On the Existence of Critical Points to the Seiberg-Witten functional

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November 21, 2018

Abstract
Let $X$ be a closed smooth 4-manifold. In the Theory of the Seiberg-Witten Equations\footnote{1 MSC 58J05, 58E50}, the configuration space is $C_\alpha = \mathcal{A}_\alpha \times \Gamma(S_+^{\alpha})$, where $\mathcal{A}_\alpha$ is a space of $u_1$-connections defined on a complex line bundle over $X$ and $\Gamma(S_+^{\alpha})$ is the space of sections of the positive complex spinor bundle over $X$. The original $SW_\alpha$-equations are 1\textsuperscript{st}-order PDE fitting into a variational principle $SW_\alpha : \mathcal{A}_\alpha \times S_\alpha \to \mathbb{R}$, which is invariant by the group action of (Gauge Group) $G_\alpha = \text{Map}(X, U_1)$ and satisfies the Palais-Smale Condition, up to gauge equivalence. The Euler-Lagrange equations of the functional $SW_\alpha$ are 2\textsuperscript{nd}-order PDE and the solutions of the original $SW_\alpha$-equations are stable critical points. Our aim is to prove the existence of solutions to the Euler-Lagrange equations of the functional $SW_\alpha$ by the method of the Minimax Principle.

1 Introduction

Although the physical meaning of the Seiberg-Witten equations ($SW_\alpha$-eq.) is yet to be discovered, the mathematical meaning is rather deep and highly efficient to understand one of the most basic phenomenon of differential topology in four dimension, namely, the existence of non-equivalent differential smooth structures on the same underlying topological manifold. The Seiberg-Witten equations arised through the ideas of duality described in Witten\footnote{2 connections, gauge fields, 4-manifolds}. It is conjectured that the Seiberg-Witten equations are dual to Yang-Mills equations ($YM$-eq.). The duality is at the quantum level, since one of its necessary condition is the equality of the expectations values for the dual theories. In topology, this means that fixed a 4-manifold its Seiberg-Witten invariants are equal to Donaldson invariants. A good reference for $SW_\alpha$-eq. is\footnote{1}
were empty, but a finite number of them. Jost-Peng-Wang, in [7], used this integral to define a functional, which we refer as the $SW_\alpha$ functional; their main result is to prove that the functional satisfies the Palais-Smale (PS) condition up to a gauge equivalence.

In our context, we refer to the $SW_\alpha$-equation as the Euler-Lagrange equation of the $SW_\alpha$ functional. These equations are 2nd-order differential equations and from now on they will be called $SW_\alpha$-equations.

Since the (PS)-condition is satisfied in the quotient space, the main aim is to describe the weak homotopy type of the moduli space $A_\alpha \times G_\alpha \Gamma(S^+_\alpha)$ in order to prove the existence of solutions to the $SW_\alpha$-equation.

2 Basic Set Up

From a duality principle applicable to SUSY theories in Quantum Field Theory, Seiberg-Witten, in [7], discovered a nice coupling of the self-dual (SD) equation, of a $U_1$ Yang-Mills Theory, to the Dirac equation. In order to describe this coupling it is necessary a particular isomorphism relating the space $\Omega^2_+(X)$, of self-dual 2-forms, and the bundle $End^0(S^-_\alpha)$ ([9]).

The space of $Spin^c$-structures on $X$ is identified as $Spin^c(X) = \{\alpha \in H^2(X, \mathbb{Z}) | w_2(X) = \alpha \text{ mod } 2\}$.

For each $\alpha \in Spin^c(X)$, there is a representation $\rho_\alpha : SO_4 \to Cl_4$, induced by a $Spin^c$ representation, and consequently, a pair of vector bundles $(S^+_\alpha, L_\alpha)$ over $X$ (see [8]), where

- $S_\alpha = P_{SO_4} \times_{\rho_\alpha} V = S^+_\alpha \oplus S^-_\alpha$.
  The bundle $S^+_\alpha$ is the positive complex spinors bundle (fibers are $Spin_4^c$-modules isomorphic to $\mathbb{C}^2$)

- $L_\alpha = P_{SO_4} \times_{det(\alpha)} \mathbb{C}$.
  It is called the determinant line bundle associated to the $Spin^c$-structure $\alpha$. ($c_1(L_\alpha) = \alpha$)

Thus, given $\alpha \in Spin^c(X)$ we associate a pair of bundles

$$\alpha \in Spin^c(X) \leftrightarrow (L_\alpha, S^+_\alpha)$$

From now on, we fixed

- a Riemannian metric $g$ over $X$

- a Hermitian structure $h$ on $S_\alpha$.

Remark 2.0.1. Let $E \to X$ be a vector bundle over $X$ ([4]);

1. The space of sections of $E$ (usually denoted by $\Gamma(E)$) is denoted by $\Omega^0(E)$;

2. The space of p-forms (1 $\leq$ p $\leq$ 4) with values in $E$ is denoted by $\Omega^p(E)$.

3. For each fixed covariant derivative $\nabla$ on $E$, there is a 1st-order differential

$\nabla$ on $E$, connection 1-form $A \leftrightarrow \nabla^A$ covariant derivative
tial operator $d^\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$

For each class $\alpha \in Spin^c(X)$ corresponds a $U_1$-principal bundle over $X$, denoted $P_\alpha$, with $c_1(P_\alpha) = \alpha$. Also, we consider the adjoint bundles

$$Ad(U_1) = P_{U_1} \times_{Ad} U_1, \quad ad(u_1) = P_{U_1} \times_{ad} u_1.$$\[
\text{Ad}(U_1) \text{ is a fiber bundle with fiber } U_1, \text{ and } \text{ad}(u_1) \text{ is a vector bundle with fiber isomorphic to the Lie Algebra } u_1.\]

Once a covariant derivative is fixed on $\text{ad}(u_1)$, it induces the sequence

$$\Omega^0(ad(u_1)) \xrightarrow{d^\nabla} \Omega^1(ad(u_1)) \xrightarrow{d^\nabla} \Omega^2(ad(u_1)) \xrightarrow{d^\nabla} (\ast)$$

$$(\ast) \xrightarrow{d^\nabla} \Omega^3(ad(u_1)) \xrightarrow{d^\nabla} \Omega^4(ad(u_1))$$

The 2-form of curvature $F_\nabla$, induced by the connection $\nabla$, is the operator

$$F_\nabla = d^\nabla \circ d^\nabla : \Omega^0(ad(u_1)) \to \Omega^2(ad(u_1))$$

Since $\text{Ad}(U_1) \sim X \times U_1$ and $\text{ad}(u_1) \sim X \times u_1$, the spaces $\Omega^0(ad(u_1))$ and $\Gamma(\text{Ad}(U_1))$ are identified, respectively, to the spaces $\Omega^0(X, i\mathbb{R})$ and $\text{Map}(X, U_1)$. It is well known from the theory (see in [3]) that a $u_1$-connection defined on $L_\alpha$ can be identified with a section of the vector bundle $\Omega^1(ad(u_1))$, and a Gauge transformation with a section of the bundle $Ad(U_1)$.

Given a vector bundle $E$ over $(X,g)$, endowed with a metric and a covariant derivative $\nabla$, we define the Sobolev Norm of a section $\phi \in \Omega^0(E)$ as

$$\| \phi \|_{L^{k,p}} = \sum_{|i|=0}^{k} \left( \int_X |\nabla^i \phi|^p \right)^{\frac{1}{p}}$$

and the Sobolev Spaces of sections of $E$ as

$$L^{k,p}(E) = \{ \phi \in \Omega^0(X,E) \mid \| \phi \|_{L^{k,p}} < \infty \}$$

Now, consider the spaces

- $A_\alpha = L^{1,2}(\Omega^0(ad(u_1)))$
- $\Gamma(S_\alpha^+) = L^{1,2}(\Omega^0(X,S_\alpha^+))$
- $C_\alpha = A_\alpha \times \Gamma(S_\alpha^+)$
- $G_\alpha = L^{2,2}(X,U_1) = L^{2,2}(\text{Map}(X,U_1))$
The space $G_\alpha$ is the Gauge Group acting on $C_\alpha$ by the action

$$G_\alpha \times C_\alpha \to C_\alpha: \quad (g, (A, \phi)) \to (g^{-1}dg + A, g^{-1}\phi)$$

(1)

Since we are in dimension 4, the vector bundle $\Omega^2(ad(u_1))$ admits a decomposition

$$\Omega^2_+(ad(u_1)) \oplus \Omega^2_-(ad(u_1))$$

(2)
in self-dual (+) and anti-self-dual (-) parts (3).

The 1st-order (original) Seiberg-Witten equations are defined over the configuration space $C_\alpha = A_\alpha \times \Gamma(S_\alpha^+)$ as

$$\begin{cases}
D_A^+(\phi) = 0, \\
F_A^+ = \sigma(\phi)
\end{cases}$$

(3)

where

- $D_A^+$ is the $Spinc^c$-Dirac operator defined on $\Gamma(S_\alpha^+)$;

- Given $\phi \in \Gamma(S_\alpha^+)$, the quadratic form

$$\sigma(\phi) = \phi \otimes \phi^* - \frac{|\phi|^2}{2}I \quad \Rightarrow \quad \sigma(\phi) \in End^0(S_\alpha^+)$$

(4)

performs the coupling of the ASD-equation with the Dirac$^c$ operator.

Locally, if $\phi = (\phi_1, \phi_2)$, then the quadratic form $\sigma(\phi)$ is written as

$$\sigma(\phi) = \begin{pmatrix}
|\phi_1|^2 - |\phi_2|^2 & \phi_1 \cdot \phi_2 \\
\phi_2 \cdot \phi_1 & |\phi_2|^2 - |\phi_1|^2
\end{pmatrix}$$

The set of solutions of equations (3) can be described as the inverse image $\mathcal{F}^{-1}(0)$ by the map $\mathcal{F}_\alpha: C_\alpha \to \Omega^2_+(X) \oplus \Gamma(S_\alpha^-)$, defined as

$$\mathcal{F}_\alpha(A, \phi) = (F_A^+ - \sigma(\phi), D_A^+(\phi))$$

The $SW_\alpha$-equations are $G_\alpha$-invariant.
3 A Variational Principle for the Seiberg-Witten Equation

The lack of a natural Lagrangean let us to consider, in a pure formally way, the functional

\[ SW(A, \phi) = \frac{1}{2} \int_X \{ |F_A^+ - \sigma(\phi)|^2 + |D_A^+(\phi)|^2 \} dv_g \]  

The next identities, which proofs are standard, are useful to expand the functional (5).

**Proposition 3.0.2.** For each \( \alpha \in \text{Spin}^c(X) \), let \( \mathcal{L}_\alpha \) be the determinant line bundle associated to \( \alpha \) and \((A,\phi) \in \mathcal{C}_\alpha \). Also, assume that \( k_g = \text{scalar curvature of (X,g)} \). Then,

1. \( < F_A^+, \sigma(\phi) > = \frac{1}{2} < F_A^+, \phi, \phi > \)
2. \( < \sigma(\phi), \sigma(\phi) > = \frac{1}{4} |\phi|^4 \)
3. Weitzenböck formula
   \[ D^2 \phi = \nabla^* \nabla \phi + \frac{k_g}{4} \phi + \frac{F_A}{2} \phi \]
4. \( \sigma(\phi) \phi = \frac{|\phi|^2}{2} \phi \)
5. \( c_2(\mathcal{L}_\alpha \oplus \mathcal{L}_\alpha) = \int_X F_A \wedge F_A \)
6. \( |F_A^+|^2 = \frac{1}{2} |F_A|^2 - 4\pi^2 \alpha^2 \)

Consequently, after expanding the functional (5), we get the expression

\[ SW(A, \phi) = \int_X \left\{ \frac{1}{4} |F_A|^2 + |\nabla^A \phi|^2 + \frac{1}{8} |\phi|^4 + \frac{1}{4} < k_g \phi, \phi > \right\} dv_g - 2\pi^2 \alpha^2 \]  

**Definition 3.0.3.** For each \( \alpha \in \text{Spin}^c(X) \), the Seiberg-Witten Functional is the functional \( SW_\alpha : \mathcal{C}_\alpha \rightarrow \mathbb{R} \) given by

\[ SW_\alpha(A, \phi) = \int_X \left\{ \frac{1}{4} |F_A|^2 + |\nabla^A \phi|^2 + \frac{1}{8} |\phi|^4 + \frac{1}{4} < k_g \phi, \phi > \right\} dv_g \]  

where \( k_g = \text{scalar curvature of (X,g)} \).
Let $k_{g,X} = \min_{x \in X} k_g$ and

$$k_{g,X}^- = \min \{0, -k_{g,X}^+ \}$$

(8)

**Remark 3.0.4.** 1. Since $X$ is compact and $|| \phi ||_{L^4} < || \phi ||_{L^{1,2}}$, the functional is well defined on $\mathcal{C}_\alpha$.

2. The $SW_\alpha$-functional (5) is Gauge invariant.

3. From (5) and (6), it follows that

$$SW_\alpha(A, \phi) - 2\pi^2 \alpha^2 \geq 0$$

Thus, the $SW_\alpha$-functional is bounded below by $2\pi^2 \alpha^2$, where

$$\alpha^2 = Q_X(\alpha, \alpha)$$

($Q_X : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$ is the intersection form of $X$).

4. The set of classes in $Spin^c(X)$, such that there exits $(A, \phi) \in \mathcal{C}_\alpha$ attaining the minimum value $2\pi^2 \alpha^2$, is finite (9). More precisely, the minimum value $2\pi^2 \alpha^2$ is attained for those $\alpha \in Spin^c(X)$ such that

$$\alpha^2 \leq \frac{1}{4\pi^2}(k_{g,X}^- \text{vol}(X) + 2\chi(X) + 3\sigma_X)$$

where $\chi(X) = \text{Euler characteristic class of } X$ and $\sigma_X = \text{signature of } Q_X$.

**Proposition 3.0.5.** The Euler-Lagrange equations of the $SW_\alpha$-functional (5) are

$$\Delta_A \phi + \frac{|\phi|^2}{4} \phi + \frac{k_g}{4} \phi = 0$$

(9)

$$d^* F_A + 4\Phi^*(\nabla^A \phi) = 0$$

(10)

where $\Phi : \Omega^1(u_1) \to \Omega^1(\mathcal{S}_\alpha^+)$

**Proof.** Let $\phi_t : (-\epsilon, \epsilon) \to \Gamma(\mathcal{S}_\alpha^+)$ be a small perturbation of a critical point $(A, \phi) \in \mathcal{C}_\alpha$, such that $\phi_0 = \phi$ and $\frac{d\phi_t}{dt}|_{t=0} = \Lambda \in \Gamma(\mathcal{S}_\alpha^+)$. The derivative of the function $SW_\alpha(t) = SW_\alpha(A, \phi_t)$, at $(A, \phi)$, is given by

$$\frac{dSW_\alpha(t)}{dt}|_{t=0} = d(SW_\alpha)(A, \phi) \cdot \Lambda = \int_X \text{Re}\{\Delta_A \phi + \frac{|\phi|^2}{4} \phi + \frac{k_g}{4} \phi, \Lambda \} dx,$$
where $\Delta_A = (\nabla^A)^* \nabla^A$.

If for all $A \in \mathcal{I}(S^+)$, we have that $d(S\mathcal{W}_\alpha)(A,\phi), \Lambda = 0$ then

$$\Delta_A \phi + \frac{|\phi|^2}{4} \phi + \frac{k_g}{4} \phi = 0$$

The second equation is obtained by considering a smooth curve $A_t : (-\epsilon, \epsilon) \to \mathcal{A}_\alpha$ given by $A_t = A + t\Theta$, where $\Theta \in \mathcal{O}^1(u_1)$. It follows from the first order approximations

$$F_{A+t\Theta} = F_A + td_A \Theta + o(t^2)$$

and

$$\nabla^{A+t\Theta} \phi = \nabla^A \phi + t\Theta(\phi) + o(t^2),$$

that

$$dS\mathcal{W}_\alpha.\Theta = \frac{1}{4} \int_X \left\{ < F_A, d_A \Theta > + 4 < \nabla^A(\phi), \Phi(\Theta) > \right\},$$

where $\Phi : \mathcal{O}^1(u_1) \to \mathcal{O}^1(S_{\alpha}^+)$ is the linear operator $\Phi(\Theta) = \Theta(\phi)$

If $(\phi, A)$ is a critical point of $S\mathcal{W}_\alpha$, then for all $\Theta \in \mathcal{O}^1(u_1)$

$$dS\mathcal{W}_\alpha.\Theta = \frac{1}{4} \int_X < d^*_A F_A + 4 \Phi^* (\nabla^A \phi), \Theta >, $$

Hence,

$$d^*_A F_A + 4 \Phi^* (\nabla^A \phi) = 0$$

Remark 3.0.6. Locally, in a orthonormal basis $\{\eta^i\}_{1 \leq i \leq 4}$ of $T^*X$, the operator $\Phi^*$ can be written as

$$\Phi^* (\nabla^A \phi) = \sum_{i=1}^4 \nabla^A_i \phi, \phi > \eta^i, \text{ where } \nabla^A_i = \nabla^A_{X_i}, (\eta_i(X_j) = \delta_{ij})$$

The regularity of the solutions of (9) and (10) was studied by Jost-Peng-Wang in (7). They observed that the $L^\infty$ estimate of a solution $\phi$, already known to be satisfied by the stable critical points, is also obeyed by the non-stable critical points of $S\mathcal{W}_\alpha$. The estimate is the following;

**Proposition 3.0.7.** If $(A, \phi) \in \mathcal{C}_\alpha$ is a solution of (9) and (10), then

$$\| \phi \|_\infty \leq \bar{k}_{g,X}$$

where $\bar{k}_{g,X} = \max_{x \in X} \{0, -k_{g,X}^2(X)\}$
As a consequence of the estimate above, if the Riemannian metric \( g \) on \( X \) has non-negative scalar curvature then the only solutions are \((A, 0)\), where
\[
d^* F_A = 0
\]
Since \( X \) is compact,
\[
d^* F_A = 0 \iff \Delta_A F_A = 0 \quad (F_A \text{ is harmonic})
\]
It follows from the formula to the 1\(^{st}\)-Chern class
\[
\alpha([\Sigma]) = \frac{1}{2\pi i} \int_{[\Sigma]} F_A,
\]
for all class \([\Sigma] \in H^2(X, \mathbb{R})\), that \( F_A \) is the only harmonic representative of the De Rham class of \( \alpha \).

If \((A, 0)\) is a solution of the 1\(^{st}\)-order \( SW_\alpha \)-equation (minimum for \( SW_\alpha \)), then \( F_A = 0 \) and \( SW_\alpha(A, 0) = 2\pi^2 \alpha^2 \). It is known (\cite{3}) that if \( b^+_1 > 1 \), then such solutions do not exists for a dense set of the space of metrics on \( X \). Therefore, whenever \( b^+_2 > 1 \), there is a dense set of metrics on \( X \) such that
\[
SW_\alpha(A, 0) > 2\pi^2 \alpha^2.
\]

Although, for each \( \alpha \in Spin^c(X) \), the functional always attains its minimum value (\cite{1.0.12}), it may happens that
\[
\inf_{(A, \phi) \in C_\alpha} SW_\alpha(A, \phi) > 2\pi^2 \alpha^2,
\]
since
\[
\inf_{(A, \phi) \in C_\alpha} SW_\alpha(A, \phi) = 2\pi^2 \alpha^2
\]
just for a finite subset of \( Spin^c(X) \).

In the Euclidean \( \mathbb{R}^4 \), the only solution to the equations (8) and (10), up to gauge equivalence, is the trivial one \((0, 0)\).

In \cite{7}, Jost-Peng-Wang studied the analytical properties of the \( SW_\alpha \)-functional. They proved that the Palais-Smale Condition, up to gauge equivalence, is satisfied. Whenever the quotient space is a smooth manifold, one may use Minimax Principle to prove the existence of solutions to (8) and (10). (see \cite{10}).

Since the \( SW_\alpha \)-functional is \( G_\alpha \)-invariant, it induces a functional on the space \( A_\alpha \times_{G_\alpha} \Gamma(S^+_\alpha) \); the quotient space by the Gauge Group action. However, the space \( A_\alpha \times_{G_\alpha} \Gamma(S^+_\alpha) \) isn’t a manifold since the action isn’t free.

The \( G_\alpha \)-action on \( C_\alpha \) has non-trivial isotropic groups once all elements in \( C_\alpha \) are fixed by the action of the constant maps \( g: X \to U_1 \). In fact,

1. If \((A, \phi) \neq (0, 0) \Rightarrow G_{(A, \phi)} \cong U_1 \), since
\[
g.(A, \phi) = A \leftrightarrow g^{-1}dg = 0 \leftrightarrow g \text{ is constant}
\]
2. \( G_{(0,0)} \cong \mathcal{G}_\alpha \)

Therefore, we consider the Gauge Group
\[
\hat{\mathcal{G}}_\alpha = \left\{ g : X \to U_1 \mid g = \text{constant} \right\} \cong \mathcal{G}_\alpha / U_1
\]

If \( \alpha \neq 0 \), then the \( \hat{\mathcal{G}}_\alpha \)-action on \( \mathcal{C}_\alpha \) is free. Ignoring the case of the trivial bundle \( (\alpha = 0) \); from now on, instead of the \( \mathcal{G}_\alpha \)-action, we consider on \( \mathcal{C}_\alpha \) the \( \hat{\mathcal{G}}_\alpha \)-action, and so, the quotient space \( \hat{\mathcal{B}}_\alpha \) is a manifold. In the case of the trivial bundle we get an orbifold.

Let’s consider \( SW_\alpha : \mathcal{A}_\alpha \times \hat{\mathcal{G}}_\alpha \Gamma(S^+_\alpha) \to \mathbb{R} \) as the induced functional. The Palais-Smale Condition proved by Jost-Peng-Wang can be written in the following way:

**Proposition 3.0.8.** (\([7]\)) Consider a sequence \( [(\mathcal{A}_n, \phi_n)] \in \mathcal{A}_\alpha \times \hat{\mathcal{G}}_\alpha \Gamma(S^+_\alpha) \) satisfying the conditions

1. \( d(SW_\alpha)[(\mathcal{A}_n, \phi_n)] \to 0 \) strongly in \( \mathcal{A}_\alpha \times \hat{\mathcal{G}}_\alpha \Gamma(S^+_\alpha) \),
2. \( SW_\alpha(\mathcal{A}_n, \phi_n) \leq c \) for \( n \in \mathbb{N} \).

So, there exists a subsequence \( [(\mathcal{A}_{n_k}, \phi_{n_k})] \) converging in \( \mathcal{A}_\alpha \times \hat{\mathcal{G}}_\alpha \Gamma(S^+_\alpha) \) to a critical point \( [(\mathcal{A}, \phi)] \) of \( SW_\alpha \). Moreover,

\[
\lim_{n_k \to \infty} SW_\alpha([(\mathcal{A}_{n_k}, \phi_{n_k})]) = SW_\alpha([(\mathcal{A}, \phi)])
\]

Consequently, the basic Deformation Lemma of Morse Theory can be applied allowing the application of the Minimax Principle (\([10]\)).

### 4 W-Homotopy Type of \( \mathcal{A}_\alpha \times \hat{\mathcal{G}}_\alpha \Gamma(S^+_\alpha) \)

In this section, the hypothesis of the theorem ?? are checked to the space \( \mathcal{A}_\alpha \times \hat{\mathcal{G}}_\alpha \Gamma(S^+_\alpha) \), and the study of the weak homotopy type of \( \mathcal{A}_\alpha \times \hat{\mathcal{G}}_\alpha \Gamma(S^+_\alpha) \) is performed.

We start observing that:

1. The quotient spaces \( \mathcal{B}_\alpha = \mathcal{A}_\alpha / \mathcal{G}_\alpha \) and \( \Gamma(S^+_\alpha) / \mathcal{G}_\alpha \) are Hausdorff spaces (\([6]\)).
2. the \( \mathcal{G}_\alpha \)-action on \( \mathcal{A}_\alpha \) is not free since the action of the subgroup of constant maps \( g : M \to U_1, g(x) = g, \forall x \in M \), acts trivially on \( \mathcal{A}_\alpha \).

As mentioned before, instead of the \( \mathcal{G}_\alpha \)-action, we consider on \( \mathcal{C}_\alpha \) the \( \hat{\mathcal{G}}_\alpha \)-action. On \( \mathcal{A}_\alpha \), the \( \hat{\mathcal{G}}_\alpha \)-action is free, and so, the space \( \hat{\mathcal{B}}_\alpha = \mathcal{A}_\alpha / \hat{\mathcal{G}}_\alpha \) is a manifold.

The \( \hat{\mathcal{G}}_\alpha \)-action on \( \Gamma(S^+_\alpha) \) is free except on the 0-section, where the isotropic group is the full group \( \hat{\mathcal{G}}_\alpha \). The action also preserves the spheres in \( \Gamma(S^+_\alpha) \),
consequently, the quotient space is a cone over the quotient of a sphere by the $\hat{G}_\alpha$-action. Therefore, the quotient space is contractible.

It follows from the Corollary of A.0.17 that there exists the fibration

$$\Gamma(S^+_{\alpha}) \to A_\alpha \times \hat{G}_\alpha \Gamma(S^+_{\alpha}) \to \hat{B}_\alpha$$

By the contractibility of $\Gamma(S^+_{\alpha})$, it follows that

$$A_\alpha \times \hat{G}_\alpha \Gamma(S^+_{\alpha}) \htpy \sim \hat{B}_\alpha$$

In [1], they studied the homotopy type of the space $A/\mathcal{G}^*$, where $A$ is the space of connections defined on a $G$-Principal Bundle $P$ and $\mathcal{G}^*$ is a subgroup of the Gauge Group $\mathcal{G} = \Gamma(Ad(P))$. They observed that $\mathcal{G}^*$ acts freely on $A$, and so, the quotient space $\mathcal{B}^*$ is a manifold. We need to compare the $\mathcal{G}_\alpha$ and $\mathcal{G}^*_\alpha$ actions, nevertheless, they turn out to be equal. The exact sequence

$$1 \to U_1 \to \mathcal{G}_\alpha \to \mathcal{G}^*_\alpha \to 1, \quad \rho(g) = g(x_0)^{-1}g$$

implies that $\mathcal{G}^*_\alpha \htpy \mathcal{G}_\alpha / U_1 = \hat{G}_\alpha$, where the quotient $U_1$ corresponds to the constant maps in $Map(X, U_1)$. The actions are equal.

In this way, the results of [1] can be applied to the understanding of the topology of the space $A_\alpha / \hat{G}_\alpha$.

The weak homotopy type of $\mathcal{B}^*_\alpha$ has been studied in [1] and [3]; they proved the following:

**Theorem 4.0.9.** Let $\mathcal{L}_\alpha$ be a complex line with $c_1(\mathcal{L}_\alpha) = \alpha$, $\mathcal{E}U_1$ be the Universal bundle associated to $U_1$ and

$$Map^0_\alpha(X, \mathbb{C}P^\infty) = \{ f : X \to \mathbb{C}P^\infty \mid f^*(\mathcal{E}U_1) \htpy \mathcal{L}_\alpha, f(x_0) = y_0 \}.$$  

Then,

$$\mathcal{B}^*_\alpha \htpy \sim Map^0_\alpha(X, \mathbb{C}P^\infty)$$

**Corollary 4.0.10.** The space $A_\alpha \times \hat{G}_\alpha \Gamma(S^+_{\alpha})$ is path-connected and

$$\pi_n(A_\alpha \times \hat{G}_\alpha \Gamma(S^+_{\alpha}) = \pi_n(Map^0_\alpha(X, \mathbb{C}P^\infty)), \quad n \in \mathbb{N}.$$  

The set of path-connected components of $Map^0(X, \mathbb{C}P^\infty)$ is equal to the space of homotopic classes $f : X \to \mathbb{C}P^\infty$, denoted by $[X, \mathbb{C}P^\infty]$. From Algebraic Topology, we know that
1. There is a 1-1 correspondence
\[ \{ L \mid L \text{ is a complex line bundle over } X \} \leftrightarrow \text{Map}_0(X, \mathbb{C}P^\infty), \]
2. The space of isomorphic classes of complex line bundles is 1-1 with \([X, \mathbb{C}P^\infty],\]
i.e., if \(L\) is isomorphic to \(L_\alpha\) then \(f \in \text{Map}_0^\alpha(X, \mathbb{C}P^\infty)\).
3. \([X, \mathbb{C}P^\infty] = H^2(X, \mathbb{Z}).\]
   In other words, \(\pi_0(\text{Map}^0_\alpha(X, \mathbb{C}P^\infty)) = H^2(X, \mathbb{Z}).\)

   By the Minimax Principle, the SW_\alpha-functional attains its minimum value
in \(A_\alpha \times \hat{G}_\alpha \Gamma(S^+\alpha),\) and so, the equations (9) and (10) admit a solution.

**Theorem 4.0.11.** Let \(\alpha \in \text{Spin}^c(X).\) For each \(n \in \mathbb{N},\) the homotopy group
\(\pi_n(\text{Map}^0(\alpha, \mathbb{C}P^\infty)),\) \(n \in \mathbb{N}\) is isomorphic to
\[ H = H^2(X, \mathbb{Z}) \oplus \{ H^1(S^n, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}) \} \oplus H^2(S^n, \mathbb{Z}), \]
and it is computed in table 1.

**Proof.** Since
\[ \pi_n(\text{Map}^0(X, \mathbb{C}P^\infty)) \simeq H^2(S^n \times X, \mathbb{Z}), \]
we can perform the computation of \(\pi_n(\text{Map}^0_\alpha(X, \mathbb{C}P^\infty))\) by fixing a class of
\([X, \mathbb{C}P^\infty].\)
For a class \(\alpha \in H^2(X, \mathbb{Z}),\) we fix a map \(f : X \to \mathbb{C}P^\infty\) representing \(\alpha\) and
\(a \in S^n.\) Thus,
\[ \pi_n(\text{Map}^0_\alpha(X, \mathbb{C}P^\infty)) = [(S^n \times X, \{a\} \times X), (\mathbb{C}P^\infty, f(x_0))] \]
\[ = [(S^n \times X, \{a\} \times X), \mathbb{C}P^\infty] \]

However,
\[ [(S^n \times X, \{a\} \times X), \mathbb{C}P^\infty] = H^2(S^n \times X, \mathbb{Z})/H^2(\{a\} \times X, \mathbb{Z}) \]
Let \(H = H^2(S^n \times X, \mathbb{Z})/H^2(\{a\} \times X, \mathbb{Z}).\)

By Kuneth’s formula,
\[ H^2(S^n \times X, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus \{ H^1(S^n, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}) \} \oplus H^2(S^n, \mathbb{Z}) \]
and \(H^2(\{a\} \times X, \mathbb{Z}) = H^2(X, \mathbb{Z}).\) Consequently,
\[ H = \{ H^1(S^n, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}) \} \oplus H^2(S^n, \mathbb{Z}) \]
The group \(H\) is described below in the Table 1; (the symbol * stands whether
the group is 0 or not)
Table 1: \( H \)

| \( n \) | \( H^1(X, \mathbb{Z}) \) | \( H \) |
|------|-----------------|------|
| \( \neq 1, 2 \) | * | 0 |
| 1    | 0               | 0 |
| 1    | \( \neq 0 \)   | \( H^1(X, \mathbb{Z}) \) |
| 2    | *               | \( \mathbb{Z} \) |

**Theorem 4.0.12.** There exist critical points for the \( SW_\alpha \)-functional.

**Proof.** The Minimax Principle implies that the minimum value is attained in all connected component of \( A_\alpha \times G_\alpha \Gamma(S^+_\alpha) \). The table 1 shows that it is possible to construct non-contractible families of elements in \( A_\alpha \times G_\alpha \Gamma(S^+_\alpha) \), consequently, by applying the Minimax Principle, it follows that there exist stable and non-stable critical points. In other words, there exist solutions to the equations (9) and (10).

**A Quotient Spaces by the Diagonal Action**

Let \( M, N \) be smooth manifolds endowed with \( G \)-actions \( \alpha_M, \alpha_N \) (respec.). About the \( G \)-actions, we will assume that:

1. The isotropic groups of the action on \( M \) are isomorphics, i.e. there exists a Lie Group \( H \) such that for all \( m \in M \Rightarrow G_m \simeq H \).
2. The quotient spaces \( M/G \) and \( N/G \) are Hausdorff spaces.

The product action of \( G \times G \) on the manifold \( M \times N \), is defined by

\[
\alpha_M \times \alpha_N : G \times G \times (M \times N) \to M \times N,
\]

\[
\alpha_M \times \alpha_N (g_1, g_2, m, n) = (\alpha_M(g_1, m), \alpha_N(g_2, n)),
\]

or equivalently,

\[
(g_1, g_2). (m, n) = (g_1.m, g_2.n)
\]

**Definition A.0.13.** The diagonal action \( \alpha_D : G \times (M \times N) \to M \times N \) is defined as

\[
\alpha_D (g, (m, n)) = (\alpha_M(g.m), \alpha_N(g.n)),
\]

and denoted as \( g.(m,n) = (g.m, g.n) \). The quotient space is denoted by \( M \times G/N \).

**Definition A.0.14.** Let \( m \in M \) and \( n \in N \). The corresponding orbit spaces are defined as follows:
1. For the action $\alpha_M$ on $M$, let $O^M_m = \{g.m \mid g \in G\}$;
2. For the action $\alpha_N$ on $N$, let $O^N_n = \{g.n \mid g \in G\}$;
3. For the product action (P-action) $\alpha_M \times \alpha_N$ on $M \times N$, let
   \[ O^P_{(m,n)} = \{(g_1.m, g_2.n) \mid g_1, g_2 \in G\} \]
4. For the diagonal action D-action) $\alpha_D$ on $M \times N$, let
   \[ O^D_{(m,n)} = \{(g.m, g.n) \mid g \in G\} \]

The orbit of $(m, n)$, by the product action, is easily described by the orbits in $M$ and $N$ as
\[ O^P_{(m,n)} = O^M_m \times O^N_n. \]

Consequently,
\[ (M \times N)/(G \times G) = (M/G) \times (N/G), \]
what induces the fibration
\[ N/G \longrightarrow (M \times N)/(G \times G) \longrightarrow M/G. \]

In order to describe the topology of the space $M \times_G N$, we consider the commuting diagram

\[
\begin{array}{ccc}
M \times N & \xrightarrow{p_1} & M \\
\downarrow \pi^{M \times N} & & \downarrow \pi^M \\
M \times_G N & \xrightarrow{p} & M/G
\end{array}
\]

(11)

where
1. $p_1 : M \times N \to M$ is the projection on the 1st factor;
2. $\pi^{M \times N} : M \times N \to M \times G N$ is the projection induced by the quotient;
3. $p : M \times_G N \to M/G$ is the natural map induced by the projection $O^P_{(m,n)} \to O^M_m$.

From now on, we fix $[m_0] \in M/G$ in order to describe $p^{-1}([m_0])$.

From the diagram, we get that
1. $(\pi^M)^{-1}([m_0]) = O^M_{m_0}$
2. $(\pi^M \circ p_1)^{-1}([m_0]) = O^M_{m_0} \times N.$
3. \((\pi^M \times N)^{-1}([m_0, n_0]) = O^D_{[m_0, n_0]}\)

**Proposition A.0.15.** The subspace \(O^M_{m_0} \times N\) is a \(G\)-space with respect to the \(D\)-action.

**Proof.** The proof is split into two easy claims:

1. If \((m, n) \in O^M_{m_0} \times N\), then \(O^D_{(m, n)} \subset O^M_{m_0} \times N\).

   Let \(m = g.m_0\):
   
   \[ g.(m, n) = (g.g.m_0, g.n) \in O^M_{m_0} \times N \]

2. If \((m, n) \in O^M_{m_0} \times N\), then there exists \(g \in G\) and \(n' \in N\) such that \((m, n) \in O^D_{(m_0, n')}\).

   Let \(m = g.m_0\) and \(n' = g^{-1}.n\):
   
   \[ (m, n) = g.(m_0, g^{-1}.n) \Rightarrow (m, n) \in O^D_{(m_0, n')} \]

   Consequently,

   \[ p_1([m_0]) = O^M_{m_0} \times G.N \]

**Proposition A.0.16.**

\[ O^D_{(m, n)} \cap p_1^{-1}(m_0) = \{g.n \mid g \in G_{m_0}\} \]

**Proof.** Let \(m = g.m_0\); so \((m, n) = g.(m_0, g^{-1}.n) \Rightarrow O^D_{(m, n)} = O^D_{(m_0, g^{-1}.n)}.\)

Nevertheless,

\[ g.(m_0, n) \in p_1^{-1}(m_0) \iff \exists g \in G \text{ such that } g.(m, n) = (m_0, n'), \]

this implies that \(g \in G_{m_0}\) and \(n' = g.n\)

Therefore, every \(D\)-orbit meet the set \(p_1^{-1}(m_0)\). Now, we will construct a smooth invariant map. Define \(\rho : O^M_{m_0} \times N \rightarrow N/G_{m_0}\) by

\[ \rho(g.(g.m_0, n)) = g^{-1}.n, \quad \text{for all } g \in G \]

The map \(\rho\) is well defined, since whenever \(g \in G_{m_0}\) it follows that

\[ g^{-1}.n = \rho((g.m_0, n)) = \rho((m_0, n)) = n \]

This remark is consistent with A.0.16.

It is easily seen to be bijective. Consequently,
Proposition A.0.17.

\[ M \times_G N = \bigcup_{[m] \in M/G} N/G_m \]

Proof. Its follows from the discussion above, since it was concluded that

\[ p^{-1}([m_0]) = N/G_{m_0} \]

\[ \square \]

Corollary A.0.18.

1. If the G-action on M is free, then there is the fibration

\[ N \longrightarrow M \times_G N \longrightarrow M/G \] \hspace{1cm} (12)

2. Suppose that there exist a Lie Group H such that for all \( m \in M \) it is true that \( G_m \) is isomorphic to H. In this case, there is also a fibration

\[ N/H \longrightarrow M \times_G N \longrightarrow M/G \] \hspace{1cm} (13)

which fiber \( N/H \) may be is a singular space.

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