TWO AND THREE-REVOLUTION CYCLICAL SURFACES

The creation of two-revolution and three-revolution cyclical surfaces is presented in the paper. Classification and vector equations of the surfaces are given. The surfaces are created by revolution of the circle along the curves and its centre is on the curve. The curves are created by revolution of a point about any edge of the trihedron of the previous curve and this trihedron moves simultaneously along this curve. All specific forms of surfaces are illustrated in figures visualized in Maple.

1. Introduction

The point \( S_1 \) revolves about the coordinate axis \( z \) with angular velocity \( w_1 \) in the distance \( d_1 \) from the origin of the coordinate system \((O, x, y, z)\). For every value of the angle \( w_1 \) there exists only one position of the point \( R_1 \) and the trajectory of this point \( R_1 \) is the curve \( k_1 \) (circle). The trihedron \((R_1, t_1, n_1, b_1)\) defined in every point \( R_1 \in k_1 \) is determined by tangent, principal normal and binormal of the curve \( k_1 \). The point \( S_1 \) revolves at an angular velocity \( w_1 \) about any axis of the coordinate system, which is identical with the trihedron \((R_1, t_1, n_1, b_1)\) of the curve \( k_1 \), in the distance \( d_1 \) from the origin of this coordinate system which is moving simultaneously along the curve \( k_1 \). For every value of the angle \( w_2 \) there exists only one position of the point \( R_2 \). The trajectory of this point \( R_2 \) is the curve \( k_2^g \) where \( g = t, n, b \). The trihedron \((R_2, t_2, n_2, b_2)\) in every point \( R_2 \in k_2^g \) is determined by the tangent, principal normal and binormal of the curve \( k_2^g \). The point \( S_1 \) revolves about any axis of the coordinate system identical with the trihedron \((R_2, t_2, n_2, b_2)\) of the curve \( k_2^g \) at an angular velocity \( w_3 \) in the distance \( d_2 \) from the origin of this coordinate system which is moving simultaneously along the curve \( k_2^g \). For every value of the angle \( w_3 \) there exists only one position of the point \( R_3 \). The trajectory of the point \( R_3 \) is the curve \( k_3^{gh} \), where \( g = t, n, b \). The trihedron \((R_3, t_3, n_3, b_3)\) in every point \( R_3 \in k_3^{gh} \) is determined by the tangent, principal normal and binormal of the curve \( k_3^{gh} \).

The surface of the type \( P_{(uv)} \) is created by translation of the circle \( c_1 = (R_1, r_1) \) along the curve \( k_1 \), the surface of the type \( P_{(uv)}^g \) is created by translation of the circle \( c_2 = (R_2, r_2) \) along the curve \( k_2^g \) and the surface of the type \( P_{(uv)}^{gh} \) is created by translation of the circle \( c_3 = (R_3, r_3) \) along the curve \( k_3^{gh} \). The index \( g = t, n, b \) determines that the point \( S_1 \) revolves about the tangent \( t_1 \), or principal normal \( n_1 \) or binormal \( b_1 \) of the curve \( k_1 \) and the index \( h = t, n, g \) determines that the point \( S_2 \) revolves about tangent \( t_2 \), principal normal \( n_2 \) or binormal \( b_2 \) of the curve \( k_2^g \).

2. Vector functions of the curves \( k_1, k_2^g, k_3^{gh} \)

Let the curve \( k_1 \) be a circle created by revolution of the point \( S_1 = S_1(\psi_1, 0, 0, 1) \) about the axis of the coordinate system \((O, x, y, z)\) at an angular velocity \( w_1 = v \) and \( k_1 \) is determined by the vector function

\[
\mathbf{r}_1(v) = (x_1(v), y_1(v), z_1(v), 1) = S_1 \cdot T_1(w_1) = (d_1 \cos v, s \cdot q_1 \sin v, 0, 1), \quad v \in (0, 2\pi).
\]

The matrix \( T_1(w_1) \) represents the revolution of the point \( S_1 \) about the coordinate axis \( z \) given by \( (5) \) \((3^{rd} \) matrix for \( i = 1)\), where the parameter \( q_1 = \pm 1 \) determines the right-turn or left-turn revolution movement of the point \((Fig. 1, i = 1, j = z) [3]\).

We will define the trihedron \((R_1, t_1, n_1, b_1)\) of the curve \( k_1 \) in every point \( R_1 \in k_1 \) by the tangent \( t_1 \), principal normal \( n_1 \) and by binormal \( b_1 \) with their unit vectors \( t_1(v), n_1(v), b_1(v) \) by equations \((2), (3), (4)\) for \( i = 1, v \in [0, 2\pi]

\[
\begin{align*}
t_1(v) &= (a_1, b_1, c_1) = \frac{1}{h_0} \frac{d\mathbf{r}_1}{dv} = \\
&= \frac{1}{h_0} \left( \frac{dx_1}{dv}, \frac{dy_1}{dv}, \frac{dz_1}{dv} \right), \quad h_0 = \left| \frac{d\mathbf{r}_1}{dv} \right| \\
\mathbf{n}_1(v) &= (a_1, b_1, c_1) = \frac{1}{h_0} \frac{d\mathbf{n}_1}{dv} = \\
&= \frac{1}{h_0} \left( \frac{dx_1}{dv^2}, \frac{dy_1}{dv^2}, \frac{dz_1}{dv^2} \right), \quad h_0 = \left| \frac{d\mathbf{n}_1}{dv} \right| \\
h_1 &= \frac{1}{h_0} (\mathbf{t}_1(v) \times \mathbf{n}_1(v)), \quad h_0 = \left| \mathbf{t}_1(v) \times \mathbf{n}_1(v) \right|.
\end{align*}
\]

The curve \( k_2^g \) is created by revolution of the point \( S_2 \) in the distance \( d_2 \) from the origin of the coordinate system \((O, x, y, z)\) about any coordinate axis \( x, y, z \) through the angle \( w_2 \) into the point \( S_2' \) \((Fig. 1, i = 2 \text{ for } j = x, y, z)\). Angular velocity \( w_2 = mlv \)

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of the point $S_2$ is $m_1$-multiple of angular velocity $w_1$ of the point $S$. The point $S_1$ is transformed into the point $R_1$ in the coordinate system $(R_1, t_1, n_1, b_1)$. If we create a surface of the type $P^h(w_3)$, where $h = t$ (or $g = n$, or $g = b$), we will revolve the point $S_2$ about the axis $j = x, y, z$. The revolution of the point $S_2$ is represented by a matrix $T_2(w)$, $j = x, y, z$, in (5), where the parameter $q_1 = \pm 1$ determines the right-turned or left-turned revolution and the transformation of the point $S_1$ into the point $R_2$ is represented by a matrix $M_2(w)$ given by (5) [4]. The point $S_2$, will be situated always on any coordinate axis $x, y, z$.

$$T_2(w) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos w & q \sin w & 0 \\
0 & -q \sin w & \cos w & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

$$T_2(w) = \begin{bmatrix}
\cos w & 0 & q \sin w & 0 \\
0 & 1 & 0 & 0 \\
-q \sin w & 0 & \cos w & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

$$T_2(w) = \begin{bmatrix}
\cos w & q \sin w & 0 & 0 \\
-q \sin w & \cos w & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

$$M_2(w) = \begin{bmatrix}
\theta_x & \theta_y & \theta_x & 0 \\
\theta_y & \theta_z & \theta_x & 0 \\
\theta_x & \theta_y & \theta_z & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

The elements of the matrix $M_2(w)$ in (5) are the coordinates of unit vectors $t_i(v), n_i(v), b_i(v)$ of tangent $t_i$, principal normal $n_i$ and binormal $b_i$ in trihedron $(R_1, t_i, n_i, b_i), i = 1, 2, 3$. Then the vector function of the curve $k_{ij}^h$ is

$$r_3(v) = (x_{3j}(v), y_{3j}(v), z_{3j}(v), 1) = S_1 \cdot T_2(w) +$$

$$S_2 \cdot T_3(w) \cdot M_2(w), j = x, y, z.$$  \hspace{1cm} (6)

and $S_2 = S_3(d_2, 0, 0, 1)$ or $S_2 = S_3(0, 0, d_2, 1)$.

The trihedron $(R_1, t_j, n_j, b_j)$ is determined in every point $R_2 \in k_{ij}^h$ by the tangent $t_j$, principal normal $n_j$ and binormal $b_j$ with the unit vectors $r_3(v), n_j(v), b_j(v)$ expressed by equations (2), (3), (4) for $i = 2$.

The revolution $k_{ij}^h$ is created by revolution of the point $R_1$ in the distance $d_1$ from the origin of the coordinate system $(O, x, y, z)$ about any coordinate axis $x, y, z$ or $z$ through the angle $w_1$ into the point $S_1$ (Fig. 2, $i = 3$ for, $j = x, y, z$). Angular velocity $w_2 = m_2w_2 = m_2w_2$ of the point $S_3$ is $m_2$-multiple of angular velocity $w_3 = m_3w_3$ of the point $S_3$. The point $S_3$ is transformed into the point $R_3$ in the coordinate system $(R_3, t_j, n_j, b_j)$. If we create the surface of type $P^h(w_3)$, where $h = t$ (or $h = n$, or $h = b$), we will revolve the point $S_3$ about the axis $j = x, y, z$. The revolution of the point $S_3$ is represented by a matrix $T_3(w)$, $j = x, y, z$, where the parameter $q_3 = \pm 1$ determines the right-turned or left-turned revolution and transformation of the point $S_3$ into the point $R_3$ is represented by the matrix $M_3(w)$ by equations (5) [4]. The point $S_3$, will always be situated on any coordinate axis $x, y, z$.

Then the vector function of the curve $k_{ij}^h$ for $j = x, y, z$ is

$$r_3(v) = (x_{3j}(v), y_{3j}(v), z_{3j}(v), 1) = S_1 \cdot T_2(w) +$$

$$S_2 \cdot T_3(w) \cdot M_3(w), j = x, y, z.$$  \hspace{1cm} (7)

The trihedron $(R_1, t_j, n_j, b_j)$ in every point $R_1 \in k_{ij}^h$ is determined by the tangent $t_j$, principal normal $n_j$ and binormal $b_j$ with the unit vectors $t_j(v), n_j(v), b_j(v)$ by equations (2), (3), (4) for $i = 3$.

In Fig. 2 there is displayed a revolution of the point $S_1$ about the coordinate axis $z$ through the angle $w_1$ into the point $R_1$, where its revolutionary movement creates the curve $k_{ij}^h$, revolution of the point $S_2$ about the coordinate axis $z$ through the angle $w_2$ into the point $S_3$ and its transformation into the point $R_3$, where its revolutionary movement creates the curve $k_{ij}^h$. In the points $R_1, R_2, R_3$ there are displayed trihedrons $(R_1, t_j, n_j, b_j), (R_2, t_2, n_2, b_2), (R_3, t_3, n_3, b_3)$.

In Fig. 3 there are displayed for illustration only three combinations of the curves $k_{ij}^h, k_{ij}^h, k_{ij}^h, k_{ij}^h, k_{ij}^h$ and $k_{ij}^h, k_{ij}^h, k_{ij}^h$. 

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**Fig. 1 Revolution of the point about axes x, y, z**
The two-revolution cyclical surface of the type \( P^2_t(u,v) \) is created by translation of the circle \( C_2 = (R_2, r_2) \) along the curve \( k_2^t \) at an angular velocity \( w_2 = m_1 v \). The circle is always in the plane \((n_2, b_2)\) if the index \( g = 1 \), or in the plane \((t_2, b_2)\) if \( g = n \), or in the plane \((t_2, n_2)\) if \( g = b \) and its centre is the point \( R_2 \in k_2^t \). We will create it so that the circle \( c_{02} \) determined by the vector function \( c_{02}(u) = (0, r_2 \cos u, r_2 \sin u, 1) \) if \( g = t \), or \( c_{02}(u) = (r_2 \cos u, 0, r_2 \sin u, 1) \) if \( g = n \), or \( c_{02}(u) = (r_2 \cos u, r_2 \sin u, 0, 1) \) if \( g = b \) we will transform into the circle \( c_2 \) in the coordinate system \((R_2, t_2, n_2, b_2)\) using the matrix \( M_2(w_2) \) by equations (5) (Fig. 4). The vector function of the cyclical surface of the type \( P^2_t(u,v) \) is

\[
P^2_t(u,v) = r_2(v) + c_{02}(u) \cdot M_2(w_2),
\]

where \( r_2(v) \) is the vector function of the curve \( k_2^t \) determined by (6).

The three-revolution cyclical surface of the type \( P^3_t(u,v) \) is created by translation of the circle \( C_3 = (R_3, r_3) \) along the curve \( k_3 \) at an angular velocity \( w_3 = m_1 v \). The circle is always in the plane \((n_3, b_3)\) if the index \( h = t \), or in the plane \((t_3, b_3)\) if \( h = n \), or in the plane \((t_3, n_3)\) if \( h = b \) and its centre is the point \( R_3 \in k_3^t \). We will create it so that the circle \( c_{03} \) determined by the vector function \( c_{03}(u) = (0, r_3 \cos u, r_3 \sin u, 1) \) if \( h = t \), or \( c_{03}(u) = (r_3 \cos u, 0, r_3 \sin u, 1) \) if \( h = n \), or \( c_{03}(u) = (r_3 \cos u, r_3 \sin u, 0, 1) \) if \( h = b \) we will transform into the circle \( c_3 \) in the coordinate system \((R_3, t_3, n_3, b_3)\) using the matrix \( M_3(w_3) \) by equations (5). The vector function of the cyclical surface of the type \( P^3_t(u,v) \) is

\[
P^3_t(u,v) = r_3(v) + c_{03}(u) \cdot M_3(w_3),
\]

where \( r_3(v) \) is the vector function of the curve \( k_3^t \) determined by (7).
4. Classification of cyclical surfaces of the type $P_g^2(u,v), \ P_g^3(u,v)$

The two-revolution cyclical surface of the type $P_g^2(u,v)$ can be classified according to the index $g$:

Table 1: Classification of cyclical surfaces of the type $P_g^2(u,v)$

| $g$  | $t$  | $n$  | $b$  |
|------|------|------|------|
| $t$  | $P_t^2(u,v)$ | $P_t^g(u,v)$ | $P_t^b(u,v)$ |
| $n$  | $P_n^2(u,v)$ | $P_n^g(u,v)$ | $P_n^b(u,v)$ |
| $b$  | $P_b^2(u,v)$ | $P_b^g(u,v)$ | $P_b^b(u,v)$ |

The three-revolution cyclical surface of the type $P_g^3(u,v)$ can be classified according to the index $g$ and $h$:

Table 2: Classification of cyclical surfaces of the type $P_g^3(u,v)$

| $g/h$  | $t$  | $n$  | $b$  |
|--------|------|------|------|
| $t$    | $P_t^3(u,v)$ | $P_t^h(u,v)$ | $P_t^b(u,v)$ |
| $n$    | $P_n^3(u,v)$ | $P_n^h(u,v)$ | $P_n^b(u,v)$ |
| $b$    | $P_b^3(u,v)$ | $P_b^h(u,v)$ | $P_b^b(u,v)$ |

5. Illustrations of cyclical surfaces of the type $P_g(u,v), \ P_g^2(u,v), \ P_g^3(u,v)$

In Fig. 5 there are displayed three combinations of cyclical surfaces of the type $P_t(u,v), \ P_t^2(u,v), \ P_t^3(u,v)$ in fig. a), $P_n(u,v), \ P_n^2(u,v), \ P_n^3(u,v)$ in fig. b), $P_b(u,v), \ P_b^2(u,v), \ P_b^3(u,v)$ in fig. c).

In Fig. 6 there are displayed three combinations of cyclical surfaces of the type $P_t(u,v), \ P_n^2(u,v), \ P_b^3(u,v)$ in fig. a), $P_n(u,v), \ P_t^2(u,v), \ P_b^3(u,v)$ in fig. b), $P_b(u,v), \ P_t^2(u,v), \ P_n^3(u,v)$ in fig. c).

In Fig. 7 there are displayed three combinations of cyclical surfaces of the type $P_t(u,v), \ P_n^2(u,v), \ P_b^3(u,v)$ in fig. a), $P_n(u,v), \ P_t^2(u,v), \ P_b^3(u,v)$ in fig. b), $P_b(u,v), \ P_t^2(u,v), \ P_n^3(u,v)$ in fig. c).

The surfaces mentioned above can be used in design practice as constructive or ornamental structural components.

The paper was supported by the grant VEGA 1 / 4002 / 07.

Fig. 4 Transformation of the circle $c_0$ into the circle $c_i$

Fig. 5 Combinations of cyclical surfaces for $g = t$
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