A note on dual prehomomorphisms from a group into the Margolis–Meakin expansion of a group

Bernd Billhardt1 · Boorapa Singha2 · Worachead Sommanee2 · Paweena Thamkaew2 · Jukrapong Tiammee2

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Abstract
We give a category-free order theoretic variant of a key result in Auinger and Szendrei (J Pure Appl Algebra 204(3):493–506, 2006) and illustrate how it might be used to compute whether a finite X-generated group H admits a canonical dual prehomomorphism into the Margolis–Meakin expansion M(G) of a finite X-generated group G. We show that for G the Klein four-group a suitable H must be of exponent 6 at least and recapture a result of Szakács.

Keywords Margolis–Meakin expansion · E-unitary inverse monoid · Dual prehomomorphism

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Bernd Billhardt
billhardt@uni-kassel.de
Boorapa Singha
boorapas@yahoo.com
Worachead Sommanee
worachead_som@cmru.ac.th
Paweena Thamkaew
paweena_tha@ac.th
Jukrapong Tiammee
jukrapong_tia@cmru.ac.th

1 Fachbereich 10 - Mathematik und Naturwissenschaften, Institut für Mathematik, Universität Kassel, Untere Königsstraße 86, 34109 Kassel, Germany
2 Department of Mathematics and Statistics, Faculty of Science and Technology, Chiang Mai Rajabhat University, Chiang Mai 50300, Thailand

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1 Introduction

The following note considers canonical, i.e. generator preserving dual prehomomorphisms from an $X$-generated group $H$ into the Margolis–Meakin expansion $M(G)$ of an $X$-generated group $G$. It was shown by Auinger and Szendrei [1] that such mappings play an important role in constructing (finite) $F$-inverse covers for (finite) inverse monoids. We give a necessary and sufficient order theoretic condition for $M(G)$ to admit a canonical dual prehomomorphism from an $X$-generated group $H$. It can be seen as a variant of the key statement Lemma 3.1 in [1] and might be applicable on a computer. The idea is to represent the elements of both $M(G)$ and $H$ as congruence classes of words in the free monoid with involution $(X \cup X^{-1})^*$. This enables us to handle the elements of $H$ in relation to $M(G)$ by systematically going through the words in $(X \cup X^{-1})^*$. We use the slightly different view of $M(G)$, introduced in [2], to show how already known positive examples fit into the picture. Further, for $G$ the Klein four-group, we prove that a suitable group $H$ must be of exponent 6 at least and recapture a result of Szakács [6].

2 Preliminaries and notations

For all undefined notions and notations, the reader is referred to [3, 5]. Let $X$ be a nonempty set and let $G$ be an $X$-generated group with respect to an injection $\varepsilon_G : X \to G \setminus \{1_G\}$. Note that the mapping $\varepsilon_G$ can be uniquely extended to a homomorphism $\varphi_G : (X \cup X^{-1})^* \to G$, where $(X \cup X^{-1})^*$ is the free monoid with involution on $X$. For $w \in (X \cup X^{-1})^*$ we denote $w\varphi_G$ by $\overline{w}$. By the Cayley graph $\Gamma(G)$ with respect to $\varepsilon_G$, we mean the directed graph whose vertex set $V(\Gamma(G))$ is $G$ and whose edge set $E(\Gamma(G))$ is $G \times X$, where for each $g \in G, x \in X, (g,x)$ denotes an edge with initial vertex $g$ and terminal vertex $g\overline{x}$. Put

$$M(G) = \{(\Gamma, g) : \Gamma \text{ is a finite connected subgraph of } \Gamma(G) \text{ with at least one edge and } 1_G, g \in V(\Gamma)\} \cup \{ (\emptyset, 1_G) \}.$$  

There is a natural action of $G$ on the semilattice of all subgraphs of $\Gamma(G)$ with operation the set theoretic union, defined as follows: Put $g\emptyset = \emptyset$, and for each nonempty subgraph $\Gamma$ of $\Gamma(G)$ and $g \in G$, let $g\Gamma$ be the subgraph of $\Gamma(G)$ with $V(g\Gamma) = \{ gh : h \in V(\Gamma) \}$ and $E(g\Gamma) = \{ (gh, x) : (h, x) \in E(\Gamma) \}$. The graphs we consider do not have isolated vertices, whence they are solely determined by their edge sets, and we conveniently may regard them as (possibly empty) subsets of $X \times G$.

The following theorem was essentially proved in [4].
Theorem 2.1 [4] \( M(G) \) is an E-unitary inverse monoid with respect to the multiplication \((\Gamma, g)(\Gamma', h) = (\Gamma \cup g\Gamma', gh)\) with identity element \((\emptyset, 1_G)\) and maximal group homomorphic image \( G \). Further, \( M(G) \) is \( X \)-generated as inverse monoid via the injection \( \varepsilon_{M(G)} : x \mapsto ((1_G, x), \bar{x}) \).

We often represent the elements of \( M(G) \) by their corresponding images \( (w) \) in \((X \cup X^{-1})^*/\ker \varphi_{M(G)}\), where \( \varphi_{M(G)} \) denotes the unique extension of \( \varepsilon_{M(G)} \) to a homomorphism from \((X \cup X^{-1})^* \) onto \( M(G) \). Then obviously \(<\emptyset>\) corresponds to \((\emptyset, 1_G)\).

Let \( \emptyset \neq w = \prod x_i^{\eta_i}, \eta_i \in \{-1, 1\} \), be a word in \((X \cup X^{-1})^* \). To \( w \) we associate a word \( w' = h_1x_1h_2x_2 \cdots h_nx_nh_{n+1} \) in the free product \( X^* \ast G \), where \( X^* \) is the free monoid on \( X \), by replacing each \( x_i^{\eta_i} \) in \( w \) by \( g_i, x_i, g_i \), where
\[
g_i = \begin{cases} 1_G & \text{if } \eta_i = 1, \\ \overline{x_i}^{-1} & \text{if } \eta_i = -1. \end{cases}
\]

Then \( <w> \) corresponds to \((\Gamma(<w>), \overline{w}) \in M(G)\), in symbols \( <w> \cong (\Gamma(<w>), \overline{w}), \) where \( E(\Gamma(<w>)) = \{(h_1, x_1), (h_1\overline{x_1}h_2, x_2), \ldots, (h_1\overline{x_1}h_2\overline{x_2} \cdots h_n, x_n)\} \) and \( \overline{w} = \prod x_i^{\eta_i} \). Conversely, for each \((\Gamma, g) \in M(G)\) there is a unique \( <w> \) with \( <w> \cong (\Gamma, g) \) for some \( w \in (X \cup X^{-1})^* \). For details we refer to [2]. We illustrate the situation by the following example.

Example 2.1 Let \( X = \{x, y\} \) and let \( G = \{1_G, g, h, gh\} \) be the \( X \)-generated Klein four-group with \( \bar{x} := g \) and \( \bar{y} := h \). Then \( \Gamma(G) = \{(1_G, x), (1_G, y), (g, x), (g, y), (h, x), (h, y), (gh, x), (gh, y)\} \). Now, let e.g. \( w = xy^{-1}x^{-1} \in (X \cup X^{-1})^* \). We get \( w' = x\bar{y}^{-1}y\bar{y}^{-1}\bar{x}^{-1}x\bar{x}^{-1} = xhyghxg \), whence \( <w> \) corresponds to
\[
(((1_G, x), (gh, y), (h, x)), h) \in M(G).
\]

On the other hand e.g. \(((1_G, x), (g, x), (h, y)), g) \in M(G)\) corresponds to \( <x^2y^{-1}yx^{-1}> \), being equal to e.g. \( <x^2y^{-1}yx> \).
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In $M(G)$ the natural partial order is given by $\langle v \rangle \leq \langle w \rangle$ if and only if $\overline{v} = \overline{w}$ and $\Gamma(\langle w \rangle) \subseteq \Gamma(\langle v \rangle)$. The following order theoretic statements are straightforward.

**Proposition 3.1** The least upper bound $\vee_{i \in I} \langle w_i \rangle$ with respect to $\leq$ exists in $M(G)$ if and only if all $\overline{w_i}$ are equal to a given $\overline{w}$, say, and

1. $\overline{w} \neq 1_G$ and $\overline{w}$ is a vertex of the $1_G$ containing connected part of $\cap_{i \in I} \Gamma(\langle w_i \rangle)$, denoted by $\text{cp}(\cap_{i \in I} \Gamma(\langle w_i \rangle))$, in which case $\vee_{i \in I} \langle w_i \rangle \equiv (\text{cp}(\cap_{i \in I} \Gamma(\langle w_i \rangle)), 1_G)$, if the latter exists, and
2. $\overline{w} = 1_G$ in which case $\vee_{i \in I} \langle w_i \rangle \equiv (\text{cp}(\cap_{i \in I} \Gamma(\langle w_i \rangle)), 1_G) = 1_{M(G)}$, otherwise.

Note that the greatest lower bound $\wedge_{i \in I} \langle w_i \rangle$ exists in $M(G)$ for each finite set $I$ if and only if all $\overline{w_i}$ are equal to a given $\overline{w}$, say, in which case $\wedge_{i \in I} \langle w_i \rangle \equiv (\cup_{i \in I} \Gamma(\langle w_i \rangle), \overline{w})$. Note further that $\vee_{i \in I} \langle w_i \rangle$ exists if and only if the set $\{\langle w_i \rangle, i \in I\}$ has an upper bound in $M(G)$. 
Let $H$ be an $X$-generated group via an injection $\epsilon_H : X \to H \setminus \{1_H\}$. Like with $M(G)$ we may represent the elements of $H$ by their corresponding images $[w]$ in $(X \cup X^{-1})^*/\ker \varphi_H$, where $\varphi_H$ denotes the unique extension of $\epsilon_H$ to a homomorphism from $(X \cup X^{-1})^*$ onto $H$. A mapping $\psi : H \to M(G)$ is called a dual prehomomorphism if $([v][w])\psi \geq ([v])\psi([w])\psi$ and $([v])\psi = ([v])\psi^{-1}$ for all $[v],[w] \in H$, see [5]. According to [1], we call $\psi$ canonical if $([x])\psi = \langle x \rangle$ for all $x \in X$. Note that a canonical dual prehomomorphism $\psi : H \to M(G)$ always induces a generator respecting homomorphism from $H$ onto $G$, given by $[w] \mapsto \bar{\psi}$, which follows from the fact that in $M(G)$ we have that $(\Gamma(\langle v \rangle)) \subseteq (\Gamma(\langle \bar{\psi} \rangle))$ implies $\bar{v} = \bar{\omega}$ and $\psi$ respects generators. Thus $H$ necessarily must be an extension by $G$. Further $(1_H)\psi = 1_{M(G)}$, since

$$(1_H)\psi = ([xx^{-1}])\psi \geq ([x])\psi([x^{-1}])\psi = ([x])\psi(([x])\psi^{-1}) = \langle x \rangle\langle x^{-1} \rangle$$

which corresponds to $(\Gamma(\langle x \rangle), 1_G) = (((1_G, x), 1_G)$ and on the other hand $(1_H)\psi = ([x^{-1}])\psi \geq \langle x^{-1} \rangle \langle x \rangle$ is corresponding to $((\bar{x}^{-1}, x), 1_G)$. Consequently $(\Gamma(1_H)\psi) \subseteq \{(1_G, x) \cap (\bar{x}^{-1}, x) = \emptyset$ implying $(1_H)\psi = 1_{M(G)}$.

In what follows we give a necessary and sufficient condition for $M(G)$ to admit a canonical dual prehomomorphism $\psi : H \to M(G)$. Our condition is of an order theoretic form.

**Theorem 3.2** Let $G$ and $H$ be groups as defined above. Then $H$ admits a canonical dual prehomomorphism $\psi : H \to M(G)$ if and only if the following sequence of least upper bounds exists for each $[w] \in H$:

$$P_0([w]) := \vee_{[v] \equiv [w]} \langle v \rangle$$

$$P_n([w]) := P_{n-1}([w])P_{n-1}([w])$$

**Proof** Necessity: Let $\psi : H \to M(G)$ be a canonical dual prehomomorphism. Let $[w] \in H$, for some $w \in (X \cup X^{-1})^*$. Since $\psi$ is canonical we obtain $([w])\psi \geq \langle v \rangle$ for all $v \in (X \cup X^{-1})^*$ with $[v] = [w]$. Consequently $P_0([w]) = \vee_{[v] \equiv [w]} \langle v \rangle$ exists and $([w])\psi \geq P_0([w])$). Let now $[u],[v] \in H$ with $[u][v] = [w]$. Then $([w])\psi = ([u][v])\psi \geq ([u])\psi([v])\psi \geq P_0([u])P_0([v])$. Consequently $P_1([w]) = \vee_{[u][v] \equiv [w]} (P_0([u])P_0([v]))$ and $([w])\psi \geq P_1([w])$). Continuing this process we see that $P_n([w]), n \in \mathbb{N}_0$ exist.

Sufficiency: Let the condition in the assumption of Theorem 3.2 be satisfied. Note that $\{P_n([w])\}_{n \in \mathbb{N}_0}$ is increasing and will be constant after a finite number of steps, for each $[w] \in H$, since all occurring graphs are finite. Let $P([w]) := \lim_{n \to \infty} P_n([w])$, $[w] \in H$. We show that the mapping $\psi : [w] \mapsto P([w])$ defines a canonical dual prehomomorphism. Let $[u],[v] \in H$. It follows $P_1([uv]) \geq P_0([u])P_0([v]), P_2([uv]) \geq P_1([u])P_1([v]), \ldots, P_n([uv]) \geq P_{n-1}([u])P_{n-1}([v]), \ldots$ which after a finite number of steps gives $P([uv]) \geq P([u])P([v])$. Further $P([w])^{-1} = (P([w]))^{-1}$, since $\langle u \rangle \vee \langle v \rangle$ exists if and only if $\langle u \rangle^{-1} \vee \langle v \rangle^{-1}$ exists in which case $\langle u \rangle^{-1} \vee \langle v \rangle^{-1} = (\langle u \rangle \vee \langle v \rangle)^{-1}$. This fact holds in any inverse semigroup $S$ and easily follows from $s \leq t \iff s^{-1} \leq t^{-1}, s,t \in S$. Finally
ψ is canonical since from Γ(P([x])) ⊆ Γ(⟨x⟩) we infer Γ(P([x])) = Γ(⟨x⟩), whence P([x]) = ⟨x⟩. □

Note that the above defined mapping P is the least possible canonical dual prehomomorphism with respect to the pointwise order of mappings, since in the necessity proof of Theorem 3.2 we have ([w])ψ ≥ P([w]), [w] ∈ H.

**Corollary 3.3** In case \( P_0([w]) \cong (\cap_{[u]=[w]} \Gamma(⟨u⟩), \bar{w}) \in M(G) \), for all [w] ∈ H, it follows \( P_0([w]) = P_n([w]) \), for all n ∈ ℤ, whence \( ([w])ψ = P_0([w]) \) defines a canonical dual prehomomorphism \( ψ : H → M(G) \).

**Proof** Under the assumptions we obtain for arbitrary \([w_1],[w_2] \in H\) with \([w_1][w_2] = [w]\)

\[
P_0([w_1])P_0([w_2]) \cong (\cap_{[u]=[w_1]} \Gamma(⟨u⟩)) \cup \bar{w}_1 \cap_{[u]=[w_2]} \Gamma(⟨u⟩), \bar{w}) 
\leq (\cap_{[u]=[w]} \Gamma(⟨u⟩), \bar{w}) 
\cong P_0([w]),
\]

since \(\cap_{[u]=[w_1]} \Gamma(⟨u⟩)) \cup \bar{w}_1 \cap_{[u]=[w_2]} \Gamma(⟨u⟩) \supseteq \cap_{[u]=[w]} \Gamma(⟨u⟩))\). Thus we have

\[
P_1([w]) = \vee_{[w_1][w_2] = [w]} (P_0([w_1])P_0([w_2])) \leq P_0([w]) \leq P_1([w]),
\]

whence \( P_1([w]) = P_0([w]) \) follows. We conclude by induction

\[
P_0([w]) = P_1([w]) = P_2([w]) = \ldots = P([w]),
\]

proving the assertion. □

**Example 3.1** Let G be any X-generated group and let H be the free group on X. Then for any \([w] \in H\) we have \( P_0([w]) = (\Gamma(⟨r(w)⟩), \bar{w}) \in M(G) \) where \( r(w) \) is the reduced word associated to \([w]\).

**Example 3.2** Let G be the \{x\}-generated cyclic group of order n and let H be the \{x\} -generated cyclic group of order 2n. Inspecting \( \Gamma(G) \) which is an n-cycle, we directly see

\[
\cap_{[w]=[x^k]} \Gamma(⟨w⟩) = \Gamma(⟨x^k⟩), \quad 1 \leq k \leq n
\]
and

\[
\cap_{[w]=[x^l]} \Gamma(⟨w⟩) = \Gamma(⟨x^{l-2n}⟩), \quad n \leq l \leq 2n.
\]

In particular we have \(\cap_{[w]=[x^n]} \Gamma(⟨w⟩) = \emptyset\), since \([\emptyset] = [x^{2n}]\) corresponds to \(1_H\). Hence \( ψ : H → M(G) \) may be defined by \( ([x^k])ψ = ⟨x^k⟩, 1 \leq k \leq n, ([x^l])ψ = ⟨x^{l-2n}⟩, n < l < 2n, and ([x^{2n}])ψ = ⟨\emptyset⟩ \cong (\emptyset, 1_G) = 1_{M(G)}\), cf. ([2], Theorem 19).

To check whether a given extension H by a group G satisfies the condition of Theorem 3.2 it is crucial to determine \(\cap_{[v]=[w]} \Gamma(⟨v⟩) \) for any \([w] \in H\). In what follows we describe a way of doing that for finite H and G which might be implemented...
on a computer. We start to determine a finite subset $T$ of $(X \cup X^{-1})^*$ satisfying the following property: For each $w \in (X \cup X^{-1})^*$ there is $v \in T$ such that $[w] = [v]$ and $\Gamma(\langle v \rangle) \subseteq \Gamma(\langle w \rangle)$. To compute such a set $T$ we describe a simple algorithm which directly implements the defining property of $T$.

(0) Put the identity element of $(X \cup X^{-1})^*$ into $T$.

(1) For $T$, constructed so far, construct a superset $T'$ of $T$ in the following way: Put all elements of $T$ into $T'$. List the elements of $T \times X \times \{-1, 1\}$ and check for each $(w, x, \varepsilon)$ in $T \times X \times \{-1, 1\}$ if there is $u \in T$ such that $[u] = [wx^\varepsilon]$ and $\Gamma(\langle u \rangle) \subseteq \Gamma(\langle w \rangle)$. If the answer for a given $(w, x, \varepsilon)$ is yes, go to the next triple in the list. If the answer is no, put $wx^\varepsilon$ into $T'$ and go to the next triple in the list.

(2) If $T$ is a proper subset of $T'$, as constructed in (1), take $T'$ as new $T$ and start (1) again. If $T = T'$ the algorithm stops.

Note that since $H$ and $M(G)$ are finite, the computation stops after a finite number of steps. To see that in the end $T$ has the required property, we note that if a word $w'$ is dropped in (1) of the above algorithm because $[w'] = [u]$ with $\Gamma(\langle u \rangle) \subseteq \Gamma(\langle w' \rangle)$ for some $u \in T$, then for each word $w'v$, $v \in (X \cup X^{-1})^*$ we have $[w'v] = [uv]$ with $\Gamma(\langle uv \rangle) \subseteq \Gamma(\langle w'v \rangle)$, where $uv$ is in $T$ or has been dropped earlier in (1), i.e. there is some $u' \in T$ such that $[uv] = [u']$ and $\Gamma(\langle u' \rangle) \subseteq \Gamma(\langle uv \rangle)$, whence $[w'v] = [u']$ and $\Gamma(\langle u' \rangle) \subseteq \Gamma(\langle w'v \rangle)$. Consequently the final set $T$ satisfies the property that for each word $w$ in $(X \cup X^{-1})^*$ there is a word $u$ in $T$ such that $[w] = [u]$ and $\Gamma(\langle u \rangle) \subseteq \Gamma(\langle w \rangle)$. Now for a given $[w] \in H$ we get

$$\cap_{[v]=[w]} \Gamma(\langle v \rangle) = \cap_{[u]=[w]} \Gamma(\langle u \rangle),$$

where $v \in (X \cup X^{-1})^*$, $u \in T$, and in case $\bar{w} \neq 1_G$ we have to check whether the right hand intersection contains a connected subgraph with vertices $1_G$ and $\bar{w}$, to see whether $P_0([w])$ exists. Note that in case $\bar{w} = 1_G$, $P_0([w])$ always exists. If for some $[w] \in H$, $P_0([w])$ does not exist, the algorithm stops. If all $P_0([w])$, $[w] \in H$ exist, we check whether for each $[w] \in H$, $P_1([w]) = \bigvee_{[v]=[w]} (P_0([w_1])P_0([w_2]))$ exists, by going through all $1H$ factorisations of $[w]$. If $P_1([w])$ does not exist for some $[w] \in H$, the algorithm stops. In the other case we continue, checking whether $P_2([w])$ exists, and so on. After a finite number of computations we end up with $n_0 \in \mathbb{N}$ such that either $P_{n_0}([w])$ does not exist for some $[w] \in H$, in which case $H$ does not satisfy the conditions of Theorem 3.2, or $P_{n_0}([w]) = P_{n_0+1}([w])$ for all $[w] \in H$. The latter must be the case since for each $[w] \in H$ the sequence $\{P_n([w])\}_{n \in \mathbb{N}_{n_0}}$ is decreasing whence eventually constant, since all occurring graphs are finite. Further $H$ is finite. We then have $P_{n_k}([w]) = P_k([w])$ for all $k \geq n_0$, $[w] \in H$. Thus $H$ satisfies the conditions of Theorem 3.2.

Even for a small finite noncyclic $X$-generated group $G$, an $X$-generated group $H$ admitting a canonical dual prehomomorphism $\psi : H \to M(G)$ might be large. The following theorem points into this direction.
**Theorem 3.4** Let $G = \{1_G, g, h, gh\}$ be the $\{x, y\}$-generated Klein four-group with respect to $\bar{x} = g, \bar{y} = h$. Then any $X$-generated group $H$ which admits a canonical dual prehomomorphism $\psi : H \to M(G)$ must be of exponent 6 at least.

**Proof** We show that the $\{x, y\}$-generated Burnside group of exponent 4, $B(2; 4)$, does not admit a suitable $\psi : B(2; 4) \to M(G)$. Assume that $\psi$ exists. Note first that in $B(2; 4)$ we have

$$[xyx^2yx^{-1}] = [xyx^2yx^3] = [(xyx^2yx^2)x] = [x^{-1}y^{-1}x^2yx^2yx] = [x^{-1}y^{-1}(yx^2)^{-1}x] = [x^{-1}y^{-1}x^{-2}y^{-1}x] = [x^{-1}y^{-1}x^2y^{-1}x] = : [u].$$

We get $\langle xyx^2yx^{-1} \rangle \leq ([xyx^2yx^{-1}])\psi = ([x^{-1}y^{-1}x^2y^{-1}x])\psi \geq \langle x^{-1}y^{-1}x^2y^{-1}x \rangle$, whence $\Gamma([u]) \subseteq \Gamma(\langle xyx^2yx^{-1} \rangle) \cap \Gamma(\langle x^{-1}y^{-1}x^2y^{-1}x \rangle)$.

Since the intersection of both graphs does not contain a connected subgraph having at least one edge and vertex $1_G$, we conclude that $\Gamma([u]_\psi) = \emptyset$, whence $([u])_\psi = 1_{M(G)}$. We infer

$$([x^2y^{-1}x])_\psi = ([xxy^{-1}x^{-1}x^2y^{-1}x])_\psi \geq ([yx])_\psi ([x^{-1}y^{-1}x^2y^{-1}x])_\psi = ([yx])_\psi \geq \langle xy \rangle,$$

and on the other hand $([x^2y^{-1}x])_\psi \geq \langle x^2y^{-1}x \rangle$ which means

$$\Gamma(([x^2y^{-1}x])_\psi) \subseteq \Gamma(\langle xy \rangle) \cap \Gamma(\langle x^2y^{-1}x \rangle)$$

with contradiction, since the intersection on the right hand side does not contain a connected subgraph with vertices $1_G$ and $x^2y^{-1}x = gh$. 

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It is an open question whether the finite group $B(2; 6)$ admits a canonical dual prehomomorphism into $M(G)$ with $G$ the Klein four-group, or a contradiction can be achieved following the pattern in the proof of Theorem 3.4. It is also an open question whether the group $G^U$, as defined in [1], with $U$ the variety of all groups of exponent $n = 3$, respectively $n = 4$, admits a suitable mapping $\psi : G^U \to M(G)$ in this case. In our setting $G^U$ may be represented by $FG\langle x, y \rangle / \equiv$, where $\equiv$ is the congruence on the free group $FG\langle x, y \rangle$ generated by the relators $w^3 = 1$, respectively $w^4 = 1$, where $w = 1_G$, $w \in FG\langle x, y \rangle$. Since, by construction in [1], $G^U$ is a subgroup of a semidirect product of the finite groups $B(8; 3)$, respectively $B(8; 4)$ by $G$, it is finite. Obviously $B(2; 4)$ is a homomorphic image of $G^U$ in case $n = 4$. However $B(2; 4)$ itself is not of the form $G^V$ for some group variety $V$, since the only possible choice of such $V$ would be the variety of elementary Abelian 2-groups. Only if $V$ has exponent 2, the group $G^V$ has exponent $2 \cdot 2 = 4$. But in this case $G^V$ is a subgroup of a semidirect product of the free elementary Abelian 2-group of rank 8 by $G$ whence $|G^V| < 2^8 \cdot 2^2 = 2^{10} < 2^{12} = |B(2; 4)|$. Note in particular that $G^U$ has exponent 6 in case $n = 3$, and exponent 8 in case $n = 4$.

Anyway it follows from [1], Proposition 4.4., referring to a remark of V. Guba, that $\psi : G^U \to M(G)$ exists if $U$ is the variety of all groups of sufficiently large odd exponent $n$.

We continue our considerations with a theorem which also follows from a result of Szakács [6]. For sake of completeness we give an elementary direct proof.

**Theorem 3.5** Let $G$ be an $X$-generated noncyclic group, and let $H$ be a generator respecting $X$-generated extension by $G$ such that the homomorphism $H \to G$, defined by $[w] \mapsto w$ has a nontrivial Abelian kernel $K$. Then there is no canonical dual prehomomorphism $\psi : H \to M(G)$.

**Proof** We show first that under the assumptions $\Gamma(G)$ contains a subgraph consisting of two disjoint cycles connected by a path, of the form
where \( u_1, u_2, v, z_1, z_2 \) are nonempty words in \((X \cup X^{-1})^*\), labeling the respective paths.

Assume first that there is \( y \in X \) such that \( \overline{y} \) has finite order \( m \geq 2 \). Since \( G \) is noncyclic there is \( x \in X \) such that \( \overline{x} \neq \overline{v} \), for all \( n \in \mathbb{N} \). Consequently, by use of the words \( u_1 = z_1 = y, u_2 = y^{1-m}, v = x, z_2 = y^{m-1} \), we may define a graph which consists of two cycles with vertex sets \( A = \{1_G, \overline{y}, \ldots, \overline{y}^{m-1}\} \) and \( B = \{\overline{y}x, \overline{y}xy, \ldots, \overline{y}x^{y^{m-1}}\} \) connected by the edge \((\overline{y}, x)\). Since \( A \) is a subgroup of \( G \) and \( B = \overline{y}xA, \) with \( \overline{y} \not\in A \) by assumption, we obtain \( A \cap B = \emptyset \).

Assume now that there is \( x \in X \), such that \( \overline{x} \) has infinite order. Since \( K \) is nontrivial there is a nonempty reduced word \( w = y_1 \ldots y_m, \) \( m \geq 2, \) with \( y_i \in X \cup X^{-1}, 1 \leq i \leq m, \) such that \( \overline{w} = 1_G, \) and \( \Gamma(\langle w \rangle) \) forms a cycle. Let \( u_1 = y_1 = z_1, u_2 = (y_2 \ldots y_m)^{-1}, z_2 = y_2 \ldots y_m, \) and \( v' = x^n, \) where \( n \) is such that \( \overline{y}_i x^a \neq b \) for all \( a, b \) in the set \( A = \{1_G, \overline{y}_1, \ldots, \overline{y}_1 \ldots y_{m-1}\}. \) Such \( n \) exists, since the equality \( \overline{y}_1 x^a = b \) can only hold for at most one \( k \in \mathbb{N} \) by the assumption that \( \overline{x} \) has infinite order, and the set \( A \), whence \( A \times A, \) is finite. We may use \( u_1, u_2, v', z_1, z_2 \) to define a graph which consists of the two disjoint cycles with vertex sets \( A = \{1_G, \overline{y}_1, \ldots, \overline{y}_1 \ldots y_{m-1}\} \) and \( B = \{\overline{y}_1 x^q, \overline{y}_1 x^q \overline{y}_1, \ldots, \overline{y}_1 x^q \overline{y}_1 \ldots y_{m-1}\}, \) connected by the path with initial vertex \( \overline{y}_1 \) labeled by \( v' = x^n. \) Let \( p, q \in \{1, \ldots, n\} \) such that \( p \) is the least element with \( \overline{y}_1 \overline{x}^p \not\in A \) for all \( k, p \leq k \leq n, \) and such that \( q \) is the least element with \( \overline{y}_1 \overline{x}^q \in B. \) Then the path with initial vertex \( \overline{y}_1 \overline{x}^{q-1} \) and label \( v = x^{q-p+1} \) connects the cycles with vertex sets \( A \) and \( B \) precisely as shown in the graph above.

We conclude

\[
\langle u_1 v z_1 z_2 v^{-1} u_1^{-1} \rangle \lor \langle u_2 v z_1 z_2 v^{-1} u_2^{-1} \rangle = 1_{M(G)}.
\]

On the other hand we obtain

\[
[u_1 v z_1 z_2 v^{-1} u_1^{-1}] = [u_1 v z_1 z_2 v^{-1} u_1^{-1} u_2 u_2^{-1}]
= [u_1 u_1^{-1} u_2 v z_1 z_2 v^{-1} u_2^{-1}], \quad \text{since } [u_1^{-1} u_2], [v z_1 z_2 v^{-1}] \in K
= [u_2 v z_1 z_2 v^{-1} u_2^{-1}].
\]

Hence for any canonical dual prehomomorphism \( \psi : H \rightarrow M(G) \) we get

\[
([u_1 v z_1 z_2 v^{-1} u_1^{-1}]) \psi = ([u_2 v z_1 z_2 v^{-1} u_2^{-1}]) \psi
\geq \langle u_1 v z_1 z_2 v^{-1} u_1^{-1} \rangle \lor \langle u_2 v z_1 z_2 v^{-1} u_2^{-1} \rangle = 1_{M(G)},
\]

whence \( ([u_1 v z_1 z_2 v^{-1} u_1^{-1}]) \psi = 1_{M(G)}. \)
By the rule $([w_1^1 w_2^2])_\psi = 1_{M(G)} \Rightarrow ([w_2^2])_\psi \geq ([w_1^1])_\psi \Rightarrow ([w_2^2])_\psi \geq ([w_1^1])_\psi$, since $\psi$ respects inverses, we obtain with $[w_1^1] = [u_1^1 v z_1^1]$ and $[w_2^2] = [z_2^2 v^{-1} u_1^{-1}]$ that $([u_1 v z_1^{-1}])_\psi \geq ([u_1 v z_1])_\psi \geq \langle u_1 v z_1 \rangle$, which together with $([u_1 v z_2^{-1}])_\psi \geq \langle u_1 v z_2^{-1} \rangle$ leads to a contradiction. Note in particular that $\langle u_1 v z_1 \rangle \vee \langle u_1 v z_2^{-1} \rangle$ does not exist. \hfill \Box

Note that in case $K$ is trivial in Theorem 3.5, i.e., $K = \{1_H\}$, we have that $H$ is isomorphic to $G$ via the homomorphism induced by the mapping $[x] \mapsto \bar{x}$.

It is shown in [2], see also Example 3.2, that for any $\{x\}$-generated cyclic group $G$ of order $n$ there is a canonical dual prehomomorphism $\psi$ from the $\{x\}$-generated cyclic group $H$ of order $2n$ into $M(G)$. Clearly the homomorphism $[w] \mapsto \bar{w}$ has Abelian kernel. If we regard, however, $G$ as an e.g. $\{x, y\}$-generated group where $\bar{y} = x^2$, say, $n \geq 3$, then the assertion of Theorem 3.5 remains true, although $G$ is cyclic, as the following example shows for $n = 3$.

**Example 3.3** Let $G = \{1_G, g, g^2\}$ be the three element cyclic group generated by $X = \{x, y\}$ with respect to $\bar{x} = g, \bar{y} = g^2$. The Cayley graph $\Gamma(G)$ looks as follows:

\[
\begin{align*}
\Gamma(\langle G \rangle) & \quad \text{Assume } \psi \text{ exists for some } \langle x, y \rangle\text{-generated generator preserving group extension } H \text{ by } G, \text{ where } [w] \mapsto \bar{w} \text{ has Abelian kernel. It follows }
\end{align*}
\]

\[
[y^{-1} x y^2] = [x(x^{-1} x^{-1})(x y)y] = [x(x y)(x^{-1} x^{-1})y], \text{ since } \bar{x} y = x^{-1} x^{-1} = 1_G
\]

\[
= [x^2 y x^{-1}].
\]

We obtain $([y^{-1} x y^2])_\psi \geq \langle y^{-1} x y^2 \rangle, \langle x^2 y x^{-1} \rangle$

\[
\Rightarrow \Gamma(\langle [y^{-1} x y^2] \rangle_\psi) \subseteq \Gamma(\langle y^{-1} x y^2 \rangle) \cap \Gamma(\langle x^2 y x^{-1} \rangle)
\]

\[
\Rightarrow \Gamma(\langle [y^{-1} x y^2] \rangle_\psi) = \emptyset \Rightarrow ([y^{-1} x y^2])_\psi = 1_{M(G)}.
\]
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\[ \Gamma(\langle y^{-1}xy^2 \rangle) \]

\[ \Gamma(\langle x^2yx^{-1} \rangle) \]

We infer

\[
([x])\psi = ([y(y^{-1}xy^2)y^{-2}])\psi \\
\geq ([y])\psi \left( [y^{-1}xy^2] \right)\psi \left( [y^{-2}] \right)\psi \\
= ([y])\psi \left( [y^{-2}] \right)\psi \geq \langle yy^2y^{-1} \rangle,
\]

which means together with \([x])\psi = \langle x \rangle a contradiction.

Note that \(\Gamma(G)\) contains a forbidden minor in the sense of Szakács, namely

\[ \Gamma(\langle x \rangle) \]

In particular \(\Gamma(\langle x \rangle)\) is a breaking path in her terminology.

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