SHORTENING THE HOFER LENGTH OF HAMILTONIAN CIRCLE ACTIONS

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Abstract. A Hamiltonian circle action on a compact symplectic manifold is known to be a closed geodesic with respect to the Hofer metric on the group of Hamiltonian diffeomorphisms. If the momentum map attains its minimum or maximum at an isolated fixed point with isotropy weights not all equal to plus or minus one, then this closed geodesic can be deformed into a loop of shorter Hofer length. In this paper we give a lower bound for the possible amount of shortening, and we give a lower bound for the index (“number of independent shortening directions”). If the minimum or maximum is attained along a submanifold $B$, then we deform the circle action into a loop of shorter Hofer length whenever the isotropy weights have sufficiently large absolute values and the normal bundle of $B$ is sufficiently un-twisted.

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1. Introduction

A Hamiltonian circle action on a symplectic manifold \((M, \omega)\) can be viewed as a loop in the group \(\text{Ham}(M, \omega)\) of Hamiltonian diffeomorphisms, parametrized by \(\mathbb{R}/\mathbb{Z}\). As said in the abstract, we are concerned with deforming this loop, through loops in \(\text{Ham}(M, \omega)\), into a loop whose Hofer length is smaller. In short, we call such a deformation “shortening”. The reader may go directly to Section 3 for the precise statements of the new results that we prove in this paper.

One motivation for such questions comes from Riemannian geometry, where one studies the action functional \(\gamma \mapsto \int_0^1 \|\dot{\gamma}(t)\|^2 dt\) on spaces of loops or paths with fixed endpoints in a compact Riemannian manifold. Critical points are geodesics; the index of a critical point is finite and is equal to the number (counting multiplicity) of conjugate points along the geodesic; every homotopy class contains a minimal geodesic. Morse theory in this context has played a major role in the study of the topology of compact Lie groups. See Milnor’s book [17].

The infinite dimensional group \(\text{Ham}(M, \omega)\) is a central object in symplectic geometry. It has a natural Finsler metric, introduced by Hofer, which induces a non-trivial distance function. A path \(\gamma(t)\) in \(\text{Ham}(M, \omega)\) is generated by a time-dependent function \(H_t: M \to \mathbb{R}\), which plays the role of \(\dot{\gamma}(t)\); the Hofer length functional is \(\gamma \mapsto \int_0^1 \left( \max_M H_t - \min_M H_t \right) dt\).

There have been various attempts to extend to this setting notions that had been applied successfully to the actional functional of Riemannian geometry. The situation here is more difficult and it is not yet clear to what extent the analogy can be carried out. But the partial results that exist are beautiful and deep. A main theme is to express Morse properties of Hofer’s functional on a space of paths in \(\text{Ham}(M, \omega)\) in terms of the dynamics of the flows on \(M\) given by these paths.

There are several notions of “geodesic path” in \(\text{Ham}(M, \omega)\). With all of these notions, a time-independent Hamiltonian flow is geodesic. (But not every geodesic is time-independent. This is in contrast to the finite dimensional case: in a finite dimensional Lie group with a bi-invariant metric, geodesics through the origin are exactly the one-parameter subgroups.) Thus, every Hamiltonian circle action is a geodesic loop in \(\text{Ham}(M, \omega)\).

Starting with a Hamiltonian circle action \(\{\phi_t: M \to M\}_{0 \leq t \leq 1}\) on a compact connected symplectic manifold, here is a sample of known results.

1. Suppose that the circle action is semi-free near the set where the momentum map is maximal. (Semi-free means that only the trivial group and the entire circle occur as stabilizers.) Then its positive Hofer length is minimal among homotopic loops. See [19], the paragraph after Thm. 1.21, whose proof relies on [15, Thm. 1.9].

In particular, if the circle action is semi-free on \(M\) then it is length-minimizing among homotopic loops in \(\text{Ham}(M, \omega)\). This was already shown by McDuff-Slimowitz [14, Theorem 1.4].
Another consequence is that, if the circle action is semi-free near the set where the momentum map is maximal, then the circle action is non-contractible in $\text{Ham}(M, \omega)$. This was already shown by McDuff-Tolman [15, Theorem 1.1].

(2) Suppose that the momentum map attains its maximum at an isolated fixed point and that the circle action is not semi-free near the maximum. Then the circle action can be deformed to a loop in $\text{Ham}(M, \omega)$ whose Hofer length is smaller. See Ustilovsky [23, Theorem 1.2.E].

(3) If $n$ is a sufficiently large integer, then the non-effective circle action $\{\phi_{nt}\}_{0 \leq t \leq 1}$ can be deformed into a loop of smaller Hofer length. See Polterovich [18, Theorem 8.2.H].

Let $B$ be the set where the momentum map attains its maximum. If there exists a Hamiltonian diffeomorphism $g$ such that $g(B) \cap B = \emptyset$, then the non-effective circle action $\{\phi_{2t}\}_{0 \leq t \leq 1}$ can be deformed to a loop in $\text{Ham}(M, \omega)$ of smaller Hofer length. See Polterovich [18, Theorem 8.2.H].

Considering the opposite circle action gives the analogous results with “maximum” replaced by “minimum” and “positive Hofer length” replaced by “negative Hofer length”.

Many results in the literature apply to spaces of paths in $\text{Ham}(M, \omega)$ with fixed endpoints and to geodesic loops or paths that are possibly time-dependent. We refer the reader to the papers by Hofer [7], Bialy-Polterovich [3], Yi Ming Long [12], Siburg [21], Ustilovsky [23], Lalonde-McDuff [10, 11], McDuff-Slimowitz [14], McDuff [13], and Saveliev [19, 20]. Polterovich’s book [18] contains systematic explanations of many results in the subject.

**Example 1.1.** Start with $\mathbb{CP}^2$ with the circle action $[u, z, w] \mapsto [u, e^{-2\pi it}z, e^{2\pi it}w]$ and momentum map $(|w|^2 - |z|^2)/(|u|^2 + |z|^2 + |w|^2) + \text{constant}$. Take an equivariant blow-up at the point $[0, *, 0]$ where the momentum map is minimal. This yields a Hamiltonian circle action on a symplectic manifold $(M, \omega)$ whose isotropy weights where the momentum map is maximal are $-1, -2$ and whose isotropy weights where the momentum map is minimal are $1, 1$. It is an effective circle action that represents a non-trivial element of $\pi_1(M, \omega)$ (by the above item (1), applied to the minimum) but that can be deformed into a shorter loop in $\text{Ham}(M, \omega)$ (by the above item (2), applied to the maximum).

Typically, to show that a geodesic cannot be deformed to a loop of smaller Hofer length requires “hard” holomorphic curve techniques, whereas to show that a geodesic can be shortened is possible with “soft” techniques.

In this paper we use “soft” techniques to explore the behaviour of the Hofer length functional as we deform a Hamiltonian circle action within the space of loops in $\text{Ham}(M, \omega)$. Starting with a circle action is a very restrictive assumption, but it allows us to obtain new results that do not follow from general existing results. Our new results are sketched in the abstract and stated in Section 3. In brief,

* Our Theorem 3.1 is the first quantitative shortening estimate for effective Hamiltonian circle actions.
Our Theorem 3.7, Theorem 3.9, and Corollary 3.12 are the first shortening results that apply when the extrema of the momentum map are not necessarily displaceable and when the circle action is effective.

Our Theorem 3.16 is a lower bound on the index of the Hofer length functional. It confirms a prediction of Yasha Savelyev. (But see Remark 1.2.)

We now give an overview of the paper.

In Section 2 we review standard material that we will use. The purpose of this review is to remind equivariant symplectic geometers of facts about time-dependent Hamiltonian flows and to remind symplectic topologists of facts about momentum maps. In Section 3 we state our results, give examples, and pose further questions.

In Sections 4–7 we develop the main tools for the later proofs of our main results. We warm up by describing, in Section 4, Polterovich’s shortening trick for non-effective circle actions (item 3 in the above sample of known results). In our later proofs we use variations of this Polterovich trick; we give such variations in Section 5: take a Hamiltonian circle action with momentum map $H$, and assume that, near the set where $H$ is maximal, we can write $H = K + F$ where $K$ and $F$ also generate circle actions. Also assume that the sets where $K$ and $F$ are maximal have non-empty intersection, and that there exists a symplectic isotopy that disjoins the first of these sets from the second. Finally, assume that the infimum of the sum of $K$ and $F$ is equal to the sum of their infima. We deduce that the circle action can be deformed to a shorter loop in $\text{Ham}(M,\omega)$, and we estimate the amount by which this loop is shorter. Our actual statement is slightly more technical; see Lemma 5.1. The qualitative part of this result can be strengthened if $H$ decomposes, near its maximum, into a sum of more than two momentum maps; see Lemma 5.10. Sections 6 and 7 contain variations on the simple fact that a disc of area $A_1$ can be disjoined from a disc of area $A_2$ inside any disc of area greater than $A_1 + A_2$. See Lemmas 6.1 and 7.1.

In Section 8 we prove our first new result, Theorem 3.11 when the momentum map attains its maximum at an isolated fixed point and one of the isotropy weights has absolute value $\geq 2$, we give a lower bound for the amount of possible shortening, in terms of the gap between the two largest singular values of the momentum map. The proof consists of the following steps. There exists an equivariant Darboux chart whose image is the entire set of points where the momentum map is larger than the second-largest critical value; this set can be identified with an ellipsoid in $\mathbb{C}^n$. We view this ellipsoid as a family of discs in $\mathbb{C}$ parametrized by points in a subset of $\mathbb{C}^{n-1}$, where the circle action on the $\mathbb{C}$ component is non-effective. From the results of Section 5 we get shortening by an amount that depends on the size of a family of discs that can be disjoined from another such family through a Hamiltonian isotopy supported in the ellipsoid. Such an estimate, in turn, is obtained from the results of Section 7. For effective circle actions, these are the first shortening results that are quantitative.

In Sections 9 and 10 we prove our second new result (more precisely, set of results). We now allow the momentum map to attain its maximum along a submanifold $B$ of positive
dimension. To describe a neighbourhood of $B$ in $M$, we use Sternberg’s minimal coupling procedure and Weinstein’s symplectic tubular neighbourhood theorem. Let $E$ denote the normal bundle of $B$ in $M$. In the simplest case, we assume that the $k$th power of the Euler class of $E$ is zero, where $k + 1$ is the smallest absolute value of an isotropy weight in $E$. In the general case, we make a similar assumption for a sub-bundle $E'$ of $E$. The idea of the proof is to express the circle action as a composition of $k + 1$ circle actions on a neighbourhood of $B$ and to then apply the results of Section 5. For the precise results, and for some nice special cases, see Theorems 3.7 and 3.9 and Corollary 3.12. These are the first shortening results when the maximum is attained along a manifold that is not necessarily displaceable and when the circle action is effective.

Finally, in Section 11, we prove our third new result, Theorem 3.16: we give a lower bound for the index of the Hofer length functional, assuming that the momentum map attains its maximum at an isolated fixed point. Composing with an equivariant Darboux chart, we may work with a linear circle action on $\mathbb{C}^n$ that rotates the coordinates of $\mathbb{C}^n$ with speeds $-k_1, \ldots, -k_n$, where the $k_j$s are positive integers. We write the action on the $j$th coordinate as a composition of $k_j$ terms, $\phi_{j} \circ \ldots \circ \phi_{i}$, where $\phi_{-t}$ generates the scalar multiplication circle action. We obtain a family of shortenings by applying Polterovich’s trick of Section 4 to the different factors of these compositions: in each coordinate, we conjugate all but one of the $\phi_{i}$s by a translation. Varying over the two dimensional family of translations of $\mathbb{C}$, independently in each one of the $\sum (k_j - 1)$ conjugations, we obtain a family of loops in $\text{Ham}(M, \omega)$ that depends on $2 \sum (k_j - 1)$ real parameters and such that the original circle action corresponds to the origin in the parameter space $\mathbb{R}^{2 \sum (k_j - 1)}$. An explicit computation then shows that the Hofer length of these loops, as a function on $\mathbb{R}^{2 \sum (k_j - 1)}$, achieves its maximum at the origin, and its Hessian is negative definite at the origin. This confirms one direction of a prediction for the index that was made by Yasha Savelyev.

Remark 1.2. While completing this paper, we learned from Savelyev that he now expects to be able to prove this conjecture.

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2. Preliminaries

The purpose of this section is to remind equivariant symplectic geometers of facts about time-dependent Hamiltonian flows and to remind symplectic topologists of facts about momentum maps.
Hamiltonian isotopies. Let \((M, \omega)\) be a symplectic manifold. A smooth path of symplectomorphisms is a map \(t \mapsto \psi_t\), from an interval \(I\) to the group of symplectomorphisms of \((M, \omega)\), such that the map \((t, m) \mapsto \psi_t(m)\) from \(I \times M\) to \(M\) is smooth. Such a path is a Hamiltonian isotopy if its velocity vector field, defined by \(\frac{d}{dt} \psi_t = X_t \circ \psi_t\), is generated by a smooth function \(H: I \times M \to \mathbb{R}\), by Hamilton’s equations \(dH_t = -\iota(X_t) \omega\), where \(H_t(\cdot) = H(t, \cdot)\). The function \(H = \{H_t\}_{t \in I}\) is called the Hamiltonian. If the Hamiltonian isotopy \(\{\psi_t\}\) is compactly supported, its Hofer length is

\[
\text{length}\{\psi_t\}_{t \in I} = \int_I \left( \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \right) dt.
\]

A symplectomorphism is Hamiltonian if it can be connected to the identity map by a Hamiltonian isotopy.

The positive and negative Hofer length functionals. Suppose that \(M\) is compact. The Hamiltonian \(\{H_t\}\) is normalized if

\[
\int_M H_t \omega^n = 0 \quad \text{for all } t.
\]

The positive Hofer length functional associated to a Hamiltonian isotopy \(\{\varphi_t\}_{a \leq t \leq b}\) in \(\text{Ham}(M, \omega)\) is the number

\[
\ell_+(\varphi) = \int_a^b \max_M H_t \, dt
\]

where \(\{H_t\}_{a \leq t \leq b}\) is the normalized Hamiltonian that generates the isotopy. The negative Hofer length functional is, similarly,

\[
\ell_-(\varphi) = \int_a^b \max_M -H_t \, dt.
\]

The Hofer length of \(\varphi\) is then \(\ell_+(\varphi) + \ell_-(\varphi)\).

Conjugation. If \(\{\psi_t\}\) is a Hamiltonian isotopy, generated by \(\{H_t\}\), and \(b\) is a symplectomorphism, then \(\{b \psi_t b^{-1}\}\) is a Hamiltonian isotopy, generated by \(\{H_t \circ b^{-1}\}\). If \(H_t\) is normalized then \(H_t \circ b^{-1}\) is also normalized.

Reparametrization. Let \(\{\psi_t\}\) be a Hamiltonian isotopy, generated by a function \(\{H_t\}\). Let \(t = t(\tau)\) be a smooth function. Then \(\overline{\psi}_\tau := \psi_{t(\tau)}\) is a Hamiltonian isotopy, generated by the product \(\overline{H}_\tau := H_{t(\tau)} \frac{dt}{d\tau}\).

Indeed, let \(\xi_t\) be the vector field that generates the isotopy, so that \(\frac{d}{dt} \psi_t = \xi_t \circ \psi_t\) and \(dH_t = -\iota(\xi_t) \omega\). Let \(\overline{\xi}_\tau = \xi_{t(\tau)} \frac{dt}{d\tau}\). Then \(d\overline{H}_\tau = dH_{t(\tau)} \frac{dt}{d\tau} = -\iota(\overline{\xi}_\tau) \omega\) and \(\overline{d\psi}_\tau = \frac{dt}{d\tau} \psi_{t=\tau} = \overline{\xi}_\tau \circ \overline{\psi}_\tau\).

Composition. If \(\{\psi_t^K\}_{t \in I}\) is a Hamiltonian isotopy generated by \(\{K_t\}_{t \in I}\), and \(\{\psi_t^F\}_{t \in I}\) is a Hamiltonian isotopy generated by \(\{F_t\}_{t \in I}\), then \(\{\psi_t^K \circ \psi_t^F\}_{t \in I}\) is a Hamiltonian isotopy, generated by \(\{K_t + F_t(\psi_t^K)^{-1}\}_{t \in I}\). If \(K_t\) and \(F_t\) are normalized, so is \(K_t + F_t(\psi_t^K)^{-1}\).
The Hamiltonian group. A symplectomorphism is Hamiltonian if it is the time-one map of a Hamiltonian isotopy. The set of Hamiltonian symplectomorphisms is a group; it is denoted $\text{Ham}(M,\omega)$.

Paths in the Hamiltonian group. If $\{\psi_t\}$ is a smooth path in the group of symplectomorphisms of $(M,\omega)$, and for each $t$ the symplectomorphism $\psi_t$ is in the subgroup $\text{Ham}(M,\omega)$, then $\{\psi_t\}$ is a Hamiltonian isotopy. This is a result due to Banyaga [2, p.190, prop. II.3.3]. In textbooks, the proof is often omitted or is intertwined with proofs of more difficult facts. We recall the proof.

Let $\xi_t$ be the velocity vector field, defined by $\frac{d}{dt}\psi_t = \xi_t \circ \psi_t$. The path $\{\psi_t\}$ is a Hamiltonian isotopy if and only if $\iota(\xi_t)\omega$ is exact for all $t$, if and only if the $H^1(M;\mathbb{R})$-valued integral

$$\text{Flux} \{\psi_t\}_{0 \leq t \leq t} := \int_0^t \iota(\xi_t)\omega \, d\tau$$

is zero for all $t$. (We then set $H_t(x) = -\int_0^x \iota(\xi_t)\omega$.)

The evaluation of the flux on a smooth loop $\gamma: S^1 \to M$ is equal to the symplectic area of the surface $\tilde{\gamma}: S^1 \times [0, t] \to M$ given by $\tilde{\gamma}(s,\tau) := \psi_\tau(\gamma(s))$; this follows from Stokes's formula. It follows that the flux of a closed loop of symplectomorphisms is a class in $H^1(M;\mathbb{R})$ whose evaluation on elements of $H_1(M;\mathbb{Z})$ takes values in the countable set $\{\omega, H_2(M;\mathbb{Z})\}$. Such classes make up a countable subgroup of $H^1(M;\mathbb{R})$.

Suppose that $\{\psi_t\}$ is a path of symplectomorphisms and that each $\psi_t$ is Hamiltonian. Then for each $t$ the path $\{\psi_t\}_{0 \leq \tau \leq t}$ can be completed to a closed loop of symplectomorphisms by a Hamiltonian isotopy. This completion does not effect the flux. So $t \mapsto \text{Flux}\{\psi_t\}_{0 \leq \tau \leq t}$ is a continuous map that takes values in a countable subgroup of $H^1(M;\mathbb{R})$. Such a map must be constant. Since it is zero for $t = 0$, it is zero for all $t$.

Hamiltonian circle actions. Let $(M,\omega)$ be a symplectic manifold with a circle action generated by a Hamiltonian $H: M \to \mathbb{R}$. The momentum map for the circle action is the Hamiltonian $H$.

The critical points of $H$ are the fixed points for the circle action. Let $p \in M$ be such a point. The linearized isotropy action on $T_p M$ is linearly equivariantly symplectomorphic to $\mathbb{C}^n$ with the standard symplectic form and with the circle acting as a subgroup of $(S^1)^n$, namely, $e^{2\pi i t} \cdot (z_1, \ldots, z_n) = (e^{2\pi i k_1 t} z_1, \ldots, e^{2\pi i k_n t} z_n)$. The integers $k_1, \ldots, k_n$, which measure the speeds at which the linearized $S^1$-action rotates its eigenspaces, are called the isotropy weights at $p$. They are unique up to permutation. This isomorphism $T_p M \cong \mathbb{C}^n$ carries the Hessian of $H$ to the quadratic form $\pi k_1 |z_1|^2 + \ldots + \pi k_n |z_n|^2$. So if $H$ attains its maximum at $p$ then all the isotropy weights at $p$ are non-positive.

The equivariant Darboux theorem asserts that a neighbourhood of $p$ in $M$ is equivariantly symplectomorphic to a neighbourhood of the origin in $T_p M$. In particular, if $p$ is an isolated fixed point then the isotropy weights at $p$ are all non-zero. Thus, at least one isotropy weight at $p$ has absolute value greater than one if and only if every neighbourhood of $p$ contains a point whose stabilizer is finite and non-trivial. Also, the set $B_{\text{max}}$ where $H$ is maximal is a symplectic submanifold, and the circle acts on the fibres of its normal bundle with negative weights.
The momentum map $H$ is a Bott-Morse function with even indices and coindices. Assuming that $M$ is compact and connected, this implies that the level sets of $H$ are connected and that preimages of intervals are connected. Thus, the maximum of $H$ is attained along a connected component of the fixed point set. If a fixed point $p$ is a local maximum for $H$ then it is also a global maximum. If a neighbourhood of $p$ in $M$ is equivariantly symplectomorphic to the ellipsoid \( \{ z \mid \sum k_j \pi_j |z_j|^2 < \epsilon \} \) with the momentum map $- \sum k_j \pi_j |z_j|^2$, then this neighbourhood contains all the points in $M$ where $H$ is $\epsilon$-close to its maximum.

See [1, 5].

3. NEW RESULTS AND FURTHER QUESTIONS

In this section we state our main results. We prove them in Sections 8, 10, and 11. We also give some examples, corollaries, and further questions.

In each of these results, we start with a Hamiltonian circle action, viewed as a loop in the group of Hamiltonian diffeomorphisms, and we deform it to a loop whose Hofer length is smaller. Moreover, throughout the deformation, the negative Hofer length remains constant, and the positive Hofer length becomes smaller without ever becoming larger than the initial one.

**Shortening on a manifold with an isolated maximum.**

**Theorem 3.1.** Let $(M, \omega)$ be a compact connected symplectic manifold with a Hamiltonian circle action. Suppose that the momentum map attains its maximum at an isolated fixed point and that at least one of the isotropy weights at that point has absolute value greater than one. Let $d$ be a positive number that is smaller than the gap between the two largest singular values of the momentum map. Then the circle action can be deformed to a loop in Ham($M, \omega$) whose Hofer length is smaller than that of the original circle action by $2d/9$.

**Remark 3.2.** In Theorem 3.1, we can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let $H$ denote the momentum map, let $p_{\text{max}}$ denote the point where $H$ attains its maximum, let $\alpha$ denote the gap between the two largest singular values of $H$, and consider the subset of $M$ given by

\[
\{ m \mid H(m) > H(p_{\text{max}}) - \alpha \}. \tag{3.3}
\]

We can choose the deformation such that the normalized deformed Hamiltonians remain greater than $H(p_{\text{max}}) - \alpha$ on the set (3.3) and coincide with $H$ outside this set.

**Remark 3.4.** In Theorem 3.1, we actually obtain a better estimate than $2d/9$. If one of the isotropy weights is even then we can shorten by $d/4$. Otherwise, let $-k_1$ be the weight whose absolute value is largest; then we can shorten by $\frac{1}{4}(1 - \frac{1}{k_1^2})d$. When $k_1 = 3$, this becomes $\frac{2}{9}d$.

This is the first quantitative result for shortening of loops in Ham($M, \omega$) that are effective circle actions. We prove it in Section 8.

The estimate of Theorem 3.1 is far from sharp, as we see in the following example.
Example 3.5. Let $a, b$ be positive integers that are relatively prime. Let $e^{2\pi it} \in S^1$ act on $\mathbb{CP}^2$ by $[u, z, w] \mapsto [u, e^{-2\pi it}z, e^{2\pi it}w]$, with the Fubini-Study form normalized such that the momentum map is $(-a|z|^2 + b|w|^2)/(|u|^2 + |z|^2 + |w|^2) + \text{constant.}$ The Hofer length of this action is $a + b$. The isotropy weights at the maximum are $-b, -(a + b)$; the isotropy weights at the minimum are $a, a + b$. Applying the shortening result of Theorem 3.1 and the analogous result with maximum replaced by minimum, for any $\epsilon > 0$ we can deform the circle action to a loop of Hofer length $< 7/9(a + b) + \epsilon$. This upper bound is greater than one. But the action can be deformed to a loop of Hofer length one or zero, so the bound is not optimal. (To see that the action can be deformed to a loop of Hofer length one or zero, notice that the action extends to an action of $\text{PU}(3)$, that $\pi_1(\text{PU}(3)) \cong \mathbb{Z}/3\mathbb{Z}$, and that the non-trivial elements of $\pi_1(\text{PU}(3))$ are represented by the actions $[u, z, w] \mapsto [u, e^{-2\pi it}z, w]$ and $[u, z, w] \mapsto [u, z, e^{2\pi it}w]$, which have Hofer length equal to one.)

Remark 3.6. The above example is inspired by Proposition 1.3 of the paper [16] of McDuff and Tolman: if $G$ is be a compact Lie group with trivial centre and $M$ is a coadjoint orbit of $G$, then every non-trivial element of $\pi_1(G)$ is represented by a sub-circle of $G$ that acts on $M$ semi-freely.

Shortening on a manifold with an arbitrary maximum.

Recall that, on a compact connected symplectic manifold with a Hamiltonian circle action, the set where the momentum map is maximal is a symplectic submanifold.

In the following theorem, a subset $B$ of a symplectic manifold $M$ is \textit{symplectically k-displaceable} in a neighbourhood $V$ if there exist $k$ symplectomorphisms $b_1, \ldots, b_k$ of $M$, each connected to the identity through a path of symplectomorphisms supported in $V$, such that $B \cap b_1(B) \cap \ldots \cap b_k(B) = \emptyset$.

In the symplectic literature, “$B$ is displaceable” often means that there exists a Hamiltonian symplectomorphism $g$ such that $B \cap g(B) = \emptyset$. Thus, a set can be symplectically $k$-displaceable in every neighbourhood without being displaceable (even when $k = 1$).

Theorem 3.7. Let $(M, \omega)$ be a compact connected symplectic manifold with a Hamiltonian circle action. Let $B_{\max}$ be the set where the momentum map is maximal. Let $-k_1, \ldots, -k_s$ denote the distinct weights for the circle action on the normal bundle of $B_{\max}$, and let $k := \min\{k_1, \ldots, k_s\} - 1$. Suppose that $B_{\max}$ is symplectically $k$-displaceable in every neighbourhood. Then the circle action can be deformed through loops in $\text{Ham}(M, \omega)$ into a loop of smaller Hofer length.

Remark 3.8. In Theorem 3.7, we can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let $H$ denote the momentum map. For every positive number $\alpha$, we can choose the deformation such that the normalized deformed Hamiltonians remain greater than $H(B_{\max}) - \alpha$ on the set $\{m \in M \mid H(m) > H(B_{\max}) - \alpha\}$ and coincide with $H$ outside this set.

In practice, the easiest way to show that a symplectic submanifold is symplectically $k$-displaceable in every neighborhood is to show that the $k$th power of its normal bundle is
zero. \((k\)-displaceability then follows from the local normal form [9,5] and Lemma 9.9\) More generally, we have the following theorem.

**Theorem 3.9.** Let \((M, \omega)\) be a compact connected symplectic manifold with a Hamiltonian circle action. Let \(B_{\text{max}}\) be the set where the momentum map is maximal. Let \(E' \to B_{\text{max}}\) be an \(S^1\)-invariant subbundle of the normal bundle to \(B_{\text{max}}\) in \(M\), and let \(-k_1', \ldots, -k_s'\) be the distinct weights for the circle action on \(E'\). Let

\[
k' = \min\{k_1', \ldots, k_s'\} - 1.
\]

Let \(e(E') \in H^*(B_{\text{max}})\) be the Euler class of \(E'\). Suppose that

\[
e E')^{k'} = 0
\]

in \(H^*(B_{\text{max}})\). Then the circle action can be deformed through loops in \(\text{Ham}(M, \omega)\) into a loop of smaller Hofer length.

**Remark 3.11.** In Theorem 3.9, we can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let \(H\) denote the momentum map. For every positive number \(\alpha\), we can choose the deformation such that the normalized deformed Hamiltonians remain greater than \(H(B_{\text{max}}) - \alpha\) on the set \(\{m \in M \mid H(m) > H(B_{\text{max}}) - \alpha\}\) and coincide with \(H\) outside this set.

Theorem 3.9 has the following corollaries.

**Corollary 3.12.** Let \((M, \omega)\) be a compact symplectic manifold with a Hamiltonian circle action. Let \(B_{\text{max}}\) be the subset of \(M\) where the momentum map attains its maximum. Let \(E\) denote the normal bundle of \(B_{\text{max}}\) in \(M\) and \(-k_1, \ldots, -k_s\) the distinct weights for the circle action on \(E\). Suppose that one of the following conditions holds.

1. \(e(E)^{k'} = 0\), where \(e(E)\) is the Euler class of \(E\) and \(k = \min\{k_1, \ldots, k_s\} - 1\).
2. The normal bundle of \(B_{\text{max}}\) in \(M\) has a trivial sub-bundle on which the circle acts with a weight of absolute value \(\geq 2\).

   (In particular, this holds under the assumptions of Theorem 3.1.)
3. We have

\[
k_j > 1 + \frac{\dim B_{\text{max}}}{\text{codim } B_{\text{max}}}
\]

for all \(j \in \{1, \ldots, s\}\).
4. After a possible relabeling, assume that \(k_1 < \ldots < k_s\). Let \(E_j\) denote the sub-bundle where the circle acts with weight \(-k_j\). There exists \(j \in \{1, \ldots, s\}\) such that

\[
(k_j - 1) \text{rank}(E_j \oplus \ldots \oplus E_s) > \dim B_{\text{max}}.
\]

   (For \(j = 1\), this condition amounts to case [3].)

Then the circle action can be deformed through loops in \(\text{Ham}(M, \omega)\) to a loop of smaller Hofer length.
Moreover, we can choose a deformation with the properties described in Remark 3.11.

Proof of Corollary 3.12. In each of the four cases, (10.2) holds for an appropriate choice of sub-bundle $E'$:

1. Set $E' = E$. Then $k' = k$, and the assumption $e(E)^k = 0$ gives (10.2).
2. Let $E'$ be a trivial sub-bundle of $E$ on which $S^1$ acts with a weight $-k_1'$ of absolute value $k_1' \geq 2$. Then $e(E') = 0$ and $k' = k_1' - 1 \geq 1$, so (10.2) holds.
3. Let $E' = E$. Then $\text{rank } e(E') = \text{codim } B_{\text{max}}$. Our assumption on the weights give $k' > \frac{\text{dim } B_{\text{max}}}{\text{codim } B_{\text{max}}}$. So $k' \text{rank } e(E') > \text{dim } B_{\text{max}}$, and (10.2) holds for dimension reasons.
4. Let $E' = E_j \oplus \ldots \oplus E_s$. Then $k' = k_j - 1$. The assumption on the weights gives $k' \text{rank } e(E') > \text{dim } B_{\text{max}}$, and again (10.2) holds for dimension reasons.

In each of these cases, the shortening result follows from Theorem 3.9. □

Remark 3.13. The condition $e(E)^k = 0$ in Part (1) of Corollary 3.12 means that all the weights have absolute value $\geq 2$ and the normal bundle $E$ is "sufficiently untwisted". The condition $e(E')^{k'} = 0$ for a sub-bundle $E'$ in Theorem 3.9 implies that $e(E)^k = 0$, but it does not necessarily imply that $e(E)^k = 0$, because $k'$ can be larger than $k$.

Remark 3.14. At final stages of preparing this article we became aware of an interesting relationship with the work of McDuﬀ-Tolman. In the terminology of Section 1.1 of McDuﬀ-Tolman’s paper [15], the component $B_{\text{max}}$ is homologically visible if and only if the Euler class of the bundle $(E_1 \otimes \mathbb{C}^{k_1-1}) \oplus \ldots \oplus (E_s \otimes \mathbb{C}^{k_s-1})$ is non-zero. By Theorem 1.2 of [15], if $B_{\text{max}}$ is homologically visible then the circle action is non-contractible in Ham$(M, \omega)$. (Proposition 3.3 of [15] might also imply that if $B_{\text{max}}$ is homologically visible then the circle action is minimal among homotopic loops, but we did not check the details.) The condition (10.2) of our Theorem 3.9 implies that $B_{\text{max}}$ is not homologically visible. We do not know whether, in Theorem 3.9 condition (10.2) can be replaced by the condition that $B_{\text{max}}$ not be homologically visible.

We prove Theorems 3.7 and 3.9 in Section 10. These are the first shortening results that apply to situations when the maximum is not necessarily displaceable.

Example 3.15. Let $e^{2\pi it} \in S^1$ act on $\mathbb{CP}^5$ by

$$[z_0, z_1, z_2, z_3, z_4] \mapsto [z_0, z_1, e^{-2\pi it} z_2, e^{-2\pi it - 2} z_3, e^{-2\pi it - 3} z_4],$$

with momentum map $(-|z_2|^2 - 2|z_3|^2 - 3|z_4|^2)/(|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) + \text{constant}$. The momentum map attains its maximum along the submanifold $B_{\text{max}} = \{[0, z_1, 0, 0, 0]\}$, which is isomorphic to $\mathbb{CP}^1$. The isotropy weights along $B_{\text{max}}$ are $-1, -2, -3$. The momentum map attains its minimum at the isolated fixed point $[0, 0, 0, 0, z_4]$. The isotropy weights at the minimum are 1, 2, 3, 3. In the notation of item (4) of Corollary 3.12 applied at $B_{\text{max}}$, we have the following values.
We can apply item (4) of Corollary 3.12 to either $E_2 \oplus E_3$ or $E_3$. Also applying Theorem 3.1 at the minimum, we conclude that the circle action can be deformed to a loop whose positive Hofer length and negative Hofer length are both smaller. Nevertheless, the circle action is not contractible in Ham($M, \omega$): it is homotopic (in PU(5)) to the circle action $[z_0, z_1, z_2, z_3, z_4] \mapsto [e^{-2\pi it}z_0, z_1, z_2, z_3, z_4, z_5]$, which is semi-free and thus non-contractible (by item (1) on p. 2).

The index of the Hofer positive length functional.

**Theorem 3.16.** Let $(M, \omega)$ be a $2n$ dimensional compact connected manifold with a Hamiltonian circle action. Suppose that the momentum map attains its maximum at an isolated fixed point. Let $-k_1, \ldots, -k_n$ be the isotropy weights at the maximum, with possible repetitions. Then there exists a neighbourhood $D$ of the origin in $\mathbb{R}P^2(2(k_i - 1))$, and, for each $\lambda \in D$, a loop $\{\psi_t^{(\lambda)}\}_{0 \leq t \leq 1}$ in Ham($M, \omega$), such that the following properties hold. For $\lambda = 0$, the loop $\{\psi_t^{(0)}\}_{0 \leq t \leq 1}$ is the given circle action. The function $\lambda \mapsto \text{length}(\{\psi_t^{(\lambda)}\})$ is smooth, $\lambda = 0$ is a critical point of this function, and the Hessian of this function at $\lambda = 0$ is negative definite.

**Remark 3.17.** In fact, we can choose the deformation such that the negative Hofer length remains constant. Specifically, let $H$ denote the momentum map, and let $p_{\max}$ denote the point where $H$ attains its maximum. For every positive number $\alpha$, we can choose the deformation such that the normalized deformed Hamiltonian remains greater than $H(p_{\max}) - \alpha$ on the set $\{m \in M \mid H(m) > H(p_{\max}) - \alpha\}$ and coincides with $H$ outside this set.

Savelyev [19, 20], in the setup of Theorem 3.16, predicted that the index of the positive Hofer length functional on the space of loops in Ham($M, \omega$) at the given Hamiltonian circle action is equal to $\sum 2(k_i - 1)$. Theorem 3.16 is a sense in which the index of the positive Hofer length functional is at least $\sum 2(k_i - 1)$. So Theorem 3.16 confirms the $\geq$ direction of Savelyev’s prediction. (But see Remark 1.2.)

**Further questions.**

1. Are there examples where the estimate of Theorem 3.1 is sharp?
2. Can the estimate of Theorem 3.1 be improved in the presence of high isotropy weights?
   
   As noted by Polterovich, this question is also interesting for non-effective circle actions obtained by iterating an effective circle action, as it may provide information on the asymptotic norm of the homotopy class of the effective circle action (see [18]).
3. Under the assumptions of Theorem 3.7 or of Theorem 3.9, can one obtain a quantitative estimate?
4. In Theorem 3.9, can the condition (10.2) be replaced by the condition that $B_{\max}$ not be homologically visible? (Cf. Remark 3.14.)
(5) Can Theorem 3.16 be extended to cases in which the maximum is attained along a manifold of positive dimension?

(6) Savelyev’s prediction for the index of the positive Hofer length functional is an even number. Polterovich expects the index to be even in additional cases. On a compact connected manifold, a Morse function with even indices and coindices has nice topological properties: every local minimum is a global minimum, and the level sets are connected. Palais-Smale-Morse theory does not apply to the space of loops in $\text{Ham}(M, \omega)$, but we may seek analogies.

If a loop in the Hamiltonian group is a local minimum for the positive Hofer length functional, is it also a global minimum in its homotopy class?

Suppose that a homotopy class in $\pi_1(\text{Ham}(M, \omega))$ contains two circle actions that are semi-free near the set where the momentum map is maximal. (Cf. item (1) on p. 2.) Can we deform one to the other through loops of constant positive Hofer length?

4. Shortening a non-effective circle action

To warm up, we describe a procedure, which we learned from Leonid Polterovich, for shortening a non-effective Hamiltonian circle action by displacing the set where its momentum map is maximal.

Let $H: M \to \mathbb{R}$ be a time independent Hamiltonian. Suppose that $\frac{1}{2}H$ generates a circle action, $\{\phi_t\}_{0 \leq t \leq 1}$, so that $H$ generates the non-effective circle action

\[ \psi_t^H = \phi_{2t} = \phi_t \circ \phi_t. \]

Let $b: M \to M$ be a symplectomorphism. Then

\[ \psi_t^{\overline{H}} := \{\phi_t b \phi_t b^{-1}\}_{0 \leq t \leq 1} \]

is a loop in $\text{Ham}(M)$, based at the identity, and generated by the Hamiltonian

\[ \overline{H}_t = \frac{1}{2}H + \frac{1}{2}Hb^{-1}\phi_t^{-1}. \]

If $H$ is normalized, so is $\overline{H}_t$.

If $b$ can be connected to the identity through a path of symplectomorphisms, then the loop $\{\psi_t^{\overline{H}}\}$ is a deformation of the loop $\{\psi_t^H\}$. Since $\overline{H}_t(x) \leq \max H$ and $\overline{H}_t(x) \geq \min H$ for all $x \in M$ and all $t$,

\[ \text{length}\{\psi_t^{\overline{H}}\} \leq \text{length}\{\psi_t^H\}, \]

and this is a strict inequality if $\max \overline{H}_t < \max H$ for some $t$.

Denote by $B_{\text{max}}^H$ the subset of $M$ where $H$ takes its maximal value. If $B_{\text{max}}^H \cap b(B_{\text{max}}^H) = 0$ then, for $t = 0$, the maximum of $\overline{H}_0 = \frac{1}{2}H + \frac{1}{2}Hb^{-1}$ is strictly smaller than that of $H$.

We have shown that, if there exists a symplectic isotopy that displaces the set where the momentum map is maximal, then the non-effective circle action can be deformed to a loop of shorter Hofer length. A similar argument gives a quantitative result: if there
exists a symplectic isotopy that displaces the set where the momentum map is \( \epsilon \)-close to its maximum, then the non-effective circle action can be deformed to a loop whose Hofer length is shorter by \( \epsilon \) from that of the original circle action.

5. Shortening a combined action

In this section we give variations of Polterovich’s trick that apply to effective circle actions. We begin with a lemma that we will use to prove Theorems 3.1. The lemma has two parts – a qualitative part and a quantitative part.

We say that a symplectomorphism \( b \) disjoins a set \( A \) from a set \( B \) if \( b(A) \cap B = \emptyset \).

**Lemma 5.1.** Let \((M,\omega)\) be a compact symplectic manifold. Let \( H: M \to \mathbb{R} \) be the momentum map for a circle action \( \{ \psi^H_t \}_{0 \leq t \leq 1} \). Let

\[
i: (U,\omega_0) \to (M,\omega)
\]

be an open symplectic embedding. Suppose that

\[
i^*H = K + F \quad \text{on} \quad U,
\]

where \( K \) and \( F \) generate commuting circle actions \( \{ \psi^K_t \}_{0 \leq t \leq 1} \) and \( \{ \psi^F_t \}_{0 \leq t \leq 1} \) on \( U \).

Let \( \tilde{N}: U \to \mathbb{R} \) be a function that Poisson commutes with \( K \) and \( F \) and satisfies

\[
\inf_{\tilde{N}^{-1}(\alpha)} (K + F) = \inf_{\tilde{N}^{-1}(\alpha)} K + \inf_{\tilde{N}^{-1}(\alpha)} F \quad \text{for every} \quad \alpha \in \mathbb{R}.
\]

Also suppose that

\[
i(U) \text{ does not meet the set where } H \text{ is minimal}.
\]

(1) Let

\[
B^H_{\text{max}} = \{ m \in M \mid H \text{ is maximal at } m \},
\]

\[
B^K_{\text{max}} = \{ u \in U \mid K \text{ is maximal at } u \},
\]

and

\[
B^F_{\text{max}} = \{ u \in U \mid F \text{ is maximal at } u \}.
\]

Suppose that

\[
B^H_{\text{max}} \subset i(U) \quad \text{and} \quad B^K_{\text{max}} \cap B^F_{\text{max}} \neq \emptyset.
\]

Suppose that there exists a symplectomorphism \( b: U \to U \) that disjoins \( B^F_{\text{max}} \) from \( B^K_{\text{max}} \) and such that \( b \) can be connected to the identity through a path of symplectomorphisms that are compactly supported in \( U \) and that preserve the function \( \tilde{N} \).

Then the circle action \( \{ \Psi^H_t \} \) can be deformed, through loops in Ham\((M,\omega)\), to a loop \( \{ \Psi^H_t \} \) whose Hofer length is smaller.

(2) Let \( \epsilon > 0 \). Let

\[
\mathcal{N}^H_\epsilon = \{ m \in M \mid H(m) > \max H - \epsilon \},
\]

\[
\mathcal{N}^K_\epsilon = \{ u \in U \mid K(u) > \max K - \epsilon \},
\]

and

\[
\mathcal{N}^F_\epsilon = \{ u \in U \mid F(u) > \max F - \epsilon \}.
\]
Suppose that
\[(5.6) \quad \mathcal{N}_t^H \subset i(U) \quad \text{and} \quad B^K_{\max} \cap B^F_{\max} \neq \emptyset.\]

Suppose that there exists a symplectomorphism \(b: U \to U\) that disjoins \(\mathcal{N}_t^F\) from \(\mathcal{N}_t^K\) and such that \(b\) can be connected to the identity through a path of symplectomorphisms that are compactly supported in \(U\) and that preserve the function \(N\).

Then the circle action \(\{\Psi_t^H\}\) can be deformed, through loops in \(\text{Ham}(M, \omega)\), to a loop \(\overline{\psi_t^H}\) whose Hofer length is smaller than that of \(\{\Psi_t^H\}\) by at least \(\epsilon\).

In each of these two cases, the deformation can be chosen such that the deformed Hamiltonians coincide with \(H\) outside the set \(i(U)\) and attain the same infimum as \(H\) on the set \(i(U)\), so the positive Hofer length never exceeds the initial one and the negative Hofer length remains the same throughout the deformation.

**Proof.** Because \(K\) and \(F\) generate commuting circle actions, \(K \circ \psi_t^F = K\) and \(F \circ \psi_t^K = F\) for all \(t\).

Because \(B^H_{\max} \subset i(U)\), \(i^* H = K + F\), and \(B^K_{\max} \cap B^F_{\max} \neq \emptyset\),
\[(5.7) \quad B^H_{\max} = i \left( B^K_{\max} \cap B^F_{\max} \right).\]

Let \(b: U \to U\) be a compactly supported symplectomorphism of \(U\). The loop
\[
\psi_t^F = \begin{cases} 
  i \circ (\psi_t^K \circ b \psi_t^F b^{-1}) \circ i^{-1} & \text{on } U \\
  \psi_t^H & \text{outside } U
\end{cases}
\]
is generated by the Hamiltonian \(\overline{H}_t: M \to \mathbb{R}\) that is given by
\[(5.8) \quad \overline{H}_t = \begin{cases} 
  (K + Fb^{-1}(\psi_t^K)^{-1}) \circ i^{-1} & \text{on } U \\
  H & \text{outside } U.
\end{cases}
\]

If \(H\) is normalized, so is \(\overline{H}_t\).

If \(b\) can be connected to the identity through a compactly supported symplectic isotopy, then the loop \(\overline{\psi_t^H}\) is a deformation of the loop \(\{\psi_t^H\}\) in \(\text{Ham}(M, \omega)\).

By \((5.7)\) and \((5.8)\), we have \(\max \overline{H}_t \leq \max H\) for all \(t\). So the positive Hofer length of \(\{\psi_t^H\}\) does not exceed that of the circle action \(\{\psi_t^H\}\).

If \(b\) preserves \(\hat{N}\), then, by \((5.3)\), \((5.4)\), \((5.8)\), and because \(\psi_t^K\) preserves \(\hat{N}\), we have \(\min \overline{\psi_t^H} = \min H\) for all \(t\). So the negative Hofer length of \(\{\psi_t^H\}\) is equal to that of the circle action \(\{\psi_t^H\}\).

Suppose that \(b\) disjoins \(B^F_{\max}\) from \(B^K_{\max}\). Then, for \(t = 0\), we have \(\max \overline{H}_0 = \max(K + Fb^{-1}) < \max K + \max F = \max H\). This strict inequality is because \(B^K_{\max} \cap B^{Fb^{-1}}_{\max}\), being equal to \(B^K_{\max} \cap b(B^F_{\max})\), is empty. (In fact, \(\max \overline{H}_t < \max H\) for all \(t\); this follows from \(K + Fb^{-1}(\psi_t^K)^{-1} = (K + Fb^{-1})(\psi_t^K)^{-1}\) and \(B^K_{\max} \cap b(B^F_{\max}) = \emptyset\).) Because the inequality \(\max \overline{H}_t \leq \max H\) holds for all \(t\) and is strict for \(t = 0\), the positive Hofer length of \(\{\psi_t^H\}\) is strictly smaller than that of the circle action \(\{\psi_t^H\}\).
This completes the proof of the first, qualitative, part of the lemma.

To prove the second, quantitative, part of the lemma, suppose now that \( b \) disjoins \( N^F_\epsilon \) from \( N^K_\epsilon \), let \( m \) be a point in \( M \), and we will show that \( \overline{H}_t(m) \leq \max H - \epsilon \) for all \( t \).

Suppose that \( m \) is not in \( i(U) \), and, thus, also not in \( N^K_\epsilon \). Then \( \overline{H}_t(m) = H(m) \) and \( H(m) \leq \max H - \epsilon \), so
\[
\overline{H}_t(m) \leq \max H - \epsilon.
\]
Now suppose that \( m \) is in \( i(U) \), say, \( m = i(u) \) for \( u \in U \). By (5.8),
\[
\overline{H}_t(m) = K(u) + F(u')
\]
where \( m' = b^{-1}(\psi^K_t)^{-1}(u) \). If \( \overline{H}_t(m) \) is \( \epsilon \)-close to \( \max H \), then \( u \in N^K_\epsilon \) and \( u' \in N^F_\epsilon \), that is,
\[
(5.9) \quad u \in N^K_\epsilon \cap (b(N^F_\epsilon)).
\]
Because \( K = K \circ \psi^K_t \), The set \( N^K_\epsilon \) is preserved under \( \psi^K_t \), so we can rewrite (5.9) as \( u \in \Psi^K_t(N^K_\epsilon \cap b(N^F_\epsilon)) \). But, by assumption, \( N^K_\epsilon \cap b(N^F_\epsilon) \) is empty. So \( \overline{H}_t(m) \) cannot be \( \epsilon \)-close to \( \max H \), and again we conclude that \( \overline{H}_t(m) \leq \max H - \epsilon \). So \( \overline{H}_t(m) \leq \max H - \epsilon \) for all \( m \in M \) and for all \( t \). Since \( \overline{H}_t \) is normalized, we conclude that the positive Hofer lengths satisfy \( \ell_+(\{\psi^H_t\}) \leq \ell_+(\{\psi^H_t\}) - \epsilon \), as required. \( \square \)

The second, quantitative, part of Lemma 5.1 does not automatically extend to more than two summands. The first, qualitative, part does extend to more than two summands. We now give such an extension. We will use it to prove Theorems 3.7 and 3.9.

**Lemma 5.10.** Let \((M, \omega)\) be a compact symplectic manifold. Let \( H : M \to \mathbb{R} \) be the momentum map for a circle action \( \{\psi^H_t\}_{0 \leq t \leq 1} \). Let
\[
i : (U, \omega_0) \to (M, \omega)
\]
be an open symplectic embedding. Suppose that
\[
i^* H = H_{(0)} + H_{(1)} + \ldots + H_{(k)} \quad \text{on } U
\]
where \( H_{(0)}, H_{(1)}, \ldots, H_{(k)} \) generate commuting circle actions on \( U \).

Let
\[
B^H_{\max} = \{ m \in M \mid H \text{ is maximal at } m \}
\]
and, for each \( 0 \leq j \leq k \),
\[
B^{(j)}_{\max} = \{ u \in U \mid H_{(j)} \text{ is maximal at } u \}.
\]
Suppose that
\[
(5.12) \quad B^H_{\max} \subset i(U) \quad \text{and} \quad B^{(0)}_{\max} \cap B^{(1)}_{\max} \cap \ldots \cap B^{(k)}_{\max} \neq \emptyset.
\]
Let \( \hat{N} : U \to \mathbb{R} \) be a function that Poisson commutes with \( H_{(0)}, H_{(1)}, \ldots, H_{(k)} \), and such that
\[
(5.13) \quad \inf_{N^{-1}(\alpha)} (H_{(0)} + H_{(1)} + \ldots + H_{(k)}) = \sum_{j=0}^{k} \inf_{N^{-1}(\alpha)} H_{(j)} \quad \text{for every } \alpha \in \mathbb{R}.
\]
Also suppose that
\[(5.14)\quad i(U) \text{ does not meet the set where } H \text{ is minimal.}\]

Suppose that there exist symplectomorphisms \(b_1, \ldots, b_k\) of \(U\) such that
\[(5.15)\quad B_{max}^{(0)} \cap b_1(B_{max}^{(1)}) \cap \ldots \cap b_k(B_{max}^{(k)}) = \emptyset\]
and such that each \(b_j\) is connected to the identity through a path of symplectomorphisms that is compactly supported in \(U\) and that preserves \(\hat{N}\).

Then the circle action \(\{\psi_t^H\}\) can be deformed, through loops in \(\text{Ham}(M, \omega)\), to a loop \(\{\overline{\psi}_t\}\) whose Hofer length is smaller.

The deformation can be chosen such that the deformed Hamiltonians coincide with \(H\) outside the set \(i(U)\) and attain the same infimum as \(H\) on the set \(i(U)\), so the positive Hofer length never exceeds the initial one and the negative Hofer length remains the same.

Proof. Denote by
\[
\{\psi_t^{(0)}\}_{0 \leq t \leq 1}, \ldots, \{\psi_t^{(k)}\}_{0 \leq t \leq 1}
\]
the circle actions on \(U\) that are generated by \(H_{(0)}, \ldots, H_{(k)}\). Let \(b_1, \ldots, b_k\) be arbitrary compactly supported symplectomorphisms of \(U\). For each \(j \in \{0, \ldots, k\}\), let
\[(5.16)\quad \overline{\psi}_t^{(j)} := \psi_t^{(0)} \circ (b_1 \psi_t^{(1)} b_1^{-1}) \circ \cdots \circ (b_j \psi_t^{(j)} b_j^{-1}).\]

This is a loop in \(\text{Ham}(U, \omega_0)\). By induction on \(j\), one shows the following three facts:

1. \(\{\overline{\psi}_t^{(j)}\}\) is generated by the Hamiltonian
\[(5.17)\quad \overline{H}_t^{(j)} := H_{(0)} + H_{(1)} b_1^{-1} \circ (\overline{\psi}_t^{(0)})^{-1} + \ldots + H_{(j)} b_j^{-1} \circ (\overline{\psi}_t^{(j-1)})^{-1}.\]

2. Outside the support of \(b_1, \ldots, b_j\),
\[
\overline{\psi}_t^{(j)} = \psi_t^{(j)} \quad \text{and} \quad \overline{H}_t^{(j)} = H_{(0)} + H_{(1)} + \ldots + H_{(j)}.
\]

3. For all \(t\),
\[(5.18)\quad \int_U \left( \overline{H}_t^{(j)} - (H_{(0)} + H_{(1)} + \ldots + H_{(j)}) \right) \omega_0^n = 0.\]

We give more details for the third fact. By (5.17), the left hand side of (5.18) is the sum over \(m = 1, \ldots, j\) of
\[
\int_U \left( H_{(m)} b_m^{-1} (\overline{\psi}_t^{(m-1)})^{-1} - H_{(m)} \right) \omega_0^n.
\]
This \(m\)th summand is equal to
\[
\int_U \left( H_{(m)} \circ b_m^{-1} \circ (\overline{\psi}_t^{(m-1)})^{-1} - H_{(m)} \circ (\overline{\psi}_t^{(m-1)})^{-1} \right) \omega_0^n
\]
because \(\overline{\psi}_t^{(m-1)}\) preserves \(H_{(m)}\). This, in turn, is equal to
\[
\int_U \left( H_{(m)} \circ b_m^{-1} \circ (\overline{\psi}_t^{(m-1)})^{-1} \circ \overline{\psi}_t^{(m-1)} - H_{(m)} \right) \omega_0^n
\]
because \( \psi_t^{(m-1)} \) preserves the volume form \( \omega_0^n \). This last expression is equal to zero, because \( b_m^{-1} \circ (\psi_t^{(m-1)})^{-1} \circ \psi_t^{(m-1)} \) is a compactly supported diffeomorphism that preserves the volume form \( \omega_0^n \). Thus, (5.18) is true.

By the above items (1) and (2), we can define a loop in \( \text{Ham}(M,\omega) \) by

\[
\psi_t := \begin{cases} 
  i \circ \psi_t^{(k)} \circ i^{-1} & \text{on } i(U) \\
  \psi_t & \text{on } M \setminus i(U),
\end{cases}
\]

and it is generated by the Hamiltonian

\[
(5.19) \quad \overline{H}_t := \begin{cases} 
  \left( H(0) + H(1)b_t^{-1}(\psi_t^{(0)})^{-1} + \ldots + H(k)b_t^{-1}(\psi_t^{(k-1)})^{-1} \right) \circ i^{-1} & \text{on } i(U) \\
  H & \text{on } M \setminus i(U).
\end{cases}
\]

By the above item (3), if \( H \) is normalized, so is \( \overline{H}_t \).

Suppose that \( b_j \) preserve the function \( \hat{N} \). Because \( \psi_t^{(j)} \) preserve \( \hat{N} \), and by (5.16), \( \overline{H}_t \) also preserve \( \hat{N} \). From (5.11), (5.13), (5.14), and (5.19), we deduce that \( \min \overline{H}_t = \min H \).

So the negative Hofer length of \( \{ \psi_t \} \) is equal to that of the circle action \( \{ \psi^H_t \} \).

From (5.11), (5.12), and (5.19), we see that \( \max_M \overline{H}_t \leq \max_M H \), with strict inequality when \( t = 0 \) if \( b_1, \ldots, b_k \) satisfy (5.15). This implies that the positive Hofer length of \( \{ \psi_t \} \) does not exceed that of \( \{ \psi^H_t \} \), and, if \( b_1, \ldots, b_k \) satisfy (5.15), then the positive Hofer length of \( \{ \psi_t \} \) is strictly smaller than that of \( \{ \psi^H_t \} \).

The lemma is obtained by applying this argument as \( b_1, \ldots, b_k \) vary over smooth paths of symplectomorphisms that connect them to the identity, are compactly supported in \( U \), and preserve the function \( \hat{N} \), and where the initial \( b_1, \ldots, b_k \) satisfy (5.15). \( \square \)

6. Disjoining discs in \( \mathbb{C} \)

For every neighbourhood of the origin in \( \mathbb{C} \), there exists a Hamiltonian isotopy, supported in this neighbourhood, which disjoins the origin from itself. Such an isotopy can be generated by a function \( F: \mathbb{C} \to \mathbb{R} \) whose Hamiltonian vector field is zero outside the given neighbourhood, is equal to \( -\frac{\partial}{\partial x} \) on a neighbourhood of the origin, and is equal to a non-negative multiple of \( -\frac{\partial}{\partial x} \) everywhere on the \( x \)-axis. For example, if the given neighbourhood contains the set \( \{ z \mid |\pi|z|^2 \leq 3e \} \), we may take the function \( F(z) = y \rho(|\pi|z|^2/\epsilon - 1) \) where \( y \) is the imaginary part of \( z \) and where \( \rho(s) \) is equal to one for \( s \leq 0 \), to zero for \( s \geq 1 \), and to

\[
(\int_0^1 e^{-\frac{s}{1-t}} d\tau)/(\int_0^1 e^{-\frac{s-1}{1-t}} d\tau)
\]

for \( 0 \leq s \leq 1 \).

For our quantitative results we will need to keep track of the sizes of neighbourhood of the origin that get disjoined. We do so in the following lemma.

Lemma 6.1. There exists a smooth family of functions

\[
F_t^{A,\epsilon}: \mathbb{C} \to \mathbb{R} , \quad 0 \leq t \leq 1,
\]
parametrized by $A \geq 0$ and $0 < \epsilon < 1$, such that $F^{A,\epsilon}(z) = 0$ for $z$ outside the disc \{ $z \mid \pi|z|^2 < 1 + A + \epsilon$ \} and such that the Hamiltonian flow $b^{A,\epsilon}_t : \mathbb{C} \to \mathbb{C}$ of $F^{A,\epsilon}_t$, at time $t = 1$, carries the disc \{ $z \mid \pi|z|^2 \leq 1$ \} into the annulus \{ $z \mid A < \pi|z|^2 < 1 + A + \epsilon$ \}.

Remark 6.2. Smooth family means that the function $(A, \epsilon, t, z) \mapsto F^{A,\epsilon}_t(z)$ from $[0, \infty) \times (0,1) \times [0,1] \times \mathbb{C}$ to $\mathbb{R}$ is smooth in the usual sense. It follows that the function $(A, \epsilon, t, z) \mapsto b^{A,\epsilon}_t(z)$ is smooth.

Lemma 6.1 is used in the next section, in the proof of Lemma 7.1. The rest of this section is devoted to the proof of Lemma 6.1. The reader may skip the proof and proceed to the next section.

Lemma 6.3. There exists a smooth family of (time independent) functions $H^\epsilon : \mathbb{C} \to \mathbb{R}$, parametrized by $0 < \epsilon < 1$, whose Hamiltonian flows $b^\epsilon_t$ have the following properties.

- For every $t$, the diffeomorphism $b^\epsilon_t$ is the identity map outside the disc

$$\{ z \mid \pi|z|^2 < 1 + \epsilon \}. \quad (6.4)$$

- For $t > 1$, the diffeomorphism $b^\epsilon_t$ carries the disc

$$\{ z \mid \pi|z|^2 \leq 1 \} \quad (6.5)$$

into the slit disc

$$\{ z \mid \pi|z|^2 < 1 + \epsilon \} \setminus \text{the non-negative x-axis}. \quad (6.6)$$

Proof. We choose $H^\epsilon$ such that, for each $\epsilon$, the Hamiltonian vector field of $H^\epsilon$ is zero outside the disc (6.4), is equal to $-\frac{\partial}{\partial x}$ on the segment $[0,1]$ of the x-axis, and is equal to a non-negative multiple of $-\frac{\partial}{\partial x}$ on the rest of the x-axis.

For example, we may take $H^\epsilon(z) = y\rho((\pi|z|^2 - 1)/\epsilon)$ where $y$ is the imaginary part of $z$ and where $\rho(s)$ is equal to 1 for $s \leq 0$, to 0 for $s \geq 1$, and is non-negative.

For each $t$, the Hamiltonian flow generated by the function $H^\epsilon$, at times $t \geq 0$, carries the segment $[-1 - \epsilon,1]$ to a segment of the form $[-1 - \epsilon, x_t]$; if $t > 1$, then $x_t < 0$. Also, this flow carries the segment $[1,1 + \epsilon]$ to the segment $[x_t,1 + \epsilon]$, and it is the identity map outside the disc (6.4). Thus, it satisfies the requirements of the lemma. \hfill $\square$

Lemma 6.7. There exists a smooth family of functions $H^{t,\delta}_t : \mathbb{C} \to \mathbb{R}$, parametrized by $0 < \epsilon < 1$ and $0 < \delta < \epsilon$, whose Hamiltonian flows $b^{t,\delta}_t : \mathbb{C} \to \mathbb{C}$ have the following properties.

- The diffeomorphism $b^{t,\delta}_t$ is the identity map outside the disc \{ $z \mid \pi|z|^2 < t + 1 + \epsilon$ \}.

- The diffeomorphism $b^{t,\delta}_t$ carries the set

$$\{ re^{i\theta} \mid \delta \leq \pi r^2 \leq 1 + \epsilon - \delta, \; \delta \leq \theta \leq 2\pi - \delta \} \quad (6.8)$$

to the set

$$\{ re^{i\theta} \mid t + \delta \leq \pi r^2 \leq t + 1 + \epsilon - \delta, \; \delta \leq \theta \leq 2\pi - \delta \} \quad (6.9).$$
Proof. We choose $H^\epsilon,\delta_t$ whose Hamiltonian vector field vanishes outside the annulus $\{z \mid t < \pi |z|^2 < 1 + t + \epsilon\}$ and, on the set (6.9), it is equal to $\frac{1}{2\pi r} \frac{\partial}{\partial r}$. (Then, on the set (6.9), the derivative of the function $\pi |z|^2$ along this vector field is equal to one.) The Hamiltonian flow generated by the function $H^\epsilon,\delta_t$, at time $t \geq 0$, will carry the set (6.8) to the set (6.9), as required.

To be explicit, we may take $H^\epsilon,\delta_t(re^{i\theta}) = -\theta^2 \pi \rho_1^\delta(\theta) \rho_2^\epsilon (\pi r^2 - t)$ where $0 \leq \theta < 2\pi$ and $t \leq \pi r^2 \leq 1 + t + \epsilon$, where $\rho_1^\delta : [0, 2\pi] \to \mathbb{R}$ is a smooth family of functions that vanish to all orders at $\theta = 0$ and $\theta = 2\pi$ and such that $\rho_1^\delta$ is equal to one on $[\delta, 2\pi - \delta]$, and where $\rho_2^\epsilon : \mathbb{R} \to \mathbb{R}$ is a smooth family of functions that vanish outside $[0, 1 + \epsilon]$ and such that $\rho_2^\epsilon$ is equal to one on $[\delta, 1 + \epsilon - \delta]$. □

Proof of Lemma 6.1. Every closed subset of $\mathbb{C}$ which is contained in the slit disc
\[(6.10) \quad \{z \mid \pi |z|^2 < 1 + \epsilon\} \setminus \text{the non-negative x-axis}\]
is also contained in a set of the form
\[\{re^{i\theta} \mid \delta \leq \pi r^2 \leq 1 + \epsilon - \delta, \delta \leq \theta \leq 2\pi - \delta\}\]
for some $\delta > 0$. Moreover, for every closed subset of $(0, 1) \times \mathbb{C}$ which is contained in the product of the open segment $(0, 1)$ with the slit disc (6.10), there exists a smooth function $\epsilon \mapsto \delta_\epsilon$ such that the closed subset is also contained in the subset
\[(6.11) \quad \{(\epsilon, re^{i\theta}) \mid 0 < \epsilon < 1, \delta_\epsilon \leq \pi r^2 \leq 1 + \epsilon - \delta_\epsilon, \delta_\epsilon \leq \theta \leq 2\pi - \delta_\epsilon\}\]
of $(0, 1) \times \mathbb{C}$.

Let $b_t^\epsilon$ be the Hamiltonian flows of Lemma 6.3. Because $(\epsilon, z) \mapsto (\epsilon, b_t^\epsilon(z))$ is a diffeomorphism of $(0, 1) \times \mathbb{C}$, it carries the set $\{(\epsilon, z) \mid \pi |z|^2 \leq 1\}$ to a closed subset of $(0, 1) \times \mathbb{C}$. For $t > 1$, by the second item of Lemma 6.3, this closed subset is contained in the product of $(0, 1)$ with the slit disc (6.10). Fix $T > 1$, and fix a smooth function $\epsilon \mapsto \delta_\epsilon$ such that the set $\{(\epsilon, b_T^\epsilon(z)) \mid \pi |z|^2 \leq 1\}$ is contained in the set (6.11).

Let $b_t^{\epsilon,\delta}$ be the Hamiltonian flows of Lemma 6.7.

Let $\rho : [0, 1] \to [0, 1]$ be a smooth function that takes 0 to 0 and 1 to 1 and whose derivatives of all orders vanish at the endpoints 0 and 1. For example, we may take $\rho(s) = \int_0^s e^{-\frac{1}{\pi(1-t)}} ds/ \int_1^1 e^{-\frac{1}{\pi(1-t)}} ds$. The Hamiltonian flow
\[b_T^{\alpha,\epsilon} = \begin{cases} b_T^{\epsilon,\rho(2\tau)} & 0 \leq \tau \leq 1/2 \\ b_T^{\epsilon,\delta(2\tau-1)} & 1/2 \leq \tau \leq 1 \end{cases}\]
satisfies the requirements of Lemma 6.1. □

7. Disjoining a family of discs

The purpose of this section is to prove the following quantitative result, which is later used in the proof of Theorem 5.1.
Lemma 7.1. Let $W$ be a symplectic manifold and $N : W \to [0, r)$ a proper function. Let $A_1, A_2 : [0, r) \to \mathbb{R}$ be smooth functions such that the set
\[ C := \{ x \in [0, r) \mid A_1(x) \geq 0 \text{ and } A_2(x) \geq 0 \} \]
is compact. Let $U$ be an open subset of $\mathbb{C} \times W$ that contains the set
\[ \{(z, w) \mid N(w) \in C \text{ and } \pi|z|^2 \leq A_1(N(w)) + A_2(N(w))\}. \]
Then there exists a symplectic isotopy, compactly supported in $U$, that disjoins the set
\[ \{(z, w) \mid \pi|z|^2 \leq A_1(N(w))\} \]
from the set
\[ \{(z, w) \mid \pi|z|^2 \leq A_2(N(w))\} \]
and that preserves the function $(z, w) \mapsto N(w)$.

Lemma 7.1 is, more or less, a parametrized version of the fact that a disc of area $A_1$ can be disjoined from a disc of area $A_2$ inside any disc of area greater than $A_1 + A_2$. The symplectic isotopy moves the $w$ coordinate but it does not change the value of $N(w)$. For a fixed value of $N(w)$, the effect of the isotopy on the $z$ coordinate is independent of the $w$ coordinate.

Proof. There exist an open neighbourhood $V$ of $C$ whose closure in $[0, r)$ is compact and a positive number $\tau$ such that $U$ contains the set
\[ \{(z, w) \mid N(w) \in \text{closure}(V) \text{ and } \pi|z|^2 \leq A_1(N(w)) + A_2(N(w)) + \tau\}. \]
This follows from the compactness of (7.2), which, in turn, follows from the compactness of the set \( \{(s, x) \mid x \in C \text{ and } 0 \leq s \leq A_1(x) + A_2(x)\} \) and the properness of the map \((z, w) \mapsto (\pi|z|^2, N(w))\) from $\mathbb{C} \times W$ to $[0, \infty) \times [0, r)$.

Let $\rho : [0, r) \to \mathbb{R}$ be a smooth function that vanishes outside $V$ and such that $\rho|_{C} \equiv 1$. Let $F_{t}^{A_1, A_2}(z)$ be a smooth family of functions, defined for $z \in \mathbb{C}$, $t \in [0, 1]$, and $A_1, A_2 \in \mathbb{R}$, such that, when $A_1 \geq 0$ and $A_2 \geq 0$, the time-dependent function $z \mapsto F_{t}^{A_1, A_2}$ vanishes outside the set given by $\pi|z|^2 \leq A_1 + A_2 + \tau$ and its flow $b_{t}^{A_1, A_2}$; at time $t = 1$, carries the set $\{z \mid \pi|z|^2 \leq A_1\}$ into the set $\{z \mid A_2 < \pi|z|^2 \leq A_1 + A_2 + \tau\}$. For example, we may set
\[ F_{t}^{A_1, A_2}(z) := F_{t}^{A, \epsilon}(z/\sqrt{A_1 + \epsilon/2}) \]
where $A = \frac{A_2}{A_1 + \epsilon/2}$ and $\epsilon = \frac{\tau/2}{A_1 + \epsilon/2}$ and where $F_{t}^{A, \epsilon}$ is as in Lemma 6.1.

Let
\[ H_t(z, w) = \rho(N(w))F_{t}^{A_1(N(w)), A_2(N(w))}(z). \]
This function vanishes outside the compact set
\[ \{(z, w) \mid N(w) \in \text{closure}(V) \text{ and } \pi|z|^2 \leq A_1 + A_2 + \tau\}. \]
Its Hamiltonian flow preserves the function $N(w)$. We will show, for every $N_0$, that this flow, at time $t = 1$, disjoins
\[ \{(z, w) \mid \pi|z|^2 \leq A_1(N_0) \text{ and } N(w) = N_0\} \]
from

\{(z, w) \mid \pi|z|^2 \leq A_2(N_0) \text{ and } N(w) = N_0\}.

When \(N_0 \in C\), the restriction of the Hamiltonian flow of \(H_t\) to the level set \(\{N(w) = N_0\}\) is

\[b_t(z, w) = \left(\bar{b}_t^{A_1, A_2}, b_t^N(z, w, t)(w)\right)\]

where \(A_1 = A_1(N_0)\), where \(A_2 = A_2(N_0)\), where \(\bar{b}_t^{A_1, A_2}\) is the Hamiltonian flow of \(F_t^{A_1, A_2}\) on \(\mathbb{C}\), where \(b_t^N\) is the Hamiltonian flow of \(N\) on \(W\) (with time parameter \(T\)), and where \(T(z, w, t)\) is some real valued function of \(z, w, t\). At \(t = 1\), this flow carries the set

\[\{(z, w) \mid \pi|z|^2 \leq A_1(N_0) \text{ and } N(w) = N_0\}\]

into the set

\[\{(z, w) \mid A_2(N_0) < \pi|z|^2 \leq A_1(N_0) + A_2(N_0) + \bar{\epsilon} \text{ and } N(w) = N_0\}\]

so it disjoins (7.3) from (7.4).

When \(N_0 \not\in C\), one of the sets (7.3) and (7.4) is empty. So the flow trivially disjoins the set (7.3) from the set (7.4).

\[\Box\]

8. Shortening on a manifold with an isolated maximum

In this section we prove Theorem 3.1 and Remarks 3.2 and 3.4. We recall the statement:

Let \((M, \omega)\) be a compact connected symplectic manifold with a Hamiltonian circle action. Suppose that the momentum map attains its maximum at an isolated fixed point and that at least one of the isotropy weights at that point has absolute value greater than one. Let \(d\) be a positive number that is smaller than the gap between the two largest singular values of the momentum map. Then the circle action can be deformed to a loop in \(\text{Ham}(M, \omega)\) whose Hofer length is smaller than that of the original circle action by \(2d/9\).

We can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let \(H\) denote the momentum map, let \(p_{\text{max}}\) denote the point where \(H\) attains its maximum, let \(\alpha\) denote the gap between the two largest singular values of \(H\), and consider the subset of \(M\) given by

\[\{m \mid H(m) > H(p_{\text{max}}) - \alpha\}.
\]

We can choose the deformation such that the normalized deformed Hamiltonians remain greater than \(H(p_{\text{max}}) - \alpha\) on the set (8.1) and coincide with \(H\) outside this set.

We actually obtain a better estimate than \(2d/9\). If one of the isotropy weights is even then we can shorten by \(1/4\). Otherwise, let \(-k_1\) be the weight whose absolute value is largest; then we can shorten by \(1/4(1 - 1/k_1)\). When \(k_1 = 3\), this becomes \(2d/9\).
Proof of Theorem 3.1 and Remarks 3.2 and 3.4. Let \(-k_1, \ldots, -k_n\) be the isotropy weights at \(p_{\max}\). The equivariant Darboux theorem implies that the integers \(k_j\) are positive. We assume that at least one of them is greater than one. Without loss of generality,

\[(8.2) \quad k_1 = a + b\]

where \(a\) and \(b\) are positive integers.

The point \(p_{\max}\) is the only critical point of \(H\) in the set (8.1), and the momentum map \(H\) is proper as a map from this set to the ray \((H(p_{\max}) - \alpha, \infty)\). There exists an equivariant symplectomorphism from the subset of \(\mathbb{C}^n\) given by

\[N^H_\alpha = \{z \mid \pi(k_1|z_1|^2 + \ldots + k_n|z_n|^2) < \alpha\},\]

with the circle action generated by the momentum map \(H(z_1, \ldots, z_n) = -k_1\pi|z_1|^2 - \ldots - k_n|z_n|^2\) to the subset (8.1) of \(M\); denote it

\[i: N^H_\alpha \to M.\]

This follows from [9, Proposition 2.8] (which should have been stated with \(|z_j|^2\) instead of the typo \(|z_j|\)).

Let \(N(z_2, \ldots, z_n) := \pi(k_2|z_2|^2 + \ldots + k_n|z_n|^2)\).

We will apply Lemma 5.1 with \(U = N^H_\alpha\),

\[K(z_1, \ldots, z_n) := H(p_{\max}) - \pi b|z_1|^2, \quad \text{and} \quad F(z_1, \ldots, z_n) := -\pi a|z_1|^2 - N(z_2, \ldots, z_n).\]

Let \(0 < d < \alpha\) and let \(s = \frac{3}{9}d\). By Lemma 5.1, to shorten by amount \(s\), it is enough to disjoin the set \(N^s_K = \{z \mid \pi b|z_1|^2 < s\}\) from the set \(N^s_F = \{z \mid \pi a|z_1|^2 + N(z_2, \ldots, z_n) < s\}\) by a compactly supported symplectic isotopy of \(N^H_\alpha\) that preserves the function

\[\hat{N}(z_1, z_2, \ldots, z_n) := N(z_2, \ldots, z_n).\]

Write the sets \(N^s_K\) and \(N^s_F\) as

\[N^s_K = \{z \mid \pi|z_1|^2 < \frac{s}{b}\} \quad \text{and} \quad N^s_F = \{z \mid \pi|z_1|^2 < \frac{s - N}{a}\}.\]

By Lemma 7.1 with \(A_1(N) \equiv s/b\) and \(A_2(N) = (s - N)/a\), for any open neighbourhood \(U\) of the set

\[(8.3) \quad \{z \mid \pi|z_1|^2 \leq \frac{s}{b} + \frac{s - N}{a}\}\]

there exists a symplectic isotopy, compactly supported in \(U\), that disjoins \(N^s_F\) from \(N^s_K\) and that preserves the function \(\hat{N}\). So it is enough to show that the set (8.3) is contained in the set \(N^H_\alpha\).

Because \(a + b = k_1\), we can rewrite the set (8.3) as

\[(8.4) \quad \{z \mid \frac{ab}{sk_1}\pi|z_1|^2 + \frac{b}{sk_1}N \leq 1\}.\]
Also, we rewrite the set $N^H_\alpha$ as
\[
\{ z \mid k_1 \alpha \pi |z_1|^2 + \frac{1}{\alpha} N < 1 \}. 
\] (8.5)

It remains to show that the set (8.4) is contained in the set (8.5). This is equivalent to requiring that
\[
\frac{ab}{sk_1} \geq \frac{k_1}{\alpha} \quad \text{and} \quad \frac{b}{sk_1} \geq \frac{1}{\alpha}.
\]

Because $a \leq k_1$, the first of these inequalities implies the second.

If $k_1$ is even, write $k_1 = a + b$ where $b = a$. Then $ab/k_1^2 = 1/4$, which is greater than $2/9$.

If $k_1$ is odd, write $k_1 = a + b$ where $b = a + 1$. Then $ab/k_1^2 = 1/4(1 - 1/k_1^2)$ which, because $k_1 \geq 3$, is greater than or equal to $1/4(1 - 1/9) = 2/9$. In either case, if $s = 2/9d$ and $d < \alpha$, then $s < (ab/k_1^2)\alpha$, which is equivalent to the required inequality .

9. LOCAL NORMAL FORM NEAR A FIXED COMPONENT, AND ITS CONSEQUENCES

In this section we recall a local normal form for a neighborhood of a fixed component in a manifold with Hamiltonian circle action, and we give some consequences that we will use in the next section.

Lemma 9.1. Let
\[
\pi: E \to B
\]
be a symplectic vector bundle, i.e., a vector bundle with a fibrewise symplectic form. There exists on the total space $E$ a closed two-form $\omega_\Theta$ with the following properties.

1. The pullback of $\omega_\Theta$ to the fibres of $E$ coincides with the fibrewise symplectic forms.
2. The pullback of $\omega_\Theta$ by the zero section of $E$ is zero.
3. At the points of the zero section, the fibres of $E$ are $\omega_\Theta$-orthogonal to the zero section.

Moreover, if a compact Lie group $K$ acts on $E$ by bundle automorphisms, then $\omega_\Theta$ can be chosen to be $K$-invariant.

Proof. Let $2n = \text{rank}E$, and let
\[
G = \text{Sp}(\mathbb{R}^{2n}).
\]
Let $P \to B$ be the principal $G$ bundle (with a right $G$ action) whose fibre $E_b$ over a point $b$ of $B$ consists of the set of linear symplectic isomorphisms from $\mathbb{R}^{2n}$ to $E_b$.

We identify $E$ with the associated bundle
\[
P \times_G \mathbb{R}^{2n},
\]
in which $[pa, z] = [p, az]$ for all $p \in P$, $z \in \mathbb{R}^{2n}$, and $a \in G$.

Let $\Theta \in \Omega^1(P, g)$ be a connection one-form; use the same symbol, $\Theta$, to denote its pullback to $P \times \mathbb{R}^{2n}$. Let $\omega_0$ be the standard symplectic form on $\mathbb{R}^{2n}$. Let $\Phi_{\mathbb{R}^{2n}}: \mathbb{R}^{2n} \to g^*$ be the quadratic momentum map for the $G$-action on $\mathbb{R}^{2n}$. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $g^*$ and $g$. Then $\langle \Phi_{\mathbb{R}^{2n}}, \Theta \rangle$ is a $G$-invariant real valued one-form on $P \times \mathbb{R}^{2n}$. The difference $\omega_0 - d \langle \Phi_{\mathbb{R}^{2n}}, \Theta \rangle$ descends to a closed two-form $\omega_\Theta$ on $P \times_G \mathbb{R}^{2n}$ with the desired properties.
If a compact Lie group $K$ acts on $E$ by bundle automorphisms, then, by averaging, we can choose the connection one-form $\Theta$ to be $K$-invariant. The resulting two-form $\omega_\Theta$ is then $K$-invariant.

□

Remark 9.2. The connection on $E$ that is induced from $\Theta$ determines a splitting

$$TE = \pi^*TB \oplus \pi^*E.$$  

With respect to this splitting, $\omega_\Theta$ can be written as a block matrix

$$\begin{pmatrix} X & 0 \\ 0 & A \end{pmatrix}$$

where $X$ is an alternating bilinear form on the fibres of $\pi^*TB$ and $A$ is the fibrewise symplectic form on $E$. At the points of the zero section, $X = 0$.

Let $(B, \omega_B)$ be a symplectic manifold, let $\pi : E \to B$ be a symplectic vector bundle, and let a compact Lie group $K$ act on $E$ by bundle automorphisms that descend to symplectomorphisms of $(B, \omega_B)$. Let $\omega_\Theta$ be a closed two-form as in Lemma 9.1. The corresponding coupling form is

$$(9.3) \quad \omega_E := \pi^*\omega_B + \omega_\Theta.$$  

It has the following properties.

1. Its pullback to the fibres coincides with the fibrewise symplectic forms.
2. Its pullback by the zero section coincides with $\omega_B$.
3. At the points of the zero section, the fibres of $E$ are $\omega_E$-orthogonal to the zero section.

Consequently, $\omega_E$ is non-degenerate near the zero section, and the normal bundle of the zero section in $(E, \omega_E)$ is isomorphic to $E \to B$ as symplectic vector bundles.

Remark 9.4. The construction that we just described is Shlomo Sternberg’s minimal coupling [22]. For more details, see chapter 1 of the book [4] by Guillemin, Lerman, and Sternberg.

Now let $B$ be a connected component of the fixed point set of a circle action on a symplectic manifold $(M, \omega)$, and let $E$ be the normal bundle of $B$ in $M$. Then $B$ is a symplectic submanifold, and $E$ can be identified with the symplectic orthocomplement of $TB$ in $TM|_B$. Let $\omega_E$ be a coupling form on $E$, as described above. With this setting, we have the following symplectic tubular neighbourhood theorem.

There exists a neighbourhood $U$ of the zero section in $E$ and an equivariant symplectic open embedding $(U, \omega_E) \to (M, \omega)$ whose restriction to the zero section is the identity map on $B$ and, under the natural identification of $TE|_B$ with $TM|_B$, whose differential is the identity map at every point of $B$.

(This follows from the classical tubular neighborhood theorem in differential topology, combined with Theorem 4.1 of Alan Weinstein’s paper [24], keeping track of a group action as explained in the last paragraph of Section 3 of [24].)
Lemma 9.6. Let $E \to B$ be a symplectic vector bundle with a fibrewise circle action. Suppose that all the weights are negative; denote the weights by $-k_1, \ldots, -k_s$ and let $$k = \min\{k_1, \ldots, k_s\} - 1.$$ Then there exist $k + 1$ commuting fibrewise circle actions on $E$, with quadratic momentum maps $$H_{(j)} : E \to \mathbb{R}, \quad 0 \leq j \leq k$$ that are negative outside the zero section and whose sum $$H_{(0)} + H_{(1)} + \ldots + H_{(k)}$$ is a fibrewise momentum map for the given circle action. Also, there exist arbitrarily small neighborhoods $U$ of the zero section whose intersections with the fibres of $E$ are connected and that satisfy

$$\inf_U \sum_{j=0}^k H_{(j)} = \sum_{j=0}^k \inf_U H_{(j)}.$$  

Proof. After eliminating repetitions, we may assume that the positive numbers $k_1, \ldots, k_s$ are distinct. Let $E_j$ denote the $-k_j$th weight space for the circle action on $E$, so that $E$ decomposes as $E_1 \oplus \ldots \oplus E_s$. Choose a compatible fibrewise Hermitian metric on each of the bundles $E_1, \ldots, E_s$. The function

$$z \mapsto \pi \|z_j\|^2,$$

for $z = (z_1, \ldots, z_s) \in E_1 \oplus \ldots \oplus E_s$, is a momentum map for the scalar multiplication circle action on the $E_j$ factor. The quadratic fibrewise momentum map for the given circle action on $E$ is

$$-k_1 \pi \|z_1\|^2 - \ldots - k_s \pi \|z_s\|^2.$$  

For each $j$, because $k_j \geq k + 1$, we can decompose $k_j$ as

$$k_j = a_{0j} + a_{1j} + \ldots + a_{kj}$$

where $a_{ij}$ are positive integers. We get a decomposition of (9.8) into the sum

$$H_{(0)} + H_{(1)} + \ldots + H_{(k)},$$

where

$$H_{(i)}(z) = -a_{i1} \pi \|z_1\|^2 - \ldots - a_{is} \pi \|z_s\|^2$$

generates the fibrewise circle action on $U$ that rotates the $j$th coordinate with speed $-a_{ij}$. Because the numbers $a_{ij}$ are positive, the functions $H_{(j)}$ are negative outside the zero section.

Finally, every neighborhood of the zero section contains a neighborhood of the form $U = \bigcap_{j=1}^s \{ z \mid \pi \|z_j\|^2 < \epsilon \}$. This neighborhood $U$ satisfies (9.7), and the intersections of $U$ with the fibres of $E$ are connected.
Lemma 9.9. Let \((B, \omega_B)\) be a symplectic manifold, let \(E \to B\) be a symplectic vector bundle, let \(\omega_E\) be a coupling form (cf. (9.3)), and let \(U \subset E\) where \(\omega_E\) is non-degenerate. Let \(k\) be a positive integer. Suppose that the Euler class \(e(E)\) of \(E\) satisfies \(e(E)^k = 0\) in \(H^*(B)\). Then there exist smooth functions \(g_1, \ldots, g_k: U \to \mathbb{R}\) whose Hamiltonian vector fields are nowhere all tangent to the zero section of \(E\).

Proof. Let \(\xi_1, \ldots, \xi_k\) be sections of \(E \to B\) that are nowhere all vanishing. (Choose the first \(k - 1\) sections arbitrarily; perturb them so that the intersection of their zero-sets is a smooth submanifold \(\hat{B}\) of \(B\). The homology class of \(\hat{B}\) is the Poincaré dual to \(e(E)^{k-1}\) in \(H_*(B)\). Because \(e(E)^k = 0\), the Euler class of the bundle \(E|_{\hat{B}} \to \hat{B}\) is zero. So this bundle has a non-vanishing section. Take \(\xi_k\) to be a smooth extension of this section to a section of \(E \to B\).)

Fix \(j \in \{1, \ldots, k\}\). Let \(g_j: E \to \mathbb{R}\) be the function whose restriction to the fibre over \(b \in B\) is the linear functional \(\iota(\xi_j(b))\omega_E\). Identifying \(E\) with the vertical subbundle of \(TE|_B\), the Hamiltonian vector field of \(g_j\) at the points of the zero section.

Because \(\xi_1, \ldots, \xi_k\) are vertical and nowhere vanishing, they are nowhere all tangent to the zero section. \(\square\)

In preparation for the proof of Theorem 3.9, which involves a subbundle of the normal bundle, we now give a refinement of the minimal coupling construction. Let \((B, \omega_B)\) be a symplectic manifold, let \(\pi: E \to B\) be a symplectic vector bundle, and let a compact Lie group \(K\) act on \(E\) by bundle automorphisms. Let \(E'\) be a \(K\)-invariant symplectic sub-bundle of \(E\).

Let \(E''\) be the symplectic orthocomplement of \(E'\) in \(E\). Then \(E = E' \oplus E''\), and we have a pullback diagram:

\[
\begin{array}{c}
E' \xrightarrow{\pi'} E \\
\downarrow p' \quad \downarrow \pi'' \\
E'' \xrightarrow{\pi''} B
\end{array}
\]

Let \(\omega_{E'}\) and \(\omega_{E''}\) denote two-forms on \(E'\) and \(E''\) with the properties listed in Lemma 9.1. The corresponding coupling forms on \(E'\) and on \(E\) are

\[\omega_{E'} = \pi'^* \omega_B + \omega_{E'}\]

and

\[
\omega_E = \pi^* \omega_B + p'^* \omega_{E'} + p''^* \omega_{E''}
\]

We say that such a coupling form \(\omega_E\) on \(E\) is compatible with the coupling form \(\omega_{E'}\) on \(E'\).

The proof of Theorem 3.9 will use the following lemma. In it, we view \(E''\) as the subset of \(E\) consisting of the set of points whose fibrewise \(E'\) coordinate is zero, i.e., the preimage under \(p'\) of the zero set of \(E'\).
Lemma 9.12. Let \((B, \omega_B)\) be a symplectic manifold, let \(E \to B\) be a symplectic vector bundle, and let \(E'\) and \(E''\) be symplectic sub-bundles such that \(E = E' \oplus E''\). Let \(\omega_E\) be a coupling form on \(E'\) and let \(\omega_E\) be a compatible coupling form on \(E\) (cf. \((9.12)\)). Let \(U'\) be a neighbourhood of the zero section in \(E'\) where \(\omega_E\) is non-degenerate, and let \(U\) be a neighbourhood of the zero section in \(E\) where \(\omega_E\) is non-degenerate and that is contained in \(p^{-1}(U')\).

Let \(g'_1, \ldots, g'_k: U' \to \mathbb{R}\) be smooth functions whose Hamiltonian vector fields in \(U'\) are nowhere all tangent to the zero section of \(E'\). Let \(g_1, \ldots, g_k: U \to \mathbb{R}\) be their pullbacks under \(p'|_U: U \to U'\). Then, at each point of the zero section of \(E\), the Hamiltonian vector fields of \(g_1, \ldots, g_k\) are tangent to \(E'\) and are not all tangent to the zero section of \(E\).

Proof. The connections on \(E'\) and on \(E''\) give decompositions
\[
TE' = \pi''TB \oplus \pi'^*E'
\]
and
\[
TE = \pi^*TB \oplus \pi^*E' \oplus \pi^*E''
\]

A vector in \(TE'\) is tangent to the zero section exactly if it is based at a point of the zero section and, under the decomposition \((9.13)\), is represented by a block vector of the form
\[
\begin{bmatrix}
* \\
0
\end{bmatrix}.
\]

Similarly, a vector in \(TE\) is tangent to the zero section exactly if it is based at a point of the zero section and, under the decomposition \((9.14)\), is represented by a block vector of the form
\[
\begin{bmatrix}
* \\
0 \\
0
\end{bmatrix}.
\]

Fix a point \(b\) of \(B\); identify it with the corresponding points of the zero sections of \(E'\) and of \(E\). Let \(\xi'_1, \ldots, \xi'_k\) be the Hamiltonian vector fields of \(g'_1, \ldots, g'_k\) in \(U'\). Under the decomposition \((9.13)\), their values at the point \(b\) are represented by block vectors, which we write as
\[
\begin{pmatrix}
\xi'_{1H}(b) \\
\xi'_{1V}(b)
\end{pmatrix}, \ldots, 
\begin{pmatrix}
\xi'_{kH}(b) \\
\xi'_{kV}(b)
\end{pmatrix}.
\]

To prove the lemma, it is enough to show that, under the decomposition \((9.14)\), the values at \(b\) of the Hamiltonian vector fields of the pullbacks \(g_1, \ldots, g_k\) are represented by the block vectors
\[
\begin{pmatrix}
\xi'_{1H}(b) \\
\xi'_{1V}(b)
\end{pmatrix}, \ldots, 
\begin{pmatrix}
\xi'_{kH}(b) \\
\xi'_{kV}(b)
\end{pmatrix}.
\]
Indeed, from the assumption that, for every \( b \in B \), not all the vectors \( (9.17) \) have the form \( (9.15) \), we deduce that, for every \( b \in B \), not all the vectors \( (9.18) \) have the form \( (9.16) \).

We proceed to prove that, at each point of the zero section, if the Hamiltonian vector fields \( \xi_1', \ldots, \xi_k' \) of \( g_1', \ldots, g_k' \) have the form \( (9.17) \), then the Hamiltonian vector fields of the pullbacks \( g_1, \ldots, g_k \) have the form \( (9.18) \).

Let

\[
\tilde{\xi}_1', \ldots, \tilde{\xi}_k'
\]

denote the vector fields on \( U \) that, under the decomposition \( (9.14) \), are represented by the block vectors \( (9.18) \). Under the same decomposition \( (9.14) \), the two-form \( p''^*\omega_{\Theta''} \) on \( E \) is represented by a block matrix of the form

\[
(9.19)
\begin{pmatrix}
X & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & A
\end{pmatrix}.
\]

At the points of the zero section, \( X = 0 \) (cf. Remark \( (9.2) \)), so, at these points, the vectors \( (9.18) \) are in the null space of the bilinear form given by \( (9.19) \).

Fix \( j \in \{1, \ldots, k\} \). We have just shown that

\[
(9.20)
\iota(\tilde{\xi}_j')p''^*\omega_{\Theta''} = 0 \quad \text{at the points of the zero section of } E.
\]

We now compute:

\[
\begin{align*}
dg_j &= p'' dg_j' \quad \text{by the definition of } g_j \\
&= -p'' \iota(\xi_j')\omega_{E'} \quad \text{by the definition of } \xi_j' \\
&= -\iota(\tilde{\xi}_j')p''^*\omega_{E'} \quad \text{because } p'_j \tilde{\xi}_j' = \xi_j' \\
&= -\iota(\tilde{\xi}_j') (\omega_E - p''^*\omega_{\Theta''}) \quad \text{by } (9.12).
\end{align*}
\]

By \( (9.21) \) and \( (9.20) \),

\[
dg_j = -\iota(\tilde{\xi}_j')\omega_E \quad \text{at the points of the zero section}.
\]

This implies that the Hamiltonian vector field of \( g_j \) is equal to the vector field \( \tilde{\xi}_j' \) at the point of the zero section, as required. \( \square \)

10. Shortening on a Manifold with an Arbitrary Maximum

In this section we prove Theorems 3.7 and 3.9 with Remarks 3.8 and 3.11.

We begin by recalling the statement of Theorem 3.7 and Remark 3.8.

Let \( (M, \omega) \) be a compact connected symplectic manifold with a Hamiltonian circle action. Let \( B_{\max} \) be the set where the momentum map is maximal. Let \( -k_1, \ldots, -k_s \) denote the distinct weights for the circle action on the normal bundle of \( B_{\max} \), and let \( k := \min\{k_1, \ldots, k_s\} - 1 \). Suppose that \( B_{\max} \) is symplectically \( k \)-displaceable in every neighbourhood. Then the circle action can be deformed through loops in \( \text{Ham}(M, \omega) \) into a loop of smaller Hofer length.
We can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let $H$ denote the momentum map. For every positive number $\alpha$, we can choose the deformation such that the normalized deformed Hamiltonians remain greater than $H(B_{\text{max}}) - \alpha$ on the set $\{m \in M \mid H(m) > H(B_{\text{max}}) - \alpha\}$ and coincide with $H$ outside this set.

**Proof of Theorem 3.7 and Remark 3.8.** Let $E$ denote the normal bundle of $B_{\text{max}}$ in $(M, \omega)$. Let $\omega_E$ be a coupling form on $E$ (cf. (9.3)). By the local normal form theorem (9.5) there exists a neighborhood $U$ of the zero section in $E$ and there exists an $S^1$ equivariant open symplectic embedding $i: (U, \omega_E) \to (M, \omega)$ whose restriction to the zero section is the identity map on $B_{\text{max}}$ and, under the natural identification of $TE|_B$ with $TM|_B$, whose differential is the identity map at every point of $B$. Given a positive number $\alpha$, we may choose the neighborhood $U$ such that $i(U) \subset \{m \in M \mid H(m) > H(B_{\text{max}}) - \alpha\}$.

Apply Lemma 9.6 to obtain $k + 1$ commuting fibrewise circle actions on $E$, with quadratic momentum maps $H'_j: E \to \mathbb{R}$, $0 \leq j \leq k$, with the following properties. These momentum maps are negative outside the zero section. Their sum $H'_0 + H'_{(1)} + \ldots + H'_{(k)}$ is a fibrewise momentum map for the circle action on $E$. After possibly shrinking the set $U$, we may assume that the intersections of $U$ with the fibers of $E$ are connected and that

$$\inf_U \sum_{j=0}^k H'_{(j)} = \sum_{j=0}^k \inf_U H'_{(j)}.$$

Let $H: M \to \mathbb{R}$ be the momentum map on $M$. We have

$$i^*H = \left( H(B_{\text{max}}) + H'_0 + H'_{(1)} + \ldots + H'_{(k)} \right) |_U = H(0) + H_{(1)} + \ldots + H_{(k)},$$

where $H(0) = H(B_{\text{max}}) + H'_0 |_U$ and where $H_{(j)} = H'_j |_U$ for each $1 \leq j \leq k$. (On each fibre of $E$, the left and right hand sides of (10.1) are momentum maps with the same value at the origin, so they are equal.)

We are now in the setup of Lemma 5.10 with the function $\tilde{N}: U \to \mathbb{R}$ being constant. The sets $B_{\text{max}}^{(j)}$ where the functions $H_{(j)}$ attain their maximum are all equal to the zero section of $E$. This implies the assumption (5.12) of Lemma 5.10. It also reduces the last assumption of Lemma 5.10 to the assumption that the zero section is symplectically $k$-displaceable in $U$. This assumption then follows from that $B_{\text{max}}$ is symplectically $k$-displaceable in $i(U)$. Theorem 3.7 and Remark 3.8 then follows from Lemma 5.10. \qed
Next, we prove Theorem 3.9 and Remark 3.11. We recall the statement:

Let $(M,\omega)$ be a compact connected symplectic manifold with a Hamiltonian circle action. Let $B_{\text{max}}$ be the set where the momentum map is maximal. Let $E' \to B_{\text{max}}$ be an $S^1$-invariant subbundle of the normal bundle to $B_{\text{max}}$ in $M$, and let $-k'_1, \ldots, -k'_s'$ be the distinct weights for the circle action on $E'$. Let $k' = \min\{k'_1, \ldots, k'_s'\} - 1$.

Let $e(E') \in H^*(B_{\text{max}})$ be the Euler class of $E'$. Suppose that
\begin{equation}
(10.2) \quad e(E')^{k'} = 0
\end{equation}
in $H^*(B_{\text{max}})$. Then the circle action can be deformed through loops in $\text{Ham}(M,\omega)$ into a loop of smaller Hofer length.

We can choose the deformation such that the positive Hofer length never exceeds the initial one and the negative Hofer length remains constant. Specifically, let $H$ denote the momentum map. For every positive number $\alpha$, we can choose the deformation such that the normalized deformed Hamiltonians remain greater than $H(B_{\text{max}}) - \alpha$ on the set $\{m \in M \mid H(m) > H(B_{\text{max}}) - \alpha\}$ and coincide with $H$ outside this set.

Proof of Theorem 3.9 and Remark 3.11. Let $E$ denote the normal bundle of $B_{\text{max}}$ in $(M,\omega)$. Let $E''$ be the symplectic orthocomplement of $E'$ in $E$, so $E = E' \oplus E''$. Recall the pullback diagram (9.10):
\begin{equation}
(10.3)
\begin{array}{ccc}
E & \xrightarrow{\nu''} & E'' \\
\downarrow \nu' & & \downarrow \pi'' \\
E' & \xrightarrow{\pi'} & B.
\end{array}
\end{equation}
Let $\omega_{E''}$ be a coupling form on $E'$ (cf. (9.3)), and let $\omega_E$ be a compatible coupling form on $E$ (cf. (9.12)).

Applying Lemma 9.6, we get $k' + 1$ commuting fibrewise circle actions on the fibres of $E'$, with quadratic momentum maps
\[ H'_{(j)}: E' \to \mathbb{R}, \quad 0 \leq j \leq k', \]
that are negative outside the zero section, and whose sum
\[ H'_{(0)} + \ldots + H'_{(k)} \]
is a fibrewise momentum map for the circle action on $E'$. We also get a neighborhood $U'$ of the zero section of $E'$ where $\omega_{E'}$ is non-degenerate, whose intersection with the fibres of $E'$ are connected, and on which
\[ \sup_{U'} \sum_{j=0}^{k} H'_{(j)} = \sum_{j=0}^{k} \sup_{U'} H'_{(j)}. \]
By the local normal form theorem (cf. (9.5)) there exists a neighborhood $U$ of the zero section in $E$ and an $S^1$ equivariant symplectic open embedding
\[ i: (U, \omega_E) \to (M, \omega) \]
whose restriction to the zero section is the identity map on $B$ and, under the natural identification of $TE|_B$ with $TM|_B$, whose differential is the identity map at every point of $B$. We choose such a neighborhood $U$ whose intersections with the fibres of $E$ are connected and that is contained in $p'^{-1}U'$. Given a positive number $\alpha$, we may also arrange that $i(U) \subset \{ m \in M \mid H(m) > H(B_{\text{max}}) - \alpha \}$.

Let $H'' : E'' \to \mathbb{R}$ denote the fibrewise quadratic momentum map for the circle action on $E''$. Let $H : M \to \mathbb{R}$ be the momentum map on $M$. Then
\[ i^*H = (H(B_{\text{max}}) + p''^*H'_{(0)} + \ldots + p''^*H'_{(k)} + p''''H'') |_{U}, \]
where
\[ H_{(0)} = H(B_{\text{max}}) + p''^*H'_{(0)} + p''''H'' \]
and where
\[ H_{(j)} = p''^*H'_{(j)} \quad \text{for all} \quad 1 \leq j \leq k. \]
(The left and right hand sides of (10.4) are fibrewise momentum maps on an ellipsoid bundle that coincide along the zero section, so they are equal.)

We are now in the setup of Lemma 5.10 with $\tilde{N} = p''^*H''$. The set $B_{\text{max}}^{(0)}$ where $H_{(0)}$ attains its maximum is the zero section, and, for $1 \leq j \leq k$, the set $B_{\text{max}}^{(j)}$ where $H_{(j)}$ attains its maximum is the intersection of $U$ with the image of $E''$ in $E$. Slightly abusing notation, we write these sets as $B_{\text{max}}$ and as $E'' \cap U$. To apply Lemma 5.10 we need to find symplectomorphisms $b_1, \ldots, b_k$ of $U$, each connected to the identity through a path of compactly supported symplectomorphisms of $U$, such that
\[ B_{\text{max}} \cap b_1(E'' \cap U) \cap \ldots \cap b_k(E'' \cap U) = \emptyset. \]

By Lemma 9.9 there exist functions $g_1, \ldots, g_k$ on $U'$ whose Hamiltonian vector fields are nowhere all tangent to the zero section of $E'$. Let $g_1, \ldots, g_k$ be the product of their pullbacks to $U$ with a cutoff function that is equal to 1 near the zero section and is supported in $U$. Lemma 9.12 implies that, at each point of $B_{\text{max}}$, the Hamiltonian vector fields of $g_1, \ldots, g_k$ are tangent to $E'$ and are not all tangent to $B_{\text{max}}$. Let $\psi_t^{g_1}, \ldots, \psi_t^{g_k}$ be the corresponding flows. If $t$ is a sufficiently small positive number, the intersection $B_{\text{max}} \cap \psi_t^{g_1}(E'' \cap U) \cap \ldots \cap \psi_t^{g_k}(E'' \cap U)$ is empty. Lemma 5.10 with $b_1, \ldots, b_k$ taken to be $\psi_t^{g_1}, \ldots, \psi_t^{g_k}$, gives the result of Theorem 3.9 and Remark 3.11.

11. THE INDEX OF THE POSITIVE HOFER LENGTH FUNCTIONAL

In this section we prove Theorem 3.16 and Remark 3.17. We recall the statement, with slight change of notation ($H_M$ instead of $H$):

\[ \]
Let \((M, \omega)\) be a \(2n\) dimensional compact connected manifold with a Hamiltonian circle action. Suppose that the momentum map attains its maximum at an isolated fixed point. Let \(-k_1, \ldots, -k_n\) be the isotropy weights at the maximum, with possible repetitions. Then there exists a neighbourhood \(D\) of the origin in \(\mathbb{R}^{\sum 2(k_j - 1)}\), and, for each \(\lambda \in D\), a loop \(\{\psi_t^{(\lambda)}\}_{0 \leq t \leq 1}\) in \(\text{Ham}(M, \omega)\), such that the following properties hold. For \(\lambda = 0\), the loop \(\{\psi_t^{(0)}\}_{0 \leq t \leq 1}\) is the given circle action. The function \(\lambda \mapsto \text{length}(\{\psi_t^{(\lambda)}\})\) is smooth, \(\lambda = 0\) is a critical point of this function, and the Hessian of this function at \(\lambda = 0\) is negative definite.

In fact, we can choose the deformation such that the negative Hofer length remains constant. Specifically, let \(H_M\) denote the momentum map, and let \(p_{\text{max}}\) denote the point where \(H_M\) attains its maximum. For every positive number \(\alpha\), we can choose the deformation such that the normalized deformed Hamiltonian remains greater than \(H_M(p_{\text{max}}) - \alpha\) on the set \(\{m \in M \mid H_M(m) > H_M(p_{\text{max}}) - \alpha\}\) and coincides with \(H_M\) outside this set.

Proof of Theorem 3.16 and Remark 3.17

We set the following notation.

\[
\phi_t : \mathbb{C} \to \mathbb{C}, \quad \phi_t(z) = e^{-2\pi it}z \quad \text{for} \ t \in \mathbb{R}.
\]

\[
\beta_\lambda : \mathbb{C} \to \mathbb{C}, \quad \beta_\lambda(z) = z + \lambda \quad \text{for} \ \lambda \in \mathbb{C}.
\]

Let \(p_{\text{max}}\) denote the point where the momentum map \(H_M\) attains its maximum. By the equivariant Darboux theorem, near \(p_{\text{max}}\), we can identify the manifold with \(\mathbb{C}^n\), the action with the product action

\[
(\phi_{k_1 t}) \times \ldots \times (\phi_{k_n t}),
\]

and the momentum map with the function

\[
z \mapsto H_M(p_{\text{max}}) - N(z),
\]

where

\[
N(z) = \pi \left( k_1 |z_1|^2 + \ldots + k_n |z_n|^2 \right).
\]

(We will use variants of the symbol \(N\) to denote variations and combinations of norm-square functions, and we will use variants of the symbol \(H\) to denote momentum maps.)

If \(k\) is an integer greater than 1, the non-effective circle action \(\phi_{kt}\) on \(\mathbb{C}\) has the deformation

\[
\phi_t^{(\lambda)} := \phi_t \circ \beta_{\lambda_{k-1}} \circ \phi_t \circ \ldots \circ \beta_{\lambda_2} \circ \phi_t \circ \beta_{\lambda_1} \circ \phi_t \circ \beta_{-\lambda_1} \ldots \beta_{-\lambda_{k-1}},
\]

parametrized by \((\lambda) = (\lambda_1, \ldots, \lambda_{k-1}) \in \mathbb{C}^{k-1}\) and generated by

\[
\mathcal{P}_t^{(\lambda)} := H + H_{-\lambda_{k-1}}^{-1} \phi_t^{-1} + H_{-\lambda_{k-2}}^{-1} \phi_t^{-1} \beta_{-\lambda_{k-1}}^{-1} \phi_t^{-1} + \ldots + H_{-\lambda_1}^{-1} \phi_t^{-1} \beta_{-\lambda_2}^{-1} \phi_t^{-1} \ldots \beta_{-\lambda_{k-2}}^{-1} \phi_t^{-1},
\]

where \(H(z) = -\pi |z|^2\). (We will use the symbol \((\lambda)\), with brackets, for a parameter in \(\mathbb{C}^{k-1}\) or in \(\mathbb{C}^{\sum (k_i - 1)}\) and the symbol \(\lambda\), without brackets, for a parameter in \(\mathbb{C}\).)

Applying such a deformation to each factor of (11.1), we get a family of deformations of the product action. This family is parametrized by elements of \(\mathbb{C}^{\sum (k_i - 1)}\). When we view the
parameter \( (\lambda) \) as an element \( ((\lambda)_1, \ldots, (\lambda)_n) \) of the \( n \)-fold product \( \mathbb{C}^{k_1 - 1} \times \cdots \times \mathbb{C}^{k_n - 1} \), the deformation is
\begin{equation}
(11.5) \quad \overline{\phi}_{(\lambda)_1} \times \cdots \times \overline{\phi}_{(\lambda)_n},
\end{equation}
and it is generated by
\begin{equation}
(11.6) \quad H_t^{(\lambda)}(z) := \overline{H}_t^{(\lambda)}(z_1) + \cdots + \overline{H}_t^{(\lambda)}(z_n).
\end{equation}

With the help of cut-off functions, this deformation can be plugged into the original manifold to obtain a family of deformations of the original circle action, parametrized by \( \lambda \)s in a neighbourhood \( D \) of the origin in \( \mathbb{C}^{\sum (k_i - 1)} \), and generated by a function that in a neighborhood of \( p \) can be identified with \( z \mapsto H_M(p_{max}) + H_t^{(\lambda)}(z) \) and that attains its maximum in that neighborhood. We give the details later.

We now turn to the relevant computation on \( \mathbb{C}^n \). We consider the function
\begin{equation}
(\lambda) \mapsto \int_0^1 \left( \max_{z \in \mathbb{C}^n} H_t^{(\lambda)}(z) \right) dt
\end{equation}
from \( \mathbb{C}^{\sum (k_i - 1)} \) to \( \mathbb{R} \). By (11.4) and (11.6), this function is smooth, is everywhere non-positive, and vanishes at the origin. We would like to show that its Hessian at the origin is negative definite.

By (11.6), it is enough to consider each of the functions \( (\lambda)_i \mapsto \int_0^1 \max_{z \in \mathbb{C}} H_t^{(\lambda)} dt \) separately. Omitting the index \( i \) and multiplying by \(-1/\pi\), we consider the function
\begin{equation}
(11.7) \quad (\lambda) \mapsto \int_0^1 \left( \min_{z \in \mathbb{C}} \frac{1}{\pi} \overline{H}_t^{(\lambda)}(z) \right) dt
\end{equation}
from \( \mathbb{C}^{k - 1} \) to \( \mathbb{R} \). By (11.4), it is smooth, everywhere non-negative, and vanishes at the origin. We need to show that its Hessian at the origin is positive definite.

The function (11.7) is a function of \( 2k - 2 \) variables, namely, of the real and imaginary parts of \( \lambda_1, \ldots, \lambda_{k-1} \). To compute its Hessian, we only need to vary two variables at a time. First suppose that the two variables are the real and imaginary parts of the same \( \lambda_i \), say, of \( \lambda_b \) where \( 1 \leq b \leq k - 1 \). We set \( \lambda_b = \lambda \) and \( \lambda_i = 0 \) for all \( i \neq b \). Setting \( b = k - a \), we have \( k = a + b \) with \( a \) and \( b \) positive integers. The original non-effective circle action on \( \mathbb{C} \) can be written as
\[ \phi_{kt} = \phi_{at} \circ \phi_{bt}, \]
and the deformed action in these directions is
\[ \overline{\phi}_{\lambda}^t = \phi_{at} \circ \beta_{\lambda} \circ \phi_{bt} \circ \beta_{-\lambda}. \]

We compute directly the Hamiltonian that generates the loop \( \overline{\phi}_t^\lambda \). The rotation \( \phi_{bt} \) is generated by the Hamiltonian \( bH \), where \( H(z) = -\pi |z|^2 \). Its conjugation \( \beta_{\lambda} \phi_{bt} \beta_{-\lambda} \) is generated by \( bH \beta_{-\lambda} \). Composing with \( \phi_{at} \), we get that \( \overline{\phi}_t^\lambda \) is generated by the Hamiltonian
\[ \overline{H}_t^\lambda = aH + bH \beta_{-\lambda} \phi_{-at}. \]
(Of course, this is just (11.4) when we set \( \lambda_i = \lambda \) and \( \lambda_i = 0 \) for \( i \neq a \).) So
\[
-\frac{1}{\pi} \mathcal{H}_t^\lambda(z) = a|z|^2 + b \left| e^{2\pi i a t} z - \lambda \right|^2
\]
\[
= (a + b)|z|^2 + b|\lambda|^2 - 2b \text{Re}(e^{-2\pi i a t} \bar{z} \lambda)
\]
\[
= kx^2 + ky^2 - 2b \text{Re}(e^{-2\pi i a t} \lambda) - 2by \text{Im}(e^{-2\pi i a t} \lambda) + b|\lambda|^2
\]
where \( z = x + iy \). The minimal value of this function as \( z \) varies in \( \mathbb{C} \) is
\[
b(1 - \frac{b}{k})|\lambda|^2.
\]
This is independent of \( t \), so integrating over \( t \in [0, 1] \) gives again \( b(1 - \frac{b}{k})|\lambda|^2 \). Because \( 1 \leq b \leq k - 1 \), the coefficient \( b(1 - \frac{b}{k}) \) is positive, so the Hessian of this function (as a function of the real and imaginary parts of \( \lambda \)) is positive definite, as required.

Now suppose that one of the two variables that we vary is the real or imaginary part of \( \lambda_i \) and the other is the real or imaginary part of \( \lambda_j \) where \( i \) and \( j \) are different, say, \( i = c \) and \( j = b + c \) where \( b \) and \( c \) are positive integers and \( b + c \) is smaller than \( k \). We set \( \lambda_c = \mu \), \( \lambda_{b+c} = \lambda \), and \( \lambda_i = 0 \) for all \( i \) other than \( c \) and \( b + c \).

Writing \( k = a + b + c \) with \( a, b, c \) positive integers, the deformed action is the family of loops of symplectomorphisms, parametrized by \( \lambda \) and \( \mu \) in \( \mathbb{C} \), given by
\[
\phi_t^{\lambda, \mu} = \phi_{at} \circ \beta_\lambda \circ \phi_{bt} \circ \beta_\mu \circ \phi_{ct} \circ \beta_{-\mu - \lambda}.
\]

We now compute the Hamiltonian that generates the deformed family \( \phi_t^{\lambda, \mu} \). The rotation \( \phi_{ct} \) is generated by the Hamiltonian \( cH \), where \( H(z) = -\pi|z|^2 \). Its conjugation \( \beta_\mu \phi_{ct} \beta_{-\mu} \) is generated by \( cH \beta_{-\mu} \). The composition \( \phi_{bt} \circ \beta_\mu \phi_{ct} \beta_{-\mu} \) is generated by \( bH + (cH \beta_{-\mu}) \phi_{-bt} \). Further conjugating by \( \beta_\lambda \), we get that \( \beta_\lambda \phi_{bt} \beta_\mu \phi_{ct} \beta_{-\mu - \lambda} \) is generated by \( bH \beta_{-\lambda} + cH \beta_{-\mu} \phi_{-bt} \beta_{-\lambda} \).

Finally, composing with \( \phi_{at} \), we get that \( \phi_t^{\lambda, \mu} \) is generated by the Hamiltonian
\[
\mathcal{H}_t^{\lambda, \mu}(z) = a|z|^2 + b \left| e^{2\pi i a t} z - \lambda \right|^2 + c \left| e^{2\pi i(a+b)t} z - e^{2\pi i b t} \lambda - \mu \right|^2.
\]
(Of course, this is just (11.4) when we set \( \lambda_c = \mu \), \( \lambda_{b+c} = \lambda \), and \( \lambda_i = 0 \) for all \( i \) other than \( c \) and \( b + c \).) So
\[
-\frac{1}{\pi} \mathcal{H}_t^{\lambda, \mu}(z) = a|z|^2 + b \left| e^{2\pi i a t} z - \lambda \right|^2 + c \left| e^{2\pi i(a+b)t} z - e^{2\pi i b t} \lambda - \mu \right|^2.
\]

The second summand is \( b \) times
\[
|z|^2 + |\lambda|^2 - 2x \text{Re}(e^{-2\pi i a t} \lambda) - 2y \text{Im}(e^{-2\pi i a t} \lambda),
\]
where \( z = x + iy \).

The third summand is \( c \) times
\[
|\lambda|^2 + |\mu|^2 + 2x \text{Re}(e^{-2\pi i b t} \lambda \mu)
\]
\[
- 2x \text{Re}(e^{-2\pi i a t} \lambda + e^{-2\pi i(a+b)t} \mu) - 2y \text{Im}(e^{-2\pi i a t} \lambda + e^{-2\pi i(a+b)t} \mu)).
\]
So we have

$$\frac{-1}{\pi}H^{\lambda,\mu}(z) = k(x^2 + y^2) - 2x(blah_x) - 2y(blah_y) + \text{(the rest)}$$

where

$$blah_x = \Re \left( (b + c)e^{-2\pi i at} + c e^{-2\pi (a+b)t} \mu \right),$$

$$blah_y = \Im \left( (b + c)e^{-2\pi i at} + c e^{-2\pi (a+b)t} \mu \right),$$

and

$$\text{the rest} = (b + c)|\lambda|^2 + c|\mu|^2 + 2e\Re \left( e^{-2\pi ibt} \lambda \mu \right).$$

By completing the squares, we find that the minimal value of (11.8) is

$$\frac{-1}{k}(blah_x)^2 - \frac{1}{k}(blah_y)^2 + \text{(the rest)}.$$

Explicitly,

$$\min_{\mathcal{C}} \left( -\frac{1}{\pi}H^{\lambda,\mu} \right) = (b + c)(1 - \frac{b + c}{k})|\lambda|^2 + c(1 - \frac{c}{k})|\mu|^2$$

$$+ 2e\Re \left( e^{-2\pi ibt} \times \text{something that is independent of } t \right).$$

So

$$\int_{0}^{1} \min_{\mathcal{C}} \left( -\frac{1}{\pi}H^{\lambda,\mu} \right) dt = (b + c)(1 - \frac{b + c}{k})|\lambda|^2 + c(1 - \frac{c}{k})|\mu|^2.$$

The matrix of second derivatives of this function, as a function of the real and imaginary parts of $\lambda$ and $\mu$, is diagonal. (Of course, the elements on the diagonal are exactly the ones that we found before: the diagonal elements corresponding to the real and imaginary parts of $\lambda_j$ are $j(1 - \frac{j}{k}).$)

Thus, the matrix of second derivatives of the function (11.7) at the origin is a diagonal matrix with positive entries. So the Hessian of the function (11.7) at the origin is positive definite, as required.

This completes our computation in $\mathbb{C}^n$. We now return to our manifold $M$. Fix a symplectomorphism $f : U \rightarrow M$ from a neighborhood $U$ of the origin in $\mathbb{C}^n$ onto a neighborhood of $p$ in $M$ such that $f(0) = p$ and such that $(f^*H_M)(z) = H_M(p) - N(z)$, where $N(\cdot)$ is given in (11.2). Given a positive number $\alpha$, we may choose $U$ such that $i(U) \subset \{ m \in M \mid H_M(m) > H_M(p_{\max}) - \alpha \}$.

Let $\varepsilon$ be a positive number that is sufficiently small so that the set $\{ z \in \mathbb{C}^n \mid |N(z)| \leq \varepsilon \}$ is contained in $U$. Then the subset of $M$ where $H_M$ is $\varepsilon$-close to its maximum $H_M(p)$ is contained in $f(U)$. (See the facts about Hamiltonian circle actions in §2.)

Let $\rho : [0, \infty) \rightarrow [0, 1]$ be a smooth function that is equal to one on $[0, \varepsilon/2]$ and is equal to zero outside $[0, \varepsilon]$. Use the same symbol, $\rho$, to denote the function $z \mapsto \rho(N(z))$ on $\mathbb{C}^n$. Then the function $H_{t\rho}^{\mu}(z)$ is equal to $H_{t\lambda}(z)$ when $N(z) \leq \varepsilon/2$ and is equal to $-N(z)$ when $N(z) > \varepsilon$.
The function on $M$

\[
H_{M_t}^{(\lambda)} := \begin{cases} 
H_M(p_{\text{max}}) + H_t^{(\rho \cdot \lambda)}(\cdot) \circ f^{-1}, & \text{if } H_M \text{ is } \tau \text{-close to its maximum } H_M(p_{\text{max}}) \\
H_M, & \text{elsewhere}
\end{cases}
\]

is smooth and it generates a deformation of our loop.

If the original momentum map $H_M$ is normalized, so is the deformed Hamiltonian (11.9).

The argument is similar to (5.18).

Let $D$ be a neighborhood of the origin in $\mathbb{C}^{(k_i-1)}$ that is small enough so that, for every $\lambda \in D$, the following facts are true.

- For every $t$, the subset of $\mathbb{C}^n$ where $H_t^{(\lambda)}(z)$ attains its maximum is contained in $\{z \mid N(z) \leq \tau/4\}$.
- This maximum is greater than $-\tau/4$.
- For every $\nu$ and $z$ such that $0 \leq \nu \leq 1$ and $\tau/2 \leq N(z) \leq \tau$, we have $H_t^{(\nu \cdot \lambda)}(z) \leq -\tau/4$ for all $t$.

This is possible because $H_t^{(\lambda)}(z)$ is smooth in the variables $\lambda$, $z$, and $t$, is quadratic in $z$, and is equal to $-N(z)$ when $\lambda = 0$.

Then, for every $\lambda$ and $t$, the functions $H_t^{(\rho \cdot \lambda)}(z)$ and $H_t^{(\lambda)}(z)$ have the same maximal value, and they attain it at the same (unique) value of $z$.

Denote by $\psi_t^{(\lambda)}$ the family of loops in $\text{Ham}(M, \omega)$ that are generated by the functions (11.9). By our choice of $D$, for every $\lambda \in D$, the positive Hofer length of the corresponding loop is

\[
\ell_+\left(\left\{\psi_t^{(\lambda)}\right\}\right) = H_M(p_{\text{max}}) + \int_0^1 \max_{z \in \mathbb{C}^n} H_t^{(\lambda)}(z) dt.
\]

By this and our computation in $\mathbb{C}^n$, the function

\[
\lambda \mapsto \ell_+\left(\left\{\psi_t^{(\lambda)}\right\}\right)
\]

from $D$ to $\mathbb{R}$ takes its maximal value at the origin and its Hessian at the origin is negative definite. This completes the proof of Theorem 3.16 and Remark 3.17. \qed

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