Stable recovery of noncompactly supported electromagnetic potentials in unbounded domain

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Communicated by: H. Ammari

Funding information
Agence Nationale de la Recherche, Grant/Award Number: ANR-17-CE40-0029

MSC CLASSIFICATION
35R30; 35J15

We consider the inverse problem of determining an electromagnetic potential appearing in an infinite cylindrical domain from boundary measurements. More precisely, we prove the stable recovery of a general class of magnetic field and electric potential from boundary measurements. Assuming the knowledge of the unknown coefficients close to the boundary, we obtain other results of stable recovery with measurements restricted to some portion of the boundary. Our approach combines construction of complex geometric optics solutions and Carleman estimates suitably designed for our stability results stated in an unbounded domain.

KEYWORDS
electromagnetic potential, elliptic equations, inverse problems, partial data, stability estimate, unbounded domain

1 | INTRODUCTION

1.1 | Statement of the problem

Let $\Omega$ be an open set of $\mathbb{R}^3$ corresponding to a closed waveguide. More precisely, we assume that there exists $\omega$ a $C^3$ bounded, open and simply connected set of $\mathbb{R}^2$ such that $\Omega = \omega \times \mathbb{R}$. For $A \in W^{1,\infty}(\Omega)^3$, we define the magnetic Laplacian $\Delta_A$ given by

$$\Delta_A = \Delta + 2iA \cdot \nabla + i \text{ div } (A) - |A|^2.$$ 

For $q \in L^\infty(\Omega)$ such that $0$ is not in the spectrum of the operator $-\Delta_A + q$ acting on $L^2(\Omega)$ with Dirichlet boundary condition, we can introduce the boundary value problem

$$\begin{cases} (-\Delta_A + q)u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Gamma := \partial\Omega. \end{cases} \quad (1.1)$$

Recall that $\Gamma = \partial\omega \times \mathbb{R}$ and that the outward unit normal vector $\nu$ to $\Gamma$ takes the form

$$\nu(x',x_3) = (\nu'(x'), 0), \quad x = (x', x_3) \in \Gamma,$$

where $\nu'$ is the outward unit normal vector of $\partial\omega$. In the present paper, we consider the simultaneous stable recovery of the magnetic field associated with $A$ and the electric potential $q$ from the full and partial knowledge of the
Dirichlet-to-Neumann (DN in short) map:

\[ \Lambda_{A,q} : H^\frac{3}{2}(\partial\Omega) \to H^\frac{3}{2}(\partial\Omega) \]

\[ f \mapsto (\partial_n + iA \cdot \nu)u|_{\partial\Omega}. \]  

(1.2)

where \( \partial_n \) is the normal derivative. Let \( \Gamma_0 \subset \partial\omega \) be an arbitrary open set. The restriction \( \Lambda A, q' \) of \( \Lambda_{A,q} \) on \( \Gamma_0 \times \mathbb{R} \) is defined by

\[ \Lambda'_{A,q} : H^\frac{3}{2}(\partial\Omega) \to H^\frac{3}{2}(\Gamma_0 \times \mathbb{R}) \]

\[ f \mapsto (\partial_n + iA \cdot \nu)u|_{\Gamma_0 \times \mathbb{R}}. \]  

(1.3)

1.2 Motivations

The problem addressed in this article is connected with the electrical impedance tomography (EIT in short) method as well as its applications in different scientific areas (e.g., medical imaging and geophysical prospection). We refer to Uhlmann\(^1\) for a review of this problem. Our formulation of this problem in an unbounded closed waveguide can be associated with problems of transmission to long distance or transmission through structures having important ratio length-to-diameter (e.g., nanostructures). The main objective of our study is to determine in a stable way an electromagnetic impurity perturbing the guided propagation (see, for instance, in Chang and Lin\(^2\) and Kane et al\(^3\)).

1.3 Known results

There have been many works so far devoted to the study of the Calderón problem initially stated in Calderón.\(^4\) The first positive answer to this problem can be found in Sylvester and Uhlmann\(^5\) where the authors used an approach based on the construction of complex geometric optics (CGO in short) solutions. We refer also to previous studies\(^6–8\) for some alternative constructions of CGO solutions. Motivated by this result, many authors investigated several aspects of this problem. One of the first results devoted to the recovery of electromagnetic potentials can be found in Sun\(^9\). Here, the authors stated a uniqueness result under a smallness assumption of the associated magnetic field. This smallness assumption has been removed by Nakamura et al\(^10\) for smooth coefficients and improved in terms of regularity by Krupchyk and Uhlmann.\(^11\) Since then, Tolmasky\(^12\) considered magnetic potentials lying in \( C^1 \), Salo\(^13\) treated the case of magnetic potentials lying in a Dini class, and Krupchyk and Uhlmann\(^11\) considered this problem with bounded electromagnetic potentials. One of the first results of stability for this problem can be found in Tzou,\(^14\) and, without being exhaustive, we refer to previous studies\(^15–18\) for some recent improvements of such results and to previous studies\(^19–22\) for the stable recovery of several classes of coefficients appearing in an elliptic equation.

All the above-mentioned results are stated in a bounded domain. There have been only a few works devoted to the recovery of coefficients for elliptic equations in an unbounded domain. Among these results, several works have been devoted to the recovery of coefficients of an elliptic equation in a slab (see, e.g., previous studies\(^23–25\)), and we refer to Choulli et al\(^26,27\) for the recovery of periodic coefficients in an infinite waveguide. As far as we know, the first results dealing with the unique recovery of general class of noncompactly supported and nonperiodic coefficients, appearing in an unbounded cylindrical domain, can be found in Kian.\(^28,29\) More recently, Soussi\(^30\) proved the stable recovery of an electric potential similar to the class of coefficients under consideration in Kian.\(^28\) To the best of our knowledge, the results of Soussi\(^30\) correspond to the first proof of stable recovery of coefficients similar to those considered by Kian\(^28\) from full and partial data. We mention also the works\(^31–38\) dealing with similar problems in a different class of PDEs.

1.4 Statement of the main results

Taking into account the well-known obstruction to the recovery of the electromagnetic potentials (see, e.g., Kian\(^29\), section 1.4), we study the stable recovery of the magnetic field and the electric potential appearing in (1.1). More precisely, for \( A = (a_1, a_2, a_3) \), we consider the recovery of the magnetic field corresponding to the two-form valued distribution \( dA \) defined by

\[ dA := \sum_{1 \leq j < k \leq 3} (\partial_n a_k - \partial_n a_j)dx_j \wedge dx_k \]

and the electric potential \( q \). In our first result, we prove the stable recovery of the magnetic field.
Theorem 1.1. For \( j = 1, 2 \), let \( A_j \in W^{2,1}(\Omega)^3 \cap W^{2,\infty}(\Omega)^3 \) satisfy the condition

\[
\partial_x^\alpha A_1(x) = \partial_x^\alpha A_2(x), \quad x \in \partial \Omega, \quad \alpha \in \mathbb{N}^3, \quad |\alpha| \leq 1
\]

and assume that 0 is not in the spectrum of the operator \( -\Delta_j^* + q_j \) acting in \( L^2(\Omega) \) with Dirichlet boundary condition. Assume also that there exist \( M > 0 \), \( s \in (0, 1/2) \) and \( f \in L^7(\mathbb{R}^3; \mathbb{R}^3) \) a decreasing function such that the following conditions are fulfilled:

\[
\int_\Omega (x_3)^s |A_j(x) - A_j(x)|^2 \, dx + \left\| r \mapsto r^3 f(r) \right\|_{L^7(1, +\infty)} \leq M,
\]

\[
\sum_{j=1}^2 \left[ \|A_j\|_{W^{2,\infty}(\Omega)^3} + \|A_j\|_{H^2(\Omega)^3} + \|q_j\|_{L^\infty(\Omega)} \right] \leq M,
\]

\[
|A_1(x) - A_2(x)| \leq f(|x|), \quad x \in \Omega.
\]

Then, there exist \( C > 0 \) depending only on \( \Omega \), \( s \), \( f \) and \( M \) and \( s_1 \in (0, 1) \) depending only on \( s \) such that the following estimate

\[
\|dA_1 - dA_2\|_{L^2(\Omega)} \leq C \left[ \ln \left( 3 + \|A_{1,q_1} - A_{1,q_2}\|_{H^3(\Omega)} \right) \right]^{s_1},
\]

holds true.

Assuming that the divergence of the magnetic potential under consideration is known, we prove also the stable recovery of the electric potential.

Theorem 1.2. Let the condition of Theorem 1.1 and conditions (1.4)–(1.6) be fulfilled. Assume also that

\[
d\text{div}(A_1) = d\text{div}(A_2).
\]

Moreover, let \( q_j \in H^2(\Omega) \cap L^2(\Omega), \ j = 1, 2, \) satisfy the following condition:

\[
q_1(x) = q_2(x), \quad x \in \partial \Omega,
\]

\[
\int_\Omega (x_3)^s |q_1(x) - q_2(x)| \, dx + \|q_1 - q_2\|_{H^s(\Omega)} \leq M,
\]

with \( s \in (0, 1) \). Then, there exists a constant \( s_2 > 0 \) depending only on \( s \) such that the following estimate

\[
\|q_1 - q_2\|_{L^2(\Omega)} \leq C \left[ \ln \left( \ln \left( e^3 + \|A_{1,q_1} - A_{1,q_2}\|_{H^3(\partial \Omega)} \right) \right) \right]^{s_2},
\]

holds true, with \( C > 0 \) depending only on \( \Omega \), \( s \), and \( M \).

Let us mention that the condition (1.8) was introduced in order to stably recover the full magnetic potential \( A \) and to deduce the above result. For a bounded domain, it is well known that one can prove simultaneously the stable recovery of the magnetic field and the electric potential without imposing (1.8) (see, e.g., Tzou14). This approach requires an argument based on the gauge invariance of the DN map and some results related to the Hodge decomposition of one form which are only available for a bounded domain. Since we deal here with an unbounded domain, we have imposed condition (1.8) in order to avoid difficulties related to the extension of such arguments to unbounded domains.

Now, we give two partial data results with restriction of the measurements to an arbitrary subset of the boundary. The statement of these results requires some definitions and assumptions that we need to recall first. Let \( \mathcal{W}_0 \subset \omega \) be an arbitrary neighborhood of the boundary \( \partial \omega \) such that \( \partial \mathcal{W}_0 = \partial \omega \cup \Gamma^s \) with \( \partial \omega \cap \Gamma^s = \emptyset \). We assume that \( \Gamma^s \) is \( C^2 \). Let \( \Gamma_0 \subset \partial \omega \subset \partial \mathcal{W}_0 \) be an arbitrary (not empty) open set of \( \partial \omega \) and let \( \mathcal{O}_0 = \mathcal{W}_0 \times \mathbb{R} \). For a given \( M > 0 \), we introduce the admissible sets of coefficients:

\[
\mathcal{A}(M,A_0,\mathcal{O}_0) = \{ A \in C^2(\mathcal{\bar{\Omega}},\mathbb{R}^3); \|A\|_{C^2(\Omega)} \leq M \text{ and } A = A_0 \text{ in } \mathcal{O}_0 \},
\]

\[
\mathcal{Q}(M,q_0,\mathcal{O}_0) = \{ q \in L^\infty(\mathcal{\bar{\Omega}},\mathbb{R}^3); \|q\|_{L^\infty(\mathcal{\bar{\Omega}})} \leq M \text{ and } q = q_0 \text{ in } \mathcal{O}_0 \}.
\]
For $j = 1, 2$, let $q_j \in Q(M, q_0, \mathcal{O}_0)$ and let $A_j \in \mathcal{A}(M, A_0, \mathcal{O}_0)$ satisfy the conditions of Theorem 1.1. Then, there exist $C > 0$ depending only on $\Omega$, $s$, and $M$ and $s_1$ depending only on $s$ such that the following estimate

$$\|dA_1 - dA_2\|_{L^2(\Omega)} \leq C \left[ \ln \left( 3 + \left\| \mathcal{A}_{A_1,q_1} - \mathcal{A}_{A_2,q_1} \right\|_{L^2(\partial \Omega), L^2(\Gamma_\omega \times R)}^{-1} \right) \right]^{-s_1}$$

holds true.

For $j = 1, 2$, let $A_j \in \mathcal{A}(M, A_0, \mathcal{O}_0)$ and $q_j \in Q(M, q_0, \mathcal{O}_0)$ satisfy the conditions of Theorem 1.2. Then, there exists a constant $s_2 > 0$ depending only on $s$ such that the following estimate

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C \left[ \ln \left( e^3 + \left\| \mathcal{A}_{A_1,q_1} - \mathcal{A}_{A_2,q_2} \right\|_{L^2(\partial \Omega), L^2(\Gamma_\omega \times R)}^{-1} \right) \right]^{-s_2}$$

holds true, with $C > 0$ depending only on $\Omega$, $s$, and $M$.

To the best of our knowledge, Theorems 1.1 and 1.2 correspond to the first results of stable recovery of the magnetic field and the electric potential associated with noncompactly supported electromagnetic potentials from boundary measurements. Indeed, while the uniqueness of this problem can be found in Kian,29 the stability issue for this problem has not been treated so far. We mention that in contrast to bounded domains, in unbounded domains, the transition from a uniqueness result to a stability estimate requires some careful analysis and one has to deal with several difficulties strongly related to the loss of compactness of the closure of the domain. For instance, in order to obtain such stability estimates, we need to impose the extra assumptions (1.5)–(1.6) and (1.10) to the electromagnetic potentials under consideration. Roughly speaking, conditions (1.5)–(1.6) and (1.10) claim that the difference of the electromagnetic potentials under consideration admit some decay at infinity. It is not clear how one can get a stability result associated with the results of Kian29 without assuming such extra assumptions.

In the spirit of Ben Joud,15 in Theorem 1.3, we treat the stable recovery of electromagnetic potentials that are known close to the boundary from some suitable subboundary of $\partial \Omega$. Our approach requires both results of Theorems 1.1 and 1.2 and some extension of the arguments of Ben Joud15 to an unbounded cylindrical domain. This includes a weak unique continuation result, stated in Lemma 4.1, that we derive for unbounded cylindrical domains.

In contrast to similar results stated in bounded domains (see, e.g., Tzou14), in Theorem 1.2, we assume the knowledge of the divergence of the magnetic potentials under consideration in order to prove the recovery of the electric potentials. This is related to the fact that, in contrast to bounded domains, it is not clear how one can exploit the gauge invariance associated with our problem, introduced in Kian,29 section 1.4 for showing the stable recovery of the electric potential without assuming the knowledge of the divergence of the magnetic potential.

Let us mention that, for all $n \geq 4$, our approach can be applied to any unbounded domain $\Omega \subset \mathbb{R}^n$ of the form $\Omega = \omega \times \mathbb{R}$ with $\omega$ a bounded $C^3$ set of $\mathbb{R}^{n-1}$. In the same way, by applying a projection argument inspired by the analysis of Bellassoued et al12 and Kian36 (see also Kian29, section 7 for more details), for all $n \geq 4$ and all $1 \leq p \leq n - 3$, one can extend our results to any unbounded domain $\Omega \subset \mathbb{R}^n$ of the form $\Omega = \omega \times \mathbb{R}^p$, with $\omega$ a $C^3$ bounded set of $\mathbb{R}^{n-p}$.

### 1.5 Outlines

This paper is organized as follows. In Section 2, we introduce some class of CGO solutions, suitably designed for our problem, that we build by mean of Carleman estimates. In Section 3, we complete the proof of the results with full boundary measurements stated in Theorems 1.1 and 1.2. Section 4 will be devoted to the results stated in Theorem 1.3 using partial boundary measurements. Finally, in Appendices A1–C1 we prove several intermediate results including an interpolation result, a Carleman estimate, and a weak unique continuation property.

### 2 CGO SOLUTIONS

In this section, we introduce a class of CGO solutions suitably designed for our problem stated in an unbounded domain for magnetic Schrödinger equations. More precisely, we consider CGO solutions $u_j \in H^2(\Omega_1)$, $j = 1, 2$, satisfying $\Delta A_j u_1 + q_1 u_1 = 0$, $\Delta A_j u_2 + q_2 u_2 = 0$ in $\Omega$ for $A_j \in W^{1,\infty}(\Omega)^3 \cap H^2(\Omega)^3$ and $q_j \in L^{\infty}(\Omega)$ satisfying (1.4)–(1.6). In a similar way to Kian,28,29 we consider first $\theta \in S^1 := \{ y \in \mathbb{R}^2 : |y| = 1 \}$, $\zeta' \in \theta^1 \setminus \{0\}$, with $\theta^1 := \{ y \in \mathbb{R}^2 : y \cdot \theta = 0 \}$, $\xi := (\zeta', \xi_3) \in \mathbb{R}^3$, ...
with $\xi_3 \neq 0$. Then, we define $\eta \in S^2 := \{ y \in \mathbb{R}^3 : |y| = 1 \}$ by

$$
\eta = \frac{(\xi', -\frac{\xi_1}{\xi_3})}{\sqrt{|\xi'|^2 + \frac{|\xi_1|^2}{\xi_3^2}}}.
$$

Clearly, we have

$$
\eta \cdot \xi = (\theta, 0) \cdot \xi = (\theta, 0) \cdot \eta = 0.
$$

We fix also $\psi \in C_0^\infty((-2, 2); [0, 1])$ satisfying $\psi = 1$ on $[-1, 1]$, and, for $\rho > 1$, we introduce solutions $u_j \in H^2(\Omega)$ of $\Delta_{A_j} u_1 + q_1 u_1 = 0$, $\Delta_{A_j} u_2 + q_2 u_2 = 0$ in $\Omega$ of the form

$$
u_j(x', x_3) = e^{i \theta \cdot x'} \left( \psi \left( \rho^{-\frac{1}{2}} x_3 \right) b_j e^{i \omega_\xi \cdot x + w_1(x', x_3)} \right), \quad x' \in \omega, x_3 \in \mathbb{R},
$$

$$u_2(x', x_3) = e^{-i \theta \cdot x'} \left( \psi \left( \rho^{-\frac{1}{2}} x_3 \right) b_2 e^{i \omega_\xi \cdot x + w_2(x', x_3)} \right), \quad x' \in \omega, x_3 \in \mathbb{R}.
$$

Here, for $j = 1, 2$, $b_j \in W^{2, \infty}(\Omega)$ and the remainder term $w_{j, \rho} \in H^2(\Omega)$ satisfies the decay property

$$ho^{-1} \| w_{1, \rho} \|_{H^2(\Omega)} + \| w_{2, \rho} \|_{H^2(\Omega)} + \rho \| w_{1, \rho} \|_{L^2(\Omega)} \leq C(\| x \|^2 + 1) \left( 1 + \frac{|\xi'|^2}{|\xi|^2} \right) \rho^{\frac{3}{2}},$$

$$ho^{-1} \| w_{2, \rho} \|_{H^2(\Omega)} + \| w_{2, \rho} \|_{H^2(\Omega)} + \rho \| w_{2, \rho} \|_{L^2(\Omega)} \leq C \left( 1 + \frac{|\xi'|^2}{|\xi|^2} \right) \rho^{\frac{3}{2}},$$

with $C > 0$ depending on $\Omega$ and $\| A_j \|_{W^{\frac{5}{2}, \infty}(\Omega)^\dagger} + \| q_j \|_{L^2(\Omega)}$, $j = 1, 2$. We summarize this construction as follows.

**Theorem 2.1.** For $j = 1, 2$ and for all $\rho > \rho_2$, with $\rho_2$ the constant of Proposition 2.4, the equations $\Delta_{A_j} u_1 + q_1 u_1 = 0$ and $\Delta_{A_j} u_2 + q_2 u_2 = 0$ admit a solution $u_j \in H^2(\Omega)$ of the form (2.15)–(2.16) with $w_{j, \rho}$ satisfying the decay property (2.17).

### 2.1 Principal parts of the CGO

In this section, we consider $A_j \in W^{2, \infty}(\Omega)^\dagger$, $j = 1, 2$ satisfying (1.4). From now on, for all $r > 0$, we define $B_r := \{ x \in \mathbb{R}^3 : |x| < r \}$ and $B_r' := \{ x' \in \mathbb{R}^2 : |x'| < r \}$. In order to define $b_j$, $j = 1, 2$, we start by introducing a suitable extension of the coefficients $A_j$, $j = 1, 2$. Following Bellassoued et al., lemma 2.2 we define $\tilde{A}_j \in W^{2, \infty}(\mathbb{R}^3)^\dagger$, $j = 1, 2$, and we fix $r_0 > 0$ such that

$$
\tilde{A}_j(x) = A_j(x), \quad x \in \Omega \setminus \text{supp}(\tilde{A}_j) \subset B_{r_0} \times \mathbb{R},
$$

$$\tilde{A}_1(x) = \tilde{A}_2(x), \quad x \in \mathbb{R}^3 \setminus \Omega \| \tilde{A}_j \|_{W^{\frac{5}{2}, \infty}(\mathbb{R}^3)^\dagger} \leq C(\| A_1 \|_{W^{2, \infty}(\Omega)^\dagger} + \| A_2 \|_{W^{2, \infty}(\Omega)^\dagger}),
$$

with $C > 0$ depending only on $\Omega$. Here, $r_0$ will only depend on $\Omega$.

Following Kian, we fix $\tilde{\theta} = (\theta, 0) \in \mathbb{R}^3$, and we define

$$
\Phi_1(x) := \frac{-i}{2\pi} \int_{\mathbb{R}^2} \frac{(\tilde{\theta} + in) \cdot \tilde{A}_1(x - s_1 \tilde{\theta} - s_2 n)}{s_1 + is_2} ds_1 ds_2,
$$

$$
\Phi_2(x) := \frac{-i}{2\pi} \int_{\mathbb{R}^2} \frac{(-\tilde{\theta} + in) \cdot \tilde{A}_2(x + s_1 \tilde{\theta} - s_2 n)}{s_1 + is_2} ds_1 ds_2.
$$

According to Kian, one can check that $\Phi_j \in W^{2, \infty}(\Omega)$. Assuming that condition (1.6) is fulfilled, we can even prove the following estimates.

**Lemma 2.2.** Assume that condition (1.6) is fulfilled. Then, for all $R \geq r_0$, there exists $C > 0$ depending on $\Omega$ and $R$ such that the following estimates

$$
\| \Phi_1 \|_{W^{\frac{5}{2}, \infty}(B_R' \times \mathbb{R})} + \| \Phi_2 \|_{W^{\frac{5}{2}, \infty}(B_R' \times \mathbb{R})} \leq C(\| A_1 \|_{W^{2, \infty}(\Omega)^\dagger} + \| A_2 \|_{W^{2, \infty}(\Omega)^\dagger}) \left( 1 + \frac{|\xi'|^2}{|\xi|^2} \right),
$$
\begin{equation}
\|\Phi_1 + \Phi_2\|_{L^\infty(\mathbb{R}^3)} \leq C \left( \|A_1\|_{L^\infty(\Omega)} + \|A_2\|_{L^\infty(\Omega)} + \|r \mapsto r f(r)\|_{L^3(1, +\infty)} \right),
\end{equation}

(2.21)

hold true.

\textbf{Proof.} We will prove the estimate (2.20) only for $\Phi_1$, the proof for $\Phi_2$ being similar. For this purpose, we fix $R \geq r_0$. For $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2$, we have

\[ |\partial_x^\alpha \Phi_1(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{|\partial_x^\alpha \tilde{A}_1(x - s_1 \tilde{\theta} - s_2 \eta)|}{|s_1 + i s_2|} ds_1 ds_2. \]

On the other hand, using the fact that $\text{supp}(\tilde{A}_1) \subset B'_r \times \mathbb{R}$, one can check that, for all $x \in B'_r \times \mathbb{R}$, we get

\[ |\partial_x^\alpha \tilde{A}_1(x - s_1 \tilde{\theta} - s_2 \eta)| = 0, \quad (s_1, s_2) \geq \frac{2R}{|\eta_1, \eta_2|}, \quad x \in \Omega, \]

where $\eta = (\eta_1, \eta_2, \eta_3)$. It follows that

\begin{align*}
|\partial_x^\alpha \Phi_1(x)| &\leq \frac{1}{2\pi} \|\tilde{A}_1\|_{W^{2,\infty}(\mathbb{R}^3)^3} \left( \int_{|s_1, s_2| \leq \frac{2R}{|\eta_1, \eta_2|}} \frac{1}{|s_1 + i s_2|} ds_1 ds_2 \right) \\
&\leq C \|\tilde{A}_1\|_{W^{2,\infty}(\mathbb{R}^3)^3} (|\eta_1, \eta_2|)^{-1}.
\end{align*}

(2.22)

Recalling that

\[ (\eta_1, \eta_2) = \frac{\xi'}{\sqrt{|\xi'|^2 + \frac{|\xi''|^4}{\xi''}}}, \]

and applying (2.18), we deduce (2.20) from (2.22).

Now, let us consider (2.21). Let us first observe that, according to (2.18), we have

\[ \Phi_1(x) + \Phi_2(x) = \frac{-i}{2\pi} \int_{\mathbb{R}^3} (\tilde{\theta} + i \eta) \cdot A(x - s_1 \tilde{\theta} - s_2 \eta) ds_1 ds_2, \]

where $A = A_1 - A_2$ is extended by zero to $\mathbb{R}^3$. Therefore, applying (1.6) and the fact that $f$ is a decreasing function, we deduce that, for all $x \in \mathbb{R}^3$, we have
\[ |\Phi_1(x) + \overline{\Phi_2(x)}| \]
\[ \leq C \left( (\|A_1\|_{L^\infty(\Omega)} + \|A_2\|_{L^\infty(\Omega)}) \int_{B'_1} \frac{1}{|s_1 + i s_2|} ds_1 ds_2 + \int_{R^2 \setminus B'_1} \left| f(|x - s_1 \tilde{\theta} - s_2 \eta|) \right| ds_1 ds_2 \right) \]
\[ \leq C \left( \|A_1\|_{L^\infty(\Omega)} + \|A_2\|_{L^\infty(\Omega)} + \left( \int_{R^2} |f(|x - s_1 \tilde{\theta} - s_2 \eta|)| \frac{1}{|s_1 + i s_2|} ds_1 ds_2 \right) \right) \]

This completes the proof of the lemma.

Fixing

\[ b_1(x) = e^{\Phi_1(x)}, b_2(x) = e^{\Phi_2(x)}, \]

we obtain

\[ (\tilde{\theta} + i \eta) \cdot \nabla b_1 + i((\tilde{\theta} + i \eta) \cdot \tilde{A}_1(x))b_1 = 0, \quad (-\tilde{\theta} + i \eta) \cdot \nabla b_2 + i((-\tilde{\theta} + i \eta) \cdot \tilde{A}_2(x))b_2 = 0, \quad x \in \mathbb{R}^3. \]

(2.24)

Here, using the fact that \( \tilde{\omega} \subset B'_{r_0} \) and applying (2.20)–(2.21), we obtain

\[ \left\| b_j \right\|_{W^{2,\infty}(B_{r_0} \setminus \mathbb{R}^3)} \leq C \left( \|A_1\|_{W^{2,\infty}(\Omega)} + \|A_2\|_{W^{2,\infty}(\Omega)} \right) \left( 1 + \frac{|\xi|}{|\xi_3|} \right), \quad j = 1, 2. \]

(2.25)

Using these properties of the expressions \( b_j, j = 1, 2 \), we will complete the construction of the solutions \( u_j \) of the form (2.15)–(2.16). For this purpose, we will use some suitable Carleman estimates that will extend the one introduced in Kian\(^{29}\) (see also Dos Santos Ferreira et al\(^{39}\) and Salo and Tzou\(^{40}\)).

### 2.2 General Carleman estimate

Let us first introduce a weight function depending on two parameters \( s, \rho \in (1, +\infty) \) with \( \rho > s > 1 \). Following Kian,\(^{29}\) section 2.1 for \( \theta \in \mathbb{S}^2 \), we consider the perturbed weight

\[ \phi_{\pm,s}(x', x_3) := \pm \rho \theta \cdot x' - s(x' \cdot \theta) \frac{1}{s}, \quad x = (x', x_3) \in \Omega. \]

(2.26)

We introduce also the weighted operator

\[ P_{A,q,\pm,s} := e^{-\phi_{\pm,s} \cdot (\Delta + 2iA \cdot \nabla + q)} e^{\Phi_{\pm,s}}. \]

Then, we can consider the following Carleman estimate.
Proposition 2.3. Let \( A \in L^\infty(\Omega)^3 \cap L^\infty(\Omega)^3 \) and \( q \in L^\infty(\Omega) \). Then, there exist \( s_1 > 1 \), and for \( s > s_1 \), \( \rho_1(s) \) such that for any \( v \in H^2(\Omega) \cap H^1_0(\Omega_1) \), the estimate

\[
\rho \int_{\partial_\Omega \times \mathbb{R}^3} |\partial_\nu v|^2 |\theta \cdot v| d\sigma(x) + \omega^2 \int_{\Omega_1} |\Delta v|^2 dx + s \int_{\Omega_1} |\nabla v|^2 dx + \rho \int_{\partial_\Omega \times \mathbb{R}^3} |\partial_\nu v|^2 |\theta \cdot v| d\sigma(x)
\]

\[
\leq C \left[ \left\| P_{A,q} \pm V \right\|^2_{L^2(\Omega_1)} + \rho \int_{\partial_\Omega \times \mathbb{R}^3} |\partial_\nu v|^2 |\theta \cdot v| d\sigma(x) \right]
\]

holds true for \( s > s_1 \), \( \rho \geq \rho_1(s) \) with \( C \) depending only on \( \Omega \) and \( \|q\|_{L^\infty(\Omega)} + \|A\|_{L^\infty(\Omega)^3} \).

In this subsection, we will apply Proposition 2.3 in order to derive Carleman estimates in negative order Sobolev space required for the construction of the CGO solutions. For this purpose, we recall some preliminary tools. Following Kian,29,41 (see also Dos Santos Ferreira et al39 and Salo and Tzou40), for all \( m \in \mathbb{R} \), we consider the space \( H^m_\rho(\mathbb{R}^3) \) defined by

\[
H^m_\rho(\mathbb{R}^3) = \{ u \in S'(\mathbb{R}^3) : (|\xi|^2 + \rho^2)^{m-\frac{1}{2}} \hat{u} \in L^2(\mathbb{R}^3) \},
\]

with the norm

\[
\|u\|^2_{H^m_\rho(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|\xi|^2 + \rho^2)^m |\hat{u}(\xi)|^2 d\xi.
\]

In the above formula, for all tempered distribution \( u \in S'(\mathbb{R}^3) \), \( \hat{u} \) denotes the Fourier transform of \( u \) which, for \( u \in L^1(\mathbb{R}^3) \), corresponds to

\[
\hat{u}(\xi) := \mathcal{F}u(\xi) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} u(x)dx.
\]

From now on, for \( m \in \mathbb{R} \) and \( \xi \in \mathbb{R}^3 \), we fix

\[
\langle \xi, \rho \rangle = (|\xi|^2 + \rho^2)^{\frac{1}{2}}
\]

and \( (D_x, \rho)^m \hat{u} \) given by

\[
\langle D_x, \rho \rangle^m \hat{u} = \mathcal{F}^{-1}(\langle \xi, \rho \rangle^m \hat{u}).
\]

For \( m \in \mathbb{R} \), we introduce also the class of symbols

\[
S^m_\rho = \{ c_\rho \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) : |\partial_\xi^\alpha \partial_\rho^\beta c_\rho(x, \xi)| \leq C_{\alpha,\beta} \langle \xi, \rho \rangle^{-|\beta|}, \alpha, \beta \in \mathbb{N}^3 \}.
\]

In light of Hörmander,42, theorem 18.1.6 for any \( m \in \mathbb{R} \) and \( c_\rho \in S^m_\rho \), we define \( c_\rho(x, D_x) \), with \( D_x = -i\nabla \), by

\[
c_\rho(x, D_x) y(x) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} c_\rho(x, \xi) \hat{y}(\xi) e^{ix\cdot\xi} d\xi, \quad y \in S(\mathbb{R}^3).
\]

For all \( m \in \mathbb{R} \), we set also \( OpS^m_\rho := \{ c_\rho(x, D_x) : c_\rho \in S^m_\rho \} \) and

\[
OpS^\infty_\rho = \bigcap_{m<0} OpS^m_\rho.
\]

We introduce also

\[
P_{A,q} \pm := e^{\mp \rho \pm q} (\Delta_A + q) e^{\pm \rho \pm q}.
\]

According to Kian,29 proposition 2.4 there exist \( C > 0 \), \( \rho_\ast > 1 \), depending only on \( \Omega \) and \( \|q\|_{L^\infty(\Omega)} + \|A\|_{L^\infty(\Omega)^3} \), such that, for all \( v \in C^\infty_0(\Omega_1) \), the following estimate

\[
\rho^{-1} \|v\|_{H^{-1}(\mathbb{R}^3)} \leq C \left\| P_{A,q} \pm v \right\|_{H^{-1}(\mathbb{R}^3)}, \quad \rho > \rho_\ast,
\]

holds true. Using this estimate we can build CGO solutions lying in \( H^2(\Omega) \). However, in order to improve the smoothness of our CGO solutions into functions lying in \( H^2(\Omega) \), we will consider the following extension of Kian,29, proposition 2.4
Proposition 2.4. Let $A \in W^{1,\infty}(\Omega)^3$ and $q \in L^\infty(\Omega)$. Then, there exists $\rho_2 > 1$, depending only on $\Omega$ and $\|q\|_{L^\infty(\Omega)} + \|A\|_{W^{1,\infty}(\Omega)^3}$, such that for all $v \in C^0_0(\Omega)$, we have
\[
\rho^{-1} \|v\|_{L^2(\mathbb{R}^3)} \leq C \left\| P_{A,q} \pm v \right\|_{L^2(\mathbb{R}^3)}, \quad \rho > \rho_2,
\] (2.28)
with $C > 0$ depending on $\Omega$ and $\|q\|_{L^\infty(\Omega)} + \|A\|_{W^{1,\infty}(\Omega)^3}$.

Proof. Without loss of generality, we will only show this result for $P_{A,q}v$. We fix
\[
S_{A,q,+} = e^{-\varphi} \left( \Delta A + q \right) w_{A,-},
\]
and we split $S_{A,+}$ into three terms:
\[
S_{A,q,+} = P_1 + P_2 + P_3,
\]
where we recall that
\[
P_1 = \Delta + \rho^2 - 2s\rho(x' \cdot \theta) + s^2(x' \cdot \theta)^2 + s, \quad P_2 = 2(\rho - s(x' \cdot \theta))\theta \cdot \nabla - 2s,
\]
\[
P_3 = 2iA \cdot \nabla + 2iA \cdot \varphi_{A,+} + q - |A|^2 + i \text{ div } (A) = 2iA \cdot \nabla + 2(\rho - s(x' \cdot \theta))A' \cdot \theta + q - |A|^2 + i \text{ div } (A).
\]
We choose $\omega$ a bounded $C^2$ open set of $\mathbb{R}^3$ such that $\omega \subset \Omega$, and we extend the function $A$ and $q$ to $\mathbb{R}^3$ with $q = 0$ on $\mathbb{R}^3 \setminus \Omega$ and $A \in W^{1,\infty}(\mathbb{R}^3)^3$ satisfying
\[
\|A\|_{W^{1,\infty}(\mathbb{R}^3)^3} \leq C \|A\|_{W^{1,\infty}(\Omega)^3},
\]
where $C > 0$ depends only on $\Omega$. We consider also $\Omega = \omega \times \mathbb{R}$. We prove first the following estimate:
\[
\rho^{-1} \|v\|_{L^2(\mathbb{R}^3)} \leq C \left\| S_{A,q,+} v \right\|_{L^2(\mathbb{R}^3)}, \quad v \in C^0_0(\Omega).
\] (2.29)

For this purpose, we set $w \in H^1(\mathbb{R}^3)$ such that $\text{supp}(w) \subset \Omega$, and we consider
\[
\langle D_x, \rho \rangle^{-2}(P_1 + P_2)(D_x, \rho)^2 w.
\]
In all this proof, $C > 0$ denotes a constant depending on $\Omega$ and $\|A\|_{W^{1,\infty}(\Omega)^3} + \|q\|_{L^\infty(\Omega)}$. According to the properties of composition of pseudodifferential operators (e.g., Hörmander, theorem 18.1.8), we have
\[
\langle D_x, \rho \rangle^{-2}(P_1 + P_2)(D_x, \rho)^2 = P_1 + P_2 + R_{\rho}(x, D_x),
\] (2.30)
where $R_{\rho}$ is defined by
\[
R_{\rho}(x, \xi) = \nabla \xi(\xi, \rho)^{-2} \cdot D_x(p_1(x, \xi) + p_2(x, \xi))(\xi, \rho)^2 + \langle \xi, \rho \rangle^{-2} (1),
\]
with
\[
p_1(x, \xi) = -|\xi|^2 + \rho^2 - 2s(\xi' \cdot \theta) + s^2(\xi' \cdot \theta)^2 + s, \quad p_2(x, \xi) = 2[(\rho - s(\xi' \cdot \theta))\theta \cdot \xi' - 2s], \quad (\xi', \xi_3) \in \mathbb{R}^2 \times \mathbb{R}.
\]
Therefore, one can check that
\[
\|R_{\rho}(x, D_x)w\|_{L^2(\mathbb{R}^3)} \leq C \|w\|_{L^2(\mathbb{R}^3)},
\] (2.31)
Moreover, in view of (2.27) applied to $w$, with $\Omega$ replaced by $\Omega$ and $A = 0$, $q = 0$, we obtain
\[
\|P_1 w + P_2 w\|_{L^2(\mathbb{R}^3)} \geq C \left( s^{1/2} \rho^{-1} \|\Delta w\|_{L^2(\mathbb{R}^3)} + s^{1/2} \|\nabla w\|_{L^2(\mathbb{R}^3)} + s^{1/2} \rho \|w\|_{L^2(\mathbb{R}^3)} \right).
\]
Combining this with (2.30)–(2.31) and choosing $\rho$ sufficiently large, we get

$$
\| (P_1 + P_2)(D_x, \rho)^2 w \|_{H^{2\gamma}(\mathbb{R}^3)}
= \| (D_x, \rho)^{-2}(P_1 + P_2)(D_x, \rho)^2 w \|_{L^2(\mathbb{R}^3)}
\geq C^\gamma (\rho^{-1}\| \Delta w \|_{L^2(\mathbb{R}^3)} + \| \nabla w \|_{L^2(\mathbb{R}^3)} + \rho \| w \|_{L^2(\mathbb{R}^3)}) .
$$

Meanwhile, since $w \in H^2(\tilde{\Omega}) \cap H^1_0(\tilde{\Omega})$, the elliptic regularity (e.g., Chouli et al33, lemma 2.2) implies

$$
\| w \|_{H^2(\mathbb{R}^3)} = \| w \|_{H^2(\tilde{\Omega})} \leq C(\| \Delta w \|_{L^2(\tilde{\Omega})} + \| w \|_{L^2(\tilde{\Omega})}).
$$

In view of the previous estimate, for $s$ sufficiently large, we have

$$
\| (P_1 + P_2)(D_x, \rho)^2 w \|_{H^{2\gamma}(\mathbb{R}^3)} \geq C^\gamma s^{-1}\| w \|_{H^2(\mathbb{R}^3)}. \tag{2.32}
$$

In addition, we find

$$
\| P_3(D_x, \rho)^2 w \|_{H^{2\gamma}(\mathbb{R}^3)}
\leq \| 2i(\rho - s(x' \cdot \theta))A \cdot \theta + (q - |A|^2)(D_x, \rho)^2 w \|_{H^{2\gamma}(\mathbb{R}^3)} + 2 \| A \cdot \nabla(D_x, \rho)^2 w \|_{H^{2\gamma}(\mathbb{R}^3)}
\leq \rho^{-1} C \| (D_x, \rho)^2 w \|_{L^2(\mathbb{R}^3)}
\leq C \rho^{-1}\| w \|_{H^2(\mathbb{R}^3)}, \tag{2.33}
$$

For the first term on the right-hand side of this inequality, we have

$$
\| 2i(\rho - s(x' \cdot \theta))A \cdot \theta + (q - |A|^2)(D_x, \rho)^2 w \|_{H^{2\gamma}(\mathbb{R}^3)} \leq \rho^{-1} \| 2i(\rho - s(x' \cdot \theta))A \cdot \theta + (q - |A|^2)(D_x, \rho)^2 w \|_{L^2(\mathbb{R}^3)}
\leq \rho^{-1} \| A \|_{W^{1,\infty}(\Omega_0)} \| \nabla(D_x, \rho)^2 w \|_{H^{-\gamma}(\mathbb{R}^3)} \tag{2.34}
$$

with $C$ depending only on $\| A \|_{W^{1,\infty}(\Omega_0)}$ + $\| q \|_{L^\infty(\Omega_0)}$. For the second term on the right-hand side of (2.33), we get

$$
\| A \cdot \nabla(D_x, \rho)^2 w \|_{H^{2\gamma}(\mathbb{R}^3)} \leq \rho^{-1} \| A \cdot \nabla(D_x, \rho)^2 w \|_{H^{-\gamma}(\mathbb{R}^3)}
\leq \rho^{-1} \| A \|_{W^{1,\infty}(\Omega_0)} \| \nabla(D_x, \rho)^2 w \|_{H^{-\gamma}(\mathbb{R}^3)} \tag{2.35}
$$

Finally, for the last term on the right-hand side of (2.33), we find

$$
\| \mathrm{div}(A)(D_x, \rho)^2 w \|_{H^{2\gamma}(\mathbb{R}^3)} \leq \rho^{-2} \| \mathrm{div}(A)(D_x, \rho)^2 w \|_{L^2(\mathbb{R}^3)}
\leq 3 \rho^{-1} \| A \|_{W^{1,\infty}(\Omega_0)} \| w \|_{H^2(\mathbb{R}^3)}. \tag{2.36}
$$

Combining the estimates (2.33)–(2.36), we obtain

$$
\| P_3(D_x, \rho)^2 w \|_{H^{2\gamma}(\mathbb{R}^3)} \leq C \rho^{-1}\| w \|_{H^2(\mathbb{R}^3)},
$$
and applying (2.32) for \( s > 1 \) sufficiently large, we have

\[
\| S_{A,q,s} (D_x, \rho)^2 w \|_{H^s_x (\mathbb{R}^3)}^2 \geq C s^{\frac{3}{2}} \rho^{-1} \| w \|_{H^{s}_f (\mathbb{R}^3)}, \tag{2.37}
\]

Now, let us fix \( \omega_j, j = 1, 2 \), two open subsets of \( \partial \omega \) such that \( \partial \omega \subset \omega_1, \overline{\omega_1} \subset \omega_2 \), and \( \overline{\omega_2} \subset \partial \omega \). We consider \( \psi_0 \in C^\infty (\partial \omega) \) satisfying \( \psi_0 = 1 \) on \( \overline{\omega_2}, \nu \in C^\infty (\Omega), w (x', x_3) = \psi_0 (x') (D_x, \rho)^{-2} v (x', x_3), \) and \( \psi_1 \in C^\infty (\omega_1) \) satisfying \( \psi_1 = 1 \) on \( \omega \). Then, we have

\[
(1 - \psi_0) (D_x, \rho)^{-1} v = (1 - \psi_0) (D_x, \rho)^{-2} \psi_1 v,
\]

where \( \psi_1 v = (x', x_3) \mapsto \psi_1 (x') v (x', x_3) \). In view of Hörmander,\(^42\), theorem 18.1.8 using the fact that \( 1 - \psi_0 \) is vanishing in a neighborhood of supp(\( \psi_1 \)), we have \( (1 - \psi_0) (D_x, \rho)^{-1} \psi_1 \in Op S^{-\infty}_p \), and we get

\[
\rho^{-1} \| v \|_{L^2_x (\mathbb{R}^3)} = \rho^{-1} \| (D_x, \rho)^{-2} v \|_{H^s_x (\mathbb{R}^3)} \leq \| (1 - \psi_0) (D_x, \rho)^{-2} \psi_1 v \|_{H^s_x (\mathbb{R}^3)} \leq \rho^{-1} \| w \|_{H^s_x (\mathbb{R}^3)} + \rho^{-1} \| (1 - \psi_0) (D_x, \rho)^{-2} \psi_1 v \|_{H^s_x (\mathbb{R}^3)} \leq \rho^{-1} \| w \|_{H^s_x (\mathbb{R}^3)} + \frac{C \| v \|_{L^2_x (\mathbb{R}^3)}}{\rho^2}.
\]

In the same way, we obtain

\[
\| P_{A,-\lambda} v \|_{H^s_x (\mathbb{R}^3)} \geq \| P_{A,-\lambda} (D_x, \rho)^2 w \|_{H^s_x (\mathbb{R}^3)} - \| P_{A,-\lambda} (D_x, \rho)^2 (1 - \psi_0) (D_x, \rho)^{-2} \psi_1 v \|_{H^s_x (\mathbb{R}^3)} \geq \| P_{A,-\lambda} (D_x, \rho)^2 w \|_{H^s_x (\mathbb{R}^3)} - \| (1 - \psi_0) (D_x, \rho)^{-2} \psi_1 v \|_{H^s_x (\mathbb{R}^3)} \geq \| P_{A,-\lambda} (D_x, \rho)^2 w \|_{H^s_x (\mathbb{R}^3)} - \frac{C \| v \|_{L^2_x (\mathbb{R}^3)}}{\rho^2}.
\]

Combining these estimates with (2.37), we deduce that (2.29) holds true for a sufficiently large value of \( \rho \). Then, fixing \( s \), we deduce (2.28). \( \square \)

2.3 | Remainder term of the CGO solutions

In this subsection, we will construct the remainder term \( w_{r,j}, j = 1, 2 \), appearing in (2.15)–(2.16) and satisfying the decay property (2.17). For this purpose, we will combine the Carleman estimate (2.28) with the properties of the expressions \( b_j, j = 1, 2 \), in order to complete the construction of these solutions. In this subsection, we assume that \( \rho > \rho_2 \), with \( \rho_2 \) the constant introduced in Proposition 2.4, and we fix \( A_j \in (W^{2,\infty} (\Omega))^2, j = 1, 2 \), satisfying (1.4)–(1.5). The proof for the existence of the remainder term \( w_{r,1} \) and \( w_{r,2} \), being similar, will only show the existence of \( w_{r,1} \). Let us first remark that \( w_{r,1} \) should be a solution of the equation:

\[
P_{A_1, q_1} w = e^{-\delta_{x'} \cdot x} (-\Delta_{A_1} + q_1) e^{\delta_{x'} \cdot x} w = e^{\delta_{x'} \cdot x} F_{1, \rho} (x), x \in \Omega. \tag{2.38}
\]

with \( F_{1, \rho} \) defined, for all \( x = (x', x_3) \in B'_{r_0+1} \times \mathbb{R} \) (we recall that \( B'_r = \{ x' \in \mathbb{R}^2 : |x'| < r \} \)), by

\[
F_{1, \rho} (x) = -e^{-\delta_{x'} \cdot x} e^{i \delta_{x'} \cdot x} \left( e^{i \delta_{x'} \cdot x} e^{i \delta_{x'} \cdot x} \psi \left( \rho^{-\frac{1}{2}} x_3 \right) b_1 e^{-i \delta_{x'} \cdot x} \right)
= -\left( \left( |\xi|^2 + q_1 \right) \psi \left( \rho^{-\frac{1}{2}} i x_3 \right) - 2i \delta_{x'} \rho^{-\frac{1}{2}} \psi' \left( \rho^{-\frac{1}{2}} i x_3 \right) - 2i \delta_{x'} \rho^{-\frac{1}{2}} \psi' \left( \rho^{-\frac{1}{2}} i x_3 \right) \right) b_1 e^{-i \delta_{x'} \cdot x}
- \left[ -\rho^{-\frac{1}{2}} \psi'' \left( \rho^{-\frac{1}{2}} i x_3 \right) b_1 - 2i \delta_{x'} \rho^{-\frac{1}{2}} \psi' \left( \rho^{-\frac{1}{2}} i x_3 \right) + i 2 \delta_{x'} \cdot \nabla b_1 - \Delta_{A_1} b_1 \psi \left( \rho^{-\frac{1}{2}} i x_3 \right) \right] e^{-i \delta_{x'} \cdot x}. \tag{2.39}
\]

Here, we have used (2.24), and we consider \( q_1 \) as a function extended by zero to \( \mathbb{R}^3 \). We fix \( \phi \in C^\infty (B'_{r_0+1} : [0, 1]) \) satisfying \( \phi = 1 \) on \( B'_{r_0+1} \), and we define

\[
G_{\rho} (x', x_3) := \phi (x') F_{1, \rho} (x', x_3), x' \in \mathbb{R}^2, x_3 \in \mathbb{R}.
\]
It is clear that \( G_\rho \in L^2(\mathbb{R}^3) \) and in view of (2.20) and the fact that
\[
\| \psi \left( \rho^{-\frac{1}{2}} x_3 \right) \|_{L^2(B_{\rho^{-1}, x_3} \times \mathbb{R})} + \| \psi \left( \rho^{-\frac{1}{2}} x_3 \right) \|_{L^2(B_{\rho^{-1}, x_3} \times \mathbb{R})} + \| \psi'' \left( \rho^{-\frac{1}{2}} x_3 \right) \|_{L^2(B_{\rho^{-1}, x_3} \times \mathbb{R})} \leq C \rho^\frac{5}{2},
\]
we deduce that
\[
\| G_\rho \|_{L^2(\mathbb{R}^3)} \leq C(1 + 1 + \frac{|\xi'|}{|\xi|}) \rho^\frac{5}{2}, \quad (2.40)
\]
with \( C > 0 \) depending on \( \Omega \) and \( M \). From now on, we denote by \( C > 0 \) a constant depending only on \( \Omega \) and \( M \) that may change from line to line. Applying (2.28), we will complete the construction of the remainder term \( w_{1, \rho} \) by using a classical duality argument. More precisely, applying (2.28), we consider the linear form \( T_\rho \) defined on \( Q := \{ P_{A_1 \rho^\frac{1}{4}} w : w \in C^\infty_0(\Omega) \} \) by
\[
T_\rho(P_{A_1 \rho^\frac{1}{4}} w) := \frac{\langle G_\rho, e^{-i\rho x_3} \rangle}{H_{H_2, 1}(\mathbb{R}^3, H_2^1(\mathbb{R}^3))}, \quad v \in C^\infty_0(\Omega).
\]
Here, and from now on, we define the duality bracket \( \langle \cdot, \cdot \rangle_{H_2, 1(\mathbb{R}^3), L^2(\mathbb{R}^3)} \) in the complex sense, which means that
\[
\langle v, w \rangle_{H_2, 1(\mathbb{R}^3), L^2(\mathbb{R}^3)} = \langle v, w \rangle_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \overline{v}w dx, \quad v \in L^2(\mathbb{R}^3), \quad w \in H^2(\mathbb{R}^3).
\]
Applying again (2.28), for all \( v \in C^\infty_0(\Omega) \), we obtain
\[
\| T_\rho(P_{A_1 \rho^\frac{1}{4}} - v) \| \leq \| G_\rho \|_{L^2(\mathbb{R}^3)} \| e^{-i\rho x_3} \|_{L^2(\mathbb{R}^3)} \leq C \rho \| G_\rho \|_{L^2(\mathbb{R}^3)} \rho^{-1} \| v \|_{L^2(\mathbb{R}^3)} \leq C \rho \| G_\rho \|_{L^2(\mathbb{R}^3)} \| P_{A_1 \rho^\frac{1}{4}} - v \|_{H_2, 1(\mathbb{R}^3)}.
\]
with \( C > 0 \) depending on \( \Omega \) and \( \| q_1 \|_{L^\infty(\Omega)} + \| A_1 \|_{W^{3, \infty}(\Omega)^3} \). Thus, applying the Hahn–Banach theorem, we deduce that \( T_\rho \) admits an extension as a continuous linear form on \( H_2^2(\mathbb{R}^3) \) whose norm will be upper bounded by \( C \rho \| G_\rho \|_{L^2(\mathbb{R}^3)} \). Therefore, there exists \( w_{1, \rho} \in H_2^2(\mathbb{R}^3) \) such that
\[
\langle P_{A_1 \rho^\frac{1}{4}} - v, w_{1, \rho} \rangle_{H_2, 1(\mathbb{R}^3), L^2(\mathbb{R}^3)} = T_\rho(P_{A_1 \rho^\frac{1}{4}} - v) = \frac{\langle G_\rho, e^{-i\rho x_3} \rangle}{H_{H_2, 1}(\mathbb{R}^3, H_2^1(\mathbb{R}^3))}, \quad v \in C^\infty_0(\Omega), \quad (2.41)
\]
\[
\| w_{1, \rho} \|_{H_2^2(\mathbb{R}^3)} \leq C \rho \| G_\rho \|_{L^2(\mathbb{R}^3)}, \quad (2.42)
\]
From (2.41) and the fact that, for all \( x \in \Omega \), \( G_\rho(x) = F_{1, \rho}(x) \), we obtain
\[
\langle P_{A_1 \rho^\frac{1}{4}} - v, w_{1, \rho} \rangle_{D'(\Omega), C^\infty_0(\Omega)} = \langle e^{i\rho x_3} F_{1, \rho}, v \rangle_{D'(\Omega), C^\infty_0(\Omega)}.
\]
It follows that \( w_{1, \rho} \) solves \( P_{A_1 \rho^\frac{1}{4}} - w_{1, \rho} = e^{i\rho x_3} F_{1, \rho} \) in \( \Omega \) and \( u_1 \) given by (2.15) is a solution of \( \Delta A_1 u + q_1 u = 0 \) in \( \Omega \) lying in \( H^2(\Omega) \). In addition, from (2.42), we deduce that
\[
\rho^{-1} \| w_{1, \rho} \|_{H^2(\Omega)} + \| w_{1, \rho} \|_{H^2(\Omega)} + \rho \| w_{1, \rho} \|_{L^2(\Omega)} \leq C(1 + 1 + \frac{|\xi'|}{|\xi|}) \rho^\frac{5}{2},
\]
which implies the decay property (2.17). This completes the proof of Theorem 2.1.

## 3 STABILITY RESULTS ON THE WHOLE BOUNDARY

This section is devoted to the proof of our results with full boundary measurements stated in Theorems 1.1 and 1.2.
3.1 Recovery of the magnetic field

In this subsection, we will prove Theorem 1.1. In all this proof, \( C \) and \( c \) will be two positive constants depending only on \( \Omega \) and \( M \) that may change from line to line.

For \( j = 1, 2 \), we fix \( u_j \in H^2(\Omega) \) a solution of \( \Delta A_j u_1 + q_1 u_1 = 0, \Delta A_j u_2 + q_2 u_2 = 0 \) in \( \Omega \) of the form (2.15)–(2.16) with \( \rho > \rho_2 \) and with \( w_{j, \rho} \) satisfying (2.17). These solutions satisfy the following property:

**Lemma 3.1.** There exists \( C > 0 \) such that the following estimates

\[
\| u_1 \|_{H^2(\Omega)} \leq C e^{D^+} \left( 1 + \frac{|\xi'|}{|\xi_3|} \right) (1 + |\xi|^2),
\]

\[
\| u_2 \|_{H^2(\Omega)} \leq C e^{D^+} \left( 1 + \frac{|\xi'|}{|\xi_3|} \right) (1 + |\xi|^2).
\]

hold true for any solutions \( u_1 \) and \( u_2 \) given by (2.15) and (2.16). Here, \( D := \sup_{x' \in \tilde{\Omega}} |x'|. \)

**Proof.** Using the expression of \( u_1 \), we can easily deduce that

\[
\| u_1 \|_{L^2(\Omega)} \leq \| e^{\varphi} \|_{L^2(\Omega)} \| \psi \left( \rho^{-\frac{1}{2}} x_3 \right) b_1 e^{\rho q x^3} + w_{1, \rho}(x', x_3) \|_{L^2(\Omega)}.
\]

Setting \( D := \sup_{x' \in \tilde{\Omega}} |x'| \) and using (2.17) and (2.25), we get

\[
\| u_1 \|_{L^2(\Omega)} \leq C e^{D^+} \left( 1 + \frac{|\xi'|}{|\xi_3|} \right) (1 + |\xi|^2).
\]

By simple computations of \( \nabla u_1 \) and \( \partial_{x_i} \partial_{x_j} \), for \( i, j = 1, 2, 3 \), and by the same arguments used previously, we obtain

\[
\| \nabla u_1 \|_{L^2(\Omega)} \leq e^{D^+} \left( 1 + \frac{|\xi'|}{|\xi_3|} \right)(1 + |\xi|^2),
\]

and

\[
\| \partial_{x_i} \partial_{x_j} u_1 \|_{L^2(\Omega)} \leq e^{D^+} \left( 1 + \frac{|\xi'|}{|\xi_3|} \right)(1 + |\xi|^2).
\]

In the same way, we get

\[
\| u_2 \|_{L^2(\Omega)} \leq C e^{D^+} \left( 1 + \frac{|\xi'|}{|\xi_3|} \right)(1 + |\xi|^2),
\]

and

\[
\| u_2 \|_{H^2(\Omega)} \leq C e^{D^+} \left( 1 + \frac{|\xi'|}{|\xi_3|} \right)(1 + |\xi|^2).
\]

This completes the proof. \( \Box \)
Fixing \( q = q_1 - q_2 \) extended by zero to an element of \( L^\infty(\mathbb{R}^2) \) and applying a classical integration by parts argument, we deduce the following identity:

\[
\left\langle (\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2})u_1, u_2 \right\rangle_{L^2(\partial \Omega)} = i \int_{\mathbb{R}^2} (A \cdot \nabla u_1) \overline{u}_2 \, dx - i \int_{\mathbb{R}^2} u_1 (A \cdot \nabla \overline{u}_2) \, dx + \overline{\eta} u_1 \overline{u}_2, \tag{3.43}
\]

where \( \overline{\eta} = |A_2|^2 - |A_1|^2 + q \).

By simple computations, we get

\[
\nabla u_1 \overline{u}_2 - u_1 \nabla \overline{u}_2
= 2\rho(\bar{\theta} + i\eta)\psi \left( \rho^{-\frac{i}{2}} x_3 \right)^2 b_1 \overline{b}_2 e^{-i\xi \cdot x} + \rho(2\bar{\theta} + i\eta)\psi \left( \rho^{-\frac{i}{2}} x_3 \right) \left( b_1 e^{i\xi \cdot x} w_{2,\rho} + \overline{b}_2 e^{-i\xi \cdot x} \overline{w}_{1,\rho} \right)
- i\xi \psi \left( \rho^{-\frac{i}{2}} x_3 \right) b_1 \left( \psi \left( \rho^{-\frac{i}{2}} x_3 \right) b_2 e^{-i\xi \cdot x} + e^{i\xi \cdot x} w_{2,\rho} \right) + \psi \left( \rho^{-\frac{i}{2}} x_3 \right) e^{-i\xi \cdot x} \left( \overline{b}_2 \nabla b_1 - b_1 \nabla \overline{b}_2 \right)
+ \psi \left( \rho^{-\frac{i}{2}} x_3 \right) \left[ e^{i\xi \cdot x} \left( \nabla b_1 w_{2,\rho} - b_1 \nabla w_{2,\rho} \right) + e^{-i\xi \cdot x} \left( \overline{b}_2 \nabla w_{1,\rho} - \overline{b}_2 \nabla \overline{w}_{1,\rho} \right) \right]
+ \rho^{-\frac{i}{2}} \partial_3 \psi \left( \rho^{-\frac{i}{2}} x_3 \right) \left( b_1 e^{i\xi \cdot x} w_{2,\rho} - b_2 e^{-i\xi \cdot x} \overline{w}_{1,\rho} \right) + 2\rho \bar{\theta} w_{1,\rho} w_{2,\rho} + \nabla w_{1,\rho} \nabla w_{2,\rho} - w_{1,\rho} \nabla w_{2,\rho}.
\]

According to (2.17), (2.20), Lemma 3.1, and the fact that \( A \in L^1(\mathbb{R}^2) \), multiplying this expression by \(-i\rho^{-1}2^{-1}\), we find

\[
\left| \int_{\mathbb{R}^2} (A \cdot (\bar{\theta} + i\eta))\psi \left( \rho^{-\frac{i}{2}} x_3 \right)^2 \exp \left( \Phi_1 + \overline{\Phi}_2 \right) e^{-i\xi \cdot x} \, dx \right|
\leq C\rho^{-1} \left| \int_{\Omega} \left( \psi \left( \rho^{-\frac{i}{2}} x_3 \right) b_1 e^{i\xi \cdot x} w_{1,\rho}(x', x_3) \right) \left( \psi \left( \rho^{-\frac{i}{2}} x_3 \right) b_2 e^{-i\xi \cdot x} + w_{2,\rho}(x', x_3) \right) \, dx \right|
+ C \left( 1 + \left| \frac{\xi'}{|\xi|} \right|^2 \left( 1 + |\xi|^2 \right) \rho^{-\frac{1}{2}} + \| \Lambda_{A_1,q_1} - \Lambda_{A_2,q_2} \|_{B_{H^1(\partial \Omega)} L^2(\partial \Omega)} \right) \exp \rho \right)
\leq C \left( 1 + \left| \frac{\xi'}{|\xi|} \right|^2 \left( 1 + |\xi|^2 \right) \rho^{-\frac{1}{2}} + \| \Lambda_{A_1,q_1} - \Lambda_{A_2,q_2} \|_{B_{H^1(\partial \Omega)} L^2(\partial \Omega)} \right) \exp \rho, \tag{3.44}
\]

where

\[
c = 2(\sup_{x' \in \partial \Omega} |x'| + 1).
\]

On the other hand, using (1.5), (2.21), and the fact that \( \psi = 1 \) on \([-1, 1] \) and \( 0 \leq \psi \leq 1 \), we get

\[
\left| \int_{\mathbb{R}^2} (A \cdot (\bar{\theta} + i\eta))\psi \left( \rho^{-\frac{i}{2}} x_3 \right)^2 \exp \left( \Phi_1 + \overline{\Phi}_2 \right) e^{-i\xi \cdot x} \, dx \right|
\geq \left| \int_{\mathbb{R}^2} (A \cdot (\bar{\theta} + i\eta)) \exp \left( \Phi_1 + \overline{\Phi}_2 \right) e^{-i\xi \cdot x} \, dx \right| - C \int_{\mathbb{R}^2} |A| \left( 1 - \psi \left( \rho^{-\frac{i}{2}} x_3 \right) \right)^2 \, dx
\geq \left| \int_{\mathbb{R}^2} (A \cdot (\bar{\theta} + i\eta)) \exp \left( \Phi_1 + \overline{\Phi}_2 \right) e^{-i\xi \cdot x} \, dx \right| - C \int_{|x| \geq \frac{1}{2}} \nabla \left( \psi \left( \rho^{-\frac{i}{2}} x_3 \right) \right)^2 \, dx
\geq \left| \int_{\mathbb{R}^2} (A \cdot (\bar{\theta} + i\eta)) \exp \left( \Phi_1 + \overline{\Phi}_2 \right) e^{-i\xi \cdot x} \, dx \right| - C \rho^{-\frac{1}{2}} \int_{\Omega} \left( |A_1| + |A_2| \right) \, dx
\geq \left| \int_{\mathbb{R}^2} (A \cdot (\bar{\theta} + i\eta)) \exp \left( \Phi_1 + \overline{\Phi}_2 \right) e^{-i\xi \cdot x} \, dx \right| - 2CM \rho^{-\frac{1}{2}}.
\]
Combining this with (3.44), we obtain
\[
\left| \int_{\mathbb{R}^3} (A \cdot (\vec{\theta} + i\eta)) \exp \left( \Phi_1 + \Phi_2 \right) e^{-ix \cdot \xi} dx \right| 
\leq \left| \int_{\mathbb{R}^3} (A \cdot (\vec{\theta} + i\eta)) \psi \left( x - x_0 \right)^2 \exp \left( \Phi_1 + \Phi_2 \right) e^{-ix \cdot \xi} dx \right| + 2CM \rho^{-\frac{1}{4}}.
\] (3.45)

Now, let us observe that
\[
\Phi := \Phi_1 + \Phi_2 = -\frac{i}{2\pi} \int_{\mathbb{R}^2} (\vec{\theta} + i\eta) \cdot A(x - s_1 \vec{\theta} - s_2 \eta) ds_1 ds_2.
\]

Therefore, we have
\[
\left| \int_{\mathbb{R}^3} (A \cdot (\vec{\theta} + i\eta)) e^\Phi e^{-ix \cdot \xi} dx \right| \leq C \left( 1 + \left| \frac{x_1}{x_3} \right|^2 \right) (1 + \left| \xi \right|^2) \rho^{-\frac{1}{4}} + \left\| A_{1,q_1} - A_{2,q_2} \right\|_{B(\mathbb{R}^3)} e^{\sigma}.
\]

Applying Kian\textsuperscript{29} lemma 4.1 we deduce from the above estimate that
\[
\left| (\vec{\theta} + i\eta) \cdot F(A)(\xi) \right| \leq C \left( 1 + \left| \frac{x_1}{x_3} \right|^2 \right) (1 + \left| \xi \right|^2) \rho^{-\frac{1}{4}} + \left\| A_{1,q_1} - A_{2,q_2} \right\|_{B(\mathbb{R}^3)} e^{\sigma}.
\] (3.46)

In the same way, replacing \( \eta \) by \(-\eta\) in the construction of the CGO \( u_j, j = 1, 2 \), we obtain
\[
\left| (\vec{\theta} - i\eta) \cdot F(A)(\xi) \right| \leq C \left( 1 + \left| \frac{x_1}{x_3} \right|^2 \right) (1 + \left| \xi \right|^2) \rho^{-\frac{1}{4}} + \left\| A_{1,q_1} - A_{2,q_2} \right\|_{B(\mathbb{R}^3)} e^{\sigma}.
\]

Combining these two estimates with the fact that \( (\vec{\theta}, \eta) \) is an orthonormal basis of \( \xi^\perp = \{ y \in \mathbb{R}^3 : y \cdot \xi = 0 \} \), we find
\[
\left| \zeta \cdot F(A)(\xi) \right| \leq C |\zeta| \left( 1 + \left| \frac{x_1}{x_3} \right|^2 \right) (1 + \left| \xi \right|^2) \rho^{-\frac{1}{4}} + \left\| A_{1,q_1} - A_{2,q_2} \right\|_{B(\mathbb{R}^3)} e^{\sigma}, \zeta \in \xi^\perp. \quad (3.47)
\]

Moreover, for \( 1 \leq j < k \leq 3 \), fixing \( \zeta = \xi_j e_j - \xi_k e_k \), with
\[
e_j = (0, \ldots, 0, \underbrace{1}_{\text{position } j}, 0, \ldots, 0), \quad e_k = (0, \ldots, 0, \underbrace{1}_{\text{position } k}, 0, \ldots, 0),
\]

(3.47) implies
\[
\left| \xi_k F(a_j)(\xi) - \xi_j F(a_k)(\xi) \right| \leq C \left( 1 + \left| \frac{x_1}{x_3} \right|^2 \right) (1 + \left| \xi \right|^2) \rho^{-\frac{1}{4}} + \left\| A_{1,q_1} - A_{2,q_2} \right\|_{B(\mathbb{R}^3)} e^{\sigma}, \quad (3.48)
\]

where \( A = (a_1, a_2, a_3) \). Recall that so far, we have proved (3.48) for any \( \xi = (\xi', \xi_3) \in \mathbb{R}^2 \times \mathbb{R} \) with \( \xi' \neq 0 \) and \( \xi_3 \neq 0 \). Then, we deduce from (3.48) that
\[
\left| F(\partial_{\xi_j} a_j - \partial_{\xi_k} a_k)(\xi) \right| \leq C \left( 1 + \left| \frac{x_1}{x_3} \right|^2 \right) (1 + \left| \xi \right|^2) \rho^{-\frac{1}{4}} + \left\| A_{1,q_1} - A_{2,q_2} \right\|_{B(\mathbb{R}^3)} e^{\sigma}.
\]

From now on, we fix \( R > 1 \), \( \gamma := \left\| A_{1,q_1} - A_{2,q_2} \right\|_{B(\mathbb{R}^3)} \), and we consider the set
\[
D_R = \{ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi| \leq R, |\xi_3| \geq R^{-4} \}. 
\]
We obtain the estimate
\[ |F(\partial_x a_j - \partial_x a_k)(\xi)| \leq C(R^{13} \rho^{-\frac{7}{2}} + R^{13} \gamma e^{\gamma}). \] 
\( \xi \in D_R. \)

It follows that
\[ \int_{D_R} |F(\partial_x a_j - \partial_x a_k)(\xi)|^2 d\xi \leq C(R^{29} \rho^{-\frac{7}{2}} + R^{29} \gamma^2 e^{2\gamma}). \] (3.49)

On the other hand, using the fact that \( A_1 - A_2 \in W^{1,1}(\Omega)^3 \) satisfies (1.4), we obtain
\[ \|F(\partial_x a_j - \partial_x a_k)\|_{L^\infty(\mathbb{R}^3)} \leq 2\|A\|_{W^{1,1}(\mathbb{R}^3)} \leq 2\|A_1 - A_2\|_{W^{1,1}(\Omega)^3} \leq 2M. \]

Therefore, we have
\[ \int_{\mathbb{R}^3 \setminus D_R} |F(\partial_x a_j - \partial_x a_k)(\xi)|^2 d\xi \leq 4M^2 \int_{-R}^{R} \int_{B_R} d\xi' d\xi \leq CR^{-2}. \] (3.50)

In the same way, using the fact that \( A_j \in H^2(\Omega)^3 \) and applying (1.4), we deduce that \( A \in H^2(\mathbb{R}^3)^3 \) and (1.5) implies that
\[ \|A\|_{H^2(\mathbb{R}^3)} \leq 2M. \]

Applying this estimate, we deduce that
\[ \int_{\mathbb{R}^3 \setminus D_R} |F(\partial_x a_j - \partial_x a_k)(\xi)|^2 d\xi \leq CR^{-2} \int_{\mathbb{R}^3} (1 + |\xi|^2) |F(\partial_x a_j - \partial_x a_k)(\xi)|^2 d\xi \leq 2CMR^{-2}. \] (3.51)

Combining (3.49)–(3.51), we get
\[ \int_{\mathbb{R}^3} |F(\partial_x a_j - \partial_x a_k)(\xi)|^2 d\xi \leq C(R^{-2} + R^{29} \rho^{-\frac{7}{2}} + R^{29} \gamma^2 e^{2\gamma}). \]

and by Plancherel formula, it follows
\[ \|dA\|_{L^2(\Omega)} \leq C(R^{-1} + R^{29/2} \rho^{-\frac{7}{2}} + R^{29/2} \gamma e^{\gamma}). \]

Choosing \( R = \rho^{-\frac{\gamma}{2}}, \) we get
\[ \|dA\|_{L^2(\Omega)} \leq C(\rho^{-\frac{7}{2}} + (\rho^{-\frac{7}{2}})^{29/2} \gamma e^{\gamma}) \leq C(\rho^{-\frac{7}{2}} + \gamma e^{\gamma/3}). \] (3.52)

Now, let us recall a classical result already stated in Soussi.\(^3^0\)

**Lemma 3.2.** Let \( a \in (0, 1] \) and \( b > 0. \) Then, there exists \( C > 0 \) depending only on \( b, \) such that
\[ \inf_{\rho > 1} \rho^{-\frac{7}{2}} + ae^b \leq C(\log(3 + a^{-1}))^{-\frac{7}{2}}. \]

Combining (3.52) with Lemma 3.2, for \( \gamma \leq 1, \) we obtain
\[ \|dA\|_{L^2(\mathbb{R}^3)} \leq C(\log(3 + \gamma^{-1}))^{-\frac{7}{2}}. \] (3.53)

In the same way, for \( \gamma \geq 1, \) we have
\[ \|dA\|_{L^2(\mathbb{R}^3)} \leq 2M \log(4) (\log(3 + \gamma^{-1}))^{-\frac{7}{2}} \leq C(\log(3 + \gamma^{-1}))^{-\frac{7}{2}}. \]
Combining this estimate with (3.53), we deduce that (1.7) holds true for \( \gamma > 0 \).

For \( \gamma = 0 \), (3.52) implies that \( \|dA\|_{L^2(\Omega)} \leq C \rho^{-\frac{1}{2}} \). Since \( \rho > 1 \) is arbitrary, we can send \( \rho \) to \( +\infty \) and deduce (1.7) for \( \gamma = 0 \). This completes the proof.

### 3.2 Recovery of the electric potential

In this subsection, we assume that (1.4)–(1.8) hold true, and we will show (1.11). In all this proof, \( C \) and \( c \) will be two positive constants depending only on \( \Omega \) and \( M \) that may change from line to line. For this purpose, we start by proving the following estimate:

\[
\|A_1 - A_2\|_{L^2(\Omega)} \leq C \ln \left( 3 + \|A_{\gamma_1} - A_{\gamma_2}\|^{-1}_{B_1(\Omega)} \right) - r_1, 
\](3.54)

with \( r_1 > 0 \) depending only on \( s \). For this purpose, let us fix \( \xi \in \mathbb{R}^3 \setminus \{0\} \) and consider \( \eta_1, \eta_2 \in \mathbb{S}^2 \) such that \( \{\xi/|\xi|, \eta_1, \eta_2\} \) is an orthonormal basis of \( \mathbb{R}^3 \). Using the notation of the previous section, we deduce that

\[
F(A)(\xi) = \frac{(F(A)(\xi) \cdot \xi)^2}{|\xi|^2} + (F(A)(\xi) \cdot \eta_1) \eta_1 + (F(A)(\xi) \cdot \eta_2) \eta_2.
\]

However, from condition (1.4), we deduce that \( A \in H^1(\mathbb{R}^3)^3 \) and condition (1.8) implies that

\[
F(A)(\xi) \cdot \xi = -iF(\text{div}(A))(\xi) = -i \int_{\Omega} [\text{div}(A_1) - \text{div}(A_2)] e^{-i\xi \cdot x} dx = 0.
\]

Thus, we have

\[
F(A)(\xi) = (F(A)(\xi) \cdot \eta_1) \eta_1 + (F(A)(\xi) \cdot \eta_2) \eta_2,
\]

and applying (3.47), we deduce that

\[
|F(A)(\xi)| \leq C \left( 1 + \frac{|\xi|^2}{|\xi|^3} \right)^2 (1 + |\xi|^2) \left[ \rho^{-\frac{1}{2}} + \|A_{\gamma_1} - A_{\gamma_2}\|_{B_1(\Omega)}^{-1} e^{\rho} \right].
\]

Combining this estimate with the arguments used at the end of the proof of Theorem 1.1, we deduce (3.54).

Applying estimate (A1) (see Lemma A1 in Appendix A1) and (3.54), we obtain

\[
\|A_1 - A_2\|_{L^2(\Omega)} \leq C \|A_1 - A_2\|_{H^1(\Omega)^3} \|A_1 - A_2\|_{L^2(\Omega)^3}
\]

\[
\leq C(2M)^\frac{3}{2} \|A_1 - A_2\|_{L^2(\Omega)}
\]

\[
\leq C \ln \left( 3 + \|A_{\gamma_1} - A_{\gamma_2}\|^{-1}_{B_1(\Omega)} \right) - r_2, 
\](3.55)

with \( r_2 > 0 \) depending only on \( s \). Using the above estimate, we will now complete the proof of Theorem 1.2. For this purpose, applying (3.43) and the estimates (2.20) and (2.21), we obtain

\[
\left| \int_{\mathbb{R}^3} q(x) \psi \left( \rho^{-\frac{1}{2}} \chi_3 \right)^2 \exp \left( \Phi_1 + \Phi_2 \right) e^{-i\xi \cdot x} dx \right|
\]

\[
\leq C \left( 1 + \frac{|\xi|^2}{|\xi|^3} \right)^2 (1 + |\xi|^2) \left[ \rho \ln \left( 3 + \|A_{\gamma_1} - A_{\gamma_2}\|^{-1}_{B_1(\Omega)} \right) - r_2 \right.
\]

\[
+ \rho^{-\frac{1}{2}} + \|A_{\gamma_1} - A_{\gamma_2}\|_{B_1(\Omega)}^{-1} e^{\rho} \right].
\]

Recalling that

\[
\Phi_1 + \Phi_2 = -i \int_{\mathbb{R}^2} \frac{(\tilde{\theta} + i\eta) \cdot (A_1 - A_2)(x - s_1 \hat{\theta} - s_2 \eta)}{s_1 + i s_2} ds_1 ds_2
\]
and repeating the arguments used in Lemma 2.2, we deduce that

\[ \left\| \Phi_1 + \overline{\Phi}_2 \right\|_{L^\infty(\Omega)} \leq C \left\| A_1 - A_2 \right\|_{L^\infty(\Omega)^i} |(\eta_1, \eta_2)|^{-1} \leq C \left( 1 + \frac{|\xi'|}{|\xi_3|} \right) \left\| A_1 - A_2 \right\|_{L^\infty(\Omega)^i}. \]

Moreover, applying conditions (1.5)–(1.6), (2.21), and the mean value theorem, we obtain

\[ \left| \exp \left( \Phi_1 + \overline{\Phi}_2 \right) - 1 \right| \leq e^{\mathcal{M}} \| \Phi_1 + \overline{\Phi}_2 \|_{L^\infty(\Omega)} \leq C \left( 1 + \frac{|\xi'|}{|\xi_3|} \right) \left\| A_1 - A_2 \right\|_{L^\infty(\Omega)^i}. \]

Combining this with (3.55), we obtain

\[ \| \exp \left( \Phi_1 + \overline{\Phi}_2 \right) - 1 \|_{L^\infty(\Omega)} \leq C \left( 1 + \frac{|\xi'|}{|\xi_3|} \right) \ln \left( 3 + \| A_{q_1, q_1} - A_{q_2, q_2} \|_{B(1, \omega, \lambda, \Omega)}^{-1} \right)^{-2}. \] (3.57)

By inserting \( \int_{\mathbb{R}^3} q(x) \psi \left( \rho^{-\frac{1}{2}} x_3 \right)^2 e^{-i\xi \cdot x} \, dx \), we get

\[ \int_{\mathbb{R}^3} q(x) \psi \left( \rho^{-\frac{1}{2}} x_3 \right)^2 \exp \left( \Phi_1 + \overline{\Phi}_2 \right) e^{i\xi \cdot x} \, dx \]

\[ = \int_{\mathbb{R}^3} q(x) \psi \left( \rho^{-\frac{1}{2}} x_3 \right)^2 e^{-i\xi \cdot x} \, dx + \int_{\mathbb{R}^3} q(x) \psi \left( \rho^{-\frac{1}{2}} x_3 \right) \left( \exp \left( \Phi_1 + \overline{\Phi}_2 \right) - 1 \right) e^{-i\xi \cdot x} \, dx. \]

It follows that

\[ \left| \int_{\mathbb{R}^3} q(x) \psi \left( \rho^{-\frac{1}{2}} x_3 \right)^2 e^{-i\xi \cdot x} \, dx \right| \]

\[ \leq \left| \int_{\mathbb{R}^3} q(x) \psi \left( \rho^{-\frac{1}{2}} x_3 \right)^2 \exp \left( \Phi_1 + \overline{\Phi}_2 \right) e^{i\xi \cdot x} \, dx \right| + \| q \|_{L^1(\Omega)} \| \exp \left( \Phi_1 + \overline{\Phi}_2 \right) - 1 \|_{L^\infty(\Omega)} \]

\[ \leq C \left( 1 + \frac{|\xi'|}{|\xi_3|} \right)^2 \left( 1 + |\xi|^2 \right) \left[ \rho \ln \left( 3 + \| A_{q_1, q_1} - A_{q_2, q_2} \|_{B(1, \omega, \lambda, \Omega)}^{-1} \right)^{-2} + \rho^{-\frac{1}{2}} + \| A_{q_1, q_1} - A_{q_2, q_2} \|_{B(1, \omega, \lambda, \Omega)} e^{\rho} \right]. \]

Combining this estimate with the arguments used in the proof of Theorem 1.1 and in Soussi,\(^{30}\), theorem 1.1 one can check that the estimate (1.11) holds true.

4 STABILITY RESULTS FROM MEASUREMENTS ON SOME SUBSET OF THE BOUNDARY

This section is devoted to the proof of Theorems 1.3 and 1.4 by using an approach inspired by Ben Joud.\(^{15}\) We will only prove Theorem 1.3, and we refer the reader to Soussi\(^{30}\), theorem 1.2 for the proof of Theorem 1.4. For this purpose, for \( j = 1, 2 \), we fix \( A_j \in \mathcal{A}(M, A_0, \mathcal{O}_0) \) and \( q_j \in \mathcal{Q}(M, q_0, \mathcal{O}_0) \), and we consider again CGO solutions taking the form

\[ u_1(x', x_3) = e^{i\theta \cdot x'} \left( \psi \left( \rho^{-\frac{1}{2}} x_3 \right) b_1 e^{i\omega \cdot x - i\xi \cdot x} + w_{1, \rho}(x', x_3) \right), \quad x' \in \omega, \quad x_3 \in \mathbb{R}, \]

\[ u_2(x', x_3) = e^{-i\theta \cdot x'} \left( \psi \left( \rho^{-\frac{1}{2}} x_3 \right) b_2 e^{i\omega \cdot x} + w_{2, \rho}(x', x_3) \right), \quad x' \in \omega, \quad x_3 \in \mathbb{R}, \]

where \( w_{j, \rho} \in H^2(\Omega) \) satisfies the decay property

\[ \rho^{-\frac{1}{2}} \| w_{1, \rho} \|_{H^2(\Omega)} + \| w_{1, \rho} \|_{H^2(\Omega)} + \rho \| w_{1, \rho} \|_{L^2(\Omega)} \leq C \left| \xi_3 \right|^2 + 1 \left( 1 + \frac{|\xi'|}{|\xi_3|} \right)^2 \].
\[ \rho^{-1} \| w_{2,\rho} \|_{H^1(\Omega)} + \| w_{2,\rho} \|_{H^1(\Omega)} + \rho \| w_{2,\rho} \|_{L^2(\Omega)} \leq C \left( 1 + \frac{|x'|}{|x_3|} \right)^{-\frac{1}{2}}. \]

In view of Lemma 3.1, we have

\[ \| u_j \|_{H^1(\Omega)} \leq C \omega (D + 1)^\rho \left( 1 + \frac{|x'|}{|x_3|} \right)^{1 + |\alpha|^2} ; \quad j = 1, 2. \quad (4.58) \]

with \( D := \sup_{x \in \partial} |x'|. \)

We recall also that since \( q := q_1 - q_2 = 0 \) in \( \Omega_0 \), we can extend \( q \) to \( H^1(\mathbb{R}^3) \) by assigning it the value 0 outside of \( \Omega \), and we denote by \( q \) this extension. In this part, we need to set \( \mathcal{W}_j ; j = 1, 2, 3 \), such that

\[ \overline{\mathcal{W}}_{j+1} \subset \mathcal{W}_j, \quad \overline{\mathcal{W}}_j \subset \mathcal{W}_0 \quad \text{and} \quad \partial \omega \subset \partial \mathcal{W}_j. \]

Let \( \mathcal{O}_j = \mathcal{W}_j \times \mathbb{R} \) for \( j = 0, 1, 2, 3 \). The main idea of the proofs of Theorems 1.3 and 1.4 is to combine the estimate of the Fourier transform of \( dA \) and \( q \) with the weak unique continuation property which is given in the following lemma.

**Lemma 4.1.** Let \( A_1 \in C^2(\overline{\Omega}) \), \( q_1 \in L^\infty(\Omega) \), and \( M > 0 \) such that \( \| q \|_{L^\infty(\Omega)} \leq M \) and let \( w \in H^2(\Omega) \) solve

\[ \begin{cases} (-\Delta A_1 + q_1)w(x) = F(x) & \text{in} \ \Omega, \\ w = 0 & \text{on} \ \partial \Omega, \end{cases} \quad (4.59) \]

where \( F \in L^2(\Omega) \). Then, there exist positive constants \( C, \alpha_1, \alpha_2, \) and \( \lambda_0 \) such that we have the following estimate:

\[ \| w \|_{H^1(\partial \omega \setminus \Omega)} \leq C \left( e^{-2\lambda \rho} \| w \|_{H^2(\Omega)} + e^{2\lambda \rho} \left( \| \partial \nabla u \|_{L^2(\mathbb{R}^3)} + \| F \|_{L^2(\partial \omega)} \right) \right), \quad (4.60) \]

for any \( \lambda \geq \lambda_0 \). Here, the constants \( C, \alpha_1, \) and \( \alpha_2 \) depend on \( \Omega, M, \lambda_0, \) and \( \mathcal{O}_j \), and they are independent of \( A_1, q_1, F, w, \) and \( \lambda \).

### 4.1 Recovery of the magnetic field

**Proof of Theorem 1.3.** Let \( w \in H^2(\Omega) \) be the solution of

\[ \begin{cases} -\Delta A_1 w + q_1 w = 0 & \text{in} \ \Omega, \\ w = u_2 := h & \text{on} \ \partial \Omega. \end{cases} \quad (4.61) \]

Then, \( u = w - u_2 \) solves

\[ \begin{cases} -\Delta A_1 u + q_1 u = 2i A(x) \cdot \nabla u_2 + V(x) u_2(x) & \text{in} \ \Omega, \\ u = 0 & \text{on} \ \partial \Omega, \end{cases} \quad (4.62) \]

where \( V(x) = \text{div}(A) - \tilde{q}(x) \). Let \( \Theta \) be a cut-off function satisfying \( 0 \leq \Theta \leq 1 \), \( \Theta \in C^\infty(\mathbb{R}^2) \) and

\[ \Theta(x') = \begin{cases} 1 & \text{in} \ \omega \setminus \mathcal{W}_2, \\ 0 & \text{in} \ \mathcal{W}_3. \end{cases} \quad (4.63) \]

We set

\[ \tilde{u}(x', x_3) = \Theta(x') u(x', x_3), \quad x' \in \omega, x_3 \in \mathbb{R}. \]

We remark that \( \tilde{u} \) solves

\[ \begin{cases} (-\Delta A_1 + q_1)\tilde{u}(x', x_3) = 2i \Theta(x') A(x') \cdot \nabla u_2 + \Theta(x') V(x) u_2(x) + P_1(x', D)u(x) & \text{in} \ \Omega, \\ \tilde{u} = 0 & \text{on} \ \partial \Omega, \end{cases} \]

with \( P_1(x', D) \) is given by

\[ P_1(x', D)u = -[\Delta', \Theta]u - iA_1[V, \Theta]u - i[V, \Theta]A_1 u, \]
where \( V' = (\partial_{x_1}, \partial_{x_2})^T \), \( \Delta' = \partial_{x_1}^2 + \partial_{x_2}^2 \) and
\[
\tilde{\Theta}(x', x_3) = \Theta(x'), \ x' \in \omega, x_3 \in \mathbb{R}.
\]

Moreover, for an arbitrary \( \tilde{v} \in H^2(\Omega) \), an integration by parts leads to
\[
\int_{\Omega} (-\Delta A_i + q_1) \tilde{u}(x) \tilde{v}(x) dx = \int_{\Omega} \tilde{u}(x) (-\Delta A_i + q_1) \tilde{v}(x) dx.
\]

On the other hand, we have
\[
\int_{\Omega} (-\Delta A_i + q_1) \tilde{u}(x) \tilde{v}(x) dx = \int_{\mathbb{R}^d} \int_\omega \left( 2i \Theta(x') A(x) \cdot \nabla u_2 + \Theta(x') V(x) u_2(x) + P_1(x', D) u(x) \right) \tilde{v}(x) dx' dx_3.
\]

(4.64)

Choosing \( \tilde{v} = u_1 \), we have \( (-\Delta A_i + q_1) \tilde{v} = 0 \) in \( \Omega \), and by the fact that \( A = 0 \) and \( q = 0 \) in \( \mathcal{O}_0 \), we get
\[
i \int_{\Omega} \text{div}(A u_2 \overline{u_1(x)}) dx + i \int_{\Omega} A(x) \cdot u_1(x) \nabla u_2 dx - \int_{\Omega} (A_1^2 - A_2^2) u_2 u_1(x) dx - \int_{\Omega} qu_2 \overline{u_1(x)} dx = -\int_{\Omega} P_1(x', D) u(x) \overline{u_1(x)} dx.
\]

(4.65)

By integrating by parts and using the fact that \( A = 0 \) and \( q = 0 \) in \( \mathcal{O}_0 \), we can easily obtain
\[
i \int_{\Omega} A(x) \cdot [u_1(x) \overline{\nabla u_2} - u_2 \overline{\nabla u_1(x)}] dx = \int_{\Omega} (A_1^2 - A_2^2) u_2 u_1(x) dx + \int_{\Omega} qu_2 \overline{u_1(x)} dx - \int_{\Omega} P_1(x', D) u(x) \overline{u_1(x)} dx.
\]

(4.66)

Furthermore, using the fact that \( P_1(x', D) u \) is supported on \( \overline{\mathcal{O}_2} \setminus \mathcal{O}_3 \), we find
\[
\int_{\Omega} |P_1(x', D) u \overline{u_1(x)}| dx \leq \| u \|_{H^1(\mathcal{O}_2 \setminus \mathcal{O}_3)} \| \overline{u_1(x)} \|_{L^2(\Omega)}
\leq C \left( 1 + \frac{|x'|}{|x_3|} \right) (1 + |\xi|^2) e^{\omega(D+1)} \| u \|_{H^1(\mathcal{O}_2 \setminus \mathcal{O}_3)}.
\]

In a similar way to Theorem 1.1, using (4.66), we obtain
\[
\left| \int_{\mathbb{R}^1} \left( A \cdot (\tilde{\sigma} + i\eta) \right) e^{\Phi} e^{-ikx} dx \right| \leq C \left( 1 + \frac{|\xi'|}{|\xi_3|} \right)^2 (1 + |\xi|^2) \left[ \rho^{-\frac{1}{2}} + e^{\omega(D+1)} \| u \|_{H^1(\mathcal{O}_2 \setminus \mathcal{O}_3)} \right].
\]

Applying Kian\textsuperscript{29}, lemma 4.1 and following the same steps of the proof of Theorem 1.1, we deduce that
\[
\left| F(\partial_{x_j} a_j - \partial_{x_j} a_k)(\xi) \right| \leq C \left( 1 + \frac{|\xi'|}{|\xi_3|} \right)^2 (1 + |\xi|^3) \left[ \rho^{-\frac{1}{2}} + e^{\omega(D+1)} \| u \|_{H^1(\Omega)} + e^{\omega(D+1)} \left( \| \partial_{x_j} u \|_{L^2(\mathbb{R} \times \mathbb{R})} + \| (\Delta A_i + q_1) u \|_{L^2(\mathcal{O}_0)} \right) \right].
\]

Combining this last estimate with (4.60), we get
\[
\left| F(\partial_{x_j} a_j - \partial_{x_j} a_k)(\xi) \right| \leq C \left( 1 + \frac{|\xi'|}{|\xi_3|} \right)^2 (1 + |\xi|^3) \left[ \rho^{-\frac{1}{2}} + e^{\omega(D+1)} \left( e^{-\omega(D+1)} \| u \|_{H^1(\Omega)} + e^{\omega(D+1)} \left( \| \partial_{x_j} u \|_{L^2(\mathbb{R} \times \mathbb{R})} + \| (\Delta A_i + q_1) u \|_{L^2(\mathcal{O}_0)} \right) \right) \right].
\]
Since $\partial u = (\Lambda'_{A_{1},q_{1}} - \Lambda'_{A_{2},q_{2}})(h)$, where $h$ is given by (4.61), we have
\[
\|\partial_{u}\|_{L^{2}(\Gamma_{0} \times R)} \leq C\|\Lambda'_{A_{1},q_{1}} - \Lambda'_{A_{2},q_{2}}\|_{B(H^{2}(\partial \Omega), H^{2}(\Gamma_{0} \times R))}\|h\|_{H^{2}(\Omega)}.
\]
Moreover, since $A_{j} \in A(M, A_{0}, \Omega_{0})$ and $q_{j} \in Q(M, q_{0}, \Omega_{0})$, $j = 1, 2$, we have $(\Delta_{A_{j}} + q_{1})u = 0$ on $\Omega_{0}$, and it follows by (4.58):
\[
\left|\mathcal{F}(\partial_{\xi_{j}} a_{j} - \partial_{\xi_{j}} a_{k})(\xi)\right| \leq C\left(1 + \frac{|\xi|}{|\xi_{j}|}\right)^{3}(1 + |\xi_{j}|^{3})^{2}\left(\rho^{-\frac{3}{4}} + e^{2\rho(D+1)}\left(\rho^{-\frac{3}{4}} + e^{\rho(D+1)}\right)\right).
\] (4.67)

Let $D' := D + 1$ and $\lambda = \tau \rho$. Choosing $r$ sufficiently large, it becomes easy to find constants $\alpha_{3}$ and $\alpha_{4}$ such that
\[
e^{2^{D'}\rho - \lambda} = e^{\rho(2^{D'} - \tau a_{1})} \leq e^{-a_{1}\rho} \text{ and } e^{2^{D'}\rho + \lambda} = e^{\rho(2^{D'} + \tau a_{1})} \leq e^{a_{1}\rho}.
\] (4.68)

Combining (4.67) and (4.68), we conclude that
\[
\left|\mathcal{F}(\partial_{\xi_{j}} a_{j} - \partial_{\xi_{j}} a_{k})(\xi)\right| \leq C\left(1 + \frac{|\xi|}{|\xi_{j}|}\right)^{3}(1 + |\xi_{j}|^{3})^{2}\left(\rho^{-\frac{3}{4}} + e^{\rho(D+1)}\right)\|\Lambda'_{A_{1},q_{1}} - \Lambda'_{A_{2},q_{2}}\|_{B(H^{2}(\partial \Omega), H^{2}(\Gamma_{0} \times R))}.
\] (4.69)

From now on, we fix $R > 1$, $\gamma' := \|\Lambda'_{A_{1},q_{1}} - \Lambda'_{A_{2},q_{2}}\|_{B(H^{2}(\partial \Omega), H^{2}(\Gamma_{0} \times R))}$, and we consider the set
\[
D_{R} = \{\xi \in B_{R} : |\xi_{3}| \geq R^{-4}\}.
\]

We obtain the estimate
\[
\left|\mathcal{F}(\partial_{\xi_{j}} a_{j} - \partial_{\xi_{j}} a_{k})(\xi)\right| \leq C(R^{2^{D}}\rho^{-\frac{3}{4}} + R^{2^{D}}\gamma'e^{a_{1}\rho}), \quad \xi \in D_{R}.
\]

It follows that
\[
\int_{D_{R}}\left|\mathcal{F}(\partial_{\xi_{j}} a_{j} - \partial_{\xi_{j}} a_{k})(\xi)\right|^{2} d\xi \leq C(R^{45}\rho^{-\frac{3}{4}} + R^{45}\gamma'e^{2a_{1}\rho}).
\] (4.70)

Combining (4.70)–(3.51), we get
\[
\int_{D_{R}}\left|\mathcal{F}(\partial_{\xi_{j}} a_{j} - \partial_{\xi_{j}} a_{k})(\xi)\right|^{2} d\xi \leq C(R^{-2} + R^{45}\rho^{-\frac{3}{4}} + R^{45}\gamma'e^{2a_{1}\rho}),
\]

and by Plancherel formula, it follows
\[
\|dA\|_{L^{2}(\Omega)} \leq C(R^{-1} + R^{45/2}\rho^{-\frac{3}{4}} + R^{45/2}\gamma'e^{a_{1}\rho}).
\]

Choosing $R = \rho^{-\frac{1}{2}}$, we get
\[
\|dA\|_{L^{2}(\Omega)} \leq C(\rho^{-\frac{1}{2}} + (\rho^{-\frac{1}{2}})^{45/2}\gamma'e^{a_{1}\rho}) \leq C(\rho^{-\frac{1}{2}} + \gamma'e^{(a_{1}+1)\rho}).
\] (4.71)

Then, repeating the arguments used at the end of the proof of Theorem 1.1, we can deduce (1.3) from (4.71). □
ACKNOWLEDGEMENT
This work was partially supported by the Agence Nationale de la Recherche (Project MultiOnde) Grant ANR-17-CE40-0029.

CONFLICT OF INTEREST
This work does not have any conflicts of interest.

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In this appendix, we consider the following interpolation result.

**Lemma A1.** Let $h \in W^{1, \infty}(\Omega) \cap L^2(\Omega)$. Then, there exists $C > 0$ depending only on $\Omega$ such that

$$
\|h\|_{L^\infty(\Omega)} \leq C \|h\|_{W^{1, \infty}(\Omega)}^{\frac{1}{2}} \|h\|_{C^{0,1}_{L1}(\Omega)}^{\frac{1}{2}}.
$$

(A1)

**Proof.** Let us first observe that this result is well known for $\Omega$ bounded (see, e.g., Choulli and Kian\textsuperscript{43}, lemma appendix B.1), but we have not find any proof of it for unbounded domains. For this reason, we decided do give the full proof of this result. Let us fix $\psi \in C_0^\infty(-2, 2)$ satisfying $\psi = 1$ on $[-1, 1]$ and fix $y_3 \in \mathbb{R}$. Denote also by $\mathcal{O}$ a smooth open-bounded subset of $\Omega$ such that $\omega \times [-2, 2] \subset \mathcal{O}$. Now, let us consider the function $h_{y_3} : x = (x', x_3) \mapsto \psi(y_3 + x_3)h(x', x_3 + y_3)$. Applying (A1) for $\Omega = \mathcal{O}$ and $h = h_{y_3}$, we obtain that

$$
\|h_{y_3}\|_{L^\infty(\mathcal{O})} \leq C \|h_{y_3}\|_{W^{1, \infty}(\mathcal{O})}^{\frac{1}{2}} \|h_{y_3}\|_{C^{0,1}_{L1}(\mathcal{O})}^{\frac{1}{2}},
$$

(A2)

with $C > 0$ depending only on $\mathcal{O}$. In the same way, we have

$$
\|h_{y_3}\|_{W^{1, \infty}(\mathcal{O})} \leq \|\psi h\|_{W^{1, \infty}(\mathcal{O})} \leq \|\psi\|_{W^{1, \infty}(\mathbb{R})} \|h\|_{W^{1, \infty}(\Omega)}.
$$
\[ \| h(x, x_3 + y_3) \|_{L^2(\Omega)}^2 \leq \| \psi \|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} \int_{\mathbb{R}} |h(x', x_3 + y_3)|^2 dx' dx_3 = \| \psi \|_{L^2(\mathbb{R})}^2 \| h \|_{L^2(\Omega)}^2. \]

Combining these two estimates with (A2), we obtain

\[ \| h(x, x_3 + y_3) \|_{L^2(\Omega)} \leq C \| \psi \|_{W^{1,\infty}(\mathbb{R})} \| h \|_{W^{1,\infty}(\Omega)} \| h \|_{L^2(\Omega)}. \]

Since the right-hand side of the above identity is independent of \( y_3 \in \mathbb{R} \), we can take the sup with respect to \( y_3 \in \mathbb{R} \) in order to deduce that

\[ \| h \|_{L^2(\Omega)} \leq \sup_{y_3 \in \mathbb{R}} \| h(x, x_3 + y_3) \|_{L^2(\Omega)} \leq C \| \psi \|_{W^{1,\infty}(\mathbb{R})} \| h \|_{W^{1,\infty}(\Omega)} \| h \|_{L^2(\Omega)}. \]

This estimate clearly implies (A1).

\[ \square \]

**APPENDIX B: CARLEMAN ESTIMATE**

The main goal of this appendix is to prove a Carleman estimate for the magnetic Schrödinger operator \(-\Delta_A + q\) in an infinite cylindrical domain in order to deduce the weak unique continuation property given in Lemma 4.1. As we are dealing with weighted inequalities, we borrow, from Soussi\(^30\) (see also Imanuvilov and Yamamoto\(^44\) lemma 2.3 Imanuvilov\(^45\) lemma 1.2 and Krupchyk and Uhlmann\(^22\) theorem 2.4), the following result that guarantees the existence of the weigh function.

**Lemma B1.** There exists a function \( \psi_0 \in C^3(\overline{\Omega}) \) such that

1. \( \psi_0(x') > 0 \) for all \( x' \in \Omega_0 \);
2. there exists \( a_0 > 0 \) such that \( |\nabla \psi_0(x')| \geq a_0 \) for all \( x' \in \overline{\Omega}_0 \);
3. \( \partial_{\nu} \psi_0(x') \leq 0 \) for all \( x' \in \partial\Omega_0 \setminus \Gamma_0 \); and
4. \( \psi_0(x') = 0 \) for all \( x' \in \partial\Omega_0 \setminus \Gamma_0 \).

Here, \( \nabla' \) denotes the gradient with respect to \( x' \in \mathbb{R}^2 \) and \( \partial_{\nu} \), the normal derivative with respect to \( \partial\Omega_0 \), that is, \( \partial_{\nu} := \nu \cdot \nabla' \), where \( \nu \) stands for the outward normal vector to \( \partial\Omega_0 \).

Thus, putting \( \psi(x) = \psi(x', x_3) := \psi_0(x') \) for all \( x = (x', x_3) \in \overline{\Omega}_0 \), it is apparent that the function \( \psi \in C^3(\overline{\Omega}_0) \) satisfies the four following conditions:

\begin{itemize}
  \item[(C1)] \( \psi(x) > 0, x \in \Omega_0 \).
  \item[(C2)] \( |\nabla \psi(x)| \geq a_0 \) for all \( x \in \overline{\Omega}_0 \).
  \item[(C3)] \( \partial_{\nu} \psi(x) \leq 0 \) for all \( x \in \partial\Omega_0 \setminus (\Gamma_0 \times \mathbb{R}) \).
  \item[(C4)] \( \psi(x) = 0 \) for all \( x \in \Gamma_0 \times \mathbb{R} \).
\end{itemize}

Here, \( \nu \) is the outward unit normal vector to the boundary \( \partial\Omega_0 \). Evidently, \( \nu = (\nu', 0) \), so we have \( \partial_{\nu} \psi = \partial_{\nu} \psi_0 \) as the function \( \psi \) does not depend on \( x_3 \).

Next, for \( \beta \in (0, +\infty) \), we introduce the following weigh function:

\[ \varphi(x) = \varphi(x') = e^{\beta \psi(x')}; \ x \in \Omega_0. \tag{B1} \]

Then, \( \varphi \) satisfies some properties given in the following result.

**Lemma B2.** There exists a constant \( \beta_0 \in (0, +\infty) \) depending only on \( \psi \) such that the following statements hold uniformly in \( \Omega_0 \) for all \( \beta \in [\beta_0, +\infty] \).

1. \( |\nabla \varphi| \geq a := \beta_0 a_0 \).
2. \( \nabla |\nabla \varphi|^2 \cdot \nabla \varphi \geq C_0|\nabla \varphi|^3 \).
3. \( H(\varphi)\xi \cdot \xi + C_1 \beta |\nabla \varphi||\xi|^2 \geq 0; \ \xi \in \mathbb{R}^3 \).
4. \( |\Delta |\nabla \varphi| | \leq C_2|\nabla \varphi|^3 \).
5. \( \Delta \varphi \geq 0 \).

Here, \( C_0, C_1, \), and \( C_2 \) are positive constants depending only on \( \psi \) and \( a_0 \) and \( H(\varphi) \) denotes the Hessian matrix of \( \varphi \) with respect to \( x \in \Omega_0 \).
Now, we may state the following Carleman estimate for the operator $\Delta_A + q$.

**Theorem B.3.** Let $u \in H^1_0(\Omega) \cap H^2(\Omega)$, $M_1, M_2 > 0$ and let $A \in W^{1,\infty}$ and $q \in L^\infty(\Omega)$ satisfy $\|A\|_{W^{1,\infty}(\Omega)} \leq M_1$ and $\|q\|_{L^\infty(\Omega)} \leq M_2$. Then, there exists $\beta_0 \in (0, +\infty)$ such that for every $\beta \geq \beta_0$, there is $\lambda_0 = \lambda_0(\beta) \in (0, +\infty)$ depending only on $\beta$, $a_0$, $\omega$, $\Gamma_0$, $\beta$, and $\lambda_0$. \(\lambda_0 \leq \lambda_0(\beta) \in (0, +\infty)\) depending only on $\beta$, $a_0$, $\omega$, $\Gamma_0$, $\beta$, and $\lambda_0$.

**Proof.** For the proof, we can simply show the following inequality:

$$
\lambda \int_{\Omega} e^{2\lambda \rho} (\lambda^2 |u|^2 + |\nabla u|^2) \, dx \leq C \left( \int_{\Omega} e^{2\lambda \rho}|(\Delta_A + q)u|^2 \, dx + \lambda \int_{\Gamma \times \mathbb{R}} e^{2\lambda \rho} |\partial_x u|^2 \, d\sigma \right),
$$

(B2)

holds for all $\lambda \geq \lambda_0$ and some positive constant $C$ that depends only on $a_0$, $\omega$, $\Gamma_0$, $\beta$, and $\lambda_0$.

In fact, we have

$$
\Delta_A + q = \Delta + P_0,
$$

with $P_0$ is a first-order operator given by

$$
P_0 = 2iA \cdot \nabla + \text{Id}(A) - A \cdot A + q.
$$

As $\|A\|_{W^{1,\infty}(\Omega)} \leq M_1$ and $\|q\|_{L^\infty(\Omega)} \leq M_2$, we get

$$
|P_0 u| \leq C(|u| + |\nabla u|).
$$

By (B3), we get

$$
\lambda \int_{\Omega} e^{2\lambda \rho} (\lambda^2 |u|^2 + |\nabla u|^2) \, dx \\
\leq C \left( \int_{\Omega} e^{2\lambda \rho}|(\Delta_A + q)u|^2 \, dx + \lambda \int_{\Gamma \times \mathbb{R}} |\partial_x u|^2 e^{2\lambda \rho} \, d\sigma \right) \\
\leq C \int_{\Omega} e^{2\lambda \rho}|(\Delta_A + q)u|^2 \, dx + C \int_{\Omega} e^{2\lambda \rho}|u|^2 \, dx + C \lambda \int_{\Gamma \times \mathbb{R}} |\partial_x u|^2 e^{2\lambda \rho} \, d\sigma
$$

\(\forall \lambda \geq \tau_0 \).

Thus, we have

$$
\lambda^3 \left( 1 - \frac{C}{\lambda^3} \right) \int_{\Omega} e^{2\lambda \rho} |u|^2 \, dx + \lambda \left( 1 - \frac{C}{\lambda} \right) \int_{\Omega} e^{2\lambda \rho} |\nabla u|^2 \, dx \\
\leq C \int_{\Omega} e^{2\lambda \rho}|(\Delta_A + q)u|^2 \, dx + C \lambda \int_{\Gamma \times \mathbb{R}} |\partial_x u|^2 e^{2\lambda \rho} \, d\sigma.
$$

Let $\lambda' \leq \lambda_0'$ such that $\forall \lambda \geq \lambda_0', 1 - \frac{C}{\lambda} \geq \frac{1}{2}$, and $1 - \frac{C}{\lambda} \geq \frac{1}{2}$. For any $\lambda \geq \text{max}(\lambda_0, \lambda'_0)$, we have

$$
\frac{1}{2} \lambda^3 \int_{\Omega} e^{2\lambda \rho} |u|^2 \, dx + \frac{1}{2} \lambda \int_{\Omega} e^{2\lambda \rho} |\nabla u|^2 \, dx \leq C \int_{\Omega} e^{2\lambda \rho}|(\Delta_A + q)u|^2 \, dx + C \lambda \int_{\Gamma \times \mathbb{R}} |\partial_x u|^2 e^{2\lambda \rho} \, d\sigma.
$$

The proof of the estimate (B3) is stated in Soussi.30
APPENDIX C: WEAK UNIQUE CONTINUATION PROPERTY

This appendix is devoted to the proof of the weak unique continuation property stated in Lemma 4.1. Let \( \psi_0 \) be the function defined in Lemma B1. Since \( \psi_0(x') > 0 \) for all \( x' \in \mathcal{W}_0 \), there exists a constant \( \kappa > 0 \) such that

\[
\psi_0(x') \geq 2\kappa; \quad x' \in \mathcal{W}_2 \setminus \mathcal{W}_3.
\]  

(C1)

Moreover, as \( \psi_0(x') = 0, x' \in \Gamma^\sharp \), there exist \( \mathcal{W}^\sharp \) a small neighborhood of \( \Gamma^\sharp \) such that

\[
\psi_0(x') \leq \kappa; \quad x' \in \mathcal{W}^\sharp. \quad \mathcal{W}^\sharp \cap \overline{\mathcal{W}_1} = \emptyset.
\]  

(C2)

Let \( \mathcal{W}^\sharp \subset \mathcal{W}^\sharp \) be an arbitrary neighborhood of \( \Gamma^\sharp \). To apply (B2), it is necessary to introduce a function \( \Theta \) satisfying \( 0 \leq \Theta \leq 1, \Theta \in C^\infty(\mathbb{R}^2) \) and

\[
\Theta(x') = \begin{cases} 
1 & \text{in } \mathcal{W}_0 \setminus \mathcal{W}^\sharp, \\
\sim & \text{in } \mathcal{W}^\sharp.
\end{cases}
\]

(C3)

Let \( w \) be a solution to (4.59). Setting

\[
w_1(x', x_3) = \Theta(x') w(x', x_3), \quad x' \in \omega, x_3 \in \mathbb{R},
\]

we get

\[
\begin{cases}
- \Delta_{A_1} w + q_1 w_1(x) = \Theta(x) F(x) + Q_1(x, D) w & \text{in } \mathcal{O}_0, \\
w_1 = 0 & \text{on } \partial \mathcal{O}_0,
\end{cases}
\]

where \( Q_1(x, D) \) is a first-order operator supported in \( \mathcal{W}^\sharp \setminus \mathcal{W}^\sharp \) and given by

\[
Q_1(x, D) w = -[\Delta', \Theta] w - iA_1[\nabla, \Theta] w - i[\nabla, \Theta A_1] w.
\]

By applying Carleman estimate (B2) to \( w_1 \), we obtain

\[
\lambda \int_{\mathcal{O}_0} e^{2\lambda(x)} \left( \lambda^2 |w_1|^2 + |\nabla w_1|^2 \right) dx 
\leq C \left( \int_{\mathcal{O}_0} e^{2\lambda(x)} \left( |Q_1(x, D) w|^2 + |F(x)|^2 \right) dx + \lambda \int_{\Gamma \times \mathbb{R}} |\partial_x w_1|^2 e^{2\lambda(x)} d\sigma_x \right). \tag{C4}
\]

Let \( \mathcal{O}^\sharp = \mathcal{W}^\sharp \times \mathbb{R} \) and \( \mathcal{O} = \mathcal{W} \times \mathbb{R} \). Using the fact that \( Q_1(x, D) \) is a first-order operator supported in \( \overline{\mathcal{O}^\sharp} \setminus \mathcal{O} \) and by (C2), we get

\[
\int_{\mathcal{O}_0} e^{2\lambda(x)} |Q_1(x, D) w|^2 dx 
\leq \int_{\mathcal{O}_0} e^{2\lambda(x)} |Q_1(x, D) w|^2 dx 
\leq e^{2\lambda(x)} \int_{\mathcal{O}^\sharp \setminus \mathcal{O}} |Q_1(x, D) w|^2 dx 
\leq C e^{2\lambda(x)} \int_{\mathcal{O}^\sharp \setminus \mathcal{O}} (|w|^2 + |\nabla w|^2) dx.
\]

On the other hand, by using the definition of \( \Theta \) given by (C3), the estimate (C4) becomes

\[
\lambda \int_{\mathcal{O}_0} e^{2\lambda(x)} (\lambda^2 |w|^2 + |\nabla w|^2) dx 
\leq C \left( e^{2\lambda(x)} \int_{\mathcal{O}^\sharp \setminus \mathcal{O}} (|w|^2 + |\nabla w|^2) dx 
+ \int_{\mathcal{O}_0} e^{2\lambda(x)} |F(x)|^2 dx + \lambda \int_{\Gamma \times \mathbb{R}} |\partial_x w|^2 e^{2\lambda(x)} d\sigma_x \right).
\]
Moreover, by the fact that \( \mathcal{O}_2 \setminus \mathcal{O}_3 \subset \mathcal{O}_0 \setminus \mathcal{O} \) and by (C1), we easily obtain that

\[
e^{2\lambda e^{2\beta \psi}} \int_{\mathcal{O}_2 \setminus \mathcal{O}_3} (\lambda^2 |w|^2 + |\nabla w|^2) \, dx \leq C \left( e^{2\lambda e^{2\beta \psi}} \int_{\mathcal{O}_0 \setminus \mathcal{O}} (|w|^2 + |\nabla w|^2) \, dx + \int_{\mathcal{O}_2} e^{2\lambda |F(x)|^2} \, dx + \lambda \int_{\Gamma \times \mathbb{R}} |\partial_\nu w|^2 e^{2\lambda \psi} \, d\sigma_x \right).
\]

Thus, we have

\[
\lambda \int_{\mathcal{O}_2 \setminus \mathcal{O}_3} (\lambda^2 |w|^2 + |\nabla w|^2) \, dx \leq C \left( e^{-2\lambda (e^{2\beta \psi} - e^{2\beta \psi_0})} \int_{\mathcal{O}_0 \setminus \mathcal{O}} (|w|^2 + |\nabla w|^2) \, dx + e^{2\lambda (e^{2\beta \psi_0} - e^{2\beta \psi})} \left( \int_{\mathcal{O}_0} |F(x)|^2 \, dx + \lambda \int_{\Gamma \times \mathbb{R}} |\partial_\nu w|^2 \, d\sigma_x \right) \right).
\]

Let \( \alpha_1 = (e^{2\beta \psi} - e^{2\beta \psi_0}) > 0 \) and \( \alpha_2 = (e^{2\beta \psi} - e^{2\beta \psi_0}) > 0 \). We conclude that for any \( \lambda > \lambda^* \), we have

\[
\lambda \int_{\mathcal{O}_2 \setminus \mathcal{O}_3} (\lambda^2 |w|^2 + |\nabla w|^2) \, dx \leq C \left( e^{-2\lambda \alpha_1} \int_{\mathcal{O}_0 \setminus \mathcal{O}} (|w|^2 + |\nabla w|^2) \, dx + e^{2\lambda \alpha_2} \left( \int_{\mathcal{O}_0} |F(x)|^2 \, dx + \int_{\Gamma \times \mathbb{R}} |\partial_\nu w|^2 \, d\sigma_x \right) \right).
\]

Then, we have

\[
\|w\|^2_{H^1(\mathcal{O}_2 \setminus \mathcal{O}_3)} \leq C \left( e^{-2\lambda \alpha_1} \|w\|^2_{H^1(\mathcal{O})} + e^{2\lambda \alpha_2} \left( \|F\|^2_{L^2(\mathcal{O}_0)} + \|\partial_\nu w\|^2_{L^2(\Gamma \times \mathbb{R})} \right) \right),
\]

which completes the demonstration.