IDEMPOTENT IDEALS AND
THE IGUSA-TODOROV FUNCTIONS

A. GATICA, M. LANZILOTTA, M. I. PLATZECK

ABSTRACT. Let $\Lambda$ be an artin algebra and $\mathfrak{A}$ a two-sided idempotent ideal of $\Lambda$, that is, $\mathfrak{A}$ is the trace of a projective $\Lambda$-module $P$ in $\Lambda$. We consider the categories of finitely generated modules over the associated rings $\Lambda/\mathfrak{A}, \Lambda$ and $\Gamma = \text{End}_\Lambda(P)^{op}$ and study the relationship between their homological properties via the Igusa-Todorov functions.

1. Introduction

Throughout this paper we assume that $\Lambda$ is an artin algebra and all $\Lambda$-modules are in $\text{mod}\Lambda$, the category of finitely generated left $\Lambda$-modules.

In [8] Igusa and Todorov introduced two functions $\phi$ and $\psi$ which turned out to be powerful tools to study the finitistic dimension of some classes of algebras. On the other hand, associated to an idempotent ideal $\mathfrak{A}$ of $\Lambda$, there is an exact sequence of categories $\text{mod}\Lambda/\mathfrak{A} \to \text{mod}\Lambda \leftarrow \text{mod}\Gamma$, where $P$ is a projective module such that $\mathfrak{A} = \tau_P \Lambda$ is the trace of $P$ in $\Lambda$, $\Gamma = \text{End}_\Lambda(P)^{op}$ and $e_P = \text{Hom}_\Lambda(P, -)$ is the evaluation functor. In [1] the authors studied the relation between the homological properties of the three categories involved: $\text{mod}\Lambda/\mathfrak{A}, \text{mod}\Lambda$ and $\text{mod}\Gamma$. Our objective in this paper is to study the behaviour of the Igusa-Todorov functions in this situation. For a finitely generated $\Lambda$-module $X$, we will denote $\phi(X)$ by $\phi^\Lambda_l(X)$, and the supremum of these numbers for $X$ in $\text{mod}\Lambda$ is the $\phi_l$ dimension of $\Lambda$, denoted by $\phi_l \text{dim}(\Lambda)$. Additionally, $\text{add}X$ denotes the full subcategory of $\text{mod}\Lambda$ consisting of summands of finite direct sums of $X$.

First we consider the inclusion of $\text{mod}\Lambda/\mathfrak{A}$ in $\text{mod}\Lambda$. To compare the values of the Igusa-Todorov functions in a $\Lambda/\mathfrak{A}$-module $X$ in both categories we need the further assumption that the idempotent ideal $\mathfrak{A}$ is a strong idempotent ideal, in the sense defined in [1]. We recall that the ideal $\mathfrak{A}$ is a strong idempotent ideal if the morphism $\text{Ext}^i_{\Lambda/\mathfrak{A}}(X, Y) \to \text{Ext}^i_\Lambda(X, Y)$ induced by

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the canonical isomorphism $\text{Hom}_{\Lambda/\mathfrak{A}}(X, Y) \to \text{Hom}_{\Lambda}(X, Y)$ is an isomorphism for all $i \geq 0$ and all $X, Y$ in $\text{mod}\Lambda/\mathfrak{A}$. We prove that $\phi^\Lambda_{\mathfrak{A}}(X) \leq \phi^\Lambda(X)$ for all $X \in \text{mod}\Lambda/\mathfrak{A}$, whenever $\mathfrak{A}$ is a strong idempotent ideal of finite projective dimension. Thus in this case the $\phi_l$ dimension of $\Lambda/\mathfrak{A}$ is bounded by the $\phi_l$ dimension of $\Lambda$.

In order to compare the behaviour of the Igusa-Todorov functions under the functor $\text{mod}\Lambda \xrightarrow{\varepsilon P} \text{mod}\Gamma$, we recall that $\varepsilon P$ induces an equivalence between the full subcategory of $\text{mod}\Lambda$ consisting of the $\Lambda$-modules $X$ having a presentation in $\text{add}P$, and $\text{mod}\Gamma$. We prove that both functions $\phi$ and $\psi$ are preserved under $\varepsilon P$ for modules having a resolution in $\text{add}P$. As a consequence we obtain that when all $\Lambda$-modules with a presentation in $\text{add}P$ have also a resolution in $\text{add}P$, then $\phi_l\dim\Gamma \leq \phi_l\dim\Lambda$ and $\psi_l\dim\Gamma \leq \psi_l\dim\Lambda$ (Theorem 4.7).

Then we obtain information about the $\phi$ dimension of $\Lambda$ from the $\phi$ dimensions of the algebras $\Lambda/\mathfrak{A}$ and $\Gamma$. We prove several inequalities, which are interesting when either the global dimension of $\Lambda/\mathfrak{A}$ or the global dimension of $\Gamma$ are finite.

To prove these results we use, in one hand, the characterization of the Igusa-Todorov function $\phi$ in terms of the bifunctor $\text{Ext}(\cdot, \cdot)$ given in [5]. On the other hand, the full subcategory $T$ of $\text{mod}\Lambda$ introduced in [1] consisting of the modules $T$ such that $\text{Ext}^i_{\Lambda}(\Lambda/\mathfrak{A}, T) = 0$ for all $i \geq 1$, is very useful for our purposes. Consider the full subcategories $P_0$ and $P_\infty$ of $\text{mod}\Lambda$, where $P_0$ consists of the modules whose projective cover is in $\text{add}P$, and $P_\infty$ of those having a projective resolution in $\text{add}P$. We use the fact, proven in section 3, that $(P_0, \text{mod}\Lambda/\mathfrak{A})$ is a torsion pair in $\text{mod}\Lambda$ whose properties are inherited by the pair $(P_\infty, \text{mod}\Lambda/\mathfrak{A})$ in the category $\tilde{T}$ dual of $T$.

2. Preliminaries

Let $\Lambda$ be an artin algebra, $M$ and $N$ in $\text{mod}\Lambda$. We denote by $\tau_M N$ the trace of $M$ in $N$, that is, the submodule of $N$ generated by the homomorphic images of maps from $M$ to $N$. Moreover, $P_0(M)$, $I_0(M)$ denote the projective cover and injective envelope of $M$, and $\Omega^n(M)$, $\Omega^{-n}(M)$ the $n^{th}$ syzygy and the $n^{th}$ cosyzygy of $M$, respectively. Finally, $\text{pd}M$ denotes the projective dimension of $M$ and $\text{gld}\Lambda$ stands for the global dimension of $\Lambda$.

We start by recalling some definitions and results from [1] which will be used throughout the paper. Let $\mathfrak{A}$ be an idempotent ideal of $\Lambda$, $P_0$ the projective cover of $\mathfrak{A}$, and $P = \Lambda e$ where $e$ is an idempotent element of $\Lambda$ such that $\text{add}P = \text{add}P_0$. Then $\mathfrak{A} = \Lambda e \Lambda = \tau_P \Lambda$ is the trace of $P$ in $\Lambda$, $\text{mod}\Lambda/\mathfrak{A}$ is a Serre subcategory of $\text{mod}\Lambda$ and this inclusion induces an exact sequence of categories $\text{mod}\Lambda/\mathfrak{A} \to \text{mod}\Lambda \xrightarrow{\varepsilon P} \text{mod}\Gamma$, where $\Gamma = \text{End}_{\Lambda}(P)^{op}$ and $\varepsilon P = \text{Hom}_{\Lambda}(P, -)$ is the evaluation functor.
These subcategories will be useful for our purposes, and are defined as follows: \( \mathbb{P}_k \) is the full subcategory of \( \text{mod}\Lambda \) consisting of the \( \Lambda \)-modules \( X \) having a projective resolution \( \cdots \to P_1 \to P_0 \to X \to 0 \) with \( P_i \) in \( \text{add}\mathbb{P}_k \) for \( 0 \leq i \leq k \). The full subcategory \( \mathbb{I}_k \) is defined dually.

Then \( \text{Hom}_\Lambda(P, -) \) induces equivalences \( \mathbb{P}_1 \to \text{mod}\Gamma \) and \( \mathbb{I}_1 \to \text{mod}\Gamma \). Moreover, the morphism of connected sequences of functors \( \text{Ext}^i_\Lambda(X, Y) \to \text{Ext}^i_\Gamma((P, X), (P, Y)) \) induced by \( \text{Hom}_\Lambda(P, -) \) is an isomorphism for \( i = 1, \cdots, k \), whenever \( X \in \mathbb{P}_{(k+1)} \) or \( Y \in \mathbb{I}_{(k+1)} \) (Theorem 3.2, [1]).

We next turn our attention to the definition of the Igusa-Todorov functions, defined in [5]. Let \( K_0 \) denote the abelian group generated by all symbols \([M]\), where \( M \) in \( \text{mod}\Lambda \), modulo the relations a) \( |C| = |A| + |B| \) if \( C \simeq A \oplus B \) and b) \( |P| = 0 \) if \( P \) is projective. That is, \( K_0 \) is the free abelian group generated by the isomorphism classes of indecomposable finitely generated nonprojective \( \Lambda \)-modules. Let \( \Omega : K_0 \to K_0 \) denote the group homomorphism induced by the syzygy, that is, \( \Omega([M]) := [\Omega(M)] \), and let \( < \text{add}\mathbb{M} > \) be the subgroup of \( K_0 \) generated by the indecomposable summands of \( M \). When we apply the homomorphism \( \Omega \) to this subgroup the rank does not increase: \( \text{rank} \Omega(< \text{add}\mathbb{M} >) \leq \text{rank} < \text{add}\mathbb{M} > \), and there is then an integer \( n \) such that \( \Omega : \Omega^s(< \text{add}\mathbb{M} >) \to \Omega^{s+1}(< \text{add}\mathbb{M} >) \) is an isomorphism for all \( s \geq n \). Then the Igusa-Todorov functions \( \phi \) and \( \psi \) are defined as follows: \( \phi(M) \) is the smallest non-negative integer \( n \) with this property, and \( \psi(M) := \phi(M) + \sup\{\text{pd}X | X \text{ is a direct summand of } \Omega^{p(M)}(M) \text{ with } \text{pd}X < \infty\} \). Since we will need also the dual notions, we will denote the Igusa-Todorov functions \( \phi \) and \( \psi \) by \( \phi_\tau \) and \( \psi_\tau \), respectively. Using the cozyzygy we can define \( \phi_r(M) \) and \( \psi_r(M) \) in an analogous way. Then \( \phi_r(M) = \phi_\tau(DM) \) and \( \psi_r(M) = \psi_\tau(DM) \), for any \( M \) in \( \text{mod}\Lambda \).

Let \( \phi_{\text{dim}\Lambda} = \sup\{\phi(M) | M \in \text{mod}\Lambda\} \) and \( \psi_{\text{dim}\Lambda} = \sup\{\psi(M) | M \in \text{mod}\Lambda\} \). Moreover, for a subcategory \( \mathcal{X} \) of \( \text{mod}\Lambda \) we indicate by \( \phi_\tau\text{dim}\mathcal{X} \) and \( \psi_\tau\text{dim}\mathcal{X} \) the supremum of the sets \( \{\phi(X) | X \in \mathcal{X}\} \) and \( \{\psi(X) | X \in \mathcal{X}\} \), respectively. Analogous notions are defined for \( \phi_r \) and \( \psi_r \).

We will also need the characterization of the function \( \phi \) in terms of the bifunctor \( \text{Ext}^d_\Lambda(-, -) \) given in [5]. We recall first that a pair \( (X, Y) \) of objects in \( \text{add}\mathbb{M} \) is called \( d \)-division of \( M \) if the following three conditions hold:

(a) \( \text{add}(X) \cap \text{add}(Y) = 0 \)
(b) \( \text{Ext}^d_\Lambda(X, -) \not\simeq \text{Ext}^d_\Lambda(Y, -) \) in \( \text{mod}\Lambda \)
(c) \( \text{Ext}^{d+1}_\Lambda(X, -) \simeq \text{Ext}^{d+1}_\Lambda(Y, -) \) in \( \text{mod}\Lambda \).

Dually, a pair \( (X, Y) \) of objects in \( \text{add}\mathbb{M} \) is called \( d \)-injective division of \( M \) if (a) and the following two conditions hold:

(b') \( \text{Ext}^d_\Lambda(-, X) \not\simeq \text{Ext}^d_\Lambda(-, Y) \) in \( \text{mod}\Lambda \)
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(c’) \( \text{Ext}^{d+1}_\Lambda(\mathbf{-}, X) \simeq \text{Ext}^{d+1}_\Lambda(\mathbf{-}, Y) \) in \( \text{mod}\Lambda \).

Then \( \phi_l(M) = \max \{ \{d \in \mathbb{N} : \text{there is a } d\text{-division of } M \} \cup \{0\} \} \) \( (5) \), Theorem 3.6), and \( \phi_r(M) = \max \{ \{d \in \mathbb{N} : \text{there is a } d\text{-injective division of } M \} \cup \{0\} \} \).

3. Torsion theories associated to an idempotent ideal

It is interesting to notice that the idempotent ideal \( \mathfrak{A} \) determines two torsion pairs \( (\text{mod}\Lambda/\mathfrak{A}, I_0) \) and \( (P_0, \text{mod}\Lambda/\mathfrak{A}) \) in \( \text{mod}\Lambda \), in the sense defined by Dickson in \( [4] \), as we state in the following proposition.

Proposition 3.1. Let \( \mathfrak{A} \) be an idempotent ideal of \( \Lambda \), \( \mathfrak{A} = \tau P \Lambda \), where \( P \) is a projective \( \Lambda \)-module. Then

(a) \( (\text{mod}\Lambda/\mathfrak{A}, I_0) \) is a torsion pair in \( \text{mod}\Lambda \).

(b) \( (P_0, \text{mod}\Lambda/\mathfrak{A}) \) is a torsion pair in \( \text{mod}\Lambda \).

Proof. (a) To prove this we observe that \( I_0 \) consists of the modules with socle in \( \text{add}(S_1 \oplus \cdots \oplus S_r) \), and that a \( \Lambda \)-module \( M \) is in \( \text{mod}\Lambda/\mathfrak{A} \) if and only if \( S_1, \cdots, S_r \) are not composition factors of \( M \). Then \( \text{Hom}_\Lambda(M, Y) = 0 \) for \( M \in \text{mod}\Lambda/\mathfrak{A} \) and \( Y \in I_0 \). Moreover, if \( \text{Hom}_\Lambda(M, Y) = 0 \) for all \( Y \in I_0 \), then in particular \( \text{Hom}_\Lambda(M, I_0(S_1 \oplus \cdots \oplus S_r)) = 0 \), so that \( S_1, \cdots, S_r \) are not composition factors of \( M \), and \( M \) is thus a \( \Lambda/\mathfrak{A} \)-module. Finally, suppose that \( \text{Hom}_\Lambda(M, Y) = 0 \) for each \( M \in \text{mod}\Lambda/\mathfrak{A} \). Then \( \text{Hom}_\Lambda(S, Y) = 0 \) for any simple \( S \) not isomorphic to \( S_1, \cdots, S_r \). Thus the only simples in the socle of \( Y \) are amongst \( S_1, \cdots, S_r \), and therefore \( Y \in I_0 \). This shows that \( (\text{mod}\Lambda/\mathfrak{A}, I_0) \) is a torsion pair in \( \text{mod}\Lambda \).

The statement (b) follows by duality. \( \square \)

In the sequel we will consider the full subcategory \( T \) of \( \text{mod}\Lambda \) introduced and studied in section 5 of \( [1] \), consisting of the modules \( T \) such that the group \( \text{Ext}^i_\Lambda(\mathfrak{A}/\mathfrak{A}, T) = 0 \) for all \( i \geq 1 \). Dually, we define the subcategory \( \overline{\text{T}} = D(\text{T}_\Lambda) \) consisting of the \( \Lambda \)-modules \( X \) such that \( \text{Ext}^i_\Lambda(X, D(\mathfrak{A}/\mathfrak{A})) = 0 \) for all \( i \geq 1 \).

The notion of torsion pairs in abelian categories defined by Dickson was extended to pretriangulated categories by Beligianis and Reiten (see \( [2] \), Ch. II, Definition 3.1). Additive categories with kernels and cokernels are examples of pretriangulated categories, as shown in section 1, Example 2 of the same paper, and in this case torsion pairs are defined as follows.

Definition 3.2. \( (2) \) A pair of subcategories \( (\mathcal{X}, \mathcal{Y}) \) in an additive category \( \mathcal{C} \) with kernels and cokernels and closed under isomorphisms is a torsion pair if the following conditions hold:

T1) \( \text{Hom}_\mathcal{C}(X, Y) = 0 \) for all \( X \in \mathcal{X}, \ Y \in \mathcal{Y} \)

T2) For every \( C \in \mathcal{C} \) there is an exact sequence \( 0 \rightarrow X_C \rightarrow C \rightarrow Y_C \rightarrow 0 \) with \( X_C \in \mathcal{X}, \ Y_C \in \mathcal{Y} \) (glueing sequence).
We observe that for the torsion pairs \((\text{mod} \Lambda / \mathfrak{A}, \mathbb{I}_0)\) and \((\mathbb{P}_0, \text{mod} \Lambda / \mathfrak{A})\) above considered the glueing sequences for a module \(X\) in \(\text{mod} \Lambda\) are \(0 \to \tau_{\Lambda/\mathfrak{A}} X \to X \to X/\tau_{\Lambda/\mathfrak{A}} X \to 0\) and \(0 \to \tau_{\mathfrak{A}} X \to X \to X/\tau_{\mathfrak{A}} X \to 0\) respectively.

We now turn our attention to the subcategories \(\mathbb{T}\) and \(\bar{\mathbb{T}}\) of \(\text{mod} \Lambda\), and study the restriction of these torsion pairs to \(\mathbb{T}\) and \(\bar{\mathbb{T}}\) respectively, under the assumption that the ideal \(\mathfrak{A}\) is strong idempotent.

**Proposition 3.3.** Let \(\mathfrak{A}\) be a strong idempotent ideal of \(\Lambda\). Then

(a) \(\mathbb{I}_\infty = \mathbb{I}_0 \cap \mathbb{T}\), and the pair \((\text{mod} \Lambda / \mathfrak{A}, \mathbb{I}_\infty)\) of subcategories of \(\mathbb{T}\) satisfies conditions T1) and T2) of Definition 3.2 of torsion pair. Moreover, for \(T\) in \(\mathbb{T}\) the glueing sequence is \(0 \to \tau_{\Lambda/\mathfrak{A}} T \to T \to T/\tau_{\Lambda/\mathfrak{A}} T \to 0\).

(b) \(\mathbb{P}_\infty = \mathbb{P}_0 \cap \bar{\mathbb{T}}\), and the pair \((\text{mod} \Lambda / \mathfrak{A}, \mathbb{P}_\infty)\) of subcategories of \(\bar{\mathbb{T}}\) satisfies conditions T1) and T2) of Definition 3.2 of torsion pair. Moreover, for \(T \in \bar{\mathbb{T}}\) the glueing sequence is \(0 \to \tau_{\mathfrak{A}} T \to T \to T/\tau_{\mathfrak{A}} T \to 0\).

**Proof.** Since \(\mathbb{I}_\infty \subseteq \mathbb{I}_0\) then condition T1) in the definition of torsion pair holds.

Assume now that \(\mathfrak{A}\) is a strong idempotent ideal. Then \(\text{mod} \Lambda / \mathfrak{A} \subset \mathbb{T}\). In fact, if \(X \in \text{mod} \Lambda / \mathfrak{A}\) then \(\text{Ext}^i_{\Lambda}(\Lambda / \mathfrak{A}, X) \cong \text{Ext}^i_{\Lambda/\mathfrak{A}}(\Lambda / \mathfrak{A}, X) = 0\) for all \(i \geq 0\), so \(X \in \mathbb{T}\). On the other hand, we know that a module \(Y\) is in \(\mathbb{I}_\infty\) if and only if \(\text{Ext}^i_{\Lambda}(\Lambda / \mathfrak{A}, Y) = 0\) for all \(i \geq 0\), by [1], Proposition 2.6. Thus, it follows from the definition of \(\mathbb{T}\) that \(\mathbb{I}_\infty = \mathbb{I}_0 \cap \mathbb{T}\).

Therefore, for \(T \in \mathbb{T}\) the exact sequence \(0 \to \tau_{\Lambda/\mathfrak{A}} T \to T \to T/\tau_{\Lambda/\mathfrak{A}} T \to 0\) has \(\tau_{\Lambda/\mathfrak{A}} T\) in \(\text{mod} \Lambda / \mathfrak{A}\) and \(T/\tau_{\Lambda/\mathfrak{A}} T\) in \(\mathbb{I}_\infty\) and is then a glueing sequence for \(T\). This proves condition T2) and ends the proof of (a). The proof of (b) is similar.

\[\square\]

**Remark 3.4.** Though we do not know whether the subcategories \(\mathbb{T}\) and \(\bar{\mathbb{T}}\) have kernels and cokernels, we observe that the category \(\mathbb{T}\) is not in general abelian as the following simple example shows. Let \(\Lambda\) be the path algebra of the quiver \(1 \to 2\), and \(P = S_2\), the simple projective module associated to the vertex 2. Then, \(\mathfrak{A} = \tau_P \Lambda \simeq S_2 \oplus S_2\) is a projective \(\Lambda\) module and therefore it is a strong idempotent ideal. Moreover, \(\mathbb{T} = \text{add}\{S_1/S_2, S_1\}\) and \(\Lambda/\mathfrak{A} \simeq S_1\).

Consider the exact sequence \(0 \to S_2 \to S_1 \xrightarrow{f} S_1 \to 0\). Then \(f\) is a map in \(\mathbb{T}\), and \(\text{Ker}_T(f) = 0\), because there are no nonzero maps from objects in \(\mathbb{T}\) to \(S_2\). Thus \(\text{Coker}_T(\text{Ker}_T(f)) = \text{Coker}_T(0 \to S_1/S_2) = (S_1/S_2 \xrightarrow{id} S_1/S_2)\). However, \(\text{Ker}_T(\text{Coker}_T(f)) = \text{Ker}_T(S_1 \to 0) = (S_1 \xrightarrow{id} S_1)\).

In connection with the torsion pairs and subcategories above considered we prove two technical lemmas which will be useful throughout the paper.

**Lemma 3.5.** Let \(\mathfrak{A}\) be an idempotent ideal. Then
Lemma 3.6. Let \( \mathfrak{A} \) be a strong idempotent ideal. Then
\[
\begin{align*}
(a) \quad \text{Ext}^j_{\mathfrak{A}}(-, X/\tau_{\mathfrak{A}}X)_{|_{\text{mod}\Lambda/\mathfrak{A}}} = 0 \quad &\text{for all } X \in \mathcal{T} \text{ and } j \geq 1. \\
(b) \quad \text{Ext}^j_{\mathfrak{A}}(-, X)_{|_{\text{mod}\Lambda/\mathfrak{A}}} = 0 \quad &\text{for all } X \in \mathcal{T} \text{ and } j \geq 1.
\end{align*}
\]

Proof. (a) The result follows directly by applying the exact functor \( \text{Hom}_\Lambda(P, -) \) to the exact sequence \( 0 \to \tau_{\mathfrak{A}}X \to X \to X/\tau_{\mathfrak{A}}X \to 0 \).

(b) We recall from [1], Theorem 3.2 c), that there is a functorial isomorphism \( \text{Ext}^j_{\mathfrak{A}}(-, Y)_{|_{\text{mod}\Lambda/\mathfrak{A}}} \simeq \text{Ext}^j_{\mathfrak{A}}((-P, -),(P, Y))_{|_{\text{mod}\Lambda/\mathfrak{A}}} \) for all \( Y \) in \( \text{mod}\Lambda/\mathfrak{A} \). The result follows now using (a).

By duality we obtain the statements (c) and (d). \( \square \)

When we further assume that the ideal \( \mathfrak{A} \) is strong idempotent we get the following result.

**Lemma 3.6.** Let \( \mathfrak{A} \) be a strong idempotent ideal. Then
\[
\begin{align*}
(a) \quad \text{Ext}^j_{\mathfrak{A}}(-, X/\tau_{\mathfrak{A}}X)_{|_{\text{mod}\Lambda/\mathfrak{A}}} = 0 \quad &\text{for all } X \in \mathcal{T} \text{ and } j \geq 1. \\
(b) \quad \text{Ext}^j_{\mathfrak{A}}(-, X)_{|_{\text{mod}\Lambda/\mathfrak{A}}} = 0 \quad &\text{for all } X \in \mathcal{T} \text{ and } j \geq 1.
\end{align*}
\]

Proof. (a) Let \( X \) in \( \mathcal{T} \) and \( Z \) in \( \text{mod}\Lambda/\mathfrak{A} \). Then \( X/\tau_{\mathfrak{A}}X \in \mathcal{I}_\infty \) and using (d) of the previous lemma we conclude that \( \text{Ext}^j_{\mathfrak{A}}(Z/\mathfrak{A}, X/\tau_{\mathfrak{A}}X) \simeq \text{Ext}^j_{\mathfrak{A}}(\tau_{\mathfrak{A}}Z, X/\tau_{\mathfrak{A}}X) = 0 \), since \( \tau_{\mathfrak{A}}Z = 0 \) because \( Z \) is a \( \Lambda/\mathfrak{A} \)-module.

(b) Let \( X_1, X_2 \in \mathcal{T} \) be such that \( \text{Ext}^j_{\mathfrak{A}}(-, X_1) \simeq \text{Ext}^j_{\mathfrak{A}}(-, X_2) \). So \( L_i = X_1/\tau_{\mathfrak{A}}X_1, I_{i2} = X_2/\tau_{\mathfrak{A}}X_2 \in \mathcal{I}_\infty \). Then we obtain from (a) that \( \text{Ext}^j_{\mathfrak{A}}(-, L_i)_{|_{\text{mod}\Lambda/\mathfrak{A}}} = 0 \) for \( i = 1, 2 \) and for all \( j \geq 1 \).

Let \( Z \in \mathcal{T} \). Applying the functor \( \text{Hom}_\Lambda(-, L_i) \) to the glueing sequence
\[
0 \to \tau_{\mathfrak{A}}Z \to Z \to Z/\tau_{\mathfrak{A}}Z \to 0
\]
the corresponding long exact sequence yields isomorphisms
\[
\text{Ext}^j_{\mathfrak{A}}(Z, L_i) \simeq \text{Ext}^j_{\mathfrak{A}}(\tau_{\mathfrak{A}}Z, L_i),
\]
for \( i = 1, 2 \) and \( j \geq 1 \).

On the other hand, by (b) of the previous lemma we know that \( \text{Ext}^j_{\mathfrak{A}}(\tau_{\mathfrak{A}}Z, X_i) \simeq \text{Ext}^j_{\mathfrak{A}}(\tau_{\mathfrak{A}}Z, L_i) \) for \( i = 1, 2, j \geq 1 \) because \( \mathfrak{A} \) is a strong idempotent ideal, so \( Z \) in \( \mathcal{T} \) implies that \( Z \) is a \( \Lambda/\mathfrak{A} \)-module. Then, in the commutative diagram
Let $\phi$ be bounded by the finite projective dimension then the $\text{Ext}$ implies $\text{Ext}$. Lemma 4.1. Let $X_1, X_2 \in \text{mod} \Lambda$ and $t \geq 1$. Then $\text{Ext}^t_{\Lambda/\mathfrak{A}}(X_1, -) \simeq \text{Ext}^t_{\Lambda/\mathfrak{A}}(X_2, -)$ implies $\text{Ext}^{t+r}_{\Lambda}(X_1, -) \simeq \text{Ext}^{t+r}_{\Lambda}(X_2, -)$.

**Proof.** Note first that $\text{Ext}^t_{\Lambda/\mathfrak{A}}(X_j, -) \simeq \text{Ext}^t_{\Lambda}(X_j, \Omega^{-r}(-))$, for $j = 1, 2$. Since $\Omega^{-r}(\text{mod} \Lambda) \subset \mathcal{T}$, from Lemma 5.5, [1], using Proposition 1.1, [1], it follows that $\text{Ext}^t_{\Lambda/\mathfrak{A}}(X_j, \mathfrak{A}(\Omega^{-r}(-))) \simeq \text{Ext}^t_{\Lambda}(X_j, \Omega^{-r}(-))$, for all $1 \leq i$, and $j = 1, 2$.

Now $\text{Ext}^{t+r}_{\Lambda/\mathfrak{A}}(X_1, -) \simeq \text{Ext}^{t}_{\Lambda}(X_1, \Omega^{-r}(-)) \simeq \text{Ext}^{t}_{\Lambda/\mathfrak{A}}(X_1, \mathfrak{A}(\Omega^{-r}(-))) \simeq \text{Ext}^{t}_{\Lambda/\mathfrak{A}}(X_2, \mathfrak{A}(\Omega^{-r}(-))) \simeq \text{Ext}^{t+r}_{\Lambda}(X_2, -)$.

**Lemma 4.2.** Let $\mathfrak{A}$ be an idempotent ideal such that $\text{pd}(\Lambda/\mathfrak{A}) = r < \infty$. Let $X_1, X_2 \in \text{mod} \Lambda$ and $t \geq 1$. Then $\text{Ext}^t_{\Lambda/\mathfrak{A}}(-, X_1) \simeq \text{Ext}^t_{\Lambda/\mathfrak{A}}(-, X_2)$ implies $\text{Ext}^{t+r}_{\Lambda}(-, X_1) \simeq \text{Ext}^{t+r}_{\Lambda}(-, X_2)$.

**Proof.** It follows from Lemma 4.1 by duality, using that $\text{Ext}^j_{\Lambda/\mathfrak{A}}(-, Y) \simeq \text{Ext}^j_{\Lambda/\mathfrak{A}}(DY, D(-))$.

We prove next that when the ideal $\mathfrak{A}$ is a strong idempotent ideal of finite projective dimension then the $\phi$ dimension of the factor algebra $\Lambda/\mathfrak{A}$ is bounded by the $\phi$ dimension of $\Lambda$.

**Theorem 4.3.** Let $\mathfrak{A}$ be a strong idempotent ideal of $\Lambda$ such that $\text{pd}(\Lambda/\mathfrak{A}) = r < \infty$. Then

(a) $\phi^\Lambda_{\Lambda/\mathfrak{A}}(X) \leq \phi^\Lambda_\Lambda(X)$ for all $X \in \text{mod} \Lambda$.

(b) $\phi_{\text{dim} \Lambda/\mathfrak{A}} \leq \phi_{\text{dim} \Lambda}$.
Proof. Let $d$ a positive integer and assume that $X = X_1 \oplus X_2$ is a $d$-division of the $\Lambda/\mathfrak{A}$-module $X$. This is, $\text{Ext}^d_{\Lambda/\mathfrak{A}}(X_1, -) \not\cong \text{Ext}^d_{\Lambda/\mathfrak{A}}(X_2, -)$ and $\text{Ext}^{d+1}_{\Lambda/\mathfrak{A}}(X_1, -) \cong \text{Ext}^{d+1}_{\Lambda/\mathfrak{A}}(X_2, -)$. Since $\mathfrak{A}$ is a strong idempotent ideal then $\text{Ext}^d_{\Lambda}(X_1, -) \not\cong \text{Ext}^d_{\Lambda}(X_2, -)$. On the other hand, we know that $\phi_l(X) = \max\{d \in \mathbb{N} : \text{there is a } d\text{-division of } X \text{ in } \text{mod} \Lambda \cup \{0\}\}$, by Theorem 3.6 in [5]. Thus to prove that $\phi_l(X) \geq d$, it is enough to find $l > d$ such that $\text{Ext}^l_{\Lambda}(X_1, -) \cong \text{Ext}^l_{\Lambda}(X_2, -)$. In fact, if $l_0$ is minimal with this property, then $X = X_1 \oplus X_2$ is an $(l_0 - 1)$-division of the $\Lambda$-module $X$.

Since we assumed that $\text{Ext}^{d+1}_{\Lambda/\mathfrak{A}}(X_1, -) \cong \text{Ext}^{d+1}_{\Lambda/\mathfrak{A}}(X_2, -)$ then, using Lemma 4.1, we obtain that $\text{Ext}^{d+r+1}_{\Lambda}(X_1, -) \cong \text{Ext}^{d+r+1}_{\Lambda}(X_2, -)$.

Hence, $l = d + r + 1 > d$ satisfies $\text{Ext}^l_{\Lambda}(X_1, -) \cong \text{Ext}^l_{\Lambda}(X_2, -)$. So we found $l$ as required, proving that $\phi_l(X) \geq d$. This proves (a), and (b) follows immediately. \hfill \Box

We observe that $\phi_l$ can be replaced by $\phi_r$ in the previous theorem, since $\phi_r(X) = \phi_l(DX)$.

These results apply to any convex subcategory $\Delta$ of a quiver algebra $\Lambda = kQ/I$, where $Q$ is a finite quiver and $I$ is an admissible ideal of the path algebra $kQ$. That is, $\Delta = kQ/(kQ \cap I)$, where $Q'$ is a full convex subquiver of $Q$. In this case $\Delta = \Lambda/\mathfrak{A}$, where $\mathfrak{A}$ is the trace of the projective module $P = \bigoplus_{i \in Q_0} P_i$. In this situation it is known that $\mathfrak{A}$ is a strong idempotent ideal ([9], Ch. II, Lemma 3.7) and we obtain the following corollary.

Corollary 4.4. Let $\Delta$ be a full convex subcategory of the quiver algebra $\Lambda$, and let $\mathfrak{A}$ be the idempotent ideal such that $\Delta = \Lambda/\mathfrak{A}$. If $\mathfrak{A}$ has finite projective dimension then $\phi_\Delta \dim \Delta \leq \phi_\Lambda \dim \Lambda$.

Now we turn our attention to $\Gamma$-modules. We study the behaviour of both Igusa-Todorov functions $\phi$ and $\psi$ under the functor $\text{Hom}_\Lambda(P, -) : \text{mod} \Lambda \to \text{mod} \Gamma$ restricted to the subcategories $\mathbb{P}_\infty$ and $\mathbb{I}_\infty$ of $\text{mod} \Lambda$.

Proposition 4.5. For a $\Lambda$-module $Y \in \mathbb{P}_\infty$ the following properties hold:

(a) there exists a $d$-division of $Y$ if and only if there exists a $d$-division of $\text{Hom}_\Lambda(P, Y)$.

(b) $\phi_l(Y) = \phi_l^\Gamma(\text{Hom}_\Lambda(P, Y))$.

(c) $\psi_l(Y) = \psi_l^\Gamma(\text{Hom}_\Lambda(P, Y))$.

Proof. (a) Since $\text{Hom}_\Lambda(P, -) : \text{mod} \Lambda \to \text{mod} \Gamma$ induces an equivalence of categories $\mathbb{P}_1 \to \text{mod} \Gamma$ and $Y \in \mathbb{P}_\infty \subseteq \mathbb{P}_1$, it follows that $Y = Y_1 \oplus Y_2$ if and only if $\text{Hom}_\Lambda(P, Y) = \text{Hom}_\Lambda(P, Y_1) \oplus \text{Hom}_\Lambda(P, Y_2)$. The statement follows from the fact that $Y_1, Y_2 \in \mathbb{P}_\infty$ implies $\text{Ext}^l_{\Lambda}(Y_j, -) \cong \text{Ext}^l_{\Gamma}(\text{Hom}_\Lambda(P, Y_j), \text{Hom}_\Lambda(P, -))$, for $j = 1, 2$ and for all $l \geq 0$ ([1], Theorem 3.2(c)).
(b) This is a direct consequence of (a), using that $\phi_0^l(Y) = n$ if and only if $n = \max\{d \in \mathbb{N} : \text{there exists a } d\text{-division of } Y\}$ by Theorem 3.6 in [5].

(c) Let $X \in \text{mod} \Gamma$. Then $\psi_l^Y(X) = n + l$, with $n = \phi^l_0(X)$ and $l = \text{pd} Z_1$, where $Z_1$ is the largest summand of $\Omega^n(X)$ of finite projective dimension. We write $\Omega^n(X) = Z_1 \oplus Z_2$.

Let $Y \in \mathbb{P}_\infty \subseteq \mathbb{P}_1$ be such that $X = \text{Hom}_\Lambda(P,Y)$. Since $\phi^l_0(Y) = \phi_1^l(X)$, we only need to prove that $l$ is the projective dimension of largest summand $Y_1$ of $\Omega^n(Y)$ of finite projective dimension. Let $\Omega^n(Y) = Y_1 \oplus Y_2$ and let

$$\cdots P_{n+1} \to P_n \to \cdots \to P_0 \to Y \to 0$$

be a minimal projective resolution of $Y$ in $\text{mod} \Lambda$. Since $Y \in \mathbb{P}_\infty$ then

$$\cdots \to \text{Hom}_\Lambda(P,P_n) \to \cdots \to \text{Hom}_\Lambda(P,P_0) \to \text{Hom}_\Lambda(P,Y) \to 0$$

is a minimal projective resolution of $X = \text{Hom}_\Lambda(P,Y)$ in $\text{mod} \Gamma$, as follows from [1], Lemma 3.1.

Therefore $\Omega^n(X) \cong \text{Hom}_\Lambda(P,\Omega^n(Y))$. This is, $Z_1 \oplus Z_2 \cong \text{Hom}_\Lambda(P,Y_1) \oplus \text{Hom}_\Lambda(P,Y_2)$. Since $Y \in \mathbb{P}_\infty$, then $\Omega^n(Y)$ and all direct summands of $\Omega^n(Y)$ are also in $\mathbb{P}_\infty$. Thus $\text{pd}L = \text{pd} \text{Hom}_\Lambda(P,L)$ for any summand $L$ of $\Omega^n(Y)$ (see [1], Corollary 3.3). From this we conclude that the projective dimensions of the largest summands of finite projective dimension of $\Omega^n(X)$ and $\Omega^n(Y)$ coincide, as desired. \hfill \Box

We state the corresponding result for the $\phi$-injective dimension in the next proposition.

**Proposition 4.6.** For a $\Lambda$-module $Y \in \mathbb{I}_\infty$ the following properties hold:

(a) there exists a $d$-division of $Y$ if and only if there exists a $d$-division of $\text{Hom}_\Lambda(P,Y)$.

(b) $\phi^l_0(Y) = \phi^l_0(\text{Hom}_\Lambda(P,Y))$.

(c) $\psi^l_0(Y) = \psi^l_0(\text{Hom}_\Lambda(P,Y))$.

**Proof.** The result follows using that $\text{Hom}_\Lambda(P,-) : \mathbb{I}_1 \to \text{mod} \Gamma$ is an equivalence of categories, the fact that $\text{Ext}^l_\Lambda(-,Y) \cong \text{Ext}^l_\Lambda(\text{Hom}_\Lambda(P,-),\text{Hom}_\Lambda(P,Y))$ for all $Y \in \mathbb{I}_\infty$, and $i \geq 0$ ([1], Theorem 3.2, (b)), and dualizing arguments in the proof of the previous proposition. \hfill \Box

Since $\text{Hom}_\Lambda(P,-) : \text{mod} \Lambda \to \text{mod} \Gamma$ induces equivalences of categories $\mathbb{P}_1 \to \text{mod} \Gamma$ and $\mathbb{I}_1 \to \text{mod} \Gamma$, the previous propositions yield the following result.

**Theorem 4.7.**

(a) If $\mathbb{P}_1 = \mathbb{P}_\infty$ then $\phi_l \dim \Gamma \leq \phi_l \dim \Lambda$ and $\psi_l \dim \Gamma \leq \psi_l \dim \Lambda$.

(b) If $\mathbb{I}_1 = \mathbb{I}_\infty$ then $\phi_r \dim \Gamma \leq \phi_r \dim \Lambda$ and $\psi_r \dim \Gamma \leq \psi_r \dim \Lambda$.

Our next objective is to find bounds for the $\phi$ dimension of $\Lambda$ in terms of the $\phi$ dimensions of $\Lambda/\mathfrak{m}$ and $\Gamma$. 
Lemma 4.8. Let \( \mathfrak{A} \) be an idempotent ideal. Then

(a) \( \phi_r(\text{dim}(\Lambda)) \leq \text{pd}_A(\Lambda/\mathfrak{A}) + \phi_r(\text{dim}(\mathcal{T})) \)

(b) \( \phi_0(\text{dim}(\Lambda)) \leq \text{pd}(\Lambda/\mathfrak{A}) + \phi_0(\text{dim}(\mathcal{T})) \).

Proof. If \( \text{pd}_A(\Lambda/\mathfrak{A}) = \infty \) there is nothing to prove. Assume \( \text{pd}_A(\Lambda/\mathfrak{A}) = t < \infty \) and let \( X \in \text{mod}(\Lambda) \). Then by Lemma 5.5 in [1], \( \Omega^{-t}(X) \in \mathcal{T} \). The lemma follows now by repeated use of the dual of the inequality of Lemma 1.3 in [7], \( \phi_r(X) \leq t + \phi_r(\Omega^{-t}(X)) \). This proves (a), and (b) follows by duality.

Observe now that when \( \mathfrak{A} \) is a strong idempotent ideal then \( \mathfrak{A} \) is in \( P_{\infty} \) ([1], Theorem 2.1'), so \( \text{pd}_A(\mathfrak{A}) \leq \text{pd}_T(\text{Hom}_A(P, \mathfrak{A})) \leq \text{gld}(A) \). Thus \( \text{pd}_A(\Lambda/\mathfrak{A}) \leq \text{gld}(A) + 1 \). Since being a strong idempotent ideal is a symmetric condition we obtain that \( \text{pd}(\Lambda/\mathfrak{A}) \leq \text{gld}(A) + 1 \), as observed in [1] at the end of section 5.

Proposition 4.9. Let \( \mathfrak{A} \) be a strong idempotent ideal. Then

\[ \phi_r(\text{dim}(\mathcal{T})) \leq \max\{ \text{gld}(A) + 1, \phi_r(\text{dim}(\Lambda/\mathfrak{A})) + \text{pd}(\Lambda/\mathfrak{A}) \} \]

Proof. Let now \( r = \text{pd}(\Lambda/\mathfrak{A}) \), \( T \in \mathcal{T} \) and consider the glueing sequence

\[ 0 \to \tau_{\Lambda/\mathfrak{A}}T \to T \to T/\tau_{\Lambda/\mathfrak{A}}T \to 0 \in \mathcal{T} \]

Since \( T/\tau_{\Lambda/\mathfrak{A}}T \) is in \( I_{\infty} \), we know by [1], Corollary 3.3 b) that \( \text{injdim}_A(T/\tau_{\Lambda/\mathfrak{A}}T) = \text{injdim}_A(\text{Hom}_A(P, T/\tau_{\Lambda/\mathfrak{A}}T)) \). Then, the corresponding long exact sequence of functors yields isomorphisms of functors \( \delta_i : \text{Ext}_A^i(-, \tau_{\Lambda/\mathfrak{A}}T) \to \text{Ext}_A^i(-, T) \) for \( i > \text{gld}(A) + 1 \).

It is enough to show that \( \phi^A_r(T) \leq \phi^A_r(\tau_{\Lambda/\mathfrak{A}}T) + r \), whenever \( \phi^A_r(T) > \text{gld}(A) + 1 \). With this purpose we assume that \( d = \phi^A_r(T) > \text{gld}(A) + 1 \) and that \( T = T_1 \oplus T_2 \) is a \( d \)-injective-division in \( \text{mod}(\Lambda) \). We start by proving that \( \tau_{\Lambda/\mathfrak{A}}T = \tau_{\Lambda/\mathfrak{A}}T_1 \oplus \tau_{\Lambda/\mathfrak{A}}T_2 \) is a \( d \)-injective division in \( \text{mod}(\Lambda/\mathfrak{A}) \), for some \( j \) such that \( d + 1 \leq j + r \). Since \( T = T_1 \oplus T_2 \) is a \( d \)-injective-division of \( T \), then \( \text{Ext}_A^j(-, T_1) \neq \text{Ext}_A^j(-, T_2) \) and \( \text{Ext}_A^{j+1}(-, T_1) \cong \text{Ext}_A^{j+1}(-, T_2) \). Therefore, since \( T_1 \) and \( T_2 \) are also in \( \mathcal{T} \) we have that \( \text{Ext}_A^{d}(-, \tau_{\Lambda/\mathfrak{A}}T_1) \neq \text{Ext}_A^{d}(-, \tau_{\Lambda/\mathfrak{A}}T_2) \) and \( \text{Ext}_A^{d+1}(-, \tau_{\Lambda/\mathfrak{A}}T_1) \cong \text{Ext}_A^{d+1}(-, \tau_{\Lambda/\mathfrak{A}}T_2) \) in \( \text{mod}(\Lambda) \), by the isomorphisms above.

Now, since \( \mathfrak{A} \) is a strong idempotent ideal we deduce from the last isomorphism that \( \text{Ext}_A^{d+1}(-, \tau_{\Lambda/\mathfrak{A}}T_1) \cong \text{Ext}_A^{d+1}(-, \tau_{\Lambda/\mathfrak{A}}T_2) \) in \( \text{mod}(\Lambda/\mathfrak{A}) \).

On the other hand, \( \text{pd}(\Lambda/\mathfrak{A}) \leq \text{gld}(A) + 1 \) as we observed just before the statement of the proposition. Since we assumed that \( \text{gld}(A) + 1 < d \) we obtain that \( \text{pd}(\Lambda/\mathfrak{A}) + 1 = r + 1 \leq d \). We conclude then, from Lemma 4.2 that \( \text{Ext}_A^{d-\tau}(-, \tau_{\Lambda/\mathfrak{A}}T_1) \neq \text{Ext}_A^{d-\tau}(-, \tau_{\Lambda/\mathfrak{A}}T_2) \) in \( \text{mod}(\Lambda/\mathfrak{A}) \). This fact and the isomorphism \( \text{Ext}_A^{d+1}(-, \tau_{\Lambda/\mathfrak{A}}T_1) \cong \text{Ext}_A^{d+1}(-, \tau_{\Lambda/\mathfrak{A}}T_2) \) obtained above imply
that \( \tau_{\Lambda/\mathfrak{A}}T = \tau_{\Lambda/\mathfrak{A}}T_1 \oplus \tau_{\Lambda/\mathfrak{A}}T_2 \) is a j-injective division in \( \text{mod}\Lambda/\mathfrak{A} \), for some \( j \) such that \( d - r \leq j \leq d \). Thus \( d \leq j + r \leq \phi_r \dim \Lambda/\mathfrak{A} + r \). This proves that \( \phi_r(T) \leq \phi_r \dim \Lambda/\mathfrak{A} + r \), provided \( \text{gld} \Gamma + 1 \geq d = \phi_r(T) \), and ends the proof of the proposition. \( \square \)

**Proposition 4.10.** Let \( \mathfrak{A} \) be a strong idempotent ideal. Then
\[
\phi_r \dim(\Lambda) \leq \text{pd}_{\Lambda}(\Lambda/\mathfrak{A}) + \max\{ \text{gld}(\Gamma) + 1, \text{pd}(\Lambda/\mathfrak{A})_{\Lambda} + \phi_r \dim(\Lambda/\mathfrak{A}) \}.
\]

**Proof.** The result follows from Lemma 4.8 and Proposition 4.9. \( \square \)

**Corollary 4.11.** Let \( \mathfrak{A} \) be a strong idempotent ideal. Assume \( \text{gld}(\Gamma) < \infty \), then \( \phi_r \dim(\Lambda/\mathfrak{A}) \) is finite if and only if \( \phi_r \dim(\Lambda) \) is finite.

Since convex subalgebras of \( \Lambda \) are obtained as factors of \( \Lambda \) by a strong idempotent ideal, the previous results apply to them. In particular, we obtain the following corollary.

**Corollary 4.12.** Let \( \Delta \) be a full convex subalgebra of \( \Lambda \), and let \( \mathfrak{A} = \tau_P(\Lambda) \) be the idempotent ideal such that \( \Delta = \Lambda/\mathfrak{A} \). If \( \Gamma \) has finite global dimension, then \( \phi_r \dim(\Delta) \) is finite if and only if \( \phi_r \dim(\Lambda) \) is finite.

**Example 4.13.** Let \( \Lambda = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix} \), where \( A \) and \( B \) are artin algebras, \( \text{gld}(B) < \infty \) and \( M \) is a \( B \)-\( A \)-bimodule. Then \( \phi_r \dim(\Lambda) \) is finite if and only if \( \phi_r \dim(\Lambda) \) is finite.

In particular we obtain that a one point co-extension of \( A \) has finite \( \phi_r \dim \) if and only if \( A \) does.

Next we illustrate the previous Proposition with the following example.

**Example 4.14.** Let \( \Lambda \) be the algebra given by the quiver
\[
\begin{array}{c}
1 \\
\alpha \\
\beta \\
\gamma \\
\delta \\
\epsilon \\
3 \\
4 \\
5
\end{array}
\]
with relations \( \alpha \beta = \beta \alpha = 0, \mu \gamma = 0, \delta \gamma = 0, \epsilon \mu = 0 \). Let \( P = P_3 \oplus P_4 \oplus P_5 \), and let \( \mathfrak{A} = \tau_P(\Lambda) \). Then \( \mathfrak{A} \simeq P_3 \oplus P_4 \oplus P_5 \oplus S_5^2 \), so \( \text{pd} \mathfrak{A} = 1 \). Moreover, since there is an exact sequence \( 0 \rightarrow P_3 \rightarrow P_4 \rightarrow S_5 \rightarrow 0 \) we obtain that \( S_5 \in \mathbb{P}_\infty \), so \( \mathfrak{A} \in \mathbb{P}_\infty \) and is thus a strong idempotent. Then the quiver of \( \Lambda/\mathfrak{A} \) is
\[
\begin{array}{c}
1 \\
\alpha \\
\beta \\
3
\end{array}
\]
with radical square zero, so $\Lambda/\mathfrak{A}$ is selfinjective and therefore $\phi, \dim \Lambda/\mathfrak{A} = 0$. On the other hand $\Gamma$ is the hereditary algebra with quiver

\[
\begin{array}{ccc}
3 & \theta & 4 \\
& \Gamma & 5 \\
\end{array}
\]

and we get

$$\phi, \dim (\Lambda) \leq 2 + \max \{ 1 + 1, 2 + 1 \} = 5.$$ 

We now turn our attention to the case when $\text{gld}(\Lambda/\mathfrak{A})$ is finite.

**Proposition 4.15.** Let $\mathfrak{A}$ be a strong idempotent ideal and assume that $\text{gld}(\Lambda/\mathfrak{A})$ is finite. Then

(a) $\phi, (T) \leq \max \{ \text{gld}(\Lambda/\mathfrak{A}) + \text{pd}(\Lambda/\mathfrak{A}) \Lambda + 1, \phi, \text{dim}(T/\tau \Lambda/\mathfrak{A}T) \}$ for any $T \in \text{mod} \Lambda$.

(b) $\phi, \text{dim}(\mathbb{T}) \leq \max \{ \text{gld}(\Lambda/\mathfrak{A}) + \text{pd}(\Lambda/\mathfrak{A}) \Lambda + 1, \phi, \text{dim}(\Gamma) \}$.

**Proof.** Let $s = \text{gld}(\Lambda/\mathfrak{A})$, $r = \text{pd}(\Lambda/\mathfrak{A}) \Lambda$ and let $X$ be a $\Lambda/\mathfrak{A}$-module.

We claim that $\text{Ext}^i(\Lambda/\mathfrak{A}, X) = 0$ for all $i \geq s + r + 1$. In fact, since $\text{gld}(\Lambda/\mathfrak{A}) = s$ we know that $\text{Ext}^j(\Lambda/\mathfrak{A}, X) = 0$ for all $j \geq s + 1$. Then $\text{Ext}^j(\Lambda/\mathfrak{A}, X) = 0$ by Lemma 4.2, for $j \geq s + 1$. So the claim holds.

Let now $T$ in $\text{mod} \Lambda$ and consider the sequence $0 \to \tau \Lambda/\mathfrak{A}T \to T \to T/\tau \Lambda/\mathfrak{A}T \to 0$. The corresponding long exact sequence of functors yields an isomorphism of functors $\delta_i : \text{Ext}^i(\Lambda/\mathfrak{A}, X) \to \text{Ext}^i(\Lambda/\mathfrak{A}, \tau \Lambda/\mathfrak{A}T)$ for each $i \geq s + r + 1$.

We prove next that a $d$-division of $T$ in $\text{mod} \Lambda$ yields a $d$-division $T/\tau \Lambda/\mathfrak{A}T$ in $\text{mod} \Lambda$.

In fact, let $d = \phi, \text{dim}(T)$ and let $T = T_1 \oplus T_2$ be a $d$-division of $T$. This is, $\text{Ext}^i(\Lambda/\mathfrak{A}, X) \not\cong \text{Ext}^{i+1}(\Lambda/\mathfrak{A}, X)$ and $\text{Ext}^{i+1}(\Lambda/\mathfrak{A}, X) \cong \text{Ext}^{i+1}(\Lambda/\mathfrak{A}, X)$. Assume now $d \geq s + r + 1$. The functorial isomorphisms $\text{Ext}^i(\Lambda/\mathfrak{A}, X) \cong \text{Ext}^i(\Lambda/\mathfrak{A}, X)$, $k = 1, 2$ and $i \geq s + r + 1$ induced by $\delta_i$ show that $T/\tau \Lambda/\mathfrak{A}T = T_1/\tau \Lambda/\mathfrak{A}T_1 \oplus T_2/\tau \Lambda/\mathfrak{A}T_2$ is a $d$-division of $T/\tau \Lambda/\mathfrak{A}T$ in $\text{mod} \Lambda$. Therefore, $\phi, (T/\tau \Lambda/\mathfrak{A}T) \geq d$. This ends the proof of (a).

Assume finally that $T \in \mathbb{T}$. Therefore $T/\tau \Lambda/\mathfrak{A}T \in \mathbb{I}_\infty$. We know by Proposition 4.6 (b) that $\phi, \text{Hom}(\Lambda, P, T/\tau \Lambda/\mathfrak{A}T) \geq d$. This proves that $\phi, \text{dim}(\Gamma) \leq \max \{ s + r + 1, \phi, \text{dim}(\Gamma) \}$. \qed

**Proposition 4.16.** Let $\mathfrak{A}$ be a strong idempotent ideal. Then $\phi, \text{dim}(\Lambda) \leq \text{pd}(\Lambda/\mathfrak{A}) \Lambda + \max \{ \text{gld}(\Lambda/\mathfrak{A}) + \text{pd}(\Lambda/\mathfrak{A}) \Lambda + 1, \phi, \text{dim}(\Gamma) \}$.

**Proof.** If $\text{gld}(\Lambda/\mathfrak{A}) = \infty$ there is nothing to prove. If $\text{gld}(\Lambda/\mathfrak{A})$ is finite, the result follows from the previous proposition and Lemma [L.8] \qed

Next we obtain another bound for $\phi, \text{dim}(\Lambda)$ with different methods.
Lemma 4.17. Let \( \mathfrak{A} \) be a strong idempotent ideal. Let \( 0 \to X \to Y \to Z \to 0 \) be an exact sequence in \( \text{mod}\Lambda \) with \( X \in \mathfrak{P}_\infty \) and \( Z \in \text{mod}\Lambda/\mathfrak{A} \) such that \( \text{pd}_{\Lambda/\mathfrak{A}} Z \) is finite. Then

(a) \( \Omega^n(Y) \in \mathfrak{P}_\infty \) for \( n \geq \text{pd}_{\Lambda/\mathfrak{A}} Z \).

(b) \( \phi^n(Y) \leq \text{pd}_{\Lambda/\mathfrak{A}} Z + \phi_\text{dim}\Gamma \).

Proof. (a) Let \( \cdots P_n \to \cdots \to P_2 \to P_1 \to P_0 \to X \to 0 \) and \( \cdots Q_n \to \cdots \to Q_2 \to Q_1 \to Q_0 \to Z \to 0 \) be minimal projective resolutions in \( \text{mod}\Lambda \).

Since \( \mathfrak{A} \) is a strong idempotent ideal then \( \cdots Q_n/\mathfrak{A}Q_n \to \cdots \to Q_2/\mathfrak{A}Q_2 \to Q_1/\mathfrak{A}Q_1 \to Q_0/\mathfrak{A}Q_0 \to Z \to 0 \) is a minimal projective resolution of \( Z \) in \( \text{mod}\Lambda/\mathfrak{A} \) (Theorem 1.6 iii) in [1]). Then \( Q_n/\mathfrak{A}Q_n = 0 \) for \( n > s = \text{pd}_{\Lambda/\mathfrak{A}} Z \).

So \( Q_n = \mathfrak{A}Q_n = \tau_\mathfrak{A}Q_n \) is in \( \text{add} P \) for \( n > s \) and therefore \( \Omega^n(Z) \in \mathfrak{P}_\infty \) for \( n > s \).

On the other hand, we assumed that \( X \) is in \( \mathfrak{P}_\infty \), so that \( \Omega^n(X) \) is also in \( \mathfrak{P}_\infty \). Let \( n > s \). Since \( \cdots P_r \oplus Q_r \to \cdots \to P_1 \oplus Q_1 \to P_0 \oplus Q_0 \to Y \to 0 \) is a projective resolution of \( Y \) and \( P_r \oplus Q_r \in \text{add} P \) for \( r \geq n \), then \( \Omega^n(Y) \in \mathfrak{P}_\infty \).

This proves (a).

(b) Let \( s = \text{pd}_{\Lambda/\mathfrak{A}} Z \). By a) we know that \( \Omega^n(Y) \) is in \( \mathfrak{P}_\infty \), for all \( n \geq s \). Thus \( \phi^n(Y) \leq \phi^n(\Omega^n(Y)) + s = \phi^n(\text{Hom}_\Lambda(P, \Omega^n(Y))) + s \leq \phi_\text{dim}\Gamma + s \), where the first inequality is given by Lemma 1.3 in [4], and the equality follows from Proposition 1.3(b).

Proposition 4.18. Let \( \mathfrak{A} \) be a strong idempotent ideal. Then

(a) \( \phi_\text{dim}(\overline{\mathfrak{T}}) \leq \text{gld}(\Lambda/\mathfrak{A}) + \phi_\text{dim}\Gamma \).

(b) \( \phi_\text{dim}(\Lambda) \leq \text{pd}(\Lambda/\mathfrak{A})_\Lambda + \phi_\text{dim}(\Gamma) \).

Proof. (a) Taking supremum on \( T \in \overline{\mathfrak{T}} \) and using (b) of the previous lemma applied to the glueing sequence \( 0 \to \tau_\mathfrak{A}T \to T \to T/\tau_\mathfrak{A}T \to 0 \), we get \( \phi_\text{dim}(\overline{\mathfrak{T}}) \leq \phi_\text{dim}(\overline{\mathfrak{T}}) + \text{gld}(\Lambda/\mathfrak{A}) \).

(b) If \( \text{gld}\Lambda/\mathfrak{A} = \infty \) there is nothing to prove. If \( \text{gld}\Lambda/\mathfrak{A} \) is finite, by Lemma 4.18(b), we get that \( \phi_\text{dim}(\Lambda) \leq \text{pd}(\Lambda/\mathfrak{A})_\Lambda + \phi_\text{dim}(\overline{\mathfrak{T}}) \leq \text{pd}(\Lambda/\mathfrak{A})_\Lambda + \text{gld}(\Lambda/\mathfrak{A}) + \phi_\text{dim}(\Gamma) \).

References

[1] M. Auslander, M. I. Platzeck, G. Todorov. Homological theory of idempotent ideals. *Trans. Am. Math. Soc.*, vol. 332, n. 2, 667-692, (1992).

[2] A. Beligiannis, I. Reiten. Homological and homotopical aspects of torsion theories. *Mem. Amer. Math. Soc.*, vol. 188, n. 883, viii+207 pp., (2007).

[3] H. Cartan, S. Eilenberg. Homological algebra. *Princeton Landmarks in Mathematics*. Princeton University Press, xvi+390 pp., (1999).

[4] S. E. Dickson. A torsion theory for abelian categories. *Trans. Amer. Math. Soc.* 121, 223-235, (1966).

[5] S. Fernandes, M. Lanzilotta, O. Mendoza. The \( \phi \)-dimension: a new homological measure. *Algebras and Representation Theory* vol. 18,(2), 463-476, (2015).
[6] F. Huard, M. Lanzilotta, Self-injective right artinian rings and Igusa-Todorov functions, *Algebras and Representation Theory*, 16 (3), pp. 765-770, (2012).

[7] F. Huard, M. Lanzilotta, O. Mendoza. An approach to the Finitistic Dimension Conjecture. *J. of Algebra* 319, 3918-3934, (2008).

[8] K. Igusa, G. Todorov. On the finitistic global dimension for artin algebras. *Representation of algebras and related topics*, Fields Institute Communications 45 (American Mathematical Society, Providence, RI), 201-204, (2005).

[9] S. Michelena. Sobre un problema de clasificación y cohomología de Hochachild de extensiones locales, Tesis Doctor en Matemática, Universidad Nacional del Sur, (1998).

[10] D. Xu, Homological dimensions and strongly idempotent ideals. *J. Algebra* 414, 175-189, (2014).

María Andrea Gatica:
Instituto de Matemática de Bahía Blanca,
Universidad Nacional del Sur,
Av. Alem 1253, B8000CPB,
Bahía Blanca, ARGENTINA.
mariaandrea.gatica@gmail.com

Marcelo Lanzilotta:
Instituto de Matemática y Estadística Rafael Laguardia (IMERL),
Universidad de la República.
J. Herrera y Reissig 565 C.P. 11300, Montevideo, URUGUAY.
marclan@fing.edu.uy

María Inés Platzeck:
Instituto de Matemática de Bahía Blanca,
Universidad Nacional del Sur,
Av. Alem 1253, B8000CPB,
Bahía Blanca, ARGENTINA.
platzeck@uns.edu.ar