WHERE IS $f(z) / f'(z)$ UNIVALENT?

MILUTIN OBRADOVIĆ, SAMINATHAN PONNUSAMY †, AND KARL-JOACHIM WIRTHS

Abstract. Let $S$ denote the family of all univalent functions $f$ in the unit disk $D$ with the normalization $f(0) = 0 = f'(0) - 1$. There is an intimate relationship between the operator $P_f(z) = f(z)/f'(z)$ and the Danikas-Ruscheweyh operator $T_f := \int_0^z (tf'(t)/f(t)) dt$. In this paper we mainly consider the univalence problem of $F = P_f$, where $f$ belongs to some subclasses of $S$. Among several sharp results and non-sharp results, we also show that if $f \in S$, then $F \in U$ in the disk $|z| < r$ with $r \leq r_0 \approx 0.360794$ and conjecture that the upper bound for such $r$ is $\sqrt{2} - 1$.

1. Introduction and Main Results

Let $B$ denote the class of analytic functions $\omega(z)$ in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in D$. If $f, g$ are two analytic functions in $D$, then we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an $\omega \in B$ such that $f(z) = g(\omega(z))$. We also note that if $g$ is univalent, then it is easy to show that $f \prec g$ if and only if $f(0) = g(0)$ and $f(D) \subset g(D)$.

We consider the family $A$ of all functions $f$ analytic in $D$ with the normalization $f(0) = 0 = f'(0) - 1$. By $S$, $S \subset A$, we denote the class of univalent functions in $D$. Certain special subclasses of $S$ possess various remarkable features due to their geometrical properties. By $C$, $K$, and $S^*$ we denote the subclasses of $S$ which consist of convex, close-to-convex, and starlike functions, respectively. For $\beta \in [0, 1)$, let $S^*(\beta)$ denote the usual normalized class of all (univalent) starlike functions of order $\beta$. Analytically, $f \in S^*(\beta)$ if $f \in A$ and satisfies the condition

$$\frac{zf'(z)}{f(z)} < \frac{1 + (1 - 2\beta)z}{1 - z}, \quad z \in D.$$ 

It is well-known that $C \subset S^*(1/2)$, and $S^* := S^*(0)$. At this point it is interesting to note that a function belonging to $S^*(1/2)$ may not be convex in $|z| < R$ for any $R > \sqrt{2\sqrt{3} - 3} = 0.68 \ldots$, see [3] Theorem 1. We say that $f \in A$ is starlike in $|z| < r$ (i.e. to say $f \in S^*$ in $|z| < r$) for some $0 < r \leq 1$, if $f(|z| < r)$ is starlike with respect to the origin. This means that the last subordination condition is satisfied for $|z| < r$ instead of the full disk $|z| < 1$. Similar convention will be followed for other classes. We refer to [3, 4, 11] for a detailed discussion on these classes. Also

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† Corresponding author.

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let us introduce some notations and definitions as follows:

\[
\mathcal{U} = \{ f \in \mathcal{A} : |U_f(z)| < 1 \text{ for } z \in \mathbb{D} \}, \quad U_f(z) = f'(z) \left( \frac{z}{f(z)} \right)^2 - 1,
\]

\[
\mathcal{C}(-1/2) = \left\{ f \in \mathcal{A} : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2}, \text{ for } z \in \mathbb{D} \right\}, \text{ and }
\]

\[
\mathcal{G} = \left\{ f \in \mathcal{A} : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2}, \text{ for } z \in \mathbb{D} \right\}.
\]

According to Aksentév’s theorem \cite{1} (see also \cite{10}), the strict inclusion \( \mathcal{U} \subsetneq \mathcal{S} \) holds. Moreover, \( \mathcal{C}(-1/2) \subset \mathcal{K} \), and functions in \( \mathcal{G} \) are proved to be starlike in \( \mathbb{D} \), see for eg. \cite{12}, Example 1, Equation (16)]. See also \cite{7} for further details and investigation on the class \( \mathcal{G} \).

This article concerns with the operator

\[
F(z) := P_f(z) = \frac{f(z)}{f'(z)}
\]

for locally univalent functions \( f \in \mathcal{A} \). The main problem is to consider the univalence and starlikeness of \( P_f \) when \( f \) belongs to some of the subclasses of \( \mathcal{S} \) defined above.

Among others our interest in the operator \( P_f \) arose from the fact that there exists an intimate relation between this one and the Danikas-Ruscheweyh (\cite{2}) operator

\[
T_f(z) := \int_0^z \frac{tf'(t)}{f(t)} \, dt = z + \sum_{n=1}^{\infty} \frac{n}{n+1} c_n(f) z^{n+1} \quad (f \in \mathcal{S}),
\]

where \( c_n(f) (n \geq 1) \) denote the logarithmic coefficients of \( f \in \mathcal{S} \) defined by

\[
\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n(f) z^n.
\]

The conjecture that \( T_f \in \mathcal{S} \) for each \( f \in \mathcal{S} \) remains open.

The relation between \( (1) \) and \( (2) \) becomes obvious, when one considers the equivalent operators in the \( w \)-plane where \( w = f(z) \). Let \( g(w) = f^{-1}(w) \) be the function inverse to \( f \). If we transform the operator \( P_f \) to the \( w \)-plane, we get the operator

\[
Q(g)(w) = wg'(w) = q(w).
\]

A similar consideration concerning the Danikas-Ruscheweyh operator results in

\[
S(g)(w) = \int_0^w \frac{g(u)}{u} \, du = s(w).
\]

Now it is immediately seen that

\[
Q^{-1}(q)(w) = \int_0^w \frac{q(u)}{u} \, du = S(q)(w) \quad \text{and} \quad S^{-1}(s)(w) = ws'(w) = Q(s)(w).
\]
Where is \( f(z)/f'(z) \) univalent?

2. Preliminaries and two examples

We remark that if \( f \in S \) then \( (z/f(z)) \neq 0 \) in \( \mathbb{D} \) and hence, \( f \) can be represented as Taylor’s series of the form

\[
f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}.
\]

According to the well-known Area Theorem [4, Theorem 11 on p.193 of Vol. 2], for \( f \in S \) of the form (3), one has

\[
\sum_{n=2}^{\infty} (n-1) |b_n|^2 \leq 1
\]

but this condition is not sufficient for the univalence of \( f \). On the other hand, if \( f \in A \) of the form (3) satisfies the condition

\[
\sum_{n=2}^{\infty} (n-1) |b_n| \leq 1,
\]

then \( f \in \mathcal{U} \). The condition (5) is also necessary if \( b_n \geq 0 \) for \( n \geq 1 \). The constant 1 is the best possible in the sense that if

\[
\sum_{n=2}^{\infty} (n-1) |b_n| = 1 + \varepsilon,
\]

for some \( \varepsilon > 0 \), then there exists an \( f \) which is not univalent in \( \mathbb{D} \).

Let us continue the discussion with two examples. Consider

\[
f_1(z) = \frac{z(1 - \frac{z}{2})}{(1 - z)^2}, \quad \text{and} \quad f_2(z) = z - \frac{z^2}{2}.
\]

Then \( f_1 \in C(-1/2) \) and \( f_2 \in G \). Define

\[
F_j(z) = P_{f_j}(z) = \frac{f_j(z)}{f_j'(z)}, \quad \text{for} \quad j = 1, 2,
\]

so that

\[
F_1(z) = z - \frac{3}{2} z^2 + \frac{1}{2} z^3 \quad \text{and} \quad F_2(z) = \frac{z(1 - \frac{z}{2})}{1 - z}.
\]

(1) We have that

\[
F_1'(z) = \frac{3}{2} z^2 - 3z + 1 = \frac{3}{2} (z - r_+) (z - r_-), \quad r_{\pm} = 1 \pm \frac{\sqrt{3}}{3}
\]

and therefore \( F_1'(r_-) = 0 \), where \( r_- = 1 - \frac{\sqrt{3}}{3} = 0.4226497 \ldots \). We claim that \( \Re(F_1'(z)) > 0 \) for \( |z| < r_- \). To do this, we observe that

\[
\Re(F_1'(re^{i\theta})) = 3r^2 \cos^2 \theta - 3r \cos \theta + 1 - \frac{3}{2} r^2,
\]

then it is easy to show that \( \Re(F_1'(re^{i\theta})) > 0 \) for \( -1 \leq \cos \theta \leq 1 \) and \( 0 \leq r < r_- \). It means that \( F_1 \) is univalent in the disc \( |z| < r_- \).
(2) It is a simple exercise to see that $F_2 \in \mathcal{U}$. In fact,
\[
\frac{z}{F_2(z)} = \frac{1 - z}{1 - \frac{z}{2}} = 1 - \frac{z}{2} = 1 - z - \sum_{n=2}^{\infty} b_n z^n, \quad b_n = \frac{1}{2^n},
\]
so that $z/F_2(z)$ is non-vanishing in $\mathbb{D}$ and thus,
\[
-z \left(\frac{z}{F_2(z)}\right)' + \frac{z}{F_2(z)} - 1 = \left(\frac{z}{F_2(z)}\right)^2 F_2'(z) - 1 = \left(\frac{z}{1 - \frac{z}{2}}\right)^2
\]
from which we easily see that $|U_{F_2}(z)| < 1$ for $z \in \mathbb{D}$. Indeed, by a direct computation, we see that the function $w = (z/2)/(1 - (z/2))$ maps $\mathbb{D}$ onto the disk $|w - (1/3)| < 2/3$ so that $w \in \mathbb{D}$ and thus, $w^2 \in \mathbb{D}$. This observation gives that $|U_{F_2}(z)| < 1$ in $\mathbb{D}$ and hence, $F_2 \in \mathcal{U}$. Alternately, using the series expansion for $F_2$, we find that
\[
\sum_{n=2}^{\infty} (n - 1)|b_n| = \sum_{n=2}^{\infty} (n - 1)\frac{1}{2^n} = 1
\]
and, by the sufficient condition (3), it follows that $F_2 \in \mathcal{U}$.

3. Main results

Let $\omega \in \mathcal{B}$. Then by the Schwarz lemma it follows that $|\omega(z)| \leq |z|$ for $z \in \mathbb{D}$ and by the Schwarz-Pick lemma we have
\[
|\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2} \quad \text{for } z \in \mathbb{D}.
\]
Clearly, $\frac{\omega(z)}{z}$ is analytic in $\mathbb{D}$ and $|\omega(z)/z| \leq 1$ in $\mathbb{D}$. The Schwarz-Pick lemma, namely, (3), applied to $\omega(z)/z$ shows that
\[
|z \omega'(z) - \omega(z)| \leq \frac{|z|^2 - |\omega(z)|^2}{1 - |z|^2}.
\]
These three inequalities will be used frequently in the proof of our main results.

**Theorem 1.** If $f \in \mathcal{S}^*(\beta)$, then $P_f \in \mathcal{U}$ in the disk $|z| < 1/(1 + \sqrt{2(1-\beta)})$. The result is sharp (as for univalence) as the function $z/(1 - z)^{2(1 - \beta)}$ shows.

**Proof.** Each $f \in \mathcal{S}^*(\beta)$ and $F = P_f$ defined by (1) can be written as
\[
\frac{zf'(z)}{f(z)} = \frac{1 + (1 - 2\beta)\omega(z)}{1 - \omega(z)} \quad \text{and} \quad F(z) = \frac{z(1 - \omega(z))}{1 + (1 - 2\beta)\omega(z)}.
\]
where $\omega \in \mathcal{B}$. Clearly, $\frac{\omega(z)}{z}$ is analytic in $\mathbb{D}$ and $|\omega(z)/z| \leq 1$ in $\mathbb{D}$. Using the last two relations, we observe that
\[
U_f(z) = -z \left(\frac{z}{F(z)}\right)' + \frac{z}{F(z)} - 1 = \frac{zf'(z)}{f(z)} - z \left(\frac{zf'(z)}{f(z)}\right)' - 1
\]
and thus,
\[ U_F(z) = 2(1 - \beta) \left( \frac{\omega(z)}{1 - \omega(z)} - \frac{z\omega'(z)}{(1 - \omega(z))^2} \right) = 2(1 - \beta) \left( \frac{(\omega(z) - z\omega'(z)) - \omega^2(z)}{(1 - \omega(z))^2} \right) \]
from which and (7), we obtain that
\[ |U_F(z)| \leq 2(1 - \beta) \left( \frac{|\omega(z) - z\omega'(z)|}{(1 - |\omega(z)|)^2} + \frac{|\omega(z)|^2}{(1 - |\omega(z)|)^2} \right) \leq 2(1 - \beta) \left( \frac{|\omega(z)|^2}{(1 - |\omega(z)|)^2} + \frac{|\omega(z)|^2}{(1 - |\omega(z)|)^2} \right) = \frac{2(1 - \beta)|z|^2}{1 - |z|^2} \left( \frac{1 + |z|}{1 - |z|} \right) = \frac{2(1 - \beta)|z|^2}{(1 - |z|)^2} \]
which can easily seen to be less than 1 if \(|z| < 1/(1 + \sqrt{2(1 - \beta)})\). Thus, \(F\) belongs to \(U\) in the disk \(|z| < 1/(1 + \sqrt{2(1 - \beta)})\).

To prove the sharpness part, we consider \(k_\beta(z) = z/(1 - z)^2(1 - \beta)\) and define
\[ F_\beta(z) = P_{k_\beta}(z) = \frac{k_\beta(z)}{k_\beta'(z)}. \]
Then we see that \(k_\beta \in S^*(\beta)\) and
\[ F_\beta(z) = \frac{z(1 - z)}{1 + (1 - 2\beta)z} \quad \text{and} \quad \frac{z}{F_\beta(z)} = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta) \sum_{n=1}^{\infty} z^n. \]
Define \(G_\beta(z) = \frac{1}{r} F_\beta(rz)\) and observe that
\[ \frac{z}{G_\beta(z)} = 1 + 2(1 - \beta) \sum_{n=1}^{\infty} r^n z^n. \]
According to (3), the function \(G_\beta\) is in \(U\) (and hence is univalent in \(D\)) if and only if
\[ 2(1 - \beta) \sum_{n=2}^{\infty} (n - 1)r^n \leq 1, \quad \text{i.e.} \quad \frac{2(1 - \beta) r^2}{(1 - r)^2} \leq 1. \]
The gives the condition \(0 < r \leq r_1 = 1/(1 + \sqrt{2(1 - \beta)})\). Thus, the function \(F_\beta\) is univalent in the disk \(|z| < r_1\) and not in any larger larger disk with center at the origin. Note also that
\[ F_\beta'(z) = \frac{1 - 2z - (1 - 2\beta)z^2}{(1 + (1 - 2\beta)z)^2} \]
and thus, \( F_\beta'(r_1) = 0 \). Moreover,
\[ U_{F_\beta}(z) = \frac{1 - 2z - (1 - 2\beta)z^2}{(1 - z)^2} - 1 \]
showing that \( U_{F_\beta}(r_1) = -1 \). Thus, the number \(r_1\) is best both for univalence and also for \(U\). The proof is complete. \(\square\)
**Corollary 1.** If $f \in S^*$, then $P_f \in U \cap S^*$ in the disk $|z| < \sqrt{2} - 1$. The result is sharp (as for univalence) as the Koebe function $z/(1 - z)^2$ shows.

**Proof.** It suffices to prove the starlikeness part since $P_f \in U$ follows from Theorem 1 by taking $\beta = 0$. Thus, for the proof of the second part, it suffices to observe by (4) that

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| = \frac{2z\omega'(z)}{1 - \omega^2(z)} \leq \frac{2|z|\omega'(z)}{1 - |\omega(z)|^2} \leq \frac{2|z|}{1 - |z|^2}$$

which is again less than 1 provided $|z| < \sqrt{2} - 1$. In particular, $F$ is starlike in the disk $|z| < \sqrt{2} - 1$. Sharpness part follows from the discussion in Theorem 1 with $\beta = 0$. \hfill \Box

**Corollary 2.** If $f \in S^*(1/2)$, then $P_f \in U \cap S^*$ in the disk $|z| < 1/2$. The result is sharp as the function $z/(1 - z)$ shows.

**Proof.** Choose $\beta = 1/2$ in Theorem 1 and observe that it suffices to prove the starlikeness part. As in the proof of Theorem 1 for each $f \in S^*(1/2)$, we have

$$zf'(z) = \frac{1}{1 - \omega(z)}$$

and $F(z) = z(1 - \omega(z))$ for some $\omega \in B$. By (5) and the fact that $|\omega(z)| \leq |z|$, we obtain

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| = \left| \frac{-z\omega'(z)}{1 - \omega(z)} \right| \leq \frac{|z|\omega'(z)}{1 - |\omega(z)|^2} \leq \frac{|z|(1 + |\omega(z)|)}{1 - |z|^2} \leq \frac{|z|(1 + |z|)}{1 - |z|^2} = \frac{|z|}{1 - |z|}$$

which is less than 1 if $|z| < 1/2$. Note that for $f(z) = z/(1 - z)$, one has $F(z) = z - z^2$ and thus, $|F'(z) - 1| = 2|z| < 1$ for $|z| < 1/2$ and $F'(1/2) = 0$. Thus, $F$ is univalent in the disk $|z| < 1/2$ and not in any larger disk with center at the origin. Also, it is easy to see that $F(z)$ is starlike for $|z| < 1/2$. The desired conclusion follows. \hfill \Box

**Corollary 3.** If $f \in S^*(1/2)$ such that $f''(0) = 0$, then $P_f$ is starlike in the disk $|z| < r_2$, where $r_2 \approx 0.543689$ is the root of the equation $\phi_2(r) = 0$, where

$$\phi_2(r) = r^3 + r^2 + r - 1.$$ 

**Proof.** Clearly, we just need to apply Corollary 2 with $|\omega(z)| \leq |z|^2$. This will lead to the inequality

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| \leq \frac{|z|(1 + |z|^2)}{1 - |z|^2}$$

which is clearly less than 1 if $|z|^3 + |z|^2 + |z| - 1 < 0$. The result follows. \hfill \Box

**Corollary 4.** Let $f$ belong to either $S^*(1/2)$ or $C(-1/2)$, such that $f''(0) = 0$. Then $F \in U$ in the disk $|z| < 1/\sqrt{3}$. 


Proof. It known that [9, p. 68] if $C(−1/2)$ with $f''(0) = 0$, then $f \in S^*(1/2)$. In view of this result, it suffices to prove the corollary when $f$ belongs to $S^*(1/2)$ with $f''(0) = 0$. However, using the proof of Theorem 1 with $\beta = 1/2$ and $|\omega(z)| \leq |z|^2$, we easily obtain that

$$|U_F(z)| \leq \frac{|z|^2}{1-|z|^2} \left(\frac{1+|\omega(z)|}{1-|\omega(z)|}\right) \leq \frac{|z|^2}{1-|z|^2} \left(\frac{1+|z|^2}{1-|z|^2}\right)$$

which is less than 1 provided $1 - 3|z|^2 > 0$ and this gives the disk $|z| < 1/\sqrt{3}$. The proof is complete. □

A locally univalent function $f \in A$ is said to belong to $G(\alpha)$, for some $\alpha \in (0, 1]$, if it satisfies the condition

(9) $\text{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2}$, $z \in \mathbb{D}$.

Thus, we have $G := G(1)$.

**Theorem 2.** If $f \in G(\alpha)$ for some $\alpha \in (0, 1]$, then $P_f$ is starlike in the disk $|z| < 1 + \alpha - \sqrt{\alpha(1+\alpha)}$.

**Proof.** Let $f \in G(\alpha)$ and $F$ be given by (1). Then we have (see eg. [5, Theorem 1])

$$\frac{zf'(z)}{f(z)} < \frac{(1+\alpha)(1-z)}{1+\alpha-z}, \quad z \in \mathbb{D},$$

and thus, we may write

$$\frac{zf'(z)}{f(z)} = \frac{(1+\alpha)(1-\omega(z))}{1+\alpha-\omega(z)} \quad \text{and} \quad F(z) = P_f = \frac{z(1+\alpha-\omega(z))}{(1+\alpha)(1-\omega(z))}$$

for some $\omega \in B$. By a computation, we obtain that

$$\frac{zf'(z)}{F(z)} - 1 = \frac{\alpha z \omega'(z)}{(1-\omega(z))(1+\alpha-\omega(z))}$$

and, as before, it follows from the Schwarz-Pick lemma that

$$\left|\frac{zf'(z)}{F(z)} - 1\right| \leq \frac{\alpha |z| |\omega'(z)|}{(1+\alpha-|\omega(z)|)(1-|\omega(z)|)} \leq \frac{\alpha |z|}{(1+\alpha-|z|)(1-|z|)}$$

which is less than 1 provided $\phi_3(|z|) > 0$, where $\phi_3(r) = r^2 - 2(1+\alpha)r + 1 + \alpha$. Thus, we conclude that $P_f$ is starlike in the disk $|z| < r_3(\alpha) = 1 + \alpha - \sqrt{\alpha(1+\alpha)}$, where $r_3(\alpha)$ is the root of the equation $\phi_3(r) = 0$ in the interval $(0, 1]$. The theorem follows. □

Taking $\alpha = 1$ gives

**Corollary 5.** If $f \in G$, then $P_f$ is starlike in the disk $|z| < 2 - \sqrt{2} \approx 0.585786$.

The same reasoning gives as in Corollary 3 the following.
Corollary 6. If \( f \in \mathcal{G}(\alpha) \) such that \( f''(0) = 0 \) and for some \( \alpha \in (0, 1] \), then \( P_f \) is starlike in \( |z| < r_4(\alpha) \), where \( r_4(\alpha) \) is the root in the interval \((0, 1]\) of the equation \( \phi_4(r) = 0 \),
\[
\phi_4(r) = r^4 - \alpha r^3 - (2 + \alpha) r^2 - \alpha r + 1 + \alpha.
\]

Proof. In this case, the corresponding inequality for \( f \in \mathcal{G}(\alpha) \) in Theorem 2 becomes
\[
\left| \frac{z F'(z)}{F(z)} - 1 \right| = \alpha |z| \left( 1 - \frac{|\omega(z)|}{1 + \alpha - |\omega(z)|} \right) < \frac{\alpha |z|^2}{1 + \alpha - |\omega(z)|^2}
\]
which is less than 1 if \( \phi_4(|z|) > 0 \). The result follows. \( \square \)

Setting \( \alpha = 1 \) gives

Corollary 7. If \( f \in \mathcal{G} \) such that \( f''(0) = 0 \), then \( P_f \) is starlike in \( |z| < r_4 \), where \( r_4 \approx 0.64731 \) is the root in the interval \((0, 1]\) of the equation \( r^4 - r^3 - 3r^2 - r + 2 = 0 \).

Theorem 3. If \( f \in \mathcal{G}(\alpha) \) for some \( \alpha \in (0, 1] \), then \( F \in \mathcal{U} \) in the disk \( |z| < r_5(\alpha) \), where \( r_5(\alpha) = \sqrt{-\alpha + \sqrt{(1+\alpha)^2 + 1}} \).

Proof. Let \( f \in \mathcal{G}(\alpha) \) and \( F = P_f \) be given by (1). Then, following the proof of Theorem 2 one has
\[
\frac{z}{F(z)} - 1 = -\frac{\alpha \omega(z)}{1 + \alpha - \omega(z)}
\]
and, using this relation, we find that
\[
U_F(z) = -\frac{\alpha \omega(z)}{1 + \alpha - \omega(z)} + \frac{\alpha(1 + \alpha) \omega'(z)}{(1 + \alpha - \omega(z))^2}
\]
\[
= \frac{\alpha[(1 + \alpha)(z \omega'(z) - \omega(z)) + \omega^2(z)]}{(1 + \alpha - \omega(z))^2}
\]
so that, by (1), we easily have as before that
\[
|U_F(z)| \leq \frac{\alpha}{1 + \alpha - |\omega(z)|^2} \left( 1 + \alpha \right) \left( \frac{|z|^2 - |\omega(z)|^2}{1 - |z|^2} \right) + |\omega(z)|^2
\]
\[
= \frac{\alpha}{1 - |z|^2} \left( -\frac{\alpha + |z|^2}{1 + \alpha - |\omega(z)|^2} \right) = \frac{\alpha \phi(t)}{1 - r^2},
\]
where we put \( |z| = r, |\omega(z)| = t \) and
\[
\phi(t) = -\frac{(\alpha + r^2)t^2 + (1 + \alpha)r^2}{(1 + \alpha - t)^2}, \quad 0 \leq t \leq r.
\]

We compute that
\[
\phi'(t) = \frac{2(1 + \alpha)}{(1 + \alpha - t)^3} \left[ -2(\alpha + r^2)t + r^2 \right],
\]
and it is easy to see that \( \phi \) attains its maximum value \( \phi(t_0) \), where \( t_0 = \frac{r^2}{\alpha + r^2} \) and \( \phi''(t_0) < 0 \). A calculation gives

\[
\phi(t_0) = \frac{r^2(\alpha + r^2)}{\alpha(1 + \alpha + r^2)}
\]

and thus, we have

\[
|U_F(z)| \leq \frac{\alpha \phi(t_0)}{1 - r^2} = \frac{r^2(\alpha + r^2)}{(1 - r^2)(1 + \alpha + r^2)}
\]

which is less than 1 if \( 2r^4 + 2\alpha r^2 - (1 + \alpha) < 0 \). This gives that \( |U_F(z)| < 1 \) for \( 0 < r < r_5(\alpha) \), where \( r_5(\alpha) \) is the root of the equation \( 2r^4 + 2\alpha r^2 - (1 + \alpha) = 0 \), that lies in the interval \((0, 1)\). The conclusion follows. \( \square \)

The choice \( \alpha = 1 \) yields the following.

**Corollary 8.** If \( f \in G \), then \( F \) belongs to the class \( U \) in the disk \( |z| < \sqrt{\frac{\sqrt{5} - 1}{2}} \approx 0.78615 \).

**Theorem 4.** Let \( f \in S \) with \( a_2 = f''(0)/2! \). Then \( F \) belongs to \( U \) in the disk \( |z| < r_6(|a_2|) \), where \( r_6(|a_2|) \) is the root of the equation \( \phi_5(r) = 0 \) that lies in the interval \((0, 1)\), where

\[
\phi_5(r) = (a+1-\frac{1}{4}b^2)r^{10}-(5a+5-\frac{5}{4}b^2)r^8+(19a+10-\frac{19}{4}b^2)r^6+(9a-10-\frac{9}{4}b^2)r^4+5r^2-1
\]

with \( b = |a_2| \) and \( a = \frac{2\pi^2 - 12}{3} \approx 2.57974 \).

**Proof.** Let \( f \in S \) and following the idea of [6, Theorem 4], we consider

\[
\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n(f)z^n,
\]

where \( c_n(f) \) (\( n \geq 1 \)) denote the logarithmic coefficients of \( f \) with \( c_1(f) = a_2 \). Further, for \( f \in S \) the following sharp inequality is known from the work of Roth [13, Theorem 1.1]

\[
\sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^2 |c_n(f)|^2 \leq \frac{2\pi^2 - 12}{3} = a.
\]

By (10), we obtain

\[
\frac{zf'(z)}{f(z)} - 1 = \sum_{n=1}^{\infty} nc_n(f)z^n
\]

which by the relation [8] gives that

\[
U_F(z) = -\sum_{n=1}^{\infty} n(n-1)c_n(f)z^n
\]
and thus, by the Cauchy-Schwarz inequality, we obtain that

\[
|U_F(z)| = \left| \sum_{n=2}^{\infty} n(n-1)c_n(f)z^n \right| \\
\leq \left( \sum_{n=2}^{\infty} \left( \frac{n}{n+1} \right)^2 |c_n(f)|^2 \right)^{\frac{1}{2}} \left( \sum_{n=2}^{\infty} (n^2-1)^2 |z|^{2n} \right)^{\frac{1}{2}} \\
\leq \left( a - \frac{1}{4} |c_1(f)|^2 \right)^{\frac{1}{2}} \left( |z|^4 (|z|^6 - 5|z|^4 + 19|z|^2 + 9) \right)^{\frac{1}{2}} \\
\left( a - \frac{1}{4} |c_1(f)|^2 \right) |z|^4 (|z|^6 - 5|z|^4 + 19|z|^2 + 9) < (1 - |z|^2)^5.
\]

which is less than 1 whenever,

\[
\left( a - \frac{1}{4} |c_1(f)|^2 \right) |z|^4 (|z|^6 - 5|z|^4 + 19|z|^2 + 9) < (1 - |z|^2)^5.
\]

If we put \( r = |z| \), then the last inequality is equivalent to \( \phi_5(r) := \phi_5(r, |a_2|) < 0 \), where \( \phi_5(r) \) is as in the statement. The desired result follows.

\[\square\]

**Corollary 9.** Let \( f \in S \) with \( f''(0) = 0 \), and \( a = \frac{2\pi^2 - 12}{3} \). Then \( F \) belongs to \( \mathcal{U} \) in the disk \( |z| < r_6 \), where \( r_6 \approx 0.360794 \) is the root of the equation

\[
(a + 1)r^{10} - 5(a + 1)r^8 + (19a + 10)r^6 + (9a - 10)r^4 + 5r^2 - 1 = 0,
\]

that lies in the interval \((0, 1)\).

**Proof.** Set \( a_2 = 0 \) in Theorem 4. \[\square\]

It is a simple exercise to see that the values \( r_6(|a_2|) \), as the roots of the equation \( \phi_5(r) = 0 \), increase with increasing values of \( |a_2| \in [0, 2] \). For a ready reference, we included in Table 1 a list of values of \( r_6(|a_2|) \) for certain choices of \( |a_2| \). This observation shows that if \( f \in S \), then \( F \in \mathcal{U} \) in the disk \( |z| < r \) and the lower bound for \( r \) by Corollary 9 is \( r_6 \approx 0.360794 \). We end the discussion with a conjecture that the upper bound for the value of \( r \) is \( \sqrt{2} - 1 \) which is attained by the Koebe function.

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| Values of \( |a_2| \) | Values of \( r_6(|a_2|) \) | Values of \( |a_2| \) | Values of \( r_6(|a_2|) \) |
|---|---|---|---|
| 0.25 | 0.361166 | 1.25 | 0.370874 |
| 0.5 | 0.362294 | 1.5 | 0.375923 |
| 0.75 | 0.364226 | 1.75 | 0.382504 |
| 1 | 0.367042 | 2 | 0.391124 |

Table 1. Values of \( r_6(|a_2|) \) for different values of \( |a_2| \).
Where is \( f(z)/f'(z) \) univalent?

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M. Obradović, Department of Mathematics, Faculty of Civil Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia.

E-mail address: obrad@grf.bg.ac.rs

S. Ponnusamy, Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and Security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India.

E-mail address: samy@isichennai.res.in, samy@iitm.ac.in

K.-J. Wirths, Institut für Analysis und Algebra, TU Braunschweig, 38106 Braunschweig, Germany

E-mail address: kjwirths@tu-bs.de