ON NON-PERTURBATIVE ANDERSON LOCALIZATION FOR $C^\alpha$ POTENTIALS
GENERATED BY SHIFTS AND SKEW-SHIFTS

JACKSON CHAN, MICHAEL GOLDSTEIN, WILHELM SCHLAG

ABSTRACT. In this paper we address the question of proving Anderson localization (AL) for the operator

$$[H(x,\omega)\psi](n) := -\varphi(n+1) - \varphi(n-1) + V(T^n_x)\psi(n), \quad n \in \mathbb{Z}$$

where $T : \mathbb{T}^2 \to \mathbb{T}^2$ is either the shift or the skew-shift and $V$ is only $C^\alpha(\mathbb{T}^2)$ for some $\alpha > 0$. We show that under the assumption of positive Lyapunov exponents, (AL) takes place for a.e. frequency, phase, and energy.

1. STATEMENT OF THE MAIN RESULTS

Consider the one-dimensional difference Schrödinger equation

$$[H(x,\omega)\psi](n) := -\varphi(n+1) - \varphi(n-1) + V(T^n_x)\psi(n) = E\psi(n), \quad n \in \mathbb{Z}$$

where $V(x)$ is a real-valued Hölder continuous function on the two-dimensional torus $\mathbb{T}^2$, and $T_\omega : \mathbb{T}^2 \to \mathbb{T}^2$ is an ergodic transformation which in this paper will be either the shift $T_\omega(x,y) = (x,y) + \omega, \omega \in \mathbb{T}^2$, or the skew-shift $T_\omega(x,y) = (x+y,y+\omega), \omega \in \mathbb{T}$. Let $M_{[a,b]}(x,\omega,E)$ be the monodromy matrix of (1.1) on the interval $[a,b]$, i.e.

$$M_{[a,b]}(x,\omega,E) = \prod_{n=a}^{b} \begin{bmatrix} V(T^n_x) - E & -1 \\ 1 & 0 \end{bmatrix}$$

Let $L(\omega,E)$ be the Lyapunov exponent of the cocycle $M_{[1,N]}(x,\omega,E), N > 0$, i.e.,

$$L(\omega,E) = \lim_{N \to \infty} N^{-1} \int_{\mathbb{T}^2} \log \|M_N(x,\omega,E)\| \, dx$$

**Theorem 1.1.** Let $V(x)$ be $C^\alpha$-smooth with some $\alpha > 0$. Assume that $L(\omega,E) > 0$ for all $\omega$ and all $E \in \mathbb{R}$. Then there exists $Q$ with $\text{mes } Q = 0$ such that for any $\omega \notin Q$ there exists $E_\omega$ with $\text{mes } E_\omega = 0$ such that for a.a. $x \in \mathbb{T}$ and all $E \notin E_\omega$ the following assertion holds: if $[H(x,\omega)\psi](n) = E\psi(n), n \in \mathbb{Z}$, for some polynomially bounded function $\psi \neq 0$, then

$$|\psi(n)| \leq C(x,\omega,E) \exp(-L(\omega,E)|n|/2)$$

for all $n \in \mathbb{Z}$.

If we take the disorder to be large, then we arrive at the following theorem ($\kappa = \kappa(\alpha) > 0$ is a small constant):

**Theorem 1.2.** Let $V(x)$ be $C^\alpha$-smooth with some $\alpha > 0$. Let $L(\omega,\lambda,E)$ be the Lyapunov exponent with potential $\lambda V(x), \lambda \in \mathbb{R}$. There exists $\lambda_0 = \lambda_0(V)$ such that for each $|\lambda| > \lambda_0$, there exists a set $E_{\lambda} \subset \mathbb{R}$ such that $\text{mes } \{\lambda^{-1}E\lambda\} < \lambda^{-\kappa}$, and $L(\omega,\lambda,E) > \frac{1}{2}\log |\lambda|$ for all $\omega$ and all $E \notin E_{\lambda}$. Moreover, for each $|\lambda| > \lambda_0$ there exists $Q_\lambda$ with $\text{mes } Q_\lambda = 0$ such that for each $\omega \notin Q_\lambda$ there exists $E_{\lambda,\omega}$ with $\text{mes } E_{\lambda,\omega} = 0$ such that for a.a. $x$ the following assertion holds: for all $E \notin E_{\lambda} \cup E_{\lambda,\omega}$, if $[H(\lambda,x,\omega)\psi](n) = E\psi(n), n \in \mathbb{Z}$ for some polynomially bounded function $\psi(n)$, then

$$|\psi(n)| \leq C(\lambda, x, \omega, E) \exp(-L(\lambda, \omega, E)|n|/2), n \in \mathbb{Z}$$

Here $H(\lambda, x, \omega)$ stands for the Schrödinger operator $[H(x,\omega)]$ with $\lambda V(x)$ in the role of $V(x)$.

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The main novel feature in these theorems is the low regularity of the potential function \( V \). While much is known about the case of analytic \( V \), see Bourgain-Goldstein [BonGol] and Bourgain-Goldstein-Schlag [BonGolSch] and for the case of almost Mathieu, Jitmoriskaya [Jit], comparatively little is known about non-analytic category. Klein [Kle] proved (AL) for positive Lyapunov exponents, and potentials from the Gevrey class. Bjerklov [Bje] showed (AL) and positive Lyapunov exponents for \( C^1 \) potentials, large disorder, and off a set of energies of positive measure. Chan [Cha] proved (AL) for large disorder, for generic \( C^3 \) potentials in a suitable sense and for a.e. energy.

The methods of this paper are in spirit related to those of [BonGol] and [BonGolSch]. Thus, the main ingredients are large deviation theorems and the elimination of resonant frequencies. Analyticity has so far played a crucial role in obtaining these tools. Hence, we need to take a very different route here. A basic principle in this paper is to reduce matters to the study of the eigenvalues as functions parameterized by the phase (the so-called Rellich functions). Firstly, we note that by Weyl's comparison theorem the determinant of the Hamiltonian \( H_{[-N,N]}(x,\omega) - E \) of \([1]\) at energy \( E \) restricted to the interval \([-N,N]\) is comparable to the product of the determinants of the Hamiltonians at energy \( E \) corresponding to \([n_j, n_{j+1}]\) where \([-N,N] = \bigcup_{j=1}^{J}[n_j, n_{j+1}]\); albeit, their ratio can be very large, namely \( \eta^{-J} \) where \( \eta \) is the distance of the spectrum of \( H_{[-N,N]}(x,\omega) \) to \( E \). This is one source of energy removal: evidently we will need to control \( \eta \). Secondly, first order eigenvalue perturbation shows that for \( C^1 \) potentials these functions are again \( C^1 \) (in fact, they also inherit the Hölder regularity of the potential). Thus, we can apply the ergodic theorem to these individual functions and then sum up over all of them to obtain a large deviation theorem for the sum of shifts of logarithms of Dirichlet determinants. A second source of energy removal arises at this stage: we will need to exclude those energies that serve as critical values of the Rellich functions.

We conclude this introduction with a heuristic discussion of the wider context of our results. In particular, we shall mention some natural ramifications that Theorems \([1]\) and \([2]\) above appear to possess.

(1) It seems natural to combine the methods of this paper with those based on subharmonicity (developed in [BonGol], [BonGolSch], [GolSch1], [GolSch2]) to show that for the case of analytic potentials the elimination of resonances for the bulk of energies (as in this work) suffices to obtain complete (AL) at all energies.

(2) The crucial component needed to make progress in (1) consists of a count of the number of the Dirichlet eigenvalues of the operator \( H_{[-N,N]}(x,\omega) \) which fall into the set \( E_\omega \). Recall that the latter is the set of “forbidden” energies, which appear in the elimination of resonances in this work.

(3) It is not hard to prove that the Hausdorff dimension of the set \( E_\omega \) is equal to zero in case of analytic potentials (as well as for \( C^K \)-smooth potentials with large \( K \)). It seems that there is a possibility to use this fact and to modify the method of [GolSch2], which is based on the multi-scale (or avalanche principle) expansion of the function \( \log |f_{[-N,N]}(z,\omega,E)| \), to evaluate the averaged (in phases) number of eigenvalues falling into \( E_\omega \). Recall that the expansion itself is valid for the shift and skew-shift, provided the Lyapunov exponent is positive (see [GolSch2]).

(4) It is not clear to what extent the results of Theorems \([1]\) and \([2]\) are optimal for smooth potentials. For example, it is unclear whether the Hausdorff dimension of the set \( E_\omega \) also vanishes for Hölder continuous potentials. In this context, it seems natural to ask the following question:

\begin{quote}
Are there any smooth potentials \( V(x) \), \( x \in \mathbb{T} \), with \( L(E,\omega) > 0 \) for all \( E,\omega \) and with purely singular continuous spectrum for a.a. \( \omega \)? In other words, the spectrum of \( H(x,\omega) \) for such a potential would fall into the set \( E_\omega \) from Theorem \([1]\) for a.a. \( \omega \).
\end{quote}

On the other hand, it looks promising to modify the technology of [Cha] to show that for “generic” smooth potentials the set \( E_\omega \) does not contribute anything to the spectrum, i.e., that complete (AL) takes place in Theorem \([1]\).

(5) The large deviations estimates and the process of elimination of resonances which are developed in Sections 2, 3 of this paper can also be established for the case of quasi-periodic Schrödinger operators on the lattice. Moreover, not only the multidimensional Laplacian, but also its long-range versions can be treated in this fashion. Taking this into account, it seems plausible to establish (AL) for
higher-dimensional quasi-periodic lattice models in the case of analytic potentials at large values of \( \lambda \). In particular, this would prove the absolute continuity of the spectrum of these models in the regime of small values of \( \lambda \).

In the work [GolKle] the ideas of this work are modified for the analysis of localization at almost all energies in the case of random potentials with fast decaying correlations and in particular for the potentials generated by the doubling map on circle.

2. LARGE DEVIATION ESTIMATES FOR THE AVERAGES OF SHIFTS AND SKEW-SHIFTS OF LOGARITHMS OF \( C^1 \)-SMOOTH FUNCTIONS

In this section, we develop a general framework of averaging of functions of the form \( \log |f(x) - \xi| \) over orbits of the shift and skew-shift. The reader will find the relevant quantitative ergodic theorems in the appendices. In the process we shall need to remove those values of \( \xi \) for which the function \( \log |f(x) - \xi| \) becomes too singular. This is comparatively easy: it will only require Fubini’s theorem. Throughout, we assume that the potential is \( C^1 \) for the sake of simplicity. The generalization to Hölder classes is elementary, see Appendix C.

**Definition 2.1.** Suppose \( f \in C^m(\mathbb{T}^2) \). If \( \alpha + \beta \leq m \), let \( B_{\alpha,\beta}(f) := \max_{x \in \mathbb{T}^2} |\partial_{x_1}^\alpha \partial_{x_2}^\beta f(x)| \). Also, if \( k \leq m \), let \( B_k(f) := \max_{\alpha + \beta \leq k} B_{\alpha,\beta}(f) \). In particular, \( B_0(f) = \max_{x \in \mathbb{T}^2} |f(x)| \). Throughout this paper, we let

\[
S_f(\xi,\delta) := \{ x \in \mathbb{T}^2 : |f(x) - \xi| < \delta \}
\]

denote level sets of \( f \).

Let \( f \in C^1(\mathbb{T}^2) \). Our first goal is to estimate

\[
(2.1) \quad \# \left\{ k \in \mathbb{N} : 1 \leq k \leq N, |f(T^k x) - \xi| < \delta \right\} = \# \left\{ k \in \mathbb{N} : 1 \leq k \leq N, T^k x \in S_f(\xi,\delta) \right\}
\]

where \( \xi \in \mathbb{R} \) is a parameter, \( 0 < \delta < 1 \), \( T : \mathbb{T}^2 \to \mathbb{T}^2 \) is the shift \( T(x_1, x_2) = (x_1 + \omega_1, x_2 + \omega_2) \), or the skew-shift \( T(x_1, x_2) = (x_1 + x_2, x_2 + \omega) \) (addition here is always mod \( \mathbb{Z}^2 \)). In order to answer (2.1), we will need to use quantitative ergodic properties of these transformations. As a preliminary step, we introduce the following functions for the purpose of mollifying given \( C^1 \) functions.

**Definition 2.2.** Given \( \tau > 0 \), let \( h_\tau \in C^4(\mathbb{R}) \) be 1-periodic such that

- \( h_\tau \geq 0 \)
- \( \text{supp} h_\tau \subset \bigcup_{k \in \mathbb{Z}} [k - \tau, k + \tau] \)
- \( \int_0^1 h_\tau(y) dy = 1 \)
- \( \max_{y \in \mathbb{R}} \left| \left( \frac{d}{dy} \right)^m h_\tau(y) \right| \lesssim \tau^{-(m+1)} \) for \( m \leq 4 \).

Moreover, we set \( \tilde{h}_\tau(x_1, x_2) = h_\tau(x_1)h_\tau(x_2) \).

The following lemma is a well-known quantitative statement concerning the mollifiers of a given \( C^1 \) function.

**Lemma 2.3.** Given \( \varphi \in C^3(\mathbb{T}^2) \) and \( \tau > 0 \), define

\[
\psi(x_1, x_2) := \int_{\mathbb{T}^2} \varphi(y_1, y_2) \tilde{h}_\tau(x_1 - y_1, x_2 - y_2) dy
\]

Then \( \psi \in C^4(\mathbb{T}^2) \) satisfies

1. \( \max_{x \in \mathbb{T}^2} |\varphi(x) - \psi(x)| \lesssim B_1(\varphi) \tau \)
2. \( B_4(\psi) \lesssim B_0(\varphi) \tau^{-4} \)

Turning to the dynamics, we will of course need a Diophantine condition. Throughout this paper, constants will be allowed to depend on the constants appearing in this definition.
The following proposition is a quantitative version of the ergodic theorem for smooth (i.e., $C^4$) functions.

**Proposition 2.5.** For sufficiently large $N$, one has (with $T$ being either the shift or skew-shift)

$$\left| \frac{1}{N} \sum_{m=1}^{N} \psi (T^m x) - \langle \psi \rangle \right| \lesssim B_4(\psi) N^{-\sigma}$$

for all $x \in \mathbb{T}^2$. The constant $\sigma$ depends on the parameters in Definition 2.4.

**Proof.** The proof can be found in Appendix A. \qed

We now turn to estimating (2.1). It will be convenient to work with $C^1$ functions instead of indicators of level sets. The following lemma introduces the standard transition between the two.

**Lemma 2.6.** Given $\delta > 0$ small, let $\chi_\delta \in C^1(\mathbb{R})$ be such that

- $0 \leq \chi_\delta \leq 1$
- $\chi_\delta(y) = 1$ for $y \in [-\delta, \delta]$
- $\text{supp} \chi_\delta \subset [-2\delta, 2\delta]$
- $\max_{x \in \mathbb{R}} |\chi_\delta'(y)| \lesssim \delta^{-1}$

Then the following holds:

1. \# \{ $1 \leq k \leq N : T^k x \in S_f(\xi, \delta)$ \} \leq \sum_{k=1}^{N} \chi_\delta \left( f(T^k x) - \xi \right)
2. $\text{mes } S_f(\xi, \delta) \leq (\chi_\delta(f(\cdot) - \xi)) = \int_{\mathbb{T}^2} \chi_\delta(f(x) - \xi) \ dx \leq \text{mes } S_f(\xi, 2\delta)$

for any real $\xi$ and positive integer $N$.

We can now apply Proposition 2.5 to deduce the required bound on (2.1).

**Corollary 2.7.** Let $f \in C^1(\mathbb{T}^2)$ and suppose $\omega$ is Diophantine. Then for any $\xi \in \mathbb{R}$, $\delta > 0$, one has

$$\frac{1}{N} \# \{ 1 \leq k \leq N : T^k x \in S_f(\xi, \delta) \} \lesssim \text{mes } S_f(\xi, 2\delta) + (1 + B_1(f)) \delta^{\frac{\sigma}{2}}$$

for all $x \in \mathbb{T}^2$ provided $N \geq \delta^{-\frac{\omega}{2}}$. Here $\sigma > 0$ is the small constant from the ergodic theorem, see Proposition 2.5.

**Proof.** Using the notations of Lemma 2.6, we have to estimate \( \frac{1}{N} \sum_{k=1}^{N} \chi_\delta \left( f(T^k x) - \xi \right) \). Note that $\varphi(x) = \chi_\delta(f(x) - \xi)$ is $C^1$ on $\mathbb{T}^2$, $B_1(\varphi) \lesssim B_1(f) \delta^{-1}$. By Lemma 2.6, given $\tau > 0$, there is $\psi \in C^4(\mathbb{T}^2)$ such that

1. \( \max_{x \in \mathbb{T}^2} |\varphi(x) - \langle \psi \rangle| \lesssim B_1(f) \delta^{-1} \tau \)
2. \( B_4(\psi) \lesssim \tau^{-6} \)

Due to (1), \( |\langle \varphi \rangle - \langle \psi \rangle| \lesssim B_1(f) \delta^{-1} \tau \) and \( \frac{1}{N} \sum_{k=1}^{N} \varphi(T^k x) - \frac{1}{N} \sum_{k=1}^{N} \psi(T^k x) \lesssim B_1(f) \delta^{-1} \tau \) for all $x \in \mathbb{T}^2$. Therefore,

$$\frac{1}{N} \# \{ 1 \leq k \leq N : T^k x \in S_f(\xi, \delta) \} \leq \frac{1}{N} \sum_{k=1}^{N} \varphi(T^k x) \lesssim \langle \varphi \rangle + B_4(\psi) N^{-\sigma} + B_1(f) \delta^{-1} \tau \lesssim \text{mes } S_f(\xi, 2\delta) + \tau^{-4} N^{-\sigma} + B_1(f) \delta^{-1} \tau.$$

The assertion follows if we take $\tau = \delta^{\frac{2}{\sigma}}$. \qed

Since we are not making a non-degeneracy assumption on $f$ (in particular, $f$ may be constant) it will be necessary to remove certain values of $\xi$ for which $S_f(\xi, \delta)$ is very large. This can be done easily by means of Fubini’s theorem.
Corollary 2.8. Let $\omega$ be Diophantine. Given $\delta > 0$, there exists a set $\mathcal{E}_\delta \subset \mathbb{R}$, $\text{mes} \mathcal{E}_\delta \leq \delta^{\frac{1}{2}}$ such that for any $\xi \notin \mathcal{E}_\delta$, one has $\frac{1}{N} \# \{1 \leq k \leq N : T^k x \in S_f(\xi, \delta)\} \leq (1 + B_1(f)) \delta^{\frac{1}{2}}$ for all $x \in \mathbb{T}^2$ provided $N \geq \delta^{-\frac{1}{4}}$.

Proof. Clearly, $\text{mes} \{ (x, \xi) \in \mathbb{T}^2 \times \mathbb{R} : x \in S_f(\xi, 2\delta)\} = 4\delta$. By Fubini’s Theorem, there exists $\mathcal{E}_\delta \subset \mathbb{R}$, $\text{mes} \mathcal{E}_\delta \leq 4\delta^{\frac{1}{2}}$, such that $S_f(\xi, 2\delta) \leq \delta^{\frac{1}{2}}$ for $\xi \notin \mathcal{E}_\delta$. The assertion now follows from Corollary 2.7. □

Remark 2.9. Let $\mathcal{E}_\delta$ be as follows

$$\mathcal{E}_\delta = \{ \xi : \text{mes} S_f(\xi, \delta) > \delta^{\frac{1}{2}} \}$$

Given an arbitrary subset $\mathcal{E} \subset \mathbb{R}$ and $r > 0$ introduce

$$|\mathcal{E}|(r) = \{ \xi \in \mathbb{R} : \text{dist}(\xi, \mathcal{E}) < r \}$$

Note that if $\xi \in \mathcal{E}_\delta$ and $|\xi_1 - \xi| < r$, then

$$S_f(\xi_1, \delta) \subset S_f(\xi, \delta + r)$$

Define

$$\mathcal{E}_{\delta, \delta_1} = \{ \xi : \text{mes} S_f(\xi, \delta) > \delta_1^{\frac{1}{2}} \}$$

Then $\text{mes} \mathcal{E}_{\delta, \delta} \leq \delta^{\frac{1}{2}}$. On the other hand, $|\mathcal{E}_\delta|(r) \subset \mathcal{E}_{\delta, \delta}$ for $r < \delta$. In particular, $\text{mes} |\mathcal{E}_\delta|(r) \leq \delta^{\frac{1}{2}}$.

Our next goal is to estimate

$$(2.2) \quad \text{mes} \left\{ x : \left| \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k x) - \xi| - \langle \log |f(\cdot) - \xi| \rangle \right| > \delta \right\}.$$ 

This is of course motivated by the large deviation theorems in [BonGol], [GolSch1], and [GolSch2]. As a first step, note that

$$(2.3) \quad \frac{1}{N} \sum_{1 \leq k \leq N} \log |f(T^k x) - \xi| + \frac{1}{N} \# \{1 \leq k \leq N : T^k x \in S_f(\xi, \delta)\} \cdot \log \left( \min_{1 \leq k \leq N} |f(T^k x) - \xi| \right)$$

$$\leq \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k x) - \xi| \leq \frac{1}{N} \sum_{1 \leq k \leq N} \log |f(T^k x) - \xi|$$

since $\delta < 1$. The following lemma shows that the term involving the minimum in (2.3) can be controlled at the expense of removing an exponentially small set of $x$. In what follows, we use the notation

$$(2.4) \quad [-B_0(f), B_0(f)] := \mathcal{J}_0(f)$$

where $f$ will be a given $C^1$ function.

Lemma 2.10. Let $\kappa > 0$ be arbitrary. There exists $\mathcal{E}(N) \subset \mathbb{R}$, $\text{mes} \mathcal{E}(N) < \exp \left( -\frac{N^\kappa}{2} \right)$ such that for any $\xi \notin \mathcal{E}(N)$, one has

$$(2.5) \quad \text{mes} \left\{ x \in \mathbb{T}^2 : \log \left( \min_{1 \leq k \leq N} |f(T^k x) - \xi| \right) < -N^\kappa \right\} \leq \exp \left( -\frac{1}{4} N^\kappa \right).$$

provided $N \geq N_0(\kappa)$.

Proof. This follows from Fubini’s theorem. Indeed, with $\mathcal{J}_0(f)$ as above,

$$(2.6) \quad \int_{\mathcal{J}_0(f)} \text{mes} \left\{ x \in \mathbb{T}^2 : \log \left( \min_{1 \leq k \leq N} |f(T^k x) - \xi| \right) < -N^\kappa \right\} \, d\xi$$

$$\leq \sum_{k=1}^{N} \int_{\mathcal{J}_0(f)} \text{mes} \left\{ x \in \mathbb{T}^2 : |f(T^k x) - \xi| < -N^\kappa \right\} \, d\xi$$

$$\leq 2N e^{-N^\kappa}$$
Hence, we can remove a set of $\xi$ of measure not exceeding $e^{-\frac{1}{2}N^\kappa}$ so that the integrand in (2.6) is at most $e^{-\frac{1}{2}N^\kappa}$, as claimed. \hfill $\square$

Remark 2.11. One can see that the following version of Lemma 2.10 holds: For any $x_2^{(0)} \in \mathbb{T}$ there exists $\mathcal{E}^{(1)}(x_2^{(0)}, N) \subset \mathbb{R}$ with measure $\leq \exp(-N^\kappa/2)$ such that for any $\xi \notin \mathcal{E}^{(1)}(x_2^{(0)}, N)$ one has

$$\text{mes} \left\{ x_1 \in \mathbb{T} : \log \left( \min_{1 \leq k \leq N} \left| f(T^k(x_1, x_2^{(0)}) - \xi) \right| \right) < -N^\kappa \right\} \leq (-\frac{1}{4}N^\kappa) .$$

Combining this lemma with (2.3) we obtain the following:

Corollary 2.12. Let $\delta > 0$ and $\kappa > 0$ be small. Then for all $\xi \notin \mathcal{E}(N)$ there exists $B(\xi) \subset \mathbb{T}^2$, $\text{mes} B(\xi) \leq \exp(-\frac{1}{4}N^\kappa)$ such that for any $x \notin B(\xi)$ one has

$$\frac{1}{N} \sum_{1 \leq k \leq N} \log |f(T^k x) - \xi| - \frac{1}{N^{1-\kappa}} \sup_{y \in \mathbb{T}^2} \left( \# \left\{ 1 \leq k \leq N : T^k y \in S_f(\xi, \delta) \right\} \right) \leq \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k x) - \xi| \leq \frac{1}{N} \sum_{1 \leq k \leq N} \log |f(T^k x) - \xi|$$

Here $N \geq N_0(\kappa)$ is a positive integer.

In order to bound the averages on the left and right-hand sides here we introduce the following auxiliary function.

Definition 2.13. Henceforth, given $\delta > 0$ small we define $\rho = \rho_\delta \in C^2(\mathbb{R})$ to be such that

- $\rho(y) = |y|$ if $|y| \geq \delta$ and $\rho(y) \geq |y|$ for all $y$
- $\frac{\delta}{2} \leq \rho(y) \leq \delta$ if $y \in (-\delta, \delta)$
- $\max_{y \in \mathbb{R}} |\rho''(y)| \lesssim \delta^{-1}$

The significance of this definition can be seen from the following lemma.

Lemma 2.14. If $f \in C^3(\mathbb{T}^2)$, and $0 < \delta < 1$, then for $\xi \notin \mathcal{E}_{25, \delta}$

$$\frac{1}{N} \sum_{k=1}^{N} \log \rho \left( f(T^k x) - \xi \right) \leq \frac{1}{N} \sum_{1 \leq k \leq N} \log |f(T^k x) - \xi| \leq \frac{1}{N} \sum_{k=1}^{N} \log \rho \left( f(T^k x) - \xi \right) + (1 + B_1(f)) \delta^\frac{1}{2}$$

Proof. The first inequality is clear since $\delta < 1$. For the second, note that

$$\frac{1}{N} \sum_{1 \leq k \leq N} \log |f(T^k x) - \xi| - \frac{1}{N} \# \left\{ 1 \leq k \leq N : T^k x \in S_f(\xi, \delta) \right\} |\log(\delta/2)| \leq \frac{1}{N} \sum_{k=1}^{N} \log \rho \left( f(T^k x) - \xi \right)$$

By Corollary 2.12

$$\frac{1}{N} \# \left\{ 1 \leq k \leq N : T^k x \in S_f(\xi, \delta) \right\} \cdot \left| \log \frac{\delta}{2} \right| \lesssim (1 + B_1(f)) \delta^\frac{1}{2} \left| \log \frac{\delta}{2} \right|$$

and the lemma follows. \hfill $\square$

Note that we also need to exclude a set of $\xi$ in order to prevent the averages $|\log |f(\cdot) - \xi)|$ in (2.3) from being too large.

Lemma 2.15. Given $R > 0$ there exists $\mathcal{L}_R \subset J_0(f)$, $\text{mes} \mathcal{L}_R \lesssim B_0 R^{-1}$, such that for any $\xi \notin \mathcal{L}_R$, one has

1. $\int_{\mathbb{T}^2} |\log |f(x) - \xi||^2 dx \leq (\log B_0)^2 R$
2. $\int_{\mathbb{T}^2} \log |f(x) - \xi| dx - \int_{\mathbb{T}^2 \setminus S_f(\xi, \delta)} \log |f(x) - \xi| dx \leq (\log B_0) R^\frac{1}{2} \left[ \text{mes} S_f(\xi, \delta) \right]^\frac{1}{2}$
Proof. Since \(|\log |f(x) − ξ||^2 > 0\), by Fubini’s theorem,
\[
\int_{\mathcal{J}_0(f)} \int_{\mathbb{T}^2} |\log |f(x) − ξ||^2 \, dx \, dξ = \int_{\mathbb{T}^2} \int_{\mathcal{J}_0(f)} |\log |f(x) − ξ||^2 \, dξ \, dx
\]
\[
\lesssim B_0 (\log B_0)^2
\]
Hence, there exists \(\mathcal{L}_R \subset \mathcal{J}_0(f)\), \(\text{mes} \mathcal{L}_R \leq B_0 R^{-1}\) such that \(\int_{\mathbb{T}^2} |\log |f(x) − ξ||^2 \, dx \leq (\log B_0)^2 R\) for \(\xi \notin \mathcal{L}_R\). This proves (1). To prove (2), we use Cauchy-Schwarz:
\[
\left| \int_{\mathbb{T}^2} \log |f(x) − ξ| \, dx - \int_{\mathbb{T}^2 \setminus S_f(ξ, δ)} \log |f(x) − ξ| \, dx \right| \leq \int_{S_f(ξ, δ)} |\log |f(x) − ξ|| \, dx
\]
\[
\leq \left[ \int_{\mathbb{T}^2} |\log |f(x) − ξ||^2 \, dx \right]^{\frac{1}{2}} \text{mes } S_f(ξ, δ)^{\frac{1}{2}}
\]
\[
\leq (\log B_0) R^{\frac{1}{2}} \text{mes } S_f(ξ, δ)^{\frac{1}{2}}
\]
and the lemma follows. \(\square\)

Now the same for the regularized functions \(\rho (f(x) − ξ)\):

Corollary 2.16. There exists \(\mathcal{M}_δ \subset \mathcal{J}_0(f)\), \(\text{mes } \mathcal{M}_δ \lesssim B_0 δ^{\frac{1}{4}}\), such that for any \(\xi \notin \mathcal{M}_δ\) one has
\[
\left| \int_{\mathbb{T}^2} \log \rho (f(x) − ξ) \, dx - \int_{\mathbb{T}^2} \log |f(x) − ξ| \, dx \right| \lesssim (\log B_0) δ^{\frac{1}{4}}.
\]

Proof. Using the notations of the previous lemma, suppose \(\xi \notin \mathcal{L}_R \cup \mathcal{E}_δ\) where \(\mathcal{E}_δ\) is as in Remark 2.9. Then
\[
\left| \int_{\mathbb{T}^2} \log \rho (f(x) − ξ) \, dx - \int_{\mathbb{T}^2} \log |f(x) − ξ| \, dx \right| \leq \int_{S_f(ξ, δ)} |\log \rho (f(x) − ξ)| \, dx + \int_{S_f(ξ, δ)} |\log |f(x) − ξ|| \, dx
\]
\[
\leq [\text{mes } S(ξ, δ)] \left| \frac{\log δ}{2} \right| + (\log B_0) R^{\frac{1}{4}} \text{mes } S_f(ξ, δ)^{\frac{1}{4}}
\]
\[
\leq δ^{\frac{1}{4}} \left| \frac{\log δ}{2} \right| + (\log B_0) R^{\frac{1}{4}} δ^{\frac{1}{4}}.
\]
Take \(R = δ^{-\frac{1}{2}}\), \(\mathcal{M}_δ := \mathcal{L}_δ^{-\frac{1}{4}} \cup \mathcal{E}_δ\). Then \(\left| \int_{\mathbb{T}^2} \log \rho (f(x) − ξ) \, dx - \int_{\mathbb{T}^2} \log |f(x) − ξ| \, dx \right| \lesssim (\log B_0) δ^{\frac{1}{4}}\) for any \(\xi \notin \mathcal{M}_δ\). Moreover, \(\text{mes } \mathcal{M}_δ \lesssim B_0 δ^{\frac{1}{4}}\). \(\square\)

We are finally ready to state a large deviation theorem for averages of \(C^1\) functions, albeit at the expense of removing some dangerous level sets (i.e., values of \(ξ\)).

Theorem 2.17. Let \(f \in C^1 (\mathbb{T}^2)\) and suppose \(ω\) satisfies a Diophantine condition. Then there is a sufficiently small \(κ > 0\) so that for all large \(N ≥ N_0(κ)\) there exists \(\mathcal{T}(N) \subset \mathcal{J}_0(f)\), \(\text{mes } \mathcal{T}(N) < N^{-κ}\), such that for any \(ξ \in \mathcal{J}_0(f) \setminus \mathcal{T}(N)\) one has
\[
\text{mes } \left\{ x : \left| \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k x) − ξ| - \langle \log |f(⋅) − ξ| \rangle \right| > N^{-κ} \right\} \leq \exp (-N^κ)
\]
(2.7)

Moreover, one has
\[
\sup \left. \frac{1}{N} \sum_{k \in [1, N] \setminus \mathcal{B}} \log |f(T^k x) − ξ| \leq \langle \log |f(⋅) − ξ| \rangle + N^{-κ}
\]
(2.8)

for any \(ξ \in \mathcal{J}_0(f) \setminus \mathcal{T}(N)\).
Proof. Let $\rho$ be as in Definition 2.13 with $\delta$ to be specified later. Then
\[
\left| \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k x) - \xi| - \langle \log |f(\cdot) - \xi| \rangle \right| \leq \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k x) - \xi| - \frac{1}{N} \sum_{k=1}^{N} \log \rho(f(T^k x) - \xi) \\
+ \frac{1}{N} \sum_{k=1}^{N} \log \rho(f(T^k x) - \xi) - \int_{T^2} \log \rho(f(y) - \xi) dy \\\n+ \int_{T^2} \log \rho(f(y) - \xi) dy - \int_{T^2} \log |f(y) - \xi| dy.
\]

Let $B_{N,\xi} = \{y \in T^2 : \min_{1 \leq k \leq N} |f(T^k y) - \xi| \leq e^{-N\kappa}\}$, $\mathcal{E}(N)$ be as in Lemma 2.10 and $\mathcal{E}_{28,\delta}$ be as in Remark 2.9. If $\xi \notin \mathcal{E}(N) \cup \mathcal{E}_{28,\delta}$, and $x \notin B_{N,\xi}$, then
\[
\left| \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k x) - \xi| - \frac{1}{N} \sum_{k=1}^{N} \log \rho(f(T^k x) - \xi) \right| \leq \frac{1}{N} \# \left\{ 1 \leq k \leq N : T^k x \in S_{\varphi}(\xi, \delta) \right\} \left[ \log \min_{1 \leq k \leq N} |f(T^k x) - \xi| + \log \frac{\delta}{2} \right] \leq (1 + B_1(f)) \delta^\frac{1}{2} \left[ N^\kappa + \log \frac{\delta}{2} \right].
\]

Moreover, mes $B_{N,\xi} \leq \exp\left(-\frac{1}{4}N\kappa\right)$. Let $\varphi(y) = \log \rho(f(y) - \xi)$. By Lemma 2.13 for any $\tau > 0$, there is $\psi \in C^1(T^2)$ such that
\[
(1) \max_{y \in T^2} |\varphi(y) - \psi(y)| \lesssim B_1(\varphi) \tau \leq B_1(f) \delta^{-1}\tau, \\
(2) B_4(\psi) \lesssim B_0(\varphi) \tau^{-4} \leq (\log(\delta/2) + 1 + B_0(f)) \tau^{-4}
\]

Then with some $\sigma > 0$,
\[
\left| \frac{1}{N} \sum_{k=1}^{N} \varphi(T^k x) - \langle \varphi \rangle \right| \leq \left| \frac{1}{N} \sum_{k=1}^{N} \varphi(T^k x) - \frac{1}{N} \sum_{k=1}^{N} \psi(T^k x) \right| + \left| \frac{1}{N} \sum_{k=1}^{N} \psi(T^k x) - \langle \psi \rangle \right| + |\langle \psi \rangle - \langle \varphi \rangle| \lesssim B_1(f) \delta^{-1}\tau + (\log(\delta/2) + 1 + B_0(f)) \tau^{-4} N^{-\sigma}
\]
provided $N$ is sufficiently large (see Proposition 2.10). By Corollary 2.16 there exists $\mathcal{M}_\delta$, with mes $\mathcal{M}_\delta \lesssim B_0 \delta^\frac{1}{2}$, such that for any $\xi \notin \mathcal{M}_\delta$
\[
\left| \int_{T^2} \log \rho(f(y) - \xi) dy - \int_{T^2} \log |f(y) - \xi| dy \right| \leq (\log B_0) \delta^\frac{1}{2}.
\]

Take $\delta^2 = \tau = N^{-\frac{1}{\kappa}}$, and let $\mathcal{T}(N) := \mathcal{E}(N) \cup \mathcal{E}_{28,\delta} \cup \mathcal{M}_\delta$. Then
\[
\text{mes } \mathcal{T}(N) \lesssim \exp\left(-\frac{N\kappa}{2}\right) + \left(N^{-\frac{1}{\kappa}}\right)^{\frac{1}{2}} + B_0 \left(N^{-\frac{1}{\kappa}}\right)^{\frac{1}{2}} < N^{-\frac{1}{\kappa}}.
\]

Finally, we conclude from the preceding that if $\xi \notin \mathcal{T}(N)$ and $x \notin B_{N,\xi}$ then
\[
\left| \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k x) - \xi| - \langle \log |f(\cdot) - \xi| \rangle \right| < N^{-\kappa}
\]
provided $\kappa$ was chosen sufficiently small.
The uniform upper bound (2.8) is implicit in the preceding. Indeed, fixing \( B \subset [1,N] \) with \( \# B < N^{1-2\kappa} \), we obtain as above
\[
\frac{1}{N} \sum_{k \in [1,N] \setminus B} \log |f(T^k x) - \xi| - (\log |f(\cdot) - \xi|)
\]
(2.10)
\[
= \frac{1}{N} \sum_{k \in [1,N] \setminus B} \log \left| f(T^k x) - \xi \right| - \frac{1}{N} \sum_{k \in [1,N] \setminus B} \log \rho \left( f(T^k x) - \xi \right)
\]
(2.11)
\[
+ \frac{1}{N} \sum_{k \in [1,N] \setminus B} \log \rho \left( f(T^k x) - \xi \right) - \int_{T^2} \log \rho \left( f(y) - \xi \right) dy
\]
(2.12)
\[
+ \int_{T^2} \log \rho \left( f(y) - \xi \right) dy - \int_{T^2} \log |f(y) - \xi| dy
\]
It was shown above that for all \( \xi \notin \mathcal{T}(N) \) we have (2.12) \( \leq N^{-\kappa} \) uniformly in \( x \). Moreover, with \( \varphi \) and \( \psi \) as in (2.10) above,
\[
\left| \frac{1}{N} \sum_{k \in [1,N] \setminus B} \varphi(T^k x) - \langle \varphi \rangle \right|
\leq \left| \frac{1}{N} \sum_{k \in [1,N] \setminus B} \varphi(T^k x) - \frac{1}{N} \sum_{k \in [1,N] \setminus B} \psi(T^k x) \right| + \left| \frac{1}{N} \sum_{k \in [1,N] \setminus B} \psi(T^k x) - \langle \psi \rangle \right| + |\langle \psi \rangle - \langle \varphi \rangle|
\leq \frac{1}{N} \sum_{k \in [1,N] \setminus B} |\varphi(T^k x) - \psi(T^k x)| + \left| \frac{1}{N} \sum_{k=1}^N \psi(T^k x) - \langle \psi \rangle \right| + \frac{1}{N} \sum_{k \in B} |\psi(T^k x)| + |\langle \psi \rangle - \langle \varphi \rangle|
\lesssim B_1(f)\delta^{-1}\tau + (|\log \delta/2| + 1 + B_0(f))\tau^{-1}[N^{-\alpha} + N^{-2\kappa}]
\]
which implies that (2.11) is controlled uniformly in \( x \). Finally,
\[
(2.11) = \frac{1}{N} \sum_{k \in [1,N] \setminus B} \log \frac{|f(T^k x) - \xi|}{\rho(|f(T^k x) - \xi|)} \leq 0
\]
where the last inequality follows from the fact that \( |y| \leq \rho(y) \).

**Remark 2.18.** In the previous proof we can relax the Diophantine assumption on \( \omega \). Indeed, in the case of shift \( T x = x + \omega \), it suffices to require that \( \omega \) is \( (N, \gamma_1, \gamma_2) \)-Diophantine for some \( \gamma_1, \gamma_2 > 0 \) and \( N \) the same as in (2.4). This follows from the fact that the main ergodic theorem for the shift holds under this weaker Diophantine assumption, see Remark A.15 in Appendix A. For the skew shift \( T\omega(x_1, x_2) = (x_1 + x_2, x_2 + \omega) \) it suffices to require that \( \omega \in \mathbb{T}_{c,1,N} \), see Remark A.14 in Appendix A.

**Remark 2.19.** Inspection of the proof of Theorem 2.4 shows that the set \( \mathcal{T}(N) = \mathcal{T}'(N) \cup \mathcal{T}''(N) \) with the following properties:
- \( \text{mes } \mathcal{T}'(N) \lesssim \exp(-N^{\kappa/2}) \)
- \( \text{mes } \mathcal{T}(N) \lesssim N^{-\kappa} \) and \( \mathcal{T}'(N) \) can be chosen the same for \( N^{1/2} \leq N' \leq N \). In particular, the following version of (2.8) holds:
\[
\sup_{\frac{1}{2} \leq N \leq N} \sup_{\mathcal{B} \subset [1,N]} \sup_{x \in T^2} \sum_{k \in [1,N] \setminus \mathcal{B}} \log \left| f(T^k x) - \xi \right| \leq \langle \log |f(\cdot) - \xi| \rangle + N^{-\kappa}
\]
for any \( \xi \in [-B_0(f), B_0(f)] \setminus \mathcal{T}(N) \). Moreover, invoking Remark 2.16 yields
\[
\text{mes } [\mathcal{T}(N)](r) \lesssim N^{-\kappa/2}
\]
where \( r = \exp(-N^{\kappa}) \).
Lemma 2.22. Let \( f(x,\omega) \in \mathbb{T}^2 \) in Theorem 2.17 derive only from Lemma 2.10, since all other estimates are uniform in \( x \in \mathbb{T}^2 \). Taking into account Remark 2.21 (for Lemma 2.10) one obtains the following version of the first statement of Theorem 2.17 (which we will use in Section 3 for the case of the skew shift): for any \( x_2 \in \mathbb{T} \) there exists \( T^{(1)}(x_2, N) \) such that

\[
\text{mes} \left\{ x_1 \in \mathbb{T} : \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k(x_1, x_2)) - \xi| - \log |f(\xi) - \xi| > N^{-\kappa} \right\} \leq \exp(-N^\kappa)
\]

provided \( \xi \in J_0(f) \setminus T^{(1)}(x_2, N) \), where mes \( T^{(1)}(x_2, N) \leq N^{-\kappa} \).

The method of proof of Theorem 2.17 is quite robust and applies to other dynamics as well. For our applications of Theorem 2.17 to localization, we need the following modifications involving functions that depend also on \( \omega \). Let \( f \in C^1(\mathbb{T}^2 \times \mathbb{T}^2) \) and write \( T_\omega : \mathbb{T}^2 \to \mathbb{T}^2 \) to indicate the dependence on \( \omega \). As before, we define \( S_{f(\cdot, \omega)}(\xi, \delta) = \{ x \in \mathbb{T}^2 : |f(x, \omega) - \xi| < \delta \} \). In analogy with Corollary 2.21, we now have the following result.

Corollary 2.22. Let \( f \in C^1(\mathbb{T}^2 \times \mathbb{T}^2) \).

- Let \( T_\omega : \mathbb{T}^2 \to \mathbb{T}^2 \) be the shift. Assume that \( \omega_0 = (N, \gamma_1, \gamma_2) \)-Diophantine. Then for any \( \xi \in \mathbb{R} \), any small \( \delta > 0 \), and \( N > \frac{-20}{\delta} \), one has

\[
\frac{1}{N} \# \{ 1 \leq k \leq N : T^k x \in S_{f(\cdot, \omega)}(\xi, \delta) \} \lesssim S_{f(\cdot, \omega)}(\xi, 2\delta) + (1 + B_1(f)) \delta^\frac{1}{2}
\]

- Let \( T_\omega : \mathbb{T}^2 \to \mathbb{T}^2 \) be the skew-shift. Assume \( \omega_0 \in \mathbb{T}_{c, \varepsilon_1, N} \). Then (2.14) is valid, provided \( N > \frac{-20}{\delta} \), \( \delta \) is small, and \( |\omega - \omega_0| < N^{-3} \).

Proof. The proof is basically the same as that of Corollary 2.21. More precisely, with \( \chi_\delta \) as in Lemma 2.16,

- \( \# \{ 1 \leq k \leq N : T^k x \in S_{f(\cdot, \omega)}(\xi, \delta) \} \leq \sum_{k=1}^{N} \chi_\delta (f(T^k x, \omega) - \xi) \)
- \( 0 \leq \langle \chi_\delta (f(\cdot, \omega) - \xi) \rangle \leq \text{mes} S_{f(\cdot, \omega)}(\xi, 2\delta) \)

Here \( \delta > 0 \) is any small number, \( \xi \) an arbitrary real number, \( \omega \in \mathbb{T}^2 \), and \( N \) a positive integer. Now one proceeds as in Corollary 2.21 using the ergodic theorem, i.e., Propositions A.1 and A.11 from Appendix A. The point to notice here is that the constants in the ergodic theorem are uniform in \( |\omega - \omega_0| < N^{-1} \).

We can again remove a set of exceptional \( \xi \) for which the measure \( \text{mes} S_{f(\cdot, \omega)}(\xi, 2\delta) \) is too large; as in Corollary 2.21, this is an easy consequence of Fubini’s theorem with the added feature that the set we remove can be chosen to be the same for all \( \omega \) close to a given \( \omega_0 \).

Lemma 2.23. Let \( \omega_0 \in \mathbb{T}^2, \xi \in \mathbb{R}, \delta > 0 \). Then

\[
S_{f(\cdot, \omega)}(\xi, \delta) \subset S_{f(\cdot, \omega_0)}(\xi, 2\delta)
\]

for all \( |\omega - \omega_0| < B_1(f)^{-1}\delta \). In particular, there exists \( \mathcal{E}_{\delta, \omega_0} \subset \mathbb{R} \), \( \mathcal{E}_{\delta, \omega_0} \lesssim \delta^\frac{1}{2} \) such that for any \( \xi \notin \mathcal{E}_{\delta, \omega_0} \), one has

\[
\text{mes} S_{f(\cdot, \omega)}(\xi, \delta) \lesssim \delta^\frac{1}{2}
\]

provided \( |\omega - \omega_0| < B_1(f)^{-1}\delta \).

Proof. Clearly, \( |f(x, \omega) - f(x, \omega_0)| \leq B_1(f) |\omega - \omega_0| < \delta \). Thus, if \( |f(x, \omega) - \xi| < \delta \) then also \( |f(x, \omega_0) - \xi| \leq |f(x, \omega_0) - f(x, \omega)| + |f(x, \omega) - \xi| < 2\delta \) and the lemma follows. The second statement follows from

\[
\int \text{mes} S_{f(\cdot, \omega)}(\xi, \delta) d\xi = 2\delta
\]

and Fubini’s theorem.

Combining the previous two statements yields the following:

Corollary 2.24. Given \( \delta > 0 \), let \( \mathcal{E}_{\delta, \omega_0} \subset \mathbb{R} \) be as in the previous lemma.
• Let \( T_x : \mathbb{T}^2 \to \mathbb{T}^2 \) be the shift. Assume \( \omega_0 \) is \((N, \gamma_1, \gamma_2)\)-Diophantine. Then for any \( \xi \notin \mathcal{E}_{\delta, \omega_0} \), one has

\[
\frac{1}{N} \# \left\{ 1 \leq k \leq N : T^k_x x \in S_{f(x, \omega)}(\xi, \delta) \right\} \lesssim (1 + B_1(f)) \delta^{\frac{1}{4}}
\]

for all \( x \in \mathbb{T}^2 \) and \( |\omega - \omega_0| < ((1 + B_1(f))N)^{-1} \) provided \( N \geq \delta^{-\frac{1}{20}} \) with \( \sigma > 0 \) a sufficiently small constant depending on \( \omega_0 \).

• Let \( T_x : \mathbb{T}^2 \to \mathbb{T}^2 \) be the skew-shift. Assume \( \omega_0 \in T_{c+1, N} \). Then \( (\ref{eq:2.15}) \) is valid provided \( |\omega - \omega_0| < (1 + B_1(f))^{-1} N^{-3} \), \( N > \delta^{-\frac{20}{7}} \) with some small \( \sigma > 0 \).

**Proof.** We can apply Corollary 2.23 for large \( N \), since \( |\omega - \omega_0| \leq \frac{\varepsilon_0}{2} N^{-1} \) for the shift, and \( |\omega - \omega_0| \leq \frac{\varepsilon_0}{2} N^{-3} \) for the skew-shift. Furthermore, since \( |\omega - \omega_0| < \delta/B_1(f) \) we can apply the previous lemma to conclude that

\[
\mes S_{f(x, \omega)}(\xi, 3\delta) \lesssim \delta^{\frac{1}{4}}
\]

for all \( \xi \notin \mathcal{E}_{\delta, \omega_0} \).

\[\square\]

**Corollary 2.24.** Using the notation of Corollary 2.23, one has

\[
\frac{1}{N} \sum_{k=1}^{N} \log \rho \left( f(T^k_x x, \omega) - \xi \right) \leq \frac{1}{N} \sum_{1 \leq k \leq N \atop T^k_x x \notin S_{f(x, \omega)}(\xi, \delta)} \log \left| f(T^k_x x, \omega) - \xi \right|
\]

for any \( \xi \notin \mathcal{E}_{\delta, \omega_0} \), where \( \delta = N^{-\frac{7}{20}} \).

We now present a somewhat sharper version of Lemma 2.23 on large values of certain logarithmic integrals.

**Lemma 2.25.** Given \( R > 0 \), \( \omega_0 \in \mathbb{T}^2 \), \( \eta > 0 \), there exists

\[
\mathcal{L}_{\omega_0, \eta, R} \subset (\omega_0 - \eta, \omega_0 + \eta) \times \mathcal{J}_0(f), \quad \mes \mathcal{L}_{\omega_0, \eta, R} \lesssim (1 + B_0(f)) \eta \exp(-\sqrt{R}/2)
\]

such that for \( (\omega, \xi) \in \mathbb{T}^2 \times (\omega_0 - \eta, \omega_0 + \eta) \times \mathcal{J}_0(f) \setminus \mathcal{L}_{\omega_0, \eta, R} \) one has

1. \( \int_{\mathbb{T}^2} \log \| f(x, \omega) - \xi \|^2 dx \leq R \)
2. \( \int_{\mathcal{J}} \left| \log \| f(x, \omega) - \xi \| \right| dx \leq R^{\frac{3}{2}} (\mes S)^{\frac{3}{4}} \)

where \( S \) is an arbitrary measurable set in part (2). Moreover, an analogous statement holds with \( \omega = \omega_0 \) fixed. In that case we only need to remove sets of \( \xi \).

**Proof.** The function \( \Phi(y) = \exp(\sqrt{1 + y}) \) is convex on \( y > 0 \). Then, by Jensen inequality

\[
\mes \left\{ (\omega, \xi) : \int_{\mathbb{T}^2} \left| \log \| f(x, \omega) - \xi \| \right|^2 dx > R \right\}
= \mes \left\{ (\omega, \xi) : \Phi \left( \frac{1}{4} \int_{\mathbb{T}^2} \left| \log \| f(x, \omega) - \xi \| \right|^2 dx \right) > \Phi \left( \frac{1}{4} R \right) \right\}
\leq \mes \left\{ (\omega, \xi) : \int_{\mathbb{T}^2} \Phi \left( \frac{1}{4} \left| \log \| f(x, \omega) - \xi \| \right|^2 \right) dx > \Phi \left( \frac{1}{4} R \right) \right\}
\leq [\Phi \left( \frac{1}{4} R \right)]^{-1} \int_{\mathcal{J}} \int_{\mathbb{T}^2} \Phi \left( \frac{1}{4} \left| \log \| f(x, \omega) - \xi \| \right|^2 \right) dx d\xi d\omega
\]

Note that \( \Phi(y^2) \leq \exp(y + 1) \) for all \( y \geq 0 \). Hence,

\[
\Phi \left( \frac{1}{4} \left| \log \| f(x, \omega) - \xi \| \right|^2 \right) \leq \exp \left( \frac{1}{2} \left| \log \| f(x, \omega) - \xi \| \right| + 1 \right)
\lesssim (1 + B_0(f))^{\frac{3}{4}} \left| f(x, \omega) - \xi \right|^{-\frac{1}{4}}
\]

\[\square\]
Inserting this into (2.16) yields
\[
\mes \left\{ (\omega, \xi) : \int_{T^2} \left| \log |f(\omega, \xi) - \xi| \right|^2 \, d\omega > R \right\} \lesssim e^{-\frac{1}{2}\sqrt{\pi}} (1 + B_0(f))^{\frac{1}{2}} \int_{T^2} \int_{\mathbb{T}^2} |f(\omega, \xi) - \xi|^{\frac{1}{2}} \, d\omega \, d\xi \\
\lesssim e^{-\frac{1}{2}\sqrt{\pi}} (1 + B_0(f)) \eta,
\]
which proves (1). Finally, claim (2) follows from (1) by Cauchy-Schwarz. The final statement of the lemma follows by the same arguments but without averaging in \( \omega \).

We can now formulate the analogue of Corollary 2.16 for the case of functions which depend on \( \omega \), but with exceptional sets that do not depend on \( \omega \) as long as \( |\omega - \omega_0| \) is sufficiently small.

**Corollary 2.26.** Given \( N \), let \( \delta = N^{-\frac{1}{4}} \) and \( \eta = \frac{1}{2} (1 + B_1(f))^{-1} N^{-1} \). For any \( \omega_0 \in T^2 \) let \( \mathcal{E}_{\delta,\omega_0} \) be as in Lemma 2.22. There exists \( \mathcal{M}_{\delta,\omega_0} \subset (\omega_0 - \eta, \omega_0 + \eta) \times J(f) \), \( \mes \mathcal{M}_{\delta,\omega_0} \leq \exp(-N^{\sigma_1}) \), where \( \sigma_1 > 0 \) is some small constant, such that for any \( (\omega, \xi) \in (\omega_0 - \eta, \omega_0 + \eta) \times (J(f) \setminus \mathcal{E}_{\delta,\omega_0}) \) one has
\[
\left| \int_{T^2} \log |f(\omega, \xi) - \xi| \, d\omega - \int_{T^2} \log |f(\omega, \xi)| \, d\omega \right| \lesssim \delta^\frac{1}{2}.
\]

**Proof.** Suppose \( \xi \notin \mathcal{E}_{\delta,\omega_0} \), \( (\omega, \xi) \notin \mathcal{L}_{\omega_0, \eta, R} \) (see Lemma 2.25). By Corollary 2.28,
\[
\mes S_{f(\cdot, \omega)}(\xi, \delta) \leq \delta^\frac{1}{2} \quad \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta).
\]
Hence
\[
\left| \int_{T^2} \log |f(\omega, \xi)| \, d\omega \right| \leq \left| \int_{T^2} \log |f(\omega, \xi) - \xi| \, d\omega \right| + \left| \int_{T^2} \log |f(\omega, \xi)| \, d\omega \right|
\]
\[
\leq \int_{S_{f(\cdot, \omega)}(\xi, \delta)} \log |f(\omega, \xi)| \, d\omega + \int_{S_{f(\cdot, \omega)}(\xi, \delta)} \log |f(\omega, \xi) - \xi| \, d\omega
\]
\[
\leq \left[ \mes S_{f(\cdot, \omega)}(\xi, \delta) \right] \left| \log \frac{\delta}{2} \right| + R^\frac{1}{2} \left[ \mes S_{f(\cdot, \omega)}(\xi, \delta) \right]^{\frac{1}{2}
\]
\[
\leq \delta^\frac{1}{2} \left| \log \frac{\delta}{2} \right| + R^\frac{1}{2} \delta^\frac{1}{2}
\]
Take \( R = \delta^{-\frac{1}{4}} \), \( \mathcal{M}_{\delta,\omega_0} := \mathcal{L}_{\omega_0, \eta, R} \).

We are now ready to prove the analogue of Theorem 2.17 for functions depending on \( \omega \). The reader should take note of the fact that we first remove a large (i.e., of size \( N^{-\gamma} \)) set of exceptional parameters \( \xi \) which only depends on \( \omega_0 \) – after that we proceed to remove exponentially small sets in \( (x, \omega, \xi) \). In the following theorem we use the notion of \( (N, \gamma_1, \gamma_2) \)-Diophantine \( \omega \), cf. Remark 2.18 and Remark A.2.

**Theorem 2.27.** Let \( f(x, \omega) \) be \( C^1 \)-smooth. Let \( T_\omega : T^2 \to T^2 \) be a shift (a skew-shift). Given large \( N \) assume that \( \omega \) is \( (N, \gamma_1, \gamma_2) \)-Diophantine \( \omega \in \mathbb{T}_{\varepsilon_1, N} \) for some small \( \gamma_1, \gamma_2 > 0 \) (for some small \( \varepsilon_1 > 0 \)). Then there exists \( T(N) \subset J(f) \), \( \mes T(N) < N^{-\gamma} \), such that
\[
\mes \left\{ (x, \omega, \xi) \in T^2 \times (\omega_0 - \eta, \omega_0 + \eta) \times (J \setminus T(N)) : \left| \frac{1}{N} \sum_{k=1}^{N} \log |f(T_\omega^k x, \omega) - \xi| - \langle \log |f(\cdot, \omega) - \xi| \rangle \right| > N^{-\gamma} \right\} \leq \exp(-N^\gamma)
\]
provided \( \eta = (1 + B_1(f))^{-1} N^{-3} \). Here \( \gamma > 0 \) is a small constant that depends on the Diophantine condition.
Proof. We proceed as in the proof of Theorem 2.17. Thus, let \( \rho \) be as in Definition 2.13 with \( \delta \) to be specified later. Then

\[
\begin{align*}
\frac{1}{N} \sum_{k=1}^{N} \log |f(T^k_\omega x, \omega) - \xi| - \langle \log |f(\cdot, \omega) - \xi| \rangle \\
\leq \frac{1}{N} \sum_{k=1}^{N} \log |f(T^k_\omega x, \omega) - \xi| - \frac{1}{N} \sum_{k=1}^{N} \log \rho(f(T^k_\omega x, \omega) - \xi) \\
+ \frac{1}{N} \sum_{k=1}^{N} \log \rho(f(T^k_\omega x, \omega) - \xi) - \int_{T^2} \log \rho(f(y, \omega) - \xi) \, dy \\
+ \int_{T^2} \log \rho(f(y, \omega) - \xi) \, dy - \int_{T^2} \log |f(y, \omega) - \xi| \, dy
\end{align*}
\]

Let \( B_{N,\xi,\omega} = \{ y \in \mathbb{T}^2 : \min_{1 \leq k \leq N} |f(T^k_\omega y, \omega) - \xi| \leq e^{-N^\kappa} \} \) where \( \kappa > 0 \) is small. Moreover, let \( E(N, \omega) \) be as in Lemma 2.10 applied to \( f(\cdot, \omega) \) and \( E_{\delta,\omega_0} \) be as in Lemma 2.22. If \( \xi \notin E(N, \omega) \cup E_{\delta,\omega_0}, \) and \( x \notin B_{N,\xi,\omega}, \) then

\[
\begin{align*}
\frac{1}{N} \sum_{k=1}^{N} \log |f(T^k_\omega x, \omega) - \xi| - \frac{1}{N} \sum_{k=1}^{N} \log \rho(f(T^k_\omega x, \omega) - \xi) \\
\leq \frac{1}{N} \# \{ 1 \leq k \leq N : T^k_\omega x \in S_f(\xi, \delta) \} \left[ \log \min_{1 \leq k \leq N} |f(T^k_\omega x, \omega) - \xi| + |\log \delta| \right] \\
\lesssim (1 + B_1(f)) \frac{\delta^2}{2} \left[ N^\kappa + |\log \frac{\delta}{2}| \right]
\end{align*}
\]

Moreover, \( \text{mes } B_{N,\xi,\omega} \leq \exp \left( -\frac{1}{4} N^\kappa \right) \). Let \( \varphi(y) = \log \rho(f(y, \omega) - \xi) \). By Lemma 2.23, for any \( \tau > 0 \), there is \( \psi \in C^4(\mathbb{T}^2) \) such that

1. \( \max_{y \in \mathbb{T}^2} |\varphi(y) - \psi(y)| \lesssim B_1(\varphi) \tau \leq B_1(f) \delta^{-1} \tau \)
2. \( B_4(\psi) \lesssim B_0(\varphi) \tau^{-4} \leq (|\log(\delta)/2| + 1 + B_0(f)) \tau^{-4} \)

Then with some \( \sigma > 0 \),

\[
\begin{align*}
\frac{1}{N} \sum_{k=1}^{N} \varphi(T^k_\omega x) - \langle \varphi \rangle \\
\leq \frac{1}{N} \sum_{k=1}^{N} \varphi(T^k_\omega x) - \frac{1}{N} \sum_{k=1}^{N} \psi(T^k_\omega x) + \frac{1}{N} \sum_{k=1}^{N} \psi(T^k_\omega x) - \langle \psi \rangle + |\langle \psi \rangle - \langle \varphi \rangle| \\
\lesssim B_1(f) \delta^{-1} \tau + (|\log \delta/2| + 1 + B_0(f)) \tau^{-4} N^{-\sigma}
\end{align*}
\]

provided \( N \) is sufficiently large (see Proposition 2.24). By Corollary 2.26, there exists \( M_{\delta,\omega_0} \), with \( \text{mes } M_{\delta,\omega_0} \lesssim \exp(-N^{\sigma_1}) \), such that for any \( (\omega, \xi) \notin M_{\delta,\omega_0} \) one has

\[
\int_{T^2} \log \rho(f(y, \omega) - \xi) \, dy - \int_{T^2} \log |f(y, \omega) - \xi| \, dy \leq \delta^2.
\]

Take \( \delta^2 = \tau = N^{-\frac{\sigma}{2}} \), and let \( T(N) := E_{\delta,\omega_0} \). Then with some small \( \gamma > 0 \),

\[
\text{mes } T(N) \lesssim \delta^2 < N^{-\gamma}.
\]

Finally, we conclude from the preceding that if \( \xi \notin T(N) \cup E(N, \omega), \) \( (\omega, \xi) \notin M_{\delta,\omega_0}, \) and \( x \notin B_{N,\xi,\omega} \) then

\[
\frac{1}{N} \sum_{k=1}^{N} \log |f(T^k_\omega x) - \xi| - \langle \log |f(\cdot) - \xi| \rangle < N^{-\gamma}
\]

provided \( \gamma \) was chosen sufficiently small. \( \square \)
3. Large Deviation Theorems in Frequencies and Elimination of Resonances in a General Setting

Let \( f \in C^1 (\mathbb{T}^2 \times \mathbb{T}^2) \). Let \( T_\omega : \mathbb{T}^2 \to \mathbb{T}^2 \) be the shift (the skew-shift). We begin this section with some simple statements concerning the introduction of perturbations into the results of the previous section.

**Lemma 3.1.** For any \( x, \varepsilon_k, \omega \in \mathbb{T}^2 \), \( \tau_k, \omega_1 \in \mathbb{T}^2 \) \((x, \varepsilon_k \in \mathbb{T}^2, \tau_k, \omega, \omega_1 \in \mathbb{T}\)) one has
\[
\# \left\{ 1 \leq k \leq N : |f(T^k_\omega x + \varepsilon_k, \tau_k + \omega_1) - \xi| < \delta \right\} \leq \# \left\{ 1 \leq k \leq N : |f(T^k_\omega x, \omega_1) - \xi| < \Delta + B_1(f) \max_k (|\varepsilon_k| + |\tau_k|) \right\}
\]

In particular, if \( \varepsilon := \max_k \{|\varepsilon_k| + |\tau_k|\} < \frac{\delta}{B_1(f)} \), then
\[
\# \left\{ 1 \leq k \leq N : |f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi| < \delta \right\} \leq \# \left\{ 1 \leq k \leq N : T^k_\omega x \in S_{f(\omega_1)}(\xi, 2\Delta) \right\}
\]

**Corollary 3.2.** Let \( N \) be large and assume that \( \omega_0 \) is \((N, \gamma_1, \gamma_2)\)-Diophantine for the case of the shift, see Remark A.2 (or \( \omega_0 \in \mathcal{S}_{e, e_1} \) for the skew-shift, see Remark A.1 in Appendix A). Given \( \delta \geq N^{-\frac{3}{2}} \), assume that \( \varepsilon := \max_k \{|\varepsilon_k| + |\tau_k|\} < \frac{\delta}{B_1(f)} \). Then, with \( \omega_1 \) fixed,

1. the estimate
\[
\frac{1}{N} \# \left\{ 1 \leq k \leq N : |f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi| < \delta \right\} \lesssim \text{mes} S_{f(\omega_1)}(\xi, 3\delta) + 1 + B_1(f) \frac{\delta^2}{2}
\]
holds for all \( x \in \mathbb{T}^2 \), provided \(|\omega - \omega_0| < N^{-3}, \xi \in \mathcal{J}_0(f)\)

2. there exists \( \mathcal{E}_{\omega_0, \omega_1, \delta} \subset \mathbb{R} \), \( \text{mes} \mathcal{E}_{\omega_0, \omega_1, \delta} \lesssim \frac{\delta^2}{2} \) such that for any \( \xi \in \mathcal{J}_0(f) \setminus \mathcal{E}_{\omega_0, \omega_1, \delta} \), \(|\omega - \omega_0| < N^{-3}\)

one has
\[
\frac{1}{N} \# \left\{ 1 \leq k \leq N : |f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi| < \delta \right\} \lesssim \left[ 1 + B_1(f) \right] \frac{\delta^2}{2}
\]
for all \( x \in \mathbb{T}^2 \) (\( \mathcal{E}_{\omega_0, \omega_1, \delta} \) does not depend on \( \omega, \varepsilon, \tau_k \)). Furthermore, if \(|\omega_0 - \omega_1| < \frac{\delta}{B_1(f)} \), then \( \mathcal{E}_{\omega_0, \omega_1, \delta} \) can be chosen to depend only on \( \omega_0, \omega_1 \).

**Proof.** Recall that
\[
\frac{1}{N} \# \left\{ 1 \leq k \leq N : T^k_\omega x \in S_{f(\omega_1)}(\xi, 2\delta) \right\} \lesssim \text{mes} S_{f(\omega_1)}(\xi, 3\delta) + 1 + B_1(f) \frac{\delta^2}{2}
\]
for any \(|\omega - \omega_0| < N^{-2}\) and any \( x \in \mathbb{T}^2 \) due to Corollary 2.2. Therefore, (1) follows from Lemma 3.1. Assumption (2) is a consequence of Lemma 2.22. □

**Remark 3.3.** As we have noted in Remark 2.2, the estimates of Corollary 2.8 can be stated in a slightly stronger form which we need in our applications. Namely, the set \( \mathcal{E}_\delta \) in that corollary satisfies
\[
\text{mes} \left[ \mathcal{E}_\delta \right](\rho) \lesssim \delta^{\frac{2}{3}}, \quad \rho \leq \delta
\]
where \([\mathcal{E}_\delta](\rho) = \{ \xi : \text{dist}(\xi, \mathcal{E}_\delta) \} \leq \rho\). For the same reason the set \( \mathcal{E}_{\omega_0, \omega_1, \delta} \) in Corollary 3.2 obeys
\[
\text{mes} \left[ \mathcal{E}_{\omega_0, \omega_1, \delta} \right](\rho) \lesssim \delta^{\frac{2}{3}}, \quad \rho \leq \delta
\]

**Lemma 3.4.** Assume \( \varepsilon < \frac{1}{2} \left( \frac{\delta}{B_1(f)} \right) \). If \(|f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi| > \delta\) then
\[
\left| \log |f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi| - \log |f(T^k_\omega x, \omega_1) - \xi| \right| \lesssim B_1(f) \varepsilon \delta^{-1}
\]
uniformly in \( \omega, \omega_1 \).

**Proof.**
\[
1 - \frac{B_1(f) \varepsilon}{\delta} \leq \frac{|f(T^k_\omega x, \omega_1) - \xi|}{|f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi|} \leq 1 + \frac{B_1(f) \varepsilon}{\delta}.
\]
By assumption, \( B_1(f) \varepsilon \delta^{-1} < \frac{1}{2} \) and the assertion follows. □
We shall also need the following analogue of (2.17): given \( x \in \mathbb{T}^2, \ |\varepsilon_k|, |\tau_k| < 1 \), let
\[
J_N(x, \xi, \delta) = \left\{ 1 \leq k \leq N : |f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi| < \delta \right\}.
\]
Then
\[
\frac{1}{N} \sum_{1 \leq k \leq N} \log |f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi| + \frac{1}{N} \left[ \# J_N(x, \xi, \delta) \right] \log \left( \min_{1 \leq k \leq N} |f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi| \right)
\]
\[
\leq \frac{1}{N} \sum_{k=1}^N \log |f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi|
\]
\[
\leq \frac{1}{N} \sum_{k \in J_N(x, \xi, \delta)} \log |f(T^k_\omega x + \varepsilon_k, \omega_1 + \tau_k) - \xi|.
\]

As in Lemma 2.14, Fubini's theorem immediately yields the following statement (recall the definition of \( J_0(f) \) in (2.1)):

**Lemma 3.5.** Given \( N \in \mathbb{N}, \varepsilon_k, \tau_k : \mathbb{T}^2 \to \mathbb{T}^2, \ k = 1, 2, \ldots, N, \omega_1 \) and any small \( \kappa > 0 \) there exists \( \mathcal{E}_\omega(N, \varepsilon_k, \{ \tau_k \}) \subset J_0(f), \) \( \text{mes} \mathcal{E}_\omega(N, \varepsilon_k, \{ \tau_k \}) \leq \exp \left( -\frac{1}{2} N^\kappa \right) \) such that for any \( \xi \in J_0(f) \setminus \mathcal{E}_\omega(N, \varepsilon_k, \{ \tau_k \}) \) one has
\[
\text{mes} \left\{ x \in \mathbb{T}^2 : \log \left( \min_{1 \leq k \leq N} |f(T^k_\omega x + \varepsilon_k(x), \omega_1(x) + \tau_k(x)) - \xi| \right) < -N^\kappa \right\} < \exp \left( -\frac{1}{4} N^\kappa \right).
\]
Furthermore, just as in Remark 2.3, the set \( \mathcal{E}_\omega(N, \varepsilon_k, \{ \tau_k \}) \) satisfies
\[
\text{mes} \mathcal{E}_\omega(N, \varepsilon_k, \{ \tau_k \})(\rho) \leq \exp(-N^\kappa/4)
\]
for any \( \rho \leq \exp(-N^\kappa) \).

The following result is a perturbed version of Theorem 2.1. Note that in the statement of the following theorem we introduce two different sets of \( \xi \) which need to be removed. This is due to the fact that in later applications we wish to sum over the perturbations \( \varepsilon_k \) and \( \tau_k \).

**Proposition 3.6.** Let \( f(x, \omega) \) be \( C^1 \)-smooth. Let \( T_\omega : \mathbb{T}^2 \to \mathbb{T}^2 \) be the shift (or the skew-shift). Let \( N \) be large. Assume that \( \omega_0 \in \mathcal{D}_{\gamma_1, \gamma_2} \) is \( \omega \)-Diophantine (or \( \omega \in \mathcal{T}_{\gamma_1, N} \) for the skew-shift). Let
\[
\varepsilon_k(x, \omega), \tau_k(x, \omega), \ (x, \omega) \in \mathbb{T}^2 \times \mathbb{T}^2 \ (\ (x, \omega) \in \mathbb{T}^2 \times \mathbb{T}),
\]
\[
\text{max}_{x \in \mathbb{T}} B_0(\varepsilon_k) \lesssim \frac{N^{-1}}{B_0(\varepsilon_k)}, \ \text{max}_{x \in \mathbb{T}} B_0(\tau_k) \lesssim \frac{N^{-1}}{B_0(\tau_k)}.
\]
Moreover, let \( |\omega_0 - \omega_1| < [1 + B_1(f)]N^3 \). Then there exist \( \mathcal{E}_{\omega_0}(N), \mathcal{E}_{\omega_0, \omega_1}(N, \varepsilon_k, \{ \tau_k \}) \subset J_0(f), \) such that
\[
\text{mes} \mathcal{E}_{\omega_0}(N) \lesssim N^{-\kappa}, \ \text{mes} \mathcal{E}_{\omega_0, \omega_1}(N, \varepsilon_k, \{ \tau_k \}) \lesssim \exp \left( -\frac{1}{2} N^\kappa \right),
\]
and so that for any \( \xi \in J_0(f) \setminus (\mathcal{E}_{\omega_0}(N) \cup \mathcal{E}_{\omega_0, \omega_1}(N, \varepsilon_k, \{ \tau_k \})) \)
\[
\text{mes} \left\{ x \in \mathbb{T}^2 : \left| \sum_{k=1}^N \log \left| f(T^k_\omega x + \varepsilon_k(x), \omega_1 + \tau_k(x)) - \xi \right| - \log |f(\cdot, \omega_1) - \xi| \right| > N^{-\kappa} \right\}
\]
\[
\leq \exp(-N^\kappa).
\]
Here \( \kappa > 0 \) is some small constant. Moreover, the sets \( \mathcal{E}_{\omega_0}(N), \mathcal{E}_{\omega_0, \omega_1}(N, \varepsilon_k, \{ \tau_k \}) \) obey
\[
\text{mes} \mathcal{E}_{\omega_0}(N)(\rho) \lesssim N^{-\kappa}, \ \text{mes} \mathcal{E}_{\omega_0, \omega_1}(N, \varepsilon_k, \{ \tau_k \})(\rho) \lesssim \exp(-N^\kappa/2)
\]
for any \( \rho < \exp(-N^\kappa) \).
Proof. We shall reduce this theorem to Theorem 2.17 applied to the function $f(\cdot, \omega_1)$ (see also Remark 2.18). Let $J_N(x, \xi, \delta)$ be as in Remark 3.2. Set $\delta = N^{-\frac{\sigma}{2}}$ with $0 < \sigma < 1$. Due to Corollary 5.2 for all $x \in \mathbb{T}^2$, $\xi \notin \mathcal{E}_{\omega_0, \omega_1, \delta}$

$$N^{-1} \# J_N(x, \xi, \delta) \lesssim (1 + B_1(f)) N^{-\frac{\sigma}{2}}$$

Let $\mathcal{E}_{\omega_1}(N, \{\epsilon_k\}, \{\tau_k\})$ be as in Lemma 3.5 and let $\mathcal{B}_{\omega_1}(N, \xi)$ be the set defined in Remark 3.7. Then for any $\xi \in \mathcal{J}_0(f) \setminus \mathcal{E}_{\omega_1}(N, \{\epsilon_k\}, \{\tau_k\})$ and any $x \in \mathbb{T}^2 \setminus \mathcal{B}_{\omega_1}(N, \xi)$ one has due to Lemma 3.1

$$1 \leq \exp \left( \frac{1}{N} \sum_{k=1}^N \log |f(T_{\omega_1}^k x + \epsilon_k(x), \omega_1 + \tau_k(x)) - \xi| - \frac{1}{N} \sum_{1 \leq k \leq N \setminus k \notin J_N(x, \xi, \delta)} \log |f(T_{\omega_1}^k x + \epsilon_k(x), \omega_1 + \tau_k(x)) - \xi| \right) \lesssim N^{-\frac{\sigma}{2}}$$

By Lemma 3.4

$$\frac{1}{N} \sum_{1 \leq k \leq N \setminus k \notin J_N(x, \xi, \delta)} \log |f(T_{\omega_1}^k x, \omega_1) - \xi| - \frac{1}{N} \sum_{1 \leq k \leq N} \log |f(T_{\omega_1}^k x, \omega_1) - \xi| \lesssim N^{-\frac{\sigma}{2}}$$

for any $\xi, x$ as above, where $\epsilon := \max \{\epsilon_k, |\tau_k(x)|\}$. Let $\mathcal{E}_{\omega_1}(N, \{0\} k, \{0\} k)$ be defined as in Lemma 3.5 and let $\mathcal{F}_{\omega_1}(N, \xi)$ be the set defined in Remark 2.20 both times applied to $f(x, \omega_1)$ with $\epsilon_k = \tau_k = 0$. Then for any $\xi \in \mathcal{J}_0(f) \setminus \mathcal{E}_{\omega_1}(N, \{0\} k, \{0\} k)$ and any $x \notin \mathcal{F}_{\omega_1}(N, \xi)$ one has

$$\frac{1}{N} \sum_{k \notin \mathcal{F}_{\omega_1}(x, \xi, \delta)} \log |f(T_{\omega_1}^k x, \omega_1) - \xi| \lesssim N^{-\frac{\sigma}{2}}$$

The main part of the theorem now follows from Theorem 2.17 and Remark 2.18 applied to $f(\cdot, \omega_1)$. For the final statement we use Remark 2.17 and estimate (3.4). \qed

Remark 3.7. Inspection of the proof of Proposition 3.6 in view of Remark 2.20 shows that the following version of the statement holds: With $\epsilon_k, \tau_k$ as before,

$$\operatorname{mes} \left\{ x_1 \in \mathbb{T} : \left| \frac{1}{N} \sum_{k=1}^N \log |f(T_{\omega_1}^k (x_1, x_2) + \epsilon_k(x_1, x_2), \omega_1 + \tau_k(x_1, x_2)) - \xi| - \langle \log |f(\cdot, \omega_1) - \xi| \rangle \right| > N^{-\kappa} \right\} \leq \exp(-N^\kappa)$$

for all $x_2 \in \mathbb{T}$, $|\omega_0 - \omega_1| < \left[ (1 + B_1(f)) N^3 \right]^{-1}$, $\xi \in J_0(f) \setminus (\mathcal{E}_{\omega_0}(x_2, N) \cup \mathcal{E}_{\omega_0, \omega_1}(x_2, N, \{\epsilon_k\}, \{\tau_k\}))$, where

$$\operatorname{mes} \left[ \mathcal{E}_{\omega_0}(x_2, N) \right](\rho) \leq N^{-\kappa}, \quad \operatorname{mes} \left[ \mathcal{E}_{\omega_0, \omega_1}(x_2, N, \{\epsilon_k\}, \{\tau_k\}) \right](\rho) \leq \exp\left(-\frac{1}{2} N^\kappa\right),$$

$\rho = \exp(-N^\kappa)$.

Now we are going to apply Theorem 2.17 to the evaluation of the measure of those frequencies $\omega$ for which so-called resonances occur. In the general setting of Section 2 we define a resonance by means of the following inequality, where $\kappa > 0$ is small and fixed:

$$\frac{1}{N} \sum_{1 \leq k \leq N} \log |f(T_{\omega}^k (T_{\omega}^N x_0), \omega) - \xi| - \langle \log |f(\cdot, \omega) - \xi| \rangle > N^{-\kappa}$$

where $\bar{N} \gg N$. The goal is to show that the measure of those $(\omega, \xi)$ for which (3.6) occurs for some $e^{N^\sigma} > \bar{N} \gg N$ is small for any fixed $x_0 \in \mathbb{T}^2$ (here $\sigma > 0$ is another small constant).

Theorem 3.8. Fix $x_0 \in \mathbb{T}^2$ and $N$ large.

1. Let $T_\omega : \mathbb{T}^2 \to \mathbb{T}^2$ be a shift and let $\omega_0$ be $(N, \gamma_1, \gamma_2)$-Diophantine with some choice of small $\gamma_1, \gamma_2 > 0$, $|\omega_1 - \omega_0| < \left[ (1 + B_1(f)) N^3 \right]^{-1}$. Given $\bar{N} > B_1(f) N^2$, there exist sets

$$\mathcal{E}_{\omega_0}(N), \tilde{\mathcal{E}}_{\omega_0, \omega_1}(N, \bar{N}) \subset J_0(f),$$

...
Invoking now Remark 3.7 (instead of Proposition 3.6) one obtains the statement.

Proof. \( \xi \neq \mu \) /\( \text{mes} \), \( \text{mes}(\xi) \geq \frac{1}{N} \), \( \text{mes} \{ \theta \in [0, 1]^2 : |1 \sum_{k=1}^{N} \log f(x_0 + N \omega_1 + \theta + k(\omega_1 + \theta/\bar{N}), \omega_1 + \theta/\bar{N}) - \xi| - (\log |f(\cdot, \omega_1) - \xi|) > N^{-\kappa} \} < \exp (-N^\kappa) \)

where \( \kappa > 0 \) is some small constant. The constants (but not the sets \( \mathcal{E}_\omega(N), \bar{\mathcal{E}}_{\omega_1}(N, \bar{N}) \)) are uniform in the choice of \( x_0 \).

(2) Let \( T_\omega : \mathbb{T}^2 \to \mathbb{T}^2 \) be the skew-shift and let \( \omega_0 \in \mathbb{T}_{\varepsilon_1, N} \) with some small \( \varepsilon_1 > 0 \), \( |\omega_1 - \omega_0| < (1 + B_1(\varepsilon))^{-1} N^{-3} \). Given \( \bar{N} > B_1(\varepsilon)N^4 \) there exist \( \mathcal{E}_\omega(N), \bar{\mathcal{E}}_{\omega_1}(N, \bar{N}) \subset \mathcal{J}_0(f) \) with \( \text{mes} \{ \mathcal{E}_\omega(N) \}(\rho) < N^{-\kappa}, \text{mes} \{ \bar{\mathcal{E}}_{\omega_1}(N, \bar{N}) \}(\rho) < \exp (-N^\kappa) \), \( \rho = \exp (-N^\kappa) \), so that for any \( \xi \notin \mathcal{E}_\omega(N) \cup \bar{\mathcal{E}}_{\omega_1}(N, \bar{N}) \) one has

\[
\text{mes} \left\{ \theta \in \mathbb{T} : \left| \frac{1}{N} \sum_{k=1}^{N} \log |f(T_\omega^k(T_\omega^N(x_0, \omega_1 + \theta/\bar{N}))(x_0))| - \xi| - (\log |f(\cdot, \omega_1) - \xi|) > N^{-\kappa} \right\} < \exp (-N^\kappa)
\]

Here \( \bar{N} = \bar{N}(\bar{N} - 1)/2, \omega := \omega_1 + \theta/\bar{N}, T^N_{\omega_1 + \theta/\bar{N}}(x_0) = (x_1^0 + N x_2^0 + \bar{N} \omega_1 + \theta, x_2^0 + \bar{N} \omega_1 + \theta/\bar{N}), (x_1^0, x_2^0) = x_0. \)

Proof. (1) Define \( g : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{R} \) by \( g(\theta, \omega) = f(x_0 + N \omega_1 + \theta, \omega_1 + \omega) \). Set \( \varepsilon_k(\theta) := k(\bar{N} - \bar{N}), \tau_k(\theta) := \bar{N}/\bar{N}, k = 1, 2, \ldots, N \) then

\[
f(x_0 + N \omega_1 + \theta + k(\omega_1 + \theta/\bar{N}), \omega_1 + \theta/\bar{N}) = g(\theta + k \omega_1 + \varepsilon_k(\theta), \tau_k(\theta)).
\]

Note that, for any \( \theta \in [0, 1], \xi \in \mathbb{R}, (\log |f(\cdot, \omega_1) - \xi|) = (\log |g(\cdot, \xi) - \xi|). \)

Set \( \bar{\mathcal{E}}_{\omega_1}(N, \bar{N}) := \mathcal{E}_{\omega_1}(N, \{ \varepsilon_k \}, \{ \tau_k \}) \).

Then

\[
\text{mes} \left\{ \theta \in [0, 1]^2 : \left| \frac{1}{N} \sum_{k=1}^{N} \log |f(x_0 + N \omega_1 + \theta + k(\omega_1 + \theta/\bar{N}), \omega_1 + \theta/\bar{N})| - \xi| - (\log |f(\cdot, \omega_1) - \xi|) > N^{-\kappa} \right\} < \exp (-N^\kappa)
\]

for \( \xi \notin \mathcal{E}_{\omega_1}(N) \cup \mathcal{E}_{\omega_1}(N, \{ \varepsilon_k \}, \{ \tau_k \}) \), where these sets are as in Proposition 3.6.

(2) Note that with \( \omega = \omega_1 + \theta/\bar{N} \) one has

\[
T^k_{\omega}(T^N_\omega(x_1^0, x_2^0)) = T^k_{\omega}(y_1^0 + \theta, y_2^0 + \bar{N} \theta/\bar{N}) + \varepsilon_k
\]

where

\[
y_1^0 = x_1^0 + N x_2^0, y_2^0 = x_2^0 + N \omega_1,
\]

\[
\varepsilon_k = T^k_{\omega}(0, 0).
\]

Invoking now Remark 3.7 (instead of Proposition 3.6) one obtains the statement. \( \square \)
Lemma 3.9. Let \( f(x, \omega) \) be \( C^1 \)-smooth. Let \( T_\omega \) be the shift (or the skew-shift). Given large \( N \) there exists a set \( \mathcal{J}(N) \subset \{(m_1, m_2) : 1 \leq m_1, m_2 \leq \tilde{N}_1 \} \), \( \tilde{N}_1 = N^2 \) (resp. \( \mathcal{J}(N) \subset [1, \tilde{N}_1] \), \( \tilde{N}_1 = N^4 \)) and subsets \( \mathcal{E}_m(N) \subset \mathcal{J}_0(f) \), for every \( m = (m_1, m_2) \in [1, N^2]^2 \setminus \mathcal{J}(N) \) (resp. \( m \in [1, \tilde{N}_1] \setminus \mathcal{J}(N) \)), such that

1. Using the notations (resp. (B.2)),
\[
\operatorname{mes} \bigcup_{m \in \mathcal{J}(N)} \mathcal{E}_m(\tilde{N}_1) \leq N^{-\kappa}
\]
for some small \( \kappa \).

2. For any \( m \notin \mathcal{J}(N) \)
\[
\operatorname{mes} \mathcal{E}_m(N)(\rho) \leq N^{-\kappa}, \quad \rho = \exp(-N^\kappa)
\]

3. Let \( \tilde{N} \geq \tilde{N}_1 \) and let \( \omega_j^{(\bar{N})} \) be as in Lemma (resp. as in (B.2)). If \( \omega_j^{(\bar{N})} \in \mathcal{P}_m(\bar{N}) \) for some \( m \notin \mathcal{J}(N) \) then there exists \( \mathcal{E}_{m,j}(N, \bar{N}) \subset [B_0(f), B_0(f)] \) with \( \operatorname{mes} \mathcal{E}_{m,j}(N, \bar{N}) \leq \exp(-N^\gamma) \) such that for any \( \xi \in [-B_0(f), B_0(f)] \setminus (\mathcal{E}_m(N) \cup \mathcal{E}_{m,j}(N, \bar{N})) \) one has
\[
\mathcal{E}_m(N)(\rho) \leq (\operatorname{mes} \mathcal{P}_j^{(\bar{N})}) \exp(-N^\gamma)
\]
where \( \gamma > 0 \) is some small constant.

Proof. The proof is basically the same for the shift and skew-shift. So, assume that \( T_\omega \) is the shift with \( \omega \in \mathbb{T}^2 \). By Corollary 3.6 there exists \( \mathcal{J}(N) \subset \{1, \ldots, N^2\} \) such that

- \( \operatorname{mes} \bigcup_{m \in \mathcal{J}(N)} \mathcal{P}_m(N^2) \leq N^{-\kappa} \)
- each \( \omega_j^{(N^2)} \) with \( m \notin \mathcal{J}(N) \) is \((N, \gamma_1, \gamma_2)\)-Diophantine for some small \( \gamma_1, \gamma_2 > 0 \).

Let \( m \notin \mathcal{J}(N) \) and let \( \mathcal{E}_m(N) = \mathcal{E}_{\omega_j^{(N^2)}, \omega_0}^{(N^2)}(N) \) be as in Theorem 3.8 with \( \omega_m^{(N^2)} \) in place of \( \omega_0 \) and any \( \omega_j^{(N)} \in \mathcal{P}_m(N^2) \) in place of \( \omega_1 \). Since \( |\omega_j^{(N)} - \omega_m^{(N^2)}| \lesssim N^{-2} \) this is legitimate. Then by Theorem 3.8 there exists \( \mathcal{E}_{m,j}(N, \bar{N}) := \mathcal{E}_{\omega_0, \omega_1}(N, \bar{N}) \) such that for any \( \xi \in [-B_0(f), B_0(f)] \setminus (\mathcal{E}_m(N) \cup \mathcal{E}_{m,j}(N, \bar{N})) \)

one has
\[
\mathcal{E}_{\omega_j^{(N)} + \theta/\bar{N}, \bar{N}}(\rho, \theta) \in \mathbb{T}^2 : \frac{1}{N} \sum_{k=1}^N \log |f(x_0 + \bar{N}\omega_j^{(N)} + \theta + k\omega, \omega) - \xi| - \langle f(\cdot, \omega_j^{(N)}) - \xi \rangle > N^{-\gamma}
\]
\[\leq N^{-2} \exp(-N^\gamma)\]

and the lemma follows. For the case of the skew-shift we use Corollary 3.6.

The following proposition is the main result of this section. It captures the mechanism needed for the elimination of bad \( \omega \) in the sense of 3.10. The exceptional set of \( \xi \) which appears in the proposition will be converted into an exceptional set of energies in Section 3.5.

Proposition 3.10. Let \( f(x, \omega) \) be \( C^1 \)-smooth. Let \( T_\omega \) be the shift (or the skew-shift). Fix \( x_0 \in \mathbb{T}^2 \). Given large \( N \) there exists \( Q(N) \subset \mathbb{T}^2 \) (resp. \( Q(N) \subset \mathbb{T} \)) and for each \( \omega \notin Q(N) \) a subset \( \mathcal{E}_\omega(N) \subset [-B_0(f), B_0(f)] \) such that

- \( \operatorname{mes} Q(N) \leq N^{-\kappa} \), \( \operatorname{mes} [\mathcal{E}_\omega(N)](\rho) \leq N^{-\kappa}, \rho = \exp(-N^{2\kappa}) \)
- For each \( \omega \notin Q(N) \), \( \xi \in [-B_0(f), B_0(f)] \setminus \mathcal{E}_\omega(N) \) and any \( N^2 < \bar{N} < \exp(N^\beta) \) (resp. \( N^4 < \bar{N} < \exp(N^\beta) \)) one has
\[
\frac{1}{N} \sum_{k=1}^N \log |f(T^k(T^\bar{N} x_0), \omega) - \xi| - \langle f(\cdot, \omega) - \xi \rangle | < N^{-\gamma}
\]
We refer to the eigenvalues of Hermitian matrices, see for example Appendix C in Cha or Bha.

\[ (4.1) \]

Consider the Schrödinger equation

Moreover, inspection of the sets

By (3.8), the set (3.9) consists of a union of at most 25 sets. Hence

Clearly, \( \text{mes} Q'(N) \leq 4 \text{mes} \left( \bigcup_{m \in \mathcal{J}(N)} P_m^{(N^2)} \right) \). Assume that \( \omega \in \mathbb{T}^2 \setminus Q'(N) \) and let for instance \( m, \tilde{N}, j \) be such that \( \omega \in P_j^{(\tilde{N})} \), \( j/\tilde{N} \in P_m^{(N^2)} \), and \( m \notin \mathcal{J}(N) \). Set

By (3.8), the set (3.9) consists of a union of at most 25 sets. Hence

On the other hand, since \( \text{mes} \mathcal{E}_{m,j}(N, \tilde{N}) \leq \text{mes} (-N^\kappa) \),

provided \( \beta < \kappa \) and \( N \) is large. Due to the estimate (3.7) and Fubini’s theorem there exists \( Q''(N) \subset \mathbb{T}^2 \setminus Q'(N) \) with \( \text{mes} Q''(N) \leq \exp(-N^{\gamma}/4) \) such that for any \( \omega \in \mathbb{T}^2 \setminus (Q'(N) \cup Q''(N)) \) any \( \xi \in \mathcal{E}_{\omega}'(N) \cup \mathcal{E}_{\omega}''(N) \) and any \( N^2 < \tilde{N} < \exp(N^{\beta}) \) we have

Set \( Q(N) = Q'(N) \cup Q''(N) \), \( \mathcal{E}_{\omega}(N) = \mathcal{E}_{\omega}'(N) \cup \mathcal{E}_{\omega}''(N) \). Then \( \text{mes} Q(N) < N^{-\kappa} \) and \( \text{mes} E_{\omega}(N) < N^{-\kappa} \). Moreover, inspection of the sets \( \mathcal{E}_{m}(N) \), \( \mathcal{E}_{m,j}(N, \tilde{N}) \) in (3.3) and (3.4) shows that, due to Theorem 3.8, \( \text{mes} \mathcal{E}_{m,j}(N, \tilde{N})(\rho) \leq N^{-\kappa} \) provided \( \rho < \exp(-N^\kappa) \).

4. LARGE DEVIATION THEOREMS AND ELIMINATION OF RESONANCES FOR DIRICHLET DETERMINANTS

Let \( T = T_\omega : \mathbb{T}^2 \to \mathbb{T}^2 \) be the shift or the skew-shift and let \( V(x) \in C^1(\mathbb{T}^2) \) be a real-valued function. Consider the Schrödinger equation

Let \( H_{[a,b]}(x, \omega) \) be the operator defined by (4.1) on the interval \([a,b]\) with Dirichlet boundary condition \( \psi(a - 1) = \psi(b + 1) = 0 \). Let \( f_{[a,b]}(x, \omega, E) \) be the characteristic determinant of \( H_{[a,b]}(x, \omega) \), i.e.,

We refer to \( f_{[a,b]}(x, \omega, E) \) as the Dirichlet determinant. Let \( E_1^{[a,b]}(x, \omega) < E_2^{[a,b]}(x, \omega) < \cdots < E_N^{[a,b]}(x, \omega) \) be the eigenvalues of \( H_{[a,b]}(x, \omega) \) with corresponding normalized eigenfunctions \( \psi_1^{[a,b]}(x, \omega), \ldots, \psi_N^{[a,b]}(x, \omega) \). We reserve the notations \( H_N(x, \omega), f_N(x, \omega), E_j^{(N)}(x, \omega), \) and \( \psi_j^{(N)}(x, \omega) \) for \([a,b] = [1,N]\).

The following lemma is a simple consequence of Weyl’s comparison theorem for the eigenvalues of Hermitian matrices, see for example Appendix C in Cha or Bha.
Lemma 4.1. Assume \(1 = a_1 < b_1 < b_1 + 1 = a_2 < b_2 < \ldots < a_n < b_n \leq N\). Then for any \(x \in \mathbb{T}^2\), \(E \in \mathbb{C}\) one has
\[
\begin{align*}
\left| \log |f_N(x, \omega, E)| - \sum_{k=1}^{n} \log |f_{[a_k, b_k]}(x, \omega, E)| \right| & \lesssim (n + N - b_n) \log[(B_0(V) + 1)\eta^{-1}].
\end{align*}
\]
where
\[
\eta = \text{dist}(E, \{E_j^{(N)}(x, \omega) : 1 \leq j \leq N\} \cup \{E_j^{[a_k, b_k]}(x, \omega) : 1 \leq k \leq n, 1 \leq j \leq \ell_k\})
\]
with \(\ell_k = b_k - a_k + 1\).

Proof. See Appendix C in [ Cha].

Theorem 4.2. Let \(T_\omega\) be the shift or skew-shift on \(\mathbb{T}^2\) and suppose \(V \in C^1(\mathbb{T}^2)\) is a real-valued function. Let \(N\) be large and assume that \(\omega\) is \((N, \gamma_1, \gamma_2)\)-Diophantine (resp. \(\omega \in \mathbb{T}_{c, e, N}\)). Then there exists \(\mathcal{E}_\omega(N) \subset [-B_0(V) - 2, B_0(V) + 2]\) with \(\text{mes}(\mathcal{E}_\omega(N)) < N^{-\kappa}\) such that for any
\[
E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \mathcal{E}_\omega(N)
\]
one has
\[
\text{mes}\left\{ x \in \mathbb{T}^2 : \left| \log |f_N(x, \omega, E)| - \langle \log |f_N(\cdot, \omega, E)| \rangle \right| > N^{1-\kappa}\right\} \lesssim \exp(-N^\kappa)
\]
where \(\kappa > 0\) is a small constant depending on the Diophantine condition.

Proof. The proof is the same for shift and skew-shift. So, assume \(T_\omega\) is a shift, \(\omega \in \mathbb{T}^2\). Let \(\ell \asymp N^\beta\) be an integer with some small \(\beta > 0\). Let \(n = [N\ell^{-1}]\), \(k = N - m\ell\). Then by Lemma 4.1 with \(a_m = (m-1)\ell + 1\), \(b_m = m\ell\), \(1 \leq m \leq n\), \(a_{n+1} = b_n + 1\), \(b_{n+1} = N\) one has
\[
\left| \log |f_N(x, \omega, E)| - \sum_{m=1}^{n} \log |f_{[a_m, b_m]}(x \omega, E)| \right| \lesssim N^{1-\beta/2}
\]
provided \(N\) is large and
\[
\begin{align*}
\min_{1 \leq j \leq N} |E_j^{(N)}(x, \omega) - E| & \geq \exp(-N^{\frac{\beta}{2}}) \\
\min_{j, m} |E_j^{[a_m, b_m]}(x, \omega)| - E| & \geq \exp(-N^{\frac{\beta}{2}})
\end{align*}
\]
There exists \(\tilde{\mathcal{E}}_\omega(N)\) with \(\text{mes}(\tilde{\mathcal{E}}_\omega(N)) < \exp(-N^{\frac{\beta}{4}}/4)\) such that for any \(E \notin \tilde{\mathcal{E}}_\omega(N)\) one has
\[
\text{mes}\left\{ x \in \mathbb{T}^2 : (4.3) \text{ fails} \right\} \lesssim \exp(-N^{\frac{\beta}{4}}/4)
\]
Note that
\[
\begin{align*}
\log |f_N(x, \omega, E)| = & \sum_{j=1}^{N} \log |E_j^{(N)}(x, \omega) - E| \\
\log |f_{\ell}(x, \omega, E)| = & \sum_{j=1}^{\ell} \log |E_j^{(\ell)}(x, \omega) - E|
\end{align*}
\]
Let \(S\) be the set in (4.3). Due to Lemma 4.2, one has
\[
\begin{align*}
\langle \log |E_j^{(N)}(\cdot, \omega) - E| \rangle - \int_{\mathbb{T}^2 \setminus S} \log |E_j^{(N)}(x, \omega) - E| \, dx & \leq \exp(-N^{\frac{\beta}{4}}/10) \quad \forall 1 \leq j \leq N \\
\langle \log |E_j^{(\ell)}(\cdot, \omega) - E| \rangle - \int_{\mathbb{T}^2 \setminus S} \log |E_j^{(\ell)}(x, \omega) - E| \, dx & \leq \exp(-N^{\frac{\beta}{4}}/10) \quad \forall 1 \leq j \leq \ell
\end{align*}
\]
provided \(E\) does not fall into some set of measure \(\lesssim \exp(-N^{\frac{\beta}{4}}/4)\). We may assume that \(\tilde{\mathcal{E}}_\omega(N)\) contains that set. In particular, due to (4.3),
\[
\begin{align*}
N^{-1} \langle \log |f_N(\cdot, \omega, E)| \rangle - \ell^{-1} \langle \log |f_{\ell}(\cdot, \omega, E)| \rangle & \lesssim N^{-\frac{\beta}{2}} \asymp \ell^{-\frac{\beta}{4}}
\end{align*}
\]
Since $\omega$ is $(N, \gamma_1, \gamma_2)$–Diophantine, it follows that $\{\ell \omega\}$ is $(N, \gamma_1, \gamma_2/2)$–Diophantine provided $\beta < \gamma_2/2$. Recall that the functions $E_j^{(\ell)}(x, \omega)$ are $C^1$–smooth with

$$B_0(E_j^{(\ell)}) \leq 2 + B_0(V), \quad B_1(E_j^{(\ell)}) \lesssim \ell(1 + B_1(V)).$$

Therefore, Theorem 2.17 applies to each average

$$\text{Estimate (4.9) follows from Lemma 4.1. Applying (4.9) entry-wise, we conclude that (4.10) follows for any}$$

for all $1 \leq j \leq \ell$.

Let $E_\omega(\ell, j, n)$ stand for the set $T(N)$ from Theorem 2.17 applied to the function $f(x) = E_j^{(\ell)}(x, \omega)$ and the shift $T_\ell \omega$. Then

$$\text{mes} \left\{ x \in \mathbb{T}^2 : \left| \frac{1}{n} \sum_{m=1}^{n} \log |E_j^{(\ell)}(T_\ell^m x, \omega) - E| - \langle \log |E_j^{(\ell)}(\cdot, \omega) - E| \rangle > N^{1-\beta/2} \right| \right\} \leq \exp(-N^{\beta/2})$$

for any $E \notin \bigcup_{j=1}^{\ell} E_\omega(\ell, j, n)$. □

Let $M_{[a,b]}(x, \omega, E)$ be the monodromy matrices of equation (4.1). We reserve the notation $M_N(x, \omega, E)$ for $[a, b] = [1, N]$. Recall that

$$(4.8) \quad M_{[a,b]}(x, \omega, E) = \begin{bmatrix} f_{[a,b]}(x, \omega, E) & -f_{[a+1,b]}(x, \omega, E) \\ f_{[a,b-1]}(x, \omega, E) & -f_{[a+1,b+1]}(x, \omega, E) \end{bmatrix}$$

Note that Lemma 4.1 implies the following assertion.

**Lemma 4.3. One has**

- For any intervals $[s_i, t_i]$, $i = 1, 2$,

$$(4.9) \quad \log |f_{[s_1,t_1]}(x, \omega, E)| \leq \log |f_{[s_2,t_2]}(x, \omega, E)| + (|s_2 - s_1| + |t_2 - t_1|) \log[(1 + B_0(V))\eta^{-1}]$$

where

$$\eta = \min \left\{ \frac{1}{2}, \text{dist}(E, \text{spec } H_{[s_2,t_2]}(x, \omega)) \right\}$$

- For the monodromies, one has

$$(4.10) \quad 0 \leq \log \|M_{[a,b]}(x, \omega, E)\| - \log |f_{[a,b]}(x, \omega, E)| \lesssim \log[(1 + B_0(V))\eta^{-1}]$$

where

$$\eta = \min \left\{ \frac{1}{2}, \text{dist}(E, \text{spec } H_{[a,b]}(x, \omega)) \right\}$$

Proof. Estimate (4.9) follows from Lemma 4.1. Applying (4.9) entry-wise, we conclude that (4.10) follows from (4.8). □

For any $x, \omega, E$ clearly

$$0 \leq \log \|M_N(x, \omega, E)\| \leq N \log(1 + B_0(V))$$

Next, we can draw the following conclusion from Lemma 4.3. Recall that $L(\omega, E)$ is the Lyapunov exponent.

**Proposition 4.4.** There exists $\mathcal{F}_\omega(N)$ with $\text{mes } \mathcal{F}_\omega(N) \leq \exp(-N^\kappa/4)$ such that for any

$$E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \mathcal{F}_\omega(N)$$

one has

$$(4.11) \quad \left| N^{-1} \langle \log \|M_N(\cdot, \omega, E)\| \rangle - N^{-1} \langle \log |f_N(\cdot, \omega, E)| \rangle \right| \leq N^{-\kappa}$$

$$(4.12) \quad \left| N^{-1} \langle \log \|M_N(\cdot, \omega, E)\| \rangle - \ell^{-1} \langle \log \|M_\ell(\cdot, \omega, E)\| \rangle \right| \leq \ell^{-1/2}$$
for any $\ell \asymp N^{-\beta}$. Here $\kappa, \beta > 0$. In particular, given $\ell$, there exists $\mathcal{F}_\omega(\ell)$ with $\text{mes} \mathcal{F}_\omega(\ell) \leq \exp(-\ell^\kappa)$ such that for any $E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \mathcal{F}_\omega(\ell)$ one has
\begin{equation}
| \ell^{-1} \log |f_\ell(\cdot, \omega, E)| - L(\omega, E)| \lesssim \ell^{-\frac{1}{2}} \tag{4.13}
\end{equation}

Proof. We shall use the notations from the proof of Theorem 3.2. Thus,
\begin{equation}
|N^{-1} \log \|M_N(\cdot, \omega, E)\| - \ell^{-1} \langle \log \|M_\ell(\cdot, \omega, E)\| \rangle | \leq \ell^{-\frac{1}{2}} \tag{4.13}
\end{equation}
for all $E \notin \mathcal{E}_\omega(N)$ (see (4.7) in the proof). Assume $E \notin \mathcal{E}_\omega(N)$. Let $S$ be the set in (4.5) so that (4.13) is valid whenever $x \notin S$ ($S$ depends on $E$). Due to Lemma 4.8 one has
\begin{equation}
| \log \|M_N(x, \omega, E)\| - \log |f_N(x, \omega, E)| | \lesssim N^{-\beta/2} \tag{4.14}
\end{equation}
provided $x \notin S$. Due to (4.6),
\begin{equation}
|N^{-1} \log |f_N(\cdot, \omega, E)| - N^{-1} \int_{T^2 \setminus S} \log |f_N(\cdot, \omega, E)| \, dx | \lesssim \exp(-N^{\beta/2}/20) \tag{4.15}
\end{equation}
Since
\begin{equation}
0 \leq \log \|M_N(x, \omega, E)\| \leq N \log(1 + B_0(V)) \tag{4.16}
\end{equation}
for any $x$, see (4.8),
\begin{equation}
|N^{-1} \log \|M_N(\cdot, \omega, E)\| - N^{-1} \int_{T^2 \setminus S} \log \|M_N(\cdot, \omega, E)\| \, dx | \lesssim \log(1 + B_0(V)) \text{mes } S \lesssim \exp(-N^{\beta/2}/20) \tag{4.17}
\end{equation}
and (4.11), (4.12) follow. Given $\ell$, set $\ell_t = \left(\ell(t)\right)^N, N_t = \ell_{t+1}, t = 0, 1, \ldots, \mathcal{F}_\omega(\ell) = \bigcup_{t=0}^{\infty} \mathcal{E}_\omega(N_t).$ Then (4.13) is valid for $E \notin \mathcal{F}_\omega(\ell)$. \hfill \Box

Corollary 4.5. Let $T_\omega$ be the shift or the skew-shift on $T^2$ and suppose $V \in C^1(T^2)$ is a real-valued function. Let $N$ be large and assume that $\omega$ is $(N, \gamma_1, \gamma_2)$-Diophantine (resp. $\omega \in T_{c, \varepsilon, N}$). Then there exist $\mathcal{E}_\omega(N) \subset [-B_0(V) - 2, B_0(V) + 2]$ with $\text{mes } \mathcal{E}_\omega(N) < N^{-\kappa}$ such that for any
\begin{equation}
E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \mathcal{E}_\omega(N) \tag{4.18}
\end{equation}
one has
\begin{equation}
\text{mes } \{ x \in T^2 : | \log \|M_N(x, \omega, E)\| - \langle \log \|M_N(\cdot, \omega, E)\| \rangle | > N^{-\kappa} \} \lesssim \exp(-N^{-\kappa}) \tag{4.19}
\end{equation}
\begin{equation}
\sup_{x \in T^2} \log \|M_N(x, \omega, E)\| \leq \langle \log \|M_N(\cdot, \omega, E)\| \rangle + N^{-\kappa} \tag{4.20}
\end{equation}
\begin{equation}
\text{mes } \{ x \in T^2 : | \log \|M_N(x, \omega, E)\| - \langle \log |f_N(\cdot, \omega, E)| \rangle | > N^{-\kappa} \} \lesssim \exp(-N^{-\kappa}) \tag{4.21}
\end{equation}
\begin{equation}
| \log \|M_N(\cdot, \omega, E)\| - \langle \log |f_N(\cdot, \omega, E)| \rangle | \lesssim N^{-\kappa} \tag{4.22}
\end{equation}

Proposition 4.6. Let us use the notations of Theorem 3.2. Then for any $E \notin \mathcal{E}_\omega(N)$ the following holds: if for some $x_1 \in T^2$, $\text{dist}(E, \text{spec } H_N(x_1, \omega)) > (1 + B_1(V)) \exp(-N^{-\kappa}/2)$, then
\begin{equation}
\log |f_N(x_1, \omega, E)| > \langle \log |f_N(\cdot, \omega, E)| \rangle - 2N^{-1}\kappa \tag{4.23}
\end{equation}
Proof. Due to Theorem 4.2 there exists $x \in T$ such that $|x - x_1| < \exp(-N^{-\kappa})$ and
\begin{equation}
\log |f_N(x, \omega, E)| > \langle \log |f_N(\cdot, \omega, E)| \rangle - N^{-1}\kappa \tag{4.24}
\end{equation}
Since $E_j^{(N)}(x, \omega)$ are $C^1$-smooth with $B_1(E_j^{(N)}) \lesssim B_1(V)$, one obtains
\begin{equation}
|E_j^{(N)}(x, \omega) - E_j^{(N)}(x_1, \omega)| \lesssim B_1(V) \exp(-N^{-\kappa}) \tag{4.25}
\end{equation}
Hence,
\begin{equation}
\sup_{1 \leq j \leq N} \frac{|E_j^{(N)}(x, \omega) - E|}{|E_j^{(N)}(x_1, \omega) - E|} \leq 1 + C \left[ \text{dist}(E, \text{spec } H_N(x, \omega)) \right]^{-1} \exp(-N^{-\kappa}) \tag{4.26}
\end{equation}
The proposition follows from \(4.18\) and \(4.19\). □

The following proposition — which is a consequence of our main elimination method of Proposition \(3.10\) — shows that we can insure that the large deviation theorem holds for a fixed phase \(x_0\) as long as we shift it by an amount \(\tilde{N}\omega\) with \(\tilde{N} \gg N\); of course this requires the removal of a small set of frequencies \(\omega\) and energies \(E\) depending on \(x_0\).

**Proposition 4.7.** Let \(V \in C^1(\mathbb{T}^2)\) and fix \(x_0 \in \mathbb{T}^2\). Given large \(N\), there exist a set \(\mathcal{P}(N)\) and for each \(\omega \notin \mathcal{P}(N)\) a subset \(\mathcal{R}_\omega(N) \subset [-B_0(V) - 2, B_0(V) + 2]\) so that

- with \(\rho = \exp(-N^\kappa)\), we have
  
  \[
  \text{mes } \mathcal{P}(N) < N^{-\kappa}, \text{mes } |\mathcal{R}_\omega(N)|(\rho) < N^{-\kappa}
  \]

- for each \(\omega \notin \mathcal{P}(N)\), \(E \in [-B_0(V) - 2, B_0(V) + 2]\) \(\mathcal{R}_\omega(N)\), and any \(N^3 < \tilde{N} < \exp(N^\beta)\) there is the bound
  
  \[
  \left| N^{-1} \log |f_N(x_0 + \tilde{N}\omega, \omega, E)| - N^{-1} \langle \log |f_N(\cdot, \omega, E)| \rangle \right| \lesssim N^{-\gamma}
  \]

Here \(\kappa, \beta, \gamma > 0\) are small constants.

**Proof.** We consider the case of the shift \(T_\omega\), \(\omega \in \mathbb{T}^2\) and shall use the notation from the proof of Theorem \(4.2\). By Proposition \(3.10\) applied to \(f(x, \omega) = E_j^{(t)}(x, \omega)\) there exists a set \(Q_j(N) \subset \mathbb{T}^2\) and a subset \(\mathcal{E}_{\omega,j}(N) \subset [-B_0(V) - 2, B_0(V) + 2]\) with \(\text{mes } Q_j(N) < N^{-\kappa}\) and \(\text{mes } |\mathcal{E}_{\omega,j}(N)|(\rho) < N^{-\kappa}, \rho = \exp(-N^\kappa)\) so that for each \(\omega \in \mathbb{T}^2 \setminus Q_j(N)\) and any

\[
E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \mathcal{E}_{\omega,j}(N)
\]

as well as \(N^2 \leq \tilde{N} \leq \exp(N^\beta)\) one has

\[
\left| \frac{1}{n} \sum_{k=1}^n \log |E_j^{(t)}(x_0 + \tilde{N}\omega + k\omega, \omega) - E| - \langle \log |E_j^{(t)}(\cdot, \omega) - E| \rangle \right| \lesssim N^{-\gamma}
\]

Hence,

\[
\left| \frac{1}{n} \sum_{k=1}^n \log |f_t(x_0 + \tilde{N}\omega + k\omega, \omega) - E| - \langle \log |f_t(\cdot, \omega) - E| \rangle \right| \lesssim \ell N^{-\gamma} \lesssim N^{-\gamma/2}
\]

for \(\omega \in \mathbb{T}^2 \setminus \bigcup_{j=1}^\ell Q_j(N), E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \bigcup_{j=1}^\ell \mathcal{E}_{\omega,j}(N)\). As in the proof of Theorem \(4.2\),

\[
\left| \log |f_N(x_0 + \tilde{N}\omega, \omega, E)| - \sum_{k=1}^n \log |f_t(x_0 + \tilde{N}\omega + k\ell\omega, \omega, E)| \right| \lesssim N^{1-\gamma/2}
\]

provided

\[
\min_{1 \leq j \leq \ell, \ell \leq n} \min_{N^2 \leq N \leq \exp(N^3)} |E_j^{(t)}(x_0 + \tilde{N}\omega, \omega) - E| \geq \exp(-N^\gamma/2)
\]

\[
\min_{1 \leq j \leq \ell, 0 \leq k \leq n} \min_{N^2 \leq N \leq \exp(N^3)} |E_j^{(t)}(x_0 + \tilde{N}\omega + k\ell\omega, \omega) - E| \geq \exp(-N^\gamma)
\]

Given \(\omega\), let \(\mathcal{E}'_{\omega}(N)\) be the set of \(E \in [-B_0(V) - 2, B_0(V) + 2]\) such that \(4.20\) fails. Clearly, \(\text{mes } |\mathcal{E}'_{\omega}(N)|(\rho) < \exp(-N^{\gamma/2}/2), \rho = \exp(-N^\gamma)\) provided \(\beta \ll \gamma\). Finally, define

\[
\mathcal{P}(N) = \bigcup_{j=1}^\ell Q_j(N), \quad \mathcal{R}_\omega(N) = \mathcal{E}'_{\omega}(N) \cup \bigcup_{j=1}^\ell \mathcal{E}_{\omega,j}(N)
\]

and the proposition is proved. □
5. Estimates on the Green function and the proof of Theorem 1.1

By Cramer’s rule,

\[ G_{[a,b]}(x, \omega, E)(m, n) := (H_{[a,b]}(x, \omega) - E)^{-1}(m, n) \]
(5.1)

\[ = \frac{f_{[a,m-1]}(x, \omega, E)f_{[n+1,b]}(x, \omega, E)}{f_{[a,b]}(x, \omega, E)} \]

for all \( a \leq m \leq n \leq b \). To evaluate the Green function \( G_{[a,b]}(x, \omega, E) \) we need to obtain appropriate estimates on the Dirichlet determinants in (5.1) which are uniform in \( x, m, n \). To derive such estimates we need to modify the proof of Theorem 4.2 slightly; in fact, we refer the reader to Remark 2.19 for these matters. We shall use the notations from the proof of Theorem 4.2. Note that \( f \) for any \( \gamma \) modify the proof of Theorem 4.2 slightly; in fact, we refer the reader to Remark 2.19 for these matters. We shall use the notations from the proof of Theorem 4.2. Note that for any \( x, \omega, E \) and \( 1 \leq N' \leq N \)

\[ \log \| M_{N'}(x, \omega, E) \| \leq \sum_{m=1}^{n'} \log \| M_\ell(T_{E_{\omega}}^m x, \omega, E) \| + \ell \log |2B_0(V) + 4| \]

where \( n' = \lfloor \ell^{-1} N' \rfloor \). This is because

\[ M_{N'}(x, \omega, E) = M_{[n',N']}(x, \omega, E) \prod_{m=n'}^{1} M_{[a_m,b_m]}(x, \omega, E) \]

where \( a_m, b_m \) are as in the proof of Theorem 4.2. Assume that \( \omega \) is \((N, \gamma_1, \gamma_2)\)–Diophantine. Recall that due to Remark 2.19 one then has

\[ \sup_{n^{-\frac{1}{2}} \leq n' \leq n} \sup_{\# B \leq (n')^{1-2n} x \in T^2} |n| \sum_{m \in [1,n'] \setminus B} \log \| E^{(1)}_j(x + (m - 1)\ell \omega, \omega, E) \| \leq \langle \log \| E^{(1)}_j(\cdot, \omega, E) \| \rangle + n^{-\kappa} \]

for any \( E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \tilde{E}_\omega(\ell, j, n) \) where \( \text{mes } \tilde{E}_\omega(\ell, j, n) \leq 2n^{-\frac{n}{2}} \). Since \( \ell \leq N^3 \) and taking \( \beta \ll \kappa \) one has the following assertion

**Lemma 5.1.** Let \( f \in C^1(T^2) \) and assume that \( \omega \) is \((n, \gamma_1, \gamma_2)\)–Diophantine with some large \( n \) (resp. \( \omega \in T_{c, \varepsilon, N} \) for the case of the skew-shift). There exists a set \( E'_\omega(n) \), \( \text{mes } E'_\omega(n) < n^{-\frac{n}{2}} \) so that

\[ \sup_{n^{-\frac{1}{2}} \leq n' \leq n} \sup_{\# B \leq (n')^{1-2n} x \in T^2} \frac{1}{n} \sum_{m \in [1,n'] \setminus B} \log \| f_t(T^m_{\omega_x} x, \omega, E) \| \leq \langle \log \| f_t(\cdot, \omega, E) \| \rangle + n^{-\kappa} \]

Similar estimates hold for \( f_{[a,\ell+b]} \) with \( |a|, |b| \leq 1 \). Moreover, the average on the right-hand side of (5.2) can be kept the same for all \( f_{[a,\ell+b]} \), \( |a|, |b| \leq 1 \).

**Proof.** Adding up (5.2) over \( 1 \leq j \leq \ell \) one obtains (5.3). The same arguments are valid for \( f_{[a,\ell+b]} \). The last part follows from Lemma 4.3. \( \square \)

To proceed we need to compare the following two sums:

\[ \sum_{m=1}^{n'} \log \| M_\ell(x + (m - 1)\ell \omega, \omega, E) \| \]

versus

\[ \sum_{m=1}^{n'} \log \| f_\ell(x + (m - 1)\ell \omega, \omega, E) \| \]

**Lemma 5.2.** Under the assumptions of the previous lemma there exists \( \tilde{E}_\omega(\ell, n) \subset [-B_0(V) - 2, B_0(V) + 2] \) with \( \text{mes } \tilde{E}_\omega(\ell, n) \leq n^{-\kappa_1} \) such that for any \( E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \tilde{E}_\omega(\ell, n) \), \( n^{-\frac{n}{2}} \leq n' \leq n \), and any \( x \in T^2 \) one has

\[ \# \{ 1 \leq m \leq n' : \log \| M_\ell(T^m_{\omega_x} x, \omega, E) \| > \log \| f_\ell(T^m_{\omega_x} x, \omega, E) \| + \log (|B_0(V) + 1|n) \} \leq (n')^{1-\kappa_2} \]

Here \( 0 < \mu \ll \kappa_1, \kappa_2, \kappa \ll 1, \ell \asymp n^{2\mu} \).
Proof. We consider the case of a shift $T_\omega$, $\omega \in \mathbb{T}^2$. Due to Lemma 4.3
\[ \log \| M_\ell(x + m\ell \omega, \omega, E) \| \leq \log | f_\ell(x + m\ell \omega, \omega, E) | + \log (B_0(V) + 1)n \]
unless
\[ (5.4) \quad \min_j | E - E_j^{(\ell)}(x + m\ell \omega, \omega) | < n^{-\kappa} \]
Due to Corollary 2.8, one can find $\tilde{E}_\omega(\ell, j, n)$ with $\tilde{E}_\omega(\ell, j, n) \leq n^{-\kappa/2}$ such that for any $E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \tilde{E}_\omega(\ell, j, n)$ and any $x \in \mathbb{T}^2$ one has
\[ \# \{ 1 \leq m \leq n' : | E - E_j^{(\ell)}(x + m\ell \omega, \omega) | < n^{-\kappa} \} \leq (n')^{1-\kappa/2} \]
Let $\bar{E}_\omega(\ell, n) = \bigcup_j \tilde{E}_\omega(\ell, j, n)$. Then $\bar{E}_\omega(\ell, n) \leq n^{-\kappa/3}$ and for any $E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \bar{E}_\omega(\ell, n)$, $x \in \mathbb{T}^2$ we have
\[ \# \{ 1 \leq m \leq n' : (5.4) \text{ fails} \} \leq \ell(n')^{1-\kappa/3} \]
as desired. $\square$

We can now prove the following uniform upper bound for $\log \| M_N(x, \omega, E) \|$.

**Proposition 5.3.** Let $V \in C^1(\mathbb{T}^2)$, and assume $N$ is large. Let $T_\omega$ be a shift or a skew-shift. Suppose that $\omega$ is $(N, \gamma_1, \gamma_2)$-Diophantine (resp. $\omega \in \mathbb{T}_{c,r,N}$). There exists $\bar{E}_\omega'(N) \subset [-B_0(V) - 2, B_0(V) + 2]$ with $\bar{E}_\omega'(N) < N^{-\infty}$ such that
\[ \sup_{N \uparrow N_0} \sup_{x \in \mathbb{T}^2} \frac{1}{N} \log \| M_N'(x, \omega, E) \| \leq \frac{1}{N} \log (\| M_N(\cdot, \omega, E) \|) + N^{-\kappa} \]
for all $E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \bar{E}_\omega'(N)$.

**Proof.** We shall use the notations from the proof of Theorem 4.2. Let $E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \bar{E}_\omega(\ell, n)$ where $\ell = N^\delta$, $n = \lfloor N\ell^{-1} \rfloor$, and $\bar{E}_\omega(\ell, n)$ is defined in Lemma 5.2. Then for any $n^\frac{1}{\delta} \leq n' \leq n$, $x \in \mathbb{T}^2$ one has
\[ \sum_{m=1}^{n'} \log \| M_\ell(T_\omega^{(m-1)\ell}x, \omega, E) \| \leq \sum_{m \in [1, n']} \log | f_\ell(T_\omega^{(m-1)\ell}x, \omega, E) | + n' \log | (B_0(V) + 1)n \| + (n')^{1-\kappa_1} \sup_{y} \log \| M_\ell(y, \omega, E) \| \]
where
\[ B_\ell(x, \omega, E) = \{ 1 \leq m \leq n' : \log | M_\ell(T_\omega^{(m-1)\ell}x, \omega, E) | > \log | f_\ell(T_\omega^{(m-1)\ell}x, \omega, E) | + \log | (B_0(V) + 1)n | \}\]
and $\# B_\ell(x, \omega, E) \leq (n')^{1-\kappa}$. Combining (5.5) with Lemma 5.1 yields
\[ \frac{1}{n'\ell} \sum_{m=1}^{n'} \log \| M_\ell(T_\omega^{(m-1)\ell}x, \omega, E) \| \leq \frac{1}{\ell} \log | f_\ell(\cdot, \omega, E) | + \frac{1}{\ell} \log | (B_0(V) + 1)n | + \ell^{-1} n^{-\kappa/2} \]
Finally, by Proposition 1.2
\[ \ell^{-1} \log | f_\ell(\cdot, \omega, E) | \leq N^{-1} \log | f_N(\cdot, \omega, E) | + \ell^{-1/2} \]
The proposition is proved. $\square$

Combining Propositions 4.7 and 5.3 yields the following.

**Proposition 5.4.** Let $V \in C^1(\mathbb{T}^2)$ and fix $x_0 \in \mathbb{T}^2$. Let $T_\omega$ be the shift or the skew-shift. Assume that $L(\omega, E) \geq L_0 > 0$ for any $\omega$ and $E \in (E_1, E_2)$. Given large $N$ there exists a set $P(N)$ and for each $\omega \in P(N)$ a subset $P(\omega)(N) \subset [-B_0(V) - 2, B_0(V) + 2]$ so that
(a) with \( \rho = \exp(-N^\kappa) \) one has
\[
\mes \mathcal{P}(N) < N^{-\kappa}, \quad \mes [\mathcal{K}_\omega(N)](\rho) < N^{-\kappa}
\]

(b) for each \( \omega \notin \mathcal{P}(N) \), \( E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \mathcal{K}_\omega(N) \cap (E_1, E_2) \) and \( N^3 \leq \bar{N} \leq \exp(N^\kappa) \) one has
\[
|G_N(x_0 + \bar{N}, \omega, E)(m, n)| \leq \exp(-L_0|m - n|/2)
\]
for any \( |m - n| > N/2 \)

(c) with \( N_0 = N^3 \), \( N_0 \ll N_1 \leq \exp(N^\beta) \), for any \( \omega \notin \mathcal{P}(N) \), and any
\[
E \in [-B_0(V) - 2, B_0(V) + 2] \setminus [\mathcal{K}_\omega(N)](\rho) \cap (E_1, E_2)
\]
we have \( \text{dist}(\text{spec} H_{[N_0, N_1]}(x, \omega), E) > \exp(-N^\kappa) \) and
\[
|G_{[N_0, N_1]}(x, \omega, E)|(m, n) \leq \exp(-L_0|m - n|/3)
\]
for any \( |m - n| > N/2 \). Here, \( \beta, \kappa \) are as in Proposition 4.7

Proof. Let \( \mathcal{P}(N), \mathcal{E}_\omega(N) \subset [-B_0(V) - 2, B_0(V) + 2], \omega \in \mathbb{T}^2 \setminus \mathcal{P}(N) \) be as in Proposition 4.7 Then every
\[
E \in [-B_0(V) - 2, B_0(V) + 2] \cap \mathcal{E}_\omega(N)
\]
satisfies, for any \( \bar{N} \) as above,
\[
|f_N(T^N_{\omega}(x_0), \omega, E)| > \exp(\left< \log |f_N(\cdot, \omega, E)| \right> - N^{1-\gamma})
\]
\[
> \exp(NL(\omega, E) - N^{1-\gamma}) > \exp(NL(\omega, E)/2)
\]
provided \( N \) is large. Here we used Proposition 4.3. Due to Proposition 6.11 each \( \omega \notin \mathcal{P}(N) \) is \( (N, \gamma_1, \gamma_2) \)-Diophantine (resp. \( \omega \in T_{c,e,N} \)). Therefore, by Proposition 5.10 there exists \( \mathcal{E}_\omega(N) \), mes \( \mathcal{E}_\omega(N) < N^{-\kappa} \) such that for any
\[
E \in [-B_0(V) - 2, B_0(V) + 2] \cap (E_1, E_2) \setminus \mathcal{E}_\omega(N)
\]
any \( x \in \mathbb{T}^2 \) and any interval \([s, t] \) with \( N^{1/2} < t - s \leq N \) one has
\[
|f_{[s,t]}(x, \omega, E)| \leq \exp(\left< |t - s|N^{-1}(\log |f_N(\cdot, \omega, E)|) + (t - s)^{1-\kappa} \right>) \leq \exp((t - s)L(\omega, E) + 2(t - s)^{1-\kappa})
\]
Assume \( \omega \notin \mathcal{P}(N) \), and
\[
E \in [-B_0(V) - 2, B_0(V) + 2] \cap (E_1, E_2) \setminus \left( \mathcal{E}_\omega(N) \cup \mathcal{E}_\omega(N) \right)
\]
Then 5.11, 5.7. 5.8 imply 5.10 when \( a = 1 \leq N^{1/2} < m < n < N - N^{1/2} \) and \( n - m > N^{1-\kappa} \). To prove 5.6 when \( 1 \leq m \leq N^{1/2} \) or \( N - N^{1/2} < n < N \) one can use the trivial upper bound
\[
|f_{[s,t]}(x, \omega, E)| \leq \exp(2(s - t)(2 + B_0(V)))
\]
Thus (b) holds. We invoke now the following general fact which is valid for general discrete Schrödinger equations: if for some \( E \) the estimate
\[
|G_{[a,b] + N}(x, \omega, E)(m, n)| \leq \exp(-L|m - n|)
\]
holds for all \( a' \in [a, b], |m - n| > N/2 \), then \( E \notin \text{spec} H_{[a,b]}(x, \omega) \) provided \( N > (\log(b - a))^2 \), \( b - a > R_0(L) \) where \( R_0(L) \) is a suitable constant (see for example Appendix C in [Cha]). From this and (b) we conclude that \( E \notin \text{spec} H_{[N_0, N_1]}(x_0, \omega) \) for any \( \omega \notin \mathcal{P}(N) \),
\[
E \in [-B_0(V) - 2, B_0(V) + 2] \cap (E_1, E_2) \setminus \mathcal{K}_\omega(N)
\]
That means, in particular, if
\[
E \in [-B_0(V) - 2, B_0(V) + 2] \cap (E_1, E_2) \setminus \mathcal{K}_\omega(N)(\rho)
\]
then \( (E - \rho, E + \rho) \cap \text{spec} H_{[N_0, N_1]}(x_0, \omega) = \emptyset \).

Finally, to complete the proof of (c) we refer to yet another general fact about Schrödinger equations: if for some \( E \) the estimate 5.9 holds for all \( a' \in [a, b] \) and \( \text{dist}(E, \text{spec} H_{[a,b]}(x, \omega)) > \exp(-N^\kappa) \), then
\[
|G_{[a,b]}(x, \omega, E)(m, n)| \leq \exp(-L_1|m - n|/2)
\]
for all $|m - n| > N/2$, provided $N > (\log(b - a))^2$, $b - a > R_0(L_1)$. 

**Remark 5.5.** A similar statement is valid for the Green functions $G_N(x_0 - \tilde{N}, \omega, E)(m, n)$, $N^3 \leq \tilde{N} \leq \exp(N^a)$ and $G_{[-N_1, -N_0]}(x_0, \omega, E)(m, n)$. We will use the same notations $P(N), K_\omega(N)$ for the exceptional sets of $\omega \in \mathbb{T}^d$ and $E \in [-B_0(V) - 2, B_0(V) + 2]$ needed to guarantee the estimates for these Green functions as for the Green function in Proposition 5.4.

**Corollary 5.6.** Assume that some function $\psi(n)$, $n \in \mathbb{Z}$ obeys

$$\tag{5.10} - \psi(n + 1) - \psi(n - 1) + V(T^n_{\omega}x_0)\psi(n) = E\psi(n) \text{ for } -N_1 \leq n \leq N_1$$

for some $\omega \notin P(N)$, $E \in [-B_0(V) - 2, B_0(V) + 2] \setminus K_\omega(N)$ where $P(N), K_\omega(N)$ are as in Proposition 5.4 and $\exp(N^a) \geq N_1$. Assume in addition that

$$\max_{|n| \leq N_1} |\psi(n)| \leq 1$$

then for any $N^3 < |n| < N_1$ holds

$$|\psi(n)| \lesssim \exp(-L_0 \min(|n| - N^3, N_1 - |n|)/2).$$

If $\psi(n)$ in addition satisfies the Dirichlet boundary conditions on $[-N_1 + 1, N_1 - 1]$, i.e., $\psi(-N_1) = \psi(N_1) = 0$, then

$$|\psi(n)| \leq \exp(-L_0(|n| - N^3)/2)$$

for any $N^3 < |n| \leq N_1$.

**Proof.** Let $N^3 < n < N_1$. Then

$$\psi(n) = \sum_{m \in \{N^3, N_1\}} G_{[N^3, N_1]}(x_0, \omega, E)(n, m)\psi(m).$$

Both estimates follow now from Proposition 5.4 (see also Remark 5.5). 

**Proof of Theorem 1.1.** We shall use the notations from the Proposition 5.4. Fix $x_0$. Given $N$, define $N_t := \exp(N_t^\beta), t = 1, 2, \ldots, N_0 = N_1, \mathcal{P}(N) = \bigcup_{t \geq 0} P(N_t)$, $\mathcal{K}_{\omega}(N) = \bigcup_{t \geq 0} \left[ K_{\omega}(N_t) \right] (\rho_{N_t})$, for $\omega \notin \mathcal{P}(N)$ where $P(N_t), K_{\omega}(N_t)$ are as in Remark 5.5, $\rho_{N_t} = \exp(-(N_t)^\kappa)$. Then

$$\tag{5.11} \arrayenv{\text{mes}}{\mathcal{P}(N)} \leq \sum N_t^{-\kappa} \lesssim N^{-\kappa}$$

$$\text{mes} \mathcal{K}_{\omega}(N) \leq \sum N_t^{-\kappa} \lesssim N^{-\kappa}$$

If $\psi(n)$ obeys

$$\tag{5.12} - \psi(n + 1) - \psi(n - 1) + V(T^n_{\omega}x_0)\psi(n) = E\psi(n) \quad n \in \mathbb{Z}^1$$

$$|\psi(n)| \leq |n|^2$$

with $\omega \notin \mathcal{P}(N)$ and $E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \mathcal{K}_{\omega}(N)$, then due to Corollary 5.6

$$\tag{5.13} |\psi(n)| \leq \min_{s = n, n + 1} N_s^2 \exp(-L_0 \min\left(\left(|n| - N_{s-1}^3, N_{s+1} - |n|\right)\right)) \leq \exp(-L_0|n|/4)$$

where $N_t \leq n < N_{t+1}$. Theorem 1.1 follows from (5.13) and (5.11). 

□
6. Skew shifted $C^1$–potentials at large disorder

Consider

$$H(x, y, \lambda, \omega)\psi(n) := -\psi(n - 1) - \psi(n + 1) + \lambda V(T^n(x, y))\psi(n), \quad n \in \mathbb{Z}^1,$$

where $V(x, y), (x, y) \in \mathbb{T}^2$ is a real valued $C1$–function, $T = T_\omega : \mathbb{T}^2 \to \mathbb{T}$ is the skew–shift $T_\omega(x, y) = (x + y, y + \omega)$, $\lambda$ is a parameter. Let $f_N(x, y, \lambda, \omega, E)$ be the characteristic determinant of the operator $H_N(x, y, \lambda, \omega)$ which is the restriction of $H(x, y, \lambda, \omega)$ on $[1, N]$ with Dirichlet boundary conditions, i.e.,

$$f_N(x, y, \lambda, \omega, E) = \det \begin{vmatrix} \lambda V_1 - E & -1 & 0 & 0 & \cdots & 0 \\ -1 & \lambda V_2 - E & -1 & 0 & \cdots & 0 \\ 0 & -1 & \lambda V_3 - E & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -1 & \lambda V_N - E \end{vmatrix}$$

where $V_j = V(T^j(x, y))$. Recall that the monodromy matrices are as follows

$$M_N(x, y, \lambda, \omega, E) = \begin{bmatrix} f_N(x, y, \lambda, \omega, E) & -f_{N-1}(T(x, y), \lambda, \omega, E) \\ f_{N-1}(x, y, \lambda, \omega, E) & -f_{N-2}(T(x, y), \lambda, \omega, E) \end{bmatrix}$$

Consider also the following diagonal matrix

$$D_N(x, y, \lambda, \omega, E) = \text{diag}(\lambda V_1 - E, \ldots, \lambda V_N - E)(x, y)$$

Lemma 6.1. There exists $\lambda_0 = \lambda_0(B_0(V))$ such that for $|\lambda| \geq \lambda_0$ and with $N \approx \lambda^{1/2}$ the following assertion holds: there exists $\mathcal{E}_{\omega, \lambda}(N) \subset [-\lambda B_0(V) - 2, \lambda B_0(V) + 2]$ with $\mathcal{E}_{\omega, \lambda}(N) < N^{-\kappa}$ such that for any $E \in [-B_0(V) - 2, B_0(V) + 2] \setminus \mathcal{E}_{\omega, \lambda}(N)$ one has

$$\frac{1}{N} \log |f_N(\cdot, \lambda, \omega, E)| > \frac{1}{2} \log |\lambda|$$

and

$$\text{mes} \left\{ (x, y) \in \mathbb{T}^2 : \left| \frac{1}{N} \log |f_N(x, y, \lambda, \omega, E)| - \frac{1}{N} \log |f_N(\cdot, \lambda, \omega, E)| \right| > N^{-\kappa} \right\} \leq N^{-\kappa}$$

Proof. The matrix in the right-hand side of (6.2) can be written in the form $D_N(x, y, \lambda, \omega, E) + B_N$, where $D_N$ is given by (6.4). Clearly $\|B_N\| = 2$ and

$$\frac{1}{N} \log |\det D_N(x, y, \lambda, \omega, E)| = \log |\lambda| + N^{-1} \sum_{m=1}^{N} \log |V(T^m(x, y))| - E/\lambda|.$$

By Lemma 2.2.2 and Theorem 2.1.7 there exists $L_{\lambda, \omega}(N) \subset [-B_0(V) - 2/\lambda, B_0(V) + 2/\lambda]$ with $\text{mes} L_{\omega, \lambda}(N) < N^{-\kappa}$ such that for any $E/\lambda \in [-B_0(V) - 2/\lambda, B_0(V) + 2/\lambda] \setminus L_{\omega, \lambda}(N)$ holds

$$\text{mes} \left\{ (x, y) \in \mathbb{T}^2 : \min_{1 \leq m \leq N} |V(T^m(x, y))| - E/\lambda| \leq |\lambda|^{-1/2} \right\} \leq |\lambda|^{-1/4}$$

$$\text{mes} \left\{ (x, y) \in \mathbb{T}^2 : \left| N^{-1} \sum_{m=1}^{N} \log |V(T^m(x, y))| - E/\lambda| - \langle \log |V(\cdot) - E/\lambda| \rangle \right| > N^{-\kappa} \right\} \leq \exp(-N^\kappa) = \exp(-\lambda^{-\kappa})$$

$$\left| \langle \log |V(\cdot) - E/\lambda| \rangle - \int_{\mathbb{T}^2 \setminus B_{\omega, \lambda}(N)} \log |V(x) - E/\lambda| \, dx \right| \leq N^{-\kappa}$$

Then

$$\|D_N(x, y, \lambda, \omega, E)^{-1}\| \leq \lambda^{-1/2} \text{ for any } (x, y) \in \mathbb{T}^2 \setminus B_{\omega, \lambda}(N)$$
where \( B_{\omega, \lambda}(N) \) is the set in (6.6), \( \text{mes} \ B_{\omega, \lambda}(N) < \lambda^{-1/4} \)

\[
\frac{1}{N} \log |\det D_N(\cdot, \lambda, \omega, E)| > \frac{1}{2} \log |\lambda| .
\]

Note that (6.10) implies

\[
\|D_N^{-1} B_N\| < 2|\lambda|^{-1/2}
\]

\[
(1 + 2|\lambda|^{-1/2})^N > |\det(1 + D_N^{-1} B)| > (1 - 2|\lambda|^{-1/2})^N > \exp(-4N|\lambda|^{-1/2}) \geq 1
\]

\[
\log |f_N| = \log |\det(D_N + B_N)| = \log |\det D_N| + O(|\lambda|^{-1/2})
\]

Let \( E_j^{(N)}(x, y) < \cdots < E_0^{(N)}(x, y, \omega) \) be the eigenvalues \( H_N(x, y, \lambda, \omega) \). Let \( S \) be the union of the sets in (6.3) and (6.7). Then due to Lemma 2.25 there exists \( \bar{\omega}, \lambda(N) \) with \( \bar{\omega}, \lambda \) such that for any \( E \) \( \bar{\omega}, \lambda(E) \) and (6.6) and (6.7). Then due to Lemma 2.25, there exists \( \bar{\omega}, \lambda(j, N) \) with \( \bar{\omega}, \lambda(j, N) \leq \exp(-\lambda^c) \) such that

\[
\left| \frac{1}{\mathcal{T}_2 \setminus S} \log |\lambda^{-1} E_j^{(N)}(\cdot, \omega) - \lambda^{-1} E| - \int_{\mathcal{T}_2 \setminus S} \log |\lambda^{-1} E_j^{(N)}(x, y, \omega) - \lambda^{-1} E| \, dx \, dy \right| \leq N^{-\kappa}
\]

provided \( \lambda^{-1} E \in [-B_0, B_0] \setminus \mathcal{E}_{\omega, \lambda}(N) \), where \( \mathcal{E}_{\omega, \lambda}(N) = L_{\omega, \lambda}(N) \cup \left( \bigcup_j [\bar{\omega}, \lambda(j, N) \right) \). The lemma now follows from (6.7), (6.8), and (6.11).

**Proof of Theorem 1.12.** Let \( \lambda_0 \) be as in Lemma 6.1 \( |\lambda| \geq \lambda_0, \ell = [\lambda^{1/2}] \) and let \( \mathcal{E}_{\omega, \lambda}^{(*)} \) be the set in the statement of Lemma 6.1 then

\[
\frac{1}{\ell} \left( \frac{1}{\ell} (\log |f_N(\cdot, \lambda, \omega, E)|) > \frac{1}{2} \log |\lambda|ight.
\]

for any \( E \in [-B_0(V - 2, B_0(V + 2] \setminus \mathcal{E}_{\omega, \lambda}(\ell) \). By Proposition 4.4 there exists \( \mathcal{E}_{\omega, \lambda}^{(*)} \) with \( \mathcal{E}_{\omega, \lambda}^{(*)} \leq \exp(-\ell^c) \) such that for any

\[
\left| \frac{1}{\ell} \left( \frac{1}{\ell} (\log |f_N(\cdot, \lambda, \omega, E)|) - L(\lambda, \omega, E) \right) \right| < \ell^{-1/2}
\]

Thus \( L(\lambda, \omega, E) > 1/3 \log |\lambda| \) for any \( E \in [-\lambda B_0(V - 2, B_0(V + 2] \setminus \mathcal{E}_{\omega, \lambda}^{(*)} \), where \( \mathcal{E}_{\omega, \lambda}^{(*)} = \mathcal{E}_{\omega, \lambda}^{(*)} \cup \mathcal{E}_{\omega, \lambda}(\ell) \). That proves the first part in Theorem 1.12. The second part now follows from Theorem 1.1.

**Appendix A: Quantitative Ergodic Theorem**

In this section, let \( \psi \in C^4(T^2) \). We first derive a quantitative ergodic theorem for the shift \( T_\omega : T^2 \to T^2 \) which is defined as \( T_\omega(x_1, x_2) = (x_1 + \omega_1, x_2 + \omega_2) \). We recall from Definition 2.4 that \( \omega \) is said to satisfy a Diophantine condition provided

\[
\| k_1 \omega_1 + k_2 \omega_2 \| > c ((|k_1| + |k_2|)^{-A} \text{ for all } (k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\})
\]

where \( c > 0, A > 2 \) are fixed.

**Proposition A.1.** Suppose \( \bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) \) satisfies the Diophantine assumption (A.1). Then for all \( \omega = (\omega_1, \omega_2) \) with \( |\omega - \bar{\omega}| < \frac{c}{2} N^{-1} \)

\[
\left| \frac{1}{N} \sum_{m=1}^N \psi(T_{\omega}^m x) - \left< \psi \right> \right| \leq B_4(\psi) N^{-\sigma}
\]

for all \( x \in T^2, N \geq N_0(c, A) \). Here, \( \sigma = \frac{1}{1 + A} \).
Proof. The Fourier series
\[
\psi(T^m \omega x) = \sum_{k_1, k_2} c(k_1, k_2) e(k_1(x_1 + m\omega_1) + k_2(x_2 + m\omega_2))
\]
where
\[
c(k_1, k_2) = \int_{T^2} \psi(x) e(-k_1x_1 - k_2x_2) \, dx
\]
converges absolutely. Indeed, integrating by parts yields
\[
|c(k_1, k_2)| \lesssim B_4(\psi) \left(1 + |k_1|^2\right)^{-1} \left(1 + |k_2|^2\right)^{-1}
\]
\[
c(0, 0) = \int_{T^2} \psi(x) \, dx = \langle \psi \rangle
\]
\[
\frac{1}{N} \sum_{m=1}^N \psi(T^m \omega 1 - (\psi) = \sum_{(k_1, k_2) \neq 0} c(k_1, k_2) e(k_1x_1 + k_2x_2) \frac{1}{N} \sum_{m=1}^N e(m(k_1\omega_1 + k_2\omega_2))
\]
Recall the simple bound
\[
\left| \frac{1}{N} \sum_{m=1}^N e(m\theta) \right| \leq \frac{2}{1 + N|\theta|}
\]
Moreover, for all \(|k| \leq N^\sigma\) with \(\sigma = \frac{1}{1+A}\),
\[
\|k_1\omega_1 + k_2\omega_2\| \geq \|k_1\omega_1\| + k_2\omega_2\| - (|k_1| + |k_2|)\|\omega - \tilde{\omega}\|
\]
\[
\geq c|k|^{-A} - |k| \frac{c}{2N} > \frac{c}{2}|k|^{-A}
\]
From this we conclude that
\[
\left| \frac{1}{N} \sum_{m=1}^N \psi(T^m \omega 1 - (\psi) \leq \sum_{(k_1, k_2) \neq 0} |c(k_1, k_2)| \left| \frac{1}{N} \sum_{m=1}^N e(m(k_1\omega_1 + k_2\omega_2)) \right|
\]
\[
\lesssim \sum_{(k_1, k_2) \neq 0} B_4(\psi) \left(1 + |k_1|^2\right)^{-1} \left(1 + |k_2|^2\right)^{-1} \left(1 + N\|k_1\omega_1 + k_2\omega_2\|\right)^{-1}
\]
\[
\leq \sum_{(k_1, k_2) \neq 0} B_4(\psi) \left(1 + |k_1|^2\right)^{-1} \left(1 + |k_2|^2\right)^{-1} \left(1 + Nc(|k_1| + |k_2|)^{-A}\right)^{-1}
\]
\[
\leq \sum_{|k_1| \leq N^\sigma} B_4(\psi) \left(1 + |k_1|^2\right)^{-1} \left(1 + |k_2|^2\right)^{-1} \left(1 + cN^{1+A}\right)^{-1}
\]
\[
+ \sum_{|k_1| > N^\sigma \ OR \ |k_2| > N^\sigma} B_4(\psi) \left(1 + |k_1|^2\right)^{-1} \left(1 + |k_2|^2\right)^{-1}
\]
\[
\lesssim B_4(\psi) N^{-\sigma}.
\]
as claimed. \(\Box\)

Remark A.2. Inspection of the previous proof reveals that Proposition A.1 only requires the weaker condition
\[
(A.3) \quad \|k \cdot \omega\| = \|k_1\omega_1 + k_2\omega_2\| \geq N^{-\gamma_1} \quad \forall \ 1 \leq |k| \leq N^{\gamma_2}
\]
were \(\gamma_1, \gamma_2 > 0\) are any small numbers (but fixed; one then also needs to change the power on the right-hand side of (A.2) according to the choice of these constants). Note that this is stable under perturbations \(|\omega - \tilde{\omega}| < N^{-1}\). We refer to \(\omega\) as in (A.3) as \((N, \gamma_1, \gamma_2)\)-Diophantine. Proposition A.1 holds for \(\omega\) being \((N, \gamma_1, \gamma_2)\)-Diophantine, with \(\sigma = \frac{1}{2} \min \gamma_1, \gamma_2\).
Lemma A.3. Let \(0 < \mu < \frac{1}{2}\). Then
- for any \(k \in \mathbb{Z}^2\) and \(\theta \in \mathbb{T}\), \(\operatorname{mes}\{\omega \in \mathbb{T}^2 : \|k \cdot \omega + \theta\| < \mu\} = 2\mu\)
- for any \(\theta \in \mathbb{T}\), \(\operatorname{mes}\{\omega \in \mathbb{T}^2 : \min_{1 \leq |k| \leq N_0} \|k \cdot \omega + \theta\| < \mu\} \lesssim N_0\mu\)

Lemma A.4. For any \(\omega, \omega_0 \in \mathbb{T}^2\), \(\min_{1 \leq |k| \leq N_0} \|k \cdot \omega\| - \min_{1 \leq |k| \leq N_0} \|k \cdot \omega_0\| \lesssim N_0|\omega - \omega_0|

Lemma A.5. Given \(\bar{N} \in \mathbb{N}\), \(j = (j_1, j_2) \in [1, \bar{N}]^2 \subset \mathbb{Z}^2\) set \(\omega_j^{(\bar{N})} = (\frac{j_1}{\bar{N}}, \frac{j_2}{\bar{N}})\). Then for any \(N_0 \in \mathbb{N}\), \(\mu > 0\) one has
\[
\min_{1 \leq |k| \leq N_0} \|k \cdot \omega\| > \mu/2
\]
for any \(|\omega - \omega_j^{(\bar{N})}| < \frac{\mu}{2N_0}\) with \(j \not\in J(\bar{N}, N_0, \mu)\).

Proof. If \(j = (j_1, j_2) \in J(\bar{N}, N_0, \mu)\), then due to Lemma A.4 one has
\[
\min_{1 \leq |k| \leq N_0} \|k \cdot \omega\| \leq \mu + \frac{N_0}{\bar{N}}
\]
for any
\[
\omega \in \mathcal{P}_j^{(\bar{N})} = (j_1/\bar{N}, (j_1 + 1)/\bar{N}) \times (j_2/\bar{N}, (j_2 + 1)/\bar{N}).
\]
Hence,
\[
\operatorname{mes}\left[\bigcup_{j \in J(\bar{N}, N_0, \mu)} \mathcal{P}_j^{(\bar{N})}\right] \leq \operatorname{mes}\left\{\omega \in \mathbb{T}^2 : \min_{1 \leq |k| \leq N_0} \|k \cdot \omega\| \leq \mu + \frac{N_0}{\bar{N}}\right\} \lesssim N_0^2(\mu + N_0/\bar{N})
\]
due to Lemma A.3. The bound A.4 now follows from A.6. The second assertion follows from Lemma A.3. \(\square\)

Corollary A.6. Given \(N > 0\) there exists \(J(N) \subset \{(j_1, j_2) : 1 \leq j_1, j_2 \leq N^2\}\) such that (with \(\mathcal{P}_j^{(N^2)}\) as in A.5)

1. for any \(\bar{N} \geq N^2\) and any \(\omega_j^{(\bar{N})} = (j_1/\bar{N}, j_2/\bar{N}) \in \bigcup_{k \in \mathcal{J}(N)} \mathcal{P}_k^{(N^2)}\) one has
\[
\min_{1 \leq |k| \leq N^{\kappa/4}} \|k\omega_j^{(\bar{N})}\| \geq N^{-\kappa}
\]
2. \(\operatorname{mes}\left[\bigcup_{k \in \mathcal{J}(N)} \mathcal{P}_k^{(N^2)}\right] \leq N^{-\kappa/2}\)

Here \(\kappa > 0\) is a small constant.

Proof. Using the notations of Lemma A.5, set \(\mathcal{J}(N) = J(N^2, N^2, N^{-\kappa})\). If \(\omega_j^{(\bar{N})} \in \bigcup_{k \in \mathcal{J}(N)} \mathcal{P}_k^{(N^2)}\), then
\[
|\omega_j^{(\bar{N})} - \omega_k^{(N^2)}| \lesssim N^{-2}\]
for some \(k \not\in \mathcal{J}(N)\). Therefore, (1), (2) follow from Lemma A.5. \(\square\)

Let \(\psi \in C^4(\mathbb{T}^2)\). Let \(\omega_1, \omega_2\) satisfy
\[
\|k\omega_i\| \geq c|k|^{-A}\]
for any \(k \in \mathbb{Z} \setminus \{0\}\), where \(c > 0\), \(A > 2\).

To deal with the skew-shift, we need the following well-known estimate.
Lemma A.7. Let $\alpha \in (0, 1)$. If
\[
\left| \frac{\alpha - p}{q} \right| \leq \frac{1}{q^2}
\]
where $p, q \in \mathbb{N}$, $(p, q) = 1$, then for any $N \in \mathbb{N}$, one has
\[
\sum_{k=1}^{N} \min (N, \|k\alpha\|^{-1}) \lesssim (q + N + N^2q^{-1}) \max\{1, \log q\}.
\]

Proof. This is Lemma 4.1 in [Nat].

Suppose $\alpha$ satisfies
\begin{equation}
(A.7) \quad \|k\alpha\| > c|k|^{-(1+\varepsilon)} \text{ for } k \in \mathbb{Z} \setminus \{0\}
\end{equation}
where $0 < c < 1$ and $\varepsilon > 0$. Let $\left\{ \frac{p_s}{q_s} \right\}_{s=1}^{\infty}$ be the convergents of $\alpha$. Recall that
\[
|\alpha - \frac{p_s}{q_s}| \leq \frac{1}{q_sq_{s+1}} \leq \frac{1}{q_s^2} \quad \forall s \geq 1.
\]
Note
\[
\alpha q_s^{-(1+\varepsilon)} < \|q_{s-1}\alpha\| = |q_{s-1}\alpha - p_{s-1}| \leq q_{s-1}^{-1}.
\]
Hence
\[
q_{s} < c^{-1}q_{s-1}^{1+\varepsilon}.
\]
If $q_{s-1} < N \leq q_{s}$, then $q_{s} < c^{-1}N^{1+\varepsilon}$. Combining this with Lemma A.7, one has

Lemma A.8. Suppose $\omega$ satisfies (A.7). Then for any $N \geq N_0(c, \varepsilon)$
\[
\sum_{k=1}^{N} \min (N, \|k\alpha\|^{-1}) \lesssim c^{-1}N^{1+\varepsilon} \log N.
\]

We also need the following version of Weyl’s inequality (a more general version can be found in [Nat].)

Lemma A.9. Suppose $\alpha$ satisfies (A.7). Let $S = \sum_{k=1}^{N} e\left((k^2\alpha + k\beta)\right)$ where $N \in \mathbb{N}$, $\beta \in \mathbb{R}$. Then $|S| \lesssim N^{1+\varepsilon}$ for $N \geq N_0(c, \varepsilon)$.

Proof.
\[
|S|^2 = \sum_{k=1}^{N} e\left(k^2\alpha + k\beta\right) \sum_{\ell=1}^{N} e\left(-\ell^2\alpha - \ell\beta\right)
\]
\[
= \sum_{k=1}^{N} \sum_{\ell=1}^{N} e\left((k^2 - \ell^2)\alpha + (k - \ell)\beta\right)
\]
\[
= \sum_{k=1}^{N} \sum_{m=k-N}^{-1} e\left(m(2k-m)\alpha + m\beta\right)
\]
\[
= \sum_{m=1-N}^{N} \sum_{k=1}^{N} e\left(m(2k-m)\alpha + m\beta\right)
\]
\[
\leq \sum_{m=1-N}^{N} \sum_{k=1}^{N} e\left(k(-2m\alpha)\right) + N + \sum_{m=1}^{N-1} \sum_{k=m+1}^{N} e\left(k(2m\alpha)\right)
\]
\[
\lesssim N + \sum_{m=1}^{N} \min (N, \|m(2\alpha)\|^{-1})
\]
\[ \|m(2\alpha)\| > c|2m|^{-(1+\varepsilon)} = \left(c2^{-(1+\varepsilon)}\right)|m|^{-(1+\varepsilon)} \]

So \(2\alpha\) satisfies (A.7) if \(c\) is replaced by \(c2^{-(1+\varepsilon)}\). By Lemma A.8

\[ |S|^2 \lesssim N + c^{-1}2^{1+\varepsilon}N^{1+\varepsilon} \log N \lesssim N^{1+2\varepsilon}. \]

\[ \square \]

Remark A.10. We require \(N \geq N_0(c, \varepsilon)\) only to make sure that

- \(\log c^{-1}N^{1+\varepsilon} \lesssim \log N\)
- \(c^{-1}N^{1+\varepsilon} \log N \lesssim N^{1+2\varepsilon}\).

Hence we can choose \(N_0 \asymp c^{-1}\). Moreover, for the purpose of Lemma A.6, there is no need in condition (A.7) for all \(k \in \mathbb{Z} \setminus \{0\}\). Indeed, assume that

(A.8) \[ \|k\omega\| \geq cN^{-(1+\varepsilon)} \text{ for all } 0 < |k| \leq N. \]

If \(q_{s-1} \leq N < q_s\), then \(\|q_s-1\omega\| \geq cN^{-(1+\varepsilon)}\). Since \(\|q_s-1\omega\| \leq q_s^{-1}\), that implies \(q_s \leq c^{-1}N(1+\varepsilon)\) thus, \(\omega - \frac{q_s^{-1}}{q_s} \leq q_s^{-2}\), and \(N \leq q_s \leq c^{-1}N(1+\varepsilon)\). Therefore, Lemma A.8 holds and Lemma A.9 follows. We denote by \(T_{c,\varepsilon,N}\) the set of all \(\omega\) for which (A.8) is valid (see Appendix B).

Now suppose \(T : \mathbb{T}^2 \to \mathbb{T}^2\) is a skew-shift \(T(x_1, x_2) = (x_1 + x_2, x_2 + \omega)\) with \(\alpha = \frac{\omega}{2}\) satisfying (A.7).

Proposition A.11.

\[ \left| \frac{1}{N} \sum_{m=1}^{N} \psi(T^m x) - \langle \psi \rangle \right| \lesssim B_4(\psi) N^{-\frac{\varepsilon}{2}} \]

for all \(x \in \mathbb{T}^2, N \geq N_1(c, \varepsilon)\).

Proof.

\[ T^m x = \left( x_1 + m x_2 + \frac{m(m-1)}{2} \omega, x_2 + m \omega \right) \]

\[ \frac{1}{N} \sum_{m=1}^{N} \psi(T^m x) - \langle \psi \rangle = \sum_{(k_1, k_2) \neq 0} c(k_1, k_2) e(k_1 x_1 + k_2 x_2) \frac{1}{N} \sum_{m=1}^{N} e \left( m^2 k_1 \omega + m \left( k_1 x_2 - k_1 \frac{\omega}{2} + k_2 \omega \right) \right) \]

\[ \left| \frac{1}{N} \sum_{m=1}^{N} \psi(T^m x) - \langle \psi \rangle \right| \leq \sum_{k_2 \neq 0} \left| c(0, k_2) \right| \frac{1}{N} \sum_{m=1}^{N} e(mk_2 \omega) + \sum_{k_1 \neq 0, k_2 \in \mathbb{Z}} \left| c(k_1, k_2) \right| \frac{1}{N} \sum_{m=1}^{N} e \left( m^2 k_1 \omega + m \left( k_1 x_2 - k_1 \frac{\omega}{2} + k_2 \omega \right) \right) \]

\[ \lesssim \sum_{k_2 \neq 0} B_4(\psi) \left( 1 + |k_2|^2 \right)^{-1} (1 + N\|k_2 \omega\|)^{-1} + \sum_{k_2 \in \mathbb{Z}} B_4(\psi) \left( 1 + |k_2|^2 \right)^{-1} \]

\[ \cdot \sum_{k_1 \neq 0} (1 + |k_1|^2)^{-1} \frac{1}{N} \sum_{m=1}^{N} e \left( m^2 k_1 \omega + m \left( k_1 x_2 - k_1 \frac{\omega}{2} + k_2 \omega \right) \right) \]

As in the proof of Proposition A.11,

\[ \sum_{k_2 \neq 0} B_4(\psi) \left( 1 + |k_2|^2 \right)^{-1} (1 + N\|k_2 \omega\|)^{-1} \lesssim B_4(\psi) N^{-\frac{\varepsilon}{2}} \]

provided \(N \geq N_1(c, \varepsilon)\).
We want to apply Lemma A.9 to estimate the second sum. Note that \( \alpha = k_1 \frac{\omega}{2} \) satisfies A.7 if \( c \) is replaced by \( c|k_1|^{-1+\epsilon} \). For \( N \geq \left\lfloor c|k_1|^{-1+\epsilon} \right\rfloor^{-\frac{2}{3}} \),

\[
\sum_{k_1 \neq 0} (1 + |k_1|^2)^{-1} N^{-\frac{1}{2} + \epsilon} \left| \sum_{m=1}^{N} e \left( m^2 k_1 \frac{\omega}{2} + m \beta \right) \right| \lesssim N^{\frac{1}{2} + \epsilon}
\]

provided \( N \geq N(c, \epsilon) \).

**Appendix B: Metric estimates for the typical rational approximation rate**

Given \( \omega \in (0, 1) \) denote by \([a_1, a_2, \ldots] = [a_1(\omega), a_2(\omega), \ldots] \) its continued fraction, \( a_j \in \mathbb{Z}, a_j \geq 1 \). Take arbitrary integers \( k_j \geq 1, j = 1, 2, \ldots, n \) and put

\[
\mathcal{E} \left( \frac{1}{k_1}, \frac{2}{k_2}, \ldots, \frac{n}{k_n} \right) = \{ \omega \in (0, 1) : a_j(\omega) = k_j, 1 \leq j \leq n \}
\]

Denote by \( \omega_s = p_s/q_s \) the convergents for \( \omega = [a_1, a_2, \ldots] \). Recall that

\[
\omega_s = \frac{p_s}{q_s} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_s}}}}
\]

\[
q_s = a_s q_{s-1} + q_{s+2}, \quad q_0 = 1, q_{-1} = 0
\]

\[
p_s = a_s p_{s-1} + p_{s-2}, \quad p_0 = 0, p_{-1} = 1
\]

The set \( \mathcal{E} \left( \frac{1}{k_1}, \frac{2}{k_2}, \ldots, \frac{n}{k_n} \right) \) consists of an interval,

\[
\mathcal{E} \left( \frac{1}{k_1}, \frac{2}{k_2}, \ldots, \frac{n}{k_n} \right) = \left[ \frac{p_n}{q_n}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right], \quad \frac{p_s}{q_s} = [k_1, \ldots, k_s]
\]

In particular,

\[
\left| \mathcal{E} \left( \frac{1}{k_1}, \ldots, \frac{n}{k_n}, \frac{n+1}{k} \right) \right| = k^{-2}(1 + (kq_n)^{-2}q_{n-1})^{-1}(1 + k^{-1} + (kq_n)^{-1}q_{n-1})(1 + q_{n-1}^{-1}q_{n-1}) \left| \mathcal{E} \left( \frac{1}{k_1}, \frac{2}{k_2}, \ldots, \frac{n}{k_n} \right) \right|
\]

\[
\frac{1}{3k^2} \left| \mathcal{E} \left( \frac{1}{k_1}, \frac{2}{k_2}, \ldots, \frac{n}{k_n} \right) \right| \leq \left| \mathcal{E} \left( \frac{1}{k_1}, \ldots, \frac{n}{k_n}, \frac{n+1}{k} \right) \right| \leq \frac{2}{k^2} \left| \mathcal{E} \left( \frac{1}{k_1}, \frac{2}{k_2}, \ldots, \frac{n}{k_n} \right) \right|
\]
Hence \( \frac{1}{32e} \leq \text{mes}\{\omega : a_{n+1}(\omega) = k\} \leq \frac{2}{e} \) and

\[
\prod_{j=1}^r \frac{c}{k_j} \leq \text{mes}\{\omega : a_{n_j}(\omega) \geq k_j, j = 1, 2, \ldots, r\} \leq \prod_{j=1}^r \frac{C}{k_j}
\]  

(B.1)

**Proposition B.1.**

\[
\text{mes}\{\omega \in (0, 1) : \max_{1 \leq t \leq s} a_t \geq K\} \leq CsK^{-1}
\]

\[
\text{mes}\{\omega \in (0, 1) : \omega_s = p_s q_s^{-1}, \log q_s > s_0 K \text{ for some } s \leq s_0\} \leq Cs_0 e^{-K}
\]

**Proof.** Due to (B.1)

\[
\text{mes}\{\omega \in (0, 1) : \max_{1 \leq t \leq s} a_t \geq K\} \leq \sum_{1 \leq t \leq s} \text{mes}\{\omega : a_t(\omega) \geq K\} \leq CsK^{-1}
\]

Note that \( q_t = (a_t + 1)q_{t-1} \). If \( \log q_s > K \) for some \( s \leq s_0 \), then \( \max_{1 \leq t \leq s_0} \log(a_t(\omega) + 1) \geq K \). Therefore

\[
\text{mes}\{\omega \in (0, 1) : \omega_s = p_s q_s^{-1}, \log q_s \geq K \text{ for some } s \leq s_0\} \leq Cs_0 e^{-K}
\]

\[
\square
\]

Note also that (B.1) implies for arbitrary \( k(n) \)

\[
\text{mes}\{\omega : a_n(\omega) \leq k(n), n = n_0 + 1, \ldots, n_0 + m\} \leq \prod_{n=n_0+1}^{n+m} \left(1 - \frac{c}{k(n)}\right)
\]

In particular

\[
\text{mes}\{\omega : a_n(\omega) \leq k(n), n = n_0 + 1, n_0 + m\} \leq \exp\left(-\frac{cm}{k(n_0)}\right)
\]

\[
\text{mes}\{\omega : a_n(\omega) \leq n_0 \log n_0(\log \log n_0)(\log \log \log n_0), n = n_0 + 1, \ldots, n_0 + m\}
\]

\[
\leq \exp(-C(\log \log n_0))
\]

provided \( m > n_0(\log n_0)(\log \log n_0)^2 \).

**Lemma B.2.** Let \( \omega_s = \frac{a_s}{q_s} \) be the convergents for \( \omega \). Take \( 1 \leq m < q_s, s \geq 1 \). Then \( \|m\omega\| \geq \frac{a_{s+1}}{q_{s+1}} \).

**Proof.** By induction in \( s \). If \( s = 1 \), then can assume that \( a_1 = q_s \geq 2 \). Since \( \frac{a_2}{q_2} = \omega_2 < \omega < \omega_1 = \frac{1}{a_1} \), and 1 \( \leq a_1 - 1 \) one has \( \frac{a_2}{q_2} \leq \omega_1 < 1 - \frac{1}{a_1} \). Thus \( \|m\omega\| \geq \min\left(\frac{a_2}{q_2}, \frac{1}{a_1}\right) \geq \frac{a_2}{q_2} \), which is (1) for \( s = 1 \). Let \( s \geq 2 \) and set \( m = aq_s + b, 0 \leq b < q_{s-1}, a < \frac{a_{s+1}}{q_{s+1}} \leq a_s + 1 \). Thus \( a \leq a_s \).

**Case 1** \( a = 0 \). Then \( m < q_{s-1} \) and by the reductive hypothesis,

\[
\|m\omega\| \geq \frac{a_s}{q_s} \geq \frac{a_{s+1}}{q_{s+1}}
\]

The final inequality holds since \( q_{s+1} \geq a_{s+1}q_s \) and \( a_s \geq 1 \).

**Case 2** \( 1 \leq a < a_s, b \neq 0 \). Then
\[
\| m \omega \| \geq \| b \omega \| - \| a q_{s-1} \omega \| \geq \frac{a_s}{q_s} - a \| q_{s-1} \omega \| \\
\geq \frac{a_s}{q_s} - a \geq \frac{1}{q_s} \geq \frac{a_{s+1}}{q_{s+1}}
\]

**Case 3** \( m = a q_{s-1} + 1 \leq a \leq a_s \). (i.e. \( b = 0 \)) One has

\[
(-1)^{s-1} \omega_{s+1} < (-1)^{s-1} \omega < (-1)^{s-1} \omega_s \quad \text{for } s = 1, 2, \ldots
\]

Hence, for \( s \geq 2 \), and \( a \) as above,

\[
(-1)^{s-1} a q_{s-1} \omega_{s-1} < (-1)^{s-1} a q_{s-1} \omega_{s+1} < (-1)^{s-1} a q_{s-1} \omega < (-1)^{s-1} a q_{s-1} \omega_s.
\]

where

\[
| a q_{s-1} \omega_{s-1} - a q_{s-1} \omega_s | = \frac{a}{q_s} \leq \frac{a_s}{q_s} < 1.
\]

Therefore,

\[
\| a q_{s-1} \omega \| \geq \min(\| a q_{s-1} \omega_{s+1} - a q_{s-1} \omega_{s-1} \|, 1 - \| a q_{s-1} \omega_s - a q_{s-1} \omega_{s-1} \|)
\]

On one hand

\[
| a q_{s-1} \omega_{s+1} - a q_{s-1} \omega_{s-1} | = a q_{s-1} \left| \frac{p_{s+1}}{q_{s+1}} - \frac{p_{s-1}}{q_{s-1}} \right| = a \frac{q_{s+1}}{q_{s+1}} \geq \frac{a_{s+1}}{q_{s+1}},
\]

and on the other hand,

\[
1 - \| a q_{s-1} \omega_s - a q_{s-1} \omega_{s-1} \| = 1 - a q_{s-1} \frac{1}{q_s} q_{s-1} = 1 - \frac{a}{q_s}
\]

\[
\geq \frac{q_s - a_s}{q_s} = \frac{a_s (q_s - 1) + q_{s-2}}{q_s} \geq \frac{1}{q_s} \geq \frac{a_{s+1}}{q_{s+1}}
\]

**Case 4** \( m = a s q_{s-1} + b < q_s, 1 \leq b < a_{s-2} \) Then

\[
\| m \omega \| \geq \| (q_s - m) \omega \| - \| q_s \omega \| = \| (q_{s-2} - b) \omega \| - \| q_s \omega \|
\]

\[
\geq \frac{a_{s-1}}{q_{s-1}} - \frac{1}{q_{s-1}} \geq \frac{1}{q_{s-1}} - \frac{1}{q_{s+1}} = \frac{a_{s+1} q_s}{q_{s-1} q_{s+1}} \geq \frac{a_{s+1}}{q_{s+1}}.
\]

\[\square\]

Given \( N \in \mathbb{N}, \varepsilon > 0 \) let

\[
\mathcal{J}(N, \varepsilon) = \{ \omega \in [0, 1] : q_{s-1}(\omega) \leq N < q_s(\omega) \quad \text{for some } s \text{ and } a_s(\omega) > N^{\varepsilon} \}
\]

**Lemma B.3.** \( \text{mes } \mathcal{J}(N, \varepsilon) \leq N^{-\varepsilon/2}, \text{ provided } N > N_0(\varepsilon) \).
Proof. Recall that \( q_s(\omega) \geq 2^{\delta s} \). Therefore \( q_{s-1}(\omega) \leq N \) implies \( s \lesssim \log N \). Hence,
\[
\text{mes } J(N, \varepsilon) \leq \sum_{s \leq \log N} \text{mes } \{ \omega \in [0, 1] : a_s(\omega) > N^{-\varepsilon} \} \lesssim N^{-\varepsilon} \log N
\]
due to Proposition B.1.

**Definition B.4.** Given \( N \in \mathbb{N} \), and \( 0 < c, \varepsilon \ll 1 \). Let
\[
T_{c,\varepsilon,N} = \left\{ \omega \in \mathbb{T} : \| k\omega \| \geq cN^{-(1+\varepsilon)} \text{ for any } 0 < |k| \leq N \right\}.
\]

**Lemma B.5.** \( T \setminus T_{c,\varepsilon,N} \subset J(N, \varepsilon) \), provided \( N > N_0(\varepsilon) \). In particular, \( \text{mes } (T \setminus T_{c,\varepsilon,N}) \leq N^{-\varepsilon/2} \).

**Proof.** Assume that \( \| k\omega \| < cN^{-(1+\varepsilon)} \) for some \( \omega \in \mathbb{T} \), \( 0 < k \leq N \). Find \( s \) so that \( q_{s-1}(\omega) \leq N < q_s(\omega) \). Then \( k < q_s(\omega) \) and by Lemma B.2
\[
\| k\omega \| \geq \frac{c}{q_s(\omega)} N \varepsilon. \quad \text{Hence, } q_s(\omega) \geq (2c)^{-1} N^{1+\varepsilon} \text{. Therefore } a_s(\omega) + 1 \geq q_s(\omega)/q_{s-1}(\omega) \geq (2c)^{-1} N^{1+\varepsilon}/N \geq c^{-1} N^{\varepsilon} \geq N^{\varepsilon} + 1.
\]

Given \( \bar{N} \) set \( \omega_j^{(\bar{N})} = j/\bar{N} \)
\[
\mathcal{P}_j^{(\bar{N})} = \left( \omega_j^{(\bar{N})}, \omega_{j+1}^{(\bar{N})} \right), \quad j = 1, 2, \ldots
\]

**Corollary B.6.** Let \( N, \bar{N} \in \mathbb{N} \), \( \bar{N} > N^3 \), \( 0 < \varepsilon, c \ll 1 \). Then
\[
\# \left\{ 1 \leq j \leq \bar{N} : \min_{1 \leq |k| \leq N} \| k\omega_j^{(\bar{N})} \| < cN^{-(1+\varepsilon)} \right\} \lesssim \bar{N} N^{-\varepsilon/2}.
\]

Denote the set on the left-hand side of \( \text{(B.3)} \) by \( J(\bar{N}, N, c, \varepsilon) \), then
\[
\min_{1 \leq |k| \leq N} \| k\omega \| \geq \frac{c}{2} N^{-(1+\varepsilon)}
\]
for any \( |\omega - \omega_j^{(\bar{N})}| < 1/\bar{N} \), with \( j \notin J(\bar{N}, N, c, \varepsilon) \).

**Proof.** If \( j \in J(\bar{N}, N, c, \varepsilon) \) then by Lemma A.4
\[
\min_{1 \leq |k| \leq N} \| k\omega \| \leq N/\bar{N} + cN^{-(1+\varepsilon)} \leq 2cN^{-(1+\varepsilon)}
\]
for any \( \omega \in \mathcal{P}_j^{(\bar{N})} \). Hence,
\[
\text{mes } \left( \bigcup_{j \notin J(\bar{N}, N, c, \varepsilon)} \mathcal{P}_j^{(\bar{N})} \right) \leq \text{mes } (T \setminus T_{2c,\varepsilon,N}) \leq N^{-\varepsilon/2}.
\]

Estimate \( \text{(B.3)} \) follows from \( \text{(B.4)} \). The second assertion follows from Lemma A.3.

### Appendix C: \( C^{\alpha} \)-smooth potentials

Here we discuss the modifications needed for the case of \( C^{\alpha} \)-smooth potential, \( 0 \leq \alpha < 1 \), in Theorems 1.1 and 1.2. Let \( \varphi(x), x = (x_1, x_2) \in \mathbb{T}^2 \) be a \( C^{\alpha} \)-smooth function, i.e.,
\[
B_\alpha(\varphi) := \sup_{x \neq y} |x - y|^{-\alpha} |\varphi(x) - \varphi(y)| < +\infty
\]
where \( 0 < \alpha \leq 1 \). Given \( \tau > 0 \), let \( h_\tau(x) \in C^1(\mathbb{R}) \) be as in Definition 2.2 i.e., \( h_\tau(x) \) is 1-periodic,
- \( h_\tau \geq 0 \)
- \( \text{supp } h_\tau \subset \bigcup_{k \in \mathbb{Z}} [k - \tau, k + \tau] \)
- \( \int_0^1 h_\tau(y) \, dy = 1 \)
Then for any $\xi$ due to Proposition A.1 and Remark A.2. The assertion follows if we take here $|\langle \psi \rangle| \lesssim \tau^{-1}$ for $m \leq 4$

Set $\tilde{h}_r(x_1, x_1) = h_r(x_1)h_r(x_2)$.

Lemma C.1. Define

(C.2) \[
\psi(x_1, x_2) := \int_{T^2} \varphi(y_1, y_2)\tilde{h}_r(x_1 - y_1, x_2 - y_2) \, dy
\]

Then $\psi \in C^4(T^2)$ satisfies

1. $\max_{x \in T^2} |\varphi(x) - \psi(x)| \leq B_\alpha(\varphi)\tau^\alpha$
2. $B_4(\psi) \lesssim B_0(\varphi)\tau^{-4}$

Proof. Note that

\[
\psi(x_1, x_2) = \int_{\|y_1 - x_1\| \leq \tau, \|y_2 - x_2\| \leq \tau} \varphi(x_1 - y_1, x_2 - y_2)\tilde{h}_r(y_1, y_2) \, dy_1 dy_2
\]

\[
\varphi(x_1, x_2) = \int_{\|y_1 - x_1\| \leq \tau, \|y_2 - x_2\| \leq \tau} \varphi(x_1, x_2)\tilde{h}_r(y_1, y_2) \, dy_1 dy_2
\]

Therefore, (1) follows from (C.1). Part (2) follows just from the definition (C.1). \qed

Similarly to the $C^1$ case (see Corollary 2.1) we proceed with the following.

Corollary C.2. Let $f \in C^\alpha(T^2)$, $T_\omega(x) = x + \omega$. Let $N \in \mathbb{N}$ be large and let $\omega$ be $(N, \gamma_1, \gamma_2)$-Diophantine. Then for any $\xi \in \mathbb{R}$, $0 < \delta < 1$ one has

\[
\frac{1}{N} \# \{1 \leq k \leq N : T^k_\omega x \in S_f(\xi, \delta) \} \lesssim \text{mes } S_f(\xi, 2\delta) + (1 + B_\alpha(f))\delta^{\frac{1}{2}}
\]

provided $N > \delta^{-\min(\gamma_1, \gamma_2)}$. Here $\sigma = \frac{1}{2} \min(\gamma_1, \gamma_2) > 0$ is a constant, $S_f(\xi, \delta') = \{x \in T^2 : |f(x) - \xi| < \delta'\}$.

Proof. Using the notations of Lemma 2.10 we have to estimate $\frac{1}{N} \sum_{k=1}^n \chi_\delta(f(T^k_\omega \xi) - \xi)$. Note that $\varphi(x) = \chi_\delta(f(x) - \xi)$ is $C^\alpha$-smooth with $B_\alpha(\varphi) \lesssim \delta^{-1}B_\alpha(f)$. Define $\psi(x)$ as in Lemma C.1 then, due to C.1, $|\langle \psi \rangle - \langle \varphi \rangle| \leq B_\alpha(\varphi)\tau^\alpha \leq \delta^{-1}B_\alpha(f)\tau^\alpha$ and

\[
\max_x \left| \frac{1}{N} \sum_{k=1}^n \varphi(T^m x) - \frac{1}{N} \psi(T^m x) \right| \leq \delta^{-1}B_\alpha(f)\tau^\alpha.
\]

As in Corollary 2.17 one obtains

\[
\frac{1}{N} \# \{1 \leq k \leq N : T^k_\omega x \in S_f(\xi, \delta) \} \leq \frac{1}{N} \sum_{k=1}^n \varphi(T^k_\omega x)
\]

\[
\leq \frac{1}{N} \sum_{k=1}^n \psi(T^k_\omega x) + \delta^{-1}B_\alpha(f)\tau^\alpha
\]

\[
\leq \langle \psi \rangle + B_4(\psi)N^{-\sigma} + \delta^{-1}B_\alpha(f)\tau^\alpha
\]

\[
\leq \langle \varphi \rangle + B_0(\varphi)\tau^{-5}N^{-\sigma} + 2\delta^{-1}B_\alpha(f)\tau^\alpha
\]

due to Proposition A.1 and Remark A.2. The assertion follows if we take here $\tau = \delta^{\frac{1}{2\sigma}}$. \qed

One can see that the rest of the auxiliary assertions needed for the proof of Theorem 2.17 do not rely on the smoothness of the function $f(x)$, and therefore does not require any modifications. Thus, Theorem 2.17 as well as the remarks after it hold for $f \in C^\alpha(T^2)$.

The modifications needed for the rest of the Section 2 and whole Section 3 consist only in stronger restrictions on the interval in which $\omega$ runs. For instance, the assertions in Theorem 3.5 are valid, provided $|\omega_1 - \omega_0| < (1 + B_\alpha(f))^{-1}N^{-\frac{1}{2}}$. 


Section 4, 5, 6 rely only on the application of Theorem 2.17 and Theorem 3.8 to \( \log |E_j^{(N)}(x, w) - E| \), where \( E_1^{(N)}(x, \omega) < \cdots < E_N^{(N)}(x, \omega) \) are the eigenvalues of \( H_N(x, \omega) \). The only fact needed for the validity of these applications is as follows:

**Lemma C.3.** Suppose \( V(x) \in C^\alpha(T^2) \), let \( T = T_\omega \) be the shift (or the skew-shift). Let \( E_1^{(N)}(x, \omega) < \cdots < E_N^{(N)}(x, \omega) \) be the eigenvalues of \( H_N(x, \omega) \). Then the functions \( E_j^{(N)} \) are \( C^\alpha \)-smooth and \( B_\alpha(E_j^{(N)}) \lesssim N^2 B_\alpha(V) \), \( j = 1, 2, \cdots, N \).

**Proof.** Note that

\[
\|H_N(x, \omega) - H_N(\tilde{x}, \tilde{\omega})\| \lesssim B_\alpha(V) N^2 (|x - \tilde{x}|^\alpha + |\omega - \tilde{\omega}|^\alpha)
\]

Recall that due to the minimax principle, if \( A_i, i = 1, 2 \), are Hermitian operators in \( \mathbb{C}^N \), and \( E_1^{(i)} < E_2^{(i)}, \cdots \) are the eigenvalues of \( A_i, i = 1, 2 \), then

\[
|E_j^{(1)}, -E_j^{(2)}| \leq \|A_1 - A_2\|
\]

for \( j = 1, 2, \ldots, N \), and the lemma follows. \( \square \)

**References**

[Bha] Bhatia, R. Perturbation bounds for matrix eigenvalues. Pitman research notes in mathematics series 162, Longman, 1987.

[Bje] Bjerklof, K. Positive Lyapunov exponents for continuous quasiperiodic Schrödinger equations. J. Math. Phys. 47, no. 2, (2006)

[Bou] Bourgain, J. Green’s function estimates for lattice Schrödinger operators and applications. Annals of Mathematics Studies, 158. Princeton University Press, Princeton, NJ, 2005.

[BouGol] Bourgain, J., Goldstein, M. On nonperturbative localization with quasi-periodic potential. Ann. of Math. (2) 152 (2000), no. 3, 835–879.

[BouGolSch] Bourgain, J., Goldstein, M., Schlag, W. Anderson localization for Schrödinger operators on \( \mathbb{Z} \) with potentials given by the skew-shift. Comm. Math. Phys. 230 (2001), no. 3, 583–621.

[Cha] Chan, J. Method of variations of potential of quasi-periodic Schrödinger equation, preprint 2005, to appear in GAFA.

[GolKle] Goldstein, M., Klein, S. Anderson localization for random potentials with fast decaying correlations. In preparation.

[GolSch1] Goldstein, M., Schlag, W. Hölder continuity of the integrated density of states for quasiperiodic Schrödinger equations and averages of shifts of subharmonic functions. Ann. of Math. (2) 154 (2001), no. 1, 155–203.

[GolSch2] Goldstein, M., Schlag, W. Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues, Preprint 2005.

[Jit] Jitomirskaya, S. Metal–insulator transition for the almost Mathieu operator. Annals of Math. 150, no. 3 (1999).

[Khin] Khinchin, A. Ya. Continued Fractions, Dover, 1992.

[Kle] Klein, S. Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function. J. Funct. Anal. 218 (2005), no. 2, 255–292.

[Nat] Nathanson, M. Additive Number Theory I, Springer.

**First and second authors:** Department of Mathematics, University of Toronto, Toronto, Ontario, Canada

E-mail address: jchan@math.toronto.edu, gold@math.toronto.edu

**Third author:** Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

E-mail address: schlag@math.uchicago.edu