New convergence theorems for maximal monotone operators in Banach spaces

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Abstract—The purpose of this paper is to introduce a new hybrid iterative scheme for resolvents of maximal monotone operators in Banach spaces by using the notion of generalized \( f \)-projection. Next, we apply this result to the convex minimization and variational inequality problems in Banach spaces. The results presented in this paper improve and extend important recent results in the literature.

I. INTRODUCTION

Maximal monotone operators have frequently proven to be a key class of objects in Optimization and Analysis. Constructing iterative algorithms to approximate zeros of maximal monotone operators is a very active topic in applied mathematics. The problem for finding a zero point of a maximal monotone operator is defined as follows: Given a Banach space \( E \) and a maximal monotone operator \( T \), we consider the problem for finding a point \( u \in E \) such that:

\[
0 \in T(u).
\]

The set of all points \( u \in E \) such that \( 0 \in T(u) \) denote by \( T^{-1}(0) \). This problem is very important in optimization theory and related fields. For example, if \( F : E \to (-\infty, \infty] \) is a proper lower semicontinuous convex function. In this case, the equation \( 0 \in \partial F(u) \) is equivalent to the problem of minimizing \( F \) over \( E \).

The proximal point algorithm (PPA) is one of the popular methods for solving (1), which was first proposed by Martinet [1] and further developed by Rockafellar [2] in a framework of maximal monotone operators in a Hilbert space. A variety of problems, for example, convex programming and variational inequalities can be formulated as finding a zero point of maximal monotone operators. Many research study and extend the proximal methods for various models (see, for example, [3], [4] and [5]). Both numerical experiments and theoretical analysis have demonstrated that the PPA is robust and has nice convergence properties. The proximal point algorithm (PPA) according to Rockafellar [2] generates a sequence \( \{x_n\} \) via the rule

\[
x_0 \in H, \quad x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, 3, \ldots
\]

where \( J_{r_n} = (I + r_n T)^{-1} \) and \( \{r_n\} \subset (0, \infty) \), then the sequence \( \{x_n\} \) converges weakly to an element of \( T^{-1}(0) \).

If \( T = \partial F \) where \( F : E \to (-\infty, \infty] \) is a proper lower semicontinuous convex function, then (2), is reduced to

\[
x_{n+1} = \arg\min_{y \in H} \{F(y) + \frac{1}{2r_n} \|x_n - y\|^2\}, \quad n = 1, 2, 3, \ldots
\]

To get the results of strong convergence, Solodov and Sviater [4] modified the proximal point algorithm and projection in a Hilbert space. In 2003, Kohsaka and Takahashi [13] obtained a strong convergence theorem for maximal monotone operators in a Banach space, which extended the result of Solodov and Sviater in a Hilbert space.

On the other hand, Alber [6], [7] introduced and studied the generalized projections \( \pi_C : E^* \to C \) and \( \Pi_E : E \to C \) in uniformly convex and uniformly smooth Banach spaces. In 2005, Li [9] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces. Later, Wu and Huang [10] introduced a new generalized \( f \)-projection operator in Banach spaces. They extended the definition of the generalized projection operators introduced by Alber [7] and proved some properties of the generalized \( f \)-projection operator. In 2009, Fan et al. [11] presented some basic results for the generalized \( f \)-projection operator, and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces. Recently, Li et al. [12] proved some property of the generalized \( f \)-projection operator and proved strong convergence theorems for relatively nonexpansive mappings in Banach spaces.

In 2005, Matsushita and Takahashi [14] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping in a Banach space \( E \). They obtained a strong convergence theorem for relatively nonexpansive mapping in a Banach space. In 2010, Li et al. [12] introduced and proved the strong convergence theorem for approximation of fixed point of relatively nonexpansive mapping using the properties of generalized \( f \)-projection operator in a uniformly smooth real Banach space which is also uniformly convex. We remark here that the results of Li et al. [12] extended and improved on the results of Matsushita and Takahashi [14]. Recently, Saewan and Kumam [15] extended the ideal of the generalized \( f \)-projection operator to hybrid Ishikawa iteration process for finding a common element of the fixed point set for two countable families of weak relatively nonexpansive mappings and the set of solutions of finding a common element of the fixed point set for two countable families of weak relatively nonexpansive mappings and the set of solutions of the system of generalized Ky Fan inequalities in a uniformly convex and uniformly smooth Banach space.

Motivated by the previously known results, we introduce a new hybrid iterative scheme of the generalized \( f \)-projection
Let $E$ be a real Banach space with dual $E^*$ $E$ is said to be **strictly convex** if \( \| \frac{x+y}{2} \| < 1 \) for all $x, y \in E$ with $\| x \| = \| y \| = 1$ and $x \neq y$. The modulus of convexity of $E$ is the function $\delta : [0, 2] \rightarrow [0, 1]$ defined by

$$
\delta(\varepsilon) = \inf \{ 1 - \frac{\| x + y \|}{2} : x, y \in E, \| x \| = \| y \| = 1, \| x - y \| \geq \varepsilon \}.
$$

A Banach space $E$ is said to be uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Let $U = \{ x \in E : \| x \| = 1 \}$ be the unit sphere of $E$. Then a Banach space $E$ is said to be smooth if the limit $\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t} = 0$ exists for each $x, y \in U$. $E$ is also said to be uniformly smooth if the limit exists uniformly in $x, y \in U$. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing of $E^*$ and $E$, the normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$
J(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \| x \|^2, \| x^* \| = \| x \| \}.
$$

If $E$ is a Hilbert space, then $J = I$, where $I$ is the identity mapping and $\langle \cdot, \cdot \rangle$ denotes an inner product on $E$. Consider the functional defined by

$$
\phi(x, y) = \| x \|^2 - 2 \langle x, Jy \rangle + \| y \|^2, \quad \text{for } x, y \in E, \quad (3)
$$

where $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping.

As well known that if $C$ is a nonempty closed convex subset of a Hilbert space $H$ and $P_C : H \rightarrow C$ is the metric projection of $H$ onto $C$. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [6], [7] recently introduced the generalized projection $\Pi_C : E \rightarrow C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$
\phi(\bar{x}, x) = \inf_{y \in C} \phi(x, y). \quad (4)
$$

The existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping $J$ (see, for example, [6], [8], [17], [16]). It is obvious from the definition of function $\phi$ that

$$
(||y|| - ||x||)^2 \leq \phi(x, y) \leq (||y|| + ||x||)^2, \quad \forall x, y \in E. \quad (5)
$$

We also known that

$$
\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle. \quad (6)
$$

If $E$ is a Hilbert space, then $\phi(x, y) = ||y - x||^2$ and $\Pi_C$ becomes the metric projection of $E$ onto $C$.

**Remark 1:** Let $E$ be a Banach space. Then we know that

1. if $E$ is an arbitrary Banach space, then $J$ is monotone and bounded;
2. if $E$ is a strictly convex, then $J$ is strictly monotone;
3. if $E$ is a smooth, then $J$ is single valued and semicontinuous;
4. if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$;
5. if $E$ is reflexive, smooth and strictly convex, then the normalized duality mapping $J$ is single valued, one-to-one and onto;
6. if $E$ is uniformly smooth, then $E$ is smooth and reflexive;
7. $E$ is uniformly smooth if and only if $E^*$ is uniformly convex;

see [8] for more details.

**Remark 2:** If $E$ is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From 3, we have $\| x \| = \| y \|$. This implies that $\langle x, Jy \rangle = \| x \|^2 = \| Jy \|^2$. From the definition of $J$, one has $Jx = Jy$. Therefore, we have $x = y$; see [8], [16] for more details.

**Lemma 1:** (Kamimura and Takahashi [17]). Let $E$ be a uniformly convex and smooth Banach space and let $\{ x_n \}$ and $\{ y_n \}$ be two sequences of $E$. If $\phi(x_n, y_n) \rightarrow 0$ and either $\{ x_n \}$ or $\{ y_n \}$ is bounded, then $x_n - y_n \rightarrow 0$.

Let $G : C \times E^* \rightarrow \mathbb{R} \cup \{ +\infty \}$ be a functional defined as follows:

$$
G(y, x) = \| y \|^2 - 2\langle y, x \rangle + \| x \|^2 + 2\rho \| y \|, \quad (7)
$$

where $y \in C$, $x \in E^*$, $\rho$ is positive number and $f : C \rightarrow \mathbb{R} \cup \{ +\infty \}$ is proper, convex and lower semicontinuous. From definitions of $G$ and $f$, it is easy to see the following properties:

1. $G(y, x)$ is convex and continuous with respect to $x$ when $y$ is fixed;
2. $G(y, x)$ is convex and lower semicontinuous with respect to $y$ when $x$ is fixed.

Let $E$ be a real Banach space with its dual $E^*$. Let $C$ be a nonempty closed convex subset of $E$. We say that $\pi_C^f : E^* \rightarrow 2^C$ is generalized $f$-projection operator if

$$
\pi_C^f x = \{ u \in C : G(u, x) = \inf_{y \in C} G(y, x), \forall y \in E^* \}. \quad (8)
$$

**Lemma 2:** (Wu and Hung [10]). Let $E$ be a reflexive Banach space with its dual $E^*$ and $C$ be a nonempty closed convex subset of $E$. The following statements hold:

1. $\pi_C^f \omega$ is nonempty closed convex subset of $C$ for all $\omega \in E^*$;
2. if $E$ is smooth, then for all $\omega \in E^*$, $x \in \pi_C^f \omega$ if and only if
   \[ \langle x - y, \omega - Jx \rangle + \rho \| f(y) - f(x) \| \geq 0, \forall y \in C; \]
3. if $E$ is strictly convex and $f : C \rightarrow \mathbb{R} \cup \{ +\infty \}$ is positive homogeneous (i.e., $f(tx) = t \cdot f(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then $\pi_C^f$ is single valued mapping.
Fan et al. [11] show that the condition $f$ is positive homogeneous which appeared in [11, Lemma 2.1 (iii)] can be removed.

**Lemma 3:** (Fan et al. [11]). Let $E$ be a reflexive Banach space with its dual $E^*$ and $C$ be a nonempty closed convex subset of $E$. If $E$ is strictly convex, then $\pi_{C,x}$ is single valued.

Recall that $T$ is single valued mapping when $E$ is a smooth Banach space. There exists a unique element $\omega \in E^*$ such that $\omega = Jx$ where $x \in E$. This substitution for (7) give

$$G(y, Jx) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 + 2\rho f(y).$$

(8)

It is obvious from the definition of $G$ that

$$G(y, Jx) = G(y, Jz) + \phi(z, x) + 2\langle y - z, Jz - Jx \rangle,$$  (9)

for all $x, y, z \in E$.

Next, we consider the second generalized $f$-projection operator in Banach spaces.

**Definition 1:** (Li et al. [12]). Let $E$ be a real smooth Banach space and $C$ be a nonempty closed convex subset of $E$. We say that $\Pi_{C,x} : E \to 2^C$ is generalized $f$-projection operator if

$$\Pi_{C,x}x = \{u \in C : G(u, Jx) = \inf_{y \in C} G(y, Jx)\}, \forall x \in E.$$  

**Lemma 4:** (Deimling [18]). Let $E$ be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex functional. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that $f(x) \geq \langle x, x^* \rangle + \alpha$, $\forall x \in E$.

**Lemma 5:** (Li et al. [12]). Let $E$ be a reflexive smooth Banach space and $C$ be a nonempty closed convex subset of $E$. The following statements hold:

1) $\Pi_{C,x}x$ is nonempty closed convex subset of $C$ for all $x \in E$;
2) for all $x \in E$, $\hat{x} \in \Pi_{C,x}x$ if and only if

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \forall y \in C;$$  (10)

3) if $E$ is strictly convex, then $\Pi_{C,x}$ is single valued mapping.

**Lemma 6:** (Li et al. [12]). Let $E$ be a reflexive smooth Banach space and $C$ be a nonempty closed convex subset of $E$ and let $x \in E$, $\hat{x} \in \Pi_{C,x}x$. Then

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \forall y \in C.$$  

**Remark 3:** Let $E$ be a uniformly convex and uniformly smooth Banach space and $f(x) = 0$ for all $x \in E$. Then Lemma 6 reduces to the property of the generalized projection operator considered by Alber [6].

**Lemma 7:** (Li et al. [12]). Let $E$ be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous mapping with convex domain $D(f)$. If $\{x_n\}$ is a sequence in $D(f)$ such that $x_n \to \hat{x} \in D(f)$ and $\lim_{n \to \infty} G(x_n, Jy) = G(\hat{x}, Jy)$, then $\lim_{n \to \infty} \|x_n\| = \|\hat{x}\|$.

An operator $T \subset E \times E^*$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone $T$ is said to be maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. If $T$ is maximal monotone, then the solution set $T^{-1}0$ is closed and convex. Let $E$ be a reflexive, strictly convex and smooth Banach space, it is known that $T$ is a maximal monotone if and only if $R(T + rT) = E^*$ for all $r > 0$. Define the resolvent of $T$ by $J_r = (T + rT)^{-1} J$ for all $r > 0$, $J_r$ is a single-valued mapping from $E$ to $D(T)$.

**Theorem 1:** Let $C$ be a nonempty closed convex subset of $E$ and let $T \subset E \times E^*$ be a monotone operator satisfying $D(T) \subset C \subset J^{-1}(\{r > 0 : R(J + rT) = E^*)\}$. Let $r > 0$, let $J_r$ and $T_r$ be the resolvent and the Yosida approximation of $T$, respectively. Then the following hold:

(i) $\phi(\hat{x}, J \hat{x}) + \phi(J_r \hat{x}, x) \leq \phi(u, x), \forall x \in C, u \in T^{-1}0$;
(ii) $(J_r x, T_r x) \in T_r \forall x \in C$.

**Lemma 9:** (Kohsaka and Takahashi [13]). Let $E$ be a smooth, strictly convex and reflexive Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $T \subset E \times E^*$ be a monotone operator with $T^{-1}0 \neq \emptyset$, and let $J_r = (T + rT)^{-1} J$ for each $r > 0$. Then

$$G(p, J_r x, J_r \hat{x}) \leq G(p, J_r \hat{x}, \hat{x}) \leq G(p, J_r \hat{x}, \hat{x}) + 2r(p - J_r \hat{x}, -T_r \hat{x}) \geq G(p, J_r x, J_r \hat{x}).$$

The proof is complete.

**III. MAIN RESULTS**

**Theorem 1:** Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $T \subset E \times E^*$ be a maximal monotone operator satisfying $D(T) \subset C$ and let $J_r = (T + rT)^{-1} J$ for all $r > 0$. Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset \text{int}(D(f))$. Assume that $T^{-1}0 \neq \emptyset$, for arbitrary point $x_1 \in C_1$ with $C_1 = C$, generate a sequences $\{x_n\}$ by

$$y_n = J^{-1}(\alpha_n x_n + (1 - \alpha_n) J x_n), \quad C_{n+1} = \{z \in C_n : G(z, J x_n) \leq G(x_n, J x_n)\},$$  (11)

$$x_{n+1} = \Pi_{C_{n+1}} x_1$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{x_n\}$ is a sequence in $(0, \infty)$. If $\lim \inf_{n} (1 - \alpha_n) > 0$ and $\lim \sup_{n} \alpha_n = \infty$, then $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0} x_1$.

**Proof.** We first show that $C_{n+1}$ is closed and convex for each $n \geq 1$. Clearly $C_1$ is closed and convex. Suppose that $C_n$ is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know that $G(z, J x_n) \leq G(z, J x_n)$ is equivalent to

$$2 \langle z, J x_n - J y_n \rangle \leq \|x_n\|^2 - \|y_n\|^2.$$
Therefore, \( C_{n+1} \) is closed and convex. This implies that 
\( \Pi_{C_{n+1}} x_1 \) is well defined.

Next, we will show by induction that \( T^{-10} \subset C_n \) for all \( n \in \mathbb{N} \). It is obvious that \( T^{-10} \subset C_{1n} \supset C \). Suppose that \( T^{-10} \subset C_n \) for some \( n \in \mathbb{N} \). Let \( q \in T^{-10} \), put \( v_n = J_n x_n \) for all \( n \geq 1 \) and we have

\[
G(q, Jy_n) = G(q, \alpha_n J_n x_n + (1 - \alpha_n) Jv_n) \\
\leq \|q\|^2 - 2\alpha_n \langle q, J_n x_n \rangle + (1 - \alpha_n) \|Jv_n\|^2 + 2\rho f(x_n) \\
\leq \|q\|^2 - 2\alpha_n \|J_n x_n\| - 2(1 - \alpha_n) \langle q, Jv_n \rangle \\
+ \alpha_n \|J_n x_n\|^2 + (1 - \alpha_n) \|Jv_n\|^2 + 2\rho f(q) \\
= \alpha_n G(q, J_n x_n) + (1 - \alpha_n) G(q, Jv_n) \\
\leq \alpha_n G(q, J_n x_n) + (1 - \alpha_n) G(q, J_n x_n) \\
= G(q, J_n x_n).
\]

(12)

So, \( p \in C_{n+1} \). That is \( T^{-10} \subset C_{n+1} \). Consequently, \( T^{-10} \subset C_n \), for all \( n \in \mathbb{N} \). This implies that \( \{x_n\} \) is well defined. Since \( f : E \to \mathbb{R} \) is convex and lower semicontinuous, from Lemma 4, we known that there exist \( x^* \in E^* \) and \( \alpha \in \mathbb{R} \) such that

\[
f(x) \geq \langle x, x^* \rangle + \alpha, \forall x \in E.
\]

For \( x_n \in E \), it follows that

\[
G(x_n, Jx_1) = \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\rho f(x_n) \\
\geq \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\rho \langle x_n, x^* \rangle \\
+ 2\rho \alpha \\
= \|x_n\|^2 - 2\|x_n\| \|Jx_1 - \rho x^*\| + \|x_1\|^2 \\
+ 2\rho \alpha \\
= (\|x_n\|^2 - \|Jx_1 - \rho x^*\|^2) + \|x_1\|^2 \\
- \|Jx_1 - \rho x^*\|^2 + 2\rho \alpha.
\]

(13)

For each \( q \in T^{-10} \subset C_n \) and \( x_n = \Pi_{C_n} x_1 \), by the definition of \( C_n \) it follows from (13) that

\[
G(q, Jx_1) \geq G(x_n, Jx_1) \\
\geq (\|x_n\|^2 - \|Jx_1 - \rho x^*\|^2) + \|x_1\|^2 \\
- \|Jx_1 - \rho x^*\|^2 + 2\rho \alpha.
\]

This implies that \( \{x_n\} \) is bounded and so are \( \{y_n\} \) and \( \{G(q, J_n x_1)\} \).

By the fact that \( x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n \) and \( x_n = \Pi_{C_n} x_1 \), it follows by Lemma 6, we get

\[
0 \leq (\|x_n\|^2 - \|x_{n+1}\|^2)^2 \\
\leq \phi(x_{n+1}, x_n) \\
\leq G(x_{n+1}, Jx_1) - G(x_n, Jx_1).
\]

(14)

This implies that \( \{G(x_n, Jx_1)\} \) is nondecreasing. So, \( \lim_{n \to \infty} G(x_n, Jx_1) \) exist.

For any \( m > n \), \( x_n = \Pi_{C_n} x_1, x_m = \Pi_{C_m} x_1 \in C_m \subset C_n \) and from (14), we have

\[
\phi(x_m, x_n) \leq G(x_m, Jx_1) - G(x_n, Jx_1).
\]

Taking \( m, n \to \infty \), we have \( \phi(x_m, x_n) \to 0 \). From Lemma 1 we get that \( \|x_n - x_m\| \to 0 \). Hence, \( \{x_n\} \) is a Cauchy sequence. Since \( E \) is a Banach space and \( C_n \) is closed and convex, we can assume that there exists \( p \in C \) such that \( x_n \to p \in C \) as \( x_n \to \infty \). In particular, since \( \lim_{n \to \infty} G(x_n, Jx_1) \) exist from (14), we also have

\[
\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0.
\]

(15)

It follow from Lemma 1 that

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]

(16)

Since \( J \) is uniformly norm-to-norm continuous on bounded subsets of \( E \), we obtain that

\[
\lim_{n \to \infty} \|Jx_{n+1} - Jx_n\| = 0.
\]

(17)

From definition of \( C_{n+1} \) and \( x_{n+1} = \Pi_{C_{n+1}} x_1 \), we have

\[
G(x_{n+1}, Jy_n) \leq G(x_{n+1}, Jx_{n+1}).
\]

Therefore, we obtain that

\[
\|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n \rangle + \|y_n\|^2 + 2\rho f(x_{n+1}) \\
\leq \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_{n+1} \rangle + \|x_{n+1}\|^2 + 2\rho f(x_{n+1})
\]

then

\[
\|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n \rangle + \|y_n\|^2 \\
\leq \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_{n+1} \rangle + \|x_{n+1}\|^2
\]

so,

\[
\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n).
\]

From Lemma 1 and (15), it follows that

\[
\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0,
\]

(18)

and

\[
\lim_{n \to \infty} \|Jx_{n+1} - Jy_n\| = 0.
\]

(19)

From (12), we have

\[
G(q, Jv_n) \geq \frac{1}{1 - \alpha_n}(G(q, Jy_n) - \alpha_n G(q, Jx_n))
\]

and from Lemma 9, we observe that

\[
\phi(v_n, x_n) \leq G(q, Jx_n) - G(q, Jv_n) \\
\leq G(q, Jx_n) - G(q, Jx_{n+1}) \\
\leq G(q, Jx_n) - \frac{1}{1 - \alpha_n} G(q, Jy_n) - \alpha_n G(q, Jx_n) \\
= \frac{1}{1 - \alpha_n} G(q, Jx_n) - G(q, Jy_n) \\
= \frac{1}{1 - \alpha_n} (\|x_n\|^2 - \|y_n\|^2 - 2\langle q, Jx_n - Jy_n \rangle) \\
\leq \frac{1}{1 - \alpha_n} (\|x_n\|^2 - \|y_n\|^2 + 2\langle q, Jx_n - Jy_n \rangle) \\
\leq \frac{1}{1 - \alpha_n} (\|x_n\|^2 - \|y_n\|^2 + 2\|q\| \|Jx_n - Jy_n\|) \\
\leq \frac{1}{1 - \alpha_n} (\|x_n\|^2 - \|y_n\|^2 + 2\|q\| \|Jx_n - Jy_n\|).
\]

(20)

Applied from (18), (19) and \( \lim \inf_{n \to \infty} (1 - \alpha_n) > 0 \), we get

\[
\lim_{n \to \infty} \phi(v_n, x_n) = 0.
\]

(21)
From Lemma 1, we get
\[ \lim_{n \to \infty} \|v_n - x_n\| = 0. \] (22)

Since \( J \) is uniformly norm-to-norm continuous, we have
\[ \lim_{n \to \infty} \|Jx_n - Jv_n\| = 0. \] (23)

Noticing the condition \( r_n > 0 \), it follows that
\[ \lim_{n \to \infty} \frac{1}{r_n} \|Jx_n - Jv_n\| = 0. \] (24)

Therefore,
\[ \lim_{n \to \infty} \|T_n x_n\| = \lim_{n \to \infty} \frac{1}{r_n} \|Jx_n - JJ_r x_n\| = \lim_{n \to \infty} \frac{1}{r_n} \|Jx_n - Jv_n\| = 0. \] (25)

For \( (w, w^*) \in T \), from the monotonicity of \( T \), we have \( \langle w - v_n, w^* - T_n x_n \rangle \geq 0 \) for all \( n \geq 0 \). Letting \( n \to \infty \), we get \( \langle w - p, w^* \rangle \geq 0 \). From the maximality of \( T \), we have \( p \in T^{-10} \). Finally, we show that \( p = \Pi_{T^{-10} x_1} \). Since \( F \) is closed and convex set from Lemma 5, we have \( \Pi_{T^{-10} x_1} \) is a single value, denote by \( v \). From \( x_n = \Pi_{C_n} x_1 \) and \( v \in F \subset C_n \), we also have
\[ G(x_n, Jx_1) \leq G(v, Jx_1), \forall n \geq 1. \]

By definition of \( G \) and \( f \), we know that, for each given \( x \), \( G(y, Jx) \) is convex and lower semicontinuous with respect to \( y \). So
\[ G(p, Jx_1) \leq \liminf_{n \to \infty} G(x_n, Jx_1) \leq \limsup_{n \to \infty} G(x_n, Jx_1) \leq G(v, Jx_1). \]

From definition of \( \Pi_{T^{-10} x_1} \) and \( p \in T^{-10} \), we can conclude that \( v = p = \Pi_{T^{-10} x_1} \) and \( x_n \to p \) as \( n \to \infty \). This completes the proof.

Taking \( f(y) = 0 \) for all \( y \in E \), we have \( G(y, Jx) = \phi(y, x) \) and \( \Pi_{C} x = \Pi_{C} \). From Theorem 1 we obtain the following corollary.

**Corollary 1:** Let \( C \) be a nonempty closed convex subset of a uniformly smooth Banach space \( E \). Let \( T \subset E \times E^* \) be a maximal monotone operator satisfying \( D(T) \subset C \) and let \( J_{r_n} = (J + r_n T)^{-1} J \) for all \( r_n > 0 \). Assume that \( T^{-10} \neq \emptyset \). For arbitrary point \( x_1 \in C_1 \) with \( C_1 = C \), generate a sequences \( \{x_n\} \) by
\[
\begin{cases}
  y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JJ_r x_n), \\
  C_{n+1} = \{z \in C_n : \phi(z, Jy_n) \leq \phi(z, Jx_n)\}, \\
  x_{n+1} = \Pi_{C_{n+1}} x_1
\end{cases}
\] (26)

where \( \{\alpha_n\} \) is a sequence in \([0,1]\) and \( \{r_n\} \) is a sequence in \((0,\infty)\). If \( \liminf_{n \to \infty} (1 - \alpha_n) > 0 \) and \( \limsup_{n \to \infty} r_n = \infty \), then \( \{x_n\} \) converges strongly to \( \Pi_{T^{-10} x_1} \).

### IV. APPLICATIONS

**A. Convex minimization problem**

In this section, we study the problem for finding a minimizer of a proper lower semicontinuous convex function in Banach spaces.

A proper function \( F : E \to (-\infty, \infty] \) is said to be convex if
\[ F(\alpha x + (1 - \alpha) y) \leq \alpha F(x) + (1 - \alpha) F(y), \] (27)
for all \( x, y \in E \) and \( \alpha \in (0,1) \).

The function \( F \) is said to be lower semicontinuous if the set \( \{x \in E : F(x) \leq r\} \) is closed in \( E \) for all \( r \in \mathbb{R} \). For a proper lower semicontinuous convex function \( F : E \to (-\infty, \infty] \), the subdifferential \( \partial F \) of \( F \) is defined by
\[ \partial F(x) = \{x^* \in E^* : F(x) + \langle y - x, x^* \rangle \leq F(y), \forall y \in E\}. \] (28)
for all \( x \in E \). It is easy to see that \( 0 \in \partial F(u) \) if and only if \( F(u) = \min_{x \in E} F(x) \). Rockafellar [20] proved that the subdifferential mapping \( \partial F \subset E \times E^* \) of \( F \) is a maximal monotone operator.

**Lemma 10:** (Takahashi [21]) Let \( E \) be a Banach space, let \( F : (-\infty, \infty) \) be a proper lower semicontinuous convex function and let \( g : E \to \mathbb{R} \) be a continuous convex function. Then
\[ \partial(F + g)(x) = \partial F(x) + \partial g(x), \] (29)
for all \( x \in E \).

**Theorem 2:** Let \( C \) be a nonempty closed convex subset of a uniformly smooth Banach space \( E \). Let \( F \to (-\infty, \infty) \) be a proper lower semicontinuous convex function and let \( f : E \to \mathbb{R} \) be a convex and lower semicontinuous function with \( C \subset \text{int}(D(f)) \). Assume that \( (\partial F)^{-1} \neq \emptyset \), for arbitrary point \( x_1 \in C_1 \) with \( C_1 = C \), generate a sequences \( \{x_n\} \) by
\[
\begin{cases}
  z_n = \arg\min_{u \in E} F(u) + \frac{1}{2r_n} \|u\|^2 - \frac{1}{r_n} \langle u, Jx_1 \rangle, \\
  y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\
  C_{n+1} = \{z \in C_n : \phi(z, Jy_n) \leq \phi(z, Jx_n)\}, \\
  x_{n+1} = \Pi_{C_{n+1}} x_1
\end{cases}
\] (30)

where \( \{\alpha_n\} \) is a sequence in \([0,1]\) and \( \{r_n\} \subset (0,\infty) \). If \( \liminf_{n \to \infty} (1 - \alpha_n) > 0 \) and \( \limsup_{n \to \infty} r_n = \infty \), then \( \{x_n\} \) converges strongly to \( \Pi_{T^{-10} x_1} \).

**Proof.** We known that \( (\partial F)^{-1} \) is a maximal monotone operator. For \( w \in E \) and \( r > 0 \), let \( J_r \) be the resolvent of \( (\partial F) \), we have
\[ J_r w \in J_r(w) + \partial r(\partial F)(J_r w). \]

Hence
\[ 0 \in (\partial F)(J_r w) + \frac{1}{r} J_r(w) - \frac{1}{r} J_r w = \partial F + \frac{1}{2r} \|u\|^2 - \frac{1}{r} \langle u, J_r w \rangle, \]
that is
\[ J_r w = \arg\min_{u \in E} F(u) + \frac{1}{2r} \|u\|^2 - \frac{1}{r} \langle u, J_r w \rangle. \]

Since \( z_n = J_r x_n \) for all \( n = 1, 2, 3, \ldots \) By Theorem 1, \( \{x_n\} \) converges strongly to \( \Pi_{T^{-10} x_1} \).
B. Variational inequality problem

Let $E$ be a real Banach space and let $C$ be a nonempty closed and convex subset of $E$ and $A : C \to E^*$ be an operator. The variational inequality problem for an operator $A$ is to find $x \in C$ such that

$$\langle y - x, Ax \rangle \geq 0, \quad \forall y \in C. \quad (31)$$

The set of solution of 31 is denote by $VI(A, C)$.

Let $A$ be a monotone mapping of $C$ into $E^*$ which is said to be hemicontinuous if for all $x, y \in C$, the mapping $h$ of $[0, 1]$ into $E^*$, defined by $h(t) = A((1-t)x + ty)$, is continuous with respect to the weak* topology of $E^*$. We define by $N_C(v)$ the normal cone to $C$ at a point $v \in C$, that is,

$$N_C(v) = \{ x^* \in E^* : \langle y - v, x^* \rangle \leq 0, \quad \forall y \in C \}. \quad (32)$$

**Lemma 11:** (Rockafellar [19]). Let $C$ be a nonempty, closed convex subset of a Banach space $E$ and $A$ a monotone, hemicontinuous operator of $C$ into $E^*$. Let $T \subset E \times E^*$ be an operator defined as follows:

$$TV = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & \text{otherwise}. \end{cases} \quad (33)$$

Then $T$ is maximal monotone and $T^{-1}(0) = VI(A, C)$.

**Theorem 3:** Let $C$ be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $A : C \to E^*$ be a single-value, monotone and hemicontinuous operator and let $f : E \to R$ be a convex and lower semicontinuous function with $C \subset \text{int}(D(f))$. Assume that $VI(A, C) \neq \emptyset$. For arbitrary point $x_1 \in C_1$ with $C_1 = C$, generate a sequences $\{ x_n \}$ by

$$\begin{cases} z_n = VI(A + \frac{1}{r_n}(J - Jx_n), C), \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\ C_{n+1} = \{ z \in C_n : G(z, Jy_n) \leq G(z, Jx_n) \}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n = 1, 2, 3, \ldots \end{cases} \quad (34)$$

where $\{ \alpha_n \}$ is a sequence in $[0, 1]$ and $\{ r_n \} \subset (0, \infty)$. If $\lim inf_{n \to \infty} (1 - \alpha_n) > 0$ and $\lim inf_{n \to \infty} r_n = \infty$, then $\{ x_n \}$ converges strongly to $\Pi_{VI(A, C)} x_1$.

**Proof.** For $w \in E$ and $r > 0$, by Theorem 11, we have

$$Jw \in J(J(w)) + rT(J(w)).$$

Hence

$$-A(J(w)) + \frac{1}{r} (Jw - J(J(w))) \in NC(J(w)).$$

It follows that

$$\langle y - J(w), A(J(w)) + \frac{1}{r} (J(J(w)) - Jw) \rangle \geq 0,$$

for all $y \in C$. That is $J(w) = J(J(w))$. Since $z_n = J_{r_n} x_n$ for all $n = 1, 2, 3, \ldots$, by Theorem 1, $\{ x_n \}$ converges strongly to $\Pi_{VI(A, C)} x_1$.

V. Conclusion

In this paper, we extend and improve the iterative scheme for resolvents of maximal monotone operators from using the generalized projection to using the generalized $J$-projection. We apply this result to the convex minimization and variational inequality problems in Banach spaces.

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