Perturbative renormalisation for not-quite-connected bialgebras

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Abstract

We observe that the Connes–Kreimer Hopf-algebraic approach to perturbative renormalisation works not just for Hopf algebras but more generally for filtered bialgebras $B$ with the property that $B_0$ is spanned by group-like elements (e.g. pointed bialgebras with the coradical filtration). Such bialgebras occur naturally both in Quantum Field Theory, where they have some attractive features, and elsewhere in Combinatorics, where they cover a comprehensive class of incidence bialgebras. In particular, the setting allows us to interpret Möbius inversion as an instance of renormalisation.

1 Introduction

Kreimer [12] made the crucial discovery that the combinatorics underlying the BPHZ renormalisation scheme in perturbative quantum field theory can be encoded in a Hopf algebra, and his seminal joint work with Connes [3], [4] highlighted the significance of this through deep connections to many areas of mathematics, constituting a starting point for numerous further developments.

Subsequent work by Ebrahimi-Fard, Guo, Manchon [6] and others provided a more algebraic formulation of the Connes–Kreimer approach, expressing it abstractly in the setting of a connected graded Hopf algebra $H$ and a Rota–Baxter algebra $A$, as briefly recalled in Section 2 below.

The present note makes the observation that the same construction works when the Hopf algebra is replaced by a filtered bialgebra $B$ with the property that $B_0$ is spanned by group-like elements. Interest in this observation resides in the fact that there are natural examples in perturbative QFT where the bialgebra $B$ contains interesting combinatorics of physical relevance, invisible in the quotient Hopf algebra $H$. In Combinatorics, the notion covers incidence bialgebras, and we show that in this setting Möbius inversion becomes a special instance of renormalisation.

In Section 2 we quickly run through the Hopf case, to set up notation, and to facilitate the generalisation to bialgebras. This generalisation comes in two versions. In Section 3 we take the simplest approach, requiring only that $B_0$ is spanned by group-like elements $x$, but assuming that the Feynman rules $\phi : B \to A$ satisfy $\phi(x) = 1$. However, this assumption on the Feynman rules is strictly speaking not
realistic physically. In Section 4 we give a second version, where $B$ is assumed to be a polynomial algebra generated by elements whose comultiplication has group-like components in degree 0. (See [4,1] for the precise condition.) This condition, which is essentially satisfied automatically for bialgebras defined in terms of combinatorial data, ensures the existence of a residue operator, which allows to calibrate the Feynman rules, now allowed to take invertible values on group-like elements, so as to reduce to the previous case, $\phi(x) = 1$. In Section 5 the relevance of the bialgebra generalisation is substantiated through interpretation in perturbative QFT, and with some related examples from Combinatorics. Finally in Section 6 we establish the renormalisation principle for coalgebras, show that incidence coalgebras of Möbius categories (and more generally, of Möbius decomposition spaces) constitute examples, and show that Möbius inversion is a special case of renormalisation.

2 Hopf algebra renormalisation

2.1 Connes–Kreimer Hopf-algebraic renormalisation [4]. (Convenient self-contained accounts are given in [16], generous with mathematical preliminaries, and in [7], emphasising physical background and perspectives.) Let $H$ be a connected graded Hopf algebra, let $(A, R)$ be a Rota–Baxter algebra, with idempotent Rota–Baxter operator $R$ of weight 1. Put $A_+ := \text{Ker}(R)$ (a unital subalgebra of $A$) and $A_- := \text{Im}(R)$ (a non-unital subalgebra of $A$); we have $A = A_- \oplus A_+$.

The space $\text{Lin}(H, A)$ of linear maps is a monoid under the convolution product, with $e := \eta_A \circ \varepsilon$ as neutral element. Linear maps $\phi : H \to A$ with $\phi(1) = 1$ are referred to as (regularised) Feynman rules; these form a group. The Feynman rules that are furthermore algebra homomorphisms form a subgroup, the group of $A$-valued characters of $H$.

2.2 Proposition. ([4]) For each Feynman rule $\phi$, denote by $\phi_-$ the linear map defined recursively by

$$\phi_- := e + R(\phi_- \ast (e - \phi)).$$

The renormalised Feynman rule

$$\phi_+ := \phi_- \ast \phi$$

maps $H_+$ into $A_+$. Furthermore, if $\phi$ is a character, then so are $\phi_-$ and $\phi_+$.

(The equation $\phi = \phi_-^{-1} \ast \phi_+$ constitutes a Birkhoff decomposition of $\phi$, as has been observed and exploited by Connes and Kreimer; see also [5].)

For a proof of the proposition, see Manchon [16]. In the proof, the antipode of $H$ does not play any role (although other formulations may exploit it, cf. Kreimer’s interpretation of $\phi_-$ as a twisted antipode). More important is the fact that in a connected graded bialgebra, the comultiplication takes the following form (for $x \in H_+$):

$$\Delta(x) = 1 \otimes x + \sum_{\langle x \rangle} x' \otimes x'' + x \otimes 1.$$
The middle sum is restricted Sweedler notation for all the terms of the comultiplication of positive degree strictly less than the degree of $x$. This splitting makes the recursive definition of $\phi_-$ meaningful: $e$ and $\phi$ agree on $H_0$, so this case is the basis of the recursive definition, $\phi_-(1) = e(1) = 1$. For $x$ of degree $n > 0$, we have $e(x) = 0$, and hence
\[
\phi_-(x) = -R\phi(x) - R\left(\sum_{(x)} \phi_-(x')\phi(x'')\right) - 0,
\] (3)
where the middle sum involves only elements of degree strictly less than $n$, and hence are determined inductively.

The Rota–Baxter axiom is needed only for the character property: to show that $\phi_-(xy) = \phi_-(x)\phi_-(y)$ one can clearly assume that both $x$ and $y$ are of positive degree, and then exploit (2) in an inductive argument where the Rota–Baxter property turns out to be exactly what is needed.

### 2.3 Example: QFT.
In perturbative quantum field theory [1] (see Section 5 for further details), $H$ is a Hopf algebra of certain 1PI Feynman graphs (excluding graphs with no inner lines), $\phi : H \to A := \mathbb{C}[\left[t, t^{-1}\right]$ is a dimension-regularised Feynman rule, and $R$ is taking pole part, the minimal subtraction scheme. Other regularisations and renormalisation schemes fit the description too.

## 3 Bialgebra renormalisation I

We now weaken the hypotheses.

### 3.1 Hypothesis I.
Instead of a connected graded Hopf algebra, we work with a filtered bialgebra $B$ with the property that $B_0$ is spanned by group-like elements.

### 3.2 Remark on pointedness.
One can show that any bialgebra $B$ satisfying Hypothesis I is in fact pointed. Conversely, for every pointed bialgebra $B$, for the coradical filtration we have that $B_0$ is spanned by group-like elements. Hence an alternative to Hypothesis I is to work with pointed bialgebras. The condition in Hypothesis I is preferred over pointedness because in some cases of interest, the filtration may be different from the coradical filtration.

In many important cases, $B$ will actually be graded. (The more general hypothesis will be important to cover many examples from Combinatorics, cf. 6.2, 6.3 below.) Even if $B$ is only filtered, some of the main arguments exploit degree:

### 3.3 Auxiliary notion of degree.
Denote the filtration with subscripts:

\[
B_0 \subset B_1 \subset B_2 \subset \cdots \subset B.
\]

\footnote{See Sweedler [18] for the notions of pointedness and the coradical filtration (not needed in what follows).}
Put \( B(0) := B_0 \), and for each \( i \geq 0 \), choose a linear complement \( B(i + 1) \) of \( B_i \subset B_{i+1} \) so as to write \( B_{i+1} = B_i \oplus B(i + 1) \). Altogether \( B_n = \oplus_{i=0}^n B(i) \) and

\[
B = \oplus_{n=0}^\infty B(n),
\]

defining a notion of degree. We shall assume that each \( B(i + 1) \) is chosen inside \( \text{Ker} \varepsilon \); this is possible since \( 1 \oplus \text{Ker} \varepsilon = B \) and \( 1 \in B_0 \). We put \( B_+ := \oplus_{n=1}^\infty B(n) \subset \text{Ker} \varepsilon \).

With this auxiliary notion of degree, we can write the comultiplication of a homogeneous degree-\( n \) element \( x \) according to degree splitting. Denote by \( \Delta_{p,q}(x) \) the projection of \( \Delta(x) \) onto \( B(p) \otimes B(q) \), then we can write

\[
\Delta(x) = \sum_{p+q \leq n} \Delta_{p,q}(x).
\]

Note that this may involve terms of lower degree than expected, hence the summation over \( p + q \leq n \), in contrast to the graded case where only terms with \( p + q = n \) contribute.

**3.4 Lemma.** Assuming Hypothesis I, and with respect to the auxiliary degree, for \( \deg(x) = n > 0 \) we have

\[
\Delta(x) = \Delta_{0,n}(x) + \sum_{(x)} x' \otimes x'' + \Delta_{n,0}(x).
\]

The middle part (indicated with restricted Sweedler notation as in (2)) involves only \( x' \) and \( x'' \) of positive degree strictly less than \( n \).

The non-trivial statement is that no degree splittings of type \( 0 + m \) or \( m + 0 \) occur with \( m < n \). This is a consequence of counitality, together with the assumption that \( B_+ \subset \text{Ker} \varepsilon \).

**3.5 Lemma.** The ideal \( I = \langle 1 - x \mid x \text{ group-like} \rangle \subset B \) is also a (filtered) two-sided co-ideal, and the quotient bialgebra \( H := B/I \) is connected, hence Hopf.

**3.6 Lemma.** For \( A \) a unital algebra, the linear maps \( \phi \in \text{Lin}(B,A) \) such that \( \phi(x) = 1 \) for all group-like elements \( x \), form a group under convolution. The subgroup of multiplicative maps is isomorphic to the group of \( A \)-valued characters of \( H \).

The convolution inverse of such a \( \phi \) is given by the series expansion \( \phi^{-1} = \sum_{n \geq 0} (e - \phi)^n \), which is convergent for every \( x \), by induction on the grading, and because \( e \) and \( \phi \) agree on \( B_0 \).

**3.7 Proposition.** If \( \phi(x) = 1 \) for all group-like elements \( x \), then the definitions of \( \phi_- \) and \( \phi_+ \) from Proposition 2.2 make sense, and the conclusions there hold again.
Proof. The proof goes mostly as in the connected case, but using (4) instead of (2). Again, since $e$ and $\phi$ agree on $B_0$, the basis of the recursion is clear, and we have $\phi_-(x) = e(x) = 1$ for all group-like elements $x$. For the same reason, for $x$ homogeneous of degree $n > 0$, the $\Delta_{n,0}$ part of the comultiplication is zero. For the middle part, note again that since $B_+ \subset \text{Ker} \, \varepsilon$, we are left with $-R\left( \sum (x) \phi_-(x') \phi(x'') \right)$ just as in the connected case. For the $\Delta_{0,n}$ part, observe again that $e$ and $\phi_-$ agree on the left-hand tensor factor. Now $\phi_- \otimes \phi$ can be taken in two steps:

$$B \otimes B \xrightarrow{e \otimes \text{id}} k \otimes B \xrightarrow{\eta \otimes \phi} A \otimes A.$$ 

By counitality of $\Delta$, the first step yields $1 \otimes x$, and therefore the second step yields $1 \otimes \phi(x)$, which finally multiplies to $\phi(x)$, hence altogether this part gives $-R\phi(x)$, just as in the connected case. Therefore formula (3) holds true again. The proof of the character property, $\phi_-(xy) = \phi_-(x) \phi_-(y)$, follows the proof in the connected case [16], by induction on $\text{deg}(x) + \text{deg}(y)$. The only new ingredient needed is the following lemma (trivial in the connected case), which is easily proved by induction. □

3.8 Lemma. With notation as above, if $x$ is group-like then for all $y$,

$$\phi_-(xy) = \phi_-(yx) = \phi_-(y).$$

3.9 Example. For $\text{deg}(x) = 1$, we have

$$\phi_-(x) = -R(\phi(x)) \quad \text{and} \quad \phi_+(x) = \phi(x) - R(\phi(x))$$

just like in the connected case, even though $x$ cannot be assumed to be primitive.

4 Bialgebra renormalisation II

The assumption that the (regularised) Feynman rules assign value 1 to every group-like element is perhaps not realistic from the viewpoint of physics (cf. Section 5 below for discussion). We proceed to show that weaker hypotheses are possible on the Feynman rules, provided some further conditions are imposed on the bialgebra, to allow reduction to the previous case.

4.1 Hypothesis II. We assume that our filtered bialgebra $B$ is defined from combinatorial data in the following precise sense. $B$ is the free vector space on a set $C$ of homogeneous ‘combinatorial elements’, which is closed under multiplication, and also closed under comultiplication in the sense that for $x \in C$, all the terms in $\Delta(x)$ belong to $C \times C$. In this situation, the key requirement we make is that for $x \in C$ of degree $n$, we have that both $\Delta_{0,n}(x)$ and $\Delta_{n,0}(x)$ are ‘indecomposable group-like’, meaning that

$$\Delta_{0,n}(x) = \text{in}(x) \otimes x \quad \text{and} \quad \Delta_{n,0}(x) = x \otimes \text{out}(x)$$

where both $\text{in}(x)$ and $\text{out}(x)$ are group-like elements. It follows that the elements $x \in C_0 = C \cap B_0$ are precisely the group-like elements. Hence Hypothesis II implies Hypothesis I.
The terminology ‘in’ and ‘out’ is motivated mainly by Example 5.2: for \( x \) a forest in the bialgebra of operadic trees, \( \text{in}(x) \) is the set of leaves and \( \text{out}(x) \) is the set of roots. The terms \( \text{in}(x) \otimes x + x \otimes \text{out}(x) \) constitute the \text{skew-primitive} part of the comultiplication\(^2\) playing the role of the primitive part in the connected case, \( 1 \otimes x + x \otimes 1 \).

4.2 Lemma. For \( x, y \in C \), if \( xy \in C_0 \) then \( x \in C_0 \) and \( y \in C_0 \).

Proof. If we had \( \deg(x) > 0 \) then \( \varepsilon(x) = 0 \) and hence \( \varepsilon(xy) = 0 \), in contradiction with the fact that \( xy \) is group-like. □

4.3 Lemma. The assignments \( \text{in} : C \to C_0 \) and \( \text{out} : C \to C_0 \) are idempotent monoid homomorphisms.

Proof. We do the case of \( \text{out} \). We have \( \Delta_{n,0}(xy) = xy \otimes \text{out}(xy) \). On the other hand, the \( n+0 \) part of \( \Delta(x)\Delta(y) = (\cdots + x \otimes \text{out}(x))(\cdots + y \otimes \text{out}(y)) \) contains the term \( xy \otimes \text{out}(y) \text{out}(y) \). It cannot contain other \( n+0 \) terms by Lemma 4.2 □

4.4 Residue. The monoid homomorphism \( \text{out} : C \to C_0 \) extends to an algebra homomorphism \( \text{res} : B \to B_0 \).

4.5 Calibration. For \( \phi \in \text{Lin}(B, A) \), define \( \tilde{\phi} : C \to A \) by

\[
\tilde{\phi} := \frac{\phi \circ \text{res}}{\phi(\text{res}(x)) \text{ divides } \phi(x)}. \tag{5}
\]

For this to make sense, we need to assume that for all \( x \in C \),

\( \phi(\text{res}(x)) \text{ divides } \phi(x). \) \( \tag{5} \)

Extend linearly to \( \tilde{\phi} : B \to A \). Since \( \text{res} \) is a projection onto \( B_0 \), clearly \( \tilde{\phi} \) sends group-like elements to 1. Clearly it is again multiplicative if \( \phi \) is so.

Now the following is an immediate consequence of Proposition 3.7.

4.6 Proposition. If \( B \) satisfies Hypothesis II, and if \( \phi : B \to A \) satisfies \( \tag{5} \), then with notation as above, the recursive definition

\[
\phi_- := e + R(\phi_- \ast (e - \tilde{\phi})). \tag{6}
\]

is meaningful, and the renormalised Feynman rule

\[
\phi_+ := \phi_- \ast \phi
\]

maps \( B_+ \) to \( A_+ \). If \( \phi \) is character, then so are \( \phi_- \) and \( \phi_+ \).

It should be noted that the Bogoliubov counter term \( \phi_- \) depends only on the calibrated Feynman rule \( \tilde{\phi} \), not on \( \phi \) itself, but that the final renormalised Feynman rule \( \phi_+ \) does take the full information in \( \phi \) into account.

\(^2\)The notion of skew-primitive is well established in the Hopf algebra literature; see for example [2].
5 Examples

5.1 Perturbative quantum field theory. An inner line in a Feynman graph is called a bridge if removing it would increase the number of connected components. A graph is called 1PI (1-particle irreducible) if it contains no bridges. We call a connected graph a star when it contains no inner lines (hence is 1PI).

In perturbative quantum field theory [4], $H$ is a Hopf algebra of certain 1PI Feynman graphs, but excluding stars. The comultiplication is given by

$$\Delta(\Gamma) = 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma + \Gamma \otimes 1,$$

like for example

$$\Delta(\gamma) = 1 \otimes \gamma + \gamma \otimes \gamma + \gamma \otimes 1.$$

Note that the comultiplication is defined ‘by hand’ to have the form (2). In a uniform description, the natural last term would be $\Gamma \otimes \Gamma/\Gamma = \Gamma \otimes \text{res} \Gamma$, but the star res $\Gamma$ was excluded! One is led, as Manchon [16], not to exclude the stars. But then the small graphs $\gamma$ should be allowed to be stars too and in the end the comultiplication looks like this:

$$\Delta(\gamma) = \gamma \otimes \gamma + \gamma \otimes \gamma + \gamma \otimes 1.$$

It is now a bialgebra $B$ rather than a Hopf algebra, since the stars (and disjoint unions of stars) are group-like. One possible grading is by loop number, with which it is clear that $B$ satisfies Hypothesis II: for $x$ a graph, $\text{in}(x)$ is the set of vertices, and $\text{out}(x)$ is the residue of $x$. The connected quotient is the usual Hopf algebra $H$.

Mathematically this bialgebra has some pleasant features: for one thing, all the left-hand tensor factors have the same set of vertices as the original graph, and all the right-hand tensor factors have the same residue as the original graph. Furthermore, the residue of each left-hand tensor factor matches precisely the set of vertices of the right-hand tensor factor.

Physically, an attractive feature of $B$ is that it contains all the terms of the (bare) Lagrangian; these are the stars, including stars like $\gamma$ for mass terms.

The Feynman rules (as defined in text books, independently of Hopf algebra viewpoints) naturally assign non-trivial amplitudes also to stars; these cannot be seen at the level of $H$. What the Feynman rules exactly are in the Hopf algebra interpretation is more subtle than outlined in [2,3] (see Kreimer [13], §1.3): in reality the Feynman rules depend on external momenta (and possibly other physical parameters), but it is a basic feature that this dependence is the same for a graph and for its residue, up to a scalar function (the so-called form factor). Hence the divisibility assumption (5) is validated, and these parameters cancel out in $\bar{\phi}$, which then clearly sends group-like elements to 1. Kreimer writes in fact

\footnote{Manchon does actually not allow the $\gamma$ to be stars, but keeps the quotienting in the left-hand tensor factor. Note also that the $\varepsilon$ he indicates is not in fact the counit.}
§3.3: “Our Feynman rules [...] are normalized to evaluate the tree-level term to unity.” The calibration step 4.5 may be seen as a transparent formalisation of this assumption, building it into the abstract framework which makes sense also beyond the case of graphs.

The bialgebra of Feynman graphs has the prospect of formulating more aspects of renormalisation inside it than are possible inside $H$. For instance, one may wish to write down a Dyson–Schwinger equation such as

$$
\Delta\left(\begin{array}{c}
\end{array}\right) = 1 \otimes \begin{array}{c}
\end{array} \otimes \begin{array}{c}
\end{array} + 1 \otimes \begin{array}{c}
\end{array} \otimes \begin{array}{c}
\end{array} \otimes 1
$$

which, as it stands, makes sense and has a solution (the Green function) in $B$ (or rather in the completion of $B$), but not in $H$.

5.2 Trees — combinatorial versus operadic. In the usual Connes–Kreimer Hopf algebra of rooted trees [3], also called the Butcher–Connes–Kreimer Hopf algebra, the trees are combinatorial trees (such as $\begin{array}{c}
\end{array}$, $\begin{array}{c}
\end{array}$, $\begin{array}{c}
\end{array}$), and the admissible cuts used to define the comultiplication actually delete edges rather than cutting them:

$$
\Delta\left(\begin{array}{c}
\end{array}\right) = 1 \otimes \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \otimes \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \otimes 1
$$

For an operadic interpretation of Hopf algebra renormalisation (as hinted at in many papers by Kreimer and his collaborators, e.g. [4]), the natural trees to consider are operadic trees (i.e. with open-ended edges (leaves)). These form a bialgebra rather than a Hopf algebra, cf. [10]: the comultiplication is exemplified by

$$
\Delta\left(\begin{array}{c}
\end{array}\right) = \left|\begin{array}{c}
\end{array}\right| \otimes \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \otimes \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \otimes \left|\begin{array}{c}
\end{array}\right|
$$

The nodeless trees and forests are the group-like elements. The bialgebra of operadic trees [10] is easily seen to satisfy Hypothesis II: for $x$ a forest, $\text{in}(x)$ is the forest consisting of its leaves, and $\text{out}(x)$ is the forest consisting of its roots. There is a bialgebra homomorphism from operadic trees to combinatorial trees given by taking core (i.e. shaving off leaves and root [10]): this is a more drastic quotient than just collapsing group-like elements.

In analogy with the case of graphs, note that in the comultiplication formula in $B$, every left-hand tensor factor has the same leaf profile as the original tree or forest, while each of the right-hand tensor factors has the same root profile as the original tree or forest; again in each term, the root profile of the left-hand tensor factor matches the leaf profile of the right-hand tensor factor.

We mention in passing that these strict typing constraints in the comultiplication formula were found important in recent work [8] establishing a Faà di Bruno formula for the Green function in the bialgebra of operadic trees, in analogy with similar formulae found by van Suijlekom [19] in the case of graphs. The Green function is the sum of all operadic trees, weighted by symmetry factors (and it is crucial for these symmetry factors to come out right to use operadic trees rather
than combinatorial trees); it appears as solution to a certain abstract combinatorial Dyson–Schwinger equation in the category of groupoids. Further relationships with Category Theory and Logic are explored in [10] and [11].

6 Coalgebra renormalisation and Möbius inversion

In the above account of bialgebra renormalisation, the multiplication in $B$ did not play the most important role: all the results can be stated for coalgebras instead of bialgebras — except of course that the notions of multiplicativity and characters do not make sense any more. We have:

6.1 Proposition. Let $C$ be a coalgebra satisfying Hypothesis II, and assume $\phi : C \to A$ satisfies (5). Then the recursive definition

$$\phi_- := e + R(\phi_- \ast (e - \tilde{\phi}))$$

makes sense, and

$$\phi_+ := \phi_- \ast \phi$$

maps $C_+$ to $A_+$.

Note that since there is no multiplicativity to establish, it is not essential that $R$ be Rota–Baxter, it suffices to be idempotent.

6.2 Incidence coalgebras of Möbius categories. Two classical settings for incidence (co)algebras and Möbius inversion are locally finite posets (Rota et al.), and monoids with the finite decomposition property (Cartier–Foata). An elegant common generalisation is Leroux’s notion of Möbius category (for which we refer to [15] for a modern treatment). Very briefly, a Möbius category is a category $\mathcal{C}$ subject to some finiteness conditions to make the following constructions make sense. The incidence coalgebra of $\mathcal{C}$ is the vector space $C$ spanned by the arrows of $\mathcal{C}$ (the arrows are the combinatorial elements in the sense of 4.1), with coalgebra structure given by the formula

$$\Delta(f) = \sum_{b \circ a = f} a \otimes b.$$  

The sum is over all pairs of composable arrows whose composite is the arrow $f$.

(The classical settings are special cases of Möbius categories, by interpreting a poset as a category in which there is one arrow $x \to y$ whenever $x \leq y$, and by interpreting a monoid as a category with only one object (the monoid elements being then the arrows).)

A coalgebra filtration of $C$ is given by the maximum length of effective chains of arrows (i.e. not involving identity arrows) that compose to a given arrow. (The Möbius condition is equivalent to the existence of this filtration.) Clearly $C_0$ is spanned by the identity arrows, and these are group-like since the only factorisation is $id = id \circ id$. Finally, for any arrow $f : x \to y$ the trivial factorisations $f = f \circ id_x$ and $f = id_y \circ f$ constitute the only $0 + n$ and $n + 0$ splittings, so as to verify Hypothesis II. In conclusion:
6.3 Proposition. The incidence coalgebra of a Möbius category always satisfies Hypothesis II.

6.4 Remark. The following configuration of arrows in a Möbius category illustrates the need for filtering rather than bona fide grading:

Clearly deg\((f) = 3\), but \(\Delta(f)\) contains the term \(a \otimes b\) of degree splitting \(1 + 1\).

6.5 Möbius inversion. A very special case of renormalisation in the coalgebra setting is Möbius inversion. Let \(C\) be the incidence coalgebra of a Möbius category. Let \(A = k\) be the trivial Rota–Baxter algebra (the ground field with \(R\) the identity map). Take \(\phi\) to be the zeta function, \(\zeta(x) = 1\) for all combinatorial elements \(x\). Then \(\phi_+\) is the Möbius function \(\mu\) (i.e. the inverse to \(\zeta\) in the convolution algebra \(\text{Lin}(C, k)\)). Indeed, the standard formula for Möbius inversion (see [17] for the poset case)

\[
\mu(\text{id}) = 1, \quad \mu(x) = - \sum_{ab = x \atop b \neq \text{id}} \mu(a)
\]

can be written

\[
\mu = \varepsilon + \mu \ast (\varepsilon - \zeta)
\]

which is precisely (7). (From a renormalisation viewpoint, this is a very degenerate case: \(A = \{0\}\), and the ‘renormalised zeta function’ is just \(\phi_+ = \varepsilon\).)

In the incidence coalgebra situation, dividing out by the co-ideal spanned by \(1 - x\) for \(x\) group-like corresponds to considering certain reduced incidence coalgebras, but unlike in the bialgebra case this does not imply the existence of an antipode. When the antipode exists, of course Möbius inversion is nothing more than applying the antipode, but the Möbius inversion formula is more general.

6.6 Möbius decomposition spaces. A far-reaching generalisation of the notion of Möbius category was introduced recently in [9] under the name Möbius decomposition space. We shall not reproduce the definition here, but only mention three facts: (1) The above notions and results generalise readily to Möbius decomposition spaces; (2) a monoidal structure on a Möbius decomposition space makes the resulting incidence coalgebra a bialgebra; and (3) in fact the bialgebras of graphs and trees are examples of incidence bialgebras of monoidal Möbius decomposition spaces.

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