_2(K_1, K_1) \|A_\gamma \psi\|_{L^2}^2.
\]

Multiplying \(115_1\) and \(115_2\) by \(w\) and \(\epsilon A_\gamma \psi\) respectively and using \(116\) yields

\[
\nu \|w\|^2 + \epsilon^2 \|A_\gamma \psi\|_{L^2}^2 \leq c(\|v_1^*\| + |A_\gamma \phi_2^*|_{L^2} + \|v_2^*\| + M_2(K_1, K_1)) \|(w, \psi)\|_{Y}^2
\]

\(117\)

which gives

\[
(\kappa_1 - c(2K_1 + M_2(K_1, K_1))) \|(w, \psi)\|_{Y}^2 \leq 0,
\]

\(118\)

and \(\|(w, \psi)\|_{Y} = 0\) assuming \(111\), where \(\kappa_1 = \min(\nu, \epsilon^2)\), and the theorem is proved.

5.3. Asymptotic behavior. Hereafter, we assume that

\[
Q_0 : Y \to H_1
\]

\(119\)

satisfies

\[
|Q_0(v_1, \phi_1) - Q_0(v_2, \phi_2)|_{L^2} \leq L_1|((v_1, \phi_1) - (v_2, \phi_2))|_Y, \forall (v_1, \phi_1), (v_2, \phi_2) \in Y,
\]

\(120\)

for some fixed constant \(L_1 > 0.\)

Theorem 5.4. We assume that \(117\), \(120\) and \(108\)-\(109\) are satisfied. We also suppose that \(\nu, \epsilon\) are large enough such that \(111\) and \(109\) are satisfied. We also assume that

\[
-2\nu + K_1 + K_2^2 + L_1\lambda_1 - L_1\lambda_1^{-1} + \frac{L_1\lambda_1^{-1}}{1 - \tau^*} < 0, \quad -\epsilon^2 + \kappa_0\lambda_1^{-2}\epsilon + K_2^2 + \frac{L_1\lambda_1^{-1}}{1 - \tau^*} < 0.
\]

\(121\)

Then any weak solution \((v, \phi)\) to \(11\) converges to the unique solution \((v^*, \phi^*)\) to \(93\) exponentially as \(t\) goes to \(\infty\). More precisely, we have the following estimate

\[
|(v, \phi)(t) - (v^*, \phi^*)|_{Y}^2 \leq Ce^{-\theta t} \left( |(v, \phi)(t) - (v^*, \phi^*)|_{Y}^2 + \int_{-\tau}^{0} |(v_1, \phi_1(t) - (v^*, \phi^*)|_{Y}^2 dt \right),
\]

\(122\)

for all \(t \geq 0\), where \(C > 0\) is a constant and \(\theta > 0\) is given by \(133\).
Let \((w, \psi) = (v, \phi) - (v^*, \phi^*)\). Then \((w, \psi)\) satisfies
\[
\begin{aligned}
\frac{dw}{dt} + \nu A w + B_0(v, w) + B_0(w, v^*) - R_0(\epsilon A_\gamma \psi, \phi) - R_0(\epsilon A_\gamma \phi^*, \psi) \\
= Q_0((v, \phi)(t - \tau(t))) - Q_0((v^*, \phi^*)), \\
\frac{d\psi}{dt} + \epsilon A_\gamma \psi + \alpha f_\gamma(\phi) - \alpha f_\gamma(\phi^*) + B_1(w, \phi) + B_1(v^*, \psi) = 0.
\end{aligned}
\tag{123}
\]
Let
\[
\mathcal{Y}(t) = |(w, \psi)|^2 = |w|_{L^2}^2 + \epsilon \|\psi\|^2.
\]
Recall that
\[
|b_0(w, v^*, w)| \leq c\|w\|^2\|v^*\| \leq K_1\|w\|^2, 
\tag{124}
\]
\[
\langle R_0(\epsilon A_\gamma \phi^*, \psi), w \rangle \leq \epsilon c\|w\|^2\|\psi\|^2/2|A_\gamma \psi|_{L^2}^{1/2}|A_\gamma \phi^*|_{L^2}
\leq \frac{\epsilon^2}{2}|A_\gamma \psi|_{L^2}^2 + c\|w\|^2|A_\gamma \phi^*|_{L^2}^2,
\tag{125}
\]
\[
|(B_1(v^*, \psi), \epsilon A_\gamma \psi)| \leq \epsilon c\|v^*\|\|\psi\|^2/2|A_\gamma \psi|_{L^2}^{3/2}
\leq \frac{\epsilon^2}{2}|A_\gamma \psi|_{L^2}^2 + c\|v^*\|^2|A_\gamma \psi|_{L^2}^2 \tag{126}
\]
\[
(\alpha f_\gamma(\phi) - \alpha f_\gamma(\phi^*), \epsilon A_\gamma \psi) \geq -\kappa_0\lambda_1^{-2}c|A_\gamma \psi|_{L^2}^2, \quad \text{(see (108))}
\tag{127}
\]
\[
2|<-Q_0((v^*, \phi^*)) + Q_0((v, \phi)(t - \tau(t)), w)| \leq 2L_1|\psi(t - \tau(t))|_{L^2} |w|_{L^2} \leq L_1\lambda_1^{-1}\|w\|^2 + L_1|(w, \psi)(t - \tau(t))|_{L^2}^2. \tag{128}
\]
Multiplying (123) by \(w\) and (123) by \(\epsilon A_\gamma \psi\) and using (124)-(128) gives
\[
\frac{d\mathcal{Y}}{dt} + (2\nu - K_1 - K_1^2 - L_1\lambda_1^{-1})\|w\|^2 + (\epsilon^2 - \kappa_0\lambda_1^{-2}\epsilon - K_1^2)|A_\gamma \psi|_{L^2}^2
\leq L_1|(w, \psi)(t - \tau(t))|_{L^2}^2. \tag{129}
\]
Let \(\gamma(t) = t - \tau(t)\) and \(\mu > 0\) such that \(\gamma^{-1}(t) \leq t + \mu\) for all \(t \geq -\tau(0)\). Then setting \(\eta = s - \gamma(s) = s\) (s)\), we obtain that for \(\theta > 0\)
\[
\int_0^t e^{\theta s} |(w, \psi)(t - \tau(t))|_{L^2}^2 ds = \int_{-\tau(0)}^{t - \tau(t)} e^{\theta \gamma^{-1}(\eta)} |(w, \psi)(\eta)|_{L^2}^2 \frac{1}{\gamma(\gamma^{-1}(\eta))} d\eta 
\leq \frac{e^{\theta \mu}}{1 - \tau} \int_{-\tau}^t e^{\theta \eta} |(w, \psi)(\eta)|_{L^2}^2 d\eta \tag{130}
\]
\[
\leq \frac{e^{\theta \mu}}{1 - \tau} \int_{-\tau}^t e^{\theta \eta} \mathcal{Y}(\eta) d\eta.
\]
It follows from (129) and (130) that
\[
\frac{d}{dt}(e^{\theta t} Y) = \theta e^{\theta t} Y + e^{\theta t} \frac{dY}{dt}
\leq \theta e^{\theta t} |w|^2 + \theta e^{\theta t} \|\psi\|^2 + e^{\theta t} (-2\nu + K_1 + K_1^2 + L_1 \lambda_1^{-1}) \|w\|^2
\leq \theta e^{\theta t} \|w\|^2 + e^{\theta t} \|\psi\|^2 + e^{\theta t} L_1 |(w, \psi)(t - \tau(t))|^2.
\] (131)

If we choose \( \theta \) such that \( \theta \lambda_1^{-1} - 2\nu + K_1 + K_1^2 + L_1 \lambda_1^{-1} + \frac{L_1 \lambda_1^{-1}}{1 - \tau} e^{\theta \mu} < 0 \),
\[
-\epsilon^2 + \kappa_0 \lambda_1^{-2} \epsilon + K_1^2 + \theta \lambda_1^{-1} \epsilon + \frac{L_1 \lambda_1^{-1}}{1 - \tau} e^{\theta \mu} < 0,
\] (132)
then we derive that
\[
e^{\theta t} Y(t) \leq Y(0) + \frac{L_1 e^{\theta \mu}}{1 - \tau} \int_{-r}^0 e^{\theta s} |(w, \psi)(s)|^2 ds
\leq Y(0) + \frac{L_1 e^{\theta \mu}}{1 - \tau} \int_{-r}^0 e^{\theta s} Y(s) ds,
\] (133)
and (122) follows. It is clear that (132) is satisfied for \( \theta > 0 \) small enough provided that (121) holds true.

Acknowledgments. The author would like to thank the anonymous referees whose comments help to improve the contain of this article.

REFERENCES

[1] T. Blesgen, A generalization of the Navier-Stokes equation to two-phase flow, Physica D (Applied Physics), 32 (1999), 1119–1123.
[2] G. Caginalp, An analysis of a phase field model of a free boundary, Arch. Rational Mech. Anal., 92 (1986), 205–245.
[3] T. Caraballo and X. Han, A survey on Navier-Stokes models with delays: Existence, uniqueness and asymptotic behavior of solutions, Discrete Contin. Dyn. Syst. Ser. S, 8 (2015), 1079–1101.
[4] T. Caraballo, A. M. Marquez-Duran and J. Real, Pullback and forward attractors for a 3D LANS-\( \alpha \) model with delay, Discrete Contin. Dyn. Syst., 15 (2006), 559–578.
[5] T. Caraballo and J. Real, Navier-Stokes equations with delays, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 457 (2001), 2441–2453.
[6] T. Caraballo and J. Real, Asymptotic behavior of two-dimensional Navier-Stokes equations with delays, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 459 (2003), 3181–3194.
[7] T. Caraballo and J. Real, Attractors for 2D Navier-Stokes models with delays, J. Differential Equations, 205 (2004), 271–297.
[8] R. Datko, Representation of solutions and stability of linear differential-difference equations in a Banach space, J. Differential Equations, 29 (1978), 105–166.
[9] E. Feireisl, H. Petzeltova, E. Rocca and G. Schimperna, Analysis of a phase-field model for two-phase compressible fluid, Math. Models Methods Appl. Sci., 20 (2010), 1129–1160.
[10] C. G. Gal and M. Grasselli, Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D, Ann. Inst. H. Poincare Anal. Non Lineaire, 27 (2010), 401–436.
[11] C. G. Gal and M. Grasselli, Longtime behavior for a model of homogeneous incompressible two-phase flows, Discrete Contin. Dyn. Syst., 28 (2010), 1–39.
[12] C. G. Gal and M. Grasselli, Trajectory attractors for binary fluid mixtures in 3D, Chin. Ann. Math. Ser. B, 31 (2010), 655–678.
[13] P.C. Hohenberg and B. I. Halperin, Theory of dynamical critical phenomena, Rev. Modern Phys., 49 (1977), 435–479.
3294  THEODORE TACHIM MEDJO

[14] T. Tachim Medjo, Pullback attractors for a non-autonomous homogeneous two-phase flow model, J. Diff. Equa., 253 (2012), 1779–1806.
[15] H. Mei, G. Yin and F. Wu, Properties of stochastic integro-differential equations with infinite delay: regularity, ergodicity, weak sense Fokker-Planck equations, Stochastic Process. Appl., 126 (2016), 3102–3123.
[16] S. A. Messaoudi, A. Fareh and N. Doudi, Well posedness and exponential stability in a wave equation with a strong damping and a strong delay, J. Math. Phys., 57 (2016), 111501, 13 pp.
[17] C. Niche and G. Planas, Existence and decay of solutions in full space to Navier-Stokes equations with delays, Nonlinear Anal., 74 (2011), 244–256.
[18] A. Onuki, Phase transition of fluids in shear flow, J. Phys. Condens. Matter, 9 (1997), 6119–6157.
[19] T. Tachim Medjo, Attractors for a two-phase flow model with delays, Differential Integral Equations, 29 (2016), 1071–1092.
[20] T. Taniguchi, The exponential behavior of Navier-Stokes equations with time delay external force, Discrete Contin. Dyn. Syst., 12 (2005), 997–1018.
[21] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, volume 68. Appl. Math. Sci., Springer-Verlag, New York, second edition, 1997.
[22] X. S. Wang and J. Wu, Seasonal migration dynamics: periodicity, transition delay and finite-dimensional reduction, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 468 (2012), 634–650.

Received September 2016; revised February 2017.

E-mail address: tachimt@fiu.edu