Energetic Variation with the Anderson Hamiltonian

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Abstract

We study the variation problem associated with the Anderson Hamiltonian in 2-dimensional torus in the paraccontrolled distribution framework. We obtain the existence of minimizers by the direct method in the calculus of variations, and show that the Euler-Lagrange equation of the energy functional is an elliptic singular stochastic partial differential equation with the Anderson Hamiltonian. We also establish the $L^2$ estimates and Schauder estimates for the minimizer as weak solution of the elliptic singular stochastic partial differential equation.

Keywords: singular SPDEs, energetic variation, paraccontrolled distribution, Anderson Hamiltonian, regularity.

1 Introduction

The Anderson Hamiltonian was originally introduced by Anderson [1] as a Schrödinger operator with a random potential. In this famous work, Anderson showed that the lower part of the electron spectrum gives rise to localized eigenfunctions in a Gaussian potential. In condensed matter physics, this phenomenon is called Anderson localization.

Consider the Anderson Hamiltonian in 2-dimensional torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$

\[
\mathcal{H} u := -\Delta u + \mu u - \xi \circ u,
\]

where $\mu > 0$, $\Delta$ denote the Laplacian with periodic boundary conditions, $u : T^2 \to \mathbb{R}$, $\xi$ is a spatial white noise on the $T^2$, and the symbol $\circ$ stands for the wick product. We study the variation problem of the energy functional with the Anderson Hamiltonian $\mathcal{H}$

\[
E(u) := \frac{1}{2} \langle u, \mathcal{H} u \rangle - \int_{T^2} F(u(x))dx,
\]

where the nonlinear term $F(s) = \int_0^s f(t)dt$, with a continuous function $f$ from $\mathbb{R}$ to $\mathbb{R}$. The corresponding Euler-Lagrange equation of the energy functional $E(u)$ is formally given by the following elliptic singular stochastic partial differential equation (SPDE):

\[
- (\Delta - \mu) u = f(u) + \xi \circ u.
\]
Since spatial white noise is very rough when dimension $d \geq 2$, the stochastic partial differential equation associated with the Anderson Hamiltonian is singular. However, how to rigorously understand singular SPDEs mathematically had been an open problem for a long time. In recent years, some new mathematical theories, such as regularity structures by Harier [11] or paracontrolled distributions by Gubinelli, Imkeller and Perkowski [8], have been developed to study singular SPDEs. Both regularity structures and paracontrolled distribution frameworks are developed from the theory of controlled rough paths, and allow a pathwise description of the singular SPDEs. Comparing with regularity structures, the paracontrolled distribution relies on harmonic analysis tools, including Littlewood-Paley decomposition, Besov space, paraproduct calculus. So it is natural to generalize some classical PDE methods to study singular SPDEs in the paracontrolled distribution framework.

The construction of the Anderson Hamiltonian on $\mathbb{T}^2$ by paracontrolled distribution were studied by Allez and Chouk [2]. By using the tools of regularity structures, Labbé [14] constructed the Anderson Hamiltonian in $d \leq 3$ with both periodic and Dirichlet boundary conditions. Asymptotics of the eigenvalues of the Anderson Hamiltonian is considered by Chouk and van Zuijlen [3]. In [10], Gubinelli, Ugurcan and Zachhuber extended the construction of the Anderson Hamiltonian on $\mathbb{T}^2$ by Allez and Chouk [2] for $d = 3$, and used these results to study the semilinear Schrödinger equations and wave equations for the Anderson Hamiltonian in two and three dimensions on $\mathbb{T}^2$ and $\mathbb{T}^3$. For the parabolic Anderson model, the discrete parabolic Anderson model has been well understood during the past decades, has seen in the surveys [6, 13], and references therein. The well-posedness of the continuous parabolic Anderson model equation was also considered in [8, 11, 12] by different ways, including regularity structures, paracontrolled distribution, and the transformation method with a elaborate renormalisation procedure.

In this present paper, we study the variation problem associated with the Anderson Hamiltonian in 2-dimensional torus $\mathbb{T}^2$. It can be viewed as a continuation of our work for the parabolic case [17]. We hope to build a bridge between the variation problem and the singular SPDE in the paracontrolled distribution framework. Since the Anderson Hamiltonian is a self-adjoint operator in a function space defined by paracontrolled distribution, we study the corresponding Gelfand triple, and define the energy functional via the quadratic form of the Anderson Hamiltonian. Moreover, we show that the Euler-Lagrange equation of the energy functional $E(u)$ is an elliptic singular SPDE (1.3). So the minimizer of the energy functional $E(u)$ is an weak solution of the elliptic singular SPDE (1.3). We assume that the nonlinear term $f(u)$ satisfies the following assumption: For every $s \in \mathbb{R}$:

$$F(s) = \int_0^s f(t)dt \geq C_0 - C_1 |s|^k, \quad k > 2,$$

(1.4)

$$|f'(s)| \leq l + |s|^{k-2},$$

(1.5)

where $C_0, C_1, l > 0$ are positive constants. By the direct method of calculus of variation, we prove the existence of minimizers as the following Theorem.

**Theorem 1.1.** Under the assumption (1.4), there exists at least one minimizer $u \in \mathcal{D}_\alpha$ for the energy functional $E(u)$. Moreover, the minimizer $u$ is a weak solution of the elliptic singular SPDE (1.3).

The definition of spaces $\mathcal{D}_\alpha$ and $\mathcal{D}_\alpha$ are given in Definition 3.1. For the explicit definition of the energy functional $E(u)$ we refer to Definition 3.2. We refer Subsection 3.3 for the proof of Theorem 1.1.

Since the minimizer $u$ is a weak solution of the elliptic singular SPDE, we split the original
singular SPDE in the following elliptic system:

\[
\begin{aligned}
- (\Delta - \mu)\phi &= \Phi(u), \\
- (\Delta - \mu)u^\vartheta &= f(u) + \Psi(u),
\end{aligned}
\]  

(1.6)

where \( \Phi \) contains all of irregular but linear terms, and \( \Psi \) contains all the regular terms and the nonlinear terms. Then we establish \( L^2 \) estimates and Schauder estimates for the minimizer in the following theorem.

**Theorem 1.2.** (See Theorem 4.1 and 4.2) Under the assumption (1.3), the minimizer \( u \in \mathcal{D}_\vartheta^{\alpha,2} \). In addition, if the nonlinear term satisfies the assumption (1.5), then the minimizer \( u \in \mathcal{C}^\alpha \), with reminders \( R(u) \in \mathcal{C}^{2\alpha} \) and \( u^\vartheta \in \mathcal{C}^{3\alpha} \).

Throughout the paper, we use the notation \( a \lesssim b \) if there exists a constant \( C > 0 \), independent of the variables under consideration, so that \( a \leq C \cdot b \), and we denote \( a \asymp b \) if \( a \lesssim b \) and \( b \lesssim a \). The Fourier transform on the torus \( \mathbb{T}^d \) is defined with \( \hat{u}(k) := \mathcal{F}_{\mathbb{T}^d} u(k) = \sum_{k \in \mathbb{Z}^d} e^{2\pi i k \cdot x} u(x) \), and the inverse Fourier transform on the torus \( \mathbb{T}^d \) is given by \( \mathcal{F}_{\mathbb{T}^d}^{-1} e^{-2\pi i k \cdot x} \hat{u}(k) = \sum_{k \in \mathbb{Z}^d} \hat{u}(k) \).

We denote the space of Schwartz functions on \( \mathbb{T}^d \) by \( \mathcal{S}(\mathbb{T}^d) \). The space of tempered distributions on \( \mathbb{T}^d \) is denoted by \( \mathcal{S}'(\mathbb{T}^d) \). We also denote \( \mathcal{L} := -\Delta + \mu \).

This paper is organized as follows: In Section 2, we revisit some basic notation and estimates of the singular SPDEs. In Section 3, we define the energy functional \( E(u) \) by the quadratic form of the Anderson Hamiltonian on its domain \( \mathcal{D}_\vartheta^{\alpha,1} \), and show that the energy functional is a \( C^1 \) map from \( \mathcal{D}_\vartheta^{\alpha,1} \) to \( \mathbb{R} \), and the singular SPDE (1.3) is the corresponding Euler-Lagrange equation of energy functional \( E(u) \). We also obtain the existence of minimizer as a weak solution to \( \mathcal{L}u = 0 \) and show that the energy functional is coercive. In Section 4, we establish the \( L^2 \) estimates and Schauder estimates for the minimizer in the calculus of variations. In Section 5, we decomposed the singular SPDE (1.3) into a simpler elliptic system, and establish the \( L^2 \) estimates and Schauder estimates for the minimizer \( u \). This paper ends with some summary and discussion in Section 6.

## 2 Preliminaries

### 2.1 Besov space and Bony’s paraproduct

In this subsection, we introduce some basic notations and useful estimates about Littlewood-Paley decomposition, Besov space and Bony’s paraproduct. For more details, we refer to [3, 8, 9].

Littlewood-Paley decomposition can describe the regularity of (general) functions via the decomposition of a (general) function into a series of smooth functions with different frequencies. In order to do this, we introduce the following dyadic partition.

Let \( \varphi : \mathbb{R}^d \rightarrow [0, 1] \) be a smooth radial cut-off function so that

\[
\varphi(x) = \begin{cases} 
1, & |x| \leq 1 \\
\text{smooth}, & 1 < |x| < 2 \\
0, & |x| \geq 2.
\end{cases}
\]

Denote \( \varrho(x) = \varphi(x) - \varphi(2^{-1}x) \) and \( \chi(x) = 1 - \sum_{j \geq 0} \varrho(2^{-j}x) \). Then \( \chi, \varrho \in C_c^\infty(\mathbb{R}^d) \) are nonnegative radial functions, so that

1. \( \text{supp}(\chi) \subset B_1(0) \) and \( \text{supp}(\varrho) \subset \{ x \in \mathbb{R}^d : \frac{1}{2} \leq |x| \leq 2 \} \);
2. \( \chi(x) + \sum_{j \geq 0} \varrho(2^{-j}x) = 1, \quad x \in \mathbb{R}^n \).
3. \( \text{supp}(\chi) \cap \text{supp}(\varrho(2^{-j}x)) = \emptyset \) for \( j \geq 1 \) and \( \text{supp}(\varrho(2^{-i}x)) \cap \text{supp}(\varrho(2^{-j}x)) = \emptyset \) for \( |i-j| \geq 2 \).

**Definition 2.1.** For \( u \in \mathcal{S}'(\mathbb{T}^d) \) and \( j \geq -1 \), the Littlewood-Paley blocks of \( u \) are defined as

\[
\Delta_j u = \mathcal{F}_{\mathbb{T}^d}^{-1}(\varrho_j \mathcal{F}_{\mathbb{T}^d} u),
\]

where \( \varrho_{-1} = \chi \) and \( \varrho_j = \varrho(2^{-j} \cdot) \) for \( j \geq 0 \).

**Definition 2.2.** For \( \alpha \in \mathbb{R}, p, q \in [1, \infty] \), we define

\[
B^\alpha_{p,q}(\mathbb{T}^d) = \left\{ u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} = \left( \sum_{j \geq -1} (2^{j\alpha} \|\Delta_j u\|_{L^p(\mathbb{T}^d)})^q \right)^{1/q} < \infty \right\}.
\]

For \( \alpha \in \mathbb{R} \), the Hölder-Besov space on \( \mathbb{T}^d \) is denoted by \( \mathfrak{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{T}^d) \). We remark that if \( \alpha \in (0, \infty) \setminus \mathbb{N} \), then the Hölder-Besov space \( \mathfrak{C}^\alpha \) is equal to the Hölder space \( C^\alpha(\mathbb{T}^d) \). The Sobolev space \( H^\alpha \) is the same as the Besov space \( B^\alpha_{2,2}(\mathbb{T}^d) \).

We need the following Bernstein inequality in \( L^2 \) estimates.

**Lemma 2.1.** Let \( \mathcal{B} \) be a unit ball, \( n \in \mathbb{N}_{0} \), and \( 1 \leq p \leq q \leq \infty \). Then for every \( \lambda > 0 \) and \( u \in L^p \) with \( \text{supp}(\mathcal{F} u) \subset \lambda \mathcal{B} \), we have

\[
\max_{\mu \in \mathbb{N}^n; |\mu| = n} \|\partial^{\mu}_x u\|_{L^q} \lesssim C_{n,p,q,\mathcal{B}} \lambda^{n+\frac{d}{4}} \|u\|_{L^p}.
\]

The Besov embedding theorem is useful in regularity estimates.

**Lemma 2.2.** Let \( 1 \leq p_1 \leq p_2 \leq \infty \), \( 1 \leq q_1 \leq q_2 \leq \infty \), and \( \alpha \in \mathbb{R} \). Then we have

\[
B^\alpha_{p_1,q_1}(\mathbb{T}^d) \hookrightarrow B^\alpha_{p_2,q_2}(\mathbb{T}^d).
\]

Now we introduce localization operators \( \mathcal{U}_> \), \( \mathcal{U}_< \) from [9] Subsection 2.3. By a delicate high-low frequency decomposition, these localizers allow to decompose a distribution \( f \) into a sum of two components: the higher regular component \( \mathcal{U}_> f \), and the less regular component \( \mathcal{U}_< f \). More precisely, let \((w_k)_{k \geq -1}\) be a smooth partition of unity on \( \mathbb{T}^d \) with \( \sum_{k \geq -1} w_k = 1 \). Let \((L_k)_{k \geq -1} \subset [-1, \infty)\) be a sequence. For every \( f \in \mathcal{S}'(\mathbb{T}^d) \), we define the following localization operators

\[
\mathcal{U}_> f = \sum_{k} w_k \Delta_{>L_k} f, \quad \mathcal{U}_< f = \sum_{k} w_k \Delta_{\leq L_k} f,
\]

where \( \Delta_{>L_k} = \sum_{l; l > L_k} \Delta_l \) and \( \Delta_{\leq L_k} = \sum_{j; j \leq L_k} \Delta_j \).

**Lemma 2.3.** Let \( L > 0 \). There exists a sequence \((L_k)_{k \geq -1}\) so that for every \( \alpha, \delta, \gamma > 0 \),

\[
\|\mathcal{U}_> f\|_{\mathfrak{C}^{-\gamma-\delta}} \lesssim_{\alpha, \delta} 2^{-\delta L} \|f\|_{\mathfrak{C}^{-\alpha}}, \quad \|\mathcal{U}_< f\|_{\mathfrak{C}^{-\alpha+\gamma}} \lesssim_{\alpha, \delta} 2^{\gamma L} \|f\|_{\mathfrak{C}^{-\alpha}}.
\]

Now we introduce the Bony’s paraproduct. Let \( f \) and \( g \) be tempered distributions in \( \mathcal{S}'(\mathbb{T}^d) \). By Littlewood-Paley blocks, the product \( fg \) can be (formally) decomposed as

\[
f g = \sum_{j \geq -1} \sum_{i \geq -1} \Delta_i f \Delta_j g = f < g + f \circ g + f \succ g,
\]

where

\[
f < g + f \succ g = \sum_{j \geq -1} \sum_{i \geq -1} \Delta_i f \Delta_j g \quad \text{and} \quad f \circ g = \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g.
\]

We have following paraproduct estimates in the Bony’s paraproduct (See [8] Lemma 2.1 and [10] Proposition A.1).
Lemma 2.4. For every $\beta \in \mathbb{R}$, we have
\[
\| f \prec g \|_{\mathcal{E}^\beta} \lesssim \| f \|_{L^\infty} \| g \|_{\mathcal{E}^\beta},
\]
\[
\| f \prec g \|_{H^\beta} \lesssim \| f \|_{L^2} \| g \|_{\mathcal{E}^{\beta+\kappa}} + \| f \|_{L^\infty} \| g \|_{H^\beta} \quad \text{for all } \kappa > 0.
\]
If $\beta \in \mathbb{R}$, $\alpha < 0$, we have
\[
\| f \prec g \|_{\mathcal{E}^{\alpha+\beta}} \lesssim \| f \|_{\mathcal{E}^\alpha} \| g \|_{\mathcal{E}^\beta},
\]
\[
\| f \prec g \|_{H^{\alpha+\beta}} \lesssim \| f \|_{H^\alpha} \| g \|_{\mathcal{E}^{\beta+\kappa}} + \| f \|_{\mathcal{E}^\alpha} \| g \|_{H^\beta} \quad \text{for all } \kappa > 0.
\]
Moreover, if $\alpha + \beta > 0$, then
\[
\| f \circ g \|_{\mathcal{E}^{\alpha+\beta}} \lesssim \| f \|_{\mathcal{E}^\alpha} \| g \|_{\mathcal{E}^\beta},
\]
\[
\| f \circ g \|_{H^{\alpha+\beta}} \lesssim \| f \|_{H^\alpha} \| g \|_{\mathcal{E}^\beta}.
\]

The following commutator estimate is also crucial in paracontrolled distribution (See [8 Lemma 2.4] and [10 Proposition A.2])

Lemma 2.5. Assume that $\alpha \in (0, 1)$ and $\beta, \gamma \in \mathbb{R}$ are such that $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$. Then for $f, g, h \in C^\infty(\mathbb{T}^d)$, the trilinear operator
\[
C(f, g, h) = (f \prec g) \circ h - f(g \circ h)
\]
has the following estimate
\[
\| C(f, g, h) \|_{\mathcal{E}^{\alpha+\beta+\gamma}} \lesssim \| f \|_{\mathcal{E}^\alpha} \| g \|_{\mathcal{E}^\beta} \| h \|_{\mathcal{E}^\gamma}.
\]
Thus $C$ can be uniquely extended to a bounded trilinear operator from $\mathcal{E}^\alpha \times \mathcal{E}^\beta \times \mathcal{E}^\gamma$ to $\mathcal{E}^{\alpha+\beta+\gamma}$.

For $H^\alpha$ space, we also have
\[
\| C(f, g, h) \|_{H^{\alpha+\beta+\gamma}} \lesssim \| f \|_{H^\alpha} \| g \|_{H^\beta} \| h \|_{\mathcal{E}^\gamma}.
\]
It implies that $C$ can be uniquely extended to a bounded trilinear operator from $H^\alpha \times H^\beta \times \mathcal{E}^\gamma$ to $H^{\alpha+\beta+\gamma}$.

For $f, g, h \in C^\infty(\mathbb{T}^d)$, we define the trilinear operator
\[
D(f, g, h) = \langle f, h \circ g \rangle - \langle f \prec g, h \rangle.
\]
(2.1)
We have the following estimate from [10 Lemma A.6].

Lemma 2.6. Let $\alpha \in (0, 1)$, $\beta, \gamma \in \mathbb{R}$ such that $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$. Then we have
\[
| D(f, g, h) | \lesssim \| f \|_{H^\alpha} \| g \|_{H^\beta} \| h \|_{\mathcal{E}^\gamma}.
\]
Thus $D$ can be uniquely extended to a bounded trilinear operator from $H^\alpha \times H^\beta \times \mathcal{E}^\gamma$ to $\mathbb{R}$.

The following estimate from [2 Proposition A.2] is useful in this paper.

Lemma 2.7. Let $f \in H^\alpha$, $g \in \mathcal{E}^\beta$ with $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$. Then
\[
\| \mathcal{L}(f \prec g) - f \prec (\mathcal{L} g) \|_{H^{\alpha+\beta+2}} \lesssim \| f \|_{H^\alpha} \| g \|_{\mathcal{E}^\beta}.
\]

We also need the following interpolations result for Besov space.

Lemma 2.8. Let $\eta$ be time weights, $\gamma > 0$, $\theta \geq 0$, and $\psi \in C_\eta \mathcal{E}^\gamma$. Then for any $\alpha \in [0, \gamma]$, we have
\[
\| \psi \|_{\mathcal{E}^\alpha} \lesssim \| \psi \|_{L^\infty} \| \psi \|_{\mathcal{E}^\gamma}^{1-\alpha/\gamma}.
\]
(2.2)
Lemma 2.9. Let $\beta \in (0, 1)$ and $\psi \in H^\beta$. Then for arbitrary $\delta > 0$, we have
\[ \|\psi\|_{H^\beta} \leq \delta \|\nabla \psi\|_{L^2}^2 + C_\delta \|\psi\|_{L^2}^2. \] (2.3)

**Proof:** Since $\|\psi\|_{H^\beta} \simeq \|\psi\|_{B_{\infty,2}^\beta}$, by Bernstein inequality (Lemma 2.1), Hölder inequality and weighted Young inequality, we have
\[ \|\psi\|_{H^\beta}^2 = \sum_{i \geq -1} 2^{2\beta k} \|\Delta_i \psi\|_{L^2}^2 \]
\[ = \sum_{i \geq -1} 2^{2\beta k} \|\Delta_i \psi\|_{L^2}^2 \|\Delta_i \psi\|_{L^2}^{2(1-\beta)} \]
\[ \leq \left[ \sum_{i \geq -1} 2^{2k} \|\Delta_i \psi\|_{L^2}^2 \right]^{2\beta} \left[ \sum_{i \geq -1} \|\Delta_i \psi\|_{L^2}^2 \right]^{2(1-\beta)} \]
\[ \leq \|\nabla \psi\|_{L^2}^\beta \|\psi\|_{L^2}^{1-\beta} \]
\[ \leq \delta \|\nabla \psi\|_{L^2}^2 + C_\delta \|\psi\|_{L^2}^2. \] (2.4)

This completes the proof. ■

### 2.2 Renormalization and paracontrolled distributions

The spatial white noise $\xi$ on $\mathbb{T}^2$ is a centered Gaussian process with value in $\mathcal{S}'(\mathbb{T}^2)$ so that for all $f, g \in \mathcal{S}(\mathbb{T}^2)$, we have $\mathbb{E}[\xi(f)\xi(g)] = \langle f, g \rangle_{L^2(\mathbb{T}^2)}$. Let $(\hat{\xi}(k))_{k \in \mathbb{Z}^2}$ be a sequence of i.i.d. centered complex Gaussian random variables with covariance
\[ \mathbb{E}[(\hat{\xi}(k)\hat{\xi}(l))] = \delta(k-l), \]
and $\hat{\xi}(k) = \bar{\hat{\xi}}(-k)$. Then the spatial white noise $\xi$ on $\mathbb{T}^2$ can be defined as the sum of random series
\[ \xi(x) = \sum_{k \in \mathbb{Z}^2} \hat{\xi}(k)e^{2\pi ik \cdot x}. \]

Moreover, the spatial white noise $\xi$ take value in $\mathcal{C}^{-1,\kappa}$ for all $\kappa > 0$. Since $\xi$ is only a distribution, the product $u\xi$ is ill-defined in classic sense. How to let singular term $u\xi$ make sense is a main challenge in studying the Anderson Hamiltonian. We set a smooth approximation $\xi_\varepsilon$ of $\xi$. More precisely, we set $\varphi: \mathbb{T}^2 \to \mathbb{R}^+$ be a smooth function with $\int_{\mathbb{T}^2} \varphi dt = 1$, and define $\xi^\varepsilon = e^{-\varepsilon \varphi(\cdot) * \xi}$ for $\varepsilon > 0$ as the mollification of $\xi$. Now we take
\[ \vartheta = (-\Delta + \mu)^{-1}\xi = \int_0^\infty e^{t(\Delta - \mu)} \xi dt, \]
where \( (e^{t(\Delta - \mu)})_{t \geq 0} \) denotes the semigroup generated by \( \Delta - \mu \). Then \( \|\vartheta\|_{1-\kappa} \lesssim \|\xi\|_{-1-\kappa} \). In order to obtain a well-defined area \( \vartheta \circ \xi \), we have to renormalize the product by “subtracting an infinite constant” as following arguments (See [8, Lemma 5.8]).

**Lemma 2.10.** If \( \vartheta = (-\Delta + \mu)^{-1} \xi \), then

\[
\lim_{\epsilon \to 0} E[\|\vartheta \diamond \xi - (\vartheta \circ \xi)\|_{p-2c}] = 0
\]

for all \( p \geq 1 \) and \( \kappa > 0 \) with the renormalization constant

\[
C_\epsilon = E(\vartheta \circ \xi) = \sum_{k \in \mathbb{Z}^2} \frac{|\mathcal{F}_\vartheta \varphi(\epsilon k)|^2}{|k|^2 + \mu}.
\]

By Bony’s paraproducts, we define the following paracontrolled ansatz:

**Definition 2.3.** Let \( \alpha \in (2/3,1) \) and \( \beta \in (0,\alpha] \) be such that \( 2\alpha + \beta > 2 \). We say a pair \( (u,u') \in H^\alpha \times H^\beta \) is called paracontrolled by \( \vartheta \) if

\[
\|u - u' \prec \vartheta + R(u)\|_{H^\gamma} \leq \frac{1}{2} \|u\|_{H^\gamma}.
\]

### 3 Energy functional with the Anderson Hamiltonian

#### 3.1 The Anderson Hamiltonian

In order to define domains of the Anderson Hamiltonian and the energy functional \( E(u) \), we introduce following function spaces by paracontrolled distributions.

**Definition 3.1.** Let \( 0 \leq \alpha < 1 - \kappa, \beta \in \{1,2\} \). We define

\[
\mathcal{D}^{\alpha,\beta}_\vartheta := \{ u \in H^\alpha : \text{there exists } u^\vartheta := u - u \prec \vartheta - R(u) \in H^\beta \},
\]

where \( R(u) \) is given by

\[
R(u) = L^{-1}( -u \prec \mathcal{H}_\vartheta \xi + u \succ \mathcal{H}_\vartheta \xi + u \succ \mathcal{H}_\vartheta (\vartheta \circ \xi) + u \prec \mathcal{H}_\vartheta (\vartheta \circ \xi) - (L^* u) \prec \vartheta - \nabla u \prec \nabla \vartheta).
\]

The space \( \mathcal{D}^{\alpha,\beta}_\vartheta \) is equipped with the inner product

\[
\langle u,v \rangle_{\mathcal{D}^{\alpha,\beta}_\vartheta} := \langle u,v \rangle_{H^\gamma} + \langle u^\vartheta,v^\vartheta \rangle_{H^\beta}, \quad u,v \in \mathcal{D}^{\alpha,\beta}_\vartheta.
\]

We recall the following theorem for the Anderson Hamiltonian \( \mathcal{H} \) from [2] Theorem 1.1] or [10] Lemma 2.29).

**Theorem 3.1.** The Anderson Hamiltonian \( \mathcal{H} \) is a self-adjoint operator from \( \mathcal{D}^{\alpha,\beta}_\vartheta \) to \( L^2 \).

We remark that since the regularity of functions depends only on its high-frequency components, the domain in Definition 3.1 is equivalent to the domain of the Anderson Hamiltonian \( \mathcal{H} \) in [2] or [10] even though we use different high-low frequency decomposition. The following estimates for the Anderson Hamiltonian \( \mathcal{H} \) in this subsection is essential in this paper.

**Lemma 3.2.** Let \( \alpha \in (2/3,1), \kappa \in (0,1-\alpha) \), and \( \gamma \in [0,\alpha] \). Then for every \( u \in H^\gamma \), we have

\[
\|u \prec \vartheta + R(u)\|_{H^\gamma} \leq \frac{1}{2} \|u\|_{H^\gamma}.
\]
Proof: Note that the term \( u \times \partial + R(u) \) satisfies the following equation
\[
\mathcal{L}(u - u \times \partial - R(u)) = \Phi(u) = u \times \mathcal{U}_x + u \times \mathcal{U}_x + u \times \mathcal{U}_x(\partial \circ \xi) + u \times \mathcal{U}_x(\partial \circ \xi). \quad (3.5)
\]
By Lemma 2.3, we employ the Localization operators \( \mathcal{U}_x \) and \( \mathcal{U}_x \) with the parameter \( L \) such that
\[
\| \mathcal{U}_x \xi \|_{C^{-2\gamma + \delta}} \lesssim 2^{-\left(1 - \kappa - \gamma - \delta\right)L} \| \xi \|_{C^{-1 - \kappa}},
\]
where \( \delta \in (0, 1 - \kappa - \alpha) \). Then by Bony’s paraproduct estimate, we have
\[
\| u \times \mathcal{U}_x \xi \|_{C^{-2\gamma + \delta}} + \| u \times \mathcal{U}_x(\partial \circ \xi) \|_{C^{-3\gamma + \delta}} \lesssim \| \mathcal{U}_x \xi \|_{C^{-2\gamma + \delta}} \| u \|_{H^\gamma} \lesssim 2^{-\left(1 - \kappa - \gamma - \delta\right)L} \| \xi \|_{C^{-1 - \kappa}} \| u \|_{H^\gamma}. \quad (3.6)
\]
Similarly, we employ the Localization operators \( \mathcal{U}_x \) and \( \mathcal{U}_x \) with the parameter \( K \) such that
\[
\| \mathcal{U}_x(\partial \circ \xi) \|_{C^{-2\gamma + \delta}} \lesssim 2^{-\left(2 - 2\kappa - \gamma - \delta\right)K} \| \partial \circ \xi \|_{C^{-2\kappa}}.
\]
Then
\[
\| u \times \mathcal{U}_x(\partial \circ \xi) \|_{H^{2\gamma}} + \| u \times \mathcal{U}_x(\partial \circ \xi) \|_{H^{3\gamma}} \lesssim \| \mathcal{U}_x(\partial \circ \xi) \|_{C^{-2\gamma + \delta}} \| u \|_{H^\gamma} \lesssim 2^{-\left(2 - 2\kappa - \gamma - \delta\right)K} \| \partial \circ \xi \|_{C^{-2\gamma}} \| u \|_{H^\gamma}. \quad (3.7)
\]
Note that the stochastic terms \( \xi \) and \( \partial \circ \xi \) can be constructed such that
\[
\| \xi \|_{C^{-1 - \kappa}} \lesssim 1, \quad \| \partial \circ \xi \|_{C^{-2\kappa}} \lesssim 1.
\]
Now we choose \( L, K > 1 \), such that
\[
\| u \times \partial + R(u) \|_{H^\gamma} \lesssim \| \Phi(u) \|_{H^{2\gamma}} \lesssim 2^{-\left(1 - \kappa - \gamma - \delta\right)L} + 2^{-\left(2 - 2\kappa - \gamma - \delta\right)K} \| u \|_{H^\gamma} \leq \frac{1}{2} \| u \|_{H^\gamma}.
\]
The proof is completed.

Similar to [2, 10], we introduce the \( \Gamma \) map and \( \Gamma_x \) map as follows.
\[
\Gamma f = f + (\Gamma_x f) \times \partial + R(\Gamma_x f), \quad f \in H^\gamma. \quad (3.8)
\]
The renormalization Lemma 2.10 implies that \( (\xi_x, \partial \circ \xi_x) \rightarrow (\xi, \partial \circ \xi) \) in \( C^{-1 - \kappa} \times C^{-2\kappa} \) as \( \epsilon \rightarrow 0 \). We can also define the bounded approximation linear operator \( \Gamma_x \) by
\[
\Gamma_x f = f + (\Gamma_x f) \times \partial_x + R_x(\Gamma_x f), \quad f \in H^\gamma, \quad (3.9)
\]
where \( R_x \) is given by
\[
R_x(u) = \mathcal{L}^{-1}(-u \times \mathcal{U}_x + u \times \mathcal{U}_x + u \times \mathcal{U}_x(\partial_x \circ \xi_x) + u \times \mathcal{U}_x(\partial_x \circ \xi_x) - (\mathcal{L} u) \times \partial_x - \nabla u \times \nabla \partial_x). \quad (3.10)
\]

Lemma 3.3. Let \( 0 \leq \gamma \leq \alpha \). Then \( \Gamma \) and \( \Gamma_x \) are injective bounded linear operators from \( H^\gamma \) to \( H^\gamma \). For every \( f \in H^\gamma \), we have
\[
\| \Gamma f \|_{H^\gamma} \leq 2 \| f \|_{H^\gamma}, \quad \| \Gamma_x f \|_{H^\gamma} \leq 2 \| f \|_{H^\gamma}. \quad (3.11)
\]
Moreover, \( \Gamma \) can be approximated by \( \Gamma_x \) in the following sense
\[
\lim_{\epsilon \rightarrow 0} \| \Gamma - \Gamma_x \|_{L(H^\gamma, H^\gamma)} = 0. \quad (3.12)
\]
**Proof:** By Lemma 3.2, for every \( f \in H^\gamma \), we have

\[
\|\Gamma f\|_{H^\gamma} \leq \|f\|_{H^\gamma} + \|\Gamma f\|_{H^\gamma} + R(\Gamma f)\|f\|_{H^\gamma} + \|\Gamma f\|_{H^\gamma} \leq \|f\|_{H^\gamma} + \frac{1}{2}\|\Gamma f\|_{H^\gamma}.
\]

Thus \( \Gamma \) is an injective bounded linear operator from \( H^\gamma \) to \( H^\gamma \). For every \( f \in H^\gamma \), we have

\[
\|f\|_{H^\gamma} = \|\Gamma f\|_{H^\gamma} \leq 2\|f\|_{H^\gamma}.
\]

The same estimates in Lemma 3.2, for every \( f \in H^\gamma \) we have

\[
\|f\|_{H^\gamma} = \|\Gamma f\|_{H^\gamma} \leq 2\|f\|_{H^\gamma}.
\]

Moreover, by Lemma 3.2, it follows that

\[
\|\Gamma f - \Gamma f\|_{H^\gamma} = \|\Gamma f - \Gamma f\|_{H^\gamma} \leq \|f - f\|_{H^\gamma} + \|f - f\|_{H^\gamma} + \|f - f\|_{H^\gamma} + \|f - f\|_{H^\gamma} \leq \frac{1}{2}\|\Gamma f - \Gamma f\|_{H^\gamma} + C'\|\xi - \xi\|_{\mathcal{H}^\gamma} + \|\xi - \xi\|_{\mathcal{H}^\gamma} + \|\xi - \xi\|_{\mathcal{H}^\gamma} + \|\xi - \xi\|_{\mathcal{H}^\gamma},
\]

for some constant \( C' > 0 \). Since \( \lim_{\epsilon \to 0} \|\xi - \xi\|_{\mathcal{H}^\gamma} = 0 \), \( \lim_{\epsilon \to 0} \|\xi - \xi\|_{\mathcal{H}^\gamma} = 0 \), the map \( \Gamma \) can be approximated by \( \Gamma_\epsilon \) as

\[
\lim_{\epsilon \to 0} \|\Gamma - \Gamma_\epsilon\|_{L(H^\gamma, H^\gamma)} = 0.
\]

This completes the proof. \( \blacksquare \)

Lemma 3.4 implies that for every \( u^\beta \in H^\gamma \), we can uniquely determine

\[
u = \Gamma u^\beta = u - \varphi + \mathcal{R}(u) + u^\beta,
\]

so that

\[
\|u\|_{H^\gamma} = \|\Gamma u\|_{H^\gamma} \leq 2\|u^\beta\|_{H^\gamma}.
\]

Using linear maps \( \Gamma \) and \( \Gamma_\epsilon \), we have the following result for space \( \mathcal{D}^{\alpha,2} \).

**Lemma 3.4.** Let \( 0 \leq \alpha < 1 \). The space \( \mathcal{D}^{\alpha,\beta} (\beta \in \{1, 2\}) \) is a Hilbert space. Moreover, \( \mathcal{D}^{\alpha,\beta} \) is dense in \( H^\alpha \).

**Proof:** Since \( \langle \cdot, \cdot \rangle_{\mathcal{D}^{\alpha,\beta}} \) is an inner product, in order to prove that the space \( \mathcal{D}^{\alpha,\beta} \) is a Hilbert space, it remains to show that the space \( \mathcal{D}^{\alpha,\beta} \) is complete. Assume that \( (u_n)_{n \geq 1} \) is a Cauchy sequence in \( \mathcal{D}^{\alpha,\beta} \). Then there exists \( u \in H^\alpha \), \( u^\beta \in H^\beta \) such that

\[
\lim_{n \to \infty} \|u_n - u\|_{H^\alpha} = 0, \quad \lim_{n \to \infty} \|u_n - u^\beta\|_{H^\beta} = 0.
\]

Then by (3.11), we have

\[
\lim_{n \to \infty} \|u_n - \varphi + \mathcal{R}(u_n) - u - \varphi - \mathcal{R}(u)\|_{H^\alpha} = 0.
\]

Thus \( u^\beta = u - \varphi - \mathcal{R}(u) \in H^\beta \), and the space \( \mathcal{D}^{\alpha,\beta} \) is complete.

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9
Now we turn to prove the density of $D_{\theta}^{\alpha,\beta}$. For every $f \in H^\alpha$, we can decompose

$$f = f \prec \vartheta + \mathcal{R}(f) + \Gamma^{-1}f.$$  \hfill (3.17)

Since $H^\beta$ is dense in $H^\alpha$, we can choose a sequence $\{w_n\} \subset H^\beta$ so that

$$w_n \to \Gamma^{-1}f \text{ in } H^2 \text{ as } n \to \infty.$$  \hfill (3.18)

Moreover, by map $\Gamma$ we can define $f_n := \Gamma(w_n) \in D_{\theta}^{\alpha,\beta}$ such that

$$f_n \to f \text{ in } H^\alpha \text{ as } n \to \infty.$$  \hfill (3.19)

The proof is completed. 

\textbf{Lemma 3.5.} Let $\alpha \in (2/3, 1)$. Then for every $u \in H^\alpha$, we have

$$\|R(u)\|_{H^{2\alpha}} \lesssim_{K,L} \|u\|_{H^\alpha}.$$  

If $u \in H^1$, we have

$$\|R(u)\|_{H^1} \lesssim_{K,L} \|u\|_{H^\alpha}.$$  

\textbf{Proof:} By paraproduct estimates and $L^p$ estimates, we have

$$\|(\mathcal{L}u) \prec \vartheta\|_{H^{2\alpha-2}} \lesssim \|\mathcal{L}u\|_{H^{\alpha-2}}\|\vartheta\|_{\varphi^{1-\kappa}}$$  \hfill (3.20)

and

$$\|\nabla u \prec \nabla \vartheta\|_{H^{2\alpha-2}} \lesssim \|\nabla u\|_{H^{\alpha-1}}\|\nabla \vartheta\|_{\varphi^{-\kappa}}$$  \hfill (3.21)

By the localization operators and paraproduct estimates, we get

$$\|u \succ \mathcal{W}_\varphi \xi + u \succ \mathcal{W}_\varphi (\vartheta \circ \xi)\|_{H^{2\alpha-2}} + \|u \prec \mathcal{W}_\varphi \xi + u \prec \mathcal{W}_\varphi (\vartheta \circ \xi)\|_{H^{2\alpha-2}}$$

$$\lesssim (\|\mathcal{W}_\varphi \xi\|_{\varphi^{1-\kappa}} + \|\mathcal{W}_\varphi (\vartheta \circ \xi)\|_{\varphi^{1-\kappa}})\|u\|_{H^{\alpha-1}} + (\|\mathcal{W}_\varphi \xi\|_{\varphi^{1-\kappa}} + \|\mathcal{W}_\varphi (\vartheta \circ \xi)\|_{\varphi^{1-\kappa}})\|u\|_{H^{\alpha-1}}$$

$$\lesssim (2^{-(-1-k)\alpha L})\|\xi\|_{\varphi^{-1-\kappa}} + 2^{-(2-2\kappa\alpha)L}\|\vartheta \circ \xi\|_{\varphi^{-2\kappa}}\|u\|_{H^{\alpha}}$$

$$+ (2^{2\alpha-1+k\alpha})L\|\xi\|_{\varphi^{-1-\kappa}} + 2^{(2\alpha-2+2\kappa\alpha)K}\|\vartheta \circ \xi\|_{\varphi^{-2\kappa}}\|u\|_{L^2}.$$  \hfill (3.22)

Combining with above estimates (3.20-3.22), and using the $L^p$ estimates, we have

$$\|R(u)\|_{H^{2\alpha}} \lesssim \|\mathcal{L}u \prec \vartheta\|_{H^{2\alpha-2}} + \|\nabla u \prec \nabla \vartheta\|_{H^{2\alpha-2}} + \|u \succ \mathcal{W}_\varphi \xi + u \succ \mathcal{W}_\varphi (\vartheta \circ \xi)\|_{H^{2\alpha-2}}$$

$$+ \|u \prec \mathcal{W}_\varphi \xi - u \prec \mathcal{W}_\varphi (\vartheta \circ \xi)\|_{H^{2\alpha-2}}$$

$$\lesssim C_{K,L}\|u\|_{H^\alpha}.$$  

By similar estimates, we also have

$$\|u \succ \mathcal{W}_\varphi \xi + u \succ \mathcal{W}_\varphi (\vartheta \circ \xi)\|_{L^1} + \|u \prec \mathcal{W}_\varphi \xi + u \prec \mathcal{W}_\varphi (\vartheta \circ \xi)\|_{L^1}$$

$$\lesssim (\|\mathcal{W}_\varphi \xi\|_{\varphi^{-1}} + \|\mathcal{W}_\varphi (\vartheta \circ \xi)\|_{\varphi^{-1}})\|u\|_{H^\alpha} + (\|\mathcal{W}_\varphi \xi\|_{\varphi^{-1}} + \|\mathcal{W}_\varphi (\vartheta \circ \xi)\|_{\varphi^{-1}})\|u\|_{H^\alpha}$$

$$\lesssim (2^{-(-1-k)\alpha L})\|\xi\|_{\varphi^{-1-\kappa}} + 2^{-(2-2\kappa\alpha)L}\|\vartheta \circ \xi\|_{\varphi^{-2\kappa}}\|u\|_{H^{\alpha}}$$

$$+ (2^{2\alpha-1+k\alpha})L\|\xi\|_{\varphi^{-1-\kappa}} + 2^{(2\alpha-2+2\kappa\alpha)K}\|\vartheta \circ \xi\|_{\varphi^{-2\kappa}}\|u\|_{L^2}.$$  \hfill (3.23)
Combining with above estimates (3.20-3.23), and using the $L^p$ estimates, we have

\[
\|R(u)\|_{H^1} \lesssim \|\mathcal{L}u \prec \partial\|_{H^{-1}} + \|\nabla u \prec \nabla \partial\|_{H^{-1}} + \|u \succ \mathcal{U}\xi + u \succ \mathcal{U}_\theta(\partial \circ \xi)\|_{H^{-1}}
\]

\[
+ \|u \prec \mathcal{U}_\prec\xi - u \prec \mathcal{U}_\prec(\partial \circ \xi)\|_{H^{-1}} 
\lesssim C_{K,L}\|u\|_{H^\alpha}.
\]

The proof is completed.

Now we define $u \circ \xi$ by above the renormalization argument and paracontrolled distributions of singular term $\partial \circ \xi$. If $u \in \mathcal{D}_\theta^{\alpha,2}$, then we define the Wick product $u \circ \xi$ as following

\[
u \circ \xi = u \prec \xi + u \succ \xi
\]

\[
u = u \prec \xi + u \succ \xi + (u \prec \partial) \circ \xi + R(u) \circ \xi + u^\partial \circ \xi
\]

\[
u = u \prec \xi + u \succ \xi + C(u, \partial, \xi) + u(\partial \circ \xi) + R(u) \circ \xi + u^\partial \circ \xi
\]

\[
u = \lim_{\varepsilon \to 0}(u \prec \xi + u \succ \xi + C(u, \partial, \xi_e) + u(\partial \circ \xi_e - C_e) + R_e(u) \circ \xi_e + u^\partial \circ \xi_e). \quad (3.24)
\]

Thus the Anderson Hamiltonian $\mathcal{H} u$ can be written as

\[
\mathcal{H} u = \mathcal{L} u - u \circ \xi
\]

\[
= \mathcal{L}(u - \partial + R(u)) + \mathcal{L} u^\partial - u \circ \xi
\]

\[
= \mathcal{L} u^\partial - u^\partial \circ \xi - R(u) \circ \xi - u \prec \mathcal{U}_\circ\xi - u \succ \mathcal{U}_\sim\xi - u \succ \mathcal{U}_\prec(\partial \circ \xi)
\]

\[
- u \prec \mathcal{U}_\prec(\partial \circ \xi) - C(u, \partial, \xi) - u \circ (\partial \circ \xi)
\]

\[
:= \mathcal{L} u^\partial - u^\partial \circ \xi - G(u). \quad (3.25)
\]

Now we show that $\mathcal{H}$ is a linear operator from $\mathcal{D}_\theta^{\alpha,2}$ to $L^2$ in the following Lemma.

**Lemma 3.6.** Let $\alpha \in (\frac{2}{3}, 1)$, and $u \in \mathcal{D}_\theta^{\alpha,2}$. Then we have

\[
\|\mathcal{H} u\|_{L^2} \lesssim \|u^\partial\|_{H^2} + \|u\|_{L^2} \lesssim \|u\|_{\mathcal{D}_\theta^{\alpha,2}}, \quad (3.26)
\]

and

\[
\|u^\partial\|_{H^2} \lesssim \|\mathcal{H} u\|_{H^2} + \lambda\|u\|_{L^2}, \quad (3.27)
\]

where $\lambda > 0$ is a constant depending on $\|\xi\|_{\mathcal{F}_{-1-\alpha}}$ and $\|\partial \circ \xi\|_{\mathcal{F}_{1-\alpha}}$.

**Proof:** By paraproduct estimates and Lemma [3.5] we have

\[
\|u^\partial \circ \xi\|_{L^2} \lesssim \|\xi\|_{\mathcal{F}_{-1-\alpha}} \|u^\partial\|_{H^{1+\alpha}}, \quad (3.28)
\]

\[
\|R(u) \circ \xi\|_{L^2} \lesssim \|\xi\|_{\mathcal{F}_{-1-\alpha}} \|R(u)\|_{H^{2\alpha}} \lesssim \|u\|_{H^{\alpha}}, \quad (3.29)
\]

\[
\|\mathcal{U}_\alpha(\partial \circ \xi) \prec u\|_{L^2} \lesssim \|\partial \circ \xi\|_{\mathcal{F}_{1-\alpha}} \|u\|_{H^{\alpha}}, \quad (3.30)
\]

\[
\|u \prec \mathcal{U}_\prec\xi + u \succ \mathcal{U}_\circ\xi\|_{L^2} \lesssim \|\xi\|_{\mathcal{F}_{-1-\alpha}} \|u\|_{H^{\alpha}}, \quad (3.31)
\]

\[
\|u \prec \mathcal{U}_\prec(\partial \circ \xi)\|_{L^2} \lesssim \|\partial \circ \xi\|_{\mathcal{F}_{1-\alpha}} \|u\|_{H^{\alpha}}. \quad (3.32)
\]

The commutator estimate Lemma [2.5] implies that

\[
\|C(u, \partial, \xi)\|_{L^2} \lesssim \|u\|_{H^{\alpha}} \|\xi\|_{\mathcal{F}_{-1-\alpha}} \|\theta\|_{\mathcal{F}_{1-\alpha}} \lesssim \|u\|_{H^{\alpha}}. \quad (3.33)
\]

Combining above estimates (3.29-3.33), we have

\[
\|G(u)\|_{L^2} \leq 2\|u\|_{H^{\alpha}}. \quad (3.34)
\]
Lemma 3.11 implies $\|u\|_{H^\alpha} \lesssim \|u^\delta\|_{H^\alpha}$.

Thus by interpolation inequality and weighted Young inequality we obtain

$$\|H u - Lu^\delta\|_{L^2} \lesssim \|u^\delta \circ \xi\|_{L^2} + \|G(u)\|_{L^2} \lesssim \|u^\delta\|_{H^{1+\alpha}} + 2\|u^\delta\|_{H^\alpha} \leq C_\delta \|u^\delta\|_{L^2} + \delta \|u^\delta\|_{H^2}.$$ 

Thus

$$\|H u\|_{H^2} \lesssim \|Lu^\delta\|_{L^2} + \|H u - Lu^\delta\|_{L^2} \leq \|u^\delta\|_{L^2} + \|u^\delta\|_{H^2}. \quad (3.35)$$

Choosing $\delta$ small enough, we have

$$\|u^\delta\|_{H^2} \lesssim \|Lu^\delta\|_{L^2} \leq \|H u\|_{H^2} + \|H u - Lu^\delta\|_{L^2} \leq \|H u\|_{H^2} + \lambda \|u\|_{L^2}, \quad (3.36)$$

where $\lambda > 0$ is a constant depending on $\|\xi\|_{\ell^{q-1}}$ and $\|\vartheta \circ \xi\|_{\ell^{q-2}}$.

We turn to prove that the Anderson Hamiltonian $\mathcal{H}$ is a closed and symmetric operator.

**Lemma 3.7.** The Anderson Hamiltonian $\mathcal{H} : \mathcal{D}_{\vartheta}^{\alpha, 2} \to L^2$ is a closed and symmetric operator. Moreover, for every $u \in \mathcal{D}_{\vartheta}^{\alpha, 2}$, $\mathcal{H} u$ can be approximated as

$$\lim_{\epsilon \to 0} \|\mathcal{H} u - \mathcal{H}_\epsilon u\|_{L^2} = 0, \quad (3.37)$$

where $\mathcal{H}_\epsilon := L - \xi - C_\epsilon$ is a self-adjoint operator from $H^2$ to $L^2$, and $u_\epsilon = \Gamma_\epsilon(u^\delta) \in H^2$.

**Proof:** Suppose that $(u_n)_{n \geq 1} \subset \mathcal{D}_\vartheta^{\alpha, 2}$ and $f \in L^2$ such that

$$\lim_{n \to \infty} \|u_n - u\|_{L^2} = 0, \quad \lim_{n \to \infty} \|\mathcal{H} u_n - f\|_{L^2} = 0$$

Then by Lemma 3.13 $(u_n^\delta)_{n \geq 1}$ is a Cauchy sequence in $H^2$, and $\lim_{n \to \infty} \|u_n^\delta - u^\delta\|_{H^2} = 0$ for some $u^\delta \in H^2$ such that $\Gamma(u^\delta) = u$. Thus $u \in \mathcal{D}_\vartheta^{\alpha, 2}$, and

$$\lim_{n \to \infty} \|\mathcal{H} u_n - f\|_{L^2} \leq \lim_{n \to \infty} \|\mathcal{H} (u - u_n)\|_{L^2} + \lim_{n \to \infty} \|\mathcal{H} u_n - f\|_{L^2} \leq \lim_{n \to \infty} \|u^\delta - u_n^\delta\|_{H^2} + \lim_{n \to \infty} \|\mathcal{H} u_n - f\|_{L^2} = 0. \quad (3.38)$$

Thus the Anderson Hamiltonian $\mathcal{H}$ is closed.

Now we approximate $\mathcal{H}$ by the self-adjoint operator $\mathcal{H}_\epsilon = L - \xi - C_\epsilon$. For every $u \in \mathcal{D}_\vartheta^{\alpha, 2}$, we set $u_\epsilon = \Gamma_\epsilon(u^\delta)$. Then $u_\epsilon \in H^2$, and $u$ can be approximated as

$$\lim_{\epsilon \to 0} \|u - u_\epsilon\|_{H^2} = 0, \quad \lim_{\epsilon \to 0} \|u^\delta - u_\epsilon^\delta\|_{H^2} = 0. \quad (3.39)$$

Moreover, $\mathcal{H}_\epsilon u_\epsilon$ can be written as

$$\mathcal{H}_\epsilon u_\epsilon = L u_\epsilon - \xi u_\epsilon - C_\epsilon u_\epsilon$$

$$= L u^\delta - u^\delta \vartheta \circ \xi - R_\epsilon(u_\epsilon) \vartheta \circ \xi - u_\epsilon \vartheta \varphi_\leq \xi \varphi_\leq \xi - u_\epsilon \vartheta \varphi_\leq \xi \delta_\leq \varphi_\leq \delta_\leq \varphi_\leq \xi \varphi_\delta$$

$$= L u^\delta - u^\delta \vartheta \circ \xi - G_\epsilon(u_\epsilon), \quad (3.40)$$

where $R_\epsilon(u)$ is defined in (3.10). Then

$$\|\mathcal{H} u - \mathcal{H}_\epsilon u\|_{L^2} \lesssim \|G(u) - G_\epsilon(u_\epsilon)\|_{L^2} \lesssim \|G(u) - G(u_\epsilon)\|_{L^2} + \|G(u_\epsilon) - G_\epsilon(u_\epsilon)\|_{L^2}.$$
Lemma 3.3 implies that \( \lim_{\epsilon \to 0} \| u - u_{\epsilon} \|_{H^s} \lesssim \lim_{\epsilon \to 0} \| \Gamma(u^\vartheta) - \Gamma_{\epsilon}(u^\vartheta) \|_{H^s} = 0 \). Thus
\[
\lim_{\epsilon \to 0} \| G(u) - G_{\epsilon}(u_{\epsilon}) \|_{L^2} = \lim_{\epsilon \to 0} \| G(u - u_{\epsilon}) \| \lesssim \lim_{\epsilon \to 0} \| \Gamma(u^\vartheta) - \Gamma_{\epsilon}(u^\vartheta) \|_{H^s} = 0.
\]
Since \( \xi_{\epsilon} \to \xi \) in \( C^{-1-\kappa} \), \( \vartheta_{\epsilon} \to \vartheta \) in \( C^{1-\kappa} \), and \( \vartheta \xi_{\epsilon} - C_{\epsilon} \to \vartheta \vartheta \xi_{\epsilon} \) in \( C^{-2\kappa} \) as \( \epsilon \to 0 \), by similar estimates for \( R(u) \) and \( G(u) \) we have
\[
\lim_{\epsilon \to 0} \| G(u_{\epsilon}) - G_{\epsilon}(u_{\epsilon}) \|_{L^2} \lesssim (\| \xi_{\epsilon} \|_{C^{-1-\kappa}} + \| \vartheta - \vartheta_{\epsilon} \|_{C^{1-\kappa}} \| \xi_{\epsilon} \|_{C^{-1-\kappa}} + \| \vartheta \vartheta - \vartheta_{\epsilon} \circ \vartheta_{\epsilon} \|_{C^{-2\kappa}}) \| u_{\epsilon} \|_{H^s} = 0.
\]
Combining above estimates, we obtain
\[
\lim_{\epsilon \to 0} \| \mathcal{H} u - \mathcal{H}_{\epsilon} u_{\epsilon} \|_{L^2} = 0. \tag{3.41}
\]
By approximation (3.39), for every \( u, v \in \mathcal{D}_0^{\alpha,2} \) we have
\[
\langle v, \mathcal{H} u \rangle = \lim_{\epsilon \to 0} \langle v, \mathcal{H}_{\epsilon} u \rangle = \lim_{\epsilon \to 0} \langle u, \mathcal{H}_{\epsilon} v \rangle = \langle u, \mathcal{H} v \rangle. \tag{3.42}
\]
Thus the Anderson Hamiltonian \( \mathcal{H} \) is symmetric. \( \blacksquare \)

### 3.2 The Fréchet derivative of the energy functional

In this subsection, we define the energy functional \( E(u) \) by the quadratic form of the Anderson Hamiltonian \( \mathcal{H} \), and derive the Fréchet derivative of the energy functional of \( E(u) \) on its domain \( \mathcal{D}_0^{\alpha,1} \).

For every \( v, u \in \mathcal{D}_0^{\alpha,2} \), we define the bilinear form \( B_{\mathcal{H}}(v, u) = \langle v, \mathcal{H} u \rangle \), and estimate \( B_{\mathcal{H}}(v, u) \) with norm \( \| \cdot \|_{\mathcal{D}_0^{\alpha,1}} \) as follows.

**Theorem 3.8.** For every \( v, u \in \mathcal{D}_0^{\alpha,2} \), we have
\[
B_{\mathcal{H}}(v, u) \lesssim \| v \|_{\mathcal{D}_0^{\alpha,1}} \| u \|_{\mathcal{D}_0^{\alpha,1}}. \tag{3.43}
\]
Moreover, there exists constants \( c, \lambda > 0 \) depending on \( \| \xi \|_{C^{-1-\kappa}} \) and \( \| \vartheta \circ \vartheta \|_{C^{-2\kappa}} \), such that
\[
c \| u \|^2_{\mathcal{D}_0^{\alpha,1}} - \lambda \| u \|^2_{L^2} \leq B_{\mathcal{H}}(u, u). \tag{3.44}
\]

**Proof:** For every \( v, u \in \mathcal{D}_0^{\alpha,2} \), by integration by part we have
\[
B_{\mathcal{H}}(v, u) = \langle v, \mathcal{L} u^\vartheta \rangle - \langle v, u^\vartheta \circ \xi \rangle - \langle v, G(u) \rangle = \langle v^\vartheta, \mathcal{L} u^\vartheta \rangle + \langle v \vartheta + R(v), \mathcal{L} u^\vartheta \rangle - D(v, \xi, u^\vartheta) - \langle v \xi, u^\vartheta \rangle - \langle v, G(u) \rangle = \langle \nabla v^\vartheta, \nabla u^\vartheta \rangle + \mu \langle v^\vartheta, u^\vartheta \rangle + \langle \mathcal{L} R(v), u^\vartheta \rangle - D(v, \xi, u^\vartheta) + \langle (\mathcal{L}(v \vartheta) - v \xi), u^\vartheta \rangle - \langle v, G(u) \rangle, \tag{3.45}
\]
where
\[
D(v, \xi, u^\vartheta) = \langle v, u^\vartheta \circ \xi \rangle - \langle v \xi, u^\vartheta \rangle. \tag{3.46}
\]

Lemma 3.3 implies that
\[
\langle \mathcal{L} R(v), u^\vartheta \rangle \lesssim \| R(v) \|_{H^{2\kappa-2}} \| u^\vartheta \|_{H^s} \lesssim \| v \|_{H^{s'}} \| u^\vartheta \|_{H^s}. \tag{3.47}
\]
By Lemma 3.3 and Lemma 2.6, we obtain

\[ D(v, \xi, u^\theta) \lesssim \|\xi\|_{\mathcal{F}^{-1,-\alpha}} \|v\|_H^\alpha \|u^\theta\|_H^\alpha \]
\[ \lesssim \|\xi\|_{\mathcal{F}^{-1,-\alpha}} (\|v\|_{H^2}^2 + \|u^\theta\|_{H^2}^2). \]  

(3.48)

Using Lemma 2.7 we have

\[ \langle (\mathcal{L}(v - \vartheta) - v - \xi), u^\theta \rangle \leq \|\mathcal{L}(v - \vartheta) - v - \xi\|_{L^2} \|u^\theta\|_{L^2} \lesssim \|\xi\|_{\mathcal{F}^{-1,-\alpha}} \|v\|_{H^\alpha} \|u^\theta\|_{H^\alpha} \]  

(3.49)

The estimate (3.34) implies that

\[ \langle v, G(u) \rangle \leq \|v\|_{L^2} \|G(u)\|_{L^2} \lesssim \|v\|_{L^2} \|u\|_{H^\alpha}. \]  

(3.50)

Combining above estimates (3.47)-(3.50), we conclude that

\[ \langle v, \mathcal{H} u \rangle - \langle \nabla v^\theta, \nabla u^\theta \rangle - \mu \langle v^\theta, u^\theta \rangle \lesssim (\|v^\theta\|_{H^\alpha} + \|v\|_{H^\alpha}) (\|u^\theta\|_{H^\alpha} + \|u\|_{H^\alpha}). \]  

(3.51)

Thus

\[ \langle v, \mathcal{H} u \rangle \lesssim \|v\|_{\mathcal{G}^{\alpha,1}} \|u\|_{\mathcal{G}^{\alpha,1}}. \]

When \( u = v \), by \( \Gamma \) map and interpolation inequality, it follows that

\[ c \|u\|_{\mathcal{G}^{\alpha,1}}^2 - \lambda \|u\|_{L^2}^2 \leq \langle u, \mathcal{H} u \rangle, \]  

(3.52)

where \( c, \lambda \) are positive constants depending on \( \|\xi\|_{\mathcal{F}^{-1,-\alpha}} \) and \( \|\vartheta \circ \xi\|_{\mathcal{F}^{-2\alpha}} \).

By Theorem 3.8 we can define \( B_{\mathcal{H}}(v, u) \) as the quadratic form given by the Anderson Hamiltonian \( \mathcal{H} \).

**Definition 3.2.** For every \( v, u \in \mathcal{G}^{\alpha,1}_\vartheta \), we define the bilinear form \( B_{\mathcal{H}}(v, u) \) as

\[
B_{\mathcal{H}}(v, u) = \langle \nabla v^\theta, \nabla u^\theta \rangle + \mu \langle v^\theta, u^\theta \rangle + \langle \mathcal{L} R(v), u^\theta \rangle - D(v, \xi, u^\theta) - \langle \mathcal{L}(v - \vartheta) - v - \xi), u^\theta \rangle - \langle v, G(u) \rangle.
\]

(3.53)

When \( v, u \in \mathcal{G}^{\alpha,2}_\vartheta \), by 3.14 it follows that \( B_{\mathcal{H}}(v, u) = \langle v, \mathcal{H} u \rangle \). The \( \Gamma \) map implies that \( \mathcal{G}^{\alpha,2}_\vartheta \) is dense in \( \mathcal{G}^{\alpha,1}_\vartheta \). Thus by a density argument, the estimates (3.43) and (3.44) still hold, and the quadratic form \( B_{\mathcal{H}}(v, u) \) is closed, symmetric, and semi-bounded from \( \mathcal{G}^{\alpha,1}_\vartheta \times \mathcal{G}^{\alpha,1}_\vartheta \) to \( \mathbb{R} \).

Now we show the self-adjointness of the operator \( \mathcal{H} + \lambda I \) by the Friederich’s extension. We remark that this result also implies the self-adjointness of the Anderson Hamiltonian \( \mathcal{H} \) (Theorem 3.1).

**Theorem 3.9.** The operator \( \mathcal{H} + \lambda I \) is a positive self-adjoint operator from \( \mathcal{G}^{\alpha,2}_\vartheta \) to \( L^2 \), where \( \lambda \) is the constant given in Theorem 3.8

**Proof:** By Theorem 3.8 we have \( c \|u\|_{\mathcal{G}^{\alpha,1}_\vartheta}^2 \leq \langle u, (\mathcal{H} + \lambda I) u \rangle \leq C \|u\|_{\mathcal{G}^{\alpha,1}_\vartheta}^2 \) for every \( u \in \mathcal{G}^{\alpha,2}_\vartheta \).

Then by Friederichs extension theorem (see e.g. [16, Theorem XI. 7.2]), the operator \( \mathcal{H} + \lambda I \) is a positive self-adjoint operator from \( \mathcal{G}^{\alpha,2}_\vartheta \) to \( L^2 \).

Now we define the energy functional

\[ E(u) = \frac{1}{2} B_{\mathcal{H}}(u, u) - \int_{\mathbb{T}^2} F(u) dx, \quad u \in \mathcal{G}^{\alpha,1}_\vartheta. \]

(3.54)
Recall that the energy functional $E(u)$ is Fréchet differentiable at $u \in \mathcal{D}^{\alpha,1}$ if there exists a continuous linear functional $DE(u) : \mathcal{D}^{\alpha,1} \rightarrow \mathbb{R}$ such that for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, u) > 0$ so that

$$|E(u + v) - E(u) - DE(u)(v)| < \epsilon \|v\|_{\mathcal{D}^{\alpha,1}}$$

for every $\|v\|_{\mathcal{D}^{\alpha,1}} < \delta$.

We call $L(u)$ is the Fréchet derivative of the energy functional $E(u)$.

Now we consider the Fréchet derivative of the energy functional $E(u)$.

**Theorem 3.10.** The energy functional $E(u)$ is well defined and of class $C^1$ on $\mathcal{D}^{\alpha,1}$. Its Fréchet derivative at point $u$ is given by

$$DE(u)(v) = B_{\mathcal{F}}(v, u) - \int_{T^2} vf(u)dx \quad \text{for every } v \in \mathcal{D}^{\alpha,1}.$$  \hspace{1cm} (3.55)

**Proof:** By Sobolev embedding theorem, for every $p \geq 1$ we can choose $\alpha$ large enough such that $\mathcal{D}^{\alpha,1} \hookrightarrow L^p$. Thus the energy functional $E(u)$ is meaningful and continuous on $\mathcal{D}^{\alpha,1}$.

Now we show that $E$ is differentiable in the sense of Fréchet and that its Fréchet derivative is given by (3.55). Since $\mathcal{D}^{\alpha,1} \hookrightarrow L^p$, the nonlinear term in (3.54) is well-studied in Calculus of Variations (see e.g. [4, Corollary 1.1.7]). The nonlinear term is $C^1$ from $\mathcal{D}^{\alpha,1}$ to $\mathbb{R}$, and its Fréchet derivative at $u \in \mathcal{D}^{\alpha,1}$ is

$$D \left( \int_{T^2} F(u)dx \right)(v) = \int_{T^2} vf(u)dx, \quad v \in \mathcal{D}^{\alpha,1}.$$ \hspace{1cm} (3.56)

It is enough to show that the map $u \rightarrow B_{\mathcal{F}}(u, u)$ is differentiable. For all $u, v \in \mathcal{D}^{\alpha,1}$, by Theorem 3.8 it holds that

$$|B_{\mathcal{F}}(u + v, u + v) - B_{\mathcal{F}}(u, u) - 2B_{\mathcal{F}}(v, u)| = |B_{\mathcal{F}}(v, v)| \lesssim \|v\|^2_{\mathcal{D}^{\alpha,1}}.$$ 

Thus

$$|B_{\mathcal{F}}(u + v, u + v) - B_{\mathcal{F}}(u, u) - 2B_{\mathcal{F}}(v, u)| = 0 \quad \text{as } \|v\|_{\mathcal{D}^{\alpha,1}} \rightarrow 0.$$ \hspace{1cm} (3.57)

Combining with (3.56) and (3.57), we obtain that

$$DE(u)(v) = B_{\mathcal{F}}(v, u) - \int_{T^2} vf(u)dx, \quad v \in \mathcal{D}^{\alpha,1}.$$ 

Moreover, the Fréchet derivative $u \rightarrow DE(u)$ is continuous from $\mathcal{D}^{\alpha,1}$ to its dual $(\mathcal{D}^{\alpha,1})^* = \mathcal{D}^{\alpha,1}$. This finishes the proof of theorem. \hspace{1cm} \blacksquare

By the Fréchet derivative of $E(u)$, we define the weak solution of singular SPDE (1.3) as follows.

**Definition 3.3.** We say $u \in \mathcal{D}^{\alpha,1}$ is a weak solution of singular SPDE (1.3) if

$$B_{\mathcal{F}}(v, u) - \int_{T^2} vf(u)dx = 0, \quad v \in \mathcal{D}^{\alpha,1}.$$ 

### 3.3 Existence of minimizer

In this subsection, we show the existence of minimizer by direct method in the calculus of variations.
Proof: [Proof of Theorem 1.1] By Theorem 3.8, the energy functional $E(u)$ satisfies
\[
E(u) = B_{\mathcal{H}}(u, u) - \int_{\mathbb{T}^2} F(u)dx
\geq c_1\|u\|_{L^2}^2 - \lambda\|u\|_{L^2} + C\|u\|_{L^k}^k - C_0
\geq c_2(\|u\|_{H^1}^2 + \|u\|_{H^1}^2) - \lambda\|u\|_{L^2} + C\|u\|_{L^k}^k - C_0.
\] (3.58)

Then we choose $\alpha$ large enough so that $H^{\alpha-\delta} \hookrightarrow L^k$ for some small $\delta > 0$. Thus by interpolation inequality, we have
\[
E(u) \geq c\|u\|^2_{H^1} - C'
\] (3.59)
for some constant $C' > 0$. Thus the energy functional $E(u)$ is bounded from below and coercive on $\mathcal{D}^{\alpha,1}_{\vartheta}$.

Now we turn to prove the weak lower-semicontinuity of $E(u)$. By Theorem [3.9] since $\mathcal{H} + \lambda I$ is a positive self-adjoint operator, the map $u \to B_{\mathcal{H}}(u, u) + \lambda\|u\|_{L^2}^2$ is a convex functional from $\mathcal{D}^{\alpha,1}_{\vartheta}$ to $\mathbb{R}$. So for every $u_n, u \in \mathcal{D}^{\alpha,1}_{\vartheta}$, the energy functional $E(u_n)$ satisfies
\[
E(u_n) = \frac{1}{2}B_{\mathcal{H}}(u_n, u_n) - \int_{\mathbb{T}^2} F(u_n)dx
\geq \frac{1}{2}(B_{\mathcal{H}}(u_n, u) + \lambda\|u_n\|_{L^2}^2) + B_{\mathcal{H}}(u_n - u, u) + \lambda(u_n - u, u) - \lambda\|u_n\|_{L^2}^2 - \int_{\mathbb{T}^2} F(u_n)dx.
\]

For any sequence $\{u_n\}_{n=1}^{\infty}$ with $u_n \rightharpoonup u$ weakly in $\mathcal{D}^{\alpha,1}_{\vartheta}$, we have
\[
\lim_{n \to \infty} B_{\mathcal{H}}(u_n - u, u) + \lambda(u_n - u, u) = 0.
\] (3.60)

Since the weakly convergent sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $\mathcal{D}^{\alpha,1}_{\vartheta}$, the Sobolev compact embedding theorem implies that $u_n \rightarrow u$ strongly in $L^k$ as $n \rightarrow \infty$. Thus
\[
\lim_{n \to \infty} (-\lambda\|u_n\|_{L^2}^2 - \int_{\mathbb{T}^2} F(u_n)dx) = -\lambda\|u\|_{L^2}^2 - \int_{\mathbb{T}^2} F(u)dx.
\] (3.61)

From (3.60) and (3.61), we get
\[
E(u) \leq \liminf_{n \to \infty} E(u_n).
\] (3.62)

It implies the weak lower-semicontinuity of $E(u)$.

Since $E(u)$ is bounded from below, there exists a minimizing sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{D}^{\alpha,1}_{\vartheta}$ such that
\[
\lim_{n \to \infty} E(u_k) = \inf_{w \in \mathcal{D}^{\alpha,1}_{\vartheta}} E(w).
\]

The coercivity of $E(u)$ implies that the minimizing sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $\mathcal{D}^{\alpha,1}_{\vartheta}$. Since $\mathcal{D}^{\alpha,1}_{\vartheta}$ is reflexive, there exists $u \in \mathcal{D}^{\alpha,1}_{\vartheta}$, such that $u_n \rightharpoonup u$ weakly in $\mathcal{D}^{\alpha,1}_{\vartheta}$ as $n \rightarrow \infty$. Then the weak lower-semicontinuity implies that $u$ is a minimizer of $E(u)$. Moreover, by Theorem 3.10 the Fréchet derivative of $E(u)$ at the minimizer $u$ is 0, i.e.
\[
B_{\mathcal{H}}(v, u) - \int_{\mathbb{T}^2} vf(u)dx = 0, \quad v \in \mathcal{D}^{\alpha,1}_{\vartheta}.
\]

Thus the minimizer $u$ is a weak solution of the Euler-Lagrange equation (1.3). This completes the proof. 

\[\blacksquare\]
4 Regularity of the minimizer

In this section, we study the regularity of the minimizer $u$. We establish $L^2$ estimates in Theorem 4.1 and Schauder estimates for $u$.

4.1 $L^2$ estimates

**Theorem 4.1.** Under the assumption \((1.4)\), the minimizer $u$ given in Theorem 1.1 satisfies $u \in \mathcal{D}_0^{α,2}$.

**Proof:** Let $u \in \mathcal{D}_0^{α,1}$ be a weak solution for the elliptic singular SPDE \((1.3)\). Recall definition (3.2), for every $v \in \mathcal{D}_0^{α,1}$, the reminder term $u^\vartheta$ satisfies

$$\langle \nabla v^\vartheta, \nabla u^\vartheta \rangle = - \mu(v^\vartheta, u^\vartheta) - \langle \mathcal{L} R(v), u^\vartheta \rangle - \langle (\mathcal{L}(v \vartriangle) - v \vartriangle \xi), u^\vartheta \rangle$$

$$+ \langle v, G(u) \rangle + \int_{T^2} v f(u) dx,$$

where

$$D(v, \xi, u^\vartheta) = \langle v, u^\vartheta \circ \xi \rangle - \langle v \vartriangle, u^\vartheta \rangle.$$

We substitute $v^\vartheta = 2^{2(1-\kappa)k} \Delta_k u^\vartheta$, $v = \Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta)$ into \((1.1)\), and deduce

$$\langle \nabla (2^{2(1-\kappa)k} \Delta_k u^\vartheta), \nabla u^\vartheta \rangle$$

$$= - \mu(2^{2(1-\kappa)k} \Delta_k u^\vartheta, 2^{2(1-\kappa)k} \Delta_k u^\vartheta) - \langle \mathcal{L} R(\Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta)), u^\vartheta \rangle + D(\Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta), u^\vartheta)$$

$$- \langle (\mathcal{L} \Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta) \vartriangle \vartheta) - \Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta) \vartriangle \xi \rangle, u^\vartheta \rangle + \langle \Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta), G(u) \rangle$$

$$+ \int_{T^2} \Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta) f(u) dx.$$  \hspace{1cm} (4.2)

By Bernstein inequality Lemma 2.1, we have

$$\|2^{2(1-\kappa)k} \Delta_k u^\vartheta\|_{L^2}^2 \lesssim \langle \nabla (2^{2(1-\kappa)k} \Delta_k u^\vartheta), \nabla u^\vartheta \rangle.$$  \hspace{1cm} (4.3)

Since $\|\Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta)\|_{H^\vartheta} \leq 2^{2(1-\kappa)k} \Delta_k u^\vartheta \|_{H^\vartheta}$, by Lemma 3.3 and 3.4, we have

$$\langle \mathcal{L} R(\Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta)), u^\vartheta \rangle \lesssim \|\mathcal{L} R(\Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta))\|_{H^\vartheta} \|u^\vartheta\|_{H^\vartheta}$$

$$\lesssim \|\Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta)\|_{H^\vartheta} \|u^\vartheta\|_{H^\vartheta}.$$  \hspace{1cm} (4.4)

$$D(\Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta), \xi, u^\vartheta) \lesssim \|\xi\|_{\vartheta} \|2^{2(1-\kappa)k} \Delta_k u^\vartheta\|_{H^\vartheta} \|u^\vartheta\|_{H^\vartheta}.$$  \hspace{1cm} (4.5)

$$\langle (\mathcal{L} \Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta) \vartriangle \vartheta) - \Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta) \vartriangle \xi \rangle, u^\vartheta \rangle \lesssim \|2^{2(1-\kappa)k} \Delta_k u^\vartheta\|_{H^\vartheta} \|u^\vartheta\|_{H^\vartheta}.$$  \hspace{1cm} (4.6)

$$\langle \Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta), G(u) \rangle \leq \|2^{2(1-\kappa)k} \Delta_k u^\vartheta\|_{L^2} \|G(u)\|_{L^2} \lesssim \|2^{2(1-\kappa)k} \Delta_k u^\vartheta\|_{L^2} \|u\|_{H^\vartheta}.$$  \hspace{1cm} (4.7)

By Sobolev embedding, we obtain

$$\int_{T^2} \Gamma(2^{2(1-\kappa)k} \Delta_k u^\vartheta) f(u) dx \lesssim \|\Gamma(2^{2k} \Delta_k u^\vartheta)\|_{L^2} \|f(u)\|_{L^2} \lesssim \|2^{2(1-\kappa)k} \Delta_k u^\vartheta\|_{L^2} \|u\|_{H^\vartheta}.$$  \hspace{1cm} (4.8)

We combine above estimates \((1.3)-(1.8)\), and sum over $k$ to get

$$\|u^\vartheta\|_{H^2-\kappa}^2 + \mu \|u^\vartheta\|_{H^1-\kappa}^2 \lesssim \|u^\vartheta\|_{H^2-\kappa} \|u\|_{H^\vartheta} + \|u^\vartheta\|_{H^1}.$$  \hspace{1cm} (4.9)
After using the weighted Young inequality and choosing δ small enough to absorb \( \|u^\vartheta\|_{H^{2-\kappa}} \) into the left hand side, we have
\[
\|u^\vartheta\|_{H^{2-\kappa}}^2 \lesssim \|u^\vartheta\|_{H^1}^2 + \|u^\vartheta\|_{H^1}^2.
\] (4.10)
Thus reminder term \( u^\vartheta \in H^{2-\kappa} \). Now we substitute \( v^\vartheta = 2^{2k}\Delta_k u^\vartheta \), \( v = \Gamma(2^{2k}\Delta_k u^\vartheta) \) into (4.11), and deduce
\[
\langle \nabla (2^{2k}\Delta_k u^\vartheta), \nabla u^\vartheta \rangle
\]
\[
= \mu(2^{2k}\Delta_k u^\vartheta, 2^{2k}\Delta_k u^\vartheta) - \langle \mathcal{L} R(\Gamma(2^{2k}\Delta_k u^\vartheta)), u^\vartheta \rangle + D(\Gamma(2^{2k}\Delta_k u^\vartheta), \xi, u^\vartheta)
\]
\[
- \langle (\mathcal{L}(\Gamma(2^{2k}\Delta_k u^\vartheta)) \prec v) - \Gamma(2^{2k}\Delta_k u^\vartheta) \prec v, u^\vartheta \rangle + \langle \Gamma(2^{2k}\Delta_k u^\vartheta), G(u) \rangle
\]
\[
+ \int_{\mathbb{T}^2} \Gamma(2^{2k}\Delta_k u^\vartheta)f(u)dx.
\] (4.11)
We estimate
\[
\|2^{2k}\Delta_k u^\vartheta\|_{L^2}^2 \lesssim \langle \nabla (2^{2k}\Delta_k u^\vartheta), \nabla u^\vartheta \rangle.
\] (4.12)
\[
\langle \mathcal{L} R(\Gamma(2^{2k}\Delta_k u^\vartheta)), u^\vartheta \rangle \lesssim \| \mathcal{L} R(\Gamma(2^{2k}\Delta_k u^\vartheta)) \|_{H^{-1}} \| u^\vartheta \|_{H^1}^2 \lesssim \|2^{2k}\Delta_k u^\vartheta\|_{L^2} \| u^\vartheta \|_{H^{2-\kappa}}.
\] (4.13)
\[
D(\Gamma(2^{2k}\Delta_k u^\vartheta), \xi, u^\vartheta) \lesssim \| \xi \|_{H^{-1-\kappa}} \|2^{2k}\Delta_k u^\vartheta\|_{L^2} \| u^\vartheta \|_{H^{2-\kappa}}.
\] (4.14)
\[
\langle (\mathcal{L}(\Gamma(2^{2k}\Delta_k u^\vartheta)) \prec v) - \Gamma(2^{2k}\Delta_k u^\vartheta) \prec v, u^\vartheta \rangle \lesssim \| \xi \|_{H^{-1-\kappa}} \|2^{2k}\Delta_k u^\vartheta\|_{L^2} \| u^\vartheta \|_{H^{2-\kappa}}.
\] (4.15)
\[
\langle \Gamma(2^{2k}\Delta_k u^\vartheta), G(u) \rangle \lesssim \|2^{2k}\Delta_k u^\vartheta\|_{L^2} \| G(u) \|_{L^2} \lesssim \|2^{2k}\Delta_k u^\vartheta\|_{L^2} \| u \|_{H^{\kappa}}.
\] (4.16)
By Sobolev embedding, we obtain
\[
\int_{\mathbb{T}^2} \Gamma(2^{2k}\Delta_k u^\vartheta)f(u)dx \lesssim \| \Gamma(2^{2k}\Delta_k u^\vartheta) \|_{L^2} \| f(u) \|_{L^2} \lesssim \|2^{2k}\Delta_k u^\vartheta\|_{L^2} \| u \|_{H^{\kappa}}.
\] (4.17)
We combine above estimates, and sum over k to get
\[
\|u^\vartheta\|_{H^{2-\kappa}} + \mu \|u^\vartheta\|_{H^{1-\kappa}}^2 \lesssim \|u^\vartheta\|_{H^{2-2\kappa}} (\|u\|_{H^{\kappa}} + \|u^\vartheta\|_{H^1}).
\] (4.18)
After using the weighted Young inequality and choosing δ small enough to absorb \( \|u^\vartheta\|_{H^{2-2\kappa}} \) into the left hand side, we have
\[
\|u^\vartheta\|_{H^{2}}^2 \lesssim \|u\|_{H^{\kappa}}^2 + \|u^\vartheta\|_{H^{1}}^2.
\] (4.19)
The proof is completed.

### 4.2 Schauder estimates

By \( L^2 \) estimates, the minimizer \( u \in \mathcal{D}_{\vartheta}^{1,2} \). Motivated by the definition of the space \( \mathcal{D}_{\vartheta}^{1,2} \), we introduce the ansatz \( u = \phi + u^\vartheta \), where \( \phi = u - \vartheta + R(u) \). Then the original equation (4.13) can be decomposed into a simpler elliptic system
\[
\begin{aligned}
\mathcal{L} \phi &= \Phi(u), \\
\mathcal{L} u^\vartheta &= f(u) + \Psi(u),
\end{aligned}
\] (4.20)
where
\[
\Phi(u) := u \prec \mathcal{U}_{\vartheta} \xi + u \succ \mathcal{U}_{\vartheta} \xi + u \succ \mathcal{U}_{\vartheta} (\vartheta \circ \xi) + u \prec \mathcal{U}_{\vartheta} (\vartheta \circ \xi),
\]
\[
\Psi(u) := R(u \circ \xi + C(u, \vartheta, \xi) + (\vartheta \circ \xi) \circ u + u^\vartheta \circ \xi + u \prec \mathcal{U}_{\vartheta} \xi + u \succ \mathcal{U}_{\vartheta} \xi
\] + \( u \succ \mathcal{U}_{\vartheta} (\vartheta \circ \xi) + u \prec \mathcal{U}_{\vartheta} (\vartheta \circ \xi). \)
Here \( \Phi \) is the collection of all terms of negative regularity, and \( u^\vartheta \) the collection of all other regular terms (belonging to \( L^\infty \)). By estimating \( \Phi(u) \) and \( \Psi(u) \), we establish the following Schauder estimates for \( u, R(u) \), and \( u^\vartheta \).
Theorem 4.2. If the nonlinear term satisfies assumptions (1.4) and (1.5), then the minimizer $u$ given in Theorem 3.1 satisfies $u \in C^\alpha$, with reminders $R(u) \in C^{2\alpha}$ and $u^\theta \in C^{3\alpha}$.

**Proof:** We prove in three steps.

**Step 1. Bound for $u$ in $C^\alpha$**

First, we estimate $\Phi(u)$ in $C^{-2+\kappa}$, and derive a priori for $\phi$ in $C^\kappa$ by elliptic Schauder estimates. By Besov embedding (Lemma 2.2), we have $H^\alpha = B^\alpha_{2,2} \rightarrow C^{-1}$ and $u \in \dot{C}^{\alpha-1}$. Then the paraproduct estimates imply that

$$
\|u \prec \mathcal{U}_\alpha \xi\|_{\dot{C}^{-2+\kappa}} + \|u \succ \mathcal{U}_\alpha \xi\|_{\dot{C}^{-2+\kappa}} \lesssim 2^{-(\alpha-2\kappa)}\|\xi\|_{\dot{C}^{-1-\kappa}}\|u\|_{\dot{C}^{\alpha-1}},
$$

and

$$
\|u \prec \mathcal{U}_\alpha (\vartheta \circ \xi)\|_{\dot{C}^{-2+\kappa}} + \|u \succ \mathcal{U}_\alpha (\vartheta \circ \xi)\|_{\dot{C}^{-2+\kappa}} \lesssim 2^{-(1-\kappa+\alpha)}\|\vartheta \circ \xi\|_{\dot{C}^{-2}}\|u\|_{\dot{C}^{\alpha-1}}.
$$

Then by Schauder estimates, we have

$$
\|\phi\|_{\dot{C}^\kappa} \lesssim \|\Phi(u)\|_{\dot{C}^{-2+\kappa}} \lesssim (2^{-(1-\kappa)L} + 2^{-(2-3\kappa)K})\|u\|_{\dot{C}^{\alpha-1}} \lesssim \|u\|_{H^\alpha}. \tag{4.21}
$$

Since $u^\theta \in H^2$, the Sobolev embedding theorem implies that $u^\theta \in C^\alpha$, and $u = \phi + u^\theta \in C^\kappa$. Now we estimate $\Phi(u)$ in $C^{-2+\alpha}$, and derive a bound for $\phi$ in $C^\alpha$ by Schauder estimates. By Bony’s paraproduct estimates, we have

$$
\|u \prec \mathcal{U}_\alpha \xi + u \succ \mathcal{U}_\alpha \xi\|_{\dot{C}^{-2+\alpha}} \lesssim \|\mathcal{U}_\alpha \xi\|_{\dot{C}^{\alpha-2}}\|u\|_{L^\infty} \lesssim 2^{-(1-\kappa-\alpha)L}\|\mathcal{U}_\alpha \xi\|_{\dot{C}^{-1-\kappa}}\|u\|_{L^\infty},
$$

and

$$
\|u \prec \mathcal{U}_\alpha (\vartheta \circ \xi) + u \succ \mathcal{U}_\alpha (\vartheta \circ \xi)\|_{\dot{C}^{-2+\alpha}} \lesssim \|\mathcal{U}_\alpha (\vartheta \circ \xi)\|_{\dot{C}^{\alpha-2}}\|u\|_{L^\infty} \lesssim 2^{-(2-3\kappa-\alpha)K}\|\mathcal{U}_\alpha (\vartheta \circ \xi)\|_{\dot{C}^{-2}}\|u\|_{L^\infty}.
$$

Then by above estimates, we have

$$
\|\phi\|_{C^\alpha} = \|\mathcal{L}^{-1}(\Phi(u))\|_{\dot{C}^\alpha} \lesssim \|\Phi(u)\|_{\dot{C}^{-2+\alpha}} \lesssim \|u\|_{L^\infty} \lesssim \|\phi\|_{C^\kappa} + \|u^\theta\|_{C^\kappa} \lesssim \|u\|_{\dot{C}^{\alpha-2}}^\theta. \tag{4.22}
$$

Thus $u = \phi + u^\theta \in C^\alpha$.

**Step 2. Bound for $R(u)$ in $C^{2\alpha}$**

Since $R(u)$ satisfies

$$
\mathcal{L}R(u) = -\mathcal{L}(u \prec \vartheta) + \Phi(u) = -\mathcal{L}u \prec \vartheta - \nabla u \prec \nabla \vartheta - u \prec \xi + \Phi(u). \tag{4.23}
$$

Since $\|\xi\|_{\dot{C}^{\alpha-2}} \lesssim 1$. Thus by paraproduct estimates, we have

$$
\|\nabla u \prec \nabla \vartheta\|_{\dot{C}^{2\alpha-2}} \lesssim \|\nabla u\|_{\dot{C}^{\alpha-1}}\|\nabla \vartheta\|_{C^\alpha} \lesssim \|u\|_{\dot{C}^{\alpha}}. \tag{4.24}
$$

Recall the choosing of $L, K$, by paraproduct estimates, we get

$$
\| - u \prec \xi + \Phi(u)\|_{\dot{C}^{2\alpha-2}} \lesssim \|u \prec \mathcal{U}_\alpha \xi + u \succ \mathcal{U}_\alpha (\vartheta \circ \xi)\|_{\dot{C}^{2\alpha-2}} + \|u \prec \mathcal{U}_\alpha \xi + u \succ \mathcal{U}_\alpha (\vartheta \circ \xi)\|_{\dot{C}^{2\alpha-2}} \lesssim (\|\mathcal{U}_\alpha \xi\|_{\dot{C}^{\alpha-2}} + \|\mathcal{U}_\alpha (\vartheta \circ \xi)\|_{\dot{C}^{\alpha-2}})\|u\|_{\dot{C}^{\alpha}} + (\|\mathcal{U}_\alpha \xi\|_{\dot{C}^{\alpha-2}} + \|\mathcal{U}_\alpha (\vartheta \circ \xi)\|_{\dot{C}^{2\alpha-2}})\|u\|_{L^\infty} \lesssim (2^{-(1-\kappa-\alpha)L} + 2^{-(2-3\kappa-\alpha)K})\|u\|_{\dot{C}^{\alpha}} + (2^{(2\alpha-1+\kappa)L} + 2^{-(2-2\kappa-2\alpha)K})\|u\|_{L^\infty} \lesssim \|u\|_{\dot{C}^{\alpha}}. \tag{4.25}
$$

Thus $R(u) \in C^{2\alpha}$.
Combining with above estimates (4.24)-(4.26), and using the Schauder estimates, we have
\[ \|R(u)\|_{\mathcal{E}^{2\alpha}} \lesssim \| -\mathcal{L} u < \vartheta - \nabla u < \nabla \vartheta - u \| \lesssim \xi + \Phi(u)\|_{\mathcal{E}^{2\alpha-2}} \lesssim \|u\|_{\mathcal{E}^\alpha}. \] (4.27)

**Step 3. Bound for \( u^\vartheta \) in \( \mathcal{E}^{3\alpha} \)**

We derive a bound for \( u^\vartheta \) in \( \mathcal{E}^{3\alpha} \). By paraproduct estimates and a priori estimates (4.22), (4.27), we have
\[ \|R(u) \circ \xi\|_{\mathcal{E}^{3\alpha-2}} \lesssim \|\xi\|_{\mathcal{E}^{1-\kappa}} \|R(u)\|_{\mathcal{E}^{2\alpha}} \lesssim \|u\|_{\mathcal{E}^\alpha} \] (4.28)
\[ \|u^\vartheta \circ \xi\|_{\mathcal{E}^{3\alpha-2}} \lesssim \|u\|_{\mathcal{E}^{2\alpha}}, \] (4.29)
\[ \|\mathcal{W}_\xi(\vartheta \circ \xi)\|_{\mathcal{E}^{3\alpha-2}} \lesssim \|\vartheta \circ \xi\|_{\mathcal{E}^{2\alpha-2}} \|u\|_{\mathcal{E}^\alpha} \lesssim \|u\|_{\mathcal{E}^\alpha}. \] (4.30)
\[ \|((\vartheta \circ \xi) \circ \alpha)\|_{\mathcal{E}^{3\alpha-2}} \lesssim \|u\|_{\mathcal{E}^\alpha}. \] (4.31)

The commutator estimate Lemma 2.5 implies that
\[ \|C(u, \vartheta, \xi)\|_{\mathcal{E}^{3\alpha-2}} \lesssim \|u\|_{\mathcal{E}^\alpha} \|\xi\|_{\mathcal{E}^{1-\kappa}} \|\vartheta\|_{\mathcal{E}^\alpha} \lesssim \|u\|_{\mathcal{E}^\alpha}. \] (4.32)

According to Lemma 2.3, we have
\[ \|u \prec \mathcal{W}_\xi(\vartheta \circ \xi)\|_{\mathcal{E}^{3\alpha-2}} + \|u \succ \mathcal{W}_\xi(\vartheta \circ \xi)\|_{\mathcal{E}^{3\alpha-2}} \lesssim \int \|u\|_{L^\infty} \|\vartheta \circ \xi\|_{\mathcal{E}^{3\alpha-2}} + \int \|\vartheta \circ \xi\|_{\mathcal{E}^{3\alpha-2}} \|u\|_{L^\infty} \lesssim \|u\|_{\mathcal{E}^{3\alpha}} \|u\|_{L^\infty}, \] (4.33)
and
\[ \|u \succ \mathcal{W}_\xi(\vartheta \circ \xi)\|_{\mathcal{E}^{3\alpha-2}} \leq \|u \succ \mathcal{W}_\xi(\vartheta \circ \xi)\|_{\mathcal{E}^{3\alpha-2}} + \|\mathcal{W}_\xi(\vartheta \circ \xi)\|_{\mathcal{E}^{3\alpha-2}} \|u\|_{\mathcal{E}^{\alpha}} \lesssim \int (2^{(3\alpha-1-\kappa)L} + 1) \|u\|_{\mathcal{E}^{\alpha}}. \] (4.34)

By 4.21, and the assumption (1.3) of \( f \), we have
\[ \|f(u)\|_{\mathcal{E}^{3\alpha-2}} \lesssim (l + \|u\|_{L^\infty}^{k-2}) \|u\|_{\mathcal{E}^\alpha}. \] (4.35)

Combining with above estimates, and using the interpolation inequality in Lemma 2.8, and weighted Young inequality, for every \( \delta > 0 \) we have
\[ \|\Phi(u)\|_{\mathcal{E}^{3\alpha-2}} \lesssim \|u\|_{\mathcal{E}^{\alpha}} + \|u^\vartheta\|^2_\mathcal{E}^{3\alpha} \|u^\vartheta\|_{L^\infty}^{1/3} \lesssim \delta \|u\|_{\mathcal{E}^\alpha} + \|u^\vartheta\|_{\mathcal{E}^{3\alpha}} + C\delta \|u^\vartheta\|_{\mathcal{E}^\alpha}. \]

Then by Schauder estimates and choosing \( \delta \) small enough, we obtain
\[ \|u^\vartheta\|_{\mathcal{E}^{3\alpha}} = \|\mathcal{L}^{-1}(f(u) + \Phi(u))\|_{\mathcal{E}^{3\alpha}} \lesssim 1 + \|u^\vartheta\|_{\mathcal{E}^\alpha} + \|u\|_{\mathcal{E}^\alpha} + \|\vartheta\|_{L^\infty}^{k-2} \|u\|_{\mathcal{E}^\alpha}. \] (4.36)

The proof is completed.

5 Conclusions

This paper is an attempt to build a bridge between the variation problem and the singular SPDE in the paracontrolled distribution framework. We have defined the energy functional \( E(u) \) associated with the Anderson Hamiltonian by the quadratic form of the Anderson Hamiltonian on the space of paracontrolled distributions \( \mathcal{D}^{0,1}_\vartheta \). Then we have shown that the energy functional
$E(u)$ is a $C^1$ map from $\mathcal{D}_0^{\alpha,1}$ to $\mathbb{R}$, and the Euler-Lagrange equation of the energy functional $E(u)$ is the elliptic singular SPDE. By the direct method of calculus of variation, we proved the existence of minimizers. Since the minimizer $u$ is a weak solution of the elliptic singular SPDE, we use the structure of the singular SPDE, and establish the $L^2$ estimates and Schauder estimates for the minimizer $u$.

We only consider the Anderson Hamiltonian in 2-dimensional torus $T^2$ in this paper. We point out that in the 3-dimensional case, since the regularity of spatial white noise is much more singular, the (rigorous) definition of the Anderson Hamiltonian and its domain is more complex. Thus the assumption of the nonlinear term will have more restriction, and the regularity estimates will be technically more involved. It will be the subject of future work.

There are other directions we plan to explore in our future work. By Dirichlet and Neumann Besov spaces, we can replace the periodic boundary condition by Dirichlet or Neumann boundary condition. We can also use some spacial weight and study the variation problem in the whole space. In addition, the energy functional $E(u)$ associated with the Anderson Hamiltonian can help us construct a Lyapunov function of the continuous parabolic Anderson model, and it will be a powerful tool to study the long-time behavior of the continuous parabolic Anderson model.

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