Relativistic gas in a Schwarzschild metric

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Abstract. A relativistic gas in a Schwarzschild metric is studied within the framework of a relativistic Boltzmann equation in the presence of gravitational fields, where Marle’s model for the collision operator of the Boltzmann equation is employed. The transport coefficients of the bulk and shear viscosities and thermal conductivity are determined from the Chapman–Enskog method. It is shown that the transport coefficients depend on the gravitational potential. Expressions for the transport coefficients in the presence of weak gravitational fields in the non-relativistic (low temperature) and ultra-relativistic (high temperature) limiting cases are given. Apart from the temperature gradient the heat flux has two relativistic terms. The first one, proposed by Eckart, is due to the inertia of energy and represents an isothermal heat flux when matter is accelerated. The other, suggested by Tolman, is proportional to the gravitational potential gradient and indicates that—in the absence of an acceleration field—a state of equilibrium of a relativistic gas in a gravitational field can be attained only if the temperature gradient is counterbalanced by a gravitational potential gradient.

Keywords: Boltzmann equation
1. Introduction

The research on relativistic gases by using the Boltzmann equation is an old subject in the literature. We can state that the statistical description of a relativistic gas began with the works of Jüttner in 1911 and 1926, when he succeeded in deriving the equilibrium distribution functions for a relativistic gas that obeys the Maxwell–Boltzmann [1], Fermi–Dirac and Bose–Einstein statistics [2]. The covariant formulation of the Boltzmann equation came later and was first proposed by Lichnerowicz and Marrot [3] in 1940. The establishment of the non-equilibrium distribution function and of the transport coefficients of a relativistic gas by using the Chapman–Enskog method was achieved by Israel [4] and Kelly [5] in the 1960s. The work of these pioneers was followed by several research papers in the literature concerning the study of gases in non-equilibrium states by using the Boltzmann equation in special relativity. However, there exist only few papers in the literature concerning research on relativistic gases in the presence of gravitational fields within the framework of the Boltzmann equation. The first works were due to Chernikov [6, 7], who analyzed the equilibrium distribution functions in some specific metric tensors, and Bernstein [8], who determined the bulk viscosity for a relativistic gas in a Friedmann–Robertson–Walker metric.

Some years ago the influence of gravity on the heat transport in a rarefied gas was analyzed in the works [9, 10] on the basis of a non-relativistic Boltzmann equation, where it was shown that a gravitational field parallel (anti-parallel) to the temperature gradient increases (decreases) the heat transport. These results are related with the one derived by Tolman [11, 12] from a general relativity theory, who showed that a state of equilibrium of a relativistic gas in a gravitational field can be attained only if the temperature gradient is counterbalanced by a gravitational potential gradient.
Recently, a relativistic gas in a gravitational field was studied by using the Boltzmann equation in the work [13], where the dependence of the heat flux on the gravity—the so-called Tolman’s law—was obtained. In this work no metric tensor was assigned—although the components of the metric tensor appear explicitly in the equilibrium distribution function—and the dependence on the gravitational field was connected with the Christoffel symbol written in the Newtonian approximation.

The aim of this work is to analyze a relativistic gas in a Schwarzschild metric within the framework of the Boltzmann equation in the presence of gravitational fields. The Marle model [14, 15] of the collision operator of the Boltzmann equation is used and the transport coefficients of the shear and bulk viscosities and thermal conductivity are determined by the method of Chapman–Enskog. It is shown that the transport coefficients in the presence of a gravitational field are larger in comparison to their values in the absence of it. Furthermore, the heat flux obtained has two relativistic terms, aside from the temperature gradient. One of them—caused by the inertia of energy—represents an isothermal heat flux when matter is accelerated and was suggested by Eckart [16]. The other is proportional to the gravitational potential gradient and was proposed by Tolman [11, 12]. It expresses that a state of equilibrium of a relativistic gas in a gravitational field and in the absence of an acceleration field can be attained only if the temperature gradient is counterbalanced by a gravitational potential gradient.

This paper is structured as follows. In section 2 the Marle model of the Boltzmann equation and the equilibrium distribution function in the Schwarzschild metric are introduced as well as the definitions and balance equations for the particle four-flow and energy–momentum tensor. The non-equilibrium distribution function, obtained by the use of the Chapman–Enskog method, is the subject of section 3. In this section the Euler equations of a relativistic gas in a gravitational field are obtained and the constitutive equations with the corresponding transport coefficients are determined for a viscous heat-conducting relativistic gas. In section 4 we state the main conclusions of this work and in the appendices we introduce the isotropic Schwarzschild metric and show how to evaluate the integrals when the distribution function depends on the components of the metric tensor.

2. Boltzmann equation

Let us consider a single non-degenerate gas in a Riemannian space with line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, where $g_{\mu\nu}$ is the metric tensor. A particle with rest mass $m$ is characterized by the space–time coordinates $(x^\mu) = (x^0 = ct, \mathbf{x})$ and by the momentum four-vector $(p^\mu) = (p_0, \mathbf{p})$, where $c$ denotes the speed of light.

The constant length of the momentum four-vector $g_{\mu\nu} p^\mu p^\nu = m^2 c^2$ implies that $p_0 = p_0 - g_{0i} p^i / g_{00}$, where

$$p_0 = \sqrt{g_{00} m^2 c^2 + (g_{0i} g_{0j} - g_{00} g_{ij}) p^i p^j}. \quad (1)$$

The components of the four-velocity with $U^\mu U_\mu = c^2$ are

$$U^\mu = (\Gamma c, \Gamma \mathbf{v}), \quad \Gamma = \frac{1}{\sqrt{g_{00} (1 + (g_{0i} / g_{00}) (v^i / c))^2 - (v^2 / c^2)}}, \quad (2)$$

and in a comoving frame $\mathbf{v} = 0$ yield $U^\mu = (c / \sqrt{g_{00}}, 0)$. 

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For the isotropic Schwarzschild metric (A.2) we have
\[ p^0 = \frac{p_0}{g_0}, \quad p_0 = \sqrt{g_0}\sqrt{m^2c^2 + g_1|p|^2}, \tag{3} \]
\[ U^\mu = \left( \frac{c}{\sqrt{g_0}}, 0 \right), \quad \sqrt{-g} = \sqrt{g_0g_1}, \tag{4} \]
where \( g = \det(g_{\mu\nu}) \).

In the phase space spanned by the space–time and momentum coordinates, the state of a relativistic gas is characterized by the one-particle distribution function \( f(x^\alpha, p^\alpha) = f(x, p, t) \), such that \( f(x, p, t) \, d^3x \, d^3p \) gives the number of particles in the volume element \( d^3x \) about \( x \) and with momenta in a range \( d^3p \) about \( p \) at time \( t \).

The space–time evolution of the one-particle distribution function is governed by the Boltzmann equation. Here, for simplicity, we shall use a model equation for the Boltzmann equation, which replaces the collision term \( Q(f, f) \) of the Boltzmann equation by a collision model \( J(f) \). Two important model equations for relativistic gases were proposed by Marle [14, 15] and Anderson and Witting [17]. The Marle model equation is given by
\[ J(f) = -\frac{m}{\tau}(f - f^{(0)}), \tag{5} \]
while the Anderson–Witting model equation reads
\[ J(f) = -\frac{U_L^\mu p_\mu}{c^2\tau}(f - f^{(0)}). \tag{6} \]

In the above equations, \( \tau \) denotes a mean free time and \( f^{(0)} \) the Maxwell–Jüttner distribution function. In the Marle model, the Eckart decomposition for the particle four-flow and energy–momentum tensor is used, while in the Anderson and Witting model, the Landau–Lifshitz decomposition for these fields is employed. Note that in (6) the four-velocity \( U_L \) refers to the Landau–Lifshitz decomposition.

The collision terms of the model equations must fulfill the same properties as those of the true collision term of the Boltzmann equation, namely,
(1) for the summational invariant \( \psi = p^\mu \), \( Q(f, f) \) and \( J(f) \) must satisfy the relationship
\[ \int \psi Q(f, f) \frac{d^3p}{p_0} = 0, \quad \text{hence} \quad \int \psi J(f) \frac{d^3p}{p_0} = 0; \tag{7} \]
(1) the \( \mathcal{H} \)-theorem, or equivalently the tendency of the one-particle distribution function to the equilibrium distribution function, must hold,
\[ \int Q(f, f) \ln f \frac{d^3p}{p_0} \leq 0, \quad \text{hence} \quad \int J(f) \ln f \frac{d^3p}{p_0} \leq 0. \tag{8} \]

For the proof of these properties one is referred, e.g., to the book [18].

Here, we shall use the Marle model of the Boltzmann equation, which in the presence of gravitational fields reads
\[ p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = -\frac{m}{\tau}(f - f^{(0)}). \tag{9} \]
Above, $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols, and in a comoving frame the Maxwell–Jüttner distribution function becomes
\[
f^{(0)} = \frac{n}{4\pi kTm^2cK_2(\zeta)} \exp \left(-\frac{c\sqrt{m^2c^2 + g_1|p|^2}}{kT}\right),
\] (10)
thanks to $(3)_2$ and $(4)_1$. Here, $n, T$ and $k$ represent the particle number density, temperature and Boltzmann’s constant, respectively. Furthermore, $K_2(\zeta)$ is the modified Bessel function of the second kind,
\[
K_2(\zeta) = \left(\frac{\zeta}{2}\right)^n \frac{\Gamma(1/2)}{\Gamma(n+1/2)} \int_1^\infty e^{-\zeta y} (y^2 - 1)^{n-1/2} dy,
\] (11)
which is a function of $\zeta = mc^2/kT$, representing the ratio of the particle rest energy $mc^2$ and the thermal energy of the gas $kT$. The limiting case $\zeta \gg 1$ corresponds to a gas in the non-relativistic regime at low temperatures, while $\zeta \ll 1$ refers to an ultra-relativistic gas at high temperatures.

The macroscopic state of a relativistic gas in a gravitational field may be described by the two first moments of the distribution function, the particle four-flow $N^\mu$ and the energy–momentum tensor $T^\mu_{\nu}$. Their definitions in terms of the one-particle distribution function are given by
\[
N^\mu = c \int p^\mu f \sqrt{-g} \frac{d^3 p}{p_0}, \quad T^\mu_{\nu} = c \int p^\mu p^\nu f \sqrt{-g} \frac{d^3 p}{p_0}.
\] (12)
The respective balance equations are obtained from the Boltzmann equation, yielding
\[
N^\mu_{;\mu} = 0, \quad T^\mu_{\nu;\mu} = 0,
\] (13)
where the semicolon denotes a covariant derivative.

It is usual in the thermodynamic theory of relativistic fluids to decompose the particle four-flow and the energy–momentum tensor in terms of quantities that appear in the theory of non-relativistic fluid dynamics, namely, particle number density $n$, energy per particle $e$, hydrostatic pressure $p$, non-equilibrium pressure $\varpi$, pressure deviator $P^\mu_{\nu}$ (the traceless part of the pressure tensor) and heat flux $q^\mu$. For Marle’s model equation the decomposition used is that of Eckart (see, e.g., [16, 18]),
\[
N^\mu = nU^\mu,
\] (14)
\[
T^\mu_{\nu} = P^\mu_{\nu} - (p + \varpi) \Delta^\mu_{\nu} + \frac{1}{c^2} (q^\mu U^\nu + q^\nu U^\mu) + \frac{en}{c^2} U^\mu U^\nu,
\] (15)
where $\Delta^\mu_{\nu}$ is the projector
\[
\Delta^\mu_{\nu} = g^\mu_{\nu} - \frac{1}{c^2} U^\mu U^\nu.
\] (16)

From the insertion of the Maxwell–Jüttner distribution function (10) into the definition of the energy–momentum tensor (12)$_2$, and integration of the resulting equation, it follows that the energy per particle and the hydrostatic pressure read
\[
e = mc^2 \left[\frac{K_2}{K_2 - \frac{1}{\zeta}}\right], \quad p = nkT,
\] (17)
respectively.
3. Chapman–Enskog method

In the Chapman–Enskog method the distribution function is written as \( f = f^{(0)} (1 + \varphi) \), where \( \varphi \)—the deviation from the Maxwell–Jüttner distribution function—is considered as a small quantity, i.e., \( |\varphi| < 1 \). Furthermore, the Maxwell–Jüttner distribution function is inserted on the left-hand side of the Boltzmann equation (9) and the representation \( f = f^{(0)} (1 + \varphi) \) on its right-hand side. By performing the derivatives it follows that

\[
- \frac{m}{\tau} \left( f - f^{(0)} \right) = f^{(0)} \left\{ \frac{p'}{n} \frac{\partial n}{\partial x^\nu} + \frac{p'}{T} \left[ \frac{1}{K_2} - \frac{p' U_{\tau}}{K_2 T} \right] \frac{\partial T}{\partial x^\nu} 
- \frac{p' p'}{kT} \frac{\partial U_{\tau}}{\partial x^\nu} - \frac{c^2}{2kT} \frac{dg}{dr} p' p' p' \delta_{ij} \delta_{kl} \frac{x^j}{r} + \frac{c^2}{kT} g_{ji} \Gamma_{\nu}^{i} \frac{p' p' p'}{U_{\tau} U_{\tau}} + \frac{c}{2 \sqrt{-g} k} \frac{dg_0}{dr} \delta_{ij} \frac{x^j}{r} \right\} = - \frac{m}{\tau} f^{(0)} \varphi.
\]

Hence, the deviation from the Maxwell–Jüttner distribution \( \varphi \) is determined as a function of gradients and derivatives of the metric tensor components.

3.1. Euler’s equations

A relativistic fluid in the absence of the gradients of temperature and four-velocity represents an Eulerian fluid. The determination of Euler’s equations proceeds as follows. First, the multiplication of (18) by \( \sqrt{-g} \frac{d^3 p}{p_0} \) and integration of the resulting equation leads to the balance equation of the particle number density, namely,

\[
U^\nu \frac{\partial n}{\partial x^\nu} + n U^{\nu, \nu} = 0.
\]

Next, the multiplication of (18) by \( p^\mu \sqrt{-g} \frac{d^3 p}{p_0} \) and subsequent integration implies an equation which is used to derive the balance equations for the energy density and momentum density of an Eulerian fluid. The energy density balance equation is obtained through the projection \( U_{\mu} \), yielding

\[
n c_v U^\nu \frac{\partial T}{\partial x^\nu} + p U^{\nu, \nu} = 0,
\]

where \( c_v \) is the heat capacity per particle at constant volume,

\[
c_v = \frac{\partial e}{\partial T} = k \left( c^2 + 5 \frac{K_3}{K_2} \tilde{\zeta} - \frac{K_2 c^2}{K_3} \tilde{\zeta} - 1 \right).
\]

The momentum density balance equation results from the projection \( \Delta^{\nu}_{\mu} \),

\[
mn \frac{K_3}{K_2} U^{\nu, \nu} - \Delta^{\nu}_{\mu} \frac{\partial p}{\partial x^\mu} - mn \frac{K_3}{K_2} \frac{1}{1 - \Phi^2/4c^4} g_{\nu i} \frac{\partial \Phi}{\partial x^i} = 0,
\]

where we have introduced the gravitational potential

\[
\Phi = - \frac{GM}{r}, \quad \text{with} \quad \frac{\partial \Phi}{\partial x^k} = \frac{GM}{r^2} \delta_{kj} \frac{x^j}{r}.
\]

We note that (21) is a function of the ratio \( |\Phi(R)|/c^2 \), which we can estimate at the surface of some bodies.
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(1) Earth: \( M \approx 5.97 \times 10^{24} \text{ kg} \); \( R \approx 6.38 \times 10^6 \text{ m} \); \( |\Phi(R)|/c^2 \approx 7 \times 10^{-10} \).
(2) Sun: \( M \approx 1.99 \times 10^{30} \text{ kg} \); \( R \approx 6.96 \times 10^8 \text{ m} \); \( |\Phi(R)|/c^2 \approx 2.2 \times 10^{-6} \).
(3) White dwarf: \( M \approx 1.02M_\\odot \); \( R \approx 5.4 \times 10^6 \text{ m} \); \( |\Phi(R)|/c^2 \approx 2.8 \times 10^{-4} \).
(4) Neutron star: \( M \approx M_\\odot \); \( R \approx 2 \times 10^4 \text{ m} \); \( |\Phi(R)|/c^2 \approx 7.5 \times 10^{-2} \).

The above estimates imply that in most cases the approximation \( |\Phi|/c^2 \ll 1 \) is valid.

Hence, the momentum density balance equation (22) in the non-relativistic limiting case \( \zeta \gg 1 \) and in the presence of a weak gravitational field reads

\[
\begin{align*}
\rho \left( 1 + \frac{5}{2\zeta} + \cdots \right) U^\mu U_\nu \gamma_{\mu\nu} - \Delta^{\mu\nu} \frac{\partial p}{\partial x^\mu} \\
- \rho \left( 1 + \frac{5}{2\zeta} + \cdots \right) \left( 1 + \frac{\Phi^2}{4c^4} + \cdots \right) g^{ij} \frac{\partial \Phi}{\partial x^i} = 0.
\end{align*}
\]

Note that without the underlined terms (24) reduces to the usual form of Newton’s second law for a non-relativistic gas in the presence of a weak gravitational field.

3.2. Viscous and heat-conducting relativistic gas

In order to determine the energy–momentum tensor with the deviation of the Maxwell–Jüttner distribution function \( \varphi \)—given by (18)—we first note that according to (5) and (7) the particle four-flow evaluated with the equilibrium and non-equilibrium distribution functions must have the same representation, i.e., \( N^\mu = N_E^\mu = nU^\mu \). From now on the index \( E \) will denote the value of a quantity at equilibrium, i.e., when evaluated with the Maxwell–Jüttner distribution function. However, in Marle’s model the pressure and the energy per particle have different values at equilibrium and non-equilibrium states. This is not the case for the Anderson and Witting model, equation (6), or for the full Boltzmann equation (see, e.g., [18]).

In the following we shall write the constitutive equations in a comoving frame where the components of the projector become

\[
\begin{align*}
\Delta^{00} &= 0, \\
\Delta^{ij} &= g^{ij} = -\frac{1}{g_t} \delta^{ij} = -\frac{1}{(1 - |\Phi|/2c^2)^4} \delta^{ij}.
\end{align*}
\]

The knowledge of the energy–momentum tensor is obtained from the insertion of the representation of the distribution function \( f = f^{(0)}(1 + \varphi) \)—with \( \varphi \) given by (18)—into its definition (12) and subsequent integration of the resulting equation.

Let us first analyze the projections

\[
\begin{align*}
\varpi + p &= -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu}, \\
ne &= \frac{1}{c^2} U_\mu U_\nu T^{\mu\nu},
\end{align*}
\]

which give the following relationships:

\[
\begin{align*}
\varpi + (p - p_E) &= -\frac{\tau k p E}{3c^4_t (1 - |\Phi|/2c^2)^4} \left[ 20 K_E^3 K_F^3 + 3 \zeta_E + 2 \zeta_E^2 \left( \frac{K_E^3}{K_F^3} \right)^3 \right. \\
&\left. - 13 \zeta_E \left( \frac{K_E^3}{K_F^3} \right)^2 - 2 \zeta_E^2 \frac{K_E^3}{K_F^3} \right] \frac{\partial U^j}{\partial x^j}.
\end{align*}
\]

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\[ (e - e_E) = \frac{\tau m c^2 k}{c^E_0 \xi_E (1 - |\Phi|/2c^2)^4} \left[ 20 \frac{K^E_3}{K^E_2} + 3 \xi_E + 2 \xi_E^2 \left( \frac{K^E_3}{K^E_2} \right)^3 \right] - 13 \xi_E \left( \frac{K^E_3}{K^E_2} \right)^2 - 2 \xi_E^2 \left( \frac{K^E_3}{K^E_2} \right) \frac{\partial U^j}{\partial x^j}. \] 

(28)

In the above equations \( K^E_n \equiv K_n(\xi_E) \) was introduced.

We can relate the pressure difference \( (p - p_E) \) with the energy difference \( (e - e_E) \) by noting that the pressure difference can be written as a temperature difference through \( (p - p_E) = nk(T - T_E) = nmc^2(1/\xi - 1/\xi_E) \). If we expand the energy difference \( (e - e_E) \) about \( T_E \) and neglect terms up to second order in \( (T - T_E) \)—since we are only interested in processes close to equilibrium with gradients of first order—it follows that \( e - e_E = c^E_0(T - T_E) \). Hence, these last two relationships together with (26) furnish the following:

\[ p - p_E = \frac{\tau p_E k^2}{(c^E_0)^2 (1 - |\Phi|/2c^2)^4} \left[ 20 \frac{K^E_3}{K^E_2} + 3 \xi_E + 2 \xi_E^2 \left( \frac{K^E_3}{K^E_2} \right)^3 \right] - 13 \xi_E \left( \frac{K^E_3}{K^E_2} \right)^2 - 2 \xi_E^2 \left( \frac{K^E_3}{K^E_2} \right) \frac{\partial U^j}{\partial x^j}. \] 

(29)

Now we can obtain from (27) and (29) the final expression for the non-equilibrium pressure,

\[ \varpi = -\eta \frac{\partial U^j}{\partial x^j}, \] 

(30)

where \( \eta \) denotes the coefficient of bulk viscosity, which is given by

\[ \eta = \frac{\tau k^2 p_E}{3(c^E_0)^2 (1 - |\Phi|/2c^2)^4} \left[ 20 \frac{K^E_3}{K^E_2} - 13 \left( \frac{K^E_3}{K^E_2} \right)^2 \xi_E - 2 \frac{K^E_3}{K^E_2} \xi_E^2 + 3 \xi_E \right] + 2 \left( \frac{K^E_3}{K^E_2} \right)^3 \xi_E^2 \left( 4 - \xi_E - 5 \frac{K^E_3}{K^E_2} \xi_E + \left( \frac{K^E_3}{K^E_2} \right)^2 \xi_E^2 \right). \] 

(31)

The heat flux and the traceless part of the pressure tensor are obtained from the projections

\[ q^\sigma = \Delta^\sigma_\mu U^\nu T^{\mu\nu}, \quad \mathbb{P}^{\sigma\tau} = \left[ \Delta^\sigma_\mu \Delta^\tau_\nu - \frac{1}{3} \Delta^{\sigma\tau} \Delta_{\mu\nu} \right] T^{\mu\nu}, \] 

(32)

and the constitutive equations follow as

\[ q^i = -\lambda \delta^{ij} \left[ \frac{\partial T}{\partial x^j} - \frac{T}{c^2} U^\sigma U_\sigma^j + \frac{T}{c^2} \frac{1}{1 - \Phi^2/4c^4} \frac{\partial \Phi}{\partial x^j} \right], \] 

(33)

\[ \mathbb{P}^{ij} = -\mu \left[ (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) - \frac{2}{3} \delta^{ij} \delta^{kl} \right] \frac{\partial U^k}{\partial x^l}. \] 

(34)

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The coefficients $\lambda$ and $\mu$ are identified with the thermal conductivity and shear viscosity, respectively, and their expressions read

$$
\lambda = \frac{\tau p E}{(1 - |\Phi|/2c^2)^4} \frac{k}{m} \zeta_E \left( \zeta + \frac{5 K_3 E}{K_2 E} - \left( \frac{K_3 E}{K_2 E} \right)^2 \zeta E \right), \quad (35)
$$

$$
\mu = \frac{\tau p E}{(1 - |\Phi|/2c^2)^8} \frac{K_3 E}{K_2 E} \zeta E. \quad (36)
$$

Equations (30) and (34) represent the constitutive equations of a Newtonian relativistic fluid, while (33) represents the generalized Fourier law.

Let us analyze the transport coefficients given by (31), (35) and (36). All these coefficients depend on the gravitational potential through the component of the metric tensor $g_1(r) = (1 + GM/2c^2 r)^4$, and we may infer from their expressions that the values become larger in the presence of a gravitational field. The increase of the transport coefficients due to the gravitational field is rather small for stellar objects that are not too compact, i.e., for stellar objects where $|\Phi|/c^2 \ll 1$. In the absence of the gravitational potential the metric tensor reduces to the one of a Minkowski space–time. In this case $g_1 = 1$ and the expressions for the transport coefficients reduce to those of Marle [15, 18].

In the limiting case of weak gravitational fields, $|\Phi|/c^2 \ll 1$, and low temperatures, $\zeta E \gg 1$, the transport coefficients (31), (35) and (36) become

$$
\eta = \frac{5}{6} \frac{p E T \zeta E}{\zeta E} \left[ 1 + \frac{21}{2} \frac{|\Phi|}{c^2} + \cdots \right] \left[ 1 + \frac{2|\Phi|}{c^2} + \cdots \right], \quad (37)
$$

$$
\mu = \frac{4}{\zeta E} \frac{p E T}{\zeta E} \left[ 1 + \frac{5}{2} \frac{\zeta E}{c^2} + \cdots \right] \left[ 1 + \frac{4|\Phi|}{c^2} + \cdots \right], \quad (38)
$$

$$
\lambda = \frac{5 k p E T}{2m} \left[ 1 + \frac{3}{2} \frac{\zeta E}{c^2} + \cdots \right] \left[ 1 + \frac{2|\Phi|}{c^2} + \cdots \right]. \quad (39)
$$

The above expressions correspond to a non-relativistic gas in a weak gravitational field.

The transport coefficients (31), (35) and (36) in the limiting case of weak gravitational fields, $|\Phi|/c^2 \ll 1$, and high temperatures, $\zeta E \ll 1$, read

$$
\eta = \frac{p E T \zeta E}{54} \left[ 1 + \frac{31}{12} + \frac{9}{2} \ln \left( \frac{\zeta E}{2} \right) + \frac{9}{2} \gamma \right] \left[ 1 + \frac{2|\Phi|}{c^2} + \cdots \right], \quad (40)
$$

$$
\mu = \frac{4 p E T}{\zeta E} \left[ 1 + \frac{\zeta E}{8} + \cdots \right] \left[ 1 + 4 \frac{|\Phi|}{c^2} + \cdots \right], \quad (41)
$$

$$
\lambda = \frac{4 c^2 p E T}{T \zeta E} \left[ 1 - \frac{\zeta E}{8} + \cdots \right] \left[ 1 + \frac{2|\Phi|}{c^2} + \cdots \right]. \quad (42)
$$

Here, the expressions refer to a gas in a weak gravitational field and in the ultra-relativistic regime.

As a remark we call attention to the fact that in the literature of neutron stars (see, e.g., [19]) the dependence of the transport coefficients on the gravitational potential is not taken into account.
3.3. Fourier’s law

Let us analyze Fourier’s law in more detail. It has the following three terms.

(1) The usual dependence of the heat flux in the gradient of temperature \( \partial T / \partial x^j \).

(2) A relativistic term proportional to \(- (T/c^2) U^\sigma U_j \), which represents an isothermal heat flux when matter is accelerated. It is a small term that acts in a direction opposite to the acceleration and is due to the inertia of energy. It was proposed by Eckart [16] within an irreversible thermodynamic theory.

(3) The third term, \((T/c^2)(1/(1 - \Phi^2/4c^4))(\partial \Phi / \partial x^j)\), is proportional to the gravitational potential gradient. It was proposed by Tolman [12] and indicates that in the absence of the acceleration term a state of equilibrium of a relativistic gas in a gravitational field can be attained if the temperature gradient is counterbalanced by a gravitational potential gradient. In a weak gravitational field the equilibrium condition reads

\[
\frac{1}{T} \nabla T = - \frac{g}{c^2},
\]

which is the so-called Tolman’s law, where \( g \) is the gravitational field.

By considering that the temperature field depends only on the radial coordinate \( T = T(r) \) and that there is no heat flux or acceleration field, it follows from (33) that the temperature field in a gravitational field must fulfil the differential equation

\[
\frac{1}{T(r)} \frac{dT(r)}{dr} = - \frac{1}{c^2} \frac{1}{1 - \Phi(r)^2/4c^4} \frac{d\Phi(r)}{dr}.
\]

The solution of the above differential equation for the boundary condition \( T(R) = 1 \)—where \( R \) is the radius of the spherical source—is

\[
T(r) = \frac{(1 - |\Phi(R)|/2c^2)(1 + |\Phi(R)|/2c^2(1 - R/r))}{(1 + |\Phi(R)|/2c^2)(1 - |\Phi(R)|/2c^2(1 - R/r))},
\]

which is a decreasing function of \( r \) for \( r > R \). In a weak gravitational field where \(|\Phi(R)|/c^2\) is a small quantity, (45) can be expressed as

\[
T(r) = 1 - \frac{|\Phi(R)|}{c^2} \left( 1 - \frac{R}{r} \right) + \frac{|\Phi(R)|^2}{2c^2} \left( 1 - \frac{R}{r} \right)^2 + \cdots.
\]

It is worth calling attention to the fact that if we use the momentum density balance equation of an Eulerian fluid (22) in order to eliminate the acceleration term we get that the heat flux (33) becomes

\[
q^j = - \lambda \delta^j_\nu \left[ \frac{\partial T}{\partial x^\nu} - \frac{TK_2}{nmc^2K_3} \frac{\partial \rho}{\partial x^\nu} \right].
\]

We can transform the dependence of the pressure gradient in (47) into a dependence on the temperature gradient and particle number density gradient through the use of the equation of state \( p = nkT \), and in this new description the presence of the gravitational potential gradient disappears. This indicates that the term related with the gravitational potential gradient in the heat flux is associated with the acceleration term. However, in the work [13] the acceleration term was eliminated by the use of the momentum balance equation, and Fourier’s law has a dependence on the gravitational field and the gradient.

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of the particle number density. We presume that the difference in the results is due to the fact that in the referred work there is no dependence of the equilibrium distribution function on the metric tensor components.

4. Conclusions

In this work we have analyzed a relativistic gas in a Schwarzschild metric within the framework of the Boltzmann equation in the presence of gravitational fields. The model of Marle for the collision operator of the Boltzmann equation was used and the deviation from the equilibrium distribution function was determined from the Chapman–Enskog method. Among other results we have presented the following. (i) Apart from the temperature gradient the heat flux has two relativistic terms; one due to the inertia of the energy is connected with an acceleration, while the other is proportional to the gravitational potential gradient. (ii) The transport coefficients depend on the gravitational potential and become larger than their values in the absence of it. (iii) Expressions for the transport coefficients in the limiting cases of low and high temperatures for weak gravitational fields were given. (iv) The radial temperature field of the gas is a decreasing function of the distance of the source of the gravitational field, when the heat flux and acceleration field vanish. (v) The momentum balance equation for an Eulerian fluid in the presence of a weak gravitational field and in the non-relativistic regime reduces to the usual expression of Newton’s second law.

As a final comment it is worth calling attention to the fact that the present theory should not be valid in the case where \( \Phi \to 2c^2 \). In this case the transport coefficients diverge and we are in the presence of the horizon of a Schwarzschild black hole. This limiting case will be the subject of a future investigation.

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Appendix A. Schwarzschild metric

The Schwarzschild metric is the solution of Einstein’s field equation for a spherical symmetrical non-rotating and uncharged source of a gravitational field with total mass \( M \). In terms of the spherical coordinates \((\tilde{r}, \theta, \varphi)\) it is given by

\[
d s^2 = \left(1 - \frac{2GM}{c^2 \tilde{r}}\right) (d\tilde{x}^0)^2 - \frac{1}{\left(1 - 2GM/c^2 \tilde{r}\right)} d\tilde{r}^2 - \tilde{r}^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2),
\]

where \( G \) is the gravitational constant. The isotropic Schwarzschild metric (see, e.g., [20]) follows by introducing a new radial coordinate \( r^2 = \delta_{ij} x^i x^j \) through the relationship \( \tilde{r} = r \left(1 + GM/2c^2 r\right)^2 \), so that (A.1) becomes

\[
d s^2 = g_0(r) (d\tilde{x}^0)^2 - g_1(r) \delta_{ij} dx^i \, dx^j.
\]
Above, we have introduced the abbreviations

\[ g_0(r) = \frac{(1 - GM/2c^2r)^2}{(1 + GM/2c^2r)^2}, \quad g_1(r) = \left(1 + \frac{GM}{2c^2r}\right)^4. \]  
(A.3)

For the isotropic metric (A.2) the Christoffel symbols read

\[ \Gamma^0_{00} = 0, \quad \Gamma^0_{ij} = 0, \quad \Gamma^k_{ij} = 0 \quad (i \neq j \neq k), \quad \Gamma^0_{0i} = 0, \]  
(A.4)

\[ \Gamma^i_{00} = \frac{1}{2g_1(r)} \frac{dg_1(r)}{dr} x^i \frac{x^j}{r}, \quad \Gamma^i_{0i} = \frac{1}{2g_1(r)} \frac{dg_1(r)}{dr} \delta_{ij} x^i \frac{x^j}{r}, \]  
(A.5)

\[ \Gamma^i_{0j} = -\frac{1}{2g_1(r)} \frac{dg_1(r)}{dr} x^i \frac{x^j}{r} \quad (i \neq j), \]  
(A.6)

where the underlined indices are not summed and

\[ \frac{dg_0(r)}{dr} = \frac{2GM}{c^2r^2} \left(1 + \frac{GM}{2c^2r}\right)^3, \quad \frac{dg_1(r)}{dr} = -\frac{2GM}{c^2r^2} \left(1 + \frac{GM}{2c^2r}\right)^3. \]  
(A.7)

**Appendix B. Evaluation of integrals**

To evaluate the integral

\[ Z = \sqrt{+g} \int e^{-(1/kT)U^\lambda p_\lambda} \frac{d^3p}{p_0}, \]  
(B.1)

we choose a comoving frame so that the above integral reduces to

\[ Z = 4\pi \sqrt{g_0 g_1} \int_0^\infty e^{-c_0/(kT \sqrt{g_0})} |p|^2 \frac{d|p|}{p_0}, \]  
(B.2)

through the introduction of spherical coordinates \(d^3p = |p|^2 \sin \theta \, d|p| \, d\theta \, d\varphi\) and by integrating in the angles \(0 \leq \theta \leq \pi\) and \(0 \leq \varphi \leq 2\pi\). If we use the relationship (3) and change the variable of integration through \(p_0 = mc y \sqrt{g_0}\) we have that

\[ |p|^2 = \frac{m^2c^2}{g_1} (y^2 - 1), \quad d|p| = \frac{mc}{\sqrt{g_1} (y^2 - 1)^{1/2}} \frac{dy}{y}, \]  
(B.3)

and (B.2) reduces to

\[ Z = 4\pi m^2c^2 \int_1^\infty e^{-\zeta y} (y^2 - 1)^{1/2} dy = 4\pi m k T K_1(\zeta), \]  
(B.4)

i.e., \(Z\) is given in terms of the modified Bessel function of second kind.

For the evaluation of the integral

\[ Z_1 = \sqrt{-g} \int \frac{e^{-(1/kT)U^\lambda p_\lambda} \, d^3p}{U^\tau p_\tau} \frac{1}{p_0}, \]  
(B.5)

we change the variables and introduce

\[ p_0 = mc \sqrt{g_0} \cosh t, \quad |p| = \frac{mc}{\sqrt{g_1}} \sinh t, \]  
(B.6)
and get
\[ Z_1 = 4\pi m \int_0^\infty \frac{\cosh^2 t - 1}{\cosh t} e^{-\zeta \cosh t} \, dt = 4\pi m \left[ K_1(\zeta) - K_i(\zeta) \right], \]  
(B.7)

thanks to the definition
\[ K_i(\zeta) = \int_\zeta^\infty K_{i-1}(t) \, dt = \int_0^\infty \frac{e^{-\zeta \cosh t}}{\cosh^n t} \, dt. \]  
(B.8)

Following the same methodology it is easy to obtain
\[ Z^\mu = \sqrt{-g} \int p^\mu e^{-(1/kT)U^\lambda p_\lambda} \frac{d^3p}{p_0} = 4\pi m^2 kT^2(\zeta) U^\mu, \]  
(B.9)

\[ Z_1^\mu = \sqrt{-g} \int p^\mu e^{-(1/kT)U^\lambda p_\lambda} \frac{d^3p}{U^\tau p_\tau} = 4\pi m^2 K_1(\zeta) U^\mu, \]  
(B.10)

and to derive further expressions for \( Z^{\mu_1 \cdots \mu_n} \) and \( Z_1^{\mu_1 \cdots \mu_n} \).

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