The Néron-Severi group of a proper seminormal complex variety

Luca Barbieri-Viale¹, Andreas Rosenschon², V. Srinivas³

¹ Dipartimento di Matematica, Università di Padova, Padova, Italy, e-mail: barbieri@math.unipd.it
² Mathematisches Institut, Ludwig-Maximilians-Universität, München, Germany, e-mail: Andreas.Rosenschon@mathematik.uni-muenchen.de
³ Tata Institute of Fundamental Research, Mumbai, India, e-mail: srinivas@math.tifr.res.in

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Abstract We prove a Lefschetz (1, 1)-Theorem for proper seminormal varieties over the complex numbers.

1 Introduction

Let $X$ be a proper variety over $\mathbb{C}$. The Néron-Severi group $\text{NS}(X)$ is the image of the map $\text{Pic}(X) \to H^2(X, \mathbb{Z})$ which associates to a line bundle its cohomology class. If $X$ is smooth, the Lefschetz (1, 1)-theorem [17] states

$$\text{NS}(X) = F^1 \cap H^2(X, \mathbb{Z}) = \{ x \in H^2(X, \mathbb{Z}) \mid x_C \in F^1 H^2(X, \mathbb{C}) \}.$$  

In the smooth case one also knows the Néron-Severi group is isomorphic to the subgroup $H^2_{\text{ZL}}(X, \mathbb{Z}) \subseteq H^2(X, \mathbb{Z})$ of Zariski locally trivial elements, which in turn is identified with the Zariski cohomology group $H^1_{\text{Zar}}(X, H^1_X)$, yielding a formula

$$\text{NS}(X) = H^2_{\text{ZL}}(X, \mathbb{Z}) = H^1_{\text{Zar}}(X, H^1_X).$$

If $X$ is singular, $H^2(X, \mathbb{Z})$ carries a mixed Hodge structure, and it makes sense to consider the question whether the above characterizations of the Néron-Severi group hold in this case as well; see Bloch’s letter to Jannsen in [15, Appendix A]. This is false. If $X$ is an irreducible singular variety one always has inclusions

$$\text{NS}(X) \subseteq H^2_{\text{ZL}}(X, \mathbb{Z}) = H^1_{\text{Zar}}(X, H^1_X).$$
\[ \text{NS}(X) \subseteq F^1 \cap H^2(X, \mathbb{Z}) \]

and one knows of examples where both inclusions are strict \([3, 4]\). Barbieri-Viale and Srinivas \([4, \text{Question 4}]\) raised the question whether the correct formulation of the Lefschetz theorem for a projective normal variety \(X\) is

\[
\text{NS}(X) = F^1 \cap H^2_{\text{Zar}}(X, \mathcal{H}^1_X) = \{ x \in H^1_{\text{Zar}}(X, \mathcal{H}^1_X) \mid x_C \in F^1 H^2(X, \mathbb{C}) \}. \tag{1}
\]

This has been proved recently by Biswas-Srinivas \([8, \text{Theorem 1.1}]\). They also give an example of a non-normal variety for which this Lefschetz theorem fails.

Our main theorem states that the above description (1) also holds in the irreducible seminormal case. Recall that a variety is seminormal if all of its local rings are seminormal (see \([19]\) for the precise definition of a seminormal ring). Geometrically a variety \(X\) is seminormal if any finite birational morphism \(X' \to X\) that is bijective on closed points is an isomorphism. For example, the nodal cubic curve is seminormal but the cuspidal cubic curve is not seminormal. For further discussion of the notion of seminormality from a scheme-theoretic viewpoint we refer to, for instance, \([13]\).

We in fact prove the following more general result.

**Theorem 1.** Let \(X\) be a proper seminormal variety over \(\mathbb{C}\). Then

\[ \text{NS}(X) = F^1 \cap H^2_{\text{Zar}}(X, \mathcal{H}^1_X). \]

Our methods differ considerably from the ones used in \([8]\). Instead of working with a resolution of singularities we use simplicial techniques. Recall from \([11]\) that the mixed Hodge structure on \(H^2(X, \mathbb{Z})\) is obtained from a smooth proper hypercovering \(\pi_\bullet : X_\bullet \to X\) and the cohomological descent isomorphism \(H^*(X, \mathbb{Z}) \cong H^*(X_\bullet, \mathbb{Z}_\bullet)\). The basic idea behind our proof is to use such a hypercovering and cohomological descent to compare cohomology classes of line bundles on \(X\) with cohomology classes of simplicial line bundles on \(X_\bullet\). The assumption that \(X\) is seminormal is equivalent to \(\pi_\bullet_1 \mathcal{O}_{X_\bullet} \cong \mathcal{O}_X\), and implies \(\text{Pic}^0(X) \cong \text{Pic}^0(X_\bullet)\), see Lemmas 8 and 9.

Let \(X_\bullet\) be a smooth proper simplicial scheme over \(\mathbb{C}\). We study the largest sub Hodge structure of \(H^2(X_\bullet, \mathbb{Z}_\bullet)/W_0\) (modulo torsion) of level \(\leq 1\). By a standard argument \([11, \S 10]\), this Hodge structure \(E\) corresponds to a 1-motive \(L(E)\), which is given by an extension class map. It is a special case of the main result of \([2]\) (see also \([18]\)) that the 1-motive \(L(E)\) can be obtained from the Picard functors of the components \(X_p\) of \(X_\bullet\). More precisely, \(E\) is the Hodge realization of the isogeny class of the Lefschetz 1-motive \(L(X_\bullet)\) defined by the map

\[ \ker \{ \text{NS}(X_0) \to \text{NS}(X_1) \} \to \frac{\ker \{ \text{Pic}^0(X_1) \to \text{Pic}^0(X_2) \}}{\text{im} \{ \text{Pic}^0(X_0) \to \text{Pic}^0(X_1) \}} \]

where \(\ker^0\) denotes the connected component of the kernel of the underlying homomorphism of algebraic groups.
We give an alternate characterization of the Hodge realization of $L(X_\bullet)$. Let $H_1^1(X_\bullet,\mathcal{H}^1_\bullet(Z))$ be the simplicial (Zariski) sheaf whose $p$-th component is the sheaf associated to the presheaf that sends a Zariski open subset $U \subseteq X_p$ to the (singular) cohomology group $H_1^1(U,\mathbb{Z})$. It follows from a Leray spectral sequence argument that $H_1^1_{\text{Zar}}(X_\bullet, H_1^1(\mathbb{Z}))$ carries a mixed Hodge structure as a subgroup of $H_2(X_\bullet,\mathbb{Z})/W_0$. We show $H_1^1_{\text{Zar}}(X_\bullet, H_1^1(\mathbb{Q}))$ is canonically isomorphic to $E_\mathbb{Q}$; thus it is the Hodge realization of the isogeny class of $L(X_\bullet) = L(E)$.

To prove our Lefschetz theorem we consider the simplicial Picard group $\text{Pic}(X_\bullet) = H_1^1_{\text{Zar}}(X_\bullet, \mathcal{O}^*_X) = H_1^{\text{an}}(X_\bullet, \mathcal{O}^*_X)$. Let $\text{Pic}^0(X_\bullet)$ be its connected component of the identity and define the simplicial Néron-Severi group as the quotient $\text{NS}(X_\bullet) := \text{Pic}(X_\bullet)/\text{Pic}^0(X_\bullet)$. The Néron-Severi group $\text{NS}(X_\bullet)$ is the image of two cycle class maps, the analytic $c_1 : \text{Pic}(X_\bullet) \to H^2(X_\bullet,\mathbb{Z})$ given by the simplicial exponential sequence, and the map $\tilde{c}_1 : \text{Pic}(X_\bullet) \to H^1_{\text{Zar}}(X_\bullet, \mathcal{H}^1_\bullet(Z))$ defined by ‘pushing the exponential sequence to the Zariski site’. We compare the inclusions of the Néron-Severi group into the group of Zariski locally trivial classes on $X$ and $X_\bullet$ to obtain

$$\frac{F^1 \cap H^2_{\text{ZL}}(X,\mathbb{Z})}{\text{NS}(X)} = \ker \{ \delta : \text{NS}(X_\bullet)/\text{NS}(X) \to H^1_{\text{Zar}}(X_\bullet, \mathcal{H}^1_\bullet(Z)) \}.$$

If $X$ is seminormal we show $\ker \delta = 0$. This last step involves a non-trivial geometric argument (Lemma 13 below) and does not seem to follow immediately from cohomological considerations.

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**Notations.** If $X$ is a complex algebraic variety and $\mathcal{F}$ is an analytic sheaf on $X$, we write $H^p(X,\mathcal{F})$ for the corresponding cohomology groups; that is, unless specified otherwise, $H^p$ denotes cohomology in the analytic sense. Similarly, if $X_\bullet$ is a simplicial scheme and $\mathcal{F}_\bullet$ is a simplicial sheaf, $\mathbb{H}^p(X_\bullet,\mathcal{F}_\bullet)$ denotes the hypercohomology with respect to the analytic topology.

2 Lefschetz 1-motive

In this section we define the Lefschetz 1-motive. Let $X_\bullet$ be a smooth proper simplicial scheme over $\mathbb{C}$. We are interested in the largest Hodge substructure $E$ of $\text{im}(H^2(X_\bullet,\mathbb{Z}) \to H^2(X_\bullet,\mathbb{Q})/W_0)$ of level $\leq 1$. This Hodge substructure is given by the extension

$$0 \to \text{gr}_1^W \mathbb{H}^2 \to E \to \mathbb{H}^{(1,1)}_Z \to 0,$$  \hspace{1cm} (2)
where $H^{(1,1)} := (\text{gr} W_2 H^2)^{(1,1)}$ [10, 2.3.7] and $E$ is the pullback of $H^{(1,1)}_2 \subseteq \text{gr}^W H^2$ along the projection map $H^2 / W_0 \to \text{gr}^W H^2$ [11, 2.3.7]. A standard argument [11, §10] shows that $E$ is the Hodge realization of the 1-motive

$$L(E) := [H^{(1,1)}_2 \to J^1(\text{gr} W H^2)], \quad (3)$$

where $e$ is the extension class map. It is implicit in [2] that the 1-motive $L^1(X_\bullet)$ is algebraically defined, see Proposition 1 below for a precise statement. For the convenience of the reader we briefly sketch the argument.

Let $X_\bullet$ be a proper simplicial scheme over an algebraically closed field $k$ of characteristic 0, and let $X'_\bullet$ be a smooth proper hypercovering of $X_\bullet$. On each component $X'_p$ we have an exact sequence

$$0 \to \text{Pic}^0(X'_p) \to \text{Pic}(X'_p) \to \text{NS}(X'_p) \to 0 \quad (4)$$

which is compatible with the simplicial structure. Hence we have complexes

$$(\text{Pic}^0)^* : \cdots \to \text{Pic}^0(X'_{p-1}) \to \text{Pic}^0(X'_p) \to \text{Pic}^0(X'_{p+1}) \to \cdots$$

$$\text{Pic}^* : \cdots \to \text{Pic}(X'_{p-1}) \to \text{Pic}(X'_p) \to \text{Pic}(X'_{p+1}) \to \cdots$$

$$\text{NS}^* : \cdots \to \text{NS}(X'_{p-1}) \to \text{NS}(X'_p) \to \text{NS}(X'_{p+1}) \to \cdots$$

where the maps are the alternating sums of the maps induced by the face maps $\delta^i_p : X'_{p+1} \to X'_p$, $0 \leq i \leq p + 1$. If $H^p(-)$ denotes the cohomology of any of the above complexes, there is a boundary map

$$\lambda^p : H^{p-1}(\text{NS}^*) \to H^1(\text{Pic}^0)^*).$$

The quotient $H^p((\text{Pic}^0)^*)^0$ of the connected component of the identity of the group of $p$-cycles of $(\text{Pic}^0)^*$, viewed as an algebraic group scheme over $k$, by $p$-boundaries is a subgroup of $H^p((\text{Pic}^0)^*)$. The pullback square

$$\begin{array}{ccc}
L & \to & H^1((\text{Pic}^0)^*^0) \\
\downarrow & & \downarrow \\
H^0(\text{NS}^*) & \to & H^p((\text{Pic}^0)^*)
\end{array}$$

defines a finitely generated abelian group $L$, and an effective 1-motive

$$L(X_\bullet) := [L \to H^p((\text{Pic}^0)^*)^0], \quad (5)$$

[2, §1] which we will refer to as the Lefschetz motive.

**Remark 1.** Analogously to $L(X_\bullet)$ one defines 1-motives $L^p(X_\bullet) := [L^{p-1} \to H^p((\text{Pic}^0)^*)^0]$ for $p \geq 1$. The 1-motives $L^p(X_\bullet)$ have been investigated by the first author in [1].
If \( \mathcal{F}_\bullet \) is a simplicial sheaf on \( X_\bullet \) with respect to a topology \( \tau \), we write \( E^{p,q}_2(\mathcal{F}_\bullet) \) for the \( E^{p,q}_2 \)-term of the spectral sequence computing the cohomology of \( \mathcal{F}_\bullet \) from the cohomology of its components

\[
E^{p,q}_1 = H^p_f(X_p, \mathcal{F}_p) \Rightarrow H^{p+q}(X_\bullet, \mathcal{F}_\bullet).
\]

Following our convention, if we do not specify \( \tau \) it is understood that we are considering the analytic topology; in some cases when there is no risk of confusion, we will also suppress the subscript \( \tau \).

**Lemma 1.** Let \( X_\bullet \) be a smooth proper simplicial scheme over \( \mathbb{C} \).

\( (i) \) \( E_2^{p,q}(\mathbb{Q}_\bullet) \cong g^{p,q}_f H^{p+q}(X_\bullet, \mathbb{Q}_\bullet) \).

\( (ii) \) \( E_2^{p,0}(\mathbb{Z}_\bullet)_{\text{Zar}} \cong H^p_{\text{Zar}}(X_\bullet, \mathbb{Z}_\bullet) \). Hence \( W_0 H^p(X_\bullet, \mathbb{Q}_\bullet) \cong H^p_{\text{Zar}}(X_\bullet, \mathbb{Q}_\bullet) \).

**Proof.** The spectral sequence (6) for the sheaf \( \mathbb{Q}_\bullet \) on the analytic site is a spectral sequence of mixed Hodge structures. By our assumption each component \( X_p \) is smooth and proper, thus the groups \( H^q(X_p, \mathbb{Q}) \) carry a pure Hodge structure of weight \( q \). This implies (i). The corresponding spectral sequence for the constant Zariski sheaf \( \mathbb{Z}_\bullet \) degenerates at \( E_2 \), which immediately gives (ii). \( \square \)

From Lemma 1 it is easy to see that source and target of the 1-motives \( \mathbb{L}(X_\bullet) \) and \( \mathbb{L}(E) \) are isomorphic up to torsion. We need to show that under these isomorphisms the maps \( e \) and \( \lambda \) coincide.

By [7, Theorem 3.4] the derived category \( D^b(\mathit{MHS}) \) of bounded complexes of mixed Hodge structures is equivalent to the triangulated category of bounded mixed Hodge complexes [7, Definition 3.2].

Let \( K \to K' \) be a morphism of bounded mixed Hodge complexes, where \( H^i(K) \) and \( H^i(K') \) are pure of weight \( i \). Let \( \text{Ker} := \text{ker}\{H^2(K) \to H^2(K')\} \) and \( \text{Coker} := \text{coker}\{H^1(K) \to H^1(K')\} \). The distinguished triangle

\[
\to C(K \to K') \xrightarrow{+1} K \to K' \to
\]

shows that an element \( u \in \text{Hom}_{\mathit{MHS}}(\mathbb{Z}, \text{Ker}(1)) \) gives rise to an extension class \( e_u \in \text{Ext}^1_{\mathit{MHS}}(\mathbb{Z}, \text{Coker}(1)) \). Explicitly \( e_u \) is given by the extension

\[
0 \to \text{Coker}(1) \to H^1(C(K \to K')) \to \text{Ker}(1) \to 0.
\]

Let \( u' : \mathbb{Z} \to K(1)[2] \) be a lift of \( u \), and let \( u'' : \mathbb{Z} \to K'(1)[2] \) be the composition of \( u \) with \( K \to K' \). Then \( u'' \) induces an element of \( \text{Ext}^1_{\mathit{MHS}}(\mathbb{Z}, \text{Coker}(1)) \) which coincides with \( e_u \) [2, Remark 5.5].

Consider the case when \( K = R\Gamma(X_0, \mathbb{Z}) \) and \( K' = R\Gamma(X_1, \mathbb{Z}) \) are the Hodge complexes of \( X_0 \) and \( X_1 \), i.e. the underlying complex of abelian groups is the usual chain complex of topological spaces with the analytic topology [7, Section 4.3]. Taking the difference of the face maps \( \delta^0_1, \delta^1_1 : X_1 \to X_0 \) induces a map

\[
\delta^*_1 : R\Gamma(X_0, \mathbb{Z}) \to R\Gamma(X_1, \mathbb{Z}).
\]
Here $H^i(R\Gamma(X_0, \mathbb{Z})$ are the usual singular cohomology groups and, by definition,
\[ H^i_D(X_p, \mathbb{Z}(k)) := \text{Hom}_{D^b(MHS)}(\mathbb{Z}, R\Gamma(X_p, \mathbb{Z})(k)[j]) \]
are the Deligne-Beilinson cohomology groups of $X_p$. From Lemma 1 we have
\[ H^2(X_\bullet)(1, 1) \cong \text{Hom}_{MHS}(\mathbb{Q}, \text{ker}(H^2(K) \to H^2(K'))(1)), \]
\[ J^1(\text{gr} W H^2(X_\bullet)) \subseteq \text{Ext}_{MHS}(\mathbb{Z}, \text{coker}(H^1(K) \to H^1(K'))(1)). \]
Now $H^2_D(X_p, \mathbb{Z}(1)) \cong \text{Pic}(X_p)$ [7, Section 5.4], and the map $\delta_0^* : \text{Pic}(X_0) \to \text{Pic}(X_1)$ is the map obtained by composition in the derived category as above. By definition of $\lambda$, if an element $u \in H^0(\text{NS}_\bullet)$ lifts to $\text{Pic}(X_0)$, then $\delta_0^*(u) \in \text{Pic}^0(X_1)$ modulo the image of $\text{Pic}^0(X_0)$. Thus the above description of the extension class map shows the required compatibility. In summary, we have obtained the following.

**Proposition 1.** Let $X_\bullet$ be a smooth proper simplicial scheme over $\mathbb{C}$. Then up to isogeny $\mathbb{L}(X_\bullet) \cong \mathbb{L}(E)$. In particular, the Hodge realization of the isogeny class of $\mathbb{L}(X_\bullet)$ is $E_\mathbb{Q}$.

We need a different description of the Hodge realization of the 1-motive $\mathbb{L}(X_\bullet)$. Let $X_\bullet$ be any simplicial scheme and let $\omega_\bullet$ be the change of topology map from the analytic to the Zariski site of $X_\bullet$. Consider the simplicial Zariski sheaf
\[ \mathcal{H}_\bullet^n(\mathbb{Z}(n)) := \mathbb{R}^n(\omega_\bullet)_*(\mathbb{Z}(n)). \]
The $p$-th component of this sheaf is the Zariski sheaf $\mathcal{H}_p^n(\mathbb{Z}(n))$ associated to the presheaf $U \mapsto H^p(U, \mathbb{Z}(n))$, where $U \subseteq X_p$. For example, $\mathcal{H}_0^n(\mathbb{Z}(n))$ is the constant Zariski sheaf $\mathbb{Z}(n)$. The component spectral sequence (6) is
\[ E_1^{p,q} = H^p_{\text{zar}}(X_p, \mathcal{H}_p^n(\mathbb{Z}(n))) \Rightarrow H^{p+q}(X_\bullet, \mathcal{H}_\bullet^n(\mathbb{Z}(n))). \] (7)

The Leray spectral sequence with respect to $\omega_\bullet$ has the form
\[ L_2^{p,q} = H^p_{\text{zar}}(X_\bullet, \mathcal{H}_\bullet^n(\mathbb{Z}(n))) \Rightarrow H^{p+q}(X_\bullet, \mathbb{Z}(n)). \] (8)

There are versions of (7) and (8) for rational, real and complex coefficients; if there is no risk of confusion, we will omit the coefficients from the notation.

**Lemma 2.** Let $X_\bullet$ be a smooth proper simplicial scheme over $\mathbb{C}$. The Leray spectral sequence (8) induces an exact sequence
\[ 0 \to H^1_{\text{zar}}(X_\bullet, \mathcal{H}_\bullet^n(\mathbb{Q})) \to \frac{H^2(X_\bullet, \mathcal{H}_\bullet^n(\mathbb{Q}))}{W_0 H^2(X_\bullet, \mathcal{H}_\bullet^n(\mathbb{Q}))} \to H^0_{\text{zar}}(X_\bullet, \mathcal{H}_\bullet^n(\mathbb{Q})). \]
Proof. Let $L^p_q$ be the filtration induced by the spectral sequence. We show the differentials $d_{2,1}^i$ are trivial for $i \geq 0$. In particular, $L^1_{2,1} = L^1_{0,1}$, $L^2_{0,2} = L^2_{0,1}$, and the exact sequence claimed in the lemma is of the form $0 \to L^1_{0,1} \to H^2(X) \to L^2_{0,1}$. Note that the edge map
ti : L^0_{2,i} = H^i_{Zar}(X, \mathcal{H}^1(Q)) \to H^i(X, Q)$
coincides with the pullback map $H^i_{Zar}(X, Q) \to H^i(X, Q)$. The component spectral sequences of $(Q \otimes Q)$ and $Q$ induce the commutative diagram

$$E^2_{i,0}((Q \otimes Q)_{Zar}) \xrightarrow{\cong} E^0_{i,0}(Q)$$

$$\downarrow \quad \downarrow$$

$$H^i_{Zar}(X, Q) \xrightarrow{t^i} H^i(X, Q)$$

By Lemma 1 we can identify $t^i$ with the inclusion $W_0 H^i(X) \to H^i(X)$. Thus $L^0_{2,i} = L^0_{0,1}$. Our claim follows since $L^0_{2,0} = \text{coker } d_{2,1}^1$ and $L^0_{2,0} = \text{coker } d_{2,1}^1$. □

Lemma 3. Let $X$ be a smooth proper simplicial scheme over $\mathbb{C}$. There is an extension of rational mixed Hodge structures

$$0 \to gr^W H^2(X, Q) \to H^2_{Zar}(X, H^1(Q)) \to \ker \{NS(X_0) \to NS(X_1)\} \to 0$$

which is compatible with the mixed Hodge structure on $H^2(X, Q)$. In particular, $H^1_{Zar}(X, H^1(Q))$ carries a mixed Hodge structure with the properties:

- $W_0 H^1_{Zar}(X, H^1(Q)) = 0$,
- $W_1 H^1_{Zar}(X, H^1(Q)) = gr^W_1 H^2(X, Q)$,
- $W_2 H^1_{Zar}(X, H^1(Q)) = H^1_{Zar}(X, H^1(Q))$, 
- $gr^W H^1_{Zar}(X, H^1(Q)) = \ker \{NS(X_0) \to NS(X_1)\} = gr^W_2 H^2(X, Q)^{(1,1)}$.

Proof. We claim we have a commutative diagram with exact rows

$$0 \to H^1_{Zar}(X, H^1(Q)) \to H^2(X, Q) \to H^2_{Zar}(X, H^2(Q))$$

whose vertical maps are induced by the componentwise spectral sequences, and whose horizontal maps are coming from the Leray spectral sequences along the change of topology maps for $X$ and its components $X_p$. To see that this diagram commutes, consider first the constant simplicial scheme $(X_0)$. The map $X \to (X_0)$ clearly induces such a commutative diagram with the bottom row replaced by the corresponding $E^2_{0,i}$-terms. Since for any simplicial sheaf $\mathcal{F}$,

$$E^2_{0,i}(\mathcal{F}) = \ker \{H^i(X_0, \mathcal{F}_0) \to H^i(X_1, \mathcal{F}_1)\} \subseteq E^0_{0,i}(\mathcal{F}) = H^i(X_0, \mathcal{F}_0),$$
this implies our claim. Now we use that by Lemma 1 we have the formula
\[ E^{0,2}(Q) = \ker\{H^2(X_0, Q) \to H^2(X_1, Q)\} = \gr_2 \mathbb{H}^2(X_\bullet, Q_\bullet). \]

Hence
\[ \ker\alpha = \ker\beta = \gr_1 \mathbb{H}^2(X_\bullet, Q_\bullet) \]
and a diagram chase shows $\mathbb{H}^1_{\text{Zar}}(X_\bullet, \mathcal{H}^1_\bullet(Q))$ surjects onto the group
\[ E^{0,1}(\mathcal{H}^1_\bullet(Q)) = \ker\{\text{NS}(X_0)_Q \to \text{NS}(X_1)_Q\} = \gr_2 \mathbb{H}^2(X_\bullet, Q_\bullet)^{(1,1)}. \]  

In particular we have a commutative diagram of mixed Hodge structures
\[
\begin{array}{ccc}
E^{0,1}(\mathcal{H}_\bullet^1(Q)) & \to & \mathbb{H}_\bullet^1(X_\bullet, \mathcal{H}_\bullet^1(Q)) \\
\downarrow & & \downarrow \\
0 & \to & \gr_1 \mathbb{H}^2(X_\bullet, Q_\bullet) \\
\end{array}
\]
\[ \mathbb{H}_{\text{Zar}}^1(X_\bullet, \mathcal{H}_\bullet^1(Q)) \to \mathbb{H}^2(X_\bullet, Q_\bullet)/W_0 \to \gr_2 \mathbb{H}^2(X_\bullet, Q_\bullet) \to 0. \]  

\[ \square \]

**Proposition 2.** Let $X_\bullet$ be a smooth proper simplicial scheme over $\mathbb{C}$. Then $\mathbb{H}_{\text{Zar}}^1(X_\bullet, \mathcal{H}_\bullet^1(Q))$ is isomorphic to the Hodge realization of the isogeny class of the Lefschetz 1-motive $L(X_\bullet)$.

**Proof.** By (2), (9) and (10), the mixed Hodge structure $E_Q$ is canonically isomorphic to $\mathbb{H}_{\text{Zar}}^1(X_\bullet, \mathcal{H}_\bullet^1(Q))$. Now use Proposition 1. \(\square\)

### 3 Cohomology classes of simplicial line bundles

Let $X_\bullet$ be a smooth proper simplicial scheme over an algebraically closed field $k$. By [6, 4.1] the $fppf$-sheafification of the simplicial Picard functor $T \mapsto \text{Pic}(X_\bullet \times_k T)$ is representable, and its connected component of the identity $\text{Pic}^0(X_\bullet)$ is a semi-abelian variety over $k$. Explicitly, the semi-abelian variety $\text{Pic}^0(X_\bullet)$ is given by the extension
\[ 0 \to T(X_\bullet) \to \text{Pic}^0(X_\bullet) \to A(X_\bullet) \to 0, \]  
\[ \text{(11)} \]
where $A(X_\bullet)$ is the connected component of $\ker\{\text{Pic}^0(X_0) \to \text{Pic}^0(X_1)\}$, and
\[ T(X_\bullet) = E^{1,0}_2(\mathcal{O}_{X_\bullet}) = \mathbb{H}_{\text{Zar}}^1(X_\bullet, k_\bullet) \]  
\[ \text{(12)} \]
is the connected $k$-torus which is the $E^{1,0}_2$-term of the spectral sequence computing $\text{Pic}(X_\bullet)$. We define the simplicial Néron-Severi group as the quotient
\[ \text{NS}(X_\bullet) := \text{coker}\{\text{Pic}^0(X_\bullet) \to \text{Pic}(X_\bullet)\}. \]

From the above description of $\text{Pic}^0(X_\bullet)$ it is easy to see that $\text{NS}(X_\bullet)$ injects into $\text{NS}(X_0)$ up to a finite group; in particular, the simplicial Néron-Severi group is a finitely generated abelian group.
If $X_\bullet$ is a smooth proper simplicial scheme over $\mathbb{C}$ we have two cycle class maps: The simplicial exponential sequence induces the analytic cycle map

$$c_1 : \text{Pic}(X_\bullet) \to \mathbb{H}^2(X_\bullet, \mathbb{Z}_\bullet(1)),$$

and if $\omega_\bullet$ is the change of topology map from the analytic to the Zariski site, the restriction of $(\omega_\bullet)_* (\mathcal{O}_{X_\bullet})_{\text{an}} \to R^1(\omega_\bullet)_* \mathbb{Z}_\bullet(1)$ to the subsheaf $(\mathcal{O}_{X_\bullet})_{\text{Zar}}$ induces

$$\tilde{c}_1 : \text{Pic}(X_\bullet) \to \mathbb{H}^1_{\text{Zar}}(X_\bullet, \mathbb{H}^1_{\text{Zar}}(\mathbb{Z}_\bullet(1))).$$

Note that the map $\mathcal{O}_{X_\bullet} \to \mathbb{H}^1_{\text{Zar}}(\mathbb{Z}_\bullet(1))$ defining $\tilde{c}_1$ coincides with the composition

$$\mathcal{O}_{X_\bullet} \to \mathcal{F}^{1,1}_\bullet \to \mathbb{H}^1_{\text{Zar}}(\mathbb{Z}_\bullet(1)),$$

where $\mathcal{F}^{1,1}_\bullet$ is the simplicial sheaf whose $p$-th component is the Zariski sheaf associated to the presheaf defined by sending an open subset $U \subseteq X_p$ to

$$U \mapsto \mathcal{F}^{1,1}_p(U) := \text{Hom}_{\text{MHS}}(\mathbb{Z}(0), H^1(U, \mathbb{Z}(1)) \subseteq H^1(U, \mathbb{Z}(1)).$$

In particular, the map $d\log : \mathcal{O}_{X_p}^*(U) \to \mathcal{F}^{1,1}_p(U)$, $f \mapsto df/f$ induces the following exact sequence

$$0 \to \mathbb{C}_\bullet \to \mathcal{O}_{X_\bullet}^* \to \mathcal{F}^{1,1}_\bullet \to 0.\quad (15)$$

**Lemma 4.** Let $X_\bullet$ be a smooth proper simplicial scheme over $\mathbb{C}$.

(i) $\mathbb{H}^0_{\text{Zar}}(X_\bullet, \mathcal{F}^{1,1}_\bullet) = 0$

(ii) $\mathbb{H}^0_{\text{Zar}}(X_\bullet, \mathcal{F}^{1,1}_\bullet) = \ker\{\text{Pic}(X_0) \to \text{Pic}(X_1)\}$.

(iii) $\mathbb{H}^0_{\text{Zar}}(X_\bullet, \mathbb{H}^1_{\text{Zar}}/\mathcal{F}^{1,1}_\bullet) = \ker\{H^1(X_0, \mathcal{O}_{X_0}) \to H^1(X_1, \mathcal{O}_{X_1})\}$.

**Proof.** For any smooth proper variety $X_p$ the sheaf $\mathcal{F}^{1,1}_p$ has no global sections, thus (i). Clearly, $H^r(X_p, \mathcal{F}^{1,1}_p) = 0$ for $r \geq 2$. Since $H^1(X_p, \mathcal{F}^{1,1}_p) = \text{Pic}(X_p)$ [12, Theorem 1.3] (ii) is immediate from the componentwise spectral sequence. For (iii) use that the quotient sheaf $\mathbb{H}^1_{\text{Zar}}/\mathcal{F}^{1,1}_\bullet$ is a constant sheaf with value $H^1(X_p, \mathcal{O}_{X_p})$, see [5]. □

**Lemma 5.** Let $X_\bullet$ be a smooth proper simplicial scheme over $\mathbb{C}$. Then

$$\text{Pic}(X_\bullet) = \ker c_1 = \ker \tilde{c}_1.$$

**Proof.** For $c_1$ this follows from the simplicial exponential sequence and the fact that the Hodge filtration on $\mathbb{H}^r(X_\bullet, \mathbb{C}_\bullet)$ is defined by truncations of the simplicial De Rham complex [11, 8.1.12]; in particular, $\mathbb{H}^2(X_\bullet, \mathcal{O}_{X_\bullet}) = \mathbb{H}^2(X_\bullet)/F^1$.

For $\tilde{c}_1$ note that the restriction of $\tilde{c}_1$ to Pic$^0(X_\bullet)$ and the composition $\ker \tilde{c}_1 \to \text{Pic}(X_\bullet) \to \text{NS}(X_\bullet)$ are trivial, since NS$(X_\bullet)$ and im $\tilde{c}_1$ are finitely generated abelian groups, and $\ker \tilde{c}_1$ is divisible by Lemma 4. □
Lemma 6. Let $X_\bullet$ be a smooth proper simplicial scheme over $\mathbb{C}$. Then
\[ F^1 H^2(X_\bullet) \cap H^2(X_\bullet, \mathbb{Z}(1)) \cong NS(X_\bullet) \subseteq H_{\text{Zar}}^1(X_\bullet, H_1^4(\mathbb{Z}(1))), \]
where ‘intersection’ means the inverse image under $H^2(X_\bullet, \mathbb{Z}) \to H^2(X_\bullet, \mathbb{C})$.

Proof. Since $H^2(X_\bullet, \mathcal{O}_{X_\bullet}) \cong H^2(X_\bullet, \mathbb{C})/F^1$ the exponential sequence shows
\[ \text{im} \{ \text{Pic}(X_\bullet) \to H^2(X_\bullet, \mathbb{Z}(1)) \} = \ker \{ H^2(X_\bullet, \mathbb{Z}(1)) \to H^2(X_\bullet, \mathbb{C})/F^1 \}. \]
Hence $NS(X_\bullet) = F^1 \cap H^1(X_\bullet, \mathbb{Z})$. The remaining claim is a consequence of the fact that the kernel of both maps $c_1$ and $\tilde{c}_1$ is isomorphic to $\text{Pic}^0(X_\bullet)$.

Define
\[ \mathcal{I}(X_\bullet) := \text{coker} \{ NS(X_\bullet) \to H_{\text{Zar}}^1(X_\bullet, H_1^4(\mathbb{Z})) \}. \]
Note that $\mathcal{I}(X_\bullet)$ is the quotient of a finitely generated abelian group and $\mathcal{I}(X_\bullet)$ inherits a mixed Hodge structure from $H_{\text{Zar}}^1(X_\bullet, H_1^4(\mathbb{Q}))$.

Lemma 7. Let $X_\bullet$ be a smooth proper simplicial scheme over $\mathbb{C}$.
\[ W_0 \mathcal{I}(X_\bullet) = 0, \]
\[ W_1 \mathcal{I}(X_\bullet) = \text{gr}^W_1 H^2(X_\bullet, \mathbb{Q}), \]
\[ W_2 \mathcal{I}(X_\bullet) = \mathcal{I}(X_\bullet), \]
\[ \text{gr}_2^W \mathcal{I}(X_\bullet) = \text{gr}_2^W H^2(X_\bullet, \mathbb{Q}) \cap (F^1 \cap (H^2(X_\bullet, \mathbb{Q})/W_0)) = \text{im} \lambda_\mathcal{Q}. \]

Proof. The functor $W_p(-)$ is exact, thus all claims about the weight subspaces follow from Lemma 3. For the last claim we use the isomorphisms
\[ \text{gr}_2^W \mathcal{I}(X_\bullet) \cong \frac{\text{gr}_2^W H_{\text{Zar}}^1(X_\bullet, H_1^4(\mathbb{Q}))}{F^1 \cap H_{\text{Zar}}^1(X_\bullet, H_1^4(\mathbb{Q}))} \cong \frac{\text{gr}_2^W H^2(X_\bullet, \mathbb{Q})^{(1,1)}}{F^1 \cap (H^2(X_\bullet, \mathbb{Q})/W_0)} \cong \text{im} \lambda_\mathcal{Q}. \]
Here the first and the last identification are clear, and the second one follows from Lemma 1, Lemma 3 and Proposition 1. In particular, we note
\[ \ker \lambda_\mathcal{Q} \cong F^1 \cap H_{\text{Zar}}^1(X_\bullet, H_1^4(\mathbb{Q})) \cong F^1 \cap (H^2(X_\bullet, \mathbb{Q})/W_0). \]

4 (1, 1)-classes on proper seminormal varieties

We first give a characterization of seminormality in terms of hypercoverings; we believe this is known to experts although we could not find a reference in the literature.

Lemma 8. Let $X$ be a proper reduced scheme over $\mathbb{C}$, and let $\pi_\bullet : X_\bullet \to X$ be a smooth proper hypercovering. The following statements are equivalent.
(i) $X$ is seminormal.
(ii) $\pi_\bullet \mathcal{O}_{X_\bullet} \cong \mathcal{O}_X.$
Proof. Let $X_{\text{sn}}$ be the seminormalization of $X$. Then $X_{\text{sn}}$ satisfies the universal property that any morphism $Y \to X$ with $Y$ seminormal factors uniquely through $X_{\text{sn}}$ [16, Chapter I, Proposition 7.2.3.3]. Since $X_{\text{sn}} \to X$ is bijective, the functor from $X_{\text{sn}}$-schemes to $X$-schemes is faithful. The structure maps $X_i \to X$ factor through $X_0$, thus by the universal property of seminormalization through $X_{\text{sn}}$. Furthermore, the maps $X_i \to X_{\text{sn}}$ are compatible with the simplicial structure, i.e. $X_\bullet$ is a simplicial $X_{\text{sn}}$-scheme in a canonical way, and $\pi_\bullet$ factors as

$$X_\bullet \to X_{\text{sn}} \to X.$$ 

Let $\tilde{X} = \text{Spec}(\pi_\bullet, \mathcal{O}_{X_\bullet})$, and consider the factorization of $\pi_\bullet : X_\bullet \to X$ as

$$X_\bullet \to \tilde{X} \to X.$$ 

We claim that $X_{\text{sn}} = \tilde{X}$. Note that $\tilde{X} \to X$ clearly factors through $X_{\text{sn}}$ (with the obvious notion, $X_{\text{sn}} = \tilde{X}$). On the other hand, if $(X_0)_{\text{an}}/(X_1)_{\text{an}}$ denotes the topological quotient given by the equivalence relation determined by $(X_1)_{\text{an}}$, we have continuous proper maps

$$X_{\text{an}} \leftarrow (X_0)_{\text{an}}/(X_1)_{\text{an}} \to (\tilde{X})_{\text{an}}.$$ 

Here the first map is a bijection (and thus a homeomorphism) by definition of a hypercovering, and the second map results from the fact that $X_0$ is an $\tilde{X}$-scheme. The continuous section $X_{\text{an}} \to (\tilde{X})_{\text{an}}$ implies that the finite birational morphism $\tilde{X} \to X$ must be a bijection on closed points, and therefore $X_{\text{sn}} \to X$ factors through $\tilde{X} \to X$. $\square$

Assume $X$ is a proper reduced scheme over $\mathbb{C}$, and $\pi_\bullet : X_\bullet \to X$ is a smooth proper hypercovering. We consider divisor classes on $X$ and $X_\bullet$. By [6, Remark 4.11] Pic$^0(X_\bullet)$ is independent of the choice of $\pi_\bullet$. Lemma 6 shows that the same holds for $\text{NS}(X_\bullet)$. Hence Pic$(X_\bullet)$ is independent of the choice of a hypercovering. Lemma 7 and its proof show that the 1-motive $\mathbb{L}(X_\bullet)$ as well is independent of the choice of $\pi_\bullet$.

Lemma 9. Let $X$ and $\pi_\bullet : X_\bullet \to X$ be as above. Then

(i) $\text{NS}(X) \subseteq \text{NS}(X_\bullet)$ and $\text{NS}(X_\bullet)/\text{NS}(X) \cong \text{Pic}(X_\bullet)/\text{Pic}(X)$ which is a finite lattice (i.e. isomorphic to $\mathbb{Z}^k$ for some integer $k$).

(ii) If Pic$(X_\bullet)$ is generated by line bundles on $X$, $\text{NS}(X) = F^1 \cap H^2(X, \mathbb{Z})$.

(iii) $\pi^!_\bullet : \text{Pic}^0(X) \to \text{Pic}^0(X_\bullet)$ is surjective, with kernel a $\mathbb{C}$-vector space.

(iv) If $H^p(X, \mathcal{O}_X) = H^p(X, \mathbb{C})/F^1$ for $p = 1, 2$, then Pic$(X) \cong \text{Pic}(X_\bullet)$.

(v) If $X$ is seminormal, Pic$^0(X) \cong \text{Pic}^0(X_\bullet) \cong H^1(X, \mathbb{C})/(F^1 + H^1(X, \mathbb{Z}))$.

Proof. Consider the map $H^p(X, \mathbb{C}) \to H^p(X, \mathcal{O}_X)$ induced by the inclusion $\mathbb{C} \to \mathcal{O}_X$. The isomorphism $\mathbb{H}^p(X_\bullet, \mathcal{O}_{X_\bullet}) \cong \text{gr}^p \mathbb{H}^p(X_\bullet, \mathbb{C})$ and descent imply the map $\pi^!_\bullet : H^p(X, \mathcal{O}_X) \to H^p(X_\bullet, \mathcal{O}_{X_\bullet})$ is surjective for all $p \geq 0$. 
Comparing the exponential sequences for $X$ and $X_\bullet$ using GAGA we obtain a commutative diagram with exact rows

$$
\begin{array}{cccccccc}
H^1(X, \mathbb{Z}) & \rightarrow & H^1(X, \mathcal{O}_X) & \rightarrow & \text{Pic}(X) & \rightarrow & H^2(X, \mathbb{Z}) & \rightarrow & H^2(X, \mathcal{O}_X) \\
\cong & \downarrow & \text{onto} & \downarrow & \cong & \downarrow & \text{onto} & \\
\rightarrow H^1(X_\bullet, \mathbb{Z}) & \rightarrow H^1(X_\bullet, \mathcal{O}_{X_\bullet}) & \rightarrow & \text{Pic}(X_\bullet) & \rightarrow & H^2(X_\bullet, \mathbb{Z}) & \rightarrow & H^2(X_\bullet, \mathcal{O}_{X_\bullet}) \\
\end{array}
$$

Now (i)-(iv) follow from a diagram chase using descent and Lemma 6. For (v) note that if $X$ is seminormal we have an isomorphism $\pi_\bullet^* \mathcal{O}^\bullet_{X_\bullet} \cong \mathcal{O}^\bullet_X$ by Lemma 8. The Leray spectral sequence of $\pi_\bullet^*$ implies $\pi_\bullet^* : \text{Pic}(X) \rightarrow \text{Pic}(X_\bullet)$ is injective; combined with (iii) this shows $\text{Pic}^0(X) \cong \text{Pic}^0(X_\bullet)$. \ \square

Let $X$ be a proper reduced scheme over $\mathbb{C}$ with smooth proper hypercovering $\pi_\bullet : X_\bullet \rightarrow X$. Consider the commutative diagram with exact rows

$$
\begin{array}{cccc}
0 & \rightarrow & \text{NS}(X) & \rightarrow & H^1_{\text{Zar}}(X, \mathbb{Z}) & \rightarrow & \text{I}(X)_0 & \rightarrow & 0 \\
\pi_\bullet^* & \downarrow & \pi_\bullet^* & \downarrow & \pi_\bullet^* & \\
0 & \rightarrow & \text{NS}(X_\bullet) & \rightarrow & H^1_{\text{Zar}}(X_\bullet, \mathcal{H}_1^\bullet) & \rightarrow & \text{I}(X_\bullet) & \rightarrow & 0
\end{array}
$$

(17)

Here the top row is the exact sequence constructed in [3] (modified to allow for $X$ being reducible). The bottom row is obtained from Lemma 6. The quotient $\text{I}(X)_0$ is known to be a lattice of finite rank. The above diagram is a diagram of mixed Hodge structures. From the snake lemma we have an induced map

$$
\delta : \frac{\text{NS}(X_\bullet)}{\text{NS}(X)} \rightarrow \frac{H^1_{\text{Zar}}(X_\bullet, \mathcal{H}_1^\bullet)}{\text{Im} H^1_{\text{Zar}}(X, \mathcal{H}_1^\bullet) \cap H^2_{\text{Zar}}(X, \mathbb{Z})}.
$$

Lemma 10. Let $X$ be a proper reduced scheme over $\mathbb{C}$. Then

$$
\text{NS}(X)_Q = F^1 \cap H^2_{\text{Zar}}(X, \mathbb{Q}) \leftrightarrow \ker \delta_Q = 0.
$$

Remark 2. Since $X$ is proper, it follows from the exponential sequence and GAGA that the torsion in $H^2(X, \mathbb{Z})$ is always algebraic. The above lemma implies that $\text{NS}(X) = F^1 \cap H^2_{\text{Zar}}(X, \mathbb{Z})$, provided $\ker \delta_Q = 0$.

Proof. By descent and Lemma 2, the kernel of $H^2_{\text{Zar}}(X, \mathbb{Q}) \rightarrow H^1_{\text{Zar}}(X_\bullet, \mathcal{H}_1^\bullet(\mathbb{Q}))$ in (17) is precisely $\text{Wd} H^2_{\text{Zar}}(X)$. Lemma 6 and Lemma 7 show that

$$
\ker \delta_Q = (F^1 \cap (H^2_{\text{Zar}}(X, \mathbb{Q})/W_0))/\text{NS}(X)_Q.
$$

The claim follows since $F^1 \cap W_0 = 0$. \ \square

To prepare for the proof of our main theorem we need three lemmas.

Lemma 11. Let $\pi_\bullet : X_\bullet \rightarrow X$ be a smooth proper hypercovering. Let $A$ denote either $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ or $\mathbb{C}^*$. Then the restriction map

$$
\mathbb{H}^p_{\text{Zar}}(X_\bullet, A_\bullet)_X \rightarrow \mathbb{R}^p \pi_\bullet^* A_\bullet
$$

is surjective for $p \geq 0$. In particular, the sheaves $\mathbb{R}^p \pi_\bullet^* Z_\bullet$ are $\mathbb{Z}$-constructible in the Zariski topology (see [3] for the definition).
Proof. We will only consider \( A = \mathbb{C}^* \); the other cases are similar. If \( U \subseteq X \) is Zariski open, let \( U_\bullet = \pi_\bullet^{-1}(U) \subseteq X_\bullet \) be the inverse image, and let \( Y_\bullet = X_\bullet \setminus U_\bullet \) be the complement. We need to show the following map (of Zariski cohomology groups) is injective

\[
\mathbb{H}^{p+1}_{\text{Zar}}(X_\bullet, \mathbb{C}_*^\bullet) \rightarrow \mathbb{H}^{p+1}_\bullet(X_\bullet, \mathbb{C}_*^\bullet). \tag{19}
\]

In the spectral sequences computing these groups \( E_1^{pq} = 0 \) for \( q \geq 1 \). Hence

\begin{align*}
\mathbb{H}^{p+1}_Y(X_\bullet, \mathbb{C}_*^\bullet) &= \mathbb{H}^{p+1}(H^0_Y(X_\bullet, \mathbb{C}_*^\bullet))^*, \\
\mathbb{H}^{p+1}_\bullet(X_\bullet, \mathbb{C}_*^\bullet) &= \mathbb{H}^{p+1}(H^0(X_\bullet, \mathbb{C}_*^\bullet))^*,
\end{align*}

where \( H^0_Y(X_\bullet, \mathbb{C}_*^\bullet)^* \) and \( H^0_{\text{Zar}}(X_\bullet, \mathbb{C}_*^\bullet)^* \) are the complexes of global sections induced by the simplicial structure. Consider the map of complexes

\[
H^0_Y(X_\bullet, \mathbb{C}_*^\bullet)^* \rightarrow H^0(X_\bullet, \mathbb{C}_*^\bullet)^*. \tag{20}
\]

We will show (20) is split injective, this implies (19) is injective as claimed.

Note that for each \( n \) the group \( H^1(X_n, \mathbb{C}_*^\bullet) \) is the free abelian group on connected components of \( X_n \), tensored with \( \mathbb{C}_*^\bullet \). Similarly, \( H^1_{Y_n}(X_n, \mathbb{C}_*^\bullet) \) is the free abelian group on connected components of \( X_n \), tensored with \( \mathbb{C}_*^\bullet \). There is an obvious inclusion between these groups, and a natural splitting. We need to show this splitting is compatible with the differentials. That is, given a \( C_\bullet \)-term in the kernel of the splitting at level \( n \) (corresponding to a component \( Z \) of \( X_n \) which is not supported in \( Y_n \)), the face maps send this term into a \( C_\bullet \)-term in the kernel of the splitting at level \( n-1 \) (corresponding to a component \( W \) of \( X_{n-1} \) which is not supported in \( Y_{n-1} \)).

Recall that \( Y_n = X_n \setminus U_n \), where \( U_n \) is the inverse image of \( U \) under the structure morphism \( \pi_n : X_n \rightarrow X \). Consider a \( C_\bullet \)-term in \( H^1(X_n, \mathbb{C}_*^\bullet) \) that corresponds to a component \( Z \) of \( X_n \) which is not supported in \( Y_n \). Thus \( Z \cap U_n \neq \emptyset \) and \( \pi_n(Z) \cap U \neq \emptyset \). Assume \( W \) is a component of \( X_{n-1} \) such that there is some face map \( Z \rightarrow W \). Then this face map is in fact a morphism of \( X \)-schemes, and the image \( \pi_{n-1}(W) \subset X \) contains the image \( \pi_n(Z) \subset X \). In particular, \( W \) is \textit{not} supported in \( Y_{n-1} \), and the \( C_\bullet \)-term in \( H^1(X_{n-1}, \mathbb{C}_*^\bullet) \) corresponding to \( W \) lies in the kernel of the splitting at level \( n-1 \). \( \square \)

Lemma 12. Let \( \pi_\bullet : X_\bullet \rightarrow X \) be a smooth proper hypercovering. The map of simplicial sheaves \( O_{X_\bullet} \rightarrow F_{1,1} \) induces an injective map \( \mathbb{R}^1\pi_\bullet O_{X_\bullet} \rightarrow \mathbb{R}^1\pi_\bullet F_{1,1} \).

Proof. If \( A \) is an abelian group, we write \( A_X \) for the constant sheaf \( A \) on \( X \). From the description of \( \text{Pic}^0(X_\bullet) \) in (11) and (12) we have an injective map \( \mathbb{H}^1_{\text{Zar}}(X_\bullet, \mathbb{C}_*^\bullet) \rightarrow \text{Pic}^0(X_\bullet)X \) which fits into the commutative diagram

\[
\begin{array}{c}
\mathbb{H}^1_{\text{Zar}}(X_\bullet, \mathbb{C}_*^\bullet) \rightarrow \text{Pic}^0(X_\bullet)X \\
\downarrow \quad \downarrow \\
\mathbb{R}^1\pi_\bullet \mathbb{C}_*^\bullet \rightarrow \mathbb{R}^1\pi_\bullet O_{X_\bullet} \rightarrow \mathbb{R}^1\pi_\bullet F_{1,1}
\end{array}
\]
whose vertical maps are the sheafification maps. The bottom row is induced by (15) and is exact. We claim \( \text{Pic}^0(X_\bullet) \to \mathbb{R}^1_{\pi_\bullet}O_{X_\bullet} \) is the zero map. Let \( O_{X,x} \) be the local ring of a point \( x \in X \). Consider the commutative diagram

\[
\begin{array}{ccc}
\text{Pic}^0(X_\bullet) & \to & \text{Pic}(X_\bullet) \\
\downarrow & & \downarrow \\
\text{Pic}(\pi^{-1}_\bullet(\text{Spec}(O_{X,x}))) & \to & (\mathbb{R}^1_{\pi_\bullet}O_{X_\bullet})_x
\end{array}
\]

By Lemma 9(iii) \( \pi_\bullet^\ast : \text{Pic}^0(X) \to \text{Pic}^0(X_\bullet) \) is surjective. Thus any element of \( \text{Pic}^0(X_\bullet) \) lifts to an element of \( \text{Pic}(X) \) and has trivial image in \( H^0(X, \mathbb{R}^1_{\pi_\bullet}O_{X_\bullet}) \); this shows the map in question is trivial on stalks.

To finish the proof, note that by the previous lemma \( H^1_{\text{Zar}}(X_\bullet, \mathbb{C}_\bullet)_X \to \mathbb{R}^1_{\pi_\bullet}C_{\bullet} \) is surjective, thus \( \mathbb{R}^1_{\pi_\bullet}O_{X_\bullet} \to \mathbb{R}^1_{\pi_\bullet}F_{\bullet,1} \) is injective. \( \square \)

**Lemma 13.** Let \( f : Y \to Z \) be a morphism between \( k \)-varieties, where \( k \) is an algebraically closed field, and \( Y \) is non-singular and proper over \( k \). Let \( z \in Z \) be a (closed) point, \( F = f^{-1}(z) \) the reduced fiber, \( \tilde{F} \) the seminormalization of \( F \). Let \( Y_z = \text{Spec}(O_{Z,z}) \times_Z Y \). Consider the commutative triangle of groups

\[
\begin{array}{ccc}
\text{Pic}^0(Y) & \xrightarrow{\alpha} & \text{Pic}(Y_z) \\
\downarrow & & \downarrow \\
\text{Pic}(\tilde{F}) & & \end{array}
\]

Then \( \alpha \) and \( \beta \) have the same kernels, and isomorphic images. Further, the image of \( \beta \) is contained in \( \text{Pic}^0(\tilde{F}) \).

**Proof.** It clearly suffices to prove that \( \alpha \) and \( \beta \) have the same kernels. Let \( \widehat{O}_{z,Z} \) denote the completion of \( O_{z,Z} \) and set \( \widehat{Y}_z = \text{Spec}(\widehat{O}_{z,Z}) \times_Z Y \). Since cohomology commutes with flat base change, we see at once that

\[
\text{Pic}(Y_z) \to \text{Pic}(\widehat{Y}_z)
\]

is injective. If \( \mathfrak{M}_z \subseteq O_{z,Z} \) is the maximal ideal, let

\[
F_n = \text{Spec} \left( \frac{O_{z,Z}}{\mathfrak{M}_z^n} \right) \times_Z Y.
\]

The Formal Function Theorem [14, III, Theorem 11.1] implies the natural map

\[
\text{Pic}(\widehat{Y}_z) \to \lim_{n} \text{Pic}(F_n)
\]

is injective (in fact this map is an isomorphism by [14, II, Ex.9.6]).

It suffices to show that, for any \( n \geq 1 \), the maps of abelian groups

\[
\text{Pic}^0(Y) \to \text{Pic}(F_n)
\]
and
\[ \text{Pic}^0(Y) \rightarrow \text{Pic}(\tilde{F}) \]
have the same kernel. The kernel of the second map is the group of \( k \)-points of a reduced group subscheme of \( \text{Pic}^0(Y) \), which is hence an extension of a finite group by an abelian variety.

In fact, since \( Y, F_n \) and \( \tilde{F} \) are all proper over \( k \), the respective Picard groups are \( k \)-points of group schemes over \( k \), and the homomorphisms between Picard groups correspond to homomorphisms of group schemes. Here \( \text{Pic}^0(Y) \) is in fact an abelian variety.

The morphism \( \tilde{F} \rightarrow F_n \) factorizes as
\[ \tilde{F} \rightarrow (F_1)_{\text{red}} \rightarrow F_1 \rightarrow F_n, \]
where the first arrow is the seminormalization map \([19]\), and the other two are infinitesimal extensions (bijective closed immersions defined by nilpotent ideal sheaves). The kernels of the induced homomorphisms on Picard schemes
\[ \text{Pic}(F_n) \rightarrow \text{Pic}(F_1) \rightarrow \text{Pic}((F_1)_{\text{red}}) \rightarrow \text{Pic}(\tilde{F}) \]
are vector group schemes (with affine spaces over \( k \) as underlying scheme). This holds for \((F_1)_{\text{red}} \rightarrow F_n\) by \([9, \text{VI, Proposition 4.15}]\). For \( \tilde{F} \rightarrow (F_1)_{\text{red}} \) we use that by Lemma 9(iii) (note that the map \( X_\bullet \rightarrow X \) factors through the seminormalization), the map \( \text{Pic}((F_1)_{\text{red}}) \rightarrow \text{Pic}(\tilde{F}) \) is an isomorphism on torsion, thus the kernel is an affine group scheme which is torsion free, i.e. a vector group scheme.

The lemma now follows from the fact that if \( G \) is an extension of a finite group by an abelian variety over \( k \), and \( H \) is a vector group scheme over \( k \), then any homomorphism of groups schemes \( G \rightarrow H \) is trivial. \( \square \)

Let \( \pi_\bullet : X_\bullet \rightarrow X \) be a smooth proper hypercovering. If \( \mathcal{F}_\bullet \) is a sheaf on \( X_\bullet \), we write \( E_2^{p,q}(R^1\pi^*_\bullet \mathcal{F}_\bullet) \) for the sheaf corresponding to the \( E_2^{p,q} \)-term of the componentwise spectral sequence computing the sheaf \( R^1\pi^*_\bullet \mathcal{F}_\bullet \). For example,
\[ E_2^{0,1}(R^1\pi^*_\bullet \mathcal{O}^*_X) = \ker \{ R^1\pi_0^* \mathcal{O}^*_{X_0} \rightarrow R^1\pi_1^* \mathcal{O}^*_{X_1} \}. \]

We are ready to prove our main theorem.

**Proof.** (of Theorem 1)

By Lemma 10 and Remark 2 it suffices to show \( \ker \delta = 0 \). Our first step is to use a Leray spectral sequence argument to produce an injective map
\[ \text{NS}(X_\bullet)/\text{NS}(X) \rightarrow H^0(X, E_2^{0,1}(R^1\pi^*_\bullet \mathcal{O}^*_X))). \]

Consider the commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \rightarrow & H^1(X, R^0\pi_0^* \mathcal{O}^*_{X_0}) & \rightarrow & \text{Pic}(X_0) & \rightarrow & H^0(X, R^1\pi_0^* \mathcal{O}^*_{X_0}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^1(X, R^0\pi_1^* \mathcal{O}^*_{X_1}) & \rightarrow & \text{Pic}(X_1) & \rightarrow & H^0(X, R^1\pi_1^* \mathcal{O}^*_{X_1}) \\
\end{array}
\]

induced by the Leray spectral sequences of \( \pi_0 \) and \( \pi_1 \). Evidently the quotient \( \ker \{ \text{Pic}(X_0) \to \text{Pic}(X_1) \} / \ker \alpha \) injects into \( H^0(X, E_{2}^{0,1}(R\pi_1^* O_{X_1}^*)) \).

On each component we have an exact sequence analogous to (15), hence
\[
\ker \alpha = \ker \{ H^1(X, R^0 \pi_0^* F_{0}^{1,1}) \to H^1(X, R^0 \pi_1^* F_{1}^{1,1}) \}.
\]

Let \( C = \text{coker} \{ R^0 \pi_0^* F_{0}^{1,1} \to R^0 \pi_0^* F_{0}^{1,1} \} \) and let \( Q = \text{coker} \{ C \to R^0 \pi_1^* F_{1}^{1,1} \} \). The sheaves \( C \) and \( Q \) are subsheaves of \( R^0 \pi_0^* F_{0}^{1,1} \) and \( R^0 \pi_1^* F_{1}^{1,1} \) respectively and have no global sections by Lemma 4. It follows that
\[
\ker \alpha = H^1(X, R^0 \pi_1^* F_{1}^{1,1}). \tag{22}
\]

Consider the commutative diagram with exact rows
\[
\begin{array}{ccc}
0 & \to & H^1(X, R^0 \pi_1^* O_{X_1}^*) \\
\downarrow & & \downarrow \delta \\
0 & \to & H^1(X, R^0 \pi_1^* F_{1}^{1,1}) \\
\end{array}
\]
\[
\begin{array}{ccc}
\to & \text{Pic}(X_0) & \to \text{Pic}(X_1) \\
\downarrow \gamma & & \downarrow \\
\to & H^0(X, R^1 \pi_1^* O_{X_1}^*) & \to H^0(X, R^1 \pi_1^* F_{1}^{1,1})
\end{array}
\]

Here \( \gamma \) is injective by Lemma 12. Using Lemma 4 we can identify the middle vertical with the map \( \text{Pic}(X_0) \to \text{ker} \{ \text{Pic}(X_0) \to \text{Pic}(X_1) \} \) induced by the componentwise spectral sequence. Thus \( \ker \delta = T(X_1) \) and \( \text{coker} \delta = F \) is a finite group. Since \( X \) is seminormal, \( R^0 \pi_1^* O_{X_1}^* = O_{X}^* \) and \( H^1(X, R^0 \pi_1^* O_{X_1}^*) \cong \text{Pic}(X) \). An obvious diagram chase, combined with (22), gives an exact sequence
\[
0 \to \text{Pic}(X) / T(X_1) \to \ker \alpha \to F, \tag{23}
\]
where \( F \) is a finite group. Since \( X \) is seminormal \( \pi_1^* : \text{Pic}^0(X) \cong \text{Pic}^0(X_1) \) by Lemma 9. The above diagram shows the restriction of this map to the torus \( T(X_1) \subseteq \text{Pic}^0(X_1) \) is an isomorphism. Hence we have inclusions
\[
\text{Pic}(X) / T(X_1) \subseteq \text{Pic}(X_0) / T(X_1) \subseteq \ker \{ \text{Pic}(X_0) \to \text{Pic}(X_1) \}, \tag{24}
\]
where the second map is induced by the componentwise spectral sequence computing \( \text{Pic}(X_1) \). From (23) and (24) we see that the induced map
\[
\text{coker} \{ \text{Pic}(X) / T(X_1) \to \text{Pic}(X_0) / T(X_1) \} \to H^0(X, E_{2}^{0,1}(R\pi_1^* O_{X_1}^*)) \tag{25}
\]
has finite kernel. The source of this map is isomorphic to \( \text{NS}(X_1) / \text{NS}(X) \) and torsion free by Lemma 9. Thus (25) is injective and we have (21).

We claim the injective map (21) fits into the commutative square
\[
\begin{array}{ccc}
\text{NS}(X_1) / \text{NS}(X) & \overset{\text{into}}{\longrightarrow} & H^0(X, E_{2}^{0,1}(R\pi_1^* O_{X_1}^*)) \\
\downarrow \delta & & \downarrow \\
\text{im} H^2_{\text{Zar}}(X, H^1) & \to & H^0_{\text{Zar}}(X, R\pi_1^* H^1)
\end{array}
\]
\[
\tag{26}
\]
To explain the bottom row in the above diagram, we note that the spectral sequence of $X_{\text{an}} \to X_{\text{Zar}}$ induces a (surjective) map

$$H^2_{\text{Zar}}(X, \mathbb{Z}) \to \ker \{H^1_{\text{Zar}}(X, \mathcal{H}_X^1) \to H^3_{\text{Zar}}(X, \mathcal{H}_X^0) \}. \quad (27)$$

This map fits into the commutative square of mixed Hodge structures

$$
\begin{array}{ccc}
H^2_{\text{Zar}}(X, \mathbb{Z}) & \to & H^1_{\text{Zar}}(X, \mathcal{H}_X^1) \\
\downarrow & & \downarrow \\
\mathbb{H}^1_{\text{Zar}}(X, \mathcal{H}_X^1) & \to & \mathbb{H}^1_{\text{Zar}}(X, \mathcal{H}_X^1)
\end{array}
$$

where the kernel of the left vertical map is $W_0$. Thus we have an injective map

$$H^2_{\text{Zar}}(X, \mathbb{Z})/W_0 \to H^1_{\text{Zar}}(X, \mathcal{H}_X^1)/K,$$

where $K = \ker \{H^1_{\text{Zar}}(X, \mathcal{H}_X^1) \to H^1_{\text{Zar}}(X, \mathcal{H}_X^1)\}$. Now the bottom row of (26) is obtained from the Leray spectral sequence of $\pi_\ast$ using that the sheaf map $\mathcal{H}_X^1 \to \mathbb{R}^0 \pi_\ast \mathcal{H}_X^1$ induces an injection of $H^1_{\text{Zar}}(X, \mathcal{H}_X^1)/K$ into $H^1_{\text{Zar}}(X, \mathcal{H}_X^1)$.

Next we show the right vertical map in (26) is injective. Since the quotient $\text{NS}(X_\ast)/\text{NS}(X)$ is torsion free, it suffices to show the following map is injective

$$\gamma : H^0(X, E^{0,1}_2(\mathbb{R}^1 \pi_\ast \mathcal{O}_{X_\ast}))_Q \to H^0(X, E^{0,1}_2(\mathbb{R}^1 \pi_\ast \mathcal{H}_X^1))_Q. \quad (28)$$

Let Pic($X_0)_X$ be the constant sheaf Pic($X_0$) on $X$. We have an exact sequence

$$\text{Pic}(X_0)_X \to R^1 \pi_0_\ast \mathcal{O}_{X_0} \to R^1 \pi_0_\ast \mathcal{H}_X^1 \to 0$$

and we define $\mathcal{F}^0$ to be the image of Pic($X_0)_X$ in $R^1 \pi_0_\ast \mathcal{O}_{X_0}$. Starting with the map Pic($X_1)_X \to R^1 \pi_1_\ast \mathcal{O}_{X_1}$, one defines analogously $\mathcal{F}^1$. Then, by definition,

$$\ker \gamma = \ker \{H^0(X, \mathcal{F}^0)_Q \to H^0(X, \mathcal{F}^1)_Q\}.$$

We show $\ker \{\mathcal{F}^0 \to \mathcal{F}^1\}_Q = 0$. Let $x \in X$ be any point. For $i = 0, 1$ define $F^i_x = (\pi^{-1}_i(x))_{\text{red}} \subseteq X_i$, i.e. $F^i_x$ is the seminormalization of the fiber of $x$ over $\pi_i$ (with its reduced structure). Set $(X_i)_x = \text{Spec}(\mathcal{O}_{X,x}) \times_X X_i$. By definition, the stalk of $\mathcal{F}^i$ at $x$ is the image of Pic($X_i$) $\to$ Pic(($(X_i)_x$) for $i = 0, 1$. By Lemma 13 the map $\mathcal{F}^i_x \to \text{Pic}(F^i_x)$ is injective, and clearly factors through $\text{Pic}^0(F^i_x)$. We have so the following commutative square

$$
\begin{array}{ccc}
\mathcal{F}^0_x & \text{into} & \text{Pic}^0(F^0_x) \\
\downarrow & & \downarrow \\
\mathcal{F}^1_x & \to & \text{Pic}^0(F^1_x)
\end{array}
$$

Now $F^i_x \to \{x\}$ is a proper hypercovering. Hence $E^{1,0}_\infty = \ker \{H^1(F^0_x, \mathbb{Z}) \to H^1(F^1_x, \mathbb{Z})\}$ is a quotient of $\mathbb{H}^1(F^1_x, \mathbb{Z}) = H^1(\{x\}, \mathbb{Z}) = 0$. This implies the map $\text{Pic}^0(F^0_x) \to \text{Pic}^0(F^1_x)$ has finite kernel, and $\ker \{\mathcal{F}^0 \to \mathcal{F}^1\}_Q = 0$ as claimed. ◯
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