The integral formalism and the generating function of grand confluent hypergeometric function

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Abstract

Biconfluent Heun (BCH) function, a confluent form of Heun function\(^1\), is the special case of Grand Confluent Hypergeometric (GCH) function\(^1\) this has a regular singularity at \(x = 0\), and an irregular singularity at \(\infty\) of rank 2.

In this paper I apply three term recurrence formula (3TRF) \(^{18}\) to the integral formalism of GCH function including all higher terms of \(A_n\)’s and the generating function for the GCH polynomial which makes \(B_n\) term terminated. I show how to transform power series expansion in closed forms of GCH equation to its integral representation analytically.

This paper is 10th out of 10 in series “Special functions and three term recurrence formula (3TRF)”. See section 6 for all the papers in the series. The previous paper in the series describes the power series expansion in closed forms of GCH equation and its asymptotic behaviours\(^{25}\).

Keywords: Biconfluent Heun Equation, Generating function, Integral form, Three term recurrence formula

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1. Introduction

\[
\frac{x^2}{dx^2}y + (\mu x^2 + \varepsilon x + \nu) \frac{dy}{dx} + (\Omega x + \varepsilon \omega) y = 0
\]

\(^1\) is Grand Confluent Hypergeometric (GCH) differential equation where \(\mu, \varepsilon, \nu, \Omega\) and \(\omega\) are real or imaginary parameters \(^{17, 25}\). GCH ordinary differential equation is of Fuchsian types with the one regular and one irregular singularities. In contrast, Heun equation of Fuchsian types has the four regular singularities. Heun equation has the four kind of confluent forms: (1) Confluent Heun (two regular and one irregular singularities), (2) Doubly confluent Heun (two irregular singularities), (3) Biconfluent Heun (one regular and one irregular singularities), (4)\(^{\text{1}}\)

\(^{\text{1}}\)For the canonical form of BCH equation \(^{\text{2}}\), replace \(\mu, \varepsilon, \nu, \Omega\) and \(\omega\) by \(-2, -\beta, 1 + \alpha, \gamma - \alpha - 2\) and \(1/2(\delta \beta + 1 + \alpha)\) in \(^{\text{1}}\). For DLFM version \(^{15}\) or in ref. \(^{14}\), replace \(\mu\) and \(\omega\) by 1 and \(-\tau/\varepsilon\) in \(^{\text{1}}\).

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Triconfluent Heun equations (one irregular singularity). BCH equation is derived from the GCH equation by changing all coefficients.\[2\]

In previous paper I construct analytic solutions of GCH function for all higher terms of $A_n$'s\[25\] by applying three term recurrence formula\[18\]; for power series expansions of infinite series and polynomial, its asymptotic behaviors and boundary conditions for an independent variable $x$.

In this paper I consider an integral form of GCH function and the generating function for the GCH polynomial which makes $B_n$ term terminated. Since the integral form of GCH function is constructed, the GCH function is able to be transformed to other well-known special functions analytically such as Bessel, Kummer and Hypergeometric functions, etc.

I already obtained approximative normalized wave function of the spin free Hamiltonian involving only scalar potential for the $q - \bar{q}$ system up to first order of extremely small mass of quark\[17\]. According to this paper we might be possible to obtain the analytic normalized wave function from the generating function for the GCH polynomial.

This new wave function has infinite eigenvalues\[17\]. Because a GCH differential equation consists of three recursive coefficients\[18\]. In contrast any differential equations having two recursive coefficients have only one eigenvalue; i.e. the wave function for hydrogen-like atom.

We can apply GCH function into modern physics\[3 \ 4 \ 5 \ 6 \ 7\]. Section 4 contain three additional examples using power series expansion in closed forms and its integral form of GCH function.

2. Integral formalism

2.1. Polynomial which makes $B_n$ term terminated

In this article Pochhammer symbol $(x)_n$ is used to represent the rising factorial: $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. There is a generalized hypergeometric function which is written by

$$I_j = \sum_{i=1}^{\beta} (-\beta)_i h_{i-1}(1 + \frac{j}{2} + \frac{j}{2})_i (\frac{j}{2} + \frac{j}{2})_{i-1} \gamma^{-i-1}$$

$$= \sum_{i=0}^{\beta} B(i_j + \frac{j}{2} + \frac{j}{2}, l + 1) B(i_{j-1} - 1 + \frac{j}{2} + \frac{j}{2}, l + 1)((-\beta) + i_j) \gamma^{-i}$$

(2)

By using integral form of beta function,

$$B(i_j + \frac{j}{2} + \frac{j}{2}, l + 1) = \int_0^1 dt_j i_j^{i_j + \frac{j}{2} + \frac{j}{2}} (1 - t_j)^l$$

(3a)

$$B(i_{j-1} + \frac{j}{2} + \frac{j}{2}, l + 1) = \int_0^1 du_j u_j^{i_{j-1} + \frac{j}{2} + \frac{j}{2}} (1 - u_j)^l$$

(3b)

Substitute (3a) and (3b) into (2), and divide $(i_j + \frac{j}{2} + \frac{j}{2})(i_{j-1} - 1 + \gamma + \frac{j}{2} + \frac{j}{2})$ into the new (2).

$$\frac{1}{(i_j + \frac{j}{2} + \frac{j}{2})(i_{j-1} - 1 + \gamma + \frac{j}{2} + \frac{j}{2})} \sum_{i=0}^{\beta} (-\beta)_i h_{i-1}(1 + \frac{j}{2} + \frac{j}{2})_i \gamma^{-i-1}$$

$$= \int_0^1 dt_j i_j^{i_j + \frac{j}{2} + \frac{j}{2}} \int_0^1 du_j u_j^{i_{j-1} + \frac{j}{2} + \frac{j}{2}} (z_j, \mu_j)^{i_{j-1}} \sum_{i=0}^{\infty} \frac{(-\beta) - i_{j-1}}{(1)_i} \gamma^{-i-1} (1 - t_j)(1 - u_j)^l (4)$$
Confluent hypergeometric polynomial of the first kind is defined by

\[ F_{\beta, \gamma}(y; z) = \frac{\Gamma(\beta + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(-\beta_0)_n}{(\gamma)_n n!} z^n = \frac{\beta_0!}{2\pi i} \int \frac{dv}{v^{\beta_0+1}(1-v)^\gamma} \]

(5)

Replace \( \beta_0, \gamma, v \) and \( z \) by \( \beta_j - i_{j-1}, 1, v_j \) and \( z(1-t_j)(1-u_j) \) in (5), and divide \( \Gamma(\beta_j + 1 - i_{j-1}) \) on the new (6).

\[
\frac{F_{\beta_j-i_{j-1}}(y = 1; z(1-t_j)(1-u_j))}{\Gamma(\beta_j + 1 - i_{j-1})} = \frac{1}{2\pi i} \int \frac{dv_j}{v_j^{\beta_j+1-i_j}(1-v_j)} \\
= \sum_{i=0}^{\infty} \left( (-\beta_j - i_{j-1})_i \right)_i \left[ z(1-t_j)(1-u_j) \right]^i
\]

(6)

Substitute (6) into (4).

\[
K_j = \frac{1}{(i_{j-1} + \frac{1}{2} + \frac{1}{2})(i_j-1 - 1 + \gamma + \frac{1}{2} + \frac{1}{2})} \sum_{i_{j-1}}^{\beta_j} \left( (-\beta_j)_i (\frac{1}{2} + \frac{1}{2})_i (\frac{1}{2} + \gamma + \frac{1}{2})_i \right) \\
= \int_0^1 dt_j \left( \int_0^1 du_j \gamma^{2i+j} \right) \frac{1}{2\pi i} \int dv_j \left\{ \exp \left( -\frac{v_j}{1-t_j} \gamma(1-t_j)(1-u_j) \right) \right\}
\]

(7)

In Ref. [25] the general expression of power series of GCH equation for polynomial which makes \( B_n \) term terminated is given by; \( \lambda \) is indicial roots which are 0 or 1 - \( \nu \)

\[
y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots
\]

\[
= c_0 x^\nu \left\{ \sum_{i=0}^{\beta_0} \frac{(-\beta_0)_i}{(1 + \frac{1}{2})_i_0 (\gamma + \frac{1}{2})_i_0} z^i + \sum_{i=0}^{\beta_0} \frac{(i_0 + \frac{1}{2} + \frac{1}{2})_i}{(i_0 + \frac{1}{2} + \gamma + \frac{1}{2})_i (\gamma + \frac{1}{2})_i} \left( \frac{(i_0 + \frac{1}{2} + \frac{1}{2})_i}{(i_0 + \frac{1}{2} + \gamma + \frac{1}{2})_i (\gamma + \frac{1}{2})_i} \right) \right\} \nonumber
\]

\[
+ \sum_{i=0}^{\infty} \left\{ \sum_{i=0}^{\beta_0} \frac{(i_0 + \gamma + \frac{1}{2})_i}{(i_0 + \frac{1}{2} + \gamma + \frac{1}{2})_i} \left( \frac{(i_0 + \gamma + \frac{1}{2})_i}{(i_0 + \frac{1}{2} + \gamma + \frac{1}{2})_i} \right) \right\}
\]

(8)

where

\[
\begin{aligned}
z &= -\frac{1}{2} \mu x^2 \\
\hat{\epsilon} &= -\frac{1}{2} \mu x \\
\gamma &= \frac{1}{2} (1 + \nu) \\
\Omega &= -\mu(2\beta_i + i + \lambda) \quad \text{as} \quad i = 0, 1, 2, \cdots \quad \text{and} \quad \beta_i = 0, 1, 2, \cdots \\
\text{As} \quad i \leq j \rightarrow \beta_j \leq \beta_j
\end{aligned}
\]
The general expression of the integral representation of the GCH polynomial which makes $B_n$ term terminated is given by

$$y(x) = c_0 x^L \left\{ \sum_{h=0}^{\beta_0} \frac{(-\beta_0)_h}{(1 + \frac{i}{2})_h(y + \frac{i}{2})_h} z^h + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left( \int_0^1 dt_{n-j} t_{n-j}^{\frac{n-j-1+\xi}{2}} \int_0^1 du_{n-j} u_{n-j}^{(n-j-2+\xi)/2} \right) \right\} \right\}$$

$$\times \frac{1}{2\pi i} \oint\frac{dv_{n-j}}{v_{n-j}^\beta(1 - v_{n-j})} \left\{ w_{n-j} \frac{\exp(-\nu_{n-j} \sum_{j=0}^{n-1} (1 - t_{n-j})(1 - u_{n-j})/2)}{(1 - v_{n-j})} \left( \frac{w_{n-j} \frac{\exp(-\nu_{n-j} \sum_{j=0}^{n-1} (1 - t_{n-j})(1 - u_{n-j})/2)}{(1 - v_{n-j})} \left( n - j - 1 + \omega + \lambda \right) \right) \right\}$$

where

$$w_{a,b} = \left\{ \begin{array}{ll}
\left\lfloor \frac{b}{a} \right\rfloor & \text{if } a \leq b \\
\infty & \text{if } a > b
\end{array} \right. \text{ where } a \leq b$$

**Proof of Theorem.** In (5) sub-power series $y_0(x), y_1(x), y_2(x)$ and $y_3(x)$ of the GCH polynomial which makes $B_n$ term terminated are given by

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+1} = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots$$

where

$$y_0(x) = c_0 x^L \sum_{h=0}^{\beta_0} \frac{(-\beta_0)_h}{(1 + \frac{i}{2})_h(y + \frac{i}{2})_h} z^h$$

$$y_1(x) = c_0 x^L \left\{ \sum_{h=0}^{\beta_1} \frac{(i_0 + \frac{1}{2} + \frac{1}{2})_h (i_0 - \frac{1}{2} + \gamma + \frac{1}{2})_h}{(i_0 + \frac{1}{2})_h(y + \frac{1}{2})_h} \right\}$$

$$\times \sum_{h=0}^{\beta_1} \frac{(-\beta_1)_h}{(i_0 + \frac{1}{2})_h(y + \frac{1}{2} + \frac{1}{2})_h} z^h$$

$$y_2(x) = c_0 x^L \left\{ \sum_{h=0}^{\beta_1} \frac{(i_0 + \frac{1}{2} + \frac{1}{2})_h (i_0 - \frac{1}{2} + \gamma + \frac{1}{2})_h}{(i_0 + \frac{1}{2})_h(y + \frac{1}{2})_h} \right\}$$

$$\times \sum_{h=0}^{\beta_1} \frac{(-\beta_1)_h}{(i_0 + \frac{1}{2})_h(y + \frac{1}{2} + \frac{1}{2})_h} z^h$$

$$y_3(x) = c_0 x^L \left\{ \sum_{h=0}^{\beta_1} \frac{(i_0 + \frac{1}{2} + \frac{1}{2})_h (i_0 - \frac{1}{2} + \gamma + \frac{1}{2})_h}{(i_0 + \frac{1}{2})_h(y + \frac{1}{2})_h} \right\}$$

$$\times \sum_{h=0}^{\beta_1} \frac{(-\beta_1)_h}{(i_0 + \frac{1}{2})_h(y + \frac{1}{2} + \frac{1}{2})_h} z^h$$

$^2$ $y_2(x)$ means the sub-power series in (5) contains one term of $A_s'$, $y_3(x)$ means the sub-power series in (5) contains two terms of $A_s'$, $y_4(x)$ means the sub-power series in (5) contains three terms of $A_s'$, etc.
\[ y_2(x) = c_0 x^t \left\{ \sum_{i=0}^{\beta_0} \frac{(i_0 + \frac{i}{2} + \frac{\omega}{2})(-\beta_0)_{i_0}}{(i_0 + \frac{1}{2} + \frac{\omega}{2})(i_0 + \frac{1}{2} + \gamma + \frac{\omega}{2})(1 + \frac{\omega}{2})_{i_0}(y + \frac{\omega}{2})_{i_0}} \right. \\
\left. \times \sum_{i=0}^{\beta_1} \frac{(i_1 + \frac{1}{2} + \frac{\omega}{2})(-\beta_1)_{i_1}(\frac{3}{2} + \frac{\omega}{2})_{i_1}(\gamma + \frac{1}{2} + \frac{\omega}{2})_{i_1}}{(i_1 + 1 + \frac{1}{2})(i_1 + \gamma + \frac{1}{2})(-\beta_1)_{i_1}(\frac{3}{2} + \frac{\omega}{2})_{i_1}(\gamma + \frac{1}{2} + \frac{\omega}{2})_{i_1}} \right\} \times \sum_{i=1}^{\beta_2} \frac{(-\beta_2)_{i_2}(2 + \frac{1}{2})_{i_2}(\gamma + 1 + \frac{1}{2})_{i_2}}{(-\beta_2)_{i_2}(2 + \frac{1}{2})_{i_2}(\gamma + 1 + \frac{1}{2})_{i_2}} x^i e^{i^2} \right\} \right\} (11c) \\
y_3(x) = c_0 x^t \left\{ \sum_{i=0}^{\beta_0} \frac{(i_0 + \frac{i}{2} + \frac{\omega}{2})(-\beta_0)_{i_0}}{(i_0 + \frac{1}{2} + \frac{\omega}{2})(i_0 + \frac{1}{2} + \gamma + \frac{\omega}{2})(1 + \frac{\omega}{2})_{i_0}(y + \frac{\omega}{2})_{i_0}} \right. \\
\left. \times \sum_{i=0}^{\beta_1} \frac{(i_1 + \frac{1}{2} + \frac{\omega}{2})(-\beta_1)_{i_1}(\frac{3}{2} + \frac{\omega}{2})_{i_1}(\gamma + \frac{1}{2} + \frac{\omega}{2})_{i_1}}{(i_1 + 1 + \frac{1}{2})(i_1 + \gamma + \frac{1}{2})(-\beta_1)_{i_1}(\frac{3}{2} + \frac{\omega}{2})_{i_1}(\gamma + \frac{1}{2} + \frac{\omega}{2})_{i_1}} \right\} \times \sum_{i=1}^{\beta_2} \frac{(-\beta_3)_{i_3}(\frac{3}{2} + \frac{1}{2})_{i_3}(\gamma + \frac{1}{2} + \frac{\omega}{2})_{i_3}}{(-\beta_3)_{i_3}(\frac{3}{2} + \frac{1}{2})_{i_3}(\gamma + \frac{1}{2} + \frac{\omega}{2})_{i_3}} x^i e^{i^3} \right\} (11d) \]

Put \( j = 1 \) in (7). Take the new (7) into (11b).
Put $j = 2$ in (7). Take the new (7) into (11c).

$$
y_2(x) = c_0 x^i \int_0^1 dt_2 t_2^{\gamma + \frac{i}{2}} \int_0^1 du_2 u_2^{\gamma - 1 + \frac{i}{2}} \frac{1}{2\pi i} \oint_{\gamma + \frac{i}{2} + \frac{\omega}{2}} \exp \left( -\frac{\gamma}{1 - t_2} z(1 - t_2)(1 - u_2) \right) \left( w_{2,2} \partial w_{2,2} + \frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \\
\times \left\{ \sum_{\beta_0 = 0}^{\beta_0} \frac{(i_0 + \frac{1}{2} + i_0)}{(i_0 + \frac{1}{2} + i_0) (i_0 - \frac{1}{2} + \gamma + \frac{1}{2})} \frac{(-\beta_0)_n}{(\beta_0)_n} \right\} e^2 \\
\times \left\{ \sum_{\beta_1 = 0}^{\beta_1} \frac{(-\beta_1)_n (i_1 + \frac{1}{2})_n (i_1 - \frac{1}{2})_n (i_1 + \frac{1}{2} + \frac{3}{2})_n}{(-\beta_1)_n (i_1 + \frac{1}{2} + \frac{3}{2})_n (i_1 + \frac{1}{2} + \frac{3}{2})_n} \right\} w_{1,2}^i \right\} e^2
$$

(13)

where $w_{2,2} = z \int_{\eta = 2}^{\bar{\eta}_{2,2}} \xi (\eta \gamma i \eta)$

Put $j = 1$ and $z = \bar{z}_{2,2}$ in (7). Take the new (7) into (13).

$$
y_2(x) = c_0 x^i \int_0^1 dt_1 t_1^{\gamma + \frac{i}{2}} \int_0^1 du_1 u_1^{\gamma - 1 + \frac{i}{2}} \frac{1}{2\pi i} \oint_{\gamma + \frac{i}{2} + \frac{\omega}{2}} \exp \left( -\frac{\gamma}{1 - t_1} z(1 - t_1)(1 - u_1) \right) \left( w_{1,2} \partial w_{1,2} + \frac{1}{2} + \frac{\lambda}{2} + \frac{\omega}{2} \right) \\
\times \left\{ \sum_{\beta_0 = 0}^{\beta_0} \frac{(i_0 + \frac{1}{2} + i_0)}{(i_0 + \frac{1}{2} + i_0) (1 + \frac{1}{2})_n (1 + \frac{1}{2} + \gamma + \frac{1}{2})_n} \frac{(-\beta_0)_n}{(-\beta_0)_n} \right\} e^2 \\
\times \left\{ \sum_{\beta_1 = 0}^{\beta_1} \frac{(-\beta_1)_n (i_1 + \frac{1}{2})_n (i_1 - \frac{1}{2})_n (1 + \frac{1}{2} + \gamma + \frac{1}{2})_n}{(-\beta_1)_n (i_1 + \frac{1}{2} + \frac{3}{2})_n (i_1 + \frac{1}{2} + \frac{3}{2})_n} \right\} w_{1,2}^i \right\} e^2
$$

(14)

where $w_{1,2} = z \int_{\eta = 1}^{\bar{\eta}_{1,2}} \xi (\eta \gamma i \eta)$

By using similar process for the previous cases of integral forms of $y_1(x)$ and $y_2(x)$, the integral form of sub-power series expansion of $y_3(x)$ is

$$
y_3(x) = c_0 x^i \int_0^1 dt_3 t_3^{\gamma + \frac{i}{2}} \int_0^1 du_3 u_3^{\gamma - 1 + \frac{i}{2}} \frac{1}{2\pi i} \oint_{\gamma + \frac{i}{2} + \frac{\omega}{2}} \exp \left( -\frac{\gamma}{1 - t_3} z(1 - t_3)(1 - u_3) \right) \left( w_{3,3} \partial w_{3,3} + 1 + \frac{1}{2} + \frac{\omega}{2} \right) \\
\times \left\{ \sum_{\beta_0 = 0}^{\beta_0} \frac{(i_0 + \frac{1}{2} + i_0)}{(i_0 + \frac{1}{2} + i_0) (1 + \frac{1}{2})_n (1 + \frac{1}{2} + \gamma + \frac{1}{2})_n} \frac{(-\beta_0)_n}{(-\beta_0)_n} \right\} e^3 \\
\times \left\{ \sum_{\beta_1 = 0}^{\beta_1} \frac{(-\beta_1)_n (i_1 + \frac{1}{2})_n (i_1 - \frac{1}{2})_n (1 + \frac{1}{2} + \gamma + \frac{1}{2})_n}{(-\beta_1)_n (i_1 + \frac{1}{2} + \frac{3}{2})_n (i_1 + \frac{1}{2} + \frac{3}{2})_n} \right\} w_{1,3}^i \right\} e^3
$$

(15)
where

\[
\begin{cases}
  w_{3,3} = z \prod_{j=3}^{3} t_j u_j v_j \\
  w_{2,3} = z \prod_{j=2}^{3} t_j u_j v_j \\
  w_{1,3} = z \prod_{j=1}^{3} t_j u_j v_j
\end{cases}
\]

By repeating this process for all higher terms of integral forms of sub-summation \( y_m(x) \) terms where \( m \geq 4 \), we obtain every integral forms of \( y_m(x) \) terms. Since we substitute (11a), (12), (14), (15) and including all integral forms of \( y_m(x) \) terms where \( m \geq 4 \) into (10), we obtain (9).

Let \( \lambda = 0 \) and \( c_0 = \frac{\Gamma(\beta_0 + \gamma)}{\Gamma(\gamma)} \) in (9). Apply (5) into the new (9).

**Remark 1.** The integral representation of GCH equation of the first kind for polynomial which makes \( B_n \) term terminated about \( x = 0 \) as \( \Omega = -2\mu(\beta_i + \frac{\gamma}{2}) \) where \( i, \beta_i = 0, 1, 2, \ldots \) is

\[
y(x) = QW_{\beta_i}(\beta_i = -\frac{\Omega}{2\mu} - \frac{i}{2}; \gamma = \frac{1}{2}(1 + \nu); \bar{\nu} = -\frac{1}{2}ex; z = -\frac{1}{2}ux^2)
\]

\[
= F_{\beta_i}(\gamma; z) + \sum_{n=1}^{\infty} \left\{ \prod_{j=1}^{n-1} \int_{0}^{1} dt_{n-j} t_{n-j}^{\frac{1}{2}(n-j)-1} \int_{0}^{1} dt_{n-j} u_{n-j}^{\gamma+\frac{1}{2}(n-j)-2} \times \frac{1}{2\pi i} \oint dv_{n-j} \exp \left( \frac{v_{n-j}^{\psi_0}}{v_{n-j}^{\psi_0+1}} \right) F_{\beta_i}(\gamma; w_{1,n}) \right\} \bar{\mu}^n
\]

(16)

Confluent hypergeometric polynomial of the second kind is defined by

\[
A_{\psi_0}(\gamma; z) = \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} \frac{(-\psi_0)_n z^n}{n! (2 - \gamma)_n}
\]

\[
= \psi_0! \frac{1}{2\pi i} \oint dv_0 \exp \left( -\frac{v_0^{\psi_0}}{v_0^{\psi_0+1}} \right)
\]

(17)

Put \( c_0 = \left( -\frac{1}{2\mu} \right)^{1-\gamma} \frac{\Gamma(\psi_0+2-\gamma)}{\Gamma(2-\gamma)} \) as \( \lambda = 1 - \nu = 2(1 - \gamma) \) on (9) with replacing \( \beta_i \) by \( \psi_i \). Apply (17) into the new (9).

**Remark 2.** The integral representation of GCH equation of the second kind for polynomial which makes \( B_n \) term terminated about \( x = 0 \) as \( \Omega = -2\mu(\psi_i + 1 - \gamma + \frac{\gamma}{2}) \) where \( i, \psi_i = 0, 1, 2, \ldots \)
is

\[ y(x) = RW_\psi \left( \psi_i = -\frac{\Omega}{2x} + \gamma - 1 - \frac{i}{2}, \gamma = \frac{1}{2}(1 + \gamma); \tilde{e} = -\frac{1}{2}ex; \ z = -\frac{1}{2}\mu x^2 \right) \]

\[ = e^{1-\gamma} \int_0^1 \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} dt_{n-j} t_{n-j}^{\frac{1}{4}(\gamma-\beta)-\gamma} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{4}(\gamma-\beta)-1} \right\} \]

\[ \times \frac{1}{2\pi i} \oint \frac{dv_{n-j}}{v_{n-j}^{\frac{1}{4}(\gamma-\beta)-1} (1 - v_{n-j})} \]

\[ \times \left( w_{n-j,\mu} \theta_{w_{n-j,\mu}} + \frac{1}{2}(n - j + 1 - 2\gamma + \omega) A_{\phi_0}(\gamma; w_{1,n}) \right) \]  \[ \{ e^0 \} \]  \[ (18) \]

\subsection{2.2. Infinite series}

\textbf{Theorem 2.} The general expression of the integral representation of GCH equation for infinite series is given by

\[ y(x) = \sum_{m=0}^{\infty} y_m(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \]

\[ = c_0 x^1 \int_0^1 \frac{(\frac{\Omega}{2x} + \frac{i}{2})_n}{(1 + \frac{i}{2})_n (\gamma + \frac{i}{2})_n} z^n \]

\[ + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} dt_{n-j} t_{n-j}^{\frac{1}{4}(\gamma-\beta)-1} \int_0^1 du_{n-j} u_{n-j}^{\frac{1}{4}(\gamma-\beta)-2+\frac{i}{2}} \right\} \]

\[ \times \frac{1}{2\pi i} \oint \frac{dv_{n-j}}{v_{n-j}^{\frac{1}{4}(\gamma-\beta)-1} (1 - v_{n-j})} \]

\[ \times \left( w_{n-j,\mu} \theta_{w_{n-j,\mu}} + \frac{1}{2}(n - j + 1 - \omega + \lambda) \right) \]

\[ \int_0^1 \frac{(\frac{\Omega}{2x} + \frac{i}{2})_n}{(1 + \frac{i}{2})_n (\gamma + \frac{i}{2})_n} w_{n,\mu}^0 e^{\tilde{e} t} \]  \[ \{ e^0 \} \]

\[ (19) \]

\textbf{Proof of Theorem.} There is a generalized hypergeometric function which is written by

\[ L_j = \sum_{i=j}^{\infty} \frac{(\frac{\Omega}{2x} + \frac{i}{2} + \frac{j}{2})_j}{(\frac{\Omega}{2x} + \frac{i}{2} + \frac{j}{2} - \frac{i}{j})_j} (1 + \frac{i}{2} + \frac{j}{2})_i (\gamma + \frac{i}{2} + \frac{j}{2})_i (i+j-1)_i e^{i+j} \]

\[ = \sum_{i=j}^{\infty} \frac{B(i+j-1 + \frac{i}{2} + \frac{j}{2} + 1) B(i+j-1 + \gamma + \frac{i}{2} + \frac{j}{2} + l + 1) (\frac{\Omega}{2x} + \frac{i}{2} + \frac{j}{2} + i+j-1)}{(i+j-1 + \frac{i}{2} + \frac{j}{2} - \frac{i}{j} - 1 + \gamma + \frac{i}{2} + \frac{j}{2})_i l!} \]

\[ (20) \]
Substitute (3a) and (3b) into (20), and divide \((i_j-1+\frac{1}{2}+\frac{1}{2})(i_j-1+\gamma+\frac{1}{2}+\frac{1}{2})\) into the new (20).

\[
\frac{1}{(i_j-1+\frac{1}{2}+\frac{1}{2})(i_j-1+\gamma+\frac{1}{2}+\frac{1}{2})} \sum_{i=j-1}^{\infty} \left( \frac{\nu_j}{\nu_j} + \frac{1}{2} + \frac{1}{2} \right)_{i_j} (1 + \frac{1}{2} + \frac{1}{2})_{i_j-1} (\frac{1}{2} + \gamma + \frac{1}{2})_{i_j-1} e^{\nu_j} \\
= \int_{0}^{1} dt_j t_j^{i_j-1+\frac{1}{2}} \int_{0}^{1} du_j u_j^{i_j-2+\frac{1}{2}} \left( z(t_j, u_j) \right)^{i_j-1} \sum_{l=0}^{\infty} \left( \frac{\nu_j}{\nu_j} + \frac{1}{2} + \frac{1}{2} + i_j-1 \right)_{l} (1-z)(1-u_j)^l \tag{21}
\]

Kummer function of the first kind is defined by

\[
M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} z^n = e^z M(b-a, b, -z) \\
= -\frac{1}{2\pi i} \frac{\Gamma(1-a) \Gamma(b)}{\Gamma(b-a)} \int dv e^{i\nu(1-v)} (1-v)^{b-a-1} \\
= \frac{\Gamma(a)}{2\pi i} \int dv e^{i\nu/v} v^{b-1} \left( 1 - \frac{z}{v} \right)^{-a} \\
= \frac{1}{2\pi i} \frac{\Gamma(1-a) \Gamma(b)}{\Gamma(b-a)} \int dv e^{i\nu/v} (1-v)^{-b} \tag{22}
\]

Replace \(a, b\) and \(z\) by \(\frac{\nu_j}{\nu_j} + \frac{1}{2} + \frac{1}{2} + i_j-1\), \(1\) and \(z(1-t_j)(1-u_j)\) in (22). Take the new (22) into (21).

\[
Q_j = \frac{1}{(i_j-1+\frac{1}{2}+\frac{1}{2})(i_j-1+\gamma+\frac{1}{2}+\frac{1}{2})} \sum_{j=i_{j-1}}^{\infty} \left( \frac{\nu_j}{\nu_j} + \frac{1}{2} + \frac{1}{2} \right)_{i_j} (1 + \frac{1}{2} + \frac{1}{2})_{i_j-1} (\frac{1}{2} + \gamma + \frac{1}{2})_{i_j-1} e^{\nu_j} \\
= \int_{0}^{1} dt_j t_j^{i_j-1+\frac{1}{2}} \int_{0}^{1} du_j u_j^{i_j-2+\frac{1}{2}} \frac{1}{2\pi i} \int dv_j e^{-\nu_j/v_j} (1-v_j)^{-1} \left( z(t_j, u_j) \right)^{i_j-1} \tag{23}
\]
In Ref. [25] the general expression of power series of GCH equation for infinite series is given by

\[ y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots \]

\[ y(x) = c_0 x^2 \left\{ \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \right\} x^{i} + \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \times \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \times \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \times (25a) \]

In (24) sub-power series \( y_0(x), y_1(x), y_2(x) \) and \( y_3(x) \) of the GCH equation for infinite series using 3TRF about \( x = 0 \) are

\[ y_0(x) = c_0 x^2 \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} x^{i} \]

\[ y_1(x) = c_0 x^2 \left\{ \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \times \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \times \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \times (25b) \]

\[ y_2(x) = c_0 x^2 \left\{ \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \times \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \times \sum_{i=0}^{\infty} \frac{(\frac{1}{16} + \frac{x}{2})_0}{(1 + \frac{x}{4})_0(y + \frac{1}{2})_0} \times (25c) \]

(25a)
\[ y_3(x) = c_0 x^3 \left\{ \sum_{n=0}^{\infty} \frac{(i_0 + \frac{i}{2} + \frac{w}{2})}{(i_0 + \frac{i}{2} + \frac{w}{2})(i_0 - \frac{i}{2} + \gamma + \frac{w}{2})(1 + \frac{i}{2})_0(y + \frac{w}{2})_2} \right. \]

\[ \times \sum_{m=0}^{\infty} \frac{(i_2 + \frac{i}{2} + \frac{w}{2})}{(i_2 + \frac{i}{2} + \frac{w}{2})(i_2 - \frac{i}{2} + \gamma + \frac{w}{2})(1 + \frac{i}{2})_0(y + \frac{w}{2})_2} \left\{ \frac{(\gamma + \frac{w}{2})_n(y + \frac{w}{2})_n}{(\gamma + \frac{w}{2})_n(y + \frac{w}{2})_n} \right\} e^{2} \]
Put $j = 1$ and $z = \frac{m}{2}$. Take the new (23) into (27).

$$y_2(x) = c_0 x^4 \int_0^1 dt_2 t_2^{\frac{3}{2}} \int_0^1 du_2 u_2^{\gamma - 1 + \frac{1}{2}} \int_0^1 dv_2 \frac{1}{2\pi i} \oint \frac{v_2^{-(\frac{\gamma}{2} + 1) + 1} (1 - v_2)}{v_2^{-(\frac{\gamma}{2} + \frac{1}{2}) + 1}} \left( w_2, \partial_v w_2, + \left( \frac{1}{2} + \frac{\lambda}{2} + \omega \right) \right)$$

$$\times \sum_{l=0}^{\infty} \frac{(\Omega + \lambda)_l}{(1 + \frac{2}{3})_l (y + \frac{2}{3})_l} \frac{u_1^{l+2}}{u_1^{l+1}}$$

(28)

where $w_{1,2} = z \sum_{l=1}^{2} i \lambda u_l v_l$

By using similar process for the previous cases of integral forms of $y_1(x)$ and $y_2(x)$, the integral form of sub-power series expansion of $y_3(x)$ is

$$y_3(x) = c_0 x^4 \int_0^1 dt_3 t_3^{\frac{3}{2}} \int_0^1 du_3 u_3^{\gamma - 1 + \frac{1}{2}} \int_0^1 dv_3 \frac{1}{2\pi i} \oint \frac{v_3^{-(\frac{\gamma}{3} + 1) + 1} (1 - v_3)}{v_3^{-(\frac{\gamma}{3} + \frac{1}{3}) + 1}} \left( w_3, \partial_v w_3, + \left( \frac{1}{2} + \frac{\lambda}{2} + \omega \right) \right)$$

$$\times \sum_{l=0}^{\infty} \frac{(\Omega + \lambda)_l}{(1 + \frac{2}{3})_l (y + \frac{2}{3})_l} \frac{u_1^{l+2}}{u_1^{l+1}}$$

(29)

where

\[
\begin{align*}
    w_{3,3} &= z \sum_{l=3}^{3} i \lambda u_l v_l \\
    w_{2,3} &= z \sum_{l=2}^{3} i \lambda u_l v_l \\
    w_{1,3} &= z \sum_{l=1}^{3} i \lambda u_l v_l
\end{align*}
\]

By repeating this process for all higher terms of integral forms of sub-summation $y_m(x)$ terms where $m \geq 4$, we obtain every integral forms of $y_m(x)$ terms. Since we substitute (25a), (26), (28), (29) and including all integral forms of $y_m(x)$ terms where $m \geq 4$ into (24), we obtain (19).

Let $\lambda = 0$ and $c_0 = \frac{\Gamma(y - \frac{5}{2})}{\Gamma(y)}$ in (19). And apply (22) into the new (19).
Remark 3. The integral representation of GCH equation of the first kind for infinite series about \(x = 0\) for infinite series is

\[
y(x) = QW \left( y = \frac{1}{2}(1 + \nu); \tilde{e} = -\frac{1}{2}\tilde{e}x; z = -\frac{1}{2}x^2 \right) = \frac{\Gamma(\gamma - \frac{\nu}{2})}{\Gamma(\gamma)} \left( M \left( \frac{\Omega}{2\mu}, \gamma, \frac{\Omega}{2\mu} \right) + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \int_0^1 dt_{n-j} t_{n-j}^{\frac{\nu}{2}(n-j)-1} \int_0^1 du_{n-j} u_{n-j}^{\frac{\nu}{2}(n-j)-2} \times \frac{1}{2\pi i} \int dv_{n-j} \exp \left( -\frac{w_{n-j}}{1-v_{n-j}} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j}) \right) \left( w_{n-j,\mu,\beta} w_{n-j,\mu,\beta} + \frac{1}{2}(n - j + 1 + \omega) \right) \right\} \right) \times M \left( \frac{\Omega}{2\mu}, \gamma, w_{1,a} \right) \right)^{\frac{\nu}{2}} \right) \right)
\]

Put \(c_0 = \left( -\frac{1}{2\mu} \right)^{1-\gamma} \frac{\Gamma(1-\frac{\nu}{2})}{\Gamma(2-\gamma)}\) as \(\lambda = 1 - \nu = 2(1 - \gamma)\) on (19). And apply (22) into the new (19).

Remark 4. The integral representation of GCH equation of the second kind for infinite series about \(x = 0\) for infinite series is

\[
y(x) = RW \left( y = \frac{1}{2}(1 + \nu); \tilde{e} = -\frac{1}{2}\tilde{e}x; z = -\frac{1}{2}x^2 \right) = \frac{1}{2\pi i} \int_0^1 dt_{n-j} t_{n-j}^{\frac{\nu}{2}(n-j)-1} \int_0^1 dv_{n-j} \exp \left( -\frac{w_{n-j}}{1-v_{n-j}} w_{n-j+1,n}(1-t_{n-j})(1-u_{n-j}) \right) \left( w_{n-j,\mu,\beta} w_{n-j,\mu,\beta} + \frac{1}{2}(n - j + 1 + \omega) \right) \times M \left( \frac{\Omega}{2\mu} + 1 - \gamma, 2 - \gamma, w_{1,a} \right) \right)^{\frac{\nu}{2}} \right) \right)
\]

3. Generating function for the polynomial which makes \(B_a\) term terminated

Now let’s investigate the generating function for the GCH polynomials of the first and second kinds.

Definition 1. I define that

\[
\begin{aligned}
& s_{a,b} = \begin{cases} 
  s_a \cdot s_{a+1} \cdot s_{a+2} \cdots s_{b-2} \cdot s_{b-1} \cdot s_b & \text{where } a > b \\
  s_a & \text{only if } a = b
\end{cases} \\
& w_{a,b}^s = \sum_{\ell=0}^b t^{\ell} s_{a+\ell} \text{ where } a \leq b
\end{aligned}
\]

where

\[
a, b \in \mathbb{N}_0
\]
And I have
\[\sum_{\beta(n)\not=} \beta_i \gamma_i = \frac{\beta_i}{1 - s_i} \tag{33}\]

Acting the summation operator \(\sum_{\beta(n)\not=} \beta_i \gamma_i \prod_{n=1}^{\infty} \left( \sum_{\beta(n)\not=} \beta_i \gamma_i \right)\) on \(\mathcal{F}\) where \(|s_i| < 1\) as \(i = 0, 1, 2, \cdots\) by using (32) and (33).

**Theorem 3.** The general expression of the generating function for the GCH polynomial which makes \(B_n\) term terminated is given by

\[\sum_{\beta(n)\not=} \beta_i \gamma_i \prod_{n=1}^{\infty} \left( \sum_{\beta(n)\not=} \beta_i \gamma_i \right) \mathcal{F}(x)\]

\[= \prod_{k=1}^{\infty} \frac{1}{1 - s_{k,\infty}} \mathcal{F}(\lambda; s_{0,\infty}; z) + \prod_{k=1}^{\infty} \frac{1}{1 - s_{k,\infty}} \int_0^1 dt_1 t_1^{1+z+\frac{1}{2}} \int_0^1 du_1 u_1^{1+z+\frac{1}{2}} \times \exp\left(-s_{1,\infty}\frac{z(1-t_1)(1-u_1)}{\lambda}\right) \left(\partial^{\gamma+1} + \left(\frac{\lambda}{2} + \frac{1}{2}\right)\right) \mathcal{F}(\lambda; s_{0,\infty}; w_{1,\gamma}) \]
Acting the summation operator on (11a),

\[
\sum_{\beta_0=0}^{\infty} \frac{s_{00}}{\beta_0!} \prod_{n=1}^{\infty} \left\{ \sum_{\beta_n=\beta_{n-1}}^{\infty} s_n^\beta \right\} y_0(x)
\]

\[
= \prod_{k=1}^{\infty} \frac{1}{(1 - s_{k,\infty})} \sum_{\beta_0=0}^{\infty} \frac{s_{00}}{\beta_0!} \left( c_0 x^1 \sum_{i_0=0}^{\infty} \frac{(-\beta_0)_{i_0}}{(1 + \frac{i_0}{2})_{i_0} (y + \frac{i_0}{2})_{i_0}} s_{i_0,n} \right)
\]

(36)

Acting the summation operator on (12),

\[
\sum_{\beta_0=0}^{\infty} \frac{s_{00}}{\beta_0!} \prod_{n=1}^{\infty} \left\{ \sum_{\beta_n=\beta_{n-1}}^{\infty} s_n^\beta \right\} y_1(x)
\]

\[
= \prod_{k=2}^{\infty} \frac{1}{(1 - s_{k,\infty})} \int_{0}^{1} dt_1 t_1^{z+\frac{1}{2}} \int_{0}^{1} du_1 u_1^{z+\frac{1}{2}} \frac{1}{2\pi i} \oint dv_1 \exp \left( \frac{-n_{1,v_1} z (1-t_1)(1-u_1)}{v_1 (1-v_1)} \right) \sum_{\beta_1=0}^{\infty} \left( \frac{s_{1,\infty}}{v_1} \right)^{\beta_1}
\]

\[
\times \left( w_{1,1} \partial_{w_{1,1}} + \left( \frac{\omega}{2} + \frac{3}{2} \right) \right) \sum_{\beta_0=0}^{\infty} \frac{s_{00}}{\beta_0!} \left( c_0 x^1 \sum_{i_0=0}^{\infty} \frac{(-\beta_0)_{i_0}}{(1 + \frac{i_0}{2})_{i_0} (y + \frac{i_0}{2})_{i_0}} s_{i_0,n} \right)
\]

(37)

Replace \( \beta_1, \beta_1, \) and \( s_i \) by \( \beta_1, \beta_0 \) and \( \frac{s_{1,\infty}}{v_1} \) in (33). Take the new (33) into (37).

\[
\sum_{\beta_0=0}^{\infty} \frac{s_{00}}{\beta_0!} \prod_{n=1}^{\infty} \left\{ \sum_{\beta_n=\beta_{n-1}}^{\infty} s_n^\beta \right\} y_1(x)
\]

\[
= \prod_{k=2}^{\infty} \frac{1}{(1 - s_{k,\infty})} \int_{0}^{1} dt_1 t_1^{z+\frac{1}{2}} \int_{0}^{1} du_1 u_1^{z+\frac{1}{2}} \frac{1}{2\pi i} \oint dv_1 \exp \left( \frac{-n_{1,v_1} z (1-t_1)(1-u_1)}{v_1 (1-v_1)} \right) \sum_{\beta_1=0}^{\infty} \left( \frac{s_{1,\infty}}{v_1} \right)^{\beta_1}
\]

\[
\times \left( w_{1,1} \partial_{w_{1,1}} + \left( \frac{\omega}{2} + \frac{3}{2} \right) \right) \sum_{\beta_0=0}^{\infty} \frac{1}{\beta_0!} \left( \frac{s_{0,\infty}}{v_1} \right)^{\beta_0} \left( c_0 x^1 \sum_{i_0=0}^{\infty} \frac{(-\beta_0)_{i_0}}{(1 + \frac{i_0}{2})_{i_0} (y + \frac{i_0}{2})_{i_0}} s_{i_0,n} \right)
\]

(38)

By using Cauchy’s integral formula, the contour integrand has poles at \( v_1 = 1 \) or \( s_{1,\infty} \), and \( s_{1,\infty} \).
is only inside the unit circle. As we compute the residue there in (38) we obtain

\[
\prod_{k=1}^{\infty} \frac{1}{1 - s_{k,\infty}} \mathcal{P}_0 \{ \sum_{\beta=0}^{\infty} \beta^0 \} \left( \sum_{\beta_{n-1}}^{\infty} \beta^0_n \right) y_1(x)
\]

\[
= \prod_{k=1}^{\infty} \frac{1}{1 - s_{k,\infty}} \int_0^1 dt_1 \int_0^1 du_1 u_1^{-\frac{1}{2} + i} \exp \left( -\frac{s_{1,\infty}}{1 - s_{1,\infty}} (1 - t_1)(1 - u_1) \right)
\]

\[
\times \left( u_1^\gamma \partial u_1^\gamma + \left( \frac{\omega}{2} + \frac{1}{2} \right) \right) \sum_{\beta_{n-1}}^{\infty} \beta^0_n \left( c_0 x \right) \beta^0_n \left( 1 + \frac{1}{2} \right)_{\beta_{n-1}} \left( \frac{(-\beta_0)_{n-1}}{2} \right) \left( u_1^\gamma \right) \hat{E}
\]

(39)

where

\[
w_1^\gamma = \z s_{1,\infty} \prod_{k=1}^{\infty} \left( 1 - s_k \right)
\]

Acting the summation operator \( \sum_{\beta_{n-1}}^{\infty} \beta^0_n \prod_{k=1}^{\infty} \left( \sum_{\beta_{n-1}}^{\infty} \beta^0_n \right) \) on (40).

\[
\prod_{k=1}^{\infty} \frac{1}{1 - s_{k,\infty}} \int_0^1 dt_1 \int_0^1 du_1 u_1^{-\frac{1}{2} + i} \exp \left( -\frac{s_{1,\infty}}{1 - s_{1,\infty}} (1 - t_1)(1 - u_1) \right)
\]

\[
\times \int_0^1 dv_2 \int_0^1 dv_2 \exp \left( -\frac{s_{2,\infty}}{v_2(1 - v_2)} \right) \sum_{\beta_{n-1}}^{\infty} \beta^0_n \left( 1 + \frac{1}{2} + \frac{\omega}{2} \right)
\]

\[
\times \left( \frac{1}{2\pi i} \int_0^{2\pi i} \frac{w_{1,2}(1 - t_1)(1 - u_1)}{v_1(1 - v_1)} \right)
\]

\[
\times \sum_{\beta_{n-1}}^{\infty} \beta^0_n \left( c_0 x \right) \beta^0_n \left( 1 + \frac{1}{2} \right)_{\beta_{n-1}} \left( \frac{(-\beta_0)_{n-1}}{2} \right) \left( u_1^\gamma \right) \hat{E}
\]

(40)
By using Cauchy’s integral formula, the contour integrand has poles at $s_{2,\infty}$, $s_{2,\infty}$ is only inside the unit circle. As we compute the residue there in (41) we obtain

\[
\sum_{\beta=0}^{\infty} \frac{s_{n}}{\beta_{n}!} \prod_{j=1}^{k} \int_{0}^{1} dt_{j} t_{j}^{2} \int_{0}^{1} du_{2} u_{2}^{-1+\frac{i}{2}} = \prod_{k=3}^{\infty} \frac{1}{1-s_{k,\infty}} \int_{0}^{1} dt_{2} t_{2}^{2} \int_{0}^{1} du_{2} u_{2}^{-1+\frac{i}{2}}
\]

\[
\times \frac{1}{2\pi i} \int_{0}^{1} dv_{2} \exp \left( -\frac{s_{2}}{(1-v_{2})(1-u_{2})} \right) \left( w_{2,2} \partial w_{2,2} + \left( \frac{1}{2} + \frac{A}{2} + \frac{\omega}{2} \right) \right)
\]

\[
\times \int_{0}^{1} dt_{1} t_{1}^{-1+\frac{i}{2}} \int_{0}^{1} du_{1} u_{1}^{-1+\frac{i}{2}} \frac{1}{2\pi i} \int_{0}^{1} dv_{1} \exp \left( -\frac{s_{1,2}}{(1-v_{1})(1-u_{1})} \right) \frac{1}{v_{1}(1-v_{1})}
\]

\[
\times \sum_{\beta=0}^{\infty} \frac{s_{1,2}}{\beta_{n}!} \left( w_{1,2} \partial w_{1,2} + \left( \frac{A}{2} + \frac{\omega}{2} \right) \right) \sum_{\beta=0}^{\infty} \frac{s_{n}}{\beta_{n}!} \left( c_{0,1} x^{1} \sum_{l=0}^{\beta_{l}} \frac{(-\beta_{l})_{l}}{(1+\frac{\omega}{2})_{l}} \right) \frac{1}{(1+\frac{\omega}{2})_{l}} w_{1,2}^{l} \left( \gamma \right)^{l} \tag{41}
\]

By using Cauchy’s integral formula, the contour integrand has poles at $v_{2} = 1$ or $s_{2,\infty}$, and $s_{2,\infty}$ is only inside the unit circle. As we compute the residue there in (41) we obtain

\[
\sum_{\beta=0}^{\infty} \frac{s_{n}}{\beta_{n}!} \prod_{j=1}^{k} \int_{0}^{1} dt_{j} t_{j}^{2} \int_{0}^{1} du_{2} u_{2}^{-1+\frac{i}{2}} = \prod_{k=2}^{\infty} \frac{1}{1-s_{k,\infty}} \int_{0}^{1} dt_{2} t_{2}^{2} \int_{0}^{1} du_{2} u_{2}^{-1+\frac{i}{2}}
\]

\[
\times \frac{1}{2\pi i} \int_{0}^{1} dv_{2} \exp \left( -\frac{s_{2,\infty}}{(1-v_{2})(1-u_{2})} \right) \left( w_{2,2} \partial w_{2,2} + \left( \frac{1}{2} + \frac{A}{2} + \frac{\omega}{2} \right) \right)
\]

\[
\times \int_{0}^{1} dt_{1} t_{1}^{-1+\frac{i}{2}} \int_{0}^{1} du_{1} u_{1}^{-1+\frac{i}{2}} \frac{1}{2\pi i} \int_{0}^{1} dv_{1} \exp \left( -\frac{s_{1,2}}{(1-v_{1})(1-u_{1})} \right) \frac{1}{v_{1}(1-v_{1})}
\]

\[
\times \left( w_{1,2} \partial w_{1,2} + \left( \frac{A}{2} + \frac{\omega}{2} \right) \right) \sum_{\beta=0}^{\infty} \frac{s_{n}}{\beta_{n}!} \left( c_{0,1} x^{1} \sum_{l=0}^{\beta_{l}} \frac{(-\beta_{l})_{l}}{(1+\frac{\omega}{2})_{l}} \right) \frac{1}{(1+\frac{\omega}{2})_{l}} w_{1,2}^{l} \left( \gamma \right)^{l} \left( \gamma \right)^{l} \tag{42}
\]

where

\[
w_{2,2}^{l} = z s_{2,\infty} \prod_{l=2}^{2} \int_{0}^{1} du_{l} \quad \hat{w}_{1,2} = z s_{1,2} \prod_{l=1}^{2} \int_{0}^{1} du_{l}
\]
Replace $\beta_1, \beta_2$ and $s$ by $\beta_1, \beta_0$ and $\frac{s_1}{v_1}$ in (33). Take the new (33) into (42).

\[
\sum_{\beta_0=0}^{\infty} \frac{1}{\beta_0!} \prod_{n=1}^{\infty} \left( \sum_{\beta_n=0}^{\infty} s_0^\beta \right) y_2(x) = \prod_{k=2}^{\infty} \frac{1}{(1-s_k,\infty)} \int_0^1 dt_1 t_1^{\frac{1}{2}+\frac{1}{2}} \int_0^1 dt_2 u_2^{r-1+\frac{1}{2}} \times \exp \left( -\frac{s_{2,0}}{1-s_{2,0}} z(1-t_2)(1-u_2) \right) \left( w_{2,2}^* \partial_{w_2} + \left( 1 + \frac{1}{2} + \frac{3}{4} \right) \right)
\]

By using Cauchy’s integral formula, the contour integrand has poles at $v_1 = 1$ or $s_1$, and $s_1$ is only inside the unit circle. As we compute the residue there in (43), we obtain

\[
\sum_{\beta_0=0}^{\infty} \frac{1}{\beta_0!} \prod_{n=1}^{\infty} \left( \sum_{\beta_n=0}^{\infty} s_0^\beta \right) y_2(x) = \prod_{k=2}^{\infty} \frac{1}{(1-s_k,\infty)} \int_0^1 dt_1 t_1^{\frac{1}{2}+\frac{1}{2}} \int_0^1 dt_2 u_2^{r-1+\frac{1}{2}} \times \exp \left( -\frac{s_{2,0}}{1-s_{2,0}} z(1-t_2)(1-u_2) \right) \left( w_{2,2}^* \partial_{w_2} + \left( 1 + \frac{1}{2} + \frac{3}{4} \right) \right)
\]

where

\[
w_{1,2}^* = zs_{1,0} \prod_{l=1}^{2} h_{u_l}
\]
Acting the summation operator \( \sum_{\beta_0=0}^{\infty} \prod_{n=1}^{\infty} \sum_{\beta_n=0}^{\infty} \beta_n^{s_n} \) on (33),

\[
\sum_{\beta_0=0}^{\infty} \prod_{n=1}^{\infty} \left\{ \sum_{\beta_n=0}^{\infty} s_n^{\beta_n} \right\} \ y_3(x)
\]

\[
= c_0 x^3 \sum_{k=3}^{\infty} \frac{1}{(1-s_k,\infty)} \int_0^1 dt_3 t_3^{r_3} \int_0^1 du_3 u_3^{r_3} \ exp \left( -s_{3,\infty} \right. \ \frac{1-(1-t_3)(1-u_3)}{(1-s_3,\infty)} \ \left. \right) 
\]

\[
\times \left( w_{3,3}^* \beta_3 + \frac{1}{2} + \omega \right) 
\]

\[
\times \left( w_{2,2}^* \beta_2 + \frac{1}{2} + \omega \right) 
\]

\[
\times \left( w_{1,1}^* \beta_1 + \frac{1}{2} + \omega \right) 
\]

\[
(45)
\]

where

\[
w_{3,3}^* = z s_{3,\infty} \prod_{l=3}^{l=3} t_l \mu_l \\
w_{2,2}^* = z s_{2,\infty} \prod_{l=2}^{l=2} t_l \mu_l \\
w_{1,1}^* = z s_{1,\infty} \prod_{l=1}^{l=1} t_l \mu_l
\]

By repeating this process for all higher terms of integral forms of sub-summation \( y_m(x) \) terms where \( m > 3 \), we obtain every \( \sum_{\beta_0=0}^{\infty} \prod_{n=1}^{\infty} \sum_{\beta_n=0}^{\infty} s_n^{\beta_n} \) \( y_m(x) \) terms. Since we substitute (36), (39), (44), (45) and including all \( \sum_{\beta_0=0}^{\infty} \prod_{n=1}^{\infty} \sum_{\beta_n=0}^{\infty} s_n^{\beta_n} \) \( y_m(x) \) terms where \( m > 3 \) into (33), we obtain (34)

\[\Box\]

**Remark 5.** The generating function for the GCH polynomial which makes \( B_n \) term terminated
of the first kind about \( x = 0 \) as \( \Omega = -2\mu(\beta_1 + \frac{1}{2}) \) where \( i, \beta_i = 0, 1, 2, \cdots \) is

\[
\sum_{\beta_0=0}^{\infty} \frac{\beta_0}{\beta_0!} \left\{ \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n!} \right\} QW_{\beta_i} (\beta_i = -\Omega \frac{1}{2\mu} - i \frac{1}{2} (1 + \gamma); \; \tilde{e} = -\frac{1}{2} \exp; \; z = -\frac{1}{2} \mu x^2)
\]

\[
= \prod_{k=1}^{\infty} \frac{1}{(1 - s_{k,0})} A \left( s_{0,0}; z \right)
\]

\[
+ \sum_{k=2}^{\infty} \prod_{k=1}^{\infty} \frac{1}{(1 - s_{k,0})} \int_0^1 dt_1 t_1^{\gamma-2} \int_0^1 du_1 u_1^{\gamma-2} T_1 (s_{1,0}; t_1, u_1, z) \left( w_{1,1}^{\nu} \partial_{w_{1,1}} + \frac{\omega}{2} \right) A \left( s_0; w_{1,1} \right) \tilde{e}
\]

\[
+ \sum_{k=2}^{\infty} \prod_{k=1}^{\infty} \frac{1}{(1 - s_{k,0})} \int_0^1 dt_1 t_1^{\gamma-2} \int_0^1 du_1 u_1^{\gamma-2} T_1 (s_{1,0}; t_1, u_1, z) \left( w_{1,1}^{\nu} \partial_{w_{1,1}} + \frac{1}{2} (n - 1 + \omega) \right)
\]

\[
\times \left\{ \sum_{n=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{(1 - s_{j,0})} \int_0^1 dt_{n-j} t_{n-j}^{\gamma-2} \int_0^1 du_{n-j} u_{n-j}^{\gamma-2} w_{n-j}^{*} \partial_{w_{n-j}} \right\} A \left( s_0; w_{1,n} \right) \tilde{e}^n
\]

(46)

where

\[
\begin{align*}
T_1 (s_{1,0}; t_1, u_1, z) &= \exp \left( -\frac{s_{1,0}}{1 - s_{1,0}} z (1 - t_1) (1 - u_1) \right) \\
T_n (s_{n,0}; t_n, u_n, z) &= \exp \left( -\frac{s_{n,0}}{1 - s_{n,0}} z (1 - t_n) (1 - u_n) \right) \\
T_{n-j} (s_{n-j}; t_{n-j}, u_{n-j}, w_{n-j}^{*}) &= \exp \left( \frac{-w_{n-j}^{*}(1-t_{n-j})(1-u_{n-j})}{(1-s_{n-j})} \right)
\end{align*}
\]

and

\[
\begin{align*}
A \left( s_{0,0}; z \right) &= (1 - s_{0,0})^{-\gamma} \exp \left( \frac{z s_{0,0}}{(1 - s_{0,0})} \right) \\
A \left( s_0; w_{1,1}^{*} \right) &= (1 - s_0)^{-\gamma} \exp \left( -\frac{w_{1,1}^{*}}{w_{1,1}} s_0 \right) \\
A \left( s_0; w_{1,n}^{*} \right) &= (1 - s_0)^{-\gamma} \exp \left( -\frac{w_{1,n}^{*}}{w_{1,n}} s_0 \right)
\end{align*}
\]

Proof. The generating function for confluent Hypergeometric polynomial of the first kind is written by

\[
\sum_{\beta_0=0}^{\infty} \frac{\beta_0}{\beta_0!} F_{\beta_0}(\gamma; z) = (1 - t)^{-\gamma} \exp \left( -\frac{zt}{1 - t} \right)
\]

(47)

Replace \( t \) by \( s_{0,0} \) in (47).

\[
\sum_{\beta_0=0}^{\infty} \frac{\beta_0}{\beta_0!} F_{\beta_0}(\gamma; z) = (1 - s_{0,0})^{-\gamma} \exp \left( -\frac{z s_{0,0}}{(1 - s_{0,0})} \right)
\]

(48)
Replace $t$ and $z$ by $s_0$ and $w_{1,1}^*$ in (47).

$$\sum_{\beta_0 = 0}^{\infty} \frac{\beta_0!}{\beta_0!} \tilde{F}_{\beta_0}(y; w_{1,1}^*) = (1 - s_0)^{-\gamma} \exp \left( - \frac{w_{1,1}^* s_0}{1 - s_0} \right)$$

(49)

Replace $t$ and $z$ by $s_0$ and $w_{1,n}^*$ in (47).

$$\sum_{\beta_0 = 0}^{\infty} \frac{\beta_0!}{\beta_0!} \tilde{F}_{\beta_0}(y; w_{1,n}^*) = (1 - s_0)^{-\gamma} \exp \left( - \frac{w_{1,n}^* s_0}{1 - s_0} \right)$$

(50)

Put $c_0 = \frac{\Gamma(\gamma s_0)}{\Gamma(\gamma)}$ as $\lambda = 0$ in (34). And substitute (48), (49) and (50) into the new (34).

\[ \square \]

**Remark 6.** The generating function for the GCH polynomial which makes $B_n$ term terminated of the second kind about $x = 0$ as $\Omega = -2\mu(\psi_i + 1 - \gamma + \frac{\psi}{2})$ where $i, \psi_i = 1, 2, \cdots$ is

$$\sum_{\psi_0 = 0}^{\infty} \frac{\psi_0!}{\psi_0!} \prod_{n=1}^{\infty} \left\{ \sum_{\psi_0=0}^{\infty} \frac{\psi_0!}{\psi_0!} \right\} \tilde{R}_W(\psi_0 = -\frac{\Omega}{2\mu} + \gamma - \frac{i}{2}, \gamma = \frac{1}{2} (1 + \gamma); \quad \tilde{e} = -\frac{1}{2} e x; \quad z = -\frac{1}{2} \mu x^2)$$

$$= \frac{1}{(1 - s_0, \infty)} B(s_0, \infty; z)$$

$$+ \sum_{n=1}^{\infty} \left\{ \prod_{k=1}^{n-1} \frac{1}{(1 - s_0, \infty)} \right\} \int_0^1 \frac{dt_1}{t_1-\gamma} \int_0^1 \frac{dt_1}{t_1-\gamma} \frac{1}{\tilde{u}_1 \tilde{u}_1^{(1)}} \tilde{F}_1 \left( s_{1,1} \infty; t_1, u_1, 0 \right) w_{1,1}^* \tilde{u}_1 \tilde{u}_1^{(1)} \tilde{F}_1 \left( s_{1,1} \infty; t_1, u_1, 0 \right) w_{1,1}^*$$

$$\times \prod_{j=1}^{n-1} \left\{ \int_0^1 \frac{dt_n}{t_n-\gamma} \frac{1}{t_n-\gamma} \right\} \int_0^1 \frac{dt_n}{t_n-\gamma} \frac{1}{\tilde{u}_n \tilde{u}_n^{(n)}} \tilde{F}_n \left( s_{n-1, n} \infty; t_n, u_n, 0 \right) w_{n-1, n}^* \tilde{u}_n \tilde{u}_n^{(n)}$$

$$\times \left( \frac{1}{2} (n - j + 1 + \gamma) \right) B(s_0, w_{1,n}^*) \left\{ \tilde{e} \right\}$$

(51)

where

$$\tilde{F}_1 \left( s_{1,1} \infty; t_1, u_1, 0 \right) = \exp \left( -\frac{s_{1,1}}{1 - s_{1,1}} (1 - t_1) (1 - u_1) \right)$$

$$\tilde{F}_n \left( s_{n, n} \infty; t_n, u_n, 0 \right) = \exp \left( -\frac{s_{n, n}}{1 - s_{n, n}} (1 - t_n) (1 - u_n) \right)$$

$$\tilde{F}_{n-1} \left( s_{n-1, n} \infty; t_{n-1}, u_{n-1}, 0 \right) = \exp \left( -\frac{s_{n-1, n}}{1 - s_{n-1, n}} (1 - t_{n-1}) (1 - u_{n-1}) \right)$$

and

$$B(s_0, \infty; z) = (1 - s_0)^{\gamma - 2} \exp \left( -\frac{s_{0, 0}}{1 - s_{0, 0}} \right)$$

$$B(s_0, w_{1,1}^*) = (1 - s_0)^{\gamma - 2} \exp \left( -\frac{w_{1,1}^* s_0}{1 - s_0} \right)$$

$$B(s_0, w_{1,n}^*) = (1 - s_0)^{\gamma - 2} \exp \left( -\frac{w_{1,n}^* s_0}{1 - s_0} \right)$$

21
Proof. The generating function for confluent hypergeometric polynomial of the second kind is written by

\[
\sum_{\psi_0=0}^{\infty} \frac{\phi_0^n}{\psi_0!} A_{\psi_0}(\gamma; z) = (1 - t)^{-2} \exp\left( -\frac{zt}{1 - t} \right)
\] (52)

Replace \(t\) by \(s_{0,\infty}\) in (52).

\[
\sum_{\psi_0=0}^{\infty} \frac{s_{0,\infty}^{\psi_0}}{\psi_0!} A_{\psi_0}(\gamma; z) = (1 - s_{0,\infty})^{-2} \exp\left( -\frac{zs_{0,\infty}}{1 - s_{0,\infty}} \right)
\] (53)

Replace \(t\) and \(z\) by \(s_0\) and \(w_{1,1}^*\) in (52).

\[
\sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} A_{\psi_0}(\gamma; w_{1,1}^*) = (1 - s_0)^{-2} \exp\left( -\frac{w_{1,1}^* s_0}{1 - s_0} \right)
\] (54)

Replace \(t\) and \(z\) by \(s_0\) and \(w_{1,n}^*\) in (52).

\[
\sum_{\psi_0=0}^{\infty} \frac{s_0^{\psi_0}}{\psi_0!} A_{\psi_0}(\gamma; w_{1,n}^*) = (1 - s_0)^{-2} \exp\left( -\frac{w_{1,n}^* s_0}{1 - s_0} \right)
\] (55)

Put \(c_0 = \left( -\frac{1}{2} \right)^{1-\gamma} \frac{\Gamma(\lambda_m + 2 - \gamma)}{\Gamma(2 - \gamma)}\) as \(\lambda = 1 - \nu = 2(1 - \gamma)\) on (34) with replacing \(\beta_0, \beta_{n-1}\) and \(\beta_n\) by \(\psi_0, \psi_{n-1}\) and \(\psi_n\). Substitute (53), (54) and (55) into the new (34). \(\square\)

4. Application

I show integral forms and generating functions for GCH polynomials of the first and second kinds in this paper. We can apply this new special function into many physics areas. I show three examples of GCH equation as follows:

4.1. the rotating harmonic oscillator

For example, there are quantum-mechanical systems whose radial Schrödinger equation may be reduced to a Biconfluent Heun function [25, 11, 12], namely the rotating harmonic oscillator and a class of confinement potentials. Its radial Schrödinger equation is

\[
\Psi''(r) + \left( \frac{2\lambda_m + 1}{2\omega} - \frac{r - 1)^2}{4\omega^2} - \frac{l_m(l_m + 1)}{2r^2} \right) \Psi(r) = 0
\] (56)

where \(0 \leq r < \infty\), \(\lambda_m\) is the eigenvalue, \(l_m\) is the rotational quantum number and \(\omega\) is a coupling parameter.

The wave function for the rotating harmonic oscillator is given by [25]

\[
\Psi(r) = N r^{l_m + \frac{3}{2}} \exp\left( -\frac{r - 1)^2}{2\omega} \right) QW_{\beta_1} \left( \beta_1 = \frac{l_m - l_m - 1 - i}{2}, \omega = l_m + 1, \gamma = l_m + \frac{3}{2} \right)
\]

\[
; \beta = -\frac{r}{2\omega}; z = \frac{r^2}{2\omega}
\] (57)
We obtain the integral form of (58) from (16).

\[
Q_{W_{\beta_i}} \left( \beta_i = \frac{\lambda_m - l_m - 1 - i}{2}, \omega = l_m + 1, \gamma = l_m + \frac{3}{2}; \bar{\varepsilon} = -\frac{r}{2\omega}; z = \frac{r^2}{2\omega} \right)
\]

\[
= \frac{\Gamma(\gamma + \beta_0)}{\Gamma(\gamma)} \left( \sum_{i=0}^{\beta_\gamma} \frac{(-\beta_\gamma)^{l_m}}{(1)_{l_m}(\gamma)_{1}} + \sum_{i=0}^{\beta_\gamma} \frac{(i_0 + \frac{3}{2})}{(i_0 + \frac{3}{2})}\frac{(-\beta_\gamma)^{l_m}}{(1)_{l_m}(\gamma)_{1}} \right)
\times \sum_{i=0}^{\beta_\gamma} \frac{(-\beta_\gamma)^{l_m}}{(1)_{l_m}(\gamma)_{1}} \left( \prod_{k=1}^{n-1} \left( i_k + \frac{3}{2} \right) \right) (1 + \frac{3}{2}) \left( -\beta_\gamma(1 + \frac{3}{2})_{l_m}(\gamma + \frac{3}{2})_{l_m} \right)
\times \sum_{i=0}^{\beta_\gamma} \frac{(-\beta_\gamma)^{l_m}}{(1 + \frac{3}{2})_{l_m}(\gamma + \frac{3}{2})_{l_m}} \bar{\varepsilon}^n \right) \right)
\]

(58)

\[N \text{ is normalized constant. GCH function with three recursive coefficients has infinite eigenvalues as we see (58) which is } \beta_i = \frac{\lambda_m - l_m - 1 - i}{2} \text{ where } i, \beta_i = 0, 1, 2, \cdots .\]

We obtain the integral form of (58) from (16).

\[
Q_{W_{\beta_i}} \left( \beta_i = \frac{\lambda_m - l_m - 1 - i}{2}, \gamma = l_m + \frac{3}{2}; \bar{\varepsilon} = -\frac{r}{2\omega}; z = \frac{r^2}{2\omega} \right)
\]

\[
= F_{\beta_i}(\gamma; z) + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} \left\{ \frac{1}{2\pi} \int d\nu_{n-j} \int_{0}^{1} dt_{n-j} \int_{0}^{1} du_{n-j} \right\} \right)
\times \frac{1}{2\pi} \int d\nu_{n-j} \quad \nu_{n-j} \left( 1 - \nu_{n-j} \right)
\times \left( \bar{w}_{n-j} \partial_{w_{n-j}} + \frac{1}{2} \left( \nu_{n-j} \right) \right) \left( F_{\beta_i}(\gamma; W_{1,n}) \right) \bar{\varepsilon}^n
\]

(59)

where

\[
w_{a,b} = \frac{z}{z} \prod_{i=0}^{b} t_{i} u_{i} v_{i}
\]

(60)

We can transform GCH function into all other well-known special functions having two recursive coefficients because a \( F_{1} \) (or \( F_{\beta} \)) function recurs in each of sub-integral forms of GCH function in (59).

4.2. Confinement potentials

Following Chaudhuri and Mukherjee, there is the radial Schrödinger equation, 25, 13, 14, 11

\[
\Psi'(r) + \left( \frac{2\mu}{\hbar^2} \right) \left( E + \frac{a}{r} - br - cr^2 \right) - \frac{l(l + 1)}{r^2} \Psi(r) = 0
\]

(61)

with E being the energy.
The wave function for confinement potentials is given by (see section 4.2 in Ref [25])

\[ \Psi(r) = N_F r^l \exp \left( -\frac{1}{2} r^2 \alpha_F - \beta_F r \right) QW_\beta \left( \beta_i = \frac{1}{4\alpha_F} \left( \beta_F^2 + \frac{2\mu}{\hbar^2} \right) \right) - \frac{1}{2} \left( i + l + \frac{3}{2} \right), \omega = -\frac{\mu a}{\hbar^2 \beta_F} + l + 1, \gamma = l + \frac{3}{2}; \tilde{\epsilon} = -\beta_F r; z = \alpha_F r^2 \]

(62)

where

\[
\begin{align*}
QW_\beta \left( \beta_i = \frac{1}{4\alpha_F} \left( \beta_F^2 + \frac{2\mu}{\hbar^2} \right) \right) - \frac{1}{2} \left( i + l + \frac{3}{2} \right), \omega = -\frac{\mu a}{\hbar^2 \beta_F} + l + 1 \\
, \gamma = l + \frac{3}{2}; \tilde{\epsilon} = -\beta_F r; z = \alpha_F r^2 \\
\end{align*}
\]

\[
\begin{align*}
&= \frac{\Gamma(y + \beta_0)}{\Gamma(y)} \left\{ \sum_{i=0}^{\beta_0} (-\beta_0)_i \omega^i \sum_{n=0}^{\beta_0} \left( \frac{\gamma}{2} \right)_n (\gamma - \frac{1}{2})_n \right\} + \frac{i}{\omega} \left\{ \sum_{n=0}^{\beta_0} \left( \frac{\gamma}{2} \right)_n (\gamma - \frac{1}{2})_n \right\}
\end{align*}
\]

\[
\begin{align*}
&= \frac{\beta_0}{\omega} \left\{ \sum_{i=0}^{\beta_0} \left( \frac{\gamma}{2} \right)_n (\gamma - \frac{1}{2})_n \right\}
\end{align*}
\]

(63)

N is normalized constant. Energy E is given by

\[
E = \frac{\hbar^2}{2\mu} \left( \beta_i + \frac{i + l + \frac{3}{2}}{2} \right) - \beta_F^2 \]

where i, \beta_i = 0, 1, 2, \cdots \quad (64)

GCH function with three recursive coefficients has infinite eigenvalues as we see (63). We obtain the integral form of (62) from (16).

\[
\begin{align*}
QW_\beta \left( \beta_i = \frac{1}{4\alpha_F} \left( \beta_F^2 + \frac{2\mu}{\hbar^2} \right) \right) - \frac{1}{2} \left( i + l + \frac{3}{2} \right), \omega = -\frac{\mu a}{\hbar^2 \beta_F} + l + 1 \\
, \gamma = l + \frac{3}{2}; \tilde{\epsilon} = -\beta_F r; z = \alpha_F r^2 \\
\end{align*}
\]

\[
\begin{align*}
&= F_\beta(y; z) + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left( \int_0^1 dt_{n-j} \int_{-\beta_{n-j}^2}^{\beta_{n-j}^2} u_{n-j}^2 \right) \right\} \int_0^1 dt_{n-j} \int_0^1 dt_{n-j} u_{n-j}^2 \\
&\times \left( \frac{w_{n-j}}{1 - w_{n-j}} \right) \frac{1}{\beta_{n-j}^2 (1 - v_{n-j})} \frac{1}{\beta_{n-j}^2 + 1} \frac{1}{2\beta_F} \left( n - j + l - \frac{a}{2\beta_F} \right) \left( F_\beta(y; w_{n-j}) \right) \right\}
\end{align*}
\]

(65)
where
\[
w_{a,b} = \begin{cases} 
z \prod_{l=a}^{b} l!u_lv_l & \text{only if } a > b \\
 \end{cases}
\]  

(66)

Again, \( iF_1 \) (or \( F_{\beta_0} \)) function recurs in each of sub-integral forms in (65).

4.3. Two interacting electrons in a uniform magnetic field and a parabolic potential

There are three identical charged particles on a plane under a perpendicular magnetic field and interacting through Coulomb repulsion. Its solution is also Biconflent Heun function which is the special case of GCH function. In “Two interacting electrons in a uniform magnetic field and a parabolic potential: The general closed-form solution”, the author consider a system of two interacting electrons of mass \( m^* \) and charge \( e \) in two space dimensions subjected to both a uniform magnetic field along the direction perpendicular to the plane and an external parabolic potential. Its Hamiltonian is given by

\[
H = \sum_{j=1}^{2} \left\{ \frac{1}{2m^*} \left[ \mathbf{p}(\mathbf{r}_j) + \frac{e}{c} \mathbf{A}(\mathbf{r}_j) \right]^2 + U(|\mathbf{r}_j|) \right\} + \frac{e^2}{\varepsilon_\infty |\mathbf{r}_1 - \mathbf{r}_2|} 
\]  

(67)

where \( U(|\mathbf{r}_j|) = 1/2m^* \omega^2 \mathbf{r}_j^2 \) is the single particle confinement potential, and \( \mathbf{A}(\mathbf{r}_j) \) is the vector potential of the magnetic field. By introducing the relative and center-of-mass coordinates, \( \mathbf{r} = |\mathbf{r}_1 - \mathbf{r}_2| \) and \( \mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2 \), respectively, (67) can be decoupled as the sum of two single particle Hamiltonians. by setting \( \Psi(\mathbf{r}) = e^{im \phi} F(\mathbf{r}) \), the equation for the radial part of the relative motion in the cylindrical coordinates is

\[
\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} - \frac{\sigma^2 m^2}{\rho^2} R - \rho \frac{\dot{R}}{\rho} R + \ddot{\epsilon} R = 0
\]  

(68)

where

\[
\ddot{\epsilon} = (2E_r - m\hbar \omega)/\hbar \omega \\
u = 2\mu e^2/\varepsilon_\infty \hbar^2 \dot{\gamma} \\
\rho = \tilde{\gamma} \tilde{r} \\
\tilde{\gamma}^2 = \mu \omega/\hbar
\]  

(69)

Put \( R(\rho) = \rho^{a|m|} e^{-\rho^2/2} F(\rho) \) in (68).

\[
\rho F'' + (a - 2\rho^2) F' + (d\rho - u) F = 0
\]  

(70)

where

\[
a = 2\sigma|m| + 1 \\
d = \ddot{\epsilon} - (2\sigma|m| + 1)
\]  

(71)
\( (70) \) is generally called the Biconfluent Heun equation in canonical form. If we compare \( (70) \) with \( (1) \), all coefficients on the above are correspondent to the following way.

\[
\begin{align*}
\mu & \leftrightarrow -2 \\
\varepsilon & \leftrightarrow 0 \\
\nu & \leftrightarrow a = 2|x|m + 1 \\
\Omega & \leftrightarrow d = \tilde{c} - (2|x|m + 1) \\
\omega & \leftrightarrow -\frac{\mu}{0} \\
x & \leftrightarrow \rho \\
\end{align*}
\]

Let’s investigate function \( \Psi(r) \) as \( n \) and \( r \) go to infinity. I assume that \( F(\rho) \) is infinite series in \( (70) \). In Ref. [25], the asymptotic behavior of \( \lim_{n \to 1} \) for the case of \( |x| \ll |\mu| \) is

\[
\lim_{n \to 1} y(x) = 1 + \sqrt{-\frac{\pi}{2\mu x^2}} \text{Erf}\left(\sqrt{\frac{1}{2\mu x^2}}\right) \exp\left(-\frac{1}{2\mu x^2}\right) \quad \text{where} \quad -\infty < x < \infty \tag{73}
\]

Replacing \( y(x) \) and \( \mu \) by \( F(\rho) \) and \( -2 \) in \( (73) \). Take the new \( (73) \) into \( \Psi(r) = e^{im\phi}R(\rho) = \rho^{|\mu|} e^{-\rho^2/2} F(\rho) e^{im\phi} \) putting \( \rho = \tilde{\gamma}r \).

\[
\lim_{n \to 1} \Psi(r) \approx (\tilde{\gamma}r)^{|\mu|} \exp\left(-\frac{(\tilde{\gamma}r)^2}{2}\right) \left(1 + \tilde{\gamma} \sqrt{\pi} \text{Erf}(\tilde{\gamma}r) \ r \exp\left(\tilde{\gamma}^2 r^2\right)\right) \exp\left(im\phi\right) \tag{74}
\]

In \( (74) \) if \( r \to \infty \), then \( \lim_{n \to 1} \Psi(r) \to \infty \). It is unacceptable that wave function \( \Psi(r) \) is divergent as \( r \) goes to infinity in the quantum mechanical point of view. Therefore the function \( F(\rho) \) must to be polynomial in \( (70) \) in order to make the wave function \( \Psi(r) \) being convergent even if \( r \) goes to infinity.

In Ref. [25], the Frobenius solutions of GCH polynomial which makes \( B_n \) term terminated of the first and second kinds are given by

\[
\begin{align*}
\Psi(x) &= \text{QW}_{\beta_1} \left(\beta_1 = -\frac{\Omega}{2\mu} - \frac{i}{2}, \omega, \gamma = \frac{1}{2}(1 + v); \ \tilde{\varepsilon} = -\frac{1}{2}ev; \ z = -\frac{1}{2}e^2\right) \\
&= \Gamma(\gamma + \beta_0) \left\{ \sum_{n=0}^{\beta_0} (\gamma)_n \tilde{e}^n + \sum_{n=0}^{\beta_0} (\gamma + \frac{1}{2})_n (\gamma - \frac{1}{2})_n \tilde{e}^{n+1} \right\} \times \prod_{i=1}^{n-1} \left\{ \frac{(\gamma + \frac{1}{2})_n (\gamma + \frac{1}{2})_n (1 + \frac{1}{2})_n}{(1 + \frac{1}{2})_n (\gamma + \frac{1}{2})_n (\gamma + \frac{1}{2})_n} \tilde{e}^{n+1} \right\} \\
\end{align*}
\]

\[
\begin{align*}
&\times \sum_{n=0}^{\beta_0} \left\{ \frac{(\gamma + \frac{1}{2})_n (\gamma + \frac{1}{2})_n (1 + \frac{1}{2})_n}{(1 + \frac{1}{2})_n (\gamma + \frac{1}{2})_n (\gamma + \frac{1}{2})_n} \tilde{e}^{n+1} \right\} \\
&\times \sum_{n=0}^{\beta_0} \left\{ \frac{(\gamma + \frac{1}{2})_n (\gamma + \frac{1}{2})_n (1 + \frac{1}{2})_n}{(1 + \frac{1}{2})_n (\gamma + \frac{1}{2})_n (\gamma + \frac{1}{2})_n} \tilde{e}^{n+1} \right\} \\
&\times \sum_{n=0}^{\beta_0} \left\{ \frac{(\gamma + \frac{1}{2})_n (\gamma + \frac{1}{2})_n (1 + \frac{1}{2})_n}{(1 + \frac{1}{2})_n (\gamma + \frac{1}{2})_n (\gamma + \frac{1}{2})_n} \tilde{e}^{n+1} \right\} \tag{75}
\]

26
\[ y(x) = RW_y \left( \psi_i = -\frac{\Omega}{2\mu} + \gamma - i \frac{i}{2}, \omega, \gamma = \frac{1}{2}(1 + \nu); \bar{\varepsilon} = -\frac{i}{2} \varepsilon x; z = -\frac{i}{2} \mu x^2 \right) \]

\[ = z^{-1/2} \Gamma(\nu_0 + 2 - \gamma) \left( \sum_{n=0}^{\infty} \frac{(-\psi_0)_{i_n}}{(1)_{\nu_0}(2 - \gamma)_{i_n}} \right) \]

\[ + \sum_{n=2}^{\infty} \left( \prod_{l=0}^{\infty} \frac{(i_0 + 1 - \gamma + \frac{n}{2})}{(i_0 + \frac{1}{2})} \right) \left( \psi_0(1 + \frac{1}{2})_{i_n} \right) \left( \sum_{i=0}^{\infty} \frac{(-\psi_0)_{i_n}}{(1)_{\nu_0}(2 - \gamma)_{i_n}} \right) x^2 \]

\[ \times \prod_{l=0}^{\infty} \left( \prod_{k=1}^{\infty} \frac{(i_k + 1 - \gamma + \frac{n}{2})}{(i_k + \frac{1}{2})} \right) \left( \psi_0(1 + \frac{1}{2})_{i_n} \right) \left( \sum_{i=0}^{\infty} \frac{(-\psi_0)_{i_n}}{(1)_{\nu_0}(2 - \gamma)_{i_n}} \right) x^2 \]

\[ = R W_{\phi} \left( \psi_i = -\frac{\Omega}{2\mu} + \gamma - i \frac{i}{2}, \omega, \gamma = \frac{1}{2}(1 + \nu); \bar{\varepsilon} = -\frac{i}{2} \varepsilon x; z = -\frac{i}{2} \mu x^2 \right) \]

\[ = \frac{\Gamma(\nu_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \left( \sum_{n=0}^{\infty} \frac{(-\psi_0)_{i_n}}{(1)_{\nu_0}(2 - \gamma)_{i_n}} \right) \]

\[ + \sum_{n=2}^{\infty} \left( \prod_{l=0}^{\infty} \frac{(i_0 + 1 - \gamma + \frac{n}{2})}{(i_0 + \frac{1}{2})} \right) \left( \psi_0(1 + \frac{1}{2})_{i_n} \right) \left( \sum_{i=0}^{\infty} \frac{(-\psi_0)_{i_n}}{(1)_{\nu_0}(2 - \gamma)_{i_n}} \right) \]

Since (72) is put in (75) and (76), \( R W_{\phi} (\psi_i, \omega, \gamma; \bar{\varepsilon}; z) \to \infty \) as \( r \to 0 \) because of \( \gamma = \sigma|m| + 1 \). And \( Q W_{\beta} (\beta_i, \omega, \gamma; \bar{\varepsilon}; z) \to 0 \) as \( r \to 0 \). So I choose \( Q W_{\beta} (\beta_i, \omega, \gamma; \bar{\varepsilon}; z) \) as an eigenfunction for (70). Apply (72) into (75) with replacing \( y(x) \) by \( F(\rho) = \hat{\psi} r \).

\[ F(\rho) = Q W_{\beta} \left( \beta_i = \frac{d}{4} - i \frac{i}{2}, \gamma = \sigma|m| + 1; \bar{\varepsilon} = \frac{\mu e^2}{2\varepsilon_0 h^2} r; z = \frac{\mu \omega}{h} r^2 \right) \]

\[ = \frac{\Gamma(\gamma + \beta_0)}{\Gamma(\gamma)} \left( \sum_{n=0}^{\infty} \frac{(-\beta_0)_{i_n}}{(1)_{\nu_0}(\gamma)_{i_n}} \right) \left( \sum_{i=0}^{\infty} \frac{1}{(i_0 + \frac{1}{2})(i_0 - \frac{1}{2} + \gamma)} \right) \left( \psi_0(1 + \frac{1}{2})_{i_n} \right) x^2 \]

\[ \times \prod_{l=0}^{\infty} \left( \prod_{k=1}^{\infty} \frac{(i_k + 1 - \gamma + \frac{n}{2})}{(i_k + \frac{1}{2})} \right) \left( \psi_0(1 + \frac{1}{2})_{i_n} \right) \left( \sum_{i=0}^{\infty} \frac{(-\beta_0)_{i_n}}{(1)_{\nu_0}(\gamma)_{i_n}} \right) \]

\[ \times \prod_{l=0}^{\infty} \left( \prod_{k=1}^{\infty} \frac{(i_k + 1 - \gamma + \frac{n}{2})}{(i_k + \frac{1}{2})} \right) \left( \psi_0(1 + \frac{1}{2})_{i_n} \right) \left( \sum_{i=0}^{\infty} \frac{(-\beta_0)_{i_n}}{(1)_{\nu_0}(\gamma)_{i_n}} \right) \]

Put (77) in \( \Psi(r) = \rho^{i|m|} e^{-r^2/2} F(\rho) e^{im\phi} \) where \( \rho = \hat{\psi} r = \sqrt{\frac{\mu}{\hbar}} r \). The wave function in a uniform magnetic field and a parabolic potential is given by

\[ \Psi(r) = N \left( \frac{\mu \omega}{h} \right)^{\frac{m}{2}} \exp \left( -\frac{\mu \omega}{h} r^2 \right) Q W_{\beta} \left( \beta_i = \frac{d}{4} - i \frac{i}{2}, \gamma = \sigma|m| + 1; \bar{\varepsilon} = \frac{\mu e^2}{2\varepsilon_0 h^2} r; z = \frac{\mu \omega}{h} r^2 \right) e^{im\phi} \]

\[ N \text{ is normalized constant. Energy } E_r \text{ is given by } \]

\[ E_r = \hbar \omega \left( 2\beta_i + i + \sigma|m| + \frac{1}{2} \right) + \frac{1}{2} \hbar \omega_c \quad \text{where } i, \beta_i = 0, 1, 2, \cdots \]
GCH function with three recursive coefficients has infinite eigenvalues as we see (77). We obtain the integral form of (77) from (16).

\[
F(\rho) = QW_{\beta}(\rho, \gamma; \nu) + \sum_{n=1}^{\infty} \left\{ \prod_{j=0}^{n-1} \left( \int_0^1 dt_n \frac{t_n^{(n-j)-1}}{(1-t_n)(1-u_n)} \right) F_{\beta}(\beta, \gamma; \nu, w_{1,n}) \right\} \tilde{\epsilon}^n
\]

where

\[
w_{a,b} = \begin{cases} 
\left( \prod_{l=a}^{b} t_l u_l \right) & \text{if } a > b \\
\left( \prod_{l=a}^{b} t_l \right) & \text{if } a = b \\
\left( \prod_{l=a}^{b} u_l \right) & \text{if } b > a \\
\text{only if } a > b 
\end{cases}
\]

(81)

1F1 (or F_\beta) function again recurs in each of sub-integral forms in (80).

5. Conclusion

In Ref. [25] I show the power series expansion in closed forms and its asymptotic behaviors of the GCH equation for infinite series and polynomial. In this paper I construct integral forms of GCH equation for infinite series and polynomial which makes B_n term terminated including all higher terms of A_n’s. Indeed, the generating functions for GCH polynomial are derived in mathematical rigour. And I show how to derive the power series expansions in closed forms and its integral forms of analytic wave functions and its eigenvalues in four examples: (1) Schrödinger equation with the rotating harmonic oscillator and a class of confinement potentials, (2) The spin free Hamiltonian involving only scalar potential for the \(q - \bar{q}\) system [25], (3) The radial Schrödinger equation with Confinement potentials, (4) Two interacting electrons in a uniform magnetic field and a parabolic potential.

In general, most of wave-functions in physics are quantized with specific eigenvalues. As we see three examples on the above, all solutions are quantized with certain eigenvalues and its analytic wave-functions have polynomial expansions in closed forms. There are infinite eigenvalues because its differential equations have three recursive coefficients [17, 18, 25]. In contrast most of well-known wave functions only have one eigenvalue because its differential equations have two recursive coefficients.

We can express representations in closed form integrals in an easy way since we have power series expansions with Pochhammer symbols in numerators and denominators. We can transform any special functions, having three term recursive relation, into all other well-known special functions with two recursive coefficients because a 1F1 (GCH function) or 2F1 (Mathieu, Lame, Heun, functions [20, 21, 22, 23]) function recurs in each of sub-integral forms of them. It means all analytic solutions in the three-term recurrence can be described as hypergeometric function: understanding the connection between other special functions is important in the mathematical and physical points of views as we all know.

The analytic integral form of the GCH function with three recursive coefficients are derived from power series expansion in closed forms of the GCH function. And generating function for
the GCH polynomial is constructed from the integral form of the GCH polynomial. The generating function is really helpful in order to derive orthogonal relations, recursion relations and expectation values of any physical quantities. For the case of hydrogen-like atoms, the normalized wave function is derived from the generating function for associated Laguerre polynomials. And expectation values of physical quantities such as position and momentum are constructed by applying the recursive relation of associated Laguerre polynomials.

6. Series “Special functions and three term recurrence formula (3TRF)"

This paper is 10th out of 10.

1. “Approximative solution of the spin free Hamiltonian involving only scalar potential for the $q - \bar{q}$ system” [17] - In order to solve the spin-free Hamiltonian with light quark masses we are led to develop a totally new kind of special function theory in mathematics that generalize all existing theories of confluent hypergeometric types. We call it the Grand Confluent Hypergeometric Function. Our new solution produces previously unknown extra hidden quantum numbers relevant for description of supersymmetry and for generating new mass formulas.

2. “Generalization of the three-term recurrence formula and its applications” [18] - Generalize three term recurrence formula in linear differential equation. Obtain the exact solution of the three term recurrence for polynomials and infinite series.

3. “The analytic solution for the power series expansion of Heun function” [19] - Apply three term recurrence formula to the power series expansion in closed forms of Heun function (infinite series and polynomials) including all higher terms of $A_n$'s.

4. “Asymptotic behavior of Heun function and its integral formalism”, [20] - Apply three term recurrence formula, derive the integral formalism, and analyze the asymptotic behavior of Heun function (including all higher terms of $A_n$'s).

5. “The power series expansion of Mathieu function and its integral formalism”, [21] - Apply three term recurrence formula, analyze the power series expansion of Mathieu function and its integral forms.

6. “Lame equation in the algebraic form” [22] - Applying three term recurrence formula, analyze the power series expansion of Lame function in the algebraic form and its integral forms.

7. “Power series and integral forms of Lame equation in Weierstrass’s form and its asymptotic behaviors” [23] - Applying three term recurrence formula, derive the power series expansion of Lame function in Weierstrass’s form and its integral forms.

8. “The generating functions of Lame equation in Weierstrass’s form” [24] - Derive the generating functions of Lame function in Weierstrass’s form (including all higher terms of $A_n$’s). Apply integral forms of Lame functions in Weierstrass’s form.

9. “Analytic solution for grand confluent hypergeometric function” [25] - Apply three term recurrence formula, and formulate the exact analytic solution of grand confluent hypergeometric
function (including all higher terms of $A_n$’s). Replacing $\mu$ and $\varepsilon\omega$ by 1 and $-q$, transforms the grand confluent hypergeometric function into Biconfluent Heun function.

10. “The integral formalism and the generating function of grand confluent hypergeometric function” [26] - Apply three term recurrence formula, and construct an integral formalism and a generating function of grand confluent hypergeometric function (including all higher terms of $A_n$’s).

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