On the number of star-shaped classes in optimal colorings of Kneser graphs

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Abstract
A family of sets is called star-shaped if all the members of the family have a point in common. The main aim of this paper is to provide a negative answer to the following question raised by Aisenberg et al., for the case \( k = 2 \). Do there exist \( (n - 2k + 2) \)-colorings of the \( (n, k) \)-Kneser graphs with more than \( k - 1 \) many non-star-shaped color classes?

Keywords
chromatic number, Kneser graph, line graph

1 | INTRODUCTION

In this paper all graphs are finite, undirected, and simple. A (proper) coloring of a graph is an assignment of colors to its vertices such that no two adjacent vertices have the same color. The chromatic number \( \chi(H) \) of a graph \( H \) is the smallest number of colors in a coloring of \( H \). For a given coloring of a graph \( H \), a subgraph of \( H \) is called colorful if all its vertices receive different colors. The \( (n, k) \)-Kneser graph \( KG(n, k) \) is a graph whose vertices correspond to all \( k \)-subsets of the set \( [n] = \{1, 2, \ldots, n\} \), and where two vertices are adjacent if and only if the two corresponding sets are disjoint. Finally, a family \( \mathcal{F} \) of sets is called star-shaped if \( \bigcap \mathcal{F} \neq \emptyset \).

Almost 70 years ago, a purely combinatorial conjecture was raised by Kneser which in the language of graph theory is equivalent to say \( \chi(KG(n, k)) = n - 2k + 2 \) for \( n \geq 2k - 1 \). About two decades later, Lovász [13] proved Kneser’s conjecture by taking an extraordinary approach, using tools from algebraic topology. Since then, a lot of attention has been drawn to study various problems related to this conjecture, including a large number of new proofs, such as [5, 10, 12, 14], and many generalizations, such as [2, 4, 6, 7, 9, 11, 15]. In particular, Aisenberg et al. [1] gave a new proof of this conjecture using a simple counting argument based on the Hilton–Milner theorem, for all but finitely many cases.\(^1\) Indeed, their main idea was based on the fact that if

\(^1\)Here, we assume that \( k \) is a fixed number.
there was an \((n - 2k + 1)\)-coloring of \(KG(n, k)\), then it would contain a star-shaped color class \([1, \text{Lemma 8}]\) provided that \(n\) is large enough. Then removing this vertex would lead to a \(((n - 1) - 2k + 1)\)-coloring of \(KG(n - 1, k)\). So, to verify the conjecture one just needs to check the validity of the conjecture for finitely many base cases which can be checked, for instance, using the topological arguments provided by Lovász for these cases. Actually, they showed if there was such a coloring then it would contain many star-shaped color classes \([1, \text{Lemma 9}]\), again provided \(n\) is large enough. Motivated by this result, they showed that there is an optimal coloring (a coloring with \((n - 2k + 2)\) colors) of \(KG(n, k)\) with \((k - 1)\) non-star-shaped color classes provided that \(n \geq 3k - 3\) and then they raised the following question.

**Question 1** (Aisenberg et al. [1]). Do there exist \((n - 2k + 2)\)-colorings of the \((n, k)\)-Kneser graphs with more than \(k - 1\) many non-star-shaped color classes?

The main objective of this paper is to give a negative answer to the above question for the case \(k = 2\). As a corollary, this result leads to a purely combinatorial proof for the colorful \(K_{l,m}\)-theorem [16, Theorem 2] for the case \(KG(n, 2)\). To state it precisely, some definitions are required. For a given graph \(G\) and a proper coloring of \(G\), a subgraph \(H\) of \(G\) is called colorful if its vertices receive different colors. A graph \(G\) is said to have the colorful \(K_{l,m}\)-property if for any optimal coloring of \(G\) and any bipartition \(\{A, B\}\) of the color set there exists a colorful complete bipartite subgraph \(K_{l,m}\) of \(G\), where \(l = |A|\) and \(m = |B|\), such that the colors in \(A\) are on one side, and the colors in \(B\) are on the other side. The colorful \(K_{l,m}\)-theorem provides a sufficient “topological condition”4 for a graph to satisfy the colorful \(K_{l,m}\)-property. We can probably say that Kneser graphs are the main family of graphs that are known to pass this condition, and consequently have the colorful \(K_{l,m}\)-property. In this paper, we present a completely combinatorial proof that \(KG(n, 2)\) fulfills the colorful \(K_{l,m}\)-property. A similar result for the existence of a colorful complete tripartite subgraph in \(KG(n, 2)\) will also be provided.

## 2 | MAIN RESULTS

First note that we can view \(KG(n, 2)\) as the complement of the line graph of the complete graph \(K_n\) and therefore proper colorings of \(KG(n, 2)\) are exactly partitions of the edge set of the complete graphs \(K_n\) into stars and triangles. More generally, as mentioned here [8], if \(G\) is the complement of the line graph of a graph \(H\), then proper colorings of \(G\) are exactly partitions of the edge set of \(H\) into stars and triangles. A star is a tree with a vertex, the center of the star, connected to all other vertices. A single-edge tree is in particular a star. In this particular case, we will always assume that exactly one of the two vertices has been identified as the center, so as to be in a position of always speak of the center of a star without ambiguity. A triangle is a circuit of length 3. We call such a partition of \(H\) into stars and triangles an ST-partition.

**Lemma 1.** Consider an optimal ST-partition \(\mathcal{P}\) of a graph \(H\). Then, at most one vertex of each triangle in \(\mathcal{P}\) can be the center of a star.

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2For instance, \(n \geq k^4\) works.

3If \(k\) is small enough (\(k = 2\) or 3), then the base cases can be checked by hand or a computer. However, for \(k \geq 4\) we do not know whether there exists a “purely combinatorial way” to establish the conjecture.

4See [16, Theorem 2] or [3, Theorem 4] for more details.
Proof. We proceed by a contradiction. Suppose that $T$ is a triangle in $\mathcal{P}$ with at least two vertices $x, y$ such that each of them is also the center of a star, name it $S_x$, and $S_y$, respectively. Then we can add the two edges of $T$ incident to $x$ to $S_x$, and the remaining edge of $T$ to $S_y$, and finally remove $T$ from $S$. This leads to a new ST-partition with fewer elements which contradicts the optimality of $\mathcal{P}$. □

Lemma 2. Consider an optimal ST-partition $\mathcal{P}$ of a graph $H$. If there is a triangle $T$ in $\mathcal{P}$, then each vertex $x$ of $H$ which does not belong to any triangle of $\mathcal{P}$ and it is connected to at least two vertices of $T$ is the center of a star.

Proof. We will use a proof by contradiction. Suppose $x$ is not the center of any star. If $a, b$ are two vertices of $T$ which are connected to $x$, then each of the edges $\{a, x\}$ and $\{b, x\}$ must belong to a star or a triangle in $\mathcal{P}$. This implies that $a$ and $b$ must both be the center of stars, as no triangle in $\mathcal{P}$ having $x$ as its vertices, which contradicts the previous lemma. □

Theorem 1. There is no optimal ST-partition of the complete graph $K_n$ with more than one triangle. Moreover, if $n \geq 3$, any optimal ST-partition of $K_n$ contains exactly one triangle.

Proof. Suppose the contrary and choose the minimum integer $n$ with the property that there is an optimal ST-partition $\mathcal{P}$ of the complete graph $K_n$ with more than one triangle. Let $T_1, \ldots, T_k$ be the list of all the triangles in $\mathcal{P}$ where $k \geq 2$. Let $A = \bigcup_{i=1}^{k} V(T_i)$ be the set of the vertices of these triangles, $B \subseteq [n] \setminus A$ be the set of the rest vertices which are centers of stars, and finally put $C = [n] \setminus (A \cup B)$. By Lemma 2, we have $C = \emptyset$. Moreover, $B$ also must be empty. Indeed, suppose $B \neq \emptyset$. Without loss of generality assume $n \in B$. Let $S_n = \{e \in E(K_n) : n \in e\}$ and define $\mathcal{P}' = \{P \setminus S_n : P \in \mathcal{P}\}$. Note that $\mathcal{P}'$ is an ST-partition of $K_{n-1}$ with the same number of triangles as $n$ does not belong to $A$. In addition, $\mathcal{P}'$ is an optimal ST-partition. Indeed, from one side we have $|\mathcal{P}'| \leq |\mathcal{P}| - 1 = n - 3$ as $n \in B$. On the other hand, we must have $|\mathcal{P}'| \geq n - 3$ since otherwise $\mathcal{P}'$ would induce a proper coloring of $KG(n-1, 2)$ with less than $n - 3$ colors which contradicts $\chi(KG(n-1, 2)) = n - 3$. Hence, $\mathcal{P}'$ is an optimal ST-partition of $K_{n-1}$ with at least two triangles. But, this contradicts the minimality of $n$, and therefore we must have $B = \emptyset$. In summary, we have shown that any vertex of $K_n$ belongs to at least one of these triangles $T_1, \ldots, T_k$. Now, we consider two cases:

Case I. First, we consider the case that there are exactly two triangles in $\mathcal{P}$, say $T_1$ and $T_2$. We have two possibilities.

(1) $n = 6$. In other words $T_1$, and $T_2$ are vertex disjoint:
Note that in this case we have $n = 6$ and therefore $|\mathcal{P}| = 4$ (since $\mathcal{P}$ is an optimal ST-partition of $\mathcal{K}_6$). But on the other hand, we have

$$|\mathcal{P}| \geq 2 + 3 = 5$$

as $T_1, T_2 \in \mathcal{P}$ and at least three vertices of $T_1 \cup T_2$ are centers. To see the latter claim, consider the edges $e_1, e_2, e_3$ of $\mathcal{K}_6$ which are shown below.

![Diagram of edges e1, e2, e3](image)

Each of these edges must belong to a triangle or a start in $\mathcal{P}$. But none of them can belong to a triangle as there are no other triangles in $\mathcal{P}$ except $T_1$ and $T_2$. So, each of them must belong to a star, and hence one of the endpoints of each of $e_1, e_2$, and $e_3$ must be the center of a star. So, there are at least three stars in $\mathcal{P}$ as these edges are pairwise disjoint. Hence $\mathcal{P}$ must have at least five elements which contradicts the optimality of $\mathcal{P}$.

(2) $n = 5$. In other words, if $T_1$ and $T_2$ have a vertex in common

![Diagram of edges e1 and e2](image)

then we have

$$|\mathcal{P}| \geq 2 + 2 = 4,$$

as $T_1, T_2 \in \mathcal{P}$, and at least two vertices of $T_1 \cup T_2$ are centers. Again to see why the latter claim is true, it is enough to consider the edges $e_1$ and $e_2$ which are depicted below.

![Diagram of edges e1 and e2](image)
and repeat the same argument. So, the size of \( \mathcal{P} \) is at least four. But, this is again impossible as \( \mathcal{P} \) is an optimal ST-partition of \( K_5 \) which implies \( |\mathcal{P}| = 3 \).

**Case II.** Finally, suppose there are at least three triangles in \( \mathcal{P} \). Then, this implies that no vertex of a triangle is a center of a star. Indeed, if there was a triangle \( T \) in \( \mathcal{P} \) which one of its vertices was a center of a star, then all the other triangles in \( \mathcal{P} \) except at most one of them are attached to this triangle at \( x \) (otherwise removing this vertex would lead to an optimal ST-partition for \( K_{n-1} \) with at least two triangles which would contradict the minimality of \( n \)) as shown below.

![Diagram](image)

Now, consider the edges \( e_1 \) and \( e_2 \) of \( K_n \) as shown below.

![Diagram](image)

At most one of these edges can be covered with that only one possible triangle \( \mathcal{P} \) which is not attached to \( x \). So, at least one of the endpoints of \( e_1 \) or \( e_2 \) must be the center of a star. But then, we would have a triangle with two of its vertices (\( x \) and of the endpoints of \( e_1 \) or \( x \) and of the endpoints of \( e_2 \)) being the center of stars which would contradict Lemma 1. So, all members of \( \mathcal{P} \) are triangles. Thus, we must have

\[
3(n - 2) = \binom{n}{2},
\]

because \( |\mathcal{P}| = n - 2 \) (as it is optimal) and in an ST-partition no edge can belong to two different triangles (or to a triangle and a star). This equation just has two solutions \( n = 3 \) or \( 4 \), but on the other hand having at least three triangles in \( \mathcal{P} \) implies \( n \geq 6 \) which is
again a contradiction. So, we have shown that any optimal ST-partition of $\mathcal{K}_n$ contains at most one triangle. Moreover, we show that every optimal ST-partition of $\mathcal{K}_n$ contains exactly one triangle provided that $n \geq 3$. Indeed, the size of any optimal ST-partition $\mathcal{P}$ of $\mathcal{K}_n$ without a triangle is exactly $n - 1$ as at least one of the endpoints of each edge must be the center of a star. Thus, any optimal ST-partition $\mathcal{P}$ of $\mathcal{K}_n$ where $n \geq 3$ has exactly one triangle (as $|\mathcal{P}| = n - 2$ for $n \geq 3$).

\textbf{Corollary 1.} For each $n \geq 3$, $KG(n, 2)$ has the colorful $K_{l,m}$-property.

\textit{Proof:} The assertion is obviously true for $n = 3$. So, without loss of generality we can assume $n \geq 4$. Let $c : V(KG(n, 2)) \to \{1, ..., n - 2\}$ be a proper coloring of $KG(n, 2)$, and $\{X_1, X_2\}$ be a bipartition of the color set $\{1, ..., n - 2\}$. As we saw before, this coloring induces an optimal ST-partition $\mathcal{P}$ of $\mathcal{K}_n$. By Theorem 1, $\mathcal{P}$ contains exactly one triangle, say $T$ with vertex set $\{1, 2, 3\}$. Moreover, no vertex of this triangle is a center of a star. Since otherwise, we could delete this vertex and find a minimal ST-partition for $\mathcal{K}_{n-1}$ without any triangle. Then, this would imply that $n - 1 \leq 2$, which contradicts our assumption. Hence, in particular, for any fixed $1 \leq j \leq 3$ each edge of $\mathcal{K}_n$ whose one of the endpoints is $j$ and the other one is in $[n]\setminus\{3\}$ belongs to a different class in $\mathcal{P}$. In other words, for each $1 \leq j \leq 3$, $c([i, j]) \neq c([k, j])$, where $4 \leq i < k \leq n$. Thus, for each fixed $1 \leq j \leq 3$, the set 

$$\{c([i, j]) : i \in [n]\setminus\{j\}\}$$

is equal to the whole coloring set $[n - 2]$. Note that $c(\{2, 3\})$ either belongs to $X_1$ or $X_2$ as $\{X_1, X_2\}$ is a bipartition of the color set. Without loss of generality assume $c(\{2, 3\}) \in X_2$. Now, $\mathcal{K}_{A_1,A_2}$ is the desired subgraph in $KG(n, 2)$, where

$$A_1 = \{\{1, i\} : c(\{1, i\}) \in X_1\},$$

$$A_2 = \{\{2, j\} : c(\{2, j\}) \in X_2 \text{ and } \{1, j\} \notin A_1\} \cup \{\{2, 3\}\}.$$ 

\qed

It is natural to wonder whether one can obtain a similar result about the existence of a colorful complete multipartite subgraph in $KG(n, 2)$ with more than two parts. Let us first have a look at the following remark to find the right formulation of such a result.

\textbf{Remark 1.} To have a complete $t$-partite subgraph in $KG(n, 2)$ which all of its parts are nonempty, we obviously need $n \geq 2t$. Indeed, according to Corollary 1, this condition is even sufficient for $KG(n, 2)$ to satisfy the colorful $K_{l,m}$-property when $t = 2$. However, for $t \geq 3$, in general this condition is insufficient to guarantee even the existence of any colorful complete $t$-partite subgraph in $KG(n, 2)$ of size $n - 2$. Indeed, given an arbitrary optimal proper coloring of $KG(n, 2)$, we can only hope to find a colorful complete $t$-partite subgraph $K_{X_1,...,X_t}$ in $KG(n, 2)$ where $|X_1| + \cdots + |X_t| \leq n - t$. In particular, this implies that, in general, the set $\{X_1,...,X_t\}$
cannot be a partition for the set of colors. To see this, consider the following optimal proper coloring of $KG(n, 2)$:

$$c([i < j]) = \begin{cases} 
3 & \text{if } j \leq 3, \\
 j & \text{if } i \leq 3 \text{ and } j \geq 4, \\
i & \text{if } i \geq 4.
\end{cases}$$

We can see this coloring as a coloring of edges of the complete graph $K_n$. With this interpretation in mind, one can check that there is no colorful cycle in $K_n$, that is, a cycle whose all edges receive different colors. Now, if there is a colorful complete $t$-partite subgraph $\mathcal{K}_{X_1, ..., X_t}$ in $KG(n, 2)$, then, as none of $X_i$ forms a cycle in $K_n$, we must have

$$|\bigcup X_i| \geq |X_i| + 1 \quad \forall \ i = 1, ..., t.$$ 

Now, since the sets $\bigcup X_1, ..., \bigcup X_t$ are pairwise disjoint, we must have $(|X_1| + 1) + \cdots + (|X_t| + 1) \leq n$ which implies $X_1| + \cdots + |X_t| \leq n - t$ which verifies our claim. The situation is even worse, if we want to determine the place of colors. Actually, it is easy to check that, under this coloring, there is no colorful complete 3-partite $\mathcal{K}_{X_1,X_2,X_3}$ subgraph in $KG(n, 2)$ with $c(X_1) = \{3, 4\}$, $c(X_2) = n - 1$, and $c(X_3) = \{n\}$. However, if we do not allow that none of the parts contains the color of that triangle class, then we have the following result for the existence of a colorful complete tripartite subgraph in $KG(n, 2)$.

**Corollary 2.** Let $n \geq 6$ be a natural number and $c : V(KG(n, 2)) \to [n - 2]$ be an optimal proper coloring of $KG(n, 2)$. Then, for every partition $\{X_1, X_2, X_3\}$ of the set $[n - 2] \setminus \{c(T)\}$ there is a colorful complete tripartite $\mathcal{K}_{A_1,A_2,A_3}$ subgraph in $KG(n, 2)$ such that $c(A_i) = X_i$, where $T$ is the non-star-shaped class (or equivalently, the triangle in the induced ST-partition of $\mathcal{K}_n$) and $c(T)$ is the color of this class.

**Proof.** Let $\mathcal{P}$ be the optimal ST-partition which is induced by $c$. Again, by Theorem 1, $\mathcal{P}$ contains exactly one triangle, say $T$ with vertex set $\{1, 2, 3\}$ and also as mentioned before no vertex of this triangle can be a center of a star. Now, the proof of Corollary 1 reveals that $\mathcal{K}_{A_1,A_2,A_3}$ is the desired subgraph in $KG(n, 2)$ where

$$A_1 = \{\{1, i\} : c(\{1, i\}) \in X_i\},$$

$$A_2 = \{\{2, i\} : c(\{2, i\}) \in X_2 \text{ and } \{1, i\} \notin A_1\},$$

$$A_3 = \{\{3, i\} : c(\{3, i\}) \in X_3 \text{ and } \{1, i\} \notin A_1 \text{ & } \{2, i\} \notin A_2\}.$$

The preceding result cannot be extended for the case with more than three components, even if the color of the triangle class is taken out of the demanding list. Consider the proper coloring provided in Remark 1 once more to see this. It is easy to check that, under that coloring, $KG(n, 2)$ does not contain any colorful complete 4-partite $\mathcal{K}_{X_0,X_1,X_2,X_3}$ subgraph with $c(X_i) = \{n - i\}$ for $i = 0, ..., 3$. 


3 | CONCLUSIONS AND OPEN PROBLEMS

In summary, in this paper, we provided a negative response to Question 1 for the case \( k = 2 \) which leads to a complete combinatorial proof for the \( K_{l,m} \)-theorem for \( KG(n, 2) \). So, it is left open to explore if the answer to Question 1 for the other cases \( k > 2 \) is negative or not. Also, it would be intriguing to know if there is a “pure combinatorial” proof for the fact that \( KG(n, k) \) has the colorful \( K_{l,m} \)-property when \( k > 2 \). We firmly believe that one could most likely develop a combinatorial proof for the case \( k = 3 \) by using similar ideas.

In light of our discussion about the existence of a colorful complete \( t \)-partite subgraph in \( KG(n, 2) \), it seems entirely natural to investigate this question for arbitrary graphs. To make it precise, we need a definition. For a given natural number \( t \geq 2 \), we say a graph \( G \) has the colorful \( t \)-partite property if for any optimal coloring \( c \) of \( G \) and any partition \( \{X_1, \ldots, X_t\} \) of the color sets into \( t \) parts, \( G \) contains a colorful complete \( K_{A_1, \ldots, A_t} \) subgraphs such that \( c(A_i) = X_i \) for all \( i = 1, \ldots, t \). For example, for each \( 2 \leq t \leq n \), the complete graph \( K_n \) with \( n \) vertices is an example of a graph with the colorful \( t \)-partite property. Now, we are in a position to mention our questions.

**Question 2.** Is there any “simple” classification of all graphs satisfying the colorful \( t \)-partite property?

This question is most likely too ideal to elicit a positive response. A more realistic query might be:

**Question 3.** Is there any ‘nontrivial’ sufficient condition that guarantees a graph to satisfy the colorful \( t \)-partite property?

Note that, for \( t = 2 \), one possible answer to this question could be the colorful \( K_{l,m} \)-theorem [16, Theorem 2] or [3, Theorem 4]. But, what if \( t \geq 3 \)?

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