Gradient Estimates for Nonlinear Diffusion Semigroups by Coupling Methods

Yongsheng Song*

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Abstract

Our purpose is to obtain gradient estimates for certain nonlinear partial differential equations by coupling methods. First we derive uniform gradient estimates for a certain semilinear PDEs based on the coupling method introduced in Wang (2011) and the theory of backward SDEs. Then we generalize Wang’s coupling to the $G$-expectation space and obtain gradient estimates for nonlinear diffusion semigroups, which correspond to the solutions of a certain fully nonlinear PDEs.

Key words: gradient estimates, coupling methods, $G$-expectation, nonlinear PDEs

MSC-classification: 35K55, 60H10, 60J60

1 Introduction

The classical Feynman-Kac formula established the connection between linear partial differential equations and stochastic processes, which is the starting point for the study of PDEs by probabilistic methods. For example, probabilistic tools, such as Malliavin calculus, the method of coupling, etc, can be used to obtain the regularity property for PDEs.

In 1990, Pardoux and Peng ([7]) introduced the general nonlinear BSDEs, based on which [8] and [9] gave a probabilistic formula for a certain quasi-linear parabolic partial differential equations. This is the so-called generalized Feynman-Kac formula, which provides a way to study quasi-linear PDEs by probabilistic methods. [3] established Bismut-Elworthy formula for backward SDEs by the method of Malliavin calculus. As in the linear case, this formula provides gradient estimates for the solutions to the associated PDEs.

A natural question is how to give a probabilistic interpretation for fully nonlinear PDEs, which is one of the main motivations for Shige Peng to establish the fully nonlinear expectation theory. $G$-expectation is a typical time-consistent sublinear expectation. In the $G$-expectation space, the nonlinear semigroups associated with SDEs driven by $G$-Brownian motion are solutions to certain fully nonlinear PDEs.

In this article, we derive gradient estimates for certain nonlinear partial differential equations by coupling methods. First, in Section 3, we obtain uniform gradient estimates for a certain semi-linear PDEs based on the coupling method introduced in [21] and the theory of backward SDEs. Then, in Section 4, we generalize Wang’s coupling to the $G$-expectation space and obtain gradient estimates for nonlinear diffusion semigroups. Our main results are Theorem 3.1 and Theorem 4.1.

*Academy of Mathematics and Systems Science, CAS, Beijing, China, yssong@amss.ac.cn. Research supported by by NCMIS; Youth Grant of NSF (No. 11101406); Key Project of NSF (No. 11231005); Key Lab of Random Complex Structures and Data Science, CAS (No. 2008DP173182).
2 Some Definitions and Notations about $G$-expectation

We review some basic notions and definitions of the related spaces under $G$-expectation. The readers may refer to [16, 17, 18, 19, 21] for more details.

Let $\Omega_T = C_0([0, T]; \mathbb{R}^d)$ be the space of all $\mathbb{R}^d$-valued continuous paths $\omega = (\omega(t))_{t \geq 0} \in \Omega$ with $\omega(0) = 0$ and let $B_t(\omega) = \omega(t)$ be the canonical process.

Let us recall the definitions of $G$-Brownian motion and its corresponding $G$-expectation introduced in [17]. Set

$$L_{ip}(\Omega_T) := \{ \varphi(\omega(t_1), \cdots, \omega(t_n)) : t_1, \cdots, t_n \in [0, T], \varphi \in C_b,Lip((\mathbb{R}^d)^n), n \in \mathbb{N} \},$$

where $C_b,Lip(\mathbb{R}^d)$ is the collection of bounded Lipschitz functions on $\mathbb{R}^d$.

We are given a function $G : \mathbb{S}_d \mapsto \mathbb{R}$ satisfying the following monotonicity, sublinearity and positive homogeneity:

**A1.** $G(a) \geq G(b)$, if $a, b \in \mathbb{S}_d$ and $a \geq b$;

**A2.** $G(a + b) \leq G(a) + G(b)$, for each $a, b \in \mathbb{S}_d$;

**A3.** $G(\lambda a) = \lambda G(a)$ for $a \in \mathbb{S}_d$ and $\lambda \geq 0$.

**Remark 2.1** When $d = 1$, we have $G(a) := \frac{1}{2}(\sigma^2 a^+ - \sigma^2 a^-)$, for $0 \leq \sigma^2 \leq \sigma^2$.

For each $\xi(\omega) \in L_{ip}(\Omega_T)$ of the form

$$\xi(\omega) = \varphi(\omega(t_1), \omega(t_2), \cdots, \omega(t_n)), \quad 0 = t_0 < t_1 < \cdots < t_n = T,$$

we define the following conditional $G$-expectation

$$\mathbb{E}_t[\xi] := u_k(t, \omega(t); \omega(t_1), \cdots, \omega(t_{k-1}))$$

for each $t \in [t_{k-1}, t_k)$, $k = 1, \cdots, n$. Here, for each $k = 1, \cdots, n$, $u_k = u_k(t, x; x_1, \cdots, x_{k-1})$ is a function of $(t, x)$ parameterized by $(x_1, \cdots, x_{k-1}) \in (\mathbb{R}^d)^{k-1}$, which is the solution of the following PDE ($G$-heat equation) defined on $[t_{k-1}, t_k) \times \mathbb{R}^d$:

$$\partial_t u_k + G(\partial^2_x u_k) = 0$$

with terminal conditions

$$u_k(t_k, x; x_1, \cdots, x_{k-1}) = u_{k+1}(t_k, x; x_1, \cdots, x_{k-1}, x), \text{ for } k < n$$

and $u_n(t_n, x; x_1, \cdots, x_{n-1}) = \varphi(x_1, \cdots, x_{n-1}, x)$.

The $G$-expectation of $\xi(\omega)$ is defined by $\mathbb{E}[\xi] = \mathbb{E}_0[\xi]$. From this construction we obtain a natural norm $\|\xi\|_{L^p_G} := \mathbb{E}[|\xi|^p]^{1/p}$. The completion of $L_{ip}(\Omega_T)$ under $\|\cdot\|_{L^p_G}$ is a Banach space, denoted by $L^p_G(\Omega_T)$. The canonical process $B_t(\omega) := \omega(t), t \geq 0$, is called a $G$-Brownian motion in this sublinear expectation space $(\Omega, L^1_G(\Omega), \mathbb{E})$.

**Definition 2.2** A process $\{M_t\}$ with values in $L^1_G(\Omega_T)$ is called a $G$-martingale if $\mathbb{E}_s(M_t) = M_s$ for any $s \leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both $G$-martingales, we call $\{M_t\}$ a symmetric $G$-martingale.
Theorem 2.3  ([2, 6]) There exists a weakly compact subset \( P \subset M_1(\Omega_T) \), the set of probability measures on \((\Omega_T, \mathcal{B}(\Omega_T))\), such that
\[
\mathbb{E}[\xi] = \sup_{P \in P} E_P[\xi] \text{ for all } \xi \in L_p(\Omega_T).
\]
P is called a set that represents \( \mathbb{E} \).

Let \( P \) be a weakly compact set that represents \( \mathbb{E} \). For this \( P \), we define capacity
\[
c(A) := \sup_{P \in P} P(A), \quad A \in \mathcal{B}(\Omega_T).
\]
c defined here is independent of the choice of \( P \) (see Remark 2.7 in [22] for details). We say a set \( A \subset \Omega_T \) is polar if \( c(A) = 0 \). A property holds “quasi-surely” (q.s. for short) if it holds outside a polar set.

Definition 2.4 A function \( \eta(t, \omega) : [0, T] \times \Omega_T \to \mathbb{R} \) is called a step process if there exists a time partition \( \{t_i\}_{i=0}^n \) with \( 0 = t_0 < t_1 < \cdots < t_n = T \), such that for each \( k = 0, 1, \cdots, n - 1 \) and \( t \in (t_k, t_{k+1}] \)
\[
\eta(t, \omega) = \xi_{t_k} \in L_{ip}(\Omega_{t_k}).
\]
We denote by \( M^0(0, T) \) the collection of all step processes.

For a step process \( \eta \in M^0(0, T) \), we set the norm
\[
\|\eta\|_{H_p} := \mathbb{E}[\int_0^T |\eta_s|^2 ds]^{p/2}, \quad \|\eta\|_{M_p}^p := \mathbb{E}[\int_0^T |\eta_s|^p ds]
\]
and denote by \( H_p^0(0, T) \) and \( M_p^0(0, T) \) the completion of \( M^0(0, T) \) with respect to the norms \( \| \cdot \|_{H_p} \) and \( \| \cdot \|_{M_p} \), respectively.

Definition 2.5 (i) We say that a map \( \xi(\omega) : \Omega_T \to \mathbb{R} \) is quasi-continuous if for all \( \varepsilon > 0 \), there exists an open set \( G \) with \( c(G) < \varepsilon \) such that \( \xi(\cdot) \) is continuous on \( G^c \).

(ii) We say that a process \( M_t(\omega) : \Omega_T \times [0, T] \to \mathbb{R} \) is quasi-continuous if for all \( \varepsilon > 0 \), there exists an open set \( G \) with \( c(G) < \varepsilon \) such that \( M(\cdot) \) is continuous on \( G^c \times [0, T] \).

Remark 2.6 (i) Different from Wiener probability space, counterexamples can be given to show that not all \( B \)-measurable functions on \( \Omega_T \) are c-quasi continuous.

(ii) For \( \eta \in M^1_G(0, T) \), it’s easy to see that the process \( \int_0^t \eta_s(\omega)ds \) has a c-quasi continuous version; Also, [22] shows that any \( G \)-martingale has a c-quasi continuous version.

Theorem 2.7 ([2]) For \( p \geq 1 \),
\[
L_p^G(\Omega_T) = \{ X \in L^0 : X \text{ has a q.c. version, } \lim_{n \to \infty} \mathbb{E}[|X|^p 1_{\{|X| > n\}}] = 0 \},
\]
where \( L^0 \) denotes the space of all \( \mathbb{R} \)-valued \( B \)-measurable functions on \( \Omega_T \).
3 Gradient Estimates for Quasi-linear PDEs

We consider the forward-backward stochastic differential equations below

\[ X_t^x = x + \int_0^t \sigma(X_s^x)dB_s + \int_0^t b(X_s^x)ds, \quad (3.1) \]
\[ Y_t^x = \varphi(X_T^x) + \int_t^T g(Y_s^x, Z_s^x)ds - \int_t^T Z_s^x dB_s, \quad (3.2) \]

where \( B \) is a \( d \)-dimensional standard Brownian motion. Set \( u(T, x) = Y_T^x \). By the generalized Feynman-Kac formula given in [8] and [9], \( u \) is the solution to the following PDE

\[ \partial_t u - \mathcal{L}u - g(u, \sigma^* Du) = 0, \quad (3.3) \]
\[ u(0, x) = \varphi(x), \quad (3.4) \]

where the generator

\[ \mathcal{L}f = \frac{1}{2} \text{tr}[\sigma \sigma^* D^2 f] + b \cdot Df. \]

Hypothesis (P).

(i) \( \sigma : \mathbb{R}^d \to \mathbb{S}_d \), \( b : \mathbb{R}^d \to \mathbb{R}^d \) are Lipschitz continuous.

\[ |\sigma(x) - \sigma(x')| \leq L_\sigma |x - x'|; \]
\[ |b(x) - b(x')| \leq L_b |x - x'|; \]

(ii) There exists \( \Lambda_\sigma \geq \lambda_\sigma > 0 \) such that \( \lambda_\sigma I \leq \sigma(x) \leq \Lambda_\sigma I; \)

(iii) There exists \( K_g, L_g > 0 \) such that \( |g(y, z) - g(y', z')| \leq K_g |y - y'| + L_g |z - z'|. \)

For convenience, we denote by \( \beta_\sigma := \frac{A_\sigma}{L_\sigma} \) and \( g_0 = g(0, 0). \)

Below is the main result in this section.

**Theorem 3.1** Assume that Hypothesis (P) holds. Let \( u(T, x) = Y_T^x \). There exists a constant \( C > 0 \) depending on \( \Lambda_\sigma, \lambda_\sigma, K_g, L_g, g_0 \) such that

\[ |u(T, x) - u(T, y)| \leq C(\|\varphi\|_\infty + |g_0|/\mu) \frac{e^{\mu T}}{\sqrt{(1 - e^{-LT})/L}} |x - y|, \]

where \( \mu = K_g + 4L_g^2; \quad L = 2(L_g\Lambda_\sigma + L_b + 2L_g^2). \)

**Remark 3.2** The constant \( C \) above can be chosen as \( C = 4C_5 C_\beta \sigma C_g \frac{\Lambda_\sigma^2}{L_\sigma^2} \) with \( C_5 := \left( c_p \frac{2}{5} \right)^\frac{1}{\beta} \left( \frac{1}{\beta} \right)^\frac{1}{\beta}, \quad C_\beta \sigma = \frac{1}{\sqrt{2}} d_{32(\beta_\sigma^2 + \frac{1}{4})^2} + 1, \quad C_g = 1 + \frac{K_g}{L_g}, \) where \( c_p, d_p \) are constants depending only on \( p \) in Lemma 3.6 and Lemma 3.7, respectively.

**Corollary 3.3** Assume that Hypothesis (P) holds. Let \( u(T, x) := E[\varphi(X_T^x)] \). There exists a constant \( C > 0 \) depending on \( \Lambda_\sigma, \lambda_\sigma \) such that

\[ |u(T, x) - u(T, y)| \leq C\|\varphi\|_\infty \frac{1}{\sqrt{(1 - e^{-LT})/L}} |x - y|, \]

where \( L = 2L_b + 4L_g^2. \)
Proof. This is a special case of Theorem 3.1 with \( g \equiv 0 \). So we can assume \( g \equiv 0 \), \( K_g = 0 \) and \( L_g = \frac{1}{n} \), which implies that \( C_g = 1 \). So

\[
|u(T, x) - u(T, y)| \leq 4C_5 C_{\beta_0} \frac{\Lambda^2_{\sigma}}{X^2_{\sigma}} \| \varphi \|_{\infty} \frac{e^{\mu T}}{\sqrt{1 - e^{-LT}}/L} |x - y|,
\]

where \( \mu = \frac{4}{n^2} \), \( L = 2\left( \frac{L_3}{n} + L_b + 2L_{2,\sigma}^3 \right) \). Letting \( n \) go to infinity, we get the desired result. \( \square \)

3.1 Construction of the Coupling

Let \( \xi_t := \frac{\alpha}{\alpha - 1} e^{\theta (1 - e^{L_{T-t}})} =: \frac{\alpha}{\alpha - 1} \xi_0^0 \) for some \( \alpha \geq 2 \), where

\[
\theta = \frac{\lambda^2_{\sigma} \Lambda^{-1}_{\sigma}}{2}, \quad L = 2L_{\alpha/2}^g, \quad L_{\alpha/2}^g = L_0 L_\sigma + L_b + \frac{\alpha - 1}{2} L_{2,\sigma}^2.
\]

3.1.1 the Coupling before time \( T \)

Consider the coupling below which is adapted from [24]

\[
dX^x_t = \sigma(X^x_t)dB_t + b^x(t, X^x_t)dt, \quad X^x_0 = x; \tag{3.5}
\]

\[
d\hat{X}^y_t = \sigma(\hat{X}^y_t)dB_t + b^y(t, \hat{X}^y_t)dt + \frac{1}{\xi_t} \sigma(X^x_t)(X^x_t - \hat{X}^y_t)dt, \quad \hat{X}^y_0 = y. \tag{3.6}
\]

Let \( \varepsilon_t \) be an adapted measurable process such that \( |\varepsilon_t| \leq L_g \). Set

\[
B^z_t = B_t - \int_0^t \varepsilon_s ds.
\]

Rewrite (3.5, 3.6)

\[
dX^x_t = \sigma(X^x_t)dB^z_t + b^x(t, X^x_t)dt, \quad X^x_0 = x; \tag{3.7}
\]

\[
d\hat{X}^y_t = \sigma(\hat{X}^y_t)dB^z_t + b^y(t, \hat{X}^y_t)dt + \frac{1}{\xi_t} \sigma(X^x_t)(X^x_t - \hat{X}^y_t)dt, \quad \hat{X}^y_0 = y, \tag{3.8}
\]

where \( b^z(t, x) = \sigma(x)\varepsilon_t + b(x) \).

Then \( \hat{X}_t := X^x_t - \hat{X}^y_t \) satisfies the equation below

\[
d\hat{X}_t = \hat{\sigma}(t)dB^z_t + \hat{b}^z(t, \hat{X}_t)dt - \frac{1}{\xi_t} \sigma(X_t^x)\hat{X}_t dt, \quad \hat{X}_0 = x - y. \tag{3.9}
\]

where \( \hat{\sigma}(t) = \sigma(X^x_t) - \sigma(\hat{X}^y_t) \), \( \hat{b}^z(t) = b^z(t, X^x_t) - b^z(t, \hat{X}^y_t) \).

Set \( \hat{B}^z_t = B^z_t + \int_0^t \frac{\sigma(X^x_s) - \sigma(\hat{X}^y_s)}{\xi_s} ds \). Rewrite equations (3.7, 3.9) as

\[
dX^x_t = \sigma(X^x_t)dB^z_t + b^z(t, X^x_t)dt - \frac{1}{\xi_t} \sigma(X^x_t)\sigma(\hat{X}^y_t)^{-1} \sigma(X^x_t)\hat{X}_t dt, \quad X^x_0 = x; \tag{3.10}
\]

\[
d\hat{X}^y_t = \sigma(\hat{X}^y_t)dB^z_t + b^z(t, \hat{X}^y_t)dt, \quad \hat{X}^y_0 = y; \tag{3.11}
\]

\[
d\hat{X}_t = \hat{\sigma}(t)dB^z_t + \hat{b}^z(t)dt - \frac{1}{\xi_t} \sigma(X^x_t)\sigma(\hat{X}^y_t)^{-1} \sigma(X^x_t)\hat{X}_t dt, \quad \hat{X}_0 = x - y. \tag{3.12}
\]
By Itô’s formula, we have

\[ H_t := |\dot{X}_t|^2 = |x - y|^2 + \int_0^t 2\tilde{\sigma}(s)^* \dot{X}_s \cdot dB^\varepsilon_s + \int_0^t 2\dot{X}_s \cdot d\tilde{\nu}(s) ds \]

\[ - \int_0^t \frac{2\tilde{\sigma}(s)^* \sigma(X^\varepsilon_t) \dot{X}_s}{\xi_s} ds + \int_0^t \text{tr}[\tilde{\sigma}(s)^* \tilde{\sigma}(s)] ds, \]

\[ = |x - y|^2 + \int_0^t 2\tilde{\sigma}(s)^* \dot{X}_s \cdot d\tilde{B}^\varepsilon_s + \int_0^t 2\dot{X}_s \cdot d\tilde{\nu}(s) ds \]

\[ - \int_0^t \frac{2\tilde{\sigma}(s)^* \sigma(X^\varepsilon_t) \dot{X}_s}{\xi_s} ds + \int_0^t \text{tr}[\tilde{\sigma}(s)^* \tilde{\sigma}(s)] ds, \quad t \in (0, T). \]

So, setting \( L_{b^\varepsilon} := L_q L_\sigma + L_b \), we have

\[ dH_t \leq 2\tilde{\sigma}(t)^* \dot{X}_t \cdot dB^\varepsilon_t + (2L_{b^\varepsilon} + L_\sigma^2 - \frac{2\lambda_\sigma}{\xi_t}) H_t dt, \quad t \in (0, T), \]

\[ dH_t \leq 2\tilde{\sigma}(t)^* \dot{X}_t \cdot d\tilde{B}^\varepsilon_t + (2L_{b^\varepsilon} + L_\sigma^2 - \frac{2\lambda_\sigma^2 \Lambda_\sigma^{-1}}{\xi_t}) H_t dt, \quad t \in (0, T), \]

and, for \( p \geq 1 \),

\[ dH_t^p \leq 2pH_t^{p-1}\tilde{\sigma}(t)^* \dot{X}_t \cdot dB^\varepsilon_t + (2L_{b^\varepsilon} + (2p - 1)L_\sigma^2 - \frac{2\lambda_\sigma}{\xi_t}) pH_t^p dt \]

\[ = 2pH_t^{p-1}\tilde{\sigma}(t)^* \dot{X}_t \cdot dB^\varepsilon_t + (2L_{b^\varepsilon} - \frac{2\lambda_\sigma}{\xi_t}) pH_t^p dt. \]

\[ dH_t^p \leq 2pH_t^{p-1}\tilde{\sigma}(t)^* \dot{X}_t \cdot d\tilde{B}^\varepsilon_t + (2L_{b^\varepsilon} - \frac{2\lambda_\sigma^2 \Lambda_\sigma^{-1}}{\xi_t}) pH_t^p dt. \]

where \( L_{b^\varepsilon} = L_{b^\varepsilon} + (p - \frac{1}{2})L_\sigma^2 \). Then, setting \( \tilde{H}_t = e^{2\varepsilon t} H_t \),

\[ \frac{\tilde{H}_t^p}{\xi_t^{2p-1}} \leq \frac{\tilde{H}_s^p}{\xi_s^{2p-1}} + \int_s^t \frac{2p e^{2\varepsilon r} p e^{2\varepsilon r}}{\xi_r^{2p-1}} H_r^{p-1}\tilde{\sigma}(r)^* \dot{X}_r \cdot dB^\varepsilon_r \]

\[ + \int_s^t \frac{2p e^{2\varepsilon r}}{\xi_r^{2p-1}} ((\varrho + L_{b^\varepsilon}) \xi_r - \frac{2p - 1}{2p} \xi_r' dr, \] (3.13)

\[ \frac{\tilde{H}_t^p}{\xi_t^{2p-1}} \leq \frac{\tilde{H}_s^p}{\xi_s^{2p-1}} + \int_s^t \frac{2p e^{2\varepsilon r} p e^{2\varepsilon r}}{\xi_r^{2p-1}} H_r^{p-1}\tilde{\sigma}(r)^* \dot{X}_r \cdot d\tilde{B}^\varepsilon_r \]

\[ + \int_s^t \frac{2p e^{2\varepsilon r} p e^{2\varepsilon r}}{\xi_r^{2p-1}} ((\varrho + L_{b^\varepsilon}) \xi_r - \frac{\lambda_\sigma}{\xi_r} \xi_r' dr. \] (3.16)

3.1.2 Extension of the Coupling to \( T \)

Setting \( \varrho = \frac{p - 1}{p} L_{b^\varepsilon} \), we have \( \varrho + L_{b^\varepsilon} = \frac{2p - 1}{p} L_{b^\varepsilon} \). By the definition of \( \xi \), for any \( \varphi \geq p \geq 1 \), we have

\[ \frac{2p - 1}{p} L_{b^\varepsilon} \xi_r - \lambda_\sigma - \frac{2p - 1}{2p} \xi_r' \leq \frac{2p - 1}{p} L_{b^\varepsilon} \xi_r - \frac{\lambda_\sigma^2}{\Lambda_\sigma} - \frac{2p - 1}{2p} \xi_r' \leq -\theta. \]

(3.17)

So, by (3.13)(3.14), we have

\[ E^\varepsilon [\int_0^T \frac{e^{2\varepsilon \varrho r} |\dot{X}_r|^2 dr}{\xi_r^{2p}}] \leq \frac{1}{2p\theta} \frac{|x - y|^{2p}}{\xi_0^{2p-1}}, \]
where $E^\varepsilon$ is the expectation under $P^\varepsilon$ with \( \frac{dP^\varepsilon}{dP} := e^{\int_0^T \varepsilon_s dB_s - \frac{1}{2} \int_0^T |\varepsilon_s|^2 ds} \).

Actually, for any $p \geq 1$, there exists $s(p) \in (0, T)$ such that $\frac{2p-1}{p} L_p \varepsilon_t - \lambda_{\sigma} - \frac{2p-1}{2p} \xi'_r \leq -\frac{\theta}{2p}$ for any $s \in [s(p), T)$. So

\[
E^\varepsilon \left[ \int_{s(p)}^t e^{2p\theta |X_s|^2P} \right] \frac{e^{2p\theta |X_s|^2P}}{\xi'^2_r} \frac{\xi'^2_r}{\xi'^2_r} \leq \frac{1}{\theta} E^\varepsilon \left[ \frac{e^{2p\theta |X_{s(p)}|^2P}}{\xi'^2_r} \right] < \infty.
\]

Consequently, for any $p \geq 1$, there exists $C(p) > 0$ such that

\[
E^\varepsilon \left[ \int_0^t |X_t|^p \xi'^2_r \right] \frac{\xi'^2_r}{\xi'^2_r} \leq C(p).
\]

So there exists a $\mathbb{R}^d$-valued adapted measurable process \{\( g_t \) \( t \in [0, T] \)} such that

\[
E^\varepsilon \left[ \int_0^T |g_s|^p \xi'^2_r \right] \frac{\xi'^2_r}{\xi'^2_r} < \infty \quad \text{and} \quad g_s = \frac{1}{\xi'^2_r} (X^x_s - \hat{X}^y_s), \quad s \in [0, T).
\]

Let \( (\hat{X}_t)_{t \in [0, T]} \) be the solution to the equation below

\[
d\hat{X}_t = \sigma(\hat{X}_t) dB_t + b(\hat{X}_t) dt + \sigma(\hat{X}^x_t) g_t dt, \quad \hat{X}_0 = y.
\]

Clearly, we have \( \hat{X}^y_t = \hat{X}_t, \quad t \in [0, T) \). So \( \hat{X} \) is a continuous extension of \( \hat{X}^y \) to \( [0, T) \). In the sequel, we shall still write \( \hat{X}^y \) for \( \hat{X} \).

**Proposition 3.4** Let $X^x$ and $\hat{X}^y$ be the solutions to equations (3.5) and (3.19). We have $X^x_T = \hat{X}^y_T$.

**Proof.** For $\omega$ such that $X^x_T(\omega) \neq \hat{X}^y_T(\omega)$, we have

\[
\int_0^T |g_s(\omega)|^p \xi'^2_r \frac{\xi'^2_r}{\xi'^2_r} \leq \int_0^T \xi'^2_r (X^x_s(\omega) - \hat{X}^y_s(\omega)) |\xi'^2_r| \xi'^2_r = \infty.
\]

Noting that $E^\varepsilon \left[ \int_0^T |g_s|^p \xi'^2_r \right] \frac{\xi'^2_r}{\xi'^2_r} \leq C(p)$, we conclude that $X^x_T = \hat{X}^y_T$ a.s. \( \Box \)

**3.1.3 Estimates for the Drift**

Set $h_s = -\frac{1}{\xi'^2_r} \sigma(\hat{X}^y_s)^{-1} \sigma(\hat{X}^y_s)(X^x_s - \hat{X}^y_s)$. Choose a sequence of stopping times $\tau_n$ such that $\tau_n \uparrow T$, $\tau_n \leq T - \frac{1}{n}$ and $h^n_s := h_s 1_{[0, \tau_n]}$ is bounded. Denote by $\tilde{E}^n$ the expectation under the probability $\tilde{P}^n$ with \( \frac{dP^n}{dP} = e^{\int_0^T h^n dB_s - \frac{1}{2} \int_0^T |h^n|^2 ds} =: U^n_T \).

**Proposition 3.5**

\[
\tilde{E}^n \exp \left\{ \frac{\theta^2}{8 \lambda_{\sigma}^2} \int_{\tau_n}^\tau |\hat{X}_s|^2 \xi'^2_r \frac{\xi'^2_r}{\xi'^2_r} ds \right\} \leq \exp \left\{ \frac{\theta}{8 \lambda_{\sigma}^2 \xi_0} |x - y|^2 \right\}.
\]

\[
E^\varepsilon [\|U^n_{T}^{-1}\|^{1+\delta}] \leq \exp \left\{ \frac{\theta \sqrt{1 + \delta^{-1}}}{8 \lambda_{\sigma}^2 \xi_0 (1 + \sqrt{1 + \delta^{-1}})} |x - y|^2 \right\},
\]

where $\delta := \frac{\theta^2}{4 \lambda_{\sigma}^2 \beta_2^2 + 8 \lambda_{\sigma} \beta_2}$.
The estimates above are from Lemma 2.2 in [24]. For readers’ convenience, we give the sketch of the proof.

**Proof.** For \( p = 1, (3.15, 3.16) \) with \( q = 0 \) shows that

\[
\int_0^t \frac{|\dot{X}_r|^2}{\xi_r^2} dr \leq \frac{|x - y|^2}{2\theta \xi_0} + \int_0^t \frac{1}{\theta \xi_r} \hat{\sigma}(r)^* \dot{X}_r \cdot d\tilde{B}_r^n.
\]

Set \( \tilde{B}_t^n := B_t^n - \int_0^t h^n_s ds \). By Girsanov transformation, we know that \( \tilde{B}_t^n \) is a standard Brownian motion under \( \tilde{E}^n \). Noting that

\[
\int_0^{\tau_n} \frac{|\dot{X}_r|^2}{\xi_r^2} dr \leq \frac{|x - y|^2}{2\theta \xi_0} + \int_0^{\tau_n} \frac{1}{\theta \xi_r} \hat{\sigma}(r)^* \dot{X}_r \cdot d\tilde{B}_r^n,
\]

we get

\[
\tilde{E}^n[\exp\{a \int_0^{\tau_n} \frac{\dot{X}_r^2}{\xi_r^2} dr\}] \leq \exp\left(\frac{a|x - y|^2}{2\theta \xi_0}\right)\left(\tilde{E}^n[\exp\left\{\frac{8a^2 \Lambda^2}{\theta^2} \int_0^{\tau_n} \frac{\dot{X}_r^2}{\xi_r^2} dr\right\}\right)^{1/2}.
\]

Taking \( a = \frac{\theta^2}{8 \Lambda^2} \), we get the first estimate.

By the definition of \( U_T^{h,n} \), we have

\[
E^\varepsilon[|U_T^{h,n}|^{1+b}] = \tilde{E}^n[\exp\{\delta \int_0^T h_s^n \cdot dB^\varepsilon_s - \delta \int_0^T |h_s^n|^2 ds\}].
\]

Noting that \( M_t := \int_0^t h_s^n \cdot dB^\varepsilon_s - \int_0^t |h_s^n|^2 ds \) is a martingale under \( \tilde{P}^n \), we have, for any \( q > 1 \),

\[
E^\varepsilon[|U_T^{h,n}|^{1+b}] \leq \tilde{E}^n[\exp\{\delta (M_T + \frac{\delta}{2} (M_T)_{T})\}]
\]

\[
\leq (\tilde{E}^n[\exp\{\frac{\delta q (q+1)}{2(q-1)} (M_T)_{T}\}])^{(q-1)/q}
\]

\[
\leq (\tilde{E}^n[\exp\{\frac{\delta q (q+1)}{2(q-1)} \beta^2 \int_0^T \frac{\dot{X}_r^2}{\xi_r^2} ds\}])^{(q-1)/q}.
\]

Taking \( q = 1 + \sqrt{1 + \delta - 1} \), we get

\[
\frac{\delta q (q+1)}{2(q-1)} \beta^2 = \frac{(\delta + \sqrt{\delta + 3}) \beta^2}{\sqrt{\Lambda^2}} = \frac{\theta^2}{8 \Lambda^2}.
\]

By the first estimate, we get the second desired result. \( \square \)

So for any adapted measurable process \( \varepsilon \) with \( |\varepsilon_t| \leq L_g \),

\[
\tilde{P}^\varepsilon := U_T.P^\varepsilon, \text{ with } U_T := \exp\left\{\int_0^T h_s dB^\varepsilon_s - \frac{1}{2} \int_0^T |h_s|^2 ds\right\}
\]

is a probability, under which \( \tilde{B}_t^\varepsilon = B_t^\varepsilon - \int_0^t h_s ds \) is a standard Brownian motion. We write \( \tilde{P}, \tilde{B}_t \) for \( \tilde{P}^0, \tilde{B}_t^0 \).

Let \( u_t = \int_0^T h_s dB^\varepsilon_s - \frac{1}{2} \int_0^T |h_s|^2 ds \).

**Lemma 3.6** We have the following estimates:

\[
(E^\varepsilon[|u_T|^2])^{\frac{2}{2}} \leq C_\alpha 2 \Lambda^2 \frac{|x - y|}{\lambda^2} + O(|x - y|^2), \quad (3.20)
\]

\[
(\tilde{E}^\varepsilon[|u_T|^2])^{\frac{2}{2}} \leq C_\alpha 2 \Lambda^2 \frac{|x - y|}{\lambda^2} + O(|x - y|^2), \quad (3.21)
\]

where \( C_\alpha := (c_\alpha^2) \frac{\beta}{\lambda^2} \frac{1}{\alpha} \frac{1}{\alpha^2} \) and \( c_\lambda \) is the constant for BDG inequalities.
Proof. Set \( q = \frac{\alpha - 1}{\alpha} L_0^\theta \) for \( p \leq \frac{\alpha}{2} \). By (3.13), (3.15) and (3.17),

\[
E^\varepsilon[\int_0^T e^{2p\varphi r} |\dot{X}_r|^{2p} dr] \leq \frac{1}{2p\theta} \frac{|x - y|^{2p}}{\xi_0^{2p-1}},
\]

\[
E^\varepsilon[\int_0^T e^{2p\varphi r} |\dot{Y}_r|^{2p} dr] \leq \frac{1}{2p\theta} \frac{|x - y|^{2p}}{\xi_0^{2p-1}}.
\]

So \( E^\varepsilon[\int_0^T e^{2p\varphi r} |h_r|^{2p} dr] \leq \frac{q^{2p}}{2p\theta} \frac{|x - y|^{2p}}{\xi_0^{2p-1}} \). Then

\[
(E^\varepsilon[|u_T|^p])^{1/p} \leq (E^\varepsilon[\int_0^T h_s dB_s^\varepsilon |^p])^{1/p} + \frac{1}{2}(E^\varepsilon[\int_0^T |h_s|^2 ds])^{1/p} + \frac{1}{2}(E^\varepsilon[\int_0^T |h_s|^2 ds])^{1/p} \leq c_p^{1/p} \sqrt{(E^\varepsilon[\int_0^T |h_s|^2 ds])^{1/p}}.
\]

For \( p = \frac{\alpha}{2} \), we have \( q = \frac{\alpha - 2}{\alpha} L_0^\theta \alpha/2 \) and

\[
(E^\varepsilon[\int_0^T |h_s|^2 ds]^{\alpha/2})^{1/\alpha} = (E^\varepsilon[\int_0^T e^{-2\alpha s} e^{2\alpha s} |h_s|^2 ds]^{\alpha/2})^{1/\alpha} \leq (E^\varepsilon[\int_0^T e^{2\alpha r} h_r^\alpha dr]^{1/\alpha} \frac{\xi_0^{2\alpha}}{\alpha^{\alpha \theta}} \leq \frac{1}{\alpha} \frac{1}{\alpha^{\alpha \theta}} \frac{2\alpha^2}{\lambda_0^2} \frac{|x - y|}{(1 - e^{-L^2})/L}.
\]

By arguments above, we get (3.20). (3.21) can be obtained in the same way. \( \square \)

### 3.2 Gradient Estimates

For \( \varphi \in C_b(\mathbb{R}^d) \), we consider backward SDEs below

\[
Y_t^x = \varphi(X_t^x) + \int_t^T g(Y_s^x, Z_s^x) ds - \int_t^T Z_s^x dB_s^\varepsilon,
\]

\[
\tilde{Y}_t^y = \varphi(\tilde{X}_t^y) + \int_t^T g(\tilde{Y}_s^y, \tilde{Z}_s^y) ds - \int_t^T \tilde{Z}_s^y d\tilde{B}_s^\varepsilon.
\]

Set \( u(T, x) := Y_0^x \). Clearly, we have \( u(T, y) := \tilde{Y}_0^y \). Rewrite (3.24), (3.25)

\[
Y_t^x = \varphi(X_t^x) + \int_t^T [g(Y_s^x, Z_s^x) - \varepsilon_s Z_s^x] ds - \int_t^T Z_s^x dB_s^\varepsilon,
\]

\[
\tilde{Y}_t^y = \varphi(\tilde{X}_t^y) + \int_t^T [g(\tilde{Y}_s^y, \tilde{Z}_s^y) - (\varepsilon_s - h_s) \tilde{Z}_s^y] ds - \int_t^T \tilde{Z}_s^y d\tilde{B}_s^\varepsilon.
\]
Let $k_s$ be an adapted measurable process with $|k_t| \leq K_g$. Set $V_t = \exp \{ \int_0^t k_s ds \}$ and $W_t = U_t V_t$. Applying Itô’s formula, we get

\[
V_t Y_t^x = V_T \varphi(X_T^x) + \int_0^T V_s [g(Y_s^x, Z_s^x) - k_s Y_s^x - \varepsilon_s Z_s^x] ds - \int_0^T V_s Z_s^x dB_s^x,
\]

\[
W_t \tilde{Y}_t^y = W_T \varphi(X_T^x) + \int_0^T W_s [g(\tilde{Y}_s^y, \tilde{Z}_s^y) - k_s \tilde{Y}_s^y - \varepsilon_s \tilde{Z}_s^y] ds - \int_0^T W_s (\tilde{Z}_s^y + \tilde{Y}_s^y h_s) dB_s^c.
\]

So

\[
\tilde{Y}_0 = V_T \varphi(X_T^x)(1 - U_T) + (1 - U_T) \int_0^T V_s [g(Y_s^x, Z_s^x) - k_s Y_s^x - \varepsilon_s Z_s^x] ds
\]

\[+ \int_0^T W_s [\dot{g}(s) - k_s \dot{\tilde{Y}}_s - \varepsilon_s \dot{\tilde{Z}}_s] ds - \int_0^T [V_s Z_s^x - W_s (\tilde{Z}_s^y + \tilde{Y}_s^y h_s)] dB_s^c
\]

\[+ \int_0^T U_t h_t \int_0^t W_s [g(Y_r^x, Z_r^x) - k_s Y_r^x - \varepsilon_s Z_r^x] ds dB_r^c,
\]

where $\tilde{Y}_s = Y_s^x - \tilde{Y}_s^y$, $\tilde{Z}_s = Z_s^x - \tilde{Z}_s^y$, $\dot{g}(s) = g(Y_s^x, Z_s^x) - g(\tilde{Y}_s^y, \tilde{Z}_s^y)$. Choosing processes $k^0, \varepsilon^0$ such that $\dot{g}(s) - k^0 \dot{\tilde{Y}}_s - \varepsilon^0 \dot{\tilde{Z}}_s \leq 0$, we have

\[
\tilde{Y}_0 \leq \exp \left[ \int_0^T (1 - U_T) [V_T \varphi(X_T^x)] ds \right]
\]

\[+ \int_0^T V_s [g(Y_s^x, Z_s^x) - k_s Y_s^x - \varepsilon_s Z_s^x] ds - \int_0^T V_s Z_s^x dB_s^x
\]

\[+ \int_0^T W_s [\dot{g}(s) - k_s \dot{\tilde{Y}}_s - \varepsilon_s \dot{\tilde{Z}}_s] ds - \int_0^T [V_s Z_s^x - W_s (\tilde{Z}_s^y + \tilde{Y}_s^y h_s)] dB_s^c
\]

\[+ \int_0^T U_t h_t \int_0^t W_s [g(Y_r^x, Z_r^x) - k_s Y_r^x - \varepsilon_s Z_r^x] ds dB_r^c,
\]

where $g_0 = g(0, 0)$.

### 3.2.1 Estimates for the Backward SDEs

From [1], we have the following estimates:

**Proposition 3.7** Let $(Y_t^x, Z_t^x)$ be the solution to the Backward SDE (3.24). Set $\mu = K_g + 4L_g^2$. Then there exists $d_p > 0$ depending on $p$, such that

\[
(E^0 \left[ \sup_{t \in [0, T]} e^{\mu t} |Y_t^x|^p \right]^{\frac{1}{p}}) \leq e^{\mu T} d_p \left[ ||\varphi||_{\infty} + |g_0|/\mu \right],
\]

\[
(E^0 \left[ \int_0^T e^{2\mu s} |Z_s^x|^2 ds \right]^{p/2})^{\frac{1}{p}} \leq e^{\mu T} d_p \left[ ||\varphi||_{\infty} + |g_0|/\mu \right].
\]

### 3.2.2 Proof to the Gradient Estimates

**Lemma 3.8** Let $v : \mathbb{R}^d \to \mathbb{R}$. Assume that there exists a smooth function $F : \mathbb{R} \to \mathbb{R}$ such that $F(0) = 0$ and

\[
|v(x) - v(y)| \leq F(|x - y|), \text{ for any } x, y \in \mathbb{R}^d.
\]

Then

\[
|v(x) - v(y)| \leq F'(0)|x - y|.
\]

**Proof.** Fix $x, y \in \mathbb{R}^d$. Set $x_t = x + t(y - x)$ and $f(t) = v(x_t)$ for $t \in [0, 1]$. Then

\[
|f(t) - f(s)| \leq F(|t - s||x - y|) = F'(0)|x - y||t - s| + o(|t - s|),
\]

where $o(|t - s|)$ represents a quantity that tends to zero faster than $|t - s|$. This completes the proof.
by which we get the desired result. □

**Proof to Theorem 3.1** By Proposition 3.7, we have the following estimates

\[
(e^{KsT} \| \varphi \|_\infty + |g_0| \int_0^T e^{KsT} \| \partial_t u_T - 1 \| \leq (\| \varphi \|_\infty + |g_0|/\mu) e^{\mu T} (\| \partial_t u_T \|_{L^1} + \| u_T \|_{L^1}),
\]

where \( \| \cdot \|_{L^1} \) denotes the norms under \( P e^0 \). Below we shall denote by \( \| \cdot \|_{L^p} \) the norms under \( \tilde{P} e^0 \).

\[
2L_\gamma E^0 [1 - U_T] \int_0^T e^{KsT} \| Z_s^x \| ds
\]

\[
\leq \frac{1}{\sqrt{2}} (1 - U_T)^{1+\delta} (E^0 [\int_0^T e^{2\mu s} \| Z_s^x \| ds]^{2+\delta})^{\frac{\delta}{2+\delta}}
\]

\[
\leq \frac{1}{\sqrt{2}} d_{1+\delta} (\| \varphi \|_\infty + |g_0|/\mu) e^{\mu T} (\| \partial_t u_T \|_{L^1}^{1+\frac{\delta}{2}} + \| u_T \|_{L^1}^{1+\frac{\delta}{2}});
\]

\[
2K_\beta E^0 [1 - U_T] \int_0^T e^{KsT} \| Y_s^x \| ds
\]

\[
\leq \frac{K_\beta}{2L_\gamma^2} E^0 [1 - U_T] \sup_{t \in [0,T]} (e^{\mu T} |Y_t^x|)
\]

\[
\leq \frac{K_\beta}{2L_\gamma^2} d_{1+\delta} (\| \varphi \|_\infty + |g_0|/\mu) e^{\mu T} (\| \partial_t u_T \|_{L^1}^{1+\frac{\delta}{2}} + \| u_T \|_{L^1}^{1+\frac{\delta}{2}}).
\]

So by (3.28)(3.29), we have

\[
|u(T, x) - u(T, y)| \leq C_\beta C_\gamma (\| \varphi \|_\infty + |g_0|/\mu) e^{\mu T} (\| \partial_t u_T \|_{L^1}^{1+\frac{\delta}{2}} + \| u_T \|_{L^1}^{1+\frac{\delta}{2}}),
\]

where \( C_\beta = \frac{1}{\sqrt{2}} d_{\frac{1}{2}+\delta} + 1 \) and \( C_\gamma = 1 + \frac{K_\beta}{L_\gamma^2} \).

By Proposition 3.5, we have

\[
\| \partial_t u_T \|_{L^1}^{1+\frac{\delta}{2}} \leq (E^0 [U_T^{1+\delta}]^{\frac{1}{1+\delta}} E^0 [\| u_T \|^{2+\delta}]^{\frac{1}{2+\delta}} \leq \exp\left\{ \frac{1}{\Lambda_\delta^2 C_\gamma^2} |x - y|^2 \right\} \| u_T \|_{L^1}^{1+\delta}.
\]

Choose \( \alpha \geq 4 + 2\delta \) in the definition of \( \xi \). By Lemma 3.6 and Lemma 3.8, we have

\[
|u(T, x) - u(T, y)| \leq 4C_\alpha C_\beta C_\gamma \frac{\Lambda_\delta^2}{\Lambda_\delta^2} (\| \varphi \|_\infty + |g_0|/\mu) e^{\mu T} \frac{\sqrt{\xi_0}}{\sqrt{\xi_0}}.
\]

Setting \( \alpha = 5 \), we get the desired result.

### 4 Gradient Estimates for Nonlinear Semigroups

In this section, we shall derive uniform gradient estimates for semigroups associated with stochastic differential equations below

\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt,
\]

where \( B \) is a \( d \)-dimensional \( G \)-Brownian motion with \( G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}(A \gamma) \) for some bounded, closed, convex subset \( \Gamma \subset \mathcal{S}_d^+ \). The generator of the diffusion process will be

\[
L f = G(\sigma^* D^2 f \sigma) + b \cdot D f.
\]
Then those in the last section.

First, we shall introduce the construction of the coupling. The arguments below are similar to 4.1 Construction of the Coupling

Consider the coupling $\beta$ for convenience, we denote by $\beta_\sigma := \frac{\Lambda_\sigma}{\lambda_\sigma}$, $\beta_\Gamma := \frac{\Lambda_\Gamma}{\gamma_\Gamma}$.

The main result of this section is below.

**Theorem 4.1** Assume that Hypothesis (G) holds. Let $u(T, x) := \mathbb{E}[\varphi(X_T^x)]$. Then

$$|u(T, x) - u(T, y)| \leq C \frac{\|\varphi\|_\infty}{\sqrt{(1 - e^{-LT})/L}} |x - y|,$$

where $C = \frac{2\Lambda_\sigma^2}{\lambda_\sigma^2 \lambda_\Gamma}$, $L = 2L_b + \Lambda_\Gamma^2 L_\sigma^2$.

### 4.1 Construction of the Coupling

First, we shall introduce the construction of the coupling. The arguments below are similar to those in the last section.

Let $\xi_t = \frac{2(\lambda_\sigma^2 \Lambda_\sigma^2 - \theta)}{(1 - e^{L(t-T)})}$ for some $\theta \in [\lambda_\sigma^2 \Lambda_\sigma^{-1}/2, \lambda_\sigma^2 \Lambda_{\sigma}^{-1}]$, where $L = 2L_b + \Lambda_\Gamma^2 L_\sigma^2$.

Consider the coupling

$$dX_t = \sigma(X_t) d\tilde{B}_t + b(X_t) dt, \quad X_0 = x; \quad (4.2)$$
$$dY_t = \sigma(Y_t) d\tilde{B}_t + b(Y_t) dt + \frac{1}{\xi_t} \sigma(X_t) (X_t - Y_t) dt, \quad Y_0 = y. \quad (4.3)$$

Then $Z_t := X_t - Y_t$ satisfies the equation below

$$dZ_t = \dot{\sigma}(t) d\tilde{B}_t + \dot{b}(t) dt - \frac{1}{\xi_t} \sigma(X_t) Z_t dt, \quad Z_0 = x - y. \quad (4.4)$$

where $\dot{\sigma}(t) = \sigma(X_t) - \sigma(Y_t), \quad \dot{b}(t) = b(X_t) - b(Y_t)$.

Set $\tilde{B}_t = B_t + \int_0^t \frac{\sigma(Y_s)^{-1} \sigma(X_s)}{\xi_s} dZ_s$. Rewrite equations (4.2, 4.4) as

$$dX_t = \sigma(X_t) d\tilde{B}_t + b(X_t) dt - \frac{1}{\xi_t} \sigma(X_t) \sigma(Y_t)^{-1} \sigma(X_t) Z_t dt, \quad X_0 = x; \quad (4.5)$$
$$dY_t = \sigma(Y_t) d\tilde{B}_t + b(Y_t) dt, \quad Y_0 = y; \quad (4.6)$$
$$dZ_t = \dot{\sigma}(t) d\tilde{B}_t + \dot{b}(t) dt - \frac{1}{\xi_t} \sigma(X_t) \sigma(Y_t)^{-1} \sigma(X_t) Z_t dt, \quad Z_0 = x - y. \quad (4.7)$$

By Itô's formula, we have

$$|Z_t|^2 = |x - y|^2 + \int_0^t 2\dot{\sigma}(s)^* Z_s \cdot d\tilde{B}_s + \int_0^t 2Z_s \cdot \dot{b}(s) ds - \int_0^t \frac{2Z_s^* \sigma(X_s) Z_s}{\xi_s} ds$$
$$+ \int_0^t \text{tr}[\dot{\sigma}(s)^* \dot{\sigma}(s) d\langle B \rangle_s], \quad t \in (0, T).$$
So
\[ d|Z_t|^2 \leq 2\sigma(t)^*Z_t \cdot dB_t + (2L_b + L^2_\sigma \Lambda_t^2 - \frac{2\lambda_r}{\xi_t})|Z_t|^2 dt, \quad t \in (0, T), \]
\[ d|Z_t|^2 \leq 2\sigma(t)^*Z_t \cdot \tilde{B}_t + (2L_b + L^2_\sigma \Lambda_t^2 - \frac{2\lambda_r}{\xi_t})|Z_t|^2 dt, \quad t \in (0, T). \]

Then
\[ \frac{|Z_t|^2}{\xi_t} \leq \frac{|x - y|^2}{\xi_0} + \int_0^t \frac{2}{\xi_r} \dot{\sigma}(r)^*Z_r \cdot dB_r + \int_0^t \frac{2}{\xi_r} |Z_r|^2 (\frac{L}{2} \xi_r - \lambda_r - \frac{1}{2} \xi_r')dr, \quad (4.8) \]
\[ \frac{|Z_t|^2}{\xi_t} \leq \frac{|x - y|^2}{\xi_0} + \int_0^t \frac{2}{\xi_r} \dot{\sigma}(r)^*Z_r \cdot \tilde{B}_r + \int_0^t \frac{2}{\xi_r} |Z_r|^2 (\frac{L}{2} \xi_r - \lambda_r - \frac{1}{2} \xi_r')dr. \quad (4.9) \]

### 4.2 Extension of $Y$ to $T$

By the definition of $\xi$, we have $\frac{L}{2} \xi_r - \lambda_r - \frac{1}{2} \xi_r' \leq \frac{L}{2} \xi_r - \frac{\lambda_r^2}{\Lambda_r} - \frac{1}{2} \xi_r' = -\theta$. So
\[ \mathbb{E}[\int_0^t \frac{|Z_r|^p}{\xi_r^p} dr] \leq \frac{|x - y|^2}{2\theta \xi_0}, \]
which, however, does NOT imply that $\frac{Z_t}{\xi_t} \in [M^p_G(0, T)]^d$. This is different from the linear case.

Similar to the arguments in Section 3.1.1, we can show that, for any $p \geq 1$, there exists $C(p) > 0$ such that
\[ \mathbb{E}[\int_0^t \frac{|Z_r|^p}{\xi_r^p} dr] \leq C(p), \quad \text{for any } t \in [0, T). \]

Note that, for any $t \in (0, T)$ and $p \geq 1$, $\frac{Z_r}{\xi_r} 1_{[0,t]}(r) \in [M^p_G(0, T)]^d$, and that
\[ \mathbb{E}[\int_s^t \frac{|Z_r|^p}{\xi_r^p} dr] \leq C(\alpha p)^\frac{1}{\alpha} (t - s)^\frac{1}{\alpha} \] for any $\alpha > 1$.

So there exists
\[ g \in [M^p_G(0, T)]^d \text{ such that } g_s = \frac{1}{\xi_s}(X_s - Y_s), \quad s \in [0, T). \]

Let $(\bar{Y}_t)_{t \in [0, T]}$ be the solution to the equation below
\[ d\bar{Y}_t = \sigma(\bar{Y}_t)dB_t + b(\bar{Y}_t)dt + \sigma(X_t)g dt, \quad \bar{Y}_0 = y. \quad (4.11) \]

Clearly, we have $Y_t = \bar{Y}_t, \quad t \in [0, T)$. So $\bar{Y}$ is a continuous extension of $Y$ to $[0, T]$. In the sequel, we shall still write $Y$ for $\bar{Y}$.

The following result is the counterpart of Proposition 3.4 in the $G$-expectation space.

**Proposition 4.2** Let $X$ and $Y$ be the solutions to equations (4.2) and (4.11). We have $X_T = Y_T$, q.s.
4.3 Girsanov Transformation with Bounded Drifts

For a bounded process $h \in [M_G^2(0,T)]^d$, set $\tilde{B}_t := B_t - \int_0^t h_s ds$. We try to find a sublinear expectation under which $\tilde{B}_t$ is a $G$-Brownian motion. Hu et al (2014) constructed a sublinear expectation $\tilde{E}$ with such property in an extended $\tilde{G}$-expectation space $(\tilde{\Omega}_T, L^G_{\tilde{G}}, \tilde{E})$ with $\tilde{\Omega}_T := C_0([0,T]; \mathbb{R}^{2d})$ and

$$
\hat{G}(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr } \left[ A \gamma I_d \gamma^{-1} \right], \quad A \in \mathbb{S}_{2d}.
$$

Let $\hat{B}_t = (B_t, B'_t)$ be the canonical process in the extended space. By the definition of $\hat{G}$, we have $\langle \hat{B}_t, x \rangle = \hat{E}_t[x]$, for $x \in L_{ip}(\Omega_T)$, where

$$
\hat{E}_t[x] = \mathbb{E}_t[U_t^h x], \quad \text{for } x \in L_{ip}(\Omega_T),
$$

for any $\varphi \in C_b(\mathbb{R}^d)$, set $u(t,x) := \mathbb{E}[\varphi(x + \hat{B}_t)]$. We shall prove that $u$ is the viscosity solution to the $G$-heat equation

$$
\partial_t u - G(D_x^2 u) = 0,
$$

$$
u(0,x) = \varphi(x).
$$

By the definition of $\hat{E}$, we have

$$
u(t + s, x) = \hat{E}[\varphi(x + \hat{B}_{s+t})] = \mathbb{E}[U_{t+s}^h \varphi(x + \hat{B}_{t+s})] = \mathbb{E}[U_s^h \mathbb{E}_s[U_{t+s}^h \varphi(x + \hat{B}_{t+s})]] = \mathbb{E}[U_s^h u(t, x + \hat{B}_s)] = \hat{E}_s[u(t, x + \hat{B}_s)].
$$

By Itô’s formula, we have $U_t^h \tilde{B}_t = \int_0^t U_r^h dB_r$. So

$$
\hat{E}_s[\tilde{B}_t] = \mathbb{E}_s[U_t^h \tilde{B}_t] = \frac{1}{U_s^h} \int_0^s U_r^h dB_r = \tilde{B}_s,
$$

which implies that $\tilde{B}_t$ is a (symmetric) martingale under $\tilde{E}$. Particularly, we have

$$
\tilde{E}[\tilde{B}_1] = 0, \quad \text{and } \frac{1}{2} \tilde{E}[\langle A \tilde{B}_1, \tilde{B}_1 \rangle] = \frac{1}{2} \tilde{E}[\text{tr}[A(B)_1]], \quad \text{for } A \in \mathbb{S}_d.
$$

On the one hand, we have $\frac{1}{2} \text{tr}[A(B)_1] \leq G(A)$, and consequently

$$
\frac{1}{2} \tilde{E}[\langle A \tilde{B}_1, \tilde{B}_1 \rangle] \leq G(A).
$$

On the other hand, by the representation of the $G$-expectation (see [2]), we have

$$
\frac{1}{2} \tilde{E}[\langle A \tilde{B}_1, \tilde{B}_1 \rangle] \geq \frac{1}{2} \sup_{\gamma \in \Gamma} \mathbb{E}_P[U_t^h \text{tr}[A(\gamma)_1]] = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[A \gamma] = G(A),
$$

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where $P_{\gamma}$ is a probability on $\hat{\Omega}_{T}$ such that the canonical process $\hat{B}_{t}$ is a martingale with

$$\langle \hat{B} \rangle_t = \begin{bmatrix} \gamma & I_d \\ I_d & \gamma^{-1} \end{bmatrix} t.$$ 

Combining the above arguments, we can prove that $u$ is the viscosity solution to the $G$-heat equation.

### 4.4 Localizations and Estimates

#### 4.4.1 Localizations

To apply localization procedure, one has to show that the corresponding stopping times are quasi-continuous, which is not obvious in the $G$-expectation space. In this section, we shall prove the quasi-continuity of hitting times for processes of certain forms. The following result generalizes Theorem 4.1 in [23].

**Lemma 4.3** Let $X_t = \int_0^t Z_s \cdot dB_s + \int_0^t \eta_s ds + \int_0^t \text{tr}[\zeta_s d(B)_s]$ with $Z \in [H^1_G(0,T)]^d$ and $\eta, \zeta^{i,j} \in M^1_G(0,T)$. Assume $\int_0^t \eta_s ds + \int_0^t \text{tr}[\zeta_s d(B)_s]$ is non-decreasing and

$$\int_0^t \text{tr}[Z_s Z_s^* d(B)_s] + \int_0^t \eta_s ds + \int_0^t \text{tr}[\zeta_s d(B)_s]$$

is strictly increasing. For $a > 0$, $\tau_a := \inf\{t \geq 0 | X_t > a\} \wedge T$ is quasi-continuous.

**Proof.** Set $\tau_a := \inf\{t \geq 0 | X_t \geq a\} \wedge T$. By Lemma 3.3 in [23], it suffices to show that $[\tau_a > \tau_a]$ is a polar set. Define

$$S_a(X) = \{\omega \in \Omega_T | \text{there exists } (r,s) \in Q_T \text{ s.t. } X_t(\omega) = a \text{ for all } t \in [s,r] \},$$

where

$$Q_T = \{(r,s)| T \geq r > s \geq 0, \ r,s \text{ are rational} \}.$$ 

It is clear that

$$[\tau_a > \tau_a] \subset S_a(X) \cup \bigcup_{r \in \mathbb{Q} \cap [0,T]} [X_{r \land \tau_a} < X_{r \land \underline{\tau}_a}],$$

where $\mathbb{Q}$ denotes the totality of rational numbers. By the assumption, we know that $S_a(X)$ is a polar set. Noting that $X_{r \land \tau_a} \leq X_{r \land \underline{\tau}_a}$ and $\mathbb{E}[X_{r \land \underline{\tau}_a} - X_{r \land \tau_a}] \leq 0$, we conclude that $[\tau_a > \tau_a] \subset S_a(X)$ is also a polar set. □

**Corollary 4.4** Let $X_t = \int_0^t Z_s \cdot dB_s + \int_0^t \eta_s ds$ with $Z^{i,j} \in H^2_G(0,T)$ and $\eta^i \in M^2_G(0,T)$. Assume $\int_0^t \text{tr}[Z_s^* Z_s d(B)_s]$ is strictly increasing. Then there exists a sequence of quasi-continuous stopping times $\tau_n$ such that $\tau_n \uparrow T$ and $(X_{t \wedge \tau_n})_{t \in [0,T]}$ is bounded.

**Proof.** Applying Itô’s formula to $|X_t|^2$, we have

$$|X_t|^2 = \int_0^t 2X_s Z_s dB_s + \int_0^t 2X_s \cdot \eta_s ds + \int_0^t \text{tr}[Z_s^* Z_s d(B)_s].$$

Set $Y_t := \int_0^t 2X_s Z_s dB_s + \int_0^t 2|X_s| \cdot \eta_s ds + \int_0^t \text{tr}[Z_s^* Z_s d(B)_s]$ and $\tau_n := \inf\{t \geq 0 | Y_t > n\} \wedge T$. By Lemma 3.3, we get the desired result. □
By Girsanov transformation with bounded drifts, \( \mathbb{E} \) Corollary 4.7 this, we get the desired result.

Proposition 4.5

\[
\tilde{\mathbb{E}}^n[\exp\left\{ \frac{\theta^2}{8\Lambda^2} \int_0^{\tau_n} |Z_s|^2 ds \right\}] \leq \exp\left\{ \frac{\theta}{8\Lambda^2} |x - y| \right\}.
\]

\[
\mathbb{E}[|U^n_T|^{1+\delta}] \leq \exp\left\{ \frac{\theta\sqrt{1+\delta^{-1}}}{8\Lambda^2(1+\sqrt{1+\delta^{-1}})} |x - y| \right\},
\]

where \( \delta := \frac{\theta^2}{4\Lambda^2\beta^2 + 4\Lambda\beta^2} \).

Proof. By Girsanov transformation with bounded drifts, we know that \( \tilde{B}^n_t \) is a \( G \)-Brownian motion under \( \tilde{\mathbb{E}} \). Noting that \( M_t := \int_0^{t \wedge \tau_n} h_s \cdot dB'_s - \int_0^{t \wedge \tau_n} \text{tr}[h_s h_s^* d(B'_s)] \) is a symmetric martingale under \( \tilde{\mathbb{E}}^n \), the proof is similar to that of Proposition 3.5. \( \square \)

### 4.5 Girsanov Transformation with Unbounded Drifts

Proposition 4.6 \( \tilde{B}_t = B_t - \int_0^t h_s ds \) is a \( G \)-Brownian motion under \( \tilde{\mathbb{E}} \).

Proof. Noting that

\[
|e^x - e^y| \leq e^m |x - y| + e^{-m} (1+\delta)x + e^{-m} (1+\delta)y,
\]

we have

\[
\mathbb{E}[|U^h_T - U^n_T|] \leq e^m \mathbb{E}[|u^h_T - u^n_T|] + e^{-m} \mathbb{E}[|U_T^n|^{1+\delta}] + e^{-m} \mathbb{E}[|U_T^h|^{1+\delta}],
\]

where \( u^h_t = \int_0^t h_s dB'_s - \frac{1}{2} \int_0^t \text{tr}[h_s h_s^* d(B'_s)] \). By the Monotone Convergence Theorem under \( G \)-expectation (Theorem 31, Denis, et al (2011)), we have

\[
\lim_{n \to \infty} \mathbb{E}\left[ \int_0^T |h_s - h^n_s|^2 ds \right] = 0.
\]

So we conclude that

\[
\lim_{n \to \infty} \mathbb{E}[|u^h_T - u^n_T|] = 0.
\]

First letting \( n \) go to infinity, then letting \( m \) go to infinity, we get that

\[
\mathbb{E}[|U^h_T - U^n_T|] \to 0
\]

by Proposition 3.5.

For a function \( \varphi \in C_b, Lip(\mathbb{R}^k) \) and a partition of \([0, T]\): \( 0 \leq t_1 < t_2 < \cdots < t_k \leq T \), set \( \xi = \varphi(\tilde{B}_{t_1}, \cdots, \tilde{B}_T) \) and \( \xi^n = \varphi(\tilde{B}^n_{t_1}, \cdots, \tilde{B}^n_T) \).

\[
|\mathbb{E}[U^h_T \xi] - \mathbb{E}[U^{h^n}_T \xi^n]| \leq |\mathbb{E}[U^h_T \xi] - \mathbb{E}[U^{h^n}_T \xi^n]| + |\mathbb{E}[U^{h^n}_T \xi] - \mathbb{E}[U^{h^n}_T \xi^n]| \to 0.
\]

By Girsanov transformation with bounded drifts, \( \mathbb{E}[U^{h^n}_T \xi^n] = \mathbb{E}[\varphi(B_{t_1}, \cdots, B_T)] \) for each \( n \). By this, we get the desired result. \( \square \)

Corollary 4.7 \( \mathbb{E}[\varphi(X^T_T)] = \tilde{\mathbb{E}}[\varphi(Y^T_T)] \) for any \( \varphi \in C_b(\mathbb{R}^d) \).
4.6 Proof to the Gradient Estimates

Proof to Theorem 4.1. (4.8-4.9) shows that
\[
\int_0^t \frac{|Z_r|^2}{\xi^2_r} dr \leq \frac{|x-y|^2}{2\theta^2\xi_0} + \int_0^t \frac{1}{\theta^2\xi_r} \hat{\sigma}(r)^* Z_r \cdot dB_r,
\]
\[
\int_0^t \frac{|Z_r|^2}{\xi^2_r} dr \leq \frac{|x-y|^2}{2\theta^2\xi_0} + \int_0^t \frac{1}{\theta^2\xi_r} \hat{\sigma}(r)^* Z_r \cdot d\tilde{B}_r,
\]
where \( \theta' = \theta + \lambda - \lambda^2 \Lambda^{-1} \). So
\[
\mathbb{E}\left[\int_0^T \text{tr}[h_s h_s^* d(B')_s]\right] \leq \frac{\beta^2}{2\lambda^2 \theta^2 \xi_0} |x-y|^2,
\]
\[
\tilde{\mathbb{E}}\left[\int_0^T \text{tr}[h_s h_s^* d(B')_s]\right] \leq \frac{\beta^2}{2\lambda^2 \theta^2 \xi_0} |x-y|^2.
\]
By Corollary 4.7,
\[
||\mathbb{E}[\varphi(X^y_T)] - E[\varphi(X^y_T)]|| = ||\tilde{\mathbb{E}}[\varphi(X^y_T)] - E[\varphi(X^y_T)]||
\leq \|\varphi\|_{\infty} \mathbb{E}[|U^h_T - 1|]
\leq \|\varphi\|_{\infty} (\tilde{\mathbb{E}}[|u^h_T|] + \mathbb{E}[|u^h_T|])
\leq 2\|\varphi\|_{\infty} \left( \frac{\beta^2}{4\lambda^2 \theta \xi_0} |x-y|^2 + \sqrt{\frac{\beta^2}{2\lambda^2 \theta^2 \xi_0}} |x-y| \right).
\]
So, by Lemma 3.8
\[
||\mathbb{E}[\varphi(X^y_T)] - E[\varphi(X^y_T)]|| \leq \sqrt{\frac{2\beta^2}{\lambda^2 \theta \xi_0}} \|\varphi\|_{\infty} |x-y|.
\]
Taking \( \theta = \frac{\lambda^2 \Lambda^{-1}}{2} \), we get
\[
||\mathbb{E}[\varphi(X^y_T)] - E[\varphi(X^y_T)]|| \leq \frac{2\Lambda^2}{\lambda^3 \Lambda \sqrt{(1-e^{-LT})/L}} |x-y|.
\]
\[\square\]

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