OPEN ACCESS

E₆ GUT through effects of dimension-5 operators

Chao-Shang Huang¹, Wen-Jun Li² and Xiao-Hong Wu³

¹ Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, People’s Republic of China
² Institute of Theoretical Physics, College of Physics and Material Science, Henan Normal University, Xinxiang 453007, People’s Republic of China
³ Institute of Modern Physics, School of Science, East China University of Science and Technology, Meilong Road 130, Shanghai 200237, People’s Republic of China

E-mail: xhwu@ecust.edu.cn

Keywords: grand unification theory, E₆ group, gauge coupling unification

Abstract

In the effective field theory framework, quantum gravity can induce effective dimension-5 operators, which have important impacts on grand unified theories. Interestingly, one of main effects is the modification of the usual gauge coupling unification condition. We investigate the gauge coupling unification in E₆ under modified gauge coupling unification condition at scales M_X in the presence of one or more dimension-5 operators. It is shown that non-supersymmetric models of E₆ unification can be obtained and can easily satisfy the constraints from the proton lifetime. For constructing these models, we consider several maximal subgroups H = SO(10) × U(1), H = SU(3) × SU(3) × SU(3), and H = SU(2) × SU(6) of E₆ and the usual breaking chains for a specific maximal subgroup, and derive all of the Clebsch–Gordan coefficients Φ_{r,s}^{(t)} associated with E₆ breaking to the standard model, which are given in appendix A.

1. Introduction

It is well-known that the problem of quantum gravity has not been solved yet, although superstring theory presents a beautiful promise. Can we examine the effects of quantum gravity nowadays? The answer is positive. The reason is that the scale of unification M_G ~ 2 × 10¹⁶ GeV in the minimal supersymmetric (SUSY) standard model (SM), as implicated by the experiment measurements [1–4], is smaller than the Planck scale M_P (M_P = (8πG_N)⁻¹/₂ ~ 2.4 × 10¹⁸ GeV), where quantum gravity should come in, by about two orders of magnitudes so that one can build a field theoretical description of the unification of particle interactions without a full solution to the problem of quantum gravity. To describe the effects of quantum gravity, we could use the effective field theory approach in which non-renormalizable higher dimension operators are introduced. The d ≥ 5 operators induced by gravity should enter the Lagrangian, which are suppressed by factors of (M_p)⁻⁻(d−4) with coefficients at the order of ~O(1). They are only subject to the constraints of the symmetries (gauge invariance, supersymmetry in SUSY models, etc) of the low energy theory.

As it has been shown in [5–27] that the presence of higher dimension operators may have substantial impacts on grand unified theory (GUT) and its phenomenology for gauge groups SU(5) and SO(10). These operators modify the usual gauge coupling unification condition [5–10, 20, 27]. It is estimated that the effects of dimension-5 operators can be more important than two-loop corrections in the renormalization group (RG) analysis of gauge coupling unification [7]. With one or more dimension-5 operators, it is possible to achieve unification at scales M_X much different than usually expected [27]. Higher dimension operators can lead to an acceptable value of sin²θ_W [5, 11, 12, 20], affect SUSY particle spectrum in SUSY GUT and supergravity [14–24] and the analysis of proton decay [25, 26]. Therefore, we should include them in the researches of GUT and SUSY GUT. In particular, in some cases, such studies are dramatically important. For example, the gauge coupling unification in the minimal SU(5) without supersymmetry cannot be realized and the minimal SUSY SU(5) has
been already excluded by the limit (larger than $10^{34} \text{y}$) of the proton lifetime from Super-Kamiokande [64], but they can be realized if effects of $d \geq 5$ operators are included (see, for example, [20, 27]).

The exceptional group $E_6$ is an attractive unification group among well-known unification groups. The main reasons are as follows. Firstly, from the viewpoint of superstring theory, the gauge and gravitational anomaly cancellation occurs only for the gauge groups $SO(32)$ or $E_8 \times E_8$ [31–33] and compactification on a Calabi–Yau manifold with an SU(3) holonomy results in the breaking $E_8 \rightarrow SU(3) \times E_6$ [34]. Secondly, in terms of the phenomenology of low energy effective theories originated from $E_6$ GUT, there are several attractive features [35–42]. Moreover, if we assume the dynamical symmetry breaking scenario, we would have several constraints on the possible GUT models. It has been pointed out [43–45] that $E_6$ is uniquely selected among many GUT groups, if one demands that (1) the theory is automatically anomaly free, (2) every generation of quark/lepton fields belongs to a single irreducible representation (irrep) of the GUT group, and (3) the Higgs fields, which are necessary for inducing the symmetry breaking down to $SU_C(3) \times U_{em}(1)$, fall in the representations that can be provided by the fermion bilinears. Recently some models and their low energy phenomenology originated from $E_6$ GUT generated further interests [46–54].

Effects of higher dimension operators to the unification of gauge couplings have also been investigated in the grand unified gauge group $E_6$ [20, 55]. In the [20], $E_6 \rightarrow SU(3) \times SU(3) \times SU(3)$ has been studied and the corresponding effective contributions to gauge kinetic terms have been given. However, their contributions to gauge kinetic terms in the SM have been not given, although it is not difficult to derive them from the results shown in table 4 of the paper. Moreover, the numerical analysis on the unification of gauge couplings and corresponding physical discussions have not been carried out in the paper. To construct $E_6$ unification models with effects of dimension-5 operators in [55], the author considers only the maximal subgroup $H = SO(10) \times U(1)$ and gives the Clebsch–Gordan coefficients $\Phi_{ij}^{kl}$ associated with $E_6$ breaking to the SM for that case. Nevertheless, same as the [20], the numerical analysis on the unification of gauge couplings and corresponding physical discussions have not been carried out.

In this paper, we investigate the unification of gauge couplings for the grand unified gauge group $E_6$ through effects of dimension-5 operators. Surveying the branching rules for the GUT group $E_6$ [56], we see that there are several maximal subgroups containing $G_{321}$ (e.g., $H = SO(10) \times U(1)$, $H = SU(3) \times SU(3) \times SU(3)$, $H = SU(2) \times SU(6)$, and $H = E_6$) and for a specific maximal subgroup there are usually several breaking chains. We consider all common maximal classical subgroups with $SU_C(3) \times U(1)_{em}$ and the usual breaking chains for a specific maximal subgroup. We show that the gauge coupling unification condition is modified due to the effects of dimension-5 operators, which are the lowest higher dimension operators in $E_6$ GUT, and the gauge coupling unification at a higher scale can be realized without SUSY. Furthermore, we find that the constraints from the proton lifetime can be safely satisfied even at the higher unification scale $M_G$ near the Planck scale $M_{Pl}$.

In section 2, we set up the notation to study effects of dimension-5 operators and point out how dimension-5 operators modify the usual gauge coupling unification condition. Section 3 is devoted to the numerical analysis of RG evolution of the gauge couplings and the corresponding physical results for these cases. In section 4, we discuss the constraints from the proton lifetime. Summary and conclusions are given in section 5. All of the Clebsch–Gordan coefficients $\Phi_{ij}^{kl}$ associated with $E_6$ breaking to the SM, in different bases $\{s, z\}$, up to a uniform normalization constant for different representations $r$, are derived, and results are given in appendix A. In appendix B, we present the structure constants of $E_6$ explicitly, which are mostly used by physicists and in the study of GUT.

2. Dimension-5 operators and modified gauge coupling unification condition

Dimension-5 operators are singlets of the grand unified gauge group $G$, and are formed from gauge field strengths $G_{\mu \nu}$ and Higgs multiplets $H_k$ of $G$,

$$\mathcal{L} = \frac{\epsilon_k}{M_{Pl}} G^a_{\mu \nu} G^{b \mu \nu} H^a_k, \quad (1)$$

where $a, b$ are group indices and $k$ labels different multiplets. Therefore, the terms relevant to our discussions of gauge coupling unification in the Lagrangian at the unification scale are

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu \nu} G^{a \mu \nu} + \frac{\epsilon_k}{4M_{Pl}} G^a_{\mu \nu} G^{b \mu \nu} H^a_k, \quad (2)$$

$^{4}$ SU(5) GUT can be obtained by adding more Higgs representations or discrete symmetries [28–30].
Now $G = E_8$, it is evident from equation (1) that the representations, to which Higgs fields $H_k$ belong, can only be contained in the symmetric product of two adjoints$^5$,

$$ (78 \times 78)_{\text{symmetric}} \equiv 1 + 650 + 2430. \quad (3) $$

For a specific irrep $r$, $r = 1, 650, 2430$, in equation (3), we denote the Higgs multiplet by a $d$-dimensional symmetric matrix $\Phi^{(r)}$, with $d = d(G)$, the dimension of the adjoint representation (rep) $G$ (we use the same letter $G$ to denote the group and its adj. rep. for simplicity), and $d = 78$ for $G = E_8$. For our purpose, we find that all possible $\Phi^{(r)}$ are invariant under the SM gauge group $G_{321} \equiv SU(3) \times SU(2) \times U(1)_Y$. That is, each of them is a SM singlet and then the matrix is largely simplified: it contains only a few independent entries.

There are different ways to define the hypercharge $Y$, which are consistent with the SM (see, e.g., the $[59, 60]$). We consider two cases: (1) $U_Y(1)$ is a subgroup of $H_2$; (2) $U_Y(1)$ is a subgroup of $E_6$. Hereafter, we shall call the case (1) as 'normal embedding', and the case (2) as 'flipped embedding'. For example, for $H = SU(3) \times SU(3) \times SU(3)$,

(1) Normal embedding, $G_{321} \subset SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_Y \subset SU(3)_C \times SU(3)_L \times SU(3)_R \subset E_6$

$$ 78 \overset{G_{321}}{\longrightarrow} (8, 1, 1) \oplus (1, 8, 1) \oplus (1, 1, 8) \oplus (3, 3, 3) \oplus (3, 3, \bar{3}) $$

(2) Flipped embedding, $G_{321} \subset SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_Y \subset SU(3)_C \times SU(3)_L \times SU(3)_R \subset E_6$

$$ 78 \overset{G_{321}}{\longrightarrow} (8, 1, 1) \oplus (1, 3, 0) \oplus (1, 1, 0) \oplus (3, 2, \frac{3}{2}) \oplus (3, 1, \frac{3}{2}) \oplus (1, 1, 0) $$

$$ \oplus (3, 2) \oplus h.c. \oplus (3, 1) \oplus h.c. \oplus (1, 1) \oplus h.c. \oplus (1, 1) $$

Therefore, $\Phi^{(r)}$, which are invariant under $G_{321}$, can be written as,

$$ (\Phi^{(r)})^{ab} = \delta^{ab} \, \text{diag}[h_1 I_8, h_2 I_8, h_3, h_4 I_{12}, h_5, h_6 I_{12}, h_7 I_8, h_8 I_2, h_9, h_{10} I_{12}, h_{11} I_8, h_{12} I_8, h_{13} I_8, h_{14} I_8, h_{15} I_1 + \delta^{a2} h^{23} h_{35} + \delta^{a2} h^{23} h_{35} + \delta^{a2} h^{23} h_{35} + \delta^{a2} h^{23} h_{35}] $$

$$ + \delta^{a1} h^{23} h_{59} + \delta^{a2} h^{23} h_{35} + \delta^{a2} h^{23} h_{35} + \delta^{a2} h^{23} h_{35}, \quad (6) $$

where $I_8$ is $n$-dimensional unit matrix, and $hij = hji$, since $\Phi^{(r)}$ is the symmetric matrix. We have ordered indices $a, b$ according to the order of terms in equation (4), i.e., $a = 1, \ldots, 8$ for $h_1, a = 9, 10, 11$ for $h_3, h_5, h_7, h_9$, $a = 12, 13, 14$ for $h_8$, $a = 15$ for $h_9$.

In order to find $\Phi^{(r)}$, we use the second order Casimir operator,

$$ C_R = -\sum_{i=1}^{d(G)} X_i^2, \quad (7) $$

$^5$ For simplicity, we use ‘×’ and ‘+’ to denote the direct product and the direct sum respectively, whenever there is no confusion.
The convention is: 

\[ h_{ijk} \text{ for the normal embedding, and} \]

\[ f_{ijk} \text{ as the totally antisymmetric structure constants. The operator acts on the tensor product } R \times R \text{ so that} \]

\[ C_{R \times R} = C_R \times 1 + 1 \times C_R + 2F, \]

where

\[ F = -\sum_{i=1}^{d(G)} X_i \times X_i, \]

It is easy to derive \( F \Phi^{(r)} = F_G(r) \Phi^{(r)} \), where \( F_G(r) = C(r) / 2 - C(G) \) is the eigenvalue of \( F \) in irrep \( r \) with \( C(r) \) and \( C(G) \) being the eigenvalues of the Casimir operator in irrep \( r \) and \( G \) respectively. \( C(r) \) depends on the normalization of the Casimir operator and consequently the choice of structure constants\(^\text{6}\). The structure constants of \( E_6 \) have been given in a Chevalley base\(^\text{61}\). For convenience of the study in GUT, they are transformed into the usual form and listed in appendix B.

The eigenvalues \( C(r) \) for several irreps \( r \) in \( E_6 \), as well as \( SO(10) \) and \( SU(6) \) are listed in table 1.

In the case of \( H = SU(6) \times SU(2) \), depending on different embeddings of the SM into \( H \), i.e., different specific assignment of the chiral fields to representations of subgroup \( H \), there are three cases. In the first case, denoted as \( H = SU(6) \times SU(2)_L \), the SM is totally in \( SU(6) \) and no relation with the factor group \( SU(2) \) of \( H \). In the second case, denoted as \( H = SU(6) \times SU(2)_L \), \( SU(2)_L \) is the factor group \( SU(2) \) of \( H \). And in the third case, denoted as \( H = SU(6) \times SU(2)_R \), \( SU(2)_R \) is the factor group \( SU(2) \) of \( H \). For all the cases, a direct but tedious calculation gives all of the Clebsch–Gordan coefficients, \( \Phi^{(r)}(z) \), associated with \( E_6 \) breaking to the SM, which are listed in appendix A.

By classifying the SM singlets \( \Phi^{(r)}(z) \) into irreps under different breaking chains as in tables in appendix A, we do not mean to imply a grand unified symmetry breaking scenario, where \( E_6 \) is broken to the SM necessarily via one or some intermediate gauge groups, though such a classification is well suited for such a scenario. However, such a classification is suitable to serve as a parametrization for the SM singlets \( \langle H^{ab}_{kz} \rangle \), the non-zero vacuum expectation values of \( H^{ab}_{kz} \), transforming in irrep \( r_{kz} \) of \( E_6 \) in terms of the basis \( \Phi_{kz}^{(r)} \equiv \Phi_{kz}^{(r)}(z) \),

\[ \langle H^{ab}_{kz} \rangle = \sum_{kz} v^{kz}_{a} \Phi_{kz}^{ab} \equiv v^{kz}_{a} \Phi_{kz}^{ab}, \]

where \( v^{kz}_{a}, v^{kz}_{b} \) are real.

Equations (2), (11) lead to an alteration of the gauge coupling unification condition,

\[ g_1^2(M_{GUT})(1 + \epsilon_1) = g_2^2(M_{GUT})(1 + \epsilon_2) = g_3^2(M_{GUT})(1 + \epsilon_3) = g^2_G / 4\pi, \]

where

\[ \epsilon_i = \sum_k \frac{C_k}{M_{Pl}} \sum_{i,z} v^{kz}_{i} \phi_{i}^{kz}, \quad i = 1, 2, 3, \quad \phi_{1}^{kz} \equiv -h_{3}^{kz}, \quad \phi_{2}^{kz} \equiv -h_{2}^{kz}, \quad \phi_{3}^{kz} \equiv -h_{1}^{kz}, \]

for the normal embedding, and \( \phi_{i}^{kz} \) are listed in tables (see appendix A). For the case of flipped embedding\(^6\), \( \phi_{i}^{kz} \) is changed into \( \phi_{i}^{kz} \equiv -h_{3}^{kz}, \quad h_{2}^{kz}, \quad h_{1}^{kz} \) are also listed in tables in appendix A.

Equation (12) is the boundary condition at the scale \( M_{GUT} \) of the RG evolution of gauge couplings, and will be used in the numerical analysis in the next section.

\(^\text{6}\) The choice in this paper leads to that for the subgroup \( SO(10) \) of \( E_6 \), \( \epsilon(r) \)'s are the same as those in \( [27, 57, 58, 62] \).

\(^\text{7}\) The convention is: \( k = 1, 2, 3 \) correspond to irreps 1, 650, 2430.
3. Grand unification in $E_6$

In this section, we study the unification of gauge couplings with one, two, or more dimension-5 operators numerically from different $650$ and $2430$ breaking chains.

As to the case of one-dimension-5 operator from $650$ or $2430$, because there are several maximal subgroups and for a specific maximal subgroup there are several breaking chains, the SM singlets ($H_{ab}^k$), the non-zero vacuum expectation values of $H_{ab}^k$, are not determined for fixed $k = 2$ or 3. Consequently it is clear from equation (13) and tables 3–10 that the ratio $\epsilon_1 : \epsilon_2 : \epsilon_3$ cannot be determined fully. So one has much freedom to choose ratios among $v^k_a$. It is a boring and not necessary task to exhaust all possibilities. Instead, we just take some breaking chains as examples. Then, the unification scale $M_X$ (set $M_X = M_{GUT}$ for simplicity) and Wilson coefficient $c_i$ are computable by means of the gauge coupling unification condition equation (12), when the running gauge couplings in the SM are given. We limit ourselves to one-loop case for running in the paper for simplicity and it is straightforward to generalize to two-loop case. Moreover, dimension-5 operators also affect analysis of proton decay [25, 26]. As analyzed in the [27] and in the next section, the absolute value of Wilson coefficient should be less than 10, i.e., $\max|c_i| \leq 10$, in order to satisfy the proton decay constraint. Hereafter, we name the unification with $\max|c_i| \leq 10$ as the successful unification.

In table 2, with only one breaking chain, five example results are given with their respective unification scale $M_X$, Higgs VEV $v$, Wilson coefficient $c_b$, gauge coupling $\alpha_{WC}$ at the unification scale, and three $\epsilon_{1,2,3}$ for the unification. The needed Wilson coefficients are too large in two cases of $650 \rightarrow 54 \rightarrow 24$ and $650 \rightarrow (35, 1) \rightarrow 1$. While in the other three cases of $2430 \rightarrow 210 \rightarrow 75$, $2430 \rightarrow (405, 1) \rightarrow 1$, and $2430 \rightarrow (189, 1) \rightarrow 1$, we can reach the successful unification. In particular, for $2430 \rightarrow (189, 1) \rightarrow 1$, $M_X$ is as large as the Planck scale and $|c_3|$ is as small as $\sim 0.015$. That means, we may have a perturbative theory up to the onset of quantum gravity. With only one-dimension-5 operator, we may also use two or more different breaking chains to achieve successful gauge coupling unification with continuously varied unification scale $M_X$, as given in figures 1, 2 for illustrations. In figure 1, we choose two breaking chains $210 \rightarrow 24$ and $210 \rightarrow 75$ for $650$, $770 \rightarrow 24$ and $770 \rightarrow 200$ for $2430$, respectively, under the subgroup embedding $SU(5) \subset SO(10) \subset E_6$. We plot the ratios of VEVs, $v_{2430}/v_{24}$ for $650$, and $v_{2430}/v_{24}$ for $2430$ as a function of continuously varied $M_X$ with successful unification. In figure 2, the ratios of VEVs, $v_{2430}/v_{24}$ for both $650$ and $2430$, as a function of $M_X$ are plotted, with $(35, 1) \rightarrow 1$ and $(189, 1) \rightarrow 8$ for $650$, and $(405, 1) \rightarrow 1$ and $(405, 1) \rightarrow 8$ for $2430$, under the subgroup embedding $SU(3)_C \subset SU(6) \times SU(2)_C \subset E_6$. As it is easy to see, from equations (12), (13) and tables in appendix I and in the [35] that for a given specific breaking chain, when two of three numbers $\{ -h_1^{k,1}, -h_2^{k,2}, -h_3^{k,3} \}$ are zero, i.e., for (45, 0), $(210, 0) \rightarrow 1$ in $H = SU(10) \times U(1)$, $(189, 1)$, $(405, 1) \rightarrow 8$, 27 in $H = SU(6) \times SU(2)_C$, $(35, 3)$, (1, 5) in $H = SU(6) \times SU(2)_C$ and $(1, 8), (8, 8), (1, 27)$ in $H = SU(5) \times SU(3) \times SU(3)$, one cannot get successful unification.

In the following, we study the unification of gauge couplings in the case of two-dimension-5 operators with two different Higgs multiplets from $650$ and $2430$ of $E_6$, respectively. In order to achieve continuously varied unification scale $M_X$, we have four variables, two VEVs of Higgs multiplets, $v_{650}$ and $v_{2430}$, and two Wilson coefficients. Without fine-tuning for the VEVs, we fix the ratio of $650$ and $2430$ in three cases, 1:5, 1:1 and 5:1, in our figures for an illustration. Also, for the VEVs of the Higgs multiplets, we assume they account for half of the average gauge boson squared mass [27]. Then, we are left with two Wilson coefficients for the unification. The maximal absolute values of Wilson coefficients as a function of unification scale $M_X$ are given in figures 3–6 for different embedding of subgroup into $E_6$.

The numerical results of the subgroup embedding $SU(5) \subset SO(10) \times U(1) \subset E_6$ are shown in figure 3. We choose two breaking chains, $650 \rightarrow 54 \rightarrow 24$ and $2430 \rightarrow 210 \rightarrow 75$, as an example for the study. For all three cases of the ratios of Higgs VEVs, we have Wilson coefficients less than 10, $\max|c_i| \leq 10$, while unification scale $M_X > 10^{10}$ GeV. The needed $\epsilon$s for the unification are also given in figure 3, which are independent of the ratios of Higgs VEVs. The $\epsilon_i$ is varied smoothly with the scale $M_X$, but the needed $\epsilon_i$ is negative and larger absolute value is required. Similar results for the other embeddings, $SU(3)_C \subset SU(6) \times SU(2)_C \subset E_6$, $SU(3)_C \subset SU(6) \times SU(2)_C \subset E_6$ and $SU(3)_C \subset SU(6) \times SU(3)_C \subset E_6$, are shown in figures 4–6, respectively. In order to obtain $\max|c_i| \leq 10$, larger unification scale $M_X$ are needed for different ratios of Higgs VEVs. A comment is that for different subgroup embedding of $E_6$, we may have different sets of values of $\epsilon$s at a given unification scale $M_X$.

In the case of two-dimension-5 operators with more than two breaking chains, we scan the maximal absolute value of Wilson coefficients as a function of unification scale $M_X$ for subgroup embeddings $SU(5) \subset SO(10) \subset E_6$ and $SU(3)_C \subset SU(6) \times SU(2)_C \subset E_6$ with random VEVs of different breaking chains and random ratios of Wilson coefficients of the two operators in figures 7 and 8. For subgroup embedding $SU(5) \subset SO(10) \subset E_6$, there are actually four different ratios among $v^k_a$, and for subgroup embedding $SU(3)_C \subset SU(6) \times SU(3)_C \subset E_6$, we have eight different ratios. As a result, the unification are much easier for the latter. Most of the points in both cases have satisfied the successful unification with $\max|c_i| \leq 10$. 
Table 2. The breaking chain, unification scale $M_X$, Higgs VEV $v$, Wilson coefficient $c_k$, gauge coupling $\alpha_G$ at the unification scale, and three $\epsilon_{1,2,3}$ for unification with only one-dimension-5 operator.

| $E_6$ embedding | Breaking chain | $M_X$(GeV) | $v$(GeV) | $\alpha$ | $1/\alpha_G$ | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ |
|-----------------|----------------|------------|----------|---------|-------------|-------------|-------------|-------------|
| $SU(5) \subset SO(10)$ | $650 \rightarrow 54 \rightarrow 24$ | $4.0 \times 10^{13}$ | $9.8 \times 10^{13}$ | $1.1 \times 10^{5}$ | 40.6 | 0.023 | 0.068 | −0.045 |
| | $2430 \rightarrow 210 \rightarrow 75$ | $3.7 \times 10^{15}$ | $7.8 \times 10^{15}$ | $-6.4$ | 43.0 | −0.103 | 0.062 | 0.021 |
| $SU(3)_L \subset SU(6) \times SU(2)_L$ | $650 \rightarrow (35, 1) \rightarrow 1$ | $4.0 \times 10^{13}$ | $9.8 \times 10^{13}$ | $1.4 \times 10^{5}$ | 41.1 | 0.011 | 0.056 | −0.056 |
| | $2430 \rightarrow (405, 1) \rightarrow 1$ | $6.4 \times 10^{16}$ | $1.4 \times 10^{17}$ | $0.63$ | 48.9 | −0.250 | −0.038 | −0.038 |
| $SU(3)_L \subset SU(6) \times SU(2)_L$ | $2430 \rightarrow (189, 1) \rightarrow 1$ | $3.6 \times 10^{18}$ | $8.1 \times 10^{18}$ | $-0.015$ | 49.0 | −0.306 | 0 | 0.051 |
Table 3. The diagonal part, $h_j$ ($j = 1, 2, \ldots, 15$) in equation (6) of the SM singlets $\Phi^r_{ij}$ in each of the irreps $r$ according to their transformation properties under the $SU(3)_c \subset SU(6) \times SU(2)_L \subset E_6$ subgroup.

| $E_6$ | SU(6)×SU(2)_{LS} | SU(3)_c | $h^r_{1_{xyz}}$ | $h^r_{2_{xyz}}$ | $h^r_{3_{xyz}}$ | $h^r_{4_{xyz}}$ | $h^r_{5_{xyz}}$ | $h^r_{6_{xyz}}$ | $h^r_{7_{xyz}}$ | $h^r_{8_{xyz}}$ | $h^r_{9_{xyz}}$ | $h^r_{10_{xyz}}$ | $h^r_{11_{xyz}}$ | $h^r_{12_{xyz}}$ | $h^r_{13_{xyz}}$ | $h^r_{14_{xyz}}$ | $h^r_{15_{xyz}}$ | $N^r_{xyz}$ |
|-------|-------------------|---------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|----------------|
| 1     | 11                | 11      | 1               | 1               | 1               | 1               | -1/2            | -1/2            | 1               | -3             | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               |
| 650   | (1, 1) 1          | 11      | 1               | 1               | 1               | -1/2            | -1/2            | 1               | -3             | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               |
|       | (35, 1) 2         | 11      | 1               | 1               | 1               | -1/2            | -1/2            | 1               | -3             | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               | 1               |
|       | (189, 1) 3        | 11      | 1               | 1               | 1               | -3             | 0               | -3/2            | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
|       | 2                 | 0               | 0               | -1/2            | 1/2             | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
|       | 82                | 0               | 1               | 1               | -1/2            | 1/2             | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
| 2430  | (1, 1) 1          | 11      | 1               | 1               | 1               | 1               | -3             | 0               | -3/2            | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
|       | (189, 1) 2        | 11      | 1               | 1               | 1               | 1               | -3             | 0               | -3/2            | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
|       | 82                | 0               | 1               | 1               | 1               | 0               | 0               | -3/2            | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
|       | (405, 1) 3        | 11      | 1               | 1               | 1               | 1               | 1               | 0               | -3             | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
|       | 82                | 0               | 1               | 1               | 1               | 0               | 0               | -3             | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
|       | 27                | 0               | 1               | 1               | 1               | 1               | 1               | 0               | -3             | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |

J. Phys. Commun. 1 (2017) 055025 C-S Huang et al
Table 4. The non-diagonal part, $h_{ij}$ in equation (6) of the SM singlets $\Phi_{i}^{\prime}$, in each of the irreps $r$ according to their transformation properties under the $SU(3)_{C} \subset SU(6) \times SU(2)_{L} \subset E_{6}$ subgroup. The entries $h_{33}^{\prime}, h_{59}^{\prime}, h_{39}^{\prime}$ for the flipped embedding are also listed here.

| $E_{6}$ | SU(6)×SU(2)$_{L}$×SU(2)$_{R}$ | SU(3)$_{C}$ | $h_{33}^{\prime}$ | $h_{59}^{\prime}$ | $h_{39}^{\prime}$ | $h_{33}^{\prime}$ | $h_{59}^{\prime}$ | $h_{39}^{\prime}$ |
|--------|--------------------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 650    | (1, 1) | 1 1 1 1 1 0 0 0 0 0 0 0 0 | 0          | 0          | 0          | 0          | 0          | 0          | 0          |
|        | (35, 1) | 1 1 1 1 1 1 1 0 0 0 0 0 0 | 3.577      | 3.577      | 3.577      | 3.577      | 3.577      | 3.577      | 3.577      |
|        | 8 2 8 2 8 2 8 2 8 2 8 2 8 2 | 0.0577      | 0.0577      | 0.0577      | 0.0577      | 0.0577      | 0.0577      | 0.0577      |
|        | (189, 1) | 1 1 1 1 1 1 1 0 0 0 0 0 0 | 4.275      | 4.275      | 4.275      | 4.275      | 4.275      | 4.275      | 4.275      |
|        | 8 2 8 2 8 2 8 2 8 2 8 2 8 2 | 4.275      | 4.275      | 4.275      | 4.275      | 4.275      | 4.275      | 4.275      |
| 2430   | (1, 1) | 1 1 1 1 1 1 0 0 0 0 0 0 0 | 0          | 0          | 0          | 0          | 0          | 0          | 0          |
|        | (189, 1) | 1 1 1 1 1 1 1 0 0 0 0 0 0 | 0          | 0          | 0          | 0          | 0          | 0          | 0          |
|        | (405, 1) | 1 1 1 1 1 1 1 0 0 0 0 0 0 | 0          | 0          | 0          | 0          | 0          | 0          | 0          |
|        | 8 2 8 2 8 2 8 2 8 2 8 2 8 2 | 0.0577      | 0.0577      | 0.0577      | 0.0577      | 0.0577      | 0.0577      | 0.0577      |
|        | 27 3 27 3 27 3 27 3 27 3 27 3 27 3 27 3 | 0.0577      | 0.0577      | 0.0577      | 0.0577      | 0.0577      | 0.0577      | 0.0577      |

As a summary of this section, we conclude that non-SUSY models of $E_{6}$ grand unification can be obtained through effects of dimension-5 operators. Comparing with the other groups like SU(5) and SO(10), it is much easier to achieve the successful unification with natural Wilson coefficient $c_{H}$ and continuously varied unification scale $M_{X}$. Thus including effects of quantum gravity provides a greater probability for building a realistic non-SUSY model in $E_{6}$ GUT.

4. The constraint from the proton decay

As it is well-known, the key predictions in GUT, despite of the details of model building, are the gauge coupling uni

Combining equations (6) and (7), we can realize the gauge symmetry breaking chain in a specific model. In order to guarantee correct use of the running gauge couplings in the SM (see the previous section), it is necessary to require $M_{H} \sim$ close to the grand unified symmetry breaking scale, i.e., the unification scale $M_{X}$, in the case of one-step breaking. For the purpose of definiteness and omitting heavy threshold effects, we take $M_{H} = M_{X}$. Then, the proton lifetime due to superheavy gauge boson exchange can be written as [25, 26, 63],

$$\tau_{p} = C M_{X}^{4} / (\alpha_{e}^{2} m_{p})$$ (15)

where $C$ is a coefficient containing all information about the flavor structure of the theory and $m_{p}$ is the mass of proton. The newest experimental bound on the proton lifetime for the channel $p \rightarrow e^{+} \pi^{0}$ is [64],

$$\tau_{p} \rightarrow e^{+} \pi^{0} > 1.6 \times 10^{34} \text{ years}$$ (16)

Combining equations (15) and (16), one has

$M_{X} > (40 \alpha_{e})^{1/2} (1/C)^{1/4} 4.3 \times 10^{15} \text{ GeV}$. (17)

$C$ is order of 1, $C \sim O(1)$. $\alpha_{e} = 1/70 \sim 1/40$ (see table 2 and figures above). Even when $C$ is as small as 0.1, the constraint from the proton decay is just $M_{X} > 8.9 \times 10^{15} \text{ GeV}$. 

8
Table 5. The diagonal part, $h_{ij}$ $(i = 1, 2, \ldots, 15)$ in equation (6) of the SM singlets $\Phi^c_{ir}$ in each of the irreps $r$ according to their transformation properties under the $SU(3)_{\text{c}} \subset SU(6) \times SU(2)_{\text{L}} \subset E_6$ subgroup.

| $E_6$ | $SU(6) \times SU(2)_{\text{R}}$ | $SU(3)_{\text{c}} \times SU(2)_{\text{L}}$ | $h^1_{i1}$ | $h^2_{i1}$ | $h^3_{i1}$ | $h^4_{i1}$ | $h^5_{i1}$ | $h^6_{i1}$ | $h^7_{i1}$ | $h^8_{i1}$ | $h^9_{i1}$ | $h^{10}_{i1}$ | $h^{11}_{i1}$ | $h^{12}_{i1}$ | $h^{13}_{i1}$ | $h^{14}_{i1}$ | $h^{15}_{i1}$ | $N_{ir}$ |
|-------|---------------------------------|---------------------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1     |                                 |                                 | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         |
| 650   | $(1, 1) 1$                       |                                 | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         | 1         |
|       | $(35, 1) 2$                      |                                 | 1         | 0         | $-\frac{4}{7}$ | 0         | $-\frac{7}{11}$ | 0         | 0         | $-\frac{1}{7}$ | 0         | 0         | $-\frac{1}{7}$ | 0         | 0         | $-\frac{1}{7}$ | 0         | 0         | $-\frac{1}{7}$ |
|       | $(189, 1) 3$                     |                                 | 1         | 0         | $\frac{2}{7}$ | 0         | $-\frac{15}{56}$ | 0         | $-\frac{5}{7}$ | 1         | $-\frac{2}{7}$ | 0         | $-\frac{2}{7}$ | 1         | $-\frac{2}{7}$ | 0         | $-\frac{2}{7}$ | 1         | $-\frac{2}{7}$ |
| 2450  | $(1, 1) 1$                       |                                 | 1         | 1         | $\frac{35}{7}$ | 1         | $\frac{7}{8}$ | 1         | 1         | $\frac{7}{8}$ | 1         | 1         | $\frac{7}{8}$ | 1         | 1         | $\frac{7}{8}$ | 1         | 1         | $\frac{7}{8}$ |
|       | $(189, 1) 2$                     |                                 | 1         | 1         | 0         | $-\frac{6}{7}$ | 0         | $\frac{11}{11}$ | 0         | $-\frac{6}{7}$ | 1         | $-\frac{2}{7}$ | 0         | $-\frac{2}{7}$ | 1         | $-\frac{2}{7}$ | 0         | $-\frac{2}{7}$ | 1         | $-\frac{2}{7}$ |
|       | $(405, 1) 3$                     |                                 | 1         | 1         | 0         | $\frac{12}{7}$ | 0         | $\frac{31}{56}$ | 0         | $-\frac{4}{7}$ | 1         | $\frac{7}{7}$ | 0         | $-\frac{4}{7}$ | 1         | $\frac{7}{7}$ | 0         | $-\frac{4}{7}$ | 1         | $\frac{7}{7}$ |
| 27    | 1                               |                                 | 0         | 0         | $-\frac{2}{19}$ | 0         | 0         | $\frac{23}{456}$ | 0         | 0         | $\frac{10}{97}$ | 0         | $\frac{25}{258}$ | 0         | $\frac{135}{135}$ | 0         | $\frac{3}{37}$ | 0         | $\frac{25}{76}$ | 0         | $\frac{5}{57}$ |
Table 6. The non-diagonal part, \( h_{ij} \) in equation (6) of the SM singlets \( \Phi^I_{ij} \), in each of the irreps \( r \) according to their transformation properties under the \( SU(3)_c \times SU(2)_L \times U(1)_Y \) subgroup. The entries \( h_{55}^{20}, h_{32}^{36}, h_{95}^{63} \) for the flipped embedding are also listed here.

| \( E_6 \) | \( SU(6) \times SU(2)_L \times U(1)_Y \) | \( h_{55}^{20} \) | \( h_{32}^{36} \) | \( h_{95}^{63} \) |
|---|---|---|---|---|
| \((1, 1, 1)\)
| 650 | 11 | 0 | 0 | 0 |
| (35, 1, 2) | 11 | -1/2 | -1/2 | -1/2 |
| (189, 1, 3) | 11 | 11/2 | 11/2 | 11/2 |
| 2430 | 11 | 11 | 11 | 11 |
| (189, 1, 2) | 11 | 11 | 11 | 11 |
| (405, 1, 3) | 11 | 11 | 11 | 11 |
| \((1, 1, 1)\)
| 273 | 11 | 11 | 11 | 11 |

We can see from the table 2 and figures in the previous section that most of the cases can easily satisfy the bound equation (17). In the case of only one breaking chain, \( 2430 \rightarrow (405, 1) \rightarrow 1 \) and \( 2430 \rightarrow (189, 1) \rightarrow 1 \), have the unification scale \( 6.4 \times 10^{16} \) GeV and \( 3.6 \times 10^{18} \) GeV, respectively. Both lie well above the bound equation (17). For the case of \( 2430 \rightarrow 210 \rightarrow 75 \), the unification scale is \( 3.7 \times 10^{15} \) GeV, which is very close to the bound equation (17) and can be easily adjusted (say, to increase the value of \( v \)) to satisfy the bound. With only one-dimension-5 operator, using two (or more) different breaking chains, one can achieve successful gauge coupling unification with continuously varied unification scale \( M_{\chi} \), as long as the ratio of two vevs varies with \( M_{\chi} \). Thus, the bound equation (17) is satisfied. Looking at the results shown in the last section, the same remains in the case of two-dimension-5 operators with two different Higgs multiplets from 650 and 2430 of \( E_6 \) respectively.

5. Summary and discussion

It has been pointed out that gauge coupling unification condition is modified due to the effects of dimension-5 operators. We have investigated the gauge coupling unification in \( E_6 \) without SUSY under modified gauge coupling unification condition. For this purpose, considering several maximal subgroups \( H = SO(10) \times U(1) \), \( H = SU(3) \times SU(3) \times SU(3) \), \( H = SU(2) \times SU(2) \times U(1) \) of \( E_6 \) and the usual breaking chains for a specific maximal subgroup, we have derived and given all of the Clebsch–Gordan coefficients \( \Phi^I_{ij} \) associated with \( E_6 \) breaking to the SM. We have also presented the structure constants of \( E_6 \) in the usual form, which are mostly used by physicists and in the study of GUT. Results on the gauge coupling unification show that, for a single dimension-5 operator, realizing the gauge coupling unification under modified gauge coupling unification condition in \( E_6 \) GUT is easier than that in SO(10) GUT, since there are more maximal subgroups, and, for a specific maximal subgroup, there are more breaking chains, so that one has much freedom to choose ratios among the non-zero vacuum expectation values \( v_{ij}^\xi \) of the Higgs multiplets \( H_c \). We have also analyzed the constraint on the unification scale \( M_{\chi} \) from the newest data of the proton decay. It is shown that most of cases studied in the section IV satisfy the constraint easily.

In the effective field theory spirit, operators of dimension higher than 5 are also present, e.g., a dimension-6 operator generalization of equation (1),

\[
\mathcal{L} = \frac{c_6}{M_{Pl}} H_c H_c G_{\mu\nu} G^{\mu\nu} + \ldots
\]

After the Higgs multiplets acquire vevs at the scale \( M_{\chi} = M_{Pl}/O(1) \), they can contribute to the gauge kinetic terms as well,

\[
\mathcal{L} = \sum_{i=1}^{3} \frac{1}{4} (1 + \epsilon_i + \epsilon_i^{(6)} + \ldots) G_i^{\mu\nu} C_i^{\mu\nu} C_i^{\mu\nu}
\]
Table 7. The diagonal part, $h_j (j = 1, 2, \ldots, 15)$ in equation (6) of the SM singlets $\Phi^r_{ir}$ in each of the irreps $r$ according to their transformation properties under the SU(3)$_c \times$ SU(6) $\times$ SU(2)$_b \subset E_6$ subgroup.

| $E_6$ | SU(6)×SU(2)$_b$ | SU(3)$_c$ | $h_{1r}$ | $h_{2r}$ | $h_{3r}$ | $h_{4r}$ | $h_{5r}$ | $h_{6r}$ | $h_{7r}$ | $h_{8r}$ | $h_{9r}$ | $h_{10r}$ | $h_{11r}$ | $h_{12r}$ | $h_{13r}$ | $h_{14r}$ | $h_{15r}$ | $N_r$ |
|-------|-----------------|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|-------|
| 650   |                 |           | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      |
|       | (1, 1) 1       | 11        | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      |
|       | (35, 1) 2      | 11        | 1      | -1     | $-\frac{1}{3}$ | 0      | $-\frac{1}{2}$ | 0      | 0      | 0      | -1      | 0      | 0      | 0      | -1      | 0      | $\frac{1}{2}$ | 0      | $\frac{1}{2}$ |
|       | (35, 3) 3      | 82        | 0      | 1      | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
|       | (189, 1) 4     | 11        | 1      | 1      | -3      | $-\frac{2}{3}$ | $-\frac{1}{3}$ | 0      | 0      | 0      | 0      | 0      | 0      | 0      | -2      | $\frac{1}{3}$ | 1      | 0      |
|       | 82              | 0      | 1      | -1     | $-\frac{1}{3}$ | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | $\frac{1}{3}$ | 0      | $\frac{1}{2}$ |
| 2430  |                 |           | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      | 1      |
|       | (1, 5) 2       | 11        | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
|       | (35, 3) 3      | 11        | 0      | 0      | 0      | 0      | $-\frac{15}{2}$ | 0      | 0      | $-\frac{25}{2}$ | 25      | 15      | 0      | $-\frac{25}{2}$ | 0      | $-\frac{25}{2}$ |
|       | 82              | 0      | 0      | 0      | 0      | $-\frac{15}{2}$ | $\frac{5}{2}$ | 0      | $-\frac{15}{2}$ | $\frac{5}{2}$ | $\frac{5}{2}$ | 0      | 0      | 0      | 0      | 0      | 0      |
|       | (189, 1) 4     | 11        | 1      | 1      | -3      | $-\frac{1}{3}$ | $\frac{15}{2}$ | 0      | 0      | 0      | $-\frac{15}{2}$ | 0      | 0      | 0      | -2      | $\frac{1}{3}$ | 1      | 0      |
|       | (405, 1) 5     | 11        | 1      | 1      | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 0      | 0      | 0      | $-\frac{1}{3}$ | 0      | 0      | 0      | $-\frac{1}{3}$ | 1      | 0      | $\frac{1}{3}$ |
|       | 82              | 0      | 1      | $\frac{2}{3}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0      | 0      | 0      | -3      | 0      | 0      | 0      | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0      | $\frac{1}{3}$ |
|       | 27              | 0      | 1      | $\frac{1}{3}$ | 0      | $\frac{1}{3}$ | 0      | 0      | 0      | $\frac{1}{3}$ | 0      | 0      | 0      | 0      | $-\frac{1}{3}$ | 0      | $\frac{1}{3}$ |
So we can use $c_{k} H_{k} h$ where $v \sim \frac{M_{g}}{g_{k}}$ is the average of the $H_{i}, H_{k}$ and $c_{k}$ $(k = 1, 2, 3$ corresponding to the representation $1, 78, 650$ respectively) is the Clebsch–Gordan coefficient from the decomposition of the Kronecker product $27 \times 27$ (see (20)), instead of $H_{1} H_{2}$ in (18). For specific, set $k = 3,$ i.e., the $650,$ then we have,

$$27 \times 27 = 1 + 78 + 650.$$  \hfill (20)

So we can use $c_{k} H_{k} h,$ where $v \sim \frac{M_{g}}{g_{k}}$ is the average of the $H_{i}, H_{k}$ and $c_{k}$ $(k = 1, 2, 3$ corresponding to the representation $1, 78, 650$ respectively) is the Clebsch–Gordan coefficient from the decomposition of the Kronecker product $27 \times 27$ (see (20)), instead of $H_{1} H_{2}$ in (18). For specific, set $k = 3,$ i.e., the $650,$ then we have,

$$L = \epsilon_{i} \frac{c_{k}}{M_{Pl}} \frac{M_{k}}{g_{k}} H_{k} G_{\mu \nu} G^{\mu \nu} + \ldots, \hfill (21)$$

which leads to $\epsilon_{i}^{(6)} = \frac{c_{k} M_{k}}{M_{Pl} g_{k}} \langle H_{k} \rangle.$ For $k = 3,$ $\epsilon_{i} = \frac{c_{3}}{M_{g}} \langle H_{3} \rangle.$ Therefore, $\epsilon_{i}^{(6)} \sim \frac{c_{k} g_{k} \langle H_{i} \rangle}{M_{Pl} g_{k}} \epsilon_{i}.$ With $c_{k} \sim c_{s},$ $g_{k} \leq 1$ and $M_{k} = M_{Pl} / O(1),$ effects of dimension-6 operators might be the same order of those of dimension-5 operators. Therefore, the expansion of equation (19) might not or might be controlled perturbatively. If it is not, one cannot claim perturbative gauge-gravity unification at the Planck scale. However, the fact that the modification of the gauge coupling unification condition of equation (12) allows us in principle to adjust the unification scale to a higher scale $\sim M_{Pl}$ could at least be taken as a hint that gauge-gravity unification is a possible scenario, even if the necessary parameter values or the last piece of the evolution cannot be computed perturbatively.

There is an interesting subject in the effective theory framework. That is, the following operators of dimension-5 are also probably present,
Table 9. The diagonal part, $h_j (j = 1, 2, \ldots, 15)$ in equation (6) of the SM singlets $\Phi^r_{\alpha r}$ in each of the irreps $r$ according to their transformation properties under the $\text{SU}(3)_c \times \text{SU}(3)_L \subset E_6$ subgroup.

| $E_6$ | $\text{SU}(3)_c \times \text{SU}(3)_L$ | $h_1^L_{1,2}$ | $h_2^L_{1,2}$ | $h_3^L_{1,2}$ | $h_4^L_{1,2}$ | $h_5^L_{1,2}$ | $h_6^L_{1,2}$ | $h_7^L_{1,2}$ | $h_8^L_{1,2}$ | $h_9^L_{1,2}$ | $h_{10}^L_{1,2}$ | $h_{11}^L_{1,2}$ | $h_{12}^L_{1,2}$ | $h_{13}^L_{1,2}$ | $h_{14}^L_{1,2}$ | $h_{15}^L_{1,2}$ | $N^r_{1,2}$ |
|-------|---------------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1 1 1 | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 | $1/\sqrt{6}$ |
| 650  | $(1, 1)_1$ | 0 1 | $-\frac{2}{1}$ | 0 | $-\frac{2}{1}$ | 0 | 0 | $-\frac{1}{1}$ | 0 | 0 | 0 | $-\frac{1}{1}$ | 0 | 1 | $-\frac{1}{1}$ | $1/\sqrt{\frac{7}{6}}$ |
|      | $(1, 1)_2$ | 1 0 | $-\frac{6}{7}$ | 0 | $-\frac{10}{7}$ | 0 | 0 | $-\frac{1}{7}$ | 0 | 0 | 0 | $-\frac{1}{7}$ | 0 | 0 | $-\frac{1}{7}$ | $1/\sqrt{\frac{7}{6}}$ |
|      | $(1, 8)$ | 0 0 | 1 0 | $\frac{7}{4}$ | 0 | 0 | $\frac{3}{4}$ | 0 | 0 | 0 | $\frac{3}{4}$ | 0 | 0 | 0 | $\frac{3}{4}$ | $5/\sqrt{13}$ |
|      | $(8, 1)$ | 0 1 | $-\frac{1}{7}$ | 0 | $-\frac{3}{11}$ | 0 | 0 | 0 | $-\frac{1}{7}$ | 0 | 0 | 0 | $-\frac{1}{7}$ | 0 | 0 | $-\frac{1}{7}$ | $5/\sqrt{13}$ |
|      | $(8, 8)$ | 0 0 | $-\frac{1}{7}$ | $-\frac{25}{37}$ | $-\frac{5}{37}$ | 0 | 0 | 0 | $\frac{3}{7}$ | 0 | $\frac{25}{37}$ | 0 | 0 | 0 | 0 | $\frac{25}{37}$ |
| 2430 | $(1, 1)$ | 1 1 | 1 | $-\frac{4}{7}$ | 1 | $-\frac{4}{7}$ | 1 | 1 | $-\frac{4}{7}$ | 1 | $-\frac{4}{7}$ | 1 | $-\frac{4}{7}$ | 1 | $-\frac{4}{7}$ | $\frac{5}{\sqrt{236}}$ |
|      | $(1, 8)$ | 0 0 | 1 | 0 | $\frac{7}{2}$ | 0 | 0 | $\frac{3}{2}$ | 0 | 0 | 0 | $\frac{3}{2}$ | 0 | 0 | 0 | $\frac{3}{2}$ | $\sqrt{\frac{3}{11}}$ |
|      | $(8, 1)$ | 0 1 | $-\frac{2}{1}$ | 0 | $\frac{31}{10}$ | 0 | 0 | 0 | $\frac{31}{10}$ | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{1}{3}$ | $\sqrt{\frac{3}{11}}$ |
|      | $(8, 8)$ | 0 0 | $-\frac{1}{2}$ | $-\frac{4}{7}$ | $-\frac{4}{7}$ | $-\frac{1}{7}$ | $-\frac{1}{7}$ | 0 | $\frac{3}{7}$ | $-\frac{1}{7}$ | $-\frac{1}{7}$ | 0 | $-\frac{1}{7}$ | 0 | 0 | 0 | $\sqrt{\frac{3}{11}}$ |
|      | $(1, 27)$ | 0 0 | 1 | 0 | $-\frac{33}{10}$ | 0 | 0 | $\frac{3}{10}$ | 0 | 0 | 0 | $\frac{3}{10}$ | 0 | 0 | 0 | $-\frac{1}{3}$ | $\sqrt{\frac{3}{11}}$ |
|      | $(27, 1)$ | 0 1 | 0 | $\frac{6}{7}$ | 0 | $-\frac{33}{10}$ | 0 | 0 | $\frac{3}{10}$ | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{3}{7}$ | $\sqrt{\frac{3}{11}}$ |
Table 10. The non-diagonal part, $h_{ij}$ in equation (6) of the SM singlets $\Psi_{r,s}$ in each of the irreps $r$ according to their transformation properties under the SU(3)$_L \times$ SU(3)$_R \subset E_6$ subgroup. The entries $h_{35r,s}, h_{59r,s}, h_{93r,s}$ for the flipped embedding are also listed here.

| $E$  | SU(3)$_L \times$ SU(3)$_R$ | $h_{35r,s}$ | $h_{39r,s}$ | $h_{59r,s}$ | $h_{35r,s}$ | $h_{39r,s}$ | $h_{59r,s}$ | $h_{35r,s}$ | $h_{39r,s}$ | $h_{59r,s}$ |
|------|----------------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 650  | (1, 1)$_L$                 | $-\sqrt{6}/5$ | $\sqrt{5}/2$ | 0           | $-3/5$      | $-1/40$     | $-3/8$      | $-23\sqrt{3}/20$ | $-1/(4\sqrt{10})$ | $-9\sqrt{3}/5/8$ |
|      | (1, 1)$_R$                 | $-\sqrt{3}/2/5$ | $-7/\sqrt{10}$ | $-19/(2\sqrt{15})$ | $-4/5$      | $-17/10$    | $1/2$       | $9\sqrt{3}/5/2$ | $7/\sqrt{10}$    | $-1/(2\sqrt{15})$ | |
|      | (1, 8)$_L$                 | $\sqrt{3}/2/4$ | $-\sqrt{5}/2/4$ | $\sqrt{15}/8$ | 1           | $7/8$       | $5/8$       | $\sqrt{3}/2/4$ | $-\sqrt{5}/2/4$  | $\sqrt{15}/8$   |
|      | (1, 8)$_R$                 | $\sqrt{3}/2/5$ | $-1/\sqrt{10}$ | $\sqrt{5}/12/2$ | $-1/5$      | $-3/10$     | $-1/2$      | $\sqrt{3}/2/5$ | $-1/(\sqrt{10})$  | $\sqrt{5}/2/2$   |
|      | (8, 8)$_L$                 | $\sqrt{3}/2/2$ | 0           | 0           | 1           | $-3/8$      | $-5/8$      | $-3\sqrt{3}/2/8$ | $3\sqrt{5}/2/8$  | $\sqrt{15}/8$   |
|      | (8, 8)$_R$                 | 0           | 0           | 0           | 0           | 0           | 0           | 0           | 0           | 0           |
| 2430 | (1, 1)$_L$                 | $\sqrt{3}/2/4$ | $-\sqrt{5}/2/4$ | $\sqrt{15}/8$ | 1           | $7/8$       | $5/8$       | $\sqrt{3}/2/4$ | $-\sqrt{5}/2/4$  | $\sqrt{15}/8$   |
|      | (1, 1)$_R$                 | $\sqrt{3}/2/2$ | $23/(2\sqrt{10})$ | $-23\sqrt{3}/5/4$ | $-1/5$      | $-17/40$    | $77/8$      | $-33\sqrt{3}/2/10$ | $17/(2\sqrt{10})$ | $-9\sqrt{3}/5/8$ |
|      | (8, 1)$_L$                 | $\sqrt{3}/2/2$ | 0           | 0           | 1           | $-3/8$      | $-5/8$      | $-3\sqrt{3}/2/8$ | $3\sqrt{5}/2/8$  | $\sqrt{15}/8$   |
|      | (8, 1)$_R$                 | $\sqrt{3}/2/4$ | $-\sqrt{5}/2/4$ | $\sqrt{15}/8$ | 1           | $7/8$       | $5/8$       | $\sqrt{3}/2/4$ | $-\sqrt{5}/2/4$  | $\sqrt{15}/8$   |
|      | (1, 27)$_L$                | $\sqrt{3}/2/4$ | $-\sqrt{5}/2/4$ | $\sqrt{15}/8$ | 1           | $7/8$       | $5/8$       | $\sqrt{3}/2/4$ | $-\sqrt{5}/2/4$  | $\sqrt{15}/8$   |
|      | (27, 1)$_R$                | $-19\sqrt{3}/2/5$ | $9/\sqrt{10}$ | $-9\sqrt{3}/5/2$ | 9/5        | $279/20$    | $-27/4$     | $7\sqrt{3}/2/10$ | $33/(2\sqrt{10})$ | $-33\sqrt{3}/5/4$ |
where $a$, $b$ are group indices, $k$ labels different Higgs multiplets and $\tilde{G}^{\mu\nu}$ is dual of gauge field strength $G^{\mu\nu}$. If one assumes that when $E_8$ breaks into $G_{321}$, one of the other two $U(1)$’s ($U_7(1)$, $U_{243}(1)$) is anomalous Peccei–Quinn $U(1)$. After spontaneously breaking of anomalous Peccei–Quinn $U(1)$ symmetry, the associated pseudo Goldstone boson, axion, is to become a component of the corresponding vector boson (say, $Z_5$), since now this
$U(1)$ is a local symmetry. When $\tilde{H}_0$ acquires the non-zero vacuum expectation values, from equation (22) and the relevant Clebsch–Gordan coefficients $\Phi^{\alpha_1 \alpha_2}_{\beta_0}$, one can estimate the size of the coupling of $Z_V$. It might be instructive to study the relation of this term to strong CP and axion physics.
It is straightforward to generalize the study of the gauge coupling unification in $E_6$ including the effects of dimension-5 operators to SUSY GUT. In this case, gaugino mass ratios can be read from the tables shown in appendix A, since gauginos belong to the same multiplets as gauge bosons in SUSY. That is, for the flipped embedding,

$$M_5 : M_2 : M_1 = h^{kL}_{cL} : h^{kR}_{cR} : h^{kL}_{vL},$$  

(23)
where $M_a$ are the gaugino masses and $a = 3, 2, 1$ corresponds the SU(3), SU(2), U(1) of the SM, i.e., the gluino, wino and bino masses. The results agree with those in corresponding tables given by Martin [16].

For model building of $E_6$ GUT with effects of dimension-5 operators, there are several important problems, such as the doublet-triplet splitting, neutrino mass hierarchy, etc, which need to be answered. However, this is beyond the scope of this paper. One should study them in the future.
Acknowledgments

This research was supported in part by the Natural Science Foundation of China under grant numbers 11375248, 11647601, 11135009 and 11005033.

Appendix A. Clebsch–Gordan coefficients associated with $E_6$ breaking to the SM

All of the Clebsch–Gordan coefficients $\psi_{ij}^{(r)}$ associated with $E_6$, breaking to the SM, in different bases $\{s, z\}$, up to a uniform normalization constant for different representations $r$ have been derived and results are given in this appendix. For the subgroup $H = SO(10) \times U(1)$, the analysis and results have been given in the [55]. We start from $H = SU(6) \times SU(2)$.

Appendix B. structure constants of $E_6$

In Cartan–Weyl basis, generators of a group $G$ can be written as $\{H_j, E_a\}, j = 1, \ldots, l, l = \text{rank}(G)$, $\alpha^b$ is a root, $b = 1, \ldots, (N - l)/2$ for positive root, $N = \text{order}(G)$ = the number of generators of $G$. The Lie algebra is,

$$ [H_j, E_a] = \alpha^b E_a, \quad (24) $$

where $\alpha^b$ is $j$th component of $\alpha^b$ and

$$ [E_a, E_a] = N(\alpha^a)E_a, \quad (25) $$

with $\alpha^a = \alpha^i + \alpha^j$ fixed. The $N(\alpha^a)$ in (25) is called real SC. We call the SC directly obtained from roots in (24) as 'direct SC' for simplicity. Then all SC of a group $G$ are composed of real SC and direct SC.

The structure constants of $E_6$ have been given in a Chevalley base [61]. We transform them into the usual form mostly used by physicists and in the study of GUT (see, equation (8)). Starting from the table 'Structure constants for $E_6$', page 1526 in Vavilov’s paper [61], we obtain the real SC of $E_6$. In this paper, $\{i, j, k\}, i, j, k = 1, \ldots, 78$, which are totally antisymmetric. The part of $f[i, j, k]$, which are not zero and ordered according to $i < j < k$, is given as follows,

\[
\begin{align*}
 f[1, 4, 7] &= 1/2; f[1, 5, 6] = -1/2; f[1, 13, 16] = 1/2; f[1, 14, 15] = -1/2; \\
 f[1, 19, 22] &= 1/2; f[1, 20, 21] = -1/2; \\
 f[1, 26, 29] &= 1/2; f[1, 27, 28] = -1/2; f[1, 32, 35] = 1/2; f[1, 33, 34] = -1/2; \\
 f[1, 40, 43] &= -1/2; \\
 f[1, 41, 42] &= 1/2; f[1, 47, 50] = 1/2; f[1, 48, 49] = -1/2; f[1, 53, 56] = -1/2; f[1, 54, 55] = 1/2; \\
 f[1, 61, 64] &= 1/2; f[1, 62, 63] = -1/2; f[1, 69, 72] = 1/2; f[1, 70, 71] = -1/2; f[2, 4, 6] = 1/2; \\
 f[2, 5, 7] &= 1/2; f[2, 13, 15] = 1/2; f[2, 14, 16] = 1/2; f[2, 19, 21] = 1/2; \\
 f[2, 20, 22] &= 1/2; f[2, 26, 28] = 1/2; \\
 f[2, 27, 29] &= 1/2; f[2, 32, 34] = 1/2; f[2, 33, 35] = 1/2; f[2, 40, 42] = 1/2; \\
 f[2, 41, 43] &= 1/2; f[2, 47, 49] = 1/2; \\
 f[2, 48, 50] &= 1/2; f[2, 53, 55] = -1/2; f[2, 54, 56] = -1/2; \\
 f[2, 61, 63] &= 1/2; f[2, 62, 64] = 1/2; \\
 f[2, 69, 71] &= 1/2; f[2, 70, 72] = 1/2; f[4, 13, 18] = -1/2; \\
 f[4, 14, 17] &= 1/2; f[4, 19, 24] = -1/2; \\
 f[4, 20, 23] &= 1/2; f[4, 26, 31] = -1/2; f[4, 27, 30] = 1/2; \\
 f[4, 32, 37] &= 1/2; f[4, 33, 36] = -1/2; \\
 f[4, 38, 41] &= -1/2; f[4, 39, 40] = 1/2; f[4, 47, 52] = -1/2; \\
 f[4, 48, 51] &= 1/2; f[4, 53, 58] = -1/2; \\
 f[4, 54, 57] &= 1/2; f[4, 59, 64] = 1/2; f[4, 60, 63] = -1/2; \\
 f[4, 67, 72] &= -1/2; f[4, 68, 71] = 1/2; \\
 f[5, 13, 17] &= -1/2; f[5, 14, 18] = -1/2; f[5, 19, 23] = -1/2; \\
 f[5, 20, 24] &= -1/2; f[5, 26, 30] = -1/2; \\
 f[5, 27, 31] &= -1/2; f[5, 32, 36] = 1/2; f[5, 33, 37] = 1/2; \\
 f[5, 38, 40] &= -1/2; f[5, 39, 41] = -1/2;
\end{align*}
\]
\[ f[5, 47, 51] = -1/2; f[5, 48, 52] = -1/2; f[5, 53, 57] = -1/2; \\
f[5, 54, 58] = -1/2; f[5, 59, 63] = 1/2; \\
f[5, 60, 64] = 1/2; f[5, 67, 71] = -1/2; f[5, 68, 72] = -1/2; \\
f[6, 15, 18] = -1/2; f[6, 16, 17] = 1/2; \\
f[6, 21, 24] = -1/2; f[6, 22, 23] = 1/2; f[6, 28, 31] = -1/2; \\
f[6, 29, 30] = 1/2; f[6, 34, 37] = 1/2; \\
f[6, 35, 36] = -1/2; f[6, 38, 43] = -1/2; f[6, 39, 42] = 1/2; \\
f[6, 49, 52] = -1/2; f[6, 50, 51] = 1/2; \\
f[6, 55, 58] = 1/2; f[6, 56, 57] = -1/2; f[6, 59, 62] = -1/2; \\
f[6, 60, 61] = 1/2; f[6, 67, 70] = 1/2; \\
f[6, 68, 69] = -1/2; f[7, 15, 17] = -1/2; f[7, 16, 18] = -1/2; \\
f[7, 21, 23] = -1/2; f[7, 22, 24] = -1/2; \\
f[7, 28, 30] = -1/2; f[7, 29, 31] = -1/2; \\
f[7, 34, 36] = 1/2; f[7, 35, 37] = 1/2; f[7, 38, 42] = -1/2; \\
f[7, 39, 43] = -1/2; f[7, 49, 51] = -1/2; f[7, 50, 52] = -1/2; \\
f[7, 55, 57] = 1/2; f[7, 56, 58] = 1/2; \\
f[7, 59, 61] = -1/2; f[7, 60, 62] = -1/2; f[7, 67, 69] = 1/2; \\
f[7, 68, 70] = 1/2; f[9, 13, 20] = -1/2; \\
f[9, 14, 19] = 1/2; f[9, 15, 22] = -1/2; f[9, 16, 21] = 1/2; \\
f[9, 17, 24] = -1/2; f[9, 18, 23] = 1/2; \\
f[9, 26, 33] = -1/2; f[9, 27, 32] = 1/2; f[9, 28, 35] = -1/2; \\
f[9, 29, 34] = 1/2; f[9, 30, 37] = 1/2; \\
f[9, 31, 36] = -1/2; f[9, 47, 54] = 1/2; f[9, 48, 53] = -1/2; \\
f[9, 49, 56] = -1/2; f[9, 50, 55] = 1/2; \\
f[9, 51, 58] = 1/2; f[9, 52, 57] = -1/2; f[9, 73, 76] = -1/2; \\
f[9, 74, 75] = 1/2; f[10, 13, 19] = 1/2; \\
f[10, 14, 20] = 1/2; f[10, 15, 21] = 1/2; f[10, 16, 22] = 1/2; \\
f[10, 17, 23] = 1/2; f[10, 18, 24] = 1/2; \\
f[10, 26, 32] = -1/2; f[10, 27, 33] = -1/2; f[10, 28, 34] = -1/2; \\
f[10, 29, 35] = -1/2; f[10, 30, 36] = 1/2; \\
f[10, 31, 37] = 1/2; f[10, 47, 53] = 1/2; f[10, 48, 54] = 1/2; \\
f[10, 49, 55] = -1/2; f[10, 50, 56] = -1/2; \\
f[10, 51, 57] = 1/2; f[10, 52, 58] = 1/2; f[10, 73, 75] = -1/2; \\
f[10, 74, 76] = -1/2; f[13, 28, 39] = -1/2; \\
f[13, 29, 38] = 1/2; f[13, 30, 43] = -1/2; f[13, 31, 42] = 1/2; \\
f[13, 32, 45] = -1/2; f[13, 33, 44] = 1/2; \\
f[13, 49, 60] = 1/2; f[13, 50, 59] = -1/2; f[13, 51, 62] = 1/2; \\
f[13, 52, 61] = -1/2; f[13, 53, 66] = -1/2; \\
f[13, 54, 65] = -1/2; f[13, 71, 76] = 1/2; f[13, 72, 75] = -1/2; \\
f[14, 28, 38] = 1/2; f[14, 29, 39] = 1/2; \\
f[14, 30, 42] = 1/2; f[14, 31, 43] = 1/2; f[14, 32, 44] = -1/2; \\
f[14, 33, 45] = -1/2; f[14, 49, 59] = -1/2; \\
f[14, 50, 60] = -1/2; f[14, 51, 61] = -1/2; f[14, 52, 62] = -1/2; \\
f[14, 53, 65] = 1/2; f[14, 54, 66] = -1/2; \\
f[14, 71, 75] = -1/2; f[14, 72, 76] = -1/2; f[15, 26, 39] = 1/2; \\
f[15, 27, 38] = -1/2; f[15, 30, 41] = 1/2; \\
f[15, 31, 40] = -1/2; f[15, 34, 45] = -1/2; f[15, 35, 44] = 1/2; \\
f[15, 47, 60] = -1/2; f[15, 48, 59] = 1/2; \\
f[15, 51, 64] = 1/2; f[15, 52, 63] = -1/2; f[15, 55, 66] = 1/2; \\
f[15, 56, 65] = 1/2; f[15, 69, 76] = -1/2;
\[
\begin{align*}
 f[15, 70, 75] & = -1/2; f[16, 26, 38] = -1/2; f[16, 27, 39] = -1/2; \\
 f[16, 30, 40] & = -1/2; f[16, 31, 41] = -1/2; \\
 f[16, 34, 44] & = -1/2; f[16, 35, 45] = -1/2; f[16, 47, 59] = 1/2; \\
 f[16, 48, 60] & = 1/2; f[16, 51, 63] = -1/2; \\
 f[16, 52, 64] & = -1/2; f[16, 55, 65] = -1/2; f[16, 56, 66] = 1/2; \\
 f[16, 69, 75] & = 1/2; f[16, 70, 76] = 1/2; \\
 f[17, 26, 43] & = 1/2; f[17, 27, 42] = -1/2; f[17, 28, 41] = -1/2; \\
 f[17, 29, 40] & = 1/2; f[17, 36, 44] = 1/2; \\
 f[17, 37, 44] & = -1/2; f[17, 47, 62] = -1/2; f[17, 48, 61] = 1/2; \\
 f[17, 49, 64] & = -1/2; f[17, 50, 63] = 1/2; \\
 f[17, 57, 66] & = -1/2; f[17, 58, 65] = -1/2; f[17, 67, 76] = -1/2; \\
 f[17, 68, 75] & = 1/2; f[18, 26, 42] = -1/2; \\
 f[18, 27, 43] & = -1/2; f[18, 28, 40] = 1/2; f[18, 29, 41] = 1/2; \\
 f[18, 36, 44] & = 1/2; f[18, 37, 45] = 1/2; \\
 f[18, 47, 61] & = 1/2; f[18, 48, 62] = 1/2; f[18, 49, 63] = 1/2; \\
 f[18, 50, 64] & = 1/2; f[18, 57, 65] = 1/2; \\
 f[18, 58, 66] & = -1/2; f[18, 67, 75] = 1/2; f[18, 68, 76] = 1/2; \\
 f[19, 26, 45] & = -1/2; f[19, 27, 44] = 1/2; \\
 f[19, 34, 39] & = 1/2; f[19, 35, 38] = -1/2; f[19, 36, 43] = -1/2; \\
 f[19, 37, 42] & = 1/2; f[19, 47, 66] = 1/2; \\
 f[19, 48, 65] & = 1/2; f[19, 55, 60] = -1/2; f[19, 56, 59] = 1/2; \\
 f[19, 57, 62] & = 1/2; f[19, 58, 61] = -1/2; \\
 f[19, 71, 74] & = 1/2; f[19, 72, 73] = -1/2; f[20, 26, 44] = -1/2; \\
 f[20, 27, 45] & = -1/2; f[20, 34, 38] = -1/2; \\
 f[20, 35, 39] & = -1/2; f[20, 36, 42] = 1/2; f[20, 37, 43] = 1/2; \\
 f[20, 47, 65] & = -1/2; f[20, 48, 66] = 1/2; \\
 f[20, 55, 59] & = 1/2; f[20, 56, 60] = 1/2; f[20, 57, 61] = -1/2; \\
 f[20, 58, 62] & = -1/2; f[20, 71, 73] = -1/2; \\
 f[20, 72, 74] & = -1/2; f[21, 28, 45] = -1/2; f[21, 29, 44] = 1/2; \\
 f[21, 32, 39] & = -1/2; f[21, 33, 38] = 1/2; \\
 f[21, 36, 41] & = 1/2; f[21, 37, 40] = -1/2; f[21, 49, 66] = 1/2; \\
 f[21, 50, 65] & = 1/2; f[21, 53, 60] = -1/2; \\
 f[21, 54, 59] & = 1/2; f[21, 57, 64] = 1/2; f[21, 58, 63] = -1/2; \\
 f[21, 69, 74] & = -1/2; f[21, 70, 73] = 1/2; \\
 f[22, 28, 44] & = -1/2; f[22, 29, 45] = -1/2; \\
 f[22, 32, 38] & = 1/2; f[22, 33, 39] = 1/2; f[22, 36, 40] = -1/2; \\
 f[22, 37, 41] & = -1/2; f[22, 49, 65] = -1/2; f[22, 50, 66] = 1/2; \\
 f[22, 53, 59] & = 1/2; f[22, 54, 60] = 1/2; \\
 f[22, 57, 63] & = -1/2; f[22, 58, 64] = -1/2; f[22, 69, 73] = 1/2; \\
 f[22, 70, 74] & = 1/2; f[23, 30, 45] = -1/2; \\
 f[23, 31, 44] & = 1/2; f[23, 32, 43] = -1/2; f[23, 33, 42] = 1/2; \\
 f[23, 34, 41] & = 1/2; f[23, 35, 40] = -1/2; \\
 f[23, 51, 66] & = 1/2; f[23, 52, 65] = 1/2; f[23, 53, 62] = -1/2; \\
 f[23, 54, 61] & = 1/2; f[23, 55, 64] = 1/2; \\
 f[23, 56, 63] & = -1/2; f[23, 67, 74] = -1/2; f[23, 68, 73] = 1/2; \\
 f[24, 30, 44] & = -1/2; f[24, 31, 45] = -1/2; \\
 f[24, 32, 42] & = 1/2; f[24, 33, 43] = 1/2; f[24, 34, 40] = -1/2; \\
 f[24, 35, 41] & = -1/2; f[24, 51, 65] = -1/2;
\end{align*}
\]
\[ f[24, 52, 66] = 1/2; f[24, 53, 61] = 1/2; f[24, 54, 62] = 1/2; \]
\[ f[24, 55, 63] = -1/2; f[24, 56, 64] = -1/2; \]
\[ f[24, 67, 73] = 1/2; f[24, 68, 74] = 1/2; f[26, 47, 78] = 1/2; \]
\[ f[26, 48, 77] = 1/2; f[26, 55, 68] = -1/2; \]
\[ f[26, 56, 67] = 1/2; f[26, 57, 70] = -1/2; f[26, 58, 69] = 1/2; \]
\[ f[26, 63, 74] = 1/2; f[26, 64, 73] = -1/2; \]
\[ f[27, 47, 77] = -1/2; f[27, 48, 78] = 1/2; f[27, 55, 67] = 1/2; \]
\[ f[27, 56, 68] = 1/2; f[27, 57, 70] = -1/2; f[27, 64, 74] = -1/2; \]
\[ f[28, 49, 78] = 1/2; f[28, 50, 77] = 1/2; \]
\[ f[28, 53, 68] = -1/2; f[28, 54, 67] = 1/2; f[28, 57, 72] = -1/2; \]
\[ f[28, 58, 71] = 1/2; f[28, 61, 74] = -1/2; \]
\[ f[29, 47, 78] = 1/2; f[29, 50, 77] = -1/2; f[29, 51, 72] = -1/2; \]
\[ f[29, 52, 67] = 1/2; f[29, 53, 70] = 1/2; f[30, 54, 69] = -1/2; \]
\[ f[30, 55, 72] = -1/2; f[30, 56, 71] = 1/2; \]
\[ f[30, 59, 74] = 1/2; f[30, 60, 73] = -1/2; f[31, 51, 77] = -1/2; \]
\[ f[31, 52, 78] = 1/2; f[31, 53, 69] = -1/2; f[31, 54, 70] = -1/2; \]
\[ f[31, 57, 71] = 1/2; f[31, 56, 72] = 1/2; f[31, 59, 73] = -1/2; \]
\[ f[32, 49, 68] = 1/2; f[32, 50, 67] = -1/2; f[32, 51, 70] = -1/2; \]
\[ f[32, 52, 69] = 1/2; f[32, 53, 78] = -1/2; \]
\[ f[32, 54, 77] = -1/2; f[32, 63, 76] = 1/2; f[32, 64, 75] = -1/2; \]
\[ f[33, 49, 67] = -1/2; f[33, 50, 68] = -1/2; f[33, 51, 70] = 1/2; f[33, 53, 77] = 1/2; \]
\[ f[33, 54, 78] = -1/2; f[33, 63, 75] = -1/2; f[34, 47, 68] = -1/2; f[34, 48, 67] = 1/2; \]
\[ f[34, 51, 72] = -1/2; f[34, 52, 71] = 1/2; f[34, 55, 78] = 1/2; f[34, 56, 77] = 1/2; f[34, 61, 76] = -1/2; \]
\[ f[34, 62, 75] = 1/2; f[35, 47, 67] = 1/2; f[35, 48, 68] = 1/2; f[35, 51, 71] = 1/2; f[35, 52, 72] = 1/2; \]
\[ f[35, 55, 77] = -1/2; f[35, 56, 78] = 1/2; f[35, 61, 75] = 1/2; f[36, 47, 70] = -1/2; \]
\[ f[36, 48, 69] = 1/2; f[36, 49, 72] = -1/2; \]
\[ f[36, 50, 71] = 1/2; f[36, 57, 78] = 1/2; f[36, 58, 77] = 1/2; \]
\[ f[36, 59, 76] = -1/2; f[36, 60, 75] = 1/2; f[37, 47, 69] = 1/2; f[37, 48, 70] = 1/2; f[37, 49, 71] = 1/2; \]
\[ f[37, 50, 72] = 1/2; f[37, 57, 77] = -1/2; f[37, 58, 75] = 1/2; f[37, 60, 76] = 1/2; f[38, 51, 74] = -1/2; f[38, 52, 73] = 1/2; \]
\[ f[38, 57, 76] = 1/2; f[38, 58, 75] = -1/2; f[38, 59, 78] = -1/2; f[38, 60, 77] = -1/2; f[38, 65, 68] = -1/2; \]
The other parts of \( f[i, j, k] \) either can be obtained from equation (26) by using their antisymmetric properties, or are equal to zero.

The positive roots of \( E_6 \) in Dynkin basis have been given (see, table 20 in [56]). It is straightforward to get the direct SC, \( f_{\text{sc}}[i, j, k], i, j, k = 1, \ldots, 78 \). The part of \( f_{\text{sc}}[i, j, k] \), which are not zero and ordered according to \( i < j < k \), is given in equation (27). The other parts of \( f_{\text{sc}}[i, j, k] \) can either be obtained from equation (27) by using their antisymmetric properties, or are equal to zero.

\[
\begin{align*}
\text{fsc}[1, 2, 3] &= 1; \quad \text{fsc}[3, 4, 5] = \frac{1}{2}; \quad \text{fsc}[3, 6, 7] = -\frac{1}{2}; \quad \text{fsc}[3, 13, 14] = \frac{1}{2}; \\
\text{fsc}[3, 15, 16] &= -\frac{1}{2}; \\
\text{fsc}[3, 19, 20] &= \frac{1}{2}; \quad \text{fsc}[3, 21, 22] = -\frac{1}{2}; \quad \text{fsc}[3, 26, 27] = \frac{1}{2}; \quad \text{fsc}[3, 28, 29] = -\frac{1}{2}; \\
\text{fsc}[3, 32, 33] &= \frac{1}{2}; \\
\text{fsc}[3, 34, 35] &= -\frac{1}{2}; \quad \text{fsc}[3, 40, 41] = -\frac{1}{2}; \quad \text{fsc}[3, 42, 43] = \frac{1}{2}; \quad \text{fsc}[3, 47, 48] = \frac{1}{2}; \\
\text{fsc}[3, 49, 50] &= -\frac{1}{2}; \\
\text{fsc}[3, 53, 54] &= \frac{1}{2}; \quad \text{fsc}[3, 55, 56] = -\frac{1}{2}; \quad \text{fsc}[3, 61, 62] = \frac{1}{2}; \quad \text{fsc}[3, 63, 64] = -\frac{1}{2}; \\
\text{fsc}[3, 69, 70] &= \frac{1}{2}; \\
\text{fsc}[3, 71, 72] &= -\frac{1}{2}; \quad \text{fsc}[4, 5, 8] = \frac{\sqrt{3}}{2}; \quad \text{fsc}[6, 7, 8] = \frac{\sqrt{3}}{2}; \quad \text{fsc}[8, 13, 14] = \frac{1}{2\sqrt{3}}; \\
\text{fsc}[8, 15, 16] &= \frac{1}{2\sqrt{3}};
\end{align*}
\]
\[
\begin{align*}
\text{fsc}[8, 17, 18] &= -\frac{1}{\sqrt{3}}; \quad \text{fsc}[8, 19, 20] = \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 21, 22] = \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 23, 24] = -\frac{1}{\sqrt{3}}; \\
\text{fsc}[8, 26, 27] &= \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 28, 29] = \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 30, 31] = -\frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 32, 33] = \frac{1}{2\sqrt{3}}; \\
\text{fsc}[8, 34, 35] &= \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 36, 37] = -\frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 38, 39] = \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 40, 41] = -\frac{1}{2\sqrt{3}}; \\
\text{fsc}[8, 42, 43] &= -\frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 47, 48] = \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 49, 50] = \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 51, 52] = -\frac{1}{2\sqrt{3}}; \\
\text{fsc}[8, 53, 54] &= \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 55, 56] = \frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 57, 58] = -\frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 59, 60] = \frac{1}{2\sqrt{3}};
\end{align*}
\]

\[
\text{fsc}[8, 61, 62] = -\frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 63, 64] = -\frac{1}{2\sqrt{3}}; \quad \text{fsc}[8, 67, 68] = \frac{1}{\sqrt{3}}; \\
\text{fsc}[8, 69, 70] = -\frac{1}{2\sqrt{3}}; \\
\text{fsc}[8, 71, 72] = -\frac{1}{2\sqrt{3}}; \quad \text{fsc}[9, 10, 11] = 1; \quad \text{fsc}[11, 13, 14] = -\frac{1}{2}; \quad \text{fsc}[11, 15, 16] = -\frac{1}{2}; \\
\text{fsc}[11, 17, 18] = -\frac{1}{2}; \\
\text{fsc}[11, 19, 20] = \frac{1}{2}; \quad \text{fsc}[11, 21, 22] = \frac{1}{2}; \quad \text{fsc}[11, 23, 24] = \frac{1}{2}; \quad \text{fsc}[11, 26, 27] = \frac{1}{2}; \\
\text{fsc}[11, 28, 29] = \frac{1}{2}; \\
\text{fsc}[11, 30, 31] = \frac{1}{2}; \quad \text{fsc}[11, 32, 33] = -\frac{1}{2}; \quad \text{fsc}[11, 34, 35] = -\frac{1}{2}; \quad \text{fsc}[11, 36, 37] = -\frac{1}{2}; \\
\text{fsc}[11, 47, 48] = \frac{1}{2}; \\
\text{fsc}[11, 49, 50] = \frac{1}{2}; \quad \text{fsc}[11, 51, 52] = -\frac{1}{2}; \quad \text{fsc}[11, 53, 54] = -\frac{1}{2}; \quad \text{fsc}[11, 55, 56] = -\frac{1}{2}; \\
\text{fsc}[11, 57, 58] = -\frac{1}{2}; \\
\text{fsc}[11, 73, 74] = \frac{1}{2}; \quad \text{fsc}[11, 75, 76] = -\frac{1}{2}; \quad \text{fsc}[12, 13, 14] = -\frac{\sqrt{5}}{2}; \quad \text{fsc}[12, 15, 16] = -\frac{\sqrt{5}}{2}; \\
\text{fsc}[12, 17, 18] = -\frac{\sqrt{5}}{2}; \quad \text{fsc}[12, 19, 20] = -\frac{\sqrt{5}}{2}; \quad \text{fsc}[12, 21, 22] = -\frac{\sqrt{5}}{2}; \quad \text{fsc}[12, 23, 24] = -\frac{\sqrt{5}}{2}; \\
\text{fsc}[12, 26, 27] = \frac{1}{2\sqrt{15}}; \quad \text{fsc}[12, 28, 29] = \frac{1}{2\sqrt{15}}; \quad \text{fsc}[12, 30, 31] = \frac{1}{2\sqrt{15}}; \quad \text{fsc}[12, 32, 33] = \frac{1}{2\sqrt{15}}; \\
\text{fsc}[12, 34, 35] = \frac{1}{2\sqrt{15}}; \quad \text{fsc}[12, 36, 37] = \frac{1}{2\sqrt{15}}; \quad \text{fsc}[12, 38, 39] = -\frac{2}{\sqrt{15}}; \quad \text{fsc}[12, 40, 41] = -\frac{2}{\sqrt{15}}; \\
\text{fsc}[12, 42, 43] = -\frac{2}{\sqrt{15}}; \quad \text{fsc}[12, 44, 45] = -\frac{3}{\sqrt{5}}; \quad \text{fsc}[12, 47, 48] = \frac{1}{2\sqrt{15}}; \quad \text{fsc}[12, 49, 50] = \frac{1}{2\sqrt{15}}; \\
\text{fsc}[12, 51, 52] = \frac{1}{2\sqrt{15}}; \quad \text{fsc}[12, 53, 54] = \frac{1}{2\sqrt{15}}; \quad \text{fsc}[12, 55, 56] = \frac{1}{2\sqrt{15}}; \quad \text{fsc}[12, 57, 58] = \frac{1}{2\sqrt{15}}; \\
\text{fsc}[12, 59, 60] = -\frac{2}{\sqrt{15}}; \quad \text{fsc}[12, 61, 62] = -\frac{2}{\sqrt{15}}; \quad \text{fsc}[12, 63, 64] = -\frac{2}{\sqrt{15}}; \quad \text{fsc}[12, 65, 66] = -\frac{3}{\sqrt{5}}; \\
\text{fsc}[12, 67, 68] = \frac{1}{\sqrt{15}}; \quad \text{fsc}[12, 69, 70] = \frac{1}{\sqrt{15}}; \quad \text{fsc}[12, 71, 72] = \frac{1}{\sqrt{15}}; \quad \text{fsc}[12, 73, 74] = -\frac{\sqrt{5}}{2}; \\
\text{fsc}[12, 75, 76] = -\frac{\sqrt{5}}{2}; \quad \text{fsc}[25, 26, 27] = \frac{\sqrt{2}}{\sqrt{5}}; \quad \text{fsc}[25, 28, 29] = \frac{\sqrt{2}}{\sqrt{5}}; \quad \text{fsc}[25, 30, 31] = \frac{\sqrt{2}}{\sqrt{5}}; \\
\text{fsc}[25, 32, 33] = \frac{\sqrt{2}}{\sqrt{5}}; \quad \text{fsc}[25, 34, 35] = \frac{\sqrt{2}}{\sqrt{5}}; \quad \text{fsc}[25, 36, 37] = \frac{\sqrt{2}}{\sqrt{5}}; \quad \text{fsc}[25, 38, 39] = \frac{\sqrt{2}}{\sqrt{5}}.
\]
As said above, all SC of $E_6$, $f_{ijk}$, are the sum of real SC and direct SC. That is, $ff_{[i, j, k]} = f_{ijk}$. It has been checked that $ff_{[i, j, k]}$ satisfy the Jacobi identities.

ORCID iDs

Xiao-Hong Wu @ https://orcid.org/0000-0003-3836-075X

References

[1] Langacker P and Luo M X 1991 Phys. Rev. D 44 817
[2] Giunti C, Kim C W and Lee U W 1991 Mod. Phys. Lett. A 6 1745
[3] Amaldi U, de Boer W and Furstenau H 1991 Phys. Lett. B 260 447
[4] Ellis J R, Kelley S and Nanopoulos D V 1991 Phys. Lett. B 260 131
[5] Hill C T 1984 Phys. Lett. 135B 47
[6] Shaﬁ Q and Wetterich C 1984 Phys. Rev. Lett. 52 875
[7] Calmet X, Hsu S D H and Reeb D 2008 Phys. Rev. Lett. 101 171802
[8] Hall J L and Sarid U 1993 Phys. Rev. Lett. 70 2673
[9] Datta A, Pakvasa S and Sarkar U 1995 Phys. Rev. D 52 550
[10] Huizuo K, Kawamura Y, Kobayashi T and Puolamaki K 1999 Phys. Lett. B 468 111
[11] Rizzo T G 1994 Phys. Lett. B 342 163
[12] Mariano T and Senjanovic G 1982 Phys. Rev. D 25 3092
[13] Dasgupta T, Mamides P and Nath P 1995 Phys. Rev. D 52 5366
[14] Anderson G, Chen C H, Gunion J F, Lykken J D, Moroi T and Yamada Y 1996 eConf C 960625 SUP107
[15] Amundson J et al 1996 eConf C 960625 SUP106

8 In equation (8), all SC of a group $G$ is denoted by $f_{ijk}$. We now denote all SC of $E_6$ by $ff_{[i, j, k]}$. It has been checked that $ff_{[i, j, k]}$ satisfy the Jacobi identities.
[16] Martin S P 2009 Phys. Rev. D 79 095019
[17] Balazs C, Li T, Nanopoulos D V and Wang F 2010 J. High Energy Phys. 2010 3
[18] Balazs C, Li T, Nanopoulos D V and Wang F 2011 J. High Energy Phys. 2011 96
[19] Wang F 2011 Nucl. Phys. B 851 104
[20] Chakrabortty J and Raychaudhuri A 2009 Phys. Lett. B 673 57
[21] Lykken J D and Willenbrock S 1994 Phys. Rev. D 49 4902
[22] Howl R and King S F 2007 Phys. Lett. B 652 331
[23] Ellis J R, Enqvist K, Nanopoulos D V and Tamvakis K 1985 Phys. Lett. 155B 381
[24] Drees M 1985 Phys. Lett. B 158 409
[25] Tobe K and Wells J D 2004 Phys. Lett. B 588 99
[26] Nath P and Fileviez Perez P 2007 Phys. Rep. 441 191
[27] Calmet X, Hsu S D H and Reeb D 2010 Phys. Rev. D 81 035007
[28] Georgi H and Jarlskog C 1979 Phys. Lett. 86B 297
[29] Fileviez Perez P 2007 Karlsruhe 2007, SUSY 2007 pp 678–81 arXiv:0710.1321 [hep-ph]
[30] Arbelaez C, Cárcamo Hernández A E, Kovalenko S and Schmidt I 2015 Phys. Rev. D 92 115015
[31] Green M B and Schwarz J H 1984 Phys. Lett. B 149 117
[32] Green M B, Schwarz J H and Witten E 1987 Supersstring Theory. Vol. 1: Introduction (Cambridge Monographs On Mathematical Physics) (Cambridge: Cambridge University Press) p 469
[33] Green M B, Schwarz J H and Witten E 1987 Supersstring Theory. Vol. 2: Loop Amplitudes, Anomalies, And Phenomenology (Cambridge Monographs On Mathematical Physics) (Cambridge: Cambridge University Press) p 596
[34] Candelas P, Horowitz G T, Strominger A and Witten E 1985 Nucl. Phys. B 258 46
[35] King S F, Moretti S and Nevzorov R 2006 Phys. Rev. D 73 035009
[36] Kawase H and Maekawa N 2010 Prog. Theor. Phys. 123 941
[37] Athron P, Binjomaid M and King S F 2013 Phys. Rev. D 87 115023
[38] Rosner J L 2014 Phys. Rev. D 90 015005
[39] Nevzorov R and Pakvasa S 2014 Phys. Lett. B 728 210
[40] Nevzorov R and Thomas A W 2015 Phys. Rev. D 92 075007
[41] Rojas E and Erler J 2015 J. High Energy Phys. 2015 63
[42] Nevzorov R 2015 PoS EPS-HEP2015 p 381
[43] Barbieri R and Nanopoulos D V 1980 Phys. Lett. B 95 43
[44] Barbieri R, Nanopoulos D V and Masiero A 1981 Phys. Lett. B 104 194
[45] Ramond P 1998 arXiv:hep-th/9809459
[46] Athron P, Harries D, Nevzorov R and Williams A G 2016 J. High Energy Phys. 2016 128
[47] Joglekar A and Rosner J L 2017 Phys. Rev. D 96 015026
[48] Nevzorov R and Pakvasa S 2016 Nucl. Part. Phys. Proc. 273–275 690
[49] Dhuria M, Hati C and Sarkar U 2016 Phys. Rev. D 93 015001
[50] Athron P, Harries D, Nevzorov R and Williams A G 2016 Phys. Lett. B 760 19
[51] Harada J 2016 Fortsch. Phys. 64 510
[52] Athron P, Thomas A W, Underwood S J and White M J 2017 Phys. Rev. D 95 035023
[53] Belanger G, Da Silva J and Tran H M 2017 Phys. Rev. D 95 115017
[54] Nevzorov R and Thomas A W 2017 Phys. Lett. B 774 123
[55] Huang C S 2014 Mod. Phys. Lett. A 29 1450150
[56] Slansky R 1981 Phys. Rep. 79 1
[57] Chamoun N, Huang C S, Liu C and Wu X H 2002 Nucl. Phys. B 624 81
[58] Chamoun N, Huang C S, Liu C and Wu X H 2010 J. Phys. G: Nucl. Part. Phys. 37 105016
[59] Harada J 2003 J. High Energy Phys. 2003 11
[60] London D and Rosner J L 1986 Phys. Rev. D 34 1530
[61] Vavilov N A 2004 J. Math. Sci. 120 1513
[62] Girardi G, Sciarrino A and Sorba P 1981 Nucl. Phys. B 182 477
[63] Dorsner I and Fileviez Perez P 2005 Phys. Lett. B 625 88
[64] Abe K et al (Super-Kamiokande Collaboration) 2017 Phys. Rev. D 96 012003