Finite difference and averaging operators in
generalized entropies

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Abstract. In the two-parameter generalization of thermostatistics based on the Sharma-Taneja-Mittal entropy not only the generalized entropic functional \(S_{a;b}\) but also a new functional \(I_{a;b}\) plays a fundamental role. These functionals are related to the finite difference and averaging operators arising in finite difference calculus.

1. Introduction

In the last decades great attention has been devoted to a variety of generalized entropies [1] from which probability distributions with power-law tails can be derived by applying the Jaynes maximum entropy principle. Among the many, some researchers have studied the two parameter family of entropies \(S_{a;b}\) given by

\[
S_{a;b}[p] \equiv -\sum_i p_i^a - \frac{p_i^b}{a - b} = -\sum_i s_{a,b}(p_i),
\]

with real parameters \(a\) and \(b\). \(S_{a;b}\) is introduced for the first time by Sharma-Taneja [2] and Mittal [3] (STM-entropy), independently in the fields of information theory, and successively reconsidered by Borges-Roditi [4] and Kaniadakis-Lissia-Scarfone [5] in the fields of statistical physics. As firstly noted in [6] and later in [4], some generalized entropies can be obtained by acting finite-difference operators on a generating function \(g^p(s) = \sum i p_i^s\). This holds true also for the entropy \(S_{a,b}\) that can be obtained by applying the two-parameter Chakrabarti and Jagannathan difference quotient operator \(D_{a,b}\) [7] (a two parameter generalization (36) of Jackson derivative) on the generating function \(g^p(s)\).

As it has been shown in [8], not only the finite-difference quotient operator but also its associated averaging operator plays a fundamental role for characterizing generalized entropies. Acting this averaging operator on the generating function \(g^p(s)\) produces the quantity

\[
I_{a;b}[p] \equiv \sum_i \frac{p_i^a + p_i^b}{2} = \sum_i s_{a,b}(p_i).
\]
In Eqs. (1) and (2), we introduced, for the sake of notational simplicity, the two functions

\[ s_{a,b}(x) \equiv \frac{x^a - x^b}{a - b} \quad \xrightarrow{(a,b) \to (1,1)} \quad x \ln(x) , \quad (3) \]

\[ t_{a,b}(x) \equiv \frac{x^b + x^b}{2} \quad \xrightarrow{(a,b) \to (1,1)} \quad x . \quad (4) \]

Of course the above limiting values don’t depend on the order of taking the limits. The functional \( I_{a,b} \) seems to play an important role in the formulation of statistical mechanics based on the entropy \( S_{a,b} \). For instance, by considering two statistically independent composite systems \( A \cup B \), with \( p^{A \cup B} = p^A p^B \), the entropy \( S_{a,b} \) can be factorized according to [9]

\[ S_{a,b} [p^{A \cup B}] = S_{a,b} [p^A] I_{a,b} [p^B] + I_{a,b} [p^A] S_{a,b} [p^B] , \quad (5) \]

and similarly for \( I_{a,b} \)

\[ I_{a,b} [p^{A \cup B}] = I_{a,b} [p^A] I_{a,b} [p^B] + \left( \frac{a - b}{2} \right) S_{a,b} [p^A] S_{a,b} [p^B] . \quad (6) \]

Moreover, the functional \( I_{a,b} \) appears repeatedly in the generalized thermodynamic relations among several quantities like, for example, the entropy \( S_{a,b} \), the free-energy \( F_{a,b} \) and the partition function \( Z_{a,b} \) [10]. In the limit of \((a, b) \to (1, 1)\), \( I_{a,b} \) reduces to unity, since from Eq. (4) we see that

\[ \lim_{(a, b) \to (1,1)} I_{a,b}[p] = \sum_i p_i = 1 . \quad (7) \]

Consequently it disappears in the standard thermostatistics.

The paper is organized as follows. In the next section 2, we introduce briefly the generalized entropy \( S_{a,b} \) and the associated Legendre structures by emphasizing the importance of the functional \( I_{a,b} \). The two-parameter finite difference and averaging operators are introduced in section 3, and then in section 4, we relate the functionals \( S_{a,b} \) and \( I_{a,b} \) with the finite difference calculus on a one-dimensional lattice. The final section 5 is devoted to our conclusions.

2. Sharma-Taneja-Mittal entropy

As it has been shown in [5], the entropy \( S_{a,b} \) can be rewritten in the form

\[ S_{a,b}[p] = - \sum_i p_i \ln_{(a,b)} (p_i) , \quad (8) \]

where the two-parameter logarithmic function

\[ \ln_{(a,b)} (x) \equiv \frac{x^{a-1} - x^{b-1}}{a - b} \quad \xrightarrow{(a,b) \to (1,1)} \quad \ln(x) , \quad (9) \]

with real parameters \( a \) and \( b \), fulfills the relations \( \ln_{(a,b)}(x) = \ln_{(b,a)}(x) = -\ln_{(2a-2a)}(1/x) \).

Formula (8) mimics the standard expression of the Boltzmann-Gibbs-Shannon entropy (recovered in the limit of \((a, b) \to (1, 1)\)) by replacing the ordinary logarithm with its two-parameter generalization.
Entropy $S_{a;b}$ includes some well known one-parameter generalized entropies as special cases. For example, setting $a = q$ and $b = 1$, we recover the Tsallis entropy \[ S_q[p] = \sum_i \frac{p_i^q - p_i}{1 - q}, \] (10)

by setting $a = q_A$ and $b = q_A^{-1}$, $S_{a;b}$ reduces to the Abe entropy \[ S_{q_A}[p] = \sum_i \frac{p_i^{q_A} - p_i^{q_A}}{q_A - q_A}, \] (11)

whilst, by setting $a = 1 - \kappa$ and $b = 1 + \kappa$, we obtain the Kaniadakis entropy \[ S_\kappa[p] = \sum_i \frac{p_i^{1-\kappa} - p_i^{1+\kappa}}{2\kappa}. \] (12)

Maximizing $S_{a;b}$, constrained by the normalization of probabilities and the linear mean energy, according to

\[
\frac{\delta}{\delta p} \left( S_{a;b}[p] - \gamma \sum_i p_i - \beta U[p] \right) = 0,
\] (13)

where

\[
U[p] = \sum_i p_i E_i,
\] (14)

leads to the following probability distribution function

\[
p_i^{\text{ME}} = \alpha \exp_{(a,b)} \left[ -\frac{1}{\lambda} (\gamma + \beta E_i) \right].
\] (15)

Here we introduced the generalized exponential $\exp_{(a,b)}(x)$ as the inverse function of the generalized logarithm

\[
\exp_{(a,b)} \left( \ln_{(a,b)}(x) \right) = \ln_{(a,b)} \left( \exp_{(a,b)}(x) \right) = x.
\] (16)

We note that, in general, the two-parameter exponential $\exp_{(a,b)}(x)$ cannot be expressed in terms of elementally analytic functions. In Eq. (15), $\gamma$ and $\beta$ are the Lagrange multipliers associated to the two constraints, whilst $\alpha$ and $\lambda$ depend on the parameters $a$ and $b$ throughout the relations

\[
\alpha = \left( \frac{a}{b} \right)^{\frac{1}{a-1}}, \quad \lambda = \left( \frac{a^{b-1}}{b^{a-1}} \right)^{\frac{1}{b-a}}.
\] (17)

They are related, each to the other, by means of

\[
a \alpha^{a-1} = b \alpha^{b-1} = \lambda, \quad \ln_{(a,b)} \left( \frac{1}{\alpha} \right) = \frac{1}{\lambda}.
\] (18)
The Legendre structure concerning the two parameter generalized entropy $S_{a,b}$ has been advanced in [10]. In particular, the generalized free energy $F_{a,b}$ and partition function $Z_{a,b}$, have been introduced according to

$$F_{a,b} = -\frac{I_{a,b} + \gamma - \left(\frac{a+b}{2} - 1\right) \beta U}{\left(\frac{a+b}{2}\right) \beta},$$

(19)

$$\ln_{(e,o)}(Z_{a,b}) = \frac{a+b}{2} \left[ I_{a,b} + \gamma - \left(\frac{a+b}{2} - 1\right) \beta U \right],$$

(20)

which satisfy formally the ordinary thermodynamical relations

$$F_{a,b} = U - \frac{1}{\beta} S_{a,b}, \quad \frac{dF_{a,b}}{d(1/\beta)} = -S_{a,b},$$

(21)

$$S_{a,b} = \ln_{(e,o)}(Z_{a,b}) + \beta U, \quad \frac{d}{d\beta} \ln_{(e,o)}(Z_{a,b}) = -U.$$  

(22)

In the limit of $(a, b) \to (1, 1)$, these mathematical structures reduce to the standard thermodynamic relations, in particular we obtain the well known relations

$$F = -\frac{1}{\beta} + \gamma, \quad \ln(Z) = 1 + \gamma,$$

(23)

for the standard free energy and the partition function of thermal systems.

3. Finite difference quotient and averaging operators

For the sake of simplicity, let us consider an one dimensional space. As well known, there are several difference operators associated with the differential operator $d/dx$. For instance, the forward, central, and backward difference operators for equally spaced points are given by

$$\Delta_{+x} f(x) = f(x + \epsilon) - f(x),$$

$$\Delta_{x} f(x) = f(x + \epsilon/2) - f(x - \epsilon/2),$$

$$\Delta_{-x} f(x) = f(x) - f(x - \epsilon),$$

(24)

respectively. Difference quotient operators are defined in terms of these quantities according to

$$D_{+x} f(x) = \frac{\Delta_{+x} f(x)}{\epsilon} = \frac{f(x + \epsilon) - f(x)}{\epsilon}, \quad \text{forward}$$

(25)

$$D_{x} f(x) = \frac{\Delta_{x} f(x)}{\epsilon} = \frac{f(x + \epsilon/2) - f(x - \epsilon/2)}{\epsilon}, \quad \text{central}$$

$$D_{-x} f(x) = \frac{\Delta_{-x} f(x)}{\epsilon} = \frac{f(x) - f(x - \epsilon)}{\epsilon}, \quad \text{backward}$$

All of these difference quotient operators reduce to the single differential operator $d/dx$ in the limit of $\epsilon \to 0$. In short, the finite difference calculus has a richness of varieties. In addition to the difference quotient operators, one can consider also the related forward, central, and backward averaging operators given by

$$M_{+x} f(x) = \frac{f(x + \epsilon) + f(x)}{2},$$

$$M_{x} f(x) = \frac{f(x + \epsilon/2) + f(x - \epsilon/2)}{2},$$

$$M_{-x} f(x) = \frac{f(x) - f(x - \epsilon)}{2},$$

(26)
respectively. We note that all these quantities reduce to the identity operator \( \hat{I} f(x) = f(x) \), in the \( \epsilon \to 0 \) limit. Therefore, in the finite difference calculus both averaging and difference quotient operators are important tools, whereas in the continuous calculus, only the differential operator \( d/dx \) is relevant.

Furthermore, for non-equally spaced points, averaging operators play an important role because it is non-commutative with respect to the corresponding difference quotient operator. In order to see this non-commutativity, let us consider the following difference quotient operator

\[
D_n f(x_n) = \frac{f(x_{n+1}) - f(x_n)}{\epsilon_n},
\]

and the corresponding averaging operator

\[
M_n f(x_n) = \frac{f(x_{n+1}) + f(x_n)}{2},
\]

for non-equally spaced points \( x_n \), with the separation \( \epsilon_n = x_{n+1} - x_n \).

Applying first the averaging operator and then the difference quotient operator to a function \( f(x_n) \) it leads to

\[
D_nM_n f(x_n) = D_n \left( \frac{f(x_{n+1}) + f(x_n)}{2} \right) = \frac{f(x_{n+2}) - f(x_{n+1})}{2 \epsilon_n} + \frac{f(x_{n+1}) - f(x_n)}{2 \epsilon_n}.
\]

Next reversing the order of the operations, we obtain

\[
M_nD_n f(x_n) = M_n \left( \frac{f(x_{n+1}) - f(x_n)}{\epsilon_n} \right) = \frac{f(x_{n+2}) - f(x_{n+1})}{2 \epsilon_{n+1}} + \frac{f(x_{n+1}) - f(x_n)}{2 \epsilon_n}.
\]

Comparing these results clearly show the non-commutativity of the two operators in a non-equally spaced lattice: \( D_nM_n \neq M_nD_n \). The origin of this non-commutativity is in the different denominators (\( \epsilon_n \) and \( \epsilon_{n+1} \) respectively) of the first terms in the right hand sides of Eqs. (29) and (30). In the case of equally spaced points, all \( \epsilon_n \) reduce to the same value \( \epsilon \), and consequently the averaging and difference quotient operators become commutative.

Finally, by considering the action of the operators \( D_n \) and \( M_n \) on the product of two functions, we can derive the following Leibniz rules

\[
D_n \left[ f(x) h(x_n) \right] = D_n f(x_n) M_n h(x_n) + M_n f(x_n) D_n h(x_n),
\]

\[
M_n \left[ f(x) h(x_n) \right] = M_n f(x_n) M_n h(x_n) + \left( \frac{\epsilon_n}{2} \right)^2 D_n f(x_n) D_n h(x_n),
\]

for arbitrary functions \( f(x) \) and \( h(x) \). These relations clearly show how these two operators are intimately related each into the other.

4. Relationship between generalized entropies and finite difference quotient operators

A possible way to obtain generalized entropies from difference quotient operators has been illustrated by Abe in [6] by introducing a generating function

\[
g^p(s) = \sum_i p_i^s.
\]

For example, by applying the Jackson derivative

\[
D^q f(x) = \frac{f(qx) - f(x)}{(q - 1)x},
\]
to the generating function (33), the Tsallis entropy is obtained as follows

\[ S[q] = -D_s^q g^p(s) \bigg|_{s=1} = \frac{1}{q-1} \sum_i p_i^q - p_i^1 . \]  

(35)

In a similar way, by applying the two-parameter difference quotient operator introduced by Chakrabarti and Jagannathan in [7]

\[ D_x^{a,b} f(x) = \frac{f(a x) - f(b x)}{(a - b) x} , \]  

(36)

Borges and Roditi [4] have derived the STM-entropy according to

\[ S_{a,b}[p] = -D_s^{a,b} g^p(s) \bigg|_{s=1} = - \sum_i p_i^a - p_i^b = - \sum_i s_{a,b}(p_i) . \]  

(37)

All these entropic forms reduce, in a suitable limit, to the standard Boltzmann-Gibbs-Shannon entropy, which is obtained as

\[ S[p] = -\frac{d}{ds} g^p(s) \bigg|_{s=1} = - \sum_i p_i \ln(p_i) . \]  

(38)

since, both Jackson derivative (for \( q \to 1 \)) and Chakrabarti-Jagannathan derivative (for \((a, b) \to (1, 1)\)) reduce to the standard differential operator \( d/ds \).

We remark that the operators \( D_x^q \) and \( D_x^{a,b} \) are generalizations of the difference quotient operator (27) in a non-equally spaced lattice. In fact, for instance, by setting \( x_n = b x \), \( x_{n+1} = a x \) in Eq. (27) (so that \( \epsilon_n = (a - b) x \)) we obtain

\[ D_n f(x) = \frac{f(a x) - f(b x)}{(a - b) x} = D_x^{a,b} f(x) . \]  

(39)

In a similar way, we can emphasized the role of the functional \( \mathcal{I}_{a,b} \) which can be obtained by applying the two-parameter averaging operator \( M_s^{a,b} \) to the same generating function \( g^p(s) \)

\[ \mathcal{I}_{a,b}[p] = M_s^{a,b} g^p(s) \bigg|_{s=1} = \sum_i \frac{p_i^a + p_i^b}{2} = \sum_i \mathcal{I}_{a,b}(p_i) . \]  

(40)

Here, the averaging operator \( M_x^{a,b} \) is also the generalization of the averaging operator (28) in a non-equally spaced lattice.

4.1. Leibniz rule and the generalized additivity
As shown in [8], the generalized additivity of \( S_{a,b} \) is rooted in the Leibniz product rule concerning with the corresponding difference quotient operator. For the Chakrabarti-Jagannathan derivative \( D_x^{a,b} \), the Leibniz rule can be written, according to Eq. (31), in the following symmetric form

\[ D_x^{a,b} \left[ f(x) h(x) \right] = D_x^{a,b} f(x) \ M_x^{a,b} h(x) + M_x^{a,b} f(x) \ D_x^{a,b} h(x) . \]  

(41)

Therefore, by acting \( D_s^{a,b} \) on the generating function \( g^{q^a_n}(s) = \sum_{i,j} p_{ij}^a = \sum_i p_i^a \sum_j p_{ji}^b \), we obtain

\[ - \sum_{i,j} D_s^{a,b} p_{ij}^a \bigg|_{s=1} = - \sum_i D_s^{a,b} p_i^a \sum_j M_s^{a,b} p_{ji}^b \bigg|_{s=1} - \sum_i M_s^{a,b} p_i^a \sum_j D_s^{a,b} p_{ji}^b \bigg|_{s=1} . \]  

(42)
By recalling the definitions (3) and (4), we can write

\[ S_{a;b}[p^{A\cup B}] = \sum_i s_{a;b}(p_i) \sum_j t_{a;b}(p_{j|i}) + \sum_j s_{a;b}(p_{j|i}) \sum_i t_{a;b}(p_i) , \]  

(43)

that is the two-parameter generalization of Shannon additivity [13, 14] for the STM-entropy. In the limit of \((a, b) \to (1, 1)\), this equation reduces to the standard Shannon additivity [13],

\[ S[p^{A\cup B}] = S[p^A] + \sum_i p_i \sum_j s(p_{j|i}) , \]  

(44)

whilst, for statistically independent systems with \(p_{ij} = p_ip_j\), it reduces to Eq. (5). Thus, the generalized additivity for the two-parameter generalized entropies \(S_{a;b}\) is nothing but a consequence of the Leibniz rule for the two-parameter difference quotient operator \(D^a_b\).

In a similar way, from the relation

\[ M^a_b[f(x)h(x)] = M^a_b[f(x)] M^a_b[h(x)] + \left(\frac{a-b}{2}\right)^2 D^a_b[f(x)] D^a_b[h(x)] , \]  

(45)

we can obtain the additivity rule for the functional \(I_{a;b}\) by the action of \(M^a_b\) on the generator \(g^{A\cup B}(s)\),

\[ \sum_{i,j} M^a_b p_{ij}^s \big|_{s=1} = \sum_i M^a_b p_i^s \big|_{s=1} \sum_j M^a_b p_{j|i}^s \big|_{s=1} + \left(\frac{a-b}{2}\right)^2 \sum_i D^a_b p_i^s \big|_{s=1} \sum_j D^a_b p_{j|i}^s \big|_{s=1} . \]  

(46)

As a consequence, we can derive the relation

\[ I_{a;b}[p^{A\cup B}] = \sum_i t_{a;b}(p_i) \sum_j t_{a;b}(p_{j|i}) + \left(\frac{a-b}{2}\right)^2 \sum_i s_{a;b}(p_i) \sum_j s_{a;b}(p_{j|i}) , \]  

(47)

which reduces to Eq. (6) for statistically independent systems. In the limit of \((a, b) \to (1, 1)\), this last relation reduces to a merely trivial identity: \(1 = 1\).

5. Conclusion

In the two-parameter generalization of thermostatistics based on the Sharma-Taneja-Mittal entropy, we have shown that not only the entropic functional \(S_{a;b}\) but also the complementary functional \(I_{a;b}\) plays a fundamental role. As it has been shown, the functional \(I_{a;b}\) disappears in the limit of \((a, b) \to (1, 1)\). This fact suggest that a consistent formulation of the statistical mechanics for complex systems could require further “informations” carried out by the functional \(I_{a;b}\), with respect to the ”ingredients” which are essential to describe ideal and non-interacting systems. Although the physical meaning of the functional \(I_{a;b}\) is still not yet clear, there are some evidences about its relevance in the developed of a generalized thermostatistics based on the entropic form \(S_{a;b}\).

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