Exact projector Hamiltonian, local integrals of motion, and many-body localization with topological order

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(Dated: May 6, 2020)

In this work, we construct an exact projector Hamiltonian with interactions, which is given by a sum of mutually commuting operators called stabilizers. The model is based on the recently studied Creutz-ladder of fermions, in which flat-band structure and strong localization are realized. These stabilizers are local integrals of motion from which many-body localization (MBL) is realized. All energy eigenstates are explicitly obtained even in the presence of local disorders. All states are MBL states, that is, this system is a full many-body localized (FMBL) system. We show that this system has a topological order and stable gapless edge modes exist under the open boundary condition. By the numerical study, we investigate stability of the FMBL and topological order.

PACS numbers:

I. INTRODUCTION

Topological order and many-body localization (MBL) are most intensively studied topics in condensed matter physics in the last decade$^{1,2}$. In particular, certain studies suggest that in some systems, all eigenstates are MBL including those at high energies$^{3,4}$. This phenomenon, which is called full MBL (FMBL), is protected from disturbance due to the existence of local integrals of motion (LIOMs)$^6$–$14$. Furthermore, some FMBL systems have topologically-ordered states at all energies$^{6,8,12,17}$ and under the open boundary condition (OBC), all states exhibit degeneracy due to the presence of gapless edges modes, which exist under the OBC. Actually, the above properties were numerically verified for some typical topological models$^{18}$–$20$. Furthermore, the above properties, which concern to all energy eigenstates, are closely related to a concept in the modern quantum information called stabilizer$^{21}$. Recently, study on LIOMs in topologically-ordered localized states was given, in which a notion of topological LIOMs (tLIOMs) was proposed$^{22}$.

Stabilizers are mutually commuting local operators, and if Hamiltonian consists of terms including only the stabilizers, this Hamiltonian is called projector Hamiltonian. Obviously, such Hamiltonian commutes with all the stabilizers, and the stabilizers play a role of LIOMs if the system is FMBL$^4,5$.

The above idea might be applicable for various interesting systems, e.g., majorana-fermion system$^{23}$–$25$. However, LIOMs, stabilizers and local gapless edge modes can be constructed only perturbatively except for some toy models$^{23,24}$. Therefore, discovery of nontrivial models, in which LIOMs, stabilizers and edge modes can be constructed explicitly, is certainly welcome. In this work, we shall present one of them based on a flat-band system.

This work is also motivated by recent theoretical studies on a fermion system in the Creutz-ladder$^{17}$–$27,34$, which is to be realized by cold atomic gases$^{35}$ and can be constructed in photonic crystals$^{36}$. With specific inter- and intra-chain hopping amplitudes, the system has a flat-band dispersion and all eigenstates are localized within a plaquette (FMBL). Moderate interactions between fermions do not change this localization properties, in particular, at low fillings. In Ref.$^{17}$, we showed that the Hamiltonian of the Creutz-ladder fermion can be rewritten in terms of linear-combination operators of the original fermions, and the transformed Hamiltonian has only local terms in terms of these operators. And very recently, some theoretical works have reported the presence of the MBL in some flat-band systems with interactions$^{37}$–$39$. In this paper, we show that composites of the new operators, which appear in the Hamiltonian, are stabilizers, and they form LIOMs. This finding presents a new point of view for the localization properties of the Creutz-ladder model.

The original Creutz-ladder model is understood as a projector system rather straightforwardly. In this work, we extend the model in a nontrivial way by adding certain specific interactions. Even in this case, the system does not lose properties of the projector Hamiltonian. We explicitly construct stabilizes and study the structure of the energy eigenstates including the gapless edge mode under the OBC. This study sheds light on perturbative construction of LIOMs proposed in Ref.$^4$. In addition to analytical study on the models, numerical study is quite useful to reveal physical properties of models extended by adding extra hopping terms, generic interactions between fermions, etc. Stability of the topological nature and MBL can be investigated by calculating various quantities.

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This paper is organized as follows. In Sec. II, we explain the properties of the original Creutz-ladder model as the projector Hamiltonian, and explicitly construct gapless edge modes. Then, we introduce interactions, with which non-trivial entanglements between fermions are generated. We show that stabilizers can be constructed explicitly. By using these stabilizers, we study energy eigenstates in detail. Topological order parameter is given explicitly. Section III gives results of numerical study on the models in Sec. II. Existence of the gapless edge modes is clearly shown for the non-interaction case, and effects of extra hopping terms, which break the topological nature of the system, are also studied. Then, we study non-ergodic and localized properties of interacting models by investigating time evolution of the system. Section IV is devoted for discussion and conclusion.

II. MODEL AND LIOMS

A. Creutz-ladder model as projector system

In this section, we introduce a target model and explain its stabilizer structure. Although we can directly introduce the stabilizer system, we would like to explain its physical background. To this end, we start with the simple Creutz ladder model, Hamiltonian of which is given as follows,

\[ H_{\text{CL}} = \sum_{j} \left[ -i\tau_{1}(a_{j+1}^{\dagger}a_{j} - b_{j+1}^{\dagger}b_{j}) - \tau_{0}(a_{j+1}^{\dagger}b_{j} + b_{j+1}^{\dagger}a_{j}) + \text{h.c.} \right] \]

where \( a_{j} \) and \( b_{j} \) are fermion annihilation operator at site \( j \), and \( \tau_{1} \) and \( \tau_{0} \) are intra-ladder and inter-chain hopping amplitudes, respectively. In the case of \( \tau_{1} = \tau_{0} \), the system has flat-band dispersions. Furthermore, all energy eigenstates are localized in a plaquette. To see this, it is useful to introduce the following operators,

\[ w_{A,j} = (a_{j} + ib_{j})/\sqrt{2}, \quad w_{B,j} = (a_{j} - ib_{j})/\sqrt{2}, \]

where the above \( w \)-particles satisfy the fermionic commutation relations, \( \{w_{\alpha,j}, w_{\beta,k}\} = \delta_{\alpha\beta}\delta_{jk} \), etc. By using \( (w_{A,j}, w_{B,j}) \) for \( H_{\text{flat}} \equiv H_{\text{CL}}|_{\tau_{1} = \tau_{0}} \),

\[ H_{\text{flat}} = \sum_{j} 2 \left[ -i\tau_{0}w_{A,j+1}^{\dagger}w_{B,j} + i\tau_{0}w_{B,j}^{\dagger}w_{A,j+1} \right], \]

and therefore,

\[ H_{\text{flat}}w_{A,j}^{\dagger}|0\rangle = 2i\tau_{0}w_{B,j-1}^{\dagger}|0\rangle, \]

\[ H_{\text{flat}}w_{B,j}^{\dagger}|0\rangle = -2i\tau_{0}w_{A,j+1}^{\dagger}|0\rangle. \]

Eq. (4) implies that the \( w \)-particle changes its species by the nearest-neighbor (NN) hopping, and its location fluctuates around its original position by successive hoppings. The Hamiltonian \( H_{\text{flat}} \) can be diagonalized by the following operators,

\[ W_{j}^{+} = \frac{1}{\sqrt{2}}(-iw_{A,j+1} + w_{B,j}), \]

\[ W_{j}^{+\dagger} = \frac{1}{\sqrt{2}}(iw_{A,j+1}^{\dagger} + w_{B,j}^{\dagger}), \]

\[ W_{j}^{-} = \frac{1}{\sqrt{2}}(-iw_{A,j+1} - w_{B,j}), \]

\[ W_{j}^{-\dagger} = \frac{1}{\sqrt{2}}(iw_{A,j+1}^{\dagger} - w_{B,j}^{\dagger}), \]

where it is easy to prove that there is one-to-one correspondence between \((a_{j}, b_{j})\) and \((W_{j}^{+}, W_{j}^{-})\) under the periodic boundary condition (PBC), and \((W_{j}^{+}, W_{j}^{-})\) satisfy the fermionic commutation relations,

\[ \{W_{j}^{\alpha\dagger}, W_{k}^{\beta}\} = \delta_{\alpha\beta}\delta_{jk}, \quad (\alpha, \beta = \pm), \quad \text{etc.} \]

Under the PBC, we have \( \sum_{j}[W_{j}^{+}W_{j}^{+\dagger} + W_{j}^{-}W_{j}^{-\dagger}] = \sum_{j}(a_{j}^{\dagger}a_{j} + b_{j}^{\dagger}b_{j}) \). In terms of the \( W \)-particle, the Hamiltonian \( H_{\text{flat}} \) in Eq. (3) is diagonalized

\[ H_{\text{flat}} = \sum_{j}[2\tau_{0}W_{j}^{+\dagger}W_{j}^{+} + 2\tau_{0}W_{j}^{-\dagger}W_{j}^{-}]. \]

From Eqs. (5) and (6), it is obvious that \( H_{\text{flat}} \) is a projector Hamiltonian (stabilizer Hamiltonian), i.e.,

\[ [H_{\text{flat}}, W_{j}^{+}W_{j}^{+\dagger}] = [H_{\text{flat}}, W_{j}^{-}W_{j}^{-\dagger}] = 0, \]

\[ [W_{j}^{\alpha\dagger}W_{j}^{\alpha}, W_{k}^{\beta\dagger}W_{k}^{\beta}] = 0, \quad (\alpha, \beta = \pm). \]

We define stabilizers, \( K_{\alpha j}^{\pm} \), as

\[ K_{\alpha j}^{+} = W_{j}^{+\dagger}W_{j}^{+}, \quad K_{\alpha j}^{-} = W_{j}^{-\dagger}W_{j}^{-}, \]

with properties

\[ [K_{\alpha j}^{\alpha}, K_{\beta k}^{\beta}] = 0, \quad (\alpha, \beta = \pm), \]

and then, all energy eigenstates of the Hamiltonian \( H_{\text{flat}} \) are obtained from the eigenstates of \( K_{\alpha j}^{\pm} \) with eigenvalue 0 or 1.

It is straightforward to generalize the above system by introducing random hopping amplitudes by \( \tau_{0} \rightarrow \tau_{j} \). Hamiltonian of the random-hopping system is given by

\[ H_{\text{rh}} = \sum_{j} \tau_{j}[2W_{j}^{+\dagger}W_{j}^{+} + 2W_{j}^{-\dagger}W_{j}^{-}]. \]

Although the flat-band character is destroyed by the random hopping, the system \( H_{\text{rh}} \) in Eq. (12) is still solvable,
and all the states are localized. We can also add the interactions such as
\[ H_1 = \sum \sum c_{jk}(W^+_j W^+_j)(W^+_k W^+_k) + \cdots, \] (13)
where the model is still solvable as a ‘classical system’ with variables \( W_j \) = ±1. For example,
\[ \sum \left[ W^+_j W^+_j + W^-_j W^-_j \right]^2 = \frac{1}{2} \sum \left[ a^+_j a^+_j b^+_j b^-_j + a^-_j a^-_j b^-_j b^+_j \right] + \text{(scattering terms)} \] (14)
where (scattering terms) = \[ \sum_j (a^+_j a^+_j b^+_j b^-_j) \], etc., which induce scattering of a and b-particles.

From the above construction of the model, \( H_{\text{tot}} = H_{\text{rh}} + H_1 \), there exists a \( Z_2 \) topological order parameter such as
\[ Z_2 = (-1)^{\sum_j (W^+_j W^+_j + W^-_j W^-_j)}. \] (15)

Here, since \( Z_2, H_{\text{tot}} \) = 0 holds, both \( H_{\text{tot}} \) and \( Z_2 \) can be diagonalized simultaneously. \( Z_2 \) operator has two eigenvalues ±1, and then all energy eigenstates are doubly degenerate characterized by \( Z_2 \) eigenvalue, i.e., topological \( Z_2 \) symmetry appears. It is straightforward to define a string-order parameter, \( O_{ab}(i, k) \), which is a hallmark of the topological state. It is given by \( O_{ab}(i, k) = (-1)^{\sum_j (a^+_j a^+_j + b^-_j b^-_j)} \), and in a specific state and disorder realization, the string-order parameter takes random values \( O_{ab}(i, k) = \pm 1 \).

In particular, the above \( Z_2 \) order parameter plays an important role for the system with the OBC. In this case, \( Z_2 \to (-1)^{\sum_j (a^+_j a^+_j + b^-_j b^-_j)} \). It is interesting to see if gapless edge modes appear in the system \( H_{\text{tot}} \) under the OBC as a hallmark of the topological state. We expect that the edge modes are fermionic, and then, the degenerate states belong to the different \( Z_2 \) sectors. To see the edge modes, we consider the ladder of length \( L \) with \( j = 0, 1, \cdots, L \), and define
\[ H_{\text{op}} = \sum_{j=0}^{L-1} t_j \left[ -2W^+_j W^+_j + 2W^-_j W^-_j \right] \]
\[ + \sum_{j,k=0}^{L-1} c_{jk}(W^+_j W^+_j)(W^+_k W^+_k) + \cdots. \] (16)

From the fact that \( W^+_j \) is an operator defined on the plaquette with \( j \)-th and \( (j + 1) \)-th rungs, \( W^+_j \to 1/2 \) \( -(a_{-1} + b_{-1}) \) as \( a_{-1} = b_{-1} = 0 \) due to the present OBC. Then, an edge operator is expected to be \( G_0 = (a_0 + i b_0)/\sqrt{2} = \omega_{A,0} \). Here, since \( \{G_0^+, G_0\} = 1, (G_0)^2 = 0 \), \( G_0 \) is fermionic. Furthermore, even for the interacting model of \( H_{\text{op}} \), this operator \( G_0 \) satisfies four criteria of the gapless (fermionic) edge mode [23, 26]: (I) The \( G_0 \) commutes with \( H_{\text{op}} \), \( [H_{\text{op}}, G_0] = 0 \), (II) The \( G_0 \) anti-commutes with \( Z_2 \), (III) \( \{G_0^+, G_0\} = 1 \), (IV) \( G_0 \) acts on the edge portion of the system. Especially, the criterion (II) guarantees that \( G_0 \) is a mapping operator between even and odd sectors characterized by the \( Z_2 \) operator. From the above, it is obvious that \( G_0 \) represents the gapless edge mode dictated by the topological order. Similarly for the \( j = L \) edge, \( W^+_L \to 1/2 (a_L - i b_L) \) as \( a_{L+1} = b_{L+1} = 0 \), and then \( G_L = (a_L - i b_L)/\sqrt{2} = \omega_{L,1} \) forms the gapless edge mode. By the analytical calculation with \( H_{\text{op}} \) in Eq. (16), we can explicitly verify the above conclusion.

However, careful examination is required to see if the above operators of the gapless modes under the OBC really produce physically meaningful states. Operation of \( G_0 \), \( G_0^+ \) to the energy eigenstates may vanish them. In fact, the practical calculation shows,
\[ G_0 W_{0^+}^+|0\rangle = G_0 W_{0^-}^+|0\rangle = G_0 W_{0^+}^+ W_{0^+}^+|0\rangle = 0. \] (17)
where \( |0\rangle \) is the empty state. On the other hand, application of the creation operator \( G_0^+ \) produces the state \( G_0^+|0\rangle = 1/\sqrt{2}(a_0 - i b_0)|0\rangle \) and
\[ G_0^+ W_{0^+}^+|0\rangle = 1/\sqrt{2}(a_0 - i b_0)(a_1^+ + b_1^+) + 2i a_0 b_0|0\rangle. \] (18)
and similarly for \( G_0^+ W_{0^-}^+|0\rangle \) and \( G_0^+ W_{0^+}^+ W_{0^+}^+|0\rangle \). All the states reside on the \( j = 0 \) plaquette are doubly degenerate, and these degenerate states have different \( Z_2 \)-values with each other.

From the view point of MBL, \( K_{0,1}^\pm \) is nothing but LIOMs, the existence of which indicates that all energy eigenstates are localized. In the present system, inclusion of arbitrary hoppings of a and b-particles break the above LIOM structure, but it is plausible to expect that the system maintains its localization properties as in the ordinary one-dimensional systems. Our recent study on the Creutz-ladder revealed that the inclusion of random on-site potentials induces a crossover from the flat-band localization to the Anderson-type localization as a result of the breaking of the above LIOM picture.

It is interesting to extend the above discussion on the projector Hamiltonian with LIOMs to interacting systems that have non-trivial LIOMs. Stability of MBL coming from LIOMs, then, can be investigated. This will be studied in the following subsection.

### B. Interacting models

In Sec. IIA, we considered the genuine flat-band Creutz-ladder model of fermions. There, the stabilizers \( \{K_{0,1}^\pm\} \) have only single-plaquette entanglements, and therefore, each energy eigenstate is a simple product state. From this aspect, \( \{K_{0,1}^\pm\} \) have properties of \( \ell \)-bit operators (local-bit operators) [3, 3].
In this subsection, we shall introduce a projector Hamiltonian with non-trivial interactions and entanglements. The interaction, which we consider, is given as follows,

\[ H_N = \sum_j g N_j \{ W_j^{+1} W_j^- + W_j^{-1} W_j^+ \}, \tag{19} \]

where \( N_j \equiv [W_j^{+1} W_j^- + W_j^{-1} W_j^+] = K_{0j}^+ + K_{0j}^- \), and \( g \) is an arbitrary parameter. Total Hamiltonian of the target model is \( H_T = H_{th} + H_N \). The reason to consider the interaction \( H_N \) in Eq. (19) will be explained shortly.

We shall show that the system \( H_T \) is a projector Hamiltonian with topological order. We first seek stabilizers by extending \( K_{0j}^\alpha \) of the Creutz-ladder model. To this end, we calculate the commutator between \( K_{0j}^\alpha \) and \( H_N \) to obtain

\[ [K_{0j}^+, H_N] = g N_j \{ W_j^{+1} W_j^- - W_j^{-1} W_j^+ \}. \tag{20} \]

On the other hand, we introduce the following operators \( O_j \),

\[ O_j = N_j \{ W_j^{+1} W_j^- + W_j^{-1} W_j^+ \}, \tag{21} \]

and practical calculation shows

\[ [O_j, H_{th}] = 4 t_j N_j \{ W_j^{+1} W_j^- - W_j^{-1} W_j^+ \}. \tag{22} \]

From Eqs. (20) and (22), we can prove

\[ [K_{0j}^+, -\frac{g}{4t_j} O_j, H_T] = 0. \tag{23} \]

Similar calculation to the above shows

\[ [K_{0j}^-, \frac{g}{4t_j} O_j, H_T] = 0. \tag{24} \]

Then we define operators \( K_j^\alpha (\alpha = \pm) \) as,

\[ K_j^+ = K_{0j}^+ - \frac{g}{4t_j} O_j, \]
\[ K_j^- = K_{0j}^- + \frac{g}{4t_j} O_j. \tag{25} \]

Practical calculation shows that \( K_j^\alpha (\alpha = \pm) \) commute with each other, \([K_j^\alpha, K_j^\beta]\) = 0, and therefore, they are stabilizers. Finally, we can prove

\[ H_T = \sum_j 2 t_j (-K_j^+ + K_j^-). \tag{26} \]

Here we should stress that the above results are satisfied exactly for arbitrary \( t_j \)'s and \( g \). More complicated systems with Hamiltonian composed of products of the stabilizers are also solvable. But they may exhibit different behaviors from \( H_T \) in Eq. (26) concerning to dephasing and temporal developing of entanglement entropy, etc., if non-local (long-range) terms of the stabilizers exist in the Hamiltonian.

C. Eigenstates and topological order

In this section, we study the energy eigenstate of the Hamiltonian \( H_T \) in Eq. (26). This study is useful for later discussion on the topological order, in particular, existence of gapless edge modes. Energy eigenstates are constructed from eigenstates of the stabilizers as a product state.

Let us focus on states in \( j \). From Eqs. (21) and (25), the states at site \( j \) have non-trivial entanglement with the state at site \( (j - 1) \). We first introduce notations for the states;

\[ W_j^+ |0\>_+ = 0, W_j^+ |0\>_- = |1\>_+, \]
\[ W_j^- |0\>_+ = 0, W_j^- |0\>_- = |1\>_-. \tag{27} \]

For the case of \( N_{j-1} = 0 \), \( K_j^\alpha = K_{0j}^\alpha = W_j^\alpha W_j^\alpha \), and then, the eigenstates of \( K_j^\alpha \) are \(|0\>_+ \) and \(|1\>_+ \), with eigenvalue 0 and 1, respectively. Similarly for \( K_j^- \). In particular, the \( N_j = 1 \) sector is spanned by the states \{|0\>_+, |0\>+, |1\>_-, |1\>_+\}. On the other hand, the cases of \( N_{j-1} = 1 \) and 2 are non-trivial. Let us set the stabilizers as

\[ K_j^+ = N_j^+ - \lambda \{ W_j^{+1} W_j^- + W_j^{-1} W_j^+ \}, \]
\[ K_j^- = N_j^- - \lambda \{ W_j^{+1} W_j^- + W_j^{-1} W_j^+ \}, \tag{28} \]

where \( \lambda = g/(4t_j) \) (\( g/(2t_j) \)) for \( N_{j-1} = 1 \) (2). Eigenstates of the above stabilizers are obtained as follows for the \( N_j = 1 \) sector,

\[ \psi_1 \propto x_1 |1\>_+ |0\>_+ - \lambda |0\>_+ |1\>_+ - \lambda |0\>_+ |1\>_+, \]
\[ \psi_2 \propto \lambda |1\>_+ |0\>_+ + x_1 |0\>_+ |1\>_+. \tag{29} \]

with \( x_1 = \frac{1}{2} (1 + \sqrt{1 + 4\lambda^2}) \) and \( x_2 = \frac{1}{2} (1 - \sqrt{1 + 4\lambda^2}) \). The above result obviously indicates that unitary rotation in the two-dimensional vector space \(|0\>_+ - |1\>_+ \} \rightarrow \{ \psi_1, \psi_2 \} \) is induced by the interaction. This unitary rotation is given by operator such as

\[ U(\theta) = e^{i(W^+ W^- - W^- W^+)}, \tag{31} \]

with a suitable angle \( \theta \). Here, \([W^+ W^- - W^- W^+]^2 = -1 \) for the present sector \([N_j = 1]\), and therefore

\[ U(\theta) = \cos \theta + i \sin \theta \cdot (W^+ W^- - W^- W^+). \tag{32} \]

From Eqs. (29) and (30), value of \( \theta \) is determined for \( \psi_{1(2)} \), i.e., \( \tan \theta = \lambda/x_1 \). In Ref. [6], methods of constructing LIOMs was discussed. There, unitary rotation of subsystem plays a central role. The above discussion
that all states in the jth interaction term induces the rotation in the W-operator plane for the N1 eigenstates may not be constructed by simply applying the stabilizers appear in the Hamiltonian $H_{\text{liom}}$. In the previous case for $H_{\text{liom}}$ the above gapless edge modes with the OBC logical order exists in the full stabilizers are simply $\{|1\rangle_+|1\rangle_-|0\rangle_+|0\rangle_-\}$.

Let us turn to the system with the OBC, $j = 0, \cdots, L$. In this case, Hamiltonian of the OBC is derived from $H_{\text{liom}}$, and for the $j = 0$ boundary, the corresponding term is given by $2t_{0j}(-K_{0j}^W + K_{0j}^L)$ as $N_{j-1} = 0$. Then, the discussion on the edge modes given in the previous section is directly applied to the $j = 0$ boundary and we find that all states in the $j = 0$ plaquette are doubly degenerate. On the other hand for the $j = L - 1$ plaquette, the full stabilizers appear in the Hamiltonian $H_{\text{liom}}$. By fixing the $W$-particle number at $j = L - 2$, the above discussion on the eigenstates can be applied to the $j = L - 1$ plaquette. There, we show that the existence of the interaction term induces the rotation in the $\{W^+, W^-\}$ operator plane for the $N_{L-1} = 1$ case. Then by using the rotated $W$’s, similar argument to the case of $g = 0$ can be applied to the interacting case. However from Eq. (35), a rotation of $\{W^+, W^-\}$ means simply a change of relative weight of $w_{A,j+1}$ and $w_{B,j}$. This indicates that the edge mode $G_L = (a_L - ib_L)/\sqrt{2}$ in the $g = 0$ case also works in the interacting case. Practical calculation shows,

$$\{G^+_{L}, W_{L-1}^-\} = \{G^-_{L}, W_{L-1}^+\} = 0,$$

then, degenerate states are obtained by applying the operator $G^+_{L}$ to the states at $j = L$.

If we introduce interactions expressed in terms of $K_j^+$ and $K_j^-$ such as $\sum c_{lk} K_l^+ K_k^+ + \cdots$, there appear correlations between states residing on different sites. Even in this case, the above gapless edge modes with the OBC survive. On the other hand for general interactions, which are not written in terms of $K_j^+$ and $K_j^-$, energy eigenstates may not be constructed by simply applying the unitary rotation $U(\theta)$ in Eq. (31).

The topological order for $H_{\text{liom}}$ is also defined as the previous case for $H_{\text{opt}}$, and the suitable order parameter characterizing the above topological states is a ‘generalization’ of Eq. (14).

$$\tilde{Z}_2 = (-1)^{\sum_{l=0}^N K_l^\alpha},$$

where we should note $|K_j^\alpha|^2 \neq K_j^\alpha$ in general, but from Eq. (26), $K_j^+ + K_j^- = K_{0j}^W + K_{0j}^L = W_j^+ + W_j^- + W_j^+ - W_j^-$. $\tilde{Z}_2$ operator also commutes with $H_{\text{liom}},$ $[\tilde{Z}_2, H_{\text{liom}}] = 0$, therefore, each eigenstate are doubly degenerate, i.e, $Z_2$ topological order exists in $H_{\text{liom}}$. Furthermore, even if we add interactions expressed in terms of $K_j^+$ and $K_j^-$ to $H_{\text{liom}}$, $\tilde{Z}_2$ operator acts as $Z_2$ topological order parameter, and $Z_2$ topological order exists.

### III. NUMERICAL STUDIES

In Sec. II, we introduced models and studied their physical properties concerning MBL and topological order. In this section, we shall study the target systems in detail by employing numerical exact diagonalization [4].

#### A. Edge modes in random Creutz-ladder models

We first study the edge modes existing in the random Creutz-ladder model with the OBC. We calculate the energy eigenvalues of the model Eq. (12) by fixing random hoppings $\{t_j\}$ and verify that there are two gapless edge modes. It is interesting to see (in)stability of the edge modes in the existence of other type of hopping terms. To study this problem, we consider the following two kinds of inter-chain hopping,

$$H_{\text{IR1}} = v_{1j} \sum_j (a_j^\dagger b_j + b_j^\dagger a_j)$$
$$= v_{1j} \sum_j (w_{A,j} w_{B,j} - w_{B,j}^\dagger w_{A,j}),$$
$$H_{\text{IR2}} = v_{2j} \sum_j (a_j^\dagger b_j - b_j^\dagger a_j)$$
$$= v_{2j} \sum_j (w_{A,j}^\dagger w_{A,j} - w_{B,j}^\dagger w_{B,j}),$$

where $v_1$ and $v_2$ are real parameters. We use the $w_{A,B}$-particle representation as the gapless edge modes in the OBC are nothing but $G_0 = w_{A,0}, G_L = w_{B,L}$, and therefore, the topological nature of the system becomes clear in the description by the $w_{A,B}$-particle. We call the system given by the Hamiltonian $H_{\text{irh}} + H_{\text{IR1,2}}$ extended Creutz-ladder model, where $H_{\text{irh}}$ in Eq. (12) is expressed as

$$H_{\text{irh}} = \sum_j 2t_j \left[ - i w_{A,j+1}^\dagger w_{B,j} + i w_{B,j+1}^\dagger w_{A,j} \right].$$

In the practical calculation, we sometimes employ spin-1/2 operators as $w_{A(B),j}^{(i)} \rightarrow S_{A(B),j}^{(i)}$, $n_{A(B),j}^{(1/2)} \rightarrow S_{A(B),j}^{z}$ related by a Jordan-Wigner (JW) transformation. As numerical bases, it is sufficient to use the product states of spin-$z$ component base for each sites; the total spin-$z$ component in the system is not conserved. Since in the $w$-particle Hamiltonian under the JW transformation, the total spin-$z$ component in the system is conserved. Then, in practical calculation, we fix the total spin-$z$ component to reduce the numerical cost.

We display the energy eigenvalues for the case of the uniform and random hoppings in Appendix A. For the uniform-hopping case, addition of $H_{\text{IR1}}$ destroys the flat-band structure, but gapless edge modes survive. On the other hand by adding $H_{\text{IR2}}$, the flat-band structure remains but gapless edge modes split into two gap edge modes with energy $\pm \Delta \propto \pm v_2$. For the random-hopping case, similar results are obtained, in particular, for the gapless edge modes. In fact in
the presence of $H_{\text{IR1}}$, the gapless edge modes are obtained perturbatively. It is straightforward to verify that

$$G_0 = \omega_{A,0} + \frac{u_j}{t_{j-1}} \omega_{A,1} + \frac{u_j}{t_{j-1}} \omega_{A,2} + \cdots$$

commutes with the total Hamiltonian, $H_{\text{IR1}} + \hat{H}_{\text{IR1}}$. Even for $\tau_0 \neq \tau_1$, the “chiral symmetry” $(a_j \leftrightarrow b_j)$ is not broken and zero-energy edge modes exist. Therefore in the present system, the chiral symmetry protects gapless edge modes.

We examine the density profile of the $w_{A,B}$-particles in the many-body wave function to verify that the gapless modes are really localized in the edges. In terms of the many-body wave function, the signal of the presence of some localized zero-energy edge modes may be a “filling anomaly” [41, 42]. For example, for our model with the OBC and in the vicinity of the half filling, there exist four-fold degenerate states with different particle numbers indicating the existence of zero-energy edge modes. In terms of the many-body wave function to verify that the gapless edge modes are really localized in the edges. In terms of the many-body wave function to verify that the gapless edge modes are really localized in the edges.

From the results of the three different fillings in Fig. 1 (a), the “phase transition” to a non-topological state takes place there. We numerically estimate the critical coupling, $u_c$, as $u_c \sim 2$. We note that similar dynamics have been reported in a diamond flat band system with interactions, where fate of the localization dynamics was studied [43].

$H_S = \sum_j \frac{u}{4} \left[ (n_j^A - \frac{1}{2})(n_j^B - \frac{1}{2}) + (n_j^B - \frac{1}{2})(n_j^A + \frac{1}{2}) \right],$

where $n_j^{A(B)} = w_{A(B),j}^\dagger w_{A(B),j}$. It should be remarked here that the interactions in Eq. (34) as well as the hopping amplitudes in Eqs. (31) cannot be expressed in terms of the stabilizers $K_0^A$ and $K_0^B$, and therefore, the system described by the total Hamiltonian loses the projector nature and the edge modes tend to unstable for sufficiently large $u$. Figure (a) displays the typical result for it. The four-fold degenerate states existing for $u = 0$ split into three states with different particle filling for $u = 3.0$. Therefore, a “phase transition” to a non-topological state takes place there. We numerically estimate the critical coupling, $u_c$, as $u_c \sim 2$. We note that similar dynamics have been reported in a diamond flat band system with interactions, where fate of the localization dynamics was studied [43].
Next, we shall examine the stability of edge modes by studying time evolution of the states that include the edge modes. In particular, we focus on effects of the inter-chain hopping, $H_{\text{IR1}}$ in Eq. (34), on the stability. That is, we focus on $H_{\text{flat}} + H_{\text{IR1}}$.

As the initial state for the time evolution, we consider the following states,

$$\Psi_0 = G_0^\dagger \Phi_0, \quad \Psi_L = G_L^\dagger \Phi_q,$$  \hspace{1cm} (36)

with

$$\Phi_q = \prod_{i=1}^{2qL} A_q |0\rangle,$$  \hspace{1cm} (37)

where $A_q = w_{A,(2q)^{-1}(i-1)+1}$, and $q$ is given by $q = 1/2$ for the density-wave like bulk state, which we consider in the following. By numerical methods, we obtain the wave function at $t$, $\Psi(t)$, and calculate the following quantity to observe the (in)stability of the edge modes,

$$P_0(t) = \langle \Psi_0(t) | G_0^\dagger G_0 | \Psi_0(t) \rangle$$  \hspace{1cm} (38)

If $P_0(t)$ decreases from unity, the edge modes lose their coherence [coherence of $|a\rangle$ and $|b\rangle$]. On the other hand, if it keeps a value close to unity for large $t$, the edge modes and topological nature of the system are stable against the decoherence.

Numerical results are shown in Fig. 2 for various values of $v_1$. There, we calculate $P_0(t)$ and also the particle density at typical sites in the bulk. Figure 2(a) shows that as $v_1$ increases, the value of $P_0(t)$ ($t \to \infty$) is getting small, in other words, the initial edge mode is robust up to $v_1 \simeq 0.2$. On the other hand, Fig. 2(b) shows that the particle density at the central regime of the system oscillates around 0.5 for any value of $v_1$. Hence, the dynamics of the localized edge states is clearly different from that of the bulk dynamics.

In addition to the effects of $H_{\text{IR1}}$, we investigate the effects of the interaction of Eq. (35) with a weak but finite $v_1 = 0.1$, i.e., the model is $H_{\text{flat}} + H_{\text{IR1}} + H_{\text{S}}$. The results are displayed in Fig. 2(c). The stability of the localized edge states is gradually lost as increasing the interaction $u$. On the other hand, for the $u/4 = 2.5$ case, the decay is somewhat weakened. Possible reason for this behavior is that the entanglement spreading is faster compared to the other cases, and the spreading saturates in a finite system. Anyway, detailed analysis is interesting and this is a future work.

B. (Non)ergodicity of interacting model

In this subsection, we shall study MBL for the system $H_T$ [Eq. (29)] including the NN interactions. Even in the presence of the interaction of $H_N$ [Eq. (19)], the total Hamiltonian can be described by the local stabilizer form. Accordingly, if we consider the dynamics for a certain quantum state based on this system, the information of the quantum state can be conserved. Here, as a typical example, we investigate time evolution where the initial states is given by Eq. (37). To this end, as explained in the previous subsection, the $w$-particle representation is employed and for $H_T$, we use

$$N_j = (w_{A,j+1}^\dagger w_{A,j} + w_{B,j}^\dagger w_{B,j}),$$

$$W_j + W_j^{-1} = (w_{A,j+1}^\dagger w_{A,j} + w_{B,j}^\dagger w_{B,j}).$$

To study the dynamics, we calculate the expectation value of the particle density in the central regime of the system as a function of time, which is set to unity in the initial state.

Figure 3 is the result of the time evolution. For Fig. 3(a), the clear oscillating behavior is observed for any value of $g$, which indicates the particle-density revivals. This means that the information of the density distribution at the initial state is preserved, and therefore, the ergodicity is broken. Furthermore, we add $H_{\text{IR1}}$ and investigate the effect of $H_N$. Here, we set $v_1 = 0.1$ and employ the same initial state and calculate the same observable in Fig. 3(a). The result of the dynamics is shown in Fig. 3(b). Interestingly enough, even for the presence of $H_{\text{IR1}}$ term, the revival oscillation gets recovered as increasing the strength of $H_N$. This shows that as increasing the interaction, the nature as stabilizer of the projector Hamiltonian $H_T$ gets dominant in the dynamics of the system. Strong correlations between NN plaquettes generated by $H_N$ enhances the non-ergodicity. Very recently, interesting observation concerning to oscillating behavior in MBL states, a quantum time crystal, was given by using notion of extensive dynamical symmetries [14]. The above numerical results suggest that they exist in the present system, $H_T$ [Eq. (29)]. This is an interesting future problem.

IV. CONCLUSION AND DISCUSSION

In this work, we studied the Creutz-ladder model of fermions and its extensions from the view point of the projector system. It is known that the Creutz-ladder model has a flat-band dispersion and exhibits the strong localization properties for specific choice of the intra-chain and inter-chain couplings. We revealed that the model with the above hopping amplitudes has a projector Hamiltonian. The stabilizers are explicitly obtained. We showed that they play a role of LIOMs, which dictate localization of all energy eigenstates. Furthermore, we showed that the systems have topological nature and there exist gapless edge modes under the OBC. Numerical study indicates that the above single-particle properties are preserved even in the existence of interactions up to moderate coupling constants.

Next, we generalized the Creutz-ladder model with preserving its projector nature. The new model has strong correlations between particles in NN plaquettes.
Dynamics for the system $H$ of the stabilizes have non-trivial entanglement and they stay at the center of the system. (a) Dynamics for the system $H_T$. Data are almost the same for various values of $g$. (b) Dynamics for the system $H_T + H_{RL}$. Here, $v_1 = 0.1$. For all data, $L = 8$. The hopping amplitude in $H_T$ is set to be uniform: $t_j = \tau_0$. The unit of time is $\hbar/(2\tau_0)$, and unit of energy $2\tau_0$.

The stabilizers are explicitly obtained and they work on particles in NN plaquettes. This generalized model retains the topological nature, and there exist gapless edge modes. Typical energy eigenstates are analytically obtained, and they give an important insight on construction of LIOMs in generic systems.

Finally, by numerical methods, we investigated time-evolution of the states with and without edge modes in order to examine the (non)ergodicity of the system. The inter-chain hopping amplitude, which breaks the chiral symmetry, plays an important role for the time evolution.

We would like to comment that the LIOMs in the present model, in particular those in the interacting model, are not simple LIOMs but the tLIOMs (topological LIOMs). In recent work [22], the notion of tLIOMs and s-bit was introduced. In order to explain tLIOMs, let us consider the following spin model, which was studied in Refs. [18, 19],

$$H = \sum_j [\lambda_j \sigma_j^{x-1} \sigma_j^{x} \sigma_j^{z+1} + h_j \sigma_j^{z}],$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (39)$$

where $\lambda_j$ and $h_j$ are random variables. In the limit for $\lambda_j = 0$, the energy eigenstates are simple product states of eigenstates of $\sigma_j^{z}$, and then $\{\sigma_j^{z}\}$ are stabilizers. This case is a trivial limit with no correlations, and all the states are topologically trivial. On the other hand for the limit $h_j = 0$, $H$ again reduces to a projector system with stabilizers $\{\sigma_j^{x-1} \sigma_j^{x} \sigma_j^{z+1}\}$. In this case, eigenstates of the stabilizes have non-trivial entanglement and they have a topological order. Important observation is that the stabilizers in the latter case cannot be transformed to $\ell$-bits $\{\tau_j^{\ell}\}$ by any local unitary operator $U$ such as $U(\sigma_j^{x-1} \sigma_j^{x} \sigma_j^{z+1}) U^\dagger = \tau_j^{\ell}$, where $U_0$ is local unitary. The stabilizers in the latter case are called tLIOMs.

In the present model, the stabilizers $\{K_j^{\alpha}\}$ are tLIOMs. The model similar to the system Eq. (39) is realized by adding random chemical potentials to the Hamiltonian $H_T$ [Eq. (20)] such as $\mu_j a_j^\dagger a_j + \mu_j b_j^\dagger b_j$. Then for the limit $t_j = 0$, the system reduces a projector system with stabilizers $\{a_j^\dagger a_j\}$ and $\{b_j^\dagger b_j\}$. All energy eigenstates are trivial states without any correlations, and $\{a_j^\dagger a_j\}$ and $\{b_j^\dagger b_j\}$ are $\ell$-bits.

Our findings in the present work are quite interesting and useful for the investigation of the full localization and topological order from the view point of the projector Hamiltonian. The present models are feasible by recent experiments on cold atomic gases, and we hope that our findings will be verified by experiments. In particular, the time evolution of the systems is very interesting and important for the quantum information physics.

Appendix A: Single-particle spectrum of various extended models of Creutz-ladder fermion system

As we explain in the main text, we study a single-particle spectrum for various extended models of the Creutz-ladder fermions. Through this investigation, we can observe relevant terms in the Hamiltonian for preserving/ breaking topological character of the Creutz-ladder model with the specific hopping amplitude, $\tau_0 = \tau_1$.

We consider the OBC, and results are shown in

FIG. 3: Short-time exact dynamics of the particle density at the center of the system. (a) Dynamics for the system $H_T$. Data are almost the same for various values of $g$. (b) Dynamics for the system $H_T + H_{RL}$. Here, $v_1 = 0.1$. For all data, $L = 8$. The hopping amplitude in $H_T$ is set to be uniform: $t_j = \tau_0$. The unit of time is $\hbar/(2\tau_0)$, and unit of energy $2\tau_0$.

FIG. A.1: Energy spectra in the $w$-particle representation under the OBC. (a) Uniform coupling $2t_j = 1$. Flat-band dispersion and zero-energy modes exist in the band center. (b) Single-shot disordered spectra for random coupling: $2t_j = 1 + \delta_j$ with the uniformly-distributed random values $\delta_j \in [-0.5, 0.5]$. (c) $v_1 = 0.3$. Flat bands disappear, but zero-energy modes exist in the band center. (d) $v_2 = 0.3$. Flat bands exist, but zero-energy modes split into inter-band states with finite energy.
Fig. A.1. Fig. A.1 (a) is the energy spectrum for the non-random uniform hopping amplitude with $\tau_0 = \tau_1$. There exist two flat bands and two energy-zero modes between the flat bands. Figure A.1 (b) shows the result of the random hopping amplitude, and there still exist two gapless modes. The above two cases are analytically studied in the main text and the numerical results are what we expected. Figure A.1 (c) and (d) are results for the models with uniform coupling $\tau_1 = \tau_0$ and including inter-chain hopping such as $(v_1 = 0.3, v_2 = 0)$ and $(v_2 = 0.3, v_1 = 0)$, respectively. These results are interesting. In the case (c), there still exist two gapless modes in the band center. On the other hand in the case (d), two flat bands are preserved, whereas the inter-band modes split into two gapful modes.

From the above observation, we conclude that the symmetry under $a_j \leftrightarrow b_j$ (chiral symmetry) is essential ingredient for the existence of the gapless modes. In the main text, we show that these gapless modes are confined near the boundaries of the system.

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