Upper semi-continuity of entropy map for nonuniformly hyperbolic systems

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Abstract

We prove that entropy map is upper semi-continuous for $C^1$ nonuniformly hyperbolic systems with domination, while it is not true for $C^{1+\alpha}$ nonuniformly hyperbolic systems in general. This goes against common intuition that conclusions are parallel between $C^1$ domination systems and $C^{1+\alpha}$ systems.

Keywords: entropy map, upper semi-continuity, dominated splitting, Pesin set

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(Some figures may appear in colour only in the online journal)

1. Introduction

The entropy map of a continuous transformation $f$ on a metric space $M$ is defined by $\mu \mapsto h_\mu(f)$ on the set $\mathcal{M}_{in}(M)$ of all $f$-invariant measures and is generally not continuous (see [13]). However, it is still worth our effort to investigate the upper semi-continuity of the entropy map since, for instance, it implies the existence of invariant measures of maximal entropy. It has been shown that the entropy map is upper semi-continuous for expansive homeomorphisms of compact metric spaces [22], and then it is generalized to entropy expansive maps [3] as well as asymptotic entropy expansive maps [14]. In 1989 Newhouse [16] proved: (i) for any $C^\infty$ maps the entropy map is upper semi-continuous; (ii) for $C^{1+\alpha}$ nonuniformly hyperbolic diffeomorphisms the entropy map, when restricted on the set of hyperbolic measures with the same ‘hyperbolic rate’, is also upper semi-continuous. In the present paper, we remove the assumption on ‘hyperbolic rates’ in [16] to show that for $C^1$ nonuniformly hyperbolic systems with a domination property, the entropy map is upper semi-continuous.
Definition 1.1. Let $M$ be a compact Riemannian manifold and $f : M \to M$ be a $C^1$ diffeomorphism. Given $\lambda_-, \lambda_+ \gg \varepsilon > 0$, and for all $k \in \mathbb{N}$, we define $\Lambda_k = \Lambda(\lambda_-, \lambda_+; \varepsilon), k \geq 1$, to be all points $x \in M$ for which there is a splitting $T_x M = E^s_x \oplus E^u_x$ with the invariance property $D_x f^m(E^s_x) = E^s_{f^m x}$ and $D_x f^m(E^u_x) = E^u_{f^m x}$ and satisfying:

(a) $\|Df^m|_{E^s_{f^m x}}\| \leq e^{\varepsilon \ell_k - (\lambda_- - \varepsilon) m}$, $\forall m \in \mathbb{Z}$, $n \geq 1$;

(b) $\|Df^{-m}|_{E^u_{f^m x}}\| \leq e^{\varepsilon \ell_n - (\lambda_+ - \varepsilon) m}$, $\forall m \in \mathbb{Z}$, $n \geq 1$;

(c) $\tan(\text{ang}(E^s_{f^m x}, E^u_{f^m x})) \geq e^{-\varepsilon \ell_k} e^{-\varepsilon m}$.

$\Lambda = \Lambda(\lambda_-, \lambda_+; \varepsilon) = \bigcup_{k=1}^{\infty} \Lambda_k$ is called a Pesin set.

Denote by $\mathcal{M}_{inv}(M)$ the set of all $f$–invariant probability measures on $M$ endowed with weak* topology and $\mathcal{M}_{inv}(\Lambda)$ the subset of $\mathcal{M}_{inv}(M)$ supported on $\Lambda$, i.e. $\mu \in \mathcal{M}_{inv}(\Lambda) \iff \mu(\Lambda) = 1$. Any invariant measure $\nu$ with non-zero Lyapunov exponents is called a hyperbolic measure. For any ergodic hyperbolic measure, we could define a Pesin set associated to it in the following way. Let $\Omega$ be the Oseledec basin of $\nu$ where all Lyapunov exponents exist, by Oseledec theorem [17] $\nu(\Omega) = 1$. Denote by $E^s$ and $E^u$ the direct sum of the Oseledec splittings with respect to negative and positive Lyapunov exponents respectively. Let $\lambda_0$ be the norm of the largest Lyapunov exponent of vectors in $E^s$ and $\lambda_0$ be the smallest one in $E^u$ and choose $0 < \varepsilon \ll \min[\lambda_0, \lambda_0]$. Then $\Omega$ is contained in the Pesin set $\Lambda = \Lambda(\lambda_0, \lambda_0; \varepsilon)$ in definition 1.1, i.e. $\nu \in \mathcal{M}_{inv}(\Lambda)$. Observe that the splitting $T_x M = E^s(x) \oplus E^u(x)$ in definition 1.1 is not necessary to be continuous with $x \in \Lambda$, and the angle between $E^s$ and $E^u$ may approach to zero along orbits in $\Lambda$. This discontinuity leads to an obstacle for the continuity property of the entropy map on $\mathcal{M}_{inv}(\Lambda)$. In the present paper, we make an assumption that there is a domination between $E^s$ and $E^u$, which ensures both continuity of the splittings and the uniformly bounded angles from below between them. To be precise, a continuous splitting $T_x M = E^s(x) \oplus E^u(x)$, $x \in \Lambda$ is dominated if there are constants $C > 0$ and $0 < \lambda < 1$ such that $\frac{\|Df^m v\|}{\|Df^m u\|} \leq C |\lambda|^m$ for any $v \in E^s(x)$ and $u \in E^u(x)$ with $\|v\| = 1$, $\|u\| = 1$.

Here is our main theorem in the paper.

Theorem 1.2. Let $f$ be a $C^1$ diffeomorphism of a compact Riemannian manifold $M$. Let $\Lambda = \Lambda(\lambda_0, \lambda_0; \varepsilon)$ be a Pesin set with a dominated splitting $T_x M = E^s(x) \oplus E^u(x), x \in \Lambda$. Then the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous on $\mathcal{M}_{inv}(\Lambda)$.

Remark 1.3. The dominated splitting of $\Lambda$ can be extended to its closure, but the closure $\bar{\Lambda}$ may support non-hyperbolic measures. An application of theorem 1.2 in the three-dimensional case is a class of $C^1$ robustly transitive partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ introduced by Mañé [12], where the Lyapunov exponent of central bundle is non-zero. For higher dimensions, Bonatti and Viana (section 6.4 of [2]) constructed a $C^1$ open set $\mathcal{U}$ of diffeomorphisms on $\mathbb{T}^4$ with dominated splittings but admit no hyperbolic bundles, where the closure of the Pesin set $\bar{\Lambda}$ coincides with $\mathbb{T}^4$ and $\Lambda$ supports a hyperbolic SRB measure [21].

Remark 1.4. The topological pressure $P(\phi)$ of a continuous potential $\phi$ is

$$\sup_{\mu \in \mathcal{M}_{inv}(M)} \left\{ h_\mu(f) + \int \phi d\mu \right\}.$$ 

The upper semi-continuity of the entropy map on $\mathcal{M}_{inv}(\Lambda)$ implies the existence of equilibrium states (invariant measures $\mu$ with $h_\mu(f) + \int \phi d\mu = P(\phi)$) provided there exists a sequence
\{\mu_i\}_{i=1}^{\infty}$ in a closed subset of $\mathcal{M}_{\text{inv}}(\Lambda, f)$ such that $\lim_{n \to \infty} (h_{\mu_i}(f) + \int \phi d\mu_i) = P(\phi)$. This condition is satisfied by many systems with some hyperbolicity including Mané’s example [12] and Bonatti–Viana’s example [2], where the closure of a certain open neighborhood of the unique maximal entropy measure is contained in $\mathcal{M}_{\text{inv}}(\Lambda)$ by theorems 6.7 and 6.8 in [7]. In Mané’s example [12] and Bonatti–Viana’s example [2], the non-uniformly hyperbolic sets $\Lambda$ exist for diffeomorphisms in a $C^1$ neighborhood and the method in the present paper can be applied to all systems in it at the same time, instead of just one system. Then it is not hard to show the upper semi-continuity of the topological entropy. In fact, the topological entropy is locally constant in these examples by [5] and [6].

Lack of domination may cause no upper semi-continuity of the entropy map for the $C^r$ diffeomorphism for any $2 \leq r < \infty$ by a version of Downarowicz–Newhouse example [8]. This goes against common intuition that conclusions are usually parallel between $C^1$ domination systems and $C^{1+\alpha}$ systems (the stable manifold theorem and entropy formula in $C^{1+\alpha}$ Pesin theory also hold for $C^{1+\alpha}$ domination systems [1, 20]).

Moreover, recall that the upper semi-continuity of the entropy map is obtained for $C^1$ diffeomorphisms away from tangencies in [11]. However, due to the nonuniformity of hyperbolicity of $(f, \Lambda)$ in theorem 1.2, the system $(f, \tilde{\Lambda})$ may be approximated by having homoclinic tangencies of some periodic points whose index is different from $\dim E^s$, see such examples in [2].

Generally, the difference between metric entropy and the metric entropy with respect to a partition with small diameter is bounded by local entropy in [3]. In section 2, we give an integral form of local entropy to improve Bowen’s classical inequality in theorem 3.5 of [3]. In section 3 we will prove theorem 1.2. By using a class of $C^r$ $(2 \leq r < \infty)$ diffeomorphisms studied in [8] we illustrate in section 4 that the entropy map may not be upper semi-continuous for nonuniformly hyperbolic systems without domination.

## 2. Preliminaries

Let $M$ be a compact metric space and $f$ a continuous map on $M$. Let $\mu$ be an $f$-invariant probability measure and $\xi = \{B_1, \cdots, B_k\}$ a finite partition of $M$ into measurable sets. The entropy of $\xi$ with respect to $\mu$ is

$$H_\mu(\xi) = - \sum_{i=1}^{\xi} \mu(B_i) \log \mu(B_i).$$

The entropy of $f$ with respect to $\mu$ and $\xi$ is given by

$$h_\mu(f, \xi) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\xi^n) = \inf_{n} \frac{1}{n} H_\mu(\xi^n)$$

where $\xi^n = \bigvee_{i=0}^{n-1} f^{-i} \xi$. The entropy of $f$ with respect to $\mu$ is given by

$$h_\mu(f) = \sup_{\xi} h_\mu(f, \xi)$$

where $\xi$ is taken over all finite partitions of $M$. A partition $\alpha = \{A_0, A_1, \cdots, A_k\}$ is called a compact partition if $A_1, \cdots, A_k$ are disjoint compact subsets and $A_0 = M \setminus \bigcup_{i=1}^{k} A_i$. For any given finite partition $\xi$ and any $\varepsilon > 0$, we could construct a finite compact partition $\alpha$ such that $h_\mu(f, \xi) < h_\mu(f, \alpha) + \varepsilon$. The construction is as follows: starting from $\xi = \{B_1, \cdots, B_k\}$, by
the regularity of \( \mu \), we can get compact sets \( A_i \) by slightly shrinking \( B_i \) such that \( \mu(B_i \backslash A_i) \) is small enough. Let \( A_0 = \mathcal{M} \backslash \bigcup_{i=1}^{k} A_i \) then \( \alpha = \{ A_0, A_1, \ldots, A_k \} \) is a compact partition. Clearly \( \mu(A_0) = \sum_{i=1}^{k} \mu(B_i \backslash A_i) \) is small. Let \( H_\mu(\xi|\alpha) = -\sum_{i,j} \mu(A_j \cap B_i) \log \frac{\mu(A_j \cap B_i)}{\mu(A_j)} \) which is called the conditional entropy of \( \xi \) given \( \alpha \) and can be arbitrarily small by the choice of \( \alpha \). By the fact that \( h_\mu(f, \xi) \leq h_\mu(f, \alpha) + H_\mu(\xi|\alpha) \) (theorem 4.12 (iv) in [22]) we get \( h_\mu(f, \xi) < h_\mu(f, \alpha) + \epsilon' \). Taking supremum over all finite partitions \( \xi \), since \( \epsilon' \) is arbitrarily small, we get that

\[
 h_\mu(f) = \sup_{\alpha} h_\mu(f, \alpha) ,
\]

where \( \alpha \) is taken over all finite compact partitions of \( \mathcal{M} \).

Let \( F \) be a subset of \( \mathcal{M} \). A set \( E \subseteq \mathcal{M} \) is called a \((n, \delta)\)-spanning set of \( F \subseteq \mathcal{M} \) with respect to \( f \) if \( \forall x \in F, \exists y \in E \) such that \( d(f^i(x), f^i(y)) \leq \delta, 0 \leq i < n \). Denote \( r_\delta(f, \delta, f) \) the minimal cardinality of sets which \((n, \delta)\)-spans \( F \) with respect to \( f \). Denote \( r(f, \delta, f) = \lim \sup_{n \to +\infty} \frac{1}{n} \log r_\delta(f, \delta, f) \) and the topological entropy of \( f \) on \( F \) is defined by

\[
 h_{\text{top}}(f, F) = \lim_{\delta \to 0} r(f, \delta, f).
\]

In particular, the topological entropy of \( f \) on \( \mathcal{M} \), \( h_{\text{top}}(f, \mathcal{M}) \), is denoted by \( h_{\text{top}}(f) \).

For each \( x \in \mathcal{M}, n \in \mathbb{N}, r \in \mathbb{R}^+ \), denote \( B_r(x, r, f) = \{ y \in \mathcal{M} : d(f^i(x), f^i(y)) \leq r, 0 \leq i < n \} \), and \( B_{\infty}(x, r, f) = \{ y \in \mathcal{M} : d(f^i(x), f^i(y)) \leq r, i \geq 0 \} \). When \( f \) is a homeomorphism, one may also define \( B_{\infty}(x, r, f) = \{ y \in \mathcal{M} : d(f^i(x), f^i(y)) \leq r, -n < i < n \} \) and \( B_\infty(x, r, f) = \{ y \in \mathcal{M} : d(f^i(x), f^i(y)) \leq r, i \in \mathbb{Z} \} \). Denote

\[
 h_{\text{loc}}^*(x, r, f) = h(f, B_\infty(x, r, f)).
\]

Further let

\[
 h_{\text{loc}}(x, r, f) = \lim_{\delta \to 0} \lim_{n \to +\infty} \frac{1}{n} \log r_\delta(B_{\infty}(x, r, f), \delta, f).
\]

It’s obvious that \( h_{\text{loc}}^*(x, r, f) \leq h_{\text{loc}}(x, r, f) \).

**Proposition 2.1.** Let \( \mathcal{M} \) be a compact Riemannian manifold and \( f : \mathcal{M} \to \mathcal{M} \) be a diffeomorphism preserving a measure \( \mu \in \mathcal{M}_{\text{inv}}(\mathcal{M}) \). Then

\[
 h_\mu(f) - h_\mu(f, \xi) \leq \int h_{\text{loc}}(x, r, f) \mu(dx)
\]

for any finite partition \( \xi \) with \( \text{diam}(\xi) \leq r \).

**Proof.** We start from any given compact partition \( \alpha = \{ A_0, A_1, \ldots, A_k \} \), where \( A_1, \ldots, A_k \) are disjoint compact subsets and \( A_0 = \mathcal{M} \backslash \bigcup_{i=1}^{k} A_i \). Let

\[
 \delta_0 = \frac{1}{4} \min \{ d(A_i, A_j), 1 \leq i, j \leq k, i \neq j \}.
\]

For \( m \in \mathbb{N} \) take \( \delta_0 \in (0, \delta_0) \) such that \( d(x, y) < \delta_1 \) implies that \( d(f^i(x), f^i(y)) < \delta_0, 0 \leq i < m \). Denote \( \alpha_m^n = \bigcup_{i=0}^{m-1} f^{-i} \alpha \), then
\[
\frac{1}{n} H_{\mu}(\alpha^n) - \frac{1}{n} H_{\mu}(\xi^n) \leq \frac{1}{n} H_{\mu}(\alpha^n) - \frac{1}{n} H_{\mu}(\xi^n)
\]

where \( \xi^n = \bigvee_{i=0}^{m-1} f^{-i} \xi \)

\[
\leq \frac{1}{n} \sum_{B \in (\xi^n)^n} \mu(B) \sum_{A \in \alpha^n} \mu_{\beta}(A) \log \left( \frac{\mu(A \cap B)}{\mu(B)} \right)
\]

\[
\leq \frac{1}{n} \sum_{B \in (\xi^n)^n} \mu(B) \log N_{\delta}(\alpha^n).
\]  \hspace{1cm} (1)

where \( N_{\delta}(\alpha^n) = \# \left\{ A \in \alpha^n : A \cap B \neq \emptyset \right\} \).

Fix \( B \in (\xi^n)^n \), let \( E \) be a \((n, \delta_{B}, f^{m})\)-spanning set of \( B \) with respect to \( f^{m} \) with minimal cardinality. For every \( y \in E \), by the choice of \( \delta_{B} \), the number of elements of \( \alpha^n \) which intersect with \( B_{y}(y, \delta_{B}, f^{m}) \) could not exceed \( 2^{n} \). Since \( \text{diam}(\xi) < r \), \( B \subseteq B_{x}(x, r) \) for any given \( x \in B \). From these we get

\[
N_{\delta}(\alpha^n) \leq r_{d}(B, \delta_{B}, f^{m}) \cdot 2^{n}
\]

\[
\leq r_{d}(B_{x}(x, r), \delta_{B}, f^{m}) \cdot 2^{n}
\]

\[
\leq r_{m}(B_{x}(x, r), \delta_{B}, f) \cdot 2^{n}
\]

for any point \( x \in B \). Thus by (1) we get

\[
\frac{1}{n} H_{\mu}(\alpha^n) - \frac{1}{n} H_{\mu}(\xi^n)
\]

\[
\leq \frac{1}{n} H_{\mu}(\alpha^n) - \frac{1}{n} H_{\mu}(\xi^n)
\]

\[
\leq \frac{m}{mn} \sum_{B \in (\xi^n)^n} \int_{B} \log r_{m}(B_{x}(x, r), \delta_{B}, f) d\mu(x) + \log 2
\]

\[
= m \int \frac{1}{mn} \log r_{m}(B_{x}(x, r), \delta_{B}, f) d\mu(x) + \log 2.
\]

When \( n \) is large enough, \( \frac{1}{mn} \log r_{m}(B_{x}(x, r), \delta_{B}, f) \) is less than or equal to \( h_{\text{top}}(f) + 1 \), which is a finite number for a diffeomorphism on a compact manifold. Applying Fatou Lemma we have

\[
\frac{1}{n} H_{\mu}(\alpha^n) - \frac{1}{n} H_{\mu}(\xi^n)
\]

\[
= \lim_{n \to +\infty} \left( \frac{1}{n} H_{\mu}(\alpha^n) - \frac{1}{n} H_{\mu}(\xi^n) \right)
\]

\[
\leq m \cdot \limsup_{n \to +\infty} \int \frac{1}{mn} \log r_{m}(B_{x}(x, r), \delta_{B}, f) d\mu(x) + \log 2
\]

\[
\leq m \int \limsup_{n \to +\infty} \frac{1}{mn} \log r_{m}(B_{x}(x, r), \delta_{B}, f) d\mu(x) + \log 2
\]

\[
\leq m \int \log h_{\text{loc}}(x, f) d\mu(x) + \log 2
\]
for any compact partition \( \alpha \) and any \( m \in \mathbb{N} \). Note that \( h_\mu(f^m, \xi^m) = mh_\mu(f, \xi), \forall m \in \mathbb{N} \). It follows that

\[
h_\mu(f) - h_\mu(f, \xi) \leq \int h_{\text{loc}}(x, r, f) d\mu(x).
\]

**Remark 2.2.** In [3], the right-hand side of the inequality in proposition 2.1 is

\[
\lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{1}{n} \sup_{x \in M} \log r_n(B_\delta(x, r, f), \delta, f),
\]

which is called the local entropy of \( f \). It is obvious that this quantity is not smaller than \( \int h_{\text{loc}}(x, r, f) d\mu(x) \) which we call the local entropy of \( f \) with respect to \( \mu \). Proposition 2.1 is crucial for the proof of theorem 1.2 because it enables us to deal with local entropy on sets with large measure instead of the whole Pesin set. The hyperbolicity assumption of measures guarantees some kind of ‘uniform hyperbolicity’ (average along the orbit) for sets with a large measure (proposition 3.1) for all ‘nearby’ measures. Then applying domination property (lemma 3.2) and the classical Pliss Lemma we can get small local entropy for points in a large measured set. In this way we could control the difference between metric entropy and the metric entropy with respect to a partition with small diameter for all nearby \( \nu \) in proposition 3.5, which is a necessary step to prove theorem 1.2.

For a continuous map \( f \) on a compact metric space \( M \), a measure \( \mu \in \mathcal{M}_{\text{inv}}(M) \) and a Borel set \( A \), by Birkhoff Ergodic Theorem, the set of points for which the limit of

\[
\bar{\nu}_x = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(f^i(x))
\]

whenever exists. By Birkhoff Ergodic Theorem, \( \int \chi_A(x) d\nu(x) = \int \bar{\nu}_x d\nu(x) \). Let \( E = \{ x : \bar{\nu}_x > 1 - \gamma \} \), then

\[
\nu(A) = \int_E \bar{\nu}_x d\nu(x) + \int_{M \setminus E} \bar{\nu}_x d\nu(x) \leq \nu(E) + (1 - \gamma)(1 - \nu(E)).
\]

Choose \( \sigma = \frac{1}{2} \gamma \eta \), then \( \nu(A) > 1 - \sigma \) implies that \( \nu(E) > 1 - \frac{1}{2} \eta \). 

**Remark 2.4.** Lemma 2.3 is also true for \( f^{-1} \).

### 3. Proof of theorem 1.2

In this section, we prepare several lemmas and propositions and then prove theorem 1.2.

Recall that \( f : M \to M \) is a \( C^1 \) diffeomorphism, which has a Pesin set \( \Lambda = \Lambda(\lambda_1, \lambda_2; \epsilon) \) with a dominated splitting \( T_M = E^s(x) \oplus E^u(x), x \in \Lambda \). The weak* topology on \( \mathcal{M}_{\text{inv}}(M) \) could be given by metric \( D : \mathcal{M}_{\text{inv}}(M) \times \mathcal{M}_{\text{inv}}(M) \to \mathbb{R}, D(\mu, \nu) = \sum_{i=1}^{\infty} \frac{|\int f_\mu \omega_i - \int f_\nu \omega_i|}{2^i \| \omega_i \|} \), where \( \{ f_\mu \}_{\mu \in \mathcal{M}} \).
is a dense subset of the set of all continuous functions on $M$. The weak$^*$ topology is independent of the choice of the specific metric $D$. We can obtain the induced metric on $\mathcal{M}_{\text{inv}}(\Lambda)$ by restricting $D$ on $\mathcal{M}_{\text{inv}}(\Lambda)$. Denote $B_\rho(\mu) = \{ \nu \in \mathcal{M}_{\text{inv}}(M) | D(\nu, \mu) \leq \rho \}$.

**Proposition 3.1.** Given $\mu \in \mathcal{M}_{\text{inv}}(\Lambda)$ and $0 < \eta < 1$ there exist $\rho > 0$, $L > 0$ such that the set $T_L$ of points $x$ with the following properties:

\[
\lim_{k \to +\infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{L} \log \left\| Df^{-L} \right\|_{E^{\gamma}(x)} < -\lambda_\mu + 5\varepsilon,
\]

\[
\lim_{k \to +\infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{L} \log \left\| Df^{i} \right\|_{E^{\gamma}(x)} < -\lambda_\mu + 5\varepsilon.
\]

satisfies $\nu(T_L) > 1 - \eta$ for any $\nu \in B_\rho(\mu) \cap \mathcal{M}_{\text{inv}}(\Lambda)$.

**Proof.** For an integer $n \geq 1$ set

\[ A_n^\varepsilon = \left\{ x \in \Lambda : \frac{1}{m} \log \left\| Df^{-m} \right\|_{E^{\gamma}(x)} < -\lambda_\mu + 2\varepsilon, \forall m \geq n \right\}. \]

Then $A_1^\varepsilon \subseteq \cdots \subseteq A_n^\varepsilon \subseteq A_{n+1}^\varepsilon \subseteq \cdots$, and $\mu\left( \bigcup_{n=1}^{\infty} A_n^\varepsilon \right) = 1$. Choose $c$ large enough such that

\[ \log c > \max \{ \sup_{x \in A} \log \left\| Df^{-m} \right\|_{E^{\gamma}(x)} | m \geq 1 \} \]

and let $0 < \eta < 1$ be given in the condition of the proposition. Take $\gamma < \frac{\varepsilon}{2(\lambda_\mu + 2\varepsilon)}$ with $0 < \gamma < 1$ and take $\sigma = \frac{1}{2}\gamma$ as in lemma 2.3. Clearly $0 < \sigma < 1$. Take $N$ such that $\mu\left( \bigcup_{n=1}^{N} A_n^\varepsilon \right) > 1 - \sigma$. Let

\[ U_N^\varepsilon = \left\{ x \in \Lambda : \frac{1}{N} \log \left\| Df^{-N} \right\|_{E^{\gamma}(x)} < -\lambda_\mu + 2\varepsilon \right\}. \]

then $\bigcup_{n=1}^{N} A_n^\varepsilon \subseteq U_N^\varepsilon$. Since $U_N^\varepsilon$ is open in $\Lambda$, $\nu(U_N^\varepsilon) > 1 - \sigma$ for any $\nu \in \mathcal{M}_{\text{inv}}(\Lambda)$ close enough to $\mu$. Denote $f_{U_N^\varepsilon}(x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{U_N^\varepsilon}(f_i(x))$ whenever exists and

$U_N = \{ x \in \Lambda : f_{U_N^\varepsilon}(x) > 1 - \gamma \}$. By lemma 2.3, we get that $\nu(U_N^\varepsilon) > 1 - \frac{1}{2}\eta$.

Following the same procedure for $f^{-1}$ and $\frac{1}{n} \log \left\| Df^{-n} \right\|_{E^{\gamma}(x)}$ we get $N$ and thus (2).

Then we get a constant $\rho > 0$ such that $\nu(T_L) > 1 - \eta$ for any $\nu \in B_\rho(\mu) \cap \mathcal{M}_{\text{inv}}(\Lambda)$.

Now we prove that the set $T_L$ satisfies (2) and (3). For $x \in T_L$ and $i \in \mathbb{Z}^+$, choose a sequence of integers $\{n_i\}_{i=1}^{\infty}$

\[ (i - 1)L = n_{i+1}^j < n_i^j < n_{i-1}^j < \cdots < n_i^1 = iL \]

by the following procedure.
\[ n^j_{j+1} = \begin{cases} 
  n^j_i - N, & n^j_i \geq (i-1)L + N \text{ and } f^{n^j_i}(x) \in U^j_N \\
  n^j_i - 1, & \text{otherwise.} \end{cases} \]

where \(1 \leq j \leq \ell\). Write \(\{n^j_1, \ldots, n^j_{\ell}\}\) as a disjoint union \(A_1 \bigcup B_1 \bigcup C_i\), where

\[ A_i = \{n^j_i \geq (i-1)L + N, \text{ and } f^{n^j_i}(x) \in U^j_N\}, \]
\[ B_i = \{n^j_i \geq (i-1)L + N \text{ and } f^{n^j_i}(x) \not\in U^j_N\}, \]
\[ C_i = \{(i-1)L < n^j_i < (i-1)L + N\}. \]

It’s obvious that \(0 \leq \#C_i < N\) and thus
\[
\log\left\| Df^{-i} \right\|_{E^i(f^{\pi_i}(x))} \leq \sum_{j=1}^{\ell} \log\left\| Df^{-(n^j_i-n^j_{i+1})} \right\|_{E^i(f^{\pi_i}(x))} \leq N(-\lambda_u + 2\varepsilon) \cdot \#A_i + \log c \cdot \#B_i + \log c \cdot \#C_i < N(-\lambda_u + 2\varepsilon) \cdot \#A_i + \log c \cdot (N + \#B_i).
\]

By the definition of \(\overline{U^j_N}\), for any \(k\) large enough, \(\sum_{i=1}^{k} (N \cdot \#A_i + N) \geq kL(1 - \gamma)\) and \(\sum_{i=1}^{k} \#B_i \leq kLy\). Therefore,
\[
\frac{1}{k} \sum_{i=1}^{k} \frac{1}{L} \log\left\| Df^{-i} \right\|_{E^i(f^{\pi_i}(x))} \leq \frac{1}{kL} \left( N(-\lambda_u + 2\varepsilon) \sum_{i=1}^{k} \#A_i + \log c \cdot \sum_{i=1}^{k} \#B_i + \log c \cdot Nk \right) \leq \frac{1}{kL} \left( (-\lambda_u + 2\varepsilon)(kL(1 - \gamma) - kN) + \log c \cdot kLy + \log c \cdot Nk \right) < (-\lambda_u + 2\varepsilon) \left( 1 - \gamma - \frac{N}{L} \right) + \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon < -\lambda_u + 4\varepsilon.
\]

Hence,
\[
\lim_{k \to +\infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{L} \log\left\| Df^{-i} \right\|_{E^i(f^{\pi_i}(x))} \leq -\lambda_u + 5\varepsilon, \quad \forall x \in T_L
\]
and we get (2).

Replace \(f\) and \(E^{u}(x)\) by \(f^{-1}\) and \(E^{s}(x)\) respectively, we get (3) analogously. \(\square\)
The following lemma comes from Burns and Wilkinson [4], which uses locally invariant fake foliations to avoid the assumption of dynamical coherence, a construction that goes back to Hirsch, Pugh, and Shub [9]. Given a foliations \( \mathcal{F} \) and a point \( y \) in the domain, we denote \( \mathcal{F}(y) \) as the leaf through \( y \) and by \( \mathcal{F}(y, \rho) \) we mean the neighborhood of radius \( \rho > 0 \) around \( y \) inside the leaf.

**Lemma 3.2.** Let \( f: M \to M \) be a \( C^1 \) diffeomorphism. Assume that \( \Delta \) is an \( f \)-invariant compact set and its tangent space admits an \( f \)-dominated splitting: \( T_x M = E^s(x) \oplus E^u(x) \), \( \forall x \in \Delta \). Suppose that angles between \( f \)-invariant subbundles \( E^s \) and \( E^u \) are bounded from zero by \( \theta \) for every \( x \in \Delta \). Then for any \( 0 < \zeta < \frac{\theta}{2} \), \( \exists \rho > 0 \), for any \( x \in \Delta \), the neighborhood \( B(x, \rho) \) admits foliations \( \mathcal{F}^s_x \) and \( \mathcal{F}^u_x \), such that for any \( y \in B(x, \rho) \) and \( \sigma \in \{ s, u \} \),

(i) almost tangency: leaf \( \mathcal{F}^\sigma_x(y) \) is \( C^1 \) and \( \mathcal{F}^\sigma_x(y) \) lies in a cone of width \( \zeta \) of \( E^\sigma(x) \);

(ii) local invariance: \( f \mathcal{F}^\sigma_x(y, r_0) \subseteq \mathcal{F}^\sigma_x(f y) \), \( f^{-1} \mathcal{F}^\sigma_x(y, r_0) \subseteq \mathcal{F}^\sigma_{x^{-1}}(f^{-1} y) \).

By lemma 3.2 (i) one can define local product structure on the \( r \)-neighborhoods of every \( x \in \Delta \), for a small \( r > 0 \), as used in [11].

For \( y, z \in B(x, \rho) \), write \( [y, z]_a = a \) if \( \mathcal{F}^s_x(y) \) intersects \( \mathcal{F}^s_x(z) \) at \( a \in B(x, \rho) \). By transversality of (i), the intersection point \( a \) is unique whenever it exists. We could find \( r_1 \in [0, r_0] \) independent of \( x \) such that \( [y, z]_a \) is well defined whenever \( y, z \in B(x, r_1) \), and for any \( y \in B(x, r_1) \) there exists \( y_a \in \mathcal{F}^s_x(x) \) such that \( [y, y_a]_a = y \). Lemma 3.2 implies that the locally invariant foliations \( \mathcal{F}^s_a \) are transverse with angles uniformly bounded from below. Therefore, \( \exists \ell > 0 \) independent of \( x \) such that for any \( y \in B(x, \ell r) \) we have \( y_u \in \mathcal{F}^u_x(x, \ell r) \) for \( \ell r < r_1 \). Furthermore, by locally invariance of foliations we get that \( y \in B_{\ell 2}(x, \ell r) \) implies that \( f^{\pm 1}(y_u) = (f^{\pm 1} y)_u \), where we recall that \( B_{\ell 2}(x, \ell r) = \{ y \in M : d(f^i y, f^j x) \leq \ell r, -2 < i < 2 \} \).

Also note that \( y_{u_a} = x \) for \( y \in B(x, \ell r) \) implies that \( y \in \mathcal{F}^s_{y_u}(x, \ell r) \), therefore \( y_u = y_a = x \) implies that \( y = x \).

Since there exists domination on the Pesin set \( \Lambda = \Lambda(\lambda_0, \lambda_u; \epsilon) \), we could extend it to the closure of \( \Lambda \), then the process above could be done on \( \bar{\Lambda} \). Therefore, we get \( \ell \) independent of \( x \) in \( \Lambda \) such that for \( y \in B(x, \ell r) \), \( y_u = y_a = x \) implies that \( y = x \).

**Lemma 3.3 (Pliss [19]).** Let \( a_1 \leq c_2 < c_1 \) and \( \theta = \frac{c_1 - c_2}{c_1 - a_1} \). For given real numbers \( a_1, \ldots, a_N \), with \( \sum_{i=1}^N a_i < c_2N \) and \( a_i > a_\ell \) for every \( i \), there exists \( \ell \geq N0 \) and \( 1 \leq n_1 < n_2 < \cdots < n_\ell \leq N \), such that

\[
\sum_{i=k+1}^{n_j} a_i \leq c_1(n_j - k), \quad 0 \leq k < n_j, \quad 1 \leq j \leq \ell.
\]

By (2) of proposition 3.1, for every \( x \in T_L \) and every \( k \) large enough,

\[
\sum_{i=1}^k \log \left\| Df^{-L} \left| f^{-Lx(x)} \right| \right\| \leq (-\lambda_a + 5\epsilon) Lk.
\]

Take \( a_\ell = \inf_{x \in \Lambda} \{ \log \| Df^{-L} \| E^s(x) \} \), \( c_1 = (-\lambda_a + 6\epsilon)L \), \( c_2 = (-\lambda_a + 5\epsilon)L \). Applying lemma 3.3 it is easy to find an infinite sequence

\[1 \leq n_1 < n_2 < \cdots < n_j < \cdots\]

such that
\[ \sum_{j=k+1}^{n_k} \log \left| Df^{i-1} \right|_{E_i^{(j-1)}} \leq (-\lambda + 6i)(n_j - k)L, \]

\[ 0 \leq k < n_j, \quad j = 1, 2 \ldots. \]

Choose \( r^* > 0 \) and \( \zeta > 0 \) such that \( \|D_x f^{-1}\| \leq e^{r^*} \) and \( \|D_x f^y\| \leq e^{r^*} \) for \( d(x,y) < r^* \), \( \zeta \leq \frac{\Lambda(u,v)}{2} \), \( \|u\| = \|v\| = 1 \).

Take \( r^* = M_0^{-1}r' \), where \( M_0 = \sup_{x \in M} \{ \|D_x f^{-1}\|, \|D_x f\| \} \). Then \( f^{-i} \mathcal{F}^{n_i}_{j,i} \big( f^{n_i}x, r^* \big) \subseteq \mathcal{F}^{n_i}_{j,i} \big( f^{n_i}x, r \big) \), \( 0 \leq i < L \).

Now we are able to take \( r > 0 \) such that \( B_{x,0}(x,r) = \{ x \} \). Let \( r = \min \{ r', r^*, r'' \} \). Note that

\[ f^{-\lambda_0 - \lambda kL} \mathcal{F}^{n_i}_{j,i} \big( f^{n_i}x, \epsilon \big) \big( f^{n_i}x, e^{-\lambda_0 - \lambda kL} \epsilon \big) \]

for \( 0 \leq k < n_j, j = 1, 2 \ldots \). In particular,

\[ f^{-\lambda_0 - \lambda kL} \mathcal{F}^{n_i}_{j,i} \big( f^{n_i}x, \epsilon \big) \big( f^{n_i}x, e^{-\lambda_0 - \lambda kL} \epsilon \big) \]

by the choice of \( r' \), for \( y \in B_{x,0}(x,r) \), \( f^i(y_\nu) = (f^i y)_\nu \), \( \forall i \in \mathbb{N} \), thus \( \nu_i = f^{-\lambda_0 - \lambda kL} ( f^{n_i}x, \epsilon \big( f^{n_i}x, e^{-\lambda_0 - \lambda kL} \epsilon \big) \big) \), \( \forall j \in \mathbb{N} \). Therefore \( \nu \) belongs to the intersection of all \( \mathcal{F}^{n_i}_{j,i} \big( f^{n_i}x, \epsilon \big) \big( f^{n_i}x, e^{-\lambda_0 - \lambda kL} \epsilon \big) \big) \) over all \( j \) which reduces to \( \{ x \} \). Analogously, for \( y \in B_{x,0}(x,r) \), we get that \( \nu_i = x \). Thus \( y \in B_{x,0}(x,r) \) implies that \( y = x \).

To conclude, we have obtained the following:

**Claim 3.4.** For any \( \mu \in \mathcal{M}_\text{inv}(\Lambda) \) and any \( \sigma > 0 \) there exist \( r > 0 \), \( \rho > 0 \) and a measurable set \( T \) with \( \nu(T) > 1 - \rho \) for any \( \nu \in B_{\rho}(\mu) \cap \mathcal{M}_\text{inv}(\Lambda) \) such that

\[ B_{x,0}(x,r) = \{ x \}, \quad \forall x \in T. \]

Claim 3.4 says that fixing a small \( r > 0 \), for \( \nu \) close to \( \mu \), \( h^{\text{inv}}_{\text{inv}}(x, r, f) = 0 \) on a set with large \( \nu \)-measure. To estimate the difference between \( h_\mu(f) \) and \( h_\mu(f, \xi) \), by proposition 2.1 we need to deal with \( h_{\text{inv}}(x, r, f) \). One always has that \( h^{\text{inv}}_{\text{inv}}(x, r, f) \leq h_{\text{inv}}(x, r, f) \) but the inverse inequality is generally not true. However, we are going to show that \( h_{\text{inv}}(x, r, f) \) is still small on a set with large measure. Combining claim 3.4 with proposition 2.1 we aim to deduce the following proposition.

**Proposition 3.5.** Let \( \mu \in \mathcal{M}_\text{inv}(\Lambda) \) and \( \tau > 0 \). There exist \( r > 0 \) and \( \rho > 0 \) such that

\[ h_\mu(f) - h_\mu(f, \xi) \leq \tau \]

for any \( \nu \in B_{\rho}(\mu) \cap \mathcal{M}_\text{inv}(\Lambda) \) and any finite partition \( \xi \) with \( \text{diam}(\xi) \leq r \).

**Proof.** Let \( C = h_{\text{top}}(f, \Lambda) \) and \( C_0 = \sup_{x \in M} \{ \|D_x f^{-1}\| + 1, \|D_x f\| + 1 \}^{2 \dim M} \). We assume that \( C_0 \) is an integer (if not, replace it with \( \lceil C_0 \rceil + 1 \)). It is clear that both of them are finite. We assume that \( C > 0 \), otherwise the entropy map for \( f \) is upper semi-continuous and we complete the proof. Take \( \eta = \frac{\tau}{22} \), \( \gamma = \frac{r}{2 \log C_0} \) in lemma 2.3, we get \( \sigma = \frac{\eta}{\gamma} \leq \frac{\eta}{\gamma} \). By claim 3.4 we get \( r > 0 \), \( \rho > 0 \) and a Borel set \( T \) such that for any \( \nu \in B_{\rho}(\mu) \) which supported on \( \Lambda \), we have \( \nu(T) > 1 - \sigma \) and
We could assume that $T$ is compact by the regularity of measure.

Fix $\delta > 0$, for any $x \in T$, by (4) we get $N(x) > 0$ as well as an open neighborhood $V(x)$ of $x$ such that $\forall y \in V(x)$,

$$B_{x} \subseteq (x, \frac{\delta}{3})$$

By compactness of $T$, $\exists \ x_1, \ldots, x_k$ such that $T \subseteq \bigcup_{i=1}^{k} V(x_i) := W$. Then $\nu(W) > 1 - \sigma$.

We define a sequence $0 = n_0 < n_1 < \cdots < n_k = n$ of integers as follows. Let $n_1 = N_0$ and $n_k = n$.

We could assume that $n_j = n_1 < n - N_0$ has been defined with $f^{n}(x) \in W$. If $\min \{ \ell : \ell' > n_j, f^{\ell}(x) \in W \} \leq n - N_0$, we take $n_{j+1} = \min \{ \ell : \ell' > n_j, f^{\ell}(x) \in W \}$. Otherwise, we take $k = j + 1$ and $n_k = n$.

Now we begin to calculate $r_n(B_n(x, r), \delta)$ for any $x \in W$. We first state a lemma from [3].

Lemma 3.6. (Lemma 2.1 in [3]) Suppose $0 = t_0 < t_1 < \cdots < t_k = n$ and $E_i$ is a $(l_i + l_i - 1, \alpha)$-spanning set of $f^{l_i}$ for $0 \leq i < k$. Then

$$r_n(f, 2\alpha) \leq \prod_{0 \leq i < k} \text{card } E_i.$$
where the second inequality comes from the definition of $\tilde{W}$ for every $n$ large enough.

Therefore, fix any $\delta > 0$, $\forall x \in W'$,

$$
\limsup_{n \to +\infty} \frac{1}{n} \log r_n(B_n(x, r), \delta)
\leq \lim_{n \to +\infty} \left( \frac{1}{n} \log \kappa + \frac{2N_0 - 1}{n} \log C_0 + \gamma \log C_0 \right)
= \gamma \log C_0.
$$

By the choice of $\gamma$ we get that

$$
\limsup_{n \to +\infty} \frac{1}{n} \log r_n(B_n(x, r), \delta) \leq \frac{\tau}{2}, \ \forall x \in W', \forall \delta > 0.
$$

Therefore,

$$
h_{\text{loc}}(x, r, f) = \lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log r_n(B_n(x, r), \delta) \leq \frac{\tau}{2}, \ \forall x \in W'.
$$

For any measurable partition $\xi$ with $\text{diam}(\xi) \leq r$ by proposition 2.1 it holds that

$$h_\nu(f) - h_\nu(f, \xi) \leq \int h_{\text{loc}}(x, r, f) d\nu(x)
$$

and thus

$$
h_\nu(f) - h_\nu(f, \xi) \leq \int_{W} h_{\text{loc}}(x, r, f) d\nu(x) + \int_{M \setminus W} h_{\text{loc}}(x, r, f) d\nu(x)
\leq \frac{\tau}{2} + \eta \cdot C \leq \frac{\tau}{2} + \frac{\tau}{2} \leq \tau.
$$

This completes the proof of proposition 3.5.

We are now turning to the proof of theorem 1.2. \qed

**Proof of theorem 1.2.** For any given $\mu \in M_{\text{inv}}(\Lambda)$ we will show that the entropy map is upper semi-continuous at $\mu$. For $\mu$ and a real $r > 0$, we can choose two constants $r > 0$, $\rho > 0$ as in proposition 3.5 and a partition $\xi$ with $\text{diam}(\xi) < r$ and $\mu(\partial \xi) = 0$. From proposition 3.5, we know that

$$h_\nu(f) - h_\nu(f, \xi) \leq \tau, \ \forall \nu \in B_\rho(\mu) \cap M_{\text{inv}}(\Lambda).$$

Since $h_\nu(f) = \sup_\xi h_\nu(f, \xi)$,

$$h_\nu(f, \xi) - h_\nu(f) \leq 0.$$

Note for a fixed $n$ and a partition $\xi$ with $\mu(\partial \xi) = 0$, $\frac{1}{n} H_n(f, \xi^n)$ is continuous at $\mu$. Thus $h_\nu(f, \xi) = \inf_n \frac{1}{n} H_n(f, \xi^n)$ is upper semi-continuous at $\mu$. Shrink $\rho > 0$ if necessary, we get

$$h_\nu(f, \xi) - h_\mu(f, \xi) \leq \tau, \ \nu \in M_{\text{inv}}(\Lambda) \cap B_\rho(\mu).$$
Therefore,
\[ h_x(f) - h_y(f) = (h_x(f) - h_x(f, \xi)) + (h_x(f, \xi) - h_y(f, \xi)) + (h_y(f, \xi) - h_y(f)) \]
\[ \leq \tau + \tau \]
\[ \leq 2\tau, \quad \nu \in M_{\text{in}}(\Lambda) \cap B_\nu(\mu), \]
which shows that the entropy map is upper semi-continuous at \( \mu \).

4. \( C'(2 \leq r < \infty) \) diffeomorphisms without domination

In this section, by some brief analysis of techniques in [8] we illustrate examples of \( C'(2 \leq r < \infty) \) nonuniformly hyperbolic system without domination for which the entropy map is not upper semi-continuous. For a detailed proof, readers may refer to [8].

We denote \( \text{Diff}^r(M)(2 \leq r < +\infty) \) as the set of \( C^r \) diffeomorphisms on a smooth surface \( M \). We can choose an open subset \( U \subset \text{Diff}^r(M) \) such that each \( f \) in it has a hyperbolic basic set \( \Delta(f) \) with the same adapted neighborhood \( U \subset M \) which has persistent homoclinic tangencies, i.e. there exist \( x, y \in \Delta(f) \) such that \( W^s(x) \) and \( W^u(y) \) have tangencies. This is according to chapter 6 of [18]. Let \( \tilde{H}(f) \) be the set of hyperbolic periodic points \( p \) which are homoclinic related to \( \Delta(f) \) (i.e. \( W^s(\Delta(f)) \setminus \Delta(f) \) and \( W^u(\mathcal{O}(p)) \setminus \mathcal{O}(p) \) have nonempty transverse intersections and vice versa). Denote \( \chi(p) = \frac{1}{\pi(p)} \min[\log|\lambda^{-1}_s(p)|, \log|\lambda_u(p)|] > 0 \), where \(|\lambda_s(p)| < 1 \) and \(|\lambda_u(p)| > 1 \) are the norms of the two eigenvalues of \( D_{f^k}f \) respectively, and \( \pi(p) \) the least period of \( p \). Let \( \mu_p \) be the periodic measure supported on the orbit of \( p \), i.e. \( \mu_p = \frac{1}{\pi(p)} \sum_{i=1}^{\pi(p)-1} \delta_{f^i(p)} \). For an ergodic hyperbolic measure \( \mu \) on \( M \), let \( \chi(\mu) = \min \{ |\chi_s(\mu)|, |\chi_u(\mu)| \} \), where \( \chi_s(\mu), \chi_u(\mu) \) are the two Lyapunov exponents of \( \mu \). In the sequel, by \( (C', \epsilon) \)-perturbation we mean that the perturbation is done in the \( \epsilon \)-neighborhood in \( C' \) topology. By \( C' \) perturbation, we mean \((C', \epsilon)\)-perturbation for any sufficiently small \( \epsilon \).

Fix any \( f \in U \) and any \( p \in \tilde{H}(f) \). Take \( C' \) neighborhood \( V \) of \( f \) in \( U \) such that any \( g \in V \) has a continuation \( g_{|p} \) of \( p_f \) and we omit \( g \) in the subscript for the simplicity of our symbols.

For any \( g \in V \) we first \( C' \)-perturb \( g \) to get a homoclinic tangency for \( \mathcal{O}(p) \) (see lemmas 8.3 and 8.4 in [15]). We may assume \( g \) is both \( r - \) shrinking \( |\lambda_s(g)p| |\lambda_u(g)p| < 1 \) and nonresonant meaning that for any pair of positive integers \( n \) and \( m \) the number \( |\lambda_s^n(g)p| |\lambda_u^m(g)p| \) is different from 1. Then according to proposition 5 and lemma 3 in [10] by a further \( C' \) small perturbation, one can get an interval \( I \) of tangencies between \( W^s(p) \) and \( W^u(p) \). Near this interval we take one more \( C' \) small perturbation which we still denote by \( g \) to create a curve \( J \subset W^u(p) \) with \( N \) bumps as in figure 1.

This perturbation can be done as follows. Denote \( I = \{ a_1 \leq x \leq a_2 : y = A \cos|x-c| \} \). To keep the perturbation to be \( C' \)-small, we only require that \( A' \cdot \omega' \leq \epsilon \), where \( \epsilon \) could be arbitrarily small. For any fixed small \( \epsilon > 0 \), let \( A = \epsilon \left( \frac{a_2 - a_1}{a_2 - a_1} \right)^N \), \( \omega = \frac{x-N}{a_2 - a_1} \), \( c = \frac{a_1 + a_2}{2} \). Without loss of generality, we assume that \( p \) is a fixed point.

To create small hyperbolic basic sets, consider a small rectangle \( D_N \) close to \( I \) with distance less than \( 2^{-\frac{N}{2}} \) and consider the iterations of \( g^{k+1} \) (where \( k \) denote the first \( k \) iterations near \( p \)). To obtain a periodic hyperbolic basic set \( \Delta(p,N) \) by transversal intersections, it is required that
\[
A \cdot |\lambda_s|^l \geq 1, \quad |\lambda_u|^l \leq A,
\]
where by $a \geq b$ we mean that $a \geq \text{const} \cdot b$, and the const is independent of $N$ and $k(N)$ ( $a < b$ is defined similarly), and by $a \approx b$ we mean that both $a \geq b$ and $a \leq b$. In this way we get an $N$-horseshoe with topological entropy $\frac{\log N}{k+T}$. Note that $A = e^{\left(\frac{a_2-a_1}{\log N}\right)} \approx \frac{1}{N}$, so to get (5), $k$ should be large enough such that

$$k \approx -\frac{\log A}{\chi(p)} = \frac{\log (N' \cdot \text{const})}{\chi(p)} = \frac{r \log N}{\chi(p)} + \text{const}$$

and thus

$$h_{\text{top}}(\Delta(p,N), g) = \frac{\log N}{r \frac{\log N}{\chi(p)} + \text{const} + T}.$$ 

For $n \in \mathbb{N}$, choose $N(n)$ large enough such that

$$h_{\text{top}}(\Delta(p,N(n)), g) > \frac{\chi(p)}{r} - \frac{1}{n}.$$ 

By the variational principle, there exists an ergodic measure $\nu_n$ supported on $\Delta(p,N(n))$ such that $h_{\nu_n}(g) > \frac{\chi(p)}{r} - \frac{1}{n}$. By estimating one sees that $D[g^{k+T}]$ expands unstable direction in $\Delta(p,N)$ about $N$ times and contracts the stable direction about $1/N$ times, so $\chi(\mu_n)$ of any ergodic measure $\mu_n$ on $\Delta(p,N(n))$ will be close to $\frac{\log N}{k+T} \approx \frac{\chi(p)}{r}$. Moreover, since by iterations of $g$, $\Delta(p,N)$ spends most of time near $p$, $\mu_n$ is close to the periodic measure $\mu_p$. Let $N(n)$ be larger if necessary such that

$$\chi(\mu_n) > \frac{\chi(p)}{r} - \frac{1}{n} \quad \text{and} \quad d(\mu_n, \mu_p) < \frac{1}{n}.$$ 

Denote $\tilde{\Delta}(p,n) = \Delta(p,N(n))$. 

Figure 1. Creation of small basic sets.
To conclude, for any diffeomorphism \( g \in \mathcal{V} \) and the continuation \( p \), through any \( C^1 \) small perturbation we get a diffeomorphism \( g_n \) satisfying property \( \mathcal{S}_n \):

(i) there exists a hyperbolic basic set \( \mathcal{A}(p, n) \) and an ergodic measure \( \nu_n \) on \( \mathcal{A}(p, n) \) such that

\[
h_{\nu_n}(g_n) > \frac{\chi(p)}{r} - \frac{1}{n},
\]

(ii) for any ergodic measure \( \mu_n \) on \( \mathcal{A}(p, n) \), we have

\[
\chi(\mu_n) > \frac{\chi(p)}{r} - \frac{1}{n} \quad \text{and} \quad d(\mu_n, \mu_p) < \frac{1}{n}.
\]

Denote \( \mathcal{V}_n \) as the subset of \( \mathcal{V} \) satisfying property \( \mathcal{S}_n \). It is obvious that property \( \mathcal{S}_n \) is an open property. From the above discussion, we see that \( \mathcal{V}_n \) is an open dense subset of \( \mathcal{V} \). Let

\[
\mathcal{R} = \bigcap_{n \geq 1} \mathcal{V}_n,
\]

then \( \mathcal{R} \) is a residual subset of \( \mathcal{V} \). For any \( g \in \mathcal{R} \) and the continuation \( p \), there exists a sequence of ergodic measures \( \{\nu_n\} \) such that \( \nu_n \to \mu_p \) and \( \chi(\nu_n) > \frac{1}{r} \chi(p) > 0 \). By definition 1.1, \( \{\nu_n\} \) and \( \mu_p \) are supported on a same Pesin set \( \Lambda = \Lambda\left(\frac{1}{r} \chi(p), \frac{1}{r} \chi(p); \varepsilon\right) \). But at the same time, \( h_{\nu_n}(g) \to \frac{1}{r} \chi(p) > 0 \) while \( h_{\mu_p}(g) = 0 \), which implies that the entropy map of \( f \in \mathcal{R} \) is not upper semi-continuous at \( \mu_p \) on the Pesin set \( \Lambda \).

Note that although for each fixed \( n \), \( \nu_n \) is supported on \( \mathcal{A}(p, n) \) which is uniformly hyperbolic and the angles between \( E^s \) and \( E^u \) are uniformly bounded from below by about \( A \cdot \omega \simeq \frac{1}{\delta(\mu_p) \gamma^{-1}} \), the angles of the Oseledec splittings for the sequence \( \nu_n \), \( n \geq 1 \), may be arbitrary small as \( n \) goes to infinity. Therefore there is no domination between \( E^s \) and \( E^u \) over \( \Lambda \).

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