TOPOLOGICAL CHARACTERIZATION OF THE HYPERBOLIC MAPS IN THE SINE FAMILY

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Abstract. The purpose of this paper is to establish a topological characterization of all the hyperbolic maps in the Sine family \( \{ \lambda \sin(z) \mid \lambda \neq 0 \} \) which have super-attracting cycles.

1. Introduction

We assume that the readers are familiar with the paper [6]. The topological characterization theorem of all post-critically finite rational maps, which was established by Thurston in early 1980’s, plays a central role within complex dynamics. Since then, Hubbard has asked to what extent the result generalizes. For example, does it generalize to certain family of entire functions? or to rational functions which are no longer post-critically finite? Indeed, Hubbard writes in the introduction in his recent Teichmüller theory text [9]: “In particular, we hope Thurston’s theorem \( \cdots \) might be extended to mappings that are not post-critically finite.”

Thurston’s proof depends essentially on the fact that the branched covering map of the sphere to itself is of finite degree and that the dimension of the underlying Teichmüller space is finite. The former does not hold for transcendental entire functions and the later does not hold for holomorphic maps with post-critical sets being infinite. Nevertheless, by getting around these difficulties, Hubbard, Schleicher and Shishikura extended the Thurston’s theorem to exponential family [10]; Cui, Jiang and Sullivan extended it to sub-hyperbolic rational maps [3][4][5][16]; the author extended it to certain family of rational maps with bounded type Siegel disks [14]. Besides these, the reader may also refer to [2] for some relative knowledge in this aspect.

Let \( f_\lambda(z) = \lambda \sin(z), \lambda \neq 0 \). The main interest of the iteration of this family lies in the fact that it exhibits the typical features of quadratic polynomials and exponential maps both of which have been extensively studied since 1980’s. For instance, in the parameter plane of the Sine family, one can find infinitely many periodic escaping rays, and between these rays, one can also find infinitely many embedded copies of the Mandelbrot set. To understand

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how the rays and the Mandelbrot copies are organized together, a key tool is the topological characterization of all hyperbolic maps in the Sine family. The purpose of this paper is to establish such a characterization.

We say $f_\lambda$ is hyperbolic if every critical point of $f_\lambda$ is attracted to some attracting periodic cycle. By the symmetry, if $f_\lambda$ is hyperbolic, it has either one or two attracting periodic cycles. In the latter case, the two periodic cycles are symmetric about the origin. Note that $f_\lambda$ has no asymptotic values and thus $f_\lambda$ has exactly two singular values, $-\lambda$ and $\lambda$. Since the order of $f_\lambda$ is one, by [8] it follows that $f_\lambda$ has neither wandering domains nor Baker domains.

We say $\lambda \neq 0$ is a hyperbolic parameter if $f_\lambda$ is hyperbolic. The set of hyperbolic parameters is an open set in the plane. The components of this set are called hyperbolic components of the parameter plane. Among all these hyperbolic components, there is a very special one which is the punctured unit disk. The dynamics of the maps corresponding to the parameters in this component had been clearly understood. For instance, it was shown in [7] that for $0 < |\lambda| < 1$, $f_\lambda$ has an immediate attracting basin at the origin which is actually the unique Fatou component of $f_\lambda$ and is therefore unbounded and contains all the critical points of $f_\lambda$. In an unpublished work [15], we show that for hyperbolic parameters $\lambda$ with $|\lambda| > 1$, there are exactly two escaping rays landing on the origin. This allows us to construct certain puzzle pieces $V \subset U$ such that the restriction $f_\lambda^m : V \to U$ can be thickened into a polynomial-like map, where $m$ is the length of the attracting periodic cycle of $f_\lambda$. It follows that for hyperbolic maps $f_\lambda$ with $|\lambda| > 1$, all the Fatou component are bounded, and each Fatou component contains at most one critical point. In particular, one can show that for each hyperbolic component other than the punctured unit disk, there is a unique parameter $\lambda_0$, called the center of the hyperbolic component, such that one of the critical points of $f_{\lambda_0}$ is periodic. In this case we call $f_{\lambda_0}$ a center hyperbolic map. By the symmetry, $f_{\lambda_0}$ is a center hyperbolic map if and only if it has either one or two super-attracting cycles. Through a standard quasiconformal surgery, any hyperbolic map $f_\lambda$ with $|\lambda| > 1$ can be transformed to a map $f_{\lambda_0}$ such that $\lambda_0$ is the center of the hyperbolic component containing $\lambda$. It turns out that the center hyperbolic map carries all the essential topological data and dynamical properties of all the maps with the parameters belonging to the same hyperbolic component. This means to topologically characterize the hyperbolic maps $f_\lambda$ with $|\lambda| > 1$, it suffices to do this for all the center hyperbolic maps.

Let $\mathbb{C}$ denote the complex plane. Let $f : \mathbb{C} \to \mathbb{C}$ be a finitely or infinitely branched covering map. We will call $\Omega_f = \{ x \in S^2 \mid \deg_f(x) \geq 2 \}$ the critical set and $P_f = \bigcup_{k \geq 1} f^k(\Omega_f)$ the post-critical set. Every point in $\Omega_f$ is called a critical point of $f$. 
Definition 1.1. Let $\Sigma^c_{geom}$ denote the class of all center hyperbolic maps in the Sine family, that is, a map $f_\lambda$ belongs to $\Sigma^c_{geom}$ if and only if it has at least one periodic critical point.

Definition 1.2. Let $\Sigma^c_{top}$ denote the class of all the infinitely branched coverings $f : \mathbb{C} \to \mathbb{C}$ such that
1. $f(-z) = -f(z)$ and $f(z + \pi) = -f(z)$,
2. $\Omega_f = \{\pi/2 + k\pi \mid k \in \mathbb{Z}\}$,
3. $f$ maps the imaginary axis homeomorphically to a straight line $L$ passing through the origin, and moreover, $f$ maps the strip $S = \{x + iy \mid 0 < x < \pi, -\infty < y < \infty\}$ two-to-one onto one of the two half planes separated by $L$,
4. at least one of the critical points of $f$ is periodic.

Definition 1.3. Let $f, g \in \Sigma^c_{top}$. We say $f$ and $g$ are combinatorially equivalent to each other if there exist a pair of plane homeomorphisms $\phi, \psi : \mathbb{C} \to \mathbb{C}$ such that (1) $\phi f = g \psi$, and (2) $\psi$ is isotopic to $\phi$ rel $P_f$.

It is clear that $\Sigma^c_{geom} \subset \Sigma^c_{top}$. The main result of the paper is as follows.

Main Theorem. Let $f \in \Sigma^c_{top}$. Then there is a unique $g \in \Sigma^c_{geom}$ such that $f$ and $g$ are combinatorially equivalent to each other.

Since $P_f$ is a finite set, we may assume that $f$ is quasi-regular. As in [6], the basic idea of our proof is to consider the iteration of the pull back operator $\sigma_f$ defined on the Teichmüller space $T_f$ modeled on $(\mathbb{C}, P_f)$. Let $\mu(z)$ be a Beltrami coefficient on the complex plane. We say $\mu$ is admissible if
$$\mu(z) = \mu(-z) = \mu(z + \pi).$$

Let $\mu_0$ be an admissible Beltrami coefficient on the complex plane with $\|\mu_0\|_{\infty} = \kappa < 1$. Let $\mu_n$ denote the pull back of $\mu_0$ by $f^n$. Let $\phi_n : \mathbb{C} \to \mathbb{C}$ denote the quasi-conformal homeomorphism which fixes $0$ and $\pi$ and which solves the Beltrami equation given by $\mu_n$. We will show that there is a sequence of $\lambda_n \neq 0$ such that $\phi_n \circ f \circ \phi_n^{-1}(z) = \lambda_n \sin(z)$ (Lemma 4.3).

For $b > 0$ let $T_{f,b}$ be the subset of $T_f$ consisting of all the elements $[\mu]$ such that if $\phi : \mathbb{C} \to \mathbb{C}$ is the quasiconformal homeomorphism which fixes $0, \pi$ and the infinity and solves the Beltrami equation given by $\mu$, then the spherical distance between any two distinct points in $\phi(P_f) \cup \{\infty\}$ is not less than $b$.

We will prove that for any $\tau \in T_{f,b}$, there is a $0 < \delta < 1$ depending only on $b$ such that $\|d\phi|_{\tau}\| < \delta$ (Lemma 2). Consider the curve segment $\mu_t = [(1-t)\mu_0 + t\mu_1]$ in $T_f$. It is clear that $\mu_t$ is admissible for all $0 \leq t \leq 1$. The existence part of the Main Theorem will follow if one can show that $\mu_n \in T_{f,b}$ where $b > 0$ is some constant depending only on $\kappa$. This is the heart of the whole paper. Compared with [6], the difference here is that the map $f$ is of infinite degree and thus the
short geodesic argument cannot be applied directly. The key in our proof is Lemma 5.6 which says that the sequence \( \lambda_n \) is bounded away from 0 and the infinity. This lemma allows us to adapt the short geodesic argument in [6] to our situation and prove that \( \mu_n \in T_{f,b} \) for some constant \( b > 0 \) depending only on \( \kappa \). This proves the existence part of the Main Theorem. The uniqueness part follows easily from the strict contraction property of the pull back operator when restricted on a bounded subset of \( T_f \).

The core argument used here is different from the one used in [10], where the authors invented a very brilliant argument, which is called "limit quadratic differential argument". Using this argument, they actually showed that, no matter whether the geometry is bounded or not, the push forward operator always strictly decreases the norm of the quadratic differential. An interesting question is how to adapt the "limit quadratic differential argument" to deal with the situation in the present paper, or more generally, to deal with entire functions with finite forward critical orbits. To my knowledge, the main issue in such adaption is caused by the presence of the critical points which does not occur for the exponential family. More precisely, near the critical point, the distribution of the mass of the quadratic differential becomes essentially different after the operation of the push forward operator, and this may break down the "limit quadratic differential argument" presented in [10]. Seeking a way to overcome this problem will be much desirable in the further study.

The organization of the paper is as follows. We fix a \( f \in \Sigma_{top} \) throughout the whole paper. In §2, we define the Teichmüller space \( T_f \) and present a basic background of the Teichmüller theory. In §3, we introduce the pull back operator \( \sigma_f : T_f \to T_f \). The contents in both §2 and §3 are quite standard, see [6], [10] and [16]. In particular, the presentation in these two sections follows almost the same line as in [16]. We included them here just for the completeness of the proof and the readers’ convenience. In §4, we prove that the pull back operator will produce a sequence of complex structures \( \mu_n \) and a sequence of complex numbers \( \lambda_n \). In §5, we prove that the sequence \( \lambda_n \) is bounded away from the zero and the infinity. This is the key lemma of the whole paper. In §6, we prove the bounded geometry of \( \mu_n \) by adapting the short geodesic argument in [6]. In §7, we prove the Main Theorem.

2. The Teichmüller space \( T_f \)

Let \( f \in \Sigma_{top} \) and be fixed throughout the following. In this section, we will define the Teichmüller space and lay the foundation of the Teichmüller theory which suffices for our later use.

For more detailed knowledge about the Teichmüller theory, we refer the reader to [9] and [11].

**Definition 2.1.** The Teichmüller space \( T_f \) is the Teichmüller space modeled on \((\mathbb{C}, P_f)\).
The Teichmüller space $T_f$ can be constructed as the space of all the Beltrami coefficients defined on $\mathbb{C}$ module the following equivalent relation: let $\mu$ and $\nu$ be two Beltrami coefficients defined on $\mathbb{C}$ and let

$$\phi_\mu : \mathbb{C} \to S$$

be two quasiconformal homeomorphisms which solve the Beltrami equations given by $\mu$ and $\nu$, respectively. We say $\mu$ and $\nu$ are equivalent to each other if there exists a holomorphic isomorphism $h : R \to S$ such that the map $\phi_\mu$ and $h \circ \phi_\nu$ are isotopic to each other rel $P_f$, that is, there is a continuous family of quasiconformal homeomorphisms $g_t : \mathbb{C} \to S$, $0 \leq t \leq 1$, such that

1. $g_0 = \phi_\mu$,
2. $g_1 = h \circ \phi_\nu$,
3. $g_t(z) = \phi_\mu(z) = (h \circ \phi_\nu)(z)$ for all $0 \leq t \leq 1$ and $z \in P_f$.

In the following we use $[\mu]$ to denote the element in $T_f$ represented by $\mu$.

Now let us give a brief description of the relative background about the Teichmüller space $T_f$.

Let $M(\mathbb{C})$ denote the space of all the measurable Beltrami differentials $\mu(z) \frac{dz}{\overline{dz}}$ on $\mathbb{C}$ with $\|\mu\|_\infty < \infty$. Then $M(\mathbb{C})$ has a natural complex analytic structure, and moreover, it is a Banach analytic manifold with respect to the norm $\|\cdot\|_\infty$. Let $B(\mathbb{C}) \subset M(\mathbb{C})$ denote the unit ball. Then $B(\mathbb{C})$ consists of all the Beltrami coefficients on $\mathbb{C}$. Let

$$P : B(\mathbb{C}) \to T_f$$

be the projection map given by $\mu \mapsto [\mu]$.

**Lemma 2.1** (See Chapter 6 of [9]). There exists a unique complex analytic structure on $T_f$ such that with respect to this structure, the map $P$ is complex analytic, and moreover, the map $P$ is a holomorphic split submersion.

Let $\mu$ be a Beltrami coefficient defined on $\mathbb{C}$. Let

$$\phi_\mu : \mathbb{C} \to \mathbb{C}$$

be a quasiconformal homeomorphism which solves the Beltrami equation given by $\mu$. Let

$$M_\mu = \{ \xi(z) \frac{dz}{\overline{dz}} \mid \xi(z) \text{ is measurable and } \|\xi\|_\infty < \infty \}$$

be the linear space of all the Beltrami differentials defined on $\mathbb{C}$. Let

$$A_\mu = \{ q(z)dz^2 \mid q(z) \text{ is holomorphic and } \int_{\mathbb{C}\setminus\phi_\mu(P_f)} |q(z)|dz \wedge d\overline{z} < \infty \}$$

be the linear space of all the integrable holomorphic quadratic differentials defined on $\mathbb{C} \setminus \phi_\mu(P_f)$. It is easy to see that any $q \in A_\mu$ can have only simple poles.
A Beltrami differential \( \xi(z) \frac{dz}{dz} \in M_\mu \) is called infinitesimally trivial if
\[
\int_{C \setminus \phi_\mu(P_f)} \xi(z) q(z) |dz|^2 = 0
\]
holds for all \( q(z) dz^2 \in A_\mu \).

Let \( N_\mu \subset M_\mu \) be the subspace of all the infinitesimally trivial Beltrami differentials. Then the tangent space of \( T_f \) at \([\mu]\) is isomorphic to the quotient space \( M_\mu / N_\mu \).

Let \( \mu \) be a Beltrami coefficient defined on \( \mathbb{C} \). Let \( \xi \) be a tangent vector of \( T_f \) at \([\mu]\) which is identified with a Beltrami differential \( \xi(z) \frac{dz}{dz} \) defined on \( \mathbb{C} \).

**Definition 2.2.** The Teichmüler norm of the tangent vector \( \xi \) is defined to be
\[
\|\xi\| = \sup \left| \int_{C \setminus \phi_\mu(P_f)} q(z) \xi(z) |dz|^2 \right|,
\]
where the sup is taken over all \( q(z) dz^2 \in A_\mu \) with \( \int_{\phi_\mu(C \setminus P_f)} |q(z)| |dz|^2 = 1 \).

**Definition 2.3.** Let \([\mu],[\nu] \in T_f \). The Teichmüller distance \( d_T([\mu],[\nu]) \) is defined to be
\[
\frac{1}{2} \inf \log K(\phi_\mu \circ \phi_\nu^{-1})
\]
where \( \phi_\mu \) and \( \phi_\nu \) are quasi-conformal mappings with Beltrami coefficients \( \mu \) and \( \nu \) and the inf is taken over all \( \mu \) and \( \nu \) in the same Teichmüller classes as \( \mu \) and \( \nu \), respectively.

**Lemma 2.2** (see Chapter 5 of [11]). Let \( \mu \) and \( \nu \) be two Beltrami coefficients defined on \( \mathbb{C} \). Then
\[
d_T([\mu],[\nu]) = \inf \int_0^1 \|\tau'(t)\| dt
\]
where inf is taken over all the piecewise smooth curves \( \tau(t) \) in \( T_f \) such that \( \tau(0) = [\mu] \) and \( \tau(1) = [\nu] \).

3. The pull back operator \( \sigma_f \)

Since \( P_f \) is a finite set, we may assume that \( f \) is a quasiregular map throughout the following.

Remind that for a Beltrami coefficient \( \mu \) defined on \( \mathbb{C} \), the pull back of \( \mu \) by \( f \), which is denoted by \( f^*(\mu) \), is defined to be
\[
(f^* \mu)(z) = \frac{\mu_f(z) + \mu(f(z)) \theta(z)}{1 + \mu_f(z) \mu(f(z)) \theta(z)}
\]
where \( \theta(z) = \overline{f_z} / f_z \) and \( \mu_f(z) = f_z / f_z \). It is important to note that if \( \mu \) depends complex analytically on \( t \), then so does \( f^*(\mu) \).

**Lemma 3.1.** The map \( f^* \) induces a complex analytic operator \( \sigma_f : T_f \rightarrow T_f \).
Proof. Since $P_f$ is forward invariant and contains all the critical values, it follows that the map $\sigma_f$ is well defined. Note that by (1) the map
\[ f^* : B(\mathbb{C}) \to B(\mathbb{C}) \]
is complex analytic. Since by Lemma 2.1 the projection map
\[ P : B(\mathbb{C}) \to T_f \]
is a holomorphic split submersion, it follows that $\sigma_f$ is analytic also. This completes the proof of the lemma. \[ \square \]

Let $\phi_\mu, \phi_{\tilde{\mu}} : \mathbb{C} \to \mathbb{C}$ denote the quasiconformal homeomorphisms which fix 0, $\pi$, and the infinity and which solve the Beltrami equations given by $\mu$ and $\tilde{\mu}$, respectively. Let
\[ g = \phi_{\mu} \circ f \circ \phi_{\bar{\mu}}^{-1}. \]
It is clear that $g$ is an entire function and the following diagram commutes.

\[
\begin{array}{ccc}
(C, P_f) & \rightarrow & (C, \phi_{\bar{\mu}}(P_f)) \\
\downarrow f & & \downarrow g \\
(C, P_f) & \rightarrow & (C, \phi_{\mu}(P_f))
\end{array}
\]

In the next section we will show that $g(z) = \lambda \sin(z)$ for some $\lambda \neq 0$. Now suppose that $\xi$ is a tangent vector of $T_f$ at $\tau = [\mu]$. This means that there is a smooth curve of Beltrami coefficients $\gamma(t)$ defined on $\mathbb{C} \setminus P_f$, such that $\gamma(0) = \mu$ and
\[
(2) \quad \xi = \frac{d}{dt} \bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_{\bar{\mu}}^{-1}}.
\]
Let $d\sigma_f \big|_\tau$ denote the tangent map of $\sigma_f$ at $\tau$. Let $\tilde{\xi} = d\sigma_f \big|_\tau(\xi)$.

**Lemma 3.2.** Let $\xi$ and $\tilde{\xi}$ be as above. Then
\[
(3) \quad \tilde{\xi}(w) = \xi(g(w)) \frac{g'(w)}{g'(w)}.
\]

Proof. Note that
\[
\tilde{\xi} = \frac{d}{dt} \bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_{\bar{\mu}}^{-1}} = \frac{d}{dt} \bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_{\bar{\mu}}^{-1} \circ \phi_{\mu} \circ \phi_{\bar{\mu}}^{-1}} = \frac{d}{dt} \bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_{\bar{\mu}}^{-1} \circ g}.
\]
Since $g$ is an entire function, by (1) we have
\[
\mu_{\phi_{\gamma(t)} \circ \phi_{\bar{\mu}}^{-1} \circ g}(w) = \mu_{\phi_{\gamma(t)} \circ \phi_{\bar{\mu}}^{-1}}(g(w)) \frac{g'(w)}{g'(w)}
\]
The lemma then follows from (2). \[ \square \]
Let $\tilde{q} = \tilde{q}(w) dw^2$ be a non-zero integrable holomorphic quadratic differential defined on $\mathbb{C} \setminus \phi_\tilde{q}(P_f)$. Define
\begin{equation}
q(z) = \sum_{g(w) = z} \frac{\tilde{q}(w)}{|g'(w)|^2}.
\end{equation}
It is easy to see that $q = q(z) dz^2$ is a holomorphic quadratic differential defined on $\mathbb{C} \setminus \phi_\mu(P_f)$. We call $q$ the push forward of $\tilde{q}$.

**Proposition 3.1.** For $q$ and $\tilde{q}$ given as above, we have
\begin{equation}
\int_{\mathbb{C} \setminus \phi_\mu(P_f)} |q(z)| \, |dz|^2 \leq \int_{\mathbb{C} \setminus \phi_\mu(P_f)} |\tilde{q}(w)| \, |dw|^2.
\end{equation}

**Proof.** By the definition of $\tilde{q}$, we have
\begin{equation}
\int_{\mathbb{C} \setminus \phi_\mu(P_f)} |q(z)| \, |dz|^2 = \int_{\mathbb{C} \setminus \phi_\mu(P_f)} \left| \sum_{g(w) = z} \frac{\tilde{q}(w)}{|g'(w)|^2} \right| |dz|^2.
\end{equation}
Since
\begin{equation}
\left| \sum_{g(w) = z} \frac{\tilde{q}(w)}{|g'(w)|^2} \right| \leq \sum_{g(w) = z} \frac{|\tilde{q}(w)|}{|g'(w)|^2}
\end{equation}
and
\begin{equation}
\int_{\mathbb{C} \setminus \phi_\mu(P_f)} |\tilde{q}(w)| \, |dw|^2 = \int_{\mathbb{C} \setminus \phi_\mu(P_f)} \sum_{g(w) = z} \frac{|\tilde{q}(w)|}{|g'(w)|^2} |dz|^2,
\end{equation}
Proposition 3.1 follows. \[\square\]

**Proposition 3.2.** We have the following duality of the pairing,
\begin{equation}
\int_{\mathbb{C} \setminus \phi_\mu(P_f)} \tilde{\xi}(z) q(z) \, |dz|^2 = \int_{\mathbb{C} \setminus \phi_\mu(P_f)} \xi(w) \tilde{q}(w) \, |dw|^2.
\end{equation}

**Proof.** It follows easily from (3), (4) and the fact that $|dz|^2 = |g'(w)|^2 |dw|^2$. \[\square\]

As a direct consequence of Propositions 3.1 and 3.2, we have

**Corollary 3.1.** Let $\tau \in T_f$. Then $\|d\sigma_f|_\tau\| \leq 1$.

**Remark 3.1.** Let $Q_f = P_f \cup \{0, \pi\}$ and $W_f = P_f \cup \{k\pi \mid k \in \mathbb{Z}\}$. As in §2, one may define the Teichmüller space modeled on $(\mathbb{C}, Q_f)$ or $(\mathbb{C}, W_f)$. It is not difficult to see that the map $f$ also induces an analytic pull back operator $\sigma_f$ defined on these two new Techmüller spaces. To simplify the notation, in the following we use the same notation $T_f$ to denote the Teichmüller space for each of these three cases, that is, the Teichmüller space modeled on $(\mathbb{C}, X)$ with $X = P_f, Q_f$ or $W_f$. 
4. The sequence $\lambda_n$

We need a result of [7].

**Lemma 4.1** (Lemma 1 of [7]). Let $g$ be an entire function. If there exist two homeomorphisms $\phi, \psi : \mathbb{C} \rightarrow \mathbb{C}$ such that $\phi(\sin(z)) = g(\psi(z))$, then $g(z) = a \sin(bz + c) + d$ where $a, b, c$ and $d$ are some constants and $ab \neq 0$.

Let $\mu(z)$ be a Beltrami coefficient on the complex plane. Recall that $\mu$ is admissible if

$$\mu(z) = \mu(-z) = \mu(z + \pi).$$

Let $\mu_0$ be an admissible Beltrami coefficient on the complex plane with $\|\mu_0\|_{\infty} = \kappa < 1$. Let $\mu_n$ denote the pull back of $\mu_0$ through the iterations $f^n$. For $n \geq 0$, let $\phi_n : \mathbb{C} \rightarrow \mathbb{C}$ be the quasiconformal homeomorphism which solves the Beltrami equation given by $\mu_n$ and which fixes $0, \pi$ and $\infty$. It is clear that for every $n \geq 0$, there is an entire function $g_n(z)$ such that $\phi_n \circ f(z) = g_n \circ \phi_{n+1}(z)$.

**Lemma 4.2.** For every $n \geq 0$, there exist constants $a_n, b_n, c_n$ and $d_n$ such that $g_n(z) = a_n \sin(b_n z + c_n) + d_n$ and $a_n b_n \neq 0$.

**Proof.** By Lemma 4.1, it suffices to prove that there exist a pair of homeomorphisms $\theta, \sigma : \mathbb{C} \rightarrow \mathbb{C}$ such that $\theta(f(z)) = \sin(\sigma(z))$. Since $f \in \Sigma_c^{\text{top}}$, it follows from the definition that $f$ maps the imaginary axis homeomorphically to a straight line $L$ and maps the strip

$$S = \{x + iy \mid 0 < x < \pi, -\infty < y < \infty\}$$

two-to-one onto one of the two half planes separated by $L$. Let us denote this half plane by $H$. Let $\theta : \mathbb{C} \rightarrow \mathbb{C}$ be a homeomorphism which maps $L$ to the imaginary axis and maps $H$ to the left half plane, and maps the critical value of $f$ to $\{1, -1\}$, and moreover, $\theta(-z) = -\theta(z)$. Then by lifting $\theta$ through the relation $\theta(f(z)) = \sin(\sigma(z))$ one can obtain a homeomorphism $\sigma : S \rightarrow S$. It is easy to see that $\sigma(z) + \pi = \sigma(z + \pi)$ and $\sigma(-z) = -\sigma(z)$ hold for all $z$ belonging to the imaginary axis. We can then periodically extend $\sigma$ to a homeomorphism of the plane to itself by the relation $\sigma(z + \pi) = \sigma(z) + \pi$. This completes the proof of Lemma 4.2. \qed

**Lemma 4.3.** For every $n \geq 0$, there is a $\lambda_n \neq 0$ such that

$$\phi_n \circ f(z) = g_n \circ \phi_{n+1}(z)$$

where $g_n(z) = \lambda_n \sin(z)$.

**Proof.** We claim that $\mu_n$ is admissible for every $n \geq 1$, that is,

$$\mu_n(z) = \mu_n(z + \pi) = \mu_n(-z).$$

(5)

This completes the proof of Lemma 4.3. \qed
Let us prove this by induction. Note that it is true for \( n = 0 \). Suppose it is true for \( \mu_{n} \). From (3) and the symmetry of \( f \), it follows that it is also true for \( \mu_{n+1} \). This proves the claim.

Next let us prove that for every \( n \geq 0 \), \( \phi_{n}(z) = -\phi_{n}(z) \) and \( \phi_{n}(z + \pi) = \phi(z) + \pi \). To see this, by (5) it follows that \( \phi_{n}(-z) = a \phi_{n}(z) + b \) for some constants \( a \) and \( b \) where \( a \neq 0 \). Since \( \phi_{n}(0) = 0 \) it follows that \( b = 0 \). It then follows that \( a^{2} = 1 \). It is clear that \( a = -1 \) since otherwise \( \phi_{n} \) is not a homeomorphism. The first assertion is proved. To prove the second assertion, by (5) again it follows that \( \phi_{n} \) is quasiconformal, it follows that \( \text{mod}(\phi_{n}) \rightarrow \infty \) as \( k \rightarrow \infty \). Since \( \phi_{n} \) is quasiconformal, it follows that \( \text{mod}(\phi_{n}(A_{k})) \rightarrow \infty \) as \( k \rightarrow \infty \). This implies that \( a = 1 \). This is because if \( a \neq 1 \), then as \( k \rightarrow \infty \), \( \phi_{n}(k \pi) \rightarrow \infty \), and the modulus of any annulus which separates \( \{0, \infty\} \) and \( \{k \pi, (k + 1) \pi\} \) such that the \( \text{mod}(A_{k}) \rightarrow \infty \) as \( k \rightarrow \infty \). Since \( \phi_{n} \) is quasiconformal, it follows that \( \text{mod}(\phi_{n}(A_{k})) \rightarrow \infty \) as \( k \rightarrow \infty \). This implies that \( a = 1 \). This is a contradiction. So we get \( a = 1 \). By the normalization condition \( \phi_{n}(\pi) = \pi \), it follows that \( b = \pi \). This proves the second assertion.

By Lemma 112 it follows that for every \( n \geq 0 \), there exist \( \alpha_{n}, \beta_{n}, \gamma_{n} \) and \( \lambda_{n} \) with \( \lambda_{n}, \alpha_{n} \neq 0 \) such that

\[
g_{n}(z) = \lambda_{n} \sin(\alpha_{n} z + \beta_{n}) + \gamma_{n}.
\]

By the first assertion we proved above, it follows that \( g_{n}(-z) = -g_{n}(z) \). By a simple calculation, it follows that \( \gamma_{n} = 0 \) and \( \beta_{n} = k \pi \) for some \( k \in \mathbb{Z} \). By changing the sign of \( \lambda_{n} \), if necessary, we may assume that \( \beta_{n} = 0 \).

All the above arguments imply that \( g_{n}(z) = \lambda_{n} \sin(\alpha_{n} z) \). It suffices to prove that \( \alpha_{n} = 1 \). Since \( \phi_{n}(z + \pi) = \phi_{n}(z) + \pi \) holds for every \( n \geq 0 \) and since the set of the zeros of \( f \) is \( \{k \pi \mid k \in \mathbb{Z}\} \), it follows that the set of the zeros of \( g_{n} \) is also \( \{k \pi \mid k \in \mathbb{Z}\} \). This implies that \( \alpha_{n} = 1 \) or \(-1 \). By changing the sign of \( \lambda_{n} \) we may assume that \( \alpha_{n} = 1 \). The lemma has been proved. \( \square \)

Let \( \phi_{n} : \mathbb{C} \rightarrow \mathbb{C} \) be the sequence of quasiconformal homeomorphisms in the proof of the above theorem.

**Lemma 4.4.** \( \phi_{n+1}(\pi/2 + k \pi) = \pi/2 + k \pi \) for all \( k \in \mathbb{Z} \) and \( n \geq 0 \).

**Proof.** By the proof of Lemma 113 it follows that \( \phi_{n+1}(-z) = -\phi_{n+1}(z) \) and \( \phi_{n+1}(z + \pi) = \phi_{n+1}(z) + \pi \). Thus it suffices to show that \( \phi_{n+1}(\pi/2) = \pi/2 \).

Since \( \phi_{n+1} \) maps a critical point of \( f \) to some critical point of \( g_{n} \), it follows from Lemma 113 that \( \phi_{n+1}(\pi/2) = \pi/2 + k_{0} \pi \) for some integer \( k_{0} \). Since \( \phi_{n+1}(z) = \phi_{n+1}(z + \pi) \), it follows from that \( \phi_{n+1}(-\pi/2) = -\pi/2 - k_{0} \pi \). Since \( \phi_{n+1}(-z) = -\phi_{n+1}(z) \), it follows that \( \phi_{n+1}(\pi/2) = \phi_{n+1}(\pi/2) + \pi \).

Thus we have

\[-\pi/2 - k_{0} \pi + \pi = \pi/2 + k_{0} \pi.\]

This implies that \( k_{0} = 0 \). This completes the proof of Lemma 4.4. \( \square \)
5. The bounds of the sequence $\lambda_n$

In §4, let $\mu_0$ be the standard complex structure on the complex plane $\mathbb{C}$. It is clear that $\mu_0$ is admissible. Thus we get a sequence of complex structures $\mu_n$ by pulling back $\mu_0$ through the iteration of $f$. Let $\phi_n$ be the sequence of quasiconformal homeomorphisms and $\lambda_n$ be the sequence of complex numbers obtained in §4. The main purpose of this section is to prove Lemma 5.6 which says that $\lambda_n$ is bounded away from the origin and the infinity. This is the key lemma of the whole paper. Before that, we need some preliminary lemmas.

For a Beltrami coefficient $\mu$ defined on $\mathbb{C}$, let $\phi_\mu : \mathbb{C} \to \mathbb{C}$ denote the quasiconformal homeomorphism of the plane which fix 0 and $\pi$. For a non-peripheral curve $\gamma \subset \mathbb{C} \setminus X$, let $\|\gamma\|_{u,X}$ denote the hyperbolic length of the simple closed geodesic in $\mathbb{C} \setminus \phi_\mu(X)$ which is homotopic to $\phi_\mu(\gamma)$. We say $\gamma$ is a $(\mu, X)$-simple closed geodesic if $\phi_\mu(\gamma)$ is a simple closed geodesic in $\mathbb{C} \setminus \phi_\mu(X)$.

Recall that $Q_f = P_f \cup \{0, \pi\}$ and $W_f = P_f \cup \{k\pi \mid k \in \mathbb{Z}\}$ (see Remark 3.1). By using the same argument as in the proof of Proposition 7.2 of [6], we have

**Lemma 5.1.** Let $T_f$ denote the Teichmüller space modeled on $(\mathbb{C}, X)$ with $X = P_f, Q_f$ or $W_f$. Let $\gamma \subset \mathbb{C} \setminus X$ be a non-peripheral curve. Then the map $\omega_\gamma : T_f \to \mathbb{R}$ given by

$$\omega_\gamma([\mu]) = \log \|\gamma\|_{u,X}$$

is a Lipschitz function with Lipschitz constant 2.

**Lemma 5.2.** Let $X = P_f, Q_f$ or $W_f$. Then there is a $1 < C_1 < \infty$ such that for every $n \geq 0$ and any non-peripheral curve $\gamma \subset \mathbb{C} \setminus X$, one has

$$\|\gamma\|_{x, \mu_{n+1}} / C_1 \leq \|\gamma\|_{x, \mu_n} \leq C_1 \|\gamma\|_{x, \mu_{n+1}}.$$

**Proof.** By Corollary 3.1 it follows that

$$d_{T_f}([\mu_n], [\mu_{n+1}]) \leq d_{T_f}([\mu_0], [\mu_1]).$$

Now lemma 5.2 follows from Lemma 5.1. □

Note that $f$ has either one or two periodic cycles containing critical points. In the later case, the two cycles are symmetric about the origin and thus has the same length. We call such cycle a critical periodic cycle. Let $m \geq 1$ denote the length of the critical periodic cycle(s). Let us label the points in the critical periodic cycle by

$$x_0, \ldots, x_{m-1}$$

such that $f(x_{i+1}) = x_i$ for $0 \leq i \leq m-2$ and $f(x_0) = x_{m-1}$ where $x_0 = k_0\pi/2$ for some integer $k_0 \in \mathbb{Z}$. Since $x_0$ and $-x_0$ are both periodic, we can always assume that $k_0$ is an even integer. It follows that

$$\lambda_n = \phi_n(x_{m-1}).$$
For the convenience of our later discussion, let us define the constant
\[ \kappa_0 = \frac{1}{10^4 m}. \]

Let \( d(\cdot, \cdot) \) denote the distance with respect to the Euclidean metric in the plane. Note that from the proof of Lemma 4.3, we have \( \phi_n(k\pi) = k\pi \) for all \( n \geq 0 \) and \( k \in \mathbb{Z} \).

**Lemma 5.3.** There exists an \( 0 < \epsilon_0 < 1 \) such that for any integers \( k \in \mathbb{Z} \) and \( 0 \leq i \leq m - 1 \), if \( d(k\pi, \phi_n(x_i)) < \epsilon_0 \kappa_0 \) for some \( n \geq 0 \), then either
\[ d(k\pi, \phi_l(x_i)) < \epsilon_0 \kappa_0 \]
for all \( l \geq n \) or
\[ \epsilon_0 \kappa_0 \leq d(k\pi, \phi_l(x_i)) < \kappa_0 \]
for some \( l \geq n \).

**Proof.** Let \( X = W_f \). For \( \kappa_0 \) defined above, it is easy to see that there exists a \( \delta_0 > 0 \) depending only on \( \kappa_0 \) such that for any \( n \geq 0 \) and any non-peripheral curve \( \gamma \subset \mathbb{C} \setminus X \) which separates \( \{k\pi, x_i\} \) and \( \{(k+1)\pi, \infty\} \) for some \( 0 \leq i \leq m - 1 \), one has \( \|\gamma\|_{\mu_n, X} > \delta_0 \) provided that \( d(k\pi, \phi_n(x_i)) \geq \kappa_0 \).

Note that there exists a short simple closed \( (\mu_n, X) \)-geodesic which separates \( \{k\pi, x_i\} \) and \( \{(k+1)\pi, \infty\} \) such that \( \|\gamma\|_{\mu_n, X} \) can be as small as wanted provided that \( d(k\pi, \phi_n(x_i)) \) is small. This implies that there exists an \( \epsilon_0 > 0 \) small such that when \( d(k\pi, \phi_n(x_i)) < \epsilon_0 \kappa_0 \), then one can find a short simple closed \( (\mu_n, X) \)-geodesic which separates \( \{k\pi, x_i\} \) and \( \{(k+1)\pi, \infty\} \) such that \( \|\gamma\|_{\mu_n, X} < \delta_0/C_1 \) where \( C_1 > 1 \) is the constant in Lemma 5.2. It follows from Lemma 5.2 that such \( \epsilon_0 \) is the desired number. The proof of Lemma 5.3 is completed.

Note that \( \lambda_n = \phi_n(x_{m-1}) \).

**Lemma 5.4.** There exist \( C_2 > 1 \) and \( A > 1 \) independent of \( n \) such that for every \( n \geq 0 \), if \( \kappa_0 \leq |\lambda_n| \leq 1/\kappa_0 \), then
\[ 1/A < |\lambda_{n+1}| < A, \]
and if \( |\lambda_n| < \kappa_0 \), then
\[ |\lambda_n|^{C_2} < |\lambda_{n+1}| < |\lambda_n|^{1/C_2}, \]
and if \( |\lambda_n| > 1/\kappa_0 \), then
\[ |\lambda_n|^{1/C_2} < |\lambda_{n+1}| < |\lambda_n|^{C_2}. \]

**Proof.** Let \( X = Q_f \). Then there is a constant \( L > 1 \) independent of \( n \) such that if \( \kappa_0 \leq |\lambda_n| \leq 1/\kappa_0 \), one has
\[ \|\xi_1\|_{\mu_n, X} > L^{-1} \quad \text{and} \quad \|\xi_2\|_{\mu_n, X} < L, \]
where \( \xi_1 \) and \( \xi_2 \) are respectively the shortest \( (\mu_n, X) \)-simple closed geodesics which separate \( \{0, x_{m-1}\} \) and \( \{\pi, \infty\} \), and \( \{0, \pi\} \) and \( \{x_{m-1}, \infty\} \).
By Lemma 5.2 it follows that
\[ \|\xi_1\|_{\mu_n+1, X} > (C_1 L)^{-1} \quad \text{and} \quad \|\xi_2\|_{\mu_n+1, X} < C_1 L. \]
This implies the existence of \( A \) such that the first inequality holds.

Now suppose that \( |\lambda_n| < \kappa_0 \). Let \( \eta_1 \) and \( \eta_2 \) be respectively the shortest \((\mu_n, X)\) and \((\mu_n+1, X)\)-simple closed geodesics which separate \( \{0, x_{m-1}\} \) and \( \{\pi, \infty\} \). Then one has
\[ \|\eta_1\|_{\mu_n, X}^{-1} = -\log |\lambda_n| + O(1) \]
and
\[ \|\eta_2\|_{\mu_n+1, X}^{-1} = -\log |\lambda_{n+1}| + O(1) \]
where \( O(1) \) is used to denote some constant with upper and lower bounds independent of \( n \). By Lemma 5.2 there is a constant \( 1 < C < \infty \) independent of \( n \) such that
\[ C^{-1} \|\eta_1\|_{\mu_n, X} \leq \|\eta_2\|_{\mu_n+1, X} \leq C \|\eta_1\|_{\mu_n, X}. \]
This implies the existence of \( C_2 \) so that the second inequality holds. The same argument can be used to prove the third inequality. The proof of Lemma 5.4 is completed.

-----

**Lemma 5.5.** There exists a monotone increasing function \( \beta : (0, +\infty) \to (0, +\infty) \) such that for all \( n \geq 0 \), if \( |\lambda_n| < M \) then \( |\phi_n(x_k)| < \beta(M) \) for all \( 0 \leq k \leq m-1 \).

**Proof.** We may assume that \( n \geq m \) where \( m \) is the length of the critical periodic cycle of \( f \). Suppose \( |\lambda_n| < M \). Since \( \lambda_n = \phi_n(x_{m-1}) \) and \( f(x_{i+1}) = x_i \) for \( 0 \leq i \leq m-2 \), it follows that
\[ \phi_{n-m+k+1}(x_k) = g_{n-m+k+1} \circ \cdots \circ g_{n-1}(\lambda_n) \]
holds for \( 0 \leq k \leq m-1 \). This, together with Lemma 5.4, implies that there is a constant \( 0 < \alpha(M) < \infty \) depending only on \( M \) such that
\[ |\phi_{n-m+k+1}(x_k)| \leq \alpha(M). \]
Let \( X = Q_f \). Let \( \gamma \) be the shortest \((\mu_n, X)\)-simple closed geodesic which separates \( (0, \pi) \) and \( \{x_k, \infty\} \). Then the above inequality implies that
\[ \|\gamma\|_{\mu_n-m+k+1, X} \]
has a positive lower bound depending only on \( M \). This and Lemma 5.2 imply that \( \|\gamma\|_{\mu_n, X} \) has a positive lower bound depending only on \( M \). Lemma 5.4 then follows.

-----

**Lemma 5.6.** There exist \( 0 < \delta < M < \infty \) independent of \( n \) such that
\[ \delta \leq |\lambda_n| \leq M \quad \text{for all} \quad n \geq 0. \]
Proof. Recall that the critical periodic cycle is labeled by
\[ x_0, \cdots, x_{m-1} \]
such that \( f(x_{i+1}) = x_i \) for \( 0 \leq i \leq m-2 \) and \( f(x_0) = x_{m-1} \) where
\[ x_0 = \frac{\pi}{2} + k_0 \pi \]
for some even integer \( k_0 \in \mathbb{Z} \). By Lemma \ref{lem:4.4} it follows that
\[ (6) \quad \phi_n(x_0) = x_0 \]
holds for all \( n \geq 0 \). This is the key of the whole proof.

By Lemma \ref{lem:4.3} we have
\[ (7) \quad g_n \circ \cdots \circ g_{n+m-1}(x_0) = x_0 \]
for every \( n \geq m \). By Lemma \ref{lem:5.4} if \( |\lambda_n| \) is small, then
\[ |\lambda_{n+l}|, \quad 0 \leq l \leq m-1, \]
are all small. Since \( g_{n+l}(z) = \lambda_{n+l} \sin(z) \) for all \( l \geq 0 \), it follows from the above equation that there exists a \( \delta > 0 \) so that \( |\lambda_n| \geq \delta \) for all \( n \geq 0 \). It remains to prove the sequence \( \{\lambda_n\} \) is contained in a compact subset of \( \mathbb{C} \).

Let us first claim that there is an \( 0 < M_1 < \infty \) independent of \( n \) such that
\[ |\phi_n(x_1)| < M_1 \]
holds for all \( n \geq 0 \). Let us prove the claim now. In fact, by Lemma \ref{lem:4.3} and \ref{lem:6.3} one has
\[ (8) \quad \lambda_n \sin(\phi_{n+1}(x_1)) = \phi_n(f(x_1)) = \phi_n(x_0) = x_0. \]

Let \( C_2 > 1 \) and \( A > 1 \) be the constants in Lemma \ref{lem:5.4}. Let \( \epsilon_0 > 0 \) and \( \kappa_0 > 0 \) be the constants in Lemma \ref{lem:5.3} Let
\[ D_0 > \max \left\{ |\lambda_0|^{C_2}, \left( \frac{|x_0|}{\epsilon_0 \kappa_0} \right)^{C_2}, A \right\} \]
be large enough such that if
\[ |\sin(z)| \leq \frac{|x_0|}{D_0^{1/C_2}}, \]
then
\[ d(z, k\pi) < \epsilon_0 \kappa_0 \]
for some integer \( k \in \mathbb{Z} \).

Assume that there exists an \( n \) such that \( |\lambda_n| > D_0 \). Otherwise one can take \( M = D_0 \). By the choice of \( D_0 \), one has \( |\lambda_0| < D_0^{1/C_2} \). It follows from Lemma \ref{lem:5.4} and the choice of \( D_0 \) that there is a least integer \( n_0 \geq 1 \) such that
\[ D_0^{1/C_2} \leq |\lambda_{n_0}| < D_0. \]
Since \( \lambda_{n_0} \sin(\phi_{n_0+1}(x_1)) = x_0 \), by the choice of \( D_0 \) it follows that
\[ (9) \quad D_0^{1/C_2} \leq |\lambda_{n_0}| < D_0. \]

Since \( |\lambda_{n_0}| \sin(\phi_{n_0+1}(x_1)) = x_0 \), by the choice of \( D_0 \) it follows that
\[ (10) \quad d(\phi_{n_0+1}(x_1), k\pi) < \epsilon_0 \kappa_0 \]
where $k \in \mathbb{Z}$. It is easy to see that

$$|k| < \beta(D_0^{C_2}) + 1.$$  

Let us prove (11) now. In fact, By (9) and Lemma 5.2 it follows that

$$|\lambda_{n_0+1}| < D_0^{C_2}.$$  

By Lemma 5.5 it follows that

$$|\phi_{n_0+1}(x_1)| < \beta(D_0^{C_2}).$$  

Since $0 < \epsilon_0 \kappa_0 < 1$ is very small, (11) then follows from (10) and (12).

Now by (10) and Lemma 5.3, either

$$d(\phi_{l+1}(x_1), k\pi) < \epsilon_0 \kappa_0$$  

holds for all $l \geq n_0$ or there is a least integer $m_1 > n_0$ such that

$$\epsilon_0 \kappa_0 \leq d(\phi_{m_1+1}(x_1), k\pi) < \kappa_0.$$  

Since $\kappa_0$ is small, it follows that $|\sin(\phi_{m_1+1}(x_1))| > \epsilon_0 \kappa_0/2$. It follows from (8) that

$$|\lambda_{m_1}| < \frac{2|x_0|}{\epsilon_0 \kappa_0} \leq D_0^{1/C_2}.$$  

By Lemma 5.3 and the choice of $D_0$ there is a least integer $n_1 > m_1$ such that

$$D_0^{1/C_2} \leq |\lambda_{n_1}| < D_0.$$  

We claim that there is a $M_1$ which depends only on $D_0, C_2$ such that for every $n_0 \leq l \leq n_1$, one has

$$|\phi_l(x_1)| < M_1.$$  

Let us prove the claim. Note that $\phi_{n_0}(x_1) \leq \beta(D_0)$ by (9), and when $n_0 \leq l \leq m_1$, from the above argument we have $d(\phi_{l+1}(x_1), k\pi) < \kappa_0$ where $k$ is some integer with $|k| < \beta(D_0^{C_2}) + 1$ by (11), and when $m_1 < l \leq n_1$, we have $|\lambda_l| < D_0$ and thus by Lemma 5.5 we have $|\phi_l(x_1)| < \beta(D_0)$. This implies (14) by taking $M_1 = (\beta(D_0^{C_2}) + 1)\pi + 1$.

Now in (9), we may replace $n_0$ by $n_1$ and repeat the procedure from (9) to (13). By induction, we either stop at some $n_k$ such that

$$d(\phi_{l+1}(x_1), k\pi) < \epsilon_0 \kappa_0$$  

holds for all $l \geq n_k$ where $k \in \mathbb{Z}$ with $|k| < \beta(D_0^{C_2}) + 1$ or get a sequence $n_0 < n_1 < n_2 < \cdots$. From the above argument it follows that for both the cases,

$$|\phi_n(x_1)| < M_1$$  

holds for all $n \geq 0$. This proves the claim proposed in the beginning of the proof.

Now one can replace (8) by

$$\lambda_n \sin(\phi_{n+1}(x_2)) = \phi_n(x_1).$$
Note that in the argument to deduce (15), what we need is (8) and the uniform bound of $\phi_n(x_0)$ which is identically equal to $x_0$ for all $n$. Since $|\phi_n(x_1)| < M_1$ for some $0 < M_1 < \infty$ independent of $n$, we can use (10) and the same argument as above to get a constant $0 < M_2 < \infty$ independent of $n$ such that

$$|\phi_n(x_2)| < M_2$$

for all $n \geq 0$. By induction, we finally get an $0 < M < \infty$ independent of $n$ such that

$$|\lambda_n| = |\phi_n(x_{m-1})| \leq M$$

holds for all $n \geq 0$. This completes the proof of Lemma 5.6.

As a direct consequence of Lemmas 5.5 and 5.6, we get

Corollary 5.1. There exist $0 < \delta < M < \infty$ independent of $n$ such that

$$\delta \leq \phi_n(x_k) \leq M$$

holds for all $0 \leq k \leq m - 1$ and $n \geq 0$.

6. Bounded Geometry

Consider the Teichmüller space $T_f$ modeled on $(\mathbb{C}, X)$ with $X = P_f$. For $b > 0$ let $T_{f,b}$ be the subset of $T_f$ consisting of all the elements $[\mu]$ such that if $\phi : \mathbb{C} \to \mathbb{C}$ is the quasiconformal homeomorphism which fixes 0 and $\pi$ and which solves the Beltrami equation given by $\mu$, then the spherical distance between any two distinct points in $\phi(P_f) \cup \{\infty\}$ is not less than $b$. The main tool we used is the short geodesic argument in §8 of [6]. The situation here, however, is different from the case of rational maps which are of finite degrees. We will see that it is Lemma 5.6 which allows us to adapt the short geodesic argument in [6] to the present paper.

Let $\gamma$ be a non-peripheral curve in $\mathbb{C} \setminus P_f$. Let $\mu$ be a Beltrami coefficient on $\mathbb{C}$. Let $\phi_{\mu} : \mathbb{C} \to \mathbb{C}$ be the quasiconformal homeomorphism which solves the Beltrami equation given by $\mu$. Recall that $\|\gamma\|_{\mu,P_f}$ is used to denote the hyperbolic length of the simple closed geodesic in $\mathbb{C} \setminus \phi_{\mu}(P_f)$ which is homotopic to $\phi_{\mu}(\gamma)$. Also recall that $\gamma$ is called a $(\mu,P_f)$-geodesic if $\phi_{\mu}(\gamma)$ is a simple closed geodesic in $\mathbb{C} \setminus \phi_{\mu}(P_f)$.

For a hyperbolic Riemann surface $R$ and a simple closed geodesic $\gamma$ in $R$, we use $l_R(\gamma)$ to denote the hyperbolic length of $\gamma$ with respect to the hyperbolic metric in $R$.

Recall that $\{x_0, \ldots, x_{m-1}\}$ is the critical periodic cycle of $f$ where

$$x_0 = k_0 \pi + \pi/2$$

with $k_0$ being some even integer, and moreover, $f(x_{i+1}) = x_i$ for $0 \leq i \leq m - 2$ and $f(x_0) = x_{m-1}$. Let

$$F_n(z) = g_n \circ \cdots \circ g_{n+m-1}(z).$$

Then we have $F_n(x_0) = x_0$. As a direct consequence of Lemma 5.6 we have
Lemma 6.1. The sequence \( \{F_n\} \) is compact. More precisely, for any subsequence \( \{F_{n_k}\} \), one can take a subsequence \( F_{n_{k'}} \) of \( F_{n_k} \) and a non-holomorphic entire function \( F \) such that \( F_{n_{k'}} \) converge uniformly to \( F \) in any compact subset of the complex plane.

For \( r > 0 \), let \( V = B_r(x_0) \) denote the Euclidean disk with center \( x_0 \) and radius \( r \). For every \( n \geq 0 \), let

\[
\Omega_n = \{ z \mid F_n'(z) = 0 \}
\]

denote the set of the critical points of \( F_n \). Let \( P_n = F_n(\Omega_n) \) denote the set of the critical values of \( F_n \). It is easy to see that \( P_n = \phi_n(P_f) \). Therefore one can choose \( r > 0 \) so that

\[
( \bigcup_{n \geq 0} P_n ) \cap \partial B_r(x_0) = \emptyset.
\]

Let \( U_n \) denote the component of \( F_n^{-1}(V) \) which contains \( x_0 \). Then \( U_n \) is simply connected and the map

\[
F_n : U_n \rightarrow V
\]

is a holomorphic branched covering map. By Lemma 6.1 we have

Lemma 6.2. There exist an \( R > 1 \) and an integer \( N \geq 1 \) independent of \( n \) such that

\[
B_{1/R}(x_0) \subset U_n \subset B_R(x_0).
\]

and

\[
|F_n^{-1}(P_n) \cap U_n| \leq N.
\]

Let \( \mu_n \) and \( \phi \) be as defined in the beginning of §5. Let

\[
X_n = U_n \setminus \phi_{m+n}(P_f), \quad Y_n = V \setminus \phi_n(P_f), \quad Z_n = \mathbb{C} \setminus \phi_n(P_f).
\]

By the Collaring Theorem (for instance, see Theorem A.1 of [10]), we have

Lemma 6.3. There exist constants \( \epsilon > 0 \) and \( C > 0 \) such that if \( \gamma \) and \( \xi \) are simple closed geodesics in \( Z_n \) and \( Z_{m+n} \) which encloses \( \phi_n(x_0) = x_0 \) in their inside with \( l_{Z_n}(\gamma) < \epsilon \) and \( l_{Z_{m+n}}(\xi) < \epsilon \), then \( \gamma \) belongs to \( Y_n \), and \( \xi \subset X_n \), and moreover,

\[
l^{-1}_{X_n}(\eta) < l^{-1}_{Z_{m+n}}(\xi) < l^{-1}_{X_n}(\eta) + C
\]

and

\[
l^{-1}_{Y_n}(\omega) < l^{-1}_{Z_n}(\gamma) < l^{-1}_{Y_n}(\omega) + C,
\]

where \( \omega \) and \( \eta \) are respectively the simple closed geodesics in \( X_n \) and \( Y_n \) which are homotopic to \( \gamma \) and \( \xi \).
Let $W_n = U_n \setminus F_n^{-1}(P_n)$. It follows that $W_n \subset X_n$ and by Lemma [6.2], $|X_n \setminus W_n| \leq N$ for some integer $N \geq 0$ independent of $n$. By Theorem 7.1 of [6], for every short simple closed geodesic $\eta$ in $X_n$ with length $l_{X_n}(\eta) < \epsilon$, we have

\begin{equation}
\l_{X_n}^{-1}(\eta) < \sum_{\eta'} l_{W_n}^{-1}(\eta') + C
\end{equation}

where the sum is taken over all the simple closed geodesic in $W_n$ which are homotopic to $\eta$ in $X$ and which have length less than $\epsilon$.

Let

$$s_n = \sum_{\gamma} l_{Z_n}^{-1}(\gamma)$$

where the sum is taken over all the simple closed geodesics in $Z_n$ which have length less than $\epsilon$.

**Lemma 6.4.** There is an $0 < M < \infty$ such that $0 \leq s_n \leq M$.

**Proof.** By Proposition 7.2 of [6] or Lemma 5.1 of this paper, it suffices to prove that there exists a constant $0 < M < \infty$ such that $0 \leq s_{km} \leq M$ for all $k \geq 0$. Let us prove this as follows. By Lemma 6.3 we have

$$s_{(k+1)m} = \sum_{\gamma} l_{Z_{(k+1)m}}^{-1}(\xi) \leq \sum_{\eta} l_{X_{km}}(\eta) + C.$$

by (18) we have

$$\l_{X_{km}}^{-1}(\eta) < \sum_{\eta'} l_{W_{km}}^{-1}(\eta') + C$$

Since $F_{km} : W_{km} \to Y_{km}$ is a holomorphic covering map, it follows that $F_{km}(\eta')$ must be a short simple closed geodesic in $Y_{km}$ which enclose $x_0$ in its inside. Since $F$ is holomorphic in the inside of each $\eta'$ and since for any two distinct $\eta'$, one must be contained in the inside of the other, it follows that $F_{km}(\eta'_1) \neq F_{km}(\eta'_2)$ if $\eta'_1 \neq \eta'_2$. This implies that

$$\sum_{\eta'} l_{W_{km}}^{-1}(\eta') \leq \frac{1}{2} \sum_{\omega} l_{Y_{km}}^{-1}(\omega) + C$$

where the second sum is taken over all the short simple closed geodesics $\omega$ which enclose $x_0$ in their inside and with length less than $\epsilon$. These, together with Lemma 6.3, implies that

$$s_{(k+1)m} \leq \frac{1}{2} s_{km} + C.$$

This implies that $\{s_{km}\}$ is bounded and thus completes the proof of Lemma 6.4.

As a direct consequence of Lemma 6.4, we have
Corollary 6.1. There exists a \( \delta > 0 \) such that

\[
d(\phi_n(x_0), \phi_n(x)) = d(x_0, \phi_n(x)) > \delta
\]

holds for all \( n \geq 0 \) and \( x \in P_f \) with \( x \neq x_0 \).

Lemma 6.5. There exists a positive number \( b > 0 \) independent of \( n \) such that

\[
[\mu_n] \in T_{f,b} \quad \text{for all} \quad n \geq 0.
\]

Proof. Suppose the lemma were not true. Then there would be \( x_i \) and some \( x \) which may not belong to the same periodic cycle as \( x_i \), and an monotonically increasing integer sequence \( n_k \) such that as \( k \to \infty \),

\[
d(\phi_{n_k}(x_i), \phi_{n_k}(x)) \to 0
\]

where \( d(\cdot, \cdot) \) denotes the distance with respect to the Euclidean metric in the plane. Since

\[
\lambda_n \sin(\phi_{n+1}(x_{l+1})) = \phi_n(x_l), \quad 0 \leq l \leq m - 2,
\]

and since \( |\lambda_n| \) has a uniform upper bound by Lemma 5.6, we have

(19)

\[
d(\phi_{n_k-i}(x_0), \phi_{n_k-i}(f^i(x))) \to 0.
\]

Note that \( \phi_n(x_0) = x_0 \) by Lemma 4.4. So by (19) and by renewing the notations, we may assume that there exist some \( 1 \leq i \leq m - 1 \) and a monotonically increasing integer sequence \( n_k \) such that as \( k \to \infty \),

\[
d(x_0, \phi_{n_k}(x)) \to 0.
\]

This is a contradiction with Corollary 6.1. The proof of Lemma 6.5 is completed. \( \square \)

7. Proof of the Main Theorem

Let \( \tilde{q}(w)dw^2 \) be a holomorphic quadratic differential defined on \( \mathbb{C} \) with \( L^1 \)-norm equal to 1. Let \( q(w) = a \sin(bw + c) + d \) with \( ab \neq 0 \). Recall that by the push forward of \( \tilde{q}(w)dw^2 \) through \( g \), one can define a new quadratic differential \( q(z)dz^2 \), see (1).

Lemma 7.1. Let \( \tilde{q} \) and \( q \) as above. Then one has \( \|q\| < \|\tilde{q}\| \).

Proof. Since \( \tilde{q} \) is integrable, it follows that

\[
\tilde{q}(w) = \frac{\alpha}{w^k} + o\left(\frac{1}{|w|^k}\right)
\]

holds in a neighborhood of the infinity where \( \alpha \neq 0 \) and \( k \geq 3 \) is some integer.

Let \( \theta = \frac{\pi}{2}(1 - 1/k) \). Take \( r \gg 1 \) such that \( r \sin(\pi/2k) = b^{-1}k_0\pi \) for some integer \( k_0 > 0 \). Let \( w = re^{i\theta} \). It follows that \( \arg(w - 2b^{-1}k_0\pi) = \frac{\pi}{2}(1 + 1/k) \). Thus we have

\[
w^{-k} = -(w - 2b^{-1}k_0\pi)^{-k}.
\]
Since when $|w| \gg 1$ is large enough, $\tilde{q}(w)$ is dominated by $|w|^{-k}$. The last equation implies that when $|w|$ is large enough, the push forward operator will decrease the norm of $\tilde{q}$ due to the values of $\tilde{q}$ near $w$ and $w - 2b^{-1}k_0 \pi$ are almost negative of each other. The proof of Lemma 7.1 is completed.

Let us prove the existence part of the Main Theorem. Recall that $\mu_0$ is the standard complex structure on the complex plane and $\mu_n$ is the sequence of the pull backs of $\mu_0$ through the iterations of $f$. Let

$$\nu_0(t) = (1 - t)\mu_0 + t\mu_1.$$  

Then $\nu_0(t)$ is a smooth curve of Beltrami coefficients on the complex plane. Moreover, $\nu_0(t)$ is admissible for every $0 \leq t \leq 1$. Let $\nu_n(t)$ be the pull back of $\nu_0(t)$ through $f^n$. It follows that $[\nu_n(t)]$ is a smooth curve in $T_f$. We claim that there is a $0 < \delta < 1$ such that

$$\|d_T([\nu_n(t)])\| \leq \delta$$

holds for all $n \geq 0$ and $0 \leq t \leq 1$. Let us prove the claim now. For $0 \leq t \leq 1$ and $n \geq 0$, let $\phi_{n,t}$ denote the quasiconformal homeomorphism of the complex plane which solves the Beltrami equation given by $\nu_n(t)$ and which fixes 0 and $\pi$. Since $\nu_n(t)$ is admissible also, it follows that $\phi_{n,t} \circ f \circ \phi_{n+1,t}^{-1}(z) = \lambda_{n,t} \sin(z)$ where $\lambda_{n,t}$ is some complex number. Note that $[\nu_{n,0}] = [\mu_n]$ and $[\nu_{n,1}] = [\mu_{n+1}]$. By Corollary 3.1 we have

$$d_T([\nu_{n,1}], [\mu_n]) \leq d_T([\nu_{n,1}], [\mu_{n+1}]).$$

Since $\lambda_{n,0} = \lambda_n$ is bounded away from the origin and the infinity by Lemma 5.6, it follows from (21) that there exists a $1 < C < \infty$ such that

$$1/C \leq \lambda_{n,t} \leq C$$

for all $n \geq 0$ and $0 \leq t \leq 1$. Now (20) follows from Lemma 7.1 and a compact argument and the claim has been proved. Let $L_0$ denote the Teichmüller length of the curve $[\nu_0(t)]$. Then we have

$$d_T([\mu_n], [\mu_{n+1}]) \leq \delta^n L_0.$$  

This implies that $[\{\mu_n\}]$ is a Cauchy sequence in $T_f$ and thus has a limit point $\tau$. It is clear that $\tau$ is a fixed point of $\sigma_f$. This proves the existence part of the Main Theorem.

Now let us prove the uniqueness part. Suppose there exist two distinct fixed points of $\sigma_f$. Let $\nu_0(t), 0 \leq t \leq 1$, be a smooth path connecting the two points. Then we get a sequence of paths $\nu_n(t), 0 \leq t \leq 1$, which belong to a compact subset of $T_f$. By Lemma 7.1 and a compact argument, it follows easily that there exists a $0 \leq \delta < 1$ such that (20) holds for all $n \geq 0$ and $0 \leq t \leq 1$. This implies that the Teichmüller length of the path $[\nu_n(t)], 0 \leq t \leq 1$, goes to zero as $n \to \infty$. It follows that the two fixed points coincide and this is a
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contradiction. This proves the uniqueness part of the Main Theorem and the proof of the Main Theorem is completed.

References

[1] L. V. Ahlfors, Lectures on Quasiconformal Mappings, Van Nostrand, 1966.
[2] D. Brown, Spider Theory to Explore Parameter Spaces, Cornell University Ph.D. thesis, Stony Brook thesis preprint server.
[3] G. Cui, Y. Jiang, and D. Sullivan, On geometrically finite branched covering-I. Locally combinatorial attracting, Complex Dynamics and Related Topics, New Studies in Advanced Mathematics, 2004, The International Press, 1-14.
[4] G. Cui, Y. Jiang, and D. Sullivan, On geometrically finite branched covering maps-II. Realization of rational maps, Complex Dynamics and Related Topics, New Studies in Advanced Mathematics, 2004, The International Press, 15-29.
[5] G. Cui and L. Tan, A characterization of hyperbolic rational maps, preprint, arXiv:mathDS/0703380.
[6] A. Douady and J. H. Hubbard, A proof of Thurston’s topological characterization of rational functions, Acta Math., Vol. 171, 1993, 263-297.
[7] P. Domínguez, G. Sienra, A study of the dynamics of $\lambda \sin z$, International Journal of Bifurcation and Chaos, Vol. 12, No. 12 (2002) 2869-2883.
[8] A. Eremenko and M. Lyubich, Dynamical properties of some classes of entire functions, Ann. Inst. Fourier 42 (1992) 989-1020.
[9] J. H. Hubbard, Teichmüller Theory and Applications to Geometry, Topology, and Dynamics, Volume I: Teichmüller Theory, Matrix Edition, June 2006.
[10] J. H. Hubbard, D. Schleicher, and M. Shishikura, Exponential Thurston maps and limits of quadratic differentials, Journal of Amer. Math. Soc. 22 (2009), 77-117.
[11] F. Gardiner, Teichmüller Theory and Quadratic Differentials, John Wiley & Sons, 1987.
[12] C. McMullen, Complex dynamics and Renormalization, Ann. of Math. Studies, 79, 1994.
[13] L. Tan, On Cui’s method and applications of Thurston theorem, Complex Dynamics and Related Topics, New Studies in Advanced Mathematics, 2004, The International Press, 465-494.
[14] G. Zhang, Dynamics of Siegel rational maps with prescribed combinatorics, preprint, arXiv:0811.3041.
[15] G. Zhang, Polynomial Dynamics of the Hyperbolic maps in the Sine Family, In preparation.
[16] G. Zhang and Y. Jiang, Combinatorial characterization of sub-hyperbolic rational maps, to appear in Advances in mathematics, DOI: 10.1016/j.aim.2009.03.009.

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