Varieties of semiassociative relation algebras and tense algebras

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Abstract. It is well known that the subvariety lattice of the variety of relation algebras has exactly three atoms. The (join-irreducible) covers of two of these atoms are known, but a complete classification of the (join-irreducible) covers of the remaining atom has not yet been found. These statements are also true of a related subvariety lattice, namely the subvariety lattice of the variety of semiassociative relation algebras. The present article shows that this atom has continuum many covers in this subvariety lattice (and in some related subvariety lattices) using a previously established term equivalence between a variety of tense algebras and a variety of semiassociative $r$-algebras.

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1. Introduction

Varieties have been a major focus of research in relation algebras for a number of years. For example, several of the early landmark results in the subject focus on the variety of representable relation algebras. In 1955, Tarski showed in [25] that the class of all representable relation algebras is indeed a variety. A year later, in [19], Lyndon gave the first (equational) basis for this variety. In 1964, Monk showed in [24] that this variety is not finitely based.

The lattice of subvarieties of the variety of all relation algebras was first studied extensively by Jónsson in [14], although Tarski did publish some results much earlier in [26]. In [26], Tarski showed, using a result from Jónsson and Tarski [16], that this lattice has exactly three atoms. These atoms are generated by $A_1$, $A_2$, and $A_3$ (the minimal subalgebras of the full relation algebras on a 1-element set, a 2-element set, and a 3-element set, respectively). As the variety of relation algebras is congruence distributive, we only need to look for join-irreducible varieties to find the other varieties of small height. It follows from results in [16] that the variety generated by $A_1$ has
no join-irreducible covers. In [3], Andréka, Jónsson, and Németi showed that the variety generated by $A_2$ has exactly one join-irreducible cover. The variety generated by $A_3$ is known to have at least 20 finitely generated join-irreducible covers and at least one infinitely generated join-irreducible cover; generators of these covers are listed by Jipsen in [11]. However, it is not yet known if there are more covers. The problem of completely classifying these covers is posed (in various forms) by Jónsson and Maddux in [20], by Hirsch and Hodkinson in [10], and by Givant in [8]. Further results on this lattice can be found in Jónsson [14] and Andréka, Givant, and Németi [1], for example.

Semiassociative relation algebras arise fairly naturally in the study of relation algebras and the calculus of relations; see Maddux [21], for example. As such, the subvariety lattice of the variety of all semiassociative relation algebras has attracted interest from researchers in this field. In [3], Andréka, Jónsson, and Németi show that the properties of the subvariety lattice of the variety of relation algebras that we mentioned above also hold in this lattice. In [13], Jipsen, Kramer, and Maddux show that $A_3$ has a countably infinite family of covers (in the semiassociative case) by constructing a term equivalence and a countably infinite family of covers of the variety generated by $T_0$ (the two element Boolean algebra with a pair of identity operators). In 1999, the problem of showing that $T_0$ has continuum many covers was posed by Peter Jipsen in a conversation with the second author relating to Kowalski [17]. In Section 3 of the present article we will solve this problem, and hence conclude that $A_3$ has continuum many covers in the subvariety lattice of the variety of semiassociative relation algebras.

2. Preliminaries

2.1. Relation-type algebras

For an extensive introduction to the theory and history of relation algebras, we refer the reader to Hirsch and Hodkinson [10], Maddux [22] and Givant [9]; shorter introductions can be found in Chin and Tarski [7] and Jónsson [14]. For sources that cover nonassociative and semiassociative relation algebras, we refer the reader to Maddux [23], Maddux [21], Hirsch and Hodkinson [10], and Maddux [22]. Subvariety lattices are discussed in Jónsson [14], Andréka, Jónsson, and Németi [3], Jipsen [11], Andréka, Givant, and Németi [2], Jipsen, Kramer, and Maddux [13], Andréka, Givant, and Németi [1], and Givant [8]. We will begin by recalling some definitions from [23] and [13].

**Definition 2.1.** An algebra $A = \langle A; \lor, \land, \cdot, \cdot^\ast, 0, 1, e \rangle$ is called a nonassociative relation algebra iff $\langle A; \lor, \land, \cdot, 0, 1 \rangle$ is a Boolean algebra, $\cdot$ is a binary operation, $e$ is an identity element for $\cdot$, and the triangle laws hold in $A$, i.e.,

$$(x \cdot y) \land z = 0 \iff (x^\ast \cdot z) \land y = 0 \iff (z \cdot y^\ast) \land x = 0,$$

for all $x, y, z \in A$. A nonassociative relation algebra $A$ is called a semiassociative relation algebra iff $A$ satisfies $(x \cdot 1) \cdot 1 \approx x \cdot 1$. A nonassociative relation algebra is called reflexive iff $x \leq x \cdot x$, for all $x \in A$. A nonassociative
Varieties of semiassociative relation algebras and tense algebras

relation algebra $A$ is called symmetric iff $A$ satisfies $x \sim x$. A nonassociative relation algebra $A$ is called subadditive iff $x \cdot (x' \land y) \leq x \lor y$, for all $x, y \in A$.

The triangle laws are equivalent to an equation modulo the other axioms of nonassociative relation algebras (see Theorem 1.4 of [23]), so the classes of all nonassociative relation algebras, semiassociative relation algebras, and reflexive subadditive symmetric semiassociative relation algebras are varieties.

**Notation 2.2.** The subvariety lattices of the varieties of nonassociative relation algebras, semiassociative relation algebras and reflexive subadditive symmetric semiassociative algebras will be denoted by $\Lambda_{NA}$, $\Lambda_{SA}$ and $\Lambda_{RSA}$, respectively.

**2.2. Tense algebras**

For an introduction to the theory and history of tense (or temporal) algebras, we refer the reader to Kracht [18] and Blackburn, de Rijke, and Venema [4]. We refer the reader to Hirsch and Hodkinson [10] and Maddux [22] for an introduction to the theory of Boolean algebras with conjugated operators; Jónsson and Tarski [15] is a shorter introduction. We will need the following definitions from [13].

**Definition 2.3.** An algebra $A = \langle A; \lor, \land, ^{'}, f, g, 0, 1 \rangle$ is called a tense algebra iff $\langle A; \lor, \land, ^{'}, 0, 1 \rangle$ is a Boolean algebra, and $f$ and $g$ are a conjugate pair, i.e.,

$$f(x) \land y = 0 \iff x \land g(y) = 0,$$

for all $x, y \in A$. A tense algebra $A$ is called total iff $f(x) \lor g(x) = 1$, for all $x \in A$ with $x \neq 0$.

A concrete example of a tense algebra is the complex algebra of a frame, i.e., a directed graph. Recall that the complex algebra of a frame $\langle U; R \rangle$ is the algebra $\mathbf{Cm}(U; R) := \langle \wp(U); \cup, \cap, c, f_{R}, g_{R}, \emptyset, U \rangle$, where $\wp(U)$, $c$, $f_{R}$, and $g_{R}$ respectively denote the powerset of $U$, complementation relative to $U$, the image operation defined by $R$, and the preimage operation defined by $R$. Thus, we have

$$f_{R}(X) = \{ u \in U \mid (x, u) \in R, \text{ for some } x \in X \},$$

$$g_{R}(X) = \{ u \in U \mid (u, x) \in R, \text{ for some } x \in X \},$$

for all $X \subseteq U$.

The fact that $f$ and $g$ form a conjugate pair can be expressed by equations (see Theorem 1.15 of [15]), so the class of all tense algebras is a variety. The class of all total tense algebras is not a variety, but the variety it generates is finitely based (see Jipsen [12]).

**Notation 2.4.** The subvariety lattice of the variety of tense algebras will be denoted by $\Lambda_{TA}$. The subvariety lattice of the variety generated by the class of all total tense algebras will be denoted by $\Lambda_{TTA}$.

The variety of reflexive subadditive symmetric semiassociative $r$-algebras (reflexive subadditive symmetric semiassociative relation algebras without
the requirement of an identity element) is known to be term equivalent to \( \Lambda_{\text{TTA}} \) (see Theorem 7 of [13]). This observation can be used to establish the following result (see Section 4 of [13]). Here we use \( \text{Var}(A) \) to denote variety generated by an algebra \( A \).

**Proposition 2.5.** \( \text{Var}(A_3) \) has at least as many covers (in \( \Lambda_{\text{RSA}} \)) as \( \text{Var}(T_0) \) (in \( \Lambda_{\text{TTA}} \)).

We conclude this section with some standard results that we use later. We call a binary relation \( R \) on a set \( U \) total when \((x, y) \in R \) or \((y, x) \in R\), for all \( x, y \in U \); we do not require that \( x \neq y \), so total relations are reflexive. The following results can be found in Section 1 of [13] and Chapter 2 of [11].

**Proposition 2.6.**
1. Let \( R \) be a total binary relation on a set \( U \). Then \( \text{Cm}(\langle U; R \rangle) \) is a total tense algebra.
2. Let \( A \) be a total tense algebra. Then \( A \) is discriminator.

Lastly, we state some basic results on discriminator varieties; we refer the reader to Lemma 9.2 and Theorem 9.4 of [6]. Here we use \( \mathbb{I}, \mathbb{S}, \) and \( \mathbb{U} \) for the usual class operators of taking all isomorphic copies, subalgebras, and ultraproducts of a class of similar algebras, respectively.

**Proposition 2.7.** Let \( \mathcal{K} \) be a class of similar non-trivial algebras with a common discriminator term. Then

1. every element of \( \mathcal{K} \) is simple,
2. all directly indecomposable and subdirectly irreducible elements of \( \text{Var}(\mathcal{K}) \) are simple,
3. the class of simple (and therefore the class of subdirectly irreducible) elements of \( \text{Var}(\mathcal{K}) \) is precisely \( \mathbb{I} \mathbb{S} \mathbb{U}(\mathcal{K}) \).

### 3. Uncountable collections of varieties

In this section we will construct continuum many varieties that cover \( \text{Var}(T_0) \) in \( \Lambda_{\text{TTA}} \) and hence show that \( \text{Var}(A_3) \) has continuum many covers in \( \Lambda_{\text{RSA}} \). These varieties are generated by the complex algebras of an uncountable collection of frames with a very strong resemblance to the recession frames that were defined by Blok in [5] and by Jipsen, Kramer, and Maddux in [13]. Roughly speaking, in the terminology of [13], these frames are \( \omega \)-recession frames with minor modifications that are combined like a \( \mathbb{Z} \)-recession frame. Before we make this more explicit, we will need to introduce some notation.

**Notation 3.1.** Let \( \mathbb{E} \) and \( \mathbb{O} \) denote \( \{2n \mid n \in \mathbb{N}\} \) and \( \{2n + 1 \mid n \in \mathbb{N}\} \), respectively (where \( \mathbb{N} := \{p \in \mathbb{Z} \mid p > 0\} \)). For each \( S \subseteq \mathbb{O} \), let \( S_{\mathbb{E}} := S \cup \mathbb{E} \).

Now we can define the frames we will be working with. We use \( a_{p,m} \) for an arbitrary vertex (or node) because vertices correspond to atoms of complex algebras, and to match the notation used in [13].
Definition 3.2. Let \( a_{p,m} \) be a (distinct) vertex, for each \( p \in \mathbb{Z} \) and \( m \in \mathbb{N} \). Define \( V := \{a_{p,m} \mid p \in \mathbb{Z}, m \in \mathbb{N}\} \). For each \( S \subseteq \emptyset \), let
\[
R_S := \{(a_{p,m}, a_{q,n}) \mid p > q \text{ or } (p = q \text{ and } m \geq n)\} \\
\cup \{(a_{p,1}, a_{p+1,m}) \mid p \in \mathbb{Z}, m \in S\} \\
\cup \{(a_{p,m}, a_{p,m+1}) \mid p \in \mathbb{Z}, m \in \mathbb{N}\}
\]
and let \( F_S := \text{Cm}(\langle V; R_S \rangle) \).

As is usually the case with frames, a diagram is easier to work with than a written description. Thus, we will usually refer to Figure 1.

This graph drawing uses some conventions from [13]. To reduce clutter, we exclude loops, as there are loops at every vertex. The thick vertical arrow indicates that a vertex points at each vertex below it. For example, \( a_{1,1} \) points at \( a_{0,1}, a_{0,2}, \) and \( a_{-1,1} \). The thick horizontal arrows indicate that a vertex points at each vertex to its right. For example, \( a_{1,3} \) points at \( a_{1,2} \) and \( a_{1,1} \). Dashed arrows indicate edges whose inclusion depends on the choice of \( S \). For example, \( a_{0,1} \) points at \( a_{1,3} \) and \( a_{1,1} \) points at \( a_{2,3} \) and \( a_{2,3} \) when \( 3 \in S \), but \( a_{0,1} \) and \( a_{1,1} \) do not point at \( a_{1,3} \) and \( a_{2,3} \), respectively, when \( 3 \notin S \).
For another example, we will list all of the vertices that $a_{0,1}$ points at. When $m \in \mathbb{N}$ is even, $a_{0,1}$ always points at $a_{1,m}$. If $m \in \mathbb{O}$, then $a_{0,1}$ points at $a_{1,m}$ when $m \in S$. If $p < 0$ and $m \in \mathbb{N}$, then $a_{0,1}$ always points at $a_{p,m}$.

Lastly, $a_{0,1}$ always points at $a_{0,1}$ and $a_{0,2}$.

Firstly, we will need to check that these relations are total.

**Lemma 3.3.** Let $S \subseteq \mathbb{O}$. Then $R_S$ is total.

**Proof.** Let $p, q \in \mathbb{Z}$ and let $m, n \in \mathbb{N}$. As $m \geq m$, we have $(a_{p,m}, a_{p,m}) \in R_S$, so $R_S$ is reflexive. Assume that $a_{p,m} \neq a_{q,n}$.

Using Proposition 2.6 and Proposition 2.7(1), we get the following result.

**Corollary 3.4.** Let $S \subseteq \mathbb{O}$. Then every element of $\mathbb{S}(F_S)$ is simple.
Lemma 3.7. \( S \)

Proof. It is clear that while \( B \)

It follows that \( B \)

Under forming complements, it will be enough to show that \( B \)

Clearly, \( S \)

Then it is clear that \( D \)

Thus, \( D \)

By additivity, to show that \( D \)

By distributivity, to show that \( D \)

From De Morgan’s laws and the observations above, to show the closure of \( S \)

Combining these results, we find that \( B \)

From De Morgan’s laws and the observations above, to show the closure of \( B \)

As \( A_{p,m}^c = \bigcup\{A_{p,n} \mid n < m\} \cup U_{p+1} \cup D_{p-1} \cup S_{p,m+1} \cup S_{m,p-1} \cup \bar{S}_{p,m} \),

\[ U_p^c = D_{p-1}, \quad D_p^c = U_{p+1}, \quad S_{p,m}^c = \bigcup\{A_{p,n} \mid n < m\} \cup S_{p,m} \cup U_{p+1} \cup D_{p-1} \]

and \( \bar{S}_{p,m}^c = \bigcup\{A_{p,n} \mid n < m\} \cup S_{p,m} \cup U_{p+1} \cup D_{p-1} \)， it follows that \( B \)

It follows that \( B \)

By additivity, to show that \( B \)

So we only need to check that \( f_B(X), g_S(X) \in B \) if \( X \in S \). This follows from Lemma 3.6 so \( B \) is indeed a subuniverse of \( F_S \), which is what we wanted.

This result allows us to make the following definition.

Notation 3.8. Let \( S \subseteq \emptyset \). The subalgebra of \( F_S \) with universe \( B \) will be denoted by \( B_S \).

Before we show that these algebras have the properties we want, we will show that they are generated by any element of the form \( V_p \), for some \( p \in \mathbb{Z} \).
**Lemma 3.9.** Let $S \subseteq \mathcal{O}$, let $X \subseteq V$ and assume that there is a maximal $p \in \mathbb{Z}$ with $V_p \cap X \neq \emptyset$, say $q$. Then

1. $f_S^2(X) \cap f_S^5(X)^c = V_{q+2}$ if $a_{q,1} \in X$,
2. $f_S^5(X) \cap f_S^3(X)^c = V_{q+2}$ if $a_{q,1} \notin X$.

**Proof.** Firstly, assume that $a_{q,1} \in X$. Based on Figure 1, $f_S^2(X) = D_{q+1}$ and $f_S^5(X) = D_{q+2}$, which implies that $f_S^2(X) \cap f_S^5(X)^c = V_{q+2}$. Thus, (1) holds.

Now, assume that we have $a_{q,1} \notin X$. Similarly to the previous case, $f_S^5(X) = D_{q+1}$ and $f_S^5(X) = D_{q+2}$, hence $f_S^5(X) \cap f_S^3(X)^c = V_{q+2}$. Thus, (2) also holds. \qed

We will need the following argument later, so we isolate it here. Firstly, we define some terms (in the signature $\{\lor, \land, \lnot, f, g, 0, 1\}$ of tense algebras).

**Definition 3.10.**

1. Let $\beta(x) := f^4(x) \land f^2(x)'$.
2. Let $\sigma(x) := f(x) \land (x \lor g^2(\beta(x)) \lor f^4(g^{10}(\beta(x)) \land g^8(\beta(x)'))')$.
3. Let $\nu_3(x) := f(\sigma(x)) \land f(x)'$.
4. Let $\nu_4(x) := f(\nu_3(x)) \land f(\sigma(x))'$.
5. (For each $n \geq 5$, let $\nu_n := f(\nu_{n-1}(x)) \land f(\nu_{n-2}(x))'$.)

**Lemma 3.11.** Let $S \subseteq \mathcal{O}$ and let $p \in \mathbb{Z}$. Then we have $\sigma(A_{p,1}) = A_{p,2}$ and $\nu_n(\sigma(x)) = A_{p,n}$, for all $n \geq 3$.

**Proof.** By Lemma 3.9(1), we have $\beta(A_{p,1}) = V_{p+2}$. So, based on Figure 1, $g_S^2(\beta(A_{p,1})) = U_{p+1}$. Similarly, $g_S^8(V_{p+2}) = U_{p-2}$ and $g_S^{10}(V_{p+2}) = U_{p-3}$, hence $f_S^2(g_S^{10}(\beta(A_{p,1})) \land g_S^8(\beta(A_{p,1})))^c = f_S^2(V_{p-3}) = D_{p-1}$. By Lemma 3.6(1), we have $\sigma(A_{p,1}) = A_{p,2}$, as claimed.

Next, we will use a (strong) inductive argument for the second claim. By Lemma 3.6(1) and Lemma 3.6(2), $\nu_3(A_{p,1}) = f_S(A_{p,2}) \cap f_S(A_{p,1})^c = A_{p,3}$ and $\nu_4(A_{p,1}) = f_S(A_{p,3}) \cap f_S(A_{p,2})^c = A_{p,3}$ as $\sigma(A_{p,1}) = A_{p,2}$. Let $n \geq 5$ and assume that $\nu_m(A_{p,1}) = A_{p,m}$, for all $4 \leq m \leq n$. From this assumption and Lemma 3.6(2), it follows that $\nu_{n+1}(A_{p,1}) = f_S(A_{p,n}) \cap f_S(A_{p,n-1})^c = A_{p,n+1}$. Thus, $\nu_m(A_{p,1}) = A_{p,m}$, for all $m \geq 3$, as claimed. \qed

**Lemma 3.12.** Let $S \subseteq \mathcal{O}$ and let $p \in \mathbb{Z}$. Then $\mathcal{B}_S$ is the subuniverse of $\mathcal{B}_S$ generated by $V_p$.

**Proof.** Let $\mathcal{V}_p$ denote the subuniverse of $\mathcal{B}$ generated by $V_p$. By Lemma 3.7 it will be enough to show that $\mathcal{B}_S \subseteq \mathcal{V}_p$.

Firstly, we claim that $V_q \in \mathcal{V}_p$, for every $q \in \mathbb{Z}$. Based on Figure 1, Lemma 3.6(3) and Lemma 3.6(5), we have $f_S^{2m}(V_p) = D_{p+m}$, for each $m \in \mathbb{N}$. This implies that $V_{p+m+1} = f_S^{2m+2}(V_p) \cap f_S^{2m}(V_p)^c \in \mathcal{V}_p$, for every $m \in \mathbb{N}$, hence we have $V_q \in \mathcal{V}_p$, for every $q \geq m+2$. Similarly, if $q \in \mathbb{Z}$ and $m \in \mathbb{N}$, then $g_S^{2m}(V_q) = U_{q-m}$, hence $V_{q-m-1} = g_S^{2m+2}(V_q)^c \cap g_S^{2m}(V_q) \in \mathcal{V}_p$. Thus, we must have $V_q \in \mathcal{V}_p$, for all $q \in \mathbb{Z}$, as claimed.

Based on Figure 1, Lemma 3.6(3) and Lemma 3.6(5), $f_S^2(V_{q-1}) = D_q$, for each $q \in \mathbb{Z}$. Similarly, we also have $g_S^2(V_{q+1}) = U_q$, for every $q \in \mathbb{Z}$. Hence, by the previous result, we have $D_q, U_q \in \mathcal{V}_p$, for all $q \in \mathbb{Z}$.
By Lemma 3.6(13) and the previous result, \( A_{p,1} = g(U_{q+1}) \cap U_{q+1}^c \in \mathcal{V}_p \), for all \( q \in \mathbb{Z} \). So, by Lemma 3.11 we have \( A_{p,n} \in \mathcal{V}_p \), for all \( p \in \mathbb{Z} \) and \( n \in \mathbb{N} \).

Based on Lemma 3.6(3), \( S_{q,m} = f_S(D_{q-1}) \cap (\bigcup \{ A_{q,n} \mid n < m \} \cup D_{q-1}^c) \). Hence, by the above results, we must have \( S_{q,m} \in \mathcal{V}_p \), for all \( q \in \mathbb{Z} \) and \( m \in \mathbb{N} \).

Clearly, we must have \( \bar{S}_{q,m} = (\bigcup \{ A_{q,n} \mid n < m \} \cup D_{q-1} \cup U_{q+1} \cup S_{q,m}^c) \), for all \( q \in \mathbb{Z} \) and \( m \in \mathbb{N} \). So, based the above results, we must have \( \bar{S}_{q,m} \in \mathcal{V}_p \), for all \( q \in \mathbb{Z} \) and \( m \in \mathbb{N} \).

Based on these results, \( \mathcal{V}_p = \mathcal{B}_S \), which is what we wanted to show. \( \square \)

Now we will shift our focus to varieties. To make use of Proposition 2.7(3), we will need a number of intermediate results.

**Lemma 3.13.** Let \( S \subseteq \mathcal{O} \) and let \( X \in \mathcal{B}_S \). Then \( f_S(X) \neq V \) or \( f_S(X^c) \neq V \).

**Proof.** By Lemma 3.7, \( X \) and \( X^c \) can be represented as unions of finite subsets of \( \mathcal{S}_S \). Clearly, only one of the unions will involve an element of \( \{ U_p \mid p \in \mathbb{Z} \} \). So, based on Figure 1, we have \( f_S(X) \neq V \) or \( f_S(X^c) \neq V \), as required. \( \square \)

**Lemma 3.14.** Let \( S \subseteq \mathcal{O} \) and let \( X \in \mathcal{B}_S \) such that \( X \neq \emptyset \) and \( f_S(X) \neq V \). Then there is a maximal \( p \in \mathbb{Z} \) with \( V_p \cap X \neq \emptyset \).

**Proof.** By Lemma 3.7, \( X \) can be represented as the union of a finite subset of \( \mathcal{S} \). By assumption, \( f_S(X) \neq V \), so Lemma 3.6(iii) tells us that such a representation cannot involve an element of \( \{ U_p \mid p \in \mathbb{Z} \} \). Since \( X \neq \emptyset \), there is maximal \( p \in \mathbb{Z} \) with \( V_p \cap X \neq \emptyset \), as claimed. \( \square \)

Using Lemma 3.9 and the preceding pair of results, it is easy to verify our previous claim that \( \mathcal{B}_S \) is the subalgebra of \( \mathcal{F}_S \) generated by its atoms, or by any element of \( \mathcal{B}_S \), for each \( S \subseteq \mathcal{O} \). The following result will allow us to obtain similar results for ultrapowers.

**Lemma 3.15.** Let \( S \subseteq \mathcal{O} \) and let \( p, q \in \mathbb{Z} \).

1. We have \( \mathcal{B}_S = \{ t^{\mathcal{B}_S}(V_p) \mid t \text{ is a unary term} \} \).
2. If \( t \) and \( s \) are unary terms with \( t^{\mathcal{B}_S}(V_p) = s^{\mathcal{B}_S}(V_p) \), then we have \( t^{\mathcal{B}_S}(V_q) = s^{\mathcal{B}_S}(V_q) \).
3. There is an automorphism of \( \mathcal{B}_S \) that maps \( V_p \) to \( V_q \).

**Proof.** The first statement is an immediate consequence of Lemma 3.7, while (2) is evident from the self similarity of \( (V; R_S) \).

From (1) and (2), it follows that we can define a map \( \mu : \mathcal{B}_S \to \mathcal{B}_S \) by setting \( \mu(t^{\mathcal{B}_S}(V_p)) = t^{\mathcal{B}_S}(V_q) \), for every unary term \( t \). Combining (1) and (2), we find that \( \mu \) is a bijection. Based on (2), \( \mu \) is an endomorphism of \( \mathcal{B}_S \). Thus, \( \mu \) is an automorphism of \( \mathcal{B}_S \), so (3) holds. \( \square \)

**Lemma 3.16.** Let \( S \subseteq \mathcal{O} \), let \( I \) be a non-empty set, let \( \mathcal{U} \) be an ultrafilter over \( I \), and let \( X \in \mathcal{B}_S^I \) with \( X/\mathcal{U} \neq 0 \) and \( X/\mathcal{U} \neq 1 \). Then \( \mathcal{B}_S \) embeds into the subalgebra of \( \mathcal{B}_S^I/\mathcal{U} \) generated by \( X/\mathcal{U} \).
Proof. By Lemma 3.13 and Loš’s Theorem, \( f(X/\mathcal{U}) \neq 1 \) or \( f(X/\mathcal{U}') \neq 1 \). Without loss of generality, we can assume that \( f(X/\mathcal{U}) \neq 1 \). By Lemma 3.9 and Lemma 3.14, either \( \{i \in I \mid f_X(X(i)) \cap f_Y(X(i)) = V_p \} \) or \( \{i \in I \mid f_X(X(i)) \cap f_Y(X(i)) = V_p \} \) is an element of \( \mathcal{U} \). Based on Lemma 3.15(3), \( B_S \) embeds into the subalgebra of \( B_S/X/\mathcal{U} \) generated by \( X/\mathcal{U} \), as claimed. \( \square \)

Note that we only needed the fact that ultrafilters are prime filters, hence this result applies to principal ultrafilters.

**Lemma 3.17.** Let \( S \subseteq \emptyset \). Then \( \text{Var}(B_S) \) covers \( \text{Var}(T_0) \) in \( \Lambda_{TTA} \). Further, \( \text{Var}(B_S) \) is join-irreducible.

Proof. It is clear that \( \{\emptyset, V\} \) is a subuniverse of \( B_S \), and that the corresponding subalgebra of \( B_S \) is isomorphic to \( T_0 \). By Jónsson’s Theorem, \( \text{Si}(\text{Var}(T_0)) = \mathbb{I}(T_0) \), so by Lemma 3.4 we must have \( \text{Var}(T_0) \subseteq \text{Var}(B_S) \). Let \( V \in \Lambda_{TTA} \) with \( \text{Var}(T_0) \subseteq V \subseteq \text{Var}(B_S) \). Clearly, \( \text{Si}(V) \subseteq \text{Si}(\text{Var}(B_S)) \). By Proposition 2.7(3), \( \text{Si}(\text{Var}(B_S)) = \mathbb{ISU}(B_S) \), hence \( \text{Si}(V) \subseteq \mathbb{ISU}(B_S) \).

Since \( \text{Var}(T_0) \subseteq V \), there is some \( A \in \text{Si}(V) \) with more than 2 elements. By Lemma 3.16, \( B_S \) embeds into \( A \), so \( B_S \in V \), hence \( V = \text{Var}(B_S) \). Thus, \( \text{Var}(B_S) \) is a cover of \( \text{Var}(T_0) \) in \( \Lambda_{TTA} \), as required. Based on these arguments, it is evident that \( \text{Var}(B_S) \) is also join-irreducible. \( \square \)

Lastly, we will need to show that these varieties are distinct. The following pair of results effectively reduce this problem to showing that \( A_S \) and \( A_T \) are not elementarily equivalent, for all distinct \( S, T \subseteq \emptyset \).

**Lemma 3.18.** Let \( S, T \subseteq \emptyset \) and let \( \mu: B_S \to B_T \) be a homomorphism. Then \( \mu \) is an isomorphism.

Proof. Since \( B_S \) and \( B_T \) are non-trivial, the kernel of \( \mu \) must be non-zero. From Corollary 3.4 \( B_S \) is simple, so the kernel of \( \mu \) is the identity relation. This implies that \( \mu \) is an embedding, so \( \emptyset \subseteq \mu(V_0) \subseteq V \). By Lemma 3.9 Lemma 3.12 Lemma 3.13 and Lemma 3.14 we must have \( \mu[\mathcal{B}_S] = \mathcal{B}_T \). Hence, \( \mu \) is surjective, and \( \mu \) is an isomorphism, as required. \( \square \)

Using Lemma 3.11 we will construct some useful first-order formulae (again, in the signature \( \{\lor, \land, ', f, g, 0, 1\} \) of tense algebras). To avoid confusion, we will use \( \gamma \) for logical disjunction and \( \lambda \) for logical conjunction.

**Definition 3.19.**
1. Let \( \alpha(x) := x \neq 0 \land (\forall y: x \land y \approx 0 \land x \land y \approx x) \).
2. Let \( \varphi(x) := \alpha(x) \land \neg(\exists w, y, z: \alpha(w) \land \alpha(y) \land \alpha(z) \land f(x) \land g(x) \approx w \lor w \lor z) \).
3. For each \( n \geq 3 \), let \( \tau_n(x) := \varphi(x) \land \nu_n(x) \land f(g(x) \land g(x') \lor 0) \).

**Lemma 3.20.** Let \( S \subseteq \emptyset \), let \( n \geq 3 \) and let \( X \in \mathcal{B}_S \). Then
1. \( B_S \models \varphi[X] \) if and only if \( X = A_{p,1} \), for some \( p \in \mathbb{Z} \).
2. \( B_S \models \tau_n[X] \) if and only if \( n \in S_2 \) and \( X = A_{p,1} \), for some \( p \in \mathbb{Z} \).

Proof. If \( T \) is a tense algebra and \( x \in T \), then \( T \models \alpha[x] \) if and only if \( x \) is an atom, hence \( B \models \alpha[X] \) if and only if \( X = A_{p,n} \), for some \( p \in \mathbb{Z} \) and \( n \in \mathbb{N} \). By Lemma 3.6(1) and Lemma 3.6(9), \( f_S(A_{p,n}) \cap g_S(A_{p,n}) = A_{p,1} \cup A_{p,2} \cup S_{p,1} \).
if \( n = 1 \) and \( p \in \mathbb{Z} \). Similarly, \( f_S(A_{p,n}) \cap g_S(A_{p,n}) = A_{p,n-1} \cup A_{p,n} \cup A_{p,n+1} \)
if \( n > 1 \) and \( p \in \mathbb{Z} \). Combining these results, we see that (1) holds.

Based on Lemma 3.6(9) and Lemma 3.6(13), we have \( g_S(A_{p,1}) = U_p \)
and \( g_S^2(A_{p,1}) = g_S(U_p) = A_{p-1,1} \cup U_p \), for each \( p \in \mathbb{Z} \). So, by Lemma 3.6(1),
\[ f_S(g_S^2(A_{p,1}) \cap g_S(A_{p,1})^c) = f_S(A_{p-1,1} \cup A_{p-1,2} \cup D_{p-2} \cup S_{p,1}) \]
when \( p \in \mathbb{Z} \). By Lemma 3.11 \( \tau_n(A_{p,1}) \cap f_S(g_S^2(A_{p,1}) \cap g_S(A_{p,1})^c) \neq \emptyset \) if and only
if \( n \in S_0 \), as \( \tau_n(A_{p,1}) = A_{p,n} \). Based on this and \( (1) \), \( (2) \) also holds. \( \square \)

Now we have the results we need to show that our varieties are distinct.

**Lemma 3.21.** Let \( S, T \subseteq \emptyset \) with \( S \neq T \). Then \( \text{Var}(B_S) \neq \text{Var}(B_T) \).

**Proof.** Without loss of generality, we can assume that \( S \not\subseteq T \), since \( S \neq T \). Let \( n \in N \) with \( n \in S \) and \( n \notin T \). By Lemma 3.20 \( B_S \models \exists x: \tau_n(x) \) and
\( B_T \models \exists x: \tau_n(x) \), hence \( B_S \) and \( B_T \) are not elementarily equivalent. Thus, \( B_S \) and \( B_T \) are not isomorphic, so by Lemma 3.18 \( B_S \) does not embed into \( B_T \).
Based on Proposition 2.7(3), Corollary 3.4 and Lemma 3.16 \( B_T \notin \text{Var}(B_S) \),
so \( \text{Var}(B_S) \neq \text{Var}(B_T) \), as claimed. \( \square \)

Now we just need to put on the finishing touches.

**Theorem 3.22.** \( \text{Var}(T_0) \) has \( 2^{\aleph_0} \) join-irreducible covers in \( \text{Var}(T_{TTA}) \) and \( \text{Var}(T_{TA}) \).

**Proof.** Let \( C \) denote the set of join-irreducible covers of \( \text{Var}(T_0) \) in \( \text{Var}(T_{TTA}) \).
Combining Lemma 3.17 and Lemma 3.21 we find that \( |C| \geq 2^{\aleph_0} \). It is easy
to see that there are at most \( 2^{\aleph_0} \) sets of equations in a countable signature
(up to replacing variables), hence \( |C| \leq |\text{Var}(T_{TTA})| \leq 2^{\aleph_0} \). Therefore \( |C| = 2^{\aleph_0} \),
which is what we wanted to show. \( \square \)

Combining Proposition 2.5 and Theorem 3.22 we obtain our main result.

**Theorem 3.23.** \( \text{Var}(A_3) \) has exactly \( 2^{\aleph_0} \) join-irreducible covers in \( \text{Var}(T_{RSA}), \text{Var}(A_{SA}), \text{Var}(A_{NA}). \)

We can also use Theorem 3.22 to obtain similar results on the subvariety
lattices of other varieties of \( r \)-algebras and nonassociative relation algebras;
see Section 4 of [13] for more details.

4. Concluding remarks

In the previous section we saw that \( \text{Var}(T_0) \) has \( 2^{\aleph_0} \) covers in \( \text{Var}(T_{TA}) \) and \( \text{Var}(T_{TTA}), \)
and that \( \text{Var}(A_3) \) has \( 2^{\aleph_0} \) covers in \( \text{Var}(A_{SA}) \) and \( \text{Var}(A_{RSA}) \). However, the problem
of completely characterizing these covers remains open. Based on a computer
search, it seems that there are finite semiassociative relation algebras that
generate covers of \( \text{Var}(A_3) \) in \( \text{Var}(A_{SA}) \) that are not in the list in [11]. In fact,
based on this search, it seems reasonable to conjecture that there are finite
algebras with arbitrarily large atom sets that generate covers of \( \text{Var}(A_3) \) in \( \text{Var}(A_{SA}), \) so the problem
of completely characterizing covers of \( \text{Var}(A_3) \) in \( \text{Var}(A_{SA}) \) could prove to be quite difficult.
Using Lemma 3.6 and the term equivalence in [13], it is not too difficult to show that the semiassociative relation algebras corresponding to the tense algebras constructed above are not relation algebras. (For example, \(A_{0,1}(A_{0,1}A_{1,1})\) and \((A_{0,1}A_{1,1})A_{1,1}\) are always distinct, so associativity fails.) Thus, the results obtained above do not appear to provide any information about the covers of \(\text{Var}(B_3)\) in the lattice \(\Lambda_{RA}\) of subvarieties of the variety of relation algebras.

Based on these observations, the following problems seem like reasonable starting points for further research in this area.

**Problem 1.** Classify the covers of \(\text{Var}(T_0)\) in \(\Lambda_{TA}\) (or \(\Lambda_{TTA}\)).

**Problem 2.** Classify the covers of \(\text{Var}(A_3)\) in \(\Lambda_{SA}\) (or \(\Lambda_{RSA}\)).

**Problem 3.** Determine whether or not \(\text{Var}(A_3)\) has infinitely many finitely generated covers in \(\Lambda_{SA}\).

**Problem 4.** Determine the number of covers of \(\text{Var}(A_3)\) in \(\Lambda_{RA}\).

**Problem 5.** Classify the covers of \(\text{Var}(A_3)\) in \(\Lambda_{RA}\).

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