An $S$-type eigenvalue localization set for tensors

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Abstract

An $S$-type eigenvalue localization set for a tensor is given by breaking $N = \{1, 2, \ldots, n\}$ into disjoint subsets $S$ and its complement. It is shown that the new set is tighter than those provided by L. Qi (Journal of Symbolic Computation 40 (2005) 1302-1324) and Li et al. (Numer. Linear Algebra Appl. 21 (2014) 39-50). As applications of the results, a checkable sufficient condition for the positive definiteness of tensors and a checkable sufficient condition of the positive semi-definiteness of tensors are given.

Keywords: Tensor eigenvalue, Localization set, Positive definite, Positive semi-definite

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1. Introduction

Eigenvalue problems of tensors have become an important topic of study in numerical multilinear algebra, and they have a wide range of practical applications; see [2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 18, 19, 21, 22, 24, 25]. Here we call $\mathcal{A} = (a_{i_1\cdots i_m})$ a complex (real) tensor of order $m$ dimension $n$, denoted by $\mathcal{A} \in C^{[m,n]} (R^{m,n})$, if

$$a_{i_1\cdots i_m} \in C (R),$$

where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$. Moreover, if there are a complex number $\lambda$ and a nonzero complex vector $x = (x_1, x_2, \ldots, x_n)^T$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

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then $\lambda$ is called an eigenvalue of $A$ and $x$ an eigenvector of $A$ associated with $\lambda$ [16, 20], where $Ax^{m-1}$ is an $n$ dimension vector whose $i$th component is

$$(Ax^{m-1})_i = \sum_{i_2, \ldots, i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \quad (N = \{1, 2, \ldots, n\})$$

and

$$x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T.$$ 

If $x$ and $\lambda$ are all real, then $\lambda$ is called an H-eigenvalue of $A$ and $x$ an H-eigenvector of $A$ associated with $\lambda$ [20, 21, 27].

One of many practical applications of eigenvalues of tensors is that one can identify the positive (semi-)definiteness for an even-order real symmetric tensor by using the smallest H-eigenvalue of a tensor, consequently, can identify the positive (semi-)definiteness of the multivariate homogeneous polynomial determined by this tensor, for details, see [11, 20].

Because it is not easy to compute the smallest H-eigenvalue of tensors when the order and dimension are large, ones always try to give a set including all eigenvalues in the complex [20, 13, 14, 15]. In particular, if this set for an even-order real symmetric tensor is in the right-half complex plane, then we can conclude that the smallest H-eigenvalue is positive, consequently, the corresponding tensor is positive definite.

In [20], Qi gave an eigenvalue localization set for real symmetric tensors, which is a generalization of the well-known Geršgorin’s eigenvalue localization set of matrices [6, 23]. This result can be easily generalized to general tensors [13, 26].

**Theorem 1.** [13, 26, 20] Let $A = (a_{i_1 \cdots i_m}) \in C^{[m,n]}$. Then

$$\sigma(A) \subseteq \Gamma(A) := \bigcup_{i \in N} \Gamma_i(A),$$

where $\sigma(A)$ is the set of all the eigenvalues of $A$,

$$\Gamma_i(A) = \{z \in \mathbb{C} : |z - a_{i \cdots i}| \leq r_i(A)\}, \quad r_i(A) = \sum_{i_2, \ldots, i_m \in N, \delta_{i_1 \cdots i_m} = 0} |a_{i_1 i_2 \cdots i_m}|$$

and

$$\delta_{i_1 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$
Although it is easy to get $\Gamma(A)$ in the complex by computing $n$ sets $\Gamma_i(A)$, $\Gamma(A)$ does’nt always capture all eigenvalues of $A$ very precisely. To obtain tighter sets than $\Gamma(A)$, Li et al. [13] extended the Brauer’s eigenvalue localization set of matrices [1, 23] and gave the following Brauer-type eigenvalue localization set for tensors.

**Theorem 2.** [13] Let $A = (a_{i_1 \cdots i_m}) \in C^{[m,n]}$, $n \geq 2$. Then

$$\sigma(A) \subseteq K(A) := \bigcup_{i,j \in N, j \neq i} K_{i,j}(A),$$

where

$$K_{i,j}(A) = \{ z \in \mathbb{C} : (|z - a_{i \cdots i}| - r_i^2(A)) |z - a_{j \cdots j}| \leq |a_{i \cdots j} |r_j(A) \}$$

and

$$r_i^2(A) = \sum_{\delta_{i_1,i_2,\ldots,i_m} = 0, \delta_{j_1,j_2,\ldots,j_m} = 0} |a_{i_1 \cdots i_m}| = r_i(A) - |a_{i \cdots j}|.$$

Furthermore, $K(A) \subseteq \Gamma(A)$.

As Theorem 2 shows, we need compute $n(n-1)$ sets $K_{i,j}(A)$ to give the set $K(A)$, however $K(A)$ captures all eigenvalues of $A$ more precisely than $\Gamma(A)$. To reduce computations, Li et al. give an $S$-type eigenvalue localization set by breaking $N$ into disjoint subsets $S$ and $\bar{S}$, where $\bar{S}$ is the complement of $S$ in $N$.

**Theorem 3.** [13, Theorem 2.2] Let $A = (a_{i_1 \cdots i_m}) \in C^{[m,n]}$, $n \geq 2$, and $S$ be a nonempty proper subset of $N$. Then

$$\sigma(A) \subseteq K^S(A) := \left( \bigcup_{i \in S, j \notin S} K_{i,j}(A) \right) \bigcup \left( \bigcup_{i \notin S, j \neq i} K_{i,j}(A) \right).$$

The set $K^S(A)$ in Theorem 3 consists of $2|S|(n - |S|)$ sets $K_{i,j}(A)$, where $|S|$ is the cardinality of $S$. It is obvious that $2|S|(n - |S|) \leq n(n-1)$, and then

$$K^S(A) \subseteq K(A) \subseteq \Gamma(A), \quad (1)$$

for details, see [13]. In this paper, by the technique in [13] we give a new eigenvalue localization set involved with a proper subset $S$ of $N$, and prove that the new set is tighter than $\Gamma(A), K(A)$ and $K^S(A)$. As an application, we give some checkable sufficient conditions for the positive (semi-)definiteness of tensors.
2. A new $S$-type eigenvalue localization set

we begin with some notation. Given an nonempty proper subset $S$ of $N$, we denote
\[
\Delta N := \{(i_2, i_3, \ldots, i_m) : \text{each } i_j \in N \text{ for } j = 2, \ldots, m\},
\]
\[
\Delta S := \{(i_2, i_3, \ldots, i_m) : \text{each } i_j \in S \text{ for } j = 2, \ldots, m\},
\]
and then
\[
\overline{\Delta S} = \Delta N \setminus \Delta S.
\]
This implies that for a tensor $A = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, we have that for $i \in S$,
\[
r_i(A) = r_{\Delta S}^i(A) + r_{\overline{\Delta S}}^i(A),
\]
where
\[
r_{\Delta S}^i(A) = \sum_{(i_2, \ldots, i_m) \in \Delta S} |a_{ii_2 \cdots i_m}|, n_{\overline{\Delta S}}^i(A) = \sum_{(i_2, \ldots, i_m) \in \overline{\Delta S}} |a_{ii_2 \cdots i_m}|
\]

**Theorem 4.** Let $A = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$, and $S$ be a nonempty proper subset of $N$. Then
\[
\sigma(A) \subseteq \Omega^S(A) := \left( \bigcup_{i \in S, j \in \overline{S}} \Omega_{i,j}^S(A) \right) \bigcup \left( \bigcup_{i \in \overline{S}, j \in S} \Omega_{i,j}^S(A) \right),
\]
where
\[
\Omega_{i,j}^S(A) := \{ z \in \mathbb{C} : (|\lambda - a_{i \cdots i}|)(|\lambda - a_{j \cdots j}| - r_j^S(A)) \leq r_i(A)r_{\Delta S}^j(A) \}
\]
and
\[
\Omega_{i,j}^S(A) := \{ z \in \mathbb{C} : (|\lambda - a_{i \cdots i}|)(|\lambda - a_{j \cdots j}| - r_j^S(A)) \leq r_i(A)r_{\Delta S}^j(A) \}
\]

**Proof.** For any $\lambda \in \sigma(A)$, let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$ be an associated eigenvector, i.e.,
\[
A x^{m-1} = \lambda x^{[m-1]}.
\]
Let $|x_p| = \max_{i \in S} |x_i|$ and $|x_q| = \max_{i \in \overline{S}} |x_i|$ . Then, at least one of $x_p$ and $x_q$ is nonzero. We next divide into three cases to prove.
Case I: \(x_p x_q \neq 0\) and \(|x_q| \geq |x_p|\), that is, \(|x_q| = \max_{i \in N} |x_i|\). By (2), we have

\[
(\lambda - a_{q \cdots q}) x_q^{m-1} = \sum_{(i_2, \cdots, i_m) \in \Delta^S} a_{q i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} + \sum_{(i_2, \cdots, i_m) \in \Delta^S \setminus \Delta^S_0} a_{q i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.
\]

Taking modulus in the above equation and using the triangle inequality gives

\[
|\lambda - a_{q \cdots q}| |x_q|^{m-1} \leq \sum_{(i_2, \cdots, i_m) \in \Delta^S} |a_{q i_2 \cdots i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{(i_2, \cdots, i_m) \in \Delta^S \setminus \Delta^S_0} |a_{q i_2 \cdots i_m}| |x_{i_2}| \cdots |x_{i_m}|
\]

\[
\leq \sum_{(i_2, \cdots, i_m) \in \Delta^S} |a_{q i_2 \cdots i_m}| |x_p|^{m-1} + \sum_{(i_2, \cdots, i_m) \in \Delta^S \setminus \Delta^S_0} |a_{q i_2 \cdots i_m}| |x_q|^{m-1}
\]

\[
= r_q^{\Delta^S} (\mathcal{A}) |x_p|^{m-1} + r_q^{\Delta^S} (\mathcal{A}) |x_q|^{m-1}.
\]

Hence,

\[
(\lambda - a_{q \cdots q}) |x_q|^{m-1} \leq r_q^{\Delta^S} (\mathcal{A}) |x_q|^{m-1},
\]

(3)

On the other hand, by (2), we also get that

\[
(\lambda - a_{p \cdots p}) x_p^{m-1} = \sum_{i_2, \cdots, i_m \in N, \delta q_{i_2 \cdots i_m} = 0} a_{p i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}
\]

\[
|\lambda - a_{p \cdots p}| |x_p|^{m-1} \leq r_p (\mathcal{A}) |x_p|^{m-1}.
\]

(4)

Multiplying (3) with (4) gives

\[
(\lambda - a_{p \cdots p}) |x_p|^{m-1} \leq r_p (\mathcal{A}) |x_q|^{m-1},
\]

(5)

which leads to \(\lambda \in \Omega_p^{\Delta^S} (\mathcal{A}) \subseteq \Omega^S (\mathcal{A})\).

Case II: \(x_p x_q \neq 0\) and \(|x_p| \geq |x_q|\), that is, \(|x_p| = \max_{i \in N} |x_i|\). Similar to the proof of Case I, we can obtain that

\[
(\lambda - a_{p \cdots p}) |x_p|^{m-1} \leq r_p (\mathcal{A}) |x_q|^{m-1},
\]

and

\[
|\lambda - a_{q \cdots q}| |x_q|^{m-1} \leq r_q (\mathcal{A}) |x_p|^{m-1}.
\]
This gives

\[(\lambda - a_{q\ldots q}) \left( |\lambda - a_{p\ldots p} - \overline{r}_p(\mathcal{A})| \right) \leq r_q(\mathcal{A})r^\Delta_q(\mathcal{A}).\]

Hence, \(\lambda \in \Omega_{q,p}^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}).\)

Case III: \(|x_p||x_q| = 0\), without loss of generality, let \(|x_p| = 0\) and \(|x_q| \neq 0\). Then by (3),

\[|\lambda - a_{q\ldots q}| - r\overline{\Delta}_q(\mathcal{A}) \leq 0.
\]

hence for any \(i \in S\),

\[(|\lambda - a_{i\ldots i})| \left( |\lambda - a_{q\ldots q}| - r\overline{\Delta}_q(\mathcal{A}) \right) \leq r_i(\mathcal{A})r^\Delta_i(\mathcal{A}),\]

which leads to \(\lambda \in \Omega^S_i(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}).\) The conclusion follows from Cases I, II and III.

To compare the sets \(\Gamma(\mathcal{A})\) in Theorem 1, \(\mathcal{K}(\mathcal{A})\) in Theorem 2, \(\mathcal{K}^S(\mathcal{A})\) in Theorem 3 with \(\Omega^S(\mathcal{A})\) in Theorem 4, two lemmas in [14] are listed as follows.

**Lemma 5.** [14] Lemmas 2.2 and 2.3 \(\{\) I\(\) Let \(a, b, c \geq 0\) and \(d > 0\). If \(\frac{a}{b+c+d} \leq 1\), then \(a - (b + c) \leq \frac{a - b}{c + d} \leq \frac{a}{b+c+d}\).

(II) Let \(a, b, c \geq 0\) and \(d > 0\). If \(\frac{a}{b+c+d} \geq 1\), then \(a - (b + c) \geq \frac{a - b}{c + d} \geq \frac{a}{b+c+d}\).

**Theorem 6.** Let \(\mathcal{A} = (a_{i_1\ldots i_m}) \in C^{[m,n]}, n \geq 2\). And let \(S\) be a nonempty proper subset of \(N\). Then

\[\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).\]

**Proof.** By (I), \(\mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})\) holds. We only prove \(\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})\). Let \(z \in \Omega^S(\mathcal{A})\). Then

\[z \in \bigcup_{i \in S, j \in \bar{S}} \Omega^S_{i,j}(\mathcal{A}) \text{ or } z \in \bigcup_{i \in S, j \in \bar{S}} \Omega^S_{i,j}(\mathcal{A}).\]
Without loss of generality, suppose that \( z \in \bigcup_{i \in S, j \in \bar{S}} \Omega_{i,j}^S(A) \) (we can prove it similarly if \( z \in \bigcup_{i \in \bar{S}, j \in S} \Omega_{i,j}^S(A) \)). Then there are \( p \in S \) and \( q \in \bar{S} \) such that \( z \in \Omega_{p,q}^S(A) \), i.e.,

\[
(|z - a_{p\cdots p}| - r_{q}^S(A)) \leq r_p(A) r_q^S(A).
\] (5)

If \( r_p(A) r_q^S(A) = 0 \), then \( r_p(A) = 0 \), or \( r_q^S(A) = 0 \). When \( r_q^S(A) = 0 \), we have \( |a_{qp\cdots p}| = 0 \), \( r_q^S(A) = r_p^p(A) \) and

\[
|z - a_{p\cdots p}| (|z - a_{q\cdots q}| - r_p^p(A)) = |z - a_{p\cdots p}| \left( |z - a_{q\cdots q}| - r_q^S(A) \right) \\
\leq r_p(A) r_q^S(A) \\
= r_p(A) |a_{qp\cdots p}| \\
= 0,
\]

which implies that \( z \in \mathcal{K}_{q,p}(A) \subseteq \bigcup_{i \in S, j \in \bar{S}} \mathcal{K}_{i,j}(A) \subseteq \mathcal{K}^S(A) \), consequently, \( \Omega^S(A) \subseteq \mathcal{K}^S(A) \). When \( r_p(A) = 0 \), we have

\[
|z - a_{p\cdots p}| (|z - a_{q\cdots q}| - r_p^p(A)) \leq |z - a_{p\cdots p}| \left( |z - a_{q\cdots q}| - r_q^S(A) \right) \\
\leq r_p(A) r_q^S(A) \\
= 0 \\
= r_p(A) |a_{qp\cdots p}|.
\]

This also leads to \( z \in \bigcup_{i \in S, j \in S} \mathcal{K}_{i,j}(A) \subseteq \mathcal{K}^S(A) \), and \( \Omega^S(A) \subseteq \mathcal{K}^S(A) \).

If \( r_p(A) r_q^S(A) > 0 \), then we can equivalently express Inequality (5) as

\[
\frac{|z - a_{q\cdots q}| - r_q^S(A)}{r_p(A)} \leq 1,
\] (6)

which implies

\[
\frac{|z - a_{q\cdots q}| - r_q^S(A)}{r_q^S(A)} \leq 1,
\] (7)
or

\[
\frac{|z - a_{p\ldots p}|}{r_p(A)} \leq 1. \tag{8}
\]

Let \( a = |z - a_{t\ldots t}|, b = r_q^{\Delta^S}(A), c = r_q^{\Delta^S}(A) - |a_{qp\ldots p}| \) and \( d = |a_{qp\ldots p}|. \) When Inequality (7) holds and \( d = |a_{qp\ldots p}| > 0, \) from the part (I) in Lemma 5 and Inequality (8) we have

\[
\frac{|z - a_{q\ldots q}| - r_q^p(A)}{|a_{qp\ldots p}|} \frac{|z - a_{p\ldots p}|}{r_p(A)} \leq \frac{|z - a_{q\ldots q}| - r_q^{\Delta^S}(A)}{r_q^{\Delta^S}(A)} \frac{|z - a_{p\ldots p}|}{r_p(A)} \leq 1
\]
equivalently,

\[
|z - a_{p\ldots p}| (|z - a_{q\ldots q}| - r_q^p(A)) \leq r_p(A)|a_{qp\ldots p}|.
\]

This implies \( \Omega^S(A) \subseteq K^S(A). \) When Inequality (7) holds and \( d = |a_{qp\ldots p}| = 0, \) we easily get

\[
|z - a_{q\ldots q}| - r_q^p(A) \leq 0 = |a_{qp\ldots p}|.
\]

Hence,

\[
|z - a_{p\ldots p}| (|z - a_{q\ldots q}| - r_q^p(A)) \leq 0 = r_p(A)|a_{qp\ldots p}|.
\]

This also implies \( \Omega^S(A) \subseteq K^S(A). \) On the other hand, when Inequality (8) holds, we only need to prove \( \Omega^S(A) \subseteq K^S(A) \) under the case that

\[
\frac{|z - a_{q\ldots q}| - r_q^{\Delta^S}(A)}{r_q^{\Delta^S}(A)} > 1. \tag{9}
\]

Note that Inequality (9) yields

\[
\frac{|z - a_{q\ldots q}|}{r_q(A)} > 1.
\]

Hence, when Inequality (8) holds and \( |a_{pq\ldots q}| > 0, \) we have from Lemma 5 and Inequality (8) that

\[
\frac{|z - a_{q\ldots q}|}{r_q(A)} \frac{|z - a_{p\ldots p}|}{|a_{pq\ldots q}|} \leq \frac{|z - a_{q\ldots q}|}{r_q^{\Delta^S}(A)} \frac{|z - a_{p\ldots p}|}{r_p(A)} \leq 1
\]
equivalently,

\[
|z - a_{q\ldots q}| (|z - a_{p\ldots p}| - r_q^p(A)) \leq r_q(A)|a_{pq\ldots q}|.
\]
This implies \( z \in \mathcal{K}_{p,q}(A) \subseteq \bigcup_{i \in S, j \in \bar{S}} \mathcal{K}_{i,j}(A) \subseteq \mathcal{K}^S(A) \) and \( \Omega^S(A) \subseteq \mathcal{K}^S(A) \). And when Inequality (8) holds and \( |a_{pq} \cdots q| = 0 \), we easily get
\[
|z - a_{p \cdots p}| - r^q_p(A) \leq 0 = |a_{pq} \cdots q|.
\]
Hence,
\[
|z - a_{q \cdots q}| \left( |z - a_{p \cdots p}| - r^q_p(A) \right) \leq 0 = r_q(A)|a_{pq} \cdots q|.
\]
This also implies \( \Omega^S(A) \subseteq \mathcal{K}^S(A) \).

**Remark 1.** For a complex tensor \( A \in \mathbb{C}^{[m,n]} \), \( n \geq 2 \), the set \( \mathcal{K}^S(A) \) consists of \( 2^{|S|} (n - |S|) \) sets \( \mathcal{K}_{i,j}(A) \), and the set \( \Omega^S(A) \) consists of \( |S| (n - |S|) \) sets \( \Omega^S_{i,j}(A) \) and \( |S| (n - |S|) \) sets \( \Omega^S_{\bar{S},j}(A) \), where \( S \) is a nonempty proper subset of \( N \). Hence, under the same computations, \( \Omega^S(A) \) captures all eigenvalues of \( A \) more precisely than \( \mathcal{K}^S(A) \).

### 3. Sufficient conditions for positive (semi-)definiteness of tensors

As applications of the results in Sections 2, we in this section provide some checkable sufficient conditions for the positive definiteness and positive semi-definiteness of tensors, respectively. Before that, we give some definitions in [5, 12, 28].

**Definition 1.** [5, 28] A tensor \( A = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]} \) is called a (strictly) diagonally dominant tensor if for \( i \in N \),
\[
|a_{i \cdots i}| \geq (>) r_i(A) \tag{10}
\]

**Definition 2.** [12] A tensor \( A = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]} \) with \( n \geq 2 \) is called a quasi-doubly (strictly) diagonally dominant tensor if for \( i, j \in N, j \neq i \),
\[
\left( |a_{i \cdots j}| - r^j_i(A) \right) |a_{j \cdots j}| \geq (>) r_j(A)|a_{i \cdots j}| \tag{11}
\]

**Definition 3.** Let \( A = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]} \) with \( n \geq 2 \) and \( S \) be a nonempty proper subset of \( N \). \( A \) is called an \( S\text{-QDSDD} \) \( (S\text{-QDSDD}) \) tensor if for each \( i \in S \) and each \( j \in S \), Inequality (11) holds and
\[
\left( |a_{j \cdots j}| - r^j_i(A) \right) |a_{i \cdots i}| \geq (>) r_i(A)|a_{j \cdots j}| \tag{12}
\]
**Definition 4.** Let $A = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \geq 2$ and $S$ be a nonempty proper subset of $N$. $A$ is called an $S$-$SDD_0$ ($S$-$SDD$) tensor if for each $i \in S$ and each $j \in \bar{S}$,

$$|a_{i\cdots i}| \left( |a_{j\cdots j}| - r^\Delta_j(A) \right) \geq (>) r^\Delta_i(A) r^\Delta_{j^S}(A),$$

and

$$|a_{j\cdots j}| \left( |a_{i\cdots i}| - r^\Delta_i(A) \right) \geq (>) r^\Delta_j(A) r^\Delta_{i^S}(A).$$

Next, we give the relationships between (strictly) diagonally dominant tensors, quasi-doubly (strictly) diagonally dominant tensors, $S$-$QDSDD_0$ ($S$-$QDSDD$) tensors and $S$-$SDD_0$ ($S$-$SDD$) tensors.

**Proposition 1.** If $A = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ is a strictly diagonally dominant tensor, then $A$ is a quasi-doubly strictly diagonally dominant tensor. If $A$ is a diagonally dominant tensor, then $A$ is a quasi-doubly diagonally dominant tensor.

**Proof.** If $A$ is a strictly diagonally dominant tensor, then for any $i \in N$,

$$|a_{i\cdots i}| > r_i(A),$$

equivalently,

$$|a_{i\cdots i}| - r^\Delta_i(A) > |a_{ij\cdots j}|.$$

Hence, for $i, j \in N$, $j \neq i$,

$$|a_{i\cdots i}| > r_i(A),$$

and

$$|a_{j\cdots j}| - r^\Delta_j(A) > |a_{ji\cdots i}|,$$

which implies that the strict inequality (11) holds, i.e., $A$ is a quasi-doubly strictly diagonally dominant tensor by Definition 2. Similarly, we can prove that if $A$ is a diagonally dominant tensor, then $A$ is a quasi-doubly diagonally dominant tensor.

**Proposition 2.** If $A = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ is a quasi-doubly strictly diagonally dominant tensor, then $A$ is an $S$-$QDSDD$ tensor. If $A$ is a quasi-doubly diagonally dominant tensor, then $A$ is an $S$-$QDSDD_0$ tensor.
Proof. If $A$ is a quasi-doubly strictly diagonally dominant tensor, then for $i, j \in N$, $j \neq i$, the strict inequality (11) holds, i.e.,

$$(|a_{i\cdots i}| - r^i_j(A)) |a_{j\cdots j}| > r_j(A) |a_{ij\cdots j}|.$$  

For a given nonempty proper subset $S$ of $N$, we easily get that for each $i \in S$ and each $j \in \bar{S}$,

$$(|a_{i\cdots i}| - r^i_j(A)) |a_{j\cdots j}| > r_j(A) |a_{ij\cdots j}|,$$

and

$$(|a_{j\cdots j}| - r^j_i(A)) |a_{i\cdots i}| > r_i(A) |a_{ji\cdots i}|.$$  

Hence, $A$ is an $S$-QDSDD tensor. Similarly, we can prove that a quasi-doubly diagonally dominant tensor is an $S$-QDSDD tensor. \hfill \Box

Proposition 3. Let $A = (a_{i_1\cdots i_m}) \in C^{[m,n]}$ and $S$ be a nonempty proper subset of $N$. If $A$ is an $S$-QDSDD tensor, then $A$ is an $S$-SDD tensor. If $A$ is an $S$-QDSDD$_0$ tensor, then $A$ is an $S$-SDD$_0$ tensor.

Proof. We only prove that an $S$-QDSDD tensor is an $S$-SDD tensor, and by a similar way, we can prove that an $S$-QDSDD$_0$ tensor is an $S$-SDD$_0$ tensor.

Let $A$ be an $S$-QDSDD tensor. It is easy to see from Definition 3 that either for any $i \in S$,

$$|a_{i\cdots i}| > r_i(A),$$

or for any $j \in \bar{S}$,

$$|a_{j\cdots j}| > r_j(A).$$

Without loss of generality, we next suppose that for any $j \in \bar{S}$, Inequality (16) holds. Hence, for any $j \in \bar{S}$,

$$|a_{j\cdots j}| - r^j_j(A) > |a_{ji\cdots i}|$$

and

$$|a_{j\cdots j}| - r^j_j(A) > r^j_j(A).$$

Case I: for $i \in S$ such that Inequality (15) holds, i.e.,

$$|a_{i\cdots i}| - r^i_i(A) > r^i_i(A),$$

and

$$|a_{i\cdots i}| - r^i_i(A) > r^i_i(A).$$
by combining with Inequalities (16) and (18) we easily get that for this \( i \in S \) and each \( j \in \bar{S} \), Inequalities (13) and (14) hold.

Case II: for \( i \in S \) such that

\[
|a_{i\ldots i}| \leq r_i(A),
\]

by Definition 3 we can get that \( 0 < |a_{i\ldots i}| \leq r_i(A), \)

\[
0 < |a_{i\ldots i} - r_i^1(A)| \leq |a_{i\ldots j}|, \tag{20}
\]

and

\[
0 < |a_{i\ldots i} - r_i^{\Delta^S}(A)| \leq r_i^{\Delta^S}(A). \tag{21}
\]

Let \( a = |a_{i\ldots i}|, b = r_i^{\Delta^S}(A), c = r_i^{\Delta^S}(A) - |a_{i\ldots j}|, d = |a_{i\ldots j}|, e = |a_{j\ldots j}|, f = r_j^{\Delta^S}(A), g = r_j^{\Delta^S}(A) - |a_{j\ldots j}| \) and \( h = |a_{j\ldots i}| \). If \( r_j(A) \neq 0 \) for some \( j \in \bar{S} \), then by Inequality (11), Inequality (20) and by Lemma 5, we have

\[
\frac{|a_{i\ldots i} - r_i^{\Delta^S}(A)|}{r_i^{\Delta^S}(A)} \geq \frac{|a_{i\ldots i} - r_i^j(A)|}{|a_{i\ldots j}|} \frac{|a_{i\ldots j}|}{r_j(A)} > 1,
\]

and \( |a_{j\ldots j}| \left( |a_{i\ldots i} - r_i^{\Delta^S}(A)| \right) > r_j(A)r_i^{\Delta^S}(A) \), i.e., Inequality (14) holds. Similarly, by Inequality (11) and by Lemma 5, we can also get

\[
\frac{|a_{i\ldots i}| |a_{j\ldots j}| - r_i^{\Delta^S}(A)}{r_i^{\Delta^S}(A)} \geq \frac{|a_{i\ldots i} - r_i^j(A)|}{|a_{i\ldots j}|} \frac{|a_{i\ldots j}|}{r_j(A)} > 1,
\]

where \( \frac{|a_{j\ldots j}| - r_j^{\Delta^S}(A)}{r_j^{\Delta^S}(A)} = +\infty \) if \( r_j^{\Delta^S}(A) = 0 \), and \( |a_{i\ldots i}| \left( |a_{j\ldots j} - r_j^{\Delta^S}(A)| \right) > r_i(A)r_j^{\Delta^S}(A) \), i.e., Inequality (13) holds. On the other hand, if \( r_j(A) = 0 \) for some \( j \in \bar{S} \), then \( r_j^{\Delta^S}(A) = r_j^{\Delta^S}(A) = 0 \). Obviously, Inequalities (13) and (14) also hold. The conclusion follows from Cases I and II.

As shown in [13, 14], by using eigenvalue localization sets for tensors, one can give some corresponding checkable sufficient conditions of the positive (semi-)definiteness of tensors. Here we call a tensor \( A = (a_{i_1\ldots i_m}) \in R^{[m,n]} \) symmetric if

\[
a_{i_1\ldots i_m} = a_{\pi(i_1\ldots i_m)}, \forall \pi \in \Pi_m,
\]

where \( \pi \) is a permutation of \( \{1, 2, \ldots, m\} \).
where $\Pi_m$ is the permutation group of $m$ indices. And an even-order real symmetric tensor is called positive (semi-)definite, if its smallest H-eigenvalue is positive (nonnegative). Next, a new checkable sufficient condition of the positive (semi-)definiteness of tensors is obtained by using Theorem 4.

**Theorem 7.** Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ with $n \geq 2$ and $S$ be a nonempty proper subset of $N$. If $\mathcal{A}$ is an even-order symmetric $S$-SDD ($S$-SDD$_0$) tensor with $a_{k \ldots k} \geq (\geq) 0$ for all $k \in N$, then $\mathcal{A}$ is positive (semi-)definite.

**Proof.** We need only prove that $\mathcal{A}$ is positive semi-definite, and by a similar way, we can prove that $\mathcal{A}$ is positive definite. Let $\lambda$ be an H-eigenvalue of $\mathcal{A}$. Suppose on the contrary that $\lambda < 0$. From Theorem 4, we have $\lambda \in \Omega_S(\mathcal{A})$ which implies that there are $i_0, i_1 \in S$, $j_0, j_1 \in \bar{S}$ such that $\lambda \in \Omega_{i_0,j_0}(\mathcal{A})$ or $\lambda \in \Omega_{j_1,i_1}(\mathcal{A})$, that is,

$$|\lambda - a_{i_0 \ldots i_0}| (|\lambda - a_{j_0 \ldots j_0}| - r_{j_0}^S(\mathcal{A})) \leq r_{i_0}(\mathcal{A}) r_{j_0}^{\Delta^S}(\mathcal{A})$$

or

$$|\lambda - a_{j_1 \ldots j_1}| (|\lambda - a_{i_1 \ldots i_1}| - r_{i_1}^S(\mathcal{A})) \leq r_{j_1}(\mathcal{A}) r_{i_1}^{\Delta^S}(\mathcal{A}).$$

On the other hand, since $\mathcal{A}$ is an $S$-SDD$_0$ tensor with $a_{k \ldots k} \geq 0$ for all $k \in N$, we have that for $i_0, i_1 \in S$, $j_0, j_1 \in \bar{S},$

$$|\lambda - a_{i_0 \ldots i_0}| (|\lambda - a_{j_0 \ldots j_0}| - r_{j_0}^S(\mathcal{A})) > |a_{i_0 \ldots i_0}| (|a_{j_0 \ldots j_0}| - r_{j_0}^S(\mathcal{A})) \geq r_{i_0}(\mathcal{A}) r_{j_0}^{\Delta^S}(\mathcal{A})$$

and

$$|\lambda - a_{j_1 \ldots j_1}| (|\lambda - a_{i_1 \ldots i_1}| - r_{i_1}^S(\mathcal{A})) > |a_{j_1 \ldots j_1}| (|a_{i_1 \ldots i_1}| - r_{i_1}^S(\mathcal{A})) \geq r_{j_1}(\mathcal{A}) r_{i_1}^{\Delta^S}(\mathcal{A}).$$

These lead to a contradiction. Hence, $\lambda \geq 0$, and $\mathcal{A}$ is positive semi-definite. The conclusion follows. \[\square\]

According to Theorem 7, Proposition 1, Proposition 2 and Proposition 3, we easily obtain the following results which were also obtained in [5, 11, 13, 28].

**Corollary 1.** An even-order strictly diagonally dominant symmetric tensor with all positive diagonal entries is positive definite. And an even-order diagonally dominant symmetric tensor with all nonnegative diagonal entries is positive semi-definite.
Corollary 2. An even-order quasi-doubly strictly diagonally dominant symmetric tensor with all positive diagonal entries is positive definite. And an even-order quasi-doubly diagonally dominant tensor symmetric tensor with all nonnegative diagonal entries is positive semi-definite.

Corollary 3. An even-order $S$-$QDSDD$ tensor with all positive diagonal entries is positive definite. And an even-order $S$-$QDSDD_0$ symmetric tensor with all nonnegative diagonal entries is positive semi-definite.

Example 1. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ be a real symmetric tensor with elements defined as follows:

\[
\begin{array}{l}
    a_{1111} = 5, a_{2222} = 6, a_{3333} = 3.3, a_{1112} = -0.1, a_{1113} = 0.1, a_{1122} = -0.2, \\
    a_{1123} = -0.2, a_{1133} = 0, a_{1222} = -0.1, a_{1223} = 0.3, a_{1233} = 0.1, \\
    a_{1333} = -0.1, a_{2223} = 0.1, a_{2233} = -0.1, a_{2333} = 0.2.
\end{array}
\]

Let $S = \{1, 2\}$. Obviously $\bar{S} = \{3\}$. By computations, we get that

\[
\begin{align*}
|a_{1111}| \left( |a_{3333}| - r_3^1(\mathcal{A}) \right) &= -0.5000 < 0.3800 = |a_{3111}|r_1(\mathcal{A}). \\
|a_{3333}| \left( |a_{1111}| - r_3^{\bar{S}}(\mathcal{A}) \right) &= 4.0000 > 3.8000 = r_1(\mathcal{A})r_3^{\bar{S}}(\mathcal{A}), \\
|a_{2222}| \left( |a_{3333}| - r_3^{\bar{S}}(\mathcal{A}) \right) &= 4.8000 > 4.5000 = r_2(\mathcal{A})r_3^{\bar{S}}(\mathcal{A})
\end{align*}
\]

and

\[
|a_{3333}| \left( |a_{2222}| - r_2^{\bar{S}}(\mathcal{A}) \right) = 5.6100 > 0.7000 = r_3(\mathcal{A})r_2^{\bar{S}}(\mathcal{A}),
\]

i.e., $\mathcal{A}$ is an $S$-$SDD$ tensor. By Theorem 7, $\mathcal{A}$ is positive definite.
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