Generalized Conformal Representations of Orthogonal Lie Algebras *

Xiaoping Xu¹ and Yufeng Zhao²

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1. Corresponding author, Hua Loo-Keng Key Mathematical Laboratory, Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, P. R. China.
2. LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China

Abstract

The conformal transformations with respect to the metric defining $o(n, \mathbb{C})$ give rise to a nonhomogeneous polynomial representation of $o(n + 2, \mathbb{C})$. Using Shen’s technique of mixed product, we generalize the above representation to a non-homogenous representation of $o(n + 2, \mathbb{C})$ on the tensor space of any finite-dimensional irreducible $o(n, \mathbb{C})$-module with the polynomial space, where a hidden central transformation is involved. Moreover, we find a condition on the constant value taken by the central transformation such that the generalized conformal representation is irreducible. In our approach, Pieri’s formulas, invariant operators and the idea of Kostant’s characteristic identities play key roles. The result could be useful in understanding higher-dimensional conformal field theory with the constant value taken by the central transformation as the central charge. Our representations virtually provide natural extensions of the conformal transformations on a Riemannian manifold to its vector bundles.

1 Introduction

A quantum field is an operator-valued function on a certain Hilbert space, which is often a direct sum of infinite-dimensional irreducible modules of a certain Lie algebra (group). The Lie algebra of two-dimensional conformal group is exactly the Virasoro algebra, which is infinite-dimensional. The minimal models of two-dimensional conformal field theory were constructed from direct sums of certain infinite-dimensional irreducible modules of the Virasoro algebra, where a distinguished module called, the vacuum module, gives rise to a vertex operator algebra. When $n \geq 3$, $n$-dimension conformal groups (depending on the metric) are finite-dimensional. Higher-dimensional conformal field

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theory is not so well understood partly because we are lack of enough knowledge on the infinite-dimensional irreducible modules of orthogonal Lie algebras that are compatible to the natural conformal representations. This motivates us to study explicit infinite-dimensional irreducible representations of the orthogonal Lie algebra \( o(n+2, \mathbb{C}) \) by using the non-homogeneous polynomial representations arising from conformal transformations with respect to the metric defining \( o(n, \mathbb{C}) \) and Shen’s technique of mixed product (cf. [Sg1-3]) (also known as Larsson functor (cf. [La])).

It is well known that \( n \)-dimensional projective group gives rise to a non-homogenous representation of the Lie algebra \( sl(n+1, \mathbb{C}) \) on the polynomial functions of the projective space. Using Shen’s mixed product for Witt algebras, the authors [ZX] generalized the above representation of \( sl(n+1, \mathbb{C}) \) to a non-homogenous representation on the tensor space of any finite-dimensional irreducible \( gl(n, \mathbb{C}) \)-module with the polynomial space. Moreover, the structure of such a representation was completely determined by employing projection operator techniques (cf. [Gm1]) and the well-known Kostant’s characteristic identities (cf. [K]).

In this paper, we generalize the conformal representation of of \( o(n+2, \mathbb{C}) \) to a non-homogenous representation of \( o(n+2, \mathbb{C}) \) on the tensor space of any finite-dimensional irreducible \( o(n, \mathbb{C}) \)-module with the polynomial space by Shen’s idea of mixed product for Witt algebras. It turns out that a hidden central transformation is involved. More importantly, we find a condition on the constant value taken by the central transformation such that the generalized conformal representation is irreducible. In our approach, Pieri’s formulas, invariant operators and the idea of Kostant’s characteristic identities play key roles. The result could be useful in understanding higher-dimensional conformal field theory with the constant value taken by the central transformation as the central charge. Our representations virtually provide natural extensions of the conformal transformations on a Riemannian manifold to its vector bundles.

Characteristic identities have a long history. The first person to exploit them was Dirac [D], who wrote down what amounts to the characteristic identity for the Lie algebra \( o(1,3) \). This particular example is intimately connected with the problem of describing the structure of relativistically invariant wave equations. It had been shown by Kostant [K] (also cf. [Gm2]) that the characteristic identities for semi-simple Lie algebras also hold for infinite dimensional representations.

The \( n \)-dimensional conformal group with respect to Euclidean metric \((\cdot, \cdot)\) is generated by the translations, rotations, dilations and special conformal transformations

\[
\bar{x} \mapsto \frac{x - \left<x, \bar{x}\right> \bar{b}}{\left<\bar{b}, \bar{b}\right>\left<x, \bar{x}\right> - 2\left<\bar{b}, \bar{x}\right> + 1}.
\] (1.1)
Let $\mathcal{A} = \mathbb{C}[x_1, \ldots, x_n]$. The Witt algebra $\mathcal{W}(n) = \sum_{i=1}^{n} \mathcal{A} \partial_{x_i}$ with the commutator of differential operators as its Lie bracket. The conformal transformations with respect to the metric defining $o(n, \mathbb{C})$ give rise to a nonhomogeneous polynomial representation of $\vartheta: \mathfrak{o}(n + 2, \mathbb{C}) \to \mathcal{W}(n)$, acting on $\mathcal{A}$. Let $E_{r,s}$ be the square matrix with 1 as its $(r,s)$-entry and 0 as the others. Acting on the entries of the elements of the Lie algebra $\mathfrak{gl}(n, \mathcal{A})$, $\mathcal{W}(n)$ becomes a Lie subalgebra of the derivation Lie algebra of $\mathfrak{gl}(n, \mathcal{A})$. In particular,

$$\widehat{\mathcal{W}}(n) = \mathcal{W}(n) \oplus \mathfrak{gl}(n, \mathcal{A}).$$

becomes a Lie algebra with the Lie bracket

$$[d_1 + A_1, d_2 + A_2] = [d_1, d_2] + [A_1, A_2] + d_1(A_2) - d_2(A_1)$$

for $d_1, d_2 \in \mathcal{W}(n)$ and $A_1, A_2 \in \mathfrak{gl}(n, \mathcal{A})$. Shen [S1] found a monomorphism $\Im: \mathcal{W}(n) \to \widehat{\mathcal{W}}(n)$ defined by

$$\Im(\sum_{i=1}^{n} f_i \partial_{x_i}) = \sum_{i=1}^{n} f_i \partial_{x_i} + \sum_{i,j=1}^{n} \partial_{x_i}(f_j) \otimes E_{i,j}.$$\hspace{1cm}(1.4)

Moreover, he [S1-S3] used the monomorphism $\Im$ to develop a theory of mixed products for the modules of Lie algebras of Cartan type, which is also known as the Larsson functor (cf. [L]) in the case of Witt algebras. Rao [R] constructed some irreducible weight modules over the derivation Lie algebra of the algebra of Laurent polynomials based on Shen’s mixed product. Lin and Tan [LT] did the similar thing over the derivation Lie algebra of the algebra of quantum torus. The second author [Z] determined the module structure of Shen’s mixed product over Xu’s nongraded Lie algebras of Witt type in [X].

Let $M$ be any $o(n, \mathbb{C})$-module and let $b \in \mathbb{C}$. It can be verified that $\Im(\vartheta(o(n + 2, \mathbb{C}))) \subset \vartheta(o(n + 2, \mathbb{C})) + o(n, \mathcal{A}) + \mathcal{A} \sum_{i=1}^{n} E_{i,i}$, which enable us to define an $o(n + 2, \mathbb{C})$-module structure on $M = \mathcal{A} \otimes_{\mathcal{C}} M$, where the hidden central transformation $(\sum_{i=1}^{n} E_{i,i})|_{M}$ takes the constant value $b$. We call such a module $M$ a generalized conformal module of $o(n + 2, \mathbb{C})$. Geometrically, the tensor modules yield natural extensions of the conformal transformations on a Riemannian manifold to its vector bundles. For any two integers $m$ and $n$, we use the notation

$$\overline{m,n} = \begin{cases} \{m, m + 1, \ldots, n\} & \text{if } m \leq n, \\ \emptyset & \text{otherwise.} \end{cases}$$\hspace{1cm}(1.5)

Moreover, we denote by $\mathbb{N}$ the set of nonnegative integers.

Suppose that $\mathcal{E}$ is a vector space over $\mathbb{R}$ with a basis $\{\varepsilon_i \mid i \in \overline{1,m}\}$ and the inner product $(\sum_{i=1}^{m} a_i \varepsilon_i, \sum_{r=1}^{m} c_r \varepsilon_r) = \sum_{i=1}^{m} a_i c_i$. As usual, we take the set of simple positive roots $\Pi_{2m} = \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_{m-1} + \varepsilon_m \mid i \in \overline{1, m-1}\}$ of $o(2m, \mathbb{C})$ and the set of simple positive roots $\Pi_{2m+1} = \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_m \mid i \in \overline{1, m-1}\}$ of $o(2m + 1, \mathbb{C})$. Set

$$\Lambda^+_n = \{\mu \in \mathcal{E} \mid [2(\mu, \nu)/(\nu, \nu)] \in \mathbb{N} \text{ for } \nu \in \Pi_n\}, \quad n = 2m, 2m + 1.$$\hspace{1cm}(1.6)
It is well known that any finite-dimensional irreducible \( o(n, \mathbb{C}) \)-module is a highest-weight irreducible module \( V(\mu) \) with highest weight \( \mu \in \Lambda^+_n \) (e.g., cf. [Hu]). For \( \mu = \sum_{i=1}^{m} \mu_i \varepsilon_i \in \Lambda^+_n \), we define \( \iota_\mu \in \mathbb{N} \) by
\[
\mu_1 = \mu_2 = \cdots = \mu_{\iota_\mu} \neq \mu_{\iota_\mu + 1}.
\] (1.7)

Using Pieri’s formulas, invariant operators and the idea of Kostant’s characteristic identities, we prove:

**Main Theorem.** Let \( 0 \neq \mu = \sum_{i=1}^{m} \mu_i \varepsilon_i \in \Lambda^+_n \) with \( n = 2m, 2m + 1 \geq 3 \). The generalized conformal module \( \hat{V}(\mu) \) of \( o(2m, \mathbb{C}) \) is irreducible if \( b \in \mathbb{C} \setminus \{m - 1 - N/2, \mu_1 + 2m - \iota_\mu - 1 - N\} \). Moreover, the generalized conformal module \( \hat{V}(\mu) \) of \( o(2m + 1, \mathbb{C}) \) is irreducible if \( b \in \mathbb{C} \setminus \{m - N/2, \mu_1 + 2m - \iota_\mu - N\} \).

The generalized conformal module \( \hat{V}(0) \) of \( o(n, \mathbb{C}) \) is irreducible if and only if \( b \notin -N \). When \( b = 0 \), \( \hat{V}(0) \) is isomorphic to the natural conformal \( o(n + 2, \mathbb{C}) \)-module \( \mathcal{A} \), on which \( A(f) = \vartheta(A)(f) \) for \( A \in o(n + 2, \mathbb{C}) \) and \( f \in \mathcal{A} \). The subspace \( \mathbb{C} \) forms a trivial \( o(n + 2, \mathbb{C}) \)-submodule of the conformal module \( \mathcal{A} \) and the quotient space \( \mathcal{A}/\mathbb{C} \) forms an irreducible \( o(n + 2, \mathbb{C}) \)-module.

In some singular cases of \( \mu \neq 0 \), the condition can be relaxed slightly. We speculate that the module \( \hat{V}(0) \) may play the same role in higher-dimensional conformal field theory as that of the vacuum module of the Virasoro algebra plays in two-dimensional conformal field theory. The central charge in two-dimensional conformal field theory may be replaced by the constant \( b \) in the above theorem for higher-dimensional conformal field theory.

The paper is organized as follows. In Section 2, we prove the main theorem for \( o(2m, \mathbb{C}) \) (\( m \) will be replaced by \( n \) customarily). Section 3 is devoted to the proof of the main theorem for \( o(2m + 1, \mathbb{C}) \) (\( m \) will also be replaced by \( n \) customarily).

## 2 Generalized Conformal Representations of \( D_{n+1} \)

Let \( n > 1 \) be an integer. The orthogonal Lie algebra
\[
o(2n, \mathbb{C}) = \sum_{1 \leq p < q \leq n} \mathbb{C}(E_{p,n+q} - E_{q,n+p}) + \mathbb{C}(E_{n+p,q} - E_{n+q,p}) + \sum_{i,j=1}^{n} \mathbb{C}(E_{i,j} - E_{n+j,n+i}). \tag{2.1}
\]

We take the subspace
\[
\mathcal{H} = \sum_{i=1}^{n} \mathbb{C}(E_{i,i} - E_{n+i,n+i}) \tag{2.2}
\]
as a Cartan subalgebra and define \( \{ \varepsilon_i \mid i \in \overline{1,n} \} \subset \mathcal{H}^* \) by

\[
\varepsilon_i(E_{j,j} - E_{n+j,n+j}) = \delta_{i,j}.
\]

(2.3)

The inner product \((\cdot, \cdot)\) on the \(\mathbb{Q}\)-subspace \(L_{\mathbb{Q}} = \sum_{i=1}^{n} \mathbb{Q}\varepsilon_i\) is given by

\[
(\varepsilon_i, \varepsilon_j) = \delta_{i,j} \quad \text{for } i, j \in \overline{1,n}.
\]

(2.5)

Then the root system of \(o(2n, \mathbb{C})\) is

\[
\Phi_{D_n} = \{ \pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n \}.
\]

(2.6)

We take the set of positive roots

\[
\Phi^+_{D_n} = \{ \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n \}.
\]

(2.7)

In particular,

\[
\Pi_{D_n} = \{ \varepsilon_1 - \varepsilon_2, ..., \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n \} \text{ is the set of positive simple roots.}
\]

(2.8)

Recall the set of dominate integral weights

\[
\Lambda^+ = \{ \mu \in L_{\mathbb{Q}} \mid (\varepsilon_{n-1} + \varepsilon_n, \mu), (\varepsilon_i - \varepsilon_{i+1}, \mu) \in \mathbb{N} \text{ for } i \in \overline{1,n-1} \}.
\]

(2.9)

According to (2.5),

\[
\Lambda^+ = \{ \mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \mid \mu_i \in \mathbb{Z}/2; \mu_i - \mu_{i+1}, \mu_{n-1} + \mu_n \in \mathbb{N} \}.
\]

(2.10)

Note that if \( \mu \in \Lambda^+ \), then \( \mu_{n-1} \geq |\mu_n| \). Given \( \mu \in \Lambda^+ \), there exists a unique sequence \( S(\mu) = \{n_0, n_1, ..., n_s\} \) such that

\[
0 = n_0 < n_1 < n_2 < \cdots < n_{s-1} < n_s = n
\]

(2.11)

and

\[
\mu_i = \mu_j \Leftrightarrow n_r < i, j \leq n_{r+1} \text{ for some } r \in \overline{0,s-1}.
\]

(2.12)

For \( \lambda \in \Lambda^+ \), we denote by \( V(\lambda) \) the finite-dimensional irreducible \( o(2n, \mathbb{C}) \)-module with highest weight \( \lambda \). The \( 2n \)-dimensional natural module of \( o(2n, \mathbb{C}) \) is \( V(\varepsilon_1) \) with the weights \( \{ \pm \varepsilon_i \mid i \in \overline{1,n} \} \). The following result is well known (e.g., cf. [FH]):

**Lemma 2.1 (Pieri’s formula).** Given \( \mu \in \Lambda^+ \) with \( S(\mu) = \{n_0, n_1, ..., n_s\} \),

\[
V(\varepsilon_1) \otimes \mathbb{C} V(\mu) \cong \bigoplus_{i=1}^{s} (V(\mu - \varepsilon_{n_i}) \oplus V(\mu + \varepsilon_{1+n_{i-1}}))
\]

(2.13)
if \( \mu_{n-1} + \mu_n > 0 \) and
\[
V(\varepsilon_1) \otimes \mathbb{C} V(\mu) \cong \bigoplus_{i=1}^{s-2+\delta_{\mu,n,0}} V(\mu - \varepsilon_n) \oplus \bigoplus_{i=1}^s V(\mu + \varepsilon_{1+n-i})
\] (2.14)
when \( \mu_{n-1} + \mu_n = 0 \).

Denote by \( U(\mathcal{G}) \) the universal enveloping algebra of a Lie algebra \( \mathcal{G} \). The algebra \( U(\mathcal{G}) \) can be imbedded into the tensor algebra \( U(\mathcal{G}) \otimes \mathbb{C} U(\mathcal{G}) \) by the associative algebra homomorphism \( d: U(\mathcal{G}) \to U(\mathcal{G}) \otimes \mathbb{C} U(\mathcal{G}) \) determined by
\[
d(u) = u \otimes 1 + 1 \otimes u \quad \text{for } u \in \mathcal{G}.
\] (2.15)

Note that the Casimir element of \( o(2n, \mathbb{C}) \) is
\[
\omega = \sum_{1 \leq i < j \leq n} [(E_{i,n+j} - E_{j,n+i})(E_{n+j,i} - E_{n+i,j}) + (E_{n+j,i} - E_{n+i,j})(E_{i,n+j} - E_{j,n+i})]
\]
\[
+ \sum_{i,j=1}^{n} (E_{i,j} - E_{n+j,n+i})(E_{j,i} - E_{n+i,n+j}) \in U(o(2n, \mathbb{C})).
\] (2.16)

Set
\[
\bar{\omega} = \frac{1}{2}(d(\omega) - \omega \otimes 1 - 1 \otimes \omega) \in U(o(2n, \mathbb{C})) \otimes \mathbb{C} U(o(2n, \mathbb{C})).
\] (2.17)

By (2.16),
\[
\bar{\omega} = \sum_{1 \leq i < j \leq n} [(E_{i,n+j} - E_{j,n+i}) \otimes (E_{n+j,i} - E_{n+i,j}) + (E_{n+j,i} - E_{n+i,j}) \otimes (E_{i,n+j} - E_{j,n+i})]
\]
\[
+ \sum_{i,j=1}^{n} (E_{i,j} - E_{n+j,n+i}) \otimes (E_{j,i} - E_{n+i,n+j}).
\] (2.18)

Denote
\[
\rho = \frac{1}{2} \sum_{\nu \in \Phi_D^+} \nu.
\] (2.19)

Then
\[
(\rho, \nu) = 1 \quad \text{for } \nu \in \Pi_D
\] (2.20)
(e.g., cf. [Hu]). By (2.8),
\[
\rho = \sum_{i=1}^{n-1} (n - i) \varepsilon_i.
\] (2.21)

For any \( \mu \in \Lambda^+ \), we have
\[
\omega|_{V(\mu)} = (\mu + 2\rho, \mu) \text{Id}_{V(\mu)}.
\] (2.22)

Denote
\[
\ell^+(\mu) = \dim V(\mu + \varepsilon_i) \quad \text{if } \mu + \varepsilon_i \in \Lambda^+
\] (2.23)
and
\[ \ell^-_i(\mu) = \dim V(\mu - \varepsilon_i) \quad \text{if} \quad \mu - \varepsilon_i \in \Lambda^+. \] (2.24)

Observe that
\[ (\mu + \varepsilon_i + 2\rho, \mu + \varepsilon_i) - (\mu + 2\rho, \mu) - (\varepsilon_1 + 2\rho, \varepsilon_1) = 2(\mu_i + 1 - i) \] (2.25)
and
\[ (\mu - \varepsilon_i + 2\rho, \mu - \varepsilon_i) - (\mu + 2\rho, \mu) - (\varepsilon_1 + 2\rho, \varepsilon_1) = 2(1 + i - 2n - \mu_i) \] (2.26)
for \( \mu = \sum_{r=1}^n \mu_r \varepsilon_r \) by (2.21). Moreover, the algebra \( U(o(2n, \mathbb{C})) \otimes_{\mathbb{C}} U(o(2n, \mathbb{C})) \) acts on \( V(\varepsilon_1) \otimes_{\mathbb{C}} V(\mu) \) by
\[ (\xi_1 \otimes \xi_2)(v \otimes u) = \xi_1(v) \otimes \xi_2(u) \quad \text{for} \quad \xi_1, \xi_2 \in U(o(2n, \mathbb{C})), \ v \in V(\varepsilon_1), \ u \in V(\mu). \] (2.27)

By Lemma 2.1, (2.17) and (2.23)-(2.26), we obtain:

Lemma 2.2. Let \( \mu = \sum_{i=1}^n \mu_i \varepsilon_i \in \Lambda^+ \) with \( S(\mu) = \{n_0, n_1, \ldots, n_s\} \). If \( \mu_{n-1} + \mu_n > 0 \), the characteristic polynomial of \( \tilde{\omega}|_{V(\varepsilon_1) \otimes_{\mathbb{C}} V(\mu)} \) is
\[ \prod_{i=1}^s (t - \mu_{1+n_{i-1}} + n_{i-1})^{l_{1+n_{i-1}}(\mu)} (t + \mu_{n_i} + 2n - n_i - 1)^{l_{n_i}(\mu)}. \] (2.28)

When \( \mu_{n-1} + \mu_n = 0 \), the characteristic polynomial of \( \tilde{\omega}|_{V(\varepsilon_1) \otimes_{\mathbb{C}} V(\mu)} \) is
\[ \prod_{i=0}^{s-1} (t - \mu_{1+n_i} + n_i)^{l_{1+n_i}(\mu)} [\prod_{j=1}^{s-2+\delta_{n_n,0}} (t + \mu_{n_j} + 2n - n_j - 1)^{l_{n_j}(\mu)}]. \] (2.29)

We remark that the above lemma is equivalent to special detailed version of Kostant’s characteristic identity. Set
\[ A = \mathbb{C}[x_1, x_2, \ldots, x_{2n}]. \] (2.30)

Then \( A \) forms an \( o(2n, \mathbb{C}) \)-module with the action determined via
\[ E_{i,j}|_A = x_i \partial_{x_j} \quad \text{for} \quad i, j \in \overline{1, 2n}. \] (2.31)

The corresponding Laplace operator and dual invariant are
\[ \Delta = \sum_{r=1}^n \partial_{x_r} \partial_{x_{n+r}}, \quad \eta = \sum_{i=1}^n x_i x_{n+i}. \] (2.32)

Denote
\[ D = \sum_{r=1}^{2n} x_r \partial_{x_r}, \quad J_i = x_i D - \eta \partial_{x_{n+i}}, \quad J_{n+i} = x_{n+i} D - \eta \partial_{x_i}, \] (2.33)
\[ A_{i,j} = x_i \partial_{x_j} - x_{n+j} \partial_{x_{n+i}}, \quad B_{i,j} = x_i \partial_{x_{n+j}} - x_j \partial_{x_{n+i}}, \quad C_{i,j} = x_{n+i} \partial_{x_j} - x_{n+j} \partial_{x_i} \]  

(2.34) for \( i, j \in \overline{1,n} \). Then

\[ \mathcal{C}_{2n} = \mathbb{C}D + \sum_{r=1}^{2n} (\mathbb{C}\partial_{x_r} + \mathbb{C}J_r) + \sum_{i,j=1}^{n} \mathbb{C}A_{i,j} + \sum_{1 \leq i < j \leq n} (\mathbb{C}B_{i,j} + \mathbb{C}C_{i,j}) \]  

(2.35) is the Lie algebra of 2n-dimensional conformal group over \( \mathbb{C} \) with \( \eta \) as the metric.

Set

\[ \mathcal{L}_0 = \sum_{i,j=1}^{n} \mathbb{C}A_{i,j} + \sum_{1 \leq i < j \leq n} (\mathbb{C}B_{i,j} + \mathbb{C}C_{i,j}), \quad \mathcal{J} = \sum_{i=1}^{2n} \mathbb{C}J_i, \quad \mathcal{D} = \sum_{i=1}^{2n} \mathbb{C}\partial_{x_i}. \]  

(2.36) Then \( \mathcal{L}_0 = o(2n, \mathbb{C})|_{\mathcal{A}} \). We can easily verify the following Lie brackets:

\[
\begin{align*}
[\mathcal{J}, \mathcal{J}] &= \{0, \mathcal{D}, \mathcal{D}\} = \{0\}, \quad [\partial_{x_k}, J_{n+i}] = C_{i,k}, \quad [\partial_{x_{n+k}}, J_i] = B_{i,k}, \\
[\partial_{x_k}, J_i] &= \delta_{k,i}D + A_{i,k}, \quad [\partial_{x_{n+k}}, J_{n+i}] = \delta_{k,i}D - A_{k,i}, \\
[\partial_{x_k}, A_{i,j}] &= \delta_{k,i}\partial_{x_j}, \quad [\partial_{x_k}, B_{i,j}] = \delta_{k,i}\partial_{x_{n+j}} - \delta_{k,j}\partial_{x_{n+i}}, \quad [\partial_{x_k}, C_{i,j}] = 0, \\
[\partial_{x_{n+k}}, A_{i,j}] &= -\delta_{k,j}\partial_{x_{n+i}}, \quad [\partial_{x_{n+k}}, B_{i,j}] = 0, \quad [\partial_{x_{n+k}}, C_{i,j}] = \delta_{k,j}\partial_{x_j} - \delta_{k,i}\partial_{x_i}, \\
[J_k, A_{i,j}] &= -\delta_{k,j}J_i, \quad [J_k, B_{i,j}] = 0, \quad [J_k, C_{i,j}] = \delta_{k,i}J_{n+j} - \delta_{k,j}J_{n+i}, \\
[J_{n+k}, A_{i,j}] &= \delta_{k,i}J_{n+j}, \quad [J_{n+k}, B_{i,j}] = \delta_{k,i}J_j - \delta_{k,j}J_i, \quad [J_{n+k}, C_{i,j}] = 0, \\
[D, J_k] &= J_k, \quad [D, J_{n+k}] = J_{n+k}, \quad [\partial_{x_k}, D] = \partial_{x_k}, \quad [\partial_{x_{n+k}}, D] = \partial_{x_{n+k}}.
\end{align*}
\]  

(2.37)-(2.43) for \( i, j, k \in \overline{1,n} \). Recall that the split

\[
o(2n + 2, \mathbb{C}) = \sum_{1 \leq r < s \leq n+1} \mathbb{C}(E_{r,n+1+s} - E_{s,n+1+r}) + \mathbb{C}(E_{n+1+r,s} - E_{n+1+s,r}) \\
+ \sum_{i,j=1}^{n+1} \mathbb{C}(E_{i,j} - E_{n+1+j,n+1+i}).
\]  

(2.44) By (2.37)-(2.43), we have the Lie algebra isomorphism \( \vartheta : o(2n + 2, \mathbb{F}) \rightarrow \mathcal{C}_{2n} \) determined by

\[
\vartheta(E_{i,j} - E_{n+1+j,n+1+i}) = A_{i,j}, \quad \vartheta(E_{r,n+1+s} - E_{s,n+1+r}) = B_{r,s}, \\
\vartheta(E_{n+1+r,s} - E_{n+1+s,r}) = C_{r,s}, \quad \vartheta(E_{n+1,n+1} - E_{2n+2,2n+2}) = -D, \\
\vartheta(E_{n+1,i} - E_{n+1+i,2n+2}) = \partial_{x_k}, \quad \vartheta(E_{i,2n+2} - E_{n+1+n+1+i}) = -\partial_{x_{n+i}}, \\
\vartheta(E_{i,n+1} - E_{2n+2,j+n+1}) = -J_i, \quad \vartheta(E_{2n+2,i} - E_{n+1+n+1+n+1}) = J_{n+i}
\]  

(2.45)-(2.48) for \( i, j \in \overline{1,n} \) and \( 1 \leq r < s \leq n \).
Recall the Witt algebra $W_{2n} = \sum_{i=1}^{2n} A\partial_{x_i}$, and Shen [Sg1-3] found a monomorphism $\mathfrak{S}$ from the Lie algebra $W_{2n}$ to the Lie algebra of semi-product $W_{2n} + gl(2n, A)$ defined by

$$\mathfrak{S} \left(\sum_{i=1}^{2n} f_i \partial_{x_i}\right) = \sum_{i=1}^{2n} f_i \partial_{x_i} + \sum_{i,j=1}^{n} \partial_{x_i} (f_j) E_{i,j}. \quad (2.49)$$

Note that $C_{2n} \subset W_{2n}$ and

$$\mathfrak{S}(A_{i,j}) = A_{i,j} + E_{i,j} - E_{n+j,n+i}, \quad \mathfrak{S}(B_{r,s}) = B_{r,s} + E_{r,n+s} - E_{s,n+r}, \quad \mathfrak{S}(\partial_{x_i}) = \partial_{x_i}, \quad (2.50)$$

$$\mathfrak{S}(C_{r,s}) = C_{r,s} + E_{n+r,s} - E_{n+s,r}, \quad \mathfrak{S}(\partial_{x_{n+i}}) = \partial_{x_{n+i}}, \quad \mathfrak{S}(D) = D + \sum_{p=1}^{2n} E_{p,p} \quad (2.51)$$

$$\mathfrak{S}(J_i) = J_i + \sum_{p=1}^{n} x_n p (E_{i,n+p} - E_{p,n+i}) + \sum_{q=1}^{n} x_q (E_{i,q} - E_{n+q,n+i}) + x_i \sum_{p=1}^{2n} E_{p,p}, \quad (2.52)$$

$$\mathfrak{S}(J_{n+i}) = J_{n+i} + \sum_{p=1}^{n} x_n p (E_{n+i,n+p} - E_{p,i}) + \sum_{q=1}^{n} x_q (E_{n+i,q} - E_{n+q,i}) + x_{n+i} \sum_{p=1}^{2n} E_{p,p} \quad (2.53)$$

for $i, j \in \overline{1,n}$ and $1 \leq r < s \leq n$. Moreover,

$$\widehat{C}_{2n} = C_{2n} + o(2n, A) + A \sum_{p=1}^{2n} E_{p,p} \quad (2.54)$$

forms a Lie subalgebra of $W_{2n} + gl(2n, A)$ and $\mathfrak{S}(C_{2n}) \subset \widehat{C}_{2n}$. In particular, the element $\sum_{p=1}^{2n} E_{p,p}$ is a hidden central element.

Let $M$ be an $o(2n, \mathbb{C})$-module and let $b \in \mathbb{C}$ be a fixed constant. Then

$$\widehat{M} = A \otimes_{\mathbb{F}} M \quad (2.55)$$

becomes a $\widehat{C}_{2n}$-module with the action:

$$(d + f_1 A + f_2 \sum_{p=1}^{2n} E_{p,p}) (g \otimes v) = (d(g) + bf_2 g) \otimes v + f_1 g \otimes A(v) \quad (2.56)$$

for $f_1, f_2, g \in A$, $A \in o(2n, \mathbb{C})$ and $v \in M$. Moreover, we make $\widehat{M}$ a $C_{2n}$-module with the action:

$$\xi(w) = \mathfrak{S}(\xi)(w) \quad \text{for } \xi \in C_{2n}, \ w \in \widehat{M}. \quad (2.57)$$

Furthermore, $\widehat{M}$ becomes an $o(2n + 2, \mathbb{F})$-module with the action

$$A(w) = \mathfrak{S}(\partial(A))(w) \quad \text{for } A \in o(2n + 2, \mathbb{C}), \ w \in \widehat{M} \quad (2.58)$$

(cf. (2.45)-(2.48)).

**Lemma 2.3** If $M$ is an irreducible $o(2n, \mathbb{C})$-module, then the space $U(J)(1 \otimes M)$ is an irreducible $o(2n + 2, \mathbb{C})$-submodule of $\widehat{M}$. 
Proof. Recall $o(2n, \mathbb{C})|_A = \mathcal{L}_0$. Moreover, (2.41) and (2.42) give

$$[\mathcal{L}_0, \mathcal{J}] = \mathcal{J}. \quad (2.59)$$

Note that $D$ is the degree operator (cf. (2.33)) and

$$[D, \mathcal{J}] \subset \mathcal{L}_0 + CD \quad (2.60)$$

by (2.37) and (2.38). According to (2.35) and (2.36),

$$\vartheta(o(2n + 2, \mathbb{C})) = C_{2n} = \mathcal{L}_0 + D + \mathcal{J} + CD. \quad (2.61)$$

By (2.43) and (2.59)-(2.61), $U(\mathcal{J})(1 \otimes M)$ forms an $o(2n + 2, \mathbb{C})$-submodule of $\hat{M}$. Let $W$ be a nonzero $o(2n + 2, \mathbb{C})$-submodule of $U(\mathcal{J})(1 \otimes M)$. Note

$$\partial_{x_i}| \hat{M} = \partial_{x_i} \otimes 1 \quad \text{for } i \in \overline{1, 2n}. \quad (2.62)$$

By repeatedly applying the above operators to $W$, we can prove

$$W \bigcap (1 \otimes M) \neq \{0\}. \quad (2.63)$$

However, $W \bigcap (1 \otimes M)$ is a nonzero $\mathcal{L}_0$-submodule of $1 \otimes M$, which is an irreducible $\mathcal{L}_0$-module. Therefore, $1 \otimes M \subset W$. As a $\mathcal{C}_{2n}$-module, $W \supset U(\mathcal{J})(1 \otimes M)$. $\square$

Write

$$x^\alpha = \prod_{i=1}^{2n} x_i^{\alpha_i}, \quad J^\alpha = \prod_{i=1}^{2n} J_i^{\alpha_i} \quad \text{for } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n}) \in \mathbb{N}^{2n}. \quad (2.64)$$

For $k \in \mathbb{N}$, we set

$$\mathcal{A}_k = \text{Span}_\mathbb{C}\{x^\alpha \mid \alpha \in \mathbb{N}^{2n}, \sum_{i=1}^{2n} \alpha_i = k\}, \quad \hat{M}_{(k)} = \mathcal{A}_k \otimes_\mathbb{C} M \quad (2.65)$$

and

$$(U(\mathcal{J})(1 \otimes M))_{(k)} = \text{Span}_\mathbb{C}\{J^\alpha(1 \otimes M) \mid \alpha \in \mathbb{N}^{2n}, \sum_{i=1}^{2n} \alpha_i = k\}. \quad (2.67)$$

Moreover,

$$(U(\mathcal{J})(1 \otimes M))_{(0)} = \hat{M}_{(0)} = 1 \otimes M. \quad (2.68)$$

Furthermore,

$$\hat{M} = \bigoplus_{k=0}^{\infty} \hat{M}_{(k)}, \quad U(\mathcal{J})(1 \otimes M) = \bigoplus_{k=0}^{\infty} (U(\mathcal{J})(1 \otimes M))_{(k)}. \quad (2.69)$$

Next we define a linear transformation $\varphi$ on $\hat{M}$ determined by

$$\varphi(x^\alpha \otimes v) = J^\alpha(1 \otimes v) \quad \text{for } \alpha \in \mathbb{N}^{2n}, v \in M. \quad (2.70)$$
Note that $\mathcal{A}_1 = \sum_{i=1}^{2n} Cx_i$ forms the $2n$-dimensional natural $\mathcal{L}_0$-module (equivalently $\mathfrak{o}(2n, \mathbb{C})$-module). According to (2.41) and (2.42), $\mathcal{J}$ forms an $\mathcal{L}_0$-module with respect to the adjoint representation, and the linear map from $\mathcal{A}_1$ to $\mathcal{J}$ determined by $x_i \mapsto J_i$ for $i \in \mathbb{T}_1; 2n$ gives an $\mathcal{L}_0$-module isomorphism. Thus $\varphi$ can also be viewed as an $\mathcal{L}_0$-module homomorphism from $\hat{M}$ to $U(\mathcal{J})(1 \otimes M)$. Moreover,

$$\varphi(\hat{M}_{(k)}) = (U(\mathcal{J})(1 \otimes M))_{(k)} \quad \text{for } k \in \mathbb{N}. \quad (2.71)$$

**Lemma 2.4.** We have $\varphi|_{\hat{M}_{(1)}} = (b + \hat{\omega})|_{\hat{M}_{(1)}}$ (cf. (2.16)-(2.18)).

**Proof.** Let $i \in \mathbb{T}_1; n$ and $v \in M$. Expressions (2.52), (2.53) and (2.56)-(2.58) give

$$\varphi(x_i \otimes v) = J_i(1 \otimes v) = \sum_{p=1}^{n} x_{n+p} \otimes (E_{i,n+p} - E_{p,n+i})(v)$$

$$+ \sum_{q=1}^{n} x_q \otimes (E_{i,q} - E_{n+q,n+i})(v) + bx_i \otimes v, \quad (2.72)$$

$$\varphi(x_{n+i} \otimes v) = J_{n+i}(1 \otimes v) = \sum_{p=1}^{n} x_{n+p} \otimes (E_{n+i,n+p} - E_{p,i})(v)$$

$$+ \sum_{q=1}^{n} x_q \otimes (E_{n+i,q} - E_{n+q,i})(v) + bx_{n+i} \otimes v. \quad (2.73)$$

On the other hand, (2.18) and (2.31) yield

$$\hat{\omega}(x_i \otimes v) = \sum_{1 \leq p < q \leq n} [(E_{p,n+q} - E_{q,n+p}) \otimes (E_{n+q,p} - E_{n+p,q})(x_i \otimes v)$$

$$+ (E_{n+q,p} - E_{n+p,q}) \otimes (E_{p,n+q} - E_{q,n+p})(x_i \otimes v)]$$

$$+ \sum_{r,s=1}^{n} (E_{r,s} - E_{n+s,n+r}) \otimes (E_{s,r} - E_{n+r,n+s})(x_i \otimes v)$$

$$= \sum_{p=1}^{n} x_{n+p} \otimes (E_{i,n+p} - E_{p,n+i})(v) + \sum_{r=1}^{n} x_r \otimes (E_{i,r} - E_{n+r,n+i})(v), \quad (2.74)$$

$$\hat{\omega}(x_{n+i} \otimes v) = \sum_{1 \leq p < q \leq n} [(E_{p,n+q} - E_{q,n+p}) \otimes (E_{n+q,p} - E_{n+p,q})(x_{n+i} \otimes v)$$

$$+ (E_{n+q,p} - E_{n+p,q}) \otimes (E_{p,n+q} - E_{q,n+p})(x_{n+i} \otimes v)]$$

$$+ \sum_{r,s=1}^{n} (E_{r,s} - E_{n+s,n+r}) \otimes (E_{s,r} - E_{n+r,n+s})(x_{n+i} \otimes v)$$

$$= \sum_{p=1}^{n} x_p \otimes (E_{n+i,p} - E_{n+p,i})(v) + \sum_{s=1}^{n} x_{n+s} \otimes (E_{n+i,n+s} - E_{s,i})(v). \quad (2.75)$$

Comparing the above four expressions, we get the conclusion in the lemma. \hfill \Box
For $f \in \mathcal{A}$, we define the action
\[
    f(g \otimes v) = fg \otimes v \quad \text{for } g \in \mathcal{A}, \; v \in M.
\]

Then we have the $o(2n, \mathbb{C})$-invariant operator
\[
    T = \sum_{i=1}^{n} (J_{i} x_{n+i} + J_{n+i} x_{i})|_{\tilde{M}}.
\]

**Lemma 2.5.** We have $T|_{\tilde{M}(k)} = (2b + 2n + k)\eta|_{\tilde{M}(k)}$.

**Proof.** Let $f$ be a homogeneous polynomial with degree $k$ and let $v \in M$. By (2.52), (2.53) and (2.57), we have
\[
    T(f \otimes v) = \sum_{i=1}^{n} (J_{i} x_{n+i} + J_{n+i} x_{i})(f \otimes v) = \sum_{i=1}^{n} [J_{i} (x_{n+i} f \otimes v) + J_{n+i} (x_{i} f \otimes v)]
\]
\[
    = \sum_{i=1}^{n} [J_{i} (x_{n+i} f) \otimes v + \sum_{p=1}^{n} (x_{n+p} x_{n+i} f) \otimes (E_{i,n+p} - E_{p,n+i})(v)]
\]
\[
    + \sum_{q=1}^{n} (x_{q} x_{n+i} f) \otimes (E_{i,q} - E_{n+q,n+i})(v) + (x_{i} x_{n+i} f) \otimes (\sum_{p=1}^{2n} E_{p,p})(v)
\]
\[
    + J_{n+i} (x_{i} f) \otimes v + \sum_{p=1}^{n} (x_{n+p} x_{i} f) \otimes (E_{n+i,n+p} - E_{p,i})(v)
\]
\[
    + \sum_{q=1}^{n} (x_{q} x_{i} f) \otimes (E_{n+i,q} - E_{n+q,i})(v) + (x_{n+i} x_{i} f) \otimes (\sum_{p=1}^{2n} E_{p,p})(v)]
\]

\[
    = \sum_{i,p=1}^{n} [(x_{n+p} x_{n+i} f) \otimes E_{i,n+p}(v) - (x_{n+p} x_{n+i} f) \otimes E_{p,n+i}(v)] + \sum_{i,q=1}^{n} (x_{q} x_{n+i} f)
\]
\[
    \otimes (E_{i,q} - E_{n+q,n+i})(v) + \sum_{i,p=1}^{n} (x_{n+p} x_{i} f) \otimes (E_{n+i,n+p} - E_{p,i})(v) + \sum_{i,q=1}^{n} [(x_{q} x_{i} f)
\]
\[
    \otimes E_{n+i,q}(v) - (x_{q} x_{i} f) \otimes E_{n+q,i}(v)] + [2b\eta f + \sum_{i=1}^{n} [J_{i} (x_{n+i} f) + J_{n+i} (x_{i} f)] \otimes v
\]
\[
    = \sum_{i=1}^{n} [J_{i} (x_{n+i} f) + J_{n+i} (x_{i} f)] + 2b\eta f \otimes v.
\]

According to (2.33), we find
\[
    \sum_{i=1}^{n} [J_{i} (x_{n+i} f) + J_{n+i} (x_{i} f)]
\]
\[
    = \sum_{i=1}^{n} [(x_{i} D - \eta \partial_{x_{n+i}})(x_{n+i} f) + (x_{n+i} D - \eta \partial_{x_{i}})(x_{i} f)]
\]
\[
    = 2(k + 1)\eta f - 2n\eta f - \eta D(f) = (k + 2 - 2n)\eta f.
\] 

So the lemma holds. $\square$
For \( 0 \neq \mu = \sum_{i=1}^n \mu_i \varepsilon_i \in \Lambda^+ \) with \( S(\mu) = \{n_0, n_1, \ldots, n_s\} \), we define
\[
\Theta(\mu) = \begin{cases} 
\mu_1 + n - 1 - N & \text{if } \mu_{n-1} = -\mu_n > 0 \text{ and } s = 2, \\
\mu_1 + 2n - n_i - 1 - N & \text{otherwise.}
\end{cases}
\] (2.80)

**Theorem 2.6.** For \( 0 \neq \mu \in \Lambda^+ \), the generalized conformal \( o(2n + 2, \mathbb{C}) \)-module \( \hat{V}(\mu) \) defined by (2.45)-(2.58) is irreducible if \( b \in \mathbb{C} \setminus \{n - 1 - N/2, \Theta(\mu)\} \).

**Proof.** By Lemma 2.3, it is enough to prove that the homomorphism \( \varphi \) defined in (2.70) satisfies \( \varphi(\hat{V}(\mu)) = \hat{V}(\mu) \). According to (2.71), we only need to prove
\[
\varphi(\hat{V}(\mu)_{(k)}) = \hat{V}(\mu)_{(k)}
\] (2.81)
for any \( k \in \mathbb{N} \). We will prove it by induction on \( k \).

When \( k = 0 \), (2.81) holds by the definition (2.70). Consider \( k = 1 \). Write \( \mu = \sum_{i=1}^n \mu_i \varepsilon_i \in \Lambda^+ \) with \( S(\mu) = \{n_0, n_1, \ldots, n_s\} \). According to Lemma 2.2 and Lemma 2.4 with \( M = V(\mu) \), the eigenvalues of \( \varphi|_{\hat{V}(\mu)_{(1)}} \) are
\[
b + \mu_1 + n_{i-1} - n_{i-1}, b - \mu_{n_i} - 2n + n_i + 1 \quad \text{for } i \in \overline{1, s} \text{ if } \mu_{n-1} + \mu_n > 0
\] (2.82)
and
\[
b + \mu_1 + n_{i-1} - n_{i-1}, b - \mu_{n_i} - 2n + n_r + 1 \quad \text{for } i \in \overline{1, s}, r \in \overline{1, s - 2 + \delta_{m_n,0}}
\] (2.83)
when \( \mu_{n-1} + \mu_n = 0 \). Recall that \( \mu_r \in \mathbb{Z}/2 \) for \( r \in \overline{1, n} \),
\[
\mu_{s+1} - \mu_s \in \mathbb{N} \text{ for } i \in \overline{1, n-1}, \quad \mu_{n-1} \geq \lvert \mu_n \rvert
\] (2.84)
and (2.12) holds. So
\[
-\mu_1 + n_{i-1}, \mu_{n_i} + 2n - n_i - 1 \in \mu_1 + 2n - n_1 - 1 - N \quad \text{for } i \in \overline{1, s}.
\] (2.85)
If \( b \notin \mu_1 + 2n - n_1 - 1 - N \), then all the eigenvalues of \( \varphi|_{\hat{V}(\mu)_{(1)}} \) are nonzero. In the case \( \mu_n = -\mu_{n-1} > 0 \) and \( s = 2 \), \( \mu_1 = \mu_{n-1} = -\mu_n \) and the eigenvalues \( \varphi|_{\hat{V}(\mu)_{(1)}} \) are \( b + \mu_1 \) and \( b + \mu_n + n + 1 = b - \mu_1 - n + 1 \), which are not equal to 0 because of \( b \notin \Theta(\mu) = \mu_1 + n - 1 - N \).

Thus (2.81) holds for \( k = 1 \).

Suppose that (2.81) holds for \( k \leq \ell \) with \( \ell \geq 1 \). Consider \( k = \ell + 1 \). Note that
\[
\varphi(\hat{V}(\mu)_{(\ell+1)}) = \sum_{i=1}^{2n} \varphi(x_i \hat{V}(\mu)_{(\ell)}) = \sum_{i=1}^{2n} J_i[\varphi(\hat{V}(\mu)_{(\ell)})] = \sum_{i=1}^{2n} J_i(\hat{V}(\mu)_{(\ell)})
\] (2.86)
by the inductional assumption. To prove (2.81) with \( k = \ell + 1 \) is equivalent to prove
\[
\sum_{i=1}^{2n} J_i(\hat{V}(\mu)_{(\ell)}) = \hat{V}(\mu)_{(\ell+1)}.
\] (2.87)
For any $u \in \overline{V(\mu)}_{(\ell - 1)}$, Lemma 2.5 says that

$$
\sum_{i=1}^{n}[J_i(x_{n+i}u) + J_{n+i}(x_iu)] = (2b + 1 - 2n + \ell)\eta u.
$$

(2.88)

Since $b \not\in n - 1 - N/2$, $2b + 1 - 2n + \ell \neq 0$ and (2.88) gives

$$
\eta u \in \sum_{i=1}^{2n} J_i(\overline{V(\mu)}_{(\ell)}) \quad \text{for } u \in \overline{V(\mu)}_{(\ell - 1)}.
$$

(2.89)

Let $g \otimes v \in \overline{V(\mu)}_{(\ell)}$. According to (2.52)-(2.57) and Lemma 2.4,

$$
J_i(g \otimes v) = x_iD(g) \otimes v - \eta \partial_{x_i}(g) \otimes v + \sum_{p=1}^{n} x_{n+p}g \otimes (E_{i,n+p} - E_{p,n+i})(v)
$$

$$
+ \sum_{q=1}^{n} x_qg \otimes (E_{i,q} - E_{n+q,n+i})(v) + x_i g \otimes (\sum_{p=1}^{2n} E_{p,p})(v)
$$

$$
= -\eta \partial_{x_i}(g) \otimes v + g[(\ell + b + \vec{\omega})(x_i \otimes v)]
$$

(2.90)

and

$$
J_{n+i}(g \otimes v) = x_{n+i}D(g) \otimes v - \eta \partial_{x_i}(g) \otimes v + \sum_{p=1}^{n} x_{n+p}g \otimes (E_{i,n+p} - E_{p,i})(v)
$$

$$
+ \sum_{q=1}^{n} x_qg \otimes (E_{n+i,q} - E_{n+q,i})(v) + x_{n+i}g \otimes (\sum_{p=1}^{2n} E_{p,p})(v)
$$

$$
= -\eta \partial_{x_i}(g) \otimes v + g[(\ell + b + \vec{\omega})(x_{n+i} \otimes v)]
$$

(2.91)

for $i \in \overline{1,n}$. Since

$$
\eta \partial_{x_i}(g) \otimes v, \eta \partial_{x_{n+i}}(g) \otimes v \in \sum_{r=1}^{2n} J_r(\overline{V(\mu)}_{(\ell)}) \quad \text{for } i \in \overline{1,n}
$$

(2.92)

by (2.89), Expressions (2.90) and (2.91) show

$$
g[(\ell + b + \vec{\omega})(x_i \otimes v)] \in \sum_{r=1}^{2n} J_r(\overline{V(\mu)}_{(\ell)}) \quad \text{for } i \in \overline{1,2n}, g \in \mathcal{A}_\ell.
$$

(2.93)

According to Lemma 2.2 and 2.4, the eigenvalue of $(\ell + b + \vec{\omega})|_{\overline{V(\mu)}_{(\ell)}}$ are among

$$
\{b + \ell + \mu_{1+n_i-1} - n_{i-1}, b + \ell - \mu_{n_i} - 2n + n_i + 1 \mid i \in \overline{1,s}\}.
$$

(2.94)

Again

$$
-\ell - \mu_{1+n_i-1} + n_{i-1}, -\ell - \mu_{n_i} + 2n - n_i - 1 \in \mu_1 + 2n - n_i - 1 - \mathbb{N} \quad \text{for } i \in \overline{1,s}.
$$

(2.95)

If $b \not\in \mu_1 + 2n - n_i - 1 - \mathbb{N}$, then all the eigenvalues of $(\ell + b + \vec{\omega})|_{\overline{V(\mu)}_{(\ell)}}$ are nonzero. In the case $\mu_n = -\mu_{n-1} > 0$ and $s = 2$, $\mu_1 = \mu_{n-1} = -\mu_n$ and the eigenvalues $(\ell + b + \vec{\omega})|_{\overline{V(\mu)}_{(\ell)}}$ are nonzero.
are $b + \mu_1 + \ell$ and $b + \mu_n - n + 1 + \ell = b - \mu_1 - n + 1 + \ell$, which are not equal to 0 because of $b \not\in \Theta(\mu) = \mu_1 + n - 1 - N$. Hence

$$(\ell + b + \omega)(\widehat{V(\mu)})_{(1)} = \widehat{V(\mu)}_{(1)}.$$ \hfill (2.96)

By (2.93) and (2.96),

$$g(\widehat{V(\mu)})_{(1)} \subset \sum_{r=1}^{2n} J_r(\widehat{V(\mu)}_{(r)}) \quad \text{for} \ g \in \mathcal{A}_\ell,$$

equivalently, (2.81) holds for $k = \ell + 1$. By induction, (2.81) holds for any $k \in \mathbb{N}$. \hfill \Box

We remark that the $o(2n + 2, \mathbb{C})$-module $\widehat{V(\mu)}$ is $o(2n, \mathbb{C})$-finite, that is, $\widehat{V(\mu)}$ is of $(G, \mathcal{K})$-type with $G = o(2n + 2, \mathbb{C})$ and $\mathcal{K} = o(2n, \mathbb{C})$. Up to this stage, we do not known if the condition in Theorem 2.6 is necessary for the generalized conformal $o(2n + 2, \mathbb{C})$-module $\widehat{V(\mu)}$ to be irreducible if $\mu \neq 0$. In the case $\mu = 0$, the situation becomes clear.

**Theorem 2.7.** The generalized conformal $o(2n + 2, \mathbb{C})$-module $\widehat{V(0)}$ is irreducible if and only if $b \not\in -\mathbb{N}$. When $b = 0$, $\widehat{V(0)}$ is isomorphic to the natural conformal $o(2n + 2, \mathbb{C})$-module $\mathcal{A}$, on which $A(f) = \vartheta(A)(f)$ for $A \in o(2n + 2, \mathbb{C})$ and $f \in \mathcal{A}$ (cf. (2.45)-(2.48)). The subspace $\mathbb{C}$ forms a trivial $o(2n + 2, \mathbb{C})$-submodule of the conformal module $\mathcal{A}$ and the quotient space $\mathcal{A}/\mathbb{C}$ forms an irreducible $o(2n + 2, \mathbb{C})$-module.

**Proof.** Pick $0 \neq v_0 \in V(0)$. Then $\widehat{V(0)} = \mathcal{A} \otimes v_0$. Since $V(0)$ is the trivial $o(2n, \mathbb{C})$-module, (2.50)-(2.57) yield

$$\xi(f \otimes v_0) = \xi(f) \otimes v_0 \quad \text{for} \ f \in \mathcal{A}, \ \xi \in \mathcal{L}_0 + \mathcal{D} \hfill (2.98)$$

(cf. (2.36)) and

$$D(f \otimes v_0) = (b + D)(f) \otimes v_0, \hfill (2.99)$$

$$J_i(f \otimes v_0) = [x_i(D+b) - \eta \partial_{x_{n+i}}](f) \otimes v_0, \quad J_{n+i}(f \otimes v_0) = [x_{n+i}(D+b) - \eta \partial_{x_i}](f) \otimes v_0 \hfill (2.100)$$

for $f \in \mathcal{A}$ and $i \in 1, 2n$ (cf. (2.33)).

Recall the action of $o(2n + 2, \mathbb{C})$ on $\widehat{V(0)}$ by (2.58). In particular, the map $f \otimes v_0 \mapsto f$ for $f \in \mathcal{A}$ gives an $o(2n, \mathbb{C})$-module isomorphism from $\widehat{V(0)}$ to $\mathcal{A}$. Remember the $o(2n, \mathbb{C})$-invariant differential operator $\Delta = \sum_{i=1}^{n} \partial_{x_{i}} \partial_{x_{n+i}}$ and its dual $\eta = \sum_{i=1}^{n} x_{i}x_{n+i}$. Moreover, $\mathcal{A}_k$ denotes the subspace of homogeneous polynomials in $\mathcal{A}$ with degree $k$. Set

$$\mathcal{H}_k = \{ f \in \mathcal{A}_k \mid \Delta(f) = 0 \} \quad \text{for} \ k \in \mathbb{N}. \hfill (2.101)$$

Then $\mathcal{H}_k \otimes v_0$ is an irreducible $o(2n, \mathbb{C})$-submodule with the highest-weight vector $x_{i}^{k} \otimes v_0$. Indeed, $\widehat{V(0)} = \bigoplus_{m, k=0}^{\infty} \eta^m \mathcal{H}_k \otimes v_0 \hfill (2.102)$
is a direct sum of irreducible $o(2n, \mathbb{C})$-submodules. On the other hand, $U(J)(1 \otimes v_0)$ forms an irreducible $o(2n + 2, \mathbb{C})$-submodule of $\hat{V}(0)$ (cf. Lemma 2.3). By (2.100), a necessary condition for $\hat{V}(0) = U(J)(1 \otimes v_0)$ is $b \not\in -N$.

Next we assume $b \not\in -N - 1$. Let $W$ be a nonzero $o(2n + 2, \mathbb{C})$-submodule of $\hat{V}(0)$ such that
\[
W \not\subset \mathbb{C} \otimes v_0 \text{ if } b = 0. 
\tag{2.103}
\]
By repeatedly acting $D$ on $W$ if necessary, we have $1 \otimes v_0 \in W$. Note
\[
J_i(1 \otimes v_0) = bx_i \otimes v_0 \text{ for } i \in 1, 2n. 
\tag{2.104}
\]
Thus
\[
\hat{V}(0)_{(1)} \subset W 
\tag{2.105}
\]
if $b \neq 0$. When $b = 0$, (2.105) also holds because of (2.103), $D(W) \subset W$ and the irreducibility of $\hat{V}(0)_{(1)}$ as an $o(2n, \mathbb{C})$-submodule. Suppose that
\[
\hat{V}(0)_{(k)} \subset W \text{ for } k < \ell, 
\tag{2.106}
\]
where $2 \leq \ell \in \mathbb{N}$. According to (2.102),
\[
\hat{V}(0)_{(\ell)} = \bigoplus_{m=0}^{[\ell/2]} \eta^m \mathcal{H}_{\ell-2m} \otimes v_0. 
\tag{2.107}
\]
Moreover,
\[
[\Delta, \eta] = n + D. 
\tag{2.108}
\]
Set
\[
\hat{V}(0)_{(\ell, r)} = \bigoplus_{m=0}^{r} \eta^m \mathcal{H}_{\ell-2m} \otimes v_0 
\tag{2.109}
\]
for $r \in 0, [\ell/2]$. Then
\[
\hat{V}(0)_{(\ell, r)} = \{ w \in \hat{V}(0)_{(\ell)} \mid \Delta^{r+1}(w) = 0 \} 
\tag{2.110}
\]
and
\[
\Delta^r(\hat{V}(0)_{(\ell, r)}) = \mathcal{H}_{\ell-2r} \otimes v_0 
\tag{2.111}
\]
by (2.108).

Since
\[
J_1(x_1^{\ell-1} \otimes v_0) = (b + \ell - 1)x_1^{\ell} \otimes v_0 \in W 
\tag{2.112}
\]
and $b \not\in -\mathbb{N} - 1$, we have
\[
x_1^{\ell} \otimes v_0 \in W. 
\tag{2.113}
\]
Hence
\[ \hat{V}(0)_{(\ell,0)} = \mathcal{H}_\ell \otimes v_0 \subset W \] (2.114)
because $\mathcal{H}_\ell \otimes v_0$ is an irreducible $o(2n, \mathbb{C})$-submodule generated by $x_1^\ell \otimes v_0$. Recall $n \geq 2$ by our assumption. For $r \in \mathbb{Q} \cap [\ell/2]$, 
\[ J_2(x_1^{\ell-r-1} x_{n+1}^r \otimes v_0) = (b + \ell - 1)x_1^{\ell-r-1} x_2 x_{n+1}^r \otimes v_0 \in W. \] (2.115)
So $x_2 x_{n+1}^r \otimes v_0 \in W$. Moreover,
\[ \Delta^r(x_1^{\ell-r-1} x_2 x_{n+1}^r \otimes v_0) = r! \prod_{s=1}^{r}(\ell - r - s)x_1^{\ell-2r-1} x_2 \otimes v_0 \in \mathcal{H}_{\ell-2r}. \] (2.116)
Observe that (2.107) is a direct sum of irreducible $o(2n, \mathbb{C})$-submodules with distinct highest weights. So
\[ \hat{V}(0)_{(\ell)} \cap W = \bigoplus_{m=0}^{\lfloor \ell/2 \rfloor} (\eta^m \mathcal{H}_{\ell-2m} \otimes v_0) \cap W. \] (2.117)
By (2.109)-(2.111) and (2.116), $(\eta^r \mathcal{H}_{\ell-2r} \otimes v_0) \cap W$ is a nonzero $o(2n, \mathbb{C})$-submodule. Since $\eta^r \mathcal{H}_{\ell-2r} \otimes v_0$ is an irreducible $o(2n, \mathbb{C})$-module, we have $\eta^r \mathcal{H}_{\ell-2r} \otimes v_0 = (\eta^r \mathcal{H}_{\ell-2r} \otimes v_0) \cap W$. Therefore, $\hat{V}(0)_{(\ell)} \subset W$. By induction, $\hat{V}(0)_{(k)} \subset W$ for any $k \in \mathbb{N}$, equivalently, $W = \hat{V}(0)$. This proves the theorem. \qed

3 Generalized Conformal Representations of $B_{n+1}$

Let $n \geq 1$ be an integer. The orthogonal Lie algebra
\[ o(2n+1, \mathbb{C}) = o(2n, \mathbb{C}) + \sum_{i=1}^{n} (\mathbb{C}(E_{0,i} - E_{n+i,0}) + \mathbb{C}(E_{0,n+i} - E_{i,0})) \] (3.1)
(cf. (2.1)). We take (2.2) as a Cartan subalgebra and use the settings in (2.3)-(2.5). Then the root system of $o(2n+1, \mathbb{C})$ is
\[ \Phi_{B_n} = \{ \pm \varepsilon_i \pm \varepsilon_j, \pm \varepsilon_r | 1 \leq i < j \leq n; r \in \overline{1,n} \}. \] (3.2)
We take the set of positive roots
\[ \Phi_{B_n}^+ = \{ \varepsilon_i \pm \varepsilon_j, \varepsilon_r | 1 \leq i < j \leq n, r \in \overline{1,n} \}. \] (3.3)
In particular,
\[ \Pi_{B_n} = \{ \varepsilon_1 - \varepsilon_2, ..., \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n \} \] is the set of positive simple roots. (3.4)
Recall the set of dominate integral weights
\[ \Lambda^+ = \{ \mu \in L_\mathbb{Q} | 2(\varepsilon_i, \mu), (\varepsilon_i - \varepsilon_{i+1}, \mu) \in \mathbb{N} \text{ for } i \in \overline{1,n-1} \}. \] (3.5)
According to (2.5),
\[ \Lambda^+ = \{ \mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \mid \mu_i \in \mathbb{N}/2; \mu_i - \mu_{i+1} \in \mathbb{N} \text{ for } i \in \{1, \ldots, n-1\} \}. \] (3.6)
Given \( \mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \in \Lambda^+ \), there exists a unique sequence \( S(\mu) = \{n_0, n_1, n_2, \ldots, n_s\} \) such that (2.11) and (2.12) holds. Denote
\[ \rho = \frac{1}{2} \sum_{\nu \in \Phi^+_B} \nu. \] (3.7)
Then
\[ \frac{2(\rho, \nu)}{(\nu, \nu)} = 1 \quad \text{for } \nu \in \Pi_B. \] (3.8)
(e.g., cf. [Hu]). By (3.4),
\[ \rho = \sum_{i=1}^{n-1} (n-i+1/2) \varepsilon_i. \] (3.9)
For \( \lambda \in \Lambda^+ \), we denote by \( V(\lambda) \) the finite-dimensional irreducible \( o(2n+1, \mathbb{C}) \)-module with highest weight \( \lambda \). The \( (2n+1) \)-dimensional natural module of \( o(2n+1, \mathbb{C}) \) is \( V(\varepsilon_1) \) with weights \( \{0, \pm \varepsilon_i \mid i \in \{1, \ldots, n\}\} \). The following result is well known (e.g., cf. [FH]):

**Lemma 3.1 (Pieri’s formula).** Given \( \mu \in \Lambda^+ \) with \( S(\mu) = \{n_0, n_1, \ldots, n_s\} \),
\[ V(\varepsilon_1) \otimes_C V(\mu) \cong V(\mu) \oplus \bigoplus_{i=1}^{s-\delta_{\mu_0,0}-\delta_{\mu_1,1/2}} V(\mu - \varepsilon_{n_i}) \oplus \bigoplus_{r=1}^{s} V(\mu + \varepsilon_{1+n_r}). \] (3.10)

Note that the Casimir element of \( o(2n+1, \mathbb{C}) \) is
\[ \omega = \sum_{1 \leq i < j \leq n} \left[ (E_{i,n+j} - E_{j,n+i})(E_{n+j,i} - E_{n+i,j}) + (E_{n+j,i} - E_{n+i,j})(E_{i,n+j} - E_{j,n+i}) \right] + \sum_{i=1}^{n} \left[ (E_{0,i} - E_{n+i,0})(E_{i,0} - E_{0,n+i}) + (E_{i,0} - E_{0,n+i})(E_{0,i} - E_{n+i,0}) \right] + \sum_{i,j=1}^{n} (E_{i,j} - E_{n+j,n+i})(E_{j,i} - E_{n+i,n+j}) \in U(o(2n+1, \mathbb{C})). \] (3.11)
Set
\[ \tilde{\omega} = \frac{1}{2} (d(\omega) - \omega \otimes 1 - 1 \otimes \omega) \in U(o(2n+1, \mathbb{C})) \otimes_C U(o(2n+1, \mathbb{C})). \] (3.12)
By (3.11),
\[ \tilde{\omega} = \sum_{1 \leq i < j \leq n} \left[ (E_{i,n+j} - E_{j,n+i}) \otimes (E_{n+j,i} - E_{n+i,j}) + (E_{n+j,i} - E_{n+i,j}) \otimes (E_{i,n+j} - E_{j,n+i}) \right] + \sum_{i=1}^{n} \left[ (E_{0,i} - E_{n+i,0}) \otimes (E_{i,0} - E_{0,n+i}) + (E_{i,0} - E_{0,n+i}) \otimes (E_{0,i} - E_{n+i,0}) \right] + \sum_{i,j=1}^{n} (E_{i,j} - E_{n+j,n+i}) \otimes (E_{j,i} - E_{n+i,n+j}). \] (3.13)
Moreover, (2.22) also holds. Take the settings in (2.23) and (2.24). Denote

$$\ell(\mu) = \dim V(\mu).$$  \hfill (3.14)

Observe that

$$\begin{align*}
(\mu + 2\rho, \mu) - (\mu + 2\rho, \mu) - (\varepsilon_1 + 2\rho, \varepsilon_1) &= -2n, \\
(\mu + \varepsilon_i + 2\rho, \mu + \varepsilon_i) - (\mu + 2\rho, \mu) - (\varepsilon_1 + 2\rho, \varepsilon_1) &= 2(\mu_i + 1 - i)
\end{align*}$$  \hfill (3.15)

and

$$\begin{align*}
(\mu - \varepsilon_i + 2\rho, \mu - \varepsilon_i) - (\mu + 2\rho, \mu) - (\varepsilon_1 + 2\rho, \varepsilon_1) &= 2(i - 2n - \mu_i)
\end{align*}$$  \hfill (3.17)

for $\mu = \sum_{r=1}^n \mu_r \varepsilon_r$ by (3.9). Moreover, the algebra $U(o(2n + 1, \mathbb{C})) \otimes_{\mathbb{C}} U(o(2n + 1, \mathbb{C}))$ acts on $V(\varepsilon_1) \otimes_{\mathbb{C}} V(\mu)$ by

$$(\xi_1 \otimes \xi_2)(v \otimes u) = \xi_1(v) \otimes \xi_2(u)$$ for $\xi_1, \xi_2 \in U(o(2n + 1, \mathbb{C}))$, $v \in V(\varepsilon_1)$, $u \in V(\mu)$.  \hfill (3.18)

By Lemma 3.1, (3.12), (2.22) and (3.15)-(3.17), we get:

**Lemma 3.2.** Let $\mu = \sum_{i=1}^n \mu_i \varepsilon_i \in \Lambda^+$ with $S(\mu) = \{n_0, n_1, \ldots, n_s\}$. The characteristic polynomial of $\tilde{\omega}|_{V(\varepsilon_1) \otimes_{\mathbb{C}} V(\mu)}$ is

$$
(t + n)^{\ell(\mu)} \left[ \prod_{i=0}^{s - \delta_{\mu;0,0} - \delta_{\mu,1/2}} (t - \mu_{1+n_i} + n_i)^{\ell_{1+n_i} + 1}(\mu) \right] \prod_{j=1}^{s - 1} (t + \mu_{n_j} + 2n - n_j - 1)^{\ell_{n_j}(\mu)}. \hfill (3.19)
$$

We remark that the above lemma is also equivalent to special detailed version of Kostant’s characteristic identity. Set

$$\mathcal{A} = \mathbb{C}[x_0, x_1, x_2, \ldots, x_{2n}].$$  \hfill (3.20)

Then $\mathcal{A}$ forms an $o(2n + 1, \mathbb{C})$-module with the action determined via

$$E_{i,j}|_{\mathcal{A}} = x_i \partial_{x_j} \quad \text{for } i, j \in \{0, 2n\}. \hfill (3.21)$$

The corresponding Laplace operator and dual invariant are

$$\Delta = \partial_{x_0}^2 + 2 \sum_{r=1}^n \partial_{x_r} \partial_{x_{n+r}}, \quad \eta = \frac{1}{2} x_0^2 + \sum_{i=1}^n x_i x_{n+i}. \hfill (3.22)$$

Denote

$$D = \sum_{r=0}^{2n} x_r \partial_{x_r}, \quad J_i = x_i D - \eta \partial_{x_0}, \quad J_{n+i} = x_{n+i} D - \eta \partial_{x_i}, \hfill (3.23)$$

$$J_0 = x_0 D - \eta \partial_{x_0}, \quad K_i = x_0 \partial_{x_i} - x_{n+i} \partial_{x_0}, \quad K_{n+i} = x_0 \partial_{x_{n+i}} - x_i \partial_{x_0}. \hfill (3.24)$$
\[A_{i,j} = x_i \partial_{x_j} - x_{n+j} \partial_{x_{n+i+1}}, \quad B_{i,j} = x_i \partial_{x_{n+j}} - x_j \partial_{x_{n+i}}, \quad C_{i,j} = x_{n+i} \partial_{x_j} - x_{n+j} \partial_{x_i}\]  
(3.25)
Note that $C_{2n+1} \subset \mathcal{W}_{2n+1}$, and (2.50) and (2.51) except the last equation hold. Moreover,

$$\mathfrak{Z}(K_i) = K_i + E_{0,i} - E_{n+i,0}, \quad \mathfrak{Z}(D) = D + \sum_{p=0}^{2n} E_{p,p},$$

(3.38)

$$\mathfrak{Z}(K_{n+i}) = K_{n+i} + E_{0,n+i} - E_{i,0}, \quad \mathfrak{Z}(\partial x_0) = \partial x_0,$$

(3.39)

$$\mathfrak{Z}(J_0) = J_0 + \sum_{s=1}^{n} [x_s(E_{0,s} - E_{n+s,0}) + x_{n+s}(E_{0,n+s} - E_{s,0})] + x_0 \sum_{p=0}^{2n} E_{p,p},$$

(3.40)

$$\mathfrak{Z}(J_i) = J_i + \sum_{p=1}^{n} x_{n+p}(E_{i,n+p} - E_{p,n+i}) + \sum_{q=1}^{n} x_q(E_{i,q} - E_{n+q,n+i})$$

$$+ x_0(E_{i,0} - E_{0,n+i}) + x_i \sum_{p=0}^{2n} E_{p,p},$$

(3.41)

$$\mathfrak{Z}(J_{n+i}) = J_{n+i} + \sum_{p=1}^{n} x_{n+p}(E_{n+i,n+p} - E_{p,i}) + \sum_{q=1}^{n} x_q(E_{n+i,q} - E_{n+q,i})$$

$$+ x_0(E_{n+i,0} - E_{0,i}) + x_{n+i} \sum_{p=0}^{2n} E_{p,p}.$$  (3.42)

Furthermore,

$$\widehat{C}_{2n+1} = C_{2n+1} + o(2n + 1, \mathcal{A}) + \mathcal{A} \sum_{p=0}^{2n} E_{p,p}$$

(3.43)

forms a Lie subalgebra of $\mathcal{W}_{2n+1} + gl(2n + 1, \mathcal{A})$ and $\mathfrak{Z}(C_{2n+1}) \subset \widehat{C}_{2n+1}$. In particular, the element $\sum_{p=0}^{2n} E_{p,p}$ is a hidden central element.

Let $\mathcal{M}$ be an $o(2n + 1, \mathbb{C})$-module and let $b \in \mathbb{C}$ be a fixed constant. Then

$$\widehat{\mathcal{M}} = \mathcal{A} \otimes \mathcal{M}$$

(3.44)

becomes a $\widehat{C}_{2n+1}$-module with the action:

$$(d + f_1 A + f_2 \sum_{p=0}^{2n} E_{p,p})(g \otimes v) = (d(g) + b f_2 g) \otimes v + f_1 g \otimes A(v)$$

(3.45)

for $f_1, f_2, g \in \mathcal{A}$, $A \in o(2n + 1, \mathbb{C})$ and $v \in \mathcal{M}$. Moreover, we make $\widehat{\mathcal{M}}$ a $C_{2n}$-module with the action:

$$\xi(w) = \mathfrak{Z}(\xi)(w) \quad \text{for} \quad \xi \in C_{2n+1}, \quad w \in \widehat{\mathcal{M}}.$$  (3.46)

Furthermore, $\widehat{\mathcal{M}}$ becomes an $o(2n + 3, \mathbb{F})$-module with the action

$$A(w) = \mathfrak{Z}(\partial(A))(w) \quad \text{for} \quad A \in o(2n + 2, \mathbb{C}), \quad w \in \widehat{\mathcal{M}}$$

(3.47)

(cf. (2.45)-(2.48), (3.35) and (3.36)). By a proof similar to that of Lemma 2.3, we have:
Lemma 3.3  If $M$ is an irreducible $o(2n+1, \mathbb{C})$-module, then the space $U(\mathcal{J})(1 \otimes M)$ is an irreducible $o(2n + 3, \mathbb{C})$-submodule of $\hat{M}$.

Write
\[ x^\alpha = \prod_{i=0}^{2n} x_i^{\alpha_i}, \quad J^\alpha = \prod_{i=0}^{2n} J_i^{\alpha_i} \text{ for } \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{2n}) \in \mathbb{N}^{2n+1}. \quad (3.48) \]

For $k \in \mathbb{N}$, we set
\[ A_k = \text{Span}_\mathbb{C}\{x^\alpha \mid \alpha \in \mathbb{N}^{2n+1}, \sum_{i=0}^{2n} \alpha_i = k\}, \quad \hat{M}_{(k)} = A_k \otimes \mathbb{C} M \quad (3.49) \]
and
\[ (U(\mathcal{J})(1 \otimes M))_{(k)} = \text{Span}_\mathbb{C}\{J^\alpha (1 \otimes M) \mid \alpha \in \mathbb{N}^{2n+1}, \sum_{i=0}^{2n} \alpha_i = k\}. \quad (3.50) \]

Moreover,
\[ (U(\mathcal{J})(1 \otimes M))_{(0)} = \hat{M}_{(0)} = 1 \otimes M. \quad (3.51) \]

Furthermore,
\[ \hat{M} = \bigoplus_{k=0}^{\infty} \hat{M}_{(k)}, \quad U(\mathcal{J})(1 \otimes M) = \bigoplus_{k=0}^{\infty} (U(\mathcal{J})(1 \otimes M))_{(k)}. \quad (3.52) \]

Next we define a linear transformation $\varphi$ on $\hat{M}$ determined by
\[ \varphi(x^\alpha \otimes v) = J^\alpha (1 \otimes v) \quad \text{for } \alpha \in \mathbb{N}^{2n+1}, \, v \in M. \quad (3.53) \]

Note $A_1 = \sum_{i=0}^{2n} \mathbb{C} x_i$ forms the $(2n + 1)$-dimensional natural $L_0$-module (equivalently $o(2n + 1, \mathbb{C})$-module). According to (2.41), (2.42), (3.32) and (3.33), $\mathcal{J}$ forms an $L_0$-module with respect to the adjoint representation, and the linear map from $A_1$ to $\mathcal{J}$ determined by $x_i \mapsto J_i$ for $i \in \overline{0, 2n}$ gives an $L_0$-module isomorphism. Thus $\varphi$ can also be viewed as an $L_0$-module homomorphism from $\hat{M}$ to $U(\mathcal{J})(1 \otimes M)$. Moreover,
\[ \varphi(\hat{M}_{(k)}) = (U(\mathcal{J})(1 \otimes M))_{(k)} \quad \text{for } k \in \mathbb{N}. \quad (3.54) \]

Lemma 3.4. We have $\varphi|_{\hat{M}_{(1)}} = (b + \bar{\omega})|_{\hat{M}_{(1)}}$ (cf. (3.11)-(3.13)).

Proof. Recall $\hat{M}_{(1)} = A_1 \otimes \mathbb{C} M$. Let $i \in \overline{1, n}$ and $v \in M$. Expressions (3.40), (3.45) and (3.46) give
\[ \varphi(x_0 \otimes v) = \sum_{s=1}^{n} [x_s \otimes (E_{0,s} - E_{n+s,0})(v) + x_{n+s} \otimes (E_{0,n+s} - E_{s,0})(v)] + bx_0 \otimes v. \quad (3.55) \]
Moreover, (3.41), (3.45) and (3.46) imply

$$\varphi(x_i \otimes v) = \sum_{p=1}^{n} x_{n+p} \otimes (E_{i,n+p} - E_{p,n+i})(v) + x_0 \otimes (E_{i,0} - E_{0,n+i})(v)$$

$$+ \sum_{q=1}^{n} x_q \otimes (E_{i,q} - E_{n+q,n+i})(v) + bx_i \otimes v \quad (3.56)$$

for \( i \in \overline{1,n} \). Furthermore, (3.42), (3.45) and (3.46) yield

$$\varphi(x_{n+i} \otimes v) = \sum_{p=1}^{n} x_{n+p} \otimes (E_{n+i,n+p} - E_{p,i})(v) + x_0 \otimes (E_{n+i,0} - E_{0,i})(v)$$

$$+ \sum_{q=1}^{n} x_q \otimes (E_{n+i,q} - E_{n+q,n+i})(v) + bx_{n+i} \otimes v \quad (3.57)$$

for \( i \in \overline{1,n} \).

On the other hand, (3.13) and (3.21) yield

$$\tilde{\omega}(x_0 \otimes v) = \sum_{i=1}^{n} [-x_{n+i} \otimes (E_{i,0} - E_{0,n+i})(v) + x_i \otimes (E_{0,i} - E_{n+i,0})(v)], \quad (3.58)$$

$$\tilde{\omega}(x_i \otimes v) = \sum_{p=1}^{n} x_{n+p} \otimes (E_{i,n+p} - E_{p,n+i})(v) + x_0 \otimes (E_{i,0} - E_{0,n+i})(v)$$

$$+ \sum_{r=1}^{n} x_r \otimes (E_{i,r} - E_{n+r,n+i})(v), \quad (3.59)$$

$$\tilde{\omega}(x_{n+i} \otimes v) = \sum_{p=1}^{n} x_p \otimes (E_{n+i,p} - E_{n+p,i})(v) - x_0 \otimes (E_{0,i} - E_{n+i,0})(v)$$

$$+ \sum_{s=1}^{n} x_{n+s} \otimes (E_{n+i,n+s} - E_{s,i})(v). \quad (3.60)$$

Comparing the above six expressions, we get the conclusion in the lemma. \( \square \)

We use the definition (2.76) and we have the \( o(2n + 1, \mathbb{C}) \)-invariant operator

$$T = [J_0x_0 + \sum_{i=1}^{n} (J_i x_{n+i} + J_{n+i}x_i)]_{\tilde{\mathcal{M}}} \quad (3.61)$$

**Lemma 3.5.** We have \( T|_{\tilde{\mathcal{M}}^{(k)}} = (2b - 2n + k + 1)\eta \).

**Proof.** Let \( f \in \mathcal{A}_k \) and \( v \in M \). According to (3.24) and (3.40),

$$J_0x_0(f \otimes v) = x_0 \sum_{s=1}^{n} [x_s f \otimes (E_{0,s} - E_{n+s,0})(v) + x_{n+s} f \otimes (E_{0,n+s} - E_{s,0})(v)]$$

$$+ [(k + 1 + b)x_0^2 - \eta]f \otimes v - \eta x_0 \partial x_0(f) \otimes v. \quad (3.62)$$
Moreover, (2.79), (3.23), (3.41) and (3.42) give

\[
\sum_{i=1}^{n} (J_{i}x_{n+i} + J_{n+i}x_{i})(f \otimes v) = x_{0} \sum_{i=1}^{n} [x_{i}f \otimes (E_{n+i,0} - E_{0,i})(v) + x_{n+i}f \otimes (E_{i,0} - E_{0,n+i})(v)]
+ 2[(b + k + 1)(\sum_{i=1}^{n} x_{i}x_{n+i}) - n\eta](f) \otimes v - \eta(\sum_{i=1}^{2n} x_{i}\partial x_{i}(f) \otimes v).
\]

(3.63)

Thus

\[
T(f \otimes v) = [2(b + k + 1) - 2n - 1]\eta(f) \otimes v - \eta D(f) \otimes v = (2b + k + 1 - 2n)\eta f \otimes v.
\]

(3.64)

So the Lemma holds. \(\square\)

For \(0 \neq \mu = \sum_{i=1}^{n} \mu_{i}\varepsilon_{i} \in \Lambda^{+}\) with \(S(\mu) = \{n_{0}, n_{1}, ..., n_{s}\}\), we define

\[
\Theta(\mu) = \begin{cases} 
\emptyset & \text{if } \mu = (\sum_{i=1}^{n} \varepsilon_{i})/2, \\
\mu_{1} + 2n - n_{1} - N & \text{otherwise}.
\end{cases}
\]

(3.65)

**Theorem 3.6.** For \(0 \neq \mu \in \Lambda^{+}\), the generalized conformal \(o(2n + 2, \mathbb{C})\)-module \(\widehat{V}(\mu)\) defined by (2.50), (2.51) except the last equation, and (3.37)-(3.47) is irreducible if \(b \in \mathbb{C} \setminus \{n - N/2, \Theta(\mu)\}\).

**Proof.** By Lemma 3.3, it is enough to prove that the homomorphism \(\varphi\) defined in (3.53) satisfies \(\varphi(\widehat{V}(\mu)) = \widehat{V}(\mu)\). According to (3.54), we only need to prove

\[
\varphi(\widehat{V}(\mu)_{\langle k \rangle}) = \widehat{V}(\mu)_{\langle k \rangle}
\]

(3.66)

for any \(k \in \mathbb{N}\). We will prove it by induction on \(k\).

When \(k = 0\), (3.66) holds by the definition (3.53). Consider \(k = 1\). Write \(\mu = \sum_{i=1}^{n} \mu_{i}\varepsilon_{i} \in \Lambda^{+}\) with \(S(\mu) = \{n_{0}, n_{1}, ..., n_{s}\}\). According to Lemma 3.2 and Lemma 3.4 with \(M = V(\mu)\), the eigenvalues of \(\varphi|_{\widehat{V}(\mu)_{\langle 1 \rangle}}\) are among

\[
\{b - n, b + \mu_{n_{i-1}} - n_{i-1}, b - \mu_{n_{i}} - 2n + n_{i} \text{ for } i \in \overline{1, s}\}.
\]

(3.67)

Recall that \(\mu_{r} \in \mathbb{N}/2\) for \(r \in \overline{1, n}\),

\[
\mu_{i+1} - \mu_{i} \in \mathbb{N} \text{ for } i \in \overline{1, n-1}
\]

(3.68)

and (2.12) holds. So

\[
-\mu_{1+n_{i-1}} + n_{i-1}, \mu_{n_{i}} + 2n - n_{i} \in \mu_{1} + 2n - n_{1} - N \text{ for } i \in \overline{1, s}.
\]

(3.69)
If \( b \not\in \mu_1 + 2n - n_1 - \mathbb{N} \) and \( b \neq n \), then all the eigenvalues of \( \varphi|_{\overline{V(\mu)}_{(1)}} \) are nonzero. In the case \( \mu = (\sum_{i=1}^{n} \varepsilon_i)/2 \), the eigenvalues \( \varphi|_{\overline{V(\mu)}_{(1)}} \) are \( b - n \) and \( b + 1/2 \), which are not equal to 0 because of \( b \not\in n - \mathbb{N}/2 \). Thus (3.66) holds for \( k = 1 \).

Suppose that (3.66) holds for \( k \leq \ell \) with \( \ell \geq 1 \). Consider \( k = \ell + 1 \). Note that

\[
\varphi(\overline{V(\mu)}_{(\ell+1)}) = \sum_{i=0}^{2n} \varphi(x_i \overline{V(\mu)}_{(\ell)}) = \sum_{i=0}^{2n} J_i[\varphi(\overline{V(\mu)}_{(\ell)})] = \sum_{i=0}^{2n} J_i(\overline{V(\mu)}_{(\ell)})
\]

by the inductional assumption. To prove (3.66) with \( k = \ell + 1 \) is equivalent to prove

\[
\sum_{i=0}^{2n} J_i(\overline{V(\mu)}_{(\ell)}) = \overline{V(\mu)}_{(\ell+1)}.
\]

For any \( u \in \overline{V(\mu)}_{(\ell-1)} \), Lemma 3.5 says that

\[
J_0(x_0 u) + \sum_{i=1}^{n} [J_i(x_{n+i} u) + J_{n+i}(x_i u)] = (2b - 2n + \ell) \eta u.
\]

Since \( b \not\in n - \mathbb{N}/2 \), we have \( 2b - 2n + \ell \neq 0 \), and so (3.72) gives

\[
\eta u \in \sum_{i=0}^{2n} J_i(\overline{V(\mu)}_{(\ell)}) \quad \text{for } u \in \overline{V(\mu)}_{(\ell-1)}.
\]

Let \( g \otimes v \in \overline{V(\mu)}_{(\ell)} \). According to (3.40)-(3.46) and Lemma 3.4,

\[
J_0(g \otimes v) = -\eta \partial_{x_0}(g) \otimes v + g[(\ell + b + \bar{\omega})(x_0 \otimes v)],
\]

\[
J_i(g \otimes v) = -\eta \partial_{x_{n+i}}(g) \otimes v + g[(\ell + b + \bar{\omega})(x_i \otimes v)],
\]

\[
J_{n+i}(g \otimes v) = -\eta \partial_{x_i}(g) \otimes v + g[(\ell + b + \bar{\omega})(x_{n+i} \otimes v)]
\]

for \( i \in \overline{1, n} \) (cf. (2.90) and (2.91)). Since

\[
\eta \partial_{x_i}(g) \otimes v \in \sum_{r=0}^{2n} J_r(\overline{V(\mu)}_{(\ell)}) \quad \text{for } i \in \overline{0, 2n}
\]

by (3.73), Expressions (3.74)-(3.76) show

\[
g[(\ell + b + \bar{\omega})(x_i \otimes v)] \in \sum_{r=0}^{2n} J_r(\overline{V(\mu)}_{(\ell)}) \quad \text{for } i \in \overline{0, 2n}, \ g \in \mathcal{A}_\ell.
\]

According to Lemma 3.2 and 3.4, the eigenvalue of \( (\ell + b + \bar{\omega})|_{\overline{V(\mu)}_{(1)}} \) are among

\[
\{b + \ell - n, b + \ell + \mu_{1+n_{i-1}} - n_{i-1}, b + \ell - \mu_{n_i} - 2n + n_i \mid i \in \overline{1, s}\}.
\]

Again

\[
-\ell - \mu_{1+n_{i-1}} + n_{i-1}, -\ell - \mu_{n_i} + 2n - n_i \in \mu_1 + 2n - n_1 - \mathbb{N} \quad \text{for } i \in \overline{1, s}.
\]
If \( b \not\in \{n - N/2, \mu_1 + 2n - n_1 - N\} \), then all the eigenvalues of \( (\ell + b + \tilde{\omega})|_{V(\mu)}_{(1)} \) are nonzero. In the case \( \mu = (\sum_{i=1}^{n} \varepsilon_i)/2 \), the eigenvalues of \( (\ell + b + \tilde{\omega})|_{V(\mu)}_{(1)} \) are \( b + \ell - n \) and \( b + \ell + 1/2 \), which are not equal to 0 because of \( b \not\in n - N/2 \). Hence
\[
(\ell + b + \tilde{\omega})(V(\mu)_{(1)}) = V(\mu)_{(1)}.
\]
By (3.78) and (3.81),
\[
g(V(\mu)_{(1)}) \subset \sum_{r=0}^{2n} J_r(V(\mu)_{(\ell)}) \quad \text{for } g \in A_\ell,
\]
equivalently, (3.66) holds for \( k = \ell + 1 \). By induction, (3.66) holds for any \( k \in \mathbb{N} \). \( \square \)

We remark that the \( o(2n + 3, \mathbb{C}) \)-module \( V(\mu) \) is \( o(2n + 1, \mathbb{C}) \)-finite, that is, \( V(\mu) \) is of \((\mathcal{G}, \mathcal{K})\)-type with \( \mathcal{G} = o(2n + 3, \mathbb{C}) \) and \( \mathcal{K} = o(2n + 1, \mathbb{C}) \). Up to this stage, we do not know if the condition in Theorem 3.6 is necessary for the generalized conformal \( o(2n + 3, \mathbb{C}) \)-module \( V(\mu) \) to be irreducible if \( \mu \neq 0 \). In the case \( \mu = 0 \), the situation becomes clear.

**Theorem 3.7.** The generalized conformal \( o(2n + 3, \mathbb{C}) \)-module \( \hat{V}(0) \) is irreducible if and only if \( b \not\in -N \). When \( b = 0 \), \( \hat{V}(0) \) is isomorphic to the natural conformal \( o(2n+3, \mathbb{C}) \)-module \( A \), on which \( A(f) = \partial(A)(f) \) for \( A \in o(2n + 3, \mathbb{C}) \) and \( f \in A \) (cf. (2.45)-(2.48), (3.35) and (3.36)). The subspace \( \mathbb{C} \) forms a trivial \( o(2n+3, \mathbb{C}) \)-submodule of the conformal module \( A \) and the quotient space \( A/\mathbb{C} \) forms an irreducible \( o(2n+3, \mathbb{C}) \)-module.

**Proof.** Pick \( 0 \neq v_0 \in V(0) \). Then \( \hat{V}(0) = A \otimes v_0 \). We only list some facts different from the proof of Theorem 2.7. Since \( V(0) \) is the trivial \( o(2n + 1, \mathbb{C}) \)-module,
\[
J_0(f \otimes v_0) = [x_0(D + b) - \eta \partial x_0](f) \otimes v_0.
\]
Recall (3.22) and (3.49). Set
\[
\mathcal{H}_k = \{f \in A_k \mid \Delta(f) = 0\} \quad \text{for } k \in \mathbb{N}.
\]
Then \( \mathcal{H}_k \otimes v_0 \) is an irreducible \( o(2n+1, \mathbb{C}) \)-submodule also with the highest-weight vector \( x_1^k \otimes v_0 \). Moreover,
\[
[\Delta, \eta] = 1 + 2n + D.
\]
Furthermore,
\[
J_0(x_1^{\ell-r-1}x_{n+1}^r \otimes v_0) = (b + \ell - 1)x_1^{\ell-r-1}x_0x_{n+1}^r \otimes v_0
\]
and \( x_1^{\ell-2r-1}x_0 \in \mathcal{H}_{\ell-2r} \). By the similar arguments as those in the Proof of Theorem 2.7, we can prove the conclusion in Theorem 3.7. \( \square \)
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References

[BG] A. J. Bracken and H. S. Green, Vector operators and a polynomial identity for SO(n), J. Math. Phys. 12 (1971), 2099-2106.

[D] P. A. M. Dirac, Relativistic wave equations, Proc Roy. Soc. London Ser. A 155 (1936), 447-459.

[FH] W. Fulton and J. Harris, Representation Theory: A First Course, volume 129 of Graduate Texts in Mathematics, Readings in Mathematics, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1991.

[Gm1] M. D. Gould, Tensor operators and projection techniques in infinite dimensional representations of semi-simple Lie algebras, J. Phys. A: Math. Gen. 17 (1984), 1-17.

[Gm2] M. D. Gould, Characteristic identities for semi-simple Lie algebras, J. Aus. Math. Soc. Series B. Applied Mathematics 26 (1985), 257-283.

[G] H. S. Green, Characteristic identities for generators of GL(n), O(n) and Sp(n), J. Math. Phys. 12 (1971), 2106-2113.

[Ha] K. C. Hannabuss, Characteristic equations for semi-simple Lie groups, preprint, Math. Inst. Oxford (1972) (unpublished).

[Hu] J. E. Humphreys, Introduction to Lie algebras and representation theory, Springer-Verlag, New York-Heidelberg-Berlin, 1972.

[JG] P. D. Jarvis and H. S. Green, Casimir invariants and characteristic identities for generators of the general linear, special linear and orthosymplectic graded Lie algebras, J. Math. Phys. 20 (1979), 2115-2122.

[K] B. Kostant, On the tensor product of a finite and an infinite dimensional representation, J. Func. Anal. 20 (1975), 257-285.

[La] T. Larsson, Conformal fields: A class of representations of Vect(N)[J], Internat. J. Modern Phys. A 7 (1992), no. 26, 6493-6508.
[LT] W. Lin and S. Tan, Representations of the Lie algebra for quantum torus, *J. Algebra* **275** (2004), 250-274.

[OCC] D. M. O’Brien, A. Cant and A. L. Carey, On characteristic identities for Lie algebras, *Ann. Inst. H. Poincare Sect. A N.S.* **26** (1977), 405-429.

[R] S. E. Rao, Irreducible representations of the Lie algebra of the diffeomorphism of a $d$-dimensional torus, *J. Algebra* **182** (1992), 401-421.

[S1] G. Shen, Graded modules of graded Lie algebras of Cartan type (I)—mixed product of modules, *Science in China A* **29** (1986), 570-581.

[S2] G. Shen, Graded modules of graded Lie algebras of Cartan type (II)—positive and negative graded modules, *Science in China A* **29** (1986), 1009-1019.

[S3] G. Shen, Graded modules of graded Lie algebras of Cartan type (III)—irreducible modules, *Chin. Ann. of Math B* **9** (1988), 404-417.

[X] X. Xu, New generalized simple Lie algebras of Cartan type over a field with characteristic 0, *J. Algebra* **224** (2000), 23-58.

[Z] Y. Zhao, Irreducible representations of nongraded Witt type Lie algebras, *J. Algebra* **298** (2006), 540-562.

[ZX] Y. Zhao and X. Xu, Generalized projective representations for sl(n+1), *J. Algebra* **328** (2011), 132-154.