Graded Automorphisms of the Algebra of Polynomials in Three Variables

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1. INTRODUCTION

Let \( K \) be an algebraically closed field, and let \( \mathcal{A} = K[x_1, \ldots, x_n] \) be the polynomial algebra in \( n \) variables over the field \( K \). Denote by \( \varphi = (f_1, \ldots, f_n) \) the polynomial mapping \( \mathcal{A} \to \mathcal{A} \) of the form

\[
\varphi(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)),
\]

where \( f_i \in \mathcal{A} \). Denote the group of automorphisms, i.e., invertible polynomial mappings, of the algebra \( \mathcal{A} \) by \( \text{Aut}(\mathcal{A}) \). The automorphisms of the form

\[
\varphi = (x_1, \ldots, x_{i-1}, \lambda x_i + F, x_{i+1}, \ldots, x_n),
\]

where \( F \in K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n], \lambda \neq 0 \), are said to be elementary. The automorphisms that can be represented as a composition of elementary ones are said to be tame. In particular, nondegenerate linear transformations of coordinates are tame automorphisms. The nontame automorphisms are said to be wild.

The question of the existence of wild automorphisms is currently solved for \( n \leq 3 \).

- The automorphism group of the algebra \( K[x] \) is readily described, namely, all automorphisms are affine transformations: \( x \mapsto \lambda x + c \). This means that there are no wild automorphisms among them.

- In the group of automorphisms of the algebra of polynomials in two variables \( K[x,y] \) there are no wild automorphisms either [1]. Moreover, the automorphism group is the amalgamated product of two subgroups, one of which is the subgroup of affine automorphisms [2].

- For \( n = 3 \), wild automorphisms exist: it is proved in [3] that Nagata’s automorphism [4] is wild.

Thus, nontrivial systems of generators for the groups \( \text{Aut}(K[x]) \) and \( \text{Aut}(K[x, y]) \) are known.

Suppose now that there is a grading by some group \( G \) on the algebra \( \mathcal{A} \). We denote this grading by \( \Gamma \). An automorphism \( \varphi \) is said to be \( \Gamma \)-graded if \( \varphi(\mathcal{A}_g) \subseteq \mathcal{A}_g \) for any \( g \in G \), where \( \mathcal{A}_g \) is a homogeneous component of degree \( g \). The \( \Gamma \)-graded automorphisms form a subgroup of \( \text{Aut}_\Gamma(\mathcal{A}) \) in the group \( \text{Aut}(\mathcal{A}) \). We call a graded automorphism graded tame with respect to the grading \( \Gamma \) if it can be represented as a composition of elementary graded automorphisms. Otherwise, we call such an automorphism graded wild. In [5], for \( n = 4 \), an example of a \( \mathbb{Z} \)-grading that admits graded wild automorphisms is given, and it is shown that Anik’s automorphism is graded wild with respect to
this grading. In [6], all \(\mathbb{Z}\)-gradings of the algebra \(\mathbb{K}[x, y, z]\) admitting graded wild automorphisms are described. For the gradings that do not admit graded wild automorphisms, the graded elementary automorphisms form a natural system of generators of the group \(\text{Aut}_F(\mathcal{A})\). In this paper, we construct systems of generators of the groups \(\text{Aut}_F(\mathcal{A})\) for \(\mathbb{Z}\)-gradings of the algebra \(\mathcal{A} = \mathbb{K}[x, y, z]\) that admit wild automorphisms.

2. GRADED AUTOMORPHISMS

In this section, we present the results of the paper [6].

Consider a nontrivial \(\mathbb{Z}\)-grading \(\Gamma\) on the algebra \(\mathcal{A} = \mathbb{K}[x, y, z]\) under which the variables \(x, y,\) and \(z\) are homogeneous. If there are zeros among the degrees of the variables, or if the degrees of all variables are of the same sign, then, as was proved in [6], all graded automorphisms are graded tame. Below we assume that there are no zeros among the degrees of the variables and that they are not all of the same sign. Multiplying all three powers of the same degree by the same number does not change the set of automorphisms. Therefore, we may assume that
\[
(\deg_F(x), \deg_F(y), \deg_F(z)) = (a, b, -c), \quad a, b, c > 0, \quad a \geq b,
\]
and the greatest common divisor of \(a, b,\) and \(c\) is equal to one. The main result of the paper [6] is the following theorem.

**Theorem 1.** A graded wild automorphism of the algebra \(\mathbb{K}[x, y, z]\) with a grading \(\Gamma\) exists if and only if there is a pair of positive integers \(P\) and \(Q\) such that \(a = cP + bQ,\) where \(Q \geq 2.\)

Consider two subgroups of the group \(\text{Aut}_F(\mathcal{A})\):
\[
E = \{ \varphi \in \text{Aut}_F(\mathcal{A}) \mid \varphi(z) = z \}, \quad T = \{(x, y, \lambda z) \mid \lambda \neq 0 \}.
\]

**Lemma 1** [6, Lemma 3.8]. The group \(\text{Aut}_F(\mathcal{A})\) is isomorphic to a semidirect product \(E \rtimes T.\)

Geometrically, automorphisms of the algebra \(\mathcal{A}\) correspond to automorphisms of the three-dimensional affine space \(\mathbb{A}^3.\) Consider the plane \(\gamma = \{ z = 1 \}.\) The automorphisms of \(\mathbb{A}^3\) corresponding to the automorphisms of the group \(E\) preserve the plane \(\gamma.\) The algebra of regular functions \(\mathbb{K}[\gamma]\) is isomorphic to the algebra \(\mathbb{K}[u, v],\) where \(u = x|_\gamma, v = y|_\gamma.\) Thus, there is a homomorphism
\[
\alpha : E \to \text{Aut}(\mathbb{K}[u, v]),
\]
and \(\alpha\) has the following structure:
\[
\alpha(f, g, z) = (f(u, v, 1), g(u, v, 1)).
\]

**Remark 1** [6, Corollary 3.13]. The homomorphism \(\alpha\) is injective.

**Theorem 2** [6, Proposition 2.3]. Let \(\tilde{\gamma}\) be an arbitrary grading on the algebra \(\mathbb{K}[u, v].\) Then all graded automorphisms are graded tame.

**Proposition 1** [6, Corollary 2.2]. If an automorphism of the plane preserves the origin, then it decomposes into a composition of elementary ones that also preserve the origin.

We say that an automorphism \(\varphi\) of the plane \(\gamma\) lifts to a space automorphism if \(\alpha^{-1}(\varphi) \neq \emptyset.\)

Consider the grading \(\tilde{\gamma}\) on \(\mathbb{K}[u, v]\) by a cyclic group \(\mathbb{Z}_c\) such that \((\deg_{\tilde{\gamma}}(u), \deg_{\tilde{\gamma}}(v)) = (\overline{a}, \overline{b}),\) where \(\overline{a}\) and \(\overline{b}\) are the residues of \(a\) and \(b\) modulo \(c.\) It can readily be seen that the automorphism \(\alpha(\varphi)\) is \(\tilde{\gamma}\)-graded for every \(\varphi \in E.\)

We need the following results of [6].

**Lemma 2** [6, Lemma 3.14]. Let \(\varphi = (\tilde{f}, \tilde{g}) \in \text{Aut}_{\tilde{\gamma}}(\mathbb{K}[u, v]).\) In this case, \(\alpha^{-1}(\varphi) = \emptyset\) if and only if \(\tilde{f}\) contains, with nonzero coefficient, a monomial \(v^a\) such that \(aq < a\) or \(\tilde{g}\) has a nonzero free term.

**Lemma 3** [6, Lemma 3.17]. Let \(\xi_1 \in \text{Aut}_{\tilde{\gamma}}(\mathbb{K}[u, v])\) be an elementary automorphism, and let \(\xi_2 \in \text{Aut}_{\tilde{\gamma}}(\mathbb{K}[u, v])\) be a linear automorphism. Then the automorphism \(\xi_2 \circ \xi_1\) can be re-decomposed in the form \(\xi_2 \circ \xi_1 = \xi_2 \circ \xi_1 \circ \zeta_0,\) where \(\zeta_0\) and \(\zeta_2\) are linear automorphisms, \(\xi_1\) is an elementary automorphism, and \(\xi_2(u) = \lambda u.\)
3. SYSTEM OF GENERATORS

For the case in which the grading does not admit wild automorphisms, the system of generators consists of elementary graded automorphisms. Now consider the case in which the grading $\Gamma$ admits wild automorphisms.

**Remark 2.** If a grading admits wild automorphisms, then, by Theorem 1,
\[
\deg(\varphi) = a > b = \deg(\psi).
\]

Let $\varphi = (f, g, z)$, and let $\alpha(\varphi) = \tilde{\varphi} = (\tilde{f}, \tilde{g}) \in \text{Aut}_F(\mathbb{K}[u, v])$.

**Proposition 2.** The automorphism $\tilde{\varphi}$ can be written out as a composition $\xi_1' \circ \cdots \circ \xi_n'$ of graded elementary automorphisms such that, for any $j$, the linear part of polynomial $\xi_j'(u)$ is equal to $\lambda_j u$ and $\xi_j'(0, 0) = (0, 0)$.

**Proof.** Note that, due Remark 2, the automorphism $\varphi$ saves the origin; moreover, the degree of $f$ is greater than that of $g$, and thus the monomials of the form $yz^n$ have zero coefficients in the polynomial $f$.

This implies that the linear part of the polynomial $\tilde{f}$ is equal to $\lambda_1 u$. By Theorem 2 and Proposition 1, the automorphism $\tilde{\varphi}$ can be decomposed into a composition of graded elementary ones: $\tilde{\varphi} = \xi_n \circ \cdots \circ \xi_1$ such that $\xi_j(0, 0) = (0, 0)$. We may assume that $\xi_1$ is linear, possibly the identity one. Let us apply the induction on $t$, where $0 \leq t \leq n$, and show that the automorphism $\tilde{\varphi}$ can be represented as a composition $\xi_n' \circ \cdots \circ \xi_1'$ of elementary graded automorphisms such that, for any $j$ such that $n - t < j \leq n$, the linear part of $\xi_j'(u)$ is equal to $\lambda_j u$.

**Base.** For $t = 0$, the assertion is obvious.

**Step.** Write $k = n - t$. By the induction assumption, $\tilde{\varphi} = \xi_n \circ \cdots \circ \xi_1$ and, for $k < j \leq n$, the linear part of $\xi_j'(u)$ is equal to $\lambda_j u$.

If $\xi_k$ is nonlinear, then either the linear part is $\xi_k(u) = \lambda_1 u$ or $\xi_k = (\nu u + \lambda v + f(v), v)$. In the second case, write $\tau = (u + \lambda v, v)$. We can replace $\xi_k$ by $\xi_k' = \xi_k \circ \tau^{-1}$ and $\xi_k^{-1}$ by $\xi_k' \circ \tau \circ \xi_k^{-1}$. In both cases, the linear part of $\xi_k'(u)$ is equal to $\lambda_j u$.

If $\xi_k$ is linear and $k > 1$, then we can assume that $\xi_k^{-1}$ is not linear and $k > 2$. By Lemma 3, the composition $\xi_k \circ \xi_k^{-1}$ can always be written in the form $\zeta_2 \circ \zeta_1 \circ \zeta_0$, where $\zeta_0$ and $\zeta_2$ are linear, the linear part of $\zeta_2(u)$ is $\lambda_1 u$, and $\zeta_1$ is elementary. In this case, we replace $\xi_k \circ \xi_k^{-1} \circ \xi_k^{-2}$ by $\xi_k' \circ \xi_k'^{-1} \circ \xi_k'^{-2}$, where $\xi_k' = \zeta_2$, $\xi_k'^{-1} = \zeta_1$, $\xi_k'^{-2} = (\zeta_0 \circ \xi_k^{-2})$. Then the linear part of $\xi_k'(u)$ is equal to $\lambda_1 u$.

Finally, if $k = 1$, then $\xi_1' \circ \cdots \circ \xi_1'(u) = \lambda_2 \cdots \lambda_n u$ by the induction assumption. Since the linear part of $\tilde{\varphi}(u)$ is equal to $\lambda_1 u$, it follows that
\[
\xi_1(u) = \frac{\lambda_2 \cdots \lambda_n}{\lambda_1} u. \quad \Box
\]

Let us now denote the degree of the least monomial in a polynomial in one variable $f$ by $\deg(f)$. Consider the following sets of $\Gamma$-graded automorphisms of the algebra $\mathbb{K}[u, v]$:
\[
D = \{(u, \lambda v + \mu u^k) \mid ka \equiv b \pmod{c}, k > 0, \lambda \in \mathbb{K}^x\},
\]
\[
U = \{(\lambda u + f(v), v) \mid \deg(\varphi) = a, \deg(\psi) \geq \frac{a}{b}, \lambda, \mu \in \mathbb{K}^x\},
\]
\[
W = \{(\lambda u + f(v), v) \mid \deg(\varphi) = a, \deg(\psi) \geq 1, \deg(\psi) < \frac{a}{b}, \lambda, \mu \in \mathbb{K}^x\}.
\]

**Remark 3.** By Lemma 2, the automorphisms in $D$ and $U$ lift to space automorphisms, and the automorphisms in $W$ lift if and only if $f = 0$.

**Remark 4.** The sets $W$ are subgroups of the automorphism group.
Let \( \tau \in W, \theta \in D \), and let \( \tau^{-1} \circ \theta \circ \tau(u, v) = (\tilde{f}(u, v), \tilde{g}(u, v)) \).

The form of the groups \( W \) and \( D \) implies the following remark.

**Remark 5.** The linear part of the polynomial \( g \) is equal to \( \lambda v, \lambda \neq 0 \); moreover, the automorphism preserves the origin, i.e., \( (\tilde{f}(0, 0), \tilde{g}(0, 0)) = (0, 0) \).

**Lemma 4.** For elementary automorphisms \( \tau \in W \) and \( \theta \in D \), there is an automorphism \( s_{\tau, \theta} \in W \) such that the automorphism \( s_{\tau, \theta} \circ \tau^{-1} \circ \theta \circ \tau \) lifts to a space automorphism.

**Proof.** By Remark 5, the automorphism \( \tau^{-1} \circ \theta \circ \tau \) does not lift to a space automorphism if and only if the polynomial \( \tilde{f}(u, v) \) contains monomials of the form \( \nu v^m \), where \( m < a/b \) and \( \nu \neq 0 \). Among all these monomials, we choose a monomial with the minimal degree \( m_1 \). Let now \( \lambda \) be the coefficient of the monomial \( v \) in the polynomial \( \tilde{g} \). The coefficient \( \lambda \) is nonzero by Remark 5. Consider the automorphism \( s_1 = (u - (\nu/\lambda)v^{m_1}, v) \). Consider a new automorphism \( s_1 \circ \tau^{-1} \circ \theta \circ \tau \) and the image \( \tilde{f}_1(u, v) \) of the variable \( u \) under this automorphism. The minimal degree of monomials of the form \( v^m \) has increased in this automorphism. Thus, \( s_{\tau, \theta} = s_1 \circ \cdots \circ s_1 \).

Introduce the following notation: \( \tau_0 = s_{\tau, \theta} \circ \tau^{-1} \), \( S = \{ \tau \circ \theta \circ \tau \mid \tau \in W, \theta \in D \} \).

**Lemma 5.** The group \( \tilde{E} = \alpha(E) \) is generated by the subgroup \( U \) and the set \( S \).

**Proof.** Take \( \tilde{\varphi} \in \tilde{E} \). Let \( \tilde{\varphi} = \xi_n \circ \cdots \circ \xi_1 \), where \( \xi_k \) are elementary graded automorphisms, and, according to Proposition 2, we can assume \( \xi_k \) lie in \( D \cup U \cup W \). Let, among these automorphisms, there are \( m \) automorphisms in \( D \). By induction on \( m \), we will show that the automorphism \( \tilde{\varphi} \) can be represented in the form of a composition of automorphisms in \( U \) and \( S \).

*Base.* For \( m = 0 \), the automorphism \( \tilde{\varphi} \) lies in \( \langle W \cup U \rangle \). Here it lifts to a space automorphism, and hence lies in \( U \).

*Step.* Let \( \xi_{k-1}, \ldots, \xi_1 \) lie in \( U \cup W \), and let \( \xi_k \) lie in \( D \). Let \( \xi_{k-1} \circ \cdots \circ \xi_1 = \tau \circ \tau_1 \), where \( \tau \in W \) and \( \tau_1 \in U \). Represent the automorphism \( \tilde{\varphi} \) in the form \( \tilde{\varphi} = \xi_n \circ \cdots \circ \xi_{k+1} \circ \tau \circ \tau_1 \). Further, Lemma 4 implies the existence of an automorphism \( s_{\tau, \xi_k} \in W \) such that the automorphism \( s_{\tau, \xi_k} \circ \tau^{-1} \circ \xi_k \circ \tau = \tau_1 \circ \xi_k \circ \tau \) lifts to a space automorphism. Thus,

\[
\tilde{\varphi} = (\xi_n \circ \cdots \circ \xi_{k+1} \circ \tau_1 \circ s^{-1}) \circ (\tau_1 \circ \xi_k \circ \tau_1) \circ \tau_1.
\]

Here the composition \( \xi_n \circ \cdots \circ \xi_{k+1} \circ \tau \circ s^{-1} \) lifts to a space automorphism, and the induction assumption holds for it.

Thus, the following theorem is proved.

**Theorem 3.** The automorphisms of the algebra \( \text{Aut}_1(\mathcal{A}) \) are generated by the automorphisms in \( \alpha^{-1}(U) \), the automorphisms of the form \( \alpha^{-1}(\tau_0 \circ \theta \circ \tau) \), where \( \tau \in W \) and \( \theta \in D \), and the automorphisms in the group \( T = \{(x, y, \lambda z) \mid \lambda \neq 0\} \).

**4. EXAMPLES**

Let the algebra \( \mathbb{K}[x, y, z] \) be graded as follows:

\[
(\deg(x), \deg(y), \deg(z)) = (3, 1, -1).
\]

This grading admits graded wild automorphisms according to Theorem 1. With this grading, we have

\[
D = \{(u, \lambda v + \mu w^k) \mid ka \equiv b (\text{mod } c), k > 0, \lambda \in \mathbb{K}^x \},
\]

\[
U = \{(\lambda u + f(v), v) \mid \deg(f) \geq 3, \lambda, \mu \in \mathbb{K}^x \},
\]

\[
W = \{(\lambda u + f(v), v) \mid \deg(f) > 1, \deg(f) < 3, \lambda, \mu \in \mathbb{K}^x \}.
\]
Let
\[ \tau = \begin{pmatrix} u + v^2 \\ v \end{pmatrix} \in W, \quad \theta = \begin{pmatrix} u \\ v + u \end{pmatrix} \in D. \]

Then we have
\[ \tau^{-1} \circ \theta \circ \tau = \begin{pmatrix} u - u^2 - v^4 - 2uv - 2v^3 - 2uv^2 \\ v + u + v^2 \end{pmatrix}. \]

Now consider the corresponding automorphism of the polynomial algebra in three variables:
\[ \sigma = \alpha^{-1}(\tau^{-1} \circ \theta \circ \tau) = \begin{pmatrix} x - x^2z^3 - y^4z - 2xyz - 2y^3 - 2xy^2z^2 \\ y + xz^2 + y^2z \end{pmatrix}. \]

The resulting automorphism \( \sigma \) coincides with the Nagata automorphism [4]. Thus, the Nagata automorphism lies in the system of generators of the group of graded automorphisms.

Let us present the general form of an automorphism of the form \( \eta \circ \theta \circ \tau \).
Let
\[ \tau = \begin{pmatrix} \lambda_1u + \nu v^2 \\ v \end{pmatrix} \in W, \quad \theta = \begin{pmatrix} u \\ \lambda_2v + \mu u^k \end{pmatrix} \in D. \]

Then
\[ \tau^{-1} \circ \theta \circ \tau = \begin{pmatrix} u + \frac{\nu}{\lambda_1} v^2 - \frac{\nu}{\lambda_1}(\lambda_2v + \mu(\lambda_1u + \nu v^2)^k)^2 \\ \lambda_2v + \mu(\lambda_1u + \nu v^2)^k \end{pmatrix}. \]

The coefficient at \( v^2 \) in the polynomial \( \tau^{-1} \circ \theta \circ \tau(u) \) is equal to \((1 - \lambda_1^2)\nu/\lambda_1 \).

Thus, the automorphism \( \eta \circ \theta \circ \tau \) is equal to \( s_{\tau, \theta} \circ \tau^{-1} \circ \theta \circ \tau \), where
\[ s_{\tau, \theta} = \begin{pmatrix} u - \frac{(1 - \lambda_1^2)\nu}{\lambda_1 \lambda_2} v^2 \\ v \end{pmatrix}. \]

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**REFERENCES**

1. H. W. E. Jung, J. Reine Angew. Math. 184, 161 (1942).
2. A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, in *Prog. in Math.* (Birkhäuser, Basel, 2000), Vol. 190.
3. I. P. Shestakov and U. U. Umirbaev, J. Amer. Math. Soc. 17 (1), 197 (2004).
4. M. Nagata, *On Automorphism Group of k[x, y]*, in *Lect. in Math.* (Kyoto University, Tokyo, 1972), Vol. 5.
5. I. V. Arzhantsev and S. A. Gaifullin, Sb. Math. 201 (1), 1 (2010).
6. A. Trushin, J. Algebra Appl. 21 (8) (2022).