Symbolic solutions of some linear recurrences

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Abstract

A symbolic method for solving linear recurrences of combinatorial and statistical interest is introduced. This method essentially relies on a representation of polynomial sequences as moments of a symbol that looks as the framework of a random variable with no reference to any probability space. We give several examples of applications and state an explicit form for the class of linear recurrences involving Sheffer sequences satisfying a special initial condition. The results here presented can be easily implemented in a symbolic software.

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1 Introduction

Many counting problems, especially those related to lattice path counting and random walks and ballot paths [6], give rise to recurrence relations. Once a recursion has been established, Sheffer polynomials are often a simple and general tool for finding answers in closed form. Main contributions in this respect are due to Niederhausen [8, 7, 9, 10]. Further contributions are given by Razpet [13] and Di Bucchiano and Soto y Koelemeijer [1]. Most of the recent papers involving Sheffer sequences still use the language of finite linear operators [19, 12]. In this paper we propose a symbolic theory of Sheffer sequences, developed in [4], in which their main properties are encoded by using a symbolic method arising from the notion of umbra, introduced by Rota and Taylor in [16] and developed in [5]. As a matter of fact, an umbra looks as the framework of a random variable with no reference to any probability space, someway getting closer to statistical methods. This method has given rise to very efficient algorithms for computing unbiased estimators such as $k$-statistics and polykays [3] and generalizations of Sheppard’s corrections [2].

An umbral calculus consists of a set $A = \{\alpha, \beta, \ldots\}$, called the alphabet, whose elements are named umbrae and a linear functional $E$, called evaluation, defined on the polynomial
ring \( R[A] \) and taking values in \( R \), where \( R \) is a suitable ring. The linear functional \( E \) is such that \( E[1] = 1 \) and
\[
E[\alpha^i \beta^j \cdots \gamma^k] = E[\alpha^i]E[\beta^j] \cdots E[\gamma^k], \quad \text{(uncorrelation property)}
\]
for any set of distinct umbrae in \( A \) and for \( i, j, \ldots, k \) nonnegative integers.

A sequence \( a_0 = 1, a_1, a_2, \ldots \) in \( R \) is umbrally represented by an umbra \( \alpha \) when \( E[\alpha^n] = a_n \) for all nonnegative integers \( n \). The elements \( \{a_n\} \) are called moments of the umbra \( \alpha \). Special umbrae are:

i) the augmentation umbra \( \epsilon \in A \), such that \( E[\epsilon^n] = \delta_{0,n} \), for any nonnegative integer \( n \);

ii) the unity umbra \( u \in A \), such that \( E[u^n] = 1 \), for any nonnegative integer \( n \);

iii) the singleton umbra \( \chi \in A \), such that \( E[\chi^1] = 1 \) and \( E[\chi^n] = 0 \), for any nonnegative integer \( n \geq 2 \);

iv) the Bell umbra \( \beta \in A \), such that such that \( E[\beta^n] = B_n \), for any nonnegative integers \( n \), where \( \{B_n\} \) are the Bell numbers;

v) the Bernoulli umbra \( \iota \in A \), such that \( E[\iota^n] = B_n \), for any nonnegative integers \( n \), where \( \{B_n\} \) are the Bernoulli numbers.

An umbral polynomial is a polynomial \( p \in R[A] \). The support of \( p \) is the set of all umbrae occurring in \( p \). If \( p \) and \( q \) are two umbral polynomials then \( p \) and \( q \) are uncorrelated if and only if their supports are disjoint. Moreover, \( p \) and \( q \) are umbrally equivalent if and only if \( E[p] = E[q] \), in symbols \( p \cong q \).

So, in this new setting, to which we refer as the classical umbral calculus, there are two basic devices. The first one is to represent a unital sequence of scalars or polynomials by a symbol \( \alpha \), called an umbra. In other words, the sequence 1, \( a_1, a_2, \ldots \) is represented by means of the sequence 1, \( \alpha, \alpha^2, \ldots \) of powers of \( \alpha \) via an operator \( E \), resembling the expectation operator of random variables. The second device is to represent the same sequence 1, \( a_1, a_2, \ldots \) by means of distinct umbrae, as it could happen also in probability theory with independent and identically distributed random variables. More precisely, two umbrae \( \alpha \) and \( \gamma \) are similar when \( \alpha^n \) is umbrally equivalent to \( \gamma^n \), for all nonnegative integer \( n \), in symbols \( \alpha \equiv \gamma \iff \alpha^n \simeq \gamma^n \) for all \( n \geq 0 \). It is mainly thanks to these devices that the early umbral calculus [16] has had a rigorous and simple formal look.

Section 2 of this paper is devoted to resume Sheffer umbrae and their characterizations. Let us underline that a complete version of the theory of Sheffer polynomials by means of umbrae can be find in [4]. Here, we have chosen to recall terminology, notation and the basic definitions strictly necessary to deal with the object of this paper. Section 3 involves linear recurrence relations whose solutions can be expressed via Sheffer sequences. Two of the examples we propose involve Fibonacci numbers. The first example encodes a definite integral by means of a suitable substitution of a special umbra. The last example is related to counting patterns avoiding ballot paths which has a connection with Dyck paths [17].

2 Sheffer umbrae.

Auxiliary umbrae. Thanks to the notion of similar umbrae, the alphabet \( A \) has been extended with the so-called auxiliary umbrae, resulting from operations among similar
umbrae. This leads to the construction of a saturated umbral calculus [16], in which auxiliary umbrae are handled as elements of the alphabet. Two special auxiliary umbrae are:

i) the *dot-product* of $n$ and $\alpha$, denoted by the symbol $n.\alpha$, similar to the sum $\alpha^1 + \alpha^2 + \cdots + \alpha^n$, where $\{\alpha^1, \alpha^2, \ldots, \alpha^n\}$ is a set of $n$ distinct umbrae, each one similar to the umbra $\alpha$;

ii) the inverse of an umbra $\alpha$, denoted by the symbol $-1.\alpha$, such that $-1.\alpha + \alpha \equiv \epsilon$.

The computational power of the umbral syntax can be improved through the construction of new auxiliary umbrae by means of the notion of generating function of umbrae. The formal power series

$$u + \sum_{n \geq 1} \alpha^n \frac{t^n}{n!}$$

is the *generating function* of the umbra $\alpha$, and it is denoted by $e^{\alpha t}$. The notion of umbral equivalence and similarity can be extended coefficientwise to formal power series [1], that is $\alpha \equiv \beta \iff e^{\alpha t} \simeq e^{\beta t}$ (see [18] for a formal construction). Note that any exponential formal power series $f(t) = 1 + \sum_{n \geq 1} a_n t^n / n!$ can be umbrally represented by a formal power series [1]. In fact, if the sequence $1, a_1, a_2, \ldots$ is umbrally represented by $\alpha$ then $f(t) = E[e^{\alpha t}]$, that is $f(t) \simeq e^{\alpha t}$, assuming that we extend $E$ by linearity. In this case, instead of $f(t)$ we write $f(\alpha, t)$ and we say that $f(\alpha, t)$ is umbrally represented by $\alpha$. Henceforth, when no confusion occurs, we just say that $f(\alpha, t)$ is the generating function of $\alpha$. For example for the augmentation umbra we have $f(\epsilon, t) = 1$, for the unity umbra we have $f(u, t) = e^t$, and for the singleton umbra we have $f(\chi, t) = 1 + t$. Moreover for the Bernoulli umbra we have $f(\iota, t) = t/(e^t - 1)$ (see [16]) and for the Bell umbra we have $f(\beta, t) = \exp(e^t - 1)$ (see [5]).

The advantage of an umbral notation for generating functions is the representation of operations among generating functions through symbolic operations among umbrae. For example, the product of exponential generating functions is umbrally represented by a sum of the corresponding umbrae:

$$f(\alpha, t) f(\gamma, t) \simeq e^{(\alpha + \gamma) t} \quad \text{with} \quad f(\alpha, t) \simeq e^{\alpha t} \text{ and } f(\gamma, t) \simeq e^{\gamma t}. \quad (2)$$

Then for the inverse of an umbra we have $f(-1.\alpha, t) = 1/f(\alpha, t)$ and for the dot-product of $n$ and $\alpha$ we have $f(n.\alpha, t) = f(\alpha, t)^n$. Suppose we replace $R$ by a suitable polynomial ring having coefficients in $R$ and any desired number of indeterminates. Then, an umbra is said to be *scalar* if the moments are elements of $R$ while it is said to be *polynomial* if the moments are polynomials. In this paper, we deal with $R[x, y]$. We can define the dot-product of $x$ and $\alpha$ as an auxiliary umbra having generating functions $f(x.\alpha, t) = f(\alpha, t)^x$. This construction has been done formally in [5] where the moments of $x.\alpha$ have been also computed. A feature of the classical umbral calculus is the construction of new auxiliary umbrae by suitable symbolic substitutions. In $n.\alpha$ replace the integer $n$ by an umbra $\gamma$. The auxiliary umbra $\gamma.\alpha$ is the *dot-product* of $\alpha$ and $\gamma$ having generating function $f(\gamma.\alpha, t) = f(\gamma, \log f(\alpha, t))$. If we replace the umbra $\gamma$ with the Bell umbra $\beta$, we have $f(\beta.\alpha, t) = \exp[f(\alpha, t) - 1]$, so that the umbra $\beta.\alpha$ has been called $\alpha$-partition umbra. The $\alpha$-partition umbra plays a crucial role in the umbral representation of the composition of exponential generating functions. Indeed, first we can consider the polynomial $\alpha$-partition
umbra $x, \beta, \alpha$ with generating function $f(x, \beta, \alpha, t) = \exp[x(f(\alpha, t) - 1)]$. Its moments are [5]

$$E[(x,\beta,\alpha)^t] = \sum_{j=1}^{i} x^j B_{i,j}(a_1, a_2, \ldots, a_{i-j+1}),$$  \hspace{1cm} (3)

$B_{i,j}$ are the (partial) Bell exponential polynomials [14] and $a_i$ are the moments of the umbra $\alpha$. When $x$ is replaced by an umbra $\gamma$, the moments in (3) give the coefficients of the composition of the generating functions $f(\alpha, t)$ and $f(\gamma, t)$, that is $f(\gamma, \beta, \alpha, t) = f[\gamma, f(\alpha, t) - 1]$. The umbra $\gamma, \beta, \alpha$ is the composition umbra of $\alpha$ and $\gamma$. If $E[\alpha] \neq 0$, the umbra $\alpha^{<\gamma}$ is the compositional inverse of $\alpha$, with generating function $f(\alpha^{<\gamma}, t) = f^{<\gamma}(\alpha, t)$, where $f^{<\gamma}(\alpha, t)$ is the compositional inverse of $f(\alpha, t)$ that is $f[\alpha^{<\gamma}, f(\alpha, t) - 1] = f[\alpha, f(\alpha^{<\gamma}, t) - 1] = 1 + t$.

**Sheffer umbrae.** In the following let $\alpha$ and $\gamma$ be scalar umbrae, with $g_1 = E[\gamma] \neq 0$.

**Definition 2.1.** A polynomial umbra $\sigma_x$ is said to be a Sheffer umbra for $(\alpha, \gamma)$ if $\sigma_x \equiv \alpha + x, \beta, \gamma^{<\gamma}$.

In the following, we denote by $\gamma^*$ the $\gamma^{<\gamma}$-partition umbra $\beta, \gamma^{<\gamma}$. The umbra $\gamma^*$ is called the adjoint umbra of $\gamma$ and in particular $f(\gamma^*, t) = \exp[f^{<\gamma}(\gamma, t) - 1]$. The name parallels the adjoint of an umbral operator [15] since $\gamma, \alpha^*$ gives the umbral composition of $\gamma$ and $\alpha^{<\gamma}$. So a polynomial umbra $\sigma_x$ is said to be a Sheffer umbra for $(\alpha, \gamma)$ if

$$\sigma_x \equiv \alpha + x, \gamma^*.$$  \hspace{1cm} (4)

In the following, we denote a Sheffer umbra by $\sigma_x^{(\alpha, \gamma)}$ in order to make explicit the dependence on $\alpha$ and $\gamma$. From (4), the generating function of $\sigma_x^{(\alpha, \gamma)}$ is

$$f[\sigma_x^{(\alpha, \gamma)}, t] = f(\alpha, t) e^x[f^{<\gamma}(\gamma, t) - 1].$$  \hspace{1cm} (5)

Note that, given the umbra $\gamma$, any Sheffer umbra is uniquely determined by its moments evaluated at 0, since via equivalence (4) we have $\sigma_0^{(\alpha, \gamma)} \equiv \alpha$.

Among Sheffer umbrae, a special role is played by the associated umbra. Let us consider a Sheffer umbra for the umbrae $(\epsilon, \gamma)$, where $\gamma$ has a compositional inverse and $\epsilon$ is the augmentation umbra.

**Definition 2.2.** The polynomial umbra $\sigma_x \equiv x, \gamma^*$, where $\gamma^*$ is the adjoint umbra of $\gamma$, is the associated umbra of $\gamma$.

The associated umbrae are polynomial umbrae whose moments $\{p_n(x)\}$ satisfy the well-known binomial identity

$$p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$$  \hspace{1cm} (6)

for all $n = 0, 1, 2, \ldots$. So if $\sigma_x^{(\alpha, \gamma)}$ is a Sheffer umbra whose moments are the Sheffer sequence $\{s_n(x)\}$, then the moments $\{p_n(x)\}$ of the umbra $\sigma_x^{(\epsilon, \gamma)}$ represent the sequence associated to $\{s_n(x)\}$.

In the following, we recall a generalization of the well-known Sheffer identity, for the proof see [4].
Theorem 2.1 (Generalized Sheffer identity). A polynomial umbra $\sigma_x$ is a Sheffer umbra if and only if there exists an umbra $\gamma$, provided with a compositional inverse, such that

$$\sigma_{\eta + \zeta} \equiv \sigma_{\eta} + \zeta \cdot \gamma^*, \quad (7)$$

for any $\eta, \zeta \in A$.

Corollary 2.1 (The Sheffer identity). A polynomial umbra $\sigma_x$ is a Sheffer umbra if and only if there exists an umbra $\gamma$, provided with a compositional inverse, such that

$$\sigma_{x+y} \equiv \sigma_x + y \cdot \gamma^*. \quad (8)$$

Equivalence (8) gives the well-known Sheffer identity, because, setting $s_n(x+y) = E[\sigma_{x+y}^n]$, $s_k(x) = E[\sigma_x^k]$ and $p_{n-k}(y) = E[(y \cdot \gamma^*)^{n-k}]$ and by using the binomial expansion, we have

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y). \quad (9)$$

3 Solving recursions.

In this section we show how to use the language of umbrae in order to solve some recursions. The idea is to characterize the umbrae $(\alpha, \gamma)$ in (4) in such a way that the umbra $\gamma$ is characterized by the recursion and the umbra $\alpha$ is characterized by the initial condition.

Integral initial condition. Consider the following difference equation

$$q_n(x+1) = q_n(x) + q_{n-1}(x) \quad (10)$$

under the condition $\int_0^1 q_n(x)dx = 1$, for all nonnegative integers $n$. We assume $q_n(x) = 0$ for negative $n$. Let $q_n(x) = s_n(x)/n!$. Recursion (10) for $s_n(x)$ becomes

$$s_n(x+1) = s_n(x) + ns_{n-1}(x),$$

for all nonnegative $n$. This last recursion fits the Sheffer identity (8) if we set $y = 1$ and choose the sequence $\{p_n(x)\}$ such that $p_0(x) = 1, p_1(1) = 1$ and $p_n(1) = 0$ for all $n \geq 2$. This is true if we choose $p_n(x) \simeq (x \cdot \chi)^n$, that is the sequence $\{p_n(x)\}$ associated to the umbra $u$. Indeed, since we have

$$\beta \cdot \chi \equiv \chi, \beta \equiv u \quad \text{and} \quad \beta \cdot u^{<1>} \equiv u^{<1>}, \beta \equiv \chi$$

(see also [5]), the adjoint of the unity umbra $u$ is $u^* \equiv \beta \cdot u^{<1>} \equiv \chi$, and $x \cdot \chi \equiv x \cdot u^*$. So we are looking for solutions of (11) such that

$$q_n(x) \simeq \frac{(\alpha + x \cdot \chi)^n}{n!}, \quad (11)$$

for some umbra $\alpha$ depending on the initial condition $\int_0^1 q_n(x)dx = 1$. Since

$$\int_0^1 p(x)dx = E[p(-1,u)]$$
for all \( p(x) \in R[x] \), where \(-1.\iota\) is the inverse of the Bernoulli umbra (see [2]), then

\[
\int_0^1 q_n(x)dx = 1 \iff E[s_n(-1.\iota)] = n! \quad \text{for all} \quad n \geq 1 \iff -1.\iota.\chi + \alpha \equiv \bar{u},
\]

with \( \bar{u} \) an umbra whose moments are \( E[\bar{u}^n] = n! \). Thus we have \( \alpha \equiv \bar{u} + \iota.\chi \). Replacing this last result in (11), solutions of (11) are moments of the Sheffer umbra \( \bar{u} + (\iota + x.\iota)\chi \) normalized by \( n! \),

\[
q_n(x) \simeq \frac{(\bar{u} + (\iota + x.\iota)\chi)^n}{n!}.
\]

**Pascal’s recursions.** Consider the following Pascal’s recursion

\[
q_n(x) = q_n(x-1) + q_{n-1}(x)
\]

(12)

that satisfies the initial condition

\[
q_n(1-n) = \sum_{i=0}^{n-1} q_i(n-2i) \quad \text{for all} \quad n \geq 1, \quad q_0(-1) = 1.
\]

(13)

Let \( q_n(x) = s_n(x)/n! \) for all nonnegative \( n \). Recursion (12) for \( s_n(x) \) becomes

\[
s_n(x-1) = s_n(x) - n s_{n-1}(x).
\]

This last equation fits the Sheffer identity (8) if we set \( y = -1 \) and choose the sequence \( \{p_n(x)\} \) such that \( p_0(x) = 1 \), \( p_1(-1) = -1 \) and \( p_n(-1) = 0 \) for all \( n \geq 2 \). From (7), we need to have \(-1.\gamma^* \equiv -\chi \Leftrightarrow \gamma^* \equiv -1. - \chi \) and so \( p_n(x) \simeq (-x. - \chi)^n \). The umbra \(-1. - \chi\) is similar to the umbra \( \bar{u} \), defined in Example 3 since

\[
f(-1. - \chi, t) = \frac{1}{1-t} = f(\bar{u}, t).
\]

Then \( p_n(x) \simeq (-x. - \chi)^n \simeq (x.\bar{u})^n \) and solutions of (12) are therefore of the type

\[
q_n(x) \simeq \frac{(\alpha + x.\bar{u})^n}{n!}.
\]

(14)

Due to the initial condition (13), we need to consider a polynomial sequence such that

\[
q_n(x-n) \simeq \frac{[\alpha + (x-n).\bar{u}]^n}{n!} \simeq \frac{[\alpha + (x-1)\chi]^n}{n!}.
\]

(15)

The equivalence on the right of (15) follows from the following remarks. Since \( f(x.\bar{u}, t) = 1/(1-t)^x \), we have \( E[(x.\bar{u})^n] = (x)^n = x(x + 1) \cdots (x + n - 1) = (x + n - 1)_n \) and since \( f(x.\chi, t) = (1+t)^x \), we have \( E[(x.\chi)^n] = (x)_n \) (see [5]). Therefore,

\[
(x.\bar{u})^n \simeq [(x+n-1).\chi]^n \Leftrightarrow [(x-n).\bar{u}]^n \simeq [(x-1).\chi]^n
\]

and if \( \{s_n(x)\} \) is a Sheffer sequence with associated polynomials \( (x.\bar{u})^n/n! \), then \( \{s_n(x-n)\} \) is a Sheffer sequence with associated polynomials \( [(x-n).\bar{u}]^n/n! \). Note that, having written
\[ q_n(x - n) \text{ as the latter equivalence in } \left[ 15 \right], \text{ the values of } q_n(1 - n) \text{ in the initial condition } \left[ 13 \right] \text{ give exactly the moments of } \alpha \text{ normalized by } n!. \] By observing that from the latter equivalence in \left[ 15 \right]

\[ q_n(x) \simeq \frac{[\alpha + (x + n - 1)\chi]^n}{n!} \simeq \sum_{k=0}^{n} \frac{\alpha^k}{k!} \left( x + n - 1 \choose n - k \right), \]

we have a first expression of \( q_n(x) \), that is

\[ q_n(x) = \sum_{k=0}^{n} q_k (1 - k) \left( x + n - 1 \choose n - k \right). \]

From a computational point of view, this formula is very easy to implement by using the recursion of the initial condition. One could eliminate the contribution of this recursion in the following way. By using the umbral expression of \( q_n(x - n) \) we have

\[ \sum_{i=0}^{n-1} q_i (n - 2i) = \sum_{i=0}^{n-1} q_i (x - i) \big|_{x=n-i} \simeq \sum_{i=0}^{n-1} \frac{[\alpha + (n - i - 1)\chi]^i}{i!} \]

and from the initial condition we have

\[ \frac{\alpha^n}{n!} \simeq \sum_{j=0}^{n-1} \frac{\chi^j}{j!} \sum_{i=0}^{n-j-1} \frac{[(n - i - j - 1)\chi]^i}{i!}. \]

If we consider an umbra \( \delta \) such that

\[ \frac{\delta^n}{n!} \simeq \sum_{i=0}^{n} \frac{[(n - i)\chi]^i}{i!}, \]

then equivalence \( \left[ 17 \right] \) gives

\[ \alpha^n \simeq n \left( \alpha + \delta \right)^{n-1}, \text{ for all } n \geq 1. \] (18)

If we denote by \( \gamma_D \) an auxiliary umbra such that \( \gamma^n \simeq \partial_{\gamma} \gamma^n \simeq n \gamma^{n-1} \), then \( \left[ 18 \right] \) can be written as \( \alpha \equiv (\alpha + \delta)_D \). Since \( f(\gamma_D, t) = 1 + tf(\gamma, t) \) then \( f(\alpha, t) = 1 + tf(\alpha, t)(\delta, t) \) gives \( f(\alpha, t) = 1/[1 - tf(\delta, t)] \), and in umbral terms \( \alpha \equiv \bar{u}, \beta, \delta_D \). The solution of recurrence \( \left[ 12 \right] \) is therefore

\[ q_n(x) \simeq \frac{[\bar{u}, \beta, \delta_D + (x + n - 1)\chi]^n}{n!}. \]

**Remark 3.1.** We call the umbra \( \delta \) in the previous example *Fibonacci umbra*. Indeed, because

\[ \delta^n \simeq n! \sum_{k=0}^{n} \frac{(k\chi)^{n-k}}{(n-k)!} \simeq \sum_{k \geq 0} \binom{n}{k} \bar{u}^k (k\chi)^{n-k}, \]

we have \( \delta \equiv \bar{u}, \beta, \chi_D \), by using \( \left[ 3 \right] \), with \( x \) replaced by \( \bar{u} \), and Lemma 15 of \( \left[ 5 \right] \). Thus we have

\[ f(\delta, t) = \frac{1}{1 - t(1 + t)} = \frac{1}{1 - t - t^2} = \sum_{n \geq 0} F_n t^n, \]

which is the generating function of Fibonacci numbers \( \{F_n\} \) with \( F_0 = 1 \), so that \( \delta^n \simeq n! F_n \).
**Fibonacci’s numbers.** Consider the following difference equation \( F_n(m) = F_n(m - 1) + F_{n-1}(m - 2) \) under the condition \( F_n(0) = 1 \) for all nonnegative integers \( n \), looking for a polynomial extension. Replace \( m \) with \( x + n + 1 \). Then the difference equation can be rewritten as

\[
F_n(x + n + 1) = F_n(x + n) + F_{n-1}(x + n - 1).
\]

Let \( F_n(x + n) = s_n(x)/n! \) for all nonnegative \( n \). Recursion \( 19 \) for \( s_n(x) \) becomes

\[
s_n(x + 1) = s_n(x) + ns_{n-1}(x).
\]

This last recursion fits the Sheffer identity \( 8 \) if we set \( \{ \} \) such that \( \{ p_n(x) \} \) such that \( p_0(x) = 1, p_1(1) = 1 \) and \( p_n(1) = 0 \) for all \( n \geq 2 \). As in the previous paragraph, we are looking for solutions of \( 19 \) such that

\[
F_n(x + n) \simeq \frac{(\alpha + x \cdot \chi)^n}{n!}.
\]

Let us observe that equation \( 19 \), for \( x = 0 \), gives the well-known recurrence relation for Fibonacci numbers so that \( n!F_n(n) \simeq \delta^n \), where \( \delta \) is the Fibonacci umbra defined in the previous paragraph. Because \( n!F_n(n) \simeq \alpha^n \) we have \( \alpha \equiv \delta \). Solutions of \( 19 \) are such that

\[
F_n(x + n) \simeq \frac{(\delta + x \cdot \chi)^n}{n!}.
\]

Recalling \( \delta \equiv \bar{u} \beta \cdot \chi \) we find

\[
F_n(x + n) \simeq \sum_{k=0}^{n} \frac{(x \cdot \chi)^{n-k} k \cdot \chi^{k-j} (j \cdot \chi)^{k-j}}{(n-k)! (j-k)!} \simeq \sum_{k=0}^{n} \frac{(x + k) \cdot \chi^{n-k}}{(n-k)!} \simeq \sum_{k=0}^{n} \frac{x + k}{n-k},
\]

by which we can verify that the initial conditions \( F_n(0) = F_n(-n + n) = 1 \) hold.

In the following example, the recursion does not fit the Sheffer identity \( 9 \) directly, but we are able to translate the recursion in an umbral equivalence by which the umbra \( \gamma \) can be still characterized.

**Dyck paths.** A ballot path takes up steps \((u)\) and right steps \((r)\), starting at the origin and staying weakly above the diagonal. For example, \( urururur \) is a ballot path to \((3, 4)\). We denote by \( D(n, m) \) the number of ballot paths to \((n, m)\) under the conditions

1. no path goes below the diagonal (Dyck path);
2. no path contains the pattern (substring) \( urru \) (see Table 1).

The numbers \( D(n, m) \) follow the difference equation

\[
D(n, m) - D(n - 1, m) = D(n, m - 1) - D(n - 2, m - 1) + D(n - 3, m - 1)
\]

for all nonnegative \( n \) and \( m \), under the initial condition \( D(n, n) = D(n - 1, n) \), and \( D(0, 0) = 1 \). Let \( D(n, m) = s_n(m)/n! \). The recursion \( 20 \) for \( s_n(m) \) becomes

\[
s_n(m) - ns_{n-1}(m) = s_n(m - 1) - (n)_{2} s_{n-2}(m - 1) + (n)_{3} s_{n-3}(m - 1),
\]

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Table 1: The ballot path

|   | 0 | 1 | 2 | 3 | 4 | n |
|---|---|---|---|---|---|---|
| m | 1 | 7 | 22 | 46 | 82 | 132 |
| 6 | 1 | 6 | 16 | 29 | 46 | 63 |
| 5 | 1 | 5 | 11 | 17 | 23 | 23 |
| 4 | 1 | 4 | 7  | 9  | 9  |    |
| 3 | 1 | 3 | 4  | 4  |    |    |
| 2 | 1 | 2 | 2  |    |    |    |
| 1 | 1 | 1 |    |    |    |    |
| 0 | 1 |    |    |    |    |    |

and the initial condition becomes \( s_n(n) = ns_{n-1}(n) \). We are looking for solutions of Sheffer type \( E[\sigma_n^x] \simeq s_n(x) \), i.e. \( \sigma_x \equiv \alpha + x^* \gamma^* \). Because \( s_n(x) - ns_{n-1}(x) \simeq (\sigma_x - \chi)^n \), we can replace \( \sigma_x \) by \( \sigma_{x-1} + \gamma^* \) from Sheffer identity \( \text{[6]} \) with \( y = -1 \). Comparing \( (\sigma_x - \chi)^n \simeq (\sigma_{x-1} + \gamma^* - \chi)^n \) with the recursion \( \text{[21]} \), with \( x \) replaced by \( m \), a characterization of the moments \((\gamma^*)^n\) is available. Indeed we have

\[
(\sigma_{x-1} + \gamma^* - \chi)^n \simeq \sigma_{x-1}^n + n \sum_{i=0}^{n-1} \binom{n-1}{i} \sigma_{x-1}^{n-i-1} \left( \frac{(\gamma^*)^{i+1}}{i+1} - (\gamma^*)^i \right).
\]

By replacing \( x \) with \( m \) in the previous equivalence and by applying the linear functional \( E \) to both sides, we have

\[
s_n(m) - ns_{n-1}(m) \simeq s_n(m-1) + n \sum_{i=0}^{n-1} \binom{n-1}{i} s_{n-i-1}(m-1) \left[ \frac{g_{i+1}}{i+1} - g_i \right],
\]

where \( E[(\gamma^*)^i] = g_i \). By comparing \( \text{[22]} \) with \( \text{[21]} \), we have \( g_0 = 1, g_1 = 1, g_2 = 0, g_k = k! \) for \( k \geq 3 \). Then we have

\[
f(\gamma^*, t) = 1 + t + \sum_{k \geq 3} t^k = \frac{1-t^2+t^3}{1-t} \implies f(x.\gamma^*, t) = \left[ \frac{1-t^2+t^3}{1-t} \right]^x.
\]

In order to characterize the umbra \( \alpha \) we need to use the initial condition on \( s_n(x) \), that is \( s_n(n) = ns_{n-1}(n) \), in umbral terms

\[
(\alpha + n.\gamma^*)^n \simeq n(\alpha + n.\gamma^*)^{n-1}.
\]

Suppose \( \alpha \equiv \bar{u} + \zeta \). In this case the initial condition \( \text{[23]} \) gives

\[
(\zeta + n.\gamma^*)^n \simeq e^n
\]

for all nonnegative integers \( n \). By applying the binomial expansion, we have

\[
(n.\gamma^*)^n \simeq \sum_{k=1}^{n} \binom{n}{k} (-\zeta^k)(n.\gamma^*)^{n-k}.
\]

The following proposition allows us to take a step forward.
**Proposition 3.1.** If \( \gamma \) is an umbra with \( E[\gamma] = 1 \), then for \( n \geq 1 \) we have \((x.\gamma^*)^n \simeq x(\eta + x.\gamma^*)^{n-1}\), with \( \eta \) an umbra such that \( \eta^n \simeq (\gamma^{<\eta_1})^{n+1} \) for \( n \geq 1 \).

**Proof.** Compare the generating function of the umbra \( \eta + x.\gamma^* \) with the one obtained by replacing \((\eta + x.\gamma^*)^n \) with \((x.\gamma^*)^{n+1}/x\).

Due to the previous Proposition, with \( x \) replaced by \( n \), we have

\[
(n.\gamma^*)^n \simeq n(\eta + n.\gamma^*)^{n-1} \simeq \sum_{k=1}^{n} \binom{n}{k} k[\gamma^{<\eta_1}]^k(n.\gamma^*)^{n-k}.
\]

By comparing equivalence (25) with equivalence (24), we have

\[
\zeta^k \simeq -k \eta^{k-1} \simeq -k[\gamma^{<\eta_1}]^k
\]

for \( k \geq 1 \). Then, the solution of the recurrence relation (21) is

\[
s_n(x) \simeq \frac{(\bar{\eta} + \zeta + x.\gamma^*)^n}{n!}.
\]

Let us observe that

\[
s_n(x) \simeq \sum_{k=0}^{n} \binom{n}{k} (n-k)! (\eta + x.\gamma^*)^k \simeq \sum_{k=0}^{n} \binom{n}{k} (n-k) \frac{x-k}{x} (x.\gamma^*)^k
\]

because

\[
x \sum_{j=1}^{k} \binom{k}{j} j\eta^{j-1}(x.\gamma^*)^{k-j} \simeq x k (\eta + x.\gamma^*)^{k-1} \simeq k(x.\gamma^*)^k,
\]

due to Proposition 3.1. More on counting paths containing a given number of a certain pattern can be found in [17] and [11].

We conclude this section stating a generalization of Proposition 3.1 which turns out to be useful in solving the class of recursions involving Sheffer sequences satisfying the initial condition \( s_n(-cn) = \delta_{0,n} \).

**Theorem 3.1.** If \( \gamma \) is an umbra with \( E[\gamma] = 1 \), then for \( n \geq 1 \) we have \( x(\chi.\beta.\eta + x.\gamma^*)^n \simeq (x + cn)(x.\gamma^*)^n \), with \( \eta \) an umbra such that \( \eta^n \simeq (\gamma^{<\eta_1})^{n+1} \) for \( n \geq 1 \) and \( c \in R \).

**Proof.** Due to Proposition 3.1, we have

\[
\frac{(x + cn)}{x} (x.\gamma^*)^n \simeq (x.\gamma^*)^n + cn(\eta + x.\gamma^*)^{n-1} \simeq (x.\gamma^*)^n + c \sum_{j=1}^{n} \binom{n}{j} j\eta^{j-1}(x.\gamma^*)^{n-j}.
\]

The result follows by observing that \( j\eta^{j-1} \simeq \eta_0^j \) and \( c \simeq (\chi.\beta)^j \), for \( j = 1, 2, \ldots, n \), so that \( c j\eta^{j-1} \simeq (\chi.\beta)^j \eta_0^j \simeq [(\chi.\beta)\eta_0]^j \simeq (\chi.\beta.\eta_0)^j \).

If \( \gamma \) is the umbra characterized by a linear recursion and \( s_n(-cn) = \delta_{0,n} \) is the initial condition, then the previous proposition states that solutions have the form \((\chi.\beta.\eta_0 + x.\gamma^*)^n\).
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