ON PRO-$p$ ANALOGUES OF LIMIT GROUPS
VIA EXTENSIONS OF CENTRALIZERS

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Abstract. We begin a study of a pro-$p$ analogue of limit groups via extensions of centralizers and call $L$ this new class of pro-$p$ groups. We show that the pro-$p$ groups of $L$ have finite cohomological dimension, type $FP_{\infty}$ and non-positive Euler characteristic. Among the group theoretic properties it is proved that they are free-by-(torsion-free poly-procyclic) and if non-abelian do not have a finitely generated non-trivial normal subgroup of infinite index. Furthermore it is shown that every 2 generated pro-$p$ group in the class $L$ is either free pro-$p$ or abelian.

1. Introduction

Abstract limit groups have recently attracted a great deal of attention and played a key role in the solution of the famous Tarski problems ([9]–[11], [21]–[26]). V. Remeslennikov has initiated their study in [16] referring to them as $\exists$-free groups, reflecting the fact that these groups have the same existential theory as a free group, or $\omega$-residually free groups. O. Kharlampovich and A. Myasnikov have studied abstract limit groups extensively under the name fully residually free groups, i.e. groups $G$ such that for every finite subset $X$ of $G$ there is a free group $F$ together with a homomorphism of groups $\varphi : G \to F$ whose restriction to $X$ is injective (see [7] and [8]). In fact, abstract limit groups $G$ are exactly finitely generated fully residually free groups. The most relevant definition for this paper is that abstract limit groups are finitely generated subgroups of a group obtained by a finite sequence of extensions of centralizers starting with a free group of finite rank, i.e. every limit group can be viewed as a finitely generated subgroup of an inductively constructed group $B_n = B_{n-1} \ast_C A$ by forming a free amalgamated product of a group $B_{n-1}$ and a free finite rank abelian group $A$ with a cyclic amalgamation $C$ that is self-centralized in $B_{n-1}$ and the class $B_0$ contains precisely the finitely generated free groups.

The objective of this paper is to begin the study of a pro-$p$ analogue of limit groups. We define a class of pro-$p$ groups $L$ as the class that contains the finitely generated pro-$p$ subgroups of a sequence of extensions of centralizers performed in the category of pro-$p$ groups starting with a finitely generated free pro-$p$ group, see Section 3. When the term finitely generated is used for a pro-$p$ group we mean of course topologically finitely generated. Note that the geometric approach of Z. Sela [21] is not available in the pro-$p$ case and we do not know whether the class of pro-$p$ groups we define contains only fully residually free pro-$p$ groups or contains all finitely generated fully residually free pro-$p$ groups.

2000 Mathematics Subject Classification. Primary 20E18; Secondary 20E06, 20E08.

Both authors are partially supported by “bolsa de produtividade em pesquisa” from CNPq, Brazil.
We study the group theoretic structure and the homological properties of a pro-$p$ group in $L$ and show that various results that are known for abstract limit groups hold in the pro-$p$ case. For example as shown in Section 5 the pro-$p$ groups from the class $L$ are transitive commutative and the centralizer and the normalizer of a closed procyclic subgroup always coincide and are abelian. Furthermore every virtually soluble pro-$p$ group from the class $L$ is abelian. We list our main results in the following

**Theorem.** Let $G$ be a pro-$p$ group from the class $L$. Then

(1) $G$ is free-by-(torsion free poly-procyclic);
(2) $G$ is of finite cohomological dimension;
(3) $G$ is of type $FP_\infty$ and has non-positive Euler characteristic;
(4) If $G$ is non-abelian then $G$ does not have a finitely generated non-trivial normal subgroup of infinite index;
(5) If $G$ is 2-generated then $G$ is either free pro-$p$ or free pro-$p$ abelian.

In the abstract case the same results hold. The fact that an abstract limit group is of type $FP_\infty$ follows directly from Bass-Serre theory and the abstract case of (4) is proved in [1]. The properties that abstract limit groups are free-by-(torsion-free nilpotent) and of non-positive Euler characteristic were proved in [12].

The properties (1)-(3) stated above are proved in different sections of the paper: Lemma 3.1, Proposition 4.3, Corollary 4.4 and Theorem 8.1. The proof of (4) is included in Section 6, see Theorem 6.5 and the proof of (5) is the subject of Section 7.

We note that the methods traditionally used to prove statements about abstract limit groups can not be used in the pro-$p$ case. First observe that an element of a pro-$p$ group can not be expressed as a finite word of generators; this eliminates the possibility to use combinatorial methods in their original sense. The geometric version of combinatorial group theory, namely the Bass-Serre theory of groups acting on trees, is also heavily used in the theory of abstract limit groups. The profinite, in particular pro-$p$ version of Bass-Serre theory, exists (see for example [19]) but not in full strength. This theory developed by O. Melnikov, L. Ribes and P. Zalesskii [15], [30], [17], [20], [31] is one of the main tools of our paper.

We finish the paper with a section of open questions, where we list further possible properties of the class $L$.

2. Preliminaries on pro-$p$ groups acting on trees

In this section we collect properties of pro-$p$ groups acting on pro-$p$ trees and free amalgamated pro-$p$ products that will be used in the proofs later on. Further information on this subject can be found in [19].

In accordance with [19] the triple $(\Gamma,d_0,d_1)$ is a profinite graph, provided, $\Gamma$ is a boolean space and $d_0,d_1: \Gamma \rightarrow \Gamma$ are continuous with $d_i d_j = d_j (i,j \in \{0,1\})$. The elements in $V(\Gamma) := d_0(\Gamma) \cup d_1(\Gamma)$ and $E(\Gamma) := \Gamma \setminus V(\Gamma)$ are called the vertices and edges of $\Gamma$ respectively, and, for $e \in E(\Gamma)$, $d_0(e)$ and $d_1(e)$ are called the initial and terminal vertices of $e$. Note that $d_0,d_1$ are the identity when restricted to $V(\Gamma)$. When there is no danger of confusion, we shall simply write $\Gamma$ instead of $(\Gamma,d_0,d_1)$.

Let $(E^*,*) = (\Gamma/V(\Gamma),*)$ be a pointed profinite quotient space with $V(\Gamma)$ as a distinguished point. Let $\mathbb{F}_p[[E^*(\Gamma),*]]$ and $\mathbb{F}_p[[V(\Gamma)]]$ be free profinite $\mathbb{F}_p$-modules
over the pointed profinite space \((E^*(\Gamma), *)\) and over the profinite space \(V(\Gamma)\) (cf. [13]). Then we have the following complex of free profinite \(\mathbb{F}_p\)-modules
\[
0 \rightarrow \mathbb{F}_p[[E^*(\Gamma), *]] \xrightarrow{\delta} \mathbb{F}_p[[V(\Gamma)]] \xrightarrow{\epsilon} \mathbb{F}_p \rightarrow 0,
\]
where \(\delta(e) = d_1(e) - d_0(e)\) for all \(e \in E^*(\Gamma)\) and \(\epsilon(v) = 1\) for all \(v \in V(\Gamma)\).

The profinite graph \(\Gamma\) is connected if (2.1) is exact in the middle. A connected component of \(\Gamma\) is a maximal profinite subgraph that is connected. The graph \(\Gamma\) is called a pro-\(p\) tree if (2.1) is exact. We say that a pro-\(p\) group \(G\) acts on the pro-\(p\) tree \(\Gamma\) if it acts continuously on \(\Gamma\) and the action commutes with \(d_0\) and \(d_1\). We denote by \(G_t\) the stabilizer of \(t \in V(T) \cup E(T)\) in \(G\).

Let \(H, M\) and \(S\) be pro-\(p\) groups such that \(S\) embeds as a pro-\(p\) subgroup of both \(H\) and \(M\), and let \(G = H \Pi_S M\) be the amalgamated free pro-\(p\) product. We say that this amalgamated free pro-\(p\) product is proper if \(H\) and \(M\) embed in \(G\). If the above decomposition of \(G\) as an amalgamated free pro-\(p\) product is proper there is a pro-\(p\) tree \(T\) on which \(G\) acts. By definition \(V(T) = \{gH\}_{g \in G} \cup \{gM\}_{g \in G}\) and \(E(G) = \{gS\}_{g \in G}\). For \(e = gS \in E(T)\) we have \(d_0(e) = gH\) and \(d_1(e) = gM\). Note that \(T/G\) is just an edge with two vertices. Furthermore if the pro-\(p\) group \(G\) is finitely generated then \(T\) is second countable.

**Theorem 2.1.** [17] Theorem 3.2] A free pro-\(p\) product with procyclic amalgamation is proper.

**Theorem 2.2.** [19] Prop. 3.5, Cor. 3.6] Let \(G\) be a pro-\(p\) group acting on a pro-\(p\) tree \(T\) and \(N = \langle \{G_v\}_{v \in V(T)} \rangle\). Then \(G/N\) acts on the pro-\(p\) tree \(T/N\) and \(G/N\) is a free pro-\(p\) group.

**Lemma 2.3.** Let \(G\) be a pro-\(p\) group acting on a pro-\(p\) tree \(T\) such that \(T/G\) is a pro-\(p\) tree. Then \(G = \langle \{G_v\}_{v \in V(T)} \rangle\).

**Proof.** Let \(\bar{G} = \langle \{G_v\}_{v \in V(T)} \rangle\). By Theorem 2.2 \(G/\bar{G}\) acts freely on \(T/\bar{G}\). This means that we have a free resolution of \(\mathbb{F}_p\) over \(\mathbb{F}_p[[G/\bar{G}]]\)
\[
0 \rightarrow \mathbb{F}_p[[E^*(T/\bar{G}), *]] \xrightarrow{\delta} \mathbb{F}_p[[V(T/\bar{G})]] \xrightarrow{\epsilon} \mathbb{F}_p \rightarrow 0.
\]
Tensoring it with \(\tilde{\otimes}_{\mathbb{F}_p[[G/\bar{G}]]} \mathbb{F}_p\) we get the exact sequence (2.1) for \(T/G\) implying that \(H_1(G/\bar{G}, \mathbb{F}_p) = (G/\bar{G})/(G/\bar{G}, G/\bar{G})(G/\bar{G})^p = 0\). It follows that \(G/\bar{G}\) is trivial, as needed. \(\square\)

**Theorem 2.4.** [19] Thm. 3.18] Let \(G\) be a pro-\(p\) group acting on a pro-\(p\) tree \(T\). Then one of the following holds:

a) \(G\) is the stabilizer of a vertex of \(T\);
b) \(G\) has a free non-abelian pro-\(p\) subgroup \(P\) such that \(P \cap G_v = 1\) for every vertex \(v\) of \(T\);
c) There exists an edge \(e\) of \(T\) whose stabilizer \(G_e\) is normal in \(G\) and the quotient group \(G/G_e\) is solvable and isomorphic to the one of the following groups: \(\mathbb{Z}_p\) or an infinite dihedral pro-2 group \(C_2 \amalg C_2\).

The above theorem implies immediately the following result.

**Corollary 2.5.** Let \(G\) be a torsion pro-\(p\) group acting on a pro-\(p\) tree. Then \(G\) fixes a vertex.
Theorem 2.6. [15] Thm. 5.6] Let $G$ be a pro-$p$ group acting on a second countable (as topological space) pro-$p$ tree $T$ with trivial edge stabilizers. Then $G$ is a free pro-$p$ product of some vertex stabilizers $G_v$ and a free pro-$p$ group $F$.

Theorem 2.7. [31] Thm. B] Let $H = Q \amalg C A$ be a proper amalgamated free pro-$p$ product and $B$ be a normal subgroup of $A$ such that $B \cap C = 1$. Let $G = \langle B^q/g \in H \rangle$. Then $G$ is the free pro-$p$ product of groups $B^q$ where $q$ runs over a closed set of coset representatives for $C$ in $Q$.

The above theorem was stated in [31] Thm. B] using the language of fundamental groups of profinite graphs of profinite groups, and it was observed that the statement holds for pro-$C$ groups, where $C$ is a class of finite groups closed under subgroups, quotients and extensions. Note that $H$ acts on the standard pro-$p$ tree $T$ associated with the free amalgamated pro-$p$ product.

Lemma 2.8. Let $G$ be a profinite group acting on a profinite graph $S$ and let $m_1, m_2$ be elements of a connected component $C$ of $S$. If $g \in G$ with $g m_1 = m_2$, then $g$ leaves $C$ invariant, i.e. $g \in \text{Stab}_C(C)$. In other words the restriction of the factorization $S \to S/G$ to $C$ coincides with the factorization $C \to C/\text{Stab}_C(C)$.

Proof. Since $m_2 \in C \cap gC$, $C \cup gC$ is connected (cf. [19] Exercise 1.9 (i)), so $C = gC$. □

3. Definition of pro-$p$ analogues of limit groups via extensions of centralizers

First we define recursively a class of pro-$p$ groups $G_n$. Denote by $G_0$ the class of all free pro-$p$ groups of finite rank. Define inductively the class $G_n$ of pro-$p$ groups $G_n$, where $G_n$ is a free pro-$p$ amalgamated product $G_{n-1} \amalg_C A$, where $G_{n-1}$ is any group from the class $G_{n-1}$, $C$ is any self-centralized procyclic pro-$p$ subgroup of $G_{n-1}$, $A$ is any finite rank free abelian pro-$p$ group such that $C$ is a direct summand of $A$. In Lemma 3.1 we will show that every $G_n$ is torsion-free.

Definition. The class of pro-$p$ groups $L$ consists of all finitely generated pro-$p$ subgroups $H$ of some $G_n \in G_n$ where $n \geq 0$. If $n$ is minimal one with the above properties we say that $H$ has height $n$.

Examples. Note that all Demushkin pro-$p$ groups $H$ with the invariant $q = \infty$ have pro-$p$ presentation $\langle x_1, \ldots, x_d \mid [x_1, x_2] \cdots [x_{d-1}, x_d] \rangle$ where $d$ is even. Write $H_d$ for the above group. If $d = 2$ then $H_2$ is a free abelian pro-$p$ group of finite rank, so a pro-$p$ group of the class $L$.

Suppose that $d$ is divisible by 4 and define $F_1$ as the free pro-$p$ group with basis $x_1, \ldots, x_{d/2}$ and $F_2$ the free pro-$p$ group with basis $x_{d/2+1}, \ldots, x_d$. Let $C_1$ be the procyclic subgroup of $F_1$ generated by $z_1 = [x_1, x_2] \cdots [x_{d/2-1}, x_{d/2}]$ and $C_2$ be the procyclic subgroup of $F_2$ generated by $z_2 = [x_{d/2+1}, x_{d/2+2}] \cdots [x_{d-1}, x_d]$. Then $H \simeq F_1 \amalg_{C_1 \triangleleft C_2} F_2$ where the isomorphism between $C_1$ and $C_2$ sends $z_1$ to $z_2^{-1}$. Note that there is an isomorphism between $F_1$ and $F_2$ that identifies $C_1$ and $C_2$ i.e. $x_1, x_2, \ldots, x_{d/2}$ go to $x_{d/2+1}, x_{d/2+2}, \ldots, x_d$. Thus $H \simeq F \amalg_C F$ where $F$ is a free pro-$p$ group of rank $d/2$ and $C$ is a selfcentralized procyclic subgroup of $F$. Note that $H \simeq F \amalg_C F$ embeds in $T = F \amalg_C A$, where $A \simeq \mathbb{Z}_p^2$ with $A/C \simeq \langle a \rangle$, since $F \amalg_C F \simeq F \amalg_C F^a$ is the pro-$p$ subgroup of $T$ generated by $F$ and $F^a$. Thus $H$ is a pro-$p$ group of the class $L$. 


Lemma 3.1. The groups $G_n \in \mathcal{G}_n$ are always of finite cohomological dimension. In particular every pro-$p$ group from the class $\mathcal{L}$ has finite cohomological dimension and so is torsion free.

Proof. We induct on $n$, the case $n = 0$ is obvious. Suppose $n \geq 1$. Since $G_{n-1}$, $A$ and $C$ all have finite cohomological dimensions, we have that for sufficiently large $i$, $i \geq i_0$ say, all cohomology groups $H^i(G_{n-1}, F_p)$, $H^i(A, F_p)$ and $H^i(C, F_p)$ are zero. By Theorem 2.1 the free pro-$p$ product $G_n = G_{n-1} \amalg C$ is proper and hence there is corresponding long exact sequence in homology and cohomology. Then $H^{i+1}(G_n, F_p) = 0$ for $i \geq i_0$ and so by [18, Cor. 7.1.6] $G_n$ has finite cohomological dimension.

4. Free-by-(torsion free poly-procyclic) pro-$p$ groups

In this section we prove some properties of the pro-$p$ groups of the class $\mathcal{L}$ that are known to hold for abstract limit groups. The facts that abstract limit groups are free-by-(torsion-free nilpotent) and the Euler characteristic is non-positive were proved in [12]. The fact that abstract limit groups are of type $FP_{\infty}$ follows directly from Bass-Serre theory but this theory does not hold for pro-$p$ groups in general. So we find an alternative way of proving that the pro-$p$ groups of the class $\mathcal{L}$ are of type $FP_{\infty}$.

The next lemma is valid for free pro-$p$ products of free abelian pro-$p$ groups that can be expressed as an inverse limit of free pro-$p$ products of finitely many free abelian pro-$p$ groups. This is true for example for free pro-$p$ products $M = \prod_{w \in W} M_w$ of pro-$p$ groups $M_w$ indexed locally constantly by a profinite space $W$ introduced in [23] (see Proposition 2.1 there). This means that $W$ is a finite union of clopen subsets $W_\alpha$ such that $M_v = M_w$ for $v, w \in W_\alpha$.

Lemma 4.1. Let $M$ be a locally constant free pro-$p$ product

$$M = \prod_{w \in W} M_w$$

where $W$ is a profinite space and every $M_w$ is a finite rank free abelian pro-$p$ group. Then the quotients of the lower central series of $M$ are torsion-free.

Proof. It suffices to prove the lemma for $W$ finite and then take inverse limit. For $W$ finite we use the pro-$p$ version of Magnus embedding proved by Lazard [12]. Let $Y_w$ be a basis of $M_w$ as a free abelian pro-$p$ group and $X_w$ a set of the same cardinality as $Y_w$. Set $X$ as the disjoint union $\cup_{w \in W} X_w$ and consider the ring of non-commutative formal power series $R = \mathbb{Z}_p[[X]]$. The closed group generated by $Y = \{1 + x\}_{x \in X}$ is a free pro-$p$ group with a basis $Y$.

Let $I$ be the closed two-sided ideal of $R$ generated by $\{x_i x_j - x_j x_i \mid x_i, x_j \in X_w, w \in W\}$ and set $S = R/I$. Then the closed group $T$ generated by the image of $Y$ in $S$ is isomorphic to $M$ (see Theorem 5.9a in [20]). Consider the closed two-sided ideal $J$ of $S$ generated by the image of the elements of $X$ in $S$. Then the $(i-1)$-th quotient of the lower central series of $M$ is isomorphic to $((T - 1) \cap J^{i-1})/((T - 1) \cap J^i)$. Note that $((T - 1) \cap J^{i-1})/((T - 1) \cap J^i)$ embeds in the free $\mathbb{Z}_p$-module $J^{i-1}/J^i$, so $((T - 1) \cap J^{i-1})/((T - 1) \cap J^i)$ is torsion-free.

We do not know whether every $G_n \in \mathcal{G}_n$ is residually torsion-free nilpotent pro-$p$ though abstract limit group are residually torsion-free nilpotent [12]. Still we can prove that $G_n$ is residually torsion-free poly-procyclic.
Theorem 4.2. Every \( G_n \in \mathcal{G}_n \) is an inverse limit of torsion-free poly-procyclic groups and so is residually torsion-free poly-procyclic.

Proof. We use induction on \( n \). Note that \( G_0 \) is a free pro-\( p \) group of finite rank, hence residually torsion-free nilpotent pro-\( p \).

Suppose now that \( G_n = G_{n-1} \Pi_C A \). \( C \simeq \mathbb{Z}_p \), \( A = C \times B \simeq \mathbb{Z}^n_p \) and \( G_{n-1} \) is residually torsion-free poly-procyclic. Then the intersection of the kernels of all maps \( G_{n-1} \Pi_C A \to Q \Pi_C A \) whose restriction to \( G_{n-1} \) is a projection to a poly-procyclic torsion-free group \( Q \) and whose restriction to \( B \) is the identity map, is trivial. To see this let \( \psi : G_{n-1} \to \prod_{i=1}^\infty P_i \) be an embedding of \( G_{n-1} \) in a direct product of poly-procyclic pro-\( p \) groups. Denote by \( \psi_k : G_{n-1} \to \prod_{i=1}^k P_i \) the composite of \( \psi \) with the projection \( \prod_{i=1}^\infty P_i \to \prod_{i=1}^k P_i \) and put \( \psi_k(G_{n-1}) = Q_k \). Then \( G_{n-1} \) is the inverse limit of the groups \( Q_k \) and hence the (descending) intersection \( \bigcap B_k \) (that can be viewed as the inverse limit) of kernels \( B_k \) of \( \psi_k \) is trivial. The kernel \( N_k \) of the map \( G_{n-1} \Pi_C A \to Q_k \Pi_C A \) is exactly \( \langle B_k \rangle_{g \in G_n} \) and so by Theorem 2.7 it is the free pro-\( p \) product of groups \( B_k^q \) where \( q \) runs over a closed set of coset representatives for \( C \) in \( A \). But the inverse limit of the groups \( B_k \) is trivial, hence so is the inverse limit of the groups \( N_k \).

Thus it remains to show that a proper amalgamated product \( H = Q \Pi_C A \), where \( Q \) is torsion-free poly-procyclic, the image of \( C \) in \( Q \) is infinite (and so identified with \( C \simeq \mathbb{Z}_p \)) and \( C \) is a direct summand of \( A \), is residually torsion-free poly-procyclic.

Let \( \theta : H = Q \Pi_C A \to Q \) be the homomorphism that is the identity map on \( Q \) and the projection of \( A \) to \( C \) with kernel \( B \). Then the kernel of \( \theta \) is the normal closure of \( B = \ker(\theta) \cap A \) in \( H \). Since \( \ker(\theta) \cap Q^g = (\ker(\theta) \cap Q)^g = 1 \) by Theorem 2.7, \( \ker(\theta) \) is a free pro-\( p \) product of \( B^g \) over some closed subset \( S \) of representatives of the coset classes of \( Q/C \). Let \( U \) be a normal open subgroup of \( Q \) and \( \mu_U \) be the homomorphism from \( \ker(\theta) \) to the free pro-\( p \) product \( V_U = \coprod_{g \in Q/UC} B^g \) that identifies the copies of \( B^g \) when \( g \) represent the same element in the image of \( S \) in \( Q/UC \). Then \( \cap \ker(\mu_U) = 1 \) and it suffices to produce a filtration of characteristic subgroups of \( V_U \) with torsion-free finitely generated abelian quotients. By Lemma 3.1 the lower central series of \( V_U \) has this property. \( \square \)

Proposition 4.3. Every pro-\( p \) group from the class \( \mathcal{L} \) is free-by-(torsion free poly-procyclic).

Proof. It suffices to prove the proposition for \( G_n \in \mathcal{G}_n \), by induction on \( n \). The case when \( n = 0 \) is obvious, so we assume that \( n \neq 0 \). Suppose the statement is true for any group \( \in \mathcal{G}_{n-1} \). Suppose there is a counterexample \( \in \mathcal{G}_n \), and let \( G_n = G_{n-1} \Pi_C A \) be one with the minimal number of generators \( d(G_n) \). Let \( T \) be a complement of \( C \simeq \mathbb{Z}_p \) in \( A \simeq \mathbb{Z}^m_p \), so \( T \simeq \mathbb{Z}^{m-1}_p \). Let \( D \simeq \mathbb{Z}_p \) be a direct summand of \( T \), i.e. \( T = B \times D \) (with \( B \) possibly the trivial group) and let \( G_n \) be the quotient of \( G_n \) by the normal closure of \( D \). Then \( \bar{G}_n \simeq G_{n-1} \Pi_C \mathbb{Z}^{m-1}_p \). By Theorem 4.2 \( \bar{G}_n = \lim_{\leftarrow} Q_i \) is an inverse limit of torsion-free poly-procyclic pro-\( p \) groups. Thus for certain \( i \) the restriction to \( C \) of the projection \( \bar{G}_n \to Q_i \) is injective. Denote by \( \varphi \) the canonical map \( \bar{G}_n \to Q_i \). By the induction hypothesis and the minimality assumption on \( d(G_n) \) there is an epimorphism \( \mu : \bar{G}_n \to P \) to a poly-procyclic group \( P \) with the free kernel. Thus putting \( Q \) to be the quotient of \( \bar{G}_n \) modulo \( \ker(\mu) \cap \ker(\varphi) \) we have an epimorphism \( \eta : \bar{G}_n \to Q \) with free kernel whose restriction to \( C \) is injective.
Let $\theta : G_n \to \tilde{G}_n \to Q$ be the composition of $\eta$ and the natural epimorphism $\nu : G_n \to \tilde{G}_n$ and let $K$ be its kernel. Since $K \cap C = 1$, by Theorem [2.6] $K$ is the free pro-$p$ product of some conjugates of $K \cap G_{n-1}$, $K \cap A$ and a free pro-$p$ group $F$. But the restriction of $\nu$ to $G_{n-1} \Pi C B$ is injective and so $K \cap (G_{n-1} \Pi C B)$ is naturally isomorphic to the kernel of $\eta$, which is free pro-$p$. So the kernel of $\theta |_{G_{n-1}}$ is free since it is a subgroup of $K \cap (G_{n-1} \Pi C B)$. Furthermore, the intersection $K \cap A$ is at most procyclic. Thus $K$ is free pro-$p$. 

Corollary 4.4. Every pro-$p$ group $G$ from the class $\mathcal{L}$ is of type $FP_\infty$. In particular, $G$ is finitely presented.

Proof. Let $N$ be a closed normal subgroup of $G$ such that $Q = G/N$ is a torsion-free poly-procyclic pro-$p$ group, in particular is of finite rank, and $N$ is a free pro-$p$ group. Then by [5, Thm. A] $G$ is of type $FP_m$ if and only if $H_i(N, \mathbb{F}_p)$ is finitely generated as a $\mathbb{F}_p[[Q]]$-module for every $i \leq m$. Since $G$ is of type $FP_1$ (i.e. finitely generated as a pro-$p$ group) $H_1(N, \mathbb{F}_p)$ is finitely generated as a $\mathbb{F}_p[[Q]]$-module. Since $N$ is free pro-$p$ we have that $H_i(N, \mathbb{F}_p) = 0$ for $i \geq 2$. Thus $G$ is of type $FP_n$ for every $m$.

Corollary 4.5. Every non-trivial pro-$p$ group $G$ from the class $\mathcal{L}$ has infinite abelianization.

Proof. By Proposition [4.3] $G$ is residually torsion-free poly-procyclic. Any torsion-free poly-procyclic group has infinite abelianization. The result follows. 

5. Normalizers and centralizers

Theorem 5.1. Let $G$ be a pro-$p$ group from the class $\mathcal{L}$. Then for any $g \in G \setminus \{1\}$

$$N_G(\langle g \rangle) = C_G(\langle g \rangle)$$

is a free abelian pro-$p$ group of finite rank.

Proof. Note that it is sufficient to consider the case $G = G_n \in \mathcal{G}$. We argue by induction on $n$.

I. Suppose first that $n = 0$. Note that every closed subgroup of a free pro-$p$ group is free pro-$p$ [13, Cor. 7.7.5]. Therefore $N = N_{G_0}(\langle g \rangle)$ is free pro-$p$ and by [13, Proposition 8.6.3] the rank of $N$ is not bigger than the rank of $\langle g \rangle$ so that $N \simeq \mathbb{Z}_p$, in particular $N$ is abelian. Thus $N \subseteq C_{G_0}(\langle g \rangle)$, hence $C_{G_0}(\langle g \rangle) = N_{G_0}(\langle g \rangle) \simeq \mathbb{Z}_p$ as required.

II. Suppose now that $n > 0$ and that the theorem holds for $G_{n-1}$. Thus $G = G_{n-1} \Pi C A$, where $C \cong \mathbb{Z}_p$, $A \cong \mathbb{Z}_p^m$ and $C$ is a direct factor of $A$. Suppose first that $N_{G_n}(\langle g \rangle) \neq C_{G_n}(\langle g \rangle)$ and choose $t \in N_{G_n}(\langle g \rangle) \setminus C_{G_n}(\langle g \rangle)$. Thus $g^t = g^\lambda$ for some $\lambda \in \mathbb{Z}_p \setminus \{1 \cup p\mathbb{Z}_p\}$. Put $M = \langle t, g \rangle$ and observe that it is solvable but not abelian.

Since $G_n$ splits as a proper free pro-$p$ product with amalgamation, $G_n$ acts on the canonical pro-$p$ tree from Section [2] and $M$ acts on the same pro-$p$ tree by restriction. By Theorem [2.4] either $M = M_\nu$ i.e. $M$ is contained in the stabilizer $G_v$ of a vertex or there is a stabilizer of an edge $M_e$ such that $M_e$ is a normal subgroup of $M$ and either $M/M_e \simeq \mathbb{Z}_p$ or $M/M_e \simeq C_2 \Pi C_2$ (in this case $p = 2$). If $M = M_\nu$ we may assume that $M$ is a subgroup of $G_{n-1}$ or $\mathbb{Z}_p^m$ and so by induction hypothesis $M$ is abelian, a contradiction.
Let us consider the latter case. By conjugating $M_e$ if necessary we may assume that $M_e$ is in $C$. Since $M_e$ is cyclic, by [20] Corollary 2.7

\[ N_G(M_e) = N_{G_{n-1}}(M_e) \Pi_C N_A(M_e) = C \Pi_C A = A \]

and so $M \leq N_G(M_e) = A$ is abelian, a contradiction with $M$ solvable but not abelian. Thus we have proved that

\[ N_{G_n}([g]) = C_{G_n}([g]). \]

III. Finally it remains to show that $C_{G_n}(g)$ is a finitely generated abelian pro-$p$ group. By [19] Lemma 3.11 there is a minimal pro-$p$ subtree $T_1$ on which $C_{G_n}(g)$ acts. If $T_1$ is just one vertex, $C_{G_n}(g)$ is conjugate to a subgroup of either $G_{n-1}$ or $A$ and so we deduce the result from the induction hypothesis.

Suppose $|T_1| > 1$. Denote by $K$ the kernel of the action of $C_{G_n}(g)$ on $T_1$. Then $K$ is a subgroup of every edge stabilizer of the action of $C_{G_n}(g)$ on $T_1$, but any edge stabilizer is procyclic. In particular $K$ is procyclic.

There are two cases. First if $g \in K$ then $g$ is in the stabilizer $G_e$ of an edge and this case can be resolved by applying (5.1). If $g \notin K$ note that $C_{G_n}(g)/K$ acts on $T_1/K$ faithfully irreducibly and so by [19] Lemma 3.16 (b) $C_{G_n}(g)/K$ is free pro-$p$ containing a procyclic non-trivial normal subgroup $g/K$, hence procyclic (cf. [18] Proposition 8.6.3). Then $C_{G_n}(g)$ is procyclic-by-procyclic, which corresponds exactly to the case treated in II; indeed, in II it was shown that the procyclic-by-

Corollary 5.2. The group $G_n$ is transitive commutative i.e. if $[g, h] = 1 = [h, t]$ for some non-trivial elements $g, h, t$ of $G_n$ then $[g, t] = 1$.

Proof. Note that $g, t \in C_{G_n}(h)$ and $C_{G_n}(h)$ is abelian by Theorem 5.1

Corollary 5.3. If $G$ is a non-abelian pro-$p$ group from the class $\mathcal{L}$ then its center is trivial.

Proof. If $g_1, g_2 \in G$ and $h \in Z(G) \setminus \{1\}$ then $[h, g_1] = [h, g_2] = 1$ and by transitive commutativity $[g_1, g_2] = 1$ i.e. $G$ is abelian, a contradiction.

Corollary 5.4. Every virtually abelian pro-$p$ group $G$ from the class $\mathcal{L}$ is abelian.

Proof. Let $H$ be a maximal abelian subgroup of $G$. If $G \neq H$ then take $g \in G \setminus H$ and note that for some $m > 1$ we have $g^m \in H \setminus \{1\}$. Then $[g, g^m] = 1 = [g^m, H]$ and by transitive commutativity $[g, H] = 1$ i.e. $\langle H, g \rangle$ is an abelian subgroup of $G$, a contradiction with the maximality of $H$.

Corollary 5.5. Every soluble pro-$p$ group $H$ from the class $\mathcal{L}$ is abelian. If $H$ is an abelian non-procyclic subgroup of $G_n = G_{n-1} \Pi_C A$ then $H$ is conjugate in $G_n$ to a subgroup of $G_{n-1}$ or to a subgroup of $A$.

Proof. Let $T$ be the canonical pro-$p$ tree on which $G_n$ acts (see Section 2). Then by Theorem 2.4 from the preliminaries either $H$ stabilizes a vertex or there is an edge $e$ such that $H/e$ is either $\mathbb{Z}_p$ or $C_2 \Pi C_2$.

In the first case $H$ is either conjugate to a subgroup of $G_{n-1}$ or $H$ is a subgroup of an abelian vertex stabilizer. Using induction on $n$ in both cases $H$ is abelian.

Suppose that $H/e \simeq \mathbb{Z}_p$. If $H_e = 1$ there is nothing to prove, so we can assume that $H_e \neq 1$. Since $H_e$ is procyclic we conclude that $H_e \simeq \mathbb{Z}_p$. By Theorem 5.4 $N_{G_n}(H_e)$ is abelian. But $H \subseteq N_{G_n}(H_e)$ so $H$ is abelian. Moreover, substituting in
Proof. Without loss of generality we can assume that $V$ is conjugate to a subgroup of an abelian vertex group.

Suppose that $H/H_e \cong C_2 \oplus C_2$. If $H_e = 1$ then $C_2 \oplus C_2$ embeds in $H$ but by Lemma 3.1 $H$ is torsion free, a contradiction. If $H_e \neq 1$ then $H_e \cong \mathbb{Z}/2$ and $H = N_H(H_e)$. By Theorem 6.1 $N_H(H_e)$ is abelian and hence $H/H_e \cong C_2 \oplus C_2$ is abelian, a contradiction. □

6. Finitely generated normal subgroups

Lemma 6.1. Let $G$ be a pro-$p$ group with a lower central series with torsion-free quotients and $\varphi \in \text{Aut}(G)$ be an automorphism of finite order that acts trivially on the abelianization $G/[G,G]$. Then $\varphi$ is trivial.

Proof. The proof is similar to the proofs of [18 Thm. 4.5.6], [14 Lemma, p.323] where the case of a finite rank free pro-$p$ group $G$ is considered.

Let $\gamma_i(G)$ be the $i$th term of the lower central series of $G$. We prove by induction on $i$ that $\varphi$ acts trivially on $G/\gamma_i(G)$. Suppose we know that $\varphi(g) = gm$ for some $g \in G, m \in \gamma_i(G)$. We aim to show that $m \in \gamma_{i+1}(G)$. Note that since $\varphi$ acts trivially on $G/\gamma_2(G)$ we have that $\varphi$ acts trivially on $\gamma_i(G)/\gamma_{i+1}(G)$. Let $k$ be the order of $\varphi$. Then $g = \varphi^k(g) = g m_1 \varphi(m) \varphi^2(m) \cdots \varphi^{k-1}(m) \in gm^k\gamma_{i+1}(G)$. Then the image of $m$ in $\gamma_i(G)/\gamma_{i+1}(G)$ has finite order. Finally since $\gamma_i(G)/\gamma_{i+1}(G)$ is torsion-free we get that $m \in \gamma_{i+1}(G)$, as required.

Theorem 6.2. Let $H$ be a pro-$p$ group acting on a pro-$p$ tree $T$ with a finitely generated normal pro-$p$ subgroup $L$ such that $H/L \cong \mathbb{Z}/p$. Then there is an open subgroup $H_1$ of $H$ containing $L$ such that $H_1/L = (L/L) \times \mathbb{Z}/p$, where $L = \bigcap_{V \in V} U$. Let $V$ be the vertex set of $T$.

Proof. Without loss of generality we can assume that $L \neq \tilde{L}$ otherwise there is nothing to prove. Let $U$ be an open subgroup of $H$ such that $L \subseteq U$. Set $\tilde{U} = \langle U \cap V \rangle_{V \in V}$. By Theorem 2.2 the group $H/\tilde{U}$ acts on the pro-$p$ tree $T/\tilde{U}$ and since $U/\tilde{U}$ acts freely, $U/\tilde{U}$ is a free pro-$p$ group of finite rank. Let $V_U$ be the maximal subgroup of $H$ such that $U \subseteq V_U$ and $V_U/\tilde{U}$ is a free pro-$p$ group. Then $H/\tilde{U} = (V_U/\tilde{U}) \times C_d$, where $[H : V_U] = d$ and $C_d$ denotes the cyclic group of order $d$. Define $V = \bigcap_{U} V_U$.

Claim. $V = L$.

Proof of the Claim. Assume that $V \neq L$. Then $V$ is an open subgroup of $H$ and $V_U = V$ for infinitely many $U$ such that $\cap U = L$. Consider the normal subgroup $L\tilde{U}/\tilde{U}$ of the free pro-$p$ group $V/\tilde{U}$, where $V/\tilde{U}$ has rank at most $d(L)+1$. Note that $L\tilde{U}/\tilde{U}$ is finitely generated and normal in a finitely generated free pro-$p$ group. By [18 Proposition 8.6.13] either $L\tilde{U}/\tilde{U}$ is trivial or of finite index in $V/\tilde{U}$. In the first case $L \subseteq \tilde{U}$, so $L \subseteq \cap U = \tilde{L}$, a contradiction. Then we can assume that $L\tilde{U}/\tilde{U}$ has finite index in $V/\tilde{U}$, say $m_U$ and by Schreier’s formula $d(L\tilde{U}/\tilde{U}) - 1 = m_U(d(V/\tilde{U}) - 1)$.

Suppose that $d(V/\tilde{U}) \neq 1$. Then $d(L) - 1 \geq d(L\tilde{U}/\tilde{U}) - 1 = m_U(d(V/\tilde{U}) - 1) \geq m_U$. Thus $m_U$ is bounded from above and so $[H : L\tilde{U}] = [H : V][V : L\tilde{U}] = [H : V]m_U \leq [H : V](d(L) - 1) < \infty$. The intersection of $L\tilde{U}$ over $U$ is $LL = L$ and on the other hand $L\tilde{U}$ has bounded index in $H$ and $H/L \cong \mathbb{Z}/p$, a contradiction.
Finally it remains to consider the case when $d(V/\tilde{U}) = 1$. Consider a sequence of open subgroups $U_1 \supseteq U_2 \supseteq \ldots \supseteq V$ such that $V_{\tilde{U}_i} = V, d(V/\tilde{U}_i) = 1$. Since $V/\tilde{U}_i \cong \mathbb{Z}_p$ is a quotient of $V/\tilde{U}_{i+1} \cong \mathbb{Z}_p$ we deduce that $\tilde{U}_{i+1} = \tilde{U}_i$. Note that $\tilde{L}$ is the intersection of $\tilde{U}_i$’s i.e. $\tilde{L} = \cap_i \tilde{U}_i = \tilde{U}_1$. Thus $\tilde{L} = \tilde{U}_1 \subseteq L \subseteq U_1$.

Let us fix one $U$ as above i.e. $U$ is an open subgroup of $H$ that contains $L$ and write $F_U = V_U/\tilde{U}$. Furthermore we can choose $U$ such that $s = \lbrack H : V_U \rbrack > 1$ and write $M_U = H/\tilde{U} = F_U \rtimes C_s$. Note that $H$ acts by conjugation on the abelianization $F_U/[F_U, F_U] \cong \mathbb{Z}_p^{d(F_U)}$, where $d(F_U) \leq d(V_U) \leq d(L) + 1$. This gives a homomorphism

$$\varphi_U : H \to GL_{d(F_U)}(\mathbb{Z}_p)$$

such that $V_U \subseteq Ker(\varphi_U)$. Note that for a fixed $d$ there is an upper bound to the order of finite cyclic subgroups of $GL_d(\mathbb{Z}_p)$. Indeed by [2, Thm 5.2] the first principal congruence subgroup for $p$ odd and the second principal congruence subgroup for $p = 2$ is uniform and therefore is torsion-free. Thus $\lbrack H : Ker(\varphi_U) \rbrack$ is bounded by some number depending on $d(F_U)$.

Since $d(F_U)$ is bounded by $d(L) + 1$, the index $\lbrack H : Ker(\varphi_U) \rbrack$ is bounded by a number depending on $d(L)$ as well, so there is some subgroup of finite index $H_1$ in $H$ containing $L$ and such that $H_1 \subseteq Ker(\varphi_U)$ for every $U$.

By Lemma 6.1 every automorphism of finite order of $F_U$ that induces the identity map on $F_U/[F_U, F_U]$ is identity. Then for an open subgroup $U$ such that $V_U \subseteq H_1$ we have $H_1/\tilde{U} = (V_U/\tilde{U}) \rtimes (H_1/V_U) = (V_U/\tilde{U}) \rtimes (H_1/V_U)$, so either $H_1/\tilde{U}$ is abelian, in this case $V_U/\tilde{U}$ is procyclic, or $H_1/\tilde{U}$ is non-abelian and the copy of $H_1/\tilde{U}$ is the center of $H_1/\tilde{U}$, in this case $V_U/\tilde{U}$ is a non-procyclic free pro-$p$. By taking the inverse limit over $U$ we get

$$H_1/\tilde{L} = (L/\tilde{L}) \rtimes (H_1/L) \cong (L/\tilde{L}) \rtimes (H_1/L),$$

where $H_1/L \cong \mathbb{Z}_p$. \hfill \Box

**Lemma 6.3.** Let $H$ be a pro-$p$ group acting on a second countable pro-$p$ tree $T$ with procyclic edge stabilizers and finite rank free abelian pro-$p$ vertex groups. Let $L$ be a finitely generated normal closed subgroup of $H$ such that $H/L \cong \mathbb{Z}_p$ and $He = Le$ for every edge $e \in E(T)$. Assume further that there is a decomposition $H = L \rtimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ has a generator $g$ that fixes an edge of $T$ and such that for every vertex $v$ of the tree $T$ we have $H_v = L_v \rtimes \langle g^{l_v-1} \rangle$ for some $l_v \in \tilde{L} = \langle L_v \rangle$, $v \in V(T)$. Then $g$ commutes with $\tilde{L}$.

**Proof.** Note that $Le = He$ and $H_e$ procyclic implies that $L_e = 1$ for every edge $e$ of $T$. Then by Theorem 2.6

$$L \cong L_0 \times (L/\tilde{L}) \text{ and } L_0 = \prod_{v \in V_0} L_v,$$

where $V_0$ is a subset of the vertex set $V$ and $\tilde{L}$ is the normal closure of $L_0$ in $L$. 

Note that $V_0$ is finite since $L$ is finitely generated. It follows from the pro-$p$ version of the Kurosh Subgroup Theorem [15] that

\[ \widetilde{L} = \coprod_{v \in V_0} \prod_{w} L_w = \coprod_{v \in V_1} L_v, \]

where $w$ runs over closed set of representatives of the orbit $Lv$ and $V_1$ is a closed subset of $V$. In particular, this is a locally constant free pro-$p$ product.

Observe that $He = Le$ for every edge $e$ implies $Hv = Lv$ for every vertex $v$. Recall that by assumption $H_v$ is abelian and for every $v \in V$ and for $m \in L_v$ we have $m^g = m^{l_v}$ for some $l_v \in \widetilde{L}$. This implies the following

Claim 1. The action of $g$ via conjugation on $\widetilde{L}^{ab} = \prod_{v \in V_1} L_v$ is trivial.

Let $M = \langle \widetilde{L}, g \rangle = \widetilde{L} \times \langle g \rangle$ and $U$ be the closed subgroup of $H$ generated by $(g^p)^M$.

Claim 2. $L_h^v \cap U = 1$ for every $v \in V_1, h \in M$.

Indeed $L_v \cap U \subset U \cap \widetilde{L} = [g^p, \widetilde{L}]$ and $\widetilde{L} = \prod_{v \in V_1} L_v$. Since $L_v$ survives in the abelianization of $\widetilde{L}$ and by Claim 1 the group $U$ acts trivially on the abelianization of $\widetilde{L}$ we get that $[g^p, \widetilde{L}] \cap L_v = 1$. This completes the proof of the claim.

In the rest of the proof overlining means image in the quotient group $M/U$ and should not be confused with the closure. Consider the group $\overline{M} = M/U$ and the image $\overline{L} = \overline{L}/(U \cap \overline{L})$ of $\widetilde{L}$ in $\overline{M}$. Since $U$ is generated by stabilizers of edges (hence of vertices) $\overline{M}$ acts on the pro-$p$ tree $T/U$ (cf. Theorem 2.2) with vertex stabilizers $L_v U/U \simeq L_v$. As before $\overline{L}$ intersects edge stabilizers trivially and since $\overline{L}$ is generated by vertex stabilizers the same holds for $\overline{L}$. Hence

\[ \overline{L} = \coprod_{w \in W} \overline{L}_w \]

where $\overline{L}_w$ are some pairwise not conjugated vertex groups of $\overline{L}$. Then by Claim 2 $\overline{L}_w \simeq L_v$ for some vertex group $L_v$ of $L$.

Write $\overline{g}$ for the image of $g$ in $\overline{M}$. Then by Claim 1

(6.1) $\overline{g}$ acts trivially (via conjugation) on the abelianization of $\overline{L}$.

Recall that every $L_v$ is a finite rank free abelian group. Note that $(U \cap \overline{L})$ is contained in the kernel of the natural epimorphism $L \to \prod_{v \in V_0} L_v$ restricted to $\overline{L}$ and so we have a natural epimorphism $\prod_{w \in W} \overline{L}_w \to \prod_{v \in V_0} L_v$, so the free pro-$p$ product

\[ \overline{L} = \coprod_{w \in W} \overline{L}_w \]

is also locally constant. Then $\overline{L}$ has torsion-free abelianization and by Lemma 4.1 $\overline{L}$ has a torsion-free lower central series. Then since $\overline{g}$ has finite order we deduce from Lemma 6.1 that $\overline{g}$ acts trivially on $\overline{L}$. Taking an inverse limit over $U$ (i.e. $s$ goes to infinity) we get that $g$ acts trivially on $\overline{L}$ as required. □
Lemma 6.4. Let $H$ be a pro-$p$ group from the class $\mathcal{L}$ with a finitely generated normal pro-$p$ subgroup $L$ such that $H/L \simeq \mathbb{Z}_p$. Then $H$ is abelian.

Proof. Let $H = L \rtimes \mathbb{Z}_p$ be a non-abelian pro-$p$ group from the class $\mathcal{L}$ such that $L$ is finitely generated as a pro-$p$ group. Let $n$ be the smallest non-negative integer such that $H \subseteq G_n \in \mathcal{G}_n$ i.e. $n$ is the weight of $H$. We can assume that our counterexample $H$ is minimal in the sense that $(n, d(G_n))$ is smallest possible with respect to the lexicographic order, where $d(G_n)$ is the minimal number of generators of $G_n$.

The group $G_n$ acts on a second countable pro-$p$ tree $S$ with procyclic edge stabilizers. By [19, Lemma 3.11] $H$ acts irreducibly on a pro-$p$ subtree $T$ of $S$ with procyclic edge stabilizers. Let $\tilde{L} = (L_v)_{v \in V(T)}$.

Claim 1. We claim that $H$ acts faithfully on $T$.

Proof. Indeed if the action of $H$ on $T$ is not faithful since the edge stabilizers are procyclic we get that the kernel of the action is $K \simeq \mathbb{Z}_p$. Then for any $h \in H \setminus K$ the closed subgroup of $H$ generated by $K$ and $h$ is metabelian. But any soluble pro-$p$ group from the class $\mathcal{L}$ is abelian, see Corollary 5.3. Thus $K \subseteq Z(H)$, a contradiction to Corollary 5.3.

This way we can assume from now on that $H$ acts faithfully on $T$. By Theorem 6.2 there is an open subgroup $H_1$ of $H$ containing $L$ such that

$$H_1/\tilde{L} \simeq (L/\tilde{L}) \times M$$

where $M \simeq \mathbb{Z}_p$.

Claim 2. The group $H_1$ acts irreducibly on $T$.

Proof. By Claim 1 $H$ acts faithfully irreducibly on $T$. Since $H_1$ is non-trivial, by [19, Prop. 3.14] $H_1$ acts irreducibly on $T$. The claim is proved.

By Corollary 5.3 every virtually abelian pro-$p$ group from the class $\mathcal{L}$ is abelian, so $H_1$ is not abelian. Then without loss of generality we can assume that $H = H_1$.

The pro-$p$ group $H/\tilde{L}$ acts on $T/\tilde{L}$ and by [19, Lemma 3.11] contains a minimal $H/\tilde{L}$ invariant subtree $T_1$ i.e. $H/\tilde{L}$ acts irreducibly on the pro-$p$ tree $T_1$.

Let $N$ be the kernel of the action of $H/\tilde{L}$ on $T_1$ and so the quotient group $B = (H/\tilde{L})/N$ acts irreducibly and faithfully on $T_1$. By [19, Lemma 3.16] every non-trivial abelian normal subgroup $A$ of $B$ is isomorphic to $\mathbb{Z}_p$ and $C_B(A)$ is a free pro-$p$ group, hence procyclic. Let $A$ be the image of $M$ in $B$. If $A$ is trivial we get that $M$ acts trivially on $T_1$.

Suppose that $A$ is non-trivial. Then $A \simeq \mathbb{Z}_p \simeq B$. On the other hand $L/\tilde{L}$ acts freely on $T_1$, so $(L/\tilde{L}) \cap N = 1$. Then the image of $L/\tilde{L}$ in $B = \mathbb{Z}_p$ is isomorphic to $L/\tilde{L}$, so $L/\tilde{L}$ is either trivial or $\mathbb{Z}_p$. It follows that either $H/\tilde{L} \simeq \mathbb{Z}_p^2$ or $H/\tilde{L} \simeq M \simeq \mathbb{Z}_p$ and $L = \tilde{L}$. In the first case since $B \simeq \mathbb{Z}_p$ by changing $M$ with another copy of $\mathbb{Z}_p$ inside $H/\tilde{L}$ we can reduce to the case when $M \subseteq N$. Note that we have proved that either $M$ acts trivially on $T_1$ or $L = \tilde{L}$.

Case 1. Suppose that $L \neq \tilde{L}$ and $M$ acts trivially on $T_1$. Let $C$ be a connected component of the full preimage $S_1$ of $T_1$ in $T$.
Claim 3. The (set-wise) stabilizers $H_2 := Stab_H(C)$ and $L_2 := Stab_L(C)$ of $C$ are finitely generated.

Proof. Note that $S_1$ is $H$-invariant and that $C$ coincides with a connected component of the full preimage of $T_1/(H/L) \subseteq T/H$ in $T$. Indeed, if $C$ is contained properly in a connected component $C_0$ of the full preimage of $T_1/(H/L)$ in $T$ then the image of $C_0$ in $T/L$ contains $T_1$ properly contradicting that the image of $C_0$ is $T_1/(H/L)$. Similarly $C$ coincides with a connected component of the full preimage of $T_1/(H/L) = T_1/(L/L) \subseteq T/L$ in $T$.

Let $U$ be an open normal subgroup of $H$ and $V = U \cap L$. Then $L/\hat{V}$ has bounded edge stabilizers. Let $\Sigma(V)$ denote the set of all finite subgroups $K \neq 1$ of $L/\hat{V}$. Since $L/\hat{V}$ is finitely generated having open free subgroup $V/\hat{V}$ by [4] Lemma 8 there is a finite subset $S$ of $\Sigma(V)$ with $\Sigma(V) = \{K^g \mid K \in S, g \in L/\hat{V}\}$. Therefore the subset $T_{\Sigma(V)} := \{m \in T/\hat{V} \mid \exists L \in S, m \in (T/\hat{V})^L\}$, which is the union of all subtrees of fixed points $(T/\hat{V})^K$ for subgroups $K \in \Sigma(V)$ can be represented in the form $T_{\Sigma(V)} = \bigcup_{L \in S} (T/\hat{V})^L/L$ and is hence a closed $H$-invariant subgraph of $T/\hat{V}$. Therefore by [32] Prop., p.486, the quotient graph $D(V)$ obtained by collapsing each connected component of $T_{\Sigma(V)}$ to a vertex is simply connected and hence is a pro-$p$ tree on which $L/\hat{V}$ acts with trivial edge stabilizers.

Since $M$ acts trivially on $T_1$ we have $L_e = 1$ for every $e \in C$. Indeed for every $e \in E(C)$ we have $S_0e \subseteq L_e \subseteq L_e$ where $S_0$ is a procyclic subgroup of $H$ that maps surjectively to $M$ by the canonical map $H \to H/L$. Then $H_e = (L \times S_0)e = L_e$ and so $H_e/L_e \simeq \mathbb{Z}_p$ for every $e \in E(C)$. Since $H_e$ is procyclic $L_e = 1$ for all $e \in E(C)$. But by Lemma 2.8 $H_e = (H_2)_e$ for every $e \in E(C)$, so $L \cap H_2$ is of infinite index in $H_2$ i.e. $H_2/(L \cap H_2) \simeq \mathbb{Z}_p$.

Put $L_{2V} = L_2/(L_2 \cap \hat{V})$. Denote by $C_V$ the image of $C$ in $T/\hat{V}$. Since $L_e = 1$ for all $e \in E(C)$ we obtain that $(L/\hat{V})_e = 1$ for all $e \in E(C_V)$. Then by Theorem 2.6 applied for $L_{2V}$ acting on the pro-$p$ tree $C_V$ $$L_{2V} = \left( \prod_{v \in W} (L/\hat{V})_v \right) \prod (L/\tilde{L})$$ for some $W \subseteq V(C_V)$, where we used that $$(L_{2V})_v = (L/\hat{V})_v \text{ and } L/\tilde{L} = L_{2V}/\tilde{L}_{2V}.$$ Indeed for every $v \in C$ one has $L_v = (L_2)_v$, since $L_2$ is the stabilizer of the connected component $C$, taking the images of these stabilizers in $T/\hat{V}$ one gets the first equality. For the second equality note that the definition of $L_2$ implies that $L = L_2\tilde{L}$ and by the definition of $C$ the canonical map $T \to T/\tilde{L}$ sends $C$ surjectively to $T_1$.

By Lemma 2.8 $T_1 = C/(H_2 \cap \tilde{L})$ and since $T_1$ is a pro-$p$ tree, by Lemma 2.3 $\tilde{L} \cap L_2 = \tilde{L} \cap H_2 = \tilde{L} \cap H_2 = \tilde{L}_2$, where $\tilde{L}_2$ is the closed subgroup of $L_2 = L \cap H_2$ generated by stabilizers of vertices in $C$. Then $$L/\tilde{L} = (L_2\tilde{L})/\tilde{L} = L_2/(L_2 \cap \tilde{L}) = L_2/\tilde{L}_2.$$ Let $G = L_{2V} \cap \tilde{L}/\hat{V} = (L_2\tilde{V} \cap \tilde{L})/\hat{V} = (L_2 \cap \tilde{L})\tilde{V}/\hat{V} = \tilde{L}_2\tilde{V}/\hat{V}$. Since $T_1 = C_V/(L_{2V} \cap \tilde{L}/\hat{V})$ is a pro-$p$ tree we get that $G = \tilde{G}$, so $L_{2V} = \tilde{L}_2\tilde{V}/\hat{V}$. Then $$L_{2V}/\tilde{L}_{2V} = (L_2\tilde{V}/\hat{V})/(\tilde{L}_2\tilde{V}/\hat{V}) = L_2/\tilde{L}_2.$$
Observe furthermore that $W$ can be chosen any $\delta(C_V/L_{2V})$ where $\delta : C_V/L_{2V} \to C_V$ is a continuous section of the canonical projection $C_V \to C_V/L_{2V}$.

Note that the collapsing of the connected components of $T_{\Sigma(V)}$ does not affect $C_V$, i.e. we can denote by the same letter the isomorphic image of it in $D(V)$. Thus we have the following commutative diagram

\[
\begin{array}{ccc}
C_V & \to & T/\tilde{V} \to D(V) \\
\downarrow & & \downarrow \\
T_1 = C_V/(\tilde{L}_2\tilde{V}/\tilde{V}) & \to & T/\tilde{L} \to D(V)/(\tilde{L}/\tilde{V}) \\
\downarrow & & \downarrow \\
C_V/L_{2V} = C/L_2 = T_1/(L/\tilde{L}) & \to & T/L = (T/\tilde{V})/L/\tilde{V} \to D(V)/(L/\tilde{V})
\end{array}
\]

where all composite maps from left to right are injections (the lower one follows from the middle one). Then by Theorem 2.6 applied for the action of $L/\tilde{V}$ on $D(V)$

\[L/\tilde{V} \cong (\prod_{v \in V_0} (L/\tilde{V})_v) \prod_{v \in W} (L/\tilde{V})_v \prod_{v \in V_0} (L/\tilde{L}) = (\prod_{v \in V_0} (L/\tilde{V})_v) \prod_{v \in V_0} (L/\tilde{L})\]

where $W \cup V_0 = \mu(D(V)/(L/\tilde{V}))$ with $\mu : D(V)/(L/\tilde{V}) \to D(V)$ to be an extension of $\delta$ to a continuous section of the projection $D(V) \to D(V)/(L/\tilde{V})$ (see [18 Exer. 5.6.8]). It follows that the number of generators of $L_{2V}$ does not exceed the number of generators of $L/\tilde{V}$ for every $V$ and so the number of generators of $L_2$ does not exceed the number of generators of $L$.

Since $H_2/L_2$ is procyclic, $H_2$ is finitely generated as well.

Finally we observe that by going down to a subgroup of finite index of $H$ if necessary we can assume that $H_2$ is not abelian. Indeed if $H_2$ is abelian since $H_2L = H$ we get that $H/L = (L/\tilde{L}) \times M$ is abelian, so $L/\tilde{L} = \mathbb{Z}_p$. Since $L$ is not solvable (see Corollary 5.5) by Theorem 2.4 there is a free non-abelian pro-$p$ subgroup $F$ of $L$ acting freely on $T$. Since $F = F/\bar{F}$ is the inverse limit of $K/\bar{K}$ where $K$ runs through the open subgroups of $L$ that contain $F$ we get that for some $K$ the free pro-$p$ group $K/\bar{K}$ is not procyclic. Then by substituting $L$ with $K$ if necessary we get the desired property. This completes the proof of the claim.

Since $M$ fixes all edges of $T_e$ for every $e \in E(C)$ we have $S_0 e \subseteq \bar{L} e \subseteq L e$ where $S_0$ is a procyclic subgroup of $H$ that maps surjectively to $M$ by the canonical map $H \to H/\tilde{L}$. Then $H e = (L \times S_0)e = Le$ and so $H_e/L_e \simeq \mathbb{Z}_p$ for every $e \in E(C)$. Since $H_e$ is procyclic $L_e = 1$ for all $e \in E(C)$. But by Lemma 2.3 $H_e = (H_2)e$ for every $e \in E(C)$, so $L \cap H_2$ is of infinite index in $H_2$ i.e. $H_2/(L \cap H_2) \simeq \mathbb{Z}_p$.

By the definition of $C$ the canonical map $T \to T/\tilde{L}$ sends $C$ surjectively to $T_1$. By Lemma 2.3 $T_1 = C/(H_2 \cap \tilde{L})$ and since $T_1$ is a pro-$p$ tree, by Theorem 2.3 $\tilde{L} \cap H_2 = L \cap H_2$, where $L \cap H_2$ is the closed subgroup of $L \cap H_2$ generated by stabilizers of vertices in $C$. Hence replacing $H$ by $H_2$, $L$ by $L_2$ and $T$ by $C$ we may assume that

(6.2) \[T/\tilde{L} = T_1\]
except that $H/\bar{L} = (L/\bar{L}) \times \mathbb{Z}_p$ might not hold. But by Theorem 6.2 it is sufficient to change $H$ to a subgroup of finite index that contains $L$ to repair this property and by Claim 2 this does not affect the fact that $H/\bar{L}$ acts irreducibly on $T_1$. So from now on we can assume that

\[(6.3)\quad H/\bar{L} = (L/\bar{L}) \times \mathbb{Z}_p.\]

Thus we have $L_e = 1$ for every $e \in E(T)$. Then by Theorem 2.6

\[L \cong (\prod_{v \in V_0} L_v) \prod (L/\bar{L}).\]

Since $L$ is finitely generated all $L_v$ are finitely generated.

Fix $v \in V(T)$. Since $M$ fixes all vertices of $T_1$ we have $L_v = H_v$. Since $H_v/L_v \cong \mathbb{Z}_p$, we have $H_v = L_v \times \mathbb{Z}_p$. By the minimality of $n$ we deduce that $H_v$ is abelian for all $v \in V_0$.

This together with the fact that $M$ acts trivially on $T_1$ and (6.3) implies that $H$ and $L$ satisfy the assumptions of Lemma 6.3 and by Lemma 6.3 we deduce that there is some $g \in H \setminus L$ that fixes an edge of $T$ and $g$ commutes with $\bar{L}$.

Let $l$ be an element of $L$ such that the image in $H/\bar{L}$ of the closed subgroup of $H$ generated by $lg$ is $M$.

Since $M$ fixes $T_1$ we have for a fixed vertex $v$ of $T$ that

\[(lg)v = l_v v\]

for some $l_v \in \bar{L}$. Thus $H_v = L_v \times \left< l_v^{-1}g \right>$. Note that by Lemma 6.3 $g$ acts trivially (via conjugation) on $\bar{L}$, hence $g$ acts trivially on $L_v$. On the other hand the fact that $H_v$ is abelian implies that $l_v^{-1}lg$ commutes with $L_v$. Then $l_v^{-1}l$ commutes with $L_v$. Since $\bar{L}$ is a free pro-$p$ product of pro-$p$ groups (one of which is $L_v$) we deduce that $l_v^{-1}l \in L_v$ because every free factor is self-centralized (see Corollary 4.4 (a) in [19]), so $l_v v = lv$. Then

\[g v = l^{-1}l_v v = v,\]

so $g$ fixes every vertex of $T$. But the action of $H$ on $T$ is faithful, a contradiction.

**Case 2.** Suppose that $L = \bar{L}$. Let $U$ be an open normal subgroup of $H$ and $V = U \cap L$. Suppose that $V = \tilde{V}$ for all $U$ (otherwise we can continue as in Case 1 substituting $L$ with $V$ and $H$ with $U$). Then $L/\tilde{V} = L/\bar{V}$ is finite and so stabilizes some vertex of the pro-$p$ tree $T/\tilde{V}$.

Since $\cap \tilde{V} = \cap V = 1$ the inverse limit of $L/\tilde{V}$ over $\tilde{V}$ is $L$. On the other hand this inverse limit stabilizes a vertex of $T$ as every $L/\tilde{V}$ stabilizes some vertex of $T/\tilde{V}$. Thus $L$ is in a conjugate of $G_{n-1}$ or $A$. In the second case $L$ is abelian, hence $H$ is soluble and by Corollary 5.3 $H$ is abelian, a contradiction. In the first case conjugating if necessary we can suppose that $L \subseteq G_{n-1}$. Consider an epimorphism

\[\varphi : G_n = G_{n-1} \Gamma_C A \rightarrow G_{n-1} \Gamma_C \hat{A}\]

with free kernel, $\varphi$ induces the identity map on $G_{n-1}$ and an epimorphism $A \rightarrow \hat{A}$ where $\hat{A}$ is a free abelian pro-$p$ group of rank $d(A) - 1$. Thus $\ker(\varphi) \cap L = 1$. Then

\[\varphi(H) = \varphi(L) \times \mathbb{Z}_p \text{ or } |\varphi(H) : \varphi(L)| < \infty.\]

In the first case $H \cong \varphi(H)$ is a pro-$p$ subgroup of $G_{n-1} \Gamma_C \hat{A}$. Note that the weight of $G_{n-1} \Gamma_C \hat{A}$ is at most $n$ and that


\[ d(G_{n-1} \Pi_C A) < d(G_{n-1} \Pi_C A), \] in contradiction to the minimality of \((n, d(G_n))\).

In the second case \( \ker(\varphi) \cap H = H_0 \) is infinite procyclic, and \( H \subseteq N_{G_n}(H_0) \). By Theorem 5.1 \( N_{G_n}(H_0) = C_{G_n}(H_0) \) is abelian, so \( H \) is abelian, a contradiction. \( \square \)

**Theorem 6.5.** Let \( H \) be a pro-\( p \) group from the class \( \mathcal{L} \) with a non-trivial finitely generated normal pro-\( p \) subgroup \( N \) of infinite index. Then \( H \) is abelian.

**Proof.** Since \( N \) is a pro-\( p \) group from the class \( \mathcal{L} \) by Corollary 4.4 \( N \) is of type \( FP_\infty \). Then by the main result of [28] there is a finite index subgroup \( G_0 \) of \( H \) such that \( G_0 \) contains \( N \) and \( cd(G_0/N) < \infty \). In particular \( G_0/N \) is non-trivial and torsion-free.

Let \( g \in G_0 \setminus N \). The group \( T = N/\langle g \rangle \) is a pro-\( p \) group from the class \( \mathcal{L} \), so by Corollary 4.4 is of type \( FP_\infty \) and \( T/N \simeq \mathbb{Z}_p \). By Lemma 6.4 \( T \) is abelian and since there was no restriction on the choice of \( g \) in \( G_0 \setminus N \) we get that \( N \subseteq Z(G_0) \). By Corollary 5.3 \( G_0 \) is abelian and by Corollary 5.4 \( H \) is abelian. \( \square \)

**Corollary 6.6.** Let \( H \) be a pro-\( p \) group from the class \( \mathcal{L} \) with a free pro-\( p \) subgroup \( F \) of rank 2. Then \( F = N_H(F) \).

**Proof.** Suppose that \( F \neq N_H(F) \). Let \( g \in N_H(F) \setminus F \) and define \( G \) as the pro-\( p \) subgroup of \( H \) generated by \( F \) and \( g \). If \( G/F \) is infinite then \( G \) is a non-abelian pro-\( p \) group from the class \( \mathcal{L} \) with a finitely generated normal pro-\( p \) subgroup of infinite index, in contradiction to Theorem 6.5. Thus \( F \) is open in \( G \) and by Serre's result \( G \) is a free pro-\( p \) group. Since \( F \neq G \) the Schreier formula yields \( 2 = d(F) > d(G) \), so \( G \) is procyclic and \( F \) is procyclic, a contradiction. \( \square \)

**Theorem 7.** Let \( G \) be a pro-\( p \) group from the class \( \mathcal{L} \) and \( H \) a non-trivial finitely generated subgroup of \( G \). Then \( [N_G(H):H] \) is finite unless \( N_G(H) \) is abelian.

**Proof.** Suppose on the contrary that \( N_G(H) \) is non-abelian and \( [N_G(H):H] \) is infinite. Put \( N = N_G(H) \) and suppose that \( N/H \) is not torsion. Let \( t \in N \setminus H \). Suppose first that \( t \) has infinite order modulo \( H \) (note that we have supposed that such \( t \) exists). Then the closed subgroup of \( N \) generated by \( H \) and \( t \) has a normal closed finitely generated subgroup \( H \) and so by Lemma 6.4 \( H \) is abelian. In particular \( H \) is abelian. If \( t \) has finite order modulo \( H \) then by Corollary 6.4 the closed subgroup of \( N \) generated by \( H \) and \( t \) is abelian, so \([t,H] = 1\). Since \( H \) is non-trivial, by transitive commutativity any two elements of \( N \) commute, so \( N \) is abelian, a contradiction.

Thus we can suppose from now on that \( N/H \) is an infinite torsion pro-\( p \) group. Since \( H \) is a pro-\( p \) group from the class \( \mathcal{L} \) by Corollary 4.4 \( H \) is of type \( FP_\infty \). By Lemma 5.1 every group from the class \( \mathcal{L} \) has finite cohomological dimension and hence every subgroup of a group from the class \( \mathcal{L} \) has finite cohomological dimension. In particular \( cd(N) < \infty \). Then by the main result of [28] there is a finite index subgroup \( G_0 \) of \( N \) such that \( G_0 \) contains \( H \) and \( cd(G_0/H) < \infty \). In particular \( G_0/H \) is torsion-free and hence \( G_0 = H \), a contradiction to the fact that \( N/H \) is infinite. This completes the proof. \( \square \)

7. 2-GENERATED PRO-\( p \) GROUPS IN THE CLASS \( \mathcal{L} \)

**Lemma 7.1.** Let \( Q \) be a non-trivial free abelian pro-\( p \) group of finite rank and \( Y \) be the set of all epimorphisms of pro-\( p \) groups \( \varphi : Q \to Q_1 = \mathbb{Z}_p \). Let \( \hat{\varphi} \) be the continuous ring homomorphism \( \mathbb{F}_p[[Q]] \to \mathbb{F}_p[[Q_1]] \) induced by \( \varphi \). Then \( \cap_{\varphi \in Y} \ker(\hat{\varphi}) = 0 \).
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Proof. We prove by induction on \(n\) the following more general statement. Let \(R\) be a pro-\(p\) ring of characteristic \(p\), \(R[[t_1, \ldots, t_n]]\) be the commutative ring of formal power series and \(Q_{R,n}\) be the closed group generated by \(\{1+t_i\}_{1 \leq i \leq n}\). Consider the set \(W\) of all homomorphism of pro-\(p\) rings \(\hat{\varphi} : R[[t_1, \ldots, t_n]] \to R[[t_1]]\) that induce an epimorphism of pro-\(p\) groups \(\varphi : Q_{R,n} \to Q_{R,1}\). We claim that

\[
\cap_{\hat{\varphi} \in W} \ker \hat{\varphi} = 0.
\]

By writing \(R[[t_1, \ldots, t_n]] = R[[t_n]][[t_1, \ldots, t_{n-1}]]\) we see that to prove the inductive step is sufficient to consider the case \(n = 2\).

Assume from now that \(n = 2\) and define \(\theta_i : R[[t_1, t_2]] \to R[[t_1]]\) by \(\theta_i(t_1) = t_1^i\) and \(\theta_i(t_2) = t_1^i t_2^j\). Thus \(\theta_i(1+t_2) = (1+t_1)^{i+j}\) and \(\theta_i\) sends \(Q_{R,2}\) surjectively to \(Q_{R,1}\).

We prove by induction on \(j\) the following claim: for \(\ker(\theta_i) = (t_2 - t_1^i)R[[t_1, t_2]]\)

\[
\cap_{1 \leq i \leq j} \ker(\theta_i) = \prod_{1 \leq i \leq j} \ker(\theta_i)
\]

Indeed if \(f\) is in the left hand side of (7.1), then by induction \(f = (t_2 - t_1^i) f_0\) for some \(f_0 \in R[[t_1, t_2]]\). Since \(\theta_j(f) = 0\), \(\theta_j(t_2 - t_1^i) \neq 0\) for \(1 \leq i \leq j - 1\) and \(R[[t_1]]\) is a domain we deduce that \(f_0 \in \ker(\theta_j) = (t_2 - t_1^i)R[[t_1, t_2]]\). This completes the proof of the claim.

Finally let \(I\) be the augmentation ideal of \(R[[t_1, t_2]]\). By the claim \(\cap_{1 \leq i \leq j} \ker(\theta_i) \subseteq I^j\) and \(\cap_{1 \geq i} \ker(\theta_i) = 0.\)

\[\Box\]

Lemma 7.2. Let \(1 \to F \to G \to Q \to 1\) be a short exact sequence of pro-\(p\) groups with \(Q\) non-trivial pro-\(p\) abelian torsion-free, \(G\) finitely generated and \(F\) free pro-\(p\), \(1\)-generated as a closed normal subgroup of \(G\). Then either \(G\) has a pro-\(p\) subgroup \(N\) such that \(G/N \simeq \mathbb{Z}_p\) and \(N\) is finitely generated or \(F/F'F^p \simeq \mathbb{F}_p[[Q]]\) and so there is a free pro-\(p\) subgroup \(F_0\) of \(G\) such that \(F \subset F_0\), \(F_0/F \simeq \mathbb{Z}_p\) and \(G/F_0\) is torsion-free.

Proof. Since \(F\) is generated by one element as a closed normal subgroup of \(G\) we have that \(V = F/F'F^p\) is a cyclic pro-\(p\) \(\mathbb{F}_p[[Q]]\)-module, where \(Q\) acts by conjugation.

Suppose first that \(V\) is not free as a pro-\(p\) \(\mathbb{F}_p[[Q]]\)-module. Then \(V \simeq \mathbb{F}_p[[Q]]/I\) where \(I\) is some closed ideal in \(\mathbb{F}_p[[Q]]\). Let \(\varphi : Q \to Q_1 = \mathbb{Z}_p\) be a surjective homomorphism of pro-\(p\) groups and \(\hat{\varphi}\) be the continuous ring homomorphism \(\mathbb{F}_p[[Q]] \to \mathbb{F}_p[[Q_1]]\) induced by \(\varphi\). By Lemma 7.1 \(\cap \ker(\hat{\varphi}) = 0\) when \(\varphi\) runs through all such epimorphisms. In particular there is \(\varphi\) such that \(\hat{\varphi}(I) \neq 0\). Then \(\mathbb{V} \mathbb{V}_{\mathbb{F}_p[[Q]]} \simeq \mathbb{F}_p[[Q]]/I\mathbb{V}_{\mathbb{F}_p[[Q]]} \simeq \mathbb{F}_p[[Q]]/\hat{\varphi}(I)\) is finite and by Nakayama’s lemma \(V\) is finitely generated as a pro-\(p\) \(\mathbb{F}_p[[Q]]\)-module. Then the preimage \(N\) of \(\ker(\varphi)\) in \(G\) is finitely generated as a pro-\(p\) group and \(G/N \simeq \mathbb{Z}_p\).

Suppose now that \(V \simeq \mathbb{F}_p[[Q]]\). Take any element \(q \in Q\) such that the pro-\(p\) subgroup of \(Q\) generated by \(q\) is a direct factor in \(Q\). Let \(g\) be an element of the preimage of \(q\) in \(G\). Then the pro-\(p\) subgroup \(F_0\) of \(G\) generated by \(F\) and \(g\) is free pro-\(p\). Indeed \(V \simeq \mathbb{F}_p[[Q]]\) is also a free \(\mathbb{F}_p[[Q]]\)-module with the basis \(Z\), where \(Z \times (g) = Q\). Lifting \(Z\) to a closed subset \(\bar{Z}\) of \(F\) and translating (i.e. conjugating) it by \((g)\) we obtain a closed \((g)\)-invariant basis \(X\) of \(F\) as a free pro-\(p\) group where the image of \(X\) in \(V\) is \(\{Q\}\). Then \(F_0\) is a free pro-\(p\) group with a basis \(Z \cup \{g\}\).
Theorem 7.3. Every 2-generated pro-$p$ group from the class $\mathcal{L}$ is either free pro-$p$ or abelian.

Proof. Let $H$ be a 2-generated pro-$p$ subgroup of $G_n \in \mathcal{G}_n$ such that $H$ is not procyclic. We prove by induction on the minimal number of generators of $G_n$ that $H$ is either free pro-$p$ or abelian. The induction starts since the minimal number of generators for $G_n$, when $H$ exists, is 2 and in this case $G_n$ is either free pro-$p$ or abelian.

The case $n = 0$ is obvious since free pro-$p$ groups have only free pro-$p$ subgroups. Thus we can assume that $n \geq 1$. Let $G_n = G_{n-1} \ast C$, where $C = C \times B \simeq \mathbb{Z}_p^\infty$ and $B \simeq \mathbb{Z}_p$. Let $D \simeq \mathbb{Z}_p$ be a direct summand of $B$ and $M = \langle \bigcup_{g \in G_n} D \rangle$. Thus $M$ is the kernel of the epimorphism of pro-$p$ groups $\varphi : G_n = G_{n-1} \ast C \to G_{n-1} \ast (A/D) = L$ that is the identity map on $G_{n-1}$ and the canonical projection $A \to A/D$ on $A$. By Theorem 2.7 $M$ is free pro-$p$. Note that $L$ is a pro-$p$ group from the class $\mathcal{L}$ with less generators than $G_n$. By induction $\varphi(H)$ is either free pro-$p$ or abelian. Note that by Lemma 4.3 $\varphi(H)$ is torsion-free.

1. If $\varphi(H)$ is free pro-$p$ of rank 2, since $H$ is 2-generated $H \simeq \varphi(H)$ and we are done.

2. If $\varphi(H) \simeq \mathbb{Z}_p$ we get that $H$ is an extension of the free pro-$p$ group $M_0 = M \cap H$ by $\mathbb{Z}_p$. We claim that every 2-generated pro-$p$ group from the class $\mathcal{L}$ that is free-by-$\mathbb{Z}_p$ is either free pro-$p$ or abelian. Since $H$ is 2-generated $M_0$ is generated as a normal closed subgroup of $H$ by just one element. Then for $V = M_0/[M_0^p,M_0]$ we have that $V$ is a cyclic $\mathbb{F}_p[[Q]]$-module via conjugation, where $Q = H/M_0 \simeq \mathbb{Z}_p$. Suppose that $H$ is not abelian. By Theorem 6.3 $M_0$ is not finitely generated as a pro-$p$ group, so $V$ is infinite and hence $V \simeq \mathbb{F}_p[[Q]]$. This implies together with Lemma 7.2 that $H$ is a free pro-$p$ group.

3. If $\varphi(H) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ we have that $H$ is an extension of the free pro-$p$ group $M_0 = M \cap H$ by $\mathbb{Z}_p \times \mathbb{Z}_p$. Since $H$ is 2-generated, $M_0$ is generated as a normal closed subgroup of $H$ by just one element, namely the commutator of two generators of $H$. Then by Lemma 7.2 either there is a normal finitely generated pro-$p$ subgroup of infinite index in $H$, a contradiction with Theorem 5.3 or there is a normal free pro-$p$ subgroup $F_0$ of $H$ such that $F_0/M_0 \simeq \mathbb{Z}_p$. In the last case $H/F_0 \simeq \mathbb{Z}_p$ and we can argue exactly as in the case 2.

4. If $\varphi(H)$ is the trivial group then $H$ is a pro-$p$ subgroup of the free pro-$p$ group $M$, so is free pro-$p$.

\[\square\]

8. Euler characteristic

By definition a pro-$p$ group $G$ from the class $\mathcal{L}$ is a closed subgroup of some $G_n \in \mathcal{G}_n$, hence $cd(G) \leq cd(G_n) < \infty$. Since $G$ is of type $FP_\infty$, see Corollary 4.4 it has a well-defined Euler characteristic $\chi(G) = \sum_{0 \leq i \leq cd(G)} (-1)^i \dim_{\mathbb{F}_p} H_i(G,F_p)$.

Theorem 8.1. Every pro-$p$ group $G$ from the class $\mathcal{L}$ has Euler characteristic $\chi(G) \leq 0$.

Proof. Let $N$ be a closed normal subgroup of $G$ such that $Q = G/N$ is torsion-free nilpotent and $N$ is free pro-$p$ (see Proposition 4.3). Let

$$P : 0 \to P_m \to P_{m-1} \to \ldots \to P_1 \to P_0 \to \mathbb{Z}_p \to 0$$
be a free resolution of the trivial right \( \mathbb{Z}_p[[G]] \)-module \( \mathbb{Z}_p \) with \( P_i \) finitely generated, say of rank \( \alpha_i \), for all \( i \). Note that

\[
H_i(\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p) = H_i(N, \mathbb{F}_p) \quad \text{for} \quad i \geq 1.
\]

Since \( N \) is a free pro-\( p \) group

\[
(8.1) \quad H_i(\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p) = 0 \quad \text{for} \quad i \neq 1
\]

and

\[
H_1(\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p) = N/[N, N]\mathbb{F}_p = V.
\]

Since \( Q \) is a torsion-free nilpotent pro-\( p \) group and hence of finite rank, \( \mathbb{F}_p[[Q]] \) is a right and left Noetherian ring without zero divisors \([2, \text{Cor. 7.25}]\). Then \( \mathbb{F}_p[[Q]] \) is an Ore ring and has a classical ring of quotients, denoted by \( K \). Note that the abstract tensor product \( \otimes_{\mathbb{F}_p[[Q]]} \) is an exact functor and

\[
V \otimes_{\mathbb{F}_p[[Q]]} K \simeq H_1(\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p) \otimes_{\mathbb{F}_p[[Q]]} K \simeq H_1((\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p) \otimes_{\mathbb{F}_p[[Q]]} K) \simeq K^\mathbb{Z}_{\oplus}
\]

for some non-negative integer \( z \). Then using that \((P_i \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p) \otimes_{\mathbb{F}_p[[Q]]} K \simeq K^{\alpha_i}\) and \((8.1)\) we get

\[
\chi(G) = \sum_i (-1)^i \alpha_i = \sum_i (-1)^i \dim_K H_i((\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p) \otimes_{\mathbb{F}_p[[Q]]} K).
\]

Note that since \(- \otimes_{\mathbb{F}_p[[Q]]} K\) is an exact functor it commutes with homology, hence

\[
H_1((\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p) \otimes_{\mathbb{F}_p[[Q]]} K) \simeq (H_1(\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p)) \otimes_{\mathbb{F}_p[[Q]]} K.
\]

Then

\[
\chi(G) = \sum_i (-1)^i \dim_K ((H_1(\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p)) \otimes_{\mathbb{F}_p[[Q]]} K) = -\dim_K (H_1(\mathcal{P} \widehat{\otimes} \mathbb{Z}_p[[N]] \mathbb{F}_p) \otimes_{\mathbb{F}_p[[Q]]} K) = -z \leq 0.
\]

\[\square\]

**Theorem 8.2.** Let \( G \) be a pro-\( p \) group from the class \( \mathcal{L} \) that is free-by-abelian and such that its Euler characteristic \( \chi(G) = 0 \). Then \( G \) is abelian.

**Proof.** Let \( F \) be a closed normal subgroup of \( G \) such that \( F \) is free pro-\( p \) and \( Q = G/F \) is abelian. By going down to a subgroup of finite index if necessary we can assume that \( G/F \) is torsion-free. By the proof of Corollary 8.1 \( \chi(G) = -z \)

where \( \dim_K (V \otimes_{\mathbb{F}_p[[Q]]} K) = z \), \( V = H_1(F, \mathbb{F}_p) \) and \( K \) is the abstract field of fractions of \( \mathbb{F}_p[[Q]] \). Since \( \chi(G) = 0 \) we have that \( z = 0 \) and \( V \otimes_{\mathbb{F}_p[[Q]]} K = 0 \). So the annihilator \( \text{ann}_{\mathbb{F}_p[[Q]]}(V) = I \) is non-zero. By the proof of Lemma 7.2 there is an epimorphism \( \varphi : Q \to \mathbb{Z}_p \) such that \( \mathbb{F}_p[[Q]]/I \) is finitely generated as a pro-\( p \) \( \mathbb{F}_p[[\text{Ker}(\varphi)]] \)-module. Then \( V \) is finitely generated as a pro-\( p \) \( \mathbb{F}_p[[\text{Ker}(\varphi)]] \)-module.

Let \( M \) be the preimage of \( \text{Ker}(\varphi) \) in \( G \). Then \( M \) is a normal closed subgroup of \( G \), \( M \) is finitely generated and \( G/M \simeq \mathbb{Z}_p \). By Lemma 6.3 \( G \) is abelian. \[\square\]
9. Open questions

Here we list possible properties of the pro-$p$ groups $G$ from the class $\mathcal{L}$.

1. $\text{def}(G) \geq 2$ if every abelian pro-$p$ subgroup of $G$ is procyclic and $G$ itself is not procyclic, where $\text{def}(G) = \dim_{F_p} H^1(G, F_p) - \dim_{F_p} H^2(G, F_p)$ denotes the deficiency of $G$.

2. Every finitely generated subgroup of a pro-$p$ group from the class $\mathcal{L}$ is a virtual retract.

3. The Euler characteristic $\chi(G) = 0$ if and only if $G$ is abelian. In the particular case when $G$ is free-by-abelian this holds by Theorem 8.2.

4. The Howson property: the intersection of any two finitely generated pro-$p$ subgroups of $G$ is finitely generated.

5. $G$ is residually free pro-$p$.

6. $G$ is residually torsion-free pro-$p$ nilpotent.

Note that 5 implies 6. For abstract limit groups a stronger version of 5 holds as abstract limit groups are fully residually free. In the abstract case property 4 was proved in 6, 3 in 12, 2 in 29. Property 1 follows by induction on the height $h$ of an abstract limit group: one uses the fact that a freely indecomposable abstract limit group of height $h \geq 1$ is the fundamental group of a finite graph of groups with infinite cyclic edge groups such that at least one vertex group is a non-abelian limit group of height $\leq h - 1$ [1, Lemma 1.4]. The definition of height of an abstract limit group can be found in [1, Section 1] and differs from our notion of weight by allowing almost all surface groups as height zero limit groups.

Acknowledgements

The authors thank the referee for the many helpful remarks that improved the paper.

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