ASYMMETRIC COMPLETE RESOLUTIONS AND VANISHING OF EXT OVER GORENSTEIN RINGS

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ABSTRACT. We construct a class of Gorenstein local rings $R$ which admit minimal complete $R$-free resolutions $C$ such that the sequence $\{\rank_R C_i\}$ is constant for $i < 0$, and grows exponentially for all $i > 0$. Over these rings we show that there exist finitely generated $R$-modules $M$ and $N$ such that $\Ext_R^i(M, N) = 0$ for all $i > 0$, but $\Ext_R^i(N, M) \neq 0$ for all $i > 0$.

INTRODUCTION

Let $R$ be a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, and let $M, N$ denote finitely generated $R$-modules. We write $\nu_R(M)$ for the minimal number of generators of $M$.

It is well-known that $R$ is Gorenstein if and only if the following remarkable symmetry is satisfied: for any module $M$ we have $\Ext_R^i(M, k) = 0$ for all $i > 0$ if and only if $\Ext_R^{i-1}(k, M) = 0$ for all $i > 0$ (equivalently, $\pd_R(M) < \infty$ if and only if $\id_R(M) < \infty$). A natural question is whether this statement still holds when $k$ is replaced with any module $N$. More generally, does the Gorenstein property of $R$ translate into similarities in the asymptotic behavior of the sequences $\{\nu_R(\Ext_R^i(M, N))\}_{i \geq 0}$ and $\{\nu_R(\Ext_R^i(N, M))\}_{i \geq 0}$?

Since complete intersection rings are Gorenstein, a foundation is laid by the following theorem of Avramov and Buchweitz [4, 5.6]:

Theorem AB. Suppose $R$ is a complete intersection ring. Then for any pair of finitely generated $R$-modules $M$ and $N$ the sequences $\{\nu_R(\Ext_R^i(M, N))\}_{i \geq 0}$ and $\{\nu_R(\Ext_R^i(N, M))\}_{i \geq 0}$ both have polynomial growth of the same degree.

A sequence of positive integers $\{c_i\}_{i \geq 0}$ is said to have polynomial growth of degree $d$ if there exist polynomials $f(t)$ and $g(t)$, both of degree $d$ and having the same leading term, such that $g(i) \leq c_i \leq f(i)$ for all $i > 0$. (We adopt the convention that the zero polynomial has degree $-1$.) We say that $\{c_i\}_{i \geq 0}$ has exponential growth if there exist $a, b \in \mathbb{R}$ with $1 < a \leq b$ such that $a^i \leq c_i \leq b^i$ for all $i > 0$.

The numbers $\beta_R^i(M) = \nu_R(\Ext_R^i(M, k))$, with $i \geq 0$, are called the Betti numbers of $M$ and the numbers $\mu_R^i(M) = \nu_R(\Ext_R^i(k, M))$, with $i \geq 0$, are called the Bass numbers of $M$ (associated to the maximal ideal $\mathfrak{m}$).

Complete intersection rings are characterized by polynomial growth of the sequence $\{\beta_R^i(k)\}_{i \geq 0}$, see [7], [8], so Theorem AB does not extend to all Gorenstein rings. On the other hand, it is proved in [9] 8.2.2 that for every non-complete intersection $R$ the sequence $\{\beta_R^i(k)\}_{i \geq 0}$ has exponential growth.

We extend Theorem AB to Gorenstein rings with $m^3 = 0$ or with $\codim R \leq 4$, where $\codim R = \nu_R(\mathfrak{m}) - \dim R$, in the following form: for every module $M$, both
sequences \( \{ \beta_i^R(M) \}_{i \geq 0} \) and \( \{ \mu_i^R(M) \}_{i \geq 0} \) have either polynomial growth of the same degree or exponential growth. However, the main result of the paper shows that Gorenstein rings are not characterized by such symmetry in homological behavior:

**Theorem.** There exist Gorenstein rings \( R \) with \( \mathfrak{m}^4 = 0 \) and codim \( R = 6 \), and finitely generated \( R \)-modules, such that their Betti sequence is constant (respectively, has exponential growth) and their Bass sequence has exponential growth (respectively, is constant). Moreover, there exist finitely generated \( R \)-modules \( M, N \) such that

\[
\text{Ext}_R^i(M, N) = 0 \quad \text{for all } i > 0 \quad \text{and} \quad \text{Ext}_R^i(N, M) \neq 0 \quad \text{for all } i > 0.
\]

The preceeding result answers several questions from the recent literature, which we discuss next.

**Symmetry in the vanishing of Ext.** Theorem AB, restricted to the case of polynomial growth of degree \(-1\), shows that any complete intersection ring \( R \) satisfies the following property:

(\( \text{ee} \)) If \( M \) and \( N \) are finitely generated \( R \)-modules such that \( \text{Ext}_R^i(M, N) = 0 \) for all \( i \gg 0 \), then \( \text{Ext}_R^i(N, M) = 0 \) for all \( i \gg 0 \).

The authors of [4] asked whether all Gorenstein rings satisfy (\( \text{ee} \)). It was subsequently established in [11] and [15] that (\( \text{ee} \)) holds for certain classes of Gorenstein local rings \( (R, \mathfrak{m}) \) other than the complete intersection rings, for example Gorenstein rings with \( \mathfrak{m}^4 = 0 \), and Gorenstein rings with codim \( R \leq 4 \).

In [11], Huneke and Jorgensen introduce a class of Gorenstein rings, called AB rings, and prove that any AB ring satisfies (\( \text{ee} \)). In [13], the present authors constructed Gorenstein rings which are not AB, but these examples failed to disprove (\( \text{ee} \)).

**Betti numbers versus Bass numbers; complete resolutions.** The question of whether the Betti and Bass sequences of an \( R \)-module \( M \) have the same asymptotic behavior has been previously posed in the literature in the more general context of complete resolutions. A complete resolution of the \( R \)-module \( M \) is a complex \( C \) of finitely generated free \( R \)-modules with differentials \( d_i: C_i \to C_{i-1} \) such that the complexes \( C \) and \( \text{Hom}_R(C, R) \) are both exact, and such that \( C_{\geq 0} = F_{\geq 0} \) for some free resolution \( F \) of \( M \). The complex \( C \) is said to be minimal if \( d_i(C_i) \subseteq \mathfrak{m}C_{i-1} \) for all \( i \in \mathbb{Z} \). If \( R \) is Gorenstein, then every finitely generated \( R \)-module has a minimal complete resolution \( C \). Moreover, any two minimal complete resolutions of \( M \) are isomorphic, cf. [5, 8.4], hence the numbers \( \text{rank}_R C_i \) are uniquely determined. We say that \( C \) has symmetric growth if both sequences \( \{ \text{rank}_R C_i \}_{i \geq 0} \) and \( \{ \text{rank}_R C_{-i} \}_{i \geq 0} \) have exponential growth or polynomial growth of the same degree.

If \( R \) is Gorenstein and \( C \) is a minimal complete resolution of a maximal Cohen-Macaulay \( R \)-module \( M \), then \( \beta_i^R(M) = \text{rank}_R C_i \) and \( \mu_{i+d-1}^R(M) = \text{rank}_R C_{-i} \) for all \( i \geq 1 \), where \( d = \dim R \). Therefore the question about the Betti and Bass sequences can be translated into the question of whether \( C \) has symmetric growth. Similar versions of this question have been previously posed in [5, 9.2] and in [14]. In [14] we constructed doubly infinite minimal exact complexes of free modules which had asymmetric growth; however, the ring was not Gorenstein and these complexes were not complete resolutions.
The paper is organized as follows: In Section 1 we use results of Avramov [2] and Sun [22] to prove that any complete resolution has symmetric growth over Gorenstein rings with \( m^3 = 0 \) or \( \text{codim} \, R \leq 4 \).

In Section 2 we prove that there exist Gorenstein rings \( R \) with \( m^4 = 0 \) and \( \text{codim} \, R = 6 \) which admit complete resolutions \( C \) for which \( \{ \text{rank}_R \, C_i \}_{i \geq 0} \) is constant (respectively, grows exponentially), and \( \{ \text{rank}_R \, C_{-i} \}_{i \geq 0} \) grows exponentially (respectively, is constant). We do not know whether such asymmetric complete resolutions exist when \( \text{codim} \, R = 5 \).

Using the results of Section 2, we prove in Section 3 that there exist finitely generated modules \( M, N \) which give counterexamples to (ee); the ring is the same as in Section 2. The module \( N \) has minimal possible length for such a counterexample, namely length 2. The results of this section are stated in terms of Tate (co)homology: we show that the Tate cohomology groups \( \text{Ext}_R^i(M, N) \) vanish for all \( i > 0 \), but do not vanish for all \( i < 0 \).

In Appendix A we establish the structure and relevant properties of the rings \( R \) from Sections 2 and 3. These rings are similar to those constructed in [9], [13], [14].

1. Symmetric growth of complete resolutions

In this section we show that there exist certain classes of Gorenstein rings, other than the class of complete intersection rings, for which all complete resolutions have symmetric growth.

Let \( (R, m, k) \) be a local ring as in the introduction. If \( R \) is Gorenstein, then a complete resolution of a finitely generated \( R \)-module \( M \) has symmetric growth if and only if the Betti sequence \( \{ \beta_i^R(M) \}_{i \geq 0} \) and the Bass sequence \( \{ \mu^i_R(M) \}_{i \geq 0} \) have the same growth. For the convenience of the reader, we prove this in Lemma 1.2.

1.1. The asymptotic behavior of the Betti sequences remains unchanged upon passing to syzygies. When the ring is Gorenstein, the same is true for the Bass sequences. We may thus assume, whenever convenient, that \( M \) is a maximal Cohen-Macaulay module over \( R \).

We let \( M^* \) denote the \( R \)-module \( \text{Hom}_R(M, R) \). If \( D \) is a complex, then \( D^* \) denotes the complex with \( (D^*)_i = (D_{i-})^* \) for each \( i \), and with induced differentials.

1.2. Lemma. Let \( R \) be a Gorenstein local ring of dimension \( d \), let \( M \) be a finitely generated maximal Cohen-Macaulay \( R \)-module, and \( C \) a minimal complete resolution of \( M \). The following equalities hold:

\[
\begin{align*}
(1) \quad & \beta^R_i(M) = \text{rank}_R \, C_i \quad \text{for all} \quad i \geq 0, \quad \text{and} \quad \mu^{i+d-1}_R(M) = \text{rank}_R \, C_{-i} \quad \text{for all} \quad i \geq 1; \\
(2) \quad & \beta^R_i(M^*) = \mu^{i+d}_R(M) \quad \text{for all} \quad i \geq 0.
\end{align*}
\]

Proof. (1) If \( d = 0 \), the statement is clear: \( C_{\geq 0} \) is a minimal free resolution of \( M \) over \( R \), and \( C_{\leq -1} \) is a minimal injective resolution of \( M \) over \( R \). If \( d > 0 \), then let \( \mathbf{x} = x_1, \ldots, x_d \) be a maximal regular sequence for both \( R \) and \( M \). Note that \( C/\langle \mathbf{x} \rangle C = C \otimes_R R/\langle \mathbf{x} \rangle \) is a minimal complete resolution of \( \overline{M} = M/\langle \mathbf{x} \rangle M \) over the zero-dimensional Gorenstein ring \( \overline{R} = R/\langle \mathbf{x} \rangle \). The conclusion then follows from the isomorphisms \( \text{Ext}_R^i(M, k) \cong \text{Ext}_{R/\langle \mathbf{x} \rangle}^i(\overline{M}, k) \) and \( \text{Ext}_R^{i+d}(k, M) \cong \text{Ext}_{R/\langle \mathbf{x} \rangle}^{i+d}(k, \overline{M}) \), which hold for all \( i \geq 0 \), cf. for example [16] p. 140].
1.3. If $R$ is a Gorenstein ring with $\text{codim} R \geq 2$, then $R$ has multiplicity at least $\text{codim} R + 2$. Otherwise, when $\text{codim} R \leq 1$, the multiplicity is at least $\text{codim} R + 1$. In either case, when equality holds we say that $R$ is Gorenstein of minimal multiplicity. If $R$ is furthermore Artinian, then $R$ has minimal multiplicity if and only if $m^3 = 0$.

1.4. If $R$ is Gorenstein of minimal multiplicity with $\text{codim} R \geq 3$, then for each finitely generated $R$-module $M$ either $M$ has finite projective dimension, or the sequence $\{\beta_i^R(M)\}_i$ has exponential growth. Indeed, if $\dim R = 0$, then $m^3 = 0$ and the result is proved by Sjödin [20], cf. also Lescot [15]. If $\dim R > 0$, then the reduction to the zero dimensional case can be done as described in [18, 1.7].

1.5. Assume now that $R$ is Gorenstein and $\text{codim} R \leq 4$. Avramov [2] and Sun [22] classified the possible behavior of the Betti numbers of a finitely generated $R$-module $M$. They show that the Betti sequence has either polynomial growth, or exponential growth. Note that Avramov and Sun use the terminology of strong polynomial/exponential growth for describing the same concepts that we are concerned with, only that we omit the word “strong”. The classification involves the notion of virtual projective dimension. We recall this notion for the reader’s convenience: let $M$ be a finitely generated module over a local ring $R$ (not necessarily Gorenstein). If the residue field $k$ of $R$ is infinite, set $\hat{R} = \hat{R}$, the $m$-adic completion of $R$; if $k$ is finite, set $\hat{R}$ to be the maximal-ideal-adic completion of $R[Y][mR[Y]]$, where $Y$ is an indeterminate. We say that a map of local rings $\hat{R} \leftarrow Q$ is an embedded deformation of $R$ if its kernel is generated by a $Q$-regular sequence contained in the square of the maximal ideal of $Q$. The virtual projective dimension of $M$ is the number

$$\text{vpd}_R M = \min\{\text{pd}_Q (M \otimes_R \hat{R}) \mid \hat{R} \leftarrow Q \text{ is an embedded deformation of } \hat{R}\}$$

A similar invariant, called virtual injective dimension and denoted $\text{vid}_R M$, can be defined by replacing $\text{pd}_Q (M \otimes_R \hat{R})$ with $\text{id}_Q (M \otimes_R \hat{R})$ in the formula above. In [2] and [22] it is shown that if $R$ is Gorenstein with $\text{codim} R \leq 4$, then the Betti numbers of $M$ fall into one of two categories, each described by equivalent conditions:

1. $\{\beta_i^R(M)\}_i$ has polynomial growth if and only if $\text{vpd}_R M < \infty$; when $\text{vpd}_R M$ is finite it is equal to $\text{vpd}_R M = \text{depth } R - \text{depth } M + q + 1$, where $q$ is the degree of polynomial growth of the sequence;
2. $\{\beta_i^R(M)\}_i$ has exponential growth if and only if $\text{vpd}_R M = \infty$.

Using Lemma 1.2(2), we see that the Bass numbers of $M$ have the same behavior: they either have polynomial growth or they have exponential growth.

We are now ready to prove the main result of this section, which assembles the results of [2] and [22] listed above.

1.6. Theorem. Assume that $R$ is Gorenstein, and either $R$ has minimal multiplicity, or $\text{codim} R \leq 4$. If $M$ is a finitely generated $R$-module, then one of the following statements is satisfied:

1. Both sequences $\{\beta_i^R(M)\}_{i \geq 0}$ and $\{\mu_i^R(M)\}_{i \geq 0}$ have polynomial growth of the same degree.
Both sequences $\{\beta_i^R(M)\}_{i \geq 0}$ and $\{\mu_i^R(M)\}_{i \geq 0}$ have exponential growth.

Proof. Assume first that $R$ has minimal multiplicity. If $\text{codim}\, R \leq 2$, then $R$ is a complete intersection, and Theorem AB in the introduction shows that $M$ satisfies (1). Assume now $\text{codim}(R) \geq 3$. If $\text{pd}_R M = \infty$, then $\text{id}_R M = \infty$ as well, and it can be immediately seen from (1.2) and (1.4) that $M$ satisfies condition (2). If $\text{pd}_R M$ is finite, then $\text{id}_R M$ is also finite, and $M$ satisfies (1).

Assume $\text{codim}\, R \leq 4$. Since $R$ is Gorenstein, the same is true for any ring $Q$ in an embedded deformation $\tilde{R} \leftarrow Q$, and so it is clear that $\text{vid}_R M = \text{vpd}_R M + \text{depth} M$. It follows then from [2, 1.8] and (1.4) that $M$ satisfies (1) whenever $\text{vpd}_R M$ and $\text{vid}_R M$ are finite and $M$ satisfies (2) whenever they are infinite. □

Lemma 1.2 and 1.1 now yield:

1.7. Corollary. Assume $R$ is Gorenstein. If $R$ has minimal multiplicity, or if $\text{codim}\, R \leq 4$, then any minimal complete resolution over $R$ has symmetric growth. □

2. Asymmetric growth of complete resolutions

In this section we show that complete resolutions need not be symmetric when $m^4 = 0$ and $\text{codim}\, R = 6$.

Let $k$ be a field which is not algebraic over a finite field and let $\alpha \in k$ be an element of infinite multiplicative order. Throughout the whole section we consider the ring $R$ to be defined as follows.

2.1. Let $P = k[T, U, V, X, Y, Z]$ be the polynomial ring in six variables (each of degree one) and set $R = P/I$, where $I$ is the ideal generated by the following fifteen quadratic polynomials:

\[
Z^2, \quad UZ - TX - \alpha UV, \quad U^2, \quad YZ + VY, \quad UY, \quad Y^2 - TX - (\alpha - 1)UV, \\
XZ + \alpha VX, \quad UX, \quad XY, \quad X^2 - TX - TV, \quad TZ + TY + \alpha VX, \quad TU, \\
TY - VX + TV, \quad T^2 + (\alpha + 1)UV - VY, \quad V^2.
\]

Let $t, u, v, x, y, z$ denote the residue classes of the variables modulo $I$, and $m$ denote the ideal they generate.

2.2. Proposition. The ring $R$ is local, with maximal ideal $m$, and satisfies the following properties:

1. $R$ has Hilbert series $H_R(t) = 1 + 6t + 6t^2 + t^3$. More precisely, a basis of $R$ over $k$ is given by the following fourteen elements:

\[
1, \quad t, \quad u, \quad v, \quad x, \quad y, \quad z, \quad tv, \quad uv, \quad vx, \quad vy, \quad vz, \quad tx, \quad vtx
\]

2. $R$ is Gorenstein, with $\text{Socle}(R) = (tvx)$.

3. $R$ is a Koszul algebra.

We prove this proposition in the Appendix as Theorem A.1; we also provide there a relevant part of the multiplication table of $R$.

2.3. For each $i \leq 0$ we let $d_i: R^2 \to R^2$ denote the map given with respect to the standard basis of $R^2$ by the matrix

\[
\begin{pmatrix}
\alpha^{-1}v & y \\
v & \alpha^i x & z
\end{pmatrix}.
\]
Let \( d_1 : R^3 \to R^2 \) denote the map represented with respect to the standard bases of \( R^3 \) and \( R^2 \) by the matrix
\[
\begin{pmatrix}
v & y & 0 \\
x & z & tv
\end{pmatrix}.
\]

Consider a minimal free resolution of \( \text{Coker} \, d_1 \) with \( d_1 \) as the first differential:
\[
\cdots \to R^3 \xrightarrow{d_1} R^2 \to R^1 \to R^0 \to 0.
\]

2.4. **Theorem.** The sequence of homomorphisms
\[
C : \cdots \to R^3 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R^1 \to R^0 \to 0
\]
is a minimal complete resolution such the following hold:

1. The sequence \( \{\text{rank} C_i\}_{i \geq 0} \) has exponential growth.
2. \( \text{rank} C_i = 2 \) for all \( i \leq 0 \).

**Proof of (2).** We postpone the proof of (1) to the end of the section. The minimality of \( C \) is clear from the definition of the differentials \( d_i \). Moreover, the defining equations of \( R \) guarantee \( d_i d_{i+1} = 0 \) for all \( i \leq 0 \), hence \( C \) is a complex. Since the ring \( R \) is Gorenstein, the exactness of \( C \) also implies that the complex \( \text{Hom}_R(C, R) \) is exact. Therefore it remains to show that \( C \) is exact.

Let \((a, b)\) denote an element of \( R^2 \) written in the standard basis of \( R^2 \) as a free \( R \)-module. One may check that for each \( i \leq 0 \) the \( k \)-vector space \( \text{Im} \, d_i \) has the following fourteen linearly independent elements:

\[
\begin{align*}
\d_i(1, 0) &= (v, \alpha^{-i} x) & \d_i(t v, 0) &= (0, \alpha^{-i} t v x) \\
\d_i(t, 0) &= (t v, \alpha^{-i} t x) & \d_i(t x, 0) &= (t v x, 0) \\
\d_i(u, 0) &= (u v, 0) & \d_i(0, 1) &= (y, z) \\
\d_i(v, 0) &= (0, \alpha^{-i} v x) & \d_i(0, t) &= (v x - t v, t v - (\alpha + 1) v x) \\
\d_i(x, 0) &= (v x, \alpha^{-i} (t v + t x)) & \d_i(0, u) &= (0, \alpha u v + t x) \\
\d_i(y, 0) &= (v y, 0) & \d_i(0, v) &= (v y, v z) \\
\d_i(z, 0) &= (v z, -\alpha^{-i} v x) & \d_i(0, y) &= (-\alpha^{-1} u v + t x, -v y).
\end{align*}
\]

We thus have \( \text{rank}_k \text{Im} \, d_i \geq 14 \) for all \( i \leq 0 \). Since \( \text{rank}_k R^2 = 28 \), it follows that \( \text{rank}_k \text{Ker} \, d_i \leq 14 \) for all \( i \leq 0 \). We conclude that \( H_i(C) = 0 \) for \( i \leq -1 \) and \( \text{rank}_k \text{Ker} \, d_0 = 14 \).

For \( i = 1 \) the images above are also those of \( d_1 \), by replacing \( d_1(1, 0, 0) \) with \( d_1(1, 0, 0, 0) \) and so on. Here \((a, b, c)\) denotes an element of \( R^3 \) in its standard basis as a free \( R \)-module. Not more than thirteen of these images are linearly independent, since we have the relation
\[
(v x, t v + t x) = (t v, t x) + (v x - t v, t v - (\alpha + 1) v x) + (\alpha + 1)(0, v x).
\]
One can check that we are left with precisely thirteen linearly independent elements, and a fourteenth element in \( \text{Im} \, d_1 \) can be chosen to be \( d_1(0, 0, 1) = (0, t v) \). Thus \( \text{rank}_k \text{Im} \, d_1 \geq 14 \) and it follows that \( H_0(C) = 0 \). The definition of \( C_{\geq 1} \) gives that \( H_i(C) = 0 \) for all \( i \geq 1 \), and therefore \( C \) is exact. \( \square \)

Next we provide the necessary background for the proof of part (1) of the Theorem.
2.5. Let $\pi: A \to B$ be a ring homomorphism, $D$ an $A$-module, $E$ a $B$-module (with the $A$-module structure induced by $\pi$) and $\phi: D \to E$ a homomorphism of $A$-modules. Then for each $B$-module $L$ one has a natural homomorphism

$$\text{Tor}^\pi_\ell(L, \phi): \text{Tor}^A_\ell(L, D) \to \text{Tor}^B_\ell(L, E)$$

which may be computed as follows: Let $D$ be a free $A$-resolution of $D$ and $E$ a free $B$-resolution of $E$. Let $\phi: B \otimes_A D \to E$ be a lifting of $\phi$ to a homomorphism of complexes of $B$-modules. The homomorphism $\text{Tor}^\pi_\ell(L, \phi)$ is then induced in homology by the following homomorphism of complexes, which is unique up to homotopy:

$$L \otimes_A D = L \otimes_B (B \otimes_A D) \xrightarrow{L \otimes_B \phi} L \otimes_B E$$

We say that a positively graded ring $A = \bigoplus_{i \geq 0} A_i$ is standard graded if $A_0$ is a field and $A$ is generated over $A_0$ by $A_1$.

2.6. Proposition. Let $A$ be a standard graded algebra over a field $\ell$ and set $n = \bigoplus_{i \geq 1} A_i$. Let $\pi: A \to B$ be a surjective homomorphism of graded rings with $\text{Ker} \pi \subseteq n^2$. Assume that $D$ is a finitely generated graded $A$-module with a linear graded free resolution and that $E$ is a finitely generated graded $B$-module.

Suppose there exists a homomorphism of $A$-modules $\phi: D \to E$ such that the induced map $\overline{\phi}: D/nD \to E/nE$ is injective. Then the induced homomorphisms

$$\text{Tor}^A_\ell(\ell, \phi): \text{Tor}^A_\ell(\ell, D) \to \text{Tor}^B_\ell(\ell, E)$$

are injective for each $i$.

Proof. Consider a linear resolution $(D, \delta)$ of $D$, together with an augmentation map $\varepsilon: D_0 \to D$. We will construct inductively a minimal graded free resolution $E$ of $E$, with an augmentation map $\eta: E_0 \to E$, and a map of complexes of $A$-modules $\varphi: D \to E$ such that the induced map $\overline{\varphi}: D/nD \to E/nE$ is injective in each homological degree.

Let $e_1, \ldots, e_a$ denote the standard basis of $D_0 = A^a$. Then $\varepsilon(e_1), \ldots, \varepsilon(e_a)$ form a homogeneous minimal system of generators for $D$. Since the induced map $\overline{\phi}: D/nD \to E/nE$ is injective, the elements $\phi(\varepsilon(e_1)), \ldots, \phi(\varepsilon(e_a))$ are part of a minimal system of generators for $E$. This shows that we can choose $E_0 = B^b$, with $b \geq a$, and the map $\eta: B^b \to E$ can be chosen so that $\eta(f_i) = \phi(\varepsilon(e_1))$ for each $i$ with $1 \leq i \leq a$, where $f_1, \ldots, f_k$ is the standard basis of $B^b$.

If we define an $A$-module homomorphism $\varphi_0: A^a \to B^b$ such that $\varphi_0(e_i) = f_i$, then the right-hand part of the diagram below is commutative. We set $D' = \text{Ker}(\varepsilon)$ and $E' = \text{Ker}(\eta)$, and we let $\phi': D' \to E'$ denote the induced $A$-module homomorphism, which makes entire diagram commutative:

$$\begin{array}{ccccccc}
0 & \longrightarrow & D' & \longrightarrow & A^a & \longrightarrow & D & \longrightarrow & 0 \\
& & \varepsilon \downarrow & & \varphi_0 \downarrow & & \phi \downarrow & & 0 \\
0 & \longrightarrow & E' & \longrightarrow & B^b & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

We want to prove that the induced map $\overline{\phi'}: D'/nD' \to E'/nE'$ is injective.

Let $g_1, \ldots, g_a$ be the standard basis of $D_1 = A^a$. The elements $\delta_1(g_1), \ldots, \delta_1(g_a)$ form a minimal system of generators for $D'$. Let $\alpha_i \in A$ be such that

$$\varphi_0(\alpha_1 \delta_1(g_1) + \cdots + \alpha_a \delta_1(g_a)) \in nE' \subseteq n^2 B^b.$$
Since the matrix representing \( \delta_i \) has linear entries, we can think of \( \delta_i(g_i) \) as column vectors with components in \( A_1 \) and hence of \( \varphi_0(\delta_i(g_i)) \) as column vectors with components in \( B_1 \). Thus, the degree one part of the above expression is equal to zero, so we get

\[
\varphi_0 \left( \sum_i \pi_i \delta_i(g_i) \right) = \sum_i \pi_i \varphi_0(\delta_i(g_i)) = 0
\]

where \( \pi_i \) denotes the degree zero component of \( \alpha_i \). Since the homomorphism \( \pi: A \to B \) has \( \text{Ker} \pi \subseteq \mathfrak{n}^2 \), we conclude that \( \sum_i \pi_i \delta_i(g_i) \in \mathfrak{n}^2 A^n \), hence, by degree considerations, \( \sum_i \pi_i \delta_i(g_i) = 0 \). Since the elements \( \delta_i(g_i) \) in this sum are part of a homogeneous minimal system of generators for \( D' \), it follows that \( \pi_i = 0 \) for all \( i \). Therefore \( \alpha_i \in \mathfrak{n} \) for all \( i \), and this shows \( \mathcal{O} \) is injective.

Using the construction above as the induction step, we obtain then a resolution \( E \) and a homomorphism of complexes \( \varphi: D \to E \). The homomorphism of complexes \( \tilde{\varphi} = \varphi \otimes_A B: D \otimes_A B = E \) is then a lifting of \( \varphi \) and is a split injection in each degree. This gives the desired conclusion. \( \square \)

Recall that \( P \) denotes the polynomial ring \( k[T, U, V, X, Y, Z] \) and that the monomials \( U^2 \) and \( UY \) are among the generators of the ideal \( I \) defining \( R \) as \( P/I \).

2.7. Lemma. Consider the ring \( A = P/(U^2, UY) \), the \( A \)-module \( D = (U, Y)A \), and the \( R \)-module \( E = \text{ker} d_1 \). Let \( \pi: A \to R \) denote the canonical projection. The following then hold:

1. The \( A \)-module \( D \) has a linear resolution and its Poincaré series is equal to \((2 + t)(1 - t - t^2)^{-1}\).

2. There exists a homomorphism of \( A \)-modules \( \phi: D \to E \) such that the induced map \( \text{Tor}^A_1(k, \phi): \text{Tor}^A_1(k, D) \to \text{Tor}^R_1(k, E) \) is injective for each \( i \).

Proof. (1) Set \( Q = k[U, Y]/(U^2, UY) \) and let \( G \) denote a minimal free resolution of the residue field \( Q/(U, Y)Q \) over \( Q \). Note that \( A \cong Q \otimes_k k[T, V, X, Z] \), and a minimal free resolution of \( A/D \) over \( A \) is given by the complex \( D = G \otimes_k k[T, V, X, Z] \).

Since \( Q \) is a Koszul algebra (see for example [3]), the resolution \( G \) is linear, and the Poincaré series of \( A/D \) over \( A \) is

\[
\frac{P^A_{A/D}(t)}{P^Q_{Q/(U, Y)Q}(t)} = \frac{1}{1 - t - t^2} = \frac{1 + t}{1 - t - t^2}.
\]

The Poincaré series of \( D \) is then equal to \( t^{-1}(P^A_{A/D}(t) - 1) \).

(2) Set \( p = (0, 0, u) \) and \( q = (0, 0, y) \), considered as elements of \( R^3 \). It can be easily checked that \( p, q \in \text{Ker} d_1 \), hence \( Rp + Rq \subseteq E \). We define \( \phi: D \to E \) as the following composition:

\[
\phi: D \to Rp + Rq \hookrightarrow E,
\]

where the leftmost map is the restriction of the map \( \varphi: A \to R^3 \) given by \( \varphi(r) = (0, 0, \pi(r)) \). Note that \( ap + bq \in \mathfrak{m}E \) for some \( a, b \in R \) implies \( ap + bq \in \mathfrak{m}^2 R^3 \), and hence \( a, b \in \mathfrak{m} \) by degree considerations. This shows that the induced map \( \tilde{\phi}: D/\mathfrak{n}D \to E/\mathfrak{n}E \) is injective, where \( \mathfrak{n} \) denotes the maximal ideal of \( A \). We can then apply Proposition 2.6. \( \square \)

Proof of Theorem 2.4(1). We use the notation in the statement of the Lemma above. Part (1) of the Lemma shows that the sequence \( \{\text{rank}_k \text{Tor}^A_1(k, D)\}_i \) has exponential growth. From part (2) we conclude that the sequence \( \{\text{rank}_k \text{Tor}^R_1(k, E)\}_i \) has
exponential growth, as well. Note that a minimal free resolution of the $R$-module $E$ is given by the truncation $C_{\geq 2}$, hence $\text{rank}_k C_{i+2} = \text{rank}_k \Tor_R^i(k, E)$ for all $i \geq 0$.

\[ \Box \]

3. Asymmetry in the vanishing of Ext

We will use the notation introduced in the second section. In particular, the ring $R$ is the one defined in 2.1. Recall that $R$ is zero-dimensional and Gorenstein.

3.1. If $T$ is a complete resolution of the $R$-module $X$, and $Y$ is an $R$-module, then for each $i$ the Tate (co)homology groups are defined by

$$\hat{\Ext}_R^i(X, Y) = \text{H}_{-i} \text{Hom}(T, Y) \quad \text{and} \quad \hat{\Tor}_R^i(X, Y) = \text{H}_i(T \otimes_R Y).$$

Since the ring $R$ is zero-dimensional Gorenstein, the complete resolution $T$ can be chosen to agree with a minimal free resolution of $X$ in all nonnegative degrees, cf. [5, 3.1], for example. Thus for all $i > 0$ there are isomorphisms

$$\hat{\Ext}_R^i(X, Y) \cong \Ext_R^i(X, Y) \quad \text{and} \quad \hat{\Tor}_R^i(X, Y) \cong \Tor_R^i(X, Y).$$

Also, for all $i$ one has

$$\hat{\Ext}_R^{i-1}(X, Y) \cong \Tor_R^i(X^*, Y).$$

Matlis duality yields for all $i$ the following isomorphisms:

$$\Tor_R^i(Y, X^*) \cong \Ext_R^i(Y, X).$$

Recall that $d_i$ denotes the differential of the complex $C$ defined in Section 2.

3.2. Theorem. Set $M = \text{Coker} d_0$ and $N = R/(t, u - x, y - x, z - x)$. The following then hold:

1. $\hat{\Ext}_R^i(M, N) = 0$ for all $i > 0$
2. $\hat{\Ext}_R^i(M, N) \neq 0$ for all $i < 0$.
3. $\hat{\Tor}_R^i(M, N) = 0$ for all $i > 0$.
4. $\hat{\Tor}_R^i(M, N) \neq 0$ for all $i < 0$.

In view of the isomorphisms in 3.1 we conclude the Gorenstein ring $R$ does not have the property (ee):

3.3. Corollary. For $R$, $M$ and $N$ as above we have:

$$\hat{\Ext}_R^i(M, N) = 0 \text{ for all } i > 0;$$
$$\hat{\Ext}_R^i(N, M) \neq 0 \text{ for all } i > 0.$$

The proof of the Theorem is given after the following lemma, which contains one of the ingredients of the proof.

3.4. Lemma. Set $E = \ker d_1$. If $L$ is an $R$-module of length two, then the following hold:

1. $\Tor_R^i(E, L) \neq 0$ for all $i > 0$.
2. $\Ext_R^i(E, L) \neq 0$ for all $i > 0$. 
Proof. (1) Since $L$ has length two, there is an exact sequence
$$0 \to k \to L \to k \to 0.$$ 

Using the notation of Lemma 2.7 and the naturality of the maps defined in Lemma 2.8, the short exact sequence above induces long exact sequences both over $R$ and over $A$, and they can be embedded in a commutative diagram as follows:

\[
\begin{array}{cccccc}
\cdots & \text{Tor}^A_i(k, D) & \to & \text{Tor}^A_i(L, D) & \to & \text{Tor}^A_i(k, D) & \xrightarrow{\Delta^A_i} & \text{Tor}^A_{i-1}(k, D) & \to & \cdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \text{Tor}^R_i(k, E) & \to & \text{Tor}^R_i(L, E) & \to & \text{Tor}^R_i(k, E) & \xrightarrow{\Delta^R_i} & \text{Tor}^R_{i-1}(k, E) & \to & \cdots \\
\end{array}
\]

Counting from the left we have: the first, third and fourth vertical maps are injective, cf. Lemma 2.7(2). If $\text{Tor}^R(E, L) = 0$ for some $i > 0$, then the connecting homomorphism $\Delta^R_i$ is injective, so the commutativity of the rightmost square implies that $\Delta^R_i$ is injective. However, this is not possible, because Lemma 2.7(1) shows that the Betti numbers of $D$ over $A$ are strictly increasing.

(2) By Matlis duality, the $R$-module $L^*$ has length two. We can then apply part (1) to conclude $\text{Tor}^R_i(E, L^*) = 0$ for all $i > 0$ and then use the isomorphism
$$\text{Ext}^i_R(E, L^*) \cong \text{Tor}^R_i(E, L^*).$$

Proof of Theorem 3.2 (1) A complete resolution of $M$ is given by the complex $\text{Hom}_R(C, R)$. We have
$$\text{Ext}^i_R(M, N) = H_{-i}(\text{Hom}_R(\text{Hom}_R(C, R), N)) \cong H_{-i}(C \otimes_R N)$$

In negative degrees $C \otimes_R N$ is the complex
$$N^2 \left( \frac{x}{\alpha x} \frac{x}{\beta x} \right) \to N^2 \left( \frac{x}{\alpha^2 x} \frac{x}{\beta^2 x} \right) \to N^2 \to \cdots$$

Since $N \cong k[x]/(x^2)$, this complex is acyclic, hence $\text{Ext}^i_R(M, N) = 0$ for all $i > 0$.

(2) By the isomorphisms in (1) we need to show that $\text{Tor}^R_i(M^*, N) \neq 0$ for all $i > 0$. Note that $M^* = \text{Coker} d_2 = \text{Im} d_1$, hence the module $E$ in Lemma 3.4 is the first syzygy of $M^*$. Since $N$ has length 2, the Lemma shows $\text{Tor}^R_i(M^*, N) \neq 0$ for all $i \geq 2$. To show that $\text{Tor}^R_i(M^*, N) \neq 0$, consider the short exact sequence
$$0 \to k \to N \to k \to 0,$$

and the induced long exact sequence
$$\text{Tor}^R_i(M^*, N) \to \text{Tor}^R_i(M^*, k) \to M^* \otimes_R k \to M^* \otimes_R N \to M^* \otimes_R k \to 0.$$ 

If $\text{Tor}^R_i(M^*, N) = 0$, then $\beta_2 = \dim k \text{Tor}_1(M^*, k) \leq \dim_k (M^* \otimes_R k) = 3$. However, this contradicts the fact that the following five elements are part of a minimal system of generators for $\text{Ker} d_1$:

$$\{(0, 0, t), (0, 0, u), (0, 0, v), (0, 0, y), (0, 0, z)\}.$$ 

The proofs of (3) and (4) are similar. □

3.5. Remark. The following conjecture of Auslander appears in [11, p. 795] and [10]. Let $\Lambda$ be an Artin algebra. For every finitely generated $\Lambda$-module $M$ there exists an integer $n_M$ such that for all finitely generated $\Lambda$-modules $N$, if $\text{Ext}_A^i(M, N) = 0$ for all $i \gg 0$, then $\text{Ext}_A^i(M, N) = 0$ for all $i \geq n_M$. □
Original counterexamples to this conjecture were given by the authors in [13] over a class of codimension-five Gorenstein rings. The codimension-six Gorenstein rings $R$ of this paper provide other counterexamples, as follows.

For each $q \geq 1$ set

$$N_q = R/(t, u, v - \alpha^q x, v - y, v - z).$$

A computation similar to that in [13] shows that the following holds:

$$\Ext^i_R(M, N_q) = 0 \quad \text{if and only if} \quad i \neq 0, q - 1, q.$$

Subsequent to [13], Smalø [21], and independently Mori [17], gave a very simple non-commutative counterexample to the conjecture of Auslander, involving the ring $A = k\langle x, y \rangle /\langle x^2, y^2, xy - \alpha yx \rangle$, where $\alpha \in k$ has infinite multiplicative order.

However, this ring fails to supply counterexamples to (ee). Indeed, all the indecomposable modules over $A$ have been classified, see for example [19, Section 4]. They are $A$, syzygies and cosyzygies of $k$, modules of periodicity one, and non-periodic modules having bounded resolutions. The differentials in the resolutions of the latter type are described as follows: there exists a square matrix $M(t)$ with entries in $A[t]$, with $t$ an indeterminate, such that the $i$th differential of the resolution is represented by $M(\alpha^i)$. An argument similar to that of [13, 3.13] may be used to see that the ring $A$ satisfies (ee).

3.6. Remark. The paper [14] initiated the question of whether there exist homologically defined classes of rings strictly contained between the complete intersections and the Gorenstein rings. The results of [11] and [13] show that the answer is positive: one such class consists of all rings $R$ with the property (ab) that $R$ is Gorenstein and there exists an integer $n$, depending only on the ring, such that for all pairs $(M, N)$ of finitely generated $R$-modules one has

$$\Ext^i_R(M, N) = 0 \quad \text{for all} \quad i \gg 0 \quad \text{implies} \quad \Ext^i_R(M, N) = 0 \quad \text{for all} \quad i > n.$$

The class of Gorenstein rings satisfying (ab) was studied in [11]; it is proved there that this class properly contains the complete intersections. The counterexamples to Auslander’s conjecture in [13], and those in this paper, show that the Gorenstein rings satisfying (ab) are properly contained in the class of Gorenstein rings.

Other classes of Gorenstein rings previously considered are those defined by the properties:

$$\begin{align*}
\text{(te)} & \quad \Tor^R_i(M, N) = 0 \quad \text{for all} \quad i \gg 0 \quad \text{implies} \quad \Ext^i_R(M, N) = 0 \quad \text{for all} \quad i \gg 0, \\
\text{(et)} & \quad \Ext^i_R(M, N) = 0 \quad \text{for all} \quad i \gg 0 \quad \text{implies} \quad \Tor^R_i(M, N) = 0 \quad \text{for all} \quad i \gg 0,
\end{align*}$$

where $M$ and $N$ range over all finitely generated $R$-modules. In addition, (gap) is the property that if $n$ consecutive $\Ext^i_R(M, N)$ vanish, then $\Ext^i_R(M, N)$ vanishes for all $i > \dim R$, where $n$ is a fixed positive integer depending only on $R$.

Let (gor), respectively (ci), denote the property that the ring is Gorenstein, respectively a complete intersection. The known implications among these properties are summarized by a diagram in [13], which we reproduce here. There is one added improvement, which represents one of the main results of this paper, namely the irreversibility of the rightmost implication (i.e., Corollary 3.3).
We do not know whether the implications (4), (5) and (6) are reversible, or whether there exist other implications. Thus, while we know that the classes (gap), (ab), (te) and (ee) are all strictly contained between complete intersections and Gorenstein rings, we do not know whether they represent four distinct classes or fewer.

**Appendix A.**

Let $P$ and $R$ be as defined in 2.1. The purpose of this appendix is to prove Proposition 2.2. We recall its statement below:

**A.1. Proposition.** The ring $R$ is local, with maximal ideal $m$, and satisfies the following properties:

1. $R$ has Hilbert series $H_R(t) = 1 + 6t + 6t^2 + t^3$. More precisely, a basis of $R$ over $k$ is given by the following fourteen elements:

   1, t, u, v, x, y, z, tv, uv, vx, vy, vz, tx, vtx

2. $R$ is Gorenstein, with $	ext{Socle}(R) = (tvx)$.

3. $R$ is a Koszul algebra.

**A.2.** Examining the defining ideal $I$ of $R$, we see that $m^2$ is a 6-dimensional vector space with basis $tv, uv, vx, vy, vz, tx$. Set $s = tvx$. We provide below a relevant part of the multiplication table of $R$:

|   | tv | uv | vx | vy | vz | tx |
|---|----|----|----|----|----|----|
| t | 0  | 0  | s  | 0  | 0  | 0  |
| u | 0  | 0  | 0  | s  | 0  | 0  |
| v | 0  | 0  | 0  | 0  | s  | 0  |
| x | s  | 0  | s  | 0  | 0  | 0  |
| y | 0  | 0  | s  | 0  | 0  | 0  |
| z | 0  | s  | 0  | 0  | 0  | $-\alpha s$ |

It is clear from the table that $m^4 = 0$, and $m^3 = sR$. If we can prove that $s \neq 0$, then the table above shows that the socle of $R$ is 1-dimensional, hence $R$ is Gorenstein.

In order to prove that $s \neq 0$ we will use the well-known fact that every Gorenstein quotient of $P$ with socle in degree 3 corresponds uniquely to a form $F$ of degree 3 in a divided power algebra [12, 2.12]. We recall below the necessary details of this fact.

**A.3.** Let $D$ be the divided power algebra in 6 variables $T_T, T_U, T_V, T_X, T_Y, T_Z$ over $k$. We refer to Appendix A in [12] for basic properties of this algebra. If $M$ is the multi-index $M = (m_T, m_U, \ldots, m_Z) \in \mathbb{N}^6$, then the divided powers monomials
\[ T^{[M]} = T^{[m_T]} \cdots T^{[m_Z]} \] with \([M] = m_T + \cdots + m_Z = j\) form a \(k\)-basis of \(D_j\). By definition \(T^{[1]} = T\), and similarly for the other variables. Multiplication in this algebra is defined by extending by linearity the rule

\[ T \cdot N = \frac{(M+N)!}{M!N!} T^{[M+N]}, \]

where \(\frac{(M+N)!}{M!N!} = \frac{(m_T + n_T)! \cdots (m_Z + n_Z)!}{m_T! \cdots m_Z! n_T! \cdots n_Z!}\).

One defines an action of the polynomial ring \(P\) on \(D\) as follows: If \(M = (m_T, \ldots, m_Z)\) and \(N = (n_T, \ldots, n_Z)\) are non-negative multi-indices in \(\mathbb{N}^6\), then:

\[ (T^{m_T} U^{n_U} V^{n_V} X^{n_X} Y^{n_Y} Z^{n_Z}) \circ T^{[A]} = \begin{cases} T^{[N-M]} & \text{if } M \leq N \\ 0 & \text{otherwise} \end{cases} \]

and then extend by linearity.

Given a form \(F \in D\) of degree \(j\), one can define the ideal \(I_F \subseteq P\) of the elements in \(P\) which annihilate \(F\):

\[ I_F = \{ G \in P \mid G \circ F = 0 \} \]

It is well-known, cf. [12, Section 2.3], that the ring \(P/I_F\) is a Gorenstein Artinian ring with socle in degree \(j\).

**Proof of Proposition** \(\Box\) Consider the ring \(D\) as in \(\Box\) and let \(F \in D\) be the following form (or 1/6 of this form if the characteristic of \(k\) is 2 or 3):

\[-3T_2 T_3^2 + 3T_2^2 T_3 + 3T_2 T_4^2 + 6T_2 T_5 T_6 + 3T_2^2 T_6 + 3T_3^2 T_5 + 6T_3 T_4 T_5 T_6 - 3s T_2 (T_4 + T_3)^2.\]

It is easy to check that the generators of the ideal \(I\) annihilate \(F\), hence \(I \subseteq I_F\).

By \(\Box\) the ring \(P/I_F\) is Gorenstein and has socle in degree 3. Consequently, we have \(s \neq 0\), hence \(R\) is Gorenstein with socle \(sR\). This proves (1) and (2).

(3) In \(\Box\) we have written the generators of \(I\) so that the first monomial occurring in each generator is its initial term with respect to reverse lexicographic order associated to the variable ordering \(Z > U > Y > X > T > V\). Let \(J\) denote the ideal generated by these initial terms:

\[ J = (Z^2, UZ, U^2, YZ, UY, Y^2, XZ, UX, XY, X^2, TZ, TU, TY, T^2, V^2). \]

It is easy to check that the Hilbert series of \(P/J\) is equal to \(1 + 6t + 6t^2 + t^3\), and hence it is equal to the Hilbert series of \(R\). It follows that the initial ideal of \(I\) equals \(J\), and this shows that the generators of \(I\) listed above are a Gröbner basis for \(I\). By \(\Box\) this shows that \(R\) is a Koszul algebra. \(\Box\)

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