Conditions for zero-free half-planes of the Zeta Function

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Abstract

This paper investigates a possible method for deriving zero-free half planes of the Zeta Function. The Riemann Hypothesis (RH) is equivalent to a closure problem in the Hardy Space $H^2(\mathbb{D})$. Noor [6] proved a weak form of RH by relating it with invertibility of an operator $(I - S)$ in the weaker compact-open topology on $\mathbb{D}$. This paper explores its invertibility in other weaker topologies and derives precise conditions that would ensure existence of zero-free half planes of the Zeta Function. This principle is then explored in different settings like weighted $\ell^2$ spaces and $H^p$ spaces ($0 < p < 1$).
0 Introduction

The Riemann Hypothesis can be shown to be equivalent to a closure problem in $L^2(0,1)$. This was first stated in the 1950s by Nyman [7] and Beurling [3]:

**Theorem 0.1** In the space $L^2(0,1)$, define the functions $g_t := \rho \left( \frac{1}{tx} \right) - \frac{1}{t} \rho \left( \frac{1}{x} \right)$ for $t \geq 1$ (where $\rho(x)$ denotes the fractional part of $x$). Let $\nu := \text{span}\{g_t \mid t \geq 1\}$. The following statements are equivalent:

1. RH
2. $1_{(0,1)} \in \text{clos}_{L^2(0,1)}(\nu)$
3. $\text{clos}_{L^2(0,1)}(\nu) = L^2(0,1)$, that is, $\nu$ is dense in $L^2(0,1)$

It was later simplified by Báez-Duarte [1] in 2003, who showed it is enough to restrict $\nu$ from $\{t \geq 1\}$ to the countable set $\mathbb{N}$. I direct the reader to a paper by Bagchi [2] that summarizes it perfectly. A recent paper by Noor [6] transferred this closure problem to the Hardy Space $H^2(\mathbb{D})$ by means of an isometry. That is what we are going to focus on in this paper.

Noor [6] proved a weak form of RH by relating it with invertibility of an operator $(I - S)$ in the weaker compact-open topology on $\mathbb{D}$. However it did not correspond to any zero-free regions. The focus of this paper would be to generalize this idea to other weaker topologies and establish criteria that would guarantee zero-free regions of the Zeta Function. We shall do that in Section 2, summarized in Theorem 2.3.

By considering weighted Hardy Spaces (isometrically isomorphic to a weighted $\ell^2$ space), we use the ideas of Section 2 to derive numerical conditions on the weights that may give zero-free regions. We also take a look at $H^p(\mathbb{D})$ spaces with $0 < p < 1$. 
1 Background

Definition 1.1 The Hardy Space $H^2(\mathbb{D})$ on the unit disk $\mathbb{D} \subseteq \mathbb{C}$ is defined as the set

$$H^2(\mathbb{D}) := \{ f \in \text{Hol}(\mathbb{D}) \mid \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty \} \quad (1)$$

The radial limit $\tilde{f}(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ exists almost everywhere in $\theta$ and $\tilde{f} \in L^2(\mathbb{T})$. Furthermore, $H^2(\mathbb{D})$ is a Hilbert Space with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(e^{i\theta}) \tilde{g}(e^{i\theta}) \, d\theta$$

for any $f, g \in H^2(\mathbb{D})$.

Proposition 1.2 The set $\{ z^n \mid n \in \mathbb{N} \}$ forms an orthonormal (Schauder) basis of $H^2(\mathbb{D})$. Hence we have an reformulation of Definition 1.1 using Taylor coefficients:

$$H^2(\mathbb{D}) = \{ f = \sum_{n=0}^{\infty} a_n z^n \mid (a_n) \in \ell^2 \} \quad (2)$$

with the inner product $\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$

Theorem 1.3 (Noor, 2019 [6]) The following statements are equivalent:

1. RH
2. $1 \in \text{span}_{H^2(\mathbb{D})} \{ h_k \mid k = 2, 3, 4, \ldots \}$
3. $\text{span}_{H^2(\mathbb{D})} \{ h_k \mid k = 2, 3, 4, \ldots \} = H^2(\mathbb{D})$

where $h_k(z) := \frac{1}{k} \frac{1-1}{1-z} \log \left( \frac{1+z+\ldots+z^{k-1}}{k} \right)$ and 1 is the constant function.

Noor proved furthermore that:

Theorem 1.4 Let $S$ be the shift operator on $H^2(\mathbb{D})$ such that $S(f)(z) = z \cdot f(z)$ where $f \in H^2(\mathbb{D})$. Then:

$$\left\| \sum_{k=2}^{n} \mu(k)(I - S)h_k - (1 - z) \right\|_{H^2(\mathbb{D})} \to 0 \quad \text{as } n \to \infty \quad (3)$$

It can be shown that $(1 - z)$ is a cyclic vector with respect to a semigroup of operators acting on $H^2(\mathbb{D})$. So we have:

$$\text{span}_{H^2(\mathbb{D})} \{ (I - S)h_k \mid k = 2, 3, 4, \ldots \} = H^2(\mathbb{D}) \quad (4)$$
It is important to note that \((I - S)\) is not an invertible operator in the usual \(H^2(\mathbb{D})\) topology. This is because multiplication by \(\frac{1}{1-z}\) is not well defined as a map to \(H^2(\mathbb{D})\), since \((I - S)^{-1}(1) = \frac{1}{1-z} \notin H^2(\mathbb{D})\). If it were well defined and invertible, we could apply \((I - S)^{-1}\) to the convergence 3 and obtain \(\sum_{k=2}^{n} \mu(k) h_k \to 1\), thus proving the Riemann Hypothesis. Noor [6] proves a weak version of RH by changing to a weaker compact-open topology and using a Toeplitz operator on multiplication by \(\frac{1}{1-z}\) to make it a well defined continuous operator.

\[
\begin{align*}
\mathcal{H} & \quad \gamma_k = (\rho \left( \frac{n}{k} \right))_{n \in \mathbb{N}^*} \\
\mathcal{A} & \quad (Tg)(z) = \frac{((1-z)g(z)')'}{1-z} \\
\mathcal{M} & \quad \mathcal{H}^2(\Omega) \\
& \quad G_k(s) = -\frac{\zeta(s)}{s}(k^{-s} - k^{-1}) \\
& \quad E = \frac{1}{s} \\
\mathcal{H}^2(\mathbb{D}) & \quad h_k(z) = \frac{1}{k} \frac{1}{1-z} \log \left( \frac{1+z+...+z^{k-1}}{k} \right) \\
\mathcal{M} & \quad \mathcal{H}^2(\Omega) \\
\mathcal{M} & \quad \mathcal{H}^2(\mathbb{D})
\end{align*}
\]

**Figure 1: Diagram of all Isometries and Closure Problems**

A diagram of the different Hilbert Spaces used to describe this closure problem is provided above 1. The connection to RH can be seen clearly in the space \(H^2(\Omega)\), the Hardy Space on the half-plane \(\Re(s) > \frac{1}{2}\). If \(\zeta(s_0) = 0\), then pointwise evaluations at that point, \(G_k(s_0) = 0\). Thus their closed linear span can’t contain \(E\) since \(\frac{1}{s_0} \neq 0\). The isometries \(\mathcal{M}\) and \(U\) are defined in Bagchi’s paper [2]. The isometries \(\Psi\) and \(T\) can be found in Noor’s paper [6].

## 2 Invertibility of (I-S) in other spaces

We have now set up everything we need to investigate the Closure Problems further than what is currently known. The result by Noor [6] linking
invertibility of an operator with RH is certainly very interesting. The goal of the rest of this paper would be to explore the equation (3) in different settings, where the operator \((I - S)\) is invertible and hopefully gain new insights into the zeros of \(\zeta(s)\).

Looking at the diagram we see that there is a chain of isometries:

\[
H^2(\mathbb{D}) \xrightarrow{T \text{ bij}} A \xrightarrow{\varphi^{-1} \text{ bij}} H \xrightarrow{U^{-1} \text{ bij}} M \xrightarrow{\mathcal{M} \text{ not onto}} H^2(\Omega)
\] (5)

We denote the composition of these isometries to be \(\Lambda: H^2(\mathbb{D}) \rightarrow H^2(\Omega)\). For each \(s \in \Omega\), there is an evaluation map \(E_s: H^2(\Omega) \rightarrow \mathbb{C}\), that evaluates a function \(f \in H^2(\Omega)\) at the point \(s\). As \(H^2(\Omega)\) is a Reproducing Kernel Hilbert Space, these maps \(E_s\) are continuous linear functionals. Composing them with the isometry \(\Lambda: H^2(\mathbb{D}) \rightarrow H^2(\Omega)\), we have a continuous linear functional \(\Lambda(s)\) defined for each \(s \in \Omega\) by:

\[
\Lambda(s): H^2(\mathbb{D}) \rightarrow \mathbb{C}
\]

\[
z^k \mapsto f_k(s) = \frac{1}{s} \left( (k + 1)^{1-s} - k^{1-s} \right) \quad (k \in \mathbb{N}^*)
\]

\[
1 \mapsto f_0(s) = -\frac{1}{s}
\]

We can verify that these maps are well-defined and continuous directly from definition by a method which shall be used extensively in Section 3. We present it in a lemma:

**Lemma 2.1** The maps as defined in (5) are continuous linear functionals for \(s \in \Omega\).

**Proof.** From Riesz Representation Theorem, we know that in a Hilbert Space \(X\) any bounded linear functional \(\phi \in X^*\) can be uniquely represented in the form \(\phi(\cdot) = \langle \cdot, g \rangle\) for some corresponding \(g \in X\). Hence to show a certain \(\Lambda(s) \in H^2(\mathbb{D})^*\), we only need to find the corresponding \(g_s \in H^2(\mathbb{D})\).

Any \(g_s \in H^2(\mathbb{D})\) is completely determined by the projection on each element of the orthonormal (Schauder) basis \(z^k\). Let \(a_k := \langle g_s, z^k \rangle\) for \(k \in \mathbb{N}\). Then \(g_s = \sum_{k=0}^{\infty} a_k z^k\) and \((a_k)_{k \in \mathbb{N}} \in \ell^2\).

In our case of \(\Lambda(s)\), we know the values of the projections, \(a_k = \langle z^k, g_s \rangle = \Lambda(s)(z^k) = f_k(s)\) (from (5)). Hence \(\Lambda(s)\) is continuous if and only if \(f_k(s) \in \ell^2\).
For the fixed $s \in \mathbb{C}$, let $h(z) := z^{1-s}$. Now, for $k \in \mathbb{N}^*$, $f_k(s) = -\frac{1}{s}(h(k+1) - h(k))$. Now, $h(z)$ has a holomorphic branch on the convex domain $B(k+1/2; 1)$. Hence we can apply Fundamental Theorem of Calculus,

$$|h(k+1) - h(k)| = \left| \int_{\Gamma_k} h'(z)dz \right| \leq \text{length}(\Gamma_k)|1-s| \sup_{z \in \Gamma_k} |z^{-s}|$$  \hspace{1cm} (7)

where we take $\Gamma_k$ to be the straight line contour joining $k$ and $(k+1)$. Hence on $\Gamma_k$, $(k+1)^{1-R(s)} \leq |z^{-s}| \leq k^{-R(s)}$. Thus

$$f_k(s) = O(k^{-R(s)})$$  \hspace{1cm} (8)

So $(f_k(s))_{k \in \mathbb{N}}$ is an $\ell^2$-sequence for the values of $\Re(s) > \frac{1}{2}$. Hence we have proved our lemma.

**Remark 2.2** This estimate of $|f_k(s)|$ (through $|z^{-s}|$) we have used depends only on the real part of $s$. Hence this method of finding continuous functionals will always give us half-planes $\{\Omega_r = s \in \mathbb{C} \mid \Re(s) > r\}$ for values of $s$.

**Theorem 2.3** Suppose now we replace the space $H^2(\mathbb{D})$ with a Banach space $X$ having the following properties:

(C1) $z^k \in X$ for $k \in \mathbb{N}$ form a Schauder basis of $X$.

(C2) $H^2(\mathbb{D}) \subseteq X$, in particular, $h_k \in X$ for $k \in \mathbb{N}$.

(C3) $1 \in \text{span}\{h_k \mid k \in \mathbb{N}\}$

(C4) There is a set of continuous linear functionals $\{\Lambda_X^{(s)} \mid s \in \Omega_r\}$ uniquely determined by:

$$\Lambda_X^{(s)}: X \rightarrow \mathbb{C}$$

$$z^k \mapsto -\frac{1}{s}((k+1)^{1-s} - k^{1-s}) \hspace{1cm} (k \in \mathbb{N}^*)$$

$$1 \mapsto -\frac{1}{s}$$

where $\Omega_r = \{s \in \mathbb{C} \mid \Re(s) > r\}$ for some $r \in \mathbb{R}$.

(C5) Subspace norm induced from $X$ is weaker than that of $H^2(\mathbb{D})$ on the subspace $H^2(\mathbb{D})$.

If these conditions hold, $\Omega_r$ is a zero-free half-plane of $\zeta(s)$. 

5
**Proof.** Assuming these hold for a Banach Space \( X \), we have that \( \Lambda^{(s)}_X \) coincides with \( \Lambda^{(s)}_{H^2(\mathbb{D})} \) on polynomials. \( P_n^k(z) \) be sequence of polynomials such that \( P_n^k \to h_k \) in \( H^2(\mathbb{D}) \). Thus \( \Lambda^{(s)}_{H^2(\mathbb{D})}(h_k) = \lim_{n \to \infty} \Lambda^{(s)}_{H^2(\mathbb{D})}(P_n^k) = \lim_{n \to \infty} \Lambda^{(s)}_X(P_n^k) = \Lambda^{(s)}_X(h_k) \). The last equality is due to the norm on \( X \) being weaker than \( H^2(\mathbb{D}) \). Thus \( \Lambda^{(s)}_X(h_k) = \Lambda^{(s)}_{H^2(\mathbb{D})}(h_k) = G_k(s) \).

Recall from Diagram 1 that \( G_k(s) = -\zeta^{(s)}(s) \cdot (k - s - k^{-1}) \). Hence if \( s_0 \) is a zero of \( \zeta(s) \) in the half-plane \( \Omega_r \), we have \( \Lambda^{(s_0)}_X(h_k) = 0 \) for all \( k \in \mathbb{N} \). By (C3), \( -\frac{1}{s_0} = \Lambda^{(s_0)}_X(1) = 0 \), a contradiction. Hence \( \Omega_r \) is a zero-free half-plane of \( \zeta(s) \).

**Remark 2.4** (C3) will hold if \((I - S)^{-1}\) is a well-defined linear map from \( X \to X \) and a bounded operator on \( X \) and if Noor’s equation holds in the norm of \( X \), that is

\[
\left\| \sum_{k=2}^{n} \mu(k)(I - S)h_k - (1 - z) \right\|_X \to 0 \quad \text{as } n \to \infty \quad (9)
\]

Call the condition of \((I - S)\) being invertible the **Strong (C3)** condition.

**Remark 2.5** Since Noor’s equation holds in \( H^2(\mathbb{D}) \), it will continue to hold in any space \( X \subseteq \text{Hol}(\mathbb{D}) \) with elements defined using a weaker norm. Such a space would also satisfy (C2) since then \( H^2(\mathbb{D}) \subseteq X \).

### 3 Invertibility in Weighted Sequence Spaces

Our main aim of this section is to try different candidates for the Banach Space \( X \) and compare it with the **Checklist** above.

The definition of \( H^2(\mathbb{D}) = \{ \sum_{n=0}^{\infty} a_n z^n \mid (a_n) \in \ell^2 \} \) naturally leads to generalizations of considering different weighted \( \ell^2 \)-conditions on \((a_n)\).

#### 3.1 A few examples

**Example 3.1** (Smaller disks) Recall the definition [1.1] of \( H^2(\mathbb{D}) \). We can restrict the supremum to \( 0 < r < \epsilon \) to get a Hardy Space on the smaller disk.
\( \mathbb{D}_\epsilon := B(0; \epsilon) \) where 0 < \( \epsilon < 1 \).

\[
H^2(\mathbb{D}_\epsilon) := \{ f \in \text{Hol}(\mathbb{D}_\epsilon) \mid \sup_{0 < r < \epsilon} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty \} \tag{10}
\]

It is a quick check (similar to proof of Proposition 1.2) to see this is equivalent to the weighted \( \ell^2 \) definition:

\[
H^2(\mathbb{D}_\epsilon) = \{ \sum_{n=0}^\infty a_n z^n \mid (a_n \epsilon^n)_{n \in \mathbb{N}} \in \ell^2 \} \tag{11}
\]

Comparing with the Checklist, (C1) certainly holds as \( \frac{z}{\epsilon} \) form an orthonormal basis. From the integral definition 10, we see that \( H^2(\mathbb{D}) \subseteq H^2(\mathbb{D}_\epsilon) \). Thus (C2) holds trivially. (C3) also follows quite easily from definition 10 as:

\[
\left\| \left| 1 - z \right| f(z) \right\|_{H^2(\mathbb{D}_\epsilon)} \leq \frac{1}{1 - \epsilon} \| f(z) \|_{H^2(\mathbb{D}_\epsilon)}
\]

(C5) is also satisfied as the norm on \( H^2(\mathbb{D}) \) is stronger than that of \( H^2(\mathbb{D}_\epsilon) \). Hence Noor’s equation 3 carries over by Remark 2.5.

The problem arises in (C4). Repeating the same steps as in Lemma 2.1 we see that \( \frac{z}{\epsilon} \in \mathbb{C} \) holds as \( \frac{z}{\epsilon} \) form an orthonormal basis of \( X \). Also, (C2) holds as \( H^2(\mathbb{D}) \subseteq X \) which follows.

**Example 3.2** For \( \alpha > 0 \), consider the space of functions

\[
X = \{ \sum_{n=0}^\infty a_n z^n \mid (a_n n^{-\alpha})_{n \in \mathbb{N}^*} \in \ell^2 \} \tag{12}
\]

with the inner product \( \langle \sum_{n=0}^\infty a_n z^n, \sum_{n=0}^\infty b_n z^n \rangle = a_0 \overline{b_0} + \sum_{n=1}^\infty a_n \overline{b_n} n^{-2\alpha} \).

It is easy to see that (C1) holds as \( n^\alpha z^n \) for \( n \in \mathbb{N}^* \) and 1 form an orthonormal basis of \( X \). Also, (C2) holds as \( H^2(\mathbb{D}) \subseteq X \) which follows.
immediately from definition and $|n^{-2\alpha}| \leq 1$. (C5) also holds as the norm on $X$ is weaker than one on $H^2(D)$.

Now we look at (C4). Repeating the same steps as in Lemma 2.1 we see that $u_k = \langle z^k \alpha, g_s \rangle = k^\alpha \Lambda(s)(z^k) = k^\alpha f_k(s)$ (from (E)). Hence $\Lambda(s)$ is continuous if and only if $k^\alpha f_k(s) \in \ell^2$. Using our estimate (8) of $f_k(s) = O(k^{-\Re(s)})$, we require

$$\left( k^\alpha k^{-\Re(s)} \right)_{k \in \mathbb{N}^*} \in \ell^2 \quad \text{where } \alpha > 0$$

This holds for $\Re(s) > \frac{1}{2} + \alpha$. Thus we get a nice half-plane.

This time however **Strong (C3)** is false. $(I - S)^{-1}(1) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. With a slight abuse of notation, if we apply the norm of $X$ on it, we have:

$$\left\| \sum_{n=0}^{\infty} z^n \right\|_X = 1 + \sum_{n=1}^{\infty} n^{-2\alpha} < \infty \quad \text{for } \alpha > \frac{1}{2}$$

Hence we must necessarily work with $\alpha > \frac{1}{2}$. However, if we look at $f = \sum_{m=1}^{\infty} m^\delta z^m$ for some $\delta \in \mathbb{R}$, it is a quick check to see $f \in X$ if and only if $\delta < \alpha - \frac{1}{2}$. Now,

$$(I - S)^{-1} f = \sum_{n=0}^{\infty} z^n \cdot \sum_{m=1}^{\infty} m^\delta z^m$$

By ratio test we see each of the infinite sums have radius of convergence 1. Collecting the coefficient of $z^k$ on the right hand side, we see that $(I - S)^{-1} f = \sum_{k=0}^{\infty} c_k z^k$. Where $c_k = \sum_{l=1}^{k} l^\delta$. Now, from the proof of Integral test we can bound this sum below by:

$$c_k = \sum_{l=1}^{k} l^\delta \geq \int_0^k x^\delta \, dx = \frac{k^{\delta+1}}{\delta + 1}$$

Hence $(I - S)^{-1} f \in X$ if and only if

$$\sum_{k=1}^{\infty} k^2(\delta+1) k^{-2\alpha} < \infty$$

Hence we require $\delta < \alpha - \frac{3}{2}$. However choosing $\delta \in (\max(\alpha - \frac{3}{2}, 0), \alpha - \frac{1}{2})$ gives us a counterexample.

### 3.2 Weighted $\ell^2$ sequence spaces

We try to study this now in a general setting of weighted $\ell^2$ sequence spaces.
**Definition 3.3** For \( w_n \geq 1 \ (n \in \mathbb{N}) \), we define the following Hilbert Space:

\[
X = \left\{ \sum_{n=0}^{\infty} a_n z^n \mid (a_n/w_n)_{n \in \mathbb{N}} \in \ell^2 \right\}
\]

(13)

\( X \) is equipped with the inner product \( \langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}/w_n^2 \). This is isometrically isomorphic to a weighted \( \ell^2 \)-space.

It follows directly from definition that \( z^k w_k \) for \( k \in \mathbb{N} \) form an orthonormal basis for \( X \). So \( (C_1) \) holds. As \( w_n \geq 1 \), the norm of \( X \) would be automatically weaker than \( H^2(\mathbb{D}) \). Thus \( H^2(\mathbb{D}) \subseteq X \). So condition \( (C_2) \) holds. Furthermore, due to the weaker norm on \( X \), \( (C_5) \) holds. We check for Strong \( (C_3) \) and \( (C_4) \).

**Notation 3.4** To make things clearer, we relabel the conditions **Strong** \( (C_3) \) and \( (C_4) \) as:

(A) \( (I - S)^{-1} \) is well-defined and bounded operator on \( X \)

(B) There is a set of functionals \( \Lambda_X^{(s)} \) defined for values of \( s \) in a non-empty half-plane \( \Omega_r \)

The examples we considered were:

1. \( w_n = 1 \): Corresponds to \( H^2(\mathbb{D}) \). \( (I - S)^{-1} \) is not well-defined but we have a half-plane \( \Re(s) > 1/2 \) on which the functionals are defined.

2. \( w_n = (1/\epsilon)^n, \epsilon < 1 \): Corresponds to \( H^2(\mathbb{D}_\epsilon) \) in Example 3.1 \( (I - S)^{-1} \) is a well-defined bounded operator.

3. \( w_n = n^\alpha, \alpha \geq 0 \): Corresponds to \( X \) in Example 3.2 \( (I - S)^{-1} \) is not well-defined but we have the half-plane \( \Re(s) > 1/2 + \alpha \).

We can illustrate the ideas in a little diagram below:

If the decay rates of weights are too fast (left strip), we do not have a half-plane of functionals. If the decay rates are too slow (right strip), then we do not have \( (I - S)^{-1} \) as a bounded operator. Ideally we would want to try and find weights that lie in the central strip, where we have a half-plane of functionals as well as \( (I - S)^{-1} \) as a bounded operator.
We can try and look at different weights and classify them accordingly to each strip in Figure 2. But first, we take a closer look at what we require for Conditions (A) and (B) to hold.

**Proposition 3.5** Given a Hilbert Space $X$ as in 13, Condition (B) holds in $X$ if and only if

$$\left( \frac{w_k}{k^\Re(s)} \right)_{k \in \mathbb{N}^*} \in \ell^2$$

(14)

**Proof.** As we have seen $z^k w_k$ for $k \in \mathbb{N}$ form an orthonormal basis for $X$. Following our proof from Lemma 2.1, we have

$$\overline{a_k} = \langle z^k w_k, g_s \rangle = w_k \Lambda^{(s)}(z^k) = w_k f_k(s)$$

Hence $\Lambda^{(s)}$ is continuous if and only if $w_k f_k(s) \in \ell^2$. Using our estimate (8) of $f_k(s) = O(k^{-\Re(s)})$, we require

$$\left( \frac{w_k}{k^\Re(s)} \right)_{k \in \mathbb{N}^*} \in \ell^2$$

**Remark 3.6** Now, if $w_n = O(n^\alpha)$, it is immediately apparent from (14) that Condition (B) holds for a half-plane $\Re(s) > \alpha + 1/2$. This confirms what we have seen in Example 3.2. If $w_k = (1/\epsilon)^k$, as in Example 3.1, (14) certainly does not hold as the individual terms $\frac{(1/\epsilon)^k}{k^\Re(s)} \not\rightarrow 0$. Hence there is no half-plane obtained.

| Decay rates $w_n \to$ | Left Strip | Central Strip | Right Strip |
|-----------------------|------------|--------------|-------------|
| $(1/\epsilon)^n$     | $(I - S)^{-1}$ bounded | $(I - S)^{-1}$ bounded | $(I - S)^{-1}$ not bounded |
| No half-plane of functionals | A half-plane of functionals | A half-plane of functionals |

Figure 2: Weights and Conditions
Proposition 3.7  Given a Hilbert Space $X$ as in [13] let

$$r_m := \sum_{n=m}^{\infty} \frac{w_m^2}{w_n^2}$$

If Condition (A) holds in $X$, then $r_m$ are finite and bounded, that is, there is some $C > 0$ such that $|r_m| \leq C$ for all $m \in \mathbb{N}$.

Note that this is not a sufficient condition for (A).

Proof. Suppose the operator is well defined and bounded, with operator norm $C > 0$. For each $m \in \mathbb{N}$, consider $(I - S)^{-1}z_m$ in the equation above. Then:

$$\| (I - S)^{-1}z_m \|^2 = r_m/w_m^2 = r_m\|z_m\|^2_X$$

By definition of the operator norm of $(I - S)^{-1}$, $r_m \leq C^2$ for all $m \in \mathbb{N}$ \hfill \qed

Remark 3.8  In the case of Example 3.1, where $w_n = n$, we have

$$r_m = \sum_{n=m}^{\infty} \left(\frac{1}{\epsilon}\right)^{2(m-n)} = \frac{1}{1 - \epsilon^2}$$

which is constant for all $m$, hence bounded. So Condition (A) might hold. As we had proved earlier, it does hold in this case.

For $w_n = \log n$ in Example 3.2

$$r_m = m^{2c} \sum_{n=m}^{\infty} n^{-2c}$$

To be well defined we require $c > \frac{1}{2}$. Furthermore, using the approximation used in the Integral Test,

$$r_m \sim m^{2c} \int_{m}^{\infty} x^{-2c} \, dx = m^{2c} \cdot \frac{1}{2c-1} \cdot m^{1-2c} = \frac{1}{1-2c} m$$

The sequence $r_m$ is not bounded, hence condition (A) doesn’t hold.

We can use some other weights and try to classify them:
Table 1: Weights \( w_n \) and their classifications

| Weight \( w_n \) | \( r_m \) bounded | Condition (B) | Strip |
|------------------|-------------------|---------------|-------|
| 1                | \( \times \)     | \( \checkmark \) | Right |
| \( n^\alpha \)   | \( \times \)     | \( \checkmark \) | Right |
| \( n^\alpha + (\log n)^\beta \) | \( \times \) | \( \checkmark \) | Right |
| \( \exp((\log n)^{1+\alpha}), \alpha > 0 \) | \( \times \) | \( \times \) | -     |
| \( \exp(n^\alpha), 0 < \alpha < 1 \) | \( \times \) | \( \times \) | -     |
| \((1/\epsilon)^n\) | \( \checkmark \) | \( \times \) | Left  |
| \( \exp(n^\alpha), \alpha > 1 \) | \( \checkmark \) | \( \times \) | Left  |

3.3 Some results on possible weights

Suppose we have weights \( w_n \) such that \( r_m \) are bounded and there is \( r > 0 \) such that \( (w_n/n^r)_{n>0} \in \ell^2 \) (thus giving possible zero-free half plane \( \Re(s) > r \)). We are only interested in \( \frac{1}{2} < r < 1 \) since \( \zeta(s) \) has no zeroes for \( r = \Re(s) \geq 1 \).

**Proposition 3.9** Given \( w_n \) as above, let \( (n_i) \subseteq \mathbb{N} \) be a subsequence such that \( \sum \frac{1}{n_i} = \infty \). Then,

\[
\lim\inf_{i \to \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} = 0 \quad \text{and} \quad \lim\sup_{i \to \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} = \infty \quad (15)
\]

In particular, \( \lim_{i \to \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} \) does not exist for any such subsequence \( (n_i) \).

**Proof.** We prove by contradiction for each half of the result.

Suppose, \( \lim\inf_{i \to \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} = C > 0 \). Then for any \( C' < C \), there exists \( N \in \mathbb{N} \) such that for all \( i > N \), \( \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} > C' \). Thus,

\[
\sum_{i=N}^{\infty} \left( \frac{w_{n_i}}{n_i^r} \right)^2 > C'^2 \sum_{i=N}^{\infty} \left( \frac{n_i^{r-\frac{1}{2}}}{n_i^r} \right)^2 = C'^2 \sum_{i=N}^{\infty} \frac{1}{n_i} = \infty
\]

Hence we contradict our Condition (B), as \( (w_n/n^r)_{n>0} \in \ell^2 \).

Now suppose, \( \lim\sup_{i \to \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} < +\infty \). Then, there is \( C'' > 0 \) and \( M \in \mathbb{N} \)
such that for all $i > M$, $\frac{w_{ni}}{n_i^{r-\frac{1}{2}}} < C''$. This gives,

$$\sum_{i=N}^{\infty} \frac{1}{w_{ni}^{r-\frac{1}{2}}} > \frac{1}{C''2} \sum_{i=N}^{\infty} \frac{1}{n_i^{2r-1}} > \frac{1}{C''2} \sum_{i=N}^{\infty} \frac{1}{n_i} = \infty$$

Hence we contradict the fact that $r_0$ is bounded.

\[ \square \]

Remark 3.10 As we see in this proof, the two criteria of $r_m$ being bounded and Condition (B) act against each other. Looking at the trivial subsequence $n_i = i$, we deduce that for any monotonically increasing/decreasing weight $w_i$ these two criteria never hold together. Since in that case, the limsup and liminf would be the same. This concurs with the examples considered in table 1.

3.4 $H^p(\mathbb{D})$ spaces, $0 < p < 1$

Here we consider the space $H^p$ ($0 < p < 1$) consisting of the functions $f$ holomorphic in $\mathbb{D}$ for which

$$\|f\|_p := \sup_{0 < r < 1} \int_\mathbb{T} |f(rz)|^p dm(z)$$

is finite, where $dm(e^{i\theta}) = \frac{d\theta}{2\pi}$. They have been treated extensively in [4].

$$d_p(f, g) := \|f - g\|_p$$

defines a complete translation-invariant metric, which is a (Banach) norm when $p \geq 1$. By Fatou’s theorem, any $f \in H^p$ has radial limits almost everywhere and, using the same name $f$ for its boundary function a.e. defined,

$$\|f\|_p = \int_\mathbb{T} |f(z)|^p dm(z)$$

Proposition 3.11 (C5) condition holds. The induced norm of $H^p$ for $0 < p < 2$ is weaker than that of $H^2(\mathbb{D})$. Thus (C1), (C2) and Noor’s equation (9) hold.

Proof. As $p < 2$, $p' = \frac{2}{p} > 1$. Let $r > 1$ such that $\frac{1}{p'} + \frac{1}{r} = 1$. For $f \in H^2$ such that $\|f\|_{H^2} = 1$, by Hölder’s inequality,

$$\|f\|_{H^p}^p = \int_\mathbb{T} |f|^p \cdot 1 \, dm \leq \|f^p\|_{H^{2/p}} \cdot \|1\|_{H^r} = 1$$

13
Hence \( \| \cdot \|_{H^p} \) norm is weaker.

**Proposition 3.12** (C3) condition holds, that is, \((I - S)\) has a bounded inverse.

**Proof.** We use the Reverse Hölder’s inequality as stated in [5]: Let \(0 < r < 1\) and \(s\) satisfying \(\frac{1}{r} + \frac{1}{s} = 1\) (so that \(s < 0\)). For any nonnegative \(f \in L^r\) and \(g \in L^s\) (with respect to some given measure space) with \(\int g^s > 0\),

\[
\int fg \geq \left( \int f^r \right)^{1/r} \left( \int g^s \right)^{1/s}.
\]

Now, consider the measure \(m\) on the circle. Let \(0 < q < p\) and \(h \in H^p\). Take \(r = q/p\), \(s < 0\) the conjugated exponent to \(r\), \(f(z) = \left| \frac{h(z)}{z - 1} \right|^p\) and \(g(z) = |1 - z|^p\). Then,

\[
\int |f|^r = \int \left| \frac{h(z)}{z - 1} \right|^q dm(z),
\]

\[
\int fg = \int |h(z)|^p dm(z),
\]

\[
\int g^s = \int |1 - z|^ps dm(z),
\]

\[
\therefore \int \frac{|h(z)|^q}{|z - 1|} dm(z) \leq \left[ \frac{\int |h(z)|^p dm(z)}{\left( \int |1 - z|^ps dm(z) \right)^{1/s}} \right]^{p/q}.
\]

In order to \(\int |1 - z|^ps dm(z)\) to be finite, we need

\[
ps > -1 \iff -1 < s > p \iff 1 - 1 \frac{1}{s} > 1+p \iff q \frac{p}{q} = r < \frac{1}{1 + p} \iff q < \frac{p}{1 + p}.
\]

Such \(p\) can exist iff \(q < 1\). Also, in terms of \(p\) and \(q\) we have

\[
\frac{1}{s} = 1 - \frac{1}{r} = 1 - \frac{p}{q} = \frac{q - p}{q} \iff s = \frac{q}{q - p}.
\]

Hence we can conclude: Given \(0 < q < 1\), let \(p > 0\) satisfying \(p/(1 + p) > q\). Then, we have the inequality

\[
\left\| \frac{h}{1 - z} \right\|_{q}^q \leq \left( \|h\|_{p}^{p} \left\| \frac{1}{1 - z} \right\|_{-ps}^{p} \right)^{p/q} \forall h \in H^p, \ s = \frac{q}{q - p}. \tag{16}
\]
In particular, the operator of multiplication by $1/(1 - z)$ is well defined and continuous from $H^p$ to $H^q$. For our particular case of interest, set

$$H_n = \sum_{k=2}^{n} \frac{\mu(k)}{k} h_k - 1 \quad \text{and} \quad G_n = (I - S)H_n, \quad n \geq 2. \quad (17)$$

Since $G_n \to 0$ in $H^p$ for all $p > 0$ by Proposition 3.11, we have that $H_n \to 0$ in $H^q$ for any $0 < q < 1$.

**Proposition 3.13** (C4) holds, the continuous linear functionals $\Lambda^{(s)}_{H^p}$ exist for $\Re(s) > \frac{1}{p}$ for $0 < p \leq 1$.

**Proof.** We refer to Theorem 6.4 in [4, Pg 98]. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p$ for $0 < p \leq 1$. Then $|a_n| \leq C n^{1/p-1} \|f\|_{H^p}$ for some $C > 0$. Hence,

$$|\Lambda^{(s)}_{H^p} f| \leq \sum_{n=0}^{\infty} |a_n| |f_n(s)| \leq C \sum_{n=0}^{\infty} n^{1/p-1-\Re(s)} \|f\|_{H^p}$$

So, $\Lambda^{(s)}_{H^p}$ is a bounded functional if $\Re(s) > \frac{1}{p}$.

**Proposition 3.14** By Theorem 2.3, we have that $\zeta(s)$ has no zeroes for $\Re(s) > 1$.

Of course this follows trivially from much simpler methods. It should be noted that the proof of Noor’s equation in $H(D)$ [6, Lem 11] uses the Prime Number Theorem, which is equivalent to $\zeta(1 + it) \neq 0$ for $t \in \mathbb{R}$.

We end with a small result about the density of $\text{span}\{h_k \mid k \geq 2\}$ in $H^p$ ($0 < p < 1$):

**Proposition 3.15** $\text{span}\{h_k \mid k \geq 2\}$ is dense in $H^p$ ($0 < p < 1$).

**Proof.** Firstly, the operators given by

$$W_n f(z) = (1 + z + \cdots + z^{n-1}) f(z^n), \quad n \geq 1$$

are continuous in $H^p$. To justify this, we recall Littlewood’s subordination theorem [4, Theorem 1.7]:

$$f, \varphi \in \text{Hol}(\mathbb{D}), \ |\varphi(z)| \leq |z| \forall z \in \mathbb{D} \implies \int_{\mathbb{T}} |f \circ \varphi|^p \, dm \leq \int_{\mathbb{T}} |f|^p \, dm \forall p \in (0, \infty).$$
The composition operator $f \mapsto f \circ \varphi$, with $\varphi(z) = z^n$, is then automatically continuous in $H^p$ as well as the multiplication operator $f \mapsto \psi f$, where $\psi(z) = 1 + z + \cdots + z^{n-1}$.

The semigroup $\{W_n\}$ leaves $\text{span}\{h_k : k \geq 2\}$ invariant and has the constant $1$ as a cyclic vector. Indeed, the orbit of $1$ spans all the polynomials, which are dense in $H^p$.  \qed
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