Semiintegrable almost Grassmann structures

M. A. Akivis
Department of Mathematics, Jerusalem College of Technology - Mahon Lev, 21 Havaad Haleumi St., P. O. B. 16031, Jerusalem 91160, Israel; E-mail address: akivis@math.jct.ac.il

V. V. Goldberg
Department of Mathematics, New Jersey Institute of Technology, University Heights, Newark, NJ 07102, U.S.A.; E-mail address: vlgold@numerics.njit.edu

Abstract: In the present paper we study locally semiflat (we also call them semiintegrable) almost Grassmann structures. We establish necessary and sufficient conditions for an almost Grassmann structure to be $\alpha$- or $\beta$-semiintegrable. These conditions are expressed in terms of the fundamental tensors of almost Grassmann structures. Since we are not able to prove the existence of locally semiflat almost Grassmann structures in the general case, we give many examples of $\alpha$- and $\beta$-semiintegrable structures, mostly four-dimensional. For all examples we were able to integrate these systems and find closed form equations of submanifolds.

Keywords: almost Grassmann structure, locally semiintegrable, locally semiflat, webs.

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0 Introduction

In the paper \cite{2} (see also the book \cite{1}, Ch. 7) the authors constructed the real theory of almost Grassmann structures $AG(p - 1, p + q - 1)$ defined on a differentiable manifold of dimension $n = pq$ by a fibration of Segre cones $SC(p, q)$. In particular, in \cite{2} we derived the structure equations of $AG(p - 1, p + q - 1)$ and found (in a fourth-order differential neighborhood) a complete geometric object of the almost Grassmann structure totally defining its geometric structure. We also found the structure group of these structures and its differential prolongation and the conditions under which an almost Grassmann structure is locally flat or locally semiflat.

While constructing this theory, we distinguished three cases: $p = 2, q = 2$ (dim $M = 4$); $p = 2, q > 2$ (or $p > 2, q = 2$); and $p > 2, q > 2$. We constructed the fundamental geometric objects of these structures up to fourth order for each of these three cases and established connections among them. In the first case the almost Grassmann structure $AG(1, 3)$ is equivalent to the pseudoconformal structure $CO(2, 2)$. Since the four-dimensional conformal structures play an important role in general relativity, this provides a physical justification for studying the structures $AG(1, 3)$ as well as for studying the general almost Grassmann structures $AG(p - 1, p + q - 1)$.

In the present paper we study locally semiflat almost Grassmann structures (we also call them semiintegrable), and for different values $p$ and $q$ we establish necessary and sufficient conditions of $\alpha$- and $\beta$-semiintegrability of almost Grassmann structures. These conditions are expressed in terms of the fundamental tensors of almost Grassmann structures. We find the relations between 10 independent components of an almost Grassmann structure $AG(1, 3)$ and 10 independent components of an equivalent pseudoconformal structure $CO(2, 2)$.

Since we are not able to prove the existence of locally semiflat almost Grassmann structures in the general case, we give many examples of $\alpha$- and $\beta$-semiintegrable structures,

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mostly for \( p = q = 2 \). For all examples we find systems of differential equations of the families of integral submanifolds \( V_\alpha \) and \( V_\beta \) of the distributions \( \Delta_\alpha \) and \( \Delta_\beta \) of plane elements associated with almost Grassmann structures. For some examples we were able to integrate these systems and find closed form equations of submanifolds \( V_\alpha \) and \( V_\beta \).

Note that the existence of globally semiflat four-dimensional conformal structures was proved in \cite{10} (see also \cite{9}).

\section{Almost Grassmann Structures}

1. First we recall the definition of Segre varieties and Segre cones.

The Segre variety \( S(k,l) \) is an embedding of the direct product \( P^k \times P^l \) of projective spaces \( P^k \) and \( P^l \) of dimensions \( k \) and \( l \) into a projective space of dimension \((k+1)(l+1) - 1 = kl + k + l\). Analytically this embedding can be written by means of the following equations:

\[
    z^i_\alpha = t_\alpha s^i, \quad \alpha = 0, 1, \ldots, k; \quad i = 0, 1, \ldots, l, \tag{1.1}
\]

where \( t_\alpha, s^i \) and \( z^i_\alpha \) are homogeneous coordinates in the spaces \( P^k, P^l \) and \( P^{kl+k+l} \), respectively. These equations are equivalent to the condition

\[
    \text{rank}(z^i_\alpha) = 1. \tag{1.2}
\]

The Segre variety \( S(k,l) \) has the dimension \( k + l \). It is proved in algebraic geometry that the degree of this variety is

\[
    \text{deg } S(k,l) = \binom{k+l}{k}.
\]

The cone \( SC_x(k+1,l+1) \) with vertex at the point \( x \) whose projectivization is the Segre variety \( S(k,l) \) is called the Segre cone.

Now we can define the notion of almost Grassmann structure.

\textbf{Definition 1.1} Let \( M \) be a differentiable manifold of dimension \( pq \), and let \( SC(p,q) \) be a differentiable fibration of Segre cones with the base \( M \) such that \( SC_x(M) \subset T_x(M) \), \( x \in M \). The pair \((M, SC(p,q))\) is said to be an \textit{almost Grassmann structure} and is denoted by \( AG(p-1,p+q-1) \). The manifold \( M \) endowed with such a structure is said to be an \textit{almost Grassmann manifold}.

Note that the almost Grassmann structure \( AG(p-1,p+q-1) \) is equivalent to the structure \( AG(q-1,p+q-1) \) since both of these structures are generated on the manifold \( M \) by a differentiable family of Segre cones \( SC_x(p,q) \).

In \cite{2} we discussed the following examples of almost Grassmann structures: the almost Grassmann structure associated with the Grassmannian \( G(m,n) \) (in this case \( p = m + 1 \) and \( q = n - m \)); the almost Grassmann structure \( AG(1,3) \) which is equivalent to the pseudoconformal \( CO(2,2) \)-structure; and almost Grassmann associated with multidimensional webs.

2. The structural group of the almost Grassmann structure is a subgroup of the general linear group \( GL(pq) \) of transformations of the space \( T_x(M) \), which leave the cone \( SC_x(p,q) \subset T_x(M) \) invariant. We denote this group by \( G = GL(p,q) \).

To clarify the structure of this group, we consider in the tangent space \( T_x(M) \) a family of frames \( \{e^\alpha_i\}, \alpha = 1, \ldots, p; \ i = p+1, \ldots, p+q \), such that for any fixed \( i \), the vectors \( e^\alpha_i \) belong to a \( p \)-dimensional generator \( \xi \) of the Segre cone \( SC_x(p,q) \), and for any fixed \( \alpha \), the vectors \( e^\alpha_i \) belong to a \( q \)-dimensional generator \( \eta \) of \( SC_x(p,q) \). In such a frame, the equations of the cone \( SC_x(p,q) \) can be written in the form \((1.1)\) where now \( \alpha = 1, \ldots, p; \ i = p+1, \ldots, p+q, z^i_\alpha \)
are the coordinates of a vector \( z = z^i e^i_s \subset T_x(M) \), and \( t^i_s \) and \( s^i_t \) are parameters on which a vector \( z \subset SC_x(M) \) depends.

As was shown in [1], the group \( G \) of admissible transformations of the frames \( \{e^i_s\} \) keeping the Segre cone \( SC_x(p, q) \) invariant can be presented in the form:

\[
G = SL(p) \times SL(q) \times H, \tag{1.3}
\]

where \( SL(p) \) and \( SL(q) \) are special linear groups in spaces of dimensions \( p \) and \( q \), and \( H = R^* \otimes \text{Id} \) is the group of homotheties in \( T_x(M) \).

It follows that an almost Grassmann structure \( AG(m, n) \) is a \( G \)-structure of first order.

From equation (1.1) defining the Segre cone \( SC_x(p, q) \) it follows that this cone carries \((q - 1)\)-parameter family of \( p \)-dimensional generators \( \xi \) and \((p - 1)\)-parameter family of \( q \)-dimensional generators \( \eta \).

The \( p \)-dimensional generators \( \xi \) form a fiber bundle on the manifold \( M \). The base of this bundle is the manifold \( M \), and its fiber attached to a point \( x \in M \) is the set of all \( p \)-dimensional plane generators \( \xi \) of the Segre cone \( SC_x(p, q) \). The dimension of a fiber is \( q - 1 \), and it is parametrized by means of a projective space \( P_{\alpha} \), \( \dim P_{\alpha} = q - 1 \). We will denote this fiber bundle of \( p \)-subspaces by \( E_{\alpha} = (M, P_{\alpha}) \).

In a similar manner, \( q \)-dimensional plane generators \( \eta \) of the Segre cone \( SC_x(p, q) \) form on \( M \) the fiber bundle \( E_{\beta} = (M, P_{\beta}) \) with the base \( M \) and fibers of dimension \( p - 1 = \dim P_{\beta} \). The fibers are \( q \)-dimensional plane generators \( \eta \) of the Segre cone \( SC_x(p, q) \).

Consider the manifold \( M_{\alpha} = M \times P_{\alpha} \) of dimension \( pq + q - 1 \). The fiber bundle \( E_{\alpha} \) induces on \( M_{\alpha} \) the distribution \( \Delta_{\alpha} \) of plane elements \( \xi_{\alpha} \) of dimension \( q \). In a similar manner, on the manifold \( M_{\beta} = M \times P_{\beta} \) the fiber bundle \( E_{\beta} \) induces the distribution \( \Delta_{\beta} \) of plane elements \( \xi_{\beta} \) of dimension \( p \).

**Definition 1.2** An almost Grassmann structure \( AG(p - 1, p + q - 1) \) is said to be \( \alpha \)-semiintegrable if the distribution \( \Delta_{\alpha} \) is integrable on this structure. Similarly, an almost Grassmann structure \( AG(p - 1, p + q - 1) \) is said to be \( \beta \)-semiintegrable if the distribution \( \Delta_{\beta} \) is integrable on this structure. A structure \( AG(p - 1, p + q - 1) \) is called integrable if it is both \( \alpha \)- and \( \beta \)-semiintegrable.

Integral manifolds \( \tilde{V}_{\alpha} \) of the distribution \( \Delta_{\alpha} \) of an \( \alpha \)-semiintegrable almost Grassmann structure are of dimension \( p \). They are projected onto the original manifold \( M \) in the form of a submanifold \( V_{\alpha} \) of the same dimension \( p \), which, at any of its points, is tangent to the \( p \)-subspace \( \xi_{\alpha} \) of the fiber bundle \( E_{\alpha} \). Through each point \( x \in M \), there passes a \((q - 1)\)-parameter family of submanifolds \( V_{\alpha} \).

Similarly, integral manifolds \( \tilde{V}_{\beta} \) of the distribution \( \Delta_{\beta} \) of a \( \beta \)-semiintegrable almost Grassmann structure are of dimension \( q \). They are projected onto the original manifold \( M \) in the form of a submanifold \( V_{\beta} \) of the same dimension \( q \), which, at any of its points, is tangent to the \( q \)-subspace \( \eta_{\beta} \) of the fiber bundle \( E_{\beta} \). Through each point \( x \in M \), there passes a \((p - 1)\)-parameter family of submanifolds \( V_{\beta} \). If an almost Grassmann structure on \( M \) is integrable, then through each point \( x \in M \), there pass a \((q - 1)\)-parameter family of submanifolds \( V_{\alpha} \) and a \((p - 1)\)-parameter family of submanifolds \( V_{\beta} \) which were described above.

The Grassmann structure \( G(m, n) \) is an integrable almost Grassmann structure \( AG(m, n) \) since through any point \( x \) of the variety \( \Omega(m, n) \), onto which the manifold \( G(m, n) \) is mapped bijectively under the Grassmann mapping, there pass a \((q - 1)\)-parameter family of \( p \)-dimensional plane generators (which are the submanifolds \( V_{\alpha} \)) and a \((p - 1)\)-parameter family of \( q \)-dimensional plane generators (which are the submanifolds \( V_{\beta} \)). In the projective space \( P^m \), to submanifolds \( V_{\alpha} \) there corresponds a family of \( m \)-dimensional subspaces
satisfy the following structure equations:

The forms constitute a completely integrable system of forms.

Consider a differentiable manifold $M$ of dimension $pq$ endowed with an almost Grassmann structure $AG(p - 1, p + q - 1)$. Suppose that $x \in M$, $T_x(M)$ is the tangent space of the manifold $M$ at the point $x$ and that $\{e_i^\alpha\}$ is an adapted frame of the structure $AG(p - 1, p + q - 1)$. The decomposition of a vector $z \in T_x(M)$ with respect to this basis can be written in the form

$$z = \omega^i_\alpha(x)e^\alpha_i,$$

where $\omega^i_\alpha$ are 1-forms making up the co-frame in the space $T_x(M)$. If $z = dx$ is the differential of a point $x \in M$, then the forms $\omega^i_\alpha(dx)$ are differential forms defined on a first-order frame bundle associated with the almost Grassmann structure. These forms constitute a completely integrable system of forms.

As was proved in [2], the form $\theta = (\omega^i_\alpha)$ and the forms arising under its prolongation satisfy the following structure equations:

$$\begin{align*}
d\omega^i_\alpha - \omega^j_\delta \wedge \omega^\delta_\beta - \omega^j_\gamma \wedge \omega^\gamma_\delta + \omega^j_\beta \wedge \omega^\beta_\gamma &= a^\beta_\gamma^\delta \omega^j_\delta \wedge \omega^\gamma_\beta + \omega^j_\alpha \wedge \omega^\alpha_\gamma, \\
d\omega^i_\beta - \omega^j_\gamma \wedge \omega^\gamma_\beta &= \frac{q}{p + q} (\delta^i_\gamma \omega^j_\delta \wedge \omega^\gamma_\beta - p \omega^i_\delta \wedge \omega^\delta_\gamma) + b^\gamma_\delta \omega^j_\gamma \wedge \omega^\gamma_\delta, \\
d\omega^j_\beta \wedge \omega^i_\delta &= \frac{p}{p + q} (\delta^j_\delta \omega^i_\gamma \wedge \omega^\gamma_\delta - q \omega^j_\gamma \wedge \omega^\gamma_\delta) + b^j_\gamma \omega^i_\gamma \wedge \omega^\gamma_\delta, \\
d\omega^i_\gamma &= \omega^i_\alpha \wedge \omega^\alpha_\gamma, \\
d\omega^i_\beta \wedge \omega^\beta_\gamma - \omega^j_\delta \wedge \omega^\delta_\gamma + \omega \wedge \omega^\gamma_\beta &= c^{\alpha\beta\gamma} \omega^i_\delta \wedge \omega^\delta_\gamma - d^\gamma b^\delta \omega^i_\delta \wedge \omega^\delta_\gamma,
\end{align*}$$

where the matrix 1-form $\theta = (\omega^i_\alpha)$ is defined in a first-order frame bundle, the form $\omega$ is a scalar form defined in a second-order frame bundle, $\omega_\alpha = (\omega^\alpha_\beta)$ and $\omega_\beta = (\omega^\beta_\gamma)$ are the matrix 1-forms also defined in a second-order frame bundle.

The forms $\omega^i_\alpha$ and $\omega^i_\beta$ satisfy the conditions

$$\omega^i_\gamma = 0, \quad \omega^i_\delta = 0.$$

In equations (1.4) the 2-form

$$\Theta_i^\alpha = a^\beta_\gamma^\delta \omega^\delta_\beta \wedge \omega^\gamma_\alpha$$

is the torsion 2-form of the $AG(p - 1, p + q - 1)$-structure, and the forms

$$\begin{align*}
\Omega^i_\alpha &= b^\beta_\gamma^\delta \omega^\delta_\beta \wedge \omega^\gamma_\alpha, \\
\Omega^j_\beta &= b^\gamma_\delta \omega^\delta_\gamma \wedge \omega^\gamma_\beta, \\
\Phi^\alpha_i &= c^{\alpha\beta\gamma} \omega^\beta_\gamma \wedge \omega^\gamma_\alpha
\end{align*}$$

are the curvature 2-forms of this structure.

Note that Dhooghe in [4] and [5] gave the structure equations of an almost Grassmann structure in the form close to our equations (1.4). His equations differ from equations (1.4) only by the additional term $\Omega_i^\alpha$ in the right-hand side of the fourth equation. This means that on a manifold $M$, dim $M = pq$, carrying an almost Grassmann structure our equations (1.4) define a normal Cartan connection.

Moreover, in his further considerations in [4] and [5] Dhooghe assumes that the torsion form $\Omega^i_\alpha$ is identically equal to 0 not only for $p = q = 2$ but also for $p > 2$ and $q > 2$. This leads to the loss of generality. We did not make the above assumption.

In [2] (see also [1], §7.2) we proved the following facts:
a) The quantities \( a = \{ a_{\alpha j k}^{i \beta \gamma} \} \), defined by a second-order neighborhood, form a relative tensor of weight \(-1\) and satisfy the following conditions:
\[
a_{\alpha j k}^{i \beta \gamma} = -a_{\alpha k j}^{i \beta \gamma}
\] (1.8)
and
\[
a_{\alpha j k}^{i \alpha \gamma} = 0, \quad a_{\alpha k j}^{i \beta \gamma} = 0.
\] (1.9)
The tensor \( \{ a_{\alpha j k}^{i \beta \gamma} \} \) is said to be the first structure tensor, or the torsion tensor, of an almost Grassmann manifold \( AG(p-1, p+q-1) \).

b) Let us set \( b^1 = \{ b_{jkl}^{i \gamma \delta} \}, b^2 = \{ b_{akl}^{i \beta \delta} \} \) and \( b = (b^1, b^2) \). The quantities \( (a, b^1) \) and \( (a, b^2) \) form linear homogeneous objects. They represent two subobjects of the second structure object \( (a, b) \) of the almost Grassmann structure \( AG(p-1, p+q-1) \). The components \( b^1 \) and \( b^2 \) satisfy the conditions:
\[
b_{akl}^{i \beta \delta} = -b_{akl}^{i \beta \gamma}, \quad b_{jkl}^{i \gamma \delta} = -b_{jkl}^{i \beta \gamma},
\] (1.10)
\[
b_{akl}^{i \alpha \gamma} = 0, \quad b_{akl}^{i \beta \gamma} = 0,
\] (1.11)
b_{akl}^{i \alpha \delta} - b_{akl}^{i \beta \delta} + b_{akl}^{i \delta \alpha} - b_{akl}^{i \delta \beta} = 0,
\] (1.12)
and the components of the tensor \( a \) satisfy the following differential equations:
\[
2 a_{\alpha [j|kl] m}^{i [\beta |\gamma |]} a_{\alpha [i|kl] m}^{m [\beta |\delta |]} + \delta_{\alpha [j|kl]}^{i [\beta |\gamma |]} a_{\alpha [i|kl] m}^{m [\beta |\delta |]} + a_{\alpha [i|kl] m}^{m [\beta |\delta |]} = 0,
\] (1.13)
where \( a_{\alpha j k l}^{i \beta \delta} \) are the Pfaffian derivatives of \( a_{\alpha j k}^{i \beta \gamma} \) and the alternation is performed with respect to the vertical pairs of indices \( i \) and \( j \). Formulas (1.13) are analogues of the Bianchi equations in the theory of spaces with affine connection.

c) For \( p > 2 \) and \( q > 2 \), the components of \( b^2 \) and \( b^1 \), respectively, are expressed in terms of the components of the tensor \( a \) and their Pfaffian derivatives.

This implies that if for \( p > 2, q > 2 \), the torsion tensor \( a \) of an almost Grassmann structure vanishes, then its curvature tensor \( b \) vanishes as well.

d) Let us set \( c = \{ c_{ij k}^{\alpha \beta \gamma} \} \). Then \( S = (a, b, c) \) forms a linear homogeneous object, which is called the third structure object of the almost Grassmann structure \( AG(p-1, p+q-1) \). It is defined by a fourth-order differential neighborhood of \( AG(p-1, p+q-1) \). Its subobject \( a \) is a relative tensor (the torsion tensor) defined by a second-order differential neighborhood of \( AG(p-1, p+q-1) \), and the subobjects \( (a, b^1), (a, b^2) \), and \( (a, b) \) are defined by a third-order differential neighborhood of \( AG(p-1, p+q-1) \).

The third structural object \( S = (a, b, c) \) is the complete geometric object of the almost Grassmann structure \( AG(p-1, p+q-1) \), since during the prolongation of structure equations (1.4) of \( AG(p-1, p+q-1) \), all newly arising objects are expressed in terms of the components of the object \( S \) and their Pfaffian derivatives.

The components of \( c \) satisfy the conditions
\[
c_{\alpha j k}^{i \beta \gamma} = -c_{\alpha k j}^{i \beta \gamma}
\] (1.14)
and
\[
c_{[i|j|k]}^{[\alpha \beta \gamma]} = 0,
\] (1.15)
and the components of $b$ satisfy the differential equations

\[
\begin{align*}
\frac{p}{p+q} b^{\gamma \delta \epsilon}_{ijklm} - \frac{pq}{p+q} c_{ijklm}^{[\gamma \delta \epsilon]} - 2b^{\gamma \delta \epsilon}_{ijklm} a_{\alpha}^{\gamma \delta \epsilon} - 2b^{\gamma \delta \epsilon}_{ijklm} a_{\alpha}^{\gamma \delta \epsilon} = 0, \\
\frac{p}{p+q} b^{\gamma \delta \epsilon}_{ijklm} + \frac{pq}{p+q} c_{ijklm}^{[\gamma \delta \epsilon]} + 2b^{\gamma \delta \epsilon}_{ijklm} a_{\alpha}^{\gamma \delta \epsilon} = 0,
\end{align*}
\]

(1.16)

where $b^{\gamma \delta \epsilon}_{ijklm}$ and $b^{\gamma \delta \epsilon}_{ijklm}$ are the Pfaffian derivatives of $b^{\gamma \delta \epsilon}_{ijklm}$ and $b^{\gamma \delta \epsilon}_{ijklm}$, respectively. In formulas (1.15) and (1.16) the alternation is carried out with respect to the vertical pairs of indices.

e) If $p > 2$, then the components of $c$ are expressed in terms of the components of the subobject $(a, b^1)$ and their Pfaffian derivatives, and if $q > 2$, then the components of $c$ are expressed in terms of the components of the subobject $(a, b^2)$ and their Pfaffian derivatives. This implies that in this case the object $(a, b)$ satisfies certain differential equations which are other analogues of the Bianchi equations in the theory of spaces with affine connection. These equations can be obtained if we substitute for the components of $c$ in equations (1.16) their values.

f) An almost Grassmann structure $AG(1, p + q - 1)$ is said to be locally Grassmann (or locally flat) if it is locally equivalent to a Grassmann structure. For $p > 2$ and $q > 2$, an almost Grassmann structure $AG(1, p + q - 1)$ is locally Grassmann if and only if its first structure tensor $a$ vanishes. For $p = 2, q = 2$, the tensor $a_{\alpha \beta \gamma}^{\delta}$ vanishes, and the condition for the structure $AG(1, 3)$ to be locally Grassmann is the vanishing of the tensor $b$. The case $p = 2, q > 2$ (and $p > 2, q = 2$) will be considered below.

## 2 Semiintegrability of Almost Grassmann Structures

1. In this section we find geometric conditions for an almost Grassmann structure $AG(p - 1, p + q - 1)$ defined on a manifold $M$ to be semiintegrable. The conditions are expressed in terms of the complete structure object $S$ of the almost Grassmann structure $AG(p - 1, p + q - 1)$ and its subobjects $S_\alpha$ and $S_\beta$ which will be defined in this section.

In what follows, we often encounter quantities satisfying the conditions similar to conditions (1.8) for the quantities $a_{ij}^{\alpha \beta \gamma}$. For calculations with quantities of this kind, the following lemma is very useful:

**Lemma 2.1** If a system of quantities $T_{ij}^{\alpha \beta}$ is skew-symmetric with respect to the pairs of indices $(\alpha \beta)$ and $(\beta \alpha)$, namely satisfies the conditions

\[
T_{ij}^{\alpha \beta} = -T_{ij}^{\beta \alpha},
\]

(2.1)

then the following identities hold:

\[
\begin{align*}
T_{ij}^{\alpha \beta} &= T_{ij}^{\alpha \beta}, & T_{ij}^{\alpha \beta} &= T_{ij}^{\alpha \beta}, \\
T_{ij}^{\alpha \beta} &= -T_{ij}^{\alpha \beta} = T_{ij}^{\alpha \beta}, & T_{ij}^{\alpha \beta} &= -T_{ij}^{\alpha \beta} = T_{ij}^{\alpha \beta}, \\
T_{ij}^{\alpha \beta} &= 0, \quad T_{ij}^{\alpha \beta} = 0.
\end{align*}
\]

(2.2)

In these relations the symmetrization and the alternation are carried separately over the lower indices and the upper indices. In addition the following decompositions take place:

\[
T_{ij}^{\alpha \beta} = T_{ij}^{\alpha \beta} + T_{ij}^{\alpha \beta}, \quad T_{ij}^{\alpha \beta} = T_{ij}^{\alpha \beta} + T_{ij}^{\alpha \beta},
\]

(2.3)
Proof. All these identities can be proved by direct calculation with help of (2.1).\[\blacksquare\]
In addition, in the proof of the main theorem, we will use the following lemma:

**Lemma 2.2** The condition

\[ T^{[\alpha\beta\gamma]}_{(ijk)} = 0, \tag{2.4} \]

where the alternation is carried over three vertical pairs of indices, implies the condition

\[ T^{[\alpha\beta\gamma]}_{ijk} = 0, \tag{2.5} \]

where the alternation and symmetrization are carried separately over the upper triple of indices and the lower triple of indices.

Proof. To prove this, one writes down 36 terms of \( T^{[\alpha\beta\gamma]}_{ijk} \) and collects from them 6 groups of 6 terms to each of which the hypothesis (2.4) can be applied.

Next we will prove the following important result on the decomposition of the torsion tensor of an almost Grassmann structure \( AG(p - 1, p + q - 1) \):

**Theorem 2.3** The torsion tensor \( a = \{a_{ij\alpha\beta}\} \) of the almost Grassmann structure \( AG(p - 1, p + q - 1) \) decomposes into two subtensors:

\[
a = a_\alpha + a_\beta, \tag{2.6}
\]

where

\[
a_\alpha = \{a_{ij\beta\gamma}\}, \quad a_\beta = \{a_{ij\alpha\gamma}\}.
\]

Proof. Since the tensor \( a_{ij\alpha\beta} \) is skew-symmetric with respect to the pairs of indices \((\alpha, \beta)\) and \((\gamma, \delta)\), then, by Lemma 2.1, the decomposition (2.6) is equivalent to the obvious decomposition

\[
a_{ij\alpha\beta} = a_{ij\beta\gamma} + a_{ij\beta\gamma} \tag{2.7}
\]

Note that by Lemma 2.1, the subtensors \( a_\alpha \) and \( a_\beta \) can be also represented in the form

\[
a_\alpha = \{a_{ij\beta\gamma}\}, \quad a_\beta = \{a_{ij\alpha\gamma}\}.
\]

Note also that like the tensor \( a \), its subtensors \( a_\alpha \) and \( a_\beta \) are skew-symmetric with respect to the pairs of indices \((\beta, \gamma)\) and \((\gamma, \delta)\):

\[
a_{ij\alpha\beta} = -a_{ij\alpha\beta} \quad a_{ij\beta\gamma} = -a_{ij\beta\gamma} \quad a_{ij\beta\gamma} = -a_{ij\beta\gamma},
\]

and they are also trace-free, since it follows from (1.9) that

\[
a_{ij\alpha\beta} = 0, \quad a_{ij\beta\gamma} = 0, \quad a_{ij\beta\gamma} = 0, \quad a_{ij\alpha\beta} = 0, \quad a_{ij\beta\gamma} = 0. \tag{2.7}
\]

**Theorem 2.4** If \( p = 2 \), then \( a_\alpha = 0 \), and if \( q = 2 \), then \( a_\beta = 0 \).

Proof. Suppose that \( p = 2 \). Then \( \alpha, \beta, \gamma = 1, 2 \). Since, by definition and Lemma 2.1, the tensor \( a_\alpha \) is skew-symmetric with respect to the indices \( \beta \) and \( \gamma \), we have

\[
a_{ij\alpha\beta} = a_{ij\alpha\beta} = 0.
\]

But the first condition of (2.7) gives

\[
a_{ij\alpha\beta} + a_{ij\alpha\beta} = 0, \quad a_{ij\alpha\beta} + a_{ij\alpha\beta} = 0.
\]
It follows from these relations that

\[ a_{2(jk)}^{i2} = a_{1(jk)}^{i1} = 0; \]

that is, all components of the tensor \( a_\alpha \) vanish.

For the case \( q = 2 \), the proof is similar. \( \blacksquare \)

2. We introduce the following notation:

\[
\begin{aligned}
b_1^\alpha &= \{b_1^{i\gamma\delta}\}, & b_2^\alpha &= \{b_2^{i\gamma\delta}\}, & c_\alpha &= \{c_\alpha^{[\alpha\beta\gamma]}\}, \\
b_1^\beta &= \{b_1^{j\delta\gamma}\}, & b_2^\beta &= \{b_2^{j\delta\gamma}\}, & c_\beta &= \{c_\beta^{(\alpha\beta\gamma)}\}.
\end{aligned}
\]

(2.8)

Now we give necessary and sufficient conditions for an almost Grassmann structure \( AG(p-1, p+q-1) \) to be \( \alpha \)-semiintegrable or \( \beta \)-semiintegrable.

**Theorem 2.5** (i) If \( p \geq 2 \) and \( q \geq 2 \), then for an almost Grassmann structure \( AG(p-1, p+q-1) \) to be \( \alpha \)-semiintegrable, it is necessary and sufficient that the conditions

\[ a_\alpha = b_1^\alpha = b_2^\alpha = 0 \]

hold.

(ii) If \( p \geq 2 \) and \( q \geq 2 \), then for an almost Grassmann structure \( AG(p-1, p+q-1) \) to be \( \beta \)-semiintegrable, it is necessary and sufficient that the conditions

\[ a_\beta = b_1^\beta = b_2^\beta = 0 \]

hold.

**Proof.** We prove part (i) of theorem. The proof of part (ii) is similar.

Suppose that \( \theta_\alpha, \alpha = 1, \ldots, p \), are basis forms of the integral submanifolds \( V_\alpha \), \( \dim V_\alpha = p \), of the distribution \( \Delta_\alpha \) appearing in Definition 1.2. Then

\[ \omega^i_\alpha = s^i\theta_\alpha, \quad \alpha = 1, \ldots, p; \quad i = p+1, \ldots, p+q \]

(2.9)

where \( \theta_\alpha \) are basis forms on the submanifold \( V_\alpha \).

For the structure \( AG(p-1, p+q-1) \) to be \( \alpha \)-semiintegrable, it is necessary and sufficient that system (2.9) be completely integrable. Taking the exterior derivatives of equations (2.9) by means of structure equations (1.4), we find that

\[ (ds^i + s^j\omega^i_j - s^i\omega) \wedge \theta_\alpha + s^i(d\theta_\alpha - \omega^\beta_\alpha \wedge \theta_\beta) = a_{ijk}^{i\beta\gamma} s^j s^k \theta_\beta \wedge \theta_\gamma. \]

(2.10)

It follows from these equations that

\[ d\theta_\alpha - \omega^\beta_\alpha \wedge \theta_\beta = \varphi^\beta_\alpha \wedge \theta_\beta, \]

(2.11)

where \( \varphi^\beta_\alpha \) is an 1-form that is not expressed in terms of the basis forms \( \theta_\alpha \).

For brevity, we set

\[ \varphi^i = ds^i + s^j\omega^i_j - s^i\omega. \]

(2.12)

Then the exterior quadratic equation (2.10) takes the form

\[ (\delta_\alpha^\beta \varphi^i + s^i\varphi^\beta_\alpha) \wedge \theta_\beta = a_{ijk}^{i\beta\gamma} s^j s^k \theta_\beta \wedge \theta_\gamma. \]

(2.13)

From (2.13) it follows that for \( \theta_\alpha = 0 \), the 1-form \( \delta_\alpha^\beta \varphi^i + s^i\varphi^\beta_\alpha \) vanishes:

\[ \delta_\alpha^\beta \varphi^i(\delta) + s^i\varphi^\beta_\alpha(\delta) = 0. \]

(2.14)

Contracting equation (2.14) with respect to the indices \( \alpha \) and \( \beta \), we find that

\[ \varphi^i = -s^i\varphi(\delta), \quad \varphi^\beta_\alpha = \delta_\alpha^\beta \varphi(\delta), \]

(2.15)
where we set $\varphi(\delta) = \frac{1}{p} \varphi^\gamma(\delta)$.

It follows from (2.15) that on the subvariety $V_\alpha$, the 1-forms $\varphi^i$ and $\varphi_\alpha^\beta$ can be written as follows:

$$\varphi^i = -s^i \varphi + s^i \theta_\beta, \quad \varphi_\alpha^\beta = \delta_\alpha^\beta \varphi + \delta_\alpha^\beta \theta_\gamma.$$  (2.16)

Substituting these expressions into equations (2.11) and (2.12), we find that

$$d \theta_\alpha - \omega^\beta_{\alpha} \wedge \theta_\beta = \varphi \wedge \theta_\alpha + s^\beta_{\alpha} \gamma \wedge \theta_\beta.$$  (2.17)

where $s^\beta_{\alpha} = \frac{s^{i[\beta]}}{s^\alpha_{[\beta]}}$ and

$$ds^i + s^i \omega^j_j - s^i \omega = -s^i \varphi + s^i \theta_\beta.$$  (2.18)

Substituting (2.17) and (2.18) into equation (2.10), we obtain

$$-s^i s^\beta_{\alpha} - \delta_\alpha^\beta s_i^{|i|} = a_{\alpha j k} s^j s^k.$$  (2.19)

Contracting equation (2.19) with respect to the indices $\alpha$ and $\beta$, we obtain

$$-2s^i s^\gamma_{\alpha} - ps^i \gamma + s^i \gamma = 0,$$

from which it follows that

$$s^i \gamma = s^i s^\gamma,$$  (2.20)

where we set $s^\gamma = -\frac{2}{p-1} s^\alpha_{[\gamma]}$. Substituting (2.20) into (2.19), we find that

$$s^i (\delta_\alpha^\beta s^\beta_{\alpha} - \delta_\alpha^\beta s^\gamma_{\alpha} - 2s^\beta_{\alpha} s^\gamma_{\alpha}) = 2a_{\alpha j k} s^j s^k.$$  (2.21)

It follows that

$$\delta_\alpha^\beta s^\gamma_{\alpha} - \delta_\alpha^\beta s^\gamma_{\alpha} - 2s^\beta_{\alpha} s^\gamma_{\alpha} = s^\beta_{\alpha} s^j_{\alpha},$$  (2.22)

where $s^\beta_{\alpha j} = -s^\beta_{\alpha j}$. Substituting (2.22) into (2.21), we arrive at the equation

$$s^\beta_{\alpha j} (s^i_{\alpha j}) = a^i_{\alpha (jk)}.$$  (2.23)

where the alternation sign in the right-hand side is dropped by Lemma 2.1.

Contracting (2.23) with respect to the indices $i$ and $j$ and taking into account of equations (1.9) and (1.10), we obtain

$$s^\beta_{\alpha k} = 0,$$  (2.24)

from which, by (2.23), it follows that

$$a^i_{\alpha (jk)} = 0.$$  (2.25)

This proves that if an almost Grassmann structure $AG(p-1, p+q-1)$ is $\alpha$-semiintegrable, then its torsion tensor satisfies the condition (2.25), that is, $a_\alpha = 0$.

Since, by Theorem 2.4, for $p = 2$ the subtensor $a_\alpha = 0$, condition (2.25) is identically satisfied. Hence, while proving sufficiency of this condition for $\alpha$-semiintegrability, we must assume that $p > 2$.

Let us return to equations (2.17) and (2.18). Substitute into equation (2.18) the values $s^i \beta$ taken from (2.20) and set

$$\bar{\varphi} = \varphi - s^\beta \theta_\beta.$$  (2.26)

In addition, by (2.24), relations (2.22) imply that

$$s^\beta_{\alpha} = \delta_\alpha^\gamma s^\beta_{\alpha}.$$
Then equations (2.17) and (2.18) take the form

\[ d\omega^\alpha - (\omega^\alpha + \delta^\alpha \bar{\varphi}) \wedge \omega_\beta = 0 \]  

(2.27)

and

\[ ds^i + s^j \omega^j_i - s^i (\omega - \bar{\varphi}) = 0. \]  

(2.28)

Taking the exterior derivatives of (2.28), we obtain the following exterior quadratic equation:

\[ s^i \Phi + b^{[\gamma\delta]}_{jkil} s^j s^k \omega^\gamma_i \wedge \omega^\delta_k \wedge \theta^\alpha = 0, \]  

(2.29)

where

\[ \Phi = d\bar{\varphi} - \frac{(p + 1)q}{p + q} s^k \omega^\gamma_k \wedge \theta^\gamma. \]

Next, taking the exterior derivatives of (2.27), we find that

\[ \Phi \wedge \omega^\alpha + b^{[\beta\gamma\delta]}_{\alpha kl} s^{\beta} \theta^\gamma_k \wedge \theta^\delta_l \wedge \theta^\alpha = 0. \]  

(2.30)

Equation (2.29) shows that the 2-form \( \Phi \) can be written as

\[ \Phi = A^{[\gamma\delta]}_{kl} s^k \theta^\gamma_k \wedge \theta^\delta, \]  

(2.31)

where the coefficients \( A^{[\gamma\delta]}_{kl} \) are symmetric with respect to the lower indices and skew-symmetric with respect to the upper indices. Substituting this value of the form \( \Phi \) into equations (2.29) and (2.30), we arrive at the conditions

\[ b^{[\gamma\delta]}_{(jk)l} + \delta^i_{[j} A^{[\gamma\delta]}_{kl} = 0 \]  

(2.32)

and

\[ b^{[\beta\gamma\delta]}_{\alpha(kl)} + \delta^i_{[\alpha} A^{[\beta\gamma\delta]}_{kl} = 0. \]  

(2.33)

Contracting equation (2.32) with respect to the indices \( i \) and \( j \) and equation (2.33) with respect to the indices \( \alpha \) and \( \beta \), we find that

\[ 2(q + 2)A_{[\gamma}^{\delta]} + b^{[\gamma\delta]}_{kkl} + b^{[\gamma\delta]}_{ikl} + b^{[\gamma\delta]}_{i^l} + b^{[\gamma\delta]}_{k^l} = 0 \]  

(2.34)

and

\[ 2(p - 2)A_{\alpha}^{[\gamma\delta]} + b^{[\beta\gamma\delta]}_{\alpha kl} + b^{[\beta\gamma\delta]}_{\alpha^l} + b^{[\beta\gamma\delta]}_{\alpha^k} + b^{[\beta\gamma\delta]}_{\alpha^l} = 0. \]  

(2.35)

Note that for \( p = 2 \) equation (2.33) becomes an identity, and we will not obtain equation (2.35).

If we add equations (2.34) and (2.35) and apply condition (1.12), we find that

\[ A_{[\gamma}^{\delta]} = 0. \]  

(2.36)

As a result equations (2.32) and (2.33) take the form

\[ b^{[\gamma\delta]}_{(jk)l} = 0, \quad b^{[\beta\gamma\delta]}_{\alpha(kl)} = 0. \]  

(2.37)

By Lemma 2.1, conditions (2.37) are equivalent to the conditions

\[ b^{[\gamma\delta]}_{(jk)l} = 0, \quad b^{[\beta\gamma\delta]}_{\alpha(kl)} = 0. \]  

(2.38)

It follows from equations (2.36) and (2.31) that

\[ d\bar{\varphi} = \frac{(p + 1)q}{p + q} s^k \omega^\gamma_k \wedge \theta^\gamma. \]  

(2.39)
Finally, taking the exterior derivatives of equations (2.39) and applying (2.27), (2.28), and (1.4), we obtain the condition
\[ c_{(ijk)}^{[\alpha \beta \gamma]} = 0. \tag{2.40} \]
These equations will not be trivial only if \( p > 2 \). But, by Lemma 2.2, conditions (2.40) follow from integrability conditions (1.15).

Thus the system of Pfaffian equations (2.9), defining integral submanifolds of an \( \alpha \)-semiintegrable almost Grassmann structure, together with Pfaffian equations (2.28) and (2.39) following from (2.9), is completely integrable if and only if conditions (2.25) and (2.38) are satisfied. This proves part (i).

As we noted in the beginning, the proof of part (ii) is similar. We note only that the equations of integral submanifolds \( V_{\beta} \), \( \dim V_{\beta} = q \), of the distribution \( \Delta_{\beta} \) appearing in Definition 1.6 can be written in the form
\[ \omega^i_{\alpha} = s_{\alpha} \theta^i, \quad \alpha = 1, \ldots, p; \quad i = p + 1, \ldots, p + q, \tag{2.41} \]
where the 1-forms \( \theta^i \) are linearly independent on the submanifold \( V_{\beta} \).

It follows from our previous considerations and (2.8) that

1. for \( p = 2 \) we have \( b_\alpha^2 = 0 \) and \( c_\alpha = 0 \);
2. for \( q = 2 \) we have \( b_\beta^1 = 0 \) and \( c_\beta = 0 \);
3. for \( p > 2 \) we have \( c_\alpha = 0 \);
4. for \( q > 2 \) we have \( c_\beta = 0 \).

The last two results follow from conditions (1.15) and Lemma 2.2. These results combined with differential equations which the components of \( b^1 \) and \( b^2 \) satisfy imply that the tensors \( a_\alpha, a_\beta \) and the quantities \( b_\alpha^1, b_\alpha^2, b_\beta^1, b_\beta^2 \) form the following geometric objects:
\[(a_\alpha, b_\alpha^1), \quad (a_\alpha, b_\alpha^2), \quad S_\alpha = (a_\alpha, b_\alpha^1, b_\alpha^2), \]
\[(a_\beta, b_\beta^1), \quad (a_\beta, b_\beta^2), \quad S_\beta = (a_\beta, b_\beta^1, b_\beta^2), \]
which are subobjects of the second structural object and the complete structural object of the almost Grassmann structure.

The following theorem gives the conditions of \( \alpha \)- and \( \beta \)-semiintegrability in the cases when \( p = 2 \) or \( q = 2 \).

**Theorem 2.6** (i) *If \( p = 2 \), then the structure subobject \( S_\alpha \) consists only of the tensor \( b_\alpha^1 \), and the vanishing of this tensor is necessary and sufficient for the almost Grassmann structure \( AG(1, q + 1) \) to be \( \alpha \)-semiintegrable.*

(ii) *If \( q = 2 \), then the structure subobject \( S_\beta \) consists only of the tensor \( b_\beta^2 \), and the vanishing of this tensor is necessary and sufficient for the almost Grassmann structure \( AG(p - 1, p + 1) \) (which is equivalent to the structure \( AG(1, p + 1) \)) to be \( \beta \)-semiintegrable.*

(iii) *If \( p = q = 2 \), then the complete structural object \( S \) consists only of the tensors \( b_\alpha^1 \) and \( b_\beta^2 \), and the vanishing of one of these tensors is necessary and sufficient for the almost Grassmann structure \( AG(1, 3) \) to be \( \alpha \)- or \( \beta \)-semiintegrable, respectively.*
Proof. We will prove part (i). As we have already seen, for \( p = 2 \), the tensor \( a_\alpha \) as well as the quantities \( b_\alpha^2 \) and \( c_\alpha \) vanish (\( a_\alpha = b_\alpha^2 = c_\alpha = 0 \)), and the object \( b_\alpha^1 \) becomes a tensor. Thus the vanishing of this tensor is necessary and sufficient for the almost Grassmann structure \( AG(1, q + 1) \) to be \( \alpha \)-semiintegrable. The proof of part (ii) is similar. Part (iii) combines the results of (i) and (ii).

We will make two more remarks:

1. The tensors \( b_\alpha^1 \) and \( b_\beta^2 \) are defined by a third-order differential neighborhood of the almost Grassmann structure.

2. For \( p = q = 2 \), as was indicated earlier (see Subsection 1.1), the almost Grassmann structure \( AG(1, q + 1) \) is equivalent to the conformal \( CO(2, 2) \)-structure. Thus we have the following decomposition of its complete structural object: \( S = b_\alpha^1 + b_\beta^2 \) (see [1], §5.1).

Note also that in [1] and [3] the author assumed that an almost Grassmann structure \( AG(p - 1, q + 1) \) is semiintegrable if and only if one of two curvature forms \( \Omega_i^\alpha \) or \( \Omega_\alpha^\beta \) vanishes. However, our previous considerations as well as formula (3.18) below show that this will be the case only if \( p = q = 2 \).

3 Existence of Semiintegrable Almost Grassmann Structures

1. We will prove the existence of four-dimensional and \((2p)\) and \((2q)\)-dimensional semi-integrable almost Grassmann structures, where \( p, q > 2 \), by constructing examples of such structures.

In order to prove that a certain almost Grassmann structure is \( \alpha \)- or \( \beta \)-semiintegrable, we will use two methods.

The first method is to check whether conditions of \( \alpha \)- or \( \beta \)-semiintegrability outlined in Theorems 2.5 and 2.6 are satisfied.

When we apply this method, we will need the following lemma:

**Lemma 3.1** If the forms \( \omega, \omega_\alpha^\beta, \) and \( \omega_\beta^i \), occurring in equations (1.4) are principal forms, that is,

\[
\omega, \omega_\alpha^\beta, \omega_\beta^i \equiv 0 \quad (\text{mod } \omega_\alpha^\beta),
\]

then the form \( \omega \) can be reduced to 0, and the forms \( \omega_\alpha^\beta, \alpha = 1, 2, i = 3, 4 \), become principal forms with a symmetric matrix of coefficients.

**Proof.** Suppose that \( \omega \) is expressed in terms of the basis forms \( \omega_\alpha^i \):

\[
\omega = a \omega_3^1 + b \omega_3^2 + c \omega_4^1 + e \omega_2^1.
\]  

If we take exterior derivative of this equation, apply Cartan’s lemma to the obtained exterior quadratic equation, and set \( \omega_\alpha^i = 0 \), we will obtain the following Pfaffian equations:

\[
\delta a + \pi_1^3 = 0, \quad \delta b + \pi_2^3 = 0, \\
\delta c + \pi_1^4 = 0, \quad \delta e + \pi_2^4 = 0,
\]

where \( \delta \) is the symbol of differentiation with respect to fiber parameters, and \( \pi_\alpha^i = \omega_\alpha^i|_{\omega_\alpha^i = 0} \).

This implies that the quantities \( a, b, c \) and \( e \) can be reduced to 0, \( a = b = c = e = 0 \). Let us show, for example, that \( a \) can be reduced to 0. In fact, since the forms \( \pi_1^3, \pi_2^3, \pi_1^4, \pi_2^4 \),
and $\pi_3^3$ are linearly independent, we can set $\pi_3^3 = \pi_4^3 = \pi_7^3 = 0$ preserving $\pi_3^1 \neq 0$. Since now $\pi_3^1$ depends on one fiber parameter, for an appropriate choice of this parameter we have $\pi_3^1 = \delta t$. By integrating the equation $\delta a + \delta t = 0$, we obtain $a = -t + C$, where $C$ is a constant. For any value of $C$, we can take $t = C$, and as a result, we will get $a = 0$.

Conversely, if the fiber parameters are chosen in such a way that $a = 0$, we get from the above differential equations that $\pi_3^1 = 0$.

The remaining reductions $b = c = e = 0$ can be proved in a similar manner. This gives $\pi_3^3 = \pi_4^3 = \pi_7^3 = 0$. After these reductions, equation (3.1) becomes

$$\omega = 0.$$  \hspace{1cm} (3.2)

Now equations (1.4) and (3.2) imply that

$$\omega_i^a \wedge \omega_a^i = 0.$$  

Applying Cartan’s lemma to this exterior quadratic equation, we obtain that the forms $\omega_i^a$ become principal forms, and they are expressed in terms of the basis forms $\omega_a^i$ as follows:

$$\begin{align*}
\omega_1^1 &= A_1 \omega_1^1 + A_2 \omega_2^1 + A_3 \omega_1^2 + A_4 \omega_4^1, \\
\omega_2^2 &= A_2 \omega_1^1 + B_2 \omega_2^1 + B_3 \omega_1^2 + B_4 \omega_4^2, \\
\omega_1^3 &= A_3 \omega_1^3 + B_3 \omega_2^3 + C_3 \omega_1^4 + C_4 \omega_4^3, \\
\omega_2^4 &= A_4 \omega_1^1 + B_4 \omega_2^3 + C_4 \omega_1^4 + E_4 \omega_4^4.
\end{align*}$$  \hspace{1cm} (3.3)

Note that the conditions for fiber forms in Lemma 3.1 mean the group $G$ of admissible transformations of first-order frames is reduced to the identity group, $G = \{e\}$, and instead of a fibration of first-order frames associated with an $AG(1, 3)$-structure, we have a distribution of such frames. In other words, now with any point of a manifold $M^4$ on which the $AG(1, 3)$-structure is given only one frame is associated.

The normalization (3.2) and relations (3.3) implied by this normalization means that the subgroup $T(4)$ of translations contained in the prolonged group $G'$ (see [4], p. 274) is also reduced to the identity group, and $G' = G = \{e\}$. Furthermore, only one second-order frame is associated with any point $x \in M^4$. Moreover, the normalization (3.2)–(3.3) singles out a pseudo-Riemannian metric $g$ of signature $(2, 2)$ that is concordant with the almost Grassmann structure $AG(1, 3)$ given on the manifold $M^4$.

When we use the first method, we also need to use relations among the components of the tensor $b = (b^1, b^2)$ that follow from equations (1.12) and (1.13). There are 16 equations (1.12) and 256 equations (1.13). In order to get the relations along components of the tensor $b$, first we prove the following two lemmas which list 10 independent relations among relations (1.12) and 16 independent relations among relations (1.13).

**Lemma 3.2** For $p = q = 2$, equations (1.12) take the form:

$$\begin{align*}
&b_{12}^{111} - b_{33}^{111} = 0, \quad b_{11}^{22} - b_{34}^{22} = 0, \\
&b_{22}^{122} - b_{31}^{122} = 0, \quad b_{34}^{122} - b_{33}^{122} = 0, \\
&2b_{11}^{112} - b_{11}^{212} - b_{34}^{212} = 0, \quad 2b_{12}^{312} - b_{12}^{231} - b_{12}^{321} = 0, \\
&2b_{13}^{122} - b_{23}^{222} = 0, \quad 2b_{23}^{222} - b_{23}^{322} = 0.
\end{align*}$$  \hspace{1cm} (3.4)

and

$$\begin{align*}
&2b_{11}^{112} + b_{21}^{312} - 2b_{12}^{312} - b_{12}^{212} + b_{11}^{211} - b_{32}^{212} = 0, \\
&b_{11}^{321} + 2b_{13}^{221} - b_{23}^{221} - b_{23}^{321} + b_{22}^{222} - b_{33}^{321} = 0.
\end{align*}$$  \hspace{1cm} (3.5)
Proof. The proof is straightforward. Equations (3.4) are obtained from (1.12) taking \( \gamma = \delta = 1, \, k = l = 3; \gamma = \delta = 2, \, k = l = 3; \gamma = \delta = 1, \, k = l = 4, \) and \( \gamma = \delta = 2, \, k = l = 4, \) respectively. Equations (3.5) are obtained from (1.12) taking \( \gamma = 1, \, \delta = 2, \, k = l = 3; \gamma = 1, \delta = 2, \, k = l = 4; \gamma = 1, \, \delta = 1, \, k = 3, \, l = 4, \) and \( \gamma = 2, \, k = 3, \, l = 4, \) respectively. Finally, equations (3.6) are obtained from (1.12) taking \( \gamma = 1, \, \delta = 2, \, k = 3, \, l = 4 \) and \( \gamma = 2, \, \delta = 1, \, k = 3, \, l = 4, \) respectively. The remaining equations (1.12) do not give new conditions. Note that while obtaining (3.4)–(3.6), one should use conditions (1.10) and (1.11).

The remaining relations (1.12) are satisfied identically or lead to the same relations (3.4)–(3.6).

Lemma 3.3 For \( p = q = 2, \) equations (1.13) take the form:

\[
\begin{cases}
    b_{233}^{121} - b_{334}^{411} = 0, & b_{133}^{212} - b_{334}^{422} = 0, \\
    b_{244}^{121} - b_{433}^{411} = 0, & b_{244}^{212} - b_{334}^{422} = 0,
\end{cases}
\tag{3.7}
\]

\[
\begin{cases}
    b_{233}^{121} = b_{334}^{412} + b_{433}^{422}, & b_{233}^{212} = b_{334}^{412} + b_{433}^{422}, \\
    b_{234}^{121} = b_{344}^{412} + b_{434}^{422}, & b_{234}^{212} = b_{344}^{412} + b_{434}^{422}, \\
    b_{334}^{211} = b_{434}^{411} + b_{433}^{411}, & b_{344}^{211} = b_{434}^{411} + b_{433}^{411}, \\
    b_{334}^{222} = b_{434}^{422} + b_{433}^{422}, & b_{344}^{222} = b_{434}^{422} + b_{433}^{422},
\end{cases}
\tag{3.8}
\]

and

\[
\begin{cases}
    b_{334}^{312} + b_{334}^{222} = b_{434}^{422} + b_{433}^{422}, & b_{334}^{312} + b_{334}^{222} = b_{434}^{422} + b_{433}^{422}, \\
    b_{334}^{312} + b_{334}^{222} = b_{434}^{422} + b_{433}^{422}, & b_{334}^{312} + b_{334}^{222} = b_{434}^{422} + b_{433}^{422}.
\end{cases}
\tag{3.9}
\]

Proof. The proof is also straightforward. Equations (3.7) are obtained from (1.13) \( \alpha = \gamma = 2, \beta = \delta = 1, \, i = j = 4, \, k = l = 3; \alpha = \gamma = 1, \beta = \delta = 2, \, i = j = 4, \, k = l = 3; \alpha = \gamma = 2, \beta = \delta = 1, \, i = j = 3, \, k = l = 4; \) and \( \alpha = \gamma = 1, \beta = \delta = 2, \, i = j = 3, \, k = l = 4, \) respectively.

Equations (3.8) are obtained from (1.13) taking \( \alpha = \beta = \delta = 1, \, i = j = 4, \, k = l = 3; \alpha = \beta = \gamma = 2, \, \delta = 1, \, i = k = j = 4, \, j = l = 3; \alpha = \beta = \gamma = 1, \, \delta = 2, \, \alpha = j = 3, \, k = \delta = 4; \alpha = \beta = \gamma = 2, \, \delta = 1, \, i = k = j = 4, \, j = l = 3; \alpha = \beta = \gamma = 1, \, \delta = 2, \, i = j = 3, \, k = l = 4; \) and \( \alpha = \beta = \gamma = 2, \delta = 1, \, i = j = k = 4, \, l = 3, \) respectively.

Finally, equations (3.9) are obtained from (1.13) taking \( \alpha = \beta = \delta = 2, \, \gamma = 1, \, i = j = k = 3, \, l = 3; \alpha = \beta = \gamma = 1, \delta = 2, \, i = k = l = 4, \, k = 3; \alpha = \beta = \delta = 1, \, \gamma = 2, \, i = j = k = 3, \, l = 4; \) and \( \alpha = \beta = \gamma = 2, \delta = 1, \, i = j = k = 3, \, l = 3, \) respectively.

The remaining relations (1.13) are satisfied identically or lead to the same relations (3.7)–(3.9).

Theorem 3.4 The components of the tensor \( b \) of an almost Grassmann structure \( AG(1, 3) \) satisfy the following conditions:

\[
\begin{cases}
    b_{233}^{121} = b_{334}^{411} = b_{233}^{212} = b_{334}^{422} = b_{244}^{421} = b_{334}^{421} = b_{244}^{422} = b_{334}^{422} = 0,
\end{cases}
\tag{3.10}
\]

\[
\begin{cases}
    b_{133}^{121} = 0, & b_{133}^{212} = b_{433}^{412} = b_{433}^{412} = 0, \\
    b_{144}^{121} = 0, & b_{144}^{212} = b_{344}^{412} = b_{344}^{412} = 0, \\
    b_{144}^{212} = 0, & b_{144}^{212} = b_{223}^{212} = b_{223}^{212} = 0, \\
    b_{334}^{322} = 0, & b_{334}^{322} = b_{134}^{322} = b_{134}^{322} = 0.
\end{cases}
\tag{3.11}
\]
structure and the basis forms

\[ \omega_{C} \]

conformal curvature tensor

\{ \text{AG} \}

Since an \text{AG} (3.9) and the second equation of (3.6). The remaining equations of (3.13) are obtained from the last two equations of relations (3.13), we obtain

Equations (3.15) and the first two equations (3.13) imply that

Equations (3.15) and (3.16) gives relations (3.12). \( \blacksquare \)

It is easy to see from relations (3.10)–(3.12) that there are only 10 independent components of the curvature tensor \( b \) of an almost Grassmann structure \( \text{AG}(1,3) \), and that we were able to obtain relations (3.10)–(3.12) only because for \( p = q = 2 \) conditions (1.13) became conditions for the components of the curvature tensor \( b \) of an almost Grassmann structure \( \text{AG}(1,3) \).

We make the following remark on a four-dimensional almost Grassmann structure \( \text{AG}(1,3) \). Since an \( \text{AG}(1,3) \)-structure is equivalent to a \( \text{CO}(2,2) \)-structure, we can calculate the conformal curvature tensor \( C = C_{\alpha} + C_{\beta} \), where \( C_{\alpha} = \{ a_{0}, a_{1}, a_{2}, a_{3}, a_{4}, a_{5} \} \) and \( C_{\beta} = \{ b_{0}, b_{1}, b_{2}, b_{3}, b_{4}, b_{5} \} \) (see [2], Section 5.1) of the latter.

We impose the following relations between the basis forms \( \omega^{1}, \omega^{2}, \omega^{3}, \omega^{4} \) of a \( \text{CO}(2,2) \)-structure and the basis forms \( \omega_{1}^{3}, \omega_{1}^{4}, \omega_{2}^{3}, \omega_{2}^{4} \) of an equivalent \( \text{AG}(1,3) \)-structure:

\[ \begin{align*}
\omega^{1} &= \frac{1}{\sqrt{2}} \omega_{1}^{3}, & \omega^{2} &= \frac{1}{\sqrt{2}} \omega_{1}^{4}, \\
\omega^{3} &= \frac{1}{\sqrt{2}} \omega_{2}^{3}, & \omega^{4} &= \frac{1}{\sqrt{2}} \omega_{2}^{4}.
\end{align*} \quad (3.17)
\]

The factor \( \frac{1}{\sqrt{2}} \) can be explained by the fact that the metric of a \( \text{CO}(2,2) \)-structure usually is written in the form \( g = 2(\omega^{1}\omega^{4} - \omega^{2}\omega^{3}) \) and that the metric of an equivalent \( \text{AG}(1,3) \)-structure has the form \( g = \omega_{1}^{3}\omega_{2}^{4} - \omega_{2}^{3}\omega_{1}^{4} \) (see Ch. 5 and 7 of [2]), and relations (3.13) make these metrics equal.
The calculations involving the apparatus developed in Ch. 5 of the book [1] and the formulas (1.4)–(1.5), (3.4)–(3.5), give the following relations between the independent 10 components of the curvature tensor \( b \) of the structure AG(1, 3) (see Theorem 3.4) and independent 10 components of the tensor \( C \) of the equivalent pseudoconformal structure CO(2, 2):

\[
\begin{align*}
  a_0 &= b_{333}^{112}, \\
  a_1 &= b_{433}^{112} - b_{343}^{112} - b_{334}^{112}, \\
  a_2 &= b_{334}^{321} = b_{343}^{321} = b_{344}^{321} = b_{334}^{412}, \\
  a_3 &= b_{343}^{321} = b_{334}^{321} = b_{344}^{321}, \\
  a_4 &= b_{334}^{412}, \\
  b_0 &= b_{243}^{111}, \\
  b_1 &= b_{243}^{121} = b_{243}^{121} = b_{243}^{121}, \\
  b_2 &= b_{134}^{211} = b_{134}^{211} = b_{134}^{211}, \\
  b_3 &= b_{134}^{212} = b_{134}^{212} = b_{134}^{212}, \\
  b_4 &= b_{134}^{212}.
\end{align*}
\]

(3.18)

The conditions \( C_\alpha = 0, C_\beta = 0 \) and \( C = 0 \) are necessary and sufficient for a CO(2, 2)-structure (and consequently for an equivalent (AG(1, 3)-structure) to be \( \alpha \)-semiintegrable, \( \beta \)-semiintegrable and locally flat, respectively. Moreover, if we find components of \( C_\alpha \) and \( C_\beta \), then by investigating the roots of polynomials

\[ C_\alpha (\lambda) = a_0 \lambda^4 - 4a_1 \lambda^3 + 6a_2 \lambda^2 - 4a_3 \lambda + a_4 \]

and

\[ C_\beta (\mu) = b_0 \mu^4 - 4b_1 \mu^3 + 6b_2 \mu^2 - 4b_3 \mu + b_4 \]

we can make some additional conclusions on integrability of the distributions \( \Delta_\alpha (\lambda) \) and \( \Delta_\beta (\mu) \) of the fiber bundles \( E_\alpha \) and \( E_\beta \) associated with a CO(2, 2)-structure (see [1], Section 5.4).

Note that equations (3.18) show that for a four-dimensional almost Grassmann structure AG(1, 3) and for an equivalent pseudoconformal structure CO(2, 2) we have

i) The condition of \( \alpha \)-semiintegrability \( C_\alpha = 0 \) is equivalent to the condition \( b_{ijkl}^{\alpha \beta \gamma} = 0 \), that is, \( \Omega_\alpha^2 = 0 \).

ii) The condition of \( \beta \)-semiintegrability \( C_\beta = 0 \) is equivalent to the condition \( b_{ijkl}^{\beta \gamma \delta} = 0 \), that is, \( \Omega_\alpha^2 = 0 \).

iii) The condition of local flatness \( C = 0 \) is equivalent to the vanishing of the tensor \( b = \{b_\alpha^1, b_\beta^2\} \) (cf. the end of Section 1).

Note that only for a four-dimensional almost Grassmann structure AG(1, 3) the conditions of \( \alpha \)- and \( \beta \)-semiintegrability are equivalent to \( \Omega_\alpha^2 = 0 \) and \( \Omega_\beta^2 = 0 \), respectively: only for such structures the components of the tensor \( b \) satisfy relations (3.10)–(3.12) by means of which the conditions \( b_{ijkl}^{\alpha \beta \gamma \delta} = 0 \) and \( b_{ijkl}^{\beta \gamma \delta} = 0 \) of \( \alpha \)- and \( \beta \)-semiintegrability imply the vanishing the components \( b_{ijkl}^{\beta \gamma \delta} \) and \( b_{ijkl}^{\alpha \beta \gamma \delta} \) themselves.

2. The second method is the method of direct integration of equations (2.9) and (2.41) defining integral submanifolds of distributions \( \Delta_\alpha \) and \( \Delta_\beta \) of semiintegrable almost Grassmann structures. Let us describe this method in more detail.
In the proof of Theorem 2.5 we wrote the equations of the submanifolds $V_\alpha$ and $V_\beta$ in the form (2.9) and (2.41), respectively. Note that $\dim V_\alpha = p$ and $\dim V_\beta = q$. An almost Grassmann structure is $\alpha$-semiintegrable (or $\beta$-semiintegrable) if and only if the system of equations (2.9) (respectively, (2.41)) is completely integrable.

We write the matrix of basis forms of the almost Grassmann manifold $AG(p-1,p+q-1)$ in more detail:

$$
(\omega^i_\alpha) = \begin{pmatrix}
\omega^{p+1}_1 & \omega^{p+1}_2 & \ldots & \omega^{p+1}_p \\
\omega^{p+2}_1 & \omega^{p+2}_2 & \ldots & \omega^{p+2}_p \\
\vdots & \vdots & \ddots & \vdots \\
\omega^{p+q}_1 & \omega^{p+q}_2 & \ldots & \omega^{p+q}_p 
\end{pmatrix}.
$$

(3.19)

Here $\alpha = 1, \ldots, p$ is the column number, and $i = p+1, \ldots, p+q$ is the row number.

The condition (2.9) of $\alpha$-integrability means that on integral submanifolds $V_\alpha$ of the distribution $\Delta_\alpha$ (see Definition 1.2) the rows of the matrix $(\omega^i_\alpha)$ are proportional, and the entries of every nonzero row are basis forms on $V_\alpha$.

The condition (2.41) of $\beta$-integrability means that on integral submanifolds $V_\beta$ of the distribution $\Delta_\beta$ (see Definition 1.2) the columns of the matrix $(\omega^i_\beta)$ are proportional, and the entries of every nonzero row are basis forms on $V_\beta$.

For $p = q = 2$, equations (2.9) of submanifolds $V_\alpha$ can be written in the form

$$
\lambda \omega^1_1 + \omega^1_2 = 0, \quad \lambda \omega^2_1 + \lambda^2_2 = 0,
$$

(3.20)

where $\lambda = -\frac{s}{2}$ and $\omega^1_1 \wedge \omega^2_1 \neq 0$, and equations (2.41) of submanifolds $V_\beta$ can be written in the form

$$
\mu \omega^1_3 + \omega^1_4 = 0, \quad \mu \omega^2_3 + \omega^2_4 = 0,
$$

(3.21)

where $\mu = -\frac{s}{2}$ and $\omega^2_3 \wedge \omega^2_4 \neq 0$.

If $p > 2$ or $q > 2$, the systems (3.20) and (3.21) have different forms. For example, let us consider the case $p = 2$ and $q = 3$. In this case equations (2.9) can be written as follows:

$$
\begin{cases}
\lambda_1 \omega^1_1 + \omega^1_2 = 0, & \lambda_2 \omega^2_1 + \omega^2_3 = 0, \\
\lambda_1 \omega^2_1 + \omega^2_2 = 0, & \lambda_2 \omega^2_1 + \omega^2_3 = 0,
\end{cases}
$$

(3.22)

where $\lambda_1 = -\frac{s}{2}$ and $\lambda_2 = -\frac{s}{3}$ and $\omega^1_1 \wedge \omega^2_2 \neq 0$, and equations (2.41) take the form:

$$
\begin{cases}
\mu \omega^1_3 + \omega^1_4 = 0, \\
\mu \omega^2_3 + \omega^2_4 = 0, \\
\mu \omega^3_5 + \omega^3_6 = 0,
\end{cases}
$$

(3.23)

where $\mu = -\frac{s}{2}$ and $\omega^3_4 \wedge \omega^3_5 \wedge \omega^3_6 \neq 0$.

For these cases to prove that an almost Grassmann structure is $\alpha$-semiintegrable (resp. $\beta$-semiintegrable) we must prove that the system (3.20) or (3.22) (resp. the system (3.21) or (3.23)) is completely integrable. If it is possible, we integrate these systems and find $\lambda$ or $\lambda_1$ and $\lambda_2$ (resp. $\mu$) and closed form equations of submanifolds $V_\alpha$ (resp. $V_\beta$).

**3.** We next construct examples of semiintegrable and integrable almost Grassmann structures $AG(1,3)$. To prove that they are semiintegrable, we will apply one of two methods indicated above.

**Example 3.5** Suppose that $x, y, u, \text{ and } v$ are coordinates in $M^4$, and that the basis 1-forms $\omega^i_\alpha$ of an almost Grassmann structure $AG(1,3)$ are

$$
\begin{cases}
\omega^1_1 = dx + f(u)dy, & \omega^3_2 = dy, \\
\omega^1_2 = du, & \omega^3_4 = dv.
\end{cases}
$$

(3.24)
Taking exterior derivatives of equations (3.22) by means of (1.4), (1.5) and (3.22), we arrive at the following exterior quadratic equations:
\[
\begin{aligned}
(\omega + \omega^1_1 - \omega^3_1) \wedge \omega^2_1 + \omega^4_1 \wedge \omega^3_1 &= f'(u)\omega^1_1 \wedge \omega^2_1, \\
(\omega - \omega^1_1 + \omega^3_1) \wedge \omega^2_1 + \omega^4_1 \wedge \omega^3_1 &= 0, \\
(\omega + \omega^1_1 + \omega^2_1) \wedge \omega^3_1 + \omega^4_1 \wedge \omega^3_1 &= 0, \\
(\omega - \omega^3_1 + \omega^3_1) \wedge \omega^1_1 + \omega^4_1 \wedge \omega^3_1 &= 0.
\end{aligned}
\tag{3.25}
\]

First, equations (3.25) prove that the form ω is a principal form. Thus, by Lemma 3.1 we have equations (3.2) and (3.3). Second, it follows from (3.25), that the forms ω^2_1, ω^2_1, ω^4_1, ω^3_1 and ω^3_1 - ω^3_1 are principal forms. We will write their expressions as follows:
\[
\begin{aligned}
\omega^2_1 &= \alpha_1 \omega^1_1 + \alpha_2 \omega^2_1 + \alpha_3 \omega^4_1 + \alpha_4 \omega^3_1, \\
\omega^2_1 &= \beta_1 \omega^1_1 + \beta_2 \omega^2_1 + \beta_3 \omega^4_1 + \beta_4 \omega^3_1, \\
\omega^4_1 &= \gamma_1 \omega^1_1 + \gamma_2 \omega^2_1 + \gamma_3 \omega^4_1 + \gamma_4 \omega^3_1, \\
\omega^3_1 &= \delta_1 \omega^1_1 + \delta_2 \omega^2_1 + \delta_3 \omega^4_1 + \delta_4 \omega^3_1, \\
\omega^1_1 + \omega^2_1 &= \sigma_1 \omega^1_1 + \sigma_2 \omega^2_1 + \sigma_3 \omega^4_1 + \sigma_4 \omega^3_1, \\
\omega^1_1 - \omega^3_1 &= \tau_1 \omega^1_1 + \tau_2 \omega^2_1 + \tau_3 \omega^4_1 + \tau_4 \omega^3_1.
\end{aligned}
\tag{3.26}
\]

Substituting (3.2) and (3.26) into equations (3.25) and equating coefficients in the independent exterior forms ω^1_1 ∧ ω^2_1 to 0, we obtain: α_1 = α_2 = α_4 = 0; β_1 = β_2 = β_3 = β_4 = 0; γ_1 = γ_2 = γ_3 = γ_4 = 0; δ_1 = δ_3 = δ_4 = 0; σ_1 = σ_2 = σ_4 = 0; τ_1 = τ_2 = τ_3 = τ_4 = 0, and α_3 = δ_2 = σ_4 = 1/4 f'(u). As a result, equations (3.26) become
\[
\begin{aligned}
\omega^2_1 &= \frac{1}{4} f'(u)\omega^1_1, \\
\omega^3_1 &= \omega^2_1, \\
\omega^4_1 &= \omega^3_1, \\
\omega^1_1 &= -\omega^2_1 = -\omega^3_1 = -\omega^4_1 = \frac{1}{4} f'(u)\omega^2_1.
\end{aligned}
\tag{3.27}
\]

Taking exterior derivatives of (3.27) by means of (1.4), (3.21) and (3.27), we arrive at the following system of exterior quadratic equations:
\[
\begin{aligned}
-\omega^2_1 \wedge \omega^2_1 - \omega^1_1 \wedge \omega^2_1 + \Omega^1_1 &= \frac{1}{4} f'(u)^2 \omega^1_1 \wedge \omega^2_1, \\
-\omega^2_1 \wedge \omega^2_1 - \omega^1_1 \wedge \omega^2_1 + \Omega^1_1 &= 0, \\
-\omega^3_1 \wedge \omega^1_1 - \omega^2_1 \wedge \omega^3_1 + \Omega^2_1 &= 0, \\
-\omega^4_1 \wedge \omega^1_1 - \omega^2_1 \wedge \omega^4_1 + \Omega^3_1 &= \frac{1}{4} f''(u)\omega^1_1 \wedge \omega^2_1 + \frac{1}{4} f'(u)^2 \omega^2_1 \wedge \omega^3_1, \\
-\omega^3_1 \wedge \omega^1_1 - \omega^3_1 \wedge \omega^4_1 + \Omega^1_1 &= \frac{1}{4} f''(u)\omega^1_1 \wedge \omega^2_1, \\
-\omega^2_1 \wedge \omega^3_1 - \omega^3_1 \wedge \omega^1_1 + \Omega^2_1 &= \frac{1}{4} f''(u)\omega^1_1 \wedge \omega^2_1, \\
-\omega^1_1 \wedge \omega^2_1 - \omega^2_1 \wedge \omega^3_1 + \Omega^3_1 &= \frac{1}{4} f''(u)\omega^1_1 \wedge \omega^2_1.
\end{aligned}
\tag{3.28}
\]

where Ω^i_1 and Ω^i_2 are the curvature 2-forms defined by (1.5). Substituting the values of these 2-forms from (1.5) and the values of the forms ω^i_1 ∧ ω^j_1 from (3.3) into equations (3.28) and equating coefficients in independent exterior forms ω^i_1 ∧ ω^j_1 to 0, in addition to equations (3.10)-(3.12) (which hold for any AG(1,3)-structure) we obtain the following additional conditions:
\[
\begin{aligned}
b^{11}_{134} &= b^{11}_{234} = b^{11}_{343} = b^{11}_{131} = b^{11}_{132} = b^{11}_{232} = 0, \\
b^{12}_{134} &= b^{12}_{134} = b^{12}_{234} = b^{12}_{343} = b^{12}_{342} = 0, \\
b^{21}_{133} &= b^{21}_{133} = b^{21}_{343} = b^{21}_{342} = 0, \\
b^{22}_{134} &= b^{22}_{134} = b^{22}_{234} = b^{22}_{233} = 0, \\
b^{31}_{144} &= b^{31}_{144} = b^{31}_{244} = b^{31}_{344} = 0.
\end{aligned}
\]
that is, the only nonvanishing components of the object \( b = \{b^1, b^2\} \) are the components \( b^{312}_{1314}, b^{321}_{1314} \) and \( b^{312}_{3411} = -b^{412}_{4414} \).

In addition, we find that all coefficients of (3.3), except \( E_4 \) and \( C_4 \), equal to 0, and for the coefficients \( E_4 \) and \( C_4 \) we find the following values:

\[
C_4 = -\frac{1}{4} f''(u), \quad E_4 = -\frac{1}{4} (f'(u))^2.
\]

As a result, equations (3.3) become

\[
\begin{aligned}
\omega_3^1 &= 0, \quad \omega_4^1 = \frac{1}{4} f''(u) \omega_2^3 \\
\omega_3^2 &= 0, \quad \omega_4^2 = -\frac{1}{4} f''(u) \omega_1^1 - \frac{1}{4} (f'(u))^2 \omega_2^1,
\end{aligned}
\]

and the curvature 2-forms (1.7) become

\[
\begin{cases}
\Omega_1^1 = \Omega_2^2 = \Omega_3^3 = \Omega_4^4 = 0, \\
\Omega_3^1 = -\Omega_1^1 = \frac{1}{4} f''(u) \omega_1^3 \wedge \omega_2^1, \\
\Omega_3^2 = \frac{1}{4} f''(u) \omega_1^3 \wedge \omega_2^3 - \frac{1}{4} f''(u) \omega_2^3 \wedge \omega_1^4.
\end{cases}
\]

Since \( b^{\gamma \nu \delta}_{\alpha \beta \kappa \lambda} = 0 \), by (2.41), we have \( b^3_\beta = 0 \), and the almost Grassmann structure \( AG(1,3) \) under consideration is \( \beta \)-semiintegrable. This structure is not locally flat: in fact, \( b^{312}_{(333)} = b^{312}_{(444)} = \frac{1}{4} f''(u) \neq 0 \) for a general function \( f(u) \), and thus \( b^3_\alpha \neq 0 \), that is, the structure is not \( \alpha \)-semiintegrable. Note that \( f''(u) = 0 \) if and only if \( f(u) = au + b \), where \( a \) and \( b \) are constants. In this case the structure is \( \beta \)-semiintegrable, and consequently it is locally flat.

As we noted earlier, a structure \( AG(1,3) \) is equivalent to a conformal \( CO(2,2) \)-structure. Note that the \( CO(2,2) \)-structure corresponding to the \( \beta \)-semiintegrable structure \( AG(1,3) \) we have constructed in this example is self-dual.

By proving the local existence of a \( \beta \)-semiintegrable structure \( AG(1,3) \), we also proved a local existence of a self-dual \( CO(2,2) \)-structure. On the global existence of four-dimensional semiintegrable smooth compact oriented Riemannian manifolds see [10] and [11].

We will find now the metric of this \( CO(2,2) \)-structure.

To this end, we recall that the equation of the Segre cone of the \( AG(1,3) \)-structure (or the asymptotic cone of the corresponding \( CO(2,2) \)-structure) has the form (1.2) or \( \omega_1^3 \omega_2^1 - \omega_1^4 \omega_2^3 = 0 \). Thus the fundamental form of the \( CO(2,2) \)-structure is

\[
g = \omega_1^3 \omega_2^1 - \omega_1^4 \omega_2^3,
\]

or by (3.14),

\[
g = dx dv + f(u) dy dv - dy du.
\]

The quadratic form (3.31) defines on the manifold \( M^4 \) a pseudo-Riemannian metric of signature (2, 2) that is conformally equivalent to the almost Grassmann structure \( AG(1,3) \) with the basis forms (3.24).

Let us apply the second method to prove that the structure (3.24) is \( \beta \)-semiintegrable and to find conditions under which this structure is \( \alpha \)-semiintegrable.

For the structure (3.24), the equations (3.21) take the form

\[
\mu dy + dx + f(u) dy = 0, \quad \mu dv + du = 0,
\]

where \( dy \wedge dv \neq 0 \). Exterior differentiation of (3.32) gives

\[
(d\mu + f'(u) du) \wedge dy = 0, \quad (d\mu + f'(u) du) \wedge dv = 0.
\]
Since $dy \wedge dv \neq 0$, it follows from (3.33) that
\[ d\mu + f'(u)du = 0. \]  
(3.34)

The solution of (3.34) is
\[ \mu = C_1 - f(u), \]  
(3.35)
where $C_1$ is an arbitrary constant. Substituting this value of $\mu$ into equations (3.32), we find the following differential equations of submanifolds $V_\beta$:
\[ C_1 dy + dx = 0, \quad (C_1 - f(u))dv + du = 0. \]  
(3.36)

By integration we find the following closed form equations of submanifolds $V_\beta$:
\[ x = C_2 - C_1 y, \quad v = C_3 - \int \frac{du}{C_1 - f(u)}, \]  
(3.37)
where $C_2$ and $C_3$ are arbitrary constants.

Thus we proved again the structure $AG(1, 3)$ with the basis forms (3.14) is $\beta$-semiintegrable. In addition, we proved that the integral submanifolds $V_\beta$ of this structure are defined by equations (3.37), the family of these submanifolds depends on three parameters $C_1, C_2$ and $C_3$, and through any point $(x_0, y_0, u_0, v_0)$ of $M^4$ there passes an one-parameter family of submanifolds $V_\beta$.

To find conditions of $\alpha$-semiintegrability of this structure, we first specialize equations (3.20) for it:
\[ \lambda(dx + f(u)dy) + du = 0, \quad \lambda dy + dv = 0, \]  
(3.38)
where $dx \wedge dy \neq 0$. Exterior differentiation of (3.38) gives
\[ (d\lambda + \lambda^2 f'(u)dy) \wedge dx = 0, \quad (d\lambda + \lambda^2 f'(u)dy) \wedge dy = 0. \]  
(3.39)

Since $dx \wedge dy \neq 0$, it follows from (3.39) that
\[ d\lambda + \lambda^2 f'(u)dy = 0. \]  
(3.40)

Exterior differentiation of (3.40) leads to the following equation:
\[ \lambda^3 f''(u)dx \wedge dy = 0. \]  
(3.41)

Since $dx \wedge dy \neq 0$, in general the structure (3.24) is not $\alpha$-semiintegrable. It will be $\alpha$-semiintegrable if and only if $f''(u) = 0$, or
\[ f(u) = au + b. \]  
(3.42)

Substituting $f(u) = au + b$ into equation (3.40), we find that
\[ d\lambda + \lambda^2 ady = 0. \]  
(3.43)

The solution of (3.43) is
\[ \lambda = \frac{1}{ay + C_4}, \]  
(3.44)
where $C_4$ is an arbitrary constant. Substituting $f(u) = au + b$ and $\lambda$ from (3.44) into equations (3.38), we find the following differential equations of submanifolds $V_\alpha$:
\[ dx + ad(uy) + bdy + C_4 du = 0, \quad dy + (ay + C_4)dv = 0. \]  
(3.45)
By integration we find the following closed form equations of submanifolds $V_\alpha$:

$$x + auy + by + C_4y = C_5, \quad \frac{1}{a} \log(ay + C_4) + v = C_6,$$

(3.46)

where $C_5$ and $C_6$ are arbitrary constants.

Thus, we proved again that in general the structure $AG(1, 3)$ with the basis forms (3.24) is not $\alpha$-semiintegrable, and that it will be $\alpha$-semiintegrable and consequently integrable if and only if $f(u) = au + b$. If it is the case, the closed form equations of submanifolds $V_\alpha$ and $V_\beta$ are equations (3.46) and (3.37). If $f(u) = au + b$, the family of submanifolds $V_\alpha$ on the manifold $M^4$ carrying the $AG(1, 3)$-structure with the basis forms (3.24) depends on three parameters $C_1, C_5$ and $C_6$, and through any point $(x_0, y_0, w_0, v_0)$ of $M^4$ there passes a one-parameter family of submanifolds $V_\alpha$.

Let us indicate three more examples of almost Grassmann structures similar to the structure (3.24).

**Example 3.6**

$$\begin{cases} 
\omega_1^3 = dx, & \omega_2^3 = dy, \\
\omega_1^4 = du + g(x)dv, & \omega_2^4 = dv.
\end{cases}$$

(3.47)

**Example 3.7**

$$\begin{cases} 
\omega_1^3 = dx, & \omega_2^3 = h(v)dx + dy, \\
\omega_1^4 = du, & \omega_2^4 = dv.
\end{cases}$$

(3.48)

and

**Example 3.8**

$$\begin{cases} 
\omega_1^3 = dx, & \omega_2^3 = dy, \\
\omega_1^4 = du, & \omega_2^4 = k(y)du + dv.
\end{cases}$$

(3.49)

As was the case for the structure (3.24), each of the structures (3.47)-(3.49) is $\beta$-semiintegrable and in general not $\alpha$-semiintegrable. These structures will be $\alpha$-semiintegrable and consequently integrable if and only if the functions $g(x)$, $h(v)$ and $k(y)$ are linear functions of $x$, $v$, and $y$, respectively.

If we apply formulas (3.18) to Examples 3.5–3.8, then, we obtain that $C_\beta = 0$ for all these examples (that is, the corresponding structures are $\beta$-semiintegrable), and that the only nonvanishing components of the tensor $C_\alpha$ are $a_3$ for Examples 3.5 ($a_3 = -\frac{f''(u)}{4}$) and 3.7 ($a_3 = \frac{h''(v)}{4}$) and $a_1$ for Examples 3.6 ($a_1 = \frac{g''(x)}{4}$) and 3.8 ($a_1 = \frac{-k''(y)}{4}$). It follows that in the first case the polynomial $C_\alpha$ has the triple root $\lambda_1 = \lambda_2 = \lambda_3 = \infty$ and the simple root $\lambda_4 = 0$, and in the second case it has the simple root $\lambda_1 = \infty$ and the triple root $\lambda_2 = \lambda_3 = \lambda_4 = 0$. According to Section 5.4 of the book [1], this means that the fiber bundle $E_\alpha$ has two invariant distributions $\Delta_\alpha(\infty)$ and $\Delta_\alpha(0)$, and the distribution corresponding to a multiple root is integrable. Moreover, it is easy to prove that the distribution corresponding to a simple root is also integrable. For all cases the integral submanifolds $V_\alpha$ are defined by the equations $u = C_3, v = C_4$ (for $\lambda = \infty$) and by the equations $x = C_1, y = C_2$ (for $\lambda = 0$).

The next example was considered in [2]. However, since results obtained in [2] contain inaccuracies and the result is wrong (according to [3], the structure of this example is of general type), we give here a complete investigation of this example.
Example 3.9 Suppose again that \( x, y, u \) and \( v \) are coordinates in \( M^4 \), and that the basis \( 1 \)-forms \( \omega^i \) of an almost Grassmann structure \( AG(1, 3) \) are

\[
\begin{align*}
\omega_1^1 &= dx + f(u)dy, \quad \omega_2^1 = dy, \\
\omega_1^4 &= du + g(y)dv, \quad \omega_2^4 = dv.
\end{align*}
\] (3.50)

The metric of the \( CO(2, 2) \)-structure which is equivalent to the \( AG(1, 3) \)-structure with basis forms (3.40) is

\[
g = dx dv - dy du + (f(u) - g(y))dy dv.
\] (3.51)

In this case we have

\[
\begin{align*}
\omega_1^2 &= g'(y)\omega_2^1 + \frac{1}{2}f'(u)\omega_1^1 - \frac{1}{2}f'(u)g(y)\omega_2^2, \\
\omega_2^2 &= \frac{1}{2}f'(u)\omega_2^1, \quad \omega_1^1 = 0, \quad \omega_1^4 = 0, \\
\omega_1^4 &= -\omega_2^1 = \omega_3^1 = -\omega_4^1 = \frac{1}{2}f'(u)\omega_2^2; \\
C_4 &= -\frac{1}{4}f''(u), \quad E_4 = \frac{1}{2}f''(u)g(y) - \frac{1}{4}(f'(u))^2; \\
b^{312}_{1444} &= b^{312}_{1444} = \frac{1}{4}f''(u), \quad b^{312}_{3444} = -b^{312}_{4444} = \frac{1}{4}f''(u); \\
\begin{align*}
\omega_1^3 &= 0, \quad \omega_1^4 = -\frac{1}{4}f''(u)\omega_2^1, \\
\omega_2^3 &= 0, \quad \omega_2^4 = -\frac{1}{4}f''(u)\omega_2^1 + \left[\frac{1}{2}f''(u)g(y) - \frac{1}{4}(f'(u))^2\right]\omega_2^2; \\
\Omega^1 &= \Omega^2 = \omega_1^3 = \omega_1^4 = 0, \\
\Omega^3 &= -\Omega^4 = \frac{1}{4}f''(u)\omega_1^1 \land \omega_2^1, \\
\Omega^4 &= \frac{1}{4}f''(u)\omega_1^1 \land \omega_2^1 - \frac{1}{4}f''(u)\omega_2^2 \land \omega_4^1.
\end{align*}
\] (3.52)

We have \( b^{3\gamma\delta}_{\alpha k l} = 0 \), and by (2.41), \( b_1^1 = 0 \), and \( -b_{4444}^{312} = b_{3444}^{312} = -\frac{1}{4}f''(u) \neq 0 \). This implies that \( b_2^2 \neq 0 \), and the almost Grassmann structure \( AG(1, 3) \) with the basis forms (3.50) is \( \beta \)-semiintegrable but not locally flat.

If we apply the method of direct integration, we find that equations (3.21) take the form

\[
\mu dy + dx + f(u)dy = 0, \quad \mu dv + du + g(y)dv = 0,
\]
where \( dy \land dv \neq 0 \). Exterior differentiation of these equations leads to

\[
(d\mu + f'(u)du) \land dy = 0, \quad (d\mu + g'(y)dy) \land dv = 0.
\]

It follows that

\[
d\mu + f'(u)du + g'(y)dy = 0.
\]

By integration of the first equation, we find that

\[
\mu = C_1 - f(u) - g(y),
\] (3.57)

where \( C_1 \) is a constant, and the following closed form equations of submanifolds \( V_\beta \):

\[
x = \int g(y)dy - C_1 y + C_2, \quad v = C_3 - \int \frac{du}{C_1 - f(u)}.
\] (3.58)

where \( C_2 \) and \( C_3 \) are constants. Hence we proved again that the almost Grassmann structure with the base forms (3.59) is \( \beta \)-semiintegrable. In addition we proved that the family of
submanifolds $V_\alpha$ on the manifold $M^4$ carrying the $AG(1,3)$-structure with the basis forms (3.50) depends on three parameters $C_1, C_2$ and $C_3$, and through any point $(x_0, y_0, u_0, v_0)$ of $M^4$ there passes a one-parameter family of submanifolds $V_\alpha$.

If we look for conditions for $\alpha$-semiintegrability of this structure, then after writing equations (3.20) and taking their exterior derivatives we come to equations (3.39) which imply (3.40) and (3.41). So this structure is $\alpha$-semiintegrable and subsequently integrable if and only if $f(u) = au + b$. Moreover, the expression for $\lambda$ and the closed form equations of submanifolds $V_\alpha$ are (3.44) and (3.46).

Note that if $g(y) = 0$, equations (3.57) and (3.58) coincide with equations (3.35) and (3.37).

Note that the application of formulas (3.18) gives the same values for 10 components of the tensor of conformal curvature that were obtained in Example 3.5.

Next we consider an example of an $\alpha$-semiintegrable almost Grassmann structure $AG(1,3)$.

**Example 3.10** Suppose that the basis 1-forms $\omega_\alpha$ of an almost Grassmann structure $AG(1,3)$ are

\[
\begin{align*}
\omega_1^3 &= dx + p(y)du, \quad \omega_2^3 = dy, \\
\omega_1^4 &= du, \quad \omega_2^4 = dv.
\end{align*}
\]

The metric of the $CO(2,2)$-structure which is equivalent to the $AG(1,3)$-structure with basis forms (3.59) is

\[
g = dx dv - dy du + p(y) dvdu.
\]

In this case we have

\[
\begin{align*}
\omega_1^3 &= -\frac{1}{3} p'(y) \omega_1^4, \quad \omega_1^4 = 0, \\
\omega_2^3 &= -\frac{1}{4} p'(y) \omega_2^4, \quad \omega_2^4 = 0, \\
\omega_1^1 &= -\omega_2^2 = \omega_3^3 = -\omega_4^4 = -\frac{1}{4} p'(y) \omega_4^3; \\
B_4 &= -\frac{1}{4} p''(y), \quad E_4 = -\frac{1}{4} (p'(y))^2;
\end{align*}
\]

\[
b_{12}^{134} = b_{13}^{212} = b_{13}^{231} = -\frac{1}{4} p''(y);
\]

\[
\begin{align*}
\omega_1^3 &= 0, \quad \omega_1^4 = 0, \\
\omega_2^3 &= -\frac{1}{2} p''(y) \omega_2^4, \quad \omega_2^4 = -\frac{1}{4} p''(y) \omega_2^3 - \frac{1}{4} (p'(y))^2 \omega_2^3;
\end{align*}
\]

\[
\begin{align*}
\Omega_1^3 &= \Omega_2^3 = \frac{1}{2} p''(y) \omega_2^3 \land \omega_2^4, \quad \Omega_2^3 = \Omega_3^3 = \Omega_3^4 = \Omega_4^1 = 0, \\
\Omega_1^4 &= -\frac{1}{4} p''(y) (\omega_2^3 \land \omega_2^4 + \omega_3^3 \land \omega_1^4).
\end{align*}
\]

In this case $b_{jkl}^{\gamma\delta} = 0$, that is, $b_{13}^1 = 0$, and the almost Grassmann structure $AG(1,3)$ with the basis forms (3.59) is $\alpha$-semiintegrable. However, in this case $p_{jkl}^{\beta\gamma\delta} \neq 0$ since we have $b_{13}^{112} = -\frac{1}{4} p''(y)$, that is $b_{13}^1 \neq 0$, and in general the almost Grassmann structure $AG(1,3)$ with the basis forms (3.59) is not $\beta$-semiintegrable, and consequently, it is not locally flat.

If we apply the second method to the structure (3.59), we find that

\[
\lambda = -\frac{1}{p(y) + C_1};
\]

and the following closed form equations of submanifolds $V_\alpha$:

\[
x = C_2 - C_1 u, \quad v = C_3 + \int \frac{dy}{p(y) + C_3},
\]
where \( C_1, C_2 \) and \( C_3 \) are constants.

Thus the family of submanifolds \( V_\alpha \) on the manifold \( M^4 \) carrying the \( AG(1,3) \)-structure with the basis forms (3.59) depends on three parameters \( C_1, C_2 \) and \( C_3 \), and through any point \( (x_0, y_0, u_0, v_0) \) of \( M^4 \) there passes an one-parameter family of submanifolds \( V_\alpha \).

If \( p(y) = ay + b \), then we find that

\[
\mu = au + C_4,
\]

and the following closed form equations of submanifolds \( V_\beta \):

\[
\begin{align*}
x &= C_5 - ay - C_4 y - bu, \\
v &= C_6 - \frac{1}{a}(au + C_4).
\end{align*}
\]

Thus if \( p(y) = ay + b \), then the family of submanifolds \( V_\alpha \) on the manifold \( M^4 \) carrying the \( AG(1,3) \)-structure with the basis forms (3.49) depends on three parameters \( C_4, C_5 \) and \( C_6 \), and through any point \( (x_0, y_0, u_0, v_0) \) of \( M^4 \) there passes a one-parameter family of submanifolds \( V_\alpha \).

Let us indicate examples of three more almost Grassmann structures similar to the structure (3.59).

**Example 3.11**

\[
\begin{align*}
\omega_1^3 &= dx, & \omega_2^3 &= dy + q(x)dv, \\
\omega_1^4 &= du, & \omega_2^4 &= dv.
\end{align*}
\]

**Example 3.12**

\[
\begin{align*}
\omega_1^3 &= dx, & \omega_2^3 &= dy, \\
\omega_1^4 &= du + r(v)dx, & \omega_2^4 &= dv.
\end{align*}
\]

**Example 3.13**

\[
\begin{align*}
\omega_1^3 &= dx, & \omega_2^3 &= dy, \\
\omega_1^4 &= du, & \omega_2^4 &= dv + s(u)dy.
\end{align*}
\]

As was the case for the structure (3.59), each of the structures (3.70) – (3.72) is \( \alpha \)-semiintegrable and in general not \( \beta \)-semiintegrable. These structures will be \( \beta \)-semiintegrable and consequently integrable if and only if the functions \( q(x), r(v) \) and \( s(u) \) are linear functions of \( x, v, \) and \( u \), respectively.

If we apply formulas (3.18) to Examples 3.10–3.13, we obtain that \( C_\alpha = 0 \) for all these examples (that is, the corresponding structures are \( \alpha \)-semiintegrable), and that the only nonvanishing components of the tensor \( C_\beta \) are \( b_3 \) for Examples 3.10 (\( b_3 = -\frac{p'(x)}{4} \)) and 3.12 (\( b_3 = \frac{r''(u)}{4} \)) and \( b_1 \) for Examples 3.11 (\( b_1 = \frac{q'(x)}{4} \)) and 3.13 (\( b_1 = -\frac{s''(u)}{4} \)). It follows that in the first case the polynomial \( C_\beta \) has the triple root \( \mu_1 = \mu_2 = \mu_3 = \infty \) and the simple root \( \mu_4 = 0 \), and in the second case it has the simple root \( \mu_1 = \infty \) and the triple root \( \mu_2 = \mu_3 = \mu_4 = 0 \). According to Section 5.4 of the book [1], this means that the fiber bundle \( E_\beta \) has two invariant distributions \( \Delta_\beta(\infty) \) and \( \Delta_\beta(0) \), and the distribution corresponding to a multiple root is integrable. Moreover, it is easy to prove that the distribution corresponding to a simple root is also integrable. For all cases the integral submanifolds \( V_\beta \) are defined by the equations \( y = C_2, v = C_4 \) (for \( \mu = \infty \)) and by the equations \( x = C_1, u = C_3 \) (for \( \mu = 0 \)).

4. An almost Grassmann structure is associated with a web, and if a web is transversally geodesic or isoclinic, the corresponding almost Grassmann structure is \( \alpha \)- or \( \beta \)-semiintegrable (see [1], Ch. 7). Our next two examples are generated by examples of exceptional (nonalgebraizable) isoclinic webs of maximum 2-rank (see [2], [3] or [4], Ch. 8, and [5], §5.5).
Example 3.14 Suppose that the basis 1-forms $\omega^i$ of an almost Grassmann structure $AG(1,3)$ are
\[
\begin{aligned}
\omega^1 &= dx, & \omega^3 &= du, \\
\omega^2 &= -(y + v)dx + (u - x)dy, & \omega_2 &= (y + v)du + (u - x)dv.
\end{aligned}
\tag{3.73}
\]

The metric of the $CO(2,2)$-structure which is equivalent to the $AG(1,3)$-structure with basis forms (3.73) is
\[
g = 2(y + v)dx du + (u - x)(dx dv - dy du).
\tag{3.74}
\]

In this case we have
\[
\begin{aligned}
\omega_1^2 &= \frac{1}{2(u - x)}\omega_3^3, & \omega_3^3 &= -\frac{2(y + v)}{u - x}(\omega_1^1 + \omega_2^2) + \frac{1}{2(u - x)}(-\omega_1^1 + \omega_2^2), \\
\omega_1^3 &= -\frac{1}{2(u - x)}\omega_2^2, & \omega_4^3 &= 0, \\
\omega_1^4 &= \frac{3}{4(u - x)}(\omega_1^1 + \omega_2^2), & \omega_4^4 &= \frac{1}{4(u - x)}(-\omega_1^1 + \omega_2^2); \\
A_1 &= B_2 = -\frac{5}{4(u - x)^2}; & A_2 &= -\frac{7}{4(u - x)^2}, \\
b_{333}^{412} &= -\frac{8(y + v)}{(u - x)^3}; \\
\Omega^1_3 &= \Omega^1_3 = \Omega^2_3 = \Omega^3_4 = \Omega^4_3 = 0,
\end{aligned}
\tag{3.75}
\]

It is easy to check that $b_{\alpha \beta}^{i\gamma\delta} = 0$, that is, $b^\gamma_\beta = 0$, and the almost Grassmann structure $AG(1,3)$ with basis forms (3.73) is $\beta$-semiintegrable. However, in this case $b_{\alpha \beta}^{i\gamma\delta}$ is not $\beta$-semiintegrable. Thus, the almost Grassmann structure $AG(1,3)$ with basis forms (3.73) is not locally flat.

If we apply the second method for the structure (3.73), then while looking for $\alpha$-semiintegrability conditions, we find that
\[
\begin{aligned}
d\lambda &= dy + dv - \frac{y + v + \lambda}{u - x} dx + \frac{y + v - \lambda}{u - x} du.
\end{aligned}
\]

Exterior differentiation of this equation gives $\lambda dx \wedge du = 0$. Since on $V_\alpha$ we have $dx \wedge du \neq 0$, the structure (3.73) is never $\alpha$-semiintegrable.

For $\beta$-semiintegrability we find that
\[
\begin{aligned}
d\mu &= \frac{2\mu(1 - \mu)du}{u - x}, \\
dx + \mu dv &= 0, \\
dy &= -\mu dv - \frac{2\mu(y + v)}{u - x} du.
\end{aligned}
\tag{3.80}
\]

It is easy to see from (3.80) that
\[
\frac{d\mu}{2\mu(1 - \mu)} = \frac{du}{u - x} = \frac{dx}{-\mu(u - x)}.
\]
and subsequently
\[
\frac{d(u-x)}{(u-x)(1+\mu)} = \frac{d\mu}{2\mu(1-\mu)}.
\] (3.81)

The solution of equation (3.81) is
\[
\frac{\sqrt{\mu}}{\mu-1} = C_1(u-x).
\] (3.82)

The submanifolds \( V_\beta \) of the distribution \( \Delta_\beta \) are defined by the completely integrable system (3.80) where \( \mu \) can be found from equations (3.82).

Thus the family of submanifolds \( V_\beta \) on the manifold \( M^4 \) carrying the \( AG(1,3) \)-structure with the basis forms (3.63) depends on three parameters, and through any point \( (x_0, y_0, u_0, v_0) \) of \( M^4 \) there passes an one-parameter family of submanifolds \( V_\alpha \).

If we apply formulas (3.18), we find that for the tensor of conformal curvature of the equivalent \( CO(2,2) \)-structure we have \( C_3 = 0 \) and the only nonvanishing component of \( C_\alpha \) is \( a_0 = -\frac{8(v+\nu)}{(u-x)^2} \). This means that the polynomial \( C_\alpha(\lambda) \) has a quadruple root \( \lambda = 0 \), and the fiber bundle \( E_\alpha \) possesses an integrable distribution \( \Delta_\alpha(0) \). It is easy to prove that the submanifolds \( V_\alpha \) of this distribution are defined by the equations \( x = C_1, u = C_3, \) where \( C_1 \) and \( C_3 \) are constants.

**Example 3.15** Suppose that the basis 1-forms \( \omega_\alpha^i \) of an almost Grassmann structure \( AG(1,3) \) are
\[
\begin{align*}
\omega_1^3 &= dx, \quad \omega_2^3 = du, \\
\omega_1^4 &= -vdx +udy, \quad \omega_2^4 = ydu - xdv.
\end{align*}
\] (3.83)

The metric of the \( CO(2,2) \)-structure which is equivalent to the \( AG(1,3) \)-structure with basis forms (3.83) is
\[
g = (y+v)dxdu - xdxv - udyu.
\] (3.84)

In this case we have
\[
\begin{align*}
\omega_1^2 &= -\frac{1}{2x} \omega_1^3, \quad \omega_2^3 = \left( \frac{y}{x} - \frac{v}{u} \right) \left( \omega_1^3 + \omega_2^3 \right) - \frac{1}{2u} \omega_1^4 - \frac{1}{2x} \omega_2^4, \\
\omega_1^3 &= -\frac{1}{2u} \omega_2^3, \quad \omega_1^4 = -\omega_2^4 = \left( \frac{1}{4u} - \frac{1}{2x} \right) \omega_1^4 + \left( \frac{1}{2u} - \frac{1}{4x} \right) \omega_2^4, \\
\omega_1^4 &= 0, \quad \omega_3^4 = -\omega_4^4 = \left( \frac{1}{4u} + \frac{1}{2x} \right) \omega_1^4 + \left( \frac{1}{2u} + \frac{1}{4x} \right) \omega_2^4;
\end{align*}
\] (3.85)
\[
\begin{align*}
A_1 &= \frac{1}{4u^2} + \frac{1}{4xu}; \quad A_2 = \frac{1}{4xu} - \frac{1}{4x^2} - \frac{1}{4u^2}; \quad B_2 = \frac{1}{2xu} - \frac{1}{4x^2}; \\
b_{333} &= \frac{1}{u} - \frac{1}{x} \left( \frac{y}{x} - \frac{v}{u} \right), \quad b_{333} = -\frac{1}{u} \left( \frac{1}{4xu} - \frac{1}{4x^2} \right) \omega_3^4, \quad (3.86)
\end{align*}
\]
\[
\begin{align*}
\omega_1^3 &= \left( \frac{1}{4u^2} + \frac{1}{2xu} \right) \omega_1^3 + \left( \frac{1}{4u^2} - \frac{1}{4x^2} \right) \omega_2^4, \quad \omega_1^4 = 0, \\
\omega_2^3 &= \left( \frac{1}{4xu} - \frac{1}{4x^2} - \frac{1}{4u^2} \right) \omega_1^3 + \left( \frac{1}{2xu} - \frac{1}{4x^2} \right) \omega_2^4, \quad \omega_2^4 = 0;
\end{align*}
\] (3.87)
\[
\begin{align*}
\Omega_1^1 &= \Omega_2^1 = \Omega_1^2 = \Omega_2^2 = 0, \\
\Omega_1^3 &= -\Omega_1^4 = \left( \frac{1}{4u^2} - \frac{1}{4x^2} \right) \omega_1^3 \wedge \omega_2^3, \\
\Omega_1^4 &= \left( \frac{1}{u} - \frac{1}{x} \right) \left( \frac{y}{x} - \frac{v}{u} \right) \omega_1^3 \wedge \omega_2^3 + \left( \frac{1}{4x^2} - \frac{1}{4u^2} \right) (\omega_1^3 \wedge \omega_2^4 - \omega_2^3 \wedge \omega_1^4).
\end{align*}
\] (3.89)
It is easy to check that $b_{ijkl}^{(3/8)} = 0$, that is, $b_{ij}^2 = 0$, and the almost Grassmann structure $AG(1,3)$ with the basis forms (3.83) is $β$-semiintegrable. However, in this case $b_{ijkl}^{(3/8)} \neq 0$. Thus, the almost Grassmann structure $AG(1,3)$ with the basis forms (3.83) is not locally flat.

If we apply the second method for the structure (3.73), then while looking for $α$-semiintegrability conditions, we find that

$$dλ = \frac{y + λ}{u} dx - dy + \frac{v - λ}{u} du + dv.$$  \hspace{1cm} (3.90)

Exterior differentiation of (3.90) gives

$$λ = -\frac{1}{x^2 + u^2} [vx(u - x) - yu(x + u)].$$  \hspace{1cm} (3.91)

The differential equations of submanifolds $V_α$ have the form:

$$\left\{ \begin{array}{l}
(v + y)(x + u)dx - (x^2 + u^2)dy = 0, \\
(v + y)(x + u)du - (x^2 + u^2)dv = 0.
\end{array} \right.$$  \hspace{1cm} (3.92)

It is easy to prove that the integrability conditions of equations (3.92) implies that $du$ is proportional to $dx$. This is impossible since on $V_α$ we have $dx \wedge du \neq 0$. Thus, the structure $AG(1,3)$ with the basis forms (3.83) is never $α$-semiintegrable.

The $β$-submanifolds of this structure are defined by the following completely integrable system:

$$\left\{ \begin{array}{l}
μu du + dx = 0, \\
μ[(y + ν)du - xdv] + udy = 0, \\
d log(μ - 1) = \left( \frac{1}{u} - \frac{1}{x} \right) dx.
\end{array} \right.$$  \hspace{1cm} (3.93)

Thus the family of submanifolds $V_β$ on the manifold $M^4$ carrying the $AG(1,3)$-structure with the basis forms (3.83) depends on three parameters, and through any point $(x, y, u, v)$ of $M^4$ there passes an one-parameter family of submanifolds $V_β$.

If we apply formulas (3.18), we find that for the tensor of conformal curvature of the equivalent $CO(2,2)$-structure we have $C_{β3} = 0$ and the only nonvanishing components of $C_α$ are $a_0 = \left( \frac{1}{u} - \frac{1}{x} \right) \left( \frac{u}{x} - \frac{u}{x} \right)$ and $a_1 = \frac{1}{4} \left( \frac{1}{x} - \frac{1}{u} \right)$. This means that the polynomial $C_α(λ)$ has a triple root $λ_1 = λ_2 = λ_3 = 0$ and a simple root $λ_4 = \frac{x^2 + u^2}{xv - yu}$, and the fiber bundle $E_α$ possesses an integrable distribution $Δ_α(0)$. It is easy to prove that the integral submanifolds $V_α$ of this distribution are defined by the equations $x = C_1, u = C_3$, where $C_1$ and $C_3$ are constants. It is also easy to check that the distribution $Δ(λ_4)$ is not integrable.

The next example is generated by an example of an isoclinic three-web given in [3] (see Exercise 6 on p. 133).

**Example 3.16** Suppose that the basis 1-forms $ω_α^i$ of an almost Grassmann structure $AG(1,3)$ are

$$\left\{ \begin{array}{l}
ω_1^2 = (y - ν)dx + (x + u)dy, \quad ω_2^3 = (y - ν)du - (x + u)dv, \\
ω_3^4 = dy, \quad ω_4^1 = dv.
\end{array} \right.$$  \hspace{1cm} (3.94)

The metric of the $CO(2,2)$-structure which is equivalent to the $AG(1,3)$-structure with basis forms (3.94) is

$$g = (y - ν)(dx dv - dy du) + 2(x + u)dy dv.$$  \hspace{1cm} (3.95)
In this case we have

\[
\begin{align*}
\omega_1^2 &= -\frac{1}{2(y-v)}\omega_1^4, \\
\omega_2^1 &= \frac{1}{2(y-v)}\omega_2^3, \\
\omega_3^1 &= -\omega_2^2 = \frac{3}{4(y-v)}(\omega_1^4 + \omega_2^4), \\
\omega_4^1 &= \frac{1}{2(y-v)}(\omega_1^4 - \omega_2^4) - \frac{2(x+u)}{y-v}(\omega_1^4 + \omega_2^4), \\
\omega_3^3 &= -\omega_4^4 = -\frac{1}{4(y-v)}(\omega_1^4 - \omega_2^4);
\end{align*}
\]

(3.96)

\[
C_3 = -\frac{5}{4(y-v)^2}; \\
C_4 = -\frac{7}{4(y-v)^2}; \\
E_4 = -\frac{5}{4(y-v)^2};
\]

(3.97)

\[
b_{i44}^{312} = \frac{8(x+u)}{(y-v)^2};
\]

(3.98)

\[
\begin{align*}
\omega_1^3 &= 0, \\
\omega_1^4 &= -\frac{1}{4(y-v)^2}(5\omega_1^4 + 7\omega_2^4), \\
\omega_3^3 &= 0, \\
\omega_4^4 &= -\frac{1}{4(y-v)^2}(7\omega_1^4 + 5\omega_2^4);
\end{align*}
\]

(3.99)

\[
\begin{align*}
\Omega_1^3 &= \Omega_2^3 = \Omega_1^4 = \Omega_3^4 = \Omega_4^3 = 0, \\
\Omega_4^3 &= \frac{8(x+u)}{(y-v)^2}\omega_1^4 \wedge \omega_2^4.
\end{align*}
\]

(3.100)

It is easy to check that \(b^{(\beta y)}_{ijkl} = 0\), that is, \(b_\beta = 0\), and the almost Grassmann structure \(AG(1,3)\) is \(\beta\)-semiintegrable. However, in this case \(b^{(\gamma \delta)}_{ijkl} \neq 0\). Thus, the almost Grassmann structure is not locally flat.

If we apply the method of direct integration, we find that the submanifolds \(V_\beta\) are defined by the following completely integrable system of differential equations:

\[
\begin{align*}
\frac{d\mu}{2\mu(\mu-1)} &= \frac{dv}{y-v}, \\
(y-v)(\mu du + dx) - 2\mu(x+u)dv &= 0, \\
\mu dv + dy &= 0.
\end{align*}
\]

(3.101)

It is easy to see from (3.101) that

\[
\frac{d\mu}{2\mu(\mu-1)} = \frac{dv}{y-v} = -\frac{dy}{\mu(y-v)},
\]

and subsequently

\[
\frac{d(y-v)}{(y-v)(1+\mu)\sqrt{\mu}} = \frac{d\mu}{2\mu(\mu-1)}.
\]

(3.102)

The solution of equation (3.102) is

\[
\frac{\mu-1}{\sqrt{\mu}} = C_3(y-v).
\]

(3.103)

The submanifolds \(V_\beta\) of the distribution \(\Delta_\beta\) are defined by the completely integrable system (3.101) where \(\mu\) can be found from equations (3.103).

Thus the family of submanifolds \(V_\beta\) on the manifold \(M^4\) carrying the \(AG(1,3)\)-structure with the basis forms (3.94) depends on three parameters, and through any point \((x_0, y_0, u_0, v_0)\) of \(M^4\) there passes an one-parameter family of submanifolds \(V_\beta\).
If we apply formulas (3.18), we find that for the tensor of conformal curvature of the equivalent $CO(2,2)$-structure we have $C_\beta = 0$ and the only nonvanishing component of $C_\alpha$ is $a_1 = \frac{8(z+u)}{(y+v)}$. This means that the polynomial $C_\alpha(\lambda)$ has a quadruple root $\lambda = \infty$, and the fiber bundle $E_\alpha$ possesses an integrable distribution $\Delta_\alpha(\infty)$. It is easy to prove that the submanifolds $V_\alpha$ of this distribution are defined by the equations $y = C_2, v = C_4$, where $C_2$ and $C_4$ are constants.

5. The next example is generated by an example of an almost Grassmann structure $AG(1,4)$ associated with a six-dimensional Bol web considered in [8] (p. 270).

**Example 3.17** Suppose that the basis 1-forms of the direct integration.

Note that on $V_\alpha$ we have $dx \wedge du \neq 0$. Exterior differentiation of the last two equations of (3.105) gives the following exterior quadratic equations:

$$d\lambda_2 \wedge dx = 0, \quad d\lambda_2 \wedge du = 0.$$  

Since $dx \wedge du \neq 0$, it follows that $d\lambda_2 = 0$ and

$$\lambda_2 = C_2,$$

where $C_2$ is a constant. Now the last two equations of (3.105) can be written as

$$dz = -C_2dx = 0, \quad dw = -C_2du.$$  

Integration of (3.107) gives

$$z = -C_2x + C_3, \quad w = -C_2u + C_4,$$

where $C_3$ and $C_4$ are constants.

By (3.106) and (3.107), the first two equations of (3.105) become

$$dy = \{-\lambda_1 + [2v + 2uz + (1 - 2x)w + C_2u]e^{-2x}\}dx,$$

$$dv = -(\lambda_1e^{2x} + z + C_2x)du.$$  

Exterior differentiation of (3.109) leads to the following exterior quadratic equations:

$$(d\lambda_1 + 2\lambda_1 du) \wedge dx = 0, \quad (d\lambda_1 + 2\lambda_1 dx) \wedge du = 0.$$
These equations can be written as

\[ [d\lambda_1 + 2\lambda_1(dx + du)] \wedge dx = 0, \quad [d\lambda_1 + 2\lambda_1(dx + du)] \wedge du = 0. \]

Since \( dx \wedge du \neq 0 \), it follows that

\[ d\lambda_1 + 2\lambda_1(dx + du) = 0. \]

This implies that

\[ \lambda_1 = C_1 e^{-2(x+u)}. \quad (3.110) \]

Substituting this value of \( \lambda_1 \) and \( z \) from (3.108) into the second equation of (3.109), we find that

\[ dv = -(C_1 e^{-2u} + C_3)du. \]

This implies that

\[ v = \frac{1}{2} C_1 e^{-2u} - C_3 u + C_5, \quad (3.111) \]

where \( C_5 \) is a constant.

Substituting the values of \( \lambda_1, z \) and \( w \) from (3.111), (3.110) and (3.108) into the first equation of (3.109), we find that

\[ dy = [2C_5 + C_4(1 - 2x)] e^{-2x} dx. \]

The solution of this equation is

\[ y = (-C_5 + C_4 x)e^{-2x} + C_6, \quad (3.112) \]

where \( C_6 \) is a constant.

Thus two-dimensional surfaces \( V_\alpha \) are defined by the closed form equations (3.108), (3.110), (3.111) and (3.112). This completes the proof that the \( AG(1,4) \)-structure (3.104) is \( \alpha \)-semiintegrable. Thus the family of submanifolds \( V_\alpha \) on the manifold \( M^6 \) carrying the \( AG(1,4) \)-structure with the basis forms (3.104) depends on six parameters, and through any point \( (x_0, y_0, z_0, u_0, v_0, w_0) \) of \( M^6 \) there passes a two-parameter family of submanifolds \( V_\alpha \).

We will prove now that this structure is not \( \beta \)-semiintegrable. Equations (3.23) of surfaces \( V_\beta \), \( \dim V_\beta = 3 \), can be written as follows:

\[ \mu du + dx = 0, \quad \mu \omega_2^4 + \omega_1^4 = 0, \quad \mu dw + dz = 0. \quad (3.113) \]

where \( \mu = -\frac{4}{u^2} \). Taking exterior derivatives of the first and third equations of (3.113), we find that \( d\mu \wedge du = 0 \) and \( d\mu \wedge dw = 0 \). Since on surfaces \( V_\beta \) we must have \( du \wedge dv \wedge dw \neq 0 \), it follows that \( d\mu = 0 \), and

\[ \mu = C, \quad (3.114) \]

where \( C \) is a constant. Substituting this value of \( \mu \) into the first and third equations of (3.113), we find that

\[ dx = -Cdu, \quad dz = -Cdw, \quad (3.115) \]

and

\[ x = -C u + C_1, \quad z = -C w + C_2, \quad (3.116) \]

Substituting \( \mu, x \) and \( z \) from (3.114) and (3.115) into the second equation of (3.113), we obtain

\[ \frac{1}{C} e^{2(C_1 - C u)} = -dv + [C_1 + u(1-C)] dw + [(C + 2C_1 - 1)w - 2v - 2C_2 u - C_2] du. \quad (3.117) \]
Taking exterior derivative of (3.117), we find that if the surfaces \( V_\beta \) exist, we would have

\[
du \wedge [(2C - 1)dv + 2(1 - C)(Cu + C_1 - 2)]dw = 0. \tag{3.118}
\]

This is impossible since (3.118) implies that \( du \) is a linear combination of \( dv \) and \( dw \) and as a result, \( du \wedge dv \wedge dw = 0. \)

Thus the \( AG(1,4) \)-structure is not \( \beta \)-semiintegrable.

Example 3.5 can be generalized in the following manner.

Example 3.18 Suppose that \( x_\alpha, y_\alpha, \alpha = 1, \ldots, p, \) are coordinates in \( M^{2p} \), and that the basis 1-forms \( \omega^i_\alpha \) of an almost Grassmann structure \( AG(p-1, p+1) \) are

\[
\begin{align*}
\omega_1^{p+1} &= dx_1 + f(y_1)dx_2, \quad \omega_2^{p+1} = dx_2, \quad \omega_s^{p+1} = dx_s, \\
\omega_1^{p+2} &= dy_1, \quad \omega_2^{p+2} = dy_2, \quad \omega_s^{p+2} = dy_s,
\end{align*}
\]

where \( s = 3, \ldots, p \). If the structure (3.119) is \( \alpha \)-semiintegrable, then \( dy_1 \wedge dy_2 \wedge \ldots \wedge dy_p \neq 0 \), and the rows of the matrix \( (\omega^i_\alpha) \) are proportional:

\[
dx_1 + f(y)dx_2 + \lambda dy_1 = 0, \quad dx_2 + \lambda dy_2 = 0, \quad dx_s + \lambda dy_s = 0. \tag{3.120}
\]

Exterior differentiation of (3.120) gives the following exterior quadratic equations:

\[
f'(y_1)dy_1 \wedge dx_2 + d\lambda \wedge dy_1 = 0, \quad d\lambda \wedge dy_2 = 0, \quad d\lambda \wedge dy_3 = 0. \tag{3.121}
\]

The last two equations of (3.121) imply that \( d\lambda = 0 \). By (3.120), the first equation of (3.121) gives

\[
\lambda f'(y_1)dy_1 \wedge dy_2 = 0.
\]

Since \( dy_1 \wedge dy_2 \neq 0 \), it follows that \( f'(y_1) = 0 \) and \( f(y_1) = a \), where \( a \) is a constant. Thus the structure (3.119) is \( \alpha \)-semiintegrable if and only if \( f(y_1) \) is constant. If it is the case, closed form equations of integral submanifolds \( V_\alpha \) of this structure have the form

\[
x_1 + ax_2 + by_1 = C_1, \quad x_2 + by_2 = C_2, \quad x_s + by_s = C_s. \tag{3.122}
\]

Hence the submanifolds \( V_\alpha \) are flat \( p \)-dimensional submanifolds, and the family of submanifolds \( V_\alpha \) depends on \( p + 1 \) constants \( b, C_1, C_2 \) and \( C_s \).

If the structure (3.129) is \( \beta \)-semiintegrable, then \( dx_2 \wedge dy_2 \neq 0 \), and the columns of the matrix \( (\omega^i_\alpha) \) are proportional:

\[
\begin{align*}
dx_1 + f(y_1)dx_2 + \mu_1 dy_x = 0, \quad dx_s + \mu_s dx_2 = 0, \\
dy_1 + \mu_1 dy_2 = 0, \quad dy_s + \mu_s dy_2 = 0.
\end{align*}
\]

Exterior differentiation of (3.123) give the following exterior quadratic equations:

\[
\begin{align*}
(d\mu_1 + f'(y_1))dy_1 \wedge dx_2 = 0, \quad d\mu_s \wedge dx_2 = 0, \\
(d\mu_1 + f'(y_1))dy_1 \wedge dy_2 = 0, \quad d\mu_s \wedge dy_2 = 0.
\end{align*}
\]

Since \( dx_2 \wedge dy_2 \neq 0 \), it follows from (3.114) that

\[
d\mu_1 + f'(y_1)dy_1 = 0, \quad d\mu_s = 0,
\]

and

\[
\mu_1 + f(y_1) = C_1, \quad \mu_s = C_s. \tag{3.125}
\]
Substituting (3.125) into system (3.123), we find that
\[
\begin{align*}
&\begin{cases}
    dx_1 + C_1 dx_2 = 0, & dx_s + C_s dx_2 = 0, \\
    dy_1 + (C_1 - f(y_1)) dy_2 = 0, & dy_s + C_s dy_2 = 0.
\end{cases} \\
\end{align*}
\tag{3.126}
\]

The solution of this system is
\[
\begin{align*}
&\begin{cases}
    x_1 + C_1 x_2 = A_1, & x_s + C_s x_2 = A_s, \\
    \int \frac{dy_1}{C_1 - f(y_1)} + y_2 = B_1, & y_s + C_s y_2 = B_s.
\end{cases} \\
\end{align*}
\tag{3.127}
\]

Hence the two-dimensional integral submanifolds \( V_3 \) are defined by closed form equations (3.127), and the family of submanifolds \( V_3 \) depends on \( 3(p - 1) \) constants \( C_1, A_1, B_1, C_s, A_s \) and \( B_s \).

Thus if \( f(y_1) \) is not a constant, the structure \( AG(p - 1, p + 1) \) with the structure forms (3.119) is \( \beta \)-semiintegrable, and if \( f(y_1) \) is a constant, this structure is locally flat.

Example 3.10 can be also generalized.

**Example 3.19** Suppose that \( x^3, x^4, x^t, y^3, y^4, y^t, \ t = 5, \ldots, q + 2 \), are coordinates in \( M^{2q} \), and that the basis 1-forms \( \omega^a \) of an almost Grassmann structure \( AG(1, q + 1) \) are
\[
\begin{align*}
&\begin{cases}
    \omega^1 = dx^3 + p(y^3) dx^4, & \omega^{p+1} = dy^3, \\
    \omega^2 = dx^4, & \omega^4 = dy^4, \\
    \omega^t = dx^t, & \omega^t = dy^t,
\end{cases} \\
\end{align*}
\tag{3.128}
\]

where \( t = 5, \ldots, q + 2 \).

If the structure (3.128) is \( \alpha \)-semiintegrable, then \( dx^4 \wedge dy^4 \neq 0 \) and the rows of the matrix \( (\omega^a_i) \) are proportional:
\[
\begin{align*}
&\begin{cases}
    dx^3 + p(y^3) dx^4 + \lambda^3 dx^4 = 0, & dy^3 + \lambda^3 dy^4 = 0, \\
    dx^t + \lambda^t dx^4 = 0, & dy^t + \lambda^t dy^4 = 0.
\end{cases} \\
\end{align*}
\tag{3.129}
\]

Exterior differentiation of (3.129) give the following exterior quadratic equations:
\[
\begin{align*}
&\begin{cases}
    (d\lambda^3 + p'(y^3) dy^3) \wedge dx^4 = 0, & (d\lambda^3 + p'(y^3) dy^3) \wedge dy^4 = 0, \\
    d\lambda^t \wedge dx^4 = 0, & d\lambda^t \wedge dy^4 = 0.
\end{cases} \\
\end{align*}
\tag{3.130}
\]

Since \( dy^3 \wedge dy^4 \neq 0 \), equations (3.130) imply that \( d\lambda^3 + p'(y^3) dy^3 = 0 \) and \( d\lambda^t = 0 \). Thus we have
\[
\lambda^3 + p(y^3) = C^3, \quad \lambda^t = C^t. \
\tag{3.131}
\]

As a result, equations (3.129) become
\[
\begin{align*}
&\begin{cases}
    dx^3 + C^3 dx^4 = 0, & dy^3 + (C^3 - p(y^3)) dy^4 = 0, \\
    dx^t + C^t dx^4 = 0, & dy^t + C^t dy^4 = 0.
\end{cases} \\
\end{align*}
\tag{3.132}
\]

It follows from (3.132) that closed form equations of two-dimensional integral submanifolds \( V_\alpha \) are
\[
\begin{align*}
&\begin{cases}
    x^3 + C^3 x^4 = A^3, & \int \frac{dy^3}{C^3 - p(y^3)} + y^3 = B^4, \\
    x^t + C^t x^4 = A^t, & y^t + C^t y^4 = B^t.
\end{cases} \\
\end{align*}
\tag{3.133}
\]
Thus the family of submanifolds $V_\alpha$ depends on $3(q-1)$ constants $C^3, A^3, B^3, C^4, A^4$ and $B^4$.

If the structure (3.128) is $\beta$-semiintegrable, then $dy^3 \wedge dy^4 \wedge \ldots \wedge dy^{q+2} \neq 0$, and the columns of the matrix $(\omega_\alpha')$ are proportional:

$$dx^3 + p(y^3)dx^4 + \mu dy^3 = 0, \quad dx^4 + \mu dy^4 = 0, \quad dx^t + \mu dy^t = 0.$$  \hspace{0.5cm} (3.134)

Exterior differentiation of (3.134) gives the following exterior quadratic equations:

$$p'(y^3)dy^3 \wedge dx^4 + d\mu \wedge dy^3 = 0, \quad d\mu \wedge dy^4 = 0, \quad d\mu \wedge dy^t = 0.$$  \hspace{0.5cm} (3.135)

It follows from (3.135) that $d\mu = 0$ and consequently $\mu = C_0$. As a result, by (3.134), the first equation of (3.135) becomes

$$C_0p'(y^3)dy^3 \wedge dy^4 = 0.$$  

Since $dy^3 \wedge dy^4 \neq 0$, it follows that $p'(y^3) = 0$ and $p(y^3) = a$. Thus the structure (3.118) is $\beta$-semiintegrable if and only if $p(y^3)$ is a constant. If it is the case, equations (3.134) become

$$dx^3 + adx^4 + C_0dy^3 = 0, \quad dx^4 + C_0dy^4 = 0, \quad dx^t + C_0dy^t = 0,$$  \hspace{0.5cm} (3.136)

and closed form equations of integral submanifolds $V_\beta$ of this structure have the form

$$x^3 + ax^4 + C_0y^3 = C^3, \quad x^4 + C_0y^4 = C^4, \quad x^t + C_0y^t = C^t.$$  \hspace{0.5cm} (3.137)

Hence the submanifolds $V_\beta$ are flat two-dimensional submanifolds, and the family of submanifolds $V_\beta$ depends on $q + 2$ constants $a, C^0, C^3, C^4$ and $C^t$.

Thus if $p(y^3)$ is not a constant, the structure $AG(1,q+1)$ with the structure forms (3.128) is $\alpha$-semiintegrable, and if $p(y^3)$ is a constant, this structure is locally flat.

Example 3.17 can be generalized to an example of an $\alpha$-integrable almost Grassmann structure $AG(1,q+1)$.

**Example 3.20** Suppose that the basis 1-forms $\omega_\alpha^i$ of an almost Grassmann structure $AG(1,q+1)$ are

\[
\begin{align*}
\omega_1^3 &= dx, \\
\omega_1^4 &= [-2v - 2uz + (2x - 1)w|e^{-2x}]dx + dy + ue^{-2x}dz, \\
\omega_1^s &= dz^s, \\
\omega_2^3 &= du, \\
\omega_2^4 &= (zu + dv - xdw)e^{-2x}, \\
\omega_2^s &= dw^s,
\end{align*}
\]  \hspace{0.5cm} (3.138)

where $s = 5, \ldots, q + 2$.

Equations (3.105) will take the form

\[
\begin{align*}
\{\lambda_1 + [-2v - 2uz + (2x - 1)w|e^{-2x}]\}dx + dy + ue^{-2x}dz &= 0, \\
\lambda_1du + (zu + dv - xdw)e^{-2x} &= 0, \\
\lambda_2dx + dz^s &= 0, \\
\lambda_2du + dw^s &= 0, \quad s = 5, \ldots, q + 2.
\end{align*}
\]  \hspace{0.5cm} (3.139)

The proof of the fact that the $AG(1,q+1)$-structure with the structure forms (3.138) is $\alpha$-semiintegrable is similar to that for Example 3.17.
Note that using this method of generalization, we cannot find a semiintegrable $AG(2, 5)$-structure (in this case $p = q = 3$). In fact, if we set

\[
\begin{align*}
\omega_1^4 &= \omega, & \omega_2^2 &= dx_2, & \omega_3^4 &= dx_3, \\
\omega_1^5 &= dy_1, & \omega_2^5 &= dy_2, & \omega_3^5 &= dy_3, \\
\omega_1^6 &= dz_1, & \omega_2^6 &= dz_2, & \omega_3^6 &= dz_3,
\end{align*}
\]

then for $\alpha$-semiintegrability we must have

\[
\begin{align*}
\omega + \lambda dy_1 &= 0, & dx_2 + \lambda dy_2 &= 0, & dx_3 + \lambda dy_3 &= 0, \\
dz_1 + \tilde{\lambda} dy_1 &= 0, & dz_2 + \tilde{\lambda} dy_2 &= 0, & dz_3 + \tilde{\lambda} dy_3 &= 0.
\end{align*}
\]

It is easy to find from this that $\lambda = C$, $\tilde{\lambda} = \tilde{C}$, where $C$ and $\tilde{C}$ are constants, $d\omega = 0$, that is, $\omega$ is a total differential, $\omega = dx_1$. But in this case the structure in question is also $\beta$-semiintegrable, i.e., this structure is locally flat.

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