Cyclic sums of comparative indices and their applications

Julia Elyseeva

Department of Applied Mathematics, Moscow State University of Technology, Vavkovskii per. 3a, 101472, Moscow, Russia

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, CZ-61137 Brno, Czech Republic
e-mail:elyseeva@gmail.com

Abstract
In this paper we generalize the notion of the comparative index for the pair of Lagrangian subspaces which has fundamental applications in oscillation theory of symplectic difference systems and linear differential Hamiltonian systems. We introduce cyclic sums \( \mu_r^k(Y_1, Y_2, \ldots, Y_m) \), \( m \geq 2 \) of the comparative indices for the set of \( n \)-dimensional Lagrangian subspaces. We formulate and prove main properties of the cyclic sums, in particular, we state connections of the cyclic sums with the Kashiwara index. The main results of the paper connect the cyclic sums of the comparative indices with the number of positive and negative eigenvalues of symmetric matrices.

Keywords: Comparative index, Kashiwara index, Maslov index, Discrete symplectic systems, Oscillation theory

2000 MSC: 15B57, 39A21, 53D12, 37B30

1. Introduction

In this paper we develop an important concept from the matrix analysis, the comparative index \([10, 11, 2, \text{chapter 3}]\) which has fundamental applications in the oscillation and spectral theory of the symplectic difference systems

\[
y_{k+1} = S_k y_k, \quad y_k \in \mathbb{R}^{2n}, \quad S_k \in \mathbb{R}^{2n \times 2n}, \quad S_k^T J S_k = J, \quad k = 0, \ldots, N, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (1.1)
\]

as well as in the oscillation theory of the linear differential Hamiltonian systems

\[
y' = J^T \mathcal{H}(t)y, \quad t \in [a, b], \quad y = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad \mathcal{H}(t) = \mathcal{H}^T(t) \quad (1.2)
\]

which are continuous counterparts of \((1.1)\).

In \([11, 9, \text{chapter 3}]\) the comparative index was introduced for the pair of \(2n \times n\) matrices \(Y, \hat{Y}\) which obey the conditions

\[
w(Y, Y) = 0 \quad \text{rank} Y = n, \quad w(\hat{Y}, \hat{Y}) = 0 \quad \text{rank} \hat{Y} = n, \quad w(Y, \hat{Y}) := Y^T J \hat{Y}. \quad (1.3)
\]

The matrices \(Y, \hat{Y}\) whose columns form bases of Lagrangian subspaces \(L_1, \hat{L} \) in \(\mathbb{R}^{2n}\) will be referred to as frames for \(L_1, L_2\). They can also be regarded as conjoined bases of \((1.1)\) and \((1.2)\) and the matrix \(w(Y, \hat{Y})\) in \((1.3)\) is called the Wronskian of \(Y\) and \(\hat{Y}\) according to the terminology from the oscillation theory of \((1.1)\) and \((1.2)\) (see \([5]\) and \([24]\)).

The advantage of the comparative index lies in the fact that it allows to derive classical separation and comparison results for conjoined bases of \((1.1)\) and \((1.2)\) in the form of explicit relations between the multiplicities of their focal points, see \([11, 12, 14, 26, 27]\) and \([9, \text{Chapter 4}]\). Further applications of the comparative index can be found in the spectral theory of \((1.1)\) and \((1.2)\), see \([9, \text{Chapters 5,6}], [13]\) and the reference given therein. In the recent publication \([25]\) the notion of the comparative index was connected with the traditional Lidskii angles \([23]\) for symplectic matrices, in \([15]\) we use the comparative index defining the so-called oscillation numbers (see \([13, 14]\)) for continuous Lagrangian paths and connect the oscillation numbers with the Maslov index in \([13]\).

According to \([11]\), we define the comparative index for \(Y, \hat{Y}\) partitioned into \(n \times n\) blocks according to \(Y = \begin{pmatrix} X \\ U \end{pmatrix} \), \(\hat{Y} = \begin{pmatrix} \hat{X} \\ \hat{U} \end{pmatrix}\) using the notation

\[
M = (I - X^T X) w(Y, \hat{Y}), \quad T = I - M^T M, \quad \mathcal{P} = T (w(Y, \hat{Y}) X^T \hat{X}) T, \quad (1.4)
\]
We prove that (1.13) is valid for the general case symmetric matrix according to the following theorem.

**Definition 1.1.** Consider 2n \( \times \) n matrices \( Y_k, k = 1, 2, \ldots, m, m \geq 2 \) which obey condition (1.3). We define the cyclic sums (of the first kind) as follows

\[
\mu(Y_1, Y_2, \ldots, Y_m) = \mu(Y_1, Y_2) + \mu(Y_2, Y_3) + \cdots + \mu(Y_{m-1}, Y_m) + \mu(Y_m, Y_1),
\]

where \( \mu(Y_1, Y_2) = \text{rank} \, M \), \( \mu(Y_2, Y_3) = \text{ind} \, P \), \( \mu(Y_3, Y_4) = \text{ind}(\mathcal{P}) \), and so on, where ind \( A \) denotes the index, i.e., the number of negative eigenvalues of the symmetric matrix \( A = A^T \).

By a similar way we introduce the cyclic sums (of the second kind)

\[
v(Y_1, Y_2, \ldots, Y_m) = \n(Y_1, Y_2) + \n(Y_2, Y_3) + \cdots + \n(Y_{m-1}, Y_m) + \n(Y_m, Y_1),
\]

where \( \n(Y_1, Y_2) = \text{rank} \, W(Y_1, Y_2) = \text{ind}(\mathcal{W}) \), \( \n(Y_2, Y_3) = \text{ind}(\mathcal{W}) \), and so on, where ind \( A \) denotes the index, i.e., the number of negative eigenvalues of the symmetric matrix \( A = A^T \).

The main result of the paper connects \( \mu(Y_1, Y_2, \ldots, Y_m) \) with the negative and positive inertia of a \( (mn) \times (mn) \) symmetric matrix according to the following theorem.
Theorem 1.2. Define the \((mn) \times (mn)\) symmetric matrix

\[
S_{1,2,\ldots,m} = \begin{pmatrix}
0 & w_{1,2} & w_{1,3} & \cdots & w_{1,m} \\
0 & 0 & w_{2,3} & \cdots & w_{2,m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
w_{m-1,1} & w_{m-1,2} & \cdots & 0 & w_{m-1,m} \\
w_{m,1} & w_{m,2} & \cdots & w_{m,m-1} & 0
\end{pmatrix},
\]

then for cyclic sums \((1.7)\) we have

\[
\mu^c_i(Y_1, Y_2, \ldots, Y_m) = i_2(S_{1,2,\ldots,m}) = \text{ind}(\pm S_{1,2,\ldots,m}).
\]  

(1.17)

Formula \((1.17)\) presents fundamental connections between the comparative index theory and matrix linear algebra. The proof is based on the main theorem of the comparative index (see [11, Theorem 2.2]) which implies that cyclic sums \((1.7), (1.8)\) are invariant with respect to arbitrary symplectic transformations, compare with \([8, Property III (Symplectic invariance)]. A similar index result is also proved for the cyclic sums of the second kind (see Theorem \([1.2, 3.4]\) in the proof we apply to \([1.16]\) the results by Y. Tian (see \([28, Theorem 2.3]\)) concerning evaluations of inertias of block symmetric matrices. Theorems \([1.2, 3.4]\) can be used as a new tool for computations of cyclic sums \((1.7), (1.8)\) and the Kashiwara indices defined by \((1.12)\) and \((1.15)\).

The Kashiwara and the Hörmander indices \((1.7)\) are traditionally related to the Maslov index theory for Lagrangian paths, see \([24, 8, 7, 15, 29]\). We conjecture that the cyclic sums \((1.7), (1.8)\) and their properties proved in this paper will be a useful complement to the theory of the Maslov index and it’s applications which are the subject of our future investigations.

In the present paper we concentrate on new applications of the cyclic sums \((1.7), (1.8)\) in the oscillation theory of \((1.1)\). For \((1.1)\) we consider arbitrary fundamental symplectic matrices \(Z_k \in Sp(2n), k = 0, 1, \ldots, N + 1\) and show that the cyclic sums

\[
\mu^c_i(Z_0^{-1}(0 I)^T, Z_1^{-1}(0 I)^T, \ldots, Z_{N+1}^{-1}(0 I)^T) = \mu^c_i(Z_{N+1}^{-1}(0 I)^T, Z_N^{-1}(0 I)^T, \ldots, Z_0^{-1}(0 I)^T)
\]

(1.18)

and

\[
\nu^c_i(Z_0^{-1}(0 I)^T, Z_1^{-1}(0 I)^T, \ldots, Z_{N+1}^{-1}(0 I)^T) = \nu^c_i(Z_{N+1}^{-1}(0 I)^T, Z_N^{-1}(0 I)^T, \ldots, Z_0^{-1}(0 I)^T)
\]

(1.19)

are invariant with respect to \(Z_k \in Sp(2n), k = 0, 1, \ldots, N + 1\) and present the maximal and minimal numbers of focal points of conjoined bases of \((1.1)\). These numbers also coincide with the numbers of (forward) focal points of the principal solutions of \((1.1)\) at \(k = N + 1\) and \(k = 0\), respectively (see Theorem \([4.6]\)). Observe that the minimal number of focal points described by the cyclic sums of the second kind \((1.19)\) is highly important in the oscillation and spectral theory of \((1.1)\), see e.g. \([4, 5, 21, 6, 16]\), Chapters 4, 5. In particular, by the Reid Roundabout Theorem, system \((1.1)\) is conjugate on \([0, N + 1]\) if and only if the number of (forward) focal points of the principal solution at \(k = 0\) is equal to zero (see \([4, Theorem 1.1, 5, Theorem 2.36]\)). In \([5, 16]\) this result was interpreted as nonnegative definiteness of \(n(N + 1) \times n(N + 1)\) symmetric matrices associated with \((1.1)\). In this paper, applying the index results for cyclic sums (Theorems \([4.3, 5.3]\)) we present a generalization of the results in \([5, 16]\) connecting the number of (forward) focal points of the principal solution at \(k = 0\) with the index of \(n(N + 1) \times n(N + 1)\) symmetric matrices, see Theorem \([4.8]\).

The paper is organized as follows. In the next section we recall in more details the notion of the comparative index and its properties, formulate and prove main properties of cyclic sums \((1.7), (1.8)\) (see Propositions \([2.5, 2.3, 2.8]\)). In Section 4 we present the proof of Theorem \([1.2]\) formulate and prove Theorem \([3.4]\) connecting the cyclic sums \((1.8)\) with the index of symmetric matrices. In Section 5 we present applications of the cyclic sums, in particular, we prove connections \((1.13), (1.14)\) and similar connections for the cyclic sums of the second kind (see Theorem \([4.1]\)). In Section 6 we also present the above applications to the oscillation theory of \((1.1)\), in particular, we prove Theorems \([4.6, 4.8]\) and their corollaries.

2. Properties of cyclic sums

We will use the following notation. For a matrix \(A\), we denote by \(A^T, A^{-1}, A^1\), rank \(A\), \(\text{Ker} A, \text{Im} A, \text{ind} A, \text{sign} A, A \geq 0, A \leq 0\), respectively, its transpose, inverse, Moore-Penrose pseudoinverse, rank (i.e., the dimension of its image), kernel, image, index (i.e., the number of its negative eigenvalues), signature (i.e., the difference between the numbers of positive and negative eigenvalues of \(A\)), positive semidefiniteness, negative semidefiniteness and we use the notation \(E_A = I - AA^\dagger, F_A = I - A^\dagger A\) for the orthogonal projectors on to the null spaces of \(A^T\) and \(A\), respectively.
Lemma 2.1. For the convenience we collect some properties of the comparative index which we will use in the subsequent proofs. Other special cases for the comparative index are the following (see \[9, Lemma 3.14\]).}

\[\mu(Y, \hat{Y}) = \text{rank}(\hat{M}) = \text{rank}(X\hat{X} - \text{rank}(X) = \text{rank}(X^T w(Y, \hat{Y})) - \text{rank}(X).\]

Applying the formula (see \[23\])

\[\text{rank}(A B) = \text{rank}(A) + \text{rank}(E_{AB}), \ A \in \mathbb{C}^{kj}, \ B \in \mathbb{C}^{lk}\]

one can define the first addends \(\mu_1(Y, \hat{Y})\) in \[1.5, 1.6\] as follows

\[\mu_1(Y, \hat{Y}) = \text{rank}(\hat{M}) = \text{rank}(X\hat{X} - \text{rank}(X) = \text{rank}(X^T w(Y, \hat{Y})) - \text{rank}(X).\]

(see \[28, Theorem 2.3\]) and \[11\], where \((2.4)\) was derived for the proof of the comparative index properties for the special case \(k = l = n\) the comparative index \(\mu(Y, \hat{Y})\) and the dual comparative index \(\mu'(Y, \hat{Y})\) can be shortly defined as follows (see \[3, Lemma 3.14\])

\[\mu(Y, \hat{Y}) = i_\ast\left(0, \hat{X}^T (\hat{Q} - Q)\hat{X}\right), \ \mu'(Y, \hat{Y}) = i_\ast\left(0, \hat{M} \hat{X}^T (\hat{Q} - Q)\hat{X}\right),\]

where \(\hat{M}\) is given by \((2.1)\) (\(\hat{M}\) can be replaced by \(M\) in \[1.4\]). In particular, for the case \(\det X \neq 0, \det \hat{X} \neq 0\) we have

\[\mu(Y, \hat{Y}) = \mu_2(Y, \hat{Y}) = \mu'(Y, \hat{Y}) = \mu'\left(\hat{Y}, Y\right), \ \mu = \text{ind}(\hat{Q} - Q), \ Q = UX^{-1}, \ \hat{Q} = \hat{U}\hat{X}^{-1}.\]

Other special cases for the comparative index are the following

\[\mu(Y, (0 I)^T) = 0, \ \mu((0 I)^T, \hat{Y}) = \mu_1((0 I)^T, \hat{Y}) = \text{rank} \hat{X}, \ \mu((0 I)^T, \hat{Y}) = \mu_2((0 I)^T, \hat{Y}) = \text{ind}(\hat{Q} - Q).\]

For the convenience we collect some properties of the comparative index which we will use in the subsequent proofs (see \[11, p.448\] and \[2, Theorem 3.5 \text{and formula (3.34), p.165}\]).

**Lemma 2.1.** For \(Z, \hat{Z}\) and \(Y = \left(\begin{array}{c} X \\ U \end{array}\right), \ \hat{Y} = \left(\begin{array}{c} \hat{X} \\ \hat{U} \end{array}\right)\) we have the following properties.

\[(i) \ \mu(LY, L\hat{Y}C) = \mu(Y, \hat{Y}), \ l = 1, 2, \ \det C \neq 0, \ det \hat{C} \neq 0, \text{ where } L \text{ is an arbitrary symplectic lower block-triangular matrix.}\]

\[(ii) \ \mu(Y, \hat{Y}) + \mu(\hat{Y}, Y) = \text{rank} w(Y, \hat{Y}), \ \ w(Y, \hat{Y}) = Y^T J\hat{Y},\]

\[(iii) \ \mu(Y, \hat{Y}) = \text{rank} \hat{X} - \text{rank} X + \mu'(\hat{Y}, Y),\]

\[(iv) \ \mu(Y, \hat{Y}) = \mu(\hat{Z}^{-1}(0 I)^T, \hat{Z}^{-1}\hat{Y}), \ l = 1, 2,\]

\[(v) \ \mu(Y, \hat{Y}) + \mu'(Y, \hat{Y}) = \text{rank} w(Y, \hat{Y}) + \text{rank} \hat{X} - \text{rank} X.\]

\[(vi) \ 0 \leq \mu(Y, \hat{Y}) \leq \min(\text{rank} w(Y, \hat{Y}), \text{rank} \hat{X}) \leq n.\]
Observe that according to a duality principle Lemma 2.1 holds also for the dual index \( \mu^*(Y, \hat{Y}) \) (where we use that according to the definition \( \mu^*(Y, \hat{Y})^* = \mu(Y, \hat{Y}) \)), see [1] and [3], Chapter 3.

Introduce \( Z_i \in S p(2n) \) associated with \( Y_i \) in Definition 1.7, according to

\[
Y_i = Z_i(0 I)^T, \quad i = 1, 2, \ldots, m. \tag{2.8}
\]

We have the connection

\[
(I \ 0)Z_i^{-1}Y_j = -w_{i,j}, \quad w_{i,j} := w(Y_i, Y_j), \tag{2.9}
\]

where we use that \( Z_i^{-1} = -JZ_i^T J \) for \( Z_i \in S p(2n) \). Remark that \( Z_i \in S p(2n) \) are not uniquely defined by (2.3). We have \( Y_i = Z_i(0 I)^T \) for arbitrary symplectic unit lower block-triangular matrix \( L = \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \), \( Q = Q^T \). In subsequent computations we will use \( Z_i^{-1} \) taking in mind that results of the computations do not depend on the choice of \( Z_i \) in (2.8) by Lemma 2.1).

**Example 2.2.** Let \( Y_1, Y_2, Y_3 \) be \( 2n \times n \) matrices with conditions (1.3). Consider the definition of the comparative index \( \mu(Z_i^{-1} Y_1, Z_i^{-1} Y_2) \) and the dual comparative index \( \mu^*(Z_i^{-1} Y_1, Z_i^{-1} Y_2) \), where \( Z_i \) obeys (2.8).

Putting \( Y := Z_i^{-1} Y_1, \ Y := Z_i^{-1} Y_2 \) we have by (2.8), (2.1), and (2.3)

\[
\mu_1(Z_3^{-1} Y_1, Z_3^{-1} Y_2) = \text{rank}(E_{w_1}, w_{3,2}) = \text{rank}(w_{3,1} w_{3,2}) - \text{rank } w_{3,1} \\
= \text{rank}(F_{w_1}, w_{1,2}) = \text{rank}(w_{1,3} w_{1,2}) - \text{rank } w_{1,3}, \tag{2.10}
\]

where we computed the Wronskian of \( Z_3^{-1} Y_1, Z_3^{-1} Y_2 \) according to \( w(Z_3^{-1} Y_1, Z_3^{-1} Y_2) = w(Y_1, Y_2) \).

Next we consider the second addend given by (1.4), (1.5), (1.6)

\[
\mu_2(Z_3^{-1} Y_1, Z_3^{-1} Y_2) = \text{ind}(F_M D \ M), \quad M = F_{w_1}, w_{1,2}, \quad M = F_{w_3}, w_{3,2}, \\
\mu_2(Z_3^{-1} Y_1, Z_3^{-1} Y_2) = \text{ind}(F_M D \ M), \quad D = w_{1,2} w_{3,1} w_{3,2}, \tag{2.11}
\]

and define \( \mu(Z_i^{-1} Y_1, Z_i^{-1} Y_2) \) and \( \mu^*(Z_i^{-1} Y_1, Z_i^{-1} Y_2) \) given by (2.10) and (2.11) according to (2.4)

\[
\mu(Z_3^{-1} Y_1, Z_3^{-1} Y_2) = i \begin{pmatrix} 0 & M \\ M^T & D + D^T \end{pmatrix}, \quad \mu^*(Z_3^{-1} Y_1, Z_3^{-1} Y_2) = i \begin{pmatrix} 0 & M \\ M^T & D + D^T \end{pmatrix}, \tag{2.12}
\]

where we use that \( F_M D \ M \) is symmetric and then after the application of (2.4) we will have \( F_M D \ M + F_M D \ M^T F_M = 2F_M D \ M \). Remark that (2.12) can be derived directly from Theorem 1.2 for \( m = 3 \) together with other representations of the given type incorporating properties of the cyclic sums, see Section 3.

Next we formulate properties of (1.7), (1.8) based on Lemma 2.1.

**Proposition 2.3.** The cyclic sums (1.7), (1.8) obey the following properties.

(i) For arbitrary nonsingular matrices \( C_k \in \mathbb{R}^{n \times n} \), \( k = 1, 2, \ldots, m \) we have

\[
\mu^c(Y_1, Y_2, \ldots, Y_m) = \mu^c(Y_1 C_1, Y_2 C_2, \ldots, Y_m C_m), \quad \nu^c(Y_1, Y_2, \ldots, Y_m) = \nu^c(Y_1 C_1, Y_2 C_2, \ldots, Y_m C_m). \tag{2.13}
\]

(ii) According to the definition in (1.7) we have the following invariant property with respect to cyclic permutations

\[
\mu^c(Y_1, Y_2, \ldots, Y_m) = \mu^c(Y_m, Y_1, Y_2, \ldots, Y_{m-1}) = \cdots = \mu^c(Y_2, Y_3, \ldots, Y_m, Y_1). \tag{2.14}
\]

(iii) We have

\[
\mu^c(Y_1, Y_2, \ldots, Y_m) = \mu^c(Y_m, Y_{m-1}, \ldots, Y_1), \quad \nu^c(Y_1, Y_2, \ldots, Y_m) = \nu^c(Y_m, Y_{m-1}, \ldots, Y_1). \tag{2.15}
\]

(iv) We have

\[
\mu^c(Y_1, Y_2, \ldots, Y_m) + \mu^c(Y_1, Y_2, \ldots, Y_m) = \sum_{j=1}^{m-1} \text{rank } w(Y_j, Y_{j+1}) + \text{rank } w(Y_m, Y_1), \tag{2.16}
\]

\[
\nu^c(Y_1, Y_2, \ldots, Y_m) + \nu^c(Y_1, Y_2, \ldots, Y_m) = \sum_{j=1}^{m-1} \text{rank } w(Y_j, Y_{j+1}) - \text{rank } w(Y_m, Y_1)
\]

(2.16)
\textbf{Proof.} The proof of (2.13) follows from Lemma 2.1(i) and the definitions of the cyclic sums (1.7), (1.8).

The proof of (2.14) follows from (1.7). Indeed, the cyclic permutations of \( Y_k \) do not change the order of the pairs \( Y_k, Y_{k+1} \) and \( Y_m, Y_1 \) in (2.14).

For the proof of (2.15) we apply Lemma 2.1(iii)

\[
\mu(Y_k, Y_{k+1}) = \mu'(Y_{k+1}, Y_k) + \text{rank} \ X_{k+1} - \text{rank} \ X_k, \quad k = 1, \ldots, m - 1, \quad \mu(Y_m, Y_1) = \mu'(Y_1, Y_m) + \text{rank} \ X_1 - \text{rank} \ X_m.
\]  

Summing the previous identities we derive

\[
\mu(Y_1, Y_2, \ldots, Y_m) = \sum_{k=1}^{m-1} \mu(Y_k, Y_{k+1}) + \mu(Y_m, Y_1) = \sum_{k=1}^{m-1} \mu'(Y_k, Y_{k+1}) + \mu'(Y_1, Y_m) + s_m = \mu'(Y_m, Y_{m-1}, \ldots, Y_1) + s_m,
\]  

where \( s_m = \sum_{k=1}^{m-1} \Delta(\text{rank} \ X_k) + \text{rank} \ X_1 - \text{rank} \ X_m = 0 \). The proof of equality (2.15) for \( v^c(Y_1, Y_2, \ldots, Y_m) \) based on a dual form of Lemma 2.1(iii) is similar. For \( v^c(Y_1, Y_2, \ldots, Y_m) \) we have according to (1.10)

\[
v^c(Y_1, Y_2, \ldots, Y_m) = \mu^c(Y_1, Y_2, \ldots, Y_m) - \text{rank} \ w(Y_m, Y_1) = \mu^c(Y_m, Y_{m-1}, \ldots, Y_1) - \text{rank} \ w(Y_1, Y_m) = v^c(Y_1, Y_2, \ldots, Y_m),
\]  

where we used the first equality in (2.15) proved above.

The proof of (2.16) is based on Lemma 2.1(v). We have

\[
\mu(Y_k, Y_{k+1}) + \mu'(Y_1, Y_{k+1}) = \text{rank} \ X_{k+1} - \text{rank} \ X_k + \text{rank} \ w(Y_1, Y_{k+1}), \quad k = 1, \ldots, m - 1,
\]

\[
\mu(Y_m, Y_1) + \mu'(Y_m, Y_1) = \text{rank} \ X_1 - \text{rank} \ X_m + \text{rank} \ w(Y_m, Y_1).
\]

Summing the previous identities we derive

\[
\mu^c(Y_1, Y_2, \ldots, Y_m) + \mu^c(Y_1, Y_2, \ldots, Y_m) = \sum_{j=1}^{m-1} \text{rank} \ w(Y_j, Y_{j+1}) + \text{rank} \ w(Y_m, Y_1) + s_m,
\]  

where \( s_m = 0 \) is the same as in (2.17). So we proved the first identity in (2.16). Applying (1.10) we also prove the second one. The proof is completed. \( \square \)

As it was mentioned in Section 1 the main property of the cyclic sums (1.7) and (1.8) is their invariance with respect to arbitrary symplectic transformations. The proof of the invariance is based on the main theorem of the comparative index theory (see [11], Theorem 2.2, formulas (2.14), (2.15), [8], Theorem 3.5, Corollary 3.12, formulas (3.17),(3.26)).

\textbf{Theorem 2.4.} For arbitrary \( W \in \mathbb{S} p(2n) \) and \( 2n \times n \) matrices \( Y, \hat{Y} \) with condition (1.3) we have

\[
\mu(W Y, W \hat{Y}) = \mu(Y, \hat{Y}) + \mu(\hat{Y}, W^{-1}(0 I)^T) - \mu(Y, W^{-1}(0 I)^T),
\]

\[
\mu'(W Y, W \hat{Y}) = \mu'(Y, \hat{Y}) + \mu'(\hat{Y}, W^{-1}(0 I)^T) - \mu'(Y, W^{-1}(0 I)^T).
\]  

(2.18)

Based on Theorem 2.4 we prove the following result.

\textbf{Proposition 2.5} (Symplectic invariance). The cyclic sums (1.7) and (1.8) are invariant with respect to an arbitrary symplectic transformation, i.e., for arbitrary matrix \( R \in S p(2n) \)

\[
\mu^c(Y_1, Y_2, \ldots, Y_m) = \mu^c(R^{-1}Y_1, R^{-1}Y_2, \ldots, R^{-1}Y_m), \quad v^c(Y_1, Y_2, \ldots, Y_m) = v^c(R^{-1}Y_1, R^{-1}Y_2, \ldots, R^{-1}Y_m).
\]  

(2.19)

\textbf{Proof.} According to Theorem 2.4 we have for \( W := R^{-1} \)

\[
\mu(R^{-1}Y_k, R^{-1}Y_{k+1}) = \mu(Y_k, Y_{k+1}) + \mu(Y_{k+1}, R(0 I)^T) - \mu(Y_k, R(0 I)^T), \quad k = 1, \ldots, m - 1,
\]

\[
\mu(R^{-1}Y_m, R^{-1}Y_1) = \mu(Y_m, Y_1) + \mu(Y_1, R(0 I)^T) - \mu(Y_m, R(0 I)^T).
\]  

(2.20)

Summing the first identities for all \( k = 1, \ldots, m - 1 \) and the second one we derive

\[
\mu^c(R^{-1}Y_1, R^{-1}Y_2, \ldots, R^{-1}Y_m) = \sum_{k=1}^{m-1} \mu(R^{-1}Y_k, R^{-1}Y_{k+1}) + \mu(R^{-1}Y_m, R^{-1}Y_1) = \mu^c(Y_1, Y_2, \ldots, Y_m) + s_m.
\]  

(6)
where
\[ s_m = \sum_{j=1}^{m-1} \Delta(\mu(Y_j, R(0 I)^T)) + \mu(Y_1, R(0 I)^T) - \mu(Y_m, R(0 I)^T) = 0. \]

The proof for \( \mu^c(Y_1, Y_2, \ldots, Y_m) \) based on the second identity in (2.18) for the dual indices is similar. The invariance of the sums \( \nu^c(Y_1, Y_2, \ldots, Y_m) \) follows from (1.10). Indeed, we have proved the invariance of \( \mu^c(Y_1, Y_2, \ldots, Y_m) \), then the sums \( \nu^c(Y_1, Y_2, \ldots, Y_m) = \mu^c(Y_1, Y_2, \ldots, Y_m) - w(Y_1, Y_m) \) are invariant because of the obvious invariant property of the Wronskian

\[ w(R^{-1} Y_1, R^{-1} Y_m) = Y_1^T R^{-1} \mathcal{J} R^{-1} Y_m = Y_1^T \mathcal{J} Y_m = w(Y_1, Y_m). \]

The proof is completed.

**Corollary 2.6.** Putting \( R := Z_m \) in Proposition 2.3 we derive the following representations for cyclic sums (1.7), (1.8)

\[ \mu^c(Y_1, Y_2, \ldots, Y_m) = \mu^c(Z_m^{-1} Y_1, Z_m^{-1} Y_2, \ldots, Z_m^{-1} Y_{m-1}, (0 I)^T) = \sum_{j=1}^{n-2} \mu(Z_m^{-1} Y_j, Z_m^{-1} Y_{j+1}) + \text{rank } w(Y_m, Y_1), \]

\[ \nu^c(Y_1, Y_2, \ldots, Y_m) = \mu^c(Z_m^{-1} Y_1, Z_m^{-1} Y_2, \ldots, Z_m^{-1} Y_{m-1}, (0 I)^T) = \sum_{j=1}^{n-2} \mu(Z_m^{-1} Y_j, Z_m^{-1} Y_{j+1}). \]

(2.21)

Similar equalities hold for \( \mu^c(Y_1, Y_2, \ldots, Y_m) \) and \( \nu^c(Y_1, Y_2, \ldots, Y_m) \) with the dual comparative indices \( \mu^c(Z_m^{-1} Y_j, Z_m^{-1} Y_{j+1}) \) in the right-hand sides of (2.21), in particular, it follows from the representations for \( \nu^c(Y_1, Y_2, \ldots, Y_m) \)

\[ \nu^c(Y_1, Y_2, \ldots, Y_m) \geq 0. \]

(2.22)

**Remark 2.7.** (i) We have proved Proposition 2.3 applying Theorem 2.4 Indeed the invariant property (2.19) for \( \nu^c(Y_1, Y_2, Y_3) \) and Theorem 2.4 are equivalent. We have for \( m = 3 \)

\[ \nu^c(Y, \hat{Y}, W^{-1}(0 I)^T) = \mu(Y, \hat{Y}) + \mu(\hat{Y}, W^{-1}(0 I)^T) - \mu(Y, W^{-1}(0 I)^T) = \nu^c(WY, W\hat{Y}, (0 I)^T) = \mu(WY, W\hat{Y}) \]

(2.23)

where we applied (2.19) for \( R := W^{-1} \).

(ii) For the case \( m = 3 \) we derive from Corollary 2.6

\[ \mu^c(Y_1, Y_2, Y_3) = \mu^c(Z_3^{-1} Y_1, Z_3^{-1} Y_2, (0 I)^T) = \text{rank } w(Y_1, Y_3) + \mu(Z_3^{-1} Y_1, Z_3^{-1} Y_2), \]

\[ \mu^c(Y_1, Y_2, Y_3) = \mu^c(Z_3^{-1} Y_1, Z_3^{-1} Y_2, (0 I)^T) = \text{rank } w(Y_1, Y_3) + \mu(Z_3^{-1} Y_1, Z_3^{-1} Y_2), \]

(2.24)

\[ \nu^c(Y_1, Y_2, Y_3) = \nu^c(Z_3^{-1} Y_1, Z_3^{-1} Y_2, (0 I)^T) = \mu(Z_3^{-1} Y_1, Z_3^{-1} Y_2), \]

\[ \nu^c(Y_1, Y_2, Y_3) = \nu^c(Z_3^{-1} Y_1, Z_3^{-1} Y_2, (0 I)^T) = \mu(Z_3^{-1} Y_1, Z_3^{-1} Y_2), \]

(2.25)

Recall that we already computed \( \mu(Z_3^{-1} Y_1, Z_3^{-1} Y_2) \), \( \mu(Z_3^{-1} Y_1, Z_3^{-1} Y_2) \) in Example 2.2.

Next we present recurrent relations for the cyclic sums (1.7), and (1.8).

**Proposition 2.8.** (i) For cyclic sums (1.8) we have for any \( 2 \leq l < m \)

\[ \nu^c(Y_1, Y_2, \ldots, Y_m) = \nu^c(Y_1, Y_2, \ldots, Y_l) + \nu^c(Y_l, Y_{l+1}, \ldots, Y_m) + \nu^c(Y_1, \ldots, Y_m), \]

(2.26)

where we use that \( \nu^c(Y_l, Y_l) = 0 \) and

\[ \nu^c(Y_1, Y_2, \ldots, Y_m) = \sum_{j=1}^{m-1} \nu^c(Y_1, Y_j, Y_{j+1}) = \sum_{j=1}^{m-2} \nu^c(Y_j, Y_{j+1}, Y_m) \]

(2.27)

(ii) For cyclic sums given by (1.7) we have for any \( 2 \leq l < m \)

\[ \mu^c(Y_1, Y_2, \ldots, Y_m) = \mu^c(Y_1, Y_2, \ldots, Y_l) + \mu^c(Y_l, Y_{l+1}, \ldots, Y_m) - \text{rank } w(Y_1, Y_l), \]

(2.28)

where we use that \( \mu^c(Y_l, Y_l) = \text{rank } w(Y_l, Y_l) \) and

\[ \mu^c(Y_1, Y_2, \ldots, Y_m) = \sum_{j=2}^{m-1} \mu^c(Y_1, Y_j, Y_{j+1}) - \sum_{j=3}^{m-2} \text{rank } w(Y_1, Y_j) = \sum_{j=1}^{m-2} \mu^c(Y_j, Y_{j+1}, Y_m) - \sum_{j=2}^{m-2} \text{rank } w(Y_j, Y_m) \]

(2.29)
Proof. The proof of the equalities in (i) follows from the definition of cyclic sums (1.8). Indeed, applying (1.8) to the cyclic sums \( \nu_c(\cdot) \) in the right-hand sides of (2.26) we see that the addends \(-\mu(Y_1, Y)\) and \(\mu(Y_1, Y_2)\) are cancelled in the first identity, and similarly, \(\mu(Y_1, Y_m)\) and \(-\mu(Y_1, Y_m)\) are cancelled in the second one. The proof for \( \mu_c^\pm(\cdot) \) is similar.

By a similar way, we prove the first equality in (2.27) cancelling the terms \(-\mu(Y_{2j+1}, Y_m)\) and \(\mu(Y_{2j+1}, Y_m)\) for \(j = 2, \ldots, m - 1\). For the proof of the second one it is sufficient to cancel the terms \(-\mu(Y_{2j+1}, Y_m)\) and \(\mu(Y_{2j+1}, Y_m)\) for \(j = 1, \ldots, m - 2\). Remark that the second equalities in (2.26) and (2.27) can be also proved via subsequent applications of Proposition\( \ref{prop:2.3}(iii) \) to the first ones.

The proof of (ii) follows from (i) according to connection (1.10) between the cyclic sums \( \nu_c^\pm(\cdot) \) and \( \mu_c^\pm(\cdot) \) in the left and right-hand sides of the equalities in (ii) and (i). The proof is completed.

**Remark 2.9.** (i) The results in Propositions\( \ref{prop:2.3}, \ref{prop:2.3}, \ref{prop:2.3} \) were inspired by properties of the Kashiwara index (1.12) (see [2.6]). Property I we have \( \tau(L_1, L_2, L_3) = \tau(L_3, L_1, L_2) = \tau(L_2, L_3, L_1) \) and by (2.19) we have the similar property for (1.7)

\[
\mu_c^\pm(Y_1, Y_2, Y_3) = \mu_c^\pm(Y_3, Y_1, Y_2) = \mu_c^\pm(Y_2, Y_3, Y_1).
\]

At the same time instead of the property \( \tau(L_1, L_2, L_3) = -\tau(L_2, L_1, L_3) \) (see [2.6 Property I]) for \( m = 3 \) we have

\[
\mu_c^\pm(Y_1, Y_2, Y_3) = \sum_{r < t} \text{rank } w(Y_{r+1}) - \mu_c^\pm(Y_2, Y_1, Y_3), \quad \mu_c^\pm(Y_1, Y_2, Y_3) = \mu_c^\pm(Y_3, Y_2, Y_1) = \mu_c^\pm(Y_1, Y_3, Y_2),
\]

where we used the connection \( \mu_c^\pm(Y_2, Y_1, Y_3) = \mu_c^\pm(Y_3, Y_2, Y_1) = \mu_c^\pm(Y_1, Y_2, Y_3) \) according to Proposition\( \ref{prop:2.3}(ii),(iii) \) and then applied Proposition\( \ref{prop:2.3}(iv) \).

(ii) For the cyclic sum \( \nu_c^\pm(Y_1, Y_2, Y_3) \) one can easily derive from (2.31)

\[
\nu_c^\pm(Y_1, Y_2, Y_3) = \text{rank } w(Y_1, Y_2) - \nu_c^\pm(Y_2, Y_1, Y_3) = \text{rank } w(Y_2, Y_3) - \nu_c^\pm(Y_1, Y_2, Y_3) = \text{rank } w(Y_1, Y_2) + \text{rank } w(Y_2, Y_3) - \text{rank } w(Y_1, Y_3) - 2\nu_c^\pm(Y_2, Y_1, Y_3),
\]

where we use (1.10) incorporating the order of the components in the cyclic sums.

(iii) Observe that the cyclic sums (1.8) do not obey property (2.14), but one can easily derive the connections

\[
\nu_c^\pm(Y_1, Y_2, \ldots, Y_m) = \nu_c^\pm(Y_1, Y_2, \ldots, Y_m) - \text{rank } w(Y_m, Y_1) = \nu_c^\pm(Y_m, Y_1, \ldots, Y_{m-1}) - \text{rank } w(Y_m, Y_1) = \\
\vdots = \nu_c^\pm(Y_2, Y_3, \ldots, Y_m, Y_1) - \text{rank } w(Y_m, Y_1) + \text{rank } w(Y_1, Y_2)
\]

where we used (1.10) and applied Proposition\( \ref{prop:2.3}(iii) \).

(iv) Relations (2.27) for \( m = 4 \) imply the following analogs of the "cocycle condition", see [2.6 Theorem A.3.2(iii)]

\[
\nu_c^\pm(Y_1, Y_2, Y_3, Y_4) = \nu_c^\pm(Y_1, Y_2, Y_3) + \nu_c^\pm(Y_1, Y_3, Y_4) = \nu_c^\pm(Y_1, Y_2, Y_4) + \nu_c^\pm(Y_2, Y_3, Y_4),
\]

where (2.33) can be rewritten in the form

\[
\nu_c^\pm(Y_1, Y_2, Y_3) - \nu_c^\pm(Y_1, Y_2, Y_3) = \nu_c^\pm(Y_1, Y_3, Y_4) - \nu_c^\pm(Y_2, Y_3, Y_4).
\]

3. Index results for cyclic sums

In this section we present the proof of Theorem\( \ref{thm:1.2} \) which connects \( \mu_c^\pm(Y_1, Y_2, \ldots, Y_m) \) with the negative and positive inertia \( i_- \) of the \((mn) \times (mn)\) symmetric matrix given by (1.16). Based on this result we prove a similar connection for \( \nu_c^\pm(Y_1, Y_2, \ldots, Y_m) \).

The proof of Theorem\( \ref{thm:1.2} \) is based on the fact that for Lagrangian subspaces \( L_1, L_2, \ldots, L_m \) there exists a Lagrangian subspace \( L_R \) such that \( L_i \cap L_R = \{0\} \), see e.g. [2.9], [3]. For completeness we prove a similar result for the frames \( Y_1, Y_2, \ldots, Y_m \) presenting a special transformation matrix associated with \( L_R \) (see also [1]), where this result is used for \( m = 1 \) and [14] Lemma 3.5] for the case \( m = 2 \).
Lemma 3.1. Let \( Y_k = \begin{pmatrix} X_k \\ U_k \end{pmatrix} \), \( k = 1, 2, \ldots, m \) be \( 2n \times n \) matrices with condition (3.1). Then there exists \( \alpha \in \mathbb{R}, \alpha \neq \pi k/2, k \in \mathbb{Z} \) such that for the transformation matrix

\[
R_\alpha = \begin{pmatrix} \cos(\alpha) I & \sin(\alpha) I \\ -\sin(\alpha) I & \cos(\alpha) I \end{pmatrix}
\]

we have

\[
det \tilde{X}_k \neq 0, \quad k = 1, 2, \ldots, m, \quad \tilde{X}_k = \cos(\alpha)X_k - \sin(\alpha)U_k
\]

where \( \tilde{X}_k, k = 1, 2, \ldots, m \) are the upper blocks of \( \tilde{Y}_k = R^{-1}_\alpha Y_k \).

Proof. Consider the determinants

\[
P_k(\gamma) = det(X_k - \gamma U_k), \quad k = 1, 2, \ldots, m
\]

for a complex parameter \( \gamma \in \mathbb{C} \). All these determinants have a polynomial dependence in \( \gamma \in \mathbb{C} \). Since these functions are nontrivial (for example, for \( \gamma := i \)) conditions (3.2) are satisfied for all \( \gamma_0 \in \mathbb{R} \) which do not coincide with real roots of the polynomials \( det(X_k - \gamma U_k), k = 1, 2, \ldots, m \). Remark that for \( m < \infty \) we have a finite number of the roots \( det(X_k - \gamma U_k), k = 1, 2, \ldots, m, \) then such \( \gamma_0 \in \mathbb{R} \) does exists. Remark also that one can put \( \alpha := \arctan(\gamma_0) + \pi k, \) for \( R_\alpha \) given by (3.1). In this case condition \( \alpha \neq \pi k, k \in \mathbb{Z} \) is satisfied for \( \gamma_0 \neq 0 \). The proof is completed.

Applying Lemma 3.1 and Proposition 2.5 we derive the following representations for (1.7) and (1.8)

Lemma 3.2. Let \( R_\alpha \) be chosen according to (3.2), then for the cyclic sums defined by (1.7) and (1.8) we have

\[
\mu_\alpha^c(Y_1, Y_2, \ldots, Y_m) = \mu_\alpha^c(R^{-1}_\alpha Y_1, R^{-1}_\alpha Y_2, \ldots, R^{-1}_\alpha Y_m) = \sum_{k=1}^{m-1} \text{ind}(\pm(\tilde{Q}_k - \tilde{Q}_{k+1})) + \text{ind}(\pm(\tilde{Q}_m - \tilde{Q}_1))
\]

and similarly

\[
v_\alpha^c(Y_1, Y_2, \ldots, Y_m) = v_\alpha^c(R^{-1}_\alpha Y_1, R^{-1}_\alpha Y_2, \ldots, R^{-1}_\alpha Y_m) = \sum_{k=1}^{m-1} \text{ind}(\pm(\tilde{Q}_k - \tilde{Q}_{k+1})) - \text{ind}(\pm(\tilde{Q}_m - \tilde{Q}_1)),
\]

where

\[
\tilde{Q}_k = \tilde{Q}_k^1 = \tilde{U}_k \tilde{X}_k^{-1}, \quad \tilde{Y}_k = R^{-1}_\alpha Y_k = \begin{pmatrix} \tilde{X}_k \\ \tilde{U}_k \end{pmatrix}
\]

Proof. The proof follows from Proposition 2.5 (1.7), (1.8), and the definition of the comparative indices

\[
\mu(\tilde{Y}_k, \tilde{Y}_{k+1}) = \text{ind}(\tilde{Q}_{k+1} - \tilde{Q}_k), k = 1, \ldots, m, \quad \mu(\tilde{Y}_m, \tilde{Y}_1) = \text{ind}(\tilde{Q}_1 - \tilde{Q}_m)
\]

and the dual comparative indices \( \mu^c(\tilde{Y}_k, \tilde{Y}_{k+1}) = \text{ind}(\tilde{Q}_k - \tilde{Q}_{k+1}), \) \( \mu^c(\tilde{Y}_m, \tilde{Y}_1) = \text{ind}(\tilde{Q}_m - \tilde{Q}_1) \) under the nonsingularity condition (3.2) according to (2.6).

Now we present the proof of Theorem 1.2.

Proof of Theorem 1.2 Consider the matrix \( S_{1,2,\ldots,m} \) given by (1.16). By Lemmas 3.1, 3.2 we have for the Wronskians \( w(Y_i, Y_j) \) in (1.16)

\[
w(Y_i, Y_j) = w(R_\alpha Y_i, R_\alpha Y_j) = w(\tilde{Y}_i, \tilde{Y}_j) = \tilde{X}_i^T(\tilde{Q}_j - \tilde{Q}_i)\tilde{X}_j, \quad \tilde{Q}_k = \tilde{U}_k \tilde{X}_k^{-1}, k = i, j.
\]

Hence \( S_{1,2,\ldots,m} \) can be rewritten in the form

\[
S_{1,2,\ldots,m} = \text{diag}([\tilde{X}_1^T, \tilde{X}_2^T, \ldots, \tilde{X}_m^T])S_{1,2,\ldots,m} \text{diag}([\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_m]),
\]

where \( S_{1,2,\ldots,m} \) consists of \( n \times n \) blocks \( S_{1,2,\ldots,m}(i, j), i, j = 1, 2, \ldots, m \)

\[
S_{1,2,\ldots,m}(i, j) = \tilde{Q}_j - \tilde{Q}_i, \quad S_{1,2,\ldots,m}(j, i) = S^T_{1,2,\ldots,m}(i, j), \quad j \geq i.
\]

Here the symmetric matrices \( \tilde{Q}_i \) are given by (3.5) and the matrices \( \tilde{X}_j, j = 1, 2, \ldots, m \) are nonsingular according to (3.2) in Lemma 3.1. Then we have

\[
\text{ind}(\pm S_{1,2,\ldots,m}) = \text{ind}(\pm \tilde{S}_{1,2,\ldots,m}).
\]
Introduce the matrices $M_m$, $m \geq 2$ with $n \times n$ blocks such that $M_{i,i} = I$, $M_{i,i+1} = -I$, $M_{m,1} = I$, and $M_{i,j} = 0$ otherwise. In particular, we have

$$M_2 = \begin{pmatrix} I & -I \\ I & I \end{pmatrix}, \quad M_3 = \begin{pmatrix} I & -I & 0 \\ 0 & I & -I \\ I & 0 & I \end{pmatrix}, \quad M_4 = \begin{pmatrix} I & -I & 0 & 0 \\ 0 & I & -I & 0 \\ 0 & 0 & I & -I \\ I & 0 & 0 & I \end{pmatrix}.$$ 

We prove that the matrices $M_m$ are nonsingular using their partitioned form

$$M_m = \begin{pmatrix} L & N \\ I & I \end{pmatrix}, \quad L = \begin{pmatrix} I & -I & 0 & \ldots & 0 \\ 0 & I & -I & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & I \\ 0 & 0 & \ldots & 0 & I \end{pmatrix}, \quad N = (0 \ldots 0 - I)^T, \ K = (I 0 \ldots 0).$$

Then, by [3 Proposition 2.8.3] \(\det M_m = \det L \det(I - KL^{-1}N) = \det(2I) = 2^n\), where we used that $L^{-1}$ has the $n \times n$ blocks $L^{-1}(i, j) = I, j \geq i$ and $L^{-1}(i, j) = 0$ otherwise.

For arbitrary $m \geq 2$ we have for $\tilde{S}_1, \ldots, m$ given by (3.7), (3.8)

$$M_m\tilde{S}_1, \ldots, m M_m^T = \tilde{S}_1, \ldots, m, \quad \tilde{S}_1, \ldots, m = \text{diag}(2(\tilde{Q}_1 - \tilde{Q}_2), 2(\tilde{Q}_2 - \tilde{Q}_3), \ldots, 2(\tilde{Q}_m - \tilde{Q}_1)).$$

(3.9)

Indeed, using the simple structure of $M_m$ we have for the $n \times n$ blocks of $M_m\tilde{S}_1, \ldots, m$

$$M_m\tilde{S}_1, \ldots, m(k, j) = \tilde{Q}_k - \tilde{Q}_{k+1}, \quad M_m\tilde{S}_1, \ldots, m(k, j) = \tilde{Q}_k - \tilde{Q}_{j}, \quad k = 1, \ldots, m - 1, \quad j = 1, \ldots, m,$

and $M_m\tilde{S}_1, \ldots, m(m, m) = \tilde{Q}_m - \tilde{Q}_1, \ j = 1, \ldots, m$. Multiplying $M_m\tilde{S}_1, \ldots, m$ and $M_m^T$ we derive (3.9).

Applying Lemma 3.2 we have by (3.9)

$$\mu^x(Y_1, Y_2, \ldots, Y_m) = \sum_{k=1}^{m-1} \text{ind}(\pm(\tilde{Q}_k - \tilde{Q}_{k+1})) + \text{ind}(\pm(\tilde{Q}_m - \tilde{Q}_1)) = \text{ind}(\pm\tilde{S}_1, \ldots, m) = \text{ind}(\pm\tilde{S}_1, \ldots, m) = \text{ind}(\pm S_1, \ldots, m).$$

The proof is completed.

Introduce the notation for the blocks of $S_{1,2,\ldots,m-1}$ defined by (1.16) for the cyclic permutation $Y_m, Y_1, Y_2, \ldots, Y_{m-1}$

$$S_{1,2,\ldots,m-1} = \begin{pmatrix} 0 & \mathcal{W} \\ \mathcal{W}^T & S_{1,2,\ldots,m-1} \end{pmatrix}, \ m \geq 2, \ \mathcal{W} = (w_{m,1} \ w_{m,2} \ldots \ w_{m,m-1}),$$

$$S_{1,2,\ldots,m-1} = \begin{pmatrix} 0 & \mathcal{N} \\ \mathcal{N}^T & S_{1,2,\ldots,m-1} \end{pmatrix}, \ m \geq 3, \ \mathcal{N} = (w_{m,2} \ w_{m,3} \ldots \ w_{m,m-1}), \ \mathcal{K} = (w_{1,2} \ w_{1,3} \ldots \ w_{1,m-1}),$$

(3.10)

where in the first representation $\mathcal{W} \in \mathbb{R}^{n(n-1)m}$, $m \geq 2$ and for $m = 2$ we put $S_1 := 0_n$. We also have $\mathcal{K}, \mathcal{N} \in \mathbb{R}^{n(m-2)n}$, $m \geq 3$, and for $m = 3$ we put $S_2 := 0_n$.

Applying [28, Theorem 2.3], see (2.4), we derive the following corollary to Theorem 1.2

**Corollary 3.3.** Under notation (3.10) we have the following representations for cyclic sums (1.7)

$$\mu^x(Y_1, Y_2, \ldots, Y_m) = \text{rank} \mathcal{W} + \text{ind}(\pm \mathbf{F} \mathcal{W} \ S_{1,2,\ldots,m-1} \mathbf{F}) = \text{rank} \mathcal{W} + \text{ind}(\pm \mathbf{F} \mathcal{W} \ S_{1,2,\ldots,m-1} \mathbf{F}), \ m \geq 2,$$

(3.11)

$$\mu^x(Y_1, Y_2, \ldots, Y_m) = \text{rank} \mathcal{W} w_{m,1} + \text{ind}(\pm \tilde{S}_{1,2,\ldots,m-1}) = \begin{pmatrix} 0 & \tilde{M} \\ \tilde{M}^T & S_{1,2,\ldots,m-1} - \mathbf{D} - \mathbf{D}^T \end{pmatrix}, \ m \geq 3,$$

(3.12)

where the matrix $\tilde{M}$ can be replaced by

$$\mathcal{M} = \mathbf{F} w_{m,1} \mathcal{K} = (I - w_{m,1}^T w_{m,1}) (w_{1,2} w_{1,3} \ldots \ w_{1,m-1}).$$

(3.13)
Proof. Applying Proposition 2.3 ii) we have by Theorem 1.2
\[
\mu^+(Y_1, Y_2, \ldots, Y_m) = \mu^+(Y_m, Y_1, \ldots, Y_{m-1}) = \text{ind}(\pm S_{m,1,2,\ldots,m-1}), 
\]
(3.14)
where \(S_{m,1,2,\ldots,m-1}\) is given by the first matrix in (3.10). Then (3.11) is derived by the direct application of the second formula in (2.4) to (3.14).

Consider the proof of (3.12). Rewrite \(S_{m,1,2,\ldots,m-1}\) in the second equality (3.10) as follows
\[
S_{m,1,2,\ldots,m-1} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \quad A := \begin{pmatrix} 0 & w_{m,1} \\ w_{m,1}^T & 1 \end{pmatrix}, \quad B := \begin{pmatrix} N \\ K \end{pmatrix}, \quad D := S_{2,3,\ldots,m-1} 
\]
(3.15)
and then apply the first formula in (2.4) incorporating that \(\text{ind}(\pm A) = \text{rank} w_{m,1}\), see (1.9). We also compute by (2.4)
\[
D - B^T A^+ B = S_{2,3,\ldots,m-1} - K^T w_{m,1}^+ N - N^T w_{m,1}^+ K = S_{2,3,\ldots,m-1} - D - \bar{D} T
\]
and
\[
E_A B = \text{diag}(E_{w_{1}}, E_{w_{m,1}}) \begin{pmatrix} N \\ K \end{pmatrix} = \begin{pmatrix} \tilde{M} \\ M \end{pmatrix}.
\]
(3.16)
A key step in the proof is connected with properties of \(\tilde{M}\) and \(M\) defined in (3.12), (3.13). We have that there exists a nonsingular matrix \(L\) such that
\[
M = L \tilde{M}.
\]
(3.17)
This fact follows directly from the properties of the comparative index, see the proof of Theorem 2.1 in [10], the proof of Theorem 3.2 in [9, p. 151]. Indeed, it is sufficient to consider the comparative indices \(\mu(Z_m^1 Y_1, Z_m^{-1} Y_m)\) for \(j = 2, 3, \ldots, m - 1\) and use the definition of the matrices \(M_j\) and \(\tilde{M}_j\) according to (1.6) and (2.1) incorporating (2.9)
\[
M_j = (I - w_{m,1}^+ w_{m,1})w_{1,j}, \quad \tilde{M}_j = (I - w_{m,1}^+ w_{m,1})w_{m,j}.
\]
Applying the result from [9, p. 151] we see that \(M_j = L \tilde{M}_j\), where \(L\) depends only on the blocks of \(Z_m^1 Y_1\), see [9, formula (1.74)]. Since
\[
M = (M_2 M_3 \ldots M_{m-1}), \quad \tilde{M} = (\tilde{M}_2 \tilde{M}_3 \ldots \tilde{M}_{m-1})
\]
formula (3.17) is proved.

By (3.17) the matrix \(\begin{pmatrix} 0 & E_A B \\ (E_A B)^T & D - B^T A^+ B \end{pmatrix}\) with blocks (3.15), (3.16) in the right-hand side of the first equality in (2.4) can be simplified by deleting rows and columns containing \(M\), \(\tilde{M}\) (or \(M\), \(\tilde{M}\)). So we see that the dimension of \(\tilde{S}_{1,2,\ldots,m-1}\) in (3.12) is equal to \((m - 1)n \times (m - 1)n\). Finally, we also showed by (3.17) that \(\tilde{M}\) can be replaced by \(M\). The proof is completed.

Based on Corollary 3.3 we derive the following representations for the cyclic sums \(v^c_i(Y_1, Y_2, \ldots, Y_m)\).

**Theorem 3.4.** Under the notation of Corollary 3.3 we have for (1.8)
\[
v^c_i(Y_1, Y_2, \ldots, Y_m) = \text{rank}(\tilde{M}) + \text{ind}(\pm F_W S_{1,2,\ldots,m-1} F_W), \quad m \geq 2,
\]
\[
v^c_i(Y_1, Y_2, \ldots, Y_m) = \text{ind}(\pm S_{1,2,\ldots,m-1}) = \text{rank}(\tilde{M}) + \text{ind}(\pm F_{\tilde{M}}(S_{2,3,\ldots,m-1} - D - \bar{D} T) F_{\tilde{M}}), \quad m \geq 3,
\]
(3.18)
where \(S_{1,2,\ldots,m-1}\), \(\tilde{M}\) are given by (3.12) and \(\tilde{M}\) can be replaced by \(M\) defined by (3.13).

**Proof.** Representation (3.11) implies the first connection in (3.18), where we use (1.10) and apply (2.2)
\[
\text{rank} W = \text{rank} w_{m,1} + \text{rank}(\tilde{M}).
\]
(3.19)
By a similar way, the equality \(v^c_i(Y_1, Y_2, \ldots, Y_m) = \text{ind}(\pm S_{1,2,\ldots,m-1})\) in (3.18) follows from (3.12), where we use connection (1.10) between \(v^c_i(Y_1, Y_2, \ldots, Y_m)\) and \(\mu^+(Y_1, Y_2, \ldots, Y_m)\). Next we compute the index \(\text{ind}(\pm S_{1,2,\ldots,m-1})\) applying the second identity in (2.4). Observe that the matrix \(F_{\tilde{M}}\) stays the same after the replacement \(\tilde{M}\) by \(M\) because of connection (3.17), see [3, Theorem 8, Lemma 3]. The proof is completed.
Remark 3.5. (i) It follows from (3.18) that for $m \geq 3$

$$\text{ind}(\pm F_{W} S_{1,2,\ldots,m-1} F_{W}) = \text{ind}(\pm F_{M}(S_{2,3,\ldots,m-1} - D - D^{T})F_{M}),$$

(3.20)
in particular, for $m = 3$ formula (3.20) presents the second addend in the definition of the comparative index $\mu(Z_{m}Y_{1}, Z_{m}Y_{2})$, compare with (2.11).

(ii) Summing $\mu_{c}(-)$ and $\mu_{c}(\cdot)$ given by (3.3) and incorporating (3.20) we have for $m \geq 3$

$$\text{rank}(\pm F_{W} S_{1,2,\ldots,m-1} F_{W}) = \text{rank}(\pm F_{M}(S_{2,3,\ldots,m-1} - D - D^{T})F_{M})$$

$$= \sum_{j=1}^{m-1} \text{rank} w(Y_{j}, Y_{j+1}) + \text{rank} w(Y_{m}, Y_{1}) - 2 \text{rank} W, \quad W = (w_{m,1} \ w_{m,2} \ldots \ w_{m,m-1}),$$

(3.21)

where we also used Proposition 2.3(iv). Observe that rank $W$ can be presented in terms of the dimension of $L_{1} + L_{2} + \cdots + L_{m}$ (or $L_{1} \cap L_{2} \cap \cdots \cap L_{m}$) by

$$\text{dim}(L_{1} + L_{2} + \cdots + L_{m}) = \text{rank}(Y_{1} Y_{2} \ldots Y_{m}) = \text{rank}Z_{m}(Z_{m}^{-1}Y_{1} Z_{m}^{-1}Y_{2} \ldots Z_{m}^{-1}Y_{m-1} \ (0 \ I)^{T})$$

$$= \text{rank}(Z_{m}^{-1}Y_{1} Z_{m}^{-1}Y_{2} \ldots Z_{m}^{-1}Y_{m-1} \ (0 \ I)^{T}) = n + \text{rank} W,$$

(3.22)

where we used (2.8) and (2.9). Finally, substituting rank $W = \text{dim}(L_{1} + L_{2} + \cdots + L_{m}) - n$ from (3.22) and rank $w_{i,j} = n - \text{dim}(L_{i} \cap L_{j})$ into (3.21) we derive by (dim($L_{1} \cap L_{2} \cap \cdots \cap L_{m}$) = $2n - \text{dim}(L_{1} + L_{2} + \cdots + L_{m})$

$$\text{rank}(\pm F_{W} S_{1,2,\ldots,m-1} F_{W}) = \text{rank}(\pm F_{M}(S_{2,3,\ldots,m-1} - D - D^{T})F_{M})$$

$$= 2 \text{dim}(L_{1} \cap L_{2} \cap \cdots \cap L_{m}) + (m - 2)n - \sum_{j=1}^{m-1} \text{dim}(L_{j} \cap L_{j+1}) - \text{dim}(L_{n} \cap L_{1}) \geq 0,$$

(3.23)

compare with [29, Corollary 3.7] for the case $m = 3$.

(iii) One can verify that diagonal blocks of $D$ defined in (3.12) are symmetric on the image of $F_{M}$. We incorporated this fact in the right-hand side of formula (3.20) for $m = 3$, see Example 2.2.

From Theorem 1.2 and Lemma 2.1(vi) we also derive the following estimates for cyclic sums (1.7), (1.8).

**Corollary 3.6.** For the cyclic sums (1.7), (1.8) we have the estimates

$$\begin{align*}
0 & \leq r \leq \mu_{c}^{+}(Y_{1}, Y_{2}, \ldots, Y_{m}) \leq P, \quad 0 \leq r - \text{rank} w_{1,m} \leq \mu_{c}^{+}(Y_{1}, Y_{2}, \ldots, Y_{m}) \leq P - \text{rank} w_{1,m}, \\
0 & \leq r \leq \text{max}_{i,j, l,k = 1, \ldots, m} (\text{rank} w(Y_{i}, Y_{j})), \\
0 & \leq P = \sum_{j=1}^{m-1} \text{min}(\text{rank} w(Y_{j}, Y_{j+1}), \text{rank} w(R(0 \ I)^{T}, Y_{j+1})) + \text{min}(\text{rank} w(Y_{m}, Y_{1}), \text{rank} w(R(0 \ I)^{T}, Y_{1})),
\end{align*}$$

(3.24)

where $R$ is arbitrary symplectic matrix.

**Proof.** By Proposition 2.5 we have $\mu_{c}^{+}(Y_{1}, Y_{2}, \ldots, Y_{m}) = \mu_{c}^{+}(R^{-1}Y_{1}, R^{-1}Y_{2}, \ldots, R^{-1}Y_{m})$, then the upper bounds in (3.24) follow from Lemma 2.1(vi) applied to the sums of the comparative indices $\mu(R^{-1}Y_{i}, R^{-1}Y_{j})$, $j = 1, \ldots, m - 1$, $\mu(R^{-1}Y_{m}, R^{-1}Y_{1})$ according to Definition 1.1 and (1.10).

For the proof of the lower bounds we apply Theorem 1.2. By (1.16) one can chose the $2n \times 2n$ principal submatrix of $S_{1,2,\ldots,n}$ depending on the Wronskians $w(Y_{i}, Y_{j})$, $i < j$ in the form $S_{i,j} = \begin{pmatrix} 0 & w_{i,j} \\ w^{T}_{i,j} & 0 \end{pmatrix}$ with $\text{ind} S_{i,j} = \text{ind}(-S_{i,j}) = \text{rank} w(Y_{i}, Y_{j})$, (see [3, Fact 5.8.8]), then the lower bound in the first inequality (3.24) follows from the inequalities

$$\text{rank} w(Y_{i}, Y_{j}) = \text{ind}(-S_{i,j}) \leq \text{ind}(-S_{1,2,\ldots,m}) = \mu_{c}^{+}(Y_{1}, Y_{2}, \ldots, Y_{m}),$$

see [3, Fact 5.8.20]. The lower bound for (1.8) follows from (1.10).

4. Applications

4.1. Connections with the Kashiwara index

Recall that according to [29] (see Section 1) the Kashiwara index $\tau(L_{1}, L_{2}, L_{3})$ is defined as the signature $\tau(L_{1}, L_{2}, L_{3}) = \text{sign}(B(x, x)) = i_{n}(B(x, x)) - i_{n}(B(x, x))$ of the quadratic form $B(x, x)$ given by (1.1)

$$B(x, x) := B((x_{1}, x_{2}, x_{3}), (x_{1}, x_{2}, x_{3})) = w(x_{1}, x_{2}) + w(x_{2}, x_{3}) + w(x_{3}, x_{1})$$

defined on $(x_{1}, x_{2}, x_{3}) \in L_{1} \oplus L_{2} \oplus L_{3}$. This definition is generalized to the case $m \geq 3$ according to (1.15).

The main result of this section is the following theorem.
Theorem 4.1. We have the following connections for \( m = 3 \)
\[
\tau(L_1, L_2, L_3) = \mu_c(Y_1, Y_2, Y_3) - \mu_c(Y_1, Y_2, Y_3) - v_c(Y_1, Y_2, Y_3).
\] (4.1)

For the case \( m \geq 3 \) we have for (1.7), (1.8), and (1.15)
\[
\tau(L_1, L_2, \ldots, L_m) = \mu_c(Y_1, Y_2, \ldots, Y_m) - \mu_c(Y_1, Y_2, \ldots, Y_m) - v_c(Y_1, Y_2, \ldots, Y_m),
\] (4.2)

\[
\mu_c(Y_1, Y_2, \ldots, Y_m) = \frac{1}{2} \sum_{j=1}^{m-1} \text{rank } w(Y_j, Y_{j+1}) + \text{rank } w(Y_m, Y_1) + \tau(L_1, L_2, \ldots, L_m),
\] (4.3)

\[
v_c(Y_1, Y_2, \ldots, Y_m) = \frac{1}{2} \sum_{j=1}^{m-1} \text{rank } w(Y_j, Y_{j+1}) - \text{rank } w(Y_m, Y_1) + \tau(L_1, L_2, \ldots, L_m).
\]

Proof. Introduce the matrix of the quadratic form \( B(x, x) \)
\[
S_B = \frac{1}{2} \begin{bmatrix}
0 & w(Y_1, Y_2) & -w(Y_1, Y_3) \\
-w(Y_2, Y_1) & 0 & w(Y_2, Y_3) \\
w(Y_1, Y_2) & -w(Y_2, Y_3) & 0
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
0 & -w(Y_1, Y_2) & -w(Y_1, Y_3) \\
-w(Y_2, Y_1) & 0 & -w(Y_2, Y_3) \\
-w(Y_2, Y_1) & -w(Y_2, Y_3) & 0
\end{bmatrix} K
\]
\[
= \frac{1}{2} KS_{123} K, \quad K = \text{diag}[I, -I, I],
\]

where the matrix \( S_{123} \) is given by (1.16) for \( m = 3 \). Then we have
\[
\tau(L_1, L_2, L_3) = \text{sign}(B(x, x)) = i_+ B(x, x) = \text{ind}(S_B) = \text{ind}(S_{123}) = \text{ind}(S_{123}) - \text{ind}(S_{123})
\]

and by Theorem 1.2 for \( m = 3 \) we derive the first equality in (4.1). The second one follows from (1.10) for \( m = 3 \).

For the proof of (4.2) we use Proposition 2.8. By (1.10), (2.27) we have
\[
\mu_c(Y_1, Y_2, \ldots, Y_m) - \mu_c(Y_1, Y_2, \ldots, Y_m) - v_c(Y_1, Y_2, \ldots, Y_m)
\]
\[
= \sum_{j=2}^{m-1} \tau(L_1, L_j, L_{j+1}) = \sum_{j=2}^{m-1} \tau(L_1, L_j, L_{j+1}),
\]

where we also used (4.1) for the case \( v_c(Y_1, Y_j, Y_{j+1}) - v_c(Y_1, Y_j, Y_{j+1}) = \tau(L_1, L_j, L_{j+1}) \). The proof of (4.2) is completed.

Finally, by Proposition 2.8 iv) (see (2.16)) we have
\[
\mu_c(Y_1, Y_2, \ldots, Y_m) + \mu_c(Y_1, Y_2, \ldots, Y_m) = \sum_{j=1}^{m-1} \text{rank } w(Y_j, Y_{j+1}) + \text{rank } w(Y_m, Y_1),
\]

then by summing (subtracting) the last equality and \( \mu_c(Y_1, Y_2, \ldots, Y_m) - \mu_c(Y_1, Y_2, \ldots, Y_m) = \tau(L_1, L_2, \ldots, L_m) \) we derive the first equality in (4.3). The second one then follows from (1.10). The proof is completed.

Based on connections (4.3) some properties of (1.7), (1.8) can be derived from similar properties of the Kashiwara index. So we have the following corollary to Theorem 4.1

Corollary 4.2. Suppose that \( Y_1(t), Y_2(t), \ldots, Y_m(t) \) with conditions (1.8) are continuous functions of \( t \in [a, b] \). Then under the assumption
\[
\text{rank } w(Y_j(t), Y_j(t)) = \text{const}, \quad j = 1, 2, \ldots, m - 1, \quad \text{rank } w(Y_1(t), Y_m(t)) = \text{const}, \quad t \in [a, b]
\]

we have
\[
\mu_c(Y_1(t), Y_2(t), \ldots, Y_m(t)) = \text{const}, \quad v_c(Y_1(t), Y_2(t), \ldots, Y_m(t)) = \text{const}.
\]

Proof. Under assumption (4.3) the Kashiwara index \( \tau(L_1(t), L_2(t), \ldots, L_m(t)) \) (where the Lagrangian subspaces \( L_j(t) \) have the frames \( Y_j(t) \) for \( j = 1, 2, \ldots, m \)) remains constant for all \( t \in [a, b] \) by (2.8) Proposition A.3.8], then by (4.3) and (4.5) the cyclic sums \( \mu_c(Y_1(t), Y_2(t), \ldots, Y_m(t)) \) and \( v_c(Y_1(t), Y_2(t), \ldots, Y_m(t)) \) remain constant as well.

Theorem 4.1 coupled with Definition 1.1 Proposition 2.8 and the index results in Section 3 imply the following representations of the Kashiwara index.
Corollary 4.3. For the Kashiwara index \( \text{Corollary 4.15} \) we have the following representations

\[
\tau(Y_1, Y_2, \ldots, Y_m) = \sum_{j=1}^{m-1} \text{sign}(\mathcal{P}(\tilde{Y}_j, \tilde{Y}_{j+1})) + \text{sign}(\mathcal{P}(\tilde{Y}_m, \tilde{Y}_1), \quad \tilde{Y}_j = R^{-1}Y_j = \left( \frac{X_j}{U} \right))
\]

(4.7)

where \( R \in S p(2n) \) is arbitrary and the symmetric matrices \( \mathcal{P}(\tilde{Y}_j, \tilde{Y}_j) \) are defined in \( \text{Corollary 4.1} \) for the comparative indices \( \mu(\tilde{Y}_j, \tilde{Y}_j), \) i.e.,

\[
\mathcal{P}(\tilde{Y}_j, \tilde{Y}_j) = F_M W_{\tilde{X}_{\tilde{Y}_j}} F_M, \quad M = F_X W_{\tilde{X}_{\tilde{Y}_j}}, \quad w_{\tilde{X}_{\tilde{Y}_j}} := w(Y_j, Y_j),
\]

(4.8)

in particular, for \( R := Z_m, \) where \( Y_m = Z_m(0 I)^T \) we have by Corollary \( \text{2.6} \)

\[
\tau(Y_1, Y_2, \ldots, Y_m) = \sum_{j=1}^{m-1} \text{sign}(\mathcal{P}(Z_m^{-1}Y_j, Z_m^{-1}Y_{j+1})).
\]

(4.9)

Proof. By Theorem \( \text{4.1} \) and Proposition \( \text{2.5} \)

\[
\tau(Y_1, Y_2, \ldots, Y_m) = \mu_c^e(Y_1, Y_2, \ldots, Y_m) - \mu_c(\tilde{Y}_1, \tilde{Y}_2, \ldots, Y_m)
\]

\[
= \mu_c^e(R^{-1}Y_1, R^{-1}Y_2, \ldots, R^{-1}Y_m) - \mu_c(\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_m),
\]

then one can derive \( \text{(4.7)} \) substituting definition \( \text{Corollary 4.1} \) of \( \mu_c^e(R^{-1}Y_1, R^{-1}Y_2, \ldots, R^{-1}Y_m) \) into the right-hand side of the last equality and using \( \text{Corollary 4.1} \) according to

\[
\mu_c^e(\tilde{Y}_k, \tilde{Y}_{k+1}) = \text{ind}(-\mathcal{P}(\tilde{Y}_k, \tilde{Y}_{k+1})) = \text{ind}(\mathcal{P}(\tilde{Y}_k, \tilde{Y}_{k+1})) = \text{sign}(\mathcal{P}(\tilde{Y}_k, \tilde{Y}_{k+1}))
\]

\[
\mu_c(\tilde{Y}_m, \tilde{Y}_1) = \text{sign}(\mathcal{P}(\tilde{Y}_m, \tilde{Y}_1)), \quad k = 1, \ldots, m - 1,
\]

where \( \mathcal{P}(\tilde{Y}_j, \tilde{Y}_j) \) are calculated according to \( \text{Corollary 4.1} \). The proof is completed.

Corollary 4.4. Under the notation of Corollary \( \text{3.3} \) and Theorem \( \text{3.4} \) we have the following representations for the Kashiwara index \( \text{Corollary 4.15} \)

\[
\tau(Y_1, Y_2, \ldots, Y_m) = -\text{sign}(S_{1,2, \ldots, m}) = -\text{sign}(F_W S_{1,2, \ldots, m-1} F_W) = -\text{sign}(F_M(S_{2,3, \ldots, m-1} - D - D^T) F_M).
\]

(4.10)

Proof. The proof follows from Theorem \( \text{4.1} \) coupled with Theorems \( \text{1.2} \) \( \text{3.4} \) in particular, we apply Theorem \( \text{3.4} \) cancelling the same addends rank \( M \) in the representations of \( \nu_c^e(Y_1, Y_2, \ldots, Y_m) \) in \( \text{4.15} \).

Corollary 4.4 applied to the case \( m = 3 \) leads to the representations of the Kashiwara index

\[
\tau(L_1, L_2, L_3) = \text{sign}(\mathcal{P}(\tilde{Y}_1, \tilde{Y}_2)) + \text{sign}(\mathcal{P}(\tilde{Y}_2, \tilde{Y}_3)) + \text{sign}(\mathcal{P}(\tilde{Y}_3, \tilde{Y}_1))
\]

\[
= \text{sign}(\mathcal{P}(Z_1^{-1}Y_1, Z_1^{-1}Y_2)) + \text{sign}(\mathcal{P}(Z_1^{-1}Y_2, Z_1^{-1}Y_3)) = \text{sign}(\mathcal{P}(Z_1^{-1}Y_1, Z_1^{-1}Y_3)),
\]

(4.11)

where in the last row of \( \text{4.11} \) we applied Corollary \( \text{4.1} \) to the cases \( R := Z_3, R := Z_1, \) and \( R := Z_2. \)

Observe also, that the last equality in Corollary \( \text{4.1} \) for \( m = 3 \) can be simplified according to Example \( \text{2.2} \) (see \( \text{2.11} \), \( \text{2.12} \))

\[
\tau(L_1, L_2, L_3) = \text{sign}(F_M D F_M) = \text{sign}(\mathcal{P}(Z_1^{-1}Y_1, Z_1^{-1}Y_2)).
\]

(4.12)

4.2. Cyclic sums in oscillation theory of \( \text{Corollary 4.1} \)

In this section we consider discrete symplectic system \( \text{Corollary 4.1} \), where \( S_k \in S p(2n) \) is separated into \( n \times n \) blocks according to

\[
S_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix},
\]

(4.13)

Introduce \( 2n \times n \) matrix solutions \( Y_k, k = 0, 1, \ldots, N + 1 \) with conditions \( \text{1.3} \) (conjoined bases of \( \text{1.1} \)) associated with symplectic fundamental matrices \( Z_k, k = 0, 1, \ldots, N + 1 \) such that the condition \( Y_k = Z_k(0 I)^T \) holds. We define the principal solution \( Y_k^{(M)} \) of \( \text{1.1} \) at \( k = M \) by the initial condition

\[
Y_k^{(M)} = (0 I)^T, \quad M = 0, 1, \ldots, N + 1.
\]

(4.14)

According to the definition, see \( \text{Corollary 4.1} \) a conjoined basis \( Y_k = \begin{bmatrix} X_k \end{bmatrix}_{k=M} \) has a forward focal point of the multiplicity \( m_1(k) \) in the point \( k + 1 \) where \( m_1(k) = \text{rank} M_k, M_k = \begin{bmatrix} I & -X_{k+1} \end{bmatrix}_{k+1} \) \( B_k \), and this basis has a forward
focal point of the multiplicity $m_2(k)$ in the interval $(k, k + 1)$ if $m_2(k) = \text{ind}(T^T_k X_k X^T_{k+1} \mathcal{B}_k T_k)$, $T_k = I - M^T_k M_k$. The number of forward focal points in $(k, k + 1)$ is defined by $m(k) = m_1(k) + m_2(k)$. This definition can be briefly rewritten in terms of the comparative index as follows (see [1] Lemma 3.1)

$$m(k) = \mu(\mathcal{Y}_{k+1}, S_k(0 I)^T) = \mu(\mathcal{Z}^{-1}_{k+1}(0 I)^T, \mathcal{Z}^{-1}_k(0 I)^T).$$

(4.15)

The multiplicities $m^*(k)$ of backward focal points associated with conjoined bases of the so-called time-reversed symplectic system (see [2], Section 2.1.2)

$$y_k = S_k^{-1} y_{k+1}, \ k = N, N - 1, \ldots, 0$$

are defined as follows, see [11], Lemma 3.2

$$m^*(k) = \mu^*(\mathcal{Y}_k, S_k^{-1}(0 I)^T) = \mu(\mathcal{Z}^{-1}_k(0 I)^T, \mathcal{Z}^{-1}_{k+1}(0 I)^T).$$

(4.17)

Consider the total numbers of forward and backward focal points of conjoined bases of symplectic system [11]

$$l(\mathcal{Y}, 0, N + 1) = \sum_{k=0}^{N} \mu(\mathcal{Y}_{k+1}, S_k(0 I)^T) = \sum_{k=0}^{N} \mu^*(\mathcal{Z}^{-1}_{k+1}(0 I)^T, \mathcal{Z}^{-1}_k(0 I)^T)$$

(4.18)

and

$$l^*(\mathcal{Y}, 0, N + 1) = \sum_{k=0}^{N} \mu^*(\mathcal{Y}_k, S_k^{-1}(0 I)^T) = \sum_{k=0}^{N} \mu(\mathcal{Z}^{-1}_k(0 I)^T, \mathcal{Z}^{-1}_{k+1}(0 I)^T).$$

(4.19)

For any conjoined basis $\mathcal{Y}_k$ of [11] (see [3], formulas (4.62), (4.66)) the following inequalities hold

$$l(\mathcal{Y}^{(0)}, 0, N + 1) \leq l(\mathcal{Y}, 0, N + 1) \leq l(\mathcal{Y}^{(N+1)}, 0, N + 1),$$

(4.20)

$$l^*(\mathcal{Y}^{(0)}, 0, N + 1) \leq l^*(\mathcal{Y}, 0, N + 1) \leq l^*(\mathcal{Y}^{(N+1)}, 0, N + 1),$$

moreover, by [11] Lemma 3.3], [3] Theorems 4.34, 4.35]

$$l(\mathcal{Y}^{(0)}, 0, N + 1) = l^*(\mathcal{Y}^{(N+1)}, 0, N + 1), \quad l^*(\mathcal{Y}^{(0)}, 0, N + 1) = l(\mathcal{Y}^{(N+1)}, 0, N + 1).$$

(4.21)

Consider cyclic sums [17], (18) for the special case

$$m := N + 2, \ Y_k := \mathcal{Z}^{-1}_k(0 I)^T, \ k = 1, \ldots, N + 2,$$

(4.22)

where $\mathcal{Z}_k \in S p(2n)$ is a fundamental matrix of [11]. The main result of this section is the following theorem.

**Theorem 4.6.** The cyclic sums [18] are invariant with respect to a choice of a fundamental matrix $\mathcal{Z}_k \in S p(2n)$ of [11] and

$$\mu^*(\mathcal{Z}_0^{-1}(0 I)^T, \mathcal{Z}_1^{-1}(0 I)^T, \ldots, \mathcal{Z}_N^{-1}(0 I)^T) = l^*(\mathcal{Y}^{(0)}, 0, N + 1)$$

(4.23)

and

$$\mu^*(\mathcal{Z}_{N+1}^{-1}(0 I)^T, \mathcal{Z}_N^{-1}(0 I)^T, \ldots, \mathcal{Z}_1^{-1}(0 I)^T) = l(\mathcal{Y}^{(N+1)}, 0, N + 1).$$

(4.24)

By a similar way, for any choice of a symplectic fundamental matrix $\mathcal{Z}_k$ of [11]

$$\nu^*(\mathcal{Z}_0^{-1}(0 I)^T, \mathcal{Z}_1^{-1}(0 I)^T, \ldots, \mathcal{Z}_N^{-1}(0 I)^T) = l^*(\mathcal{Y}^{(N+1)}, 0, N + 1)$$

(4.25)

where $l(\mathcal{Y}^{(M)}, 0, N + 1)$ and $l^*(\mathcal{Y}^{(M)}, 0, N + 1)$ are the total numbers of forward and backward focal points of the principal solution at $M$.

*Proof.* Consider (4.19) for the case of the principal solution $\mathcal{Y}^{(0)}$ at zero, i.e., for $\mathcal{Y}^{(0)}_k = (0 I)^T$, and introduce the symplectic fundamental matrix $\mathcal{Z}^{(0)}_k$ such that $\mathcal{Y}^{(0)}_k = \mathcal{Z}^{(0)}_k(0 I)^T$. We see by the second equality in (4.19) and (2.7)

$$l^*(\mathcal{Y}^{(0)}, 0, N + 1) = \mu^*(0 I)^T, \mathcal{Z}_1^{-1}(0 I)^T, \mathcal{Z}_2^{-1}(0 I)^T, \ldots, \mathcal{Z}_N^{-1}(0 I)^T. $$

(4.26)

Next, putting $R := \mathcal{Z}^{(0)}_0$ in Proposition 3.8 we have

$$\mu^*(0 I)^T, \mathcal{Z}^{(0)}_0^{-1}(0 I)^T, \mathcal{Z}_1^{-1}(0 I)^T, \ldots, \mathcal{Z}_N^{-1}(0 I)^T$$

$$= \mu^*(0 I)^T, \mathcal{Z}_0^{-1}(0 I)^T, \mathcal{Z}_1^{-1}(0 I)^T, \ldots, \mathcal{Z}_N^{-1}(0 I)^T)$$

(4.27)

$$= \mu^*(0 I)^T, \mathcal{Z}_0^{-1}(0 I)^T, \mathcal{Z}_1^{-1}(0 I)^T, \ldots, \mathcal{Z}_N^{-1}(0 I)^T. $$

(4.28)

$$= \mu^*(0 I)^T, \mathcal{Z}^{(0)}_0^{-1}(0 I)^T, \mathcal{Z}_1^{-1}(0 I)^T, \ldots, \mathcal{Z}_N^{-1}(0 I)^T. $$

(4.29)
By (4.25) and (4.26) we have proved the first equality in (4.23). Next we use Proposition 2.3(iii)

$$
\mu_+^*(Z_0^{-1}(0) I^T, Z_1^{-1}(0) I^T, \ldots, Z_N^{-1}(0) I^T) = \mu_+^*(Z_{N+1}^{-1}(0) I^T, Z_N^{-1}(0) I^T, \ldots, Z_0^{-1}(0) I^T)
$$

(4.27)

and consider (4.18) for the principal solution $Y_{k+1}^{[N+1]}$ at $N + 1$, i.e., for $Y_{k+1}^{[N+1]} = (0) I^T$ introducing the symplectic fundamental matrix $Z_{k+1}^{[N+1]}$ such that $Y_{k+1}^{[N+1]} = Z_{k+1}^{[N+1]}(0) I^T$. We derive from the second equality in (4.18) and (2.7)

$$
l(Y_{k+1}^{[N+1]}, 0, N + 1) = \mu_+^*(Z_{N}^{-1}(0) I^T, Z_{N+1}^{-1}(0) I^T, \ldots, Z_0^{-1}(0) I^T)
$$

(4.28)

while by invariant property (2.19)

$$
\mu_+^*(Z_{N+1}^{-1}(0) I^T, Z_N^{-1}(0) I^T, \ldots, Z_0^{-1}(0) I^T)
$$

(4.29)

By (4.27), (4.28), and (4.29) we complete the proof of (4.23).

For the proof of (4.24) we see by (4.19) and (2.7)

$$
l'(Y_{k+1}^{[N+1]}, 0, N + 1) = v_c^*(Z_{N}^{-1}(0) I^T, Z_{N+1}^{-1}(0) I^T, \ldots, Z_0^{-1}(0) I^T),
$$

(4.30)

and by putting $R := Z_{N+1}$ in Proposition 2.3.

$$
v_c^*(Z_0^{-1}(0) I^T, Z_1^{-1}(0) I^T, \ldots, Z_{N+1}^{-1}(0) I^T)
$$

(4.31)

By (4.30) and (4.31) we prove the first equality in (4.24) and then again use Proposition 2.3(ii)

$$
v_c^*(Z_0^{-1}(0) I^T, Z_1^{-1}(0) I^T, \ldots, Z_{N+1}^{-1}(0) I^T) = v_c^*(Z_{N+1}^{-1}(0) I^T, Z_N^{-1}(0) I^T, \ldots, Z_0^{-1}(0) I^T).
$$

(4.32)

Finally we apply the invariant property to the right-hand side of (4.32) for $R := Z_0$ and prove

$$
v_c^*(Z_{N+1}^{-1}(0) I^T, Z_N^{-1}(0) I^T, \ldots, Z_0^{-1}(0) I^T) = v_c^*(Z_0 Z_{N+1}^{-1}(0) I^T, Z_N Z_N^{-1}(0) I^T, \ldots, Z_0 Z_0^{-1}(0) I^T),
$$

(4.33)

according to (4.18) for $Y_k := Y_k^{[0]}$. By (4.32), (4.33) the proof of (4.24) is completed.\[\square\]

**Remark 4.7.** In the proof of Theorem 4.6 we used only definitions (4.18), (4.19) of the numbers of forward and backward focal points as well as Propositions 2.3(ii) presenting the new proof of identities (4.21). Moreover, applying (4.17), (4.18) to the cyclic sums in Theorem 4.6 one can also prove inequalities (4.40) based on separation results in [11], Corollary 3.1, formulas (3.9), (3.10), [8], Section 4.2.3. For example, by (4.24), (4.18), and (4.18)

$$
l(Y_0, 0, N + 1) = v_c^*(Z_1^{-1}(0) I^T, Z_N^{-1}(0) I^T, \ldots, Z_0^{-1}(0) I^T) = l(Y_0, 0, N + 1) - \mu_+^*(Z_{N+1}^{-1}(0) I^T, Z_0^{-1}(0) I^T) \geq 0,
$$

where by Lemma 2.7(iii) $\mu_+^*(Z_{N+1}^{-1}(0) I^T, Z_0^{-1}(0) I^T) = \mu_+^*(Y_{N+1}, Z_{N+1}^{-1}(0) I^T) = \mu_+^*(Y_{M}, Y_{1}^{[N+1]}),$ compare with [11], Corollary 3.1. Observe that we also proved the lower bound in (4.20). By a similar way one can derive other equalities in [11], [8], Section 4.2.3.

Applying Theorems 1.2, 1.4 we derive the following representation for the numbers of focal points in terms of the indices of symmetric matrices.

**Theorem 4.8.** Let $Y_k^{[M]}$ be the principal solutions of (1.1) at $k = M$ for $M = 0, 1, \ldots, N + 1$ with the upper blocks $X_k^{[M]}$. Then we have the following representations for the number of focal points of (1.1) given by (4.23), (4.24)

$$
l'(Y_0, 0, N + 1) = l(Y_{N+1}^{[N+1]}, 0, N + 1) = \text{ind}(-S_{1,2,\ldots,N+2}^{[0]}), N \geq 0,
$$

$$
l(Y_0, 0, N + 1) = l'(Y_{N+1}^{[N+1]}, 0, N + 1) = \text{ind}(-S_{1,2,\ldots,N+1}^{[0]}), N \geq 1,
$$

(4.34)

$$
S_{1,2,\ldots,N+1}^{[0]} = \left(\begin{array}{c}
0 \\
M_d^T \\
S_{2,3,\ldots,N+1}^{[0]} - D_d - D_d^T
\end{array}\right).
$$
where $S_{1,2,\ldots,N+2}^{[0]}$ is the submatrix of $S_{1,2,\ldots,N+2}^{[0]}$. The $n \times n$ blocks $S_{i,j}^{[0]}$ of $S_{1,2,\ldots,N+2}^{[0]}$ are defined as follows

\[ S_{i,j}^{[0]} = X_{i,j}^{[-1]} T = -X_{i,j}^{[-1]} , \quad S_{i,j}^{[0]} = S_{i,j}^{[0]} T = S_{i,j}^{[0]} , \quad j \geq i , \quad i,j = 1, 2, \ldots, N + 2 , \]  

in particular,

\[ S_{i,i+1}^{[0]} = S_{i,i+1}^{[0]} T = S_{i,i+1}^{[0]} (i,i+1) = B_{i+1} , \]  

where $B_{i}$ is the block of $S_{i}$ according to (4.13). We also have

\[ \tilde{M}_{d} = (I - \lambda_{0}^{[N+1]} + \lambda_{0}^{[N+1]} N_{d}) M_{d} , \quad \lambda = \lambda_{0}^{[N+1]} N_{d} , \]

\[ \lambda = (X_{0}^{[0]} T X_{2}^{[0]} T \cdots X_{N}^{[0]} T) , \quad N_{d} = (X_{0}^{[N+1]} T X_{2}^{[N+1]} T \cdots X_{N}^{[N+1]} T) , \]  

where $\tilde{M}_{d}$ in (4.34) can be replaces by

\[ M_{d} = (I - \lambda_{0}^{[N+1]} + \lambda_{0}^{[N+1]} N_{d}) K_{d} . \]

**Proof.** The first equalities in (4.34) follow from (4.23) coupled with Theorem 1.2, where we derive (4.35) according to (1.16). Indeed, by (4.22) we have

\[ l'(Y_{0},0,N+1) = l(Y_{N+1},0,N+1) = \mu_{0}^{-1}(Z_{0}^{-1}(0 I)^{T}, Z_{1}^{-1}(0 I)^{T}, \ldots, Z_{N+1}^{-1}(0 I)^{T}) , \]

then by (1.16) applied to the case (4.22) we derive

\[ S_{1,2,\ldots,n}^{[0]} (i,j) = w(Z_{j-1}^{-1}(0 I)^{T} , Z_{j}^{-1}(0 I)^{T}) = w(Z_{j-1}^{-1}(0 I)^{T} , (0 I)^{T}) = X_{j-1}^{[-1]} T \]

(4.39)

Note that $S_{1,2,\ldots,n}^{[0]} (i,j) = X_{j-1}^{[-1]} = 0$ according to definition (4.14) of the principal solution at the point $M = i - 1$ and by (4.14), (1.1) $X_{j}^{[-1]} = B_{j-1} , \quad i = 1, 2, \ldots, N + 1$.

The equalities in the second row of (4.34) are derived using (4.24), Corollary 3.3 and Theorem 3.4. In more details, we have $l(Y_{0},0,N+1) = l(Y_{N+1},0,N+1) = \mu_{0}^{-1}(Z_{0}^{-1}(0 I)^{T}, Z_{1}^{-1}(0 I)^{T}, \ldots, Z_{N+1}^{-1}(0 I)^{T})$ by (4.24) and then one can calculate $\mu_{0}^{-1}(Z_{j-1}^{-1}(0 I)^{T}, Z_{j}^{-1}(0 I)^{T}, \ldots, Z_{N+1}^{-1}(0 I)^{T})$ according to the second equality in (3.18) using the definition of $\tilde{M}_{d}$, $M_{d}$, $K_{d}$, $N_{d}$ and $D_{d}$ in Corollary 3.3. Observe also that in these computations we again used connection (4.39) for the upper blocks of the principal solutions $Y_{k}^{[-1]} , Y_{k}^{[-1]}$ of (1.1). The proof is completed.

Recall that according to [4, Theorems 1.2], [9, Theorems 2.36, 2.41] systems (1.1) and (4.16) are disconjugate on $[0,N+1]$ if and only if $l(Y_{0},0,N+1) = l(Y_{N+1},0,N+1) = 0$. Based on Theorem 4.8 one can formulate the following criterion for disconjugacy of (1.1) and (4.16).

**Corollary 4.9.** Systems (1.1) and (4.16) are disconjugate on $[0,N+1]$ if and only if the matrix $S_{1,2,\ldots,N+2}^{[0]}$ defined in (4.34) is nonpositive definite, i.e. $S_{1,2,\ldots,N+2}^{[0]} \leq 0$. The last condition is equivalent to

\[ \tilde{M}_{d} = M_{d} = 0 , \quad S_{1,2,\ldots,N+2}^{[0]} - D_{d} D_{d}^{T} \leq 0 , \]  

(4.40)

where we use the notation in Theorem 4.8.

**Proof.** We have by Theorem 4.8 that the condition $l(Y_{0},0,N+1) = l(Y_{N+1},0,N+1) = 0$ is equivalent to $\text{ind}(S_{1,2,\ldots,N+2}^{[0]} \leq 0)$. Then the first claim of this corollary is proved, while conditions (4.40) follow from the last equality in (3.18).

**Acknowledgments**

This research is supported by the Ministry of Science and Higher Education of the Russian Federation under project 0707-2020-0034 and by the Czech Science Foundation under grant GA19-01246S.

**References**
References

[1] A. A. Abramov, On the computation of the eigenvalues of a nonlinear spectral problem for Hamiltonian systems of ordinary differential equations, Computational Math. Math. Phys. 41 (2001), no. 1, 27–36.

[2] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications (Wiley, New York, 1974).

[3] D.S. Bernstein, Matrix Mathematics. Theory, Facts, and Formulas with Application to Linear Systems Theory (Princeton University Press, Princeton, 2005).

[4] M. Bohner, O. Došlý, Disconjugacy and transformations for symplectic systems. Rocky Mt. J. Math. 27, 707–743 (1997).

[5] M. Bohner, O. Došlý, Positivity of block tridiagonal matrices, SIAM J. Matrix analysis, 20 (1998), 182-195.

[6] M. Bohner, W. Kratz, R. Šimon Hilscher, Oscillation and spectral theory for linear Hamiltonian systems with nonlinear dependence on the spectral parameter, Math. Nachr. 285 (2012), no. 11–12, 1343–1356.

[7] B. Booss-Bavnbek, C. Zhu, The Maslov index in symplectic Banach spaces, Mem. Amer. Math. Soc. 252 (2018), no. 1201, x+118 pp.

[8] S. E. Cappell, R. Lee, E. Y. Miller On the Maslov index Comm. Pure Appl. Math. 47 (1994), no. 2, 121–186.

[9] O. Došlý, J. Elyseeva, R. Šimon Hilscher, Symplectic Difference Systems: Oscillation and Spectral Theory, Birkhäuser Basel, 2019.

[10] Elyseeva, J.V., The Comparative Index for Conjoined Bases of Symplectic Difference Systems, in Difference Equations, Special Functions and Orthogonal Polynomials, World Scientific. (2007), pp. 135–145.

[11] Yu. Eliseeva, Comparative index for solutions of symplectic difference systems. Differ. Equ. 45 (2009), 445–459.

[12] J. Elyseeva, Comparison theorems for conjoined bases of linear Hamiltonian differential systems and the comparative index, J. Math. Anal. Appl. 444 (2016) 1260–1273.

[13] J. Elyseeva, Oscillation theorems for linear Hamiltonian systems with nonlinear dependence on the spectral parameter and the comparative index, Applied Mathematics Letters 90 (2019) 15–22.

[14] J. Elyseeva, Comparison theorems for conjoined bases of linear Hamiltonian systems without monotonicity, Monatsh. Math. 193 (2020) 305–328.

[15] J. Elyseeva, P. Šepitka, R. Šimon Hilscher, Oscillation numbers for continuous Lagrangian paths and Maslov index, Submitted February 18, 2021 (see arXiv:2107.01928 [math.SG])

[16] R. Hilscher, Disconjugacy of symplectic systems and positive definiteness of block tridiagonal matrices, Rocky Mt. J. Math.,29 (1999), no. 4, 1301-1319.

[17] L. Hörmander, Fourier integral operators, I, Acta Math. 127:1-2 (1971), 79-183.

[18] P. Howard, Y. Latushkin, A. Sukhtayev, The Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$, J. Math. Anal. Appl. 451 (2017), no. 2, 794–821.

[19] P. Howard, Hörmander’s index and oscillation theory, J. Math.Anal.Appl. 500 (2021) 125–076.

[20] M. Kashiwara, P. Schapira, Sheaves on Manifolds, in: Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer, 1980.

[21] W. Kratz, Discrete oscillation. J. Di ff. Equ. Appl. 9, 127–135 (2003).

[22] V.B. Lidskii, Oscillation theorems for canonical systems of differential equations (Russian), Dokl. Akad. Nauk SSSR (N.S.) 102 (5) (1955) 877–880, Translation in: NASA Tech. Transl. TT F-14 (1955) 696, 9 pp.

[23] G. Marsaglia, G.P.H. Styan, Equalities and inequalities for ranks of matrices, Linear and Multilinear Algebra 2 (1974) 269–292.

[24] W.T. Reid, Sturmian Theory for Ordinary Differential Equations, Springer-Verlag, 1980.

[25] P. Šepitka, R. Šimon Hilscher, Comparative index and Lidskii angles for symplectic matrices, Linear Algebra Appl. 624 (2021), 174–197.

[26] P. Šepitka, R. Šimon Hilscher, Comparative index and Sturmian theory for linear Hamiltonian systems, J. Differential Equations, 262 (2017), 914–944.

[27] P. Šepitka, R. Šimon Hilscher, Singular Sturmian comparison theorems for linear Hamiltonian systems, J. Differ. Equ. 269 (4) (2020) 2920-2955.

[28] Y. Tian, Equalities and inequalities for inertias of Hermitian matrices with applications. Linear Algebra Appl. 433, (2010), 263–296.

[29] Y. Zhou, L. Wu, C. Zhu, Hörmander index in the finite-dimensional case, Front. Math. China 13 (2018) 725–761.