An Application of Nash-Moser Theorem to Smooth Solutions of One-Dimensional Compressible Euler Equation with Gravity

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Abstract

We study one-dimensional motions of polytropic gas governed by the compressible Euler equations. The problem on the half space under a constant gravity gives an equilibrium which has free boundary touching the vacuum and the linearized approximation at this equilibrium gives time periodic solutions. But it is not easy to justify the existence of long-time true solutions for which this time periodic solution is the first approximation. The situation is in contrast to the problem of free motions without gravity. The reason is that the usual iteration method for quasilinear hyperbolic problem cannot be used because of the loss of regularities which causes from the touch with the vacuum. Interestingly, the equation can be transformed to a nonlinear wave equation on a higher dimensional space, for which the space dimension, being larger than 4, is related to the adiabatic exponent of the original one-dimensional problem. We try to find a family of solutions expanded by a small parameter. Applying the Nash-Moser theory, we justify this expansion. The application of the Nash-Moser theory is necessary for the sake of conquest of the trouble with loss of regularities, and the justification of the applicability requires a very delicate analysis of the problem.

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1 Introduction

The aim of this paper is to study one-dimensional motions of polytropic gas governed by the compressible Euler equations

\[ \rho_t + (\rho u)_x = 0, \]

\[ (\rho u)_t + (\rho u^2 + P)_x = -g\rho, \]

for \( t, x \geq 0 \) subject to the boundary condition

\[ \rho u |_{x=0} = 0. \]

Here \( \rho, u, P \) and \( g > 0 \) are density, velocity, pressure and gravitational acceleration constant respectively. Equations (1)∼(3) describe the atmosphere on the flat earth \( \{ x \leq 0 \} \) moving in one direction under the constant gravitational force downward. In this work we assume that \( P = P(\rho) = A\rho^\gamma \) for some constants \( A, \gamma \) such that \( 0 < A, 1 < \gamma \leq 2 \). Then equilibria of (1) and (2) are of the form

\[ \bar{\rho} = \begin{cases} A_1(x_+ - x)^{\frac{\gamma-1}{\gamma}}, & \text{if } 0 \leq x \leq x_+, \\ 0, & \text{if } x_+ < x, \end{cases} \]

where \( A_1 = (((\gamma - 1)g/\gamma A)^{1/(\gamma - 1)} \) and \( x_+ \) is an arbitrary positive value, which represents the stratospheric depth.

Without loss of generality, we may assume \( x_+ = 1, A_1 = 1 \) and \( A = 1/\gamma \). It can be seen easily by scale transformations of the variables. Since the interface with the vacuum would vary with the time, it is convenient to transform the equations (1) and (2) into the Lagrangian form. More precisely, we introduce the variable

\[ m = \int_0^x \rho \, dx \]

as the independent variable instead of \( x \), then equations (1) and (2) can be transformed into the following second order equation:

\[ x_{tt} + P_m = -g, \]

where \( P = \gamma^{-1}(x_m)^{-\gamma} \). Let us fix an equilibrium

\[ x = \tilde{x}(m) = 1 - A_2(m_+ - m)^{\frac{\gamma - 1}{\gamma}}, \quad 0 \leq m \leq m_+, \]

where

\[ A_2 = \left( \frac{\gamma}{\gamma - 1} \right)^{\frac{1}{\gamma - 1}} \quad \text{and} \quad m_+ = \frac{\gamma - 1}{\gamma}. \]

Then we consider small perturbations of the equilibrium in (6) by putting \( x(t, m) = \tilde{x}(m) + y \). Under such assumption, the equation (5) is reduced to

\[ y_{tt} - (\gamma P G(\frac{1}{x_m} y_m))_m = 0, \]

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where
\[ G(v) = \frac{1}{\gamma}(1 - (1 + v)^{-\gamma}). \]

Take \( \bar{x}(m) \) as the independent variable instead of \( m \), and write it as \( x \). Then the equation (7) is reduced to
\[ y_{tt} - \frac{1}{\bar{\rho}(x)}(\gamma \bar{P}(x)G(y_x))_x = 0 \]
for \( 0 < x < 1 \) and the boundary condition is
\[ y|_{x=0} = 0. \]

Equation (8) is an apparently quasilinear hyperbolic equation. But it has a singularity at \( x = 1 \). Due to the singularity, the investigation for the existence of time periodic solutions becomes a difficult and challenging problem. To the best of our knowledge, the existence problem of time periodic solutions is still open.

For the sake of comparison, let us recall the results of [5], which considered the following simplified quasilinear wave equation
\[ \begin{cases} 
 y_{tt} - (G(y_x))_x = 0 \quad \text{for } 0 < x < 1, \\
 y(t, 0) = y(t, 1) = 0.
\end{cases} \]

This problem is derived from the Euler equations
\[ \begin{align*}
 \rho_t + (\rho u)_x &= 0, \\
 (\rho u)_t + (\rho u^2 + P)_x &= 0,
\end{align*} \]
and the boundary condition
\[ \rho u|_{x=0} = \rho u|_{x=L} = 0, \]
for which the equilibria are positive constant densities. Hence there are no troubles caused by contact with vacuum. For any fixed arbitrarily long time, [5] shows that there are smooth small amplitude solutions of the problem (10) for which the periodic solutions of the linearized equation are the first-order approximation. This result was established by the usual iteration method for quasi-linear wave equations.

Therefore, similarly, we want to find smooth solutions for which a time periodic solution of the linearized equation around an equilibrium is the first approximation even for the present problem (8)(9). However, contrary to the case without gravity, the usual iteration method for quasilinear hyperbolic problem
cannot be applied directly to the present problem because of the loss of regularities which causes from the touch with the vacuum. In this work we shall apply Nash-Moser theorem to establish long time existence of smooth solutions near time-periodic solution of the linearized equation.

More precisely speaking, we introduce the variable

$$z = 1 - x$$  \hspace{1cm} (11)

and small parameter $\varepsilon$, and we shall construct approximate solutions of the form

$$\sum_{k=1}^{K} y_k(t, z) \varepsilon^k,$$

where $y_k(t, z)$ are entire functions of $t$ and $z$, while $y_1(t, z)$ is a non-trivial time periodic solution of the linearized equation. Then our aim is to find a true smooth solution $y(t, z)$ of (8)(9) on $0 \leq t \leq T$ and $0 \leq z \leq 1$, for arbitrarily fixed $T$, such that

$$y(t, z) = \sum_{k=1}^{K} y_k(t, z) \varepsilon^k + O(\varepsilon^{K+1}).$$

Of course for large $T$ we should restrict $\varepsilon$ sufficiently small. Then

$$x(t, m) = \bar{x}(m) + y(t, 1 - \bar{x}(m))$$

is a solution in the Lagrangian variable and the corresponding density distribution $\rho = \rho(t, x)$, where $x$ denotes the original Euler coordinate, satisfies

$$\rho(t, x) > 0 \quad \text{for} \quad 0 \leq x < x_F(t)$$

and

$$\rho(t, x) = 0 \quad \text{for} \quad x_F(t) \leq x,$$

where

$$x_F(t) = 1 + y(t, 0)$$

is the position of the free boundary. Since $y(t, z)$ is smooth on $0 \leq z \leq 1$, we have

$$\rho(t, x) = C(t)(x_F(t) - x)^{\frac{1}{\gamma-1}} (1 + O(x_F(t) - x)), \quad (x < x_F(t))$$

and

$$\frac{\partial}{\partial x} \left( \frac{dP}{d\rho} \right) = \frac{\partial}{\partial x} \rho^{\gamma-1} \rightarrow -C(t)$$

at $x \rightarrow x_F(t) - 0$. This condition is that of “physical vacuum boundary” so called by the most recent works \[7\](2009) and \[1\](2011). This concept can be traced back to \[8\](1996), \[9\](2000), and \[13\](2006). Hence we can say that our purpose is to find long-time smooth solutions with “physical vacuum boundary”.

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But and are interested in short-time solutions to the initial value problem for the case without external force. So the motivation, methods and results are different from those of this work.

Now we have introduced the variable

\[ z = 1 - x. \]

Moreover it is convenient to introduce the parameter

\[ \gamma = 1 + \frac{2}{N - 2}. \] (12)

Then the assumption \( 1 < \gamma \leq 2 \) is equivalent to that \( 4 \leq N < \infty \). Hence, we assume \( N \geq 4 \) in the following of this work. Moreover, the equation (8) turns out to be

\[ \frac{\partial^2 y}{\partial t^2} - \triangle y = G_I(v)\triangle y + G_{II}(v), \] (13)

where

\[ \Delta := z \frac{\partial^2}{\partial z^2} + \frac{N}{2} \frac{\partial}{\partial z}, \quad v = -\frac{\partial y}{\partial z}, \] (14)

\[ G_I(v) := DG(v) - 1 = -\frac{2N - 2}{N - 2} v + [v]_2, \] (15)

\[ G_{II}(v) := \frac{N}{2} (vDG(v) - G(v)) = -\frac{N(N - 1)}{2(N - 2)} v^2 + [v]_3 \] (16)

and \([v]_q\) denotes a convergent power series of the form \( \sum_{j \geq q} a_j v^j \). If we introduce the variable \( r \) by

\[ z = 1 - x = \frac{r^2}{4}, \]

then

\[ \Delta = z \frac{\partial^2}{\partial z^2} + \frac{N}{2} \frac{\partial}{\partial z} = \frac{\partial^2}{\partial r^2} + \frac{N - 1}{r} \frac{\partial}{\partial r} \]

is the radial part of the Laplacian operator on the \( N \)-dimensional Euclidean space \( \mathbb{R}^N \), provided that \( N \) is an integer. But we shall not assume that \( N \) is an integer in this work.

Here we would like to spend few words to explain why the usual iteration does not work although the equation (13) is apparently quasi-linear. For the sake of simplicity, let us assume \( N \) is an integer. Then a smooth function \( y \) of \( z \) can be regarded as a smooth function of \( r = ||\vec{x}|| = (\sum_j (x_j)^2)^{1/2} \), where \( \vec{x} \in \mathbb{R}^N \). Since \( y \) is smooth and spherically symmetric, we can assume that \( \partial y/\partial r = 0 \) at \( r = 0 \) and

\[ -v = \frac{\partial y}{\partial z} = \frac{2}{r} \frac{\partial y}{\partial r} \rightarrow 2 \frac{\partial^2 y}{\partial r^2} \bigg|_{r=0} \]
as \( r \to 0 \). In other words, \( v = -\partial y / \partial z \) is not of the first order, but of the second order, which is of the same order as the principal part \( \triangle y = z \partial^2 y / \partial z^2 + N \partial y / 2 \partial z \). So, the loss of regularities cannot be recovered by one step of solving a (linear) wave equation. This is the reason why we try an application of the Nash-Moser theory. Note that this trouble comes from \( z = 0 \), that is, from the touch with vacuum at the free boundary.

2 Preparatory analysis of linear problems

First let us consider the linearized problem of (13):

\[
y_{tt} - \triangle y = 0, \quad y|_{z=1} = 0. \tag{17}
\]

In \cite{4} we showed that (17) admits a time periodic solution

\[
y = y_1 = \sin(\sqrt{\lambda_n} t + \theta)\Phi^{\frac{N}{2} - 1}_{1-n}(\lambda_n z), \tag{18}
\]

where \( \theta \) is a constant, \( \lambda_n \) is the eigenvalues of the operator \( -\triangle \) with the Dirichlet boundary condition, and

\[
\Phi^{\frac{N}{2} - 1}(X) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma\left(\frac{N}{2} - 1 + k + 1\right)} X^k
\]

is an entire function such that \( \Phi^{\frac{N}{2} - 1}(\lambda_n) = 0 \). In fact,

\[
\lambda_n = \frac{1}{4} (j_{\frac{N}{2} - 1,n})^2,
\]

where \( j_{\frac{N}{2} - 1,n} \) is the \( n \)-th positive zero of the Bessel function \( J_{\frac{N}{2} - 1} \), and

\[
J_\nu(\zeta) = \left(\frac{\zeta}{2}\right)^\nu \Phi_\nu\left(\frac{\zeta^2}{4}\right).
\]

More precisely speaking, we consider the Hilbert space \( \mathcal{X} \) which consists of functions of \( 0 \leq z \leq 1 \) endowed with the inner product

\[
(y_1|y_2)_X := \int_0^1 y_1(z)\overline{y_2(z)}z^{\frac{N}{2} - 1}dz.
\]

The self-adjoint operator \( T = -\triangle \) with boundary condition is defined on

\[
\mathcal{D}(T) = \{ y \in \mathcal{X} \mid \exists \eta_n \in C_0^\infty(0,1) \text{ such that } \eta_n \to y \text{ in } \mathcal{X}, \ Q[\eta_n - \eta_m] \to 0 \text{ as } m, n \to \infty, \text{ and } -\triangle y \in \mathcal{X} \text{ in distribution sense} \}.
\]

Here

\[
Q[\eta] := \int_0^1 \left| \frac{d\eta}{dz} \right|^2 z^{\frac{N}{2} - 1}dz
\]
and \(-\Delta y = f \in \mathfrak{X}\) in distribution sense” means that for any \(\eta \in C_0^\infty(0, 1)\) there holds 
\[
(y| - \Delta \eta)_\mathcal{X} = (f|\eta)_\mathcal{X}.
\]

By [4], we have
\[
\mathcal{D}(T) = \{y \in C(0, 1) \mid y \in \mathfrak{X}, \ y(1) = 0 \text{ and } -\Delta y \in \mathfrak{X} \text{ in distribution sense}\}
\]
and the spectrum of \(T\) consists of simple eigenvalues \(\lambda_1 < \lambda_2 < \cdots\), where \(\lambda_n = j_{\nu,n}^2/4\).

Moreover, we consider the problem
\[
-\lambda y - \Delta y = f(z), \quad z \in (0, 1). \tag{19}
\]
Here \(\lambda \geq 0\) and \(f\) are given.

**Proposition 1** The inverse \(T^{-1}\) is a compact operator.

**Proof.** If \(f \in \mathfrak{X}\), the solution of the problem \((19)\) with \(\lambda = 0\) is given by the formula
\[
y(z) = \frac{2}{N-2} \left( \int_z^1 f(\zeta) d\zeta + z^{-\frac{N}{2} - 1} \int_0^z f(\zeta) \zeta^{-\frac{N}{2} - 1} d\zeta - \int_0^1 f(\zeta) \zeta^{-\frac{N}{2} - 1} d\zeta \right).
\]

Since
\[
\int_z^1 |f(\zeta)| d\zeta \leq \sqrt{\int_z^1 \zeta^{-\frac{N}{2} - 1} d\zeta} \sqrt{\int_0^1 |f(\zeta)|^2 \zeta^{-\frac{N}{2} - 1} d\zeta}
\]
\[
\leq \begin{cases} 
\sqrt{\frac{2}{N-2}} z^{-\frac{N+4}{4}} ||f||_\mathcal{X}, & \text{if } N > 4, \\
|\log z|||f||_\mathcal{X}, & \text{if } N = 4
\end{cases}
\]
and
\[
\int_0^z |f(\zeta)| \zeta^{-\frac{N}{2} - 1} d\zeta \leq \sqrt{\frac{2}{N}} z^{\frac{N}{2}} ||f||_\mathcal{X},
\]
we see that
\[
|y(z)| \leq \begin{cases} 
Cz^{-\frac{N+4}{4}} ||f||_\mathcal{X}, & \text{if } N > 4, \\
C|\log z| \cdot ||f||_\mathcal{X}, & \text{if } N = 4
\end{cases}
\]
for some constant \(C > 0\). Moreover, we have
\[
\frac{dy}{dz} = -z^{-\frac{N}{2}} \int_0^z f(\zeta) \zeta^{-\frac{N}{2} - 1} d\zeta
\]
and which implies
\[
|\frac{dy}{dz}| \leq \sqrt{\frac{2}{N}} z^{-\frac{N}{2}} ||f||_\mathcal{X}.
\]
Therefore, Ascoli-Arzelà’s theorem implies that a sequence \( y_n \) converges on each compact subset of \( (0, 1] \) when \( f \) is confined in a bounded set of \( X \). On the other hand, since

\[
\int_0^1 |y(z)|^2 z^{N-1} dz \leq \begin{cases} 
C\delta^2 \|f\|_X^2, & \text{if } N > 4, \\
C\delta^2 |\log \delta|^2 \cdot \|f\|_X^2, & \text{if } N = 4,
\end{cases}
\]

we see that \( y_n \) converges in \( X \), too. □

Therefore, \( T \) is a self-adjoint operator whose inverse is compact and the following assertion holds. See, eg., [2].

**Proposition 2** If \( \lambda \geq 0 \), then the range \( \mathcal{R}(-\lambda + T) \) is closed and

\[
\mathcal{R}(-\lambda + T) = \mathcal{N}(-\lambda + T)^\perp.
\]

Thus, if \( \lambda = \lambda_n \) is an eigenvalue with an eigenfunction \( \phi_n \), then the problem of (19) admits a solution \( y \) in \( X \) if and only if

\[
(f|\phi_n)_X = 0.
\]

**Proposition 3** If \( f(z) \) is an entire function, then there is an entire function \( y(z) \) which solves the equations of (19).

Proof. Let

\[
f(z) = \sum_{k=0}^{\infty} c_k z^k.
\]

Let \( R \) be an arbitrarily large positive number. Since \( f \) is an entire function, there is a constant \( M \) such that \( |c_k| \leq M/R^k \) for all \( k \). We seek a solution \( y(z) \) of (19) in the form

\[
y = \sum_{k=0}^{\infty} a_k z^k.
\]

Substituting (21) into (19) and comparing the coefficients, we have the formula

\[
a_{k+1} = -\frac{\lambda a_k + c_k}{(k+1)(k + \frac{N}{2})}.
\]

Taking \( a_0 \) arbitrarily, we claim that there is a constant \( M > 0 \) such that \( |a_k| \leq M/R^k \) for all \( k \). Suppose \( k > \max\{R, \lambda + 1\} \) and \( |a_k| \leq M'/R^k \), then

\[
|a_{k+1}| \leq \frac{\lambda |a_k| + |c_k|}{k^2} \leq \frac{(\lambda + 1)(M + M')}{kR^{k+1}} \leq \frac{M'}{kR^{k+1}}
\]

provided that

\[
\frac{(\lambda + 1)(M + M')}{k} \leq M'.
\]

Hence the claim follows and the radius of convergence of \( \sum a_k z^k \) is larger than \( R \). □
Proposition 4 Suppose \( \lambda > 0 \) and \( f \) is an entire function, then any solution of (19) in \( \mathfrak{X} \) is an entire function. 

Proof. The homogeneous equation \(-\lambda y - \Delta y = 0\) admits a pair of linearly independent solutions \( y_1(z) = \Phi_{N-1}(\lambda z) \) and \( y_2(z) \) such that 
\[
y_2(z) \sim (\lambda z)^{-\frac{N-1}{2}} \text{ as } z \to 0.
\]

In fact, if we take the change of variables 
\[
\lambda z = \frac{r^2}{4} \quad \text{and} \quad y = r^{-\nu} w,
\]

then the equation \(-\lambda y - \Delta y = 0\) turns out to be the following Bessel equation: 
\[
\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} + \left( 1 - \frac{\nu^2}{r^2} \right) w = 0.
\]

If \( \nu \) is not an integer then \( J_\nu \) and \( J_{-\nu} \) are linearly independent solutions. On the other hand, if \( \nu \neq 0 \) is an integer, then \( J_\nu \) and the Bessel function of the second kind \( Y_\nu \) of the form 
\[
Y_\nu(r) = \frac{2}{\pi} J_\nu(r) \log \frac{r}{2} - \frac{1}{\pi} \left( \frac{r}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (\Psi(k+1) + \Psi(\nu+k+1))}{\nu! (\nu+k)!} \left( \frac{r^2}{4} \right)^k
- \frac{1}{\pi} \left( \frac{r}{2} \right)^{-\nu-1} \sum_{k=0}^{\nu-1} \frac{(\nu-1-k)!}{k!} \left( \frac{r^2}{4} \right)^k
\]

are linearly independent solutions. Here \( \Psi(x) := D\Gamma(x)/\Gamma(x) \). See [12]. Since \( N \geq 4 \), we see \( y_2 \) does not belong to \( \mathfrak{X} \). On the other hand, there is an entire function \( y = \psi_0(z) \) which satisfies (19) due to Proposition 3. Of course \( \psi_0 \in \mathfrak{X} \). Thus any solution \( y(z) \) of (19) can be written as 
\[
y(z) = \psi_0(z) + C_1 y_1(z) + C_2 y_2(z),
\]

in which, if \( y(z) \in \mathfrak{X} \), then \( C_2 = 0 \), and therefore, \( y(z) \) is an entire function, too. \( \square \)

3 Formal solution expanded as power series of parameters

Now we construct formal power series solution of (13). Let us fix a non-trivial solution 
\[
y_1 = \sin(\sqrt{\lambda_m t} + \theta_0) \phi_m(z)
\]
of the linearized problem, where 
\[
\phi_n(z) = \frac{\Phi_{N-1}(\lambda_n z)}{||\Phi_{N-1}(\lambda_n z)||_X}
\]

(22)
is the normalized eigenfunction in the Hilbert space $X$. According to the result of [4], we know that $(\phi_n)_{n=1,2,\ldots}$ forms a complete orthonormal system in $X$. Note that $\phi_n(z)$ is an entire function of $z$. Our purpose is to find a formal solution of (13) of the form

$$y(t,z) = \sum_{k=1}^{\infty} y_k(t,z)\varepsilon^k,$$  (23)

where $\varepsilon$ stands for a small parameter. Substituting (23) into the equation (13) and comparing the coefficients, we get the following sequence of linear equations

$$\left(\frac{\partial^2}{\partial t^2} - \triangle\right)y_k = \sum_{1\leq \ell, j_1+\cdots+j_{\ell}=k} G_{I\ell}v_{j_1}\cdots v_{j_{\ell}}\triangle y_j + \sum_{2\leq \ell, j_1+\cdots+j_{\ell}=k} G_{I\ell}v_{j_1}\cdots v_{j_{\ell}},$$  (24)

where

$$v_j = -\frac{\partial y_j}{\partial z}, \quad G_I(v) = \sum_{1\leq \ell} G_{I\ell}v^\ell \quad \text{and} \quad G_{II}(v) = \sum_{2\leq \ell} G_{II}\ell v^\ell.$$  

Starting from the fixed $y_1$, we can solve the equations (24) with the boundary condition $y_k(1) = 0$ successively.

### 3.1 Solution for $k = 2$

The equation of (24) for $k = 2$ is in the form

$$\left(\frac{\partial^2}{\partial t^2} - \triangle\right)y_2 = G_{I1}v_1\triangle y_1 + G_{I12}(v_1)^2 = -\frac{2(N-1)}{N-2}(\triangle y_1 + \frac{N}{4}v_1)v_1.$$  (25)

Since $y_1$ is an entire function, we can write the right-hand side of (25) by the form

$$f_0(z) + (\cos 2\Theta)f_1(z),$$

where $\Theta := \sqrt{\lambda_n}t + \theta_0$, $f_0$ and $f_1$ are entire functions of $z$. Here we have used

$$\sin^2\Theta = \frac{1}{2} - \frac{1}{2} \cos 2\Theta.$$  

Then solutions of the problem

$$-\triangle w = f_0(z), \quad w|_{z=1} = 0$$  (26)

can be represented by

$$w = -\frac{2}{N-2} \int_0^z \left(1 - \left(\frac{\zeta}{z}\right)^{\frac{N}{2}-1}\right)f_0(\zeta)d\zeta + \frac{2}{N-2} \int_0^1 \left(1 - \zeta^{\frac{N}{2}-1}\right)f_0(\zeta)d\zeta.$$  (27)
On the other hand, we consider the problem
\[
\left(\frac{\partial^2}{\partial t^2} - \triangle\right) w = (\cos 2\Theta)f_1(z), \quad w|_{z=1} = 0.
\] (28)

We need to consider the following two cases:

Case-1: \(4\lambda_{n_0}\) is not an eigenvalue;

Case-2: there is an eigenvalue \(\lambda_q = 4\lambda_{n_0}\).

First, we consider the Case-1. Then (28) has a solution of the form
\[
w(t, z) = (\cos 2\Theta)W(z),
\]
where \(W(z)\) satisfies
\[
\left( -4\lambda_{n_0} - \triangle\right) W = f_1(z), \quad W|_{z=1} = 0.
\] (29)
According to Proposition 3, the first equation of (29) has a solution \(W_0(z)\), which is an entire function of \(z\) such that \(W_0(0) = 1\). Then, for any constant \(C\),
\[
W(z) = W_0(z) + C\Phi_{\frac{N}{2} - 1}(4\lambda_{n_0}z)
\]
is a solution of (29), too. Since \(4\lambda_{n_0}\) is not an eigenvalue, we have \(\Phi_{\frac{N}{2} - 1}(4\lambda_{n_0}) \neq 0\). Therefore, we can choose \(C\) so that
\[
W(1) = W_0(1) + C\Phi_{\frac{N}{2} - 1}(4\lambda_{n_0}) = 0,
\]
i.e. \(W(z)\) satisfies the boundary value condition.

Next, we consider Case-2, i.e. \(\lambda_q = 4\lambda_{n_0}\) for some integer \(q\). We guess that this case could not happen actually for \(N \geq 4\). More generally, we have

**Conjecture** Let \(\nu \geq 1\) and \(\theta\) be a positive zero of the Bessel function \(J_\nu\). Then
\(J_\nu(L\theta) \neq 0\) for any integer \(L \geq 2\).

Note that the conclusion is not the case if \(\nu = 1/2\), for which \(J_{1/2}(r) = \sqrt{\frac{2}{\pi r}}\sin r\). However we have not yet verified this conjecture. Therefore we should consider Case-2. By Proposition 2, there is a solution \(W_1(z)\) of
\[
(-\lambda_q - \triangle) W_1 = f_1(z) := f_1(z) - (f_1|\phi_q)\chi_{\phi q}(z),
\]
which is entire and satisfies the boundary condition. Then it is easy to see that
\[
w = (\cos 2\Theta)W_1 - \frac{1}{2\sqrt{\lambda_q}} t \cdot \sin 2\Theta \cdot (f_1|\phi_q)\chi_{\phi q}(z)
\]
satisfies (28).

Summing up, we have a solution \(y_2\) of the form
\[
y_2(t, z) = y_{20}(z) + (\cos 2\Theta)y_{21}(z),
\] (30)
for Case-1, or
\[ y_2(t, z) = y_{20}(z) + (\cos 2\Theta)y_{21}(z) + t(\sin 2\Theta)y_{22}(z), \]
for Case-2, where \( y_{20}, y_{21} \) and \( y_{22} \) are entire functions of \( z \).

Suppose Case-1. Then the 2nd order approximate solution
\[ y^{(2)}(t, z) = \varepsilon y_1(t, z) + \varepsilon^2 y_2(t, z) \]
is a time-periodic solution with period \( \Omega = \frac{2\pi}{\sqrt{\lambda_{n_0}}} \).

According to [4], we know that any non-trivial true time-periodic solution of (10), if exists, with period \( T \) should satisfy
\[ \frac{1}{T} \int_0^T y(t, x = 1) dt > 0 \]  \hspace{1cm} (31)
as an effect of nonlinearity. This is true for the approximate solution \( y^{(2)} \), since
\[ \frac{1}{T} \int_0^T y^{(2)}(t, x = 1) dt = \varepsilon^2 y_{20}(0) \]
where
\[ y_{20}(0) = \frac{N - 1}{2(N - 2)} \int_0^1 \left( \frac{d\phi_{n_0}}{dz} \right)^2 dz > 0. \]
Using integration by parts, the form of \( y_{20}(0) \) can be checked by a tedious but direct computation.

We do not know whether Case-2 actually happens and the resonance could occur or not. This is an interesting open problem.

### 3.2 Solution for \( k = 3 \)

For the sake of simplicity, we assume Case-1 for \( k = 2 \). Fixing \( y_2 \) of the form (30), we see that the right-hand side of the equation (24) is of the form
\[ (\sin \Theta)g_1(z) + (\sin 3\Theta)g_3(z), \]
where \( g_1 \) and \( g_3 \) are entire functions of \( z \). Here we have used
\[ \sin^3 \Theta = \frac{3}{4} \sin \Theta - \frac{1}{4} \sin 3\Theta. \]
The equation
\[ \left( \frac{\partial^2}{\partial t^2} - \triangle \right) w = (\sin \Theta)g_1(z) \]  \hspace{1cm} (32)
has a solution of the form
\[ w_1(t, z) = (\sin \Theta)W_1(z) - \frac{1}{2 \sqrt{\lambda_{n_0}}} t \cdot (\cos \Theta) \cdot (g_1|\phi_{n_0})\phi_{n_0}(z), \]
where $W_1(z)$ is an entire function which solves the following equation
\[
(-\lambda_n - \Delta)W_1 = \tilde{g}_1 := g_1 - (g_1|\phi_n)\phi_n.
\] (33)

Then we can claim that $w_1(t, z)$ satisfies the boundary condition for any $W_1(z)$ with arbitrary $W_1(0)$. In fact, by Proposition 2 there is a solution $W$ in $X$ satisfying the equation
\[
(-\lambda_n - \Delta)W = \tilde{g}_1
\]
and the boundary condition. Then $U := W_1 - W$ belongs to $X$ and satisfies the homogeneous equation
\[
(-\lambda_n - \Delta)U = 0.
\]
As in the proof of Proposition 2 there is a constant $C$ such that $U = C\Phi_{N-1}(9\lambda_n z)$.

Hence, $W_1 = W + U = W + C\phi_n$ satisfies the boundary condition.

On the other hand, we consider the problem
\[
\left(\frac{\partial^2}{\partial t^2} - \Delta\right)w = (\sin 3\Theta)g_3(z), \quad w|_{z=1} = 0.
\] (34)

Similar to the discussion of $k = 2$, we need to consider the following two cases:

Case-1.1: $9\lambda_n$ is not an eigenvalue;

Case-1.2: there is an eigenvalue $\lambda_q = 9\lambda_n$.

Let us consider the Case-1.1 for simplicity. The problem (34) has a solution of the form $w_3(t, z) = (\sin 3\Theta)W_3(z)$, where $W_3(z)$ satisfies
\[
(-9\lambda_n - \Delta)W_3 = g_3(z), \quad W_3|_{z=1} = 0.
\] (35)

According to Proposition 2 the first equation of (35) has a solution $\bar{W}_3(z)$, which is an entire function of $z$ such that $\bar{W}_3(0) = 1$. Then, there exists a constant $C \neq 0$ such that
\[
W_3(z) = \bar{W}_3(z) + C\Phi_{N-1}(9\lambda_n z)
\]
is a solution of (35), too. Since $9\lambda_n$ is not an eigenvalue, we have $\Phi_{N-1}(9\lambda_n) \neq 0$. Therefore, we can choose $C$ so that
\[
W_3(1) = \bar{W}_3(1) + C\Phi_{N-1}(9\lambda_n) = 0,
\]
i.e. $W_3(z)$ satisfies the boundary value condition. Thus, we have a solution $y_3(t, z)$ of (24) of the form
\[
y_3(t, z) = C_3t(\cos \Theta)\phi_n(z) + (\sin \Theta)y_{31}(z) + (\sin 3\Theta)y_{33}(z),
\] (36)

where $y_{31}$ and $y_{33}$ are entire functions of $z$ and $C_3$ is a constant.

Now we can ask whether the approximate solution $y_3$ given by (36) is time-periodic or not, or in other words, whether the resonance occurs actually or not. It depends on whether
\[
C_3 = -\frac{1}{2\sqrt{\lambda_n}}(g_1|\phi_n)
\]
vanishes or not. We guess that $C_3 \neq 0$, but we have not yet verified it.
3.3 Solutions for \( k > 3 \)

Similar to the computations for the conclusions of previous subsections, we can determine solutions \( y_k \) of (24) for \( k > 3 \) successively in the form

\[
y_k(t,z) = \sum_{M \leq k-1, \; L \leq k} t^M((\cos L\Theta)V_{k,L,M}(z) + (\sin L\Theta)W_{k,L,M}(z)), \tag{37}
\]

where \( V_{k,L,M} \) and \( W_{k,L,M} \) are entire functions of \( z \). In order to prove it, we need the following lemma.

**Lemma 1** If \( f(z) \) is an entire function of \( z \), then the problem

\[
\left( \frac{\partial^2}{\partial t^2} - \Delta \right)y = t^M(\cos L\Theta)f(z) \quad \text{or} \quad \left( \frac{\partial^2}{\partial t^2} - \Delta \right)y = t^M(\sin L\Theta)f(z) \tag{38}
\]

admits a solution of the form

\[
y(t,z) = \sum_{m=0}^{M+1} t^m(\cos L\Theta)Y_m(z) \quad \text{or} \quad y(t,z) = \sum_{m=0}^{M+1} t^m(\sin L\Theta)Y_m(z), \tag{39}
\]

respectively. Here \( Y_m(z) \) are entire functions of \( z \).

**Proof.** The proof can be done easily when \( \sqrt{\lambda_m} \neq \sqrt{\lambda_n} \) for all \( m, n \in \mathbb{Z}^+ \). Here we only consider the case that there is a \( q \in \mathbb{Z}^+ \) such that \( \sqrt{\lambda_q} = \sqrt{\lambda_n} \). This happens at least if \( L = 1 \). In this case a solution \( \psi(z) \) of

\[
\left( -\lambda_q - \Delta \right)\psi = \tilde{f} := f - (f|\phi_q)\phi_q
\]

which is an entire function of \( z \) satisfies the boundary condition. Then the problem

\[
\left( \frac{\partial^2}{\partial t^2} - \Delta \right)y = t^M(\cos L\Theta)f + (-2ML\sqrt{\lambda}t^{M-1}\sin L\Theta + M(M-1)t^{M-2}\cos L\Theta)\psi
\]

admits a solution of the form

\[
y = t^M(\cos L\Theta)\psi + A(t)(f|\phi_q)\phi_q(z),
\]

where

\[
A(t) = \frac{1}{\sqrt{\lambda_q}} \int_0^t \sin \sqrt{\lambda_q}(t - \tau)^M \cos L\Theta(\tau)d\tau,
\]

in which

\[
L\Theta(\tau) = L(\sqrt{\lambda_m} \tau + \theta_0) = \sqrt{\lambda_q} \tau + L\theta_0.
\]

In fact, \( A(t) \) is a solution of the equation

\[
\frac{d^2A}{dt^2} + \lambda_q A = t^M \cos L\Theta.
\]

We see

\[
A(t) = \frac{1}{2\sqrt{\lambda_q}} \left( \sin L\Theta \right) \frac{t^{M+1}}{M+1} + O(t^M).
\]

Then, using the mathematical induction with respect to \( M \), the assertion of the Lemma follows. \( \square \)
4 Existence of smooth solutions

In this section, we will prove the existence of smooth solutions of (13), using the Nash-Moser theorem. In Section 3, we constructed the approximate solutions

\[ y(K)(t, z) := \sum_{k=1}^{K} y_k(t, z) \varepsilon^k. \] (40)

Fixing an arbitrarily large \( T \) and an integer \( K \), we want to find a solution \( y(t, z) \) of the original problem of the form

\[ y(t, z) = y(K) + \varepsilon^K w(t, z) \]

on the time interval \( 0 \leq t \leq T \). First, we derive the problem of \( w \). By (24), it follows that \( y(K) \) satisfies

\[ \frac{\partial^2 y(K)}{\partial t^2} - \triangle y(K) = \sum_{k=1}^{K} \left( \sum_{j_1 + \cdots + j_{\ell} + j = k} G_{I\ell} v_{j_1} \cdots v_{j_{\ell}} \triangle y_j \right) \varepsilon^k + \sum_{k=1}^{K} \left( \sum_{j_1 + \cdots + j_{\ell} = k} G_{II\ell} v_{j_1} \cdots v_{j_{\ell}} \right) \varepsilon^k. \]

Let us denote \( v(K) := -\partial y(K)/\partial z \) and \( P := -\partial w/\partial z \). Then \( w \) satisfies

\[ \frac{\partial^2 w}{\partial t^2} - \triangle w = G_I(v(K) + \varepsilon^K P) \triangle w + F_I + F_{II}, \] (41)

where

\[ \varepsilon^{K+1} F_I = G_I(v(K) + \varepsilon^K P) \triangle y(K) - \sum_{k=1}^{K} \left( \sum_{j_1 + \cdots + j_{\ell} + j = k} G_{I\ell} v_{j_1} \cdots v_{j_{\ell}} \triangle y_j \right) \varepsilon^k, \] (42)

\[ \varepsilon^{K+1} F_{II} = G_{II}(v(K) + \varepsilon^K P) - \sum_{k=1}^{K} \left( \sum_{j_1 + \cdots + j_{\ell} = k} G_{II\ell} v_{j_1} \cdots v_{j_{\ell}} \right) \varepsilon^k. \] (43)

Let us denote

\[ \varepsilon a(t, z, P, \varepsilon) := G_I(v(K)(t, z) + \varepsilon^K P), \] (44)

\[ b(t, z, P, \varepsilon) := -(F_I + F_{II}) + (F_I + F_{II})|_{P=0}, \] (45)

\[ c(t, z, \varepsilon) := (F_I + F_{II})|_{P=0}. \] (46)

Then equation (41) can be written as

\[ \frac{\partial^2 w}{\partial t^2} - \left( 1 + \varepsilon a(t, z, -\partial w/\partial z, \varepsilon) \right) \triangle w + \varepsilon b(t, z, -\partial w/\partial z, \varepsilon) = \varepsilon c(t, z, \varepsilon). \] (47)
Note that $a(t, z, P, \varepsilon)$ and $b(t, z, P, \varepsilon)$ are analytic functions of

$$|t| \leq T, \quad |z| \leq T + 1, \quad |\varepsilon K| \leq \delta_0 = \delta_0(T, K), \quad |\varepsilon| \leq \varepsilon_0 = \varepsilon_0(T, K)$$

such that $b(t, z, 0, \varepsilon) = 0$, and $c(t, z, \varepsilon)$ is an analytic function of

$$|t| \leq T, \quad |z| \leq T + 1, \quad |\varepsilon| \leq \varepsilon_0 = \varepsilon_0(T, K).$$

Our goal is to seek a smooth solution $w(t, z)$ of the equation (47) such that

$$w(0, z) = w_t(0, z) = 0, w(t, 1) = 0$$

for sufficiently small $\varepsilon$. For completeness, we recall the Nash-Moser theorem as follows.

**Nash-Moser Theorem** (see [3], p.171, III.1.1.1) Let $E_0$ and $E$ be tame spaces and $P : U(\subset E_0) \to E$ a smooth tame map. Suppose that the equation for the derivative $D_P(w)h = g$ has a unique solution $h = V_P(w, g)$ for all $w$ in $U$ and all $g$, and that the family of inverses $V_P : U \times E \to E_0$ is smooth tame map. Then $P$ is locally invertible, and each local inverse $P^{-1}$ is a smooth tame map.

Now we define the spaces $E_0$, $E$ and nonlinear mapping $P : U(\subset E_0) \to E$ by

$$E_0 := \{ w \in C^\infty([0, T] \times [0, 1]) \mid w|_{t=0} = w_t|_{t=0} = 0, w|_{z=1} = 0 \},$$

$$E := C^\infty([0, T] \times [0, 1]),$$

$$P(w) := \partial^2 w \partial t^2 - (1 + \varepsilon a_1(t, z, -\partial w \partial z, \varepsilon)) \Delta w + \varepsilon b_1(t, z, -\partial w \partial z, \varepsilon).$$

Note that $P(0) = 0$ and take a neighborhood $U$ of 0 such that $|\varepsilon|^K \|\partial w / \partial z\|_{L^\infty} \leq \delta_0$ for $w \in U$. Then the equation (47) can be written by

$$P(w) = \varepsilon c(t, z, \varepsilon).$$

By definition of $P$, it is easy to see the Fréchet derivative $D_P$ of the mapping $P$ at a fixed $w \in U \subset E_0$ is of the form

$$D_P(w)h = \partial^2 h \partial t^2 - (1 + \varepsilon a_1(t, z, -\partial w \partial z, \varepsilon)) \Delta h + \varepsilon b_1(t, z, -\partial w \partial z, \varepsilon) \frac{\partial h}{\partial z}.$$ 

where

$$a_1(t, z, \varepsilon) := a(t, z, -\frac{\partial w}{\partial z}, \varepsilon),$$

$$a_2(t, z, \varepsilon) := \frac{\partial a}{\partial P}(t, z, -\frac{\partial w}{\partial z}, \varepsilon) \Delta w - \frac{\partial b}{\partial P}(t, z, -\frac{\partial w}{\partial z}, \varepsilon).$$

Suppose the following statements hold:

(S1) $P$ is a smooth tame map, $E$ being endowed with a suitable system of graded norms (for definition, see [3]) ;
(S2) for any \( w \in \mathcal{U} \subset \mathcal{E}_0, g \in \mathcal{E} \), there is a unique solution \( h := V\Psi(w, g) \) of the equation

\[
D\Psi(w, h) = g
\]  

(51)

and the mapping \( V\Psi : \mathcal{U} \times \mathcal{E} \rightarrow \mathcal{E}_0 \) is a smooth tame map.

Then it follows from Nash-Moser Theorem that \( \Psi \) is invertible in a neighborhood \( \mathcal{U} \) of 0 in \( \mathcal{E}_0 \). Thus the inverse image \( w = \Psi^{-1}(\epsilon c) \) is a solution of our problem (47) or (48), where \( \epsilon \) is sufficiently small. More precisely, we have the following results.

**Theorem 1** There is a positive constant \( \epsilon_1 = \epsilon_1(T, K) \) such that for \( |\epsilon| \leq \epsilon_1 \) there exists a smooth solution \( w = w(t, z) \) of (47) defined on \( 0 \leq t \leq T, 0 \leq z \leq 1 \) such that \( w(0, z) = w_t(0, z) = 0, w(t, 1) = 0 \) and \( w = O(\epsilon) \). In other words, there is a smooth solution \( y = y(t, z) \) of (13) such that \( y_{z=1} = 0 \) and

\[
y(t, z) = y^{(K)}(t, z) + O(\epsilon^{K+1}).
\]

First of all we must show that the linear equation (51) can be solved uniquely. But the term \( a_2 \cdot \frac{\partial h}{\partial z} \) in \( D\Psi(w)h \) could cause trouble, since this term can have same order as the principal part \( \Delta h = \frac{\partial^2 h}{\partial z^2} + \frac{N}{2} \frac{\partial h}{\partial z} \). When we try to get the energy estimate, keeping in mind that \( (-\Delta h)_X = \|\sqrt{\epsilon}\partial h/\partial z\|_X^2 \), we could not estimate \( \|\partial h/\partial z\|_X \) by \( \|\sqrt{\epsilon}\partial h/\partial z\|_X \) because of the singularity at \( z = 0 \). However we have fortunately the following observation:

**Proposition 5** For any fixed \( w \) in the neighborhood \( \mathcal{U} \subset \mathcal{E}_0 \) there is a smooth function \( a_2(t, z, \epsilon) \) of \( 0 \leq t \leq T, 0 \leq z \leq 1, |\epsilon| \leq \epsilon_0(\mathcal{U}) \) such that

\[
a_2(t, z, \epsilon) = z\hat{a}_2(t, z, \epsilon).
\]

Proof. By (50), we can write

\[
a_2(t, z, \epsilon) = \frac{\partial a}{\partial P} \cdot z^2 \frac{\partial^2 w}{\partial z^2} - \left( \frac{N}{2} \frac{\partial a}{\partial P} + \frac{\partial h}{\partial P} \right),
\]  

(52)

since

\[
\Delta w = \frac{\partial^2 w}{\partial z^2} + \frac{N}{2} \frac{\partial w}{\partial z} = \frac{\partial^2 w}{\partial z^2} - \frac{N}{2} P.
\]

It follows from (12), (13) and (14) that

\[
\epsilon \frac{\partial a}{\partial P} = \epsilon K DG_1(v^{(K)} + \epsilon K P) = \epsilon K D^2 G(v^{(K)} + \epsilon K P),
\]

\[
\epsilon^{K+1} \frac{\partial F_1}{\partial P} = \epsilon K D^2 G(v^{(K)} + \epsilon K P) \Delta y^{(K)}
\]

\[
= - \frac{N}{2} D^2 G(v^{(K)} + \epsilon K P) \cdot \psi^{(K)} + D^2 G(v^{(K)} + \epsilon K P) \cdot (z \frac{\partial^2 y^{(K)}}{\partial z^2}),
\]

\[
\epsilon^{K+1} \frac{\partial F_{11}}{\partial P} = \epsilon K DG_{11}(v^{(K)} + \epsilon K P)
\]

\[
= \epsilon K \cdot \frac{N}{2} (v^{(K)} + \epsilon K P) \cdot D^2 G(v^{(K)} + \epsilon K P),
\]

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since
\[ DG_{II}(v) = \frac{N}{2} v D^2 G(v), \]
or
\[ \varepsilon \frac{\partial F_{II}}{\partial P} = \frac{N}{2} \left[ (v^{(K)} + \varepsilon K P) D^2 G(v^{(K)} + \varepsilon K P) \right]. \]
Hence, we have
\[ \varepsilon \left( \frac{N}{2} P \frac{\partial a}{\partial P} + \frac{\partial b}{\partial P} \right) = \varepsilon \left( \frac{N}{2} P \frac{\partial a}{\partial P} - \left( \frac{\partial F_I}{\partial P} + \frac{\partial F_{II}}{\partial P} \right) \right) \]
\[ = - D^2 G(v^{(K)} + \varepsilon K P) \cdot \varepsilon \frac{\partial^2 y^{(K)}}{\partial z^2}. \]
Therefore, \( a_2(t,z,\varepsilon) = z \hat{a}_2(t,z,\varepsilon) \) by putting
\[ \hat{a}_2(t,z,\varepsilon) := D^2 G(v^{(K)} + \varepsilon K P) \cdot \frac{\partial^2}{\partial z^2} \varepsilon^{-1} (y^{(K)} + \varepsilon K w). \]
□

Thanks to Proposition 5, we can obtain the following energy inequality.

**Proposition 6** Assume that \( |\varepsilon a_1| \leq 1/2 \) uniformly for \( 0 \leq t \leq T, \ 0 \leq z \leq 1 \) and \( |\varepsilon| \leq \varepsilon_0 \). Then there is a constant \( C \) such that if \( h \in \mathcal{E}_0 \) and \( g \in \mathcal{E} \) satisfy
\[ \frac{\partial^2 h}{\partial t^2} - (1 + \varepsilon a_1) \Delta h + \varepsilon a_2 \frac{\partial h}{\partial z} = g, \quad (53) \]
then
\[ \| \frac{\partial h}{\partial t} \|_X + \| \sqrt{z} \frac{\partial h}{\partial z} \|_X \leq C \int_0^t \| g(\tau) \|_X d\tau \quad (54) \]
for \( 0 \leq t \leq T \), where
\[ \| g(t) \|_X = \left( \int_0^1 |g(\tau,z)|^2 z^{\frac{N}{2} - 1} dz \right)^{1/2}. \]

Proof. Let us consider the energy
\[ E(t) := \int_0^1 \left( (h_1)^2 + (1 + \varepsilon a_1) z (h_2)^2 \right) z^{\frac{N}{2} - 1} dz. \]
We claim that there is a constant \( A \) such that
\[ E(t)^{1/2} \leq \int_0^t e^{A(t-s)} \| g(s) \|_X ds. \]
By Proposition 5 the coefficient \( a_2(t,z,\varepsilon) \) is of the form
\[ a_2(t,z,\varepsilon) = z \hat{a}_2(t,z,\varepsilon), \]
where \( \hat{a}_2 \) is a smooth function of \( 0 \leq t \leq T, \ 0 \leq z \leq 1 \) and \( |\varepsilon| \leq \varepsilon_0 \). Then the equation (53) turns out to be

\[
h_{tt} - (1 + \varepsilon a_1) \Delta h + \varepsilon \hat{a}_2 z h_z = g. \tag{55}
\]

Multiplying equation (55) by \( h_t \) and integrating it by \( dv = z^{N-1}dz \) from \( z = 0 \) to \( z = 1 \), we obtain that

\[
\frac{1}{2} \int_0^1 \frac{\partial}{\partial t} h_t^2 dv - \int_0^1 (1 + \varepsilon a_1) \Delta h h_t dv + \int_0^1 \varepsilon \hat{a}_2 z h_z h_t dv = \int_0^1 g(t, z) h_t dv. \tag{56}
\]

Using an integration by parts under the boundary condition, we have

\[
\int_0^1 (1 + \varepsilon a_1) \Delta h h_t dv = \int_0^1 (z h_{zz} + \frac{N}{2} h_z (1 + \varepsilon a_1) h_t z^{N-1} - \int_0^1 \varepsilon h_z (a_1)_z h_t z^{N-1} dz
\]

\[
= -\int_0^1 h_z \frac{\partial}{\partial z} (1 + \varepsilon a_1) h_t z^{N-1} dz + \int_0^1 \frac{N}{2} h_z (1 + \varepsilon a_1) h_t z^{N-1} dz
\]

\[
= -\int_0^1 h_z h_{zt} (1 + \varepsilon a_1) z^{N-1} dz - \int_0^1 \varepsilon h_z (a_1)_z h_t z^{N-1} dz
\]

\[
= -\frac{1}{2} \frac{d}{dt} \int_0^1 (1 + \varepsilon a_1) h_t^2 z^{N-1} dz + \frac{1}{2} \int_0^1 \varepsilon (a_1)_t h_t^2 z^{N-1} dz
\]

\[
- \int_0^1 \varepsilon h_z (a_1)_z h_t z^{N-1} dz. \tag{57}
\]

Since \( 1 + \varepsilon a_1 \geq 1/2 \), we have \( \int_0^1 z(h_z)^2 dv \leq 2E \). Therefore, it follows from (55), (56) and (57) that

\[
\frac{1}{2} \frac{dE}{dt} = \frac{1}{2} \varepsilon \int_0^1 (a_1)_t z(h_z)^2 dv - \varepsilon \int_0^1 (a_1)_z z h_z h_t dv
\]

\[
-\varepsilon \int_0^1 \hat{a}_2 z h_z h_t dv + \int_0^1 g h_t dv
\]

\[
\leq AE + \|g(t)\|_X E^{1/2},
\]

where

\[
A := \varepsilon (\|\partial a_1/\partial t\|_{L\infty} + \sqrt{2}\|\sqrt{2}(\partial a_1/\partial z + \hat{a}_2)\|_{L\infty}).
\]

Hence, the Gronwall’s argument implies

\[
E(t)^{1/2} \leq e^{At} \left( E(0)^{1/2} + \int_0^t \|g(s)\|_X e^{-As} ds \right).
\]

Since \( E(0) = 0 \) from the initial condition for \( h \in \mathcal{C}_0 \), we get the required inequality (54). \( \square \)

As a corollary of Proposition 6, \( g = 0 \) implies \( h = 0 \) by the boundary condition and which implies that (51) has a unique solution. Moreover, this consideration of energy is sufficient to claim that the inverse \( \mathcal{V}(w, \cdot) \) of \( \mathcal{D}(w, \cdot) \).
exists. This can be verified by the standard method on solving the initial boundary value problem to linear wave equations with smooth coefficients. See, e.g., Chapter 2 of [6]. In fact, for any fixed $t_0$, if we consider the Hilbert space $H = X_1 \times X$ and the operator $A(t_0)$, whose domain $D(A(t_0))$ is

$$
D(A(t_0)) = \{ \vec{h} = (h_0, h_1)^T \in H \mid h_0 \in X_2, h_1 \in X_1, h_0|_{z=1} = h_1|_{z=1} = 0 \},
$$

by

$$
A(t_0) \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} h_1 \\ (1 + \varepsilon a_1(t_0, z))\triangle h_0 - \varepsilon \hat{a}_2(t_0, z) h_0 \end{pmatrix},
$$

then the problem

$$
d\vec{h}/dt = A(t_0) \vec{h} + \vec{g}(t), \quad \vec{h}|_{t=0} = \vec{h}_0 \in D(A(t_0)),$$

where $\vec{g}(t) = (0, g(t, \cdot))^T$, allows the application of Hille-Yosida theory. Note that $-(1 + \varepsilon a_1)\triangle h = f$ means that

$$
(\sqrt{zh}_z)(\sqrt{z}(1 + \varepsilon a_1)\phi)_z = (f|\phi)x
$$

for any test function $\phi$ or $\phi \in X_1$. Here $X_1$ denotes the space of functions $g(z) \in X$ such that $\sqrt{z}dy/dz \in X$ and $X_2$ denotes the space of functions $g \in X_1$ such that $-\triangle g \in X$. For more details we refer the reader to [6].

Next, we show that the Fréchet space $E$ is tame for some gradings of norms. For $y \in E$, $n \in \mathbb{N}$, let us define

$$
\|y\|_n^{(\infty)} := \sup_{0 \leq j + k \leq n} \left\| \left( -\partial^2/\partial t^2 \right)^j (-\Delta)^k y \right\|_{L^\infty([0,T] \times [0,1])},
$$

Then we can claim that $E$ turns out to be tame by this grading $(\| \cdot \|_n^{(\infty)})_n$ (see [3], p.136, II.1.3.6 and p.137, II.1.3.7). In fact, even if $N$ is not an integer, we can define the Fourier transformation $Fy(\zeta)$ of a function $y(z)$ for $0 \leq z < \infty$ by

$$
Fy(\zeta) := \int_0^\infty K(\zeta z) y(z) z^{-N/2 - 1} dz.
$$

Here $K(X)$ is an entire function of $X \in \mathbb{C}$ given by

$$
K(X) = 2(\sqrt{X})^{-N+1/2} J_{N-1}(4\sqrt{X}) = 2^{N/2} \Phi^{N-1}_N(X),
$$

$J_{\nu}$ being the Bessel function. Then we have

$$
F(-\Delta y)(\zeta) = 4\zeta \cdot Fy(\zeta)
$$

and the inverse of the transformation $F$ is $F$ itself. See, e.g., [11]. Then it is easy to see $E$ endowed with the grading $(\| y \|_n^{(\infty)})$ of the form is a tame direct summand of the tame space

$$
L_1^\infty(\mathbb{R} \times [0, \infty), d\tau \otimes \zeta^{-N/2 - 1} d\zeta, \log(1 + \tau^2 + 4\zeta))
$$
through the Fourier transformation

\[ \mathcal{F}y(\tau, \zeta) = \frac{1}{\sqrt{2\pi}} \int e^{-\sqrt{-1}\tau t} Fy(t, \cdot)(\zeta) dt \]

and its inverse applied to the space \( C^\infty_0((-2T, 2T) \times [0, T + 1]) \), into which functions of \( \mathcal{E} \) can be extended (see, e.g. [10], p.189, Theorem 3.13) and the space

\[ \dot{C}^\infty(\mathbb{R} \times [0, \infty)) := \{ y|\forall j \forall k \lim_{R \to \infty} \sup_{|t| \geq R, x \geq R} |(-\partial_t^2)^j (-\triangle)^k y| = 0 \}, \]

for which functions of \( \mathcal{E} \) are restrictions. For the details, see the proof of [3], p.137, II.1.3.6. Theorem.

On the other hand, let us define

\[ \| y \|^{(2)}_n := \left( \sum_{0 \leq j + k \leq n} \int_0^T \| (-\partial_t^2)^j (-\triangle)^k y \|^2_X dt \right)^{1/2}. \]

We have

\[ \sqrt{N/2} \| y \|_X \leq \| y \|_{L^\infty} \leq C \sup_{j \leq \sigma} \| (-\triangle)^j y \|_X, \]

by the Sobolev imbedding theorem (see Appendix A), provided that \( 2\sigma > N/2 \). The derivatives with respect to \( t \) can be treated more simply. Then we see that the grading \( (\| \cdot \|^{(2)}_n)_n \) is equivalent to the grading \( (\| \cdot \|^{(\infty)}_n)_n \). Hence \( \mathcal{E} \) is tame with respect to \( (\| \cdot \|^{(2)}_n)_n \). The grading \( (\| \cdot \|^{(2)}_n)_n \) is suitable for energy estimates. Note that \( \mathcal{E}_0 \) is a closed subspace of \( \mathcal{E} \) endowed with these gradings.

Now we show the statement (S1) by verifying the nonlinear mapping \( \mathcal{P} \) is tame for the grading \( (\| \cdot \|^{(\infty)}_n)_n \). To do so, we write

\[ \mathcal{P}(w) = F(t, z, Dw, w_{tt}, \triangle w), \]

where \( D = \partial/\partial z \), \( F \) is a smooth function of \( t, z, Dw, w_{tt}, \triangle w \) and linear in \( w_{tt}, \triangle w \). According to [3] (see p.142, II.2.1.6 and p.145, II.2.2.6), it is sufficient to prove the linear differential operator \( w \mapsto Dw = \partial w/\partial z \) is tame. But it is clear because of the following result.

**Proposition 7** For any \( m \in \mathbb{N} \) we have the formula

\[ \triangle^m Dw(z) = z^{N/2 - m - 1} \int_0^z \triangle^{m+1} w(\zeta) \zeta^{-N/2 + m + 1} d\zeta. \]

As a corollary it holds that, for any \( m, k \in \mathbb{N} \),

\[ \|(-\triangle)^m D^k w\|_{L^\infty} \leq \frac{1}{\prod_{j=0}^{k-1}(N/2 + m + j)} \|(-\triangle)^m D^k w\|_{L^\infty}. \]
Proof. It is easy by integration by parts in induction on \( m \) starting from the formula

\[
Dw(z) = z^{-\frac{N}{2}} \int_0^z \Delta w(\zeta)\zeta^{-\frac{N}{2}-1}d\zeta.
\]

\[\Box\]

In parallel with the results of [3] (see p.144, II.2.2.3.Corollary and p.145, II.2.2.5.Theorem), we should use the following two propositions. Proofs for these propositions are given in Appendix B.

**Proposition 8** For any positive integer \( m \), there is a constant \( C \) such that

\[
\|\Delta^m(f \cdot g)\| \leq C(\|\Delta^m f\|\|g\| + \|f\|\|\Delta^m g\|),
\]

where \( \| \cdot \| \) stands for \( \| \cdot \|_{L^\infty} \).

**Proposition 9** Let \( F(z,y) \) be a smooth function of \( z \) and \( y \) and \( C_0 \) be a positive number. Then for any positive integer \( m \), there is a constant \( C > 0 \) such that

\[
\|\Delta^m F(z,y(z))\|_0 \leq C(1 + \|y\|_m)
\]

provided that \( \|y\|_0 \leq C_0 \), where we denote

\[
\|y\|_m = \sup_{0 \leq j \leq m} \|(-\Delta)^j y\|_{L^\infty}.
\]

Now, we are going to deduce an energy estimate of the higher order derivatives of \( h \in \mathcal{E}_0 \) satisfying (51) for \( g \in \mathcal{E} \), that is,

\[
h_{tt} - (1 + \varepsilon a_1)\Delta h + \varepsilon a_2 \hat{D}h = g,
\]

where \( \hat{D} := z \cdot \partial / \partial z \). We shall use the following lemma.

**Lemma 2** If

\[
\frac{d^2 Y}{dt^2} \leq A \frac{dY}{dt} + BY + f(t), \quad Y|_{t=0} = \frac{dY}{dt}|_{t=0} = 0,
\]

then

\[
Y(t) \leq \int_0^t e^{\frac{A}{2}(t-s)} \frac{\sinh \sqrt{B + \frac{A^2}{4}(t-s)}}{\sqrt{B + \frac{A^2}{4}}} \cdot f(s)ds.
\]

Proof. Putting \( Z(t) := e^{-\lambda t}Y(t) \), it is easy to see that

\[
\frac{d^2 Z}{dt^2} \leq (A - 2\lambda) \frac{dZ}{dt} + e^{-\lambda t}f(t).
\]
Here $\lambda := \frac{A}{2} + \sqrt{B + \frac{A^2}{4}}$ is the positive root of the equation $\lambda^2 = A\lambda + B$. Since $\frac{dZ}{dt} \bigg|_{t=0} = 0$, we have

$$\frac{dZ}{dt} \leq \int_0^t e^{(A-2\lambda)(t-s)} e^{-\lambda s} f(s) ds.$$  

Using the condition $Z(0) = 0$, we have

$$Y(t) \leq e^{\lambda t} \int_0^t e^{(A-2\lambda)s} \int_0^s e^{(A+\lambda)s} f(s) ds ds'.$$

Hence, the estimate of this lemma follows. □

Now we look at the equation satisfied by the higher order derivative of $h$. Let us denote $H_m := \triangle^m h$ for $m \in \mathbb{N}$. Differentiating the equation (59), we can show that $H_m$ satisfies the equation

$$(H_m)_{tt} - (1 + \varepsilon a_1) \triangle H_m + \varepsilon b^{(m)}_1 \tilde{D} H_m + \varepsilon b_0^{(m)} H_m = \triangle^m g + \varepsilon \sum_{j=0}^{m-1} (c^{(m)}_j \tilde{D} H_j + c_0^{(m)} H_j).$$  

(60)

Here $b^{(m)}_1 = b^{(m)}_1(t, z, \varepsilon)$ and so on are smooth functions of $0 \leq t \leq T$, $|z| \leq T+1$ and $|\varepsilon| \leq \varepsilon_0$. To verify this expression, we can use the following calculus formula:

$$\triangle(Q \triangle H) = Q \triangle^2 H + 2(DQ) \tilde{D} \triangle H + (\triangle Q)(\triangle H),$$  

$$\triangle(Q \tilde{D} H) = Q \tilde{D} \triangle H + (1 + 2\tilde{D}) Q \cdot \triangle H + (\triangle - (N-2)D) Q \cdot \tilde{D} H,$$

$$\triangle(Q H) = Q \triangle H + 2(DQ) \tilde{D} H + (\triangle Q) H,$$

which can be verified by

$$\tilde{D}^2 = z \triangle - (\frac{N}{2} - 1) \tilde{D}, \quad \triangle \tilde{D} = \tilde{D} \triangle + \triangle$$

and so on. Let us write (60) as

$$H_{tt} - (1 + \varepsilon a_1) \triangle H + \varepsilon b_1 \tilde{D} H + \varepsilon b_0 H = F.$$  

Then we see that the energy

$$E(t) := \int_0^1 ((H_t)^2 + (1 + \varepsilon a_1) z(H_z)^2) \frac{dz}{z}$$

enjoys the estimate

$$\frac{1}{2} \frac{dE}{dt} \leq AE + B \int_0^1 H H_t dz + E^{1/2} \|F\|_X.$$  

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Here
\[ A := \varepsilon \|a_{1,t}\|_{L^\infty} + \sqrt{2}\|\sqrt{2}(a_{1,z} + b_1)\|_{L^\infty}, \]
\[ B := \varepsilon \|b_0\|_{L^\infty}. \]

Now we estimate the integral \( \int_0^1 HH_t d\nu \). Since \( H|_{t=0} = H|_{t=0} = 0 \), then we see
\[ U(t) := \|H\|_X^2 = \int_0^1 H^2 d\nu \]
enjoys
\[ \frac{1}{2} \frac{dU}{dt} = \int_0^1 HH_t d\nu \leq U^{1/2}E^{1/2}, \quad U|_{t=0} = 0. \]
Thus we have
\[ U(t)^{1/2} = \|H\|_X \leq \int_0^t E(s)^{1/2} ds \]
and
\[ \left| \int_0^1 HH_t d\nu \right| \leq E(t)^{1/2} \int_0^t E(s)^{1/2} ds. \]

Therefore, we get the integro-differential inequality
\[ \frac{1}{2} \frac{dE}{dt} \leq AE + BE(t)^{1/2} \int_0^t E(s)^{1/2} ds + E^{1/2}\|F(t)\|_X. \]

Taking \( Y := \int_0^t E(s)^{1/2} ds \) in Lemma 2, we have
\[ \int_0^t E(s)^{1/2} ds \leq \hat{f}(t) := \int_0^t e^{A(t-s)} \frac{\sinh \sqrt{B + \frac{A^2}{4}(t-s)}}{\sqrt{B + \frac{A^2}{4}}} \cdot \|F(s)\|_X ds. \]

Therefore, we get
\[ \frac{1}{2} \frac{dE}{dt} \leq AE + (B\hat{f}(t) + \|F(t)\|_X) E(t)^{1/2} \]
which implies the energy estimate
\[ E(t)^{1/2} \leq \int_0^t e^{A(t-s)} (B\hat{f}(s) + \|F(t)\|_X) ds. \]

Thus we have the following estimate:

**Proposition 10** We have
\[ \|\frac{\partial H}{\partial t}\|_X + \sqrt{2}\|\frac{\partial H}{\partial z}\|_X + \|H\|_X \leq C \sup_{0 \leq t \leq T} \|F(t)\|_X, \]

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where
\[ H = H_m = \Delta^m h, \]
\[ F(t) = F_m(t) = \Delta^m g + \varepsilon \sum_{j=0}^{m-1} (c_{1j}^{(m)} \Delta H_j + c_{0j}^{(m)} H_j), \]
and the constant \( C \) depends only upon
\[ A = A_m := \varepsilon \|a_{1,1}\|_{L^\infty} + \sqrt{2}\|\sqrt{2} (a_{1,z} + b_1^{(m)})\|_{L^\infty}, \]
\[ B = B_m := \varepsilon \|b_0^{(m)}\|_{L^\infty} \]
and \( T \).

In order to verify that the mapping \( \Psi : \Omega \subset \Theta_0 \times \Theta \to \Theta_0 \) is tame, we should analyze the coefficients \( b_1^{(m)}, b_0^{(m)}, c_{1j}^{(m)}, c_{0j}^{(m)} \) more concretely. The recurrence formula for these coefficients are:
\[
\begin{align*}
  b_1^{(m+1)} &= -2D a_1 + b_1^{(m)}, \\
  b_0^{(m+1)} &= -\triangle a_1 + (1 + 2\hat{\Delta}) b_1^{(m)} + b_0^{(m)}, \\
  c_{1j}^{(m+1)} &= c_{1,j-1}^{(m)} + (\triangle - (N - 2)\hat{\Delta}) c_{1,j}^{(m)} + 2Dc_{0j}^{(m)}, \quad \text{for } j \leq m - 1, \\
  c_{1m}^{(m+1)} &= -(\triangle - (N - 2)\hat{\Delta}) b_1^{(m)} - 2D b_0^{(m)} + c_{1,m-1}^{(m)}, \\
  c_{0j}^{(m+1)} &= (1 + 2\hat{\Delta}) c_{1j}^{(m)} + c_{0,j-1}^{(m)} + \triangle c_{0j}^{(m)}, \quad \text{for } j \leq m - 1, \\
  c_{0m}^{(m+1)} &= -\triangle b_0^{(m)} + (1 + 2\hat{\Delta}) c_{1,m-1}^{(m)} + c_{0,m-1}^{(m)}. 
\end{align*}
\]

Starting from \( \hat{t}_1^{(0)} = \hat{a}_2, b_0^{(0)} = 0 \), we get
\[
\begin{align*}
  b_1^{(m)} &= -2m Da_1 + \hat{a}_2, \\
  b_0^{(m)} &= -m(\triangle + m(m - 1)(1 + 2\hat{\Delta}) Da_1 + m(1 + 2\hat{\Delta}) \hat{a}_2 \\
  &= -m((2m - 1)\triangle + (m - 1)(1 - N) Da_1 + m(1 + 2\hat{\Delta}) \hat{a}_2). 
\end{align*}
\]
We remark that these coefficients, depending on \( m \), do not depend on higher derivatives of \( w \). So, we can claim the following result.

**Proposition 11** The coefficients \( A_m \) and \( B_m \) do not depend on the derivative of \( w \) of higher order than 4.

Let us take the neighborhood \( \Omega \) such that \( \sup_{0 \leq k \leq 2} \|\Delta^k w\|_{L^\infty} \leq C_0 \) for \( w \in \Omega \).

Now we analyze the coefficients \( c_{1j}^{(m)} \) and \( c_{0j}^{(m)} \). It is convenient to introduce
\[
\begin{align*}
  \gamma_{1k}^{(m)} &= c_{1,m-k}^{(m)} \quad \text{and} \quad \gamma_{0k}^{(m)} = c_{0,m-k}^{(m)} \quad \text{for } 1 \leq k \leq m.
\end{align*}
\]
Then the last term of equation (60) can be rewritten as

$$\sum (c_{ij}^{(m)} DH_j + c_{0ij}^{(m)} H_j) = \sum (\gamma_{1k}^{(m)} DH_j + \gamma_{0k}^{(m)} H_j),$$

where in the summation $k, j$ run so that $k + j = m$ under the conditions $1 \leq k$ and $j \leq m - 1$. We have the following recurrence formula:

$$\gamma_{1k}^{(m+1)} = \gamma_{1k}^{(m)} + (\Delta - (N - 2)D)\gamma_{1,k-1}^{(m)} + 2DH_{1,k-1}, \quad \text{for} \quad 2 \leq k \leq m;$$

$$\gamma_{11}^{(m+1)} = - (\Delta - (N - 2)D)b_1^{(m)} - 2Db_0^{(m)} + \gamma_{11}^{(m)}$$

$$= 4m^2(\Delta + \frac{3 - N}{2}D)Da_1 + (-4m + 1)\Delta + (2mN - 6m + N - 2)D)a_2 + \gamma_{11}^{(m)}$$

$$\gamma_{0k}^{(m+1)} = (1 + 2\tilde{D})\gamma_{1,k-1}^{(m)} + \gamma_{0k}^{(m)} + \Delta \gamma_{0,k-1}^{(m)}, \quad \text{for} \quad 2 \leq k \leq m;$$

$$\gamma_{01}^{(m+1)} = - \Delta b_0^{(m)} + (1 + 2\tilde{D})\gamma_{11}^{(m)} + \gamma_{01}^{(m)}$$

$$= m\Delta(2m - 1)\Delta + (m - 1)(1 - N)Da_1 - m(3 + 2\tilde{D})\Delta a_2 +$$

$$+ (1 + 2\tilde{D})\gamma_{11}^{(m)} + \gamma_{01}^{(m)}.$$

We start the above recurrence formula from

$$\gamma_{11}^{(1)} = (-\Delta + (N - 2)D)a_2 \quad \text{and} \quad \gamma_{01}^{(1)} = 0.$$

Let us consider the vector

$$\tilde{a} = (Da_1, \Delta a_1, (\Delta + \frac{3 - N}{2}D)Da_1, D\tilde{a}_2, \Delta \tilde{a}_2).$$

Then we have

$$\|\gamma_{11}^{(1)}\|_0 + \|\gamma_{01}^{(1)}\|_0 \leq C\|\tilde{a}\|_0.$$

Here and hereafter in this section we use the abbreviations

$$\|y\|_\ell = \sup_{0 \leq \ell \leq T} \sup_{0 \leq j \leq \ell} \|(-\Delta)^j y\|_{L^\infty},$$

and keep in mind Proposition 7. In other words, the operator $D$ performs like a second order differential operator.

Using the recurrence formula, it is easy to see

$$\|\gamma_{11}^{(m)}\|_0 \leq C\|\tilde{a}\|_0$$

for any $m$. The constant $C$ may depend upon $m$. More precisely, $\gamma_{11}^{(m)}$ is a linear combination of components of $\tilde{a}$. In fact we see

$$\gamma_{11}^{(m)} = \frac{2}{3}m(m - 1)(2m - 1)(\Delta + \frac{3 - N}{2}D)Da_1 +$$

$$+ m(-2m - 1)\Delta + (m - 1)(N - 3 + N - 2)D)a_2.$$
Therefore, we see that
\[(1 + 2\hat{D})\gamma^{(m)}_{11} \parallel_0 \leq C\parallel \vec{a} \parallel_1\]
and
\[\parallel \gamma^{(m)}_{01} \parallel_0 \leq C\parallel \vec{a} \parallel_1\] for any \(m\).
Then, keeping in mind these, the recurrence formula show that
\[\parallel \gamma^{(m)}_k \parallel_0 \leq C\parallel \vec{a} \parallel_k\] for any \(m\), since the second order differential operations occur only with descent of \(k\), provided that \(k \geq 2\).

We shall use the following

**Proposition 12** If non-negative integers \(j, k, m\) satisfy \(j + k = m\), then there is a constant \(C\) such that
\[\parallel f \parallel_j \parallel g \parallel_k \leq C(\parallel f \parallel_m \parallel g \parallel_0 + \parallel f \parallel_0 \parallel g \parallel_m).\]
A proof of this proposition will be given in Appendix B.

Now we assume that
\[\parallel \hat{D}H_j \parallel_x + \parallel H_j \parallel_x \leq C(\parallel g \parallel_j + \parallel \vec{a} \parallel_j)\]
for \(0 \leq j \leq m - 1\) and assume \(\parallel g \parallel_0 + \parallel \vec{a} \parallel_0 \leq C_0\). Then Proposition 12 implies that, for \(j + k = m\),
\[\parallel \gamma^{(m)}_k \hat{D}H_j + \gamma^{(m)}_k H_j \parallel_x \leq C\parallel \vec{a} \parallel_k(\parallel g \parallel_j + \parallel \vec{a} \parallel_j)\]
\[\leq C'(\parallel g \parallel_m + \parallel \vec{a} \parallel_m).\]
Thus the energy estimate gives us
\[\parallel \hat{D}H_m \parallel_x + \parallel H_m \parallel_x \leq C(\parallel g \parallel_m + \parallel \vec{a} \parallel_m),\]
since \(e^{xA_mT}\) and \(B_m\) are independent of higher derivatives of \(w\). By induction we can claim that the inequality
\[\parallel H_m \parallel_x \leq C(\parallel g \parallel_m + \parallel \vec{a} \parallel_m)\]
holds for any \(m \in \mathbb{N}\). We note that
\[\parallel \vec{a} \parallel_m \leq C(1 + \parallel w \parallel_{m+3}).\]
Hence by Sobolev imbedding theorem we have the tame estimate
\[\parallel h \parallel^{(2)}_n \leq C(1 + \parallel w \parallel^{(2)}_{n+\sigma} + \parallel g \parallel^{(2)}_{n+\sigma'}),\]
provided that \(\sigma = 4 + \lfloor N/4 \rfloor, \sigma' = 1 + \lfloor N/4 \rfloor\), for \(h = V\mathfrak{A}(w)g\).
Summarizing, we have verified all conditions of (S1) and (S2) and proved Theorem 1.

Appendix

A. The Sobolev imbedding theorem

For the sake of self-containedness, we prove the Sobolev imbedding theorem for our framework. Let $y \in X$ and $m \in \mathbb{N}, m \geq 1$, we denote

$$\mathcal{N}_m(y) := \|(-\Delta)^m y\|_X.$$ 

Suppose $y \in C_0^\infty(0, 1)$, then we have the expansion

$$y(z) = \sum_{n=1}^\infty c_n \phi_n,$$

where $(\phi_n)_n$ is the orthonormal system of eigenfunctions of the operator $T = -\Delta$ with the Dirichlet boundary condition at $z = 1$. Then, for $m \in \mathbb{N}$, we have

$$(-\Delta)^m y(z) = \sum_{n=1}^\infty c_n \lambda_n^m \phi_n(z)$$

and

$$\mathcal{N}_m(y) = \left(\sum_n |c_n|^2 \lambda_n^{2m}\right)^{1/2}.$$

As for prerequisites, some properties are illustrated in the sequel.

**Lemma A.1.** Let $j_{\nu,n}$ be the $n$-th positive zero of the Bessel function $J_\nu$, where $\nu = \frac{N}{2} - 1$. Then we have

$$\lambda_n = (j_{\nu,n}/2)^2 \sim \frac{\pi^2}{4} n^2 \text{ as } n \to \infty.$$

**Proof.** By the Hankel’s asymptotic expansion (see [12]), the zeros of $J_\nu$ can be determined by the relation

$$\tan(r - (\frac{\nu}{2} + \frac{1}{4})\pi) = \frac{2}{\nu^2 - \frac{1}{4}}r(1 + O(r^{-2})).$$

Then we see

$$j_{\nu,n} = (n_0 + n + \frac{\nu}{2} + \frac{3}{4})\pi + O(\frac{1}{n}) \text{ as } n \to \infty,$$

for some $n_0 \in \mathbb{Z}$. □

**Lemma A.2.** There is a constant $C = C(N)$ such that

$$|\phi_n(z)| \leq Cn^{\frac{N-1}{2}} \text{ for } 0 \leq z \leq 1.$$
Proof. We can assume that $\phi_n(z)$ is the normalization of $\Phi_\nu(\lambda_n z)$, where

$$\Phi_\nu\left(\frac{r^2}{4}\right) = J_\nu(r) - \nu \left(\frac{r^2 - \frac{\nu}{2}}{2} + O\left(\frac{1}{r^2}\right)\right).$$

Since $|\Phi_\nu(z)| \leq C$ for $0 \leq z < \infty$, it is sufficient to estimate $\|\Phi_\nu(\lambda_n z)\|_X$. Using the Hankel’s asymptotic expansion in the form

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}} \left[\cos\left(r - \frac{\nu}{2} - \frac{\pi}{4}\right)(1 + O\left(\frac{1}{r^2}\right)) + \frac{1}{r} \sin\left(r - \frac{\nu}{2} - \frac{\pi}{4}\right)(\nu^2 - \frac{1}{2} + \frac{O(1)}{r^2})\right],$$

we see that

$$\|\Phi_\nu(\lambda_n z)\|_2 = (\lambda_n)^{-\nu - 1} \int_0^{2\pi} J_\nu(r)^2 r dr = (\lambda_n)^{-\nu - 1} \left(\frac{1}{\pi} j_{\nu, n} + O(1)\right) \sim \frac{2}{\pi} (\lambda_n)^{-\nu - \frac{1}{2}}.$$

Then Lemma A.1 implies that

$$\|\Phi_\nu(\lambda_n z)\|_X^{-1} \sim \text{const.} n^{\nu + \frac{1}{2}}.$$

□

Lemma A.3. If $y \in C_0^\infty(0, 1)$ and $1 \leq j \leq m$, then $N_j(y) \leq N_m(y)$. Proof. Let $y = \sum c_n \phi_n$, we have

$$N_j(y) = \sum |c_n|^2 \lambda_n^{2j} = (\lambda_1)^{2j} \sum |c_n|^2 (\lambda_n / \lambda_1)^{2j} \leq (\lambda_1)^{2j} \sum |c_n|^2 (\lambda_n / \lambda_1)^{2m} = \lambda_1^{2j - 2m} N_m(y)^2.$$

According to [12] (see Section 15-6, p.208), we know that $j_{\nu, 1}$ is an increasing function of $\nu > 0$ and $j_{\nu, 1} = \pi$. Therefore, $\lambda_1 \geq (\pi/2)^2 > 1$ for $N \geq 2$ and which implies $N_j(y) \leq N_m(y)$. □

Lemma A.4. If $2s > N/2$, then there is a constant $C = C(s, N)$ such that

$$\|y\|_{L^\infty} \leq C N_s(y)$$

for any $y \in C_0^\infty(0, 1)$. Proof. Let $y = \sum c_n \phi_n(z)$, then Lemmas A.1 and A.2 imply that

$$|y(z)| \leq \sum |c_n| |\phi_n(z)| \leq C \sum |c_n| n^{N-1} \leq C \sqrt{\sum |c_n|^2 \lambda_n^{2s}} \sqrt{\sum n^{N-4s-1}}.$$

Since $N - 4s < 0$, the last term in the above inequality is finite. Therefore we get the required estimate. □
Now, for $R > 1$, we denote by $X(0,R)$ the Hilbert space of functions $y(z)$ of $0 \leq z \leq R$ endowed with the inner product
\[
(y_1|y_2)_{X(0,R)} = \int_0^R y_1(z)\overline{y_2(z)}z^{N-1}dz.
\]
Moreover, for $m \in \mathbb{N}$, we denote by $X_{2m}(0,R)$ the space of functions $y(z)$ of $0 \leq z \leq R$ for which the derivatives $(-\Delta)^j y \in X$ exist in the sense of distribution for $1 \leq j \leq m$. And we use the norm
\[
\|y\|_{X_{2m}(0,R)} := \left( \sum_{0 \leq j \leq m} \|(-\Delta)^j y\|_{X(0,R)}^2 \right)^{1/2}.
\]
Let us denote by $Z_{2m}(0,R)$ the closure of $C^\infty_0(0,R)$ in the space $X_{2m}(0,R)$. It is well-known in the theory of elliptic equations that there is a continuous linear extension $\Psi : X_{2m}(0,1) \to Z_{2m}(0,2)$ such that
\[
\|y\|_{X_{2m}(0,1)} \leq \|\Psi y\|_{X_{2m}(0,2)} \leq C\|y\|_{X_{2m}(0,1)}.
\]
See, e.g., [10], p.186, Theorem 3.11 and p.189, Theorem 3.12. Then, by Lemmas A.3 and A.4, the Sobolev imbedding theorem holds for $y \in Z_{2s}(0,2)$. Say, if $2s > N/2$, there is a constant $C$ such that
\[
\|y\|_{L^\infty} \leq C\|y\|_{X_{2s}(0,2)}
\]
for $y \in Z_{2s}(0,2)$. Thus the same imbedding theorem holds for $y \in C^\infty(0,1) \subset X_{2s}(0,1)$ through the above extension. The conclusion is that, if $2s > N/2$, there is a constant $C = C(s,N)$ such that
\[
\|y\|_{L^\infty} \leq C \sup_{0 \leq j \leq s} \|(-\Delta)^j y\|_X
\]
for any $y \in C^\infty([0,1])$.

**B. Nirenberg-Moser type inequalities**

Let us prove Proposition 8, Proposition 9 and Proposition 12 in the sequel.

**Proof of Proposition 8.**

First, it is easy to verify the formula
\[
\dot{D}^kDy(z) = z^{-\frac{N+k}{2}}\int_0^z \dot{D}^k \Delta y(\zeta)\zeta^{\frac{N+k}{2}-1}d\zeta, \quad (B.1)
\]
where $k \in \mathbb{N}$,
\[
\dot{D} := \sqrt{z} \frac{d}{dz} \quad \text{and} \quad D := \frac{d}{dz}.
\]
Since $\Delta = \dot{D}^2 + \frac{N-1}{2}D$, (B.1) implies
\[
\|\dot{D}^kDy\| \leq \frac{2}{N+k}\|\dot{D}^{k+2}y\| + \frac{N-1}{N+k}\|\dot{D}^kDy\|.
\]
Here and hereafter $\| \cdot \|$ stands for $\| \cdot \|_{L^\infty}$. Thus we have

$$\| \hat{D}^k D y \| \leq \frac{2}{k + 1} \| \hat{D}^{k+2} y \|. $$

Repeating this estimate, we get

$$\| \hat{D}^k D^j y \| \leq \left( \frac{2}{k + 1} \right)^j \| \hat{D}^{k+2j} y \|. \quad \text{(B.2)}$$

On the other hand, since $\hat{D}^2 = \Delta - \frac{N-1}{2} D$ and $D \Delta - \Delta D = D^2$, we have

$$\hat{D}^{2\mu} = \sum_{k=0}^{\mu} C_{k\mu} \Delta^{\mu-k} D^k \quad \text{(B.3)}$$

with some constants $C_{k\mu} = C(k, \mu, N)$. Then it follows from (B.3) and Proposition 7 that

$$\| \hat{D}^{2\mu} D^j y \| \leq C \| \Delta^{\mu+j} y \|. \quad \text{(B.4)}$$

Since

$$\Delta = \hat{D}^2 + \frac{N-1}{2} D \quad \text{and} \quad D \hat{D}^2 - \hat{D}^2 D = D^2,$$

it is easy to see that there are constants $C_{km} = C(k, m, N)$ such that

$$\Delta^m = \sum_{k=0}^{m} C_{km} \hat{D}^{2(m-k)} D^k \quad \text{(B.5)}$$

Applying the Leibnitz’ rule to $D$ and $\hat{D}$, we see

$$\Delta^m (f \cdot g) = \sum C_{k\ell jm} (\hat{D}^{2(m-k)-\ell} D^{k-\ell} f) \cdot (\hat{D}^\ell D^j g) \quad \text{(B.6)}$$

with some constants $C_{k\ell jm}$. The summation is taken for $0 \leq j \leq k \leq m$, $0 \leq \ell \leq 2(m-k)$. By estimating each term of the right-hand side of (B.6), we can obtain the assertion of Proposition 8. In fact, we consider the term

$$(\hat{D}^{\ell'} D^j f) \cdot (\hat{D}^\ell D^j g)$$

provided that $\ell' + \ell + 2(j' + j) = 2m$. By (B.2) and (B.4) we have

$$\| \hat{D}^{\ell'} D^j g \| \leq C \| \hat{D}^{\ell'+2j} g \| \leq C' \| \hat{D}^{2m} g \| \| f \|^{1-\frac{\ell'+2j}{2m}} \| g \|^{\frac{\ell'+2j}{2m}} \leq C'' \| \Delta^m g \| \| f \|^{1-\frac{\ell'+2j}{2m}} \| g \|^{\frac{\ell'+2j}{2m}}$$

for some positive constants $C$, $C'$ and $C''$. Here we have used the Nirenberg interpolation for $\hat{D} = \partial / \partial r$. The same estimate holds for $\| \hat{D}^{\ell'} D^j f \|$. Therefore we have

$$\| (\hat{D}^{\ell'} D^j f) \cdot (\hat{D}^\ell D^j g) \| \leq C \| \Delta^m f \| \| f \|^{1-\frac{\ell'+2j}{2m}} \| g \|^{\frac{\ell'+2j}{2m}} \leq C (\| \Delta^m f \| + \| f \| \| \Delta^m g \|)$$

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since $X^\theta Y^{1-\theta} \leq X + Y$.

**Proof of Proposition 9**

Suppose $F(z, y)$ is a smooth function of $z$ and $y$. Let us consider the composed function $U(z) := F(z, y(z))$. We claim that
\[
\|\triangle^m U\|_0 \leq C(1 + \|y\|_m)
\]
provided that $\|y\|_0 \leq C_0$. In fact,
\[
\triangle^m U = \sum C_{km} \dot{D}^{2(m-k)} D^k U
\]
consists of several terms of the following form:
\[
\left( \dot{D}^{K_1} \frac{\partial}{\partial y} \right)^L \left( \dot{D}^{K_L} \frac{\partial}{\partial y} \right)^\ell F \cdot (\dot{D}^{K_1}) \cdots (\dot{D}^{K_L} y) \cdot (\dot{D}^{\mu_1} D^{k_1} y) \cdots (\dot{D}^{\mu_\ell} D^{k_\ell} y),
\]
where
\[
k + k_1 + \cdots + k_\ell = \kappa,
\]
\[
K + K_1 + \cdots K_L + \mu_1 + \cdots + \mu_\ell = 2(m - \kappa).
\]
Therefore
\[
K_1 + \cdots + K_L + (\mu_1 + 2k_1) + \cdots (\mu_\ell + 2k_\ell) \leq 2m.
\]
Applying the Nirenberg interpolation to $\dot{D}$ and using (B.4), we have
\[
\|\dot{D}^{K_1} y\|_0 \leq C\|y\|^\frac{K_1}{m} \|y\|_0^{1 - \frac{K_1}{m}}.
\]
Similarly,
\[
\|\dot{D}^{\mu_1} D^{k_1} y\|_0 \leq C\|\dot{D}^{\mu_1 + 2k_1} y\|_0 \leq C\|y\|^{\frac{\mu_1 + 2k_1}{m}} \|y\|_0^{1 - \frac{\mu_1 + 2k_1}{m}},
\]
and so on. Then our claim follows obviously.

**Proof of Proposition 12**

By (B.2), (B.4), and (B.5) we have
\[
\frac{1}{C}\|\dot{D}^{2j} f\| \leq \|\triangle^j f\| \leq C\|\dot{D}^{2j} f\|.
\]
Hence, applying the Nirenberg interpolation for $\dot{D} = \partial/\partial r$, we have
\[
\|f\|_{m-\ell} \leq C\|f\|_{m-\ell} \|f\|^{n-m}_\ell,
\]
provided that $\ell \leq m \leq n$. (See, [3], p.143. II.2.2.1. Theorem.) It follows that
\[
\|f\|_i \|g\|_j \leq C(\|f\|_k \|g\|_\ell + \|f\| \|g\|_n),
\]
provided that $(i, j)$ lies on the line segment joining $(k, \ell)$ and $(m, n)$. The proof is same as that of [3], II.2.2.2. Corollary. Taking $(0, m), (m, 0)$ for $(k, \ell), (m, n)$, we have the assertion.
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