HITCHIN’S CONJECTURE FOR SIMPLY-LACED LIE ALGEBRAS IMPLIES THAT FOR ANY SIMPLE LIE ALGEBRA

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Abstract. Let \( g \) be any simple Lie algebra over \( \mathbb{C} \). Recall that there exists an embedding of \( \mathfrak{sl}_2 \) into \( g \), called a principal TDS, passing through a principal nilpotent element of \( g \) and uniquely determined up to conjugation. Moreover, \( \wedge^d(g^\ast)^g \) is freely generated (in the super-graded sense) by primitive elements \( \omega_1, \ldots, \omega_\ell \), where \( \ell \) is the rank of \( g \). N. Hitchin conjectured that for any primitive element \( \omega \in \wedge^d(g^\ast)^g \), there exists an irreducible \( \mathfrak{sl}_2 \)-submodule \( V_\omega \subset g \) of dimension \( d \) such that \( \omega \) is non-zero on the line \( \wedge^d(V_\omega) \). We prove that the validity of this conjecture for simple simply-laced Lie algebras implies its validity for any simple Lie algebra.

1. Introduction

Let \( g \) be a finite dimensional simple Lie algebra over the complex numbers \( \mathbb{C} \) with the associated connected, simply-connected complex algebraic group \( G \). Recall that there is a unique (up to conjugation) embedding of \( \mathfrak{sl}_2 \) into \( g \), called a principal TDS, such that the image passes through a principal nilpotent element of \( g \). Under the adjoint action of a principal TDS, the Lie algebra \( g \) decomposes as a direct sum of exactly \( \ell \) irreducible \( \mathfrak{sl}_2 \)-submodules \( V_1, \ldots, V_\ell \) of dimensions \( 2m_1 + 1, \ldots, 2m_\ell + 1 \) respectively, where \( \ell \) is the rank of \( g \) and \( m_1, \ldots, m_\ell \) are the exponents of \( g \).

Further, the singular cohomology \( H^*(G) = H^*(G, \mathbb{C}) \) with complex coefficients is a Hopf algebra. Let \( P(g) \subset H^*(G) \) be the graded subspace of primitive elements. Then, \( P(g) \) has a basis in degrees \( 2m_1 + 1, \ldots, 2m_\ell + 1 \). We identify \( H^*(G) \) with \( \wedge((g^\ast)^g) \) and consider \( P(g) \) as a subspace of \( \wedge((g^\ast)^g) \).

Now, N. Hitchin made the following conjecture [Hi]

Conjecture 1.1. Let \( g \) be any simple Lie algebra. For any primitive element \( \omega \in P_d \subset \wedge^d(g^\ast)^g \), there exists an irreducible subspace \( V_\omega \subset g \) of dimension \( d \) with respect to the principal TDS action such that

\[ \omega|_{\wedge^d(V_\omega)} \neq 0. \]

The main motivation for Hitchin behind the above conjecture lies in its connection with the study of polyvector fields on the moduli space \( M_G(\Sigma) \) of semistable principal \( G \)-bundles on a smooth projective curve \( \Sigma \) of any genus \( g > 2 \). Specifically, observe that the cotangent space
at a smooth point $E$ of $M_G(\Sigma)$ is isomorphic with $H^0(\Sigma, g(E) \otimes \Omega)$, where $g(E)$ denotes the associated adjoint bundle and $\Omega$ is the canonical bundle of the curve $\Sigma$. Given a bi-invariant differential form $\omega$ of degree $k$ on $G$, i.e., $\omega \in \wedge^k(g^*)^g$, and elements $\Phi_j \in H^0(\Sigma, g(E) \otimes \Omega), 1 \leq j \leq k, \omega(\Phi_1, \ldots, \Phi_k)$ defines a skew form with values in the line bundle $\Omega^k$. Dually, it defines a homomorphism

$$\Theta_\omega : H^1(\Sigma, \Omega^{1-k}) \to H^0(M_G(\Sigma), \wedge^k T),$$

where $T$ is the tangent bundle of $M_G(\Sigma)$.

Now, as shown by Hitchin, the validity of the above conjecture would imply that the map $\Theta_\omega$ is injective for any invariant form $\omega \in \wedge^k(g^*)^g$ (cf. [Hi]).

Any simple Lie algebra $\mathfrak{k}$ can be realized as the fixed point subalgebra of a diagram automorphism of an appropriate simple simply-laced Lie algebra $g$. We prove that the validity of the conjecture for $g$ implies the validity for $\mathfrak{k}$. Thus, one needs to verify the conjecture only for the simple Lie algebras of types $A, D$ and $E$. Specifically, we have the following result (cf. Theorem 2.5).

**Theorem 1.2.** If Hitchin’s conjecture is valid for any simply-laced simple Lie algebra $g$, then it is valid for any simple Lie algebra.

More precisely, if Hitchin’s conjecture is valid for $g$ of type $(A_{2\ell-1}; A_{2\ell}; D_{4}; E_6)$, then it is valid for $g$ of type $(G_\ell; B_\ell; G_2; F_4)$ respectively.

The proof relies on constructing a principal TDS in $\mathfrak{k}$ which remains a principal TDS in $g$. Moreover, we need to use the surjectivity of the space of primitive elements $P(g) \to P(\mathfrak{k})$, which allows us to lift primitive elements $\omega_\mathfrak{k} \in \wedge^d(\mathfrak{k}^*)^\mathfrak{k}$ to primitive elements $\bar{\omega}_g \in \wedge^d(g^*)^g$.

Let $K$ be the algebraic subgroup of $G$ with Lie algebra $\mathfrak{k}$, where $\mathfrak{k}$ is the fixed subalgebra under a diagram automorphism of a simple simply-laced Lie algebra $g$. Our next main result of the paper (cf. Theorem 3.1) asserts the following.

**Theorem 1.3.** The canonical map $\phi : R(G) \to R(K)$ is surjective, where $R(G)$ denotes the representation ring of $G$ (over $\mathbb{Z}$).

In particular, the canonical restriction map $\psi : S^\bullet(g^*)^g \to S^\bullet(\mathfrak{k}^*)^\mathfrak{k}$ is surjective.

Finally, we use H. Cartan’s transgression map and the surjectivity of $\psi$ to obtain the desired surjectivity of $\gamma_o : P(g) \to P(\mathfrak{k})$ and thereby the surjectivity of $\gamma : H^*(G) \to H^*(K)$ (cf. Theorem 3.5). In our view, the surjectivity of $\phi, \gamma$ and $\gamma_o$ is of independent interest.

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2. **Reduction of Hitchin’s Conjecture to Simply-laced Lie Algebras**

Let $g$ be a simple Lie algebra over $\mathbb{C}$ with the associated connected simply-connected complex algebraic group $G$ (with Lie algebra $g$).

**Definition 2.1.** A Lie algebra embedding $\varphi : \mathfrak{sl}_2 \to g$ (or its image) is called a principal TDS if $\varphi(X)$ is a principal nilpotent element of $g$, i.e., $\text{Ad} G \cdot \varphi(X)$ is the open orbit in the nilpotent cone $\mathcal{N}$ of $g$. 
Here, \( \mathfrak{sl}_2 \) is the Lie algebra of traceless \( 2 \times 2 \) matrices over \( \mathbb{C} \) with the standard basis
\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let \( \varphi' : \mathfrak{sl}_2 \to \mathfrak{g} \) be another principal TDS. Then, by a result of Kostant [Ko], Corollary 3.7, \( \varphi' \) is conjugate to \( \varphi \), i.e., there exists a \( g \in G \) such that
\[
(1) \quad \varphi' = \text{Ad} g \cdot \varphi.
\]

Decompose the adjoint representation of \( \mathfrak{g} \) with respect to a principal TDS \( \varphi \) into irreducible components:
\[
\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell,
\]
labeling them so that
\[
(2) \quad n_1 \leq \cdots \leq n_\ell, \quad \text{where} \quad n_i = \dim V_i.
\]
Then, it is known (cf. [Ko], Corollary 8.7) that
(a) \( \ell = \text{rank of} \ \mathfrak{g} \).
(b) Each \( n_i \) is an odd integer \( 2m_i + 1 \). Moreover,
\[
m_1 \leq m_2 \leq \cdots \leq m_\ell
\]
are the exponents of \( \mathfrak{g} \). (The list of exponents for any \( \mathfrak{g} \) can be found in [Bo], Planche I - IX.)
(c) Except when \( \mathfrak{g} \) is of type \( D_\ell \) (with \( \ell \) even), each \( V_i \) is an isotypical component (in particular, uniquely determined) for the principal TDS \( \varphi \), i.e., \( m_1 < m_2 < \cdots < m_\ell \).

When \( \mathfrak{g} \) is of type \( D_\ell \) (with \( \ell \) even), the exponents are:
\[
1, 3, 5, \cdots , \ell - 3, \ell - 1, \ell - 1, \ell + 1, \cdots , 2\ell - 3.
\]
Hence, the isotypical component for the highest weight \( 2\ell - 2 \) is a direct sum of two copies of the irreducible module \( V_{\mathfrak{sl}_2}(2\ell - 2) \) with highest weight \( 2\ell - 2 \).

By the identity (1), we see that the decomposition of \( \mathfrak{g} \) with respect to another principal TDS \( \varphi' \) looks like
\[
(3) \quad \mathfrak{g} = (\text{Ad} g \cdot V_1) \oplus (\text{Ad} g \cdot V_2) \oplus \cdots \oplus (\text{Ad} g \cdot V_\ell).
\]

**Definition 2.2.** Recall that the singular cohomology with complex coefficients \( H^*(G) = H^*(G, \mathbb{C}) \) is a Hopf algebra, where the product of course comes from the cup product, and the coproduct \( \Delta : H^*(G) \to H^*(G) \otimes H^*(G) \) is induced from the multiplication map \( \mu : G \times G \to G \).

Let \( P = P(\mathfrak{g}) \subset H^*(G) \) be the subspace of primitive elements, i.e.,
\[
P = \{ x \in H^*(G) \mid \Delta(x) = x \otimes 1 + 1 \otimes x \}.
\]
(Observe that \( H^*(G) \) does not depend upon the isogeny class of \( G \) and hence the notation \( P(\mathfrak{g}) \) is justified.)

Since \( \Delta \) is a graded homomorphism, \( P \subset H^*(G) \) is a graded linear subspace. It is well-known that, by a result of Hopf-Koszul-Samelson, \( P \) is concentrated in odd degrees and, moreover, the canonical map, induced from the product,
\[
\theta : \Lambda^*(P) \to H^*(G)
\]
is a graded algebra isomorphism. In particular, \( P \) generates \( H^*(G) \) as an algebra over \( \mathbb{C} \).
We can think of $\wedge(\mathfrak{g}^*)$ as the algebra of left invariant $\mathbb{C}$-valued forms on a maximal compact subgroup $G_o$ of $G$. By a result of Koszul ([K], Théorème 9.2, Chapitre IV), any $\omega \in \wedge(\mathfrak{g}^*)^\theta$ is a closed form and, moreover, the induced map (identifying $H^*(G_o)$ with the de Rham cohomology $H^*_{dR}(G_o, \mathbb{C})$)

$$
\eta : \wedge(\mathfrak{g}^*)^\theta \cong H^*(G_o) \cong H^*(G)
$$

is a graded algebra isomorphism, where the restriction map $H^*(G) \to H^*(G_o)$ is an isomorphism since $G_o$ is a deformation retract of $G$.

Via the isomorphism $\eta$, we identify the graded subspace $P \subset H^*(G)$ of primitive elements with a graded subspace (still denoted by) $P \subset \wedge(\mathfrak{g}^*)^\theta$.

For any $d \geq 1$, let $P_d$ be the subspace of $P$ of (homogeneous) degree $d$ elements. Then, by [Ko], Corollary 8.7,

$$
\dim P_d = \#\{1 \leq i \leq \ell \mid n_i = d\},
$$

where $n_i$'s (given by (2)) are the dimensions of irreducible components of $\mathfrak{g}$ under the principal TDS action.

In particular, if $\mathfrak{g}$ is not of type $D_\ell$ (with $\ell$ even), then

$$
\dim P_d \leq 1
$$

and $P_d$ is of dimension 1 if and only if $d$ is equal to one of the $n_i$'s. If $\mathfrak{g}$ is of type $D_\ell$ (with $\ell$ even),

$$
\dim P_d \leq 1 \text{ if } d \neq 2\ell - 1, \text{ and } \dim P_{2\ell-1} = 2.
$$

Fix a principal TDS. Hitchin made the following conjecture (cf. [Hi]).

**Conjecture 2.3.** Let $\mathfrak{g}$ be any simple Lie algebra. For any primitive element $\omega \in P_d \subset \wedge^d(\mathfrak{g}^*)^\theta$, there exists an irreducible subspace $V_\omega \subset \mathfrak{g}$ of dimension $d$ with respect to the principal TDS action such that

$$
\omega|_{\wedge^d(V_\omega)} \neq 0.
$$

**Remark 2.4.** (a) Unless $\mathfrak{g}$ is of type $D_\ell$ (with $\ell$ even), given $\omega \in P_d$, there exists a unique irreducible submodule $V$ of dimension $d$ in $\mathfrak{g}$ with respect to the principal TDS. Thus, $V_\omega$ is uniquely determined.

If $\mathfrak{g}$ is of type $D_\ell$ (with $\ell$ even), unless $d = 2\ell - 1$, given $\omega \in P_d$, there is a unique irreducible submodule $V$ of dimension $d$ in $\mathfrak{g}$. Thus, again $V_\omega$ is uniquely determined (for $d \neq 2\ell - 1$).

(b) A different choice of principal TDS results in the irreducible submodules being equal to $\text{Ad}_g \cdot V$, for some $g \in G$, and some irreducible submodule $V$ for the original principal TDS. But, since we are only considering forms $\omega \in \wedge^d(\mathfrak{g}^*)^\theta$ (which are, by definition, Ad$G$-invariant), $\omega|_{\wedge^d(\text{Ad}_g \cdot V)} \neq 0$ if and only if $\omega|_{\wedge^d(V)} \neq 0$.

Now, we come to the main result of this section.

**Theorem 2.5.** If Hitchin’s conjecture is valid for any simply-laced simple Lie algebra $\mathfrak{g}$, then it is valid for any simple Lie algebra.

More precisely, if Hitchin’s conjecture is valid for $\mathfrak{g}$ of type $(A_{2\ell-1}; A_{2\ell}; D_4; E_6)$, then it is valid for $\mathfrak{g}$ of type $(C_\ell; B_\ell; G_2; F_4)$ respectively.
Proof: Let \( \mathfrak{k} \) be a non simply-laced simple Lie algebra. Then, there exists a simply-laced simple Lie algebra \( g \) together with a diagram automorphism \( \sigma \) (i.e., an automorphism \( \sigma \) of \( g \) induced from a diagram automorphism of its Dynkin diagram) such that \( \mathfrak{k} \) is the \( \sigma \)-fixed point \( g^\sigma \) of \( g \). Moreover, given \( \mathfrak{k} \), we can choose \( g \) to be of type given in the statement of the theorem. (For more details, see Section 3.1 on diagram folding.) In particular, we never need to take \( g \) of type \( D_4 \) except \( D_4 \).

Choose a Borel subalgebra \( \mathfrak{b} \) of \( g \) and a Cartan subalgebra \( \mathfrak{t} \subset \mathfrak{b} \) such that they both are stable under \( \sigma \). Let \( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \subset \mathfrak{t}^* \) be the set of simple roots of \( g \), where \( \ell \) is the rank of \( g \). Since \( \sigma \) keeps \( \mathfrak{b} \) and \( \mathfrak{t} \) stable, \( \sigma \) permutes the simple roots. Let \( \{ \tilde{\beta}_1, \ldots, \tilde{\beta}_\ell \} \) be a set of simple roots taken exactly one simple root from each orbit of \( \sigma \) in \( \Delta \). Then, the fixed subalgebra \( \mathfrak{b}_\ell := \mathfrak{b}^\sigma \) is a Borel subalgebra of \( \mathfrak{t} \), \( \mathfrak{t}_\ell := \mathfrak{t}^\sigma \) is a Cartan subalgebra of \( \mathfrak{t} \) and \( \{ \beta_1, \ldots, \beta_\ell \} \) is the set of simple roots of \( \mathfrak{t} \), where \( \beta_i := \tilde{\beta}_i |_{\mathfrak{t}_\ell} \) (cf. [S1]). In particular, \( \ell \) is the rank of \( \mathfrak{k} \).

For any \( 1 \leq n \leq \ell \), choose a nonzero element \( x_n \in g_{\tilde{\beta}_n} \), where \( g_{\tilde{\beta}_n} \) is the root space of \( g \) corresponding to the root \( \tilde{\beta}_n \). Define

\[
y_n = \sum_{i=1}^{\text{ord}(\sigma)} \sigma^i(x_n),
\]

where \( \text{ord}(\sigma) \) is the order of \( \sigma \) (which is 2 except when \( g \) is of type \( D_4 \) and \( \mathfrak{k} \) is of type \( G_2 \), in which case it is 3). If \( \tilde{\beta}_n \) is fixed by \( \sigma \), then \( \sigma \) acts trivially on \( g_{\tilde{\beta}_n} \) (cf. [S1]), hence \( y_n \) is never zero. Of course, \( y_n \in \mathfrak{k} \) and, in fact, \( y_n \in \mathfrak{k}_{\beta_n} \). Define the element \( y \in \mathfrak{k} \) by

\[
y = \sum_{n=1}^{\ell} y_n.
\]

By [Ko], Theorem 5.3, \( y \) is a principal nilpotent element of \( \mathfrak{k} \) and hence there exists a principal TDS in \( \mathfrak{k} \):

\[
\varphi : \mathfrak{sl}_2 \rightarrow \mathfrak{k} \quad \text{such that} \quad \varphi(X) = y.
\]

Moreover, since

\[
y = \sum_{n=1}^{\ell} \sum_{i=1}^{\text{ord}(\sigma)} \sigma^i(x_n),
\]

again using [Ko], Theorem 5.3, we get that \( y \) is a principal nilpotent of \( g \) as well. Hence, \( \varphi \) is a principal TDS of \( g \) also. Decompose \( g \) under the adjoint action of \( \mathfrak{sl}_2 \) via \( \varphi \):

\[
g = V_1 \oplus \cdots \oplus V_\ell,
\]

where \( V_1 \oplus \cdots \oplus V_\ell \) is a decomposition of \( \mathfrak{k} \).

Take a primitive element \( \omega_d \in P_d(\mathfrak{k}) \subset \wedge^d(\mathfrak{k}^*) \), where \( P_d(\mathfrak{k}) \) is the space of primitive elements for \( \mathfrak{k} \). By (subsequent) Theorem 3.5, the canonical restriction map \( \wedge^d(g^*) \rightarrow \wedge^d(\mathfrak{k}^*) \) induces a surjection

\[
P_d(g) \rightarrow P_d(\mathfrak{k}), \quad \text{for any} \quad d > 0.
\]

Take a preimage \( \tilde{\omega}_d \in P_d(g) \) of \( \omega_d \). By (4)-(5), there exists a unique irreducible \( \mathfrak{sl}_2 \)-submodule \( V_{\omega_d} \) of \( \mathfrak{k} \) of dimension \( d \). Further, by (4)-(6), there exists a unique irreducible \( \mathfrak{sl}_2 \)-submodule \( V_{\tilde{\omega}_d} \subset g \) of dimension \( d \). (For any \( \mathfrak{k} \) not of type \( G_2 \), the uniqueness of \( V_{\tilde{\omega}_d} \) follows since we have
chosen \( g \) not of type \( D_k \); for \( \mathfrak{t} \) of type \( G_2 \), \( P_d(\mathfrak{t}) \) is nonzero if and only if \( d = 3, 11 \) (cf. §2.1). Again, for these values of \( d \), \( \dim P_d(D_4) = 1 \). Hence, \( V_{\omega_d} = V_{\tilde{\omega}_d} \). Assuming the validity of Hitchin’s conjecture for \( g \), we get that \( \tilde{\omega}_d|_{\wedge^d(V_{\omega_d})} \neq 0 \). Hence,

\[
\omega_d|_{\wedge^d(V_{\omega_d})} = \tilde{\omega}_d|_{\wedge^d(V_{\omega_d})} \neq 0.
\]

This proves the theorem. \( \square \)

3. GIT QUOTIENT \( G//AdG \) AND DIAGRAM AUTOMORPHISMS

Let \( g \) be a simple, simply-laced Lie algebra over \( \mathbb{C} \) and let \( G \) be the connected, simply-connected complex algebraic group with Lie algebra \( g \). Let \( \sigma \) be a diagram automorphism of \( g \) and let \( k = g^\sigma \) be the fixed subalgebra. Then, \( \mathfrak{k} \) is a simple Lie algebra again. Let \( K \) be the connected subgroup of \( G \) with Lie algebra \( \mathfrak{k} \). In fact, \( K = G^\sigma \) (cf. [S1]). For the connection of the root datum of \( K \) with that of \( G \), we refer, e.g., to [S1].

With this notation, we have the following main result of this section.

**Theorem 3.1.** The canonical map \( \phi : R(G) \to R(K) \) is surjective, where \( R(G) \) denotes the representation ring of \( G \) (over \( \mathbb{Z} \)).

In particular, the canonical map \( K//AdK \to G//AdG \), between the GIT quotients, is a closed embedding.

Before we come to the proof of the theorem, we need some notational preliminaries on diagram automorphisms and ‘diagram folding’ (i.e., the process of getting \( \mathfrak{k} \) from \( g \)). As in Section 2, fix a Borel subalgebra \( \mathfrak{b} \) and a Cartan subalgebra \( \mathfrak{t} \subset \mathfrak{b} \) of \( g \) stable under \( \sigma \). Then, \( \mathfrak{b}_\sigma := \mathfrak{b}^\sigma \) (resp. \( \mathfrak{t}_\sigma := \mathfrak{t}^\sigma \)) is a Borel (resp. Cartan) subalgebra of \( \mathfrak{k} \). Let \( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \subset t^* \) be the simple roots of \( g \) and let \( \{ \tilde{\beta}_1, \ldots, \tilde{\beta}_\ell \} \) be a set of simple roots taken exactly one simple root from each orbit of \( \sigma \) in \( \Delta \). Then, \( \Delta_\sigma := \{ \beta_1, \ldots, \beta_\ell \} \subset \mathfrak{t}_\sigma^* \) is the set of simple roots of \( \mathfrak{k} \), where \( \beta_i := \tilde{\beta}_i|_{\mathfrak{t}_\sigma} \). In the following diagrams, we will make a specific choice of indexing convention in each case of diagram folding.

### 3.1. Diagram Folding: Dynkin diagrams of \((g, \mathfrak{t})\).

\( (A_{2n+1}, C_{n+1}) : \)

\[
\begin{array}{cccccccc}
1 & 2 & \cdots & \cdots & n & n+1 & \sigma(n) & \sigma(2) & \sigma(1) \\
\end{array}
\]

\( \beta_i := \alpha_i|_{\mathfrak{t}_\sigma} \) for \( i \leq n+1 \) and \( \beta_{n+1} \) is a long root.

\( (A_{2n}, B_n) : \)

\[
\begin{array}{cccccccc}
1 & 2 & \cdots & \cdots & n & \sigma(n) & \sigma(2) & \sigma(1) \\
\end{array}
\]

\( \beta_i = \alpha_i|_{\mathfrak{t}_\sigma} \) for \( 1 \leq i \leq n \) and \( \beta_n \) is a short root.

\( (D_n, B_{n-1}) : \)

\[
\begin{array}{cccccccc}
1 & 2 & \cdots & \cdots & n-2 & n-1 & n \\
\end{array}
\]
\[ \sigma(n-1) = n \]

\[ \beta_i := \alpha_{i|\ell_t} \text{ for } 1 \leq i \leq n - 1 \text{ and } \beta_{n-1} \text{ is a short root.} \]

\[ \langle \rho(\varpi_1), \beta_j^\vee \rangle = \delta_{i,j} \text{ for } 1 \leq i, j \leq \ell_t. \]

In this case, we have \([S1]\)

\[ \beta_j^\vee = \sum \alpha_k^\vee, \]
where the summation runs over the orbit of \( \alpha_j \) under \( \sigma \). For \( 1 \leq j \leq \ell_t \), no \( \alpha_k \) is in the
\( \sigma \)-orbit of \( \alpha_j \) for any \( 1 \leq k \leq \ell_t \). Thus, the equation (7) follows.

(b) When \( G \) is of type \( A_{2n} \), by [S1],

\[
\beta_j^\vee = \begin{cases} 
\alpha_j^\vee + \alpha_{2n-j+1}^\vee, & \text{for } j \leq n - 1, \\
2\alpha_n^\vee + 2\alpha_{n+1}^\vee, & \text{for } j = n.
\end{cases}
\]

So, for \( 1 \leq i \leq 2n \),

\[
\langle \rho(\varpi_i), \beta_j^\vee \rangle = \begin{cases} 
\langle \varpi_i, \alpha_j^\vee \rangle + \langle \varpi_i, \alpha_{2n-j+1}^\vee \rangle, & \text{for } j \leq n - 1, \\
2\langle \varpi_i, \alpha_n^\vee \rangle + 2\langle \varpi_i, \alpha_{n+1}^\vee \rangle, & \text{for } j = n.
\end{cases}
\]

\[
= \begin{cases} 
\delta_{i,j} + \delta_{i,2n-j+1}, & \text{for } j \leq n - 1, \\
2\delta_{i,n} + 2\delta_{i,n+1}, & \text{for } j = n.
\end{cases}
\]

From this (b) follows.

(c) By [S1], following the indexing convention as in Subsection 3.1, we get that

\[
\beta_1^\vee = \alpha_2^\vee, \quad \beta_2^\vee = \alpha_4^\vee, \quad \beta_3^\vee = \alpha_3^\vee + \alpha_5^\vee, \quad \beta_4^\vee = \alpha_1^\vee + \alpha_6^\vee.
\]

Thus,

\[
\rho(\varpi_1) = \rho(\varpi_6) = \nu_4, \\
\rho(\varpi_2) = \nu_1, \\
\rho(\varpi_3) = \rho(\varpi_5) = \nu_3, \\
\rho(\varpi_4) = \nu_2.
\]

Let \( \Lambda^+(g) \subset \mathfrak{t}^* \) (resp. \( \Lambda^+(\mathfrak{k}) \subset \mathfrak{t}_\mathfrak{k}^* \)) be the set of dominant integral weights for the root system of \( g \) (resp. \( \mathfrak{k} \)) and let \( \Lambda^+(K) \subset \Lambda^+(\mathfrak{t}) \) be the submonoid of dominant characters for the group \( K \), i.e., \( \Lambda^+(K) \) is the set of characters of the maximal torus \( T_K \) (with Lie algebra \( \mathfrak{t}_K \)) of \( K \) which are dominant with respect to the group \( K \). Observe that since \( G \) is simply-connected, \( \Lambda^+(G) = \Lambda^+(g) \). Moreover, under the restriction map \( \rho: \mathfrak{t}^* \to \mathfrak{t}_\mathfrak{k}^* \),

(8) \[ \rho(\Lambda^+(g)) = \Lambda^+(K). \]

To see this, let \( \Lambda(K) \) be the character lattice of \( K \) (similarly for \( \Lambda(G) = \Lambda(g) \)). Then, by Springer’s original construction of \( \Lambda(K) \) [S1], the restriction \( \rho: \Lambda(g) \to \Lambda(K) \) is surjective. Further, from the description of the coroots of \( \mathfrak{k} \) as in [S1], \( \rho(\Lambda^+(g)) \subset \Lambda^+(\mathfrak{t}) \). Thus, we have

\[
\rho(\Lambda^+(g)) \subset \Lambda^+(\mathfrak{t}) \cap \Lambda(K) = \Lambda^+(K).
\]

Conversely, in all cases except for \( g \) of type \( A_{2n} \), by Lemma 3.2, \( \rho(\Lambda^+(g)) = \Lambda^+(\mathfrak{t}) \supset \Lambda^+(K) \), so equation (8) holds in these cases. When \( g \) is of type \( A_{2n} \), again by Lemma 3.2,

\[
\rho(\Lambda^+(g)) = (\bigoplus_{i=1}^{n-1} \mathbb{Z} \nu_i) \oplus 2\mathbb{Z} \nu_n,
\]

and

\[
\Lambda(K) = \rho(\Lambda(g)) = (\bigoplus_{i=1}^{n-1} \mathbb{Z} \nu_i) \oplus 2\mathbb{Z} \nu_n.
\]
From this again, we see that (8) is satisfied. This proves (8) in all cases.

For any \( \lambda \in \Lambda^+(g) \), let \( V(\lambda) \) be the irreducible \( G \)-module with highest weight \( \lambda \). Similarly, for \( \mu \in \Lambda^+(K) \), let \( W(\mu) \) be the irreducible \( K \)-module with highest weight \( \mu \). We denote the fundamental representations \( V(\varpi_i) \) of \( g \) by \( V_i \) and \( W(\nu_j) \) of \( \mathfrak{k} \) by \( W_j \).

**Lemma 3.3.** For any \( \lambda \in \Lambda^+(g) \), \( W(\rho(\lambda)) \) has multiplicity one in \( V(\lambda) \) as a \( \mathfrak{k} \)-module. (Observe that by (8), \( \rho(\lambda) \in \Lambda^+(K) \).)

**Proof:** Note that the Borel subalgebra \( \mathfrak{b}_\mathfrak{k} \) of \( \mathfrak{k} \) is contained in the Borel subalgebra \( \mathfrak{b} \) of \( g \). So, if \( v_\lambda \) is the highest weight vector of \( V(\lambda) \) (of weight \( \lambda \)), then \( v_\lambda \) remains a highest weight vector of weight \( \rho(\lambda) \) in \( V(\lambda) \) for the action of \( \mathfrak{k} \). Hence, \( W(\rho(\lambda)) \subset V(\lambda) \).

Multiplicity one is clear from the weight consideration. \( \blacksquare \)

3.2. **Proof of Theorem 3.1.** Let \( \{\mu_1, \ldots, \mu_N\} \subset \Lambda^+(K) \) be a set of semigroup generators of \( \Lambda^+(K) \). Then, the classes \( \{[W(\mu_j)]\}_{1 \leq j \leq N} \) generate the \( \mathbb{Z} \)-algebra \( R(K) \), where \( [W(\mu_j)] \in R(K) \) denotes the class of the irreducible \( K \)-module \( W(\mu_j) \) (cf. [P], Theorem 3.12).

We proceed separately for each of the five cases depending on the type of \( (g, \mathfrak{k}) \).

**Case I** \( (A_{2n+1}, C_{n+1}) \): By Lemmas 3.2 and 3.3, for \( 1 \leq j \leq n+1 \), \( W_j \subset V_j \) (as \( \mathfrak{k} \)-modules). Recall that \( V_1 \simeq W_1 \simeq \mathbb{C}^{2n+2} \) (so \( W_1 = V_1 \)) and \( V_j = \wedge^j V_1 \) for all \( 1 \leq j \leq 2n+1 \). Also, for \( 2 \leq j \leq n+1 \), \( W_j \) is given as the kernel of the surjective \( \mathfrak{k} \)-equivariant contraction map \( \wedge^j W_1 \to \wedge^{j-2} W_1 \). Hence, for \( 2 \leq j \leq n+1 \), in \( R(\mathfrak{k}) \) (where \( R(\mathfrak{k}) \) is the representation ring of \( \mathfrak{k} \)), by [FH], Theorem 17.5,

\[
[W_j] + [\wedge^{j-2} W_1] = [\wedge^j W_1].
\]

Thus,

\[
\phi([V_1]) = [W_1], \quad \text{and} \quad \phi([V_j]) - \phi([V_{j-2}]) = [W_j], \quad \text{for} \ 2 \leq j \leq n+1,
\]

where \( V_0 \) is interpreted as the trivial one dimensional module \( \mathbb{C} \). Thus, the class \( [W_j] \) of each fundamental representation lies in the image of \( \phi \), and hence \( \phi \) is surjective.

**Case II** \( (A_{2n}, B_n) \): By Lemmas 3.2 and 3.3, for \( 1 \leq j \leq n-1 \), \( W_j \subset V_j \) and \( W(2\nu_n) \subset V_n \) (as \( \mathfrak{k} \)-modules). Recall that \( V_1 \simeq W_1 \simeq \mathbb{C}^{2n+1} \) (so \( W_1 = V_1 \)), and \( V_j = \wedge^j V_1 \) for all \( 1 \leq j \leq 2n \). Also, \( W_j = \wedge^j W_1 \) for \( 1 \leq j \leq n-1 \) and \( W(2\nu_n) = \wedge^n W_1 \) (see, e.g., [FH], Theorem 19.14). Thus, as \( \mathfrak{k} \)-modules,

\[
W_j = V_j, \quad j \leq n-1; \quad W(2\nu_n) = V_n.
\]

Thus,

\[
[W_1], \ldots, [W_{n-1}], [W(2\nu_n)] \in \text{Image } \phi.
\]

By Lemma 3.2 (b) and the identity (8), \( \Lambda^+(K) \) is generated (as a semigroup) by \( \{\nu_1, \ldots, \nu_{n-1}, 2\nu_n\} \). Hence, \( \phi \) is surjective in this case.

**Case III** \( (D_n, B_{n-1}) \): Recall that \( V_1 \simeq \mathbb{C}^{2n} \) and \( W_1 \simeq \mathbb{C}^{2n-1} \). By Lemmas 3.2 and 3.3, for \( 1 \leq j \leq n-1 \), \( W_j \subset V_j \) (as \( \mathfrak{k} \)-modules). Since \( W_1 \subset V_1 \) (as \( \mathfrak{k} \)-modules), we get (as \( \mathfrak{k} \)-modules):

\[
V_1 = W_1 \oplus \mathbb{C}.
\]

Thus, for \( 1 \leq k \leq n-2 \), as \( \mathfrak{k} \)-modules,

\[
V_k = \wedge^k V_1 = \wedge^k (W_1 \oplus \mathbb{C}) \simeq (\wedge^k W_1) \oplus (\wedge^{k-1} W_1) = W_k \oplus W_{k-1},
\]

where the first equality is by [FH], Theorem 19.2; \( W_0 \) is interpreted as the one dimensional trivial module and the last equality is from the proof of Case II.
Since $W_{n-1} \subset V_{n-1}$ as $\mathfrak{k}$-modules, and both being spin representations have the same dimension $2^{n-1}$ (see, e.g., [GW], Section 6.2.2), we get $V_{n-1} = W_{n-1}$. Therefore,

$$\phi([V_k]) = [W_k] + [W_{k-1}] \text{ for } 1 \leq k \leq n-2, \text{ and } \phi([V_{n-1}]) = [W_{n-1}].$$

In particular, each of $[W_1], \ldots, [W_{n-1}]$ lies in the image of $\phi$, proving the surjectivity of $\phi$ in this case.

**Case IV.** $(D_4, G_2)$: The two fundamental representations $W_1$ and $W_2$ have respective dimensions 7 and 14 ([FH], Section 22.3). On the other hand, $V_1$ is eight dimensional and $V_2 = \wedge^2 V_1$. Since $\rho(\varpi_1) = \nu_1$ (by Lemma 3.2), by Lemma 3.3 we get $W_1 \subset V_1$ (as $\mathfrak{k}$-modules). So, we have the decomposition (as $\mathfrak{k}$-modules):

$$V_1 = W_1 \oplus \mathbb{C}.$$ 

Thus, as $\mathfrak{k}$-modules,

$$V_2 = \wedge^2 V_1 = \wedge^2(W_1 \oplus \mathbb{C}) \simeq (\wedge^2 W_1) \oplus W_1.$$

But, $\wedge^2 W_1 \simeq W_2 \oplus W_1$ ([FH], Section 22.3). Hence, as $\mathfrak{k}$-modules,

$$V_2 = W_2 \oplus W_1^{\wedge 2}.$$

This gives

$$\phi([V_1]) = [W_1] + 1 \text{ and } \phi([V_2]) = [W_2] + 2[W_1],$$

which proves the surjectivity of $\phi$ in this case.

**Case V.** $(E_6, F_4)$: By Lemma 3.2(c), we see that $\rho$ is surjective with kernel given by $\{a\varpi_1 + b\varpi_3 - b\varpi_5 - a\varpi_6 \mid a, b \in \mathbb{Z}\}$. Considering the images of $\varpi_i$ under $\rho$, we have as $\mathfrak{k}$-modules (by Lemmas 3.2(c) and 3.3),

$$W_1 \subset V_2, \quad W_2 \subset V_4, \quad W_3 \subset V_3, V_5, \quad W_4 \subset V_1, V_6.$$

Using [Sl], Tables 44 and 47 or [LiE], we obtain

$$\dim(W_1) = 52, \quad \dim(V_2) = 78, \quad \dim(W_2) = 1274, \quad \dim(V_4) = 2925, \quad \dim(W_3) = 273, \quad \dim(V_3) = \dim(V_5) = 351, \quad \dim(W_4) = 26, \quad \dim(V_1) = \dim(V_6) = 27.$$ 

Along with the fundamental $\mathfrak{k}$-modules, there are only three other irreducible $\mathfrak{k}$-modules of dimensions at most 1651 ([Sl], Table 44, or [LiE]). These are $\dim(W(2\nu_4)) = 324$, $\dim(W(\nu_1 + \nu_4)) = 1053$, and $\dim(W(2\nu_1)) = 1053$.

Let $U^k$ denote an arbitrary $\mathfrak{k}$-module of dimension $k$. Considering the dimensions, we get (as $\mathfrak{k}$-modules):

$$V_1 = V_6 = W_4 \oplus \mathbb{C}, \quad V_2 = W_1 \oplus U^{26}, \quad V_3 = V_5 = W_3 \oplus U^{78}, \quad V_4 = W_2 \oplus U^{1651}.$$

Now, $U^{26}$ must be either $W_4$ or the trivial module $\mathbb{C}^{26}$, and $U^{78}$ must be some combination of $W_4$, $W_1$ and $\mathbb{C}$. Since $\phi([V_1]) - 1 = [W_4]$, this implies that $[W_4]$, $[W_1]$ and $[W_3]$ are in the
image of \( \phi \). (We remark that [Sl] gives \( F_4 \subset E_6 \) branching, but we continue without these results for clarity and completeness.)

Using appropriate tensor product decompositions in [LiE], we get

\begin{align*}
(9) & \quad [W(2\nu_4)] = [W_4]^2 - [W_3] - [W_1] - [W_4] - 1, \\
(10) & \quad [W(\nu_1 + \nu_4)] = [W_1][W_4] - [W_3] - [W_4], \\
(11) & \quad [W(2\nu_1)] = [W_1]^2 - [W_2] - [W(2\nu_4)] - [W_1] - 1.
\end{align*}

Since \( W_2 \) appears in \( V_4 \) as a \( \mathfrak{t} \)-submodule exactly once by Lemma 3.3, from the above identities, we get that \([W_2]\) lies in the image of \( \phi \) if \( W(2\nu_1) \) is not a component of \( V_4 \). In fact, we prove below that \( 2\nu_1 \) is not a \( \mathfrak{t} \)-weight of \( V_4 \) at all.

In order that \( 2\nu_1 \) be a \( \mathfrak{t} \)-weight of \( V_4 \), we should have \( 2\nu_1 = \mu|_4 \), where \( \mu \) is a weight of \( V_4 \). This is only possible if there exists a weight of \( V_4 \) of the form \( \mu = a\varpi_1 + 2\varpi_2 + b\varpi_3 - b\varpi_5 - a\varpi_6 \), for some \( a, b \in \mathbb{Z} \). We claim this is impossible. Indeed, all weights of \( V_4 \) are of the form \( \varpi_4 - \sum_{i=1}^{6} d_i\alpha_i \), where \( d_i \in \mathbb{Z}^+ \). If such \( \mu \) existed, then by [Bo], Planche V,

\[
\sum_{i=1}^{6} d_i\alpha_i = \varpi_4 - \mu \\
= \varpi_4 + a(\varpi_6 - \varpi_1) - 2\varpi_2 + b(\varpi_5 - \varpi_3) \\
= (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) + (a/3)(-2\alpha_1 - \alpha_3 + \alpha_5 + 2\alpha_6) \\
-2(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 4\alpha_5 + \alpha_6) + (b/3)(-\alpha_1 - 2\alpha_3 + 2\alpha_5 + \alpha_6),
\]

from which we immediately see a contradiction since the \( \alpha_2 \) coefficient is \(-1\).

This completes the proof in this last case and hence the proof of the first part of Theorem 3.1 is completed.

To prove that \( \eta : K//\text{Ad} \ K \to G//\text{Ad} \ G \) is a closed embedding, it suffices to show that the induced map between the affine coordinate rings \( \eta^* : \mathbb{C}[G//\text{Ad} \ G] \to \mathbb{C}[K//\text{Ad} \ K] \) is surjective. But, by [P], Theorem 3.5, there is a functorial isomorphism

\[
\mathbb{C} \otimes_{\mathbb{Z}} R(G) \to \mathbb{C}[G//\text{Ad} \ G],
\]

and similarly we have an isomorphism

\[
\mathbb{C} \otimes_{\mathbb{Z}} R(K) \to \mathbb{C}[K//\text{Ad} \ K].
\]

From this the surjectivity of \( \eta^* \) follows from the surjectivity of \( R(G) \to R(K) \). This proves the theorem. \( \square \)

We give the following Lie algebra analogue as a corollary.

**Corollary 3.4.** The canonical restriction map

\[
S(\mathfrak{g}^*)^g \to S(\mathfrak{k}^*)^k
\]

is surjective.

**Proof:** By [St], §6.4, for any connected semisimple algebraic group \( H \) over \( \mathbb{C} \), the restriction map

\[
r : \mathbb{C}[H//\text{Ad} \ H] \simeq \mathbb{C}[H]^H \to \mathbb{C}[T_H]^W_H
\]

is an isomorphism of \( \mathbb{C} \)-algebras, where \( T_H \subset H \) is a maximal torus and \( W_H \) is the Weyl group of \( H \).
Similarly, the restriction map
\[ r_\circ : \mathbb{C}[h]^H \to \mathbb{C}[t_h]^W^H \]
is a graded algebra isomorphism, where \( h \) (resp. \( t_h \)) is the Lie algebra of \( H \) (resp. \( T_H \)). Thus, to prove the corollary, it suffices to show that the canonical restriction map
\[ \beta_\circ^* : \mathbb{C}[t]^W \to \mathbb{C}[t]^W^K \]
is surjective, where \( W \) (resp. \( W^K \)) is the Weyl group of \( G \) (resp. \( K \)). Since \( \beta_\circ^* \) is a graded algebra homomorphism induced from the \( \mathbb{C}^* \)-equivariant map \( \beta_\circ : t/W_K \to t/W \) (where the \( \mathbb{C}^* \)-action is the standard homothety action), it suffices to show that the tangent map between the Zariski tangent spaces at 0:
\[ (d\beta_\circ)_0 : T_0(t/W_K) \to T_0(t/W) \]
is injective. Let \( T^{\text{anal}} \) denote the analytic tangent space. Then, the canonical map
\[ T^{\text{anal}}_x(X) \to T_x(X) \]
is an isomorphism for any algebraic variety \( X \) and any point \( x \in X \).

Consider the commutative diagram:
\[
\begin{array}{ccc}
\mathfrak{t} / W_K & \xrightarrow{\beta_\circ} & \mathfrak{t} / W \\
\, \, \, \downarrow{\text{Exp}} & & \downarrow{\text{Exp}} \\
\mathfrak{t} / W_K & \xrightarrow{\beta} & \mathfrak{t} / W,
\end{array}
\]
where \( T_K \subset K \) is the maximal torus with Lie algebra \( \mathfrak{t} \) and \( \beta : T_K/W_K \to T/W \) is the canonical map. Since \( T_K, T \) are tori, \( \text{Exp} \) is a local isomorphism in the analytic category. In particular, there exist open subsets (in the analytic topology) \( 0 \in U_\mathfrak{t} \subset \mathfrak{t}/W_K, 0 \in U \subset \mathfrak{t}/W, 1 \in V_K \subset T_K/W_K \) and \( 1 \in V \subset T/W \) such that \( \beta_\circ(U_K) \subset U \) and \( \text{Exp}\,|_U : U_\mathfrak{t} \to V_K \) is an analytic isomorphism and so is \( \text{Exp}\,|_U : U \to V \). Since, by Theorem 3.1 and the isomorphism (12), \( \beta \) is a closed embedding,
\[ (d\beta)_1 : T^{\text{anal}}_1(T_K/W_K) \simeq T_1(T_K/W_K) \to T^{\text{anal}}_1(T/W) \simeq T_1(T/W) \]
is injective and hence so is \( T_0(t/W_K) \to T_0(t/W) \). This proves the corollary. □

As a consequence of Corollary 3.4, we get the following.

**Theorem 3.5.** With the notation and assumptions as in Theorem 3.1, the canonical restriction map \( \gamma : H^*(G) \to H^*(K) \) is surjective. Moreover, this induces a surjective (graded) map
\[ \gamma_\circ : P(\mathfrak{g}) \to P(\mathfrak{t}), \]
where \( P(\mathfrak{g}) \subset H^*(G) \) is the subspace of primitive elements.

**Proof:** From the definition of coproduct, it is easy to see that the following diagram is commutative:
\[
\begin{array}{ccc}
H^*(G) & \xrightarrow{\Delta_G} & H^*(G) \otimes H^*(G) \\
\downarrow{\gamma} & & \downarrow{\gamma \otimes \gamma} \\
H^*(K) & \xrightarrow{\Delta_K} & H^*(K) \otimes H^*(K).
\end{array}
\]
Thus, \( \gamma \) takes \( P(\mathfrak{g}) \) to \( P(\mathfrak{t}) \).

Let \( \mathfrak{h} \) be a reductive Lie algebra. For any \( v \in \mathfrak{h} \), define the derivation \( i(v) : S(\mathfrak{h}^*) \to S(\mathfrak{h}^*) \) given by \( i(v)(f) = f(v) \), for \( f \in \mathfrak{h}^* \). Further, define an algebra homomorphism \( \lambda : S(\mathfrak{h}^*) \to \wedge^{even}(\mathfrak{h}^*) \) by \( \lambda(f) = df \), for \( f \in \mathfrak{h}^* = S^1(\mathfrak{h}^*) \), where \( d : \wedge^1(\mathfrak{h}^*) = \mathfrak{h}^* \to \wedge^2(\mathfrak{h}^*) \) is the standard differential in the Lie algebra cochain complex \( \wedge^*(\mathfrak{h}^*) \). Now, define the \textit{transgression map}

\[
\tau = \tau_\mathfrak{h} : S^+((\mathfrak{h}^*)^\mathfrak{h}) \to \wedge^+(\mathfrak{h}^*)^\mathfrak{h}, \quad \tau(p) = \sum_j e^*_j \wedge \lambda(i(e_j)p),
\]

for \( p \in S^+((\mathfrak{h}^*)^\mathfrak{h}) \), where \( \{e_j\} \) is a basis of \( \mathfrak{h} \) and \( \{e^*_j\} \) is the dual basis of \( \mathfrak{h}^* \).

By a result of Cartan (cf. [Ca], Théorème 2; also see [L]), \( \tau \) factors through

\[
S^+((\mathfrak{h}^*)^\mathfrak{h})/(S^+(\mathfrak{h}^*)^\mathfrak{h}) : (S^+(\mathfrak{h}^*)^\mathfrak{h}) \to \wedge^+(\mathfrak{h}^*)^\mathfrak{h}
\]

to give an injective map

\[
\bar{\tau} : S^+((\mathfrak{h}^*)^\mathfrak{h})/(S^+(\mathfrak{h}^*)^\mathfrak{h}) : (S^+(\mathfrak{h}^*)^\mathfrak{h}) \to \wedge^+(\mathfrak{h}^*)^\mathfrak{h}
\]

with image precisely equal to the space of primitive elements \( P(\mathfrak{h}) \). From the definition of \( \tau \), it is easy to see that the following diagram is commutative:

\[
\begin{array}{ccc}
S^+(\mathfrak{g}^*)^\mathfrak{g} & \xrightarrow{\tau_\mathfrak{g}} & \wedge^+(\mathfrak{g}^*)^\mathfrak{g} \\
\downarrow & & \downarrow \\
S^+(\mathfrak{t}^*)^\mathfrak{t} & \xrightarrow{\tau_\mathfrak{t}} & \wedge^+(\mathfrak{t}^*)^\mathfrak{t},
\end{array}
\]

where the vertical maps are the canonical restriction maps. By using Corollary 3.4, this proves that \( P(\mathfrak{g}) \) surjects onto \( P(\mathfrak{t}) \). Since \( P(\mathfrak{t}) \) generates \( \wedge^*(\mathfrak{t}^*)^\mathfrak{t} \simeq H^*(K) \) as an algebra, we get that \( \gamma \) is surjective. This proves the theorem. \( \blacksquare \)

\textbf{Remark 3.6.} As a consequence of the above theorem, we see that the Leray-Serre homology (or cohomology) spectral sequence with coefficients in \( \mathbb{C} \) for the fibration

\[
K \to G \to G/K
\]
degenerates at the \( E^2 \)-term.

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