Rooted Trees and Symmetric Functions: 
Zhao’s Homomorphism and the Commutative Hexagon

Michael E. Hoffman
Dept. of Mathematics
U. S. Naval Academy, Annapolis, MD 21402 USA
meh@usna.edu

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Abstract
Recent work in perturbative quantum field theory has led to much study
of the Connes-Kreimer Hopf algebra. Its (graded) dual, the
Grossman-Larson Hopf algebra of rooted trees, had already been studied
by algebraists. L. Foissy introduced a noncommutative version of
the Connes-Kreimer Hopf algebra, which turns out to be self-dual.
Using some homomorphisms defined by the author and W. Zhao,
we describe a commutative diagram that relates the aforementioned
Hopf algebras to each other and to the Hopf algebras of symmetric
functions, noncommutative symmetric functions, and quasi-symmetric
functions.

1 Introduction
A. Connes and D. Kreimer \cite{1} introduced a Hopf algebra (denoted here by
$\mathcal{H}_K$) to study renormalization in quantum field theory. The Hopf algebra
$\mathcal{H}_K$ is the free commutative algebra on rooted trees, with a noncommutative
coproduct. Its graded dual $\mathcal{H}_K^*$ is isomorphic to a Hopf algebra (which we call
$kT$) studied earlier by R. Grossman and R. G. Larson \[6\] whose elements are rooted trees with a noncommutative product and a cocommutative coproduct. A noncommutative version of $\mathcal{H}_K$, denoted here by $\mathcal{H}_F$, was introduced by L. Foissy \[2, 3\]: unlike $\mathcal{H}_K$, it is self-dual. In \[10\] the author defined a Hopf algebra $k\mathcal{P}$, based on planar rooted trees in the same way $kT$ is based on rooted trees, which is isomorphic to $\mathcal{H}_F^* \cong \mathcal{H}_F$. We describe these Hopf algebras in §3 below.

The author’s earlier paper \[10\] related the Hopf algebras of the preceding paragraph to the well-known Hopf algebra $\text{Sym}$ of symmetric functions, and its more recent extensions: $\text{QSym}$, the quasi-symmetric functions, and $\text{NSym}$, the noncommutative symmetric functions (described in §2 below). In \[10\] the author gave a pair of commutative squares that relate the Hopf algebras named above, and demonstrated their usefulness for certain computations. This paper adds some important extensions to this picture. As described in §4, a Hopf algebra homomorphism $Z : \text{NSym} \to kT$ due to W. Zhao \[14\] and its dual $Z^* : \mathcal{H}_K \to \text{QSym}$ link the two commutative squares into a single diagram, which we call the commutative hexagon. We also give an explicit characterization of $Z^*$ and deduce several corollaries, including an easy proof of the surjectivity of $Z^*$ (hence the injectivity of $Z$), and a description of $Z^*$ via quasi-symmetric generating functions of posets.

## 2 Symmetric and Quasi-Symmetric Functions

Henceforth $k$ is a field of characteristic 0. Let $\mathfrak{P}$ be the subalgebra of the formal power series ring $k[[x_1, x_2, \ldots]]$ consisting of those formal power series of bounded degree, where each $x_i$ has degree 1. An element $f \in \mathfrak{P}$ is called symmetric if the coefficients in $f$ of the monomials

$$x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \quad \text{and} \quad x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} \quad \text{(2.1)}$$

agree for any sequence of distinct positive integers $n_1, n_2, \ldots, n_k$, and quasi-symmetric if the coefficients in $f$ of the monomials (2.1) agree for any strictly increasing sequence $n_1 < n_2 < \cdots < n_k$ of positive integers. The sets of symmetric and quasi-symmetric formal series (by tradition called symmetric and quasi-symmetric functions) are denoted $\text{Sym}$ and $\text{QSym}$ respectively: both are subalgebras of $\mathfrak{P}$, and evidently $\text{Sym} \subset \text{QSym}$.

As a vector space, $\text{QSym}$ is generated by the monomial quasi-symmetric functions $M_I$, which are indexed by compositions (finite sequences) $I =$
$(i_1, \ldots, i_k)$ of positive integers:

$$M_I = \sum_{n_1 < n_2 < \cdots < n_k} x_{n_1}^{i_1} x_{n_2}^{i_2} \cdots x_{n_k}^{i_k}$$

(The degree of $M_I$ is $|I| = i_1 + \cdots + i_k$; we set $M_\emptyset = 1$). Forgetting order in a composition gives a partition: Sym has a basis consisting of monomial symmetric functions

$$m_\lambda = \sum_{\pi(I) = \lambda} M_I,$$

where $\pi$ is the function from compositions to partitions that forgets order. For example, $m_{2,1,1} = M_{(2,1,1)} + M_{(1,2,1)} + M_{(1,1,2)}$.

It is well-known that Sym, as an algebra, is freely generated by several sets of symmetric functions (see, e.g., [11]), in particular (1) the elementary symmetric functions $e_k = m_{1^k}$ (where $1^k$ means 1 repeated $k$ times); (2) the complete symmetric functions

$$h_k = \sum_{|\lambda| = k} m_\lambda = \sum_{|I| = k} M_I;$$

and (3) the power-sum symmetric functions $p_k = m_k$. There is a duality between the $e_k$ and the $h_k$, reflected in the (graded) identity

$$(1 + e_1 + e_2 + \cdots)(1 - h_1 + h_2 - \cdots) = 1. \quad (2.2)$$

There is also a well-known Hopf algebra structure on Sym [4]. This structure can be defined by making the elementary symmetric functions divided powers, i.e.,

$$\Delta(e_k) = \sum_{i+j=k} e_i \otimes e_j.$$

Equivalently, the $h_i$ are required to be divided powers, or the $p_i$ primitives. For this Hopf algebra structure,

$$\Delta(m_\lambda) = \sum_{\lambda = \mu \cup \nu} m_\mu \otimes m_\nu, \quad (2.3)$$

where the sum is over all pairs $(\mu, \nu)$ such that $\mu \cup \nu = \lambda$ as multisets. For example, $\Delta(m_{2,1,1})$ is

$$1 \otimes m_{2,1,1} + m_1 \otimes m_{2,1} + m_2 \otimes m_{1,1} + m_{1,1} \otimes m_2 + m_{2,1} \otimes m_1 + m_{2,1,1} \otimes 1.$$
The Hopf algebra $\text{Sym}$ is commutative and cocommutative, so its antipode $S$ is an algebra isomorphism with $S^2 = \text{id}$. In fact, as follows from (2.2),

$$S(e_i) = (-1)^i h_i.$$

We recall a useful duality result on graded connected Hopf algebras from [10]. By an inner product on a graded connected Hopf algebra $\mathcal{A}$, we mean a nondegenerate symmetric linear function $(\cdot, \cdot) : \mathcal{A} \otimes \mathcal{A} \to k$ such that $(a, b) = 0$ for homogeneous $a, b \in \mathcal{A}$ of different degrees.

**Theorem 2.1.** Let $\mathcal{A}, \mathcal{B}$ be graded connected locally finite Hopf algebras over $k$ which admit inner products $(\cdot, \cdot)_{\mathcal{A}}$ and $(\cdot, \cdot)_{\mathcal{B}}$ respectively. Suppose there is a degree-preserving linear map $\psi : \mathcal{A} \to \mathcal{B}$ such that, for all $a_1, a_2, a_3 \in \mathcal{A}$,

\begin{align*}
(a) \quad (a_1, a_2)_{\mathcal{A}} &= (\psi(a_1), \psi(a_2))_{\mathcal{B}}; \\
(b) \quad (a_1 a_2, a_3)_{\mathcal{A}} &= (\psi(a_1) \otimes \psi(a_2), \Delta \psi(a_3))_{\mathcal{B}}; \\
(c) \quad (a_1 \otimes a_2, \Delta(a_3))_{\mathcal{A}} &= (\psi(a_1) \psi(a_2), \psi(a_3))_{\mathcal{B}}.
\end{align*}

Then $\mathcal{B}$ is isomorphic to $\mathcal{A}^*$ via the pairing $(b, a) = (b, \psi(a))_{\mathcal{B}}$.

We note that it follows from this result that a graded connected locally finite Hopf algebra $\mathcal{A}$ is self-dual provided it admits an inner product $(\cdot, \cdot)$ such that

$$(a_1 \otimes a_2, \Delta(a_3)) = (a_1 a_2, a_3)$$

for all $a_1, a_2, a_3 \in \mathcal{A}$. In particular, note that $\text{Sym}$ has an inner product with

$$(e_\lambda, m_\mu) = \delta_{\lambda, \mu}$$

for all partitions $\lambda, \mu$ (where $e_\lambda$ means $e_{\lambda_1} e_{\lambda_2} \cdots$ for $\lambda = \lambda_1, \lambda_2, \ldots$). Then by equation (2.3),

$$(e_\mu e_\nu, m_\lambda) = (e_\mu \otimes e_\nu, \Delta(m_\lambda)) = \delta_{\mu \cup \nu, \lambda}$$

so $\text{Sym}$ is self-dual.

To give $\text{QSym}$ the structure of a graded connected Hopf algebra we define the coproduct $\Delta$ by

$$\Delta(M_K) = \sum_{I \sqcup J = K} M_I \otimes M_J, \quad (2.4)$$

where $I \sqcup J$ is the juxtaposition of the compositions $I$ and $J$. While this coproduct extends that on $\text{Sym}$, it is not cocommutative, e.g.,

$$\Delta(M_{(2,1,1)}) = 1 \otimes M_{(2,1,1)} + M_{(2)} \otimes M_{(1,1)} + M_{(2,1)} \otimes M_{(1)} + M_{(2,1,1)} \otimes 1.$$
Since QSym is commutative but not cocommutative, it cannot be self-dual: in fact, its dual is the Hopf algebra NSym of noncommutative symmetric functions in the sense of Gelfand et al. \[5\]. As an algebra NSym is the noncommutative polynomial algebra \(k\langle E_1, E_2, \ldots \rangle\), with \(E_i\) in degree \(i\), and the coalgebra structure is determined by declaring the \(E_i\) divided powers.

**Theorem 2.2.** (Cf. \[5\] Theorem 6.1) The Hopf algebra NSym is dual to QSym via the pairing \(\langle E_I, M_J \rangle = \delta_{I,J}\), where \(E_{(i_1, \ldots, i_l)} = E_{i_1}E_{i_2} \cdots E_{i_l}\).

**Proof.** We use Theorem 2.1 with \(\psi : \text{QSym} \rightarrow \text{NSym}\) defined by \(\psi(M_I) = E_I\). Define inner products on NSym and QSym by \(\langle E_I, E_J \rangle = \delta_{I,J}\) and \(\langle M_I, M_J \rangle = \delta_{I,J}\) respectively. Then hypothesis (a) of Theorem 2.1 is immediate, and hypothesis (c) follows from equation (2.4). For hypothesis (b), note that each nonzero contribution to

\[
\langle E_{i_1} \cdots E_{i_k} \otimes E_{j_1} \cdots E_{j_l}, \Delta(E_{n_1} \cdots E_{n_m}) \rangle
\]

comes from a splitting of \((n_1, \ldots, n_m)\) with each \(n_a\) the sum of a part of \((i_1, \ldots, i_k)\) (or zero) and a part of \((j_1, \ldots, j_l)\) (or zero) so that the parts stay in order; and this is exactly when a nonzero contribution to

\[
(M_{(i_1, \ldots, i_k)}M_{(j_1, \ldots, j_l)}, M_{(n_1, \ldots, n_m)})
\]

occurs, as follows from the quasi-shuffle multiplication on QSym (see \[7\]). \(\square\)

There is an abelianization homomorphism \(\tau : \text{NSym} \rightarrow \text{Sym}\) sending \(E_i\) to the elementary symmetric function \(e_i\). Its dual \(\tau^* : \text{Sym} \rightarrow \text{QSym}\) is the inclusion \(\text{Sym} \subset \text{QSym}\).

**3 Hopf Algebras of Rooted Trees**

A rooted tree is a partially ordered set \(t\) (whose elements we call vertices) such that (1) \(t\) has a unique maximal vertex (the root); and (2) for any vertex \(v\), the set of vertices exceeding \(v\) is a chain. If a vertex \(v\) covers \(w\) in the partial order, we call \(v\) the parent of \(w\) and \(w\) a child of \(v\). We visualize a rooted tree as a directed graph with an edge from each vertex to each of its children: the root (uniquely) has no incoming edges. Let \(T\) be the set of (finite) rooted trees, and \(T_n = \{t \in T : |t| = n + 1\}\) the set of rooted trees with \(n + 1\) vertices. There is a graded vector space

\[
kT = \bigoplus_{n \geq 0} kT_n
\]
with the set of rooted trees as basis. We denote by \( \text{Symm}(t) \) the symmetry group of the rooted tree \( t \), i.e., the group of automorphisms of \( t \) as a poset (or directed graph).

For any forest (i.e., monomial in rooted trees) \( t_1 t_2 \cdots t_k \), there is a rooted tree \( B_+(t_1 t_2 \cdots t_k) \) given by attaching a new root vertex to each of the roots of \( t_1, t_2, \ldots, t_k \), e.g.,

\[
B_+(\bullet \quad ) = \quad .
\]

If we let \( B_+(1) = \bullet \in T_0 \), then \( B_+ \) becomes an isomorphism of graded vector spaces from the symmetric algebra \( S(kT) \) to \( kT \). Here \( S(kT) \) is graded by

\[
|t_1 \cdots t_k| = |t_1| + \cdots + |t_k|,
\]

where \(|t|\) is the number of vertices of the rooted tree \( t \).

Grossman and Larson \[6\] define a product \( \circ \) on \( kT \) as follows. For rooted trees \( t \) and \( t' \), let \( t = B_+(t_1 t_2 \cdots t_n) \) and \(|t'| = m\). Then \( t \circ t' \) is the sum of the \( m^n \) rooted trees obtained by attaching each of the \( t_i \) to some vertex of \( t' \): if \( t = \bullet \), set \( t \circ t' = t' \). For example,

\[
\begin{array}{c}
\end{array}
\]

while

\[
\begin{array}{c}
\end{array}
\]

This noncommutative product makes \( kT \) a graded algebra with two-sided unit \( \bullet \). There is a coproduct \( \Delta \) on \( kT \) defined by \( \Delta(\bullet) = \bullet \otimes \bullet \) and

\[
\Delta(B_+(t_1 t_2 \cdots t_k)) = \sum_{I \cup J = \{1, 2, \ldots, k\}} B_+(t(I)) \otimes B_+(t(J)), \tag{3.1}
\]

where \( t_1, \ldots, t_k \) are rooted trees, the sum is over all disjoint pairs \( (I, J) \) of subsets of \( \{1, 2, \ldots, k\} \) such that \( I \cup J = \{1, 2, \ldots, k\} \), and \( t(I) \) means the product of \( t_i \) for \( i \in I \) (and \( t(\emptyset) = 1 \)). As proved in \[6\], the vector space \( kT \) with product \( \circ \) and coproduct \( \Delta \) is a graded connected Hopf algebra.
It is evident from definition (3.1) that any rooted tree of the form \( B_+(t) \), i.e., any rooted tree in which the root vertex has only one child, is a primitive element of the Hopf algebra \( kT \). Let \( \mathcal{PT} \) be the set of such primitive trees, graded like \( T \) by the number of non-root vertices. Then the vector space of primitives \( \mathcal{P}(kT) \) of the Hopf algebra \( kT \) has \( \mathcal{PT} \) as basis, i.e., \( \mathcal{P}(kT) = k\mathcal{PT} \): for proof see [6, Theorem 4.1] or [8, Prop. 4.2].

As a graded algebra, the Connes-Kreimer Hopf algebra \( \mathcal{H}_K \) is \( S(kT) \) with the grading discussed above. The coproduct on \( \mathcal{H}_K \) can be described recursively by setting \( \Delta(1) = 1 \otimes 1 \) and

\[
\Delta(t) = t \otimes 1 + (\text{id} \otimes B_+)\Delta(B_-(t)), \tag{3.2}
\]

for rooted trees \( t \), where \( B_- \) is the inverse of \( B_+ \) and it is assumed that \( \Delta \) acts multiplicatively on products of rooted trees. Equation (3.2) implies that the “ladders” \( \ell_i = B_+^{-1}(\bullet) \) are divided powers, so \( \phi(e_i) = \ell_i \) defines a Hopf algebra homomorphism \( \phi : \text{Sym} \to kT \).

There is an inner product \((\cdot, \cdot)\) on \( kT \) given by

\[
(t, t') = \begin{cases} 
|\text{Symm}(t)|, & t = t', \\
0, & \text{otherwise}.
\end{cases} \tag{3.3}
\]

Since \( \text{Symm}(B_+(t)) \cong \text{Symm}(t) \) for rooted trees \( t \), \( B_+ \) is an isometry for \((\cdot, \cdot)\). In fact, we can extend this inner product to \( S(kT) = \mathcal{H}_K \) by setting

\[
(u, v) = (B_+(u), B_+(v))
\]

for forests \( u, v \). In [10, Theorem 4.2] (cf. [8, Prop. 4.4]) it is shown that Theorem 2.1 applies to \( \mathcal{H}_K \) and \( kT \) with these inner products and \( \psi = B_+ \), giving the following result.

**Theorem 3.1.** The Hopf algebra \( \mathcal{H}_K \) is isomorphic to the dual of \( kT \) via the pairing \( \langle t, u \rangle = (t, B_+(u)) \).

A planar rooted tree is a particular realization of a rooted tree in the plane, i.e., we consider

\[
\begin{array}{c}
\text{distinct planar rooted trees. In parallel to } T, \text{ we define } \mathcal{P} \text{ to be the graded set of planar rooted trees, and } k\mathcal{P} \text{ the corresponding graded vector space.}
\end{array}
\]
The tensor algebra $T(k\mathcal{P})$ can be regarded as the algebra of ordered forests of planar rooted trees, and there is a linear map $B_+: T(k\mathcal{P}) \to k\mathcal{P}$ that makes a planar rooted tree out of an ordered forest $T_1 \cdots T_k$ of planar rooted trees by attaching a new root vertex to the root of each $T_i$. With the same conventions about grading as above, $B_+$ is an isomorphism of graded vector spaces.

There is an analogue of the Grossman-Larson product for planar rooted trees. Given planar rooted trees $T, T'$ with $T = B_+(T_1 \cdots T_n)$, let $T \circ T'$ be the sum of the
\[
\begin{pmatrix} 2m + n - 2 \\ n \end{pmatrix}
\]
planar rooted trees formed by attaching $T_1, \ldots, T_n$, in order, to the vertices of $T'$. Note that if $T'$ has $m$ vertices, then it has a total of $2m - 1$ “attachment points” with a natural order: for example, if
\[
T' = \bigwedge
\]
then there are three attachment points $a_1, a_2, a_3$ for the root vertex, one attachment point $b_1$ for its left child, and one attachment point $c_1$ for its right child, with the natural order $a_1 < b_1 < a_2 < c_1 < a_3$. Thus
\[
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.
\]

Now we make $k\mathcal{P}$ a coalgebra by defining a coproduct $\Delta$ on planar rooted trees by
\[
\Delta(T) = \sum_{i=0}^{n} B_+(T_1 \cdots T_k) \otimes B_+(T_{k+1} \cdots T_n)
\]
where $B_-(T) = T_1 \cdots T_n$. In [10] the following result is proved.

**Theorem 3.2.** The product $\circ$ and coproduct $\Delta$ make $k\mathcal{P}$ a graded connected Hopf algebra.
As an algebra, the Foissy Hopf algebra $\mathcal{H}_F$ is the tensor algebra $T(kP)$. The coalgebra structure can be defined by the same equation (3.2) as for $\mathcal{H}_K$, except that rooted trees are replaced by planar rooted trees, and the forests are ordered. Application of Theorem 2.1 with $\psi = B_+$ and appropriate inner products gives the following result [10, Theorem 5.2].

**Theorem 3.3.** The Hopf algebra $(kP, \circ, \Delta)$ is dual to $\mathcal{H}_F$.

On the other hand, L. Foissy constructs in [2, §6] an inner product $(\cdot, \cdot)_F$ on $\mathcal{H}_F$ with

$$(F_1 F_2, F_3)_F = (F_1 \otimes F_2, \Delta(F_3))_F$$

for ordered forests $F_1, F_2, F_3$. This establishes the following result.

**Theorem 3.4.** The Foissy Hopf algebra $\mathcal{H}_F$ is self-dual.

## 4 The Commutative Hexagon

The “ladders” $\ell_i$, considered as elements of $\mathcal{H}_F$, are divided powers, so there is a Hopf algebra homomorphism $\Phi : \text{NSym} \rightarrow \mathcal{H}_F$ with $\Phi(E_i) = \ell_i$. If $\rho : \mathcal{H}_F \rightarrow \mathcal{H}_K$ sends each planar rooted tree to the corresponding rooted tree and forgets order in products, then $\rho \Phi = \phi \tau$. Taking duals, we have the commutative diagram of Hopf algebras

$$\begin{array}{ccc}
\text{QSym} & \overset{\Phi^*}{\longrightarrow} & kP \\
\tau^* \downarrow & & \rho^* \downarrow \\
\text{Sym} & \underset{\phi^*}{\longleftarrow} & kT
\end{array}$$

in which the maps can be described as follows. As noted earlier, $\tau^*$ is the inclusion $\text{Sym} \subset \text{QSym}$. For rooted trees $t$,

$$\phi^*(t) = \begin{cases} |\text{Symm}(B_+(\ell_\lambda))| m_\lambda, & \text{if } t = B_+(\ell_\lambda) \text{ for some partition } \lambda; \\ 0, & \text{otherwise}; \end{cases}$$

(4.2)

where $\ell_\lambda = \ell_{\lambda_1} \ell_{\lambda_2} \cdots \ell_{\lambda_k}$ for a partition $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$. On the other hand, for planar rooted trees $T$,

$$\Phi^*(T) = \begin{cases} M_I, & \text{if } T = B_+(\ell_I) \text{ for some composition } I; \\ 0, & \text{otherwise}; \end{cases}$$
where for a composition $I = (i_1, i_2, \ldots, i_k)$ we define the ordered forest $\ell_I$ of planar rooted trees by $\ell_I = \ell_{i_1} \ell_{i_2} \cdots \ell_{i_k}$. For a rooted tree $t$,

$$\rho^*(t) = |\text{Symm}(t)| \sum_{T \in \rho^{-1}(t)} T.$$

From (4.2) it follows that $\phi^*$ sends the element

$$\kappa_n = \sum_{t \in T_n} \frac{t}{|\text{Symm}(t)|} \in kT$$

to $h_n \in \text{Sym}$. Defining $\epsilon_n \in kT$ inductively by $\epsilon_0 = \bullet$ and

$$\epsilon_n = \kappa_1 \circ \epsilon_{n-1} - \kappa_2 \circ \epsilon_{n-2} + \cdots + (-1)^{n-1} \kappa_n, \quad n \geq 1,$$

the identity (2.2) implies $\phi^*(\epsilon_n) = e_n$. From [10] we cite the following.

**Proposition 4.1.** The elements $\kappa_n$ and $\epsilon_n$ are divided powers, and satisfy

1. $\epsilon_n = (-1)^n S(\kappa_n)$
2. $n! \epsilon_n = B_+(\ell_1^n)$.

W. Zhao defines a homomorphism $Z : \text{NSym} \to kT$ with the property that $Z(E_n) = \epsilon_n$ [14, Theorem 4.6]: by the preceding result, $Z$ sends the noncommutative analogue $(-1)^n S(E_n)$ of the $n$th complete symmetric function to $\kappa_n$. In fact, $Z$ unites diagram (4.1) and its dual in a commutative hexagon.

**Theorem 4.2.** The following diagram commutes:

![Diagram](4.4)
Proof. Commutativity of (4.1) and its dual show the upper and lower diamonds commute. It remains to show that the left and right triangles commute: since they are dual, it suffices to show \( \phi^*Z = \tau \). As the \( E_i \) generate \( \text{NSym} \), this follows from \( \phi^*Z(E_i) = \phi^*(\epsilon_i) = e_i = \tau(E_i) \).

The diagram (4.4) has a symmetry about the center that exchanges each Hopf algebra and homomorphism with its dual. By tracing around (4.4) starting with \( E_i \in \text{NSym} \) one sees, e.g., that \( Z^*(\ell_i) = M_{(1,\ldots,1)} = e_i \). Now \( \rho^*, \Phi^* \) and \( \phi^* \) are evidently surjective, so \( \rho^*, \Phi \) and \( \phi \) are injective. As we show below, \( Z^* \) is surjective, and so \( Z \) is injective; cf. [15, Theorem 5.1]. Thus \( \text{NSym} \) is a sub-Hopf-algebra of \( kT \) (and \( kP \)), while \( \text{QSym} \) is a quotient Hopf algebra of \( \mathcal{H}_K \) (and \( \mathcal{H}_F \)). We shall use the following characterization of \( Z^*: \mathcal{H}_K \rightarrow \text{QSym} \).

**Theorem 4.3.** For any monomial \( u \) of \( \mathcal{H}_K \), \( Z^*B_+(u) = A_+Z^*(u) \), where \( A_+: \text{QSym} \rightarrow \text{QSym} \) is the linear map that sends \( M_I \) to \( M_{I\cup(1)} \). Further, \( Z^* \) is the only homomorphism of algebras from \( \mathcal{H}_K \) to \( \text{QSym} \) with this property.

**Proof.** Let \( \Pi: kT \rightarrow k\mathcal{PT} \) be projection onto the primitive trees, i.e.,

\[
\Pi(t) = \begin{cases} 
  t, & \text{if } t = B_+(t') \text{ for some rooted tree } t'; \\
  0, & \text{otherwise}.
\end{cases}
\]

Since \( B_+^2(u) \) is a primitive tree for any monomial \( u \) of \( \mathcal{H}_K \), we have

\[
(w, B_+^2(u)) = (\Pi(w), B_+^2(u))
\]

for any \( w \in kT \) (The inner product is that given by (3.3)). Observe that

\[
\Pi(\epsilon_{i_1} \circ \epsilon_{i_2} \circ \cdots \circ \epsilon_{i_k}) = \begin{cases} 
  B_+(\epsilon_{i_1} \circ \epsilon_{i_2} \circ \cdots \circ \epsilon_{i_{k-1}}), & \text{if } i_k = 1, \\
  0, & \text{otherwise}.
\end{cases}
\]

(4.5)

Now let \( P(E_1, E_2, \ldots) \in \text{NSym} \) be any (noncommutative) polynomial in the \( E_i \). We can write

\[
P(E_1, E_2, \ldots) = \tilde{P}(E_1, E_2, \ldots)E_1 + R(E_1, E_2, \ldots)
\]

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where \( \tilde{P}(E_1, E_2, \ldots) \) and \( R(E_1, E_2, \ldots) \) are polynomials in the \( E_i \) such that every term of \( R(E_1, E_2, \ldots) \) ends in \( E_i \) with \( i > 1 \). Then

\[
\langle P(E_1, E_2, \ldots), Z^*B_+(u) \rangle = \langle P(\epsilon_1, \epsilon_2, \ldots), B_+(u) \rangle \\
= (P(\epsilon_1, \epsilon_2, \ldots), B^2_+(u)) \\
= (\Pi P(\epsilon_1, \epsilon_2, \ldots), B^2_+(u)) \\
= (B_+ \tilde{P}(\epsilon_1, \epsilon_2, \ldots), B^2_+(u)) \\
= (\tilde{P}(\epsilon_1, \epsilon_2, \ldots), B_+(u)) \\
= (\tilde{P}(E_1, E_2, \ldots), Z^*(u)).
\]

Now in view of the pairing of Theorem 2.2 it is evident that

\[
A^*_+(E_{j_1}E_{j_2} \cdots E_{j_k}) = \begin{cases} E_{j_1}E_{j_2} \cdots E_{j_{k-1}}, & \text{if } j_k = 1; \\ 0, & \text{otherwise,} \end{cases} \tag{4.6}
\]

so the preceding argument shows

\[
\langle P(E_1, E_2, \ldots), Z^*B_+(u) \rangle = \langle A^*_+P(E_1, E_2, \ldots), Z^*(u) \rangle
\]

and the first part of the theorem follows.

For uniqueness, note that \((\mathcal{H}_K, B_+)\) is the initial object in the category whose objects are pairs \((\mathcal{A}, \lambda)\) consisting of a commutative unitary algebra \( \mathcal{A} \) and a linear map \( \lambda : \mathcal{A} \to \mathcal{A} \), and whose morphisms are the obvious commutative squares (see [12, §3]). Thus there is a unique algebra homomorphism \( \rho : \mathcal{H}_K \to \text{QSym} \) with \( \rho B_+ = A_+ \rho \). Since \( Z^* \) has this property, \( \rho = Z^* \). \( \square \)

We give three corollaries of Theorem 4.3. First we show \( Z^* \) surjective.

**Corollary 4.4.** The homomorphism \( Z^* : \mathcal{H}_K \to \text{QSym} \) is surjective.

**Proof.** It suffices to show that any \( M_I \) is in the image of \( Z^* \). We proceed by induction on \(|I|\), the case \(|I| = 1\) being immediate. Suppose the result holds for \( M_I \) with \(|I| < n \). From Theorem 4.3 and the induction hypothesis, \( M_{(a_1, \ldots, a_{k-1}, 1)} \in \text{im } Z^* \) when \( a_1 + \cdots + a_{k-1} = n - 1 \). Suppose inductively that \( M_I \in \text{im } Z^* \) when \(|I| = n \) and the last part of \( I \) is at most \( m - 1 \). Since

\[
M_{(1)}M_{(a_1, \ldots, a_{k-1}, m-1)} = M_{(1,a_1,a_2,\ldots,a_{k-1},m-1)} + M_{(a_1+1,a_2,\ldots,a_{k-1},m-1)} + \cdots + M_{(a_1,\ldots,a_{k-1},1,m-1)} + M_{(a_1,\ldots,a_{k-1},m)} + M_{(a_1,\ldots,a_{k-1},m-1,1)},
\]

it follows that \( M_{(a_1, \ldots, a_{k-1}, m)} \in \text{im } Z^* \) when \( a_1 + \cdots + a_{k-1} + m = n \). \( \square \)
Our next corollary gives a description of the homomorphism $Z^*$ via quasi-symmetric generating functions of posets. Let $P$ be a finite poset, $\mathbb{Z}^+$ the set of positive integers with its usual order. Call a function $\sigma : P \to \mathbb{Z}^+$ strictly order-preserving if $\sigma(v) < \sigma(w)$ for any pair of elements $v < w$ of $P$, and let $\text{SOP}(P)$ denote the set of such functions. For any finite poset $P$, define

$$\bar{K}(P) = \sum_{\sigma \in \text{SOP}(P)} \prod_{v \in P} x_{\sigma(v)}. \quad (4.7)$$

It is evident that $\bar{K}(P) \in \text{QSym}$ and

$$\bar{K}(P \sqcup Q) = \bar{K}(P)\bar{K}(Q), \quad (4.8)$$

where $P \sqcup Q$ means the disjoint union of posets $P$ and $Q$. So we have a quasi-symmetric function $\bar{K}(t)$ for any rooted tree $t$, e.g.,

$$\bar{K}(\text{rooted tree}) = \sum_{\sigma(a) > \sigma(b) > \sigma(c) > \sigma(d)} x_{\sigma(a)}x_{\sigma(b)}x_{\sigma(c)}x_{\sigma(d)} = M_{(1,1,1,1)} + M_{(1,2,1)} + M_{(1,1,1,1)} + M_{(2,1,1)} + M_{(1,1,1,1)} = 3M_{(1,1,1,1)} + M_{(2,1,1)} + M_{(1,2,1)}.$$

**Remark.** This definition appears in [13, §7], but note that the convention there is that the root is the minimal rather than the maximal element of the poset corresponding to a rooted tree.

Now extend $\bar{K}$ to $\mathcal{H}_K$ by defining $\bar{K}(t_1 \cdots t_k)$ as $\bar{K}$ of the poset $t_1 \sqcup \cdots \sqcup t_k$ for any rooted trees $t_1, \ldots, t_k$. Equation (4.8) means that $\bar{K} : \mathcal{H}_K \to \text{QSym}$ is an algebra homomorphism.

**Corollary 4.5.** The algebra homomorphism $\bar{K} : \mathcal{H}_K \to \text{QSym}$ satisfies the equation $\bar{K} B_+(u) = A_+ \bar{K}(u)$, and hence must coincide with $Z^*$.

**Proof.** Let $u = t_1 \cdots t_k$ be a monomial in $\mathcal{H}_K$. Thought of as a poset, $B_+(u)$ is $t_1 \sqcup \cdots \sqcup t_k$ with a maximal element adjoined. But then it follows from definition (4.7) that every term in $\bar{K}(B_+(u))$ consists of a factor $x_{i_1} \cdots x_{i_p}$ of $\bar{K}(u)$ times a factor $x_i$ corresponding to the maximal element, i.e. $i > i_j$ for $j = 1, 2, \ldots, p$. But this means that $\bar{K}(B_+(u)) = A_+(\bar{K}(u))$. \qed
Finally, let $A_-: \text{QSym} \to \text{QSym}$ be the adjoint of $A_+$ with respect to the inner product on $\text{QSym}$ introduced in §2, i.e.,

$$A_-(M(a_1,\ldots,a_k)) = \begin{cases} M(a_1,\ldots,a_{k-1}), & \text{if } a_k = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $A_-A_+ = \text{id}$. Unlike $A_+$, $A_-$ is a derivation (as can be shown using the quasi-shuffle multiplication on $\text{QSym}$ and considering cases). Hence $\text{QSym}^0 = \ker A_-$ is a subalgebra of $\text{QSym}$ and in fact $\text{QSym} = \text{QSym}^0[1]$ (see the discussion in [9, §2]), so we can think of $A_-$ as $\partial/\partial M(1)$. (It’s also true that $A_-$ restricts to the derivation $p_1^\perp$ of $\text{Sym}$ described in [11, §I.5, ex. 3].) If $B_-$ is extended to $\mathcal{H}_K$ as a derivation, then we have the following.

**Corollary 4.6.** The homomorphism $Z^*$ satisfies $Z^*B_- = A_-Z^*$.

**Proof.** Since $B_-$ and $A_-$ are both derivations, it suffices to show $Z^*B_-(t) = A_-Z^*(t)$ for any rooted tree $t$. This follows by applying $A_-$ to both sides of

$$Z^*(t) = Z^*(B_+B_-(t)) = A_+Z^*B_-(t).$$

In addition, the following analogue of equation (3.2) holds:

$$\Delta(u) = u \otimes 1 + (\text{id} \otimes A_+)\Delta(A_-(u)), \quad \text{for } u \in A_+(\text{QSym}).$$

Recall that the dual $A_+^* : \text{NSym} \to \text{NSym}$ of $A_+$ is given by equation (4.6), and it is easy to see that $A_-$ has dual $A_-^*(u) = uE_1$. The following result dualizes Theorem 4.3 and Corollary 4.6.

**Proposition 4.7.** For $u \in \text{NSym},$

1. $ZA_+^*(u) = B_-\Pi Z(u);$  
2. $ZA_-^*(u) = Z(u) \circ \epsilon_1 = Z(u) \circ \ell_2.$

**Proof.** Part (2) is immediate: for part (1), use equation (4.5). □

This result should be compared to [8, Prop. 4.5]: in particular, note the parallel between the operator $t \mapsto t \circ \ell_2$ of part (2) and the “growth operator” $t \mapsto \ell_2 \circ t$ of [8, Prop. 4.5(1)] (for the growth operator see also [11, §3], [8, §2] and [10, §6.1]).
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