Abstract

Le Cam’s method, Fano’s inequality, and Assouad’s lemma are three widely used techniques to prove lower bounds for statistical estimation tasks. We propose their analogues under central differential privacy. Our results are simple, easy to apply and we use them to establish sample complexity bounds in several estimation tasks.

We establish the optimal sample complexity of discrete distribution estimation under total variation distance and $\ell_2$ distance. We also provide lower bounds for several other distribution classes, including product distributions and Gaussian mixtures that are tight up to logarithmic factors. The technical component of our paper relates coupling between distributions to the sample complexity of estimation under differential privacy.

1 Introduction

Statistical estimation tasks are often characterized by the optimal trade-off between the sample size and estimation error. There are two steps in establishing tight sample complexity bounds: An information theoretic lower bound on sample complexity and algorithmic upper bound that achieves it. Several works have developed general tools to obtain the lower bounds (e.g., [LC73, Ass83, IHM13, BR88, Dev87, HV94, CT06, SC19], and references therein), and three prominent techniques are Le Cam’s method, Fano’s inequality, and Assouad’s lemma. An exposition of these three methods and their connections is presented in [Yu97].

In several estimation tasks, individual samples have sensitive information that must be protected. This is particularly of concern in applications such as healthcare, finance, geo-location, etc. Privacy-preserving computation has been studied in various fields including database, cryptography, statistics and machine learning [War65, Dal77, DN03, WZ10, DJW13, CMS11]. Differential privacy (DP) [DMNS06], which allows statistical inference while preserving the privacy of the individual samples, has become one of the most popular notions of privacy [DMNS06, WZ10, DRV10, BLR13, MT07, DR14, KOV17]. Differential privacy has been adopted by the US Census Bureau for the 2020 census and several large technology companies, including Google, Apple, and Microsoft [EPK14, Dif17, DKY17].

Differential privacy [DMNS06]. Let $\mathcal{X}$ denote an underlying data domain of individual data samples and $\mathcal{X}^n$ be the set of all possible length-$n$ sequences over $\mathcal{X}$. For $x, y \in \mathcal{X}^n$, $d_{\text{Ham}}(x, y)$ is their

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The title of [Yu97], “Assouad, Fano, and Le Cam” is the inspiration for our title
Hamming distance, the number of coordinates they differ at. A (randomized) estimator \( \hat{\theta} : \mathcal{X}^n \to \Theta \) is \((\varepsilon, \delta)\)-differentially private (denoted as \((\varepsilon, \delta)\)-DP) if for any \( S \subseteq \Theta \), and all \( x, y \in \mathcal{X}^n \) with \( d_{\text{Ham}}(x, y) \leq 1 \), the following holds:

\[
\Pr\left( \hat{\theta}(x) \in S \right) \leq e^\varepsilon \cdot \Pr\left( \hat{\theta}(y) \in S \right) + \delta.
\] (1)

The case \( \delta = 0 \) is pure differential privacy and is simply denoted as \( \varepsilon \)-DP. We consider the problems of parameter estimation and goodness-of-fit (hypothesis testing) under differential privacy constraints.

**Setting.** Let \( \mathcal{P} \) be any collection of distributions over \( \mathcal{X}^n \), where \( n \) denotes the number of samples. Let \( \theta : \mathcal{P} \to \Theta \) be a parameter of the distribution that we want to estimate. Let \( \ell : \Theta \times \Theta \to \mathbb{R}_+ \) be a pseudo-metric that serves as our loss function for estimating \( \theta \). We now describe the minimax framework of parameter estimation, and hypothesis testing.

**Parameter estimation.** The risk of an estimator \( \hat{\theta} : \mathcal{X}^n \to \Theta \) under loss \( \ell \) is \( \max_{p \in \mathcal{P}} \mathbb{E}_{X \sim p} [\ell(\hat{\theta}(X), \theta(p))] \), the worst case expected loss of \( \hat{\theta} \) over \( \mathcal{P} \). Note here that \( X \in \mathcal{X}^n \), since \( p \) is a distribution over \( \mathcal{X}^n \).

The minimax risk of estimation under \( \ell \) for the class \( \mathcal{P} \) is

\[
R(\mathcal{P}, \ell) := \min_{\hat{\theta}} \max_{p \in \mathcal{P}} \mathbb{E}_{X \sim p} [\ell(\hat{\theta}(X), \theta(p))].
\]

The minimax risk of estimation under differentially private protocols is given by restricting \( \hat{\theta} \) to be differentially private. For \((\varepsilon, \delta)\)-DP, we study the following minimax risk:

\[
R(\mathcal{P}, \ell, \varepsilon, \delta) := \min_{\hat{\theta} \text{ is } (\varepsilon, \delta)\text{-DP}} \max_{p \in \mathcal{P}} \mathbb{E}_{X \sim p} [\ell(\hat{\theta}(X), \theta(p))].
\] (2)

For \( \delta = 0 \), the above minimax risk under \( \varepsilon \)-DP is denoted as \( R(\mathcal{P}, \ell, \varepsilon) \).

**Hypothesis testing.** Hypothesis testing can be cast in the framework of parameter estimation as follows. Let \( \mathcal{P}_1 \subset \mathcal{P} \), and \( \mathcal{P}_2 \subset \mathcal{P} \) be two disjoint subsets of distributions denoting the two hypothesis classes. Let \( \Theta = \{1, 2\} \), such that for \( p \in \mathcal{P}_i \), let \( \theta(p) = i \). For a test \( \hat{\theta} : \mathcal{X}^n \to \{1, 2\} \), and \( \ell(\theta, \theta') = \mathbb{I}\{\theta \neq \theta'\} = |\theta - \theta'| \), the error probability is the worst case risk under this loss function:

\[
P_e(\hat{\theta}, \mathcal{P}_1, \mathcal{P}_2) := \max_{i} \max_{p \in \mathcal{P}_i} \Pr\left( \hat{\theta}(X) \neq i \mid X \sim p \right) = \max_{i} \max_{p \in \mathcal{P}_i} \mathbb{E}_{X \sim p} \left[ |\hat{\theta}(X) - \theta(p)| \right].
\] (3)

**Organization.** The remainder of the paper is organized as follows. In Section 2.1, 2.2, and 2.3 we state the privatized versions of Le Cam, Fano, and Assouad’s method respectively. In Section 2.4 we discuss the applications of these results to several estimation tasks. In Section 2.5 we discuss some related and prior work. In Section 3 and 4 we prove the bounds for the applications in Section 2.4, for pure DP and approximate DP distribution estimation respectively. In Section 5, we prove the results from Section 2.1, 2.2, and 2.3.

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\(^{2}\)In the general setting, we are not assuming that the distribution over \( \mathcal{X}^n \) is from i.i.d. samples, although we will specialize to this case later.
2 Results

Le Cam’s method is used to establish lower bounds for hypothesis testing and functional estimation. Fano’s inequality, and Assouad’s lemma prove lower bounds for multiple hypothesis testing problems and can be applied to parameter estimation tasks such as estimating distributions. We present extensions of these results with differential privacy.

An observation. A coupling between distributions $p$ and $q$ over $\mathcal{X}^n$ is a joint distribution $(X,Y)$ over $\mathcal{X}^n \times \mathcal{X}^n$ whose marginals satisfy $X \sim p$ and $Y \sim q$. Our lower bounds are based on the following observation. Suppose there is a coupling $(X,Y)$ between distributions $p_1$ and $p_2$ over $\mathcal{X}^n$ with $\mathbb{E}[d_{Ham}(X,Y)] = D$. In other words, a draw from $p_1$ can be converted to a draw from $p_2$ by changing $D$ coordinates in expectation. Now any DP algorithm to distinguish $p_1$ and $p_2$ must obey (1), which we can apply for the $D$ changes in expectation to obtain the lower bounds.

2.1 DP Le Cam’s method

Le Cam’s method (Lemma 1 of [Yu97]) is widely used to prove lower bounds for composite hypothesis tests such as uniformity testing [Pan08], density estimation [Yu97, RSH18], and estimating functionals of distributions [JVHW15, WY16, PW19].

We use the expected Hamming distance between couplings of distributions in the two classes to obtain the following extension of Le Cam’s method with $(\epsilon, \delta)$-DP, which is an adaptation of a similar result in [ASZ18]. For the hypothesis testing problem described above, let $\mathcal{C}(\mathcal{P}_i)$ be the convex hull of distributions in $\mathcal{P}_i$, which are also families of distributions over $\mathcal{X}^n$.

**Theorem 1** $(\epsilon, \delta)$-DP Le Cam’s method. Let $p_1 \in \mathcal{C}(\mathcal{P}_1)$ and $p_2 \in \mathcal{C}(\mathcal{P}_2)$. Let $(X,Y)$ be a coupling between $p_1$ and $p_2$ with $D = \mathbb{E}[d_{Ham}(X,Y)]$. Then for $\epsilon \geq 0, \delta \geq 0$, any $(\epsilon, \delta)$-differentially private hypothesis testing algorithm $\hat{\theta}$ must satisfy

$$P_e(\hat{\theta}, \mathcal{P}_1, \mathcal{P}_2) \geq \frac{1}{2} \max \left\{ 1 - d_{TV}(p_1, p_2), \left(0.9e^{-10\epsilon D} - 10D\delta \right) \right\},$$

where $d_{TV}(p_1, p_2) := \sup_{A \subseteq \mathcal{X}^n} (p_1(A) - p_2(A)) = \frac{1}{2} \ell_1(p_1, p_2)$ is the total variation (TV) distance of $p_1$ and $p_2$. Choosing $p_1, p_2$ to make the RHS of (4) large gives better testing lower bounds.

The first term here is the original Le Cam’s result [LC73, LC86, Yu97, Can15] and the second term is a lower bound on the additional error due to privacy. A similar result (Theorem 1 in [ASZ18]), along with a suitable coupling was used in [ASZ18] to obtain the optimal sample complexity of testing discrete distributions. We defer the proof of this theorem to Section 5.1.

2.2 DP Fano’s inequality

Let

$$D_{KL}(p_i, p_j) := \sum_{x \in \mathcal{X}^n} p_i(x) \log \frac{p_i(x)}{p_j(x)}$$

be the KL divergence between (discrete) distributions $p_i$ and $p_j$. For continuous distributions, the summation is replaced with an integral. The following theorem, proved in Section 5.2, provides a lower bound on the risk of parameter estimation for a class of distributions $\mathcal{P}$ over $\mathcal{X}^n$ under $\epsilon$-DP.

**Theorem 2** $(\epsilon$-DP Fano’s inequality). Let $\mathcal{V} = \{p_1, p_2, ..., p_M\} \subseteq \mathcal{P}$ such that for all $i \neq j$,
(a) $\ell(\theta(p_i), \theta(p_j)) \geq \alpha$,
(b) $D_{KL}(p_i, p_j) \leq \beta$,
(c) there exists a coupling $(X, Y)$ between $p_i$ and $p_j$ such that $\mathbb{E}[d_{Ham}(X, Y)] \leq D$, then

$$R(\mathcal{P}, \ell, \varepsilon) \geq \max \left\{ \frac{\alpha}{2} \left( 1 - \frac{\beta + \log 2}{\log M} \right), 0.4 \min \left\{ \frac{M}{e^{10\varepsilon D}}, 1 \right\} \right\}. \quad (5)$$

Non-private Fano’s inequality (e.g., Lemma 3 of [Yu97]) requires only conditions (a) and (b) and provides the first term of the risk bound above. Now, if we consider the second term, which is the additional cost due to privacy, we would require $\exp(10\varepsilon D) \geq M$, i.e., $D \geq \log M/(10\varepsilon)$ to achieve a risk less than $0.4\alpha$. Therefore, for reliable estimation, the expected Hamming distance between any pair of distributions cannot be too small. In Corollary 4, we provide a corollary of this result to establish sample complexity lower bounds for several distribution estimation tasks.

**Remark.** Theorem 2 is a bound on the risk for pure differential privacy ($\delta = 0$). Our proof extends to $(\varepsilon, \delta)$-DP only for $\delta = O\left( \frac{1}{M} \right)$, which is not sufficient to establish meaningful bounds since in most problems we will require $M$ to be exponential in the problem parameters. In the next section, we provide a private analogue of Assouad’s method, which also works for $(\varepsilon, \delta)$-DP.

### 2.3 DP Assouad’s method

Our next result is a private version of Assouad’s lemma (Lemma 2 of [Yu97], and [Ass83]). Recall that $\mathcal{P}$ is a set of distributions over $\mathcal{X}^n$. Let $\mathcal{V} \subseteq \mathcal{P}$ be a set of distributions indexed by the hypercube $\mathcal{E}_k := \{\pm 1\}^k$, and the loss $\ell$ is such that

$$\forall u, v \in \mathcal{E}_k, \ell(\theta(p_u), \theta(p_v)) \geq 2\tau \cdot \sum_{i=1}^{k} I(u_i \neq v_i). \quad (6)$$

Assouad’s method provides a lower bound on the risk of estimation for distributions in $\mathcal{V}$, which is a lower bound for $\mathcal{P}$. For each coordinate $i \in [k]$, consider the following mixture distributions obtained by averaging over all distributions with a fixed value of the $i$th coordinate,

$$p_{+i} = \frac{2}{|\mathcal{E}_k|} \sum_{e \in \mathcal{E}_k, e_i = +1} p_e, \quad p_{-i} = \frac{2}{|\mathcal{E}_k|} \sum_{e \in \mathcal{E}_k, e_i = -1} p_e.$$  

Assouad’s lemma provides a lower bound on the risk by using (6) and considering the problem of distinguishing $p_{+i}$ and $p_{-i}$. Analogously, we prove the following privatized version of Assouad’s lemma by considering the minimax risk of a private hypothesis testing $\phi : \mathcal{X}^n \rightarrow \{-1, +1\}$ between $p_{+i}$ and $p_{-i}$. The detailed proof is in Section 5.4.

**Theorem 3** (DP Assouad’s method). \forall i \in [k], let $\phi_i : \mathcal{X}^n \rightarrow \{-1, +1\}$ be a binary classifier.

$$R(\mathcal{P}, \ell, \varepsilon, \delta) \geq \frac{\tau}{2} \cdot \sum_{i=1}^{k} \min_{\phi_i : i \in (\varepsilon, \delta)-DP} (\Pr_{X \sim p_{+i}}(\phi_i(X) \neq 1) + \Pr_{X \sim p_{-i}}(\phi_i(X) \neq -1)).$$

Moreover, if $\forall i \in [k]$, there exists a coupling $(X, Y)$ between $p_{+i}$ and $p_{-i}$ with $\mathbb{E}[d_{Ham}(X, Y)] \leq D$, then

$$R(\mathcal{P}, \ell, \varepsilon, \delta) \geq \frac{k\tau}{2} \cdot \left( 0.9e^{-10\varepsilon D} - 10D\delta \right). \quad (7)$$

The first bound is the classic Assouad’s Lemma and (7) is the loss due to privacy constraints. Once again note that (7) grows with decreasing $D$.
2.4 Applications

We now describe several applications of the theorems above.

Applications of Theorem 1. [ASZ18] developed a result similar to Theorem 1, which is used to establish sample complexity lower bounds for differentially private uniformity testing under total variation distance [ASZ18, ADR18], and for differentially private entropy and support size estimation [AKSZ18]. In this paper, we use Theorem 1 as a stepping stone to prove private Assouad’s method (Theorem 3).

We will apply Theorem 2 (private Fano’s inequality) and Theorem 3 (private Assouad’s lemma) to some classic distribution estimation problems. The results are summarized in Table 1 and Table 2. Before presenting the results, we describe the framework of minimax distribution estimation.

Distribution estimation framework. Let $Q$ be a collection of distributions over $X$, and for this $Q$, let $P = Q^n := \{q^n : q \in Q\}$ be the collection of $n$-fold distributions over $X^n$ induced by i.i.d. draws from a distribution over $Q$. The parameter space is $\Theta = Q$, where $\theta(q^n) = q$, and $\ell$ is a distance measure between distributions in $Q$. Let $\alpha > 0$ be a fixed parameter. The sample complexity, $S(Q, \ell, \alpha, \epsilon, \delta)$, is the smallest number of samples $n$ to make $R(Q^n, \ell, \epsilon, \delta) \leq \alpha$, i.e.,

$$S(Q, \ell, \alpha, \epsilon, \delta) = \min\{n : R(Q^n, \ell, \epsilon, \delta) \leq \alpha\}.$$  

When $\delta = 0$, we denote the sample complexity by $S(Q, \ell, \alpha, \epsilon)$. We will state our results in terms of sample complexity. The following corollary of Theorem 2 can be used to prove lower bounds on the sample complexity in this distribution estimation framework.

**Corollary 4 (\(\epsilon\)-DP distribution estimation).** Given $\epsilon > 0$, let $V = \{q_1, q_2, ..., q_M\} \subseteq Q$ be a set of distributions over $X$ with size $M$, such that for all $i \neq j$,

(a) $\ell(q_i, q_j) \geq 3\tau$,

(b) $D_{KL}(q_i, q_j) \leq \beta$,

(c) $d_{TV}(q_i, q_j) \leq \gamma$,

then

$$S(Q, \ell, \tau, \epsilon) = \Omega\left(\frac{\log M}{\beta} + \frac{\log M}{\gamma \epsilon}\right).$$

**Remark.** With only conditions (a) and (b), we obtain the first term of the sample complexity lower bound which is the original Fano’s bound for sample complexity.

We now present examples of distribution classes we consider.

**k-ary discrete distribution estimation.** Suppose $X = [k] := \{1, \ldots, k\}$, and $Q := \Delta_k$ is the simplex of $k$-ary distributions over $[k]$. We consider estimation in both total variation and $\ell_2$ distance.

**\((k,d)\)-product distributions.** Consider $X = [k]^d$, and let $Q := \Delta_k^d$ be the set of product distributions over $[k]^d$, where the marginal distribution on each coordinate is over $[k]$ and independent of the other coordinates. We study estimation under total variation distance. A special case of this is Bernoulli product distributions ($k = 2$), where each of the $d$ coordinates is an independent Bernoulli random variable.

**d-dimensional Gaussian mixtures.** Suppose $X = \mathbb{R}^d$, and $G_d := \{\mathcal{N}(\mu, I_d) : \|\mu\|_2 \leq R\}$ is the set of all Gaussian distributions in $\mathbb{R}^d$ with bounded mean and identity covariance. The bounded
| Problem                      | Upper Bounds                              | Lower Bounds                             |
|------------------------------|-------------------------------------------|------------------------------------------|
| *k*-ary                     | $\Theta\left(\frac{k}{\alpha^2} + \frac{k}{\alpha \varepsilon}\right)$ ([DHS15], Theorem 7) |                                          |
| *k*-ary, $\ell_2$ distance  | $O\left(\frac{1}{\alpha^2} + \min\left(\frac{\sqrt{\varepsilon}}{\alpha \varepsilon}, \frac{\log k}{\alpha \varepsilon}\right)\right)$ (Theorem 8) | $\Omega\left(\frac{1}{\alpha^2} + \min\left(\frac{\sqrt{\varepsilon}}{\alpha \varepsilon}, \frac{\log(ka^2)}{\alpha \varepsilon}\right)\right)$ (Theorem 8) |
| Product distribution        | $O\left(kd\log\left(\frac{kd}{\alpha}\right)\left(\frac{1}{\alpha^2} + \frac{1}{\alpha \varepsilon}\right)\right)$ ([BKSW19]) | $\Omega\left(kd\left(\frac{1}{\alpha^2} + \frac{1}{\alpha \varepsilon}\right)\right)$ (Theorem 10) |
| Gaussian mixtures            | $O\left(kd\log\left(\frac{dR}{\alpha}\right)\left(\frac{1}{\alpha^2} + \frac{1}{\alpha \varepsilon}\right)\right)$ ([BKSW19]) | $\Omega\left(kd\left(\frac{1}{\alpha^2} + \frac{1}{\alpha \varepsilon}\right)\right)$ (Theorem 11) |

Table 1: Summary of the sample complexity bounds for $\varepsilon$-DP discrete distribution estimation. Unless mentioned, the bounds are all for estimation under total variation distance.

| Problem                      | Upper Bounds                              | Lower Bounds                             |
|------------------------------|-------------------------------------------|------------------------------------------|
| *k*-ary                     | $O\left(\frac{k}{\alpha^2} + \frac{k}{\alpha \varepsilon}\right)$ (Theorem 12) | $\Omega\left(\frac{k}{\alpha^2} + \frac{k}{\alpha (\varepsilon + \delta)}\right)$ (Theorem 12) |
| *k*-ary, $\ell_2$ distance  | $O\left(\frac{1}{\alpha^2} + \min\left(\frac{\sqrt{\varepsilon}}{\alpha \varepsilon}, \frac{\log k}{\alpha \varepsilon}\right)\right)$ (Theorem 13) | $\Omega\left(\frac{1}{\alpha^2} + \min\left(\frac{\sqrt{\varepsilon}}{\alpha (\varepsilon + \delta)}, \frac{1}{\alpha^2 (\varepsilon + \delta)}\right)\right)$ (Theorem 13) |
| Product distribution $(k = 2)$ | $O\left(d\log\left(\frac{d}{\alpha}\right)\left(\frac{1}{\alpha^2} + \frac{1}{\alpha \varepsilon}\right)\right)$ ([BKSW19]) | $\Omega\left(\frac{d}{\alpha^2} + \frac{d}{\alpha (\varepsilon + \delta)}\right)$ (Theorem 15) |

Table 2: Summary of the sample complexity bounds for $(\varepsilon, \delta)$-DP discrete distribution estimation. Unless mentioned, the bounds are all for estimation under total variation distance.

mean assumption is unavoidable, since by [BKSW19], it is not possible to learn a single Gaussian distribution under pure DP without this assumption. We consider

$$Q = \mathcal{G}_{k,d} := \left\{ \sum_{j=1}^{k} w_j p_j : w_j \geq 0, w_1 + \ldots + w_k = 1, p_j \in \mathcal{G}_d \right\},$$

the collection of mixtures of $k$ distributions from $\mathcal{G}_d$.

**Applications of Theorem 2.** We apply Corollary 4 and obtain sample complexity lower bounds for the tasks mentioned above under pure differential privacy.

*k*-ary distribution estimation. Without privacy constraints, the sample complexity of *k*-ary discrete distributions under total variation, and $\ell_2$ distance is $\Theta\left(k/\alpha^2\right)$ and $\Theta\left(1/\alpha^2\right)$ respectively, achieved by the empirical estimator. Under $\varepsilon$-DP constraint, [DHS15] showed that the sample complexity of Laplace mechanism [DMNS06] for total variation distance is $O\left(\frac{k}{\alpha^2} + \frac{k}{\alpha \varepsilon}\right)$ samples. In Theorem 7,
we establish the sample complexity of this problem by providing a lower bound that matches this upper bound.

Under $\ell_2$ distance, in Theorem 8 we design estimators and establish their optimality whenever $\alpha < k^{-\frac{1}{2}}$ or $\alpha \geq k^{-\left(\frac{1}{2}-0.001\right)}$, which contains almost all the parameter range. Note that under $\ell_2$ distance, estimation without privacy has sample complexity independent of $k$, whereas an unavoidable logarithmic dependence on $k$ is introduced due to privacy requirements. The results are presented in Section 3.1.

$(k,d)$-product distribution estimation. For $(k,d)$-product distribution estimation under $\varepsilon$-DP, $[BKSW19]$ proposed an algorithm that uses $O\left(\frac{kd}{\alpha^2} + \frac{kd}{\alpha \varepsilon}\right)$ samples. In this paper, we present a lower bound of $\Omega\left(\frac{kd}{\alpha^2} + \frac{kd}{\alpha \varepsilon}\right)$, which matches their upper bound up to logarithmic factors. For Bernoulli product distributions, $[KLSU19]$ proved a lower bound of $\Omega\left(\frac{d}{\alpha \varepsilon}\right)$ under $(\varepsilon, \frac{3}{64n})$-DP, which is naturally a lower bound for pure DP. The details are presented in Section 3.2.

Estimating Gaussian mixtures. $[BKSW19]$ provided an upper bound of $\tilde{O}\left(\frac{kd}{\alpha^2} + \frac{kd}{\alpha \varepsilon}\right)$ samples. Without privacy, a tight bounds of $\Omega(kd/\alpha^2)$ was shown in $[SOAJ14, DK14, ABDH+18]$. In this paper, we prove a lower bound of $\Omega\left(\frac{kd}{\alpha^2} + \frac{kd}{\alpha \varepsilon}\right)$, which matches the upper bound up to logarithmic factors. For the special case of estimating a single Gaussian ($k = 1$), a lower bound of $n = \Omega\left(\frac{d}{\alpha \varepsilon \log d}\right)$ was given in $[KLSU19]$ for the weaker notion of $(\varepsilon, \frac{3}{64n})$-DP. Our lower bound for $k = 1$ improves their result. The details are given in Section 3.3.

Applications of Theorem 3. As remarked earlier, Theorem 2 only works for pure DP (or approximate DP with very small $\delta$). Assouad’s lemma can be used to obtain lower bounds for distribution estimation under $(\varepsilon, \delta)$-DP. For $k$-ary distribution estimation under TV distance, we get a lower bound of $\Omega\left(\frac{k}{\alpha^2} + \frac{k}{\alpha \varepsilon (\varepsilon + \delta)}\right)$. The lower bound shows that even up to $\delta = O(\varepsilon)$, the lower bound for $(\varepsilon, \delta)$-DP is the same as that of $\varepsilon$-DP.

For Bernoulli ($k = 2$) product distributions, the lower bound $\Omega\left(\frac{d}{\alpha^2} + \frac{d}{\alpha \varepsilon}\right)$ in $[KLSU19]$ obtained using fingerprinting holds for small values of $\delta = O(1/n)$. We can obtain the same lower bound for $\delta = O(\varepsilon)$, which is a larger range in most cases. We describe the details about these applications in Section 4.

2.5 Related and prior work

2.5.1 Private distribution estimation

Protecting privacy generally comes at the cost of performance degradation. Previous literature has studied various problems and established utility privacy trade-off bounds, including distribution estimation, hypothesis testing, property estimation, empirical risk minimization, etc $[CMS11, Lei11, BST14, DHS15, CDK17, ASZ18, KLSU19, ADR18, AKSZ18]$.

There has been significant recent interest in differentially private distribution estimation. $[DHS15]$ gives upper bounds for privately learning $k$-ary distributions under total variation distance. $[KLSU19, BKSW19, KV18]$ focus on high-dimensional distributions, including product distributions and Gaussian distributions. As discussed in the previous section, our proposed lower bounds improve upon their lower bounds. $[BNSV15]$ studies the problem of privately estimating a distribution in Kolmogorov distance, which is weaker than total variation distance. Upper and lower bounds for
private estimation of the mean of product distributions in $\ell_\infty$ distance, heavy tailed distributions, and Markov Random fields are studied in [BDMN05, DMNS06, SU17, BUV18, KSU20, ZKKW20].

Several estimation tasks have also been considered under the distributed notion of local differential privacy, e.g., [War65, KLN+11, EPK14, DJW13, KBR16, WHW+16, YB18, GR18, ACT19, ASZ19, AS19, ACFT19, She17].

2.5.2 Lower bounds in differential privacy

Several methods have been proposed in the literature to prove lower bounds under differential privacy. These include packing argument [HT10, Vad17, Val12, BKN10], fingerprinting [BNSV15, SU17, SU15, BSU17, BUV18, KLSU19] and coupling based arguments [ASZ18, KV18].

Packing argument is a geometric approach to prove lower bounds under pure differential privacy. Roughly speaking, packing argument tells that if there exists a collection of $M$ datasets such that the Hamming distance between any pair of datasets is at most $d$, any $\varepsilon$-DP algorithm which has disjoint output when the input dataset is different must satisfy $\varepsilon = \Omega\left(\frac{\log M}{d}\right)$. However, this packing lower bound [DR14] only considers the worst case Hamming distance between deterministic datasets. In the statistical setting where the data comes from an underlying distribution, applying this method usually provides a loose bound. For example, in the $k$-ary distribution estimation problem, this method can only give a lower bound of $n = \Omega\left(\frac{k \log (1/\alpha)}{\varepsilon}\right)$ instead of our $n = \Omega\left(\frac{k}{\alpha \varepsilon}\right)$ lower bound.

Fingerprinting [BNSV15, DSS+15, SU17, SU15, BSU17, BUV18, KLSU19, CWZ19] is another technique to prove lower bounds under DP. [SU17] derives a tight lower bound for attribute mean estimation in the database setting. [KLSU19] uses this technique to prove lower bounds on estimating Bernoulli product distributions and Gaussian distributions. However, it is not clear how to apply this method suppose the underlying distribution is a $k$-ary distribution. Besides, it can only be used under approximate DP when $\delta = O(1/n)$, where $n$ is the sample size.

Recently, coupling argument has been used to obtain differentially private lower bounds, e.g., [ASZ18] proves tight sample complexity lower bounds of DP identity testing problem under total variation distance, and [KV18] proves lower bounds of DP estimating confidence intervals of the mean of a one-dimensional Gaussian distribution. In [KV18], they construct two different Gaussian distributions with mean $\mu_0$ and $\mu_1$, and point out that the probability of $\mu_1$ in the estimated confidence interval of $\mu_0$ must be connected with $|\mu_0 - \mu_1|$ by a coupling argument. For both papers, the coupling argument implies that it is hard to differentially privately distinguish between two distributions, supposing there exists a coupling with small expected Hamming distance. This method can be viewed as another form of private Le Cam’s method (Theorem 1) and it can only be applied where binary hypothesis testing is involved. [CKM+19] provide a different coupling that provides stronger bounds than [ASZ18] for simple hypothesis testing.

[DJW13] derives analogues of Le Cam, Assouad, and Fano in the local model of differential privacy, and uses them to establish lower bounds for several problems under local differential privacy. [ACT19] proves lower bounds for various testing and estimation problems under local differential privacy using a notion of chi-squared contractions based on Le Cam’s method and Fano’s inequality.
3 $\varepsilon$-DP distribution estimation

In this section, we use Corollary 4 to prove sample complexity lower bounds for various $\varepsilon$-DP distribution estimation problems. The general idea is to construct a subset of distributions in $Q$ such that they are close in both $TV$ distance and $KL$ divergence while being separated in the loss function $\ell$. The larger the subsets we construct, the better the lower bounds we can get. In Section 3.1, we derive sample complexity lower bounds for $k$-ary distribution estimation under both $TV$ and $\ell_2$ distance that are tight up to constant factors. Tight sample complexity lower bounds up to logarithmic factors for $(k,d)$-product distributions and $d$-dimensional Gaussian mixtures are derived in Section 3.2 and 3.3 respectively.

Note that Corollary 4 requires designing a packing of distributions that any two pairs in the packing are at least $3\tau$ apart in $\ell$. A standard method to construct such distributions is using results from coding theory.

We start with some definitions. A binary code of length $k$ is a set $C \subseteq \{0,1\}^k$, and each $c \in C$ is a codeword. The weight of a binary codeword $c \in C$ is defined as $wt(c) = |\{i : c_i = 1\}|$, which is the number of 1's in $c$. We call $C$ a constant weight code if each $c \in C$ has the same weight.

The minimum distance of a code $C$ is the smallest Hamming distance between two codewords in $C$.

We now present some results on the existence of codes with certain properties, and prove them in Section 6.

Lemma 5. Let $l$ be an integer at most $k/2$ and at least 20. There exists a constant weight binary code $C$ which has code length $k$, weight $l$, minimum distance $l/4$ with $|C| \geq \left(\frac{k^{2/7}}{2^{7/8}}\right)^{7/8}$.

An $h$-ary code of length-$k$ is a subset of $\{0,1,\ldots,h-1\}^k$.

Lemma 6. There exists an $h$-ary code $H$ with code length $d$ and minimum Hamming distance $\frac{d^2}{2}$, which satisfies that $|H| \geq \left(\frac{h}{16}\right)^{\frac{d^2}{2}}$.

3.1 $k$-ary distribution estimation

We establish the sample complexity of $\varepsilon$-DP $k$-ary distribution estimation under $TV$ and $\ell_2$ distance.

Theorem 7. The sample complexity of $\varepsilon$-DP $k$-ary distribution estimation under $TV$ distance is

$$S(\Delta_k, d_{TV}, \alpha, \varepsilon) = \Theta\left(\frac{k}{\alpha^2} + \frac{k}{\alpha\varepsilon}\right).$$ \hspace{1cm} (8)

Theorem 8. The sample complexity of $\varepsilon$-DP $k$-ary distribution estimation under $\ell_2$ distance is

$$S(\Delta_k, \ell_2, \alpha, \varepsilon) = \Theta\left(\frac{1}{\alpha^2} + \frac{\sqrt{k}}{\alpha\varepsilon}\right), \text{ for } \alpha < \frac{1}{\sqrt{k}}, \text{ and}$$

$$\Omega\left(\frac{1}{\alpha^2} + \frac{\log(k\alpha^2)}{\alpha^2\varepsilon}\right) \leq S(\Delta_k, \ell_2, \alpha, \varepsilon) \leq O\left(\frac{1}{\alpha^2} + \frac{\log k}{\alpha^2\varepsilon}\right) \text{ for } \alpha > \frac{1}{\sqrt{k}}. \hspace{1cm} (9)$$

For $\ell_2$ loss, our bounds are tight within constant factors when $\alpha < \frac{1}{\sqrt{k}}$ or $\alpha > k^{-(\frac{1}{2}-0.001)}$. \hspace{1cm} (10)
3.1.1 Total variation distance

In this section, we derive the sample complexity of $\varepsilon$-DP $k$-ary distribution estimation under $TV$ distance, which is stated in Theorem 7.

Upper bound: [DHS15] provides an upper bound based on Laplace mechanism [DMNS06]. We state the algorithm and a proof here since we use it for estimation under $\ell_2$ distance.

Given a $X^n$ from an unknown distribution $p$ over $[k]$. Let $M_p(X^n)$ be the number of appearances of $x$ in $X^n$. Let $p^{\text{erm}}$ be the empirical estimator where $p^{\text{erm}}(x):=\frac{M_p(x)}{n}$. We note that changing one $X_i$ in $X^n$ can change at most two coordinates of $p^{\text{erm}}$, each by at most $\frac{1}{n}$, and thus changing one $X_i$ changes the $p^{\text{erm}}$ by at most $2/n$ in total variation distance. Therefore, by [DMNS06], adding a Laplace noise of parameter $2/n\varepsilon$ to each coordinate of $p^{\text{erm}}$ makes it $\varepsilon$-DP. For $x \in [k]$, let

$$h(x) = p^{\text{erm}}(x) + \text{Lap}\left(\frac{2}{n\varepsilon}\right),$$

where $\text{Lap}(\beta)$ is a Laplace random variable with parameter $\beta$. The final output $\hat{p}$ is the projection of $h$ on the simplex $\Delta_k$ in $\ell_2$ distance. The expected $\ell_2$ loss between $h$ and $p$ can be upper bounded by

$$(\mathbb{E}[\|h - p\|_2^2])^2 \leq \mathbb{E}\left[\|h - p\|_2^2\right] \leq \mathbb{E}\left[\|p^{\text{erm}} - p\|_2^2\right] + \mathbb{E}\left[\|h - p^{\text{erm}}\|_2^2\right],$$

where the first inequality comes from the Jensen’s inequality and the second inequality comes from the triangle inequality.

The first term $\mathbb{E}\left[\|p^{\text{erm}} - p\|_2^2\right]$ is upper bounded by $\frac{1}{n}$. For the second term, note that $\mathbb{E}\left[\|h - p^{\text{erm}}\|_2^2\right] = \sum_{i=1}^{k} \mathbb{E}[Z_i^2]$, where $\forall i, Z_i \sim \text{Lap}(\frac{2}{n\varepsilon})$. By the variance of Laplace distribution, we have $\mathbb{E}\left[\|p^{\text{erm}} - h\|_2^2\right] = O\left(\frac{k}{n\varepsilon^2}\right)$. Therefore $\mathbb{E}[\|h - p\|_2] \leq O\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{k}}{n \varepsilon}\right)$.

Note that since $\Delta_k$ is convex, $\|\hat{p} - p\|_2 \leq \|h - p\|_2$. Finally, by Cauchy-Schwarz Inequality, $\mathbb{E}[\|\hat{p} - p\|_1] \leq \sqrt{k} \cdot \mathbb{E}[\|\hat{p} - p\|_2] \leq \sqrt{k} \cdot \mathbb{E}[\|h - p\|_2] = O\left(\sqrt{\frac{k}{n}} + \frac{k}{n \varepsilon}\right)$. Therefore $\mathbb{E}[\|\hat{p} - p\|_1] \leq \alpha$ when $n = O\left(\frac{k}{\alpha^2} + \frac{k}{\alpha \varepsilon}\right)$.

Lower bound. We will construct a large set of distributions such that the conditions of Corollary 4 hold. Suppose $\alpha < 1/48$. Applying Lemma 5 with $l = k/2$, there exists a constant weight binary code $C$ of weight $k/2$, and minimum distance $k/8$, and $|C| > 2^{7k/128}$. Associate with each codeword $c \in C$, a distribution $p_c$ over $[k]$ is defined as follows:

$$p_c(i) = \begin{cases} \frac{1 + 24\alpha}{k}, & \text{if } c_i = 1, \\ \frac{1 - 24\alpha}{k}, & \text{if } c_i = 0. \end{cases}$$

We choose $\mathcal{V} = \{p_c : c \in C\}$ to apply Corollary 4. By the minimum distance property, any two distributions in $\mathcal{V}$ have a total variation distance of at least $24\alpha/k \cdot k/8 = 3\alpha$, and at most $24\alpha$. Furthermore, by using $\log(1 + x) \leq x$, we can bound the KL divergence between distributions by their $\chi^2$ distance,

$$d_{KL}(p, q) \leq \chi^2(p, q) = \sum_{x=1}^{k} \frac{(p(x) - q(x))^2}{q(x)} < 10000\alpha^2.$$

Setting $\tau = \alpha$, $\gamma = 24\alpha$, and $\beta = 10000\alpha^2$, and using $\log M > 7k/64$ in Corollary 4, we obtain $S(\Delta_k, d_{TV}, \alpha, \varepsilon) = \Omega\left(\frac{k}{\alpha^2} + \frac{k}{\alpha \varepsilon}\right)$. 

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3.1.2 $\ell_2$ distance

In this section, we derive the sample complexity of $\varepsilon$-DP $k$-ary distribution estimation under $\ell_2$ distance, which is stated in Theorem 8.

**Upper bound:** We use the same algorithm as in Section 3.1.1. Following the same argument as in Section 3.1.1, the square of expected $\ell_2$ loss of $\hat{p}$ can be upper bounded by

$$(E[\|\hat{p} - p\|_2^2])^2 \leq E\left[\|h - p\|_2^2\right] \leq E\left[\|p^{\text{ERM}} - p\|_2^2\right] + E\left[\|h - p^{\text{ERM}}\|_2^2\right] = O\left(\frac{1}{n} + \frac{k}{n^2\varepsilon^2}\right).$$

Since $\Delta_k$ is convex, we have $\|\hat{p} - p\|_2 \leq \|h - p\|_2$. Moreover, the following lemma gives another bound for $\|\hat{p} - p\|_2$ (See Corollary 2.3 in [Bas19]).

**Lemma 9.** Let $L \subset \mathbb{R}^d$ be a symmetric convex body of $k$ vertices $\{a_j\}_{j=1}^k$, and let $y \in L$ and $\tilde{y} = y + z$ for some $z \in \mathbb{R}^d$. Let $\hat{y} = \arg\min_{w \in L} \|w - \tilde{y}\|_2^2$. Then we must have

$$\|\hat{y} - \hat{y}\|_2^2 \leq 4 \max_{j \in [k]} \langle (z, a_j) \rangle.$$

From the lemma, we have $E\left[\|\hat{p} - h\|_2^2\right] \leq 4E\left[\max_{j \in [k]} |Z_j|\right]$, where $\forall j \in [k]$, $Z_j \sim \text{Lap}(\frac{2}{\alpha \varepsilon})$. Note that $E[\max|Z_j|] = O\left(\frac{\log k}{n \varepsilon}\right)$ due to the tail bound of Laplace distribution. We have $E[\|\hat{p} - p\|_2^2] = O\left(\frac{1}{n} + \frac{\log k}{n \varepsilon}\right)$. Combined with the previous analysis, $E[\|\hat{p} - p\|_2^2] = O\left(\frac{1}{n} + \min\left(\frac{k}{n^2 \varepsilon^2}, \frac{\log k}{n \varepsilon}\right)\right)$. Therefore $E[\|\hat{p} - p\|_2] \leq \frac{1}{10}\alpha$ when $n = O\left(\frac{1}{\alpha^2} + \min\left(\frac{\sqrt{k}}{\alpha \varepsilon}, \frac{\log k}{\alpha^2 \varepsilon}\right)\right)$.

**Lower bound:** We first consider the case when $\alpha < \frac{1}{\sqrt{k}}$, where we can derive the lower bound simply by a reduction. By Cauchy-Schwarz inequality, for any estimator $\hat{p}$, $E[\|\hat{p} - p\|_1] \leq \sqrt{k} \cdot E[\|\hat{p} - p\|_2]$. Therefore $S(\Delta_k, \ell_2, \alpha, \varepsilon) \geq S(\Delta_k, d_{TV}, \sqrt{k} \alpha, \varepsilon)$, which gives us $S(\Delta_k, \ell_2, \alpha, \varepsilon) = O\left(\frac{1}{\alpha^2} + \frac{\sqrt{k}}{\alpha \varepsilon}\right)$.

Now we consider $\alpha \geq \frac{1}{\sqrt{k}}$. Note that it is enough if we prove the lower bound of $\Omega\left(\frac{\log (\alpha^2 k)}{\alpha^2 \varepsilon}\right)$, since $\Omega\left(\frac{1}{\alpha^2}\right)$ is the sample complexity of non-private estimation problem for all range of $\alpha$. Similarly, we follow Corollary 4, except that we need to construct a different set of distributions.

Without loss of generality, we assume $\alpha < 0.1$. Now we use the codebook in Lemma 5 to construct our distribution set. We fix weight $l = \left\lceil \frac{1}{500 \alpha^2} \right\rceil$. Note that for any $x > 2$, $\lfloor x \rfloor > \frac{x}{2}$. Then we have $\frac{1}{10000} < \lfloor l \rfloor \leq \frac{1}{500 \alpha^2}$ since $\frac{1}{500 \alpha^2} > 2$. Therefore we get codebook $\mathcal{C}$ with $|\mathcal{C}| \geq (k \alpha^2)^{\frac{1}{500 \alpha^2}}$. Given $c \in \mathcal{C}$, we construct the following distribution $p_c$ in $\Delta_k$:

$$p_c(i) = \frac{1}{l} c_i.$$

We use $\mathcal{V}_k = \{p_c : c \in \mathcal{C}\}$ to denote the set of all these distributions. It is easy to check that $\forall p \in \mathcal{V}_k$ is a valid distribution. Moreover, for any pair of distributions $p, q \in \mathcal{V}_k$, we have $\|p - q\|_2 > \frac{1}{2\sqrt{l}} = \Omega(\alpha)$.

For any pair $p, q \in \mathcal{V}_k$, $d_{TV}(p, q) \leq 1$, which is a naive upper bound for TV distance. Finally by setting $\ell$ in Corollary 4 to be $\ell_2$ distance, we have $S(\Delta_k, \ell_2, \alpha, \varepsilon) = \Omega\left(\frac{\log |\mathcal{C}|}{\varepsilon}\right) = \Omega\left(\frac{\log (k \alpha^2)}{\alpha^2 \varepsilon}\right)$. 

\[\text{11}\]
3.2 Product distribution estimation

Recall that $\Delta_{k,d}$ is the set of all $(k, d)$-product distributions. [BKSW19] proves an upper bound of $O\left(kd \log \left(\frac{kd}{\alpha^2} \left(\frac{1}{\alpha^2} + \frac{1}{\alpha \varepsilon}\right)\right)\right)$. We prove a sample complexity lower bound for $\varepsilon$-DP $(k, d)$-product distribution estimation in Theorem 10, which is optimal up to logarithmic factors.

**Theorem 10.** The sample complexity of $\varepsilon$-DP $(k, d)$-product distribution estimation satisfies

$$S(\Delta_{k,d}, d_{TV}, \alpha, \varepsilon) = \Omega\left(\frac{kd}{\alpha^2} + \frac{kd}{\alpha \varepsilon}\right).$$

**Proof.** We start with the construction of the distribution set. First we utilize the same binary code as in Lemma 5 with weight $l = \frac{k}{2}$. Let $h := |C|$ denote the size of the codebook. Given $j \in [h]$, we construct the following $k$-ary distribution $p_j$ based on $c_j \in C$:

$$p_j(i) = \frac{1}{k} + \frac{\alpha}{k \sqrt{d}} \cdot \mathbb{I}(c_{j,i} = 1),$$

where $c_{j,i}$ denotes the $i$-th coordinate of $c_j$.

Now we have designed a set of $k$-ary distributions of size $h = \Omega\left(2^\frac{7k}{128}\right)$. To construct a set of product distributions, we use the codebook construction in Lemma 6 to get an $h$-ary codebook $H$ with length $d$ and minimum hamming distance $d/2$. Moreover, $|H| \geq \left(\frac{h}{d}\right)^d$.

Now we can construct the distribution set of $(k, d)$-product distributions. Given $b \in H$, define

$$p_b = p_{b_1} \times p_{b_2} \times \cdots \times p_{b_d}.$$  

Let $\mathcal{V}_{k,d}$ denote the set of distributions induced by $H$. We want to prove that $\forall p \neq q \in \mathcal{V}_{k,d}$,

$$d_{TV}(p, q) \geq C\alpha, \quad (11)$$

$$D_{KL}(p, q) \leq 4\alpha^2, \quad (12)$$

for some constant $C$. If we suppose these two inequalities hold, using (12), by Pinsker’s Inequality, we get $d_{TV}(p, q) \leq \sqrt{2D_{KL}(p, q)} \leq 2\sqrt{2}\alpha$. Then using Corollary 4, we can get

$$S(\Delta_{k,d}, d_{TV}, \alpha, \varepsilon) = \Omega\left(\frac{kd}{\alpha^2} + \frac{kd}{\alpha \varepsilon}\right).$$

Now it remains to prove (11) and (12). For (12), note that for any distribution pair $p, q \in \mathcal{V}_{k,d}$,

$$D_{KL}(p, q) \leq d \cdot \max_{i,j \in [k]} d_{KL}(p_i, p_j) \leq 4\alpha^2,$$

where the equality comes from the additivity of $KL$ divergence for independent distributions.

Next we prove (11). For any $b \in H$ and $\forall i \in [k]$, define set

$$S_i = \{ j \in [k] : c_{b,j} = 1 \},$$

which contains the locations of $+1$’s in the code at the $i$th coordinate of $b$. Based on this, we define a product distribution

$$p'_b = \prod_{i=1}^{d} B(\mu_i),$$
where \( \mu_i = \sum_{j \in S_i} p_b(j) \) and \( B(t) \) is a Bernoulli distribution with mean \( t \). For any \( b' \neq b \in \mathcal{H} \), we define

\[
p_b' = \prod_{i=1}^{d} B(\mu'_i),
\]

where \( \mu'_i = \sum_{j \in S_i} p_{b'}(j) \). Then we have:

\[
d_{TV}(p_b', p_b') \leq d_{TV}(p_b, p_b),
\]

since \( p_b' \) and \( p_b' \) can be viewed as a post processing of \( p_b \) and \( p_b \) by mapping elements in \( S_b \) to 1 and others to 0 at the \( i \)-th coordinate. Moreover, we have \( d_{\text{Ham}}(b, b') \geq \frac{d}{2} \), and \( \forall i \), if \( b_i \neq b'_i \), we have \( d_H(c_{b_i}, c_{b'_i}) > \frac{d}{8} \). By the definition of \( p_i \)'s, we have

\[
\|\mu_1 - \mu_2\|^2 \geq \frac{d}{2} \times \left( \frac{k}{8} \times \frac{\alpha}{k\sqrt{d}} \right)^2 = \frac{\alpha^2}{128}.
\]

By Lemma 6.4 in [KLSU19], there exists a constant \( C \) such that \( d_{TV}(p_b', p_b') \geq C\alpha \), proving (11).

### 3.3 Gaussian mixtures estimation

Recall \( \mathcal{G}_d = \{ \mathcal{N}(\mu, I_d) : \|\mu\|_2 \leq R \} \) is the set of \( d \)-dimensional spherical Gaussians with unit variance and bounded mean and \( \mathcal{G}_{k,d} = \{ p : p \) is a \( k \)-mixture of \( \mathcal{G}_d \} \) consists of mixtures of \( k \) distributions in \( \mathcal{G}_d \). [BKSW19] proves an upper bound of \( \bar{O}\left( \frac{kd}{\alpha^2} + \frac{d}{\alpha \varepsilon} \right) \) for estimating \( k \)-mixtures of Gaussians. We provide a sample complexity lower bound for estimating mixtures of Gaussians in Theorem 11, which matches the upper bound up to logarithmic factors.

**Theorem 11.** Given \( k \leq d \) and \( R \geq \sqrt{64 \log \left( \frac{8k}{\alpha} \right)} \), or \( k \geq d \) and \( R \geq \left( k \right)^{\frac{3}{2}} \cdot \sqrt{16d \log \left( \frac{8k}{\alpha} \right)} \),

\[
S(\mathcal{G}_{k,d}, d_{TV}, \alpha, \varepsilon) = \Omega \left( \frac{kd}{\alpha^2} + \frac{kd}{\alpha \varepsilon} \right).
\]

**Proof.** We first consider the case when \( k \leq d \) and \( R \geq \sqrt{64 \log \left( \frac{8k}{\alpha} \right)} \). Let \( \mathcal{C} \) denote the codebook in Lemma 5 with weight \( l = \frac{d}{2} \). Then we have \( |\mathcal{C}| \geq 2^{\frac{7k}{16}} \). Given \( c_i \) in codebook \( \mathcal{C} \), we construct the following \( d \)-dimensional Gaussian distribution \( p_i \), with identity covariance matrix and mean \( \mu_i \) satisfying

\[
\mu_{i,j} = \frac{\alpha}{\sqrt{d}} c_{i,j},
\]

where \( \mu_{i,j} \) denotes the \( j \)-th coordinate of \( \mu_i \).

Let \( h = |\mathcal{C}| \). Similar to the product distribution case, using Lemma 6, we can get an \( h \)-ary codebook \( \mathcal{H} \) with length \( d \) and minimum hamming distance \( d/2 \). Moreover, \( |\mathcal{H}| \geq \left( \frac{h}{16} \right)^{\frac{d}{2}} \).

\( \forall i \in [h] \) and \( j \in k \), define \( p_b^{(j)} = \mathcal{N}(\mu_i + \frac{\alpha}{2} e_j, I_d) \), where \( e_j \) is the \( j \)th standard basis vector. It is easy to verify their means satisfy the norm bound. For a codeword \( b \in \mathcal{H} \), let

\[
p_b = \frac{1}{k} \left( p_{b_1}^{(1)} + p_{b_2}^{(2)} + \ldots + p_{b_k}^{(k)} \right).
\]
Let $\mathcal{V}_G = \{p_b : b \in \mathcal{H}\}$ be the set of the distributions defined above. Next we prove that $\forall p_b \neq p_{b'} \in \mathcal{V}_G$,

\begin{align*}
d_{TV}(p_b, p_{b'}) &\geq C\alpha, \quad \text{(13)} \\
d_{KL}(p_b, p_{b'}) &\leq 4\alpha^2. \quad \text{(14)}
\end{align*}

where $C$ is a constant. If these two inequalities hold, using (14), by Pinsker’s Inequality, we get $d_{TV}(p_b, p_{b'}) \leq \sqrt{2d_{KL}(p_b, p_{b'})} \leq 2\sqrt{2}\alpha$. Using Corollary 4, we get

\[ S(G_{k,d}, d_{TV}, \alpha, \varepsilon) = \Omega\left(\frac{kd}{\alpha^2} + \frac{kd}{\alpha \varepsilon}\right). \]

It remains to prove (13) and (14).

For (14), note that for any distribution pair $p_b \neq p_{b'} \in \mathcal{V}_G$,

\[ d_{KL}(p_b, p_{b'}) \leq \frac{1}{k} \sum_{t=1}^{k} d_{KL}(p_{b_t}, p_{b'_t}) \leq \max_{i,j \in [h]} d_{KL}(p_i, p_j) \leq 4\alpha^2, \]

where the first inequality comes from the convexity of $KL$ divergence and the last inequality uses the fact that the $KL$ divergence between two Gaussians with identity covariance is proportional to the $\ell_2^2$ distance between their means.

Next we prove (13). Let $B_j = B_{j,1} \times \cdots \times B_{j,d}$, where

\[ B_{j,i} = \begin{cases} [\frac{R}{4}, \frac{3R}{4}], & \text{when } i = j, \\ [-\frac{R}{4}, \frac{R}{4}], & \text{when } i \neq j \text{ and } i \leq k, \\ [-\infty, \infty], & \text{when } k < i \leq d. \end{cases} \]

Then by Gaussian tail bound and union bound, for any $p \in \mathcal{V}_G$, the mass of the $j$-th Gaussian component outside $B_j$ is at most $2ke^{-\frac{1}{2}(\frac{1}{4}R)^2}$. And the mass of other Gaussian components inside $B_j$ is at most $e^{-\frac{1}{2}(\frac{1}{4}R)^2}$. Hence we have:

\begin{align*}
d_{TV}(p_b, p_{b'}) &= \frac{1}{2k} \int_{z \in \mathbb{R}^d} \left| p_{b_1}^{(1)}(z) + \cdots + p_{b_k}^{(k)}(z) - p_{b'_1}^{(1)}(z) - \cdots - p_{b'_k}^{(k)}(z) \right| \, dz \\
&\geq \frac{1}{2k} \sum_{j=1}^{k} \int_{z \in B_j} \left| p_{b_1}^{(1)}(z) + \cdots + p_{b_k}^{(k)}(z) - p_{b'_1}^{(1)}(z) - \cdots - p_{b'_k}^{(k)}(z) \right| \, dz \\
&\geq \frac{1}{2k} \cdot \sum_{j=1}^{k} \left( \int_{z \in B_j} \left| p_{b_j}^{(j)}(z) - p_{b'_j}^{(j)}(z) \right| \, dz - (k-1) \cdot e^{-\frac{1}{2}(\frac{1}{4}R)^2} \right) \\
&\geq \frac{1}{2k} \cdot \sum_{j=1}^{k} \left( \int_{z \in \mathbb{R}^d} \left| p_{b_j}^{(j)}(z) - p_{b'_j}^{(j)}(z) \right| \, dz - 3k \cdot e^{-\frac{1}{2}(\frac{1}{4}R)^2} \right) \\
&= \frac{1}{2k} \cdot \sum_{j=1}^{k} d_{TV}(p_{b_j}, p_{b'_j}) - \frac{3\alpha^2}{64k}.
\end{align*}

By Fact 6.6 in [KLSU19], there exists a constant $C_1$ such that for any pair $i \neq j \in [h]$,

\[ d_{TV}(p_i, p_j) \geq C_1\alpha. \]

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Hence we have
\[
\frac{1}{2k} \sum_{j=1}^{k} d_{TV}(p_{b_j}, p_{b_j'}) \geq \frac{C_1 \alpha}{2k} d_{\text{Ham}}(b, b') \geq \frac{C_1 \alpha}{4},
\]
where the last inequality comes from the property of the codebook. WLOG, we can assume \( \frac{3\alpha}{6k} < C_1/8 \). Taking \( C = \frac{C_1}{8} \) completes the proof of (13).

Now we consider the case when \( k \geq d \) and \( R \geq (k)^{\frac{1}{2}} \cdot \sqrt{16d \log \left( \frac{8k}{\alpha} \right)} \). Let \( r = \sqrt{16d \log \left( \frac{8k}{\alpha} \right)} \), we note that there exists a packing set \( S = \{v_1, v_2, ..., v_k\} \subset \mathbb{R}^d \) which satisfies \( \forall u, v \in S, \|u - v\|_2 \leq r \)
and \( |S| = k \) since \( R \geq (k)^{\frac{1}{2}} \cdot \sqrt{16d \log \left( \frac{8k}{\alpha} \right)} \). Consider the set of mixture distributions as following:

For a codeword \( b \in \mathcal{H} \), let \( p'_b = \frac{1}{k} (p^{(1)}_b + p^{(2)}_b + ... + p^{(k)}_b) \),
where \( \forall i \in [k], p^{(i)}_b = N(\mu_{b_i} + v_j, I_d) \). Let \( B'_j \) denote the \( \ell_2 \) ball centering at the \( v_j \) with radius \( \frac{r}{2} \). We note that by the tail bound of the Gaussian distribution, the mass of the \( j \)-th Gaussian component outside \( B_j \) is at most \( \alpha \frac{\sqrt{2k}}{64k} \). Meanwhile, the mass of other Gaussian components inside \( B_j \) is also at most \( \alpha \frac{\sqrt{2k}}{64k} \). Hence the remaining analysis follows from the previous case.

4 \((\varepsilon, \delta)\)-DP distribution estimation

In the previous section we used Theorem 2 to obtain sample complexity lower bounds for pure differential privacy. We will now use Theorem 3 to prove sample complexity lower bounds under \((\varepsilon, \delta)\)-DP.

4.1 k-ary distribution estimation

Theorem 12. The sample complexity of \((\varepsilon, \delta)\)-DP k-ary distribution estimation under total variation distance is

\[
S(\Delta_k, d_{TV}, \alpha, \varepsilon, \delta) = \Omega \left( \frac{k}{\alpha^2} + \frac{k}{\alpha(\varepsilon + \delta)} \right).
\]

In practice, the error probability is chosen to be \( \delta = O\left(\frac{1}{n}\right) \), and the privacy parameter is chosen as a small constant, \( \varepsilon = \Theta(1) \). In particular, when \( \delta \leq \varepsilon \), the theorem above shows

\[
S(\Delta_k, d_{TV}, \alpha, \varepsilon, \delta) = \Omega \left( \frac{k}{\alpha^2} + \frac{k}{\alpha \varepsilon} \right).
\]

Since the sample complexity of \( \varepsilon \)-DP is at most the sample complexity of \((\varepsilon, \delta)\)-DP, this shows that the bound above is tight for \( \delta \leq \varepsilon \). The lower bound part is proved using Theorem 3 in Section 4.1.1.

Theorem 13. The sample complexity of \((\varepsilon, \delta)\)-DP discrete distribution estimation under \( \ell_2 \) distance,

\[
\Omega \left( \frac{1}{\alpha^2} + \frac{\sqrt{k}}{\alpha(\varepsilon + \delta)} \right) \leq S(\Delta_k, \ell_2, \alpha, \varepsilon, \delta) \leq O \left( \frac{1}{\alpha^2} + \frac{\sqrt{k}}{\alpha \varepsilon} \right), \quad \text{for } \alpha < \frac{1}{\sqrt{k}}.
\]
To achieve $20^n$ with probability $\Omega(1)$, we prove the following bound on the Hamming distance between a coupling $X, Y$, through $\tau = 10 \alpha/k$. Thus, obeying (6) with $\tau = 10 \alpha/k$.

Recall the mixture distributions $p_{+i}$ and $p_{-i}$,

\[
p_{+i} = \frac{2}{|E_{k/2}|} \sum_{e \in E_{k/2}; e_i = +1} p_e^n, \quad p_{-i} = \frac{2}{|E_{k/2}|} \sum_{e \in E_{k/2}; e_i = -1} p_e^n.
\]

To apply Theorem 3, we prove the following bound on the Hamming distance between a coupling between $p_{+i}$ and $p_{-i}$.

**Lemma 14.** For any $i$, there is a coupling $(X, Y)$ between $p_{+i}$ and $p_{-i}$, such that

\[
\mathbb{E} [d_{Ham}(X, Y)] \leq \frac{20\alpha n}{k}.
\]

**Proof.** By the construction in (15), note that the distributions $p_{+i}$ and $p_{-i}$ only have a difference in the number of times $2i - 1$ and $2i$ appear. To generate $Y \sim p_{-i}$ from from $X \sim p_{+i}$, we scan through $X$ and independently change the coordinates that have the symbol $2i - 1$ to the symbol $2i$ with probability $\frac{20\alpha}{1+10\alpha}$. The expected Hamming distance is bounded by $\frac{20\alpha}{1+10\alpha} \cdot \frac{1+10\alpha}{k} \cdot n = \frac{20\alpha n}{k}$. □

Note that $\mathcal{V} \subset \mathcal{P} := \{p^n \mid p \in \Delta_k\}$. By Theorem 3, using the bound on $D$ from Lemma 17, and $\tau = 10 \alpha/k$,

\[
R(\mathcal{P}, d_{TV}, \varepsilon, \delta) \geq \frac{5\alpha}{k} \cdot k \cdot \left( 0.9 e^{-10\varepsilon D} - 10 D \delta \right) \geq 5\alpha \cdot \left( 0.9 e^{-200n\varepsilon \alpha/k} - 200 \frac{n\varepsilon \alpha \delta}{k} \right).
\]

To achieve $R(\mathcal{P}, d_{TV}, \varepsilon, \delta) \leq \alpha$, either $n\varepsilon \alpha/k = \Omega(1)$ or $n\varepsilon \alpha \delta/k = \Omega(1)$, which implies that $n = \Omega\left( \frac{k}{\alpha(\varepsilon + \delta)} \right)$.
4.1.2 Proof of Theorem 13

We first consider the case where \( \alpha < \frac{1}{\sqrt{k}} \). By Cauchy-Schwarz inequality, 
\[ S(\Delta_k, \ell_2, \alpha, \varepsilon, \delta) \geq S(\Delta_k, d_{TV}, \sqrt{k}\alpha, \varepsilon, \delta) \]
and therefore 
\[ S(\Delta_k, \ell_2, \alpha, \varepsilon, \delta) = \Omega\left( \frac{1}{\alpha^2} + \frac{1}{\alpha(\varepsilon + \delta)} \right) \].

For \( \alpha \geq \frac{1}{\sqrt{k}} \), we have \( l = \lfloor \frac{1}{16\alpha^2} \rfloor \leq k \). Therefore, \( \Delta_l \subset \Delta_k \) and \( \alpha < \frac{1}{\sqrt{l}} \). Hence,
\[ S(\Delta_k, \ell_2, \alpha, \varepsilon, \delta) \geq S(\Delta_l, \ell_2, \alpha, \varepsilon, \delta) = \Omega\left( \frac{1}{\alpha^2} + \frac{1}{\alpha(\varepsilon + \delta)} \right) \].

4.2 Binary product distribution estimation

We now consider estimation of Bernoulli product distributions under total variation distance, which is equivalent to \((k, d)\)-product distribution estimation with \( k = 2 \). A Bernoulli product distribution in \( d \) dimensions is a distribution over \( \{0, 1\}^d \) parameterized by \( \mu \in [0, 1]^d \), where the \( i \)th coordinate is distributed \( B(\mu_i) \), where \( B(\cdot) \) is a Bernoulli distribution. Let \( \Delta_{2,d} \) be the class of Bernoulli product distributions in \( d \) dimensions.

**Theorem 15.** The sample complexity of \((\varepsilon, \delta)\)-DP binary product distribution estimation satisfies
\[ S(\Delta_{2,d}, d_{TV}, \alpha, \varepsilon, \delta) = \Omega\left( \frac{d}{\alpha^2} + \frac{d}{\alpha(\varepsilon + \delta)} \right) \].

Further, when \( \delta \leq \varepsilon \),
\[ S(\Delta_{2,d}, d_{TV}, \alpha, \varepsilon, \delta) = \Omega\left( \frac{d}{\alpha^2} + \frac{d}{\alpha\varepsilon} \right) \].

**Proof.** Since \( \Theta(d/\varepsilon^2) \) is an established tight bound for non-private estimation, we only prove the second term.

We start by constructing a set of Bernoulli product distributions indexed by \( \mathcal{E}_d = \{\pm\}^d \). For all \( e \in \mathcal{E}_d \), let \( p_e = B(\mu_1^e) \times B(\mu_2^e) \times \cdots \times B(\mu_d^e) \), where
\[ \mu_i^e = \frac{1 + e_i \cdot 20\alpha}{d} \].

Let \( \mathcal{V} = \{p_e, e \in \mathcal{E}_d\} \), the set of distributions of \( n \) i.i.d samples from \( p_e \), and \( \theta(p^n_e) = p_e \). For \( u, v \in \mathcal{E}_d \), \( \ell(\theta(p^n_u), \theta(p^n_v)) = d_{TV}(p_u, p_v) \). We first prove that (6) holds for total variation distance for an appropriate \( \tau \).

**Lemma 16.** There exists a constant \( C_1 > 0 \) such that \( \forall u, v \in \mathcal{E}_d \),
\[ d_{TV}(p_u, p_v) \geq \frac{C_1\alpha}{d} \cdot \sum_{i=1}^{d} \mathbb{I}(u_i \neq v_i) \].

**Proof.** Let \( S = \{i \in [d] : u_i \neq v_i\} \), and \( S' = \{i \in S : u_i = 1\} \). WLOG, let \( |S'| \geq \frac{1}{2} |S| \) (or else we can define \( S' = \{i \in S : u_i = -1\} \)). Given a random sample \( Z \in \{\pm1\}^d \), we define an event \( A = \{\forall i \in S', Z_i = 0\} \). Now we consider the difference between the following two probabilities,
which is a lower bound of the total variation distance between \( p_u \) and \( p_v \).

\[
d_{TV}(p_u, p_v) \geq |\Pr_{Z \sim p_u} (A) - \Pr_{Z \sim p_v} (A)|
= \left( 1 - \frac{1 - 20\alpha}{d} \right)^{|S'|} - \left( 1 - \frac{1 + 20\alpha}{d} \right)^{|S'|}
\geq \frac{40\alpha}{d} \cdot |S'| \cdot \left( 1 - \frac{1 + 20\alpha}{d} \right)^{|S'|}
\geq \frac{40\alpha}{d} \cdot |S'| e^{-(1+20\alpha)} \geq \frac{C_1 \alpha}{d} \cdot d_{Ham}(u,v),
\]

where in the last inequality, we assume \( d \geq 1000 \) and \( \alpha < 0.01 \). \( \square \)

Let \( D \) be an upper bound on the expected Hamming distance for a coupling between \( p_{+i} \) and \( p_{-i} \) over all \( i \). Since \( \mathcal{N}_d \subset \Delta_{\Delta,d} \), applying Theorem 3 with Lemma 16 we have

\[
R(P, d_{TV}, \varepsilon, \delta) \geq \frac{C_1 \alpha}{2d} \cdot \left( 0.9e^{-10\varepsilon D} - 10D\delta \right) = \frac{C_1 \alpha}{2} \cdot \left( 0.9e^{-10\varepsilon D} - 10D\delta \right).
\]

Setting \( R(P, d_{TV}, \varepsilon, \delta) \leq \alpha \), we get \( D = \Omega \left( \frac{1}{\varepsilon} \right) \) or \( D = \Omega \left( \frac{1}{\delta} \right) \), or equivalently, \( D = \Omega \left( \frac{1}{\varepsilon + \delta} \right) \). Lemma 17 below shows that we can take \( D = \frac{40\alpha n}{d} \), which proves the result. \( \square \)

Lemma 17. There is a coupling between \((X, Y)\) between \( p_{+i} \) and \( p_{-i} \), such that \( \mathbb{E}[d_{Ham}(X, Y)] \leq \frac{40\alpha n}{d} \).

Proof. We generate \( Y \sim p_{-i} \) from \( X \sim p_{+i} \) as follows. If the \( i \)th coordinate of a sample \( X \) is \( +1 \), we independently flip it to \(-1\) with probability \( \frac{40\alpha}{1+20\alpha} \) to obtain a sample \( Y \). The expected Hamming distance is bounded by \( \frac{40\alpha}{1+20\alpha} \cdot \frac{1+20\alpha}{d} \cdot n = \frac{40\alpha n}{d} \). \( \square \)

5 Proof of Theorems

5.1 Proof of DP Le Cam’s method (Theorem 1)

The proof technique is similar to the proof of coupling lemma in [ASZ18]. However, we directly characterize the error probability in Theorem 1, which we then use to prove Theorem 3 (DP Assouad’s method).

Proof. From Definition 3,

\[
P_e(\hat{\theta}, P_1, P_2) \geq \frac{1}{2} \left( \Pr_{X \sim p_1} \left( \hat{\theta}(X) \neq p_1 \right) + \Pr_{X \sim p_2} \left( \hat{\theta}(X) \neq p_2 \right) \right).
\]

The first term in Theorem 1 follows from the classic Le Cam’s method (Lemma 1 in [Yu97]). For the second term, let \((X, Y)\) be distributed according to a coupling of \( p_1 \) and \( p_2 \) with \( \mathbb{E}[d_{Ham}(X, Y)] \leq D \). By Markov’s inequality, \( \Pr (d_{Ham}(X, Y) > 10D) < 0.1 \). Let \( x \) and \( y \) be the realization of \( X \) and

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Y, W := \{(x, y)|d_{\text{Ham}}(x, y) \leq 10D\} be the set of pairs of realizations with Hamming distance at most 10D. Then we have
\[
\Pr (\hat{\theta}(X) = p_2) = \sum_{x,y} \Pr (X = x, Y = y) \cdot \Pr (\hat{\theta}(x) = p_2) \\
\geq \sum_{(x,y)\in W} \Pr (X = x, Y = y) \cdot \Pr (\hat{\theta}(x) = p_2).
\] (16)

Let \(\beta_1 = \Pr (\hat{\theta}(X) = p_2)\), so we have
\[
\sum_{(x,y)\in W} \Pr (X = x, Y = y) \cdot \Pr (\hat{\theta}(x) = p_2) \leq \beta_1
\]

Next, we need the following group property of differential privacy.

**Lemma 18.** Let \(\hat{\theta}\) be a \((\varepsilon, \delta)\)-DP algorithm, then for sequences \(x,\) and \(y\) with \(d_{\text{Ham}}(x, y) \leq t\), and \(\forall S\), \(\Pr (\hat{\theta}(x) \in S) \leq e^{t\varepsilon} \cdot \Pr (\hat{\theta}(y) \in S) + \delta e^{t(\varepsilon-1)}\).

By Lemma 18, and \(\Pr (d_{\text{Ham}}(X, Y) > 10D) < 0.1\), let \(\Pr (\hat{\theta}(Y) = p_2) = 1 - \beta_2\),
\[
1 - \beta_2 = \sum_{(x,y)\in W} \Pr (x, y) \cdot \Pr (\hat{\theta}(y) = p_2) + \sum_{(x,y)\not\in W} \Pr (x, y) \cdot \Pr (\hat{\theta}(y) = p_2) \\
\leq \sum_{(x,y)\in W} \Pr (x, y) \cdot (e^{10D} \Pr (\hat{\theta}(x) = p_2) + 10D\delta \cdot e^{10(D-1)}) + 0.1 \\
\leq \beta_1 \cdot e^{10D} + 10D\delta \cdot e^{10D} + 0.1.
\]

Similarly, we get
\[
1 - \beta_1 \leq \beta_2 \cdot e^{10D} + 10D\delta \cdot e^{10D} + 0.1.
\]

Adding the two inequalities and rearranging terms,
\[
\beta_1 + \beta_2 \geq \frac{1.8 - 20D\delta e^{10D}}{1 + e^{10D}} \geq 0.9e^{-10\varepsilon D} - 10D\delta. \tag*{\qed}
\]

### 5.2 Proof of private Fano’s inequality (Theorem 2)

In this section, we prove \(\varepsilon\)-DP Fano’s inequality (Theorem 2), restated below.

**Theorem 2** (\(\varepsilon\)-DP Fano’s inequality). Let \(V = \{p_1, p_2, ..., p_M\} \subseteq \mathcal{P}\) such that for all \(i \neq j\),
(a) \(\ell(\theta(p_i), \theta(p_j)) \geq \alpha\),
(b) \(D_{\text{KL}}(p_i, p_j) \leq \beta\),
(c) there exists a coupling \((X, Y)\) between \(p_i\) and \(p_j\) such that \(\mathbb{E}[d_{\text{Ham}}(X, Y)] \leq D\),
then
\[
R(\mathcal{P}, \ell, \varepsilon) \geq \max \left\{ \frac{\alpha}{2} \left( 1 - \frac{\beta + \log 2}{\log M} \right), 0.4\alpha \min \left\{ 1, \frac{M}{e^{10\varepsilon D}} \right\} \right\}. \tag*{(5)}
\]
The proof is based on the observation that if you can change a sample from $p_i$ to $p_j$ by changing $D$ coordinates in expectation, then an algorithm that algorithm that correctly outputs a sample as from $p_i$ has to output $p_j$ with probability roughly $e^{-\varepsilon D}$. With a total of $M$ distributions in total, we show that the error probability is large as long as $\frac{M}{e^{\varepsilon D}}$ is large.

**Proof.** The first term in (5) follows from the non-private Fano’s inequality (Lemma 3 in [Yin97]). For an observation $X \in \mathcal{X}^n$,

$$\hat{p}(X) := \arg\min_{p \in \mathcal{P}} \ell(p, \hat{\theta}(X)),$$

be the distribution in $\mathcal{P}$ closest in parameters to an $\varepsilon$-DP estimate $\hat{\theta}(X)$. Therefore, $\hat{p}(X)$ is also $\varepsilon$-DP. By the triangle inequality,

$$\ell(p, \hat{\theta}(X)) = \ell(p, \hat{\theta}(X)) + \ell(p, \hat{\theta}(X)) \leq 2\ell(p, \hat{\theta}(X)).$$

Hence,

$$\max_{p \in \mathcal{P}} \mathbb{E}_{X \sim p} \left[ \ell(p, \hat{\theta}(X), \hat{\theta}(P)) \right] \geq \max_{p \in \mathcal{P}} \mathbb{E}_{X \sim p} \left[ \ell(p, \hat{\theta}(X), \hat{\theta}(p)) \right] \geq \frac{1}{2} \max_{p \in \mathcal{P}} \mathbb{E}_{X \sim p} \left[ \ell(p, \hat{\theta}(X), \hat{\theta}(p)) \right] + \frac{\alpha}{2} \mathbb{E}_{X \sim p} \left( \hat{\theta}(X) \neq p \right) \geq \frac{\alpha}{2M} \sum_{p \in \mathcal{P}} \Pr_{X \sim p} \left( \hat{p}(X) \neq p \right). \quad (17)$$

Let $\beta_i = \Pr_{X \sim p_i} \left( \hat{p}(X) \neq p_i \right)$ be the probability that $\hat{p}(X) \neq p_i$ when the underlying distribution generating $X$ is $p_i$. For $p_i, p_j \in \mathcal{V}$, let $(X, Y)$ be the coupling in condition (c). By Markov’s inequality $\Pr \left( d_{\text{Ham}}(X, Y) > 10D \right) < 1/10$.

Similar to the proof of Theorem 1 in the previous section, let $W := \{(x, y) | d_{\text{Ham}}(x, y) \leq 10D\}$ and $\Pr(x, y) = \Pr(X = x, Y = y)$. Then

$$1 - \beta_j = \Pr \left( \hat{p}(Y) = p_j \right) \leq \sum_{(x, y) \in W} \Pr(x, y) \cdot \Pr(\hat{p}(y) = p_j) + \sum_{(x, y) \notin W} \Pr(x, y) \cdot 1.$$

Therefore,

$$\sum_{(x, y) \in W} \Pr(x, y) \cdot \Pr(\hat{p}(y) = p_j) \geq 0.9 - \beta_j.$$

Then, we have

$$\Pr(\hat{p}(x) = p_j) \geq \sum_{(x, y) \in W} \Pr(x, y) \cdot \Pr(\hat{p}(x) = p_j) \geq \sum_{(x, y) \in W} \Pr(x, y) e^{-10\varepsilon D} \Pr(\hat{p}(y) = p_j) \geq (0.9 - \beta_j) e^{-10\varepsilon D}, \quad (18)$$

where (18) uses that $\hat{p}$ is $\varepsilon$-DP and $d_{\text{Ham}}(x, y) \leq 10D$. Similarly, for all $j' \neq i$,

$$\Pr(\hat{p}(x) = p_{j'}) \geq (0.9 - \beta_{j'}) e^{-10\varepsilon D}.$$
Summing over $j' \neq i$, we obtain
\[
\beta_i = \sum_{j' \neq i} \Pr \left( \hat{\beta}(X) = p_{j'} \right) \geq \left( 0.9(M - 1) - \sum_{j' \neq i} \beta_{j'} \right) e^{-10\epsilon D}.
\]
Summing over $i \in [M]$,
\[
\sum_{i \in [M]} \beta_i \geq \left( 0.9M(M - 1) - (M - 1) \sum_{i \in [M]} \beta_i \right) e^{-10\epsilon D}.
\]
Rearranging the terms
\[
\sum_{i \in [M]} \beta_i \geq \frac{0.9M(M - 1)}{M - 1 + e^{10\epsilon D}} \geq 0.8M \min \left\{ 1, \frac{M}{e^{10\epsilon D}} \right\}.
\]
Combining this with (17) completes the proof.

5.3 Proof of Corollary 4

Proof. Recall that $Q^* := \{q^n | q \in Q\}$ is the set of induced distributions over $X^n$ and $q^n \in Q^*, \theta(q^n) = q$. Then, \(\forall i \neq j \in [M], \ell(\theta(q^n_i), \theta(q^n_j)) \geq 3\tau\), and $D_{KL}(q^n_i, q^n_j) = nD_{KL}(q_i, q_j) \leq n\beta$.

The following lemma is a corollary of maximal coupling [dH12], which states that for two distributions there is a coupling of their $n$ fold product distributions with an expected Hamming distance $n$ times their total variation distance.

Lemma 19. Given distributions $q_1, q_2$ over $X$, there exists a coupling $(X, Y)$ between $q^n_1$ and $q^n_2$ such that
\[
E[d_{Ham}(X, Y)] = n \cdot d_{TV}(q_1, q_2),
\]
where $X \sim q^n_1$ and $Y \sim q^n_2$.

By Lemma 19, $\forall i, j \in [M]$, there exists a coupling $(X, Y)$ between $q^n_i$ and $q^n_j$ such that $E[d_{Ham}(X, Y)] \leq n\gamma$. Now by Theorem 2,
\[
R(Q^*, \ell, \varepsilon) \geq \max \left\{ \frac{3\tau}{2} \left( 1 - \frac{n\beta + \log 2}{\log M} \right), 1.2\tau \min \left\{ 1, \frac{M}{e^{10\epsilon n\gamma}} \right\} \right\}. \tag{19}
\]
Therefore, for $R(Q^*, \ell, \varepsilon) \leq \tau$,
\[
S(Q, \ell, \tau, \varepsilon) = \Omega \left( \frac{\log M}{\beta} + \frac{\log M}{\gamma \varepsilon} \right).
\]

5.4 Proof of private Assouad’s method (Theorem 3)

Theorem 3 (DP Assouad’s method). For each $i \in [k]$, let $\phi_i : X^n \to \{-1, +1\}$ be a binary classifier.
\[
R(P, \ell, \varepsilon, \delta) \geq \frac{\tau}{2} \sum_{i=1}^{k} \min_{\phi_i, \ell, (\varepsilon, \delta) \in D_P} \Pr_{X \sim p_{+i}}(\phi_i(X) \neq 1) + \Pr_{X \sim p_{-i}}(\phi_i(X) \neq -1).
\]
Moreover, if $\forall i \in [k]$, there exists a coupling $(X, Y)$ between $p_{+i}$ and $p_{-i}$ with $E[d_{Ham}(X, Y)] \leq D$,
\[
R(P, \ell, \varepsilon, \delta) \geq \frac{k\tau}{2} \left( 0.9e^{-10\epsilon D} - 10D\delta \right). \tag{7}
\]
Proof. The first part is from the non-private Assouad’s lemma, which we include here for completeness. Let $p \in V$ and $X \sim p$. For an estimator $\hat{\theta}(X)$, consider an estimator $\hat{E}(X) = \arg\min_{e \in \mathcal{E}_k} \ell(\hat{\theta}(X), \theta(p_e))$. Then, by the triangle inequality,

\[ \ell(\theta(\hat{p}_E), \theta(p)) \leq \ell(\hat{\theta}, \theta(\hat{p}_E)) + \ell(\hat{\theta}, \theta(p)) \leq 2\ell(\hat{\theta}, \theta(p)). \]

Hence,

\[ R(V, \ell, \varepsilon, \delta) = \min_{\hat{\theta} \text{ is } (\varepsilon, \delta)\text{-DP}} \max_{p \in V} \mathbb{E}_{X \sim p} \left[ \ell(\theta(p_E), \theta(p)) \right] \geq \frac{1}{2} \min_{\hat{\theta} \text{ is } (\varepsilon, \delta)\text{-DP}} \max_{p \in V} \mathbb{E}_{X \sim p} \left[ \ell(\theta(\hat{p}_E(X)), \theta(p)) \right] \]

(20)

For any $(\varepsilon, \delta)$-DP index estimator $\hat{E}$,

\[ \max_{p \in V} \mathbb{E}_{X \sim p} \left[ \ell(\theta(\hat{p}_E), \theta(p)) \right] \geq \frac{1}{|\mathcal{E}_k|} \sum_{e \in \mathcal{E}_k} \mathbb{E}_{X \sim p_e} \left[ \ell(\theta(p_E), \theta(p_e)) \right] \geq \frac{2\tau}{|\mathcal{E}_k|} \sum_{i=1}^{k} \sum_{e:e_i = 1} \Pr(\hat{E}_i \neq e_i | E = e) + \sum_{e:e_i = -1} \Pr(\hat{E}_i \neq -1 | E = e) \]

For each $i$, we divide $|\mathcal{E}_k| = \{\pm 1\}^k$ into two sets according to the value of $i$-th position,

\[ \max_{p \in \mathcal{V}_k} \mathbb{E}_{X \sim p} \left[ \ell(\theta(\hat{p}_E), \theta(p)) \right] \geq \frac{2\tau}{|\mathcal{E}_k|} \sum_{i=1}^{k} \left[ \sum_{e:e_i = 1} \Pr(\hat{E}_i \neq 1 | E = e) + \sum_{e:e_i = -1} \Pr(\hat{E}_i \neq -1 | E = e) \right] \]

\[ = \tau \cdot \sum_{i=1}^{k} \left( \Pr_{X \sim p_i \neq i} (\hat{E}_i \neq 1) + \Pr_{X \sim p_{-i}} (\hat{E}_i \neq -1) \right) \]

\[ \geq \tau \cdot \sum_{i=1}^{k} \min_{\phi: \phi \text{ is DP}} \Pr_{X \sim p_i \neq i} (\phi(X) \neq 1) + \Pr_{X \sim p_{-i}} (\phi(X) \neq -1)) \]

Combining with (20), we have

\[ R(\mathcal{P}, \ell, \varepsilon, \delta) \geq \frac{\tau}{2} \cdot \sum_{i=1}^{k} \min_{\phi: \phi \text{ is DP}} \Pr_{X \sim p_i \neq i} (\phi(X) \neq 1) + \Pr_{X \sim p_{-i}} (\phi(X) \neq -1)) \]

proving the first part.

For the second part. Note that for each $i \in [k]$, the summand above is the error probability of hypothesis testing between the mixture distributions $p_{+i}$ and $p_{-i}$. Hence, using Theorem 1,

\[ R(\mathcal{P}, \ell, \varepsilon, \delta) \geq \frac{k\tau}{2} \cdot \left(0.9e^{-10eD} - 10D\delta \right). \]

\[ 6 \quad \text{Proofs of existence of codes (Lemma 5 and Lemma 6)} \]

Proof of Lemma 5. This proof is a standard argument for Gilbert-Varshamov bound applied to constant weight codes. We use the following version (Theorem 7 in [GS80]).

Lemma 20. There exists a length-$k$ constant weight binary code $C$ with weight $l$ and minimum Hamming distance $2\delta$, with $|C| \geq \sum_{i=0}^{s} \binom{k}{l+i}$. 

\[ 22 \]
Applying this Lemma with $2\delta = \frac{l}{4}$, we have

$$|C| \geq \frac{(k)}{\sum_{j=0}^{\lceil l/8 \rceil} (\frac{l}{j}) \cdot (\frac{k - l}{j})} \geq \frac{l \cdot (\frac{k}{l}) \cdot (\frac{1}{l})}{\frac{l}{8} \cdot (\frac{l}{8}) \cdot (\frac{k - l}{l} - i)} \prod_{i=0}^{\frac{2l}{8} - 1} \frac{k - l - i}{l - i} \geq \frac{2\sqrt{7}\pi}{e} \cdot (0.59)^{\frac{7l}{8}} \cdot \left( \frac{k - l}{l} \right)^{\frac{2l}{8}} \geq \left( \frac{k}{2^{l/8}} \right)^{\frac{2l}{8}}. \tag{21}$$

In (21), we note that $\frac{k - l - i}{l - i}$ is monotonically increasing as $i$ increases. And the first part is obtained by the Stirling’s approximation $\sqrt{2\pi} \cdot l^{l+\frac{1}{2}} \cdot e^{-l} \leq l! \leq e \cdot l^{l+\frac{1}{2}} \cdot e^{-l}$ and the fact that $1.1^l \geq \sqrt{l}$ when $l \geq 20$. The last inequality comes from $l \leq k/2$ and $15/16 \times 0.59 > 1/2^{7/8}$. \hfill \Box

**Proof of Lemma 6.** By the Gilbert-Varshamov bound,

$$|\mathcal{H}| \geq \frac{h^d}{\sum_{j=0}^{d-1} \binom{d}{j} (h-1)^j} \geq \frac{h^d}{\frac{d}{2} \cdot \left( \frac{d}{2} \right)^* \cdot h^d} \geq \frac{h^d}{d \cdot 2^d} \geq \left( \frac{h}{16} \right)^{\frac{d}{2}}. \hfill \Box$$

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