QUASI-COMPLETE INTERSECTION HOMOMORPHISMS

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Abstract. Extending a notion defined for surjective maps by Blanco, Majadas, and Rodicio, we introduce and study a class of homomorphisms of commutative noetherian rings, which strictly contains the class of locally complete intersection homomorphisms, while sharing many of its remarkable properties.

Introduction

This paper is concerned with local properties of homomorphisms \( \varphi: R \rightarrow S \) of commutative noetherian rings. Grothendieck’s detailed investigation of the case when \( \varphi \) is flat, see [24], has produced two types of results. Some deal with the transfer of local properties between the rings \( R \) and \( S \) by maps belonging to a given class. Others establish the stability of classes of homomorphisms under operations such as composition, decomposition, flat base change, localization, or completion.

In [8, 9, 12, 13] the main results of that theory were shown to hold, in appropriate form, for homomorphisms of finite flat dimension. These include the classically important locally complete intersection, or l.c.i., homomorphisms. Here we study a class of maps that include l.c.i. homomorphisms and share many of their remarkable properties, while avoiding the strong restriction of finite flat dimension.

To introduce quasi-complete intersection, or q.c.i., homomorphisms we proceed in two steps. When \( \varphi \) is surjective, we say that it is q.c.i. if the ideal \( \text{Ker} \varphi \) has “exterior Koszul homology,” a notion introduced and studied by A. Blanco, J. Majadas Soto, and A. Rodicio Garcia in [18, 23, 31, 33, 34]. We extend the concept to general maps by using Cohen factorizations of local homomorphisms, constructed in [13]. A similar two-step approach was used in [8]–[13] to define the property of \( \varphi \) to be l.c.i., Gorenstein, quasi-Gorenstein, or Cohen-Macaulay.

Next we briefly describe the content of the paper. For Sections 1 through 6, we assume that \( R \) is a local ring, \( I \) is an ideal of \( R \), and set \( S = R/I \).

In Section 1 we define quasi-complete intersection ideals, track their behavior under flat extensions and factorizations of regular sequences, and provide examples.

Section 2 contains comparisons of the quasi-complete intersection condition with other properties. Thus, we show that \( I \) is generated by an \( R \)-regular sequence if and only if it is quasi-complete intersection and has finite projective dimension, and that if \( I \) is quasi-complete intersection, then it is quasi-Gorenstein. The latter statement is a key result of Garcia and Soto [23], for which we give a new proof.
In Section 3 we show that grade\(_R S\) behaves for quasi-complete intersections as it does for complete intersections. On the one hand, it is equal to depth \(R - \) depth \(S\); this is observed in [23] and holds even when \(I\) is quasi-Gorenstein, by [11]. Furthermore, we show that grade\(_R S\) = dim \(R - \) dim \(S\) holds in some interesting cases, conjecture that it always does, and discuss possible approaches to that conjecture.

Section 4 is devoted to quasi-complete intersection ideals generated by a single zero-divisor. A number of recent papers have have shown that such ideals are often be found in “short” local rings; see [20, 21, 26]. Using a specific ideal in a short ring constructed in [14], we give a counterexample to a characterization of quasi-complete intersection ideals, conjectured by Rodicio; see [23, 33].

A most useful property of an \(R\)-regular sequence \(b\) is that in many cases the Betti numbers of a module over \(R/(b)\) can be explicitly expressed in terms of its Betti numbers over \(R\), and vice-versa. In Sections 5 and 6 we prove that if \(I\) is quasi-complete intersection, then more complicated, but equally explicit relations hold between the Betti numbers of an \(S\)-module \(N\), computed over \(S\) and over \(R\).

In Section 7 we turn to general q.c.i. homomorphisms. We show that \(\varphi\) is l.c.i., precisely when it is q.c.i. and locally of finite flat dimension, that homomorphisms of l.c.i. rings are always q.c.i., and we investigate the transfer of various ring-theoretic properties between \(R\) and \(S\). To study the stability of the class of q.c.i. homomorphisms we use the homology theory of commutative algebras due to André [1] and Quillen [29]. This goes through a characterization of q.c.i. homomorphisms in terms of vanishing of the André-Quillen homology functors \(D_n(S|R, -)\) for \(n \geq 3\), proved in [18] when \(\varphi\) is surjective, and extended here to all maps. The developments in Section 7 follow the treatment of l.c.i. homomorphisms in [8, §5], so proofs requiring only superficial changes have been omitted. Some often used properties of André-Quillen homology are collected in Appendix A.

1. QUASI-COMPLETE INTERSECTION IDEALS

Let \((R, m, k)\) be a local ring; in detail: \(R\) is a commutative noetherian ring with unique maximal ideal \(m\), and \(k = R / m\). In addition, let \(I\) be an ideal of \(R\) and set \(S = R / I\). We discuss properties of \(I\) that interpolate between being generated by an \(R\)-regular sequence and satisfying an isomorphism \(\bigoplus_{i = 0}^{\infty} \text{Ext}_R^i(S, R) \cong S\).

Let \(a\) be a finite generating set of \(I\) and \(E\) the Koszul complex on \(a\). The homology \(H_1(E)\) has \(H_0(E) = S\) and a structure of graded-commutative \(S\)-algebra, inherited from the DG \(R\)-algebra \(E\). Thus, there is a unique homomorphism

\[
\lambda^S: \Lambda^S H_1(E) \rightarrow H_1(E)
\]

(1.0.1)
of graded \(S\)-algebras with \(\lambda^S = \text{id}^{H_1(E)}\), where \(\Lambda^S\) is the exterior algebra functor.

We say that \(I\) is quasi-complete intersection if \(H_1(E)\) is free over \(S\) and \(\lambda^S\) is bijective. This property does not depend on the choice of \(a\), see [19, 1.6.21]; in [31], such an ideal is said to have exterior projective homology.

Quasi-complete intersections can be tracked under some base changes.

**Lemma 1.1.** Let \(R'\) be a local ring and \(\rho: R \rightarrow R'\) a flat homomorphism of rings.

Assume \(IR' \not= R'\) holds; if \(I\) is quasi-complete intersection, then so is \(IR'\).

Assume \(\rho\) is faithfully flat; if \(IR'\) is quasi-complete intersection, then so is \(I\).

**Proof.** Note that \(R' \otimes_R E\) is the Koszul complex on the generating set \(\rho(a)\) of \(IR'\). As \(R'\) is a flat \(R\)-module, there is a natural isomorphism \(H_1(R' \otimes_R E) \cong R' \otimes_R H_1(E)\) of graded algebras, whence the assertions follow by standard arguments. \(\square\)
Lemma 1.2. When \( b \) is an \( R \)-regular sequence in \( I \), the ideals \( I \) of \( R \) and \( \overline{I} = I/(b) \) in \( \overline{R} = R/(b) \) are simultaneously quasi-complete intersection.

Proof. By induction, we may assume \( b = a \). Let \( E \) be the Koszul complex on a generating set \( \{a_1, \ldots, a_c\} \) of \( I \) with \( a_1 = a \). Pick \( v_1, \ldots, v_c \) in \( E_1 \) with \( \partial(v_i) = a_i \) for \( i = 1, \ldots, c \), set \( v = v_1 \), and note that \( J = (a, v)E \) is a DG ideal. Each \( y \in J \) can be written uniquely as \( y = ac + vf \) with \( e, f \in E' \), where \( E' \) is the DG \( R \)-subalgebra of \( E \), generated by \( v_2, \ldots, v_c \). Thus, \( \partial(y) = 0 \) implies \( a(\partial(e) + f) = 0 \). As \( a \) is regular, this gives \( f = -\partial(e) \), so \( y = \partial(ve) \), whence \( H_n(J) = 0 \). Thus, \( E \to E/J \) induces \( H_n(E) \cong H_n(E/J) \), as graded \( S \)-algebras. It remains to note that \( E/J \) is isomorphic to the Koszul complex on \( a = \{a_2 + b, \ldots, a_c + b\} \subseteq \overline{R} \), and \( \overline{T} = (a) \). □

We proceed to introduce some background material and examples.

1.3. The ideal \( I \) is said to be complete intersection if it has a generating set \( a \) satisfying the following equivalent conditions: (i) \( a \) is an \( R \)-regular sequence; (ii) \( H_1(E) = 0 \); (iii) \( H_n(E) = 0 \) for all \( n \geq 1 \). Thus, one evidently has:

Every complete intersection ideal is quasi-complete intersection.

The ideal \( \mathfrak{m} \) is complete intersection if and only if \( R \) is regular; see [19, 2.2.5].

1.4. By Cohen’s Structure Theorem, the \( \mathfrak{m} \)-adic-completion of \( R \) has a presentation \( \hat{R} \cong Q/J \), with \( Q \) a regular local ring. The ring \( \hat{R} \) is said to be complete intersection if in some Cohen presentation of \( \hat{R} \) the ideal \( J \) is generated by a regular sequence; this property is independent of the presentation, see [24, 19.3.2] or [19, 2.3.3].

The ideal \( \mathfrak{m} \) is quasi-complete intersection if and only if the ring \( R \) is complete intersection; this is due to Assmus, [4, 2.7], see also [19, 2.3.11].

1.5. A quasi-deformation is a pair \( R \to R' \leftarrow Q \) of homomorphisms of local rings, with \( R \to R' \) faithfully flat and \( R' \leftarrow Q \) surjective with kernel generated by a \( Q \)-regular sequence. By definition, the CI-dimension of an \( R \)-module \( M \), denoted \( \text{CI-dim}_R M \), is finite if \( \text{pd}_Q(R' \otimes_R M) \) is finite for some quasi-deformation; see [15].

If \( R \) is complete intersection, then \( \text{CI-dim}_R M \) is finite for each \( M \).

If \( \text{CI-dim}_R \mathfrak{m} \) is finite, then \( R \) is complete intersection; see [15, 1.3, 1.9].

1.6. As in [26, §1], we say that \( a \) in \( R \) is an exact zero-divisor if it satisfies

\[
R \neq (0:a)_R \cong R/(a) \neq 0. \tag{1.6.1}
\]

Equivalently, if \( (a) = (0 : b)_R \neq 0 \) and \( (b) = (0 : a)_R \neq 0 \) for some \( b \in R \); thus,

\[
\cdots \to R \xrightarrow{a} R \xrightarrow{b} R \xrightarrow{a} R \xrightarrow{0} \cdots \tag{1.6.2}
\]

is a minimal free resolution of the \( R \)-module \( R/(a) \).

An ideal generated by a non-zero element \( a \) in \( \mathfrak{m} \) is quasi-complete intersection if and only if \( a \) is either \( R \)-regular or an exact zero-divisor.

Indeed, the Koszul complex \( E \) on \( \{a\} \) has \( E_i = 0 \) for \( i \geq 2 \) and \( H_1(E) = (0 : a)_R \), so \( \lambda^R_{(a)} \) in (1.0.1) is bijective if and only if \( (0 : a)_R \) is a cyclic \( R/(a) \)-module. Thus, \( (a) \) is quasi-complete intersection, if and only if \( (0 : a)_R = 0 \) or \( (0 : a)_R \cong R/(a) \).

1.7. We say that the ideal \( I \) is quasi-Gorenstein if the homomorphism \( R \to R/I = S \) is quasi-Gorenstein, in the sense of [11]. By [16, 2.3], this is equivalent to saying that \( \text{Ext}^n_{R}(S, R) \cong S \) and \( \text{Ext}^n_{R}(S, R) = 0 \) for \( n > \text{grade}_R S \). Thus, one has:

The ideal \( \mathfrak{m} \) is quasi-Gorenstein if and only if \( R \) is a Gorenstein ring.
2. Between complete intersection and Gorenstein

In this section we present new proofs of two key results of García and Soto: The implication (3) in the next theorem is [33, Prop. 12], proved there by using André-Quillen homology. The implication (4) is obtained in [23, Rem. 8] as a consequence of a more general result concerning algebras with Poincaré duality.

**Theorem 2.1.** If $R$ is a local ring, $I$ an ideal of $R$ and $E$ the Koszul complex on a set of generators for $I$, then the following implications hold

$I$ is complete intersection $\iff$ $I$ is quasi-complete intersection and $\text{pd}_R I$ is finite

$I$ is quasi-complete intersection $\iff$ $H_1(E)$ is free over $R/I$ and $\text{CI-dim}_R I$ is finite

$I$ is quasi-Gorenstein

**Remark 2.2.** The implication (2) is irreversible, in view of 1.3 and 1.4 applied to $R = k[x]/(x^2)$ and $I = \mathfrak{m}$. Also, (4) is irreversible, due to 1.4 and 1.7 for $R = k[x,y,z]/(x^2 - y^2, y^2 - z^2, xy, xz, yz)$ and $I = \mathfrak{m}$. For (3), see Theorem 4.5.

Our proof is based on the next construction, due to Tate.

**Construction 2.3.** Let $\mathbf{a} = \{a_1, \ldots, a_c\}$ be the chosen set of generators of $I$. Recall that $E$ denotes the Koszul complex on $\mathbf{a}$, choose a basis $\mathbf{v} = \{v_1, \ldots, v_c\}$ of $E_1$ with $\partial(v_i) = a_i$ for $i = 1, \ldots, c$, and a set of cycles $\mathbf{z} = \{z_1, \ldots, z_h\}$ whose classes minimally generate $H_1(E)$. Let $W$ be an $R$-module with basis $\mathbf{w} = \{w_1, \ldots, w_h\}$ and $\Gamma_p^W$ its divided powers algebra. The products $w_1^{(j_1)} \cdots w_h^{(j_h)}$ with $j_i \geq 0$ and $j_1 + \cdots + j_h = p$ form an $R$-basis of $\Gamma_p^W$.

The *Tate complex* on $E$ and $\mathbf{z}$ is the complex $F$ of free $R$-modules with

\begin{equation}
F_n = \bigoplus_{2p+q=n} \Gamma_p^W \otimes_R E_q
\end{equation}

\begin{equation}
\partial_n^F(w_1^{(j_1)} \cdots w_h^{(j_h)} \otimes e) = w_1^{(j_1)} \cdots w_h^{(j_h)} \otimes \partial^E(e)
+ \sum_{i=1}^h w_1^{(j_1)} \cdots w_i^{(j_i)} \cdots w_h^{(j_h)} \otimes z_i e.
\end{equation}

One has $H_0(F) = S$. In [35, Thm. 2], Tate proves that if $I$ is quasi-complete intersection, then $F$ is a resolution. If $\mathbf{a}$ generates $I$ minimally, then each $z_i$ is a syzygy in the free cover $E_1 \to I$, whence $z_i \in \mathfrak{m}E_1$, so $\partial(F) \subseteq \mathfrak{m}F$ holds by (2.3.2).

**Proof of Theorem 2.1.** (1) If $I$ is quasi-complete intersection and $\text{pd}_R S$ is finite, then $F_n = 0$ for $n \gg 0$ in the minimal resolution $F$ of Construction 2.3. This forces $W = 0$, so $H_1(E) = 0$, hence $I$ is complete intersection. The converse is clear.

(2) This implication holds because finite projective dimension implies finite CI-dimension, as testified by the quasi-deformation $R \xrightarrow{\sim} R \leftarrow R$.\[\]

(3) Let $R \to R' \leftarrow Q$ be a quasi-deformation with $\text{pd}_Q(R' \otimes_R S)$ finite. Let $\mathbf{b}$ be an $R'$-regular sequence generating $\text{Ker}(Q \to R')$, and $B$ the Koszul complex on $\mathbf{b}$. Choose a subset $\mathbf{a}'$ of $Q$ mapping bijectively to $\mathbf{a}$, and let $E'$ be the Koszul complex
on $a'$. The quasi-isomorphism $B \to H_0(B) = R'$ induces a quasi-isomorphism $B \otimes Q E' \to R' \otimes Q E'$. It gives the first isomorphism in the string

$$H_1(B \otimes Q E') \cong H_1(R' \otimes Q E') \cong H_1(R' \otimes R E) \cong R' \otimes R H_1(E) \cong S' \otimes S H_1(E)$$

where we have set $S' = R' \otimes R S$. The second isomorphism is induced by $R' \otimes Q E' \cong R' \otimes R E$, and the third one is due to the flatness of $R'$ over $R$.

Since $H_1(E)$ is free over $S$, so is $H_1(B \otimes Q E')$ over $S'$. As $B \otimes Q E'$ is the Koszul complex on $b \sqcup a'$, the ideal $J = \text{Ker}(Q \to S')$ is generated by a regular sequence, due to a theorem of Gulliksen; see [25, 1.4.9]. Now 1.3 and Lemma 1.2 show that $J/(b)$ is quasi-complete intersection; then so is $I$, by Lemma 1.1, as $\text{IR}' = J/(b)$.

(4) We use the notation from Construction 2.3. Also, when $X$ is a complex of $R$-modules, $\Sigma X$ denotes its translation. For elements of $X$, we let $|x| = n$ stand for $x \in X_n$. Let $X^\vee$ denote the complex with $(X^\vee)_n = \text{Hom}_R(X_{-n}, R)$ for $n \in \mathbb{Z}$, and $(\partial X^\vee(\chi))(x) = (-1)^{|x|} \partial X(x)$ for $\chi \in X^\vee$ and $x \in X$. When $X$ is a DG $E$-module, so is $X^\vee$, with action of $E$ given by $(e \cdot \chi)(x) = (-1)^{|e||\chi|} \chi(ex)$ for $e \in E$.

Let $\omega = \{\omega_1, \ldots, \omega_h\}$ be the basis of $W^\vee$, dual to $\omega$, and $S^R(W^\vee)$ the symmetric algebra of $W^\vee$. Standard isomorphisms of $R$-modules, $(\Gamma_p W^\vee) \cong S^p(W^\vee)$, take the elements of the basis of $(\Gamma_p W)^\vee$, dual to the basis of $\Gamma_p W$ described in Construction 2.3, to the corresponding products $\omega_1^j \cdots \omega_h^j$. Thus, we have

$$\begin{align*}
(F^\vee)_n &= \bigoplus_{2p+q=n} S^R_{-p}(W^\vee) \otimes_R (E^\vee)_q \\
(2.4.1) \quad \partial E^\vee(\omega \otimes \epsilon) &= \omega \otimes \partial E^\vee(\epsilon) + \sum_{i=1}^h \omega_i \omega \otimes z_i \epsilon.
\end{align*}$$

A decreasing filtration of $F^\vee$ is given by the $R$-linear spans of the sets

$$\{(\omega_1^j \cdots \omega_h^j \otimes \epsilon) \in F^\vee \mid j_i \geq 0, j_1 + \cdots + j_h \geq -p, \epsilon \in E^\vee\}$$

for $p \leq 0$. The resulting spectral sequence $E^p_{r,q} \implies H_{p+q}(F^\vee)$ starts with

$$\begin{align*}
E^0_{p,q} &= S^R_{-p}(W^\vee) \otimes_R (E^\vee)_q - p \\
(2.4.3) \quad d^0_{p,q} &= S^R_{-p}(W^\vee) \otimes_R \partial E^\vee_{q-p}.
\end{align*}$$

Let $U$ be an $R$-module with basis $u = \{u_1, \ldots, u_h\}$. The hypothesis on $I$ yields an isomorphism $S \otimes_R \Lambda^p U \cong H_0(U)$ of graded $R$-algebras sending $u_i$ to the class of $z_i$ for $i = 1, \ldots, h$. Let $\tau^u \in (\Lambda^p U)^{-h}$ be the $R$-linear map with $\tau^u(u_1 \cdots u_h) = 1$; the map $u \mapsto u \cdot \tau^u$ is an isomorphism $\Lambda^p U \cong \Sigma^p(\Lambda^p U)^\vee$ of graded $\Lambda^p U$-modules. Similarly, the $R$-linear map $\tau^v \in (E^\vee)^{-c}$, defined by $\tau^v(v_1 \cdots v_c) = 1$, yields an isomorphism of DG $E$-modules $E \cong \Sigma^{-c}(E^\vee)$, given by $e \mapsto e \cdot \tau^v$. Thus, we get

$$\begin{align*}
H_{q-p}(E^\vee) &\cong H_{q-p+c}(E) \cong S \otimes_R \Lambda^R_{q-p+c} U \cong S \otimes_R (\Lambda^R U)^\vee_{q-p+c-h} \\
(2.4.4) \quad \text{as } R\text{-modules. Now (2.4.3) and (2.4.4) yield}
\end{align*}$$

$$\begin{align*}
E^p_{r,q} &= H_{q-p}(E^0_{r,q}) \cong S^R_{-p}(W^\vee) \otimes_R H_{q-p}(E^\vee) \\
&\cong S \otimes_R (S^R_{-p}(W^\vee) \otimes_R (\Lambda^R U)^\vee_{q-p+c-h}) \\
&\cong S \otimes_R (\Gamma_R(W) \otimes_R \Lambda^R_{-p+q}(u_{-c}) U)^\vee.
\end{align*}$$

Let $C$ be the complex with $C_n = \bigoplus_{2p+q+n} E^1_{p,q}$ and $\partial^C_n = \bigoplus_{2p+q+n} q^1_{p,q}$. Let $G$ be the Tate complex on the Koszul complex $\Lambda^p U$ with zero differentials and the set of cycles $u$. Formulas (2.4.2) and (2.3.2) show that the maps above produce an isomorphism of complexes $C \cong S \otimes_R (\Sigma^{-h} G)^\vee$. By Construction 2.3, $G$ is
an $R$-free resolution of $R$, so $G$ is homotopically equivalent to $R$. Consequently, $(\Sigma^{c-h}G)^{\vee}$ is homotopically equivalent to $\Sigma^{h-c}R$, whence $H_n(C) = 0$ for $n \neq h - c$, and $H_{h-c}(C) \cong S$. We have $C_{h-c} = E^3_{0,h-c}$ so the computation of $H_\ast(C)$ yields

$$E^2_{p,q} = H_p(E^1_{\ast,q}) \cong \begin{cases} S & \text{for } (p,q) = (0,h-c), \\ 0 & \text{otherwise}. \end{cases}$$

The convergence of the spectral sequence implies $H_{h-c}(F^{\vee}) \cong S$ and $H_n(F^{\vee}) = 0$ for $n \neq h - c$. One has $H_n(F^{\vee}) \cong \text{Ext}^n_{R}(S,R)$, see Construction 2.3, hence $\text{Ext}^{c-h}(S,R) \cong S$ and $\text{Ext}^n_{R}(S,R) = 0$ for $n \neq c - h$, so $I$ is quasi-Gorenstein. □

3. Grade

In this section $(R,\mathfrak{m},k)$ is a local ring, $I$ an ideal of $R$, and $S = R/I$.

Recall that $\text{grade}_R S$ denotes the maximal length of an $R$-regular sequence in $I$, this number is equal to the least integer $n$ with $\text{Ext}^n_R(S,R) \neq 0$.

When $I$ is quasi-complete intersection, we obtain two different expressions for $\text{grade}_R S$, and partly establish a third one. The first equality actually holds in the broader context of quasi-Gorenstein ideals, see Theorem 2.1(4), and can be obtained from [11, 6.5, 7.4, 7.5]; the short proof below comes from [23, Cor. 5].

Lemma 3.1. If $I$ is quasi-Gorenstein, then there is an equality

$$(3.1.1) \quad \text{grade}_R S = \text{depth } R - \text{depth } S.$$  

Proof. Set $d = \text{depth } S$. As $\text{Ext}^g_R(S,R) \cong S$ for $g = \text{grade}_R S$, the spectral sequence

$$E^2_{p,q} = \text{Ext}^p_S(k,\text{Ext}^q_R(S,R)) \implies \text{Ext}^{p+q}_R(k,R)$$

yields $\text{Ext}^p_R(k,R) = 0$ for $p < d + g$, and $\text{Ext}^{d+g}_R(k,R) \cong \text{Ext}^d_S(k,S) \neq 0$. □

Lemma 3.2. If $I$ is quasi-complete intersection, then there is an equality

$$(3.2.1) \quad \text{grade}_R S = \text{rank}_k(I/\mathfrak{m}I) - \text{rank}_R H_1(E),$$

where $E$ is the Koszul complex on a minimal generating set of $I$.

Proof. The grade-sensitivity of $E$, see [19, 1.6.17(b)], yields

$$\text{rank}_k(I/\mathfrak{m}I) - \text{grade}_R S = \max\{n \mid H_n(E) \neq 0\}.$$ 

As $H_\ast(E) = \Lambda^S_\ast H_1(E)$, the right-hand side equals $\text{rank}_S H_1(E)$. □

We say that a local ring $(R,\mathfrak{m},k)$ is quasi-homogeneous if there is a finitely generated graded $k$-algebra $\overline{R} = \bigoplus_{i=0}^{\infty} \overline{R}_i$, with $\overline{R}_0 = k$, such that the $\mathfrak{m}$-adic-completion $\widehat{R}$ of $R$ is isomorphic to the $(\bigoplus_{i=1}^{\infty} R_i)$-adic completion $\overline{R}$ of $R$. The ideal $I$ is called quasi-homogeneous if $I = \overline{I}$ for some homogeneous ideal $\overline{I}$ of $\overline{R}$.

Theorem 3.3. If $I$ is quasi-complete intersection, and one of the conditions

(a) $\text{rank}_k I/\mathfrak{m}I \leq \text{grade}_R S + 1$, or
(b) $I$ is quasi-homogeneous,

is satisfied, then there is an equality

$$(3.3.1) \quad \text{grade}_R S = \dim R - \dim S.$$
Proof. (a) Set $d = \dim R$ and $g = \text{grade}_R S$. Using prime avoidance, choose for $I$ a minimal generating set, whose first $g$ elements form an $R$-regular sequence $b$. By Lemma 1.2, the ideal $T = I/(b)$ of $R = R/(b)$ is quasi-complete intersection. We have $S \cong R/T$ with $\text{grade}_R S = 0$, $\dim R = d - g$, and $\text{rank}_k T/mT \leq 1$. Changing notation, we may assume $\text{grade}_R S = 0$ and $I = (a)$ for some $a \in m$.

There is nothing left to prove when $a = 0$. Else, $a$ is an exact zero-divisor, and we need to show $\dim S = d$. This is evident for $d = 0$, so we also assume $d \geq 1$.

Suppose, by way of contradiction, that $\dim R/(a) < d$ holds. Choose $q \in \text{Spec} R$ with $\dim(R/q) = d$. We must have $a \not\in q$, whence $\dim R/(a) \leq d - 1 = \dim R/(q + (a)) \leq \dim R/(a)$.

Thus, some prime ideal $p$ containing $q + (a)$ satisfies $\dim(R/p) = \dim R/(a)$. It follows that $p$ is minimal over $(a)$. Krull’s Principal Ideal Theorem now gives height $p \leq 1$; in fact, equality holds, due to the inclusion $p \supseteq q$.

Now choose a generator $b$ of $(0 : a)_R$. The exact sequence of $R$-modules

$$0 \to R/(a) \to R \to R/(b) \to 0$$

yields $(R/(b))_q \cong R_q \neq 0$, hence $b \in q \subseteq p$. Localizing at $p$ and changing notation, we get a local ring $(R, m, k)$ with $\dim R = 1$ and elements $a$ and $b$ in $m$, such that the map $R/(a) \to R$ given by $c + (a) \to bc$ is injective. A result of Fouli and Huneke, see [22, 4.1], implies that $ab$ is a parameter for $R$, which is absurd, since $ab = 0$.

(b) The ingredients of formula (3.3.1) do not change under $m$-adic completion, so we may assume $R = \tilde{R}$ and $I = LR$, with $L$ minimally generated by homogeneous elements $\bar{a}_1, \ldots, \bar{a}_\ell$. The Koszul complex $E$ on these generators is naturally bigraded. Choose in $\tilde{E}_1$ homogeneous cycles $\bar{z}_1, \ldots, \bar{z}_h$, whose homology classes minimally generate the graded $\tilde{R}$-module $H_1(E)$. Construction 2.3 then yields a complex $E$ of graded $\tilde{R}$-modules, with differentials of degree 0.

The complex of $R$-modules $R \otimes_R E$ is isomorphic to the Tate complex $F$ on $R \otimes_R E$ and $\{1 \otimes \bar{z}_1, \ldots, 1 \otimes \bar{z}_h\}$. By Construction 2.3, $F$ is a free resolution of $S$ over $R$. On the category of finite graded $\tilde{R}$-modules the functor $(R \otimes_R -)$ is exact and faithful, so $E$ is a graded free resolution of $\tilde{S} = R/L$ over $R$. As $(\tilde{E}_n)_j = 0$ for $j < n$, by counting ranks of graded free $R$-modules we get

$$\sum_{n=0}^\infty \left( \sum_{j=m}^\infty \text{rank}_R(\tilde{E}_n)_j s^j \right) t^n = \frac{\prod_{a=1}^c (1 + s^{\deg a_u} u)}{\prod_{n=1}^h (1 - s^{\deg z_n} t^2)}.$$

For every finite graded $R$-module $N$, form the formal Laurent series $H_N(s) = \sum_{j \in \mathbb{Z}} \text{rank}_N \tilde{N}_j s^n$. Counting ranks over $k$ in the exact sequence

$$\cdots \to \tilde{E}_n \to \tilde{E}_{n-1} \to \cdots \to \tilde{E}_1 \to \tilde{E}_0 \to \tilde{S} \to 0$$

one gets the first equality of formal power series in the following formula:

$$H_\tilde{S}(s) = H_R(s) \cdot \sum_{n=0}^\infty (-1)^n \left( \sum_{j=n}^\infty \text{rank}_R(\tilde{E}_n)_j s^j \right) = H_R(s) \cdot \frac{\prod_{a=1}^c (1 - s^{\deg a_u})}{\prod_{n=1}^h (1 - s^{\deg z_n})}.$$

The second equality is obtained by setting $t = -1$ in the expression (3.3.2).

By the Hilbert-Serre Theorem, $H_\tilde{S}(s)$ represents a rational function in $s$, and the order of its pole at $s = 1$ equals the Krull dimension of $\tilde{S}$. Equating the orders of the poles at $s = 1$ in the formula above, we get the second equality in the string

$$\dim S = \dim \tilde{S} = \dim R + h - c = \dim R + h - c = \dim R - \text{grade}_R S.$$
The last equality is given by (3.2.1), while the other two are standard. \( \square \)

Upon the evidence from the preceding result, we propose a conjecture with interesting applications; see the remark following 7.4.

**Conjecture 3.4.** If \( I \) is a quasi-complete intersection ideal in a local ring \( R \) and \( S = R/I \), then \( \text{grade}_R S = \dim R - \dim S \) holds.

We make a few remarks concerning possible approaches.

**Remark 3.5.** The reduction at the start of the proof of Theorem 3.3(a) shows that it suffices to prove the conjecture when \( \text{grade}_R S = 0 \). The assertion then becomes that \( I \) is contained in some prime ideal \( p \) with \( \dim(R/p) = \dim R \).

**Remark 3.6.** It follows from Lemma 1.2, Theorem 3.3(a), and induction that Conjecture 3.4 holds when \( I \) has a minimal generating set \( \{a_1, \ldots, a_c\} \), with \( a_i \) a regular element or an exact zero-divisor on \( R/(a_1, \ldots, a_{i-1}) \) for \( i = 1, \ldots, c \). One may ask whether every quasi-complete intersection ideal has such a generating set. We know of no counter-example that would rule out such a structure.

**Remark 3.7.** When \( \text{grade}_R S = 0 \) holds, see Remark 3.5, the \( R \)-module \( S \) is totally reflexive by Theorem 2.1 and Lemma 3.1; that is, there exists an exact sequence

\[
F = \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \xrightarrow{\partial_1} F_0 \rightarrow F_{-1} \rightarrow \cdots \rightarrow F_{-n} \rightarrow \cdots
\]

where each \( F_i \) is a free \( R \)-module of finite rank, \( \text{Hom}_R(F, R) \) is exact, and \( \text{Coker} \partial_1 \) is isomorphic to \( S \). It is not known whether every totally reflexive \( R \)-module \( M \) has \( \dim_R M = \dim R \). A positive answer for cyclic modules implies Conjecture 3.4.

### 4. Exact zero-divisors

In this section we focus on the simplest type of quasi-complete intersection ideals that are not generated by regular sequences—those generated by some exact zero-divisor; see 1.6. The interest of this rather special case comes from the fact that exact zero-divisors have turned up in abundance in recent studies, that they are easy to identify, and that they are amenable to explicit computations.

**Examples 4.1.** Let \( k \) be an algebraically closed field and set \( R = k[x_1, \ldots, x_e]/L \), where \( \deg(x_i) = 1 \) and \( L \) is generated by forms of degree 2 or higher.

The quadratic \( k \)-algebras \( R \) with \( \text{rank}_k R_2 = e - 1 \) are parametrized by the points of the Grassmannian of subspaces of rank \( \left( \begin{array}{c} e-1 \\ 2 \end{array} \right) \) in the \( (e+1) \)-dimensional affine space over \( k \). Conca [20, Thm. 1] proves that this Grassmannian contains a non-empty open subset \( U \), such that each \( R \in U \) contains an element \( a \in R_1 \) with \( a^2 = 0 \) and \( aR_1 = R_2 \neq 0 \). This implies \( \text{rank}_k(a) = e \) and \( (aR) \subseteq (0 : a)R \). A length count shows that equality holds, so \( a \) is an exact zero-divisor.

The Gorenstein \( k \)-algebras \( R \) with \( R_1 = 0 \neq R_3 \) are parametrized by the points of the \( (e+2) \)-dimensional projective space over \( k \). This space contains a non-empty open subset \( U \), such that each \( R \in U \) contains some exact zero-divisor \( a \in R_1 \); this follows from Conca, Rossi, and Valla [21, 2.13]; see also [26, 3.5].

In some cases, only half of the conditions (1.6.1) defining exact zero-divisors need verification. For artinian rings, this follows from the next lemma.

**Lemma 4.2.** Let \( R \) be a commutative ring and \( N \) an \( R \)-module of finite length.

If \( (0 : b)_N = aN \) holds for some elements \( a, b \in R \), then one has \( (0 : a)_N = bN \).
Proof. From \(a(bN) = b(aN) = b(0 : b)_N = 0\) one gets \((0 : a)_N \geq bN\). A length count, using this inclusion and the composition \(bN \cong N/(0 : b)_N = N/aN\), gives

\[
\ell(N) - \ell(aN) = \ell((0 : a)_N) \geq \ell(bN) = \ell(N/aN) = \ell(N) - \ell(aN).
\]

These relations imply \(\ell((0 : a)_N) = \ell(bN)\), hence \((0 : a)_N = bN\).

The situation for Cohen-Macaulay rings is only slightly more complicated.

**Proposition 4.3.** Let \((R, \mathfrak{m}, k)\) be a local ring.

For an element \(a \in \mathfrak{m}\) the following conditions are equivalent.

(i) \(a\) is an exact zero-divisor and \(R\) is Cohen-Macaulay,

(ii) \((0 : a)_R = (b) \neq R\) for some \(b \in R\), and depth \(R/(a)\) \(\geq\) \(\dim R\).

(iii) \((0 : a)_R = (b) \neq R\) for some \(b \in R\), and \(R/(a)\) is Cohen-Macaulay with \(\dim R/(a) = \dim R\).

**Proof.** (i) \(\implies\) (ii). Using (3.1.1), we get depth \(R/(a) = \dim R = \dim R\).

(ii) \(\implies\) (iii). Use the inequalities \(\dim R \geq \dim R/(a) \geq \depth R/(a)\).

(iii) \(\implies\) (i). Set \(K = (0 : b)R/(a)\) and pick \(p \in \Ass_R(K)\). From \(K \subseteq R/(a)\) we get \(p \in \Ass_R R/(a)\). The Cohen-Macaualyness of \(R/(a)\) implies that \(p\) is minimal in \(\Supp_R R/(a)\) and satisfies \(\dim(R/p) = \dim R/(a)\). We have \(\dim R/(a) = \dim R\), so \(p\) is minimal in \(\Spec R\), hence the ring \(R_p\) is artinian. As one has \((0 : (a/1))R_p = (b/1)R_p\), Lemma 4.2 yields \((a/1)R_p = (0 : (b/1))R_p\); that is, \(K_p = 0\). Since \(p\) can be chosen arbitrarily in \(\Ass_R(K)\), we conclude that \(K\) is equal to zero.

Thus, \(a\) is an exact zero-divisor. Now (3.1.1) and the hypotheses yield

\[
\depth R = \depth R/(a) = \dim R/(a) = \dim R.
\]

**Remark 4.4.** The implication (iii) \(\implies\) (i) may fail when \(\dim R/(a) \neq \dim R\).

Indeed, set \(R = k[x, y]/(xy, y^2)\), and let \(a\) and \(b\) denote the images in \(R\) of \(x\) and \(y\), respectively. The equality \((0 : a)_R = (b)\) shows that \((0 : a)_R\) is principal, and the isomorphism \(R/(a) \cong k[y]/(y^2)\) that \(R/(a)\) is Cohen-Macaualy. However, \(R\) is not Cohen-Macaualy; neither is \(a\) an exact zero-divisor, as \((0 : b)_R = (a, b) \neq (a)\).

We use a short local ring to prove that the implication (3) in Theorem 2.1 is irreversible. This settles a conjecture of Rodicio, see [33, Conj. 11], in the negative.

**Theorem 4.5.** If \(k\) is a field of characteristic different from 2, then in the ring

\[
R = \frac{k[w, x, y, z]}{(w^2, wx - y^2, wy - xz, wz, x^2 + yz, xy, z^2)}
\]

the ideal \(xR\) is quasi-complete intersection and has infinite CI-dimension.

**Proof.** Let \(a\) and \(b\) denote the images in \(R\) of \(x\) and \(y\), respectively; thus, \(xR = (a)\). A simple calculation yields \((0 : a)_R = (b)\) and \((0 : b)_R = (a)\), so \(a\) is an exact zero-divisor, and thus \((a)\) is a quasi-complete intersection ideal; see 1.6.

Assume that \(\CI\)-dim \(R\) \(\neq\) \(1\). The left \(\Ext_R(k, k)\)-module \(\Ext_R^*(R/(a), k)\), with multiplication given by Yoneda products, then is finite over a \(k\)-subalgebra of \(\Ext_R(k, k)\), generated by central elements of degree 2; see [17, 5.3]. On the other hand, it is shown in [14, p. 4] that \(\Ext_R^2(k, k)\) contains no non-zero central element of \(\Ext_R^1(k, k)\), so we must have \(\Ext_R^n(R/(a), k) = 0\) for all \(n \gg 0\). This is impossible, as the free resolution in (1.6.2) gives \(\Ext_R^n(R/(a), k) \cong k\) for each \(n \geq 0\). \(\square\)
5. Homology algebras

When \((R, m, k)\) is a local ring, and \(S = R/I\) with \(I\) a quasi-complete intersection ideal, we relate the graded \(k\)-algebras \(\text{Tor}_R^{R}(k, k)\) and \(\text{Tor}_S^{R}(k, k)\).

5.1. A system of divided powers on a graded \(R\)-algebra \(A\) is an operation that for each \(j \geq 1\) and each \(i \geq 0\) assigns to every \(a \in A_{2i}\) an element \(a^{(j)} \in A_{2ij}\), subject to certain axioms; cf. [25, 1.7.1]. A DG\( \Gamma \) algebra is a DG \(R\)-algebra \(A\) with divided powers compatible with the differential: \(\partial(a^{(j)}) = \partial(a)a^{(j-1)}\).

Let \(A\) be a DG\( \Gamma \) algebra with \(R \to A_0\) surjective and \(A_n = 0\) for \(n < 0\). Let \(A_{\geq 1}\) denote the subset of \(A\), consisting of elements of positive degree. Let \(D(A)\) denote the graded \(R\)-submodule of \(A\), generated by \(A_0 + mA_{\geq 1}\), all elements of the form \(uv\) with \(u, v \in A_{\geq 1}\), and all \(w^{(j)}\) with \(w \in A_{2i}, i \geq 1, j \geq 2\). This clearly is a subcomplex of \(A\), and it defines a complex \(Q^\ast(A) = A/D(A)\) of \(k\)-vector spaces.

Given a set \(x = \{x_i : |x| \geq 1\}\), we let \(A(x)\) denote a DG\( \Gamma \) algebra with

\[
A \otimes_R \Lambda^R_{x} \left( \bigoplus_{|x| \text{ odd}} R x \right) \otimes_R \Gamma^R_{x} \left( \bigoplus_{|x| \text{ even}} R x \right)
\]

as underlying graded algebra and differential compatible with that of \(A\) and the divided powers of \(x \in x\). For every integer \(n\) we set \(x_n = \{x \in x : |x| = n\}\).

A Tate resolution of a surjective ring homomorphism \(R \to T\) in a quasi-isomorphism \(R(x) \to T\), where \(x = \{x_i\}_{i \geq 1}\) and \(|x_i| \geq |x_j| \geq 1\) holds for all \(j \geq i \geq 1\). Such a resolution always exists: see [35, Thm. 1], [25, 1.2.4], or [7, 6.1.4].

5.2. Any Tate resolution of \(R \to k\) gives \(\text{Tor}_R^{R}(k, k)\) a structure of DG\( \Gamma \)-algebra, and this structure is independent of the choice of resolution. We set

\[
\pi_\ast(R) = Q^\ast(\text{Tor}_R^{R}(k, k)).
\]

We use the following natural isomorphisms as identifications:

\[
\pi_1(R) \cong \text{Tor}_1^{R}(k, k) \cong m/m^2.
\]

The assignment \(R \mapsto \pi_\ast(R)\) is a functor from the category of local rings and surjective homomorphisms to that of graded \(k\)-vector spaces.

Both the statement of the next theorem and its proof are reminiscent of those of [5, 1.1]. However, neither result implies the other one.

**Theorem 5.3.** Let \((R, m, k)\) be a local ring and \(I\) an ideal of \(R\).

Set \(S = R/I\) and let \(\varphi : R \to S\) be the natural map. Let \(E\) be the Koszul complex on some minimal set of generators of \(I\) and set \(H = H_1(E)\).

If \(I\) is quasi-complete intersection, then there is an exact sequence

\[
0 \to H/mH \to \pi_2(R) \xrightarrow{\pi_2(\varphi)} \pi_2(S) \xrightarrow{\delta} I/mI \to \pi_1(R) \xrightarrow{\pi_1(\varphi)} \pi_1(S) \to 0
\]

of \(k\)-vector spaces, and isomorphisms of \(k\)-vector spaces

\[
\pi_n(\varphi) : \pi_n(R) \xrightarrow{\cong} \pi_n(S) \quad \text{for} \quad n \geq 3.
\]

**Remark 5.4.** The sequence (5.3.1) can also be obtained from the Jacobi-Zariski exact sequence (A.2.1), defined by the homomorphisms \(R \to S \to k \xrightarrow{\delta} k\).

Indeed, one has isomorphisms \(\pi_n(R) \cong D_n(k[R, k])\) for \(n = 1, 2\) by [1, 1.6.1 and 15.8], \(I/mI \cong D_1(S[R, k])\) by [1, 1.6.1] and \(H/mH \cong D_2(S[R, k])\) by [1, 15.12], while the freeness of \(H_1(E)\) implies that the map \(\delta_3\) is injective, due to [30, Thm. 1].
Due to [18, Cor. 3'], the same sequence yields $D_n(k|R, k) \cong D_n(k|S, k)$ for $n \geq 3$, which implies (5.3.2) if $\text{char}(k) = 0$, see [1, 19.21], but not when $\text{char}(k) > 0$, see [2].

**Construction 5.5.** Choose a Tate resolution $S(y) \to k$ with $\partial(S(y)) \subseteq m(S(y))$: this is always possible, see [25, 1.6.4] or [7, 6.3.5].

The Tate complex $F$ from Construction 2.3 yields a Tate resolution $R(v, w) \to S$. This map can be extended to a surjective quasi-isomorphism

$$\alpha: R(v, w, x) \to S(y)$$

of DGT algebras with $x = \{x_i\}_{i=1}^3$ and $\alpha(x_i) = y_i$ for each $i$; see [25, 1.3.5]. In particular, $R(v, w, x)$ is a free, but not necessarily minimal, resolution of $k$ over $R$.

**Proof of Theorem 5.3.** The homomorphisms of DGT algebras

$$R(v, w) \to R(v, w, x) \xrightarrow{\alpha} S(y)$$

induce an exact sequence of complexes of $k$-vector-spaces:

$$(5.6.1) \quad 0 \to Q^\gamma(R(v, w)) \to Q^\gamma(R(v, w, x)) \xrightarrow{Q^\gamma(\alpha)} Q^\gamma(S(y)) \to 0$$

Due to the inclusions $\partial(R(v, w)) \subseteq mR(v, w)$ and $\partial(S(y)) \subseteq mS(y)$, the complexes at both ends of (5.6.1) have zero differentials. This gives the following expressions:

$$(5.6.2) \quad H_n(Q^\gamma(R(v, w))) = \begin{cases} kv \cong I/mI & \text{for } n = 1, \\ kw \cong H/mH & \text{for } n = 2, \\ 0 & \text{for } n \neq 1, 2, \end{cases}$$

$$(5.6.3) \quad H_n(Q^\gamma(S(y))) = ky_n \cong \pi_n(S) \quad \text{for all } n.$$  

The next statement is the key point of the argument.

**Claim.** For $G = R(v, w, x)$ one has $\partial_n(G) \subseteq mD_{n-1}(G)$ for $n \neq 2$.

Indeed, for $n \neq 3$ this follows from the exact sequence (5.6.1) and the equalities in (5.6.2) and (5.6.3). Next we prove $\partial(G_3) \subseteq mG_2 + D_2(G)$. Write $z \in \partial(G_3)$ as

$$(5.6.4) \quad z = \sum_{x \in x_2} a_x x + \sum_{w \in w} b_w w + c.$$  

with $a_x$ and $b_w$ in $R$ and $c \in D_2(G)$. One has $\partial(\alpha(z)) = \alpha(\partial(z)) = 0$, so $\alpha(z)$ is a cycle the minimal free resolution $S(y)$; this gives the inclusion below:

$$\sum_{x \in x_2} \varphi(a_x) y + \alpha(c) = \alpha(z) \in mS(y).$$

From this formula, we conclude that in (5.6.4) we have $a_x \in m$ for each $x \in x_2$.

Now we show that in (5.6.4) each $b_w$ is in $m$. Assume, by way of contradiction, that $b_w \notin m$ holds for some $w \in w$. Note that $R(v, w, x_1, x_2)$ has an $R$-basis consisting of products involving elements from $v \sqcup x_1$ and divided powers of elements from $w \sqcup x_2$. When the boundary of an element of $R(v, w, x_1, x_2)$ is written in this basis, the coefficient of $w^{(j)}$ cannot be invertible; this follows from the Leibniz rule. On the other hand, the defining properties of divided powers, see [25, 1.7.1], imply that in the expansion of $z^{(j)}$ the element $w^{(j)}$ appears with coefficient $(b_w)^j$.

In homology, this means $\text{cls}(z)^{(j)} = \text{cls}(z^{(j)}) \neq 0$. Since $\alpha$ is a quasi-isomorphism, for every $j \geq 1$ we get $\text{cls}(\alpha(z))^{(j)} \neq 0$ in $H_*(S(y_1, y_2))$. This is impossible, as for $j > \text{rank}_k(m/m^2)$ the $j$th divided power of every class of even dimension in $H_*(S(y_1, y_2))$ is equal to 0; see [5, 1.7]. This finishes the proof of the Claim.
The Claim gives, in particular, \( \partial_n(Q^\gamma(R\langle v, w, x \rangle)) = 0 \) for all odd \( n \). By [25, 3.2.1(iii) and its proof], there is a quasi-isomorphism \( \rho: R\langle v, w, x \rangle \to R\langle x' \rangle \) of DGT \( R \)-algebras, such that \( R\langle x' \rangle \) is a minimal DGT algebra resolution of \( k \), and the induced map \( Q^\gamma(R\langle v, w, x \rangle) \to Q^\gamma(R\langle x' \rangle) \) is a quasi-isomorphism. Choose, by [25, 1.8.6], a quasi-isomorphism \( \sigma: R\langle x' \rangle \to R\langle v, w, x \rangle \) of DGT \( R \)-algebras. The minimality of \( R\langle x' \rangle \) implies that the composition \( \rho \sigma : R\langle x' \rangle \to R\langle x' \rangle \) is an isomorphism, see [25, 1.9.5], hence so is the map \( Q^\gamma(\rho \sigma) \). It is equal to \( Q^\gamma(R\langle x' \rangle) \), so \( Q^\gamma(\sigma) \) is a quasi-isomorphism. Now form the composition of \( k \)-linear maps

\[
\pi_n(R) \cong H_n(Q^\gamma(R\langle x' \rangle)) \cong H_n(Q^\gamma(R\langle v, w, x \rangle)) \xrightarrow{H_n(Q^\gamma(\alpha))} H_n(Q^\gamma(S\langle y \rangle)) \cong \pi_n(S)
\]

where the first isomorphism is due to the minimality of \( R\langle x' \rangle \), the second one is \( H_n(Q^\gamma(\sigma)) \), and the third one is (5.6.3). As \( \alpha \sigma : R\langle x' \rangle \to S\langle y \rangle \) induces the identity on \( k \), the composed map is, by definition, \( \pi_n(\varphi): \pi_n(R) \to \pi_n(S) \). It follows that the homology exact sequence of the exact sequence (5.6.1) is isomorphic to

\[
\cdots \to H_n(Q^\gamma(R\langle v, w \rangle)) \to \pi_n(R) \xrightarrow{\pi_n(\varphi)} \pi_n(S) \xrightarrow{\delta_n} H_{n-1}(Q^\gamma(R\langle v, w \rangle)) \to \cdots
\]

In view of the isomorphisms in (5.6.2), it remains to prove \( \delta_3 = 0 \). This follows from the construction of the connecting isomorphism and the Claim, for \( n = 3 \). □

6. **Poincaré series**

In this section \( (R, m, k) \) is a local ring, \( I \) an ideal, \( S = R/I \), and \( N \) a finite \( S \)-module. Recall that the **Poincaré series** of \( N \) over \( S \) is the formal power series

\[
P^S_N(t) = \sum_{n=0}^{\infty} \text{rank}_k \text{Tor}^S_n(k, N) t^n.
\]

The **deviations** of \( S \) are defined using the vector spaces (5.2.1), by the formula

\[
\varepsilon_n(S) = \text{rank}_k \pi_n(S) \quad \text{for} \quad n \in \mathbb{Z}.
\]

They appear in a well-known formula, see [25, 3.1.3] or [7, 7.1.3]:

\[
P^S_k(t) = \prod_{r=0}^{\infty} \frac{(1 + t^{2r+1})^{\varepsilon_{2r+1}(S)}}{(1 - t^{2r+2})^{\varepsilon_{2r+2}(S)}}.
\]

The next theorem gives, in particular, new proofs of well-known results for ideals \( I \) generated by regular sequences. Recall that \( \text{edim} R \) stands for \( \text{rank}_k(m/m^2) \).

**Theorem 6.1.** If \( I \) is quasi-complete intersection, then the following equality holds:

\[
P^S_k(t) \cdot \frac{(1 - t)^{\text{edim} S}}{(1 - t^2)^{\text{depth} S}} = P^R_k(t) \cdot \frac{(1 - t)^{\text{edim} R}}{(1 - t^2)^{\text{depth} R}}.
\]

**Proof.** Set \( g = \text{grade}_R S, h = \text{rank}_S H_1(E), \) and \( c = \text{rank}_k I/mI \). The equalities

\[
P^S_k(t) = \prod_{r=0}^{\infty} \frac{(1 + t^{2r+1})^{\varepsilon_{2r+1}(S)}}{(1 - t^{2r+2})^{\varepsilon_{2r+2}(S)}} \cdot \frac{(1 + t)^{\varepsilon_1(S)}}{(1 - t^{2r+2})^{\varepsilon_1(S)}} \cdot \frac{(1 - t)^{\varepsilon_1(R)}}{(1 - t^2)^{\varepsilon_1(R)}} \cdot \frac{1}{(1 - t^{2r+2})^{\varepsilon_1(R)}}
\]

are obtained by applying (6.0.2) and (5.3.1) for the first one, (5.3.2) and (3.2.1) for the second, (6.0.2) and (3.1.1) for the third. Finally, \( \varepsilon_1(S) = \text{edim} S \) by (5.2.2). □
When $I$ is generated by a regular sequence and satisfies $I \cap \mathfrak{m}^2 \subseteq \mathfrak{m} \mathfrak{m}$, Nagata proved that $P^S_R(t)$ and $P^S_N(t)$ determine each other for arbitrary $S$-modules $N$; see [28, (27.3)] or [7, 3.3.5(1)]. Our next theorem yields, in particular, a new proof.

**Theorem 6.2.** If $I$ is a quasi-complete intersection ideal satisfying $I \cap \mathfrak{m}^2 \subseteq \mathfrak{m} \mathfrak{m}$, then for every finite $S$-module $N$ the following equality holds:

$$P^S_N(t) \frac{(1 - t)^{\mathrm{edim} S}}{(1 - t^2)^{\mathrm{depth} S}} = P^R_N(t) \frac{(1 - t)^{\mathrm{edim} R}}{(1 - t^2)^{\mathrm{depth} R}} \cdot \tag{6.2.1}$$

**Proof.** As $I$ is quasi-complete intersection, the map $\pi_n(\varphi) : \pi_n(R) \to \pi_n(S)$ is surjective for $n \neq 2$, by Theorem 5.3. The hypothesis $I \cap \mathfrak{m}^2 \subseteq \mathfrak{m} \mathfrak{m}$ implies that $I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2$ is injective, so the same theorem shows that that $\pi_2(\varphi)$ is surjective as well. By the definition of $\pi_*(R)$, the image of any $k$-linear right inverse $\sigma : \pi_*(R) \to \text{Tor}^\ast_* (k, k)$ of the natural surjection $\text{Tor}^\ast_* (k, k) \to \pi_*(R)$ generates $\text{Tor}^\ast_* (k, k)$ as a graded $\Gamma$-algebra over $k$. Thus, the surjectivity of $\pi_* (\varphi)$ means that the map of $\Gamma$-algebras $\text{Tor}^\ast_* (k, k) : \text{Tor}^\ast_* (k, k) \to \text{Tor}^\ast_* (k, k)$ is surjective; that is, $\varphi$ is a large homomorphism. A theorem of Levin, [27, 1.1], then gives

$$P^S_N(t) P^R_k(t) = P^R_N(t) P^S_k(t) .$$

Now replace $P^S_k(t)$ with its expression from Theorem 6.1, and simplify. \qed

**Remark 6.3.** By a theorem of Shamash, see [32, §3, Cor. (2)] or [7, 3.3.5(2)], formula (6.2.1) holds also for ideals generated by regular sequences contained in $\mathfrak{m} \text{ Ann}_R N$.

We do not know whether that formula still holds when $I$ is a quasi-complete intersection ideal satisfying $I \subseteq \mathfrak{m} \text{ Ann}_R N$.

Some hypothesis on $I$ or $N$ is needed for (6.2.1) to hold:

**Example 6.4.** For $R = k[[x]]$, $S = R/(x^2)$, and $N = S$ one has

$$P^S_N(t) \frac{(1 - t)^{\mathrm{edim} S}}{(1 - t^2)^{\mathrm{depth} S}} = 1 - t \neq 1 = P^R_N(t) \frac{(1 - t)^{\mathrm{edim} R}}{(1 - t^2)^{\mathrm{depth} R}} .$$

7. **Quasi-complete intersection homomorphisms**

In this section $\varphi : R \to S$ denotes a homomorphism of noetherian rings.

For $q \in \text{Spec} S$ we let $q \cap R$ denote the prime ideal $\varphi^{-1}(q)$ of $R$. As usual, we set $k(q) = S_q / qS_q$ and call $k(q) \otimes_R S$ the fiber of $\varphi$ at $q$. The induced homomorphism of local rings $\varphi_q : R_{q \cap R} \to S_q$ is called the localization of $\varphi$ at $q$.

By [13, 1.1], there is a commutative diagram of local homomorphisms

$$\begin{array}{ccc}
R' & \xrightarrow{\varphi'} & S' \\
\text{ } & \searrow & \text{ } \\
R_{q \cap R} & \xrightarrow{\varphi_q} & S_q
\end{array}$$

where $\varphi$ is flat, $\varphi'$ is surjective, $\sigma$ is the $qS_q$-adic completion map, and $R'/q(q \cap R)'R'$ is regular. Any such a diagram is called a Cohen factorization of $\varphi_q$.

We say that $\varphi : R \to S$ is quasi-complete intersection, or q.c.i., at $q$ if in some Cohen factorization of $\varphi_q$ the ideal $\text{Ker}(\varphi')$ is quasi-complete intersection. We first show that this property does not depend on the choice of Cohen factorization:
7.1. Independence. The homomorphism \( \varphi \) is q.c.i. at \( q \) (if and only) if \( \text{Ker} \varphi'' \) is quasi-complete intersection in every Cohen factorization \( R \xrightarrow{\varphi'} R'' \xrightarrow{\varphi''} \hat{S} \) of \( \varphi_q \).

Proof. By [13, 1.2], there is a commutative diagram of local homomorphisms

\[
\begin{array}{cccc}
R' & \xrightarrow{\varphi'} & R'' & \xrightarrow{\varphi''} & \hat{S} \\
\downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{\varphi} \\
R & \xrightarrow{\varphi} & R & \xrightarrow{\varphi'} & \hat{S} \\
\end{array}
\]

where the middle row is a Cohen factorization of \( \varphi_q \), and the vertical arrows are surjections with kernels generated by regular sequences. Lemma 1.2 shows that \( \text{Ker}(\varphi'') \) is quasi-complete intersection if and only if \( \text{Ker}(\varphi') \) is. \( \square \)

We say that \( \varphi \) is q.c.i. if it is q.c.i. at all \( q \in \text{Spec} \ S \). Recall that locally complete intersection homomorphisms, also called l.c.i. homomorphisms and quasi-Gorenstein homomorphisms are defined in similar terms; see [8] and [11].

7.2. Hierarchy. Let \( q \) be a prime ideal of \( S \).

1. \( \varphi \) is c.i. at \( q \) if and only if it is q.c.i. at \( q \) and \( \text{fd}_R S_q \) is finite.
2. If \( \varphi \) is q.c.i. at \( q \), then it is quasi-Gorenstein at \( q \). \( \square \)

Proof. Let \( R \xrightarrow{\varphi'} R' \xrightarrow{\varphi''} \hat{S} \) be a Cohen factorization and set \( I = \text{Ker}(\varphi') \).

1. If \( \text{fd}_R S_q \) is finite, then \( \text{pd}_{R'} I < \infty \) holds by [13, 3.3], so when \( I \) is q.c.i. it is c.i. by Theorem 2.1(1). The 'only if' part is clear.
2. If \( \varphi \) is q.c.i., then \( \text{Ker}(\varphi'') \) is quasi-complete intersection by 7.1, and hence quasi-Gorenstein by Theorem 2.1(4). \( \square \)

We say that \( S \) is l.c.i. if \( S_q \) is complete intersection for each \( q \in \text{Spec} \ S \); see 1.4.

Remark 7.3. When \( R \) is a regular ring, \( \varphi \) is c.i. at \( q \) if and only if \( S_q \) is complete intersection; in particular, \( S \) is l.c.i. if and only if the structure map \( Z \to S \) is q.c.i.

When the homomorphism \( \varphi \) is flat, it is q.c.i. if and only if its fibers are l.c.i. rings.

Indeed, the hypotheses above imply \( \text{fd}_R S_q < \infty \) for all \( q \in \text{Spec} \ S \), so \( \varphi \) is c.i. at \( q \) by 7.2, and then [8, 5.1] and [8, 5.2] give the desired assertions.

Next we relate certain local properties of \( R \) and \( S \). If \( \varphi \) is quasi-Gorenstein at \( q \), then the rings \( R_{q_1} \cap R \) and \( S_q \) are simultaneously Gorenstein by [11, 7.7.2]. In view of 7.2, the conclusion holds when \( \varphi \) is q.c.i. at \( q \); for surjective \( \varphi \) this is already noted in [23, Cor. 5], along with the fact that \( S_q \) is Cohen-Macaulay when \( R_{q_1} \cap R \) is.

We want to compare numerical invariants that measure the singularities of arbitrary rings. The Cohen-Macaulay defect of \( S \) at \( q \) is the non-negative number

\[ \text{cmd} S_q = \dim S_q - \text{depth} S_q, \]

which is equal to zero if and only if \( S_q \) is Cohen-Macaulay. Similarly, the complete intersection defect of \( S \) at \( q \) is the non-negative number

\[ \text{cid} S_q = \varepsilon_2(S_q) - \varepsilon_1(S_q) + \dim S_q, \]

which is equal to zero if and only if \( S_q \) is complete intersection; see [19, 2.3.3(b)].
7.4. Ascent/Descent. If \( \varphi \) is q.c.i. at \( q \), then for \( p = q \cap R \) there are inequalities
\[
\cmd S_q \leq \cmd R_p \quad \text{and} \quad \cid S_q \leq \cid R_p .
\]

Equalities hold when \( R_q \) is Cohen-Macaulay.

Remark. We conjecture that equalities always hold in 7.4, and the proof shows that this is equivalent to Conjecture 3.4. Theorem 3.3 gives some cases of equalities.

Proof. Let \( R_p \xrightarrow{\varphi} R' \xrightarrow{\varphi'} \hat{S}_q \) be a Cohen factorization. The functions \( \cmd \) and \( \cid \) are additive on flat extensions, see [19, 1.2.6, A.11] and [5, 3.6], respectively, and vanish on the regular rings \( R'/pR' \) and \( \hat{S}_q/q\hat{S}_q \). Thus, \( R_p \) and \( R' \) are simultaneously Cohen-Macaulay or complete intersection, and ditto for \( S_q \) and \( \hat{S}_q \), so we may assume that \( (R, p, k) \) is local and \( \varphi \) is surjective. From (3.1.1), we get
\[
\cmd S = \cmd R - (\dim R - \dim S - \grade_R S) .
\]

On the other hand, the definition, (5.3.1), and (3.2.1) yield
\[
\begin{align*}
\cid S &= \varepsilon_2(S) - \varepsilon_1(S) + \dim S \\
&= \varepsilon_2(R) - \varepsilon_1(R) + \rank_k(I/pI) - \rank_S H_1(E) + \dim S \\
&= \cid R + (\dim S + \grade_R S - \dim R). 
\end{align*}
\]

One has \( \dim R \geq \dim S + \grade_R S \), with equality when \( R \) is Cohen-Macaulay. \( \square \)

The next statement follows from [8, 1.5], in view of property 7.6(1) below. We give a different proof, based on results proved earlier in the paper.

7.5. Complete intersection rings. Let \( q \) be a prime ideal of \( S \), and set \( p = q \cap R \).

Any two of the following conditions imply the third one.

(a) The homomorphism \( \varphi \) is q.c.i. at \( q \).

(b) The ring \( R_p \) is c.i.

(c) The ring \( S_q \) is c.i.

Proof. As in the proof of 7.4, we may assume that \( R \) is local and \( \varphi \) is surjective.

When (a) holds, Theorem 5.3 gives \( \varepsilon_3(R) = \varepsilon_3(S) \). Vanishing of \( \varepsilon_3 \) characterizes complete intersections, see [25, 3.5.1(iii)] or [7, 7.3.3], whence (b) \( \iff \) (c).

When (b) and (c) hold, and \( x: Q \to \hat{R} \) is a surjective homomorphism with \( Q \) regular local, then both ideals \( I = \ker x \) and \( J = \ker(\hat{x}) \) are generated by regular sequences. By 1.3 and Lemma 1.2, \( J \hat{R} \) is quasi-complete intersection. Now \( J \hat{R} = I \hat{R} \), so \( I \) is quasi-complete intersection by Lemma 1.1. \( \square \)

To continue, we use a description of q.c.i. homomorphisms in terms of of vanishing of André-Quillen homology, obtained by Blanco, Majadas, and Rodicio [18] for surjective maps. Their characterization extends to arbitrary homomorphisms:

7.6. André-Quillen homology. The following hold.

(1) \( \varphi \) is q.c.i. at \( q \in \Spec S \) if and only if \( D_n(S|R, k(q)) = 0 \) for all \( n \geq 3 \).

(2) \( \varphi \) is q.c.i. if and only if \( D_n(S|R, N) = 0 \) for all \( S \)-modules \( N \) and all \( n \geq 3 \).

Proofs. (1) By A.1, \( D_n(S|R, k(q)) = 0 \) for \( n \geq 3 \) is equivalent to \( D_n(\hat{S}_q|R', -) = 0 \) for \( n \geq 3 \). As the homomorphism \( R' \to \hat{S}_q \) is surjective, [18, Cor. 3'] shows that \( D_n(\hat{S}_q|R', -) = 0 \) holds for \( n \geq 3 \) if and only if the ideal \( \ker(R' \to \hat{S}_q) \) is q.c.i.

(2) By (1), \( \varphi \) is q.c.i. if and only if \( D_n(S|R, k(q)) = 0 \) for all \( q \in \Spec S \) and \( n \geq 3 \). This is equivalent to \( D_n(S|R, N) = 0 \) for all \( N \) and \( n \geq 3 \) by [1, Suppl. 29]. \( \square \)
Further characterizations of q.c.i. homomorphisms might be possible:

**Remark 7.7.** A conjecture of Quillen for $\varphi$ of finite type, see [29, 5.6], extended to all homomorphisms in [8, p. 459], stipulates that if there is an integer $q$, such that $D_n(S|R, N) = 0$ holds for all $n \geq q$ and all $S$-modules $N$, then $\varphi$ is q.c.i.

On the other hand, André results [3, Thm., Cor.] show that in 7.5 condition (a) can be replaced by $D_3(S|R, k(q)) = 0$. This raises the question whether the latter condition implies that $\varphi$ is q.c.i. at $q$, with no additional hypothesis on $R$ or $S$.

### 7.8. Composition

Let $\psi: Q \to R$ be a ring homomorphism, with $Q$ noetherian. If $\varphi$ is q.c.i. at $q \in \text{Spec } S$ and $\psi$ is q.c.i. at $q \cap R$, then $\varphi \psi$ is q.c.i. at $q$.

### 7.9. Decomposition

Let $\psi: Q \to R$ be a ring homomorphism, with $Q$ noetherian. When $\varphi \psi$ is q.c.i. at $q \in \text{Spec } S$ the following hold.

1. If $\psi$ is q.c.i. at $q \cap R$, then $\varphi$ is q.c.i. at $q$.
2. If $\text{fd}_R S_q$ is finite, then $\varphi$ is l.c.i. at $q$ and $\psi$ is q.c.i. at $q \cap R$.

**Proof of 7.8 and 7.9.** Set $l = k(q)$, $p = q \cap R$, and $k = k(p)$.

The exact sequence (A.2.1), defined by the homomorphisms $Q \to R \to S \to l$, shows that if $D_n(S|R, l) = 0$ for $n \geq 3$, then $D_n(R|Q, k) = 0$ for $n \geq 3$ is equivalent to $D_n(S|Q, l) = 0$ for $n \geq 3$. By 7.6(1), this means that when $\varphi$ is q.c.i. at $q$, the map $\varphi \psi$ is q.c.i. at $q$ precisely when $\psi$ is q.c.i. at $p$. This proves 7.8 and 7.9(1).

For 7.9(2), note that by A.3 $D_n(S|Q, l) \to D_n(S|R, l)$ is surjective for $n = 4$ if $\text{char}(l) \neq 2$, and for $n = 3$ if $\text{char}(l) = 2$. When $\text{char}(l) = p > 0$, these $n$ satisfy $n \leq 2p - 1$, so $\varphi$ is c.i. at $q$ by A.4. Now $D_n(S|R, l) = 0$ holds for $n \geq 2$ by [8, 1.8], so (A.2.1) yields $D_n(R|Q, k) = 0$ for $n \geq 3$, whence $\psi$ is q.c.i. at $p$, by 7.6(1). □

Faithfully flat homomorphisms induce surjections on spectra, so 7.9(2) yields:

### 7.10. Flat descent

Let $\psi: Q \to R$ be a ring homomorphism, with $Q$ noetherian. When $\varphi$ is faithfully flat, $\varphi \psi$ is q.c.i. if and only if $\varphi$ is l.c.i. and $\psi$ is q.c.i. □

The proof of [8, 5.11] shows that the next result follows from Proposition 7.6(2) and standard properties of André-Quillen homology with respect to flat base change.

### 7.11. Flat base change

Let $R'$ be a noetherian ring, $\rho: R \to R'$ a homomorphism of rings, such that $S \otimes_R R'$ is noetherian, and set $\varphi' = \varphi \otimes_R R': R' \to S \otimes_R R'$.

1. If $\varphi$ is q.c.i. and $\text{Tor}^R_n(S, R') = 0$ holds for all $n \geq 1$, then $\varphi'$ is q.c.i.
2. If $\varphi'$ is q.c.i. and $\rho$ is faithfully flat, then $\varphi$ is q.c.i. □

Even when $S$ is local, if $\varphi$ is q.c.i. at the maximal ideals of $S$, it may fail to have this property at all primes of $S$; see the discussion preceding [8, 5.12], which applies here as well, as flat q.c.i. homomorphisms are l.c.i. by Remark 7.3. This explains the importance of conditions guaranteeing the localization of the q.c.i. property. We prove two such results, one placing the hypotheses on $\varphi$, the other on $R$.

Recall that a ring homomorphism is said to be essentially of finite type if it can be factored as the composition of the natural embedding into a polynomial ring in finitely many indeterminates, followed by a localization, followed by a surjection.

### 7.12. Factorizable homomorphisms

**Assume $\varphi = \phi \circ \iota$, where $\iota$ is a flat and l.c.i. ring homomorphism, and $\phi$ is a ring homomorphism essentially of finite type.**

If $\varphi$ is q.c.i. at every maximal ideal of $S$, then $\varphi$ is q.c.i.
Proof. By hypothesis, \( \phi \) factors as \( R \xrightarrow{\iota'} R'' \xrightarrow{\phi'} S \), with \( \phi' \) surjective and \( \iota' \) the canonical map \( R \to U^{-1}R[x_1, \ldots, x_n] \) for some multiplicatively closed set \( U \). The fiber of \( \iota' \) at \( p' \in \text{Spec } R'' \) is a localization of \( k(p')[x_1, \ldots, x_n] \). This ring is regular, and hence l.c.i. Since \( \iota' \) is flat, it is l.c.i. by 7.10 and 7.2, so \( \iota'' \) is l.c.i. by 7.2 and 7.8. Switching to the factorization \( \phi = \phi' \circ \iota' \), we may assume that \( \phi \) is surjective.

For \( q \in \text{Spec } S \) choose a maximal ideal \( n \), containing it, and set \( m = n \cap R' \). As \( R \to \widehat{R}_m \) is l.c.i., we have \( D_n(R_m|R, k(n)) = 0 \) for \( n \geq 2 \) by A.4. Since \( \phi \) is q.c.i. at \( n \), we have \( D_n(S_n|R, k(n)) \) for \( n \geq 3 \) by 7.6(1). Thus, the exact sequence (A.2.1) defined by \( R \to R_m \xrightarrow{\phi_n} S_n \to k(n) \) gives \( D_n(S_n|R_m', k(n)) = 0 \) for \( n \geq 3 \). As \( \phi_n \) is surjective, this implies \( D_n(S_n|R_n', k(q)) = 0 \) for \( n \geq 3 \), see [1, 4.57]. Thus, \( D_n(S|R', k(q)) = 0 \) holds for \( n \geq 3 \) by (A.1.1), so \( \phi \) is q.c.i at \( q \) by 7.6(1). In addition, \( R' \) is l.c.i. at \( q \cap R' \) by hypothesis, so \( \phi \) is q.c.i at \( q \) by 7.8.

The non-zero fiber rings of the natural homomorphisms \( R \to \widehat{R}_m \), when \( m \) ranges over the maximal ideals of \( R \), are called the formal fibers of the ring \( R \). Recall that when \( R \) is excellent, its formal fibers are regular, and hence l.c.i.

7.13. Localization. Assume that \( R_\square \rightarrow R \) has l.c.i. formal fibers for each \( q \in \text{Spec } S \).

If \( \varphi \) is q.c.i. at every \( n \in \text{Max } S \), then \( \varphi \) is q.c.i. and \( S \) has l.c.i. formal fibers.

Proof. Choose \( q \in \text{Spec } S \), a maximal ideal \( n \) of \( S \) with \( n \supseteq q \), and set \( m = n \cap R \). Set \( R^* = \widehat{R}_m \) and \( S^* = \widehat{R}_n \), and let \( R^* \to R^m \to R^* \) be a Cohen factorization of \( \varphi^* = \widehat{\varphi}_n : R^* \to S^* \). Choose \( q^* \in \text{Spec } S^* \) with \( q^* \cap S = q \), and set \( p' = q^* \cap R' \).

Since \( R^* \) and \( R' \) are complete, \( R'/mR' \) is regular, and \( R^* \to R' \) is flat, this map is c.i. at \( p' \) by [10, §3, Step 1]. By A.4, this means \( D_n(R'_m|R^*, k(p')) = 0 \) for \( n \geq 2 \).

In addition, \( R_m \to R^m \to \widehat{R}_m \) is a Cohen factorization of \( \varphi_n \), so \( \varphi^* \) is q.c.i. at \( n S^* \) by 7.1, hence \( D_n(S^*|R', k(q^*)) = 0 \) for \( n \geq 3 \) by 7.6(1). The exact sequence (A.2.1) defined by \( R \to R^* \to S^* \to k(q^*) \) now yields \( D_n(S^*|R^*, k(q^*)) = 0 \) for \( n \geq 3 \), so the composed map \( R \to S \to S^* \) is q.c.i. at \( q^* \) by 7.6(1). Since \( S \to S^* \) is flat, 7.9(2) shows that it is c.i. at \( q^* \), and that \( R \to S \) is q.c.i. at \( q \). By varying the choices of \( q, n, \) and \( q^* \) we get the desired assertions. \( \square \)

The proof of [9, 6.11] shows that the next property results from 7.13 and 7.11.

7.14. Completion. Assume that \( R_\square \rightarrow R \) has l.c.i. formal fibers for each \( q \in \text{Spec } S \).

Let \( I \subseteq R \) and \( J \subseteq S \) be ideals, such that \( \varphi(I) \subseteq J \), and let \( \varphi^* : R^* \to S^* \) be the induced map of the corresponding ideal-adic completions.

(1) If \( \varphi \) is q.c.i., then so is \( \varphi^* \).

(2) If \( I \) is contained in the Jacobson radical of \( R \) and \( \varphi^* \) is q.c.i., then so is \( \varphi \). \( \square \)

Appendix A. Andr é-Quillen homology

For an \( A \)-algebra \( B \) and a \( B \)-module \( N \), Andr é [1] and Quillen [29] constructed homology groups \( D_n(B|A, N) \) with remarkable functorial properties. We use [1] as a general reference. A few results crucial for the paper are collected below.

A.1. Let \( \beta : B \to C \) be a homomorphism of noetherian rings, let \( q \) be a prime ideal of \( C \), set \( p = q \cap B \), and let \( \widehat{C}_q \) denote the \( qC_q \)-adic completion of \( C_q \).
By [1, 4.58] and [1, 5.27], there are natural isomorphisms
\[ D_n(C|B, k(q)) \cong D_n(C_q|B, k(q)) \cong D_n(C_q|B_p, k(q)) \quad \text{for} \quad n \geq 0. \tag{A.1.1} \]

If \( B_p \to B' \to \widehat{C_q} \) is a Cohen factorization of \( \beta_q \), then \((A.1.1)\) and \([5, 1.7]\) give
\[ D_n(C|B, k(q)) \cong D_n(\widehat{C_q}|B', k(q)) \quad \text{for} \quad n \geq 2. \tag{A.1.2} \]

A.2. Let \( A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} l \) be homomorphisms of rings, where \( l \) is a field.

For \( k = k(\text{Ker}\, \gamma) \) and each \( n \geq 0 \) there is a natural exact sequence of \( l \)-modules
\[ D_n(B|A, k) \otimes_k l \to D_n(C|A, l) \to D_n(C|B, l) \xrightarrow{\delta} D_{n-1}(B|A, k) \otimes_k l \tag{A.2.1} \]

Indeed, the Jacobi-Zariski exact sequence \([1, 5.1]\) with coefficients in \( l \), defined by \( \alpha \) and \( \beta \), differs from \((A.2.1)\) only because in it \( D_n(B|A, l) \) appears in place of \( D_n(B|A, k) \otimes_k l \). However, these modules are isomorphic by \([1, 4.57]\).

A.3. For \( q \in \text{Spec}\, C \) set \( l = k(q) \) and let \( \gamma \) in \((A.2.1)\) be the canonical map.

If \( \text{fd}_R C_q \) is finite, then \( \delta_n = 0 \) holds in the following cases:
(a) \( n = 2i \) for some integer \( i \) with \( 1 \leq i < \infty \), and \( \text{char}(l) = 0 \).
(b) \( n = 2i \) for some integer \( i \) with \( 1 \leq i \leq p \), where \( p = \text{char}(l) \geq 3 \).
(c) \( n \in \{2, 3, 4\} \) and \( \text{char}(l) = 2 \).

Indeed, in view of \((A.1.1)\) these assertions come from \([8, 4.7]\), except for \( n = 3 \) in (c). This case follows from the earlier proof, because when \( \text{char}(l) = 2 \) the map \( \pi_n(\alpha) : \pi_n(A) \otimes_k l \to \pi_n(B) \) is injective for all \( n \geq 2 \), not only for even \( n \); see \([6]\).

A.4. The following conditions are equivalent.
(i) \( \phi \) is c.i. at \( q \).
(ii) \( D_2(S|R, k(q)) = 0 \).
(iii) \( D_n(S|R, k(q)) = 0 \) for all \( n \geq 2 \).
(iv) \( D_n(S|R, k(q)) = 0 \) for some \( n \) with \( 3 \leq n \leq 2m - 1 \), where \( m \) is an integer such that \( (m - 1)! \) is invertible in \( S \), and \( \text{fd}_R S_q \) is finite.

This follows from \([8, 1.2, 3.3, 4.3, \text{and 3.4}]\), via \((A.1.1)\).

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