Methods for detecting order-by-disorder transitions: the example of the Domino model

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Abstract. Detecting the zero-temperature thermal order-by-disorder (ObD) transition in classical magnetic systems is notably difficult. We propose a method to probe this transition in an indirect way. The idea is to apply adequate and suitably engineered magnetic fields to transform the zero-temperature transition into a finite-temperature sharp crossover, which should be much easier to observe and characterise with usual laboratory methods. Such a crossover should constitute proof of existence of the ObD transition in the underlying unperturbed system.

Keywords: coarsening processes, phase diagrams, kinetic Ising models, correlation functions

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1. Introduction

In condensed matter physics, fluctuations, whether thermal or quantum, usually suppress order. However, this is not a rigorous rule. Some systems undergo an ‘order-by-disorder’ (ObD) transition in which the fluctuations restore order in an otherwise disordered ground state [1, 2]. This ObD transition is, more precisely, the mechanism whereby a system with a non-trivially degenerate ground state develops long-range order as a result of the effect of classical or quantum fluctuations. Therefore, a classical system exhibiting this transition has no long-range order when the temperature is strictly zero and develops some at non-vanishing (and in the so far known cases infinitesimal) temperature.

This phenomenon was first exhibited in the classical 2D Domino model [3]. An experimental 3D realisation, in the form of Ising pyrochlores with staggered antiferromagnetic order frustrated by an applied magnetic field was recently proposed [4, 5] (concrete examples could be Nd$_2$Hf$_2$O$_7$ or Nd$_2$Zr$_2$O$_7$). Indeed, a zero-temperature ObD transition...
is relatively common in highly frustrated magnetic models [6, 7]. In this context, the
gometry of the lattice and/or the nature of the interactions make the simultaneous
minimisation of each term contributing to the energy impossible [8] (see also [9]). Two
consequences of frustration in these materials and models are the increase in the ground
state energy compared to that of the unfrustrated model and the scaling of the number
of degenerate ground states (sub-extensively) with the size of the system. A very
peculiar entropic effect is at the origin of the ObD transition: among all the degenerate
ground states, the two magnetically ordered ones dominate the Gibbs–Boltzmann mea-
sure at infinitesimal temperature, since they have an extensive number of lowest lying
excitations that allow the system to minimise the free-energy density.

Although the classical ObD transition has been intensively studied analytically, it
has been very difficult to exhibit sharp experimental evidence for it. One of the reasons
is that the transition occurs at zero temperature, and it is therefore difficult to establish
whether order is selected genuinely through the ObD mechanism or due to energetic
contributions not taken into account that actually lift the ground state degeneracy.
Detecting the ObD transition is also a hard task numerically, due to the fact that at
low temperatures the equilibration time of frustrated spin systems becomes, in general,
very long. The aim of this paper is to propose a way to probe the ObD transition in an
indirect way which might be implemented in the laboratory, for instance, in artificial
spin-ice settings [10–12]. The idea, as we explain in the main part of the article, is to
use external magnetic fields to transform the zero-temperature transition into a finite-
temperature sharp crossover, or maybe even a veritable phase transition, and then detect
the latter with the usual methods. For concreteness, we explain how this is achieved in
the context of the 2D Domino model. The procedure proposed here can be efficiently
used in numerical simulations as an efficient probe of the ObD transition, by providing
a framework in which the signatures of the transition should be much easier to detect,
as one does not need to go to very low temperatures, where the dynamics of frustrated
spin systems are generally sluggish [4, 5].

The paper is organised as follows. In section 2 we recall the definition and main
properties of the Domino model. In particular, we establish the effective 1D model that
describes its low energy properties [1], which we will use in the rest of our study. In
section 3 we add quenched disorder in the form of columnar random magnetic fields
as a first attempt to displace the ObD transition to a finite temperature. We start by
showing, with an Imry–Ma argument [13], that such a 2D disordered model cannot
have a finite-temperature phase transition but just a crossover. Still, we characterise the
pseudo-ferromagnetic order thus achieved by studying a random-field 1D effective model
using the renormalisation group approach. The next strategy, described in section 4, is
to use alternate columnar magnetic fields. With them we achieve the goal of finding a
finite critical temperature but we lose a bit of the phenomenology of the ObD transition,
as we explain in the body of the paper. As far as we know, the stochastic evolution of the
kinetic 2D Domino model has not been studied in detail yet. In each section we analyse
the quench dynamics of the pure and disordered Domino models using Monte Carlo
simulations, and we describe how the temporal evolution confirms the static behaviour
expected asymptotically. A section with our conclusions closes the article.
2. The Domino model

The Domino model is a 2D model defined on a square lattice with two kinds of ions, A and B, that carry Ising spins and are placed on alternating columns [1, 3]. There are thus three different interactions $J_{AA}$, $J_{BB}$ and $J_{AB}$ between nearest-neighbour spins. $J_{AA}$ and $J_{AB}$ are ferromagnetic ($J_{AA} > 0$, $J_{AB} > 0$), while $J_{BB}$ is antiferromagnetic ($J_{BB} < 0$). With these parameters, all plaquettes in the lattice are frustrated. The system has size $N \times N$ ($N/2$ columns A and $N/2$ columns B each of length $N$) and we assume periodic boundary conditions. Therefore, the Hamiltonian is

$$H = J_{AB} \sum_{i,j} s_{i,j} s_{i,j+1} + J_{AA} \sum_{i \text{ even}} s_{i,j} s_{i+1,j} + J_{BB} \sum_{j \text{ odd}} s_{i,j} s_{i+1,j},$$  \hspace{1cm} (1)

with $s_{i,j} = \pm 1$ the Ising spins sitting on the vertices of the square lattice. Henceforth, the rows are labeled $i = 1, 2, \ldots, N$ and the columns are labeled $j = 1, 2, \ldots, N$. We assume that the interactions respect the hierarchy

$$J_{AA} \gg |J_{BB}| > J_{AB} > 0.$$  \hspace{1cm} (2)

2.1. Ground and first excited states

With the choice of parameters in equation (2), the ground states have ferromagnetic order on each A column and antiferromagnetic order on each B column. Moreover, A and B columns are effectively uncoupled because half of the spins are up and half are down in a B column. As a consequence, it has the same cost for the A columns to be up or down. The antiferromagnetic sequence of spins on each B column can also be chosen in two equivalent ways. There are therefore $2^N$ degenerate ground states, which are frustrated because only half of the horizontal bonds with the coupling constant $J_{AB}$ can be satisfied in an optimal configuration. The ground state entropy is then sub-extensive, $S = Nk_B \ln 2$, but still much larger than the usual $O(1)$ one of, say, the 2D ferromagnetic Ising model. Moreover, looking at a typical ground state, such as that displayed in figure 1(a), we can see that the A columns are either up or down and the spins on the B columns alternate between up and down, yielding a vanishing global magnetisation $M = 0$ (in the $N \to \infty$ limit).

Let us now study the energy and entropy of the lowest excited states on top of some representative ground states. We can construct the first excited state by taking a B column sandwiched in between two A columns of the same sign and turning one of its spin from being anti-aligned to being aligned with the A spins: see the red+ in figure 1(b). We see that we lose $4|J_{BB}|$ energy and we gain $4J_{AB}$ energy from this process. This excited state has a local energy increase with respect to the ground state equal to $\epsilon_1 = 4(|J_{BB}| - J_{AB})$. The other possible flip of a B spin in the same background of parallel A columns is one in which the B spin becomes antiparallel to the A spins, but this excitation is more costly, $4|J_{BB}| + 4J_{AB}$, and can be discarded at very low temperature.

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Figure 1. (a) A typical ground state of the Domino model. Ferromagnetic interactions are represented by full lines, antiferromagnetic interactions by dashed lines. The hierarchy in equation (2) is illustrated with bold and thin lines. (b) Two possible excitations are highlighted in red and green. The red one has lower energy than the green one because it is sandwiched between two A columns of the same sign. Flipping a spin of the A columns would cost even more energy because of the hierarchy in the coupling constants in equation (2).

Another possible excitation in a B column is one in which the flipped spin is in between two anti-aligned A columns, see the green+ in figure 1(b), and has energy $\epsilon_2 = 4|J_{BB}|$. For the choice of parameters we made, this is a higher excited state than the one with excess energy $\epsilon_1$. So, at low (but finite) temperature, when only the first excited states are statistically relevant, one can expect that A column tend to be aligned for these excitations to exist (provided that $\epsilon_2$ is sufficiently larger than $\epsilon_1$).

Having identified the lowest lying excitations on top of the ground states with fully aligned A columns, one notices that their entropy is macroscopic, $S_{\text{aligned}} \propto N^2/4$, since one out of two spins in any B column can be flipped to make one such excited state. This entropic effect forces the system to have long-range ferromagnetic order of the A columns at low but non-vanishing temperature and thus exhibits the zero-temperature ObD transition [1]. Order is maintained until the critical temperature $T_c = 1/\beta_c$ (we set $k_B = 1$) given by

$$\sinh(2\beta_c J_{AB}) \sinh(\beta_c |J_{AA} + J_{BB}|) = 1$$

beyond which the system becomes a conventional paramagnet.

2.2. The effective 1D model

An effective 1D model for the low-temperature properties of the system that focuses on the A columns was derived in [1]. The argument goes as follows. First, since $J_{AA}$ is much stronger than the two other couplings, see equation (2), one assumes that the A columns are perfectly aligned and then represents them as macro-spins. Second:

—if a B chain is sandwiched in between two A chains with parallel spins, the first excitations have energy

$$\epsilon_1 = 4(|J_{BB}| - J_{AB}),$$

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and \( N/2 \) of them are possible, as explained in the previous subsection. The partition function of the B chain in this background (that we indicate with the subscript \( F \)) is

\[
Z_F \simeq [1 + \exp(-\beta \epsilon_1)]^{N/2}.
\]

—if, instead, the two A chains have opposite orientation, the second \( N/2 \) excitations have energy \( \epsilon_2 = 4|J_{BB}| \). In this other background (that we label \( AF \)) the partition function is

\[
Z_{AF} \simeq [1 + \exp(-\beta \epsilon_2)]^{N/2}.
\]

Moreover, we assume that \( |J_{BB}| \gtrsim J_{AB} \) and sufficiently far from zero so as to ensure that \( \epsilon_1 \ll \epsilon_2 \). We define the low-temperature range in which this effective model is valid with the condition \( T \ll \epsilon_2 \); in this way, the second excitations have very low probability.

We can now integrate out the spins of the B columns to get an effective nearest-neighbour coupling \( J_{\text{eff}} \) between the spins of two nearby A chains. Thinking in terms of a 1D effective Ising model, the probabilities \( P_F \) of two neighbouring A chains (of length \( N \)) being parallel, and \( P_{AF} \) of two neighbouring A chains being antiparallel, are

\[
P_F = \frac{\exp(\beta J_{\text{eff}})}{2 \cosh(\beta J_{\text{eff}})} \quad \text{and} \quad P_{AF} = \frac{\exp(-\beta J_{\text{eff}})}{2 \cosh(\beta J_{\text{eff}})},
\]

respectively. On the other hand, the same probabilities in the original model are

\[
P_F = \frac{Z_F}{Z_F + Z_{AF}} \quad \text{and} \quad P_{AF} = \frac{Z_{AF}}{Z_F + Z_{AF}}.
\]

Using these equations we find that

\[
\frac{P_F}{P_{AF}} = \exp(2\beta J_{\text{eff}}) = \frac{(1 + \exp(-\beta \epsilon_1))^{N/2}}{(1 + \exp(-\beta \epsilon_2))^{N/2}} \simeq (1 + \exp(-\beta \epsilon_1))^{N/2}
\]

since we choose \( |J_{BB}| \) of the same order as \( J_{AB} \), which makes \( \epsilon_1 = 4(|J_{BB}| - J_{AB}) \ll \epsilon_2 = 4|J_{BB}| \). In conclusion we find a temperature-dependent and \( O(N) \) effective coupling constant

\[
J_{\text{eff}}(\beta, N) = \frac{N}{4\beta} \ln[1 + \exp(-\beta \epsilon_1)]
\]

and the effective Hamiltonian of the 1D system is

\[
H_{\text{eff}}(T) = -J_{\text{eff}}(T, N) \sum_{j=1}^{N/2} s_j s_{j+1}
\]

with the new Ising macro-spins, \( s_j = \pm 1 \), representing the \( N/2 \) ferromagnetically ordered A columns. The low-temperature behaviour (\( \beta \to \infty \)) of the effective coupling constant \( J_{\text{eff}} \) in equation (5) is

\[
\lim_{\beta \to \infty} J_{\text{eff}}(\beta, N) = \frac{N}{4\beta} \exp(-\beta \epsilon_1)
\]
and tends to zero exponentially for $T \to 0$, which is coherent with the fact that the 2D Domino model is paramagnetic at $T = 0$.

We see that although the model in equation (6) is one dimensional, the coupling constant is of macroscopic order ($\propto N$), allowing for long-range order in the effective model that represents the ordering of the 2D system. In this way, as soon as $T > 0$, $J_{\text{eff}} > 0$ forcing the system into a ferromagnetic phase as a regular 2D ferromagnetic Ising model, even though only the A columns are ferromagnetically ordered: in the thermodynamic limit, the global magnetisation density $m = N^{-1}\sum_{j=1}^{N/2} s_j$ jumps from 0 to 1/2 in a discontinuous way. This approximation is valid as long as we use the hierarchy of coupling constants in equation (2) and $|J_{BB}| \gg |J_{AB}|$. Indeed, we need $J_{AA} \gg (|J_{BB}|, J_{AB})$ to consider the A columns as macro-spins and $\epsilon_1 = 4(\epsilon(J_{BB}) - J_{AB}) \ll \epsilon_2 = 4|J_{BB}|$ to keep only the first excitation accessible at the temperatures we study.

3. Columnar random fields

Let us add quenched disorder to the 2D Domino model in the form of $N/2$ columnar random magnetic fields $h_{i,j}$ that couple bilinearly to the spins, $\sum_{i,j} h_{i,j} s_{i,j}$, but only to those on the A columns and independently of the row index. In order words, $h_{i,j} = h_{j} \neq 0$ only for $j$ even. The $h_j$’s are random i.i.d. variables drawn from a Gaussian distribution $N(0, \sigma^2)$ with $\sigma \ll J_{AA}, |J_{BB}|, J_{AB}$ (and $\sigma$ taken to be typically small, e.g. $10^{-2}$, in our numerical experiments).

The Imry–Ma argument can be easily applied to show that such a disordered 2D model cannot have a phase transition, as we discuss below (section 3.1). Nevertheless, the finite size model can still present a finite-temperature crossover from a disordered low-temperature state to a quasi ferromagnetically ordered state at a higher temperature, in a way that mimics the ObD transition but at a non-zero temperature. Thermal fluctuations will eventually disorder the system again at still higher temperatures. It is easy to show that the effective 1D model derived in section 2.2 remains valid even under the random fields, equation (8), with the same effective ferromagnetic couplings between the spins of the A columns $J_{\text{eff}}$ given in equation (5), provided that $\sigma$ is very small compared to $J_{AA}, |J_{BB}|$ and $J_{AB}$, as explained above.

We hence start by analysing the crossover to the quasi ferromagnetically ordered state using the effective 1D model (section 3.1). We then study the quench dynamics of the 2D model using different initial states and final temperatures mostly in the region with quasi ferromagnetic order in section 3.2.

3.1. The 1D disordered model

We now have an effective 1D random-field Ising model (RFIM), with the A columns of size $N$ considered as $N/2$ macro-spins taking values $N s_j = \pm N$, leading to an effective coupling constant $J_{\text{eff}} \propto N$ between the spins $s_j = \pm 1$, and i.i.d. random fields that couple linearly to the Ising spins. Its Hamiltonian is

$$H_{\text{eff}}(T) = -J_{\text{eff}}(T, N) \sum_{j=1}^{N/2} s_j s_{j+1} - N \sum_{j=1}^{N/2} h_j s_j.$$  

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For each choice of the $h_j$’s we can compute the partition function, the free-energy density and the magnetisation, and then average over the different realisations of disorder.

Because at $T = 0$, $J_{\text{eff}} = 0$, see equation (5), at zero temperature the macro-spins are uncoupled and simply align with their associated magnetic field $h_j$. This single ground state still has magnetisation density $m_{\text{GS}}' = 0$ because in the infinite size limit, half of the $h_j$’s point up and half point down. Nonetheless, this ground state now has a lower energy than that of the model without disorder ($E_{\text{GS}} = 0$); more precisely,

$$E_{\text{GS}}' = -N\frac{1}{2}\sum_{j=1}^{N/2} |h_j|.$$ 

(9)

In the $N \gg 1$ limit, using $E[|h_j|] = \sqrt{2/\pi} \sigma$ and the central limit theorem

$$E_{\text{GS}}' \simeq -\sqrt{1/(2\pi)} \sigma N^2.$$ 

(10)

At very low temperatures $J_{\text{eff}}$ is very weak (see equation (7)), and the system is expected to stay in this zero magnetisation ground state until a sufficiently high temperature is reached—and $J_{\text{eff}}(T, N)$ is made strong enough—for some ferromagnetic order to appear, despite some of the spins having to be anti-aligned with their magnetic field. The energy gain by aligning the $N/2$ macro-spins is $E_F = -(N/2)J_{\text{eff}}(T_{\text{ran ObD}})$. We can estimate the crossover temperature $T_{\text{ran ObD}}$ by comparing $E_F$ and $E_{\text{GS}}'$, leading to

$$J_{\text{eff}}(T_{\text{ran ObD}}, N) \sim \frac{N}{4\beta_{\text{ran ObD}}} e^{-\beta_{\text{ran ObD}}\epsilon_1} \sim \sqrt{\frac{2}{\pi}} \sigma N^2.$$ 

(11)

where $J_{\text{eff}}$ in equation (5) has been approximated in the $\beta_{\text{ran ObD}} \epsilon_1 \ll 1$ limit. Using this equation and setting $\epsilon_1 = 1$, we find that for $\sigma = 0.01$ we should have $T_{\text{ran ObD}} \sim 0.4$ and for $\sigma = 0.005$, $T_{\text{ran ObD}} \sim 0.33$. More generally, $T_{\text{ran ObD}}$ is an increasing function of $\sigma$ that vanishes at $\sigma = 0$.

We insist upon the fact that the effective 1D RFIM that we constructed does not have a genuine phase transition, in the same way as the conventional 2D RFIM does not have one either. Still, we can use the random fields to create a sharp crossover at a finite temperature. Our intuition is that this crossover should be reminiscent of a first-order phase transition because there is no continuity in the two different states of lower energy before and after the crossover.

3.1.1. The Imry–Ma argument. We can estimate the length of the system $N_{\text{IM}}$, beyond which the pseudo-ferromagnetic phase ceases to exist, that is, when flipping a macroscopic domain can lower the energy of the system ($\Delta E < 0$), in the spirit of the Imry–Ma argument [13]. Thinking in terms of the 1D effective model with the coupling constant $J_{\text{eff}}$, reversing a domain of length $L$ implies a (surface) energy cost equal to $4J_{\text{eff}}$. The (bulk) energy difference due to the random fields is a random variable of zero mean and typical fluctuations of order $N\sigma \sqrt{L/(2\pi)}$. These two energy scales are equal for $L = N_{\text{IM}}$ given by

$$N_{\text{IM}} \sim \left( \frac{4J_{\text{eff}}\sqrt{2\pi}}{N\sigma} \right)^2 = \left( \frac{\sqrt{2\pi} \ln[1 + \exp(-\beta\epsilon_1)]}{\beta\sigma} \right)^2.$$ 

(12)

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and provide an order of magnitude of the system length, \( N_{IM} \), beyond which ferromagnetic ordering cannot be sustained. We find that \( N_{IM}(T_{\text{ObD}}) = 64 \), and we plot \( N_{IM}(T) \) as a function of \( T/\epsilon_1 \) for \( \sigma = 0.01 \) in figure 2. We note that the stronger the disorder, the smaller the \( N_{IM} \) (in the infinite disorder limit, \( \sigma \to \infty, N_{IM} \to 0 \), that is, no length scale with non-vanishing magnetisation), concluding that one has to tune \( \sigma \) to some optimal value to allow for a finite \( T_{\text{ObD}} \) with a sufficiently large \( N_{IM} \).

3.1.2. Recursive decimation. A simple way to compute the equilibrium properties of the 1D effective model with periodic boundary conditions and random fields is to use the exact renormalisation decimation procedure [14–16]. Starting from the partition function

\[
Z = \sum_{s_0, s_1, \ldots, s_{N/2-1}} \exp \left( \frac{N}{2} - \sum_{j=0}^{N/2-1} K_j s_j s_{j+1} + \sum_{j=0}^{N/2-1} H_j s_j \right),
\]

where we set \( K_j = \beta J_{\text{eff}} \) and \( H_j = \beta h_j \), we can sum over the odd spins and rewrite it in the same form

\[
Z = \sum_{s_0, s_2, \ldots, s_{N/2-2}} \exp \left( \sum_{k=0}^{N/2-1} H_{2k}s_{2k} \right) \prod_{j=0}^{N/2-1} \sum \exp \left( K_2j(s_j + s_{j+2} + H_{2j+1})s_{j+1} \right)
= \left( \prod_{k=0}^{N/2-1} c_{2k+1} \right) \sum_{s_0, s_2, \ldots, s_{N/2-2}} \exp \left( \sum_{j=0}^{N/2-1} K'_2j(s_j s_{j+2} + (H_j + H'_{2j} + H'_{2j+2}) s_j) \right)
\]

with \( K' \) the rescaled coupling constant, and \( H' \) the extra magnetic field that we add to rescale \( H \). Equating equations (13) and (14) we find the system of equations.
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\[
c_{2j+1} e^{K_{2j} + H_{2j+1}} = 2 \cosh(K_{2j} + K_{2j+1} + H_{2j+1}),
\]

\[
c_{2j+1} e^{-K_{2j} + H_{2j+1}} = 2 \cosh(-K_{2j} - K_{2j+1} + H_{2j+1}),
\]

\[
c_{2j+1} e^{-K_{2j} + H_{2j+1}} = 2 \cosh(K_{2j} - K_{2j+1} + H_{2j+1}),
\]

\[
c_{2j+1} e^{-K_{2j} + H_{2j+1}} = 2 \cosh(-K_{2j} + K_{2j+1} + H_{2j+1}),
\]

for \( j = 0, \ldots, \frac{N}{4} - 1 \). We iterate the decimation until there are only two spins left in the system: \( s_0 \) and \( s_{N/4} \), and we then compute \( Z \) for the four configurations of the decimated system. From it we derive the free energy and the mean magnetisation

\[
F = -\frac{1}{\beta} \ln Z \quad \text{and} \quad M = \left. \frac{\partial F}{\partial (\delta h)} \right|_{\delta h=0},
\]

where \( \delta h \) is an infinitesimal shift added as a global perturbing magnetic field.

In figure 3 we observe that, in all cases, the magnetisation density \( m \) smoothly increases from 0 to a value close to 1. For small \( N \), \( N \leq 128 \), the curves are non-monotonic and \( m \) decays again after reaching a maximum. Instead, for sufficiently large \( N \), say \( N \geq 512 \), \( m \) monotonically approaches 1. However, for these large system sizes there is no crossing of curves of the kind expected in a phase transition. This confirms that the random fields may destroy the 2D ferromagnetic phase in the limit \( N \to \infty \), as the Imry–Ma argument that we presented above shows that this indeed occurs. For these sizes, the curves still approach \( m = 1 \) because \( J_{\text{eff}} \) increases with temperature. However, at fixed \( T \), the ferromagnetic order is lowered as the size of the system is increased, and it will disappear in the infinite size limit (see the inserts in the two panels in figure 3).

If we ignore the fact that there is a strong size dependence in our results, we can still suppose that the magnetisation density \( m \) smoothly increases from 0 to a value close to 1. For small \( N \), \( N \leq 128 \), the curves are non-monotonic and \( m \) decays again after reaching a maximum. Instead, for sufficiently large \( N \), say \( N \geq 512 \), \( m \) monotonically approaches 1. However, for these large system sizes there is no crossing of curves of the kind expected in a phase transition. This confirms that the random fields may destroy the 2D ferromagnetic phase in the limit \( N \to \infty \), as the Imry–Ma argument that we presented above shows that this indeed occurs. For these sizes, the curves still approach \( m = 1 \) because \( J_{\text{eff}} \) increases with temperature. However, at fixed \( T \), the ferromagnetic order is lowered as the size of the system is increased, and it will disappear in the infinite size limit (see the inserts in the two panels in figure 3).

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3.2. Dynamics

To confirm the quasi ferromagnetic order reached by the ObD mechanism in a finite range of non-zero temperatures, we focus now on the quench dynamics of the bidimensional model with random columnar magnetic fields, following the evolution of different initial conditions at the target temperatures. To study the 2D model we implement a Monte Carlo simulation using the metropolis algorithm [17]. A time-step is defined as \( N^2 \) random flip attempts as the system is of size \( N \times N \). For this simulation, we took the parameters \( J_{AA} = 2, J_{BB} = -1 \) and \( J_{AB} = 0.75 \) to keep the energy of the first excitation at \( \epsilon_1 = 4(|J_{BB}| - J_{AB}) = 1 \), and to make the energy between the ground state and the second excited state much larger \( \epsilon_2 = 4|J_{BB}| = 4 \). The critical temperature between the ferromagnetic and paramagnetic phases of the pure Domino model, see equation (3), is \( T_c^{\text{pure}} \approx 1.40 \) for these parameters.

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3.2.1. Quenches from high temperatures. We now investigate the dynamics following the usual quench protocol [18–20]: starting from a completely random high-temperature initial state, $s_{i,j} = \pm 1$ with probability 1/2, we evolve it with the metropolis rule at temperature $T$ such that $T < T_{c}^{\text{ran}}$. Here, $T_{c}^{\text{ran}}$ is the pseudo-critical temperature from the high-$T$ paramagnetic phase to the would-be ferromagnetic state observed for samples of sizes smaller than the Imry–Ma length scale. For $T < T_{c}^{\text{ran}}$ the system should remain disordered, while for $T > T_{c}^{\text{ran}}$ it should tend to order ferromagnetically for the finite system sizes used.

We estimated the temperature above which no ferromagnetic ordering should be reached to be $T_{c}^{\text{ran}} \sim 1.35$ using several runs of the Monte Carlo code for different temperatures and sizes (not shown). This value is close to that found using equation (3), $T_{c}^{\text{pure}} = 1.4$, for the pure Domino model considering it should be a bit lower in our case as an effect of (weak) disorder. Also, using equation (12), we find that the Imry–Ma length is of the order $N_{IM} \sim 10^4$, ensuring that we are below this length in the simulations and that the system should tend to order ferromagnetically for the sizes accessible in numerical simulations. We recall that, for the model with random columnar fields, $T_{c}^{\text{ran}} \sim 0.4$ for $\sigma = 0.01$ and $T_{c}^{\text{ran}} \sim 0.33$ for $\sigma = 0.005$.

Snapshots. The dynamics of frustrated magnets are expected to be slower than those of the pure counterparts [21, 22] and in many cases they can also be anisotropic [23–28]. Indeed, the Domino model is essentially anisotropic and the growth of order should reflect this anisotropy. More precisely, ferromagnetic ordering along the A columns in the horizontal and vertical directions may, in principle, occur in different time scales, as well as antiferromagnetic ordering along the B columns. We focus on the growth of ferromagnetic order on A columns.

Figure 4 displays the evolution of a system with $N = 128$, quenched from a totally disordered initial condition and evolved at $T = 0.35$. Red and white cells represent up and down spins. The first row presents four snapshots of the pure Domino model at the times written below the images ($T = 0.35 < T_{c}^{\text{pure}}$ in this case). The initial state is fully disordered with as many up as down spins placed at random in the box. The system
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Figure 4. Snapshots of the system with $N = 128$ after a quench from a random initial condition across the ferromagnetic transition (pseudo in the random problem) to $T = 0.35$, which is, moreover, also lower than $T_{\text{ran}}^{\text{ObD}}$ in the disordered model. The first line shows four representative snapshots of the instantaneous state of the pure model and the second line shows the same for the model with quenched random columnar fields with $\sigma = 0.08$ and $T_{\text{ran}}^{\text{ObD}} \simeq 0.90$. The time at which the images were stored are indicated below them. The strong red color represents up spins and light red (the result of white surrounded by red boxes) represents down spins.

progressively orders and, as is clear from the later images, it goes faster in the vertical direction. A typical length of domains in the horizontal direction is also growing at a slower speed. Once flat interfaces between the up and down domains are created it will take much longer to kill them and fully order the sample ferromagnetically on all A columns. Still, the horizontal length of the FM ordered domains reached the order of the system size at the latest time, suggesting that the system will eventually fully order. More details of the configurations can be seen in the zoom in figure 5(a).

The snapshots of the pure model can be confronted with those of the model with the columnar random fields that are shown in the second row of figure 4 ($T = 0.35 < T_{\text{ran}}^{\text{ObD}} \simeq 0.90$ in this case). Globally, the evolution is similar to that of the pure model, although with some quantitative differences. Quite clearly, the horizontal extent of the ferromagnetic domains is shorter in the random model, and one can convince oneself that it is of finite extent, not scaling with system size. The reason for this is the pinning character of the random fields, which is further exhibited in figure 5(b). The random fields inhibit long-range horizontal order for $0 < T = 0.35 < T_{\text{ran}}^{\text{ObD}}$.

The plot in figure 5(b) shows the evolution of the fraction of spins of A columns that are aligned with their local columnar magnetic field. This fraction increases with time as the system approaches equilibrium and with the typical strength of the fields, $\sigma$. The blue curve is associated with the evolution of the system we follow on the second row of figure 4 and shows that, in the last snapshot ($t = 2^{20}$), more than 90%
of the spins are already aligned with their magnetic field, probing the pinning (and disordering) character of the latter. We deduce that the horizontal spin domains in these snapshots are mostly due to parts of the system where the $h_j$ have the same sign and do not represent long-range order. Therefore, the intensive magnetisation density $m_A$, defined in equation (17) below, tends to 0 when $N \to \infty$ (while the fluctuations of the extensive magnetisation are typically of the order of $N \sqrt{N}$). In other words, for the pure Domino model, a genuine ferromagnetic long-range order establishes at asymptotically long times in the low-temperature phase, implying that the size of the ordered domains will continue to grow up to the size of the system. In contrast, in the disordered case, where the ferromagnetic phase is destroyed by the random fields, at long times and low temperature the A column will mostly tend to align with the quenched magnetic field, as shown in figure 5(b). Thus, one can still find few relatively large domains of A columns with the same magnetisation for some particular realisation of the disorder (see, e.g. the last snapshot of the second row of figure 4), but this is just due to the fact that there is a finite probability (which is essentially proportional to $1/2$ to the length of these domains) that the sign of the columnar random fields is the same on those columns.

Magnetisation and correlations. We now use a higher temperature, $T_{\text{ran}}^{\text{ObD}} < T < T_{c}^{\text{ran}}$, the regime in which we expect the thermal agitation to overcome the pinning field effect and let A columns order ferromagnetically via the ObD mechanism. Instead of showing snapshots we make quantitative measurements using the typical observables that allow one to test the ordering of a magnetic system. (We recall that $T_c^{\text{ran}}$ is also a crossover in the $N \to \infty$ limit.)
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Figure 6. Monte Carlo dynamics at \( T_{\text{ObD}} < T < T_{\text{ran}} \) of the 2D Domino model with columnar random fields with \( \sigma = 0.01 \). Time evolution of the mean magnetisation density of the A columns, \( m_A(t) \) defined in equation (17), after quenches from a fully random initial condition across the paramagnetic–ferromagnetic (PM–FM) crossover at \( T_{\text{ran}} \) and above that at \( T_{\text{ran}}^{\text{ObD}} \). Different curves correspond to different sizes given in the key.

In figure 6 we show the time evolution of \( m_A \) defined as

\[
m_A(t) = \frac{2}{N^2} \left\langle \left| \sum_{k_A=1}^{N^2/2} s_{k_A}(t) \right| \right\rangle
\]

with \( \langle \ldots \rangle \) the average over many realisations of the dynamics and \( k_A \) running over the A spins indices only. The working temperature in this plot is \( T_{\text{ObD}}^{\text{ran}} < T = 1 < T_{\text{ran}} \) and the initial condition is random, typical of \( T \to \infty \). For \( N = 512 \) the magnetisation remains smaller than 0.1 until \( t \simeq 10^4 \). The analysis of the coarsening process will be carried out for such a linear system size, ensuring that the evolution remains sufficiently far from any possible equilibration.

The plot in figure 7(a) is representative of the coarsening dynamics across a second-order phase transition [18–20]. We display the horizontal correlation function of the spins sitting on the A columns

\[
C_x(x, t) = \frac{\frac{2}{N^2} \left( \sum_{i,j} s_{2i,j} s_{2(i+x),j} - \left( \sum_{i,j} s_{2i,j} \right)^2 \right)}{1 - \left( \sum_{i,j} s_{2i,j} \right)^2} = \frac{\frac{2}{N^2} \sum_{i,j} s_{2i,j} s_{2(i+x),j} - m_A(t)^2}{1 - m_A(t)^2}
\]

of a system with \( N = 512 \) for which the ferromagnetic magnetisation density of these columns at the longest time \( t \simeq 10^5 \) should be of the order of \( m_A \simeq 0.2 \), see figure 6. The system progressively orders, and this is represented by a \( C_x(x, t) \) that decays to 0 with distance in a slower manner for increasing times. These curves can be compared, for example, to those in figure 17 in [29], where similar data for the 2D Ising model.
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Figure 7. Monte Carlo dynamics after a quench (cooling) from a completely random initial condition across the PM–FM and above the ObD crossovers to $T = 1$: a square system with linear length $N = 512$. (a) The horizontal correlation function $C_x(x,t)$ as a function of $x$ for a system with columnar random fields ($\sigma = 0.01$), at different times given in the key. Inset: correlations collapse when the $x$-axis is rescaled by $R_x(t)$. (b) Evolution of the typical growing correlation length $R_x(t)$ for systems with different standard variations of the random fields given in the key.

are shown. We obtain the typical growing correlation length in the $x$-direction from the standard criterion $C_x(R_x(t), t) \sim 1/e$ (see the horizontal dotted line in figure 7(a) and the scaling plot in the inset). We then plot the evolution of $R_x(t)$ with time in panel (b). We find that at short time scales the pure and disordered Domino models have $R_x(t) \propto t^{1/2}$, as expected for the curvature-driven dynamics of a non-conserved scalar order parameter system. The various curves correspond to different strengths of the random fields, as quantified by their standard deviations $\sigma$ given in the key. At the longest time scales that we show the growth in the model with random fields saturates, to a value that decreases with increasing $\sigma$. One can expect this saturation length to have a similar parameter dependence as the Imry–Ma length $N_{IM}$ and, although we cannot make a quantitative comparison, we note that the dependence on $\sigma$ is indeed similar. The annihilation of the remaining domain walls should involve much longer time scales (see, for example [30, 31], for their study in the pure 2D Ising model), and it needs thermal activation to create a bump on the otherwise flat interfaces that, moreover, are pinned by the random fields.

Figure 8 confirms that the system orders faster vertically than horizontally. This is not surprising as the magnetic fields $h_j$ on each A column act as a driving force, since the spins that are aligned with them are in a lower energy state. (In fact, on the A columns in a fixed environment the growth of the vertical ferromagnetic domains should be ballistic and not diffusive at long times.) In panel (a) we present the vertical correlation function of the spins belonging to columns A

$$C_y(y,t) = \frac{2^N \sum_{i,j} s_{2i,j}s_{2i,j+y} - m_A(t)^2}{1 - m_A(t)^2}$$

while in panel (b) we represent the time-dependence of the corresponding growing length defined as $C_y(R_y(t), t) \sim 1/e$. The inset in (a) displays the scaling plot in which $y$ is divided by $R_y(t)$ on the horizontal axis.
3.2.2. Heating from the disordered ground state. We now investigate the dynamics across the ObD crossover itself, starting from the ground state (corresponding to \( T = 0 \)) and fixing the working temperature to \( T = 1 \) as in the sub-critical quenches discussed in section 3.2.1, where the system, for the sizes we use, should eventually approach a ferromagnetic configuration. The snapshots in figure 9 show an example of these dynamics. In the pure system (top panels) the final configuration is the one in which the system ordered ferromagnetically on the A columns with \(-1\) spins (the strong red color represents sequences of A columns being up, and the light red color represents sequences of A columns being down). In the disordered case (bottom panels) the dynamics are slower, and the stationary state has not been reached yet. Data for the pure model are gathered using, for each Monte Carlo run, an initial state chosen randomly among the collection of all possible ones. Instead, the simulations with random fields are started from the unique ground state, which is determined for each run by the magnetic fields that we draw from the Gaussian distribution. The average is then computed over random fields and/or Monte Carlo random numbers.

In figure 10 one finds the horizontal correlation functions as functions of distance, for different times, in panel (a). The curves have the same qualitative behaviour as those already shown for the quenches from the infinite temperature state. However, the growing length plotted in panel (b) is markedly different from, and much slower than, the \( t^{1/2} \) form observed in a quench across \( T_{\text{ran}} \). Still, the scaling plot in the inset of (a) is of high quality. The ObD crossover is reminiscent of a first-order transition, and the dynamics across it should then have aspects of the nucleation of a rare event with a large energetic cost. In this specific case, to flip spins on an A column and align them with those of neighbouring A columns, the system has to overcome a barrier \( \Delta E_{\text{nuc}} = 4J_{AA} \).
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Figure 9. Snapshots of a system of size $N = 128$ after a sudden increase in temperature from the disordered ground state to $T = 1$. In the pure model (first row), this temperature is below $T_{\text{pure}}^c$ and the A columns achieve complete order. In the disordered one ($\sigma = 0.01$, second row), it is in between the pseudo-critical temperatures $T_{\text{ObD}}^\text{ran}$ and $T_{\text{ran}}^c$. The times at which the images were stored are indicated below them. The strong red color represents up spins and white (surrounded by red boxes yielding light red in this scale) represents down spins.

4. Staggered columnar magnetic fields

To have a phase transition towards a ferromagnetically ordered state upon increasing temperature, circumventing the Imry–Ma argument, we no longer use random fields, but alternate columnar magnetic fields $h_j = (-1)^j h$. Because the $h_j$ are not random but staggered, the formation of macroscopic reversed ferromagnetic domains is no longer possible. The drawback is that we lose some specificity of the ObD phenomenon because the zero-temperature ground state is now antiferromagnetic as the staggered magnetic fields impose. Still, the strategy is to use these fields as a probe to exhibit the underlying conventional ObD transition. The idea is to impose an antiferromagnetic equilibrium state at very low temperature, that would be replaced by the ferromagnetic one at a first-order phase transition taking place at a finite temperature below that at which the system reaches the paramagnetic high-temperature phase.

4.1. The 1D model

The Hamiltonian of the effective 1D model under staggered local fields is

$$H_{\text{eff}}(T) = -J_{\text{eff}}(T, N) \sum_{j=1}^{N/2} s_j s_{j+1} - N h \sum_{j=1}^{N/2} (-1)^j s_j$$

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Figure 10. Heating across the ObD crossover in a square 2D system with linear length $N = 512$, and columnar random fields with $\sigma = 0.01$, evolving from one of the zero-temperature ground states. (a) Horizontal correlation $C_x(x, t)$ as a function of $x$, at $T = 1$, $\sigma = 0.01$, and for different times given in the key. Inset: a scaling plot with the $x$-axis rescaled by $R_x(t)$. (b) Evolution of the typical growing correlation length $R_x(t)$ for different values of the disorder strength in the In-linear scale since, in this time regime, the growth is closer to logarithmic than power law in the heated case.

with $J_{\text{eff}}(T, N) \propto N$, as given in equation (5).

We compute the mean magnetisation using the transfer matrix method with a matrix $T = W_1 W_2$ representing a block of two columns with $W_1$ for a column with a positive magnetic field and $W_2$ for a negative one

$$W_1 = \begin{pmatrix} e^{\frac{N}{4} \ln(1+e^{-1/T}) + \frac{Nh}{T}} & e^{\frac{N}{4} \ln(1+e^{-1/T}) - \frac{Nh}{T}} \\ e^{-\frac{N}{4} \ln(1+e^{-1/T}) - \frac{Nh}{T}} & e^{-\frac{N}{4} \ln(1+e^{-1/T}) + \frac{Nh}{T}} \end{pmatrix},$$

$$W_2 = \begin{pmatrix} e^{\frac{N}{4} \ln(1+e^{-1/T}) + \frac{Nh}{T}} & e^{-\frac{N}{4} \ln(1+e^{-1/T}) - \frac{Nh}{T}} \\ e^{-\frac{N}{4} \ln(1+e^{-1/T}) + \frac{Nh}{T}} & e^{\frac{N}{4} \ln(1+e^{-1/T}) - \frac{Nh}{T}} \end{pmatrix},$$

with $\delta h > 0$ an infinitesimal magnetic field we add to compute the magnetisation. Writing $\lambda_+$ and $\lambda_-$ the eigenvalues of $T$, the free energy per spin is

$$f = -\frac{2T}{N^2} \ln \left( \lambda_+^{N/4} + \lambda_-^{N/4} \right) \quad (21)$$

and the mean magnetisation per spin $m$ is

$$m = -\frac{\partial f}{\partial (\delta h)} \bigg|_{\delta h = 0}. \quad (22)$$

We find a transition temperature $T_{\text{ObD}}^\text{col} = 0.35$ with $h = 0.01$ (see figure 11), which corresponds to what we expect by comparing the interaction energy governed by the effective coupling constant at $T_{\text{ObD}}^\text{col}$ and the energetic contribution of the magnetic field

$$J_{\text{eff}}(T_{\text{ObD}}^\text{col}, N) = \frac{h}{2}. \quad (23)$$

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Figure 11. Mean magnetisation of the 1D model with staggered magnetic fields with amplitude $h = 0.01$, for different system sizes given in the key.

Figure 12. Snapshots of the system with $N = 128$ and $h = 0.08$ after a quench from a random initial condition to $T = 0.35 < T_{\text{ObD}}^{\text{col}}$ (first line) and $T_{\text{ObD}}^{\text{col}} < T = 1 < T_{c}^{\text{col}}$ (second line). The times at which the images were stored are indicated below them. On the latest image of the first line, the configuration is one of the ground states with alternate ordering of A columns, whereas on the latest image of the second line, the ordering of A columns is ferromagnetic.

We note that, apart from numerical constants, this is the same equation as (11), where $\sigma$ has been replaced by $h$. Here, $T_{\text{ObD}}^{\text{col}}$ for the columnar field model is also an increasing function of $h$ departing from 0. The numerical data displayed in figure 11, which represent $m$ as a function of $T$, confirm the fact that the system undergoes a first-order phase transition at $T_{\text{ObD}}^{\text{col}}$.

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Figure 13. Monte Carlo dynamics of the 2D Domino model with columnar alternate fields ($h = 0.01$). Dynamics after a quench from a completely random initial condition across the PM–FM transition. The evolution is followed at $T = 1$, that is, $T_{\text{ObD}} < T < T_{\text{col}}$. (a) Horizontal correlation $C_x(x, t)$ as a function of $x$ for different times given in the key. Inset: a scaling plot rescaling $x$ by $R_x(t)$. (b) Evolution of the typical growing correlation length $R_x(t)$. A square system of linear length $N = 512$.

4.2. Dynamics

We now turn to the analysis of the quench dynamics of the 2D model with alternate columnar magnetic fields.

Figure 12 shows the evolution of a system with $N = 128$ and staggered columnar magnetic fields of strength $h = 0.08$, quenched from a disordered initial condition and evolved at $T = 0.35 < T_{\text{ObD}}$. If we compare these snapshots with those in figure 4, we see that as for the two systems studied before (the pure Domino model and that with random magnetic fields), the system progressively orders in the vertical direction. The difference here is that the domains are not growing in the horizontal direction because of the pinning character of the alternate magnetic fields. In the last snapshot, we can see that the system reached its equilibrium state at $T = 0.35$, which is also a ground state of the pure Domino model at $T = 0$.

In figure 13 we display the horizontal correlation functions of the Domino model with columnar alternate fields of strength $h = 0.01$ for increasing times given in the key of (a). In panel (b) the growing correlation is reported and compared to the $t^{1/2}$ law, as well as to the growing correlation of the random fields case with $\sigma = 0.01$. The data confirm that the system orders ferromagnetically on the A columns in between $T_{\text{ObD}}$ and $T_{\text{col}}$.

5. Conclusion

The goal of our work was to find a way to displace the thermal ObD transition from zero to a non-vanishing temperature via suitably engineered random magnetic fields. The idea was to construct a setting in which the observation of this phenomenon can be made, at least conceptually, easier. To reach this aim we followed two routes, using the Domino model as the testing ground.
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On the one hand, we added well-tuned quenched columnar random fields. These fields lift the degeneracy of the ground states, selecting one that still has zero magnetisation but lower energy than that under no fields. Consequently, the system is stuck in this state until the temperature is high enough for it to access the large number of first excited states. In this case, the ObD crossover happens at a finite temperature $T_{\text{ObD}} > 0$, but long-range order is suppressed by this type of disorder in the thermodynamic limit. Still, we observed an ObD crossover at a finite temperature for small system sizes using various numerical and theoretical methods that were in good agreement with our predictions.

In the second approach, we used alternate columnar magnetic fields that do indeed displace the transition to a finite temperature. In this case we computed the theoretical ObD transition temperature using the transfer matrix method and we confirmed it with dynamic measurements. We also discussed some indications that the ObD transition is first order.

Even though both random and alternate fields impose the ground state, the finite-temperature crossover or transition can be used to probe the ObD phenomenon in the model without applied fields. Since the crossover or transition temperatures can be tuned at will, our procedure allows one to probe the ObD mechanism without going to too-low temperature, where other kinds of energetic contributions might interfere with it. In conclusion, we think that these methods should be useful for checking whether a system exhibits the ObD transition; for instance, in artificial spin-ice settings [10–12], in which the interactions and the disorder might be engineered at will.

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