LOCAL LARGE DEVIATION PRINCIPLE, LARGE DEVIATION PRINCIPLE FOR THE SIGNAL-TO-INTERFERENCE AND NOISE RATIO GRAPH MODELS

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Abstract. Given devices space $D$, an intensity measure $\lambda \in (0, \infty)$, transition kernel from the positive reals to the space $D$, a path-loss function which depends on some positive constant $\alpha$, and some technical constants $\tau_\lambda, \gamma_\lambda : (0, \infty) \to (0, \infty)$ we define a Marked Poisson Point Process (MPPP) and, a Signal-to-Interference and Noise Ratio (SINR) graph model. For the SINR graph model we define the empirical marked measure and the empirical connectivity measure.

For a class of SINR graphs model, we prove joint Large Deviation Principle (LDP) for the empirical marked measure and the empirical connectivity measure with speed $\lambda$ in the $\tau$-topology. In particular if $D = \mathbb{R}^{d-2} \times [-T, T]^2$, for $T > 0$, we obtain a much explicit expression for the rate function. From the joint large deviation principle we obtain an Asymptotic Equipartition Property (AEP) for network structured data modelled as an SINR graph.

We also prove a Local Large Deviation Principle (LLDP) for the class of SINR graphs on $D = \mathbb{R}^{d-2} \times [-T, T]^2$, with speed $\lambda$ from spectral potential point. Given, an empirical marked measure $\omega$, we define the so-called spectral potential $U_\omega \cdot (\omega, \cdot)$, for the SINR graph process, where $R_{T,d}$ is a properly defined constant function which depends on the device locations and the marks. We show that the Kullback action or the divergence function $I_\omega (\pi)$, with respect to the empirical connectivity measure $\pi$, is the legendre dual of the spectral potential. From the LLDP we derive a conditional LDP for the SINR graphs.

Note that, while the joint LDP is established in the $\tau$-topology, the LLDP assume no topological restriction on the space SINR graphs. Observe that all our rate functions are expressed in terms of the relative entropy or the Kullback action or Divergence function of the MPPP on the space $D$.

1. Introduction and Background

Wireless ad-hoc and sensor networks have been the topic of much recent research. Now, with the introduction of 5th generation (5G) cellular systems, several techniques, including advanced multiple access technology, massive-MIMO, full-duplex, advanced modulation and coding schemes (MCSs), and simultaneous wireless information and power transfer (SWIPT) which constitutes the next phase

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in global telecommunication standard, see Luo et al. [LSBX2019]. 5G is based on parallel processing hardware and artificial intelligence. This type of communication plays a key role in wireless networks of the next generation Bangerter et al. [BTAS2014]. This process of 5G usages comes along with unprecedented and exigent requirement of which connectivity is a vital cornerstone.

In telecommunication, wireless network comprises of a number of nodes which connect over a wireless channel. See Gupta and Kumar [GK2000]. The Signal to Inference Noise Ratio (SINR) determines whether a given pair of nodes can communicate with each other at a given time. Connectivity occurs in wireless network, if two nodes communicate, possibly via intermediate nodes and also, the information transport capacity of the network, See Ganesh & Torrisi [GT2008]. In addition, network connectivity is related to various layers, components, and metrics of wireless communication systems; however, one vital performance indicator that strongly affects other metrics as well is the SINR. See Oehmann et al. [OAVSF2015].

The SINR is of key significant to the analysis and design of wireless networks. In the process of addressing the additional requirement imposed on wireless communication, in particular, a higher availability of a highly accurate modeling of the SINR is required. Grönkvist & Hansson [GH2001] works on SINR model rely on the assumption that nodes are uniformly distributed in the plane. On the contrast, the complexity of these solution paves way for computational efficiency See, example Behzad & Rubin [BR2003].

More so, the SINR model can be made a complex model such that each transmission is given a power and then assumes a distance-dependent path loss. A transmission is deemed to be successful if the signal-to-interference-plus-noise-ratio (SINR) is more than some specified threshold. See, Andrews & Dinitz [AD2009]. In contrast, a lot of recent work has shown that packets are successfully received only when SINR exceeds a given threshold, and assumes that packet reception rate (PRR) is zero below this threshold. See example, Santi et al. [SMRDB2009]. Further study of the SINR graph model has shown that an SINR model of interference is a more realistic model of interference than the protocol model of interference: a receiver node receives a packet so long as the signal to interference plus noise ratio is above a certain threshold. See Bakshi et al. [BJN2017]. Furthermore, Manesh & Kaabouch [MK2017] stated that SINR is successful if the desired receiver surpasses the threshold. This enables the transmitted signal to be decoded with satisfactory root error probability.

The fundamental concept of SINR model determine as transceiver design on communication system that considers interference as noise. [AD2009] examine a set of transmitter receiver pairs located in the plane with each having an associated SINR requirement; and satisfies as many of the requirements as possible. In all communication systems, noise generated by circuit component in the receiver is a source of signal interruption. The ratio of the signal power to noise power is termed as SINR. The SINR is a vital indicator of communication link quality. See Jeske & Sampath [JS2004]. In the article [SMRDB2009] the wireless link scheduling problem under a graded version of the SINR interference model is revisited. Indeed, the article defines wireless link scheduling problem under the graded SINR model, where they impose an additional constraint on the minimum quality of the usable links.

Li et al. [LPNC2006] examined the statistical distribution of the SINR for the Minimum Mean Square Error (MMSE) receiver in multiple-input multiple output wireless communication. Their study decomposed SINR model into two independent random variables; the first part has an
exact gamma distribution and the second part was shown to converge in distribution to a Normal distribution and approximate by Generalized Gamma. Also, AlAmmouri et al. [AAB2017] examined the SINR and throughput of dense cellular network with stretched exponential path loss. It was established (in the article) that the area spectral efficiency, which assumes an adaptive SINR threshold, is non-decreasing with the base station density and converges to a constant for high densities. Leble & Serfaty [LS2017] investigated a microscopic quantity, the tagged empirical field and proved that LDP is at speed N by defining the rate of function as the addition of entropy term.

An accurate SINR estimation provides for both a more efficient system and a higher user-perceived quality of service.

In this paper, we prove the local large deviation and deviation principles of the Signal-To-Noise and Interference Ratio graph model (SINR). In this sequel we introduce a Marked Poisson Point Process (MPPP) and the marked SINR graph model. For a class of the marked SINR graph, we define the empirical marked measure and the empirical connectivity measure. Then, we prove a joint Large Deviation Principle (LDP) for the empirical marked measure and the empirical connectivity measure of the marked SINR graph model, with speed $\lambda$ in the $\tau-$topology. From the joint large deviation principle, we obtain an Asymptotic Equipartition Property (AEP) for network structured data modelled as an SINR graph. Further, we prove an LLDP for the SINR graph and deduce weak variant of the AEP for the SINR graph from spectral potential point. See, example Doku-Amponsah DA2017 for similar results for the critical multitype Galton-Watson process from spectral potential point.

2. Statement of the results

2.1 The SINR Model for Telecommunication Networks.

Fix a dimension $d \in \mathbb{N}$ and a measureable set $D \subset \mathbb{R}^d$ with respect to the Borel-sigma algebra $\mathcal{B}(\mathbb{R}^d)$. Denote by $m$ the Lebesgue’s measure on $\mathbb{R}^d$. Given $\lambda m : D \rightarrow [0, 1]$, an intensity measure and a probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, where $\alpha \in (0, \infty)$, and technical constants $\tau_\lambda, \gamma_\lambda : (0, \infty) \rightarrow (0, \infty)$ we define the SINR Graph as follows:

- We pick $X = (X_i)_{i \in I}$ a Poisson Point Process (PPP) with intensity measure $\lambda m : D \rightarrow [0, 1]$.
- Given $X$, we assign each $X_i$ a mark $\sigma(X_i) = \sigma_i$ independently according to the transition kernel $Q(\cdot, X_i)$.
- For any two marked points $((X_i, \sigma_i), (X_j, \sigma_j))$ we connect an edge iff $SINR(X_i, X_j, X) \geq \tau_\lambda(\sigma_j)$ and $SINR(X_j, X_i, X) \geq \tau_\lambda(\sigma_i)$, where

$$
SINR(X_j, X_i, X) = \frac{\sigma_i \ell(\|X_i - X_j\|)}{N_0 + \gamma_\lambda(\sigma_j) \sum_{i \in I \setminus \{j\}} \sigma_i \ell(\|X_i - X_j\|)}
$$

We consider $X^\lambda(\mu, Q, \ell) = \{(X_i, \sigma_i), j \in I\}, E \}$ under the joint law of the Marked PPP and the graph. We shall interpret $X^\lambda$ as an SINR graph and $(X_i, \sigma_i) := X^\lambda_i$ the mark of site $i$. We write

$$
S(D) = \{x \subset D : n(x \cap A) < \infty \text{ for any bounded } A \subset D \}.
$$

(2.1)

By $D \otimes \mathbb{R}_+$ we denote the product space $D \times \mathbb{R}_+$ and by $\mathcal{M}(D \otimes \mathbb{R}_+)$ we denote the space of positive measures on the space $D \otimes \mathbb{R}_+$ equipped with $\tau-$ topology and by $\mathcal{B}(D \otimes \mathbb{R}_+)$ we denote the space of
positive measures on $D \otimes \mathbb{R}_+$. By $\mathcal{B}(D \otimes \mathbb{R}_+)$ we denote the space of continuous linear functionals on $D \otimes \mathbb{R}_+$ and by $\mathcal{B}_+(D \otimes \mathbb{R}_+)$ the collection of all positive linear functionals on $\mathcal{B}(D \otimes \mathbb{R}_+)$. For any SINR graph $X^\lambda$ we define a probability measure, the empirical mark measure, $L^\lambda_1 \in \mathcal{M}(D \otimes \mathbb{R}_+)$, by

$$L^\lambda_1([x, \sigma_x]) := \frac{1}{\lambda} \sum_{i \in I} \delta_{\chi^\lambda_i([x, \sigma_x])}$$

and a symmetric finite measure, the empirical pair measure $L^\lambda_2 \in \mathcal{M}(D \otimes \mathbb{R}_+ \times D \otimes \mathbb{R}_+)$, by

$$L^\lambda_2([x, \sigma_x], [y, \sigma_y]) := \frac{1}{\lambda^2} \sum_{(i,j) \in E} [\delta_{(\chi^\lambda_i, \chi^\lambda_j)} + \delta_{(\chi^\lambda_j, \chi^\lambda_i)}]([x, \sigma_x], [y, \sigma_y]).$$

Note that the total mass $\|L^\lambda_1\|$ of the empirical marked measure is $1$ and total mass of the empirical pair measure is $2|E|/\lambda^2$.

The first theorem in this section, Theorem 2.1, is the LDP for the empirical marked measure of the SINR graph models.

**Theorem 2.1.** Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Then, as $\lambda \to \infty$, $L^\lambda_1$ satisfies an LDP in the space $\mathcal{M}(D \otimes \mathbb{R}_+)$ with good rate function

$$I_1(\omega) = \begin{cases} H(\omega | m \otimes Q), & \text{if } \|\omega\| = 1 \\ \infty & \text{otherwise.} \end{cases}$$

We write $R^D([x, \sigma_x], [y, \sigma_y]) := \lim_{\lambda \to \infty} \lambda R^D_\lambda([x, \sigma_x], [y, \sigma_y])$, where

$$R^D_\lambda([x, \sigma_x], [y, \sigma_y]) = \int_D \frac{\tau_\lambda(\sigma_x) \gamma_\lambda(\sigma_y)}{\tau_\lambda(\sigma_x) + \tau_\lambda(\sigma_y) + \|z\|^2/\|x-y\|^2} d\sigma_x d\sigma_y.$$

The next theorem, Theorem 2.2, is a conditional LDP for the empirical connectivity measure given the empirical marked measure, and joint LDP for the empirical marked measure and empirical connectivity measure of the SINR graph model.

**Theorem 2.2.** Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Let $Q$ be the exponential distribution with parameter $c$.

(i) Then, as $\lambda \to \infty$, conditional on the event $L^\lambda_1 = \omega$, $L^\lambda_2$ satisfies an LDP in the space $\mathcal{M}(D \otimes \mathbb{R}_+ \times D \otimes \mathbb{R}_+)$ with speed $\lambda$ and good rate function

$$I_\omega(\pi) = \begin{cases} 0, & \text{if } \pi = e^{-R^D} \omega \otimes \omega \\ \infty & \text{otherwise.} \end{cases}$$

(ii) Then as $\lambda \to \infty$, the pair $L^\lambda_2$ satisfies an LDP in the space $\mathcal{M}(D \otimes \mathbb{R}_+ \times D \otimes \mathbb{R}_+)$ with speed $\lambda$, and good rate function

$$I(\omega, \pi) = \begin{cases} H(\omega | m \otimes Q), & \text{if } \pi = e^{-R^D} \omega \otimes \omega \\ \infty & \text{otherwise.} \end{cases}$$

where

$$e^{-R^D} \omega \otimes \omega([x, \sigma_x], [y, \sigma_y]) = e^{-R^D([x, \sigma_x],[y, \sigma_y])} \omega([x, \sigma_x]) \omega([y, \sigma_y]).$$
In particular, if we assume $\sqrt{X} \tau_\lambda(\sigma_x) \to \tau(\sigma_x)$ and $\sqrt{X} \gamma_\lambda(\sigma_x) \to \gamma(\sigma_x)$, for $x \in D$ and $\sigma_x \in \mathbb{R}_+$ then we have

$$R^D([x, \sigma_x], [y, \sigma_y]) = q_\alpha(D)||y - x||^\alpha \left[\tau(\sigma_x)\gamma(\sigma_x) + \tau(\sigma_y)\gamma(\sigma_y)\right],$$

where $q_\alpha(D) := \int_D \frac{dx}{||x||^\alpha} < \infty$

we obtain a corollary, Corollary 2.3 below:

**Corollary 2.3.** Suppose $X^\lambda$ is an SIR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel $Q$ from $D = \mathbb{R}^{d-2} \times [-T, T]^2$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha \in [d - 2, d]$. Assume $\sqrt{X} \tau_\lambda(\sigma) \to \tau(\sigma)$ and $\sqrt{X} \gamma_\lambda(\sigma) \to \gamma(\sigma)$, for all $\sigma \in \mathbb{R}_+$. Let $Q$ be the exponential distribution with parameter $c$. Then, as $\lambda \to \infty$, the pair $(L_1^\lambda, L_2^\lambda)$ satisfies an LDP in the space $M(D \otimes \mathbb{R}_+ \times D \otimes \mathbb{R}_+)$ with speed $\lambda$, and good rate function

$$I_T(\omega, \nu) = \left\{ \begin{array}{ll} H(\omega \mid m \otimes Q), & \text{if } \nu = e^{-R^{d,T}} \omega \otimes \omega \\ \infty, & \text{otherwise}. \end{array} \right.$$ (2.4)

where

$$R^{d,T}([x, \sigma_x], [y, \sigma_y]) = \frac{\pi^{d/2}T^{d-\alpha} / (d - \alpha)}{\prod_{i=1}^{d/2-1} (-i + \frac{\alpha}{2})} \left[\tau(\sigma_x)\gamma(\sigma_x) + \tau(\sigma_y)\gamma(\sigma_y)\right] ||y - x||^\alpha.$$

The next theorem, Theorem 2.4, is the Asymptotic Equipartition Theorem or the Shannon-McMillian-Breiman Theorem for the class of SINR graphs obtained using Corollary 2.3.

**Theorem 2.4.** Suppose $X^\lambda$ is an SIR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel $Q$ from $D = \mathbb{R}^{d-2} \times [-T, T]^2$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha \in [d - 2, d]$. Assume $\sqrt{X} \tau_\lambda(\sigma) \to \tau(\sigma)$ and $\sqrt{X} \gamma_\lambda(\sigma) \to \gamma(\sigma)$, for all $\sigma \in \mathbb{R}_+$. Let $Q$ be the exponential distribution with parameter $c$. Then,

$$\lim_{\lambda \to \infty} -\frac{1}{\lambda^2} \log P(X^\lambda) = E_{Q \times Q} \left[ \int_D \int_D R^{d,T}([x, \sigma_x], [y, \sigma_y]) \omega^{-R^{d,T}}([x, \sigma_x], [y, \sigma_y]) dx dy \right]$$

$$+ E_{Q \times Q} \left[ \int_D \int_D -\log \left(1 - \omega^{-R^{d,T}}([x, \sigma_x], [y, \sigma_y]) \right) \omega^{-R^{d,T}}([x, \sigma_x], [y, \sigma_y]) dx dy \right]$$

with probability 1.

Let $\mathcal{G}_\lambda$ be the set of all marked PPP with intensity measure $\lambda m$, where $\lambda > 0$.

For $\omega \in M(D \otimes \mathbb{R}_+)$ we denote by $P_\omega = P\left\{ \cdot \mid L_1^\lambda = \omega \right\}$ and write

$$M_\omega = \left\{ \pi \in M(D \otimes \mathbb{R}_+ \times D \otimes \mathbb{R}_+) : \|\pi\| = \langle e^{R^{d,T}} \omega \otimes \omega \rangle \right\}.$$

Next we state the Local large Deviation Principle for SINR graph model without any topological restriction on the space $\mathcal{G}_\lambda$.

**Theorem 2.5.** Suppose $X^\lambda$ is an SIR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel $Q$ from $D = \mathbb{R}^{d-2} \times [-T, T]^2$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha \in [d - 2, d]$. Assume $\sqrt{X} \tau_\lambda(\sigma) \to \tau(\sigma)$ and $\sqrt{X} \gamma_\lambda(\sigma) \to \gamma(\sigma)$, for all $\sigma \in \mathbb{R}_+$. Let $Q$ be the exponential distribution with parameter $c$. Then,

- for any functional $\nu \in M_\omega$ and a number $\varepsilon > 0$, there exists a weak neighbourhood $B_\nu$ such that

$$P_\omega \left\{ X^\lambda \in \mathcal{G}_\lambda \mid L_1^\lambda \in B_\nu \right\} \leq e^{-\lambda M_\omega(\nu) - \lambda \varepsilon}. $$
for any \( \nu \in \mathcal{M}_\omega \), a number \( \varepsilon > 0 \) and a fine neighbourhood \( B_\nu \), we have the estimate:

\[
P_\omega \left\{ X^\lambda \in \mathcal{G}_P \mid L_2^\lambda \in B_\nu \right\} \geq e^{-\mathcal{M}_\omega(\nu) + \lambda \varepsilon}.
\]

The last result, Corollary 2.6 is the LDP for for the SINR graph model with any topological restriction on the space \( \mathcal{G}_P \).

**Corollary 2.6.** Suppose \( X^\lambda \) is an SINR graph with intensity measure \( \lambda m : D \rightarrow [0, 1] \) and a marked probability kernel \( Q \) from \( D = \mathbb{R}^{d-2} \times [-T, T]^2 \) to \( \mathbb{R}_+ \) and path loss function \( \ell(r) = r^{-\alpha} \), for \( \alpha \in [d - 2, d] \). Assume \( \sqrt{\lambda} \gamma(a) \rightarrow \tau(a) \) and \( \sqrt{\lambda} \gamma(a) \rightarrow \gamma(a) \), for all \( a \in \mathbb{R}_+ \). Let \( Q \) be the exponential distribution with parameter \( c \).

- Let \( F \) be closed subset \( \mathcal{M}_\omega \). Then we have
  \[
  \limsup_{\lambda \to \infty} \frac{1}{\lambda} \log P_\omega \left\{ X^\lambda \in \mathcal{G}_P \mid L_2^\lambda \in F \right\} \leq - \inf_{\nu \in F} I_\omega(\nu).
  \]

- Let \( O \) be open subset \( \mathcal{M}_\omega \). Then we have
  \[
  \liminf_{\lambda \to \infty} \frac{1}{\lambda} \log P_\omega \left\{ X^\lambda \in \mathcal{G}_P \mid L_2^\lambda \in O \right\} \geq - \inf_{\nu \in O} I_\omega(\nu).
  \]

**Remark 1** We observe from Corollary 2.6 that

\[
\lim_{\lambda \to \infty} P_\omega \left\{ X^\lambda \in \mathcal{G}_P \mid L_2^\lambda = e^{-R^{d,T}} \omega \otimes \omega \right\} = 1.
\]

### 3. Proof of Theorem 2.1 by Method of Types

Let \( A_1, ..., A_n \) be decomposition of \( D \subset \mathbb{R} \) and let \( (\tilde{\sigma}_1, ..., \tilde{\sigma}_n, p_1, ..., p_n, \Gamma_1, ..., \Gamma_n) \) be discretization of \( (\sigma_1, ..., \sigma_n, Q, \mathbb{R}_+) \). Thus, we have \( p_i = P\{ \Sigma = \tilde{\sigma}_i \} = Q(\cdot; \tilde{\sigma}_i, \in \Gamma_i) \), where \( \tilde{\sigma}_i \) has been chosen as a function of \( p_i \) and \( \Gamma_i \). We shall assume henceforth that \( n < \lambda \) and note by the method of types that we have

\[
P(L_1^\lambda = \omega) = \prod_{i=1}^n e^{-\lambda m \otimes Q(A_i \times \Gamma_i)}[\lambda m \otimes Q(A_i \times \Gamma_i)]^n \omega(A_i \times \Gamma_i) \]

The proof of Lemma below will use the refined Stirling’s formula

\[
(2\pi)^{\frac{1}{2}} \lambda \lambda^{\frac{1}{2}} e^{-\lambda + 1/(12\lambda + 1)} < \lambda! < (2\pi)^{\frac{1}{2}} \lambda \lambda^{\frac{1}{2}} e^{-\lambda + 1/(12\lambda)},
\]

see [1, page 52].

**Lemma 3.1.** Suppose \( X^\lambda \) is a marked PPP in a compact set \( D \times \mathbb{R}_+ \) with intensity measure \( \lambda m \otimes Q \) such that \( m \) is absolutely continuous measure on \( D \). Then,

\[
e^{-\lambda H(\omega(n) \mid m(n) \otimes Q(n))} + \theta_1(\lambda) \leq P \left\{ L_1^\lambda = \omega \right\} \leq e^{-\lambda H(\omega(n) \mid m(n) \otimes Q(n))} + \theta_2(\lambda)
\]

\[
\lim_{\lambda \to \infty} \theta_1(\lambda) = \lim_{\lambda \to \infty} \theta_2(\lambda) = 0,
\]

where \( \omega(n) \) and \( m(n) \otimes Q(n) \) are the coarsening projections of \( \omega \) and \( m \otimes Q \) on the decomposition \((A_1 \times \Gamma_1, ..., A_n \times \Gamma_n)\).
Proof. For large $\lambda$ as we have that

$$\ln P(L_1^\lambda = \omega) \leq \sum \{-\lambda m \otimes Q(A_j \times \Gamma_j)\} - \log[(2\pi)^{\frac{1}{2}} \lambda m(A_j \otimes \Gamma_j)]^{\lambda(A_j + \Gamma_j)} + \frac{1}{2} \exp^{-(\lambda m A_j + \Gamma_j)} + \frac{1}{12\lambda \omega(A_j \otimes \Gamma_j)} + \lambda \omega(A_j \otimes \Gamma_j) \log[\lambda m \otimes Q(A_j \times \Gamma_j)]$$

$$\ln P(L_1^\lambda = m) = \sum \{-\lambda m \otimes Q(A_j \otimes \Gamma_j)\} - \frac{1}{2} \log(2\pi) - \frac{1}{2} \log[\lambda m \otimes Q(A_j \times \Gamma_j)] + \frac{1}{12\lambda \omega(A_j \otimes \Gamma_j)} + \lambda \omega(A_j \otimes \Gamma_j) \log[\lambda m \otimes Q(A_j \otimes \Gamma_j)]$$

$$\ln P(L_1^\lambda = m) \leq \sum \{-\lambda(m \otimes Q(A_j \times \Gamma_j) - \omega(A_j \times \Gamma_j)) - \lambda m(A_j \times \Gamma_j) \log[\lambda m(A_j \times \Gamma_j)] - \frac{1}{2} \log[\lambda m(A_j \times \Gamma_j)] - \frac{1}{12\lambda m(A_j \times \Gamma_j) + \lambda} + \frac{1}{2} \log(2\pi)\}$$

We choose $\theta_2(\lambda)$ as

$$\theta_2(\lambda) = \frac{\log \lambda m(A_j \times \Gamma_j)}{2\lambda} - \frac{1}{12\lambda^2 m(A_j \times \Gamma_j) + \lambda} + \frac{\log(2\pi)}{2\lambda}$$

and observe that

$$\lim_{\lambda \to \infty} \theta_2(\lambda) = \lim_{\lambda \to \infty} \left[\frac{\log \lambda m(A_j \times \Gamma_j)}{2\lambda} - \frac{1}{12\lambda^2 m(A_j \times \Gamma_j) + \lambda} + \frac{\log(2\pi)}{2\lambda}\right] = 0$$

which proves the upper bound in the Lemma 3.1.

For large $\lambda$, we have the lower bound

$$\ln P(L_1^\lambda = m) \geq \sum_{j=1}^n \{-\lambda m \otimes Q(A_j \times \Gamma_j)\} - \log[(2\pi)^{\frac{1}{2}} \lambda m(A_j \otimes \Gamma_j)]^{\lambda m(A_j \otimes \Gamma_j) + \frac{1}{2}} \exp^{-(\lambda m(A_j \otimes \Gamma_j))} + \frac{1}{12\lambda m(A_j \otimes \Gamma_j) + 1} + \lambda m(A_j \otimes \Gamma_j) \log[\lambda m \otimes Q(A_j \times \Gamma_j)]$$

$$\ln P(L_1^\lambda = m) \geq \sum_{j=1}^n \{-\lambda[m \otimes Q(A_j \times \Gamma_j) - m(A_j \times \Gamma_j)] - \lambda m(A_j \times \Gamma_j) \log[\lambda m(A_j \times \Gamma_j)]$$

$$+ \lambda m(A_j \times \Gamma_j) \log[\lambda m(A_j \times \Gamma_j)] - \frac{1}{2} \log[\lambda m(A_j \times \Gamma_j)] + \frac{1}{12\lambda m(A_j \times \Gamma_j) - \frac{1}{2} \log(2\pi)}\}$$
Let $A \ll m \otimes Q(A_j \times \Gamma_j) - m(A_j \times \Gamma_j) - \lambda m(A_j \times \Gamma_j) \log \left( \frac{m(A_j \times \Gamma_j)}{\mu \otimes Q(A_j \times \Gamma_j)} \right) - \lambda \left( \frac{\log[\lambda m(A_j \times \Gamma_j)]}{2\lambda} \right) - \frac{1}{12\lambda^2 m(A_j \times \Gamma_j)} + \frac{\log(2\pi)}{2\lambda} \right}\}

We choose $\theta_1(\lambda)$ as

$$\theta_1(\lambda) = \frac{\log(\lambda m(A_j \times \Gamma_j))}{2\lambda} - \frac{1}{12\lambda^2 m(A_j \times \Gamma_j)} + \frac{\log(2\pi)}{2\lambda}$$

observe that

$$\lim_{\lambda \to \infty} \theta_1(\lambda) = \lim_{\lambda \to \infty} \left( \frac{\log(\lambda m(A_j \times \Gamma_j))}{2\lambda} - \frac{1}{12\lambda^2 m(A_j \times \Gamma_j)} + \frac{\log(2\pi)}{2\lambda} \right) = 0.$$  

This proves the lower bound of Lemma 3.

\[\square\]

4. PROOF OF THEOREM 2.2

Let $A_1 \times \Gamma_1, ..., A_n \times \Gamma_n$ be the decomposition of the space $D \times \mathbb{R}_+$. Note that, for every $(x, y) \in \hat{A}_i = A_i \times \Gamma_i$, $i = 1, 2, 3, ..., n$, $\lambda L^2(x, y)$ given $\lambda L^2(x) = \lambda \omega(x)$ is binomial with parameters $\lambda^2 \omega(x) \omega(y)/2$ and $\theta_\lambda(x, y)$. Let $Q$ be the exponential distribution with parameter $c$.

4.1 Proof of Theorem 2.2(i) by Gartner-Ellis Theorem

We recall the function $R^D_\lambda$ from the previous sections as follows:

$$R^D_\lambda([x, \sigma_x], [y, \sigma_y]) = \int_D \left[ \frac{\tau_\lambda(\sigma_x) \gamma_\lambda(\sigma_x)}{\tau_\lambda(\sigma_x) \gamma_\lambda(\sigma_x) + \|y-x\|^{\alpha}} + \frac{\tau_\lambda(\sigma_y) \gamma_\lambda(\sigma_y)}{\tau_\lambda(\sigma_y) \gamma_\lambda(\sigma_y) + \|y-x\|^{\alpha}} \right] d\mu.$$  

The next lemma is key component for the application of the Gartner-Ellis Theorem, see example.

**Lemma 4.1.** Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda\text{Leb}(x)$ : $D \to [0,1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$, conditional on the event $L^\lambda = \omega$. Let $g : D \otimes \mathbb{R}_+ \times D \otimes \mathbb{R}_+ \to \mathbb{R}$ be bounded function. Then,

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{E} \left\{ e^{\lambda g(L^\lambda)} \left| L^\lambda = \omega \right. \right\} = \frac{1}{2} \lim_{n \to \infty} \left[ \sum_{j=1}^n \sum_{i=1}^n \int_{y \in A_j \times \Gamma_j} \int_{x \in A_i \times \Gamma_i} g(x, y) e^{-R^D(x, y) \omega(dx) \omega(dy)} \right]$$

$$= \frac{1}{2} \int_{D \times \mathbb{R}_+} \int_{D \times \mathbb{R}_+} g(x, y) e^{-R^b(x, y) \omega(dx) \omega(dy)}.$$  

**Proof.**

**Calculation of Connectivity Probability by the Campbell’s Theorem:** We note that the Signal-Interference and Noise Ratio is given as

$$\text{SINR}(X_j, \check{X}_i, \check{X}) = \frac{\sigma_x(\|X_i - X_j\|)}{N_0 + \gamma_\lambda(\sigma_x) \sum_{i \in I-\{j\}} \sigma_i \ell(\|X_i - X_j\|)}$$
and the total interference is defined as
\[ I_{X, \sigma}(Y) = \sum_{i \in I} \sigma_i I_i, \]
where \( I_i = \ell(\|X_i - X_j\|) \). The probability that \( \bar{X}_i = (y, \sigma_y) \) and \( \bar{X}_j = (x, \sigma_x) \) are connected is given as
\[
P(\bar{X}_j, \bar{X}_i) = P\left[ \frac{\sigma_i \ell(\|X_i - X_j\|)}{N_0 + \gamma_\lambda(\sigma_i) \sum_{i \in I - \{j\}} \ell(\|X_i - X_j\|)} \geq \tau_\lambda(\sigma_j) \right] P\left[ \frac{\sigma_j \ell(\|X_j - X_i\|)}{N_0 + \gamma_\lambda(\sigma_j) \sum_{j \in I - \{i\}} \ell(\|X_j - X_i\|)} \geq \tau_\lambda(\sigma_i) \right]
\]
Now we have,
\[
P(\bar{X}_j, \bar{X}_i) = P\left[ \sigma_j \ell(\|X_j - X_i\|) \geq \left( (N_0 + \gamma_\lambda(\sigma_i)) \sum_{i \in I - \{j\}} \sigma_i \ell(\|X_i - X_j\|) \right) \tau_\lambda(\sigma_i) \right] P\left[ \sigma_j \geq \frac{(N_0 + \gamma_\lambda(\sigma_i)) \sum_{i \in I - \{j\}} \sigma_i \ell(\|X_j - X_i\|) \tau_\lambda(\sigma_i)}{\ell(\|X_j - X_i\|)} \right]
\]
Let \( X_i = y, X_j = x \) and \( I_{x, \sigma}(y) = \sum_{j \in I} \ell(\|X_j - y\|) \), then
\[
p([x, \sigma_x], [y, \sigma_y]) = \left[ \int_0^\infty P\left( \sigma \geq \frac{\tau_\lambda(\sigma_y)s}{\ell(\|y - x\|)} \right) P\left( (N_0 + \gamma_\lambda(\sigma_y))I_{x, \sigma}(y) \in ds \right) \right] \left[ \int_0^\infty P\left( \sigma \geq \frac{\tau_\lambda(\sigma_x)s}{\ell(\|x - y\|)} \right) P\left( (N_0 + \gamma_\lambda(\sigma_x))I_{y, \sigma}(x) \in ds \right) \right]
\]
See, example Jahnel and Koenig [JK2018]. Assuming that \( \sigma \) follow exponential distribution \((c)\) we have
\[
p([x, \sigma_x], [y, \sigma_y]) = \left[ \int_0^\infty e^{-c\tau_\lambda(\sigma_y)s} P\left( (N_0 + \gamma_\lambda(\sigma_y))I_{x, \sigma}(y) \in ds \right) \right] \left[ \int_0^\infty e^{-c\tau_\lambda(\sigma_x)s} P\left( (N_0 + \gamma_\lambda(\sigma_x))I_{y, \sigma}(x) \in ds \right) \right]
\]
Using Laplace Transform we have
\[
p([x, \sigma_x], [y, \sigma_y]) = \left[ \mathcal{L}_{N_0} + \gamma_\lambda(\sigma_y)I_{y, \sigma} \left( \frac{c\tau_\lambda(\sigma_y)s}{\ell(\|y - x\|)} \right) \right] \times \left[ \mathcal{L}_{N_0} + \gamma_\lambda(\sigma_x)I_{x, \sigma} \left( \frac{c\tau_\lambda(\sigma_x)s}{\ell(\|x - y\|)} \right) \right]
\]
Since the exterior noise and interference are independent
\[
p([x, \sigma_x], [y, \sigma_y]) = \left[ \mathcal{L}_{N_0} \left( \frac{c\tau_\lambda(\sigma_y)s}{\ell(\|y - x\|)} \right) \right] \mathcal{L}_{I_{y, \sigma}} \left( \frac{c\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\ell(\|y - x\|)} \right) \times \left[ \mathcal{L}_{N_0} \left( \frac{c\tau_\lambda(\sigma_x)s}{\ell(\|x - y\|)} \right) \right] \mathcal{L}_{I_{x, \sigma}} \left( \frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|y - x\|)} \right)
\]
Assuming there is no external noise
\[
p([x, \sigma_x], [y, \sigma_y]) = \left[ \mathcal{L}_{I_{y, \sigma}} \left( \frac{c\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\ell(\|y - x\|)} \right) \right] \times \left[ \mathcal{L}_{I_{x, \sigma}} \left( \frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|y - x\|)} \right) \right]
\]
Hence, by symmetry, we have that
\[
p([x, \sigma_x], [y, \sigma_y]) = p([y, \sigma_y], [x, \sigma_x]) = \left[ \mathcal{L}_{I_{y, \sigma}} \left( \frac{c\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\ell(\|y - x\|)} \right) \right] \times \left[ \mathcal{L}_{I_{x, \sigma}} \left( \frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|y - x\|)} \right) \right]
\]
Note that
\[
\mathcal{L}_{I_{x, \sigma}}(s) = \mathbb{E}(e^{-s I_{x, \sigma}}), \text{ for } s = \frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|y - x\|)}.
\]
by the dominated convergence theorem. Let $\mu(dz) = \lambda dz$ and recall that the battery is assumed to be

$$Q(x, d\sigma) = ce^{-\sigma}$$

$$\mathcal{L}_{I(x, \sigma)}(s) = \exp \left\{ \int_D \int_0^\infty [e^{-s\ell(||z||)} - 1] K(x, d\sigma) \mu(dz) \right\}$$

and

$$\mathcal{L}_{I(x, \sigma)}(s) = \exp \left\{ \lambda \int_D \int_0^\infty [e^{-s\ell(||z||)} - ce^{-\sigma}] d\sigma dz \right\}.$$

This gives $p([x, \sigma_x], [y, \sigma_y]) = \exp \left\{-\lambda \int_D \int_0^\infty \frac{s\ell(||z||)}{s\ell(||z||) + c} dz - \lambda \int_D \int_0^\infty \frac{t\ell(||z||)}{t\ell(||z||) + c} dz \right\}.$

Using $\ell(r) = r^{-\alpha}$ we obtain the expression

$$p([x, \sigma_x], [y, \sigma_y]) = \exp \left\{-\lambda \int_D \int_0^\infty \frac{\tau_\lambda(\sigma_x, \gamma_\lambda(\sigma_x))}{\tau_\lambda(\sigma_x, \gamma_\lambda(\sigma_x) + ||z||^\alpha/\|x-y\|^\alpha) + \gamma_\lambda(\sigma_x) + (||z||^\alpha/\|x-y\|^\alpha)} dz \right\}.$$
We denote by $\Theta$.

We recall from Lemma 4.2 that the family of measures $(\tilde{\mathcal{M}} \otimes \tilde{\mathcal{M}}) := \{ \omega \in \mathcal{M} : \lambda \omega(x) \in \mathbb{N} \text{ for all } x \in \mathbb{R}_+ \}.

\begin{equation}
\mathcal{M} \otimes \mathcal{M} := \{ \omega \in \mathcal{M} : \lambda \omega(x) \in \mathbb{N} \text{ for all } x \in \mathbb{R}_+ \},
\end{equation}

\begin{equation}
\tilde{\mathcal{M}} := \{ \pi \in \tilde{\mathcal{M}} : \lambda \pi(x, y) \in \mathbb{N} \text{ for all } x, y \in \mathbb{R}_+ \}.
\end{equation}

We denote by $\Theta := \mathcal{M} \otimes \mathcal{M}$ and $\Theta := \mathcal{M} \otimes \mathcal{M}$. With

\begin{equation}
P_{\omega}^{(\lambda)}(\eta) := \mathbb{P}\{ L_2^{\lambda} = \eta \mid L_1^{\lambda} = \omega \},
\end{equation}

\begin{equation}
P^{(\lambda)}(\omega) := \mathbb{P}\{ L_1^{\lambda} = \omega \}
\end{equation}

the joint distribution of $L_1^{\lambda}$ and $L_2^{\lambda}$ is the mixture of $P_{\omega}^{(\lambda)}$ with $P^{(\lambda)}(\omega)$ defined as

\begin{equation}
d\tilde{P}^{(\lambda)}(\omega, \eta) := dP_{\omega}^{(\lambda)}(\eta) \lambda dP^{(\lambda)}(\omega).
\end{equation}

(Biggs, Theorem 5(b), 2004) gives criteria for the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known.

The following three lemmas ensure validity of these conditions.

We recall from Lemma 4.2 that the family of measures $(P^{(\lambda)} : \lambda \in \mathbb{N})$ is exponentially tight on $\Theta.$

Lemma 4.2. The family of measures $(\tilde{P}^{(\lambda)} : \lambda \in \mathbb{R}_+)$ is exponentially tight on $\Theta \times \tilde{\mathcal{M}}$.
Define the function $I: \Theta \times \mathcal{M}_x(D \otimes \mathbb{R}_+ \times D \otimes \mathbb{R}_+) \to [0, \infty]$, by

$$I(\omega, \pi) = \begin{cases} H(\omega | m \otimes Q), & \text{if } \pi = e^{-R^D} \omega \otimes \omega \\ \infty & \text{otherwise.} \end{cases} \quad (4.3)$$

**Lemma 4.3.** $I$ is lower semi-continuous.

By (Biggins, Theorem 5(b), 2004) the two previous lemmas and the large deviation principles we have established Theorem 2.1 and Theorem 2.2 ensure that under $(\hat{P}^\lambda)$ the random variables $(\omega_\lambda, \eta_\lambda)$ satisfy a large deviation principle on $\mathcal{M}(D \otimes \mathbb{R}_+) \times \tilde{\mathcal{M}}(D \otimes \mathbb{R}_+ \times D \otimes \mathbb{R}_+)$ with good rate function $I$ which ends the proof of Theorem 2.2.

5. **Proof of Corollary 2.3 and Theorem 2.4**

5.1 **Proof of Corollary 2.3**

We observe Theorem 2.2 that if we take $D = \mathbb{R}^{d-2} \times [-T, T]$, for $T > 0$ then the pair $(L_\lambda^1, L_\lambda^2)$ obeys an LDP with speed $\lambda$ and rate function given by $I$. Now, we obtain the form of the rate function in the Corollary if we evaluate the integral $q(\mathbb{R}^{d-2} \times [-T, T])$ in equation 4.3. The evaluation to the integral will be captured in the following Lemma below:

**Lemma 5.1.** Let $d > 2$. Then, for $d - 2 < \alpha < d$

$$\int_{\mathbb{R}^{d-2} \times [-T, T]^2} \frac{1}{\|z\|^\alpha} dz_1 dz_2, \ldots, dz_d = \frac{\pi^{d/2} T^{d-\alpha}/(d-\alpha)}{\prod_{i=1}^{d/2-1} (\frac{d}{2} - i)} \quad (5.1)$$

**Proof.**

We begin the proof of Lemma 2.3 by taking a circular transformation of the left side of equation 5.2. To begin, we let $z_i = r_i \sin(\theta_i)$, $z_j = r_i \cos(\theta_j-k)$ for $i = 1, 2, 3, \ldots, k$ and $j = k+1, k+2, k+3, \ldots, d$, where $k = d/2$. Now, we obtain the Jacobian of the transformation as

$$J(r_1, r_2, r_3, \ldots, r_{d/2}, \theta_1, \theta_2, \theta_3, \ldots, \theta_{d/2})$$

$$\begin{vmatrix} \sin(\theta_1) & r_1 \cos(\theta_1) & 0 & 0 & \ldots & 0 & 0 & 0 \\ \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \sin(\theta_2) & r_2 \cos(\theta_2) & \ldots & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta_2) & -r_2 \sin(\theta_2) & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \sin(\theta_{d/2}) & r_{d/2-1} \cos(\theta_{d/2}) & 0 & 0 \\ 0 & 0 & \cos(\theta_{d/2}) & -r_{d/2-1} \sin(\theta_{d/2}) & 0 & 0 \\ 0 & 0 & \sin(\theta_{d/2}) & r_{d/2} \cos(\theta_{d/2}) & 0 & 0 \\ 0 & 0 & \cos(\theta_{d/2}) & -r_{d/2} \sin(\theta_{d/2}) \end{vmatrix}$$

Using mathematical induction, we evaluate the determinant of the matrix above to obtain

$$J(r_1, r_2, r_3, \ldots, r_{d/2}, \theta_1, \theta_2, \theta_3, \ldots, \theta_{d/2}) = \prod_{i=1}^{d/2} r_i.$$
Therefore, by symmetry we have

\[
\int_{\mathbb{R}^{d-2} \times [-T, T]^2} \frac{1}{\|z\|^d} dz_1 dz_2 dz_3 \ldots dz_d = 2^d \int_{[0, \pi/2]^{d/2}} \int_{[0, T]^{d/2-1}} \frac{d/2}{\sqrt{\sum_{i=1}^{d/2} r_i^2}} \prod_{i=1}^{d/2} r_i dr_i d\theta_i
\]

\[
= 2^d \int_{[0, \pi/2]^{d/2}} \int_{[0, T]^{d/2-1}} \frac{1}{\sqrt{\sum_{i=1}^{d/2} r_i^2}} \prod_{i=1}^{d/2} r_i^2 dr_i d\theta_i
\]

\[
= 2^d \frac{\pi^{d/2}}{2^{d/2}} \int_{[0, \pi/2]^{d/2-1}} \frac{1}{\sqrt{\sum_{i=1}^{d/2-1} r_i^2}} \prod_{i=1}^{d/2-1} r_i \int_{[0, T]} (r^2 - \frac{\alpha}{2} + i) \prod_{i=1}^{d/2} \frac{1}{\sqrt{r_i^{2\alpha}}} dr^2
\]

\[
= \frac{\pi^{d/2}}{\prod_{i=1}^{d/2-1} (\frac{\alpha}{2} - i)} T^{d-\alpha}
\]

which ends the proof of the Lemma.

5.2 Proof of Theorem 2.4

We begin the proof of the asymptotic equipartition property, by first establishing a weak law of large numbers for the empirical mark measure and the empirical connect measure of the SINR graph.

Lemma 5.2. Suppose \(X^\lambda\) is an SINR graph with intensity measure \(\lambda \text{Leb}(x) : D \to [0, 1]\) and a marked probability kernel \(Q\) from \(\mathbb{R}^{d-2} \times [-T, T]^2\) to \(\Gamma\) and path loss function \(\ell(r) = r^{-\alpha}\), for \(d - 2 < \alpha < d\). Assume \(\sqrt{\lambda} \tau(x, y) \to \tau(x, y)\) and \(\sqrt{\lambda} \gamma(x, y) \to \gamma(x, y)\), for all \((x, y) \in \mathbb{R}^d\). Let \(K\) be the exponential distribution with parameter \(c\). Then, for \(\varepsilon > 0\) we have

\[
\lim_{\lambda \to \infty} \mathbb{P} \left\{ \sup_{(x, \sigma_x) \in \mathcal{S}(\mathbb{R}^{d-2} \times [-T, T]^2 \times \Gamma)} \left| L_1^\lambda(x, \sigma_x) - m \otimes Q(x, \sigma_x) \right| > \varepsilon \right\} = 0
\]

and

\[
\lim_{\lambda \to \infty} \mathbb{P} \left\{ \sup_{([x, \sigma_x], [y, \sigma_y]) \in \mathcal{S}(\mathbb{R}^{d-2} \times [-T, T]^2 \times \Gamma)} \left| L_2^\lambda([x, \sigma_x], [y, \sigma_y]) - e^{-R_d T} m \otimes Q \otimes m \otimes Q([x, \sigma_x], [y, \sigma_y]) \right| > \varepsilon \right\} = 0
\]

Proof. Let

\[
F_1 = \left\{ \omega : \sup_{(x, \sigma_x) \in \mathcal{S}(\mathbb{R}^{d-2} \times [-T, T]^2 \times \Gamma)} |\omega(x, \sigma_x) - m \otimes Q(x, \sigma_x)| > \varepsilon \right\}
\]

and

\[
F_2 = \left\{ \varpi : \sup_{([x, \sigma_x], [y, \sigma_y]) \in \mathcal{S}(\mathbb{R}^{d-2} \times [-T, T]^2 \times \Gamma)} |\varpi([x, \sigma_x], [y, \sigma_y]) - e^{-R_d T} m \otimes Q \otimes m \otimes Q([x, \sigma_x], [y, \sigma_y])| > \varepsilon \right\}
\]

and \(F_3 = F_1 \cup F_2\). Now, observe from Theorem 2.4 that

\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{P}\left\{ (L_1^\lambda, L_2^\lambda) \in F_3^\varepsilon \right\} \leq - \inf_{(\omega, \varpi) \in F_3^\varepsilon} I_T(\omega, \varpi).\]
It suffices for us to show that $I_T$ is strictly positive. Suppose there is a sequence $(\omega_n, \varpi_n) \rightarrow (\omega, \varpi)$ such that $I_T(\omega_n, \varpi_n) \downarrow I_T(\omega, \varpi) = 0$. This implies $\omega = m \otimes Q$ and $\varpi = e^{-R^{d,T}} m \otimes Q \times m \otimes Q$ which contradicts $(\omega, \varpi) \in F^3$. This ends the proof of the Lemma. \hfill \Box

Now, the distribution of the marked PPP $P(x) = \mathbb{P}\{X^\lambda = x\}$ is given by

$$P_\lambda(x) = \prod_{i=1}^\lambda \mu \otimes Q(x_i, \sigma_i) \prod_{(i,j) \in E} \frac{e^{-\lambda R^D(|x_i, \sigma_i|, |y_j, \sigma_j|)}}{1 - e^{-\lambda R^D(|x_i, \sigma_i|, |y_j, \sigma_j|)}} \prod_{(i,j) \in E} (1 - e^{-\lambda R^D(|x_i, \sigma_i|, |y_j, \sigma_j|)}) \prod_{i=1}^\lambda (1 - e^{-\lambda R^D(|x_i, \sigma_i|, |y_i, \sigma_i|)})$$

$$-\frac{1}{\lambda^2} \log P_\lambda(x) = \frac{1}{\lambda} \left\langle -\log \lambda m \otimes Q, L_1^\lambda \right\rangle + \left\langle -\log \left(\frac{e^{-\lambda R^D}}{1 - e^{-\lambda R^D}}\right), L_2^\lambda \right\rangle + \left\langle -\log(1 - e^{-\lambda R^D}), L_1^\lambda \otimes L_1^\lambda \right\rangle$$

$$+ \left\langle -\log(1 - e^{-\lambda R^D}), L_\Delta^\lambda \right\rangle$$

Notice, $\lim_{\lambda \rightarrow \infty} \lambda R^D \rightarrow R^{d,T}$, $\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \left\langle -\log \lambda m \otimes Q, L_1^\lambda \right\rangle = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \left\langle -\log(1 - e^{-\lambda R^D}), L_\Delta^\lambda \right\rangle = 0.$

Using, Lemma 5.2 we have

$$\lim_{\lambda \rightarrow \infty} \left\langle -\log \left(\frac{e^{-\lambda R^D}}{1 - e^{-\lambda R^D}}\right), L_2^\lambda \right\rangle = \left\langle -\log \left(\frac{e^{-R^{d,T}}}{1 - e^{-R^{d,T}}}\right), e^{-R^{d,T}} m \otimes Q \times m \otimes Q \right\rangle$$

$$\lim_{\lambda \rightarrow \infty} \left\langle -\log(1 - e^{-\lambda R^D}), L_1^\lambda \otimes L_1^\lambda \right\rangle = \left\langle -\log(1 - e^{-R^{d,T}}), m \otimes Q \times m \otimes Q \right\rangle,$$

which concludes the proof of Theorem 2.4

6. PROOF OF THEOREM 2.5 AND COROLLARY 2.6

For $\omega \in \mathcal{P}(D \times \mathbb{R}_+)$ we define the spectral potential of $PPP(X^\lambda)$ conditional on the event $\{L_1^\lambda = \omega\}$, $U_Q(g, m)$ as

$$U_Q(g, \omega) = \left\langle g, e^{-R^{d,T}} \omega \otimes \omega \right\rangle. \tag{6.1}$$

The following remarkable properties holds for $U_Q$:

- (i) It is finite on $\mathcal{C}(\omega) := \left\{ g \in D \times \mathbb{R}_+ \rightarrow \mathbb{R} : e^{U_Q(g, \omega)} < \infty \right\}$
- (ii) It is monotone.
- (iii) it is additively homogeneous.
- (iv) it is convex in $g$.

For $\pi \in \mathcal{B}(D \times R_+)$, we observe that $I_\omega(\pi)$ is the Kullback action of $PPP(X^\lambda)$.

**Lemma 6.1.** The following hold for the Kullback action or divergence function $I_\omega(\pi)$:

(i) $I_\omega(\pi) = \sup_{g \in \mathcal{C}} \left\{ \langle g, \pi \rangle - \langle g, e^{-R^{d,T}} \omega \otimes \omega \rangle \right\}$

(ii) The function $I_\omega(\pi)$ is convex and lower semi-continuous on the space $\mathcal{B}(D \times R_+)$. 

(iii) For any real $\alpha$, the set $\left\{ \pi \in \mathcal{B}(D \times R_+) : I_\omega(\pi) \leq \alpha \right\}$ is weakly compact.
Now note from Lemma 6.1, for any \( \varepsilon > 0 \), there exists a function \( g \in B(D \otimes \mathbb{R}_+) \) such that
\[
I_\omega(\pi) - \frac{\varepsilon}{2} < \langle g, \pi \rangle - U_Q(g, \omega).
\]
Define the probability distribution \( P_\omega \) by
\[
P_\omega(x) = \prod_{(i,j) \in E} e^{g(x_i,x_j)} \prod_{(i,j) \in \mathcal{E}} e^{h_\lambda(x_i,x_j)},
\]
where
\[
h_\lambda(x,y) = \lambda \log \left(1 - e^{-\lambda R_0^D(x,y)} + e^{-\lambda R_1^D(x,y) + g(x,y)/\lambda}\right)
\]
Then, observe that
\[
\frac{dP_\omega(x)}{dP_\omega} = \prod_{(i,j) \in E} e^{-g(x_i,x_j)/\lambda} \prod_{(i,j) \in \mathcal{E}} e^{-h_\lambda(x_i,x_j)/\lambda}
= e^{-\lambda(\frac{1}{\lambda} g(L_2^\lambda) - \frac{1}{\lambda} h_\lambda(L_2^\lambda \otimes L_1^\lambda)) + (\frac{1}{\lambda} h_\lambda(L_2^\lambda))}
\]
Now we define the neighbourhood of \( \nu, B_\nu \) by
\[
B_\nu := \left\{ \pi \in B(D \otimes R_+ \times D \otimes R_+) : \langle g, \pi \rangle > \langle g, \nu \rangle - \varepsilon/2 \right\}
\]
Observe, Under the condition \( L_2^\lambda \subset B_\nu \), we have
\[
\frac{dP_\omega}{dP_\omega} < e^{-\langle \frac{1}{2} g, \nu \rangle + U_Q(g, \omega) + \frac{\varepsilon}{2}} < e^{-\lambda I_\omega(\nu) + \lambda \varepsilon}
\]
Hence, we have
\[
P_\omega \left\{ x^\lambda \in \mathcal{G}_P \bigg| L_2^\lambda \subset B_\nu \right\} \leq \int 1_{\{L_2^\lambda \subset B_\nu\}} d\tilde{P}_\omega(x^\lambda) \leq \int e^{-\lambda I_\omega(\nu) - \lambda \varepsilon} d\tilde{P}_\omega(x^\lambda) \leq e^{-\lambda I_\omega(\nu) - \lambda \varepsilon}.
\]
Observe that \( I_\omega(\nu) = \infty \) implies Theorem 2.6 (ii), hence it sufficient for us to establish it for a probability measure of the form \( \nu = g e^{-R_0^D} \omega \otimes \omega \), where \( g = 1 \) and for \( I_\omega(\nu) = 0 \). Fix any number \( \varepsilon > 0 \) and any neighbourhood \( B_\nu \subset \mathcal{M}(D \otimes \mathbb{R}_+) \). Now define the sequence of sets
\[
\mathcal{G}_P^\lambda = \left\{ y \in \mathcal{G}_P : L_2^\lambda(y) \in B_\nu, \left| \langle g, L_2^\lambda \rangle - \langle g, \nu \rangle \right| \leq \frac{\varepsilon}{2} \right\}
\]
Note that for all \( y \in \mathcal{G}_P^\lambda \) we have
\[
\frac{dP_\omega}{dP_\omega} > e^{-\langle \frac{1}{2} g, \nu \rangle + U_Q(g, \omega) + \frac{\varepsilon}{2}} > e^{\lambda \varepsilon}.
\]
This yields
\[
P_\omega(\mathcal{G}_P^\lambda) = \int_{\mathcal{G}_P^\lambda} dP_\omega(y) \geq \int e^{-\langle \frac{1}{2} g, \nu \rangle + U_Q(g, \omega) + \frac{\varepsilon}{2}} dP_\omega(y) \geq e^{\lambda \varepsilon} \tilde{P}_\omega(\mathcal{G}_P^\lambda).
\]
Using the law of large numbers, we have that \( \lim_{\lambda \to \infty} \tilde{P}_\omega(\mathcal{G}_P^\lambda) = 1 \). This completes of the Theorem.
6.1 Proof of Theorem 2.6

Observe that the empirical connectivity measure is a finite measure and so belongs to the ball of radius \(\langle e^{R_{T}}, \omega \otimes \omega \rangle\) in \(B^{*}(D \otimes \mathbb{R}_{+} \times D \otimes \mathbb{R}_{+})\). Hence, without loss of generality, we may assume that the set \(F\) in Theorem 2.5(i) is relatively compact. See Lemma 6.1(iii).

Let \(\varepsilon > 0\). Then, for every functional \(\nu \in F\), one can find a weak neighbourhood such that the estimate of Theorem 2.5(i) holds. Now, choose from all these neighbourhoods a finite cover of \(G_{P}\) and sum up over the estimate in theorem to get

\[
\limsup_{\lambda \to \infty} \frac{1}{\lambda} \log P_{\omega}\left\{ X^{\lambda} \in G_{P} \mid L^{\lambda}_{2} \in F \right\} \leq - \inf_{\pi \in F} I_{\omega}(\nu) + \varepsilon.
\]

As \(\varepsilon\) was arbitrarily chosen and the lower bound in Theorem 2.5(ii) implies the lower bound in Corollary 2.6(ii), we have the required results, which completes the proof of the theorem.

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