ALMOST NON-DEGENERATE ABELIAN FIBRATION

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Abstract. We provide a criterion to recognize/characterize non-degenerate abelian fibrations.

1. Introduction

Let $S$ be a scheme, irreducible, locally noetherian, regular, of residue characteristics zero, $t$ the generic point of $S$. Let $A$ be an $S$-abelian scheme, $n$ an integer $\geq 1$, $\tilde{S}$ an $S$-scheme, finite, flat over $S$, regular, with an $S$-action by $A[n] = \text{Ker}(n.\text{id}_A)$, $(\alpha, \tilde{s}) \mapsto \alpha + \tilde{s}$, such that $\tilde{S}_t = \tilde{S} \times_S t$ is an $A_t[n]$-torsor on $t$ for the fppf topology.

Consider the contracted product

$$X = A \wedge \tilde{S},$$

namely, the quotient of $A \times_S \tilde{S}$ by the finite étale $S$-equivalence relation $\sim : (\alpha + a, \tilde{a}) \sim (\alpha, \tilde{s})$, for any local $S$-sections $\alpha, a, \tilde{s}$ of $A[n], A, \tilde{S}$, respectively.

This $S$-scheme $X$, with its $S$-action by $A$, is what we call the regular $S$-minimal model of $X_t$. We call $A$ the albanese of $X/S$, for $\text{Pic}_A^{\text{et}}/S$ is the quotient of $\text{Pic}^{\text{et}}_{\tilde{S}/S}$ by an $S$-étale closed subgroup space $E$ with $E_t = 0$. As is evident from its definition, the failure for $X$ to be an $A$-torsor is the same failure for $\tilde{S}$ to be an $A[n]$-torsor : $\tilde{S}/S$ in general ramifies/degenerates.

The regular minimal model is studied in §3, Theorem 3.7, and its interest may be justified by the following characterization:

**Theorem 1.1.** Let $Y$ be a scheme, locally noetherian, regular, of residue characteristics zero. Let $Y \to S$ be a proper flat morphism, $S$ irreducible. Suppose that one geometric fiber of $Y/S$ is a quotient of an abelian variety by a finite étale equivalence relation, and suppose that for each geometric point $\overline{s} \to S$ localized at a point of codimension 1, $Y \times_S \overline{s}$ is not uniruled at one of its maximal points.

Then $Y$ is the quotient of a regular $S$-minimal model $X$ by a finite étale $S$-equivalence relation.
In particular, $Y/S$ is projective, cohomologically flat in all dimensions, and $\text{Pic}^0_{Y/S}$ is an $S$-smooth algebraic space. The proof of (1.1) consists of two parts, cf. (2.16), (3.7) iv).

**Theorem 1.2.** Let the regular minimal model claimed in (1.1) be

$$X = A^{[n]} \wedge \tilde{S}$$

and assume that $Y$ is an open subscheme of a locally noetherian regular scheme $P$, $\text{codim}(P - Y, P) \geq 2$.

Then $\tilde{S}$ admits an open immersion into a locally noetherian regular scheme $Z$, $\text{codim}(Z - \tilde{S}, Z) \geq 2$, and $A \times_S \tilde{S}$ extends to a $Z$-abelian scheme $A_Z$ such that $P$ is the quotient by a finite étale equivalence relation of an $A_Z$-scheme $Q$, where the structural morphism $h : Q \to A_Z$ verifies the following property:

Each $u \in Q$ has an affine open neighborhood $U$ and $h(u) \in A_Z$ has a decreasing sequence of affine open neighborhoods $U_i$ such that $h$ restricts to an isomorphism $h : U \cong \lim \leftarrow U_i = \cap U_i$.

In brief, $Q$ is locally isomorphic by $h$ to a “germ” of $A_Z$. The proof of (1.2) is in (3.8).

**Corollary 1.3.** Let $V$ be a scheme. Assume that $Y \to S$ and $Y \hookrightarrow P$ in (1.2) are $V$-morphisms of $V$-schemes and that $P$ is locally of finite type over $V$. Then one can conclude that $Z$ is a $V$-scheme locally of finite type and that $h$ is a local isomorphism. If moreover $P$ is separated or $V$-separated, then $h$ is an open immersion. If $P$ is proper over $V$, then $Z$ is proper over $V$ and $h$ is an isomorphism.

The proof is immediate from (1.2), cf. (3.9). Often in applications $V$ is integral and $S \to V$ is an open immersion. Then $Z \to V$, when it is proper, is an alteration of the singularities of $V$.

When in the conclusions of (1.1)+(1.2) the finite étale equivalence relation is weakened to finite flat equivalence relation, it is of interest to know if the restriction “of residue characteristics zero” might be dropped.

Studying the birational geometry of $P$ via a fibration such as $Y/S$ is an important source of motivation for this article. The works of Kodaira, Ueno, Iitaka, Fujita, Mori, Kawamata, Viehweg, Kollar and Nakayama among others are of a constant guide and inspiration in this regard. Our approach is however different, of local nature and depends in essence on purity: Zariski-Nagata, Van der Waerden, parafactoriality, absolute purity. Related applications will be detailed in a forthcoming paper.
2. BASIC FINITE ÉTALÉ EQUIVALENCE RELATION

The notion of Grothendieck topology (SGA 3, IV, SGA 4) is indispensable in any speaking of torsors and cohomologies. We use its elementary results on ét and fppf (SGA 3, IV, 6.3.1).

Definition 2.1. Let $S$ be an algebraic space, $A$ an $S$-abelian space, $X/S$ an $A$-torsor for the fppf topology and $L$ a finite étale $S$-equivalence relation on $X$. We say that $L$ is basic at a geometric point $\overline{s} \to S$ if the following condition holds:

If $E$ is an étale $\overline{s}$-subgroup of $A_{\overline{s}}$ such that the morphism

$E \times_{\overline{s}} X_{\overline{s}} \to X_{\overline{s}} \times_{\overline{s}} X_{\overline{s}}, \ (a, x) \mapsto (a + x, x)$

factors through $L_{\overline{s}}$, then $E = 0$.

We say that $L$ is basic if it is basic at every geometric point of $S$.

Note that the quotient $Y = X/L$ for the fppf topology is an algebraic space proper smooth over $S$ with geometrically irreducible fibers.

The notion of being basic has the following simple but key justification:

Over an algebraically closed field $k$, every finite étale surjective $k$-morphism from a $k$-abelian variety $A'$ to a $k$-algebraic space $Y$, $A' \to Y$, factors in a unique way up to isomorphisms as the composite of an étale isogeny of $k$-abelian varieties $A' \to A$ and a quotient $A \to Y$ by a basic finite étale $k$-equivalence relation on $A$.

Lemma 2.2. Let $S$ be an algebraic space, $A$ an $S$-abelian space, $X$ an $A$-torsor for the fppf topology. Then with each $S$-endomorphism $q$ of $X$ there is associated a unique $S$-homomorphism $p : A \to A$ such that $q$ is $p$-equivariant.

Proof. One may assume the torsor $X$ trivial.

Each $S$-morphism $q : A \to A$ is the unique composite of a translation and an $S$-homomorphism $p : A \to A$ (“Geometric Invariant Theory”, [8], Corollary 6.4). It is clear that $q$ is $p$-equivariant: for any local $S$-sections $a, x$ of $A$, $q(a + x) = p(a) + q(x)$. \qed

Lemma 2.3. Let $X, L$ be as in (2.1), $L$ basic, $X \to X/L = Y$ the quotient. Then every $Y$-endomorphism of $X$ is a $Y$-automorphism. If $X$ admits an $S$-section $x$, the only $Y$-automorphism of $X$ preserving $x$ is the identity morphism.
Proof. Let $q$ be a $Y$-endomorphism of $X$. It is equivariant relative to a unique $S$-homomorphism $p : A \to A$ (2.2). Thus, $q$ is the composite of

$$X \xrightarrow{A} p : X \to X \xrightarrow{A} A = X/\text{Ker}(p)$$

and an $S$-isomorphism

$$X/\text{Ker}(p) \xrightarrow{\sim} X.$$

Now, $p$ is étale, since $q$ is. The immersion

$$\text{Ker}(p) \times_S X \to X \times_S X, \ (a,x) \mapsto (a+x,x)$$

factors through $L$. So $\text{Ker}(p) = 0$, $L$ being basic (2.1). Hence, $p$ and $q$ are isomorphisms.

Let $Z = \text{Ker}(q, \text{id}_X)$, an open and closed sub-$Y$-algebraic space of $X$. If $q(x) = x$, that is, if $x \in Z(S)$, then $Z = X$ and $q = \text{id}_X$, as $X$ has geometrically connected fibers over $S$. \hfill \Box

**Lemma 2.4.** For $i = 1, 2$, let $X_i, L_i$ be as in (2.1), $L_i$ basic, $Y_i = X_i/L_i$, $x_i \in X_i(S)$, $y_i \in Y_i(S)$ the image of $x_i$ and $r : Y_1 \xrightarrow{\sim} Y_2$ an $S$-isomorphism such that $r(y_1) = y_2$. Then there exists a unique $r$-isomorphism $h : X_1 \xrightarrow{\sim} X_2$ satisfying $h(x_1) = x_2$.

Proof. Identify $Y_1$ with $Y_2$ via $r$. Write $Y = Y_1 = Y_2$, $y = y_1 = y_2$. One needs to prove that the finite étale $Y$-algebraic space

$$I = \text{Isom}_Y((X_1, x_1), (X_2, x_2))$$

is isomorphic to $Y$. By (2.3) the structural morphism $I \to Y$ is also a monomorphism, whence, an open and closed immersion. To show that $I = Y$, it suffices to show that $I_\mathfrak{s} = Y_\mathfrak{s}$ for each geometric point $\mathfrak{s} \to S$. Assume thus $S = \text{Spec}(k)$, the spectrum of an algebraically closed field $k$. Choose a $Y$-scheme $X$, finite étale Galois over $Y$ and connected, such that $X$ dominates $X_1$ and $X_2$, say by $q_i : X \to X_i$, $i = 1, 2$. Then $X$ is a (trivial) torsor under a $k$-abelian variety $A$ and $q_i$ is equivariant relative to a unique étale $k$-homomorphism $p_i : A \to A_i$.

With each element $g \in \text{Gal}(X/Y) = G$, there is associated a unique automorphism $a(g)$ of the $k$-abelian variety $A$ such that $g$ is $a(g)$-equivariant. The map $g \mapsto a(g)$ is a homomorphism of groups. Its kernel $a^{-1}(1) = E$ consists of translations by elements of $A(k)$.

By definition, $\text{Ker}(p_1)$ and $\text{Ker}(p_2)$ are $k$-subgroups of $E_k$. Thus $X/E$ is dominated by $X_1$ and by $X_2$. So $E_k = \text{Ker}(p_1) = \text{Ker}(p_2)$, as $L_1, L_2$ are basic (2.1).

There exist therefore a unique $Y$-isomorphism $q : X_1 \xrightarrow{\sim} X_2$ and a unique isomorphism of $k$-abelian varieties $p : A_1 \xrightarrow{\sim} A_2$ such that $q$ is $p$-equivariant and such that $qq_1 = q_2$, $pp_1 = p_2$. Identify $X_1$ with $X_2$ by
g. Identify \( A_1 \) with \( A_2 \) by \( p \). Then \( X_1 = X_2 = X/E \) is Galois over \( Y \) of Galois group \( G/E \). There is now a unique element \( h \in G/E \) satisfying \( h(x_1) = x_2 \).

**Lemma 2.5.** Let \( S \) be an algebraic space, \( A \) an \( S \)-abelian space, \( X \) an \( A \)-torsor for the fppf topology. Then there exists a canonical isomorphism

\[
X^A \wedge \text{Pic}^0_{A/S} \simeq \text{Pic}^0_{X/S},
\]

which induces isomorphisms

\[
\text{Pic}^0_{A/S} = X^A \wedge \text{Pic}^0_{A/S} \simeq \text{Pic}^0_{X/S},
\]

\[
\text{NS}_{A/S} = X^A \wedge \text{NS}_{A/S} \simeq \text{NS}_{X/S}.
\]

**Proof.** This is [10], Chapitre XIII, Proposition 1.1. \(\square\)

**Lemma 2.6.** Let \( S \) be an algebraic space, \( S' \to S \) an étale surjective morphism, \( A' \) an \( S' \)-abelian space, \( X' \) an \( A' \)-torsor for the fppf topology, \( L' \) a basic finite étale \( S' \)-equivalence relation on \( X' \) and \( X'/L' = Y' \) be given a descent data relative to \( S' \to S \).

Then there exist descent data on \( A', X', L' \) relative to \( S' \to S \) compatible with that of \( Y' \).

**Proof.** Let \( Y' = Y \times_S S' \), the \( S \)-algebraic space \( Y \) corresponding to the descent data on \( Y' \) with respect to \( S' \to S \).

It suffices to prove the existence of a finite étale morphism \( X \to Y \) verifying \( X' = X \times_Y Y' \). For then \( X \) is an \( A \)-torsor where the \( S \)-abelian space \( A \) satisfies \( \text{Pic}^0_{A/S} = \text{Pic}^0_{X/S} \) (2.5) and \( L = X \times_Y X \) satisfies \( L' = L \times_S S' \).

Assume first that \( Y \) has an \( S \)-section, say \( y \).

Replacing if necessary \( S' \) by an \( S' \)-algebraic space étale surjective over \( S' \), one may assume that there exists an \( S'' \)-section \( x' \in X'(S') \) whose image by \( X' \to Y' \) equals \( y' = y \times_S S' \). Let \( S'' = S' \times_S S' \), \( p_1, p_2 \) the two projections of \( S'' \) onto \( S' \). By (2.4) there is a unique \( Y \times_S S'' \)-isomorphism \( h : p_1^* X' \simeq p_2^* X' \) transforming \( p_1^*(x') \) to \( p_2^*(x') \). That is, \( h \) is a gluing data on \( (X', x') \) relative to \( S' \to S \). By (2.4) again, \( h \) is a descent data. Corresponding to the descent data \( h \), one has thus obtained a finite étale \( Y \)-algebraic space \( X \) with an \( S \)-section \( x \in X(S) \), \( x \) having image \( y \) in \( Y(S) \) and \( X \times_Y Y' = X' \).

Consider now the general case.
By the second projection $Y \times_S Y$ has a $Y$-algebraic space structure which admits the diagonal $Y$-section $\Delta_{Y/S}$. One finds as above a unique finite étale $Y \times_S Y = Y_Y$-algebraic space $X_Y$ with a $Y$-section $x \in X_Y(Y)$ satisfying $X_Y \times_{Y_Y} Y_Y' = X_Y \times_{Y_Y} Y_Y'$ and such that $x$ has image $\Delta_{Y/S}$ in $Y_Y(Y)$ where $Y_Y' = Y \times_S Y'$.

Let the Stein factorization of the $Y$-algebraic space $X_Y$ be

$$X_Y \to X \to Y,$$

and $c$ the canonical morphism

$$c : X_Y \to X \times_Y Y_Y.$$

The $Y$-algebraic space $X_Y$ being proper smooth, one has that $X$ is finite étale over $Y$ (SGA 1, Exposé X, Proposition 1.2). The formation of the Stein factorization $X_Y \to X \to Y$ then commutes with every base change $T \to S$.

To finish, it suffices to show that $c$ is an isomorphism. It amounts to showing that $c$ is an isomorphism for each geometric point $\mathfrak{s} \to S$. One may thus suppose that $S$ is the spectrum of an algebraically closed field. Being $S$-étale surjective, $S'$ is a non-empty disjoint union of $S$: $S' = S \cup \cdots \cup S$. Index these components as $S_i$, $i \in \pi_0(S') = \Pi$. Write $X' = \bigsqcup_{i \in \Pi} X_i / S_i$. By (2.4) the $Y$-algebraic spaces $X_i$, $i \in \Pi$, are mutually $Y$-isomorphic and so all $Y$-isomorphic to $X$. It is plain that $c$ is an isomorphism. \hfill \Box

**Lemma 2.7.** Let $S$ be a connected algebraic space, $\mathfrak{s}$ a geometric point of $S$, $X \to S$ an $S$-algebraic space proper smooth with geometrically connected fibers such that $X \times_S \mathfrak{s}$ is an $\mathfrak{s}$-abelian variety. Then $X$ is a $\text{Pic}^0_{P/S}$-torsor on $S$ for the fppf topology where $P = \text{Pic}^0_{X/S}$ is an $S$-abelian space.

**Proof.** There exists on $p_1 : X \times_S X \to X$ a unique structure of an $X$-abelian space with zero section the diagonal section $\Delta_{X/S}$ (“Geometric Invariant Theory”, [8], Theorem 6.14). It follows that $P = \text{Pic}^0_{X/S}$, being representable by an $S$-algebraic space (Artin), is an $S$-abelian space, that $\text{Pic}^0_{X/S} = \text{Pic}^0_{X/S}$, and that the canonical representation

$$\text{Aut}_S(X) \to \text{Aut}_S(\text{Pic}^0_{P/S}), \ u \mapsto \text{Pic}^0_{P/S} \text{Pic}^0_{X/S}(u)$$

is an isomorphism, as one verifies, after the base change $X \to S$, on the $X$-abelian space $X \times_S X$. The translation action of $\text{Pic}^0_{P/S}$ on itself gives rise to an $S$-action on $X$ through

$$\text{Pic}^0_{P/S} \hookrightarrow \text{Aut}_S(\text{Pic}^0_{P/S}) = \text{Aut}_S(X),$$

which provides $X/S$ with the desired $\text{Pic}^0_{P/S}$-torsor structure. \hfill \Box
Lemma 2.8. Let \(S, X, L\) be as in (2.1), \(L\) basic, \(p : X \to X/L = Y\) the quotient. Suppose that \(S\) is locally noetherian connected and that \(Y\) admits an \(S\)-section \(y\). Then \(p\) is Galois.

Proof. As \(S\) is locally noetherian connected, there is a \(Y\)-algebraic space \(\overline{X}\) finite étale Galois over \(Y\) such that \(\overline{X}\) dominates \(X\). Let \(G = \text{Gal}(\overline{X}/Y)\), \(N = \text{Gal}(\overline{X}/X)\). Let a conjugate of \(N\) in \(G\) by \(N'\), which corresponds to \(p' : X' \to Y\), \(N' = \text{Gal}(\overline{X}/X')\).

Note that \(p, p'\) are locally isomorphic on \(S\) for the étale topology: one may for verifying the claim assume that \(S\) is strictly local with closed point \(s\). Then \(X_s/Y_s\) is Galois (cf. the proof of Lemma 2.4). Choose a \(Y_s\)-isomorphism \(h_s : p_s \simeq p'_s\). Any such \(h_s\) lifts uniquely to a \(Y\)-isomorphism \(h : p \simeq p'\).

In particular, there exists a unique \(S\)-abelian space \(A'\) acting on \(X'/S\) such that \(X'\) is an \(A'\)-torsor (2.7), that \(p'\) is the quotient of \(X'\) by a basic finite étale \(S\)-equivalence relation. As in the proof of Lemma 2.6, there exist sections \(x \in X(S), x' \in X'(S)\) satisfying \(p(x) = p'(x') = y\). By Lemma 2.4, a unique \(Y\)-isomorphism \(X \simeq X'\) transforms \(x\) to \(x'\). So \(p\) is Galois. \(\square\)

Lemma 2.9. Let \(S\) be the spectra of a discrete valuation ring, \(t\) the generic point of \(S\), \(s\) the closed point, \(\overline{t}\) (resp. \(\overline{s}\)) a geometric point localized at \(t\) (resp. \(s\)). Let \(A\) be an \(S\)-abelian scheme, \(X\) an \(A\)-torsor for the fppf topology and \(L\) a finite étale \(S\)-equivalence relation on \(X\). Then the following conditions are equivalent:

1) \(L\) is basic at \(\overline{s}\).
2) \(L\) is basic at \(\overline{t}\).

Proof. One may assume \(S\) complete, \(k(s)\) algebraically closed.

Assume 1). The finite étale morphism \(X_s \to X_s/L_s\) is Galois by (2.8). Hence \(X\) is Galois over \(X/L = Y\), as \(Y\) is proper over \(S\) and \(S\) is complete. Each element \(g\) of \(\text{Gal}(X/Y)\) is equivariant relative to a unique automorphism \(a(g)\) of the \(S\)-abelian scheme \(A\). The homomorphism \(g \mapsto a(g)_s\) is injective, as \(L_s\) is basic. Then \(g \mapsto a(g)_t\) is injective, for \(\text{Aut}_t(A_t) = \text{Aut}_S(A) \hookrightarrow \text{Aut}_s(A_s)\) is. Therefore, \(L_t\) is basic and 2) holds.

Assume 2). Note that the \(A\)-torsor \(X\) is trivial, \(S\) being strictly local (“Le groupe de Brauer III”, [7], 11.7). Identify \(X\) with \(A\) by choosing an \(S\)-section of \(X\). Replacing if necessary \(S\) by its normalization in a finite extension of \(k(t)\), suppose that \(A_t\) is Galois over \(A_t/L_t = Y_t\) (2.8). Now, as \(A_t(t) = A(S)\) and \(\text{Aut}_t(A_t) = \text{Aut}_S(A)\), each element of \(\text{Gal}(A_t/Y_t)\), \(g_t : A_t \simeq A_t\) has a unique extension to an \(S\)-isomorphism
$g : A \cong A$, which is equivariant relative to a unique automorphism $a(g)$ of the $S$-abelian scheme $A$. Consider

\[ u : G_S \times_S A \to A \times_S A, \quad (g,a) \mapsto (g(a), a). \]

The $S$-morphism $u$ factors through $L$, since $u_t$ factors through $L_t$. Write

\[ u : G_S \times_S A \to L \hookrightarrow A \times_S A. \]

Observe that $v$ is an isomorphism: both compositions

\[ r_2 : L \hookrightarrow A \times_S A \ advancement{p_2} A, \]

\[ G_S \times_S A \to L \ advancement{r_2} A, \quad (g,a) \mapsto a \]

are finite étale. So $v$ is finite étale, thus, an isomorphism, as $v_t$ is.

Identify $L$ by $v$ with the graph of the $G$-action on $A$. Since $L_t$ is basic, one has that $g \mapsto a(g)$ is injective. And $g \mapsto a(g)_s$ is injective, for $\text{Aut}_S(A) \hookrightarrow \text{Aut}_s(A_s)$ is. So 1) holds. □

**Proposition 2.10.** Let $S$ be an algebraic space, $\overline{s}$ a geometric point of $S$ localized at a point $s$, $Y$ an $S$-algebraic space proper flat of finite presentation such that $Y_{\overline{s}}$ is the quotient of an $\overline{s}$-abelian variety by a basic finite étale $\overline{s}$-equivalence relation.

Then there exist an open $U$ of $S$ containing $s$, a $U$-abelian space $A$, an $A$-torsor $X$ for the fppf topology and a basic finite étale $U$-equivalence relation $L$ on $X$ such that $Y_U = X/L$.

**Proof.** It suffices to prove the proposition over an étale neighborhood of $\overline{s}$ in $S$ (2.6). One may thus assume $S$ noetherian, strictly local with closed point $s$, $k(s)$ algebraically closed. Then $Y$ is proper smooth over $S$ with geometrically connected fibers. Write $Y_s = A_s/L_s$ as the quotient of an $s$-abelian variety by a basic finite étale $s$-equivalence relation. The finite étale $s$-morphism $A_s \to Y_s$ lifts uniquely to a finite étale $S$-morphism $A \to Y$. This $S$-algebraic space $A$, being proper smooth, lifting the $s$-abelian variety $A_s$ and over $S$ strictly local, is an $S$-abelian space with zero section an $S$-section specializing to the zero of $A_s$. The finite étale $S$-equivalence relation $L = A \times_Y A$, basic at $s$, is basic at every geometric point of $S$ (2.9). □

**Lemma 2.11.** Let $S$ be the spectra of an excellent discrete valuation ring, $t$ the generic point of $S$. Let $S' \to S$ be a surjective morphism, essentially of finite type, $S'$ local, integral, of dimension 1, with generic point $t'$, such that the extension $k(t) \to k(t')$ is separable, primary, of transcendence degree $d \geq 1$.

Then there exist an $S$-scheme $S_o$, spectra of a discrete valuation ring, $S_o$ quasi-finite, surjective over $S$, and a sequence of morphisms,
$S_d \to S_{d-1}, \ldots, S_1 \to S_o$, each of which is smooth, surjective, purely of relative dimension 1, with geometrically connected fibers, such that $S''$, the normalization of $S' \times_S S_o$, is $S_o$-isomorphic to a localization of $S_d$. In particular, $S''$ is formally smooth over $S_o$.

**Proof.** This is [3], Lemma 2.13 (Faltings).

**Lemma 2.12.** Let $S$ be the spectra of a discrete valuation ring of equal characteristic zero, $t$ (resp. $s$) the generic (resp. closed) point of $S$ and $\overline{t}$ (resp. $\overline{s}$) a geometric point localized at $t$ (resp. $s$). Let $Y$ be an $S$-algebraic space proper smooth with geometrically irreducible fibers. Then the following conditions are equivalent:

1) $Y_{\overline{t}}$ is the quotient of an $\overline{s}$-abelian variety by a basic finite étale $\overline{s}$-equivalence relation.

2) $Y_{\overline{t}}$ is the quotient of a $\overline{t}$-abelian variety by a basic finite étale $\overline{t}$-equivalence relation.

**Proof.** By (2.10), 1) implies 2). Consider the converse:

One may assume $S$ complete, $k(s)$ algebraically closed. Replacing if necessary $S$ by its normalization in a finite extension of $k(t)$, one may write $Y_t = A_t / L_t$ as the quotient of a $t$-abelian variety by a basic finite étale $t$-equivalence relation. Let the normalization of $Y$ in $A_t$ be $A$, $x_1, \ldots, x_n$ the maximal points of $A_s$. Applying the previous lemma to the morphisms $\text{Spec}(\mathcal{O}_{A, x_i}) \to S$, one finds an $S$-scheme $S'$, spectra of a discrete valuation ring, $S'$ finite surjective over $S$, such that $A'$, the normalization of $A \times_S S'$, is smooth over $S'$ at the maximal points of its closed fiber. Replacing $S$ by $S'$, $A$ by $A'$, assume that $A$ is smooth over $S$ at $x_1, \ldots, x_n$. The finite surjective morphism $A \to Y$ maps each $x_i$ to the generic point $y$ of $Y_s$ and is étale at each $x_i$:

Both morphisms $\text{Spec}(\mathcal{O}_{A, x_i}) \to S$, $\text{Spec}(\mathcal{O}_{Y, y}) \to S$ are formally smooth. Hence $\text{Spec}(\mathcal{O}_{A, x_i}) \to \text{Spec}(\mathcal{O}_{Y, y})$ is formally étale, as the extension $k(y) \to k(x_i)$ is finite separable, $k(y)$ being of characteristic zero.

Now, by the purity of Zariski-Nagata (SGA 2, Éxposé X, Théorème 3.4), $A$ is finite étale over $Y$ and hence proper smooth over $S$. It is thus an $S$-abelian scheme extending the $t$-abelian variety $A_t$. The finite étale $S$-equivalence relation $L = A \times_Y A$ is basic, for it is basic at $\overline{t}$ (2.9).

**2.13. Remark.**

The implication 2) $\Rightarrow$ 1) in (2.12), already for a $\overline{t}$-hyperelliptic surface $Y_{\overline{t}}$, fails in general if $k(s)$ is of characteristic $> 0$. 

□
Proposition 2.14. Let $S/\text{Spec}(\mathbb{Q})$ be a connected scheme and $\overline{t}$ a geometric point of $S$. Let $Y$ be a proper smooth $S$-algebraic space with geometrically irreducible $S$-fibers such that $Y_{\overline{t}}$ is the quotient of a $\overline{t}$-abelian variety by a basic finite étale $\overline{t}$-equivalence relation.

Then there exist an $S$-abelian space $A$, an $A$-torsor $X$ for the fppf topology and a basic finite étale $S$-equivalence relation $L$ on $X$ such that $Y = X/L$.

Proof. By (2.12), for every affine noetherian $\mathbb{Q}$-scheme, and thus for every $\mathbb{Q}$-scheme $S$, if $E$ consists of the points $s \in S$ such that $Y_{\overline{s}}$ is the quotient of an $\overline{s}$-abelian variety by a basic finite étale $\overline{s}$-equivalence relation, then $E$ is open and closed in $S$, where $\overline{s}$ is a geometric point localized at $s$. Now apply (2.10)+(2.6). □

2.15. Let $Y \to S$ be as in (1.1), $t$ the generic point of $S$. By (2.10) there exist a $t$-abelian variety $A_t$, an $A_t$-torsor $X_t$ and a basic finite étale $t$-equivalence relation $L_t$ on $X_t$ such that $Y_t = X_t/L_t$.

Proposition 2.16. With the notations of (2.15), $X$ is finite étale over $Y$.

Proof. By the purity of Zariski-Nagata (SGA 2, Exposé X, Théorème 3.4), we may and will assume that $S$ is the spectra of a strictly henselian discrete valuation ring of equal characteristic zero. Write $s$ for the closed point of $S$. We admit for the moment that $X_s$ is irreducible (cf. 3.3). Let $x$ (resp. $y = p(x)$) denote the generic point of $X_s$ (resp. $Y_s$).

One can reduce to the case where $Y_s$ is separable:

Let $\overline{s}$ be a geometric generic point of $Y$. The image of the specialization homomorphism (SGA 1, Exposé X, 2)

$$sp : \pi_1(Y_{\overline{s}}, \overline{s}) \to \pi_1(Y, \overline{s})$$

is a normal subgroup of finite index (cf. [9], Proposition 6.3.5). The monodromy representation

$$\pi_1(Y, \overline{s}) \to \text{Coker}(sp) = G$$

corresponds to a $Y$-scheme $Y'$, connected finite étale over $Y$, Galois of Galois group $G$. Let $S' = \text{Spec} \Gamma(Y', \mathcal{O}_{Y'})$, the spectra of a discrete valuation ring. By op.cit., the total multiplicity of $Y'/S'$ is 1, because $Y$ is regular and $k(s)$ is of characteristic zero. Replacing $Y$ by $Y'$, $X$ by $X \times_Y Y'$, one thus achieves that $Y_s$ is separable.

One can further reduce to the case where $X_t/Y_t$ is Galois of cyclic Galois group:
The smooth locus of $Y/S$, $V$ is now $S$-schematically dense in $Y$. It admits $S$-sections (EGA IV, Corollaire 17.16.3). Hence $X_t$ is Galois over $Y_t$ (2.8). Let $I$ be the kernel of the restriction homomorphism
\[
\text{Gal}(X_t/Y_t) = \text{Gal}(\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}) \rightarrow \text{Gal}(k(x)/k(y)).
\]
If $X(x)$ (resp. $Y(y)$) is the henselization of $X$ at $x$ (resp. $Y$ at $y$), the group $I$ is identical to the inertia subgroup of $\text{Gal}(X(x)/Y(y))$ and in particular cyclic, as $k(y)$ is of characteristic zero. Let $Z/Y$ correspond by Galois theory to the subgroup $I$, $Z$ the normalization of $Y$ in $Y_t$. Then $Z$ is irreducible and $Z/Y$ is étale at the generic point $z$ of $Z_s$. So $Z/Y$ is étale by purity (SGA 2, Éxposé X, Théorème 3.4). It suffices to show that $X$ is étale over $Z$. Replacing $Y$ by $Z$ and changing notations, one arranges that $k(y) = k(x)$ and that $\text{Gal}(X_t/Y_t)$ be cyclic, say of order $n \geq 1$.

Fix an $S$-section $\sigma \in V(S)$.

As $S$ is strictly local of equal characteristic zero, one may identify $\mathbb{Z}/n\mathbb{Z}$ with $\mu_n$. To finish, it suffices to show that the $\mu_n$-torsor $X_t/Y_t$ extends to a $\mu_n$-torsor $X'/Y$. For then $X'$, being étale over $Y$, equals the normalization $X$ of $Y$ in $X_t$.

Describing the $\mu_n$-torsor $X_t/Y_t$ is equivalent to describing a pair $(L_t, \alpha_t)$ consisting of an invertible $Y_t$-module $L_t$ and a rigidification $\alpha_t : \mathcal{O}_{Y_t} \rightarrow L_t^\otimes n$.

Let $L$ be an invertible $Y$-module extending $L_t$, $Y$ being regular. The $n$-th tensor power of $L$, being trivial on $Y_t$, is trivial, as $Y_s$ is integral (EGA IV, Corollaire 21.4.13). Choose a rigidification $\beta : \mathcal{O}_Y \rightarrow L^\otimes n$. To $(L, \beta)$, there corresponds a $\mu_n$-torsor $p' : X' \rightarrow Y$.

The rigidifications $\alpha_t$, $\beta_t$ differ by a scalar $\lambda \in \Gamma(Y_t, \mathcal{O}_{Y_t})^\times = k(t)^\times$: $\alpha_t = \lambda \beta_t$. The two, $p_t$, $p'_t$, become isomorphic when the base $k(t)$ is extended to $k(t)(\lambda^{1/n})$. It follows in particular the existence of a $t$-abelian variety $A'_t$ acting on $X'_t/t$ such that $X'_t$ is an $A'_t$-torsor and such that $p'_t$ is the quotient of $X'_t$ by a basic finite étale $t$-equivalence relation (2.14).

As in the proof of (2.8), there exist $t$-rational points $o \in X_t(t)$ and $o' \in X'_t(t)$ verifying $p_t(o) = p'_t(o') = \sigma(t)$. By (2.4) one finds a unique $Y_t$-isomorphism $X_t \simeq X'_t$ transforming $o$ to $o'$. That is, the $\mu_n$-torsor $X'/Y$ extends $X_t/Y_t$. This concludes the proof. \qed

3. REGULAR MINIMAL MODEL

Let $k$ be an algebraically closed field. We say that a $k$-scheme $X$ does not contain $k$-rational curves if every $k$-morphism from a $k$-rational
curve $C$ to $X$ factors as $C \to \text{Spec}(k) \to X$. Here, a $k$-rational curve is a $k$-scheme $C$, integral of finite type, such that $k(\eta)/k$ is a pure extension of transcendence degree 1, $\eta$ being the generic point of $C$.

Let $F$ be a finitely generated field over $k$. We say that $F/k$ is ruled if $F$ has a $k$-subfield $K$ such that $F/K$ is separable, primary, transcendental, that $\overline{K} \otimes_K F$ is a pure extension of $\overline{K}$, where $\overline{K}$ is an algebraic closure of $K$. We say that $F/k$ is uniruled if $F$ admits a finite extension $F'/k$ that is ruled. Let $X$ be a $k$-scheme of finite type, $\eta$ a maximal point of $X$. If $k(\eta)/k$ is ruled (resp. uniruled), we say that $X/k$ is ruled (resp. uniruled) at $\eta$.

A $k$-abelian variety does not contain $k$-rational curves, thus in particular, is not uniruled at its generic point. If a connected smooth $k$-algebraic group is not ruled at its generic point, it is a $k$-abelian variety.

Let $S$ be the spectra of a discrete valuation ring, $t$ the generic point of $S$, $A$ an $S$-abelian scheme, $X_t$ an $A_t$-torsor on $t$ for the fppf topology. Recall that $X_t$ with its $A_t$-action admits a unique extension to an $S$-scheme $X$ with an $S$-action by $A$ such that $X$ is projective flat over $S$, regular, and such that the morphism $A \times_S X \to X \times_S X$, $(a, x) \mapsto (a + x, x)$, is finite surjective. Following Néron one says that $X$ is the regular $S$-minimal model of $X_t$. Note that the geometric fibers of $X/S$ do not contain rational curves.

**Lemma 3.1.** Let $S$ be a scheme, connected, locally noetherian, regular, $p : X \to S$ a proper morphism whose geometric fibers do not contain rational curves. Then every rational section of $p$ extends uniquely to a section of $p$.

*Proof.* This is [5], Lemme 2.6. □

**Lemma 3.2.** Let $S$ be the spectra of a discrete valuation ring, $t$ the generic point of $S$, $A$ an $S$-abelian scheme, $X_t$ an $A_t$-torsor. Suppose that $X_t$ extends to an $S$-scheme $X$, proper flat over $S$, regular, with irreducible closed fiber. Then $X$ is the regular $S$-minimal model of $X_t$.

*Proof.* Write $X'$ for the regular $S$-minimal model of $X_t$, $x'$ (resp. $x$) the generic point of the closed fiber of $X'/S$ (resp. $X/S$).

The identity morphism $X_t = X'_t$ extends by Lemma 3.1 to an $S$-morphism $p : X \to X'$, which is proper birational surjective, thus maps $x$ to $x'$ and is a local isomorphism at $x$, that is, at every point of $X$ of codimension $\leq 1$. As $X'$ is regular, $p$ is an isomorphism (Van der Waerden, EGA IV, Théorème 21.12.12). □
Lemma 3.3. Let $S$ be the spectra of an excellent discrete valuation ring, $t$ the generic point of $S$, $A_t$ an abelian variety over $t$, $X_t$ an $A_t$-torsor. Suppose that $X_t$ admits an extension to a proper flat $S$-scheme $X$ whose geometric closed fiber $X_\pi$ is not uniruled at one of the maximal points of $X_\pi$.

Then $X/S$ has geometrically irreducible fibers, and there is an $S$-scheme $S'$, spectra of a discrete valuation ring, $S'$ quasi-finite surjective over $S$ with generic point $t'$, such that $A_t \times_t t' \rightarrow A'_{t'}$ extends to an $S'$-abelian scheme.

Proof. Let the maximal points of the closed fiber of $X/S$ be $x_1, \ldots, x_n$. By Lemma 2.11 applied to the morphisms $\text{Spec}(\mathcal{O}_{X,x_i}) \rightarrow S$, one finds an $S$-scheme $S'$, spectra of a discrete valuation ring, $S'$ quasi-finite surjective over $S$ with generic point $t'$, such that $X'$, the normalization of $X \times_S S'$, is smooth over $S'$ at the maximal points of its closed fiber and such that $X_t \times_t t' = X'_{t'}$ admits a $t'$-rational point, say $o'$.

The smooth locus of $X'/S'$, $V'$ is thus $S'$-schematically dense in $X'$. Identify $X'_{t'}$ with $A'_{t'}$ through $o'$. Let $A'$ be the $S'$-Néron model of $A'_{t'}$ and $a : V' \rightarrow A'$ the $S'$-morphism extending the identity $X'_{t'} = A'_{t'}$.

Note that $a$ is an $S'$-dense open immersion:

One may for verifying the claim assume $S$, thus $S'$, strictly local. Let $d = \dim(A'_{t'})$, $\omega$ a basis of $\Omega^d_{A'_{t'}/V'}$. Recall (\[ \]) that $A'$ is obtained from a weak $S'$-Néron model $W'$ (loc.cit., 3.5/1), $W'$ $\omega$-minimal (op.cit, 4.3/2) at each maximal point of its $S'$-closed fiber, by solving the universal problem of extending the $S'$-birational group law of $W'$ (op.cit, 4.3/5) to an $S'$-group law (op.cit, 5.1/5). Such a weak $S'$-Néron model $W'$ can in turn be obtained as the $S'$-smooth locus of a smoothing (op.cit, 3.1/1, 3.1/3) $X''$ of $X'$. In brief, one has a diagram of $S'$-morphisms:

\[
\begin{array}{ccc}
W' & \xrightarrow{i} & A' \\
\downarrow{j} & & \downarrow{a} \\
X'' & \xrightarrow{b} & X'
\end{array}
\]

where $i, j$ are open immersions $S'$-schematically dense and $b$ is the composition of a finite sequence of blow-ups with centers lying above the complement of $V'$ in $X'$. In particular, $j, b$ are isomorphisms over $V'$. Plainly, $a$ is an $S'$-schematically dense open immersion, being equal.
to the composition
\[ V' \simeq b^{-1}(V') \simeq j^{-1}b^{-1}(V') \hookrightarrow W' \xrightarrow{i} A'. \]

Now, by assumption, \( X/S \), hence \( V'/S', A'/S' \) as well, are not uniruled at one of the maximal points of their closed fibers. It follows that \( A' \) is an \( S' \)-abelian scheme. And \( V'/S', X'/S', X/S \) have geometrically irreducible fibers. \( \square \)

**Proposition 3.4.** Let \( S \) be the spectra of a discrete valuation ring, \( t \) the generic point of \( S \), \( s \) the closed point, \( t \) (resp. \( s \)) a geometric point localized at \( t \) (resp. \( s \)). Let \( X \to S \) be proper flat, \( X \) with geometrically factorial local rings. Suppose that \( X_\tau \) is a \( \overline{t} \)-abelian variety, that \( X_\tau \) is not uniruled at one of its maximal points and that the total multiplicity of \( X/S \) is prime to the characteristic of \( k(s) \).

Then there exists a unique \( S \)-abelian scheme \( A \) acting on \( X/S \) such that \( X_t \) is an \( A_t \)-torsor and that \( X \) is the regular \( S \)-minimal model of \( X_t \).

**Proof.** Recall (SGA 2, Éxposé XIII, Commentaires) that a scheme \( X \) is said to be geometrically factorial at a point \( x \in X \) if for any morphism \( X' \to X \), étale at a point \( x' \) with image \( x \), the local ring of \( X' \) at \( x' \) is factorial.

Consider first the case where \( S \) is excellent.

By Lemma 2.7, \( P_t = \text{Pic}^0_{X_t/t} \) is a \( t \)-abelian variety, \( X_t \) is an \( A_t = \text{Pic}^0_{P_t/t} \)-torsor on \( t \) for the fppf topology. The geometric closed fiber of \( X/S \) is irreducible (Lemma 3.3). Hence, by Lemmas 2.5+3.2, to prove the proposition, it suffices to prove that \( X \) is regular and that \( A_t \) extends to an \( S \)-abelian scheme. One may thus assume \( S \) strictly local.

Let \( \overline{t} \) be a geometric generic point of \( X \). The image of the specialization homomorphism (SGA 1, Éxposé X, 2)
\[ sp : \pi_1(X_{\overline{t}}, \overline{\eta}) \to \pi_1(X, \overline{\eta}) \]
is a normal subgroup of finite index (cf. \[9\], Proposition 6.3.5). To the monodromy representation
\[ \pi_1(X, \overline{\eta}) \to \text{Coker}(sp) = G, \]
there corresponds an \( X \)-scheme \( X' \), connected finite étale over \( X \), Galois of Galois group \( G \). Let \( S' = \text{Spec } \Gamma(X', \mathcal{O}_{X'}) \), the spectra of a discrete valuation ring. Let the generic (resp. closed) point of \( S' \) be \( t' \) (resp. \( s' \)). Denote by \( \delta \) the total multiplicity of \( X/S \), which is the length of the local ring of \( X_\tau \) at its generic point. By \textit{op.cit.}, the total
multiplicity of $X'/S'$ is 1, $k(s) = k(s')$, $G = \text{Gal}(t'/t)$ is cyclic of order \( \delta \), because $X$ has geometrically factorial local rings and $\delta$ is prime to the characteristic of $k(s)$.

The smooth locus of $X'/S'$, $V'$ is $S'$-schematically dense in $X'$. Let \( \sigma' \) be a section of $V'/S'$ (EGA IV, Corollaire 17.16.3), \( A' \) the $S'$-Néron model of $A' \times_t t'$. Through $\sigma'(t')$, one identifies $A'_\nu$ with the torsor $X'_\nu = X_1 \times_t t'$. As in the previous lemma, the identity $X'_\nu = A'_\nu$ extends to an open immersion $a : V' \hookrightarrow A'$ and $A'$ is an $S'$-abelian scheme.

Now $a$ extends to an $S'$-isomorphism $X' \simeq A'$.

Write $\pi : A'^* \to S'$ for the dual abelian scheme of $A'$, $e \in A'^*(S')$ the zero section. To $a$, there corresponds a pair $(L_{V'}, \alpha_{V'})$ consisting of an invertible module $L_{V'}$ on $A'^*_{V'}$, $L_{V'}$-fiberwise algebraically equivalent to zero, and a rigidification $\alpha_{V'} : \mathcal{O}_{V'} \xrightarrow{\sim} e_{V'}^* L_{V'}$. Let $\eta'$ be the generic point of $X'$, $L_{\eta'} = L_{V'} \times_{V'} \eta'$, $D_{\eta'}$ a divisor on $A'^*_{\eta'}$ such that $\mathcal{O}(D_{\eta'}) = L_{\eta'}$ (EGA IV, Proposition 21.3.4). Let the 1-codimensional cycle, closed image of $D_{\eta'}$ in $A'^*_{\eta'}$ be $D$, which is locally principal by the theorem of Ramanujam-Samuel (EGA IV, Théorème 21.13.4). There exists then a unique invertible $X'$-module $N$ such that $L = \pi_{X'}^* N(D)$ restricts to $L_{V'}$ on $A'^*_{V'}$ (EGA IV, Corollaire 21.4.13). As $\text{prof}_{X'_{V'}(X') \geq 2}$, a unique rigidification $\alpha : \mathcal{O}_{X'} \simeq e_{X'}^* L$ extends $\alpha_{V'}$ (EGA IV, Proposition 21.1.8). Let $(L, \alpha)$ correspond to $\overline{\pi} : X' \to \text{Pic}_{A'^*/S'}$, which extends $a$, thus factors through $A' = \text{Pic}_{A'^*/S'}^0$, and is a local isomorphism at every point of $X'$ of codimension $\leq 1$. So $\overline{\pi}$ is an isomorphism from $X'$ onto $A'$ (EGA IV, Théorème 21.12.12).

In particular, $X'$ and $X$ are regular.

Let $x'$ be the generic point of $X'_s$. The image of $x'$ in $X$ is the generic point $x$ of $X_s$. Hence, $X'_s$ is finite étale over $(X_s)_{\text{red}}$. Galois of group $G = \text{Gal}(k(x')/k(x))$. As $k(s) = k(s')$, $(X_s)_{\text{red}}$ is smooth over $s$. Also, for a prime $\ell$ invertible on $S$, in the exact sequence of $\ell$-adic cohomologies

$$0 \to H^1_{X_s}(X) \to H^1(X_s) \xrightarrow{sp^1} H^1(X_{\overline{\tau}/\ell})^{\text{Gal}(\overline{\tau}/\ell)},$$

one knows that $H^1_{X_s}(X) = 0$ (absolute cohomological purity). Namely, $sp^1$ is injective.

To conclude that $P_t$ and $A_t$ extend to $S$-abelian schemes, one still needs to show that the inertia $\text{Gal}(\overline{\tau}/t)$ acts trivially on $H^1(X_{\overline{\tau}})$.

Let a generator of $G = \text{Gal}(X'/X)$ be $\zeta : X' \to X'$, which covers the generator $\sigma = \text{Spec } \Gamma(X', \zeta) : S' \to S'$ of $G = \text{Gal}(S'/S)$. Restricted to
\(X'_t\), the automorphism \(\zeta'_t\) is compatible with the \(A'_t\)-torsor structure on \(X'_t\). That is to say, there is an element \(a'_t \in A'_t(\mathbb{t}')\) expressing \(\zeta'_t\) as

\[\zeta'_t : X'_t \to X'_t, \quad z \mapsto a'_t + \sigma^*(z),\]

where the torsor action of \(a'_t\) on \(\sigma^*(z)\) is denoted by \(a'_t + \sigma^*(z)\). As \(A'/S'\) is an abelian scheme, \(a'_t\) extends uniquely to an \(S'\)-section \(a'\) of \(A'\). It is plain that \(\zeta\) has the form

\[\zeta : X' \to X', \quad z \mapsto a' + \sigma^*(z).\]

When restricted to \(X'_s\), the action of \(\zeta\) specializes to

\[\zeta_s : z \mapsto a'(s') + \sigma^*(z) = a'(s') + z,\]

the translation action by \(a'(s') \in A'(s')\), as \(k(s') = k(s)\).

One can thus equip \((X_s)_{\text{red}}\), the quotient of \(X'_s\) by \(\zeta_s\), with an \(s\)-abelian variety structure, once fixing an \(s\)-rational point of \((X_s)_{\text{red}}\) as the origin. In particular, \(H^1(X_s) = H^1((X_s)_{\text{red}})\) and \(H^1(X_T)\) have the same rank over \(\mathbb{Q}_\ell\), since \((X_s)_{\text{red}}\) and \(X_T\) have the same dimension. It follows that \(\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})\) acts trivially on \(H^1(X_T)\) in view that \(sp^1\) is injective.

The proof is now complete in the case that \(S\) is excellent.

Consider the general case.

Write \(S = \varprojlim S_j\) as the limit of a filtered projective system \((S_j)\) with surjective transition morphisms, each \(S_j\) being the spectra of a discrete valuation ring essentially of finite type over \(\mathbb{Z}\) and in particular excellent. There exist an index \(i\), a proper flat \(S_i\)-scheme \(X_i\) and an \(S\)-isomorphism \(X \simeq X_i \times_{S_i} S\). Let \(X_j = X_i \times_{S_i} S_j\), \(t_j\) the generic point of \(S_j\), \(\forall j \geq i\). For \(j \geq i\), as \(S \to S_j\) is faithfully flat, the local rings of \(X_j\) are, as those of \(X\), geometrically factorial (EGA IV, Proposition 21.13.12). One has also the equality of total multiplicities, \(\delta(X_j/S_j) = \delta(X/S)\). Hence as \(S_j\) is excellent, there follows the existence of a unique \(S_j\)-abelian scheme \(A_j\) acting on \(X_j/S_j\) such that \(X_j \times_{S_j} t_j\) is an \(A_j \times_{S_j} t_j\)-torsor and that \(X_j\) is the regular \(S_j\)-minimal model of \(X_j \times_{S_j} t_j\). Being the limit of \(X_j\), \(X\) is then regular and is the \(S\)-minimal model of the \(A \times_S t\)-torsor \(X_t\), where \(A = \varprojlim A_j\).

\[\square\]

**Lemma 3.5.** Let \(S\) be locally noetherian regular scheme of residue characteristics zero, \(U\) an open subscheme of \(S\), \(\text{codim}(S - U, S) \geq 2\). Then the functor \(A \mapsto A|U\), from the category of \(S\)-abelian schemes to the category of \(U\)-abelian schemes, is an equivalence of categories.

**Proof.** This is [6], Corollaire 4.5.

\[\square\]
Lemma 3.6. Let \( S \) be a locally noetherian algebraic space, \( U \) an open of \( S \), \( \text{prof}_{S-U}(S) \geq 2 \), \( A \) a smooth \( S \)-group space, \( X/S \) proper flat. Then every \( U \)-action of \( A_U \) on \( X_U \) extends uniquely to an \( S \)-action of \( A \) on \( X \).

Proof. The \( S \)-group \( \text{Aut}_S(X) \) is an \( S \)-algebraic space, separated, locally of finite type (Artin). Every \( U \)-homomorphism \( A_U \to \text{Aut}_U(X_U) \) therefore extends uniquely to an \( S \)-homomorphism \( A \to \text{Aut}_S(X) \) ([4], Proposition (1), 3). □

Theorem 3.7. Let \( X \) be a scheme, locally noetherian, regular, of residue characteristics zero, \( f : X \to S \) a proper flat morphism, \( S \) irreducible with generic point \( t \), \( d = \dim(X/S) \). Suppose that the geometric generic fiber of \( f \) is an abelian variety and that for each geometric point \( s \to S \) localized at a point of codimension \( 1 \), \( X \times_S s \) is not uniruled at one of its maximal points.

i) Then \( S \) is regular, \( f \) is projective, locally of complete intersection, and there is a unique \( S \)-abelian scheme \( A \) acting on \( X/S \) such that \( X_t \) is an \( A_t \)-torsor and such that the morphism \( A \times_S X \to X \times_S X \), \((a,x) \mapsto (a+x,x)\), is finite surjective unramified.

ii) Let \( S' \to S \) be a dominant morphism, \( S' \) the spectra of a discrete valuation ring, \( t' \) the generic point of \( S' \). Then the normalization of \( X \times_S S' \), \( X' \), is the regular \( S' \)-minimal model of \( X_t \times_t t' \), and the normalization morphism \( X' \to X_{S'} \) is \( A_{S'} \)-equivariant.

iii) Let \( \omega^d_{A/S} \) be the \( S \)-module of invariant differential \( d \)-forms of \( A/S \), \( \Sigma \) the points \( s \in S \) of codimension \( 1 \) such that the total multiplicity \( \delta_s \) of \( X_s/s \) is not \( 1 \), \( x \) the generic point of \( X_s \), \( \Delta = \sum_{s \in \Sigma} (\delta_s - 1)\{x_s\} \). Then

\[
f^*\mathcal{O}_S = (f^*\omega^d_{A/S})(\Delta)[d].
\]

iv) Restricted to \( V = S - \Sigma \), \( f \) is an \( A_V \)-torsor for the fppf topology of finite order. That is, \( f^{-1}(V) \) is of the form

\[
A_V^{[n]} \setminus V,
\]

for an integer \( n \geq 1 \), \( A_V^{[n]} := \text{Ker } n \cdot \text{id}_{A_V} \), an \( A_V^{[n]} \)-torsor \( \tilde{V}/V \). For every such \( \tilde{V} \), the integer \( n \) is divisible by \( \delta_s \), \( \forall s \in \Sigma \), and the \( f^*A_V^{[n]} \)-torsor \( f^*\tilde{V} \) extends uniquely to an \( f^*A^{[n]} \)-torsor \( \tilde{X}/X \). Let the Stein factorization of the \( S \)-scheme \( \tilde{X} \) be

\[
\tilde{X} \to \tilde{S} \to S.
\]

ALMOST NON-DEGENERATE ABELIAN FIBRATION
Then \( \tilde{S} \) is regular and is the normalization of \( S \) in \( \tilde{V} \). The action of \( A_V[n] \) on \( \tilde{V}/V \) extends uniquely to an action of \( A[n] \) on \( \tilde{S}/S \) and \( \tilde{X}/\tilde{S} \) is a trivial \( A_S \)-torsor extending the \( A_V \)-torsor \( f^{-1}(V) \times_V \tilde{V}/\tilde{V} \). There exists an \( A \)-equivariant \( S \)-isomorphism

\[
A \wedge^A \tilde{S} \cong X.
\]

v) The structural sheaf of \( X \) is \( f \)-cohomologically flat in all dimensions. As graded \( S \)-algebras,

\[
\wedge^1 f_* O_X \cong R^1 f_* O_X.
\]

vi) The \( S \)-group \( \text{Pic}^0_{X/S} = \text{Pic}^0_{\tilde{X}/S} \) is an \( S \)-algebraic space smooth of finite type with proper fibers. The isomorphism \( \text{Pic}^0_{X/t} = \text{Pic}^0_{A_t/t} \) (2.5) extends uniquely to an étale epimorphism

\[
\text{Pic}^0_{X/S} \rightarrow \text{Pic}^0_{A/S}.
\]

vii) Let \( S' \) be an \( S \)-scheme. Then, with respect to the action of \( A_{S'} \) on \( X_{S'/S'} \), every invertible \( X_{S'} \)-module verifies the theorem of square.

viii) Let \( \omega^1_{A/S} \) be the \( S \)-module of invariant differential 1-forms of \( A/S \), \( L_{X/S} \) the cotangent complex of \( X/S \). Then \( L_{X/S} \cong \Omega^1_{X/S} \), and there exists a canonical epimorphism

\[
\Omega^1_{X/S} \rightarrow f^* \omega^1_{A/S}
\]

whose kernel is an \( O_{\Delta} \)-module of rank 1 at the maximal points of \( \Delta \) and of tor-dimension \( \leq 1 \) over \( O_X \).

Proof. i) Note that \( f \) is faithfully flat. Hence, \( S \) is of residue characteristics zero, locally noetherian and regular (EGA, Chapitre O, Proposition 17.16.3, (i)). Thus, from 3.4, 3.5, 3.6, it follows immediately the existence and uniqueness of an \( S \)-abelian scheme \( A \) acting on \( X/S \) such that \( X_t \) is an \( A_t \)-torsor with \( X \times_S \text{Spec}(O_{S,t}) \) as its regular minimal model at each point of \( S \) of codimension 1.

Note also that \( X \times_S X \) is integral, as \( \text{Ass}(X \times_S X) = \text{Ass}(X_t \times_t X_t) \), \( X/S \) being flat. The morphism

\[
u : A \times_S X \rightarrow X \times_S X, \ (a,x) \mapsto (a+x,x)
\]
is proper and extends the isomorphism \( u_t : A_t \times_t X_t \cong X_t \times_t X_t \). So \( u \) is surjective and also has finite fibers (cf. “Geometric Invariant Theory”, [3], Chapter O, §2, p.7, first paragraph). That is, \( u \) is finite.
immersions, \(q\) is étale. As \(A\) origin of finite étale over \(X\), the stabilizer of \(A\) is pointwise dualizing on \(p\) of characteristic zero and \(u\) is finite. Then, \(u_x\) equals the composition of the projection \(A_x \rightarrow A_x/G_x\) and the bijective closed immersion \(A_x/G_x \hookrightarrow X_x\). So \(u_x\) is unramified and \(u\) is unramified.

Let \(L_t\) be an ample invertible module on \(X_t\) (Chow, Weil). As \(X\) is regular, let \(L\) be an invertible \(X\)-module extending \(L_t\).

Every such \(L\) is \(S\)-ample:

It suffices to show that \(L_X\) is ample on \(X \times_S X\) relative to \(X\). As \(u\) is finite surjective, it is also sufficient to show that \(u^*(L_X)\) is \(X\)-ample on the abelian scheme \(A_x\). For this latter, one can apply [10], Chapitre VIII, Corollaire 7, since \(u^*(L_X)\) restricted to \(A_t \times_t X_t \simeq X_t \times_t X_t\) is \(X_t\)-ample.

The morphism \(f\) is locally of complete intersection:

Every immersion of an affine open of \(X\) into an \(S\)-affine space is regular, for \(S\) is regular, \(X\) is \(S\)-flat and regular (EGA, Chapitre O, Corollaire 17.1.9, Proposition 17.3.7).

ii) Write \(s'\) for the closed point of \(S'\) and \(X'\) the regular \(S'\)-minimal model of \(X_t \times_t t'\). The identity \(X'_t = X_t \times_t t'\) extends uniquely by Lemma 3.1 to an \(S'\)-morphism \(q : X' \rightarrow X_{S'}\), which is proper surjective. One needs to show that \(q\) is \(A_{S'}\)-equivariant and finite. Let \(U\) be an open of \(A_{S'}\) and consider the canonical open immersion \(U \hookrightarrow A_{S'}\) as a point \(a \in A_{S'}(U)\). Denote by \(T_a\) the translation by \(a\) on \(X'_{U}\) and on \((X_{S'})_U\). The equality \(qT_a = T_{aq}\) holds because it holds on \(X'_{U} \times_{V} U'\), a schematically dense open of \(X'_{U}\), \(U\) being \(S'\)-smooth. So \(q\) is \(A_{S'}\)-equivariant.

Let \(s\) be a geometric point of \(s', \ z' \in X'(s)\), \(z = q(z')\). Let the stabilizer of \(z'\) (resp. \(z\)) in \(A_{S'}\) be \(G_{S'}\) (resp. \(G_z\)). These subgroups are finite étale over \(s\). The \(A_{S'}\)-equivariant morphism \(A_{S'}/G_{S'} \rightarrow A_{S'}/G_z\) is étale. As \(A_{S'}/G_{S'} \rightarrow A_{S'}/G_z\) and \(A_{S'}/G_z \rightarrow X_{S'}\) are both bijective closed immersions, \(q_{S'}\) is finite.

iii) Write \(p_1, p_2\) for the two projections of \(X \times_S X\) on \(X\). Consider \(X \times_S X\) as an \(X\)-scheme via \(p_2\) and \(u\) as an \(X\)-morphism. Let \(u_2 : A_X \rightarrow X\) be the structural morphism so that \(u_2 = p_2u\). Then, \(u_2^*\omega^d_A = u_2^*\omega^d_{A_X/X}[d] = u_2^*f^*\omega^d_{A/S}[d]\). Since \(X\) and \(S\) are regular, \(f^*\mathcal{O}_S = \omega_f[d]\) is pointwise dualizing on \(X\), with \(\omega_f = H^{-d}f^*\mathcal{O}_S\) being an invertible \(\mathcal{O}_X\)-module. By the theorem of flat base change (Verdier), one has that \(p_1^*f^*\mathcal{O}_S = p_2^*f^*\mathcal{O}_S\), i.e., \(p_1^*\omega_f[d] = p_2^*\omega_f\).
The morphism $u$ induces a homomorphism
\[ \text{Rp}_2, p_2^* f^* \omega_{A/S}^d[d] \to \text{Ru}_2, u_2^* f^* \omega_{A/S}^d[d], \]
which when composed with the trace
\[ \text{Ru}_2, u_2^* f^* \omega_{A/S}^d[d] = \text{Ru}_2, u_2^1 \mathcal{O}_X \xrightarrow{\text{Tr}_u^2} \mathcal{O}_X, \]
gives rise by duality to a morphism
\[ c_f^1 : p_1^* f^* \omega_{A/S}^d = p_2^* f^* \omega_{A/S}^d \to p_2^1 \mathcal{O}_X[-d] = p_1^* \omega_f. \]
One obtains then
\[ c_f = p_1^1(c_f^1) : f^* \omega_{A/S}^d \to \omega_f, \]
which satisfies $p_1^1(c_f) = c_f^1$ and is invertible on $X_t$, $u_t$ being invertible. There is therefore a unique effective divisor $\Delta$ on $X$ such that $c_f$ is isomorphic to the canonical injection $f^* \omega_{A/S}^d \hookrightarrow (f^* \omega_{A/S}^d)(\Delta)$.

The formations of $c_f$ and of $\Delta$ commute with étale localizations on $S$. To calculate $\Delta$, one may suppose $S$ strictly local of dimension 1 with closed point $s$.

Define as in (3.4) $X', S', t', s'$, and write $p : X' \to X$, $f' : X' \to S'$, $\varphi : S' \to S$ for the structural morphisms. One has that $p$ is finite étale cyclic of degree equal to the total multiplicity $\delta = \delta_s$ of $X_s/k(s)$, that $X'/S'$ is a (trivial) $A' = A \times_S S'$-torsor and that $S'/S$ is tamely and totally ramified along $s'$ of degree $\delta$. One then verifies by a diagram chasing that $p^*(c_f) : p^* f^* \omega_{A/S}^d \to p^* \omega_f$ factors as the composition of
\[ c_f' : f^* \omega_{A/S'}^d \xrightarrow{\sim} \omega_f = f^1 \mathcal{O}_S[-d] \]
and
\[ f''(\text{Tr}_\varphi)[d] : f'' \mathcal{O}_S[-d] \to f'' \varphi^* \mathcal{O}_S[-d] = p^* \omega_f, \]
where
\[ \text{Tr}_\varphi : \mathcal{O}_S \to \varphi^* \mathcal{O}_S \]
corresponds to the trace
\[ \text{Tr}_\varphi : \varphi_* \mathcal{O}_{S'} \to \mathcal{O}_S, \]
by duality. Now, $\varphi^* \mathcal{O}_S = \mathcal{D}_{\varphi}^{-1}$, the inverse different, and $\text{ord}_{S'}(\mathcal{D}_{\varphi}) = \delta - 1$, $S'/S$ being tame. Also $\text{Tr}_\varphi^1$ is isomorphic to the inclusion of $\mathcal{O}_{S'}$ in $\mathcal{O}_S((\delta - 1)s')$. So $p^*(c_f)$ is isomorphic to
\[ p^* f^* \omega_{A/S}^d \hookrightarrow (p^* f^* \omega_{A/S}^d)((\delta - 1)X'_s), \]
from which it follows that $p^* \Delta = (\delta - 1)X'_s$ and $\Delta = (\delta - 1)(X_s)_{\text{red}}$.

iv) The set $\Sigma$ is locally finite in $S$. Thus, $\overline{\Sigma} = \bigcup_{s \in \Sigma} \{s\}$. By definition, the morphism $f^{-1}(V) \to V$ has separable fiber at every point of
$V$ of codimension 1. Consequently, $f^{-1}(V) \times_V f^{-1}(V)$ is normal (cf. [10], Chapitre IV, Lemme 2.4), and

$$u_V : A_V \times_V f^{-1}(V) \to f^{-1}(V) \times_V f^{-1}(V)$$

is finite birational, i.e., an isomorphism. So $f^{-1}(V)/V$ is an $A_V$-torsor and, being projective over $V$, it is of finite order (op.cit, Chapitre XIII, Proposition 2.3, ii)).

Let $n$ be an integer $\geq 1$, $\tilde{V}/V$ an $A_V[n]$-torsor such that its image by the inclusion $A_V[n] \hookrightarrow A_V$ is isomorphic to the $A_V$-torsor $f^{-1}(V)/V$.

We show that the normalization of $X$ in $f^*\tilde{V}$, $\tilde{X}$ is étale over $X$, that it admits an action by $f^*A[n]$ and is an $f^*A[n]$-torsor extending the $f^*A_V[n]$-torsor $f^*\tilde{V}$ on $f^{-1}(V)$ and that $\delta_s$ divides $n, \forall s \in \Sigma$.

Since $X$ is regular, by Lemma 3.6 and the purity of Zariski-Nagata (SGA 2, Éxposée X, Théorème 3.4), one may assume that $S$ is the spectra of a discrete valuation ring and moreover, by descent, is strictly local. Let $s$ (resp. $t$) be the closed (resp. generic) point of $S$, $\delta := \delta_s$. By the period-index theorem, the index $\delta$ equals the order of the $A_t$-torsor $X_t$. Hence, $\delta$ divides $n$.

The exact sequence

$$A(S) \xrightarrow{nA} A(S) \to H^1(S, A[n]) = 0$$

says that $A(S) = A_t(t)$ is $n$-divisible. The inclusion $A_t[n] \hookrightarrow A_t$ therefore induces an injection of $H^1(t, A_t[n])$ in $H^1(t, A_t)$. In particular,

$$\delta \cdot \text{cl}(\tilde{V}_t) = \delta \cdot \text{cl}(X_t) = 0,$$

where $\text{cl}(\tilde{V}_t) \in H^1(t, A_t[n])$ (resp. $\text{cl}(X_t) \in H^1(t, A_t)$) denotes the torsor class of $\tilde{V}_t$ (resp. $X_t$).

Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$. By absolute cohomological purity, $H^2_{(s)}(S, \Lambda(1))$ (resp. $H^2_{X_s}(X, \Lambda(1))$) is a free $\Lambda$-module of rank 1 with the cycle class $\text{cl}(s)$ (resp. $\text{cl}((X_s)_{\text{red}})$) as a generator.

Now,

$$H^2_{(s)}(S, A[n]) = H^2_{(s)}(S, \Lambda(1)) \otimes_{\Lambda} H^0(S, A[n](-1)),$$

$$H^2_{X_s}(X, f^*A[n]) = H^2_{X_s}(X, \Lambda(1)) \otimes_{\Lambda} H^0(X, f^*A[n](-1)),$$

and

$$f^* : H^2_{(s)}(S, A[n]) \to H^2_{X_s}(X, f^*A[n])$$

reads as

$$f^* : \text{cl}(s) \otimes a \mapsto \text{cl}(X_s) \otimes f^*(a), \forall a \in H^0(S, A[n](-1)).$$
If \( \text{cl}(\widetilde{V}_t) \) is mapped to \( \text{cl}(s) \otimes a \) by \( H^1(t, A_t[n]) \rightarrow H^2_{\{s\}}(S, A[n]) \), then \( \delta.a = 0 \), and \( \text{cl}(X_s) \otimes f^*(a) = \text{cl}((X_s)_{\text{red}}) \otimes f^*(\delta.a) = 0 \).

It follows that \( \text{cl}(f^*\widetilde{V}_t) = f^*\text{cl}(\widetilde{V}_t) \) is contained in the kernel of

\[
H^1(X_t, f^*A_t[n]) \rightarrow H^2_{X_s}(X, f^*A[n]),
\]

since the composition

\[
H^1(t, A_t[n]) \xrightarrow{\text{cl}} H^1(X_t, f^*A_t[n]) \rightarrow H^2_{X_s}(X, f^*A[n])
\]

equals the composition

\[
H^1(t, A_t[n]) \xrightarrow{\text{cl}} H^2_{\{s\}}(S, A[n]) \xrightarrow{\text{cl}} H^2_{X_s}(X, f^*A[n]).
\]

From \( H^1_{X_s}(X, f^*A[n]) = 0 \) (absolute cohomological purity) and the local cohomology sequence

\[
H^1_{X_s}(X, f^*A[n]) \rightarrow H^1(X, f^*A[n]) \rightarrow H^1(X_t, f^*A_t[n]) \rightarrow H^2_{X_s}(X, f^*A[n])
\]

one concludes that \( \text{cl}(f^*\widetilde{V}_t) \) lies in the image of the injection

\[
H^1(X, f^*A[n]) \hookrightarrow H^1(X_t, f^*A_t[n]).
\]

In other words, the \( f^*A_t[n] \)-torsor \( f^*\widetilde{V}_t \) admits up to isomorphisms a unique extension to an \( f^*A[n] \)-torsor \( \widetilde{X}/X \), which, being finite étale over \( X \), is the normalization of \( X \) in \( f^*\widetilde{V}_t \).

Let \( \widetilde{X} \rightarrow \widetilde{S} \rightarrow S \) be the Stein factorization of the \( S \)-scheme \( \widetilde{X} \).

The \( S \)-scheme \( \widetilde{S} \) contains \( \widetilde{V} \) as an open sub-scheme : for, the open sub-scheme of \( \widetilde{X} \), \( f^{-1}(V) \times V \widetilde{V} \), as a \( V \)-scheme, has Stein factorization \( f^{-1}(V) \times V \widetilde{V} \rightarrow \widetilde{V} \rightarrow V \).

Therefore, \( \widetilde{S} \), being finite over \( S \) and normal, is the normalization of \( S \) in \( \widetilde{V} \). The action of \( A_V[n] \) on \( \widetilde{V}/V \) extends uniquely to an action of \( A[n] \) on \( \widetilde{S}/S \), for the formation of the Stein factorization \( \widetilde{X} \rightarrow \widetilde{S} \rightarrow S \) commutes with the étale base change \( A[n] \rightarrow S \). Write this action as

\[
A[n] \times \widetilde{S} \rightarrow \widetilde{S}, \ (\alpha, x) \mapsto \alpha + x.
\]

Note that the \( A_{\widetilde{V}} \)-torsor \( f^{-1}(V) \times V \widetilde{V} / \widetilde{V} \) is trivial : by the choice of \( \widetilde{V} \), there exists a canonical \( V \)-morphism

\[
\widetilde{V} \rightarrow \widetilde{V} \overset{A_V[n]}{\rightleftharpoons} A_V = f^{-1}(V),
\]

which provides \( f^{-1}(V) \times V \widetilde{V} \) with a canonical \( \widetilde{V} \)-section \( \sigma_{\widetilde{V}} \).

The inverse of the \( A_V[n] \)-equivariant \( \widetilde{V} \)-isomorphism

\[
A_{\widetilde{V}} \overset{\sim}{\rightarrow} f^{-1}(V) \times V \widetilde{V}, \ a \mapsto -a + \sigma_{\widetilde{V}}
\]
admits by Lemma 3.1 a unique extension to an $\tilde{S}$-morphism
\[ h : \tilde{X} \to A \times_S \tilde{S} = A_{\tilde{S}}, \]
which is $A[n]$-equivariant. Above each point $s$ of codimension 1 in $S$, the birational map $h^{-1}$ is a morphism (3.1), and hence, $h \times_S \text{Spec}(\mathcal{O}_{S,s})$ is an isomorphism. In particular, $h^* : Z^1(A_{\tilde{S}}) \to Z^1(\tilde{X})$ (EGA IV, 21.10.1) is defined and is a bijection.

We prove that $h$ is an isomorphism:

We may assume that $S$ is strictly local. Then, $A[n]$ is constant of value $G = A[n](S)$, $\tilde{S}$ is affine normal, and the abelian scheme $A_{\tilde{S}}$ admits ample invertible modules ([10], Chapitre XI, Théorème 1.4). So $h$ is isomorphic to the blow up of a coherent ideal $I$ of $A_{\tilde{S}}$, which can be chosen to be invariant by $G$ (cf. the proof of EGA III, Proposition 2.3.5). The $\mathcal{O}_{\tilde{X}}$-ideal $I = h^*(I)\mathcal{O}_{\tilde{X}}$ is invertible (EGA II, Proposition 8.1.11). Write the divisor $\tilde{Z} = \text{Div}(\tilde{I})$ as a sum $\tilde{Z}_0 + \tilde{Z}_1$ in such a way that every maximal point of $\text{Supp}(\tilde{Z}_0)$ (resp. $\text{Supp}(\tilde{Z}_1)$), if $\text{Supp}(\tilde{Z}_0)$ (resp. $\text{Supp}(\tilde{Z}_1)$) is non-empty, has image $t$ (resp. is of codimension 1) in $S$. There exist unique 1-codimensional cycles $Z, Z_0, Z_1$ on $A_{\tilde{S}}$ satisfying $h^*(Z) = \text{cyc}(\tilde{Z}), h^*(Z_i) = \text{cyc}(\tilde{Z}_i), i = 0, 1$. Due to the theorem of Ramanujam-Samuel (EGA IV, Théorème 21.13.4), $Z_0$ is locally principal. The cycle $Z_1$ is of the form $\pi^*(\tilde{D}_1)$ for a unique 1-codimensional cycle $\tilde{D}_1$ on $\tilde{S}$, where $\pi : A_{\tilde{S}} \to \tilde{S}$ is the projection. Let $\psi$ denote the structural morphism $\tilde{S} \to S$. Note that both $Z_1, \tilde{D}_1$ are $G$-invariant. Therefore, for a unique 1-codimensional cycle with rational coefficients $D_1$ on $S$, $\tilde{D}_1 = \psi^*(D_1)$. Choose an integer $N \geq 1$ such that $ND_1 \in Z^1(S) \simeq \text{Div}(S)$. Then $NZ_1$ is locally principal. It is now evident that $h$ is an isomorphism, since blowing up $I$ amounts to blowing up $I^N(NZ_0 + NZ_1) = J$ and $h^*(J)\mathcal{O}_{\tilde{X}} = J\mathcal{O}_{\tilde{X}}$.

Since $\tilde{X} \simeq A_{\tilde{S}}$ is faithfully flat over $\tilde{S}$ and regular (EGA, Chapitre O, Proposition 17.16.3, (i)), one obtains that $\tilde{S}$ is regular.

Next, observe that $h : \tilde{X} \simeq A \times_S \tilde{S}$, as an $S$-isomorphism, transports the action of $A[n]$ on $\tilde{X}/X$ to the diagonal action of $A[n]$ on $A \times_S \tilde{S}$:
\[ A[n] \times_S A \times_S \tilde{S} \to A \times_S \tilde{S}, \quad (\alpha, a, x) \mapsto (\alpha + a, \alpha + x). \]

In particular, $h$ induces an $A$-equivariant $S$-isomorphism
\[ h/A[n] : X \simeq A_{\tilde{A}[n]} \tilde{S}. \]

v) The structural sheaf of $X$ is $S$-cohomologically flat:
Let an integer $n \geq 1$ and let the identification 
\[ X = A^{A[n]} \sim S \]
be as in iv). One may assume $S$ strictly local. The finite étale $S$-group $A[n]$ is then constant of value $A[n](S) = G$. The quotient $\tilde{S}/G$ equals $S$, and $k(s) = \Gamma(\tilde{S}_s)^G$, $\forall \ s \in S$, as $S$ is of residue characteristics zero ("Base change for rings of invariants"). Write $H^i(Z)$ for $H^i(\mathcal{O}_Z)$, for an integer $i$, any scheme $Z$. Now, $\forall \ s \in S$, there exist $k(s)$-algebra isomorphisms 
\[ H^i(X_s) = H^i(A \times_S \tilde{S}_s)^G = H^i(A) \otimes_S H^0(\tilde{S}_s)^G = H^i(A) \otimes_S k(s), \]
because $\mathcal{O}_A$ is $S$-cohomologically flat in every dimension and because the translation action of $G = A[n](S)$ on $A/S$ induces the trivial action on $H^*(A) = \wedge^*H^1(A)$. The function $s \mapsto \dim_{k(s)}H^i(X_s)$ is therefore constant on $S$ and $H^*(X_s) = \wedge^*H^1(X_s)$.

vi) As $f$ is cohomologically flat in dimension zero, $\text{Pic}_{X/S}$ is an $S$-algebraic space (Artin). Observe that $\text{Pic}_{X/S}$ is $S$-smooth along its zero section:

For an integer $i$, an $S$-scheme $U$, $U$ affine, a quasi-coherent $U$-algebra $a$ with nilpotent local sections, note that 
\[ \log : H^i(X_U, \hat{\mathbb{G}}_m(f^*a)) \xrightarrow{\sim} H^i(X_U, \hat{\mathbb{G}}_a(f^*a)), \]
and, as $\mathcal{O}_X$ is $f$-cohomologically flat in dimension $i$, that 
\[ H^i(X_U, \hat{\mathbb{G}}_a(f^*a)) = H^i(X_U, f^*a) \xrightarrow{\sim} \Gamma(U, a \otimes_S R^if_*\mathcal{O}_X). \]
Namely, $R^if_*\hat{\mathbb{G}}_m$ is an $S$-formal Lie group with Lie algebra $R^if_*\mathcal{O}_X$. In particular, $\text{Pic}^\tau_{X/S}$ is smooth over $S$ along its zero section. So $\text{Pic}^\tau_{X/S}$ exists as an $S$-algebraic space (SGA 3, VI B, 3.10), smooth over $S$ and open in $\text{Pic}_{X/S}$.

The $S$-group $\text{Pic}^\tau_{X/S}$ is of finite type, and the $S$-monomorphism $\text{Pic}^\tau_{X/S} \hookrightarrow \text{Pic}_{X/S}$ is an open immersion (SGA 6, Éxposé XIII, Théorème 4.7).

The open immersion $\text{Pic}^\circ_{X/S} \hookrightarrow \text{Pic}^\tau_{X/S}$ is an isomorphism:

One may assume $S$ strictly local with closed point $s$. It is sufficient to show that $\text{Pic}^\tau_{X/s}(s) = \text{Pic}^\circ_{X/s}(s)$. Let $z$ be an $s$-rational point of $X_s$, the stabilizer of $z$ in $A_s$ be $G_z$, $j$ the bijective closed immersion $A'_s := A_s/G_z \hookrightarrow X_s$. The cokernel (resp. kernel) of 
\[ j^* : \text{Pic}^\tau_{X_s/s} \rightarrow \text{Pic}^\tau_{A'_s/s} \]
is a unipotent (resp. connected unipotent) $s$-group (SGA 6, Éxposé XII, Proposition 3.1). As $\text{Pic}_A^{\tau}$ is a unipotent (resp. connected unipotent) $s$-group (SGA 6, Éxposés XII, Proposition 3.1). As $\text{Pic}_A^{\tau}$ is an $s$-abelian variety, $j^*$ is epimorphic, thus smooth, and $\text{Pic}_{X_s}^{\tau}$ is connected. But $j^*$ is even étale, i.e. an isomorphism, since $\text{Pic}_{X_s}^{\tau}$ and $\text{Pic}_A^{\tau}$ are both of dimension $d$.

The canonical isomorphism $\text{Pic}_{X_t}^{\tau} \simeq \text{Pic}_A^{\tau}$ (2.5) admits a unique extension to an $S$-homomorphism $\text{Pic}_{X/S}^{\tau} \rightarrow \text{Pic}_A^{\tau}$:

To prove the existence of such a unique extension, one may assume that $S$ is the spectra of a discrete valuation ring ([4], Proposition (1), 3)). Let the closed image of the zero section in $P^r = \text{Pic}_{X/S}^{\tau}$ be $E$, which is an $S$-subgroup space, étale over $S$ ([9], Proposition 3.3.5). The quotient $Q^r = P^r/E$, for the fppf topology, is an $S$-group scheme, separated, locally of finite type, loc.cit. Thus $Q^r$, being smooth separated over $S$ with fibers proper and connected, is an $S$-abelian scheme ([10], Chapitre VIII, Corollaire 7). The identity $P^r_t = Q^r_t = \text{Pic}_A^{\tau}$ can now be extended uniquely to an $S$-isomorphism of abelian schemes $Q^r = \text{Pic}_A^{\tau}$ and an $S$-homomorphism $P^r = Q^r = \text{Pic}_A^{\tau}$.

One concludes that $\text{Pic}_{X/S}^{\tau} \rightarrow \text{Pic}_A^{\tau}$ is an étale epimorphism :

Call this homomorphism $r$. Its base change by $X \rightarrow S$, $r \times_S X$, is isomorphic to $\text{Pic}^o(u)$ where $u : A \times_S X \rightarrow X \times_S X$, $(a, x) \mapsto (a + x, x)$, since $r_t \times_t X_t = \text{Pic}^o(u_t)$. It amounts to proving that $\text{Pic}^o(u)$ is étale surjective. It suffices to observe that $\forall x \in X$, if $G_x$ denotes the stabilizer of $\Delta_{X/S}(x)$ in $A_x$, then the fiber of $u$ at $x$,

$$u_x : A_x \rightarrow A_x/G_x \hookrightarrow X_x$$

gives rise to a $k(x)$-isomorphism

$$\text{Pic}_{X_x/x}^o \sim \text{Pic}_{A'_x/x}^o,$$

where $A'_x := A_x/G_x$, and also to an étale isogeny

$$\text{Pic}_{A'_x/x}^o \rightarrow \text{Pic}_{A_x/x}^o.$$

vii) Let $S'$ be an $S$-scheme, $A' = A_{S'}$, $X' = X_{S'}$. Relative to the action of $A'$ on $X'/S'$, every invertible $X'$-module verifies the theorem of square :

Every invertible module on $A' \times_{S'} A' \times_{S'} X'$ satisfies the theorem of cube ([10], Chapitre IV, Théorème 2.1), since $X/S$ has geometrically irreducible fibers, $u$ being surjective. Then, in view that $f$ is cohomologically flat in dimension zero, one only needs to apply loc.cit, Chapitre IV, Proposition 1.4. (Recall, op.cit, Chapitre IV, Définition 3.2, that the theorem of square, for an invertible $X'$-module $L$, with
respect to the action of $A'$ on $X'/S'$, claims that for any $S'$-scheme $U$, the map

$$A(U) \to \text{Pic}_{X/S}(U), \; a \mapsto T_a^*L_U \otimes L_{U}^{-1},$$

is a homomorphism of groups, where $T_a : X_U \to X_U, \; x \mapsto a + x.$

viii) Let $e_X$ be the zero section of the abelian scheme $A_X/X$. The equality $\Delta_{X/S} = u e_X$ gives rise to the transitivity triangle of cotangent complexes

$$L e_X^*L_u \to L\Delta_{X/S} \to L e_X.$$

Its $H^{-1}$-sequence

$$0 \to H^{-1} L e_X^*L_u \to \Omega^1_{X/S} \to f^*\omega^1_{A/S} \to 0$$

is exact, since $e_X$ is a regular immersion and $u$ is unramified. As $f$ is locally of complete intersection, $L_f \in \text{Ob} \ D^{[-1,0]}_{\text{perf}}(X), \; \text{Ass}(H^{-1} L_f) \subset \text{Ass}(X) = \text{Ass}(X_t)$. So $H^{-1} L_f = 0$, as $X_t$ is smooth over $k(t)$. And $\Omega^1_{X/S} = L_f$, as well as $H^{-1} L e_X^*L_u$, is of tor-dimension $\leq 1$. Write $p : \tilde{X} = A \times_S \tilde{S} \to X$ as in iv). The above sequence pulled back by $p$ becomes

$$0 \to \Omega^1_{\tilde{S}/S} \otimes_Z \tilde{X} \to \Omega^1_{\tilde{X}/S} \to \Omega^1_{X/\tilde{S}} \to 0$$

and the remaining assertions of viii) immediately follow. \qed

3.8. Proof of Theorem 1.2.

Note that $P$ is of residue characteristics zero. Let the regular $S$-minimal model claimed in (1.1) be $X$, $A$ its Albanese and

$$X = A \wedge A[n] \sim \tilde{S}$$

as in (3.7). The finite étale morphism $A_{\tilde{S}}/A_{\tilde{S}} = A \times_S \tilde{S} \to X \to Y$ extends to a finite étale morphism $Q \to P$ by purity (Zariski-Nagata, SGA 2, Exposé X, Théorème 3.4). One has codim$(Q - A_{\tilde{S}}, Q) \geq 2$ as codim$(P - Y, P) \geq 2$. Denote by $Z$ the closed image in $\tilde{Q}$ of the zero section of $A_{\tilde{S}}/\tilde{S}$. One has thus an open immersion $\tilde{S} \hookrightarrow Z$ and a closed immersion $\zeta : Z \hookrightarrow Q$.

We proceed in several steps.

i) A $Z$-abelian scheme $A_Z$ extends $A_{\tilde{S}}$:

The abelian scheme $A_{\tilde{S}}/\tilde{S}$ base changed by $A_{\tilde{S}} \to \tilde{S}, \; A_{\tilde{S}} \times_{\tilde{S}} A_{\tilde{S}}$, has a unique extension to a $Q$-abelian scheme $A_Q$ (3.5), for $Q$ is regular of residue characteristics zero and codim$(Q - A_{\tilde{S}}, Q) \geq 2$. The $Z$-abelian scheme $A_Q \times_Q Z = A_Z$ induces $A_{\tilde{S}}$ on $\tilde{S}$.
ii) A unique morphism $h : Q \to A_Z$ extends the identity morphism of $A_{\tilde{S}}$, and $\pi h \zeta = \text{id}_Z$, where $\pi : A_Z \to Z$ denotes the projection. In particular, $q = \pi h : Q \to Z$ is surjective.

The existence and uniqueness of $h$ follow from (3.1), since $Q$ is regular and the geometric fibers of the abelian scheme $A_Z/Z$ do not contain rational curves. The identity $\pi h \zeta = \text{id}_Z$ holds, for it holds on $\tilde{S}$.

iii) The morphism $h$ is a local isomorphism at every point of $Q$ of codimension $\leq 1$:

For such a point lies in $A_{\tilde{S}}$.

iv) One has $\text{prof}_{C}(Z) \geq 2$, where $C = Z - \tilde{S}$. In particular, $\text{codim}(C, Z) \geq 2$, $\text{codim}(A_Z - A_{\tilde{S}}, A_Z) \geq 2$:

The scheme $Z$ being integral, $\text{prof}_{C}(Z) \geq 1$. Take an open $Z'$ of $Z$ and let $\ell \in \Gamma(S', \mathcal{O}_{Z'})$, where $S' = \tilde{S} \times_Z Z'$. Let $A' = A_{\tilde{S}} \times_Z Z'$, $Q' = Q \times_Z Z'$, $\zeta' = \zeta \times_Z Z'$. The pull back of $\ell$ to $A'$ has a unique extension to a section $\ell_{Q'} \in \Gamma(Q', \mathcal{O}_{Q'})$, for $Q'$ is regular and $Q' - A'$ is of codimension $\geq 2$ in $Q'$. The section $\zeta'^{*} \ell_{Q'} \in \Gamma(Z', \mathcal{O}_{Z'})$ extends $\ell$. So $\text{prof}_{C}(Z) \geq 2$.

v) The scheme $Z$ is geometrically parafactorial (EGA IV, Définition 21.13.1) along $C = Z - \tilde{S}$:

Let $Z'$ be an étale $Z$-scheme, $L$ an invertible $S'$-module, where $S' = \tilde{S} \times_Z Z'$. Write $A' = A_{\tilde{S}} \times_Z Z'$, $Q' = Q \times_Z Z'$, $\zeta' = \zeta \times_Z Z'$ as in iv). The module $L$ pulled back to $A'$ admits a unique extension to an invertible $Q'$-module $L_{Q'}$, for $Q'$ is regular and $\text{codim}(Q' - A', Q') \geq 2$. The invertible $Z'$-module $\zeta'^{*} L_{Q'}$ extends $L$. Such an extension of $L$ to $Z'$ is unique up to unique isomorphisms because $\text{prof}_{C}(Z) \geq 2$ iv).

Namely, $Z$ is geometrically parafactorial along $C$.

vi) The scheme $Z$ and the abelian scheme $A_Z$ are geometrically factorial at all points:

That $Z$ is geometrically factorial at every point follows from v) (EGA IV, Corollaire 21.13.11), since $\tilde{S}$ is regular and in particular geometrically factorial at every point. Being $\tilde{Z}$-smooth, the abelian scheme $A_Z$ is also geometrically factorial at every point ([2], Chapitre III, Proposition 2.14).

vii) We conclude the proof of (1.2) as below:

We may and do suppose $Z$ affine.

Let $U$ be an affine open subscheme of $Q$. Write $U = \lim_{\leftarrow} U_i$ as the projective limit of a filtered projective system of affine $Z$-schemes $(U_i)$, each $U_i$ being of finite type over $Z$. Choose $U_i$ so that $\Gamma(U_i, \mathcal{O}_{U_i})$ is a
sub-$\mathbb{Z}$-algebra of $\Gamma(U, \mathcal{O}_U)$. The $\mathbb{Z}$-scheme $A_Z$ being of finite presentation, there exist an index $i$ and a $\mathbb{Z}$-morphism $h_i : U_i \to A_Z$ such that $h|U : U \to A_Z$ factors as

$$h|U : U \to U_i \xrightarrow{h_i} A_Z.$$ 

For $j \geq i$, let $h_j$ be the composition

$$h_j : U_j \to U_i \xrightarrow{h_i} A_Z.$$ 

In the limit the projection $U \times_{A_Z} A_\tilde{S} \to A_\tilde{S}$ is an open immersion and $\text{codim}(U - (U \times_{A_Z} A_\tilde{S}), U) \geq 2$. There is thus an index $j \geq i$ such that for every $k \geq j$, the projection $U_k \times_{A_Z} A_\tilde{S} \to A_\tilde{S}$ is an open immersion and $\text{codim}(U_k - (U_k \times_{A_Z} A_\tilde{S}), U_k) \geq 2$. Then $h_k : U_k \to A_Z$, $\forall k \geq j$, is an open immersion (EGA IV, Théorème 21.12.12), because $h_k$ is separated of finite type, $U_k$ is integral, $A_Z$ is geometrically factorial at every point and $\text{codim}(U_k - (U_k \times_{A_Z} A_\tilde{S}), U_k) \geq 2$.

Identify $U_k$ via $h_k$ with an open subscheme of $A_Z$, $\forall k \geq j$. Passing to the limit one can identify $U$ via $h$ with the “germ” $\varprojlim U_k = \cap U_k$ of $A_Z$. Consequently, $h^* : h^{-1}\mathcal{O}_{A_Z} \to \mathcal{O}_Q$ is an isomorphism. Since $Q$ is regular at each point $u \in U$, one deduces that $A_Z$ is regular at each image point $h(u) \in A_Z$. Then $Z$ is regular at the point $\pi(h(u)) = q(u)$, the abelian scheme $\pi : A_Z \to Z$ being $Z$-flat. Finally, $Z$ is regular, as $q = \pi h : Q \to Z$ is surjective. The proof is now complete.

3.9. Proof of Corollary 1.3.

Recall (3.8) that $Z$ admits a closed $Z$-immersion into $Q$. So $Z$ is locally of finite type over $V$ if $P$ and $Q$ are. The morphism $h : Q \to A_Z$ (3.8) is then locally of finite type. It is also a local isomorphism at every point of $Q$ of codimension $\leq 1$ by (3.8) iii). As $A_Z$ is regular, $h$ is a local isomorphism (EGA IV, Théorème 21.12.12). If moreover $P$ is separated or $V$-separated, so is $Z$. In either case, $P, Q$ are separated over $Z$ and $h$ is separated. Being a separated local isomorphism, $h$ is an open immersion. If $P$, thus $Q$, are proper over $V$, then $Z/V$ is proper and $h$ is a proper open immersion, i.e., an isomorphism.

REFERENCES

1. S. Bosch, W. Lutkebohmert, M. Raynaud. Néron models. Ergebnisse der Mathematik und ihrer Grenzgebiete, 1990.
2. J. F. Boutot. Schéma de Picard Local. Lecture Notes in Mathematics, 632 (1978).
3. A. J. De Jong. Smoothness, semi-stability and alterations. Publications Mathématiques de l’IHÉS, 83 (1996), p. 51–93.
4. P. Deligne. Le lemme de Gabber. Astérisque. 127 (1985), p. 131–150.
5. A. Grothendieck. Techniques de construction en géométrie analytique. X. Construction de l’espace de Teichmüller. Séminaire Cartan, 13, no. 2 (1960–1961).
6. A. Grothendieck. Un théorème sur les homomorphismes de schémas abéliens. Invent. math. 2 (1966), p. 59–78.
7. A. Grothendieck. Le groupe de Brauer III. Dix exposés sur les cohomologie des schémas, 1968.
8. D. Mumford. Geometric Invariant Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, 1965.
9. M. Raynaud. Spécialisation du foncteur de Picard. Publications Mathématiques de l’IHÉS, 38 (1970), p. 27–76.
10. M. Raynaud. Faisceaux amples sur les schémas en groupes et les espaces homogènes. Lecture Notes in Mathematics, 119 (1971).

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