TOPOLOGICAL DYNAMICS ON MODULI SPACES II

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Abstract. Let $M$ be a Riemann surface with boundary $\partial M$ and genus greater than zero. Let $\Gamma$ be the mapping class group of $M$ fixing $\partial M$. The group $\Gamma$ acts on $\mathcal{M}_c = \text{Hom}_c(\pi_1(M), \text{SU}(2))/\text{SU}(2)$ which is the space of SU(2)-gauge equivalence classes of flat SU(2)-connections on $M$ with fixed holonomy on $\partial M$. We study the topological dynamics of the $\Gamma$-action and give conditions for the individual $\Gamma$-orbits to be dense in $\mathcal{M}_c$.

1. Introduction

Let $M$ be a Riemann surface of genus $g$ with $n$ boundary components (circles). Let

$$\{C_1, C_2, \ldots, C_n\} \subset \pi_1(M)$$

be elements in the fundamental group that correspond to these $n$ boundary components.

The space of SU(2)-gauge equivalence classes of SU(2)-connections $YM_2(\text{SU}(2))$ is the well known Yang-Mills two space of quantum field theory. Inside $YM_2(\text{SU}(2))$ is the moduli space $\mathcal{M}$ of flat SU(2)-connections. The moduli space $\mathcal{M}$ has an interpretation that relates to the representation space $\text{Hom}(\pi_1(M), \text{SU}(2))$, which is an algebraic variety. The group $\text{SU}(2)$ acts on $\text{Hom}(\pi_1(M), \text{SU}(2))$ by conjugation, and the resulting quotient space is precisely

$$\mathcal{M} = \text{Hom}(\pi_1(M), \text{SU}(2))/\text{SU}(2).$$

Note that a conjugacy class in $\text{SU}(2)$ is determined by its trace. Hence specifying a conjugacy class is the same as specifying a real number in $[-2, 2]$. To each $C_i$, assign a conjugacy class $c_i$ in $\text{SU}(2)$ $-2 \leq c_i \leq 2$ and let

$$\mathcal{C} = \{c_1, c_2, \ldots, c_n\}.$$
Definition 1.1. The relative character variety with respect to $\mathcal{C}$ is
\[ \mathcal{M}_\mathcal{C} = \{ [\rho] \in \mathcal{M} : \text{tr}(\rho(C_i)) = c_i, 1 \leq i \leq n \}. \]

The space $\mathcal{M}_\mathcal{C}$ is compact, but possibly singular. The set of smooth points of $\mathcal{M}_\mathcal{C}$ possesses a natural symplectic structure which gives rise to a finite measure $\mu$ on $\mathcal{M}_\mathcal{C}$ (see [4, 5]).

Let $\text{Diff}(M, \partial M)$ be the group of diffeomorphisms fixing $\partial M$. The mapping class group $\Gamma$ is defined to be $\pi_0(\text{Diff}(M, \partial M))$. The group $\Gamma$ acts on $\pi_1(M)$ fixing the $C_i$’s. It is known that $\mu$ is invariant with respect to the $\Gamma$-action. In [4], Goldman showed that, with respect to the measure $\mu$.

Theorem 1.2 (Goldman). The mapping class group $\Gamma$ acts ergodically on $\mathcal{M}_\mathcal{C}$.

Goldman also showed that the mapping class action is weak-mixing, but not mixing.

Since $\mathcal{M}_\mathcal{C}$ is a variety, one may also study the topological dynamics of the mapping class group action. The topological-dynamical problem is considerably more delicate. To begin with, not all orbits are dense in $\mathcal{M}_\mathcal{C}$. If $\rho \in \text{Hom}(\pi_1(M), G)$ where $G$ is a proper closed subgroup of $\text{SU}(2)$ and $\gamma \in \Gamma$, then $\gamma(\rho) \in \text{Hom}(\pi_1(M), G)$. In other words, $\text{Hom}(\pi_1(M), G)/G \subset \mathcal{M}_\mathcal{C}$ is invariant with respect to the $\Gamma$-action. The case of the one-holed torus has been dealt with in [8]:

Theorem 1.3. Suppose that $M$ is a torus with one boundary component and $\rho \in \text{Hom}(\pi_1(M), \text{SU}(2))$ such that $\rho(\pi_1(M))$ is dense in $\text{SU}(2)$. Then the $\Gamma$-orbit of the conjugacy class $[\rho] \in \mathcal{M}_\mathcal{C}$ is dense in $\mathcal{M}_\mathcal{C}$.

This paper deals with the general case of $g > 0$ and proves:

Theorem 1.4. Suppose that $M$ is a Riemann surface with boundary having genus greater than zero. Let $\rho \in \text{Hom}(\pi_1(M), \text{SU}(2))$ such that $\rho(\pi_1(M))$ is dense in $\text{SU}(2)$. Then the $\Gamma$-orbit of the conjugacy class $[\rho] \in \mathcal{M}_\mathcal{C}$ is dense in $\mathcal{M}_\mathcal{C}$.

The group $\text{SU}(2)$ double covers the group $\text{SO}(3)$:
\[ p : \text{SU}(2) \longrightarrow \text{SO}(3). \]

The group $\text{SO}(3)$ contains subgroups isomorphic to: $\text{O}(2)$, as well as the symmetry groups of the regular polyhedra: $T'$ (the tetrahedron), $C'$ (the cube), $D'$ (the dodecahedron), and their subgroups. The inverse images of $\text{O}(2)$, $T'$, $C'$, $D'$ by the projection $p$ are called $\text{Pin}(2)$, $T$, $C$, $D$, respectively. The identity component of $\text{Pin}(2)$ is called $\text{Spin}(2)$. Let $\rho \in \text{Hom}(\pi_1(M), \text{SU}(2))$. Theorem 1.4 implies that if $\rho(\pi_1(M))$
is not contained in a group isomorphic to $C, D,$ or Pin(2), then the
$\Gamma$-orbit of the conjugacy class $[\rho] \in \mathcal{M}_C$ is dense in $\mathcal{M}_C$.

Theorem 1.4 covers all moduli spaces except that of the $n$-holed sphere.

Conjecture 1.5. Theorem 1.4 holds for all Riemann surfaces with boundary.

To prove this conjecture, one must first carry out an analysis of the
four-holed sphere similar to the one performed on the one-holed torus in
[8]. The argument in [8] involves a detailed combinatorial study of cer-
tain four-term trigonometric Diophantine equations (see [2]). For the
four-holed sphere, the analysis would involve trigonometric Diophan-
tine equations with 6 terms. Provided such a result can be obtained,
the techniques provided in this paper could be adapted to extend such
a four-holed sphere result to the $n$-holed sphere.

1.1. Outline of the Proof. A pants decomposition $\mathcal{P}$ of $M$ gives rise
to a smooth open dense subset $\mathcal{M}_\mathcal{P}$ that is an integrable system inside
the moduli space $\mathcal{M}_C$. Hence, one obtains the following diagram [4]:

$$\begin{array}{ccc}
\mathcal{M}_\mathcal{P} & \xrightarrow{f_\mathcal{P}} & P \\
\downarrow i & & \downarrow i \\
\mathcal{M}_C & \xrightarrow{f_\mathcal{P}} & P'
\end{array}$$

where $\mathcal{M}_\mathcal{P}$ is a torus bundle over $P$ and $P' \subset [-2, 2]^N$, $N = \frac{1}{2}\dim(\mathcal{M}_C)$.
The subgroup $\Gamma_\mathcal{P} \subset \Gamma$ that preserves the fibres of $f_\mathcal{P}$ acts as rotations on
each fibre with angles depending on the base coordinates [4]. Section 4
gives a brief outline of this integrable system and the decomposition of
the $\Gamma$-action.

The proof of Theorem 1.4 involves two steps. Suppose that $[\rho]$ is
a generic representation (i.e. $\rho(\pi_1(M))$ is dense in SU(2)). Let $\Gamma([\rho])$
denote the $\Gamma$-orbit of $[\rho]$. The first step is to show that if $f_\mathcal{P}(\Gamma([\rho]))$
is dense in $P$, then $\Gamma([\rho])$ is dense in $\mathcal{M}_C$ (Corollary 2.5). The second step
involves proving the base density theorem, i.e., the density of $f_\mathcal{P}(\Gamma([\rho]))$
in $P$.

As the problem deals with arbitrary genus, the proof necessarily
involves induction, with the one-holed torus as the base case. To get
things started, for a generic representation $\rho$, one first shows that there
is a one-holed torus $T$ inside $M$ such that the restriction of $\rho$ to $\pi_1(T)$
is generic. This is a detailed combinatorial calculation which is outlined
in Section 3 and carried out in the Appendix.
After obtaining a generic handle, we proceed to demonstrate the base density theorem for the \((n + 2g - 2)\)-holed torus. An analysis of the case of the four-holed sphere is required to get the induction process started, and is used to prove the result for the case of the two-holed torus. From there, the case of the three-holed torus is proven which, in turn, is used to prove the case of the \((n + 2g - 2)\)-holed torus.

To complete the proof, the \(2g - 2\) holes of the \((n + 2g - 2)\)-holed torus are grouped in pairs and each pair is glued along their boundary to obtain the original surface \(M\) with genus \(g\) and \(n\) boundary components. Section 9 completes the induction process.

1.2. Some definitions. Fix a surface \(M\) with genus \(g > 0\) and \(n\) boundary components. Then \(M\) may be described as a \(2g\)-gon with \(n\) holes in the middle, with appropriate identifications. More precisely, the fundamental group \(\pi_1(M, O)\) is generated by \(S = \{A_i\}_{i=1}^{2g+n}\), subject to the relation

\[
\left(\prod_{i=1}^{g} [A_i, A_{i+g}]\right) \left(\prod_{i=2g+1}^{2g+n} A_i\right) = e.
\]

Definition 1.6.

1. A representation \(\rho\) into \(SU(2)\) is generic if the image of \(\rho\) is dense in \(SU(2)\).
2. A handle \((A, B)\) consists of two simple loops \(A, B \in \pi_1(M, O)\) crossing at \(O\), but otherwise disjoint.
3. Suppose \(G \subset SU(2)\). A representation \(\rho\) is said (resp. not) to be \(G\) if \(\text{Im}(\rho)\) is (resp. not) contained in some (resp. any) isomorphic copy of \(G\) in \(SU(2)\).
4. Associated to each simple loop \(A \in \pi_1(M, O)\) is the Dehn twist in \(A\) represented in \(\Gamma\) by a diffeomorphism of \(M\) supported on a tubular neighborhood of \(A\) in \(M\) (an annulus). The action of the Dehn twist amounts to cutting \(M\) at \(A\) and twisting one of the resulting boundary circles once in the direction of \(A\) and re-identifying the two circles.
5. With a fixed representation \(\rho\), \(X \in \pi_1(M, O)\), and \(\gamma \in \Gamma\), we write \(X\) for \(\rho(X)\) and \(\gamma(X)\) for \(\gamma(\rho)(X)\) when there is no ambiguity. A small letter will be used to denote the trace of the matrix represented by the corresponding capital letter. For example, we use \(x\) to denote \(\text{tr}(\rho(X))\). In this setting, \(\gamma(x)\) denotes \(\text{tr}(\gamma(\rho)(X))\).
6. Let \((V, d)\) be a metric space. For \(\epsilon > 0\), a set \(U\) is \(\epsilon\)-dense in \(V\) if, for each \(p \in V\), there exists a point \(q \in U\) such that \(0 < d(p, q) < \epsilon\). If \(U = V\), we simply say that \(V\) is \(\epsilon\)-dense.
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2. The Moduli Spaces as Tori Bundles

We begin by giving a brief description of the integrable system on the moduli space [5]. Suppose $M$ has genus $g \geq 1$ and $n \geq 0$ boundary components. Let

$$C = \{c_1, ..., c_n\}$$

be a fixed set of conjugacy classes with the first $m$ classes not equal to $\pm 2$ and $c_{n-m+1}, ..., c_n \in \{\pm 2\}$. Then the real dimension of $M_C$ is $6g - 6 + 2m$. Since the case of the torus is well understood, we assume throughout the remainder of the paper that $g > 1$ or $m > 0$.

2.1. Pants decompositions. There is a map

$$f_P : M_C \rightarrow [-2, 2]^N$$

that arises from a so-called pants decomposition $P$ of $M$, where $N = 3g - 3 + m$. For detailed information, see [4]. The idea is that, by cutting along $3g - 3 + m$ circles on $M$, the surface $M$ can be decomposed into $2g - 2 + m - 1$ three-holed spheres and one exceptional $(n - m + 3)$-holed sphere with $n - m$ of its boundary components assigned $\pm 2$.

Figure 1: Decomposition of $M$ for $n = m + 1$. 
Figure 1 demonstrates such a decomposition when \( n = m + 1 \), i.e., \( c_i \neq \pm 2 \), for \( 1 \leq i \leq m \), and \( c_{m+1} = \pm 2 \). In this specific case, we do not cut \( M \) along \( B \), thus obtaining \( 2g - 2 + m - 1 \) three-holed spheres and one four-holed sphere. Now in such a case, if \( c_{m+1} = 2 \), then the boundary \( C_{m+1} \) effectively does not exist, both with respect to representations and the mapping class group action. On the other hand, if \( c_{m+1} = -2 \) and \( n > 1 \), then one can simply consider the moduli space of genus \( g \) having \( n - 1 \) boundary components with \( C = (c_1, ..., -c_m) \).

**Remark 2.1.** For \( m \geq 1 \), the moduli space of a genus \( g \) surface \( M \) with \( n \) boundary components and \( C = (c_1, ..., c_m, \pm 2, ..., \pm 2) \) can be identified with the moduli space of a genus \( g \) surface \( M' \) with \( m \) boundary components and \( C' = (c_1, ..., \pm c_m) \).

In light of Remark 2.1, we will assume that either \( C_i \neq \pm I \) for all \( i \) or that \( n = 1 \) and \( C_1 = \pm I \).

Fix a decomposition \( \mathcal{P} \) of \( M \) as described above. This provides \( 3g - 3 + m \) loops \( B_1, ..., B_{3g-3+m} \in \pi_1(M, O) \). Let \([\rho] \in \mathcal{M}_C \) be given. Then

\[
\left[ f_{\mathcal{P}}([\rho]) = (b_1, ..., b_{3g-3+m}) \right]
\]

is the desired map, where \( b_i = \text{tr}(\rho(B_i)) \). Let \( \beta = (b_1, ..., b_{3g-3+m}) \).

### 2.2. The integrable system \((\mathcal{M}_\mathcal{P}, f_\mathcal{P})\)

Let \( P' \) be the image of \( f_\mathcal{P} \) and let \( P = P' \setminus \partial P' \). The map \( f_\mathcal{P} \) restricted to \( \mathcal{M}_\mathcal{P} = f_\mathcal{P}^{-1}(P) \) is a submersion [4]. Denote by \( \Gamma_\mathcal{P} \subset \Gamma \) the stabilizer of the fibres of \( f_\mathcal{P} \).

**Proposition 2.2.** The set \( P' \) consists of all \( \beta \in [-2, 2]^{3g-3+m} \) that simultaneously satisfy the \( 2g - 2 + m - 1 \) inequalities

\[
b_i^2 + b_j^2 + b_k^2 - b_ib_jb_k \leq 4,
\]

where the three curves \( B_i, B_j \) and \( B_k \) bound a triply punctured sphere in the decomposition \( \mathcal{P} \) of \( M \) (this includes the possibility of \( B_i \) being a boundary curve in \( \partial M \)). In addition, if \( B_i, B_j, B_k \) are the three boundaries of the exceptional \( (n - m + 3) \)-holed sphere with \( b_i, b_j, b_k \neq \pm 2 \), then

\[
b_i^2 + b_j^2 + b_k^2 - cb_i b_j b_k \leq 4,
\]

where \( c \) is the sign of \( \Pi_{i=m+1}^n C_i \). Suppose \( \beta \in P \). Then there is a \( \Gamma_{\mathcal{P}} \)-equivariant homeomorphism

\[
h : f_{\mathcal{P}}^{-1}(\beta) \to T^{3g-3+m}
\]

such that for all \( \xi \in f_{\mathcal{P}}^{-1}(\beta) \) and \( (n_1, ..., n_{3g-3+m}) \in \mathbb{Z}^{3g-3+m} \)

\[
h : \tau_1^{n_1} ... \tau_{3g-3+m}^{n_{3g-3+m}} \xi \mapsto \begin{bmatrix} e^{in_1 \theta_1} h_1 \\ \vdots \\ e^{in_{3g-3+m} \theta_{3g-3+m}} h_{3g-3+m} \end{bmatrix},
\]
where

\[ h(\xi) = \begin{bmatrix} h_1 \\ \vdots \\ h_{3g-3+m} \end{bmatrix}, \]

\[ \theta_j = \cos^{-1}(b_j/2), \] and \( \tau_i \) is the action of the Dehn twist in \( B_i \).

Proof. See [4].}

In short, \((\mathcal{M}_P, f_P)\) is an integrable system. The real dimensions of \( P \) and \( \mathcal{M}_P \) are \( 3g - 3 + m \) and \( 6g - 6 + 2m \), respectively. Denote by \( \mathcal{M}_C^\circ \) the subset of irreducible representations of \( \mathcal{M}_C \).

**Lemma 2.3.** \( \mathcal{M}_C^\circ \) is smooth, open, and dense in \( \mathcal{M}_C \).

Proof. The space \( \mathcal{M}_C \) corresponds to the moduli space \( \mathcal{M}_C^{ss} \) of semi-stable parabolic SU(2)-bundles on \( M \). Moreover \( \mathcal{M}_C^\circ \) corresponds to the stable subspace and is open and dense in \( \mathcal{M}_C^{ss} \) [9].

**Proposition 2.4.** The subset \( \mathcal{M}_P \) is open and dense in \( \mathcal{M}_C^\circ \).

Proof. A direct calculation from Proposition 2.2 shows that \( \mathcal{M}_C^\circ \setminus \mathcal{M}_P \) is a real algebraic subvariety with positive co-dimension. The result then follows from the fact that \( \mathcal{M}_C^\circ \) is smooth and has dimension \( 6g - 6 + 2m \).

Together Proposition 2.2 and 2.4 imply:

**Corollary 2.5.** Let \([\rho]\) \( \in \mathcal{M}_C \) and \( \Gamma([\rho]) \) be the \( \Gamma \)-orbit of \([\rho]\). If \( f_P(\Gamma([\rho])) \) is dense in \( P \), then \( \Gamma([\rho]) \) is dense in \( \mathcal{M}_C \).

3. Generic Representations and Handles

Here we adapt an idea in [3] to find a generic handle (one-holed torus) inside \( M \). Let \( \rho \in \text{Hom}(\pi_1(M, O), \text{SU}(2)) \). We first observe that moving the base point \( O \), has the effect of conjugating \( \rho \) by an element in \( \text{SU}(2) \).

**Proposition 3.1.** For any generic \( \rho \in \text{Hom}(\pi_1(M, O), \text{SU}(2)) \), there exists a handle \((A, B)\) such that \( \rho|_{(A,B)} \) is generic.

The proof of Proposition 3.1 is highly computational. Therefore, we outline its general structure here and leave the details to the Appendix. One should first consult Section 5 and [8], as the proof involves the moduli spaces of the one-holed torus and uses many ideas pertaining to those spaces.
Suppose \( \rho \) is a generic representation. For any \( i \leq g \), the group \( \langle A_i, A_{g+i} \rangle \) may very well be contained in a proper closed subgroup \( G \subset \text{SU}(2) \). However, the crucial observation is that for any \( 1 \leq j \leq 2g+m \) with \( j \neq i \) and \( j \neq g+i \), the following are also handles: \( (A_i, A_{g+i}A_j) \), \( (A_iA_j, A_{g+i}) \), \( (A_i, A_jA_{g+i}) \), \( (A_jA_i, A_{g+i}) \) (see Figures 2 and 3). The fact that \( \rho \) is generic implies that there exists \( j \) such that \( A_j \not\in G \).

![Figure 2: \( (A_i, A_{g+i}A_j) \) is a handle.](image1)

![Figure 3: \( (A_i, A_{g+i}A_j) \) is a handle.](image2)

One begins by assuming that \( \langle A_i, A_{g+i} \rangle \) is \( \text{Spin}(2) \) and then shows that there exists \( j \) such that \( \langle A_i, A_{g+i}A_j \rangle \) is not \( \text{Spin}(2) \). From there, one moves onto the groups \( \text{Pin}(2) \), \( T \), \( C \), and finally to \( D \). Note that the computational complexity increases drastically from one case to
the next, not only because the groups are more complicated, but also because it is a recursive process. For example, suppose that $\langle A_i, A_{g+i} \rangle$ is $D$ and one wants to show that there exists $j$ such that $\langle A_i, A_{g+i}A_j \rangle$ is generic. One must not only show that $\langle A_i, A_{g+i}A_j \rangle$ is not $D$, but in addition that it is neither $C$ nor $\text{Pin}(2)$.

4. The Three-holed Sphere

Suppose $M$ is a three-holed sphere. Then $\pi_1(M)$ has a presentation:

$$\langle A, B, C : ABC = I \rangle,$$

where $A, B,$ and $C$ represent the homotopy classes of the three boundaries of $M$.

**Proposition 4.1.**

1. A representation $\rho$ on a three-holed sphere is a $\text{Spin}(2)$-representation if and only if $a^2 + b^2 + c^2 - abc - 4 = 0$.
2. A representation $\rho$ on a three-holed sphere is $\text{Pin}(2)$ and not $\text{Spin}(2)$ if and only if $a^2 + b^2 + c^2 - abc - 4 \neq 0$ and at least two of the three: $A, B, AB,$ have zero trace.

**Proof.** If $\rho$ is a $\text{Spin}(2)$ representation, then up to conjugation:

$$\rho(A) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \quad \rho(B) = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix},$$

$$\rho(AB) = \begin{bmatrix} \cos(\theta + \phi) & \sin(\theta + \phi) \\ -\sin(\theta + \phi) & \cos(\theta + \phi) \end{bmatrix}.$$

Note that:

$$a^2 + b^2 + c^2 - abc - 4 =$$

$$4 \cos^2 \theta + 4 \cos^2 \phi + 4 \cos^2(\theta + \phi) - 8 \cos \theta \cos \phi \cos(\theta + \phi) - 4 = 0.$$

The other direction follows from the uniqueness of characters [4]. This proves (1).

Suppose $\rho$ is $\text{Pin}(2)$, but not $\text{Spin}(2)$, then at least two of the following $A, B,$ and $AB$ are in $\text{Spin}_-(2)$. Since $A \in \text{Spin}_-(2)$ implies $\text{tr}(A) = 0$, at least two of the three global coordinates of $[\rho]$ must be zero. See [8] for a similar proof in the case of the one-holed torus. The other direction follows similarly from the uniqueness of characters [4]. This proves (2).
5. The One-holed Torus

We briefly summarize some relevant results that appear in [4] and [8]. Suppose that $M$ is a one-holed torus. The fundamental group $\pi_1(M)$ has a presentation

$$\pi_1(M) = \langle X, Y, K | K = X Y X^{-1} Y^{-1} \rangle,$$

where $K$ represents the element corresponding to the boundary component as in Figure 4.

![Figure 4: The one-holed torus](image)

Let

$$E = \text{Hom}(\pi_1(M), \text{SU}(2))/ \text{SU}(2).$$

A representation class $[\rho] \in E$ is determined by

$$x = \text{tr}(\rho(X)), y = \text{tr}(\rho(Y)), z = \text{tr}(\rho(XY)).$$

This provides a global coordinate chart:

$$[\rho] \mapsto E \mapsto (\text{tr}(\rho(X)), \text{tr}(\rho(Y)), \text{tr}(\rho(XY))).$$

In addition, $k = \text{tr}(\rho(K))$ is given by the formula

$$k = \text{tr}(\rho(K)) = x^2 + y^2 + z^2 - xyz - 2.$$

Let

$$E_k = \{(x, y, z) \in [-2, 2]^3 : x^2 + y^2 + z^2 - xyz - 2 = k\},$$

then

$$E = \bigcup_{k \in [-2,2]} E_k.$$

For $-2 < k < 2$, the set $E_k$ is a smooth two-sphere, the set $E_2$ is a singular sphere, and $E_{-2} = (0,0,0)$.

The mapping class group $\Gamma$ is generated by the maps $\tau_X$ and $\tau_Y$ induced by the Dehn twists in $X$ and $Y$, respectively. With respect to
the global coordinate, the action can be described explicitly [4]:

\[ \tau_X(x, y, z) = (x, z, xz - y) \]

\[ \tau_Y(x, y, z) = (z, y, yz - x). \]

The action of \( \tau_X \) fixes \( x \) and \( k \), and preserves the ellipse

\[ X_k(x) = \{ x \} \times \{ (y, z) : 2 + \frac{x}{4} (y + z)^2 + \frac{2 - x}{4} (y - z)^2 = 2 + k - x^2 \}. \]

A change of coordinates transforms \( X_k(x) \) into the circle

\[ X_k(x) = \{ x \} \times \{ (\tilde{y}, \tilde{z}) : \tilde{y}^2 + \tilde{z}^2 = 2 + k - x^2 \} \]

(see [4, 8]). In this new coordinate system, \( \tau_X \) acts as a rotation by \( \cos^{-1}(x/2) \). In short, the sphere \( E_k \) is the union of circles

\[ E_k = \bigcup_x X_k(x), \]

and \( \tau_X \) rotates (up to a coordinate transformation) each level set \( X_k(x) \) by an angle of \( \cos^{-1}(x/2) \). Similarly, there is a coordinate transformation so that \( \tau_Y \) acts as a rotation of \( Y_k(y) \) by an angle of \( \cos^{-1}(y/2) \).

**Proposition 5.1.** The space of \( \text{Spin}(2) \) representation classes consists precisely of \( E_2 \). The \( \text{Pin}(2) \) representation classes consist of \( E_2 \) and the intersections of the three coordinate axes with \( E \). For each \(-2 < k < 2\), there are exactly six points corresponding to \( \text{Pin}(2) \) representation classes in \( E_k \). Moreover, a representation class \((x, y, z) \in E_k \) with \(-2 < k < 2\) and \( x \neq 0 \) is \( \text{Pin}(2) \) if and only if \( k = x^2 - 2 \).

**Proof.** See [8]. \( \square \)

**Remark 5.2.** This is the first explicit example of Proposition 2.2 with \( g = 1 \) and \( m = 1 \). One obtains a pair of pants by cutting along \( X \) (resp. \( Y \)). The important property is that if \( X_k(x) \) (resp. \( Y_k(y) \)) is a non-degenerate circle, then \( \tau_X \) (resp. \( \tau_Y \)) acts on the fibre \( X_k(x) \) (resp. \( Y_k(y) \)) as a rotation with an angle depending solely on \( x \) (resp. \( y \)) and independent of either \( k \) or \( y \) (resp. \( x \)). In particular, if a representation \( \rho \) is not \( \text{Pin}(2) \), then neither \( \tau_X \) nor \( \tau_Y \) fix \( [\rho] \).

**Remark 5.3.** Let \( \rho \) be generic. By Theorem [1.3], the \( \langle \tau_X, \tau_Y \rangle \)-orbit \( O \) of \([\rho]\) is dense in \( E_k \). Hence, there is a number \( r > 0 \) such that the set \( \{ (x, y) : (x, y, z) \in O \} \) is dense in \( R = [-r, r]^2 \). In particular, by Dehn twisting in \( \tau_X \) and \( \tau_Y \), one can always assume that \( x \) and \( y \) are simultaneously off of any finite set of values and are arbitrarily close to zero.
Remark 5.4. In addition to Theorem 1.3, it was shown in [8] that the k-values for the surjective T, C, D representations are 0, $\frac{1+\sqrt{5}}{2}$, 1. We define the set of special values:

$$S = \{\frac{1 \pm \sqrt{5}}{2}, 0, 1\}$$

In particular, if $(x, y, z)$ is not Pin(2) and $k \notin S$, then $(x, y, z)$ is generic.

6. The Four-Holed Sphere

We first review some results that appear in [1] and [4]. Suppose $M$ is a four-holed sphere. Then the fundamental group $\pi_1(M, O)$ admits a presentation

$$\langle A, B, C, D : ABCD = I \rangle.$$}

6.1. The moduli space $\mathcal{M}$. The moduli space $\mathcal{M}$ for the four-holed sphere is six-dimensional. Let $g : \mathcal{M} \rightarrow [-2, 2]^4$ be the map defined by $g([\rho]) = (a, b, c, d)$.

Set $X = AB, Y = BC$, and $Z = CA$ (see Figure 5). Then the set $E_\kappa = g^{-1}(a, b, c, d)$ satisfies the equation

$$x^2 + y^2 + z^2 + xyz = (ab + cd)x + (ad + bc)y + (ac + bd)z - (a^2 + b^2 + c^2 + d^2 + abcd - 4).$$

Let

$$I_{a,b} = \left[\frac{ab - \sqrt{(a^2 - 4)(b^2 - 4)}}{2}, \frac{ab + \sqrt{(a^2 - 4)(b^2 - 4)}}{2}\right].$$
For any $\kappa = (a, b, c, d) \in [-2, 2]^4$ with $I_{a,b} \cap I_{c,d} \neq \emptyset$, the $x$-level sets $X_{\kappa}(x) \subset E_{\kappa}$ are ellipses (possibly degenerate) in $y$ and $z$ given by:

$$\frac{2 + x}{4}((y + z) - \frac{(a + b)(d + c)}{2 + x})^2 + \frac{2 - x}{4}((y - z) - \frac{(a - b)(d - c)}{2 + x})^2 = \frac{(x^2 - abx + a^2 + b^2 - 4)(x^2 - cdx + c^2 + d^2 - 4)}{4 - x^2}.$$

There are similar descriptions for the $y$- and $z$-level sets $Y_{\kappa}(y)$ and $Z_{\kappa}(z)$ respectively (see [4]).

For $x$ in the interior of $I_{a,b} \cap I_{c,d}$ the level set $X_{\kappa}(x)$ is an ellipse centered at

$$y_c(x) = \frac{2(2(ad+bc) - x(ac+bd))}{4-x^2},$$
$$z_c(x) = \frac{2(2(ac+bd) - x(ad+bc))}{4-x^2}.$$

Figure 6 below displays several $Z_{\kappa}(z) \subset E_{\kappa}$ for a particular $\kappa$.

![Figure 6: The sphere $E_{\kappa}$ for $\kappa = (.5, -1, -.2, 1.2)$](image)

For fixed $\kappa = (a, b, c, d)$, the $x$-coordinates in $E_{\kappa}$ take on all the values inside $I_{a,b} \cap I_{c,d}$ (see Proposition 2.2 and [1]). In particular,

$$E_{\kappa} = \bigcup_{x \in I_{a,b} \cap I_{c,d}} X_{\kappa}(x).$$
By symmetry, similar constructs can be made for the $y$- and $z$-coordinates.

6.2. **The mapping class action.** In local coordinates, the actions of $\tau_X, \tau_Y, \tau_Z$ are

\[
\begin{bmatrix}
y \\
z
\end{bmatrix} \mapsto \begin{bmatrix}
ad + bc - x(ac + bd - xy - z) - y \\
ac + bd - xy - z
\end{bmatrix},
\begin{bmatrix}
z \\
x
\end{bmatrix} \mapsto \begin{bmatrix}
bd + ca - y(ba + cd - yz - x) - z \\
ba + cd - yz - x
\end{bmatrix},
\begin{bmatrix}
x \\
y
\end{bmatrix} \mapsto \begin{bmatrix}
cd + ab - z(cb + ad - zx - y) - x \\
ab + ad - zx - y
\end{bmatrix}.
\]

These actions preserve the ellipses $X_\kappa(x) \subset E_\kappa, Y_\kappa(y) \subset E_\kappa$, and $Z_\kappa(z) \subset E_\kappa$, respectively. After coordinate transformations, these are rotations by angles $2 \cos^{-1}(x/2), 2 \cos^{-1}(y/2)$, and $2 \cos^{-1}(z/2)$, respectively \[\square\].

**Remark 6.1.** *This is the second explicit example of Proposition 2.3 with $g = 0$ and $m = 4$. One obtains two pairs of pants by cutting along $X$ (resp. $Y$ or $Z$). The important property is that if $X_\kappa(x)$ (resp. $Y_\kappa(y)$ or $Z_\kappa(z)$) is a non-degenerate circle, then $\tau_X$ (resp. $\tau_Y$ or $\tau_Z$) acts on the fibre $X_\kappa(x)$ (resp. $Y_\kappa(x)$ or $Z_\kappa(z)$) as a rotation with an angle depending only on $x$ (resp. $y$ or $z$) and independent of $\kappa$.***

Let $d$ be the metric on $\mathbb{R}^3$ that generates the box topology. The metric $d$ generates the usual topology on $\mathcal{M}_C$. For a fixed $\kappa$, the coordinates provide an embedding of $E_\kappa$ into $\mathbb{R}^3$ and $E_\kappa$ inherits the metric $d$ which is described as:

\[
d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}.
\]

6.3. **Filtration on the level sets.** We introduce a filtration that is analogous to the one introduced in \[\square\] for the one-holed torus. The Dehn twist $\tau_\gamma$ acts on the (transformed) subsets $Y_\kappa(y)$ via a rotation of angle $2 \cos^{-1}(y/2)$. Thus there is a filtration of the $y$-coordinates that yield finite orbits under $\tau_\gamma$ as follows: Let $Y_n \subset (-2, 2)$ such that $y \in Y_n$ if and only if the $\tau_\gamma$-action on non-fixed points $(x, y, z) \in E_\kappa$ is periodic with period less than or equal to $n$. This gives a filtration

\[\{0\} = Y_2 \subset Y_3 \subset \ldots \subset Y_n \subset \ldots\]

For example: $Y_2 = \{0\}, Y_3 = \{0, 1, -1\}, Y_4 = \{0, 1, -1, \sqrt{2}, -\sqrt{2}\},$ etc. Note that the filtration is independent of the choice of $\kappa$ and that $Y_n$ is a finite set for every $n$. By symmetry, there exist similar filtrations $X_\kappa$ and $Z_\kappa$, with $X_n = Y_n = Z_n$ as sets. The following lemmas, though proven for the $Y_n$ filtration, apply equally to the other filtrations.
Lemma 6.2. For $\epsilon > 0$ there exists $N(\epsilon) > 0$ so that if $y \not\in Y_{N(\epsilon)}$, then the $\tau_{y}$-orbit of $(x, y, z)$ is $\epsilon$-dense in $Y_{\kappa}(y)$ for any $(x, y, z)$ in any $E_{\kappa}$.

Proof. Since the ellipses $Y_{\kappa}(y)$ are (possibly degenerate) of uniformly bounded circumferences, there exists $N(\epsilon) > 1$ such that for any $y \not\in Y_{N(\epsilon)}$, the $\tau_{y}$-orbit is $\epsilon$-dense in $Y_{\kappa}(y)$.

Throughout the remainder of the paper, the moduli spaces of four-holed spheres with $\kappa = (a, b, c, d)$, having small $|c|$ and $|d|$, will play an important role. We first analyze the case of $E_{\kappa}$ for $\kappa = (a, b, 0, 0)$.

Lemma 6.3. Suppose $(a, b, 0, 0) = \kappa \in [-2, 2]^{4}$. Then $X_{\kappa}(x) \subset E_{\kappa}$ is an ellipse (possibly degenerate) centered at $(x, 0, 0)$. Thus, $Y_{\kappa}(0)$ and $Z_{\kappa}(0)$ intersect every ellipse $X_{\kappa}(x)$.

Proof. This result follows directly from equation (1) which gives $(y_{c}(x), z_{c}(x))$, the center of the ellipse $X_{\kappa}(x)$.

Lemma 6.4. Let $a, b \in (-2, 2)$ and $\epsilon > 0$. Then there exists $\delta > 0$ so that for any $\kappa = (a, b, c, d)$ with $|c|, |d| < \delta$, every ellipse $Y_{\kappa}(y) \subset E_{\kappa}$ with $|y| \leq \delta$ has points with $x$-coordinates that come within $\epsilon$ of all possible $x$-coordinates in $E_{\kappa}$. That is,

$$\{x : (x, y, z) \in E_{\kappa}\} \subset \{x \pm \epsilon : (x, y, z) \in Y_{\kappa}(y), |y| \leq \delta\}.$$ 

Moreover, each $X_{\kappa}(x) \subset E_{\kappa}$ satisfies at least one of:

1. $X_{\kappa}(x)$ has points which realize all $y$- and $z$-coordinates in $[-\frac{\delta}{2}, \frac{\delta}{2}]$.
2. All $y$- and $z$-coordinates of $X_{\kappa}(x)$ are inside $[-\delta, \delta]$.

Proof. The result holds by the continuous dependence of $E_{\kappa}$ on $c$ and $d$, by the continuous dependence of $X_{\kappa}(x)$ on $x$, and by the geometry of $E_{\kappa}$ for $\kappa = (a, b, 0, 0)$ as described in Lemma 6.3. \qed

7. The Two-Holed Torus

In this section, we prove Theorem 1.4 for the case of $g = 1$ and $m = 2$. The fundamental group $\pi_{1}(M, \Omega)$ has a presentation

$$\langle X, Y, A, B | XYX^{-1}Y^{-1} = AB \rangle,$$

where $A$ and $B$ represent the boundary components. Let

$$K = XYX^{-1}Y^{-1}, W = AX, W' = XB, Z = XY,$$

(see Figure 7).
Assume that $\rho \in \text{Hom}(\pi_1(M,O),\text{SU}(2))$ is generic. By Proposition 3.1, we have that $\langle X, Y \rangle$ is generic. If we cut $M$ along $X$ (resp. $Y$), then we obtain a four-holed sphere with $\kappa = (a,b,x,x)$ (resp., $\kappa = (a,b,y,y)$). By Remark 2.1 and Theorem 1.3, we assume that $a,b \neq \pm 2$. For $\kappa = (a,b,x,x)$, the formula for the defining equation of $E_\kappa$ simplifies to:

$$w^2 + (w')^2 + k^2 + kw'w' = k(ab + x^2) + wx(a + b) + w'x(a + b) - a^2 - b^2 - 2x^2 - abx^2 + 4,$$

and the formulas for the Dehn twists $\tau_K, \tau_W, \tau_{W'}$ become:

$$\begin{bmatrix} w \\ w' \end{bmatrix} \xrightarrow{\tau_K} \begin{bmatrix} x(a + b) - k(x(a + b) - kw - w') - w \\ x(a + b) - kw - w' \end{bmatrix},$$

$$\begin{bmatrix} w' \\ k \end{bmatrix} \xrightarrow{\tau_W} \begin{bmatrix} x(a + b) - w(x^2 + ab - wu' - k) - w' \\ x^2 + ab - wu' - k \end{bmatrix},$$

$$\begin{bmatrix} k \\ w \end{bmatrix} \xrightarrow{\tau_{W'}} \begin{bmatrix} ab + x^2 - w'(x(a + b) - w'k - w) - k \\ x(a + b) - w'k - w \end{bmatrix}.$$ 

Note that $I_{x,x} = [x^2 - 2, 2]$.

**Lemma 7.1.** For the representation $\rho$, there exists a sequence of Dehn twists $\gamma \in \Gamma$ so that at least one of the following has non-zero trace: $\gamma(AX), \gamma(XB), \gamma(AXY), \gamma(XYB), \gamma(AY), \gamma(YB), \gamma(AYX), \gamma(YXB)$, with $\langle \gamma(X), \gamma(Y) \rangle$ generic.

**Proof.** Suppose that $AX, XB, AXY, XYB, AY, YB, AYX, \text{and } YXB$ all have zero trace. If $\tau_K$ preserves $w = w' = 0$, then $x(a + b) = 0$ (if not, note that the twist in $\tau_K$ preserves $x,y$ and $k$, so $\langle \tau_K(X), \tau_K(Y) \rangle$ remains generic). By Remark 5.3, $x$ can be assumed non-zero. Hence $a = -b$. Then the equation for the four-holed sphere $E_{(a,-a,x,x)}$ implies that:
with the case 

By Remark 5.3 and the fact that the left-hand side of equation (2) is invariant under \( \langle \tau_X, \tau_Y \rangle \), it must be that \( k - 2 - a^2 = 0 \). By the formula \( \text{tr}(AB^{-1}) + \text{tr}(AB) = \text{tr}(A) \text{tr}(B) \) (see [1, 2]) and the fact that \( K = AB \), we have that \( \text{tr}(AB^{-1}) + 2 - a^2 = -a^2 \); this implies that \( A = -B \) as matrices.

Suppose \( A = -B \). Conjugating by an element in \( \text{SU}(2) \), we may assume that

\[
X = \begin{bmatrix} x_1 + y_1i & 0 \\ 0 & x_1 + y_1i \end{bmatrix}, \quad Y = \begin{bmatrix} x_2 + y_2i & z_2 \\ -z_2 & x_2 - y_2i \end{bmatrix},
\]

and

\[
A = \begin{bmatrix} a_1 + b_1i & c_1 + d_1i \\ -c_1 + d_1i & a_1 - b_1i \end{bmatrix}.
\]

The equations: \( \text{tr}(AX) = 0, \text{tr}(AY) = 0, \text{tr}(AXY) = 0, \text{tr}(AYX) = 0 \), together with the matrix equation \( K = -A^{-2} = XYX^{-1}Y^{-1} \) and \( \det(A) = \det(X) = \det(Y) = 1 \) have solutions that must include one of the following: \( x_1 = 0, \pm 1, a_1 = 0 \) or \( a_1 = \pm \sqrt{1 - x_1^2} \). By Remark 5.3, one may assume that \( a_1 \neq \sqrt{1 - x_1^2} \) and \( x_1 \neq 0, \pm 1 \). This leaves us with the case \( a_1 = 0 \), which implies that \( K = I \), contradicting the fact that \( \langle X, Y \rangle \) is generic. \( \square \)

Combining Proposition 5.1 and Lemma 7.1, one obtains:

**Corollary 7.2.** For a generic \( \rho \), we may assume that one of: \( \langle AX, Y \rangle, \langle XB, Y \rangle, \langle AY, X \rangle, \langle YB, X \rangle \), is not \( \text{Pin}(2) \).

By Remark 7.2 and Corollary 7.2, we assume without loss of generality that the action \( \tau_Y \) does not fix \( w = \text{tr}(AX) \).

**Proposition 7.3.** Suppose \( g = 1 \) and \( m = 2 \) and \( \rho \) is a generic representation. Then the \( \Gamma \)-orbit \( \Gamma([\rho]) \) is dense in \( \mathcal{M}_C \).

**Proof.** Let \( \epsilon' > 0 \). Let \( \rho \) be a generic representation. By Proposition 3.1, \( M \) has a generic handle \( \langle X, Y \rangle \) (we adopt the notation presented in Figure 7). By Proposition 2.4, without loss of generality we may take \( [\rho_0] \in \mathcal{M}_P \) and show that there exists \( \gamma \in \Gamma \) such that \( d(\gamma([\rho]), [\rho_0]) < \epsilon' \). Let \( [\rho_0] \in \mathcal{M}_P \) be a representation class with \( x_0 = \text{tr}(\rho_0(X)) \) and \( k_0 = \text{tr}(\rho_0(K)) \). Let \( w_0 = \text{tr}(\rho_0(W)) \), \( w_0' = \text{tr}(\rho_0(W')) \), etc.

The map \( f_P \) is a submersion on \( \mathcal{M}_P \). By the implicit function theorem, there exists \( \epsilon' > \epsilon > 0 \), such that the \( \epsilon \)-tubular neighborhood of \( f_P^{-1}(x_0, k_0) \subset \mathcal{M}_C \) looks like \([x_0 - \epsilon, x_0 + \epsilon] \times [k_0 - \epsilon, k_0 + \epsilon] \times T^2 \).
Cutting along $K$ and $X$ gives a pants decomposition of $M$. Hence, by Corollary 2.5, we only need to show that there exists $\gamma \in \Gamma$ such that $|\gamma(k) - k_0| < \epsilon$ and $|\gamma(x) - x_0| < \epsilon$. In other words, the goal is to move the $x$- and $k$-coordinates near $x_0$ and $k_0$. The strategy is to use $\tau_Y$ to first move the $x$-coordinate near zero so that the moduli space of the four-holed sphere (obtained by cutting along $X$) contains $k$-coordinates near $k_0$. At the same time one needs to ensure that $\tau_W$ has enough points on its orbit so that the $k$-coordinate can actually be moved near $k_0$ (this may require a sequence of twists in $\tau_W$ followed by a sequence of twists in $\tau_K$ followed again by a sequence of twists in $\tau_W$). This can be accomplished by using the generic $\langle X, Y \rangle$ to move the $y$-coordinate to a sufficiently general position (i.e., outside of $Y_N$ for some very large $N$). By Corollary 7.2, this will produce a $\tau_Y$-orbit in $\langle Y, W \rangle$ with a sufficiently large number of points. Under such conditions, this $\tau_Y$-orbit will potentially contain points that have a large number of points on their $\tau_W$-orbit in $E(x,x,a,b)$. The exceptional case being when the $\tau_W$-action fixes $(k, w, w') \in E(x,x,a,b)$. In this case, $W_\kappa(w)$ is a point, which implies that

$$2k = ab + x^2 - ww'.$$

If this occurs, one may try to use the $\tau_{W'}$-action. However in order to use the $\tau_{W'}$-action, one must first ensure that $\langle Y, W' \rangle$ itself is not Pin(2). Hence under the assumption that $\tau_W$ fixes $(k, w, w') \in E(a,b,x,x)$, we have the following two exceptional cases:

Case 1: $\langle W', Y \rangle$ is Pin(2). This leads to either $y = 0$ or $w' = 0$. This implies that either $y = 0$ or

$$2k = ab + x^2. \quad (3)$$

Case 2: $\langle W', Y \rangle$ is not Pin(2) but $(k, w, w') \subset E(a,b,x,x)$ is fixed by $\tau_{W'}$. Then one must have

$$\begin{cases}
2k &= ab + x^2 - ww' \\
2w' &= x(a + b) - wk \\
2w &= x(a + b) - w'k
\end{cases}$$

This leads to $2(w' - w) = k(w' - w)$. Since $k \neq \pm 2$, we have $w = w' = \frac{x(a + b)}{k + 2}$. Moreover, for non-zero $w$, we have $a \neq -b$. This implies that

$$x^2 \left( 1 - \left( \frac{a + b}{k + 2} \right)^2 \right) = 2k - ab. \quad (4)$$
If \( a + b = \pm (k + 2) \) and \( k = \frac{ab}{2} \), then either \( a = \pm 2 \) or \( b = \pm 2 \). This contradicts our assumptions on \( a \) and \( b \). So,

\[
x^2 = \frac{2k - ab}{1 - \left(\frac{a+b}{k+2}\right)^2}.
\]

Now we choose a generic representation \( \langle X, Y \rangle \) that allows us to avoid the special cases presented above. The following lemma follows immediately from Remark \( 5.3 \), Corollary \( 7.2 \) and the fact that the \( \tau_X, \tau_Y \) actions fix \( k \).

**Lemma 7.4.** For any integer \( J > 0 \), there is \( \gamma \in \Gamma \) such that the \( \tau_Y \)-orbit of \( \gamma(p) \) has at least \( J \) points satisfying the following conditions:

1. The \( x \)-coordinates of these \( J \) points have \( |x| \) small enough so that \( k_0 \pm \frac{\epsilon}{2} \in I_{a,b} \cap I_{x,x} \) and \( |x| \leq \delta \), where \( \delta \) is provided in Lemma \( 6.4 \) for \( E_{(a,b,x,x)} \).
2. These \( J \) points do not belong to the subvarieties defined by equations \( (3), (2) \), \( y = 0 \), and \( w = 0 \).

Note that Condition 2 of Lemma \( 7.4 \) ensures that if \( \tau_W \) fixes \( (k, w, w') \), then \( \langle Y, W \rangle \) is not \( \text{Pin}(2) \) and \( \tau_{W'} \) does not fix \( (k, w, w') \).

Now choose \( \gamma \in \Gamma \) with \( J \) sufficiently large in Lemma \( 7.4 \) so that one of the \( J \) points on the \( \tau_Y \)-orbit of \( \gamma(p) \), denoted \( \gamma_1(p) \), has a \( \tau_W \)-orbit (or \( \tau_{W'} \)-orbit) with at least

\[
N\left(\frac{\delta}{N(\frac{\delta}{2}) + 3}\right) + 3
\]

points with distinct \( k \)-coordinates inside \( E_{(a,b,\gamma_1(x),\gamma_1(x))} \).

Since \( |\gamma_1(x)| \leq \delta \) and \( k_0 \pm \epsilon \in I_{a,b} \cap I_{\gamma_1(x),\gamma_1(x)} \), the sphere \( E_{\kappa} \) (with \( \kappa = (a, b, \gamma_1(x), \gamma_1(x)) \)), has points with \( k \)-coordinates inside \( [k_0 - \frac{\epsilon}{2}, k_0 + \frac{\epsilon}{2}] \).

Since the \( \tau_W \)-orbit (resp. \( \tau_{W'} \)) of \( \gamma_1(p) \) has at least

\[
N\left(\frac{\delta}{N(\frac{\delta}{2}) + 3}\right) + 3
\]

points with distinct \( k \)-coordinates, one such point \( \gamma_2(p) \) has \( k \)-coordinate not in \( K_{N(\frac{\delta}{N(\frac{\delta}{2})+3})} \) with non-degenerate ellipse \( K_{\kappa}(\gamma_2(k)) \subset E_{\kappa} \). Thus, the \( \tau_K \) orbit of \( \gamma_2(w) \) is \( \frac{\delta}{N(\frac{\delta}{2})+3} \)-dense. Thus, the \( \tau_{K'} \)-orbit of \( \gamma_2(p) \) has at least \( N(\frac{\delta}{2}) + 3 \) points with \( w \)-coordinates inside \( (-\delta, \delta) \). One such point, \( \gamma_3(p) \) has \( w \)-coordinate not in \( W_{N(\frac{\delta}{2})} \) with non-degenerate ellipse \( W_{\kappa}(\gamma_3(w)) \). Thus, the \( \tau_W \)-orbit of \( \gamma_3(p) \) is \( \frac{\epsilon}{2} \)-dense in \( W_{\kappa}(\gamma_3(w)) \). This fact, together with the properties of \( \delta \) provided in Lemma \( 6.4 \) imply that at least one point, \( \gamma_4([p]) \), in the \( \tau_W \)-orbit of \( \gamma_3(p) \) has \( k \)-coordinate \( \gamma_4(k) \) that comes within \( \epsilon \) of \( k_0 \).
The one-holed torus \( \langle \gamma_4(X), \gamma_4(Y) \rangle \) is generic so long as \( \gamma_4(k) \not\in S \) and \( \gamma_4(k) \neq (\gamma_4(x))^2 - 2 \) (see Proposition 5.1 and Remark 5.4). This can be accomplished by replacing \( \epsilon \) by \( \frac{\epsilon}{20} \) at the start of the argument. Since \( \langle \gamma_4(X), \gamma_4(Y) \rangle \) is generic, we may apply \( \tau_X \) and \( \tau_Y \) to obtain \( \gamma_5 \in \Gamma \) so that the \( x \)-coordinate \( \gamma_5(x) \) of \( \gamma_5([\rho]) \) is within \( \epsilon \) of \( x_0 \). Note that both \( \tau_X \) and \( \tau_Y \) fix \( \gamma_4(k) \), thus \( \langle \gamma_5(X), \gamma_5(Y) \rangle \) remains generic. The result now follows by Corollary 2.5.

To summarize, one first uses \( \tau_Y \) to obtain points with \( x \)-coordinates near zero having \( \tau_W \) (resp. \( \tau_W' \)) actions that generate points having \( \tau_K \)-actions with a sufficiently large number of points. Next, one uses \( \tau_K \)-action to get the \( w \)-coordinate to be near zero on the moduli space of the four holed sphere obtained by cutting at \( X \). This ensures that one can move the \( k \)-coordinate near \( k_0 \) (see Lemma 6.4). After these twists, the handle \( \langle \gamma(X), \gamma(Y) \rangle \) is shown to remain generic. Finally move the \( x \)-coordinate near \( x_0 \) by using Theorem 1.3.

The moduli space for the two-holed torus is 4-dimensional. By Proposition 7.3, we obtain the following (analogous to Remark 5.3):

**Remark 7.5.** Suppose \( \rho \in \mathcal{M}_C \) is generic with coordinates \( (x, y, k, w) \). Then the orbit \( \Gamma([\rho]) \) is dense in \( \mathcal{M}_C \). Hence there is a number \( r > 0 \) such that the set \( \{(x', y', k', w') : (x', y', k', w') \in \Gamma([\rho])\} \) is dense in a ball of radius \( r \) centered at \( (x, y, k, w) \). In particular, by Dehn twisting in \( \tau_X, \tau_Y, \tau_K, \tau_W \), one can always assume that \( (x, y, k, w) \) does not belong to any proper subvariety of \( \mathcal{M}_C \).

8. The \( m \)-holed torus

As \( m > 2 \) throughout this section, we assume that \( C_i \neq \pm I \) for all boundary curves \( C_i \in \partial M \).

**Proposition 8.1.** Let \( M \) be an \( m \)-holed torus and \( \rho \) a generic representation with generic handle \( \langle X, Y \rangle \), which is part of a pants decomposition \( \mathcal{P} \), i.e., \( K = XXY^{-1}Y^{-1}, X \in \mathcal{P} \). Then there is \( \gamma \in \Gamma \) so that \( \gamma(P) \neq \pm I \) for all \( P \in \mathcal{P} \). Moreover, the handle \( \langle \gamma(X), \gamma(Y) \rangle \) is generic.

**Proof.** We first treat the case of \( m = 3 \). Let \( C \) and \( D \) be two boundary loops separated from \( \langle X, Y \rangle \) by \( B = CD \). Let \( A \) be the remaining boundary loop (see Figure 8). Since \( \langle X, Y \rangle \) is generic, we have that \( K, X \neq \pm I \). The goal, therefore, is to find \( \gamma \in \Gamma \) such that \( \gamma(B) \neq \pm I \).
Suppose $B = \pm I$. Then $C = \pm D^{-1}$ and $A = \pm K$. If $AD = E \neq \pm I$, then we may apply Proposition 7.3 to the two-holed torus bounded by $E$ and $C$. Hence, if $E \neq \pm I$, there exists an element in $\gamma \in \Gamma$ that fixes $a$ with $\gamma(k) \neq \pm a$ such that $\langle \gamma(X), \gamma(Y) \rangle$ generic. This implies $\gamma(B) \neq \pm I$. The same conclusion can be drawn if $AC \neq \pm I$.

Suppose that $AC = \pm I$ and $AD = \pm I$. Since $A = \pm K$, we have that $C, D = \pm K^{-1}$. Hence, $B = CD = \pm K^{-2} = \pm I$. This is only possible if $k = 0$ or $K = \pm I$. The latter case is ruled out by the generic assumption on $\langle X, Y \rangle$. Hence $k = 0$. We will show that in this special case, we can obtain $\gamma(B) \neq \pm I$.

Suppose that $X$ commutes with $K$. If $\tau_Y(X) = XY$ also commutes with $K$, then $X, Y, K$ all belong to the same one parameter subgroup of $\text{SU}(2)$. Hence $X$ and $Y$ commute. Since $\langle X, Y \rangle$ is generic, we have that $XY$ does not commute with $K$. Suppose that $XX^{-1}K^{-1} = -I$ ($X$ and $K$ anti-commute). Then, $X = K(-X)K^{-1}$, which implies that $x = 0$. Again, since $\langle X, Y \rangle$ is generic, we may assume that $x \neq 0$. To summarize, since $\langle X, Y \rangle$ is generic, one may arrange that $X$ neither commutes nor anti-commutes with $K$ (i.e. $XX^{-1}K^{-1} \neq \pm I$).

Now consider the curve $Z = XC$. The Dehn twist in $Z$ preserves $E = \pm I$ and $c$. Hence it fixes $k = 0$ (consider the pants bounded by $C, E,$ and $K$). On the four-holed sphere bounded by $C, D, X$ and $W$, the action of $\tau_Z$ on the curve $B$ is $\tau_Z(B) = C(XX^{-1})D(C^{-1})$, so $\tau_Z(B) = \pm K^{-1}XX^{-1}K^{-1}K^{-1}X^{-1} = K^{-1}XX^{-1} \neq \pm I$. Hence $\gamma(B) \neq \pm I$. Observe that by Remark 5.3 we may assume that $x$ is initially not in $\{0, \pm 1, \pm \frac{1 \pm \sqrt{5}}{2}, \pm \sqrt{2}\}$. Since $\tau_Z$ preserves $k$ and $x$, $\langle \tau_Z(X), \tau_Z(Y) \rangle$
is also generic, note that $\langle \tau_Z(X), \tau_Z(Y) \rangle$ is not $\text{Pin}(2)$ since $k = 0$ and $x \not\in \{\pm \sqrt{2}, 0\}$. This proves the case $m = 3$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9}
\caption{Getting rid of $\pm I$ on $P$ for $m = 5$.}
\end{figure}

The above argument may be repeated iteratively, starting with loops in $P$ that bound two boundary loops and working inward towards $K$. We demonstrate using the case $m = 5$ with notation provided in Figure 9.

First consider the three-holed torus bounded by $E$, $F$, and $ACD$. Since $\langle X, Y \rangle$ is generic, if $B_1 = \pm I$, then $ACD \neq \pm I$ (otherwise $K = \pm I$). Use the previous argument for $m = 3$ to arrange $B_1 \neq \pm I$. Next, the three-holed torus bounded by $B_1$, $D$, and $AC$ is used to make $B_2 \neq \pm I$, etc.

\begin{proposition}
Suppose that $g = 1$, $m = 3$ and $\rho$ is a generic representation. Then the $\Gamma$-orbit $\Gamma([\rho])$ is dense in $\mathcal{M}_C$.
\end{proposition}

\begin{proof}
Let $\epsilon' > 0$. Suppose $[\rho], [\rho_0] \in \mathcal{M}_C$ with $[\rho]$ generic. We adopt the usual practice of omitting the symbol $\rho$ and using the subscript 0 to denote the values of $\rho_0$. We will also adopt the notation of Figure 8. By Proposition \ref{prop:generic}, without loss of generality we may take $[\rho_0] \in \mathcal{M}_P$ and show that there exists $\gamma \in \Gamma$ such that $d(\gamma([\rho]), [\rho_0]) < \epsilon'$.

The map $f_P$ is a submersion on $\mathcal{M}_P$. By the implicit function theorem, there exists $\epsilon' > \epsilon > 0$, such that the $\epsilon$-tubular neighborhood of $f_P^{-1}(x_0, k_0, b_0) \subset \mathcal{M}_C$ looks like $[x_0 - \epsilon, x_0 + \epsilon] \times [k_0 - \epsilon, k_0 + \epsilon] \times [b_0 - \epsilon, b_0 + \epsilon] \times T^3$.
By Proposition 3.1, assume that \( \langle X, Y \rangle \) is generic. By Proposition 8.1, we may assume \( X, K, A, B, C, \) and \( D \) are not \( \pm I \).

Cutting along \( X, K \) and \( B \) yields a pants decomposition. Hence, by Proposition 7.3 and Corollary 2.5, we only need to show that there exists \( \gamma \in \Gamma \) such that \( \gamma \) satisfies \( |\gamma(b) - b_0| < \epsilon \), with \( \langle \gamma(X), \gamma(Y) \rangle \) generic.

The goal is to move \( b \) near \( b_0 \). The strategy is to first move \( x \) and \( w \) near zero so that the moduli space of the four-holed sphere bounded by \( X, W, C, \) and \( D \) contains \( b \)-coordinates near \( b_0 \). At the same time, one needs to ensure that either \( \tau_Z \) or \( \tau_{Z''} \) has enough points on its orbit in \( E_{(c,d,x,w)} \), so that the \( b \)-coordinate can actually be moved near \( b_0 \). This can be accomplished by using the generic \( \langle X, Y \rangle \) to move the \( y \)-coordinate to a general position (i.e., outside of \( Y_N \) for some large \( N \)). This will potentially produce large \( \gamma \)-orbits in \( \langle Y, Z \rangle \) and \( \langle Y, Z'' \rangle \). Under such conditions, the \( \gamma \)-orbit in \( \langle Y, Z \rangle \) and \( \langle Y, Z'' \rangle \) will potentially contain points such that the \( \tau_Z \) and \( \tau_{Z''} \)-orbits at these points have a sufficiently large number of points in \( E_{(c,d,\gamma(x),\gamma(w))} \). For this strategy to work, however, we must first deal with the special case where both \( \langle Y, Z \rangle \) and \( \langle Y, Z'' \rangle \) are \( \text{Pin}(2) \) (see Remark 5.2). In addition, one must also deal with the analogous situation with the four-holed sphere (bounded by \( X, W, C, \) and \( D \)) to ensure that either \( \tau_Z \) or \( \tau_{Z''} \) does not fix the \( b \)-coordinate. We begin with a detailed analysis of these exceptional cases where both of the following hold:

1. \( \langle Y, Z \rangle \) is \( \text{Pin}(2) \) or \( \tau_Z \) fixes \( (b, z, z'') \)
2. \( \langle Y, Z'' \rangle \) is \( \text{Pin}(2) \) or \( \tau_{Z''} \) fixes \( (b, z, z'') \).

Case 1: Both \( \langle Y, Z \rangle \) and \( \langle Y, Z'' \rangle \) are \( \text{Pin}(2) \) and \( y \neq 0 \). This implies that \( z = z'' = 0 \) and \( \text{tr}(YZ''') = \text{tr}(YZ) = 0 \). Note that \( (b, z, z'') \in E_{(x,w,c,d)} \). Note that the application of \( \tau_B \) fixes the moduli space of the two-holed torus bounded by \( A \) and \( B \), i.e., \( \tau_B \) fixes: \( a, b, x, y, k, \) and \( w \). For the Dehn twist \( \tau_B \) to fix \( (b, z, z'') \) with \( z = z'' = 0 \) amounts to

\[
wd + xc = 0
\]

and

\[
xd + wc = 0.
\]

In the special case where both \( c = d = 0 \), the defining equation for \( E_{(x,w,c,d)} \) yields

\[
b^2 = xwb - x^2 - w^2 + 4.
\]
Case 2: $\langle Y, Z'' \rangle$ is Pin(2), while $\langle Y, Z \rangle$ is not. If $\tau_Z$ fixes $(b, z, z'') \in E_{(x, w, c, d)}$ then

$$2b = xw + cd. \tag{9}$$

Case 3: $\langle Y, Z \rangle$ is Pin(2) while $\langle Y, Z'' \rangle$ is not, with $\tau_{Z''}$ fixing $(b, z, z'')$, then $2b = xw + cd$. The argument for this case is symmetric to that of case 2.

Case 4: Neither $\langle Y, Z \rangle$ nor $\langle Y, Z'' \rangle$ is Pin(2). If both $\tau_Z$ and $\tau_{Z''}$ fix $(b, z, z'')$, then

$$\begin{align*}
2b &= xw + cd - zz'' \\
2z &= xd + wc - bz'' \\
2z'' &= xc + wd - bz.
\end{align*} \tag{10}$$

These lead to

$$2b - cd = wx - \frac{[(dw + xc) - \frac{b}{2}(xd + wc)][(xd + wc) - \frac{b}{2}(dw + xc)]}{(2 - \frac{b^2}{2})^2}. \tag{11}$$

A special case of the above equation is when $2b = cd$. Then equation (11) becomes

$$xw = \frac{[(dw + xc) - \frac{b}{2}(xd + wc)][(xd + wc) - \frac{b}{2}(dw + xc)]}{(2 - \frac{b^2}{2})^2}. \tag{12}$$

Note this quadratic equation is degenerate in $w$ only if $c = \pm 2$, $d = \pm 2$, $c = b = 0$, or $d = b = 0$. The case $c = b = 0$ leads to the equations $2z = xd$, $2z'' = wd$, and $zz'' = xw$. Therefore, $4zz'' = xwd^2$, i.e., $d^2 = 4$. Similarly, $d = b = 0$ leads to $c^2 = 4$. Note that the non-degenerate solutions of equation (11) together with $2b = cd$ are:

$$w = cdx/4 \tag{12}$$

and

$$x = cdw/4. \tag{13}$$

Now we carefully choose a generic representation of the two-holed torus bounded by $A$ and $B$ that allows us to avoid the special cases presented above.

Lemma 8.3. For any integer $J > 0$, there is $\gamma \in \Gamma$ such that $\gamma(\rho)$ has $\tau_Y$-orbit containing at least $J$ points satisfying all of the following:

1. Each of the $J$ points have distinct $x$ and $w$-coordinates inside $(-\delta, \delta)$, where $\delta$ is given as in Lemma 6.4 applied to the four-holed sphere that bounds $X, W, C,$ and $D$. 

2. \(|x|, |w|\) are small so that \(b_0 \pm \frac{\varepsilon}{2} \in I_{c,d} \cap I_{x,w}\).

3. If \(c \neq 0\) or \(d \neq 0\), then these \(J\) points do not belong to the subvariety defined by either equation (6) or (7).

4. If \(c = d = 0\), then these \(J\) points do not belong to the subvariety defined by equation (8).

5. The \(J\) points do not belong to the subvariety defined by equation (9).

6. The \(J\) points do not belong to the subvariety defined by equation (10) if \(2b \neq cd\) and do not belong to the subvarieties defined by equations (12) and (13) if \(2b = cd\).

Note that the \(J\) points provided in Lemma 8.3 are obtained by using the mapping class action on the two-holed torus bounded by \(A\) and \(B\). Hence the value \(b \neq \pm 2\) remains fixed.

**Proof.** We begin by noting that the two-holed torus bounded by \(A\) and \(B\) is generic.

Conditions 1 and 2 follow from Proposition 7.3 and Remark 5.3. Note that both \(|x|\) and \(|w|\) can be made simultaneously near zero, since \(x\) can always be made arbitrarily close to zero for any value of \(k\) (see Remark 7.3) and since the \(w\) values of \(E_\kappa\) for \(\kappa = (a, b, x, x')\) take on values inside \(I_{x,a} \cap I_{x,b}\), which, for \(x\) near zero, contains the value \(w = 0\).

By Remark 7.3, for \(y\) having an \(\eta\)-dense orbit for \(\eta\) sufficiently small, we may find a point with a \(\tau_Y\)-orbit with at least \(J\) points not belonging to any of the proper subvarieties described in conditions 3-6. Moreover, after a possible application of \(\tau_B\), we also have that one of \(\langle Y, Z\rangle\) or \(\langle Y, Z''\rangle\) is not Pin(2).

Conditions 3-6 of Lemma 8.3 ensure that for every point on the \(\tau_Y\)-orbit of \(\gamma(\rho)\) either \(\langle Y, Z\rangle\) is not Pin(2) and \(\tau_Z\) does not fix \((b, z, z'')\) or \(\langle Y, Z''\rangle\) is not Pin(2) and \(\tau_{Z''}\) does not fix \((b, z, z'')\). We assume without loss of generality assume the former.

Finally, we start with a point \(\gamma(\rho)\) with \(J\) sufficiently large so that one of the points on the \(\tau_Y\)-orbit of \([\gamma(\rho)]\) has a \(\tau_Z\)-action containing at least

\[
N\left(\frac{\delta}{N\left(\frac{\delta}{2}\right) + 3}\right) + 3
\]

points.

The argument now follows similarly to that presented in Proposition 7.3 to produce \(\gamma \in \Gamma\) by obtaining \(|\gamma(b) - b_0| < \varepsilon\). This completes the proof.
To summarize, one first makes $b \neq \pm 2$. Then uses the fact that the two-hole torus bounded by $A, B$ is generic to move the $x, w$-coordinates near zero. This allows the $b$-coordinate to be moved near $b_0$. Finally the genericity of the two-hole torus allows one to move the $x, w$-coordinates near $x_0, w_0$.

**Proposition 8.4.** Suppose $g = 1$ and $M$ has $m \geq 1$ boundary components with $\rho$ a generic representation. Then the $\Gamma$-orbit $\Gamma([\rho])$ is dense in $M_C$.

**Proof.** The cases $m = 1, 2, 3$ have previously been established. For $m \geq 4$, the argument follows by an induction process similar to that in the proof of Proposition 8.1.

Let $\epsilon' > 0$. Let $\rho$ be a generic representation. By Proposition 3.1, $M$ has a generic handle $\langle X, Y \rangle$

We demonstrate how to proceed in the case $m = 5$ (see the Figure 10 below). By Proposition 2.4, without loss of generality we may take $[\rho_0] \in M_P$ and show that there exists $\gamma \in \Gamma$ such that $d(\gamma([\rho]), [\rho_0]) < \epsilon'$.

The map $f_P$ is a submersion on $M_P$. By the implicit function theorem, there exists $\epsilon' > \epsilon > 0$, such that the $\epsilon$-tubular neighborhood of $f_P^{-1}(\langle x_0, k_0, (b_1)_0, (b_2)_0, (b_3)_0 \rangle) \subset M_C$ looks like $B_\epsilon(\{(x_0, k_0, (b_1)_0, (b_2)_0, (b_3)_0)\}) \times T^5$, where $B_\epsilon(S)$ denotes the $2\epsilon$-open box neighborhood of the set $S$.

---

**Figure 10:** The induction on the 5-holed torus.
By Proposition 8.1, we have that $B_1 \neq \pm I$. By induction, Proposition 8.4 is true for the four-holed torus bounded by $B_1, A, C$ and $D$. We use this four-holed torus to arrange for $x$ and $w$ (see Figure 10) to have near zero traces. This ensures that the $b_1$-coordinate of $[\rho_0]$ is accessible. We then arrange $H \neq \pm I$. We now cut $M$ at $H$ and use Proposition 8.2 to get the $b_1$-coordinate of $\gamma([\rho])$ within $\epsilon$ of $(b_1)_0$. Next, we get the $b_2$-coordinate of $\gamma([\rho])$ within $\epsilon$ of $(b_2)_0$, by using the four-holed torus obtained by cutting at $B_1$. Finally, Proposition 8.2 and Corollary 2.3 yields the result. The situation is identical for any $m > 3$.

9. Genus $g$ with $m$ boundary components

Suppose $M$ is a surface with $g > 1$ and $m \geq 0$. Again, we assume that $C_i \neq \pm I$ for all boundary curves $C_i \in \partial M$, unless $m = 1, g > 1$, and $C_1 = \pm I$.

**Proposition 9.1.** Let $M$ have generic $\langle X, Y \rangle$ that is part of a pants decomposition $\mathcal{P}$. Then there is $\gamma \in \Gamma$ so that $\gamma(\rho(P)) \neq \pm I$, for all $P \in \mathcal{P}$ with $\langle \gamma(X), \gamma(Y) \rangle$ generic. If $m = 1$ and $C_1 = \pm I$, then $\gamma(\rho(P)) \neq \pm I$ for all $P \in \mathcal{P}$ except $C_1$. Furthermore, $\langle \gamma(P), \gamma(Q) \rangle$ is generic, for all $P \in \mathcal{P}$ and $PQP^{-1}Q^{-1} = R \in \mathcal{P}$ that bound a three-holed sphere that forms a one-holed torus in $M$.

![Figure 11: A punctured genus two inside $M$.](image)

**Proof.** We first show that there is $\gamma \in \Gamma$ so that $\langle \gamma(P), \gamma(Q) \rangle$ is generic, for all $P \in \mathcal{P}$ and $PQP^{-1}Q^{-1} = R \in \mathcal{P}$ that bound a three-holed
sphere that forms a one-holed torus in $M$. Consider $P, Q,$ and $R$ as in Figure 11, where $P, R \in \mathcal{P}$ bound a pants that forms a one-holed torus in $M$.

Case 1: Suppose that $R \neq \pm I$. In this case, at least one of $p, q$ or $\text{tr}(PQ)$ is not in $\{0, \pm 2\}$. Without loss of generality, assume that $p \notin \{0, \pm 2\}$ (otherwise, Dehn twist in $Q$). Thus, we may cut $M$ at $P$ and $A$ and apply Proposition 8.2 to the resulting three-holed torus to obtain $r \notin S$ with $r \neq p^2 + 2$. Note that in the special case $A = \pm I$, we apply Proposition 7.3 along with Remark 2.1. Thus the handle $\langle \gamma(P), \gamma(Q) \rangle$ is generic.

Case 2: Suppose that $R = -I$. Then both $P$ and $Q$ are not $\pm I$. Moreover, $A \neq \pm I$ since $K \neq \pm I$. Thus, we may cut $M$ at $P$ and $A$, and then apply Proposition 8.2 to reduce the situation to Case 1.

Case 3: Suppose that $R = I$. Then, necessarily, $A = \pm K \neq \pm I$ since $\langle X, Y \rangle$ is generic. If either $P \neq \pm I$ or $Q \neq \pm I$, we may cut at $P$ (or $Q$) and apply Proposition 8.2 as in case 2. Since $PA = \pm K \neq \pm I$, if both $P = \pm I$ and $Q = \pm I$, then $\tau_{PA}(Q) = Q(PA) = \pm K \neq \pm I$. Note that the action of $\tau_{PA}$ does not affect the generic handle $\langle X, Y \rangle$. We now cut at $Q$ and apply Proposition 8.2 as in case 2. This argument can be applied independently to each of the $g - 1$ handles of $M$.

Having obtained that all handles are generic, we now apply Proposition 8.1 to the remaining curves in $\mathcal{P}$ that are interior to the $(m + g - 1)$-holed torus obtained by cutting off each of the $g - 1$ handles. In the special case $m = 1$ and $C_1 = \pm I$, Proposition 8.1 applies to those curves of $\mathcal{P}$ that are separated from $C_1$ by the curve $KC_1$. Moreover, $KC_1 \neq \pm I$ since $\langle X, Y \rangle$ is assumed generic.
Now we prove Theorem 1.4. We adopt the notation of Figure 12 for the decomposition of $M$. Let $\epsilon' > 0$. Let $\rho$ be a generic representation. By Proposition 3.1, $M$ has a generic handle $\langle X, Y \rangle$ (we adopt the notation presented in Figure 12). By Proposition 2.4, without loss of generality we may take $[\rho_0] \in \mathcal{M}_P$ and show that there exists $\gamma \in \Gamma$ such that $d(\gamma([\rho]), [\rho_0]) < \epsilon'$.

The map $f_P$ is a submersion on $\mathcal{M}_P$. By the implicit function theorem, there exists $\epsilon' > \epsilon > 0$, such that the $\epsilon$-tubular neighborhood of $f_P^{-1}(\beta) \subset \mathcal{M}_{C_i}$, where

$$\beta = (x_0, k_0, (r_1)_0, \ldots, (r_{g-1})_0, (p_1)_0, \ldots, (p_{g-1})_0, (b_1)_0, \ldots, (b_{(m-1)+(g-2)})_0),$$

looks like

$$B_{\epsilon'}(\{\beta\}) \times T^{3g+m-3},$$

where $B_{\epsilon'}(S)$ denotes the $\epsilon$-open box neighborhood of the set $S$.

By Proposition 9.1, we may assume that $p_i \notin \{0, \pm 2\}$ for all $1 \leq i \leq g - 1$. Cutting $M$ along all of the $P_i$’s results in a $(m + 2g - 2)$-holed torus (In the case of $m = 1$, $C_i = \pm I$, and $g > 1$, Remark 2.1 and Proposition 8.4 give us a generic $(2g - 2)$-holed torus).

Dehn twist each generic handle $\langle P_i, Q_i \rangle$ to obtain $p_i$ arbitrarily close to zero. Since $x$ and $z_i = tr(R_i \times X)$ can be made arbitrarily close to zero by Proposition 8.4, we are ensured that the target value $r_i$-coordinate of $\rho_0$ will be inside the $\epsilon$-neighborhood of $I_{p_i, p_i} \cap I_{x, z_i}$. Thus, there are
points on the four-holed sphere $E(p_i,p_i,x,z_i)$ with $r_i$-coordinates within $\epsilon$ of $(r_i)_0$, the $r_i$-coordinate of $\rho_0$.

We next re-apply Proposition 8.4 to the $(m + 2g - 2)$-holed torus to move each $r_i$ within $\epsilon$ of $(r_i)_0$, keeping $\langle \gamma(P_i), \gamma(Q_i) \rangle$ generic by ensuring that $\gamma(p_i) \neq 0$, $\gamma(r_i) \notin S$ and $\gamma(r_i) \neq (\gamma(p_i))^2 - 2$. We start working outward in (as suggested in Figure 12). Next, we twist on each of the $g - 1$ generic handles to get $p_i$ within $\epsilon$ of $(p_i)_0$. Finally, we cut off each handle at $R_i$ and apply Proposition 8.4 to the resulting $(m + g - 1)$-holed torus with generic $\langle X, Y \rangle$.

To summarize, we first moves the $p_i, r_i$-coordinates away from $\pm 2$. Then, we cut along the $P_i$’s and use the resulting generic $(m + 2g - 2)$-holed torus to move each $r_i$-coordinate to make each $\langle P_i, Q_i \rangle$ generic, while preserving the generiticity of $\langle X, Y \rangle$. Next, we use the generic torus $\langle P_i, Q_i \rangle$ to move the $p_i$-coordinates near zero. As $x$ and $z_i = \text{tr}(R_i X)$ can be taken arbitrarily close to zero as well, we see that there are points on the four-holed sphere $E(p_i,p_i,x,z_i)$ with $r_i$-coordinate within $\epsilon$ of $(r_i)_0$. Again by using the generic $(m + 2g - 2)$-holed torus, we may move the $r_i$-coordinate near $(r_i)_0$. After that, we move the $p_i$-coordinate near $(p_i)_0$. Finally cut at the $R_i$’s and use the generic $(m + g - 1)$-holed torus to move the $b_i$-coordinates near $(b_i)_0$ and lastly, the $x,k$-coordinates near $x_0,k_0$. Theorem 1.4 now follows from Corollary 2.3.

Appendix A. Proof of Proposition 3.1

Much of the proof is computational, repetitive, and performed with the aid of MAPLE. However, each sub-case can be worked out by hand. In what follows, we provide several typical examples for the various cases; however the MAPLE files for all cases are available online at http://vortex.bd.psu.edu/~jpp/td2/.

We begin by defining two constants:

\[
\begin{align*}
  r &= \frac{\sqrt{5} + 1}{4} \\
  s &= \frac{\sqrt{5} - 1}{4}.
\end{align*}
\]

Suppose that $\rho \subset \text{Hom}(\pi_1(M,O),\text{SU}(2))$ is generic and $\langle A_i, A_{g+i} \rangle$ is $G$ for some proper subgroup $G \subset \text{SU}(2)$. Suppose that there exists an element $A_j \in \text{SU}(2)$ such that $A_j \not\in G$. If $O$ is chosen to be the intersection point of $A_i$ and $A_{g+i}$, then for any $j$, one may construct a loop corresponding to the fundamental group element $A_{g+i}A_j$ such that the loops $(A_i,A_{g+i}A_j)$ intersect only at $O$. There is an analogous construction for $(A_i,A_jA_{g+i})$, as well as for $(A_iA_j, A_{g+i})$ and $(A_jA_i, A_{g+i})$. 


Lemma A.1. There exists a handle \((A_i, A_{g+i})\) such that either \(A_i \neq \pm I\) or \(A_{g+i} \neq \pm I\).

Proof. Suppose that \(A_i = \pm I\) for all \(1 \leq i \leq 2g\). Since \(\rho\) is generic, there exists \(A_j \neq \pm I\) for \(j > 2g\). The handle \((A_iA_j, A_{g+i})\) has the desired property. 

Lemma A.2. There exists a handle \((A_i, A_{g+i})\) such that \(\rho\big|_{(A_i, A_{g+i})}\) is not \(\text{Spin}(2)\).

Proof. Since \(\rho\) is generic, by the previous Lemma, there exists \(A_i \neq \pm I\) for \(1 \leq i \leq 2g\). Then \(A_i\) defines a one-parameter subgroup \(P\) of \(\text{SU}(2)\). Move \(O\) to the intersection point of \(A_i\) and \(A_{g+i}\). Since \(\rho\) is generic, there exists \(A_j \notin P\). This implies that \(\langle A_i, A_{g+i}A_j \rangle\) is non-Abelian, hence, not \(\text{Spin}(2)\).

Lemma A.3. There exists a handle \((A_i, A_{g+i})\) such that \(\rho\big|_{(A_i, A_{g+i})}\) is not \(\text{Pin}(2)\).

Proof. Suppose \(\langle A_i, A_{g+i} \rangle\) is \(\text{Pin}(2)\), but not \(\text{Spin}(2)\). The group \(\text{Pin}(2)\) acts on \(S^2\) via its quotient \(\text{SO}(3)\) and preserves a circle \(S^1 \subset S^2\). There are two cases.

Case 1: \(\langle A_i, A_{g+i} \rangle\) preserves a unique circle \(S^1 \subset S^2\). Since \(\rho\) is generic, there exists \(A_j\) not preserving \(S^1\). This implies that \(\langle A_i, A_{g+i}A_j \rangle\) does not preserve any circle in \(S^2\), hence, not \(\text{Pin}(2)\).

Case 2: \(\langle A_i, A_{g+i} \rangle\) preserves three circles: \(S_1, S_2, S_3\). Then \(\text{tr}(A_i) = \text{tr}(A_{g+i}) = 0\) and \(\text{tr}(A_iA_{g+i}) = 0\). This implies that \(A_i^2 = -I, A_{g+i}^2 = -I\), and \(A_iA_{g+i} = -A_{g+i}A_i\).

Suppose further that \(\langle A_i, A_{g+i}A_j \rangle\) and \(\langle A_iA_j, A_{g+i} \rangle\) are \(\text{Spin}(2)\) or \(\text{Pin}(2)\) preserving three circles. Then up to conjugacy,

\[
A_i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad A_{g+i} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad A_iA_{g+i} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\]

Let

\[
A_j = \begin{bmatrix} w + ix & y + iz \\ -y + iz & w - ix \end{bmatrix},
\]

with \(w^2 + x^2 + y^2 + z^2 = 1\) be such that \(A_j\) is not contained in the isomorphic copy of \(\langle A_i, A_{g+i} \rangle\).

Since \(\langle A_i, A_{g+i}A_j \rangle\) is either \(\text{Spin}(2)\) or \(\text{Pin}(2)\) preserving three circles, we have that \(A_iA_{g+i}A_j = \pm A_{g+i}A_j A_i\). However, this implies that \(-A_{g+i}A_j = \pm A_{g+i}A_j A_i\), so \(A_iA_j = \pm A_jA_i\). For \(A_iA_j = A_jA_i\), we see that \(y = z = 0\). For \(A_iA_j = -A_jA_i\), we see that \(x = w = 0\).

Likewise, since \(\langle A_iA_j, A_{g+i} \rangle\) is either \(\text{Spin}(2)\) or \(\text{Pin}(2)\) preserving three circles, we have that \(A_{g+i}A_j = \pm A_jA_{g+i}\). The case \(A_{g+i}A_j = \)
$A_j A_{g+i}$ leads to $x = z = 0$, while the case $A_{g+i} A_j = -A_j A_{g+i}$ leads to $w = y = 0$.

Taken together, we see that in all cases three of $w, x, y, z$ are zero. This implies that $A_j$ is equal to one of: $\pm A_i, \pm A_{g+i}, \pm A_i A_{g+i}$, or $\pm I$, which is a contradiction. 

\begin{lemma}
There exists a handle $(A_i, A_{g+i})$ such that $\rho\langle A_i, A_{g+i} \rangle$ is neither $T$ nor $\text{Pin}(2)$.
\end{lemma}

\begin{proof}
By Lemma \ref{lemma:handle1}, there exists $i$ such that $\langle A_i, A_{g+i} \rangle$ is not $\text{Pin}(2)$. Suppose $\langle A_i, A_{g+i} \rangle$ is $T$. By applying Dehn twists, and in light of the analysis carried out in \cite{[?,]}, one may assume that $\rho$ has character $(\text{tr}(A_i), \text{tr}(A_{g+i}), \text{tr}(A_i A_{g+i})) = (1, \pm 1, \pm 1)$ on $(A_i, A_{g+i})$, with signs taken together. We will only address representations with character $(1, 1, 1)$, as the arguments for $(1, -1, -1)$ are practically identical.

Up to conjugation, we may choose:

$$A_i = \frac{1}{2} \begin{bmatrix} 1 - i & 1 - i \\ -1 - i & 1 + i \end{bmatrix}, \quad A_{g+i} = \frac{1}{2} \begin{bmatrix} 1 - i & -1 + i \\ 1 + i & 1 + i \end{bmatrix}. $$

The group $T = \langle A_i, A_{g+i} \rangle$ consists of: $\pm I$, the 16 matrices of the form

$$\pm \frac{1}{2} \begin{bmatrix} 1 \pm i & \pm 1 \pm i \\ \mp 1 \pm i & 1 \mp i \end{bmatrix},$$

(signs in the second row determined by those in the first), and the 6 matrices of the form

$$\pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. $$

Let

$$A_j = \begin{bmatrix} w + ix & y + iz \\ -y + iz & w - ix \end{bmatrix},$$

with $w^2 + x^2 + y^2 + z^2 = 1$ be such that $A_j$ is not contained in the isomorphic copy of $T = \langle A_i, A_{g+i} \rangle$. 

First, consider the case where each of $\langle A_i, A_{g+i} A_j \rangle$, $\langle A_i, A_j A_{g+i} \rangle$, $\langle A_i A_j, A_{g+i} \rangle$, and $\langle A_j A_i, A_{g+i} \rangle$ are $\text{Spin}(2)$. This means that $A_i (A_{g+i} A_j) = (A_{g+i} A_j) A_i = (A_i A_j) A_{g+i}$, which implies that $A_j A_{g+i} = A_{g+i} A_j$. Similarly, $A_j A_i = A_i A_j$. This means that $A_j \in \langle A_i, A_{g+i} \rangle$, which is a contradiction.

Next, suppose that $\langle A_i, A_{g+i} A_j \rangle$ is $\text{Pin}(2)$ and that $\langle A_i A_j, A_{g+i} \rangle$ is $\text{Spin}(2)$. Then $\text{tr}(A_{g+i} A_j) = w + x + y - z = 0$ and $\text{tr}(A_i A_{g+i} A_j) = w + x - y - z = 0$ implies that $y = 0$ and $z = w + x$. The equation

$$\text{tr}(A_i A_j)^2 + \text{tr}(A_{g+i})^2 + \text{tr}(A_i A_j A_{g+i})^2 - \text{tr}(A_i A_j) \text{tr}(A_{g+i}) \text{tr}(A_i A_j A_{g+i}) - 2 = 2,$$
implies that \( z = \pm \frac{\sqrt{2}}{2} \). Finally, \( w^2 + x^2 + z^2 = 1 \) implies that either \( x \) or \( w \) is not real, a contradiction. A similar argument holds in the event that \( \langle A_i, A_{g+i} A_j \rangle \) is \( \text{Spin}(2) \) and \( \langle A_i A_j, A_{g+i} \rangle \) is \( \text{Pin}(2) \).

We indicate how to proceed in the case where \( \langle A_i, A_{g+i} A_j \rangle \) is \( T \) and \( \langle A_i A_j, A_{g+i} \rangle \) is \( \text{Spin}(2) \). Since \( \langle A_i, A_{g+i} A_j \rangle \) is \( T \), both \( \text{tr}(A_{g+i} A_j) \) and \( \text{tr}(A_i A_{g+i} A_j) \) must take on the values \( \pm 1 \) or \( 0 \). The case where both are zero corresponds to a \( \text{Pin}(2) \) representation which was already handled.

Suppose that \( \text{tr}(A_{g+i} A_j) = 0 \) and \( \text{tr}(A_i A_{g+i} A_j) = 1 \). Then \( w + x + y - z = 0 \) and \( w + x - y - z = 1 \), so \( y = -\frac{1}{2} \) and \( z = w + x - \frac{1}{2} \). The equation

\[
\text{tr}(A_i A_j)^2 + \text{tr}(A_i A_j A_{g+i})^2 - \text{tr}(A_i A_j) \text{tr}(A_i A_j A_{g+i}) - 3 = 0
\]

yields \( z = -1 \) or \( \frac{1}{2} \). If \( z = -1 \), then \( w \) is not real. This implies that \( w = \frac{1}{2}, x = \frac{1}{2}, y = -\frac{1}{2}, \) and \( z = \frac{1}{2} \). However, this implies that \( A_j \in \langle A_i, A_{g+i} \rangle \) which is a contradiction. Similar arguments hold in the other 7 cases, as well as the 8 cases where \( \langle A_i, A_{g+i} A_j \rangle \) is \( \text{Spin}(2) \) and \( \langle A_i A_j, A_{g+i} \rangle \) is \( T \).

We now address the remaining cases where \( \langle A_i, A_{g+i} A_j \rangle \) and \( \langle A_i A_j, A_{g+i} \rangle \) are either \( T \) or \( \text{Pin}(2) \). In these cases, \( \text{tr}(A_{g+i} A_j), \text{tr}(A_i A_j), \) and \( \text{tr}(A_i A_{g+i} A_j) \) must take on the values \( \pm 1 \) or \( 0 \).

We explicitly work out a \( T-\text{Pin}(2) \) case in which \( \text{tr}(A_i A_j), \text{tr}(A_i A_{g+i} A_j), \) and \( \text{tr}(A_{g+i} A_j) \) are all zero. These lead to the equations

\[
\begin{align*}
w + x - y - z &= 0 \\
w + x - y + z &= 0 \\
w + x + y - z &= 0 \\
w^2 + x^2 + y^2 + z^2 &= 1
\end{align*}
\]

Therefore, \( y = 0, z = 0 \) and \( w = -x \), which implies that \( x = \pm \frac{\sqrt{2}}{2} \), and \( w = \mp \frac{\sqrt{2}}{2} \). However, the trace of the matrix \( A_i A_{g+i} A_j \) is \( \pm \sqrt{2} \). Thus the handle \( \langle A_i, A_{g+i} A_j \rangle \) is neither \( T \) nor \( \text{Pin}(2) \). (This handle is obtained by first twisting in \( A_i \) on \( \langle A_i, A_{g+i} \rangle \), then using the loop \( A_j \).

The remaining 26 cases are obtained by varying the values of \( \text{tr}(A_i A_j), \text{tr}(A_i A_{g+i} A_j), \) and \( \text{tr}(A_{g+i} A_j) \) in \( \{0, \pm 1\} \). The arguments proceed similarly to those exemplified above and lead to:

1. matrices \( A_j \) inside \( \langle A_i, A_{g+i} \rangle \) (a contradiction),
2. a handle \( \langle A_i, A_{g+i} A_j \rangle \) that is neither \( T \) nor \( \text{Pin}(2) \),
3. one or more of \( w, x, y, z \) is not real (a contradiction).

\( \square \)
Lemma A.5. There exists a handle \((A_i, A_{g+i})\) such that \(\rho|_{A_i, A_{g+i}}\) is neither \(C\) nor \(\text{Pin}(2)\).

Proof. Assume that \((A_i, A_{g+i})\) is \(C\) but neither \(T\) nor \(\text{Pin}(2)\). By applying Dehn twists and in light of the analysis carried out in \cite{[8]}, one may assume that \(\rho\) has character \((\text{tr}(A_i), \text{tr}(A_{g+i}), \text{tr}(A_i A_{g+i})) = (1, \pm 1, \pm 1)\) on \((A_i, A_{g+i})\), with signs taken together. We will only address representations with character \((\sqrt{2}, 1, \sqrt{2})\), as the arguments for \((\sqrt{2}, -1, -\sqrt{2})\) are practically identical.

Up to conjugation, we may choose

\[
A_i = \frac{1}{2} \begin{bmatrix}
\sqrt{2} + \sqrt{2}i & 0 \\
0 & \sqrt{2} - \sqrt{2}i
\end{bmatrix}, \quad A_{g+i} = \frac{1}{2} \begin{bmatrix}
1 - i & -1 - i \\
1 - i & 1 + i
\end{bmatrix}.
\]

The group \(C = \langle A_i, A_{g+i} \rangle\) consists of: the 24 tetrahedral matrices in the previous lemma, the four matrices

\[
\pm \frac{1}{2} \begin{bmatrix}
\sqrt{2} \pm \sqrt{2}i & 0 \\
0 & \sqrt{2} \mp \sqrt{2}i
\end{bmatrix}
\]

(signs in the second row determined by those in the first), and the 20 matrices obtained by permuting two non-zero terms \((\sqrt{2})\) and two zero terms \((0 + 0i)\) in the first row of the matrix. One such example is:

\[
\frac{1}{2} \begin{bmatrix}
\sqrt{2}i & \sqrt{2} \\
-\sqrt{2} & -\sqrt{2}i
\end{bmatrix}.
\]

Let

\[
A_j = \begin{bmatrix}
w + ix & y + iz \\
-y + iz & w - ix
\end{bmatrix},
\]

with \(w^2 + x^2 + y^2 + z^2 = 1\) be such that \(A_j\) is not contained in the isomorphic copy of \(C\) generated by \(\langle A_i, A_{g+i} \rangle\). Note that \(\text{tr}(A_i A_j) = \sqrt{2}w - \sqrt{2}x, \text{tr}(A_{g+i} A_j) = w + x + y + z, \text{tr}(A_i A_j A_{g+i}) = \sqrt{2}w + \sqrt{2}y, \text{and tr}(A_j A_{g+i} A_j) = \sqrt{2} w + \sqrt{2} z\).

If each of \(\langle A_i, A_{g+i} A_j \rangle, \langle A_i, A_{g+i} A_{g+i} \rangle, \langle A_i A_j, A_{g+i} \rangle, \text{and \langle A_j A_i, A_{g+i} \rangle are Spin(2)}\), then the argument produced in the previous lemma applies.

Suppose that \(\langle A_i, A_{g+i} A_j \rangle\) is \(\text{Pin}(2)\) and \(\langle A_i A_j, A_{g+i} \rangle\) is \(\text{Spin}(2)\). This leads to the system of equations: \(w + x + y + z = 0, \sqrt{2}w + \sqrt{2}z = 0, w^2 + x^2 + y^2 + z^2 = 1, \text{and tr}(A_i A_j)^2 + \text{tr}(A_{g+i})^2 + \text{tr}(A_i A_j A_{g+i})^2 - \text{tr}(A_i A_j) \text{tr}(A_i A_j A_{g+i}) - 2 = 2\).

This system has non-real solutions. A similar argument holds in the event that \(\langle A_i, A_{g+i} A_j \rangle\) is \(\text{Spin}(2)\) and \(\langle A_i A_j, A_{g+i} \rangle\) is \(\text{Pin}(2)\).

We indicate how to proceed in the case where \(\langle A_i, A_{g+i} A_j \rangle\) is \(\text{Spin}(2)\) and \(\langle A_i A_j, A_{g+i} \rangle\) is \(C\). Since \(\langle A_i, A_{g+i} A_j \rangle\) is \(C\), \(\text{tr}(A_{g+i} A_j)\) and \(\text{tr}(A_i A_{g+i} A_j)\)
must take on the values $\pm \sqrt{2}, \pm 1$ or 0. The case where both are zero reduces to $\text{Pin}(2)$.

We explicitly work out the case $\text{tr}(A_iA_j) = \sqrt{2}$, $\text{tr}(A_iA_jA_{g+i}) = 0$, and $\langle A_i, A_{g+i}A_j \rangle$ is $\text{Spin}(2)$. This leads to the equations

$$\sqrt{2}w - \sqrt{2}x = \sqrt{2}$$
$$\sqrt{2}w + \sqrt{2}y = 0$$
$$w^2 + x^2 + y^2 + z^2 = 1$$

and

$$\text{tr}(A_{g+i}A_j)^2 + \text{tr}(A_iA_{g+i}A_j)^2 - \sqrt{2} \text{tr}(A_{g+i}A_j) \text{tr}(A_iA_{g+i}A_j) = 2.$$  

This system has complex and real solutions, with real solution $(w, x, y, z) = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which implies that $A_j \in \langle A_i, A_{g+i} \rangle$, a contradiction.

Similar arguments hold in the remaining 23 cases obtained by assigning the values $\{0, \pm 1, \pm \sqrt{2}\}$ to $\text{tr}(A_iA_j)$ and $\text{tr}(A_iA_jA_{g+i})$, as well as the 24 cases where $\langle A_i, A_{g+i}A_j \rangle$ is $C$ and $\langle A_iA_j, A_{g+i} \rangle$ is $\text{Spin}(2)$. All cases yield systems with only non-real solutions, or matrices $A_j \in \langle A_i, A_{g+i} \rangle$.

This leaves cases where $\langle A_i, A_{g+i}A_j \rangle$ and $\langle A_iA_j, A_{g+i} \rangle$ are either $C$ or $\text{Pin}(2)$. In these cases, $\text{tr}(A_{g+i}A_j)$, $\text{tr}(A_iA_j)$, and $\text{tr}(A_iA_{g+i}A_j)$ must take on the values $\pm \sqrt{2}, \pm 1$ or 0.

We will work out the case in which $\text{tr}(A_iA_j) = \sqrt{2}$, $\text{tr}(A_{g+i}A_j) = 1$, and $\text{tr}(A_iA_{g+i}A_j) = 1$. The solutions to the associated system are:

$$w = \frac{3}{4} \pm \frac{\sqrt{-11 + 8\sqrt{2}}}{4},$$
$$x = -\frac{1}{4} \pm \frac{\sqrt{-11 + 8\sqrt{2}}}{4},$$
$$y = -\frac{1}{2} \sqrt{2} + \frac{5}{4} \pm \frac{\sqrt{-11 + 8\sqrt{2}}}{4},$$
$$z = \frac{1}{2} \sqrt{2} - \frac{3}{4} \pm \frac{\sqrt{-11 + 8\sqrt{2}}}{4},$$

with signs taken together.

However, the trace of $A_iA_{g+i}A_iA_j$ is $\frac{3}{2} \pm \frac{\sqrt{-11 + 8\sqrt{2}}}{2}$. So the handle $\langle A_i, A_{g+i}A_iA_j \rangle$ is neither $C$ nor $\text{Pin}(2)$.

The remaining 123 cases are obtained by varying the values of $\text{tr}(A_iA_j)$, $\text{tr}(A_iA_{g+i}A_j)$, and $\text{tr}(A_{g+i}A_j)$ in $\{0, \pm 1, \pm \sqrt{2}\}$. The arguments proceed similarly to those exemplified above and lead to:

1. matrices $A_j$ inside $\langle A_i, A_{g+i} \rangle$ (a contradiction),
2. a handle $\langle A_i, A_{g+i}A_iA_j \rangle$ that is neither $C$ nor $\text{Pin}(2)$,
3. one or more of $w, x, y, z$ is not real (a contradiction). \hfill \Box

Lemma A.6. There exists a handle $(A_i, A_{g+i})$ such that $\rho|_{\langle A_i, A_{g+i} \rangle}$ is neither $D$ nor $C$ nor $\text{Pin}(2)$.

Proof. Assume that $\langle A_i, A_{g+i} \rangle$ is $D$. By applying Dehn twists, by the fact that $-I$ is in the center of $SU(2)$, and by the previous lemmas, we may assume one of the following 3 cases:

1. $\text{tr}(A_i) = -2s$, $\text{tr}(A_{g+i}) = 2s$, and $\text{tr}(A_iA_{g+i}) = 2s$.
2. $\text{tr}(A_i) = 2r$, $\text{tr}(A_{g+i}) = -2r$, and $\text{tr}(A_iA_{g+i}) = -2r$.
3. $\text{tr}(A_i) = 1$, $\text{tr}(A_{g+i}) = 2r$, and $\text{tr}(A_iA_{g+i}) = -1$.

Case 1:

By the uniqueness of coordinates, let

$$A_i = \begin{bmatrix} -s + \frac{1}{2}i & \frac{ir}{i} \\ \frac{ir}{i} & s - \frac{1}{2}i \end{bmatrix}, \quad A_{g+i} = \begin{bmatrix} s & -r + \frac{1}{2}i \\ r + \frac{1}{2}i & s \end{bmatrix}.$$  

The group $\langle A_i, A_{g+i} \rangle$ consists of: the 24 tetrahedral matrices (see above lemma), the 8 matrices

$$\pm \begin{bmatrix} s \pm ri & \pm \frac{1}{2} \\ \mp \frac{1}{2} & s \mp ri \end{bmatrix},$$

the 8 matrices

$$\pm \begin{bmatrix} s \pm \frac{1}{2}i & \pm ri \\ \pm ri & s \mp \frac{1}{2}i \end{bmatrix},$$

the 8 matrices

$$\pm \begin{bmatrix} s \mp ri \pm \frac{1}{2}i & s \\ \mp r \pm \frac{1}{2}i & \mp \frac{1}{2} \end{bmatrix},$$

(signs in the second row determined by those in the first), and the 72 matrices obtained by multiplying those listed above by:

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$  

Let

$$A_j = \begin{bmatrix} w + ix & y + iz \\ -y + iz & w - ix \end{bmatrix},$$

with $w^2 + x^2 + y^2 + z^2 = 1$ be such that $A_j$ is not contained in the isomorphic copy of $D$ generated by $\langle A_i, A_{g+i} \rangle$. Note that $\text{tr}(A_iA_j) = -2sw + x - 2rz$, $\text{tr}(A_iA_jA_{g+i}) = -w + 2rx + 2sz$, $\text{tr}(A_{g+i}A_j) = 2sw + 2ry - z$, and $\text{tr}(A_iA_{g+i}A_j) = -w - x - y - z$.

If each of $\langle A_i, A_{g+i}A_j \rangle$, $\langle A_i, A_jA_{g+i} \rangle$, $\langle A_iA_j, A_{g+i} \rangle$, and $\langle A_jA_i, A_{g+i} \rangle$ are $\text{Spin}(2)$, then the argument produced in Lemma A.4 applies.
Suppose that $\langle A_i, A_{g+i} A_j \rangle$ is Spin(2) and $\langle A_i, A_{g+i} \rangle$ is Pin(2). This leads to the equations

$$-2sw + x - 2rz = 0,$$
$$-w + 2rx + 2sz = 0,$$
$$w^2 + x^2 + y^2 + z^2 = 1,$$

and

$$\text{tr}(A_{g+i} A_j)^2 + \text{tr}(A_i)^2 + \text{tr}(A_i A_{g+i} A_j)^2 - \text{tr}(A_{g+i} A_j) \text{tr}(A_i) \text{tr}(A_i A_{g+i} A_j) - 2 = 2.$$

The solutions are $(w, x, y, z) = \pm (\frac{1}{2}, -r, 0, s)$. However, $A_j$ multiplied on the right by

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

is one of the matrices listed in $\langle A_i, A_{g+i} \rangle$, so $A_j \in \langle A_i, A_{g+i} \rangle$.

A similar argument holds in the case where $\langle A_i, A_{g+i} A_j \rangle$ is Pin(2) and $\langle A_i A_j, A_{g+i} \rangle$ is Spin(2).

We indicate how to proceed in the case where $\langle A_i, A_{g+i} A_j \rangle$ is Spin(2) and $\langle A_i A_j, A_{g+i} \rangle$ is $D$. Since $\langle A_{g+i}, A_i A_j \rangle$ is in $D$, $\text{tr}(A_i A_j)$ and $\text{tr}(A_i A_{g+i} A_j)$ must take on the values $\pm 2s$, $\pm 2r$, $\pm 1$ or 0. The case where both are zero reduces to a Pin(2)-Spin(2) case.

We will explicitly work out the case $\text{tr}(A_i A_j) = 2r$ and $\text{tr}(A_i A_j A_{g+i}) = -2s$. This leads to the system:

$$-2sw + x - 2rz = 2r,$$
$$-w + 2sx + 2rz = -2s,$$
$$w^2 + x^2 + y^2 + z^2 = 1,$$

and

$$\text{tr}(A_{g+i} A_j)^2 + \text{tr}(A_i A_{g+i} A_j)^2 + \text{tr}(A_i)^2 - \text{tr}(A_i) \text{tr}(A_{g+i} A_j) \text{tr}(A_i A_{g+i} A_j) = 4.$$

These equations have no real solutions.

Similar arguments hold in the other 47 cases and the 48 cases where $\langle A_i, A_{g+i} A_j \rangle$ is $D$ and $\langle A_i A_j, A_{g+i} \rangle$ is Spin(2). Note that some of these cases yield matrices $A_j$ that are in $\langle A_i, A_{g+i} \rangle$.

This leaves cases where $\langle A_i, A_{g+i} A_j \rangle$ and $\langle A_i A_j, A_{g+i} \rangle$ are either $D$ or Pin(2). In these cases, $\text{tr}(A_{g+i} A_j)$, $\text{tr}(A_i A_j)$, and $\text{tr}(A_i A_{g+i} A_j)$ must take on the values $\pm 1, \pm 2s, \pm 2r$, or 0.

First, we explicitly work out a special $D$-Pin(2) case in which $\text{tr}(A_i A_j)$, $\text{tr}(A_i A_j A_{g+i})$, and $\text{tr}(A_{g+i} A_j)$ are all zero. The solutions are $w = \mp r$, $x = 0$, $y = \pm \frac{1}{2}$, and $z = \pm s$ (signs taken together). However, the matrix $A_j$ is in $\langle A_i, A_{g+i} \rangle$, a contradiction.

We also explicitly work out the case $\text{tr}(A_i A_j) = 2r$, $\text{tr}(A_{g+i} A_j) = -2s$, and $\text{tr}(A_i A_{g+i} A_j) = 0$. The three resulting equations yield $x = 1$. 
Thus, since \( w^2 + x^2 + y^2 + z^2 = 1 \), we see that \( w = y = z = 0 \) which violates \( \text{tr}(A_{g+i}A_j) = -2s \).

The remaining 341 cases are obtained by varying the values of \( \text{tr}(A_iA_j) \), \( \text{tr}(A_iA_{g+i}A_j) \), and \( \text{tr}(A_{g+i}A_j) \) in \( \{0, \pm 1, \pm 2r, \pm 2s\} \). The arguments proceed similarly to those exemplified above and lead to:

1. matrices \( A_j \) inside \( \langle A_i, A_{g+i} \rangle \) (a contradiction),
2. a handle \( \langle A_i, A_{g+i}A_iA_j \rangle \) that is not \( \text{Pin}(2) \) nor \( C \) nor \( D \),
3. one or more of \( w, x, y, z \) is not real (a contradiction).

Case 2:

By the uniqueness of coordinates, let

\[
A_i = \begin{bmatrix} r - si & -\frac{i}{2} \\ -\frac{i}{2} & r + si \end{bmatrix}, \quad A_{g+i} = \begin{bmatrix} -r + \frac{i}{2} & -s \\ s & -r - \frac{i}{2} \end{bmatrix}.
\]

Then \( \langle A_i, A_{g+i} \rangle \) yields the 120 matrices in case 1. A study of each of the \( \text{Pin}(2) \)-\( \text{Spin}(2) \), \( D \)-\( \text{Spin}(2) \), and \( D \)-\( D \) cases follows as in the previous case. As usual, each case leads to:

1. matrices \( A_j \) inside \( \langle A_i, A_{g+i} \rangle \) (a contradiction),
2. a handle \( \langle A_i, A_{g+i}A_iA_j \rangle \) that is not \( \text{Pin}(2) \) nor \( C \) nor \( D \),
3. one or more of \( w, x, y, z \) is not real (a contradiction).

Case 3:

By the uniqueness of coordinates, let

\[
A_i = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}i & -\frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2} - \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \end{bmatrix}, \quad A_{g+i} = \begin{bmatrix} r + si & \frac{1}{2}i \\ \frac{1}{2}i & r - si \end{bmatrix}.
\]

Then \( \langle A_i, A_{g+i} \rangle \) yields the 120 matrices in case 1. A study of each of the \( \text{Pin}(2) \)-\( \text{Spin}(2) \), \( D \)-\( \text{Spin}(2) \), and \( D \)-\( D \) cases follows as in case 1. As usual, each case leads to:

1. matrices \( A_j \) inside \( \langle A_i, A_{g+i} \rangle \) (a contradiction),
2. a handle \( \langle A_i, A_{g+i}A_iA_j \rangle \) that is not \( \text{Pin}(2) \) nor \( C \) nor \( D \),
3. one or more of \( w, x, y, z \) is not real (a contradiction).

\[\square\]

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