Applications of Lie systems in dissipative Milne–Pinney equations.

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Abstract
We use the geometric approach to the theory of Lie systems of differential equations in order to study dissipative Ermakov systems. We prove that there is a superposition rule for solutions of such equations. This fact enables us to express the general solution of a dissipative Milne–Pinney equation in terms of particular solutions of a system of second-order linear differential equations and a set of constants.

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1 Introduction
An instance of nonlinear equations that has been receiving an increasing interest during the last years because of its ubiquity in physics and engineering is the presently called Milne–Pinney equation [1] [2],

\[ \ddot{x} = -\omega^2(t)x + \frac{k}{x^3}, \]

where \( k \) is a real constant with values depending up on the field in which the equation is applied. Ermakov introduced this equation when looking for a first integral for the corresponding time-dependent harmonic oscillator [3]. In a very short paper, [2], Pinney showed that the general solution of (1) can be written in terms of a pair of solutions of the corresponding harmonic oscillator and two constants. More recently it has been shown that the differential equation (1) admits a superposition rule involving any two particular solutions and two constants [4].
Lately Haas [5] started studying how to find an approximate solution for the simplest damped Pinney equation, the one with a damping term linear in the velocity. Moreover other problems with a term with a quadratic dependence on the velocity have recently been studied [6, 7, 8].

The geometric theory of Lie systems [9]-[12] has been very efficient in dealing with equation (1) and the corresponding Ermakov system [3, 4, 13]. The possibility of considering a larger class of systems that can be reduced to a Lie system has been shown recently. As a particular example we can study a generalization of the Pinney equation in which the constant $k$ is replaced by a time-dependent function. So we can deal with an equation similar to (1) but including a term proportional to the velocity because such an equation can be reduced to one like (1) in which the constant $k$ is replaced by a time-dependent term. The latter suggests revisiting the theory of Lie systems from this new perspective in order to deal with such dissipative Milne–Pinney equations. This is the aim of the paper, which is organized as follows: We study in Section 2 the possibility of removing the term linear in velocity by means of a time-dependent change of coordinates while the way of doing a similar thing by means of a time-reparametrization is analyzed in Section 3. The properties of Lie systems are used in Section 4 to establish the corresponding nonlinear superposition rules for some time-dependent generalizations of Milne–Pinney equations. Finally in Section 5 the recently proposed theory of quasi-Lie systems is applied in the study of differential equations of such type.

2 Damped harmonic oscillator with time-dependent angular frequency

In this Section we show that we can remove terms proportional to velocity in a certain sort of second-order differential equation through a simple time-dependent transformation. As a main result we deal with a dissipative Milne–Pinney equation in order to remove its dissipative term in this way. Nevertheless this time-dependent change of variables is an ad-hoc method and we just explain here how it works. Later we will explain that quasi-Lie schemes explain this transformation [14]. We also obtain this result and more information about some dissipative Milne–Pinney equations through the theory of quasi-Lie systems without this ad-hoc assumption.
As a first example we consider an harmonic oscillator with a time-dependent angular frequency, \( \omega(t) \), and a term proportional to the velocity with coefficient \( \gamma(t) \), namely
\[
\ddot{x} + \gamma(t) \dot{x} + \omega^2(t) x = 0. \tag{2}
\]
Particular instances of this equation are the ones with a damping term with constant coefficient \( \gamma_0 \),
\[
\ddot{x} + \gamma_0 \dot{x} + \omega^2(t) x = 0, \tag{3}
\]
or the interesting example, found in [15], given by the choice
\[
\gamma(t) = -\frac{d}{dt} \log[f(t)],
\]
which describes a physical model with variable mass \( m(t) = [f(t)]^{-1} \).

Consider a time-dependent transformation of the variable \( x \) of the form
\[
x = \zeta(t) y \tag{4}
\]
which leads to
\[
\ddot{x} = \zeta(t) \ddot{y} + \dot{\zeta}(t) \dot{y}, \quad \ddot{\zeta} = \zeta(t) \ddot{y} + 2 \dot{\zeta}(t) \ddot{y} + \ddot{\zeta}(t) y.
\]
This time-dependent change of variables transforms the equation (2) into
\[
\ddot{y} + \left( \gamma(t) + 2 \frac{\ddot{\zeta}(t)}{\zeta(t)} \right) \dot{y} + \left( \frac{\ddot{\zeta}(t)}{\zeta(t)} + \frac{\dot{\zeta}(t)}{\zeta(t)} + \omega^2(t) \right) y = 0. \tag{5}
\]

Therefore we can eliminate the term proportional to velocity by using a function \( \zeta(t) \) such that
\[
\gamma(t) + 2 \frac{\ddot{\zeta}(t)}{\zeta(t)} = 0
\]
or, more explicitly,
\[
\zeta(t) = \zeta_0 \exp \left( -\frac{1}{2} \int_0^t \gamma(t') \, dt' \right). \tag{6}
\]
In this case equation (2) reduces to the harmonic oscillator
\[
\ddot{y} + \Omega^2(t) \, y = 0, \tag{7}
\]
with an angular frequency $\Omega(t)$ given by

$$\Omega^2(t) = \omega^2(t) - \frac{\gamma(t)^2}{4} - \frac{\dot{\gamma}(t)}{2}$$

(8)

because, in view of $\dot{\zeta}(t) = -\frac{1}{2}\gamma(t)\zeta(t)$, we obtain

$$\frac{\ddot{\zeta}(t)}{\zeta(t)} = -\frac{\dot{\gamma}(t)}{2} - \frac{\gamma(t)}{2} \left( -\frac{\gamma(t)}{2} \right) = \frac{\gamma^2(t)}{4} - \frac{\dot{\gamma}(t)}{2},$$

and then

$$\frac{\ddot{\zeta}(t)}{\zeta(t)} + \gamma(t)\frac{\dot{\zeta}(t)}{\zeta(t)} = -\frac{\gamma^2(t)}{4} - \frac{\dot{\gamma}(t)}{2}.$$  

Hence equation (2) is transformed into the time-dependent harmonic oscillator (7) with the time-dependent angular frequency $\Omega(t)$ given by (8). Note that equation (7) is the reduced canonical form of (2) as it is indicated in [16, 17].

We have found a time-dependent transformation which enables one to remove the term proportional to velocity in (2). We study some particular instances of this method. For instance, if $\gamma(t) = \gamma_0$, the transformation

$$x = e^{-\frac{\gamma_0}{2}t} y,$$

(9)

transforms the differential equation (2) into

$$\ddot{y} + \left( \omega^2(t) - \frac{\gamma_0^2}{4} \right) y = 0.$$

(10)

So we can analyze the damped system (3) through the time-dependent harmonic oscillator (7) with a time-dependent angular frequency $\Omega(t)$ given by

$$\Omega(t) = \omega^2(t) - \frac{\gamma_0^2}{4}.$$  

This latter nonautonomous second-order differential equation has been considered quite often both in the classical and in the quantum approach (see e.g. [18, 19, 20]) and we can deal with it by means of the theory of Lie systems [13, 19, 21].

Had we started with a generalized Pinney equation with a time-dependent coupling $k(t)$,

$$\ddot{x} + \gamma(t) \dot{x} + \omega^2(t) x = \frac{k(t)}{x^3},$$

(11)
we would obtain, through the same time-dependent change of variables \( (4) \), the following equation

\[
\ddot{y} + \left( \gamma(t) + 2 \frac{\dot{\zeta}(t)}{\zeta(t)} \right) \dot{y} + \left( \frac{\ddot{\zeta}(t)}{\zeta(t)} + \gamma(t) \frac{\dot{\zeta}(t)}{\zeta(t)} + \omega^2(t) \right) y = \frac{k(t)}{\zeta^3(t) y^3}.
\]

Moreover, if \( \zeta(t) \) is defined by \( (6) \), the preceding equation simplifies to the velocity-independent differential equation

\[
\ddot{y} + \Omega^2(t) \ y = \frac{k(t)}{\zeta^4(t) y^3}.
\]  

(12)

Here \( \Omega(t) \) is given by \( (8) \) and \( \zeta \) is the function \( (6) \).

For the usually called damped Milne–Pinney equation, \( \gamma(t) = \gamma_0 \), the transformation \( (9) \) reduces the differential equation \( (3) \) to

\[
\ddot{y} + \left( \omega^2(t) - \frac{\gamma_0^2}{4} \right) y = \frac{k(t) e^{2\gamma_0 t}}{y^3}.
\]

Notice that in the particular case of equation \( (11) \) with \( \gamma(t) = \gamma_0 \) and \( k(t) = k_0 e^{-2\gamma_0 t} \), we recover a Milne–Pinney equation like \( (1) \) with \( k_0 \) and \( \Omega(t) \) instead of \( k \) and \( \omega(t) \). This is exactly the example considered in \([22]\) as associated with the Caldirola–Kanai model.

In the more general case of \( \gamma(t) \) being an arbitrary function, if \( \zeta(t) \) is chosen to be such that the term proportional to the velocity vanishes, i.e. \( k(t) = k_0 \exp(-2 \int t \gamma(t') dt') \), we also recover a Milne–Pinney equation like \( (1) \) with coefficient \( k_0 \).

To sum, for a generic function \( k(t) \) we can remove the term proportional to the velocity, but the coefficient of the nonlinear term becomes time-dependent.

3 Time-reparametrization of some second-order differential equations.

In the preceding section we studied a particular time-dependent change of variables which allows us to eliminate the term proportional to velocities in a certain kind of second-order differential equation. Now we show that this can also be done by means of a time-reparametrization.
Given the second-order differential equation,
\[ \ddot{x} = f(x, \dot{x}, t), \]
the time-reparametrization given by a new parameter \( s \) such that
\[ \frac{ds}{dt} = \alpha(t), \]  
(13)
where \( \alpha(t) \) has a constant sign, for instance \( \alpha(t) \) is positive, which defines a good reparametrization allowing us to express \( s \) as a function of \( t \) and, conversely, \( t \) as a function of \( s \), leads to
\[ x' = \frac{dx}{ds} = \frac{1}{\alpha(t)} \dot{x} \iff \dot{x} = \alpha(t) x' \]
and therefore
\[ \ddot{x} = \dot{\alpha}(t) x' + \alpha^2(t) x''. \]
So we get
\[ x'' = \frac{1}{\alpha^2(t(s))} \left( f(x, \alpha(t(s)) x', t(s)) - \dot{\alpha}(t(s)) x' \right). \]

In the particular instance of a second-order differential equation of the type
\[ \ddot{x} = a(t) \dot{x} + b(t) x + \frac{c(t)}{x^3} \]
the transformed equation is
\[ x'' = \frac{1}{\alpha(t(s))} \left( a(t(s)) \dot{x} + b(t(s)) x + \frac{c(t(s))}{x^3} \right) - \frac{\dot{\alpha}(t(s))}{\alpha^2(t(s))} x', \]
i.e.
\[ x'' = \frac{1}{\alpha(t(s))} \left( a(t(s)) - \frac{\dot{\alpha}(t(s))}{\alpha(t(s))} \right) x' + \frac{b(t(s))}{\alpha^2(t(s))} x + \frac{c(t(s))}{\alpha^2(t(s)) x^3}. \]

Note that, if the function \( \alpha(t) \) is chosen to be given by
\[ \alpha(t) = \alpha_0 \exp \left( \int_0^t a(t') \, dt' \right), \]
the term containing the new velocity disappears and an equation of Milne–Pinney type is obtained. Nevertheless the coefficient of the nonlinear term is $s$-dependent

$$x'' = \frac{b(t(s))}{\alpha^2(s)} x + \frac{c(t(s))}{\alpha^2(t(s))} x^3.$$ 

So, we have obtained a reduction process from the dissipative Milne–Pinney equation into an equation of Milne–Pinney type in a new way.

4 Lie systems of second-order differential equations.

We remark that a system of second-order differential equations

$$\ddot{x}_i = f_i(x, \dot{x}, t), \quad i = 1, \ldots, n,$$

is associated with a system of first-order differential equations with a double number of variables, namely

$$\begin{cases} 
\dot{x}_i = v_i, \\
\dot{v}_i = f_i(x, v, t),
\end{cases} \quad i = 1, \ldots, n. \tag{14}$$

Even more generally the system of $n$ second-order differential equations corresponding to the system of $2n$ first-order differential equations

$$\begin{cases} 
\ddot{x}_i = \alpha(t)v_i, \\
\dot{v}_i = f_i(x, v, t),
\end{cases} \quad i = 1, \ldots, n. \tag{15}$$

is related to

$$\ddot{x}_i = \frac{\dot{\alpha}}{\alpha} \dot{x}_i + \alpha f_i(x, \dot{x}/\alpha, t), \quad i = 1, \ldots, n.$$ 

This type of equations may appear as Schrödinger equations of position-dependent mass [23] with the von Roos prescription [24].

As particular instances, the damped time-dependent harmonic oscillator described by (3) is associated with the system of first-order differential equations

$$\begin{cases} 
\dot{x} = e^{-\gamma t} v, \\
\dot{v} = -e^{\gamma t} \omega^2(t) x,
\end{cases} \tag{16}$$
and the damped generalized Milne–Pinney equation (11) is related to the system
\[
\begin{align*}
\dot{x} &= e^{-\gamma_0 t} v \\
\dot{v} &= e^{\gamma_0 t} \left( -\omega^2(t) x + \frac{k(t)}{x^3} \right).
\end{align*}
\] (17)

Recall that the standard Milne–Pinney equation (1) is linked to the following system of first-order differential equations
\[
\begin{align*}
\dot{x} &= v, \\
\dot{v} &= -\omega^2(t) x + \frac{k}{x^3}.
\end{align*}
\] (18)

Hence we can study the second-order differential equation (1) through the system (18). The solutions of (18) are integral curves for the time-dependent vector field
\[
X(t) = v \frac{\partial}{\partial x} + \left( -\omega^2(t) x + \frac{k}{x^3} \right) \frac{\partial}{\partial v},
\] (19)

which can be written as a linear combination
\[
X(t) = X_2 - \omega^2(t) X_1.
\] (20)

of the vector fields
\[
X_1 = x \frac{\partial}{\partial v}, \quad X_2 = v \frac{\partial}{\partial x} + \frac{k}{x^3} \frac{\partial}{\partial v}.
\] (21)

These vector fields close on a real Lie algebra isomorphic to \(\mathfrak{sl}(2, \mathbb{R})\) with the vector field
\[
X_3 = \frac{1}{2} \left( x \frac{\partial}{\partial x} - v \frac{\partial}{\partial v} \right),
\] (22)

because the vector fields \(X_\alpha\) satisfy the commutation relations
\[
[X_1, X_2] = 2X_3, \quad [X_3, X_2] = -X_2, \quad [X_3, X_1] = X_1.
\]

Therefore the second-order differential equation (1) is a SODE Lie system in the sense of [13, 21] and we can use the theory of Lie systems in order to study its properties.

More generally each time-dependent vector field of the form
\[
X(t) = \beta(t) X_1 + \alpha(t) X_2,
\] (23)
where $\alpha$ and $\beta$ are arbitrary time-dependent functions, is a Lie system of the same type as the one studied in [13, 21]. Its integral curves are the solutions of the system

$$\begin{cases}
\dot{x} = \alpha(t) v, \\
\dot{v} = \beta(t) x + \alpha(t) \frac{k}{x^3}.
\end{cases}$$

(24)

This system of differential equations is a generalization of the case given in [13, 21] and is related to the second-order differential equation,

$$\ddot{x} - \frac{\dot{\alpha}(t)}{\alpha(t)} \dot{x} - \alpha(t) \beta(t) x - \alpha^2(t) \frac{k}{x^3} = 0,$$

(25)

because the derivative of the first equation is

$$\ddot{x} = \dot{\alpha}(t) v + \alpha(t) \dot{v},$$

and, in view of (24), we obtain (25).

If $\alpha$ is positive and we use the notations $\alpha = e^{-F}$ and $\beta = -qe^{F}$, equation (25) becomes the more general differential equation studied in [25],

$$\ddot{x} + \dot{F} \dot{x} + qx - \frac{1}{e^{2F}} \frac{k}{x^3} = 0.$$  

(26)

The latter equation can be studied by means of the theory of Lie systems and a superposition rule for its general solution can be found. In order to get such a rule we have to consider two copies of the Lie system,

$$\begin{cases}
\dot{x} = e^{-F} v \\
\dot{v} = -q e^{F} x,
\end{cases}$$

(27)

which corresponds to the second-order differential equation

$$\ddot{x} + \dot{F} \dot{x} + qx = 0,$$

(28)

together with one copy of (18). Thus we obtain the system of first-order differential equations

$$\begin{cases}
\dot{x} = e^{-F} v_x, \\
\dot{y} = e^{-F} v_y, \\
\dot{z} = e^{-F} v_z, \\
\dot{v}_x = -q e^{F} x + e^{-F} \frac{k}{x^3}, \\
\dot{v}_y = -q e^{F} y, \\
\dot{v}_z = -q e^{F} z,
\end{cases}$$

(18)
which corresponds to the differential equations of the integral curves for the time-dependent vector field

\[ X = e^{-F} \left( v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + \frac{k}{x^3} \frac{\partial}{\partial v_x} \right) - q e^F \left( x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial v_y} + z \frac{\partial}{\partial v_z} \right). \]

This vector field can be expressed as

\[ X = e^{-F} N_2 - q e^F N_1, \]

where \( N_1 \) and \( N_2 \) are

\[ N_1 = y \frac{\partial}{\partial v_y} + x \frac{\partial}{\partial v_x} + z \frac{\partial}{\partial v_z}, \quad N_2 = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + \frac{1}{x^3} \frac{\partial}{\partial v_x}. \]

These vector fields generate a 3-dimensional real Lie algebra with the vector field \( N_3 \) given by

\[ N_3 = \frac{1}{2} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - v_x \frac{\partial}{\partial v_x} - v_y \frac{\partial}{\partial v_y} - v_z \frac{\partial}{\partial v_z} \right). \]

In fact, they generate a Lie algebra isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \) because

\[ [N_1, N_2] = 2N_3, \quad [N_3, N_1] = N_1, \quad [N_2, N_3] = N_2. \]

The dimension of the distribution generated by these vector fields is three and the manifold of the Lie system has dimension six. There are three time-independent integrals of motion which turn out to be the Ermakov invariant \( I_1 \) of the subsystem involving variables \( x \) and \( y \), the Ermakov invariant \( I_2 \) of the subsystem involving variables \( x \) and \( z \), and the Wronskian \( W \) of the subsystem involving variables \( y \) and \( z \). They define a foliation with 3-dimensional leaves. This foliation can be used to obtain a superposition rule.

The Ermakov invariants are

\[ I_1 = \frac{1}{2} \left( (yv_x - xv_y)^2 + k \left( \frac{y}{x} \right)^2 \right), \quad I_2 = \frac{1}{2} \left( (xv_z - zv_x)^2 + k \left( \frac{z}{x} \right)^2 \right), \]

where \( I_1 \) and \( I_2 \) are non-negative constants for \( k > 0 \) and the Wronskian \( W \) is

\[ W = yv_z - zv_y. \]

It is to be remarked that the relation of the first equation (27) allows us to rewrite the invariants \( I_1 \) and \( I_2 \), respectively, as:

\[ I_1 = \frac{1}{2} \left( e^{2F(t)} (y\dot{x} - x\dot{y})^2 + k \left( \frac{y}{x} \right)^2 \right), \quad I_2 = \frac{1}{2} \left( e^{2F(t)} (x\dot{z} - z\dot{x})^2 + k \left( \frac{z}{x} \right)^2 \right). \]
In particular, for $F(t) = \gamma_0 t$, which corresponds to the above mentioned example associated with the Caldirola–Kanai model, we recover the invariant given in (2.11b) of [22].

We can obtain an explicit expression of $x$ in terms of $y, z$ and the three first integrals $I_1, I_2, W$

$$x = \frac{\sqrt{2}}{|W|} \left( I_2 y^2 + I_1 z^2 \pm \sqrt{4I_1I_2 - kW^2} \ yz \right)^{1/2}.$$ 

We remark that $W$ is a constant fixed by the two independent particular solutions of the time-dependent harmonic oscillator $x_1(t)$ and $x_2(t)$, i.e. $y(t) = x_1(t)$ and $z(t) = x_2(t)$, and only $I_1$ and $I_2$ play the role of constants in this superposition rule for the Milne–Pinney equation. This is not a surprising fact because the Milne–Pinney equation is a second-order differential equation. Note also that the values of $I_1$ and $I_2$ are nonnegative constants, but should be chosen such that $x(0)$ be real, i.e. $4I_1I_2 \geq kW^2$.

5 Ermakov systems that are not of Lie system type

In this section we show that there exist dissipative Milne–Pinney equations that can be transformed into simple Milne–Pinney equations by means of the methods developed for quasi-Lie systems. Furthermore we can deal with the so-obtained Ermakov systems as in [21] in order to find many of their properties, i.e. first integrals of motion or superposition rules. Next we can use these results to obtain properties of the initial system of differential equations by inverting the time-dependent change of variable.

Consider the family of differential equations

$$\ddot{x} = a(t)\dot{x} + b(t)x + c(t)\frac{1}{x^3}. \quad (29)$$

We are mainly interested in the case $c(t) \neq 0$ and we can assume that $c(t)$ has a constant sign for the set of values of $t$ we are considering. The case in which $c(t)$ is identically zero corresponds to the harmonic oscillator with time-dependent frequency and a dissipative term.

Usually we associate with such a second-order differential equation a system of first-order differential equations by introducing a new variable, $v$, and
relating (29) to the system of first-order differential equations

\[
\begin{align*}
\dot{x} &= v, \\
\dot{v} &= a(t)v + b(t)x + c(t)\frac{1}{x^3}.
\end{align*}
\]

(30)

This system describes the integral curves for the time-dependent vector field

\[ X(t) = a(t)X_1 + b(t)X_2 + c(t)X_3 + X_4, \]

where the vector fields \( X_1, \ldots, X_4 \), are given by

\[
X_1 = v \frac{\partial}{\partial v}, \quad X_2 = x \frac{\partial}{\partial v}, \quad X_3 = \frac{1}{x^3} \frac{\partial}{\partial v}, \quad X_4 = v \frac{\partial}{\partial x}. \]

(31)

Consider also the vector field

\[ X_5 = x \frac{\partial}{\partial x}. \]

(32)

The set of these five vector fields is a basis for a \( \mathbb{R} \)-linear space \( V \). However, they do not close on a Lie algebra because the commutator \([X_3, X_4]\) is not in \( V \). Moreover, it can be checked that there is no finite-dimensional real Lie algebra \( V' \) containing \( V \). Therefore the differential equation (30) cannot be considered as a Lie system. Nevertheless we can deal with this differential equation through a quasi-Lie scheme.

The two-dimensional linear subspace, \( W \subset V \), generated by the vector fields

\[
Y_1 = X_1 = v \frac{\partial}{\partial v}, \quad Y_2 = X_2 = x \frac{\partial}{\partial v}, \]

(33)

is a Lie algebra because these vector fields satisfy the commutation relation

\[ [Y_1, Y_2] = -Y_2. \]

(34)

What is more, as

\[
\begin{align*}
[Y_1, X_3] &= -X_3, & [Y_1, X_4] &= X_4, & [Y_1, X_5] &= 0, \\
[Y_2, X_3] &= 0, & [Y_2, X_4] &= X_5 - X_1, & [Y_2, X_5] &= -X_2,
\end{align*}
\]

(35)

the linear space \( V \) is invariant under the action of Lie algebra \( W \) on \( V \), i.e. \([W, V] \subset V\). Therefore we have found a quasi-Lie scheme to deal with the differential equation (30).
The corresponding set of time-dependent diffeomorphisms of $T\mathbb{R}$ related to the flows of time-dependent vector fields in $W$, $\alpha_1(t)Y_1 + \alpha_2(t)Y_2$, is given by
\[
\begin{align*}
  x &= x', \\
  v &= \alpha(t)v' + \beta(t)x'
\end{align*}
\]
with $\alpha(t) \neq 0$. The inverse transformation is
\[
\begin{align*}
  x' &= x, \\
  v' &= -\frac{\beta(t)}{\alpha(t)}x + \frac{1}{\alpha(t)}v.
\end{align*}
\]

These time-dependent diffeomorphisms transform the system (30) into a new one in which the time-dependent vector field determining the dynamics can be written as a linear combination of the fields of $V$ at each time
\[
X'(t) = a'(t)X_1 + b'(t)X_2 + c'(t)X_3 + d'(t)X_4 + e'(t)X_5.
\]

More explicitly the new coefficients are
\[
\begin{align*}
  a'(t) &= a(t) - \beta(t) - \frac{\dot{\alpha}(t)}{\alpha(t)}, \\
  b'(t) &= \frac{b(t)}{\alpha(t)} + a(t)\frac{\beta(t)}{\alpha(t)} - \frac{\beta^2(t)}{\alpha(t)} - \frac{\dot{\beta}(t)}{\alpha(t)}, \\
  c'(t) &= \frac{c(t)}{\alpha(t)}, \\
  d'(t) &= \alpha(t), \\
  e'(t) &= \beta(t),
\end{align*}
\]
and the integral curves for (38) are solutions of the system
\[
\begin{align*}
\frac{dx'}{dt} &= \beta(t)x' + \alpha(t)v', \\
\frac{dv'}{dt} &= \frac{\beta(t)}{\alpha(t)} \left( \frac{b(t)}{\beta(t)} + a(t) - \beta(t) - \frac{\dot{\beta}(t)}{\beta(t)} \right) x' + \left( a(t) - \beta(t) - \frac{\dot{\alpha}(t)}{\alpha(t)} \right) v' \\
&\quad + \frac{c(t)}{\alpha(t)} \frac{1}{x'^3}.
\end{align*}
\]
However, notice that only, if \( \beta(t) = 0 \), is this system associated with a second-order differential equation, more specifically with

\[
\frac{d^2 x'}{dt^2} = a(t) \frac{dx'}{dt} + b(t) x' + c(t) \frac{1}{x'^3}.
\]

The Ermakov systems studied in [21] are in the family of differential equations (30). Hence it is natural to look for sufficient conditions to be able to transform a given system of (30) into one of these Ermakov systems of the form

\[
\begin{align*}
\dot{x} &= f(t) v, \\
\dot{v} &= -\omega^2(t) x + f(t) \frac{k}{x'^3},
\end{align*}
\]

where \( k \) is a constant, which corresponds to the second-order differential equation

\[
\ddot{x} = \frac{f(t)}{f(t)} \dot{x} + f(t) \left( -\omega^2(t) x + f(t) \frac{k}{x'^3} \right).
\]

Next we compare (40) with (41) to transform equation (30) into one related to Ermakov systems. As a result we notice that \( \alpha = f \) and the time-dependent coefficients \( a(t) \) and \( c(t) \) must be such that

\[
\begin{align*}
k\alpha(t) &= c(t) \frac{1}{\alpha(t)}, \\
\frac{\dot{\alpha}(t)}{\alpha(t)} &= a(t),
\end{align*}
\]

i.e. the sign of \( k \) must coincide with that of \( c(t) \) and

\[
\omega^2(t) = -\sqrt{\frac{k}{c(t)}} b(t).
\]

This expression provides a sufficient condition in order to be able to transform a differential equation corresponding to the system (30) into one of the form of (42). Taking the time-derivative on the first condition of (43) we obtain

\[
2 k \alpha \dot{\alpha} = \dot{c}.\]

Dividing by \( \alpha^2 \) and using the second condition in (43) we get

\[
a(t) = \frac{\dot{c}(t)}{2c(t)},
\]
Thus the resulting transformation is determined by
\[
\alpha(t) = \sqrt{\frac{c(t)}{k}}, \quad \beta(t) = 0, \tag{45}
\]
for a certain constant \(k\).

Next we point out some differential equations \((29)\) appearing in the literature that can be related by means of the method developed here with a particular Lie system: the Milne–Pinney equation \([21]\).

As a first example we analyze the Chini differential equation \([26]\)
\[
\ddot{x} + \frac{\dot{p}(t)}{2p(t)} \dot{x} + \frac{q(t)}{p(t)} x = \frac{1}{p(t)} \frac{k}{x^3}, \quad \text{with} \quad p(t) > 0. \tag{46}
\]
This equation is associated with the system of first-order differential equations
\[
\begin{cases}
\dot{x} = v, \\
\dot{v} = -\frac{\dot{p}(t)}{2p(t)} v - \frac{q(t)}{p(t)} x + \frac{1}{p(t)} \frac{k}{x^3}.
\end{cases} \tag{47}
\]
This system is a particular instance of \((30)\) for the following choice of time-dependent coefficients
\[
a(t) = -\frac{\dot{p}(t)}{2p(t)}, \quad b(t) = -\frac{q(t)}{p(t)}, \quad c(t) = \frac{k}{p(t)}. \tag{48}
\]
In this case, as \(\dot{c}/(2c) = -\dot{p}/(2p) = a\), these coefficients satisfy the reducibility condition \((44)\) and we can transform this system into a Lie one through the transformation \((36)\) determined by the coefficients
\[
\alpha(t) = \frac{1}{\sqrt{p(t)}}, \quad \beta(t) = 0, \tag{49}
\]
i.e. by means of the time-dependent change of variables
\[
\begin{cases}
x = x', \\
v = \frac{1}{\sqrt{p(t)}} v'.
\end{cases} \tag{50}
\]
So equation \((47)\) becomes
\[
\begin{cases}
\frac{dx'}{dt} = \frac{1}{\sqrt{p(t)}} v', \\
\frac{dv'}{dt} = -\frac{q(t)}{\sqrt{p(t)}} x' + \frac{1}{\sqrt{p(t)}} \frac{k}{x^3}.
\end{cases} \tag{51}
\]
This system describes the integral curves for the time-dependent vector field

\[ X(t) = \frac{1}{\sqrt{p(t)}} \left[ v' \frac{\partial}{\partial x'} + \left( -q(t) x' + \frac{k}{x'^3} \right) \frac{\partial}{\partial v'} \right]. \]

Introducing the time-reparametrization

\[ \tau(t) = \int_0^t \frac{dt'}{\sqrt{p(t')}} \]  

the system (51) reduces to the Ermakov system studied in [21]

\[
\begin{align*}
\frac{dx'}{d\tau} &= v', \\
\frac{dv'}{d\tau} &= -q(t(\tau)) x' + \frac{k}{x'^3}.
\end{align*}
\]

which corresponds to the second-order differential equation

\[
\frac{d^2x'}{d\tau^2} = -q(t(\tau)) x' + \frac{k}{x'^3}.
\]

Now we can use the Ermakov invariant, superposition rules etc. for (53).

Next, by inverting the time-dependent change of variables used for the Chini equation, we obtain some kind of time-dependent superposition rule for the solutions of (46).

Walter [27] developed another interesting example:

\[
\ddot{x} + \dot{p}(t) \dot{x} + q(t) x = \frac{k}{p^2(t)x^3}.
\]

This equation is associated with the system of first-order differential equations

\[
\begin{align*}
\dot{x} &= v, \\
\dot{v} &= -\frac{\dot{p}(t)}{p(t)} v - \frac{q(t)}{p(t)} x + \frac{1}{p^2(t)} \frac{k}{x^3}.
\end{align*}
\]

This equation is a particular instance of (30) with the functions

\[
a(t) = -\frac{\dot{p}(t)}{p(t)}, \quad b(t) = -\frac{q(t)}{p(t)}, \quad c(t) = \frac{k}{p^2(t)}. \]

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In this case, as $\dot{c}/c = -2\dot{p}/p$, these functions satisfy the reducibility condition (44) and if we choose
\[ \alpha(t) = \frac{1}{p(t)}, \quad \beta(t) = 0, \] (58)
then the transformation (36) transforms the equation (56) into the Lie system
\[
\begin{cases}
\frac{dx'}{dt'} = \frac{1}{p(t')}v', \\
\frac{dv'}{dt'} = -q(t)x' + \frac{1}{p(t)} k x^3.
\end{cases}
\] (59)

Introducing a new time function $\tau$ by means of
\[ \tau(t) = \int_0^t \frac{dt'}{p(t')}, \] (60)
we obtain
\[ \frac{d^2 x}{d\tau^2} = -q(t(\tau))p(t(\tau))x' + \frac{k}{x^3}, \] (61)
which is a standard Milne–Pinney equation that can be described again through the theory of such systems as in [21].

Finally we consider the example of Colegrave and Abdalla [28]:
\[ \ddot{x} - 2\frac{\dot{p}(t)}{p(t)} \dot{x} + p^2(t)x = p^4(t) \frac{k}{x^3}, \] (62)
with an associated system of first-order differential equations
\[
\begin{cases}
\dot{x} = v, \\
\dot{v} = \frac{2\dot{p}(t)}{p(t)} v - p^2(t)x + p^4(t) \frac{k}{x^3},
\end{cases}
\] (63)
which is a particular instance of (29) with the coefficients
\[ a(t) = \frac{2\dot{p}(t)}{p(t)}, \quad b(t) = -p^2(t), \quad c(t) = k p^4(t). \] (64)

In this case, if we take $\beta(t) = 0$ and we choose as $\alpha$ a solution of the differential equation,
\[ \frac{\dot{\alpha}(t)}{\alpha(t)} - a(t) = 0, \] (65)
as
\[ \alpha(t) = p^2(t), \]
we obtain that (36) transforms the initial differential equation into
\[
\begin{aligned}
\frac{dx'}{dt} &= p^2(t)v', \\
\frac{dv'}{dt} &= -x' + p^2(t) \frac{k}{x^3},
\end{aligned}
\]
which can be studied through the formalism of Lie systems. On the introduction of a new time function
\[ \tau(t) = \int_t^t p^2(t') \, dt' \]
the last differential equation reduces to
\[ \frac{d^2 x'}{d\tau^2} = -\frac{1}{p^2(t(\tau))} x' + \frac{k}{x^3} \]
and the usual Milne–Pinney equation is recovered.

6 Conclusions and Outlook

We have studied 'damped time-dependent angular frequency harmonic oscillators' and time-reparametrizations of second-order differential equations through ad hoc transformations. As a result we have found a method to remove terms proportional to the velocity in certain set of second-order differential equations. Afterwards we revisited the theory of SODE Lie systems and quasi-Lie schemes. These theories allow us to study many (systems of) second-order differential equations. They provide methods to obtain time-dependent superposition rules, integrals of motion, solutions etc. In this paper we have applied these theories to some particular dissipative Ermakov systems to recover known properties from a new point of view.

The theory of quasi-Lie schemes can be used to deal with many systems of differential equations. We expect to analyse new differential equations in forthcoming papers, recovering known properties and finding new ones.
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References

[1] W.E. Milne, *The numerical determination of characteristic numbers*, Phys. Rev. 35, 863–67 (1930).

[2] E. Pinney, *The nonlinear differential equation y'' + p(x)y' + cy^{-3} = 0*, Proc. A.M.S. 1, 681 (1950).

[3] V.P. Ermakov, *Second-order differential equations. Conditions of complete integrability*, Univ. Isz. Kiev Series III 9, 1–25 (1880) (translation by A.O. Harin). See: Appl. Anal. Discrete Math. 2, 123–48 (2008).

[4] J.F. Cariñena and J. de Lucas, *A nonlinear superposition rule for solutions of Milne–Pinney equation*, Phys. Lett. A 372, 5385–5389 (2008).

[5] F. Haas, *Approximate solution for a damped Pinney equation*, math-ph/0712.4083.

[6] Z.E. Musielak, *Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients*, J. Phys. A: Math. Theor. 41, 055205 (2008).

[7] Z.E. Musielak, D. Roy and L.D. Swift, *Method to derive Lagrangian and Hamiltonian for a nonlinear dynamical system with variable coefficients*, Chaos, Solitons and Fractals 38, 894–902 (2008).

[8] Shang-Wu Qian, Bo-Wen Huang and Zhi-Yu Gu, *Ermakov invariant and the general solution for a damped harmonic oscillator with a force quadratic in velocity*, J. Phys. A 34, 5613–5617 (2001).
[9] S. Lie and G. Scheffers, Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen, Edited and revised by G. Scheffers, Teubner, Leipzig, 1893.

[10] J.F. Cariñena, J. Grabowski and G. Marmo, Lie–Scheffers systems: a geometric approach, Bibliopolis, Napoli, 2000.

[11] J.F. Cariñena and A. Ramos, A new geometric approach to Lie systems and physical applications, Acta Appl. Math. 70, 43–69 (2002).

[12] J.F. Cariñena, J. Grabowski and G. Marmo, Superposition rules, Lie theorem and partial differential equations, Rep. Math. Phys. 60, 237–258 (2007).

[13] J.F. Cariñena, J. de Lucas and M.F. Rañada, Recent applications of the theory of Lie systems in Ermakov systems, SIGMA 4, 031, 18 p. (2008).

[14] J.F. Cariñena, J. Grabowski and J. de Lucas, Quasi-Lie schemes: theory and applications, ArXiv: 0810.1160.

[15] I.A. Pedrosa, Canonical transformations and exact invariants for dissipative systems, J. Math. Phys. 28, 2662–2664 (1987).

[16] R. Milson, Liouville transformation and exactly solvable Schrödinger equation, Int. J. Theor. Phys. 37, 1735–1752 (1998).

[17] L.M. Berkovich, Transformation of Sturm-Liouville differential equations, Funct. Anal. Appl. 16, 190–192 (1982).

[18] T.I. Brazier and P.G.L. Leach, Invariants for dissipative systems and Noether’s theorem, Rev. Mex. Fis. 40, 378–385 (1994).

[19] J.F. Cariñena, J. de Lucas and A. Ramos, A geometric approach to time evolution operators of Lie quantum systems, To appear in Int. J. Theor. Phys. 48 (2009).

[20] J.M. Cerveró and J. Villarroel, On the quantum theory of the damped harmonic oscillator, J. Phys. A: Math. Gen. 17 2963–2971 (1984).
[21] J.F. Cariñena, J. de Lucas and M. F. Rañada, *Nonlinear superpositions and Ermakov systems*. In: *Differential Geometric Methods in Mechanics and Field Theory*, pp. 15–33, F. Cantrijn, M. Crampin and B. Langerock eds., Academia Press, Gent, 2007.

[22] A. Nassar, *Ermakov and non-Ermakov systems in quantum dissipative models*, J. Math. Phys. **27**, 755–58 (1986).

[23] M. Aktas and R. Sever, *Effective mass Schrödinger equation for exactly solvable class of one-dimensional potentials*, J. Math. Chem. **43**, 92–100 (2008).

[24] O. von Roos, *Position-dependent effective mass in semiconductor theory*, Phys. Rev. B **27**, 7547–7552 (1983).

[25] R. Redheffer, *Steen’s equation and its generalizations*, Aequationes Math. **58**, 60–72 (1999).

[26] M. Chini, *Sull’equazione del 2° ordine lineare omogenea*, Atti. Accad. Sci. Torino **33**, 737–745 (1898).

[27] J. Walter, *Bemerkungen zu dem Grenzpunktfallkriterium von N. Levinson*, Math. Z. **105**, 345–350 (1968).

[28] R.K. Colegrave and M. S. Abdalla, *Invariants for the time-dependent harmonic oscillators*, J. Phys. A: Math. Gen. **12**, 3805–3815 (1983).