All-versus-nothing proofs with \( n \) qubits distributed between \( m \) parties

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All-versus-nothing (AVN) proofs show the conflict between Einstein, Podolsky, and Rosen’s elements of reality and the perfect correlations of some quantum states. Given an \( n \)-qubit state distributed between \( m \) parties, we provide a method with which to decide whether this distribution allows an \( m \)-partite AVN proof specific for this state using only single-qubit measurements. We apply this method to some recently obtained \( n \)-qubit \( m \)-particle states. In addition, we provide all inequivalent AVN proofs with less than nine qubits and a minimum number of parties.

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I. INTRODUCTION

Einstein, Podolsky, and Rosen (EPR) showed that quantum mechanics is incomplete, in the sense that not every element of reality has a counterpart inside the theory \[1\]. EPR proposed the following criterion to identify an element of reality: “If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity” \[2\]. In practice, nondisturbance can be guaranteed when the measurements are performed on distant systems. Predictions with certainty are possible for states having perfect correlations. A quantum state \( \rho \) has \( p \) perfect correlations when there are \( p \) different observables \( O_i \) such that \( \langle O_i \rangle_\rho = 1 \).

Thirty years after EPR’s paper, Bell proved that there is an irresoluble conflict between EPR’s elements of reality and quantum mechanics \[2\]. All-versus-nothing (AVN) proofs are the most direct way to reveal this conflict. An AVN proof is based on a set of \( s \) perfect correlations of a specific quantum state. The name “all-versus-nothing” \[3\] reflects one particular feature of these proofs: If one assumes EPR elements of reality, then \( s - q \) of these perfect correlations lead to a conclusion that is the opposite of the one obtained from a subset of the other \( q \) perfect correlations. If all \( s \) correlations are essential to obtain a contradiction (i.e., if the contradiction vanishes when we remove one of them), then the AVN proof is said to be critical.

The first AVN proof was obtained by Heywood and Redhead \[4\]. However, the most famous AVN proof is Greenberger, Horne, and Zeilinger’s (GHZ) \[5–7\]. The first bipartite AVN proof with qubits is in Refs. \[8, 9\]. The first bipartite AVN proof with qubits and using only single-qubit measurements is in Refs. \[10, 11\]. The interest of the case in which the parties are restricted to perform single-qubit measurements is motivated by the practical difficulty of making general \( N \)-qubit measure-
to the same party to new parties will also allow an AVN proof. As supplementary material, we provide all inequivalent distributions between a minimum number \( m \) of parties allowing specific \( m \)-partite AVN proofs for all \( n \)-qubit graph states of \( n \leq 8 \) qubits.\(^1\)

II. AVN PROOFS AND GRAPH STATES

A. AVN proofs and stabilizer states

An AVN proof requires an \( n \)-qubit quantum state distributed between \( m \) parties. This state has a set of perfect correlations between the results of single-qubit measurements. These correlations must satisfy two requirements. First, they must allow us to define \( m \)-partite EPR’s elements of reality. This means that every single-qubit observable involved in the AVN proof must satisfy EPR’s criterion of elements of reality (i.e., its value can be predicted with certainty using only the results of single-qubit measurements on distant particles). Second, they must lead to a contradiction when EPR’s criterion of elements of reality is assumed. Therefore, the conclusion of an AVN proof is that if the quantum predictions are correct, observables which satisfy EPR’s condition cannot have predefined results since it is impossible to assign them values which simultaneously satisfy the perfect correlations predicted by quantum mechanics.

Perfect correlations are necessary to establish elements of reality and to prove that they are incompatible with quantum mechanics. Therefore the states which allow AVN proofs must be simultaneous eigenstates of a sufficient number of commuting \( n \)-fold tensor products of single-qubit operators. Indeed, the following observations lead us to the conclusion that without loss of generality, we can restrict our attention to a particular family of states.

Two different single-qubit operators \( A \) and \( B \) on the same qubit cannot commute. A necessary condition to make \( n \)-fold tensor products be commuting operators is to choose \( A \) and \( B \) to be anticommuting operators. Therefore, in an AVN proof, all single-qubit operators corresponding to the same qubit must be anticommuting operators. The maximum number of anticommuting single-qubit operators is three. Therefore, without loss of generality, we can restrict our attention to a specific set of three single-qubit anticommuting operators on each qubit, for example the Pauli matrices \( X = \sigma_x \), \( Y = \sigma_y \), and \( Z = \sigma_z \). This leads us to the concept of stabilizer state. An \( n \)-qubit stabilizer state is the simultaneous eigenstate with eigenvalue 1 of a set of \( n \) independent commuting elements of the Pauli group (i.e., the group, under matrix multiplication, of all \( n \)-fold tensor products of \( X, Y, Z \) and the identity \( \mathbb{1} \)). The \( n \) independent elements are called stabilizer generators and generate a maximally Abelian subgroup, the stabilizer group of the state \([19]\). The \( 2^n \) elements of the stabilizer group are the stabilizing operators and provide all the perfect correlations of the stabilizer state.

A further simplification is possible since any stabilizer state is local Clifford equivalent (i.e., equivalent under the local unitary operations that map the Pauli group to itself under conjugation) to a graph state \([20]\). Therefore the problem of which \( n \)-qubit pure states and distributions of qubits between the parties allow \( m \)-partite AVN proofs is reduced to the problem of which \( n \)-qubit graph states and distributions allow \( m \)-partite AVN proofs.

B. Graph states

A graph state \([21]\) is a stabilizer state whose generators can be written with the help of a graph. \(|G\rangle\) is the \( n \)-qubit state associated with the graph \( G \), which gives a recipe both for preparing \(|G\rangle\) and for obtaining \( n \)-qubit generators that uniquely determine \(|G\rangle\). On one hand, \( G \) is a set of \( n \) vertices (each representing a qubit) connected by edges (each representing an Ising interaction between the connected qubits). On the other hand, the stabilizer generator \( g_i \) is obtained by looking at the vertex \( i \) of \( G \) and the set \( N(i) \) of vertices which are connected to \( i \) and is defined by

\[
g_i = X_i \otimes_{j \in N(i)} Z_j,
\]

where \( X_i, Y_i, \) and \( Z_i \) denote the Pauli matrices acting on the \( i \)th qubit. \(|G\rangle\) is the only \( n \)-qubit state that fulfills

\[
g_i|G\rangle = |G\rangle \quad \text{for} \quad i = 1, \ldots, n.
\]

Therefore the stabilizer group is

\[
S(|G\rangle) = \{s_j, j = 1, \ldots, 2^n \}, \quad s_j = \prod_{i \in I_j(G)} g_i,
\]

where \( I_j(G) \) denotes a subset of \( \{g_i\}_{i=1}^N \). The stabilizing operators provide all the perfect correlations of \(|G\rangle\):

\[
\langle G|s_j|G\rangle = 1.
\]

Graph states associated with connected graphs have been exhaustively classified. There is only 1 two-qubit graph state (equivalent to a Bell state), only 1 three-qubit graph state (equivalent to a GHZ state), and 2 four-qubit graph states (equivalent to a GHZ and a cluster state), 4 five-qubit graph states, 11 six-qubit graph states, 26 seven-qubit graph states \([21]\), and 101 eight-qubit graph states \([22]\).

\(^1\) See EPAPS Document No. ... for all inequivalent distributions between a minimum number \( m \) of parties allowing specific \( m \)-partite AVN proofs for all \( n \)-qubit graph state of \( n \leq 8 \) qubits.
III. \( n \)-QUBIT \( m \)-PARTITE AVN PROOFS

A. Specific \( m \)-partite AVN proofs

The perfect correlations of any graph state associated with a connected graph of three or more vertices lead to contradictions with the concept of elements of reality when each qubit is distributed to a different party [17, 23, 25]. However, the first problem consists of finding whether these contradictions are specific to a given distribution of a graph state or, on the contrary, they can be obtained with a graph state of fewer qubits.

For example, take the four-party AVN proof based on the following four perfect correlations of the distribution of the four-qubit fully connected graph state \( |\text{FC}_4\rangle \) (a four-qubit GHZ state) in which each qubit belongs to a different party:

\[
\begin{align*}
X_1Z_2Z_3Z_4 &= 1, \quad (5a) \\
Z_1X_2Z_3Z_4 &= 1, \quad (5b) \\
Z_1Z_2X_3Z_4 &= 1, \quad (5c) \\
-X_1X_2X_3Z_4 &= 1. \quad (5d)
\end{align*}
\]

This is an example of an AVN proof which is nonspecific for the state \( |\text{FC}_4\rangle \), the reason being that neither the contradiction nor the definition of the elements of reality involved in this contradiction requires any choice from the party which has the fourth qubit. This party only involves in this contradiction requires any choice from a different party, then the AVN proof is not specific to a different party, which is connected to qubit 3.

For example, consider the four-party AVN proof based on the following four perfect correlations of the distribution of the four-qubit fully connected graph state \( |\text{FC}_4\rangle \) (a four-qubit GHZ state) distributed between three parties. Consider the AVN proof based on the following four correlations of the four-qubit linear cluster state \( |\text{LC}_4\rangle \) associated with the graph where qubit 1 is connected to qubit 2, which is connected to qubit 3, which is connected to qubit 4:

\[
\begin{align*}
X_1Z_2Z_3 &= Z_1X_2Z_3 = Z_1Z_2X_3 = -X_1X_2X_3. \quad (6)
\end{align*}
\]

The particular value of \( Z_4 \) is irrelevant. The same contradiction can be obtained using the perfect correlations of a three-qubit fully connected graph state \( |\text{FC}_3\rangle \) (a three-qubit GHZ state) distributed between three parties.

The next example illustrates that whether an AVN proof is specific can depend on the way in which the qubits are distributed between the parties. Consider the AVN proof based on the following four correlations of the four-qubit linear cluster state \( |\text{LC}_4\rangle \) associated with the graph where qubit 1 is connected to qubit 2, which is connected to qubit 3, which is connected to qubit 4:

\[
\begin{align*}
Y_1Y_2Z_3 &= 1, \quad (7a) \\
Z_1X_2Z_3 &= 1, \quad (7b) \\
Z_1Y_2Y_3Z_4 &= 1, \quad (7c) \\
-Y_1X_2Y_3Z_4 &= 1. \quad (7d)
\end{align*}
\]

If the qubits are distributed so that each qubit goes to a different party, then the AVN proof is not specific since the party who has the fourth qubit does not need to make any choice, neither for the contradiction nor for the definition of the elements of reality. The contradiction
\[
Y_1Y_2Z_3 = Z_1X_2Z_3 = Z_1Y_2Y_3 = -Y_1X_2Y_3 \quad (8)
\]
can be obtained from the perfect correlations of a three-qubit linear cluster state \( |\text{LC}_3\rangle \) associated with the graph where qubit 1 is connected to qubit 2, which is connected to qubit 3.

Therefore the party who has qubit 4 must choose between at least two measurements. To sum up, an AVN proof is specific for a given distribution of a graph state when at least two observables of all the qubits are involved.

Since the additional correlations needed to define the elements of reality can (together with those already used for the contradiction) involve additional contradictions, it is appropriate that the observables needed to guarantee that other observables are elements of reality (like \( X_4 \) and \( Z_4 \) in the previous example) are themselves elements of reality. Therefore hereinafter we will focus on AVN proofs in which at least two of the observables of all the qubits are elements of reality. It can be easily seen that when two Pauli observables, for example, \( X_i \) and \( Y_i \), are elements of reality, then the third Pauli observable, \( Z_i \), is also an element of reality. Therefore we shall focus only on those graph states and distributions in which the three Pauli observables of each and every one of the qubits are elements of reality.

B. When does a distribution allow a specific AVN proof?

The next problem is, given a distribution of an \( n \)-qubit graph state between \( m \) parties, how to decide whether it is one in which all single-qubit Pauli observables are elements of reality. For that purpose, it is useful to note that the \( 2^n \) perfect correlations (i.e., stabilizing operators) of an \( n \)-qubit graph state can be classified in four classes:

1. There are \( 2^{n-2} \) stabilizing operators (i.e., a quarter of the stabilizing operators of the graph state) that allow us to predict \( X_i \) from the results of measurements on other qubits: those that are products of the stabilizer generator \( g_j \) (defined in Eq. (1)), an even number (hereinafter “even” includes zero) of \( g_j \) with \( j \in N(i) \), and an arbitrary number (hereinafter “arbitrary number” includes zero) of \( g_k \) with \( k \neq i \) and \( k \notin N(i) \).

2. There are \( 2^{n-2} \) stabilizing operators that allow us to predict \( Y_i \) from the results of measurements on other
qubits: those that are products of $g_i$, an odd number of $g_j$ with $j \in N(i)$, and an arbitrary number of $g_k$ with $k \neq i$ and $k \notin N(i)$.

3. There are $2^{n-2}$ stabilizing operators that allow us to predict $Z_i$ from the results of measurements on other qubits: those that are products of an odd number of $g_j$ with $j \in N(i)$ and an arbitrary number of $g_k$ with $k \neq i$ and $k \notin N(i)$.

4. There are $2^{n-2}$ stabilizing operators that contain $I_i$: those that are products of an even number of $g_j$ with $j \in N(i)$ and an arbitrary number of $g_k$ with $k \neq i$ and $k \notin N(i)$.

Each particle can carry more than one qubit. It is therefore convenient to denote as $P(i)$ the set of qubits which are in the same particle as qubit $i$. The previous classification of the stabilizing operators is useful in the following sense: Given the distribution of an $n$-qubit graph state between $m$ parties, $X_i$ is an element of reality if and only if there exists a stabilizing operator of the graph state which satisfies the following two requirements: (1) It does not contain $g_j$ for all $j \in P(i)$ but contains an even number of $g_k$ with $k \in N(j)$ and (2) it contains $g_i$ and an even number of $g_l$ with $l \in N(i)$.

For instance, consider the four-qubit linear cluster state $|LC_4\rangle$ associated with the graph where qubit 1 is connected to qubit 2, which is connected to qubit 3, which is connected to qubit 4, distributed such that Alice has qubits 1 and 4 and Bob has qubits 2 and 3. The question is, is $X_1$ an element of reality? This is equivalent to the question, is there a stabilizing operator such that it does not contain $g_4$ [since $P(1) = \{4\}$] but contains an even number (necessarily zero) of $g_3$ [since $N(4) = \{3\}$] and $g_1$ and an even number (necessarily zero) of $g_2$ [since $N(1) = \{2\}$]? The answer is yes; the only stabilizing operator with these properties is $g_1 = X_1Z_2$.

Similarly, $Y_1$ is an element of reality if and only if there is a stabilizing operator satisfying (1) and the following condition: (3) It contains $g_1$ and an odd number of $g_l$ with $l \in N(i)$.

Finally, $Z_1$ is an element of reality if and only if there is a stabilizing operator satisfying (1) and the following condition: (4) It does not contain $g_i$ but contains an odd number of $g_l$ with $l \in N(i)$.

To decide whether a specific distribution allows a specific AVN proof, we first focus on qubit $i$ and test whether $X_i$ and $Y_i$ are elements of reality. If either is not an element of reality, then the distribution does not allow a specific AVN proof. If both are elements of reality, then we test whether $X_j$ and $Y_j$ of qubit $j$ are elements of reality, and so on for all the qubits. If all $X_i$ and $Y_i$ are elements of reality, then the distribution allows a specific AVN proof.

Indeed, there are simple cases where it can easily be seen that a distribution does not allow an AVN proof. For example, if more than $n/2$ qubits are carried by the same particle, for qubits of the particle with more than $n/2$ qubits, either requirement (1) is incompatible with (2), or (1) is incompatible with (3) and (4). An alternative proof will be provided in Sec. IV. If there is a qubit $i$ such that $N(i) \notin P(i)$ (i.e., if in the graph representing the state, qubit $i$ is connected only to qubits of the same particle), requirement (1) is incompatible with requirements (3) and (4). As supplementary material (see Footnote 1), we provide a computer program to decide whether a given $n$-qubit $m$-particle graph state allows a specific $m$-partite AVN proof.

C. Examples

As an example of the application of these rules, it is interesting to discuss whether some recently prepared 6-qubit two- and four-particle states allow specific AVN proofs, assuming the natural scenario in which each party has one particle.

Figure II contains several possible distributions of a six-qubit linear cluster state $|LC_6\rangle$. Figure II(a) represents the four-photon $|LC_6\rangle$ prepared in Ref. [15]. This distribution does not allow a specific AVN proof since qubit 1 is connected only to qubit 2 and qubit 6 is connected only to qubit 5.

Figure II(b) represents the two-photon $|LC_6\rangle$ prepared in Ref. [14]. This distribution satisfies all the requirements thus allows a specific AVN proof. Indeed, Fig. II(b) represents the only bipartite distribution of the six-qubit linear cluster state which allows a specific AVN proof [17]. Some distributions of $|LC_6\rangle$ in four particles allowing AVN proofs can be trivially obtained from Fig. II(b) by splitting qubits that belong to the same particle into several particles. For instance, a distribution...
FIG. 2. Different distributions of the six-qubit Y graph state
Between four particles. (a) corresponds to the state prepared in Ref. [13] and does not allow a specific AVN proof. (b)–(d) allow specific AVN proofs.

allowing a specific AVN proof is illustrated in Fig. 1(c). It can be easily seen that there is no distribution in four particles which allows a specific AVN proof which cannot be obtained from the distribution in Fig. 1(b).

Figure 2 contains several possible distributions of a six-qubit Y-graph state |Y\rangle. Figure 2(a) represents the four-photon |Y\rangle prepared in Ref. [13]. This distribution does not allow a specific AVN proof since qubit 1 is connected only to qubit 2 and qubit 5 is connected only to qubit 4. Figures 2(b)–(d) represent distributions of |Y\rangle between four particles allowing specific AVN proofs.

IV. AVN PROOFS WITH A MINIMUM NUMBER m OF PARTIES

A. Possible distributions between a minimum number of parties

In the previous section, we have seen that |Y\rangle admits specific AVN proofs when their qubits are suitably distributed between four particles. The question is whether |Y\rangle admits specific AVN proofs when it is distributed between three particles or less, or more generally speaking, the question is, given an n-qubit graph state, what is the minimum number of particles m which allows m-partite AVN proofs specific for this state?

The following definition will be useful for solving this problem. Let us define the reduced stabilizer of particle A’s qubits as the one obtained from the stabilizer of the original state by replacing the observables on all other particles’ qubits with identity matrices.

Definition: A distribution of n = n_{\text{max}} + n_{B} + \cdots + n_{m} qubits between m parties such that n_{\text{max}} \geq n_{B} \geq \ldots \geq n_{m} allows m-partite elements of reality if and only if n_{\text{max}} \leq n_{B} + \cdots + n_{m}.

Proof: Suppose that particle m_{i} carries qubits 1, \ldots, n_{\text{max}}, where n_{\text{max}} is the maximum number of qubits carried by any particle, and that particle m_{j} carries qubits n_{\text{max}} + 1, \ldots, n_{\text{max}} + n_{j}. If X_{1}, Y_{1}, Z_{1}, X_{2}, \ldots, Z_{\text{max}} are elements of reality, then the reduced stabilizer of m_{i}’s qubits must contain

\begin{align*}
X_{1} \otimes 1_{2} \otimes \cdots \otimes 1_{n_{\text{max}}}, \quad (10a) \\
Y_{1} \otimes 1_{2} \otimes \cdots \otimes 1_{n_{\text{max}}}, \quad (10b) \\
Z_{1} \otimes 1_{2} \otimes \cdots \otimes 1_{n_{\text{max}}}, \quad (10c) \\
1_{1} \otimes X_{2} \otimes \cdots \otimes 1_{n_{\text{max}}}, \cdots, \quad (10d) \\
1_{1} \otimes 1_{2} \otimes \cdots \otimes Z_{\text{max}}. \quad (10e)
\end{align*}

Moreover, the reduced stabilizer of m_{i}’s qubits must contain all possible products of Eqs. (10a)–(10e), that is, all possible variations with repetition of the four elements 1, X, Y, and Z, choosing n_{j}, which are 4^{n_{\text{max}}} = 2^{2n_{\text{max}}}. A similar reasoning applies to the three Pauli matrices of each and every one of m_{j}’s qubits. Therefore the reduced stabilizer of m_{i}’s qubits must also contain all possible products of

\begin{align*}
X_{n_{\text{max}} + 1} \otimes 1_{n_{\text{max}} + 2} \otimes \cdots \otimes 1_{n_{\text{max}} + n_{j}} \otimes \cdots, \quad (11a) \\
1_{n_{\text{max}} + 1} \otimes 1_{n_{\text{max}} + 2} \otimes \cdots \otimes Z_{n_{j} + n_{j}}. \quad (11b)
\end{align*}

which are 4^{n_{j}} = 2^{2n_{j}}. However, the reduced stabilizer of the sum of the parties m_{i} and m_{j} has only 2^{n_{\text{max}} + n_{j}} terms; therefore the only possibility is that n_{\text{max}} = n_{j}.

Given an n-qubit graph state, n_{\text{max}} restricts the possible minimum numbers of particles and the possible numbers of qubits per particle. Given n, Table II presents the possible minimum numbers of particles and the corresponding possible distributions. Other possible distributions are already contained between those in Table II.
TABLE I. Possible distributions of an $n$-qubit graph state between a minimum number $m$ of particles. For instance, $(2,2,1)$ denotes a distribution of $n=5$ qubits between $m=3$ particles such that particles 1 and 2 have two qubits each and particle 3 has one qubit.

| $n$ | $m$ | Distribution                  |
|-----|-----|------------------------------|
| 2   | 2   | $(1,1)$                      |
| 3   | 3   | $(1,1,1)$                    |
| 4   | 2   | $(2,2)$                      |
| 4   | 3   | $(1,1,1,1)$                  |
| 5   | 3   | $(2,2,1)$                    |
| 5   | 5   | $(1,1,1,1,1)$                |
| 6   | 2   | $(3,3)$                      |
| 3   | 3   | $(2,2,2)$                    |
| 4   | 4   | $(2,2,1,1)$                  |
| 6   | 6   | $(1,1,1,1,1,1)$              |
| 7   | 3   | $(3,3,1), (3,2,2)$           |
| 4   | 4   | $(2,2,2,1)$                  |
| 5   | 5   | $(2,2,1,1,1)$                |
| 6   | 7   | $(1,1,1,1,1,1,1)$            |
| 8   | 2   | $(4,4)$                      |
| 3   | 3   | $(3,3,2)$                    |
| 4   | 4   | $(3,3,1,1), (3,2,2,1), (2,2,2,2)$ |
| 5   | 5   | $(2,2,2,1,1)$                |
| 6   | 6   | $(2,2,1,1,1,1)$              |
| 8   | 8   | $(1,1,1,1,1,1,1,1)$          |

but in those cases the number of particles is not the minimum.

A corollary of the lemma is that there are no specific AVN proofs in which one particle has more than $n/2$ qubits (this result was used in Sec. III).

B. AVN proofs with a minimum number of parties for any graph state

Equipped with these tools, we can obtain all possible distributions with a minimum number of particles allowing specific AVN proofs for any graph state. We have obtained all which are inequivalent under single-qubit unitary operations for all graph states up to $n=8$ qubits. For this purpose, we used the classification of graph states up to $n=7$ qubits proposed in Ref. 21 and the classification of eight-qubit graph states proposed in Ref. 22. Given an $n$-qubit graph state, to obtain all the distributions between a minimum number of parties allowing specific AVN proofs we can use Table I in the following way. Suppose that $n=6$. We first test whether AVN proofs are possible for the simplest distributions permitted by Table I, that is, $m=2$ parties with two qubits each, and so on.

Applying this method, we have obtained all inequivalent distributions between a minimum number of particles for all graph states with up to eight qubits. In the supplementary material (see Footnote 2), we show all distributions between a minimum number of particles for the 19 classes of graph states with up to six qubits, the 26 classes of graph states with seven qubits, and the 101 classes of graph states with eight qubits. In addition, we provide as supplementary material (see Footnote) a computer program to obtain, given an $n$-qubit graph state, all distributions between $m$ parties and all distributions between a minimum number of parties which allow AVN proofs.

V. CONCLUSIONS

We have developed tools with which to decide whether a distribution of $n$ qubits between $m$ parties allows an $m$-partite AVN proof specific for this distribution (i.e., which cannot be obtained using a state with fewer qubits). As a result, we have obtained all inequivalent $m$-partite AVN proofs using $n$-qubit $m$-particle quantum states with $n<9$ qubits and a minimum number $m$ of parties. This enables us to obtain all inequivalent $m$-partite AVN proofs using $n$-qubit $m$-particle quantum states with $n<9$ qubits with an arbitrary number of parties.

The motivation of this work was to answer some natural questions raised by recent experimental developments allowing the preparation in the laboratory of graph states of several particles, each carrying several qubits. The results presented in this article provide tools to help experimentalists to design tests of new AVN proofs and new Bell inequalities based on these AVN proofs, similar to those reported in Refs. 12, 14, for specific states but exploiting the possibility of experimentally preparing new classes of graph states.

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Appendix A: Supplementary material

FIG. 3. Graphs associated to the 27 classes on graph states with up to six qubits inequivalent under local complementation and graph isomorphism. Figure taken from Ref. [21].

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[2] J. S. Bell, Physics (Long Island City, N.Y.) 1, 195 (1964).
[3] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
[4] P. Heywood and M. L. G. Redhead, Found. Phys. 13, 481 (1983).
[5] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in Bell’s Theorem, Quantum Theory, and Conceptions of the Universe, edited by M. Kafatos (Kluwer Academic, Dordrecht, 1989), p. 69.
[6] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. 58, 1131 (1990).
[7] N. D. Mermin, Phys. Rev. Lett. 65, 3373 (1990).
[8] A. Cabello, Phys. Rev. Lett. 86, 1911 (2001).
[9] A. Cabello, Phys. Rev. Lett. 87, 010403 (2001).
[10] A. Cabello, Phys. Rev. Lett. 95, 210401 (2005).
[11] A. Cabello, Phys. Rev. A 72, 050101(R) (2005).
[12] G. Vallone, E. Pomarico, P. Mataloni, F. De Martini, and V. Berardi, Phys. Rev. Lett. 98, 180502 (2007).
[13] J. T. Barreiro, N. K. Langford, N. A. Peters, and P. G. Kwiat, Phys. Rev. Lett. 95, 260501 (2005).
[14] R. Ceccarelli, G. Vallone, F. De Martini, P. Mataloni, and A. Cabello, Phys. Rev. Lett. 103, 160401 (2009).
[15] W.-B. Gao, X.-C. Yao, P. Xu, O. Gühne, A. Cabello, C.-Y. Lu, C.-Z. Peng, T. Yang, Z.-B. Chen, and J.-W. Pan, arXiv:0906.3390 (2009).
[16] W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, and J.-W. Pan, Nat. Phys. doi:10.1038/nphys1603 (2010).
[17] A. Cabello and P. Moreno, Phys. Rev. Lett. 99, 220402 (2007).
[18] See supplementary material at http://link.aps.org/supplemental/10.1103/PhysRevA.81.042110 for a computer program in MATHEMATICA.
[19] D. Gottesman, Phys. Rev. A 54, 1862 (1996).
[20] M. Van den Nest, J. Dehaene, and B. De Moor, Phys. Rev. A 69, 022316 (2004).
[21] M. Hein, J. Eisert, and H. J. Briegel, Phys. Rev. A 69, 062311 (2004).
[22] A. J. López Tarrida, P. Moreno and J. R. Portillo, Phys. Lett. A. 373, 2219 (2009).
[23] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Van den Nest, and H. J. Briegel, in Quantum Computers, Algorithms and Chaos, edited by G. Casati, D.L. Shepelyansky, P. Zoller, and G. Benenti (IOS Press, Amsterdam, 2006).
FIG. 4. Graphs associated to the 22 classes on seven-qubit graph states inequivalent under local complementation and graph isomorphism. Figure taken from Ref. [21].

[24] D. P. DiVincenzo and A. Peres, Phys. Rev. A 55, 4089 (1997).
[25] V. Scarani, A. Acín, E. Schenck, and M. Aspelmeyer, Phys. Rev. A 71, 042325 (2005).
[26] A. Cabello, O. Gühne, and D. Rodríguez, Phys. Rev. A 77, 062106 (2008).
TABLE II. Distributions of the qubits of a graph state (numbered as in Fig. 3) between a minimum number $m$ of particles $A, B, \ldots, F$ allowing AVN proofs specific for the graph state. The table contains all inequivalent distributions for all graph states up to six qubits.

| Graph state no. | $m$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ |
|-----------------|-----|-----|-----|-----|-----|-----|-----|
| 1               | 2   | 1   |     |     |     |     | 2   |
| 2 (GHZ$_3$)     | 3   | 1   |     |     |     |     | 2   |
| 3 (GHZ$_4$)     | 4   | 1   |     |     |     |     | 3   |
| 4 (LC$_4$)      | 2   |     | 1,4 |     | 2,3 |     |     |
| 5 (GHZ$_5$)     | 5   | 1   |     |     | 2   |     | 3   |
| 6               | 3   |     | 1   |     | 2,4 |     | 3,5 |
| 7 (LC$_5$)      | 3   |     | 1,5 |     | 2,4 |     | 3   |
| 8 (RC$_5$)      | 3   |     | 1   |     | 2,5 |     | 3,4 |
| 9 (GHZ$_6$)     | 6   | 1   |     |     | 2   |     | 3   |
| 10              | 4   |     | 1   |     | 2,4 |     | 5,6 |
| 11              | 3   |     | 1,5 |     | 2,3 |     | 4,6 |
| 12              | 3   |     | 1,5 |     | 2,4 |     | 3,6 |
| 13              | 2   |     | 1,5,6 |   | 2,3,4 |
| 14 (LC$_6$)     | 2   |     | 1,4,5 |   | 2,3,6 |
| 15              | 3   |     | 1,4 |     | 2,6 |     | 3,5 |
| 16              | 2   |     | 2,3,4 |   | 1,5,6 |
| 17              | 2   |     | 1,2,5 |   | 3,4,6 |
| 18 (RC$_6$)     | 2   |     | 1,2,4 |   | 3,5,6 |
| 19              | 2   |     | 1,2,3 |   | 4,5,6 |
TABLE III. Distributions of the qubits of a graph state (numbered as in Fig. 4) between a minimum number \( m \) of particles \( A, B, \ldots, G \) allowing AVN proofs specific for the graph state. The table contains all inequivalent distributions for all seven-qubit graph states.

| Graph state no. |  |  |  |  |  |  |  |  |  |  |
|-----------------|---|---|---|---|---|---|---|---|---|---|
| 20 (\(GHZ_7\)) | 7 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |   |   |
| 21              | 5 | 1,6 | 5,7 | 2 | 3 | 4 |   |   |   |   |
| 22              | 4 | 1,4 | 6,7 | 2,5 | 3 |   |   |   |   |   |
| 23              | 4 | 1,4 | 6,7 | 2,5 | 3 |   |   |   |   |   |
| 24              | 4 | 1,4 | 5,7 | 3,6 | 2 |   |   |   |   |   |
| 25              | 3 | 1,4,6 | 2,5,7 | 3 |   |   |   |   |   |   |
| 26              | 3 | 1,4,6 | 5,7 | 2,3 |   |   |   |   |   |   |
| 27              | 3 | 1,3,5 | 4,6,7 | 2 |   |   |   |   |   |   |
| 28              | 3 | 1,4,5 | 2,3,6 | 7 |   |   |   |   |   |   |
| 29              | 3 | 1,4,5 | 2,3 | 6,7 |   |   |   |   |   |   |
| 30 (\(LC_7\))  | 3 | 1,5,7 | 2,6 | 3,4 |   |   |   |   |   |   |
| 31              | 3 | 1,3,5 | 2,4,6 | 7 |   |   |   |   |   |   |
| 32              | 4 | 1,4 | 2,6 | 3,5 | 7 |   |   |   |   |   |
| 33              | 3 | 1,6,5 | 2,3,4 | 7 |   |   |   |   |   |   |
| 34              | 3 | 1,5,6 | 2,3 | 4,7 |   |   |   |   |   |   |
| 35              | 3 | 1,4,7 | 2,5,6 | 3 |   |   |   |   |   |   |
| 36              | 3 | 1,5,6 | 3,4 | 2,7 |   |   |   |   |   |   |
| 37              | 3 | 1,2,5 | 3,4,6 | 7 |   |   |   |   |   |   |
| 38              | 3 | 1,2,6 | 3,4 | 5,7 |   |   |   |   |   |   |
| 39              | 3 | 1,3,4 | 5,6,7 | 2 |   |   |   |   |   |   |
| 40 (\(RC_7\))  | 3 | 1,3,4 | 5,6,7 | 2 |   |   |   |   |   |   |
| 41              | 3 | 1,5,7 | 2,6 | 3,4 |   |   |   |   |   |   |
| 42              | 3 | 1,6,7 | 3,4,5 | 2 |   |   |   |   |   |   |
| 43              | 3 | 1,6,7 | 2,3,6 | 7 |   |   |   |   |   |   |
| 44              | 3 | 1,4,5 | 2,3,6 | 7 |   |   |   |   |   |   |
| 45              | 3 | 1,6,7 | 2,3,4 | 5 |   |   |   |   |   |   |
TABLE IV. Distributions of the qubits of a graph state (numbered as in Fig. [4]) between a minimum number \( m \) of particles \( A, B, \ldots, H \) allowing AVN proofs specific for the graph state. The table contains all inequivalent distributions for eight-qubit graph states nos. 46–90.

| Graph state no | \( m \) | \( A \) | \( B \) | \( C \) | \( D \) | \( E \) | \( F \) | \( G \) | \( H \) |
|----------------|------|-----|-----|-----|-----|-----|-----|-----|-----|
| 46 (GHZ\(_8\)) | 8    | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
| 47             | 6    | 1.6 | 2.7 | 3   | 4   | 5   | 6   | 7   | 8   |
| 48             | 5    | 1.5 | 2.6 | 3.7 | 4   | 8   |     |     |     |
| 49             | 5    | 1.7 | 3.5 | 4.6 | 2   | 8   |     |     |     |
| 50             | 4    | 1.6 | 2.7 | 3.5 | 4.8 |     |     |     |     |
| 51             | 4    | 1.7 | 2.5 | 3.4 | 6.8 |     |     |     |     |
| 52             | 4    | 1.5 | 2.4 | 3.6 | 7.8 |     |     |     |     |
|                | 4    | 1.5,6 | 3.4,7 | 2   | 8   |     |     |     |     |
|                | 4    | 1.5,7 | 3.6 | 4.8 | 2   |     |     |     |     |
| 53             | 4    | 1.5 | 2.6 | 3.4 | 7.8 |     |     |     |     |
|                | 4    | 1.4,6 | 2.5 | 3.7 | 8   |     |     |     |     |
|                | 4    | 1.4,6 | 5.7,8 | 2   | 3   |     |     |     |     |
| 54             | 4    | 1.7 | 2.6 | 3.5 | 4.8 |     |     |     |     |
|                | 4    | 1.4,7 | 2.6 | 3.5 | 8   |     |     |     |     |
|                | 4    | 1.5,6 | 4.7,8 | 2   | 3   |     |     |     |     |
| 55             | 3    | 1.4,6 | 2.3,5 | 7.8 |     |     |     |     |     |
| 56             | 3    | 1.4,6 | 2.3,7 | 5.8 |     |     |     |     |     |
| 57             | 3    | 1.3,4 | 2.5,7 | 6.8 |     |     |     |     |     |
| 58             | 3    | 1.3,6 | 2.4,7 | 5.8 |     |     |     |     |     |
| 59             | 3    | 1.3,6 | 4.7,8 | 2.5 |     |     |     |     |     |
| 60             | 3    | 1.5,7 | 2.3,6 | 4.8 |     |     |     |     |     |
| 61             | 3    | 1.6,4 | 2.3,5 | 7.8 |     |     |     |     |     |
| 62             | 3    | 1.5,7 | 2.3,6 | 8.4 |     |     |     |     |     |
| 63             | 3    | 1.2,5 | 3.6,8 | 4.7 |     |     |     |     |     |
| 64             | 2    | 1.2,4,7 | 3.5,6,8 |     |     |     |     |     |     |
| 65             | 2    | 1.2,4,7 | 3.5,6,8 |     |     |     |     |     |     |
| 66             | 2    | 1.2,4,7 | 3.5,6,8 |     |     |     |     |     |     |
| 67             | 2    | 1.3,5,7 | 2.4,6,8 |     |     |     |     |     |     |
| 68 (LC\(_8\)) | 2    | 1.4,5,8 | 2.3,6,7 |     |     |     |     |     |     |
| 69             | 4    | 1.5 | 2.6 | 3.7 | 4.8 |     |     |     |     |
| 70             | 4    | 1.4 | 2.7 | 3.6 | 5.8 |     |     |     |     |
| 71             | 4    | 1.5,6 | 2.4 | 8.7 | 3   |     |     |     |     |
|                | 4    | 1.5,6 | 4.7,8 | 3   | 2   |     |     |     |     |
|                | 4    | 1.5 | 2.4 | 3.6 | 7.8 |     |     |     |     |
| 72             | 4    | 1.5,7 | 2.6 | 4.8 | 3   |     |     |     |     |
|                | 4    | 1.4,6 | 5.7,8 | 2   | 3   |     |     |     |     |
|                | 4    | 1.5 | 2.6 | 3.7 | 4.8 |     |     |     |     |
| 73             | 3    | 1.4,5 | 2.3,6 | 7.8 |     |     |     |     |     |
| 74             | 3    | 1.5,7 | 3.6,8 | 2.4 |     |     |     |     |     |
| 75             | 3    | 1.5,7 | 2.4,6 | 3.8 |     |     |     |     |     |
| 76             | 3    | 1.2,4 | 3.6,8 | 5.7 |     |     |     |     |     |
| 77             | 3    | 1.4,6 | 2.3,5 | 7.8 |     |     |     |     |     |
| 78             | 3    | 1.6,7 | 2.5,4 | 3.8 |     |     |     |     |     |
| 79             | 3    | 1.3,6 | 2.4,7 | 5.8 |     |     |     |     |     |
| 80             | 3    | 1.3,6 | 2.5,7 | 4.8 |     |     |     |     |     |
| 81             | 3    | 1.4,5 | 2.3,6 | 7.8 |     |     |     |     |     |
| 82             | 3    | 1.3,7 | 2.5,8 | 4.6 |     |     |     |     |     |
| 83             | 3    | 1.4,6 | 2.5,7 | 3.8 |     |     |     |     |     |
| 84             | 3    | 1.3,5 | 4.6,8 | 2.7 |     |     |     |     |     |
| 85             | 3    | 1.5,7 | 3.4,8 | 2.6 |     |     |     |     |     |
| 86             | 2    | 1.3,6,7 | 2.4,5,8 |     |     |     |     |     |     |
| 87             | 2    | 1.4,6,7 | 2.3,5,8 |     |     |     |     |     |     |
| 88             | 2    | 1.2,4,7 | 3.5,6,8 |     |     |     |     |     |     |
TABLE V. Distributions of the qubits of a graph state (numbered as in Fig. 4) between a minimum number \( m \) of particles \( A, B, \ldots, H \) allowing AVN proofs specific for the graph state. The table contains all inequivalent distributions for eight-qubit graph states nos. 91–146.

| Graph state no | \( m \) | \( A \) | \( B \) | \( C \) | \( D \) | \( E \) | \( F \) | \( G \) | \( H \) |
|----------------|------|------|------|------|------|------|------|------|------|
| 91             | 2    | 1,3,6,7 | 2,4,5,8 |      |      |      |      |      |      |
| 92             | 2    | 1,3,5,7 | 2,4,6,8 |      |      |      |      |      |      |
| 93             | 2    | 1,2,4,6 | 3,5,7,8 |      |      |      |      |      |      |
| 94             | 2    | 1,3,5,8 | 2,4,6,7 |      |      |      |      |      |      |
| 95             | 2    | 1,3,4,7 | 2,5,6,8 |      |      |      |      |      |      |
| 96             | 2    | 1,3,5,7 | 2,4,6,8 |      |      |      |      |      |      |
| 97             | 2    | 1,3,5,7 | 2,4,6,8 |      |      |      |      |      |      |
| 98             | 2    | 1,3,5,6 | 2,4,7,8 |      |      |      |      |      |      |
| 99             | 2    | 1,4,5,8 | 2,3,6,7 |      |      |      |      |      |      |
| 100 (\( RC_8 \)) | 2    | 1,3,6,8 | 2,4,5,7 |      |      |      |      |      |      |
| 101            | 3    | 1,6,8  | 2,3,5 | 4,7  |      |      |      |      |      |
| 102            | 3    | 1,3,7  | 2,4,5 | 6,8  |      |      |      |      |      |
| 103            | 3    | 1,7,8  | 2,3,5 | 4,6  |      |      |      |      |      |
| 104            | 3    | 1,3,7  | 5,6,8 | 2,4  |      |      |      |      |      |
| 105            | 3    | 1,3,6  | 2,4,5 | 7,8  |      |      |      |      |      |
| 106            | 2    | 1,3,4,6 | 2,5,7,8 |      |      |      |      |      |      |
| 107            | 2    | 1,4,6,7 | 2,3,5,8 |      |      |      |      |      |      |
| 108            | 2    | 1,4,6,8 | 2,3,5,7 |      |      |      |      |      |      |
| 109            | 2    | 1,2,3,6 | 4,5,7,8 |      |      |      |      |      |      |
| 110            | 2    | 1,2,5,6 | 3,4,7,8 |      |      |      |      |      |      |
| 111            | 2    | 1,2,5,7 | 3,4,6,8 |      |      |      |      |      |      |
| 112            | 2    | 1,3,4,6 | 2,5,7,8 |      |      |      |      |      |      |
| 113            | 2    | 1,4,5,6 | 2,3,7,8 |      |      |      |      |      |      |
| 114            | 2    | 1,3,5,6 | 2,4,7,8 |      |      |      |      |      |      |
| 115            | 2    | 1,3,4,6 | 2,5,7,8 |      |      |      |      |      |      |
| 116            | 2    | 1,2,3,5 | 4,6,7,8 |      |      |      |      |      |      |
| 117            | 2    | 1,4,5,7 | 2,3,6,8 |      |      |      |      |      |      |
| 118            | 2    | 1,5,8,6 | 2,3,4,7 |      |      |      |      |      |      |
| 119            | 2    | 1,3,6,8 | 2,4,5,7 |      |      |      |      |      |      |
| 120            | 2    | 1,2,4,8 | 3,5,6,7 |      |      |      |      |      |      |
| 121            | 3    | 1,4,5  | 2,7,8 | 3,6  |      |      |      |      |      |
| 122            | 3    | 1,5,7  | 2,7,8 | 3,4  |      |      |      |      |      |
| 123            | 2    | 1,3,5,6 | 2,4,7,8 |      |      |      |      |      |      |
| 124            | 2    | 1,4,6,7 | 2,3,5,8 |      |      |      |      |      |      |
| 125            | 2    | 1,4,5,7 | 2,3,6,8 |      |      |      |      |      |      |
| 126            | 2    | 1,2,4,6 | 3,5,7,8 |      |      |      |      |      |      |
| 127            | 2    | 1,3,4,6 | 2,5,7,8 |      |      |      |      |      |      |
| 128            | 2    | 1,3,6,7 | 2,4,5,8 |      |      |      |      |      |      |
| 129            | 2    | 1,3,5,6 | 2,4,7,8 |      |      |      |      |      |      |
| 130            | 2    | 1,2,5,8 | 3,4,6,8 |      |      |      |      |      |      |
| 131            | 2    | 1,3,6,7 | 2,4,5,8 |      |      |      |      |      |      |
| 132            | 2    | 1,4,5,6 | 2,3,7,8 |      |      |      |      |      |      |
| 133            | 2    | 1,6,7,8 | 2,3,4,5 |      |      |      |      |      |      |
| 134            | 3    | 1,4,6  | 3,7,8 | 2,5  |      |      |      |      |      |
| 135            | 2    | 1,3,4,7 | 2,5,6,8 |      |      |      |      |      |      |
| 136            | 2    | 1,3,4,5 | 2,6,7,8 |      |      |      |      |      |      |
| 137            | 2    | 1,2,3,4 | 5,6,7,8 |      |      |      |      |      |      |
| 138            | 2    | 1,3,4,6 | 2,5,7,8 |      |      |      |      |      |      |
| 139            | 2    | 1,6,7,8 | 2,3,4,5 |      |      |      |      |      |      |
| 140            | 2    | 1,4,5,6 | 2,3,7,8 |      |      |      |      |      |      |
| 141            | 2    | 1,6,7,8 | 2,3,4,5 |      |      |      |      |      |      |
| 142            | 2    | 1,3,7,8 | 2,4,5,6 |      |      |      |      |      |      |
| 143            | 2    | 1,3,4,7 | 2,5,6,8 |      |      |      |      |      |      |
FIG. 5. Graphs associated to the 101 classes on eight-qubit graph states inequivalent under local complementation and graph isomorphism. Figure taken from Ref. [22].