GENERALIZED HYERS–ULAM STABILITY OF CUBIC TYPE FUNCTIONAL EQUATIONS IN NORMED SPACES

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Abstract. In this paper, we solve the Hyers-Ulam stability problem for the following cubic type functional equation

\[ f(rx + sy) + f(rx - sy) = rs^2 f(x + y) + rs^2 f(x - y) + 2r(r^2 - s^2) f(x) \]

in quasi-Banach space and non-Archimedean space, where \( r \neq \pm 1, 0 \) and \( s \) are real numbers.

1. Introduction

In [26], S.M. Ulam proposed the stability problem for functional equations concerning the stability of group homomorphisms. A functional equation is called stable if any approximate solution to the functional equation is near a true solution of that functional equation. In [11], D.H. Hyers considered the case of approximate additive mappings with the Cauchy difference controlled by a positive constant in Banach spaces. D.G. Bourgin [4] and T. Aoki [2] treated this problem for approximate additive mappings controlled by unbounded function. In [21], Th. M. Rassias provided a generalization of Hyers’ theorem for linear mappings which allows the Cauchy difference to be unbounded. In 1994, P. Gavruta [8] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions. During the last three decades a number of papers and research...
monographs have been published on various generalizations and applications of the Hyers–Ulam stability and generalized Hyers–Ulam stability to a number of functional equations and mappings [1, 5, 7, 13, 20].

A stability problem of Ulam for the quadratic functional equation
\[(1.1) \quad f(x + y) + f(x - y) = 2f(x) + 2f(y)\]
was first proved by F. Skof for mapping \(f: E_1 \to E_2\), where \(E_1\) is a normed space and \(E_2\) is a Banach space [24]. In the paper [6], S. Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1.1).

Let both \(E_1\) and \(E_2\) real vector spaces. K. Jun and H. Kim [12] proved that a mapping \(f: E_1 \to E_2\) satisfies the functional equation
\[(1.2) \quad f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)\]
if and only if there exists a mapping \(B: E_1 \times E_1 \times E_1 \to E_2\) such that
\[B(x, y, z) = \frac{1}{24} \left[ f(x + y + z) + f(x - y - z) 
- f(x + y - z) - f(x - y + z) \right] \]
for all \(x, y, z \in E_1\), is symmetric for each fixed one variable and additive for each fixed two variables. It is easy to see that the functional equation (1.2) is equivalent to a cubic functional equation
\[C(2x + y) + C(x - y) + 3C(y) = 3C(x + y) + 6C(x)\]
and every solution of the cubic functional equation is said to be a cubic mapping [19]. A. Najati [17] investigated the following generalized cubic functional equation:
\[(1.3) \quad f(kx + y) + f(kx - y) = kf(x + y) + kf(x - y) + 2(k^3 - k)f(x)\]
for a positive integers \(k \geq 2\).

Now, we introduce the following more generalized functional equation
\[(1.4) \quad f(rx + sy) + f(rx - sy) = rs^2 f(x + y) + rs^2 f(x - y) + 2r(r^2 - s^2)f(x)\]
where \(r \neq -1, 0, 1\) and \(s \in \mathbb{R}\). It is easy to see that the function \(f(x) = cx^3\) is a solution of the above functional equation. And if one take \(r = 2\) and \(s = 1\) in (1.4), then the functional equation is (1.2). Also if one take \(r \geq 2\) an integer and \(s = 1\) in (1.4), then the functional equation is (1.3).
In this paper, we establish the stability problem for the functional equation (1.4) for real number \( r \neq -1, 0, 1 \) and \( s \) in quasi normed spaces and non-Archimedean spaces.

2. The Hyers–Ulam Stability in quasi-Banach spaces

In this section, we investigate the generalized Hyers–Ulam stability problem for the functional equation (1.4) in quasi-Banach space. First, we introduce some basic information concerning quasi-Banach spaces which are referred in [3] and [23]. Let \( X \) be a linear space. A quasi-norm is a real-valued function on \( X \) satisfying the following:

(i) \( \|x\| \geq 0 \) for all \( x \in X \), and \( \|x\| = 0 \) if and only if \( x = 0 \);

(ii) \( \|\lambda x\| = |\lambda|\|x\| \) for any scalar \( \lambda \) and all \( x \in X \);

(iii) There is a constant \( M \geq 1 \) such that \( \|x + y\| \leq M(\|x\| + \|y\|) \) for all \( x, y \in X \).

The pair \((X, \| \cdot \|)\) is called a quasi-normed space if \( \| \cdot \| \) is a quasi-norm on \( X \). The smallest possible \( M \) is called the modulus of concavity of the quasi-norm \( \| \cdot \| \). A quasi-Banach space is a complete quasi-normed space. A quasi-norm \( \| \cdot \| \) is called a \( q \)-norm \((0 < q \leq 1)\) if \( \|x + y\|^q \leq \|x\|^q + \|y\|^q \) for all \( x, y \in X \). In this case, a quasi-Banach space is called a \( q \)-Banach space. Let \( X \) be a quasi-Banach space. Given a \( q \)-norm, the formula \( d(x, y) := \|x - y\|^q \) gives us a translation invariant metric on \( X \). By Aoki–Rolewicz Theorem [23] (see also [3]), each quasi-norm is equivalent to some \( q \)-norm. Since it is much easier to work with \( q \)-norms than quasi-norms, here and subsequently, we restrict our attention mainly to \( q \)-norms. Moreover, generalized stability theorems of functional equations in quasi-Banach spaces have been investigated by a lot of authors [14, 18, 25].

Now we introduce an abbreviation \( D_{r,s}f \) for a given mapping \( f : X \to Y \) as follows:

\[
D_{r,s}f(x, y) := f(rx + sy) + f(rx - sy) - rs^2f(x + y) - rs^2f(x - y) - 2r(r^2 - s^2)f(x)
\]

for all \( x, y \in X \), where \( r \neq -1, 0, 1 \) and \( s \) are fixed real numbers.

From now on, let \( X \) be a normed linear space with quasi-norm \( \| \cdot \| \) and \( Y \) be a \( q \)-Banach space with \( q \)-norm \( \| \cdot \| \). In this part, by using an direct method, we prove the stability theorem of the equation (1.4).
Theorem 2.1. Let $\phi : X^2 \to [0, \infty)$ be a function such that
\begin{equation}
\sum_{j=0}^{\infty} \frac{1}{r^{3jq}} \phi(r^j x, 0)^q < \infty, \quad \lim_{j \to \infty} \frac{\phi(r^j x, r^j y)}{|r|^{3j}} = 0
\end{equation}
for all $x, y \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality
\begin{equation}
\|D_{r,s} f(x, y)\| \leq \phi(x, y)
\end{equation}
for all $x, y \in X$. Then there exists a unique mapping $C : X \to Y$ satisfying (1.4) such that
\begin{equation}
\|f(x) - C(x)\| \leq \frac{1}{2} \frac{1}{r^{3j}} \sum_{j=0}^{\infty} \frac{\phi(r^j x, 0)^q}{|r|^{3jq}}
\end{equation}
for all $x \in X$.

Proof. Replacing $(x, y)$ by $(x, 0)$ in (2.2), we have
\begin{equation}
\|f(rx) - r^3 f(x)\| \leq \frac{1}{2} \phi(x, 0)
\end{equation}
for all $x \in X$. Replacing $x$ by $r^k x$ in (2.4) and then dividing both sides by $r^{3k+3}$, we get
\[\frac{1}{r^{3k}} f(r^k x) - \frac{1}{r^{3k+3}} f(r^{k+1} x) \leq \frac{1}{2} \frac{\phi(r^k x, 0)}{r^{3k}}\]
for all $x \in X$ and all integers $k \geq 0$. Then for any integers $m, k$ with $m \geq k \geq 0$, we obtain
\begin{equation}
\left\|\frac{1}{r^{3m+3}} f(r^{m+1} x) - \frac{1}{r^{3k+3}} f(r^k x)\right\|^q
\leq \sum_{j=k}^{m} \left\|\frac{1}{r^{3j+3}} f(r^{j+1} x) - \frac{1}{r^{3j}} f(r^j x)\right\|^q
\leq \frac{1}{2^q |x|^{3q}} \sum_{j=k}^{m} \frac{\phi(r^j x, 0)^q}{|r|^{3jq}}
\end{equation}
for all $x \in X$. Thus the sequence $\left\{\frac{f(r^k x)}{r^{3k}}\right\}_{k=1}^{\infty}$ is Cauchy by (2.1). Since $Y$ is complete, this sequence converges for all $x \in X$. So one can
define a mapping $C : X \to Y$ by

$$
(2.6) \quad \lim_{k \to \infty} \frac{f(r^k x)}{r^{3k}} = C(x) \quad (x \in X).
$$

It follows from (2.1) and (2.6) that

$$
\| D_{r,s} C(x,y) \| = \lim_{k \to \infty} \frac{1}{r^{3k}} \| D_{r,s} f(r^k x, r^k y) \| \leq \lim_{k \to \infty} \frac{\phi(r^k x, r^k y)}{r^{3k}} = 0
$$

for all $x, y \in X$. Hence, the mapping $C$ satisfies (1.4). Putting $k := 0$ and letting $m$ go to infinity in (2.5), we see that (2.3) holds. For the uniqueness of $C$, assume that there exists a mapping $C' : X \to Y$ satisfying (1.4) and (2.3). Then, we find that

$$
\left\| C(x) - C'(x) \right\|_q = \lim_{k \to \infty} \frac{1}{r^{3k}} \left\| f(r^k x) - C'(r^k x) \right\|_q \leq \lim_{k \to \infty} \frac{1}{2^k r^{3k}} \sum_{j=0}^{\infty} \frac{1}{r^{3qj} \phi(r^{-j} x, 0)^q} = 0
$$

for all $x \in X$, which proves the uniqueness.

**Theorem 2.2.** Let $\phi : X^2 \to [0, \infty)$ be a function such that

$$
\sum_{j=0}^{\infty} |r|^{3qj} \phi(r^{-j} x, 0)^q < \infty, \quad \lim_{j \to \infty} |r|^{3j} \phi(r^{-j} x, r^{-j} y) = 0
$$

for all $x, y, z \in X$. Suppose that $f : X \to Y$ is a mapping satisfying the inequality

$$
\| D_{r,s} f(x,y) \| \leq \phi(x,y)
$$

for all $x, y \in X$. Then there exists a unique mapping $C : X \to Y$ satisfying (1.4) such that

$$
\| f(x) - C(x) \| \leq \frac{1}{2|r|^3} \left[ \sum_{j=1}^{\infty} |r|^{3qj} \phi(r^{-j} x, 0)^q \right]^\frac{1}{q}
$$

for all $x \in X$. 

Proof. We observe that one can obtain the following inequality
\[
\|r^{3k}f\left(\frac{x}{r^k}\right) - r^{3(m+1)}f\left(\frac{x}{r^{m+1}}\right)\|^q \leq \frac{1}{2^q|\alpha|^q} \sum_{j=k}^{m} |r|^{3(j+1)q} \phi(r^{-(j+1)}x,0)^q
\]
for all \(x \in X\) and all integers \(k, m\) with \(m \geq k \geq 0\) by use of (2.2). Thus, we see that the proof may be verified by applying similar argument to that of Theorem 2.1.

In case \(r = 2\) and \(s = 1\), as a special case of Theorems 2.1 and 2.2 we have the Hyers-Ulam stability results for the cubic functional equation (1.2) (see [12]).

COROLLARY 2.3. Let \(\varepsilon \geq 0\). Suppose that a mapping \(f : X \to Y\) satisfies the inequality
\[
\|D_{r,s}f(x,y)\| \leq \varepsilon
\]
for all \(x, y \in X\). Then there exists a unique mapping \(C : X \to Y\) satisfying (1.4) such that
\[
\|f(x) - C(x)\| \leq \frac{\varepsilon}{2^{q/2}(|\alpha|^{a_i} - |\alpha|^{a_i})}
\]
for all \(x \in X\).

COROLLARY 2.4. Let \(\alpha, a_1, a_2\) be positive real numbers such that either \(a_i > 3\) or \(a_i < 3\) simultaneously for all \(i \in \{1, 2\}\). Suppose that a mapping \(f : X \to Y\) satisfies the inequality
\[
\|D_{r,s}f(x,y)\| \leq \alpha(\|x\|^{a_i} + \|y\|^{a_i})
\]
for all \(x, y \in X\). Then there exists a unique mapping \(C : X \to Y\) satisfying (1.4) such that
\[
\|f(x) - C(x)\| \leq \frac{\alpha \|x\|^{a_i}}{2^{q/2}(|\alpha|^{a_i} - |\alpha|^{a_i})} \quad (i = 1, 2)
\]
for all \(x \in X\).

3. The Hyers–Ulam Stability in non-Archimedean spaces

Hensel [10] has introduced a normed space which does not have the non-Archimedean spaces property. During the last three decades, the theory of non-Archimedean spaces has gain the interest of physicists for their research in problems coming from quantum physics, \(p\)-adic strings and superstrings [15].
A valuation is a function $| \cdot |$ from a field $\mathbb{K}$ to $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|ab| = |a| \cdot |b|$ and the triangle inequality holds, i.e.,

$$|a + b| \leq |a| + |b|, \forall a, b \in \mathbb{K}.$$ 

A field $\mathbb{K}$ is called a valued field if it equips with a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. Alternatively, if the triangle inequality is replaced by the weakly triangle inequality

$$|a + b| \leq \max\{|a|, |b|\}, \forall a, b \in \mathbb{K},$$

then the valuation $| \cdot |$ is called a non-Archimedean valuation, and the field is called a non-Archimedean field. Clearly $|1| = |−1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $| \cdot |$ taking everything except for 0 into 1 and $|0| = 0$.

**Definition 3.1.** Let $X$ be a vector space over a field $\mathbb{K}$ with a non-Archimedean valuation $| \cdot |$. A function $\| \cdot \| : X \to [0, \infty)$ is said to be a non-Archimedean norm on $X$ if it satisfies the following conditions

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|ax\| = |a|\|x\|$ (a $\in \mathbb{K}$);
(iii) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ ($x, y \in X$).

In this case $(X, \| \cdot \|)$ is called a non-Archimedean normed space. Because of the fact

$$\|x_k - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq k - 1\} \quad (k > m),$$

a sequence $\{x_m\}$ is Cauchy in the non-Archimedean normed space if and only if $\{x_{m+1} - x_m\}$ converges to zero with respect to the non-Archimedean norm. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

**Example 3.2.** Let $p$ be a prime number. For any nonzero rational number $x$, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where $a$ and $b$ are integers not divisible by $p$. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on rational $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\| \sum_{k \geq n_x} a_k p^k \| = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field (see [9, 22]).
Let $X$ be a vector space and $Y$ be a non-Archimedean Banach space. In the following, we now prove the generalized Hyers–Ulam stability of quadratic functional equation (1.4) over the non-Archimedean space. As corollaries, we obtain especially stability result over the $p$-adic field $\mathbb{Q}_p$. To avoid trivial case, we assume $|r| < 1$.

**Theorem 3.3.** Let $\phi : X^2 \to [0, \infty)$ $(\psi : X^2 \to [0, \infty))$ be a function such that
\[
\lim_{j \to \infty} \frac{\phi(r^j x, r^j y)}{|r|^{3j}} = 0 \quad (3.1)
\]
for all $x, y \in X$ and the limit
\[
\Phi(x) \equiv \lim_{k \to \infty} \max \left\{ \frac{\phi(r^j x, 0)}{|r|^{3j}} : 0 \leq j < k \right\}
\]
\[
\Psi(x) \equiv \lim_{k \to \infty} \max \left\{ |r|^{3j} \psi(r^{-j} x, 0) : 1 \leq j \leq k \right\},
\]
exists for each $x \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality
\[
\|D_{r,s} f(x, y)\| \leq \phi(x, y)
\]
\[
\left( \|D_{r,s} f(x, y)\| \leq \psi(x, y), \text{resp} \right),
\]
for all $x, y \in X$. Then there exists a mapping $C : X \to Y$ satisfying (1.4) such that
\[
\|f(x) - C(x)\| \leq \frac{1}{|2| \cdot |r|^3} \Phi(x)
\]
\[
\left( \|f(x) - C(x)\| \leq \frac{1}{|2| \cdot |r|^3} \Psi(x), \text{resp} \right),
\]
for all $x \in X$. Moreover, if
\[
\lim_{m \to \infty} \lim_{k \to \infty} \max \left\{ \frac{\phi(r^j x, 0)}{|r|^{3j}} : m \leq j < k + m \right\} = 0
\]
\[
\left( \lim_{m \to \infty} \lim_{k \to \infty} \max \left\{ |r|^{3j} \psi(r^{-j} x, 0) : m < j \leq k + m \right\} = 0, \text{resp} \right),
\]
for all $x \in X$, then the mapping $C$ is unique.

**Proof.** Replacing $(x, y)$ by $(x, 0)$ in (3.3), we have
\[
\|f(rx) - r^3 f(x)\| \leq \frac{1}{|2|} \phi(x, 0)
\]
(3.6)
for all $x \in X$. Replacing $x$ by $r^kx$ in (3.6) and then dividing both sides by $|r|^{3k+3}$, we get

$$\| \frac{1}{r^{3k+3}} f(r^{k+1}x) - \frac{1}{r^{3k}} f(r^k x) \| \leq \frac{1}{|2| \cdot |r|^3} \frac{\phi(r^k x, 0)}{|r|^{3k}}$$

(3.7)

for all $x \in X$. It follows from (3.7) and (3.1) that the sequence $\{f(r^k x)\}_{k=1}^\infty$ is Cauchy in the non-Archimedean Banach space $Y$. Since $Y$ is complete, we may define a mapping $C : X \to Y$ as

$$C(x) := \lim_{k \to \infty} f(r^k x)$$

for all $x \in X$. Using induction, one can show that

$$\| \frac{f(r^k x)}{r^{3k}} - f(x) \| \leq \frac{1}{|2| \cdot |r|^3} \max \left\{ \phi(r^j x, 0) : 0 \leq j < k \right\}$$

(3.8)

for all $k \in \mathbb{N}$ and all $x \in X$. By taking $k$ to approach infinity in (3.8) and using (3.2), one obtains (3.4). Replacing $x$, $y$ and $z$ by $r^{3k} x$, $r^{3k} y$ and $r^{3k} z$, respectively, in (3.3), we get

$$\| D_{r,s} f(r^k x, r^k y) \| \leq \frac{\phi(r^k x, r^k y)}{|r|^{3k}}$$

(3.9)

for all $x, y \in X$. Taking the limit as $k \to \infty$, we conclude that $C$ satisfies (1.4). Moreover, to prove the uniqueness, we assume that there exists a mapping $C' : X \to Y$ satisfying (1.4) and (3.4), (3.5). Then we figure out

$$\| C(x) - C'(x) \|$$

$$= \lim_{m \to \infty} \frac{1}{|r|^{3m}} \| C(r^m x) - C'(r^m x) \|$$

$$\leq \lim_{m \to \infty} \max \left\{ \frac{\| C(r^m x) - f(r^m x) \|}{|r|^{3m}} , \frac{\| f(r^m x) - C'(r^m x) \|}{|r|^{3m}} \right\}$$

$$\leq \lim_{m \to \infty} \lim_{k \to \infty} \frac{1}{|2| \cdot |r|^3} \max \left\{ \frac{\phi(r^j x, 0)}{|r|^3} : m \leq j < m + k \right\} = 0$$

for all $x \in X$. This completes the proof.

**Corollary 3.4.** Let $X$ be a non-Archimedean normed space, $t \neq 3$ and $\theta$ be positive numbers. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\| D_{r,s} f(x, y) \| \leq \theta (\| x \|^t + \| y \|^t) \quad (x, y \in X).$$
Then there exists a unique mapping $C : X \to Y$ satisfying (1.4) such that

$$
\|f(x) - C(x)\| \leq \begin{cases}
\frac{\theta}{|r|^t} \|x\|^t & \text{if } |r| > 1, t > 3 \text{ or } |r| < 1, t < 3 \\
\frac{\theta}{|r|^3} \|x\|^t & \text{if } |r| > 1, t < 3 \text{ or } |r| < 1, t > 3
\end{cases}
$$

for all $x \in X$.

Corollary 3.5. Let $t \neq 3$ and $\theta$ be positive numbers. Suppose that a mapping $f : \mathbb{Q}_p \to \mathbb{Q}_p$ with satisfies the inequality

$$
|D_p,f(x,y)|_p \leq \theta(|x|^t_p + |y|^t_p) \quad (x, y \in \mathbb{Q}_p).
$$

Then there exists a unique mapping $C : \mathbb{Q}_p \to \mathbb{Q}_p$ satisfying (1.4) such that

$$
|f(x) - C(x)|_p \leq \begin{cases}
\frac{p^t \cdot \theta}{|r|^t} \|x\|^t & \text{if } |r| > 1, t > 3 \text{ or } |r| < 1, t < 3 \\
\frac{p^3 \cdot \theta}{|r|^3} \|x\|^t & \text{if } |r| > 1, t < 3 \text{ or } |r| < 1, t > 3
\end{cases}
$$

for all $x \in \mathbb{Q}_p$.

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