How to construct a upper triangular matrix that satisfy the quadratic polynomial equation with different roots

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Abstract. Let $R$ be an associative ring with identity $1$. We describe all matrices in $T_n(R)$ the ring of $n \times n$ upper triangular matrices over $R (n \in \mathbb{N})$, and $T_\infty(R)$ the ring of infinite upper triangular matrices over $R$, satisfying the quadratic polynomial equation $x^2 - rx + s = 0$. For such propose we assume that the above polynomial have two different roots in $R$. Moreover, in the case that $R$ in finite, we compute the number of all matrices to solves the matrix equation $A^2 - rA + sI = 0$, where $I$ is the identity matrix.

1. Introduction

Let $R$ be an associative ring with identity $1$. Denote by $T_n(R)$ the $n \times n$ upper triangular groups with entries in $R$ and $T_\infty(R)$ the ring of infinite upper triangular matrices over $R$. There are several authors who have works over this spaces, for instance, Slowik [2] show how to construct an involution matrix over these spaces, Hou [1] prove the similar results for idempotent matrices and Gargate in [4] compute the number of all involutions over the incidence algebras $\mathcal{I}(X, \mathbb{K})$ where $X$ is a finite poset and $\mathbb{K}$ is a finite field. Recently Gargate [5] compute the number of coninvolution matrices over the special rings: the Gaussian Integers module $p$ and the Quartenion Integers module $p$, with $p$ an odd prime number. Remember that various special matrices satisfy some polynomial equations, for instance, idempotent matrices satisfies $x^2 - x = 0$ and involution matrices satisfies $x^2 - 1 = 0$.

In the present article the authors generalizes the results of [2] and [1] on a broader class of matrices that satisfy the polynomial equations $x^2 - rx + s = 0$ with the condition that the polynomial has two different roots in $R$. We investigate how to construct these special matrices and compute the total of these matrices when $R$ is a finite ring.

Our main results is the followings Theorem:

**Theorem 1.1.** Assume that $R$ is an associative ring with identity $1$. Let $M$ be either the group $T_n(K)$ or $T_\infty(K)$ for some $n \in \mathbb{N}$ and denote by $I$ the identity matrix of $M$. Consider the quadratic polynomial equation $x^2 - rx + s = 0$ and assume that this equation has two different roots $a, b \in R$ such that $a - b$ is not a right zero divisor. Then a matrix $A \in M$ satisfies the quadratic equation of the type

$$A^2 - rA + sI = 0,$$

if and only if $A$ is described by the following statements:

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Key words and phrases. triangular matrix, infinite triangular matrix.
(i) For all \(1 \leq i \leq n\), we have \(a_{ii} \in \{a, b\}\) with \(a, b\) different roots of the quadratic equation \(x^2 - rx + s = 0\), \(r = a + b\) and \(s = ab\).

(ii) For all pairs of indices \(1 \leq i < j \leq n\) such that \(a_{ii} = a_{jj}\), then \(a_{ij}\) equals to

\[
a_{ij} = \begin{cases} 
0 & \text{if } j = i + 1 \\
-\frac{1}{a_{ii} - \text{another root}} \sum_{p=i+1}^{j-1} a_{ip} a_{pj} & \text{if } j > i + 1.
\end{cases}
\]

(iii) For \(i < j\), such that \(a_{ii} \neq a_{jj}\), then \(a_{ij}\) can be chosen arbitrarily.

Next using the above theorem we will prove the following result

**Theorem 1.2.** Let \(R\) be an associative ring with identity 1 and \(|R| = q\) the number of the elements in \(R\). Consider the quadratic polynomial equation \(x^2 - rx + s = 0\) and assume that this equation has two different roots \(a, b \in R\) such that \(a - b\) is not a right zero divisor. Then the total number of \(n \times n\) upper triangular matrices that satisfy the quadratic equation \(A^2 - rA + sI = 0\) is equal to

\[
\sum_{n_1 + n_2 = n \atop 0 < n_i} \binom{n}{n_1 n_2} q^{n_1 n_2}.
\]

where \(n_1, n_2\) are the number of times that appears \(a, b\) in the diagonal respectively.

2. Matrix solutions of the equation \(A^2 - rA + sI = 0\)

We start our considerations we notice the following property.

**Remark 2.1.** Assume that \(R\) is an associative ring with identity 1, \(M = T_\infty(R)\) or \(M = T_n(R)\) for some \(n \in \mathbb{N}\). If \(A \in M\) is a block matrix such that

\[
A = \begin{bmatrix} B_{11} & B_{12} & B_{13} & \cdots \\
B_{21} & B_{22} & B_{23} & \cdots \\
B_{31} & B_{32} & B_{33} & \cdots \\
& \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
\end{bmatrix}
\]

where \(B_{ii}\) are square matrices and \(A\) satisfies the quadratic equation \(A^2 - rA + sI = 0\), then for all \(i\), the matrices \(B_{ii}\) satisfy the quadratic equation as well.

**Proof.** Since \(A\) satisfies the quadratic equation \(A^2 - rA + sI = 0\), we have

\[
A^2 - rA + sI = \begin{bmatrix} B_{11}^2 - rB_{11} + sI & * & * & \cdots \\
* & B_{22}^2 - rB_{22} + sI & * & \cdots \\
* & * & B_{33}^2 - rB_{33} + sI & \cdots \\
& \ddots & \ddots & \ddots \\
\end{bmatrix},
\]

and we obtain \(B_{ii}^2 - rB_{ii} + sI = 0\) for all \(i\) by comparing entries of the diagonal position in the matrix equality above. \(\square\)

Now, we can prove our first main result.

**Proof of Theorem 1.1.** Let \(A = \sum_{ij} a_{ij} E_{ij} \in M\) be a matrix that satisfies the equation (\text{H}). As we have that \(a\) and \(b\) are different roots of the quadratic equation then \(r = a + b\) and \(s = ab\).

Since \(A^2 - rA + sI = 0\) our coefficients must satisfy the equations:
\[
\begin{align*}
\begin{aligned}
a_{ii}^2 - r \cdot a_{ii} + s &= 0, \\
a_{ii}a_{i,i+i} + a_{i,i+1}a_{i+1,i+1} - r \cdot a_{i,i+1} &= 0, \\
a_{ii}a_{i,i+2} + a_{i,i+1}a_{i+1,i+2} + a_{i,i+2}a_{i+2,i+2} - r \cdot a_{i,i+2} &= 0, \\
& \vdots \\
\sum_{p=0}^{m} a_{i,i+p}a_{i+p,i+m} - r \cdot a_{i,i+m} &= 0, \\
& \vdots
\end{aligned}
\end{align*}
\]

(3)

Since \( A \) satisfies the equation (1) and \( a_{ii}^2 - r \cdot a_{ii} + s = 0 \) then \( a_{ii} \in \{a, b\} \).

We need to proved that (ii) and (iii) given in Theorem 1.1 hold. We use induction on \( j-i \).

Assume that \( j-i = 1 \). We have

\[
a_{ii}a_{i,i+i} + a_{i,i+1}a_{i+1,i+1} - r \cdot a_{i,i+1} = 0,
\]

(4)

from the family of equations (3). One can see that:

- If \( a_{ii} = a_{i+1,i+1} \), of the equation (4) we have
  \[
  2a_{ii}a_{i,i+1} - r \cdot a_{i,i+1} = 0,
  \]
  or
  \[
  a_{i,i+1} (2a_{ii} - r) = 0,
  \]
  then \( a_{i,i+1} = 0 \) since \( a_{ii} \in \{a, b\} \) and \( r = a + b \) with \( a \neq b \).

- If \( a_{ii} \neq a_{i+1,i+1} \), then of the equation (4) we obtain
  \[
  a_{i,i+1} (a_{ii} + a_{i+1,i+1} - r) = 0,
  \]
  thus \( a_{i,i+1} \) can be chosen arbitrarily, since \( r = a + b = a_{ii} + a_{i+1,i+1} \).

So the first super diagonal entries of the matrix \( A \) fulfill (ii) and (iii).

Now, suppose that \( j-i = m > 1 \) and consider the \( (i, i+m) \) entries of the equation (1), and we have the \( (m+1) \)-st family of the equation (3):

\[
\sum_{p=0}^{m} a_{i,i+p}a_{i+p,i+m} - r \cdot a_{i,i+m} = 0,
\]

or

\[
a_{i,i+m} (a_{ii} + a_{i+m,i+m} - r) + \sum_{p=1}^{m-1} a_{i,i+p}a_{i+p,i+m} = 0,
\]

(5)
• If \( a_{ii} = a_{i+m,i+m} \) then \((a_{ii} + a_{i+m,i+m} - r) \neq 0\) and we obtain that

\[
a_{i,i+m} = -\frac{1}{(a_{ii} + a_{i+m,i+m} - r)} \sum_{p=1}^{m-1} a_{i,i+p}a_{i+p,i+m}
\]

where \( r = a + b \) and

\[
(a_{ii} + a_{i+m,i+m} - r) = (a_{ii} - \text{other root}) = \begin{cases} 
  a - b & \text{if } a_{ii} = a_{i+m,i+m} = a \\
  b - a & \text{if } a_{ii} = a_{i+m,i+m} = b
\end{cases}
\]

So (ii) of Theorem 1.1 hold.

• If \( a_{ii} \neq a_{i+m,i+m} \) then we must have \( a_{ii} + a_{i+m,i+m} - r = 0 \) since \( a_{ii} \in \{a, b\} \) and \( r = a + b \). So we get

\[
\sum_{p=1}^{m-1} a_{i,i+p}a_{i+p,i+m} = 0
\]  

from equation (5).

Now, consider \( A(m, i) \) the submatrix of \( A \) defined as

\[
A(m, i) = \begin{bmatrix}
  a_{ii} & a_{i,i+1} & \cdots & a_{i,i+m} \\
  a_{i+1,i} & a_{i+1,i+1} & \cdots & a_{i+1,i+m} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{i+m,i} & a_{i+m,i+1} & \cdots & a_{i+m,i+m}
\end{bmatrix}
\]

From Remark (2.1) one can see that \( A \) satisfies the quadratic equation (1) if and only if \( A(m, i) \) also satisfies the equation (1) for all \( m \) and \( i \).

We write this matrix as a block matrix such that

\[
A(m, i) = \begin{bmatrix}
  a_{ii} & \alpha & a_{i,i+m} \\
  0 & \beta & \gamma \\
  0 & 0 & a_{i+m,i+m}
\end{bmatrix}
\]

(8)

Since \( A \) satisfies the equation (1) and by Remark (2.1) we have that the matrices

\[
A(m - 1, i) = \begin{bmatrix}
  a_{ii} & \alpha \\
  0 & \beta
\end{bmatrix}
\]

and

\[
A(m - 1, i + 1) = \begin{bmatrix}
  \beta & \gamma \\
  0 & a_{i+m,i+m}
\end{bmatrix}
\]

also satisfies the equation (1). So, we obtain that

\[a_{ii}\alpha + \alpha\beta - r\alpha = 0\]

and

\[\beta\gamma + \gamma a_{i+m,i+m} - r\gamma = 0\].

Thus

\[
(A(m, i))^2 - rA(m, i) + sI = \begin{bmatrix}
  0 & 0 & a_{ii}a_{i,i+m} + \alpha\gamma + a_{i,i+m}a_{i+m,i+m} - ra_{i,i+m} \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}
\]

since \( a_{ii} \in \{a, b\} \) with \( a, b \) roots of the equation \( x^2 - rx + s = 0 \).
As \( a_{ii} \neq a_{i+m,i+m} \) from equations 6 and 7 we have

\[
\alpha\gamma = \sum_{p=1}^{m-1} a_{i,i+p}a_{i+p,i+m} = 0.
\]

Hence,

\[
a_{ii}a_{i,i+m} + \alpha\gamma + a_{i,i+m}a_{i+m,i+m} - ra_{i,i+m} = a_{i,i+m}(a_{ii} + a_{i+m,i+m} - r) + \alpha\gamma
\]

\[
= \alpha\gamma
\]

\[
= 0
\]

since \( r = a + b = a_{ii} + a_{i+m,i+m} \).

Therefore, \( A(m,i) \) satisfies the equation 11, regardless of the value of the entry \( a_{i,i+m} \). Thus (iii) of Theorem 1.1 holds.

Assume now that the entries of \( A \) fulfill (i), (ii) and (iii) of Theorem 1.1. We shall prove that \( A \) satisfies the quadratic equation \( A^2 - rA + sI = 0 \). Since the equation 2 involves only the coefficients with indices \( p \), such that \( i \leq p \leq j \), it suffices to prove the claim for \( A(m,i) \). For \( m = 1 \) and \( m = 2 \) one can easily check now that all sub matrices \( A(1,i) \) and \( A(2,i) \) satisfy the quadratic equation \( A^2 - rA + sI = 0 \). Suppose that the claim hold for all \( 1 \leq t \leq m - 1 \), i.e. \( A(2,i), A(3,i), \ldots, A(m-1,i) \) satisfy the quadratic equation 11 for all \( i \), we need only prove that \( A(m,i) \) also satisfy the quadratic equation 11.

Consider \( A(m,i) \) as a block matrix given in the form of equation 8. Thus, we have that the quadratic equation \( A(m,i)^2 - rA(m,i) + sI = 0 \) equals

\[
\begin{bmatrix}
\alpha\beta^2 - r\alpha & \beta^2 - r\beta + s \\
0 & \beta^2 - r\beta + s
\end{bmatrix}
\]

By assumption, \( A(m-1,i) \) satisfy the equation 11 for all \( i \). So \( A(m-1,i) \) and \( A(m-1,i+1) \) satisfy the equation 11. Thus,

\[
A(m-1,i)^2 - rA(m-1,i) + sI = \begin{bmatrix}
\alpha\beta^2 - r\alpha & \beta^2 - r\beta + s \\
0 & \beta^2 - r\beta + s
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]

and

\[
A(m-1,i+1)^2 - rA(m-1,i+1) + sI = \begin{bmatrix}
\beta^2 - r\beta + s & \beta\gamma + a_{i+m,i+m} - r\gamma \\
0 & \beta^2 - r\beta + s
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

From the equations 10 and 11 above, we have \( a_{ii}^2 - ra_{ii} + s = 0 \) and \( a_{i+m,i+m}^2 - ra_{i+m,i+m} + s = 0 \) since \( a_{ii} \in \{a, b\} \) is root the equation \( x^2 - rx + s = 0 \), \( \beta^2 - r\beta + sI = 0 \) by Lemma 2.1 and

\[
a_{ii}\alpha + \alpha\beta - r\alpha = 0
\]
\[ \beta \gamma + a_{i+m,i+m} \gamma - r \gamma = 0 \quad (13) \]

Then by multiplying the equation (12) by \( \gamma \) and the equation (13) by \( \alpha \) we obtain that
\[ a_{ii} \alpha \gamma + \alpha \beta \gamma - r \alpha \gamma = 0 \]
\[ \alpha \beta \gamma + a_{i+m,i+m} \alpha \gamma - r \alpha \gamma = 0. \]

Hence,
\[ \alpha \beta \gamma = (r - a_{ii}) \alpha \gamma \]
\[ \alpha \beta \gamma = (r - a_{i+m,i+m}) \alpha \gamma. \]

- If we consider \( a_{ii} \neq a_{i+m,i+m} \) we have
  \[ \alpha \beta \gamma = (a_{ii}) \alpha \gamma \]
  since \( r = a_{ii} + a_{i+m,i+m} \), which implies that \( \alpha \gamma = 0 \).

  So, the \((1, m)\) entries of equation (9) is
  \[ a_{i,i+m} (a_{ii} + a_{i+m,i+m} - r) + \alpha \gamma = \alpha \gamma = 0. \]

Therefore, \( A(m, i) \) satisfies the quadratic equation \( A^2 - rA + sI = 0 \).

- On the other hand, if \( a_{ii} = a_{i+m,i+m} \) from (iii) of the Theorem 1.1 or the equations (5) and (6) we have
  \[ a_{ii} = -1 \left( a_{ii} + a_{i+m,i+m} - r \right) \sum_{p=1}^{m-1} a_{i,i+p} a_{i,p,i+m} = -1 \left( a_{ii} + a_{i+m,i+m} - r \right) \alpha \gamma; \]

  so, in this case the \((1, m)\) entries of equation (9) is
  \[ a_{i,i+m} (a_{ii} + a_{i+m,i+m} - r) + \alpha \gamma = \left( -\frac{\alpha \gamma}{a_{ii} + a_{i+m,i+m} - r} \right) (a_{ii} + a_{i+m,i+m} - r) + \alpha \gamma \]

  \[ = 0. \]

Therefore, \( A(m, i) \) also satisfies the quadratic equation (11).

Thus, we have proved that \( A \) satisfies the quadratic equation (11) in the upper triangular matrix ring \( M \) where \( a_{ii} \in \{a, b\} \) and \( a, b \) are different roots of the equation \( x^2 - rx + s = 0 \) if and only if \( A \) is described as in (i), (ii) and (iii) of the Theorem 1.1. \( \square \)

Follows immediately from Theorem 1.1 the results of Hou [1] and Slowik [2]:

**Corollary 2.2 (Hou [1]).** We can construct any \( n \times n \) idempotent upper triangular matrix over \( R \) that has only zeros and ones on its diagonal

(i) For all \( i \), the entries in the main diagonal \( a_{ii} \in \{0, 1\} \).

(ii) For \( i < j \), if \( a_{ii} = a_{jj} \), then \( a_{ij} \) equals to

\[
 a_{ij} = \begin{cases} 
 0 & \text{if } j = i + 1 \\
 (1 - 2a_{ii}) \sum_{p=i+1}^{j-1} a_{ip} a_{pj} & \text{if } j > i + 1.
\end{cases}
\]

(iii) For \( i < j \), if \( a_{ii} \neq a_{jj} \), then \( a_{ij} \) can be chosen arbitrarily.
**Proof.** For $a_{ii} \in \{0,1\}$ the quadratic equation (1) equals $A^2 = A$ then $A$ is an idempotent matrix.

We need to verify that equation (14) of the Theorem 1.1 yields the same possibilities for $a_{ij}$ shown in the procedure above. The equation (2) becomes

$$a_{ij} = -\frac{1}{a_{ii} - \text{another root}} \sum_{p=1}^{j-1} a_{ip}a_{pj}$$

$$= (1 - 2a_{ii}) \sum_{p=1}^{j-1} a_{ip}a_{pj}$$

for $a_{ii} \in \{0,1\}$. □

**Corollary 2.3** (Slowik [2]). We can construct any $n \times n$ involution upper triangular matrix over $R$ when $a_{ii} \in \{1,-1\}$

(i) For all $i$, the entries in the main diagonal $a_{ii} \in \{-1,1\}$.

(ii) For $i < j$, if $a_{ii} = a_{jj}$, then $a_{ij}$ equals to

$$a_{ij} = \begin{cases} 
0 & \text{if } j = i + 1 \\
-(2a_{ii})^{-1} \sum_{p=1}^{j-1} a_{ip}a_{pj} & \text{if } j > i + 1.
\end{cases} \quad (15)$$

(iii) For $i < j$, if $a_{ii} = -a_{jj}$, then $a_{ij}$ can be chosen arbitrarily.

**Proof.** For $a_{ii} \in \{-1,1\}$ the quadratic equation (11) equals $A^2 = I$ then $A$ is an Involution matrix.

We need to verify that equation (15) of the Theorem 1.1 yields the same possibilities for $a_{ij}$ shown in the procedure above. The equation (2) becomes

$$a_{ij} = -\frac{1}{a_{ii} - \text{another root}} \sum_{p=1}^{j-1} a_{ip}a_{pj}$$

$$= -\frac{1}{a_{ii} - (-a_{ii})} \sum_{p=1}^{j-1} a_{ip}a_{pj}$$

$$= -\frac{1}{2a_{ii}} \sum_{p=1}^{j-1} a_{ip}a_{pj}$$

for $a_{ii} \in \{-1,1\}$. □

3. Compute the number of all solutions for the quadratic polynomial equation

**Theorem 3.1.** Let $R$ be an associative ring with identity $1$ and $|R| = q$ the number of the elements in $R$. Then the total number of $n \times n$ upper triangular that satisfy the quadratic equation $A^2 - rA + sI = 0$ with $a_{ii} \in \{a, b\}$ on the diagonal where $\{a, b\}$ different roots of the quadratic equation $x^2 - rx - s = 0$ is equal to

$$\sum_{n_1+n_2=n}^{n_1+n_2=n} \binom{n}{n_1 n_2} \cdot q^{n_1 n_2}.$$
where \( n_1, n_2 \) are the number of times that appears \( a, b \) in the diagonal respectively, \( r = a + b \) and \( s = ab \).

**Proof of Theorem 1.2.** By Theorem 1.1, the number of possible upper triangular matrices that satisfy the quadratic equation \( A^2 - rA + sI = 0 \) with the set \( D = \{a, b\}, a \neq b \) on the diagonal depends entirely on which pairs of diagonal entries have \( a_{ii} \neq a_{jj} \). To enumerate those possibilities, consider an integer column vector \( d = (d_1, d_2, \ldots, d_n) \) the respective diagonal having each \( d_i \in D \) and denote for \( n_1, n_2 \) the numbers of \( a, b \) that appears in the diagonal respectively, such that \( n_1 + n_2 = n \) with \( 0 \leq n_i \) for \( i = 1, 2 \). By \( \Delta \) we denote the number of pairs \( (d_i, d_j) \) with \( i < j \) and \( d_i \neq d_j \). Notice that

\[
\Delta = n_1 \cdot n_2.
\]

In particular, \( \Delta \) is independent of the order in which the elements of the set \( D \) appear on \( d \). Consequently we have on the diagonal yields \( q^{\Delta} = q^{n_1 \cdot n_2} \), possible upper triangular matrices that satisfy the quadratic equation \( A^2 - rA + sI = 0 \).

Finally, all \( d_i's \) can be put on our main diagonal on

\[
\binom{n}{n_1, n_2} = \binom{n}{n_1} \cdot \binom{n - n_1}{n_2},
\]

where

\[
\binom{n}{n_1, n_2} = \frac{n!}{n_1!n_2!}.
\]

Therefore, the total number of \( n \times n \) upper triangular matrices that satisfy the quadratic equation \( A^2 - rA + sI = 0 \) with elements the set \( \{a, b\} \) with \( a \neq b \) on the diagonal is

\[
\sum_{n_1 + n_1 = n \atop 0 \leq n_i} \binom{n}{n_1, n_2} \cdot q^{n_1n_2}.
\]

\[\square\]

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