\(A_{2l}^{(2)}\) AT LEVEL \(-l - \frac{1}{2}\)

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**Abstract.** Let \(L_l = L(\mathfrak{sl}_{2l+1}, -l - \frac{1}{2})\) be the simple vertex operator algebra based on the affine Lie algebra \(\hat{\mathfrak{sl}}_{2l+1}\) at boundary admissible level \(-l - \frac{1}{2}\).

We consider a lift \(\nu\) of the Dynkin diagram involution of \(A_{2l} = \mathfrak{sl}_{2l+1}\) to an involution of \(L_l\). The \(\nu\)-twisted \(L_l\)-modules are \(A_{2l}^{(2)}\)-modules of level \(-l - \frac{1}{2}\) with an anti-homogeneous realization. We classify simple \(\nu\)-twisted highest-weight (weak) \(L_l\)-modules using twisted Zhu algebras and singular vectors for \(\hat{\mathfrak{sl}}_{2l+1}\) at level \(-l - \frac{1}{2}\) obtained by Perse.

We find that there are finitely many such modules up to isomorphism, and the \(\nu\)-twisted (weak) \(L_l\)-modules that are in category \(\mathcal{O}\) for \(A_{2l}^{(2)}\) are semi-simple.

1. Introduction

In [16], while studying the modular invariant representations of affine Lie algebras, Kac and Wakimoto introduced the notion of admissible highest-weight representations and classified these in [17]. Let \(\mathfrak{g}\) be a finite dimensional simple Lie algebra of type \(X_N\), and let \(\hat{\mathfrak{g}}\) be the corresponding un twisted affine Lie algebra of type \(X_N^{(1)}\) (see [15, Table Aff 1]). Since [16], vertex operator algebras (say \(L(\mathfrak{g}, k)\)) based on the untwisted affine Lie algebra \(\hat{\mathfrak{g}}\) at admissible levels \(k\) have received a tremendous amount of attention.

In [2], Adamović and Milas analysed the case of \(\mathfrak{g} = \mathfrak{sl}_2\) for all admissible levels \(k\), classified the weak modules of \(L(\mathfrak{sl}_2, k)\) that belong to category \(\mathcal{O}\) as \(\mathfrak{sl}_2\)-modules and showed that this category is semi-simple with finitely many equivalence classes of irreducibles. They conjectured that this holds for all untwisted affine Lie algebras. The \(\mathfrak{g} = \mathfrak{sl}_2\) case was also studied in [10, 12]. In a celebrated achievement, Arakawa proved this conjecture [4]. Before [4], several other specific cases of this conjecture were known to be true, notably in type \(C\) [1], in type \(A\) [25], in type \(B\) [24] and for \(G_2\) [5]. In [2, 1, 25, 24, 5], the technique of Zhu algebras [29, 14] and an explicit knowledge of the singular vectors at the prescribed levels was used.

It is also important to consider categories larger than \(\mathcal{O}\), namely, the categories generated by relaxed highest weight modules. For \(\mathfrak{g} = \mathfrak{sl}_2\), simple relaxed highest-weight modules at admissible levels were classified using the Zhu technology in [2]. Recently, a classification for arbitrary rank based on Mathieu’s coherent families [23] is presented in [20]. We will not pursue this direction here.

Kac-Wakimoto’s work [16] also included a discussion of affine Lie super-algebras, and indeed models related to \(\mathfrak{osp}(1|2)\) at admissible levels have been analysed in [26, 9, 27]. However, it is not clear if semi-simplicity holds beyond \(\mathfrak{osp}(1|2n)^{1}\), and [4] does not encompass affine super-algebras.

Despite all these stellar advances, the case of twisted affine Lie algebras (see [15, Table Aff 2, Aff 3]) has received little to no attention. The most natural way to access modules for \(X_N^{(r)}\) where \(r = 2, 3\) is by considering \(\nu\)-twisted modules for the VOAs based on the corresponding untwisted...
affine Lie algebras $X_{N}^{(1)}$ [13, 22]. Here, $\nu$ is a lift of the non-trivial Dynkin diagram automorphism to $g$ and $\nu$ fixes the chosen Cartan sub-algebra. $\nu$ is then extended to act on the whole VOA. We may modify $\nu$ by composing it with $\exp(2\pi i \cdot ad h)$ for certain Cartan elements $h$ with $\nu(h) = h$ [15, Eq. 8.1.2]. This way, we get different realizations of $X_{N}^{(r)}$ differing primarily in their gradings.

In this paper, we consider the case of $A_{2l}^{(2)}$ at level $-l - \frac{1}{2}$ for $l \in \mathbb{Z}_{>0}$. We use the anti-homogeneous realization of $A_{2l}^{(2)}$ obtained from an involutive lift $\nu$ of the Dynkin diagram automorphism of $g = sl_{2l+1} = A_{2l}$. Here, anti-homogeneous refers to the fact that our picture is exactly the opposite of the traditional one – our affine, i.e., 0th node for $A_{2l}^{(2)}$ is what is usually the last, i.e., $l$th node in the affine Dynkin diagram, and our horizontal subalgebra is thus $so_{2l+1} = B_l$ and not $sp_{2l} = C_l$.

We use twisted Zhu algebras [11] (see also [28]) and the singular vectors for $\tilde{A}_{2l+1}$ at level $-l - \frac{1}{2}$ obtained by Perše in [25]. Somewhat surprisingly, we find that the top spaces of the $A_{2l}^{(2)}$ modules (which are naturally modules for our horizontal subalgebra, $B_l$) are exactly the same as the top spaces for the highest-weight $L(B_l, -l + \frac{3}{2})$-modules found in [24]. Letting $h^{\vee}$ denote the dual Coxeter number [15, Ch. 6], the relation between these levels for $l > 1$ is that

$$-l - \frac{1}{2} + h_{A_{2l}^{(2)}}^{\vee} = l + \frac{1}{2} = -l + \frac{3}{2} + h_{B_l}^{\vee}. \quad (1.1)$$

Our proof of admissibility of the $A_{2l}^{(2)}$ highest weights thus obtained also uses a large portion of the corresponding proof in [24]. The proof of semi-simplicity then proceeds as in [2, 1, 25, 24, 5], etc., with appropriate changes to accommodate twisted modules.

We find that there are two inequivalent $\nu$-twisted irreducible modules for $L(sl_{2l+1}, -l - \frac{1}{2})$ with finite dimensional top spaces (Remark 4.8). Recall that $-l - \frac{1}{2}$ is a boundary admissible level [19] for $A_{2l}^{(1)}$ and correspondingly, there is exactly one (up to equivalence) irreducible with finite dimensional top space in the untwisted sector [25].

This naturally leads to the following speculations and considerations that we are currently investigating.

(1) Perhaps the most important speculation we have is that the Adamović-Milas conjecture / Arakawa’s theorem is true for twisted affine Lie algebras as well. To be precise, we speculate that given a twisted affine Lie algebra $X_{N}^{(r)}$, and an admissible level $k$ for (the untwisted) $X_{N}^{(1)}$, there exists an appropriate realization of $X_{N}^{(r)}$ and a corresponding lift $\nu$ of the (non-trivial) diagram automorphism of $X_{N}$, such that $\nu$-twisted (weak) $L(X_{N}, k)$-modules which are in category $\mathcal{O}$ as $X_{N}^{(r)}$-modules form a semi-simple category with finitely many irreducibles.

(2) In [8], it was proved that the ordinary modules for $L(X_{N}, k)$ $k$ is admissible level for the untwisted affine Lie algebra $X_{N}^{(1)}$, form a vertex tensor category; the rigidity of this category for the simply-laced cases was proved in [7]. Our results imply that in general, the category consisting of untwisted and $g$-twisted ordinary modules for $g \in \langle \nu \rangle$ will not be closed under twisted fusion.

In our present case, the untwisted and $\nu$-twisted ordinary modules form semi-simple categories, but the untwisted sector has one simple (up to equivalences) and the $\nu$-twisted one has two inequivalent simples. The aforementioned closure under twisted fusion is now forbidden by elementary considerations of tensor categories.

In general, such ordinary $g$-twisted modules are integrable in the direction of $g^{0}$ (the fixed point subalgebra of $g = X_{N}$ under $\nu$), thus it is natural to expect their (twisted) fusion to be integrable with respect to $g^{0}$, but it need not be $g$-integrable.
It will be difficult but interesting to work out the twisted fusion for our \( \nu \)-twisted modules, and perhaps also the fusion for the untwisted modules for the corresponding orbifold. Here, the structure of this orbifold \([3]\) will be important to first classify its modules.

(3) It will be very interesting to also analyse twisted quantum Drinfeld-Sokolov reductions \([18]\) of the \( \nu \)-twisted modules we have found for appropriate nilpotents \( f \in \mathfrak{sl}_{2l+1} \) fixed by \( \nu \), and compare these to twisted representations of the corresponding \( W \)-algebras. Here again, one may take a slightly different route and investigate the relation of the structure and representation theory of the affine orbifold with that of the \( W \)-algebra orbifold.

2. Twisted affine Lie algebra \( A_{2l}^{(2)} \)

2.1. Twisted affine Lie algebra \( A_{2l}^{(2)} \), basics. We will consider what we call the anti-homogeneous realization of \( A_{2l}^{(2)} \) and recall basic facts from \([15, 6]\). Consider the generalized Cartan matrices:

\[
\bar{A} = \begin{pmatrix}
0 & 1 \\
2 & -1 \\
1 & -4 \\
-2 & 2 \\
\end{pmatrix}, \ (l = 1), \quad (2.1)
\]

\[
\bar{A} = \begin{pmatrix}
0 & 1 & 2 & \cdots & l-2 & l-1 & l \\
2 & -1 & & & & & \\
1 & -2 & 2 & -1 & & & \\
2 & -1 & 2 & \ddots & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
l-2 & & & & & & \\
l-1 & & \ddots & 2 & -1 & & \\
l & & & -2 & 2 & & \\
\end{pmatrix}, \ (l > 1). \quad (2.2)
\]

We have the corresponding (affine) Dynkin diagrams:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (ls) at (0,0) {$1$}; \node (l2) at (1,0) {$2$}; \node (l0) at (0,-2) {$0$}; \node (l1) at (1,-2) {$1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \end{tikzpicture}
\end{array} \quad l=1
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (ls) at (0,0) {$1$}; \node (l2) at (1,0) {$2$}; \node (l0) at (0,-2) {$0$}; \node (l1) at (1,-2) {$1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \node (ls) at (2,0) {$2$}; \node (l2) at (3,0) {$2$}; \node (l0) at (2,-2) {$0$}; \node (l1) at (3,-2) {$1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \node (ls) at (4,0) {$2$}; \node (l2) at (5,0) {$2$}; \node (l0) at (4,-2) {$0$}; \node (l1) at (5,-2) {$1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \node (ls) at (6,0) {$2$}; \node (l2) at (7,0) {$2$}; \node (l0) at (6,-2) {$0$}; \node (l1) at (7,-2) {$1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \node (ls) at (8,0) {$2$}; \node (l2) at (9,0) {$2$}; \node (l0) at (8,-2) {$0$}; \node (l1) at (9,-2) {$1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \end{tikzpicture}
\end{array} \quad l>1
\]

Here, the \( 0 \)-th node is considered to be the affine node and the horizontal subalgebra of \( A_{2l}^{(2)} \) is of type \( B_l = \mathfrak{so}_{2l+1} \) (unlike the usual convention where it turns out to be \( C_l = \mathfrak{sp}_{2l} \)):

\[
\begin{array}{c}
\begin{tikzpicture}
\node (ls) at (0,0) {$\alpha_1$}; \node (l2) at (1,0) {$\alpha_2$}; \node (l0) at (0,-2) {$\alpha_0$}; \node (l1) at (1,-2) {$\alpha_1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \end{tikzpicture}
\end{array} \quad l=1
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (ls) at (0,0) {$\alpha_1$}; \node (l2) at (1,0) {$\alpha_2$}; \node (l0) at (0,-2) {$\alpha_0$}; \node (l1) at (1,-2) {$\alpha_1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \node (ls) at (2,0) {$\alpha_3$}; \node (l2) at (3,0) {$\alpha_3$}; \node (l0) at (2,-2) {$\alpha_0$}; \node (l1) at (3,-2) {$\alpha_1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \node (ls) at (4,0) {$\alpha_{l-2}$}; \node (l2) at (5,0) {$\alpha_{l-2}$}; \node (l0) at (4,-2) {$\alpha_0$}; \node (l1) at (5,-2) {$\alpha_1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \node (ls) at (6,0) {$\alpha_{l-1}$}; \node (l2) at (7,0) {$\alpha_{l-1}$}; \node (l0) at (6,-2) {$\alpha_0$}; \node (l1) at (7,-2) {$\alpha_1$}; \draw (ls) -- (l2); \draw (l0) -- (l1); \end{tikzpicture}
\end{array} \quad l>1
\]

We have:

\[
a = (a_0, \ldots, a_l)^t = (1, 2, \ldots, 2)^t, \quad \bar{A}a = 0 \quad (2.5)
\]

\[
a^\vee = (a_0^\vee, \ldots, a_l^\vee) = (2, 2, \ldots, 2, 1), \quad a^\vee \bar{A} = 0. \quad (2.6)
\]

We will often use the following indexing sets:

\[
I = \{1, \ldots, l\}, \quad \hat{I} = \{0, \ldots, l\}. \quad (2.7)
\]

The twisted affine Lie algebra \( A_{2l}^{(2)} \) has Kac-Moody generators \( h_i, e_i, f_i \ (i \in \hat{I}) \) and \( d \) satisfying the usual relations \([15]\). We let the Cartan subalgebra \( \mathfrak{h} \) be spanned by \( h_0, h_1, \ldots, h_l, d \). The simple
roots \( \alpha_i \ (i \in \tilde{I}) \) are elements of \( \mathfrak{h}^* \). We will sometimes denote the pairing between \( \mathfrak{h}^* \) and \( \mathfrak{h} \) by \( (\cdot, \cdot) \). This notation will be overloaded below. For \( i, j \in \tilde{I}, \ k \in I \) we have:
\[
\alpha_i(h_j) = (\alpha_i, h_j) = \tilde{A}_{ij}, \quad \alpha_0(d) = (\alpha_0, d) = 1, \quad \alpha_k(d) = (\alpha_k, d) = 0. \tag{2.8}
\]
The canonical central element \( c \in \mathfrak{h} \) of \( A_{2l}^{(2)} \) and the basic imaginary root \( \delta \) are expressed as:
\[
c = \sum_{0 \leq i \leq l} a_i^\vee h_i = 2h_0 + \cdots + 2h_{l-1} + h_l, \quad \delta = \sum_{0 \leq i \leq l} a_i \alpha_i = \alpha_0 + 2\alpha_1 \cdots + 2\alpha_l. \tag{2.9}
\]
We choose \( h_1, \ldots, h_l, c, d \) as a standard basis for \( \mathfrak{h} \). We have:
\[
\delta(c) = 0, \ \delta(h_1) = 0, \ldots, \ \delta(h_l) = 0, \ \delta(d) = 1. \tag{2.10}
\]
We also consider \( \Lambda_0^c \in \mathfrak{h}^* \) such that:
\[
\Lambda_0^c(c) = 1, \ \Lambda_0^c(h_1) = 0, \ldots, \ \Lambda_0^c(h_l) = 0, \ \Lambda_0^c(d) = 0. \tag{2.11}
\]
Note that \( \Lambda_0^c \) is \( \frac{1}{2} \Lambda_0 \), where \( \Lambda_0 \) is the fundamental weight corresponding to the 0th node. It is easy to see that \( \alpha_1, \ldots, \alpha_l, \delta, \Lambda_0^c \) form a basis of \( \mathfrak{h}^* \). The standard symmetric (non-degenerate) bilinear form on \( \mathfrak{h} \) is given by:
\[
(h_i, h_j) = (\alpha_i, h) \cdot \frac{a_i^\vee}{a_i}, \quad i \in \tilde{I}, \ h \in \mathfrak{h}, \quad (d, d) = 0. \tag{2.12}
\]
The non-degenerate map \( (\cdot, \cdot) \) leads to a (linear) isomorphism \( \iota : \mathfrak{h} \to \mathfrak{h}^* \) with \( i \in \tilde{I} \):
\[
(\iota(h), h_1) = (h, h_1), \ \text{for all } h, h_1 \in \mathfrak{h}, \tag{2.13}
\]
\[
\iota(h_i) = \frac{a_i^\vee}{a_i} \cdot \alpha_i, \quad \iota(c) = \delta, \quad \iota(d) = \Lambda_0^c. \tag{2.14}
\]
We may thus get a (non-degenerate) symmetric bilinear form on \( \mathfrak{h}^* \) by transport of structure. It satisfies \( \iota(i, j) \in \tilde{I} \) and \( k \in I \):
\[
(\alpha_i, \alpha_j) = \tilde{A}_{ij} \cdot \frac{a_i^\vee}{a_i}, \quad (\delta, \alpha_k) = (\delta, \delta) = (\Lambda_0^c, \alpha_k) = (\Lambda_0^c, \Lambda_0^c) = 0, \quad (\delta, \Lambda_0^c) = 1. \tag{2.15}
\]
The squared root lengths are therefore \( (k = 1, \ldots, l - 1) \):
\[
(\alpha_0, \alpha_0) = 4, \quad (\alpha_k, \alpha_k) = 2, \quad (\alpha_l, \alpha_l) = 1. \tag{2.16}
\]

The root system of \( A_{2l}^{(2)} \) depends on \( l \). Let \( l > 1 \). The root system of the horizontal subalgebra \( B_l \) can be realized as (with \( k = 1, \ldots, l - 1 \) and \( i, j \in I \)):
\[
\alpha_k = \epsilon_k - \epsilon_{k+1}, \quad \alpha_l = \epsilon_l, \quad \text{where} \ (\epsilon_i, \epsilon_j) = \delta_{ij}. \tag{2.17}
\]
We have:
\[
\Phi^{\text{long}} = \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq l \}, \quad \Phi^{\text{short}} = \{ \pm \epsilon_i \mid i = 1, \ldots, l \}, \tag{2.18}
\]
and the real roots for \( A_{2l}^{(2)} \) are \([6]\):
\[
\widehat{\Delta}^\text{re} = \tilde{\Phi}^{\text{long}} \cup \tilde{\Phi}^{\text{intermediate}} \cup \tilde{\Phi}^{\text{short}}
\]
\[
= \{2\alpha_s + (2m + 1)\delta \mid \alpha \in \Phi_s, m \in \mathbb{Z}\} \cup \{\alpha + m\delta \mid \alpha \in \Phi_l, m \in \mathbb{Z}\} \cup \{\alpha + m\delta \mid \alpha \in \Phi_s, m \in \mathbb{Z}\} \tag{2.19}
\]
where the squared norms of roots in the respective sets are 4, 2 and 1. With \( k = 1, \ldots, l - 1 \), the fundamental weights of the horizontal subalgebra are:
\[
\omega_k = \epsilon_1 + \cdots + \epsilon_k, \quad \omega_l = \frac{1}{2}(\omega_1 + \cdots + \omega_l). \tag{2.20}
\]
For $l = 1$, the horizontal subalgebra is $\mathfrak{sl}_2$ with simple positive root $\alpha_1$, and we have:

$$
\hat{\Delta}^{\text{re}} = \hat{\Phi}^{\text{long}} \cup \hat{\Phi}^{\text{short}} = \{ \pm 2\alpha_1 + (2m+1)\delta \mid m \in \mathbb{Z} \} \cup \{ \pm \alpha_1 + m\delta \mid m \in \mathbb{Z} \}.
$$

(2.21)

Here, note that $(\alpha_1, \alpha_1) = 1$, and thus squared norms of the roots in these sets are 4 and 1, respectively. The fundamental weight for the horizontal algebra is $\rho_1 = \frac{1}{2}\alpha_1$.

Let $\rho$ be any element of $\mathfrak{h}^*$ satisfying $\rho(h_i) = 1$ for all $i \in \hat{I}$. We may take it to be: $\rho = h^\vee \Lambda_0^\vee + \overline{\rho}$ where $\overline{\rho}$ is half the sum of positive roots of the horizontal sub-algebra and $h^\vee = a_0^\vee + \cdots + a_l^\vee = 2l + 1$ is the dual Coxeter number of $A^{(2)}_l$. For $l = 1, \overline{\rho} = \frac{1}{2}\rho_1$. If $l > 1$, we have:

$$
\overline{\rho} = \left(l - \frac{1}{2}\right) \epsilon_1 + \left( l - \frac{3}{2} \right) \epsilon_2 + \cdots + \frac{1}{2} \epsilon_l.
$$

(2.22)

Finally, recall the notion of Weyl group $W$ generated by reflections $r_i$ ($i \in \hat{I}$) satisfying $r_i(h) = h - (\alpha_i, h)h_i$ for $h \in \mathfrak{h}$ and we transfer the action to $\mathfrak{h}^*$ by $\iota$. We have $W \cdot \{ \alpha_0, \ldots, \alpha_l \} = \hat{\Delta}^{\text{re}}$ and we define $\hat{\Delta}^{\text{re}} = W \cdot \{ h_0, \ldots, h_l \}$, which is the set of real coroots. There is thus a bijection from real roots to real coroots denoted by $\iota$ such that $\alpha_i \mapsto \alpha_i^\vee = h_i$, and it is not hard to prove, using the invariance of $\langle \cdot, \cdot \rangle$ under the Weyl group that for $\lambda \in \mathfrak{h}^*, \alpha \in \hat{\Delta}^{\text{re}}$ that

$$
\langle \lambda, \alpha^\vee \rangle = \frac{2}{(\alpha, \alpha)} \langle \lambda, \alpha \rangle.
$$

(2.23)

Given $\lambda \in \mathfrak{h}^*$, define

$$
\hat{\Delta}^{\text{re}} = \{ \alpha^\vee \in \hat{\Delta}^{\text{re}} \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \}, \quad \hat{\Delta}^{\text{re}} = \{ \alpha \in \hat{\Delta}^{\text{re}} \mid \alpha^\vee \in \hat{\Delta}^{\text{re}} \}.
$$

(2.24)

**Definition 2.1.** [16] We say an element $\lambda \in \mathfrak{h}^*$ is an admissible weight if:

1. $(\lambda + \rho, \alpha^\vee) \not\in \{0, -1, -2, \cdots \}$ for all $\alpha^\vee \in \hat{\Delta}^{\text{re}}$
2. $\mathbb{Q} \hat{\Delta}^{\text{re}} = \mathbb{Q} \{ h_0, \ldots, h_l \}$.

**Remark 2.2.** The second condition can be equivalently replaced with $\mathbb{Q} \hat{\Delta}^{\text{re}} = \mathbb{Q} \{ \alpha_0, \ldots, \alpha_l \}$.

2.2. **Twisted affinizations of Lie algebras.** Suppose we are given a finite dimensional (simple) Lie algebra $\mathfrak{g}$ with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Let $\nu$ be an automorphism of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ of a finite order, say $T$. Corresponding to $\nu$, we have the eigen-decomposition

$$
\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/T\mathbb{Z}} \mathfrak{g}^j, \quad x \in \mathfrak{g}^j \iff \nu(x) = e^{2\pi ij/T} x.
$$

(2.25)

Consider the affinization:

$$
\hat{\mathfrak{g}}^{1/T} = \mathfrak{g} \otimes \mathbb{C}[t^{1/T}, t^{-1/T}] \oplus \mathbb{C}c.
$$

(2.26)

We will often drop the superscript $1/T$ since it will be clear from the context. The element $c$ is central and the other brackets are $(a, b \in \mathfrak{g}, m, n \in \frac{1}{T}\mathbb{Z})$:

$$
[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\delta_{m+n,0}(a, b)c.
$$

(2.27)

Define $\nu(t^{1/T}) = e^{-2\pi ij/T} t^{1/T}$ and extend linearly to $\mathbb{C}[t^{1/T}, t^{-1/T}]$. Also let $\nu(c) = c$. We are interested in the fixed point sub-algebra

$$
\hat{\mathfrak{g}}[\nu] = \bigoplus_{j \in \mathbb{Z}/T\mathbb{Z}} \left( \mathfrak{g}^j \otimes t^{1/T} \mathbb{C}[t, t^{-1}] \right) \oplus \mathbb{C}c.
$$

(2.28)

We shall obtain $A^{(2)}_l$ via such twisted affinization of $\mathfrak{g} = A_{2l} = \mathfrak{sl}_{2l+1}$. 
2.3. **Anti-homogeneous realization of $A_{2l}^{(2)}$.** We start by fixing some notation. Fix $l \in \mathbb{Z}_{>0}$. Consider $\mathfrak{g}_{2l+1}$ spanned by elementary matrices $E_{i,j}$ (or simply $E_{ij}$) with 1 in row $i$ and column $j$, zeros everywhere else. Let $E_i = E_{i,i+1}$, $F_i = E_{i+1,i}$, $H_i = E_{i,i} - E_{i+1,i+1}$ be the standard choices of simple root vectors and simple coroots for $\mathfrak{g} = \mathfrak{sl}_{2l+1} \subset \mathfrak{g}_{2l+1}$. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$. Let $E_0 = E_{1,2l+1} = [\cdots [E_1, E_2], E_3, \cdots , E_{2l}]$.

The anti-homogeneous realization is achieved via an involutive lift of the diagram automorphism of $\mathfrak{g}_{2l+1}$ which we now describe.

Define $\nu(E_{i,j}) = -(1)^{i-j} E_{2l+2-j,2l+2-i}$. It is straightforward to prove that $\nu$ is an involution of $\mathfrak{g}_{2l+1}$ and also of the Lie subalgebra $\mathfrak{g} = \mathfrak{sl}_{2l+1}$. Corresponding to $\nu$ we have the decomposition $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ (where the superscripts are understood as elements of $\mathbb{Z}_2$). It is clear that $\nu(H_i) = H_{2l+1-i}$, for $i = 1, \ldots , 2l$, and thus $\nu$ is an involutive lift of the Dynkin diagram automorphism of $\mathfrak{g}$. Observe that $E_0 \in \mathfrak{g}^1$.

The fixed points $\mathfrak{g}^0$ form a simple Lie algebra of type $B_l = \mathfrak{so}_{2l+1}$ with the following Chevalley generators [6].

$$e_i = E_i + E_{2l+1-i}, \quad f_i = F_i + F_{2l+1-i}, \quad h_i = H_i + H_{2l+1-i}, \quad (i = 1, \ldots , l - 1),$$

$$\overline{c} = \sqrt{2}(E_{l} + E_{l+1}), \quad \overline{t}_i = \sqrt{2}(F_i + F_{i+1}), \quad \overline{h}_i = 2(H_i + H_{i+1}).$$

(2.29)

For convenience, we denote:

$$e_l = E_l + E_{l+1}, \quad f_l = F_l + F_{l+1}, \quad h_l = H_l + H_{l+1}.$$  

(2.30)

Note again that the actual generators for $B_l$ (which will also get promoted to a subset of generators for $A_{2l}^{(2)}$ below) are indeed $e_1, \ldots , e_{l-1}, \overline{c}, h_1, \ldots , h_{l-1}, \overline{h}_i, f_1, \ldots , f_{l-1}, \overline{t}_i$. We have introduced $e_l, f_l, h_l$ only to save ourselves from keeping track of the various scalars.

Given any $a \in \mathfrak{g}$, we let

$$a = a^+ + a^-, \quad a^+ = \frac{1}{2}(a + \nu a) \in \mathfrak{g}^0, \quad a^- = \frac{1}{2}(a - \nu a) \in \mathfrak{g}^1.$$  

(2.31)

We let $\mathfrak{n}^0 = \mathfrak{n}_-^0 \oplus \mathfrak{h}^0 \oplus \mathfrak{n}_+^0$ be the triangular decompositions with respect to our choices of root vectors. Note that $\mathfrak{n}_+^0$ is spanned by $E_{i,j}$ for $1 \leq i < j \leq 2l + 1$ and that $\dim(\mathfrak{n}^0) = l$.

Later, we shall require the dimension of weight 0 space of $\mathfrak{g}^1$ as a $\mathfrak{g}^0$-module. One may calculate this directly by decomposing $\mathfrak{g}$ with respect to $\mathfrak{g}^0$. Here we present one more approach. Temporarily, let $L(\omega)$ denote irreducible $\mathfrak{g}^0 \cong \mathfrak{so}_{2l+1}$ module with highest weight $\omega$. As a $\mathfrak{g}^0$-module, $\mathfrak{g}^1 \cong L(2\omega_1)$ and is generated by the highest weight vector $E_0$. Further, $\text{Sym}^2 L(\omega_1) \cong L(2\omega_1) \oplus \mathbb{C}$ and $L(\omega_1)$ is the defining representation of dimension $2l + 1$. It can be seen that if $\omega$ is a weight of $L(\omega_1)$ then so is $-\omega$, $0$ is a weight, and every weight space is one dimensional. Thus, the 0 weight space of $\text{Sym}^2 L(\omega_1)$ has dimension $l + 1$, and the 0 weight space of $\mathfrak{g}^1 \cong L(2\omega_1)$ has dimension $l$.

Now, $\mathfrak{g}_{2l+1}[\nu]$ gives us an anti-homogeneous realization of $A_{2l}^{(2)}$. Considering the numbering from (2.3), we let the Kac-Moody generators to be the ones given in (2.29) for $i = 1, \ldots , l$. As for $h_0, e_0, f_0$, we take them to be:

$$h_0 = -H_l + \frac{1}{2}c = -(H_1 + \cdots + H_{2l}) + \frac{1}{2}c, \quad e_0 = E_{2l+1,1} \otimes c^{1/2}, \quad f_0 = E_{1,2l+1} \otimes c^{-1/2}.$$  

(2.32)

The involution $\nu$ extends to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and we have $\mathcal{U}(\mathfrak{g}^0) \subseteq \mathcal{U}(\mathfrak{g})^0 \subseteq \mathcal{U}(\mathfrak{g})$. Later, we will be interested in certain two-sided ideals $I \subset \mathcal{U}(\mathfrak{g}^0)$.

**Remark 2.3.** It is possible to achieve this realization of $A_{2l}^{(2)}$ by using the Chevalley involution [15, Eq. 1.3.4] of $A_{2l}$. However, it is convenient to use an automorphism that respects the triangular decomposition of $\mathfrak{g}$.  

6
3. Twisted Zhu algebra: Preliminaries

Let \((V, Y, 1, \omega)\) be a vertex operator algebra [21] and let \(g\) be an automorphism of finite order \(T\) of \(V\). Let \(V^j\) \((j = 0, \ldots, T - 1)\) be the subspace of eigenvalue \(e^{2\pi i j/T}\) for \(g\). Following [11], we now define the twisted Zhu algebra \(A_g(V)\) as follows. Let \(u \in V^j\) \((0 \leq j < T)\) be \(L(0)\)- and \(g\)-homogeneous element and let \(v \in V\). Define

\[
u \circ_g v = \text{Res}_{x} \left( \frac{(1 + x)^{wt_u - 1 + \delta_j + \frac{j}{T}} Y(u, x)v}{x^{1 + \delta_j}} \right), \tag{3.1}\]

\[
u \ast_g v = \begin{cases} \text{Res}_{x} \left( \frac{(1 + x)^{wt_u}}{x} Y(u, x)v \right) & \text{if } j = 0 \\ 0 & \text{if otherwise.} \end{cases} \tag{3.2}\]

where we take \(\delta_j = 1\) when \(j = 0\) and \(\delta_j = 0\) if \(j \neq 0\). Extend \(\circ_g, \ast_g\) to \(V\) linearly. Further define

\[O_g(V) = \text{Span}\{\nu \circ_g v \mid u, v \in V\}, \quad A_g(V) = V/O_g(V).\]  \tag{3.3}\]

Taking \(v = 1\) in (3.1) immediately gives us:

\[V^i \subset O_g(V) \text{ if } i \not\equiv 0 \pmod{T}.\]  \tag{3.4}\]

We will denote the image in \(A_g(V)\) of \(v \in V\) by \([v]_g\). It was shown in [11] that \(O_g(V)\) is a two-sided ideal with respect to \(\ast_g\) and that \(A_g(V)\) is an associative algebra under product \(\ast_g\) with \([1]\) as the unit and \([\omega]\) belonging to the center. When \(g = 1\), \(\circ_g, \ast_g, O_g(V), A_g(V)\) are simply denoted as \(\circ, \ast, O(V), A(V)\), respectively. We recall the following basic theorems (and their twisted analogues) from [29, 14, 11, 28].

**Theorem 3.1.** We have:

1. [14, 28] Let \(I\) be a \(g\)-stable ideal of \(V\), and suppose \(1 \not\in I, \omega \not\in I\). Then, the image of \(I\) in \(A_g(V)\), denoted as \(A_g(I)\) is a two-sided ideal. Moreover, \(A_g(V/I) \cong A_g(V)/A_g(I)\).

2. [11, Thm. 7.2] There is a bijective correspondence between the set of equivalence classes of simple \(A_g(V)\) modules and weak, \(\frac{1}{T}\)-\(\mathbb{Z}\)-gradable \(g\)-twisted \(V\)-modules (see [11, Def. 3.3], where these modules are called admissible, not to be confused with [16]).

**Remark 3.2.** The first part of the theorem above is proved for \(g = 1\) (the untwisted case) in [14]. It is not hard to extend the proof to general \(g\) [28].

Now, for the rest of the section, let \(V = V(\mathfrak{g}, k)\) be the (universal) Verma module vertex operator algebra based on \(\langle \mathfrak{g}, \langle \cdot, \cdot \rangle \rangle\) with level \(k \neq -h^\vee\) [21]. Let \(g\) be an automorphism of \(V\) order \(T \neq 1\) lifted from an automorphism \(\tilde{g}\) of \(\mathfrak{g}, \langle \cdot, \cdot \rangle\) of the same order \(T\).

**Theorem 3.3.** We have:

1. [28] There exists an (explicit) isomorphism of associative algebras \(F : A_g(V(\mathfrak{g}, k)) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{g}^0)\).

2. [14] Let \(x \in \mathfrak{g}^0\) and \(v \in V\). Then,

\[F([x(0)v]_g) = [x, [v]_g],\]  \tag{3.5}\]

where both sides are zero if \(v \in V^1 \oplus \cdots \oplus V^T\).

3. [14] Let \(x_1, x_2, \ldots, x_m \in \mathfrak{g}^0, n_1, n_2, \ldots, n_m \in \mathbb{Z}_{\geq 0}\). Then under the isomorphism above,

\[F([x_1(-n_1 - 1)x_2(-n_2 - 1)\cdots x_m(-n_m - 1)]_g) = (-1)^{n_1 + n_2 + \cdots + n_m} x_1x_2x_3\cdots x_m.\]  \tag{3.6}\]

4. The previous part immediately implies that for \(x \in \mathfrak{g}^0, n \in \mathbb{Z}_{\geq 0}\) and \(v \in V\), we have:

\[F([x(-n - 1)v]_g) = (-1)^n F([v]_g) x.\]  \tag{3.7}\]

Henceforth, we will suppress \(F(\cdots)\) and simply identify \(A_g(V(\mathfrak{g}, k))\) with \(\mathfrak{U}(\mathfrak{g}^0)\).
Definition 3.4. We have an action $L$ of $g^0$ on $\mathfrak{U}(g^0)$ given by $x_L u = [x, u]$ where $x \in g^0$, $u \in \mathfrak{U}(g^0)$. We may and do extend the action $L$ of $g^0$ to an action of $\mathfrak{U}(g^0)$.

Theorem 3.5. Suppose that the (unique) maximal $g$-submodule $J(g, k)$ of $V(g, k)$ is generated by a single $g$-homogeneous singular vector $v$. Let $\mathfrak{U}(g)$ be the $g$-module generated by $v$ where $x \in g$ acts on $v$ by $x(0)$. Let

$$\mathcal{R} = [\mathfrak{U}(g)v]_g = [(\mathfrak{U}(g)v) \cap V(g, k)^0]_g = [(\mathfrak{U}(g)v)^0]_g.$$  

(3.8)

We have the following.

1. $\mathcal{R}$ is a finite-dimensional module for $\mathfrak{U}(g^0)$ under the $L$ action.

2. Let $L(g, k) = V(g, k)/J(g, k)$ be the unique simple quotient of $V(g, k)$. Then,

$$A_g(L(g, k)) = \frac{\mathfrak{U}(g^0)}{\langle \mathcal{R} \rangle}$$  

(3.9)

where $\langle \mathcal{R} \rangle$ denotes two sided ideal of $\mathfrak{U}(g^0)$ generated by $\mathcal{R}$.

3. A $g^0$-module $M$ is a $A_g(L(g, k))$-module iff $\langle \mathcal{R} \rangle \cdot M = 0$.

Proof. This theorem is analogous to the corresponding theorems in the untwisted setup, and in the twisted setting, our proof is very similar to the proof of [28, Thm. 6.3].

All elements of $\mathcal{R}$ have the same conformal weight as that of $v$, and each conformal weight space of $V(g, k)$ is finite-dimensional, hence $\mathcal{R}$ is finite-dimensional. The fact that $\mathcal{R}$ is closed under $L$ action is immediate from (3.5).

For the second assertion, it is enough prove that $[J(g, k)]_g = \langle \mathcal{R} \rangle$. Observe that if $X$ is a subspace of $\mathfrak{U}(g^0)$ that is closed under the $L$ action and also under the right multiplication by $\mathfrak{U}(g^0)$ then, $X$ is a two-sided ideal. Indeed, for $a \in g^0$ and $x \in X$, we have $ax = [a, x] − xa$, and both terms on the right-hand side belong to $X$, giving us the closure of $X$ under the left-action. In light of (3.5), $[J(g, k)]_g$ is closed under $L$ and (3.7) implies that it is also closed under the right action of $\mathfrak{U}(g^0)$. Thus, it is a two sided ideal. Clearly, $\mathcal{R} \subset [J(g, k)]_g$, and thus $\langle \mathcal{R} \rangle \subset [J(g, k)]_g$.

For the reverse inclusion, note that $J(g, k)$ is spanned by terms of the sort

$$y = a_1(-n_1 − 1)a_2(-n_2 − 1)\cdots a_t(-n_t − 1)x,$$

(3.10)

where $a_i \in g$ are $g$-homogeneous and are arranged so that all $a_i$’s in $g^0$ are to the right, $n_i \in \mathbb{Z}_{\geq 0}$ and $x$ is a $g$-homogeneous element of $\mathfrak{U}(g)v$. We proceed by induction on $t$. The case for $t = 0$ is clear: $[x]_g \in \mathcal{R}$. Now let $t > 0$. If all $a_i$’s and $x$ are already fixed by $g$ then (3.7) immediately tells us that $[y]_g \in (\mathcal{R})$. Suppose that $y$ is fixed by $g$ (otherwise its projection is 0 anyway) and that $a_1$ is in $g^r$, $1 \leq r \leq T − 1$. We have the following relation [11]:

$$\text{Res}_x \frac{(1 + x)^{r/T}}{x^{m+1}} Y(a_1(-1)1, x)v = a_1(-m − 1)v + \frac{r}{T}a_1(-m)v + \cdots + O_g(V(g, k))$$  

(3.11)

for all $v \in V(g, k)$ and $m \geq 0$. Repeating this relation for $m = n_1, n_1 − 1, \ldots$, it is clear that for some scalar $\alpha$,

$$y \equiv_{O_g(V(g, k))} \alpha \cdot a_1(0)a_2(-n_2 − 1)\cdots a_t(-n_t − 1)x + \text{shorter terms}$$  

(3.12)

$$\equiv_{O_g(V(g, k))} \alpha \cdot a_2(-n_2 − 1)\cdots a_t(-n_t − 1)a_1(0)x + \text{shorter terms}.$$  

(3.13)

We may similarly peel off all the elements $a_2, \ldots, a_j$ which are not in $g^0$ and put them near $x$. We thus see, for some scalar $\alpha'$:

$$y \equiv_{O_g(V(g, k))} \alpha' \cdot a_{j+1}(-n_2 − 1)\cdots a_t(-n_t − 1)\cdot a_j(0)\cdots a_1(0)x + \text{shorter terms}.$$  

(3.14)

Since $y$ and $a_{j+1} \cdots a_t$ are all fixed by $g$, $a_j(0)\cdots a_2(0)a_1(0)x \in (\mathfrak{U}(g)v)^0$. Now, again, (3.7) and induction hypothesis give us that $[y]_g \in \langle \mathcal{R} \rangle$. □
Now we recall a couple of important results that form the basis of all our calculations.

**Definition 3.6.** Recall that $\mathcal{R}$ is a $\mathfrak{g}^0$ module under the $L$ action. We have already chosen a Cartan subalgebra for $\mathfrak{g}^0$, namely $\mathfrak{h}^0$. Let $\mathcal{R}_0$ be the weight 0 subspace of $\mathcal{R}$ with respect $\mathfrak{h}^0$.

**Theorem 3.7.** ([2, Lem. 3.4.3], [24, Prop. 13]) Let $L(\lambda)$ be an irreducible highest-weight $\mathfrak{g}^0$-module with highest-weight $\lambda$ and a highest-weight vector $v_\lambda$. The following statements are equivalent.

1. $L(\lambda)$ is an $A_g(L(\mathfrak{g}, k))$-module.
2. $\mathcal{R} \cdot L(\lambda) = 0$.
3. $\mathcal{R}_0 \cdot v_\lambda = 0$.

**Definition 3.8.** In the notation of the previous theorem, for every $r \in \mathcal{R}_0$ there exists a (unique) polynomial $p_r \in \mathfrak{S}(\mathfrak{h}^0)$ such that $rv_\lambda = p_r(\lambda)v_\lambda$. Define $\mathcal{P}_0 = \{p_r \mid r \in \mathcal{R}_0\}$.

We immediately have:

**Corollary 3.9.** ([25, Cor. 2.10]) There is a one-to-one correspondence between:

1. Irreducible, highest-weight $A_g(L(\mathfrak{g}, k))$ modules and
2. Weights $\lambda \in (\mathfrak{h}^0)^*$ such that $p(\lambda) = 0$ for all $p \in \mathcal{P}_0$.

We now present some calculations that will be used below. Let $a \in \mathfrak{g}^j$, $0 < j < T$, $b \in \mathfrak{g}$. Then,

\[
(a(-1)1) \circ_g (b(-1)1) = \text{Res}_x \left( \frac{(1 + x)^{j/T}}{x} \left( \sum_{n \geq 0} \frac{(j/T)^n}{n!} x^n \right) \right) (a(-1)b(-1)1 + a(0)b(-1)1x^{-1} + a(1)b(-1)1x^{-2})
\]

\[
= \text{Res}_x \left( \frac{(j/T)^n}{n!} x^n \right) \left( \sum_{n \geq 0} \frac{(j/T)^n}{n!} x^n \right) (a(-1)b(-1)1 + a(0)b(-1)1x^{-1} + a(1)b(-1)1x^{-2})
\]

\[
= a(-1)b(-1)1 + \frac{j}{T} [a,b](-1)1 + \frac{j(j-T)}{2T^2} k(a,b)1. \tag{3.15}
\]

This implies that for $a \in \mathfrak{g}^j$, $0 < j < T$ and $b \in \mathfrak{g}$,

\[
a(-1)b(-1)1 \equiv_{O_g(V)} - \frac{j}{T} [a,b](-1)1 - \frac{j(j-T)}{2T^2} k(a,b)1. \tag{3.16}
\]

Or, equivalently,

\[
[a(-1)b(-1)1]_g = -\frac{j}{T} [a,b](-1)1 - \frac{j(j-T)}{2T^2} k(a,b)[1]_g. \tag{3.17}
\]

Since $\langle \cdot, \cdot \rangle$ is $\mathfrak{g}$-invariant, both sides are zero if $b^{(T-j)} = 0$.

For general elements $a, b \in \mathfrak{g}$, we have:

\[
a(-1)b(-1)1 = (a^{(0)} + \cdots + a^{(T-1)})(-1)b^{(0)} + \cdots + b^{(T-1)}(-1)1
\]

\[
= \left( a^{(0)}(-1)b^{(0)}(-1)1 + a^{(1)}(-1)b^{(T-1)}(-1)1 + \cdots + a^{(T-1)}(-1)b^{(1)}(-1)1 \right) + \cdots \tag{3.18}
\]

where the last ellipses denote terms that are in $V(1) \oplus \cdots \oplus V(T-1)$. So, using (3.4) and (3.17)

\[
[a(-1)b(-1)1]_g = [a^{(0)}(-1)b^{(0)}(-1)1]_g - \sum_{0 < j < T} \frac{j}{T} [a^{(j)}, b^{(T-j)}](-1)1 = \sum_{0 < j < T} \frac{j(j-T)}{2T^2} k(a^{(j)}, b^{(T-j)})[1]_g. \tag{3.19}
\]

Henceforth, we will drop the subscript $g$. 


4. \( \nu \)-Twisted Zhu Algebra for \( L(\mathfrak{sl}_{2l+1}, -l - \frac{1}{2}) \)

Fix \( l \in \mathbb{Z}_{>0} \) and let \( \mathfrak{g} = \mathfrak{sl}_{2l+1} \) as before and let \( k = -l - \frac{1}{2} \). Recall that \( V(\mathfrak{g}, k) \) is the (universal) generalized Verma module VOA and \( J(\mathfrak{g}, k) \) is its (unique) maximal proper ideal.

**Theorem 4.1.** From [25] we have:

1. The vector

\[
v = \sum_{i=1}^{2l} \frac{2l - 2i + 1}{2l + 1} E_\theta(-1)H_i(-1)1 + \sum_{i=1}^{2l-1} E_{i,i+1}(-1)E_{i+1,2l+1}(-1)1 - \frac{1}{2}(2l - 1)E_\theta(-2)1 \tag{4.1}
\]

is a singular vector in \( V(\mathfrak{g}, k) \).

2. The ideal \( J(\mathfrak{g}, k) \) is generated by \( v \), that is, \( J(\mathfrak{g}, k) = \mathfrak{U}(\mathfrak{g})v \).

**Proof.** Our notation is slightly different from [25]. Negative of the singular vector given in [25] is:

\[
- \sum_{i=1}^{2l} \frac{2l - 2i + 1}{2l + 1} H_i(-1)e_\theta(-1)1 + \sum_{i=1}^{2l-1} e_{\epsilon_i - \epsilon_{i+1}}(-1)e_{\epsilon_i + \epsilon_{i+1} - \epsilon_{2l+1}}(-1)1 + \frac{1}{2}(2l - 1)e_\theta(-2)1, \tag{4.2}
\]

where they define for \( i < j \)

\[
e_{\epsilon_i - \epsilon_j} = [E_{j-1}, [E_{j-2}, [\cdots [E_{i+1}, E_i] \cdots ]]], \quad e_\theta = e_{\epsilon_1 - \epsilon_{2l+1}}.
\]

It can be seen that

\[
e_{\epsilon_i - \epsilon_j} = (-1)^{i-j}E_{i,j}, \quad e_\theta = -E_\theta. \tag{4.4}
\]

We thus get

\[
\sum_{i=1}^{2l} \frac{2l - 2i + 1}{2l + 1} H_i(-1)E_\theta(-1)1 + \sum_{i=1}^{2l-1} E_{i,i+1}(-1)E_{i+1,2l+1}(-1)1 - \frac{1}{2}(2l - 1)E_\theta(-2)1 \tag{4.5}
\]

In the first summation, \([H_i(-1), E_\theta(-1)] = 0 \) if \( 1 < i < 2l \) and \([H_i(-1), E_\theta(-1)] = E_\theta(-2) \) if \( i = 1, 2l \). We thus get the required formula. \( \square \)

**Lemma 4.2.** We have \( \nu(v) = v \).

**Proof.** We have:

\[
\nu(v)
\]

\[
= \sum_{i=1}^{2l} \frac{2l - 2i + 1}{2l + 1} E_\theta(-1)H_{2l+1-i}(-1)1 + \sum_{i=1}^{2l-1} E_{2l-i+1,2l+2}(-1)E_{1,2l-i+1}(-1)1 + \frac{2l - 1}{2}E_\theta(-2)1
\]

\[
= \sum_{i=1}^{2l} \frac{2l - 2i + 1}{2l + 1} E_\theta(-1)H_i(-1)1 + \sum_{i=1}^{2l-1} E_{i,i+1,2l+1}(-1)E_{i+1,2l+1}(-1)1 + \frac{2l - 1}{2}E_\theta(-2)1
\]

\[
= \sum_{i=1}^{2l} \frac{2l - 2i + 1}{2l + 1} E_\theta(-1)H_i(-1)1 + \sum_{i=1}^{2l-1} (E_{1,i+2}(-1)E_{i+2,2l+2}(-1)1 + \frac{2l - 1}{2}E_\theta(-2)1
\]

\[
= v, \tag{4.6}
\]

where the first equality follows by definition of \( \nu \) and the second by re-indexing the summations. \( \square \)

Our next task is to calculate enough information about \( \mathcal{R} = [\mathfrak{U}(\mathfrak{g})v] \) so that we can use Corollary 3.9. The \( \mathfrak{g} \)-weight of \( v \) is \( \theta \), and as \( \mathfrak{g} \)-module, \( \mathfrak{U}(\mathfrak{g})v \) is isomorphic to the adjoint module of \( \mathfrak{g} \) with \( v \mapsto E_\theta \). As \( \mathfrak{g}^0 \)-modules, we then have \( \mathfrak{U}(\mathfrak{g})v \cong \mathfrak{g}^0 \oplus \mathfrak{g}^1 \). Note that \( E_\theta \in \mathfrak{g}^1 \) and so \( \mathfrak{U}(\mathfrak{g}^0)v \cong \mathfrak{g}^1 \) as \( \mathfrak{g}^0 \)-modules. Since \( v \) is \( \nu \)-fixed, we have \( \mathcal{R} = [[\mathfrak{U}(\mathfrak{g})v]] = [\mathfrak{U}(\mathfrak{g}^0)v] \). From Section 2.3 we know that \( \dim(\mathcal{R}_0) = \dim((\mathfrak{g}^1)_0) = l \) and thus we seek \( l \) independent polynomials in \( \mathcal{P}_0 \).
Lemma 4.3. The projection of $v$ on the twisted Zhu algebra is given by the following formula:

$$[v] = \sum_{i=1}^{2l-1} E_{i+1,2l+1}^+ E_{1,i+1}^+.$$  \hfill (4.7)

Proof. First, it is easy to see that:

$$\sum_{i=1}^{2l} \frac{2l - 2i + 1}{2l + 1} E_0(-1)H_i(-1)1 - \frac{1}{2}(2l - 1)E_0(-2)1 \in V(g, -l - \frac{1}{2})^1.$$  \hfill (4.8)

Using (3.19), we get:

$$\sum_{i=1}^{2l-1} [E_{1,i+1}(-1)E_{i+1,2l+1}^+(-1)1]$$  \hfill (4.9)

$$= \sum_{i=1}^{2l-1} [E_{1,i+1}^+(-1)E_{i+1,2l+1}^+(-1)1] - \frac{1}{2}[[E_{1,i+1}^-, E_{i+1,2l+1}^-](-1)1] - \frac{l+1}{8}E_{1,i+1}^-, E_{i+1,2l+1}^-]1]$$  \hfill (4.10)

For $i = 1, \ldots, 2l - 1$, we have:

$$[E_{1,i+1}^-, E_{i+1,2l+1}^-] = \frac{1}{4}[E_{1,i+1} + (-1)^i E_{2l+1-i,2l+1}, E_{i+1,2l+1} + (-1)^i E_{1,2l+1-i}] = 0.$$  \hfill (4.11)

Using (3.6) for the first term, we get the required result. \hfill \Box

Lemma 4.4. Consider $E_{l+1,1}^+(-1)^i E_{2l+1,l+1}^+ \in g^0$. Let

$$v_1 = 2(E_{l+1,1} + (-1)^i E_{2l+1,l+1})L[v] \in \mathcal{U}(g^0).$$  \hfill (4.12)

Then, we have:

$$v_1 = (-1)^i \sum_{i \leq i < l} (E_{1,i+1} - (-1)^i E_{2l+1-i,2l+1})(E_{l+1,l+1} - (-1)^{l-i} E_{l+1,2l+1-i})$$  \hfill (4.13)

$$+ (-1)^i (E_{1,1} - E_{2l+1,2l+1})(E_{1,l+1} - (-1)^l E_{l+1,2l+1})$$  \hfill (4.14)

$$- \frac{(-1)^i}{2^i}(E_{l+1,l+1} - (-1)^i E_{l+1,2l+1})$$  \hfill (4.15)

$$+ \sum_{l < i \leq 2l-1} (-1)^i (E_{i+1,l+1} - (-1)^i E_{l+1,2l+1-i})(E_{1,i+1} - (-1)^i E_{2l+1-i,2l+1}).$$  \hfill (4.16)

Proof.

$$v_1 = 2(E_{l+1,1} + (-1)^i E_{2l+1,l+1})L[v]$$

$$= 2 \sum_{i=1}^{2l-1} [E_{l+1,1} - (-1)^i E_{2l+1,l+1}, E_{i+1,2l+1}^+] E_{1,i+1}^+ + 2 \sum_{i=1}^{2l-1} E_{i+1,2l+1}^+[E_{l+1,1} - (-1)^i E_{2l+1,l+1}, E_{1,i+1}^+]$$

$$= \sum_{i=1}^{2l-1} [E_{l+1,1} - (-1)^i E_{2l+1,l+1}, E_{i+1,2l+1} - (-1)^i E_{1,2l+1-i} E_{1,i+1}^+] + 2 \sum_{i=1}^{2l-1} E_{i+1,2l+1}^+[E_{l+1,1} - (-1)^i E_{2l+1,l+1}, E_{1,i+1}^+]$$
\[
\begin{align*}
    &\sum_{i=1}^{2l-1} (-1)^i(-\delta_{i,l}E_{2l+1,2l+1} + \delta_{i,l}E_{1,1} + E_{i+1,1,l+1} - (-1)^{l-i}E_{l+1,2l+1-i})E_{1,i+1}^+ \\
    &+ \sum_{i=1}^{2l-1} E_{i+1,2l+1}^+ (E_{l+1,i+1} - (-1)^{l-i}E_{2l+1-i,l+1} - \delta_{i,l}E_{1,1} + \delta_{i,l}E_{2l+1,2l+1}).
\end{align*}
\] (4.17)

Our aim is to convert the expressions so that elements from \( n_+^0 \) are to the right. Note that \( n_i^0 \) is spanned by \( E_{i,j}^+ \) for \( 1 \leq i < j \leq 2l + 1 \). We split both summations into \( i < l, i = l, i > l \) parts.

In the first summation, all terms are already in this form, but we still rewrite them with a view towards future calculations. The \( i < l \) component is:

\[
\frac{1}{2} \sum_{1 \leq i < l} (-1)^i(E_{i+1,1,l+1} - (-1)^{l-i}E_{l+1,2l+1-i})(E_{1,i+1} - (-1)^iE_{2l+1-i,2l+1})
\]

\[
= \frac{(-1)^l}{2} \sum_{1 \leq i < l} ((E_{1,i+1} - (-1)^iE_{2l+1-i,2l+1})(E_{i+1,1,l+1} - (-1)^{l-i}E_{l+1,2l+1-i}) - E_{1,l+1} + (-1)^lE_{l+1,2l+1})
\]

\[
= \frac{(-1)^l}{2} \sum_{1 \leq i < l} (E_{1,i+1} - (-1)^iE_{2l+1-i,2l+1})(E_{i+1,1,l+1} - (-1)^{l-i}E_{l+1,2l+1-i})
\]

\[
- \frac{(-1)^l(l-1)}{2} (E_{1,l+1} - (-1)^lE_{l+1,2l+1})
\] (4.18)

The \( i = l \) term is:

\[
\frac{(-1)^l}{2} (E_{1,1} - E_{2l+1,2l+1})(E_{1,l+1} - (-1)^lE_{l+1,2l+1}).
\] (4.19)

We keep the \( i > l \) terms unchanged:

\[
\frac{1}{2} \sum_{l+1 \leq i \leq 2l-1} (-1)^i(E_{i+1,1,l+1} - (-1)^{l-i}E_{l+1,2l+1-i})(E_{1,i+1} - (-1)^iE_{2l+1-i,2l+1}).
\] (4.20)

For the second summation, the \( i < l \) terms become:

\[
\sum_{1 \leq i < l} E_{i+1,2l+1}^+ (E_{l+1,i+1} - (-1)^{l-i}E_{2l+1-i,l+1})
\]

\[
= \frac{1}{2} \sum_{1 \leq i < l} (E_{i+1,2l+1} - (-1)^iE_{1,2l+1-i})(E_{l+1,i+1} - (-1)^{l-i}E_{2l+1-i,l+1})
\]

\[
= \frac{1}{2} \sum_{1 \leq i < l} ((E_{l+1,i+1} - (-1)^{l-i}E_{2l+1-i,l+1})(E_{i+1,2l+1} - (-1)^iE_{1,2l+1-i}) - E_{l+1,2l+1} + (-1)^lE_{1,l+1})
\]

\[
= \frac{1}{2} \sum_{l+1 \leq i \leq 2l-1} (E_{l+1,2l+1-i} - (-1)^{l-i}E_{i+1,l+1})(E_{2l+1-i,2l+1} - (-1)^iE_{1,l+1}) - E_{l+1,2l+1} + (-1)^lE_{1,l+1}
\]

\[
= \frac{1}{2} \left( \sum_{l+1 \leq i \leq 2l-1} (-1)^i(E_{i+1,l+1} - (-1)^{l-i}E_{l+1,2l+1-i})(E_{1,i+1} - (-1)^iE_{2l+1-i,2l+1}) \right)
\]

\[
+ \frac{(-1)^l(l-1)}{2} (E_{1,l+1} - (-1)^lE_{l+1,2l+1}).
\] (4.23)

The \( i = l \) term is:

\[
E_{l+1,2l+1}^+ (-E_{1,1} + E_{2l+1,2l+1})
\]
\[ = \frac{1}{2}(E_{l+1,2l+1} - (-1)^i E_{1,l+1})(-E_{1,1} + E_{2l+1,2l+1}) \]
\[ = \frac{1}{2}(-E_{1,1} + E_{2l+1,2l+1})(E_{l+1,2l+1} - (-1)^i E_{1,l+1}) + \frac{1}{2}(E_{l+1,2l+1} - (-1)^i E_{1,l+1}) \]
\[ = \left(\frac{1}{2}ight)(E_{1,1} - E_{2l+1,2l+1})(E_{l+1,1} - (-1)^i E_{1,l+1}) + \frac{1}{2}(E_{l+1,2l+1} - (-1)^i E_{1,l+1}). \]

The \( i > l \) term is:

\[
\sum_{1 \leq i < l} E_{i+1,2l+1}^+(E_{l+1,i+1} - (-1)^{l-i}E_{2l+1-i,l+1}) \quad (4.24)
\]
\[
= \frac{1}{2} \sum_{1 \leq i < l} (E_{i+1,2l+1} - (-1)^i E_{1,2l+1-i})(E_{l+1,i+1} - (-1)^{l-i} E_{2l+1-i,l+1}) \quad (4.25)
\]
\[
= \frac{1}{2} \sum_{1 \leq i < l} (E_{2l+1-i,2l+1} - (-1)^i E_{1,i+1})(E_{l+1,2l+1-i} - (-1)^{l-i} E_{i+1,l+1}) \quad (4.26)
\]
\[
= \left(\frac{1}{2}\right) \sum_{1 \leq i < l} (E_{i,i+1} - (-1)^i E_{2l+1-i,2l+1})(E_{l+1,i+1} - (-1)^{l-i} E_{i+1,2l+1-i}) \quad (4.27)
\]

Combining everything, we get the required formula for \( v_1 \).

Now we shall get many elements in the weight zero space \( \mathcal{R}_0 \).

**Theorem 4.5.** Let \( 1 \leq j \leq l \). Recall (2.29) and (2.30). We have:

\[ -(-1)^j (f_jf_{j-1} \cdots f_1 f_{j+1}f_{j+2} \cdots f_l)_L v_1 = h_j \left( h_j + 2 \sum_{j < i \leq l} h_i + (l - j) - \frac{1}{2} \right) + \mathcal{U}(\mathfrak{g}) n_+^0. \quad (4.28) \]

**Proof.** It is not hard to see that for every \( 1 \leq j \leq l \), \((f_jf_{j-1} \cdots f_1 f_{j+1}f_{j+2} \cdots f_l)_L v_1 \in \mathcal{R}_0 \).

Throughout this proof, it will be often beneficial for us to do the calculations in \( \mathcal{U}(\mathfrak{g}) \) or \( \mathcal{U}(\mathfrak{g})^0 \). Since we are sure that the final answer is to be in \( \mathcal{U}(\mathfrak{g})^0 \), we will carefully omit the terms not in \( \mathcal{U}(\mathfrak{g})^0 \) that appear in the intermediate steps. Recall that \( n_+^0 \) is spanned by \( E_{i,j}^+ \) for \( 1 \leq i < j \leq 2l+1 \).

The calculation corresponding to the term (4.13) is the longest and we break it into several steps. First, we consider the term \( E_{1,i+1}^+ E_{i+1,l+1} \). Let \( 1 \leq i < j \leq l \). We have:

\[
(f_jf_{j-1} \cdots f_1 f_{j+1}f_{j+2} \cdots f_l)_L (E_{1,i+1}E_{i,i+1,l+1}) = (F_jF_{j-1} \cdots F_{i+1} F_1 \cdots F_i F_{j+1}F_{j+2} \cdots F_l)_L (E_{1,i+1}E_{i+1,l+1})
\]
\[
= (-1)^{l-j} (F_jF_{j-1} \cdots F_{i+1} F_1 \cdots F_i)_L (E_{1,i+1}E_{i+1,j+1})
\]
\[
= -(-1)^{l-j} (F_jF_{j-1} \cdots F_{i+1})_L (H_iE_{i+1,j+1})
\]
\[
= (-1)^{l-j} H_iH_j + \mathcal{U}(\mathfrak{g}) n_+^+ \quad (4.29)
\]

Now let \( i = j \), but note that we only allow \( 1 \leq i < l \) in (4.13).

\[
(f_jf_{j-1} \cdots f_1 f_{j+1}f_{j+2} \cdots f_l)_L (E_{1,j+1}E_{j+1,l+1}) = (F_jF_{j-1} \cdots F_1 F_{j+1}F_{j+2} \cdots F_l)_L (E_{1,j+1}E_{j+1,l+1})
\]
\[
= (-1)^{l-j} (F_jF_{j-1} \cdots F_i)_L (E_{1,j+1}H_{j+1})
\]
\[
= -(-1)^{l-j} (F_jF_{j-1} \cdots F_i)_L (H_{j+1}E_{1,j+1} + E_{1,j+1})
\]
\[
= -(-1)^{l-j} (H_{j+1}H_j + H_j) + \mathcal{U}(\mathfrak{g}) n_+. \quad (4.30)
\]
Now let \( i > j \), again noting that we only allow \( 1 \leq i < l \) in (4.13).

\[
\begin{align}
(f_j f_{j-1} \cdots f_1 f_{j+1} \cdots f_l)_L (E_{i,i+1} E_{i+1,l+1}) \\
= (F_j F_{j-1} \cdots F_1 F_{j+1} \cdots F_l)_L (E_{1,i+1} E_{i+1,l+1}) \\
= (-1)^l (F_j F_{j-1} \cdots F_1 F_{j+1} \cdots F_l)_L (E_{1,i+1} H_{i+1}) \\
= (-1)^{l-i} (F_j F_{j-1} \cdots F_1 F_{j+1} \cdots F_l)_L (H_{i+1} E_{i+1,i+1} + E_{1,i+1}) \\
= -(-1)^{l-j} (H_{i+1} H_j + H_j) + \mathcal{U}(g)n_+. \\
\end{align}
\]

(4.31)

Note that if we place \( i = j \) in (4.31), we get (4.30), thus we combine these two equations. Combining (4.29), (4.30), (4.31) for a fixed \( 1 \leq j \leq l \), we have:

\[
\begin{align}
(f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L & \left( (-1)^j \sum_{1 \leq i < l} E_{1,i+1} E_{i+1,l+1} + (-1)^j E_{2l+1-i,2l+1} E_{l+1,2l+1-i} \right) \\
& = (-1)^j (f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L \left( \sum_{1 \leq i < l} (1 + \nu)(E_{1,i+1} E_{i+1,l+1}) \right) \\
& = (-1)^j (1 + \nu) \sum_{1 \leq i < l} (f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L (E_{1,i+1} E_{i+1,l+1}) \\
& = (-1)^j (1 + \nu) \left( \sum_{1 \leq i < j} H_i H_j - \sum_{j \leq i < l} (H_{i+1} H_j + H_j) + \mathcal{U}(g)n_+ \right) \\
& = (-1)^j H_j \left( \sum_{1 \leq i < j} H_i - \sum_{j \leq i < l} (H_{i+1} + 1) \right) + (-1)^j H_{2l+1-j} \left( \sum_{1 \leq i < j} H_{2l+1-i} - \sum_{j \leq i < l} (H_{2l-i} + 1) \right) \\
& + \mathcal{U}(g)n_+. \tag{4.32}
\end{align}
\]

Now we consider the term \( E_{1,i+1} E_{l+1,2l+1-i} \). First let \( 1 \leq i < j \leq l \). We have:

\[
\begin{align}
(f_j \cdots f_{i+1} f_i \cdots f_j f_{j-1} f_l)_L (E_{1,i+1} E_{l+1,2l+1-i}) \\
& = (f_j \cdots f_{i+1} f_i \cdots f_j F_{2l-j} \cdots F_{l+1})_L (E_{1,i+1} E_{l+1,2l+1-i}) \\
& = (f_j \cdots f_{i+1} f_i \cdots f_j f_{j-1} \cdots f_l)_L (E_{i,i+1} E_{2l+1-j,2l+1-i}) \\
& = (f_j f_{j-1} \cdots f_1 f_{i+1})_L (-H_i E_{2l+1-j,2l+1-i}) \\
& = (F_{2l+1-j} F_{2l+2-j} \cdots F_{2l-i})_L (-H_i E_{2l+1-j,2l+1-i}) \\
& = -(-1)^{l-j} H_i H_{2l+1-j} + \mathcal{U}(g)n_+. \tag{4.33}
\end{align}
\]

Now let \( i = j \), but note that \( 1 \leq i < l \).

\[
\begin{align}
(f_j \cdots f_1 f_{j+1} \cdots f_l)_L (E_{1,i+1} E_{l+1,2l+1-j}) \\
& = (f_j \cdots f_1 F_{2l-j} \cdots F_{l+1})_L (E_{1,i+1} E_{l+1,2l+1-j}) \\
& = (f_j \cdots f_1)_L (-E_{1,i+1} H_{2l-j}) \\
& = (f_j \cdots f_1)_L (-H_{2l-j} E_{1,j+1}) \\
& = (F_j F_{j-1} \cdots F_1)_L (-H_{2l-j} E_{1,j+1}) \\
& = H_{2l-j} H_j + \mathcal{U}(g)n_. \tag{4.34}
\end{align}
\]
Now let $j < i$, but again note that $1 \leq i < l$.

\[
(f_j \cdots f_i f_{i+1} \cdots f_l) L (E_{1,i+1} E_{l+1,2l+1-i}) = (f_j \cdots f_{i+1} f_{2l-i} \cdots f_{i+1}) L (E_{1,i+1} E_{l+1,2l+1-i}) = (f_j \cdots f_i f_{i+1} \cdots f_l) L (-E_{1,i+1} H_{2l-i}) = (f_j \cdots f_i f_{i+1} \cdots f_l) L (-H_{2l-i} E_{i+1}) = (-1)^{i-j} H_{2l-i} H_j + \mathcal{U}(g)n_+.
\] (4.35)

Again, note that placing $i = j$ in (4.35) gets us (4.34), thus we combine these two. Combining (4.33), (4.34), (4.35), for a fixed $1 \leq j \leq l$, we see:

\[
(f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l) L \left( -(-1)^i E_{1,i+1} E_{l+1,2l+1-i} - (-1)^i E_{2l+1-i,2l+1} E_{i+1,l+1} \right)
= (-1)^{i+1} (f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l) L \left( \sum_{1 \leq i < l} E_{1,i+1} E_{l+1,2l+1-i} + (-1)^i E_{2l+1-i,2l+1} E_{i+1,l+1} \right)
= (-1)^{i+1} (f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l) L \left( \sum_{1 \leq i < l} (1 + \nu)(E_{1,i+1} E_{l+1,2l+1-i}) \right)
= (-1)^{i+1} (1 + \nu) \left( \sum_{1 \leq i < l} (-1)^{i-j} H_i H_{2l+1-j} + \sum_{j < i} (-1)^{i-j} H_{2l-i} H_j + \mathcal{U}(g)n_+ \right)
= (-1)^{i+1} (1 + \nu) \left( \sum_{1 \leq i < j} (H_i H_{2l+1-j} + H_{2l+1-i} H_j) - \sum_{j \leq i < l} (H_{2l-i} H_j + H_{i+1} H_{2l+1-j}) \right) + \mathcal{U}(g)n_+
= (-1)^{i+1} H_j \left( \sum_{1 \leq i < j} H_{2l+1-i} - \sum_{j \leq i < l} H_{2l-i} \right) + (-1)^{i+1} H_{2l+1-j} \left( \sum_{1 \leq i < j} H_i - \sum_{j \leq i < l} H_{i+1} \right) + \mathcal{U}(g)n_+.
\] (4.36)

Finally, we put together (4.32) and (4.36):

\[
(f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l) L \left( -(-1)^i (E_{1,i+1} E_{l+1,2l+1-i}) - (-1)^i E_{2l+1-i,2l+1} (E_{i+1,l+1} - (-1)^i E_{1,i+1,2l+1-i}) \right)
= (-1)^{i+1} H_j \left( \sum_{1 \leq i < j} H_i - \sum_{j \leq i < l} (H_{i+1} + 1) \right) + (-1)^{i+1} H_{2l+1-j} \left( \sum_{1 \leq i < j} H_{2l+1-i} - \sum_{j \leq i < l} (H_{2l+1-i} + 1) \right)
+ (-1)^{i+1} H_j \left( \sum_{1 \leq i < j} H_{2l+1-i} - \sum_{j \leq i < l} H_{2l-i} \right) + (-1)^{i+1} H_{2l+1-j} \left( \sum_{1 \leq i < j} H_i - \sum_{j \leq i < l} H_{i+1} \right) + \mathcal{U}(g)n_+.
\]
In fact, in the last equality, we may now replace $\Omega(g)n_+$ with $\Omega(g^0)n^0_+$. For (4.14) and (4.15), note that

$$
(f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L(E_{1,t+1} - (-1)^j E_{l+1,2l+1})
= (f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L(E_{1,t+1} + \nu E_{l,t+1})
= (1 + \nu)((f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L E_{l,t+1})
= (1 + \nu)((-1)^{l+j+1}H_j) = (-1)^{l+j+1}h_j.
$$

The effect of applying $(f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L$ on (4.14) is thus:

$$
(f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L((-1)^j(E_{1,1} - E_{2l+1,2l+1})(E_{l,t+1} - (-1)^j E_{l+1,2l+1}))
= (-1)^{j+1}(E_{1,1} - E_{2l+1,2l+1})h_j + \Omega(g^0)n^0_+
= (-1)^{j+1}(h_1 + \cdots + h_l)h_j + \Omega(g^0)n^0_+.
$$

and for (4.15) we get:

$$
(f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L\left(-\frac{(-1)^j}{2}(E_{1,1} - (-1)^j E_{l+1,2l+1})\right) = \frac{(-1)^j}{2}h_j + \Omega(g^0)n^0_+.
$$

It is not hard to see that $(f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L$ applied to the terms (4.16) will only yield terms in $\Omega(g^0)n^0_+$. Combining (4.37), (4.39) and (4.40) we get:

$$
(f_j f_{j-1} \cdots f_1 f_{j+1} f_{j+2} \cdots f_l)_L v_1
= (-1)^j\frac{1}{2}h_j \left(\sum_{1<i<j} h_i - \sum_{j<i \leq l} h_i - (l - j)\right) - (-1)^j(h_1 + \cdots + h_l)h_j + \frac{(-1)^j}{2}h_j + \Omega(g^0)n^0_+
= (-1)^j\frac{1}{2}h_j \left(-h_j - 2 \sum_{j<i \leq l} h_i + \frac{1}{2} - (l - j)\right) + \Omega(g^0)n^0_+.
$$

Remark 4.6. There is a very illuminating way to write the polynomials in (4.28). Let $1 \leq j < l$. Note that the coroot $h_{\epsilon_j + \epsilon_{j+1}}$ is $h_j + 2h_{j+1} + \cdots + 2h_{l-1} + h_l$ which is the same as $h_j + 2\sum_{j<i \leq l} h_i$. In the case $j = l$, we write $h_l = \frac{1}{2}h_l$. All in all, we see that we have got the following polynomials:

$$
p_j = h_j \left(h_{\epsilon_j + \epsilon_{j+1}} + l - j + \frac{1}{2}\right) \quad \text{for } 1 \leq j \leq l - 1,
$$

$$
p_l = \frac{1}{4}h_l(h_l - 1).
$$

Observe that $p_l \in \mathcal{O}_0$ and they are linearly independent. Thus $\dim(\mathcal{O}_0) \geq l$. However, since $\dim(\mathcal{O}_0) \leq \dim(\mathcal{O}) = l$, we in fact have an equality and hence the $p_l$ span $\mathcal{O}_0$.

These are exactly the polynomials (up to a factor of 4 in $p_l$) obtained by Peršč [24] in relation to the top spaces of $B_l$ modules at level $-l + \frac{3}{2}$. Thus, we may immediately import relevant results from [24] on zero sets of these polynomials.
Theorem 4.7. [24, Prop. 30] For every subset \( S = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, l-1\} \) with \( i_1 < \ldots < i_k \), define:

\[
\mu_S = \sum_{j=1}^{k} \left( i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j}i_s + (-1)^{k-j+1} \left( l - \frac{1}{2} \right) \right) \omega_{i_j}, \tag{4.44}
\]

\[
\mu'_S = \omega_l + \sum_{j=1}^{k} \left( i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j}i_s + (-1)^{k-j+1} \left( l + \frac{1}{2} \right) \right) \omega_{i_j}. \tag{4.45}
\]

Then, \( \{\mu_S, \mu'_S | S \subset \{1, 2, \ldots, l-1\}\} \) provides the complete list of highest weights of irreducible highest-weight \( A_\nu(L(\mathfrak{sl}_{2(l+1)}, -l - \frac{1}{2})) \)-modules.

Remark 4.8. In (4.44) and (4.45), notice that the coefficient of each of the \( \omega_{i_j} \) (\( j = 1, \ldots, k \)) is an element of \( \frac{1}{2} + \mathbb{Z} \). This means that we have obtained exactly two weights that are dominant integral for \( B_l \). These correspond to \( S \) being the empty set: \( \mu_\phi = 0, \mu'_\phi = \omega_l \). These are precisely the highest weights of the simple ordinary (i.e., Virasoro mode \( L(0) \) acts semisimply with finite dimensional weight spaces, and weights are bounded from below) \( \nu \)-twisted modules.

5. Admissibility and complete reducibility

5.1. Admissibility. Due to the results in [22], every (weak) \( \nu \)-twisted \( L(\mathfrak{sl}_{2(l+1)}, -l - \frac{1}{2}) \)-module is naturally a module for the twisted affine Lie algebra \( A^{(2)}_{2l} \) of level \( -l - \frac{1}{2} \). As weights for \( A^{(2)}_{2l} \), the weights obtained in Theorem 4.7 become:

\[
\lambda_S = \left( -l - \frac{1}{2} \right) \Lambda_0^\phi + \mu_S, \quad \lambda'_S = \left( -l - \frac{1}{2} \right) \Lambda_0^\phi + \mu'_S. \tag{5.1}
\]

We now prove that these are admissible, see Definition 2.1.

Theorem 5.1. For every \( S \subseteq \{1, 2, \ldots, l-1\} \), the weights \( \lambda_S \) and \( \lambda'_S \) are admissible for \( A^{(2)}_{2l} \).

Proof. Recall that \( \rho = \bar{\rho} + h \Lambda_0^c \) and observe that

\[
\lambda_S + \rho = \left( l + \frac{1}{2} \right) \Lambda_0^c + \bar{\rho} + \mu_S, \quad \lambda'_S + \rho = \left( l + \frac{1}{2} \right) \Lambda_0^c + \bar{\rho} + \mu'_S. \tag{5.2}
\]

First, let \( l = 1 \). Then, the only choice for \( S \) is the empty set \( \phi \), and we have two weights, \( \mu_\phi = 0, \mu'_\phi = \omega_1 = \frac{1}{2} \alpha_1 \). Let \( \mu \) be one of these, and let \( \lambda = -\frac{3}{2} \Lambda_0^c + \mu \).

Consider \( m \in \mathbb{Z}, \bar{\alpha} = \pm \alpha_1 + m \delta \in \Phi_+^{\text{short}} \). If \( m > 0 \), then, recalling (2.23), (2.15),

\[
(\lambda + \rho, \bar{\alpha}^\vee) = \frac{2}{1} \left( \frac{3}{2} \Lambda_0^c + \frac{1}{2} \alpha_1 + \mu, \pm \alpha_1 + m \delta \right) = 3m \pm 1 \pm 2(\mu, \alpha_1) > 0 \tag{5.3}
\]

since \( (\mu, \alpha_1) = 0 \) or \( \frac{1}{2} \). If \( m = 0 \), then, \( \bar{\alpha} = \alpha_1 \), and

\[
(\lambda + \rho, \bar{\alpha}^\vee) = \frac{2}{1} \left( \frac{3}{2} \Lambda_0^c + \frac{1}{2} \alpha_1 + \mu, \alpha_1 \right) = 1 + 2(\mu, \alpha_1) > 0. \tag{5.4}
\]

Now let \( m \in \mathbb{Z}, \bar{\alpha} = \pm 2 \alpha_1 + (2m + 1) \delta \in \Phi_+^{\text{long}} \). Necessarily, \( m \geq 0 \) and, recalling (2.23), (2.15),

\[
(\lambda + \rho, \bar{\alpha}^\vee) = \frac{2}{4} \left( \frac{3}{2} \Lambda_0^c + \frac{1}{2} \alpha_1 + \mu, \pm 2 \alpha_1 + (2m + 1) \delta \right) = \frac{3}{4} (2m + 1) \pm \frac{1}{2} \pm (\mu, \alpha_1) \notin \mathbb{Z}, \tag{5.5}
\]

since \( (\mu, \alpha_1) = 0 \) or \( \frac{1}{2} \). Thus the first condition of admissibility is satisfied.
For the second condition, note that $\alpha_1, \delta - \alpha_1 \in \hat{\Phi}_{short}^\epsilon$. We have $\alpha_1, \delta - \alpha_1 \in \hat{\Delta}_{\alpha_1}^{re}$ since:

$$(\lambda_\phi, (\alpha_1)') = 2\left(-\frac{3}{2} \lambda_0, \alpha_1\right) = 0, \quad (\lambda_\phi, (\delta - \alpha_1)') = 2\left(-\frac{3}{2} \lambda_0, \delta - \alpha_1\right) = -3.$$  

We have $\alpha_1, \delta - \alpha_1 \in \hat{\Delta}_{\alpha_1}^{re}$ since:

$$(\lambda_\phi', (\alpha_1)') = 2\left(-\frac{3}{2} \lambda_0' + \frac{1}{2} \alpha_1, \alpha_1\right) = 1, \quad (\lambda_\phi', (\delta - \alpha_1)') = 2\left(-\frac{3}{2} \lambda_0' + \frac{1}{2} \alpha_1, \delta - \alpha_1\right) = -4.$$  

Now, let $l > 1$. Most of the work for this case has been already done in [24, Lem. 32]. Also, as in [24], the proof for $\lambda_S$ and $\lambda_S'$ is similar, so we only present the former.

Suppose that $\alpha_i \in \Phi_{short}^\epsilon \cup \Phi_{long}^\epsilon$, $m \in \mathbb{Z}$ such that $\tilde{\alpha} = \alpha + m\delta \in \hat{\Phi}_{short}^\epsilon \cup \hat{\Phi}_{intermediate}^\epsilon$. Then, recalling (2.23), (2.15), we get the following, exactly as in [24, Eq. 12]:

$$(\lambda_S + \rho, \tilde{\alpha'}) = \left(\left(l + \frac{1}{2}\right) \lambda_0 + \rho + \mu_S, (\alpha + m\delta)'ight) = \frac{2}{(\alpha, \alpha)} \left(m \left(l + \frac{1}{2}\right) + (\rho, \alpha) + (\mu_S, \alpha)\right). \tag{5.6}$$

In [24, Lem. 32], it was shown that the right-hand side does not belong to $\{0, -1, -2, \ldots\}$.

Now suppose $\alpha = \pm \epsilon_i \in \Phi_{short}^\epsilon (i = 1, \ldots, l)$ and $m \in \mathbb{Z}$ such that $\tilde{\alpha} = 2\alpha + (2m + 1)\delta \in \Phi_{long}^\epsilon$. We have:

$$(\lambda_S + \rho, \tilde{\alpha'}) = \frac{2}{4} \left(2m + 1\right) \left(l + \frac{1}{2}\right) + (\rho + \mu_S, 2\alpha) = (2m + 1) \left(l + \frac{1}{2}\right) + (\rho + \mu_S, \alpha). \tag{5.7}$$

Recalling (2.18), (2.20), we see that $(\mu_S, \alpha) \in \frac{1}{2}\mathbb{Z}$. Recalling (2.22), we see $(\rho, \alpha) \in \frac{1}{2}\mathbb{Z}$. Hence, $(\lambda_S + \rho, \tilde{\alpha'}) \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$, and thus not in $\{0, -1, -2, \ldots\}$.

The proof for checking the second condition of Definition 2.1 (recall Remark 2.2) is also similar to [24]. For $i = 1, \ldots, k$, denote the coefficient of $\omega_i$ in $\mu_S$ by $x_i \in \frac{1}{2} + \mathbb{Z}$.

Using (2.18) and (2.20), it is easy to see that for $i \in \{1, \ldots, l\} \setminus S$, $(\lambda_S, \alpha_i') = (\mu_S, \alpha_i') = 0$. If $i_j \in S$, $\delta - \alpha_i \in \hat{\Phi}_{intermediate}^\epsilon$. We have, again using (2.18) and (2.20):

$$(\lambda_S, (\delta - \alpha_i)') = 2 \left(\lambda_S, \delta - \alpha_i\right) = -l - \frac{1}{2} - x_i \in \mathbb{Z}. \tag{5.8}$$

Now, if $S = \{i_1, \ldots, i_k\}$ has two or more elements, consider $i_j \in S$ with $j = 1, \ldots, k - 1$. Note, $\epsilon_i - \epsilon_i(i_{j+1} + 1) = \alpha_{i_j} + \alpha_{i_{j+1}} + \alpha_{i_{j+2}} + \cdots + \alpha_{i_{j+1}} \in \hat{\Phi}_{intermediate}^\epsilon$.

$$(\lambda_S, (\alpha_{i_j} + \alpha_{i_{j+1}} + \alpha_{i_{j+2}} + \cdots + \alpha_{i_{j+1}})') = (\mu_S, \epsilon_i - \epsilon_i(i_{j+1} + 1)) = x_i + x_{i_{j+1}} \in \mathbb{Z}. \tag{5.9}$$

If $S$ has two or more elements, the observations above are enough to guarantee the second condition of admissibility. If $S$ has exactly one element, $S = \{i_1\}$, consider $\epsilon_i = \alpha_{i_1} + \alpha_{i_1+1} \cdots + \alpha_i \in \hat{\Phi}_{short}^\epsilon$. We have:

$$(\lambda_{\{i_1\}}, (\alpha_{i_1} + \alpha_{i_1+1} \cdots + \alpha_i)') = 2(\mu_{\{i_1\}}, \epsilon_{i_1}) = 2x_i \in \mathbb{Z}. \tag{5.10}$$

This, combined with the other observations is enough to handle the present case. Finally, if $S$ is empty, consider $\delta - \alpha_l = \hat{\Phi}_{short}^\epsilon$.

$$(\lambda_\phi, (\delta - \alpha_l)') = 2(\lambda_\phi, \delta - \alpha_l) = 2 \left(\left(l - \frac{1}{2}\right) - (\mu_\phi, \alpha_l)\right) = -2l - 1 \in \mathbb{Z}. \tag{5.11}$$
5.2. Semi-simplicity. Again, our proofs are parallel to the ones in \cite{2}, \cite{25}, \cite{24} etc., with statements modified to accommodate the twist. Recall the notion of category \( \mathcal{O} \) for representations of affine Kac-Moody algebras, \cite[Ch. 9]{15}.

**Theorem 5.2.** (\cite[Thm. 4.1]{16}) Let \( \mathfrak{g} \) be any affine Lie algebra and let \( M \) be a \( \mathfrak{g} \)-module from category \( \mathcal{O} \) such that its every irreducible subquotient \( L(\lambda) \) with highest weight \( \lambda \) satisfies:

- (1) \( (\lambda + \rho, \alpha^\vee) \notin \{-1, -2, \ldots\} \) for all \( \alpha^\vee \in \hat{\Delta}_+^{\text{re}} \) and
- (2) \( \Re(\lambda + \rho, c) > 0 \).

Then \( M \) is completely reducible.

It is clear that our weights \( \lambda_S, \lambda'_S \) for all \( l \geq 1 \) and \( S \subseteq \{1, \ldots, l - 1\} \) satisfy these conditions.

**Theorem 5.3.** (cf. \cite[Thm. 33]{24}) Let \( M \) be a weak \( \nu \)-twisted \( L(\mathfrak{sl}_{2l+1}, -l - \frac{1}{2}) \)-module that is in category \( \mathcal{O} \) as a \( A^{(2)}_{2l} \)-module. Then, \( M \) is completely reducible.

**Proof.** Any irreducible subquotient \( L \) of \( M \) is also a \( \nu \)-twisted \( L(\mathfrak{sl}_{2l+1}, -l - \frac{1}{2}) \)-module that is in category \( \mathcal{O} \) as a \( A^{(2)}_{2l} \)-module. Thus, the highest weight of \( L \) is \( \lambda_S \) or \( \lambda'_S \), in particular it satisfies the conditions of Theorem 5.2. So, \( M \) is completely reducible as a \( A^{(2)}_{2l} \)-module, and thus completely reducible as a (weak) \( \nu \)-twisted \( L(\mathfrak{sl}_{2l+1}, -l - \frac{1}{2}) \)-module. \( \square \)

**Theorem 5.4.** (cf. \cite[Lem. 26]{24}) Let \( M \) be an ordinary \( \nu \)-twisted \( L(\mathfrak{sl}_{2l+1}, -l - \frac{1}{2}) \)-module. Then, \( M \) is in category \( \mathcal{O} \) as a \( A^{(2)}_{2l} \)-module, in particular, \( M \) is completely reducible.

**Proof.** \( M \) is a level \( -l - \frac{1}{2} \) module for \( A^{(2)}_{2l} \) \cite{22}, in particular, the central element \( c \) of \( A^{(2)}_{2l} \) acts semi-simply on \( M \). Clearly, every conformal weight space of \( M \) which is finite dimensional by assumption is a module for \( \mathfrak{h}^0 \). Thus, \( \mathfrak{h}^0 \) acts semi-simply on \( M \) with finite dimensional weight spaces. If \( \nu \) is a highest weight vector in \( M \) of weight \( \lambda \in \mathfrak{h}^* \), then the irreducible \( A^{(2)}_{2l} \) module \( L(\lambda) \) is an irreducible subquotient of \( M \), and hence an ordinary \( \nu \)-twisted \( L(\mathfrak{sl}_{2l+1}, -l - \frac{1}{2}) \)-module. \( L(\lambda) \) has a finite dimensional lowest conformal weight space, in particular, this space is finite dimensional irreducible module for \( \mathfrak{g}^0 \). Thus, \( \lambda \) has only two choices, \( \lambda_\phi \) or \( \lambda'_\phi \) since \( \mu_\phi \) and \( \mu'_\phi \) are the only dominant integral weights for \( \mathfrak{g}^0 \) among the possible highest weights (Remark 4.8). This implies that any weight of \( M \) has to be dominated by one of \( \lambda_\phi \) or \( \lambda'_\phi \), i.e., \( \text{wt}(M) \subseteq D(\lambda_\phi) \cup D(\lambda'_\phi) \). This proves that \( M \) is in category \( \mathcal{O} \) as a \( A^{(2)}_{2l} \)-module. The last assertion is due to Theorem 5.3. \( \square \)

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