ON FROBENIUS-DESTABILIZED RANK-2 VECTOR BUNDLES OVER CURVES

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Abstract. Let $X$ be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field $k$ of characteristic $p > 0$. Let $\mathcal{M}_X$ be the moduli space of semistable rank-2 vector bundles over $X$ with trivial determinant. The relative Frobenius map $F : X \to X_1$ induces by pull-back a rational map $V : M_{X_1} \to M_X$. In this paper we show the following results.

1. For any line bundle $L$ over $X$, the rank-$p$ vector bundle $F^*L$ is stable.
2. The rational map $V$ has base points, i.e., there exist stable bundles $E$ over $X_1$ such that $F^*E$ is not semistable.
3. Let $B \subset M_{X_1}$ denote the scheme-theoretical base locus of $V$. If $g = 2$, $p > 2$ and $X$ ordinary, then $B$ is a 0-dimensional local complete intersection of length $\frac{2}{3}p(p^2 - 1)$ and the degree of $V$ equals $\frac{1}{3}p(p^2 + 2)$.

Introduction

Let $X$ be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field $k$ of characteristic $p > 0$. Denote by $F : X \to X_1$ the relative $k$-linear Frobenius map. Here $X_1 = X \times_{k,\sigma} k$, where $\sigma : \text{Spec}(k) \to \text{Spec}(k)$ is the Frobenius of $k$ (see e.g. [R] section 4.1). We denote by $\mathcal{M}_X$, respectively $\mathcal{M}_{X_1}$, the moduli space of semistable rank-2 vector bundles on $X$, respectively $X_1$, with trivial determinant. The Frobenius $F$ induces by pull-back a rational map (the Verschiebung)

$$V : \mathcal{M}_{X_1} \to \mathcal{M}_X, \quad [E] \mapsto [F^*E].$$

Here $[E]$ denotes the $S$-equivalence class of the semistable bundle $E$. It is shown [MS] that $V$ is generically étale, hence separable and dominant, if $X$ or equivalently $X_1$ is an ordinary curve. Our first result is

**Theorem 1** Over any smooth projective curve $X_1$ of genus $g \geq 2$ there exist stable rank-2 vector bundles $E$ with trivial determinant, such that $F^*E$ is not semistable. In other words, $V$ has base points.

Note that this is a statement for an arbitrary curve of genus $g \geq 2$ over $k$, since associating $X_1$ to $X$ induces an automorphism of the moduli space of curves of genus $g$ over $k$. The existence of Frobenius-destabilized bundles was already proved in [LP2] Theorem A.4 by specializing the so-called Gunning bundle on a Mumford-Tate curve. The proof given in this paper is much simpler than the previous one. Given a line bundle $L$ over $X$, the generalized Nagata-Segre theorem asserts the existence of rank-2 subbundles $E$ of the rank-$p$ bundle $F_*L$ of a certain (maximal) degree. Quite surprisingly, these subbundles $E$ of maximal degree turn out to be stable and Frobenius-destabilized.

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In the case $g = 2$ the moduli space $\mathcal{M}_X$ is canonically isomorphic to the projective space $\mathbb{P}^3_k$ and the set of strictly semistable bundles can be identified with the Kummer surface $\text{Kum}_X \subset \mathbb{P}^3_k$ associated to $X$. According to [LP2] Proposition A.2 the rational map

$$V: \mathbb{P}^3_k \rightarrow \mathbb{P}^3_k$$

is given by polynomials of degree $p$, which are explicitly known in the cases $p = 2$ [LP1] and $p = 3$ [LP2]. Let $\mathcal{B}$ be the scheme-theoretical base locus of $V$, i.e., the subscheme of $\mathbb{P}^3_\mathbb{k}$ determined by the ideal generated by the 4 polynomials of degree $p$ defining $V$. Clearly its underlying set equals

$$\text{supp} \mathcal{B} = \{ E \in \mathcal{M}_X \cong \mathbb{P}^3_k \mid F^*E \text{ is not semistable} \}$$

and $\text{supp} \mathcal{B} \subset \mathbb{P}^3_k \setminus \text{Kum}_X$. Since $V$ has no base points on the ample divisor $\text{Kum}_X$, we deduce that $\dim \mathcal{B} = 0$. Then we show

**Theorem 2** Assume $p > 2$. Let $X_1$ be an ordinary curve of genus $g = 2$. Then the 0-dimensional scheme $\mathcal{B}$ is a local complete intersection of length

$$\frac{2}{3}p(p^2 - 1).$$

Since $\mathcal{B}$ is a local complete intersection, the degree of $V$ equals $\deg V = p^3 - l(\mathcal{B})$ where $l(\mathcal{B})$ denotes the length of $\mathcal{B}$ (see e.g. [O1] Proposition 2.2). Hence we obtain the

**Corollary** Under the assumption of Theorem 2

$$\deg V = \frac{1}{3}p(p^2 + 2).$$

The underlying idea of the proof of Theorem 2 is rather simple: we observe that a vector bundle $E \in \text{supp} \mathcal{B}$ corresponds via adjunction to a subbundle of the rank-$p$ vector bundle $F_*(\theta^{-1})$ for some theta characteristic $\theta$ on $X$ (Proposition 3.1). This is the motivation to introduce Grothendieck’s Quot-Scheme $\mathcal{Q}$ parametrizing rank-2 subbundles of degree 0 of the vector bundle $F_*(\theta^{-1})$. We prove that the two 0-dimensional schemes $\mathcal{B}$ and $\mathcal{Q}$ decompose as disjoint unions $\bigsqcup \mathcal{B}_\theta$ and $\bigsqcup \mathcal{Q}_\eta$ where $\theta$ and $\eta$ vary over theta characteristics on $X$ and $p$-torsion points of $JX_1$ respectively and that $\mathcal{B}_\theta$ and $\mathcal{Q}_0$ are isomorphic, if $X$ is ordinary (Proposition 4.6). In particular since $\mathcal{Q}$ is a local complete intersection, $\mathcal{B}$ also is.

In order to compute the length of $\mathcal{B}$ we show that $\mathcal{Q}$ is isomorphic to a determinantal scheme $\mathcal{D}$ defined intrinsically by the 4-th Fitting ideal of some sheaf. The non-existence of a universal family over the moduli space of rank-2 vector bundles of degree 0 forces us to work over a different parameter space constructed via the Hecke correspondence and carry out the Chern class computations on this parameter space.

The underlying set of points of $\mathcal{B}$ has already been studied in the literature. In fact, using the notion of $p$-curvature, S. Mochizuki [Mo] describes points of $\mathcal{B}$ as “dormant atoms” and obtains, by degenerating the genus-2 curve $X$ to a singular curve, the above mentioned formula for their number ([Mo] Corollary 3.7 page 267). Moreover he shows that for a general curve $X$ the scheme $\mathcal{B}$ is reduced. In this context we also mention the recent work of B. Osserman [O2], [O3], which explains the relationship of $\text{supp} \mathcal{B}$ with Mochizuki’s theory.
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§1 Stability of the direct image $F_\ast L$.

Let $X$ be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p > 0$ and let $F : X \to X_1$ denote the relative Frobenius map. Let $L$ be a line bundle over $X$ with
\[ \text{deg } L = g - 1 + d, \]
for some integer $d$. Applying the Grothendieck-Riemann-Roch theorem to the morphism $F$, we obtain

**Lemma 1.1** The slope of the rank-$p$ vector bundle $F_\ast L$ equals
\[ \mu(F_\ast L) = g - 1 + \frac{d}{p}. \]

The following result will be used in section 3.

**Proposition 1.2** If $g \geq 2$, then the vector bundle $F_\ast L$ is stable for any line bundle $L$ on $X$.

**Proof.** Suppose that the contrary holds, i.e., $F_\ast L$ is not stable. Consider its Harder-Narasimhan filtration
\[ 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l = F_\ast L, \]
such that the quotients $E_i/E_{i-1}$ are semistable with $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ for all $i \in \{1, \ldots, l - 1\}$. If $F_\ast L$ is not semistable, we denote $E := E_1$. If $F_\ast L$ is semistable, we denote by $E$ any proper semistable subbundle of the same slope. Then clearly
\[ (1) \quad \mu(E) \geq \mu(F_\ast L). \]

In case $r = \text{rk } E > \frac{p-1}{2}$, we observe that the quotient bundle
\[ Q = \begin{cases} F_\ast L/E_{l-1} & \text{if } F_\ast L \text{ is not semistable,} \\ F_\ast L/E & \text{if } F_\ast L \text{ is semistable,} \end{cases} \]
is also semistable and that its dual $Q^\ast$ is a subbundle of $(F_\ast L)^\ast$. Moreover, by relative duality $(F_\ast L)^\ast = F_\ast(L^{-1} \otimes \omega_X^{\otimes 1-p})$ and by assumption $\text{rk } Q^\ast \leq p-r \leq \frac{p-1}{2}$. Hence, replacing if necessary $E$ and $L$ by $Q^\ast$ and $L^{-1} \otimes \omega_X^{\otimes 1-p}$, we may assume that $E$ is semistable and $r \leq \frac{p-1}{2}$.

Now, by [SB] Corollary 2, we have the inequality
\[ (2) \quad \mu_{\max}(F^\ast E) - \mu_{\min}(F^\ast E) \leq (r - 1)(2g - 2), \]
where $\mu_{\max}(F^\ast E)$ (resp. $\mu_{\min}(F^\ast E)$) denotes the slope of the first (resp. last) graded piece of the Harder-Narasimhan filtration of $F^\ast E$. The inclusion $E \subset F_\ast L$ gives, by adjunction, a nonzero map $F^\ast E \to L$. Hence
\[ \text{deg } L \geq \mu_{\min}(F^\ast E) \geq \mu_{\max}(F^\ast E) - (r - 1)(2g - 2) \geq p\mu(E) - (r - 1)(2g - 2). \]
Combining this inequality with (1) and using Lemma 1.1, we obtain
\[
g - 1 + \frac{d}{p} = \mu(F_*L) \leq \mu(E) \leq \frac{g - 1 + d}{p} + \frac{(r - 1)(2g - 2)}{p},
\]
which simplifies to
\[
(g - 1) \leq (g - 1) \left(\frac{2r - 1}{p}\right).
\]
This is a contradiction, since we have assumed \(r \leq \frac{p - 1}{2}\) and therefore \(\frac{2r - 1}{p} < 1\). □

**Remark 1.3** We observe that the vector bundles \(F_*L\) are destabilized by Frobenius, because of the nonzero canonical map \(F^*F_*L \to L\) and clearly \(\mu(F^*F_*L) > \deg L\). For further properties of the bundles \(F_*L\), see [JRXY] section 5.

**Remark 1.4** In the context of Proposition 1.2 we mention the following open question: given a finite separable morphism between smooth curves \(f : Y \to X\) and a line bundle \(L \in \text{Pic}(Y)\), is the direct image \(f_*L\) stable? For a discussion, see [B].

§2 Existence of Frobenius-destabilized bundles.

Let the notation be as in the previous section. We recall the generalized Nagata-Segre theorem, proved by Hirschowitz, which says

**Theorem 2.1** For any vector bundle \(G\) of rank \(r\) and degree \(\delta\) over any smooth curve \(X\) and for any integer \(n, 1 \leq n \leq r - 1\), there exists a rank-\(n\) subbundle \(E \subset G\), satisfying
\[
\mu(E) \geq \mu(G) - \left(\frac{r - n}{r}\right)(g - 1) - \frac{\epsilon}{rn},
\]
where \(\epsilon\) is the unique integer with \(0 \leq \epsilon \leq r - 1\) and \(\epsilon + n(r - n)(g - 1) \equiv n\delta \mod r\).

**Remark 2.2** The previous theorem can be deduced (see [I] Remark 3.14) from the main theorem of [Hr] (for its proof, see http://math.unice.fr/~ah/math/Brill/).

**Proof of Theorem 1.** We apply Theorem 2.1 to the rank-\(p\) vector bundle \(F_*L\) on \(X_1\) and \(n = 2\), where \(L\) is a line bundle of degree \(g - 1 + d\) on \(X\), with \(d \equiv -2g + 2 \mod p\): There exists a rank-2 vector bundle \(E \subset F_*L\) such that
\[
\mu(E) \geq \mu(F_*L) - \frac{p - 2}{p}(g - 1).
\]
Note that our assumption on \(d\) was made to have \(\epsilon = 0\).

Now we will check that any \(E\) satisfying inequality (4) is stable with \(F^*E\) not semistable.

(i) \(E\) is stable: Let \(N\) be a line subbundle of \(E\). The inclusion \(N \subset F_*L\) gives, by adjunction, a nonzero map \(F^*N \to L\), which implies (see also [JRXY] Proposition 3.2(i))
\[
\deg N \leq \mu(F_*L) - \frac{p - 1}{p}(g - 1).
\]
Comparing with (4) we see that \(\deg N < \mu(E)\).
(ii) \(F^*E\) is not semistable. In fact, we claim that \(L\) destabilizes \(F^*E\). For the proof note that Lemma 1.1 implies
\[
\mu(F_*(L)) - \frac{p-2}{p}(g-1) = \frac{2g-2+d}{p} > \frac{g-1+d}{p} = \deg L
\]

since \(g \geq 2\). Together with (4) this gives \(\mu(E) > \deg L\) and hence
\[
\mu(F^*E) > \deg L.
\]
This implies the assertion, since by adjunction we obtain a nonzero map \(F^*E \to L\).

Replacing \(E\) by a subsheaf of suitable degree, we may assume that inequality (4) is an equality. In that case, because of our assumption on \(d\), \(\mu(E)\) is an integer, hence \(\deg E\) is even. In order to get trivial determinant, we may tensorize \(E\) with a suitable line bundle. This completes the proof of Theorem 1. \(\square\)

§3 Frobenius-destabilized bundles in genus 2.

From now on we assume that \(X\) is an ordinary curve of genus \(g = 2\) and the characteristic of \(k\) is \(p > 2\). Recall that \(\mathcal{M}_X\) denotes the moduli space of semistable rank-2 vector bundles with trivial determinant over \(X\) and \(\mathcal{B}\) the scheme-theoretical base locus of the rational map
\[
V : \mathcal{M}_X \cong \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k \cong \mathcal{M}_X,
\]
which is given by polynomials of degree \(p\).

First of all we will show that the 0-dimensional scheme \(\mathcal{B}\) is the disjoint union of subschemes \(\mathcal{B}_\theta\) indexed by theta characteristics of \(X\).

**Proposition 3.1**

(a) Let \(E\) be a vector bundle on \(X_1\) such that \(E \in \text{supp} \mathcal{B}\). Then there exists a unique theta characteristic \(\theta\) on \(X\), such that \(E\) is a subbundle of \(F_*(\theta^{-1})\).

(b) Let \(\theta\) be a theta characteristic on \(X\). Any rank-2 subbundle \(E \subset F_*(\theta^{-1})\) of degree 0 has the following properties

(i) \(E\) is stable and \(F^*E\) is not semistable,
(ii) \(F^*(\det E) = \mathcal{O}_X\),
(iii) \(\dim \text{Hom}(E, F_*(\theta^{-1})) = 1\) and \(\dim H^1(E^* \otimes F_*(\theta^{-1})) = 5\),
(iv) \(E\) is a rank-2 subbundle of maximal degree.

**Proof:** (a) By [LS] Corollary 2.6 we know that, for every \(E \in \text{supp} \mathcal{B}\) the bundle \(F^*E\) is the nonsplit extension of \(\theta^{-1}\) by \(\theta\), for some theta characteristic \(\theta\) on \(X\) (note that \(\text{Ext}^1(\theta^{-1}, \theta) \cong k\)). By adjunction we get a homomorphism \(\psi : E \to F_*(\theta^{-1})\) and we have to show that this is of maximal rank.

Suppose it is not, then there is a line bundle \(N\) on the curve \(X_1\) such that \(\psi\) factorizes as \(E \to N \to F_*(\theta^{-1})\). By stability of \(E\) we have \(\deg N > 0\). On the other hand, by adjunction, we get a nonzero homomorphism \(F^*N \to \theta^{-1}\) implying \(p \cdot \deg N \leq -1\), a contradiction. Hence \(\psi : E \to F_*(\theta^{-1})\) is injective. Moreover \(E\) is even a subbundle of \(F_*(\theta^{-1})\), since otherwise there exists a subbundle \(E' \subset F_*(\theta^{-1})\) with \(\deg E' > 0\) and which fits into the exact sequence
\[
0 \to E \to E' \xrightarrow{\pi} T \to 0,
\]
where $T$ is a torsion sheaf supported on an effective divisor. Varying $\pi$, we obtain a family of bundles $\ker \pi \subset E'$ of dimension $> 0$ and $\det \ker \pi = \mathcal{O}_{X_1}$. This would imply (see proof of Theorem 1) $\dim \mathcal{B} > 0$, a contradiction.

Finally, since $\theta$ is the maximal destabilizing line subbundle of $F^*E$, it is unique.

(b) We observe that inequality (4) holds for the pair $E \subset F_*(\theta^{-1})$. Hence, by the proof of Theorem 1, $E$ is stable and $F^*E$ is not semistable.

Let $\varphi : F^*E \to \theta^{-1}$ denote the homomorphism, adjoint to the inclusion $E \subset F_*(\theta^{-1})$. The homomorphism $\varphi$ is surjective, since otherwise $F^*E$ would contain a line subbundle of degree $> 1$, contradicting $\text{[LS]},$ Satz 2.4. Hence we get an exact sequence

$$0 \to \ker \varphi \to F^*E \to \theta^{-1} \to 0.$$  

On the other hand, let $N$ denote a line bundle on $X_1$ such that $E \otimes N$ has trivial determinant, i.e. $N^{-2} = \det E$. Applying $\text{[LS]}$ Corollary 2.6 to the bundle $F^*(E \otimes N)$ we get an exact sequence

$$0 \to \tilde{\theta} \otimes N^{-p} \to F^*E \to \tilde{\theta}^{-1} \otimes N^{-p} \to 0,$$

for some theta characteristic $\tilde{\theta}$. By uniqueness of the destabilizing subbundle of maximal degree of $F^*E$, this exact sequence must coincide with (6) up to a nonzero constant. This implies that $N^p \otimes \tilde{\theta} = \theta$, hence $N^{2p} = \mathcal{O}_X$. So we obtain that $\mathcal{O}_X = \det(F^*E) = F^*(\det E)$ proving (ii).

By adjunction we have the equality $\dim \text{Hom}(E, F_*(\theta^{-1})) = \dim \text{Hom}(F^*E, \theta^{-1}) = 1$. Moreover by Riemann-Roch we obtain $\dim H^1(E^* \otimes F_*(\theta^{-1})) = 5$. This proves (iii).

Finally, suppose that there exists a rank-2 subbundle $E' \subset F_*(\theta^{-1})$ with $\deg E' \geq 1$. Then we can consider the kernel $E = \ker \pi$ of a surjective morphism $\pi : E' \to T$ onto a torsion sheaf with length equal to $\deg E'$. By varying $\pi$ and after tensoring $\ker \pi$ with a suitable line bundle of degree $0$, we construct a family of dimension $> 0$ of stable rank-2 vector bundles with trivial determinant which are Frobenius-destabilized, contradicting $\dim \mathcal{B} = 0$. This proves (iv). \qed

It follows from Proposition 3.1 (a) that the scheme $\mathcal{B}$ decomposes as a disjoint union

$$\mathcal{B} = \bigsqcup_{\theta} \mathcal{B}_\theta,$$

where $\theta$ varies over the set of all theta characteristics of $X$ and

$$\text{supp } \mathcal{B}_\theta = \{ E \in \text{supp } \mathcal{B} \mid E \subset F_*(\theta^{-1}) \}.$$  

Tensor product with a 2-torsion point $\alpha \in JX_1[2] \cong JX[2]$ induces an isomorphism of $\mathcal{B}_\theta$ with $\mathcal{B}_{\theta \otimes \alpha}$ for every theta characteristic $\theta$. We denote by $l(\mathcal{B})$ and $l(\mathcal{B}_\theta)$ the length of the schemes $\mathcal{B}$ and $\mathcal{B}_\theta$. From the preceding we deduce the relations

$$l(\mathcal{B}) = 16 \cdot l(\mathcal{B}_\theta)$$

for every theta characteristic $\theta$.  

\[ (7) \]
\section*{§4 Grothendieck’s Quot-Scheme.}

Let \( \theta \) be a theta characteristic on \( X \). We consider the functor \( Q \) from the opposite category of \( k \)-schemes to the category of sets defined by

\[
Q(S) = \{ \sigma : \pi_{X_1}^*(F_*(\theta^{-1})) \to G \to 0 \mid G \text{ coherent over } X_1 \times S, \text{ flat over } S, \deg G|_{X_1 \times \{s\}} = \operatorname{rk} G|_{X_1 \times \{s\}} = p - 2, \forall s \in S \}/ \cong
\]

where \( \pi_{X_1} : X_1 \times S \to X_1 \) denotes the natural projection and \( \sigma \cong \sigma' \) for quotients \( \sigma \) and \( \sigma' \) if and only if there exists an isomorphism \( \lambda : G \to G' \) such that \( \sigma' = \lambda \circ \sigma \).

Grothendieck showed in \cite{G} (see also \cite{HL} section 2.2) that the functor \( Q \) is representable by a \( k \)-scheme, which we denote by \( Q \). A \( k \)-point of \( Q \) corresponds to a quotient \( \sigma : F_*(\theta^{-1}) \to G \), or equivalently to a rank-2 subsheaf \( E = \ker \sigma \subset F_*(\theta^{-1}) \) of degree 0 on \( X_1 \). By the same argument as in Proposition 3.1 (a), any subsheaf \( E \) of degree 0 is a subbundle of \( F_*(\theta^{-1}) \). Since by Proposition 3.1 (b) (iv) the bundle \( E \) has maximal degree as a subbundle of \( F_*(\theta^{-1}) \), any sheaf \( G \in Q(S) \) is locally free (see \cite{MuSa} or \cite{L} Lemma 3.8).

Hence taking the kernel of \( \sigma \) induces a bijection of \( Q(S) \) with the following set, which we also denote by \( Q(S) \)

\[
Q(S) = \{ E \hookrightarrow \pi_{X_1}^*(F_*(\theta^{-1})) \mid E \text{ locally free sheaf over } X_1 \times S \text{ of rank 2, } \pi_{X_1}^*(F_*(\theta^{-1}))/E \text{ locally free, } \deg E|_{X_1 \times \{s\}} = 0 \forall s \in S \}/ \cong
\]

By Proposition 3.1 (b) the scheme \( Q \) decomposes as a disjoint union

\[
Q = \coprod_{\eta} Q_{\eta},
\]

where \( \eta \) varies over the \( p \)-torsion points \( \eta \in J_{X_1[p]_{\text{red}}} = \ker(V : J_{X_1} \to JX) \). We also denote by \( V \) the Verschiebung of \( J_{X_1} \), i.e. \( V(L) = F^*L \), for \( L \in J_{X_1} \). The set-theoretical support of \( Q_{\eta} \) equals

\[
\operatorname{supp} Q_{\eta} = \{ E \in \operatorname{supp} Q \mid \det E = \eta \}.
\]

Because of the projection formula, tensor product with a \( p \)-torsion point \( \beta \in J_{X_1[p]_{\text{red}}} \) induces an isomorphism of \( Q_{\eta} \) with \( Q_{\eta \otimes \beta} \). So the scheme \( Q \) is a principal homogeneous space for the group \( J_{X_1[p]_{\text{red}}} \) and we have the relation

\[
(8) \quad l(Q) = p^2 \cdot l(Q_0),
\]

since \( X_1 \) is assumed to be ordinary. Moreover, by Proposition 3.1 we have the set-theoretical equality

\[
\operatorname{supp} Q_0 = \operatorname{supp} B_{\theta}.
\]

\textbf{Proposition 4.1}

(a) \( \dim Q = 0 \).

(b) The scheme \( Q \) is a local complete intersection at any \( k \)-point \( e = (E \subset F_*(\theta^{-1})) \in Q \).
Let \( \mathcal{N}_{X_1} \) denote the moduli space of semistable rank-2 vector bundles of degree 0 over \( X_1 \). We denote by \( \mathcal{N}_{X_1}^s \) and \( \mathcal{M}_{X_1}^s \) the open subschemes of \( \mathcal{N}_{X_1} \) and \( \mathcal{M}_{X_1} \) corresponding to stable vector bundles. Recall (see \cite{La1} Theorem 4.1) that \( \mathcal{N}_{X_1}^s \) and \( \mathcal{M}_{X_1}^s \) universally corepresent the functors (see e.g. \cite{HL} Definition 2.2.1) from the opposite category of \( k \)-schemes of finite type to the category of sets defined by

\[
\mathcal{N}_{X_1}^s(S) = \{ \mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank 2 } | \mathcal{E}_{|X_1 \times \{s\}} \text{ stable, } \\
\deg \mathcal{E}_{|X_1 \times \{s\}} = 0, \forall s \in S \}/ \sim,
\]

\[
\mathcal{M}_{X_1}^s(S) = \{ \mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank 2 } | \mathcal{E}_{|X_1 \times \{s\}} \text{ stable } \forall s \in S, \\
det \mathcal{E} = \pi_S^*M \text{ for some line bundle } M \text{ on } S \}/ \sim,
\]

where \( \pi_S : X_1 \times S \to S \) denotes the natural projection and \( \mathcal{E}' \sim \mathcal{E} \) if and only if there exists a line bundle \( L \) on \( S \) such that \( \mathcal{E}' \cong \mathcal{E} \otimes \pi_S^*L \). We denote by \( \langle \mathcal{E} \rangle \) the equivalence class of the vector bundle \( \mathcal{E} \) for the relation \( \sim \).

Consider the determinant morphism

\[
\det : \mathcal{N}_{X_1} \to JX_1, \quad [E] \mapsto \det E,
\]

and denote by \( \det^{-1}(0) \) the scheme-theoretical fibre over the trivial line bundle on \( X_1 \). Since \( \mathcal{N}_{X_1} \) universally corepresents the functor \( \mathcal{N}_{X_1}^s \), we have an isomorphism

\[
\mathcal{M}_{X_1}^s \cong \mathcal{N}_{X_1}^s \cap \det^{-1}(0).
\]

**Remark 4.2** If \( p > 0 \), it is not known whether the canonical morphism \( \mathcal{M}_{X_1} \to \det^{-1}(0) \) is an isomorphism (see e.g. \cite{La2} section 3).

In the sequel we need the following relative version of Proposition 3.1 (b)(ii). By a \( k \)-scheme we always mean a \( k \)-scheme of finite type.

**Lemma 4.3** Let \( S \) be a connected \( k \)-scheme and let \( \mathcal{E} \) be a locally free sheaf of rank-2 over \( X_1 \times S \) such that \( \deg \mathcal{E}_{|X_1 \times \{s\}} = 0 \) for all points \( s \) of \( S \). Suppose that \( \text{Hom}(\mathcal{E}, \pi_X^*(F_s(\theta^{-1})) \neq 0 \). Then we have the exact sequence

\[
0 \to \pi_X^*(\theta) \to (F \times \text{id}_S)^*\mathcal{E} \to \pi_X^*(\theta^{-1}) \to 0.
\]

In particular

\[
(F \times \text{id}_S)^*(\det \mathcal{E}) = \mathcal{O}_{X_1 \times S}.
\]

**Proof:** First we note that by flat base change for \( \pi_X : X_1 \times S \to X_1 \), we have an isomorphism \( \pi_X^*(F_s(\theta^{-1})) \cong (F \times \text{id}_S)_*(\pi_X^*(\theta^{-1})) \). Hence the nonzero morphism \( \mathcal{E} \to \pi_X^*(F_s(\theta^{-1})) \) gives via adjunction a nonzero morphism \( \varphi : (F \times \text{id}_S)^*\mathcal{E} \to \pi_X^*(\theta^{-1}) \).
We know by the proof of Proposition 3.1 (b) that the fibre $\varphi(x, s)$ over any closed point $(x, s) \in X \times S$ is a surjective $k$-linear map. Hence $\varphi$ is surjective by Nakayama and we have an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow (F \times \text{id}_S)^* \mathcal{E} \longrightarrow \pi_X^*(\theta^{-1}) \longrightarrow 0,$$

with $\mathcal{L}$ locally free sheaf of rank 1. By [K] section 5, the rank-2 vector bundle $(F \times \text{id}_S)^* \mathcal{E}$ is equipped with a canonical connection $\nabla : (F \times \text{id}_S)^* \mathcal{E} \longrightarrow (F \times \text{id}_S)^* \mathcal{E} \otimes \Omega^1_{X_S/S}$.

We note that $\Omega^1_{X_S/S} = \pi_X^*(\omega_X)$, where $\omega_X$ denotes the canonical line bundle of $X$. The first fundamental form of the connection $\nabla$ is an $\mathcal{O}_{X_S}$-linear homomorphism $\psi_\nabla : \mathcal{L} \longrightarrow \pi_X^*(\theta^{-1}) \otimes \pi_X^*(\omega_X) = \mathcal{O}_{X_S}$.

The restriction of $\psi_\nabla$ to the curve $X \times \{s\} \subset X \times S$ for any closed point $s \in S$ is an isomorphism (see e.g. proof of [LS] Corollary 2.6). Hence the fibre of $\psi_\nabla$ is a $k$-linear isomorphism over any closed point $(x, s) \in X \times S$. We conclude that $\psi_\nabla$ is an isomorphism, by Nakayama’s lemma and because $\mathcal{L}$ is a locally free sheaf of rank 1.

We obtain the second assertion of the lemma, since

$$(F \times \text{id}_S)^*(\det \mathcal{E}) = \det(F \times \text{id}_S)^* \mathcal{E} = \mathcal{L} \otimes \pi_X^*(\theta^{-1}) = \mathcal{O}_{X_S}.$$

\[\square\]

**Proposition 4.4** We assume $X$ ordinary.

(a) The forgetful morphism

$$i : Q \hookrightarrow \mathcal{N}_{X_1}^s, \quad e = (E \subset F_*(\theta^{-1})) \mapsto E$$

is a closed embedding.

(b) The restriction $i_0$ of $i$ to the subscheme $Q_0 \subset Q$ factors through $\mathcal{M}_{X_1}^s$, i.e. there is a closed embedding

$$i_0 : Q_0 \hookrightarrow \mathcal{M}_{X_1}^s.$$

**Proof:** (a) Let $e = (E \subset F_*(\theta^{-1}))$ be a $k$-point of $Q$. To show that $i$ is a closed embedding at $e \in Q$, it is enough to show that the differential $(di)_e : T_e Q \to T_{[E]} \mathcal{N}_X$ is injective. The Zariski tangent spaces identify with $\text{Hom}(E, G)$ and $\text{Ext}^1(E, E)$ respectively (see e.g. [HL] Proposition 2.2.7 and Corollary 4.5.2). Moreover, if we apply the functor $\text{Hom}(E, \cdot)$ to the exact sequence associated with $e \in Q$

$$0 \longrightarrow E \longrightarrow F_*(\theta^{-1}) \longrightarrow G \longrightarrow 0,$$

the coboundary map $\delta$ of the long exact sequence

$$0 \longrightarrow \text{Hom}(E, E) \longrightarrow \text{Hom}(E, F_*(\theta^{-1})) \longrightarrow \text{Hom}(E, G) \xrightarrow{\delta} \text{Ext}^1(E, E) \longrightarrow \cdots$$

identifies with the differential $(di)_e$. Now since the bundle $E$ is stable, we have $k \cong \text{Hom}(E, E)$. By Proposition 3.1 (b) we obtain that the map $\text{Hom}(E, E) \to \text{Hom}(E, F_*(\theta^{-1}))$ is an isomorphism. Thus $(di)_e$ is injective.

(b) We consider the composite map

$$\alpha : Q \xrightarrow{i} \mathcal{N}_{X_1}^s \xrightarrow{\det} JX_1 \xrightarrow{\nabla} JX,$$
where the last map is the isogeny given by the Verschiebung on \( JX_1 \), i.e. \( V(L) = F^* L \) for \( L \in JX_1 \). The morphism \( \alpha \) is induced by the natural transformation of functors \( \alpha : Q \to JX \), defined by
\[
Q(S) \to JX(S), \quad (E \mapsto \pi_X^*(F_s(\theta^{-1}))) \mapsto (F \times \text{id}_S)^*(\text{det } E).
\]
Using Lemma 4.3 this immediately implies that \( \alpha \) factors through the inclusion of the reduced point \( \{ O_x \} \to JX \). Hence the image of \( Q \) under the composite morphism \( \text{det} \circ i \) is contained in the kernel of the isogeny \( V \), which is the reduced scheme \( JX[p]_{\text{red}} \), since we have assumed \( X \) ordinary. Taking connected components we see that the image of \( Q_0 \) under \( \text{det} \circ i \) is the reduced point \( \{ O_{X_1} \} \to JX_1 \), which implies that \( i_0(Q_0) \) is contained in \( N_{X_1}^s \cap \text{det}^{-1}(0) \cong M_{X_1}^s \). \( \square \)

In order to compare the two schemes \( B_0 \) and \( Q_0 \) we need the following lemma.

**Lemma 4.5**

1. The closed subscheme \( B \subset M_{X_1}^s \) corepresents the functor \( B \) which associates to a \( k \)-scheme \( S \) the set
\[
B(S) = \{ E \text{ locally free sheaf over } X_1 \times S \text{ of rank 2 } | \text{ } E|_{X_1 \times \{ s \}} \text{ stable } \forall s \in S, \]
\[
0 \to L \to (F \times \text{id}_S)^*E \to M \to 0, \text{ for some locally free sheaves } L, M
\]
\[
\text{over } X \times S \text{ of rank 1, } \deg L|_{X \times \{ s \}} = -\deg M|_{X \times \{ s \}} = 1 \forall s \in S,
\]
\[
\text{det } E = \pi^*S M \text{ for some line bundle } M \text{ on } S \}/ \sim .
\]

2. The closed subscheme \( B_0 \subset M_{X_1}^s \) corepresents the subfunctor \( B_0 \) of \( B \) defined by \( \langle E \rangle \in B_{0}(S) \) if and only if the set-theoretical image of the classifying morphism of \( L \)
\[
\Phi_L : S \to \text{Pic}^1(X), \quad s \mapsto L|_{X \times \{ s \}},
\]
is the point \( \theta \in \text{Pic}^1(X) \).

**Proof:** We denote by \( M_{X_1} \) the algebraic stack parametrizing rank-2 vector bundles with trivial determinant over \( X_1 \). Let \( M_{X_1}^{ss} \) denote the open substack of \( M_{X_1} \) parametrizing semistable bundles and \( M_{X_1}^{unst} \) the closed substack of \( M_{X_1} \) parametrizing non-semistable bundles. We will use the following facts about the stack \( M_{X_1} \).

- The pull-back of \( O_{\mathbb{P}^3}(1) \) by the natural map \( M_{X_1}^{ss} \to M_{X_1} \cong \mathbb{P}^3 \) extends to a line bundle, which we denote by \( O(1) \), over the moduli stack \( M_{X_1} \) and \( \text{Pic}(M_{X_1}) = \mathbb{Z} \cdot O(1) \). Moreover there are natural isomorphisms \( H^0(M_{X_1}, O(n)) \cong H^0(M_{\mathbb{P}^3}, O_{\mathbb{P}^3}(n)) \) for any positive integer \( n \) (see [BL] Propositions 8.3 and 8.4).

- The closed subscheme \( M_{X_1}^{unst} \) is the base locus of the linear system \( |O(1)| \) over the stack \( M_{X_1} \). This is seen as follows: we deduce from [S] Theorem 6.2 that \( M_{X_1}^{unst} \) is the base locus of the linear system \( |O(n)| \) for some integer \( n \). Since \( |O(n)| \) is generated by symmetric products of \( n \) sections in \( |O(1)| \), we obtain that \( M_{X_1}^{unst} \) is the base locus of \( |O(1)| \).

We need to compute the fibre product functor \( \mathcal{B} = B \times_{M_{X_1}} M_{X_1}^{s} \). Let \( \mathcal{B} : M_{X_1} \to M_X \) denote the morphism of stacks induced by pull-back under the Frobenius map \( F : X \to X_1 \). Let \( S \) be a \( k \)-scheme and consider a vector bundle \( E \in M_{X_1}^s(S) \). Since the subscheme \( B \) is defined as base locus of the linear system \( V^*|O_{\mathbb{P}^3}(1)| \), we obtain that \( (E) \in B(S) \) if and only if \( E \) lies in the base locus of \( \mathcal{B}^*|O(1)| \) — here we use the isomorphism \( |O_{\mathbb{P}^3}(1)| \cong |O(1)| \), or
equivalently \( \mathfrak{V}(\mathcal{E}) = (F \times \text{id}_S)^* \mathcal{E} \) lies in the base locus of \(|\mathcal{O}(1)|\), which is the closed substack \( \mathcal{M}_{X_1}^{\text{unst}} \).

By [Sh] section 5 the substack \( \mathcal{M}_{X_1}^{\text{unst,1}} \) of \( \mathcal{M}_{X_1}^{\text{unst}} \) parametrizing non-semistable vector bundles having a maximal destabilizing line subbundle of degree 1 is an open substack of \( \mathcal{M}_{X_1}^{\text{unst}} \). By [LS] Corollary 2.6 the vector bundle \( \mathfrak{V}(\mathcal{E}) \) lies in \( \mathcal{M}_{X_1}^{\text{unst,1}}(S) \). We then consider the universal exact sequence defined by the Harder-Narasimhan filtration over \( \mathcal{M}_{X_1}^{\text{unst,1}} \):

\[
0 \to \mathcal{L} \to (F \times \text{id}_S)^* \mathcal{E} \to \mathcal{M} \to 0,
\]

with \( \mathcal{L} \) and \( \mathcal{M} \) locally free sheaves over \( X \times S \) such that \( \deg \mathcal{L}_{|X \times \{s\}} = - \deg \mathcal{M}_{|X \times \{s\}} = 1 \) for any \( s \in S \). This proves (1).

As for (2), we add the condition that the family \( \mathcal{E} \) is Frobenius-destabilized by the theta-characteristic \( \theta \).

\[\square\]

**Proposition 4.6** There is a scheme-theoretical equality

\[ \mathcal{B}_\theta = \mathcal{Q}_0 \]

as closed subschemes of \( \mathcal{M}_{X_1} \).

*Proof:* Since \( \mathcal{B}_\theta \) and \( \mathcal{Q}_0 \) corepresent the two functors \( \mathcal{B}_\theta \) and \( \mathcal{Q}_0 \), it will be enough to show that there is a canonical bijection between the set \( \mathcal{B}_\theta(S) \) and \( \mathcal{Q}_0(S) \) for any \( k \)-scheme \( S \). We recall that

\[
\mathcal{Q}_0(S) = \{ \mathcal{E} \hookrightarrow \pi_{X_1}^*(F_* (\theta^{-1})) \mid \text{\mathcal{E} locally free sheaf over } X_1 \times S \text{ of rank 2}, \pi_{X_1}^*(F_* (\theta^{-1}))/\mathcal{E} \text{ locally free}, \det \mathcal{E} \cong \mathcal{O}_{X_1 \times S} \}/\cong
\]

Note that the property \( \det \mathcal{E} \cong \mathcal{O}_{X_1 \times S} \) is implied as follows: by Proposition 4.4 (b) we have \( \det \mathcal{E} \cong \pi_{X}^* L \) for some line bundle \( L \) over \( S \) and by Lemma 4.3 we conclude that \( L = \mathcal{O}_S \).

First we show that the natural map \( \mathcal{Q}_0(S) \to \mathcal{M}_{X_1}^{\text{st}}(S) \) is injective. Suppose that there exist \( \mathcal{E}, \mathcal{E}' \in \mathcal{Q}_0(S) \) such that \( \langle \mathcal{E} \rangle = \langle \mathcal{E}' \rangle \), i.e. \( \mathcal{E}' \cong \mathcal{E} \otimes \pi_{S}^*(L) \) for some line bundle \( L \) on \( S \). Then by Lemma 4.3 we have two inclusions

\[
i : \pi_{X}^*(\theta) \to (F \times \text{id}_S)^* \mathcal{E}, \quad i' : \pi_{X}^*(\theta) \otimes \pi_{S}^*(L^{-1}) \to (F \times \text{id}_S)^* \mathcal{E}.
\]

Composing with the projection \( \sigma : (F \times \text{id}_S)^* \mathcal{E} \to \pi_{X}^*(\theta^{-1}) \) we see that the composite map \( \sigma \circ i' \) is identically zero. Hence \( \pi_{S}^*(L) = \mathcal{O}_{X_1 \times S} \).

Therefore the two sets \( \mathcal{Q}_0(S) \) and \( \mathcal{B}_\theta(S) \) are naturally subsets of \( \mathcal{M}_{X_1}^{\text{st}}(S) \).

We now show that \( \mathcal{Q}_0(S) \subset \mathcal{B}_\theta(S) \). Consider \( \mathcal{E} \in \mathcal{Q}_0(S) \). By Proposition 3.1 (b) the bundle \( \mathcal{E}_{|X_1 \times \{s\}} \) is stable for all \( s \in S \). By Lemma 4.3 we can take \( \mathcal{L} = \pi_{X}^*(\theta) \) and \( \mathcal{M} = \pi_{X}^*(\theta^{-1}) \), so that \( \langle \mathcal{E} \rangle \in \mathcal{B}_\theta(S) \).

Hence it remains to show that \( \mathcal{B}_\theta(S) \subset \mathcal{Q}_0(S) \). Consider a sheaf \( \mathcal{E} \) with \( \langle \mathcal{E} \rangle \in \mathcal{B}_\theta(S) \) — see Lemma 4.5 (2). As in the proof of Lemma 4.3 we consider the canonical connection \( \nabla \) on \( (F \times \text{id}_S)^* \mathcal{E} \). Its first fundamental form is an \( \mathcal{O}_{X \times S} \)-linear homomorphism

\[
\psi_\nabla : \mathcal{L} \to \mathcal{M} \otimes \pi_{X}^*(\omega_X),
\]
which is surjective on closed points \((x, s) \in X \times S\). Hence we can conclude that \(\psi_\tau\) is an isomorphism. Moreover taking the determinant, we obtain
\[
\mathcal{L} \otimes \mathcal{M} = \det\((F \times \text{id}_S)^*\mathcal{E} = \pi^*_S M.
\]
Combining both isomorphisms we deduce that
\[
\mathcal{L} \otimes \mathcal{L} = \pi^*_X(\omega_X) \otimes \pi^*_S M.
\]
Hence its classifying morphism \(\Phi_{\mathcal{L} \otimes \mathcal{L}} : S \to \text{Pic}^2(X)\) factorizes through the inclusion of the reduced point \(\{\omega_X\} \hookrightarrow \text{Pic}^2(X)\). Moreover the composite map of \(\Phi_L\) with the duplication map \([2]\)
\[
\Phi_{\mathcal{L} \otimes \mathcal{L}} : S \xrightarrow{\Phi_L} \text{Pic}^1(X) \xrightarrow{[2]} \text{Pic}^2(X)
\]
coinsides with \(\Phi_{\mathcal{L} \otimes \mathcal{L}}\). We deduce that \(\Phi_L\) factorizes through the inclusion of the reduced point \(\{0\} \hookrightarrow \text{Pic}^1(X)\), since the fibre \([2]^{-1}(\omega_X)\) is reduced, since \(p > 2\). Since \(\text{Pic}^1(X)\) is a fine moduli space, there exists a line bundle \(N\) over \(S\) such that
\[
\mathcal{L} = \pi^*_X(\theta) \otimes \pi^*_S(N).
\]
We introduce the vector bundle \(\mathcal{E}_0 = \mathcal{E} \otimes \pi^*_S(N^{-1})\). Then \(\langle \mathcal{E}_0 \rangle = \langle \mathcal{E} \rangle\) and we have an exact sequence
\[
0 \longrightarrow \pi^*_X(\theta) \longrightarrow (F \times \text{id}_S)^*\mathcal{E}_0 \xrightarrow{\sigma} \pi^*_X(\theta^{-1}) \longrightarrow 0,
\]
since \(\pi^*_S M = \pi^*_S N^2\). By adjunction the morphism \(\sigma\) gives a nonzero morphism
\[
j : \mathcal{E}_0 \longrightarrow (F \times \text{id}_S)_*(\pi^*_X(\theta^{-1})) \cong \pi^*_X(F_*(\theta^{-1})).
\]
We now show that \(j\) is injective. Suppose it is not. Then there exists a subsheaf \(\tilde{\mathcal{E}}_0 \subset \pi^*_X(F_*(\theta^{-1}))\) and a surjective map \(\tau : \mathcal{E}_0 \to \tilde{\mathcal{E}}_0\). Let \(\mathcal{K}\) denote the kernel of \(\tau\). Again by adjunction we obtain a map \(\alpha : (F \times \text{id}_S)^*\tilde{\mathcal{E}}_0 \to \pi^*_X(\theta^{-1})\) such that the composite map
\[
\sigma : (F \times \text{id}_S)^*\mathcal{E}_0 \xrightarrow{\tau^*} (F \times \text{id}_S)^*\tilde{\mathcal{E}}_0 \xrightarrow{\alpha} \pi^*_X(\theta^{-1})
\]
coinsides with \(\sigma\). Here \(\tau^*\) denotes the map \((F \times \text{id}_S)^*\tau\). Since \(\sigma\) is surjective, \(\alpha\) is also surjective. We denote by \(\mathcal{M}\) the kernel of \(\alpha\). The induced map \(\overline{\tau} : \pi^*_X(\theta) = \ker \sigma \to \mathcal{M}\) is surjective, because \(\tau^*\) is surjective. Moreover the first fundamental form of the canonical connection \(\nabla\) on \((F \times \text{id}_S)^*\mathcal{E}_0\) induces an \(\mathcal{O}_{X \times S}\)-linear homomorphism \(\psi_\nabla : \mathcal{M} \to \pi^*_X(\theta)\) and the composite map
\[
\psi_\nabla : \pi^*_X(\theta) \xrightarrow{\overline{\tau}} \mathcal{M} \xrightarrow{\psi} \pi^*_X(\theta)
\]
coinsides with the first fundamental form of \(\nabla\) of \((F \times \text{id}_S)^*\mathcal{E}_0\), which is an isomorphism. Therefore \(\overline{\tau}\) is an isomorphism too. So \(\tau^*\) is an isomorphism and \((F \times \text{id}_S)^*\mathcal{K} = 0\). We deduce that \(\mathcal{K} = 0\).

In order to show that \(\mathcal{E}_0 \in \mathcal{Q}_{\mathcal{O}_S}(S)\), it remains to verify that the quotient sheaf \(\pi^*_X(F_*(\theta^{-1}))/\mathcal{E}_0\) is flat over \(S\). We recall that flatness implies locally freeness because of maximality of degree. But flatness follows from [HL] Lemma 2.1.4, since the restriction of \(j\) to \(X_1 \times \{s\}\) is injective for any closed \(s \in S\) by Proposition 3.1 (a).

Combining this proposition with relations (7) and (8), we obtain

**Corollary 4.7** We have
\[
l(\mathcal{B}) = \frac{16}{p^2} \cdot l(\mathcal{Q}).
\]
§5 Determinantal subschemes.

In this section we introduce a determinantal subscheme $\mathcal{D} \subset N_{X_1}$, whose length will be computed in the next section. We also show that $\mathcal{D}$ is isomorphic to Grothendieck’s Quot-scheme $\mathcal{Q}$. We first define a determinantal subscheme $\tilde{\mathcal{D}}$ of a variety $JX_1 \times Z$ covering $N_{X_1}$ and then we show that $\tilde{\mathcal{D}}$ is a $\mathbb{P}^1$-fibration over an étale cover of $\mathcal{D} \subset N_{X_1}$.

Since there does not exist a universal bundle over $X_1 \times M_{X_1}$, following an idea of Mukai $[\text{Mu}]$, we consider the moduli space $M_{X_1}(x)$ of stable rank-2 vector bundles on $X_1$ with determinant $\mathcal{O}_{X_1}(x)$ for a fixed point $x \in X_1$. According to $[\text{Mu}]$ the variety $M_{X_1}(x)$ is a smooth intersection of two quadrics in $\mathbb{P}^5$. Let $\mathcal{U}$ denote a universal bundle on $X_1 \times M_{X_1}(x)$ and denote

$$U_z := \mathcal{U}|_{(x) \times M_{X_1}(x)}$$

considered as a rank-2 vector bundle on $M_{X_1}(x)$. Then the projectivized bundle

$$Z := \mathbb{P}(U_z)$$

is a $\mathbb{P}^1$-bundle over $M_{X_1}(x)$. The variety $Z$ parametrizes pairs $(F_z, l_z)$ consisting of a stable vector bundle $F_z \in M_{X_1}(x)$ and a linear form $l_z : F_z(x) \to k_x$ on the fibre of $F_z$ over $x$. Thus to any $z \in Z$ one can associate an exact sequence

$$0 \to E_z \to F_z \to k_x \to 0$$

uniquely determined up to a multiplicative constant. Clearly $E_z$ is semistable, since $F_z$ is stable, and $\det E_z = \mathcal{O}_{X_1}$. Hence we get a diagram (the so-called Hecke correspondence)

$$\begin{array}{ccc}
Z & \xrightarrow{\varphi} & M_{X_1} \cong \mathbb{P}^3 \\
\pi & & \\
M_{X_1}(x) & & \\
\end{array}$$

with $\varphi(z) = [E_z]$ and $\pi(z) = F_z$. We note that there is an isomorphism $\varphi^{-1}(E) \cong \mathbb{P}^1$ and that $\pi(\varphi^{-1}(E)) \subset M_{X_1}(x) \subset \mathbb{P}^5$ is a conic for any stable $E \in M_{X_1}^s$. On $X_1 \times Z$ there exists a “universal” bundle, which we denote by $\mathcal{V}$ (see $[\text{Mu}]$ (3.8)). It has the property

$$\mathcal{V}|_{X_1 \times \{z\}} \cong E_z, \quad \forall z \in Z.$$ 

Let $\mathcal{L}$ denote a Poincaré bundle on $X_1 \times JX_1$. By abuse of notation we also denote by $\mathcal{V}$ and $\mathcal{L}$ their pull-backs to $X_1 \times JX_1 \times Z$. We denote by $\pi_{X_1}$ and $q$ the canonical projections

$$X_1 \xleftarrow{\pi_{X_1}} X_1 \times JX_1 \times Z \xrightarrow{q} JX_1 \times Z.$$ 

We consider the map $m$ given by tensor product

$$m : JX_1 \times M_{X_1} \longrightarrow N_{X_1}, \quad (L, E) \longmapsto L \otimes E.$$ 

Note that the restriction of $m$ to the stable locus $m^s : JX_1 \times M_{X_1}^s \longrightarrow N_{X_1}^s$ is an étale map of degree 16. We denote by $\psi$ the composite map

$$\psi : JX_1 \times Z \xrightarrow{\mathrm{id}_J \times \varphi} JX_1 \times M_{X_1} \xrightarrow{m} N_{X_1}, \quad \psi(L, z) = L \otimes E_z.$$ 

Let $D \mid \omega_{X_1}$ be a smooth canonical divisor on $X_1$. We introduce the following sheaves over $JX_1 \times Z$

$$\mathcal{F}_1 = q_* (\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^*(F_0(\theta^{-1}) \otimes \omega_{X_1})) \quad \text{and} \quad \mathcal{F}_0 = \oplus_{y \in D} \left( \mathcal{L}^* \otimes \mathcal{V}_{(y)}^* \otimes JX_1 \times Z \right) \otimes k^\oplus.$$
The next proposition is an even degree analogue of [LN] Theorem 3.1.

**Proposition 5.1**

(a) The sheaves \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are locally free of rank \( 4p \) and \( 4p - 4 \) respectively and there is an exact sequence

\[
0 \to \mathcal{F}_1 \xrightarrow{\gamma} \mathcal{F}_0 \to R^1 q_* (\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* (F_*(\theta^{-1}))) \to 0.
\]

Let \( \tilde{D} \subset JX_1 \times Z \) denote the subscheme defined by the 4-th Fitting ideal of the sheaf \( R^1 q_* (\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* (F_*(\theta^{-1}))) \). We have set-theoretically

\[
\text{supp} \left( \mathcal{F}_0 - \mathcal{F}_1 \right) = \{ (L, z) \in JX_1 \times Z \mid \dim \text{Hom}(L \otimes E, F_*(\theta^{-1})) = 1 \},
\]

and \( \dim \tilde{D} = 1 \).

(b) Let \( \delta \) denote the \( l \)-adic \((l \neq p)\) cohomology class of \( \tilde{D} \) in \( JX_1 \times Z \). Then

\[
\delta = c_5(\mathcal{F}_0 - \mathcal{F}_1) \in H^{10}(JX_1 \times Z, \mathbb{Q}).
\]

**Proof:** We consider the canonical exact sequence over \( X_1 \times JX_1 \times Z \) associated to the effective divisor \( \pi_{X_1}^* D \)

\[
0 \to \mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* F_*(\theta^{-1}) \xrightarrow{\otimes D} \mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* (F_*(\theta^{-1}) \otimes \omega_{X_1}) \to \mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* (F_*(\theta^{-1}) \otimes \omega_{X_1} D) \otimes k^{\otimes p} \to 0.
\]

By Proposition 1.2 the rank-\( p \) vector bundle \( F_*(\theta^{-1}) \) is stable and since

\[
1 - \frac{2}{p} = \mu(F_*(\theta^{-1})) > \mu(L \otimes E) = 0 \quad \forall (L, E) \in JX_1 \times \mathcal{M}_{X_1},
\]

we obtain

\[
\dim H^1(L^* \otimes E^* \otimes F_*(\theta^{-1}) \otimes \omega_{X_1}) = \dim \text{Hom}(F_*(\theta^{-1}), L \otimes E) = 0.
\]

This implies

\[
R^1 q_* (\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* (F_*(\theta^{-1}) \otimes \omega_{X_1})) = 0.
\]

By the base change theorems the sheaf \( \mathcal{F}_1 \) is locally free. Taking direct images by \( q \) (note that \( q_* (\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* F_*(\theta^{-1})) = 0 \) because it is a torsion sheaf), we obtain the exact sequence

\[
0 \to \mathcal{F}_1 \xrightarrow{\gamma} \mathcal{F}_0 \to R^1 q_* (\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* (F_*(\theta^{-1}))) \to 0.
\]

with \( \mathcal{F}_1 \) and \( \mathcal{F}_0 \) as in the statement of the proposition. Note that by Riemann-Roch we have

\[
\text{rk } \mathcal{F}_1 = 4p - 4 \quad \text{and} \quad \text{rk } \mathcal{F}_0 = 4p.
\]

It follows from the proof of Proposition 3.1 (a) that any nonzero homomorphism \( L \otimes E \to F_*(\theta^{-1}) \) is injective. Moreover by Proposition 3.1 (b) (iii) for any subbundle \( L \otimes E \subset F_*(\theta^{-1}) \) we have \( \dim \text{Hom}(L \otimes E, F_*(\theta^{-1})) = 1 \), or equivalently \( \dim H^1(L^* \otimes E^* \otimes F_*(\theta^{-1})) = 5 \). Using the base change theorems we obtain the following series of equivalences

\[
(L, z) \in \text{supp } \tilde{D} \iff \text{rk } \gamma_{(L, z)} < 4p = \text{rk } \mathcal{F}_0
\]

\[
\iff \dim H^1(L^* \otimes E^* \otimes F_*(\theta^{-1})) \geq 5
\]

\[
\iff \dim \text{Hom}(L \otimes E, F_*(\theta^{-1})) \geq 1
\]

\[
\iff \dim \text{Hom}(L \otimes E, F_*(\theta^{-1})) = 1.
\]
Finally we clearly have the equality $\text{supp } \psi(\tilde{D}) = \text{supp } Q$. Since $\dim Q = 0$ and since the fibers of the morphism $\varphi$ over stable vector bundles are projective lines, we deduce that $\dim \tilde{D} = 1$. This proves part (a).

Part (b) follows from Porteous’ formula, which says that the fundamental class $\delta \in H^{10}(JX_1 \times Z, \mathbb{Z})$ of the determinantal subscheme $\tilde{D}$ is given (with the notation of [ACGH], p.86) by

$$
\delta = \Delta_{4p-(4p-5),4p-4-(4p-5)}(c_t(F_0 - F_1)) \\
= \Delta_{5,1}(c_t(F_0 - F_1)) \\
= c_5(F_0 - F_1).
$$

Lemma 5.2 There is a scheme-theoretical equality

$$\tilde{D} = \psi^{-1}D.$$

Proof: We consider a point $E \in \text{supp } D$ and denote by $Z_E$ the fibre $\psi^{-1}(\text{Spec } \mathcal{O}_E)$ and by $\psi_E : Z_E \rightarrow \text{Spec } \mathcal{O}_E$ the morphism obtained from $\psi$ after taking the base change $\text{Spec } \mathcal{O}_E \rightarrow N_{X_1}^s$. The lemma now follows because the formation of the Fitting ideal and taking the higher direct image $R^1\pi_{\text{Spec } \mathcal{O}_E*}$ commutes with the base change $\psi_E$ (see [E] Corollary 20.5 and [Ha] Proposition 12.5, i.e.

$$
\psi^{-1}_E \left[ \text{Fitt}_4(R^1\pi_{\text{Spec } \mathcal{O}_E*}(\mathcal{E}^* \otimes \pi_X^*F_*(\theta^{-1}))) \right] \cdot \mathcal{O}_{Z_E} = \text{Fitt}_4(R^1\pi_{Z_E*}((\text{id}_{X_1} \times \psi_E)^*\mathcal{E}^* \otimes \pi_{X_1}^*F_*(\theta^{-1}))) \\
\text{and } (\text{id}_{X_1} \times \psi_E)^* \mathcal{E} \sim \mathcal{L} \otimes V_{X_1 \times Z_E}.
$$

□
Lemma 5.3 The subscheme $\mathcal{D} \subset \mathcal{N}^s_{X_1}$ corepresents the functor which associates to any $k$-scheme $S$ the set
\[
\mathcal{D}(S) = \{ E \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2 \mid \deg E|_{X_1 \times \{s\}} = 0 \; \forall s \in S, \\
\text{Fitt}_4 [R^1 \pi_{S*}(E^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}))) = 0]/\sim \}
\]

Proof: This is an immediate consequence of the definition of $\mathcal{D}$ and the fact that $\mathcal{N}^s_{X_1}$ universally corepresents the functor $\mathcal{N}^s_{X_1}$. 

Lemma 5.4 Let $S$ be a $k$-scheme and $\mathcal{E}$ a sheaf over $X_1 \times S$ with $\langle \mathcal{E} \rangle \in \mathcal{N}^s_{X_1}(S)$. We suppose that the set-theoretical image of the classifying morphism of $\mathcal{E}$
\[
\Phi_{\mathcal{E}} : S \rightarrow \mathcal{N}^s_{X_1}, \quad s \mapsto \mathcal{E}|_{X_1 \times \{s\}}
\]
is a point. Then there exists an Artinian ring $A$, a morphism $\varphi : S \rightarrow S_0 := \text{Spec}(A)$ and a locally free sheaf $\mathcal{E}_0$ over $X_1 \times S_0$ such that

1. $\mathcal{E} \sim (\text{id}_{X_1} \times \varphi)^* \mathcal{E}_0$
2. the natural map $\mathcal{O}_{S_0} \rightarrow \varphi_* \mathcal{O}_S$ is injective.

Proof: Since the set-theoretical support of $\text{im} \; \Phi_{\mathcal{E}}$ is a point, there exists an Artinian ring $A$ such that $\Phi_{\mathcal{E}}$ factorizes through the inclusion $\text{Spec}(A) \hookrightarrow \mathcal{N}^s_{X_1}$. By the argument, which we already used in the definition of $\mathcal{D}$, there exists a universal bundle $\mathcal{E}_0$ over $X_1 \times \text{Spec}(A)$. So we have shown property (1). As for (2), we consider the ideal $I \subset A$ defined by $I = \ker(\mathcal{O}_{\text{Spec}(A)} \rightarrow \varphi_* \mathcal{O}_S)$, where $I$ denotes the associated $\mathcal{O}_{\text{Spec}(A)}$-module. If $I \neq 0$, we replace $A$ by $A/I$ and we are done. 

Proposition 5.5 There is a scheme-theoretical equality
\[
\mathcal{D} = \mathcal{Q}.
\]

Proof: We note that $\mathcal{D}(S)$ and $\mathcal{Q}(S)$ are subsets of $\mathcal{N}^s_{X_1}(S)$ (the injectivity of the map $\mathcal{Q}(S) \rightarrow \mathcal{N}^s_{X_1}(S)$ is proved similarly as in the proof of Proposition 4.5). Since $\mathcal{D}$ and $\mathcal{Q}$ corepresent the two functors $\mathcal{D}$ and $\mathcal{Q}$, it will be enough to show that the set $\mathcal{D}(S)$ coincides with $\mathcal{Q}(S)$ for any $k$-scheme.

We first show that $\mathcal{D}(S) \subset \mathcal{Q}(S)$. Consider a sheaf $\mathcal{E}$ with $\langle \mathcal{E} \rangle \in \mathcal{D}(S)$. For simplicity we denote the sheaf $\mathcal{E}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}))$ by $\mathcal{H}$. By [Ha] Theorem 12.11 there is an isomorphism
\[
R^1 \pi_{S*} \mathcal{H} \otimes k(s) \cong H^1(X_1 \times s, \mathcal{H}|_{X_1 \times \{s\}}) \; \forall s \in S.
\]
Since we have assumed $\text{Fitt}_4 [R^1 \pi_{S*} \mathcal{H}] = 0$, we obtain $\dim H^1(X_1 \times \{s\}, \mathcal{H}|_{X_1 \times \{s\}}) \geq 5$, or equivalently $\dim H^0(X_1 \times \{s\}, \mathcal{H}|_{X_1 \times \{s\}}) \geq 1$, i.e., the vector bundle $\mathcal{E}|_{X_1 \times \{s\}}$ is a subsheaf, hence subbundle, of $F_*(\theta^{-1})$. This implies that the set-theoretical image of the classifying map $\Phi_{\mathcal{E}}$ is contained in $\text{supp} \mathcal{Q}$. Taking connected components of $S$, we can assume that the image of $\Phi_{\mathcal{E}}$ is a point. Therefore we can apply Lemma 5.4: there exists a locally free sheaf $\mathcal{E}_0$ over $X_1 \times S_0$ such that $\mathcal{E} \sim (\text{id}_{X_1} \times \varphi)^* \mathcal{E}_0$. For simplicity we write $\mathcal{H}_0 = \mathcal{E}_0^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}))$. In particular $\mathcal{H} = (\text{id}_{X_1} \times \varphi)^* \mathcal{H}_0$. Since the projection $\pi_{S_0} : X_1 \times S_0 \rightarrow S_0$ is of relative dimension 1, taking the higher direct image $R^1 \pi_{S_0*} \mathcal{H}$ commutes with the (not necessarily flat) base change $\varphi : S \rightarrow S_0$ ([Ha] Proposition 12.5), i.e., there is an isomorphism
\[
\varphi^* R^1 \pi_{S_0*} \mathcal{H}_0 \cong R^1 \pi_{S*} \mathcal{H}.
\]
Since the formation of Fitting ideals also commutes with any base change (see [E] Corollary 20.5), we obtain

\[ \text{Fitt}_4[R^1\pi_*\mathcal{H}] = \text{Fitt}_4[R^1\pi_{S_0*}\mathcal{H}_0] \cdot \mathcal{O}_S. \]

Since \( \text{Fitt}_4[R^1\pi_*\mathcal{H}] = 0 \) and \( \mathcal{O}_{S_0} \to \varphi_*\mathcal{O}_S \) is injective, we deduce that \( \text{Fitt}_4[R^1\pi_{S_0*}\mathcal{H}_0] = 0 \). Since by Proposition 3.1 (b) (iii) \( \dim \mathcal{O}_{S_0} \otimes k(s_0) = 5 \) for the closed point \( s_0 \in S_0 \), we have \( \text{Fitt}_5[R^1\pi_{S_0*}\mathcal{H}_0] = \mathcal{O}_{S_0} \). We deduce by [E] Proposition 20.8 that the sheaf \( R^1\pi_{S_0*}\mathcal{H}_0 \) is a free \( A \)-module of rank 5. By [Ha] Theorem 12.11 (b) we deduce that there is an isomorphism

\[ \pi_{S_0*}\mathcal{H}_0 \otimes k(s_0) \cong H^0(X_1 \times s_0, \mathcal{H}_{|X_1 \times \{s_0\}}) \]

Again by Proposition 3.1 (b) (iii) we obtain \( \dim \pi_{S_0*}\mathcal{H}_0 \otimes k(s_0) = 1 \). In particular the \( \mathcal{O}_{S_0} \)-module \( \pi_{S_0*}\mathcal{H}_0 \) is not zero and therefore there exists a nonzero global section \( i_0 \in H^0(S_0, \pi_{S_0*}\mathcal{H}_0) = H^0(X_1 \times S_0, \mathcal{E}_0^* \otimes \pi_{X_1}^* F_*(\theta^{-1})) \). We pull-back \( i \) under the map \( \text{id}_{X_1} \times \varphi \) and we obtain a nonzero section

\[ j = (\text{id}_{X_1} \times \varphi)^* i \in H^0(X_1 \times S, \mathcal{E}^* \otimes \pi_{X_1}^* F_*(\theta^{-1})). \]

Now we apply Lemma 4.3 and we continue as in the proof of Proposition 4.6. This shows that \( \langle \mathcal{E} \rangle \in \mathcal{Q}(S) \).

We now show that \( \mathcal{Q}(S) \subset \mathcal{D}(S) \). Consider a sheaf \( \mathcal{E} \in \mathcal{Q}(S) \). The nonzero global section \( j \in H^0(X_1 \times S, \mathcal{H}) = H^0(S, \pi_{S*}\mathcal{H}) \) determines by evaluation at a point \( s \in S \) an element \( \alpha \in \pi_{S*}\mathcal{H} \otimes k(s) \). The image of \( \alpha \) under the natural map

\[ \varphi^0(s) : \pi_{S*}\mathcal{H} \otimes k(s) \rightarrow H^0(X_1 \times \{s\}, \mathcal{H}_{|X_1 \times \{s\}}) \]

coincides with \( j_{|X_1 \times \{s\}} \) which is nonzero. Moreover since \( \dim H^0(X_1 \times \{s\}, \mathcal{H}_{|X_1 \times \{s\}}) = 1 \), we obtain that \( \varphi^0(s) \) is surjective. Hence by [Ha] Theorem 12.11 the sheaf \( R^1\pi_{S*}\mathcal{H} \) is locally free of rank 5. Again by [E] Proposition 20.8 this is equivalent to \( \text{Fitt}_4[R^1\pi_{S*}\mathcal{H}] = 0 \) and \( \text{Fitt}_5[R^1\pi_{S*}\mathcal{H}] = \mathcal{O}_S \) and we are done. \( \square \)

§6 Chern class computations.

In this section we will compute the length of the determinantal subscheme \( \mathcal{D} \subset \mathcal{N}_{X_1} \) by evaluating the Chern class \( c_5(\mathcal{F}_0 - \mathcal{F}_1) \) — see Proposition 5.1 (b).

Let \( l \) be a prime number different from \( p \). We have to recall some properties of the cohomology ring \( H^*(X_1 \times JX_1 \times Z, \mathbb{Z}_l) \) (see also [LN]). In the sequel we identify all classes of \( H^*(X_1, \mathbb{Z}_l), H^*(JX_1, \mathbb{Z}_l) \) etc. with their preimages in \( H^*(X_1 \times JX_1 \times Z, \mathbb{Z}_l) \) under the natural pull-back maps.

Let \( \Theta \in H^2(JX_1, \mathbb{Z}_l) \) denote the class of the theta divisor in \( JX_1 \). Let \( f \) denote a positive generator of \( H^2(X_1, \mathbb{Z}_l) \). The cup product \( H^1(X_1, \mathbb{Z}_l) \times H^1(X_1, \mathbb{Z}_l) \to H^2(X_1, \mathbb{Z}_l) \simeq \mathbb{Z}_l \) gives a symplectic structure on \( H^1(X_1, \mathbb{Z}_l) \). Choose a symplectic basis \( e_1, e_2, e_3, e_4 \) of \( H^1(X_1, \mathbb{Z}_l) \) such that \( e_1 e_3 = e_2 e_4 = -f \) and all other products \( e_i e_j = 0 \). We can then normalize the Poincaré bundle \( \mathcal{L} \) on \( X_1 \times JX_1 \) so that

\[ (9) \quad c(\mathcal{L}) = 1 + \xi_1 \]

where \( \xi_1 \in H^1(X_1, \mathbb{Z}_l) \otimes H^1(JX_1, \mathbb{Z}_l) \subset H^2(X_1 \times JX_1, \mathbb{Z}_l) \) can be written as

\[ \xi_1 = \sum_{i=1}^4 e_i \otimes \varphi_i \]
with \( \varphi_i \in H^1(JX_1, \mathbb{Z}_l) \). Moreover, we have by the same reasoning, applying [ACGH] p.335 and p.21

\[
\xi_1^2 = -2\Theta f \quad \text{and} \quad \Theta^2[JX_1] = 2.
\]

Since the variety \( \mathcal{M}_{X_1}(x) \) is a smooth intersection of 2 quadrics in \( \mathbb{P}^5 \), one can work out that the \( l \)-adic cohomology groups \( H^i(\mathcal{M}_{X_1}(x), \mathbb{Z}_l) \) for \( i = 0, \ldots, 6 \) are (see e.g. [Re] p. 0.19)

\[
\mathbb{Z}_l, 0, \mathbb{Z}_l^4, \mathbb{Z}_l, 0, \mathbb{Z}_l.
\]

In particular \( H^2(\mathcal{M}_{X_1}(x), \mathbb{Z}_l) \) is free of rank 1 and, if \( \alpha \) denotes a positive generator of it, then

\[
\alpha^3[\mathcal{M}_{X_1}(x)] = 4.
\]

According to [N2] p. 338 and applying reduction mod \( p \) and a comparison theorem, the Chern classes of the universal bundle \( \mathcal{U} \) are of the form

\[
c_1(\mathcal{U}) = \alpha + f \quad \text{and} \quad c_2(\mathcal{U}) = \chi + \xi_2 + \alpha f
\]

with \( \chi \in H^4(\mathcal{M}_{X_1}(x), \mathbb{Z}_l) \) and \( \xi_2 \in H^1(\mathcal{X}_1, \mathbb{Z}_l) \otimes H^3(\mathcal{M}_{X_1}(x), \mathbb{Z}_l) \). As in [N2] and [KN] we write

\[
\beta = \alpha^2 - 4\chi \quad \text{and} \quad \xi_2^2 = \gamma f \quad \text{with} \quad \gamma \in H^6(\mathcal{M}_{X_1}(x), \mathbb{Z}_l).
\]

Define \( \Lambda \in H^1(JX_1, \mathbb{Z}_l) \otimes H^3(\mathcal{M}_{X_1}(x), \mathbb{Z}_l) \) by

\[
\xi_1 \xi_2 = \Lambda f.
\]

Then we have for dimensional reasons and noting that \( H^5(\mathcal{M}_{X_1}(x), \mathbb{Z}_l) = 0 \), that the following classes are all zero:

\[
f^2, \xi_1, \alpha^4, \xi_1 f, \xi_2 f, \alpha \xi_2, \alpha \Lambda, \Theta^2 \Lambda, \Theta^3.
\]

Finally, \( Z \) is the \( \mathbb{P}^1 \)-bundle associated to the vector bundle \( \mathcal{U}_x \) on \( \mathcal{M}_{X_1}(x) \). Let \( H \in H^2(Z, \mathbb{Z}_l) \) denote the first Chern class of the tautological line bundle on \( Z \). We have, using the definition of the Chern classes \( c_i(\mathcal{U}) \) and (11),

\[
H^2 = \alpha H - \frac{\alpha^2}{2}, \quad H^4 = 0, \quad \alpha^3 H[Z] = 4
\]

and we get for the “universal” bundle \( \mathcal{V} \),

\[
c_1(\mathcal{V}) = \alpha \quad \text{and} \quad c_2(\mathcal{V}) = \frac{\alpha^2}{2} + \xi_2 + Hf.
\]

**Lemma 6.1**

(a) The cohomology class \( \alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1) \in H^{12}(JX_1 \times Z, \mathbb{Z}_l) \) is a multiple of the class \( \alpha^3 H \Theta^2 \).

(b) The pull-back under the map \( \varphi : Z \to \mathcal{M}_{X_1} \cong \mathbb{P}^3 \) of the class of a point is the class \( H^3 = \frac{\alpha^2}{2} H - \frac{\alpha^2}{2} \).
Proof: For part (a) it is enough to note that all other relevant cohomology classes vanish, since $\alpha^4 = 0$ and $\alpha \Lambda = 0$.

As for part (b), it suffices to show that $c_1(\varphi^*O_{P^3}(1)) = H$. The line bundle $O_{P^3}(1)$ is the inverse of the determinant line bundle \([KM]\) over the moduli space $M_{X_1}$. Since the formation of the determinant line bundle commutes with any base change (see \([KM]\)), the pull-back $\varphi^*O_{P^3}(1)$ is the inverse of the determinant line bundle associated to the family $\mathcal{V} \otimes \pi^*_1N$ for any line bundle $N$ of degree 1 over $X_1$. Hence the first Chern class of $\varphi^*O_{P^3}(1)$ can be computed by the Grothendieck-Riemann-Roch theorem applied to the sheaf $\mathcal{V} \otimes \pi^*_1N$ over $X_1 \times Z$ and the morphism $\pi_Z: X_1 \times Z \rightarrow Z$. We have

\[
ch(\mathcal{V} \otimes \pi^*_1N) \cdot \pi^*_1 td(X_1) = (2 + \alpha + (-\xi_2 - Hf) + \text{h.o.t.})(1 + f)(1 - f) = 2 + \alpha + (-\xi_2 - Hf) + \text{h.o.t.},
\]

and therefore G-R-R implies that $c_1(\varphi^*O_{P^3}(1)) = H$ — note that $\pi^*_Z(\xi_2) = 0$. \hfill \Box

**Proposition 6.2** We have

$$l(D) = \frac{1}{24} p^3 (p^2 - 1).$$

Proof: Let $\lambda$ denote the length of the subscheme $m^{-1}(\mathcal{D}) \subset JX_1 \times M_{X_1}$. Since the map $m^s$ is étale of degree 16, we obviously have the relation $\lambda = 16 \cdot l(\mathcal{D})$. According to Lemma 6.1 (b) we have in $H^{10}(JX_1 \times Z, \mathbb{Z}^l)$

$$[(\text{id} \times \varphi)^{-1}(pt)] = H^3 \cdot \frac{\Theta^2}{2} = \frac{1}{4} \alpha^2H\Theta^2 - \frac{1}{4} \alpha^3 \Theta^2,$$

where $pt$ denotes the class of a point in $JX_1 \times M_{X_1}$. Using Lemma 5.2 we obtain that the class $\delta = c_5(\mathcal{F}_0 - \mathcal{F}_1) \in H^{10}(JX_1 \times Z, \mathbb{Z}^l)$ equals $\lambda \cdot \left(\frac{1}{4} \alpha^2H\Theta^2 - \frac{1}{4} \alpha^3 \Theta^2\right)$. Intersecting with $\alpha$ we obtain with Lemma 6.1 (a) and (10)

(19)

$$\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1) = \frac{\lambda}{4} \alpha^3 H \Theta^2.$$

So we have to compute the class $\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1)$. By (9) and (10),

$$ch(\mathcal{L}) = 1 + \xi_1 - \Theta f$$

whereas by (14), (16) and (18),

$$ch(\mathcal{V}) = 2 + \alpha + (-\xi_2 - Hf) + \frac{1}{12}(-\alpha^3 - 6\alpha Hf) + \frac{1}{12}(\alpha^3 f - \alpha^2 Hf).$$

Moreover

$$ch(\pi^*_1(F_*(\theta^{-1}) \otimes \omega_{X_1})) \cdot \pi^*_1 td(X_1) = p + (2p - 2)f.$$
So using (14), (15) and (16),
\[
ch(V^* \otimes L^* \otimes \pi_X^*(F_*(\theta^{-1}) \otimes \omega_X)) \cdot \pi_X^*td(X_1) = 2p + [(4p - 4)f - p\alpha - 2p\xi_1] \\
+ [p\alpha\xi_1 - 2p\Theta f - (2p - 2)\alpha f - p\xi_2 - pHf] \\
+ \left[\frac{p}{12}\alpha^3 + \frac{p}{2}\alpha Hf + p\Lambda + p\Theta f\right] \\
+ \left[\frac{3p - 2}{12}\alpha^3 f - \frac{p}{12}\alpha^3 \xi_1 - \frac{p}{12}\alpha^2 Hf\right] + \left[-\frac{p}{12}\alpha^3 f\right].
\]

Hence by Grothendieck-Riemann-Roch for the morphism \(q\) we get
\[
ch(F_1) = 4p - 4 + [- (2p - 2)\alpha - 2p\Theta - pH] + \left[\frac{p}{12}\alpha H + p\Lambda + p\Theta\right] \\
+ \left[\frac{3p - 2}{12}\alpha^3 - \frac{p}{12}\alpha^2 H\right] + \left[-\frac{p}{12}\alpha^3\Theta\right].
\]

From (10) and (18) we easily obtain
\[
ch(F_0) = 4p - 2p\alpha + \frac{p}{6}\alpha^3.
\]
So
\[
ch(F_0 - F_1) = 4 + [2p\Theta - 2\alpha + pH] + \left[-\frac{p}{2}\alpha H - p\Lambda - p\alpha\Theta\right] \\
+ \left[-\frac{p + 1}{12}\alpha^3 + \frac{p}{12}\alpha^2 H\right] + \left[\frac{p}{12}\alpha^3\Theta\right].
\]

Defining \(p_n := n! \cdot ch_n(F_0 - F_1)\) we have according to Newton’s recursive formula ([F] p.56),
\[
c_5(F_0 - F_1) = \frac{1}{5} \left( p_5 - \frac{5}{6}p_2p_3 - \frac{5}{4}p_1p_4 + \frac{5}{6}p_1^2p_3 + \frac{5}{8}p_1p_2^2 - \frac{5}{12}p_1p_2 + \frac{1}{24}p_5 \right)
\]
with
\[
p_1 = 2p\Theta - 2\alpha + pH \\
p_2 = -p(\alpha H + 2\Lambda + 2\alpha\Theta) \\
p_3 = \frac{1}{2}(-(p + 1)\alpha^3 + p\alpha^2 H) \\
p_4 = 2p\alpha^3\Theta \\
p_5 = 0.
\]

Now an immediate computation using (16) and (17) gives
\[
\alpha \cdot c_5(F_0 - F_1) = \frac{p^3(p^2 - 1)}{6} \alpha^3 H\Theta^2.
\]

We conclude from (19) that \(\lambda = \frac{2}{3}p^3(p^2 - 1)\) and we are done. \(\square\)

**Remark 6.3** If \(k = \mathbb{C}\), the number of maximal subbundles of a general vector bundle has recently been computed by Y. Holla by using Gromov-Witten invariants [Ho]. His formula ([Ho Corollary 4.6]) coincides with ours.
§7 Proof of Theorem 2.

The proof of Theorem 2 is now straightforward. It suffices to combine Corollary 4.7, Proposition 5.5 and Proposition 6.2 to obtain the length $l(B)$.

The fact that $B$ is a local complete intersection follows from the isomorphism $B_0 = Q_0$ (Proposition 4.6) and Proposition 4.1. □

§8 Questions and Remarks.

(1) Is the rank-$p$ vector bundle $F^*L$ very stable, i.e. $F^*L$ has no nilpotent $\omega_X$-valued endomorphisms, for a general line bundle?

(2) Is $F^*(\theta^{-1})$ very stable for a general curve $X$? Note that very-stability of $F^*(\theta^{-1})$ implies reducedness of $B$ (see e.g. [LN] Lemma 3.3).

(3) If $g = 2$, we have shown that for a general stable $E \in \mathcal{M}_X$ the fibre $V^{-1}(E)$ consists of $\frac{1}{3}p(p^2 + 2)$ stable vector bundles $E_1 \in \mathcal{M}_{X_1}$, i.e. bundles $E_1$ such that $F^*E_1 \cong E$ or equivalently (via adjunction) $E_1 \subset F^*E$. The Quot-scheme parametrizing rank-2 subbundles of degree 0 of the rank-2 vector bundle $F^*E$ has expected dimension 0, contains the fibre $V^{-1}(E)$, but it also has a 1-dimensional component arising from Frobenius-destabilized bundles.

(4) If $p = 3$ the base locus $B$ consists of 16 reduced points, which correspond to the 16 nodes of the Kummer surface associated to $JX$ (see [LP2] Corollary 6.6). For general $p$, does the configuration of points determined by $B$ have some geometric significance?

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