THE PROBLEM OF TIME AND GAUGE INVARIANCE IN THE QUANTIZATION OF COSMOLOGICAL MODELS.

II. RECENT DEVELOPMENTS IN THE PATH INTEGRAL APPROACH

T. P. Shestakova†‡ and C. Simeone²‡

† Department of Theoretical and Computational Physics, Rostov State University, Sorge Str. 5, Rostov-on-Don, 344090, Russia

‡ Departamento de Física, Facultad de Ciencias Exactas y Naturales Universidad de Buenos Aires, Ciudad Universitaria Pabellón I - 1428, Buenos Aires, Argentina

and Instituto de Astronomía y Física del Espacio C.C. 67, Sucursal 28 - 1428 Buenos Aires, Argentina

The paper is the second part of the work devoted to the problem of time in quantum cosmology. Here we consider in detail two approaches within the scope of Feynman path integration scheme: The first, by Simeone and collaborators, is gauge-invariant and lies within the unitary approach to a consistent quantization of gravity. It is essentially based on the idea of deparametrization (reduction to physical degrees of freedom) as a first step before quantization. The other approach by Savchenko, Shestakova and Vereshkov is rather radical. It is an attempt to take into account peculiarities of the Universe as a system without asymptotic states that leads to the conclusion that quantum geometrodynamics constructed for such a system is, in general, a gauge-noninvariant theory. However, this theory is shown to be mathematically consistent and the problem of time is solved in this theory in a natural way.

1. Introduction

In Part I of our work [39] we have considered most representative approaches to the well-known problem of time in quantum cosmology which lie in the framework of canonical quantization. Unfortunately, most of these proposals can be applied only to restricted classes of models. The most interesting and promising approaches which go beyond the minisuperspace approximation are the proposals by Brown and Kuchař [8] and by Barvinsky and collaborators [1, 2, 3, 4], the later having been formulated mainly making use the Feynman path integral formalism.

The main object of the path integral approach [12, 16, 17] is a transition amplitude between two states which is obtained as the sum over all histories of the exponential of the action. For a constrained system, divergences yielding from the overcounting of paths in phase space which are physically equivalent should be avoided by imposing gauge conditions that select one path from each class of equivalence [13, 14]. In its phase space form the propagator then reads

\[
\langle q' \mid q \rangle = \int Dq^i Dp_i D\delta(\chi) \exp \left( iS[q', p_i, N] \right).
\]  

(1)

Here \( \chi = 0 \) is a gauge fixing function and \( \left| \left| \chi, \mathcal{H} \right| \right| \) is the Faddeev – Popov determinant, which makes the result independent of the gauge choice. Admissible gauge conditions are those which can be reached from any path by performing a gauge transformation which is compatible with the symmetries of the action.

Because in gravitational dynamics the Hamiltonian generates the evolution and also acts as a generator of gauge transformations, it is natural to think that a time could be defined by means of gauge fixation, so that the resulting non divergent amplitudes would include a clear notion of evolution. But the problem arises that gauges defining a time in terms of the canonical variables are the so-called canonical gauges, which can be imposed only if the constraints are linear in the canonical momenta, and the Hamiltonian constraint in the gravitational action is quadratic in the momenta [46]. This seemed to appear as an obstacle for a program of deparametrization based on this idea; however, we shall show in Section 2 that for a class of cosmological models this can be solved by associating to them an ordinary gauge system (that is a system with constraints linear in the momenta), so that fixing the gauge in the gauge system defines a time for the corresponding minisuperspace [41].

In Section 2 we shall follow Simeone and collaborators [15, 40, 41, 42, 11, 20, 21, 22, 43, 44]. The connection between fixing an admissible gauge condition and a time definition will be considered in detail in Section 2.1. In Section 2.2 we shall describe a special canonical transformation that gives rise to an action for a system with a zero Hamiltonian and a constraint which is linear in the momenta. On this way we face the problem of observations mentioned in Part I of our work: new canonical variables appear to be conserved quantities since they commute with a new Hamiltonian. So we need another canonical transformation which leads to a
time-dependent Hamiltonian. This will be discussed in Section 2.3. We shall come to a formulation in terms of true degrees of freedom in what we call the reduced phase space. It allows us to define a transition amplitude through a path integral by the usual Faddeev – Popov procedure in Section 2.4. Examples will be given in Section 2.5.

In the approaches by Barvinsky and by Simeone and collaborators time is introduced into the theory by means of a time-dependent gauge condition. In Section 3 it will be shown that time may appear as a consequence of breaking down gauge invariance of the theory, even if the gauge condition does not depend on time. In the approach presented in papers by Savchenko, Shestakova and Vereshkov [32, 33, 36, 34, 35] the authors argued that the breakdown of gauge invariance is inevitable since the Universe as a physical system does not possess asymptotic states. It prevents from imposing asymptotic boundary conditions which eventually ensure gauge invariance. This will be discussed in Section 3.1. In Section 3.2 dynamics of a simple minisuperspace model in extended phase space will be constructed and its quantum version will be explored in Section 3.3.

At last, in Section 3.4 we shall touch upon an intriguing question if irreversibility of time could be related to nontrivial topology of the Universe.

2. Path integral quantization of minisuperspaces as ordinary gauge systems

In this section we shall review our procedure for associating an ordinary gauge system to a minisuperspace model, which allows to effectively deparametrize the minisuperspace and to obtain a consistent path integral quantization. The analogy between gauge transformations and dynamical evolution reflected in equations

\[
\frac{dq^i}{d\tau} = N_{\mu}[q^i, H^\mu], \quad \frac{dp_i}{d\tau} = N_{\mu}[p_i, H^\mu]
\]

and

\[
\delta q^i = \epsilon_\mu(\tau)[q^i, H^\mu], \quad \delta p_i = \epsilon_\mu(\tau)[p_i, H^\mu],
\]

\[
\delta N_{\mu} = \frac{\partial \epsilon_\mu(\tau)}{\partial \tau} - u_{\mu}^{\nu\rho} \epsilon_\rho N_\nu
\]

is the basic idea leading to the reduction procedure identifying physical degrees of freedom and time. However, because of the lack of gauge invariance at the end points in the action of gravitation resulting from the quadratic form of the Hamiltonian constraint, admissible gauges would not be of the canonical form \(\chi(q^i, p_i, \tau) = 0\); hence in order to perform the deparametrization, we shall introduce a reformulation of the theory leading to a globally gauge invariant action [41].

2.1. Gauge fixation and deparametrization

Admissible gauge conditions are those which can be reached from any path by means of gauge transformations leaving the action unchanged, and such that only one point of each orbit is on the manifold defined by \(\chi = 0\). This requires to analyse the possibility of the Gribov problem [23, 29], that is, that depending of the form of the orbits and on the topology of the constraint surface, it may be difficult to intersect it with a gauge condition which is crossed by each orbit only once.

If it is possible to perform a canonical transformation \((q^i, p_i) \rightarrow (Q^i, P_i)\) such that the Hamiltonian \(\mathcal{H}\) is matched to one of the new momenta, in terms of the new variables the action functional would include a constraint which is linear and homogeneous in the momenta. This is equivalent to say that the canonical variables \((Q^i, P_i)\) describe an ordinary gauge system, so that canonical gauges \(\chi(Q^i, P_i, \tau) = 0\) would be admissible.

The condition that a gauge transformation moves a point of an orbit off the surface \(\chi = 0\) is fulfilled if

\[
[\chi, \mathcal{H}] \neq 0.
\]

Now, as \(Q^0\) and \(P_0\) are conjugated variables,

\[
[Q^0, P_0] = 1
\]

and if we identify \(\mathcal{H} \equiv P_0\), then a gauge condition of the form

\[
\chi \equiv Q^0 - T(\tau) = 0
\]

with \(T\) a monotonous function is a good choice. Equation (4) only ensures that the orbits are not tangent to the surface \(\chi = 0\); however, as (6) defines a plane \(Q^0 = \) constant for each \(\tau\), if at any \(\tau\) any orbit was intersected more than once (then yielding Gribov copies) at another \(\tau\) it should be \([\chi, P_0] = 0\). Therefore this gauge fixation procedure avoids the Gribov problem [40].

The connection with the identification of time is the following: as we have already seen, for a parametrized system whose canonical variables are \((q^i, p_i)\), a global phase time \(t(q^i, p_i)\) is a function fulfilling [24]

\[
[t, \mathcal{H}] > 0.
\]

As the Poisson bracket is invariant under a canonical transformation, from (5) and (7) it follows that a global phase time can be defined for a minisuperspace by imposing on its associated gauge system a gauge condition in terms of the coordinate \(Q^0\). In other words, a gauge choice for the gauge system defines a particular foliation of spacetime for the corresponding cosmological model [41]. If a gauge choice avoiding the Gribov ambiguity can be found, then a definition of time which is good everywhere is obtained. A transformation such that \(\mathcal{H} = P_0\) can always be found locally; in the next paragraphs we shall show how a canonical transformation which works in the whole phase space can be found.
2.2. Gauge-invariant action for a minisuperspace

Here we shall review our procedure to obtain a gauge-invariant action for cosmological models whose Hamiltonian constraint is such that a solution for its associated \( \tau \)-independent Hamilton–Jacobi equation can be found. Consider a complete solution [31] \( W(q^i, \alpha_\mu, E) \) of the Hamilton–Jacobi equation

\[
H \left( q^i, \frac{\partial W}{\partial q^i} \right) = E \tag{8}
\]

where \( H \) is not necessarily the original Hamiltonian constraint but it can be a scaled Hamiltonian, that is \( H = \mathcal{F}^{-1} \mathcal{H} \) with \( \mathcal{F} \) a positive definite function of \( q^i \). If \( E \) and the integration constants \( \alpha_\mu \) are matched to the new momenta \( \mathcal{P}_0 \) and \( \mathcal{P}_\mu \) respectively, then \( W(q^i, \mathcal{P}_i) \) turns to be the generator function of a canonical transformation \( (q^i, p_i) \rightarrow (\mathcal{Q}^i, \mathcal{P}_i) \) defined by the equations

\[
p_i = \frac{\partial W}{\partial q^i}, \quad \mathcal{Q}^i = \frac{\partial W}{\partial \mathcal{P}_i}, \quad \overline{K} = N \mathcal{P}_0 = NH \tag{9}
\]

where \( \overline{K} \) is a new Hamiltonian. The new coordinates and momenta verify

\[
[\mathcal{Q}^i, \mathcal{P}_0] = [\mathcal{Q}^i, H] = 0
\]
\[
[\mathcal{P}_\mu, \mathcal{P}_0] = [\mathcal{P}_\mu, H] = 0
\]
\[
[\mathcal{Q}^i, \mathcal{P}_0] = [\mathcal{Q}^i, H] = 1.
\]

The variables \( (\mathcal{Q}^i, \mathcal{P}_\mu) \) are then observables: they commute with the constraint, so that they are gauge-invariant. The resulting action

\[
\mathcal{S}[\mathcal{Q}, \mathcal{P}, N] = \int_{\tau_1}^{\tau_2} \left( \mathcal{P}_i \frac{d\mathcal{Q}^i}{d\tau} - N \mathcal{P}_0 \right) d\tau \tag{10}
\]

describes a system with a zero true Hamiltonian and a constraint which is linear and homogeneous in the momenta (hence canonical gauges would be admissible in a path integral with this action). The action \( \mathcal{S} \) is related with \( S \) by

\[
\mathcal{S}[q^i, p_i, N] = \int_{\tau_1}^{\tau_2} \left( p_i \frac{dq^i}{d\tau} - NH \right) d\tau
+ \left[ \mathcal{Q}(q^i, p_i) \mathcal{P}_i(q^i, p_i) - W(q^i, \mathcal{P}_i) \right]_{\tau_1}^{\tau_2} \tag{11}
\]

so that the gauge-invariant action \( \mathcal{S} \) differs from the original action \( S \) in end point terms [28]. These terms do not modify the dynamics, as they can be included in the action integral as a total derivative with respect to the parameter \( \tau \).

2.3. Time and true degrees of freedom

The observables \( Q^i, P_\mu \) are conserved quantities, because they commute with \( K = N \mathcal{P}_0 \). This makes impossible the characterization of the dynamical trajectories of the system by an arbitrary choice of \( Q^i \) at the end points \( \tau_1 \) and \( \tau_2 \). To obtain a set of observables such that the choice of the new coordinates is enough to characterize the dynamical evolution, non conserved variables must be defined, and a new \( \tau \)-dependent transformation leading to a non null Hamiltonian must be introduced.

Let us consider the canonical transformation generated by

\[
F(Q^i, P_\mu, \tau) = P_0 Q^i + f(Q^i, P_\mu, \tau), \tag{12}
\]

which leads to

\[
\mathcal{P}_0 = \frac{\partial F}{\partial Q^i} = P_0 = H
\]
\[
\mathcal{P}_\mu = \frac{\partial F}{\partial Q^i} = \frac{\partial f}{\partial Q^i},
\]
\[
Q^0 = \frac{\partial F}{\partial P_0} = Q^i,
\]
\[
Q^\mu = \frac{\partial F}{\partial P_\mu} = \frac{\partial f}{\partial P_\mu}. \tag{13}
\]

The generator \( f \) defines a canonical transformation in what we call the reduced phase space, which corresponds to the true degrees of freedom of the theory. The coordinates and momenta \( (Q^i, P_\mu) \) are observables because

\[
[Q^\mu, P_0] = [P_\mu, P_0] = 0,
\]

but they are not conserved quantities, because their evolution is determined by the non zero Hamiltonian

\[
K = NP_0 + \frac{\partial f}{\partial \tau} = NH + \frac{\partial f}{\partial \tau}. \tag{14}
\]

Indeed,

\[
\frac{dQ^\mu}{d\tau} = \frac{\partial K}{\partial P_\mu} - \frac{\partial^2}{\partial \tau \partial P_\mu} f(Q^\mu, P_\mu, \tau)
\]
\[
\frac{dP_\mu}{d\tau} = \frac{\partial K}{\partial Q^\mu} - \frac{\partial^2}{\partial \tau \partial Q^\mu} f(Q^\mu, P_\mu, \tau), \tag{15}
\]

so that

\[
h(Q^\mu, P_\mu, \tau) = \frac{\partial}{\partial \tau} f(Q^\mu, P_\mu, P_\mu, \tau) \tag{16}
\]

is a true Hamiltonian for the reduced system (below we shall give a prescription to choose \( f \)). For the coordinate conjugated to \( P_0 \) we have

\[
\frac{dQ^0}{d\tau} = [Q^0, K] = N[Q^0, P_0] = N. \tag{17}
\]
The transformation \( (\overline{Q}, \overline{P}) \rightarrow (Q', P') \) yields additional end point terms of the form
\[
\left[ Q^\mu P_\mu - f(\overline{Q}')(\overline{Q'}, \overline{P}, \tau) \right]_{\tau_1}^{\tau_2}.
\]

The gauge-invariant action resulting from the two successive canonical transformations \((q', p_i) \rightarrow (\overline{Q}, \overline{P}) \rightarrow (Q', P')\) is
\[
S(Q', P, N) = \int_{\tau_1}^{\tau_2} \left( P_i \frac{dq^i}{d\tau} - NP_0 - \frac{\partial f}{\partial \tau} \right) d\tau
\]
and in terms of the original variables it includes end point terms,
\[
S[q, p_i, N] = \int_{\tau_1}^{\tau_2} \left( p_i \frac{dq^i}{d\tau} - NH \right) d\tau
\]
where \(Q', P'\) and \(P_0\) must be written in terms of \(q^i\) and \(p_i\). The action \(S(Q', P, N)\) describes an ordinary gauge system with a constraint \(P_0 = 0\), so that the coordinate \(Q^0\) is pure gauge, that is, \(Q^0\) is not associated to a physical degree of freedom. This coordinate can be defined as an arbitrary function of \(\tau\) by means of a canonical gauge choice. Writing \(Q^0\) in terms of \(q^0\) and \(p_i\), we have a function of the original phase space variables whose Poisson bracket with \(H = P_0\) is positive definite; as \(H\) differs from the original Hamiltonian constraint only by a positive definite function, then we can always define a global phase time as
\[
t(q^0, p_i) \equiv Q^0(q^0, p_i)
\]
because \([t(q^0, p_i), H(q^0, p_i)] = [Q^0, P_0] = 1\), and then
\[
[t(q^0, p_i), H(q^0, p_i)] > 0.
\]
The key point that allows to define a global phase time for the minisuperspace by imposing a canonical gauge condition on the associated gauge system described by \((Q', P')\) is that in terms of these variables we have a natural choice for a function whose Poisson bracket with the constraint is non vanishing elsewhere.

2.4. Path integral

The action \(S(Q^i, P_i, N)\) is stationary when the coordinates \(Q^i\) are fixed at the boundaries. The coordinates and momenta \((Q^i, P_i)\) describe a gauge system with a linear constraint, so that this action allows to obtain the amplitude for the transition \(|Q_1', \tau_1\rangle \rightarrow |Q_2', \tau_2\rangle\) by the usual Faddeev–Popov procedure:
\[
\langle Q_2', \tau_2 | Q_1', \tau_1 \rangle = \int DQ^0 DP_0 DQ^\mu DP_\mu DN \delta(\chi, P_0) e^{iS[Q^i, P_i, N]} \]
with \(S[Q^i, P_i, N]\) the gauge invariant action (18), and where \(\chi = 0\) can be any canonical gauge condition.

The Faddeev–Popov determinant \(||\chi, P_0||\) ensures that the result does not depend on the gauge choice. If we perform the functional integration on the lapse \(N\) enforcing the paths to lie on the constraint hypersurface \(P_0 = 0\), we obtain
\[
\langle Q_2', \tau_2 | Q_1', \tau_1 \rangle = \int DQ^0 DP_\mu D\tau \delta(\chi) ||\chi, P_0|| \times \exp \left( i \int_{\tau_1}^{\tau_2} \left[ P_\mu \frac{dQ^\mu}{d\tau} - h(Q^\mu, P_\mu, \tau) \right] d\tau \right),
\]
where \(h \equiv \partial f/\partial \tau\) is the true Hamiltonian of the reduced system. The path integral gives an amplitude between states characterized by the variables which, when fixed at the boundaries, make the action stationary. As \(S\) is stationary when the \(Q^i\) are fixed, then we choose the gauge in the most general form giving \(Q^0\) as a function of the other coordinates \(Q^\mu\) and \(\tau\); thus a choice of the boundary values of the physical coordinates and \(\tau\) fixes the boundary values of \(Q^0\). With a choice \(\chi \equiv Q^0 - T(Q^\mu, \tau) = 0\) and after trivially integrating on \(Q^0\) we finally obtain
\[
\langle Q_2', \tau_2 | Q_1', \tau_1 \rangle = \int DQ^\mu DP_\mu 
\]
\[
\times \exp \left( i \int_{\tau_1}^{\tau_2} \left[ P_\mu \frac{dQ^\mu}{d\tau} - h(Q^\mu, P_\mu, \tau) \right] d\tau \right)
\]
so that we have \(\langle Q_2', \tau_2 | Q_1', \tau_1 \rangle = \langle Q_2', \tau_2 | Q_1', \tau_1 \rangle\). Now, what we are looking for is an amplitude between states characterized by the original variables of a minisuperspace. Because the original action \(S[q^i, p_i, N]\) is stationary when the coordinates \(q^i\) are fixed at the boundaries, it is usual to look for a propagator of the form
\[
\langle q_2'| q_1' \rangle,
\]
so that the states are characterized only by the coordinates. But, as we have already remarked, in cosmology it is not always possible to define a time in terms of the \(q^i\) only; then the amplitude \(\langle q_2'| q_1' \rangle\) could not in general be understood as the probability that the observables of the system take a certain values at time \(t\) if at a previous time they took other given values.

If we pretend that
\[
\langle Q_2', \tau_2 | Q_1', \tau_1 \rangle = \langle q_2'| q_1' \rangle
\]
the paths should be weighted by the action \(S\) in the same way that they are weighted by \(S\), and the quantum states \(|Q^\mu, \tau\rangle\) should be equivalent to \(|q^i\rangle\). As the path integral in the variables \((Q^i, P_i)\) is gauge invariant, this requirement is verified if it is possible to impose a–globally good– gauge condition \(\tilde{\chi} = 0\) such that \(\tau = \tau(q^i)\) is defined. But this can be fulfilled only in the case that a global time \(t(q^i)\) exists, which is not true in general. In the most general case a global phase time must necessarily involve the momenta, and then we cannot fix the gauge in such a way that \(\tau = \tau(q^i)\). Hence,
we should admit the possibility of identifying the quantum states in the original phase space not by \( q^i \) but by a complete set of functions of both the coordinates and momenta \( q^i \) and \( p_i \).

This may suggest to give up the idea of obtaining an amplitude for states characterized by the coordinates. However, while a deparametrization in terms of the momenta may be completely valid at the classical level, at the quantum level there is an obstacle which is peculiar of gravitation [4]: There are basically two representations for quantum operators, the coordinate representation and the momentum representation, in which the states are characterized by occupation numbers associated to given values of the momenta. The last one is appropriate when the theory under consideration allows for the existence of asymptotically free states, so that an interpretation in terms of creation and annihilation operators exists. In quantum cosmology these asymptotic states do not, in general, exist. The appropriate representation is then a coordinate one, in which the quantum states are represented by wave functions in terms of the coordinates. The usual Dirac–Wheeler–DeWitt quantization with momentum operators in the coordinate representation follows this line; but, as we have already noted, this formalism is devoid of a clear notion of time and evolution, unless a time in terms of only the canonical coordinates exists.

An intermediate way can then be followed: When the constraint allows for the existence of an intrinsic time, our deparametrization and path integral quantization procedure straightforwardly gives the transition amplitude for states characterized by the original coordinates; this provides a quantization with a clear distinction between time and observables. On the other hand, when only an extrinsic time exists we change from the original variables \( (q^i, p_i) \) to a set \( (\tilde{q}^i, \tilde{p}_i) \) defined in such a way that the Hamiltonian constraint of a given model has a non vanishing potential; then an intrinsic time exists in terms of the coordinates \( \tilde{q}^i \), and the action \( S[\tilde{q}^i, \tilde{p}_i, N] \) is stationary when the \( \tilde{q}^i \) are fixed at the boundaries. Therefore our procedure yields the transition amplitude for states characterized by the new coordinates, which is given by

\[
\langle \tilde{q}^i_2 | \tilde{q}^i_1 \rangle = \langle Q^\mu_2, \tau_2 | Q^\mu_1, \tau_1 \rangle.
\]

In both cases we obtain a consistent quantization with a clear distinction between time and observables. Though this seems to complicate the interpretation of the resulting propagator, the original momenta turn to appear only in the time variable, while the new coordinates corresponding to the physical degrees of freedom depend on the \( q^i \) only (a detailed discussion has been given in the context of the quantization of the Taub anisotropic cosmology; see [22, 44]).

The form of the Hamiltonian \( H \) of the reduced system depends on the choice of the function \( f \). We can choose \( f \) so that the amplitude \( \langle Q^\mu_2, \tau_2 | Q^\mu_1, \tau_1 \rangle \) is equivalent to \( \langle \tilde{q}^i_2 | \tilde{q}^i_1 \rangle \). This requires that the Hamiltonian constraint allows to define a time in terms of the coordinates \( \tilde{q}^i \) and that the end point terms vanish on the constraint surface and in the gauge \( \bar{\chi} = 0 \) defining \( \tau = \tau(\tilde{q}^i) \), that is,

\[
\left[ \bar{Q}^{\mu} \bar{P}_{\mu} - W + Q^\mu P_\mu - f \right]_{\tau_1}^{\tau_2} = 0, \quad p_\tau = 0, \bar{\chi} = 0.
\]

Because the action \( S \) is gauge-invariant, this ensures that with any gauge choice the paths are weighted in the same way by \( S \) and \( S \). This requirement gives a prescription for the generator \( f(\bar{Q}^\mu, P_\tau, \tau) \) which determines the reduced Hamiltonian \( h = \partial f / \partial \tau \). As \( f \) depends only on observables, \( h \) commutes with the complete Hamiltonian \( K = N P_0 + h \), so that

\[
\frac{dh}{d\tau} = \frac{\partial^2 f}{\partial \tau^2}.
\]

Thus a generator \( f \) linear in \( \tau \) yields a conserved Hamiltonian for the reduced system.

The reduced Hamiltonian \( h \) could be both positive or negative-definite. As we shall illustrate with the second example of the next section, in general the sign of \( h \) will be in correspondence with the sign of a non vanishing momentum of the set \( \{ \tilde{p}_i \} \) in terms of which the constraint surface splits into two sheets. The formalism will therefore include two theories for the physical degrees of freedom, each one corresponding to each sign of \( h \) associated to one of the two sheets of the constraint surface. The path integral in the reduced space will give two propagators, one for the evolution of the wave functions of each theory (see [4], and also [24] for an analogous point of view). Note that then, if our path integral is to be associated to a canonical quantization, the splitting of the formulation into two disjoint theories is in correspondence with two Schrödinger equations; so in general it does not coincide with the ordinary Wheeler–DeWitt quantization. However, the existence of two disjoint theories, one for each sheet of the constraint surface, is a general property resulting from working with a time \( t(\tilde{q}^i) \), which comes from the fact that we want to identify the path integral in the variables \( Q^i \) with a transition amplitude between states given in terms of the coordinates \( \tilde{q}^i \); the nonequivalence between the Schrödinger and the Wheeler–DeWitt quantizations, instead, depends on the model under consideration, and also on the choice of coordinates. This has been discussed in detail in [45].

### 2.5. Examples

Consider the Hamiltonian constraint of the most general empty homogeneous and isotropic cosmological model:

\[
\mathcal{H} = \frac{1}{4} e^{-3\Omega} p_\Omega^2 - k e^\Omega + e^{3\Omega} = 0.
\]

This Hamiltonian corresponds to a universe with arbitrary curvature \( k = -1, 0, 1 \) and non zero cosmological
constant; we shall assume $\Lambda > 0$. If $k = 0$ we have the de Sitter universe. The classical evolution corresponds to an exponential expansion. For both $k = 0$ and $k = -1$ the potential is never zero, and then $p_\Omega$ cannot change its sign. Instead, for the closed model $p_\Omega = 0$ is possible.

It is convenient to work with the rescaled Hamiltonian $H = e^{-\Omega}H$:

$$H = \frac{1}{4}e^{-4\Omega}p_\Omega^2 - k + \Lambda e^{2\Omega} = 0.$$  \hspace{1cm} (28)

The constraints $H$ and $H$ are equivalent because they differ only in a positive factor. The $\tau$-independent Hamilton-Jacobi equation for the Hamiltonian $H$ has the solution

$$W(\Omega, \varpi_0) = 2\eta \int d\Omega e^{2\Omega} \sqrt{\Lambda e^{2\Omega} - k - \varpi_0},$$  \hspace{1cm} (29)

which is the generating function of the canonical transformation $(\Omega, \pi_\Omega) \rightarrow (Q^0, \varpi_0)$ defined by

$$Q^0 = -\eta \Lambda^{-1} \sqrt{\Lambda e^{2\Omega} - k - \varpi_0}, \quad \varpi_0 = H,$$  \hspace{1cm} (30)

with $\eta = \text{sgn}(p_\Omega)$. Then we define the function $F = Q^0P_0 + f(\tau)$ which generates the second canonical transformation yielding a non vanishing true Hamiltonian $h = \partial f / \partial \tau$ and $Q^0 = Q^0_0$, $\varpi_0 = P_0$.

The variables $Q^0$ and $P_0$ describe the gauge system into which the model has been turned. The gauge can now be fixed by means of a $\tau$-dependent canonical condition like $\chi \equiv Q^0 - T(\tau) = 0$ with $T$ a monotonic function of $\tau$. Then we can define the time as

$$t = Q^0 \big|_{P_0 = 0} = -\eta \Lambda^{-1} \sqrt{\Lambda e^{2\Omega} - k},$$  \hspace{1cm} (31)

or, using the constraint equation,

$$t(\Omega, p_\Omega) = -\frac{1}{2} \Lambda^{-1} e^{-2\Omega} p_\Omega.$$  \hspace{1cm} (32)

which is in agreement with the time obtained by matching the model with the ideal clock [7, 10]. An important difference between the cases $k = -1$ and $k = 1$ arises: for $k = -1$ the constraint surface splits into two disjoint sheets. In this case the evolution can be parametrized by a function of the coordinate $\Omega$ only, the choice given by the sheet on which the system remains: on the sheet $p_\Omega > 0$ the time is $t = -\Lambda^{-1} \sqrt{\Lambda e^{2\Omega} + 1}$, while on the sheet $p_\Omega < 0$ we have $t = \Lambda^{-1} \sqrt{\Lambda e^{2\Omega} + 1}$. The deparametrization of the flat model is completely analogous. For the closed model, instead, the potential can be zero and the topology of the constraint surface is no more equivalent to that of two disjoint planes. Although for $\Omega = -\ln(\sqrt{\Lambda})$ we have $V(\Omega) = 0$ and $p_\Omega = 0$, at this point it is $dp_\Omega / d\tau \neq 0$. Hence in this case $\Omega$ cannot parametrize the evolution, because the system can go from $(\Omega, p_\Omega)$ to $(\Omega, -p_\Omega)$; therefore a global phase time must necessarily be defined as a function of both the coordinate and the momentum.

The system has one degree of freedom and one constraint, so that it is pure gauge. In other words, there is only one physical state: from a given point in the phase space any other point on the constraint surface can be reached by means of a finite gauge transformation. This provides a proof for the consistency of our procedure: it should be possible to verify that the transition probability written in terms of the variables which include a globally well defined time is equal to unity.

The quantization is straightforward, and the observation above is reflected in that we obtain the propagator [21]

$$\langle Q^0_2, \tau_2 | Q^0_1, \tau_1 \rangle = \exp \left( -i \int_{\tau_1}^{\tau_2} \frac{\partial f}{\partial \tau} d\tau \right),$$  \hspace{1cm} (33)

and then the probability for the transition from $Q^0_1$ at $\tau_1$ to $Q^0_2$ at $\tau_2$ is indeed

$$\langle Q^0_2, \tau_2 | Q^0_1, \tau_1 \rangle^2 = 1.$$  \hspace{1cm} (34)

When the model is open or flat the coordinates $\Omega$ and $Q^0$ are uniquely related; hence the result simply reflects that once a gauge is fixed there is only one possible value of the scale factor $a \sim e^{\Omega}$ at each $\tau$, and

$$\langle \Omega_2 | \Omega_1 \rangle^2 = 1.$$  \hspace{1cm} (35)

But in the case of the closed model, at each $\tau$ there are two possible values of the coordinate $\Omega$; instead, there is only one possible value of the momentum $p_\Omega$ at each $\tau$. Hence the transition probability in terms of $Q^0$ does not correspond to the evolution of the coordinate $\Omega$, but rather of its derivative, and the amplitude $\langle Q^0_2, \tau_2 | Q^0_1, \tau_1 \rangle$ corresponds to an amplitude $\langle p_{\Omega,2} | p_{\Omega,1} \rangle$, and we have

$$\langle p_{\Omega,2} | p_{\Omega,1} \rangle^2 = 1.$$  \hspace{1cm} (36)

The fact that the resulting amplitude is not equivalent to $\langle \Omega_2 | \Omega_1 \rangle$ is clearly not a failure of the quantization procedure, because for this model a characterization of the states in terms of only the original coordinates is not possible if we want to retain a formally right notion of time on the whole evolution.

Now let us apply our formulation to a system with true degrees of freedom; consider a Hamiltonian constraint of the form

$$\mathcal{H} = G(q^2)(p_1^2 - p_2^2) + V(q^1, q^2) = 0,$$  \hspace{1cm} (37)

where $G(q^2) > 0$. This constraint includes homogeneous and isotropic models, both relativistic and dilatonic, and also some anisotropic models, like the Bianchi type I, the Kantowski–Sachs universe and also the Taub universe (after the appropriate canonical transformation introduced in [22]). We shall restrict our analysis to the cases in which the potential $V(q^1, q^2)$ has a definite sign, so that $q^1$ is a set of coordinates including a global time;
we shall assume \( V > 0 \). We shall also suppose that coordinates

\[
x = x(\tilde{q}^1 + \tilde{q}^2), \quad y = y(\tilde{q}^1 - \tilde{q}^2)
\]

(38)
can be introduced so that \( 4(\partial x / \partial \tilde{q}^1)(\partial y / \partial \tilde{q}^1) = V/G \); then we can write the constraint in the (scaled) equivalent form

\[
H = p_x p_y + 1 = 0.
\]

(39)
The solution of the corresponding Hamilton–Jacobi equation can be chosen so that the canonical variables of the associated gauge system are given by

\[
Q^0 = \frac{y}{P},
\]

\[
Q = x + \frac{1}{P^2} y(1 - P_0 - \eta T(\tau)),
\]

\[
P_0 = p_x p_y + 1,
\]

\[
P = p_x.
\]

(40)
Thus a canonical gauge condition \( \chi \equiv Q^0 - T(\tau) = 0 \) is associated with the extrinsic time \( t = y/p_x \). We can also define an intrinsic time, which is related with the obtaining of a transition amplitude between states characterized by the coordinates. The end point terms associated to the canonical transformation \( (x, y, p_x, p_y) \rightarrow (Q^i, P_i) \) are of the form

\[
B(\tau) = 2Q^0 - Q^0 P_0 - 2\eta \frac{T(\tau)}{P}.
\]

(41)

On the constraint surface \( P_0 = 0 \) these terms clearly vanish in gauge \( \chi \equiv \eta Q^0 P - T(\tau) = 0 \) which is in correspondence with the intrinsic time and true Hamiltonian(s)

\[
t(\tilde{q}^1, \tilde{q}^2) = \eta y(\tilde{q}^1 - \tilde{q}^2), \quad h(Q, P, \tau) = \frac{\eta}{P} \frac{dT}{d\tau},
\]

(42)
with \( \eta = sgn(p_x) = sgn(\tilde{p}_1 + \tilde{p}_2) = sgn(\tilde{p}_2) \), because \( V > 0 \) ensures that \( |\tilde{p}_2| > |\tilde{p}_1| \). The propagator for the transition \( |\tilde{q}^1_1, \tilde{q}^2_1\rangle \rightarrow |\tilde{q}^1_2, \tilde{q}^2_2\rangle \) is given by

\[
\langle \tilde{q}^1_2, \tilde{q}^2_2 | \tilde{q}^1_1, \tilde{q}^2_1 \rangle = 
\int DQDP \exp \left[ i \int_{T_1}^{T_2} \left( PdQ - \frac{\eta}{P} dT \right) \right],
\]

(43)
where the end points are given by \( T_1 = \pm y(\tilde{q}^1_1 - \tilde{q}^2_1) \) and \( T_2 = \pm y(\tilde{q}^1_2 - \tilde{q}^2_2) \). Note that with the gauge choice defining an intrinsic time, the observable \( Q \) reduces to a function of only the original coordinates:

\[
Q|_{\chi=0} = x(\tilde{q}^1 + \tilde{q}^2).
\]

Hence the paths go from \( Q_1 = x(\tilde{q}^1_1 + \tilde{q}^2_1) \) to \( Q_2 = x(\tilde{q}^1_2 + \tilde{q}^2_2) \). The propagator in the reduced space is therefore that of a system with a true degree of freedom given by the coordinate \( Q \). Also, because \( \tilde{p}_2 \) does not vanish on the constraint surface, the coordinate \( \tilde{q}^2 \) is itself a global time, namely \( t^* \); hence, though \( \tilde{q}^2 \) is not the time parameter in the path integral, the transition amplitude could be written as \( \langle \tilde{x}_2, t_2^* | \tilde{x}_1, t_1^* \rangle \). Observe that by considering both possible signs of the reduced Hamiltonian, this path integral gives the transition amplitude for both theories corresponding to both sheets of the constraint surface identified by the sign of the momentum \( \tilde{p}_2 \).

3. A closed universe as a system without asymptotic states and the problem of time

Several years ago a new approach to constructing quantum geometrodynamics was proposed by Savchenko, Shestakova and Vereshkov [32, 33, 36, 34, 35]. The central part in this approach is given to a Schrödinger equation for a wave function of the Universe which contains time as an external parameter like in ordinary quantum mechanics. However, the appearance of time in the Schrödinger equation is a consequence of breaking down gauge invariance of the theory. The proposed formulation is radically distinguished from the generally accepted Wheeler–DeWitt quantum geometrodynamics, so one needs to have a strong grounds for justifying this formulation.

3.1. Asymptotic states and gauge invariance

A key point of the authors’ argumentation is the analysis of the role of asymptotic states in quantum gravity [34]. It is emphasized that any gauge-invariant quantum field theory is essentially based on the assumption about asymptotic states. Indeed, in the case of canonical quantization, in order to separate true physical degrees of freedom from “nonphysical” ones we need to resolve gravitational constraints. It can be done in the limits of perturbation theory in asymptotically flat spaces or in some special cases. But in a general situation, if the Universe has some nontrivial topology and does not possess asymptotic states, this procedure meets insurmountable mathematical difficulties.

In the path integral approach, which was accepted by the authors as most adequate, asymptotic boundary conditions ensure the BRST-invariance of a path integral and play the role of selection rules; as a consequence, the path integral does not depend on a gauge-fixing function (see [27]). Since a closed universe is a system without asymptotic states, it is not correct in this case to impose asymptotic boundary conditions in a path integral, so that the set of all possible transition amplitudes determined through the path integral inevitably involves gauge-noninvariant ones.

If the path integral is considered without asymptotic boundary conditions it should be skeletonized on a full set of gauge-noninvariant equations obtained by varying an appropriate effective action including ghost and gauge-fixing terms. Further, there are two nonequivalent ways to proceed: to make use of the Batalin–
Vilkovisky (Lagrangian) [6] or the Batalin–Fradkin–Vilkovisky (Hamiltonian) [18, 5, 19] effective action. There exists the difference in the structure of ghost sectors, which, in turn, results from the fact that the gauge group of gravity does not coincide with the group of canonical transformations generated by gravitational constraints. Two formulations based on Lagrangian and Hamiltonian effective actions could be done equivalent in the gauge-invariant sector, the latter being singled out by means of asymptotic boundary conditions. Again, here we can see a crucial role of the assumption about asymptotic states in ensuring gauge invariance. In the situation without asymptotic states one has to make a choice between these two effective actions; the authors give preference to the Lagrangian formalism since it maintains the original group of gauge transformations. Moreover, one cannot ensure the BRST-invariance of the action without imposing asymptotic boundary condition, and the BFV scheme is broken then.

This approach leads to the extended set of Einstein equations in which the constraints are broken already at the classical level. Eventually, this causes the dynamical Schrödinger equation and the appearance of time. A similar modification of the Hamiltonian constraint and the related time-dependent Schrödinger equation was discussed early by Weinberg [48] and Unruh [47]. The modification aimed to solve the cosmological constant problem, and the cosmological constant appeared to be a Hamiltonian eigenvalue. It resulted from an additional condition on a metric tensor which did not fix a gauge. It is clear then, that the modification suggested by Weinberg and Unruh did not touch other gravitational constraints and equations of motion and was considered as a remedy for a particular (though very important) problem.

Another argument in favor of the gauge-noninvariant approach is the parametrization noninvariance of the Wheeler–DeWitt equation [26, 25]. The authors consider a unified interpretation of the choice of gauge variables (parametrization) and the choice of gauge conditions; the latter ones together determine equations for the metric components $g_{\mu\nu}$, fixing a reference frame, as it is illustrated by the scheme [37]:

\[
\begin{align*}
\text{Parametrization} & : g_{\mu\nu} = v_{\mu}(\mu, \gamma_{ij}) \\
& \quad + \text{Gauge conditions} \\
\mu_{\nu} &= f_{\nu}(\gamma_{ij})
\end{align*}
\]

Here $\mu_{\nu}$ are new gauge variables, in particular, the lapse and shift functions, $N$ and $N_{ij}$, $\gamma_{ij}$ is 3-metric. Thus even if one considers $\mu_{\nu}$ as independent of $\gamma_{ij}$, different parametrizations will correspond to different reference frames. This leads to the conclusion that a transition to another gauge variable is formally equivalent to imposing a new gauge condition, and vice versa, and the parametrization noninvariance of the Wheeler–DeWitt equation is ill-hidden gauge noninvariance.

### 3.2. Hamiltonian dynamics in extended phase space

After these preliminary notes let us go into mathematical details. The authors consider a simple minisuperspace model with the gauged action

\[
S = \int dt \left\{ \frac{1}{2} v(\mu, Q^a) \gamma_{ab} \dot{Q}^a \dot{Q}^b - \frac{1}{v(\mu, Q^a)} U(Q^a) \right. \\
+ \pi \left( \dot{\mu} - f_{a} \dot{Q}^a \right) - i w(\mu, Q^a) \dot{\theta} \theta \right\}.
\]

Here $Q^a$ stands for physical variables such as a scale factor or gravitational-wave degrees of freedom and material fields, and an arbitrary parametrization of a gauge variable $\mu$ determined by the function $v(\mu, Q^a)$ is accepted. For example, in the case of isotropic universe or the Bianchi IX model $\mu$ is bound to the scale factor $r$ and the lapse function $N$ by the relation

\[
\frac{r^3}{N} = v(\mu, Q^a).
\]

The special class of gauges not depending on time is used

\[
\mu = f(Q^a) + k; \quad k = \text{const}.
\]

It is convenient to present the gauge in a differential form,

\[
\dot{\mu} = f_{,a} \dot{Q}^a, \quad f_{,a} \overset{def}{=} \frac{\partial f}{\partial Q^a} \quad (47)
\]
\[
\theta, \bar{\theta} \text{ are the Faddeev–Popov ghosts after replacement } \theta \to -i \bar{\theta}. \text{ Further,}
\]

\[
w(\mu, Q^a) = \frac{v(\mu, Q^a)}{v_{,\mu}}; \quad v_{,\mu} \overset{def}{=} \frac{\partial v}{\partial \mu}.
\]

Varying the effective action (44) with respect to $Q^a, \mu, \pi$ and $\theta, \bar{\theta}$ one gets, correspondingly, motion equations for physical variables, the constraint, the gauge condition and equations for ghosts. The extended set of Lagrangian equations is complete in the sense that it enables one to formulate the Cauchy problem. The explicit substitution of trivial solutions for ghosts and the Lagrangian multiplier $\pi$ to this set of equations turns one back to the gauge-invariant classical Einstein equations.

The path integral approach does not require the construction of a Hamiltonian formulation before deriving the Schrödinger equation, but it implies that the Hamiltonian formulation can be constructed. Indeed, in the class of gauges (47) the Hamiltonian can be obtained in a usual way, according to the rule $H = p\dot{q} - L$, where $(p, q)$ are the canonical pairs of extended phase space
3.3. Quantum geometrodynamics in extended phase space

The constraint (52) can be presented in the form $H = E$, where $E$ is a conserved quantity (a new integral of motion). Correspondingly, in a quantum theory the relation $H = E$ should be replaced by a stationary Schrödinger equation, $H\langle \Psi \rangle = E \langle \Psi \rangle$, the Hamiltonian spectrum in EPS being not limited by the unique zero eigenvalue.

So, there is no reason to require that a wave function of a closed universe should satisfy the Wheeler–DeWitt equation. Independently of our notion of gauge invariance or noninvariance of the theory, the wave function should obey some Schrödinger equation. The Schrödinger equation is derived from the path integral with the effective action (44) by a standard method originated by Feynman [16, 9]. For the present model it reads

$$i \frac{\partial \Psi(\mu, Q^a, \theta, \bar{\theta}; t)}{\partial t} = H\Psi(\mu, Q^a, \theta, \bar{\theta}; t),$$

where

$$H = -\frac{i}{\omega} \frac{\partial}{\partial \theta} - \frac{1}{2M} \frac{\partial}{\partial Q^a} MG^{\alpha \beta} \frac{\partial}{\partial Q^a} + \frac{1}{v}(U - V);$$

the operator $H$ corresponds to the Hamiltonian in EPS (49). $M$ is the measure in the path integral,

$$M(\mu, Q^a) = \frac{K}{2w} (\mu, Q^a)w^{-1}(\mu, Q^a);$$

$K$ is a number of physical degrees of freedom; the wave function is defined on extended configurational space with the coordinates $\mu, Q^a, \theta, \bar{\theta}$. $V$ is a quantum correction to the potential $U$, that depends on the chosen parametrization (45) and gauge (46):

$$V = \frac{5}{12w^2} (w^2_{,\mu} f^a_{,\nu} + 2w_{,\mu} f^a_{,\nu} + w_{,\nu} w^a) + \frac{1}{3w} (w_{,\mu,\nu} f^a_{,\chi} + 2w_{,\mu} f^a_{,\chi} + w_{,\chi} w^a) + \frac{K - 2}{6w^2} (v_{,\mu} w_{,\nu} f^a_{,\chi} + v_{,\nu} f^a_{,\chi} + v_{,\chi} w^a) - \frac{K^2 - 7K + 6}{24w^2} (v^2_{,\mu} f^a_{,\chi} + 2v_{,\mu} f^a_{,\chi} + v_{,\chi} v^a) + \frac{1 - K}{6w} (v_{,\mu} f^a_{,\chi} + 2v_{,\mu} f^a_{,\chi} + v_{,\chi} v^a).$$

Let us emphasize that the Schrödinger equation (54) – (57) is a direct mathematical consequence of a path integral with the effective action (44) without asymptotic boundary conditions. Once we reject imposing asymptotic boundary conditions, we are doomed to come to a gauge-dependent description of the Universe.
The general solution to the Schrödinger equation has the following structure:

$$\Psi(\mu, Q^a, \theta, \bar{\theta}; t) = \int \Psi_k(Q^a, t) \delta(\mu - f(Q^a) - k)(\bar{\theta} + i\theta) dk.$$ (58)

As one can see, the general solution is a superposition of eigenstates of a gauge operator,

$$\{\mu - f(Q^a)\}|k\rangle = k|\rangle;$$

$$|k\rangle = \delta(\mu - f(Q^a) - k).$$ (59)

It can be interpreted in the spirit of Everett’s “relative state” formulation. In fact, each element of the superposition (58) describes a state in which the only gauge degree of freedom $\mu$ is definite, so that time scale is determined by processes in the physical subsystem through functions $v(\mu, Q)$, $f(Q^a)$ (see (45), (46)), while $k$ being determined by initial clock setting. The function $\Psi_k(Q^a, t)$ describes a state of the physical subsystem for a reference frame fixed by the condition (46). It is a solution to the equation

$$i \frac{\partial \Psi_k(Q^a, t)}{\partial t} = H_{(phys)}[f]\Psi_k(Q^a, t),$$ (60)

$$H_{(phys)}[f] = \left[ -\frac{1}{2M} \frac{\partial}{\partial Q^b} \frac{1}{v} \bar{M} \gamma^{ab} \frac{\partial}{\partial Q^b} + \frac{1}{v}(U - V) \right]_{\mu = f(Q^a) + k}.$$ (61)

One can seek the solution to Eq.(60) in the form of superposition of stationary state eigenfunctions:

$$\Psi_k(Q^a, t) = \sum_n c_n \psi_n(Q^a) \exp(-iE_n t);$$

$$H_{(phys)}[f] \psi_n(Q^a) = E_n \psi_n(Q^a).$$ (62)

The eigenvalues $E_n$ should not be associated with energy of any material field. It results from fixing a gauge condition and characterizes a subsystem which corresponds to observation means—a reference frame (see [34, 35] for details).

Having constructed a general solution to the Schrödinger equation one can pose the question if a physical part of the wave function could obey the Wheeler–DeWitt equation under some additional conditions. A natural additional condition in EPS is the requirement of BRST invariance of the wave function. Indeed, in the BFV approach the requirement of the BRST invariance leads immediately to the Wheeler–DeWitt equation. The BRST charge has an especially simple form for the present model,

$$\Omega_{BFV} = \eta^a \mathcal{G}_a = T \theta - i\pi \rho,$$ (63)

where $\mathcal{G}_a = (\pi, T)$ is the full set of constraints, and due to arbitrariness of BFV ghosts $\{\eta^a\}$ one gets the Wheeler–DeWitt equation $T |\Psi\rangle = 0$ from the requirement $\Omega_{BFV} |\Psi\rangle = 0$.

It is not the case in the approach considered above. We should remind that the original group of transformations was the group of gauge transformations in the Lagrangian formalism. It is the reason why transformations generated by (63) do not coincide with those under which the action (44) is invariant. The BRST charge constructed accordingly the BFV prescription turns out to be irrelevant in this consideration. Instead there exists another quantity that plays the role of the BRST generator,

$$\Omega = w(Q^a, \mu) \pi \theta - H\theta = -i\pi \rho - H\theta.$$ (64)

It is easy to check that (64) generates transformations in EPS which are identical to the BRST transformations in the Lagrangian formalism. Nevertheless, it cannot be presented as a combination of constraints with infinitesimal parameters replaced by ghosts and cannot help us to obtain the Wheeler–DeWitt equation [35].

On the other hand, as the authors show, the fact that the wave function obeys the Wheeler–DeWitt equation does not mean that this wave function describes the Universe in a gauge invariant way, i. e. independently of a reference frame. If one puts $\mu = k$, $E = 0$ and restricts the class of parametrizations as was done above (see (53) and the text before) the equation for the physical part of the wave function $H_{(phys)} \Psi_k(Q^a) = E \Psi_k(Q^a)$ is reduced to the Wheeler–DeWitt equation with its parametrization noninvariance and without any visible vestige of a gauge. By construction, however, a solution to this equation corresponds to a particular choice of a gauge condition and a particular line in the Hamiltonian spectrum. It is enough then to fix parametrization to complete the choice of a reference frame. It confirms the conclusion about ill-hidden gauge-noninvariance of the Wheeler–DeWitt equation which has been done in the beginning of this section.

All the above demonstrates that this attempt to derive a gauge invariant quantum theory from a more general gauge noninvariant one arises many questions. For a system with asymptotic states we have the BFV approach where we consider constraints yet at the classical level before quantization. But even in this case making use of asymptotic boundary conditions to exclude gauge-noninvariant terms is an idealization in the sense that we neglect the problem of Gribov’s copies.

### 3.4. Topology of the Universe and the irreversibility of time

In conclusion we shall touch on one of consequences of the presented approach – the irreversibility of a transition to another reference system in the framework of gauge-noninvariant description [38]. Since the reference frame was declared to be a constituent of an integrated system as well as a physical Universe and plays a role of
a measuring device, any change in respect of the reference
frame would cause changes in an observed physical
picture. Indeed, let us consider a small variation of the
gauge-fixing function \( f(Q^a) \), so that the reference frame
will be fixed by the condition
\[
\mu = f(Q^a) + \delta f(Q^a) + k. \tag{65}
\]
Then, in a new basis corresponding to this reference
frame the wave function will take the form
\[
\Psi(\mu, Q^a, \theta, \tilde{\theta}; t) = \int \Psi_k(Q^a, t) \delta(\mu - f(Q^a) - \delta f(Q^a) - k) \delta f(Q^a) + k. \tag{66}
\]
Here the function \( \Psi_k(Q^a, t) \) satisfies Eq.(60) with a
Hamiltonian
\[
H_{(phys)}[f + \delta f] = \left[ -\frac{1}{2M} \frac{\partial}{\partial Q^a} \left( \frac{1}{v} M \gamma^{ab} \frac{\partial}{\partial Q^b} \right) + \frac{1}{v} (U - V) \right]_{\mu = f(Q^a) + \delta f(Q^a) + k}. \tag{67}
\]
It is obvious that the equation for the physical part of
the wave function with the Hamiltonian (67) cannot be
reduced in general to the equation with the Hamiltonian
(61). The measure in the subspace of physical degrees
of freedom also depends on a chosen gauge condition as
it follows from the normalization equation
\[
\int \Psi_k^*(Q^a, t) \Psi_k(Q^a, t) \delta(\mu - f(Q^a) - k') \times \delta(\mu - f(Q^a) - k) dk' dk M(\mu, Q^a) d\mu \prod_a dQ^a = \int \Psi_k^*(Q^a, t) \Psi_k(Q^a, t) M(f(Q^a) + k, Q^a) \times \prod_a dQ^a dk = 1. \tag{68}
\]
Due to smallness of \( \delta f(Q^a) \) one can write
\[
H_{(phys)}[f + \delta f] = H_{(phys)}[f] + W[\delta f] + V_1[\delta f]. \tag{69}
\]
For our minisuperspace model the operator \( W[\delta f] \) reads
\[
W[\delta f] = \left[ \frac{1}{2M^2} \frac{\partial^2}{\partial \mu^2} \delta f \frac{\partial}{\partial Q^a} \left( \frac{1}{v} M \gamma^{ab} \frac{\partial}{\partial Q^b} \right) - \frac{1}{2} \frac{\partial}{\partial Q^a} \left( \frac{1}{v} \frac{\partial M}{\partial \mu} - \frac{M \delta v}{\partial \mu} \right) \right]_{\mu = f(Q^a) + k}. \tag{70}
\]
and \( V_1[\delta f] \) is the change of quantum potential \( V \) (57)
in first order of \( \delta f \).

One can inquire how the probabilities of stationary
states (62) change under the perturbation \( W[\delta f] + V_1[\delta f] \), which is due to a small variation of the gauged
fixing function \( f(Q^a) \). The Hamiltonian (67) is Her-
mitian by construction in a space with the measure
\( M(f(Q^a) + \delta f(Q^a) + k, Q^a) \), however it is not Her-
mitian in a space with the measure \( M(f(Q^a) + k, Q^a) \) in
which the functions (62) are normalized. In this space
the operator (70) will have, in general, anti-Hermitian
part. So any transition to another reference frame must
be irreversible.

This is true for a transition to another reference
frame in the same spacetime region, and this is also
true if spacetime consists of several regions, different
reference frames being introduced in these regions. A
nontrivial topology of the Universe may be a reason why
one has to introduce various reference frames in different
spacetime regions. In particular, we can consider mutually
intersected spacetime regions ordered in time. Ev-
ery time when we move from one region to another, the
physical part of the wave function would be undergone
a non-unitary transformation followed by changing the
measure in the subspace of physical degrees of freedom
that may lead to irreversible consequences in the phys-
ic picture of the Universe. If so, taking into account
interaction with the reference frame—the measuring in-
sert representing the observer in quantum theory of
gravity— not only enables one to introduce time into
quantum geometrodynamics, but also may attach an ir-
reversible character to cosmological evolution.

4. Discussion

Among many attempts to give a solution to the problem
of time we have paid a considerable attention to two ap-
proaches, which are, as a matter of fact, very different.
The first one, by Simeone and collaborators, considered
in Section 2, is a development of the unitary approach to
quantum gravity inspired by earlier works of Barvinsky
and Hajiček. The key point here is a reduction of the
gravitational action to that of an ordinary gauge sys-
tem. Since the Hamiltonian constraint is quadratic in
the momenta, we come, in general, to two formulations
of the theory which correspond to two disjoint sheets of
the constraint surface given by the two signs of the mo-
momentum conjugated to a time variable. The proposed
procedure enables one to formulate the theory in terms
of true degrees of freedom and then return to a transi-
tion amplitude between states characterized by original
variables of phase space, so that the whole scheme is
gauge-invariant. This approach demonstrates how time
can be introduced into the theory without breaking down
its gauge invariance. Let us emphasize that the require-
ment of gauge invariance is conventionally thought to
be one of basic requirements for a physical theory.

In this sense the second approach, presented in Sec-
tion 3, is very radical. According to the analysis by
Isham [30], approaches to the problem of time can be
subdivided into three main categories: those in which
time is identified before quantizing, approaches in which
time is identified after quantizing and approaches in
which time plays no fundamental role at all. The pro-

posal by Savchenko, Shestakova and Vereshkov does not belong to any of these categories. In their scheme time naturally appears while quantizing a gravitational system, namely, while driving a Schrödinger equation from the path integral. In this consideration time has a status of an external parameter as in ordinary quantum mechanics. The price for it is a refusal from gauge invariance of the theory.

We would note that the fact that the Universe does not possess asymptotic states has not been analysed early from the viewpoint of its connection with gauge invariance. Traditionally the Universe was quantized as any gauge system with which we deal in laboratory physics. On the other hand, in modern field theory one can find indications that the role of gauge degrees of freedom may not be just auxiliary. It is enough to mention the Aharonov – Bohm effect and instanton solutions. All of them originate from nontrivial topological structure of spacetime. A future development will give an objective appraisal to the proposed approaches to the problem of time which remains to be a fundamental problem of constructing quantum gravity.

References

[1] A.O. Barvinsky and V.N. Ponomariov, Phys. Lett. B 167, 289 (1986).
[2] A.O. Barvinsky, Phys. Lett. B 175, 401 (1986).
[3] A.O. Barvinsky, Phys. Lett. B 195, 289 (1987).
[4] A.O. Barvinsky, Phys. Rep. 230, 237 (1993).
[5] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B 69, 309 (1977).
[6] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B 102, 27 (1981).
[7] S.C. Bellucci and R. Ferraro, Phys. Rev. D 52, 1963 (1995).
[8] J.D. Brown and K.V. Kuchař, Phys. Rev. D 51, 5600 (1995)
[9] K.S. Cheng, J. Math. Phys. 13, 1723 (1972).
[10] H. De Cicco and C. Simeone, Gen. Rel. Grav. 31, 1225 (1999).
[11] H. De Cicco and C. Simeone, Int. J. Mod. Phys. A 14, 5105 (1999).
[12] P.A.M. Dirac, Physik. Zeits. Sowjetunion 3, 64 (1933).
[13] L.D. Faddeev and V.N. Popov, Phys. Lett. B 25, 29 (1967).
[14] L.D. Faddeev and A.A. Slavnov, “Gauge Fields: Introduction to Quantum Theory”, Benjamin/Cummings Publishing, 1980.
[15] R. Ferraro and C. Simeone, J. Math. Phys. 38, 599 (1997).
[16] R.F. Feynman, Rev. Mod. Phys. 20, 367 (1948).
[17] R.F. Feynman and A.R. Hibbs, “Quantum Mechanics and Path Integrals”, McGraw–Hill, New York, 1965.
[18] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. B 55, 224 (1975).
[19] E.S. Fradkin and T.E. Fradkina, Phys. Lett. B 72, 343 (1978).
[20] G. Giribet and C. Simeone, Mod. Phys. Lett. A 16, 19 (2001).
[21] G. Giribet and C. Simeone, Phys. Lett. A 287, 344 (2001).
[22] G. Giribet and C. Simeone, Int. J. Mod. Phys. A 17, 2885 (2002).
[23] V.N. Gribov, Nucl. Phys. B 139, 1 (1978).
[24] P. Hájicek, Phys. Rev. D 34, 1040 (1986).
[25] J.J. Halliwell, Phys. Rev. D 38, 2468 (1988).
[26] S.W. Hawking and D.N. Page, Nucl. Phys. B 264, 185 (1986).
[27] M. Henneaux, Phys. Rep. 126, 1 (1985).
[28] M. Henneaux, C. Teitelboim and J.D. Vergara, Nucl. Phys. B 387, 391 (1992).
[29] M. Henneaux and C. Teitelboim, “Quantization of Gauge Systems”, Princeton University Press, New Jersey, 1992.
[30] C. Isham, “Canonical quantum gravity and the problem of time”, lectures presented at NATO Advanced Study Institute, Salamanca, June 1992.
[31] L.D. Landau and E.M. Lifshitz, “Mechanics”, Pergamon Press, Oxford, 1960.
[32] V.A. Savchenko, T.P. Shestakova and G.M. Vereshkov, Int. J. Mod. Phys. A 14, 4473 (1999).
[33] V.A. Savchenko, T.P. Shestakova and G.M. Vereshkov, Int. J. Mod. Phys. A 15, 3207 (2000).
[34] V.A. Savchenko, T.P. Shestakova and G.M. Vereshkov, Grav. & Cosmol. 7, 18 (2001).
[35] V.A. Savchenko, T.P. Shestakova and G.M. Vereshkov, Grav. & Cosmol. 7, 102 (2001).
[36] T.P. Shestakova, Grav. & Cosmol. 5, 297 (1999).
[37] T.P. Shestakova, in: Proceedings of the IV International Conference “Cosmion-99”, Grav. & Cosmol. 6, Supplement, 47 (2000).
[38] T.P. Shestakova, in: Proceedings of the V International Conference on Gravitation and Astrophysics of Asian-Pacific countries, Grav. & Cosmol. 8, Supplement II, 140 (2002).
[39] T.P. Shestakova and C. Simeone, “The Problem of Time and Gauge Invariance in the Quantization of Cosmological Models. I. Canonical Quantization Methods”, to be published in Grav. & Cosmol., gr-qc/0409114.
[40] C. Simeone, J. Math. Phys. 39, 3131 (1998).
[41] C. Simeone, J. Math. Phys. 40, 4527 (1999).
[42] C. Simeone, Gen. Rel. Grav. 32, 1835 (2000).
[43] C. Simeone, Gen. Rel. Grav. 34, 1887 (2002).
[44] C. Simeone, “Deparametrization and Path Integral Quantization of Cosmological Models” (World Scientific Lecture Notes in Physics 69), World Scientific, Singapore, 2002.
[45] C. Simeone, Phys. Lett. A 310, 143 (2003).
[46] C. Teitelboim, Phys. Rev. D 25, 3159 (1982).
[47] W.G. Unruh, Phys. Rev. D 40, 1048 (1989).
[48] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989).