FROM HÖRMANDER’S $L^2$-ESTIMATES TO PARTIAL POSITIVITY

TAKAHIRO INAYAMA

Abstract. In this article, using a twisted version of Hörmander’s $L^2$-estimate, we give new characterizations of notions of partial positivity, which are uniform $q$-positivity and RC-positivity. We also discuss the definition of uniform $q$-positivity for singular Hermitian metrics.

1. Introduction

In this article, we give a new characterization of partial positivity, which is called uniform $q$-positivity (cf. Definition 2.1) via Hörmander’s $L^2$-estimate. The statement is the following.

Theorem 1.1. Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $\omega = \sqrt{-1} \partial \bar{\partial} |z|^2$ be the standard Kähler metric on $D$, and $L \rightarrow D$ be a line bundle over $D$. For a smooth Hermitian metric $h$ on $L$ and a non-negative constant $c \geq 0$ on $D$, the following properties are equivalent for $1 \leq q \leq n$:

1. The summation of any distinct $q$ eigenvalues (counting multiplicity) of the Chern curvature $\sqrt{-1} \Theta_{(L,h)}$ of $(L,h)$ with respect to $\omega$ is greater than or equal to $c$.

2. For any smooth strictly plurisubharmonic function $\psi$ and any smooth $\bar{\partial}$-closed $L$-valued $(n,q)$-form $f$ with compact support, there exists $L$-valued $(n,q-1)$-form $u$ satisfying $\bar{\partial} u = f$ and

$$\int_D |u|_{(\omega,h)}^2 e^{-\psi} dV_{\omega} \leq \int_D \langle ([\sqrt{-1} \partial \bar{\partial} \psi, \Lambda_{\omega}] + c)^{-1} f, f \rangle_{(\omega,h)} e^{-\psi} dV_{\omega}.$$

The condition (2) in Theorem 1.1 allows us to add a weight $\psi$. Taking an arbitrary weight, we can estimate the curvature $\sqrt{-1} \Theta_{(L,h)}$. This type of condition was firstly introduced in [HI20], which was named as the twisted Hörmander condition. After that, in [DNW19] and [DNWZ20], Deng et al. generalized this notion and introduced the optimal $L^p$-estimate condition, which corresponded to the particular case of the twisted Hörmander condition when $p = 2$. These studies provide new characterizations of positivity based on the Hörmander-type condition, which was initially observed by Berndtsson in [Ber98]. Theorem 1.1 is a generalization for partial positivity of the result obtained by the authors in [DNW19].

2010 Mathematics Subject Classification. 32U05.

Key words and phrases. $L^2$-estimates, $q$-positivity, RC-positivity.
As a higher-rank analogue, we also establish a characterization of RC-positivity. RC-positivity is a partial positivity notion introduced by Yang in [Yan18] and a higher-rank analogue of \((\dim X - 1)\)-positivity. We use the same notation as in Theorem 1.1.

**Theorem 1.2.** Let \(E \rightarrow D\) be a vector bundle and \(h\) be a smooth Hermitian metric on \(E\). Assume that if \(\psi\) is a smooth strictly plurisubharmonic function on \(D\) and \(f\) is a smooth \(E\)-valued \((n,n)\)-form with compact support, there exists a solution of \(\bar{\partial}u = f\) satisfying

\[
\int_D |u|_{(\omega,h)}^2 e^{-\psi} dV_\omega \leq \int_D \langle ([\sqrt{\nabla^2 \psi} \otimes \text{Id}_E, A_\omega] + c)^{-1} f, f \rangle_{(\omega,h)} e^{-\psi} dV_\omega.
\]

Then we obtain

\[
\text{tr}_0 (\sqrt{-1} \Theta_{(E,h)} a, a)_h(x) \geq c |a|_{h(x)}^2
\]

for any point \(x \in D\) and any element \(a \in E_x\). Especially, \((E, h)\) is RC-positive if \(c > 0\) and RC-semi-positive if \(c = 0\).

As an application of the characterization in Theorem 1.1, we prove that uniform \(q\)-positivity is preserved with respect to the decreasing sequence (Theorem 4.1). This property is well-known in the case that \(q = 0\), that is, it is a sequence of plurisubharmonic functions. We also propose the definition of uniform \(q\)-positivity for singular Hermitian metrics (Definition 4.2).

As a further study, we propose the following problem, which generalizes the Prékopa-Berndtsson theorem ([Ber98, Theorem 1.3]).

**Problem 1.3.** Let \(U\) be a bounded domain in \(\mathbb{C}^n_z\) and \(D\) be a bounded pseudoconvex domain in \(\mathbb{C}^m_w\). Let \(\varphi\) be a smooth function on \(\overline{U}_z \times \overline{D}_w \subset \mathbb{C}^n_z \times \mathbb{C}^m_w\). We set \(\omega_0, \omega_1\) and \(\omega_2\) be the standard Kähler metrics on \(U \subset \mathbb{C}^n_z\), \(D \subset \mathbb{C}^m_w\) and \(U \times D \subset \mathbb{C}^n_z \times \mathbb{C}^m_w\), respectively. Assume that

1. \(D\) is a connected Reinhardt domain and \(\varphi(z, w_1, \ldots, w_m)\) is independent of \(\arg(w_j)\) for \(1 \leq j \leq m\).
2. The summation of any distinct \(q\) eigenvalues of \(\sqrt{-1} \partial \bar{\partial} \varphi\) with respect to \(\omega_2\) is greater than or equal to \(c\), where \(c \geq 0\) is a non-negative constant and \(1 \leq q \leq n\).

We define the function \(\tilde{\varphi}\) on \(U\) by

\[
e^{-\tilde{\varphi}(z)} := \int_{w \in D} e^{-\varphi(z,w)} d\omega_1(w).
\]

Then the summation of any distinct \(q\) eigenvalues of \(\sqrt{-1} \partial \bar{\partial} \tilde{\varphi}\) with respect to \(\omega_0\) is greater than or equal to \(c\).

We immediately see that Problem 1.3 is true in the case that \(q = 1\), which corresponds to the Prékopa-Berndtsson theorem. The proof of the Prékopa-Berndtsson theorem is based on Hörmander’s \(L^2\)-estimates, and on a partial converse of these estimates in one variable. In Section 5, we explain the reason why Theorem 1.1 can be applied to Problem 1.3.
The organization of this paper is as follows. In Section 2, we introduce definitions of $q$-positivity, uniform $q$-positivity and RC-positivity. We also explain the result of Hörmander’s $L^2$-estimate which we use in this article. In Section 3, we characterize partial positivity by using the Hörmander $L^2$-estimate. We also show the proofs of Theorem 1.1 and 1.2. In Section 4, we show applications of the main theorems. We also discuss a definition of uniform $q$-positivity for singular Hermitian metrics. In Section 5, we propose some problems.

Acknowledgment. The author would like to thank his supervisor Prof. Shigeharu Takayama for enormous supports. The author specially wishes to express his gratitude to Dr. Wang Xu for his pointing out mistakes in the first version of the manuscript. He would also be grateful to Prof. Shin-ichi Matsumura for helpful comments. This work is supported by the Program for Leading Graduate Schools, MEXT, Japan. This work is also supported by JSPS KAKENHI Grant Number 18J22119.

2. Preliminaries

Notation.

- $dV_\omega := \frac{\omega^n}{n!}$: the volume form determined by $\omega$.
- $C^k_{p,q}(X,E) := C^k(X,\wedge^p\Omega^q_X \otimes E)$ for $0 \leq k \leq +\infty$.
- $\mathcal{D}_{p,q}(X,E)$: the space of smooth sections of $\wedge^{p,q}T^*_X \otimes E$ with compact support.
- $L^2_{p,q}(X,E;\omega,h)$: the space of $L^2$ sections of $\wedge^{p,q}T^*_X \otimes E$ with respect to $\omega$ and $h$.
- $\langle \langle \alpha,\beta \rangle \rangle_{(\omega,h)} := \int_X \langle \alpha,\beta \rangle_{(\omega,h)}dV_\omega$.
- $\|\alpha\|_{(\omega,h)}^2 := \langle \langle \alpha,\alpha \rangle \rangle_{(\omega,h)}$.
- $D^*_\psi$ : the adjoint operator of $D_\psi$ with respect to $\langle \langle \cdot,\cdot \rangle \rangle_{(\omega,he^{-\psi})}$.
- $\overline{\partial}_\psi$ : the adjoint operator of $\overline{\partial}$ with respect to $\langle \langle \cdot,\cdot \rangle \rangle_{(\omega,he^{-\psi})}$.
- $\Delta_\psi := D^*_\psi D_\psi + D_\psi^* D_\psi$, $\Delta'^{\omega} = \overline{\partial}_\psi^* \overline{\partial}_\psi + \overline{\partial}_\psi \overline{\partial}_\psi$ with respect to $\langle \langle \cdot,\cdot \rangle \rangle_{(\omega,he^{-\psi})}$.
- $L_\omega$ : the operator defined by $\omega \wedge \cdot$.
- $\Lambda_\omega$ : the adjoint operator of $L_\omega$.
- $[\cdot,\cdot]$ : the graded Lie bracket.
- $\mathbb{B}^n_r := \{(z_1,\cdots,z_n) \in \mathbb{C}^n | \sum_{i=1}^n |z_i|^2 < r^2\}$.

In [AG62], Andreotti and Grauert introduced partial positivity notions and studied partially vanishing cohomology. Here, we introduce the notions of $q$-positivity and uniform $q$-positivity for smooth Hermitian metrics on line bundles.

Definition 2.1. (cf. [AG62], [Yan19, Definition 2.1]) Let $L \rightarrow X$ be a holomorphic line bundle over a complex manifold $X$ with $\dim X = n$. Let $h$ be a smooth Hermitian metric on $L$. We say that
(1) \((L, h)\) is \(q\)-(semi-)positive if the Chern curvature \(\sqrt{-1}\Theta_{(L, h)}\) has at least \((n-q)\) (semi-)positive eigenvalues at any point on \(X\). We also say that \(L\) is \(q\)-(semi-)positive if there exists a smooth Hermitian metric \(h\) on \(L\) such that \((L, h)\) is \(q\)-(semi-)positive.

(2) \((L, h)\) is uniformly \(q\)-(semi-)positive if there exists a smooth Hermitian metric \(\omega\) such that the summation of any distinct \((q+1)\) eigenvalues of the Chern curvature \(\sqrt{-1}\Theta_{(L, h)}\) with respect to \(\omega\) is (semi-)positive at any point on \(X\). We also say that \(L\) is uniformly \(q\)-(semi-)positive if there exist a smooth Hermitian metric \(h\) on \(L\) and a smooth Hermitian metric \(\omega\) such that \((L, h)\) is uniformly \(q\)-(semi-)positive with respect to \(\omega\).

A simple computation yields that uniform \(q\)-(semi-)positivity implies \(q\)-(semi-)positivity. Note that usual (semi-)positivity corresponds to \(0\)-(semi-)positivity. Conversely, it is known that the above two positivity notions are equivalent over a compact complex manifold.

**Proposition 2.2.** ([Yan19, Proposition 2.2]) Let \(X\) be a compact complex manifold and \(L\) be a \(q\)-positive line bundle. Then \(L\) is a uniformly \(q\)-positive line bundle.

Next, we also give definitions of RC-positivity and weak RC-positivity, which were introduced by Yang in [Yan18].

**Definition 2.3.** ([Yan18, Definition 3.3]) A Hermitian holomorphic vector bundle \((E, h)\) over a complex manifold \(X\) is called RC-positive (resp. RC-negative) if at any point \(x \in X\) and for any non-zero element \(a \in E_x\), there exists a vector \(v \in T_xX\) such that

\[
(\sqrt{-1}\Theta_{(E, h)}(v, v) a, a)_h > 0 \quad \text{(resp. } < 0).\]

We also call \((E, h)\) weakly RC-positive if there exists a smooth Hermitian metric \(h\) on the tautological line bundle \(\mathcal{O}_E(1)\) over \(\mathbb{P}(E^*)\) such that \((\mathcal{O}_E(1), h)\) is \((\dim X - 1)\)-positive.

Note that Griffiths positivity implies RC-positivity. Moreover, if \(\dim X = 1\), RC-positivity is equivalent to Griffiths positivity. If \(\text{rank} E = 1\), RC-positivity is the same concept as \((\dim X - 1)\)-positivity.

Finally, we mention the following result, which was initially obtained by Hörmander [Hör65]. Hörmander’s \(L^2\)-estimate is fundamental and important in several complex variables. In our paper, we use this \(L^2\)-estimate to characterize several notions of partial positivity. Here, we adopt the following form.

**Theorem 2.4.** (cf. [Dem82], [Dem, Theorem (5.1)] and [Dem-book, Theorem 6.1 in Chapter VIII]) Let \(X, \bar{\omega}\) be a complete Kähler manifold, \(\omega\) be another Kähler metric which is not necessarily complete, and \((E, h) \to X\) be a holomorphic line bundle. We also let \(A_{(\omega, h)} = [\sqrt{-1}\Theta_{(E, h)}, \Lambda_{\omega}]\) be the curvature operator in bidegree \((n, q)\) for \(q \geq 1\). Assume that \(A_{(\omega, h)}\) is positive definite everywhere on \(\wedge^{n,q}T^*X \otimes E\). Then for any \(\bar{\partial}\)-closed \(f \in\)
there exists \( u \in L^2_{(n,q-1)}(X,E;\omega,h) \) such that \( \partial u = f \) and
\[
\int_X |u|^2_{(\omega,h)}dV_\omega \leq \int_X \langle A^{-1}_{(\omega,h)}f,f \rangle_{(\omega,h)}dV_\omega,
\]
where we assume that the right-hand side is finite.

3. A characterization of partial positivity via \( L^2 \)-estimates

3.1. Uniform \( q \)-positivity. In this subsection, we discuss a characterization of uniform \( q \)-positivity in terms of \( L^2 \)-estimates. Before proving Theorem 1.1 we prepare the following lemma. The proof is a simple computation.

**Lemma 3.1.** (cf. [Dem] (4.10) and [Dem-book] Proposition (5.8) in Chapter VI) Let the notation be the same as in Theorem 1.1. We also let \( f \) be any \( \partial \)-closed \( L \)-valued \((n,q)\)-form. At a fixed point \( p \in X \), we take a coordinate \((z_1, \cdots, z_n)\) around \( p \) such that
\[
\omega = \sqrt{-1} \sum_{j=1}^n dz_j \wedge d \bar{z}_j,
\]
\[
\sqrt{-1} \Theta_{(L,h)} = \sqrt{-1} \sum_{j=1}^n \gamma_j dz_j \wedge d \bar{z}_j.
\]
We write
\[
f = \sum_{1 \leq i_1 < \cdots < i_q \leq n} f_{i_1 \cdots i_q} dz_1 \wedge \cdots \wedge d z_n \wedge d \bar{z}_{i_1} \wedge \cdots \wedge d \bar{z}_{i_q} \otimes e_L
\]
for a local holomorphic frame \( e_L \) of \( L \) around \( p \). Then we get
\[
[\sqrt{-1} \Theta_{(L,h)}; \Lambda_\omega] f = \sum_{1 \leq i_1 < \cdots < i_q \leq n} (\sum_{k=1}^q \gamma_{i_k}) f_{i_1 \cdots i_q} dz_1 \wedge \cdots \wedge d z_n \wedge d \bar{z}_{i_1} \wedge \cdots \wedge d \bar{z}_{i_q} \otimes e_L.
\]

Then we give a proof of Theorem 1.1. The idea for the proof is based on the arguments in [DNW19] Theorem 2.1 and [DNWZ20] Theorem 3.1.

**Proof of Theorem 1.1** (1) \( \Rightarrow \) (2). We have
\[
[\sqrt{-1} \Theta_{(L,h-e^{-\psi})}; \Lambda_\omega] = [\sqrt{-1} \Theta_{(L,h)}; \Lambda_\omega] + [\sqrt{-1} \partial \bar{\partial} \psi; \Lambda_\omega]
\]
for any smooth strictly plurisubharmonic function \( \psi \). We fix a smooth \( \partial \)-closed \( L \)-valued \((n,q)\)-form \( f \) with compact support. The assumption of (1) and Lemma 3.1 implies that
\[
\langle [\sqrt{-1} \Theta_{(L,h)}; \Lambda_\omega] f, f \rangle_{(\omega,h)} \geq c |f|^2_{(\omega,h)}.
\]

The curvature operator \( [\sqrt{-1} \Theta_{(L,h-e^{-\psi})}; \Lambda_\omega] \) is positive definite on \( \wedge^{n,q} T^*D \otimes L \) everywhere. Therefore, by using Theorem 2.4 we can solve the \( \partial \)-equation \( \partial u = f \) as follows
\[
\int_D |u|^2_{(\omega,h)} e^{-\psi} dV_\omega \leq \int_D \langle [\sqrt{-1} \Theta_{(L,h-e^{-\psi})}; \Lambda_\omega]^{-1} f, f \rangle_{(\omega,h)} e^{-\psi} dV_\omega
\]
\[
\leq \int_D \langle ([\sqrt{-1} \partial \bar{\partial} \psi; \Lambda_\omega] + c)^{-1} f, f \rangle_{(\omega,h)} e^{-\psi} dV_\omega < +\infty
\]
for some \( u \in L^2_{(n,q-1)}(D, L; \omega, he^{-\psi}) \).

(2) \( \implies \) (1). For any smooth strictly plurisubharmonic function \( \psi \) and any \( \partial \)-closed \( f \in \mathcal{D}_{(n,q)}(D, L) \), we get a solution \( u \in L^2_{(n,q-1)}(D, L; \omega, he^{-\psi}) \) of \( \partial u = f \) satisfying

\[
\|u\|_{(\omega, he^{-\psi})}^2 \leq \left\langle \left[ \sqrt{-1}\partial\bar{\partial}\psi, \Lambda_\omega \right] + c \right\rangle_{\omega, he^{-\psi}}^{-1} f, f \rangle_{(\omega, he^{-\psi})}.
\]

Set \( g := \left[ \sqrt{-1}\partial\bar{\partial}\psi, \Lambda_\omega \right] + c \) \(-1 f \). We obtain

\[
\|\langle g, f \rangle_{(\omega, he^{-\psi})}\|^2 = \|\langle g, \partial u \rangle_{(\omega, he^{-\psi})}\|^2 \\
\leq \|\langle \partial^* g, u \rangle_{(\omega, he^{-\psi})}\|^2 \\
\leq \|\partial^* g\|_{(\omega, he^{-\psi})}^2 \|u\|_{(\omega, he^{-\psi})}^2 \\
\leq \|\partial^* g\|_{(\omega, he^{-\psi})}^2 \|\langle g, f \rangle_{(\omega, he^{-\psi})}\|.
\]

Using the Bochner-Kodaira-Nakano identity \( \Delta''_\psi = \Delta''_\psi + \left[ \sqrt{-1}\Theta_{(L, he^{-\psi})}, \Lambda_\omega \right] \) (cf. [Dem (4.6)]), we have

\[
\|\partial^* g\|_{(\omega, he^{-\psi})}^2 = \left\langle \left( \Delta''_\psi - \partial^* \partial \right) g, g \right\rangle_{(\omega, he^{-\psi})} \\
= \left\langle \Delta'\psi g, g \right\rangle_{(\omega, he^{-\psi})} + \left\langle \left[ \sqrt{-1}\Theta_{(L, he^{-\psi})}, \Lambda_\omega \right] g, g \right\rangle_{(\omega, he^{-\psi})} - \|\partial^* g\|_{(\omega, he^{-\psi})}^2 \\
\leq \|\Delta'\psi g\|_{(\omega, he^{-\psi})}^2 + \left\langle \left[ \sqrt{-1}\Theta_{(L, h)}, \Lambda_\omega \right] g, g \right\rangle_{(\omega, he^{-\psi})} + \left\langle \left[ \sqrt{-1}\partial\bar{\partial}\psi, \Lambda_\omega \right] g, g \right\rangle_{(\omega, he^{-\psi})}.
\]

Therefore, we get

\[
\|\partial^* g\|_{(\omega, he^{-\psi})}^2 + \left\langle \left[ \sqrt{-1}\Theta_{(L, h)}, \Lambda_\omega \right] g, g \right\rangle_{(\omega, he^{-\psi})} + \left\langle \left[ \sqrt{-1}\partial\bar{\partial}\psi, \Lambda_\omega \right] g, g \right\rangle_{(\omega, he^{-\psi})} \\
\leq \|\Delta'\psi g\|_{(\omega, he^{-\psi})}^2 + \left\langle \left[ \sqrt{-1}\Theta_{(L, h)}, \Lambda_\omega \right] g, g \right\rangle_{(\omega, he^{-\psi})} + \left\langle \left[ \sqrt{-1}\partial\bar{\partial}\psi, \Lambda_\omega \right] g, g \right\rangle_{(\omega, he^{-\psi})},
\]

that is,

\[
0 \leq \|\Delta'\psi g\|_{(\omega, he^{-\psi})}^2 + \left\langle \left[ \sqrt{-1}\Theta_{(L, h)}, \Lambda_\omega \right] g, g \right\rangle_{(\omega, he^{-\psi})}.
\]

We give a proof by contradiction. In other words, we suppose that the summation of some distinct \( q \) eigenvalues of \( \sqrt{-1}\Theta_{(L, h)} \) with respect to \( \omega \) is less than \( c \) at some point \( a \in D \). We can assume that \( o = a \in D \), where \( o \) is the origin of \( \mathbb{C}^n \). Let \( \gamma_1, \cdots, \gamma_n \) be the eigenvalues of \( \sqrt{-1}\Theta_{(L, h)} \) with respect to \( \omega \), which are globally defined on \( D \). Changing the coordinate by some unitary transformation, we take a coordinate \((z_1, \cdots, z_n)\) centered at \( o \) such that

\[
\omega = \sqrt{-1} \sum dz_j \wedge d\bar{z}_j,
\]

on \( D \) and

\[
\sqrt{-1}\Theta_{(L, h)} = \sqrt{-1} \sum \gamma_j dz_j \wedge d\bar{z}_j
\]
at \( o \). Without any loss of generality, we suppose that

\[
\gamma_1(o) + \cdots + \gamma_q(o) - c < 0.
\]

We fix an open neighborhood \( U \) of \( o \) and a local holomorphic frame \( e_L \) of \( L \) on \( U \). We define

\[
F := dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \otimes e_L \in C_{(n,q)}^\infty(U, L).
\]
Then we have
\[ \langle ([\sqrt{-1}\Theta_{\omega,\psi}] - c) F, F \rangle_{\omega,\psi}(o) = \sum_{j=1}^{q} \gamma_j(o) - c |e_L|^2_h < 0. \]

We take a positive constant $\delta > 0$ such that
\[ \langle ([\sqrt{-1}\Theta_{\omega,\psi}] - c) F, F \rangle_{\omega,\psi}(o) = \left( \sum_{j=1}^{q} \gamma_j(o) - c \right) |e_L|^2_h = -2\delta. \]

Since $\langle ([\sqrt{-1}\Theta_{\omega,\psi}] - c) F, F \rangle_{\omega,\psi}(o)$ has continuous coefficients, we take a sufficiently small $r \in (0, +\infty]$ such that $B_r^o \subset U \subset D$ and
\[ \langle ([\sqrt{-1}\Theta_{\omega,\psi}] - c) F, F \rangle_{\omega,\psi} < -\delta \]
on $B_r^o$.

We take a smooth strictly plurisubharmonic function $\psi(z) = |z|^2 - \frac{z^2}{4}$ on $D$. Let $\chi$ be a cut-off function on $B_r^o$ such that $\chi$ is smooth, $0 \leq \chi \leq 1$, supp$\chi \subset B_r^o$ and $\chi|_{B_r^o} \equiv 1$. We set $v := (-1)^{n+q-1} \chi \partial \bar{\partial} \psi d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \otimes e_L$ and $g := \partial v$. Then $g$ is a $\overline{\partial}$-closed $L$-valued $(n, q)$-form with compact support and
\[ g = dz_1 \wedge \cdots \wedge d\bar{z}_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \otimes e_L \]
on $B_r^o/2$. We remark that $[\sqrt{-1}\partial \bar{\partial} (m\psi), \Lambda_\omega] f = mq f$ for $f \in \Lambda^{n,q} T_D^* \otimes L$. We define $f_{m} := ([\sqrt{-1}\partial \bar{\partial} (m\psi), \Lambda_\omega] + c) g = (mq + c) g$. It clearly holds that $f_{m}$ is an also $\overline{\partial}$-closed $L$-valued $(n, q)$-form with compact support. Then $g$ satisfies the inequality (3.1) for every $m\psi$. Considering the commutation relation $\sqrt{-1}[\Lambda_\omega, \overline{\partial}] = D_{m\psi}^*$ (cf. [Dem-book (1.1) in Chapter VII]), we have that
\[ D_{m\psi}^* g = 0 \]
on $B_r^o/2$ since $\omega$ is the standard Kähler metric and $g$ has constant coefficients on $B_r^o/2$, and
\[ |D_{m\psi}^* g|_{(\omega, h)}^2 \leq C_1 \]
for some positive constant $C_1 > 0$ which is independent of $m$ and $\psi$ on $B_r^o$.

Since $g = F$ on $B_r^o/2$, we know that $\langle ([\sqrt{-1}\Theta_{\omega,\psi}] - c) g, g \rangle_{\omega,\psi} < -\delta$ on $B_r^o/2$ and $\langle ([\sqrt{-1}\Theta_{\omega,\psi}] - c) g, g \rangle_{(\omega, h)} \leq C_2$ for some positive constant $C_2 > 0$ on $B_r^o$. Consequently,
we can compute the right-hand side of (3.1) for \( g \) and \( m\psi \) as follows:
\[
0 \leq \int_D |D_{m\psi}^*g|^2_{(\omega,h)} e^{-m\psi} d\omega + \int_D \langle \left( [\sqrt{-1}\Theta(L,h), \Lambda_\omega] - c \right) g, g \rangle_{(\omega,h)} e^{-m\psi} d\omega \\
= \int_{\mathbb{B}_r^n \setminus \overline{\mathbb{B}_{r/2}^n}} |D_{m\psi}^*g|^2_{(\omega,h)} e^{-m\psi} d\omega + \int_{\mathbb{B}_{r/2}^n} \langle \left( [\sqrt{-1}\Theta(L,h), \Lambda_\omega] - c \right) g, g \rangle_{(\omega,h)} e^{-m\psi} d\omega \\
+ \int_{\mathbb{B}_r^n \setminus \overline{\mathbb{B}_{r/2}^n}} \langle \left( [\sqrt{-1}\Theta(L,h), \Lambda_\omega] - c \right) g, g \rangle_{(\omega,h)} e^{-m\psi} d\omega \\
\leq (C_1 + C_2) \int_{\mathbb{B}_r^n \setminus \overline{\mathbb{B}_{r/2}^n}} e^{-m\psi} d\omega - \delta \int_{\mathbb{B}_{r/2}^n} e^{-m\psi} d\omega.
\]

Since \( \psi > 0 \) on \( \mathbb{B}_r^n \setminus \overline{\mathbb{B}_{r/2}^n} \), the first term goes to zero as \( m \to +\infty \) by Lebesgue’s dominated convergence theorem. The second term has a negative upper bound
\[
-\delta \int_{\mathbb{B}_{r/2}^n} e^{-m\psi} d\omega < -\delta |\mathbb{B}_{r/2}^n|
\]
which is independent of \( m \) since \( \psi < 0 \) on \( \mathbb{B}_{r/2}^n \). Taking a sufficiently large \( m >> 1 \), we get
\[
(C_1 + C_2) \int_{\mathbb{B}_r^n \setminus \overline{\mathbb{B}_{r/2}^n}} e^{-m\psi} d\omega - \delta \int_{\mathbb{B}_{r/2}^n} e^{-m\psi} d\omega < 0,
\]
which is a contradiction. \( \square \)

3.2. RC-positivity. In this subsection, we give a characterization of RC-positivity via \( L^2 \)-estimates. This is a higher-rank analogue of Theorem 1.1. Although the proof is almost identical to the proof of Theorem 1.1 we show it for the sake of completeness.

**Proof of Theorem 1.2** We take an arbitrary smooth strictly plurisubharmonic function \( \psi \) and an arbitrary \( f \in \mathcal{D}(n,n)(D,E) \). Repeating the argument in the proof of Theorem 1.1 (cf. [DNWZ20] Theorem 3.1 or [Ina20] Proposition 2.7), we obtain the following inequality
\[
0 \leq \|D_{\psi}^*g\|^2_{(\omega,h,e^{-\psi})} + \langle \left( [\sqrt{-1}\Theta(E,h), \Lambda_\omega] - c \right) g, g \rangle_{(\omega,h,e^{-\psi})},
\]
where \( g = ([\sqrt{-1} \partial \bar{\partial} \psi \otimes Id_E, \Lambda_\omega] + c)^{-1} f \).

We give a proof by contradiction. We assume that there exists some point \( x \in D \) and some element \( a \in E_x \setminus \{0\} \) such that
\[
\text{tr}_{\omega}(\sqrt{-1}\Theta(E,h)a, a)_h(x) < c|a|^2_h(x).
\]
We may assume that \( x = o \in D \). Since \( h \) has smooth coefficients, we can take a sufficiently small \( r \in (0, +\infty) \) such that \( \mathbb{B}_r^o \subseteq D, E|_{\mathbb{B}_r^o} \) is trivial, and
\[
\text{tr}_{\omega}(\sqrt{-1}\Theta(E,h)a, a)_h - c|a|^2_h < -\delta
\]
on $\mathbb{B}_r^n$ for some positive constant $\delta > 0$. Here we regard $a$ as a section of $E$ with constant coefficients.

As in the proof of Theorem 1.1, we take a smooth strictly plurisubharmonic function $\psi(z) = |z|^2 - \frac{r^2}{4}$ and a cut-off function $\chi$ such that $\text{supp} \chi \subset \mathbb{B}_r^n$ and $\chi|_{\mathbb{B}_{r/2}} \equiv 1$. We consider the following $E$-valued $(n, n)$-form with compact support

$$g = \chi dZ \wedge d\bar{Z}$$

on $D$. Here we use the notation

$$dZ = dz_1 \wedge \cdots \wedge dz_n, \quad d\bar{Z} = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

for simplicity. We also define

$$f_m := (\sqrt{-1} \partial \bar{\partial} (m \psi) \otimes I d_e, \Lambda|_\omega) + c)g = (mn + c)g.$$ 

Note that $f_m \in \mathcal{D}_{(n, n)}(D, E)$. Hence, we see that $g$ satisfies the inequality (3.2) for each $m \psi$.

We compute the terms $\langle [\sqrt{-1} \Theta_{E,h}, \Lambda|_\omega](sdZ \wedge d\bar{Z}), sdZ \wedge d\bar{Z}\rangle_{(\omega, h)}$ and $\text{tr}_\omega(\sqrt{-1} \Theta_{E,h} s, s)_h$ for any section $s$ of $E$. Note that $sdZ \wedge d\bar{Z} \in C^\infty_{(n, n)}(D, E)$. We write the curvature tensor $\sqrt{-1} \Theta_{E,h}$ as

$$\sqrt{-1} \Theta_{E,h} = \sum_{1 \leq j, k \leq n} \Theta_{j\bar{k}}dz_j \wedge d\bar{z}_k,$$

where $\Theta_{j\bar{k}}$ are operators on each $E_t$. Then we get

$$\langle [\sqrt{-1} \Theta_{E,h}, \Lambda|_\omega](sdZ \wedge d\bar{Z}), sdZ \wedge d\bar{Z}\rangle_{(\omega, h)} = \langle (\sum_{j=1}^n \Theta_{j\bar{j}}s)dz \wedge d\bar{z}, sdZ \wedge d\bar{Z}\rangle_{(\omega, h)}$$

$$= \sum_{j=1}^n (\Theta_{j\bar{j}}s, s)_h$$

and

$$\text{tr}_\omega(\sqrt{-1} \Theta_{E,h} s, s)_h = \text{tr}_\omega\left( \sum_{1 \leq j, k \leq n} (\Theta_{j\bar{k}}s, s)_hdz_j \wedge d\bar{z}_k \right)$$

$$= \sum_{j=1}^n (\Theta_{j\bar{j}}s, s)_h.$$ 

Hence, on $\mathbb{B}_{r/2}^n$, the inequality (3.4) implies that

$$\langle ([\sqrt{-1} \Theta_{E,h}, \Lambda|_\omega] - c)g, g\rangle_{(\omega, h)} < -\delta.$$ 

Then, taking a sufficiently large $m >> 1$ and repeating the argument in the proof of Theorem 1.1 again, we conclude that the inequality (3.3) contradicts the inequality (3.2), which completes the proof. □
4. Applications of a new characterization

In this section, we give some applications of Theorem 1.1. First, we prove the following theorem. Here we use the same notation as in Theorem 1.1.

**Theorem 4.1.** Let $\varphi$ be a smooth function on $D$. Suppose that there exists a sequence of smooth functions $\{\varphi_j\}_{j=1}^{\infty}$ decreasing to $\varphi$ pointwise such that the summation of any distinct $q$ eigenvalues of $\sqrt{-1}\partial\bar{\partial}\varphi_j$ with respect to $\omega$ is greater than or equal to some non-negative constant $c \geq 0$. Then the summation of any distinct $q$ eigenvalues of $\sqrt{-1}\partial\bar{\partial}\varphi$ with respect to $\omega$ is greater than or equal to $c$.

It is well-known that Theorem 4.1 holds in the case that $q = 1$, that is, $\varphi_j$ are plurisubharmonic functions.

**Proof.** We use the characterization in Theorem 1.1. Since the result is a local property, we may assume that $D$ is pseudoconvex. It is enough to show that for any smooth strictly plurisubharmonic function $\psi$ and any smooth $\partial$-closed $(n,q)$-form $f$ with compact support, there exists a solution of $\partial u = f$ satisfying

$$\int_D |u|^2 \omega_0 e^{-(\varphi + \psi)} dV_{\omega_0} \leq \int_D \langle ([\sqrt{-1}\partial\bar{\partial}\psi, \Lambda_{\omega_0}] + c)^{-1} f, f \rangle_{\omega_0} e^{-(\varphi + \psi)} dV_{\omega_0}.$$  

The assumption of $\varphi_j$ implies that we get a solution of $\partial u_j = f$ satisfying

$$\int_D |u_j|^2 \omega_0 e^{-(\varphi_j + \psi)} dV_{\omega_0} \leq \int_D \langle ([\sqrt{-1}\partial\bar{\partial}\psi, \Lambda_{\omega_0}] + c)^{-1} f, f \rangle_{\omega_0} e^{-(\varphi_j + \psi)} dV_{\omega_0}$$

$$\leq \int_D \langle ([\sqrt{-1}\partial\bar{\partial}\psi, \Lambda_{\omega_0}] + c)^{-1} f, f \rangle_{\omega_0} e^{-(\varphi + \psi)} dV_{\omega_0}$$

$$< +\infty$$

for each $j \in \mathbb{N}$. Note that the right-hand side of the above inequality has an upper bound independent of $j$ and $\{u_k\}_{k \geq j}$ forms a bounded sequence in $L^2_{(n,q-1)}(D, \mathbb{C}; \omega_0, e^{-(\varphi_j + \psi)})$. Therefore, we find a weakly convergent subsequence $\{u_{j_k}\}_{k=1}^{\infty}$ by using a diagonal argument and monotonicity of $\{\varphi_j\}_{j=1}^{\infty}$, which is the standard argument of $L^2$-solutions of $\overline{\partial}$. We have that $\{u_{j_k}\}_{k=1}^{\infty}$ weakly converges in $L^2_{(n,q-1)}(D, \mathbb{C}; \omega_0, e^{-(\varphi_j + \psi)})$ for every $j$ and the weak limit denoted by $u_\infty$ satisfies $\overline{\partial} u_\infty = f$ and

$$\int_D |u_\infty|^2 \omega_0 e^{-(\varphi + \psi)} dV_{\omega_0} \leq \int_D \langle ([\sqrt{-1}\partial\bar{\partial}\psi, \Lambda_{\omega_0}] + c)^{-1} f, f \rangle_{\omega_0} e^{-(\varphi + \psi)} dV_{\omega_0}$$

due to the monotone convergence theorem. Then we complete the proof. □

Next, by using the new characterization, we propose the definition of uniform $q$-positivity for singular Hermitian metrics. Note that we can consider the condition (2) in Theorem 1.1 without assuming that $h$ is smooth.
Definition 4.2. Let $L$ be a holomorphic line bundle over an $n$-dimensional Kähler manifold $(X, \omega)$ and $h$ be a singular Hermitian metric on $L$ such that $-\log h$ is upper semi-continuous. Set $1 \leq q \leq n$ and $c \geq 0$. We say that $(L, h)$ is uniformly $(q - 1)$-positive with respect to $\omega$ if for any point $x \in X$, there exists an open neighborhood $U$ of $x$ such that for any relatively compact pseudoconvex domain $D$ in $U$, $(L, h), \omega$ and $c$ satisfy the condition (2) in Theorem 1.1 on $D$.

Thanks to Theorem 1.1 in the case that $h$ is smooth, the above definition is equivalent to uniform $(q - 1)$-positivity. Under this formulation, we can show Theorem 4.1 without assuming the condition that $\varphi$ is smooth. The proof remains the same.

Theorem 4.3. (cf. Theorem 4.1) Let $\{\varphi_j\}_{j=1}^\infty$ be a sequence of smooth functions decreasing to a locally integrable function $\varphi \neq -\infty$. If the summation of any distinct $q$ eigenvalues of $\sqrt{-1} \partial \bar{\partial} \varphi_j$ with respect to $\omega$ is greater than or equal to $c$, $(\mathbb{C}, e^{-\varphi})$ is uniformly $(q - 1)$-positive in the sense of Definition 4.2.

The argument above has many other applications. For instance, it is known that Nakano semi-positivity can be characterized via $L^2$-estimates (cf. [DNW20, Theorem 1.1]). By using the same method, we can also show that if a sequence of smooth Nakano semi-positive Hermitian metrics $\{h_\nu\}_{\nu=1}^\infty$ increasing to a (possibly singular) Hermitian metric $h$, $h$ is also Nakano semi-positive (for Nakano semi-positivity of singular Hermitian metrics, see [Ina20, Definition 1.2]).

5. Further Study

In this section, we propose some problems. First, we discuss Problem 1.3. In [Ber98], Berndtsson generalized the Prékopa theorem [Pré73] by assuming that the plurisubharmonic function satisfies some invariant properties.

Theorem 5.1. ([Ber98, Theorem 1.3]) Let $\varphi$ be a plurisubharmonic function on $U_z \times D_w \subset \mathbb{C}_z^n \times \mathbb{C}_w^m$, where $D_w$ is pseudoconvex. Assume that one of the following conditions holds:

1. $D$ is a connected Reinhardt domain and $\varphi(z, w_1, \cdots, w_m)$ is independent of $\arg(z_j)$ for $1 \leq j \leq m$.
2. $D$ contains the origin and for any $z \in U$, $w \in D$, and $\theta \in \mathbb{R}$, we have $e^{\sqrt{-1} \theta} w \in D$ and $\varphi(z, e^{\sqrt{-1} \theta} w) = \varphi(z, w)$.

Then the function $\tilde{\varphi}$ defined on $U$ by

$$e^{-\tilde{\varphi}(z)} := \int_{w \in D} e^{-\varphi(z, w)}$$

is plurisubharmonic.

This research has been generalized in a variety of directions (cf. [Cor05], [DZZ14]). Since Theorem 5.1 can be attributed to the case that $D$ is bounded, Problem 1.3 is a generalization...
of Theorem 5.1 for partial positivity. Here we explain the reason why Theorem 1.1 is one strategy to prove Problem 1.3. Let $\pi: U \times D \to U$ be the projection map and $dZ = dz_1 \wedge \cdots \wedge dz_n, dW = dw_1 \wedge \cdots \wedge dw_m$.

Thanks to Theorem 1.1 for any smooth strictly plurisubharmonic function $\psi$ on $U$ and any $\overline{\partial}$-closed $f \in \mathcal{D}_{(n,q)}(U)$, it is enough to show that there exists a solution $\overline{\partial}u = f$ satisfying

$$
\int_U |u|^2 e^{-(\overline{\partial} + \psi)}dV_{\omega_0} \leq \int_U \langle (c + [\sqrt{-1} \, \overline{\partial}\psi, \Lambda_{\omega_0}])^{-1}f, f \rangle_{\omega_0} e^{-(\overline{\partial} + \psi)}dV_{\omega_0}.
$$

Consider the $\overline{\partial}$-closed $(n+m,q)$-form $\pi^*f \wedge dW$ on $U \times D$. By assumption of $\varphi$, we can get a solution of $\overline{\partial}\tilde{u} = \pi^*f \wedge dW$ satisfying an $L^2$-estimates. Take the $L^2$-minimal solution $\tilde{u}$. “If” $\tilde{u}$ has the form

$$
\tilde{u} = \sum_{1 \leq j_1 < \cdots < j_q-1 \leq n} \tilde{u}_{j_1 \cdots j_q-1} dZ \wedge dW \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q-1},
$$

each coefficient $\tilde{u}_{j_1 \cdots j_q-1}$ is holomorphic in $w$ and invariant under the rotation of $w$ due to the uniqueness of the minimal solution. Hence, we have $\tilde{u} = \pi^*u \wedge dW$ for some $u \in C_\infty(U)$ satisfying $\overline{\partial}u = f$ and the inequality (5.1) on $U$.

However, the fact that $\tilde{u}$ has the form (5.2) does not immediately follow from that $\tilde{u}$ is the minimal $L^2$-solution, which is pointed out by Wang Xu. While there are still technical problems, we believe that Theorem 1.1 is a valid way to solve Problem 1.3.

As an application of Theorem 1.2, we also propose the following problem. The reason why Theorem 1.2 is useful to Problem 5.2 is the same reason why Theorem 1.1 is useful to Problem 1.3.

**Problem 5.2.** Assume that $E$ is weakly RC-positive. Is $S^k E \otimes \det E$ RC-positive for every $k \geq 1$?

Weak RC-positivity of $E$ implies that $\mathcal{O}_E(1)$ is uniformly $(\dim X - 1)$-positive, where $\mathcal{O}_E(1)$ is the tautological line bundle over the projectivized bundle $\mathbb{P}(E^*)$ (cf. Proposition 2.2). This problem asserts that if $\mathcal{O}_E(1)$ is uniformly $(\dim X - 1)$-positive, $\pi_* (K_{\mathbb{P}(E^*)/X} \otimes \mathcal{O}(r+k)) \cong S^k E \otimes \det E$ is RC-positive. This is related to the following conjecture raised by Yang.

**Conjecture 5.3.** ([Yan18, Question 7.11]) Assume that $E$ is weakly RC-positive. Then $E$ is RC-positive.

**References**

[AG62] A. Andreotti and H. Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90 (1962), 193-259.

[Ber98] B. Berndtsson, *Prekopa’s theorem and Kiselman’s minimum principle for plurisubharmonic functions*, Math. Ann. 312, (1998), 785-792.

[Cor05] D. Cordero-Erausquin, *On Berndtssons generalization of Prékopas theorem*, Math. Z. 249, (2005), 401-410.
[Dem82] J.-P. Demailly, *Estimations $L^2$ pour l’opérateur $\bar{\partial}$ d’un fibré vectoriel holomorphe semi-positif au dessus d’une variété Kählerienne complète*, Ann. Sci. Ec. Norm. Sup. **15**, (1982), 457-511.

[Dem] J.-P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012.

[Dem-book] J.-P. Demailly, *Complex analytic and differential geometry*, http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf

[DNW19] F. Deng, J. Ning, and Z. Wang, *Characterizations of plurisubharmonic functions*, arXiv:1910.06518

[DNWZ20] F. Deng, J. Ning, Z. Wang, and X. Zhou, *Positivity of holomorphic vector bundles in terms of $L^p$-conditions of $\bar{\partial}$*, arXiv:2001.01762

[DZZ14] F. Deng, H. Zhang, and X. Zhou, *Positivity of direct images of positively curved volume forms*, Math. Z. **278**, (2014), no. 1-2, 347-362.

[Hör65] L. Hörmander, *$L^2$ estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. **113**, (1965), 89-152.

[HI20] G. Hosono and T. Inayama, *A converse of Hörmander’s $L^2$-estimate and new positivity notions for vector bundles*, Sci. China Math. (2020). https://doi.org/10.1007/s11425-019-1654-9.

[Ina20] T. Inayama, *Nakano positivity of singular Hermitian metrics and vanishing theorems of Demailly-Nadel-Nakano type*, arXiv:2004.05798

[Pré73] A. Prékopa, *On logarithmic concave measures and functions*, Acad. Sci. Math. (Szeged) **34**, (1973), 335-343.

[Yan18] X. Yang, *RC-positivity, rational connectedness and Yau’s conjecture*, Camb. J. Math. **6**, (2018), no. 2, 183-212.

[Yan19] X. Yang, *A partial converse to the Andreotti-Grauert theorem*, Compos. Math. **155**, (2019), 89-99.

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan

E-mail address: inayama@ms.u-tokyo.ac.jp
E-mail address: inayama570@gmail.com