Let $X$ be a hyperkähler manifold. Trianalytic subvarieties of $X$ are subvarieties which are complex analytic with respect to all complex structures induced by the hyperkähler structure. Given a 2-dimensional complex torus $T$, the Hilbert scheme $T^{[n]}$ classifying zero-dimensional subschemes of $T$ admits a hyperkähler structure. A finite cover of $T^{[n]}$ is a product of $T$ and a simply connected hyperkähler manifold $K^{[n-1]}$, called generalized Kummer variety. We show that for $T$ generic, the corresponding generalized Kummer variety has no trianalytic subvarieties. This implies that a generic deformation of the generalized Kummer variety has no proper complex subvarieties.

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1. Introduction

In [B] Beauville has constructed two series of examples of compact hyperkähler manifolds. The first one consists of the Hilbert schemes $\text{Hilb}$ of points on a K3 surface. The second is the series of the so-called generalized Kummer varieties, one in each even complex dimension. A generalized Kummer variety $K^{[n]}$ is a
subvariety of the Hilbert scheme of \( n + 1 \) points on a 2-dimensional complex torus \( K \), defined as a fiber of the Albanese map.

Let \( M \) be a hyperkähler manifold. The hyperkähler structure induces a 2-dimensional sphere of complex structures on \( M \), called **induced complex structures**. A closed subset of \( M \) is called **trianalytic** if it is complex analytic with respect to all induced complex structures.

In [V7] it was proved that the Hilbert scheme of points on a generic (in the sense of 2.8) K3 surface has no proper trianalytic subvarieties. In this paper we prove the same result for the generalized Kummer varieties. Our methods are an adaptation of those of [V7], greatly simplified by the presence of a group structure on a complex torus. We hope that this paper may serve as an introduction to the more difficult situation of the K3 surface treated in [V7].

Let \( M \) be a compact hyperkähler manifold. For a generic induced complex structure on \( M \), all complex analytic subvarieties of \( M \) are trianalytic. This implies that a generic deformation of a generalized Kummer variety has no complex subvarieties.

### 1.1. Idea of the proof

Here we give a rough sketch of the proof of our main result. The rest of this paper is independent from this Subsection.

Consider a generalized Kummer variety \( K[n] \). To prove that \( K[n] \) has no trianalytic subvarieties, we use the deformation theory of trianalytic subvarieties, developed in [V5].

Let \( M \) be a compact hyperkähler manifold and \( I \) an induced complex structure. We denote by \((M, I)\) the manifold \( M \), considered as a Kähler manifold. The cohomology of \( M \) is equipped with a natural action of the group \( SU(2) \) (see, e.g. [V3]). Let \( X \subset (M, I) \) be a closed complex subvariety. In [V3] (see also Theorem 2.6), it was proven that \( X \) is trianalytic if and only if its fundamental class \( [X] \) is \( SU(2) \)-invariant. Since the fundamental class is deformationally invariant, this implies that all complex analytic deformations of \( X \) are trianalytic.

A slightly more evolved version of this argument ([V5]) implies that the deformation space of \( X \subset (M, I) \) is hyperkähler, and the union of all deformations of \( X \subset M \) is a trianalytic subvariety of \( M \). If the subvariety \( X \subset (M, I) \) has no complex analytic deformations, it is called **rigid**. If \( M \) contains a proper trianalytic subvariety \( X \), then either \( M \) contains a proper rigid trianalytic subvariety \( X' \) (obtained as a union of all deformations of \( X \)), or \( M \) has a finite cover which is a product of two hyperkähler manifolds (see Proposition 3.1). A generalized Kummer variety \( K[n] \) is simply connected, and has \( \dim H^{2,0}(K[n]) = 1 \). Therefore, it cannot have such a finite cover. We obtain that \( K[n] \) has a proper rigid trianalytic subvariety if \( K[n] \) has a proper trianalytic subvariety.

The rest of the argument does not use the hyperkähler structure of \( K[n] \), but only uses the canonical holomorphic symplectic structure. It is well known that a hyperkähler manifold \( M \) is equipped with a canonical holomorphic symplectic structure ([Be]). A trianalytic subvariety \( X \subset (M, I) \) is **non-degenerately symplectic**, that is, the restriction of a holomorphic symplectic form from
(M, I) to X is non-degenerate outside of singularities. Given a non-degenerately symplectic subvariety X ⊂ K[n], we show that X is never rigid. Thus, X cannot be trianalytic.

A generalized Kummer variety is canonically embedded into a Hilbert scheme which is a desingularization of the symmetric power of a torus. Consider the corresponding map K[n] → T^{(n+1)} from the generalized Kummer variety to the n + 1-th symmetric power of a torus. We prove that, for X ⊂ K[n] a non-degenerate symplectic subvariety, the map X → π(X) is finite over the generic point of π(X). This is done using the basic properties of Hilbert schemes (Lemma 6.5).

Consider the subvariety Y := π⁻¹(π(X)) ⊂ K[n]. Over a generic point of π(X), the map π : Y → π(X) is a locally trivial fibration. Using the group structure on T^{n+1}, we show that this fibration admits a canonical flat connection with finite monodromy, i.e., a trivialization over a finite covering (Lemma 6.5). To simplify notations, we assume for the duration of the Introduction that this trivialization is defined globally over the generic part of π(X).

Let F be the generic fiber of π : Y → π(X). Let π(X)_0 denote the generic part of π(X), and Y_0 := π⁻¹(π(X)_0). Then Y_0 = F × π(X)_0. Denote by γ ⊂ F × π(X) the closure of X ∩ Y_0 in F × π(X). Then γ is a correspondence between F and π(X) which is finite over generic points of π(X). Given that X is irreducible, we obtain that γ is also irreducible. Assume now that the complex structure on the torus T is Mumford-Tate generic. Then, by Theorem 2.3 all complex subvarieties Z ⊂ T^{n+1} are trianalytic. Trianalytic subvarieties are hyperkähler in the neighbourhood of every smooth point. Therefore, the dimension dim_C Z is even. We obtain that the variety π(X) has no complex subvarieties of codimension 1. Consider the natural projection p : Γ → F. Let C be a family of divisors passing through every point of F. Unless p(F) is a single point, for each point x ∈ Γ there exists a divisor C_x ∈ C such that p⁻¹(C_x) is a subvariety of codimension 1 in Γ which passes through x. By construction of Γ, the natural projection p' : Γ → π(X) is finite over generic point of π(X). Taking x ∈ Γ generic, we obtain that the projection p'(p⁻¹(C_x)) of the divisor p⁻¹(C_x) to π(X) has codimension 1 in π(X). This gives a contradiction. Therefore, p(Γ) is a single point. We obtain that X ⊂ Y is the closure of a trivial section of a trivial fibration π : Y_0 → π(X)_0. Such a section is determined by the choice of f ∈ F. Varying the choice of f, we obtain a deformation of X. Since X is rigid, the map π : π⁻¹(π(X)) → π(X) is generically finite. The fiber π⁻¹(π(X)) is a product of punctual Hilbert schemes (Lemma 5.3). Therefore, π⁻¹(π(X)) is connected, and consists of a single point. We obtain that X is an irreducible component of the subvariety π⁻¹(π(X)) ⊂ K[n]. Therefore, X is rigid if and only if π(X) is a rigid subvariety of π(K[n]) ⊂ T^{(n+1)}.

Consider the natural action of T on T^{(n+1)}. Let X' ⊂ T^{(n+1)} be a subvariety obtained as a union of t(π(X)), for all t ∈ T. Clearly, if π(X) is rigid in π(K[n]), then X' is rigid in T^{(n+1)}. To prove that K[n] has no trianalytic subvarieties it remains to study rigid subvarieties in T^{(n+1)}.

Let σ : T^{n+1} → T^{(n+1)} be the natural quotient map. Consider the variety D := σ⁻¹(X'). Since T^{n+1} acts on itself by holomorphic automorphisms, the

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variety \( D \) is never rigid. Denote by \( t(D) \) a deformation of \( D \), associated with \( t \in T^{n+1} \). By diagonals of \( T^{n+1} \) we understand subvarieties given by equations of type \( x_i = x_j \) (see (4.1)). Unless \( D \) is a diagonal, \( \sigma(t(D)) \subset T^{(n+1)} \) is a deformation of \( X' \), for appropriate \( t \in T^{n+1} \). Therefore, all rigid subvarieties of \( T^{(n+1)} \) are diagonals. Since the map \( \pi : \pi^{-1}(\pi(X)) \to \pi(X) \) is generically finite, \( \pi(X) \) does not lie in the union of diagonals of \( T^{(n+1)} \). Therefore, \( X' \) is not a diagonal. We obtain that \( \sigma(X) \) is not rigid in \( T^{(n+1)} \), and hence \( X \) is not rigid in \( K^n \). This gives a rough idea of the proof of our result.

2. Definitions and the statement of the theorem

2.1. Recall that a hyperkähler manifold \( M \) is a Riemannian manifold equipped an action of the quaternion algebra \( \mathbb{H} \) in its tangent bundle such that this action is smooth and parallel with respect to the Levi-Civita connection. For excellent introductions to hyperkähler manifolds, we refer the reader to [Be] and [HKLR].

Let \( M \) be a hyperkähler manifold. Every quaternion \( h \in \mathbb{H} \) with \( h^2 = -1 \) induces an almost complex structure on \( M \). It is well-known that all these almost complex structures are integrable. We call them the induced complex structures. The set of induced complex structures is naturally identified with the complex projective line \( \mathbb{C}P^1 \). For every \( I \in \mathbb{C}P^1 \), we denote by \( M_I \) the manifold \( M \) equipped with the corresponding induced complex structure.

2.2. Every hyperkähler manifold \( M \) with any induced complex structure \( M_I \) is canonically holomorphically symplectic. Therefore if \( M \) is compact, then \( \dim H^{2,0}(M_I) \geq 1 \). A simply connected compact hyperkähler manifold \( M \) with \( \dim H^{2,0}(M_I) = 1 \) is called simple. By a theorem of Bogomolov [Bo], every compact hyperkähler manifold has a finite covering which splits into a product of several simple hyperkähler manifolds and a complex torus.

A compact hyperkähler manifold of complex dimension 2 is either a complex torus \( T \), or a K3 surface \( M \). Of these, only the K3 surfaces are simple. For every \( n > 1 \), examples of simple compact hyperkähler manifolds of complex dimension \( 2n \) were constructed by Beauville in [B].

2.3. In this paper we study one of the two classes of hyperkähler manifolds introduced by Beauville, namely, the so-called generalized Kummer varieties. For the convenience of the reader, we reproduce here their definition and main properties.

Let \( T \) be a complex torus of dimension \( k \). Consider the Hilbert scheme \( T^{[n+1]} \) of \( n + 1 \) points on \( T \). This is a complex variety of dimension \( k(n+1) \). The commutative group structure on the torus \( T \) defines a summation map \( \Sigma : T^{n+1} \to T \), which induces a summation map \( \Sigma : T^{[n+1]} \to T \).

**Definition.** The generalized Kummer variety \( K^n \) associated to the torus \( T \) is the preimage \( \Sigma^{-1}(0) \subset T^{[n+1]} \) of the zero \( 0 \in T \) of the group structure on the torus \( T \).

2.4. Assume that the complex torus \( T \) is 2-dimensional. In this case the Hilbert scheme \( T^{[n+1]} \) is smooth. The Kummer variety \( K^n \) associated to \( T \) is also smooth. Moreover, it is simply-connected, and \( \dim H^{2,0}(K^n) = 1 \).
Assume further that the torus $T$ is equipped with a hyperkähler structure. The holomorphic 2-form associated to the hyperkähler structure on the torus $T$ defines a canonical non-degenerate holomorphic 2-form on the Hilbert scheme $T^{[n+1]}$. This form gives by restriction a holomorphic symplectic form $\Omega$ on the Kummer variety $K^{[n]}$. Therefore the canonical bundle of the complex manifold $K^{[n]}$ is trivial. By the Calabi-Yau Theorem $[Y]$, every Kähler class $\alpha \in H^{1,1}(K^{[n]})$ contains a unique Ricci-flat metric. By $[B]$, every one of these metrics together with the form $\Omega$ defines a hyperkähler structure on the Kummer variety $K^{[n]}$. The Kummer variety equipped with any of these hyperkähler structures is a simple compact hyperkähler manifold.

**Caution.** There is no canonical choice for a Kähler structure on the manifold $K^{[n]}$. Therefore, unlike the holomorphic symplectic form, the hyperkähler structure on the Kummer variety $K^{[n]}$ is not defined by the hyperkähler structure on the torus $T$.

2.5. We now recall some general facts on hyperkähler manifolds introduced in $[V1]$, $[V3]$ and $[V6]$. Let $M$ be a hyperkähler manifold, and let $X \subset M$ be a closed subset.

**Definition.** ($[V3]$) The subset $X \subset M$ is called trianalytic if it is analytic for every induced complex structure $I$ on $M$.

2.6. Recall that every analytic subset $X \subset Y$ of dimension $k$ in a compact complex manifold $Y$ of dimension $n$ defines a canonical homology class $[X] \in H_{2k}(Y, \mathbb{C})$ called the fundamental class of the subset $X$. Using the Poincare duality isomorphism $H_{2k}(Y, \mathbb{C}) \cong H^{2n-2k}(Y, \mathbb{C})$, we can consider the fundamental class $[X]$ as an element of the cohomology group $H^{2n-2k}(Y, \mathbb{C})$.

Assume that the hyperkähler manifold $M$ is compact. The $\mathbb{H}$-action in the tangent bundle to $M$ induces a canonical $SU(2)$-action in de Rham algebra of the manifold $M$. By $[V3]$, this action commutes with the Laplacian and induces therefore an $SU(2)$-action in the cohomology $H^*(M, \mathbb{C})$. The following criterion for trianalyticity is proved in $[V3]$:

**Theorem** (Trianalyticity criterion). Let $I$ be an induced complex structure on $M$, and let $N \subset M_I$ be a closed analytic subvariety of the complex manifold $M_I$. Let $[N] \in H^*(M, \mathbb{C})$ be the fundamental class of the subvariety $N$. Then $N$ is trianalytic if and only if the cohomology class $[N]$ is $SU(2)$-invariant.

2.7. Trianalytic subvarieties in hyperkähler manifolds have many special properties. Of these, the most important to us will be the following theorem proved in $[V6]$.

**Desingularization Theorem.** Let $X \subset M$ be a trianalytic variety in a compact hyperkähler manifold $M$, and let $I$ be an induced complex structure on $M$.

The normalization $\tilde{X} \to M_I$ of the complex-analytic subvariety $X \subset M_I$ is smooth, and the canonical projection $\tilde{X} \to M$ induces a hyperkähler structure on the smooth manifold $\tilde{X}$.

2.8. The goal of this paper is to study trianalytic subvarieties in the Kummer variety associated to a generic hyperkähler torus of complex dimension 2. The
notion of genericity appropriate for our purposes is the following one, introduced in \[V7\].

**Definition.** Let $X$ be a compact hyperkähler manifold. An induced complex structure $I$ on $X$ is called *Mumford-Tate generic* with respect to the hyperkähler structure if for all $n > 0$, every cohomology class
\[
\alpha \in \bigoplus_p H^{p,p}(X^n) \cap H^{2p}(X^n, \mathbb{Z}) \subset H^*(X^n, \mathbb{C})
\]
is invariant under the canonical $SU(2)$-action.

As proved in \[V2\], for every compact hyperkähler manifold $X$ all the induced complex structures on $X$ except for a countable number are Mumford-Tate generic. If an induced complex structure $I$ on a hyperkähler manifold $X$ is Mumford-Tate generic, then it is obviously also Mumford-Tate generic on any power $X^l$ of the manifold $X$. Moreover, by the Trianalyticity Criterion of \[V3\] every complex-analytic subvariety $Y \subset X_I$ is trianalytic and, in particular, even-dimensional.

**2.9.** We can now formulate our main result.

**Theorem.** Let $T$ be a 2-dimensional complex torus equipped with a hyperkähler structure. Assume that the complex structure on $T$ is Mumford-Tate generic, and consider a generalized Kummer variety $K^n$ associated to $T$.

For any hyperkähler structure on $K^n$ compatible with the canonical holomorphic 2-form, every irreducible trianalytic subvariety $X \subset K^n$ is either the whole $K^n$ or a single point.

The proof of Theorem 2.9 takes up the rest of this paper.

**2.10.** We finish this section with the following corollary of Theorem 2.9.

**Corollary.** Assume that the Kummer variety $K^n$ is equipped with a complex structure $I$ which is Mumford-Tate generic with respect to some hyperkähler structure on $K^n$ compatible with the canonical holomorphic 2-form.

Then every irreducible analytic subvariety $X \subset K^n_I$ is either the whole $K^n$ or a single point.

**Proof.** Indeed, by Theorem 2.9, every analytic subvariety in $K^n_I$ is trianalytic. □

**Caution.** No matter which hyperkähler structure on the Kummer variety $K^n$ we take, the standard complex structure on $K^n$ which comes from the embedding $K^n \subset T^{[n+1]}$ into the Hilbert scheme of the torus $T$ is not Mumford-Tate generic.

**3. Reduction to the case of rigid subvarieties**

**3.1.** In this section we give our first reduction of Theorem 2.9. Namely, call a subvariety $X \subset Y$ in a complex variety $Y$ *rigid* if it admits no local deformations. In this section we prove the following.
Proposition. Let $Y$ be a complex manifold of dimension $n$ equipped with some hyperkähler structure. Assume in addition that the manifold $Y$ is simply connected and that $\dim H^{2,0}(Y) = 1$.

If the manifold $Y$ admits a subvariety $X \subset Y$ of dimension $k$, $0 < k < n$, which is trianalytic with respect to the hyperkähler structure on $Y$, then it also admits a rigid subvariety of dimension $m$, $k \leq m < n$, trianalytic with respect to the hyperkähler structure on $Y$.

All the generalized Kummer varieties $K^{[n]}$ are simply connected and have $\dim H^{2,0}(K^{[n]}) = 1$. Thus to prove Theorem 2.9 it suffices to prove that for a generic torus $T$ the associated Kummer variety $K^{[n]}$ has no proper rigid subvarieties trianalytic with respect to some hyperkähler structure.

3.2. Before we prove Proposition 3.1, we need to recall several facts about moduli spaces of subvarieties in a complex manifold. Let $Y$ be a compact complex manifold, and let $a \in H^q(Y, \mathbb{C})$ be a cohomology class of the manifold $Y$. Douady [D] had constructed a moduli space $S(Y, a)$ of complex subvarieties in $Y$ with fundamental class $a$. The Douady moduli space is equipped with the family $X \to S(Y, a)$ and the universal map $X \to Y$ which coincide near every point $s \in S(Y, a)$ with the universal family provided by the local deformation theory. If the manifold $Y$ is Kähler, the Douady moduli space $S(Y, a)$ is compact ([Fu], [L]).

3.3. The proof of Proposition 3.1 is based on the following general fact.

Proposition. Let $X$ and $Y$ be compact hyperkähler manifolds. Assume given an immersion $f : X \to Y$ which is an embedding on a dense open subset and compatible with the hyperkähler structure. Fix a complex structure $I$ on $Y$. Let $[f(X)]$ be the fundamental class of the analytic subset $f(X) \subset Y$, and let $S = S(Y, [f(X)])$ be the Douady moduli space of complex subvarieties in $Y$ with fundamental class $[f(X)]$.

Then the complex-analytic space $S$ is a compact smooth hyperkähler manifold. Moreover, the total space $\tilde{X}$ of the universal family $\tilde{X} \to S$ is also a smooth hyperkähler manifold, and the canonical map $\tilde{X} \to Y$ is an immersion compatible with the hyperkähler structure. Finally, the projection $\tilde{S} \to S$ carries a canonical flat holomorphic connection.

3.4. The proof of Proposition 3.3 begins in 3.7 and takes up the rest of this section. However, first we deduce Proposition 3.1 from Proposition 3.3. Indeed, assume given a trianalytic subvariety $X_0 \subset Y$ in a complex variety $Y$ equipped with some hyperkähler structure. Consider the normalization $X \to X_0$ of the analytic subvariety $X_0 \subset Y$. By the Desingularization Theorem of [V6] the normalization $X \to X_0$ is a smooth hyperkähler manifold, and the map $X \to Y$ is a trianalytic immersion. Moreover, it is an embedding outside of the preimage in $X$ of the subset of singular points of $X_0$. Therefore Proposition 3.3 applies to $X \to Y$. Hence the universal family $\tilde{X} = X \times S$ of deformations of $X \to Y$ is a smooth hyperkähler manifold, and the canonical map $f : \tilde{X} \to Y$ is an immersion.
Since $\tilde{X}$ is compact, the image $f(\tilde{X}) \subset Y$ is a trianalytic subvariety, and $\widetilde{X} \to Y$ is the normalization of the subvariety $f(\tilde{X}) \subset Y$. We claim that the subvariety $f(\tilde{X}) \subset Y$ is rigid. Indeed, by Proposition 3.3 every deformation $\tilde{X}' \to Y$ of $\tilde{X} \to Y$ is a hyperkähler manifold isometric (hence isomorphic) to $\tilde{X}$. In particular, we have a fibration $\tilde{X}' \cong \tilde{X} \to S$, and for every point $s \in S$ the fiber $X_s' \to Y$ of this fibration over the point $s$ is a deformation of the corresponding fiber $X_s \to Y$ of the family $\tilde{X} \to S$. Consequently the family $\tilde{X}' \to Y$ is a family of deformations of $X \to Y$. Since the family $\tilde{X} \to Y$ is universal, we must have $\tilde{X}' = \tilde{X}$. Therefore the subvariety $f(\tilde{X}) \subset Y$ is rigid.

To prove Proposition 3.3, it remains to show that $\dim \tilde{X}$ is strictly less than $n$. Indeed, assume that $\dim \tilde{X} = n$, so that the immersion $f : \tilde{X} \to Y$ is étale. By Proposition 3.3 the fibration $f : \tilde{X} \to S$ carries a flat holomorphic connection. Since its fibers are compact, we can take a finite cover $S' \to S$ such that the pullback $\tilde{X} \times_S S'$ splits into a product

$$\tilde{X} \times S' \cong X \times S'.$$

Since the manifold $Y$ is by assumption simply connected, the étale map $X \times S' \to Y$ is an isomorphism. But by assumption both $X$ and $S'$ are compact hyperkähler manifolds. Therefore $\dim H^{2,0}(X) \geq 1$ and $\dim H^{2,0}(S') \geq 1$. By the Künneth formula $\dim H^{2,0}(X \times S') \geq 2$, which contradicts $\dim H^{2,0}(Y) = 1$.

3.5. Proposition 3.3 is a simple corollary of the results of [V5], where an analogous statement was proved for not necessarily smooth trianalytic subvarieties $X \subset Y$. For the convenience of the reader, we sketch here an alternative proof using the twistor spaces.

Let $f : X \to Y$ be as in the statement of Proposition 3.3, and let $\pi : \mathcal{Z} \to \mathcal{C}P^1$ be the twistor space of the hyperkähler manifold $Y$. The complex manifold $Y_I$ is embedded into $\mathcal{Z}$ as the fiber over $I \in \mathcal{C}P^1$. Moreover, as a smooth manifold, the twistor space $\mathcal{Z}$ is canonically isomorphic to the product $\mathcal{C}P^1 \times Y$. Therefore by Künneth formula $H^*(\mathcal{Z}, \mathbb{C}) = H^*(Y, \mathbb{C}) \otimes H^*(\mathcal{C}P^1, \mathbb{C})$.

Let $[f(X)]$ be the fundamental class of the complex subvariety $f(X_I) \subset \mathcal{Z}$, and consider the Douady space $S(\mathcal{Z}, [f(X)])$ of subvarieties in $\mathcal{Z}$ with fundamental class $[f(X)]$. Since the projection $\pi : \mathcal{Z} \to \mathcal{C}P^1$ is proper, for every analytic subvariety $X_s \subset \mathcal{Z}$ the image $\pi(X_s) \subset \mathcal{C}P^1$ is also an analytic subvariety. Therefore it is either the whole $\mathcal{C}P^1$, or a union of several points.

Every point $s \in S(\mathcal{Z}, [f(X)])$ in the Douady space corresponds to a subvariety $X_s \subset \mathcal{Z}$. Since the variety $\mathcal{Z}$ is proper over $\mathcal{C}P^1$, the subset

$$S_{gen}(\mathcal{Z}, [f(X)]) \subset S(\mathcal{Z}, [f(X)])$$

of points $s \in S(\mathcal{Z}, [f(X)])$ such that $\pi(X_s) = \mathcal{C}P^1$ is open in the Douady space $S(\mathcal{Z}, [f(X)])$. Denote by $\mathcal{G} \subset S(\mathcal{Z}, [f(X)])$ the subset of points $s \in S(\mathcal{Z}, [f(X)])$ such that the image $\pi(X_s) \subset \mathcal{C}P^1$ of the corresponding subvariety $X_s \subset \mathcal{Z}$ consists of a single point. The subset $\mathcal{G} \subset S(\mathcal{Z}, [f(X)])$ is obviously a union of connected components of the complement $S(\mathcal{Z}, [f(X)]) \setminus$$

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$S_{\text{gen}}(\mathbb{P}, [f(X)])$. Therefore it is closed in the Douady space $S(\mathbb{P}, [f(X)])$, hence also a complex variety. In order to prove that the Douady space $S$ is hyperkähler, we will identify $\mathcal{G}$ with its twistor space.

The correspondence $s \mapsto \pi(X_s) \in \mathbb{C}P^1$ defines a holomorphic map $\pi : \mathcal{G} \to \mathbb{C}P^1$. Moreover, let $J = \pi(s) = \pi(X_s) \in \mathbb{C}P^1$, so that we have $X_s \subset Y_J \subset \mathbb{P}$. By the Künneth formula, the fundamental class $[X_s] \in H^*(Y_J) = H^*(Y)$ in the cohomology of the fiber $Y_J$ coincides with the class of $f(X)$ in the cohomology of $Y$. Therefore the fiber $S_J = \pi^{-1}(s)$ of the space $\mathcal{G}$ over the point $J \in \mathbb{C}P^1$ coincides with the Douady space of subvarieties of $Y_J$ with fundamental class $[f(X)]$. This applies, in particular, to the case $J = J$, so that $S_J$ coincides with the space $S$.

3.6. We first prove that the variety $\mathcal{G}$ is smooth and that the projection $\pi : \mathcal{G} \to \mathbb{C}P^1$ is submersive at every point $s \in \mathcal{G}$.

Let $X_s \subset \mathbb{P}$ be the subvariety corresponding to the point $s$. We have $[X_s] = [f(X)]$. By the Trianalyticity Criterion of $\mathcal{G}$, this implies that the submanifold $X_s \subset Y_J$ is trianalytic. Let $f_s : \tilde{X}_s \to Y_J$ be its normalization. By the Desingularization Theorem of $\mathcal{G}$ the map $f_s$ induces on $X_s$ the structure of a smooth hyperkähler manifold. Since the same applies to all deformations of $X_s \subset \mathbb{P}$ as well, the local universal moduli space for deformations of the subvariety $X_s \subset \mathbb{P}$ coincides with the local deformation space for the pair $(\tilde{X}_s, f_s)$ of the smooth manifold $\tilde{X}_s$ and the map $f_s : \tilde{X}_s \to \mathbb{P}$.

Recall that we have a canonical short exact sequence

$$0 \longrightarrow \mathcal{T}(\tilde{X}_s) \longrightarrow f_s^*(\mathcal{T}(\mathbb{P})) \longrightarrow \mathcal{N}(f_s) \longrightarrow 0$$

of holomorphic vector bundles on $\tilde{X}_s$, where $\mathcal{T}(\tilde{X}_s)$ and $\mathcal{T}(\mathbb{P})$ are tangent bundles, and $\mathcal{N}(f_s)$ is by definition the normal bundle to the map $f_s$. By general deformation theory, the formal completion of the universal local moduli space for deformations of $f_s : \tilde{X}_s \to \mathbb{P}$ is isomorphic to the formal neighborhood of $0$ in the certain cone $\mathcal{C}_s$ in the space $H^0(X_s, \mathcal{N}(f_s))$ of global sections of the normal bundle $\mathcal{N}(f_s)$.

To prove this, we split the normal bundle $\mathcal{N}(f_s)$ in two pieces in the following way. Since $X_s \subset Y_J$, we can consider the map $f_s : \tilde{X}_s \to \mathbb{P}$ as a map $f_s : \tilde{X}_s \to Y_J$ into the fiber $Y_J \subset \mathbb{P}$ over the point $J \in \mathbb{C}P^1$. Let $\mathcal{N}^\perp(f_s)$ be the normal bundle to $f_s : \tilde{X}_s \to Y_J$, and let $\mathcal{C}_s^\perp \subset H^0(\mathcal{N}^\perp(f_s))$ be the Massey cone corresponding to deformations of $f_s : X_s \to Y_J$ inside the fiber $Y_J$.

The normal bundle $\mathcal{N}(f_s)$ splits canonically

$$\mathcal{N}(f_s) = f_s^*\pi^*T_J(\mathbb{C}P^1) \oplus \mathcal{N}^\perp(f_s)$$

into the sum of two bundles. The first is the pullback to $X_s$ of the normal bundle to the fiber $Y_J \subset \mathbb{P}$, which is isomorphic to the constant rank-1 bundle $\pi^*T_J(\mathbb{C}P^1)$ whose fiber is the tangent space $T_J(\mathbb{C}P^1)$ to $\mathbb{C}P^1$ at the point $J$. 

The second is the normal bundle $N^\perp(f_s)$ to the map $f_s : X \to Y_J$. This splitting, in turn, induces a splitting
\[
H^0(\tilde{X}_s, N(f_s)) = H^0(\tilde{X}_s, f_s^* \pi^* T_J(CP^1)) \oplus H^0(\tilde{X}_s, N^\perp(f_s)) = T_J(CP^1) \oplus H^0(\tilde{X}_s, N^\perp(f_s)),
\]
and the cone $C_s$ lies in the product $T_J(CP^1) \times C^\perp_s$.

Now, by the Trianalyticity Criterion of [V3] (Theorem 2.6) the subvariety $X_s$ and all its deformations are trianalytic. Therefore the cone $C_s$ contains the direct product $T_J(CP^1) \times C^\perp_s$, hence coincides with it. This proves that the projection $\pi : S \to CP^1$ is submersive at $s \in S$.

In order to prove that the variety $S$ is smooth at $s \in S$, it remains to prove that the Massey products on the normal bundle $N^\perp(X_s)$ vanish, so that the cone $C^\perp_s$ is the whole space $H^0(X_s, N^\perp(f_s))$. This follows from the splitting of the canonical exact sequence
\[
0 \longrightarrow T(\tilde{X}_s) \longrightarrow f_s^*(T(Y_J)) \longrightarrow N^\perp(f_s) \longrightarrow 0
\]
(3.1)
of holomorphic bundles on $X_s$. This splitting was established in [V4]. As explained in [V7], it follows from the fact that all these bundles are hyperholomorphic.

3.7. Recall that a pseudo-Riemannian manifold is a manifold equipped with a nowhere degenerate symmetric 2-form in the tangent bundle, not necessarily positively defined. A pseudo-Kähler manifold is a complex manifold equipped with a pseudo-Riemannian structure $(\cdot, \cdot)$, such that the complex structure operator $I$ is orthogonal, and the corresponding skew-symmetric 2-form $(\cdot, I \cdot)$ is closed. A pseudo-hyperkähler manifold is a pseudo-Riemannian manifold equipped with three complex structures, satisfying quaternion relations, which is pseudo-Kähler with respect to these complex structures. For every pseudo-hyperkähler manifold, one can define its twistor space, exactly as one does it for a hyperkähler manifold.

3.8. By the general theory of twistor spaces developed in [HKLR], to prove that $S$ is a twistor space for a pseudo-hyperkähler structure $H$ on $S$, it remains to prove the following.

(a) There exists an antiholomorphic involution $S \to \overline{S}$ compatible with the antipodal involution on $CP^1$.

(b) For every point $s \in S$ there exists a real section $\tilde{s} : CP^1 \to S$ of the projection $\pi : S \to CP^1$ passing through $s \in S$.

(c) For every such section $\tilde{s} : CP^1 \to S$ the normal bundle $N(\tilde{s})$ is a sum of several copies of the bundle $O(1)$ on $CP^1$.

(d) There exists a relative $O(2)$-valued non-degenerate holomorphic 2-form on $S$ over $CP^1$. 

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Defined this way pseudo-hyperkähler structure $\mathcal{H}$ is unique.

**3.9.** The involution required in (a) is induced by the canonical involution on the twistor space $\mathfrak{Y}$. Indeed, this involution sends subvarieties into subvarieties and acts identically on the subspace $H^*(Y_\ell, \mathbb{C}) \subset H^*(\mathfrak{Y}, \mathbb{C})$. Therefore it preserves the fundamental class $[f(X)] \in H^*(Y_\ell, \mathbb{C}) \subset H^*(\mathfrak{Y}, \mathbb{C})$.

The claim (b) follows from the fact that every deformation $X_s$ of the manifold $X$ is trianalytic in its fiber $Y_\ell \subset \mathfrak{Y}$ of the twistor projection $\pi: \mathfrak{Y} \to \mathbb{C}P^1$. Indeed, to obtain such a real section, it suffices to take the twistor space $f_s : X_s \subset \mathfrak{Y}$ of the desingularization $\tilde{X}_s$ of the trianalytic submanifold $X_s$ and let $\tilde{s} : \mathbb{C}P^1 \to \mathcal{S}$ map a point $J' \in \mathbb{C}P^1$ into the point in $\mathcal{S}$ corresponding to the subvariety $f_s(\pi^{-1}(J) \cap X_s)$.

The normal bundle $\mathcal{N}(\tilde{s})$ to this real section coincides with the direct image $\pi_*\mathcal{N}(f_s)$ of the normal bundle $\mathcal{N}(f_s)$ to the map $f_s : X_s \to \mathfrak{Y}$. Therefore (c) follows from \[V4\]. Finally, under the identification

$$\mathcal{N}(\tilde{s}) \cong \pi_*(\mathcal{N}(f_s))$$

the holomorphic 2-form required in (d) is induced by the canonical holomorphic 2-form on the bundle $\mathcal{N}(f_s)$, and it is non-degenerate by virtue of the splitting of the exact sequence \[3.1\].

To prove that $S$ is not only pseudo-hyperkähler but hyperkähler, we need to check some positivity conditions, which is easy.

**3.10.** To finish the proof of Proposition 3.3, it remains to prove that the universal family map $\tilde{X} \to Y$ is an immersion and to construct a flat holomorphic connection on the fibration $\tilde{X} \to S$. For this we refer the reader to \[V5\], noting only that the splitting of the exact sequence \[3.1\] is crucial for both these facts.

\[\square\]

**4. Stratification by diagonals on a Hilbert scheme**

**4.1.** In order to proceed in our proof of Theorem 2.9 we need to recall several facts on the geometry of Hilbert schemes $\text{Hilb}$ of points on complex manifolds.

Let $M$ be a complex manifold of dimension $k$, and let $M^{(n)} = M^n/\Sigma_n$ be the $n$-th symmetric power of the manifold $M$. By definition the Hilbert scheme $M^{(n)}$ of 0-dimensional subschemes in $M$ of length $n$ maps into the space $M^{(n)}$. The map $M^{(n)} \to M^{(n)}$ is proper. Its fibers are isomorphic to products of punctual Hilbert schemes of dimension-0 subschemes of $\mathbb{C}^k$ concentrated at 0.

**4.2.** The variety $M^{(n)}$ is singular. However, it admits a canonical stratification with non-singular strata. The strata of this stratification are numbered by Young diagrams of length $n$, that is, sequences of positive integers $k_1 \leq k_2 \leq \ldots \leq k_l$ such that $\sum k_i = n$. The stratum $M^{(n)}_\Delta$ corresponding to a Young diagram $\Delta = (k_1 \leq \ldots \leq k_l)$ is by definition the subvariety in $M^{(n)}$ consisting of orbits
of points \((a_1, \ldots, a_n) \in M^n\) such that

\[
\begin{align*}
a_1 &= a_2 = \ldots = a_{k_1} \\
a_{k_1+1} &= a_{k_1+2} = \ldots = a_{k_1+k_2} \\
&\quad \ldots \\
a_{k_1+\ldots+k_{l-1}+1} &= a_{k_1+\ldots+k_{l-1}+2} = \ldots = a_n,
\end{align*}
\]

(4.1)

and neither of \(a_{k_1}, a_{k_1+k_2}, \ldots, a_n\) is equal to any other. Every stratum \(M_\Delta^{(n)}\) is smooth. It is isomorphic to the quotient

\[
(M_1 \times M_2 \times \cdots \times M_l \setminus \text{Diag}) / \Sigma_\Delta,
\]

where \(M_1, \ldots, M_l\) are \(l\) copies of the manifold \(M\), \(\text{Diag} \subset M_1 \times \cdots \times M_l\) is the subset of diagonals, and \(\Sigma_\Delta \subset \Sigma_l\) is the subgroup in the symmetric group on \(l\) letters consisting of transpositions which fix the sequence \(\Delta = (k_1, k_2, \ldots, k_l)\). We will call this canonical stratification on the variety \(M^{(n)}\) the stratification by diagonals.

4.3. The stratification by diagonals on the variety \(M^{(n)}\) induces a stratification on the Hilbert scheme \(M^{[n]}\). The strata \(M_\Delta^{[n]}\) are no longer necessarily smooth. The fiber of the canonical proper map \(M_\Delta^{[n]} \to M^{(n)}\) at a point \((a_1, \ldots, a_n) \in M^{(n)}\) is isomorphic to the product of punctual Hilbert schemes of subschemes in \(M\) of lengths \(k_1, \ldots, k_l\) concentrated at points \(a_{k_1}, \ldots, a_n \in M\).

Denote by \(\eta : M^l \setminus \text{Diag} \to M^{(n)}\) the canonical Galois covering with the Galois group \(\Sigma_\Delta\), and let

\[
\widehat{M}_\Delta^{[n]} = M_\Delta^{[n]} \times_{M^{(n)}} (M^l \setminus \text{Diag})
\]

be the pullback of the stratum \(M_\Delta^{[n]}\) of the Hilbert scheme \(M^{[n]}\) with respect to this covering.

4.4. The variety \(\widehat{M}^{[n]}\) admits a modular interpretation. Indeed, it is the moduli space of pairs

\[
\begin{cases}
\text{l different points } a_1, \ldots, a_l \in M \\
\text{for each } i, 1 \leq i \leq l, \text{ a subscheme } Z_i \subset M \text{ of length } k_i \text{ concentrated} \\
\text{at the point } a_i \in M.
\end{cases}
\]

This interpretation allows one to construct a canonical compactification of the moduli space \(\widehat{M}_\Delta^{[n]}\). Namely, it embeds as a dense open subset in the larger moduli space of pairs

\[
\begin{cases}
\text{l points } a_1, \ldots, a_l \in M, \text{ not necessarily different} \\
\text{for each } i, 1 \leq i \leq l, \text{ a subscheme } Z_i \subset M \text{ of length } k_i \text{ concentrated} \\
\text{at the point } a_i \in M.
\end{cases}
\]

(4.2)

This larger moduli space is compact. We denote it by \(\overline{M}_\Delta^{[n]}\).
4.5. Unfortunately, it seems that the canonical Galois covering \( \eta : \widetilde{M}_n^\Delta \to M_n^\Delta \subset M[n] \) does not extend to the compactification \( \overline{M}_n^\Delta \supset \widetilde{M}_n^\Delta \).

However, the map \( \eta \) does extend to \( \overline{M}_n^\Delta \) in a weaker sense. Recall that a meromorphic map \( f : X \to Y \) from a complex variety \( X \) to a complex variety \( Y \) is by definition an analytic subvariety \( \Gamma \subset X \times Y \) such that for an open dense subset \( U \subset X \) the canonical projection

\[
\Gamma \cap (U \times Y) \to U
\]

is one-to-one.

If the varieties \( X \) and \( Y \) are algebraic, then every algebraic map \( f : U \to Y \) from an open dense subset \( U \subset X \) trivially extends to a meromorphic map \( X \to Y \). However, for general complex varieties this is no longer true.

Lemma. The projection \( \eta : \widetilde{M}_n^\Delta \to M_n^\Delta \) extends to a meromorphic map \( \eta : \overline{M}_n^\Delta \to M[n] \) from the compactification \( \overline{M}_n^\Delta \) into the Hilbert scheme \( M[n] \).

Proof. Note that the lower arrow \( \eta : \widetilde{M}_n^\Delta \to M_n^\Delta \) in the Cartesian square

\[
\begin{array}{ccc}
\widetilde{M}_n^\Delta & \to & M_n^\Delta \\
\downarrow \pi & & \downarrow \pi \\
M^l \setminus \text{Diag} & \to & M(n),
\end{array}
\]

does extend to a holomorphic map

\[
\eta : M^l \to M(n).
\]

Let \( \overline{M}_n^\Delta \) be the fibered product given by the Cartesian square

\[
\begin{array}{ccc}
\overline{M}_n^\Delta & \to & M_n^\Delta \\
\downarrow \eta & & \downarrow \pi \\
M^l & \to & M[n],
\end{array}
\]

Both varieties \( \overline{M}_n^\Delta \) and \( \overline{M}_n^\Delta \) are proper over the complex manifold \( M^l \). Moreover, the preimage of the open subset \( (M^l \setminus \text{Diag}) \subset M^l \) in each of these varieties is canonically isomorphic to the variety \( \overline{M}_n^\Delta \).

Since by construction we have a map \( \eta : \overline{M}_n^\Delta \to M[n] \), it suffices to prove that the embedding \( \iota : \widetilde{M}_n^\Delta \to \overline{M}_n^\Delta \) extends to a meromorphic map \( \iota : M_n^\Delta \to \overline{M}_n^\Delta \). In other words, we have to prove that

- the closure \( \text{graph} \iota \subset \overline{M}_n^\Delta \times \overline{M}_n^\Delta \) of the graph

\[
\text{graph} \iota \subset \overline{M}_n^\Delta \times \overline{M}_n^\Delta \subset \overline{M}_n^\Delta \times M_n^\Delta
\]

of the embedding \( \iota : \overline{M}_n^\Delta \to \overline{M}_n^\Delta \) is an analytic subvariety.
The statement (●) is obviously local on $M^I$, therefore it suffices to prove it in a neighborhood of every point $m \in M^I$. Moreover, it is non-trivial only over a neighborhood of the subset of diagonals $\text{Diag} \subset M^I$. By induction, it suffices to prove it over the neighborhood of the smallest diagonal $M \subset M^I$, or, in other words, over subsets of the form $U^I \subset M^I$ for a neighborhood $U \subset M$ of every point $m \in M$.

But all the varieties and maps in (●) admit modular interpretations as Hilbert schemes of points on $M$ with some additional conditions. Therefore everything in (●) is functorial with respect to open embeddings. Consequently, for every open subset $U \subset M$ the statement (●) holds over $U^I \subset M^I$ if and only if it hold for $U$ in place of $M$.

It remains to notice that, since all complex manifolds of the same dimension $k$ are locally isomorphic, the statement (●) holds for every one of them if and only if it holds for an arbitrary single manifold $M$. However, for the algebraic manifold $M = \mathbb{C}^k$ the statement (●) holds trivially. Therefore it must hold for every complex manifold $M$ of dimension $k$. \[\square\]

5. Stratification of the Kummer variety

5.1. Everything said in the last section was valid for an arbitrary complex manifold $M$. The main result of this section, Lemma 5.5, is specific for the case of a complex torus. It is this result that makes the proof of Theorem 2.9 for the Kummer varieties much simpler than the proof of the analogous statement for K3 surfaces given in [V7].

5.2. Let $T$ be a complex torus. Let $T^{[n+1]}$ be the Hilbert scheme of 0-dimensional subschemes of length $n + 1$ in the torus $T$, and let $K^{[n]} \subset T^{[n+1]}$ be the associated generalized Kummer variety.

The summation map $\Sigma : T^{n+1} \rightarrow T$ factors through the symmetric power $T^{(n+1)} \rightarrow T$. Denote by $K^{(n)} \subset T^{(n+1)}$ the preimage $\Sigma^{-1}(0) \subset T^{(n+1)}$ of the zero $0 \in T$. The canonical projection $\pi : T^{[n+1]} \rightarrow T^{(n+1)}$ commutes with the summation map and defines therefore a proper map $\pi : K^{[n]} \rightarrow K^{(n)}$.

5.3. The stratification by diagonals on the variety $T^{(n+1)}$ induces a stratification on the varieties $K^{(n)}$ and $K^{[n]}$. For every Young diagram $\Delta$ of length $n + 1$, denote by

\[
\begin{align*}
K^{(n)}_{\Delta} &= T^{(n+1)}_{\Delta} \cap K^{[n]} \subset T^{(n+1)} \\
K^{[n]}_{\Delta} &= T^{[n+1]}_{\Delta} \cap K^{[n]} \subset T^{[n+1]}
\end{align*}
\]

the corresponding strata of this stratification.

The canonical Galois covering $\eta : (T^I \setminus \text{Diag}) \rightarrow T^{(n+1)}_{\Delta}$ also commutes with the summation map. Therefore it defines a Galois covering $\eta : T^I_0 \setminus \text{Diag} \rightarrow K^{[n]}_{\Delta}$.
where $T^l_0 \subset T^l$ is the kernel of the summation map $\Sigma : T^l \to T$. Let

$$\bar{K}^{[n]} = (T^l \setminus \text{Diag}) \times_{K^{[n]}} K^{[n]} \subset \bar{T}_\Delta^{[n+1]}$$

be the pullback variety. The summation map $\Sigma : \bar{T}_\Delta^{[n+1]} \to T$ extends to the compactification $\bar{T}_\Delta^{[n+1]} \supset \bar{T}_\Delta^{[n]}$, which gives a compactification

$$\bar{K}^{[n]} \subset \bar{K}^{[n]} = \Sigma^{-1}(0) \subset \bar{T}_\Delta^{[n+1]}.$$

5.4. Lemma 4.5 immediately implies that the map $\eta : \bar{K}^{[n]} \to K^{[n]}$ extends to a meromorphic map $\eta : \bar{K}^{[n]} \to K^{[n]}$. We will need the following corollary of this lemma.

**Corollary.** Let $X \subset \bar{K}^{[n]}_\Delta$ be an analytic subset. If the closure $\overline{X} \subset \bar{K}^{[n]}_\Delta$ is an analytic subset in the variety $\bar{K}^{[n]}_\Delta$, then the closure

$$\overline{\eta(X)} \subset K^{[n]}$$

of the image $\eta(X) \subset K^{[n]}$ is also an analytic subvariety.

**Proof.** Indeed, since the meromorphic map $\eta : \bar{K}^{[n]}_\Delta \to K^{[n]}_\Delta$ is given by a proper correspondence, the direct image $\overline{\eta(X)} \subset K^{[n]}_\Delta$ is an analytic subvariety. The closure $\overline{\eta(X)} \subset K^{[n]}$ is a union of irreducible components of the subvariety $\overline{\eta(X)}$.

5.5. For any integer $l \geq 2$, denote by $F_l$ the punctual Hilbert scheme of length $l$, that is, the moduli space of 0-dimensional subschemes in $\mathbb{C}^l$ of length $l$ concentrated at 0. For any Young diagram $\Delta = \langle k_1 \leq \cdots \leq k_l \rangle$ let $F_\Delta = F_{k_1} \times \cdots \times F_{k_l}$ be the product of $l$ such Hilbert schemes of lengths $k_1, \ldots, k_l$. The main result of this section is the following.

**Lemma.** There exists a direct product decomposition

$$K^{[n]}_\Delta = F_\Delta \times T^l_0.$$  \hfill (5.1)

**Proof.** Recall that the variety $\bar{T}_\Delta^{[n+1]}$ is the moduli space of data $\langle \bar{I}, \bar{Z} \rangle$. The group $T^l$ acts on these data by left translations, which induces a $T^l$-action on the moduli space $\bar{T}_\Delta^{[n+1]}$. This action is free and gives a decomposition

$$\bar{T}_\Delta^{[n+1]} \simeq F_\Delta \times T^l.$$  

This decomposition is compatible with the summation map and induces the desired decomposition (5.1). \qed

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6. Subvarieties of a stratum $K_{\Delta}^{[n]}$

6.1. Up to this point we did not need any facts on the torus $T$ except for its group structure. From now on, assume that the complex torus $T$ is 2-dimensional and equipped with a hyperkähler structure. Moreover, assume that the complex structure on $T$ is Mumford-Tate generic with respect to this hyperkähler structure in the sense of 2.8. In particular, there are no analytic subvarieties in $T$ except for $T$ itself and unions of its points.

Since $\dim T = 2$, the Hilbert scheme $T^{[n+1]}$ and the Kummer variety $K^{[n]}$ are smooth. The hyperkähler structure on the torus $T$ induces a natural holomorphic symplectic structure on the associated Kummer variety $K^{[n]}$. Fix once and for all a hyperkähler structure on $K^{[n]}$ compatible with the canonical holomorphic symplectic form.

6.2. The hyperkähler structure on the torus $T$ induces a canonical hyperkähler structure on the powers $T^l$ of $T$. The first consequence of the genericity of the hyperkähler structure on the torus $T$ is the following.

Lemma. Every analytic subvariety $X \subset T^l$ is trianalytic.

Proof. By the Trianalyticity Criterion of [V3] it suffices to prove that the fundamental class $[X] \in H^*(T^l, \mathbb{C})$ is invariant under the canonical $SU(2)$-action. But since $T$ is Mumford-Tate generic, every Hodge cohomology class in $H^*(T^l, \mathbb{C})$ is $SU(2)$-invariant. □

6.3. Applying the theory of trianalytic subvarieties developed in [V5], we get the following stronger statement.

Lemma. Let $X \subset T^l$ be an analytic subvariety, and denote by

$$\pi : \mathbb{C}^2l \to T^l$$

the universal covering map.

Then every irreducible component of the variety $X$ is a complex torus isogenous to a power of $T$. Moreover, the subvariety $X \subset T^l$ is of the form $\pi(V)$, where $V \subset \mathbb{C}^2l$ is a union of hyperplanes.

Proof. Indeed, since the subvariety $X \subset T^l$ is trianalytic by Lemma 6.3, then so is the subvariety $\pi^{-1}(X) \subset \mathbb{C}^2l$. By [V5], Corollary 5.4, every trianalytic subvariety in a hyperkähler manifold is totally geodesic. A totally geodesic subvariety in a flat manifold is a union of hyperplanes. □

6.4. Let now $X \subset K^{[n]}$ be an analytic subvariety which is trianalytic with respect to the chosen hyperkähler structure on the Kummer variety $K^{[n]}$. Say that the subvariety $X$ generically lies in the stratum $K_{\Delta}^{[n]} \subset K^{[n]}$ if

(a) $X$ lies in the closure of the stratum $K_{\Delta}^{[n]} \subset K^{[n]}$, and

(b) the intersection $X \cap K_{\Delta}^{[n]} \subset X$ is a dense open subset.
Every irreducible subvariety $X \subset K^{[n]}$ obviously lies generically in one and only one stratum of the stratification by diagonals.

**6.5.** In order to apply the genericity of the hyperkähler structure on $T$ to the study of the subvariety $X \subset K^{[n]}$, we first prove the following.

**Lemma.** Assume that the trianalytic subvariety $X \subset K^{[n]}$ lies generically in the stratum $K^{[n]}_{\Delta}$, and let $\pi : K^{[n]} \to K^{(n)}$ be the canonical proper map. Then the induced projection $\pi : X \to \pi(X)$ is finite and étale over an open dense subset $U \subset \pi(X)$.

**Proof.** It suffices to take $U \subset \pi(X) \cap K^{[n]}_{\Delta}$ and such that $\pi^{-1}(U) \subset X$ is smooth. Then the hyperkähler structure on $K^{[n]}_{\Delta}$ induces a hyperkähler structure on $\pi^{-1}(U)$. Therefore the restriction $\Omega |_{\pi^{-1}(U)}$ to $\pi^{-1}(U)$ of the canonical holomorphic 2-form $\Omega$ on $K^{[n]}$ associated to the hyperkähler structure must be non-degenerate.

On the other hand, the canonical holomorphic 2-form associated to the hyperkähler structure on the torus $T$ gives a holomorphic 2-form on the variety $T^{[n]}_0$. This form is invariant with respect to the action of the group $\Sigma_\Delta$ on $T^{[n]}_0$. Therefore it induces a holomorphic 2-form $\Omega$ on the smooth complex manifold $K^{[n]}_{\Delta} = (T^{[n]}_0 \setminus \text{Diag}) \times \Sigma_\Delta$. By assumption the subvariety $\pi^{-1}(U) \subset K^{[n]}_{\Delta}$ is smooth, therefore by [V7], Proposition 4.5, we have

$$\Omega |_{\pi^{-1}(U)} = \pi^* \tilde{\Omega} |_U.$$  

Since this form is non-degenerate, the projection $\pi : U \to \pi(U) \subset K^{[n]}_{\Delta}$ must be unramified. Since it is proper, it must therefore, indeed, be finite and étale.

**6.6.** Let now $X \subset K^{[n]}$ be an irreducible trianalytic subvariety which lies generically in the stratum $K^{[n]}_{\Delta} \subset K^{[n]}$. By the Desingularization Theorem of [V7], the normalization $\hat{X}$ of the variety $X$ is smooth, and the normalization map $\nu : \hat{X} \to X \hookrightarrow K^{[n]}$ is an immersion which induces on $\hat{X}$ a hyperkähler structure.

Choose a dense open subset $U \subset \pi(X)$ as in Lemma 6.5, and let

$$\hat{U} = \nu^{-1} \left( X \cap \pi^{-1}(U) \right) \subset \hat{X}$$

be its preimage in $\hat{X}$. Moreover, consider the pullback $\hat{U} \times_{K^{[n]}_{\Delta}} \tilde{K}^{[n]}_{\Delta}$ of the variety $\hat{U} \to K^{[n]}_{\Delta}$ with respect to the canonical Galois covering $\eta : \tilde{K}^{[n]}_{\Delta} \to K^{[n]}_{\Delta}$, and let $\hat{U}$ be any one of its irreducible components.

Recall that by assumption the subvariety $X \subset K^{[n]}$ is irreducible. Therefore both its normalization $\hat{X}$ and the open dense subset $\hat{U} \subset \hat{X}$ are irreducible, which implies that $\eta(\hat{U}) \subset K^{[n]}_{\Delta} \subset K^{[n]}$ is dense in the analytic subvariety $X \subset K^{[n]}$.

**6.7.** By definition the variety $\hat{U}$ is equipped with a canonical map $\rho : \hat{U} \to \tilde{K}^{[n]}_{\Delta}$. Under the product decomposition (5.4) the map $\rho$ decomposes $\rho = \rho_T \times \rho_F$ into
a product of a holomorphic map

\[ \rho_T : \bar{U} \to T_0^1 \setminus \text{Diag} \]

and a holomorphic map

\[ \rho_F : \bar{U} \to F_\Delta. \]

Moreover, by Lemma 6.5 the first factor \( \rho_T : \bar{U} \to T_0^1 \setminus \text{Diag} \) in this decomposition is étale onto its image. The main result of this section is the following.

**Proposition.** The canonical map \( \rho_F : \bar{U} \to F_\Delta \) is a projection onto a single point \( o \in F_\Delta \).

**Proof.** Let \( \bar{U} \subset K_\Delta^{[n]} \) be the closure of the subvariety \( \bar{U} \subset K_\Delta^{(n)} \) in the compactification \( K_\Delta^{[n]} \). This closure is an analytic subvariety. Indeed, since the manifold \( K^{[n]} \) is proper, the preimage \( \eta^{-1}(X) \subset K_\Delta^{[n]} \) of the subvariety \( X \subset K^{[n]} \) under the meromorphic map \( \eta : K_\Delta^{[n]} \twoheadrightarrow K^{[n]} \) is an analytic subvariety. Since the variety \( \bar{U} \) is irreducible, its closure \( \bar{U} \subset K_\Delta^{[n]} \) coincides with an irreducible component of the subvariety \( \eta^{-1}(X) \subset K_\Delta^{[n]} \).

Assume that the projection \( \rho_F : \bar{U} \to F_\Delta \) does not map \( \bar{U} \subset \bar{U} \) to a single point. Since \( \bar{U} \) is irreducible, this implies that \( \rho_F(\bar{U}) \subset \rho_T(\bar{U}) \subset F_t \) is a variety of positive dimension. But the punctual Hilbert scheme \( F_\Delta \) is projective. Thus we can take the preimage of the appropriate hyperplane section in \( F_t \) and obtain an analytic subvariety \( D \subset \bar{U} \) such that the intersection \( D \cap \bar{U} \subset \bar{U} \) is a non-trivial subvariety of codimension 1.

Since the projection \( \rho_T : \bar{U} \to T_0^1 \) is proper, the image \( \rho_T(\bar{U}) \subset T_0^1 \) is an analytic subvariety in the torus \( T_0^1 \). The image \( \rho_T(D) \subset \rho_T(\bar{U}) \) is also analytic. Moreover, the map \( \rho_T : \bar{U} \to \rho_T(\bar{U}) \) is finite. Therefore the intersection \( \rho_T(D) \cap \rho_T(\bar{U}) \subset \rho_T(\bar{U}) \) is a subvariety of codimension 1. Since its closure

\[ \overline{D} = \overline{\rho_T(D) \cap \rho_T(\bar{U})} \subset \rho_T(\bar{U}) \]

is a union of irreducible components of the analytic subvariety \( \rho_T(D) \subset \rho_T(\bar{U}) \), it is also an analytic subvariety, and its codimension is exactly 1.

Now, by Lemma 6.2 the subvariety \( \rho(\bar{U}) \subset T_0^1 \) is trianalytic. Therefore by Lemma 6.3 every one of its irreducible components is a complex torus isogenous to a power of the torus \( T \). Since the torus \( T \) is Mumford-Tate generic in the sense of 2.8, all its analytic subvarieties are even-dimensional, and the same is true for its powers \( T^1 \). This contradicts the existence of \( \overline{D} \subset \rho(\bar{U}) \) and proves the proposition. \( \square \)

### 7. Proof of the main theorem

**7.1.** We can now prove Theorem 2.9. Let \( X \subset K^{[n]} \) be an analytic subvariety in the Kummer variety \( K^{[n]} \) which is trianalytic with respect to the chosen
hyperkähler structure on $K^{[n]}$. Assume that $\dim X > 0$. We have to show that $X = K^{[n]}$.

By Proposition 3.4 we can assume that the subvariety $X \subset K^{[n]}$ is rigid. Assume also, by induction, that every trianalytic subvariety $Y \subset K^{[n]}$ with $\dim Y > \dim X$ coincides with the whole Kummer variety $K^{[n]}$. Let $\Delta$ be the Young diagram of length $n + 1$ such that the subvariety $X$ generically lies in the stratum $K^{[n]}_{\Delta}$ of the diagonal stratification.

**7.2.** Assume first that $\Delta$ is not the diagram $\langle 1, 1, \ldots, 1 \rangle$, so that the stratum $K^{[n]}_{\Delta} \subset K^{[n]}$ is not the maximal one, and the dimension of the punctual Hilbert scheme $F_{\Delta}$ is positive. Let the dense open subset $\overline{U} \subset X \cap K^{[n]}_{\Delta}$ and the étale covering $\overline{U} \rightarrow \overline{U}$ be as in 6.6. The map $\overline{U} \rightarrow \overline{U} \subset K^{[n]}$ factors through the quotient map $\eta : \overline{K}_{\Delta}^{[n]} \rightarrow \overline{K}^{[n]}_{\Delta}$ by means of a map

$$\rho = \rho_F \times \rho_T : \overline{U} \rightarrow F_{\Delta} \times (T^l_0 \setminus \text{Diag}) = \overline{K}^{[n]}_{\Delta},$$

and by Proposition 6.7 the map $\rho_F : \overline{U} \rightarrow F_{\Delta}$ factors through a projection to a single point $o \in F_{\Delta}$.

**7.3.** Let $a \in F_{\Delta}$ be any other point, and consider the subvariety

$$U_a = \eta \left( \{ a \times \rho_T \} \left( \overline{U} \right) \right) \subset K^{[n]}_{\Delta}.$$

We claim the following.

**Lemma.** The closure $\overline{U}_a \subset K^{[n]}$ is a trianalytic subvariety in the Kummer variety $K^{[n]}$.

**Proof.** We first prove that the subset $\overline{U}_a \subset K^{[n]}$ is analytic. Indeed, as established in the proof of Proposition 3.4, the closure $\overline{U} \subset T_0^l \setminus \text{Diag} \subset T_0^l$ is an analytic subvariety. Thus the closure in $\overline{K}^{[n]}_\Delta = F_{\Delta} \times T_0^l$ of the subset

$$\left( a \times \rho_T \right)\left( \overline{U} \right) = \{ a \} \times F_{\Delta} \times (T^l_0 \setminus \text{Diag}),$$

being equal to $\{ a \} \times \overline{U} \subset F_{\Delta} \times T_0^l$, is also analytic. Therefore $\overline{U}_a \subset K^{[n]}$ is analytic by Corollary 5.4 of Lemma 1.3.

Now, for every cohomology class $\alpha \in H^*(K^{[n]}, \mathbb{C})$ consider the Poincare pairing $\langle \alpha, \overline{U}_a \rangle$ between $\alpha$ and the fundamental class $[\overline{U}_a] \in H^*(K^{[n]}, \mathbb{C})$ of the analytic subvariety $\overline{U}_a \subset K^{[n]}$. This pairing coincides with the integral of a certain $C^\infty$ differential form $\omega_\alpha$ over the dense open subset $U_a \subset \overline{U}_a$. The map $\eta \circ (a \times \rho_T) : U_a \rightarrow U$ is finite, say, of degree $m_\alpha$, and we have

$$m_\alpha \langle \alpha, [\overline{U}_a] \rangle = m_\alpha \int_{U_a} \omega_\alpha = \int_{\overline{U}} (\eta \circ (a \times \rho_T))^* \omega_\alpha.$$

The right-hand side obviously depends continuously on the point $a \in F_{\Delta}$. Since the Poincare pairing is non-degenerate, this implies that the function

$$a \mapsto m_\alpha [\overline{U}_a] \in H^*(K^{[n]}, \mathbb{C})$$
is continuous on the punctual Hilbert scheme $F_\Delta$. Since it takes values inside the integral lattice $H^\ast(K^{[n]}, \mathbb{Z}) \subset H^\ast(K^{[n]}, \mathbb{C})$, it must be locally constant. Moreover, the punctual Hilbert scheme $F_\Delta$ is connected and irreducible by [Br]. Therefore this function is constant on $F_\Delta$. Thus

$$\overline{U}_a = \frac{m_o}{m_a} \overline{U}_a = \frac{m_o}{m_a} [X] \in H^\ast(K^{[n]}, \mathbb{C}),$$

and the subvariety $\overline{U}_a \subset K^{[n]}$ is trianalytic by the Trianalyticity Criterion of [Br] (Theorem 3.6).

**7.4.** Since by assumption $\dim F_\Delta > 0$, this lemma implies that there exist a continuum of trianalytic subvarieties in the Kummer variety $K^{[n]}$. They all have the same fundamental class. Moreover, by our assumption on the maximality of the dimension $\dim X$ and by Proposition 3.1, all these subvarieties are rigid. But the Douady moduli space of subvarieties of the manifold $K^{[n]}$ with fixed fundamental class is compact ([Br], [L]). In particular, there exists at most a finite number of rigid subvarieties, which is a contradiction.

**Remark.** The family of subvarieties $U_a \subset K^{[n]}_\Delta$ obviously depends holomorphically on the point $a \in F_\Delta$. A finer argument should produce directly a structure of a holomorphic family on the set of subvarieties $U_a \subset K^{[n]}$, thus avoiding references to the Compactness Theorem and to the cardinality argument. However, it is not immediately clear how to prove that taking closure preserves, at least generically, the holomorphic dependence on the parameter $a$.

**7.5.** We have proved that the trianalytic subvariety $X \subset K^{[n]}$ must necessarily lie generically in the maximal stratum

$$K^{[n]}_{\{1, \ldots, 1\}} \subset K^{[n]}$$

in the stratification by diagonals of the Kummer variety $K^{[n]}$. To simplify notation, we will write $K^{[n]}_{\text{gen}}$ instead of $K^{[n]}_{\{1, \ldots, 1\}}$. The stratum $K^{[n]}_{\text{gen}}$ is open and dense in $K^{[n]}$, and the map $\pi : K^{[n]}_{\text{gen}} \rightarrow K^{[n]}$ is one-to-one on $K^{[n]}_{\text{gen}} \subset K^{[n]}$.

Let $U \subset X \cap K^{[n]}_{\text{gen}}$ be a smooth open dense subset in the subvariety $X$. Let

$$\overline{U} = \eta^{-1}(U) \subset \tilde{K}^{[n]}_{\text{gen}} = T^{n+1}_0 \setminus \text{Diag}$$

be the preimage of the subset $U \subset \tilde{K}^{[n]}_{\text{gen}}$ under the Galois covering $\eta : \tilde{K}^{[n]}_{\text{gen}} \rightarrow K^{[n]}_{\text{gen}}$, and let

$$\overline{U} \subset T^{n+1}_0$$

be its closure in the complex torus $T^{n+1}_0$. By the same argument as in the proof of Lemma 5.3, the subset $\overline{U} \subset T^{n+1}_0$ is a trianalytic subvariety. Therefore by Lemma 6.3 it is an image of a union of hyperplanes in the universal covering $\mathbb{C}^{2n} \rightarrow T^{n+1}_0$.

**7.6.** We finish the proof by repeating the argument used in the case of the nongeneric stratum $K^{[n]}_{\Delta}$. Namely, if $\dim X < 2n$, applying linear translation to this union of hyperplanes gives a continuous family $\tilde{U}_a \subset \tilde{K}^{[n]}_{\text{gen}}$ of different subvarieties in $\tilde{K}^{[n]}_{\text{gen}}$. This family contains $\overline{U}$ as $U_a$ for some value of the parameter
a. As in the non-generic case, for every \( a \) the closure \( \overline{U}_a \subset K^{[n]} \) is an analytic subvariety with the same fundamental class as \( X \subset K^{[n]} \). These subvarieties are all rigid, which is impossible since the Douady space is compact. Therefore we must have \( \dim X = 2n \), or, in other words, \( X = K^{[n]} \). This finishes the proof of Theorem 2.9. \( \Box \)

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