On maximum parallel classes in packings

Douglas R. Stinson*
David R. Cheriton School of Computer Science
University of Waterloo
Waterloo, Ontario, N2L 3G1, Canada
dstinson@uwaterloo.ca

Ruizhong Wei†
Department of Computer Science
Lakehead University
Orillia, Ontario, L3V 0B9, Canada
rwei@lakeheadu.ca

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Abstract

The integer \( \beta(\rho, v, k) \) is defined to be the maximum number of blocks in any \((v, k)\)-packing in which the maximum partial parallel class (or PPC) has size \( \rho \). This problem was introduced and studied by Stinson in [7] for the case \( k = 3 \). Here, we mainly consider the case \( k = 4 \) and we obtain some upper bounds and lower bounds on \( \beta(\rho, v, 4) \). We also provide some explicit constructions of \((v, 4)\)-packings having a maximum PPC of a given size \( \rho \). For small values of \( \rho \), the number of blocks of the constructed packings are very close to the upper bounds on \( \beta(\rho, v, 4) \). Some of our methods are extended to the cases \( k > 4 \).

1 Introduction

For positive integers \( v \) and \( k \) with \( 2 \leq k \leq v-1 \), a \((v, k)\)-packing is a pair \( \mathcal{S} = (X, \mathcal{B}) \), where \( X \) is a set of \( v \) points and \( \mathcal{B} \) is a set of \( k \)-subsets of \( X \), called blocks, such that every pair of points occurs in at most one block.

The packing number \( D(v, k) \) is the maximum number of blocks in any \((v, k)\)-packing. For \( k = 4 \), we have the following result (see [8]).

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Theorem 1.1. \( D(v, 4) = \left\lceil \frac{v}{4} \right\rceil - \frac{v - 1}{3} \right\rceil - \epsilon \), where

\[
\epsilon = \begin{cases} 
1 & \text{if } v \equiv 7 \text{ or } 10 \text{ mod } 12, v \neq 10, 19 \\
1 & \text{if } v = 9 \text{ or } 17 \\
2 & \text{if } v = 8, 10 \text{ or } 11 \\
3 & \text{if } v = 19 \\
0 & \text{otherwise.}
\end{cases}
\]

When every pair of \( X \) occurs in exactly one block, the packing is a balanced incomplete block design and it is denoted as a \((v, k, 1)\)-BIBD.

A parallel class of a packing \( S \) is a set of disjoint blocks that partitions \( X \). A partial parallel class (or PPC) in \( S \) is any set of disjoint blocks in \( B \). The size of a PPC is the number of blocks in it. A PPC of size \( \rho \) is maximum if there does not exist any PPC in \( S \) of size \( \rho + 1 \).

Define \( \beta(\rho, v, k) \) to be the maximum number of blocks in any \((v, k)\)-packing in which the maximum parallel class has size \( \rho \). Stinson [7] studied \( \beta(\rho, v, 3) \) and gave some explicit constructions of \((v, 3)\)-packings with largest PPC of size \( \rho \). Upper bounds for \( \beta(\rho, v, 3) \) were also proven in [7].

In this paper, we generalize the results of [7] to the case \( k = 4 \). We give two constructions of \((v, 4)\)-packings with largest PPC of size \( \rho \) in Section 2. Then we provide three upper bounds for \( \beta(\rho, v, 4) \) in Section 3. Section 4 determines some exact values of \( \beta(\rho, v, 4) \). Some of our results can be generalized to the cases \( k > 4 \), as discussed in Section 5.

2 Constructions

Stinson [7] constructed \((v, 3)\)-packings with small maximum PPCs using Room squares. A natural generalization of that method, to construct \((v, 4)\)-packings, is obtained by using Kirkman squares.

A Kirkman square, denoted \( KS_k(n) \), is a \( t \times t \) array, say \( \mathcal{R} \), where \( t = \frac{n-1}{k-1} \), defined on an \( n \)-set \( V \), such that

1. every point of \( V \) is contained in one cell of each row and column of \( \mathcal{R} \),
2. each cell of \( \mathcal{R} \) either is empty or contains a \( k \)-subset of \( V \), and
3. the collection of blocks obtained from the non-empty cells of \( \mathcal{R} \) is an \((n, k, 1)\)-BIBD.

A \( KS_2(n) \) is known as a Room square. For \( k = 3 \), [3] and [5] proved the following.

Theorem 2.1. Let \( n \) be a positive integer, \( n \equiv 3 \) mod 6. Then there exists a \( KS_3(n) \) except possibly for \( n \in \{9, 15, 21, 141, 153, 165, 177, 189, 231, 249, 261, 285, 351, 357\} \).

Suppose \( \mathcal{R} \) is a \( KS_3(n) \). Then we can construct an \((n + \rho, 4)\)-packing in which the largest PPC has size \( \rho \), where \( \rho \leq \frac{n}{3} \), as follows. We first select \( \rho \) rows of \( \mathcal{R} \)
such that the first cell in each of these rows is non-empty (there are $\frac{n}{3}$ such rows).

Suppose these rows are denoted $r_1, \ldots, r_\rho$. For each non-empty cell in $r_i$, add a new point $s_i$ to form a block of size four. These $\frac{n}{3}$ blocks form an $(n + \rho, 4)$-packing.

The blocks arising from the cells in the first column of $R$ form a PPC with size $\rho$. Since every block in the packing contains some $s_i$, this PPC is maximum. So we have the following result.

**Theorem 2.2.** Let $n$ be a positive integer, $n \equiv 3 \pmod{6}$, and

$$n \notin \{9, 15, 21, 141, 153, 165, 177, 189, 231, 249, 261, 285, 351, 357\}.$$  

Then there is an $(n + \rho, 4)$-packing with $\frac{n \rho}{3}$ blocks, in which the largest PPC has size $\rho$.

We can improve this result slightly. If we adjoin the blocks of any 4-packing on the points $s_1, \ldots, s_\rho$, then the maximum size of a PPC is still $\rho$, because every block still contains at least one $s_i$.

**Theorem 2.3.** Let $n$ be a positive integer, $n \equiv 3 \pmod{6},$

$$n \notin \{9, 15, 21, 141, 153, 165, 177, 189, 231, 249, 261, 285, 351, 357\}.$$  

Then there is an $(n + \rho, 4)$-packing with $\frac{n \rho}{3} + D(\rho, 4)$ blocks, in which the largest PPC has size $\rho$.

The construction of $(v, 4)$-packings in Theorems 2.2 and 2.3 requires that $v - \rho \equiv 3 \pmod{6}$. We now present a modification that relaxes this condition.

A *transversal design*, denoted $TD(k, n)$, is a triple $(V, G, B)$, where $V$ is a set of $kn$ points, $G$ is a partition of $V$ into $k$ groups, each of size $n$, and $B$ is a collection of $k$-subsets of $V$, called blocks, such that every pair of points from $V$ is contained either in one group or in one block, but not both. A set of $k - 2$ mutually orthogonal latin squares (MOLS) of order $n$ is equivalent to a $TD(k, n)$. For information about MOLS, see [1].

A *transversal* of a latin square $L$ of order $n$ is a set of $n$ cells, one from each row and each column of $L$, such that these $n$ cells contain $n$ different symbols. A *common transversal* of a set of $k$ mutually orthogonal latin squares of order $n$ is a set of $n$ cells that is a transversal of each of the $k$ latin squares.

**Theorem 2.4.** Suppose there are two MOLS of order $n$ that have a common transversal, and suppose that $1 \leq \rho \leq n$. Then there is a $(3n + \rho, 4)$-packing with $n \rho + D(\rho, 4)$ blocks, in which the largest PPC has size $\rho$.

*Proof.* From the two MOLS of order $n$, we first construct the corresponding $TD(4, n)$. Then we delete $n - \rho$ points from the fourth group of the transversal design and we delete all of the blocks containing these points. The remaining $n \rho$ blocks form a $(3n + \rho, 4)$-packing. Finally, we construct a packing of size $D(\rho, 4)$ on the $\rho$ non-deleted points in the fourth group, thus obtaining a $(3n + \rho, 4)$-packing with $n \rho + D(\rho, 4)$ blocks.
The non-deleted blocks corresponding to the transversal form a PPC of size $\rho$. Since each block contains at least one point from the $\rho$ non-deleted points in the fourth group of the $TD$, the maximum size of any PPC is $\rho$.

A *self-orthogonal latin square of order* $n$ (denoted SOLS($n$)) is a latin square that is orthogonal to its transpose. It is easy to see that the main diagonal of a SOLS($n$) is a common transversal of this set of two MOLS($n$).

**Theorem 2.5.** [4] A self-orthogonal latin square of order $n$ exists for all positive integers $n \neq 2, 3$ or 6.

From Theorem 2.4 and Theorem 2.5, we have the following.

**Theorem 2.6.** If $n \geq 4, n \neq 6$, then for $1 \leq \rho \leq n$, there is a $(3n + \rho, 4)$-packing with $n\rho + D(\rho, 4)$ blocks, in which the size of largest PPC has size $\rho$.

**Remark 2.7.** Theorems 2.3 and 2.6 yield similar results. They both produce packings of size roughly equal to

$$\frac{\rho(v - \rho)}{3} + \frac{\rho(\rho - 1)}{12},$$

for appropriate values of $v$.

Theorem 2.6 yields a $(v, 4)$-packing with a PPC of size $\rho$, in which $v - \rho \equiv 0 \mod 3$. We now describe some variations where $v - \rho \equiv 1, 2 \mod 3$.

Suppose we use Theorem 2.6 to construct a $(3n + \rho, 4)$-packing with $n\rho + D(\rho, 4)$ blocks, in which the size of largest PPC has size $\rho$. Then delete a point $x$ that is not in a block of the PPC of size $\rho$ (this requires $\rho \leq n - 1$), along with the $\rho$ blocks that contain $x$. We obtain the following.

**Theorem 2.8.** If $n \geq 4, n \neq 6$, then for $1 \leq \rho \leq n - 1$, there is a $(3n - 1 + \rho, 4)$-packing with $(n - 1)\rho + D(\rho, 4)$ blocks, in which the size of largest PPC has size $\rho$.

In Theorem 2.8, we construct a $(v, 4)$-packing with a PPC of size $\rho$, in which $v - \rho \equiv 2 \mod 3$. We now handle the last case, where $v - \rho \equiv 1 \mod 3$. Again, we use Theorem 2.6 to construct a $(3n + \rho, 4)$-packing with $n\rho + D(\rho, 4)$ blocks, in which the size of largest PPC has size $\rho$. We carry out the following modifications:

1. Delete the $D(\rho, 4)$ blocks on the $\rho$ points in the last group.

2. Adjoin the blocks of a 4-packing consisting of $D(\rho + 1, 4)$ blocks constructed on the $\rho$ points in the last group and one new point.

**Theorem 2.9.** If $n \geq 4, n \neq 6$, then for $1 \leq \rho \leq n$, there is a $(3n + 1 + \rho, 4)$-packing with $n\rho + D(\rho + 1, 4)$ blocks, in which the size of largest PPC has size $\rho$. 

4
3 Upper bounds

In this section, we prove three upper bounds on $\beta(\rho, v, 4)$ (i.e., the maximum number of blocks in a $(v, 4)$-packing with largest PPC of size $\rho$).

We begin by defining some notation. Suppose $S = (X, B)$ is a $(v, 4)$-packing with $b$ blocks, in which the maximum partial parallel class has size $\rho$; hence $v \geq 4\rho$. Let $P = \{B_1, \ldots, B_\rho\}$ be a set of $\rho$ disjoint blocks and let $P = \bigcup_{i=1}^\rho B_i$. Let $T = X \setminus P$. Thus, $|P| = 4\rho$ and $|T| = v - 4\rho$.

For $x \in P$ and for $0 \leq i \leq 3$, let $T^i_x$ denote the set of blocks not in $P$ that contain the point $x$ and exactly $i$ points in $T$. Define $t^i_x = |T^i_x|$.

3.1 The First Bound

Our first bound uses the method described in [7]. We observe that, for any two points $x, y \in B_i \in P$, a block $B \in T^3_x$ and a block $B' \in T^3_y$ cannot be disjoint. Otherwise, we can delete $B_i$ and add the two blocks $B$ and $B'$ to the PPC to get a new PPC of size $\rho + 1$.

The following lemma is a straightforward consequence of this observation.

Lemma 3.1. Suppose $B_i = \{w, x, y, z\} \in P$ and $t^3_w \geq \max\{t^3_x, t^3_y, t^3_z\}$. Then one of the following two conditions holds:

1. $t^3_w \leq 3$, or
2. $t^3_w \geq 4$ and $t^3_x = t^3_y = t^3_z = 0$.

Proof. Suppose $t^3_w \geq 4$ and $B \in T^3_x \cup T^3_y \cup T^3_z$. Then there is a block $B' \in T^3_w$ such that $B \cap B' = \emptyset$. This contradicts the observation above. Hence $t^3_x = t^3_y = t^3_z = 0$ if $t^3_w \geq 4$. \qed

Since $P$ contains $\rho$ blocks, we have the following bound.

Theorem 3.2.\

$$\beta(\rho, v, 4) \leq \rho \left((8\rho - 7) + \max\left\{12, \left\lfloor \frac{v - 4\rho}{3} \right\rfloor \right\}\right).$$

Proof. Since $P$ is maximum, there is no block contained in $T$. From Lemma 3.1, there are at most $\rho \times \max\left\{12, \left\lfloor \frac{v - 4\rho}{3} \right\rfloor \right\}$ blocks having one point in $P$ and three points in $T$.

The number of pairs of points in $P$ that are not contained in a block of $P$ is

$$\binom{4\rho}{2} - 6\rho = 8\rho(\rho - 1).$$

Therefore there are at most $8\rho(\rho - 1)$ blocks not in $P$ that contain at least two points in $P$. Finally, there are $\rho$ blocks in $P$.\[5\]
Therefore, in total, there are at most
\[ 8\rho(\rho - 1) + \rho \times \max \left\{ 12, \left\lfloor \frac{v - 4\rho}{3} \right\rfloor \right\} + \rho \]
blocks in the packing.

**Corollary 3.3.** For \( v \geq 4\rho + 36 \), it holds that
\[ \beta(\rho, v, 4) \leq \frac{\rho v}{3} + \frac{20\rho^2}{3} - 7\rho. \]

**Proof.** We have that
\[ \frac{v - 4\rho}{3} \geq 12 \]
if and only if \( v \geq 4\rho + 36 \). Thus, when \( v \geq 4\rho + 36 \), Theorem 3.2 yields
\[ \beta(\rho, v, 4) \leq \rho \left( 8\rho - 7 + \frac{v - 4\rho}{3} \right) = \frac{\rho v}{3} + \frac{20\rho^2}{3} - 7\rho. \]

\[ \square \]

### 3.2 The Second Bound

We now prove a somewhat better bound by using a more precise counting argument. The next two lemmas are straightforward.

**Lemma 3.4.** For any \( x \in P \), the following two inequalities hold:
\[ 3t^3_x + 2t^2_x + t^1_x \leq v - 4\rho. \]  
and
\[ t^2_x + 2t^1_x + 3t^0_x \leq 4(\rho - 1). \]  

**Lemma 3.5.** The number of blocks \( b \) in the packing is given by the following formula:
\[ b = \rho + \sum_{x \in P} t^3_x + \frac{1}{2} \sum_{x \in P} t^2_x + \frac{1}{3} \sum_{x \in P} t^1_x + \frac{1}{4} \sum_{x \in P} t^0_x. \]  

For any \( x \in P \), define
\[ c_x = t^3_x + \frac{t^2_x}{2} + \frac{t^1_x}{3} + \frac{t^0_x}{4} = \frac{12t^3_x + 6t^2_x + 4t^1_x + 3t^0_x}{12}. \]

Then it is clear from Lemma 3.5 that the following equation holds:
\[ b = \rho + \sum_{x \in P} c_x. \]  

Our strategy is to obtain upper bounds on \( c_x \) given the constraints 1 and 2. This leads to an integer program; however, for convenience, we will consider the linear programming relaxation. For ease of notation, let us fix a point \( x \) and denote \( y_3 = t^3_x, y_2 = t^2_x, y_1 = t^1_x \) and \( y_0 = t^0_x \). We are interested in the optimal solution to the following LP:
maximize \[ 12y_3 + 6y_2 + 4y_1 + 3y_0 \]
subject to the constraints
\[ 3y_3 + 2y_2 + y_1 \leq v - 4\rho \]  \hspace{1cm} (5) \\
\[ y_2 + 2y_1 + 3y_0 \leq 4(\rho - 1) \]  \hspace{1cm} (6) \\
y_3, y_2, y_1, y_0 \geq 0

If we compute \( 4 \times (5) + (6) \), we obtain the following bound:
\[ 12y_3 + 9y_2 + 6y_1 + 3y_0 \leq 4v - 12\rho - 4. \]  \hspace{1cm} (7)

Since
\[ 12c_x = 12y_3 + 6y_2 + 4y_1 + 3y_0 \leq 12y_3 + 9y_2 + 6y_1 + 3y_0, \]
we have the following.

**Lemma 3.6.**
\[ c_x \leq \frac{v - 3\rho - 1}{3}. \]  \hspace{1cm} (8)

**Remark 3.7.** We note that we can achieve equality in (8) by taking
\[ y_3 = \frac{v - 4\rho}{3}, \quad y_2 = 0, \quad y_1 = 0, \quad \text{and} \quad y_0 = \frac{4(\rho - 1)}{3}. \]
Thus, the optimal solution to the LP is \( 4(v - 3\rho - 1) \).

We are also interested in the optimal solution to the LP in the special cases where \( y_3 \leq 3 \). Here, we just use the inequality
\[ 6y_2 + 12y_1 + 18y_0 \leq 24(\rho - 1), \]
which follows immediately from (6). Since
\[ 12c_x = 12y_3 + 6y_2 + 4y_1 + 3y_0 \leq 12y_3 + 6y_2 + 12y_1 + 18y_0, \]
we obtain the following.

**Lemma 3.8.**
\[ c_x \leq y_3 + 2\rho - 2. \]  \hspace{1cm} (9)

Since we are assuming that \( y_3 \leq 3 \), we have
\[ c_x \leq 2\rho + 1. \]  \hspace{1cm} (10)
Remark 3.9. We can achieve equality in (9) by taking
\[ y_2 = 4(\rho - 1), \quad y_1 = 0, \quad \text{and} \quad y_0 = 0. \]
This is a feasible solution to the LP provided that (5) is satisfied, i.e., if
\[ 3y_3 + 8(\rho - 1) \leq v - 4\rho, \]
which simplifies to
\[ v \geq 3y_3 + 12\rho - 8. \]
We are assuming that \( y_3 \leq 3 \), so the optimal solution to the LP is \( 24\rho + 12 \) whenever \( v \geq 12\rho + 1 \).

The following lemma is an immediate application of (10).

Lemma 3.10. Suppose that \( B_i = \{w, x, y, z\} \in \mathcal{P} \) and \( \max \{t^3_w, t^3_x, t^3_y, t^3_z\} \leq 3 \). Then
\[ c_w + c_x + c_y + c_z \leq 8\rho + 4. \] (11)

Lemma 3.11. Suppose that \( B_i = \{w, x, y, z\} \in \mathcal{P} \) and \( \max \{t^3_w, t^3_x, t^3_y, t^3_z\} \geq 4 \). Then
\[ c_w + c_x + c_y + c_z \leq \frac{v}{3} + 5\rho - \frac{19}{3}. \] (12)

Proof. Without loss of generality, assume that \( t^3_w = \max \{t^3_w, t^3_x, t^3_y, t^3_z\} \geq 4 \). Then \( t^3_x = t^3_y = t^3_z = 0 \). Hence, we have \( c_w \leq (v - 3\rho - 1)/3 \) from (5) and we obtain \( c_x, c_y, c_z \leq 2\rho - 2 \) by setting \( y_3 = 0 \) in (9). Hence,
\[ c_w + c_x + c_y + c_z \leq 3(2\rho - 2) + \frac{v - 3\rho - 1}{3} = \frac{v}{3} + 5\rho - \frac{19}{3}. \]

\[ \square \]

Theorem 3.12. Suppose \( v \geq 9\rho + 31 \). Then
\[ \beta(\rho, v, 4) \leq \frac{\rho v}{3} + 5\rho^2 - \frac{16\rho}{3}. \]

Proof. If \( v \geq 9\rho + 31 \), then
\[ \frac{v}{3} + 5\rho - \frac{19}{3} \geq 8\rho + 4. \]
Hence, from Lemmas 3.10 and 3.11
\[ c_w + c_x + c_y + c_z \leq \frac{v}{3} + 5\rho - \frac{19}{3} \]
for all \( \rho \) blocks \( \{w, x, y, z\} \in \mathcal{P} \). Now, applying (4), we obtain the upper bound
\[ \beta(\rho, v, 4) \leq \rho + \rho \left( \frac{v}{3} + 5\rho - \frac{19}{3} \right) = \frac{\rho v}{3} + 5\rho^2 - \frac{16\rho}{3}. \]

\[ \square \]
Remark 3.13. Ignoring lower order terms, the upper bound on $\beta(\rho, v, 4)$ proven in Theorem 3.12 is
\[
\frac{\rho v}{3} + 5\rho^2,
\]
while the previous bound from Corollary 3.3 was
\[
\frac{\rho v}{3} + \frac{20\rho^2}{3}.
\]

3.3 The Third Bound

The third upper bound on $\beta(\rho, v, 4)$ is based on more refined analysis of $T_x^i, 0 \leq i \leq 3$. As defined above, the blocks in the PPC are denoted as $B_i = \{a_i, b_i, c_i, d_i\},$ for $i = 1, \ldots, \rho$. We further assume that
\[
t^3_{a_1} \leq t^3_{a_2} \leq \cdots \leq t^3_{a_\rho},
\]
and
\[
t^3_{a_i} \geq \max\{t^3_{b_i}, t^3_{c_i}, t^3_{d_i}\},
\]
for $i = 1, \ldots, \rho$. Let $A = \{a_i: 1 \leq i \leq \rho\}$.

Now we will partition the blocks of $B \setminus P$ into various subsets as follows.

1. $\{T_x^3: x \in P\}$.

2. For blocks in $\bigcup_{x \in P} T_x^2$, let
   \[
   A_i = \{\{a_i, e, y, z\} \in B \setminus P: e \in P \setminus A; y, z \in T\}
   \]
   $A'_i = \{\{a_i, a_s, y, z\} \in B \setminus P: i + 1 \leq s \leq \rho; y, z \in T\}$
   $C = \{\{e, f, y, z\} \in B \setminus P: e, f \in P \setminus A; y, z \in T\}.$

   The blocks in $A_i$ contain one point in $A$, the blocks in $A'_i$ contain two points in $A$, and the blocks in $C$ contain no points in $A$.

3. For blocks in $\bigcup_{x \in P} T_x^1$, let
   $E = \{\{e, f, g, z\} \in B \setminus P: e, f, g \in P \setminus A; z \in T\},$
   and let $E'$ consist of the remaining blocks in $\bigcup_{x \in P} T_x^1$ (the blocks in $E$ contain no points in $A$ and the blocks in $E'$ contain at least one point in $A$). Further, we partition the blocks in $E'$ into subsets $E'_1, \ldots, E'_p$, where a block in $B \in E'$ is placed in $E'_i$ if $a_i \in B$ and $a_j \notin B$ for any $j < i$.

4. For blocks in $\bigcup_{x \in P} T_x^0$, let
   $F = \{\{e, f, g, h\} \in B \setminus P: e, f, g, h \in P \setminus A\},$
   and let $F'$ consist of the remaining blocks in $\bigcup_{x \in P} T_x^0$ (the blocks in $F$ contain no points in $A$ and the blocks in $F'$ contain at least one point in $A$). Further, we partition the blocks in $F'$ into subsets $F'_1, \ldots, F'_p$, where a block in $B \in F'$ is placed in $F'_i$ if $a_i \in B$ and $a_j \notin B$ for any $j < i$. 

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For future use, we define the following notation:

\[ \alpha_i = |A_i| + |A'_i|, \]
\[ \epsilon_i = |E'_i|, \quad \text{and} \]
\[ \zeta_i = |F'_i|, \]

for \( i = 1, \ldots, \rho \).

Since the various subsets of blocks defined above are disjoint, the total number of blocks in the packing is

\[ b = \sum_{x \in P} |T^3_x| + \sum_{i=1}^\rho (\alpha_i + \epsilon_i + \zeta_i) + |C| + |E| + |F| + \rho. \]  

(13)

**Lemma 3.14.** For \( i = 1, \ldots, \rho \), it holds that

\[ \alpha_i + 2\epsilon_i + 3\zeta_i \leq 4\rho - i - 3. \]  

(14)

**Proof.** Let \( 1 \leq i \leq \rho \). Denote \( U_i = P \setminus (A_i \cup \{a_j : j < i\}) \). We note that \( |U_i| = 4\rho - i - 3 \). Each block in \( |A_i| \cup |A'_i| \) contains one point in \( U_i \), each block in \( |E'_i| \) contains two points from \( U_i \), and each block in \( |F'_i| \) contains three points from \( U_i \). Further, each point from \( U_i \) occurs in at most one block of \( |A_i| \cup |A'_i| \cup |E'_i| \cup |F'_i| \). The result follows.

**Lemma 3.15.** For \( i = 1, \ldots, \rho \), it holds that

\[ t^3_{a_i} + \alpha_i + \epsilon_i + \zeta_i \leq \frac{v - i - 3}{3}. \]  

(15)

**Proof.** Let \( 1 \leq i \leq \rho \). Every block in \( T^3_i \cup |A_i| \cup |A'_i| \cup |E'_i| \cup |F'_i| \) contains the point \( a_i \) but these blocks are otherwise disjoint. Also, none of these blocks contains \( b_i, c_i \) or \( d_i \), or any \( a_j \) with \( j < i \). The result follows.

For \( 1 \leq i \leq \rho \), denote \( L_i = t^3_{a_i} + t^3_{b_i} + t^3_{c_i} + t^3_{d_i} + \alpha_i + \epsilon_i + \zeta_i \).

**Lemma 3.16.** Suppose \( v \geq 12\rho + 28 \). Then for each \( i, 1 \leq i \leq \rho \), we have

\[ L_i \leq \frac{v - i - 3}{3}. \]  

(16)

**Proof.** First, consider the case when \( t^3_{a_i} \geq 4 \). Then, \( t^3_{b_i} + t^3_{c_i} + t^3_{d_i} = 0 \) and we have

\[ L_i = t^3_{a_i} + \alpha_i + \epsilon_i + \zeta_i \]
\[ \leq \frac{v - i - 3}{3} \]

from (15).
Now we consider the case where $t_{a_i}^3 \leq 3$. Then we have

$$L_i \leq 12 + \alpha_i + \epsilon_i + \zeta_i$$
$$\leq 12 + \alpha_i + 2\epsilon_i + 3\zeta_i$$
$$\leq 12 + 4\rho - i - 3$$
$$= 4\rho - i + 9,$$

from (14). Since $v \geq 12\rho + 28$, we have $\rho \leq \frac{v - 28}{12}$ and hence

$$L_i \leq \frac{v - 28}{3} - i + 9$$
$$= \frac{v - 3i - 1}{3}$$
$$\leq \frac{v - i - 3}{3},$$

since $i \geq 1$. \hfill \Box

Lemma 3.17. If $\rho \geq 2$, $t_{a_1}^3 \geq 3$ and $t_{a_2}^3 \geq 6$, then $C = \emptyset$.

Proof. First, suppose there is a block $B = \{e, f, x, y\} \in C$, where $x, y \in T$ and $e, f \in P \setminus A$. Assume that $e \in B_i$ and $f \in B_j$, where $i < j$. Since $t_{a_1}^3 \leq t_{a_2}^3 \leq \cdots \leq t_{a_\rho}^3$, we have $t_{a_1}^3 \geq 3$ and $t_{a_2}^3 \geq 6$.

Since $t_{a_1}^3 \geq 3$, we can choose a block $B'_1 \in T_{a_1}^3$ that is disjoint from $B$ (we just choose $B'_1 \in T_{a_1}^3$ such that $x, y \not\in B$). Similarly, since $t_{a_2}^3 \geq 6$, we can choose a block $B'_2 \in T_{a_2}^3$ that is disjoint from $B$ and $B'_1$ (note that $B$ and $B'_1$ contain five points from $T$). Then, deleting $B_i$ and $B_j$ from the PPC and adjoining $B, B'_1$ and $B'_2$, we obtain $\rho + 1$ disjoint blocks, which is a contradiction. \hfill \Box

Lemma 3.18. If $\rho \geq 3$, $t_{a_1}^3 \geq 2$, $t_{a_2}^3 \geq 5$ and $t_{a_3}^3 \geq 8$, then $E = \emptyset$.

Proof. Suppose there is a block $B = \{b, c, d, x\} \in E$, where $b, c, d \in P \setminus A$, $b \in B_i$, $c \in B_j$, $d \in B_k$, and $x \in T$. Similar to the proof of Lemma 3.17, we can find $B'_1 \in T_{a_1}^3$, $B'_2 \in T_{a_2}^3$ and $B'_3 \in T_{a_3}^3$ such that $B, B'_1, B'_2, B'_3$ are disjoint. If we delete $B_i, B_j$ and $B_k$ from the PPC and adjoin $B, B'_1, B'_2$ and $B'_3$, we obtain $\rho + 1$ disjoint blocks, which is a contradiction. \hfill \Box

Lemma 3.19. If $\rho \geq 4$, $t_{a_1}^3 \geq 1$, $t_{a_2}^3 \geq 4$, $t_{a_3}^3 \geq 7$ and $t_{a_4}^3 \geq 10$, then $F = \emptyset$.

Proof. Suppose there is a block $B = \{b, c, d, e\} \in F$, where $b, c, d, e \in P \setminus A$, $b \in B_i$, $c \in B_j$, $d \in B_k$, and $e \in B_l$. Similar to the proof of Lemma 3.17, we can find $B'_1 \in T_{a_1}^3$, $B'_2 \in T_{a_2}^3$, $B'_3 \in T_{a_3}^3$ and $B'_4 \in T_{a_4}^3$ such that $B, B'_1, B'_2, B'_3, B'_4$ are disjoint. If we delete $B_i, B_j, B_k$ and $B_l$ from the PPC and adjoin $B, B'_1, B'_2, B'_3$ and $B'_4$, we obtain $\rho + 1$ disjoint blocks, which is a contradiction. \hfill \Box
Lemma 3.20. Suppose $\rho \geq 4$, $t_{a_1}^3 \geq 3$, $t_{a_2}^3 \geq 6$, $t_{a_3}^3 \geq 8$ and $t_{a_4}^3 \geq 10$, and $v \geq 12\rho + 28$. Then the number of blocks in the packing is at most
\[
\frac{\rho v}{3} - \frac{\rho(\rho + 1)}{6}
\]

Proof. Let $M = \sum_{i=1}^{\rho} L_i$. From Lemma 3.16 we have
\[
M \leq \sum_{i=1}^{\rho} \left( v - i - 3 \right) = \frac{\rho(v - 3)}{3} - \frac{\rho(\rho + 1)}{6} = \frac{\rho v}{3} - \frac{\rho^2 + 7\rho}{6}.
\]
Since $C \cup E \cup F = \emptyset$ by Lemmas 3.17, 3.18 and 3.19 the total number of blocks is
\[
M + \rho \leq \frac{\rho v}{3} - \frac{\rho^2 + 7\rho}{6} + \rho = \frac{\rho v}{3} - \frac{\rho(\rho + 1)}{6}.
\]

Lemma 3.20 provides a good upper bound on the number of blocks when the four smallest values $t_{a_i}$ are large enough, because the conditions ensure there are no blocks in $C \cup E \cup F$. On the other hand, if even one of these four values is “small,” then we will obtain a bound on the number of blocks by upper-bounding the relevant $t_{a_i}$ by a quantity that is independent of $v$. In this situation, we will just use a trivial upper bound on the number of blocks in $C \cup E \cup F$.

Lemma 3.21. Suppose $t_{a_1}^3 < 3$, $t_{a_2}^3 < 6$, $t_{a_3}^3 < 8$ or $t_{a_4}^3 < 10$. If $v \geq 12\rho + 28$, then the number of blocks in the packing is at most
\[
\frac{(\rho - 1)v}{3} + \frac{\rho(13\rho - 2)}{3} + 10
\]

Proof. Let $M = \sum_{j=1}^{\rho} L_j$. Since $v \geq 12\rho + 28$, we can apply Lemma 3.16. Choose $i \leq 4$ such that $t_{a_i}^3 \leq 3i - 1$ (at least one such value of $i$ exists).

Suppose first that $t_{a_i}^3 \geq 4$. Then $t_{a_{4i}}^3 = t_{a_{3i}}^3 = t_{a_{2i}}^3 = 0$. We have $L_j \leq \frac{v - j - 3}{3}$ for all $j$ from (16). Also, $L_i \leq 3i - 1 + 4\rho - i - 3$ from (14). Thus we have
\[
M \leq \sum_{j=1, j\neq i}^{\rho} \left( \frac{v - j - 3}{3} \right) + 3i - 1 + 4\rho - i - 3 = \frac{(\rho - 1)(v - 3)}{3} - \frac{\rho(\rho + 1)}{6} + i + 2i + 4\rho - 4 \leq \frac{(\rho - 1)(v - 3)}{3} - \frac{\rho(\rho + 1)}{6} + \frac{28}{3} + 4\rho - 4 \quad \text{since } i \leq 4
\]
On the other hand, if $t_{a_i} \leq 3$, then $t_{a_i} + t_{b_i} + t_{c_i} + t_{d_i} \leq 12$. So, by a similar argument, we obtain

$$M \leq \sum_{j=1, j \neq i}^{\rho} \left( \frac{v - j - 3}{3} \right) + 12 + 4\rho - i - 3$$

$$= \frac{(\rho - 1)(v - 3)}{3} - \frac{\rho(\rho + 1)}{6} + \frac{i}{3} - i + 4\rho + 9$$

$$< \frac{(\rho - 1)(v - 3)}{3} - \frac{\rho(\rho + 1)}{6} + 4\rho + 9 \quad \text{since } i > 0$$

$$= \frac{(\rho - 1)(v - 3)}{3} - \frac{\rho(\rho - 23)}{6} + 9.$$ 

Now, each block in $C \cup E \cup F$ contains at least one pair of points from $P \setminus A$. Further, none of these blocks contains more than one point from any block in $P$. Hence,

$$|C \cup E \cup F| \leq \left(\frac{3\rho}{2}\right) - 3\rho = \frac{9\rho(\rho - 1)}{2}.$$ 

Therefore, the total number of blocks, $b$, satisfies the following inequality:

$$b \leq \frac{(\rho - 1)(v - 3)}{3} - \frac{\rho(\rho - 23)}{6} + 9 + \frac{9\rho(\rho - 1)}{2} + \rho$$

$$= \frac{(\rho - 1)v}{3} - (\rho - 1) - \frac{\rho(\rho - 23)}{6} + 9 + \frac{9\rho(\rho - 1)}{2} + \rho$$

$$= \frac{(\rho - 1)v}{3} - \frac{\rho(\rho - 23)}{6} + 9 + \frac{9\rho(\rho - 1)}{2} + 10$$

$$= \frac{(\rho - 1)v}{3} + \frac{\rho(13\rho - 2)}{3} + 10.$$ 

\[\square\]

When $v$ is sufficiently large compared to $\rho$, the bound of Lemma 3.20 is the relevant bound.

**Theorem 3.22.** Suppose $v \geq \frac{1}{2}(27\rho^2 - 3\rho + 60)$. Then the number of blocks in the packing is at most

$$\frac{\rho v}{3} - \frac{\rho(\rho + 1)}{6}.$$ 

**Proof.** From Lemmas 3.20 and 3.21, we have

$$b \leq \max \left\{ \frac{\rho v}{3} - \frac{\rho(\rho + 1)}{6}, \frac{(\rho - 1)v}{3} + \frac{\rho(13\rho - 2)}{3} + 10 \right\}.$$ 

Since $v \geq \frac{1}{2}(27\rho^2 - 3\rho + 60)$, we have

$$\frac{\rho v}{3} - \frac{\rho(\rho + 1)}{6} - \left( \frac{(\rho - 1)v}{3} + \frac{\rho(13\rho - 2)}{3} + 10 \right) = \frac{v}{3} - \frac{\rho(27\rho - 3)}{6} - 10 \geq 0.$$ 

\[\square\]
Table 1: Some small \((v, 4)\)-packings

| \(v\) | \(D(v, 4)\) |
|------|-------------|
| 7    | 2           |
|      | 1, 2, 3, 4  |
|      | 1, 5, 6, 7  |
| 8    | 2           |
|      | 1, 2, 3, 4  |
|      | 1, 5, 6, 7  |
| 9    | 3           |
|      | 1, 2, 3, 4  |
|      | 1, 5, 6, 7  |
|      | 2, 5, 8, 9  |
| 10   | 5           |
|      | 1, 2, 3, 4  |
|      | 1, 5, 6, 7  |
|      | 2, 5, 8, 9  |
|      | 3, 6, 8, 10 |
|      | 4, 7, 9, 10 |
| 11   | 6           |
|      | 1, 2, 3, 4  |
|      | 1, 5, 6, 7  |
|      | 1, 8, 9, 10 |
|      | 2, 5, 8, 11 |
|      | 3, 6, 9, 11 |
|      | 4, 7, 10, 11|

4 Some Values of \(\beta(\rho, v, 4)\)

In this section, we determine some exact values of \(\beta(\rho, v, 4)\). First, for \(\rho = 1\), we can determine the exact values of \(\beta(1, v, 4)\) for all \(v\).

Theorem 4.1.

\[
\beta(1, v, 4) = \begin{cases} 
D(v, 4) & \text{if } 4 \leq v \leq 13 \\
13 & \text{if } 14 \leq v \leq 39 \\
\left\lfloor \frac{v-1}{3} \right\rfloor & \text{if } v \geq 40 
\end{cases}
\]

Proof. When \(4 \leq v \leq 6\), \(D(v, 4) = 1\), so \(\beta(1, v, 4) = 1\). For \(7 \leq v \leq 11\), we display the blocks of the optimal packings in Table 1. These packings do not contain any disjoint blocks. For \(v = 12\), we have \(D(12, 4) = 9\) and the optimal packing is obtained by deleting a point \(x\) and the four blocks containing \(x\) from a projective plane of order 3. This packing also does not contain disjoint blocks. For \(v = 13\), we have \(D(13, 4) = 13\) and the optimal packing is a projective plane of order 3, which does not contain disjoint blocks.

For \(13 \leq v \leq 40\), Theorem 3.2 gives the bound \(\beta(1, v, 4) \leq 13\), and for \(v \geq 40\), Theorem 3.2 gives the bound \(\beta(1, v, 4) \leq \left\lfloor \frac{v-1}{3} \right\rfloor\). For \(v \geq 40\), Theorems 2.6, 2.8 and 2.9 provide the desired packings. For \(14 \leq v \leq 39\), \(\beta(1, v, 4) = 13\) because

\[13 = \beta(1, 13, 4) \leq \beta(1, v, 4) \leq \beta(1, 40, 4) = 13\]
Proof. The upper bound follows from Theorem 3.22. The lower bounds follow from Theorems 2.6, 2.8 and 2.9.

**Theorem 4.3.** Suppose $v \geq 147$. Then $\beta(3, v, 4) \leq v - 2$. Also,

$$\beta(3, v, 4) \geq \begin{cases} v - 3 & \text{if } v \equiv 0, 1 \mod 3 \\ v - 5 & \text{if } v \equiv 2 \mod 3. \end{cases}$$

Proof. The upper bound follows from Theorem 3.22 and the lower bounds follow from Theorems 2.6, 2.8 and 2.9. Note that we use the fact that $D(4, 4) = 1$ when $v \equiv 1 \mod 3$; in this case, we apply Theorem 2.9.

**Theorem 4.4.** Suppose $v \geq 240$. Then

$$\beta(4, v, 4) \leq \left\lfloor \frac{4v - 10}{3} \right\rfloor.$$  

Also,

$$\beta(4, v, 4) \geq \begin{cases} \frac{4v - 21}{3} & \text{if } v \equiv 0 \mod 3 \\ \frac{4v - 13}{3} & \text{if } v \equiv 1 \mod 3 \\ \frac{4v - 17}{3} & \text{if } v \equiv 2 \mod 3. \end{cases}$$

Proof. The upper bound follows from Theorem 3.22. The lower bounds follow from Theorems 2.6, 2.8 and 2.9 using the fact that $D(4, 4) = D(5, 4) = 1$.

**Theorem 4.5.** Suppose $v \geq 360$. Then

$$\beta(5, v, 4) \leq \left\lfloor \frac{5v - 15}{3} \right\rfloor.$$  

Also,

$$\beta(5, v, 4) \geq \begin{cases} \frac{5v - 27}{3} & \text{if } v \equiv 0 \mod 3 \\ \frac{5v - 32}{3} & \text{if } v \equiv 1 \mod 3 \\ \frac{5v - 22}{3} & \text{if } v \equiv 2 \mod 3. \end{cases}$$

Proof. The upper bound follows from Theorem 3.22. The lower bounds follow from Theorems 2.6, 2.8 and 2.9 using the fact that $D(5, 4) = D(6, 4) = 1$.

We note that, using the bounds from Theorems 3.2 and 3.12, one can also obtain results for smaller values of $v$.

For a $(v, 4)$-packing, the largest possible value of $\rho$ is $\left\lfloor \frac{v}{4} \right\rfloor$. From the existence of $(v, 4, 1)$-BIBDs, we can determine some values of $\beta\left(\left\lfloor \frac{v}{4} \right\rfloor, v, 4\right)$. The necessary and sufficient conditions for the existence of a $(v, 4, 1)$-BIBD is $v \equiv 1, 4 \mod 12$ (see [2]). Further, for $v \equiv 4 \mod 12$, there exists a resolvable $(v, 4, 1)$-BIBD. Now we consider the maximum partial parallel classes in $(v, 4, 1)$-BIBDs, for $v \equiv 1 \mod 12$.

A $k$-group divisible design (or $k$-GDD) of type $h^n$ is a triple $(X, \mathcal{G}, \mathcal{B})$, where $X$ is a set of $hn$ points, $\mathcal{G}$ is a partition of $X$ into $h$ groups of size $n$ and $\mathcal{B}$ is a set of
blocks of size $k$, such that every pair of distinct points of $X$ occurs in exactly one block or one group, but not both. A $k$-GDD of type $h^k$ is the same as a $TD(k,h)$. When the blocks in $B$ can be partitioned into parallel classes, we say that the GDD is resolvable GDD and denote it as a $k$-RGDD.

From [6, Theorem 2.19], we have the following result.

**Lemma 4.6.** If $n \equiv 0 \mod 4$ and $n > 4$, then there exists a 4-RGDD of type $3^n$.

By adding a new point to each of the groups of a 4-RGDD of type $3^n$, we obtain a $(3n+1,4,1)$-BIBD that has a PPC of size $3n/4$ (in fact, it has many PPCs of this size).

When $v \equiv 1,4 \mod 12$, a $(v,4,1)$-BIBD is a maximum packing, so $D(v,4) = v(v-1)/12$. So we have proven the following result.

**Theorem 4.7.** Suppose $v \equiv 1,4 \mod 12$ and $v \neq 13$. Then

$$\beta\left(\left\lfloor \frac{v}{4} \right\rfloor, v, 4 \right) = \frac{v(v-1)}{12}.$$

The exceptional case in Theorem 4.7 can be handled easily.

**Theorem 4.8.** $\beta(3,13,4) = 7$.

**Proof.** The following seven blocks are a packing on 13 points with a maximum PPC of size 3:

$$\{1,2,3,4\}, \quad \{5,6,7,8\}, \quad \{9,10,11,12\},$$
$$\{1,5,9,13\}, \quad \{2,6,10,13\}, \quad \{3,7,11,13\}, \quad \{4,8,12,13\}.$$

Also, it is clear that there does not exist a packing having eight blocks and a maximum PPC of size 3.

### 5 Some results for $k > 4$

Many of the methods used in previous sections can be generalized to $k > 4$. First we consider constructions.

**Theorem 5.1.** Suppose there exist $k-2$ MOLS of order $n$ with a transversal of size $\rho \leq n$. Then there is a $((k-1)n + \rho, k)$-packing with $\rho n + D(\rho,k)$ blocks, in which the size of largest PPC has size $\rho$.

**Proof.** The proof is very similar to the proof of Theorem 2.4. We start with a $TD(k,n)$ having $\rho$ disjoint blocks. These blocks will be the PPC of size $\rho$. Let $Y$ denote the $\rho$ points in the last group that occur in a block of the PPC. Delete the blocks that do not contain a point in $Y$. The remaining $\rho n$ blocks form a packing on $(k-1)n + \rho$ points. We can also adjoin the blocks of a packing on $Y$. The resulting packing does not contain a PPC of size $\rho + 1$ because every block contains at least one point from $Y$. 

\[\square\]
**Corollary 5.2.** Suppose there are \( k - 1 \) MOLS of order \( n \). Then, for \( 1 \leq \rho \leq n \), there is a \( (k - 1)n + \rho, k \)-packing with \( n\rho + D(\rho, k) \) blocks, in which the largest PPC has size \( \rho \).

**Proof.** If there are \( k - 1 \) MOLS of order \( n \), then any \( k - 2 \) of these MOLS have a transversal of size \( n \) and hence they have a transversal of size \( \rho \) for any positive integer \( \rho \leq n \). Apply Theorem 5.1.

Next we generalize Theorem 3.2 in a straightforward manner to obtain an upper bound for \( \beta(\rho, v, k) \). For a \( (v, k) \)-packing in which \( P \) is a largest PPC of size \( \rho \), let \( P \) be the points in \( P \) and let \( T \) be the remaining points in the packing. Consider a block \( B = \{a_1, a_2, \ldots, a_k\} \in P \). For \( 1 \leq i \leq k \), let \( T_{a_i} \) denote the set of blocks that contain \( a_i \) and \( k - 1 \) points in \( T \). Denote \( t_{a_i} = |T_{a_i}| \) for \( 1 \leq i \leq k \). Observe that

\[
t_{a_i} \leq \left\lfloor \frac{v - k\rho}{k - 1} \right\rfloor
\]

for all \( i \).

Similar to Lemma 3.1 we have

**Lemma 5.3.** Suppose \( B = \{a_1, a_2, \ldots, a_k\} \in P \) and suppose \( t_{a_1} \geq \max\{t_{a_2}, \ldots, t_{a_k}\} \). Then one of the following two conditions holds:

1. \( t_{a_1} \leq k - 1 \), or
2. \( t_{a_1} \geq k \) and \( t_{a_2} = \ldots = t_{a_k} = 0 \).

**Theorem 5.4.**

\[
\beta(\rho, v, k) \leq \rho \left( \frac{k^2(\rho - 1)}{2} + 1 + \max \left\{ k(k - 1), \left\lfloor \frac{v - k\rho}{k - 1} \right\rfloor \right\} \right).
\]

**Proof.** Since \( P \) is maximum, there is no block contained in \( T \). From Lemma 5.3, there are at most \( \rho \times \max \left\{ k(k - 1), \left\lfloor \frac{v - k\rho}{k - 1} \right\rfloor \right\} \) blocks having one point in \( P \) and \( k - 1 \) points in \( T \).

The number of pairs of points in \( P \) that are not contained in a block of \( P \) is

\[
\binom{k\rho}{2} - \binom{k}{2} = \frac{k^2\rho(\rho - 1)}{2}.
\]

Therefore there are at most \( 8\rho(\rho - 1) \) blocks not in \( P \) that contain at least two points in \( P \). Finally, there are \( \rho \) blocks in \( P \).

In total, there are at most

\[
\rho \left( \frac{k^2(\rho - 1)}{2} + 1 + \max \left\{ k(k - 1), \left\lfloor \frac{v - k\rho}{k - 1} \right\rfloor \right\} \right)
\]

blocks in the packing. \( \square \)
We now consider \( \rho = 1, 2 \) for general \( k \).

**Theorem 5.5.** Suppose \( v \equiv 1 \mod (k - 1) \) and \( v \geq k(k - 1)^2 + k \). Then

\[
\beta(1, v, k) = \frac{v - 1}{k - 1}.
\]

**Proof.** Theorem 5.4 yields the bound

\[
\beta(1, v, k) \leq \left\lfloor \frac{v - 1}{k - 1} \right\rfloor
\]

when \( v \geq k(k - 1)^2 + k \). When \( v \equiv 1 \mod (k - 1) \), we can construct the desired packing by taking \( \frac{v - 1}{k - 1} \) blocks that contain a given point but are otherwise pairwise disjoint.

For \( \rho = 2 \), the upper bound from Theorem 5.4 is

\[
\beta(2, v, k) \leq \left\lfloor \frac{2(v - 2k)}{k - 1} \right\rfloor + k^2 + 2
\]

when \( v \geq k(k - 1)^2 + 2k \).

On the other hand, we can construct a packing with \( \frac{2v - 4}{k - 1} \) blocks having a maximum PPC of size 2 whenever \( v \equiv 2 \mod (k - 1) \) and \( v \geq k(k - 1) + 2 \). Let \( v = t(k - 1) + 2 \) where \( t \geq k \). We construct a packing on the points \( (\mathbb{Z}_t \times \{1, \ldots, k - 1\}) \cup \{\infty_1, \infty_2\} \). The packing has the following \( 2t \) blocks:

\[
\{(0, 1), (0, 2), (0, 3), \ldots, (0, k - 1), \infty_1\} \mod (t, -)
\]

and

\[
\{(0, 1), (1, 2), (2, 3), \ldots, (k - 2, k - 1), \infty_2\} \mod (t, -).
\]

Since \( t \geq k \), this packing contains two disjoint blocks:

\[
\{(0, 1), (0, 2), (0, 3), \ldots, (0, k - 1), \infty_1\}
\]

and

\[
\{(1, 1), (2, 2), (3, 3), \ldots, (k - 1, k - 1), \infty_2\}.
\]

It is clear that the packing does not contain three disjoint blocks because every block contains \( \infty_1 \) or \( \infty_2 \).

Thus we have proven the following.

**Theorem 5.6.** Suppose \( v \equiv 2 \mod (k - 1) \) and \( v \geq k(k - 1)^2 + 2k \). Then

\[
\frac{2v - 4}{k - 1} \leq \beta(2, v, k) \leq \frac{2v - 4}{k - 1} + k^2 - 2.
\]

Note that we proved a stronger result when \( k = 4 \), for sufficiently large \( v \), in Theorem 4.2.
6 Summary

In this paper, we studied \((v,4)\)-packings with maximum parallel classes of a pre-
specified size and thus we extended the results of \([7]\) which studied this problem for
\((v,3)\)-packings.

We presented two constructions for \((v,4)\)-packings. However, our method using
MOLS with disjoint transversals provides the greatest flexibility and we also used
it to construct \((v,4)\)-packings with \(k \geq 4\).

Using counting arguments, we gave three upper bounds for \(\beta(\rho,v,4)\). While
each successive bound improves the previous one, the latter bounds only hold for
larger values of \(v\).

Using the third upper bound, we have

\[
\beta(\rho,v,4) \leq \frac{\rho v}{3} - \frac{\rho (\rho + 1)}{6} = \frac{\rho (v - \rho)}{3} + \frac{\rho (\rho - 1)}{6}
\]

for sufficiently large values of \(v\). Our constructions give the lower bound

\[
\beta(\rho,v,4) \geq \frac{\rho (v - \rho)}{3} + D(v,4) \approx \frac{\rho (v - \rho)}{3} + \frac{\rho (\rho - 1)}{12}.
\]

So our upper and lower bounds are very close, especially when \(\rho\) is small. Also the
difference between the upper and lower bounds is a constant (for a fixed value of \(\rho\)).
However, for small values of \(v\) (or large values of \(\rho\)), the lower and/or upper bounds
could potentially be improved.

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