THE IRREDUCIBLE CONTROL PROPERTY IN MATRIX GROUPS

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Abstract. This paper concerns matrix decompositions in which the factors are restricted to lie in a closed subvariety of a matrix group. Such decompositions are of relevance in control theory: given a target matrix in the group, can it be decomposed as a product of elements in the subvarieties, in a given order? And if so, what can be said about the solution set to this problem? Can an irreducible curve of target matrices be lifted to an irreducible curve of factorisations? We show that under certain conditions, for a sufficiently long and complicated such sequence, the solution set is always irreducible, and we show that every connected matrix group has a sequence of one-parameter subgroups that satisfies these conditions, where the sequence has length less than 1.5 times the dimension of the group.

1. Two motivating examples

1.1. SL₂(C) generated by two groups of shear mappings. It is well known that every matrix g in the group SL₂(C) of complex 2 × 2-matrices with determinant 1 can be written as a product of matrices of the following forms:

\[ x_1(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \text{ and } x_2(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \text{ for } a \in \mathbb{C}, \]

which represent shears along the x-axis and along the y-axis, respectively. Let \( X_i = \{ x_i(a) \mid a \in \mathbb{C} \} \), an additive one-parameter subgroup of SL₂(C).

This paper concerns the variety of all factorisations of a target matrix \( g \) as a product of matrices in the \( X_i \), in a prescribed order, where repetitions are allowed. Since \( \dim \text{SL}_2(\mathbb{C}) = 3 \), we need at least three factors to reach all elements of SL₂(C).

For \( a, b, c \in \mathbb{C} \) we compute

\[ x_1(a)x_2(b)x_1(c) = \begin{bmatrix} 1 + ab & a + c + abc \\ b & 1 + bc \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \]

We can recover \( a, b, c \) as rational functions in \( x, y, z, w \):

\[ b = z, \quad c = (w - 1)/z, \quad \text{and } a = (x - 1)/z. \]

This implies, first, that the image of the multiplication map \( \mu_{121} : X_1 \times X_2 \times X_1 \to \text{SL}_2(\mathbb{C}) \) is three-dimensional, hence dense in SL₂(C); and second, that the pre-image of any matrix \( g \in \text{SL}_2(\mathbb{C}) \) with \( g_{21} \neq 0 \) has precisely one pre-image. Matrices with \( g_{21} = 0 \) and \( g_{11}, g_{22} \) not both 1 are not in the image of the multiplication map. Summarising, the multiplication map is dominant (has dense image) and birational (has generic fibres of cardinality 1); but it is not surjective.

This can be remedied by adding another factor: the multiplication map \( \mu_{1212} : X_1 \times X_2 \times X_1 \times X_2 \to \text{SL}_2(\mathbb{C}) \) is surjective, since an arbitrary element of SL₂(C) can be right-multiplied by some element \( x_2(-a) \) from \( X_2^{-1} = X_2 \)—which corresponds to subtracting \( a \) times the second column from the first—to make sure that the entry at position \( (2, 1) \) becomes nonzero, so that the product is in \( \text{im} \mu_{121} \).
However, the map $\mu_{1212}$ has another undesirable feature: certain fibres are not irreducible. For instance, the solution set to the system of equations

$$x_1(a)x_2(b)x_1(c)x_2(d) = \begin{bmatrix} 1 + ab + ad + cd + abcd & a + c + abc \\ b + d + bcd & 1 + bc \end{bmatrix} = I$$

is the union of the two lines in $\mathbb{C}^4$ with equations $b = d = a + c = 0$ and $c = a = b + d = 0$. When designing a system where the control parameters $a, b, c, d$ should vary with the target matrix $g$, it is desirable that pre-images of irreducible varieties are irreducible themselves.

As we will see later in §4.1, the multiplication map $\mu_{12121}$ still has his undesirable behaviour, but the multiplication map $\mu_{121212}$ and those for longer words do not: for those, the pre-image of any irreducible variety is irreducible. Note that the order of the factors is important here; e.g., since $X_1 \cdot X_1 = X_1$, the image of $\mu_{11222}$ is the same as that of $\mu_{122}$, and only two-dimensional.

Our goal is to show that this behaviour is quite typical for collections of subvarieties $(X_a)_{a \in A}$ of a matrix group $G$: under suitable conditions, for sufficiently long and sufficiently complicated words $w$ over the alphabet $A$, the corresponding multiplication map $\mu_w$ has the property that the pre-image of any irreducible variety is irreducible. It follows, for instance, that any irreducible curve worth of matrices $g$ can be lifted to an irreducible curve worth of factorisations.

1.2. The ULU-decomposition. Let $L, U \subseteq GL_n(\mathbb{C})$ be the groups of invertible lower-triangular matrices and of upper-triangular matrices with 1’s on the diagonal, respectively. By the classical LU-decomposition, the multiplication map

$$L \times U \to GL_n(\mathbb{C})$$

is an isomorphism of varieties with the open subset of $GL_n(\mathbb{C})$ where all leading principal subdeterminants are nonzero. To reach all invertible matrices, one usually adds a factor from the finite group of permutations matrices. Here, instead, we add another factor $U$, and will prove the following fact.

**Proposition 1.** The multiplication map

$$\mu : U \times L \times U \to GL_n(\mathbb{C})$$

is surjective, and, moreover, the preimage $\mu^{-1}(X)$ of every irreducible variety $X \subseteq GL_n(\mathbb{C})$ is irreducible.

Observe that the variety on the left-hand side has dimension

$$n^2 + \binom{n}{2} < 1.5 \cdot \dim GL_n(\mathbb{C}).$$

This is not a coincidence; see Theorem 8.

**Organisation.** The structure of this paper is as follows. In Section 2 we introduce the general setting and state our main results, Theorem 7 and Theorem 8. We also formulate a useful application, Proposition 6, about lifting curves of matrices to curves of factorisations. In Section 3 we prove Theorem 7, Proposition 1, and various intermediate results of independent interest. Finally, Section 4 contains the worked-out example of $SL_2(\mathbb{C})$; an example with symplectic groups discussed in [IKL20], and a proof of Theorem 8.
2. Introduction and results

Let \( G \) be a complex algebraic group and let \((X_a)_{a \in A}\) be a collection of irreducible subvarieties of \( G \), each containing the unit element \( 1 \in G \). Without loss of generality [Bor91, Proposition I.1.10], \( G \) is a closed subgroup of some \( \text{GL}_n(\mathbb{C}) \) defined by the vanishing of some polynomial equations in the \( n^2 \) matrix entries. But we will not need this concrete realisation of \( G \) as a matrix group.

Denote by \( A^* \) the set of finite sequences (words) in the alphabet \( A \). Each \( w = w_1 \ldots w_l \in A^* \) gives rise to a multiplication map

\[
\mu_w : X_w := X_{w_1} \times \cdots \times X_{w_l} \to G, \quad (x_1, \ldots, x_l) \mapsto x_1 \cdots x_l.
\]

Assume that for all \( a \in A \) there exists some \( b \in A \) such that \((X_a)^{-1} = X_b\), and that the \((X_a)_{a \in G}\) together generate \( G \) as a group.

**Definition 2.** A word \( w \) is called dominant/surjective/birational if the map \( \mu_w \) has the corresponding property. The word \( w \) is called irreducible if of all irreducible, closed subsets \( Y \subseteq G \) the pre-image \( \mu_w^{-1}(Y) \) is irreducible in \( X_w \).

In this definition, \( \mu_w \) and \( w \) are called birational if for \( g \) in an open dense subset of \( G \) the pre-image \( \mu_w^{-1}(Y) \) is irreducible in \( X_w \).

By [Bor91, Proposition I.2.2], surjective words exist; in particular, since the \( X_a \) are irreducible, our assumptions imply that \( G \) is a connected algebraic group (for algebraic groups, this is equivalent to being irreducible [Bor91, Proposition I.1.2]).

Throughout the text, except where stated otherwise, topological terms will refer to the Zariski topology, where the closed sets in \( G \) are defined by regular functions (restrictions of polynomials in the concrete model of \( G \) as a closed subgroup of \( \text{GL}_n(\mathbb{C}) \)). However, the image of \( \mu_w \) is constructible by Chevalley’s theorem [Bor91, Corollary AG11.10.2], and therefore its closure in the Zariski topology is the same as its closure in the Euclidean topology. In particular, \( \mu_w \) is dominant in the Zariski topology if and only if it is dominant in the Euclidean topology.

There is one notable exception to the rule that topological terms refer to the Zariski topology, which we state now:

**Definition 3.** A word \( w \) is called open if the map \( \mu_w \) is open in the Euclidean topology.

If \( \mu_w \) is open in the Euclidean topology, and if \( U \) is a Zariski-open subset of \( X_w \), then \( \mu_w(U) \) is Zariski-constructible and Euclidean-open, and therefore Zariski-open. Hence \( \mu_w \) is also open in the Zariski topology, and therefore, since \( G \) is irreducible in the Zariski topology, dominant. But in our proofs we will really use the Euclidean notion of openness: the image of any open ball is open.

We have the following implications for a word \( w \in A^* \) and its map \( \mu_w \):

\[
\begin{array}{cccc}
\text{irreducible} & \Rightarrow & \text{birational} & \Rightarrow \\
\downarrow & & \Downarrow & \\
\text{open} & \Rightarrow & \text{surjective} & \Rightarrow \\
\Rightarrow & \Downarrow & \Rightarrow & \Rightarrow \\
\text{dominant} & & & \\
\end{array}
\]

For the implication irreducible \( \Rightarrow \) surjective we observe that the pre-image of any point under \( \mu_w \) must be irreducible, hence in particular non-empty.
Example 4. For the subgroups $X_1, X_2$ of $\text{SL}_2(\mathbb{C})$ in $\mathbb{C}$, the word 12 is not dominant, and the word 121 is dominant and birational. The map $\mu_{121}$ is locally open around points $(x_1(b), x_2(b), x_3(c)) \in X_{121}$ where $b \neq 0$, as there its derivative has full rank 3; this is immediate from (1). But it is not open around the remaining points; e.g., no neighbourhood of $(x_1(0), x_2(0), x_3(0))$ is mapped onto a neighbourhood of $I$—indeed, no open neighbourhood of $I$ is in the image of $\mu_{121}$. So 121 is not open. As we have seen, the word $(1, 2, 1, 2)$ is surjective but not irreducible. It is open; see §4.1 where also longer words are discussed.

We now introduce the key notion in this paper.

**Definition 5.** We say that the collection $(X_a)_{a \in A}$ has the **irreducible control property** if there exists a word $u$ such that each word containing $u$ as a consecutive sub-word is irreducible.

For the one-parameter subgroups $X_1, X_2$ of $\text{SL}_2(\mathbb{C})$ from §1.1 the word 121212 in $\mathbb{C}$ has the property required of $u$ in the definition; see §1.1. So $(X_a)_{a=1,2}$ has the irreducible control property.

We use the term “control” because we can think, for each word $w \in A^*$, of the arguments of $\mu_w : X_w \to G$ as parameters that we have control over and that we want to tune so as to obtain a target element $g \in G$. The irreducible control property is clearly desirable: it tells us that for words $w = w_1 \ldots w_l$ containing $u$, the solution set to the system of equations

$$x_1 \cdots x_l \in Y \text{ on the control parameters } x_i \in X_{w_i}, \ i = 1, \ldots, l$$

is irreducible. Here is one possible application.

**Proposition 6.** Let $w$ be an irreducible word, $g \neq g'$ elements of $G$, and $C$ an irreducible curve in $G$ passing through $g$ and $g'$. Moreover, let $x, x' \in X_w$ be such that $\mu_w(x) = g$ and $\mu_w(x') = g'$. Then there exists an irreducible curve $D$ in $X_w$ passing through $x$ and $x'$ that via $\mu_w$ maps dominantly into $C$.

**Proof.** By irreducibility of $w$, $Z := \mu_w^{-1}(C)$ is irreducible. Any two points in an irreducible variety are connected by some irreducible curve (see, e.g., [Mum85, page 56]), hence $x, x' \in Z$ are connected by an irreducible curve $D \subseteq Z$. Since the image of $D$ contains two points of $C$, $\mu_w$ maps $D$ dominantly into $C$. \hfill $\Box$

We can now state our main theorems.

**Theorem 7.** Let $(X_a)_{a \in A}$ be a collection of irreducible subvarieties of an algebraic group $G$, each containing the neutral element $1 \in G$. Assume that $\bigcup_{a \in A} X_a$ generates $G$ as a group and that for each $a \in A$ there exists some $b \in A$ with $X_a^{-1} = X_b$. Suppose that for some word $u = u_1 \ldots u_l \in A^*$ the multiplication map

$$\mu_u : X_u := X_{u_1} \times \cdots \times X_{u_l} \to G, \quad (x_1, \ldots, x_l) \mapsto x_1 \cdots x_l$$

is birational. Then the collection $(X_a)_{a \in A}$ satisfies the irreducible control property. Specifically, there exists a natural number $k$ such that for each word $w \in A^*$ containing $k$ consecutive copies of $u$ any irreducible $Y \subseteq G$ has an irreducible pre-image $\mu^{-1}_w(Y) \subseteq X_w$.

**Theorem 8.** Every connected algebraic group $G$ has a collection $(X_a)_{a \in A}$ of connected, one-dimensional subgroups with the irreducible control property. Indeed, there exist $n := \dim G$ such one-parameter subgroups $X_1, \ldots, X_n$ and a word $u$ of length $< 1.5 \cdot \dim G$ (= if $G$ is the trivial group) such that each word containing $u$ as a consecutive sub-word is irreducible.
Remark 9. Theorem 8 can be interpreted as follows: if we want to use one-parameter subgroups in designing a multiplicative system that can reach all elements of the group \( G \), then for dimension reasons we need at least \( n \) of these groups to reach all elements of \( G \). At the cost of choosing (less than) 1.5 times as many, we can ensure that irreducible varieties lift to irreducible factorisation varieties. For \( \text{GL}_n(\mathbb{C}) \), the ULU-decomposition from Section 1.2 is of this form, if we write \( U \) and \( L \) as suitable products of one-parameter groups.

We do not know if the factor 1.5 is optimal—it is conceivable, for instance, that for a different choice of one-parameter subgroups of \( G \), a word of length \( n \) suffices.

Remark 10. We will only use classical facts from the huge literature on matrix decompositions, such as the LU decomposition and its generalisation, the Bruhat decomposition. Nevertheless, we would like to point out one recent paper that, although it concerns matrix decompositions of a different nature than ours, uses techniques from algebraic groups similar to our techniques: in [YL16], it is proved that every matrix is a product of Toeplitz matrices and also a product of Hankel matrices, and a bound on the number of factors is given. It would be interesting to see whether the factorisation spaces are also irreducible. A complicating factor there, is that the matrices are not required to be invertible, like they are here.

3. Proofs

In this section, we prove Theorem 7 and Proposition 1. We retain the notation from Section 2.

Lemma 11. If a word \( u \) is dominant/surjective/open, then any word \( w \) containing \( u \) as a consecutive sub-word has the same property.

Proof. Since \( 1 \in X_a \) for each \( a \in A \), we have \( \text{im} \mu_w \supseteq \text{im} \mu_u \) so dominance or surjectivity of \( u \) implies that of \( w \) (here we do not even need that the letters of \( u \) appear at consecutive positions in \( w \)). For openness, write \( w \) as a concatenation \( w_1uw_2 \) and let \((x, y, z) \in X_{w_1} \times X_u \times X_{w_2} = X_w \). Let \( U \) be an open neighbourhood of \((x, y, z) \) in the Euclidean topology. The intersection of \( U \) with \([x] \times X_{w_1} \times [z] \) is of the form \([x] \times V \times [z]\) with \( V \subseteq X_{w_2} \) open in the Euclidean topology. By openness of \( w \), \( \mu_w(V) \) contains an open neighbourhood \( O \) of \( \mu_w(y) \) in the Euclidean topology. But then \( \mu_{w_1}(x) \cdot O \cdot \mu_{w_2}(z) \) is an open neighbourhood of \( \mu_w(x, y, z) \) contained in \( \mu_w(U) \). \( \square \)

In our examples in Section 4, the \( X_a \) will be connected subgroups of \( G \). By the following lemma, we may then restrict to words without consecutive repeated letters.

Lemma 12. Suppose that each \( X_a \) is a closed, connected subgroup of \( G \). Let \( w \in A^* \) and let \( u \in A^* \) be obtained from \( w \) by replacing every run of consecutive copies of any letter \( b \) by a single \( b \). Then \( w \in A^* \) is dominant/surjective/open/irreducible if and only if \( u \) has the corresponding property.

Proof. It suffices to prove the result when \( w = w_1bw_2 \) and \( u = w_1bw_2 \). Since \( X_b \cdot X_b = X_b \), we have \( \text{im} \mu_w = \text{im} \mu_u \) and hence \( w \) is dominant/surjective iff \( u \) is. Furthermore, consider the multiplication map

\[ \phi : X_{w_1} \times X_b \times X_b \times X_{w_2} \to X_{w_1} \times X_b \times X_{w_2} = X_\mu, \quad (x, s, s', z) \mapsto (x, ss', z). \]
We have $\mu_w = \mu_\varphi \circ \varphi$ and $\varphi$ is open, so if $u$ is open, then so is $w$. On the other hand, for any Euclidean-open $O \subseteq X_u$ we have $\mu_w(O) = \mu_w(\varphi^{-1}(O))$, so if $w$ is open, then so is $u$. Now let $Y \subseteq G$ be irreducible. Then $\mu_w^{-1}(Y) = \varphi(\mu_\varphi^{-1}(Y))$, so if $w$ is irreducible, then so is $u$. Conversely, $\mu_w^{-1}(Y)$ is the image of $X_b \times \mu_w^{-1}(Y)$ under the map $((s), (x, s', z)) \mapsto (x, s', s^{-1}, z)$, so since $X_b$ is irreducible, if $u$ is irreducible, then so is $w$. □

Lemma 13. Dominant and surjective words exist.

This is well known; we recall the argument from [Bor91, Proposition I.2.2].

Proof. Let $w$ be a word such that $H := \operatorname{im} \mu_w$ has maximal dimension. Then $X_w H \subseteq H$ for all $a \in A$ and since $\bigcup_{a \in A} X_a$ is closed under inversion and generates $G$ we have $H = G$. Hence $w$ is dominant. By Chevalley’s theorem, $\operatorname{im} \mu_w$ contains an open, dense subset $U$ of $G$. Then for each $g \in G$ the set $U \cap U^{-1}g$ is nonempty, so that there exist $h, h' \in U$ with $hh' = g$. So $UU = G$ and therefore the concatenation $ww$ is surjective. □

As remarked before, any irreducible word is also surjective. But unlike surjective words, irreducible words need not exist; see the following example.

Example 14. Let $G = (\mathbb{C}^*)^2$ and let $X_1 = \{(t, t^2) | t \in \mathbb{C}^*\}$ and $X_2 = \{(t^2, t) | t \in \mathbb{C}^*\}$. Since $X_1$ and $X_2$ are subgroups, by Lemma 12 we may restrict our attention to words in which the letters 1, 2 alternate. For definiteness, consider $w = 1212$. Then $\mu_w$ is the homomorphism of tori

$$\mu_w : X_1 \times X_2 \times X_1 \times X_2 \cong (\mathbb{C}^*)^4 \to (\mathbb{C}^*)^2, \quad (t_1, t_2, t_3, t_4) \mapsto (t_1 t_2 t_3 t_4, t_1 t_2^2 t_3).$$

Here the last map is the monomial map whose exponent vectors are the rows of the $2 \times 4$-matrix

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix}.$$ 

Let $b_1, b_2 \in \mathbb{Z}^4$ be the rows of this matrix. The fact that $b_1, b_2$ do not span a saturated lattice—e.g. $b_1/3 + b_2/3 \in \mathbb{Z}^4$—implies that $\ker \mu_w = \mu_w^{-1}(1, 1)$ is not irreducible (it has 3 irreducible components). Hence $w$ is not irreducible. The same applies to longer words. ▲

An important difference between Example 14 (where the irreducible control property does not hold) and the example of $\text{SL}_2(\mathbb{C})$ in §4.1 (where it does hold) is the existence of birational words in the latter case and the non-existence of birational words in the former case. This explains the condition in Theorem 7.

We now set out to prove the existence of open words. A well-known sufficient condition for a map to be open is that its derivative is surjective at every point. This requirement will be too restrictive for our purposes. For instance, in the example of $\text{SL}_2(\mathbb{C})$ from §4.1 for any word $w = w_1 \ldots w_l$ that contains both letters 1, 2 at least once, the derivative at $(x_{w_1}(0), \ldots, x_{w_l}(0)) = (I, \ldots, I)$ of the multiplication map has rank 2 rather than 3: its image is spanned by all matrices in the Lie algebra $sl_2$ with zeroes on the diagonal.

We will show, however, that the set of such bad points is small enough in a suitable sense. To this end, for any word $u \in A^*$, we define

$$J_u := \{x \in X_u | d_x \mu_u \text{ is not surjective}\};$$
here $d_z \mu_u$ is the derivative $T_x X_u \to T_{\mu_u(x)} G$ at $x$ of the map $\mu_u$. If $u$ is not dominant, then $J_u$ is all of $X_u$; otherwise, dim $J_u \leq \dim X_u - 1$.

**Lemma 15.** For any two words $u, w \in A^*$ we have $J_{uw} \subseteq J_u \times J_w$. In particular, writing $w^n$ for the concatenation of $n$ copies of a dominant word $u$, we have dim $J_{w^n} \leq n(\dim X_u - 1)$.

**Proof.** For $(x, y) \in X_u \times X_w = X_{uw}$ we have

$$\im d_{(x,y)} \mu_{uw} \supseteq \mu_u(x) \cdot (\im d_y \mu_u) + (\im d_z \mu_u) \cdot \mu_w(y) \quad (\text{in } T_{\mu_u(x)\mu_w(y)} G),$$

so if the left-hand side has dimension less than $\dim G$, then $d_z \mu_u, d_y \mu_u$ have rank less than $\dim G$. The last statement follows from $\dim J_u \leq \dim X_u - 1$. \hfill \Box

**Proposition 16.** Open words exist. More specifically, if $u$ is a dominant word, then for any $s \in X_u$ there is an open, dense subset $(\psi(x), \psi(y)) \in \im \mu_u$ of $(x, y) \in X_u \times X_w = X_{uw}$.

**Proof.** Let $u$ be a dominant word and set $d := \dim X_u$. For a positive integer $n$ consider $w := u^n$. Fix a point $x \in X_u$ and consider an irreducible component $F \ni x$ of the fibre $\mu_u^{-1}(\mu_u(x))$. Now dim $F \geq nd - \dim G$ by properties of fibre dimension [Mum88 §8], while dim $J_u \leq n(d - 1)$ by Lemma 15. Hence if $n > \dim G$, then $F$ is not contained in $J_u$. Hence any open ball $B$ around $x$ (in the Euclidean topology) contains a $y \in F \setminus J_u$. Since $d_y \mu_u$ is surjective, $\mu_u$ maps an open ball in $B$ around $y$ onto an open neighbourhood (in the Euclidean topology) of $\mu_u(y) = \mu_u(x)$. Hence $\mu_u$ is open in the Euclidean topology. \hfill \Box

**Proposition 17.** If $w$ is an open word and $u$ is a birational word, then for any $s, t \in A^*$ the concatenation $swut$ is irreducible.

**Proof.** Let $Y \subseteq G$ be an irreducible, closed subset, and set $F := \mu_u^{-1}(Y)$. We will show that $F$ is irreducible.

There exists an open, dense subset $O$ of $X_u$ such that $\mu_u$ restricts to an isomorphism from $O$ to an open, dense subset $P$ of $G$. Let $\varphi : P \to O$ be the inverse of that isomorphism.

If $\mu_u$ maps $(x_s, x_w, x_u, x_t) \in X_{swut}$ to $y \in Y$, then

$$\mu_u(x_u) = \mu_w(x_w)^{-1} \mu_s(x_s)^{-1} y \mu_t(x_t)^{-1}$$

and therefore, if $x_u \in O$, then we have

$$x_u = \varphi(\mu_u(x_w)^{-1} \mu_s(x_s)^{-1} y \mu_t(x_t)^{-1}).$$

Let $Q$ be the open subset of $X_s \times X_w \times X_t \times Y$ defined by

$$Q := \{(x_s, x_w, x_t, y) \mid \mu_w(x_w)^{-1} \mu_s(x_s)^{-1} y \mu_t(x_t)^{-1} \in P\}.$$ 

Since $w$ is open and a fortiori dominant, $Q$ is nonempty, hence dense in $X_s \times X_w \times X_t \times Y$, hence irreducible. Define the morphism $\psi : Q \to F$ by

$$\psi(x_s, x_w, x_t, y) = (x_s, x_w, \varphi(\mu_u(x_w)^{-1} \mu_s(x_s)^{-1} y \mu_t(x_t)^{-1}), x_t).$$

The image of $\psi$ is an irreducible subset of $F$ that contains all points $(x_s, x_w, x_u, x_t) \in F$ for which $x_u$ lies in $O$. We claim that $F = \im \psi$, so that $F$ is, indeed, irreducible.

For it suffices to prove that for any $(z_s, z_w, z_u, z_t) \in F$ and any open neighbourhood $\Omega$ of $(z_s, z_w, z_u, z_t)$ in $X_{swut}$ there exists a point $(x_s, x_w, x_u, x_t) \in F \cap \Omega$ with $x_u \in O$. We can in fact take $x_s := z_s$ and $x_t := z_t$ and only vary $z_w$ and $z_u$. Indeed, the neighbourhood $\Omega$ contains $[z_s] \times B_w \times \{z_u\} \times \{z_t\}$ for small balls $B_w$ and $B_u$ around $z_w \in X_w$ and $z_u \in X_u$, respectively. As $w$ is open, $\mu_w(B_w)$ contains an open ball $B'_w$ around $\mu_w(z_w) \in G$. If we take any $x_u \in B'_w$ sufficiently
close to \(z_u\), then \(\mu_u(z_u)\mu_u(z_u)\mu_u(x_u)^{-1} \in B_u'\) and hence there exists an \(x_u \in B_u\) such that \(\mu_u(x_u)\mu_u(x_u) = \mu_u(z_u)\mu_u(x_u)\). Since \(O\) is dense in \(X_u\), we may take such an \(x_u \in O \cap B_u\) and have thus found a point \((z_u, x_u, x_u, z_u) \in F \cap \Omega\) with \(x_u \in O\).

**Proof of Theorem**\(^7\) By assumption, a birational word \(u \in A^*\) exists. By Proposition \(^{15}\), an open word \(w \in A^*\) exists; indeed, some concatenation of \(u^n\) copies of \(u\) is open. By Proposition \(^{17}\), any word in \(A^*\) containing \(u^{n+1}\) as a consecutive sub-word is irreducible. Hence \((X_u)_{u \in A}\) has the irreducible control property.

**Proof of Proposition**\(^1\) Set \(X_1 := L\) and \(X_2 := U\). By the classical LU-decomposition, the word \(u := 21\) is open and birational, and so is the word \(w := 21\) (the transpose of the LU-decomposition is the UL-decomposition). By Proposition \(^{17}\), any word containing \(ww = 2121\) is irreducible. Finally, by Lemma \(^{12}\), we may replace the two consecutive 1s by a single 1, i.e., every word containing 121 is irreducible. This proves Proposition \(^1\).

## 4. Examples

### 4.1. The case of \(\text{SL}_2(\mathbb{C})\)

Recall the subgroups \(X_1, X_2\) of \(\text{SL}_2(\mathbb{C})\) from §1.1. By Lemma \(^{12}\), we need only look at words where the letters 1, 2 alternate. In §1.1, we already saw that 121 is dominant and 1212 is surjective but not irreducible. In Example 4, we saw, moreover, that 121 is not open.

We claim that 1212 is open. By the analysis in Example 4, it is certainly locally open around points \((x_1(a), x_2(b), x_1(c), x_2(d))\) with \(b \neq 0\) or, similarly, \(c \neq 0\). Moreover, by acting with \(x_1(-a)\) from the left and \(x_2(-d)\) from the right, we see that it suffices to check local openness at the point where \(a = b = c = d = 0\). Suppose, then, that we want to solve

\[
\begin{align*}
x_1(a)x_2(b)x_1(c)x_2(d) &= \left[ \begin{array}{ccc} 1 + ab + ad + cd + abcd & a(1 + bc) + c & 1 + bc \\ d(1 + bc) + b & 1 + bc \\ \end{array} \right] \\
&= \left[ \begin{array}{ccc} x & y \\ z & w \end{array} \right] \in \text{SL}_2(\mathbb{C}),
\end{align*}
\]

where \(x, w \approx 1\) and \(y, z \approx 0\). Then we can find small \(b, c\) such that \(1 + bc = w\) (e.g., both equal to the same square root of \(w - 1\)), and after that \(a\) and \(d\) determined by \(a = (y - c)/(1 + bc)\) and \(d = (z - b)/(1 + bc)\) are also small. The last condition \(1 + ab + ad + cd + abcd = x\) now follows, since the right-hand matrix lies in \(\text{SL}_2(\mathbb{C})\). This shows that \(\mu_{1212}\) is open near \((x_1(0), x_2(0), x_1(0), x_2(0))\), so 1212 is an open word.

Next, 1212121 is open and 12122 is surjective but not irreducible. Indeed, the fibre \(\mu_{12122}^{-1}(I)\) consists of all quintuples \((q_1(a), q_2(b), q_1(c), q_2(d), q_1(e))\) with either \(c = 0 = b + d = a + e\) or \(d = 0 = b = a + c + e\).

Finally, we claim that 121212 and all larger alternating words are irreducible. Indeed, the word \(w = 121\) is open, the word \(u = 212\) is birational (by an argument similar to that for 121), and hence \(swut = s1212212t\) is irreducible by Proposition \(^{17}\) for all words \(s, t\).

### 4.2. On a question by Kutzschebauch

This paragraph concerns an example communicated to me by Frank Kutzschebauch; see [IKL20]. Let \(G := \text{Sp}_{2n}(\mathbb{C})\), the complex symplectic group preserving the symplectic form \(\langle (b, c), (d, e) \rangle := be^T - cd^T\), where \(b, c, d, e \in \mathbb{C}^n\) and \((b, c), (d, e) \in \mathbb{C}^{2n}\) are thought of as row vectors.


Let
\[ X_1 := \left\{ \begin{bmatrix} 1 & 0 \\ A & 1 \end{bmatrix} \mid A^T = A \right\} \quad \text{and} \quad X_2 := \left\{ \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} \mid B^T = B \right\}, \]
two subgroups of \(G\) isomorphic, as algebraic groups, to the vector space \(\mathbb{C}^{(n+1)^2}\).

**Theorem 18.** Let \(w\) be a word in the alphabet \(\{1, 2\}\) in which the letters 1 and 2 alternate, and assume that \(w\) is sufficiently long. Let \((b, c), (d, e) \in \mathbb{C}^{2n}\). Then the closed set
\[ \{ x \in X_w \mid (b, c) \mu_w(x) = (d, e) \} \]
is irreducible.

The following lemmas are proved by straightforward calculations.

**Lemma 19.** The map \(\mu_{121}\) maps \(X_{121}\) birationally to the closed subset
\[ Z := \left\{ \begin{bmatrix} C & D \\ E & F \end{bmatrix} \mid D^T = D \right\} \subseteq G. \]
\[ \square \]

Let \(S\) be a sufficiently general codimension-\(n\) subspace of the space of symmetric \(n \times n\)-matrices, let \(T\) be a vector space complement of \(S\) (of dimension \(n\)) and define
\[ X_3 := \left\{ \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} \mid B \in S \right\}, \quad X_4 := \left\{ \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} \mid B \in T \right\} \subseteq X_2, \]
so that the multiplication map \(X_3 \times X_4 \to X_2\) is an isomorphism.

**Lemma 20.** The map \(\mu_{1213}\) maps \(X_{1213}\) birationally onto \(G\).

In particular, by Theorem \ref{18} the collection \(X_1, X_2, X_3\) has the irreducible control property. We now use the ingredients for that theorem to prove Theorem \ref{18}.

**Proof of Theorem \ref{18}** By the previous lemma and \(X_2 \supseteq X_3\), the word \(v := 1212\) is dominant. Hence by Proposition \ref{16} \(w := \overline{v} \dim G + 1\) is open. Hence by Lemma \ref{17} and Proposition \ref{18} any word containing \(w1213\) is irreducible. In particular, this holds for any word containing \(w12134\). Since \(X_3 \times X_4 \to X_2\) is an isomorphism, it holds for any word \(u\) containing \(w1212 = \overline{v} \dim G + 2\).

Finally, for \(Y\) choose the set \(\{ g \in G \mid (b, c) g = (d, e) \}\), which is a coset of the irreducible stabiliser of \((b, c)\) in \(G\), hence irreducible. Therefore \(\mu^{-1}_w(Y)\) is irreducible, as desired.
\[ \square \]

**Remark 21.** Using direct computations it is likely possible to find much shorter open words, so that “sufficiently long” in Theorem \ref{18} becomes a much milder condition. Indeed, for \(n = 2\) and \((b, c) = (0, 0, 0, 1)\) and alternating words starting with 1, \[\text{[IKL20, Lemma 10.5]}\] states that Theorem \ref{18} holds for words of length at least 5 for all values of \((d, e)\), while for the word 1212 certain vectors \((d, e)\) have reducible fibres.
4.3. **The case of general algebraic groups.** Let \( G \) be a connected algebraic group. We will prove Theorem \([8]\).

*Proof of Theorem \([8]\).* Let \( R \) be the unipotent radical of \( G \), and let \( L \) be a Levi complement of \( R \), i.e., a subgroup of \( G \) such that the map \( L \times R \to G \) is an isomorphism of varieties, so that \( G \) is isomorphic to a semi-direct product \( L \rtimes R \); such a group exists by \([Mos55]\). Then \( L \) is a reductive group, and admits a decomposition akin to the LU decomposition in \( \text{GL}_n(\mathbb{C}) \); the details are as follows.

Let \( B_+ \) be a Borel subgroup of \( L \), \( T \) a maximal torus in \( B_+ \), \( U_+ \) the unipotent radical of \( B_+ \), and \( U_- \) the unipotent radical of the Borel group \( B_- \) opposite to \( B_+ \), i.e. such that \( B_- \cap B_+ = T \). By the Bruhat decomposition, the multiplication map \( U_- \times T \times U_+ \to L \) is an isomorphism of varieties with an open subvariety of \( L \) by \([Bor91\text{, Theorem IV.14.12}]\). (For \( L = \text{GL}_n(\mathbb{C}) \), this is just the LU-decomposition seen in \([1.2]\).) In particular, this map is open and birational.

Consequently, also the multiplication map

\[
U_- \times T \times U_+ \times R \to G
\]

is open and birational. Similarly, so is the map

\[
R \times U_+ \times T \times U_- \to G.
\]

Now \( \tilde{B}_+ := T \cdot U_+ \cdot R = R \cdot U_+ \cdot T \) is a subgroup of \( G \)—in fact, a Borel subgroup of \( G \)—and the multiplication maps \( T \times U_+ \times R \to \tilde{B}_+ \) and \( R \times U_+ \times T \to \tilde{B}_+ \) are isomorphisms of varieties. It follows that the multiplication maps

\[
U_- \times \tilde{B} \to G \text{ and } \tilde{B} \times U_- \to G
\]

are both birational and open. Hence, using Proposition \([17]\) as in the proof of Proposition \([1]\) all pre-images of irreducible varieties in \( G \) under the multiplication map

\[
U_- \times \tilde{B} \times \tilde{B} \times U_- \to G
\]

are irreducible. Now use Lemma \([12]\) to conclude that the multiplication map

\[
(2) \quad U_- \times \tilde{B} \times U_- \to G
\]

has the same property.

To find the one-parameter subgroups, we proceed as follows. Set \( l := \dim U_- = \dim U_+ \) and \( m := \dim T \) and \( k := \dim R \).

We have \( T \cong (\mathbb{C})^m \), and this yields \( m \) isomorphic copies \( X_{i+1}, \ldots, X_{i+m} \subseteq T \) of \( \mathbb{C}^* \) such that the multiplication map \( X_{i+1} \times \cdots \times X_{i+m} \to T \) is an isomorphism of varieties (and even of algebraic groups).

Furthermore, for any connected unipotent algebraic group \( H_t \), there exists a basis \( v_1, \ldots, v_p \) of the Lie algebra \( h \) of \( H \) such that the one-parameter subgroups \( H_t := \exp(Cv_t) \) (which are algebraic subgroups!) have the property that the product map \( H_1 \times \cdots \times H_p \to H \) is an isomorphism of varieties.

Applying the previous paragraph to the groups \( U_- \), \( U_+ \), \( R \) of dimensions \( l, l, k \), and combining these with the one-parameter subgroups \( X_{i+1}, \ldots, X_{i+m} \) of \( T \), we find one-parameter subgroups such that the composition of the multiplication maps:

\[
(X_1 \times \cdots \times X_l) \times (X_{i+1} \times \cdots \times X_{i+m}) \times (X_{i+m+1} \times \cdots \times X_{2i+m}) \times (X_{2i+m+1} \times \cdots \times X_{2i+m+k}) \times (X_1 \times \cdots \times X_l) \to U_- \times T \times U_+ \times R \times U_- \to G
\]

has the property that all preimages of irreducible varieties in \( G \) are irreducible, and that, indeed, all words in the alphabet \( \{1, \ldots, 2l + m + k\} \) containing the word
(1, 2, ..., 2l + m + k, 1, 2, ..., l) are irreducible. Finally, we observe that the word above has length
\[3l + m + k < (3/2)(2l + m + k) = 1.5 \cdot \dim G.\]

\[\square\]

**Remark 22.** The proof yields a slightly better bound than \(1.5 \cdot \dim G\), namely, \(\dim G + (\dim L)/2\), where \(L\) is the Levi complement of \(G\), or even \(\dim G + (\dim L - l)/2\), where \(l\) is the rank of \(G\) (and of \(L\)), namely, the dimension of a maximal torus. However, if \(G\) is restricted to the classical groups (roughly speaking, the general linear groups, the orthogonal groups, and the symplectic groups), so that \(L = G\), then \(l\) is proportional to the square root of \(\dim G\), so that this bound is not much better than \(1.5 \cdot \dim G\). It would be interesting to show the existence of irreducible words of shorter length, say around \(\dim G\).

**Example 23.** For \(SL_2(\mathbb{C})\), if in addition to \(X_1, X_2\) from \S 1.1 we take the multiplicative one-parameter subgroup
\[X_3 := \left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \mid t \in \mathbb{C}^* \right\},\]
then the proof above says that the word 2312, and all words containing it, are irreducible.

**References**

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