Recent developments in spectral theory of the focusing NLS soliton and breather gases: the thermodynamic limit of average densities, fluxes and certain meromorphic differentials; periodic gases

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Abstract

In this paper we consider soliton and breather gases for one dimensional integrable focusing nonlinear Schrödinger equation (fNLS). We derive average densities and fluxes for such gases by studying the thermodynamic limit of the fNLS finite gap solutions. Thermodynamic limits of quasimomentum, quasienergy and their connections with the corresponding \( g \)-functions were also established. We then introduce the notion of periodic fNLS gases and calculate for them the average densities, fluxes and thermodynamic limits of meromorphic differentials. Certain accuracy estimates of the obtained results are also included. Our results constitute another step towards the mathematical foundation for the spectral theory of fNLS soliton and breather gases that appeared in work of El and Tovbis (2020 Phys. Rev. E 101 052207).

Keywords: focusing nonlinear Schrödinger equation, soliton and breather gases, thermodynamic limit

(Some figures may appear in colour only in the online journal)
1. Introduction and statement of results

Solitons and breathers represent well known localized solutions in many integrable systems. Due to their ‘elastic’ interaction, they can also be viewed as ‘quasi-particles’ of complex statistical objects called soliton and breather gases. The nontrivial relation between the integrability and randomness in these gases falls within the framework of ‘integrable turbulence’, introduced by Zakharov in [32]. The latter was motivated by the complexity of many nonlinear wave phenomena in physical systems that can be modeled by integrable equations. In view of the growing evidence of wide spread presence of the integrable gases (fluids, nonlinear optical media, etc), see [10, 12] and references therein, they present a fundamental interest for nonlinear science.

In this paper we consider soliton and breather gases for the focusing nonlinear Schrödinger equation (fNLS)

\[ i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \tag{1.1} \]

where \( x, t \in \mathbb{R} \) are the space-time variables and \( \psi : \mathbb{R}^2 \to \mathbb{C} \) is the unknown complex-valued function.

One of the central objects in the spectral theory of soliton and breather gases is the nonlinear dispersion relations (NDR), defining the density of states (DOS) \( u(z) \) and its temporal analog (density of fluxes) \( v(z) \). The NDR for the fNLS breather gas are defined by geometry (a Schwarz symmetric branchcut (band) \( \gamma_0 \) and a compact \( \Gamma^+ \subset \mathbb{C}^+ \setminus \gamma_0 \) and a spectral scaling function \( \sigma(z) \geq 0, z \in \Gamma^+ \). The compact \( \Gamma^+ \) is the locus of accumulation of shrinking spectral bands in the thermodynamic limit of some finite gap solutions of (1.1), whereas \( \sigma(z) \) represents the ratio of scaled logarithmic bandwidth and the density of the bands. Description of the thermodynamic limit as well as further discussion and some details about the derivation of the NDR can be found in section 2.
The NDR for the solitonic component of the fNLS breather gas have the form ([10])

\[ \int_{\Gamma^+} \left[ \log \frac{|w - \bar{z}|}{|w - z|} + \log \frac{R_0(z)R_0(w) + zw - \delta_0^2}{R_0(z)R_0(w) + zw - \delta_0^2} \right] u(w) d\lambda(w) + \sigma(z) u(z) = \text{Im} R_0(z), \]  
\[ \int_{\Gamma^+} \left[ \log \frac{|w - \bar{z}|}{|w - z|} + \log \frac{R_0(z)R_0(w) + zw - \delta_0^2}{R_0(z)R_0(w) + zw - \delta_0^2} \right] v(w) d\lambda(w) + \sigma(z) v(z) = -2\text{Im}[zR_0(z)], \]  

where \( R_0(z) = \sqrt{z^2 - \delta_0^2} \) with the branchcut \( \gamma_0 = [-\delta_0, \delta_0] \subset i\mathbb{R} \) being the exceptional (permanent) band. \( z \in \Gamma^+ \), \( \sigma \) is a continuous non negative function on \( \Gamma^+ \) and \( \lambda \) is some reference measure on \( \Gamma^+ \) that reflects the density of accumulating in the thermodynamic limit bands. In the case of \( \delta_0 = 0 \) the fNLS breather gas reduces to the fNLS soliton gas with the NDR

\[ \int_{\Gamma^+} \log \frac{|w - \bar{z}|}{|w - z|} u(w) d\lambda(w) + \sigma(z) u(z) = \text{Im} z, \]  
\[ \int_{\Gamma^+} \log \frac{|w - \bar{z}|}{|w - z|} v(w) d\lambda(w) + \sigma(z) v(z) = -4\text{Im} z \text{ Re} z. \]  

We routinely assume that \( \lambda \) is the standard area measure if \( \Gamma^+ \) (or its connected component) is a 2D region, or the arclength measure if \( \Gamma^+ \) (or its connected component) is a contour.

It has to be noted that the properties of \( u, v \) are completely defined by \( \gamma_0 \), the compact \( \Gamma^+ \) and the function \( \sigma \).

Rigorous mathematical analysis of equations (1.4) and (1.5) was reported in [21]. It was shown there ([21], corollary 1.7) that, subject to certain mild conditions, each of these equations has a (unique) solution and, moreover, the solution \( u(z) \) of (1.4) is non negative. Moreover, in the case of a 1D compact \( \Gamma^+ \), it was shown that \( u, v \) inherit some smoothness from \( \Gamma^+, \sigma \). Most of these results (but not \( u \geq 0 \)) will hold if we replace the right-hand side of (1.4) with any sufficiently smooth (at least two times continuously differentiable) function. Similar results are expected for more general equations (1.2) and (1.3) but this work have not been completed yet.

In the present paper we will assume the existence and uniqueness of solutions \( u, v \), where \( u \geq 0 \), to (1.2) and (1.3). To be more precise, we will assume that the solutions \( u, v \) of (1.2) and (1.3), as well as that of (1.4) and (1.5), belong to \( L^1(\Gamma^+) \). This assumption is not too restrictive: for example, it was proved in [21] that \( \sigma u \in L^1(\Gamma^+) \) (with respect to the reference measure \( \lambda \)) for any \( u \) satisfying (1.4); moreover, the requirement \( \sigma > 0 \) on \( \Gamma^+ \) implies the continuity of \( u \). Sometimes, the solutions \( u, v \) will be assumed to have certain smoothness provided \( \Gamma^+ \) and \( \sigma \) possess the appropriate smoothness.

Equations (1.2) and (1.3) were derived in [10] as thermodynamic limits of the two \( N \times N \) linear systems, see (2.24), of equations satisfied (respectively) by the solitonic wavenumbers \( k_m \) and the frequencies \( \omega_m \), \( m = 1, \ldots, N \), of a finite gap (nonlinear multi phase) solution \( \psi_N \) to (1.1). A finite gap solution \( \psi_N \) is defined by a hyperelliptic Schwarz symmetrical Riemann surface \( \mathfrak{R}_N \) of the genus \( 2N \) together with a collection of \( 2N \) initial phases, see section 2 for some details. As it is known ([1, 17]), \( k_m \) and \( \omega_m \) are defined as periods of certain 2nd kind
meromorphic differentials \( dp_N, dq_N \) on \( \mathcal{R}_N \), called quasimomentum and quasienergy respectively. In fact, the \( N \times N \) systems of linear equations defining \( k_m, \omega_m \) are imaginary parts of Riemann bilinear identities for the normalized holomorphic differentials on \( \mathcal{R}_N \) and \( dp_N, dq_N \) respectively.

The differentials \( dp_N, dq_N \) and their higher analogs are interesting and important objects, since their Laurent expansions at \( z = \infty \) contain valuable information about \( \psi_N \). For example,

\[
\frac{dp_N}{dz} = 1 - \sum_{m=1}^{\infty} \frac{I_{mN}}{z^{m+1}},
\]

where \( I_{mN}, m \in \mathbb{N} \), represent average densities of \( \psi_N \) that are conserved with the \( t \) evolution (\([17]\)). As it is well known, the coefficients \( I_{mN} \in \mathbb{R} \). Similar Laurent coefficients for \( \frac{dp_N}{dz} \) and for the corresponding higher meromorphic differentials on \( \mathcal{R}_N \) represent the average fluxes of the fNLS and higher NLS hierarchy flows.

One of the main subjects of this paper is the derivation of the average densities \( I_{mN} \) as well as of the corresponding average fluxes in the thermodynamic limit \( N \to \infty \), followed by the calculation of the thermodynamic limits of \( \frac{dp_N}{dz} \) and other meromorphic differentials. We also provide the corresponding error estimates.

Let \( d_{m0}^{m+1}, m \in \mathbb{N} \), denote the coefficient of \( z^{-m} \) in the Laurent expansion of \( R_0(z) \) at infinity, where \( R_0(z) \) is defined below (1.3). That is, \( d_m = 0 \) when \( m \) is even and

\[
d_m = -\frac{1}{m} \frac{m!}{(m+1)!}
\]

when \( m \) is odd. Note if we set \( m = 2n + 1 \), then \( d_{2n+1} = -2^{-2n} C_n \), where \( C_n \) is the Catalan number (see DLMF 26.5.1). We start with the following theorem for the fNLS breather gas.

**Theorem 1.1.**

(a) Fix some \( m \in \mathbb{N} \). Then for a sufficiently large \( N \) in the thermodynamic limit of the breather gases we have

\[
I_{mN} = \left[ \frac{m}{2 \pi i} \sum_{|j|=1}^{N} U_{jN} \int_{B_j} \left[ \frac{z^{m-1} R_0(z) + dz}{R_0(z)} \right] + md_{m0}^m \right] (1 + O(N^{-1} \delta)),
\]

where: the polynomial \( \frac{1}{f(z)} \) consists of the non-negative powers of the Laurent series expansion of \( f(z) \) at infinity; \( \delta > 0 \) denotes the largest length among all the shrinking bands; \( B_j \) are the cycles on \( \mathcal{R}_N \) shown at figure 1, and; \( U_{jN} = \frac{1}{2} k_j \) with the wavenumbers \( k_j \) described in section 2.

(b) If the measures \( \lambda_N := \sum_{j=1}^{N} \frac{U_{jN}}{2} \delta(z - z_{jN}), \) where \( z_{jN} \in \Gamma^+ \) denotes the center of the \( j \)th bands of \( \mathcal{R}_N \), \( j = 1, \ldots, N \), and \( \delta(z) \) denotes the delta-function, are weakly* convergent to the measure \( u(z) \delta(z) \) on \( \Gamma^+ \), where \( u(z) \) solves (1.2), then the thermodynamic limit of \( I_{mN} \) is given by

\[
I_m := \lim_{N \to \infty} I_{mN} = 2m \int_{\Gamma^+} u(z) \Im F_m(z) dz + md_{m0}^m,
\]
where

\[ F_m(z) = \int_0^z \frac{\zeta_m^{m-1} R_0(\zeta)}{R_0(\zeta)} \, d\zeta. \] (1.10)

The proof of theorem 1.1 can be found in section 3. As it was mentioned above, soliton gas can be obtained from breather gas by shrinking the exceptional band \( \gamma_0 \), that is, by taking limit \( \delta_0 \to 0 \). In this case the statements of theorem 1.1 are given in the following corollary.

**Corollary 1.2.** In the case of the fNLS soliton gas, i.e. when \( \gamma_0 \) is one of the shrinking bands, the equations (1.8) and (1.9) become

\[ I_{m,N} = \left( \frac{m}{2\pi i} \sum_{j=1}^{N} U_{j,N} \right) \int_{B_j} \zeta_m^{m-1} d\zeta \left( 1 + O(N^\beta \delta) \right), \]

\[ I_m = 2 \int_{\Gamma^+} u(\zeta) \text{Im} \zeta_m d\lambda(\zeta) \] (1.11)

respectively, where \( \delta \) denotes the length of the largest band.

Similar results for averaged fluxes and their higher time analogs are discussed in subsection 3.4.

To estimate the rate of convergence of \( I_{m,N} \to I_m \) for a fixed \( m \in \mathbb{N} \) one needs to know the rate of convergence of the measures \( \lambda_N \) in theorem 1.1, to \( u \, d\lambda \) on \( \Gamma^+ \). This question was considered in section 3.2 for the soliton gas with a 1D compact (contour) \( \Gamma^+ \) under the following additional assumptions: (i) \( \sigma > 0 \) on \( \Gamma^+ \); (ii) the solution \( u(z) \) of (1.4) is \( \alpha \)-Hölder continuous on \( \Gamma^+ \) with some \( \alpha \in (0, 1] \), and; (iii) the density \( \varphi \) of the centers of the shrinking bands provides \( O(N^{-\beta}) \), \( \beta > 0 \), approximation for the arclength measure \( \lambda \), see section 2, including (2.27), for details.

Under these assumptions the fact that the measures \( \lambda_N \) provide \( O(N^{-\varrho}) \) approximations to \( u \, d\lambda \) on \( \Gamma^+ \), where \( \varrho = \min \{ \alpha, \beta \} \), follows directly from theorem 3.7.

**Corollary 1.3.** Under the conditions of theorem 3.7,

\[ |I_m - I_{m,N}| = O(N^{-\varrho}) \] (1.12)

for any \( m \in \mathbb{N} \) provided \( \varrho < 1 \). The accuracy in (1.12) is \( O(\ln N/N) \) if \( \varrho = 1 \).

Theorem 3.7 also implies that, in the thermodynamic limit, \( NU_{j,N} \) is approximated by \( \frac{1}{2\pi i} \varphi(z,j,N) u(z,j,N) \) with the accuracy \( O(N^{-\varrho}) \) as \( N \to \infty \), thus providing the error estimate of the transition from the discrete NDR (2.24) (systems of linear equations) to its continuous counterpart (1.4) and (1.5) (integral equations). The accuracy is \( O(\ln N) \) in the case \( \varrho = 1 \).

Next, we consider the thermodynamic \( \lim_{N \to \infty} \frac{\Delta p}{\Delta z} \) of the density of the quasimomentum meromorphic differential, which we define as

\[ \frac{dp}{dz} = 1 - \sum_{m=1}^{\infty} \frac{I_m}{z^{m+1}} \] (1.13)

in a neighborhood of \( z = \infty \) on the main sheet. It is clear that a requirement such as \( u \in L^1(\Gamma^+) \) is sufficient for the series in (1.13) to be convergent in a neighborhood of \( z = \infty \). In the remaining part of the introduction we restrict ourselves to the case when the compact \( \Gamma^+ \) is
a contour (or a collection of contours). Moreover, we: call the connected parts of $\Gamma = \Gamma^+ \cup \Gamma^-$ 'superbands'; define $\mathcal{R}_\infty$ as the limiting hyperelliptic Riemann surface to the sequence $\mathcal{R}_N$, $N \in \mathbb{N}$, where the branchcuts of $\mathcal{R}_\infty$ coincide with the superbands of $\Gamma$.

The general complex NDR for the fNLS breather gas were obtained in [10]. The imaginary parts of the general complex NDR form the NDR (1.2) and (1.3) for the solitonic components of the fNLS breather gas, whereas their real parts form the NDR for the corresponding carrier components. The first general complex NDR equation has the form:

$$i \int_{\Gamma^+} \left[ \log \frac{\bar{w}}{w} + \log \frac{R_0(z)R_0(w) + zw - \delta_0^2 + i\pi \chi(w)}{R_0(z)R_0(\bar{w}) + \bar{z}w - \delta_0^2 + i\pi \chi(\bar{w})} \right] u(w) |dw|$$

$$+ i \sigma(z) u(z) = R_0(z) + \bar{u}(z)$$

(1.14) ([10], equation (25)), where: (i) $\bar{u}(z)$ is the 'carrier DOS' function that is defined as a smooth interpolation of the carrier wavenumbers (see section 2), i.e. $\bar{u}(z)$ interpolates the values $\pm \delta_j$ at $z_j \in \Gamma^+$, $j = 1, \ldots, N$, and; (ii) $\chi(z)$ is the indicator function of the arc $(z_\infty, z)$ of $\Gamma^+$. Here $z_\infty$ denotes the beginning of the oriented curve $\Gamma^+$.

Let us consider first the soliton gas. To further simplify the situation, take $\Gamma^+ \subset i\mathbb{R}^+$. Then we prove that

$$\frac{dp}{dz} = 1 - 2 \pi i C_{\Gamma} [u] \quad \text{on} \quad \hat{\mathbb{C}} \setminus \Gamma,$$  

(1.15)

where: $\Gamma = \Gamma^+ \cup \Gamma^-$, $\Gamma^- = \overline{\Gamma^+}$; the DOS $u$ (see (1.4)) has odd (anti-Schwarz symmetrical) continuation to $\Gamma^-$, and; $C_{\Gamma}$ denotes the Cauchy transform on $\Gamma$. Thus, $\frac{dp}{dz}$ is analytic in $\hat{\mathbb{C}} \setminus \Gamma$ and has a jump $-2 \pi i u(z)$ on $\Gamma$.

It is an interesting observation that $\frac{dp}{dz}$ defined by (1.15) in the case of a condensate (i.e. $\sigma \equiv 0$) coincides with the density of the quasimomentum differential $\frac{dp_{\mathcal{R}_\infty}}{dz}$ of $\mathcal{R}_\infty$. Indeed, it was proved in [21], theorem 6.1, that in the case of a bound state (i.e. $\Gamma \subset i\mathbb{R}$) soliton condensate the DOS $u(z) = \frac{1}{\pi} \left( \frac{dp_{\mathcal{R}_\infty}}{dz} \right)_+$, where $(\frac{dp_{\mathcal{R}_\infty}}{dz})_+$ denotes the boundary value on the positive side of $\Gamma$. Then it is easy to check that $\frac{dp}{dz}$ and $\frac{dp_{\mathcal{R}_\infty}}{dz}$ have the same jump $-2 \pi i u(z)$ on $\Gamma$ and the same asymptotics at $z = \infty$ on $\hat{\mathbb{C}}$. Since $u(z)$ has at most $z^{-\frac{1}{2}}$ singularities on $\Gamma$ (at the endpoints), we proved that for a bound state condensate

$$\frac{dp}{dz} \equiv \frac{dp_{\mathcal{R}_\infty}}{dz}.$$  

(1.16)

As discussed in section 2, this result is also valid for the Korteweg–de Vries (KdV) soliton condensates.

In general, the requirement $\Gamma^+ \subset i\mathbb{R}^+$ from (1.15) can be removed if we introduce $\bar{u}(z) := u(z) e^{-i\theta(z)}$, where $\theta(z) = \arg dz$ along $\Gamma^+$ at $z \in \Gamma^+$ traverses it in the positive direction. Then (theorem 3.15) equation (1.15) becomes

$$\frac{dp}{dz} = 1 - 2 \pi i C_{\Gamma} [\bar{u}] \quad \text{on} \quad \hat{\mathbb{C}} \setminus \Gamma,$$  

(1.17)

so that (1.15) is a particular case of (1.17).

Let $\Gamma^+$ be a given contour. Then equations (1.4) and (1.17) allow one to express any two of the functions $\frac{dp}{dz}$, $\sigma(z)$, $u(z)$ in terms of the remaining one. Similar results can be obtained for the quasienergy density $\frac{dq}{dz}$ as well as for the thermodynamic limits of the higher meromorphic differentials on $\mathcal{R}_\infty$. 

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Let $\gamma \in \mathbb{C}$ be a simple Schwarz symmetric curve that consists of the superbands and connecting them gaps. Together with the density $\frac{dp}{dz}$ we consider its ‘antiderivative’ $2g_\varepsilon(z)$, satisfying $\frac{d}{dz}[2g_\varepsilon(z)] = 1 - \frac{dp}{dz}$. The notation $2g_\varepsilon$ emphasizes connection with the $g$-functions associated with the Riemann–Hilbert problems (RHPs) for the finite gap solutions $\psi_N$ of the fNLS (1.1), see section 2. Indeed, it follows from (1.17) that

$$
2g_\varepsilon(z) := z - \int_0^z \frac{dp}{\mu} = -\int_\Gamma u(\mu) \, \arg \mu |d\mu| - i \int_\Gamma \ln(\mu - z) u(\mu) |d\mu|,
$$

(1.18)

so that

$$
2g_\varepsilon(\infty) = -\int_\Gamma u(\mu) \, \arg \mu |d\mu|.
$$

(1.19)

If $0 \in \Gamma$, the integral in (1.18) start from the positive side of $\Gamma$, i.e. from $0 \in \Gamma_+$. Note that $2g_\varepsilon$ is analytic in $\mathbb{C}\setminus \Gamma$ but, if $\Gamma$ consists of several superbands, not necessarily single valued there. However, similarly to the $g$-function from section 2, $2g_\varepsilon(z)$ is analytic and single valued in $\mathbb{C}\setminus \gamma$.

Next, in theorem 3.18, we obtain a similar to (1.18) expression for the thermodynamic limit of $2g_\varepsilon$ for the breather gas. As a consequence, combining it with (1.14), we derive the following corollary for both soliton and breather gases.

**Corollary 1.4.** For any $z \in \Gamma$ we have

$$
g_{x+}(z) + g_{x-}(z) = \bar{u}(z) - i\sigma(z)u(z) + z,
$$

(1.20)

where $\bar{u}(z)$ is defined in (1.14). That is,

$$
\sigma(z)u(z) = \text{Im}(z - 2g_\varepsilon(z)), \quad \bar{u}(z) = \text{Re}[g_{x+}(z) + g_{x-}(z) - z].
$$

(1.21)

It is shown in section 3 that the above mentioned results regarding the DOS $u(z)$ and the quasimomentum differential $dp$ after appropriate reformulation will be valid for the density of fluxes $v(z)$, see (1.3), and the quasienergy differential $dq$. They will also be valid for higher meromorphic differentials associated with the flows of the fNLS hierarchy. Moreover, the corresponding results will still be valid if we replace the right-hand side in (1.4) by $xz + 2\varepsilon^2 + f_0(z)$, where $f_0(z)$ is a sufficiently smooth Schwarz symmetrical function, defining the initial phases of a particular family of finite gap solutions $\psi_N, N \to \infty$, to the fNLS (1.1), that is, defining a particular realization of a soliton gas. Various sets of sufficient conditions on $f_0(z)$ (or even $f_0(z, N)$) will be studied elsewhere. It is expected that similar results will be also valid for breather gases.

In section 4 we introduce a new class of periodic soliton and breather fNLS gases. These are the gases whose density of bands $\varphi(z)$ and the scaled bandwidth $v(z)$, see section 2 for details, are determined through the semiclassical ($\varepsilon \to 0$) limit of the fNLS spectral data with some periodic potential $q(x)$.

In this paper we assume that $q(x)$ is an even, nonnegative, continuous and single humped $2L$-periodic, $L > 0$, function, such that $M = q(0)$ and $m = q(L) \geq 0$ are the maximum and the minimum of $q$ respectively. In particular, $q(x)$ must be strictly monotonically decreasing on $[0, L]$. It follows from results of [4, 5] that:

- the bands of the Lax spectrum of the corresponding Zakharov–Shabat (ZS) operator are confined to the ‘cross’ $\mathbb{R} \cup [-iM, iM]$ in the limit $\varepsilon \to 0$;
- $\mathbb{R}$ is a single band and if $m > 0$ then $[-im, im]$ is also (asymptotically) a single band, and;
• the leading $\varepsilon \to 0$ order of the Floquet discriminant (the trace of the monodromy matrix for ZS) on $z \in [im, iM]$ is
\[
\varphi(z) = -i z \int_{0}^{q^{-1}(z)} \frac{dx}{\sqrt{q^2(x) - m^2}},
\]
and
\[
\nu(z) = \frac{\pi L}{2} \int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx.
\]
respectively, are calculated in section 4.1.

The above calculations serve as a motivation to call fNLS soliton gases with $\Gamma^+ = [0, iM]$ (the case of $m = 0$) and $\varphi, \nu$ given by (1.24) as periodic fNLS soliton gases. The gases corresponding to $m > 0$ with the same $\varphi, \nu$ and $\Gamma^+ = [im, iM]$ will be called periodic fNLS breather gases with the permanent band $\gamma_0 = [-im, im]$.

The DOS $u(z)$ of a periodic gas must be proportional to its density of bands $\varphi(z)$, a fact that is intuitively clear since the fNLS evolution of a periodic potential remains periodic for all times. In section 4, see theorems 4.1 and 4.3, we prove this fact and calculate the coefficient of proportionality. Namely, the DOS $u(z)$, $z \in \Gamma^+$, of a periodic gas with a potential $q(x)$ is given by
\[
u(z) = \frac{\pi L}{2} \int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx.
\]
It is worth noting that for the considered periodic gases the density of fluxes $\nu(z) \equiv 0$, as the right hand sides of (1.5) and (1.3) are identical zeros.

As an example, consider the potential $q(x) = Q$ on $[0, L]$ and assume $q^{-1}(z) = L$ for any $z \in \Gamma^+$, where $\Gamma^+ = [0, iQ]$. Then (1.25) immediately yields
\[
u(z) = \frac{-i z}{\pi \sqrt{Q^2 + z^2}}, \quad z \in [0, iQ],
\]
which is the well known DOS of the genus zero bound state soliton condensate, see [10], example 1. Of course, $q(x) = Q$ does not satisfy the monotonicity assumption we put on $q(x)$, but that can be mitigated by modifying $q(L) = 0$ just at one point $x = L$ and then approximating (with respect to any integral norm) the ‘modified’ discontinuous $q(x)$ by smooth and strictly monotonic functions. That will justify $u(z)$ given by (1.26). Note that the corresponding $\nu(z) \equiv 0$. 

The latter statement was obtained in [4] through formal WKB arguments. Some rigorous results about the localization of the spectral bands can be found in [5].

Equations (1.22) and (1.23) show that if we take $\Gamma^+ = [im, iM]$, the number of spectral bands will grow like $O(1/\varepsilon)$ whereas the size of these bands will decay exponentially in $1/\varepsilon$. Thus, the semiclassical fNLS evolution of the periodic potential $q(x)$ should provide a realization of a soliton ($m = 0$) or a breather ($m > 0$) fNLS gas. Spectral characteristics of such gases, namely, the density of bands $\varphi(z)$ and the scaled logarithmic bandwidth $\nu(z)$, $z \in \Gamma^+$, see section 2, given by
\[
\varphi(z) = \frac{-i z \int_{0}^{q^{-1}(z)} \frac{dx}{\sqrt{q^2(x) - m^2}}}{\int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx}
\]
and
\[
\nu(z) = \frac{\pi L}{2} \int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx
\]
respectively, are calculated in section 4.1.

The above calculations serve as a motivation to call fNLS soliton gases with $\Gamma^+ = [0, iM]$ (the case of $m = 0$) and $\varphi, \nu$ given by (1.24) as periodic fNLS soliton gases. The gases corresponding to $m > 0$ with the same $\varphi, \nu$ and $\Gamma^+ = [im, iM]$ will be called periodic fNLS breather gases with the permanent band $\gamma_0 = [-im, im]$. 

The DOS $u(z)$ of a periodic gas must be proportional to its density of bands $\varphi(z)$, a fact that is intuitively clear since the fNLS evolution of a periodic potential remains periodic for all times. In section 4, see theorems 4.1 and 4.3, we prove this fact and calculate the coefficient of proportionality. Namely, the DOS $u(z)$, $z \in \Gamma^+$, of a periodic gas with a potential $q(x)$ is given by
\[
\varphi(z) = \frac{-i z \int_{0}^{q^{-1}(z)} \frac{dx}{\sqrt{q^2(x) - m^2}}}{\int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx}
\]
and
\[
\nu(z) = \frac{\pi L}{2} \int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx
\]
respectively, are calculated in section 4.1.

The above calculations serve as a motivation to call fNLS soliton gases with $\Gamma^+ = [0, iM]$ (the case of $m = 0$) and $\varphi, \nu$ given by (1.24) as periodic fNLS soliton gases. The gases corresponding to $m > 0$ with the same $\varphi, \nu$ and $\Gamma^+ = [im, iM]$ will be called periodic fNLS breather gases with the permanent band $\gamma_0 = [-im, im]$. 

The DOS $u(z)$ of a periodic gas must be proportional to its density of bands $\varphi(z)$, a fact that is intuitively clear since the fNLS evolution of a periodic potential remains periodic for all times. In section 4, see theorems 4.1 and 4.3, we prove this fact and calculate the coefficient of proportionality. Namely, the DOS $u(z)$, $z \in \Gamma^+$, of a periodic gas with a potential $q(x)$ is given by
\[
\varphi(z) = \frac{-i z \int_{0}^{q^{-1}(z)} \frac{dx}{\sqrt{q^2(x) - m^2}}}{\int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx}
\]
and
\[
\nu(z) = \frac{\pi L}{2} \int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx
\]
respectively, are calculated in section 4.1.
On the other hand, we can set \( q(L) = m \) for any \( m \in (0, Q) \) and repeat the previous reasoning. We will still have the same \( u(z) \) given by (1.26) but only for \( z \in [im, iQ] \). Thus, the same DOS \( u \) simultaneously satisfies the NDR (1.4) for soliton gas on \( \Gamma^+ = [0, iQ] \) and the NDR (1.2) for breather gas on \( \Gamma^+ = [im, iQ] \). In both cases \( \nu \equiv 0 \) and, thus, \( \sigma \equiv 0 \). Theorems 4.1 and 4.3 imply that the above mentioned property of DOS \( u(z) \) is valid for any \( q(x) \) with \( m > 0 \).

We then calculate (theorem 4.5) the average densities \( I_k \) for the periodic soliton and breather gases, which turned out to be zero for any even \( k \) and proportional to the \((k + 1)\)th moment of \( q(x) \) for any odd \( k \):

\[
I_k = \frac{(-1)^{k+1}k!}{L} \int_0^L q^{k+1}(x)dx, \quad k = 1, 3, 5, \ldots \quad (1.27)
\]

Moreover,

\[
2g_s(z) = \frac{1}{L} \int_0^L \left(z - \sqrt{z^2 + q^2(x)}\right)dx + \frac{1}{L} \int_0^L \sqrt{q^2(x) - m^2} dx, \quad z \in \mathbb{C} \setminus \Gamma. \quad (1.28)
\]

According to corollary 1.4, that implies

\[
\sigma(z)u(z) = \frac{1}{L} \int_{q^{-1}(z)}^L \sqrt{z^2 + q^2(x)}dx, \quad \tilde{u}(z) = \frac{1}{L} \int_0^L \sqrt{q^2(x) - m^2} dx \quad (1.29)
\]

on \( \Gamma^+ \) for both soliton and breather periodic gases.

In addition to the bound state condensate (1.26), a few more particular examples of periodic soliton gases were considered in section 4.

Finally, in appendix A, lemma A.5, we find the thermodynamic limit of the asymptotic behavior of the coefficients of the systems of linear equation (2.24) for \( k_j, \omega_j \) together with error estimates. This result is later used in theorem 3.7, where we show that DOS \( u(z) \) in the thermodynamic limit can be approximated by a piece-wise constant function \( \tilde{u}(z) \) that is defined through the wavenumbers \( k_j \).

2. Background

The simplest solution of equation (1.1) is a plane wave

\[
\psi = q e^{2i\xi^2}, \quad (2.1)
\]

where \( q > 0 \) is the amplitude of the wave.

It is well known that the fNLS is an integrable equation [33]; the Cauchy (initial value) problem for (1.1) can be solved using the inverse scattering transform method for different classes of initial data (potentials). The scattering transform connects a given potential with its scattering data expressed in terms of the spectral variable \( z \in \mathbb{C} \). In particular, the scattering data consisting of one pair of spectral points \( z = a \pm ib \), where \( b > 0 \), and a (norming) constant \( c \in \mathbb{C} \), defines the famous soliton solution

\[
\psi_S(x, t) = 2ib \text{sech}[2b(x + 4at - x_0)]e^{-2i(ax + 2(a^2 + b^2)t + i\xi_0)}, \quad (2.2)
\]
to the fNLS. This solution represents a solitary traveling wave (pulse on a zero background) with \( c \) defining the initial position \( \chi_0 \) of its center and the initial phase \( \phi_0 \). It is characterized by two independent parameters: \( b = \text{Im} \ z \) determines the soliton amplitude \( 2b \) and \( a = \text{Re} \ z \) determines its velocity \( s = -4a \). Scattering data that consists of several points \( z_j \in \mathbb{C}^+ \) (and their complex conjugates \( \bar{z}_j \)), \( j \in \mathbb{N} \), together with their norming constants corresponds to multi-soliton solutions. Assuming that at \( t = 0 \) the centers of individual solitons are far from each other, we can represent the fNLS time evolution of a multi-soliton solution as propagation and interaction of the individual solitons.

It is well known that the interaction of solitons in multi-soliton fNLS solutions reduces to only two-soliton elastic collisions, where the faster soliton (corresponding to \( z_m \)) gets a forward shift \[33\]

\[
\Delta_{mj} = \frac{1}{\text{Im}(z_m)} \log \left| \frac{z_m - z_j}{z_m - \bar{z}_j} \right|
\]

and the slower ‘\( z_r \)-soliton’ is shifted backwards by \(-\Delta_{mj}\).

Suppose now we have a ‘large ensemble’ (a ‘gas’) of solitons (2.2) whose spectral characteristics \( z \) are distributed over a compact set \( \Gamma^+ \subset \mathbb{C}^+ \) according to some non negative measure \( \mu \). Assume also that the locations (centers) of these solitons are distributed uniformly on \( \mathbb{R} \) and that \( \mu(\Gamma^+) \) is small, i.e. the gas is dilute. Let us consider the speed of the trial \( z \)—soliton in the gas. Since it undergoes rare but sustained collisions with other solitons, the speed \( s_0(z) = -4 \text{Re} \ z \) of a free solution must be modified as

\[
s(z) = s_0(z) + \frac{1}{\text{Im} \ z} \int_{\Gamma^+} \log \left| \frac{w - z}{w - \bar{z}} \right| [s_0(w) - s_0(w)]d\mu(w). \tag{2.3}
\]

Similar modified speed formula was first obtain by Zakharov [31] in the context of the KdV equation. Without the ‘dilute’ assumption, i.e. with \( \mu(\Gamma^+) = O(1) \), equation (2.3) for \( s(z) \) turns into the integral equation

\[
s(z) = s_0(z) + \frac{1}{\text{Im} \ z} \int_{\Gamma^+} \log \left| \frac{w - z}{w - \bar{z}} \right| s(z) - s(w) \ d\mu(w) \tag{2.4}
\]

known as the equation of state for the fNLS soliton gas, an analogous of which was first obtained in [11] using purely physical reasoning. Here \( s(z) \) has the meaning of the speed of the ‘element of the gas’ associated with the spectral parameter \( z \) (note that when \( \mu(\Gamma^+) = O(1) \) we cannot distinguish individual solitons).

A similar equation in the KdV context was obtained earlier in [13]. If we now assume some dependence of \( s \) and \( u \) on space time parameters \( x, t \) (here \( d\mu = u \ d\lambda \) with \( \lambda \) being the Lebesgue measure) that occurs on very large spatiotemporal scales, then we complement the equation of state (2.4) by the continuity equation for the DOS

\[
\partial_t u + \partial_x (su) = 0, \tag{2.5}
\]

which was first suggested in [11] and derived in [10]. Equations (2.4) and (2.5) form the kinetic equation for a dynamic (non-equilibrium) fNLS soliton gas. The kinetic equation for the KdV soliton gas was derived in [13]. It is remarkable that recently the kinetic equation having similar structure was derived in the framework of the ‘generalized hydrodynamics’ for quantum many-body integrable systems, see, for example \[8, 9, 29\].

It is easy to observe that (2.4) is a direct consequence of (1.2) and (1.3), where \( s(z) = \frac{4}{\text{Im} \ z} \). Indeed, after multiplying (1.4) by \( s(z) \), substituting \( v(z) = s(z)u(z) \) into (1.5), subtracting the second equation from the first one and dividing both parts by \( \text{Im} \ z \) we obtain exactly (2.4).
In this paper we mostly consider the NDR (1.2) and (1.3) and (1.4) and (1.5) for equilibrium soliton gases, that is, we do not assume any dependence of $u, v$ on the space-time variables $x, t$.

A mathematical albeit formal (i.e. without, for example, error estimates) derivation of the equation of state (2.4) was presented in the recent paper [10]. The first step in this process is derivation of equations (1.4) and (1.5), which describe the DOS $u$ and its temporal analog $v$. The derivation is based on the idea of thermodynamic limit for a family of finite gap solutions of the fNLS, which was originally developed for the KdV equation in [13]. Finite-gap solutions are quasi-periodic functions in $x, t$ that can be spectrally represented by a finite number of Schwarz symmetrical arcs (bands) on the complex $z$ plane. Here Schwarz symmetry means that either a band $\gamma$ coincides with its Schwarz symmetrical image $\bar{\gamma}$ or if $\gamma$ is a band then $\bar{\gamma}$ is another band. Assume additionally that there is a complex constant (initial phase) associated with each band that also respects the Schwarz symmetry, i.e. Schwarz symmetrical bands have Schwarz symmetrical phases. Given a finite set of Schwarz symmetrical bands with the corresponding band that also respect the Schwarz symmetry, i.e. Schwarz symmetrical bands have Schwarz symmetrical phases. Given a finite set of Schwarz symmetrical bands with the corresponding phases, a finite-gap solution to the fNLS can be written explicitly in terms of the Riemann theta functions on the hyperelliptic Riemann surface $\mathcal{R}$, where the bands are the branchcuts of $\mathcal{R}$, see, for example [1], and references therein.

For convenience of the further brief exposition, we will consider the hyperelliptic Riemann surface $\mathcal{R} = \mathcal{R}_N$ to be of genus $2N$, which equals the number of bands minus one. The one exceptional band $\gamma_0$ will be crossing $\mathcal{R}$, whereas the remaining $N$ bands $\gamma_j \subset \mathbb{C}^+$, $j = 1, \ldots, N$, and their Schwarz symmetrical $\gamma_{-j} := \bar{\gamma}_j \subset \mathbb{C}^-$. Definition of a particular finite-gap solution of the fNLS starts with a smooth Schwarz symmetrical function $f_0 = f_0(z; N)$ that is defined on $\gamma_j$, $j = 0, \pm 1, \pm N$. Given $f_0$, the corresponding finite gap solution of the fNLS can be defined through the solution of the following matrix RHP (see, for example [7, 27]).

**RHP 2.1.** Find a matrix-valued function $Y(z)$, such that $Y(z)$ is: (i) analytic together with its inverse $Y^{-1}(z)$ on $\mathbb{C} \setminus \bigcup_{j=0}^N \gamma_j$; (ii) satisfies the jump condition

$$Y_+(z) = Y_-(z) i \sigma_j e^{2i(f(z)x)} \quad \text{on} \quad \gamma_j, \quad j = 0, \pm 1, \ldots, \pm N,$$

where $\sigma_j$, $j = 1, 2, 3$, denote standard Pauli matrices, $f(z) = f_0(z) + x z + 2iz^2$ and the orientation of the bands $\gamma_j$ is shown on figure 1 below; (iii) $\lim_{z \to \infty} Y(z) = 1$, and; (iv) the boundary values $Y_+(z)$ on the positive and negative sides respectively are locally $L^2$ on all the bands $\gamma_j$.

It is well known that the RHP 2.1 has a unique solution and that the solution to the fNLS (1.1) is given by

$$\psi(x, t) = -2(Y_1)_{1, 2}, \quad \text{where} \quad Y(z) = 1 + Y_1 z^{-1} + \cdots$$

is the expansion of $Y(z)$ at infinity ([16]). Note that $Y_1$ depends on $x, t$.

The next step in finding $Y(z)$ is to reduce the jump matrix on each $\gamma_j$ to a constant (in $z$, but not in $x, t$) matrix. Assume that there exists a simple piecewise smooth symmetrical contour $\Gamma$ such that $\Gamma = \bigcup_{j=0}^N \gamma_j \cup \bigcup_{j=1}^N \gamma_j$, where the arcs $\gamma_j$ will be called ‘gaps’, connecting the consecutive bands, see figure 1. Then the reduction to the ‘constant jumps’ RHP can be done with the help of the so-called $g$-function, that is, by the transformation

$$Y(z) = e^{-2i(f(z)x) t} Z(z) e^{2i(f(z)x^t)},$$

(2.8)
The spectral bands $\gamma_{\pm j}$ and the cycles $A_{\pm j}, B_{\pm j}$. The 1D Schwarz symmetrical curve $\Gamma$ consists of the bands $\gamma_{\pm j}, j = 0, \ldots, N$, and the gaps $c_{\pm j}$ between the bands (the gaps are not shown on this figure).

where $g = g(z; x, t, N)$ is an unknown function, analytic at $\bar{C} \setminus \Gamma$. Then $Z(z)$ satisfies jump conditions

$$Z_+(z) = Z_-(z)e^{2i(g_+(z)-g_-(z))\sigma_3}$$

for each $\gamma_{j}$. Similarly, we have $Z_+(z) = Z_-(z)e^{-2i(g_+(z)-g_-(z))\sigma_3}$ on each $c_{j}$.

The reduction to a piece-wise constant jump matrix for $Z(z)$ will be achieved if we define $g(z)$ as a solution of the following scalar RHP.

**RHP 2.2.** Find a function $g$ satisfying the following requirements: (1) $g$ is analytic in $\bar{C} \setminus \Gamma$; (2) it satisfies the jump conditions

$$g_+(z) + g_-(z) = f(z) + W_j \quad \text{on} \quad \gamma_{j}, \quad |j| = 0, 1, \ldots, N,$$

$$g_+(z) - g_-(z) = \Omega_j \quad \text{on} \quad c_{j}, \quad |j| = 1, \ldots, N,$$

where $f(z)$ is given in RHP 2.1, $W_0 = 0$ and the real constants $W_j, \Omega_j, |j| = 1, \ldots, N$, subject to the symmetries $W_{-j} = W_j, \Omega_{-j} = \Omega_j$, are to be defined, and; (3) the boundary values $g_{\pm}(z)$ are locally $L^2$ functions.

According to Sokhotsky–Plemelj formula, the solution of the RHP (2.10), if exists, is given by

$$g(z) = \frac{R(z)}{2\pi i} \left[ \sum_{|j|=0}^{N} \int_{\gamma_{j}} \frac{f(\zeta) + W_j}{(\zeta - z)R_+(\zeta)} d\zeta + \sum_{|j|=1}^{N} \int_{c_{j}} \frac{\Omega_j}{(\zeta - z)R_+(\zeta)} d\zeta \right].$$

(2.11)

Here

$$R(z) = \prod_{j=1}^{2N+1} (z - \alpha_j)(z - \bar{\alpha}_j),$$

(2.12)
where \( \alpha_{2j}, \alpha_{2j+1} \) are the respective endpoints of the oriented main arcs \( \gamma_j, j = 1, \ldots, N, \alpha_1 \) and \( \alpha_1 \) are the endpoints of \( \gamma_0 \) and \( R_+ \) denotes the value of \( R \) on the positive (left) side of the each \( \gamma_j \). To show that (2.11) satisfies the RHP 2.2, one has to show that the right-hand side of (2.11) is analytic at infinity, that is, the system

\[
\sum_{|j|=1}^{N} W_j \int_{A_j} \frac{\zeta^m d\zeta}{R(\zeta)} + \sum_{|j|=1}^{N} \Omega_j \int_{C_j} \frac{\zeta^m d\zeta}{R(\zeta)} = - \sum_{|j|=0}^{N} f(\zeta) \frac{\zeta^m d\zeta}{R(\zeta)}
\]

(2.13)

for unknown real constants \( W_j, \Omega_j \), where \( m = 0, 1, \ldots, 2N - 1 \), obtained from (2.11) by expanding \( \frac{1}{z} \) in powers of \( \frac{1}{z} \) as \( z \to \infty \), has a solution. Here \( A_j, C_j \) are negatively oriented loop contours on \( \mathcal{R}_N \) containing the arcs \( \gamma_j, \zeta_j \) respectively and no other arcs; to allow loop contour integration, we have to assume that \( f \tilde{\delta}(z; N) \) has an analytic extension around each band \( \gamma_j \). We note that (2.11) can be rewritten as

\[
2g(z) = \frac{R(z)}{2\pi i} \left[ \sum_{|j|=0}^{N} \int_{A_j} \frac{[f(\zeta) + W_j] d\zeta}{(\zeta - z)R_+} + \sum_{|j|=1}^{N} \int_{C_j} \frac{\Omega_j d\zeta}{(\zeta - z)R_+} \right],
\]

(2.14)

where \( z \) is outside of each loop.

Since the bands \( \gamma_j \) and the gaps \( \zeta_j \) do not depend on \( x, t \), differentiating the RHP 2.2 in \( x, t \) would transform it to similar RHPs for \( g_x, g_t \) with constants \( W_{x,j}, \Omega_{x,j} \) or \( W_{t,j}, \Omega_{t,j} \), where \( f(z) \) should be replaced by \( f_x(z) = z \) or \( f_t(z) = 2z^2 \) respectively. Thus, we obtain the corresponding expressions (2.14) for \( 2g_x \) and \( 2g_t \).

Let us define the quasi-momentum \( dp_N \) and quasi-energy \( dq_N \) differentials on \( \mathcal{R}_N \) as real normalized (all the periods of \( dp_N, dq_N \) are real) meromorphic differentials of the second kind with the only poles at \( z = \infty \) on both sheets. These differentials are (uniquely) defined (see e.g. [2, 17, 20, 26]) by local expansions

\[
dp_N \sim [\pm 1 + \mathcal{O}(z^{-2})] dz, \quad dq_N \sim [\pm 4z + \mathcal{O}(z^{-2})] dz \quad (2.15)
\]

near \( z = \infty \) on the main and second sheet respectively. One can observe that \( dp_N, dq_N \)—the quasi-momentum and the quasi-energy differentials—can be expressed as

\[
dl_N = [1 - 2g\tilde{g}(z)] dz, \quad dq_N = [4z - 2g\tilde{g}(z)] dz. \quad (2.16)
\]

Indeed, the right-hand sides of (2.16) have the required behavior at infinity and, as one can easily see ([27]), all their the periods are real.

Introduce

\[
k_j = 2(\Omega_{x,j} - \Omega_{x,j-1}), \quad \tilde{k}_j = 2W_{x,j}, \quad |j| = 1, \ldots, N,
\]

\[
\omega_j = 2(\Omega_{t,j} - \Omega_{t,j-1}), \quad \tilde{\omega}_j = 2W_{t,j}, \quad |j| = 1, \ldots, N. \quad (2.17)
\]

Then it can be easily shown (see also [10]) that the wave numbers \( k_j, \tilde{k}_j \) and the frequencies \( \omega_j, \tilde{\omega}_j \) of a quasi-periodic finite gap solution \( \psi_N \) determined by \( \mathcal{R}_N \), can be expressed as

\[
k_j = -\oint_{A_j} dp_N, \quad \omega_j = -\oint_{A_j} dq_N, \quad j = 1, \ldots, N, \quad (2.18)
\]

\[
\tilde{k}_j = \oint_{B_j} dp_N, \quad \tilde{\omega}_j = \oint_{B_j} dq_N, \quad j = 1, \ldots, N \quad (2.19)
\]
where the cycles $A_j, B_j$ are shown on figure 1.

Note that the wavenumbers and frequencies defined by (2.18) and those defined by (2.19) are of essentially different nature: in the limit of $\gamma_j$ shrinking to a point, we have

$$k_j, \omega_j \to 0, \quad \text{whereas} \quad \tilde{k}_j, \tilde{\omega}_j = \mathcal{O}(1), \quad j = 1, \ldots, N,$$

see [10]. Motivated by these properties, $k_j, \omega_j$ are called solitonic wavenumbers and frequencies whereas the remaining $\tilde{k}_j, \tilde{\omega}_j$ are called carrier wavenumbers and frequencies.

The standard normalized holomorphic differentials $w_j$ of $\mathcal{R}_N$ are defined by

$$w_j = \left[ P_j(\zeta) / R(\zeta) \right] d\zeta, \quad \oint_{A_i} w_j = \delta_{ij}, \quad i, j = \pm 1, \ldots, \pm N, \quad (2.21)$$

where the polynomials

$$P_j(\zeta) = \kappa_{j,1} \zeta^{2N-1} + \kappa_{j,2} \zeta^{2N-2} + \cdots + \kappa_{j,2N}$$

have complex coefficients and the radical $R$ given by (2.12) defines the hyperelliptic surface $\mathcal{R}_N$. Then, according to (2.14)–(2.22) (see also [10]), the wavenumbers and frequencies satisfy the systems

$$\tilde{k}_j + \sum_{|m|=1}^N k_m \oint_{B_m} \frac{P_j(\zeta) d\zeta}{R(\zeta)} = -2 \oint_{\hat{\gamma}} \zeta P_j(\zeta) d\zeta / R(\zeta),$$

$$\tilde{\omega}_j + \sum_{|m|=1}^N \omega_m \oint_{B_m} \frac{P_j(\zeta) d\zeta}{R(\zeta)} = -4 \oint_{\hat{\gamma}} \zeta^2 P_j(\zeta) d\zeta / R(\zeta),$$

$$|j| = 1, \ldots, N, \quad (2.23)$$

where $\hat{\gamma}$ is a large clockwise oriented contour containing $\Gamma$. In fact, every equation in (2.23) is the Riemann bilinear identity for the differentials $w_j$ and $d\rho_N, d\rho_{2N}$ respectively.

Taking imaginary parts of (2.23) and using the residues in the right-hand side, we obtain the systems

$$\sum_{|m|=1}^N k_m \text{Im} \oint_{B_m} \frac{P_j(\zeta) d\zeta}{R(\zeta)} = 4\pi \text{Re} \kappa_{j,1},$$

$$\sum_{|m|=1}^N \omega_m \text{Im} \oint_{B_m} \frac{P_j(\zeta) d\zeta}{R(\zeta)} = 8\pi \text{Re} \left( \kappa_{j,1} \sum_{k=1}^{2N+1} \text{Re} \alpha_k + \kappa_{j,2} \right),$$

$$|j| = 1, \ldots, N, \quad (2.24)$$

where the latter summation is taken over all the endpoints in $\mathbb{C}^+$, which define the solitonic wavenumbers and frequencies. We call (2.24) the solitonic NDR. Indeed, the NDR indirectly connect (through the Riemann surface $\mathcal{R}_N$) the solitonic wavenumbers and frequencies of the finite gap solution $\psi_N$, i.e. (2.24) represents NDR. Once the solitonic wavenumbers and frequencies were obtained, the corresponding carrier quantities can be reconstructed by taking the real part of (2.23).

Equation (2.24) together with (2.20) are our starting point for deriving equations (1.2) and (1.3). We want to point out that the matrix of the systems (2.24) is negative-definite and,
therefore, each of the systems (1.2) and (1.3) has a unique solution. The negative-definiteness of the matrix of the systems (1.2) and (1.3) follows from the properties of the Riemann period matrix $\tau$ of the Riemann surface $\mathcal{N}_0$ (Im $\tau$ is positive definite).

Suppose now that we start shrinking each band to a point. Then we will be taking the finite gap solution to its multi-soliton solution limit, where the phases should be transformed into the corresponding norming constants. The idea of the thermodynamical limit consists of increasing the number $2N+1$ of bands simultaneously with shrinking the size $2\delta_j$ of each band $\gamma_j$ (with the exception of the permanent band $\gamma_0$ in the breather gas) at some exponential rate with respect to $N$, so that the centers $z_j$ of the bands $\gamma_j \subset \mathbb{C}^+$, $j = 1, \ldots, N$, will be filling a certain compact set $\Gamma^+ \subset \mathbb{C}^+$ with some limiting probability density $\varphi(z)$. In particular, we assume

$$|\delta_j| = e^{-N(\nu(z_j) + \alpha(1))},$$

(2.25)

uniformly on $\Gamma^+$, where $\nu(z)$, called the scaled bandwidth function, is some nonnegative continuous function on $\Gamma^+$. In the case when $\nu(z) = 0$ on some subset of $\Gamma^+$, we still assume that the corresponding $\delta_j \to 0$ much faster than $N^{-1}$, perhaps, see appendix A for details. Moreover, we assume the distance between any two bands to be much larger than the size of the shrinking bands, that is, it must be of order at least $O(N^{-1})$ uniformly on $\Gamma^+$.

Under these assumptions we derive the leading order behavior of the coefficients of the linear system (2.24), see lemma A.5 and (A.53). The expression for the off-diagonal entries from (A.53) provides the kernel of the integral operator in (1.2) and (1.3), whereas the expression for the diagonal entries defines the secular (non-integral) term in the left hand sides of (1.2) and (1.3). Here the spectral scaling function

$$\sigma(z) = \frac{2\nu(z)}{\varphi(z)} \quad \text{and} \quad u(z) = \frac{1}{\pi} \vartheta(z) \varphi(z), \quad v(z) = \frac{1}{\pi} \bar{\vartheta}(z) \varphi(z),$$

(2.26)

where $\vartheta(z)$, $\bar{\vartheta}(z)$ are some smooth functions on $\Gamma^+$ interpolating the values $\frac{N_j}{\pi^2}$, $-\frac{N_{j+1}}{\pi^2}$ at $z = z_j/N$, $j = 1, \ldots, N$, respectively, see also [10].

The breather gas is obtained when in the thermodynamic limit all the bands except $\gamma_0$ are shrinking ($\delta_j \to 0$), whereas the exceptional band $\gamma_0$ approaches some permanent limiting position as $N \to \infty$. In particular, in this paper we assume that the endpoints of $\gamma_0$ approach $\pm \delta_0 \in i\mathbb{R}$ respectively. Being considered alone, the permanent spectral band $\gamma_0$ corresponds to the plane wave solution (2.1) with $q = |\delta_0|$. The band $\gamma_0$, together with Schwarz symmetrical points of discrete spectrum $z, \bar{z}$ correspond to a soliton on the plane wave (carrier) background, also known as a breather. It is remarkable that the kernel in the integral equations (1.2) and (1.3), being divided by $\text{Im} R_0(z)$, provides an elegant expression for the phase shift of two interacting breathers; some equivalent (see [24]) but considerably more complicated expressions for this phase shift were recently obtained in [19, 22]. Therefore, equations (1.2) and (1.3) represent nonlinear dispersive relations for the breather gas. It is easy to check that equations (1.2) and (1.3) coincide with (1.4) and (1.5) in the limit $\delta_0 \to 0$. Thus, soliton gas can be considered as a particular case of the breather gas, see [10] for details.

In the case of subexponential rate of shrinking of bands $\gamma_j$ in the thermodynamic limit, the function $\sigma(z)$ turns to be zero and we obtain a breather (or soliton, if $\delta_0 \to 0$ condensate ([10])). As it was mentioned in remark 1.4 from [21], the term ‘condensate’ reflects the fact that for a given $\Gamma^+$ a certain quadratic energy $J_{\sigma}(\mu^*_\sigma)$ is minimized in $\sigma$ when $\sigma \equiv 0$ on $\Gamma^+$. Here $d\mu^*_\sigma(z) = u(z)d\lambda(z)$ denotes the minimizing measure for a given $\sigma$.

Proving the transition from linear systems (2.24) to the corresponding integral equations (1.2) and (1.3) requires the following additional assumption on the probability measure $\varphi(z)$. Let us divide the compact $\Gamma^+ = \bigcup_{j=1}^N \Gamma_j$ into disjoint ‘regions of attraction’
\[ \Gamma_j, \gamma_j \in \Gamma_j, \text{ in such a way that all points in } \Gamma^+ \text{ that are closer to } z_{jN} \text{ than to any other } z_{kN}, k \neq j, \text{ belong to } \Gamma_j. \]

Then

\[ \lambda(\Gamma_j) = \frac{1}{N^\rho(z_{jN})} + O\left( \frac{1}{N^{1+\beta}} \right) \quad \text{as } N \to \infty \quad (2.27) \]

with some \( \beta > 0 \) uniformly in \( \Gamma^+ \).

### 3. The thermodynamic limit of averaged densities and fluxes

It is well known that the averaged conserved quantities (densities and fluxes) of the finite gap solutions are closely related to certain abelian differentials of the second kind \([23]\) of the corresponding hyperelliptic Riemann surface. In particular, for the case of fNLS, expressions for the averaged densities and fluxes can be found in \([17]\), where they were expressed as the coefficients of the expansion at infinity of \( d_{pN}, d_{qN} \). Let us first consider the expansion of \( d_{pN} \) at \( z = \infty \):

\[ d_{pN} = \left( 1 - \sum_{m=1}^{\infty} \frac{I_{mN}}{z^{m+1}} \right) dz. \]

**Theorem 1.1**, part (a), gives the leading behavior of \( I_{mN} \) in the thermodynamic limit. The analytic function \( \rho_N \), defined in the appendix \( A \), equation \((A.2)\), and studied in lemma \( A.1 \), plays an important part in the proof.

#### 3.1. Proof of theorem 1.1 part (a)

**Proof.** The definition of \( \rho_N \) (see also in \((A.2)\)) is

\[ R(z) = R_0(z)P(z)(1 + \rho_N(z)), \quad \text{where } P(z) := \prod_{|j|=1}^{N} (z - z_j) \quad (3.1) \]

and \( R(z) \) is given by \((2.12)\). Since \( \rho_N, R_0 \) and \( g_x \) are all analytic at \( \infty \), we can write

\[ 2g_x(z) = \sum_{m=0}^{\infty} \frac{G_{mN}}{z^m} \quad (3.2) \]

and

\[ \frac{2g_x(z)}{(1 + \rho_N(z))R_0(z)} = \sum_{m=0}^{\infty} \frac{\tilde{G}_{mN}}{z^{m+1}}. \quad (3.3) \]

Based on the representation \((2.14)\) of the \( g \)-function, differentiated with respect to \( x \), we have

\[ \tilde{G}_{mN} = \sum_{k=0}^{m} R_k Y_{m-k}, \quad m = \{0\} \cup \mathbb{N}, \quad (3.4) \]

where

\[ Y_k := - \frac{1}{2\pi i} \left( \sum_{[j]=0}^{N} \oint_{\gamma_j} (\zeta + W_{jN}) + \sum_{[j]=1}^{N} \oint_{\beta_j} U_{jN} \right) \zeta^{2N+k} \frac{d\zeta}{R(\zeta)}, \quad k = \{0\} \cup \mathbb{N}, \quad (3.5) \]
\[ W_{j,N} = \frac{1}{2} \mathring{k}_j := \frac{1}{2} \int_{B_j} dp_N, \quad |j| = 0, 1, \ldots, N, \]
\[ U_{j,N} = \frac{1}{2} k_j := -\frac{1}{2} \int_{A_j} dp_N, \quad |j| = 1, 2, \ldots, N \]

and \( R_k \) are the coefficients of
\[ P(z) = \frac{R(z)}{(1 + \rho_N(z))R_0(z)} = \sum_{k=0}^{2N} R_k z^{-2N-k}. \quad (3.6) \]

Let us introduce notations
\[ \mathcal{A}(f) := -\frac{1}{2\pi i} \sum_{|j|=0}^{N} \int_{A_j} (\zeta + W_{j,N}) \frac{f(\zeta) d\zeta}{R(\zeta)}, \]
\[ \mathcal{B}(f) := -\frac{1}{2\pi i} \sum_{|j|=1}^{N} \int_{B_j} U_{j,N} \frac{f(\zeta) d\zeta}{R(\zeta)}. \quad (3.7) \]

Then, using equations (3.4), (3.6) and (3.5), we have
\[ \tilde{G}_{m,N} = (\mathcal{A} + \mathcal{B}) \left( \sum_{k=0}^{m} R_k \zeta^{-2N+m-k} \right) = (\mathcal{A} + \mathcal{B})(\zeta^m P(\zeta)), \quad (3.8) \]

where we have used the fact that
\[ (\mathcal{A} + \mathcal{B})(\zeta^l) = 0, \quad l = 0, 1, \ldots, 2N-1, \]

that follows from (2.13).

Applying lemma A.1 and noting that \( W_{0,N} = 0 \), we have
\[ \mathcal{A}(\zeta^m P(\zeta)) = \mathcal{A} \left( \frac{\zeta^m R(\zeta)}{R_0(\zeta)(1 + \rho_N(\zeta))} \right) = -\frac{1}{2\pi i} \int_{A_0} \frac{\zeta^{m+1} d\zeta}{R_0(\zeta)} + O(\rho(\delta, N)), \quad (3.9) \]

where \( \rho(\delta, N) \) is given by (A.3).

Denote \( D_j = \{ z : |z - z_j| \leq \sqrt{2}|\delta_j| \} \). Again, by lemma A.1, we have
\[ \mathcal{B} \left( \frac{\zeta^m R(\zeta)}{R_0(\zeta)(1 + \rho_N(\zeta))} \right) = -\frac{1}{2\pi i} \sum_{|j|=1}^{N} U_{j,N} \int_{B_j} \frac{\zeta^m}{R_0(\zeta)(1 + \rho_N(\zeta))} d\zeta \]
\[ = -\frac{1}{2\pi i} \sum_{|j|=1}^{N} U_{j,N} \left( \int_{B_j \setminus D_j} + \int_{B_j \cap D_j} \right) \frac{\zeta^m}{R_0(\zeta)(1 + \rho_N(\zeta))} d\zeta. \quad (3.10) \]

Consider first the case when \( \gamma_0 \) is separated from \( \Gamma \), i.e. when \( R_0^{-1}(z) \) is bounded on all \( B_j \) uniformly in \( N \). Then, similarly to (3.9), each integral \( \int_{B_j \setminus D_j} \) can be approximated by the corresponding integral \( \int_{B_j} \) with the accuracy \( O(\rho(\delta, N)) + O(\delta) \), where the second term is an estimate of \( \int_{D_j} R_0^{-1}(\zeta) d\zeta. \)
According to lemma A.1, part (b)
\[
\int_{B_j \cap D_j} \frac{\zeta^m}{R_0(\zeta)(1 + \rho_N(\zeta))} \, d\zeta = \int_{B_j \cap D_j} \frac{\zeta^m(\zeta - z_j)}{R_0(\zeta)R_j(\zeta)} (1 + \rho_*(\delta,N)) \, d\zeta
\]
(3.11)

Note that \(|\frac{\zeta^m}{R_0(\zeta)}| \leq 1\) on the contour \(B_j\) that locally is orthogonal to \(\gamma_j\) and crosses it at \(z_j\). Thus, the integrand in (3.11) is (uniformly in \(j,N\)) bounded and so the integral in (3.11) is of order \(O(\delta)\). Thus, we obtain
\[
B \left( \frac{\zeta^m R(\zeta)}{R_0(\zeta)(1 + \rho_N(\zeta))} \right) = - \frac{1}{2\pi i} \sum_{j=1}^{N} U_{j,N} \oint_{B_j} \frac{\zeta^m \, d\zeta}{R_0(\zeta)} (1 + O(\rho_*(\delta,N)) + O(\delta)).
\]
(3.12)

Together with (3.9) this yields
\[
\tilde{G}_{m,N} = - \frac{1}{2\pi i} \left[ \sum_{j=1}^{N} U_{j,N} \oint_{B_j} \frac{\zeta^m \, d\zeta}{R_0(\zeta)} + \oint_{A_0} \frac{\zeta^{m+1} \, d\zeta}{R_0(\zeta)} \right] (1 + O(\rho_*(\delta,N)) + O(\delta)).
\]
(3.13)

Consider now the case when \(z_j\) is on the distance \(O(1/N)\) from \(\gamma_j\). Then the integrand in (3.11) is not bounded on \(B_j \cap D_j\). The corresponding calculations show that in this case the \(O(\delta)\) term in (3.12) and (3.13) should be replaced by \(O(N^{1/2}\delta)\).

Going back to expansion (3.2), we have
\[
\sum_{m=0}^{\infty} \frac{G_{m,N}}{z^m} = (1 + \rho_N(z))R_0(z) \sum_{m=0}^{\infty} \frac{\tilde{G}_{m,N}}{z^{m+1}}.
\]
(3.14)

Note that
\[
R_0(z) = \sum_{k=1}^{\infty} d_k z^{-k},
\]
(3.15)

where \(d_k\) is defined in (1.7) and \(d_{-1} = 1\).

By the Cauchy’s estimates and lemma A.1 we conclude that all the Taylor coefficients of \(\rho_N(z)\) at \(z = \infty\) are of the order \(O(\rho_*(\delta,N))\). This estimate will be uniform for all the Taylor coefficients provided that the compact \(\Gamma\) is contained inside the unit circle \(|z| = 1\).

Thus, we have
\[
G_{m,N} = \left( \sum_{k=1}^{m-1} \tilde{G}_{m-1-k,N} d_k \right) (1 + O(\rho_*(\delta,N))
\]
\[
= - \frac{1}{2\pi i} \frac{1}{m-1} \left[ \sum_{j=1}^{N} U_{j,N} \oint_{B_j} \frac{\zeta^{m-1-k} \, d\zeta}{R_0(\zeta)} + \oint_{A_0} \frac{\zeta^{m-1-k} \, d\zeta}{R_0(\zeta)} \right] d_k \delta_0^{k+1} (1 + O(\delta))
\]
\[
= - \frac{1}{2\pi i} \left[ \sum_{j=1}^{N} U_{j,N} \oint_{B_j} \frac{[\zeta^{m-1} R_0(\zeta)]_+ \, d\zeta}{R_0(\zeta)} + \oint_{A_0} \frac{\zeta^{m-1} R_0(\zeta)]_+ \, d\zeta}{R_0(\zeta)} \right] (1 + O(\delta)),
\]
where we have taken account that $O(\delta)$ is much larger than $O(\rho, \delta, N)$. Note that
\[ \frac{1}{2\pi i} \oint_{A_0} \frac{\zeta (\zeta^{-1} R_0(\zeta)+)}{R_0(\zeta)} \, d\zeta = d_m c_m^{n+1}, \quad m \in \mathbb{N}. \]

Then we use the relation that
\[ \int_{B_j} \left[ \zeta \left( \frac{1}{\zeta} R_0(\zeta) \right) \right]_{+} \, d\zeta = 0, \]

and, taking into account that $O(\delta)$ should be replace by $O(N^{1/2}/\delta)$ when $z_j$ is on the distance $O(1/N)$ from $\gamma_0$, equation (1.8) follows.

\[ \text{□} \]

Remark 3.1. It follows from (3.4) and (3.13) that
\[ 2 g_x(\infty) = \left[ \frac{1}{2\pi i} \sum_{|j| = 1} U_{j,N} \oint_{B_j} \frac{d\zeta}{R_0(\zeta)} \right] (1 + O(N^{1/2}/\delta)). \]

Corollary 3.2. In the thermodynamic limit of NLS soliton gas we have
\[ I_{m,N} = \left( \frac{m}{2\pi i} \sum_{|j| = 1} U_{j,N} \oint_{B_j} \frac{d\zeta}{R_0(\zeta)} \right) (1 + O(N^{1/2}/\delta)), \]

where we used the notations from theorem 1.1.

3.2. Approximation of $u$

The accuracy of approximation of $I_m$ with $I_{m,N}, N \to \infty$, depends on the rate of convergence of measures $\lambda_N$ to $\lambda$ $d\lambda$. In this subsection we study this rate of convergence under some additional assumptions on the thermodynamic limit.

First, here and henceforth we assume that the condition (2.27) (with some $\beta > 0$) about the continuous and positive on $\Gamma^+$ probability density of bands $\varphi(z)$ is valid uniformly in $\Gamma^+$.

Remark 3.3. Even though some results of this subsection can be extended to 2D compact sets $\Gamma^+$, in order to simplify our exposition in this subsection we assume that $\Gamma^+$ is a contour.

Lemma 3.4. For any $N \in \mathbb{N}$ and any Schwarz symmetrical hyperelliptic Riemann surface $\mathfrak{R}_N$, the matrix $M_N = -\text{Im} f_{\hat{B}_k} \frac{P_{-k}^{\mathfrak{R}_N}}{R_{\mathfrak{R}_N}}$, where $\hat{B}_k = B_k \cup B_{-k}$, is symmetric and positive definite. Moreover, the large $N$ asymptotics of $-M_N$ is given by (A.53).

This lemma follows from the well known properties of the Riemann period matrix of $\mathfrak{R}_N$. The last statement is the subject of lemma A.5.

Next, we assume that $\nu(z) \in C(\Gamma^+)$ and that
\[ \min_{\nu \in \Gamma^+} \nu(\eta) = \nu_0 > 0 \]

(3.20)
on the compact $\Gamma^+$. The results for the rest of this subsection are obtained under this assumption. In particular, it implies that the 1st NDR equation (1.4) for the fNLS soliton gas written, according to (2.26), as

$$\int_{\Gamma^+} \ln |\mu - j| \mu, (\mu) \phi(\mu) d\mu] + 2\nu(\eta) u_\nu(\eta) = \Im \eta,$$

(3.21)

where $u_\nu := \frac{\phi}{\nu}$, is a Fredholm integral equation of the second type; therefore, the existence (and uniqueness) of the solution $u_\nu$ is guaranteed. The same is true for the corresponding breather gas equation (1.2). The fact that $u_\nu \geq 0$ on $\Gamma^+$ in the case of the soliton gas was proven in [21].

**Lemma 3.5.** Under the condition (3.20), the spectral radius of $NM_N^{-1}$ in the thermodynamic limit does not exceed $\frac{\pi}{2\nu_0} + O(1/N)$ for all sufficiently large $N \in \mathbb{N}$.

**Proof.** The first system (2.24) can be written as

$$- \sum_{|k| = 1}^N \hat{u}_k \Im \frac{1}{N} \oint_{B_k} P_\nu(\zeta) d\zeta \Im z_j \quad j = 1, \ldots, N,$$

(3.22)

where $\hat{u}_k = U_{j, k} = \frac{\hat{u}_k}{\nu}$, According to lemma 3.4, the matrix $\frac{1}{N} M_N := M_N(\nu)$ is positive definite for any $\nu \geq 0$ and all $N \in \mathbb{N}$, that is the spectrum of $M_N(0)$ is positive for all $N \in \mathbb{N}$. Next, according to lemma A.5,

$$\text{diag } M_N(\nu) = \text{diag } M_N(\nu - \nu_0) + \left[ \frac{2\nu_0}{\pi} + O(1/N) \right] I_N,$$

(3.23)

where $I_N$ denotes the identity matrix of size $N$. Note that both matrices $M_N(\nu)$ and its asymptotic limit $M_N^0(\nu)$, given through (A.53), are symmetric and therefore, both have real spectrum. Arrange the eigenvalues of both matrices in the descending order. Then, according to Weyl’s perturbation theorem, see theorem VI.2.1, p 156 [3], the distance between the corresponding eigenvalues is uniformly bounded by $O(1/N)$. That means that any possible negative eigenvalue of $M_N^0(\nu - \nu_0)$ must be of order $O(1/N)$, since the matrix $M_N(\nu - \nu_0)$ is positive definite all $N \in \mathbb{N}$. Then the spectrum of $M_N^0(\nu)$ and of $M_N(\nu)$ are bounded from below by $2\nu_0/\pi + O(1/N)$. Thus, we obtained the desired spectral estimate for $M_N^{-1}(\nu)$. \hfill \Box

**Remark 3.6.** It is well known (see, for example [25]) that the integral operator $G$ in (3.21) (applied to $u(\mu) = u_{\mu}(\mu) \phi(\mu)$) expressing the Green’s potential of $u$ is positive definite. Therefore, arguments similar to those used in lemma 3.5 show that the spectrum of the operator $G + \sigma$ is situated on $[\sigma_0, \infty)$, where

$$\sigma(\eta) \geq \sigma_0 > 0 \quad \text{on } \Gamma.$$

(3.24)

Thus, we conclude that the operator $(G + \sigma)^{-1}$ has inverse bounded by $\sigma_0^{-1}$ in the appropriate functional space. Let for a fixed large $N \in \mathbb{N}$ the points $z_j := z_{j, N} \in \Gamma^+$, where $j = 1, \ldots, N$, denote the centers of the bands that are distributed on $\Gamma^+$ according to $\phi(z)$. We want to know how well the values of

$$\hat{u}_j = \frac{1}{\pi} \phi(z_j) NU_{j, N},$$

(3.25)
approximate \( u(z_j) \). Another question is the approximation of the solution \( u(z) \) of (3.21) by a piecewise constant function

\[
\hat{u}(z) = \hat{u}_N(z) := \sum_{j=1}^{N} \hat{u}_j \chi_j(z),
\]

where \( \chi_j \) is a characteristic function of a simple arc of \( \Gamma_j^+ \) containing \( z_j \) and going half way to the neighboring points \( z_{j \pm 1} \).

**Theorem 3.7.** Let the assumptions (3.20) and (2.27) with some \( \beta > 0 \) hold. If \( u(z) \) is \( \alpha \)-Hölder continuous on \( \Gamma^+ \) with some \( \alpha \in (0, 1] \). Then:

(a) the discrete measure \( \lambda_N \) from theorem 1.1, part (b), weakly* converges to \( \mu \lambda \) with accuracy \( O(N^{-\beta}) \), where \( \beta = \min\{\alpha, \beta\} \); if \( \alpha = 1 \), then the accuracy \( O(N^{-\beta}) \) should be replaced by \( O(\ln N) \);

(b) \( |u(z) - \hat{u}_N(z)| = O(N^{-\beta}) \) as \( N \to \infty \) uniformly on \( \Gamma^+ \). If \( \alpha = 1 \), then the accuracy \( O(N^{-\beta}) \) should be replaced by \( O(\ln N) \).

**Proof. (a)** Substitution of \( \sum_{j=1}^{N} u(z_j) \chi_j(z) \) into (3.21) yields

\[
\sum_{j=1}^{N} u(z_j) \int_{\Gamma_j^+} g(w, z_j) |dw| + \sigma(z) \sum_{j=1}^{N} u(z_j) \chi_j(z) = \phi(z) + E_1(z),
\]

where: \( g(w, z) \) denotes the kernel of the integral operator and \( \phi(z) \) denotes the right-hand side in (3.21). \( \Gamma_j^+ = \text{supp} \chi_j \) and \( E_1(z) \) is the error term. It is straightforward to check that \( E_1(z) = O(N^{-\alpha}) \) uniformly on \( \Gamma^+ \).

We now choose \( z = z_k, k = 1, \ldots, N \), so that (3.27) becomes

\[
\sum_{j \neq k} u(z_j) \int_{\Gamma_j^+} g(w, z_k) |dw| + u(z_k) \left[ \sigma(z_k) + \int_{\Gamma_k^+} g(w, z_k) |dw| \right] = \phi(z_k) + E_1(z_k).
\]

Using the mean value theorem, we can write \( \int_{\Gamma_j^+} g(w, z_k) |dw| = g(w_j) |\Gamma_j| \), where \( |\Gamma_j| \) is the arclength of \( \Gamma_j \) and \( w_j \in \Gamma_j \). Then the error \( E_2 \) in replacing \( \sum_{j \neq k} u(z_j) \int_{\Gamma_j^+} g(w, z_k) |dw| \) with \( \sum_{j \neq k} u(z_j) g(z_j, z_k) |\Gamma_j| \) can be estimated as

\[
\sum_{j \neq k} u(z_j) \int_{\Gamma_j^+} g(w, z_k) |dw| - g(z_j, z_k) |\Gamma_j| = \sum_{j \neq k} u(z_j) |g(w_j, z_k) - g(z_j, z_k) |\Gamma_j|
\]

\[
\leq \max_j \left( u(z_j) N |\Gamma_j| \right) \frac{1}{N} \sum_{j \neq k} \max_{w_j \in \Gamma_j} |g(w_j, z_k) - g(z_j, z_k)|
\]

(3.29)

We now want to split the terms in the last into two categories: those that are 'close' to \( z_k \) and those that are 'away' from \( z_k \). To arrange such a split we notice that for a given \( \Gamma^+ \) there exits some \( s > 0 \) such that \( g(w, z) \) becomes a monotonic function of \( w \) for any fixed \( z \in \Gamma^+ \) whenever \( |w - z| < s \) and all \( w \) are 'on the same side' of \( z \). Denote part of \( \Gamma^+ \) that is \( s \)-close to \( z_k \) by \( D_k \). We then split the latter sum into \( J_1 := \{ j : z_j \in D_k, j \neq k \} \) and the remaining part \( J_2 \). If \( j \in J_1 \), then \( |g(w_j, z_k) - g(z_j, z_k)| = O(1/N) \) uniformly over \( \Gamma^+ \). This part of sum in (3.29) can be estimated by \( O(1/N) \) uniformly in \( z_k \in \Gamma^+ \). For any remaining \( j \in J_2 \) we get
Remark 3.8. Theorem 3.7 justifies transition from the first system of linear equation (2.24) for the wavenumbers \( k_j \) to the integral equation (3.21) in the NDR. Similar result should be valid for the second system of linear equation (2.24). Assumption (3.20) was used in the proof of theorem 3.7.

Remark 3.9. Theorem 3.7 should be valid for breather gases as well.
3.3. Proof of theorem 1.1 part (b)

According to theorem 3.7 and remark 3.9, we can now prove part (b) of theorem (1.1).

Proof of Theorem 1.1, part (b). By the assumption of theorem 1.1, part (b), we have

$$\lambda_N \xrightarrow{\ast} u \d\lambda, \quad N \to \infty,$$

where $$\lambda_N = \sum_{j=1}^{N} \frac{U_{j,N}}{m} \delta(z-z_{j,N}).$$ Since $$F_m$$ is continuous on $$\Gamma^+$$, we have

$$m \int_{\Gamma^+} \left( 2 \Im F_m(z) - \oint_{A_0} \d\lambda_N(z) \right) \to 2m \int_{\Gamma^+} u(\zeta) \Im F_m(\zeta) \d\lambda(\zeta), \quad N \to \infty,$$

(3.37)

where we used $$U_{j,N} = U_{-j,N}$$, equation (1.8), and the fact that $$\oint_{A_0} F_m(z) = 0$$, $$m \in \mathbb{N}$$. Substitute back to (1.8), the result (1.9) follows. \qed

Remark 3.10. The assumptions of theorem 3.7 imply the weak convergence of $$\lambda_N$$ to $$u \lambda$$. One of these assumptions is the requirement that $$\Gamma^+$$ is 1D compact (contour).

Corollary 3.11. In the thermodynamic limit of fNLS soliton gas, we have

$$I_m = 2 \int_{\Gamma^+} u(\zeta) \Im \zeta^m \d\lambda(\zeta), \quad m \in \mathbb{N}. \quad (3.38)$$

Remark 3.12. Let $$2g$$ be the limiting g-function defined in section 3.5 below. Repeating arguments of theorem 1.1 for (3.13), we calculate the Taylor coefficients $$G_m$$ of $$2g_s/R_0 = \sum_{m=0}^{\infty} \frac{G_m}{\pi q_{m+1}}$$ for breather gas as

$$\tilde{G}_m = -2 \int_{\Gamma^+} u(\zeta) \Im \zeta^m \d\lambda(\zeta) = \frac{1}{2\pi i} \oint_{A_0} \zeta^{m+1} \frac{\d\zeta}{R_0(\zeta)}. \quad (3.39)$$

Example. Consider a special DOS (see equation (4.27)):

$$u(z) = \frac{|z|}{\pi L} \int_{0}^{\alpha L} \frac{dx}{\sqrt{q^2(x) + z^2}}, \quad z \in \Gamma^+ \subset i\mathbb{R}^+, \quad (3.40)$$

where $$q(x) = Q \chi_{[0,\alpha L]} + q \chi_{(\alpha L,L]}, Q \geq q \geq 0, \alpha \in (0,1], \ x \in [0,L]$$ and $$q(x) = q(-x)$$ for $$x \in [-L,0]$$. Then $$q(x+2L) = q(x)$$ and $$\Gamma^+ = [iq,iQ]$$. Such a DOS $$u$$ is a periodic breather/soliton gas that will be discussed in section 4.2 below. In fact, for such $$q(x)$$, we have

$$u(z) = \frac{-iz\alpha}{\pi \sqrt{Q^2 + z^2}}, \quad z \in \Gamma^+. \quad (3.41)$$

Since $$R_0(\zeta) = \sqrt{\zeta^2 + q^2} = \sum_{k=-1}^{\infty} d_k(\zeta)^{k+1} \zeta^{-k}$$ in a neighborhood of $$\zeta = \infty$$, we have $$R_0(\zeta)^{-1} = \frac{1}{d_0(\zeta)} = -\sum_{k=-1}^{\infty} d_k(\zeta)^{k+1} \zeta^{-(k+2)}$$ in a neighborhood of $$\zeta = \infty$$. Then

$$\left[ \zeta^{m-1} R_0(\zeta) \right]_{+} R_0(\zeta)^{-1} = - \left( \sum_{k=-1}^{m-1} d_k(\zeta)^{k+1} \zeta^{-(m-k)} \right) \left( \sum_{k=-1}^{\infty} kd_k(\zeta)^{k+1} \zeta^{-(k+2)} \right)$$

$$= \zeta^{m-1} - d_m(iz)^{m+1} \zeta^{-2} + O(\zeta^{-4}),$$
where we used the fact that all coefficients of $R_0(\zeta)R_0^{-1}(\zeta) - 1$ vanish. Plugging into (1.10), and taking the imaginary part, we have

$$\text{Im} F_m(\zeta) = \begin{cases} 0, & m \text{ is even}, \\ -i \left( \frac{1}{m} z^m + d_m(iq)^{m+1} \zeta^{-1} + O(\zeta^{-3}) \right), & m \text{ is odd}. \end{cases} \quad (3.42)$$

Thus, applying formula (1.9), we have, for odd $m$,

$$I_m = \frac{m}{2} \oint \frac{(-i)^{3} z \alpha}{\pi \sqrt{Q^2 + z^2}} \left( \frac{1}{m} z^m + d_m(iq)^{m+1} \zeta^{-1} + O(\zeta^{-3}) \right) \, dz + md_m(iq)^{m+1}$$

$$= \frac{m}{2\pi} \oint (1 + (Q/z)^2)^{-1/2} \left( \frac{1}{m} z^m + d_m(iq)^{m+1} \zeta^{-1} \right) \, dz + md_m(iq)^{m+1}$$

$$= -m \text{ Res}_{z=0} \left\{ (1 + (Q/z)^2)^{-1/2} \left( \frac{1}{m} z^m + d_m(iq)^{m+1} \zeta^{-1} \right) \right\} + md_m(iq)^{m+1}$$

$$= -m \frac{i^{m+1} m!!}{(m+1)!!} (Q^{m+1} + (1 - \alpha)q^{m+1}) = -m \frac{i^{m+1} m!!}{(m+1)!!} (q^{m+1}(x)), \quad (3.43)$$

where $d_m$ is defined in (1.7) and $\oint$ denotes the integral over a clockwise loop enclose $\Gamma$ and $\langle \cdot \rangle$ denotes the average over the period. For even $m$, $I_m = 0$.

It is well-known that the densities $f_k$ for NLS conserved quantities satisfy the following recursion relation (see [30]):

$$f_{n+1} = \frac{1}{2} \sum_{k=n}^{n-1} f_k f_{n-k} - q(x) \left( \frac{f_n}{q(x)} \right)_{x}, \quad n \in \mathbb{N}$$

$$f_1 = \frac{1}{2} |q(x)|^2. \quad (3.44)$$

Let $q(x)$ be a piece-wise constant and periodic function. Computing the limiting average of $f_n$ over $[-T, T]$, where $T \to \infty$ (see, for example, equations (3.3a) and (3.3b) in [16]), we obtain

$$\langle f_m \rangle = a_m \langle |q(x)|^{m+1} \rangle, \quad m \in \mathbb{N}, \quad (3.45)$$

where $a_1 = \frac{1}{2}$, and for $m \geq 2$:

$$a_m = \begin{cases} 0, & m \text{ is even}, \\ \frac{1}{2} \sum_{k=1}^{m-2} a_k a_{m-1-k}, & m \text{ is odd}. \end{cases}$$

Note that for a periodic function the limiting average coincides with the average over the period. It is easy to check that $a_m = -d_m$. Comparing (3.43) with (3.45), we obtain

$$\langle f_m \rangle = (-1)^{\frac{m+1}{2}} m^{-1} I_m, \quad m \in \mathbb{N}. \quad (3.46)$$
3.4. Averaged conserved fluxes under the fNLS and higher times flows

It is well known that the complete integrability of the fNLS equation implies there are infinitely many conserved quantities [33]. In fact, the fNLS flow is the second flow in the so-called focusing ZS hierarchy (for a full detailed description of the hierarchy see [14]). The RHP 2.1 for finite gap fNLS solution can be extended to include higher ZS-hierarchy flows if the function \( f \) in (2.6) is replaced by

\[
f(z|t) = f_0(z) + \sum_{l=1}^{\infty} t_l^\ell z^j,
\]

where \( f_0(z) \) is the same as in (2.6) and \( t = (t_1, t_2, \ldots) \), with \( t_1 = x, t_2 = 2t \) and \( t_3, t_4, \ldots \) denoting higher flows times. Here \( x, t \) are space-time variables for the fNLS. By letting all \( t_l = 0, l \geq 3 \), one recovers the \( f \) for the fNLS equation. In general, for \( f \)th flow in the ZS hierarchy one takes

\[
t_1 = x, \quad t_l = T_l \delta_{l,j}, \quad l = 2, 3, \ldots,\]

where \( \delta_{l,j} \) is the Kronecker delta, \( t \) is the time variable for the \( j \)th hierarchy equation and \( T_l > 0 \) is a certain constant. For example, \( j = 3 \) corresponds to the mKdV equation with \( t_3 = 4t \).

Similarly to the fNLS quasienergy meromorphic differential \( dq_N = dq_{2,N} \), one can (uniquely) define the second kind real normalized meromorphic differentials \( dq_{j,N} \) of the expansion

\[
dq_{j,N}(z) = [\pm j T_j z^{j-1} + O(z^{-2})]dz,
\]

of the main and the secondary sheet respectively. According to [17], the averaged conserved fluxes of the \( j \)th flow are the coefficients \( I_{m,j,N} \) of the expansion

\[
dq_{j,N} = \left( j T_j z^{j-1} - \sum_{m=1}^{\infty} \frac{I_{m,j,N}}{z^m+1} \right) dz.
\]

One can now follow the approach described in section 2 to obtain the breather gas second NDR equation

\[
\int_{\Gamma^+} \left[ \log \left| \frac{w - z}{w - z'} \right| + \log \left| \frac{R_0(z)R_0(w) + zw - \delta_0^2}{R_0(z)R_0(w) + zw - \delta_0^2} \right| \right] v_j(w) d\lambda(w) + \sigma(z)v_j(z) = -\text{Im}[j T_j z^{j-1}],
\]

for the \( j \)th hierarchy flow, where \( v_j(z) \) denotes the \( j \)th order analogue of the density of fluxes \( v(z) = v_2(z) \). By taking the limit \( \delta_0 \to 0 \), we obtain the soliton gas second NDR equation

\[
\int_{\Gamma^+} \log \left| \frac{w - z}{w - z'} \right| v_j(w) d\lambda(w) + \sigma(z)v_j(z) = -\text{Im}[j T_j z^{j-1}],
\]

for the \( j \)th hierarchy flow. It is worth noting that these equations in the \( j = 2 \) become the NDR equations (1.3) and (1.5) respectively.

As in the case of the fNLS flow (see (2.18)), the periods \( \omega_{l,j} = -\frac{\delta_{l,j}}{\delta_{l,j}} dq_{l,N}, l = 1, \ldots, N \), represent the solitonic frequencies of the \( j \)th flow, \( j = 2, 3, \ldots \). Then the average conserved fluxes in the \( j \)th flow (of ZS-hierarchy) are summarized at the following theorem
Theorem 3.13.

(a) Fix $m \in \mathbb{N}$, then for a sufficiently large $N$ in the thermodynamic limit of the breather gases for the $j$th fNLS flow the averaged conserved fluxes are

\[
I_{m, j N} = \frac{m}{2\pi i} \sum_{|j|=1}^{N} \int_{B_2} [\frac{\zeta^{m-1} R_0(\zeta)}{R_0(\zeta)} + \oint_{A_0} T_j \zeta \frac{\zeta^{m-1} R_0(\zeta)}{R_0(\zeta)} + d\zeta] \times (1 + O(N^{1/2} \delta)), \quad \text{where } U_{j,N}^{(i)} := \frac{1}{2} \omega_{i,j};
\]

(b) Let $v_j$ solves (3.51). If the measures $\lambda_{j,N} := \sum_{l=1}^{N} \frac{\delta(z - z_{LN})}{\delta(z - z_{LN})}$, where $z_{LN} \in 1^+$ denotes the center of the $l$th bands of $\mathcal{N}$, $l = 1, \ldots, N$, and $\delta(z)$ denotes the delta-function, are weakly* convergent to the (signed) measure $v_j(z)\,d\lambda(z)$ on $1^+$ then the thermodynamic limit of $I_{m,N}$ is given by

\[
I_{m, j} = 2m \int_{\Gamma^+} v_j(\zeta) \text{Im} F_m(\zeta) \,d\lambda(\zeta) + \frac{m}{2\pi i} \int_{A_0} T_j \zeta \frac{\zeta^{m-1} R_0(\zeta)}{R_0(\zeta)} + d\zeta, \quad m, j \in \mathbb{N},
\]

where $F_m(\zeta)$ is defined in (1.10). Moreover, the second integral term in (3.54) can be explicitly computed in terms of $d_k$, it reads

\[
\frac{1}{2\pi i} \int_{A_0} T_j \zeta \frac{\zeta^{m-1} R_0(\zeta)}{R_0(\zeta)} + d\zeta = -T_j \left( \sum_{k=m}^{m+j-1} (m + j - k - 2)d_k d_{m+j-k-2} \right) \delta^{m+j}. \tag{3.55}
\]

**Proof.** The proof is essentially the same as the proof of theorem 1.1 except that in the current proof we modify the definition of operator $A$, see (3.7), as

\[
A_j(f) := -\frac{1}{2\pi i} \sum_{|j|=0}^{N} \oint_{A_j} (T_j \zeta^j + U_{j,N}^{(i)}) \frac{f(\zeta) d\zeta}{R(\zeta)}.
\]  

Applying the residue theorem to the integral in (3.55):

\[
[z^{m-1} R_0(z)] + R_0(z)^{-1} = \sum_{l=1}^{\infty} \left( \sum_{k=-l,l}^{l-1} d_k (l - 2 - k) d_{l-2-k} \right) \delta_l^{m-1},
\]

one obtains

\[
\frac{1}{2\pi i} \int_{A_0} T_j \zeta \frac{\zeta^{m-1} R_0(\zeta)}{R_0(\zeta)} + d\zeta = T_j \text{Res} \left\{ \zeta \frac{\zeta^{m-1} R_0(\zeta)}{R_0(\zeta)} + R_0(\zeta)^{-1}; \zeta = \infty \right\}
\]

\[
= -T_j \left( \sum_{k=m}^{m+j-1} (m + j - k - 2)d_k d_{m+j-k-2} \right) \delta_0^{m+j},
\]

which completes the proof. \(\square\)
Taking the limit \( \delta_0 \to 0 \), we get the corresponding formulae for the soliton gas.

**Corollary 3.14.** In the case of the soliton gas for the \( j \)th fNLS flow, i.e. when \( \gamma_0 \) is one of the shrinking bands, the equations (3.53) and (3.54) become

\[
I_{m,i} = \frac{m}{2\pi i} \sum_{n=1}^{N} U_{i}^{[n]} \int_{B_i} \zeta^{m-1} \, d\zeta \left( 1 + O(N^{1/2}) \right),
\]

(3.57)

\[
I_{m,j} = 2 \int_{\Gamma^+} v_j(\zeta) \text{Im} \zeta^m \, d\lambda(\zeta), \quad j = 2, 3, \ldots,
\]

(3.58)

respectively.

### 3.5. Thermodynamic limit of the quasimomentum differentials \( dp_N \) and related questions

In this subsection we express the thermodynamic limit \( dp \) of \( dp_N \), as well as its ‘antiderivative’ \( 2g_{z}(z) \), see (1.18), for fNLS soliton and breather gases in terms of the DOS \( u(z) \). We remind that in this subsection we assume that \( \Gamma^+ \) is a contour and \( d\lambda(z) = |dz| \). We first show that \( dp/dz \) is analytic in \( \mathbb{C} \setminus \Gamma \) and find its boundary behavior on \( \Gamma \). The obtained results allow us to express the spectral scaling function \( \sigma(z) \) in terms of the average value and the jump of \( dp/dz \) on \( \Gamma \). Similar results are valid for quasienery as well as for the higher fNLS flows.

Given a compact piece-wise smooth contour \( \Gamma^+ \subset \mathbb{C}^+ \) define the real valued function \( \theta(\mu) \) on \( \Gamma^+ \) by \( d\mu = |d\mu|e^{i\theta(\mu)} \), where \( d\mu \) is a differential in the positive direction on \( \Gamma^+ \). Taking into account the orientation of \( \Gamma \), the same relation on \( \Gamma^- \) is \( d\mu = -|d\mu|e^{-i\theta(\mu)} \). Introducing a new function \( \tilde{u} = u e^{-i\theta} \), we observe that \( u|d\mu| = \tilde{u} \, d\mu \) on \( \Gamma^+ \). This equation can be extended to \( \Gamma \) if \( \tilde{u} \) is Schwarz symmetrically continued on \( \Gamma^- \).

In the following theorem 3.15 we express the thermodynamic limit \( \frac{dp}{dz} \) of the quasimomentum density \( dp_N \), defined as \( \frac{dp}{dz} = 1 - \sum_{m=1}^{\infty} \frac{dU_{i}^{[m]}}{m+1} \) in (1.13), in terms of the Cauchy transform of \( \tilde{u} \). Also, the finite Hilbert transform (FHT) \( H_{\Gamma} \) in theorem 3.15 and in this paper is defined as

\[
H_{\Gamma}[f](z) = \frac{1}{\pi} \text{p.v.} \int_{\Gamma} \frac{f(w)dw}{w-z}
\]

(3.59)
on \( \Gamma \).

**Theorem 3.15.** Let \( \Gamma \subset \mathbb{C} \) be a simple, compact, piece-wise smooth Schwarz symmetrical contour and the DOS \( u(z) \) is the solution of (1.4). Then:

(a) \( \frac{dp}{dz} \) is analytic in \( \mathbb{C} \setminus \Gamma \) and

\[
\frac{dp}{dz} = 1 + 2\pi C_{\Gamma}[-\tilde{u}] \quad \text{on} \quad \mathbb{C} \setminus \Gamma,
\]

(3.60)

where \( C_{\Gamma} \) denotes the Cauchy transform on \( \Gamma \), and;

(b) the jump \( \Delta \frac{du}{dz} \) of \( \frac{du}{dz} \) over \( \Gamma \) is \( 2\pi \tilde{u} \) whereas the average \( \frac{dU_{i}^{[m]}}{dz} \) defined as \( \frac{dU_{i}^{[m]}}{dz} = \frac{1}{2} \left( \frac{dU_{i}^{[m]}}{dz} \right)_+ + \left( \frac{dU_{i}^{[m]}}{dz} \right)_- \) is

\[= 1 - \pi H_{\Gamma}[\tilde{u}].\]
Proof.

(a) The compactness of $\Gamma^+$ implies that the series (1.13) for $\frac{dp}{dz}$ has non zero radius of convergence. Then, according to (1.13),

$$\frac{dp}{dz} = 1 - 2 \int_{\Gamma^+} u(\mu)|d\mu| \sum_{m=1}^{\infty} \frac{\text{Im} \mu^m}{\varepsilon^{m+1}} = 1 - \frac{1}{i} \int_{\Gamma} u(\mu)|d\mu| \sum_{m=1}^{\infty} \mu^m$$

$$= 1 + \frac{2\pi}{2\pi i} \int_{\Gamma} \frac{i\mu|d\mu}{\mu - z} = 1 + 2\pi C_{\Gamma}[\tilde{u}]$$  (3.61)

where we use $\mu^m - \bar{\mu}^m = 2i \text{Im} \mu^m$ and $u(z)$ has anti-Schwarz symmetric extension of in $\mathbb{C}$. Taking into account the orientation of $\Gamma$, we have $d\mu = -|d\mu|e^{-i\theta(\mu)}$ on $\Gamma^-$ so that $\tilde{u}$ is Schwarz symmetrically continued on $\Gamma^-$. Formula (3.61) is valid in $\mathbb{C}\setminus\Gamma$. Thus, we showed (3.60).

(b) According to (3.60), $\Delta \frac{dp}{dz} = 2\pi \tilde{u}$ on $\Gamma$ and $(\frac{dp}{dz})_{av} = 1 - i\pi H_{\Gamma}[\tilde{u}]$. \qed

As an immediate consequence of theorem 3.15, we calculate

$$-2g_{s}(z) + z = \int_{0}^{\tilde{z}} dp = z + i \int_{\Gamma}[\ln(\mu - z) - \ln \mu]u(\mu)|d\mu|$$

$$= z + \int_{\Gamma} u(\mu) \arg \mu |d\mu| + i \int_{\Gamma} \ln(\mu - z)u(\mu)|d\mu|,$$  (3.62)

which is valid for any $z \in \mathbb{C}\setminus\Gamma$. As it was noted in the introduction, $2g_{s}(z)$ is analytic but, in general, multiple-valued on $\mathbb{C}\setminus\Gamma$, however, it is single valued on $\mathbb{C}\setminus\gamma$, where $\gamma \in \mathbb{C}$ be a simple Schwarz symmetric curve that consists of the superbands and connecting them gaps.

According to (3.62), $2g_{s}(\infty) = -\int_{\Gamma} u(\mu) \arg \mu |d\mu|$, so the very last term in (3.62) represents the part of the Laurent expansion of $2g_{s}$ at infinity in the negative powers of $z$.

Considering the average $g_{s+} + g_{s-}$ of the boundary values, we obtain

$$g_{s+}(z) + g_{s-}(z) = -2 \int_{\Gamma^+} u(\mu) \arg \mu |d\mu| + i \int_{\Gamma^+} \left[ \ln \frac{\mu - z}{\mu - z} + i\pi \chi_{\gamma}(\mu) \right] u(\mu)|d\mu|,$$  (3.63)

where $\chi_{\gamma}(\mu)$ is the indicator function of the arc $(z_{\infty}, z)$ of $\Gamma^+$. Here $z_{\infty}$ denotes the beginning of the oriented curve $\Gamma^+$.

In the corollary below the ‘carrier DOS’ function $\tilde{u}(z)$ was defined in [10] as a smooth Schwarz symmetrical interpolation of the carrier wavenumbers $k_j$ on $\Gamma$, that satisfies equation (25) [10], see also (1.14). Comparing (3.63) with (1.14), in which the limit $\delta_0 \to 0$ is taken, and observing that $\text{Im} g_{s}(z)$ is continuous on $\mathbb{C}$, we obtain corollary 1.4 for the case of a soliton gas.

Corollary 3.16. In the conditions of theorem 3.15 we have

$$\sigma(z) = \frac{-2\pi \text{Im} \int_{0}^{\tilde{z}} \left( \frac{dp}{dz}_{av} \right) dz}{\Delta \frac{dp}{dz}(z)} \cdot e^{-i\theta(z)},$$  (3.64)

where $\sigma(z)$ from (1.4) is the relative density of bandwidth.

Proof. Indeed, WLOG, we assume $0 \in \Gamma$. Integrating the latter equation along $\Gamma$, we obtain

$$\int_{0}^{\tilde{z}} \left( \frac{dp}{dz}_{av} \right) dz = z + i \int_{\Gamma} \ln(w - z)u(w)|dw| - i \int_{\Gamma} \ln(w)u(w)|dw|,$$  (3.65)
where we changed the order of integration in $H_\Gamma[\tilde{u}]$. Taking the imaginary part in the latter equation together with (1.4) yields (3.64).

To extend the obtained above result from soliton to breather fNLS gases, we observe that, according to (3.16) and (1.9),

$$2g_x(z) = \sum_{m=0}^{\infty} \frac{G_m}{z^m},$$

where

$$G_m = -2 \int_{\Gamma+} u(\zeta) \text{Im} F_m(\zeta) d\lambda(\zeta) - z \delta_0^{m+1}$$

in the case of a breather gas and

$$G_m = -2 \int_{\Gamma+} u(\zeta) \text{Im} \zeta^m d\lambda(\zeta),$$

in the case of a soliton gas, see (3.16) and (3.38).

The following theorem 3.18 provides the expression for $2g_x(z)$ in the case of an fNLS breather gas. As a consequence, we obtain a proof of corollary 1.4.

**Remark 3.17.** Under the assumptions of theorem 1.1 part (b), it follows from (3.18) that in the thermodynamic limit of fNLS breather gas, we have

$$2g_x(\infty) = -2 \int_{\Gamma+} u(\zeta) \arg \left( \zeta + \sqrt{\zeta^2 + \delta_0^2} \right) d\lambda(\zeta),$$

and in the soliton gas case, we have

$$2g_x(\infty) = -2 \int_{\Gamma+} u(\zeta) \arg \zeta d\lambda(\zeta).$$

**Theorem 3.18.** Let $\Gamma \subset \mathbb{C}$ be a simple, compact, piece-wise smooth Schwarz symmetrical contour $\gamma_0 = [-i0, i0]$ be the permanent band, the DOS $u(\zeta)$ solves (1.2) and $2g_x$ is defined by $2g_x/R_0 = \sum_{m=0}^{\infty} \frac{\delta_0^{m+1}}{z^m}$ with $G_m$ given by (3.39). Then

$$2g_x(z) = i \int_{\Gamma} u(\zeta) \ln \frac{R_0(\zeta)R_0(z) + \zeta^2 - \delta_0^2}{\zeta - z} |d\zeta| + z - R_0(z),$$

where $z \in \mathbb{C} \setminus (\Gamma \cup \gamma_0)$. Moreover, substituting $z = \infty$ in (3.70) we obtain $2g_x(\infty)$ given by (3.68).

**Proof.** Using (3.39), we start obtain

$$\frac{2g_x}{R_0} = \sum_{m=0}^{\infty} \frac{G_m}{z^{m+1}} = \lim_{m \to \infty} \left[ \int_{\Gamma+} 2u(\zeta) \text{Im} \frac{\mu^m}{R_0(\mu)} |d\zeta| - \frac{1}{2\pi i} \int_{\Gamma} \sum_{m=0}^{\infty} \frac{\delta_0^m}{R_0(\zeta)} d\zeta \right]$$

$$= i \int_{\Gamma} u(\zeta) \left[ \sum_{m=0}^{\infty} \frac{\mu^m}{R_0(\mu)} |d\zeta| - \frac{1}{2\pi i} \int_{\Gamma} \sum_{m=0}^{\infty} \frac{\delta_0^m}{R_0(\zeta)} d\zeta \right]$$

$$= -i \int_{\Gamma} u(\zeta) \left[ \sum_{m=0}^{\infty} \frac{\mu^m}{R_0(\mu)} |d\zeta| + \frac{1}{2\pi i} \int_{\Gamma} \sum_{m=0}^{\infty} \frac{\delta_0^m}{R_0(\zeta)} d\zeta \right]$$

$$= \frac{z}{R_0(z)} - 1 + \frac{i}{R_0(z)} \int_{\Gamma} u(\zeta) \ln \frac{R_0(\zeta)R_0(z) + \zeta^2 - \delta_0^2}{\zeta - z} |d\zeta|,$$
where we used the anti-derivative (A.62) and the anti Schwarz symmetry of $u$ in the latter transformation. Multiplying (3.71) by $R_0$ yields (3.70). Note that $z - R_0(z)$ is an odd function near $z = \infty$ and therefore does not contribute to $2g_\gamma(\infty)$. Therefore, to recover (3.68), one needs to divide both the numerator and the denominator of the logarithm in (3.71) by $z$ and use the anti Schwarz symmetry of $u$.

It is straightforward to check that in the limit $\delta_0 \to 0$ equation (3.70) for the breather gas turns into (3.62) for the soliton gas. We also note that $2g_\gamma(z)$ is analytic in $\mathbb{C}(\Gamma \cup \gamma_0)$ and $\text{Im} g_\gamma(z)$ is continuous in $\mathbb{C}\backslash \gamma_0$. As a consequence of theorem 3.18, we obtain corollary 1.4 for the case of a breather gas.

**Remark 3.19.** Define $dp/dz = 1 - \sum_{m=1}^\infty I_m/z^{m+1}$ with $I_m$ given by equation (1.11), use the result of theorem 3.18 and assume that $I_m = -mG_m$ (which can be regarded as the limit of (3.16)). Then we have

$$
\frac{dp}{dz} = 1 - 2g_{xc} = 1 + 2 \pi C_{\Gamma}[\bar{u}] + \left(\frac{z}{R_0(z)} - 1 - i \int_{\Gamma} \frac{u(\zeta)|d\zeta|}{R_0(\zeta)} \right). \tag{3.72}
$$

Comparing (3.72) with $dp/dz$ for the soliton gas from theorem 3.15, one finds that they differ by a ‘breather correction’ term in the round brackets in (3.72). It is easy to see that in the limit $\delta_0 \to 0^+$ this correction term becomes 0 and $dp/dz$ for the breather gas reduces to that for the soliton gas.

**Remark 3.20.** Similar to theorem 3.15 results can be obtained for the meromorphic differentials $dq_j$ (defined as the thermodynamic limit of $dq_j/\nu_j$ given by equation (3.49)), $j = 2, 3, \ldots$, where we replace $u(z)$ by the corresponding $v_j(z)$ and 1 by $J^j \nu^{j-1}$. In particular,

$$
\frac{dq_j}{dz} = J^j \nu^{j-1} = 2 \pi C_{\Gamma}[\bar{v}_j], \quad j = 1, 2, \ldots, \tag{3.73}
$$

where $j = 1$ corresponds to equation (3.60). Obviously, $\frac{du}{dz}$ is analytic on $\mathbb{C}(\Gamma \cup \gamma_0)$ and the jump of $\frac{du}{dz}$ over $\Gamma$ is $2\pi \bar{v}_j$. The average of boundary values of $\frac{du}{dz}$ on $\Gamma$ is $1 - i\pi H_{\Gamma}[\bar{v}_j]$. We suppose now that $\Gamma \subset \mathbb{iR}$, i.e. we have a bound state gas. Then $\bar{v}_j = -i\nu_j$, so (3.73) becomes

$$
\frac{dq_j}{dz} = J^j \nu^{j-1} = -2i\pi C_{\Gamma}[v_j]. \tag{3.74}
$$

Thus the jump of $\frac{du}{dz}$ on $\Gamma$ is $-2i\nu v_j$ and its average value on $\Gamma$ is $1 - \pi H_{\Gamma}[v_j]$.

4. Periodic gases

By periodic soliton or breather fNLS gases we understand gases whose spectral characteristics $\Gamma^+$, $\varphi(z)$ and $\nu(z)$ can be generated as the semiclassical limit of the direct fNLS spectral problem with periodic potentials. Such problem was considered in the recent paper [4], where the authors considered the (formal) semiclassical limit of the direct fNLS spectral problem with the real, even, continuous single lobe potential $q(x)$ with a period $2L > 0$, that is, $q(x + 2L) = q(x)$. Following [4], WLOG, we assume $\max q(x) = q(0) = M$, $\min q(x) = q(\pm L) = m > 0$
and \( q(x) \) is monotonically decreasing on \([0, L]\). Then the semiclassical (\( \varepsilon \to 0^+ \)) limit of the Floquet discriminant (trace of the monodromy matrix) is given by
\[
\varphi(\varepsilon) = \frac{2}{\varepsilon} \cos \left( \frac{S_2(\varepsilon)}{\varepsilon} \right) 
\]
\[= 2 \cos \left( \frac{S_1(\varepsilon)}{\varepsilon} \right) \cosh \left( \frac{S_2(\varepsilon)}{\varepsilon} \right) \tag{4.1} \]
where \( \varepsilon \in [im, iM] \) is the original (ZS) spectral variable, \( \lambda := -\varepsilon^2 \in [m^2, M^2] \),
\[
S_1(\varepsilon) = \int_{-\rho(\varepsilon)}^{\rho(\varepsilon)} |q^2(x) - \lambda| \, dx, \quad S_2(\varepsilon) = \int_{\rho(\varepsilon)}^{L} |q^2(x) - \lambda| \, dx, \tag{4.2} \]
and \( x = \pm \rho(\varepsilon), |x| \leq L \Rightarrow q^2(x) = \lambda \).

The Lax spectrum (the bands) of the spectral problem are defined by the requirements
\[
\Delta = \sum_{m} \Gamma_m \quad \text{with} \quad \Gamma_m = \Gamma^+ = \Gamma^- \tag{4.3} \]
where additional \( \varepsilon \)-scaled bands are accumulating as \( \varepsilon \to 0 \). Equation (4.1) implies that the centers of bands \( \lambda_n \in \Gamma, \Gamma = \Gamma^+ \cup \Gamma^- \), are given by \( \cos \left( \frac{S_2(\varepsilon)}{\varepsilon} \right) = 0 \) or
\[
S_1(\lambda_n) = 2 \int_{0}^{\rho(\lambda_n)} \sqrt{|q^2(x) - \lambda_n|} \, dx = \pi \varepsilon (n + \frac{1}{2}), \quad \text{where} \quad n \in \mathbb{N}. \tag{4.4} \]

Then the total number of bands \( N \) is the integer part of
\[
N = \text{Int part} \left\{ \frac{2}{\pi \varepsilon} \int_{0}^{L} \sqrt{|q^2(x) - m^2|} \, dx - \frac{1}{2} \right\}. \tag{4.5} \]

By definition, the density
\[
\varphi(\lambda) = \lim_{\Delta \to 0} \lim_{N \to \infty} \frac{\# \text{ of } \lambda_n \text{ in } \Delta \text{ nbhd of } \lambda}{2N \Delta}. \tag{4.6} \]

Since
\[
\lim_{N \to \infty} \frac{\# \text{ of } \lambda_n \text{ in } [\lambda_2, \lambda_1]}{N(\lambda_1 - \lambda_2)} = \frac{\int_{0}^{\rho(\lambda_2)} \sqrt{|q^2(x) - \lambda_2|} \, dx - \int_{0}^{\rho(\lambda_1)} \sqrt{|q^2(x) - \lambda_1|} \, dx}{(\lambda_1 - \lambda_2) \int_{0}^{L} \sqrt{|q^2(x) - m^2|} \, dx}
\]
\[
= \frac{\int_{0}^{\rho(\lambda_1)} \frac{\lambda_1 - \lambda_2}{\sqrt{|q^2(x) - \lambda_1|}} + \int_{0}^{\rho(\lambda_2)} \sqrt{|q^2(x) - \lambda_2|} \, dx}{(\lambda_1 - \lambda_2) \int_{0}^{L} \sqrt{|q^2(x) - m^2|} \, dx}, \tag{4.7} \]
we obtain
\[
\varphi(\lambda) = \frac{\int_{0}^{\rho(\lambda)} \frac{dx}{\sqrt{|q^2(x) - m^2|}}}{2 \int_{0}^{L} \sqrt{|q^2(x) - m^2|} \, dx}, \tag{4.8} \]
provided that \( \rho(\lambda) \) is differentiable or at least Hölder class with the exponent \( \alpha > \frac{1}{2} \). Transition from \( \varphi(\lambda) \) to \( \varphi(z) \) yields the density of bands function
\[
\varphi(z) = \frac{|z| \int_{0}^{z} \frac{dx}{\sqrt{|q^2(x) + z^2|}}}{\int_{0}^{L} \sqrt{|q^2(x) - m^2|} \, dx}, \tag{4.9} \]
Note that the numerator is almost identical to the density (A.6) from [28].

Finally, we consider the scaled bandwidth function $\nu(z)$ (with a slight abuse of notation we sometimes write $\nu(\lambda)$). It is asymptotically defined by

$$
\frac{dw}{d\lambda} |_{\lambda=\lambda_0} \Delta \lambda = 4,
$$

(4.10)

where $\Delta \lambda$ is the bandwidth. Then

$$
\Delta \lambda = \frac{2 \pi S_1(\lambda_0)}{\cosh \frac{2m \Delta}{\lambda}}.
$$

(4.11)

so that, according to (4.5),

$$
\nu(z) = \frac{\pi S_1(\lambda)}{2 \int_0^L \sqrt{|q^2(x) + z^2|} \, dx} = \frac{\pi \int_{-L}^L \sqrt{|q^2(x) + z^2|} \, dx}{2 \int_0^L \sqrt{|q^2(x) - m^2|} \, dx}.
$$

(4.12)

This formula is near identical to (A.7) from [28]. Now we can easily express the ‘relative density of bandwidth’ function

$$
\sigma(z) = \frac{2r(z)}{\varphi(z)} = \frac{\pi \int_{-L}^L \sqrt{|q^2(x) + z^2|} \, dx}{\int_{-L}^L \sqrt{|q^2(x) - m^2|} \, dx},
$$

(4.13)

which, together with the compact set $\Gamma^+$, determines the NDR for soliton and breather gases.

Before finishing this subsection we want to emphasize that for us the asymptotic formula (4.1) is a motivation to introduce a soliton (or a breather) gas with $\Gamma^+$, $\varphi(z)$ and $\nu(z)$ ‘parametrized’ by $q(x)$ according to (4.3), (4.9) and (4.12) respectively. Moreover, as it will be shown below, the case of $m = 0$ corresponds to the soliton gas on $\Gamma = [-iM, iM]$, whereas the case of $m \neq 0$ corresponds to the breather gas on $\Gamma = \Gamma^+ \cup \Gamma^-$ given by (4.3) with additional ‘stationary’ band on $[-im, im]$. In this approach, the requirements on the ‘parametrizing’ function $q(x)$ can be relaxed, in particular, $q(x)$ can have finitely many jump discontinuities on the period. We also want to mention that in the rest of the section we consider the odd continuation of $\varphi(z)$ from $\Gamma^+$ into $\Gamma$, so the $|z|$ in (4.9) can be replaced by $\Im z$.

4.1. Density of states $u(z)$ for the periodic soliton gas

In this subsection we will solve the NDR equation for the DOS $u(z)$ in the particular case of $q(L) = m = 0$. As it turns out, $u(z)$ is proportional to $\varphi(z)$ given by (4.9). The NDR equation (1.4) for the DOS $u(z)$ can be written as

$$
-i \int_{-iM}^{iM} \ln |\mu - z| r(\mu) |\varphi(\mu)| \, d\mu + 2\nu(z) r(z) = -iz,
$$

(4.14)

where $r(z) = \frac{a(z)}{\varphi(z)}$ and $z \in [-iM, iM]$. We will show that (4.14) is satisfied by $r = \text{const}$.

**Theorem 4.1.** If $\varphi(z)$ and $\nu(z)$ are given by (4.9) and (4.12) where $q(x)$ is monotonically decreasing on $[0, L]$ and $q(L) = 0$ then equation (4.14) has a constant solution

$$
r = \frac{1}{\pi L} \int_0^L q(x) \, dx,
$$

(4.15)
so that the DOS
\[ u(z) = \frac{|z|}{\pi L} \int_0^{q^{-1}(|z|)} \frac{dx}{\sqrt{q^2(x) + z^2}}, \quad z \in \Gamma^+. \] (4.16)

**Proof.** As it was proven in [21], there exists a unique solution for the integral equation (4.14). Assume that \( r(z) = r \), where \( r > 0 \) is a constant. If we can find \( v \) satisfying (4.14), we will prove the theorem.

Changing variables \( \mu = iy, z = i\xi \) in (4.14) and then differentiating in \( \xi \), we obtain
\[ \int_{-M}^{M} \varphi(iy) dy + 2 \frac{d}{d\xi} \varphi_i(\xi) = 1/r. \] (4.17)

Using (4.12) and (4.9) we obtain
\[ 2 \frac{d}{d\xi} \varphi_i(\xi) = \frac{\pi \xi}{\int_{-M}^{M} q(x) dx} \] (4.18)

and
\[ \int_{-M}^{M} q(x) dx \int_{-M}^{M} \frac{\varphi(iy) dy}{y - \xi} = \int_{-M}^{M} \left( \int_{-M}^{M} \frac{dx}{\sqrt{q^2(x) - y^2}} \right) \] (4.19)

Changing the limit of integration in the second integral \( I \), we obtain
\[ I = \int_{-M}^{M} \left( \int_{-M}^{M} \frac{dx}{\sqrt{q^2(x) - y^2}} \right) \] (4.20)

Since the inner integral is zero when \( |\xi| < q(x) \) and \( \frac{i\pi}{\sqrt{q^2(x) - \xi^2}} \) otherwise, we obtain
\[ I = \int_{-M}^{M} \frac{i\pi dx}{\sqrt{q^2(x) - \xi^2}} = -\pi \left( \int_{-M}^{M} \frac{dx}{\sqrt{q^2(x) - q^2(x)}} \right). \] (4.21)

where one has to consider the proper branch of \( \sqrt{\xi^2 - q^2(x)} \) to obtain the correct sign. We complete calculating (4.19) by observing
\[ \int_{-M}^{M} \left( \int_{-M}^{M} \frac{dx}{\sqrt{q^2(x) - y^2}} \right) = \pi L. \] (4.22)

Substituting now (4.18), (4.19) into (4.17), we obtain
\[ r = \frac{1}{\pi L} \int_{-M}^{M} q(x) dx. \] 

Consider the family of even potentials \( q_k(x) \) with the period \( kL, k \geq 1 \), generated by \( q(x) \), where \( q_k(x) \equiv q(x) \) on \([0, L]\) and \( q_k(x) \equiv 0 \) on \([L, kL]\). We can extend theorem 4.1 from \( q(x) = q_1(x) \) to \( q_k(x) \) by considering small deformations \( \tilde{q} \) of \( q_k \) on \([L - \varepsilon, kL]\) so that \( \tilde{q}_k \) is
monotonically decreasing and \( \hat{q}_k(kL) = 0 \). Then theorem 4.1 is valid for \( \hat{q}_k \). Thus, in the small deformation limit (for a fixed \( k > 0 \)) we obtain the following result.

**Corollary 4.2.** For described above periodic potentials \( q_k(x) \) with the period \( kL, k \geq 1 \), formulae (4.15) and (4.16) become

\[
\begin{align*}
  r_k &= \frac{1}{\pi kL} \int_0^L q(x)dx, \\
  u_k(z) &= \frac{|z|}{k\pi L} \int_0^\infty \frac{q^{-1}(\xi)}{\sqrt{q^2(x) + z^2}} dx
\end{align*}
\]  

(4.23)

respectively.

Another way to prove corollary 4.2 is to repeat the steps of theorem 4.1, taking into account that the density \( \varphi(z) \) does not depend on \( k \) and

\[
\nu_k(z) = \nu(z) + \frac{\pi(k-1)L|z|}{2 \int_0^L q(x)dx}
\]  

(4.24)

4.2. Density of states for periodic breather gas

The results of the previous section 4.1 do not work in the case when \( m > 0 \) and the bands are located on the interval \( \Gamma^+ = [\text{im}, iM] \) and its Schwarz symmetrical \( \Gamma^- \). Since \( [-\text{im}, \text{im}] \) is a single band of the Lax spectrum of \( q(x) \) ([4]), it makes sense to assume that the case \( m > 0 \) corresponds to the breather gas. Indeed, the following theorem 4.3 shows that a constant \( r(z) = r \) satisfies the NDR (1.2) for the breather gas written as

\[
\text{Re} \int_{\Gamma} \ln \left( \frac{R_0(z)R_0(\mu) + z\mu + m^2}{\mu - z} \right) R(\mu) \varphi(\mu) |d\mu| + 2\nu(z)r(z) = -iR_0(z),
\]  

(4.25)

where \( z \in \Gamma, \Gamma = [-iM, iM] \) and \( R_0(z) = \sqrt{z^2 + m^2} \).

**Theorem 4.3.** If \( \varphi(z) \) and \( \nu(z) \) are given by (4.9) and (4.12) where \( q(x) \) is monotonically decreasing on \([0, L]\) and \( q(L) = m > 0 \) then the integral equation (4.25) has a constant solution

\[
r = \frac{1}{\pi L} \int_0^L \sqrt{q^2(x) - m^2} dx,
\]  

(4.26)

so that the corresponding DOS

\[
u(z) = \frac{|z|}{\pi L} \int_0^\infty \frac{q^{-1}(\xi)}{\sqrt{q^2(x) + z^2}} dx, \quad z \in \Gamma^+,
\]  

(4.27)

is given by the same expression as in the soliton gas case, see (4.16).

**Proof.** Using (A.62), we obtain

\[
R_0(z) \int_{|z|} \frac{d\zeta}{R_0(\zeta)(\zeta - z)} - \ln(\pm im) = -\ln \left( \frac{R_0(z)R_0(\mu) + z\mu + m^2}{\mu - z} \right),
\]  

(4.28)

where \( \pm \ln \mu > 0 \) respectively. Note that the term \(-\ln(\pm im) = \mp \frac{\pi}{2} - \ln m \) can be ignored when substituting the left-hand side in the integral in (4.25) because: (i) we need only the real part of this integral and so the \( \mp \frac{\pi}{2} \) term should be ignored; (ii) \( u = r \varphi \) is an odd function and so the integral of \(-\ln m u(\mu) \) is zero. We now replace the logarithmic term in (4.25) by the
remaining (first) term of the left-hand side of (4.28). Now, integrating by parts the obtained integral in (4.25), we get

\[-\text{Re} \left[ r R_0(z) \int_{\Gamma} \frac{S_1(\mu) d\mu}{R_0(\mu)(\mu - z)} \right] + 2\pi r S_2(z) = -i DR_0(z), \quad (4.29)\]

where \( D = \int_{\Gamma} \sqrt{q^2(x) - m^2} \, dx \). A few words to explain (4.29). First, the antiderivative of \( 2D \varphi \) is \(-iS_1\), see (4.2) and (4.9).

Second, the secular (not integral) terms that appear in integration by parts become zero. Indeed, it is obvious that \( S_1(\pm iM) = 0 \) as well as the integral in (4.28), evaluated at \( \mu = \pm im \), is zero. Finally, \(|d\mu| = -i d\mu\) explains the sign of the integral term in (4.29).

Note that the radical \( \sqrt{q^2(x) + \mu^2} \) in \( S_1 \) must be positive along the contour of integration, i.e. on the left (positive) shore of \( \Gamma \), which corresponds to the branch \( \sqrt{q^2(x) + \mu^2} \to -\mu \) as \( \mu \to \infty \).

Similarly to theorem 4.1, substituting (4.2) into (4.29) and changing the order of integration, we obtain

\[ I = -\int_{\Gamma} \frac{S_1(\mu) d\mu}{R_0(\mu)(\mu - z)} = 2 \int_{\Gamma} \int_{0}^{L} \frac{d\mu}{R_0(\mu)(\mu - z)} \left( e^{-i\mu} \sqrt{q^2(x) + \mu^2} \right) dx \]

\[ = 2 \int_{0}^{L} dx \left( \int_{0}^{\mu} + \int_{\mu}^{\infty} \right) \sqrt{\frac{\mu^2 + q^2(x)}{\mu^2 + m^2 - \mu - z}} d\mu \]

\[ = -2\pi i L + \frac{2\pi i}{R_0(z)} \int_{0}^{L} \sqrt{q^2(x) + \mu^2} \chi(z) dx, \quad (4.30)\]

where \( \chi \) is the characteristic function of the union of the segment \([i\mu, iM]\) with its complex conjugate. In (4.30) we use the standard \((\lim_{\mu \to \infty} \sqrt{q^2(x) + \mu^2} = \mu)\) branch of the radical and thus the sign changes after the second equality.

For \( z \in \mathbb{C}^+ \), the latter term in (4.30) becomes

\[-\frac{2\pi}{R_0(z)} \int_{0}^{L} \sqrt{|q^2(x) + \mu^2|} dx. \quad (4.31)\]

Substituting (4.30) and (4.31) into (4.29) complete the proof of the theorem for \( z \in \Gamma^+ \). The case of \( z \in \Gamma^- \) follows from symmetry considerations. \( \square \)

**Remark 4.4.** An analog of corollary 4.2 is valid for the periodic breather gas with expression for \( r_k \) in (4.23) being replaced by

\[ r_k = \frac{1}{\pi k L} \int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx. \quad (4.32)\]

Moreover, the analog of (4.24) for the breather gas is

\[ \nu_k(z) = \nu(z) + \frac{\pi(k - 1) L |z|}{2 \int_{0}^{L} \sqrt{q^2(x) - m^2} \, dx}. \quad (4.33)\]
4.3. Conserved densities for periodic gases

In this subsection, we compute the averaged densities \( I_m \) for periodic gases. We use results from subsections 4.1 and 4.2 and from section 3 to derive some formula for computing the averaged densities \( I_m \). Based on those formulae, we study the relation between \( g \)-function with the DOS \( u \) in the periodic gases situation.

**Theorem 4.5.** If \( \varphi(z) \) and \( \nu(z) \) are given by (4.9) and (4.12) where \( q(x) \) is monotonically decreasing on \([0, L]\) and \( q(L) = m > 0 \), then for any odd \( k \in \mathbb{N} \),

\[
I_k = \frac{(-1)^{\frac{k+1}{2}}kd_k}{L} \int_0^L q^{k+1}(x)dx, \tag{4.34}
\]

and \( I_k = 0 \) for any even \( k \in \mathbb{N} \), where \( d_k \) is defined in (1.7).

Moreover,

\[
2g_z(z) = \frac{1}{L} \int_0^L \left( \sqrt{z^2 + q^2(x)} \right) dx + \frac{1}{L} \int_0^L \sqrt{q^2(x) - m^2} dx, \quad z \in \overline{C}\setminus\Gamma. \tag{4.35}
\]

**Proof.** Using the result of theorem 4.3, formula (4.27) and equation (3.42), we have, for any odd \( k \in \mathbb{N} \),

\[
I_k = \frac{2ik}{\pi L} \int_m^M \int_0^{-1/(\overline{\zeta})} \frac{\zeta}{q^2(x) + \zeta^2} \left( \frac{1}{k} \zeta + d_k(imm)^{k+1} \zeta^{-1} + O(\zeta^{-3}) \right) dx \, d\zeta + kd_k(km)^{k+1}
\]

\[
= \frac{2ik}{\pi L} \int_0^L \int_m^{q(x)} \frac{\zeta}{q^2(x) + \zeta^2} \left( \frac{1}{k} \zeta + d_k(imm)^{k+1} \zeta^{-1} + O(\zeta^{-3}) \right) d\zeta \, dx + kd_k(km)^{k+1}
\]

\[
= \frac{2k}{\pi L} \int_0^L 2\pi i \cdot 4 \text{Res}_{\zeta=-\infty} \left( (\zeta^{k+1} + kd_k(km))((q^2(x) + \zeta^2)^{-1/2}) \right) dx + kd_k(km)^{k+1}
\]

\[
= \frac{i^{k+3}}{L} \int_0^L (-kd_kq^{k+1}(x) + kd_km^{k+1}) dx + i^{k+1}kd_km^{k+1}
\]

\[
= \frac{i^{k+1}kd_k}{L} \int_0^L q^{k+1}(x)dx. \tag{4.36}
\]

While when \( k \equiv 0 \), due to (3.42), \( I_k = 0 \).

Then, by definition, we have

\[
2g_{z^*} = \sum_{k=1}^{\infty} \frac{kd_k}{L} \int_0^L (i\zeta)^{k+1}(x)z^{-(k+1)}dx = 1 - \frac{z}{L} \int_0^L \frac{dx}{\sqrt{z^2 + q^2(x)}}, \quad z \in \overline{C}\setminus\Gamma, \tag{4.37}
\]

and the jump of \( 2g_{z^*} \) on \( \Gamma^+ \) is

\[
2g_{z^*-} - 2g_{z^*} = -\frac{2|z|}{L} \int_0^{-1/(\overline{\zeta})} \frac{dx}{\sqrt{z^2 + q^2(x)}} = -2i\pi u(z), \quad z \in \Gamma^+. \tag{4.38}
\]

Applying the anti-symmetric property of \( u \), the jump on \( \Gamma^- \) can be derived similarly.

Now, by integrating \( 2g_{z^*} \) and taking into account of the boundary behavior (3.68), we obtain
4.4. Examples of periodic soliton and breather gases

This subsection consider examples of periodic soliton and breather gases for some choices of \( q(x) \). We calculate the corresponding density of bands \( \varphi \), scaled bandwidth \( \nu \) and the relative scaled bandwidth \( \sigma_k \) of the \( k \)-dilution of the gas. As above, by \( k \)-dilution we mean the increase the period from \( 2L \) to \( 2kL \) with \( k \geq 1 \) while keeping \( q(x) = m \) for \( L \leq x \leq kL \). At the same time, the scaled bandwidth \( \nu(z) \) given by (4.12) will grow linearly in \( k \) for large \( k \). The limit \( k \to \infty \) corresponds to the semiclassical limit of the decaying potential, provided \( m = 0 \), otherwise, we get a potential with a constant background (leading to the breather gas). Therefore, equation (4.9) can be used to calculate the semiclassical spectral density for potentials with constant background, whereas (4.24) and (4.33) show that in the limit \( k \to \infty \) stationary semiclassical periodic gas is approaching super exponential (ideal gas) limit.

Let us calculate some examples.

**Example 1.** Consider the box potential with the width \( 2L \), height \( Q \) and period \( 2kL \), \( k \geq 1 \). Then \( M = Q \), \( m = 0 \) and \( \rho(\lambda) = L \). The latter formula can be justified by considering \( k \)-dilution of the gas. Thus \( S_1(\lambda) = 2L\sqrt{Q^2 - \lambda} \), \( S_1(m) = 2LQ \) and, by (4.9) and (4.12)

\[
\varphi(z) = \frac{|z|}{Q\sqrt{Q^2 + z^2}}, \quad \nu_k(z) = \frac{\pi(k - 1)|z|}{2Q}, \quad \sigma_k(z) = \frac{\pi(k - 1)\sqrt{Q^2 + z^2}}{2Q}.
\]

Note that \( k = 1 \) corresponds to the condensate \( q(x) = Q \) that according to theorem 4.1, has DOS \( u(z) = r \varphi(z) \equiv r \frac{|z|}{\pi\sqrt{Q^2 + z^2}} \), which is a well known DOS for the soliton condensate on \( \Gamma^+ = [0, iQ] \). In the case of \( k > 1 \), we have

\[
\sigma_k(z) = \frac{\pi(k - 1)\sqrt{Q^2 + z^2}}{2Q},
\]

i.e. this is exactly the same \( \sigma(z) \) that was obtained in [10] when \( r = \frac{Q}{\pi} \) is replaced by \( r_k = \frac{Q}{k\pi} < r \), and the \( k \)-diluted DOS \( u_k(z) = u(z)/k \).

Consider now situation when we fix some \( m \in (0, Q) \) and consider the \( k \)-dilution of the corresponding breather gas. Then
\[ \nu_k(z) = \frac{\pi(k-1)\sqrt{|z|^2 - m^2}}{2\sqrt{Q^2 - m^2}}, \quad \sigma_k(z) = \pi(k-1)\sqrt{\left(1 - \frac{m^2}{|z|^2}\right)(Q^2 + z^2)}. \] (4.42)

In this case, according to theorem 4.3,
\[ r_k = \frac{Q + (k-1)m}{k\pi}, \quad u_k(z) = u_k^k(\varphi(z) = \frac{1 + (k-1)m}{k\pi} \frac{|z|}{\sqrt{Q^2 + z^2}}. \] (4.43)

To compute the invariants in both cases within the sense of the thermodynamic limit, applying formula (4.34), we have
\[ I_n = \begin{cases} 0, & n \text{ even,} \\ \frac{1}{k}\sum_{n=0}^{k-1} d_\nu(n), & n \text{ odd.} \end{cases} \] (4.44)

**Example 2.** For the parabolic potential \( q(x) = \sqrt{1-x} \) we calculate \( p(\lambda) = 1 - \lambda \), so that
\[ S_1(\lambda) = \frac{4}{3}(1 - \lambda)^2, \quad \varphi(z) = 3|z|^2 \sqrt{1 + z^2}, \quad \nu_k(z) = \frac{\pi|z|}{4k}(2|z|^2 + 3(k - 1)). \] (4.45)

In this case we have
\[ \sigma_k(z) = \frac{\pi}{2} \cdot \frac{2|z|^2 + 3(k - 1)}{\sqrt{1 + z^2}}. \] (4.46)

In this example, the limiting averaged invariants are
\[ I_n = \begin{cases} 0, & n \text{ even,} \\ (-1)^{n+1} \frac{2n+1}{k(n+3)}, & n \text{ odd.} \end{cases} \] (4.47)

Note for \( k = 1 \), according to theorem 4.1, the DOS \( u = r\varphi = \frac{2}{3}|z|^2 \sqrt{1 + z^2} \).

**Remark 4.7.** From the formula (4.34), it is evident that the limiting averaged invariants \( I_n \) of the \( k \)-dilution of periodic soliton gases are simply \( I_n/k \), as illustrated by (4.44) with \( m = 0 \) and (4.47).

**Example 3.** For the semicircle potential \( q(x) = \sqrt{1-x^2} \) we calculate \( p(\lambda) = \sqrt{1-\lambda} \), so that
\[ S_1(\lambda) = 2 \int_0^{\sqrt{1-\lambda}} \sqrt{1-\lambda - x^2} \, dx \\
= \frac{1}{2} \int_0^{\sqrt{1-\lambda}} \sqrt{1-\lambda - x^2} \, dx = \frac{\pi(1-\lambda)}{2} = \frac{\pi(1 + z^2)}{2} \]

Then
\[ S_1(0) = \frac{\pi}{2}, \quad S_1(z) = \pi z \quad \text{and so} \quad \varphi(z) = 2|z|. \] (4.48)

Finally we obtain
\[ S_2(\lambda) = \int_{\sqrt{1-\lambda}}^1 \sqrt{x^2 - (1-\lambda)} \, dx = \frac{\sqrt{\lambda}}{2} + \frac{1-\lambda}{2} \ln \frac{\sqrt{1-\lambda}}{1 + \sqrt{\lambda}}. \] (4.49)
replacing $\lambda$ by $-z^2$, we have

$$
\nu(z) = \frac{1}{2} \left( |z| + (1 + z^2) \ln \frac{\sqrt{1 + z^2}}{1 + |z|} \right), \quad \sigma(z) = \frac{1}{4} \left( 1 + \frac{1 + z^2}{|z|} \ln \frac{\sqrt{1 + z^2}}{1 + |z|} \right).
$$

Moreover, according to theorem 4.1, we have

$$
r = 1/4, \quad u(z) = r \varphi = |z|/2.
$$

In this example, the limiting averaged invariants are

$$
I_n = \begin{cases} 
0, & n \text{ even}, \\
\frac{(-2)^{\frac{n+1}{2}}nN}{n+2}, & n \text{ odd}.
\end{cases}
$$

Data availability statement

No new data were created or analysed in this study.

Appendix A. Some error estimates

We start with constructing an approximation for the period matrix (the matrix of the system (2.24)) entries of $\mathcal{R}_N$. Let $\delta = \max_{|j| = 1} |\delta_j|$. We also use notation: $R_m(z) = \sqrt{(z - z_m)^2 - \delta^2_m}$.

Let us recall a few facts about the thermodynamic limit. First, we assume that for a large $N \in \mathbb{N}$ all the bands (except the stationary band in the breather gas case) shrink around their centers $z_j, j = \pm 1, \ldots, \pm N$ much faster than $N^{-1}$. In fact, we assume they are shrinking exponentially fast in $N$, even though some error estimates stay valid for algebraically fast shrinking.

Secondly, we assume that in the thermodynamic limit the shrinking bands are segments and all the bands are $O(N^{-1})$ spaced. That is, there exists a constant $\varphi_0 > 0$ such that for all $j, k$

$$
\min_{j \neq k} |z_j - z_k| \geq \frac{3}{N\varphi_0}.
$$

(A.1)

By the same argument we can also require that $\frac{1}{N\varphi_0}$ is the lower bound of the distances between any $z_l$ and the exceptional band $\gamma_0$.

The function $\rho_N(z)$, defined by

$$
R(z) = R_0(z) \prod_{|j|=1}^N (z - z_j)(1 + \rho_N(z)),
$$

is analytic in $\mathcal{C} \setminus \bigcup_{|j|=1}^N \gamma_j$ and $\rho_N(\infty) = 0$. The following lemma shows that in the thermodynamic limit $\rho_N(z)$ approaches zero uniformly away from $\Gamma$.

Lemma A.1. Under the thermodynamic limit assumptions for the breather gas, including (A.1), for any sufficiently large $N$ we have:

(a)
A.3

\[ |\rho_N(z) \leq 3\sqrt{2} e^{\phi_0^2 K^2 N^2} \ln N =: \rho_0(\delta, N) \] (A.3)

as long as \( z \) is away from the shrinking bands, namely,

\[ |z - z_j| > \sqrt{2}|\delta_j| \quad \text{for all } j = \pm 1, \ldots, \pm N; \] (A.4)

A.5 (b) If \( |z - z_j| < \sqrt{2}|\delta_j| \) for some \( j \), \( 1 \leq |j| \leq N \), then

\[ (1 + \rho_N(z))^{-1} = \frac{z - z_j}{R(z)} (1 + O(\rho_0)). \] (A.5)

Proof. Part (a). Since

\[ R(z) = R_0(z) \prod_{|j| = 1}^N (z - z_j) \prod_{|j| = 1}^N \left( 1 - \frac{\delta_j^2}{(z - z_j)^2} \right)^{\frac{1}{2}}, \] (A.6)

we need to estimate

\[ 1 + \rho_N(z) = \prod_{|j| = 1}^N \left( 1 - \frac{\delta_j^2}{(z - z_j)^2} \right)^{\frac{1}{2}} = e^{\frac{1}{2} \sum_{|j| = 1}^N \ln(1 - x_j)}, \] (A.7)

where \( x_j = \frac{\delta_j^2}{|z - z_j|^2} \). Using the obvious inequality \(|e^x - 1| \leq |x|e^{|x|}\), we have

\[ |\rho_N(z)| \leq \frac{1}{2} \sum_{|j| = 1}^N \ln(1 - x_j)|e^{\frac{1}{2} \sum_{|j| = 1}^N \ln(1 - x_j)}|. \] (A.8)

According to (A.4), all \( |x_j| \leq \frac{1}{2} \).

Consider now

\[ |\ln(1 - x)|^2 = \ln^2|1 - x| + \arg^2(1 - x) |\ln(1 - x)|^2 + \arcsin^2|x|. \] (A.9)

One can easily show that

\[ |\ln(1 - |x|)| \leq \frac{|x|}{1 - |x|}, \quad \text{arcsin } |x| \leq \frac{|x|}{\sqrt{1 - |x|^2}} \leq \frac{|x|}{1 - |x|}, \] (A.10)

so that

\[ |\ln(1 - x)| \leq \sqrt{2} \frac{|x|}{1 - |x|} \leq 2\sqrt{2}|x| \] (A.11)

provided \(|x| \leq \frac{1}{2} \).

Then

\[ \frac{1}{2} \sum_{|j| = 1}^N \ln(1 - x_j)| \leq \sqrt{2} \sum_{|j| = 1}^N |x_j| \leq \sqrt{2}\delta^2 \rho_N(z), \quad \text{where } \rho_N(z) = \sum_{|j| = 1}^N \frac{1}{|z - z_j|^2}. \] (A.12)

Thus, condition (A.4) implies

\[ |\rho_N(z)| \leq \sqrt{2}\delta^2 \rho_N(z)e^{\sqrt{2}\delta^2 \rho_N(z)}. \] (A.13)
If \( d > 0 \) is the distance between \( z \) and \( \Gamma \) then \( r_N(z) \leq \frac{2N}{\sqrt{d}} \), so \( |\rho_N(z)| \to 0 \) very fast as \( N \to \infty \). Consider now \( |z - z_j| = \sqrt{2} \delta \) and the worse case scenario where the \( 2N \) centers \( z_\delta \) pack the plane in hexagonal pattern (circles packing pattern) centered at \( z_j \). Then we can estimate

\[
r_N(z) \leq 6N^2 \varphi_0^3 \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) \leq 3N^2 \ln N \varphi_0^3,
\]

(A.14)

where \( n \) is the smallest integer satisfying \( n \geq \frac{2N}{\sqrt{d}} \). For a large \( N \), (A.13) and (A.14) imply \( \rho_N \leq 1 \) and so we obtain (A.3).

To prove part (b) we notice that

\[
1 + \rho_N(z) = \frac{R_f(z)}{z - z_j} \prod_{|k|=1,k \neq j} \left( 1 - \frac{\delta^2_j}{(z - z_k)^2}\right)^{\frac{1}{2}},
\]

(A.15)

so that

\[
(1 + \rho_N)^{-1} = \frac{z - z_j}{R_f(z)}(1 + \tilde{\rho}_N(z))^{-1},
\]

(A.16)

where \( 1 + \tilde{\rho}_N \) denotes the product in (A.15). Now part b) follows from the fact that part a) is applicable to \( \tilde{\rho}_N \).

Remark A.2. In the case of soliton gas we introduce \( \tilde{\delta} = \max_{|j|=0} \delta_j \). Lemma A.1 is also valid in this case if we replace \( \delta \) by \( \tilde{\delta} \) and \( R_0(z) \) by \( z - z_0 \).

Remark A.3. We want to state separately a useful estimate based on (A.8) and (A.11):

\[
\left| \prod_{j=1}^N (1 - x_j) \right| \leq \sum_{j=1}^N \ln(1 - x_j) |e^{2N} \sum_{j=1}^N \ln(1 - x_j)| \leq 2N \sum_{j=1}^N |x_j| e^{2N} \sum_{j=1}^N |x_j| \leq 2N \sum_{j=1}^N |x_j| e^{2N} \sum_{j=1}^N |x_j|
\]

(A.17)

provided \( |x_j| < \frac{1}{2} \) for all \( j \).

A.1. Approximation of the normalized holomorphic differentials \( w_j \)

In the limiting case when all the bands \( \gamma_j \) except, possibly, \( \gamma_0 \), shrink to points \( z_j \), \( j = \pm 1, \ldots, \pm N \), that is, \( \delta = 0 \), it is easy to check that

\[
w_j(z) = w_j(z, 0) = -\frac{R_0(z)dz}{2\pi i R_0(z)(z - z_j)} = -\frac{R_0(z)\prod_{k \neq j}(z - z_k)dz}{2\pi i R(z)}
\]

(A.18)

Let us fix some \( j = \pm 1, \ldots, \pm N \). In general, we have

\[
w_j(z, \delta) = \frac{\mu_j(\delta)\prod_{k \neq j}(z - \mu_k(\delta))dz}{R(z)}
\]

(A.19)

where \( \mu_j = x_{j,1} \), see (2.22). As it follows from (A.18), \( \mu_k(0) = z_k \) for all \( k \neq j \) and \( \mu_j(0) = -\frac{R_0(z)}{2\pi i R(z)} \). Also, it is a well known fact (see, for example [15]) that for any nondegenerate genus \( 2N \) hyperelliptic Riemann surface \( \Sigma \) there exists a unique collection of the normalized holomorphic differentials \( w_j \), \( j = 0, \pm 1, \ldots, \pm N \). Here we assume that \( \delta_j^2 = \phi_j(\delta^2) \), where all \( \phi_j \in C^2[0, \delta] \) and their norms are uniformly bounded with respect to \( N \). We also assume \( \phi_0(0) = 0 \).
In the following lemma we estimate the deformation of $\mu_k(\delta)$ for sufficiently small $\delta$. Here we assume that the set of all $z_j$ is bounded and (A.1) holds.

We use notations $\vec{\mu}(\delta)$, $\vec{\eta}$ for $2N$ dimensional vectors $\vec{\mu}(\delta) = (\mu_{-N}(\delta), \ldots, \mu_N(\delta))'$ and $\vec{\eta} = (z_{-N}, \ldots, z_N)'$.

**Lemma A.4.** Let us fix an arbitrary $j = \pm 1, \ldots, \pm N$ and consider all $\mu_k(\delta)$ defined by (A.19). Then for all $j = \pm 1, \ldots, \pm N$, all $k = \pm 1, \ldots, \pm N$ and all $\delta$ satisfying

$$\delta = o(N^{-6}),$$

(A.20)
in the thermodynamic limit we have

$$|\mu_k(\delta) - z_k| \leq CN^4 \delta^2, \quad k \neq j \quad \text{and} \quad |\mu_j(\delta) + \frac{R_0(z_j)}{2\pi i}| \leq CN^4 \delta^2,$$

(A.21)

where the constant $C > 0$ does not depend on $N, j, k, \delta$.

**Proof.** WLOG, we can assume that all contours $\hat{\gamma}_k$ satisfy the condition (A.4) from lemma A.1 and, thus, the estimate (A.3) from this lemma is valid on $\hat{\gamma}_k$.

Fix some $j = \pm 1, \ldots, \pm N$. By definition,

$$F_k(z; \vec{\mu}, \delta) := \oint_{\hat{\gamma}_k} w_j(z, \delta) = \delta_{k,j},$$

(A.22)

where $\delta_{k,j}$ denotes the Kronecker symbol. Then

$$\frac{d\vec{\mu}}{d\delta} = -\left( \frac{\partial \vec{F}_j}{\partial \vec{\mu}} \right)^{-1} \cdot \frac{\partial \vec{F}_j}{\partial \delta^2}$$

(A.23)

where $\vec{F}_j$ is the $j$th column of the matrix $F_{kj}$. By implicit function theorem, $\vec{\mu}(\delta)$ is uniquely defined and differentiable in some neighborhood of $\vec{\mu}(0)$, provided that $\frac{\partial \vec{F}}{\partial \vec{\mu}}$ is invertible at $\delta = 0$. We start with calculating the latter matrix and its inverse.

Indeed,

$$\left. \frac{\partial F_{kj}}{\partial \mu_m} \right|_{\delta=0} = \frac{R_0(z_j)\delta_{km}}{R_0(z_m)(z_j - z_m)} = \frac{\delta_{kj}}{z_j - z_m} \quad \text{for} \quad m \neq j$$

(A.24)

when $m \neq j$ and

$$\left. \frac{\partial F_{kj}}{\partial \mu_j} \right|_{\delta=0} = \oint_{\hat{\gamma}_k} \frac{\partial w_j(z, \delta)}{\partial \mu_j} \bigg|_{\delta=0} = -\oint_{\hat{\gamma}_k} \frac{dz}{R_0(z)(z - z_j)} = \frac{2\pi i \delta_{kj}}{R_0(z_j)}$$

(A.25)

So, the matrix $\left. \frac{\partial F}{\partial \vec{\mu}} \right|_{\delta=0}$ is the sum of the main diagonal

$$\text{diag} \left. \frac{\partial \vec{F}}{\partial \vec{\mu}} \right|_{\delta=0} = \text{diag} \left( \frac{R_0(z_j)}{R_0(z_m)(z_j - z_m)}, \ldots, -\frac{2\pi i}{R_0(z_j)}, \ldots, \frac{R_0(z_j)}{R_0(z_N)(z_j - z_N)} \right)$$

(A.26)

and the $j$th column $\left( \frac{1}{z_j - z_{-N}}, \ldots, \frac{1}{z_j - z_N} \right)'$, where the $j$th entry should be taken zero. Thus, $\left. \frac{\partial F}{\partial \vec{\mu}} \right|_{\delta=0}$ is an invertible matrix, so that the implicit function theorem is applicable to (A.22) for any fixed $j$. 

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The inverse \( \left( \frac{\partial F_j}{\partial \mu} \right)^{-1} \bigg|_{\delta=0} \) has the same structure as \( \frac{\partial^2 F_j}{\partial \mu^2} \bigg|_{\delta=0} \) with

\[
\operatorname{diag} \left( \frac{\partial F_j}{\partial \mu} \right)^{-1} \bigg|_{\delta=0} = \operatorname{diag} \left( \frac{R_0(z,-N)(z_j - z_{-N})}{R_0(z_j)} \cdot \frac{R_0(z)}{R_0(z_j)} \cdot -2\pi i \cdot \frac{R_0(z_N)(z_j - z_N)}{R_0(z_j)} \right),
\] (A.27)

and the \( j \)th column \( (R_0(z,-N), \ldots, R_0(z_N))' \), where the \( j \)th entry should be taken zero. Note that

\[
\left( \frac{\partial F_j}{\partial \mu} \right)^{-1} \bigg|_{\delta=0} = O(N)
\] (A.28)

as \( N \to \infty \) uniformly in all the entries.

Our goal is estimate the growth of \( \phi(\delta) = \tilde{\mu}(\delta) - \tilde{\mu}(0) \) in a neighborhood \( W_\mu(N) \) of \( \tilde{\mu}(0) \). We define \( W_\mu(N) \) as a centered at \( \tilde{\mu}(0) \) ‘scaled cube’, of size \( O(N^{-1}) \) in the direction of each component. We also introduce the neighborhood \( W(N) = W_\mu(N) \times W_\mu(N) \), where \( W_\mu(N) = [0, o(N^{-1})] \). We now estimate the factors in the right-hand side of (A.23) for \( \tilde{\mu} \in W_\mu(N) \).

First assume that \( m \neq j \). Then we have

\[
\frac{\partial F_{kj}}{\partial \mu_m} = \frac{\partial w_j(z, \delta)}{\partial \mu_m} = -\mu_j(\delta) \int_{\nu_k} \frac{dz}{R(z)} \left[ \int_{\nu_k} \frac{dz}{R_0(z)R_j(z)R_m(z)} + \int_{\nu_k} \frac{dz}{R_0(z)R_j(z)R_m(z)} \right].
\] (A.29)

Direct calculation shows that

\[
-\mu_j(\delta) \int_{\nu_k} \frac{dz}{R_0(z)R_j(z)R_m(z)} = -\mu_j(\delta) \frac{\partial F_{kj}}{\partial \mu_m} \bigg|_{\delta=0} - \frac{\partial \mu_j(\delta)}{\partial \mu_m} \int_{\nu_k} \frac{dz}{R_0(z)R_j(z)R_m(z)}
\]

\[
= -\mu_j(0) \int_{\nu_k} \frac{dz}{R_0(z)(z - z_j)(z - z_m)} \left[ 1 - \frac{\delta_j}{(z - z_j)^2} \right]^{1/2} \left[ 1 - \frac{\delta_m}{(z - z_m)^2} \right]^{1/2} \left[ 1 - \frac{\delta_j}{(z - z_j)^2} \right]^{1/2} \left[ 1 - \frac{\delta_m}{(z - z_m)^2} \right]^{1/2}.
\] (A.30)

We roughly estimate the second term in the right-hand side of (A.30) as \( 2\pi \varepsilon(\delta) N^2 \varphi_0^3 \), where

\[
\varepsilon(\delta) := \max_k \varepsilon_k(\delta) = \max_k \{ |\mu_k(\delta) - \mu_k(0)| \}
\] (A.31)

for all \( j, k \) and all \( \xi \in [0, \delta] \). We also estimate the last term in (A.30) as \( |R_0(z_j)| \varphi_0^3 N^3 \delta^2 \). Then

\[
-\mu_j(\delta) \int_{\nu_k} \frac{dz}{R_0(z)R_j(z)R_m(z)} = \left( \frac{R_0(z_j)}{R_0(z_m)(z_j - z_m)} \right) \frac{\delta_{kj}}{z_j - z_m} \leq \varphi_0^3 N^2 (2\pi \varepsilon(\delta) + \varphi_0 N |R_0(z_j)| \delta^2).\]

(A.32)

Using lemma A.1 and remark A.3, the product in the last term from (A.29) can be estimated as
\[
\left| \prod_{j \neq m,j} \frac{z - \mu_j(\delta)}{R_j(z)} - 1 \right| \leq \left| \prod_{j \neq m,j} \frac{z - \mu_j(\delta)}{R_j(z)} (1 + \tilde{\rho}_N)^{-1} - 1 \right| \\
\leq \left| \prod_{j \neq m,j} \left( 1 - \frac{\varepsilon_j(\delta)}{(z - z_j)} \right) (1 + \tilde{\rho}_N)^{-1} - 1 \right| \\
\leq 2 e^{2 \varepsilon(\delta)N^2} \left[ \rho_0 + \sqrt{2\varepsilon(\delta)\varphi_0 N^2} \right] \quad (A.33)
\]

provided \( \varepsilon(\delta)N^2 \to 0 \) as \( N \to \infty \). Here \( \tilde{\rho} \) is the same as in \( (A.2) \) except the factors \( R_m \) and \( R_j \) were removed from \( R \). Of course, \( \tilde{\rho}_N \) also satisfies the estimate \( (A.3) \).

Thus, the last term from \( (A.29) \) can be estimated by

\[
2\varphi_0^2N^2(R_0(z_j)[\rho_0 + 2\varepsilon(\delta)\varphi_0 N^2]). \quad (A.34)
\]

So, using \( (A.3) \),

\[
\frac{\partial F_{kj}}{\partial \mu_m} - \frac{\partial F_{kj}}{\partial \mu_j} \bigg|_{\delta=0} \leq 4\varphi_0^2N^2[(\pi + 2\varphi_0 N^2)\varepsilon(\delta) + 3\sqrt{2\varphi_0^2 N^2} \ln N][R_0(z_j)|\delta^2|] \quad (A.35)
\]

for sufficiently large \( N \) provided \( \varepsilon(\delta)N^2 \rightarrow 0 \) as \( N \rightarrow \infty \). Of course, the latter condition holds when \( \tilde{\rho} \in W_\delta(N) \).

Direct calculations show that in the case \( m = j \) we have

\[
\frac{\partial F_{kj}}{\partial \mu_j} = \frac{1}{R_j(z)} \sum_{m \neq j} R^2(z) \varphi'_m(\delta^2) = \frac{1}{2} \sum_m \varphi'_m(\delta^2) R_m(z) \quad (A.36)
\]

Applying to \( (A.36) \) similar estimates as to \( (A.35) \), we obtain that the estimate \( (A.35) \) also covers the case \( m = j \), which makes it uniform in indices \( k, j, m \).

Now we calculate

\[
\frac{\partial}{\partial \delta^2} \frac{1}{R(z)} = \frac{1}{2R(z)} \sum_{m=-N}^{N} R^2(z) \varphi'_m(\delta^2) = \frac{1}{2} \sum_m \varphi'_m(\delta^2) R_m(z) \quad (A.37)
\]

We remind that by assumption \( \varphi'_0(0) = 0 \). Then

\[
\frac{\partial F_{kj}}{\partial \delta^2} = \frac{1}{\theta} \frac{\partial F_{kj}}{\partial \delta^2} = \frac{1}{\theta} \sum_m \varphi'_m(\delta^2) R_m(z) \quad (A.38)
\]

We also have

\[
\frac{\partial F_{kj}}{\partial \delta^2} \bigg|_{\delta=0} = 2R_0(z_j) \varphi'_0(0) \left[ \frac{1}{z_k - z_j} + \frac{z_k}{R_0(z_k)} \right] \quad (A.39)
\]

in the case \( k \neq j \) and
One more term in (A.41) to estimate is
\[
\left[ \frac{\mu_j(0)}{2} \sum_m \oint_{\gamma_k} \frac{\Pi_{\bar{m}, j} \left( 1 - \frac{z(\delta)}{z_m} \right)}{R_m^2(z) R_0(z)} \phi'_m(\delta) d\gamma \right] \leq 4 |R_0(z_j)| \varphi_0 N \phi' \varphi_0 N^4. \tag{A.45}
\]
where we used the same considerations as in estimate (A.33). Finally, the last term in (A.41) to estimate is
\[
\left[ \frac{\mu_j(0)}{2} \sum_m \oint_{\gamma_k} \left[ \Pi_{\bar{m}, j} \frac{1 - \frac{z(\delta)}{z_m}}{R_m^2(z) R_0(z)} \phi'_m(\delta) \right] \right] \phi'_m(0) - \frac{R_0(z_j) \phi'_m(0)}{4 \left( \frac{1}{R_0(z)} \right)} \right| \mid_{z=z_j}^n . \tag{A.40}
\]
in the case \( k = j \). Similarly to (A.29), we have
\[
\left| \frac{\mu_j(0)}{2} \sum_m \oint_{\gamma_k} \frac{\Pi_{\bar{m}, j} \left( 1 - \frac{z(\delta)}{z_m} \right) \tilde{\rho}_N \phi'_m(\delta) d\gamma}{R_m^2(z) R_0(z) R_f(z)} \right| \leq 2 |R_0(z_j)| \phi'_0 \varphi_0 N^4. \tag{A.42}
\]
where
\[
\phi' = \max_m \sup \left| \phi'_m(\delta) \right|. \tag{A.43}
\]
The next term in (A.41) to estimate is
\[
\left[ \sum_{m} \oint_{\gamma_k} \frac{\Pi_{\bar{m}, j} \left( 1 - \frac{z(\delta)}{z_m} \right) \phi'_m(\delta) d\gamma}{R_m^2(z) R_0(z) R_f(z)} \right] \leq 2 \pi \varepsilon(\delta) \phi' \varphi_0^3 N^4. \tag{A.44}
\]
One more term in (A.41) to estimate is
\[
\left[ \frac{\mu_j(0)}{2} \sum_m \oint_{\gamma_k} \left[ \Pi_{\bar{m}, j} \left( 1 - \frac{z(\delta)}{z_m} \right) \phi'_m(\delta) d\gamma \right] \phi'_m(0) \right] - \frac{R_0(z_j)}{4 \left( \frac{1}{R_0(z)} \right)} \right| \mid_{z=z_j}^n . \tag{A.46}
\]
Since the sum of the left hand sides of (A.42)–(A.46) gives the absolute value of (A.41), we obtain
Approximation of the coefficients of the linear system (2.23) is, in fact, an approximation of the period matrix of the hyperelliptic Riemann surface $\mathcal{R}$.

According to (A.28), $A^{-1} = O(N^3)$ uniformly in $k$, $j$. That will also be true for $\frac{\partial F_j}{\partial \delta^2}$ provided the error term (A.47) is of the same or a smaller order. (Here and henceforth all the estimates are entry wise.) But the condition $(\vec{\mu}, \vec{\delta}) \in W(N)$ implies the required estimate. Then the error term (A.47) is of the order $O(N^{-3})$ uniformly in $k$, $j$.

Let $\frac{\partial F_j}{\partial \delta^2} = A_0 + \Delta A$, where $A_0 = \frac{\partial F_j}{\partial \delta^2} |_{\delta = 0}$. Then we can rewrite (A.23) as

$$\frac{\partial \vec{\mu}}{\partial \delta^2} = -(I + A_0^{-1} \Delta A)^{-1} A_0^{-1} \frac{\partial F_j}{\partial \delta^2}.$$  \hfill (A.48)

According to (A.28), $A_0^{-1} = O(N)$. It also follows then from (A.20) and (A.27) and $\vec{\mu} \in W_\mu(N)$ that $A_0^{-1} \frac{\partial F_j}{\partial \delta^2} = O(N^3)$ and $\Delta A = O(N^{-3})$ uniformly in $k$, $j$. Thus, $A_0^{-1} \Delta A = O(N^{-2})$ uniformly in $k$, $j$. Now, by Gershgorin circle theorem, see, for example [18], $(I + A_0^{-1} \Delta A)^{-1} = O(1)$. So, condition $\vec{\mu} \in W_\mu(N)$ implies that

$$\frac{\partial \vec{\mu}}{\partial \delta^2} = O(N^4)$$ \hfill (A.49)

uniformly in $k$, $j$ when $(\vec{\mu}, \vec{\delta}) \in W(N)$.

According to the mean value theorem,

$$\mu_k(\delta) - \mu_k(0) = \frac{\partial \mu_k}{\partial \delta^2}(\xi_k)\delta^2$$ \hfill (A.50)

for all $k$, where $\xi_k \in (0, \delta)$. Let us start deforming $\delta$ from $\delta = 0$ as long as $(\vec{\mu}(\delta), \vec{\delta}) \in W(N)$. Then, according to (A.50),

$$|\mu_k(\delta) - \mu_k(0)| = o(N^{-k}),$$ \hfill (A.51)

that is, $\delta \in W_\delta(N)$ guarantees that $\vec{\mu} \in W_\mu(N)$. Thus, (A.21) follow from (A.51) and we proved the lemma. \hfill $\Box$

**A.2. Approximation of periods of $\mathcal{R}$**

Approximation of the coefficients of the linear system (2.23) is, in fact, an approximation of the period matrix of the hyperelliptic Riemann surface $\mathcal{R}$. Since $\mathbf{B}$-cycles are crossing small shrinking bands $\gamma_k$, the following formula

$$\int_{\mathcal{U}} \frac{\phi(\zeta) d\zeta}{\sqrt{\zeta^2 + \delta^2}} = -2 \ln |\phi(0)| + O(1)$$ \hfill (A.52)

where $\delta \to 0$ and for $\phi(\zeta)$ is continuous and Lipschitz at $\zeta = 0$, will be used in the calculations below. Here $\delta \in \mathbb{C}$ and $\mathcal{U}$ denotes a fixed segment in a neighborhood of $\zeta = 0$ passing through the origin and intersecting the segment $[-i\delta, i\delta]$ transversely from left to right. Note that in (A.52) we assume that $\sqrt{\zeta^2 + \delta^2}$ is positive on $\mathbb{R}$, that is, the branch cut of $\sqrt{\zeta^2 + \delta^2}$ goes from $i\delta$ to $-i\delta$ through infinity.
Let us denote the cycle $\tilde{B}_k = B_k \cup B_{-k}$, $k = 1, \ldots, N$. To simplify the exposition of the following lemma A.5, we assume that $\Gamma^+$ is a 1D compact (contour).

**Lemma A.5.** Under the thermodynamic limit assumptions for all $k, j = 1, \ldots, N$ we have

$$
\oint_{\tilde{B}_k} \frac{P(\zeta) d\zeta}{R(\zeta)} = \frac{1}{-i\pi} \left[ \ln \frac{R_0(z_j)R_0(z_k) + z_jz_k - \delta_0}{R_0(z_j)R_0(z_k) + z_jz_k - \delta_0} - \ln \frac{z_k - z_j}{z_k - z_j} \right] + h(k - j) + O\left(N^2\delta^3\right)
$$

when $k \neq j$ and

$$
\oint_{\tilde{B}_j} \frac{P(\zeta) d\zeta}{R(\zeta)} = -\frac{2}{i\pi} \ln |\delta_j| + O(1)
$$

in the leading order as $N \to \infty$ provided that $\delta = o(N^{-6})$. Here $h$ denotes the Heaviside function $h(\xi) := \frac{1}{2} (1 + \text{sign } \xi)$. Equation (A.53) also stay true when $\delta_0 \to 0$ provided $\delta \ll |\delta_0|$.\[\tag{A.53}\]

**Proof.** Consider first the case $j \neq k$. Deforming the contours $B_k \cup B_{-k}$ and using the fact that the values of the integral over each sheet of $\mathcal{H}_N$ are equal, we obtain

$$
\oint_{\tilde{B}_k} w_j = \oint_{\tilde{B}_k} \frac{P(\zeta) d\zeta}{R(\zeta)} = 2 \oint_{\gamma_k} \frac{P(\zeta) d\zeta}{R(\zeta)} + h(k - j) = 2\mu j \oint_{\gamma_k} \frac{d\zeta}{R_0(\zeta)R_j(\zeta)} \prod_{m \neq j} \frac{z - \mu_m}{R_m(\zeta)} + h(k - j), \quad (A.54)
$$

where the contour connecting $\tilde{z}_k$ and $z_k$ in the latter integral is bent, if necessary, to be at least $\frac{1}{N^{\gamma_0}}$ away from any band $\gamma_j$ with $j \neq k$, see (A.1). We can also assume that the lengths of these contours for all $N$ are uniformly bounded. The requirement $\delta = o(N^{-6})$ is needed to use lemma A.4. Now, it follows from (A.53), (A.3) and (A.21) that

$$
\left| \prod_{m \neq j} \frac{z - \mu_m}{R_m(\zeta)} - 1 \right| \leq C_1 N^6 \delta^2, \quad N \to \infty, \quad (A.55)
$$

for some $C_1 > 0$ uniformly in $k$, $j$ as long as $z$ is $\frac{1}{N^{\gamma_0}}$ away from and band $\gamma_j$. The latter requirement is violated for $\frac{z - \mu_j}{R_j(\zeta)}$ near $z = z_j$ and for $\frac{z - \mu_{-j}}{R_{-j}(\zeta)}$ near $z = z_{-j}$. Therefore, we split the latter integral in (A.54) into three parts: small $\varepsilon > 0$ neighborhoods of each of the endpoints $z_k, z_{-k}$ and the rest of the contour. The value of $\varepsilon$ should satisfy $\varepsilon = o(N^{-1})$ and $\delta = o(\varepsilon)$, but its exact order will be determined below. Using the decomposition

$$
\frac{z - \mu_{\pm k}}{R_{\pm k}(\zeta)} = \frac{z - \tilde{z}_{\pm k}}{R_{\pm k}(\zeta)} + \frac{\tilde{z}_{\pm k} - \mu_{\pm k}}{R_{\pm k}(\zeta)}, \quad (A.56)
$$

Lemma A.4, (A.52) and (A.55) and the rough estimate

$$
\left| \frac{1}{R_0(\zeta)R_j(\zeta)} \right| \leq \varphi_0 N^2 \quad (A.57)
$$

on the contour of integration, we estimate the integrals over the neighborhoods of $z_k, z_{-k}$ as...
that correspond to the first and second terms of (A.56) respectively. This estimate is uniform in \( \varepsilon, k, j \). In the first term of (A.58) we used the fact that both \( \frac{-z_k}{R_0(z)} \), \( \frac{z_k-\bar{z}_j}{R_0(z)} \) are bounded near the points \( \bar{z}_k, z_k \).

According to (A.55) and (A.57), the integral over the remaining part of the contour can be represented as

\[
2\mu_j(\delta) \int \frac{dz}{R_0(z)R_j(z)} + O(N^8\delta^2).
\]

Because of

\[
\frac{1}{R_j(z)} = \frac{1}{z-z_j} \left[ 1 + \mathcal{O}\left(\frac{\delta^2}{\varepsilon^2}\right) \right] \tag{A.60}
\]

and (A.21), we can rewrite (A.59) as

\[
-\frac{2R_0(z_j)}{2\pi i} \int \left[ 1 + \mathcal{O}\left(\frac{\delta}{\varepsilon}\right) \right] \frac{dz}{R_0(z)(z-z_j)} + O(N^8\delta^2) \tag{A.61}
\]

In view of the anti-derivative

\[
\int \frac{d\zeta}{R_0(\zeta)(\zeta-\eta)} = -\frac{1}{R_0(\eta)} \ln \frac{R_0(\zeta)R_0(\zeta) + \zeta - \eta^2}{\eta - \zeta} \tag{A.62}
\]

we obtain the leading order term of the first equation (A.53) if we substitute the limiting values \( z_k, \bar{z}_k \) in the anti-derivative (A.62) in (A.61). Since these limits of integration are distance \( O(\varepsilon) \) away from the endpoints of the integral in (A.59) we have introduced an error of \( O(N^2\varepsilon) \), see estimate (A.57). The error coming from the \( \mathcal{O}\left(\frac{\delta^2}{\varepsilon^2}\right) \) term of (A.61) can be estimated as \( O\left(\frac{N^2\varepsilon^2}{\varepsilon^2}\right) \).

Now, to find the best value of \( \varepsilon \) in the partition \( [\bar{z}_k, z_k] \), we equate the errors \( O(N^2\varepsilon) \) and \( O(N^2\delta^2) \) from (A.61). That yields \( \varepsilon = \delta^2 \). Thus, the error in (A.53) is the maximum of \( O(N^2\delta^2), O(N^8\delta^2) \). In view of (A.20), the first term is larger. Thus, we have completed the proving of the first equation (A.53).

Consider now

\[
\oint_{\partial_0} \psi_j = 2\mu_j \int_{z_j}^{z_j} \frac{dz}{R_0(z)R_j(z)} \prod_{m \neq j} \frac{z-\mu_m}{R_m(z)} = -\frac{2R_0(z_j)}{2\pi i} \int_{z_j}^{z_j} \frac{dz}{R_0(z)R_j(z)}(1 + O(N^8\delta^2)) \tag{A.63}
\]

where we have used lemma A.4 and the estimate from (A.55). Now the second equation (A.53) follows from (A.52) taken with the opposite sign. This choice of the sign comes from the fact that the branch of \( R_j(z) \) in (A.63) corresponds to \( -\sqrt{\zeta^2 + \delta^2} \) in (A.52). \( \square \)

**Remark A.6.** Higher accuracy in the second equation of (A.53) can be achieved if we consider higher order terms in the small \( \delta_j \) expansion of the elliptic integral \( \int_{\zeta_j}^{\zeta_j} \frac{d\zeta}{R_0(\zeta)R_j(\zeta)} \) from (A.63).
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