SPHERICALLY SYMMETRIC EINSTEIN-SCALAR-FIELD EQUATIONS FOR WAVE-LIKE DECAYING NULL INFINITY

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Abstract. We show that the spherically symmetric Einstein-scalar-field equations for wave-like decaying initial data at null infinity have unique local solutions and unique global solutions for small initial data. We also generalize Christodoulou’s global generalized solutions to the wave-like decaying initial data. We emphasize that this decaying condition is sharp.

1. Introduction

1.1. Spherically symmetric Einstein-scalar-field equations. Let $g$ be a 4-dimensional Lorentzian metric, $\phi$ be a real function. The Einstein-scalar-field equations are the following systems

\[
\begin{aligned}
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= 8\pi T_{\mu\nu} \\
T_{\mu\nu} &= \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial_\alpha \phi
\end{aligned}
\]

(1.1)

The spherically symmetric solutions of (1.1) were studied extensively by Christodoulou [2][3][4][5][6][7] using slight different notation of the following spherically symmetric Bondi-Sachs metrics [1][12]

\[
g = -e^{2\beta(u,r)} \frac{V(u,r)}{r} du^2 - 2e^{2\beta(u,r)} dudr + r^2 (d\theta^2 + \sin^2 \theta d\psi^2),
\]

(1.2)

where $u$ is referred as the retarded time,

$V(u, r) > 0,$

and $\beta$, $V$ satisfy the following boundary conditions

$\beta(u, \infty) = 0, \quad \lim_{r \to \infty} \frac{V(u, r)}{r} = 1.$

(1.3)

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Using the expressions of the Ricci curvature (c.f. Appendix), we obtain the Einstein-scalar-field equations

\[\begin{align*}
\frac{-V \partial \beta}{r \partial u} + \frac{1}{2r} \frac{\partial V}{\partial u} &= 8\pi \left[ \left( \frac{\partial \phi}{\partial u} \right)^2 + \frac{V}{r} \frac{\partial \phi}{\partial r} \right], \\
-2 \frac{\partial^2 \beta}{\partial u \partial r} + \frac{V \partial^2 \beta}{r \partial r^2} + \frac{1}{r} \frac{\partial \beta}{\partial r} \frac{\partial V}{\partial r} + \frac{V \partial \beta}{r^2} &= 8\pi \frac{\partial \phi}{\partial u} \frac{\partial \phi}{\partial r}, \\
\frac{\partial \beta}{\partial r} &= \frac{2\pi}{r} \left( \frac{\partial \phi}{\partial r} \right)^2, \\
\frac{\partial V}{\partial r} &= e^{2\beta}. \tag{1.4}
\end{align*}\]

The function \(\phi\) satisfies the wave equation by the twice contracted Bianchi identity

\[\begin{align*}
-2 \left( \frac{\partial^2 \phi}{\partial u \partial r} + \frac{1}{r} \frac{\partial \phi}{\partial u} \right) + V \left( \frac{\partial^2 \phi}{r \partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \frac{\partial V}{\partial r} \right) &= 0. \tag{1.5}
\end{align*}\]

Given function \(f(u, r)\), denote \(\bar{f}(u, r)\) its average

\[\bar{f}(u, r) = \frac{1}{r} \int_0^r f(u, r') dr'.\]

Denote \(D\) the derivative along the incoming light rays parameterized by \(u\)

\[D = \frac{\partial}{\partial u} - \frac{V}{2r} \frac{\partial}{\partial r}.\]

In [2], Christodoulou introduced

\[h = \frac{\partial (r \phi)}{\partial r}\]

and showed that the spherically symmetric Einstein-scalar-field equations (1.4), (1.5) under boundary conditions (1.3) are equivalent to the system [2]

\[\begin{align*}
\beta &= -2\pi \int_r^\infty (h - \bar{h})^2 \frac{dr'}{r'}, \\
g &= e^{2\beta}, \\
\bar{g} &= \frac{1}{r} \int_0^r g dr' = \frac{V}{r}, \\
Dh &= \frac{1}{2r} (g - \bar{g}) (h - \bar{h}). \tag{1.6}
\end{align*}\]

Let \(r = \chi(u)\) be the integral curves of \(D\). They are the incoming light rays called characteristics and satisfy the ordinary differential equation [2]

\[\frac{dr}{du} = \frac{d\chi(u)}{du} = -\frac{1}{2} \bar{g}(u, r). \tag{1.7}\]

1.2. The Bondi mass and the Bondi-Christodoulou mass. Denote

\[m_B(u) = \frac{r}{2} \left( 1 - \frac{V}{r} \right) = \frac{r}{2} (1 - \bar{g}). \tag{1.8}\]

Following from Bondi [1], the Bondi mass and the final Bondi mass are defined as

\[M_B(u) = \lim_{r \to \infty} m_B(u), \quad M_{B1} = \lim_{u \to \infty} M_B(u).\]
In [2], Christodoulou defined a mass function
\[ m(u, r) = \frac{r}{2} \left( 1 - \frac{g}{\bar{g}} \right) \] (1.9)
and proved that it satisfies
\[
\begin{cases}
  \frac{\partial m}{\partial r} = 2\pi \frac{\bar{g}}{g} (h - \bar{h})^2 \geq 0, \\
  Dm(u, r) = -\pi \frac{g}{\bar{g}} \left( \int_{0}^{r} \bar{g}(h - \bar{h})^2 \frac{dr'}{r'} \right)^2.
\end{cases}
\] (1.10)
He also defined
\[ M(u) = \lim_{r \to \infty} m(u, r), \quad M_1 = \lim_{u \to \infty} M(u). \]
In general, \( M(u) \) may not be equal to \( M_B(u) \). So it is reasonable to call \( M(u) \) the Bondi-Christodoulou mass and \( M_1 \) the final Bondi-Christodoulou mass.

If the metric \( g \) is regular in \([0, \infty)\) at each \( u \), then (1.6), (1.10) give
\[ g \leq 1, \quad m(u, r) \geq m(u, 0) = 0. \]
It follows that
\[ m_B(u) \geq m(u), \quad M_B(u) \geq M(u), \quad M_B \geq M_1. \]

1.3. **Existence and uniqueness.** Throughout the paper, we always assume
\[ 0 < \epsilon \leq 2. \]
We summarize the main results proved by Christodoulou in [2, 3, 4, 5, 6] as the following three theorems. (They are proved for \( \epsilon = 2 \) in the papers. But the proofs can be extended to the case \( 0 < \epsilon \leq 2 \), see [9].)

**Theorem 1.1.** For any initial data \( \bar{h}(r) \in C^1[0, \infty) \) which satisfies
\[ \bar{h}(r) = O\left( \frac{1}{r^{1+\epsilon}} \right), \quad \frac{\partial \bar{h}}{\partial r}(r) = O\left( \frac{1}{r^{2+\epsilon}} \right) \]
as \( r \to \infty \), there exists \( u_0 > 0 \) and a unique classical solution
\[ h(u, r) \in C^1([0, u_0] \times [0, \infty)) \]
of (1.6) which satisfies the initial condition \( h(0, r) = \bar{h}(r) \) and
\[ h(u, r) = O\left( \frac{1}{r^{1+\epsilon}} \right), \quad \frac{\partial h}{\partial r}(u, r) = O\left( \frac{1}{r^{2+\epsilon}} \right) \]
uniformly in \( u \in [0, u_0] \) as \( r \to \infty \).
Theorem 1.2. For any initial data $\hat{h}(r) \in C^1[0, \infty)$ which satisfies

$$\hat{h}(r) = O\left(\frac{1}{r^{1+\epsilon}}\right), \quad \frac{\partial \hat{h}}{\partial r}(r) = O\left(\frac{1}{r^{2+\epsilon}}\right)$$

as $r \to \infty$, there exists $\delta_0 > 0$ such that if

$$\inf \sup_{b > b_0, r \geq 0} \left\{ \left(1 + \frac{r}{b}\right)^{1+\epsilon} |h(r)| + \left(1 + \frac{r}{b}\right)^{2+\epsilon} \left|b \frac{\partial h}{\partial r}(r)\right| \right\} \leq \delta_0,$$

there exists a global classical solution

$$h(u, r) \in C^1([0, \infty) \times [0, \infty))$$

of (1.6) which satisfies the initial condition $h(0, r) = \hat{h}(r)$ and

$$|h(u, r)| \leq \frac{C}{(1 + u + r)^{1+\epsilon}}, \quad \left|\frac{\partial h}{\partial r}(u, r)\right| \leq \frac{C}{(1 + u + r)^{2+\epsilon}}.$$

The corresponding spacetime is future causally geodesically complete with vanishing final Bondi-Christodoulou mass $M_1$. Furthermore, if the final Bondi-Christodoulou mass $M_1$ is positive, a black hole forms and, in the region exterior to the Schwarzschild sphere $r = 2M_1$, the spacetime metric tends to the Schwarzschild metric.

In [3, 4], Christodoulou introduced the following definition of global generalized solutions and proved their existence and uniqueness.

Definition 1.1. Denote $I = [0, \infty) \times (0, \infty)$ the complement of the central line. A generalized solution to the equation (1.6) is a function

$$h(u, r) \in C^1(I)$$

such that, for each $u \in [0, \infty)$, $h(u, r) \in L^2(0, \infty)$ and the quantity $\int_0^\infty h^2 dr$ is bounded by a continuous function of $u$. Moreover, $h$ satisfies the following properties.

1. $h$ satisfies the equation (1.6) in $I$ and takes $h(0, r) = \hat{h}$ at $u = 0$ as the initial data, $\hat{h}$, $g$, $\bar{g}$ are continuous functions in $I$.
2. At each $u$, for arbitrary $r_1$,

$$\frac{\bar{g}}{g} \in L^1(0, r_1).$$

3. For almost all $u$, and any $(u_1, r_1) \in I$,

$$\xi = \lim_{\delta \to 0} \int_{\delta}^{r_1} \bar{g}(h - \hat{h}) \frac{dr'}{r'}$$

exists and

$$\frac{\partial^2 \xi}{\partial r^2} \in L^2\left((0, u_1) \times (0, r_1)\right).$$

4. $\bar{h}$, $m$ are weakly differentiable in $I$, and

$$D\bar{h} = \frac{\xi}{2r}, \quad Dm = -\frac{\pi}{g} \xi^2.$$
For each \((u_1, r_1) \in I\), denote \(r_0 = \chi_{u_1}(0; r_1)\),  
\[I(u_1, r_1) = \left\{ (u, r) | 0 < r < \chi_{u_1}(u; r), \ 0 < u < u_1 \right\}.\]

Then the following integral identity holds 
\[
\int_0^{r_1} \frac{g}{g_{(u_1, r)}}(u_1, r) dr + 2\pi \int_{I(u_1, r_1)} \frac{g\xi^2}{g^2 r} dr du + \frac{1}{2} \int_0^{u_1} g(u, 0) du = \int_0^{r_0} \frac{g}{g}(0, r) dr.
\]

**Theorem 1.3.** For any initial data \(\hat{h}(r) \in C^1[0, \infty)\) which satisfies  
\[
\hat{h}(r) = O\left(\frac{1}{r^{1+\epsilon}}\right), \quad \partial_r \hat{h}(r) = O\left(\frac{1}{r^{2+\epsilon}}\right)
\]
as \(r \to \infty\), there exists at least one global generalized solution which has the same data as a classical solution coincides with it in the domain of existence of the latter.

On the other hand, by using the double null coordinate system  
\[g = -\Omega^2 du dv + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),\]

Christodoulou solved the characteristic (lightlike) initial value problem for small bounded variation norm and proved the following theorem [7].

**Theorem 1.4.** If there exists universal constant \(\delta_1 > 0\) such that  
\[
\int_{u_0}^{\infty} \left| \frac{\partial h}{\partial v}(u_0, v) \right| dv \leq \delta_1,
\]

then the unique spherically symmetric global solution exists for metric (1.11).

This result was extended to more general cases by Luk-Oh and Luk-Oh-Yang [9, 10]. In particular, Luk, Oh and Yang proved the following theorem in [10].

**Theorem 1.5.** The unique symmetrically symmetric global solution exists for metric (1.11) if  
\[
\int_u^v \left| h(u_0, v') \right| dv' \leq \epsilon (v - u)^{1-\gamma}, \quad \left| h(u_0, v) \right| + \left| \frac{\partial h}{\partial v}(u_0, v) \right| \leq \epsilon,
\]

for any \(v \geq u \geq u_0\), where \(\gamma > 0\) is certain positive constant. Moreover, the resulting spacetime is future causally geodesically complete.

We remark that the \(u\) and \(r\)-slices in metric (1.2) provide null and timelike hypersurfaces respectively. It yields characteristic (timelike) initial value problem. Thus the issue to solve the Einstein-scalar-field equations in metric (1.2) is different from that in metric (1.11), and another characteristic in (1.2) may not be given by level set of certain function \(v\).
1.4. Wave-like decaying null infinity. Gravitational waves can be described by the Bondi-Sachs metrics
\[-\left( e^{2\beta} \frac{V}{r} - r^2 h_{AB} U^A U^B \right) du^2 - 2e^{2\beta} dudr - 2r^2 h_{AB} U^B dudx^A + r^2 g_2 \]
where
\[g_2 = h_{AB}(u, x^A, x^B) dx^A dx^B\]
is certain Riemannian metrics on unit 2-sphere. Intuitively, when the waves arrive at infinity, the metrics should decay to the Minkowski metric wave-like. For instance, as \(r \to \infty\)
\[g_{\text{Bondi-Sachs}} = g_{\text{Minkowski}} + O\left( \frac{\sin(r)}{(1 + r)^{1+\delta}} \right), \quad \delta > -1.\]
Note that this kind of error terms satisfy
\[(1 + r)^{1+\delta} \left| \frac{\partial}{\partial r} \left( \frac{\sin(r)}{(1 + r)^{1+\delta}} \right) \right| < C\]
for any nonnegative integer \(k \geq 0\). Motivated by this, the second author posed the wave-like decaying spatial infinity and found that, for the Bondi-Sachs metrics, the past limit of the Bondi energy-momentum are not equal to the ADM total energy-momentum [13, 8]
\[\lim_{u \to -\infty} M_{\text{Bondi},0} \neq E_{\text{ADM}}, \quad \lim_{u \to -\infty} M_{\text{Bondi},k} \neq P_{\text{ADM},k},\]
where \(1 \leq k \leq 3\). This violates physicist’s expectation that they should be equal to each other, see, e.g. [11].

Therefore it is important to investigate wave-like boundary conditions and study new physical properties for gravitational waves. In particular, for the spherically symmetric Einstein-scalar-field equations, it is interesting to study existence and uniqueness of (1.6) for wave-like decaying initial condition, e.g.
\[\tilde{h}(r) = \frac{\sin(r)}{(1 + r)^{1+\epsilon}}.\]

1.5. Main results. Let \(h(u, r)\) be a continuous function and possess a continuous partial derivative with respect to \(r\) defined on \([0, u_1] \times [0, \infty)\) if \(u_1 < \infty\), or, on \([0, \infty) \times [0, \infty)\) if \(u_1 = \infty\). We call that \(h(u, r)\) satisfies the wave-like decaying condition at a retarded time \(u\) if
\[\sup_{r \geq 0} \left\{ (1 + r)^{1+\epsilon} \left( |h(u, r)| + \left| \frac{\partial h}{\partial r}(u, r) \right| \right) \right\} < \infty. \quad (1.12)\]
Here \(h(u, r)\) is replaced by \(\tilde{h}(r)\) for initial data.

In this paper we first prove the following local existence and uniqueness of classical solutions.
**Theorem 1.6.** For any initial data \( \hat{h}(r) \in C^1[0, \infty) \) which satisfies \((1.12)\), then there exists \( u_0 > 0 \) and a unique classical solution
\[
h(u, r) \in C^1\left([0, u_0] \times [0, \infty)\right)
\]
of \((1.6)\) which satisfies the initial condition \( h(0, r) = \hat{h}(r) \) and the decay property \((1.12)\) uniformly in \( u \in [0, u_0] \).

We then prove the following global existence and uniqueness for small initial data.

**Theorem 1.7.** Consider initial data \( \hat{h}(r) \in C^1[0, \infty) \) which satisfies \((1.12)\). Denote
\[
d_0 := \inf_{b > 0} \sup_{r \geq 0} \left\{ \left(1 + \frac{r}{b}\right)^{1+\epsilon} \left(|h_0(r)| + \left| b \frac{\partial h_0}{\partial r}(r) \right| \right) \right\}.
\]
Then there exists \( \delta > 0 \) such that if \( d_0 < \delta \), there exists a unique global classical solution
\[
h(u, r) \in C^1\left([0, \infty) \times [0, \infty)\right)
\]
of \((1.6)\) which satisfies the initial condition \( h(0, r) = \hat{h}(r) \) and the decay property
\[
|h(u, r)| \leq \frac{C}{\left(1 + \frac{u}{2} + r\right)^{1+\epsilon}}, \quad \left| \frac{\partial h}{\partial r}(u, r) \right| \leq \frac{C}{\left(1 + \frac{u}{2} + r\right)^{1+\epsilon}}
\]
for some constant \( C \) depending on \( \epsilon \) only. Moreover, the corresponding spacetime is future causally geodesically complete with vanishing final Bondi mass \( M_{B1} \).

Finally, we extend Christodoulou’s global generalized solutions to the case of wave-like decaying initial data.

**Theorem 1.8.** Given initial data \( \hat{h}(r) \in C^1[0, \infty) \) which satisfies \((1.12)\), there exists at least one global generalized solution which has the same data as a classical solution coincides with it in the domain of existence of the latter.

We remark that the wave-like decaying condition is sharp for the well-posedness of Einstein-scalar-field equations and we prove our main theorems by adopting Christodoulou’s idea. Due to the slowly wave-like decaying condition, we need to derive some key estimates in more delicate way.

The paper is organized as follows. In Section 2, we prove the three lemmas which contain the main estimates for the solutions of the Einstein-scalar-field equations. In Section 3, we prove Theorem 1.6. In Section 4, we prove Theorem 1.7. In Section 5, we prove Theorem 1.8.
2. Main estimates

In this section, we follow from Christodoulou's argument in [2] to prove three lemmas for the wave-like decaying solutions. As the solutions have a slowly decaying partial derivative with respect to $r$, we need to derive much more fine estimates.

Let $t \in [0, 1]$, $u_1 \in [0, \infty)$ or $u_1 = \infty$. Denote $X_{u_1,t}$, $Y_{u_1,t}$ the spaces of continuous functions which possess continuous partial derivatives with respect to $r$ defined on $[0, u_1] \times [0, \infty)$ if $u_1 < \infty$, or, on $[0, \infty) \times [0, \infty)$ if $u_1 = \infty$ such that the following norms are finite

$$
\|f\|_{X_{u_1,t}} = \sup_{u \in [0, u_1] \text{ or } [0, \infty)} \sup_{r \geq 0} \left\{ \left( 1 + \frac{tu_1 + r}{2} \right)^{1+\epsilon} \left( |f(u,r)| + \left| \frac{\partial f}{\partial r}(u,r) \right| \right) \right\},
$$

$$
\|f\|_{Y_{u_1,t}} = \sup_{u \in [0, u_1] \text{ or } [0, \infty)} \sup_{r \geq 0} \left\{ \left( 1 + \frac{tu_1 + r}{2} \right)^{1+\epsilon} |f(u,r)| \right\}.
$$

It is obvious that $X_{u_1,t}$, $Y_{u_1,t}$ are Banach spaces equipped with the corresponding norms.

For any initial data $h(0, r) = \tilde{h}(r) \in X_{0,t}$, denote

$$
\|h\|_{X_{0,t}} = \sup_{r \geq 0} \left\{ (1 + r)^{1+\epsilon} \left( |h(0, r)| + \left| \frac{\partial h}{\partial r}(0, r) \right| \right) \right\} = d > 0. \quad (2.1)
$$

We construct a sequence of approximate solutions by setting

$$
h_0(u, r) = h\left(0, \frac{tu_1}{2} + r\right). \quad (2.2)
$$

For $h_n \in X_{u_1,t}$, define $h_{n+1}$ to be the solution of the equation

$$
D_n h_{n+1} - \frac{1}{2r} (g_n - \bar{g}_n) h_{n+1} = -\frac{1}{2r} (g_n - \bar{g}_n) \tilde{h}_n \quad (2.3)
$$

with the initial condition

$$
h_{n+1}(0, r) = h(0, r),
$$

where $g_n$ is the $g$-function corresponding to $h_n$ and $D_n$ is the $D$-operator corresponding to $g_n$.

Denote

$$
c = \begin{cases} 
6 \epsilon|1-\epsilon|, & \epsilon \neq 1, \\
24, & \epsilon = 1,
\end{cases} \quad (2.4)
$$

$$
c_t = \begin{cases} 
1, & t = 0, \\
\exp \left( \frac{2(1 + \epsilon) \pi^2 u^2}{3} \right), & 0 < t \leq 1,
\end{cases} \quad (2.5)
$$

$$
a(u) = \begin{cases} 
u, & u < \infty, \\
\frac{1}{\nu t}, & u = \infty, \; t \neq 0,
\end{cases} \quad (2.6)
$$
and

\[ F_t(u, x) = c_t^2 \left( 2 + 8\pi c^2 x^2 a(u) \right) \left( d + \frac{24 + 6\epsilon + 4\pi c^3 x^3 a(u)}{3\epsilon} \right) \exp \left( \frac{8\pi c^2 x^2 a(u)}{3} \right), \]

\[ C_t(u, x) = \frac{32\pi (\epsilon + 1)}{c_t^2} c_t^2 a(u) (3 + 4\pi c^2 x^2) \exp \left( \frac{4\pi c^2 x^2 a(u)}{3} \right). \]

Lemma 2.1. Let initial data \( \tilde{h}(r) \) satisfy (2.4). If there exists some constant \( x > 0 \) such that

\[ \| h_n(u, r) \|_{X_{u_1, t}} \leq x, \]

then it satisfies that

\[ \| h_{n+1}(u, r) \|_{X_{u_1, t}} \leq F_t(u_1, x). \] (2.7)

Proof: First we prove the following claim case by case.

Claim I: For \( 0 < \epsilon \leq 2 \),

\[ |(h_n - \tilde{h}_n)(u, r)| \leq \frac{c\epsilon r(1 + \frac{tu}{2})^{1-\epsilon}}{\left(1 + \frac{tu}{2} + r\right)^2}, \] (2.8)

where \( c \) is given by (2.4).

Indeed, it is easy to find

\[ |\tilde{h}_n(u, r)| \leq \frac{1}{r} \int_0^r |h_n(u, r)|dr \]

\[ \leq \frac{x}{\epsilon r} \left[ \frac{1}{(1 + \frac{tu}{2})^\epsilon} - \frac{1}{(1 + \frac{tu}{2} + r)^\epsilon} \right] \]

\[ \leq \frac{x}{\epsilon r (1 + \frac{tu}{2})^\epsilon} \left[ 1 - \frac{(1 + \frac{tu}{2})^2}{(1 + \frac{tu}{2} + r)^2} \right] \]

(2.9)

Case I: \( 0 < \epsilon < 1 \).
(i) For \( r \geq 1 + \frac{tu}{2} \), we have
\[
|h_n - \bar{h}_n(u,r)| \leq |h_n| + |\bar{h}_n| \\
\leq \frac{x}{(1 + \frac{tu}{2} + r)^{1+\epsilon}} + \frac{2x}{\epsilon(1 + \frac{tu}{2})^{\epsilon} (1 + \frac{tu}{2} + r)} \\
\leq \frac{6xr}{\epsilon(1 + \frac{tu}{2})^{\epsilon} (1 + \frac{tu}{2} + r)^2} \\
\leq \frac{6xr(1 + \frac{tu}{2})^{1-\epsilon}}{\epsilon(1 + \frac{tu}{2} + r)^2}. 
\]
(2.10)

(ii) For \( 0 \leq r \leq 1 + \frac{tu}{2} \), we have
\[
|h_n - \bar{h}_n(u,r)| \leq \frac{1}{r} \int_{0}^{r} \int_{r'} |\partial h_n| ds dr' \\
\leq \frac{x}{\epsilon r} \int_{0}^{r} \left[ \frac{1}{(1 + \frac{tu}{2} + r')^{\epsilon}} - \frac{1}{(1 + \frac{tu}{2} + r)^\epsilon} \right] dr' \\
\leq \frac{x}{\epsilon(1 - \epsilon)r} \left[ (1 + \frac{tu}{2} + r)^{1-\epsilon} - (1 + \frac{tu}{2})^{1-\epsilon} - \frac{(1 - \epsilon)r}{(1 + \frac{tu}{2} + r)^\epsilon} \right] \\
\leq \frac{x(1 + \frac{tu}{2})^{1-\epsilon}}{\epsilon(1 - \epsilon)r(1 + \frac{tu}{2} + r)} \left[ F(r) + 5r^2 \right],
\]
where
\[
F(r) = (1 + \frac{tu}{2} + r)^{3-\epsilon} - (1 + \frac{tu}{2})^{\epsilon-1} - (1 + \frac{tu}{2} + r)^2 \\
- (1 - \epsilon)r(1 + \frac{tu}{2}) - 5r^2.
\]
By analyzing the monotonicity of \( F(r) \), we obtain
\[
F'''(r) \geq 0 \Rightarrow F''(r) \leq F'' \left( 1 + \frac{tu}{2} \right) = (3 - \epsilon)(2 - \epsilon)2^{1-\epsilon} - 12 \leq 0 \\
\Rightarrow F'(r) \leq F'(0) = 0 \Rightarrow F(r) \leq F(0) = 0.
\]
This gives
\[
|h_n - \bar{h}_n(u,r)| \leq \frac{5xr(1 + \frac{tu}{2})^{1-\epsilon}}{\epsilon(1 - \epsilon)(1 + \frac{tu}{2} + r)^2}. 
\]
(2.11)
Thus, (2.10) and (2.11) give that, for any \( r \geq 0 \),

\[
| (h_n - \bar{h}_n)(u, r) | \leq \frac{6 \epsilon \tau (1 + \frac{tu}{2})^{1-\epsilon}}{\epsilon (1 - \epsilon) (1 + \frac{tu}{2} + r)^2}.
\]

**Case II**: \( \epsilon = 1 \).

Since

\[
| \frac{\partial h_n}{\partial r} | \leq \frac{x}{(1 + \frac{tu}{2} + r)} \leq \frac{x}{(1 + \frac{tu}{2} + r)^{1+\frac{1}{2}}}.
\]

Then we can use the estimate for \( \epsilon = \frac{1}{2} \) in Case I and obtain

\[
| (h_n - \bar{h}_n)(u, r) | \leq \frac{24 \epsilon \tau (1 + \frac{tu}{2})^{1-\epsilon}}{(1 + \frac{tu}{2} + r)^2}.
\]

**Case III**: \( 1 < \epsilon \leq 2 \).

Using the same argument as in Case I and choose \( F \) as follows.

\[
F(r) = \left(1 + \frac{tu}{2} + r\right)^2 - \left(1 + \frac{tu}{2} + r\right)^3 - (1 - \epsilon) r \left(1 + \frac{tu}{2}\right)^{-1} - 5r^2.
\]

By analyzing the monotonicity of \( F(r) \), we obtain

\[
F''' \geq 0 \implies F''(r) \leq F'' \left(1 + \frac{tu}{2}\right) = -(3 - \epsilon)(2 - \epsilon)2^{\epsilon-1} - 8 < 0
\]

\[
\implies F'(r) \leq F'(0) = 0 \implies F(r) \leq F(0) = 0.
\]

This gives

\[
| (h_n - \bar{h}_n)(u, r) | \leq \frac{6 \epsilon \tau (1 + \frac{tu}{2})^{1-\epsilon}}{\epsilon (1 - \epsilon) (1 + \frac{tu}{2} + r)^2}.
\]

Therefore, (2.8) is proved by combining the above three cases.

Denote

\[
k = \exp \left( -\frac{2 \pi c^2 x^2}{3} \right) \leq 1.
\]
Then (1.6) and (2.8) imply that
\[
\bar{g}_n(u, r) \geq g_n(u, 0) = \exp \left[ -4\pi \int_0^\infty (h_n - \bar{h}_n)^2 \frac{dr}{r} \right] \geq \exp \left[ -\frac{2\pi c^2 x^2}{3 \left(1 + \frac{tu}{2}\right)^{2\epsilon}} \right] \geq k. \tag{2.13}
\]

Next we prove the following claim.

**Claim II:**
\[
(g_n - \bar{g}_n)(u, r) \leq \frac{4\pi c^2 x^2}{3} \frac{r^2}{\left(1 + \frac{tu}{2}\right)^{2\epsilon-1} \left(1 + \frac{tu}{2} + r\right)^3}. \tag{2.14}
\]

Indeed, using (2.8) and the following equation
\[
\frac{\partial g_n}{\partial r} = \frac{4\pi g_n (h_n - \bar{h}_n)^2}{r},
\]
we obtain
\[
(g_n - \bar{g}_n)(u, r) \leq \frac{1}{r} \int_0^r \int_{r'}^r \frac{\partial g_n}{\partial s} ds dr' \leq \frac{4\pi c^2 x^2}{r \left(1 + \frac{tu}{2}\right)^{2\epsilon-2}} \left\{ \frac{1}{2} \int_0^r \frac{1}{\left(1 + \frac{tu}{2} + r'\right)^2} - \frac{1}{\left(1 + \frac{tu}{2} + r\right)^2} \right\} dr' - \frac{1 + \frac{tu}{2}}{3} \int_0^r \frac{1}{\left(1 + \frac{tu}{2} + r'\right)^3} - \frac{1}{\left(1 + \frac{tu}{2} + r\right)^3} \right\} dr'
\]
\[
\leq \frac{4\pi c^2 x^2}{3} \frac{r^2}{\left(1 + \frac{tu}{2}\right)^{2\epsilon-1} \left(1 + \frac{tu}{2} + r\right)^3}.
\]
Therefore the claim is proved.

Now (2.9) and (2.14) give
\[
\left| -\frac{1}{2r} (g_n - \bar{g}_n) \bar{h}_n \right| \leq \frac{4\pi c^2 x^3}{3\epsilon} \frac{r}{\left(1 + \frac{tu}{2}\right)^{3\epsilon-1} \left(1 + \frac{tu}{2} + r\right)^4} \leq \frac{4\pi c^2 x^3}{3\epsilon} \frac{1}{\left(1 + \frac{tu}{2}\right)^{2\epsilon+1} \left(1 + \frac{tu}{2} + r\right)^{1+\epsilon}}. \tag{2.15}
\]

In the following we estimate $h_{n+1}$ and $\frac{\partial h_{n+1}}{\partial r}(u, r)$. These could be done by using the characteristic $r(u) = \chi_n(u; r_1)$ through the line $r = r_1$ at $u = u_1$. 
From (1.7), (2.13), we have
\[ r(u) = r_1 + \int_u^{u_1} \frac{1}{2} \bar{g}_n du' \geq r_1 + \frac{1}{2} k(u_1 - u) \geq r_1 + \frac{1}{2} tk(u_1 - u). \] (2.16)

Denote \( r_0 = r(0) \). Then (2.16) gives
\[ |h(0, r_0)| \leq d(1 + r_0)^{1+\epsilon} \leq c_1 d(1 + r_0)^{1+\epsilon}, \] (2.17)
\[ \left| \frac{\partial h}{\partial r}(0, r_0) \right| \leq d(1 + r_0)^{1+\epsilon} \leq c_1 d(1 + r_0)^{1+\epsilon}. \] (2.18)

A straightforward computation shows
\[ \int_0^{u_1} \frac{du}{(1 + \frac{tu_1}{2})^{2+\epsilon}} \leq a(u_1). \]

Using (2.14), (2.15), we obtain
\[ \int_0^{u_1} \left[ \frac{g_n - \bar{g}_n}{r} \right]_{\chi_n} \chi_n du \leq \frac{4\pi c^2 x^2 a(u_1)}{3}, \] (2.19)
and
\[ \int_0^{u_1} \left[ \left| -\frac{1}{2r} (g_n - \bar{g}_n) \bar{h}_n \right| \right]_{\chi_n} \chi_n du \]
\[ \leq \frac{4\pi c^2 x^3}{3\epsilon} \int_0^{u_1} \frac{du}{\left(1 + \frac{tu_1}{2}\right)^{2+\epsilon} \left(1 + \frac{u}{2} + r(u)\right)^{1+\epsilon}} \]
\[ \leq \frac{4\pi c^2 x^3}{3\epsilon} \int_0^{u_1} \frac{du}{\left(1 + \frac{tu_1}{2}\right)^{2+\epsilon} \left(1 + \frac{u}{2} + r_1 + \frac{tk}{2}(u_1 - u)\right)^{1+\epsilon}} \]
\[ \leq \frac{4\pi c^2 x^3 c_1 a(u_1)}{3\epsilon} \frac{1}{\left(1 + \frac{tu_1}{2} + r_1\right)^{1+\epsilon}}. \] (2.20)

where \( c_1, a(u_1) \) are given by (2.5), (2.6) respectively.

Now integrating (2.3) along the characteristic \( \chi_n \), we have
\[ h_{n+1}(u_1, r_1) = h(0, r_0) \exp \left\{ \int_0^{u_1} \left[ \frac{g_n - \bar{g}_n}{2r} \right]_{\chi_n} \chi_n du \right\} \]
\[ + \int_0^{u_1} \left[ -\frac{g_n - \bar{g}_n}{2r} \right]_{\chi_n} \chi_n \exp \left\{ \int_u^{u_1} \left[ \frac{g_n - \bar{g}_n}{2r} \right]_{\chi_n} \chi_n du \right\} du. \] (2.21)
Substituting (2.17), (2.19) and (2.20) into (2.21), we obtain
\[
\begin{align*}
(1 + \frac{tu_1}{2} + r_1)^{1+\varepsilon} |h_{n+1}(u_1, r_1)| \\
\leq c_t \left( d + \frac{4\pi c^2 x^3 a(u_1)}{3\epsilon} \right) \exp \left( \frac{4\pi c^2 x^2 a(u_1)}{3} \right).
\end{align*}
\] (2.22)

Note that \( \frac{\partial h_{n+1}}{\partial r} \) satisfies the following equation (c.f. (9.16) in \([2]\))
\[
D_n \left( \frac{\partial h_{n+1}}{\partial r} \right) - \frac{1}{r} (g_n - \bar{g}_n) \frac{\partial h_{n+1}}{\partial r} = f'_n,
\] (2.23)
where
\[
f'_n = -\frac{1}{2} \frac{\partial^2 g_n}{\partial r^2} h_n - \frac{1}{2} \frac{\partial g_n}{\partial r} \frac{h_n - \bar{h}_n}{r} + \frac{1}{2} \frac{\partial^2 g_n}{\partial r^2} h_{n+1},
\]
\[
\frac{\partial^2 g_n}{\partial r^2} = -\frac{2}{r^2} (g_n - \bar{g}_n) + \frac{4\pi}{r^2} (h_n - \bar{h}_n)^2 g_n.
\]

From (2.9) and (2.14), we have
\[
|\frac{\partial^2 g_n}{\partial r^2}| \leq \frac{8\pi c^2 x^2}{(1 + \frac{tu}{2})^{2\varepsilon - 1} \left( 1 + \frac{tu}{2} + r \right)}.
\] (2.24)

Using (2.8), (2.9), (2.14) and (2.22), we obtain
\[
|f'_n| \leq \frac{1}{2} \left| \frac{\partial^2 g_n}{\partial r^2} \right| |h_n| + \frac{1}{2} \left| \frac{\partial g_n}{\partial r} \right| \left| h_n - \bar{h}_n \right| + \frac{1}{2} \left| \frac{\partial^2 g_n}{\partial r^2} \right| |h_{n+1}|
\leq \left( \frac{8}{\epsilon} + 2 \right) c^3 x^3 + 8\pi c^2 x^2 c_t \left( d + \frac{4\pi c^2 x^3 a(u_1)}{3\epsilon} \right) \exp \left( \frac{4\pi c^2 x^2 a(u_1)}{3} \right)
\leq \frac{\left( \frac{8}{\epsilon} + 2 \right) c^3 x^3 + 8\pi c^2 x^2 c_t \left( d + \frac{4\pi c^2 x^3 a(u_1)}{3\epsilon} \right) \exp \left( \frac{4\pi c^2 x^2 a(u_1)}{3} \right)}{(1 + \frac{tu}{2})^{2\varepsilon - 1} \left( 1 + \frac{tu}{2} + r \right)^{1+\varepsilon}}.
\]

Using the same argument as in (2.20), we have
\[
\int_0^{u_1} \left[ |f'_n| \chi_n \right] du \leq \frac{c_1 c_t a(u_1)}{(1 + \frac{tu_1}{2} + r_1)^{1+\varepsilon}},
\] (2.25)
where
\[
c_1 = \left( \frac{8}{\epsilon} + 2 \right) c^3 x^3 + 8\pi c^2 x^2 c_t \left( d + \frac{4\pi c^2 x^3 a(u_1)}{3\epsilon} \right) \exp \left( \frac{4\pi c^2 x^2 a(u_1)}{3} \right).
\]

Integrating (2.23) along the characteristic \( \chi_n \), and using (2.18), (2.25), we obtain
\[
\left( 1 + \frac{tu_1}{2} + r_1 \right)^{1+\varepsilon} \left| \frac{\partial h_{n+1}}{\partial r} (u_1, r_1) \right|
\leq c_t \exp \left( \frac{4\pi c^2 x^2 a(u_1)}{3} \right) \left[ d + \left( \frac{8}{\epsilon} + 2 \right) c^3 x^3 a(u_1) \right]
\leq c_t \exp \left( \frac{4\pi c^2 x^2 a(u_1)}{3} \right) \left[ d + \left( \frac{8}{\epsilon} + 2 \right) c^3 x^3 a(u_1) \right]
\]
(2.26)
Thus (2.22) and (2.26) indicate
\[ \|h_{n+1}\|_{X_{u_1,t}} \leq F_t(u_1, x). \]
Therefore the proof of the lemma is complete. Q.E.D.

Lemma 2.2. Let initial data \( \ddot{h}(r) \) satisfy (2.1). If there exists some constant \( x > 0 \) such that
\[ \|h_n(u, r)\|_{Y_{u_1,t}} \leq x, \]
then it satisfies that
\[ \|(h_{n+1} - h_n)(u, r)\|_{Y_{u_1,t}} \leq C_t(u_1, x)\|(h_n - h_{n-1})(u, r)\|_{Y_{u_1,t}}. \] (2.27)

Proof: Note that the following equation holds (c.f. (9.27) in [2])
\[
D_n(h_{n+1} - h_n) - \frac{1}{2} \frac{\partial \bar{g}_n}{\partial r} (h_{n+1} - h_n)
= \frac{1}{2}(\bar{g}_n - \bar{g}_{n-1}) \frac{\partial h_n}{\partial r}
+ \frac{1}{2} \left( \frac{\partial \bar{g}_n}{\partial r} - \frac{\partial \bar{g}_{n-1}}{\partial r} \right) h_n
+ f_n - f_{n-1},
\] (2.28)
where
\[ f_n - f_{n-1} = -\frac{1}{2} \frac{\partial \bar{g}_n}{\partial r} (\bar{h}_n - \bar{h}_{n-1}) - \frac{1}{2} \left( \frac{\partial \bar{g}_n}{\partial r} - \frac{\partial \bar{g}_{n-1}}{\partial r} \right) \bar{h}_{n-1}. \]

Now (2.10) gives
\[ |h_n - \bar{h}_n + h_{n-1} - \bar{h}_{n-1}| \leq \frac{2c xr}{(1 + tu_2^2)^{\epsilon - 1}(1 + tu_2^2 + r)^2}. \] (2.29)
On the other hand, using the definition, we can prove
\[ |h_n - \bar{h}_n - h_{n-1} + \bar{h}_{n-1}| \leq 4\|h_n - h_{n-1}\|_{Y_{u_1,t}}. \] (2.30)
Multiplying (2.29) by (2.30), we have
\[ |(h_n - \bar{h}_n)^2 - (h_{n-1} - \bar{h}_{n-1})^2| \leq \frac{8cxr \|h_n - h_{n-1}\|_{Y_{u_1,t}}^2}{\epsilon (1 + tu_2^2)^{2\epsilon - 1}(1 + tu_2^2 + r)^3}. \] (2.31)
As
\[ x_1, x_2 < 0 \implies |e^{x_1} - e^{x_2}| \leq |x_1 - x_2|, \]
we obtain

\[ |g_n - g_{n-1}| \leq 4\pi \left| \int_r^\infty \frac{(h_n - \tilde{h}_n)^2 - (h_{n-1} - \tilde{h}_{n-1})^2}{r^{2\epsilon}} \, dr' \right| \]

\[ \leq \frac{32\pi cx}{\epsilon \left(1 + \frac{tu}{r^2}\right)^{2\epsilon}} \left[ \int_r^\infty \frac{dr'}{1 + \frac{tu}{r^2} + r^2 \epsilon} \right]^3 \|h_n - h_{n-1}\|_{Y_{1,t}} \tag{2.32} \]

This implies

\[ |\tilde{g}_n - \tilde{g}_{n-1}| \leq \frac{16\pi cx}{\epsilon \left(1 + \frac{tu}{r^2}\right)^{2\epsilon}} \left(1 + \frac{tu}{r^2} + r^2 \epsilon\right)^3 \|h_n - h_{n-1}\|_{Y_{1,t}}. \tag{2.33} \]

By (2.33), (2.31) and (2.32), we obtain

\[ \left| \frac{\partial (g_n - g_{n-1})}{\partial r} \right| = \left[ \frac{4\pi (h_n - \tilde{h}_n)^2 g_n}{r} - \frac{4\pi (h_{n-1} - \tilde{h}_{n-1})^2 g_{n-1}}{r} \right] \]

\[ \leq \frac{4\pi}{r} \left[ |g_n| (h_n - \tilde{h}_n)^2 - (h_{n-1} - \tilde{h}_{n-1})^2 | \right] \]

\[ + |g_n - g_{n-1}| |h_n - h_{n-1}| \]

\[ \leq \frac{8\pi(4cx + 8\pi c^3x^3)}{\epsilon \left(1 + \frac{tu}{r^2}\right)^{2\epsilon}} \left(1 + \frac{tu}{r^2} + r^2 \epsilon\right)^3 \|h_n - h_{n-1}\|_{Y_{1,t}}. \]

Thus

\[ \left| \frac{\partial (\tilde{g}_n - \tilde{g}_{n-1})}{\partial r} \right| \leq \frac{1}{r^2} \int_0^r \int_{r'}^r \left| \frac{\partial (g_n - g_{n-1})}{\partial s} \right| dsdr' \]

\[ \leq \frac{4\pi(4cx + 8\pi c^3x^3)}{\epsilon \left(1 + \frac{tu}{r^2}\right)^{2\epsilon+1}} \left(1 + \frac{tu}{r^2} + r^2 \epsilon\right)^{3 \epsilon} \|h_n - h_{n-1}\|_{Y_{1,t}}. \tag{2.34} \]

Using (2.14), we have

\[ |f_n - f_{n-1}| \leq \frac{1}{2} \left| \frac{\partial g_n}{\partial r} \right| |\tilde{h}_n - \tilde{h}_{n-1}| + \frac{1}{2} \left| \frac{\partial g_{n-1}}{\partial r} \right| \left| \frac{\partial g_n}{\partial r} - \frac{\partial g_{n-1}}{\partial r} \right| \]

\[ \leq \frac{4\pi c^2x\left(\frac{1}{r} + \frac{1}{r^2}\right)(6cx + 8\pi c^3x^3)}{\left(1 + \frac{tu}{r^2}\right)^{3\epsilon-1}} \left(1 + \frac{tu}{r^2} + r^2 \epsilon\right)^{3 \epsilon} \|h_n - h_{n-1}\|_{Y_{1,t}}. \tag{2.35} \]

From (2.33), (2.34) and the assumption on \( h_n \), we obtain

\[ |g_n - g_{n-1}| \left| \frac{\partial h_n}{\partial r} \right| \leq \frac{16\pi cx^2}{\epsilon \left(1 + \frac{tu}{r^2}\right)^{2\epsilon}} \left(1 + \frac{tu}{r^2} + r^2 \epsilon\right)^{2 \epsilon} \|h_n - h_{n-1}\|_{Y_{1,t}}. \tag{2.36} \]
\[ \left| \frac{\partial (\tilde{g}_n - \tilde{g}_{n-1})}{\partial r} \right| h_n \leq \frac{4\pi x(4\pi c^2 x^2 + 8\pi x^3)}{\epsilon (1 + \frac{u^2}{2})} \left[ \left( 1 + \frac{u^2}{2} + r \right)^{3+\epsilon} \right] \| h_n - h_{n-1} \|_{Y_{u_1,t}}. \] (2.37)

Now applying (2.35), (2.36) and (2.37) to (2.28), we can prove
\[
\left| D_n(h_{n+1} - h_n) - \frac{1}{2} \frac{\partial \tilde{g}_n}{\partial r}(h_{n+1} - h_n) \right| \leq \frac{32\pi \left( \frac{1}{\epsilon} + \frac{1}{\epsilon^2} \right) (3 + 4\pi c^2 x^2) c^3 x^2}{\left( 1 + \frac{tu}{2} \right)^{2+1} \left( 1 + \frac{tu}{2} + r \right)} \| h_n - h_{n-1} \|_{Y_{u_1,t}}.
\]

Integrating (2.28) along the characteristic \( \chi_n \) and using the initial condition
\[(h_{n+1} - h_n)(0,r) = 0,\]
we obtain
\[
\left( 1 + \frac{tu}{2} + r \right)^{1+\epsilon} \left| (h_{n+1} - h_n)(u_1, r_1) \right| \leq C_t(u_1, x) \| h_n - h_{n-1} \|_{Y_{u_1,t}}.
\]
Thus,
\[
\| (h_{n+1} - h_n) \|_{Y_{u_1,t}} \leq C_t(u_1, x) \| h_n - h_{n-1} \|_{Y_{u_1,t}}.
\]
Therefore the proof of the lemma is complete. Q.E.D.

**Lemma 2.3.** If \( t = 0 \) and \( x > 2d \), there exists \( \gamma > 0 \) such that for any
\[ u_0 \in (0, \gamma], \]
the sequence \( \{ h_n \} \) is uniformly bounded by \( x \) in \( X_{u_0,0} \) and contracts in \( Y_{u_0,0} \).
Moreover, both \( \{ h_n \} \) and \( \left\{ \frac{\partial h_n}{\partial r} \right\} \) are equibounded and equicontinuous in \( I_0 = [0, u_0] \times [0, \infty) \).

**Proof:** Note that, in inequalities (2.7) and (2.27), \( F_0(u, x) \) and \( C_0(u, x) \) are strictly monotonically increasing with respect to \( u < \infty \), and
\[ F_0(0, x) = 2d < x, \quad F_0(\infty, x) = \infty, \quad C_0(0, x) = 0, \quad C_0(\infty, x) = \infty. \]
There are \( \gamma_1 > 0, \gamma_2 > 0 \) such that
\[ F_0(\gamma_1, x) = x, \quad C_0(\gamma_2, x) = \frac{1}{2}. \]
Then (2.7) and (2.27) imply that, for any
\[ u_0 \in (0, \gamma], \quad \gamma = \min \left\{ \gamma_1, \gamma_2 \right\}, \]
the following inequalities hold
\[
\| h_n \|_{X_{u_0,0}} \leq x \implies \left\{ \begin{array}{l}
\| h_{n+1} \|_{X_{u_0,0}} \leq x, \\
\| h_{n+1} - h_n \|_{Y_{u_0,0}} \leq \frac{1}{2} \| h_n - h_{n-1} \|_{Y_{u_0,0}}.
\end{array} \right.
\]
This also indicates that \( \{ h_n \} \) is equibounded in \( I_0 \). Since
\[
\left| \frac{\partial h_{n+1}}{\partial r}(u, r) \right| \leq \frac{x}{(1 + r)^{1+\epsilon}},
\]
we have
\[
\left| \frac{\partial h_n+1}{\partial u}(u, r) \right| \leq \frac{1}{2} |\bar{g}_n| |\frac{\partial h_n+1}{\partial r}| + \frac{1}{2r} |g_n - \bar{g}_n| |\bar{g}_n| + \frac{1}{2r} |g_n - \bar{g}_n| |\bar{h}_n| \\
\leq \frac{1}{2} \left( 1 + \frac{4\pi c^2 x^2}{3} + \frac{8\pi c^2 x^2}{3\epsilon} \right) \frac{x}{(1 + r)^{1+\epsilon}}. \tag{2.38}
\]

Therefore \( \{\frac{\partial h_n+1}{\partial r}\} \) and \( \{\frac{\partial h_n+1}{\partial u}\} \) are equibounded in \( I_0 \), which implies that \( \{h_n\} \) is equicontinuous in \( I_0 \).

In the following we show that \( \{\frac{\partial h_n}{\partial r}\} \) is equicontinuous in \( I_0 \). For any
\[ u_1 \in [0, u_0], \quad 0 \leq r_1 < r_2, \]
denote \( \chi_n(u; r_1) \) and \( \chi_n(u; r_2) \) the two characteristics through the line \( u = u_1 \) at \( r = r_1 \) and \( r = r_2 \) respectively. Denote
\[
k' = \exp \left( \frac{4\pi c^2 x^2}{3k} \right), \tag{2.39}
\]
where \( k \) is given by (2.12). From (2.13) and (4.29) of [3], we obtain
\[
\chi_n(u; r_2) - \chi_n(u; r_1) \leq (r_2 - r_1) \sup_{s \in [r_1, r_2]} \exp \left\{ \frac{1}{2} \int_{u_1}^u \left[ \frac{\partial g_n}{\partial r} \right]_{\chi_n(u'; s)} du' \right\}
\leq k'(r_2 - r_1).
\]
Therefore, if we denote
\[
\Theta(f)(u) = f(u, \chi_n(u; r_1)) - f(u, \chi_n(u; r_2))
\]
for any differentiable function \( f \), we can find that
\[
|\Theta(f)(u)| \leq \sup \left| \frac{\partial f}{\partial r} \right| |\chi_n(u; r_1) - \chi_n(u; r_2)| \\
\leq k' \sup \left| \frac{\partial f}{\partial r} \right| (r_2 - r_1).
\]

Define
\[
\psi(u) = \frac{\partial h_n+1}{\partial r}(u, \chi_n(u; r_1)) - \frac{\partial h_n+1}{\partial r}(u, \chi_n(u; r_2)). \tag{2.40}
\]
Differentiating (2.40) and using (2.23), we obtain
\[
\psi'(u) - \frac{\left( g_n - \bar{g}_n \right)(u, \chi_n(u; r_1))}{\chi_n(u; r_1)} \psi(u) = \sum_{i=1}^{4} A_i, \tag{2.41}
\]
where

\[ A_1 = \frac{\partial h_{n+1}}{\partial r}(u, \chi_n(u; r_2)) \Theta \left( \frac{g_n - \bar{g}_n}{r} \right)(u), \]

\[ A_2 = \frac{1}{2} \Theta \left( \frac{\partial^2 \bar{g}_n}{\partial r^2}(h_{n+1} - \bar{h}_{n+1}) \right)(u), \]

\[ A_3 = \frac{1}{2} \Theta \left( \frac{\partial^2 \bar{g}_n}{\partial r^2} (\bar{h}_{n+1} - \bar{h}_n) \right)(u), \]

\[ A_4 = -\frac{1}{2} \Theta \left( \frac{\partial \bar{g}_n}{\partial r} (h_n - \bar{h}_n) \right)(u). \]

Now we estimate \( A_i \) for \( 1 \leq i \leq 4 \). Using (2.7), (2.24) and \( g_n - \bar{g}_n = \partial \bar{g}_n \partial r \), we obtain

\[ |A_1| \leq \frac{x}{\left( 1 + \chi_n(u; r_2) \right)^{1+\epsilon}} \frac{\partial^2 \bar{g}_n}{\partial r^2} |k'| |r_2 - r_1| \]

\[ \leq \frac{8\pi c^2 x^3 k'}{\left( 1 + \chi_n(u; r_2) \right)^{1+\epsilon}} |r_2 - r_1| \]

\[ \leq 8\pi c^2 x^3 k' |r_2 - r_1|. \]

To estimate \( A_2 \) we need to estimate the 3rd partial derivative of \( \bar{g}_n \) with respect to \( r \). It is straightforward that

\[ \frac{\partial^3 \bar{g}_n}{\partial r^3} = \frac{6}{r^3} (g_n - \bar{g}_n) - \frac{16\pi}{r^3} (h_n - \bar{h}_n)^2 g_n \]

\[ + \frac{8\pi}{r^2} (h_n - \bar{h}_n) \frac{\partial (h_n - \bar{h}_n)}{\partial r} g_n \]

\[ + \frac{16\pi^2}{r^3} (h_n - \bar{h}_n)^4 g_n. \]  \hspace{1cm} (2.42)

Then

\[ \left| \frac{\partial^3 \bar{g}_n}{\partial r^3} \right| \leq \frac{1}{r} (40\pi c^2 x^2 + 16\pi^2 c^4 x^4). \]  \hspace{1cm} (2.43)

Thus, (2.8), (2.42), (2.43) give

\[ |A_2| \leq \frac{k'}{2} \left| \frac{\partial^3 \bar{g}_n}{\partial r^3} \right| |h_{n+1} - \bar{h}_{n+1}| \]

\[ + \frac{\partial^2 \bar{g}_n}{\partial r^2} \left| \frac{\partial (h_{n+1} - \bar{h}_{n+1})}{\partial r} \right| |r_2 - r_1| \]

\[ \leq (28\pi c^3 x^3 + 8\pi^2 c^5 x^5) k' |r_2 - r_1|. \]
Now we use Lemma 2.2 to estimate $A_3$. Denote $I_1 = [0, u_0] \times [0, 1]$.

As $C_0(u_0, x) \leq \frac{1}{2}$ in Lemma 2.2, we know that $h_{n+1} - h_n \to 0$ uniformly. Using (2.24), we obtain

$$\frac{\partial^2 \tilde{g}_n}{\partial r^2}(h_{n+1} - h_n) \to 0$$

uniformly. This leads to

$$\lim_{n \to +\infty} \sup_{(u, r) \in I_1} \left| \frac{\partial^2 \tilde{g}_n}{\partial r^2}(\tilde{h}_{n+1} - \tilde{h}_n) \right| \leq \lim_{n \to +\infty} \sup_{(u, r) \in I_1} \left| \frac{\partial^2 \tilde{g}_n}{\partial r^2}(h_{n+1} - h_n) \right| = 0.$$

This implies that

$$\frac{\partial^2 \tilde{g}_n}{\partial r^2}(\tilde{h}_{n+1} - \tilde{h}_n) \to 0$$

uniformly. Therefore the sequence

$$\left\{ \frac{\partial^2 \tilde{g}_n}{\partial r^2}(\tilde{h}_{n+1} - \tilde{h}_n) \right\}$$

is equicontinuous in $I_1$, and, for any $\eta > 0$, there exists $t_1$ such that

$$|r'_2 - r'_1| = |\chi_n(u; r_1) - \chi_n(u; r_2)| \leq t_1 \implies |A_3| \leq \frac{\exp(-4\pi c^2 x^2)}{3u_0} \eta.$$

The same argument gives

$$|A_3| \leq (48\pi c^3 x^3 + 16\pi^2 c^4 x^5)k'|r_2 - r_1|$$

for $(u, r) \in [0, u_0] \times [1, \infty)$. Take

$$s_1 = \min \left\{ \frac{t_1}{k'}, \frac{\exp(-4\pi c^2 x^2)}{3k'(48\pi c^3 x^3 + 16\pi^2 c^4 x^5)u_0} \eta \right\},$$

then for any $(u, r_1), (u, r_2) \in I_0$,

$$|r_2 - r_1| \leq s_1 \implies |r'_2 - r'_1| \leq k's_1 \leq t_1 \implies |A_3| \leq \frac{\exp(-4\pi c^2 x^2)}{3u_0} \eta.$$

Finally, we estimate $A_4$. As

$$\frac{\partial}{\partial r} \left( \frac{\partial \tilde{g}_n h_n - \tilde{h}_n}{r} \right) = \frac{\partial^2 \tilde{g}_n}{\partial r^2} \frac{h_n - \tilde{h}_n}{r} + \frac{1}{r} \frac{\partial \tilde{g}_n}{\partial r} \frac{\partial (h_n - \tilde{h}_n)}{\partial r}$$

$$- \frac{1}{r^2} \frac{\partial \tilde{g}_n}{\partial r} (h_n - \tilde{h}_n),$$

it is bounded in $I_0$ and we find

$$|A_4| \leq 6\pi c^3 x^3 k'|r_2 - r_1|.$$

On the other hand,

$$\frac{|(g_n - \tilde{g}_n)(u, \chi_n(u; r_1))|}{\chi_n(u; r_1)} \leq 4\pi c^2 x^2.$$
Integrating (2.41), we obtain
\[
\psi(u_1) = \psi(0) \exp \left[ \int_0^{u_1} \frac{(g_n - \bar{g}_n)(u, \chi_n(u; r_1))}{\chi_n(u; r_1)} du \right] 
+ \int_0^{u_1} \left\{ \exp \left[ \int_{u'}^{u_1} \frac{(g_n - \bar{g}_n)(u', \chi_n(u'; r_1))}{\chi_n(u'; r_1)} du' \right] \sum_{i=1}^4 A_i \right\} du.
\]
Therefore, the above estimates indicate that if \(|r_2 - r_1| \leq s_1\) then
\[
|\psi(u_1)| \leq \exp(4\pi c^2 x^2) \left[ |\psi(0)| + (42 \pi c^3 x^3 + 8 \pi^2 c^5 x^5) k' u_0 (r_2 - r_1) \right] + \frac{\eta}{3}.
\]

Now we estimate \(\psi(0)\). In \(I_0\), we define
\[
\omega(t) = \sup_{|r''_1 - r''_2| \leq t} \left\{ \left| \frac{\partial h_{n+1}}{\partial r}(0, \chi_n(0; r_1)) - \frac{\partial h_{n+1}}{\partial r}(0, \chi_n(0; r_2)) \right| \right\},
\]
where
\[
r''_1 = \chi_n(0; r_1), \quad r''_2 = \chi_n(0; r_2).
\]
From the basic analysis, we know that \(\frac{\partial h_{n+1}}{\partial r}(0, r)\) is uniformly continuous if and only if
\[
\lim_{t \to 0} \omega(t) = 0.
\]
Thus, for any \(\eta > 0\), there exists \(t_2 > 0\) such that, for any \(t \leq t_2\),
\[
\omega(t) < \frac{\exp(-4\pi c^2 x^2)}{3} \eta.
\]
Denote
\[
s_2 = \frac{\exp(-4\pi c^2 x^2)}{3(42 \pi c^3 x^3 + 8 \pi^2 c^5 x^5) k' u_0}.
\]
Choosing
\[
s = \min \left\{ s_1, \frac{t_2}{k'}, s_2 \right\},
\]
we find that
\[
|r_2 - r_1| \leq s \implies \begin{cases}
|r'_1 - r'_2| \leq t_1, \\
|r''_1 - r''_2| \leq k's \leq t_2 \implies |\psi(u_1)| \leq \eta.
\end{cases}
\]
Thus we have
\[
|\psi(u_1)| \leq \eta.
\]
This implies that
\[
\left| \frac{\partial h_{n+1}}{\partial r}(u_1, r_1) - \frac{\partial h_{n+1}}{\partial r}(u_1, r_2) \right| \leq \eta.
\]
Hence \(\left\{ \frac{\partial h_{n+1}}{\partial r} \right\}\) is equicontinuous with respect to \(r\) in \(I_0\).
The equicontinuous of \( \left\{ \frac{\partial h_{n+1}}{\partial r} \right\} \) with respect to \( u \) can be proved by the equiboundedness of \( D_n \left( \frac{\partial h_{n+1}}{\partial r} \right) \). Therefore the proof of the lemma is complete. Q.E.D.

3. Local existence and uniqueness of classical solutions

In this section we prove local existence and uniqueness of classical solutions.

**Theorem 3.1.** For any initial data \( \hat{h}(r) \in C^1[0, \infty) \) which satisfies (1.12), then there exists \( u_0 > 0 \) and a unique classical solution

\[
h(u, r) \in C^1 \left( [0, u_0] \times [0, \infty) \right)
\]

of (1.6) which satisfies the initial condition \( h(0, r) = \hat{h}(r) \) and the decay property (1.12) uniformly in \( u \in [0, u_0] \).

**Proof:** We use the main estimates for the case \( t = 0 \) derived in lemmas in the last section. Let \( u_0 < \gamma \) in Lemma 2.3. By Arzela-Ascoli theorem, there exists a subsequence \( \left\{ h_{n_i} \right\} \) and function \( \hat{h} \) such that

\[
h_{n_i} \to \hat{h}, \quad \frac{\partial h_{n_i}}{\partial r} \to \frac{\partial \hat{h}}{\partial r}
\]

uniformly in any compact subset \([0, u_0] \times [0, r_0]\). Moreover, \( \hat{h} \in X_{u_0,0} \), and

\[
\| \hat{h} \|_{X_{u_0,0}} \leq x. \tag{3.2}
\]

Therefore the convergence of (3.1) is uniform in \( I_0 \), and \( \left\{ \frac{\partial \hat{h}}{\partial r} \right\} \) is equibounded in \( I_0 \).

By the contraction principle,

\[
h_n \to h
\]

in \( Y_{u_0,0} \). Therefore

\[
h = \hat{h} \in X_{u_0,0}.
\]

Using (2.8), (2.9) and the arguments in the proofs of Lemma 2.1, Lemma 2.3, we can deduce that

\[
\hat{h}_{n_i} \to \hat{h}, \quad A_{n_i} = \int_r^\infty \left( h_{n_i} - \hat{h}_{n_i} \right)^2 \frac{dr'}{r'} \to A = \int_r^\infty \left( h - \hat{h} \right)^2 \frac{dr'}{r'}
\]

uniformly in \( I_0 \), which implies that

\[
g_{n_i} = \exp \left( -4\pi A_{n_i} \right) \to g = \exp \left( -4\pi A \right), \quad \tilde{g}_{n_i} \to \tilde{g}
\]
uniformly in $I_0$, and
\[
\frac{\partial \tilde{h}_{ni}}{\partial r} \to \frac{\partial h}{\partial r}, \quad \frac{\partial \tilde{g}_{ni}}{\partial r} \to \frac{\partial g}{\partial r}, \quad \frac{\partial \bar{g}_{ni}}{\partial r} \to \frac{\partial \bar{g}}{\partial r}
\]
uniformly in $I_0$.

Let $\chi_{ni}(u; r_1)$, $\chi(u; r_1)$ be the characteristics corresponding to $h_{ni}$, $h$ through $r = r_1$ at $u = u_1$. Then
\[
\frac{d(\chi_{ni} - \chi)}{du} = -\frac{1}{2} \left[ \bar{g}_{ni}(u, \chi_{ni}) - \bar{g}(u, \chi) \right].
\]
And it satisfies the initial condition
\[
(\chi_{ni} - \chi)(u_1) = \chi_{ni}(u_1, r_1) - \chi(u_1, r_1) = r_1 - r_1 = 0.
\]
We obtain
\[
(\chi_{ni} - \chi)(u) = -\frac{1}{2} \int_{u_1}^{u} \left[ \bar{g}_{ni}(u', \chi_{ni}) - \bar{g}(u', \chi) \right] du'.
\]
Thus, using the convergence of $\bar{g}_{ni}(u', \chi)$ to $\bar{g}(u', \chi)$, we can show
\[
\chi_{ni}(u; r_1) \to \chi(u; r_1) \quad \text{(3.3)}
\]
uniformly in $I_0$.

Applying (2.21) to $h_{ni}$, taking $i \to \infty$ and using (2.14), (2.15), (3.3), we obtain
\[
h(u_1, r_1) = h(0, \chi(0; r_1)) \exp \left\{ \int_{0}^{u_1} \left[ \frac{g - \bar{g}}{2r} \right] \chi \right\} 
\]
\[
+ \int_{0}^{u_1} \left[ -\frac{g - \bar{g}}{2r} \right] \chi \exp \left\{ \int_{u}^{u_1} \left[ \frac{g - \bar{g}}{2r} \right] \chi \right\} du
\]
in $I_0$. This implies that $h$ satisfies (1.6) in the integral sense in $I_0$. As $\frac{\partial h}{\partial r}$ is continuous in $I_0$, it follows that $h$ is continuously differential with respect to $u$ in $I$. Therefore, $h$ satisfies (1.6) in the differential sense in $I_0$ and satisfies the initial condition.

Now we prove the uniqueness. Similar to [2], we define
\[
\Delta(u) = \sup_{r \geq 0} \left\{ (1 + r)^{1+\epsilon} |h_1 - h_2| \right\}.
\]
Let $D_1, D_2$ are the $D$-operator corresponding to the solutions $h_1, h_2$ respectively. We have

$$
D_1(h_1 - h_2) = D_1 h_1 - D_2 h_2 + \frac{1}{2}(\bar{g}_1 - \bar{g}_2) \frac{\partial h_2}{\partial r}
$$

$$
= \frac{1}{2r} (g_1 - \bar{g}_1 - g_2 + \bar{g}_2)(h_2 - \bar{h}_2) + \frac{1}{2r} (g_1 - \bar{g}_1)(h_1 - \bar{h}_1 - h_2 + \bar{h}_2) + \frac{1}{2} (\bar{g}_1 - \bar{g}_2) \frac{\partial h_2}{\partial r}.
$$

From the proofs of Lemma 2.1, Lemma 2.2, we can derive, for $i = 1, 2$,

$$
|h_i - \bar{h}_i)(u, r)| \leq \frac{cc_0r}{(1 + r)^2}, \quad |(g_i - \bar{g}_i)(u, r)| \leq \frac{4\pi c_2 c_0^2}{3(1 + r)^2}.
$$

Therefore we obtain

$$
|h_1 - \bar{h}_1 + h_2 - \bar{h}_2| \leq |h_1 - \bar{h}_1| + |h_2 - \bar{h}_2| \leq \frac{2cc_0r}{(1 + r)^2},
$$

$$
|h_1 - \bar{h}_1 - h_2 + \bar{h}_2| \leq \frac{\Delta}{(1 + r)^{1+\epsilon}} + \frac{2\Delta}{\epsilon (1 + r)} \leq \frac{3\Delta}{\epsilon (1 + r)}.
$$

Multiply them, we obtain

$$
|(h_1 - \bar{h}_1)^2 - (h_2 - \bar{h}_2)^2| \leq \frac{6cc_0\Delta r}{\epsilon (1 + r)^3}.
$$

Therefore

$$
|g_1 - g_2| \leq 4\pi \left| \int_r^\infty (h_1 - \bar{h}_1)^2 \frac{dr'}{r'} - \int_r^\infty (h_2 - \bar{h}_2)^2 \frac{dr'}{r'} \right|
$$

$$
\leq \frac{24\pi cc_0 \Delta}{\epsilon} \int_r^\infty \frac{dr}{(1 + r)^3} \leq \frac{12\pi cc_0 \Delta}{\epsilon (1 + r)^2}.
$$

And

$$
|\bar{g}_1 - \bar{g}_2| \leq \frac{1}{r} \int_0^r |g_1 - g_2| dr \leq \frac{12\pi cc_0 \Delta}{\epsilon (1 + r)}.
$$

The above two inequality give

$$
|g_1 - \bar{g}_1 - g_2 - \bar{g}_2| \leq |g_1 - g_2| + |\bar{g}_1 - \bar{g}_2| \leq \frac{24\pi cc_0 \Delta}{\epsilon (1 + r)}.
$$

Finally, we obtain

$$
|D_1(h_1 - h_2)| \leq \frac{20\pi c_2 c_0^2 \Delta}{\epsilon (1 + r)^{1+\epsilon}}.
$$

Integrating it along the characteristic $\chi_1$ which intersects the line $u = u_1$ at $r = r_1$ and using the initial condition

$$
(h_1 - h_2)(0, r) = 0,
$$

we get
we obtain
$$\Delta(u_1) \leq \frac{20\pi c^2 r_0^2}{\epsilon} \int_0^{u_1} \Delta(u) du.$$ 

By the Gronwall inequality, we have
$$\Delta(u) = 0$$ 

for any $u \leq u_1 \in [0, u_0]$. Thus the uniqueness holds and the proof of the theorem is complete. Q.E.D.

4. Global existence and uniqueness for small initial data

In this section, we prove the global existence and uniqueness for small initial data.

**Theorem 4.1.** Consider initial data $\tilde{h}(r) \in C^1[0, \infty)$ which satisfies (1.12). Denote
$$d_0 := \inf_{b > 0} \sup_{r \geq 0} \left\{ \left( 1 + \frac{r}{b} \right)^{1+\epsilon} \left( |h_0(r)| + \left| b \frac{\partial h_0}{\partial r}(r) \right| \right) \right\}.$$ 

Then there exists $\delta > 0$ such that if $d_0 < \delta$, there exists a unique global classical solution
$$h(u, r) \in C^1([0, \infty) \times [0, \infty))$$

of (1.6) which satisfies the initial condition $h(0, r) = \tilde{h}(r)$ and the decay property
$$|h(u, r)| \leq \frac{C}{\left( 1 + \frac{r}{2} + r \right)^{1+\epsilon}}, \quad \left| \frac{\partial h}{\partial r}(u, r) \right| \leq \frac{C}{\left( 1 + \frac{r}{2} + r \right)^{1+\epsilon}}$$

for some constant $C$ depending on $\epsilon$ only. Moreover, the corresponding spacetime is future causally geodesically complete with vanishing final Bondi mass $M_{B1}$.

**Proof:** Let $u_1 = \infty$, $t = 1$ in Lemma 2.1 and Lemma 2.2. It is clearly that $h_0 \in X_{0,1}$ where $h_0$ given by (2.2). Suppose
$$\|h_0\|_{X_{0,1}} = d.$$ 

Let $\{h_n\}$ be the sequences constructed by (2.3) and $h_n$ satisfies
$$\|h_n\|_{X_{\infty,1}} \leq x$$

for some $x > 0$. Then, by (2.7), we have
$$\|h_{n+1}\|_{X_{\infty,1}} \leq F_1(\infty, x).$$

Now we define a function
$$G(x) = \frac{x}{2 + \frac{8\pi c^2 x^2}{\epsilon}} \exp \left( - \frac{4\epsilon + 4\epsilon^2 + 8\pi c^2 x^2}{3\epsilon} \right) - \frac{24 + 6\epsilon + 4\pi c^3 x^3}{3\epsilon^2 c^3 x^3}$$
for \( x \geq 0 \). It is easy to see that
\[
G(x) \leq \frac{x}{2},
\]
and
\[
F_1(\infty, x) \leq x \iff G(x) \geq d.
\]
The direct computation shows that
\[
G(0) = 0, \quad G'(0) = \frac{1}{2}, \quad G(\infty) = -\infty, \quad G'(\infty) = -\infty.
\]
Therefore there exists some \( x_1 > 0 \) such that \( G(x) \) attains its maximum value \( G_0 > 0 \) at \( x_1 > 0 \), and \( G(x) \) is strictly monotonically increasing in \([0, x_1]\).

Choose \( d < G_0 \). Then there exists \( x_0 > 0 \) such that
\[
d = G(x_0) < \frac{x_0}{2}.
\]
Thus we obtain a sequence \( \{h_n\} \) such that, for any \( x \in [x_0, x_1] \),
\[
\|h_n\|_{\infty, 1} \leq x.
\]
For this \( x \), Lemma 2.2 gives
\[
\|(h_{n+1} - h_n)\|_{Y_{\infty, 1}} \leq C_1(\infty, x)\|h_n - h_{n-1}\|_{Y_{\infty, 1}},
\]
where
\[
C_1(\infty, x) = \frac{32\pi(\epsilon + 1)}{\epsilon^3}e^2x^2(3 + 4\pi cx)\exp \left( \frac{2\epsilon + 2\epsilon^2 + 4}{3\epsilon} \pi c^2x^2 \right).
\]
As \( C_1(\infty, x) \) is strictly monotonically increasing for \( x \geq 0 \) and \( C_1(\infty, 0) = 0 \), we can find \( x_2 > 0 \) such that
\[
C_1(\infty, x_2) = \frac{1}{2}.
\]
Thus, for any \( x \in (0, x_2) \),
\[
\|h_{n+1} - h_n\|_{Y_{\infty, 1}} < \frac{1}{2}\|h_n - h_{n-1}\|_{Y_{\infty, 1}}.
\]

Now define
\[
\delta = \max_{x \in [0, x_2]} G(x) > 0.
\]
If \( d < \delta \), then we can find \( x < x_2 \) such that
\[
G(x) \geq d.
\]
Then using the same argument as the proof of Theorem 1.6, we can find \( h \) such that
\[
h_n \to h, \quad \frac{\partial h_n}{\partial r} \to \frac{\partial h}{\partial r}
\]
uniformly in \( I_0 \). Moreover, \( h \) is a global classical solution of (1.6).
Now let \( h(0, r) = \tilde{h}(r) \) be the initial data \( \tilde{h}(r) \) with \( d_0 < \delta \). Then there exists \( b > 0 \) such that
\[
\sup_{r \geq 0} \left\{ \left( 1 + \frac{r}{b} \right)^{1+\epsilon} \left( |\tilde{h}(r)| + \left| b \frac{\partial \tilde{h}}{\partial r}(r) \right| \right) \right\} < \delta.
\]
Define new initial data
\[
\tilde{h}(0, r) = h(0, br).
\]
Then
\[
\|\tilde{h}(0, r)\|_{X_{\infty, 0}} = \sup_{r \geq 0} \left\{ \left( 1 + \frac{r}{b} \right)^{1+\epsilon} \left( |\tilde{h}(r)| + \left| b \frac{\partial \tilde{h}}{\partial r}(r) \right| \right) \right\} < \delta.
\]
Thus there exists a global classical solution \( \tilde{h}(u, r) \) of (1.6) in \( I_0 \) with initial data \( \tilde{h}(0, r) \). By scaling group invariance of (1.6) \cite{2},
\[
h(u, r) = \tilde{h} \left( \frac{u}{a}, \frac{r}{a} \right)
\]
is also a global classical solution of (1.6) in \( I_0 \) with initial data \( h(0, r) \). Furthermore, \( h \) satisfies
\[
|h(u, r)| \leq \frac{C}{\left( 1 + \frac{u}{\frac{\pi}{2}} + r \right)^{1+\epsilon}}, \quad \left| \frac{\partial h}{\partial r}(u, r) \right| \leq \frac{C}{\left( 1 + \frac{u}{\frac{\pi}{2}} + r \right)^{1+\epsilon}},
\]
for some constant \( C \).

Next, using the similar argument as the proof of Lemma 2.1, we can show
\[
|h - \overline{h}| \leq \frac{C_1 r}{\left( 1 + \frac{u}{\frac{\pi}{2}} \right)^{\epsilon-1} \left( 1 + \frac{u}{\frac{\pi}{2}} + r \right)^2}, \quad (4.1)
\]
for some constant \( C_1 = C_1(C, \epsilon) \). This implies that
\[
\int_r^\infty (h - \overline{h})^2 \frac{dr'}{r'} \leq \frac{C_1^2}{2 \left( 1 + \frac{u}{\frac{\pi}{2}} \right)^{2\epsilon-2} \left( 1 + \frac{u}{\frac{\pi}{2}} + r \right)^2}.
\]
Therefore
\[
g(u, r) = e^{2\beta} \geq \exp \left[ - \frac{2\pi C_1^2}{\left( 1 + \frac{u}{\frac{\pi}{2}} \right)^{2\epsilon-2} \left( 1 + \frac{u}{\frac{\pi}{2}} + r \right)^2} \right].
\]
Using the inequality \( e^{-x} \geq 1 - x \) for \( x \geq 0 \), we obtain
\[
0 \leq 1 - g(u, r) \leq \frac{2\pi C_1^2}{\left( 1 + \frac{u}{\frac{\pi}{2}} \right)^{2\epsilon-2} \left( 1 + \frac{u}{\frac{\pi}{2}} + r \right)^2}.
\]
Thus,

\[
2 \pi C^2 r \leq \frac{2 \pi C^2 r}{\left(1 + \frac{u}{2}\right)^{2x-2} \left(1 + \frac{u}{2} + r\right)^2} \leq \frac{2 \pi C^2}{\left(1 + \frac{u}{2}\right)^{2x-2} \left(1 + \frac{u}{2} + r\right)}.
\]

This implies

\[
\lim_{r \to \infty} r(1 - g) = 0.
\]

Therefore

\[
\lim_{r \to \infty} g = 1, \quad \lim_{r \to \infty} g^{-1} = 1.
\]

From (1.8), (1.9), we obtain

\[
m(u, r) = \frac{1}{g} \left[ \frac{r}{2}(1 - \bar{g}) - \frac{r}{2}(1 - g) \right] = \frac{1}{g} \left[ m_B(u) - \frac{r}{2}(1 - g) \right].
\]

In terms of (4.2), we obtain

\[
M(u) = \lim_{r \to \infty} m(u, r) = \lim_{r \to \infty} m_B(u, r) = M_B(u).
\]

Thus, for wave-like decaying solutions, the Bondi-Christodoulou mass is equal to the Bondi mass.

Now similar to the proof of Lemma 2.1 we can show

\[
|h| \leq \frac{1}{r} \int_0^r h dr' \leq \frac{2C}{\epsilon \left(1 + \frac{u}{2}\right)^x \left(1 + \frac{u}{2} + r\right)}.
\]

Then

\[
|\bar{h} - h| \leq |h| + |\bar{h}| \leq \frac{C'}{\epsilon \left(1 + \frac{u}{2}\right)^x \left(1 + \frac{u}{2} + r\right)},
\]

where \(C' = \frac{2C}{\epsilon} + C\). From (1.10), we obtain

\[
0 \leq m(u, r) \leq \frac{2 \pi C'^2}{\epsilon^2 \left(1 + \frac{u}{2}\right)^{2x} \left(1 + \frac{u}{2} + r\right)^2} \int_0^r \frac{dr'}{\left(1 + \frac{u}{2} + r'\right)^2} = \frac{2 \pi C'^2 r}{\epsilon^2 \left(1 + \frac{u}{2}\right)^{2x+1} \left(1 + \frac{u}{2} + r\right)}.
\]

Let \(r \to \infty\), we get

\[
0 \leq M_B(u) = M(u) = \lim_{r \to \infty} m(u, r) \leq \frac{2 \pi C'^2}{\epsilon^2 \left(1 + \frac{u}{2}\right)^{2x+1}}.
\]
This implies that the final Bondi mass
\[ M_{B1} = \lim_{u \to \infty} M_B(u) = 0. \]
The uniqueness can be proved as same as Theorem 1.6.

Finally, we adopt the argument in [10] to prove the completeness of causal geodesics of the global classical solution. The following lemmas are indeed Lemma 6.1-Lemma 6.11 in [10]. We provide them here as there are slight differences on the proofs because of using different coordinate systems.

Let \( I \) be an interval and \( \gamma : I \to \mathcal{M} \) be a future pointing causal geodesic on the above spacetime \( \mathcal{M} \). For any function \( f \) on \( \mathcal{M} \), its restriction on \( \gamma \) and its derivatives are
\[
f(s) = f(\gamma(s)), \quad \dot{f}(s) = \frac{d}{ds}f(s), \quad \ddot{f}(s) = \frac{d^2}{ds^2}f(s).
\]
Under the Bondi-Sachs coordinate system, we can write
\[
\gamma(s) = (u(s), r(s), \theta(s), \psi(s)), \quad \dot{\gamma}(s) = (\dot{u}(s), \dot{r}(s), \dot{\theta}(s), \dot{\psi}(s)).
\]
Note that these are only defined away from the axis \( \Gamma = \{r = 0\} \). Same as in [10], we denote
\[
C^2 = -g_{\alpha\beta}\dot{\gamma}^\alpha(s)\dot{\gamma}^\beta(s),
\]
\[
J^2 = r^4(g_{S^2,\theta\theta}\dot{\theta}^2 + g_{S^2,\psi\psi}\dot{\psi}^2).
\]
As pointed out in [10], \( C^2, J^2 \) are conserved and \( C^2 > 0 \) when \( \gamma(s) \) is a time-like geodesic. Let
\[
U = \bar{g}\dot{u} + 2\dot{r}
\]
Then
\[
g\dot{u}U = C^2 + r^{-2}J^2. \tag{4.4}
\]
For future pointing causal geodesics,
\[
\dot{u} \geq 0, \quad U \geq 0.
\]
Note that the line \( r = 0 \) is complete. This is because
\[
g, \bar{g} \geq k > 0 \implies \int_0^u (\bar{g}\dot{u})^{\frac{1}{2}}(u,0)du \geq ku \to \infty
\]
as \( u \to \infty \). Moreover, let
\[
E(s) = g\dot{u}^2 + g\dot{r} = \frac{1}{2}(g\dot{u}^2 + gU). \tag{4.5}
\]
It is clearly that, away from the axis \( \Gamma \), \( E(s) \) is nonnegative. Thus Lemma 6.1, Lemma 6.2, Lemma 6.3 and Lemma 6.4 in [10] hold true also in the current case. They are
Lemma 4.1. Any future pointing causal geodesic $\gamma : [0, s_f) \to \mathcal{M}$ can be continued past $s_f$ if there exists a compact subset $K \subseteq \mathcal{M}$ such that

$$\{\gamma(s) \in \mathcal{M} : s \in [0, s_f)\} \subseteq K.$$  

Lemma 4.2. If $\gamma(s) : [0, s_f) \to \mathcal{M}$ is incomplete, then either $C \neq 0$ or $J \neq 0$.

Lemma 4.3. If $\gamma : [0, s_f) \to \mathcal{M}$ is incomplete, then set $\{s : \gamma(s) = 0\}$ is a discrete subset of $[0, s_f)$ (with a possible accumulation point at $s_f$).

Lemma 4.4. Let $\gamma : [0, s_f) \to \mathcal{M}$ be a future geodesic with either $C \neq 0$ or $J \neq 0$, then $E(s) > 0$ for $s \in [0, s_f)$.

Now we derive the equations of each parameter of geodesic equations.

Lemma 4.5. If geodesic $\gamma$ lies outside the axis $\Gamma$, then

$$\ddot{u}(s) = \left[ -g^{-1}g_{u} + \frac{1}{2}g^{-1}(\tilde{g}\tilde{g})_r \right] \dot{u}^2 - g^{-1}r^{-3}J^2,$$

$$\ddot{r}(s) = g^{-1}\left[ -\frac{1}{2}(\tilde{g}\tilde{g})_u + \tilde{g}g_{u} - \frac{1}{2}\tilde{g}(\tilde{g}\tilde{g})_r \right] \dot{r}^2 - g^{-1}\left[(\tilde{g}\tilde{g})_r \dot{u} \dot{r} + g_r r^2 - \tilde{g}r^{-3}J^2\right],$$

$$\ddot{U}(s) = -\frac{1}{2}g^{-1}(\tilde{g}\tilde{g})_u \dot{u} \dot{U} - g^{-1}g_{u}U \ddot{r} + \frac{1}{2}g^{-1}\tilde{g}r^{-3}J^2,$$

$$\ddot{E}(s) = \frac{1}{2}(\tilde{g}\tilde{g})_u \dot{u}^2 + g_{a} \dot{u} \dot{r} = \frac{1}{2}\left(g_{a} \dot{u} U + \tilde{g}g_{a} \dot{u}^2\right),$$

where $f_u, f_r$ are the partial derivatives of $f$ with respect to $u$ and $r$.

Proof: Substitute the Christoffel symbols in Appendix into the geodesic equations

$$\dddot{\gamma}^\lambda = -\Gamma^\lambda_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta,$$

we can derive the first two equations. They imply

$$\ddot{U}(s) = g_{u} \dot{u}^2 + \tilde{g} \dot{u} \dot{r} + \tilde{g} \ddot{u} + 2\ddot{r}$$

$$= -\frac{1}{2}g^{-1}(\tilde{g}\tilde{g})_u \dot{u} \dot{U} - g^{-1}g_{u}U \ddot{r} + \frac{1}{2}g^{-1}\tilde{g}r^{-3}J^2,$$

$$\ddot{E}(s) = \tilde{g} \ddot{u} + g_{a} \dot{u} \dot{r} + gg_{u} + \tilde{g} \ddot{r} + g\dot{r}$$

$$= (g_{a} \dot{u} + g_r \dot{r}) \ddot{u} + g(\tilde{g}_{u} \dot{u} + \tilde{g}_r \dot{r}) \ddot{u} + \ddot{g} \ddot{u}$$

$$+ (g_{a} \dot{u} + g_r \dot{r}) \ddot{r} + \ddot{g} \ddot{r}$$

$$= \frac{1}{2}(\tilde{g}\tilde{g})_u \dot{u}^2 + g_{a} \dot{u} \dot{r}$$

$$= \frac{1}{2}\left(g_{a} \dot{u} U + \tilde{g}g_{a} \dot{u}^2\right).$$

Q.E.D.
Lemma 4.6. If $\gamma(s): [0, s_f) \to \mathcal{M}$ is incomplete with $C \neq 0$ or $J \neq 0$, then there exists some constant $C > 0$ such that, for any $s \in [0, s_f)$,

$$E(s) \geq \frac{C}{s_f - s}. \quad (4.7)$$

Proof: Since

$$1 \geq g, \quad \bar{g} \geq k > 0,$$

we have

$$\dot{u} + U \leq \frac{2}{k^2} E(s).$$

In the following we denote $C_1$, $C_2$ and $C_3$ are certain positive constants. As the solution and its derivative satisfy

$$|h_u|, \ |\bar{h}_u| \leq C_1.$$

Thus, using (4.1), we obtain

$$|g_u| \leq 4\pi |g| \int_r^{\infty} 2|h - \bar{h}| |h_u - \bar{h}_u| \frac{dr'}{r} \leq C_2,$$

$$|\bar{g}_u| \leq \frac{1}{r} \int_0^r |g_u| dr' \leq C_2.$$

Therefore

$$\dot{E}(s) \leq 2C_3 (\dot{u}^2 + U^2) \leq \frac{8C_3}{k^4} E^2(s).$$

Then the lemma follows by applying the same argument as proving Lemma 6.6 in [10]. Q.E.D.

Lemma 4.7. If $\gamma(s): [0, s_f) \to \mathcal{M}$ is incomplete, then for any $r_0 > 0$ and any $s_0 \in [0, s_f)$, there exists $s \in [s_0, s_f)$ such that $r(s) < r_0$.

Proof: We prove it by contradiction. If there exist some constant $r_0 > 0$ and $s_0 \in [0, s_f)$ such that $r(s) \geq r_0$ for all $s \in [s_0, s_f)$, then Lemma 4.5 gives

$$\frac{d}{ds} (gU) = (g\bar{g})_u \dot{u}^2 + (g\bar{g})_r \ddot{r} + (g\bar{g}) \ddot{u} + 2g_u \dot{u} \dot{r} + 2g_r \ddot{r}^2 + 2g_r \dot{r}$$

$$= \left[(g\bar{g})_u - (g\bar{g}) g^{-1} g_u + \frac{1}{2} (g\bar{g}) g^{-1}(g\bar{g})_r - (g\bar{g})_u + 2\bar{g} g_u \right] \ddot{u} + \left[2g_u - (g\bar{g})_r \right] \dddot{r} + \bar{g} r^{-3} J^2$$

$$= \left[g_u - \frac{1}{2} (g\bar{g})_r \right] \left(C^2 + r^{-2} J^2 \right) + \bar{g} r^{-3} J^2.$$

As $C^2$ and $J^2$ are conserved, $g, \bar{g}$ and their derivatives are uniformly bounded, there exists constant $C_4 > 0$ such that

$$C_4 \geq U \geq 0.$$
This gives
\[ C_4 - k\dot{u} - 2\dot{r} \geq 0. \]
Integrating it from \( s_0 \) to \( s \), we obtain
\[ C_4(s - s_0) - ku(s) + ku(s_0) - 2r(s) + 2r(s_0) \geq 0. \]
Thus,
\[ u(s) \leq \frac{1}{k} \left[ C_4(s_f - s_0) + ku(s_0) - 2r_0 + 2r(s_0) \right]. \]
As \( \dot{u} \geq 0 \), we conclude that \( u \) is uniformly bounded. Similarly, \( r \) is also uniformly bounded. Thus \( \gamma(s) \) lies in a compact set in \( \mathcal{M} \). This contradicts Lemma 4.1 and the proof of lemma is complete. Q.E.D.

The following lemma and its proof are the same as Lemma 6.8 in [10].

**Lemma 4.8.** Assume \( \gamma(s) : [0, s_f) \to \mathcal{M} \) is incomplete. Then \( J \neq 0 \).

**Lemma 4.9.** There exists \( r_0 > 0 \) such that if \( J \neq 0 \) and at some time \( s_0 \),
\[ \dot{u}(s_0), U(s_0) \leq \frac{2}{kr(s_0)} J, \quad \dot{r}(s_0) \leq 0, \quad r(s_0) < r_0, \]
where \( k \) is defined by (2.12), then
\[ \ddot{u}(s_0) < 0, \quad \dot{U}(s_0) > 0. \]
Moreover, if \( \dot{r}(s_0) = 0 \), then
\[ \ddot{r}(s_0) > 0. \]

**Proof:** It is straightforward that
\[ |g_r|, |g_u|, |\bar{g}_r|, |\bar{g}_u| \leq C_5 \]
for some constant \( C_5 > 0 \). Since both \( C^2 \) and \( J^2 \) are conserved, there exists \( \ddot{r} \) such that for any \( 0 < r \leq \ddot{r} \)
\[ C^2 + \frac{1}{r^2} J^2 \leq \frac{2}{r^2} J^2. \]
Take
\[ r_0 = \min \left\{ \frac{k^4}{16C_5^2}, \ddot{r} \right\}. \]
Then (4.6) gives
\[ g\ddot{u}(s_0) = \left[ -g_u + \frac{1}{2}(gg)_{\dot{r}} \right] \ddot{u}^2 - r^{-3} J^2 \]
\[ \leq \left( |g_u| + \frac{1}{2} |g||\bar{g}_r| + \frac{1}{2} |g_r||\bar{g}| \right) - \frac{4}{k^2r^2(s_0)} J^2 - \frac{1}{r^3(s_0)} J^2 \]
\[ \leq \frac{8C_5}{k^2r^2(s_0)} J^2 - \frac{16C_5}{k^4r^2(s_0)} J^2 \]
\[ \leq -\frac{8C_5}{k^3r^2(s_0)} J^2 < 0. \]
Therefore
\[ \ddot{u}(s_0) < 0. \]

The remaining conclusions can be proved in the same way by using (4.6).

Q.E.D.

Lemma 4.10. Assume \( \gamma(s) : [0, s_f) \to \mathcal{M} \) is incomplete and \( J \neq 0 \). Then there exists \( r_0 > 0 \) such that for every \( s_0 \in [0, s_f) \) the geodesic \( \gamma(s) \) exits the cylinder with radius \( r_0 \) at some time to the future of \( s_0 \), that is, there exists \( s_1 \in (s_0, s_f) \) such that \( r(s_1) > r_0 \).

Proof: Let \( r_0 \) be given in Lemma 4.9. Suppose the geodesic \( \gamma(s) \) lies in the cylinder with radius \( r_0 \) for \( s \in [s_0, s_f) \).

Step 1. We claim that, for any \( s \in [s_0, s_f) \),
\[ \dot{r}(s) \leq 0. \] (4.8)

If there exists \( s' \in [s_0, s_f) \) such that
\[ \dot{r}(s') > 0, \]
then Lemma 4.7 implies that there exists \( s'' > s' \) such that
\[ \dot{r}(s'') < 0. \]

Take
\[ s^* = \sup \left\{ s : s' \leq s \leq s'', \dot{r}(s) \geq 0 \right\}. \]

Then
\[ \dot{r}(s^*) = 0 \]
and, for \( s^* < s \leq s'' \),
\[ \dot{r}(s) < 0. \]

Thus (4.8) implies
\[ k\ddot{u}(s^*) \leq U(s^*) = \ddot{u}(s^*) \]
and (4.3) implies
\[ \frac{k^2}{2} \leq g\ddot{u}^2(s^*) \leq C^2 + \frac{1}{r^2(s^*)} J^2 \leq \frac{2}{r^2(s^*)} J^2. \]

Therefore, we obtain
\[ U(s^*) \leq \dot{u}(s^*) \leq \frac{2}{kr(s^*)} J. \]

By Lemma 4.9 we have
\[ \ddot{r}(s^*) > 0 \implies \dot{r}(s) > 0, \quad s > s^*. \]
This gives contradiction. Hence (4.8) holds.

**Step 2.** We show that there exists \( t_0 \in (s_0, s_f) \) such that
\[
\frac{k}{2} \dot{u}(t_0) \leq U(t_0).
\] (4.9)
If not, for all \( s \in (s_0, s_f) \)
\[
\frac{k}{2} \dot{u} > U,
\]
then
\[
\dot{r}(s) = \frac{1}{2} (U - \bar{g} \dot{u}) \leq -\frac{k}{4} \dot{u}(s).
\]
Integrating it from \( s_0 \) to \( s \), we obtain that \( u \) is uniformly bounded, which violates Lemma 4.1. Hence (4.9) holds.

**Step 3.** Let \( t_0 \) be given in (4.9), we claim that, for \( s \in [t_0, s_f) \),
\[
\frac{k}{2} \dot{u}(s) \leq U(s).
\] (4.10)
Define
\[
s^* = \sup \left\{ s : \frac{k}{2} \dot{u}(s') \leq U(s'), \ \forall s' \in [t_0, s] \right\}.
\]
If \( s^* = s_f \), then (4.10) holds. If not, by continuity,
\[
\frac{k}{2} \dot{u}(s^*) \leq U(s^*).
\]
Since
\[
\dot{r}(s) \leq 0,
\]
we have
\[
U(s^*) \leq \dot{u}(s^*).
\]
Similar to **Step 1**, we can show that \( \dot{u}(s^*) \) and \( U(s^*) \) satisfy the conditions in Lemma 4.9. Then
\[
\dot{u}(s^*) < 0, \quad \dot{U}(s^*) > 0.
\]
Thus, there exists \( t_1 > s^* \) such that, for all \( s \in [s^*, t_1] \),
\[
\frac{k}{2} \dot{u}(s) \leq \frac{k}{2} \dot{u}(s^*) \leq U(s^*) \leq U(s).
\]
This contradicts the definition of \( s^* \). Hence (4.10) holds.

**Step 4.** The above arguments and Lemma 4.9 imply that, for all \( s \in [t_0, s_f) \),
\[
\dot{u}(s) < 0, \quad U(s) \leq \dot{u}(s).
\]
Therefore \( \dot{u} \) and \( U(s) \) are uniformly bounded. This contradicts Lemma 4.1. Thus the proof of lemma is complete. Q.E.D.
Lemma 4.11. Assume $\gamma(s) : [0, s_f) \to \mathcal{M}$ is incomplete and $r(s) > 0$ for all $s \in [0, s_f)$. Suppose there exists a sequence $\{s_n\}$ with $s_n \to s_f$ such that $\dot{r}(s_n) = 0$. Then

$$\lim_{n \to \infty} (\dot{u}U)(s_n) = \infty.$$ 

Proof: By (4.3), we have

$$k\dot{u}(s_n) \leq U(s_n) \leq \dot{u}(s_n).$$

Then (4.5) gives

$$E(s_n) \lesssim \sqrt{(\dot{u}U)(s_n)}.$$ 

Therefore the conclusion follows from Lemma 4.6 and the proof of lemma is complete. Q.E.D.

Now we prove future geodesic completeness. If $\gamma(s)$ is incomplete, then Lemma 4.8 indicates $\mathbf{J} \neq 0$. Take $r_0$ sufficiently small such that Lemma 4.10 holds. Lemma 4.7 and Lemma 4.10 imply that $\gamma(s)$ intersects the cylinder $r = r_0$ infinitely many times. Thus we can find a sequence $s_n \to s_f$ satisfying the conditions in Lemma 4.11. As (4.4) shows that $\dot{u}(s_n)$ and $U(s_n)$ are uniformly bounded. This contradicts Lemma 4.11. So the proof of theorem is complete. Q.E.D.

Remark 4.1. The past causal geodesics are also complete by reversing the time-orientation.

5. Generalized solutions for large data

In this section, we generalize Christodoulou’s generalized solutions of (1.6) to the wave-like decaying condition (1.12). As most proofs in [3, 4] can go through without any change, we only provide the proof of Lemma 5.1.

The $\alpha$-regularized equation is given as follows

$$D_{\alpha}h_{\alpha} = \frac{1}{2(r + \alpha)}(g_{\alpha} - \bar{g}_{\alpha})(h_{\alpha} - \bar{h}_{\alpha})$$

where

$$\bar{h}_{\alpha} = \frac{1}{r + \alpha} \int_0^r h_{\alpha} dr',$$

$$g_{\alpha} = \exp \left[ -4\pi \int_0^\infty (h_{\alpha} - \bar{h}_{\alpha})^2 \frac{dr'}{r'} \right],$$

$$\bar{g}_{\alpha} = \frac{1}{r + \alpha} \int_0^r g_{\alpha} dr'.$$

Clearly,

$$\bar{g}_{\alpha}(u, 0) = 0.$$
Denote the differential operator
\[ D_\alpha = \frac{\partial}{\partial u} - \frac{1}{2} \bar{g}_\alpha \frac{\partial}{\partial r}. \]
The integral curve of \( D_\alpha \), denoted by \( r = \chi_\alpha \), satisfies the ODE
\[ \frac{dr}{du} = -\frac{1}{2} \bar{g}_\alpha. \]
Define the \( \alpha \)-local mass function
\[ m_\alpha = \frac{r + \alpha}{2} \left( 1 - \frac{\bar{g}_\alpha}{g_\alpha} \right). \]
From [3], we know that \( m_\alpha \) is a monotonically nondecreasing function with respect to \( r \) and we can derive
\[ m_\alpha = \frac{\alpha}{2} + 2\pi \int_0^r \frac{\bar{g}_\alpha}{g_\alpha} (h_\alpha - \bar{h}_\alpha)^2 dr'. \]
Then the \( \alpha \)-total mass \( M_\alpha(u) \) can be defined [3]
\[ M_\alpha(u) = \lim_{r \to \infty} m_\alpha(u, r). \]

It is straightforward that
\[ D_\alpha \bar{h}_\alpha = \frac{\xi_\alpha}{2(r + \alpha)}, \quad D_\alpha m_\alpha = -\frac{\pi \xi_\alpha}{g_\alpha}, \]
where
\[ \xi_\alpha = \int_0^r \frac{\bar{g}_\alpha (h_\alpha - \bar{h}_\alpha)}{r' + \alpha} dr'. \]
Then the following identity holds
\[ \int_{\delta}^{r_1} \frac{g_\alpha}{\bar{g}_\alpha} (u_1, r) dr + 2\pi \int \int_{I_{\delta,\alpha}(u_1, r_1)} \frac{g_\alpha}{\bar{g}_\alpha} \frac{\xi_\alpha^2}{r + \alpha} dr du \\ + \frac{1}{2} \int_0^{u_1} g_\alpha(u, \delta) du = \int_{\delta}^{r_0,\alpha} \frac{g_\alpha}{\bar{g}_\alpha} (0, r) dr, \]
where \( r_{0,\alpha} = \chi_{\alpha,u_1}(0; r_1) \),
\[ I_{\delta,\alpha}(u_1, r_1) = \{ (u, r) | 0 < u < u_1, \delta < r < \chi_{\alpha,u_1}(u; r_1) \}. \]

**Lemma 5.1.** For any initial data \( \bar{h}(r) \in C^1[0, \infty) \) which satisfies (1.12), then, for any \( \alpha > 0 \), there exists unique global classical solution
\[ h_\alpha(u, r) \in C^1([0, \infty) \times [0, \infty)) \]
of (5.1) which satisfies the initial condition \( h(0, r) = \bar{h}(r) \) and the decay property (1.12) at each \( u \geq 0 \).
Proof: For any $\alpha > 0$, the existence of unique global classical solution can be proved by the same argument as in \[3\]. In the following we prove that the solution preserves the wave-like decay at null infinity.

Let $\chi_\alpha(u; r_1)$ be the characteristic through $(u_1, r_1)$ and $r_{0,\alpha} = \chi_\alpha(0; r_1)$. Integrating (5.1) along $\chi_\alpha(u; r_1)$, we obtain

$$h_\alpha(u_1, r_1) = h(0, r_{0,\alpha}) + \int_0^{u_1} \left[ \frac{g_\alpha - \bar{g}_\alpha}{2(r + \alpha)} (h_\alpha - \bar{h}_\alpha) \right] \chi_\alpha du. \quad (5.2)$$

Let $M_{0,\alpha} = M_\alpha(0)$, and

$$x(u) = \sup_{r \geq 4M_{0,\alpha}} \left| \left( \frac{r}{4M_{0,\alpha}} \right)^{1+\epsilon} |h_\alpha(u, r)| \right|. \quad (5.3)$$

As pointed out in \[3\], for any differentiable function $f$ such that

$$\left\{ \begin{array}{l} f, \frac{\partial f}{\partial r} \in L^2(0, \infty) \\
\lim_{r \to \infty} rf^2(r) = 0 \Rightarrow r_1 f^2(r_1) \leq \int_{r_1}^\infty r^2 \left( \frac{\partial f}{\partial r} \right)^2 dr.
\end{array} \right. \quad (5.4)$$

Taking $r_1 = 4M_{0,\alpha}$ and $f = \bar{h}_\alpha$ in the inequality, we obtain

$$4M_{0,\alpha} \bar{h}_\alpha^2(4M_{0,\alpha}) \leq \int_{4M_{0,\alpha}}^\infty (r + \alpha)^2 \left( \frac{\partial \bar{h}_\alpha}{\partial r} \right)^2 dr = \int_{4M_{0,\alpha}}^\infty (h_\alpha - \bar{h}_\alpha)^2 dr \leq \frac{M_{0,\alpha}}{\pi}. \quad (5.5)$$

This gives

$$\bar{h}_\alpha^2(4M_{0,\alpha}) \leq \frac{1}{2\pi^2}. \quad (5.6)$$

Thus, for $r \geq 4M_{0,\alpha}$,

$$|\bar{h}_\alpha(r)| = \frac{1}{r + \alpha} \left( 4M_{0,\alpha} + \alpha \right) \bar{h}_\alpha(4M_{0,\alpha}) + \int_{4M_{0,\alpha}}^r h_\alpha(r') dr' \leq \frac{1}{r + \alpha} \left[ 4M_{0,\alpha} + \frac{\alpha}{2\pi^2} + \int_{4M_{0,\alpha}}^r x \left( \frac{4M_{0,\alpha}}{r'} \right)^{1+\epsilon} dr' \right] \quad (5.7)$$

$$\leq \left( 1 + \frac{\alpha}{4M_{0,\alpha}} \right) \frac{1}{2\pi^2} + \frac{x}{\epsilon} \frac{4M_{0,\alpha}}{r}. \quad (5.8)$$

On the other hand,

$$\frac{g_\alpha - \bar{g}_\alpha}{2(r + \alpha)} = \frac{m_\alpha}{(r + \alpha)^2} g_\alpha \leq \frac{M_{0,\alpha}}{r^2}. \quad (5.9)$$
Thus (5.2), (5.3) and (5.4) imply that
\[
x(u) \leq \sup_{r \geq 4M_0,\alpha} \left[ \left( \frac{r}{4M_0,\alpha} \right)^{1+\epsilon} \left| h_0(r) \right| \right] \\
+ \frac{1}{8\pi^2} \left( 1 + \frac{\alpha}{4M_0,\alpha} \right) \frac{u}{u'} \\
+ \frac{\epsilon + 1}{16\epsilon M_0,\alpha} \int_0^u x(u')du' \\
\leq e^{\tau} \sup_{r \geq 4M_0,\alpha} \left[ \left( \frac{r}{4M_0,\alpha} \right)^{1+\epsilon} \left| h_0(r) \right| \right] \\
+ \frac{e^{\tau}}{8\pi^2} \left( 1 + \frac{\alpha}{4M_0,\alpha} \right) \frac{u}{u'}
\]
where
\[
\tau = \frac{(1 + \epsilon)u}{16\epsilon M_0,\alpha}
\]
This implies the wave-like decaying for \( h_\alpha \).

Next we show the wave-like decaying for \( \frac{\partial h_\alpha}{\partial r} \). This can be done by using the same argument to the following identity (c.f. (4.8) in [3]).

\[
\frac{\partial h_\alpha}{\partial r}(u_1, r_1) = \frac{\partial h}{\partial r}(0, r_0,\alpha) + \int_0^{u_1} \left[ \frac{g_\alpha - \bar{g}_\alpha}{r + \alpha} \frac{\partial h_\alpha}{\partial r} \right] \chi_\alpha \]

\[
+ \int_0^{u_1} \left\{ \frac{1}{2(r + \alpha)^2} \left[ -3(g_\alpha - \bar{g}_\alpha) + 4\pi g_\alpha (h_\alpha - \bar{h}_\alpha)^2 \right] (h_\alpha - \bar{h}_\alpha) \right\} \chi_\alpha du.
\]

Therefore, the proof of the lemma is complete. Q.E.D.

**Lemma 5.2.** For any \( u_0, \delta > 0 \), three families of functions
\[
\{ h_\alpha | \alpha \in (0, \frac{\delta}{2}] \}, \quad \{ \bar{h}_\alpha | \alpha \in (0, \frac{\delta}{2}] \}, \quad \{ \frac{\partial h_\alpha}{\partial r} | \alpha \in (0, \frac{\delta}{2}] \}
\]
are equicontinuous in \([0, u_0] \times [\delta, \infty)\).

**Lemma 5.3.** There exists a subsequence \( \{ h_{\alpha_\delta} \} \) of the sequence \( \{ h_\alpha \} \) such that \( \{ h_{\alpha_\delta} \} \) converges to a differentiable continuous function \( h \in C^1(I) \) on each compact subset of \( I = [0, \infty) \times (0, \infty) \) uniformly while the sequence \( \{ \frac{\partial h_{\alpha_\delta}}{\partial r} \} \) converges to \( \frac{\partial h}{\partial r} \) uniformly. Moreover, \( h \) satisfies (1.6) in \( I \) and, for arbitrary \( r_1 > 0 \),
\[
h \in L^2(0, \infty), \quad \frac{g}{g} \in L^1(0, r_1).
\]

**Lemma 5.4.** At each \( u \), the function
\[
\xi(r) = \lim_{\delta \to 0} \int_{r-\delta}^r \frac{\tilde{g}(h - \bar{h})}{r'} dr'
\]
exists. And, for arbitrary \( r_0 > 0 \), the measurable function

\[
\frac{g^{\frac{1}{2}} \xi}{gr^{\frac{1}{2}}} \in L^2\left([0, \infty) \times (0, r_0]\right).
\]

**Lemma 5.5.** At almost all \( u \), \( \frac{\xi}{g^{\frac{1}{2}}} \) is a continuous function of \( r \) and uniformly bounded such that

\[
\lim_{r \to 0} \frac{\xi}{g^{\frac{1}{2}}}(u, r) = 0.
\]

Also, for arbitrary \( u_0 > 0 \),

\[
\sup_{r \geq 0} \left| \frac{\xi}{g^{\frac{1}{2}}}(u, r) \right| \in L^2(0, u_0), \quad \lim_{r \to 0} \int_{0}^{u_0} \frac{\xi^2}{g}(u, r) du = 0.
\]

**Lemma 5.6.** At almost all \( u \),

\[
\lim_{\delta \to 0} \left( \delta DA(\delta) \right) = 0.
\]

**Lemma 5.7.** \( \bar{h} \) and \( m \) are weakly differentiable in \( I \) and

\[
D\bar{h} = \frac{\xi}{2r}, \quad Dm = -\frac{\pi}{g^2} \xi^2.
\]

**Theorem 5.1.** Given initial data \( \tilde{h}(r) \in C^1[0, \infty) \) which satisfies (1.12), there exists at least one global generalized solution which has the same data as a classical solution coincides with it in the domain of existence of the latter.

**Proof:** From the above lemmas, we know that there exists a solution \( h \in C^1(I) \) which satisfies (1)-(4) in Definition 1.1. By (5.41) in [3] we obtain

\[
D\left( \int_{0}^{r} \frac{g}{\bar{g}} dr \right) = -2\pi \int_{0}^{r} \frac{g\xi^2}{rg^2} dr - \frac{1}{2} g(\delta).
\]

By using the dominated convergence theorem and letting \( \delta \to 0 \), we know that \( \int_{0}^{r} \frac{g}{\bar{g}} dr \) is weakly differentiable in \( I \) and satisfies that

\[
D\left( \int_{0}^{r} \frac{g}{\bar{g}} dr \right) = -2\pi \int_{0}^{r} \frac{g\xi^2}{rg^2} dr - \frac{1}{2} g(0).
\]

Thus, integrating (5.5) along the characteristic \( \chi_{u_1}(u; r_1) \), we have

\[
\int_{0}^{r_1} \frac{g}{\bar{g}}(u_1, r) dr + 2\pi \int_{I(u_1, r_1)} \frac{g\xi^2}{g^2 r} dr du + \frac{1}{2} \int_{0}^{u_1} g(u, 0) du = -\int_{0}^{r_0} \frac{g}{\bar{g}}(0, r) dr.
\]

This is indeed (5) in Definition 1.1 so that global generalized solutions exist. The proof of uniqueness is the same as that in [4]. Therefore the proof of the theorem is complete.

Q.E.D.
6. Appendix: Spherically Symmetric Bondi-Sachs Metrics

The nontrivial metric components

\[
g_{uu} = -e^{2\beta} \frac{V}{r}, \quad g_{ur} = -e^{2\beta}, \quad g_{\theta\theta} = r^2, \quad g_{\psi\psi} = r^2 \sin^2 \theta,
\]
\[
g_{ur} = -e^{-2\beta}, \quad g_{rr} = e^{-2\beta} \frac{V}{r}, \quad g_{\theta\theta} = r^{-2}, \quad g_{\psi\psi} = r^{-2} \sin^{-2} \theta.
\]

The nontrivial Christoffel symbols

\[
\Gamma^u_{uu} = 2 \frac{\partial \beta}{\partial u} \frac{V}{r} - \frac{1}{2r} \frac{\partial V}{\partial r} + \frac{V}{2r^2}, \\
\Gamma^u_{\theta\theta} = re^{-2\beta}, \quad \Gamma^u_{\psi\psi} = r \sin^2 \theta e^{-2\beta}, \\
\Gamma^r_{uu} = \frac{V^2}{r^2} \frac{\partial \beta}{\partial r} + \frac{V}{2r^2} \frac{\partial V}{\partial r} - \frac{V^2}{2r^3} - \frac{1}{r} \frac{\partial V}{\partial u} + \frac{V}{2r} \left( \frac{\partial V}{\partial r} \right)^2, \\
\Gamma^r_{ru} = \frac{V}{r} \frac{\partial \beta}{\partial r} + \frac{1}{2r} \frac{\partial V}{\partial r} - \frac{V}{2r^2}, \\
\Gamma^r_{rr} = 2 \frac{\partial \beta}{\partial r}, \quad \Gamma^r_{\theta\psi} = -Ve^{-2\beta} \sin \theta, \quad \Gamma^r_{\theta\theta} = -Ve^{-2\beta}, \\
\Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\psi\psi} = -\sin \theta \cos \theta, \quad \Gamma^\psi_{\psi r} = \frac{1}{r}, \quad \Gamma^\psi_{\theta \psi} = \cot \theta.
\]

The nontrivial components of the Riemann curvature tensors

\[
R^u_{\theta\theta} = -2re^{-2\beta} \frac{\partial \beta}{\partial r}, \\
R^r_{\theta\psi} = e^{-2\beta} \left( \frac{\partial \beta}{\partial r} - \frac{1}{2} \frac{\partial V}{\partial r} + \frac{V}{2r} \right), \\
R^u_{uu} = \frac{2}{r} \frac{\partial^2 \beta}{\partial r \partial u} - \frac{V}{r} \frac{\partial^2 \beta}{\partial r^2} - \frac{V}{r^2} - \frac{1}{2r} \frac{\partial V}{\partial r} + \frac{V}{2r^2} \left( \frac{\partial V}{\partial r} \right)^2, \\
R^r_{ur} = \frac{2V}{r} \frac{\partial^2 \beta}{\partial r \partial u} + \frac{V^2}{r^2} \frac{\partial^2 \beta}{\partial r^2} + \frac{V \delta \beta \delta V}{r^2} - \frac{V^2 \delta \beta \delta V}{r^3} - \frac{V V \delta \beta \delta V}{r^3} - \frac{V^2 \delta \beta \delta V}{r^3} - \frac{V^2}{r^4} + \frac{V^2}{r^4}, \\
R^\theta_{u\theta} = R^\phi_{u\psi} = \frac{V^2}{r^3} \frac{\partial \beta}{\partial r} + \frac{V}{2r^2} \frac{\delta V}{\partial r} - \frac{V \delta \beta \delta V}{2r^2} - \frac{1}{2r^2} \frac{\partial V}{\partial u} - \frac{V^2}{2r^4}, \\
R^\theta_{r\theta} = R^\phi_{r\psi} = \frac{2}{r} \frac{\partial \beta}{\partial r}, \\
R^\theta_{ur} = R^\phi_{ur} = \frac{V}{r^2} \frac{\partial \beta}{\partial r} + \frac{1}{2r^2} \frac{\partial V}{\partial r} - \frac{V}{2r^3}, \\
R^\theta_{\psi\psi} = \sin^2 \theta \left( 1 - \frac{V}{r} e^{-2\beta} \right).
\]
The nontrivial components of the Ricci curvature tensors

\[
R_{uu} = -\frac{2V}{r} \frac{\partial^2 \beta}{\partial u \partial r} + \frac{V^2}{r^2} \frac{\partial^2 \beta}{\partial r^2} + \frac{V}{r^2} \frac{\partial \beta}{\partial r} - \frac{V \partial \beta}{r^2} \frac{\partial V}{\partial r} + \frac{1}{2r} \frac{\partial V}{\partial u},
\]

\[
R_{ur} = -2 \frac{\partial^2 \beta}{\partial u \partial r} + \frac{V}{r} \frac{\partial \beta}{\partial r} - \frac{1}{r} \frac{\partial \beta}{\partial r} \frac{\partial V}{\partial r} + \frac{V \partial \beta}{r^2} \frac{\partial V}{\partial r},
\]

\[
R_{rr} = \frac{4 \partial \beta}{r} \frac{\partial V}{\partial r},
\]

\[
R_{\theta\theta} = \sin^{-2} \theta R_{\psi\psi} = 1 - e^{-2\beta} \frac{\partial V}{\partial r}.
\]

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