Class field theory for products of open curves over a local field

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We investigate the class field theory for products of open curves over a local field. In particular, we determine the kernel of the reciprocity homomorphism.

1 Introduction

In [5] and [6], we study the class field theory for open (=non proper) curves over a local field. Here, a local field means a complete discrete valuation field with finite residue field. The aim of this note is to extend the above results to products of those open curves (over a local field).

To state our results precisely, we use the following notation:

• $k$: a local field with char$(k) = p \geq 0$,
• $X_1, \ldots, X_n$: proper and smooth curves over $k$,
• $X_i \subset X_i$: a nonempty open subscheme in $X_i$ with $X_i(k) \neq \emptyset$ for each $1 \leq i \leq n$, and
• $X = X_1 \times \cdots \times X_n \subset \overline{X} = \overline{X}_1 \times \cdots \times \overline{X}_n$.

First, we introduce an abelian group $C(X)$ which is called the idèle class group for $X$ as in [5] (Def. 2.3), and the reciprocity map

$$\rho_X : C(X) \rightarrow \pi_1^{ab}(X)$$

is defined using the 2-dimensional local class field theory(Def. 2.9). Next, we determine the prime to $p$-part of the kernel Ker$(\rho_X)$ of $\rho_X$ under some assumptions.

Theorem 1.1 (Thm. 3.5). Let $X$ and $\overline{X}$ be as above. Assume the following conditions (Red) or (Split) for each $\overline{X}_i$:

(Red) the Jacobian variety Jac($\overline{X}_i$) of $\overline{X}_i$ has potentially good reduction,
(Split) the special fiber of the connected component of the Néron model of \( \text{Jac}(X_i) \)
is an extension of an abelian variety by a split torus.

Then, the kernel \( \text{Ker}(\rho_X) \) is the maximal \( l \)-divisible subgroup of \( C(X) \) for all prime number \( l \neq p \).

The conditions in Thm. 1.1 are the same one in \[16\], Thm. 1.1. In op. cit., for the case where \( X_i = X \), the kernel of \( \rho_X \) is determined as in Thm. 1.1 as in the unramified class field theory. Our main contribution is to show that the proof of op. cit. can also be applied to open curves over the local field has positive characteristic.

Notation

In this note, we basically follow the notation and the definitions in \[6\]. We fix the following notation: A variety over a field \( F \) means a separated and connected scheme of finite type over \( \text{Spec}(F) \). For a variety \( X \), a closed integral subscheme of \( X \) of dimension one is called a curve in \( X \).

Following \[6\], Def. 3.1, we call a pair \( X \subset \overline{X} \) of
\begin{itemize}
  \item \( \overline{X} \): a smooth, proper and connected curve over a field \( F \), and
  \item \( X \): a nonempty open subscheme of \( \overline{X} \)
\end{itemize}
an open curve over \( F \).

For a field \( F \),
\begin{itemize}
  \item \( \text{char}(F) \): the characteristic of \( F \),
  \item \( \overline{F} \): a separable closure of \( F \),
  \item \( G_F := \text{Gal}(\overline{F}/F) \): the Galois group of the extension \( \overline{F}/F \),
  \item \( F_{\text{ab}} \): the maximal abelian extension of \( F \) in \( \overline{F} \),
  \item \( G_{F_{\text{ab}}} := \text{Gal}(F_{\text{ab}}/F) \): the Galois group of \( F_{\text{ab}}/F \), and
  \item \( K_2(F) \): the Milnor \( K \)-group of \( F \) (cf. \[11\]; \[3\], Chap. IX).
\end{itemize}

For an extension \( E/F \) of fields, the embedding \( F^\times \hookrightarrow E^\times \) induces a homomorphism
\[ j_{E/F} : K_2(F) \to K_2(E). \]

If the extension \( E/F \) is finite, then we also have the norm map
\[ N_{E/F} : K_2(E) \to K_2(F). \]

For a complete discrete valuation field \( K \), we define
\begin{itemize}
  \item \( v_K : K^\times \to \mathbb{Z} \): the valuation of \( K \),
  \item \( O_K := \{ f \in K \mid v_K(f) \geq 0 \} \): the valuation ring of \( K \),
  \item \( \mathfrak{m}_K := \{ f \in K \mid v_K(f) > 0 \} \): the maximal ideal of \( O_K \),
\end{itemize}
\[ k_K := O_K / \mathfrak{m}_K : \text{the residue field of } K, \]
\[ U_K := O_K^\times : \text{the group of units in } O_K, \]
\[ U^n_K := 1 + \mathfrak{m}_K^n : \text{the higher unit groups, and} \]
\[ \partial_K : K_2(K) \to k_K^\times : \text{the tame symbol map defined by} \]
\[ \partial_K(\{ f, g \}) := (-1)^{v_K(f) v_K(g)} f^{v_K(g)} g^{-v_K(f)} \mod \mathfrak{m}_K, \tag{3} \]

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2 Idèle class groups

In this section, we use the following notation:

- \( X \) : a variety over a field \( F \),
- \( X_0 \) : the set of closed points in \( X \),
- \( \text{Cu}(X) \) : the set of the normalizations \( \phi : C \to X \) of a curve in \( X \) (cf. Notation).
  For simplicity, we often refer to the domain \( C \) of the normalization \( \phi : C \to X \) in \( \text{Cu}(X) \) as an element of \( \text{Cu}(X) \) and write \( C \in \text{Cu}(X) \).
- \( F(x) \) : the residue field at \( x \in X_0 \).

For each \( \phi : C \to X \in \text{Cu}(X) \),
- \( \overline{C} \) : the regular compactification of \( C \) which is the smooth and proper curve over \( F \) containing \( C \) as a dense open subscheme (by a resolution of singularities),
- \( \overline{\phi} : \overline{C} \to \overline{X} \) : the canonical extension of \( \phi \),
- \( C_\infty := \overline{C} \setminus C \), and
- \( F(C)_x = \text{Frac}(\mathcal{O}_{\overline{C}, x}) \) : the completion of the function field \( F(C) \) of \( C \) at \( x \in \overline{C}_0 \).

Idèle group

**Definition 2.1.** The idèle group of \( X \) is defined by
\[ I(X) = \bigoplus_{x \in X_0} F(x)^\times \oplus \bigoplus_{C \in \text{Cu}(X)} \bigoplus_{x \in C_\infty} K_2(F(C)_x). \]
Lemma 2.2. Let $\varphi : X' \to X$ be a morphism of varieties over $F$. Then, there is a canonical homomorphism $\varphi_* : I(X') \to I(X)$.

Proof. We define the homomorphism

$$\varphi_* : I(X') \to I(X); (\xi_{x'}) \mapsto (\varphi_{*}^{x'} \to x)(\xi_{x'})$$

by defining $\varphi_{*}^{x'} \to x$ for each component as follows:

- For $x' \in X'_0$ and $x = \varphi(x') \in X_0$, the norm homomorphism of the field extension $F(x')/F(x)$ gives
  $$\varphi_{*}^{x'} \to x \colon F(x')^\times \to F(x)^\times.$$

- For $C' \in \text{Cu}(X')$, and $x' \in C'_\infty$, when $C'$ is the normalization of a curve $Z'$ in $X'$, we denote by $Z = \overline{\varphi(Z')}$ the scheme theoretic closure of $\varphi(Z')$, and $C \to Z$ is the normalization of $Z$. The morphism $\varphi$ induces $\overline{\varphi}_C : \overline{C'} \to \overline{C}$.

  - If $x := Z$ is a closed point of $X$, then we have
    $$\varphi_{*}^{x'} \to x : K_2(F(C')_{x'}) \xrightarrow{\partial_{F(C')_{x'}}} F(x')^\times \xrightarrow{N_{F(x')/F(x)}} F(x)^\times,$$
    where $\partial_{F(C')_{x'}}$ is the tame symbol map (defined in (3)).

  - If $Z$ is a curve in $X$ (and thus $C \in \text{Cu}(X)$), and $x := \overline{\varphi}_C(x')$ is in $C_0$, then we use
    $$\varphi_{*}^{x'} \to x : K_2(F(C')_{x'}) \xrightarrow{\partial_{F(C')_{x'}}} F(x')^\times \xrightarrow{N_{F(x')/F(x)}} F(x)^\times.$$

  - If $Z$ is a curve in $X$ (and hence $C \in \text{Cu}(X)$) and $x := \overline{\varphi}_C(x')$ is in $C_\infty$, then $\varphi$ induces the (finite) morphism $\varphi_C : C' \to C$ of curves and this gives the norm map
    $$\varphi_{*}^{x'} \to x := N_{F(C')_{x'}/F(C)} : K_2(F(C')_{x'}) \to K_2(F(C)_{x}).$$

These homomorphisms define $\varphi_* : I(X') \to I(X)$. \qed
Idèle class group

For each $\phi : C \to X \in \text{Cu}(X)$, we define a homomorphism

$$\partial_C : K_2(F(C)) \to I(C) = \bigoplus_{x \in C_0} F(x)^{\times} \oplus \bigoplus_{x \in C_{\infty}} K_2(F(C)_x); \xi \mapsto (\partial_{C,x}(\xi))_x$$

as follows:

- For $x \in C_0$, the inclusion $F(C) \hookrightarrow F(C)_x$ induces $j_{F(C)_x/F(C)} : K_2(F(C)) \to K_2(F(C)_x)$ (cf. [1]). We have the composite

$$\partial_{C,x} : K_2(F(C)) \xrightarrow{j_{F(C)_x/F(C)}} K_2(F(C)_x) \xrightarrow{\partial_{F(C)_x}} F(x)^{\times},$$

where $\partial_{F(C)_x}$ is the tame symbol map [2].
- For $x \in C_{\infty}$, the inclusion $F(C) \hookrightarrow F(C)_x$ induces

$$\partial_{C,x} := j_{F(C)_x/F(C)} : K_2(F(C)) \to K_2(F(C)_x).$$

These homomorphisms give the required $\partial_C : K_2(F(C)) \to I(C)$. Composing with the sum of $\phi_* : I(C) \to I(X)$ (Lem. [22] (i)), we obtain a homomorphism

$$\partial : \bigoplus_{C \in \text{Cu}(X)} K_2(F(C)) \xrightarrow{\oplus \partial_C} I(C) \xrightarrow{\sum_C \phi_*} I(X).$$

**Definition 2.3.** The cokernel

$$C(X/F) = \text{Coker} \left( \partial : \bigoplus_{C \in \text{Cu}(X)} K_2(F(C)) \to I(X) \right)$$

of $\partial$ defined above is called the idèle class group of $X$. When the base field $F$ is not particularly important, we also write as $C(X)$ for simplicity.

When $X$ is a projective smooth variety over $F$, a normalization $\phi : C \to X$ in $\text{Cu}(X)$ extends to $\bar{\phi} : \bar{C} \to X$ which is also in $\text{Cu}(X)$. Thus, $C = \bar{C}$. In this case, we have $C(X) = \text{SK}_1(X)$ in terms of the algebraic $K$-theory (cf. [2]).

**Lemma 2.4.** Let $\varphi : X' \to X$ be a morphism of varieties over $F$. Then, there is a canonical homomorphism $\varphi_* : C(X') \to C(X)$.

**Proof.** We show that $\varphi_* : I(X') \to I(X)$ (Lem. [22]) induces $\varphi_* : C(X') \to C(X)$. Take the normalization $\phi' : C' \to X' \in \text{Cu}(X')$ of a curve $Z'$ in $X'$. 

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5
If the scheme theoretic closure of the image $Z = \varphi(Z')$ is a curve in $X$, then we denote by $\phi : C \to Z \hookrightarrow X$ in $\text{Cu}(X)$ the normalization of $Z$, and $\varphi_C : C' \to C$ the morphism induced from $\varphi$. Consider the following diagram:

$$
\begin{array}{c}
K_2(F(C')) \xrightarrow{\partial_{C'}} I(C') \xrightarrow{\phi_*'} I(X')
\end{array}
$$

$$
\begin{array}{c}
N_{F(C')/F(C)} \downarrow \quad (\varphi_C)* \downarrow \quad \varphi_*
\end{array}
$$

where the left vertical map is the norm map. Since the norm maps and the tame symbol maps are commutative, the left square of the diagram (4) is commutative. The commutativity of the right square in (4) follows from Lem. 2.2.

If the image $x = \varphi(Z)$ is in $X_0$, then the curve $C'$ is defined over $F' := F(x)$. By the Weil reciprocity law on the Milnor $K$-groups (cf. [1], Sect. 5), the sequence

$$
K_2(F(C')) \xrightarrow{\partial_{C'}} I(C') = \bigoplus_{x' \in C' \cap X_0} F(x')^\times \xrightarrow{(N_{C'/F'})_*} (F')^\times
$$

is a complex, where $N_{C'/F'} = \sum_{x'} N_{F(x')/F'}$ is the sum of the norm maps. We have the following diagram

$$
\begin{array}{c}
K_2(F(C')) \xrightarrow{\partial_{C'}} I(C') \xrightarrow{\phi_*'} I(X')
\end{array}
$$

$$
\begin{array}{c}
0 \xrightarrow{} I(x) = F(x)^\times \xrightarrow{\iota_*} I(X),
\end{array}
$$

where $\varphi_x : C' \to x$ is induced from $\varphi$ and the right horizontal map is given by the closed immersion $\iota_x : x \hookrightarrow X$. The composite $(\varphi_x)_* \circ \partial_{C'} = N_{C'/F'} \circ \partial_{C'} = 0$ makes the above diagram commutative.

From the commutative diagrams (4) and (5), the map $\varphi_* : I(X') \to I(X)$ defines $\varphi_* : C(X') \to C(X)$.

By Lem. [24] the structure map $\gamma : X \to \text{Spec}(F)$ induces

$$
N_X := \gamma_* : C(X) \to C(\text{Spec}(F)) = F^\times.
$$

**Tame idèle class group**

We introduce a quotient $C^t(X)$ of $C(X)$ which comes to classify *tame coverings* through the reciprocity map (11).
Definition 2.5. The cokernel

\[ C^t(X) = C^t(X/F) = \text{Coker} \left( \bigoplus_{C \in \text{Cu}(X)} \bigoplus_{x \in C_\infty} U^1 K_2(F(C)_x) \to C(X) \right) \]

is called the tame idèle class group of \( X \), where the map above is induced from the inclusion \( U^1 K_2(F(C)_x) \to K_2(F(C)_x) \) for \( C \in \text{Cu}(X) \) and \( x \in C_\infty \).

Lemma 2.6. Let \( \bar{F} = F^{p^\infty} \) be the perfect closure of \( F \) with \( \text{char}(F) = p \), that is, the field adjoined with all \( p^r \)-th roots to \( F \) for all \( r \geq 1 \). Then, we have an isomorphism

\[ C^t(X/F)/l^m \cong C^t(X \otimes_F \bar{F}/F)/l^m, \]

for any prime number \( l \neq p \) and \( m \in \mathbb{Z}_{>0} \).

Proof. We denote by \( \bar{X} = X \otimes_F \bar{F} \) the base change to \( \bar{F} \) of \( X \). Since the extension \( \bar{F}/F \) is purely inseparable, the projection \( \bar{X} \to X \) is a homeomorphism ([III, Prop. 3.2.7 (c)]). In particular, there are bijections \( X_0 \approx \bar{X}_0 \) and \( \text{Cu}(X) \approx \text{Cu}(\bar{X}) \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{C \in \text{Cu}(X)} K_2(F(C)) & \to & \bigoplus_{x \in X_0} F(x)^\times \bigoplus_{C \in \text{Cu}(X)} K_2(F(C)_x)/U^1 K_2(F(C)_x) \\
\downarrow & & \downarrow \\
\bigoplus_{C \in \text{Cu}(\bar{X})} K_2(\bar{F}(C)) & \to & \bigoplus_{x \in \bar{X}_0} \bar{F}(x)^\times \bigoplus_{C \in \text{Cu}(\bar{X})} K_2(\bar{F}(C)_x)/U^1 K_2(\bar{F}(C)_x),
\end{array}
\]

(7)

where the vertical maps are given by the inclusion map \( F \to \bar{F} \). Note that the cokernel of the horizontal maps in the diagram (7) are \( C^t(X/F) \) and \( C^t(\bar{X}/\bar{F}) \).

For each \( C \in \text{Cu}(X) \), the corresponding curve in \( \bar{X} \) is \( \bar{C} := C \otimes_F \bar{F} \in \text{Cu}(\bar{X}) \). The extension \( \bar{F}(\bar{C}) = F(C)\bar{F} \) of \( F(C) \) is a constant field extension. For each \( r > 0 \), we have

\[ N_{F(C)F^{-r^\infty}/F(C)} \circ j_{F(C)F^{-r^\infty}/F(C)} = [F(C)F^{-r^\infty} : F(C)] : K_2(F(C)) \to K_2(F(C)), \]

where \( j_{F(C)F^{-r^\infty}/F(C)} \) is the map induced from the inclusion \( F(C) \to F(C)F^{-r^\infty} \) (cf. [I]). Applying \( - \otimes_{\mathbb{Z}} \mathbb{Z}/l^m \), we have \( K_2(F(C))/l^m \cong K_2(F(C)F^{-r^\infty})/l^m \). The left vertical map in the above diagram (7) becomes bijective after applying \( - \otimes_{\mathbb{Z}} \mathbb{Z}/l^m \). Moreover, we have

\[ K_2(F(C)_x)/U^1 K_2(F(C)_x) \cong K_2(F(x)) \oplus F(x)^\times \quad (C \in \text{Cu}(X), x \in C_\infty), \]

and

\[ K_2(\bar{F}(C)_x)/U^1 K_2(\bar{F}(C)_x) \cong K_2(\bar{F}(x)) \oplus \bar{F}(x)^\times \quad (C \in \text{Cu}(\bar{X}), x \in C_\infty) \]
(cf. [3], Chap. IX, Prop. 2.2). In the same way as above, \( \tilde{F}(x \otimes_F \tilde{F}) \simeq F(x)\tilde{F} \) for each \( x \in X_0 \) or \( x \in C_{\infty} \) for some \( C \in \text{Cu}(X) \). The right vertical map in (7) is an isomorphism after applying \(- \otimes_{\mathbb{Z}} \mathbb{Z}/l^m\). Hence, there is an isomorphism

\[
C^t(\tilde{X}/\tilde{F})/l^m \simeq C^t(X/F)/l^m.
\]

\( \square \)

Corresponding to the tame idèle class group \( C^t(X) \), we define the \textit{abelian tame fundamental group} of \( X \) as a quotient of \( \pi^t_{1\text{ab}}(X) \).

**Definition 2.7.** The \textbf{abelian tame fundamental group} is defined by

\[
\pi^t_{1\text{ab}}(X) = \pi^t_{1\text{ab}}(X/k) = \text{Coker} \left( \bigoplus_{C \in \text{Cu}(X)} \bigoplus_{x \in C_{\infty}} I^1_{F(C)_x} \to \pi^t_{1\text{ab}}(X) \right)
\]

where \( I^1_{F(C)_x} \) is the \( p \)-Sylow subgroup of the abelian Galois group \( G^\text{ab}_{F(C)_x} \) for \( \phi : C \to X \in \text{Cu}(X), x \in C_{\infty} \), and the map above is induced from the composition

\[
I^1_{F(C)_x} \hookrightarrow G^\text{ab}_{F(C)_x} \hookrightarrow \pi^t_{1\text{ab}}(C) \xrightarrow{\phi_*} \pi^t_{1\text{ab}}(X).
\]

**Reciprocity map**

In the rest of this section, we consider

- \( k \): a local field,
- \( X \): a \textit{smooth} variety over \( k \), and
- \( \gamma : X \to \text{Spec}(k) \): the structure map.

For the variety \( X \), we introduce the reciprocity map

\[
\rho_X : C(X) := C(X/k) \to \pi^t_{1\text{ab}}(X).
\]  

(8)

First, we define a group homomorphism

\[
\tilde{\rho}_X : I(X) \to \pi^t_{1\text{ab}}(X) ; (\xi_x)_x \mapsto \sum_x \tilde{\rho}_{X,x}(\xi_x)
\]

by introducing the following homomorphisms \( \tilde{\rho}_{X,x} \):

- For \( x \in X_0 \), we denote by \( \iota_x : x \hookrightarrow X \) the closed immersion. The reciprocity map \( \rho_{k(x)} : k(x)^\times \to G^\text{ab}_{k(x)} \) of the local class field theory (for the residue field \( k(x) \)) gives

\[
\tilde{\rho}_{X,x} : k(x)^\times \xrightarrow{\rho_{k(x)}} G^\text{ab}_{k(x)} \xrightarrow{(\iota_x)_*} \pi^t_{1\text{ab}}(X).
\]
For a morphism \( \phi : C \to X \subset \text{Cu}(X) \) and \( x \in C_\infty \), the completion \( k(C)_x \) is a 2-dimensional local field (cf. [6], Def. 2.1) with residue field \( k(x) \). The reciprocity map \( \rho_{k(C)_x} : K_2(k(C)_x) \to C_k^{ab}(C) \subset \pi_1^{ab}(\text{Spec}(k(C))) \) induces

\[
\tilde{\rho}_{X,x} : K_2(k(C)_x) \xrightarrow{\rho_{k(C)_x}} C_k^{ab}(C) \subset \pi_1^{ab}(\text{Spec}(k(C))) \xrightarrow{(j_C)_*} \pi_1^{ab}(X),
\]

where the last homomorphism is induced from \( j_C : \text{Spec}(k(C)) \leftarrow C \xrightarrow{\phi} X \).

For a morphism \( \varphi : X' \to X \) of varieties over \( k \), from the construction of \( \tilde{\rho}_X \) above and some properties of the reciprocity map of 2-dimensional local class field theory (cf. [6], Prop. 2.2), we have the following commutative diagram:

\[
\begin{array}{ccc}
I(X') & \xrightarrow{\tilde{\rho}_{X'}} & \pi_1^{ab}(X') \\
\varphi_* & & \varphi_* \\
I(X) & \xrightarrow{\tilde{\rho}_X} & \pi_1^{ab}(X).
\end{array}
\]  \quad (9)

Lemma 2.8. \( \tilde{\rho}_X : I(X) \to \pi_1^{ab}(X) \) factors through \( C(X) \).

Proof. (The case \( \text{dim}(X) = 0, 1 \)) If \( \text{dim} X = 0 \), then \( I(X) = k(X)^{\times} = C(X) \) and there is nothing to show. In the case where \( X \) is a (smooth) curve, the assertion follows from Sect. 2 of [3]. The main ingredient of the discussion in op. cit. is the reciprocity law of \( k(X) \) ([13], Chap. II, Prop. 1.2).

(The case \( \text{dim}(X) > 1 \)) For a general variety \( X \), for each \( \phi : C \to X \subset \text{Cu}(X) \), the maps \( \tilde{\rho}_C \) and \( \tilde{\rho}_X \) defined above give the diagram

\[
\begin{array}{ccc}
K_2(k(C)) & \xrightarrow{\partial_C} & I(C) \xrightarrow{\tilde{\rho}_C} \pi_1^{ab}(C) \\
\downarrow{\phi_*} & & \downarrow{\phi_*} \\
I(X) & \xrightarrow{\tilde{\rho}_X} & \pi_1^{ab}(X)
\end{array}
\]

with \( \tilde{\rho}_C \circ \partial_C = 0 \) from the case of \( \text{dim} X = 1 \) discussed above. Since the above diagram (9) is commutative, the assertion \( \tilde{\rho}_X \circ \partial = 0 \) follows.

Definition 2.9. The induced map \( \rho_X : C(X) \to \pi_1^{ab}(X) \) from \( \tilde{\rho}_X \) by Lem. 2.8 is called the reciprocity map for \( X \).
The commutative diagram (9) and Lem. 2.8 say that for each \( \varphi : X' \to X \), the reciprocity maps make the following diagram commutative:

\[
\begin{array}{ccc}
C(X') & \xrightarrow{\rho_{X'}} & \pi_{1}^{ab}(X') \\
\downarrow{\varphi_*} & & \downarrow{\varphi_*} \\
C(X) & \xrightarrow{\rho_X} & \pi_{1}^{ab}(X)
\end{array}
\]

In particular, the structure map \( \gamma : X \to \text{Spec}(k) \) gives the following commutative diagram:

\[
\begin{array}{ccc}
0 & \xrightarrow{} & C(X)^{\text{geo}} & \xrightarrow{} & C(X) & \xrightarrow{N_X} & k^\times \\
& & \downarrow{\rho_X} & & \downarrow{\rho_X} & & \downarrow{\rho_k} \\
0 & \xrightarrow{} & \pi_{1}^{ab}(X)^{\text{geo}} & \xrightarrow{} & \pi_{1}^{ab}(X) & \xrightarrow{\gamma_*} & G_{k}^{ab},
\end{array}
\]

where \( N_X \) is defined in (6) and \( \rho_k = \rho_{\text{Spec}(k)} \) is the reciprocity map of \( k \) and the groups \( C(X)^{\text{geo}} \) and \( \pi_{1}^{ab}(X)^{\text{geo}} \) are defined by the exactness of the horizontal rows.

From the 2-dimensional local class field theory ([9], see also [6], Prop. 2.5), the reciprocity map \( \rho_X \) induces a map from the tame class group (Def. 2.5) to the tame fundamental group (Def. 2.7) as the following commutative diagram indicates:

\[
\begin{array}{ccc}
C(X) & \xrightarrow{\rho_X} & \pi_{1}^{ab}(X) \\
\downarrow{=} & & \downarrow{=} \\
C^t(X) & \xrightarrow{} & \pi_{1}^{\text{t, ab}}(X),
\end{array}
\]

where the vertical maps are the quotient maps. The induced map \( C^t(X) \to \pi_{1}^{\text{t, ab}}(X) \) is also denoted by \( \rho_X \). As in (10), we have the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \xrightarrow{} & C^t(X)^{\text{geo}} & \xrightarrow{} & C^t(X) & \xrightarrow{N_X} & k^\times \\
& & \downarrow{\rho_X} & & \downarrow{\rho_X} & & \downarrow{\rho_k} \\
0 & \xrightarrow{} & \pi_{1}^{\text{t, ab}}(X)^{\text{geo}} & \xrightarrow{} & \pi_{1}^{\text{t, ab}}(X) & \xrightarrow{\gamma_*} & G_{k}^{ab},
\end{array}
\]

3 Products of curves

**Somekawa \( K \)-groups**

First, we recall the definition of the Mackey products and that of the Somekawa \( K \)-groups following [12], [16] and [8]: Recall that a Mackey functor \( A \) over a field \( F \) is a
contravariant functor from the category of étale schemes over $F$ to that of abelian groups equipped with a covariant structure for finite morphisms satisfying some conditions (for the precise definition, see [12] Section 3, or [8], Sect. 2). For a Mackey functor $A$ over $F$, we denote by $A(E)$ its value $A(\text{Spec}(E))$ for a field extension $E$ over $F$.

**Definition 3.1.** For Mackey functors $A_1, \ldots, A_n$ over $F$, their **Mackey product** $A_1 \otimes \cdots \otimes A_n$ is defined as follows: For any finite field extension $E/F$,

$$
(A_1 \otimes \cdots \otimes A_n)(E) := \left( \bigoplus_{E'/E: \text{finite}} A_1(E') \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A_n(E') \right) / R,
$$

(13)

where $R$ is the subgroup generated by elements of the following form:

(PF) For any finite field extensions $E \subset E_1 \subset E_2$, and if $x_{i_0} \in A_{i_0}(E_2)$ and $x_i \in A_i(E_1)$ for all $i \neq i_0$, then

$$
j^*(x_1) \otimes \cdots \otimes x_{i_0} \otimes \cdots \otimes j^*(x_n) - x_1 \otimes \cdots \otimes j^*(x_{i_0}) \otimes \cdots \otimes x_n,
$$

where $j = j_{E_2/E_1}: \text{Spec}(E_2) \to \text{Spec}(E_1)$ is the canonical map.

For the Mackey product $A_1 \otimes \cdots \otimes A_n$, we write $\{x_1, \ldots, x_n\}_{E/F}$ for the image of $x_1 \otimes \cdots \otimes x_n \in A_1(E) \otimes \cdots \otimes A_n(E)$ in the product $(A_1 \otimes \cdots \otimes A_n)(E)$. For any field extension $E/F$, the canonical map $j = j_{E/F}: F \rightarrow E$ induces the pull-back

$$
\text{Res}_{E/F} := j^* : (A_1 \otimes \cdots \otimes A_n)(E) \rightarrow (A_1 \otimes \cdots \otimes A_n)(F).
$$

If the extension $E/F$ is finite, then the push-forward

$$
N_{E/F} := j_* : \left( A_1 \otimes \cdots \otimes A_n \right)(E) \rightarrow \left( A_1 \otimes \cdots \otimes A_n \right)(F)
$$

is given by $N_{E/F}(\{x_1, \ldots, x_n\}_{E'/E}) = \{x_1, \ldots, x_n\}_{E'/F}$ on symbols.

**Definition 3.2.** Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be homotopy invariant Nisnevich sheaves with transfers over $F$. By considering these sheaves as Mackey functors ([8], Sect. 2), the **Somekawa $K$-group** $K(F; \mathcal{F}_1, \ldots, \mathcal{F}_n)$ attached to $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is defined by a quotient

$$
K(F; \mathcal{F}_1, \ldots, \mathcal{F}_n) := \left\{ (\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n)(F) \right\} / R,
$$

(14)

where $R$ is a subgroup which produces “the Weil reciprocity law” (for the precise definition, see [8] Def. 5.1).
For semi-abelian varieties $A_1, \ldots, A_n$ over $F$ by considering them as Nisnevich sheaves (cf. [8], Sect. 2.15), this $K$-group coincides with that of defined in [14] (8, Rem. 5.2).

In the rest of this section, we use

- $F$: a field with $\text{char}(F) = p \geq 0$,
- $X_i \subset \overline{X}_i$ : open curves over a field $F$ (cf. Notation) with $X_i(F) \neq \emptyset$ for $i = 1, \ldots, n$,
- $X = X_1 \times \cdots \times X_n \subset \overline{X} = \overline{X}_1 \times \cdots \times \overline{X}_n$, and
- $\text{Jac}(X_i)$ : the generalized Jacobian variety of $X_i$.

**Lemma 3.3** (cf. [16], Thm. 2.2). Assume that $F$ is a perfect field. Then, there is an isomorphism

$$C^i(X) \simeq \bigoplus_{r=0}^n \bigoplus_{1 \leq i_1 < \cdots < i_r \leq n} K(F; \text{Jac}(X_{i_1}), \ldots, \text{Jac}(X_{i_r}), G_m),$$

where $C^i(X)$ is the tame idèle class group of $X$ (Def. 2.5).

**Proof.** The tame idèle class group $C^i(X)$ is isomorphic to Wiesend’s tame ideal class group $C_1(X)$ in the sense of [17] (see also [17], Rem. 5.5). From [17], Thm. 1.3, we obtain

$$C_1(X) \simeq \text{Hom}_{\text{DM}}(\mathbb{Z}(-1)[-1], M(X)) \simeq \text{Hom}_{\text{DM}}(\mathbb{Z}, M(X)(1)[1]),$$

(15)

where $\text{DM} = \text{DM}^{\text{eff}}_{\text{Nis}}(F)$ is the triangulated tensor category of Voevodsky’s motivic complexes ([15]). Following [15] and [8], Sect. 9, we define $h^\text{Nis}_0(X_i) := h^\text{Nis}_0(L(X_i))$. By [8], Thm. 12.3, the far right in (15) has a description

$$\text{Hom}_{\text{DM}}(\mathbb{Z}, M(X)(1)[1]) \simeq K(F; h^\text{Nis}_0(X_1), \ldots, h^\text{Nis}_0(X_n), G_m),$$

(16)

where the right is the Somekawa $K$-group for homotopy invariant Nisnevich sheaves with transfers ([8], Def. 5.1). From [8], Lem. 11.2, $h^\text{Nis}_0(X_i)$ coincides with the presheaf of relative Picard groups with respect to $(\overline{X}_i, X_i)$. From the assumption $X_i(F) \neq \emptyset$, we have

$$h^\text{Nis}_0(X_i) \simeq \mathbb{Z} \oplus \text{Jac}(X_i).$$

Here, $\text{Jac}(X_i)$ is the generalized Jacobian variety regarded as a Nisnevich sheaf with transfers (cf. [8], 2.15). Since $K(F; h^\text{Nis}_0(X_1), \ldots, h^\text{Nis}_0(X_n), G_m)$ is a quotient of the Mackey product (as noted in [14]), we have

$$K(F; h^\text{Nis}_0(X_1), \ldots, h^\text{Nis}_0(X_n), G_m) \simeq K(F; \mathbb{Z}, h^\text{Nis}_0(X_2), \ldots, h^\text{Nis}_0(X_n), G_m) \oplus K(F; \text{Jac}(X_1), h^\text{Nis}_0(X_2), \ldots, h^\text{Nis}_0(X_n), G_m) \simeq K(F; h^\text{Nis}_0(X_2), \ldots, h^\text{Nis}_0(X_n), G_m) \oplus K(F; \text{Jac}(X_1), h^\text{Nis}_0(X_2), \ldots, h^\text{Nis}_0(X_n), G_m).$$

Inductively, we obtain the decomposition of the $K$-group in (16):

$$K(F; h^\text{Nis}_0(X_1), \ldots, h^\text{Nis}_0(X_n), G_m) \simeq \bigoplus_{r=0}^n \bigoplus_{1 \leq i_1 < \cdots < i_r \leq d} K(F; \text{Jac}(X_{i_1}), \ldots, \text{Jac}(X_{i_r}), G_m).$$
The assertion follows from these isomorphisms.

\[\square\]

**Lemma 3.4.** Let \( l \) be a prime number \( l \neq p \) and \( m \in \mathbb{Z}_{>0} \). We assume that \( F \) is perfect, and the following condition:

\((\text{Surj})\) For any finite extensions \( F'' \supset F' \supset F \), the norm map

\[ C^t(X_1 \otimes_F F'') \gogeo /l^m \to C^t(X_1 \otimes_F F') \gogeo /l^m \]

is surjective for each \( 1 \leq i \leq n \).

Then, we have

\[ C^t(X) \gogeo /l^m \simeq \bigoplus_{i=1}^n C^t(X_i) \gogeo /l^m. \]

**Proof.** From Lem. 3.3 and \( K(F; \mathbb{G}_m) \simeq F^\times \), we have

\[ C^t(X) \gogeo \simeq \bigoplus_{r=1}^n \bigoplus_{1 \leq i_1 < \cdots < i_r \leq n} K(F; \text{Jac}(X_{i_1}), \ldots, \text{Jac}(X_{i_r}), \mathbb{G}_m) \quad \text{and} \]

\[ C^t(X_i) \gogeo \simeq K(F; \text{Jac}(X_i), \mathbb{G}_m) \quad \text{for } i = 1, \ldots, n. \]

It is enough to show

\[ K(F; \text{Jac}(X_1), \text{Jac}(X_2), \ldots, \text{Jac}(X_t), \mathbb{G}_m)/l^m = 0 \]

for \( t \geq 2 \). Without loss of generality, we may assume \( t = 2 \). We consider \( M := K(-; \text{Jac}(X_2), \mathbb{G}_m) \) as a Mackey functor over \( F \) defined by

\[ E/F \mapsto M(E) = K(E; \text{Jac}(X_2), \mathbb{G}_m). \]

From the very definition of the \( K \)-group (Def. 3.2), we have the surjective map

\[ \left( \text{Jac}(X_1) \otimes M \right)(F) \to K(F; \text{Jac}(X_1), \text{Jac}(X_2), \mathbb{G}_m). \]

For any element of the form \( \{ x', y' \}_{F'/F} \in (\text{Jac}(X_1) \otimes M)(F) \), there exists a finite (separable) extension \( F''/F' \) such that \( \text{Res}_{F''/F'}(x') = l^mx'' \) for some \( x'' \in \text{Jac}(X_1)(F'') \).

From the assumption (\text{Surj}) and \( M(F') \simeq C^t(X_2 \otimes_F F') \gogeo \), the norm map \( N_{F''/F'} : M(F'')/l^m \to M(F')/l^m \) is surjective. Hence, there exists \( y'' \in M(F'') \) such that \( N_{F''/F'}(y'') = y'/l^m \).

From the “projection formula” (PF) in Def. 3.1 (the definition of the Mackey functor)
product \((13)\), we have
\[
\{x', y'\}_{F'/F} = \{x', N_{F'/F}(y') - l^m z'\}_{F'/F} = \{x', N_{F'/F}(y') - \{x', l^m z'\}_{F'/F}
\]
\[
+\left\{\text{Res}_{F'/F}(x'), y''\right\}_{F''/F} - \{\text{Res}_{F'/F}(x'), y''\}_{F''/F} - \{x', z'\}_{F'/F}
\]
\[
= \left\{\text{Res}_{F'/F}(x'), y''\right\}_{F''/F} - \{x', z'\}_{F'/F}ight).
\]
This implies \(\left(\text{Jac}(X_1) \otimes M\right)(F)/l^m = K(F; \text{Jac}(X_1), \text{Jac}(X_2), \mathbb{G}_m)(F)/l^m = 0. \)

\(\Box\)

**Proof of the main theorem**

Now, we suppose the base field \(F = k\) is a local field (with \(\text{char}(k) = p \geq 0\)), and devote to prove the following theorem.

**Theorem 3.5.** Assume the following conditions (Red) or (Split) for each \(X_i\):

(Red) the Jacobian variety \(\text{Jac}(X_i)\) of \(X_i\) has potentially good reduction,

(Split) the special fiber of the connected component of the Néron model of \(\text{Jac}(X_i)\)

is an extension of an abelian variety by a split torus.

Then, the kernel \(\text{Ker}(\rho_{X_i})\) is the maximal \(l\)-divisible subgroup of \(C(X)\) for all prime number \(l \neq p\).

The theorem above is proved by the following results on the reciprocity map \(\rho_{X_i}\) for the curve \(X_i\).

**Theorem 3.6** \([5\text{ and }6]\). For each open curve \(X_i \subset \overline{X}_i\) over \(k\), we have

(i) \(\pi_1^{ab}(X_i)/\text{Im}(\rho_{X_i}) \simeq \mathbb{Z}^{r_i}\), where \(\text{Im}(\rho_{X_i})\) is the topological closure of the image \(\text{Im}(\rho_{X_i})\) and \(r_i = r_i(\overline{X}_i)\) is the rank of \(\overline{X}_i\) (cf. [13], Chap. II, Def. 2.5).

(ii) \(\text{Ker}(\rho_{X_i})\) is \(l\)-divisible for any prime \(l \neq p\).

To prove Thm. 3.5 we prepare some lemmas.

**Lemma 3.7.** Let \(K\) be a complete discrete valuation field of characteristic \(p \geq 0\). Then, \(U^1K_2(K)\) is \(l\)-divisible for all prime \(l \neq p\).

Proof. Recall that \(U^1K_2(K)\) is generated by \(\{U^1_{K}, K^\times\} \subset K_2(K)\). As \(K\) is complete, the unit group \(U^1_{K}\) is \(l\)-divisible \([3],\text{ Chap. I, Cor. 5.5}\). Therefore, \(U^1K_2(K)\) is also \(l\)-divisible. \(\Box\)
Lemma 3.8. For any \( m \in \mathbb{Z}_{>0} \) and a prime \( l \neq p \), \( C(X)/l^m \to \pi_1^{ab}(X)/l^m \) induced from \( \rho_X \) is injective.

Proof. Consider the following commutative diagram with exact rows (cf. (10)):

\[
\begin{array}{ccccccccc}
0 & \rightarrow & C(X)^{\text{geo}} & \rightarrow & C(X) & \xrightarrow{N_X} & k^\times & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \rho_X & & \downarrow \\
0 & \rightarrow & \pi_1^{ab}(X)^{\text{geo}} & \rightarrow & \pi_1^{ab}(X) & \xrightarrow{\rho_X} & \mathcal{C}_k^{ab} & \rightarrow & 0.
\end{array}
\]

The existence of a \( k \)-rational point of \( X \) implies that the horizontal sequences in the above diagram are split. By local class field theory, the induced homomorphism \( k^\times/l^m \to \mathcal{C}_k^{ab}/l^m \) from \( \rho_k \) is injective (in fact, bijective). Thus, to show the assertion it is enough to prove that \( \rho_X : C(X)^{\text{geo}}/l^m \to \pi_1^{ab}(X)^{\text{geo}}/l^m \) (we use the same notation as \( \rho_X : C(X) \to \pi_1^{ab}(X) \) for simplicity) is injective.

From the 2-dimensional local class field theory and the commutative diagrams (10) and (12), we have

\[
\bigoplus_{C \in \text{Cu}(X)} k(C_x) \rightarrow C(X)^{\text{geo}} \rightarrow C^t(X)^{\text{geo}} \rightarrow 0,
\]

\[
\bigoplus_{C \in \text{Cu}(X)} I_{k(C)}^1 \rightarrow \pi_1^{ab}(X)^{\text{geo}} \rightarrow \pi_1^{t,ab}(X)^{\text{geo}} \rightarrow 0.
\]

From Lem. 3.7 we have \( U^1 K_2(k(C)_x)/l^m = 0 \) for each \( C \in C(X) \) and \( x \in C_\infty \) so that \( C(X)^{\text{geo}}/l^m \cong C^t(X)^{\text{geo}}/l^m \). The wild inertia subgroup \( I_{k(C)}^1 \) is pro-\( p \) so that we have \( \pi_1^{ab}(X)/l^m \cong \pi_1^{t,ab}(X)/l^m \). Hence, the assertion is reduced to showing that \( \rho_X : C(X)^{\text{geo}}/l^m \to \pi_1^{t,ab}(X)^{\text{geo}}/l^m \) is injective.

Let \( \bar{k} = k^{-p^\infty} \) be the perfect closure of \( k \). We denote by \( \bar{X} = X \otimes_k \bar{k} \) the base change to \( \bar{k} \) of \( X \). We also denote by

\[
C^t(\bar{X}/\bar{k})^{\text{geo}} = \text{Ker} \left( C^t(\bar{X}/\bar{k}) \xrightarrow{N_{\bar{X}}} \bar{k}^\times \right)
\]

as in (10). From Lem. 2.6 and \( k^\times/l^m \simeq \bar{k}^\times/l^m \) we have an isomorphism \( C^t(X/k)^{\text{geo}}/l^m \cong C^t(\bar{X}/\bar{k})^{\text{geo}}/l^m \). It is well-known that we have \( \pi_1(\bar{X}) \cong \pi_1(X) \) (cf. [1], Exp. IV, Proof
of Thm. 6.1). we obtain $\rho_{\tilde{X}}$ below:

$$
\begin{align*}
C^t(\tilde{X}/\tilde{k})^{\text{geo}}/l^m & \xrightarrow{\rho_{\tilde{X}}} \pi_1^{\text{lab}}(\tilde{X}/\tilde{k})^{\text{geo}}/l^m \\
\simeq & \quad \simeq \\
C^t(X/k)^{\text{geo}}/l^m & \xrightarrow{\rho_X} \pi_1^{\text{lab}}(X/k)^{\text{geo}}/l^m,
\end{align*}
$$

where $\pi_1^{\text{lab}}(\tilde{X}/\tilde{k})^{\text{geo}} = \text{Ker}(\pi_1^{\text{lab}}(\tilde{X}) \to \tilde{G}_k^{\text{ab}})$. Thus the assertion is reduced to showing that $\rho_{\tilde{X}}$ is injective. In the following, we write $C^t(\tilde{X})^{\text{geo}} = C^t(\tilde{X}/\tilde{k})^{\text{geo}}$ and $\pi_1^{\text{lab}}(\tilde{X})^{\text{geo}} := \pi_1^{\text{lab}}(\tilde{X}/\tilde{k})^{\text{geo}}$ for simplicity.

**Claim 1.** The condition (Surj) in Lem. 3.4 holds. Namely, for any finite extensions $\tilde{k}'' \supset \tilde{k}' \supset \tilde{k}$, put $\tilde{X}' = \tilde{X} \otimes_{\tilde{k}} \tilde{k}'$ and $\tilde{X}'' = \tilde{X} \otimes_{\tilde{k}} \tilde{k}''$. Then, the norm map

$$
C^t(\tilde{X})^{\text{geo}}/l^m \to C^t(\tilde{X}'')^{\text{geo}}/l^m
$$

is surjective.

**Proof.** (We follow the argument of [16], Prop. 1.7.) Consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \to & C^t(\tilde{X})^{\text{geo}}/l^m & \xrightarrow{\rho_{\tilde{X}}} & \pi_1^{\text{lab}}(\tilde{X})^{\text{geo}}/l^m & \to & (\mathbb{Z}/l^m)^{r_i} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \to & C^t(\tilde{X}')^{\text{geo}}/l^m & \xrightarrow{\rho_{\tilde{X}'}^{\prime}} & \pi_1^{\text{lab}}(\tilde{X}')^{\text{geo}}/l^m & \to & (\mathbb{Z}/l^m)^{r_i} & \to & 0.
\end{array}
$$

Since the middle vertical map is surjective, the assertion follows from the above diagram. \[\square\]
From Lem. 3.4 we have the following commutative diagram:

\[
\begin{array}{ccc}
C^n(\tilde{X})_{\text{geo}}/l^m & \xrightarrow{\rho_{\tilde{X}}} & \pi_1^{\text{ab}}(\tilde{X})_{\text{geo}}/l^m \\
\oplus_{i=1}^n C^i(\tilde{X}_i)_{\text{geo}}/l^m & \xrightarrow{\oplus\rho_{\tilde{X}_i}} & \oplus_{i=1}^n \pi_1^{\text{ab}}(\tilde{X}_i)_{\text{geo}}/l^m,
\end{array}
\]

where the right vertical map is given by the projection \(\tilde{X} \to \tilde{X}_i\). Since the bottom horizontal map is injective in the above diagram, so is \(\rho_{\tilde{X}_i}\) and the assertion follows from this. \(\square\)

**Proof of Thm. 3.5.** Using Lem. 3.8, the same proof of Thm. 4.6 in [6] works well. We give a sketch of the proof. Let \(N'\) be the set of \(m \in \mathbb{Z}_{\geq 1}\) which is prime to \(p\). For an abelian group \(G\), we denote by

\[G_L := \lim_{\leftarrow m \in N'} G/m.\]

From Lem. 3.8, the map \(\rho_{X,L} : C(X)_L \to \pi_1^{\text{ab}}(X)_L\) induced from \(\rho_X\) is injective. Consider the following diagram:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & \xrightarrow{\psi} & \text{Ker}(\rho_X) & \xrightarrow{\text{Ker}(\psi)} \\
0 & \xrightarrow{\text{Ker}(\rho_X)} & C(X) & \xrightarrow{\rho_X} \pi_1^{\text{ab}}(X) \\
0 & \xrightarrow{\phi} & \text{Ker}(\phi) & \text{Ker}(\psi) \\
0 & \xrightarrow{\phi} & C(X)_L & \xrightarrow{\rho_{X,L}} \pi_1^{\text{ab}}(X)_L,
\end{array}
\]

where \(\psi : C(X) \to C(X)_L\) and \(\phi : \pi_1^{\text{ab}}(X) \to \pi_1^{\text{ab}}(X)_L\) are natural maps.

For any prime \(l \neq p\), \(\text{Ker}(\psi)\) is \(l\)-divisible by using Lem. 7.7 in [7] (cf. [8], Claim in the proof of Thm. 4.6). On the other hand, we have a short exact sequence

\[
\bigoplus_{C \in Cu(X)} \bigoplus_x I^1_{k(C)x} \to \text{Ker}(\phi) \to \text{Ker}(\phi') \to 0,
\]

where \(\phi' : \pi_1^{\text{lab}}(X) \to \pi_1^{\text{lab}}(X)_L\) is the natural map. The wild inertia subgroup \(I^1_{k(C)x}\) is pro-\(p\). From the finitely generatedness of \(\pi_1^{\text{lab}}(X)\), \(\text{Ker}(\phi')\) is \(l\)-torsion free and so is \(\text{Ker}(\phi)\). From the top exact sequence in (17), \(\text{Ker}(\rho_X)\) is \(l\)-divisible. \(\square\)
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