An efficient algorithm for contextual bandits with knapsacks, and an extension to concave objectives

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Abstract

We consider a contextual version of multi-armed bandit problem with global knapsack constraints. In each round, the outcome of pulling an arm is a scalar reward and a resource consumption vector, both dependent on the context, and the global knapsack constraints require the total consumption for each resource to be below some pre-fixed budget. The learning agent competes with an arbitrary set of context-dependent policies. This problem was introduced by Badanidiyuru et al. [2014], who gave a computationally inefficient algorithm with near-optimal regret bounds for it. We give a computationally efficient algorithm for this problem with slightly better regret bounds, by generalizing the approach of Agarwal et al. [2014] for the non-constrained version of the problem. The computational time of our algorithm scales logarithmically in the size of the policy space. This answers the main open question of Badanidiyuru et al. [2014]. We also extend our results to a variant where there are no knapsack constraints but the objective is an arbitrary Lipschitz concave function of the sum of outcome vectors.

1 Introduction

Multi-armed bandits (e.g., Bubeck and Cesa-Bianchi, 2012) are a classic model for studying the exploration-exploitation tradeoff faced by a decision-making agent, which learns to maximize cumulative reward through sequential experimentation in an initially unknown environment. The contextual bandit problem [Langford and Zhang, 2008], also known as associative reinforcement learning [Barto and Anandan, 1985], generalizes multi-armed bandits by allowing the agent to take actions based on contextual information: in every round, the agent observes the current context, takes an action, and observes a reward that is a random variable with distribution conditioned on the context and the taken action. Despite many recent advances and successful applications of bandits, one of the major limitations of the standard setting is the lack of “global” constraints that are common in many important real-world applications. For example, actions taken by a robot arm may have different levels of power consumption, and the total power consumed by the arm is limited by the capacity of its battery. In online advertising, each advertiser has her own budget, so that her advertisement cannot be shown more than a certain number of times. In dynamic pricing, there are a certain number of objects for sale and the seller offers prices to a sequence of buyers with the goal of maximizing revenue, but the number of sales is limited by the supply.

Recently, a few papers started to address this limitation by considering very special cases such as a single resource with a budget constraint [Ding et al., 2013, Guha and Munagala, 2007, György et al., 2007, Madani et al., 2004, Tran-Thanh et al., 2010, Tran-Thanh et al., 2012], and application-specific bandit problems such as the ones motivated by online advertising [Chakrabarti and Vez, 2012, Pandey and Olston, 2006], dynamic pricing [Babaioff et al., 2015, Besbes and Zeevi, 2009] and crowdsourcing [Badanidiyuru et al., 2012, Singla and Krause, 2013, Slivkins and Vaughan, 2013]. Subsequently, Badanidiyuru et al. [2013] introduced a general problem capturing most previous formulations. In this problem, which they called Bandits with Knapsacks (BwK), there are $d$ differ-
ent resources, each with a pre-specified budget. Each action taken by the agent results in a $d$-dimensional resource consumption vector, in addition to the regular (scalar) reward. The goal of the agent is to maximize the total reward, while keeping the cumulative resource consumption below the budget. The BwK model was further generalized to the BwCR (Bandits with convex Constraints and concave Rewards) model by Agrawal and Devanur [2014], which allows for arbitrary concave objective and convex constraints on the sum of the resource consumption vectors in all rounds. Both papers adapted the popular Upper Confidence Bound (UCB) technique to obtain near-optimal regret guarantees. However, the focus was on the non-contextual setting.

There has been significant recent progress [Agarwal et al., 2014, Dudík et al., 2011] in algorithms for general (instead of linear [Abbasi-yadkori et al., 2012, Chu et al., 2011]) contextual bandits where the context and reward can have arbitrary correlation, and the algorithm competes with some arbitrary set of context-dependent policies. Dudík et al. [2011] achieved the optimal regret bound for this remarkably general contextual bandits problem, assuming access to the policy set only through a linear optimization oracle, instead of explicit enumeration of all policies as in previous work [Auer et al., 2002, Beygelzimer et al., 2011]. However, the algorithm presented in [Dudík et al., 2011] was not tractable in practice, as it makes too many calls to the optimization oracle. [Agarwal et al., 2014] presented a simpler and computationally efficient algorithm, with a running time that scales as the square-root of the logarithm of the policy space size, and achieves an optimal regret bound.

Combining contexts and resource constraints, Agrawal and Devanur [2014] also considered a static linear contextual version of BwCR where the expected reward was linear in the context. Wu et al. [2015] considered the special case of random linear contextual bandits with a single budget constraint, and gave near-optimal regret guarantees for it. Badanidiyuru et al. [2014] extended the general contextual version of bandits with arbitrary policy sets to allow budget constraints, thus obtaining a contextual version of BwK, a problem they called Resourceful Contextual Bandits (RCB). We will refer to this problem as CBwK (Contextual Bandits with Knapsacks), to be consistent with the naming of related problems defined in the paper. They gave a computationally inefficient algorithm, based on [Dudík et al., 2011], with a regret that was optimal in most regimes. Their algorithm was defined as a mapping from the history and the context to an action, but the computational issue of finding this mapping was not addressed. They posed an open question of achieving computational efficiency while maintaining a similar or even a sub-optimal regret.

Main Contributions. In this paper, we present a simple and computationally efficient algorithm for CBwK/RCB, based on the algorithm of [Agarwal et al., 2014]. Similar to [Agarwal et al., 2014], the running time of our algorithm scales as the square-root of the logarithm of the size of the policy set, thus resolving the main open question posed by [Badanidiyuru et al., 2014]. Our algorithm even improves the regret bound of [Badanidiyuru et al., 2014] by a factor of $\sqrt{d}$. Another improvement over [Badanidiyuru et al., 2014] is that while they need to know the marginal distribution of contexts, our algorithm does not. A key feature of our techniques is that we need to modify the algorithm in [Agarwal et al., 2014] in a very minimal way — in an almost blackbox fashion — thus retaining the structural simplicity of the algorithm while obtaining substantially more general results.

We extend our algorithm to a variant of the problem, which we call Contextual Bandits with concave Rewards (CBwR): in every round, the agent observes a context, takes one of $K$ actions and then observes a $d$-dimensional outcome vector, and the goal is to maximize an arbitrary Lipschitz concave function of the average of the outcome vectors; there are no constraints. This allows for many more interesting applications, some of which were discussed in [Agrawal and Devanur, 2014]. This setting is also substantially more general than the contextual version considered in [Agrawal and Devanur, 2014], where the context was fixed and the dependence was assumed to be linear.

Organization. In Section 2, we define the CBwK problem, and state our regret bound as Theorem 1. The algorithm is detailed in Section 3 and an overview of the regret analysis is in Section 4. In Section 5, we present CBwR, the problem with concave rewards, state the guaranteed regret bounds, and outline the differences in the algorithm and the analysis. Complete proofs and other details are provided in the appendices.

1 In particular, each arm is associated with a fixed vector and the resulting outcomes for this arm have expected value linear in this vector.

2 Access to the policy set is via an “arg max oracle”, as in [Agarwal et al., 2014].
2 Preliminaries and Main Results

**CBwK.** The CBwK problem was introduced by Badanidiyuru et al. [2014], under the name of Resourceful Contextual Bandits (RCB). We now define this problem.

Let $A$ be a finite set of $K$ actions and $X$ be a space of possible contexts (the analogue of a feature space in supervised learning). To begin with, the algorithm is given a budget $B \in \mathbb{R}_+$. We then proceed in rounds: in every round $t \in [T]$, the algorithm observes context $x_t \in X$, chooses an action $a_t \in A$, and observes a reward $r_t(a_t) \in [0, 1]$ and a $d$-dimensional consumption vector $v_t(a_t) \in [0, 1]^d$. The objective is to take actions that maximize the total reward, $\sum_{t=1}^T r_t(a_t)$, while making sure that the consumption does not exceed the budget, i.e., $\sum_{t=1}^T v_t(a_t) \leq B1$. The algorithm stops either after $T$ rounds or when the budget is exceeded in one of the dimensions, whichever occurs first. We assume that one of the actions is a “no-op” action, i.e., it always gives a reward of $0$ and a consumption vector of all $0$s. Furthermore, we make a stochastic assumption that the context, the reward, and the consumption vectors $(x_t, \{v_t(a) : a \in A\})$ for $t = 1, 2, \ldots, T$ are drawn i.i.d. (independent and identically distributed) from a distribution $D$ over $X \times [0, 1]^A \times [0, 1]^{dA}$. The distribution $D$ is unknown to the algorithm.

**Policy Set.** Following previous work [Agarwal et al. 2014, Badanidiyuru et al. 2014, Dudik et al. 2011], our algorithms compete with an arbitrary set of policies. Let $\Pi \subseteq A^X$ be a finite set of policies that map contexts $x \in X$ to actions $a \in A$. We assume that the policy set contains a “no-op” policy that always selects the no-op action regardless of the context. With global constraints, distributions over policies in $\Pi$ could be strictly more powerful than any policy in $\Pi$ itself. Our algorithms compete with this more powerful set, which is a stronger guarantee than simply competing with fixed policies in $\Pi$. For this purpose, define $\mathcal{C}(\Pi) := \{P \in [0, 1]^\Pi : \sum_{\pi \in \Pi} P(\pi) = 1\}$ as the set of all convex combinations of policies in $\Pi$. For a context $x \in X$, choosing actions with $P \in \mathcal{C}(\Pi)$ is equivalent to following a randomized policy that selects action $a \in A$ with probability $P(a|x) = \sum_{\pi \in \Pi : \pi(x) = a} P(\pi)$; we therefore also refer to $P$ as a (mixed) policy. Similarly, define $\mathcal{C}_0(\Pi) := \{P \in [0, 1]^\Pi : \sum_{\pi \in \Pi} P(\pi) \leq 1\}$ as the set of all non-negative weights over $\Pi$, which sum to at most 1. Clearly, $\mathcal{C}(\Pi) \subset \mathcal{C}_0(\Pi)$.

**Benchmark and Regret.** The benchmark for this problem is an optimal static mixed policy, where the budgets are required to be satisfied in expectation only. Let $R(P) := \mathbb{E}_{(x,r,v) \sim D} [\mathbb{E}_{\pi \sim P} [r(\pi(x))]]$ and $V(P) := \mathbb{E}_{(x,r,v) \sim D} [\mathbb{E}_{\pi \sim P} [v(\pi(x))]]$ denote respectively the expected reward and consumption vector for policy $P \in \mathcal{C}(\Pi)$. We call a policy $P \in \mathcal{C}(\Pi)$ a feasible policy if $TV(P) \leq B1$. Note that there always exists a feasible policy in $\mathcal{C}(\Pi)$, because of the no-op policy. Define an optimal policy $P^* \in \mathcal{C}(\Pi)$ as a feasible policy that maximizes the expected reward:

$$P^* = \arg \max_{P \in \mathcal{C}(\Pi)} TR(P) \quad \text{s.t.} \quad TV(P) \leq B1. \quad (1)$$

The reward of this optimal policy is denoted by $\text{OPT} := TR(P^*)$. We are interested in minimizing the regret, defined as

$$\text{regret}(T) := \text{OPT} - \sum_{t=1}^T r_t(a_t). \quad (2)$$

**AMO.** Since the policy set $\Pi$ is extremely large in most interesting applications, accessing it by explicit enumeration is impractical. For the purpose of efficient implementation, we instead only access $\Pi$ via a maximization oracle. Employing such an oracle is common when considering contextual bandits with an arbitrary set of policies [Agarwal et al. 2014, Dudik et al. 2011, Langford and Zhang 2008]. Following previous work, we call this oracle an “arg max oracle”, or AMO.

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3 More generally, different dimensions could have different budgets, but this formulation is without loss of generality: scale the units of all dimensions so that all the budgets are equal to the smallest one. This preserves the requirement that the vectors are in $[0, 1]^d$.

4 The policies may be randomized in general, but for our results, we may assume without loss of generality that they are deterministic. As observed by Badanidiyuru et al. [2014], we may replace randomized policies with deterministic policies by appending a random seed to the context. This blows up the size of the context space which does not appear in our regret bounds.

5 E.g., consider two policies that both give reward 1, but each consume 1 unit of a different resource. The optimum solution is to mix uniformly between the two, which does twice as well as using any single policy.
Definition 1. For a set of policies Π, the arg max oracle (AMO) is an algorithm, which for any sequence of contexts and rewards, \((x_1, r_1), \ldots, (x_t, r_t) \in X \times [0, 1]^A\), returns

\[
\arg\max_{\pi \in \Pi} \sum_{\tau=1}^t r_\tau(\pi(x_\tau))
\]

(3)

Main Results. Our main result is a computationally efficient low-regret algorithm for CBwK. Furthermore, we improve the regret bound of Badanidiyuru et al. [2014] by a \(\sqrt{d}\) factor; they present a detailed discussion on the optimality of the dependence on \(K\) and \(T\) in this bound.

Theorem 1. For the CBwK problem, \(\delta > 0\), there is a polynomial-time algorithm that makes \(O(d\sqrt{KT \ln(|\Pi|)})\) calls to AMO, and with probability at least \(1 - \delta\) has regret

\[
\text{regret}(T) = O \left( \frac{\text{OPT}}{B} + 1 \right) \sqrt{KT \ln(dT|\Pi|/\delta)}
\]

Note that the above regret bound is meaningful only for \(B > \Omega(\sqrt{KT \ln(dT|\Pi|/\delta)})\), therefore in the rest of the paper we assume that \(B > c' \sqrt{KT \ln(dT|\Pi|/\delta)}\) for some large enough constant \(c'\). We also extend our results to a version with a concave reward function, as outlined in Section 5. For the rest of the paper, we treat \(\delta > 0\) as fixed, and define quantities that depend on \(\delta\).

3 Algorithm for the CBwK problem

From previous work on multi-armed bandits, we know that the key challenges in finding the “right” policy are that (1) it should concentrate fast enough on the empirically best policy (based on data observed so far), (2) the probability of choosing an action must be large enough to enable sufficient exploration, and (3) it should be efficiently computable. Agarwal et al. [2014] show that all these can be addressed by solving a properly defined optimization problem, with help of an AMO. We have the additional technical challenge of dealing with global constraints. As mentioned earlier, one complication that arises right away is that due to the knapsack constraints, the algorithm has to compete against the best mixed policy in \(\Pi\), rather than the best pure policy. In the following, we will highlight the main technical difficulties we encounter, and our solution to these difficulties.

Some definitions are in place before we describe the algorithm. Let \(H_t\) denote the history of chosen actions and observations before time \(t\), consisting of records of the form \((x_\tau, a_\tau, r_\tau(a_\tau), v_\tau(a_\tau), p_\tau(a_\tau))\), where \(x_\tau, a_\tau, r_\tau(a_\tau), v_\tau(a_\tau)\) denote, respectively, the context, action taken, reward and consumption vector observed at time \(\tau\), and \(p_\tau(a_\tau)\) denotes the probability at which action \(a_\tau\) was taken. (Recall that our algorithm selects actions in a randomized way using a mixed policy.) Although \(H_t\) contains observation vectors only for chosen actions, it can be “completed” using the trick of importance sampling: for every \((x_\tau, a_\tau, r_\tau(a_\tau), v_\tau(a_\tau), p_\tau(a_\tau)) \in H_t\), define the fictitious observation vectors \(\hat{r}_\tau \in [0, 1]^A, \hat{v}_\tau \in [0, 1]^{d \times A}\) by:

\[
\hat{r}_\tau(a) := \frac{r_\tau(a_\tau)}{p_\tau(a_\tau)} \mathbb{I}\{a_\tau = a\},
\]

\[
\hat{v}_\tau(a) := \frac{v_\tau(a_\tau)}{p_\tau(a_\tau)} \mathbb{I}\{a_\tau = a\}.
\]

Clearly, \(\hat{r}_\tau, \hat{v}_\tau\) are unbiased estimator of \(r_\tau, v_\tau\): for every \(a, \mathbb{E}_{a_\tau}[\hat{r}_\tau(a)] = r_\tau(a), \mathbb{E}_{a_\tau}[\hat{v}_\tau(a)] = v_\tau(a)\), where the expectations are over randomization in selecting \(a_\tau\).

With the “completed” history, it is straightforward to obtain an unbiased estimate of expected reward vector and expected consumption vector for every policy \(P \in C(\Pi)\):

\[
\hat{R}_t(P) := \mathbb{E}_{\tau \sim [t], \pi \sim P} [\hat{r}_\tau(\pi(x_\tau))] ,
\]

\[
\hat{V}_t(P) := \mathbb{E}_{\tau \sim [t], \pi \sim P} [\hat{v}_\tau(\pi(x_\tau))] .
\]

The convenient notation \(\tau \sim [t]\) above, indicating that \(\tau\) is drawn uniformly at random from the set of integers \(\{1, 2, \ldots, t\}\), simply means averaging over time up to step \(t\). It is easy to verify that \(\mathbb{E}[\hat{R}_t(P)] = R(P)\), and \(\mathbb{E}[\hat{V}_t(P)] = V(P)\).
Given these estimates, we construct an optimization problem (OP) which aims to find a mixed policy that has a small "empirical regret", and at the same time provides sufficient exploration over "good" policies. The optimization problem uses a quantity $\text{Reg}_t(P)$, "the empirical regret of policy $P$", to characterize good policies. Agarwal et al. [2014] define $\text{Reg}_t(P)$ as simply the difference between the empirical reward estimate of policy $P$ and that of the policy with the highest empirical reward. Thus, good policies were characterized as those with high reward. For our problem, however, a policy could have a high reward while its consumption violates the knapsack constraints by a large margin. Such a policy should not be considered a good policy. A key challenge in this problem is therefore to define a single quantity that captures the "goodness" of a policy by appropriately combining rewards and consumption vectors.

We define quantities $\text{Reg}(P)$ (and the corresponding empirical estimate $\hat{\text{Reg}}_t(P)$ up to round $t$) of $P \in C(\Pi)$ by combining the regret in reward and constraint violation using a multiplier "Z". The multiplier captures the sensitivity of the problem to violation in knapsack constraints. It is easy to observe from (1) that increasing the knapsack size from $B$ to $(1+\epsilon)B$ can increase the optimal to atmost $(1+\epsilon)\text{OPT}$. It follows that if a policy violates any knapsack constraint by $\gamma$, it can achieve at most $\frac{\gamma}{B} \text{OPT}$ more reward than $\text{OPT}$. More precisely,

**Lemma 2.** For any $b$, let $\text{OPT}(b)$ denote the value of an optimal solution of (1) when the budget is set as $b$. Then, for any $b \geq 0$, $\gamma \geq 0$,

$$\text{OPT}(b + \gamma) \leq \text{OPT}(b) + \frac{\text{OPT}(b)}{b} \gamma. \tag{4}$$

We use this observation to set $Z$ as an estimate of $\frac{\text{OPT}}{B}$. We do this by using the outcomes of the first $T_0 := \frac{12K_T}{B} \ln \frac{d||\Pi||}{\delta}$ rounds, during which we do pure exploration (i.e., play an action in $A$ uniformly at random). For notational convenience, in our algorithm description we will index these initial $T_0$ exploration rounds as $t = -(T_0 - 1), -(T_0 - 2), \ldots, 0$, so that the major component of the algorithm can be started from $t = 1$ and runs until $t = T - T_0$. The following lemma provides a bound on the $Z$ that we estimate. Its proof appears in Appendix B.

**Lemma 3.** For any $B$, using the first $T_0 = \frac{12K_T}{B} \ln \frac{d||\Pi||}{\delta}$ rounds of pure exploration, one can compute a quantity $Z$ such that with probability at least $1 - \delta$,

$$\max\{\frac{4\text{OPT}}{B}, 1\} \leq Z \leq \frac{24\text{OPT}}{B} + 8.$$

Now, to define $\text{Reg}(P)$ and $\hat{\text{Reg}}_t(P)$, we combine regret in reward and constraint violation using the constant $Z$ as computed above. In these definitions, we use a smaller budget amount

$$B' := B - T_0 - c \sqrt{KT \ln(T||\Pi||/\delta)},$$

for a large enough constant $c$ to be specified later. Here, the budget needed to be decreased by $T_0$ to account for budget consumed in the first $T_0$ exploration rounds. We use a further smaller budget amount to ensure that with high probability $(1 - \delta)$ our algorithm will not abort before the end of time horizon $(T - T_0)$, due to budget violation. For any vector $v \in \mathbb{R}^d$, let $\phi(v, B')$ denote the amount by which the vector $v$ violates the budget $B'$, i.e.,

$$\phi(v, B') := \max_{j=1, \ldots, d} \left( v_j - \frac{B'}{T} \right)^+. $$

Let $P'$ denote the optimal policy when budget amount is $B'$, i.e.,

$$P' := \arg \max_{P \in C(\Pi)} TR(P) \quad \text{s.t.} \quad TV(P) \leq B' \mathbf{1}.$$ 

And, let $P_t$ denote the empirically optimal policy for the combination of reward and budget violation, defined as:

$$P_t := \arg \max_{P \in C(\Pi)} \hat{R}_t(P) - Z \phi(\hat{V}_t(P), B'). \tag{5}$$

We define

$$\text{Reg}(P) := \frac{1}{Z^2}(R(P') - R(P) + Z \phi(V(P), B')), $$
Algorithm 1 requires solving (OP) at the end of every epoch. Agarwal et al. [2014] gave an algorithm that solves (OP) using access to the AMO. We use a similar algorithm, except that calls to the AMO are now replaced by calls to a knapsack constrained optimization problem over the empirical distribution. This optimization problem is identical in structure to the optimization problem defining $P_t$ in [5], which we need to solve also. We can solve both of these problems using AMO, as outlined below.

### 3.1 Computation complexity: Solving (OP) using AMO

Algorithm [1] requires solving (OP) at the end of every epoch. Agarwal et al. [2014] gave an algorithm that solves (OP) using access to the AMO. We use a similar algorithm, except that calls to the AMO are now replaced by calls to a knapsack constrained optimization problem over the empirical distribution. This optimization problem is identical in structure to the optimization problem defining $P_t$ in [5], which we need to solve also. We can solve both of these problems using AMO, as outlined below.
A linear optimization problem over \( [0, 1]^{d+2} \); we represent a point in this domain as \((x, y, \lambda)\), where \(x\) and \(\lambda\) are scalars and \(y\) is a vector in \(d\) dimensions. Let

\[ K_1 := \{(x, y, \lambda) : x = \tilde{R}_t(P), y = \tilde{V}_t(P) \text{ for some } P \in \mathcal{C}(\Pi), \lambda \in [0, 1]\}, \]

be the set of all reward, consumption vectors achievable on the empirical outcomes up to time \(t\), through some policy in \(\mathcal{C}(\Pi)\). Let

\[ K_2 := \{(x, y, \lambda) : y \leq (B'/T + \lambda)1 \} \cap [0, 1]^{d+2}, \]

be the constraint set, given by relaxing the knapsack constraints by \(\lambda\). Now (5) is equivalent to

\[ \max x - Z\lambda \text{ such that } (x, y, \lambda) \in K_1 \cap K_2. \quad (6) \]

Recently, Lee et al. [2015, Theorem 49] gave a fast algorithm to solve problems of the kind above, given access to oracles that solve linear optimization problems over \(K_1\) and \(K_2\). The algorithm makes \(\tilde{O}(d)\) calls to these oracles, and takes an additional \(\tilde{O}(d^3)\) running time. A linear optimization problem over \(K_1\) is equivalent to the AMO; the linear function defines the “rewards” that the AMO optimizes for. A linear optimization problem over \(K_2\) is trivial to solve. As an aside, a solution \(Q \in \mathcal{C}_0(\Pi)\) output by this algorithm has support equal to the policies output by the AMO during the run of the algorithm, and hence has size \(\tilde{O}(d)\).

Using this, (OP) can be solved using \(\tilde{O}(d\sqrt{KT\ln(\|\Pi\|)})\) calls to the AMO at the end of every epoch, and (5) can be solved using \(\tilde{O}(d)\) calls, giving a total of \(\tilde{O}(d\sqrt{KT\ln(\|\Pi\|)})\) calls to AMO. The complete algorithm to solve (OP) is in Appendix [C].

4 Regret Analysis

This section provides an outline of the proof of Theorem [I], which provides a bound on the regret of Algorithm [I]. \[^{6}\] (A complete proof is given in Appendix [D].) The proof structure is similar to the proof of Agarwal et al. [2014, Theorem 2], with major differences coming from the changes necessary to deal with mixed policies and constraint violations. We defined the algorithm to minimize Reg (through the first constraint in the optimization problem (OP)),

\[^{6}\] Alternately, one could use the algorithms of Vaidya [1989a,b] to solve the same problem, with a slightly weaker polynomial running time.

\[^{7}\] Here, \(O\) hides terms of the order \(\log^{O(1)}(d/\epsilon)\), where \(\epsilon\) is the accuracy needed of the solution.

\[^{8}\] These rewards may not lie in \([0, 1]\) but an affine transformation of the rewards can bring them into \([0, 1]\) without changing the solution.
and the first step is to show that this implies a bound on Reg as well. The alternate definitions of Reg and \( \hat{\text{Reg}} \) require a different analysis than what was in \cite{agarwal2014crowdsourcing}, and this difference is highlighted in the proof outline of Lemma 5 below. Once we have a bound on Reg, we show that this implies a bound on the actual reward \( R \), as well as the probability of violating the knapsack constraints.

We start by proving that the empirical average reward \( \hat{R}_t(P) \) and consumption vector \( \hat{V}_t(P) \) for any mixed policy \( P \) are close to the true averages \( R(P) \) and \( V(P) \) respectively. We define \( m_0 \) such that for initial epochs \( m < m_0 \), \( \mu_m = \frac{1}{2K} \). Recall that \( \mu_m \) is the minimum probability of playing any action in epoch \( m + 1 \), defined in Step II of Algorithm 1. Therefore, for these initial epochs the variance of importance sampling estimates is small, and we can obtain a stronger bound on estimation error. For subsequent epochs, \( \mu_m \) decreases, and we get error bounds in terms of max variance of the estimates for policy \( P \) across all epochs before time \( t \), defined as \( V_t(P) \). In fact, the second constraint in the optimization problem (OP) seeks to bound this variance.

The precise definitions of above-mentioned quantities are provided in Appendix D.

**Lemma 4.** With probability \( 1 - \frac{1}{\tau} \), for all policies \( P \in \mathcal{C}(\Pi) \),

\[
\max\{|\hat{R}_t(P) - R_t(P)|, \|\hat{V}_t(P) - V(P)\|_\infty\} \leq \left\{ \begin{array}{ll}
\sqrt{\frac{2Kd_t}{t}} & t \in \text{epoch } m_0, t \geq t_0 \\
\frac{d_t}{\mu_{m-1}} & t \in \text{epoch } m, m > m_0
\end{array} \right.
\]

Here, \( d_t = \ln(16t^2|\Pi|(d + 1)/\delta) \), \( t_0 := \min\{t \in \mathbb{N} : \frac{d_t}{\mu_{m-1}} \leq \frac{1}{4\mu_m} \} \), \( m_0 := \min\{m \in \mathbb{N} : \frac{d_t}{\mu_m} \leq \frac{1}{4\mu_m} \} \).

Now suppose the error bounds in above lemma hold. A major step is to show that, for every \( P \in \mathcal{C}(\Pi) \), the empirical regret \( \hat{\text{Reg}}_t(P) \) and the actual regret \( \text{Reg}(P) \) are close in a particular sense.

**Lemma 5.** Assume that the events in Lemma 4 hold. Then, for all epochs \( m \geq m_0 \), all rounds \( t \geq t_0 \) in epoch \( m \), and all policies \( P \in \mathcal{C}(\Pi) \),

\[
\text{Reg}(P) \leq 2\hat{\text{Reg}}_t(P) + c_0K\mu_m, \quad \text{and} \quad \hat{\text{Reg}}_t(P) \leq 2\hat{\text{Reg}}_t(P) + c_0K\mu_m,
\]

for \( \text{Reg}(P), \hat{\text{Reg}}_t(P) \) as defined in Section 3 and \( c_0 \) being a constant smaller than 150.

**Proof Outline.** The proof of above lemma is by induction, using the second constraint in (OP) to bound the variance \( V_t(P) \). Below, we prove the base case. This proof demonstrates the importance of appropriately choosing \( Z \). Consider \( m = m_0 \) and \( t \geq t_0 \) in epoch \( m \). For all \( P \in \mathcal{C}(\Pi) \),

\[
(Z + 1)(\hat{\text{Reg}}_t(P) - \text{Reg}(P)) = \hat{R}_t(P_t) - \hat{R}_t(P) - R(P') + R(P) - Z[\phi(\hat{V}_t(P_t), B') - \phi(\hat{V}_t(P), B') + \phi(V(P), B')].
\]

We can assume that \( B \geq c'\sqrt{KT\ln(dT|\Pi|/\delta)} \) for any constant \( c' \) (otherwise the regret guarantees in Theorem 1 are meaningless). Then, we have that \( B \geq 2T_0 + 2c'\sqrt{KT\ln(T|\Pi|/\delta)} = 2(B - B') \) implying \( B' \geq B' \). Also, observe that since \( B \geq B' \), \( \text{OPT}(B) \geq \text{OPT}(B') \). Then, by Lemma 2 and choice of \( Z \) as specified by Lemma 3 we have that for any \( \gamma \geq 0 \)

\[
\text{OPT}(B' + \gamma) \leq \text{OPT}(B') + \frac{\gamma}{2}.
\]

Now, since \( P' \) is defined as the optimal policy for budget \( B' \), we obtain that \( R(P') = \text{OPT}(B') \). Also, by definition of \( \phi(V(P_t), B') \), we have that \( R(P_t) \leq \text{OPT}(B' + \phi(V(P_t), B')) \), and therefore,

\[
R(P') \geq R(P_t) - \frac{\gamma}{2} \phi(V(P_t), B') \geq R(P_t) - \frac{Z}{2} \phi(V(P_t), B') + Z\phi(V(P), B').
\]

Substituting in (7), we can upper bound \( (Z + 1)(\hat{\text{Reg}}_t(P) - \text{Reg}(P)) \) by

\[
\hat{R}_t(P_t) - \hat{R}_t(P) - R(P_t) + Z\phi(V(P_t), B') + R(P) - Z[\phi(\hat{V}_t(P_t), B') - \phi(\hat{V}_t(P), B') + \phi(V(P), B')]
\leq |\hat{R}_t(P_t) - R(P_t)| + |\hat{R}_t(P) - R(P)| + Z\|\hat{V}_t(P_t) - V(P_t)\|_\infty + Z\|\hat{V}_t(P) - V(P)\|_\infty
\]
Recall that with a smaller budget. More precisely, we show that for every $m \geq m_0$ in terms of variance $\mathcal{V}(\cdot)$, bound on variance provided by the second constraint in (OP). The second constraint in (OP) provides a bound on the variance of any policy $P$ in any past epoch, in terms of $\hat{\text{Reg}}(\tau)$. This completes the base case. The remaining proof is by induction, using the bounds provided by Lemma 4 for epochs $m > m_0$ in terms of variance $\mathcal{V}(\cdot)$, bound on variance provided by the second constraint in (OP). The second constraint in (OP) provides a bound on the variance of any policy $P$ in any past epoch, in terms of $\hat{\text{Reg}}(\tau)$. Given the above lemma, the first constraint in (OP) which bounds the estimated regret $\hat{\text{Reg}}(Q)$ for the chosen mixed policy $Q_t$ directly implies an upper bound on $\text{Reg}(Q)$ for this mixed policy. Specifically, we get that for every epoch $m$, for mixed policy $Q_m$ that solves (OP),

$$\text{Reg}(Q_m) \leq (c_0 + 2)K\psi_{m}.$$  

Next, we bound the regret in epoch $m$ using above bound on Reg($Q_{m-1}$). For simplicity of discussion, here we outline the steps for bounding regret for rewards sampled from policy $Q_{m-1}$ in epoch $m$. Note that this is not precise in following ways. First, $Q_{m-1} \in \mathcal{C}m(\Pi)$ may not be in $\mathcal{C}(\Pi)$ and therefore may not be a proper distribution (the actual sampling process puts the remaining probability on default policy $P_t$ to obtain $Q_t$ at time $t$ in epoch $m$). Second, the actual sampling process picks an action from smoothed projection $Q_{t,m-1}$ of $Q_t$. However, we ignore these technicalities here in order to get across the intuition behind the proof; these technicalities are dealt with rigorously in the complete proof provided in Appendix E.

The first step is to use the above bound on $\text{Reg}(Q_{m-1})$ to show that expected reward $R(Q_{m-1})$ in epoch $m$ is close to optimal reward $R(P^*)$. Since $\phi(\cdot, B')$ is always non-negative, by definition of $\text{Reg}(Q)$, for any $Q$

$$(Z + 1)\text{Reg}(Q) \geq R(P') - R(Q) \geq R(P) - R(Q) - \frac{\text{OPT}}{B}(B - B'),$$

where we used Lemma 2 to get the last inequality. If the algorithm never aborted due to constraint violation in Step 10 the above observation would bound the regret of the algorithm by

$$\sum_m (R(P) - R(Q_{m-1}))(\tau_m - \tau_{m-1}) \leq \sum(Z + 1)(c_0 + 2)K\psi_{m-1}(\tau_m - \tau_{m-1}) + \frac{\text{OPT}}{B}(B - B').$$

Then, using that $Z \leq O(\frac{\text{OPT}}{B})$, $B - B' = O(\sqrt{KT \ln(dT ||\Pi||/\delta)}$, and properly chosen scaling factors ($\psi$ and $\mu_{m-1}$) result in the desired bound of $O(\frac{\text{OPT}}{B}\sqrt{KT \ln(dT ||\Pi||/\delta)})$ for expected regret. An application of Azuma-Hoeffding inequality obtains the high probability regret bound as stated in Theorem 1.

To complete the proof, we show that in fact, with probability $1 - \frac{\delta}{2}$, the algorithm is not aborted in Step 10 due to constraint violation. This involves showing that with high probability, the algorithm’s consumption (in steps $t = 1, \ldots, T_0$) above $B'$ is bounded above by $c\sqrt{KT \ln(||\Pi||/\delta)}$, and since $B' + c\sqrt{KT \ln(||\Pi||/\delta)} + T_0 = B$, we obtain that the algorithm will satisfy the knapsack constraint with high probability. This also explains why we started with a smaller budget. More precisely, we show that for every $m$,

$$\phi(\mathbf{V}(Q_m), B') \leq 4(c_0 + 2)K\psi_m$$

(9)

Recall that $\phi(\mathbf{V}(P), B')$ was defined as the maximum violation of budget $\frac{B'}{T}$ by vector $\mathbf{V}(P)$. To prove the above, we observe that due to our choice of $Z$, $\phi(\mathbf{V}(P), B')$ is bounded by $\text{Reg}(P)$ as follows. By Equation 5, for all $P \in \mathcal{C}(\Pi)$, $R(P') \geq R(P) - \frac{\delta}{2}\phi(\mathbf{V}(P), B')$, so that

$$(Z + 1)\text{Reg}(P) = R(P') - R(P) + Z\phi(\mathbf{V}(P), B') \geq \frac{\delta}{4}\phi(\mathbf{V}(P), S).$$

Then, using the bound of $\text{Reg}(Q_m) \leq (c_0 + 2)K\psi_{m}$, we obtain the bound in Equation 2. Summing this bound over all epochs $m$, and using Jensen’s inequality and convexity of $\phi(\cdot, B')$, we obtain a bound on the max violation of budget constraint $\frac{B'}{T}$ by the algorithm’s expected consumption vector $\frac{1}{T}\sum_m \mathbf{V}(Q_{m-1})(\tau_m - \tau_{m-1})$. This is converted to a high probability bound using Azuma-Hoeffding inequality.
5 The CBwR problem

In this section, we consider a version of the problem with a concave objective function, and show how to get an efficient algorithm for it. The **CBwR problem** is identical to the CBwK problem, except for the following. The outcome in a round is simply the vector \( v \), and the goal of the algorithm is to maximize \( f \left( \frac{1}{T} \sum_{t=1}^{T} v_t(a_t) \right) \), for some concave function \( f \) defined on the domain \([0, 1]^d\), and given to the algorithm ahead of time. The optimum mixed policy is now defined as

\[
P^* = \arg \max_{P \in \mathcal{C}(\Pi)} f(V(P)).
\]  

(10)

The optimum value is \( \text{OPT} = f(V(P^*)) \) and we bound the average regret, which is

\[
\text{avg-regret} := \text{OPT} - f \left( \frac{1}{T} \sum_{t=1}^{T} v_t(a_t) \right).
\]

The main result of this section is an \( O(1/\sqrt{T}) \) regret bound for this problem. Note that the regret scales as \( 1/\sqrt{T} \) rather than \( \sqrt{T} \) since the problem is defined in terms of the average of the vectors rather than the sum. We assume that \( f \) is represented in such a way that we can solve optimization problems of the following form in polynomial time.

For any given \( a \in \mathbb{R}^d \),

\[
\max f(x) + a \cdot x : x \in [0, 1]^d.
\]

**Theorem 6.** For the CBwR problem, if \( f \) is \( L \)-Lipschitz w.r.t. norm \( \| \cdot \| \), then there is a polynomial time algorithm that makes \( \tilde{O}(d\sqrt{KT \ln(|\Pi|)}) \) calls to AMO, and with probability at least \( 1 - \delta \) has regret

\[
\text{avg-regret}(T) = O \left( \frac{\| 1 \|_L}{\sqrt{T}} \left( \sqrt{K \ln(T|\Pi|/\delta)} + \sqrt{\ln(d/\delta)} \right) \right).
\]

**Remark.** A special case of this problem is when there are only constraints, in which case \( f \) could be defined as the negative of the distance from the constraint set. Further, one could handle both concave objective function and convex constraints as follows. Suppose that we wish to maximize \( h(\frac{1}{T} \sum_{t=1}^{T} v_t(a_t)) \), subject to the constraint that \( \frac{1}{T} \sum_{t=1}^{T} v_t(a_t) \in S \), for some \( L \)-Lipschitz concave function \( h \) and a convex set \( S \). Further, suppose that we had a good estimate of the optimum achieved by a static mixed policy, i.e.,

\[
\text{OPT'} := \max_{P \in \mathcal{C}(\Pi)} h(V(P)) \quad \text{s.t.} \quad V(P) \in S.
\]  

(11)

For some distance function \( d(\cdot, S) \) measuring distance of a point from set \( S \), define

\[
f(v) := \min \{ h(v) - \text{OPT'}, -Ld(v, S) \}.
\]

5.1 Algorithm

Since we don’t have any hard constraints and don’t need to estimate \( Z \) as in the case of CBwK, we can drop Steps 2–5 and Step 10 in Algorithm 1 and set \( T_0 = 0 \). The optimization problem (OP) is also the same, but with new definitions of \( \text{Reg}(P), P_t \) and \( \text{Reg}_t(P) \) as below. Recall that \( P^* \) is the optimal policy as given by Equation (10), and \( L \) is the Lipschitz factor for \( f \) with respect to norm \( \| \cdot \| \). We now define the regret of policy \( P \in \mathcal{C}(\Pi) \) as

\[
\text{Reg}(P) := \frac{1}{\| 1 \|_L} (f(V(P^*)) - f(V(P))).
\]

The best empirical policy is now given by

\[
P_t := \arg \max_{P \in \mathcal{C}(\Pi)} f(V_t(P)),
\]  

(12)

and an estimate of the regret of policy \( P \in \mathcal{C}(\Pi) \) at time \( t \) is

\[
\hat{\text{Reg}}_t(P) := \frac{1}{\| 1 \|_L} (f(V_t(P_t)) - f(V_t(P))).
\]

\[\text{This problem has nothing to do with contexts and policies, and only depends on the function } f.\]
Another difference is that we need to solve a convex optimization problem to find $P_t$ (as defined in (12)) once every epoch. A similar convex optimization problem needs to be solved in every iteration of a coordinate descent algorithm for solving (OP) (details of this are in Appendix C.2). In both cases, the problems can be cast in the form

$$\min g(x) : x \in C,$$

where $g$ is a convex function, $C$ is a convex set, and we are given access to a linear optimization oracle, that solves a problem of the form $\min c \cdot x : x \in C$. In (12), for instance, $C$ is the set of all $V_t(P)$ for all $P \in \mathcal{C}(\Pi)$. A linear optimization oracle over this $C$ is just an AMO as in Definition 1. We show how to efficiently solve such a convex optimization problem using cutting plane methods [Vaidya, 1989a, Lee et al., 2015], while making only $\tilde{O}(d)$ calls to the oracle. The details of this are in Appendix C.2.

### 5.2 Regret Analysis: Proof of Theorem 6

We prove that Algorithm 1 and (OP) with the above new definition of $\hat{\text{Reg}}_t(P)$ achieves regret bounds of Theorem 6 for the CBwR problem. A complete proof of this theorem is given in Appendix C. Here, we sketch some key steps.

The first step of the proof is to use constraints in (OP) to prove a lemma akin to Lemma 5 showing that the empirical regret $\hat{\text{Reg}}_t(P)$ and actual regret $\text{Reg}(P)$ are close for every $P \in \mathcal{C}(\Pi)$. Therefore, the first constraint in (OP) that bounds the empirical regret $\hat{\text{Reg}}_t(Q_m)$ of the computed policy implies a bound on the actual regret $\text{Reg}(Q_m) = \frac{1}{mL} \left( f(V(P^*)) - f(V(Q_m)) \right)$. Ignoring the technicalities of sampling process (which are dealt with in the complete proof), and assuming that $Q_m$ is the policy used in epoch $m$, this provides a bound on regret in every epoch. Regret across epochs can be combined using Jensen’s inequality which bounds the regret in expectation. Using Azuma-Hoeffding’s inequality to bound deviation of expected reward vector from the actual reward vector, we obtain the high probability regret bound stated in Theorem 6.

### References

Yasin Abbasi-yadkori, Dávid Pál, and Csaba Szepesvári. Improved algorithms for linear stochastic bandits. In NIPS, 2012.

Alekh Agarwal, Daniel Hsu, Satyen Kale, John Langford, Lihong Li, and Robert E. Schapire. Taming the monster: A fast and simple algorithm for contextual bandits. In ICML 2014, June 2014. URL http://arxiv.org/abs/1402.0555. Full version on arXiv.

Shipra Agrawal and Nikhil R. Devanur. Bandits with concave rewards and convex knapsacks. In Proceedings of the Fifteenth ACM Conference on Economics and Computation, EC ’14, 2014.

Peter Auer, Nicolò Cesa-Bianchi, Yoav Freund, and Robert E. Schapire. The nonstochastic multiarmed bandit problem. SIAM Journal on Computing, 32(1):48–77, 2002.

Moshe Babaioff, Shaddin Dughmi, Robert D. Kleinberg, and Aleksandrs Slivkins. Dynamic pricing with limited supply. ACM Trans. Economics and Comput., 3(1):4, 2015. doi: 10.1145/2559152. URL http://doi.acm.org/10.1145/2559152

Ashwinkumar Badanidiyuru, Robert Kleinberg, and Yaron Singer. Learning on a budget: posted price mechanisms for online procurement. In Proc. of the 13th ACM EC, pages 128–145. ACM, 2012.

Ashwinkumar Badanidiyuru, Robert Kleinberg, and Aleksandrs Slivkins. Bandits with knapsacks. In FOCS, pages 207–216, 2013.

Ashwinkumar Badanidiyuru, John Langford, and Aleksandrs Slivkins. Resourceful contextual bandits. In Proceedings of The Twenty-Seventh Conference on Learning Theory (COLT-14), pages 1109–1134, 2014.
Andrew G. Barto and P. Anandan. Pattern-recognizing stochastic learning automata. *IEEE Transactions on Systems, Man, and Cybernetics*, 15(3):360–375, 1985.

Omar Besbes and Assaf Zeevi. Dynamic pricing without knowing the demand function: Risk bounds and near-optimal algorithms. *Operations Research*, 57(6):1407–1420, 2009.

Alina Beygelzimer, John Langford, Lihong Li, Lev Reyzin, and Robert E. Schapire. Contextual bandit algorithms with supervised learning guarantees. In *Proc. of the 14th AISTats*, pages 19–26, 2011.

Sébastien Bubeck and Nicolò Cesa-Bianchi. Regret analysis of stochastic and nonstochastic multi-armed bandit problems. *Foundations and Trends in Machine Learning*, 5(1):1–122, 2012.

Deepayan Chakrabarti and Erik Vee. Traffic shaping to optimize ad delivery. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, EC ’12, 2012.

Wei Chu, Lihong Li, Lev Reyzin, and Robert E. Schapire. Contextual Bandits with Linear Payoff Functions. *Journal of Machine Learning Research - Proceedings Track*, 15:208–214, 2011.

Wenkui Ding, Tao Qin, Xu-Dong Zhang, and Tie-Yan Liu. Multi-armed bandit with budget constraint and variable costs. In *Proc. of the 27th AAAI*, pages 232–238, 2013.

Miroslav Dudík, Daniel Hsu, Satyen Kale, Nikos Karampatziakis, John Langford, Lev Reyzin, and Tong Zhang. Efficient optimal learning for contextual bandits. In *Proc. of the 27th UAI*, pages 169–178, 2011.

Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer-Verlag, New York, 1988.

Sudipto Guha and Kamesh Munagala. Approximation algorithms for budgeted learning problems. In *STOC*, pages 104–113, 2007.

András György, Levente Kocsis, Ivett Szabó, and Csaba Szepesvári. Continuous time associative bandit problems. In *Proc. of the 20th IJCAI*, pages 830–835, 2007.

John Langford and Tong Zhang. The epoch-greedy algorithm for contextual multi-armed bandits. In *Advances in Neural Information Processing Systems 20*, pages 1096–1103, 2008.

Yin Tat Lee, Aaron Sidford, and Sam Chiu wai Wong. A faster cutting plane method and its implications for combinatorial and convex optimization. In *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*, pages 1049–1065. IEEE, 2015. URL [http://arxiv.org/abs/1508.04874v1](http://arxiv.org/abs/1508.04874v1) Full version on arXiv.

Omid Madani, Daniel J Lizotte, and Russell Greiner. The budgeted multi-armed bandit problem. In *Learning Theory*, pages 643–645. Springer, 2004.

Sandeep Pandey and Christopher Olston. Handling advertisements of unknown quality in search advertising. In *Advances in Neural Information Processing Systems*, pages 1065–1072, 2006.

Ralph Tyrell Rockafellar. *Convex analysis*. Princeton university press, 2015.

Adish Singla and Andreas Krause. Truthful incentives in crowdsourcing tasks using regret minimization mechanisms. In *Proc. of the 22nd WWW*, pages 1167–1178, 2013.

Aleksandrs Slivkins and Jennifer Wortman Vaughan. Online decision making in crowdsourcing markets: Theoretical challenges (position paper). *CoRR*, abs/1308.1746, 2013.

Long Tran-Thanh, Archie C. Chapman, Enrique Munoz de Cote, Alex Rogers, and Nicholas R. Jennings. Epsilon-first policies for budget-limited multi-armed bandits. In *Proc. of the 24th AAAI*, 2010. URL [http://www.aaai.org/ocs/index.php/AAAI/AAAI10/paper/view/1817](http://www.aaai.org/ocs/index.php/AAAI/AAAI10/paper/view/1817)
Long Tran-Thanh, Archie C. Chapman, Alex Rogers, and Nicholas R. Jennings. Knapsack based optimal policies for budget-limited multi-armed bandits. In AAAI, 2012.

Pravin M Vaidya. A new algorithm for minimizing convex functions over convex sets. In Foundations of Computer Science, 1989., 30th Annual Symposium on, pages 338–343. IEEE, 1989a.

Pravin M Vaidya. Speeding-up linear programming using fast matrix multiplication. In Foundations of Computer Science, 1989., 30th Annual Symposium on, pages 332–337. IEEE, 1989b.

Huasen Wu, R. Srikant, Xin Liu, and Chong Jiang. Algorithms with logarithmic or sublinear regret for constrained contextual bandits. CoRR, abs/1504.06937, 2015. URL http://arxiv.org/abs/1504.06937
**A Concentration Inequalities**

**Lemma 7.** (Freedman’s inequality for martingales [Beygelzimer et al., 2011]) Let \( X_1, X_2, \ldots, X_T \) be a sequence of real-valued random variables. Assume for all \( t \in \{1, 2, \ldots, T\}, |X_t| \leq R \) and \( \mathbb{E}[X_t | X_1, \ldots, X_{t-1}] = 0 \). Define \( S := \sum_{t=1}^T X_t \) and \( V := \sum_{t=1}^T \mathbb{E}[X_t^2 | X_1, \ldots, X_{t-1}] \). For any \( \rho \in (0, 1) \) and \( \lambda \in [0, 1/R] \), with probability at least \( 1 - \rho \),

\[
S \leq (e - 2)\lambda V + \frac{1}{\lambda} \ln \frac{1}{\rho}.
\]

**Lemma 8.** (Multiplicative version of Chernoff bounds) Let \( X_1, \ldots, X_n \) denote independent random samples from a distribution supported on \([a, b]\) and let \( \mu := \mathbb{E}[\sum X_i] \). Then, for all \( \epsilon > 0 \),

\[
\Pr \left( \left| \sum_{i=1}^n X_i - \mu \right| \leq \epsilon \mu \right) \leq \exp \left( -\frac{\mu \epsilon^2}{3(b-a)^2} \right).
\]

**Corollary 9.** Let \( X_1, \ldots, X_n \) denote independent random samples from a distribution supported on \([a, b]\) and let \( \bar{\mu} := \frac{1}{n} \mathbb{E}[\sum X_i] \). Then, for all \( \rho > 0 \), with probability at least \( 1 - \rho \),

\[
\left| \frac{1}{n} \sum_{i=1}^n X_i - \bar{\mu} \right| \leq (b - a) \sqrt{\frac{3\bar{\mu} \log(1/\rho)}{n}}.
\]

**Proof.** Given \( \rho > 0 \), use Lemma 8 with

\[
\epsilon = (b - a) \sqrt{\frac{3 \log(1/\rho)}{\mu}},
\]

to get that the probability of the event \( \left| \sum_{i=1}^n X_i - \mu \right| > \epsilon \mu = (b - a) \sqrt{3\bar{\mu} \log(1/\rho)} \) is at most

\[
\exp \left( -\frac{\mu \epsilon^2}{3(b-a)^2} \right) = \exp \left( -\log(1/\rho) \right) = \rho.
\]

**B Setting Z (Proof of Lemma 3)**

We use the first few rounds to do a pure exploration, that is \( \alpha_t \) is picked uniformly at random from the set of arms, and use the outcomes from these results to compute an estimate of OPT. Let

\[
\bar{r}_t(a) := r_t(a) \cdot \mathbb{1}\{a = a_t\},
\]

\[
\bar{v}_t(a) = v_t(a) \cdot \mathbb{1}\{a = a_t\}.
\]

Note that \( \bar{r}_t(a) \in [0,1], \bar{v}_t(P) \in [0,1]^d \). Since \( \alpha_t \) is picked uniformly at random from the set of arms,

\[
\mathbb{E}[\bar{r}_t(a) | H_{t-1}] = \frac{1}{K} \mathbb{E}[r_t(a)], \quad \text{and} \quad \mathbb{E}[\bar{v}_t(a) | H_{t-1}] = \frac{1}{K} \mathbb{E}[v_t(a)].
\]

For any policy \( P \in C_0(\Pi) \), let

\[
r(P) := \mathbb{E}_{(x,r,v) \sim D, \pi \sim P} [r(\pi(x))]
\]
be the actual and estimated means of reward and consumption for a given policy \( P \). Interpreting a policy \( \pi \in \Pi \) as a (degenerated) distribution of policies in \( \Pi \), we slightly abuse notation, defining \( r(\pi), \hat{r}_t(\pi), v(\pi), \) and \( \bar{v}_t(\pi) \) similarly. Observe that for any \( P \in \mathcal{C}_0(\Pi) \),

\[
\mathbb{E}[\hat{r}_t(P)|H_{t-1}] = r(P), \quad \text{and} \quad \mathbb{E}[\bar{v}_t(P)|H_{t-1}] = v(P).
\]

**Lemma 10.** For all \( \delta > 0 \), let \( \eta := \sqrt{3K \log((d+1)|\Pi|/\delta)} \). Then for any \( t \), with probability \( 1 - \delta \), for all \( P \in \mathcal{C}_0(\Pi) \),

\[
|\hat{r}_t(P) - r(P)| \leq \eta \sqrt{r(P)/t},
\]

\[
\forall j, |\bar{v}_t(P)_j - v(P)_j| \leq \eta \sqrt{v(P)_j/t}.
\]

**Proof.** We will first show the first inequality holds with probability \( 1 - \delta/(d + 1) \). The same analysis can be applied to each of the \( d \) dimensions of the consumption vector. The lemma follows by a direct use of the union bound.

Fix a policy \( \pi \in \Pi \). Consider the random variables \( X_\tau = \bar{r}_\tau(\pi(x_\tau)) \), for \( \tau \in [t] \). Note that \( X_\tau \in [0, 1] \), \( \mathbb{E}[X_\tau] = \frac{1}{K} r(\pi) \), and \( \frac{1}{t} \sum_{\tau \in [t]} X_\tau = \frac{1}{K} \hat{r}_t(\pi) \). Applying Corollary 5 to these variables, we get that with probability \( 1 - \delta/(d + 1)|\Pi| \),

\[
|\frac{1}{K} \hat{r}_t(\pi) - \frac{1}{K} r(\pi)| \leq \sqrt{3 \log((d+1)|\Pi|/\delta) \sqrt{r(\pi)/Kt}}.
\]

Equivalently,

\[
|\hat{r}_t(\pi) - r(\pi)| \leq \eta \sqrt{r(\pi)/t}.
\]

Applying a union bound over all \( \pi \in \Pi \), we have, with probability \( 1 - \delta/(d + 1) \), that Equation (13) holds for all \( \pi \in \Pi \). In the rest of the proof, we assume Equation (13) holds.

Now consider a policy \( P \in \mathcal{C}_0(\Pi) \).

\[
|\hat{r}_t(P) - r(P)| \leq \mathbb{E}_{\pi \sim P}[|\hat{r}_t(\pi) - r(\pi)|] \\
\leq \mathbb{E}_{\pi \sim P}[\eta \sqrt{r(\pi)/t}] \\
\leq \eta \sqrt{\mathbb{E}_{\pi \sim P}[r(\pi)]/t} \\
= \eta \sqrt{r(P)/t}.
\]

The inequality in the third line follows from the concavity of the square root function.

We solve a relaxed optimization problem on the sample to compute our estimate. Define \( \hat{\text{OPT}}_t^\gamma \) as the value of optimal mixed policy in \( \mathcal{C}_0(\Pi) \) on the empirical distribution up to time \( t \), when the budget constraints are relaxed by \( \gamma \):

\[
\hat{\text{OPT}}_t^\gamma := \max_{P \in \mathcal{C}_0(\Pi)} \text{s.t.} \quad \frac{T_t(P)}{\mathbb{E}_{\pi \sim P}[\bar{v}_t(\pi)]} \leq (B + \gamma)1
\]

\[
(14)
\]

Let \( P_t \in \mathcal{C}_0(\Pi) \) be the policy that achieves this maximum in (13). Let (as earlier) \( P^* \) denote the optimal policy w.r.t. \( \mathcal{D} \), i.e., the policy that achieves the maximum in the definition of \( \text{OPT} \).

Lemma 3 is now an immediate consequence of the following lemma, for \( \gamma \) and \( t \) as in the lemma, by setting \( Z = \max\{\frac{\text{SOPT}_t^\gamma}{B}, 1\} \).
Lemma 11. Suppose that for the first $t := 12K \ln(\frac{(d+1)|\Pi|}{\delta}) T/B$ rounds the algorithm does pure exploration, pulling each arm with equal probability, and let $\gamma := \frac{B}{T}$. Then with probability at least $1 - \delta$,

$$OPT \leq \max\{2\hat{OPT}_t, B\} \leq 2B + 6OPT.$$

Proof. Let $\eta = \sqrt{3K \log((d + 1)|\Pi|/\delta)}$ be as in Lemma 10. Observe that then $\eta/\sqrt{t} = \sqrt{B/4T}$ and $\eta\sqrt{BT}/t = \gamma$.

By Lemma 10, with probability $1 - \delta$, we have that $\hat{v}_t(P^*) \leq B + \gamma T$, and therefore $P^*$ is a feasible solution to the optimization problem (14), and hence $\hat{OPT}^\gamma_t \geq \hat{v}_t(P^*)$. Again from Lemma 10,

$$T\hat{r}_t(P^*) \geq OPT - \eta\sqrt{OPT/t} = OPT - (\sqrt{OPTB}/2).$$

Now either $B \geq OPT$ or otherwise $OPT - (\sqrt{OPTB}/2) \geq OPT/2$.

In either case, the first inequality in the lemma holds.

On the other hand, again from Lemma 10

$$\forall j, v(P_t)_j - \eta\sqrt{v(P_t)_j/t} \leq \hat{v}(P_t)_j \\
\leq (B + \gamma)/T \\
= 3B/2T \\
= 9B/4T - \eta\sqrt{9B/4Tt}.$$

The second inequality holds since $P_t$ is a feasible solution to (14). The function $f(x) = x - \sqrt{c}x$ is increasing in the interval $[c/4, \infty]$ and therefore $v(P_t)_j \leq 9B/4T$, and $P_t$ is a feasible solution to the optimization problem (1), with budgets multiplied by $9/4$. This increases the optimum value of (1) by at most a factor of $9/4$ and hence $Tr(P_t) \leq 9OPT/4$.

Also from Lemma 10

$$\hat{OPT}^\gamma_t = T\hat{r}_t(P_t) \leq Tr(P_t) + \eta T\sqrt{r(P_t)/t} \\
\leq 9OPT/4 + \sqrt{9OPTB/16}.$$

Once again, if OPT $\geq B$, we get from the above that $\hat{OPT}^\gamma_t \leq 3OPT$. Otherwise, we get that $\hat{OPT}^\gamma_t \leq 9OPT/4 + 3B/4$.

In either case, the second inequality of the lemma holds.

C Implementation details: Solving Optimization Problem (OP) by Coordinate Descent

At the end of every epoch $m$ of Algorithm 1, we solve an optimization problem (OP) to find $Q_m \in C_0(\Pi)$. The same optimization problem is used for both CBwK and CBwR, although with different definitions of $\hat{Reg}(\cdot)$. In this section, we show how to solve the optimization problem (OP) using a Coordinate Descent descent algorithm along with AMO, for both CBwK and CBwR.

In this optimization problem (OP), described in Section 3, $Q \in C_0(\Pi)$ was expressed as $\alpha Q'$ for some $Q' \in C(\Pi)$. It is easy to see that any $Q \in C_0(\Pi)$ can also be expressed as a linear combination of multiple mixed policies in $C(\Pi)$:

$$Q = \sum_{P \in C(\Pi)} \alpha_P(Q) P,$$
for some constants \( \{ \alpha_P(Q) \} \), so that
\[
\forall P \in C(\Pi) : \alpha_P(Q) \geq 0 \quad \text{and} \quad \sum_{P \in C(\Pi)} \alpha_P(Q) \leq 1.
\]

Note that the coefficients \( \{ \alpha_P(Q) \} \) may not be unique. Now, consider the following variant of (OP):

**Optimization Problem (OP')**

Given: \( H_t, \mu_m, \) and \( \psi \).

Let \( b_P := \frac{\hat{\text{Reg}}_t(P)}{\psi_{P_m}}, \forall P \in C(\Pi) \) where \( \psi := 100 \).

Find \( Q = \left( \sum_{P \in C(\Pi)} \alpha_P(Q) P \right) \in C_0(\Pi) \), such that
\[
\sum_{P \in C(\Pi)} \alpha_P(Q) b_P \leq 2K,
\]
\[
\forall P \in C(\Pi) : \hat{E}_{x \in H_t, \pi \sim P} \left[ \frac{1}{Q^{\mu_m}(\pi(x)|x)} \right] \leq b_P + 2K.
\]

**Lemma 12.** The two optimization problems, (OP) and (OP'), are equivalent.

**Proof.** It suffices to prove that, any feasible solution to one problem provides a feasible solution to the other. To see this, first note that any solution \( Q \in C_0(\Pi) \) to (OP) is trivially a solution to (OP').

For the other direction, suppose we are given a solution \( Q \in C_0(\Pi) \) to (OP'). Set \( Q' = \alpha^{-1} \sum_{P \in C(\Pi)} \alpha_P(Q) P \) with \( \alpha = \sum_{P \in C(\Pi)} \alpha_P(Q) \); clearly, \( Q' \in C(\Pi) \). Then, by Jensen’s inequality, as well as the second condition of (OP'), we have
\[
\alpha \hat{\text{Reg}}_t(Q') \leq \alpha \left( \sum_{P \in C(\Pi)} \frac{\alpha_P(Q)}{\sum_{P \in C(\Pi)} \alpha_P(Q)} \hat{\text{Reg}}_t(P) \right) = \sum_{P \in C(\Pi)} \alpha_P(Q) \hat{\text{Reg}}_t(P) = \mu_m \psi \sum_{P \in C(\Pi)} \alpha_P(Q) b_P \leq 2K \psi \mu_m.
\]

Thus, first constraint of (OP) is satisfied. Also, since \( \alpha Q' = Q \), the second constraint of (OP) is trivially satisfied. Therefore, \( \alpha Q' \) is a feasible solution to (OP).

In the rest, we show how to solve (OP') using a coordinate descent algorithm, which assigns a non-zero weight \( \alpha_P(Q) \) to at most one new policy \( P \in C(\Pi) \) in every iteration.

Let us fix \( m \) and use shorthand \( \mu \) for \( \mu_m \). Problem (OP') is of the same form as the optimization problem in \cite{Agarwal2014}, except that the policy set being considered is \( C(\Pi) \) instead of \( \Pi \). We can solve it using Algorithm 2: a coordinate descent algorithm similar to \cite{Agarwal2014,Algorithm 2}.

The lemma below bounds the number of iterations in this algorithm.

**Lemma 13.** The number of times Step 8 of the algorithm is performed is bounded by \( 4 \ln (1/(K \mu))/\mu \).

**Proof.** This follows by applying the analysis of Algorithm 2 in \cite{Agarwal2014} (refer to Section 5) with policy set being \( C(\Pi) \) instead of \( \Pi \). (Their analysis holds for any value of constant \( \mu \), and constants \( b_\pi \) for policies in the policy set being considered).
Algorithm 2 Coordinate Descent Algorithm for Solving (OP)

**Input** History $H_t$, minimum probability $\mu > 0$, initial weights $Q_{init} \in C_0(\Pi)$.

1: $Q \leftarrow Q_{init}$.
2: loop
3: Define, for all $P \in C(\Pi)$,
   
   \[ V_P(Q) := \mathbb{E}_{\pi \sim P} \left[ \hat{E}_{x \sim H_t} \left[ \frac{1}{Q^\mu(\pi(x)|x)} \right] \right], \]
   
   \[ S_P(Q) := \mathbb{E}_{\pi \sim P} \left[ \hat{E}_{x \sim H_t} \left[ \frac{1}{(Q^\mu(\pi(x)|x))^2} \right] \right], \]
   
   \[ D_P(Q) := V_P(Q) - (2K + b_P). \]
4: if $\sum_{P \in C(\Pi)} \alpha_P(Q)(2K + b_P) > 2K$ then
5: Replace $Q$ by $cQ$ so that $Q \in C(\Pi)$, where
   
   \[ c := \frac{2K}{\sum_{P \in C(\Pi)} \alpha_P(Q)(2K + b_P)} < 1. \] (15)
6: end if
7: if there is a $P \in C(\Pi)$ for which $D_P(Q) > 0$ then
8: Update the coefficient for $P$ by
   
   \[ \alpha_P(Q) \leftarrow \alpha_P(Q) + \frac{V_P(Q) + D_P(Q)}{2(1 - K\mu)S_P(Q)}. \]
9: else
10: Halt and output the current set of weights $Q$.
11: end if
12: end loop

Now, since in epoch $m$, $\mu = \mu_m \geq \sqrt{\frac{d_{\text{run}}}{K\tau_m}}$, $d_t = \ln(16t^2||\Pi||(d + 1)/\delta)$. This proves that the algorithm converges in at most $O(\sqrt{KT \ln(T||\Pi||/\delta) \ln(T \ln(T||\Pi||))}) = \tilde{O}(\sqrt{KT \ln(||\Pi||)})$ iterations of the loop.

Next, we discuss how to implement each iteration of the loop. In every iteration in Step 8, we need to identify a $P$ for which $D_P(Q) > 0$, for which we need to access the policy space using AMO. Also, in the beginning before the loop is started, one needs to compute $P_1$ by solving an optimization problem over the policy space. Below, we provide an implementation of these optimization problems using AMO. Since $\hat{\text{Reg}}_t(P)$ is define differently for CBwK and CBwR, the implementation details and number of calls to AMO differ. But importantly, as we show in Lemma 15 and Lemma 16 in both cases each iteration of Algorithm 2 can be implemented using $\tilde{O}(d)$ of AMO calls. Using these results with the above lemma, we obtain that

**Lemma 14.** For CBwK and CBwR, (OP) can be solved using $\tilde{O}(d \sqrt{KT \ln(||\Pi||)})$ calls to the AMO at the end of every epoch.

As an aside, a solution $Q \in C_0(\Pi)$ output by this algorithm has support bounded by the number of calls to AMO during the run of the algorithm (AMO maximizes a linear function, therefore always returns a pure policy). Therefore, the results in the subsections below also prove that the policies returned by this algorithm have small support, and can be compactly represented.
C.1 AMO-based Implementation for CBwK

Lemma 15. For CBwK, Algorithm 2 can be implemented using $\tilde{O}(d)$ call to AMO (Definition 1) in the beginning before the loop is started, and $O(d)$ calls for each iteration of the loop thereafter.

Proof. In the beginning before the loop is started, one needs to compute $P_i$ which means solving the following problem on the policy space:

$$\arg\max_{P \in \mathcal{C}(\Pi)} \hat{R}_t(P) - Z \phi(\hat{V}_t(P), B').$$  \hspace{1cm} (16)

Using the definition of $\phi(\cdot, \cdot)$, observe that this is same as

$$\arg\max_{P \in \mathcal{C}(\Pi), \lambda} \frac{\hat{R}_t(P) - Z\lambda}{\hat{V}_t(P)} \leq \left( \frac{B'}{T} + \lambda \right) 1.$$ \hspace{1cm} (17)

In every iteration of the loop, we need to identify a $P$ for which $D_P(Q) > 0$. (All the other steps of the algorithm can be performed efficiently for $Q$ with sparse support.) Now,

$$D_P(Q) = V_P(Q) - (2K + b_P)$$
$$= V_P(Q) - (2K + \frac{\text{Reg}(P)}{\psi \mu})$$
$$= \left( \frac{1}{T} \sum_{i=1}^t \sum_{x} P(\pi) \left( \frac{\psi \mu(Z+1)}{\hat{V}_t(x) | x \in \mathcal{T}} \right) \right) - \left( 2K + \frac{\hat{R}_t(P) - Z \phi(\hat{V}_t(P), B') - (\hat{R}_t(P) - Z \phi(\hat{V}_t(P), B'))}{\psi \mu(Z+1)} \right).$$

Finding $P$ such that $D_P(Q) > 0$ requires solving $\arg\max_{P \in \mathcal{C}(\Pi)} D_P(Q)$. Once again, using the definition of $\phi(\cdot, \cdot)$, this is equivalent to the following problem:

$$\arg\max_{P \in \mathcal{C}(\Pi), \lambda \geq 0} \frac{1}{T} \sum_{i=1}^t \sum_{x} P(\pi) \left( \psi \mu(Z+1) \frac{\phi(\hat{V}_t(x) | x \in \mathcal{T})}{\hat{V}_t(x) | x \in \mathcal{T}} + \hat{r}_i(\pi) \right) \leq (\frac{B'}{T} + \lambda) 1.$$ \hspace{1cm} (18)

Both problems (17) and (18) are of the following form:

$$\max x - Z\lambda \text{ such that } (x, y, \lambda) \in K_1 \cap K_2.$$ \hspace{1cm} (19)

where

$$K_2 := \{(x, y, \lambda) : y \leq (B'/T + \lambda) 1 \} \cap [0, 1]^{d+2},$$

and

$$K_1 := \begin{cases} (x, y, \lambda) : x = \hat{R}_t(P), y = \hat{V}_t(P) \text{ for some } P \in \mathcal{C}(\Pi), \lambda \in [0, 1] \end{cases}, \text{ for } (17),$$

$$K_1 := \begin{cases} (x, y, \lambda) : x = \frac{1}{T} \sum_{i=1}^t \sum_{x} P(\pi) \left( \psi \mu(Z+1) \frac{\phi(\hat{V}_t(x) | x \in \mathcal{T})}{\hat{V}_t(x) | x \in \mathcal{T}} + \hat{r}_i(\pi) \right), \ y = \hat{V}_t(P) \text{ for some } P \in \mathcal{C}(\Pi), \lambda \in [0, 1] \end{cases}, \text{ for } (18).$$

Recently, Lee et al. (2013, Theorem 49) gave a fast algorithm to solve problems of the form (19), given access to oracles that solve linear optimization problems over $K_1$ and $K_2$. The algorithm makes $\tilde{O}(d)$ calls to these oracles, and takes an additional $\tilde{O}(d^5)$ running time. A linear optimization problem over $K_1$ is equivalent to the AMO; the linear function defines the “rewards” that the AMO optimizes for. A linear optimization problem over $K_2$ is trivial to solve.

Therefore, each of these problems can be solved using $\tilde{O}(d)$ calls to AMO.

\[\square\]

\[10\] Alternately, one could use the algorithms of Vaidya (1989a, b) to solve the same problem, with a slightly weaker polynomial running time.

\[11\] Here, $\tilde{O}$ hides terms of the order $\log^{O(1)}(d/\epsilon)$, where $\epsilon$ is the accuracy needed of the solution.

\[12\] These rewards may not lie in $[0, 1]$ but an affine transformation of the rewards can bring them into $[0, 1]$ without changing the solution.
C.2 AMO-based Implementation for CBwR

**Lemma 16.** For CBwR, Algorithm 2 can be implemented using $\tilde{O}(d)$ call to AMO (Definition 1) in the beginning before the loop is started, and $O(d)$ calls for each iteration of the loop thereafter.

*Proof.* In each iteration, we need to identify a $P$ for which $D_P(Q) > 0$, where

$$D_P(Q) = V_P(Q) - (2K + b_P)$$

$$= V_P(Q) - (2K + \frac{\text{Reg}_c(P)}{\psi\mu})$$

$$= \left(\frac{1}{t} \sum_{i=1}^{t} \sum_{\pi} P(\pi) \frac{1}{Q(\pi(x_i)|x_i)}\right) - \left(2K + \frac{f(\hat{V}_t(P), S) - f(\hat{V}_t(P_t), S)}{\psi\mu \|1_d\|}\right).$$

Finding $P$ such that $D_P(Q) > 0$ requires solving $\arg \max_{P \in C(\Pi)} D_P(Q)$, which is essentially minimizing a convex function over a convex set

$$C = \{y \in [0, 1]^{d+1} : y = \left[\frac{1}{t} \sum_{i=1}^{t} \sum_{\pi} \left(\frac{P(\pi)}{Q(\pi(x_i)|x_i)}\right) : \hat{V}_t(P), \exists P \in C(\Pi)\},$$

using only a linear optimization oracle (AMO) over $C$.

Similarly, the problem of finding $P_t$, i.e., solving

$$\arg \max_{P \in C(\Pi)} f(\hat{R}_t(P)),$$

using access to AMO, can also be formulated as minimizing a convex function over a convex set, using only a linear optimization oracle. In fact, below we show that given any convex function $g$, a convex set $C$, both over the domain $[0, 1]^d$, and access to a linear optimization oracle over $C$, that solves a problem of the form $\min_{x \in C} c \cdot x : x \in C$, the problem $\min_{x \in C} g(x) : x \in C$ can be solved using $\tilde{O}(d)$ calls to the linear optimization oracle. This completes the proof. \hfill \Box

**Lemma 17.** Suppose that we are given a convex function $g$, a convex set $C$ with non-empty relative interior, both over the domain $[0, 1]^d$, and access to a linear optimization oracle over $C$, that solves a problem of the form $\min_{x \in C} c \cdot x : x \in C$. Then, the problem $\min_{x \in C} g(x) : x \in C$ can be solved using $\tilde{O}(d)$ calls to the linear optimization oracle and an additional $O(d^2)$ running time.

The proof of this lemma uses tools from convex optimization. We show how to solve this convex optimization problem using cutting plane methods [Vaidya, 1989; Lee et al., 2015]. We first show a simple variant of these cutting plane algorithms that can be used to solve a convex optimization problem such as the one above, given access to a separation oracle over the convex set $C$, and a subgradient oracle for the function $g$. Then we define a dual optimization problem, and show that a separation oracle for the dual constraint set can be implemented using a linear optimization oracle over $C$; thus we can solve the dual problem using cutting plane methods. Finally, we show that once the dual problem is solved, for the primal problem, it is sufficient to optimize over the convex hull of the vectors in $C$ returned by the linear optimization oracle over $C$, during the run of the algorithm. Since the number of such vectors is only $\tilde{O}(d)$, this can then be done efficiently.

A *separation oracle* for a convex set $C$ is such that given a point $x$, it returns either

- that $x \in C$, or
- a separating hyperplane, given by $a$ and $b$ s.t. $a \cdot x \geq b$ but $a \cdot y < b \forall y \in C$.

Cutting plane methods solve a convex optimization problem of the form ‘find $x \in C$, or return that $C$ is empty’, given access to a separation oracle for $C$. We first outline how to use cutting plane methods to solve an optimization problem of the form ‘$\min g(x)$ s.t. $x \in C$’, given access to a subgradient oracle for $f$ and a separation oracle for $C$. (One can use binary search on the optimum value to reduce it to a feasibility problem, but we show here how one can directly
use a cutting plane algorithm.) Given any point \( x \), we first run the separation oracle for \( C \) with input \( x \), and return a separating hyperplane if that is what the oracle returns. If the separation oracle for \( C \) returns that \( x \in C \), then we return a separating hyperplane of the following form, with \( y \) as the variable.

\[
\nabla g(x) \cdot y < \nabla g(x) \cdot x,
\]

where \( \nabla g \) is any subgradient of \( g \) at \( x \). This is a valid inequality for \( y = x^* := \arg \min g(x) : x \in C \), and \( x \neq x^* \), due to the convexity of \( g \). If the set of inequalities we return during the run of the algorithm becomes infeasible, then it must include an inequality of this kind for some point \( x \) with \( \|x - x^*\| \leq \epsilon \), where \( \epsilon \) is the accuracy of the solution.

We do this until the cutting plane algorithm returns that the set is empty, at which point we find the point \( \arg \min g(x) : x \in C \), and \( x \) was queried during the run of the cutting plane algorithm. We return this as the optimum point.

The cutting plane algorithm outlined above cannot be applied directly to our problem since we do not have a separation oracle for \( C \). It is well known that separation and (linear) optimization are polynomial time equivalent for convex sets, using the ellipsoid method [Grötschel et al. 1988]. Since we have a linear optimization oracle for \( C \) we could use this reduction to get a separation oracle. We show a more efficient method here, by using this oracle to solve the dual optimization problem. Define the Fenchel conjugate of \( g \) as

\[
g^*(\theta) := \max_x \{ \theta \cdot x - g(x) \},
\]

and let the support function of the set \( C \) be

\[
h_C(\theta) := \max_x \{ \theta \cdot x : x \in C \}.
\]

**Lemma 18.**

\[
- \min g^*(\theta) + h_C(\theta) \leq \min g(x) : x \in C.
\]

This holds with equality if \( C \) has a non-empty relative interior. The former optimization problem is called the dual of the latter.

**Proof.** The proof follows from the fact that \( h_C \) is the Fenchel conjugate of the indicator function of \( C \) (which is 0 inside \( C \) and \( \infty \) outside). This is a special case of Theorem 13.1 in [Rockafellar 2015].

A subgradient oracle for \( g^* \) can be implemented if we can solve the unconstrained optimization problem, \( \max \theta \cdot x - g(x) \). We assume that \( g \) is represented in such a way that we can solve this in polynomial time. A subgradient for \( h_C \) is simply the \( \arg \max \) in its definition, and this is essentially what the linear optimization oracle gives us.

We use the cutting plane algorithm of [Lee et al. 2015] to solve the problem, \( \min g^*(\theta) + h_C(\theta) \), as outlined above. The algorithm runs in time \( \tilde{O}(d^3) \) time and makes \( \tilde{O}(d) \) calls to the separation/subgradient oracle. Let \( x_1, x_2, \ldots x_N \in C \) denote the subgradients of \( h_C \) returned during the run of this algorithm. Then this run of the algorithm would remain unchanged if \( C \) were to be replaced with \( \text{Conv}(x_1, x_2, \ldots x_N) \), the convex hull of \( x_1, x_2, \ldots x_N \). Therefore the optima of these two convex programs are close to each other, and by strong duality, so are the optima of their duals. Hence

\[
\min g(x) : x \in \text{Conv}(x_1, x_2, \ldots x_N)
\]

is a good approximation to \( \min g(x) : x \in C \) (the problem we originally set out to solve). Further, this convex program can be solved efficiently since \( N = \tilde{O}(d) \).

**D Regret Analysis for Section 4: CBwK**

The regret analysis is structurally similar to that of [Agarwal et al. 2014], but differs in many important details as we also need to consider budget constraints.
The following quantities, already defined in the main text, are repeated here for convenience:

\[ B' = B - T_0 - c\sqrt{KT\ln(T\|\Pi|/\delta)}, \]
\[ \phi(v, B') = \max_{j=1,\ldots,d} \left( v_j - \frac{B'}{T} \right)^+, \]
\[ \text{Reg}(P) = \frac{1}{(Z + 1)} \left( R(P') - R(P) + Z\phi(V(P), B') \right), \]
\[ \widehat{\text{Reg}}_{t}(P) = \frac{1}{(Z + 1)} \left[ \hat{R}_t(P) - Z\phi(\hat{V}_t(P_t), B') - \left( \hat{R}_t(P) - Z\phi(\hat{V}_t(P), B') \right) \right]. \]

Fix the epoch schedule \( 0 = \tau_0 < \tau_1 < \tau_2 < \ldots \), such that \( \tau_m < \tau_{m+1} \leq 2\tau_m \) for \( m \geq 1 \). The following quantities are defined for convenience: for \( m \geq 1 \),

\[ d_t := \ln(16t^2\|\Pi|(d + 1)/\delta), \]
\[ m_0 := \min\{m \in \mathbb{N} : \frac{d_m}{\tau_m} \leq \frac{1}{4K} \}, \]
\[ t_0 := \min\{t \in \mathbb{N} : \frac{d_t}{t} \leq \frac{1}{4K} \}, \]
\[ \rho := \sup_{m \geq m_0} \sqrt{\frac{\tau_m}{\tau_{m-1}}}. \]

A few quick observations are in place. First, the quantity \( \mu_m \), as defined in Algorithm \( \mathbb{I} \) can be rewritten as \( \mu_m = \min\left\{ \frac{1}{2K} : \sqrt{\frac{d_m}{K\tau_m}} \right\} \). For \( m \geq m_0, \mu_m = \sqrt{\frac{d_m}{K\tau_m}} \). Furthermore, \( d_t/t \) is non-increasing in \( t \) and \( \mu_m \) is non-increasing in \( m \). Finally, \( \rho \leq \sqrt{2} \) since \( \tau_{m+1} \leq 2\tau_m \).

Finally, recall that Algorithm \( \mathbb{I} \) consists of two phases. The first phase consists of pure exploration of \( T_0 = \frac{12kT}{B} \ln \frac{3m}{\delta} \) steps to estimate \( Z \) (see Appendix \( \mathbb{B} \)), followed by a second phase that explores adaptively. The total regret of Algorithm \( \mathbb{I} \) is the sum of regret in the two phases. Most of this appendix is devoted to the regret analysis of the second phase. Note that the number of time steps in second phase is \( T' = T - T_0 \). For simplicity, we use \( T \) instead of \( T' \) in the proofs below. Since \( T_0 = o(T) \), this changes regret bounds by at most a constant factor.

### D.1 Technical Lemmas

**Definition 2** (Variance estimates). Define the following for any probability distributions \( P, Q \in C(\Pi) \), any policy \( \pi \in \Pi \), and \( \mu \in [0, 1/K] \):

\[ \text{Var}(Q, \pi, \mu) := \mathbb{E}_{x \sim D_X} \left[ \frac{1}{Q^\mu(\pi(x)|x)} \right], \]
\[ \hat{\text{Var}}_m(Q, \pi, \mu) := \mathbb{E}_{x \sim H_{\tau_m}} \left[ \frac{1}{Q^\mu(\pi(x)|x)} \right], \]
\[ \text{Var}(Q, P, \mu) := \mathbb{E}_{x \sim P} [\text{Var}(Q, \pi, \mu)], \]
\[ \hat{\text{Var}}_m(Q, P, \mu) := \mathbb{E}_{x \sim P} [\hat{\text{Var}}_m(Q, \pi, \mu)], \]

where \( \mathbb{E}_{x \sim H_{\tau_m}} \) denotes average over records in history \( H_{\tau_m} \).

Furthermore, let \( m(t) := \min\{m \in \mathbb{N} : t \leq \tau_m \} \), be the index of epoch containing round \( t \), and define

\[ V_t(P) := \max_{0 \leq m < m(t)} \{ \text{Var}(\hat{Q}_m, P, \mu_m) \}, \]

for all \( t \in \mathbb{N} \) and \( P \in C(\Pi) \).

**Definition 3.** Define \( E \) as the event that the following statements hold
• For all probability distributions \( P, Q \in \mathcal{C}(\Pi) \) and all \( m \geq m_0 \),

\[
\Var(Q, P, \mu_m) \leq 6.4 \Var_m(Q, P, \mu_m) + 81.3 K ,
\]

(21)

• For all \( P \in \mathcal{C}(\Pi) \), all epochs \( m \) and all rounds \( t \) in epoch \( m \), and any choices of \( \lambda_{m-1} \in [0, \mu_{m-1}] \),

\[
|\hat{R}_t(P) - R(P)| \leq \mathcal{V}_t(P) \lambda_{m-1} + \frac{d_t}{t \lambda_{m-1}} ,
\]

(22a)

\[
\|\hat{V}_t(P) - V(P)\|_\infty \leq \mathcal{V}_t(P) \lambda_{m-1} + \frac{d_t}{t \lambda_{m-1}} ,
\]

(22b)

**Lemma 19.** \( \Pr(\mathcal{E}) \geq 1 - (\delta/2) \).

**Proof.** Lemma 10 in [Agarwal et al., 2014] can be readily applied to show that, with probability \( 1 - \delta/4 \),

\[
\Var(Q, \pi, \mu_m) \leq 6.4 \Var_m(Q, \pi, \mu_m) + 81.3 K
\]

for all \( Q \in \mathcal{C}(\Pi) \) and \( \pi \in \Pi \). Now, taking expectations on both side over \( \pi \sim P \), we get the first condition.

For the second condition, the proof is similar to the proof of Lemma 11 in [Agarwal et al., 2014], but with some changes to account for distribution over policies. Fix component \( j \) of the consumption vector, policy \( \pi \in \Pi \) and time \( t \in [T] \). Then,

\[
\hat{V}_t(\pi)_j - V(\pi)_j = \frac{1}{t} \sum_{i=1}^t Y_i ,
\]

where \( Y_i := v_i(\pi(x_i))_j - \hat{v}_i(\pi(x_i))_j \).

Round \( i \) is in epoch \( m(i) \leq m \), so

\[
|Y_i| \leq \frac{1}{Q_{m(i)-1}^\mu(\pi(x_i)|x_i)} \leq \frac{1}{\mu_{m(i)-1}} \leq \frac{1}{\mu_{m-1}} ,
\]

by definition of fictitious reward vector \( \hat{v}_t \). Furthermore, \( \mathbb{E}[Y_i|H_{t-1}] = 0 \) and

\[
\mathbb{E}[Y_i^2|H_{t-1}] \leq \mathbb{E}[v_i(\pi(x_i))^2|H_{t-1}] \leq \Var(\hat{Q}_{m(i)-1}, \pi, \mu_{m(i)-1})
\]

from the definition of fictitious reward and of \( \Var(Q, \pi, \mu) \).

Let \( U(\pi) := \frac{1}{t} \sum_{i=1}^t \Var(Q_{m(i)-1}, \pi, \mu_{m(i)-1}) \geq \frac{1}{t} \sum_{i=1}^t \mathbb{E}[Z_i^2|H_{t-1}] \). Then, by Freedman’s inequality (Lemma 7) and a union bound to the sums \((1/t) \sum_{i=1}^t Y_i \) and \((1/t) \sum_{i=1}^t (-Y_i)\), we have that with probability at least \( 1 - 2\delta/(16t^2(d+1)|\Pi|) \), for all \( \lambda_{m-1} \in [0, \mu_{m-1}] \),

\[
\frac{1}{t} \sum_{i=1}^t Y_i \leq (e - 2)U(\pi)\lambda_{m-1} + \frac{\ln(16t^2(d+1)|\Pi|/\delta)}{t \lambda_{m-1}} , \quad \text{and}
\]

\[
\frac{1}{t} \sum_{i=1}^t Y_i \leq (e - 2)U(\pi)\lambda_{m-1} + \frac{\ln(16t^2(d+1)|\Pi|/\delta)}{t \lambda_{m-1}} .
\]

Taking union bound over all choices of \( t \leq T \) and \( \pi \in \Pi \), we have that, with probability at least \( 1 - \frac{\delta}{4(d+1)} \), for all \( \pi \) and \( t \),

\[
\hat{V}_t(\pi)_j - V(\pi)_j \leq (e - 2)U(\pi)\lambda_{m-1} + \frac{d_t}{t \lambda_{m-1}} , \quad \text{and}
\]

\[
V(\pi)_j - \hat{V}_t(\pi)_j \leq (e - 2)U(\pi)\lambda_{m-1} + \frac{d_t}{t \lambda_{m-1}} .
\]

(23)
Note that
\[
\mathbb{E}_{\pi \sim P}[U(\pi)] = \frac{1}{t} \sum_{i=1}^{t} \mathbb{E}_{\pi \sim P}[V(Q_{m(i)-1}, \pi, \mu_{m(i)-1})]
\]
\[
= \frac{1}{t} \sum_{i=1}^{t} V(Q_{m(i)-1}, P, \mu_{m(i)-1})
\]
\[
\leq \frac{1}{t} \sum_{i=1}^{t} \mathcal{V}_t(P) = \mathcal{V}_t(P),
\]
by definition of Var($Q, P, \mu$). Also, by definition, $\mathbb{E}_{\pi \sim P}[\mathcal{V}(\pi)_j] = \mathcal{V}(P)_j, \mathbb{E}_{\pi \sim P}[\hat{\mathcal{V}}(\pi)_j] = \hat{\mathcal{V}}(P)_j$. Therefore, taking expectation with respect $\pi \sim P$ on both sides of Equations (22) and (24), we get that, with probability $1 - \frac{\delta}{4(d+1)}$, for all $P \in \mathcal{C}(\Pi)$
\[
\hat{\mathcal{V}}_t(P)_j - \mathcal{V}(P)_j \leq (e-2)\mathcal{V}_t(P)\lambda_{m-1} + \frac{d_t}{t\lambda_{m-1}}, \quad \text{and} \quad (25a)
\]
\[
\mathcal{V}(P)_j - \hat{\mathcal{V}}_t(P)_j \leq (e-2)\mathcal{V}_t(P)\lambda_{m-1} + \frac{d_t}{t\lambda_{m-1}}, \quad (25b)
\]
Note that Equation (22a) for rewards can be similarly proved to hold with probability $1 - \frac{\delta}{4(d+1)}$. Applying a union bound over reward and the $d$ dimensions of the consumption vector, we have that Equation (22a) holds for all $t$ and all $P \in \mathcal{C}(\Pi)$ with probability $1 - \frac{\delta}{4}$. \hfill $\square$

**Lemma 20.** Assume event $\mathcal{E}$ holds. Then for all $m \leq m_0$, and all rounds $t$ in epoch $m$,
\[
|\hat{R}_t(P) - R(P)| \leq \max\left\{ \frac{\sqrt{4Kd_t\mathcal{V}_t(P)}}{t}, \frac{4Kd_t}{t} \right\},
\]
\[
\|\hat{\mathcal{V}}_t(P) - \mathcal{V}(P)\|_\infty \leq \max\left\{ \frac{\sqrt{4Kd_t\mathcal{V}_t(P)}}{t}, \frac{4Kd_t}{t} \right\}. \quad (26b)
\]

**Proof.** We only prove the second inequality, as the first may be thought of as a one-dimensional special case of the second. By definition of $m_0$, for all $m' < m_0$, we have $\mu_{m'} = \frac{1}{2K}$. Therefore $\mu_{m-1} = \frac{1}{2K}$. First consider the case when $\sqrt{\frac{d_t}{\mathcal{V}_t(P)}} < \mu_{m-1} = \frac{1}{2K}$. Then, substitute $\lambda_{m-1} = \sqrt{\frac{d_t}{\mathcal{V}_t(P)}}$, we get
\[
\|\hat{\mathcal{V}}_t(P) - \mathcal{V}(P)\|_\infty \leq \sqrt{\frac{4Kd_t\mathcal{V}_t(P)}{t}}.
\]
Otherwise,
\[
\mathcal{V}_t(P) \leq \frac{4K^2d_t}{t}
\]
Substituting $\lambda_{m-1} = \mu_{m-1} = \frac{1}{2K}$, we get
\[
\|\hat{\mathcal{V}}_t(P) - \mathcal{V}(P)\|_\infty \leq \frac{4Kd_t}{t}.
\]
\hfill $\square$

**Lemma 21.** Assume event $\mathcal{E}$ holds. Then, for all $m$, all $t$ in round $m$, all choices of distributions $P \in \mathcal{C}(\Pi)$
\[
|\hat{R}_t(P) - R(P)| \leq \begin{cases} \max\left\{ \frac{\sqrt{4Kd_t\mathcal{V}_t(P)}}{t}, \frac{4Kd_t}{t} \right\}, & m \leq m_0 \\ \mathcal{V}_t(P)\mu_{m-1} + \frac{d_t}{\mu_{m-1}}, & m > m_0 \end{cases}, \quad (27a)
\]
\[
\|\hat{\mathcal{V}}_t(P) - \mathcal{V}(P)\|_\infty \leq \begin{cases} \max\left\{ \frac{\sqrt{4Kd_t\mathcal{V}_t(P)}}{t}, \frac{4Kd_t}{t} \right\}, & m \leq m_0 \\ \mathcal{V}_t(P)\mu_{m-1} + \frac{d_t}{\mu_{m-1}}, & m > m_0 \end{cases}. \quad (27b)
\]

24
Proof. Follows from the definition of event $\mathcal{E}$ and Lemma 20.

**Lemma 22.** Assume event $\mathcal{E}$ holds. Then for $t \geq t_0$ in epoch $m_0$,

$$|\hat{R}_t(P) - R(P)| \leq \sqrt{\frac{8Kd_t}{t}}, \quad (28a)$$

$$||\hat{V}_t(P) - V(P)||_\infty \leq \sqrt{\frac{8Kd_t}{t}}. \quad (28b)$$

Proof. Follows from Lemma 21 using that $\forall t \geq t_0$ and $\frac{4Kd_t}{t} \leq 1$ for $t \geq t_0$ in epoch $m_0$.

**Lemma 23.** Assume event $\mathcal{E}$ holds. For any round $t \in [T]$, and any policy $P \in \mathcal{C}(\Pi)$, let $m \in \mathbb{N}$ be the epoch achieving the max in the definition of $\mathcal{V}_t(P)$. Then,

$$\mathcal{V}_t(P) \leq \begin{cases} 2K \theta_1 K + \frac{\text{Reg}_m(P)}{\theta_2 \mu_m} & \text{if } \mu_m = \frac{1}{2K}, \\ \theta_1 K + \frac{\text{Reg}_m(P)}{\theta_2 \mu_m} & \text{if } \mu_m < \frac{1}{2K}, \end{cases}$$

where $\theta_1 = 94.1$ and $\theta_2 = \psi/6.4 = 100/6.4$ are universal constants.

Proof. Fix a round $t$ and a policy distribution $P \in \mathcal{C}(\Pi)$. Let $m = m(t)$ be the epoch achieving the max in the definition of $\mathcal{V}_t(P)$ (Definition 2), so $\mathcal{V}_t(P) = V(Q_m(P), \mu_m)$ and $\mu_m = 1/(2K)$, which immediately implies $\mathcal{V}_t(P) \leq 2K$ by definition.

If $\mu_m < 1/(2K)$, then $\mu_m = \min\{\frac{1}{2K}, \sqrt{\frac{d_t}{K \tau_m}}\} = \sqrt{\frac{d_t}{K \tau_m}}$, and we have

$$V(Q_m(P), \mu_m) \leq 6.4V(Q_m(P), \mu_m) + 81.3K$$

$$\leq 6.4V(Q_m(P), \mu_m) + 81.3K$$

$$\leq 6.4 \left(2K + \frac{\text{Reg}_m(P)}{\psi \mu_m}\right) + 81.3K$$

$$= \theta_1 K + \frac{\text{Reg}_m(P)}{\theta_2 \mu_m},$$

where the first step is from Equation (21) (which holds in event $\mathcal{E}$); the second step is from the observation that $Q_m(\pi) \geq Q_m(\pi)$ for all $\pi \in \Pi$; the third step is from the constraint in (OP) that $Q_m$ satisfies; and the last step follows from the universal constants $\theta_1$ and $\theta_2$ defined earlier.

**Lemma 24.** Assume event $\mathcal{E}$ holds. Define $c_0 := 4\rho(1 + \theta_1)$. For all epochs $m \geq m_0$, all rounds $t \geq t_0$ in epoch $m$, and all policies $P \in \mathcal{C}(\Pi)$,

$$\text{Reg}(P) \leq 2\text{Reg}_t(P) + c_0 K \mu_m$$

$$\text{Reg}_t(P) \leq 2\text{Reg}_t(P) + c_0 K \mu_m,$$

for $\text{Reg}(P), \text{Reg}_t(P)$ as defined in Section 5.

Proof. Proof is by induction. For base case $m = m_0$, and $t \geq t_0$ in epoch $m$.

Consider $m = m_0$, and $t \geq t_0$ in epoch $m$. For all $P \in \mathcal{C}(\Pi)$,

$$(Z + 1)(\text{Reg}_t(P) - \text{Reg}(P)) = \hat{R}_t(P_t) - \hat{R}_t(P) - R(P') + R(P)$$

$$-Z \left(\phi(\hat{V}_t(P_t), B') - \phi(\hat{V}_t(P_t), B') + \phi(V(P), B')\right). \quad (29)$$

25
W.l.o.g., we can assume that $B \geq 2(\frac{12KT}{\delta} \ln \frac{d(H)}{\delta}) + 2c\sqrt{KT\ln(T||/\delta)}$, because otherwise $B = O(\sqrt{KT\ln(dT||/\delta)}$ and the regret bound of Theorem 1 is trivial. Under this assumption, $B \geq 2(B - B')$, so that $B' \geq B/2$. Also, observe that since $B \geq B'$, $\text{OPT}(B) \geq \text{OPT}(B')$. Then, by Lemma 2 and choice of $Z$ as specified by Lemma 3, we have that for any $\gamma \geq 0$

$$\text{OPT}(B' + \gamma) \leq \text{OPT}(B') + \frac{Z}{2}\gamma. \quad (30)$$

Now, since $P'$ is optimal policy for budget $B'$, we obtain that $R(P') = \text{OPT}(B')$. Also, by definition of $\phi(V(P), B')$, $R(P)$ can violate any budget constraint by at most $\phi(V(P), B')$, which gives $R(P) \leq \text{OPT}(B' + \phi(V(P), B'))$. Therefore, using (30) with $\gamma = \phi(V(P), B')$,

$$R(P') \geq R(P) - \frac{Z}{2}\phi(V(P), B') \geq R(P) - Z\phi(V(P), B').$$

Substituting bounds from Lemma 4, we obtain,

$$(Z + 1)(\hat{\text{Reg}}(P) - \text{Reg}(P)) \leq \hat{R}_t(P) - \hat{R}_t(P) - R(P) + Z\phi(V(P)B') + R(P) - Z\left(\phi(V_t(P), B') - \phi(V_t(P), B') + \phi(V(P), B')\right)$$

$$\leq |\hat{R}_t(P) - R(P)| + |\hat{R}_t(P) - R(P)| + Z\|\hat{V}_t(P) - V(P)\|_\infty + Z\|\hat{V}_t(P) - V(P)\|_\infty. \quad (31)$$

For the other side, by definition of $P_t$, we have that $\hat{R}_t(P_t) - Z\phi(\hat{V}(P_t), B') \geq \hat{R}_t(P) - Z\phi(V(P), B')$ for any $P \in C(P)$. Substituting in (29), and using that $\phi(V(P'), B') = 0$, we get

$$(Z + 1)(\hat{\text{Reg}}(P) - \text{Reg}(P)) \geq \hat{R}_t(P') - \hat{R}_t(P) - R(P) + R(P) - Z\left(\phi(\hat{V}_t(P'), B') - \phi(\hat{V}_t(P), B') + \phi(V(P), B')\right)$$

$$\geq -|\hat{R}_t(P') - R(P')| - |\hat{R}_t(P) - R(P)| - Z\|\hat{V}_t(P') - V(P')\|_\infty - Z\|\hat{V}_t(P) - V(P)\|_\infty. \quad (31)$$

Therefore,

$$(Z + 1)(\text{Reg}(P) - \hat{\text{Reg}}(P)) \leq |\hat{R}_t(P') - R(P')| + |\hat{R}_t(P) - R(P)| + Z\|\hat{V}_t(P') - V(P')\|_\infty + Z\|\hat{V}_t(P) - V(P)\|_\infty. \quad (32)$$

Substituting bounds from Lemma 4, we obtain,

$$|\hat{\text{Reg}}(P) - \text{Reg}(P)| \leq 2\sqrt{\frac{SKd_t}{1}} \leq c_0K\mu,$$

for $c_0 \geq 4\sqrt{2}$. The base case then follows from the non-negativity of $\hat{\text{Reg}}(P)$ and $\text{Reg}(P)$.

Now, fix some epoch $m > m_0$. We assume as the inductive hypothesis that for all epochs $m' < m$, all rounds $t'$ in epoch $m'$, and all $P \in P$,

$$\text{Reg}(P) \leq 2\hat{\text{Reg}}_{t'}(P) + c_0K\mu_{m'},$$

$$\hat{\text{Reg}}_{t'}(P) \leq 2\text{Reg}(P) + c_0K\mu_{m'}.$$
Fix a round $t$ in epoch $m$ and policy $P \in C(\Pi)$. Using Equation (32) and Equation (22) (which holds under event $\mathcal{E}$),

$$\text{Reg}(P) - \hat{\text{Reg}}_{t}(P) \leq \frac{1}{(Z + 1)} \left( |\hat{R}_{t}(P') - R(P')| + |\hat{R}_{t}(P) - R(P)| \right)$$

$$+ Z\|\hat{\mathbf{V}}_{t}(P') - \mathbf{V}(P')\|_{\infty} + Z\|\hat{\mathbf{V}}_{t}(P) - \mathbf{V}(P)\|_{\infty} \leq (V_{t}(P) + V_{t}(P'))\mu_{m-1} + \frac{2d_{t}}{t\mu_{m-1}}. \quad (33)$$

Similarly, using Equation (31),

$$\hat{\text{Reg}}_{t}(P) - \text{Reg}(P) \leq (V_{i}(P_{t}) + V_{i}(P))\mu_{m-1} + \frac{2d_{t}}{t\mu_{m-1}} \quad (34)$$

By Lemma 22 there exist epochs $m', m'' < m$ such that

$$V_{i}(P) \leq \theta_{1}K + \frac{\hat{\text{Reg}}_{t}(P)}{\theta_{2}\mu_{m'}} \mathbb{I}\left\{ \mu_{m'} < \frac{1}{2K} \right\},$$

$$V_{i}(P') \leq \theta_{1}K + \frac{\hat{\text{Reg}}_{t}(P')}{\theta_{2}\mu_{m''}} \mathbb{I}\left\{ \mu_{m''} < \frac{1}{2K} \right\}. \quad (35)$$

If $\mu_{m'} < 1/(2K)$, then $m_{0} \leq m' \leq m - 1$, and the inductive hypothesis implies

$$\frac{\hat{\text{Reg}}_{\tau_{m'}}(P)}{\theta_{2}\mu_{m'}} \leq \frac{2\text{Reg}(P) + c_{0}K\mu_{m'}}{\theta_{2}\mu_{m'}} = \frac{c_{0}K}{\theta_{2}},$$

where the last step uses the fact that $\mu_{m'} \geq \mu_{m-1}$ for $m' \leq m - 1$. Therefore, no matter whether $\mu_{m'} < 1/(2K)$ or not, we always have

$$V_{i}(P)\mu_{m-1} \leq \left( \theta_{1} + \frac{c_{0}}{\theta_{2}} \right) K\mu_{m-1} + \frac{2}{\theta_{2}}\text{Reg}(P). \quad (36)$$

If $\mu_{m''} < 1/(2K)$, then $m_{0} \leq m'' \leq m - 1$, and the inductive hypothesis implies

$$\frac{\hat{\text{Reg}}_{\tau_{m''}}(P')}{\theta_{2}\mu_{m''}} \leq \frac{2\text{Reg}(P') + c_{0}K\mu_{m''}}{\theta_{2}\mu_{m''}} = \frac{c_{0}K}{\theta_{2}},$$

where the last step uses the fact that $\text{Reg}(P') = 0$. Therefore, no matter whether $\mu_{m''} < 1/(2K)$ or not, we always have

$$V_{i}(P')\mu_{m-1} \leq \left( \theta_{1} + \frac{c_{0}}{\theta_{2}} \right) K\mu_{m-1}. \quad (36)$$

Combining Equations (33), (35) and (36) gives

$$\text{Reg}(P) \leq \frac{1}{1 - 2/\theta_{2}} \left( \hat{\text{Reg}}_{t}(P) + 2(\theta_{1} + \frac{c_{0}}{\theta_{2}})K\mu_{m-1} + \frac{2d_{t}}{t\mu_{m-1}} \right). \quad (37)$$

Since $m > m_{0}$, the definition of $\rho$ ensures that $\mu_{m-1} \leq \rho\mu_{m}$. Also, since $t > \tau_{m-1}$, $\frac{d_{t}}{t\mu_{m-1}} \leq \frac{K\mu_{m-1}}{\mu_{m-1}} \leq \rho K\mu_{m}$. Applying these inequalities and the facts $c_{0} = 4(1 + \theta_{1})$ and $\theta_{2} \geq 8\rho$ in Equation (37), we have thus proved

$$\text{Reg}(P) \leq 2\hat{\text{Reg}}_{t}(P) + c_{0}K\mu_{m}. \quad (38)$$

The other part can be proved similarly. By Lemma 23 there exist epochs $m'' < m$ such that

$$V_{i}(P_{t}) \leq \theta_{1}K + \frac{\hat{\text{Reg}}_{t}(P_{t})}{\theta_{2}\mu_{m''}} \mathbb{I}\left\{ \mu_{m''} < \frac{1}{2K} \right\}. \quad (39)$$
If $\mu_{m''} < 1/(2K)$, then $m_0 \leq m'' \leq m - 1$, and the inductive hypothesis together with Equation (38) imply

$$\frac{\widehat{\text{Reg}}_{\tau_m''}(P_t)}{\theta_2 \mu_{m''}} \leq \frac{2\text{Reg}(P_t) + c_0 K \mu_{m''}}{\theta_2 \mu_{m''}} \leq \frac{2(2\widehat{\text{Reg}}(P_t) + c_0 K \mu_{m''})}{\theta_2 \mu_{m''}} + c_0 K \mu_{m''}.$$  

Since $\widehat{\text{Reg}}(P_t) = 0$ by definition, the above upper bound is simplified to

$$\frac{\widehat{\text{Reg}}_{\tau_m''}(P_t)}{\theta_2 \mu_{m''}} \leq \frac{3c_0 K \mu_{m''}}{\theta_2} = \frac{3c_0 K}{\theta_2}.$$  

Therefore, no matter whether $\mu_{m''} < 1/(2K)$ or not, we always have

$$\mathcal{V}_t(P_t) \mu_{m-1} \leq (\theta_1 + \frac{3c_0}{\theta_2}) K \mu_{m-1}. \quad (39)$$

Combining Equations (34), (35) and (39) gives

$$\widehat{\text{Reg}}_t(P) \leq (1 + \frac{2}{\theta_2}) \text{Reg}(P) + 2(\theta_1 + \frac{2c_0}{\theta_2}) K \mu_{m-1} + \frac{2d_t}{K \mu_{m-1}}. \quad (40)$$

Since $m > m_0$, the definition of $\rho$ ensures that $\mu_{m-1} \leq \rho \mu_m$. Also, since $t > \tau_m - 1$, $\frac{d_t}{\mu_{m-1}} \leq \frac{K \mu_{m-1}}{\mu_{m-1}} \leq \rho K \mu_m$. Applying these inequalities and the facts $c_0 = 4\rho(1 + \theta_1)$ and $\theta_2 \geq 8\rho$ in Equation (37), we have thus proved the second part in the inductive statement:

$$\widehat{\text{Reg}}_t(P) \leq \text{Reg}(P) + c_0 K \mu_m,$$

and hence the whole lemma.

\[\square\]

### D.2 Main Proof

We are now ready to prove Theorem 1. By Lemma 19 event $\mathcal{E}$ holds with probability at least $1 - \delta/2$. Hence, it suffices to prove the regret upper bound whenever $\mathcal{E}$ holds.

Recall from the description of $\mathcal{I}$ in Section 3 that the algorithm samples action $a_t$ taken at time $t$ in epoch $m$ from smoothed projection $\tilde{Q}_{m-1}$ of $Q_t$, where $Q_t$ is constructed by assigning all the remaining weight from $Q_{m-1}$ to $P_t$. From the discussion in Appendix C we can represent $Q_t$ as a linear combination of $P \in \mathcal{C}(\Pi)$ as follows:

$$Q_t = \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_t) P = \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_{m-1}) P + (1 - \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_{m-1})) P_t.$$  

Let $\mu_t = \mu_{m(t)}$, where $m(t)$ denotes the epoch in which time step $t$ lies: $m(t) = m$ for $t \in [\tau_{m-1} + 1, \tau_m]$. Then

$$R(P^*) - \frac{1}{T} \sum_{t} R(\tilde{Q}_t) = \frac{1}{T} \sum_{t} \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_t)(R(P^*) - R(P))$$

$$= \frac{1}{T} \sum_{t} \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_t)(R(P^') - R(P)) + (R(P^*) - R(P'))$$

$$= \frac{1}{T} \sum_{t} \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_t)(Z + 1)\text{Reg}(P) - Z\phi(V(P), B') + (R(P^*) - R(P')),$$

$$\leq \frac{(Z + 1)}{T} \sum_{t} \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_t)\text{Reg}(P) + (R(P^*) - R(P')), \quad (41)$$
The last inequality simply follows from the non-negativeness of the function \( \phi(\cdot, \cdot) \). Now, by observation in Lemma 2 and using \( B' \geq B/2, B' = B - \frac{12K_T}{B} \ln \frac{d|\Pi|}{\delta} - c\sqrt{KT \ln(T|\Pi|/\delta)} \),

\[
R(P^*) - R(P') \leq \frac{\text{OPT}(B') (B - B')}{B'} \leq 2 \cdot \frac{\text{OPT}(B)}{B} (T_0 + c \sqrt{\frac{K}{T} \ln(Td|\Pi|/\delta)}).
\]

To bound first term in (41), note that for \( m \leq m_0, \mu_{m-1} = \frac{1}{2K} \). So, trivially, for \( t \) in epoch \( m \leq m_0 \),

\[
\sum_{P \in \mathcal{C}(\Pi)} \alpha_P(\tilde{Q}_t) \text{Reg}(P) \leq c_0 K \psi \mu_{m-1}.
\] 

Suppose \( \mathcal{E} \) holds. Then, Lemma 24 implies that for all epochs \( m \geq m_0 \), all rounds \( t \geq t_0 \) in epoch \( m \), and all policies \( P \in \mathcal{C}(\Pi) \), we have

\[
\text{Reg}(P) \leq 2\hat{\text{Reg}}_m(P) + c_0 K \mu_m.
\]

Therefore, for \( t \) in such epochs \( m \), using the first condition in OP (from Section C), we get

\[
\sum_{P \in \mathcal{C}(\Pi)} \alpha_P(\tilde{Q}_t) \text{Reg}(P) \leq \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(\tilde{Q}_t)(2\hat{\text{Reg}}_m(P) + c_0 K \psi \mu_{m-1})
= \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_m)(2\hat{\text{Reg}}_m(P) + c_0 K \psi \mu_{m-1})
\leq (c_0 + 2) K \psi \mu_{m-1}.
\] 

The equality in above holds because by definition, \( \tilde{Q}_t \) assigns remaining weight from \( Q_{m-1} \) to \( P_t \), and \( \hat{\text{Reg}}_m(P_t) = 0 \). Substituting in Equation (41), we get

\[
R(P^*) - \frac{1}{T} \sum_t R(\tilde{Q}_t)
\leq \frac{(Z + 1) K \psi (c_0 + 2)}{T} \sum_m \mu_{m-1} (\tau_m - \tau_{m-1}) + \frac{2c \cdot \text{OPT}_T}{B} \sqrt{\frac{K}{T} \ln(T|\Pi|/\delta)}.
\] 

Applying an upper bound (Lemma 16 of [Agarwal et al., 2014]) on the sum over \( \mu_{m-1} \) above gives

\[
\sum_m \mu_{m-1} (\tau_m - \tau_{m-1}) \leq 4 \left( \ln \frac{16T^2(d + 1)|\Pi|}{\delta} + \frac{T}{K} \ln \frac{64T^2(d + 1)|\Pi|}{\delta} \right)
\] 

(45)

Substituting these bounds, and using \( Z \leq 24 \text{OPT}_T/B + 8 \) from Lemma 3, we get

\[
R(P^*) - \frac{1}{T} \sum_t R(\tilde{Q}_t) = O \left( \frac{\text{OPT}_T}{B} \left( \sqrt{\frac{K}{T} \ln \frac{d|\Pi|}{\delta}} + \sqrt{\frac{1}{T} \ln \frac{d}{\delta} + \frac{K}{T} \ln \frac{d|\Pi|}{\delta}} \right) \right)
\]

Next, we show that \( \frac{1}{T} \sum_t r_t(a_t) \), is close to \( \frac{1}{T} \sum_t R(\tilde{Q}_t) \). Recall that the algorithm samples \( a_t \) from \( \tilde{Q}_t^\ast \). Define the random variable at step \( t \) by

\[
Y_t := r_t(a_t) - \left( \sum_{\pi \in \Pi} (1 - K \mu_{t}) \tilde{Q}_t(\pi) r_t(\pi(x_t)) + \mu_t \sum_a r_t(a) \right).
\]

It is easy to see \( E[Y_t | H_{t-1}] = 0 \), so the Azuma-Hoeffding inequality for martingale sequences implies that, with probability at least \( 1 - \delta/2 \),

\[
\epsilon := \sqrt{\frac{1}{2T} \ln \frac{4}{\delta}} \geq \frac{1}{T} \sum_{t=1}^T Y_t.
\]
By definition of $Y_t$, we have with probability at least $1 - \delta/2$ that
\[
\left| \frac{1}{T} \sum_t r_t(a_t) - \frac{1}{T} \sum_t R(\bar{Q}_t) \right| \leq \epsilon + \frac{K}{T} \sum_{t=1}^T \mu_t, \tag{46}
\]
which implies, together with the triangle inequality and Equation (45), that (assuming $\mathcal{E}$ holds) with probability $1 - \delta/2$,
\[
R(P^*) - \frac{1}{T} \sum_t r_t(a_t) = O \left( \frac{\text{OPT}}{B} \left( \sqrt{\frac{K}{T}\ln(|\Pi|/\delta)} + \sqrt{\frac{T}{\delta}} \ln \frac{1}{\delta} + \frac{K}{T} \ln \frac{|\Pi|}{\delta} \right) \right) \tag{47}
\]
By Lemma 19 event $\mathcal{E}$ holds with probability at least $1 - \delta/2$. Therefore, by multiplying by $T$ on both sides and adding $T_0 = \frac{12KT}{B} \ln \frac{2|\Pi|}{\delta}$ (an upper bound of cumulative regret incurred in the first $T_0$ steps of Algorithm 1), we have that the algorithm will have a regret bounded by
\[
\tilde{O} \left( \frac{\text{OPT}}{B} \sqrt{KT \ln (|\Pi|)} + \frac{12KT}{B} \ln \frac{d|\Pi|}{\delta} \right)
\]
with probability at least $1 - \delta$ and complete the proof of Theorem 1 if the algorithm never aborted due to constraint violation in Step 10. But, from Lemma 25 the event that the budget constraint is violated happens with probability at most $1 - \delta/2$. Combining this with the bounds on reward given by (47), and that $\mathcal{E}$ holds with probability $1 - \delta/2$, we obtain that the regret bound in Theorem 1 holds with probability $1 - \frac{8\delta}{2}$.

**Lemma 25.** With probability at least $1 - \delta/2$, the algorithm is not aborted in Step 10 due to budget violation.

**Proof.** The proof involves showing that with high probability, the algorithm’s consumption over $B'$, in steps $t = 1, \ldots, T - T_0$, is bounded above by $c\sqrt{KT \ln (|\Pi|/\delta)}$ for large enough universal constant $c$. And, since $B' + T_0 + c\sqrt{KT \ln (|\Pi|/\delta)} = B$, we obtain that the algorithm will satisfy the knapsack constraint with high probability. This also explains why we started with a smaller budget.

More precisely, show that assuming $\mathcal{E}$ holds, in every epoch $m$, for every $t$ in epoch $m$,
\[
\phi(V(\bar{Q}_t), B') \leq 4(c_0 + 2)K\psi\mu_m \tag{48}
\]
Recall that $\phi(V(P), B')$ was defined as the maximum violation of budget $\frac{B'}{T}$ by vector $V(P)$. To prove above, we observe that our choice of $Z$ ensures that $\phi(V(P), B')$ is bounded by $\text{Reg}(P)$ as follows. By Equation (30), for all $P \in \mathcal{C}(\Pi)$
\[
R(P') \geq R(P) - \frac{Z}{2} \phi(V(P), B'),
\]
so that
\[
(Z + 1) \text{Reg}(P) = R(P') - R(P) + Z \phi(V(P), B') \geq \frac{Z}{2} \phi(V(P), B').
\]
Summing over $P \in \mathcal{C}(\Pi)$, with weights $\alpha_P(\bar{Q}_t)$, and using $Z \geq 1$
\[
\sum_{P \in \mathcal{C}(\Pi)} \alpha_P(\bar{Q}_t) \phi(V(P), B') \leq \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(\bar{Q}_t) \text{Reg}(P).
\]
Now, $\phi(\cdot, B')$ is a convex function, therefore, applying Jensen’s inequality,
\[
\phi(V(\bar{Q}_t), B') \leq \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(\bar{Q}_t) \text{Reg}(P).
\]
Substituting from Equation (42) and (43), we obtain the bound in Equation (48). Averaging (48) over all $t$ and using Jensen’s
\[
\phi\left( \frac{1}{T} \sum_t V(\bar{Q}_t), B' \right) \leq \frac{1}{T} \sum_t \phi(V(\bar{Q}_t), B') \leq \frac{1}{T} \sum_t 4(c_0 + 2)K\psi\mu_m(t)
\]
The sum on the right hand side can be bounded using (45):

$$
\phi \left( \frac{1}{T} \sum_t V(\hat{Q}_t), B' \right) \leq \frac{16(c_0 + 2)K\psi}{T} \left( \ln \frac{16T^2||\Pi||}{\delta} + \sqrt{\frac{T}{K} \ln \frac{64T^2||\Pi||}{\delta}} \right)
$$

Also, we can use arguments similar to those used for deriving (46) to obtain that for every $i = 1, \ldots, d$, with probability at least $1 - \frac{\delta}{2}$,

$$
\left| \frac{1}{T} \sum_t [v_t(a_t)]_i - \frac{1}{T} \sum_t [V(\hat{Q}_t)]_i \right| \leq \epsilon + \frac{(K + 1)}{T} \sum_{t=1}^T \mu_t,
$$

(49)

where $\epsilon = \sqrt{\frac{1}{T} \ln \frac{d}{\delta}}$.

Using these bounds along with Equation (45), we get that with probability $1 - \frac{\delta}{2}$,

$$
\phi \left( \frac{1}{T} \sum_t v_t(a_t), B' \right) \leq \phi \left( \frac{1}{T} \sum_t v_t(a_t), B' \right) + \left| \frac{1}{T} \sum_t v_t(a_t) - \frac{1}{T} \sum_t V(\hat{Q}_t) \right|_\infty
$$

$$
\leq O \left( \sqrt{\frac{K}{T} \ln \frac{T||\Pi||}{\delta}} + \sqrt{\frac{1}{T} \ln \frac{d}{\delta}} + \frac{K}{T} \ln \frac{T||\Pi||}{\delta} \right)
$$

Therefore, for large enough constant $c$, and large enough $T \geq \max\{K, d\}$,

$$
\phi \left( \frac{1}{T} \sum_t v_t(a_t), B' \right) \leq c \sqrt{\frac{K}{T} \ln \frac{T||\Pi||}{\delta}},
$$

and by definition of $\phi(\cdot, B')$, this implies that with probability $1 - \frac{\delta}{2}$, for all $j = 1, \ldots, d$,

$$
\sum_t v_t(a_t)_j \leq B' + c \sqrt{K T \ln \frac{T||\Pi||}{\delta}}
$$

Therefore, algorithm will not exceed $B = B' + c \sqrt{K T \ln (T||\Pi||/\delta)}$ with probability $1 - \frac{\delta}{2}$ assuming $\mathcal{E}$ holds.

---

### E Regret Analysis for Section 5: CBwR

The analysis is structurally similar to that in Appendix D. Here, we only describes the differences and omit the most of the identical steps.

The first difference is in the definition of regrets, which have been define in Section 5 for $P \in C(\Pi)$,

$$
\text{Reg}(P) = \frac{1}{\|1_d\|_L} \left( f(V(P_*)) - f(V(P)) \right)
$$

$$
P_t = \arg \max_{P \in C_0(\Pi)} f(\hat{V}_t(P))
$$

$$
\widehat{\text{Reg}}_t(P) = \frac{1}{\|1_d\|_L} \left( f(\hat{V}_t(P_t)) - f(\hat{V}_t(P)) \right).
$$

Other convenience quantities ($d_t, m_0, t_0$, and $\rho$) are defined in the same as in Appendix D except that the factor $d+1$ is replaced by $d$ in $d_t$.
Definition 4 (Variance estimates). Define the following for any probability distributions $P, Q \in C(\Pi)$, any policy $\pi \in \Pi$, and $\mu \in [0, 1/K]$:

\[
V(Q, \pi, \mu) := E_{x \sim \mathcal{D}_X} \left[ \frac{1}{Q^\mu(\pi(x)|x)} \right],
\]

\[
\hat{V}_m(Q, \pi, \mu) := \hat{E}_{x \sim H_m} \left[ \frac{1}{Q^\mu(\pi(x)|x)} \right],
\]

\[
V(Q, P, \mu) := E_{\pi \sim \mathcal{P}}[V(Q, \pi, \mu)],
\]

\[
\hat{V}_m(Q, P, \mu) := E_{\pi \sim \mathcal{P}}[\hat{V}_m(Q, \pi, \mu)].
\]

where $\hat{E}_{x \sim H_m}$ denote average over records in history $H_m$.

Furthermore, let $m(t) := \min\{m \in \mathbb{N} : t \leq \tau_m\}$, be the index of epoch containing round $t$, and define

\[
\mathcal{V}_t(P) := \max_{0 \leq m < m(t)} \{ V(\hat{Q}_m, P, \mu_m) \},
\]

for all $t \in \mathbb{N}$ and $P \in C(\Pi)$.

Definition 5. Define $\mathcal{E}$ as event that the following statements hold

- For all probability distributions $P, Q \in C(\Pi)$ and all $m \geq m_0$,

\[
V(Q, P, \mu_m) \leq 6.4\hat{V}_m(Q, P, \mu_m) + 81.3K. \tag{50}
\]

- For all $P \in C(\Pi)$, all epochs $m$ and all rounds $t$ in epoch $m$, any $\delta \in (0, 1)$, and any choices of $\lambda_{m-1} \in [0, \mu_{m-1}]$,

\[
\frac{1}{\|1_d\|} \|\hat{V}_t(P) - V(P)\| \leq \mathcal{V}_t(P)\lambda_{m-1} + \frac{d_t}{t\lambda_{m-1}}. \tag{51}
\]

Lemma 26. $\Pr(\mathcal{E}) \geq 1 - (\delta/2)$.

Proof. The proof is identical to that for Lemma 19 up to Equation (25), which gives concentration on a fixed dimension of the observation vector. Now, apply union bound on all $d$ dimensions, we have that, with probability $1 - \frac{\delta}{2}$, for all $t$ and all $P \in C(\Pi)$, we have

\[
\|1_d\|^{-1}\|\hat{V}_t(P) - V(P)\| \leq (e - 2)\mathcal{V}_t(P)\lambda_{m-1} + \frac{d_t}{t\lambda_{m-1}}.
\]

Lemma 27. Assume event $\mathcal{E}$ holds. Then for all $m \leq m_0$, and all rounds $t$ in $m$,

\[
\|1_d\|^{-1}\|\hat{V}_t(P) - V(P)\| \leq \max\{\sqrt{\frac{4Kd_t\mathcal{V}_t}{t}}, \frac{4Kd_t}{t}\}, \tag{52}
\]

Proof. By definition of $m_0$, for all $m' < m_0$, $\mu_{m'} = 1/(2K)$. Therefore, $\mu_{m-1} = 1/(2K)$. First consider the case when $\sqrt{\frac{d_t}{\mathcal{V}_t}} < \mu_{m-1} = \frac{1}{2K}$. Then, substitute $\lambda_{m-1} = \sqrt{\frac{d_t}{\mathcal{V}_t}}$, to get

\[
\|1_d\|^{-1}\|\hat{V}_t(P) - V(P)\| \leq \sqrt{\frac{4d_t\mathcal{V}_t}{t}}.
\]

Otherwise,

\[
\mathcal{V}_t < \frac{4K^2d_t}{t}
\]
Substituting $\lambda_{m-1} = \mu_{m-1} = \frac{1}{KK}$, we get

\[\|1d\|^{-1}\|\hat{V}_t(P) - V(P)\| \leq \frac{4Kd_1}{t}.\]

**Lemma 28.** Assume event $\mathcal{E}$ holds. Then, for all $m$, all $t$ in round $m$, all choices of distributions $P \in \mathcal{C}(\Pi)$

\[\left(\|1d\|^{-1}\|\hat{V}_t(P) - V(P)\|\right) \leq \begin{cases} \max\left\{\frac{2Kd_1V_t(P)}{t}, \frac{4Kd_1}{t}\right\}, & m \leq m_0 \\ V_t(P)\mu_{m-1} + \frac{d_1}{\mu_{m-1}}, & m > m_0. \end{cases}\]

**Proof.** Follows from definition of event $\mathcal{E}$ and Lemma 27.

**Lemma 29.** Assume event $\mathcal{E}$ holds. For any round $t \in \mathbb{N}$, and any policy $P \in \mathcal{C}(\Pi)$, let $m \in \mathbb{N}$ be the epoch achieving the max in the definition of $V_t(P)$. Then,

\[V_t(P) \leq \begin{cases} 2K \theta_1 K + \frac{\overline{R}_{reg_{\mu}}(P)}{\theta_2\mu_m} & \text{if } \mu_m = \frac{1}{2K}, \\ \mu_m & \text{if } \mu_m < \frac{1}{2K}, \end{cases}\]

where $\theta_1 = 94.1$ and $\theta_2 = \psi/6A = \cdots$ are universal constants.

**Proof.** Identical to that of Lemma 23.

**Lemma 30.** Assume event $\mathcal{E}$ holds. Define $c_0 := 4\rho(1 + \theta_1)$. For all epochs $m \geq m_0$, all rounds $t \geq t_0$ in epoch $m$, and all policies $P \in \mathcal{C}(\Pi)$,

\[
\begin{align*}
\overline{R}e_g(P) & \leq 2\overline{R}_{\hat{V}_t}(P) + c_0 K \mu_m \\
\overline{R}_{\hat{V}_t}(P) & \leq 2\overline{R}_{\hat{V}_t}(P) + c_0 K \mu_m.
\end{align*}
\]

**Proof.** We start with two useful inequalities that show the closeness of $\overline{R}e_g(P)$ and $\overline{R}_{\hat{V}_t}(P)$. One on hand, using the triangle inequality, the $L$-smoothness of the reward function $f$, and the definition of $f$, we have

\[
\begin{align*}
\|1d\|^{-1}\left(\overline{R}_{\hat{V}_t}(P) - \overline{R}e_g(P)\right) &= \frac{1}{L} \left(f(\hat{V}_t(P)) - f(\hat{V}_t(P)) - f(V(P)) + f(V(P))\right) \\
&\leq \frac{1}{L} \left(f(V(P)) - f(\hat{V}_t(P)) + f(\hat{V}_t(P)) - f(V(P))\right) \\
&\leq \frac{1}{L} \left|f(V(P)) - f(\hat{V}_t(P))\right| + \frac{1}{L} \left|f(\hat{V}_t(P)) - f(V(P))\right| \\
&\leq \|V(P) - \hat{V}_t(P)\| + \|\hat{V}_t(P) - V(P)\|. \tag{54}
\end{align*}
\]

Similarly, one can prove the opposite direction, using the definition of $P^*$ instead:

\[
\begin{align*}
\|1d\|^{-1}\left(\overline{R}e_g(P)\overline{R}_{\hat{V}_t}(P)\right) &= \frac{1}{L} \left(-f(\hat{V}_t(P)) + f(\hat{V}_t(P)) + f(V(P^*)) - f(V(P))\right) \\
&\leq \frac{1}{L} \left(f(\hat{V}_t(P)) - f(V(P)) + f(V(P^*)) - f(\hat{V}_t(P^*))\right) \\
&\leq \frac{1}{L} \left|f(\hat{V}_t(P)) - f(V(P))\right| + \frac{1}{L} \left|f(V(P^*)) - f(\hat{V}_t(P^*))\right| \\
&\leq \|\hat{V}_t(P) - V(P)\| + \|V(P^*) - \hat{V}_t(P^*)\|. \tag{55}
\end{align*}
\]

33
We now prove the lemma by mathematical induction on \( m \). For the base case, we have \( m = m_0 \) and \( t \geq t_0 \) in epoch \( m_0 \). Then, from Lemma 28 using the facts that \( V_t \leq 2K \), and that \( 4Kd_t \leq 1 \) for \( t \geq t_0 \) in epoch \( m_0 \), we get, for all \( P \in \mathcal{C}(\Pi) \) that

\[
\|1_d\|^{-1}\|\hat{V}_t(P) - V(P)\| \leq \max \left\{ \sqrt{\frac{4Kd_tV_t(P)}{t}}, \frac{4Kd_t}{t} \right\} \leq \sqrt{\frac{8Kd_t}{t}}.
\]

Combining this with Equations 54 and 55, we prove the base case:

\[
\left| \hat{\text{Reg}}_t(P) - \text{Reg}(P) \right| \leq 2\sqrt{\frac{8Kd_t}{t}} \leq c_0K\mu_{m_0}.
\]

For the induction step, fix some epoch \( m > m_0 \) and assume for all epochs \( m' < m \), all rounds \( t' \geq t_0 \) in epoch \( m' \), and all distributions \( P \in \mathcal{C}(\Pi) \) that,

\[
\text{Reg}(P) \leq 2\hat{\text{Reg}}_{t'}(P) + c_0K\mu_{m'}
\]

\[
\hat{\text{Reg}}_{t'}(P) \leq 2\text{Reg}(P) + c_0K\mu_{m'}.
\]

Then, from Equations 54 and 55 as well as Lemma 28 we have the following inequalities

\[
\text{Reg}(P) - \hat{\text{Reg}}_{t'}(P) \leq (V_t(P) + V_t(P'))\mu_{m-1} + \frac{2d_t}{t\mu_{m-1}}
\]

\[
\hat{\text{Reg}}_{t'}(P) - \text{Reg}(P) \leq (V_t(P) + V_t(P_t))\mu_{m-1} + \frac{2d_t}{t\mu_{m-1}},
\]

which are the analogues of Equations 33 and 34 in the proof of Lemma 24. The rest of the proof is the same.

\[\square\]

### E.1 Main Proof

We are now ready to prove Theorem 6. By Lemma 26 event \( \mathcal{E} \) holds with probability at least \( 1 - \delta/2 \). Hence, it suffices to prove the regret upper bound whenever \( \mathcal{E} \) holds.

Recall from Section 3 that the algorithm samples \( a_t \) at time \( t \) in epoch \( m \) from smoothed projection \( \hat{Q}_{m-1} \) of \( Q_{m-1} \). Also, recall from Appendix C that \( Q_m \) for any \( m \) is represented as a linear combination of \( P \in \mathcal{C}(\Pi) \) as follows:

\[
Q_m = \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_m)P = \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_m)P + (1 - \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(Q_m))P_t. \quad (Q_m \text{ assigns all the remaining weight from } Q_m \text{ to } P_t).
\]

Let \( \hat{Q}_t = \hat{Q}_{m(t)-1} \), \( \mu_t = \mu_{m(t)-1} \), where \( m(t) \) denotes the epoch in which time step \( t \) lies: \( m(t) = m \) for \( t \in [\tau_{m-1} + 1, \tau_m] \). Then,

\[
f(V(P^*)) - f\left(\frac{1}{T} \sum_t V(\hat{Q}_t)\right) \leq f(V(P^*)) - \frac{1}{T} \sum_t f(V(\hat{Q}_t))
\]

\[
= \frac{1}{T} \sum_t \left( f(V(P^*)) - f(V(\hat{Q}_t)) \right)
\]

\[
\leq \frac{1}{T} \sum_t \left( f(V(P^*)) - \sum_{P \in \mathcal{C}(\Pi)} \alpha_P(\hat{Q}_t)f(V(P)) \right)
\]

\[
= \frac{\|1_d\|L}{T} \sum_{P \in \mathcal{C}(\Pi)} \sum_{m} \alpha_P(\hat{Q}_t)\text{Reg}(P),
\]

where we have used Jensen’s inequality twice.

With identical reasoning as in the proof in Appendix D.2 we can prove, using the above inequality, that

\[
f(V(P^*)) - f\left(\frac{1}{T} \sum_t V(\hat{Q}_t)\right) \leq \frac{\|1_d\|LK\psi(c_0 + 2)}{T} \sum_{m} \mu_{m-1}(\tau_m - \tau_{m-1}). \tag{56}
\]
We next show that \( f(\frac{1}{T} \sum_t \mathbf{V}(\hat{Q}_t)) \) is close enough to the regret that we are interested in. Specifically, fix a component \( i \in [d] \) and let \([v]_i \) be the \( i \)th component of vector \( v \). Recall that the algorithm samples \( a_t \) from \( \hat{Q}_t^{\mu_t} \). Define the random variable at step \( t \) by
\[
Z_t := [v_t(a_t)]_i - \sum_{\pi \in \Pi} (1 - K \mu_t) \hat{Q}_t(\pi)[v_t(\pi(x_t))]_i + \mu_t \sum_a [v_t(a)]_i.
\]

It is easy to see \( E[Z_t | H_{t-1}] = 0 \), so the Azuma-Hoeffding inequality for martingale sequences implies that, with probability at least \( 1 - \delta/(2d) \),
\[
\epsilon := \sqrt{\frac{1}{2T} \ln \frac{4d}{\delta}} \geq \frac{1}{T} \sum_{t=1}^T Z_t.
\]

Applying a union bound over \( i \in [d] \), we have with probability at least \( 1 - \delta/2 \) that
\[
\left\| \frac{1}{T} \sum_t v_t(a_t) - \frac{1}{T} \sum_t \mathbf{V}(\hat{Q}_t) \right\| \leq \|\mathbf{1}_d\| \left( \epsilon + \frac{K}{T} \sum_{t=1}^T \mu_t \right), \tag{57}
\]

which implies, together with the \( L \)-smoothness of \( f \), that
\[
f(\frac{1}{T} \sum_t \mathbf{V}(\hat{Q}_t)) - f(\frac{1}{T} \sum_t v_t(a_t)) \leq \|\mathbf{1}_d\| L \left( \epsilon + \frac{K}{T} \sum_{t=1}^T \mu_t \right). \tag{58}
\]

Combining (56) and (58), we get
\[
f(\frac{1}{T} \sum_t v_t(a_t)) \leq \|\mathbf{1}_d\| L \left( \frac{K \psi(c_0 + 4)}{T} \sum_{m=1}^\infty \mu_m (\tau_m - \tau_{m-1}) + \epsilon \right). \tag{59}
\]

Applying the same upper bound for \( \sum_{m=1}^\infty \mu_m (\tau_m - \tau_{m-1}) \) as in Appendix D, we get
\[
f(\frac{1}{T} \sum_t v_t(a_t)) \leq \|\mathbf{1}_d\| L \left( \frac{K \psi(c_0 + 4)}{T} \left( \frac{\tau_{m_0}}{2K} + \sqrt{\frac{8d\tau_{m_0}(T)\tau_{m_0}(T)}{K}} \right) + \epsilon \right). \tag{60}
\]

Now substituting the same bounds for \( \tau_{m_0} \) and \( d_{m_0}(T) \), as well as the value of \( \epsilon \), one gets the final regret upper bound, as stated in the theorem:
\[
\text{avg-regret}(T) = f(v(P^*)) - f(\frac{1}{T} \sum_t v_t(a_t)) \\
\leq \|\mathbf{1}_d\| L \psi(4c_0 + 16) \left( \frac{K}{T} \ln \frac{16T^2\|\Pi\|}{\delta} + \sqrt{\frac{K}{T} \ln \frac{64T^2\|\Pi\|}{\delta}} \right) + \|\mathbf{1}_d\| L \sqrt{\frac{1}{2T} \ln \frac{4d}{\delta}}
\]
\[
= O \left( \|\mathbf{1}_d\| L \left( \sqrt{\frac{K}{T} \ln \frac{\|\Pi\|}{\delta}} + \sqrt{\frac{1}{T} \ln \frac{d}{\delta}} \right) \right).
\]

Note that a regret bound of above order is trivial unless \( T \geq K \ln(T\|\Pi\|/\delta) \). Making that assumption, we get the following bound in a simpler form:
\[
\text{avg-regret}(T) = O \left( \|\mathbf{1}_d\| L \left( \sqrt{\frac{K}{T} \ln \frac{\|\Pi\|}{\delta}} + \sqrt{\frac{1}{T} \ln \frac{d}{\delta}} \right) \right).
\]