Matrix Compact Sets and Operator Approximation Properties

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Abstract

The relationship between the operator approximation property and the strong operator approximation property has deep significance in the theory of operator algebras. The original definitions of Effros and Ruan, unlike the classical analogues, make no mention of compact operators or compact sets. In this paper we introduce “compact matrix sets” which correspond to the two different operator approximation properties, and show that a space has the operator approximation property if and only if the “operator compact” operators are contained in the closure of the finite rank operators. We also investigate when the two types of compactness agree, and introduce a natural condition which guarantees that they do.

1 Introduction

Since the inception of functional analysis, questions revolving around the approximation property have been a fruitful and important area of investigation. With the development of the theory of operator spaces and the realization that they are “non-commutative Banach spaces,” the various versions of the approximation property have had a similar impact on the field, and dig to the heart of important questions in operator algebras.

The origin of the approximation property was the following fundamental result in the theory of operators on Hilbert spaces: if $X$ and $Y$ are Hilbert spaces, then the compact operators from $Y$ to $X$ are the norm closure of the finite-rank maps. An obvious question to ask was whether this result extended to more general Banach spaces. One direction of this—that the closure of the finite rank maps sits inside the compact maps—is straightforward. The converse direction—that for every Banach space $Y$ every compact operator in $\mathcal{B}(Y,X)$ can be norm approximated by finite rank maps—is the approximation property for $X$. The initial conjecture was that every Banach space satisfied the approximation property.

Grothendieck [10] attacked this question, and although he was unable to prove or disprove the conjecture, he did provide a number of equivalent formulations of the approximation property:

Theorem 1.1 Let $X$ be a Banach space. The following are equivalent:

i. the identity map $id : X \rightarrow X$ can be approximated uniformly on compact sets by finite rank maps,

ii. For all Banach spaces $Y$, the finite rank maps are dense in $\mathcal{B}(Y,X)$ with the topology of uniform convergence on compact sets,
iii. For all Banach spaces $Y$, the finite rank maps are dense in $B(X,Y)$ with the topology of uniform convergence on compact sets.

iv. If $\omega \in X^* \otimes X$ and $\omega(x) = 0$ for all $x \in X$ then $\tau(\omega) = 0$, where $\tau(\sum \chi_i \otimes x_i) = \sum \chi_i(x_i)$.

v. Given any Banach space $Y$, any compact map in $B(Y,X)$ can be approximated uniformly by finite rank maps.

It was not until 1973 that Enflo [9] provided a counterexample to the conjecture. A complete discussion of the classical results and their implications can be found in Lindenstrauss and Tzafriri [15].

Following Ruan’s abstract characterization of concrete operator spaces (closed linear spaces of operators on a Hilbert space) as matrix normed spaces [21], it was realized that there were many parallels between the theory of Banach spaces and the theory of operator spaces. The difficulty then, as now, lay in unwinding classical definitions and results so that they can be expressed in terms of concepts which have well-developed analogues in the operator space theory.

In their article on operator approximation properties [3], Effros and Ruan observed that classically the following notions of convergence all agree:

i. $\varphi_\nu$ converges to $\varphi$ uniformly on compact sets in $V$,

ii. $\text{id} \otimes \varphi_\nu : c_0 \tilde{\otimes} V \to c_0 \tilde{\otimes} W$ converges point-norm to $\text{id} \otimes \varphi$,

iii. $\text{id} \otimes \varphi_\nu : X \tilde{\otimes} V \to X \tilde{\otimes} W$ converges point-norm to $\text{id} \otimes \varphi$, for any Banach space $X$,

iv. $\text{id} \otimes \varphi_\nu : \ell^\infty(S) \tilde{\otimes} V \to \ell^\infty(S) \tilde{\otimes} W$ converges point-norm to $\text{id} \otimes \varphi$ for any set $S$,

where $V$ and $W$ are Banach spaces, $\varphi_\nu$ a bounded net of maps in $B(V,W)$, $\varphi$ in $B(V,W)$ and $\tilde{\otimes}$ denotes the minimal Banach space tensor product.

They then noted that in the category of operator spaces there were the following analogues to the above:

ii. $\text{id} \otimes \varphi_\nu : K \tilde{\otimes}_{op} V \to K \tilde{\otimes}_{op} W$ converges point-norm to $\text{id} \otimes \varphi$,

iii. $\text{id} \otimes \varphi_\nu : X \tilde{\otimes}_{op} V \to X \tilde{\otimes}_{op} V$ converges point-norm to $\text{id} \otimes \varphi$, for any operator space $X$,

iv. $\text{id} \otimes \varphi_\nu : B(H) \tilde{\otimes}_{op} V \to B(H) \tilde{\otimes}_{op} W$ converges point-norm to $\text{id} \otimes \varphi$ for any Hilbert space $H$. 
where $V$ and $W$ are operator spaces, $\varphi_\nu$, a bounded net of maps in $\mathcal{CB}(V, W)$, $\varphi$ in $\mathcal{CB}(V, W)$ and $\otimes_{op}$ denotes the operator space minimal tensor product.

With this observation, the following definition is natural:

**Definition 1.1**

$V$ has the operator approximation property if the identity map $\text{id} : V \to V$ can be approximated by completely bounded finite rank maps in the stable point norm topology.

They then proved the following theorem, in direct analogy with Grothendieck's Theorem 1.1:

**Theorem 1.2** Let $V$ be an operator space. The following are equivalent:

1. $V$ has the operator approximation property.
2. For all operator spaces $W$, the finite rank maps in $\mathcal{CB}(W, V)$ are dense in the stable point norm topology.
3. For all operator spaces $W$, the finite rank maps in $\mathcal{CB}(V, W)$ are dense in the stable point norm topology.
4. If $\omega \in V^* \otimes_{op} V \subseteq \mathcal{CB}(V, V)$ and $\omega(v) = 0$ for all $v \in V$ then $\tau(\omega) = 0$, where $\tau$ is the matricial trace $\tau(\omega) = \omega(\text{id})$, $\text{id} : V \to V$.

So it appears that the topology (ii) is the "correct" one for the operator space version of the approximation property. But what about the other alternatives? It is straightforward that (iii) and (iv) are equivalent and imply (ii), and this was recognized by Effros and Ruan. That (ii) does not imply (iii) or (iv) is due to some deep results of Kirchberg [12], and it is these that lead to the applications in operator algebra theory. The approximation property that we get when we use the topology (iii) or (iv) instead is called the strong operator approximation property.

But what of the classical topology (i) that was the principal one used by Grothendieck? Moreover, what of the analogue of the original definition of the approximation property. Effros and Ruan saw no obvious operator space analogue of topology (i) or Theorem 1.1 (v) and they wrote [4]:

It would be of considerable interest to find an analogue of (i) for operator spaces. A related problem is to formulate an operator space version of Grothendieck’s result that a Banach space $V$ has the approximation property if and only if any compact operator $K : W \to V$ is a uniform limit of finite rank operators.
Now that we know that the operator approximation property and the strong operator approximation property are in general distinct, we might ask if there is an analogue of topology (i) which is equivalent to topologies (iii) and (iv); and can we find an analogue of Theorem 1.1 (v) for the strong operator approximation property? If we can answer these questions we should be able to shed light on the important question of when the operator approximation property and the strong operator approximation property agree.

In this paper we are able to answer all these questions completely. We introduce two distinct notions of a “matrix compact set” which give the results we want for the two different approximation properties, and develop a condition which we call subcoexactness which implies the equivalence of these two types of matrix compactness and hence the equivalence of the two approximation properties. Moreover it turns out that subcoexactness is a natural condition, and is related to local reflexivity.

In the next section we will answer the first of Effros and Ruan’s questions, defining an ’operator compact’ matrix set and showing that the appropriate version of convergence on these matrix sets is equivalent to the stable point-norm topology of Effros and Ruan. In Section 3 we review an alternative way of thinking about operator spaces in terms of bimodules, due to Barry Johnson, introduce a version of compactness in this context and prove a bimodule version of Grothendieck’s result in the case of some special spaces. In the fourth section we use these two ideas to prove the analogue of Grothendieck’s result for the operator approximation property. Section 5 contains a discussion of the strong operator approximation case: we introduce another sort of compactness, called “complete compactness” and show that completely uniform convergence on these matrix sets is equivalent to the strongly stable point-norm topology; we discuss the implications of Kirchberg’s work; and finally prove the analogue of Grothendieck’s result for the strong operator approximation property. In the final section we introduce coexactness and subcoexactness, which are properties dual to exactness, and show that if an operator space is subcoexact then our two notions of compactness agree. We conclude by showing that subcoexactness is a natural condition, and is related to local reflexivity.

We will now introduce some notation and conventions. Given a topological space $X$ we denote the continuous functions, the bounded continuous functions, the functions vanishing at infinity and the functions vanishing off compact sets by $C(X)$, $C_b(X)$, $C_\infty(X)$ and $C_c(X)$ respectively. We denote the sequences vanishing at 0, the sequences which are 0 for all but finitely many values, the bounded sequences, absolutely summable sequences and square summable sequences of complex numbers as $c_0$ and $c_c$, $\ell^\infty$, $\ell^1$ and $\ell^2$ respectively. We will denote the standard basis in these spaces by $\{e_k\}$, where $e_k$ is the sequence which is zero everywhere but the $k$th element. As is usual, given a Hilbert space $H$, we let $B(H)$, $K(H)$, $T(H)$ and $F(H)$ be the bounded, compact, trace-class and finite rank operators on $H$ respectively.
When $H$ is $\ell^2$, we will often simply write $B$, $K$, $T$ and $F$. If $H$ is $\mathbb{C}^n$, we will write $M_n$ and $T_n$ for the bounded and trace-class operators respectively. Given a basis $\{\xi_\nu\}_{\nu \in \Lambda}$ for $H$, we will let $e_{\mu,\nu}$ be the partial isometry that takes $\operatorname{span}\{\xi_\nu\}$ to $\operatorname{span}\{\xi_\mu\}$. If $\Gamma \subseteq \Lambda$ we denote by $p_\Gamma$ the projection from $H$ to $\operatorname{span}\{\xi_\nu : \nu \in \Gamma\}$. If $\Lambda = \mathbb{N}$, then we let $p_n = p_{\{1,\ldots,n\}}$.

If $V$ is a vector space, we denote by $M_{nm}(V)$ the vector space of $n \times m$ matrices with entries in $V$. By $M_\infty(V)$ we mean the space of infinite matrices in $V$ which have only finitely many non-zero entries. It will occasionally be useful to think of $M_{nm}(V) \cong M_n(M_m(V)) \cong M_{m,n}(V)$ as being indexed by tuples $(i,j)$ where $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and we will denote this as $M_{n \times m}(V)$. We can multiply on the left and right by rectangular scalar matrices in the obvious way. Given a map $\varphi : V \to W$ between two vector spaces we define $\varphi_n : M_n(V) \to M_n(W)$ by

$$\varphi_n(v) = [\varphi(v_{i,j})].$$

We identify $M_n(L(V,W))$ with $L(V,M_n(W))$ by mapping the matrix $[\varphi_{i,j}]$ to the function $v \mapsto [\varphi_{i,j}(v)]$.

A (non-degenerate) pairing of two vector spaces $V$ and $W$ is a bilinear function

$$\langle \cdot, \cdot \rangle : V \times W \to \mathbb{C}$$

such that if $\langle v, w \rangle = 0$ for all $v \in V$, then $w = 0$; and if $\langle v, w \rangle = 0$ for all $w \in W$, then $v = 0$. So each element $v \in V$ (respectively $w \in W$) determines a linear functional $v : W \to \mathbb{C}$ (respectively $w : V \to \mathbb{C}$) by

$$v(w) = w(v) = \langle v, w \rangle.$$ 

Given such a pairing we get a matrix pairing of $M_n(V)$ and $M_m(V)$ which is a map

$$\langle \langle \cdot, \cdot \rangle \rangle : M_n(V) \times M_m(W) \to M_{n \times m}$$

where the $(i,k), (j,l)$-th entry of $\langle \langle v, w \rangle \rangle$ given by $\langle v_{i,j}, w_{k,l} \rangle$, or equivalently

$$\langle \langle v, w \rangle \rangle = v_m(w) = w_n(v).$$

We will assume that the reader is familiar with the literature on operator spaces—an unfamiliar reader might find the following references useful: [1, 17, 19, 20, 21, 24]. We will denote the completely bounded maps between two operator spaces $V$ and $W$ by $\mathcal{CB}(V,W)$, and the complete and incomplete operator space minimal tensor products by $V \otimes_{\text{op}} W$ and $V \otimes_{\text{op}} W$, and the corresponding Banach space tensor products by $V \bar{\otimes} W$ and $V \bar{\otimes} W$. 

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If $A$ is a $C^*$-algebra, with $V$ and $W$ $A$-bimodules (respectively operator $A$-bimodules), we denote the bounded (resp. completely bounded) $A$-bilinear maps from $V$ to $W$ by $B_A(V,W)$ (resp. $CB_A(V,W)$).

We define a matrix set $K = (K_n)$ in a vector space $V$ to be a family of sets $K_n \subseteq M_n(V)$ for $n \in \mathbb{N}$. We call $K_n$ the $n$-th level of $K$. If $V$ is a vector space we will write $V$ for the matrix set $(M_n(V))$. We say that one matrix set $K$ is a subset of another $L$ if $K_n \subseteq L_n$ for all $n \in \mathbb{N}$, and define intersection and union of matrix sets by taking intersections, or unions respectively, at each level. If $V$ is a topological vector space, then we say $K$ is closed if it is closed at each level, and we define open matrix sets analogously. If $\varphi : V \to W$, then given $K \subseteq V$ we get the matrix image of $K$,

$$\varphi(K) = (\varphi_n(K_n)) \subseteq W,$$

and given $L \subseteq W$ we have matrix inverse image

$$\varphi^{-1}(L) = (\varphi^{-1}(L_n)) \subseteq V.$$

Following [8], we say that a matrix set $X$ is (absolutely) matrix convex if

$$\sum_{i=1}^{k} \alpha_i v_i \beta_i \in X_n$$

whenever $v_i \in X_{n_i}$, $\alpha_i \in M_{n_i,n_i}$ and $\beta_i \in M_{n_i,n}$.

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2 Operator Compactness

The first element of our program is to find a definition of a compact matrix set which is suitable for our purposes. In Webster and Winkler [25], Webster [24], Le
Merdy [14], Weaver [23] a matrix set \( X \) is called matrix compact if it is compact at every level and completely bounded in the sense that 

\[
\sup_n \{ \| x \| : x \in X_n \} < \infty
\]

Unfortunately, although it appears to be very useful for investigating duality theory, this definition does not give us the results we want with regards to the operator approximation property. As we will see, it does play a role with regards to the strong operator approximation property.

To find the “correct” definition, we need to look at how we would prove Theorem 1.1. There, the critical notion is that if \( V \) is a Banach space, compact sets can be characterized as being closed subsets of the closed convex hulls of sequences which converge to zero in \( V \) (see [15], Proposition 1.e.2). Without loss of generality, we can take absolutely convex hulls instead of convex hulls. We can think of sequences which converge to 0 as being elements of \( c_0(V) = c_0 \otimes V \), and given an \( x \in c_0 \otimes V \), we can define

\[
\text{co } x = \{ v \in V : v = (\tau \otimes \text{id})(x), \tau \in c_1, \| \tau \|_1 \leq 1 \}.
\]

So we are saying that \( K \) is compact if there is some \( x \in c_0 \otimes V \) such that 

\[
K \subseteq \text{co } x.
\]

Substituting \( K \) for \( c_0 \) at strategic places in a definition for Banach spaces often leads to the correct definition of a concept for operator spaces. Let \( V \) be an operator space and let \( x \in \mathcal{K}(V) \cong \mathcal{K} \otimes_{\text{op}} V \), and define the absolutely matrix convex hull of \( x \) to be \( \text{co } x \) where

\[
(\text{co } x)_k = \{ v \in M_k(V) : v = (\sigma \otimes \text{id})(x), \sigma \in M_k(M_\infty), \| \sigma \|_T \leq 1 \}.
\]

and we are thinking of \( M_\infty \) as sitting inside \( T \). This is an absolutely matrix convex set, since if \( v \in (\text{co } x)_k, w \in (\text{co } x)_k \), then \( v = (\sigma \otimes \text{id})(x), w = (\tau \otimes \text{id})(x) \) and

\[
v \oplus w = (\sigma \oplus \tau \otimes \text{id})(x)
\]

where \( \sigma \oplus \tau \in M_{k+l}(M_\infty) \) and the unit ball of \( T \) is absolutely matrix convex. Similarly

\[
\alpha v \beta = (\alpha \sigma \beta \otimes \text{id})(x)
\]

and \( \alpha \sigma \beta \in M_l(M_\infty) \) and again, the unit ball of \( T \) is absolutely matrix convex.

We also note that if \( \sigma \in M_k(T) \), then \( (\sigma \otimes \text{id})(x) \in \text{co } (x) \), since if we let \( \sigma_n(v) = \sigma(p_n v p_n) \), then \( p_n v p_n \rightarrow v \) as \( n \rightarrow \infty \) and so \( (\sigma_n \otimes \text{id})(x) \rightarrow (\sigma \otimes \text{id})(x) \) as \( n \rightarrow \infty \).
Definition 2.1
If $K$ is a matrix subset of an operator space $V$, then we say $K$ is operator compact if $K$ is closed and there is some $x \in \mathcal{K}(V)$ such that $K \subseteq \text{co}x$.

Example 2.1
Consider the matrix unit ball of $T_n$. Let $\tau \in M_n(T_n) \cong CB(M_n, M_n)$ be the identity map, or equivalently
$$
\tau = \left[ \begin{array}{ccc}
\tau_{1,1} & \cdots & \tau_{1,n} \\
\vdots & \ddots & \vdots \\
\tau_{n,1} & \cdots & \tau_{n,n}
\end{array} \right] = \sum_{i,j=1}^{n} e_{i,j} \otimes \tau_{i,j}
$$
where $\tau_{i,j}(a) = a_{i,j}$. Then given any $\sigma$ in the matrix unit ball of $T_n$, we have that
$$\sigma(a) = \sigma(\tau(a)) = (\sigma \otimes \text{id})(\sum_{i,j=1}^{n} e_{i,j} \otimes \tau_{i,j})(a)$$
and so $\sigma \in \text{co}(\tau)$. Hence this matrix unit ball is operator compact.

Our claim is that operator compactness is the correct definition to solve the questions posed by Effros and Ruan.

Definition 2.2
Let $V, W$ be operator spaces, $\varphi_\nu, \varphi \in CB(V, W)$. We say that the net $\varphi_\nu$ converges to $\varphi$ completely uniformly on operator compact sets if for all operator compact sets $X \subset V$ and $\varepsilon > 0$ there is an $N$ such that
$$\sup\{\| \varphi_\nu(x) - \varphi_n(x) \|_n : x \in X_n, n \in \mathbb{N} \} < \varepsilon,$$
for all $\nu \geq N$.

Our aim is to show that this topology is the same as the topology (ii), which we call the stable point-norm topology.

Proposition 2.1 Let $V, W$ be operator spaces, $\varphi_\nu, \varphi \in CB(V, W)$. Then the following are equivalent:

i. $\varphi_\nu \rightarrow \varphi$ in the stable point norm topology on $CB(V, W)$.

ii. $\varphi_\nu \rightarrow \varphi$ completely uniformly on operator compact sets of $V$. 

Proof:

(i $\implies$ ii): Let $X$ be any operator compact set, and $x \in \mathcal{K}(V)$ be so that $X \subseteq \text{co}\ x$. We know that given any $\varepsilon$ then for $\nu$ sufficiently large we have

$$\| (\varphi_\nu \otimes \text{id})(x) - (\varphi \otimes \text{id})(x) \| \leq \frac{\varepsilon}{2\|x\|}$$

and so we have

$$\sup\{ \| \varphi_\nu(v) - \varphi(v) \| : v \in \text{co} \ x \}$$

$$= \sup\{ \| (\text{id} \otimes \varphi_\nu)(\sigma \otimes \text{id})(x) - (\text{id} \otimes \varphi)(\sigma \otimes \text{id})(x) \| : \sigma \in M_k(M_\infty), \|\sigma\|_T \leq 1 \}$$

$$= \sup\{ \| (\sigma \otimes \text{id})(\text{id} \otimes \varphi_\nu)(x) - (\sigma \otimes \text{id})(\text{id} \otimes \varphi)(x) \| : \sigma \in M_k(M_\infty), \|\sigma\|_T \leq 1 \}$$

$$\leq \| (\varphi_\nu \otimes \text{id})(x) - (\varphi \otimes \text{id})(x) \|$$

$$\leq \varepsilon/2.$$ 

Taking closures we then get that for $\nu$ sufficiently large we have

$$\sup_{v \in X} \| \varphi_\nu(v) - \varphi(v) \| < \varepsilon.$$ 

(ii $\implies$ i): Given $x \in \mathcal{K}(V)$ we notice that $\pi_n(x) \in \text{co} \ x$ for all $n$, that $\text{co} \ x$ is operator compact, and so if we choose $\nu$ sufficiently large we have

$$\| (\varphi_\nu \otimes \text{id})(x) - (\varphi \otimes \text{id})(x) \|_{\infty} = \sup_n \| p_n(\varphi_\nu \otimes \text{id})(x) - (\varphi \otimes \text{id})(x)p_n \|$$

$$= \sup_n \| (\varphi_\nu \otimes \text{id})(x_n) - (\varphi \otimes \text{id})(x_n) \|$$

$$< \varepsilon.$$ 

So we have found a simple answer to the first question posed by Effros and Ruan. To prove the analogue of Grothendieck’s result, however, we will need to look much more deeply. We define an operator compact map $\varphi$ between two operator spaces $V$ and $W$ as being one for which the image of the matrix unit ball is compact.

Completely bounded finite rank maps are operator compact, for as we will see in Corollary 6.2 the unit balls of finite dimensional operator spaces are operator compact. Although it is clear that the operator compact maps are an ideal under composition in the completely bounded maps, it is not clear whether the operator compact maps are closed in the completely uniform topology. The difficulty is that there is little control over the $x$ which we take the matrix convex hull of—and indeed, as we will see in the final section, in some cases there is little prospect of gaining any control.

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3 C*-Operator Spaces

The missing technology that we will need is the concept of a C*-operator space, introduced by Barry Johnson [11]. If the C*-algebra is $K$, then these spaces form a category which is equivalent to that of operator spaces, but has the advantage over operator spaces that often one can adapt Banach space methods to work for C*-operator spaces, where it would not necessarily even be possible to formulate a strategy for operator spaces. In particular, this will be crucial in our analysis of the operator approximation property.

If $A$ is a C*-algebra then Johnson defines an $A$-operator space $V$ to be an essential $A$-bimodule with a norm which is absolutely $A$-convex, i.e. if $v_1, v_2$ lie in the unit ball of $V$ then so does

$$v = \alpha_1 v_1 \beta_1 + \alpha_2 v_2 \beta_2$$

(1)

where $\|\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^*\| \leq 1$ and $\|\beta_1^* \beta_1 + \beta_2^* \beta_2\| \leq 1$.

The natural morphisms are the continuous bi-$A$-linear maps, i.e. continuous maps $\varphi : V \to W$ such that $\varphi(\alpha v \beta) = \alpha \varphi(v) \beta$ for all $v \in V, \alpha, \beta \in A$. We will denote the space of all such maps by $B_A(V,W)$.

A recent result of Magajna [16, Theorem 2.1] tells us that an $A$-bimodule has an operator bimodule structure—that is the bimodule has an operator space structure and the bimodule action is completely contractive—if and only if

$$\|\alpha_1 v_1 \beta_1 + \alpha_2 v_2 \beta_2\| \leq \|\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^*\|^{1/2} \max\{\|v_1\|, \|v_2\|\}\|\beta_1^* \beta_1 + \beta_2^* \beta_2\|^{1/2}.$$

In other words, every C*-operator space has an operator bimodule structure (indeed, it potentially has many). In particular Magajna considers the minimal operator $A$-bimodule whose norms are given by

$$\|v\|_n = \sup\{\|\alpha v \beta\| : \alpha \in M_{1,n}(A), \beta \in M_n(A), \|\alpha\| \leq 1, \|\beta\| \leq 1\}.$$

Given a C*-operator space $V$, we denote the corresponding minimal operator bimodule by $\min V$. Magajna does not look at morphisms explicitly, however the following lemma is immediate from his definition, and is natural given the theory of operator spaces.

**Lemma 3.1** If $V$ and $W$ are $A$-operator spaces, then $\varphi \in B_A(V,W)$ if and only if $\varphi \in CB_A(\min V, \min W)$. Indeed the completely bounded norm is equal to the bounded norm.
Proof:

We simply note that

\[
\| \varphi_n(v) \|_n = \sup \{ \| \alpha \varphi(v) \beta \| : \\
\alpha \in M_{1,n}(A), \beta \in M_{n,1}(A), \| \alpha \| \leq 1, \| \beta \| \leq 1 \}
\]

\[
= \sup \{ \| \varphi(\alpha v \beta) \| : \\
\alpha \in M_{1,n}(A), \beta \in M_{n,1}(A), \| \alpha \| \leq 1, \| \beta \| \leq 1 \}
\]

\[
\leq \| \varphi \| \sup \{ \| \alpha v \beta \| : \\
\alpha \in M_{1,n}(A), \beta \in M_{n,1}(A), \| \alpha \| \leq 1, \| \beta \| \leq 1 \}
\]

\[
= \| \varphi \| \| v \|_n,
\]

so \( \| \varphi \| \leq \| \varphi \|_\infty \leq \| \varphi \|. \)

\[\square\]

If \( A \) is an injective C*-algebra, then this immediately gives us a Hahn-Banach theorem for the category of \( A \)-operator spaces.

**Proposition 3.2** Let \( A \) be an injective C*-algebra. Then if \( V, W \) are \( A \)-operator spaces such that \( V \) is a bi-\( A \)-invariant subspace of \( W \), then given any continuous bi-\( A \)-linear functional

\[
\varphi : V \to A
\]

then there is a continuous bi-\( A \)-linear functional

\[
\bar{\varphi} : W \to A
\]

such that \( \bar{\varphi} | V = \varphi \) and \( \| \bar{\varphi} \| = \| \varphi \|. \)

**Proof:**

We can consider \( \varphi \in CB_A(\min V, \min A) \), and use Wittstock’s Hahn-Banach theorem [26, Theorem 3.1] for operator bimodules to find a completely bounded extension

\[
\bar{\varphi} : W \to A.
\]

But by the previous lemma \( \bar{\varphi} \) is also a bounded extension of \( \varphi \) with \( \| \bar{\varphi} \| = \| \bar{\varphi} \|_\infty = \| \varphi \| \). \[\square\]

In particular, we shall use this result with \( A = B(H) \). However, we will also need a Hahn-Banach theorem in the case where \( A = K \), and \( K \) is not injective. Fortunately we have another approach which delivers us such a theorem. In this
special case we can rephrase our convexity axiom in much more familiar language. A set $X$ in a $K$-operator space is $K$-convex if and only if it satisfies

$$v + w \in X \quad \text{for all orthogonal } v, w \in X, \quad (AKC1)$$

$$\alpha v \beta \in X \quad \text{for all } v \in X, \alpha, \beta \in K, \|\alpha\|, \|\beta\| \leq 1. \quad (AKC2)$$

We say that $v_1, \ldots, v_n \in V$ are orthogonal if there exist orthogonal projections $e_1, \ldots, e_n \in K$ such that $e_i v_i e_i = v_i$. A $K$-norm is a norm $\| \cdot \|$ which satisfies

$$\|v + w\| = \max\{\|v\|, \|w\|\} \quad \text{for all orthogonal } v, w \in X, \quad (AKN1)$$

$$\|\alpha v \beta\| \leq \|\alpha\|\|v\|\|\beta\| \quad \text{for all } v \in X, \alpha, \beta \in K. \quad (AKN2)$$

Clearly the unit balls of $K$-norms are $K$-convex.

Given the parallels between these axioms and Ruan’s axioms for the matrix norms of operator spaces, it is not surprising that $K \otimes_{op} V$ is a $K$-operator space where $K$ operates via

$$\alpha (a \otimes v) \beta = \alpha a \beta \otimes v.$$  

Moreover the natural map $\phi \mapsto \text{id} \otimes \phi = \phi_\infty$ is an isometric isomorphism between completely bounded maps and continuous bi-$K$-linear maps on the corresponding $K$-operator spaces.

Hence $K : V \mapsto K(V)$ takes operator spaces to $K$-operator spaces, and $K : \phi \mapsto \phi_\infty$ takes completely bounded maps to continuous bi-$K$-linear maps. In other words $K$ is a functor.

We define

$$\mathcal{F}(V) = \{e_{1,1} v e_{1,1} : v \in V\}.$$ 

Then $V = \mathcal{F}(V)$ is an operator space when we identify $M_n(V)$ with

$$\{p_n v p_n : v \in V\}$$

via the map

$$\tau_n : [v_{i,j}] \mapsto \sum_{i,j=1}^n e_{1,j} v_{i,j} e_{1,i} ,$$

and give it the norm inherited from $V$. That this is in fact an operator space norm follows immediately from (AKN1) and (AKN2). If

$$\phi : V \to W$$
is a continuous bi-$\mathcal{K}$-linear then $\varphi|_V$ is a completely bounded map from $V$ to $W = \mathcal{F}(W)$, since

$$(\varphi|_V)_n([v_{i,j}]) = \varphi|_{\mathcal{F}_n(V)}(\tau_n([v_{i,j}]))$$

and again, this is an isometric isomorphism of the appropriate mapping spaces. Hence $\mathcal{F}$ is a functor from $\mathcal{K}$-operator spaces to operator spaces.

The next proposition, to the author’s knowledge first explicitly stated by Johnson, but not explicitly proved, is now part of the folklore of the subject.

**Proposition 3.3** The functors $\mathcal{K}$ and $\mathcal{F}$ implement an equivalence of categories between the category of operator spaces with completely bounded maps and the category of $\mathcal{K}$-operator spaces with continuous bi-$\mathcal{K}$-linear maps.

The proof is essentially a matter of resolving semantic differences, and is not of particular interest. For this reason we relegate it to an Appendix. Note that there are many possible ways of implementing this equivalence. The point of such a formal result, is that we can quickly transfer results from the theory of operator spaces to $\mathcal{K}$-operator spaces and vice-versa. In particular, it allow us to quickly prove a Hahn-Banach theorem for $\mathcal{K}$-operator spaces.

**Proposition 3.4** If $V$, $W$ are $\mathcal{K}$-operator spaces such that $V$ is a bi-$\mathcal{K}$-invariant subspace of $W$, then given any continuous bi-$\mathcal{K}$-linear functional

$$\varphi : V \to \mathcal{K}$$

then there is a continuous bi-$\mathcal{K}$-linear functional

$$\tilde{\varphi} : W \to \mathcal{K}$$

such that $\tilde{\varphi}|_V = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$.

**Proof:**

We have shown that we can find $\psi = \mathcal{F}(\varphi)$ which is a completely bounded linear functional on $V = \mathcal{F}(V)$. The bi-$\mathcal{K}$-invariance of $V$ in $W$ implies that $V$ is a subspace of $W = \mathcal{F}(W)$, and so we get by the Hahn-Banach theorem for operator spaces that there is a completely bounded linear functional $\psi$ on $W$ extending $\psi$. We then push this back to the original category to get a bi-$\mathcal{K}$-linear functional $\tilde{\varphi}$ which extends $\varphi$. It remains only to note that the norms are preserved by the functors.  

We are now in a position to discuss the appropriate notion of compactness in these categories.
Definition 3.1
Let $V$ be an $A$-operator space. We say that $X \subset V$ is $A$-compact if $X$ is closed and

$$X \subseteq \text{co}_A \{x_i\}$$

where $x_i \to 0$ and $\text{co}_A$ indicates the closed $A$-convex hull.

We say that a bounded bi-$A$-linear map $\varphi : V \to W$ is an $A$-compact operator if the image of the unit ball of $V$ is $A$-compact in $W$.

We now have analogues of Proposition 2.1:

Proposition 3.5 Let $V$, $W$ be $A$-operator spaces, $\varphi_\nu$, $\varphi \in B_A(V,W)$. Then the following are equivalent:

i. $\text{id} \otimes \varphi_\nu \to \text{id} \otimes \varphi$ point-norm in $B_A(c_0(V),c_0(W))$.

ii. $\varphi_\nu \to \varphi$ uniformly on the compact sets of $V$.

Proof:

(i $\implies$ ii): If $X$ is any $A$-compact set, $X \subseteq \text{co}_A \{x_i\}$, then

$$\sup\{\|\varphi_\nu(v) - \varphi(v)\| : v \in \text{co}_A \{x_i\}\}$$

$$\leq \sup\{\|\varphi_\nu(\sum_{i=1}^n \alpha_i x_i \beta_i) - \varphi(\sum_{i=1}^n \alpha_i x_i \beta_i)\| : \|\sum_{i=1}^n \alpha_i^* \alpha_i\|, \|\sum_{i=1}^n \beta_i \beta_i^*\| \leq 1\}$$

$$\leq \sup\{\|\sum_{i=1}^n \alpha_i(\varphi_\nu(x_i) - \varphi(x_i))\beta_i\| : \|\sum_{i=1}^n \alpha_i^* \alpha_i\|, \|\sum_{i=1}^n \beta_i \beta_i^*\| \leq 1\}$$

$$\leq \sup\{\|\varphi_\nu(x_i) - \varphi(x_i)\|\}$$

since balls are $A$-convex. Taking closures then gives the result.

(ii $\implies$ i): Sequences converging to 0 are $A$-compact, and so the result is immediate. \(\square\)

We would like to define an analogue of the operator approximation property in these categories, but there is an obstacle revolving around what is the appropriate analogue of a finite rank map. Our interest in later sections is going to concentrate on $A$-operator spaces of the form $A \otimes_{\text{op}} V$ for some operator space $V$. As a result we say that a bi-$A$-linear map

$$\varphi : W \to V$$

has finite $A$-rank if the range is bi-$A$-linearly isometrically isomorphic to $A \otimes_{\text{op}} E$, where $E$ is a finite dimensional operator space. In particular if $V = A \otimes_{\text{op}} V$, then
\( \varphi \) may be written as

\[
\varphi = \sum_{i=1}^{k} \varphi_i \otimes v_i
\]

where \( v_i \in V \) and \( \varphi_i \in \mathcal{B}_A(W, A) \).

**Definition 3.2**

We say that an \( A \)-operator space \( V \) has the \( A \)-approximation property if the identity map can be approximated uniformly on \( A \)-compact sets by finite \( A \)-rank maps in \( \mathcal{B}_A(V, V) \).

With this definition and 3.5, we have the analogous version of Effros and Ruan’s result.

**Theorem 3.6** The following conditions are equivalent to a \( A \)-operator space \( V \) having the \( A \)-operator approximation property:

i. For all \( A \)-operator spaces \( W \), the finite \( A \)-rank maps are dense in \( \mathcal{B}_A(W, V) \)
   with topology of uniform convergence on \( A \)-compact sets.

ii. For all \( A \)-operator spaces \( W \), the finite \( A \)-rank maps are dense in \( \mathcal{B}_A(V, W) \)
   with topology of uniform convergence on \( A \)-compact sets.

**Proof:**

Clearly either of these implies the \( A \)-operator approximation property as a special case.

On the other hand if finite \( A \)-rank maps \( \varphi_\nu \) converge to \( \text{id} \), then for any \( \varphi \in \mathcal{B}_A(W, V) \) (resp. \( \mathcal{B}_A(V, W) \)) the maps \( \varphi_\nu \circ \varphi \) (resp. \( \varphi \circ \varphi_\nu \)) are finite \( A \)-rank maps. But

\[
\varphi_\nu \circ \varphi \to \varphi
\]

and

\[
\varphi \circ \varphi_\nu \to \varphi
\]

uniformly on \( A \)-compact sets \( \square \)

We now have all the pieces to prove a version of Theorem 1.1 in the context of certain \( C^* \)-operator spaces. The spaces we work with are determined by the restriction on those categories of \( C^* \)-operator spaces for which we have a Hahn-Banach theorem available, and consideration of what we need the results for in the sequel. More general results may be possible.
Theorem 3.7 Let \( A \) be either an injective \( C^* \)-algebra or \( K \). Let \( \mathcal{V} = A \hat{\otimes}_{\text{op}} V \) for some operator space \( V \). Then \( \mathcal{V} \) has the \( A \)-approximation property if and only if for any \( A \)-operator space \( W \) and any \( A \)-compact map \( \varphi \in \mathcal{B}_A(W, \mathcal{V}) \), the map \( \varphi \) can be approximated uniformly by \( A \)-finite rank maps.

The proof is really just an adaptation of the classical proof to a bi-module setting.

**Proof:**

First assume that \( \mathcal{V} \) has the \( A \)-approximation property. Since \( \varphi \) is \( A \)-compact, there is a sequence \( x = (x_i) \in c_0 \hat{\otimes} \mathcal{V} \) such that \( \varphi(B) \subseteq \overline{\mathcal{C}_A}\{x_i\} \), where \( B \) is the unit ball of \( \mathcal{W} \). Now we know that for any \( \varepsilon > 0 \) we can find a finite rank map \( \psi \in \mathcal{B}_A(\mathcal{V}, \mathcal{V}) \) such that

\[
\| \psi(v) - v \| < \varepsilon
\]

for all \( v \in \overline{\mathcal{C}_A}\{x\} \) and so

\[
\| \psi \circ \varphi - \varphi \| < \varepsilon.
\]

Conversely, let \( X \) be any \( A \)-compact set in \( \mathcal{V} \), and without loss of generality, we may assume that

\[
X = \overline{\mathcal{C}_A}\{x_i\}
\]

for some sequence \( \{x_i\} \in c_0(\mathcal{V}) \), and that \( x_i \neq 0 \) for all \( i \). Now let

\[
U = \overline{\mathcal{C}_A}\left\{ \frac{x_i}{\|x_i\|^{1/2}} \right\},
\]

and so the identity map

\[
id : U \mathcal{V} \to \mathcal{V},
\]

where \( U \mathcal{V} \) is the vector space \( \mathcal{V} \) with the \( A \)-convex norm determined by \( U \), is \( A \)-compact. Therefore we can find finite \( A \)-rank maps

\[
\varphi_k : U \mathcal{V} \to \mathcal{V}
\]

which approximate \( \text{id} \) in norm; in particular, we may let each map be of the form

\[
\varphi_k(v) = \sum_{i=1}^{n_k} \psi_{k,i}(v) \otimes v_{k,i}
\]
where $\psi_{k,i} \in \mathcal{B}_A(U \mathcal{V}, A)$. So all we need do is show we can approximate elements of $\mathcal{B}_A(U \mathcal{V}, A)$ by elements of $\mathcal{B}_A(V, A)$ uniformly on $X$, i.e. given $\delta > 0$, then if for all $k, i$ we can find $\psi'_{k,i} \in \mathcal{B}_A(V, A)$ such that
\[
\|\psi'_{k,i}(v) - \psi_{k,i}(v)\| < \delta
\]
for all $v \in X$, then if we let
\[
\varphi'_k(v) = \sum_{i=1}^{n_k} \psi'_{k,i}(v) \otimes v_{k,i}
\]
we have
\[
\|\varphi'_k(v) - v\| \leq \|\varphi'_k(v) - \varphi_k(v)\| + \|\varphi_k(v) - v\| < 2\delta
\]
for all $k$ sufficiently large, whence the result.

So given any $\psi \in \mathcal{B}_A(U \mathcal{V}, A)$ we may assume $\|\psi\|_U = 1$ without loss of generality. We note that $\|x_i\|_{\frac{1}{2}} \in U$ for all $i$, and so we have that
\[
\left\| \frac{x_i}{\|x_i\|^{1/2}} \right\|_U \leq 1
\]
for all $i$. But this is equivalent to saying that
\[
\|x_i\|_U \leq \|x_i\|^{1/2}
\]
for all $i$, and so $\|x_i\|_U \to 0$ as $i \to \infty$. Now choose an $N$ so that $\|x_i\| < \delta^2/2$ for all $i > N$. Then let
\[
\mathcal{V}_\delta = \text{span}_A \{x_i\}_{i=1}^N,
\]
and let $\psi_\delta$ be the restriction of $\psi$ to $\mathcal{V}_\delta$. $\psi_\delta$ is in $\mathcal{B}_K(\mathcal{V}_\delta, K)$ since $\mathcal{V}_\delta$ is a finite bi-$A$-linear span. On the other hand
\[
\|\psi_\delta\| \leq \|\psi_\delta\|_U = 1,
\]
since $U$ sits inside the unit ball of $\mathcal{V}$. But by the Hahn-Banach theorem for $A$-operator spaces, we can find a $\psi' \in \mathcal{B}_A(V, A)$ which agrees with $\psi_\delta$ on $\mathcal{V}_\delta$ and
\[
\|\psi'\| = \|\psi_\delta\| \leq 1.
\]
Now for any $x_i$ we have either that $\psi'(x_i) = \psi(x_i)$ (if $i \leq N$), or
\[
\|\psi(x_i) - \psi'(x_i)\| \leq \|\psi(x_i)\| + \|\psi'(x_i)\| \leq \|\psi\|\|x_i\|_U + \|\psi'\||x_i| \leq \delta/2 + \delta^2/2 \leq \delta
\]
for $\delta$ small.

and it is easy to see that this implies $\|\psi(x) - \psi'(x)\| < \delta$ for all $x \in X$. \hfill \Box
4 The Operator Approximation Property

The point of the previous section was to introduce notions of compactness which allows us to use more classical techniques. What we must now do is to relate those notions with the ideas from Section 2 so that we can get the result we need. Our first step is to relate Propositions 2.1 and 3.5 by showing that across the equivalence of categories that the convergence is the same.

Proposition 4.1 Let $V, W$ be operator spaces, $\varphi_\nu, \varphi \in CB(V,W)$. Then the following are equivalent:

i. $\varphi_\nu \to \varphi$ in the stable point norm topology on $CB(V,W)$.

ii. $\varphi_\nu \to \varphi$ completely uniformly on operator compact sets of $V$.

iii. $id \otimes (\varphi_\nu)_\infty \to id \otimes \varphi_\infty$ point-norm in $B_K(c_0(K(V)), c_0(K(V)))$.

iv. $(\varphi_\nu)_\infty \to \varphi_\infty$ uniformly on the compact sets of $K(V)$.

v. $(\varphi_\nu)_\infty \to \varphi_\infty$ uniformly on the $K$-compact sets of $K(V)$.

However to prove this, we will need the following lemma about tensor products of commutative C*-algebras with operator spaces.

Lemma 4.2 If $V$ is an operator space, and $X$ is a locally compact Hausdorff space, then $C_0(X) \hat{\otimes} V \cong C_0(X) \hat{\otimes}_{op} V$ as Banach spaces.

Proof (Lemma 4.2):
Recall that there exists a C*-algebra $A$ such that $V$ embeds completely isometrically in $A$, and that $C_0(X) \hat{\otimes} A \cong C_0(X) \hat{\otimes}_{op} A$ as Banach spaces. However, the minimal tensor products respect inclusion, and so $C_0(X) \hat{\otimes} V$ and $C_0(X) \hat{\otimes}_{op} V$ are isometrically isomorphic as normed vector spaces, and so their completions agree.

Proof (Proposition 4.1):
(i $\iff$ ii): Proposition 2.1
(i $\implies$ iii): We observe that

$$K(V) \cong K \hat{\otimes}_{op} V \cong K \hat{\otimes}_{op} K \hat{\otimes}_{op} V \supseteq c_0 \hat{\otimes}_{op} K \hat{\otimes}_{op} V \cong c_0 \hat{\otimes} (K \hat{\otimes}_{op} V)$$

as Banach spaces by Lemma 4.2. So if $\varphi_\nu \to \varphi$ in the stable point norm topology, then

$$id \otimes id \varphi_\nu \to id \otimes id \varphi$$
in the point-norm topology on $c_0 \otimes K(V)$.

(iii $\iff$ iv): This is the classical result, since $K(V)$, $K(W)$ are Banach spaces.

(iv $\implies$ i): Follows since a point in $K(V)$ is compact.

(iii $\iff$ v): Proposition 3.3 with $A = K$. □

**Corollary 4.3** An operator space $V$ has the operator approximation property if and only if $K(V)$ has the $K$-approximation property.

We still need a little more, however, since we need to relate operator compact operators to $K$-compact operators so that we can use Theorem 3.7. In fact we can show that $K$-compact convex sets correspond to sets which are operator compact.

To see this, we assume that

$$X \subseteq \text{co}_K(\{x_1, x_2, \ldots\}).$$

Then we let $\lambda$ be an isomorphism of $K \otimes K$ with $K$, where, for simplicity’s sake, we will assume $\lambda$ is induced by a bijection $\mu: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ via

$$\lambda(e_{(i,j)}, (k,l)) = e_{\mu(i,j),\mu(k,l)}$$

and let $x = \lambda(\text{diag}(x_1, x_2, \ldots))$.

Then if $v \in \pi_n(\text{co}_K(\{x_1, x_2, \ldots\})) \subseteq M_n(V)$ is given by

$$v = p_n \sum_{i=1}^{k} \alpha_kx_k\beta_kp_n$$

we consider the map $\phi: K \otimes K \to M_n$ given by

$$w \mapsto \begin{bmatrix} \beta_1p_n \\ \vdots \\ \beta_kp_n \\ 0 \\ \vdots \end{bmatrix} 
\begin{bmatrix} p_n\alpha_1 \\ \cdots \\ p_n\alpha_k \\ 0 \\ \cdots \end{bmatrix} w$$

so that $v = \phi(\text{diag}(x_1, x_2, \ldots)) = \phi(\lambda^{-1}(x))$. But $\phi \circ \lambda^{-1}: K \to M_m$ is given by

$$w \mapsto [a_{i,j}]w[b_{i,j}]$$

where

$$a_{\mu(i,j),\mu(s,t)} = \begin{cases} \alpha_{i,j}, & \text{if } i = 1, j \leq n, s \leq k \\ 0, & \text{otherwise,} \end{cases}$$
and
\[ b_{\mu(i,j),\mu(s,t)} = \begin{cases} 
\alpha_{i,j}, & \text{if } s = 1, t \leq n, i \leq k \\
0, & \text{otherwise},
\end{cases} \]

and \( m = \max \mu(1, \{1, \ldots, n\}) \). So \( \phi \circ \lambda^{-1} \) is an element of \( \mathcal{CB}(\mathcal{K}, M_n) \) or, rewriting, an element of \( M_n(\mathcal{T}) \), and \( \|\phi \circ \lambda^{-1}\|_{\mathcal{T}} \leq 1 \). So the image of \( \text{co}_{\mathcal{K}}(\{x_1, x_2, \ldots\}) \) sits inside \( \text{co}(x) \). But \( \pi_n \) is continuous, so if \( v_\nu \in \text{co}_{\mathcal{K}}(\{x_1, x_2, \ldots\}) \) converges to \( v \in \text{co}_{\mathcal{K}}(\{x_1, x_2, \ldots\}) \) then \( \pi_n(v_\nu) \to \pi_n(v) \in \text{co}(x) \). Hence \( X \subseteq \text{co}(x) \).

So we have that across the identification of categories
\[ \mathcal{K}\text{-compactness} \Rightarrow \text{operator compactness}. \]

Given that \( \mathcal{K}\)-compact convex sets give rise to matrix convex sets, and the fact that the images of unit balls will be convex in the appropriate sense, then it is clear that under the equivalence of categories \( \mathcal{K}\)-compact mappings become operator compact mappings.

**Theorem 4.4** An operator space \( V \) has the operator approximation property, if and only if for any operator space \( W \) and any operator compact map \( \varphi \in \mathcal{CB}(W,V) \), \( \varphi \) can be approximated completely uniformly by finite rank maps.

**Proof:**

First assume that \( V \) has the operator approximation property. Since \( \varphi \) is operator compact, there is an \( x \in \mathcal{K}(V) \) such that \( \varphi(B) \subseteq \text{co} x \), where \( B \) is the matrix unit ball of \( W \). Now we know that for any \( \varepsilon > 0 \) we can find a finite rank map \( \psi \in \mathcal{CB}(V,V) \) such that
\[ \|(\text{id} \otimes \psi)(x) - x\| < \varepsilon. \]

Therefore
\[ \|\psi(v) - v\|_{\infty} < \varepsilon \]

for all \( v \in \text{co} x \) and so
\[ \|\psi \circ \varphi - \varphi\|_{\text{cb}} < \varepsilon. \]

The converse follows from Theorem 3.7. We observe that if we have a \( \mathcal{K} \)-compact map \( \varphi_\infty \) then \( \varphi \) is an operator compact map and so we can approximate it by finite rank maps \( \varphi_\nu \) and therefore the finite \( \mathcal{K} \)-rank maps \( (\varphi_\nu)_\infty \) approximate \( \varphi_\infty \). Theorem 3.7 tells us that \( \mathcal{K}(V) \) must then satisfy the \( \mathcal{K} \)-approximation property, and Corollary 4.3 then gives our result.

\[ \Box \]
5 The Strong Operator Approximation Property

Effros and Ruan in their original paper [6] defined an operator space \( V \) as having the **strong operator approximation property** if the identity map \( \text{id} : V \to V \) can be approximated by finite rank mappings \( \varphi_\nu \) in the *strongly stable point-norm topology*, i.e. if \( \varphi_\nu \otimes \text{id} \to \text{id} \otimes \text{id} \) point-norm in \( V \otimes_{\text{op}} \mathcal{B}(H) \) for any Hilbert space \( H \). More generally for \( \varphi_\nu, \varphi \in \mathcal{CB}(V, W) \) for some \( V, W \) operator spaces, we say that \( \varphi_\nu \to \varphi \) in the strongly stable point-norm topology if \( \varphi_\nu \otimes \text{id} \to \varphi \otimes \text{id} \) in the point-norm topology on \( \mathcal{CB}(V \otimes_{\text{op}} \mathcal{B}(H), W \otimes_{\text{op}} \mathcal{B}(H)) \) for all Hilbert spaces \( H \).

Since \( K \otimes_{\text{op}} V \hookrightarrow \mathcal{B}(\ell^2) \otimes_{\text{op}} V \), it is easy to see that the strong operator approximation property implies the operator approximation property. Effros and Ruan posed the question as to whether or not the strong operator approximation property was equivalent to the operator approximation property, conjecturing that it was not. Kirchberg [12, 13] showed that this is in fact the case. A sketch of the argument is as follows: the strong operator approximation property is the same as the general slice map property which, for C*-algebras, implies exactness. Extensions of C*-algebras with the operator approximation property have the operator approximation property, but there is an extension of cone \( C_*^r(SL(2, \mathbb{Z})) \) by \( K \) (both of which have the operator approximation property) which is not exact, and so cannot have the strong operator approximation property. He also showed that if the operator space is locally reflexive, then they do agree.

So it seems that there should be another, different, notion of compactness which corresponds to the strong operator approximation property in the same way that operator compactness corresponds to the operator approximation property. An initial guess might be that this should be the matrix compactness used in the duality results mentioned at the start of Section 2, and this turns out to be correct. However to utilize this we need to reformulate the definition in terms of what is essentially an operator space version of total boundedness.

Defining an analogue of total boundedness for operator spaces runs into immediate difficulties, since we only really know what balls centered at the origin look like. However we can avoid this by noting that a set \( K \) is totally bounded if \( K \) is bounded and if for every \( \varepsilon > 0 \) there is a finite dimensional subspace \( V_\varepsilon \) such that every point of \( K \) lies within \( \varepsilon \) of a point in \( V_\varepsilon \).

This implies total boundedness: given any \( \varepsilon > 0 \), we can find a finite dimensional subspace \( V_{\varepsilon/3} \) so that for every \( x \in K \) there is a point \( v \in V_{\varepsilon/3} \) such that \( \|x - v\| < \varepsilon/3 \). Since \( V_{\varepsilon/3} \) is finite dimensional and \( K \) is closed and bounded, we can cover

\[ S = \{v \in V_{\varepsilon/3} : d(K, v) < \varepsilon/3\} \]

by finitely many \( \varepsilon/3 \) balls, centered at \( v_1, \ldots, v_k \in S \). But then for any \( x \in K \), we can find a \( v \in S \) so that \( \|x - v\| \leq \varepsilon/3 \) and there is an \( i \) so that \( v_i \) lies within \( \varepsilon/3 \)
of \(v\), and hence \(x\) lies within \(\varepsilon\) of one of the \(v_i\), and so \(K\) is totally bounded.

Conversely, if \(K\) is totally bounded, for any \(\varepsilon > 0\), choose \(v_1, \ldots, v_n\) be the centers of \(\varepsilon\)-balls which cover \(K\). Then

\[
V_\varepsilon = \text{span}\{v_1, \ldots, v_n\}
\]

is a finite dimensional subspace which meets our criterion.

Hans Saar, a student of Wittstock, in his thesis [22] implicitly noticed this. He worked with compact maps, but if you look at his conditions on the maps, they imply the following about the images of unit balls.

**Definition 5.1**

A matrix point \(v = (v_n)\) in a vector space \(V\) is a sequence of points \(v_i \in M_n(V)\) for \(i \in \mathbb{N}\).

A matrix set \(K\) in an operator space \(V\) is said to be completely compact if \(K\) is closed, completely bounded and if for all \(\varepsilon > 0\), there exists a finite dimensional subspace \(V_\varepsilon\) of \(V\) such that for every matrix point \((x_n) \in K\) we have a matrix point \((v_n) \in V_\varepsilon\), such that \(\|x_n - v_n\|_n < \varepsilon\) for all \(n\).

I would like to thank Zhong-Jin Ruan for bringing Saar’s work to my attention and for providing this definition.

A matrix set which is operator compact is automatically completely compact. First we note that \(\text{co}(x)\) is strongly operator compact, since for any \(\varepsilon > 0\), we choose \(n\) sufficiently large that \(\|x - p_nxp_n\| \leq \varepsilon\) (we can do this since \(K(V)\) is the completion of \(M_\infty(V)\)) and let \(V_\varepsilon\) be the subspace spanned by the entries of \(p_nxp_n\). Then given any matrix point \((v_1, v_2, \ldots)\) in \(\text{co}(x)\) we let \(v_i = (\sigma_i \otimes \text{id})(x)\) and so if we let \(v'_i = (\sigma_i \otimes \text{id})(p_nxp_n)\), we have

\[
\|v'_i - v_i\| = \|(\sigma_i \otimes \text{id})(p_nxp_n - x)\| \leq \|(p_nxp_n - x)\| \leq \varepsilon.
\]

Taking closures we get that any point in \(\text{co}(x)\) must lie within \(\varepsilon\) of \(V_\varepsilon\). We extend this to arbitrary operator compact sets \(K \subseteq \text{co}(x)\) by using the \(V_\varepsilon\) that you use for \(\text{co}(x)\). We will shortly show that the converse is false in general. We will investigate conditions under which the two definitions agree in Section 6.

Completely compact sets are also matrix compact, since for each level \(K_n\) of a completely compact set \(K\), and for every \(\varepsilon > 0\), we have that every element of \(K_n\) lies within \(\varepsilon\) of the finite dimensional subspace \(M_n(V_\varepsilon)\), and so \(K_n\) is closed and totally bounded. The converse is not true in general. If we take \(V = \ell^2(\mathbb{N})_c\), with standard basis \(\{e_k\}\) and let \(X_n\) be the unit ball of \(M_n(\text{span}\{e_1, \ldots, e_n\})\). Then this set is matrix compact, but is not completely compact, since given any finite dimensional subspace of \(V\), we can find an \(e_m\) such that \(d(e_m, V) > 1 - \varepsilon\).
However if the set $K$ in question is matrix convex, as will suffice for our discussion, then matrix compactness implies complete compactness. To see this we choose a finite $\varepsilon/2$-net $\{v_i\}$ for $K_1$ and let $V_\varepsilon = \text{span}\{v_i\}$. Now consider the $n$th level of $K$ and assume that there is an $x \in K_n$ such that $d(x, M_n(V_\varepsilon)) > \varepsilon$. Then in particular, $x$ is at least $\varepsilon$ distant from any matrix of the form $v = [v_{i,j}]$ where the $v_{i,j}$ are taken from the $\varepsilon/2$-net. But then we have that

We would like first to show that completely compact matrix sets are indeed related to the strong operator approximation property. To do this we need to introduce the topology of completely uniform convergence on completely compact sets:

**Definition 5.2**

If $V$ and $W$ are operator spaces, we say that a sequence of maps $\varphi_\nu \in \mathcal{CB}(V,W)$ converges to $\varphi \in \mathcal{CB}(V,W)$ completely uniformly on completely compact sets if for all completely compact sets $X \subset V$ and $\varepsilon > 0$ there is an $N$ such that

$$\sup\{\|((\varphi_\nu)_n(x) - \varphi_n(x))_n\|_n : x \in X_n, \ n \in \mathbb{N}\} < \varepsilon, \ \forall \nu \geq N$$

Clearly if $\varphi_\nu \to \varphi$ completely uniformly on completely compact sets, then it converges completely uniformly on operator compact sets, and hence in the stable point-norm topology. I am once again indebted to Zhong-Jin Ruan for pointing out the following:

**Lemma 5.1** If $\varphi_\nu$, $\varphi \in \mathcal{CB}(V,W)$ for some $V, W$ operator spaces, and $\varphi_\nu \to \varphi$ completely uniformly on completely compact sets then $\varphi_\nu \to \varphi$ in the strongly stable point-norm topology.

**Proof:**

Fix a Hilbert space $H$, some $a \in V \overset{\text{op}}{\otimes} B(H)$, and some $\varepsilon > 0$. Let $a_\eta = \sum_{i=1}^{k_\eta} v_\eta^i \otimes \alpha_\eta^i$ such that $a_\eta \to a$. We let $V^n = \text{span}\{v_1^\eta, \ldots, v_{k_\eta}^\eta\}$, and $P_n(\alpha) = p_n \alpha p_n$, so if $\|a - a_\eta\| < \varepsilon$, then

$$\|\text{id} \otimes P_n(a) - \text{id} \otimes P_n(a_\eta)\| < \varepsilon$$

So the matrix set ($\{a_n = \text{id} \otimes P_n(a)\}$) is completely compact, and hence for all $\nu$ bigger than some $\nu_0$, we have that

$$\|\varphi_\nu \otimes \text{id}(a) - \varphi \otimes \text{id}(a)\| = \sup_n \|((\varphi_\nu)_n(a_n) - \varphi_n(a_n))\| < \delta.$$

We would like to prove a converse result. What we will prove is actually slightly stronger. We will say that $\varphi_\nu \in \mathcal{CB}(V,W)$ converges to $\varphi \in \mathcal{CB}(V,W)$ completely
uniformly on matrix compact sets if for all matrix compact sets $X \subset V$ and $\varepsilon > 0$ there is an $N$ such that

$$\sup\{(\varphi_\nu)_n(x) - \varphi_n(x)\|_n : x \in X_n, \ n \in \mathbb{N}\} < \varepsilon, \ \forall \nu \geq N$$

Since completely compact sets are matrix compact, uniform convergence on matrix compact sets implies uniform convergence on completely compact sets.

Lemma 5.2 If $\varphi_\nu, \varphi \in CB(V, W)$ for some $V, W$ operator spaces, and $\varphi_\nu \to \varphi$ in the strongly stable point-norm topology then $\varphi_\nu \to \varphi$ completely uniformly on matrix compact sets.

Proof:

Let $X$ be a matrix compact set in $V$. Then for each level $X_n$, we can find a sequence of points in $x_{n,i} \in M_n(V)$ which converge to zero, such that

$$X_n \subseteq \text{co}\{x_{n,1}, \ldots, x_{n,m}, \ldots\}.$$

Let $x_n$ be the matrix in $B(\ell^2) \otimes_{op} M_n(V) \cong B(\ell^2) \otimes_{op} V$ given by

$$x_n = \text{diag}(x_{n,1}, \ldots, x_{n,m}, \ldots)$$

and let $x$ be the element of $B(\ell^2) \otimes_{op} B(\ell^2) \otimes_{op} V$ given by

$$x = \text{diag}(x_1, \ldots, x_n, \ldots).$$

Now we observe that any element $v \in \text{co}\{x_{n,1}, \ldots, x_{n,m}, \ldots\}$ can be written as $(\sigma \otimes \text{id})(x)$ for some $\sigma \in CB(B(\ell^2) \otimes_{op} B(\ell^2), M_n)$, with $\|\sigma\|_{CB} \leq 1$. Now since $\varphi_\nu \to \varphi$ in the strongly stable point-norm topology, we have that this implies by \cite{6} that

$$\text{id} \otimes \varphi_\nu \to \text{id} \otimes \varphi$$

point-norm in $CB(Z \otimes_{op} V, Z \otimes_{op} W)$ for any operator space $Z$, so in particular, we can find an $N$ such that

$$\|((\text{id} \otimes \varphi_\nu)(x) - (\text{id} \otimes \varphi)(x))\| < \varepsilon$$

for all $\nu \geq N$, and so

$$\|(\varphi_\nu)_n(v) - \varphi_n(v)\| = \|((\text{id} \otimes \varphi_\nu)(v) - (\text{id} \otimes \varphi)(v))\|$$

$$= \|(\sigma \otimes \text{id})(\text{id} \otimes \varphi_\nu)(x) - (\text{id} \otimes \varphi)(x))\|$$

$$< \varepsilon$$

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for all \( \nu \geq N \), and for any \( n \in \mathbb{N} \) and \( v \in \text{co}(\{x_{n,1}, \ldots, x_{n,m}, \ldots\}) \). So \( \varphi_\nu \) converges to \( \varphi \) completely uniformly on the matrix set
\[
(\text{co}(\{x_{n,1}, \ldots, x_{n,m}, \ldots\}))
\]
and an \( \varepsilon/3 \) argument gives us convergence on \( X \).

So we have proved that these three forms of convergence are equivalent. This means that any of these three can be substituted for the type of convergence in the strong operator approximation property. Kirchberg’s result then tells us that there is an operator space where the strongly stable point-norm topology is different from the stable point-norm topology. Hence by the above result and Proposition 2.1 the topology of completely uniform convergence on strongly operator compact sets does not agree with the topology of completely uniform convergence on operator compact sets. This implies:

**Lemma 5.3** There is a completely compact set \( X \) in some operator space \( V \) such that \( X \) is not operator compact.

**Proof:**

Let \( V \) be a space where the operator approximation property holds, but not the strong operator approximation property. Assume that there was not such \( X \) in this \( V \). Then the topology of completely uniform convergence on strongly operator compact sets agrees with the topology of completely uniform convergence on operator compact sets, as there is no difference in the classes of matrix sets, and so we have a contradiction by our previous discussion.

Turning this around we can give a condition for when the operator approximation property will imply the strong operator approximation property.

**Proposition 5.4** If an operator space \( V \) satisfies the operator approximation property and every completely compact set is operator compact, then \( V \) satisfies the strong operator approximation property.

**Corollary 5.5** If a C*-algebra \( A \) satisfies the operator approximation property and every completely compact set is operator compact, then \( A \) is exact.

This may not be a complete characterization, however, since it is conceivable that the operator and completely compact sets may be different, but the topologies of completely uniform convergence agree.

We need to start relating completely compact sets with C*-compact sets, just as in the previous section. As one might expect, we have the following result:
Proposition 5.6 Let \(V, W\) be operator spaces and \(\varphi, \varphi \in \mathcal{CB}(V, W)\). Then the following are equivalent:

i. \(\varphi_\nu \to \varphi\) in the strongly stable point-norm topology on \(\mathcal{CB}(V, W)\).

ii. \(\varphi_\nu \to \varphi\) completely uniformly on completely compact sets of \(V\).

iii. \(\varphi_\nu \to \varphi\) completely uniformly on matrix compact sets of \(V\).

iv. \(\text{id} \otimes (\text{id} \otimes \varphi_\nu) \to \text{id} \otimes (\text{id} \otimes \varphi)\) point-norm in
\[
\mathcal{B}_{\mathcal{B}(H)}(c_0 \bar{\otimes} (\mathcal{B}(H) \bar{\otimes}_{\text{op}} V), c_0 \bar{\otimes} (\mathcal{B}(H) \bar{\otimes}_{\text{op}} W))
\]
for every \(H\).

v. \(\text{id} \otimes \varphi_\nu \to \text{id} \otimes \varphi\) uniformly on compact sets of \(\mathcal{B}(H) \bar{\otimes}_{\text{op}} V\) for every \(H\).

vi. \(\text{id} \otimes \varphi_\nu \to \text{id} \otimes \varphi\) uniformly on \(\mathcal{B}(H)\)-compact sets of \(\mathcal{B}(H) \bar{\otimes}_{\text{op}} V\) for every \(H\).

Proof:

(i \iff ii \iff iii): Lemma 5.1 and 5.2

(i \implies iv): we observe that for any \(H\)
\[
c_0 \bar{\otimes} (\mathcal{B}(H) \bar{\otimes}_{\text{op}} V) \cong c_0 \bar{\otimes}_{\text{op}} (\mathcal{B}(H) \bar{\otimes}_{\text{op}} V) \cong (c_0 \bar{\otimes}_{\text{op}} \mathcal{B}(H)) \bar{\otimes}_{\text{op}} V
\]
as Banach spaces, by Lemma 4.2 and so if \(\varphi_\nu \to \varphi\) in the strongly stable point-norm topology, then in particular
\[
(\text{id} \otimes \text{id}) \otimes \varphi_\nu \to (\text{id} \otimes \text{id}) \otimes \varphi
\]
point-norm on \((c_0 \bar{\otimes}_{\text{op}} \mathcal{B}(H)) \bar{\otimes}_{\text{op}} V\).

(iv \iff v): This is just the classical result.

(v \implies i): Follows since for any \(H\) a point in \(\mathcal{B}(H) \bar{\otimes}_{\text{op}} V\) is compact.

(iv \iff vi): Proposition 3.5.

Corollary 5.7 An operator space \(V\) has the strong operator approximation property if and only if \(\mathcal{B}(H) \bar{\otimes}_{\text{op}} V\) has the \(\mathcal{B}(H)\)-approximation property for every Hilbert space \(H\).

We call a map \(\varphi \in \mathcal{CB}(W, V)\) completely compact (resp. matrix compact) if the image of the matrix unit ball of \(\varphi\) is completely compact (resp. matrix compact). Saar\cite{22} showed that the completely compact maps are a closed two-sided ideal under composition. Since classical compact maps are a closed two-sided ideal under composition we see that matrix compact maps are also a closed two-sided ideal.
under composition, for if $\varphi_\nu \to \varphi$, then the $(\varphi_\nu)_n$ are all compact, and hence so is $\varphi_n = \lim(\varphi_\nu)_n$ for all $n$.

Again we want to use Theorem 3.7 to show that the strong operator approximation theorem is equivalent to the density of the finite rank maps in the completely compact maps. Again we do this by looking at the appropriate C*-compact sets.

We have a bijection between operator spaces $V$ and the collection of $\mathcal{B}(H)$-operator spaces of the form $\mathcal{B}(H) \otimes_{op} V$ where $H$ is any Hilbert space. To see this all one has to notice is that the spaces $M_n \otimes_{op} V$ give the $n$th matrix norm on the operator space $V$.

Under this correspondence, a matrix set $X$ in $V$ is matrix convex if and only if the sets

$$X_H = \{ v \in \mathcal{B}(H) \otimes_{op} V : \gamma^* v \gamma \in X_n \}$$

where the $\gamma : \mathbb{C}^n \to H$ are are isometries are C*-convex. Note that $X_{\mathbb{C}^n} = X_n$. We say that a matrix convex set in $V$ is $\mathcal{B}(H)$-compact if for every Hilbert space $H$, $X_H$ is $\mathcal{B}(H)$-compact.

**Lemma 5.8** Any $\mathcal{B}(H)$-compact matrix convex set is completely compact.

**Proof:**

Let $X = X_{\ell^2}$ and let $X \subseteq \text{co}_{\mathcal{B}(\ell^2)} \{x_i\}$. Given any $\varepsilon > 0$, we can find

$$y_i = \sum_{j=1}^{k_i} \alpha_{i,j} \otimes v_{i,j}$$

such that

$$\|x_i - y_i\| < \varepsilon/3$$

and we know that there is some $N$ such that $\|x_i\| < \varepsilon/3$ for all $i > N$. Given this we let

$$V_\varepsilon = \text{span}\{v_{i,j} : i \leq N, 1 \leq j \leq k_i \}.$$ 

Then for any matrix point $(a_n) \in X$ we regard $a_n \in \mathcal{B}(\ell^2)$ by putting it in the top left corner, and observe that it lies in $X$. Hence given any $\varepsilon$, we can find a $\mathcal{B}(\ell^2)$-convex combination of $x_i$ which lies within $\varepsilon/3$ of $a_n$, say

$$z_n = \sum_{i=1}^{k} \alpha_i x_i \beta_i$$

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and then we note that

$$z'_n = \sum_{i=1}^{k} \alpha_i y_i \beta_i$$

lies within $\varepsilon/3$ of $z_n$ and hence

$$z''_n = \sum_{i=1}^{\min(k,N)} \alpha_i y_i \beta_i$$

satisfies

$$\|a_n - z''_n\| \leq \|a_n - z'_n\| + \| \sum_{i=\min(k,N)}^{k} \alpha_i y_i \beta_i < \|a_n - z_n\| + \|z_n - z'_n\| + \varepsilon/3 < \varepsilon$$

and moreover $z''_n \in \mathcal{B}(\ell^2) \otimes_{\text{op}} V_\varepsilon$. Hence

$$\|a_n - P_n z''_n P_n\| < \varepsilon$$

and so $X$ is completely compact. \(\square\)

Thus if $\text{id} \otimes \varphi$ is $\mathcal{B}(H)$-compact for every $H$, then $\varphi$ is completely compact.

**Theorem 5.9** An operator space $V$ has the strong operator approximation property, if and only if for any operator space $W$ and any completely compact map $\varphi \in \mathcal{C}B(W, V)$, $\varphi$ can be approximated completely uniformly by finite rank maps.

**Proof:**

First assume that $V$ has the strong operator approximation property. Since $\varphi$ is operator compact, there exists a completely compact set $X$ in $V$ such that $\varphi(B) \subseteq X$, where $B$ is the matrix unit ball of $W$. Since $V$ has the strong approximation property, for any $\varepsilon > 0$ there is a finite rank map $\psi \in \mathcal{C}B(V, V)$ such that

$$\|\psi(v) - v\| < \varepsilon$$

for all $v \in X$ and so

$$\|\psi \circ \varphi - \varphi\|_{cb} < \varepsilon.$$

The converse follows from Theorem 3.7. We observe that if we have that $\text{id} \otimes \varphi$ is a $\mathcal{B}(H)$-compact map for every $H$, then $\varphi$ is a completely compact map and so we can approximate it by finite rank maps $\varphi_\nu$, and therefore the finite $\mathcal{B}(H)$-rank maps $\text{id} \otimes \varphi_\nu$ approximate $\text{id} \otimes \varphi$. Theorem 3.7 tells us that $\mathcal{B}(H) \otimes V$ must then satisfy the $\mathcal{B}(H)$-approximation property, and Corollary 5.7 then gives our result. \(\square\)
6 Subcoexact Operator Spaces

Given the results of the previous section we would like to be able to say when the operator compact and completely compact matrix sets in an operator space agree. Heuristically, what happens in the classical case is that the two types of compactness agree because in a finite dimensional space we can always find a finite sequence of points whose convex hull is as “close” to the unit ball as we like. By taking better and better finite dimensional approximations to our compact set (in the classical sense of the definition of complete compactness), and covering more and more closely we can build a sequence which converges to zero and whose hull contains our original set.

More precisely, what we mean by “close” in the above is that any point in the convex hull of \( \{x_1, x_2, \ldots, x_n\} \) lies in the \( 1 + \varepsilon \) ball of the space \( V \), or equivalently, the map

\[
\varphi : \ell^1_n \to V
\]

defined by

\[
\varphi : e_i \mapsto x_i
\]

has norm at most \( 1 + \varepsilon \). Really we should think of this as an isomorphism \( \psi \) from \( \ell^1_n/\ker \varphi \) to \( V \), and it satisfies \( \|\psi\| \|\psi^{-1}\| \leq 1 + \varepsilon \). Recall that the Banach-Mazur distance between two finite-dimensional Banach spaces \( V \) and \( W \) is given by

\[
d_b(V, W) = \inf\{\|\varphi\| \|\varphi^{-1}\| : \varphi \in B(V, W) \text{ is an isomorphism}\}.
\]

Our heuristic can then be restated as saying that for any finite dimensional Banach space \( V \) and any \( \varepsilon > 0 \) there is an \( n \) and a \( W \subset \ell^1_k \) such that \( d_b(V, \ell^1_k/W) < 1 + \varepsilon \). As we will see, and as we would expect from the work of Pisier [20], the analogous statement for operator spaces is not necessarily true.

To start us down this road, we will look at quotients of \( \mathcal{T}_n \).

**Lemma 6.1** Let \( V = \mathcal{T}_n/W \), that is \( V \) is a finite quotient of the \( n \) by \( n \) trace class operators, then the matrix unit ball of \( V \) is operator compact.

**Proof:**

By Example 2.1, we know that the unit ball of \( \mathcal{T}_n \) lies inside \( \text{co}(\tau) \). So if \( v \) is in the \( m \)th level of the matrix unit ball of \( V \), then there is some \( \sigma \) in the \( m \)th level of the matrix unit ball of \( \mathcal{T}_n \) such that \( v = \pi_m(\sigma) \), where \( \pi \) is the quotient map. Then

\[
v = (\text{id} \otimes \pi)(\sigma \otimes \text{id})(\sum_{i,j=1}^{n} e_{i,j} \otimes \tau_{i,j}) = (\sigma \otimes \text{id})(\sum_{i,j=1}^{n} e_{i,j} \otimes \pi(\tau_{i,j}))
\]
so the unit ball lies in $\text{co} \pi_n(\tau)$.

**Corollary 6.2** If $V$ is any finite dimensional operator space, then the matrix unit ball of $V$ is operator compact.

**Proof:**
For $V$ is completely isomorphic (but maybe not completely isometric) via $\varphi$ to $T_n/W$ for any $n$ and some $W$ depending on $n$. Without loss of generality we may take $\|\varphi\|_{cb} = 1$. Hence if we let $x = \varphi_n^{-1}(\pi_n(\tau))$, we have that the matrix unit ball $X$ of $V$ has image $\varphi(X)$ inside the hull of $\pi_n(\tau)$ and so $X \subseteq \text{co}(x)$.

**Corollary 6.3** If $V$ is any finite dimensional operator space, then any closed, completely bounded matrix set is operator compact.

We notice that we have absolutely no control over the norm of $x$ in the corollary, but would like to try. We recall that the *completely bounded (or Pisier-)* Banach-Mazur distance between two finite dimensional operator spaces $V$ and $W$ is

$$d_{cb}(V, W) = \inf \{ \|\varphi\|_{cb}\|\varphi^{-1}\|_{cb} : \varphi \in CB(V, W) \text{ is a complete isomorphism} \}.$$ 

So clearly, no matter how cleverly we were to choose our $\varphi$, we must have $\|x\| \geq d_{cb}(V, T_n/W)$. But we still can vary $n$, so one might expect that by taking larger $n$ we might be able to find a better $x$. We will say that $V$ is *$\lambda$-coexact* if for every $\varepsilon > 0$, we can find $n$ and a subspace $W \subset T_n$ such that

$$d_{cb}(V, T_n/W) < \lambda + \varepsilon.$$ 

This is a concept dual to Pisier’s *$\lambda$-exactness*: a finite dimensional space $V$ is *$\lambda$-exact* if for every $\varepsilon > 0$, we can find an $n$ and a subspace $W$ of $M_n$ such that

$$d_{cb}(V, M_n/W) < \lambda + \varepsilon.$$ 

We note that for finite dimensional $V$, it is immediate from duality that $V$ being *$\lambda$-coexact* implies that $V^*$ is *$\lambda$-exact*, and $V$ being *$\lambda$-exact* implies that $V^*$ is *$\lambda$-coexact*. Since for any given $\lambda$ there are operator spaces that are not *$\lambda$-exact*, there must be operator spaces that are not *$\lambda$-coexact*.

**Example 6.1**
Pisier showed that for $r \geq 3$,

$$d_{cb}(\max \ell^1_r, W) \geq \frac{r}{2\sqrt{r-1}}$$

and so $\max \ell^1_r$ and $T_r$ are not *$\lambda$-exact* for any $\lambda \leq \frac{r}{2\sqrt{r-1}}$. Hence $(\max \ell^1_r)^* = \min \ell^\infty_r$ and $M_r$ are not *$\lambda$-coexact* for any $\lambda \leq \frac{r}{2\sqrt{r-1}}$. 

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For infinite dimensional spaces, we say \(V\) is \(\lambda\)-exact if every finite-dimensional space is \(\lambda\)-exact. We define \(d_{\text{ex}}(V)\) to be the infimum of the \(\lambda\) for which \(V\) is \(\lambda\)-exact, or \(\infty\) if there are no such \(\lambda\). We might be tempted to say that \(V\) is \(\lambda\)-coexact if every finite-dimensional space is \(\lambda\)-coexact, but this does not quite measure what we are interested in. Our original aim was to control the size of the \(x\) for which the unit ball of a finite dimensional subspace \(W\) of \(V\) is contained in \(\overline{\text{co}}x\), but clearly now that we are in a larger space, so we might be able to get a better \(x\) by choosing it sitting in \(V\), rather than \(W\). However, \(x\) must still be in the image of a finite dimensional space, so what we want is the following:

**Definition 6.1**

Let \(V\) be an operator space. We say that \(V\) is \(\lambda\)-subcoexact if for every finite dimensional subspace \(W\) of \(V\) there is another finite dimensional subspace \(X\) containing \(W\) and which is \(\lambda\)-coexact. We define \(d_{\text{scw}}(V)\) to be the infimum of the \(\lambda\) for which \(V\) is \(\lambda\)-subcoexact, or \(\infty\) if there is no such \(\lambda\).

This definition give us sufficient control that we can prove the following theorem, which is essentially an adaptation of the classical result to the operator space situation. Before we proceed, we need to observe that if \(V\) is \(\lambda\)-subcoexact, then any quotient of it by a finite dimensional space is also \(\lambda\)-subcoexact, for if \(W \subseteq V/X\), then \(\pi^{-1}(W)\) is still finite dimensional, and so it is contained in a finite-dimensional \(\lambda\)-coexact subspace \(Z\) which contains \(X\), and clearly \(W \subseteq Z/X \cong (T/Y)/X\), whence the result.

**Theorem 6.4** Let \(V\) be a \(\lambda\)-subcoexact operator space for some \(\lambda < \infty\). If \(K\) is a completely compact subset of \(V\), then \(K\) is operator compact.

**Proof:**

We will construct a sequence of points \(x_i \in M_{n_i}(V)\). Let \(K_1 = K\). Given \(K_i \subseteq V/W_i\), where \(W_i\) is finite dimensional, and \(K_i\) is completely compact, we choose a finite dimensional subspace \(V_i\) so that every matrix point \(x \in 2K_i\) has a matrix point \(v \in V_i\) within \(4^{-i}\) of it. Now since \(V/W_i\) is \(\lambda\)-coexact, we can find a subspace \(U_i\) containing \(V_i\) which is \(\lambda\)-coexact. Moreover, there is an \(x'_i\) sitting inside \(M_{n_i}(U_i)\) so that the completely bounded matrix set

\[
\{v \in M_k(V_i) : d(v, 2K_i) < 4^{-i}\} \subseteq \overline{\text{co}}x'_i
\]

since this set sits in a finite dimensional space, and since \(U_i\) is \(\lambda\)-coexact we may choose this \(x'_i\) so that

\[
\|x'_i\| \leq \lambda \sup \{\|x\| : x \in 2K_i + 4^{-i} + \varepsilon_i\}.
\]
We now let $x_i$ be an element of $M_{n_i}(V)$ in the preimage of $x'_2$ such that

$$\|x_i\| - \|x'_i\| < \varepsilon_i.$$ 

Finally $K_{i+1} = K_i + V_i \subseteq (V/W_i)/V_i = V/W_{i+1}$.

We note that if $K = \sup\{\|x\| : x \in K\}$, then $\sup\{\|x\| : x \in K_i\} \leq 2^{-i}K$, and so

$$\|x_k\| \leq 2^{-i+1}\lambda K + 4^{-i} + 2\varepsilon_i$$

and so if we choose $\varepsilon_i \to 0$, we have that

$$x = \text{diag}(x_1, x_2, \ldots)$$

is in $K \otimes_{\text{op}} V$.

We claim that $K \subseteq \text{co}x$ and so is operator compact. Given $v \in K$, we can find $v_1 \in \text{co}x_1$ so that

$$\|2v - v_1\| \leq 1/4 + \varepsilon_1$$

and then since $2v - v_1 \in K_2$, we can find $v_2 \in \text{co}x_2$ so that

$$\|2(2v - v_1) - v_2\| \leq 4^{-2} + 2\varepsilon_2.$$ 

Repeating this construction, we build a sequence of points $v_i$ so that

$$\|2^n v - 2^{-n-1}v_1 - \cdots - v_n\| \leq 4^{-n} + 2\varepsilon_n$$

and hence

$$\|v - (2^{-1}v_1 + 2^{-2}v_2 + \cdots + v_n)\| \leq 2^{-3n} + 2^{-n+1}\varepsilon_2.$$ 

But $w_n = 2^{-1}v_1 + 2^{-2}v_2 + \cdots + v_n$ is in $\text{co}x$, since $\sum_{i=1}^{n} 2^{-i} < 1$, and $w_n \to v$, so $v \in \text{co}x$. \qed

We have some immediate corollaries:

**Corollary 6.5** Let $V$ be a $\lambda$-subcoexact operator space, where $\lambda < \infty$. Then $V$ satisfies the operator approximation property if and only if $V$ satisfies the strong operator approximation property.

**Corollary 6.6** If $A$ is a $\lambda$-subcoexact $C^*$-algebra with $\lambda < \infty$, and $A$ satisfies the operator approximation property, then $A$ is exact.
Corollary 6.7 There are operator spaces which are not $\lambda$-subcoexact for any $\lambda < \infty$.

Example 6.2
The trace-class operators $\mathcal{T}$ are 1-subcoexact, since Effros and Ruan showed in [7] that for every finite dimensional subspace $V$ of $\mathcal{T}$ and every $\varepsilon > 0$, $V$ is contained in a subspace $W$ for which $d_{cb}(W, \mathcal{T}_n) \leq \varepsilon$ for some $n$. They called spaces which satisfied this property $J$ spaces, hence any $J$ space is 1-subcoexact.

At this point it is not entirely clear that subcoexactness is a natural enough condition to warrant further consideration. Is it a concept which has wide application, or is it merely an ad-hoc construction which is useful for this particular application? It is the author’s belief that subcoexactness will be an important property, and to conclude this paper we include a result which indicates another potential application.

Proposition 6.8 If $V$ is an operator space, and $V^{**}$ is subcoexact then $V$ is locally reflexive.

Proof: Let $W$ be an arbitrary finite dimensional operator space and $\varphi : W \to V^{**}$. We want to approximate $\varphi$ in the point-weak-* topology by complete contractions $\varphi_\nu : W \to V$.

We first assume that $W = \mathcal{T}_n/X$ for some $X$, so $W^* = X^\perp \subseteq M_n$, and we know that $M_n(V^*) = \mathcal{T}_n(V)^*$, so that

\[
\begin{align*}
(M_n \otimes_{op} V)^* & \cong \mathcal{T}_n \otimes_{op} V^* \\
\downarrow & \\
(X^\perp \otimes_{op} V)^* & \cong W \otimes_{op} V^*
\end{align*}
\]

and the bottom row is a complete isometry. Hence

$\mathcal{CB}(W, V^{**}) \cong (W \otimes_{op} V^*)^* \cong (X^\perp \otimes_{op} V)^{**} \cong \mathcal{CB}(W, V)^{**}$

and by the matrix bipolar theorem, we know that the unit ball of $\mathcal{CB}(W, V)$ is weakly dense in the unit ball of $\mathcal{CB}(W, V)^{**}$, so that there is a net of complete contractions $\varphi_\nu \in \mathcal{CB}(W, V)$ which converges to $\varphi$, which means that for all $x \in W$ and $\psi \in V^*$, we have

\[
\langle \langle \psi, \varphi_\nu(x) \rangle \rangle = \langle \langle x \otimes \psi, \varphi_\nu \rangle \rangle \to \langle \langle x \otimes \psi, \varphi \rangle \rangle = \langle \langle \psi, \varphi(x) \rangle \rangle.
\]

Hence $\varphi_\nu \to \varphi$ point-weak-*.
So now if $V^{**}$ is 1-coexact, for any $\varepsilon > 0$, we can find a subspace $Z$ of $V^{**}$ containing $\varphi(W)$, and a complete isomorphism $\theta : Z \to \mathcal{T}_n/X$ for some $n$ and for some $X$, such that $\|\theta\|_{cb}\|\theta^{-1}\|_{cb} \leq 1 + \varepsilon$. So if we let $\|\theta^{-1}\| = 1$, we have that $\theta^{-1}$ can be approximated by $\theta_\nu : \mathcal{T}_n/X \to V$, but if we let

$$\psi_\nu = \theta_\nu \circ \varphi : W \to V$$

we have that $\psi_\nu \to \varphi$ weak-*, and $\|\psi_\nu\| \leq 1 + \varepsilon$. We let $\varphi_{\nu,\varepsilon} = \frac{\psi_\nu}{1+\varepsilon}$.

Hence given any $x \in W$ and $\psi \in V^*$, for any $\varepsilon > 0$ we have

$$\|\langle\langle \psi, \varphi_{\nu,\delta}(x) \rangle\rangle - \langle\langle \psi, \varphi(x) \rangle\rangle\| \leq \frac{1}{1+\delta}(\|\langle\langle \psi, \varphi_{\nu,\delta}(x) \rangle\rangle - \langle\langle \psi, \varphi(x) \rangle\rangle\| + \delta\|\langle\langle \psi, \varphi(x) \rangle\rangle\|)$$

and so if $\nu$ is large enough and $\delta$ small enough, this is smaller than $\varepsilon$. Hence $\varphi_{\nu} \to \varphi$ point-weak-* as required. □

### A Operator Spaces and $\mathcal{K}$-Operator Spaces

We now give the proof of the following proposition.

**Proposition A.1** The functors $\mathcal{K}$ and $\mathcal{F}$ implement an equivalence of categories between the category of operator spaces with completely bounded maps and the category of $\mathcal{K}$-operator spaces with continuous bi-$\mathcal{K}$-linear maps.

**Proof:**

We have an isomorphism of operator spaces $\iota_V : \mathcal{F}(\mathcal{K}(V)) \to V$ given by $[v_{i,j}] \to v_{1,1}$ (noting that all other entries are 0): this is clearly an isomorphism of vector spaces, and if $v = [v_{k,l}] \in M_n(\mathcal{F}(\mathcal{K}(V)))$, then

$$\|\iota_V(v)\|_n = \|\iota_V([v_{k,l}])\|_n = \|[v_{k,l}]_{1,1}\|_n$$

but

$$\|v\|_n = \|[v_{k,l}]_{1,1}\| = \sup_m \|[v_{k,l}]_{1,1}\|_m = \|[v_{k,l}]_{1,1}\|_n$$

and so $\iota_V$ is a completely isometric isomorphism. Furthermore it is a natural transformation, since if $\varphi \in CB(V,W)$, then we have that

$$\iota_W(\mathcal{F}(\mathcal{K}(\varphi))([v_{i,j}])) = \varphi(v_{1,1}) = (\mathcal{F}(\mathcal{K}(\varphi)))(\iota_V([v_{i,j}]))$$

So we have a natural equivalence $\mathcal{F} \circ \mathcal{K} \cong id$. 35
Similarly we have an isomorphism of $\mathcal{K}$-operator spaces $\tau : \mathcal{K}(\mathcal{F}(V)) \to \mathcal{V}$ given by $(v_{i,j}) \to \sum e_{i,1}v_{i,j}e_{1,j}$, which again is bijective, is bi-$\mathcal{K}$-linear, since if $v = (v_{i,j}) \in \mathcal{K}(\mathcal{F}(V))$, then
\[
\tau(\alpha v \beta) = \sum_{i,j,k,l} e_{i,1} \alpha_i \kappa_v \beta_{k,l} e_{1,j} = \sum_{k,l} (\alpha e_{k,1}) v_{k,l} (e_{1,l} \beta) = \alpha \tau(v) \beta,
\]
and is an isometric isomorphism since
\[
\|\tau(v)\| = \|\sum_{i,j} e_{i,1} v_{i,j} e_{1,j}\| = \sup_n \|p_n(\sum_{i,j} e_{i,1} v_{i,j} e_{1,j})p_n\|
\]
\[
= \sup_n \|\sum_{i,j=1}^n e_{i,1} v_{i,j} e_{1,j}\| = \|v\|.
\]
Again, this is a natural transformation, and so we have a natural equivalence $\mathcal{K} \circ \mathcal{F} \cong \text{id}$.

Hence the two categories are equivalent. \qed

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