NEUMANN BOUNDARY VALUE PROBLEM IN DOMAINS OF THE HEISENBERG GROUP $\mathbb{H}_n$

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Abstract. Existence and uniqueness of the solution of the Neumann problem for the Kohn-Laplacian on the Korányi ball of the Heisenberg group $\mathbb{H}_n$ are discussed. Explicit representations of Green’s type function (Neumann function) for the half space and Korányi ball in $\mathbb{H}_n$ for circular functions have been obtained. These functions are then used on above regions in $\mathbb{H}_n$ to solve the inhomogeneous Neumann boundary value problem for circular data.

1. Introduction

The boundary value problems have much broader utility in physics, electrostatics and magnetic field etc. These boundary conditions have a physical interpretation where we keep the ends of our rod at freezing without regulating the heat flow in or out of the endpoints. The concept of Green type function for Neumann problem or Neumann function has been considered by several authors in different domains of $\mathbb{R}^n$ [1, 13, 15].

The classical Neumann problem is to find a solution $u : \Omega \rightarrow \mathbb{R}$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ such that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and

$$
\begin{align*}
\Delta u &= 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial n} &= f \text{ on } \partial \Omega,
\end{align*}
$$

where $n$ is the outward normal to $\Omega$ and $f$ is a given continuous function on $\partial \Omega$. Firstly it should be noted that (1) cannot have a solution for every $f$. This is clear from the physical interpretation of (1) as a steady-state heat conduction problem. There are no sources in the region $\Omega$ and the heat flow is prescribed on $\partial \Omega$. These conditions are consistent with the steady state only if the total heat flow through $\partial \Omega$ vanishes. Accordingly, a solution of (1) will exist only if

$$
\int_{\partial \Omega} f(\xi) d\xi = 0.
$$

The Neumann function for the sphere in $\mathbb{R}^3$ has been constructed using the classical method of images and expressed in terms of eigenvalues associated with the surface, leading to an analogue of the Poisson integral as a solution to the Neumann problem for the sphere [13]. The Neumann function for the positive half space of $\mathbb{R}^n$ was obtained in [1]. The fundamental solution

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for the sub-Laplacian on the Heisenberg group was first given by Folland [2]. The Dirichlet problem on the Heisenberg group and existence of unique solution was discussed in [5]. The general Green function is not known for any domain in $\mathbb{H}_n$, $n > 1$.

Our aim is to find a kernel function $N_D(\eta, \xi)$ for a given domain $D \subseteq \mathbb{H}_n$ which solves the inhomogeneous Neumann boundary value problem for the subelliptic operator $L_0$,

$$L_0 u = f \text{ in } D,$$

$$\frac{\partial}{\partial n_0} u = g \text{ on } \partial D,$$

and the solution is given by the representation formula

$$u(\eta) = \int_D N_D(\eta, \xi)f(\xi)dv(\xi) - \int_{\partial D} N_D(\eta, \xi)g(\xi)d\sigma(\xi),$$

where $f$ and $g$ are continuous circular functions.

In subsequent sections, we consider the Neumann boundary value problem for Korányi ball and half space in the Heisenberg group $\mathbb{H}_n$ with circular boundary data and obtain Green’s type function (or Neumann function) for Neumann BVP by means of the fundamental solution for Laplacian, Kelvin transform and Heisenberg spherical harmonics. A representation formula for the solution of the circular Neumann problem is also given. We also obtain a necessary and sufficient condition for the existence of solution of the Neumann boundary value problem for bounded domain in $\mathbb{H}_n$.

2. The Heisenberg Group $\mathbb{H}_n$ and Horizontal Normal Vectors

2.1. The Heisenberg Group. The Heisenberg group $\mathbb{H}_n$ is the set of all pairs $[z, t] \in \mathbb{C}^n \times \mathbb{R}$ with the operation

$$[z, t].[z', t'] = [z + z', t + t' + 2\Im(z\bar{z}')]$$

where $z = (z_1, \ldots, z_n), z.z' = z_1\bar{z}_1' + \ldots + z_n\bar{z}_n'$. A basis of left invariant vector fields on $\mathbb{H}_n$ is given by $\{Z_j, \bar{Z}_j, T : 1 \leq j \leq n\}$ where,

$$Z_j = \partial_{z_j} + i\bar{z}_j \partial_t;$$

$$\bar{Z}_j = \partial_{\bar{z}_j} - iz_j \partial_t;$$

$$T = \partial_t.$$

If we write $z_j = x_j + iy_j$ and define,

$$X_j = \partial_{x_j} + 2y_j \partial_t,$$

$$Y_j = \partial_{y_j} - 2x_j \partial_t,$$

then,

$$Z_j = \frac{1}{2}(X_j - iY_j),$$

and $\{X_j, Y_j, T\}$ is a basis. The sublaplacian $L$ on $\mathbb{H}_n$ is explicitly given by

$$L = -\sum_{j=1}^{n} (X_j^2 + Y_j^2).$$
Let $L_0$ denote the slightly modified subelliptic operator $-\frac{1}{4}L$. The natural gauge on $\mathbb{H}_n$ is given by

$$N(z, t) = (|z|^4 + t^2)^{\frac{1}{4}}.$$ 

The fundamental solution for $L_0$ on $\mathbb{H}_n$ with pole at identity is given in [2] as

$$g_e(\xi) = g_e([z, t]) = a_0(|z|^4 + t^2)^{-\frac{n}{2}},$$

where, 

$$a_0 = 2^{n-2} \left( \Gamma\left(\frac{n}{2}\right) \right)^2 \pi^{n+1},$$

is the normalization constant and $\xi = [z, t]$. The fundamental solution of $L_0$ with pole at $\eta$ is given by

$$g_\eta(\xi) = g_e(\xi^{-1} \eta).$$

From [11], for $\eta = [z', t']$ and $\xi = [z, t]$,

$$g_\eta(\xi) = a_0 |C(\eta, \xi) - P(\eta, \xi)|^{-n},$$

where

$$C(\eta, \xi) = |z|^2 + |z'|^2 + i(t - t')$$

and

$$P(\eta, \xi) = 2z \bar{z}'.$$

For an integrable function $f$ on $\mathbb{H}_n$, we denote the average of $f$ by

$$\bar{f}([z, t]) = \frac{1}{2\pi} \int_0^{2\pi} f([e^{i\theta} z, t]) d\theta.$$ 

A function $f$ is said to be circular if $f([z, t]) = \bar{f}([z, t])$ for $[z, t] \in \mathbb{H}_n$.

As in [11], the average of the fundamental solution with pole at $\eta$ is given by

$$\bar{g}_\eta(\xi) = a_0 |C(\eta, \xi)|^{-n} F\left(\frac{n}{2}, \frac{n}{2}, n; \frac{|P(\eta, \xi)|^2}{|C(\eta, \xi)|^2}\right),$$

where $F$ being the Gaussian hypergeometric function [14].

2.2. Horizontal Normal Vectors. As in [9], for every $c > 0$, we define a left invariant Riemannian metric $M_c$ on $\mathbb{H}_n$ by requiring that at every point the vector fields $cT, X_j, Y_j (1 \leq j \leq n)$ forms an orthonormal system in the tangent space. As in [6] and [9], we also define a singular Riemannian metric $(M_0)$ as follows. On the linear span of $X_j, Y_j (1 \leq j \leq n)$ we define an inner product $\langle \cdot, \cdot \rangle_0$ by the condition that the vectors $X_j, Y_j$ form an orthonormal system. For vectors not in this span we say that they have infinite length. A vector is said to be horizontal if it has finite length. The horizontal gradient $\nabla_0 F$ of a function $F$ on $\mathbb{H}_n$ is defined as the unique horizontal vector such that

$$\langle \nabla_0 F | V \rangle_0 = V.F$$

for all horizontal vector $V$. With respect to metric $(M_0)$ we have a gradient, $\nabla_0 F$, can be explicitly written as

$$\nabla_0 F = \sum_j \{(X_j F) X_j + (Y_j F) Y_j\}.$$
The horizontal normal unit vector to a hypersurface \( \{ F = 0 \} \) will be defined by
\[
\frac{\partial}{\partial n_0} = \frac{1}{||\nabla_0 F||_0} \nabla_0 F.
\]
More precisely, this is the horizontal normal pointing outwards for the domain \( \{ F < 0 \} \).

3. Uniqueness and Existence of a Solution

In this section, we establish the uniqueness and existence of the Neumann boundary value problem for bounded domains in \( H_n \).

3.1. Uniqueness of the Interior Neumann problem. Let \( D \) be a bounded domain with smooth boundary in \( \mathbb{H}_n \). We show that solution of the interior Neumann problem,
\[
L_0 u = 0 \quad \text{in} \quad D, \\
\frac{\partial}{\partial n_0} u = g \quad \text{on} \quad \partial D, 
\]
where \( g \) is a continuous function, is unique upto a constant.

**Theorem 3.1.** Two solutions of the interior Neumann problem can differ only by a constant.

**Proof.** The difference \( u := u_1 - u_2 \) of two solutions for the Neumann problem is a harmonic function continuous up to the boundary satisfying the homogeneous boundary condition \( \frac{\partial}{\partial n_0} = 0 \) on \( \partial D \) in the sense of uniform convergence. For the interior Neumann problem, suppose that \( u \) is not constant in \( D \). Then there exists some closed ball \( B \) contained in \( D \) such that
\[
\int_B |\nabla_0 u|^2 \, d\eta > 0.
\]
Since \( D \) has boundary of class \( C^1 \) (and hence \( D \) is a \( \mathbb{H} \)-Caccioppoli set ([3], Definition 2.11)), therefore, by Divergence theorem ([3], Corollary 7.7) applied to the interior \( D_h := \{ \eta - h n_0(\eta) : \eta \in \partial D \} \) with sufficiently small \( h > 0 \), we derive
\[
\int_B |\nabla_0 u|^2 \, d\eta \leq \int_{D_h} |\nabla_0 u|^2 \, d\eta = \int_{\partial D_h} u \frac{\partial u}{\partial n_0} \, ds.
\]
Passing to the limit \( h \to 0 \), we obtain the contradiction \( \int_B |\nabla_0 u|^2 \, d\eta \leq 0 \).
Hence, \( u \) must be constant. \( \square \)

3.2. Surface Potentials.

**Definition 3.2.** Weakly singular kernel: A kernel \( K \) is said to be weakly singular kernel if it is defined and continuous for all \( \eta, \xi \in \mathbb{H}_n, \eta \neq \xi \), and there exist positive constants \( M \) and \( \alpha \in (0, m] \) such that
\[
|K(\eta, \xi)| \leq M(N(\xi^{-1}\eta))^{\alpha-m}, \quad \eta, \xi \in D, \eta \neq \xi.
\]

**Definition 3.3.** Given a function \( \phi \in C(\partial D) \), for \( \eta \in \mathbb{H}_n \setminus \partial D \), the functions
\[
m(\eta) := \int_{\partial D} \phi(\xi) g_\eta(\xi) \, d\xi,
\]
where
and
\[ v(\eta) := \int_{\partial D} \phi(\xi) \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} \, ds(\xi), \tag{5} \]
are called, respectively, single-layer and double-layer potential with density \( \phi \). Both potentials are \( L_0 \)-harmonic.

In the following results, we determine the limiting values of single layer and double layer potentials.

**Theorem 3.4.** For \( \partial D \) of class \( C^2 \), the double-layer potential \( v \) with continuous density \( \phi \) can be continuously extended from \( D \) to \( \bar{D} \) and from \( \mathbb{H}_n \setminus \bar{D} \) to \( \mathbb{H}_n \setminus D \) with limiting values
\[ v_\pm(\eta) = \int_{\partial D} \phi(\xi) \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} \, ds(\xi) \pm \phi(\eta), \quad \eta \in \partial D, \tag{6} \]
where \( v_\pm(\eta) = \lim_{\eta \to 0^\pm} v(\eta \pm \eta_0(\eta)) \). Moreover, the integral in (6) exists as an improper integral.

**Proof.** For \( \xi = [z, t] \) and \( \eta = [\zeta', t'] \in \mathbb{H}_n \), we have
\[ \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} = -n a_0 \frac{a_0}{z} (N(\xi^{-1}\eta))^{(-2n-2)}(4|z|^6 + 2|z|^2|z'|^4 + 8|z|^2 z^2 z'^2 + 4|z|^4|z'|^2
\[ -4z^2|z'|^2 - 12|z|^4 4z^2 - 8i z^2 z'^2 t - 4iz z' z^2 t - 12i|z|^2 z^2 t + 4|z|^2 t^2 - 4|z|^2 t'). \]
So,
\[ \left| \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} \right| \leq C_1 |a_0|.n(N(\xi^{-1}\eta))^{(-2n-2)}, \quad \eta \neq \xi, \tag{7} \]
for some constant \( C_1 > 0 \) i.e., the integral in (6) has a weakly singular kernel. Therefore, by (12), Theorem 2.30) the integral exists for \( \eta \in \partial D \) as an improper integral and represents a continuous function on \( \partial D \). In a sufficiently small neighborhood \( U \) of \( \partial D \), we can represent every \( \eta \in U \) uniquely in the form \( \eta = \zeta + \eta_0(\zeta) \), where \( \zeta \in \partial D \) and \( h \in [-\eta_0, \eta_0] \) for some \( \eta_0 > 0 \). Now, we write the double-layer potential \( v \) with density \( \phi \) in the form
\[ v(\eta) = \phi(\zeta)w(\eta) + u(\eta), \quad \eta = \zeta + \eta_0(\zeta) \in U \setminus \partial D, \]
where,
\[ w(\eta) = \int_{\partial D} \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} \, ds(\xi), \tag{8} \]
and
\[ u(\eta) = \int_{\partial D} \{\phi(\xi) - \phi(\zeta)\} \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} \, ds(\xi). \tag{9} \]
For \( \eta \in \partial D \) i.e., \( h = 0 \), the integral in (9) exists as an improper integral and represents a continuous function on \( \partial D \). By using Gaveau’s Green formula [5], we have \( w(\eta) = 1 \) for \( \eta \in \partial D \), therefore, to establish the theorem it suffices to show that
\[ \lim_{h \to 0} u(\zeta + \eta_0(\zeta)) = u(\zeta), \quad \zeta \in \partial D, \]
uniformly on $\partial D$.

Denoting $\partial D(\zeta; r) = \partial D \cap B[\zeta; r]$, where $B[\zeta; r] = \{ \xi \in \mathbb{H}_n : N(\xi^{-1} \zeta) \leq r \}$. Take $r < N(\eta^{-1}) = \alpha$(say) for $r$ sufficiently small, $\alpha - r$ is lower bound of $N(\xi^{-1})$.

Consider
\[
\int_{\partial D(\zeta; r)} \left| \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} \right| \, ds(\xi) \leq C_1 \int_{\partial D(\zeta; r)} (N(\xi^{-1}) - 2n-2) \, ds(\xi), \eta \neq \xi
\]

\[
\leq C_1 \int_{\partial D(\zeta; r)} \frac{1}{(\alpha - r)^{2n+2}} \, ds(\xi)
\]

\[
\leq C_1 \left( \frac{1}{(\alpha - 1)^{2n+2}} m(\partial D(\xi; r)) \right)
\]

\[
= C_2 \text{(say)}.
\]

From the mean value theorem we obtain,
\[
\left| \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} - \frac{\partial g_\xi(\xi)}{\partial n_0(\xi)} \right| \leq C_3 \frac{N(\eta^{-1})}{(N(\xi^{-1}))^{2n+2}} \nabla_0 g_\zeta \cdot \left( \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} \right)
\]

Hence we can estimate
\[
\int_{\partial D(\zeta; r)} \left| \frac{\partial g_\eta(\xi)}{\partial n_0(\xi)} \right| \, ds(\xi) \leq C_3 \frac{N(\eta^{-1})}{r^{2n+2}},
\]

for some constant $C_3 > 0$. Now we can combine (10) and (11) to show that
\[
|u(\eta) - u(\xi)| \leq C \left\{ \max_{B[\zeta; r]} |\phi(\xi) - \phi(\zeta)| + \frac{N(\eta^{-1})}{r^{2n+2}} \right\}
\]

for some constant $C > 0$ and all sufficiently small $r$.

Given $\epsilon > 0$ we can choose $r > 0$ such that
\[
\max_{\xi \in B[\zeta; r]} |\phi(\xi) - \phi(\zeta)| \leq \frac{\epsilon}{2C},
\]

for all $\zeta \in \partial D$, since $\phi$ is uniformly continuous on $\partial D$. Then taking, $\delta < \frac{\epsilon r^{2n+2}}{2C}$, we see that $|u(\eta) - u(\zeta)| < \epsilon$ for all $N(\eta^{-1}) < \delta$.

**Theorem 3.5.** Let $\partial D$ be of class $C^2$. Then for a single-layer potential $m$ with continuous density $\phi$, we have

\[
\frac{\partial m_+}{\partial n_0}(\eta) = \int_{\partial D} \phi(\xi) \frac{\partial g_\eta(\xi)}{\partial n_0(\eta)} \, ds(\xi) \pm \phi(\eta), \quad \eta \in \partial D,
\]

where $\frac{\partial m_+}{\partial n_0}(\eta) := \lim_{h \to 0^+} n_0(\eta) \nabla_0 (m(\eta \pm h n_0(\eta)))$.

The limit is to be understood in the sense of uniform convergence on $\partial D$ and the integral exists as an improper integral.

**Proof.** Let $v$ denote the double-layer potential with density $\phi$ and let $U$ be as in the proof of Theorem 3.3. Then for $\eta = \zeta + h n_0(\zeta) \in U \setminus \partial D$, we can write
\[
n_0(\zeta) \nabla_0 (m(\eta)) + v(\eta) = \int_{\partial D} \{ n_0(\xi) - n_0(\eta) \} (\nabla_0)\xi (g_\eta(\xi)) \phi(\xi) \, ds(\xi),
\]

where we have made use of $(\nabla_0)\eta (g_\eta(\xi)) = - (\nabla_0)\xi (g_\eta(\xi))$.

Analogous to the single-layer potential in Theorem 3.3 the right hand side
can be seen to be continuous in $U$. The proof can be now completed by applying Theorem 3.4. □

**Theorem 3.6.** Let $\partial D$ be of class $C^2$. Then a double-layer potential $v$ with continuous density $\phi$ satisfies

$$\lim_{h \to +0} n_0(\eta) \{ \nabla_0(v(\eta + hn_0(\eta))) - \nabla_0(v(\eta - hn_0(\eta))) \} = 0,$$

uniformly for all $\eta \in \partial D$.

**Proof.** The proof is similar in structure to the proof of Theorem 3.4. □

### 3.3. Neumann Boundary Value Problem: Existence

Green’s formula shows that each harmonic function can be represented as a combination of single-layer and double-layer potentials. For boundary value problems we try to find a solution in the form of one of these two potentials. To this end we introduce two integral operators $K, K'$:

$$(K\phi)(\eta) := \int_{\partial D} \phi(\xi) \frac{\partial g_0(\eta)}{\partial n_0(\xi)} \, d\xi, \quad \eta \in \partial D,$$

and

$$(K'\psi)(\eta) := \int_{\partial D} \psi(\xi) \frac{\partial g_0(\eta)}{\partial n_0(\xi)} \, d\xi, \quad \eta \in \partial D.$$

Because of (7) the integral operators $K$ and $K'$ have weakly singular kernels and therefore are compact. As seen by interchanging the order of integration, $K$ and $K'$ are adjoint with respect to the dual system $\langle C(\partial D), C(\partial D) \rangle$ defined by

$$\langle \phi, \psi \rangle := \int_{\partial D} \phi \psi \, d\xi, \quad \phi, \psi \in C(\partial D).$$

**Theorem 3.7.** The null spaces of the operators $I + K$ and $I + K'$ have dimension one.

**Proof.** Let $\phi$ be a solution of $\phi + K\phi = 0$ and again define double layer potential $v$ as in (5). Then by (3), $v_+ = K\phi + \phi = 0$ on $\partial D$, since $v(\eta) = o(1)$, for $N(\eta)$ sufficiently large. From the uniqueness of the exterior Dirichlet problem it follows that $v = 0$ in $\mathbb{H}_n \setminus D$. From Theorem 3.6 we see that $\frac{\partial v_+}{\partial n} = 0$ on $\partial D$ and from the uniqueness of the interior Neumann problem, it follows that $v$ is constant on $D$. From (10) we deduce that $\phi$ is constant on $\partial D$. Therefore, Null space of $(I + K) \subset \text{span} \{c_0\}$ where $c_0$ is a constant and by using Green’s identity, we get

$$\int_{\partial D} \frac{\partial g_0(\xi)}{\partial n_0(\xi)} \, d\xi = 1,$$

so we have $c_0 + Kc_0 = 0$. Thus Null space of $(I + K) = \text{span} \{c_0\}$. By First Fredholm theorem ([12], Theorem 4.15), Null space of $(I + K')$ also has dimension one. □

**Theorem 3.8.** The single-layer potential

$$m(\eta) := \int_{\partial D} \psi(\xi) g_0(\xi) \, d\xi, \quad \eta \in D$$
with continuous density $\psi$ is a solution of the interior Neumann problem \[3\] provided that $\psi$ is a solution of the integral equation
\[\psi(\eta) + \int_{\partial D} \psi(\xi) \frac{\partial g_h(\xi)}{\partial n_0(\eta)} \, ds(\xi) = g(\eta), \quad \eta \in \partial D.\]

**Proof.** This follows from Theorem 3.5. \[\square\]

**Theorem 3.9.** The interior Neumann problem is solvable if and only if
\[\int_{\partial D} g \, ds = 0,\]
is satisfied.

**Proof. Necessary Part**
Using Gaveau’s Green formula \[5\] for a solution $u$ of \[3\] and $v = 1$, we have
\[\int_{\partial D} g \, ds = 0.\]

**Sufficient Part**
Its sufficiency follows from the fact that by Theorem 3.7 it coincides with the solvability condition of the Fredholm alternative for the inhomogeneous integral equation i.e., $\psi + K'\psi = g$.

By Fredholm alternative for integral equations of the second kind that was first obtained by Fredholm \[4\] and restated as a corollary in \((12), Corollary\ 4.18\) the solution of the interior Neumann problem exist if $\int_{\partial D} g \, ds = 0$. \[\square\]

4. Neumann function for korányi ball
In this section we construct an explicit representation formula for Neumann function for Laplacian over the korányi ball in $\mathbb{H}_n$ by means of the fundamental solution for Laplacian, Kelvin tranform and spherical harmonics. Then, making use of this Neumann function, the solution of the Neumann problem for the Poisson equation is given explicitly.

Let $B$ denotes the Korányi ball in $\mathbb{H}_n$, i.e, $B = \{\xi = [z,t] \in \mathbb{H}_n : N(\xi) < 1\}$. For a function $f$ on $\mathbb{H}_n$, the Kelvin transform is defined as in \[8\] by
\[Kf = N^{-2n}f \circ h,\]
where $h$ is the inversion defined as
\[h([z,t]) = \left[ \frac{-z}{|z|^2 - it}, \frac{-t}{|z|^4 + t^2} \right],\]
for $[z,t] \in \mathbb{H}_n \setminus \{e\}$.

For complex $\alpha, \beta$, the functions $C_m^{(\alpha,\beta)}$ $(m = 0, 1, 2, \ldots)$ are defined by the generating function
\[(1 - z\bar{\varsigma})^{-\alpha}(1 - z\varsigma)^{-\beta} = \sum_{m=0}^{\infty} z^m C_m^{(\alpha,\beta)}(\varsigma, \bar{\varsigma}), \quad z, \varsigma \in \mathbb{C}, \quad |z| < |\varsigma|^{-1}.

It follows immediately that
\[C_m^{(\alpha,\beta)}(\varsigma, \varsigma) = \sum_{p=0}^{m} \frac{\alpha_{m-p}(\beta)_p}{(m-p)!p!} (\varsigma)^{m-p} \varsigma^p, \quad \varsigma \in \mathbb{C}.\]
Special case $C_{m}^{(a,\alpha)}(\zeta,\bar{\zeta})$ is denoted by Gegenbauer polynomial. In [7], by using the function $C_{m}^{(a,\alpha)}$, the fundamental solution and its Kelvin transform are expressed as

$$g_{n}^{-1}(\xi^{-1}) = (|z|^{2} + t^{2})^{-n/2} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{m,k,l} t^{-k-l}(|z|^{2} + it)^{-m-l}(|z|^{2} - it)^{-m-k}$$

$$\times C_{m}^{(\frac{n}{2}+\frac{m}{2},\frac{n}{2}+\frac{k}{2}+k)}(t + i|z|^{2})C_{m}^{(\frac{n}{2}+\frac{m}{2},\frac{n}{2}+k)}(t' + i|z'|^{2})Y_{k,l,j}(z')$$,

where $N_{k,l}$ is dimension of the space $H_{k,l}$. The space $H_{k,l}$ of complex (solid) spherical harmonics of bidegree $(k, l)$ on $\mathbb{C}^{n}$ consists of all polynomials $P$ in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$, homogeneous of degree $k$ in the $z_j$'s and homogeneous of degree $l$ in the $z_j$'s satisfying

$$\sum_{j=1}^{n} \frac{\partial^{2}}{\partial z_{j} \partial \bar{z}_{j}}P = 0.$$  

For each $k, l$ the $Y_{k,l,j}$'s form a basis for $H_{k,l}$, where $Y_{k.l,j}$'s have the form

$$Y_{k,l,j}(z) = \sum_{q=0}^{r} c_{q}|z|^{2q}z_{1}^{k-q}(\bar{z}_{1})^{l-q},$$

$z = (z_{1}, z^{*}) \in \mathbb{C}^{n}$, $|z^{*}|^{2} = |z_1|^{2} + \ldots + |z_n|^{2}$, where, $r = \min(k, l)$ and $c_{0}, \ldots, c_{r}$ are constants, $c_{0} = 1$ and $c_{q}$'s are determined by the relation

$$(k - q)(l - q)c_{q} + (q + 1)(n + q - 1)c_{q+1} = 0, \quad 0 \leq q < r.$$  

Now,

$$\tilde{Y}_{k,l,j} = \frac{1}{2\pi} \int_{0}^{2\pi} Y_{k,l,j}(ze^{i\theta})d\theta$$

$$= \sum_{q=0}^{k} c_{q}|z|^{2q}|z_{1}|^{2(k-q)}$$

$$= Y_{k,l,j}(z),$$

and $c_{q}$'s are determined by the relation

$$(k - q)^{2}c_{q} + (q + 1)(n + q - 1)c_{q+1} = 0, \quad 0 \leq q < k.$$  

An easy computation yields the following expressions,

$$\tilde{g}_{n}^{-1}(\xi^{-1}) = (|z|^{4} + t^{2})^{-n/2} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{m,k,l}(|z|^{4} + t^{2})^{-m-k}C_{m}^{(\frac{n}{2}+k,\frac{n}{2}+k)}(t + i|z|^{2})$$

$$\times Y_{k,l,j}(z)C_{m}^{(\frac{n}{2}+k,\frac{n}{2}+k)}(t' + i|z'|^{2})\widetilde{Y}_{k,l,j}(z'),$$
The homogeneous Neumann problem is explicitly given in [10] as

$$K(\bar{g}_{\eta^{-1}}(\xi)) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{N_k} a_{m;k} C_m^{\left(k, \frac{n}{2} + k\right)}(t + i|z|^2) Y_{k,j}(z) C_m^{\left(k, \frac{n}{2} + k\right)}(t' + i|z'|^2) Y_{k,j}(z').$$

The homogeneous Neumann problem is

$$L_0 N_B(\eta, \xi) = \delta_\eta \text{ in } B, \quad (12)$$
$$\frac{\partial}{\partial n_0} N_B(\eta, \xi) = 0 \text{ on } \partial B. \quad (13)$$

Here $L_0$ denotes the Laplacian and $\frac{\partial}{\partial n_0}$ denotes the outward pointing horizontal normal unit vector on the boundary of the Korányi ball and it is explicitly given in [10] as

$$\frac{\partial}{\partial n_0} = \frac{1}{|z|}(\bar{A}E + AE),$$

where $E = \sum z_j Z_j$ and $A = |z|^2 + it$. Now we try to look for a solution of problem (12) and (13) together in the following form

$$N_B(\eta, \xi) = \bar{g}_{\eta^{-1}}(\xi^{-1}) + K(\bar{g}_{\eta^{-1}}(\xi)) + h_m(\eta, \xi),$$

where $\bar{g}_{\eta^{-1}}(\xi^{-1})$ is the fundamental solution for the Laplacian, namely $L_0 \bar{g}_{\eta^{-1}}(\xi^{-1}) = \delta_\eta$ and $K(\bar{g}_{\eta^{-1}}(\xi))$ is harmonic in $B$. So, the function $h_m(\eta, \xi)$ is a circular function that satisfies

$$L_0 h_m(\eta, \xi) = 0, \quad \xi \in B, \quad (14)$$
$$\frac{\partial}{\partial n_0} h_m(\eta, \xi) = - \frac{\partial}{\partial n_0} \left(\bar{g}_{\eta^{-1}}(\xi^{-1}) + K(\bar{g}_{\eta^{-1}}(\xi))\right), \quad \xi \in \partial B. \quad (15)$$

Suppose that there exists a solution for the problem (14) and (15) together in the following form i.e, in the form of Heisenberg harmonics

$$h_m(\eta, \xi) = \sum_{m-k \geq 0} \sum_{j=1}^{N_k} b_{m;k}(z', t') C_m^{\left(k, \frac{n}{2} + k\right)} \left(t + i|z|^2\right) Y_{k,j}(z),$$

where $b_{m;k}(z', t')$ are functions of $\eta = (z', t')$ to be determined.

Now, we claim that $N_B(\eta, \xi) = \bar{g}_{\eta^{-1}}(\xi^{-1}) + K(\bar{g}_{\eta^{-1}}(\xi)) + h_m(\eta, \xi)$ works as Neumann function when applied to circular functions. Since all the three series of $\bar{g}_{\eta^{-1}}(\xi^{-1})$, $K(\bar{g}_{\eta^{-1}}(\xi))$ and $h_m(\eta, \xi)$ are absolutely and uniformly convergent for $\xi \neq \eta$ and $K(\bar{g}_{\eta^{-1}}(\xi))$ and $h_m(\eta, \xi)$ are harmonic. Therefore,

$$L_0 N_B(\eta, \xi) = L_0 \bar{g}_{\eta^{-1}}(\xi^{-1}) = \delta_\eta.$$

We can easily calculate that $E(Y_{k,j}(z)) = kY_{k,j}(z)$ and $E(Y_{k,j}(z)) = kY_{k,j}(z)$, so

$$\frac{\partial}{\partial n_0}(Y_{k,j}(z)) = 2|z|kY_{k,j}(z).$$
Similarly, we have

\[
\frac{\partial}{\partial n_0} \bar{g}_{\eta^{-1}}(\xi^{-1}) = (|z|^4 + t^2)^{-n/2} \sum_{m=0}^{\infty} \sum_{k,l=0}^{\infty} a_{m;k}(|z|^4 + t^2)^{-m-k} C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t + i|z|^2)
\]

\[
\times C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t' + i|z'|^2)Y_{i,j}(\sigma) \left( \frac{\partial}{\partial n_0}(Y_{i,j}(\sigma)) + 2|z|Y_{i,j}(\sigma)(-m - 2k - n) \right)
\]

\[
= (|z|^4 + t^2)^{-n/2} \sum_{m=0}^{\infty} \sum_{k,l=0}^{\infty} a_{m;k}(|z|^4 + t^2)^{-m-k} C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t + i|z|^2)
\]

\[
\times C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t' + i|z'|^2)Y_{i,j}(\sigma)Y_{i,j}(\sigma)(-m - 2k - n).
\]

Similarly, we have

\[
\frac{\partial}{\partial n_0} K(\bar{g}_{\eta^{-1}}(\xi)) = \sum_{m=0}^{\infty} \sum_{k,l=0}^{\infty} a_{m;k} C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t + i|z|^2)C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t' + i|z'|^2)
\]

\[
\times Y_{i,j}(\sigma)Y_{i,j}(\sigma)(m + k).
\]

So, on the boundary of Korányi ball i.e. at $N(\xi) = 1$, we get

\[
\frac{\partial}{\partial n_0} (\bar{g}_{\eta^{-1}}(\xi^{-1}) + K(\bar{g}_{\eta^{-1}}(\xi))) = -2n|z| \sum_{m=0}^{\infty} \sum_{k,l=0}^{\infty} a_{m;k} C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t + i|z|^2)
\]

\[
\times C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t' + i|z'|^2)Y_{i,j}(\sigma)Y_{i,j}(\sigma)(m + k).
\]

Also, at $N(\xi) = 1$, we have

\[
\frac{\partial}{\partial n_0} h_m(\eta, \xi) = \sum_{m-2k \geq 0 \text{ and even}} \sum_{j=1}^{N_k} b_{m;k}(\sigma', t') C_j^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t + i|z|^2)Y_{i,j}(\sigma)Y_{i,j}(\sigma)(m + k).
\]

Therefore, to satisfy (15), choose $b_{m;k}(\sigma', t')$ such that

\[
b_{m;k}(\sigma', t') = \frac{2n}{m} a_{m;k} C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t' + i|z'|^2)Y_{i,j}(\sigma), \ m \in \mathbb{N}.
\]

Notice that when $m = 0$ so $k = 0$ and $C_j^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t' + i|z'|^2) = 1$, $Y_{0,j}(\sigma) = 1$ thus we get

\[
h_m(\eta, \xi) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j=1}^{N_k} b_{0;k}(\sigma', t') C_j^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t + i|z|^2)Y_{i,j}(\sigma)C_m^{(\frac{m}{2}+k, \frac{m}{2}+k)}(t' + i|z'|^2)Y_{i,j}(\sigma) + b_0,
\]

where $b_0$ is a constant.

Since these variations over $m, k$ are infinite, so we have

\[
\frac{\partial}{\partial n_0} h_m(\eta, \xi) = -\frac{\partial}{\partial n_0} (\bar{g}_{\eta^{-1}}(\xi^{-1}) + K(\bar{g}_{\eta^{-1}}(\xi))) \ , \ \xi \in \partial B.
\]

Hence, $N_B(\eta, \xi) = \bar{g}_{\eta^{-1}}(\xi^{-1}) + K(\bar{g}_{\eta^{-1}}(\xi))) + h_m(\eta, \xi)$ works as Neumann function for $B$ when applied to circular functions.

The inhomogeneous circular Neumann boundary value problem for $B$ can be solved by using Neumann function $N_B(\eta, \xi)$ which is proved as follows.
Theorem 4.1. The inhomogeneous Neumann boundary problem

$$L_0 u = f \text{ in } B,$$
$$\frac{\partial}{\partial n_0} u = g \text{ on } \partial B,$$

(16)

where $f$ and $g$ are continuous circular functions, is solvable if and only if

$$\int_B f(\xi) dv(\xi) = \int_{\partial B} g(\xi) d\sigma(\xi),$$

and the solution is given by the representation formula

$$u(\eta) = \int_B N_B(\eta, \xi) f(\xi) dv(\xi) - \int_{\partial B} N_B(\eta, \xi) g(\xi) d\sigma(\xi).$$

Proof. Necessary Part

Using Gaveau’s Green formula [5] for a solution $u$ of inhomogeneous Neumann problem (16) and $v = 1$, we get the required solvability condition.

Sufficient Part

Firstly, we consider the inhomogeneous boundary value problem

$$L_0 u_1 = f \text{ in } B,$$
$$u_1 = 0 \text{ on } \partial B,$$

(17)

where $f$ is a continuous function.

This problem is solvable for every continuous function $f$ on the Korányi ball in $\mathbb{H}_n$. Hence, $\frac{\partial u_1}{\partial n_0}$ has some value on $\partial B$.

Next, we consider the homogeneous Neumann problem

$$L_0 u_2 = 0 \text{ in } B,$$
$$\frac{\partial}{\partial n_0} u_2 = g' \text{ on } \partial B,$$

(18)

where $g' = g - \frac{\partial u_1}{\partial n_0}$ and $g$ is a continuous function.

By Theorem 3.9 interior Neumann problem (18) is solvable if $\int_{\partial B} g' = 0$, i.e., $\int_{\partial B} g = \int_{\partial B} \frac{\partial u_1}{\partial n_0} = \int_B f$ (by Gaveau’s Green identity).

Let $u = u_1 + u_2$.

Since $L_0 u = f$ and $\frac{\partial u_1}{\partial n_0} + g - \frac{\partial u_1}{\partial n_0} = g$, therefore, $u$ is a solution of inhomogeneous Neumann boundary problem (16) if

$$\int_B f(\xi) dv(\xi) = \int_{\partial B} g(\xi) d\sigma(\xi).$$

Representation formula

From Green’s second identity [5], we have

$$\int_B (uL_0 v - vL_0 u) dv(\xi) = \int_{\partial B} \left( u \frac{\partial v}{\partial n_0} - v \frac{\partial u}{\partial n_0} \right) d\sigma(\xi),$$

take $u$ is a solution of (16) and $v = N_B(\eta, \xi)$ in the above identity, we get

$$u(\eta) = \int_B N_B(\eta, \xi) f(\xi) dv(\xi) - \int_{\partial B} N_B(\eta, \xi) g(\xi) d\sigma(\xi),$$

where $f$ and $g$ are continuous circular functions.

This is the required representation formula for a solution of interior Neumann problem on the Korányi ball in the Heisenberg group and holds for circular data only. $\square$
5. Neumann function for half space

In this section we construct an explicit representation formula for Neumann function for the Laplacian over the half space in $\mathbb{H}_n$ by means of the fundamental solution for the Laplacian. The solution of the Neumann problem for the Poisson equation has been obtained using the Neumann function construction.

We denote, by $H$, the half space $\{ \xi = |z, t| \in \mathbb{H}_n : t > 0 \}$. We seek for a function $N_H(\eta, \xi)$ which satisfies

$$L_0 N_H(\eta, \xi) = \delta_\eta \text{ in } H,$$

$$\frac{\partial}{\partial n_0} N_H(\eta, \xi) = 0 \text{ on } \partial H.$$

Here $L_0$ denotes the Laplacian and $\frac{\partial}{\partial n_0}$ denotes the outward pointing horizontal normal unit vector on the boundary of the half space and it is explicitly given in \[11\] as

$$\frac{\partial}{\partial n_0} = \frac{i}{|z|}(E - \bar{E}),$$

where $E = \sum z_j Z_j$.

We claim that $N_H(\eta, \xi) = \bar{g}_\eta(\xi) + \bar{g}_\eta^*(\xi)$ acts as the Neumann function for $\xi \neq \eta$ when applied to circular functions, $\eta^*$ being the reflection of the point $\eta$ with respect to the boundary of the half space i.e. $\eta^* = [z', -t']$.

We have $L_0 N_H(\eta, \xi) = L_0(\bar{g}_\eta(\xi)) = \delta_\eta$, as $\bar{g}_\eta^*(\xi)$ is harmonic in $H$.

Now, we have

$$\bar{g}_\eta(\xi) = a_0 |C_\nu(\eta, \xi)|^{-n} F \left( \frac{n}{2}, \frac{n}{2} ; n ; \frac{|P(\eta, \xi)|^2}{|C_\nu(\eta, \xi)|^2} \right),$$

$$\bar{g}_\eta^*(\xi) = a_0 |C_{-\nu}(\eta, \xi)|^{-n} F \left( \frac{n}{2}, \frac{n}{2} ; n ; \frac{|P(\eta, \xi)|^2}{|C_{-\nu}(\eta, \xi)|^2} \right),$$

where,

$$C_{\pm \nu}(\eta, \xi) = |z|^2 + |z'|^2 + i(t \mp t'),$$

$$P(\eta, \xi) = 2z \bar{z}'. $$

Also,

$$E(|C_\nu(\eta, \xi)|^2) = \sum_{j=1}^{n} z_j (\partial z_j + i \bar{z}_j \partial t) \left( z_j^2 z_j^2 + |z'|^4 + t^2 + t'^2 + 2(z_j \bar{z}_j |z'|^2 - t t') \right)$$

$$= 2|z|^2 (|z|^2 + |z'|^2 + i(t - t')).$$

Similarly,

$$E(|C_{-\nu}(\eta, \xi)|^2) = 2|z|^2 (|z|^2 + |z'|^2 - i(t - t')).$$

So,

$$\frac{\partial}{\partial n_0} (|C_\nu(\eta, \xi)|^2) = -4|z|(t - t').$$

Hence by using the “elementary relations” of Gaussian hypergeometric function \[11\], we get

$$\frac{\partial}{\partial n_0}(\bar{g}_\eta(\xi)) = 2n|z| a_0(t - t') |C_\nu(\eta, \xi)|^{-n-2} F \left( \frac{n}{2} + 1, \frac{n}{2} ; n ; \frac{|P(\eta, \xi)|^2}{|C_\nu(\eta, \xi)|^2} \right).$$
By doing similar steps, one can easily show that
\[
\frac{\partial}{\partial n_0}(\bar{g}_\eta(\xi)) + \bar{g}_\eta^*(\xi) = 2n|z|a_0(t + t')|C_{-\nu}(\eta, \xi)|^{-n-2}F\left(\frac{n}{2} + 1, \frac{n}{2}; n; \frac{|P(\eta, \xi)|^2}{|C_{-\nu}(\eta, \xi)|^2}\right),
\]
and on the boundary of half space i.e, at \( t = 0 \), we have
\[
|C_{\nu}(\eta, \xi)| = |C_{-\nu}(\eta, \xi)|.
\]
Therefore,
\[
\frac{\partial}{\partial n_0}(\bar{g}_\eta(\xi) + \bar{g}_\eta^*(\xi)) = 0 \text{ on } \partial H.
\]
Hence \( N_H(\eta, \xi) = \bar{g}_\eta(\xi) + \bar{g}_\eta^*(\xi) \) acts as the Neumann function for \( H \) when applied to circular data.

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