Crossover of Level Statistics between Strong and Weak Localization in Two Dimensions

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We investigate numerically the statistical properties of spectra of two-dimensional disordered systems by using the decimation method applied to the Anderson model. Statistics of the spacings calculated for system size up to \(1024 \times 1024\) lattice sites exhibits a crossover between Wigner and Poisson distributions. We perform a self-contained finite-size scaling analysis to find a single-valued one-parameter function \(\gamma(L/\xi)\) which governs the crossover. The scaling parameter \(\xi(W)\) is deduced and compared with the localization length. \(\gamma(L/\xi)\) does not show critical behavior and has two asymptotic regimes corresponding to weakly and strongly localized states.

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The single-parameter scaling theory of Anderson localization suggests that there exists no conducting state in an infinite two-dimensional (2D) disordered system at zero temperature [1], since all one-electron wave functions are localized even for arbitrarily small fluctuations of a random potential [2]. The corresponding energy levels are distributed completely randomly. One expects that there are no spectral correlations at any energy scale. Therefore, for any non-vanishing disorder the distribution of neighboring level spacings $P(s)$ is given by the Poisson law $P_p(s) = \exp(-s)$ in the thermodynamic limit. (Energy is measured in units of the mean level spacing $\Delta$.)

For finite systems $P(s) \approx P_p(s)$ only if the disorder $W$ is strong enough such that the localization length $\xi$ is shorter than the system size $L$. Indeed, for fixed $W \neq 0$ the level spacing distribution was numerically shown to approach $P_p(s)$ with increasing $L$ [3]. With decreasing $W$ the electron states change gradually from “strongly localized” to “weakly localized”, and the energy levels become more correlated. When $\xi \gg L$, their statistics is well described by the Wigner-Dyson theory for eigenvalues of random matrices [4].

As a consequence of the transition from strong to weak localization, $P(s)$ is expected to undergo a continuous crossover between $P_p(s)$, and the Wigner surmise $P_W(s) = \pi s/2 \exp(-\pi s^2/4)$. In the weakly localized and non-diffusive regimes the spectral fluctuations were studied in terms of the two-level correlation function by the use of sophisticated analytical techniques [5,6]. Small corrections to the Wigner-Dyson statistics were obtained in [7]. Although the absence of the metal-insulator transition for statistics of the spectra in 2D-system was already mentioned by Shklovskii et al [8], so far extensive investigations in the intermediate regime, where the deviations from both of the limiting distributions are considerable, have not yet been carried out. An important question is which scaling law governs the crossover of the level statistics. For three-dimensional (3D) disordered systems such a scaling law shows critical behavior at a finite disorder which is the signature of a metal–insulator transition [8,10]. If no delocalization transition exists in 2D [11], $P(s)$ should not exhibit critical behavior at any finite disorder. In this paper we provide strong numerical evidence that this is indeed the case.
The scaling behavior of $P(s)$ with $L$ and $W$ is determined by using the results for the energy spectrum that were obtained by a combination of the decimation method and a Lanczos procedure. Due to the achieved large system sizes, the scaling analysis could be performed very accurately. Independently of other methods, we calculate a one–parameter scaling variable $\gamma(L/\xi)$ related to $P(s)$ which governs a crossover between the two above limits. It turns out to be monotonic without any critical point $W_c$, for which $\gamma(L,W_c) = \text{const}$, that would indicate a delocalization transition. A similar variable $\gamma(L/\xi_{TMM})$ was claimed in [8] to behave as a single-valued function, where $\xi_{TMM}$ is the localization length obtained from the transfer-matrix method (TMM) [12]. We find that in the strongly localized regime the deviation from the Poissonian is described by a linear law $\gamma \propto L/\xi$. In the weakly localized regime $\gamma$ tends towards the Wigner limit. From spectral statistics we determine the scaling parameter $\xi$, which can be interpreted as the localization length. Its dependence on $W$ is consistent with the results of previous calculations using the TMM [12,13].

We consider the Anderson model (AM) with diagonal disorder $H = \sum_i \varepsilon_i \left| i \right\rangle \left\langle i \right| + t \sum_{<i,j>} \left| i \right\rangle \left\langle j \right|$, where $i$ labels the $L^2$ sites of a square lattice of the linear size $L$. The on-site energies $\varepsilon_i$ are independently distributed at random within an interval of width $W$. The second sum is taken over all pairs of nearest-neighbor sites. It corresponds to the kinetic energy. For the calculations we used a decimation algorithm described in [3,14], which enables us to investigate the level statistics near the band center $E_0$ for systems containing up to $10^5$–$10^6$ lattice sites. The algorithm consists of a sequence of iterative steps [14], in which the spectrum of a system is computed by using the amplitudes of wave functions and corresponding eigenvalues of four smaller systems of half of the linear size. Each step consists of (i) a unitary transformation of the secular matrix to the basis set of the smaller lattices, (ii) a truncation of the corresponding Hilbert space to a fixed number $m$ of states near $E_0 = 0$, and (iii) a diagonalization of the truncated Hamiltonian. The iteration starts with the original 2D lattice of the AM. The latter is divided into many independent small square cells of linear size $L_o$, which is defined by the disorder. Subsequent larger squares are constructed by coupling four smaller squares via hopping elements $t$ only
between corresponding boundary sites. In the last step of the iteration, periodic boundary
conditions are imposed. Using ensembles of up to \( k \approx 1000 \) different realizations of the
random potential, we calculate \( P(s, L, W) \) at \( E_0 \).

As expected, we find the Poisson distribution for large \( L \) and large \( W \) and the Wigner
distribution for small \( W \) and small \( L \). The crossover between the two limiting cases can be
detected for fixed disorder with increasing \( L \), or for fixed \( L \) with increasing \( W \). Figure 1
shows that for \( W = 4 \) the level spacing distribution is only slightly different from the
Wigner surmise. On the other hand, \( P(s) \) for \( W = 7 \) already resembles strongly the Poisson
distribution. This can be explained by the considerably different values of the localization
length for these two disorders. In the former case, \( \xi \gg L \) while in the latter case \( \xi \ll L \) (see
the inset of Fig. 3 below). In order to test our method we also calculated \( P(s) \) by using
a Lanczos procedure [15], especially optimized for large sparse matrices [10]. We were able
to compute the exact spectrum for systems up to \( L^2 = 256^2 \). The results of the decimation
method for \( P(s) \) are in good agreement with those obtained by the exact diagonalization
(inset of Fig. 1). For larger systems the Lanczos procedure failed even for a very narrow
energy interval due to limitations of computing time, while we could enlarge the system up
to \( L^2 \sim 10^6 \) lattice sites by using the decimation algorithm. The striking advantage of the
latter is that it allows to go beyond the limits of conventional diagonalization procedures,
at the expense of being restricted to a small (and constant) number of energy levels around
the energy of interest.

The decimation scheme is applicable as long as \( V \ll \Delta \), where \( V \) is the typical matrix
element between states of the coupled blocks (see [3,14]). Therefore one should start the
calculations with such \( L_o \) for a given \( W \), so that this condition is fulfilled. The choice of \( m \)
defines the relative error of the retained eigenvalues \( \delta E/\Delta \sim V^2/(\Delta^2 m) \ll 1 \) [14], resulting
from the neglect of the discarded levels. The errors of the consecutive iterations steps are
mostly dominated by those of the first step, which diminish with decreasing coupling, i.e.
with increasing \( W \). In order to control the precision of \( P(s) \) we choose \( L_o \) and \( m \) depending
on \( W \) so that after several iterations \( V/\Delta \lesssim 0.01 \). The initial size grows from \( L_o = 16 \) for
$W \geq 8$ to $L_o = 256$ for $W \leq 5$ and $m$ ranges typically from 25 to 100. For the sizes smaller than $L_o$ the exact diagonalization is applied.

In order to investigate the scaling behavior of the level statistics quantitatively, we consider the integral $I(s) \equiv \int_0^s P(s')ds'$. It describes the probability to have a spacing smaller than $s$. Numerically, $I(s, L, W)$ can be obtained with a higher accuracy than $P(s, L, W)$.

Figure 2 shows how $I(s)$ changes from $I_W(s) = 1 - \exp(-\pi s^2/4)$ to $I_P(s) = 1 - \exp(-s)$ for fixed disorder in approaching the thermodynamic limit. For example, the small-$s$ behavior of $I(s, L, W)$ changes from quadratic for small to linear for larger system sizes. Also, the probability to find large spacings is enhanced, when increasing $L$.

The next step is to determine the deviation of the level statistics from the Poissonian limit due to the finiteness of the system. We introduce the function

$$\gamma(L, W) \equiv \frac{1}{N} \int_0^{s_0} [I(s, L, W) - I_P(s)] ds,$$

(1)

which is similar to the quantities considered in Ref. [8–10]. The normalization constant $N$ is chosen such that $\gamma = 1$, if $I(s) = I_W(s)$. For the scaling analysis the choice of the upper limit $s_0$ of the integral does not play an important role [10]. By optimizing the numerical accuracy we fixed it to $s_0 = 1.25\Delta$. For an infinite 2D disordered system $\gamma = 0$ for all $W \neq 0$. For finite systems, $\gamma$ is expected to change continuously from unity to zero with increasing $L$ [16].

We computed $\gamma$ for $W$ from 4 to 12 and $L$ from 8 to 1024 (inset of Fig. 2). One observes that $\gamma$ decreases with increasing $L$ for all $W$. For weak disorder it is close to unity and decreases slowly, as the system doubles in size, while for larger disorder the decrease is more rapid. Correspondingly, for a fixed $L$, $\gamma$ diminishes as the disorder becomes stronger. The data for $W \gtrsim 7$ saturate towards $\gamma = 0$. Altogether, $\gamma(L, W)$ undergoes a smooth monotonic crossover from unity to zero as $L$ grows, with no critical behavior, at least for the disorders considered here. This is markedly different from the 3D case, where a critical disorder $W_c \neq 0$ exists, below which the electron states are delocalized, and $\gamma$ should increase with $L$ [8,10].

We have estimated the error of $\gamma$, $\delta\gamma/\gamma \approx (\delta\gamma/\gamma)_{\text{stat}} + (\delta\gamma/\gamma)_{\text{syst}}$. The former is of order of
\((kmI(s_o))^\text{-}1/2 < 2\%\). The latter can be roughly estimated as \(I(s_o)/(N\gamma) \delta E/\Delta\) and is about 0.5\% near \(L_o/\xi \gtrsim 1\). In the strong localization it decreases as \(0.02 L_o/\xi \exp(-L_o/\xi)\). Thus, the statistical error due to the finite number of samples is always larger than the systematic one resulting from the approximation.

Assuming that \(\gamma\) obeys a one-parameter scaling law \(\gamma(L,W) = f(L/\xi)\), one can find the scaling parameter \(\xi(W)\) such that all numerical data fall onto a common curve. We fixed the data for \(W = 7\) and then shifted the data corresponding to other \(W\) to the left and to the right in a logarithmic scale, until the set of data of a given \(W\) overlapped for at least one value of \(L\) with those corresponding to another \(W\). One can determine \(\xi\) by this procedure up to a common numerical factor. The latter corresponds to a shift of all of the data in the inset of Fig. 2 by the same amount along the \(x\)-axis. We found this factor by adjusting \(\xi\) at large \(W\) to the localization length, which was earlier found by using the TMM [13]. For this fit we choose \(\xi_{TMM}(W = 10)\) which was smaller than the width of the quasi-1D strips used in the TMM calculations. As shown in Fig. 3 the data of \(\gamma(L/\xi)\) for all pairs of \(\{L,W\}\) collapse onto an one-branch curve within the numerical accuracy. Thus, our calculations confirm by a completely independent procedure that the crossover from the weak to the strong localization in 2D, which corresponds in the language of the level statistics to the crossover from the Wigner surmise to the Poissonian distribution, is governed by one-parameter scaling. The left, very flat part of the curve in Fig. 3, obtained for small disorders, corresponds to the weak localization. Here, the data for \(\ln \gamma\) approach zero because the scaling parameter \(\xi(W)\) becomes large compared to the size of the system. The small, but increasing deviation from \(\gamma = 1\) for \(L/\xi \ll 1\) is in agreement with analytical results obtained earlier for the corrections to the Wigner-Dyson statistics in the case \(s_0 \sim \Delta\) [7]. Details of a quantitative comparison which involves also a calculation of the conductance of the system are given elsewhere [17].

When \(L/\xi \gg 1\) the data are well described by \(\gamma = \gamma_o(\xi/L)\), where \(\gamma_o \approx 1.9\). An analogous linear behavior was found previously for the Lyapunov exponents [12]. The fact
that for strong localization the conductance $g \propto \exp(-2L/\xi)$ suggests that $\gamma \propto -(\ln g)^{-1}$ for $L/\xi \gg 1$. The intermediate part of the graph, where $L \sim \xi$, shows the smooth crossover between the two limits. Since $\gamma(L/\xi)$ consists of only a decaying branch, we conclude that there is no metal-insulator transition at least for $W \geq 4$. The inset of Fig. 3 shows $\xi(W)$. For $W < 6$ one observes a faster increase of $\xi$ with decreasing $W$ than the localization length computed via TMM in [12]. On the other hand, we find good agreement with the values of $\xi_{TMM}$, reported recently in Ref. [13]. It is reasonable to identify our scaling parameter $\xi$ with the localization length of the electron states. The disorder dependence of $\xi$ is consistent with the theoretical prediction of an essential singularity of the localization length at $W = 0$ [18] in the form $\xi(W) = a \exp[(W_o/W)^2]$, where the length scale $a$ is of order of the lattice constant. By fitting the calculated data of $\xi$ to this formula we found $a = 1.1 \pm 0.25$ and $W_o = 11 \pm 3$.

In conclusion, we performed a finite size scaling analysis of the fluctuations in one-electron spectra of 2D disordered systems by using a combination of exact diagonalization and decimation algorithm. It enables us to treat systems up to $1024^2$ lattice sites. The nearest-neighbor level spacing statistics described by the variable $\gamma$ (Eq. 1) indicates a smooth crossover from strong to weak localization when the disorder and the system size decrease. The crossover is found to obey a one-parameter scaling law which does not show critical behavior. This confirms the earlier conjecture [1] that in 2D there are no true metallic states in the thermodynamic limit. Our present results for the scaling parameter $\xi$ extracted from the level statistics are quantitatively consistent with those obtained previously for the localization length by using a completely different numerical approach [13], and conclusions based on perturbative calculations [18].

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FIGURES

FIG. 1. The probability density of neighboring spacings $P(s)$ for system size $L^2 = 256^2$ at different disorders. Solid and dotted lines: Wigner surmise and Poisson law, respectively. Inset: $P(s)$ at $W = 6$ obtained by the decimation and the Lanczos method.

FIG. 2. Total probability distribution of spacings $I(s)$ at the disorder $W=6$ for various sizes. Solid and dotted lines: $I_W(s)$ and $I_P(s)$, respectively. Inset: $\gamma(L)$ for different disorders $W = 4.0 (\ominus), 4.5 (\star), 5.0 (\circ), 5.5 (+), 6.0 (\bullet), 7.0 (\triangle), 8.0 (\times), 9.0 (\square), 10.0 (\triangledown), 12.0 (\circ)$.

FIG. 3. The parameter $\gamma$ as a function of $L/\xi(W)$. Straight line: limit of strong localization. Symbols for $W$ are the same as in the inset of Fig. 2. Inset: disorder dependence of the scaling parameter $\xi (\bullet)$ along with the results obtained for the localization length by TMM $[12] (\times)$ and $[13] (\triangle)$.
Decimation & Lanczos
