Among the zoology of different capacities of quantum channels, the entanglement assisted classical capacity \( C_E \) plays an important role. This quantity has been introduced in \( \mathfrak{3} \) to measure the amount of classical information that can be sent through the channel in the presence of an unlimited quantity of prior entanglement between sender and receiver. \( C_E \) and its quantum counterpart \( Q_E = C_E/2 \) (i.e. the amount of qubits that can be sent in the presence of an unlimited quantity of prior entanglement) give upper bounds to the classical and quantum capacities of the channel, including the unassisted capacities whose values are yet to be determined. Moreover, it has been conjectured \( \mathfrak{3} \) that the entanglement assisted classical capacity defines a class of equivalences since all channels with the same \( C_E \) seem to be able to efficiently simulate one another. Unlike the case of most of the other capacities, it has a closed expression in terms of the quantum mutual information

\[
I(N, g) = S(g) + S(N[g]) - S((N \otimes 1)[\Phi_g]) ,
\]

where \( S(g) = -\text{Tr}[g \log_2 g] \) is the Von Neumann entropy, \( N \) is the map that describes the communication channel and \( \Phi_g \) is a purification of the input density matrix \( g \). The value of \( C_E \) is the maximum of \( I(N, g) \) over all the possible inputs \( g \) to the channel \( \mathfrak{4} \).

The entanglement assisted capacity for bosonic Gaussian channels was analyzed in \( \mathfrak{5} \), where it was shown that the maximization in the expression of \( C_E \) can be performed over Gaussian states. These channels are important because they are the basic building blocks of bosonic communication schemes and because they allow one to describe infinite dimensional systems with techniques from finite dimensional linear algebra. In this paper we derive \( C_E \) for multimode bosonic channels in the presence of loss and average input energy constraint, and use these results and the techniques developed to provide upper and lower bounds for other channel capacities. We calculate \( C_E \) for the multimode channel as the sum of the entanglement assisted capacities of the single modes maximized over non-squeezed Gaussian states. In fact, \( C_E \) is additive and we show that squeezing the input states does not increase the \( C_E \) of a single mode. For generic values of the channel quantum efficiency \( \eta \) we cannot provide an analytical expression for \( C_E \), but we give a general characterization and a numerical solution. For \( \eta = 1/2 \), the value of \( C_E \) can be analytically solved and, interestingly, shown to coincide with the wideband lossless channel capacity \( \mathfrak{5} \).

**Broadband lossy channel.**— In the Heisenberg picture the \( i \)th mode of the lossy channel with quantum efficiency \( \eta_i \) evolves as

\[
a'_i = \sqrt{\eta_i} a_i + \sqrt{1 - \eta_i} b_i ,
\]

where \( a_i, a'_i \) and \( b_i \) are the annihilation operators of the input, output and noise modes respectively. The loss map \( \mathcal{N}_i \) for the \( i \)th mode arises by tracing away the noise mode \( b_i \) (in the vacuum state) and the global loss map \( \mathcal{N} \) is the tensor product \( \otimes_i \mathcal{N}_i \). The channel described by \( \mathcal{N} \) maps Gaussian input states into Gaussian output states and is hence a Gaussian channel.

The calculation of \( C_E \) for the multimode lossy channel stems from the following three facts: i) the additivity property of the entanglement assisted capacity, from which the \( C_E \) of the channel is calculated as the sum of the \( C_E \) of each mode \( \mathfrak{5, 6} \), i.e.

\[
C_E = \max_{\psi_i \in \mathcal{H}_j} \left\{ \sum_i I(N_i, \psi_i) \right\} ,
\]

where \( \mathcal{H}_j \) is the Hilbert space of the \( j \)th mode of the channel, and the max is taken over the states \( \psi_i \) that satisfy the average energy constraint

\[
\sum_i \hbar \omega_i N_i = \mathcal{E} ,
\]

with \( \omega_i \) the frequency of the \( i \)th mode and \( N_i \) its average number of photons; ii) the Holevo-Werner theorem according to which the maximum of \( I(N_i, \psi_i) \) for Gaussian channels can be evaluated on Gaussian input states \( \mathfrak{5} \); iii) the fact that squeezing the input does not increase \( C_E \), so that it can be estimated on non-squeezed inputs: as shown in the appendix, the maximum value of \( I(N_i, \psi_i) \) (fixing the energy in the \( i \)th mode) is obtained when \( \psi_i \) does not contain any squeezing and is given by

\[
c_E(N_i, \eta_i) = g(N_i) + g(\eta_i N_i) - g((1 - \eta_i) N_i) ,
\]
where the function $g$ is defined as
\[ g(x) \equiv (x + 1) \log_2(x + 1) - x \log_2(x) , \] (6)
for $x \neq 0$ and $g(0) = 0$. The total entanglement assisted capacity is then
\[ C_E = \max_{N_j} \sum_i c_E(N_i, \eta_i) , \] (7)
where the maximum is taken over the sets $\{N_j\}$ satisfying the energy constraint (11).

The maximization (7) can be performed using the Lagrange multiplier procedure, which, for $\eta \neq 0,1$, gives the following equation (8)
\[ (1 + \frac{1}{N_j}) \left( 1 + \frac{1}{\eta_j N_j} \right)^{\eta_j} = e^{\omega_j/\Omega} \left( 1 + \frac{1}{(1 - \eta_j)N_j} \right)^{1-\eta_j} , \]
where $1/(\Omega \ln 2)$ is the Lagrange multiplier that must be chosen to satisfy the constraint (11). In general this equation is difficult to solve analytically, but we can still give some characterization of the solution, at least when all the quantum efficiencies coincide (i.e. $\eta_j = \eta$ for all $j$). In this case the solution of Eq. (8) is a function of $\omega_j/\Omega$ and $\eta$, i.e. $N_j = F(\omega_j/\Omega, \eta)$. To derive $\Omega$ we use Eq. (11) that becomes
\[ \frac{\mathcal{E}}{\hbar} = \sum_i \omega_i F(\omega_i/\Omega, \eta) \sim \int_0^{\infty} \frac{d\omega}{\delta \omega} \omega F(\omega/\Omega, \eta) , \] (9)
where we have replaced the sum over the mode index $i$ with an integral over the mode frequencies, assuming that the minimum frequency interval $\delta \omega$ of the channel is small. With a variable change in the integral (9), we find that $\Omega = \sqrt{2\pi P/[f(\eta) \hbar]}$ where $P = E \delta \omega/(2\pi)$ is the wideband channel input power during the transmission time $T = 2\pi/\delta \omega$ and
\[ f(\eta) \equiv \int_0^{\infty} dx \; x \; F(x, \eta) . \] (10)
The value of $C_E$ is then obtained placing the solution of Eq. (8) to evaluate the sum (11), i.e.
\[ C_E \simeq \int_0^{\infty} \frac{d\omega}{\delta \omega} c_E(F(\omega/\Omega, \eta), \eta) . \] (11)
Performing again a change of integration variables, we finally find
\[ C_E = T \frac{1}{\ln 2} \sqrt{\frac{f(\bar{\eta})}{3} \hbar} \mathcal{C}(\eta) , \] (12)
where
\[ \mathcal{C}(\eta) \equiv \frac{\ln 2}{\pi} \sqrt{\frac{3}{2 f(\bar{\eta})}} \int_0^{\infty} dx \; c_E(F(x, \eta), \eta) . \] (13)
Notice that, even without knowing the explicit form of the function $\mathcal{C}(\eta)$, Eq. (12) gives the exact dependence on the input power of the entanglement assisted capacity for the channel (10). In particular, the entanglement assisted capacity per unit time of channel use $R_E \equiv C_E/T$ is proportional to the rate $R_C = \frac{1}{\ln 2} \sqrt{\frac{P}{3 \hbar}}$ of the wideband noiseless channel without prior entanglement (8), i.e. $R_E = R_C \mathcal{C}(\eta)$.

**General properties of $C_E$.**— The form of $\mathcal{C}(\eta)$ is not easily determined analytically, but we can still calculate it for some values of $\eta$. First of all, for $\eta = 0$ all the $c_E(N_i, \eta)$ are null and $\mathcal{C}(0) = 0$: no photons arrive, and no bits are transferred. Interestingly, for $\eta = 1/2$ Eq. (8) can be solved analytically and has solution
\[ N_j = \frac{1}{e^{\omega_j/\Omega} - 1} , \] (14)
In this case, $f(1/2) = \pi^2/6$ and $C(1/2) = 1$, and hence the entanglement assisted capacity for the $\eta = 1/2$ wideband channel equals the unassisted capacity of the noiseless wideband channel $TR_C$ (7); prior entanglement is sufficient to restore perfect transmission for a 50% lossy channel (this result holds also for the single mode channel—see appendix). The solution can be linearized around $\eta = 1/2$ and the first order Taylor expansion of $\mathcal{C}(\eta)$ can be obtained as
\[ \mathcal{C}(\eta) = \frac{3}{2} \left( \eta - \frac{1}{2} \right) + 1 + \mathcal{O}(\eta^2) . \] (15)
The case $\eta = 1$ can be completely solved too, given that the Lagrange equation has the same solution (14) of the case $\eta = 1/2$. Here, since $c_E(N_i, 1) = 2c_E(N_i, 1/2)$, we find $C(1) = 2C(1/2) = 2$: the entanglement assisted capacity for the noiseless channel is twice the unassisted capacity as predicted by the superdense coding effect (11). In Fig. 1, $\mathcal{C}(\eta)$ is numerically evaluated and plotted along with the linearization (14). The fact that $\mathcal{C}(\eta) > 1$ for $\eta > 1/2$ shows that, even in the presence of noise, prior entanglement allows one to transmit more bits than those actually sent in the channel (i.e. $TR_C$) thanks again to the superdense coding effect. A similar effect has been shown also for the erasure channel (8) (12).

An interesting class of lower bounds, that provides a good analytical approximation for $C_E$ can be obtained by considering the set (parametrized by $\zeta > 0$)
\[ N_j = \frac{\zeta^2}{e^{\omega_j/\Omega_0} - 1} , \] (16)
where $\Omega_0 = 6\ln 2R_C/\pi$. Using Eq. (16), we find the bound
\[ \mathcal{C}(\eta) \geq [\Lambda(\zeta^2) + \Lambda(\eta \zeta^2) - \Lambda((1 - \eta)\zeta^2)]/[\zeta \Lambda(1)] , \] (17)
where $\Lambda(\eta) = \int_0^{\infty} dx \; g \left( \frac{u}{e^{-u} - 1} \right)$. In particular, the case $\zeta = 1$ (see Fig. 1k) corresponds to employing the exact solution for $\eta = 1/2, 1$ of Eq. (14) for any value of $\eta$. 
states bound in Eq. (18), we follow the suggestion of [6] and we density where

\[ C \geq \max_{i} E \left[ g(\eta N_i) - g((1 - \eta)N_i) \right]. \]  

(19)

Capacity bounds.— The classical capacity \( C \) and the quantum capacity \( Q \) measure respectively the number of bits and qubits that can be sent reliably through the channel per channel use (without the aid of prior entanglement). Unlike the case of \( C_E \), for \( \eta \neq 1 \) a closed expression for \( C \) is not known nor it is known whether this quantity is additive [3]: it may be that entangling successive uses of the channel one can increase the amount of information transmitted. Limiting the analysis to unentangled coding procedures, a lower bound for \( C \) can be obtained as

\[ C \geq \max_{p_i(\mu)} \sum_{\mu} \mathcal{X}(p_i(\mu), p_i(\mu)), \]  

(18)

where \( g_i = \int d\mu \; p_i(\mu) p_i(\mu) \) describes a message in which the “\( \mu \)th letter” \( p_i(\mu) \) in the \( i \)th mode has probability density \( p_i(\mu) \) and where \( \mathcal{X} \) is the Holevo information

\[ S(\mathcal{N}[q]) - \int d\mu \; p_i(\mu) S(\mathcal{N}[p_i(\mu)]). \]  

To estimate the lower bound in Eq. (18), we follow the suggestion of [4] and we evaluate \( \mathcal{X}(p_i(\mu), p_i(\mu)) \) for the \( i \)th mode using coherent states \( p_i(\mu) = |\mu\rangle_i \langle \mu| \) weighted with Gaussian probability distribution \( p_i(\mu) = \exp[-|\mu|^2/N_i]/(\pi N_i) \), \( N_i \) being the average number of photons of the mode. In this case, Eq. (18) becomes \( C \geq \max_{N_i} \sum \mu g(\eta N_i) \) where again the maximum must be taken under the average energy constraint [4]. The corresponding Lagrange equation has solution given by Eq. (16) with \( \zeta = 1/\sqrt{\eta} \), so that \( C \geq T \sqrt{\eta} R_C \) (see Fig. 1b). Notice that for \( \eta = 1 \) the equality holds, since the noiseless channel is known to be additive and we reobtain the results of [2]. A closed expression for \( Q \) is also not known. However, for \( \eta \leq 1/2 \) the no-cloning theorem can be used to show that \( Q = 0 \), as in the case of the erasure channel [12,4]. For \( \eta > 1/2 \), a lower bound can be obtained evaluating the coherent information \( J(N, q) = S(N[q]) - S((N[1]) \Phi) \) on entangled non-squeezed Gaussian inputs [14, 15]. In fact, random quantum codes can send quantum information down a noisy channel at a rate given by the coherent information [10]. In Fig. 1b this bound is plotted by solving numerically the corresponding Lagrange equation, which maximizes the expression

\[ Q \geq \max_{N_i} \sum \mu g(\eta N_i) - g((1 - \eta)N_i). \]  

(19)

Conclusions.— Up to now only few realistic channels have been analyzed at the quantum level. In this paper we studied the wideband bosonic channel with loss, calculating the entanglement assisted capacities \( C_E \) and \( Q_E \) and we gave upper and lower bounds on the classical and quantum capacities of this channel. The capacity \( C_E \) was shown to scale with the square root of the input power as shown previously for the classical capacities in the noiseless case. Moreover, we saw that the superdense coding effect allows the sender to increase the information transferred above the entropy of the input state if the quantum efficiency is \( \eta > 1/2 \).

This work was funded by the ARDA, NRO, NSF, and by ARO under a MURI program.

Appendix.— In [4] it has been shown that, for a given value of the correlation matrix \( \alpha \), the quantum mutual information \( I(N, q) \) for a single mode \( a \) achieves its maximum value on the Gaussian state

\[ q = \frac{h}{2\pi} \int dz \exp \left[ -i(\Delta q, \Delta p) \cdot z^T - z \cdot \alpha \cdot z^T/2 \right] , \]  

(20)

where \( z \) is a real bidimensional linear vector and \( q \) and \( p \) the two orthogonal quadratures \( q = \sqrt{h/2}(a + a^\dagger) \), \( p = -i \sqrt{h/2}(a - a^\dagger) \). In order to evaluate the effect of the squeezing on the quantum mutual information of the single mode channel, it is convenient to introduce the following parametrization for the correlation matrix \( \alpha \):

\[ \alpha = \frac{h}{2} \begin{bmatrix} n_0 e^r & c \\ c & n_0 e^{-r} \end{bmatrix} , \]  

(21)

where \( r \) is the squeezing parameter. These parameters are related through the average number of photons \( N \) by
ident only in this last case. Since the eigenvalues \( \eta > 1 \) are related with the average number of photons \( N \) as

\[
\lambda_+ + \lambda_- = 2N + 1 - m ,
\]

parameter \( |c| \) measures the purity of the initial state. Choosing the maximum value of \( c \) corresponds to sending a single pure state and conveys no information. Finally, since \( I(N, \varphi) \) is an increasing function of \( n_0 \), it can be further maximized by choosing \( n_0 = 2N + 1 \) (i.e. its maximum allowed value achieved when \( \langle q \rangle = \langle p \rangle = 0 \)— see Fig. 2). With this choice, Eq. (23) becomes \( \gamma_{opt}(\eta) = \eta N \), which maximizes the quantum mutual information as

\[
c_E(N, \varphi) \equiv \max \_{\varphi \in \{w \in \mathbb{C} \}} I(N, \varphi) = g(N) + g(\eta N) - g((1 - \eta)N) ,
\]

as reported in Eq. (25).

\[
\begin{align*}
1 & \quad [13] \quad P. \quad Hausladen, \quad R. \quad Jozsa, \quad B. \quad Schumacher, \quad M. \quad Westmoreland, \quad IEEE \quad Trans. \quad Inf. \quad Theory \quad 44, \quad 2724 \quad (1998). \\
2 & \quad [11] \quad C. \quad H. \quad Bennett \quad and \quad S. \quad J. \quad Wiesner, \quad Phys. \quad Rev. \quad Lett. \quad 69, \quad 1992. \\
3 & \quad [10] \quad Notice \quad that, \quad since \quad we \quad have \quad replaced \quad the \quad summations \quad with \quad the \quad integrals, \quad Eq. \quad (12) \quad is \quad valid \quad up \quad to \quad corrections \quad of \quad order \quad 1/T .
\end{align*}
\]

\[
\begin{align*}
11 & \quad [12] \quad C. \quad H. \quad Bennett \quad and \quad S. \quad J. \quad Wiesner, \quad Phys. \quad Rev. \quad Lett. \quad 69, \quad 2881 \quad (1992). \\
12 & \quad [16] \quad S. \quad Lloyd, \quad Phys. \quad Rev. \quad A \quad 55, \quad 1613 \quad (1997).
\end{align*}
\]