No existence of the geometric potential for a Dirac fermion on a two-dimensional curved surface of revolution

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For a free particle that non-relativistically moves on a curved surface, there are curvature-induced quantum potentials that significantly influence the surface quantum states, but the experimental results in topological insulators, whenever curved or not, indicate no evidence of such a potential, implying that there does not exist such a quantum potential for the relativistic particles, constrained on the surface or not. Within the framework of Dirac quantization scheme, we demonstrate a general result that for a Dirac fermion on a two-dimensional curved surface of revolution, no curvature-induced quantum potential is permissible.

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I. INTRODUCTION

The discovery of topological insulators has initialized a new era of condensed matter physics [1–7]. However, on one hand, the surface quantum states are experimentally observed in two-dimensional curved or flat topological insulators in which no curvature-induced quantum potential is included. On the other hand, it seems well established that, once a particle is constrained to remain on a curved surface, there may be such a curvature-induced potential, conveniently called as the geometric potential [8–12]. Explicitly, for a non-relativistic particle on the curved surface, the geometric potentials have been theoretically and experimentally explored [8–12], respectively. By the surface quantum states of the topological insulators, we mean the two-dimensional states for relativistic spin 1/2 particles, usually with zero mass, and is is the Dirac fermions as commonly called. There may be no existence of the geometric potential for a Dirac fermion on two-dimensional curved surfaces. The present paper deals with this problem.

The main aim of the present study is to show that for a two-dimensional surface of revolution, such as sphere [6], Beltrami pseudosphere [7], cylinder [13], helicoid [14], etc. [15] Dirac quantization scheme offers a clear theoretical framework to demonstrate no presence of the geometric potential for the Dirac fermions.

For a particle that is constrained to remain on an (N − 1)-dimensional curved surface $\Sigma^{N-1}$ in the flat space $\mathbb{R}^N$ ($N = 2, 3, 4, \ldots$), whether the momentum and the kinetic energy operators must be extended to include the possible contribution of curvatures has been controversial for quite a long time. Part of the problem is the not unique form of the Hamiltonian after quantization, and for a review see [16]. Take a non-relativistic particle on a surface for instance, Ikegami, Nagaoka, Takagi, and Tanzawa in 1992 showed that a quantum potential must certainly be arisen [17], but Kleinert and Shabanov in 1997 demonstrated that no additional potential is permissible from the so-called proper Dirac quantization of a free particle on an (N − 1)-dimensional sphere $S^{N-1}$ in $\mathbb{R}^N$ [18]. However, since 2000, the disputes over such a geometric potential exists for the non-relativistic motion on curved hypersurface have gradually diminished. Especially, during 2010 to 2015, the physical consequences resulting from extrinsic-curvature-dependent geometric potential [8–10] and geometric momentum [19–27] are experimentally confirmed [11, 12, 29], respectively. Nevertheless, for the constrained particle that moves relativistically, whether there is curvature-induced quantum potential remains an open problem. [30–39] There is a mini-review on this subject available in a recent paper [27].

Our principle to explore this problem is simple if not the simplest: All symmetries expressed by the Poisson or Dirac brackets in classical mechanics preserve in quantum mechanics; and so the Hamiltonian itself is also determined by the symmetries. [20, 27] This is an enlargement of the Dirac quantization scheme. Let us first see what the usual Fundamental quantum conditions (FQCs) are for a particle that moves in flat space $\mathbb{R}^N$. In this simplest case, our principle is the conventional Dirac quantization scheme in which FQCs $[x_i,x_j]=0$, $[x_i,p_j]=ih\delta_{ij}$, and $[p_i,p_j]=0$ suffice, which are defined by the commutation relations between positions $x_i$ and momenta $p_j$ ($i, j, k, l = 1, 2, 3, \ldots, N$) where $N$ denotes the number of dimensions of the flat space in which the particle moves [40]. In position representation,
the momentum operator takes simple form as \( \mathbf{p} = -i\hbar \nabla \) where \( \nabla \equiv \mathbf{e}_i \partial / \partial x_i \) is the ordinary gradient operator, and \( \mathcal{N} \) mutually orthogonal unit vectors \( \mathbf{e}_i \) span the \( \mathcal{N} \) dimensional Euclidean space \( \mathcal{R}^\mathcal{N} \). Hereafter the Einstein summation convention over repeated indices is used. Once the particle is constrained to remain on a hypersurface \( \Sigma^{\mathcal{N}-1} \) embedded in \( \mathcal{R}^\mathcal{N} \), the FQCs become [41],

\[
[x_i, x_j] = 0, \quad [x_i, p_j] = i\hbar (\delta_{ij} - n_in_j), \quad \text{and} \quad [p_i, p_j] = -i\hbar \left\{ (n_in_{k,j} - n_jn_{k,i})pk \right\}_{\text{Hermitian}},
\]

where \( O_{\text{Hermitian}} \) stands for a Hermitian operator of an observable \( O \), and the equation of surface \( f(x) = 0 \) can be so chosen that \( \nabla f(x) = 1 \) so \( \mathbf{n} \equiv \nabla f(x) = \mathbf{e}_i n_i \) being the normal at a local point on the surface. FQCs (1) can by no mean give the unique form of the momentum operators, and thus the construction of the unambiguous form of the Hamiltonian operator is certainly impossible within the FQCs [16]. It is then reasonable to introduce more quantum conditions that together with the FQCs must be utilized as first principles. Remember that in classical mechanics, the classical brackets between \( (x, H)_{cb} \) and \( (p, H)_{cb} \) can be easily computed, where the subscripts ”\( cb \)” mean classical brackets, Poisson and Dirac brackets for instance. In quantum mechanics, commutation relations \( [x, H] = i\hbar \{(x, H)_{cb}\}_{\text{Hermitian}} \) and \( [p, H] = -i\hbar \{(p, H)_{cb}\}_{\text{Hermitian}} \) or their derived relations without operator-ordering problem (c.f. (2)) are hypothesized to be requirements upon the form of the Hamiltonian operator \( H \). Our principle for the constrained particle that moves non-relativistically leads to the curvature-induced geometric potential [26]. The first application of the principle to relativistic motion is for a particle that is on 2D surface of sphere [27]. The present paper is the second application to the relativistic motion.

Once the motion is relativistically, we have following Dirac brackets containing classical brackets between \( (x, H) \) and \( (p, H) \) in the following, \( p_i = H [x_i, H]_{D}/c^2 \) and \( \mathbf{n} \wedge [p, H]_{D} = 0 \) [27, 28], where \( [f, g]_D \) denotes Dirac bracket for two classical quantities \( f \) and \( g \). The meaning of \( \mathbf{n} \wedge [p, H] - [p, H] \wedge \mathbf{n} = 0 \).

This set of DQCs imposes restrictions on the form of the Hamiltonian operator. The FQCs and DQCs are the manifestation of our principle for the particle moves relativistically on a hypersurface. To note that the form of generally covariant momentum applicable to the spin particle is easily attainable with a simple inclusion of the spin-connection contribution into the geometric momentum that is originally applicable to the spinless particle [27]. The generally covariant geometric momentum is in general [27]

\[
\mathbf{p} = -i\hbar (\nabla_\Sigma + \frac{M\mathbf{n}}{2} + i\mathbf{x}^\mu \Omega_\mu) = -i\hbar (\nabla_\Sigma + \frac{M\mathbf{n}}{2}) + \hbar \mathbf{x}^\mu \Omega_\mu.
\]

where \( \mathbf{x}^\mu \equiv \partial x^\mu / \partial \xi^\mu \) with \( \xi^\mu = (\xi^1, \xi^2, ..., \xi^{\mathcal{N}-1}) \) being local parameters of the surface \( f(x) = 0 \) and \( x = x(\xi^1, \xi^2, ..., \xi^{\mathcal{N}-1}) \), \( \Omega_\mu = (-i/8) \omega^\mu_{ab} [\gamma_a, \gamma_b] \) in which \( \omega^\mu_{ab} \) are the spin-connections [7, 42–44] and \( \gamma_a \) \( (a, b = 0, 1, 2, ..., \mathcal{N}) \) are Dirac spin matrices, and \( \nabla_\Sigma \equiv \mathbf{e}_i (\delta_{ij} - n_in_j) \partial_j = \nabla - \mathbf{n} \partial_n = \mathbf{x}^\mu \partial_\mu \) is the the gradient operator, and the mean curvature \( M \equiv -\nabla_\Sigma \cdot \mathbf{n} \) is defined by the sum of the all principal curvatures. Without the spin-connection term, \( \mathbf{p} \) (3) reduces to be \(-i\hbar (\nabla_\Sigma + M\mathbf{n}/2) \) [19–22, 26, 29, 45–47]. The rest problem is then to determine the general form of the quantum potential from DQCs (2). Unfortunately it turns out to be a formidable task for we encounter great computational difficulties. However, for a Dirac fermion on a curved surface of revolution \( \Sigma^2 \) in the flat space \( \mathcal{R}^3 \), the calculations are straightforward, and we can show that no quantum potential is admissible.

This paper is organized as follows. In section II, we are going to show that how to apply both FQCs (1) and DQCs (2) for the Dirac fermion on a curved surface of revolution \( \Sigma^2 \), resulting in no geometric potential. In Section III we conclude the present study.

II. A DIRAC FERMION ON A CURVED SURFACE OF REVOLUTION

The curved surface of revolution is with \( u \in \mathcal{R}, \quad v \in [0, 2\pi) \)

\[
x = u \cos v; \quad y = u \sin v; \quad z = f(u).
\]

The metric tensor and the natural diagonal zweibein on a curved surface of revolution are

\[
g_{\mu\nu} = \text{diag} (1 + f''(u), u^2); \quad e^\mu_\mu = \text{diag} \left( \sqrt{1 + f''(u)}, u \right).
\]
The nonzero components of spin connection are
\[\omega^{12}_v = -\omega^{21}_v = \frac{1}{\sqrt{1 + f'^2(u)}}.\] (6)

The generally covariant derivatives are then
\[\nabla_u = \partial_u \quad \text{and} \quad \nabla_v = \partial_v - i \frac{\sigma_z}{2\sqrt{1 + f'^2(u)}}.\] (7)

In final, the relativistic Hamiltonian operator \(H_0 = i\hbar\gamma^a\nabla_a = i\hbar\gamma^a \xi^\mu \nabla_{\mu}\) [42] without geometric potential becomes
\[H_0 = -i\hbar(\sigma_z(\frac{1}{\sqrt{1 + f'^2(u)}} \partial_u) + \frac{1}{2u\sqrt{1 + f'^2(u)}}) + \sigma_y \frac{1}{u} \partial_v).\] (8)

In quantum mechanics, the general form of the Hamiltonian must be assumed to be \(H = H_0 + V_G\) where \(V_G\) will be discussed shortly.

The generally covariant geometric momenta (3) now give,
\[\begin{align*}
p_x &= -i\hbar \left( \frac{\cos v}{1 + f'^2(u)} \partial_u - \frac{\sin v}{u} \partial_v - \frac{\cos v(f'^2(u) + f'^4(u) + u f'(u) f''(u))}{2u(1 + f'^2(u))^2} \right) + \frac{\hbar \sin v}{u} \frac{\sigma_z}{2\sqrt{1 + f'^2(u)}}. \\
p_y &= -i\hbar \left( \frac{\sin v}{1 + f'^2(u)} \partial_u + \frac{\cos v}{u} \partial_v - \frac{\sin v(f'^2(u) + f'^4(u) + u f'(u) f''(u))}{2u(1 + f'^2(u))^2} \right) - \frac{\hbar \cos v}{u} \frac{\sigma_z}{2\sqrt{1 + f'^2(u)}}. \\
p_z &= -i\hbar \left( \frac{f'(u)}{1 + f'^2(u)} \partial_u + \frac{f'(u) + f^2(u) + u f''(u)}{2u(1 + f'^2(u))^2} \right).
\end{align*}\] (9a-9c)

The spin-connections are equivalent to a gauge potential \(A\) whose components are explicitly,
\[A_x = -\frac{\hbar \sin v}{u} \frac{\sigma_z}{2\sqrt{1 + f'^2(u)}}, \quad A_y = \frac{\hbar \cos v}{u} \frac{\sigma_z}{2\sqrt{1 + f'^2(u)}}, \quad A_z = 0.\] (10)

It is compatible with previous results on relationship between spin-connections for fermions on curved surface and gauge fields [5–7, 42, 43].

Considering the orthogonality and completeness of the 2×2 matrices \((I, \sigma_x, \sigma_y, \sigma_z)\), we can assume that the geometric potential \(V_G\) takes the following most general form,
\[V_G = a_0 I + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z,\] (11)

where \((a_0, a_x, a_y, a_z)\) are ansatz functions of \(u\) and \(v\) to be determined via requirements (2). Three commutators \([p_i, H]\) and the results are, respectively,
\[\begin{align*}
[p_x, H] &= \frac{\hbar^2}{2u^2(1 + f'^2(u))^4} (\sigma_y f'^2(u) (1 + 3f'^2(u) + 3f'^4(u) + f'^6(u)) (2 \cos v \partial_v - \sin v) \\
&\quad - \sigma_x u \cos v f'(u) (1 + f'^2(u))^{3/2} (f''(u) + uf^3(u) + 2uf''(u) \partial_u) \\
&\quad + \sigma_x u^2 \cos v (1 + f'^2(u))^{1/2} f''(u) (3f'^2(u) - 1)) + [p_x, V_G], \quad (12a)
\\
[p_y, H] &= \frac{\hbar^2}{2u^2(1 + f'^2(u))^4} (\sigma_y f'^2(u) (1 + 3f'^2(u) + 3f'^4(u) + f'^6(u)) (2 \sin v \partial_v + \cos v) \\
&\quad - \sigma_x u \sin v f'(u) (1 + f'^2(u))^{3/2} (f''(u) + uf^3(u) + 2uf''(u) \partial_u) \\
&\quad + \sigma_x u^2 \sin v (1 + f'^2(u))^{1/2} f''(u) (3f'^2(u) - 1)) + [p_y, V_G], \quad (12b)
\\
[p_z, H] &= \frac{\hbar^2}{2u^2(1 + f'^2(u))^2} \left( \sigma_x u (uf''(u) \partial_u + (1 + f'^2(u)) (f''(u) + uf^3(u)) - 4uf'(u)f''(u)) \\
&\quad + \sigma_y f'(u) (1 + f'^2(u))^{3/2} \partial_v) + [p_z, V_G]. \quad (12c)
\end{align*}\]
During the calculations, we find that $n \wedge [p,H_0] - [p,H_0] \wedge n = 0$. It strongly implies that no geometric potential is necessarily introduced. To see it, let us first compute the following commutation relations $[p_i,V_G]$,

\[
[p_x,V_G] = -i\hbar \left( \frac{\cos v}{1 + f'^2(u)} \partial_u V_G - \frac{\sin v}{u} \partial_v V_G \right) + i\hbar \frac{\sin v}{u} \frac{1}{\sqrt{1 + f'^2(u)}} (a_x \sigma_y - a_y \sigma_x),
\]

\[
[p_y,V_G] = -i\hbar \left( \frac{\sin v}{1 + f'^2(u)} \partial_u V_G + \frac{\cos v}{u} \partial_v V_G \right) - i\hbar \frac{\cos v}{u} \frac{1}{\sqrt{1 + f'^2(u)}} (a_x \sigma_y - a_y \sigma_x),
\]

\[
[p_z,V_G] = -i\hbar \frac{f'(u)}{1 + f'^2(u)} \partial_v V_G.
\]

Three components of the vector equations $n \wedge [p,H] - [p,H] \wedge n = 0$ reduce to $n \wedge [p,V_G] - [p,V_G] \wedge n = 0$. Explicitly, we have,

\[
2i\hbar \left( \frac{\sin v}{\sqrt{1 + f'^2(u)}} \partial_u V_G + \frac{\cos v}{u \sqrt{1 + f'^2(u)}} \partial_v V_G + \frac{\cos v}{u} \frac{1}{(1 + f'^2(u))} (a_x \sigma_y - a_y \sigma_x) \right) = 0,
\]

\[
2i\hbar \left( -\frac{\cos v}{\sqrt{1 + f'^2(u)}} \partial_u V_G + \frac{\sin v}{u \sqrt{1 + f'^2(u)}} \partial_v V_G + \frac{\sin v}{u} \frac{1}{(1 + f'^2(u))} (a_x \sigma_y - a_y \sigma_x) \right) = 0,
\]

\[
2i\hbar \left( \frac{f'(u)}{u \sqrt{1 + f'^2(u)}} \partial_v V_G + \frac{f'(u)}{u (1 + f'^2(u))} (a_x \sigma_y - a_y \sigma_x) \right) = 0.
\]

After simplification, we have,

\[
\left( \sin v \partial_u + \frac{\cos v}{u} \partial_v \right) V_G + \frac{\cos v}{u \sqrt{1 + f'^2(u)}} (a_x \sigma_y - a_y \sigma_x) = 0
\]

\[
\left( -\cos v \partial_u + \frac{\sin v}{u} \partial_v \right) V_G + \frac{\sin v}{u \sqrt{1 + f'^2(u)}} (a_x \sigma_y - a_y \sigma_x) = 0
\]

\[
\partial_v V_G + \frac{1}{\sqrt{1 + f'^2(u)}} (a_x \sigma_y - a_y \sigma_x) = 0
\]

The general solutions are $(a_0, a_x, a_y, a_z) = (c_1,0,0,c_2)$ where $c_1$ and $c_2$ are two constants, i.e., $V_G = c_1 + c_2 \sigma_z = \text{diag}(c_1 + c_2, c_1 - c_2)$. These constants can be set as zero as it is matter of shifting the reference point of the energy. In other words, there is no geometric potential.

Three examples are in the following.

Example one: A Dirac fermion on Torus. The toroidal surface is with two local coordinates $\theta \in [0, 2\pi), \varphi \in [0, 2\pi)$,

\[
x = (R + r \sin \theta) \cos \varphi; \ y = (R + r \sin \theta) \sin \varphi; \ z = r \cos \theta; \ (R > r \neq 0),
\]

where $\varphi$ is the azimuthal angle and $\theta$ the polar angle, and $R$ and $r$ are the outer and inner radii of the torus, respectively. Three equations for geometric potential $V_G$ are, respectively,

\[
\left( \frac{\sin \varphi}{r} \partial_\theta + \frac{\cos \theta \cos \varphi}{R + r \sin \theta} \partial_\varphi \right) V_G + \frac{\cos^2 \theta \cos \varphi}{R + r \sin \theta} (a_x \sigma_y - a_y \sigma_x) = 0
\]

\[
\left( -\frac{\cos \varphi}{r} \partial_\theta + \frac{\cos \theta \sin \varphi}{R + r \sin \theta} \partial_\varphi \right) V_G + \frac{\cos^2 \theta \sin \varphi}{R + r \sin \theta} (a_x \sigma_y - a_y \sigma_x) = 0
\]

\[
\partial_v V_G + \cos \theta (a_x \sigma_y - a_y \sigma_x) = 0
\]

The orthogonality and completeness of the $2 \times 2$ matrices $(I, \sigma_x, \sigma_y, \sigma_z)$ imply that there is no geometric potential, i.e., $V_G = 0$.

Example two: A Dirac fermion on Catenoid. The catenoid is with two local coordinates $\theta \in [0, 2\pi), \rho \in R$,

\[
x = a \cosh \frac{\rho}{\alpha} \cos \theta; \ y = a \cosh \frac{\rho}{\alpha} \sin \theta; \ z = \rho, \ (a > 0)
\]
where $a$ is the constant. Three equations for geometric potential $V_G$ are, respectively,

\[
\left\{\begin{array}{l}
\left(\tanh \frac{\rho}{a} + a \tan \theta \partial_\rho \right) V_G - \tanh^2 \frac{\rho}{a} (a_x \sigma_y - a_y \sigma_x) = 0 \\
\left(\tanh \frac{\rho}{a} - a \cot \theta \partial_\rho \right) V_G - \tanh^2 \frac{\rho}{a} (a_x \sigma_y - a_y \sigma_x) = 0 \\
\partial_\theta V_G - \tanh \frac{\rho}{a} (a_x \sigma_y - a_y \sigma_x) = 0
\end{array}\right. (19a)
\]

Again, there is no geometric potential, i.e., $V_G = 0$.

Example three: A fermion on the symmetric ellipsoid. The symmetric ellipsoid is with two local coordinates

\[
\theta \in (0, \pi), \varphi \in (0, 2\pi),
\]

\[
x = a \sin \theta \cos \varphi; y = a \sin \theta \sin \varphi; z = c \cos \theta,
\]

where $a$ and $c$ are constant. Three equations for geometric potential $V_g$ are, respectively,

\[
\left(\sin \varphi \partial_\theta + \cos \theta \cos \varphi \partial_\varphi \right) V_g + \frac{a \cos^2 \theta \cos \varphi}{\sin \theta \sqrt{(a \cos \theta)^2 + (c \sin \theta)^2}} (a_x \sigma_y - a_y \sigma_x) = 0 \quad (20a)
\]

\[
\left(- \cos \varphi \partial_\theta + \cos \theta \sin \varphi \partial_\varphi \right) V_g + \frac{a \cos^2 \theta \sin \varphi}{\sin \theta \sqrt{(a \cos \theta)^2 + (c \sin \theta)^2}} (a_x \sigma_y - a_y \sigma_x) = 0 \quad (20b)
\]

\[
\frac{c}{a} \partial_\varphi V_g + \frac{c \cos \theta}{\sqrt{(a \cos \theta)^2 + (c \sin \theta)^2}} (a_x \sigma_y - a_y \sigma_x) = 0 \quad (20c)
\]

We see $V_G = 0$ as well.

## III. DISCUSSIONS AND CONCLUSIONS

Surface quantum states exist in many systems. Surface plasmon polaritons and topological insulators are two typical ones. So, we must understand the behaviors of free particle constrained on an $(N-1)$-dimensional curved surface $\Sigma^{N-1}$ embedded in $N$-dimensional flat space $\mathbb{R}^N$. Once the (quasi-)particle moves non-relativistically, we have the geometric potential. Once it moves relativistically instead, there may be no such geometric potential. The present study explicitly shows that an enlargement of the Dirac quantization scheme can give definite results that for the relativistic spin $1/2$ particles constrained on the two-dimensional curved surfaces of revolution, no geometric potential exists.

There are three problems that remain open. One is that usually we take it for granted that non-relativistic motion in flat space is the limit of the relativistic motion in it. Once the motion is constrained, the relation in between is not clear. The second is that in present paper we deal with spin $1/2$ particle, what about an arbitrary boson or fermion under constrained and relativistic motion has been unknown yet. The third is: we are confident that Dirac quantization scheme is sufficient, but the number of quantum conditions must be finite. Though FQCs and DQCs are sufficient for non-relativistic particles in general and relativistic spin $1/2$ particles for some types of surfaces as shown in present paper, we do not know whether they are sufficient for the relativistic particles constrained on $\Sigma^{N-1}$.

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[1] C. L. Kane and E. J. Mele, *Phys. Rev. Lett.* **95**, 146802 (2005).
[2] L. Fu, C. L. Kane, and E. J. Mele, *Phys. Rev. Lett.* **98**, 106803 (2007).
[3] M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).
[4] X. L. Qi and S. C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011).
[5] M. A. H. Vozmediano, M. I. Katsnelson, F. Guinea, *Phys. Rep.* **496**, 109-148 (2010).
[6] D.-H. Lee, *Phys. Rev. Lett.* **103**, 196804 (2009).
[7] A. Iorio and G. Lambiase, *Phys. Rev. D* **90**, 025006 (2014).
[8] H. Jensen and H. Koppe, *Ann. Phys.* **63**, 586-591 (1971).
[9] R. C. T. da Costa, *Phys. Rev. A* **23**, 1982-1987 (1981); *Phys. Rev. A* **25**, 2893-2900 (1982).
[10] G. Ferrari and G. Cuoghi, *Phys. Rev. Lett.* **100**, 230403 (2010).
[11] A. Szameit, F. Dreisow, M. Heinrich, R. Keil, S. Nolte, A. Tönnemann, and S. Longhi, *Phys. Rev. Lett.* **104**, 150403 (2010).
[12] J. Onoe, T. Ito, H. Shima, H. Yoshioka, and S. Kimura, *Europhys. Lett.* **98**, 27001 (2012).
[13] J. H. Bardarson, P. W. Brouwer, and J. E. Moore, *Phys. Rev. Lett.* **105**, 156803 (2010).
[14] M. Watanabe, H. Komatsu, N. Tsuji, and H. Aoki, *Phys. Rev. B* **92**, 205425 (2015).
[15] G. Ferrari and G. Cuoghi, *Phys. Rev. Lett.* **100**, 230403 (2008).
[16] A. V. Golovnev, *Int. J. Geom. Meth. Mod. Phys.* **3**, 655 (2006).
[17] M. Ikegami, Y. Nagaoka, S. Takagi, and T. Tanzawa, *Prog. Theor. Phys.* **88**, 229-249 (1992).
[18] H. Kleinert, and S. V. Shabanov, *Phys. Lett. A*, 232(1997)327-332.
[19] Q. H. Liu, C. L. Tong, and M. M. Lai, *J. Phys. A* **40**, 4161-4168 (2007).
[20] Q. H. Liu, L. H. Tang, D. M. Xun, *Phys. Rev. A* **84**, 042101 (2011).
[21] Q. H. Liu, *J. Math. Phys.* **54**, 122113 (2013).
[22] Q. H. Liu, *J. Phys. Soc. Jap.* **82**, 104002 (2013).
[23] D. M. Xun, Q. H. Liu, X. M. Zhu, *Ann. Phys.* (NY) **338**, 123-133 (2013).
[24] D. M. Xun, Q. H. Liu, *Ann. Phys.* (NY) **341**, 132-141 (2014).
[25] Z. S. Zhang, S. F. Xiao, D. M. Xun, and Q. H. Liu, *Commun. Theor. Phys.* **63**, 19-24 (2015).
[26] D. K. Lian, L. D. Hu, Q. H. Liu, *Ann. Phys.* (Berlin) **530**, 1700415 (2018).
[27] Q. H. Liu, Z. Li, X. Y. Zhou, Z. Q. Yang, W. K. Du, *Eur. Phys. J. C*, 79, 71(2019).
[28] L. D. Hu, D. K. Lian, Q. H. Liu, *Eur. Phys. J. C* **76**, 655 (2016).
[29] R. Spittel, P. Uebel, H. Bartelt, and M. A. Schmidt, *Optics Express*, 23, 12174-12188 (2015).
[30] P. Maraner, J. K. Pachos, *Ann. Phys.* **323**, 2044-2072 (2008).
[31] S. Matsutani, H. Tsuru, *Phys. Rev. A* **46**, 1144-1147 (1992).
[32] M. Burgess and B. Jensen, *Phys. Rev. A* **48**, 1861 (1993).
[33] S. Matsutani, *Prog. Theor. Phys.* **91**, 1005-1037 (1994).
[34] S. Matsutani, *J. Phys. A: Math. Gen.* **30**, 4019 (1997).
[35] S. Matsutani, *Rev. Math. Phys.* **12**, 431 (2000).
[36] V. Atanasov and A. Saxena, *J. Phys.: Condens. Matter* **23**, 175301 (2011).
[37] M. A. Olpak, *Mod. Phys. Lett. A* **27**, 250016 (2012).
[38] V. Atanasov, A. Saxena, *Phys. Rev. B* **92**, 035440 (2015).
[39] F. T. Brandt, J. A. Sánchez-Monroy, *Phys. Lett. A* **380**, 3036-3043 (2016).
[40] P. A. M. Dirac, *The principles of quantum mechanics*, 4th ed (Oxford Univ, Oxford, 1967), pp. 87, 112-114.
[41] S. Weinberg, *Lectures on Quantum Mechanics*, 2nd ed., (Cambridge University Press, Cambridge, 2015), pp. 335-340.
[42] D. R. Brill, and J. A. Wheeler, *Rev. Mod. Phys.* **29**, 465-479 (1957).
[43] N. Ogawa, *Mod. Phys. Lett. A* **12**, 1583–1588 (1997).
[44] A. A. Abrikosov, *Int. J. Mod. Phys. A* **17**, 885-889 (2002).
[45] H. Panahi and L. Jahangiri, Ann. Phys. **372**, 57-67(2016).
[46] Y. L. Wang, H. Jiang, and H. S. Zong, *Phys. Rev. A* **96**, 022116 (2017).
[47] M. S. Shikakhwa, *Commun. Theor. Phys.* **70**, 263–267 (2018).