Restoration of Symmetry by Interactions and Nonreliability of the Perturbative Renormalization Group Approach

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We discuss examples of (1+1)-dimensional models where the perturbative renormalization group (RG) indicates a tendency to restore the symmetry in the strong coupling limit. We show that such restoration does occur sometimes, but the perturbative RG cannot be reliably used to detect it.

Enlarged symmetry phenomenon at a fixed point corresponds to a situation where the microscopic Hamiltonian has a lower symmetry than the fixed point Hamiltonian. Any deviations from the full symmetry behaviour scale to zero under the renormalization group (RG) flow when the critical point is reached. Such phenomenon occurs, for instance, in frustrated spin systems near two dimensions. In the continuum limit, the microscopic Hamiltonian associated with Heisenberg spins on the triangular lattice is a non linear sigma model with a O(3)×O(2) symmetry. A RG study of this model in d = 2 + ε dimensions show that the symmetry is dynamically enlarged at the stable fixed point to O(3)×O(3) ~ O(4) [3]. The phase transition of canted Heisenberg spin systems belongs thus, near two dimensions, to the N = 4 Wilson-Fisher universality class.

A similar enlarged symmetry scenario has been proposed by Zhang in the context of high-Tc cuprates [5]. The author suggests that the phase diagram of these compounds can be deduced from an SO(5) symmetry that unifies antiferromagnetism and D-wave superconductivity. It was also argued that even though the SO(5) symmetry is only approximate for microscopic models such as the Hubbard or t-J models, it becomes exact under the RG flow toward a bicritical point. Microscopic electron models with an exact SO(5) symmetry has been constructed [4] and this symmetry occurs in the low energy sector of two-chain Hubbard systems [3,6]. In particular, Lin, Balents, and Fisher [6], using a one-loop perturbative RG approach, predict that at half filling the weakly-interacting two-leg ladder has an exact SO(8) symmetry at low energy and belongs to a D-Mott phase where the short range pairing correlations have an approximate D-wave symmetry. The crucial point of their analysis stems from the fact that some coupling constants of the microscopic Hamiltonian flow to strong coupling but with ratios converging to fixed values and the effective model corresponds to the SO(8) Gross-Neveu model in the strong coupling limit.

The question that we shall adress, in this letter, is whether one can reliably deduce restoration of a symmetry by interactions in the strong coupling limit from weak coupling RG computations. In the following, we shall consider several (1+1)-dimensional models using a non-perturbative approach to investigate the possible dynamically enlarged symmetry in the strong coupling limit. Let us start with the U(1)-symmetric Thirring model defined by the following Hamiltonian density:

\[ H = \bar{\Psi}_i (\gamma^\mu \partial_\mu + g_\perp \gamma^\perp) \Psi_i + \frac{1}{4} \sum_{\alpha=1}^3 g_\alpha J^\alpha J^{\alpha \mu} + \frac{1}{4} \sum_{\alpha=1}^3 g_\alpha J^\alpha J^{\alpha \mu} \] (1)

with \( g_1 = g_2 = g_\perp \), and \( g_3 = g_\parallel \). The (iso)spin currents are expressed in terms of the fermion fields \( \Psi_i \) as: \( J^\mu = (1/2) \bar{\Psi}_i \gamma^\mu \sigma_{i,j} \Psi_j \), \( \sigma_{i,j} \) being the Pauli matrices and our Euclidean conventions for the gamma matrices are: \( \gamma^0 = \sigma^2, \gamma^1 = \sigma^3 \). The Thirring model \( (1) \) is exactly solvable by the Bethe ansatz and the solution was carefully studied [7,8]. This gives us the first opportunity to test whether restoration of symmetry occurs and if the one-loop approximation in the RG equations can serve as a reliable tool to detect it. The RG equations for this model are of the Kosterlitz-Thouless form:

\[ \dot{g}_\parallel = \frac{g_\parallel^2}{2\pi}, \quad \dot{g}_\perp = \frac{g_\parallel g_\perp}{2\pi} \] (2)

where the dots indicate derivatives with respect to the logarithm of the length scale: \( \dot{g} = dg/d\ln l \). The phase portrait derived from these equations is presented on Fig. 1: there are regions where the RG trajectories flowing to strong coupling converge giving appearance that the original U(1) symmetry is enlarged to SU(2). The strong coupling regime can be separated into two regions. The first one corresponds to \( g_\parallel > g_\perp > 0 \) (marked A on Fig. 1) where the Thirring model is equivalent to the sine-Gordon model. We shall be interested only in the region where the symmetry is apparently restored. In this region the spectrum of the sine-Gordon model contains only kinks and anti-kinks with equal masses and no bound states. All information about interaction between
the Casimir operator \( \xi \) is defined by:
\[
\xi = \frac{\pi \mu}{\pi - \mu}, \quad \mu = \arccos[g_{\parallel}/\cos g_{\perp}] \approx \sqrt{g_{\parallel}^2 - g_{\perp}^2} \quad (3)
\]
where the latter equality holds at small bare couplings. The S-matrix is an analytic function of \( \xi \) at \( \xi \to 0 \) and corrections to the isotropic limit are order of \( \xi \) at small \( \xi \).

\[ g_{\parallel} = \text{the number of generators of the group } G \]

\( \xi \) is much stronger in the other region, namely \( g_{\perp} > |g_{\parallel}| \). Following Jarediz, Nersesyan and Wiegmann (JNW) who studied it \( \xi \), we shall call it region C (see Fig. 1). The JNW solution can be generalized for higher representations of the SU(2) group and also for other semisimple groups \( \xi \). All these integrable models are deformations of the Wess-Zumino-Novikov-Witten model by marginally relevant perturbations. Let operators \( J^a \) and \( J^b \) be the left- and right currents of a semisimple group \( G \) satisfying the level \( k \) Kac-Moody algebra:
\[
[J^a(x), J^b(y)] = \frac{i}{4\pi} \delta^{ab} \delta'(x-y) + i f^{abc} J^c(y) \delta(x-y)
\]
with a similar relation for the right spin current \( \bar{J}^\alpha \) whereas \( J \) and \( \bar{J} \) commute each other. In Eq. \( f^{abc} \), \( f^{abc} \) are the structure constant of the algebra \( f^{abc} = \varepsilon^{abc} \) for the SU(2) group. The corresponding Hamiltonian density can be written in terms of the currents (Sugawara form):
\[
\mathcal{H} = \frac{2\pi}{k + e_v} \sum_{a=1}^{G} J^a J^a + \sum_{a,b=1}^{G} g_{ab} J^a \bar{J}^b \quad (5)
\]
where \( G \) is the number of generators of the group \( G \) and the Casimir operator \( (c_v) \) in the adjoint representation is defined by: \( c_v \delta_{ab} = f^{abc} f^{\bar{a}bc} \). For each \( G \) and \( k \) there is at least one single-parametric family of coupling constants \( g^{ab} \) (we do not consider the common factor as a parameter) at which the model is integrable. In particular, for O(2n) groups there are two families of solutions.

It turns out, however, that restoration of the symmetry is much stronger in the other region, namely \( g_{\perp} > |g_{\parallel}| \). Following Jarediz, Nersesyan and Wiegmann (JNW) who studied it \( \xi \), we shall call it region C (see Fig. 1). The JNW solution can be generalized for higher representations of the SU(2) group and also for other semisimple groups \( \xi \). All these integrable models are deformations of the Wess-Zumino-Novikov-Witten model by marginally relevant perturbations. Let operators \( J^a \) and \( J^b \) be the left- and right currents of a semisimple group \( G \) satisfying the level \( k \) Kac-Moody algebra:
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There is one problem, however, for all groups higher than SU(2) the parity is broken: \( g_{ab} \neq g_{ba} \) (to get an idea one may look at Refs. \( \xi \), \( \xi \) where some related Hamiltonians are written explicitly). For the SU(2) group there is no trouble and the model \( \xi \) becomes a generalization of the Thirring model with two coupling constants \( g_{\parallel} \) and \( g_{\perp} \). At \( k = 2 \) the isotropic SU(2)-invariant version of the model \( \xi \) becomes the O(3) Gross-Neveu model. This model with \( g_{\parallel} = 0 \) is equivalent to the model suggested by Egger and Gogolin \( \xi \) in the context of theory of carbon nanotubes:
\[
\mathcal{H} = -\frac{i}{2} \sum_{a=1}^{3} (r_a \partial_x r_a - l_a \partial_x l_a) + g_{r} r_3 l_3 (r_1 l_1 + r_2 l_2) \quad (6)
\]
where \( r_a, l_a \) are right and left moving real (Majorana) fermions. The equivalence between the SU(2)-invariant version model \( \xi \) with \( k = 2, g_{\parallel} = 0 \) and model \( \xi \) stems from the fact that the SU(2) spin currents can be written in terms of three real fermions \( \xi \), \( \xi \), \( \xi \): \( J^a = -i x^{abc} l_b l_c / 2 \) with a similar relation for the right-moving current. The RG equations of the model \( \xi \) in the first loop approximation does not depend on \( k \). Thus in the region of small bare couplings the phase portrait of the model is still the one presented on Fig. 1.

The Bethe ansatz equations for the SU(2)-based model \( \xi \) were obtained by Wiegmann \( \xi \) and Kirillov and Reshetikhin \( \xi \) and are given by
\[
|e_k(x_a + f/2\mu)e_k(x_a - f/2\mu)|^N = \prod_{b=1}^{M} e_2(x_a - x_b) \quad (7)
\]
where \( 2N \) is the number of fermions, \( M \) is the number of up spins. In the region A the functions \( e_\mu(x) \) are hyperbolic whereas in the region C they are trigonometric:
\[
e_\mu(x) = \frac{\sin[\mu(x - i n/2)]}{\sin[\mu(x + i n/2)]}.
\]

In the region C the parameters \( \mu \) and \( 0 < f < \pi \) are related to the bare coupling constants (which are supposed to be small) \( \xi \):
\[
\mu^2 = -g_{\parallel}^2 + g_{\perp}^2, \quad \cot f = g_{\parallel}/\sqrt{-g_{\parallel}^2 + g_{\perp}^2}.
\]

The results of the previous analysis show that in the region A the symmetry is not restored in the model \( \xi \). The region C for the case \( k > 1 \) has never been studied, but it is easy to see that the results obtained in Ref. \( \xi \) for the case \( k = 1 \) are straightforwardly generalizable for greater \( k \)'s. In the region C the models \( \xi \), \( \xi \) have the spectral gap \( m \approx \Lambda \exp[-\pi(\mu - f)/2\mu] \). The important result obtained in Ref. \( \xi \) is that in the strong coupling limit the anisotropy of the original Hamiltonian
only to exponentially small corrections to physical quantities \( \sim \exp(-\pi^2/\mu) \). From the technical point of view the most important difference between the Bethe ansatz equations in the two regions comes from the fact that in the sine-Gordon region they contain hyperbolic and in the region C - trigonometric functions. In the latter case the Thermodynamic Bethe Ansatz (TBA) equations differ from the equations at the isotropic point \( (\mu \to 0) \) only by the fact that real parts of the spectral parameters \( x_a \) belong to a finite interval \( 0 < \Re x < 2\pi/\mu \) and the kernels in the integral equations are periodic with the period \( 2\pi/\mu \). As a matter of fact, one can obtain TBA for the sector C from the isotropic TBA replacing the kernels by their periodic generalizations:

\[
K_C(x) = \sum_{l=-\infty}^{+\infty} K_{\text{isot}}(x - 2\pi l/\mu).
\]

At temperatures comparable with the spectral gap the thermodynamics is dominated by a small part of the available parameter interval and the difference between periodic and isotropic kernels becomes insignificant (exponentially small in \( \pi/\mu \), to be precise, because kernels in TBA equations are exponential functions of \( x \).

As we have mentioned, one application of this result is the Egger-Gogolin model of the carbon nanotubes. In the strong coupling limit, one can conclude that the model \( \ref{eq:egger-gogolin} \) is equivalent to the O(3) Gross-Neveu model, which belongs to the SU(2) group by: \( \Omega^a_5 \) with \( a \in \{1,2\} \). There is another application: the model \( \ref{eq:egger-gogolin} \) with \( k \to \infty \) corresponds to the model of classical Heisenberg antiferromagnet on a triangular lattice. At \( T = 0 \), as consequence of frustration, the system orders in a spiral state with the order parameter consisting of Heisenberg antiferromagnet on a triangular lattice. At temperatures comparable with the spectral gap the \( \rho_a \) is equivalent to the O(3) Gross-Neveu model with the Hamiltonian density \( (\ref{eq:hamiltonian}) \) with \( k \to \infty \). Therefore, according to the arguments given above, the anisotropic PC model acquires an enlarged SU(2)\times SU(2) symmetry in the strong coupling limit. In this limit its solution coincides with the solution of the isotropic Principal Chiral Model (see Refs. \ref{ref:principal-chiral}).

The third example we consider is directly related to the work of Lin, Balents, and Fisher \ref{ref:lin-balents-fisher}. It is the anisotropic Gross-Neveu model with the Hamiltonian density \( (U > 0) \)

\[
\mathcal{H} = \sum_{i=1}^{N} \bar{\psi}_{\alpha i} \gamma^\mu \partial_{\mu} \psi_{\alpha i} + \frac{1}{2} \sum_{a=1}^{3} \rho^{-1} J^a_{\mu} J^a_{\mu} \tag{\ref{eq:hamiltonian}}
\]

where the color index \( k \) tends to infinity and the current is given by: \( J^a_{\mu} = (1/2) \sum_{i=1}^{k} \bar{\psi}_{\alpha i} \gamma^\mu \sigma_{\alpha\beta} \psi_{\beta i} \). Since the bare coupling constants of model \( \ref{eq:hamiltonian} \) verify \( \rho^{-1} > \rho_{\parallel}^{-1} \), the anisotropic PC model \( \ref{eq:hamiltonian} \) belongs to the region C of the phase portrait of the SU(2)-invariant model \( \ref{eq:principal-chiral} \) with \( k \to \infty \). One can obtain the RG equations of the model \( \ref{eq:hamiltonian} \):

\[
\dot{U} = -\frac{2(N-1)}{\pi} U^2 - \frac{g^2}{2\pi} \tag{\ref{eq:rg-equations}}
\]

\[
\dot{g} = -\frac{2N-1}{\pi} g U + \frac{g_1 g}{\pi} \tag{\ref{eq:rg-equations}}
\]

\[
\dot{g}_1 = \frac{N}{2\pi} g^2 \tag{\ref{eq:rg-equations}}
\]

These equations indicate a restoration of the symmetry in the strong coupling limit: the model has an isotropic behaviour at strong coupling with \( U \) changing its sign and flowing to the strong coupling. We shall see however, that this isotropy is not reflected in the excitation spectrum.

One can, first, bosonize the Dirac fermion \( \psi_0 \) with the introduction of a bosonic field \( \Phi \) to get another formulation of \( \ref{eq:hamiltonian} \):

\[
\mathcal{H} = \frac{1}{16\pi} (\partial_{\mu} \Phi)^2 + \sum_{a=1}^{N} \bar{\psi}_{\alpha a} \gamma^\mu \partial_{\mu} \psi_{\alpha a} - \tilde{g} \cos \beta \Phi(\bar{\psi}_a \psi_a) + U(\bar{\psi}_a \psi_a)^2 \tag{\ref{eq:bosonized-hamiltonian}}
\]

where the delta constraint indicates that the gauge field \( \sigma^a \Omega^a_\mu = -2ig^{-1} \partial_{\mu} g \) has a zero field strength. One has a fermionic representation of this constraint with the identity \( \ref{eq:fermionic-representation} \):

\[
\delta (F_{\mu\nu}) = \lim_{k \to +\infty} \int D\bar{\Psi}_{\mu} D\Psi_{\nu} \exp \left( -\int d^2 x \sum_{i=1}^{k} \bar{\Psi}_{\mu\alpha} (\gamma^\mu \delta_{\alpha\beta} \partial_{\mu} + ig^2 \Omega^\mu_{\mu} \sigma_{\alpha\beta}) \Psi_{\nu\beta} \right) \tag{\ref{eq:fermionic-representation}}
\]
where \( \tilde{g} \sim g \) and for \( g_1 > 0 \) the scaling dimension of the cosine-operator is \( d = 2\beta^2 < 1 \). The operator containing the cosine term is then relevant and one may think that it will generate a mass gap for all branches of the spectrum. Thus the mass gaps in this case would be generated by the repulsive interaction in the 0-channel; this mechanism was discussed by Shelton and one of the authors in context of doped ladder models in Ref. [2] where a model very similar to (16) with \( N = 2 \) was considered. The model (16) has \( O(2) \times O(2N) \) symmetry. According to the one-loop RG equations, this symmetry is enlarged at strong coupling to \( O(2N + 2) \). Let us consider the model (18) in the large \( N \) limit to test this possible restoration of symmetry. In this case we can use the mean-field approach to decouple the interaction between 0 and \( a \)-sectors: \( \mathcal{H} = \mathcal{H}_{SG} + \mathcal{H}_{GN} \)

\[
\mathcal{H}_{SG} = \frac{1}{16\pi} (\partial_\mu \Phi)^2 - M \cos \beta \Phi
\]

\[
\mathcal{H}_{GN} = \sum_{a=1}^{N} \bar{\psi}_a \gamma_\mu \partial_\mu \psi_a - m_0 (\bar{\psi}_a \psi_a) + U(\bar{\psi}_a \psi_a)^2
\]

where \( M = \tilde{g} < (\bar{\psi}_e \psi_e) > \) and \( m_0 = \tilde{g} < \cos \beta \Phi > \). Using the exact expression of vacuum expectation values of exponential fields in the sine-Gordon model [22], one can relate \( m_0 \) to the soliton mass \( m_{SG} \) of the sine-Gordon model (17): \( m_0 = \tilde{g} C_d m_{SG}^d \) where the numerical prefactor \( C_d \) is known [22]:

\[
C_d = \frac{\pi \Gamma(1-d/2)}{2(2-d) \sin[\pi d/(2-d)] \Gamma(d/2)} \left\{ \frac{2\sqrt{\pi}}{\Gamma[1/(2-d)] \Gamma[1-d/2]} \right\}^{2-d}.
\]

Using the large \( N \) limit of model (18), one obtains the following expressions:

\[
M = \frac{\tilde{g} N}{2\pi m_0} m \ln(\Lambda/m)
\]

\[
m = \frac{1 + [2UN/\pi] \ln(\Lambda/m)}{2\pi m_0}.
\]

In order to close the different equations of the large \( N \) limit of the model (18), we need a relation between \( M \) and \( m_{SG} \) which is given by:

\[
M = D_d m_{SG}^{2-d}
\]

\[
D_d = \frac{2\Gamma(d/2)}{\pi \Gamma(1-d/2)} \left\{ \frac{\sqrt{\pi} \Gamma[1/(2-d)]}{2 \Gamma[2(2-d)]} \right\}^{2-d}.
\]

We shall assume that in the limit \( N \rightarrow \infty \), \( UN = U^* \), \( \tilde{g}^2 N^d = g^* \) remain finite and that \( U^* \ln(\Lambda/m) >> 1 \). In that case, we obtain the mass spectrum of the model in the large \( N \) limit:

\[
m_{SG} \simeq \left( \frac{\tilde{g}^2 N C_d}{4U^* D_d} \right)^{1/(2-2d)}
\]

\[
m \ln(\Lambda/m) \simeq 2\pi D_d (g^*)^{1/(2-2d)} (C_d/4D_d U^*)^{(2-d)/(2-2d)}.
\]

As we see, the ratio of the masses \( m_{SG}/m \sim \sqrt{N} \) is large. Therefore, this result indicates that the symmetry is not restored by interactions in the strong coupling limit.

From our discussion of generalized Thirring models, we conclude that there are two possible sectors where the symmetry is apparently restored in the strong coupling regime. In the first one, called the sine-Gordon sector, the parameter characterizing anisotropy does not renormalize to zero at strong coupling. Its value depends on the bare coupling constants and is unlikely to be small outside the region of weak bare interactions. There is another region, however, the region \( C \), or the trigonometric region where the influence of the anisotropy is exponentially small. Here the restored symmetry has more chances to survive outside of regions of weak bare couplings. Both regions are indistinguishable within the perturbative RG approach in the infrared limit. We have also given an example where the RG equations indicate isotropy, but the nonperturbative approach based on large \( N \)-approximation shows striking anisotropy.

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