Quantum perfect correlations and Hardy’s nonlocality theorem*

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August 12, 1999

Abstract

In this paper the failure of Hardy’s nonlocality proof for the class of maximally entangled states is considered. A detailed analysis shows that the incompatibility of the Hardy equations for this class of states physically originates from the fact that the existence of quantum perfect correlations for the three pairs of two-valued observables $(D_{11}, D_{21})$, $(D_{11}, D_{22})$, and $(D_{12}, D_{21})$ [in the sense of having with certainty equal (different) readings for a joint measurement of any one of the pairs $(D_{11}, D_{21})$, $(D_{11}, D_{22})$, and $(D_{12}, D_{21})$], necessarily entails perfect correlation for the pair of observables $(D_{12}, D_{22})$ [in the sense of having with certainty equal (different) readings for a joint measurement of the pair $(D_{12}, D_{22})$]. Indeed, the set of these four perfect correlations is found to satisfy the CHSH inequality, and then no violations of local realism will arise for the maximally entangled state as far as the four observables $D_{ij}$, $i, j = 1$ or 2, are concerned. The connection between this fact and the impossibility for the quantum mechanical predictions to give the maximum possible theoretical violation of the CHSH inequality is pointed out. Moreover, it is generally proved that the fulfillment of all the Hardy nonlocality conditions necessarily entails

*This paper has been originally published in: J.L. Cereceda, Found. Phys. Lett. 12(3), 211-231 (1999).
a violation of the resulting CHSH inequality. The largest violation of this latter inequality is determined.

Key words: perfect correlation, maximally entangled state, local realism, Hardy’s nonlocality theorem, Bell’s inequality.

1 Introduction

A very remarkable feature of Hardy’s nonlocality proof [1] is that it goes through for any entangled states of a $2 \times 2$ system except those which are maximally entangled such as the singlet state of two spin-$\frac{1}{2}$ particles. At first sight this might appear rather surprising in view of the fact that maximally entangled states yield the maximum violation predicted by quantum mechanics of the Clauser-Horne-Shimony-Holt (CHSH) inequality [2]. Referring himself to this failure, Hardy states that [1], “the reason for this is that the proof relies on a certain lack of symmetry that is not available in the case of a maximally entangled state.” Indeed, it has been found [1,3-6] that the set of Hardy equations (see Eqs. (8a)-(8d) below) upon which the nonlocality contradiction is constructed is incompatible for the case of maximal entanglement in the sense that for this case the fulfillment of all three conditions (8a), (8b), and (8c) precludes the fulfillment of condition (8d), and vice versa. From a mathematical point of view this is the reason for the failure, and this would be the end of the story.

In this paper I would like to account for this failure from a somewhat different perspective which provides a fuller mathematical understanding of the structure of Hardy’s theorem. This will allow us to gain some insight into the physical cause of the inability of completely entangled states to produce a Hardy-type nonlocality contradiction. So, after introducing in Sec. 2 some general results concerning the conditions needed to achieve perfect correlations for $2 \times 2$ systems, in Sec. 3 it will be shown that the fulfillment of all three conditions (8a)-(8c) in the case of a maximally entangled state necessarily entails perfect correlation between the two measurement outcomes (one for each particle) obtained in any one of the four possible combinations of joint measurements $(D_{1k}, D_{2l})$, $k, l = 1$ or $2$, one might actually perform
on both particles, where $D_{1k}$ and $D_{2l}$ are single-particle observables associated with particles 1 and 2, respectively. However, as we shall see, the CHSH inequality is fulfilled for such maximal-entanglement-induced perfect correlations, and, thereby, no violations of local realism will arise for the maximally entangled state as long as the four observables $D_{1k}$ and $D_{2l}$ involved in the CHSH inequality make conditions (8a), (8b), and (8c) hold. Indeed, the fulfillment of the CHSH inequality for such perfect correlations means that all of them can be consistently explained in terms of a local hidden-variable model (see, for instance, Appendix D in Ref. 7 for an explicit example of such a model that accounts for the perfect correlations of two spin-$\frac{1}{2}$ particles in the singlet state). This is ultimately the physical reason why Hardy’s nonlocality argument does not work for the maximally entangled case. Moreover, as will become clear, the failure of Hardy’s argument for the maximally entangled state is, interestingly enough, closely related to the fact that the quantum mechanical predictions cannot give the maximal possible theoretical violation of the CHSH inequality. In Sec. 4, the general case of less-than-maximally entangled state is considered, and it is shown how the fulfillment of all the Hardy conditions (8a)-(8d) necessarily leads to a violation of the resulting CHSH inequality. The largest extent of this violation is determined. Finally, in Sec. 5, examples are given illustrating the fact that maximally entangled states yield the maximum quantum mechanical violation of the CHSH inequality, while this inequality is necessarily obeyed for such states if these latter are constrained to satisfy the conditions (8a), (8b), and (8c).

2 Perfect correlations for $2 \times 2$ systems

Hardy’s nonlocality proof involves an experimental set-up of the Einstein-Podolsky-Rosen-Bohm type [8, 9]: two correlated particles 1 and 2 fly apart in opposite directions from a common source such that each of them subsequently impinges on an appropriate measuring device which can measure either one of two physical observables at a time—$D_{11}$ or $D_{12}$ for (the apparatus measuring) particle 1, and $D_{21}$ or $D_{22}$ for (the one measuring) particle 2. Conventionally, it is supposed that the measurement of each one of these
observables gives the possible outcomes “+1” and “−1” (this is the case that arises, for example, in the realistic situation [10] in which a photon is detected behind a two-channel polarizer, with the value +1 (−1) assigned to detections corresponding to the transmitted (reflected) photon), so that we shall generally assume that the operators associated with such observables are of the form, \( \hat{D}_{ij} = |d_{ij}^+\rangle \langle d_{ij}^+| - |d_{ij}^-\rangle \langle d_{ij}^-| \), with \( i, j = 1 \) or 2, and where \( \{|d_{ij}^+\rangle, |d_{ij}^-\rangle\} \) constitutes an orthonormal basis for the Hilbert space pertaining to particle \( i \). On the other hand, according to the Schmidt decomposition theorem (see, for instance, Ref. 11 for a recent account of this topic), we can always write the quantum pure state of our two-particle system as a sum of two biorthogonal terms

\[
|\eta\rangle = c_1 |u_1\rangle |u_2\rangle + c_2 |v_1\rangle |v_2\rangle ,
\]

for some suitably chosen orthonormal basis \( \{|u_i\rangle, |v_i\rangle\} \) for particle \( i \), with the real coefficients \( c_1 \) and \( c_2 \) satisfying the relation \( c_1^2 + c_2^2 = 1 \). Now, by expressing the eigenvectors \( |d_{ij}^+\rangle \) and \( |d_{ij}^-\rangle \) in terms of the basis vectors \( |u_i\rangle \) and \( |v_i\rangle \)

\[
|d_{ij}^+\rangle = e^{i\alpha_{ij}} \cos \beta_{ij} |u_i\rangle + e^{i\gamma_{ij}} \sin \beta_{ij} |v_i\rangle ,
\]

\[
|d_{ij}^-\rangle = -e^{-i\gamma_{ij}} \sin \beta_{ij} |u_i\rangle + e^{-i\alpha_{ij}} \cos \beta_{ij} |v_i\rangle ,
\]

one can evaluate the quantum probability distributions \( P_\eta(D_{1k} = m, D_{2l} = n) \), with \( m, n = \pm 1 \) and \( k, l = 1 \) or 2, that a joint measurement of the observables \( D_{1k} \) and \( D_{2l} \) on particles 1 and 2, respectively, gives the outcomes \( D_{1k} = m \) and \( D_{2l} = n \) when the particles are described by the state vector (1). These are given by

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1 Of course in a real experiment it may well happen that neither one of the two photons of a given pair emitted by the source is registered by the detection system (or else that only one of them is detected), even though the two photons have nearly opposite directions. This is mainly due (although not exclusively) to the low efficiency of the available detectors. The usual way of circumventing this problem is to assume that the subensemble of actually detected pairs is a representative sample of the whole ensemble of emitted photon pairs.
\[ P_\eta(D_{1k} = +1, D_{2l} = +1) = c_1^2 \cos^2 \beta_{1k} \cos^2 \beta_{2l} + c_2^2 \sin^2 \beta_{1k} \sin^2 \beta_{2l} + \frac{1}{2} c_1 c_2 \cos \delta_{1k2l} \sin 2\beta_{1k} \sin 2\beta_{2l} , \]  

(3a)

\[ P_\eta(D_{1k} = -1, D_{2l} = -1) = c_1^2 \sin^2 \beta_{1k} \sin^2 \beta_{2l} + c_2^2 \cos^2 \beta_{1k} \cos^2 \beta_{2l} + \frac{1}{2} c_1 c_2 \cos \delta_{1k2l} \sin 2\beta_{1k} \sin 2\beta_{2l} , \]  

(3b)

\[ P_\eta(D_{1k} = +1, D_{2l} = -1) = c_1^2 \cos^2 \beta_{1k} \sin^2 \beta_{2l} + c_2^2 \sin^2 \beta_{1k} \cos^2 \beta_{2l} - \frac{1}{2} c_1 c_2 \cos \delta_{1k2l} \sin 2\beta_{1k} \sin 2\beta_{2l} , \]  

(3c)

\[ P_\eta(D_{1k} = -1, D_{2l} = +1) = c_1^2 \sin^2 \beta_{1k} \cos^2 \beta_{2l} + c_2^2 \cos^2 \beta_{1k} \sin^2 \beta_{2l} - \frac{1}{2} c_1 c_2 \cos \delta_{1k2l} \sin 2\beta_{1k} \sin 2\beta_{2l} , \]  

(3d)

where \( \delta_{1k2l} = \delta_{1k} - \delta_{2l} \), with \( \delta_{1k} = \gamma_{1k} - \alpha_{1k} \) and \( \delta_{2l} = \alpha_{2l} - \gamma_{2l} \). Of course, the above probabilities add up to unity

\[ \sum_{m,n=\pm 1} P_\eta(D_{1k} = m, D_{2l} = n) = 1 . \]  

(4)

We will note at this point that for the special case in which \( |c_1| = |c_2| = 2^{-1/2} \) (i.e., when state (1) happens to be totally entangled), the following two equalities, \( P_\eta(D_{1k} = +1, D_{2l} = +1) = P_\eta(D_{1k} = -1, D_{2l} = -1) \) and \( P_\eta(D_{1k} = +1, D_{2l} = -1) = P_\eta(D_{1k} = -1, D_{2l} = +1) \), hold true. Now, the expectation value of the product of the measurement outcomes of \( D_{1k} \) and \( D_{2l} \) is defined (in an obvious notation) as

\[ E_\eta(D_{1k}, D_{2l}) = P_{++} + P_{--} - P_{+-} - P_{-+}. \]  

(5)

Substituting expressions (3a)-(3d) into Eq. (5) gives

\[ E_\eta(D_{1k}, D_{2l}) = \cos 2\beta_{1k} \cos 2\beta_{2l} + 2c_1 c_2 \cos \delta_{1k2l} \sin 2\beta_{1k} \sin 2\beta_{2l}. \]  

(6)
(Note, incidentally, that for either $c_1 = 0$ or $c_2 = 0$ (i.e., for product states) the quantum correlation function (6) factorizes with respect to the parameters $\beta_{1k}$ and $\beta_{2l}$, and consequently it will be unable to yield a violation of Bell's inequality.) In general the correlation function (6) takes on values in the range between $-1$ and $+1$. We are interested in determining the conditions under which this function attains its extremal values $\pm 1$. By direct inspection of Eq. (6) it follows at once that whenever we have $\beta_{1k} = n_{1k}\pi/2$ and $\beta_{2l} = n_{2l}\pi/2$ ($n_{1k}, n_{2l} = 0, \pm 1, \pm 2, \ldots$), perfect correlations happen for any state of the form (1) irrespective of the values of $c_1$ and $c_2$. This corresponds to the case that the operators $\hat{D}_{1k}$ and $\hat{D}_{2l}$ have eigenvectors that arise from the Schmidt decomposition of the entangled state (see Eq. (1)). For this case the operators $\hat{D}_{1k}$ and $\hat{D}_{2l}$ take the form $\hat{D}_{1k} = \mu_{1k} |u_1\rangle \langle u_1| + \eta_{1k} |v_1\rangle \langle v_1|$ and $\hat{D}_{2l} = \mu_{2l} |u_2\rangle \langle u_2| + \eta_{2l} |v_2\rangle \langle v_2|$ (with $\mu_{1k}, \eta_{1k}, \mu_{2l},$ and $\eta_{2l}$ being sign factors fulfilling $\mu_{1k}\eta_{1k} = \mu_{2l}\eta_{2l} = -1$), and then, from the very structure of the state vector (1), it is apparent that a measurement of $\hat{D}_{1k}$ on particle 1 will uniquely determine the outcome of a measurement of $\hat{D}_{2l}$ on particle 2, and vice versa. On the other hand, for the case in which $c_1, c_2 \neq 0$, and $\beta_{1k} \neq n_{1k}\pi/2$, $\beta_{2l} \neq n_{2l}\pi/2$, it follows that in order for the correlation function (6) to take on its extremal values $\pm 1$ it is necessary that, (i) the state (1) be maximally entangled, and (ii) $\delta_{1k2l} = n_{1k2l}\pi$, $n_{1k2l} = 0, \pm 1, \pm 2, \ldots$. Indeed, whenever the equality $2c_1c_2 \cos \delta_{1k2l} = \mp 1$ holds, Eq. (6) reduces to

$$E_\eta(D_{1k}, D_{2l}) = \cos 2(\beta_{1k} \pm \beta_{2l}) ,$$

which attains the value $+1$ or $-1$ for $\beta_{1k} \pm \beta_{2l} = m_{1k2l}\pi/2$, $m_{1k2l} = 0, \pm 1, \pm 2, \ldots$.

### 3 Hardy’s nonlocality conditions: implications for the maximally entangled case

In terms of joint probabilities, a generic two-particle state $|\eta\rangle$ of the form (1) will show Hardy-type nonlocality contradiction if the following four condi-
tions are simultaneously fulfilled for the state $|\eta\rangle$ [1, 3-6].

\begin{align}
\eta(D_{11} = -1, D_{21} = -1) &= 0, \quad (8a) \\
\eta(D_{11} = +1, D_{22} = +1) &= 0, \quad (8b) \\
\eta(D_{12} = +1, D_{21} = +1) &= 0, \quad (8c) \\
\eta(D_{12} = +1, D_{22} = +1) &> 0. \quad (8d)
\end{align}

Taking into account the aforementioned equalities, $\eta(D_{1k} = +1, D_{2l} = +1) = \eta(D_{1k} = -1, D_{2l} = -1)$ and $\eta(D_{1k} = +1, D_{2l} = -1) = \eta(D_{1k} = -1, D_{2l} = +1)$, which are valid only in the case that the state $|\eta\rangle$ is maximally entangled, it is immediate to see that the fulfillment of Eqs. (8a), (8b), and (8c) for such a special case implies perfect correlation between the measurement outcomes of $D_{11}$ and $D_{21}$, $D_{11}$ and $D_{22}$, and $D_{12}$ and $D_{21}$, respectively. So, for example, if condition (8a) is fulfilled for the maximally entangled state we have that $\eta(D_{11} = -1, D_{21} = -1) = \eta(D_{11} = +1, D_{21} = +1) = 0$. Therefore, the probability $\eta^{\pm}(D_{11}, D_{21}) = \eta(D_{11} = +1, D_{21} = -1) + \eta(D_{11} = -1, D_{21} = +1)$ of getting different readings for a measurement of $D_{11}$ and $D_{21}$ is unity, and thus the measurement outcomes for such observables are perfectly correlated in that whenever the outcome $+1$ ($-1$) is observed for particle 1 then with certainty the outcome $-1$ ($+1$) will be observed for particle 2, and vice versa. Mathematically, this is expressed by the fact that $E_{\eta}(D_{11}, D_{21}) = -1$ (see Eq. (5)). A similar conclusion applies to the measurement outcomes of $D_{11}$ and $D_{22}$, and also to the measurement

\footnote{The same type of contradiction is obtained if, in Eqs. (8a)-(8d), we convert all the +1’s into −1’s, and vice versa. Likewise, the same holds true if we reverse the sign only to the outcomes of $D_{1k}$ for each of the Eqs. (8a)-(8d). Indeed, as may be easily checked, the following set of conditions

\begin{align}
\eta(D_{11} = +1, D_{21} = -1) &= 0, \\
\eta(D_{11} = -1, D_{22} = +1) &= 0, \\
\eta(D_{12} = -1, D_{21} = +1) &= 0, \\
\eta(D_{12} = -1, D_{22} = +1) &> 0,
\end{align}

will also lead to a Hardy-type nonlocality contradiction. Naturally, by symmetry, the same is true if we reverse the sign only to the outcomes of $D_{2l}$ for each of the Eqs. (8a)-(8d).}
outcomes of $D_{12}$ and $D_{21}$, if conditions (8b) and (8c) are to be satisfied for the maximally entangled state; namely, $E_\eta(D_{11}, D_{22}) = E_\eta(D_{12}, D_{21}) = -1$.

Now we are going to show that the fulfillment of all three conditions (8a)-(8c) for the maximally entangled state also implies perfect correlation for the observables $D_{12}$ and $D_{22}$, in the sense of having in all cases different measurement outcomes for $D_{12}$ and $D_{22}$. Clearly, this makes it impossible the fulfillment of the remaining condition in Eq. (8d). First of all we note that, as a general rule, the fulfillment of Eqs. (8a), (8b), and (8c), requires, respectively, that $\delta_{1121} = n_{1121} \pi$, $\delta_{1122} = n_{1122} \pi$, and $\delta_{1221} = n_{1221} \pi$. This is so because the derivative of $P_\eta(D_{1k} = m, D_{2l} = n)$ with respect to the variable $\delta_{1k2l}$ must be zero at the minimum value $P_\eta = 0$. For concreteness, and without any loss of generality, from now on we shall take the choice $n_{1121} = n_{1122} = n_{1221} = 0$, so that $\delta_{1121} = \delta_{1122} = \delta_{1221} = 0$. Recalling that $\delta_{1k2l} = \delta_{1k} - \delta_{2l}$, this in turn implies that the relative phases $\delta_{ij}$ are constrained to obey the relation $\delta_{11} = \delta_{12} = \delta_{21} = \delta_{22}$. Therefore, we deduce that the angle $\delta_{1222} = \delta_{12} - \delta_{22}$ must equally be zero. Of course the fact that the cosine function $\cos \delta_{1222}$ takes the value $+1$ [or else $-1$] is a necessary (although not a sufficient) condition in order for the probability in Eq. (8d) to reach its minimum value 0. The constraint $\delta_{1222} = 0$ having been established, all what we need to reach the desired conclusion is the following straightforward mathematical result [6], whose proof is given in the Appendix.

\textit{Lemma}—For the case that $\cos \delta_{1k2l} = +1$, the necessary and sufficient condition in order for the probability (3a) to vanish is

$$\tan \beta_{1k} \tan \beta_{2l} = -c_1/c_2.$$  \hspace{1cm} (9a)

Analogously, for the case that $\cos \delta_{1k2l} = +1$, the vanishing of the probability in Eq. (3b) is equivalent to requiring that

$$\tan \beta_{1k} \tan \beta_{2l} = -c_2/c_1.$$  \hspace{1cm} (9b)
For the case of maximal entanglement we have $|c_1| = |c_2|$, and then, supposing for example that the coefficients $c_1$ and $c_2$ are of the same sign, both of conditions (9a) and (9b) reduce to the single one:

$$\tan \beta_{1k} \tan \beta_{2l} = -1.$$  \hspace{1cm} (10)

Thus, taking into account that $\delta_{1121} = 0$, $\delta_{1122} = 0$, and $\delta_{1221} = 0$, we can apply the previous lemma to conclude that, if conditions (8a), (8b), and (8c) are to be satisfied for the maximally entangled state, we must have the relations

$$\tan \beta_{11} \tan \beta_{21} = -1,$$  \hspace{1cm} (11a)
$$\tan \beta_{11} \tan \beta_{22} = -1,$$  \hspace{1cm} (11b)
$$\tan \beta_{12} \tan \beta_{21} = -1.$$  \hspace{1cm} (11c)

An immediate but crucial consequence of relations (11a)-(11c) is

$$\tan \beta_{12} \tan \beta_{22} = -1.$$  \hspace{1cm} (11d)

(This is obtained by simply multiplying Eqs. (11b) and (11c), and then substituting Eq. (11a) into the left-hand side of the product.) Relation (11d), together with the above deduced constraint $\delta_{1222} = 0$, allows one to finally conclude that the fulfillment of all three conditions (8a)-(8c) by the maximally entangled state necessarily implies that both probabilities $P_\eta(D_{12} = +1, D_{22} = +1)$ and $P_\eta(D_{12} = -1, D_{22} = -1)$ are equal to zero, thus contradicting the condition in Eq. (8d). Clearly, this in turn means that the correlation function $E_\eta(D_{12}, D_{22})$ has to take the extremal value $-1$. Naturally, the result $E_\eta(D_{12}, D_{22}) = -1$ also follows directly by applying Eqs. (6) and (11d). Indeed, for the case considered we have $2c_1 c_2 \cos \delta_{1222} = +1$, and then, by Eq. (6), $E_\eta(D_{12}, D_{22}) = \cos 2(\beta_{12} - \beta_{22})$.

Now, from Eq. (11d), the parameters $\beta_{12}$ and $\beta_{22}$ ought to satisfy the relation $\beta_{12} - \beta_{22} = m_{1222}\pi/2$, with $m_{1222}$ being an odd integer $\pm 1, \pm 3, \ldots$ . Hence the result $E_\eta(D_{12}, D_{22}) = -1$, as claimed.

\footnote{Naturally the coincidence of both conditions (9a) and (9b) in the case of maximal entanglement stems from the fact that, for this case, $P_\eta(D_{1l} = +1, D_{2l} = +1) = P_\eta(D_{1l} = -1, D_{2l} = -1)$.}
We will note, incidentally, that the fulfillment of condition (8d) for the maximally entangled state precludes the simultaneous fulfillment of all three conditions (8a)-(8c). Indeed, in order to have $E_\eta(D_{12}, D_{22}) \neq -1$ for the maximally entangled state it is necessary that (supposing $\delta_{1222} = 0$, and $\text{sgn } c_1 = \text{sgn } c_2$),

$$\tan \beta_{12} \tan \beta_{22} \neq -1,$$  \hspace{1cm} (12)

and thus at least one of the conditions in Eqs. (11a)-(11c) cannot be satisfied. As an example of this, consider the case where both conditions (11a) and (11b) are satisfied. Yet, this in turn implies that $\tan \beta_{22} = \tan \beta_{21}$, and then, from Eq. (12) we have that $\tan \beta_{12} \tan \beta_{21} \neq -1$, in disagreement with Eq. (11c).

The results following Eqs. (11a)-(11d) can be conveniently summarized as follows: whenever we have $E_\eta(D_{11}, D_{21}) = E_\eta(D_{11}, D_{22}) = E_\eta(D_{12}, D_{21}) = -1$ for the maximally entangled state, then necessarily we must have $E_\eta(D_{12}, D_{22}) = -1$ as well. An analogous argument could be established to conclude that, whenever $E_\eta(D_{11}, D_{21}) = E_\eta(D_{11}, D_{22}) = E_\eta(D_{12}, D_{21}) = +1$ for the maximally entangled state, then $E_\eta(D_{12}, D_{22}) = +1$. As we have seen, this very fact makes it impossible the simultaneous fulfillment of all the Hardy conditions (8a)-(8d) for the nontrivial case of maximal entanglement (of course, this impossibility holds trivially for the case of product states).

Let us now consider Bell’s inequality in the form of the Clauser-Horne-Shimony-Holt (CHSH) inequality [2]. This can be written as

$$|E_\eta(D_{11}, D_{21}) + E_\eta(D_{11}, D_{22}) + E_\eta(D_{12}, D_{21}) - E_\eta(D_{12}, D_{22})| \leq 2.$$  \hspace{1cm} (13)

It is evident that, as it should be expected, the perfect correlations we have just derived for the maximally entangled state do satisfy the CHSH inequality. Indeed, for this case we have $\Delta = 2$, where the quantity $\Delta$ is defined by $\Delta \equiv |E_\eta(D_{11}, D_{21}) + E_\eta(D_{11}, D_{22}) + E_\eta(D_{12}, D_{21}) - E_\eta(D_{12}, D_{22})|$. The validity of the CHSH inequality for the simplest case of a $2 \times 2$ system described by a pure state is a necessary and sufficient condition for the existence of a (deterministic) local hidden-variable model for the observable correlations of all combinations of measurements $(D_{1k}, D_{2l})$ independently
performed on both subsystems \cite{12}. Therefore, we can say with complete
certainty that all the perfect correlations \( E_\eta(D_{11}, D_{21}) = E_\eta(D_{11}, D_{22}) = E_\eta(D_{12}, D_{21}) = E_\eta(D_{12}, D_{22}) = \pm 1 \) can be consistently explained by a local
hidden-variable model. Indeed, it is easy to see that all these perfect
correlations are compatible with the assumption of local realism, so that they
can be reproduced by a local realistic model. So, consider for example the
case when all the perfect correlations above are \(-1\), and suppose that a joint
measurement of the observables \( D_{11} \) and \( D_{21} \) is carried out. Owing to the
perfect correlation \( E_\eta(D_{11}, D_{21}) = -1 \), the measurement results for such ob-
servables should be, respectively, \(+1\) and \(-1\) (or else, \(-1\) and \(+1\)). Suppose
now that, instead of \( D_{21} \), the observable \( D_{22} \) had been measured on particle
2. According to local realism, the result obtained in a measurement of \( D_{1k} \)
on particle 1 cannot depend in any way on which observable \( D_{2l} \) happens
to be measured on the other, spatially separated particle 2, and vice versa.
Thus, taking into account the constraint \( E_\eta(D_{11}, D_{22}) = -1 \), we conclude
that, had the observable \( D_{22} \) been measured, the outcome \(-1\) (+1) would
have been observed whenever the measurement result for \( D_{11} \) is found to be
\(+1\) (\(-1\)). Applying a symmetrical reasoning to the pair of observables \( D_{12} \)
and \( D_{21} \), and noting that \( E_\eta(D_{12}, D_{21}) = -1 \), we finally deduce that, had
\( D_{12} \) been measured on particle 1, the outcome \(+1\) (\(-1\)) would have been observed whenever the measurement result for \( D_{21} \) is found to be \(-1\) (+1).
In short, by applying local realism, and for the case that the measu-
rement results for \( D_{11} \) and \( D_{21} \) happen to be, respectively, \(+1\) and \(-1\) (\(-1\) and
\(+1\)), it is concluded that the measurement results for \( D_{12} \) and \( D_{22} \) would
have been, respectively, \(+1\) and \(-1\) (\(-1\) and \(+1\)). But of course this is con-
sistent with the fact that \( E_\eta(D_{12}, D_{22}) = -1 \). The compatibility of these
perfect correlations with the assumption of local realism implies that the
CHSH inequality should be satisfied by such perfect correlations, in ac-

\footnote{It should be noted, however, that for the case in which the \( 2 \times 2 \) system is described
by a mixed state the fulfillment of the CHSH inequality is not in general a sufficient
condition for the existence of a local hidden-variable model that reproduces the results of
more complex (“nonideal”) measurements. See, for example, the introductory part of the
article in Ref. 13, and references therein.}
dance with our numerical result that the parameter $\Delta$ has to be equal to 2 for the maximally entangled state, if this is to satisfy each of the Hardy equations (8a), (8b), and (8c). It is worth noting, however, that no local hidden-variable model exists that accommodates all the perfect correlations $E_\eta(D_{11}, D_{21}) = E_\eta(D_{11}, D_{22}) = E_\eta(D_{12}, D_{21}) = -E_\eta(D_{12}, D_{22}) = \pm 1$, in spite of the fact that there indeed exists a classical model accounting for each of these perfect correlations separately. This is because for this case the parameter $\Delta$ turns to be greater than 2 (in fact, it attains the maximum possible theoretical value $\Delta = 4$), and then the violation of the CHSH inequality excludes the existence of a local hidden-variable model that reproduces simultaneously all four correlation functions involved in it.

Clearly, the parameter $\Delta$ takes on the value 4 if, and only if, $E_\eta(D_{11}, D_{21}) = E_\eta(D_{11}, D_{22}) = E_\eta(D_{12}, D_{21}) = E_\eta(D_{12}, D_{22}) = \pm 1$. From the discussion leading to Eq. (7) it follows that, for the general case in which $\beta_{1k} \neq n_{1k}\pi/2$ and $\beta_{2l} \neq n_{2l}\pi/2$, it is necessary that the state $|\eta\rangle$ be maximally entangled, if we want that the quantum correlation function $E_\eta(D_{1k}, D_{2l})$ attains the value $+1$ or $-1$. But, as we have shown, the impossibility for the maximally entangled state to simultaneously satisfy all the Hardy equations (8a)-(8d) resides in the fact that, whenever we have $E_\eta(D_{11}, D_{21}) = E_\eta(D_{11}, D_{22}) = E_\eta(D_{12}, D_{21}) = E_\eta(D_{12}, D_{22}) = \pm 1$ for such a state, then necessarily $E_\eta(D_{12}, D_{22}) = \pm 1$.

5Precisely speaking, what we have done so far is to show that the four given perfect correlations $E_\eta(D_{11}, D_{21}) = \pm 1$, $E_\eta(D_{11}, D_{22}) = \pm 1$, $E_\eta(D_{12}, D_{21}) = \pm 1$, and $E_\eta(D_{12}, D_{22}) = \pm 1$ (with each of them taking simultaneously either the $+$ or $-$ sign) do not contradict each other when local realism is assumed to hold. On the other hand, Bell’s local model for pairs of spin-$\frac{1}{2}$ particles in the singlet state [14] (see also Sec. II and Appendix D of Ref. 7) provides perhaps the simplest example of a local hidden-variable model capable of reproducing the perfect correlations $E_\eta(n_1, n_2) = \pm 1$ that arise in the special case that $n_1 = n_2$ or $n_1 = -n_2$, where $n_1$ and $n_2$ are the directions along which the spin is measured. Therefore, our demonstration supplemented by Bell’s model (or suitable generalizations of it) allows one to fully describe all the quantum perfect correlations $E_\eta(D_{11}, D_{21}) = E_\eta(D_{11}, D_{22}) = E_\eta(D_{12}, D_{21}) = E_\eta(D_{12}, D_{22}) = \pm 1$ entirely in terms of a local realistic model.
The interesting observation then is that the failure of Hardy’s nonlocality theorem [1] for the maximally entangled case just prevents the quantum prediction for the parameter $\Delta$ from reaching the maximum theoretical value $\Delta = 4$. Indeed, for this case the quantum prediction for $\Delta$ falls to 2, and then all the relevant quantum correlation functions $E_\eta(D_{11}, D_{21})$, $E_\eta(D_{11}, D_{22})$, $E_\eta(D_{12}, D_{21})$, and $E_\eta(D_{12}, D_{22})$ can be interpreted in terms of a classical model based on the assumption of local realism.

On the other hand, as already expounded by Krenn and Svozil [16], a maximum violation of the CHSH inequality by the value 4 would correspond to a two-particle analog of the GHZ argument for nonlocality [7]. Indeed, as we have seen, if we have $E_\eta(D_{11}, D_{21}) = E_\eta(D_{11}, D_{22}) = E_\eta(D_{12}, D_{21}) = \pm 1$, then, according to local realism, we must have the prediction $E_\eta(D_{12}, D_{22}) = \pm 1$, which directly contradicts the (hypothetical) result $E_\eta(D_{12}, D_{22}) = \mp 1$.

As we are dealing with perfect correlations, this contradiction would apply to each of the pairs in the ensemble. The failure of Hardy’s theorem for the maximally entangled state can thus also be read as a proof of the fact that it is not possible to construct a GHZ-type nonlocality argument for a $2 \times 2$ system (unless hypothetical extremely nonclassical correlations are assumed to hold [16]).

4 Hardy’s nonlocality conditions: implications for the general case

Let us now consider the case of less-than-maximally entangled state, that is, one for which $|c_1| \neq |c_2|$ in Eq. (1). For this case the equalities $P_\eta(D_{1k} = +1, D_{2l} = +1) = P_\eta(D_{1k} = -1, D_{2l} = -1)$ and $P_\eta(D_{1k} = +1, D_{2l} = -1) = P_\eta(D_{1k} = -1, D_{2l} = +1)$ are no longer valid, and then the fulfillment of conditions (8a), (8b), and (8c) does not imply any perfect correlation between the measurement results of $D_{11}$ and $D_{21}$, $D_{11}$ and $D_{22}$, and $D_{12}$ and $D_{21}$, respectively. According to the lemma in Sec. 3, the fulfillment of the Hardy equations (8a), (8b), and (8c) is equivalent to requiring, respectively, (as
before, we assume that $\delta_{1121} = \delta_{1122} = \delta_{1221} = 0$)

$$\tan \beta_{11} \tan \beta_{21} = -c_2/c_1, \quad (14a)$$

$$\tan \beta_{11} \tan \beta_{22} = -c_1/c_2, \quad (14b)$$

$$\tan \beta_{12} \tan \beta_{21} = -c_1/c_2. \quad (14c)$$

From Eqs. (14a)-(14c) we get

$$\tan \beta_{12} \tan \beta_{22} = -(c_1/c_2)^3. \quad (14d)$$

Clearly, whenever $c_1, c_2 \neq 0$ and $|c_1| \neq |c_2|$, we have from Eq. (14d) that $\tan \beta_{12} \tan \beta_{22} \neq -c_1/c_2$, and, therefore, it is concluded that, for the nonmaximally entangled case, the fulfillment of conditions (8a)-(8c) automatically implies the fulfillment of condition (8d). It should be noted at this point that, for any given $c_1$ and $c_2$, the Eqs. (14a)-(14c) do not uniquely determine the four parameters $\beta_{ij}$, so that we can always choose one of them in an unrestricted way [6]. So, in what follows, we shall take $\beta_{12}$ to be equal to $\beta_0$, with $\beta_0$ being a variable taking on any arbitrary value. Naturally, once the parameter $\beta_{12}$ is given, the remaining three are forced to accommodate. Indeed, from Eqs. (14a)-(14c), we obtain immediately

$$\tan \beta_{11} = (c_2/c_1)^2 \tan \beta_0, \quad (15a)$$

$$\tan \beta_{21} = -(c_1/c_2) \cot \beta_0, \quad (15b)$$

$$\tan \beta_{22} = -(c_1/c_2)^3 \cot \beta_0. \quad (15c)$$

We now show explicitly how the fulfillment of all the Hardy conditions (8a)-(8d) relates to the violation of the CHSH inequality. Since such conditions cannot all be satisfied within a local and realistic framework, it is to be expected that the fulfillment of Eqs. (8a)-(8d) entails a violation of the resulting CHSH inequality. That this is indeed the case can be demonstrated in a rather general way as follows. For this purpose, it is convenient to express the parameter $\Delta$ in terms of the set of probabilities $P_{ij}(D_{1k}, D_{2l}) = P_{ij}(D_{1k} = +1, D_{2l} = +1) + P_{ij}(D_{1k} = -1, D_{2l} = -1)$, with $k, l = 1, 2$. Provided with such probabilities, the quantity $\Delta$ can be written
\[ \Delta = 2 \left| P_\eta(D_{11}, D_{21}) + P_\eta(D_{11}, D_{22}) + P_\eta(D_{12}, D_{21}) - P_\eta(D_{12}, D_{22}) - 1 \right|. \] (16)

Expression (16) is, as it stands, completely general. Now, for the particular case where conditions (8a)-(8d) are satisfied, we have\(^6\)

\[ \Delta = 2 \left| P_\eta(D_{11} = +1, D_{21} = +1) + P_\eta(D_{11} = -1, D_{22} = -1) + P_\eta(D_{12} = +1, D_{22} = +1) - P_\eta(D_{12} = -1, D_{22} = -1) - 1 \right|. \] (17)

Using Eqs. (3a) and (3b) in (17) (with \( \cos \delta_{1k} = +1 \)), and taking into account the constraints in Eqs. (14a)-(14d) and (15a)-(15c), we find, after a bit lengthy but straightforward calculation,

\[
\Delta = 2 \left| \frac{(2c_1^2 - 1)^2}{1 + \frac{(1-c_1^2)^2}{c_1^2} \tan^2 \beta_0 + \frac{c_1^4}{1-c_1^2} \cot^2 \beta_0} + \frac{(2c_1^2 - 1)^2}{1 + \frac{(1-c_1^2)^3}{c_1^2} \tan^2 \beta_0 + \frac{c_1^6}{(1-c_1^2)^2} \cot^2 \beta_0} - \frac{(2c_1^2 - 1)^2 \cos^2 \beta_0}{1 - c_1^2 + c_1^2 \cot^2 \beta_0} \right|.
\] (18)

The parameter \( \Delta \) given by (18) is represented graphically in Fig. 1 as a function of \( c_1^2 \) and \( \beta_0 \) for the ranges of variation \( 0 \leq c_1^2 \leq 1 \) and \( 0^\circ \leq \beta_0 \leq 90^\circ \). From Fig. 1, it can be seen that \( \Delta \) is greater than 2 for all values of \( c_1^2 \) and \( \beta_0 \) except for \( c_1^2 = 0, 1 \), and 0.5 (that is, product and maximally entangled states), and \( \beta_0 = n\pi/2, n = 0, \pm 1, \pm 2, \ldots \). This latter exception arises because, if \( \beta_0 = n\pi/2 \), then by Eqs. (15a)-(15c) we must have necessarily \( \beta_{ij} = n_{ij}\pi/2 \) for each \( i, j = 1, 2 \), and for some \( n_{ij} = 0, \pm 1, \pm 2, \ldots \) (with

\(^6\)Actually, Eq. (17) also applies to the case that the probability \( P_\eta(D_{12} = +1, D_{22} = +1) \) in Eq. (8d) is equal to zero.)
\[ \Delta = 2 + 4\tau - 5, \] with \( \tau \) being the golden mean \( \frac{1}{2}(1 + \sqrt{5}) \). The quantity \( \Delta_g - 2 \) corresponds to approximately 43.5% of the maximum violation predicted by quantum mechanics of the CHSH inequality.
It is worthwhile to mention that the graph for $\Delta$ in Fig. 1 has quite the same shape as the graphical representation of the probability $P_\eta(D_{12} = +1, D_{22} = +1)$ in Eq. (8d) (this latter graph can be found in Ref. 6), the only relevant difference being the respective ranges of variation of the values taken by such functions, namely, $[2, 2 + 4\tau^{-5}]$ for $\Delta$, and $[0, \tau^{-5}]$ for $P_\eta(D_{12} = +1, D_{22} = +1)$. Indeed, we have proved with the aid of a computer program that the whole expression (18) is connected with the probability function $P_\eta(D_{12} = +1, D_{22} = +1)$ (which is given explicitly by the fourth term on the right-hand side of Eq. (18)) through the simple relation

$$\Delta = 2 + 4P_\eta(D_{12} = +1, D_{22} = +1).$$  \hspace{1cm} (19)

In particular this means that the parameter $\Delta$ in Eq. (18) is maximum whenever $P_\eta(D_{12} = +1, D_{22} = +1)$ so is. Likewise, $\Delta$ takes its minimum value 2 whenever $P_\eta(D_{12} = +1, D_{22} = +1)$ vanishes. This close relationship between $\Delta$ and $P_\eta(D_{12} = +1, D_{22} = +1)$ was to be expected since the value of $P_\eta(D_{12} = +1, D_{22} = +1)$ can be regarded as a direct measure of the degree of “nonlocality” inherent in the Hardy equations (8a)-(8d).

It will further be noted, incidentally, that Hardy’s argument for nonlocality can equally be cast in the form of a simple inequality involving the four probabilities in Eqs. (8a)-(8d) [17]:

$$P_\eta(D_{12} = +1, D_{22} = +1) \leq P_\eta(D_{11} = -1, D_{21} = -1) + P_\eta(D_{11} = +1, D_{22} = +1) + P_\eta(D_{12} = +1, D_{21} = +1).$$  \hspace{1cm} (20)

Quantum mechanics predicts a maximum violation of inequality (20) for the values $P_\eta(D_{11} = -1, D_{21} = -1) = P_\eta(D_{11} = +1, D_{22} = +1) = P_\eta(D_{12} = +1, D_{21} = +1) = 0$, and $P_\eta(D_{12} = +1, D_{22} = +1) = \tau^{-5}$. The point to be stressed here is that, for this same set of values, quantum mechanics predicts a violation of the CHSH inequality which is four times bigger than that obtained for the inequality (20). It is therefore concluded that, in order to achieve a more conclusive, clear-cut experimental verification of Hardy’s nonlocality theorem [1], one could try to measure the observable probabilities in Eq. (17), once the conditions $P_\eta(D_{11} = -1, D_{21} = -1) = P_\eta(D_{11} =
Bell inequalities should be satisfied by any realistic theory fulfilling a very broad and general locality condition according to which the real factual situation of a system must be independent of anything that may be done with some other system which is spatially separated from, and not interacting with, the former [8, 20]. When applied to our particular situation, this requirement essentially means that the effect of the choice of the observable $D_{ij}$ to be measured on particle $i$, $i = 1, 2$, cannot influence the result obtained with another remote measuring device acting on the other particle. For this class of theories the ensemble (measurable) probability of jointly obtaining the result $D_{1k} = \pm 1$ for particle 1 and the result $D_{2l} = \pm 1$ for particle 2, has the functional form [2, 21, 22]

$$P_{HV}(D_{1k} = \pm 1, D_{2l} = \pm 1) = \int_{\Lambda} d\lambda \rho(\lambda) P(D_{1k} = \pm 1|\lambda)P(D_{2l} = \pm 1|\lambda).$$

(21)

In Eq. (21), $\lambda$ is a set of variables (with domain of variation $\Lambda$) representing the complete physical state of each individual pair of particles 1 and 2 emerging from the source, $\rho(\lambda)$ is the (normalized) hidden-variable distribution function for the initial joint state of the particles, and $P(D_{ij} = \pm 1|\lambda)$ is the probability that an individual particle $i$ in the state $\lambda$ gives the result $\pm 1$ for a measurement of $D_{ij}$. Note that both $\rho(\lambda)$ and $\Lambda$ are independent of the actual setting $j = 1$ or 2 corresponding, respectively, to a measurement of $D_{i1}$ or $D_{i2}$ on particle $i$. As is well known, classical prob-
abilities of the form (21) lead to validity of the inequality $\Delta \leq 2$ (and, generally speaking, to validity of any other Bell-type inequality). This is usually proved by invoking certain algebraic theorems (see, for instance, Ref. 22 for a derivation of Bell’s inequality in the context of actual optical tests of local hidden-variable theories). Since the fulfillment of all the Hardy conditions (8a)-(8d) implies the quantum mechanical violation of the inequality $\Delta \leq 2$ (see Fig. 1), it is concluded that no set of probabilities of the form (21) exists which generally reproduces the quantum prediction (18). The most remarkable exception to this statement is the case where $|c_1| = |c_2|$. For this case quantum mechanics predicts $\Delta = 2$, and then, as was discussed in Sec. 3, a rather trivial classical model of the type considered can be constructed which accounts for each of the quantum perfect correlations $E_{\eta}(D_{11}, D_{21}) = E_{\eta}(D_{11}, D_{22}) = E_{\eta}(D_{12}, D_{21}) = E_{\eta}(D_{12}, D_{22}) = \pm 1$.

5 Concluding remarks

A final comment is in order about the fact that maximally entangled states yield the maximum quantum mechanical violation of Bell’s inequality, while they are unable to exhibit Hardy-type nonlocality. The explanation for this seeming contradiction simply relies on the fact that the rather stringent constraints (11a)-(11d) implied by the Hardy equations (8a)-(8c) in the case of a maximally entangled state, are not at all present in the derivation of

\[8\] Recently, Barnett and Chefles (see Ref. 23) have shown how Hardy’s original theorem can be extended to reveal the nonlocality of all pure entangled states without inequalities. This is accomplished by considering generalized measurements (that is, measurements beyond the standard von Neumann type considered here) which perform unambiguous discrimination between nonorthogonal states.

\[9\] It is to be noticed that Bell’s illustrative model in Ref. 14 is a deterministic one in the sense that the hidden variable $\lambda$ (which, in Bell’s concrete model, is a unit vector in three-dimensional space) uniquely determines the outcome for any spin measurement on either particle. Eq. (21) above, on the other hand, defines a less restrictive (and, therefore, more general) type of local hidden-variable theory which is characterized by the fact that now the set of hidden variables $\lambda$ describing the joint state of the particles only determines the probability $P(D_{ij} = \pm 1 | \lambda)$ of obtaining a result $\pm 1$ when the observable $D_{ij}$ is measured on particle $i, i = 1, 2$. 

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Bell’s inequality. Indeed, in the case of Bell’s theorem, all the parameters $\beta_{1k}$, $\beta_{2l}$, $\delta_{1k}$, and $\delta_{2l}$ ($k, l = 1, 2$) are treated as independent variables which can assume any arbitrary value regardless of the quantum state $|\eta\rangle$ at issue, so that the Bell inequality will in fact be maximally violated for a suitable choice of $\beta_{ij}$ and $\delta_{ij}$ (provided $|c_1| = |c_2|$). So, consider the case in which $2c_1c_2 \cos \delta_{1k2l} = +1$ for each $k$ and $l$. For this case the quantum prediction for $\Delta$ is given by

$$\Delta = |\cos 2(\beta_{11} - \beta_{21}) + \cos 2(\beta_{11} - \beta_{22}) + \cos 2(\beta_{12} - \beta_{21}) - \cos 2(\beta_{12} - \beta_{22})|,$$

(22)

which attains the value $\Delta = 2\sqrt{2}$ whenever $\beta_{11} - \beta_{21} = -\pi/8$, $\beta_{11} - \beta_{22} = \pi/8$, $\beta_{12} - \beta_{21} = \pi/8$, and $\beta_{12} - \beta_{22} = 3\pi/8$. Only if the parameters $\beta_{ij}$ are constrained to obey the relations (11a)-(11d), as demanded by the Hardy equations (8a)-(8c) in the case of maximal entanglement, we have that $\beta_{1k} - \beta_{2l} = m_{1k2l}\pi/2$ for each $k$ and $l$, and for some odd integer $m_{1k2l}$ (for instance, $\beta_{11} - \beta_{21} = -\pi/2$, $\beta_{11} - \beta_{22} = \pi/2$, $\beta_{12} - \beta_{21} = \pi/2$, and $\beta_{12} - \beta_{22} = 3\pi/2$), and then $\Delta = 2$. Consider now the particular case in which $|c_1| = |c_2| = 2^{-1/2}$, $\beta_{11} = \pi/4$, $\beta_{12} = -3\pi/4$, $\beta_{21} = 3\pi/4$, and $\beta_{22} = -\pi/4$ (with the parameters $\delta_{1k}$ and $\delta_{2l}$ taking on any arbitrary value). For this case the quantum prediction for $\Delta$ becomes

$$\Delta = |\cos(\delta_{11} - \delta_{21}) + \cos(\delta_{11} - \delta_{22}) + \cos(\delta_{12} - \delta_{21}) - \cos(\delta_{12} - \delta_{22})|,$$

(23)

which attains the value $2\sqrt{2}$ for $\delta_{11} - \delta_{21} = -\pi/4$, $\delta_{11} - \delta_{22} = \pi/4$, $\delta_{12} - \delta_{21} = \pi/4$, and $\delta_{12} - \delta_{22} = 3\pi/4$. However, the fulfillment of the Hardy equations (8a), (8b), and (8c) requires, respectively, that $\delta_{1121} = n_{1121}\pi$, $\delta_{1122} = n_{1122}\pi$, and $\delta_{1221} = n_{1221}\pi$. In particular, the choice $\delta_{1121} = \delta_{1122} = \delta_{1221} = 0$ entails that $\delta_{1222} = 0$, and therefore, for such values, the quantity $\Delta$ takes again the value 2.

\[10\] Of course a similar conclusion applies to the case that $\text{sgn } c_1 \neq \text{sgn } c_2$, so that $2c_1c_2 \cos \delta_{1k2l} = -1$. For this case quantum mechanics predicts, $\Delta = |\cos 2(\beta_{11} + \beta_{21}) + \cos 2(\beta_{11} + \beta_{22}) + \cos 2(\beta_{12} + \beta_{21}) - \cos 2(\beta_{12} + \beta_{22})|$. Now the fulfillment of Eqs. (8a)-(8c) for the maximally entangled state requires that $\tan \beta_{1k} \tan \beta_{2l} = 1$ for each $k$ and $l$. This in turn implies that $\beta_{1k} + \beta_{2l} = m_{1k2l}\pi/2$ for some odd integer $m_{1k2l}$, and thus $\Delta = 2$.  

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To summarize, we have shown that whenever the Hardy equations (8a)-(8c) are fulfilled for the maximally entangled state then perfect correlations develop between the measurement outcomes $D_{1k} = m$ and $D_{2l} = n$ obtained in any one of the four possible combinations of joint measurements $(D_{1k}, D_{2l})$ one might actually perform on both particles. As a result, for such observables $D_{ij}$, the quantity $\Delta$ turns out to be equal to 2, and hence no violations of local realism will arise in those circumstances. This is in contrast with the situation entailed by Bell’s theorem (with inequalities) where no constraints such as Eqs. (8a)-(8c) need be fulfilled, and then all the relevant parameters can be varied freely. On the other hand, for the nonmaximally entangled case, we have generally shown that the fulfillment of conditions (8a)-(8c) necessarily makes the parameter $\Delta$ greater than 2, the greatest value of $\Delta$ predicted by quantum mechanics being as large as $\Delta_g = 2 + 4\tau^{-5}$.

As was emphasized in Sec. 4, this result could have some relevance from an experimental point of view, since it indicates that experiments based on the inequality $\Delta \leq 2$ (with $\Delta$ given by Eq. (17)) would be more efficient in order to exhibit Hardy’s nonlocality than those based on inequality (20).

Acknowledgments — The author wishes to thank Agustín del Pino for his interest and many useful discussions on the foundations of quantum mechanics. He would also like to thank an anonymous referee for his valuable suggestions which led to an improvement of an earlier version of this paper.

\footnote{Remember that the fulfillment of conditions (8a)-(8c) for the nonmaximally entangled state automatically entails the fulfillment of the remaining condition in Eq. (8d), provided $\beta_0 \neq n\pi/2$.}
APPENDIX

The demonstration of the lemma is as follows. Here we give only the proof that the vanishing of the probability function (3a) for the case that \( \cos \delta_{1k2l} = +1 \), is equivalent to the fulfillment of relation (9a), the proof concerning the equivalence of relation (9b) and the vanishing of Eq. (3b) being quite similar. We first show necessity, namely, that the vanishing of the probability in Eq. (3a) implies relation (9a), provided that \( \cos \delta_{1k2l} = +1 \). So, equating expression (3a) to zero, and putting \( \cos \delta_{1k2l} = +1 \), we get

\[
c_1^2 \cos^2 \beta_{1k} \cos^2 \beta_{2l} + c_2^2 \sin^2 \beta_{1k} \sin^2 \beta_{2l} = -2c_1c_2 \cos \beta_{1k} \cos \beta_{2l} \sin \beta_{1k} \sin \beta_{2l},
\]

(A1)

or, equivalently,

\[
\left( \frac{c_1}{c_2} \right)^2 \left[ \tan \beta_{1k} \tan \beta_{2l} \right]^{-1} + \tan \beta_{1k} \tan \beta_{2l} = -2 \left( \frac{c_1}{c_2} \right).
\]

(A2)

Now, making the identifications \( \tan \beta_{1k} \tan \beta_{2l} \equiv x \) and \( c_1/c_2 \equiv a \), Eq. (A2) can be rewritten in the form

\[
x^2 + 2ax + a^2 = 0,
\]

(A3)

or,

\[
(x + a)^2 = 0.
\]

(A4)

Obviously, Eq. (A4) is satisfied only for \( x = -a \), and, therefore, it is concluded that the vanishing of the probability (3a) (with \( \cos \delta_{1k2l} = +1 \)) necessarily entails that \( \tan \beta_{1k} \tan \beta_{2l} = -c_1/c_2 \).

The proof of sufficiency, namely, that the fulfillment of relation (9a) implies the vanishing of the probability (3a) when \( \cos \delta_{1k2l} = +1 \), is quite immediate, and it will not be detailed here.
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