PP-wave String Interactions from String Bit Model

Jian-Ge Zhou

Physics Department, University of Lethbridge, Lethbridge, Alberta, Canada T1K 3M4

Abstract

We construct the string states $|O^I_p >^J$, $|O^J_q >^J_{J_1,J_2}$ and $|O^J_{0,J_2} >^J_{J_1,J_2}$ in the Hilbert space of the quantum mechanical orbifold model so as to calculate the three point functions and the matrix elements of the light-cone Hamiltonian from the interacting string bit model. With these string states we show that the three point functions and the matrix elements of the Hamiltonian derived from the interacting string bit model up to $g_s^2$ order precisely match with those computed from the perturbative SYM theory in BMN limit.
1 Introduction

Recently, Berenstein, Maldacena, and Nastase (BMN) [1] argued that the IIB superstring theory on pp-wave background with RR-flux is dual to a sector of $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory containing operators with large $R$-charge $J$. The argument was based on the exact solvability of the Green-Schwarz strings on pp-wave background obtained from $AdS_5 \times S^5$ in the Penrose limit [2]. In BMN limit, the effective 't Hooft coupling is $\lambda' = g_{YM}^2 N/J^2$. Under this limit, the duality allows one to compute the free string spectrum from the perturbative super Yang-Mills theory. Moreover, it was shown in [3] and [4] that in BMN limit some non-planar diagrams of arbitrary genus survive, so besides the effective 't Hooft coupling is $\lambda'$, the string interactions in pp-wave also involve genus parameter $g_2 = J^2/N$. In [4], it was proposed that the interaction amplitude for a single string to split into two strings (or two strings joining into one string) is related to the three point function of the corresponding operators in the dual CFT, and $g_2 \sqrt{N}$ was identified with the effective coupling between a wide class of excited string states on the pp-wave background. By this proposal, the authors in [4] computed the second order correction to the anomalous dimension of the BMN operator from free planar three point functions and found exact agreement with the computation of the torus contributions to the two point function. Other related discussions on the string interaction on pp-wave background can be found in [5]–[19].

More recently, Verlinde proposed a string bit model [20] for interacting strings on the pp-wave background in terms of the supersymmetric quantum mechanics with a symmetric product target-space [21], [22]. In this interacting string bit model, the 't Hooft coupling is $\lambda^2 = g_{YM}^2 N$, but the effective string coupling is identified with the genus parameter $g_2 = J^2/N$. In [21], Verlinde presented some evidence that this effective interacting string bit model would reproduce the complete perturbation expansion of the $\mathcal{N} = 4$ SYM theory in the BMN limit. In [21], Verlinde made an assumption that the three-point function in the string bit model should be identified with that in the free SYM theory, so it would be interesting to see how we can use the operator $\Sigma$ to verify this assumption, and whether
we can reproduce the results derived from the perturbation expansion of the \( \mathcal{N} = 4 \) SYM theory in the BMN limit, which would give a consistency check for the interacting string bit model on pp-wave background.

Motivated by the above, in the present paper we construct the string states \(|O^I_p >_J, O^{J_1}_{q_1} >_{J_1,J_2}\) and \(|O^{J_2}_{0} >_{J_1,J_2}\) in the Hilbert space of the quantum mechanical orbifold model so as to calculate the three point functions and the matrix elements of the light-cone Hamiltonian from the interacting string bit model. The Hilbert space of the quantum mechanical orbifold model is decomposed into the direct sum of the Hilbert spaces of the twisted sectors [23], and each twisted sector describes the states of several strings, so the construction of the string states \(|O^I_p >_J, O^{J_1}_{q_1} >_{J_1,J_2}\) and \(|O^{J_2}_{0} >_{J_1,J_2}\) can be realized by the fact that the vacuum state of a twisted sector corresponds to a ground state twist operator. We show that the three point functions and the matrix elements of the light-cone Hamiltonian derived from the interacting string bit model up to \(g^2\) order precisely match with those computed from the perturbative SYM theory in BMN limit. In our calculation, instead of assuming that the three-point function in the string bit model is the same as that in the free SYM theory, we derive this by exploiting the operator \(\Sigma\), that is, we carry out our calculation from the first principles of the the quantum mechanical orbifold model.

The paper is organized as follows. In the next section, we review some basic results for the interacting string bit model. In Section 3, we develop some approach to consistently construct the string states \(|O^I_p >_J, O^{J_1}_{q_1} >_{J_1,J_2}\) and \(|O^{J_2}_{0} >_{J_1,J_2}\) in the Hilbert space of the quantum mechanical orbifold model. In Section 4, the three point functions and the matrix elements between single and double string states at the order \(g_2\) and the matrix element between two single string states at the order \(g^2_2\) are calculated from the interacting string bit model. In Section 5, we present our summary and discussion.
2 The interacting string bit model

Let us recapitulate some basic results for the interacting string bit model [21]. One can introduce \( J \) copies of supersymmetric phase space coordinates \( \{ p^i_n, x^i_n, \theta^a_n, \tilde{\theta}^a_n \} \), with \( n = 1, \ldots, J \), satisfying canonical commutation relations

\[
[p^i_n, x^j_m] = i\delta^{ij}\delta_{mn}, \quad \{ \theta^a_n, \theta^b_m \} = \frac{1}{2}\delta^{ab}\delta_{mn}, \quad \{ \tilde{\theta}^a_n, \tilde{\theta}^b_m \} = \frac{1}{2}\delta^{ab}\delta_{mn} .
\] (1)

These \( J \) copies can be regarded as obtained by the quantization of the \( J \)-th symmetric product \( \text{Sym}_J\mathcal{M} \) of the plane wave target space \( \mathcal{M} \). The Hilbert space of this quantum mechanical orbifold can be decomposed into the direct sum of “twisted sectors”

\[
\mathcal{H} = \bigoplus_{[\gamma]} \mathcal{H}_{[\gamma]}
\] (2)

labeled by conjugacy classes \([\gamma]\) of the symmetric group \( S_J \) described by

\[
[\gamma] = (1)^{J_1}(2)^{J_2}\ldots(s)^{J_s}
\] (3)

where \( J_n \) is the multiplicity of the cyclic permutation \((n)\) of \( n \) elements. In each twisted sector, one should keep only the states invariant under the centralizer subgroup \( C_g \) of \( g \)

\[
C_g = \prod_{n=1}^{s} S_{J_n} \times Z_{n}^{J_n}
\] (4)

where each factor \( S_{J_n} \) permutes the \( J_n \) cycles \((n)\), while each \( Z_n \) acts within one particular cycle \((n)\).

The Hilbert space \( \mathcal{H}_{[\gamma]} \) of each twist sector can be decomposed into the grade \( J_n \)-fold symmetric tensor products of the Hilbert spaces \( \mathcal{H}_n \) which correspond to the cycles of length \( n \)

\[
\mathcal{H}_{(J_n)} = \bigotimes_{n=1}^{s} S_{J_n} \mathcal{H}_n = \bigotimes_{n=1}^{s} \left( \mathcal{H}_n \bigotimes \mathcal{H}_n \ldots \bigotimes \mathcal{H}_n \right)^{S_{J_n}}
\] (5)

where the space \( \mathcal{H}_n \) is \( Z_n \) invariant subspace of the Hilbert space of the quantum mechanical orbifold model of 16\( n \) bosonic fields \( p^i_n \) and \( x^i_n \), and 16\( n \) fermionic fields \( \theta^a_n \) and
\[ \tilde{\theta}_n. \] The resulting Hilbert space of the quantum mechanical orbifold model is a sum over multi-string Hilbert spaces [24]–[26].

Consider the operator \( \Sigma_{mn} \) that implement a simple transposition of two string bits via

\[ \Sigma_{mn} X_m = X_n \Sigma_{nm}, \quad \Sigma_{mn} X_k = X_k \Sigma_{nm} \quad k \neq m, n \quad (6) \]

with \( X_n = \{ p^i_n, x^i_n, \theta^a_n, \tilde{\theta}^a_n \} \). By acting with \( \Sigma_{mn} \) on a given multi-string sector, we get a different multi-string sector via [23]

\[ \Sigma_{mn} : \mathcal{H}_\gamma \to \mathcal{H}_{\tilde{\gamma}} \quad \text{with } \tilde{\gamma} = \gamma \cdot (mn). \quad (7) \]

When two sites \( m \) and \( n \) in the sector \( \gamma \) correspond to one single string with length \( J \) or two separate ones with lengths \( J_1 \) and \( (J - J_1) \), the new sector \( \tilde{\gamma} \) corresponds to either splitting the single string in two pieces of length \( (m - n) \) and \( (J - m + n) \), or joining the two strings to one of length \( J \).

The light-cone supersymmetry generators and Hamiltonian of the free string theory are [21],[22]

\[ Q_0 = Q_a^{(0)} + \lambda Q_a^{(1)}, \quad \tilde{Q}_0 = \tilde{Q}_a^{(0)} - \lambda \tilde{Q}_a^{(1)}, \quad H_0 = H^{(0)} + \lambda H^{(1)}_\gamma + \lambda^2 H^{(2)}_\gamma \quad (8) \]

with

\[ Q_a^{(0)} = \sum_n \left( p^i_{\gamma i} \theta_n - x^i_{\gamma i} \right), \quad Q_a^{(1)} = \sum_n \left( x^i_{\gamma i} - x^i_n \right) \gamma_i \theta_n \]

\[ H^{(0)} = \sum_n \frac{1}{2} \left( p^2_{i,n} + x^2_{i,n} \right) + 2i \tilde{\theta}_n \Pi \theta_n \], \quad H^{(1)}_\gamma = -\sum_n i (\theta_n \theta_{\gamma(n)} - \tilde{\theta}_n \tilde{\theta}_{\gamma(n)}) \quad (9) \]

\[ H^{(2)}_\gamma = \sum_n \frac{1}{2} (x_{\gamma(n)} - x_n^i)^2 \quad (10) \]

and\(^1\)

\[ \lambda^2 = \frac{g^2_Y N}{8\pi^2}. \quad (11) \]

\(^1\)From [1],[4] and [28], we know that for free string, the \( \lambda^2 \) is identified as \( \frac{\alpha}{2\pi} \) with \( g_Y^2 = 4\pi g \), so the parameter \( \lambda^2 \) should be \( \frac{g_Y^2 M N}{8\pi^2} \), which is different from that in [21] with extra factor \( \frac{1}{8\pi^2} \).
The inner product that realizes the combinatorics of the free gauge theory amplitudes in the string bit language takes the form [22]

\[ \langle \psi_1 | \psi_2 \rangle_{g_2} = \langle \psi_1 | S | \psi_2 \rangle_0 \] (12)

where \( S \) is defined as \( S = e^{g_2 \Sigma} \) with

\[ \Sigma \equiv \frac{1}{F^2} \sum_{m<n} \Sigma_{mn} . \] (13)

The free supersymmetry generators can be split into two terms [22]

\[ Q_0 = Q_0^> + Q_0^< \] (14)

and the general supersymmetry generators to all orders in \( g_2 \) are assumed to be

\[ Q_0 = Q_0^> + S^{-1}Q_0^< S \] (15)

where the > superscript indicates the terms that contain fermionic annihilation operators only, while < denotes terms with only fermionic creation operators. The interacting light-cone Hamiltonian can be extracted from the supersymmetry algebra

\[ \delta^{IJ} \{ Q_0^a, Q_0^b \} = \delta^{\hat{a}\hat{b}} H + J^{\hat{a}\hat{b}} \] (16)

where \( J^{\hat{a}\hat{b}} \) is a suitable contraction of gamma matrices with the \( SO(4) \times SO(4) \) Lorentz generators \( J^{ij} \) [2].

The matrix elements of \( H \) can be defined by [22]

\[ \langle \psi_2 | (\delta^{\hat{a}\hat{b}} H + J^{\hat{a}\hat{b}}) | \psi_1 \rangle_{g_2} = \delta^{IJ} \langle \psi_2 | S \{ Q_0^a, Q_0^b \} | \psi_1 \rangle_0 \] (17)

from which the first and second order interaction terms can be read off

\[ H_1 = g_2 (U_1 + U_2) , \quad H_2 = g_2^2 (V_1 + V_2 + V_3) \] (18)

with

\[ U_1 = H_0 \Sigma + \Sigma H_0 , \quad U_2 = -Q_0^> \Sigma Q_0^< \] (19)

and

\[ V_1 = \frac{1}{2} (H_0 \Sigma^2 + \Sigma^2 H_0) , \quad V_2 = -\frac{1}{2} Q_0^> \Sigma^2 Q_0^< , \quad V_3 = [Q_0^> , \Sigma] [\Sigma , Q_0^<] \] (20)

Here we should point out that \( H_1 \) and \( H_2 \) are derived from the bosonic matrix elements of \( H \) [22].
3 Construction of the string states in the interacting string bit model

To construct the twisted vacuum states, let us first introduce the ground state twist operator $\Sigma(n)$ that translates the string bit $X_I$ within the individual string $(n)$ by one unit

$$\Sigma(n)X_I = X_{I+1} \Sigma(n) \quad I = 1, 2, \ldots, n.$$  \hspace{1cm} (21)

The untwisted vacuum state $|0>$ is defined by

$$a^i_m|0>=0, \quad \beta^a_m|0>=0$$  \hspace{1cm} (22)

with

$$x^i_m = \frac{1}{\sqrt{2}}(a^i_m + a^{i+}_m) \quad p^i_m = \frac{i}{\sqrt{2}}(a^i_m - a^{i+}_m)$$

$$\theta_n = \frac{1}{2} \left( \beta_n + \beta^+_n \right) \quad \Pi \theta_n = \frac{i}{2} \left( \tilde{\beta}_n - \tilde{\beta}^+_n \right)$$

$$\tilde{\theta}_n = \frac{1}{2} \left( \tilde{\beta}_n + \tilde{\beta}^+_n \right) \quad \tilde{\theta}_n \Pi = \frac{i}{2} \left( \beta_n - \beta^+_n \right).$$  \hspace{1cm} (23)

and the untwisted vacuum state $|0>$ is normalized as

$$<0|0>=1.$$  \hspace{1cm} (24)

Then the twisted vacuum state $|n>$ is defined as

$$|n>= \Sigma(n)|0>$$  \hspace{1cm} (25)

with

$$a^i_m|n>=0, \quad \beta^a_m|n>=0, \quad <n|n>=1$$  \hspace{1cm} (26)

which can be easily seen from the definition of $\Sigma(n)$.

The arbitrary group element $\gamma \in S_J$ has the decomposition

$$(n_1)(n_2) \ldots (n_l)$$  \hspace{1cm} (27)
where each cycle of length $n_\alpha$ has a definite set of indices ordered up to a cyclic permutation and generates the action of the subgroup $Z_{n_\alpha}$. Due to this decomposition, the operator $\Sigma(\gamma)$, $V_\gamma$ can be expressed

$$\Sigma(\gamma) = \prod_{\alpha=1}^l \Sigma(n_\alpha), \quad V_\gamma = \prod_{\alpha=1}^l V(n_\alpha)$$

(28)

where $V_\gamma$ is the operator defined in the twisted sector $H_\gamma$.

To define an invariant operator $V_{[\gamma]}$, we first introduce the operator $V_\gamma$ corresponding to a fixed element $\gamma \in S_J$. Under the group action, $V_\gamma$ transforms into $V_{h^{-1}\gamma h}$ [31], so the invariant operator $V_{[\gamma]}$ is the sum over all operators from a given conjugacy class

$$V_{[\gamma]} = \frac{1}{J!} \sum_{h \in S_J} V_{h^{-1}\gamma h}.$$  

(29)

Any correlation function of the operators invariant under centralizer subgroup $C_g$ should be invariant with respect to the global action of the symmetric group [31]

$$< V_{g_1} V_{g_2} \ldots V_{g_l} > = < V_{h^{-1}g_1h} V_{h^{-1}g_2h} \ldots V_{h^{-1}g_lh} >$$

(30)

and the correlation function

$$< V_{g_1} V_{g_2} V_{g_3} >$$

(31)

does not vanish only if\(^2\)

$$g_3 g_2 g_1 = 1.$$  

(32)

The one-string state operator $O_p^J$ considered in [1]

$$O_p^J = \frac{1}{\sqrt{J^N J^{J+2}}} \sum_{l=1}^J e^{2\pi ipl/J} Tr(\phi Z^l \psi Z^{J-l})$$

(33)

should be identified in the interacting string bit model as [21]

$$O_p^J = \frac{1}{J} \left( \sum_{k=1}^J a_k^+ e^{-2\pi ipk/J} \right) \left( \sum_{l=1}^J b_l^+ e^{2\pi ipl/J} \right)$$

(34)

\(^2\)We should also include $g_1 g_2 g_3 = 1$, however, the action of the operator $\Sigma_{mn}$ on the Hilbert space is defined $\Sigma_{mn} : H_\gamma \rightarrow H_\gamma$ with $\tilde{\gamma} = \gamma \cdot (mn)$, so only $g_3 g_2 g_1 = 1$ is selected [21].

7
which describes the one-string state and is invariant under the centralizer subgroup $Z_J$.

To construct the invariant state $|O^J_p\rangle$, we introduce the operator $O^J_{p,\gamma_1}$

$$O^J_{p,\gamma_1} = \frac{1}{J} \left( \sum_{k=1}^{J} a^+_{\gamma_1(k)} e^{-2\pi i pk/J} \right) \left( \sum_{l=1}^{J} b^+_{\gamma_1(l)} e^{2\pi i pl/J} \right)$$

(35)

where $\gamma_1$ indicates one-cycle, i.e., one-string state. The operator $O^J_{p,\gamma_1}$ is invariant under the transformation of the centralizer subgroup $Z_J$, and normalized as

$$<0|O^J_{p,\gamma_1} O^J_{p,\gamma_1} |0> = 1.$$  

(36)

Then the invariant single string state $|O^J_p\rangle$ can be defined as

$$|O^J_p\rangle = \frac{\alpha}{J!} \sum_{h \in S_J} O^J_{p,h^{-1}\gamma_1 h} \sum(h^{-1}\gamma_1 h) |0>$$

(37)

where $\alpha$ is the normalization factor which can be determined by

$$<O^J_p|O^J_p> = 1.$$  

(38)

By the centralizer subgroup $Z_J$, the normalization of the state $|O^J_p\rangle$ determines

$$\alpha = \sqrt{\frac{J!}{J}}.$$  

(39)

The two-string state operator $O^{J_1}_{q,\gamma_2}$ can be constructed as

$$O^{J_1}_{q,\gamma_2} = \frac{1}{J_1} \left( \sum_{k=1}^{J_1} a^+_{\gamma_2(k)} e^{-2\pi i qk/J_1} \right) \left( \sum_{l=1}^{J_1} b^+_{\gamma_2(l)} e^{2\pi i pl/J_1} \right)$$

(40)

where $\gamma_2$ is decomposed as $(J_1)(J_2)$ with $J_2 = J - J_1$, that is, two cycles, and the operator $O^{J_1}_{q,\gamma_2}$ is invariant under the transformation of the centralizer subgroup $Z_{J_1} \otimes Z_{J_2}$. Then the invariant two-string state $|O^{J_1}_q\rangle$ can be described as

$$|O^{J_1}_q\rangle = \frac{\beta}{J_1!} \sum_{h \in S_J} O^{J_1}_{q,h^{-1}\gamma_2 h} \sum(h^{-1}\gamma_2 h) |0>$$

(41)

The normalization of the state $|O^{J_1}_q\rangle$ gives

$$\beta = \sqrt{\frac{J_1!}{J_1 (J - J_1)}}.$$  

(42)
where we have used the centralizer subgroup $Z_{J_1} \otimes Z_{J_2}$. The two-string state (41) corresponds to the state in $\mathcal{N} = 4$ SYM theory [1] and [4]

$$\frac{1}{J_1} \frac{1}{N^{J_1+2}} \sum_{l=1}^{J_1} e^{2\pi i q l / J_1} \text{Tr}(\phi Z^l \psi Z^{J_1-l}) |0> .$$

(43)

The other type two-string state operator $O_{0,\gamma_2}^{J_1 J_2}$ can be constructed by

$$O_{0,\gamma_2}^{J_1 J_2} = \frac{1}{\sqrt{J_1 (J - J_1)}} \left( \sum_{k=1}^{J_1} a_{\gamma_2 (k)}^+ \right) \left( \sum_{l=J_1 + 1}^{J} b_{\gamma_2 (l)}^+ \right)$$

(44)

which is invariant under the transformation of the centralizer subgroup $Z_{J_1} \otimes Z_{J_2}$, and normalized as

$$<0|O_{0,\gamma_2}^{J_1 J_2} O_{0,\gamma_2}^{J_1 J_2} |0> = 1.$$ 

(45)

The corresponding normalized and invariant two-string state is

$$|O_{0,\gamma_2}^{J_1 J_2} > = \frac{\beta}{J_1} \sum_{h \in S_J} O_{0,\gamma_2 (h)}^{J_1 J_2} \Sigma (h^{-1} \gamma_2 h) |0>$$

(46)

where the centralizer subgroup $Z_{J_1} \otimes Z_{J_2}$ is exploited. The two-string state $|O_{0,\gamma_2}^{J_1 J_2} >$ corresponds to the following state in $\mathcal{N} = 4$ SYM theory [1] and [4]

$$\frac{1}{N^{J_1+2}} \text{Tr}(\phi Z^{J_1}) \text{Tr}(\psi Z^{J_1-J_1}) |0> .$$

(47)

Up to now, we have constructed the consistent string states $|O_p^J > _{J_1 J_2} > _{J_1 J_2}$ and $|O_0^{J_1 J_2} > _{J_1 J_2}$ in the interacting string bit model. In the next section, we will use them to calculate the three point functions and the matrix elements of the light-cone Hamiltonian in $g_2$ and $g_2^2$ order.

4 Three point functions and matrix elements of the light-cone Hamiltonian in $g_2$ and $g_2^2$ order

Exploiting the above constructed string states, we first consider the three point function between the single string state $|O_p^J >$ and the two-string state $|O_q^{J_1} >$ induced by the
action $\Sigma$

\[ C_{pqx} = \left\langle O_{p}^{J} \right| \Sigma \left| O_{q}^{J} \right\rangle_{J_{1}J_{2}} \tag{48} \]

with $x = J_{1}/J$.

Inserting (13), (35) and (41) into (48), we have

\[ C_{pqx} = \frac{\alpha \beta}{(J!)^{2}} \sum_{h,h' \in S_{J}} \left\langle 0 \left| \Sigma_{(h^{-1} \gamma_{1} h)} O_{p,h^{-1} \gamma_{1} h}^{J+} \right. \right. \]
\[ \cdot O_{q,h^{-1} \gamma_{2} h'}^{J} \left( h^{-1} \gamma_{2} h' \right) \left| 0 \right\rangle. \tag{49} \]

By exploiting (30), the three point function (49) can be recast into

\[ C_{pqx} = \frac{1}{J^{2} \sqrt{J J_{1} (J - J_{1})}} \sum_{h \in S_{J}} \sum_{m<n} \left\langle 0 \left| \Sigma_{(h^{-1} \gamma_{1} h)} O_{p,h^{-1} \gamma_{1} h}^{J+} \right. \right. \]
\[ \cdot \Sigma_{mn} O_{q,\gamma_{2} h'}^{J} \left( \gamma_{2} \right) \left| 0 \right\rangle. \tag{50} \]

Since the state $\left| O_{q}^{J_{1}} \right\rangle_{J_{1}J_{2}}$ is two-string state, to obtain the single string state $\left| O_{p}^{J} \right\rangle_{J}$, the variables $m$ and $n$ should take the following values

\[ 1 \leq m \leq J_{1}, \quad J_{1} + 1 \leq n \leq J \tag{51} \]

and the above three-point function can be rewritten as

\[ C_{pqx} = \frac{g_{2} \lambda^{2}}{J^{2} \sqrt{J J_{1} (J - J_{1})}} \sum_{h \in S_{J}} \sum_{m=1}^{J_{1}} \sum_{n=J_{1}+1}^{J} \left\langle 0 \left| \Sigma_{(h^{-1} \gamma_{1} h)} O_{p,h^{-1} \gamma_{1} h}^{J+} \right. \right. \]
\[ \cdot \Sigma_{mn} O_{q,\gamma_{2} h'}^{J} \left( \gamma_{2} \right) \left| 0 \right\rangle. \tag{52} \]

Applying the centralizer subgroup $Z_{J_{1}} \otimes Z_{J_{2}}$, Eq. (52) can be reduced to

\[ C_{pqx} = \frac{J_{1} (J - J_{1})}{J^{2} \sqrt{J J_{1} (J - J_{1})}} \sum_{h \in S_{J}} \left\langle 0 \left| \Sigma_{(h^{-1} \gamma_{1} h)} O_{p,h^{-1} \gamma_{1} h}^{J+} \right. \right. \]
\[ \cdot \Sigma_{J_{1}J} O_{q,\gamma_{2} h'}^{J} \left( \gamma_{2} \right) \left| 0 \right\rangle. \tag{53} \]

By the centralizer subgroup $Z_{J}$ and the condition (32), we arrive at

\[ C_{pqx} = \frac{1}{J} \sqrt{\frac{J_{1} (J - J_{1})}{J}} \left\langle 0 \left| \Sigma_{\gamma_{1}} O_{p,\gamma_{1}}^{J+} \right. \right. \]
\[ \cdot \Sigma_{J_{1}J} O_{q,\gamma_{2} h'}^{J} \left( \gamma_{2} \right) \left| 0 \right\rangle. \tag{54} \]
with
\[ \gamma_2 \gamma_{J_1} J \gamma_1 = 1. \] (55)

To simplify the following the calculation, we choose
\[ \gamma_1 = (1, 2, 3, \ldots, J - 1, J). \] (56)
then (55) gives
\[ \gamma_2^{-1} = \gamma_{J_1} J \cdot \gamma_1 = \left( \frac{123 \ldots J_1 - 2, J_1 - 1, J}{J_1 \text{ times}} \right) \left( \frac{J_1, J_1 + 1, \ldots, J - 2, J - 1}{J - J_1 \text{ times}} \right) \] (57)
and the operators \( O_{J_1}^{J_1} \) and \( O_{J_1}^{J_1} \) are simply given by
\begin{align*}
O_{p, \gamma_1}^{J_1} &= \frac{1}{J} \left( \sum_{m=1}^{J} a_m e^{2\pi i pm/J} \right) \left( \sum_{l=1}^{J} b_l^{+} e^{-2\pi i l/p/J} \right) \\
O_{q, \gamma_2}^{J_1} &= \frac{1}{J_1} \left( \sum_{m=1}^{J_1 - 1} a_m^{+} e^{2\pi i qm/J_1} + a_1^{+} \right) \left( \sum_{l=1}^{J_1 - 1} b_l^{+} e^{2\pi i q l/J} + b_1^{+} \right) \quad (58)
\end{align*}
where we have used the fact that the operators \( O_{p, \gamma_1}^{J_1} \) and \( O_{q, \gamma_2}^{J_1} \) are invariant under the transformation of the centralizer subgroup \( Z_J, Z_{J_1} \otimes Z_{J_2} \) respectively.

Plugging (58) into (54), we get
\[ C_{pqx} = \frac{1}{J_J} \sqrt{\frac{J - J_1}{J_1}} \frac{\sin^2 \pi px}{\sin \pi (\frac{p}{J} - \frac{q}{J_1})} \] (59)
where we have used the relation \( \langle 0 | \Sigma_{\gamma_1} \Sigma_{J_1, J} \Sigma_{\gamma_1}^{-1} \gamma_{J_1, J} | 0 \rangle = 1 \) which can be derived by (7) and (55).

In the BMN limit, \( \frac{p}{J} \), \( \frac{q}{J_1} \) are very small [1], so (59) can be written as
\[ C_{pqx} = \sqrt{\frac{1 - x}{J_J} \frac{\sin^2 \pi px}{\pi^2 (p - \frac{q}{J_1})^2}} \] (60)
which agrees with the three point function calculated from the perturbative SYM theory [4]. Here we should stress that the three-point function calculated from the interacting string bit model is (59). Only in the case of the small \( \frac{p}{J} \) and \( \frac{q}{J_1} \), the (59) can be rewritten as (60).
The other three-point function including the operator $O_{0,\gamma_2}^{J_1,J_2}$ can be calculated in the way like for $C_{pqx}$

$$C_{px} = \langle O_p^J | \Sigma | O_0^{J_1,J_2} \rangle_{J_1,J_2}$$

$$= \frac{\alpha \beta}{(J!)^2} \sum_{h,h' \in S_J} \langle 0 | \Sigma_{(h-1,\gamma_1 h)} O_p^{J_1,J_2} \Sigma_{(h'\gamma_2 h')} | 0 \rangle \cdot O_{0,h'\gamma_2 h'}^{J_1,J_2}.$$  

(61)

In the similar way, we have

$$C_{px} = g_2 \lambda^2 \frac{1}{J} \sqrt{\frac{J_1(J-J_1)}{J}} \langle 0 | \Sigma_{\gamma_1} O_p^{J_1,J_2} \rangle \right. \right.$$  

$$\left. \cdot \sum_{J,J_2} O_{0,\gamma_2}^{J_1,J_2} | 0 \rangle.$$  

(62)

Inserting (13), (58), (44), (56), (57) into (62), we obtain that

$$C_{px} = \langle O_p^J | \Sigma | O_0^{J_1,J_2} \rangle_{J_1,J_2} = \frac{\sin^2 \pi p x}{\pi^2 \sqrt{J p^2}}$$  

(63)

which is exactly the same as that derived from the perturbative SYM theory [4].

The matrix element of $\Sigma^2$ between the single string states represents the one-loop contribution due to successive splitting and joining, which can be obtained by factorization

$$\langle O_p^J | \Sigma^2 | O_q^J \rangle_{J} = \sum_{k,x} C_{pkx} C_{qkx} + \sum_{x} C_{px} C_{qx} = 2 A_{pq}$$  

(64)

where the explicit form of $A_{pq}$ is given in [4] and [3].

Now let us consider the matrix element between single and double string states at $g_2$ order, which corresponds to an operator mixing term in the gauge theory. From (18), we have

$$\langle O_p^J | H_1 | O_q^J \rangle_{J_1,J_2} = g_2 \langle O_p^J | U_1 | O_q^J \rangle_{J_1,J_2} + g_2 \langle O_p^J | U_2 | O_q^J \rangle_{J_1,J_2}$$  

(65)

where $U_1$ and $U_2$ are defined in (19).

The string states $|O_p^J\rangle_{J_2}$, $|O_q^{J_1}\rangle_{J_1,J_2}$ and $|O_0^{J_1,J_2}\rangle_{J_1,J_2}$ are eigenstates of the free Hamiltonian $H_0$, the first term in (65) can be easily obtained by exploiting (60)

$$g_2 \langle O_p^J | U_1 | O_q^J \rangle_{J_1,J_2} = g_2 \lambda \left( p^2 + q^2 x^2 \right) C_{pqx}$$  

(66)
where $C_{pqx}$ is the three point function defined in (60).

To calculate the second term in (65)

\[
\langle O_p | U_2 | O_q \rangle_{J_1 J_2} = -g_2 \langle O_p | Q^>_{\gamma_1} \Sigma Q^<_{\gamma_2} | O_q \rangle_{J_1 J_2}
\]  

(67)

we give the explicit form for $Q^>_{\gamma_1}$ and $Q^<_{\gamma_2}$

\[
Q^<_{\gamma_1} = \lambda \sum_{m=0}^{J-1} \left[ (a_{m+1}^{i\dagger} + a_{m+1}^i) - (a_{m}^{i\dagger} + a_{m}^i) \right] \gamma^i \beta_m
\]

\[
Q^>_{\gamma_2} = \lambda \sum_{m=0}^{J-1} \beta_m \gamma^i \left[ (a_{m+1}^{i\dagger} + a_{m+1}^i) - (a_{m}^{i\dagger} + a_{m}^i) \right]
+ \left( \beta_{J_1-1} - \beta_{J_1} \right) \gamma^i \left[ (a_{J_1}^{i\dagger} + a_{J_1}^i) - (a_{J_1}^{i\dagger} + a_{J_1}^i) \right]
\]

(68)

where we have identified the last site $m = J$ with the 0-th site $m = 0$. Then the second term can be recast into

\[
- \langle O_p | Q^>_{\gamma_1} \Sigma Q^<_{\gamma_2} | O_q \rangle_{J_1 J_2} = X_1 + X_2
\]

(69)

with

\[
X_1 = -g_2 \lambda^2 \sum_{m=1}^{J} \left( \sum_{i=1}^{a_{m+1}^{i\dagger} - a_{m+1}^i} \gamma^i \beta_m \right) \left( \sum_{n=1}^{a_{n+1}^{i\dagger} - a_{n+1}^i} \right) \gamma^i \beta_m
+ \left( \beta_{J_1}^{i\dagger} - \beta_{J_1}^i \right) \gamma^i \left( a_{J_1}^{i\dagger} - a_{J_1}^i \right) | O_q \rangle_{J_1 J_2}
\]

\[
X_2 = -g_2 \lambda^2 \sum_{m=1}^{J} \left( \sum_{i=1}^{a_{m+1}^{i\dagger} - a_{m+1}^i} \gamma^i \beta_m \right) \left( \sum_{n=1}^{a_{n+1}^{i\dagger} - a_{n+1}^i} \right) \gamma^i \beta_m
+ \left( \beta_{J_1}^{i\dagger} - \beta_{J_1}^i \right) \gamma^i \left( a_{J_1}^{i\dagger} - a_{J_1}^i \right) | O_q \rangle_{J_1 J_2}
\]

(70)

After taking the inner product and keeping track of the action of the centralizers, we have\(^3\)

\[
X_1 = -\frac{1}{2} g_2 \lambda^2 pq x C_{pqx}
\]

\[
X_2 = -\frac{1}{2} g_2 \lambda^2 pq x C_{pqx}
\]

(71)

\(^3\)When we calculate $X_2$ some contribution from normal ordering arises, which corresponds to a vacuum fluctuation and can be cancelled by the hopping terms in the Hamiltonian [22].
where (60) has been used in the calculation.

Inserting (66), (69) and (71) into (65), we have
\[ \langle O_p^J | H_1 | O_q^{J_1} \rangle_{J_1, J_2} = g_2 \lambda' \left( p^2 + \frac{q^2}{x^2} - \frac{pq}{x} \right) \right) C_{pqx}. \tag{72} \]

Similarly, we have
\[ \langle O_p^J | H_1 | O_q^{J_1 + J_2} \rangle_{J_1, J_2} = g_2 \lambda' p^2 C_{pqx}. \tag{73} \]

In gauge theory, the mixing between single and double trace operators is \[29], \[30]\]
\[ \langle O_p^J(0) : \bar{O}_q^{J_1} O_q^{J_1} : (x) \rangle = g_2 C_{pqx} \left( 1 - \lambda' \ln(x\Lambda)^2 \left( p^2 + \frac{q^2}{x^2} - \frac{pq}{x} \right) \right) \tag{74} \]
which precisely matches (72).

Finally, we calculate the matrix element between two single string states at \( g_2^2 \) order
\[ \langle O_p^J | H_2 | O_q^J \rangle_J = \langle O_p^J | V_1 | O_q^J \rangle_J + \langle O_p^J | V_2 | O_q^J \rangle_J + \langle O_p^J | V_3 | O_q^J \rangle_J \tag{75} \]
where \( V_1, V_2 \text{ and } V_3 \) are defined in (20).

From (19) and (20), we see \( V_1 \text{ and } V_2 \) have similar structure to \( U_1 \text{ and } U_2 \) with replacing \( \Sigma \text{ by } \Sigma^2 \). By exploiting the factorization (64), we find
\[ \langle O_p^J | V_1 | O_q^J \rangle_J = g_2^2 \lambda' (p^2 + q^2) A_{pq}, \quad \langle O_p^J | V_2 | O_q^J \rangle_J = -g_2^2 \lambda' pq A_{pq}. \tag{76} \]
where \( A_{pq} \) is the matrix element of the interaction term \( \Sigma^2 \) defined in (64).

The matrix element \( \langle O_p^J | V_3 | O_q^J \rangle_J \) can be expressed by the factorization
\[ \langle O_p^J | V_3 | O_q^J \rangle_J = g_2^2 \sum_{r, J_1} \langle O_p^J | \Sigma | O_{r q}^{J_1} \rangle_J \langle O_{r q}^{J_1} | [Q_0^\pm, \Sigma] Q_0^\pm | O_q^J \rangle_J \]
\[ - g_2^2 \sum_{r, J_1} \langle O_p^J | [Q_0^\pm, \Sigma] Q_0^\pm | O_{r q}^{J_1} \rangle_J \langle O_{r q}^{J_1} | \Sigma | O_q^J \rangle_J \]
\[ + g_2^2 \sum_{J_1} \langle O_p^J | [\Sigma| O_{0 q}^{J_1 J_2} \rangle_J \langle O_{0 q}^{J_1 J_2} | [Q_0^\pm, \Sigma] Q_0^\pm | O_q^J \rangle_J \]
\[ - g_2^2 \sum_{J_1} \langle O_p^J | [Q_0^\pm, \Sigma] Q_0^\pm | O_{0 q}^{J_1 J_2} \rangle_J \langle O_{0 q}^{J_1 J_2} | \Sigma | O_q^J \rangle_J \tag{77} \]
where in the large $J$, we have used the relation $[\Sigma, [\Sigma, Q^i_0]] = 0$.

By exploiting (66), (69) and (71), (77) can be written as

$$
\langle J^i_p | V_3 | O^j_q \rangle_J = g_2^2 \lambda' \sum_{r,x} \left( p \left( \frac{r}{x} - p \right) - \frac{r}{x} \left( r - \frac{r}{x} \right) \right) C_{prx} C_{qrx} - g_2^2 \lambda' \sum_x p^2 C_{px} C_{qx} = \frac{g_2^2 \lambda'}{4\pi^2} B_{pq}
$$

(78)

where the explicit form of $B_{pq}$ can be found in [29] and [30]. Inserting (76) and (78) into (75), we arrive at

$$
\langle J^i_p | H_2 | O^j_q \rangle_J = g_2^2 \lambda' \left( (p^2 + q^2 - pq) A_{pq} + \frac{1}{4\pi^2} B_{pq} \right)
$$

(79)

which agrees with the result obtained from perturbative Yang-Mills calculation [29],[30]

$$
\langle J^i_p (0) \bar{O}^j_q (x) \rangle_J = (\delta_{pq} + g_2^2 A_{pq}) \left[ 1 - (p^2 + q^2 - pq) \lambda' \ln(x\Lambda)^2 \right] - \frac{g_2^2 \lambda'}{4\pi^2} B_{pq} \ln(x\Lambda)^2.
$$

(80)

From (60), (63), (72), (73) and (79), we find that the three-point functions $\langle J^i_p | \Sigma | O^j_{0i} \rangle_{J_J}$ and $\langle J^i_p | \Sigma | O^j_{0J} \rangle_{J_J}$, the matrix elements $\langle J^i_p | H_1 | O^j_q \rangle_J$, $\langle J^i_p | H_1 | O^j_{0J} \rangle_J$, $\langle J^i_p | H_2 | O^j_q \rangle_J$ obtained from the interacting string bit model precisely match to those derived from the $\mathcal{N} = 4$ SYM theory. Instead of identifying the three-point function in the string bit model with those in the free SYM theory [21], in the above we have exploited the operator $\Sigma$ to carry out our calculation.

5 Summary

So far, we have developed new approach to consistently construct the string states $|O^J_p>_{J}$, $|O^J_q>_{J}$ and $|O^J_0>_{J}$ in the Hilbert space of the quantum mechanical orbifold model in order to calculate the three-point functions, and the matrix elements of the light-cone Hamiltonian from the interacting string bit model. Since the Hilbert space of the quantum mechanical orbifold model can be decomposed into the direct sum of the Hilbert spaces of the twisted sectors, and each twisted sector describes the states of several strings, the construction of the string states $|O^J_p>_{J}$, $|O^J_q>_{J}$ and $|O^J_0>_{J}$ has been realized.
by the fact that the vacuum state of a twisted sector can be described by the ground state twist operator. We have shown that for the three-point functions, and the matrix elements of the light-cone Hamiltonian up to $g_2^2$ order, the results obtained by interacting string bit model precisely match with those computed from the perturbative SYM theory in BMN limit. We should emphasize that in our calculation, instead of assuming that the three-point functions in the string bit model are the same as those in the free SYM theory, we have derived those from the first principles of the the quantum mechanical orbifold model.

In (1), we have introduced $J$ copies of supersymmetric phase space coordinates, and the operator $O_p^J$ is identified in the interacting string bit model as (34). In [3] and [4], it was shown that the sum should start from $l = 0$ instead of $l = 1$ and this small difference has drastic consequences. In particular the operator with sum beginning at $l = 1$ does not reduce to a chiral primary for $p \to 0$. Thus the operator $O_p^J$ should be defined

$$O_p^J = \frac{1}{J} \left( \sum_{k=0}^{J} a_k^+ e^{-2\pi i p k / J} \right) \left( \sum_{l=0}^{J} b_l^+ e^{2\pi i p l / J} \right).$$ (81)

One may wonder if our results would be changed by introducing $J + 1$ copies of supersymmetric phase space coordinates, and the operator $O_p^J$ instead defined as (81), but after some calculation, we find that is not the case. In the above, the interacting string bit model has been constructed by the harmonic oscillators. It would be interesting to construct the interacting string bit model in terms of Cuntz oscillators [32] and see whether both approaches match with each other. We hope to return to these issues in near future.

**Acknowledgments**

I thank M. Walton for useful discussion. This work was supported by NSERC.
References

[1] D. Berenstein, J. Maldacena, H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang Mills*, hep-th/0202021.

[2] R. R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, Nucl. Phys. B 625, 70 (2002), hep-th/0112044; R. R. Metsaev, A. A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, hep-th/0202109.

[3] C. Kristjansen, J. Plefka, G. W. Semenoff, M. Staudacher, *A New double-scaling limit of $N=4$ super Yang-Mills theory and pp-wave strings*, hep-th/0205033.

[4] N.R. Constable, D.Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov, W. Skiba, *PP-wave string interactions from perturbative Yang-Mills theory*, hep-th/0205089.

[5] D. Berenstein and H. Nastase, *On lightcone string field theory from super Yang-Mills and holography*, hep-th/0205048.

[6] M. Spradlin, A. Volovich, *Superstring interactions in a pp-wave background*, hep-th/0204146.

[7] R. Gopakumar, *String Interactions in PP-waves*, hep-th/0205174.

[8] Y. Kiem, Y. Kim, S. Lee, J. Park, *PP-wave/Yang-Mills Correspondence: An explicit check*, hep-th/0205279.

[9] M. Huang, *Three point functions of $N=4$ super Yang Mills from light cone string field theory in pp-wave*, hep-th/0205311; *String interactions in PP-wave from $N = 4$ super Yang-Mills*, hep-th/0206248.

[10] C. Chu, V.V. Khoze, G. Travaglini, *Three-point functions in N=4 super Yang-Mills theory and pp-wave*, hep-th/0206005.
[11] P. Lee, S. Moriyama, J. Park, *Cubic Interactions in PP-wave light cone string field theory*, hep-th/0206065.

[12] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background II*, hep-th/0206073.

[13] C. Chu, V.V. Khoze, G. Travaglini, *PP-wave string interactions from n-point correlators of BMN operators*, hep-th/0206167.

[14] I. R. Klebanov, M. Spradlin, A. Volovich, *New Effects in Gauge Theory from pp-wave Superstrings*, hep-th/0206221.

[15] U. Gursoy, *Vector operators in the BMN correspondence*, hep-th/0208041.

[16] C. Chu, V.V. Khoze, M. Petrini, R. Russo, A. Tanzini, *A note on string interaction on pp-wave background*, hep-th/0208148.

[17] N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff, M. Staudacher, *BMN correlators and operator mixing in N=4 Super Yang-Mills theory*, hep-th/0208178.

[18] J. H. Schwarz, *Comments on superstring interactions in a plane-wave background*, hep-th/0208179.

[19] A. Pankiewicz, *More comments on superstring interactions in the pp-wave background*, hep-th/0208209.

[20] R. Giles, C. B. Thorn, *A Lattice Approach To String Theory*, Phys. Rev. D **16**, 366 (1977); C. B. Thorn, *Supersymmetric quantum mechanics for string-bits*, Phys. Rev. D **56**, 6619 (1997), hep-th/9707048; C. B. Thorn, *A Fock Space Description Of The 1/N-C Expansion Of Quantum Chromodynamics*, Phys. Rev. D **20**, 1435 (1979).

[21] H. Verlinde, *Bits, matrices and 1/N*, hep-th/0206059.

[22] D. Vaman, H. Verlinde, *Bit strings from N=4 gauge theory*, hep-th/0209215.
[23] C. Vafa, E. Witten, *Strong coupling test of S-duality*, Nucl.Phys. **B431** (1994) 3, hep-th/9408074.

[24] L. Motl, *Proposals on nonpertubative superstring interactions*, hep-th/9701025.

[25] T. Banks, N. Seiberg, *Strings from matrices*, Nucl. Phys. **B497** (1997) 41, hep-th/9702187.

[26] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix string theory*, Nucl. Phys. B **500** (1997) 43, hep-th/9703030.

[27] G. Bonelli, *Matrix strings in pp-wave backgrounds from deformed super Yang-Mills theory*, hep-th/0205213.

[28] D.J. Gross, A. Mikhailov, R. Roiban, *Operators with large R charge in N=4 Yang-Mills theory*, hep-th/0205066.

[29] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff, M. Staudacher, *BMN correlators and operator mixing in N=4 super Yang-Mills theory*, hep-th/0208178.

[30] N.R. Constable, D.Z. Freedman, M. Headrick, S. Minwalla, *Operator mixing and the BMN correspondence*, hep-th/0209002.

[31] G. Arutyunov, S. Frolov, *Four graviton scattering amplitude from S^N R^8 supersymmetric orbifold sigma model*, Nucl.Phys. **B524** (1998) 159, hep-th/9712061.

[32] R. Gopakumar, D. J. Gross, *Mastering the Master Field*, Nucl.Phys. **B451** (1995) 379, hep-th/9411021.