Behavior of Neutrinos in Stochastic Magnetic Fields

G. Domokos and S. Kovesi–Domokos

Dipartimento di Fisica, Università di Firenze
Florence, Italy

and

The Henry A. Rowland Department of Physics and Astronomy
The Johns Hopkins University
Baltimore, MD 21218

If massive neutrinos possess magnetic moments, they can undergo spin flip in a magnetic field. The magnetic fields needed for a meaningful measurement of neutrino moments could be very high and may occur in astronomical objects such as some supernovae or active galactic nuclei: they are typically chaotic ones. We develop the general theory of the passage of neutrinos through such fields. We also develop a simple model which becomes solvable in the high energy limit. Both helicities occur with equal probability, independently of the initial distribution. Observational consequences are discussed.

Keywords: neutrinos, high energy interactions
PACS: 13.10+q, 13.15+g, 13.35.Hb

1 Introduction

The existence and magnitude of magnetic moments of neutrinos is one of the important outstanding questions of particle physics. It is well known

1Permanent address. E–mail: SKD@HAAR.PHA.JHU.EDU
that in the standard model of electroweak interactions neutrinos are purely left handed; hence, they are massless and cannot possess magnetic moments either. However, in some minimal extensions of the standard model and also in some grand unified models, e.g. in the SO(10) model, there are both left and right handed neutrinos present. Therefore, there is the possibility giving them mass and also, chirality flipping (Pauli type) interactions with the electromagnetic field. In such models, higher order electroweak interactions give rise to anomalous magnetic moments via loops in which a W and a charged lepton is circulating. This was first noticed by Lee and Shrock [1] a long time ago. The magnetic moment thus arising, however, is a very small one, due, mainly, to the large mass of the W boson. It is given by the expression,

$$\mu_\nu = \frac{3G_F e m_\nu}{8\sqrt{2}\pi^2} \approx 3 \times 10^{-19} \frac{m_\nu}{1\text{eV}} \mu_B.$$  

Here $G_F$ stands for the Fermi coupling constant and $\mu_B$ is the Bohr magneton.

There exist fairly reliable upper limits on the masses of neutrinos; using those, the magnitudes of the moments predicted by this formula come out to be several orders of magnitude lower than the current observational upper limits, see, e.g. [2] for a recent review. (Current limits are typically around $10^{-12}\mu_B$ for flavor diagonal moments and a few orders of magnitude higher for transition moments.) As a consequence, any observation of a neutrino magnetic moment indicates the presence of some significant physics beyond the standard model. Using the observational upper limits on the moments, however, one concludes that such a measurement is likely to take place by means of a neutrino telescope. Indeed, the characteristic length associated with a magnetic dipole moment of magnitude $\mu$ placed in a magnetic field of strength $B$ is given by $L = 1/\mu B$. (Roughly speaking, this is the distance over which helicity flip occurs with a substantial probability.) In convenient units,

$$\frac{L}{km} \approx 3 \times 10^{-2} \frac{\mu_B \text{Gauss}}{\mu B}.$$  

For a magnetic moment of $10^{-12}\mu_B$ moving even in a field of 1 Megagauss, the characteristic length is about $3 \times 10^4$km. Clearly, no terrestrial experiment can be designed for such a measurement. By contrast, an active galactic nucleus (AGN) or a supernova, to quote but two examples, provides us with the right laboratory for measuring small magnetic moments either because of the high magnetic fields and/or by the large size of the object in question.

The trouble is that magnetic fields in astronomical objects are generally chaotic ones and the description of the motion of a magnetized neutrino in such an environment is hard to describe by the methods particle physicists
are used to. The main purpose of the present work is to develop the general formalism necessary for solving such a problem. Section 2 is devoted to the development of this formalism, while in the subsequent one (Sec. 3) we describe the relevant dynamics in front form. A solvable model (involving a reduced number of neutrino flavors) is described in Section 4. Finally, the results and their observational consequences are discussed in Sec. 5.

2 General Formalism

The problem of investigating the behavior of a microscopic system in a random environment is often faced in condensed matter physics. The classic example is, of course, Brownian motion, but similar problems occur in the statistical theory of turbulence and many other situations as well. Typically, such systems are modeled by an equation of motion of the form:

$$\partial_t X + F[X] = f,$$

(1)

where $X$ is an element of a vector space (the phase space of dynamical variables), the functional $F[.]$ is often, but not necessarily, local and time independent. Finally, $f$ represents a random force perturbing the otherwise deterministic system.

The modern theory of such systems was developed by Martin, Siggia and Rose [3] using a canonical formalism and by De Dominicis and Peliti [4] using a functional integral formalism. For the purpose of generalization to a relativistic system, the formalism developed in ref. [4] is more suitable.

There is an important difference, however, between systems described by an equation like (1) and ones we are interested in. In (1) the random perturbing force appears as an inhomogeneous term on the right hand side of the otherwise deterministic dynamical equation, whereas for a particle in a random magnetic field, the latter appears typically in the form $\mathbf{B} \cdot \Sigma \varphi$, where $\Sigma$ is a spin operator and $\varphi$ is an amplitude describing the propagation of the particle\(^2\). Hence, the formalism has to be modified in order to take this circumstance into account.

We use a Hamiltonian formalism; manifest covariance is of little value, since the rest frame of the astronomical object provides us with a preferred frame of reference. The Hamiltonian is taken to be of the form:

$$H = H_0 + \mu \mathbf{M} \cdot \mathbf{B},$$

(2)

\(^2\)In this work we are interested in neutral particles, hence interaction terms proportional to the electric charge are absent.
where $H_0$ governs the propagation of the particle in the absence of a magnetic field. The quantity $\mathbf{M}$ describes the coupling of the particle to the magnetic field and it may contain degrees of freedom other than just spin: e.g. various flavors a particle comes in may have different types of magnetic couplings. Correspondingly, $\varphi$ is a vector in the space of helicities and flavors. Finally, $\mu$ is the magnitude of some average of the magnetic moments involved: it carries the necessary dimension and the absence of a magnetic coupling can be described as the limit $\mu \to 0$.

The equation governing the propagation of the particle thus reads:

$$\partial_t \varphi + H_0 \varphi + \mu \mathbf{M} \cdot \mathbf{B} \varphi = 0$$

This equation is a deterministic one: there is no randomness involved. Assuming that one finds a solution of (3) for an arbitrary magnetic field and one is given the distribution of $\mathbf{B}$ over the ensemble of the magnetic fields of interest, one can determine the average behavior of the solution as well as the fluctuations around the average.

Formally, let $\varphi(x|\mathbf{B})$ denote the solution of (3) as a functional of the magnetic field, where $x$ stands for the spatial coordinates and the time. Assume further that one is given the characteristic functional, $Z[j]$ of the distribution of the magnetic fields over the ensemble. Then the average behavior of $\varphi$ is given by the expression:

$$\langle \varphi(x|\mathbf{B}) \rangle = \varphi \left(x \mid \frac{-i\delta}{\delta j} \right) Z[j]_{j=0}. \quad (4)$$

The fluctuations around the average are given by similar expressions involving higher functional derivatives.

Equation (4) can be useful provided two things are specified:

- The characteristic functional, $Z[j]$
- The solution of equation (3) for an “arbitrary” magnetic field.

It is clear that, for any system but the very simplest ones, the two items specified above cannot be given in a closed form. (For a simple system to be discussed in Section 4, one can give analytic expressions in the limit of high energies.) What one needs instead is a meaningful approximation scheme permitting the calculation of the quantities needed.

First of all, we notice that for any physically acceptable ensemble of magnetic fields, the characteristic functional can be written as a functional integral, viz.

$$Z[j] = \int \mathcal{D}\mathbf{B} \exp -S[\mathbf{B}] \times \exp i(\mathbf{j} \cdot \mathbf{B}). \quad (5)$$
Here $S$ stands for a generalized entropy functional and $\langle \ldots \rangle$ denotes integration over continuous variables and summation over discrete ones. The functional measure is normalized in such a way that $Z[0] = 1$. The generalized entropy has two important properties:

1. $S \geq 0$
2. $\partial_i \frac{\delta}{\delta j} Z[j] = 0$.

The first property follows from the fact that $Z$ is the characteristic functional of a probability distribution, the second one means that the magnetic field is solenoidal, as it should be. In general, the entropy functional is determined from solving Maxwell’s equations for some random current distribution; in the next section we shall consider a simple case of a static field. Thereafter, one has to decide on physical grounds the expression of the entropy functional.

The simplest, physically reasonable approximation is to consider a Gaussian distribution,

$$S = \frac{1}{2} \langle BC^{-1}B \rangle,$$

(6)

where $C$ is the correlation operator. In what follows, we shall assume a distribution of this type.

Next, we construct the generating functional for the moments of $\phi$. We have to average over the manifold of solutions of equation (3), so we have:

$$Z[u] = \int \mathcal{D}B \mathcal{D}\phi \exp \left\{ \frac{1}{2} \langle BC^{-1}B \rangle \exp i\langle u\phi \rangle \right\} \times \delta \left\{ i\partial_t \phi + H_0\phi + \mu B \cdot M\phi \right\} \text{Det} \left\{ i\partial_t + H_0 + \mu B \cdot M \right\},$$

(7)

where $u$ is the source of $\phi$. The determinant is inserted in order to get the correct measure on the manifold of solutions. One can use the integral representation of the $\delta$-functional, $\delta[\cdot]$ in order to cast (7) into a form to which techniques familiar from quantum field theory are directly applicable. We write:

$$\delta \left\{ i\partial_t \phi + H_0\phi + \mu B \cdot M\phi \right\} = \int \mathcal{D}\phi^\dagger \exp i\langle \phi^\dagger \left( i\partial_t \phi + H_0\phi + \mu B \cdot M\phi \right) \rangle$$

(8)

It is to be noted that in (8), $\phi^\dagger$ is not the Hermitean conjugate of $\phi$, but an independent functional argument; the notation is used because of the transformation properties of $\phi^\dagger$ under any symmetry group the argument of the delta–functional may have. Using (8) the generating functional for the
moments of $\varphi$ can be brought to a form familiar from quantum field theory, viz.

\[
Z[u, u^\dagger] = \int DB D\varphi D\varphi^\dagger \exp \left(-\frac{1}{2} (BC^{-1} B)\right) \times \exp \left(i \langle (\varphi^\dagger u + u^\dagger \varphi) \rangle\right) \times \exp \left(i \langle \varphi^\dagger (i\partial_t \varphi + H_0 \varphi + \mu B \cdot M \varphi) \rangle\right) \times \text{Det} \left(i\partial_t + H_0 + \mu B \cdot M\right).
\]

(We introduced a source for $\varphi^\dagger$ as well, in order to be able to generate Green’s functions.)

Equation (9) is recognized as being formally equivalent to a quantum field theory for the variables $\varphi$ and $\varphi^\dagger$, the variable $B$ merely serving as a mediator of interactions. The functional determinant can be handled by familiar techniques, typically, by introducing Fadeev–Popov ghost fields. The Gaussian integration over $B$ can be carried out explicitly; however, this may not be the most desirable form of (9).

We conclude that the problem of describing the propagation of a neutral particle (a neutrino in particular) in a random magnetic field can be handled in a way familiar in quantum field theory. Familiar approximation methods, such as loop expansions, etc. are readily applicable to this problem as they are to any quantum field theory.

So far, we have not specified the form of the dynamics entering the preceding equations. This is an important question: casting the dynamics into an appropriate form leads to significant simplifications.

3 Dynamics in Front Form.

We use the front form of dynamics [6]. As explained in a previous work [7], this formulation of dynamics is advantageous in a situation in which one considers the propagation of high energy particles ($E \gg m$, where $m$ is the rest mass) and in which certain discrete symmetries, such as $C$ and $P$ play no significant role. Clearly, the propagation of high energy neutrinos falls into this category. In what follows, we assume an arbitrary flavor structure: all physical observables (masses, magnetic moments, etc.) are matrices in flavor space. However, flavor indices need not be explicitly exhibited.

We begin with the usual Dirac Lagrangian of a particle in an external electromagnetic field, $F_{\mu\nu}$:

\[
L = \overline{\psi} \left(\gamma^\mu \partial_\mu + m + \frac{1}{2} \mu F^{\mu\nu} \sigma_{\mu\nu}\right) \psi
\]
We work in the rest frame of the magnetic field. Assuming the field to be a static one, we can set $F_{0i} = 0, F_{ij} = \epsilon_{ijk}B_k$. In the case of interest one has to solve the Dirac equation in an arbitrary static magnetic field, since we want to average the solution over an ensemble of the $B_i$. No explicit solution is known for such a problem. However, we proceed to show that in the high energy limit the problem can be solved in a closed form.

We introduce a coordinate system in which two of the coordinates are null directions corresponding to characteristic lines of a relativistic wave equation, viz.:

$$
t = \frac{1}{\sqrt{2}}(x^0 - x^3), \quad z = \frac{1}{\sqrt{2}}(x^0 + x^3) \quad \text{and} \quad x^A; \quad (A = 1, 2). \tag{11}
$$

Correspondingly, the metric is of the form,

$$
g_{zt} = g_{tz} = 1, \quad g_{AB} = -\delta_{AB}, \tag{12}
$$

and all other components vanish. The momentum components conjugate to $z$ and $t$ are,

$$
k \sim p_3 \sqrt{2}, \quad h \sim \frac{m^2 + \vec{p}^2}{2k} \quad (p_3 \to \infty, \quad \vec{p} \text{ finite}), \tag{13}
$$

respectively. In this equation, $\vec{p}$ is the momentum transverse to $p_3$.

A Dirac spinor can be decomposed along the null directions given in (11) by introducing the mutually orthogonal projectors,

$$
\begin{align*}
P_t &= \frac{1}{2}\gamma^t\gamma^t, \\
P_z &= \frac{1}{2}\gamma^z\gamma^z
\end{align*} \tag{14}
$$

In what follows, we use the shorthand,

$$
\phi = P_t\psi, \quad \chi = P_z\psi \tag{15}
$$

It is a straightforward matter to decompose according to the conjugate null directions and express it in terms of the variables $\phi$ and $\chi$. The purpose of such an exercise is a very simple one. If, for the sake of definiteness, $t$ is regarded the ”time” variable describing the dynamics of the system, only $\phi$ obeys an equation containing $\partial_t$. Hence, the conjugate component of the Dirac spinor obeys only an equation of constraint. The constraint can be, in turn, solved before one attempts to attack the problem of dynamics.

\footnote{In physical terms, this means that the characteristic time scale of change of the field is large compared to the time of passage of neutrinos. It is a straightforward matter to generalize the formalism for arbitrary electromagnetic fields.}
After carrying out the decomposition of (10) according to the null direc-
tions, one finds:

\[
L = \sqrt{2} \left[ \phi^\dagger \left( i\partial_t - i\sqrt{2}\mu\epsilon^{AB}\gamma_A B_B \right) \phi \right. \\
+ \chi^\dagger \left( i\partial_z - i\sqrt{2}\mu\epsilon^{AB}\gamma_A B_B \right) \chi \\
+ \frac{1}{\sqrt{2}} \left[ \phi^\dagger \gamma^z \left( i\gamma^A \partial_A + m - \frac{i}{\sqrt{2}}\mu B_3 \epsilon_{ABC} \gamma^A \gamma^B \right) \phi \right. \\
+ \chi^\dagger \gamma^t \left( i\gamma^A \partial_A + m - \frac{i}{\sqrt{2}}\mu B_3 \epsilon_{ABC} \gamma^A \gamma^B \right) \phi \left. \right]\]  
(16)

Variation of (16) with respect to \( \chi^\dagger \) gives the constraint:

\[
\left( i\sqrt{2}\partial_z - 2i\mu\epsilon^{AB}\gamma_A B_B \right) \chi + \frac{1}{\sqrt{2}} \gamma^t \left( i\gamma^A \partial_A + \frac{i}{\sqrt{2}}\mu \epsilon_{ABC} \gamma^A \gamma^B B_3 + m \right) \phi = 0. 
\]  
(17)

The solution of (17) with the correct Hermiticity properties, cf. [7] reads:

\[
\chi(z) = -\int dz' \langle z|\Omega \left( B^A \right) |z'\rangle \frac{1}{\sqrt{2}} \gamma^t \left( i\gamma^A \partial_A + m - i\mu \frac{1}{\sqrt{2}} B_3 \epsilon_{ABC} \gamma^A \gamma^B \right) \phi (z') 
\]  
(18)

Here all arguments in which (13) is local, \( i.e. \ x^A \ \text{and} \ t \) have been suppressed. The matrix element of the operator \( \Omega \) is given by:

\[
\langle z|\Omega \left( B^A \right) |z'\rangle = \frac{i}{\sqrt{2}} \exp \left( \mu \sqrt{2} \int_{z'}^{z''} \epsilon_{ABC} \gamma^A B^B (z'') \right) \frac{1}{2} \epsilon (z - z') 
\]  
(19)

Let us notice that solving the constraint eliminates two components of the original, four component Dirac spinor. Therefore, instead of the original Dirac matrices one can go over to \( 2 \times 2 \) Pauli matrices. One easily verifies that \( -i\epsilon^{AB} \gamma_B \rightarrow \sigma^A \) gives the correct representation. We also introduce the Hermitean operator, \( p_A = -i\partial_A \) for the transverse degrees of freedom.

It is now a matter of straightforward algebra to use (18) in order to eliminate the variable \( \chi \) from the Lagrangian. We merely quote the result; it can be conveniently written in Hamiltonian form as follows.

\[
L = \pi \partial_t \phi - H \\
H = -2\mu \phi^\dagger \sigma^A B_A \phi \\
+ \phi^\dagger \left( -i\sigma_B \epsilon^{BC} p_C + m - \mu \sqrt{2} B_3 \sigma_3 \right) \\
\times \Omega \left( -B^A \right) \\
\times \left( i\sigma_R \epsilon^{RS} p_S + m - \mu \sqrt{2} B_3 \sigma_3 \right) \phi \]  
(20)
The canonical momentum is given by $\pi = i\sqrt{2} \phi^\dagger$. (Of course, the odd looking factor of $\sqrt{2}$ in the definition of the canonical momentum can be eliminated by rescaling the time variable.)

In (20) we omitted all the symbols of integration over the longitudinal coordinate, $z$; terms in which the operator $\Omega$ does not appear explicitly are local in all variables.

The Hamiltonian appearing in (20) is exact. However, it is given by a rather complicated, non local and non linear expression: this is the cost we have to pay for explicitly eliminating the constraint. We now argue that one can introduce physically reasonable simplifications, as a result of which the problem becomes a manageable one. We notice that the exponential appearing in (19), regarded as a matrix in the space of spinors is unitary, hence all matrix elements are bounded. Therefore, one expects that at large values of $|z - z'|$ the exponent oscillates rapidly and contributes little to the Hamiltonian. The dominant contribution is, probably, coming from small values of the difference of longitudinal coordinates. Hence, it appears reasonable to approximate the exponential in (19) by 1. It is to be emphasized, however that the linearization of the effective Hamiltonian in the magnetic field is not necessarily a harmless approximation. Many instances are known when a similar truncation of a dynamical system leads to the loss of important physics. The validity of this approximation therefore needs further study.

We now observe that, after having made the approximation just discussed, there remain three types of terms in (20):

1. The local term, $\propto \sigma^A B_A$: this is purely a helicity flip term in terms of the lightlike helicity defined by $\sigma_3$.

2. Transverse spin–orbit coupling terms, $\propto \sigma_A \epsilon^{AB} p_B$: these terms also give rise to the transverse kinetic energy term, since $\sigma^A \epsilon_{AB}^B \sigma_{RS} p_S = p_A p^A$.

3. “Mass” terms, $\propto \lambda_{eff} = m - \mu \sqrt{2} \sigma_3 B^3$. In this picture, the longitudinal component of the magnetic field plays the same role as the mass does. One has to recall in particular that, in the case of Majorana neutrinos, the two helicity states do not necessarily have the same mass.

### 4 A Solvable Model

We now turn to the description of a solvable model. Two further approximations are made in order to accomplish this. First, we consider a model with
a reduced number of flavors: we suppress the flavor matrix structure in preceding equations altogether. This means that in the case of Dirac neutrinos we are dealing with a single flavor of two helicities, one corresponding to an active, the other one to a sterile neutrino. In the case of Majorana neutrinos we have genuinely two flavors. One hopes that, despite the reduction in the number of flavors, the qualitative insight gained by solving the model will facilitate the study of more realistic models. Second, we consider the high energy limit of the theory developed so far: this is particularly easy in the front form of the dynamics.

We observe that the terms in items 2) and 3) in the preceding Section are proportional to $\epsilon(z - z')$. Hence, in a Fourier representation, viz. upon writing

$$\phi(t, z, x^A) = \int dk \tilde{\phi}(t, k, x^A) \exp(-ikz) \quad (21)$$

and

$$\epsilon(z) = \frac{P}{2\pi i} \int \frac{dk}{k} \exp(-ikz) \quad (22)$$

one finds that at high energies ($k \gg m$) the terms in items 2) and 3) are negligible with respect the local term in item 1). (In the last equation $P$ stands for the principal value.)

Thus, if one is interested in high energy neutrinos, the Hamiltonian can be approximated by a local one, given by the first term in $(20)$.

For this reason, it is convenient to work in a coordinate representation for the density matrix. The equation of motion then reads:

$$-i\partial_t \langle z, \vec{x} | \rho(t) | z', \vec{x}' \rangle = \mu \sqrt{2} \vec{\sigma} \cdot \vec{B} \left( \vec{x}, \frac{z - t}{\sqrt{2}} \right) \langle z, \vec{x} | \rho(t) | z', \vec{x}' \rangle$$

$$- \mu \sqrt{2} \langle z, \vec{x} | \rho(t) | z', \vec{x}' \rangle \vec{\sigma} \cdot \vec{B} \left( \vec{x}', \frac{z' - t}{\sqrt{2}} \right) \quad (23)$$

In this equation, $\vec{x}$ stands for the transverse part of the coordinate and $\vec{\sigma} \cdot \vec{B}$ is the two dimensional scalar product in transverse space. Of course, the coordinate $x^3$ had to be expressed by $z$ and $t$; hence the $t$-dependence in the magnetic field.

We choose the initial condition so as to describe a neutrino produced at $\vec{x} = 0$ and with a fixed value of $k$:

$$\langle z, \vec{x} | \rho(0) | z', \vec{x}' \rangle = \delta^2(\vec{x}) \delta^2(\vec{x}') \frac{\exp ik(z - z')}{2\pi k} \rho_s(0), \quad (24)$$

where $\rho_s(0)$ is the initial value of the spin density matrix.

The variable $k$ being large, the function $\exp ik(z - z')$ is rapidly oscillating unless $z \approx z'$: it is permissible to put $z = z'$ in the coefficient of the
exponential in (24). Further, in the approximation used, the dynamics described by eq. (23) is independent of $k$ and of $\vec{x}$. Therefore, the dependence of $\rho(t)$ on these variables is entirely determined by the initial condition and the dynamical equation reduces to an equation involving the spin density matrix alone, as in non relativistic spin dynamics. From now on, we omit the the subscript $s$ and we have:

$$-i\partial_t \rho(t) = \mu \sqrt{2} \left[ \vec{\sigma} \cdot \vec{B} \left( \frac{z-t}{\sqrt{2}} \right), \rho(t) \right]$$

(Here and in what follows, $\vec{x} = 0$ is understood.)

This equation can be solved by the standard time ordered series, viz.

$$\rho(t) = \rho(0) + i\mu \sqrt{2} \int_0^t dt' \left[ \vec{\sigma} \cdot \vec{B} \left( \frac{z-t'}{\sqrt{2}} \right), \rho(0) \right] + \frac{(i\mu \sqrt{2})^2}{2!} \int_0^t dt' dt'' T \left( \left[ \vec{\sigma} \cdot \vec{B} \left( \frac{z-t'}{\sqrt{2}} \right), \left[ \vec{\sigma} \cdot \vec{B} \left( \frac{z-t''}{\sqrt{2}} \right), \rho(0) \right] \right) \right) + \cdots$$

(26)

We choose the initial condition as:

$$\rho(0) = \frac{1}{2} \left( 1 + S \sigma_3 \right), \quad (S^2 \leq 1),$$

(27)

since neutrinos are produced with a definite helicity. (In the case of Dirac neutrinos, $S = \pm 1$, depending on whether a neutrino or anti neutrino is produced. In the case of Majorana neutrinos, $S$ may assume any value between the limits stated above, depending on the production mechanism.)

Next, we average the solution, (26) over the magnetic field. We choose the generating functional of the moments as follows:

$$Z[j] = \int \mathcal{D}B \exp \left[ \frac{1}{2} \int \left. d^3x d^3x' B_i(x) C_{ij}^{-1}(x,x') B_j(x') \right] \times \exp \int d^3x j_i(x) B_i(x); \right.$$  

$$C_{ij}^{-1} = \frac{L}{4\pi \langle B^2 \rangle} \left( \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \left( L^{-2} - \nabla^2 \right)^2 \delta^3(x-x').$$

(28)

In the last equation, $L$ and $\langle B^2 \rangle$ stand for the correlation length and mean square magnetic field, respectively. The measure is normalized such that $Z[0] = 1$. The transverse projector is needed in order to make the correlation
functions solenoidal. With the choice of the tensor $C^{-1}$ given in (28), the leading term in the long distance behavior of the correlation function is $\propto \exp -|x - x'|$. In order to average equation (26) over the magnetic field, one integrates over $B_3$ and sets the third component of the source equal to zero. The transverse generating functional reads:

$$Z_T = \int \mathcal{D}\vec{B} \exp -\frac{L}{8\pi \langle B^2 \rangle} \int d^3x \left[ B^A(x) \left( \delta_{AB} - \frac{1}{2} \frac{\partial_A \partial_B}{\nabla^2} \right) B^B \right]$$

$$\times \exp i \int d^3x \vec{j}(x) \cdot \vec{B}(x)$$

(29)

We now notice that in the equation (26), terms containing odd powers of $\mu$ are also odd in $B^A$. Therefore, in the limit $\vec{j} \to 0$ the average of those terms vanishes. The even terms in the series are obtained by taking the appropriate functional derivatives of (29). All of them are expressed in terms of multiple time integrals of $C_{ij}(|t - t'|)$ and its powers: those integrations are easily performed. It is sufficient to illustrate the procedure for the second order term in (26).

Carrying out the integrations, one gets:

$$-\mu^2 \frac{1}{2} S \left\langle \int_0^t dt' dt'' \left[ \vec{\sigma} \cdot \vec{B}, \left[ \vec{\sigma} \cdot \vec{B}, \sigma_3 \right] \right] \right\rangle = -\mu^2 \sigma_3 \langle B^2 \rangle t L \left( 1 - \exp -\frac{t}{L} \right)$$

For large times the result in the last equation is just proportional to $t$. The higher order terms follow a similar pattern. The end result is:

$$\langle \rho(t) \rangle \sim \frac{1}{2} \left( 1 + S \sigma_3 \exp -\frac{t}{T} \right),$$

(30)

with

$$\frac{1}{T} = 2\mu^2 \langle B^2 \rangle L.$$  (31)

5 Discussion

The solvable model described in the last section leads to the remarkable result that in the random field the behavior of the helicities is an ergodic one: irrespective of what the initial density matrix was, for $t \gg T$, the helicities are equally distributed. In the case of Dirac neutrinos, this is rather uninteresting: roughly 1/2 of them is a sterile one. Unfortunately, however, calculated neutrino fluxes emerging from such astrophysical objects as an AGN usually cannot be trusted to an accuracy which would permit an observational testing of the result. In the case of Majorana neutrinos, however,
the two helicity states correspond to two different flavors. Given the fact that the neutrinos produced arise mostly from pion decay, the presence of the other flavor in roughly equal proportion is an observationally testable result. We conjecture that the situation is similar if all neutrino flavors are properly taken into account. Should this conjecture be verified by future calculations, one would have a very important observational prediction. If neutrinos are Majorana particles and they possess substantial (say, $10^{-12} \mu_B$) transition moments, the flavor distribution of neutrino events observed in a high energy neutrino telescope would be practically uniform. For this reason it would be of utmost importance to

- observe high energy neutrinos emerging from point sources,
- develop flavor sensitive detection techniques in neutrino telescopes.

We remark that the question of flavor change in the case of magnetized Majorana neutrinos can be investigated in terrestrial neutrino experiments as well, see [8]. The two types of experiments referred to above are complementary to each other. If both types indicate the presence of flavor flipping, one would have a window on post standard model physics as well as the structure of important point sources of high energy neutrinos.

From the theoretical point of view, the main result of this work is the development of a consistent formalism in order to treat the propagation of high energy neutrinos in a random magnetic field (more generally, in a random environment). This problem has been treated before, see for instance, ref. [3] and the virtually complete bibliography quoted there. (The authors quoted in those references concentrate mostly on the early universe and the role of sterile neutrinos in it.) The main advantage of the formalism developed in refs. [3, 4] and adapted to highly relativistic systems in this paper is twofold: it is guaranteed to be free of internal inconsistencies and it allows the development of controllable approximation methods.

A final comment is in order. Assuming that one observes flavor conversion in some AGN, one may ask whether the theory developed here can be used to test some properties of the magnetic fields in the source. We argue that the answer is in the affirmative to some extent. The measurable quantity characterizing the magnetic field is the coefficient given by equation (31). One notices that this expression contains the quantities characterizing the

---

4In most of the works quoted, an average is taken at the level of the evolution equations rather than their solutions. It has been known for some time that averaging the evolution equations over a random variable is an incomplete procedure (important physics may be missed). It may also lead to internal contradictions. For a lucid exposition, see [10]. A notable exception is ref. [11], where a simplified equation is given a correct treatment.
magnetic field only in the combination $\mu^2 \langle B \rangle^2 L$. This is not an accident. There is an important class of distributions of the magnetic field characterized by the following:

1. The magnetic field has a zero mean value,

2. The distribution is approximately Gaussian (higher cumulants are approximately zero),

3. the distribution contains only one length scale.

Every such distribution leads to a damping coefficient of a form which can differ from (31) only in a numerical factor. In essence, this follows from dimensional analysis.

In fact, the magnetic moment and the magnetic field strength enter the Hamiltonian in the combination $\mu B$; since $\langle B \rangle = 0$, this combination must enter a physical observable quadratically. Finally, in order to form a quantity of inverse length (the damping coefficient), the expression has to be multiplied by a characteristic length in the distribution.

As another example, Enquist et. al., [12] derived a damping coefficient of this form, even though they use a field distribution quite different from ours. (The distribution in [12] has a vanishing correlation length, but a finite domain size.) Thus measuring such quantities as the damping coefficient can only yield information about the class of the magnetic field distribution, not about its details.

**Acknowledgement**

This work was performed during the authors’ visit at the Dipartimento di Fisica, Università di Firenze. We wish to thank Roberto Casalbuoni, Director of the Department for the hospitality extended to us. We thank Bianca Monteleoni for useful conversations on observational neutrino astrophysics and Kari Enqvist for some useful critical remarks.

**References**

[1] B.W. Lee and R.E. Shrock, Phys. Rev. D16, 1444 (1977); W.J. Marciano and A.I. Sanda, Phys. Lett. 67B, 303 (1977).

[2] C.W. Kim and A.Pevsner, “Neutrinos in Physics and Astrophysics”. Harwood Academic Publishers, Chur 1993; Ch. 11

---

5We thank Kari Enquist for pointing out the similarity of the damping coefficients in refs. [3] and [12].
[3] P.C. Martin, E.D. Siggia and H.A. Rose, Phys.Rev. A8, 423, (1973).

[4] C. De Dominicis and L. Peliti, Phys. Rev. B18, 353, (1978).

[5] G. Domokos and S. Kovesi-Domokos, preprint hep-ph/9703265.

[6] P.A.M. Dirac, Rev. Mod. Phys. 21 (1949) 392

[7] G. Domokos and S. Kovesi–Domokos, “High Energy Neutrino Interactions: Single Particle Theory” Jour. Phys. G, to be published.

[8] G. Domokos and S. Kovesi-Domokos, Phys. Rev. D55, 2526 (1997).

[9] P. Elmfors, K. Enqvist, G. Raffelt and G. Sigl, preprint hep-ph/9703214 (1997).

[10] N.G. van Kampen, “Stochastic Processes in Physics and Chemistry”. North Holland, Amsterdam 1982, Ch. XIV.

[11] A. Nicolaidis, Physics Letters B262, 303 (1991).

[12] K. Enquist, A.I. Rez and V.B. Semikhoz, Nuclear Physics B436, 49 (1995)