Topological \( s \)-wave pairing superconductivity with spatial inhomogeneity: Mid-gap-state appearance and Anderson’s theorem

Yuki Nagai,\(^1\) Yukihiro Ota,\(^1\) and Masahiko Machida\(^1\)

\(^1\)CCSE, Japan Atomic Energy Agency, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8587, Japan

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Topological superconductivity is qualified by the presence of the surface gapless excitations, relevant to a bulk topological state (bulk-boundary correspondence). We study mid-gap states, as well as the gapless states, in a two-dimensional time-reversal-symmetry-broken topological \( s \)-wave superconductor with spatial inhomogeneity. Employing a local potential model for the inhomogeneity, we calculate the quasiparticle excitation spectrum. We show the appearance of the mid-gap bound states in the vicinity of the potential region. Moreover, we find that the bound states for line potential connect with gapless states, changing the potential into hard-wall one (a line defect). This connection is notable, since typically the bulk-boundary correspondence predicts only the presence of the gapless states. The mid-gap-state appearance suggests that Anderson’s theorem in this model is broken and the mechanism is similar to that in unconventional superconductivity, such as \( p \)-wave.

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I. INTRODUCTION

Superconductivity leads to interesting phenomena and applications, but the properties in materials typically suffer from fluctuations, such as impurity scattering. A celebrated prediction on the robustness against impurities is Anderson’s theorem; no significant reduction of \( T_c \) by non-magnetic impurities occurs in \( s \)-wave superconductivity. The derivation relies on the fact that a full quasiparticle spectrum is similar to that of topological defects. Nagai et al.\(^{13–15}\) show that in a 2D topological superconductor with on-site \( s \)-wave pairing the \( T_c \) reduction via non-magnetic impurities is significant, using a self-consistent \( T \)-matrix approach. The reduction is more pronounced, when the number of the gapless surface modes increases. Thus, this topological \( s \)-wave pairing superconductor seems to lie outside Anderson’s theorem, owing to the gapless modes. Therefore, it is worth asking how the MGS induced by an impurity are related to the surface gapless modes, and how the unconventional feature appears, even in the on-site \( s \)-wave pairing.

In this article, we study the quasiparticle excitations in a topological superconductor, with spatial inhomogeneity. We focus on a 2D topological superconducting model\(^{14}\) with on-site \( s \)-wave pairing, the Rashba spin-orbital coupling, and the Zeeman magnetic field. The model for the inhomogeneity is constructed by adding a local potential term to the uniform system. Our description corresponds to the quasiparticle behaviors under a single impurity. We examine two kinds of the potential functions, line-type potential and point-type potential. The quasiparticle spectrum is calculated by the diagonalization of the tight-binding Hamiltonian, changing the Zeeman magnetic field and the potential height, with the use of the numerical construction of the projector onto a low-energy subspace\(^{17,18}\).

The main result is the appearance of the MGS, in the presence of the potential. We show that the resultant states are locally bound in the vicinity of the potential region. The result implies a link of the MGS with the gapless surface modes. This link is definite in the case of the line-type potential function; we obtain a smooth connection of the MGS into the zero-energy states, changing the potential into hard-wall potential (i.e., a line defect). The appearance of the MGS (and the gapless states) is shown for different magnetic fields and potential heights, via characterizing the lowest energy in the quasiparticle spectrum. To obtain better understanding for the results, we derive a low-energy effective theory. We find that the present system has both a chiral-\( p \)-wave character and an \( s \)-wave character. Specifically, the \( p \)-wave component is primary for a high magnetic field. To sum up, the MGS appearance supports the prediction on the \( T_c \) reduction in Ref.\(^{16}\). We claim that the violation of Anderson’s theorem in our system has the same mechanism as in
The article is organized as follows. In Sec. II, we show a model for the two-dimensional topological s-wave superconductor. The quasiparticle spectrum for line-type potential is shown in Sec. III A, and subsequently the point-type potential is examined in Sec. III B. In Sec. III C, we derive a low-energy effective theory to obtain better understanding for the numerical results. In Sec. IV, the summary is given.

II. MODEL

The system is described by a tight-binding model on an $N_x \times N_y$ square lattice, with on-site s-wave pairing and the Rashba spin-orbit coupling. Moreover, the Zeeman magnetic field is applied along the direction perpendicular to the $xy$-plane. The mean-field Hamiltonian is

$$\hat{H} = -t \sum_{\langle j,j' \rangle} \sum_{\sigma} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j',\sigma} - \mu \sum_{j} \sum_{\sigma} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma}$$

$$-\frac{\alpha}{2} \sum_{j} \left[ (\hat{c}_{j+e_x,\downarrow}^\dagger \hat{c}_{j,\uparrow} - \hat{c}_{j+e_x,\uparrow}^\dagger \hat{c}_{j,\downarrow}) + i (\hat{c}_{j+e_y,\downarrow}^\dagger \hat{c}_{j,\uparrow} - \hat{c}_{j+e_y,\uparrow}^\dagger \hat{c}_{j,\downarrow}) \right] + h.c.$$ 

$$-h \sum_{j} \left( \epsilon_{j,\uparrow} \hat{c}_{j,\uparrow}^\dagger - \epsilon_{j,\downarrow} \hat{c}_{j,\downarrow}^\dagger \right)$$

$$+ \sum_{j} (\Delta \hat{c}_{j,\uparrow}^\dagger \hat{c}_{j,\downarrow}^\dagger + h.c.) + \sum_{j} \sum_{\sigma} V_j \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma},$$

where the nearest neighbor hopping matrix element $t (> 0)$, the chemical potential $\mu$, the spin-orbit coupling constant $\alpha (> 0)$, the magnitude of the Zeeman magnetic field $h$, and the superconducting gap $\Delta$. The lattice constant is normalized by 1. The annihilation and creation operators of electrons with spin $\sigma (= \uparrow, \downarrow)$ are, respectively, $\hat{c}_{j,\sigma}$ and $\hat{c}_{j,\sigma}^\dagger$ on site $j = (j_x, j_y)$. In the third term, i.e., Rashba spin-orbit coupling, the symbol $\epsilon_x$ means the unit vector along $x$-axis: $\epsilon_x = (1, 0)$. Similarly, $\epsilon_y$ is the unit vector along $y$-axis. The last term is the potential term. We focus on two kinds of the potential functions; (i) line-type potential ($V_j = V_j^{(L)}$) and (ii) point-type potential ($V_j = V_j^{(P)}$), where

$$V_j^{(L)} = V \delta_{j_x, N_x/2},$$

$$V_j^{(P)} = V \delta_{j_x, N_x/2} \delta_{j_y, N_y/2}. $$

The potential height $V$ is positive (repulsive potential). We will use the Fourier-transformed model along $y$-axis, for case (i). Tight-binding Hamiltonian $\hat{H}$ is equivalent to a 2D topological superconducting model in ultra-cold atomic gases, if the potential term is absent. Our model also describes a proximity-induced superconducting system on the interface between semiconductors and superconductors. In the junction systems, the potential term can be implemented by controlling the chemical potential via local gate voltage.

Let us concisely summarize the topological properties of our system in clean limit ($V_j = 0$ for any $j$). The spatial uniformity allows us to write down the tight-binding Hamiltonian in momentum space. The electron annihilation and creation operators are, respectively, $\hat{c}_{k,\sigma}$ and $\hat{c}_{k,\sigma}^\dagger$. They are bound as the 4-component vectors, $\Psi_k = (\hat{c}_{k,\uparrow}, \hat{c}_{k,\downarrow}, \hat{c}_{k,\uparrow}, \hat{c}_{k,\downarrow})^\dagger$ and $\Psi_k^\dagger = (\hat{c}_{k,\uparrow}, \hat{c}_{k,\downarrow}, \hat{c}_{k,\uparrow}, \hat{c}_{k,\downarrow})$. We find that $\hat{H}|_{V=0} = (1/2) \sum_k \Psi_k^\dagger \hat{H}_k \Psi_k$, with the Bogoliubov-de Gennes (BdG) Hamiltonian,

$$\hat{H}_k = s^0 + \frac{s^3}{2} (\varepsilon_k - h) \tau^3 + \frac{s^0 - s^3}{2} (\varepsilon_k + h) \tau^3$$

$$+ s^1 \alpha \ell_{1,k} \tau^0 + s^2 \alpha \ell_{2,k} \tau^3$$

$$+ s^3 (i \Delta \tau^++ h.c.),$$

where $\varepsilon_k = -\mu - 2t(\cos k_x + \cos k_y)$, $\ell_{1,k} = \sin k_y$, and $\ell_{2,k} = -\sin k_x$. The ith component of the $2 \times 2$ Pauli matrices ($i = 1, 2, 3$) is written by $s^i$ for spin. Similarly, the Pauli matrices for the Nambu space are written by $\tau^i$. The $2 \times 2$ identity matrices for spin and the Nambu space are, respectively, $s^0$ and $\tau^0$. The ladder operators in the Nambu space are defined by $\tau^{\pm} = (1/2)(\tau^1 \pm i\tau^2)$. The diagonalization of Eq. (4) leads to the bulk spectrum

$$E(k) = \sqrt{\varepsilon_k^2 + \alpha^2 |\ell_k|^2 + h^2 + |\Delta|^2 + 2\xi_k},$$

with $\ell_{1,k} = (\ell_{1,k}, \ell_{2,k})$ and $\xi_k = \sqrt{\varepsilon_k^2 + \alpha^2 |\ell_k|^2 + (\varepsilon_k + |\Delta|^2)h^2}$. The topological phase transition occurs when the energy gap of bulk spectrum closes at specific points in the Brillouin zone. According to Table I in Ref. [11] the topological property of the superconducting state for $\mu > 2t$ and $h < 3t$ changes from a trivial phase to a non-trivial one, when $h$ is greater than a critical value,

$$h > (4t - \mu)^2 + \Delta^2. $$

III. RESULTS

We numerically diagonalize tight-binding Hamiltonian $\hat{H}$, to examine the quasiparticle excitations in this system. The pair potential is $\Delta = 0.35t$ and the chemical potential is $\mu = 3.5t$, throughout this article. We accurately evaluate the eigenvalues and the eigenstates within $(-\Delta, \Delta)$, with a method by Sakurai and Sugura. The authors in Ref. [17] propose a practical way of solving a generalized eigenvalue problem, with a restricted eigenvalue domain. This approach leads to a numerical construction of a small-size effective Hamiltonian, keeping the essential properties of the original large-size Hamiltonian. The central idea is an efficient calculation of the projector onto an aimed energy domain. The contour integral for the projector on complex plane is approximated by numerical quadrature. The application to different superconducting issues and the detail implementation way are shown in Ref. [18].
A. Line-type potential

We concentrate on line-type potential \(2\). The set up is shown in Fig. 1(a). The translational symmetry along \(y\)-axis leads to a \(k_y\)-resolved Hamiltonian from Eq. 1. The number of the lattice size along \(x\)-axis is \(N_x = 120\). We impose the periodic boundary condition along \(x\)-axis. We study about the case of the Zeeman-magnetic-field strength to be \(h = t\), for a while. We find that, from Eq. (6), a topological superconducting state occurs in the clean-limit bulk system. Figure 2 shows that the potential with \(V = 100t\) induces the zero-energy states. Since the system without the potential is in a topological phase, these zero-energy states are regarded as the gapless bound states on the surfaces of the potential wall. In other words, the line-type potential with \(V = 100t\) is a line defect, i.e., wall. This ultra-high potential barrier completely divides the system into the two subsystems, left and right parts; the left part is separated from the right part, by vacuum.

![FIG. 1. Schematic diagrams of a system with (a) line-type potential and (b) point-type potential.](image)

The result for the ultra-high-barrier line-type potential motivates us to study lower-barrier cases. Figure 3 shows the energy dispersions, with \(V = 1t, 5t, \) and \(10t\). We find that for \(V = 5t\) and \(10t\) the quasiparticle excitations occur, within the energy domain \((-\Delta, \Delta)\). Thus, the line-type potentials with either high or intermediate barriers induce the MGS. Changing \(V\) means the continuous deformation of the energy spectrum of the system. Therefore, these MGS can continuously connect with the zero-energy states, when \(V\) increases. Another evidence for the MGS to be related to the zero-energy states is obtained, calculating the local density of states (LDOS). Figure 4 shows the local density of states at energy \(E = 7.4559 \times 10^{-2}t\), for \(V = 5t\). This energy corresponds to the lowest absolute eigenvalue of the Hamiltonian. We find that the corresponding states are located on the surfaces of the potential wall. The spin-imbalanced behavior in the LDOS attributes to the Zeeman magnetic field and the chemical-potential value; the first and the second terms in Eq. (4) imply that the down-spin component mainly contributes to the LDOS near zero energy. The occurrence of the MGS indicates an unconventional superconducting property of this system. This low-energy behavior is quite similar to the in-gap excitations in a \(p\)-wave superconductor with unitarity-limit non-magnetic impurities.7,8

![FIG. 2. (Color online) Quasiparticle spectrum for a ultra-high line-type potential function \((V = 100t)\), along \(y\)-axis. See Fig. 1(a), as well. The pair potential is \(\Delta = 0.35t\), and the chemical potential is \(\mu = 3.5t\). Since we set the Zeeman magnetic field as \(h = t\), the system without the potential stays at a topological phase [See, Eq. (6)]. The gapless states appear at \(k_y = \pm \pi\).](image)

![FIG. 3. (Color online) Quasiparticle spectrum for line-type potential, with different barrier heights. The other physical parameters are the same as in Fig. 2. When \(V \geq 5t\), the mid-gap states appear.](image)
h-V diagram, with $h = 1t$. The critical magnetic field for the topological transition is $h \approx 0.61t$, from Eq. (6). We focus on the case when $h > 0.61t$ (i.e., topological phase). For a high-barrier region ($V > 5t$), we find the occurrence of the MGS, which is indicated by the dark-colored area in Fig. 4(a). Figure 4(b) shows that the energy of the MGS is proportional to $1/V$. For a ultra-low barrier region ($V < 1t$), the lowest absolute eigenvalue does not significantly reduce. Indeed, we can find that this quantity is equal to the lowest energy of the bulk states. Therefore, no MGS is excited by the ultra-low potential. We obtain a curious result when $1t < V < 5t$. The lowest absolute eigenvalue is zero at the specific potential height, depending on $h$. Figure 4(b) also shows this behavior manifestly.

In terms of a topological number, let us explain the behaviors on the h-V diagram. The topological number for classifying the present 2D topological superconductor is the Thouless-Kohmoto-Nightingale-Nijs invariant. The bulk topological superconducting state for $(3t >) h > 0.6t$ and $\mu > 2t$ has the topological number $N = 1$, as seen in Ref. 11. Since the potential height is moderate, we may consider a local property of the topological state on the line (effective 1D superconducting system). The effective chemical potential on the line can be $\mu - V$, owing to the local filling change via the potential. We define the local topological number on the line as $M$. Repeating a similar argument to the derivation of Eq. (6), we find that this number takes a non-zero value (i.e., $M = 1$) when

$$h > \sqrt{(4t - (\mu - V))^2 + \Delta^2}. \quad (7)$$

Let us divide the h-V diagram into three regions, using Eqs. (6) and (7), as shown in Fig. 5. The red solid line is defined by Eq. (6), whereas the blue dashed one is by Eq. (7). We find that the latter line qualitatively reproduces the line of the zero-energy bound states in Fig. 5(a).

Now, let us summarize phase diagram, as seen in Fig. 5(a). In phase I, the absence of the MGS is found, since the system is topologically trivial ($N = M = 0$). On the boundary between phases I and III, the energy gap in bulk spectrum closes. In phase II, there is no MGS. This comes from the fact that the two topological number $N$ is the same as $M$ (i.e., $N = M = 1$). On the boundary between phases II and III, the energy gap of the “bulk” spectrum in the effective 1D system closes. In phase III, the occurrence of the MGS is found in the vicinity of the potential wall, since the topological number $N$ is different from $M$ (i.e., $N = 1$ and $M = 0$). Taking the ultra-high potential limit in phase III leads to the occurrence of the two zero-energy bound states, one of which is located on the left side of the potential barrier, and the other of which is on the right side. Reducing the potential height leads to the hybridization of these two zero-energy states, with hopping proportional to $1/V$; as a result, the degenerate zero eigenvalue is split. These arguments are sketched in Fig. 5(b).

### B. Point-type potential

Let us study point-type potential, as seen in Fig. 2(b). We set $N_p = N_y = 120$, with the periodic boundary condition. Our calculations focus on the case for $h > 0.61t$ (topological phase). Figure 7 shows the spin-resolved LDOS at energy $E = 9.8290 \times 10^{-2}t$, with $h = 2t$ and $V = 100t$. This energy is the lowest absolute eigenvalue of the Hamiltonian. We find that the bound states occur around the point potential, with non-zero energy. Figure 8 shows the $h$-dependence of the lowest absolute eigenvalue of the Hamiltonian, with different heights $V = 5t$ and $100t$. The curve for $V = 0t$ is calculated by exact bulk energy spectrum. This curve predominates the plots for $V \neq 0$. In contrast to the line-type potential, the point-type potential does not induce the zero-energy states. It indicates that the point-type potential is not a topological defect. However, the point-type potential induces the MGS at the potential center, although the link with the zero-energy surface states in a point defect is elusive. As mentioned in the case for the line-type potential, the appearance of the MGS is a concrete sign of the unconventional superconductors.

### C. Low-energy effective theory

All the above calculations show that the present system has the unconventional superconducting property when the topological phase occurs. Let us derive a low-energy effective theory to understand the numerical results. Our approach is similar to a low-energy theory in the semiconductor-superconductor junction systems.
FIG. 5. (Color online) (a) Eigenvalue of tight-binding Hamiltonian \[ 1 \], with the lowest absolute value, for line-type potential. The magnetic field \( h \) (the horizontal axis) and the potential height \( V \) (the vertical axis) are changed. We set the pair potential \( \Delta = 0 \). The black color means the existence of the zero-energy states. (b) Section in (a), with a fixed magnetic field strength (\( h = 1t \)). The value at \( V = 0.1t \) corresponds to the bulk spectral gap. Therefore, the behavior for \( V > 2t \) indicates the presence of the mid-gap states.

but we take a high-order correction to our low-energy description.

We focus on the spatial uniform case. We seek the low-energy region for constructing the effective theory, via the examination of the normal part BdG Hamiltonian [Eq. 4, with \( \Delta = 0 \)]. The energy eigenvalues in the particle subspace (\( \tau^3 = 1 \)) are \( E_{n,k}^\pm = \varepsilon_k \pm \sqrt{\hbar^2 + \alpha^2 |\ell_k|^2} \). We focus on the case when the chemical potential is located at the intermediate energy between the higher band and the lower band. All the previous calculations in this article correspond to this case. Hence, the Fermi surface is determined by the solution of \( 0 = E_{n,k}^- \). Since \( E_{n,k}^+ - E_{n,k}^- \geq 2\hbar \), the higher-energy contributions are negligible, when \( \hbar \rightarrow \infty \). A high magnetic-field value indicates that the electron spin is almost polarized upward. Thus, the low-energy subspace is attainable by the spin-up projector (\( \frac{1}{2} \))\( |s^0 + s^3\rangle \otimes \tau^0 \).

Our low-energy effective theory is built up by a perturbation approach, in the vicinity of the Fermi surface. The order of the perturbative expansions can be reasonably evaluated by \( \alpha/\hbar \), since the spin-orbit coupling mixes the low-energy sector (spin up) with the high-energy sector (spin down). Before the perturbative expansions, we perform a basis transformation to take a higher-order correction. This procedure is similar to the Tani-Foldy-Wouthuysen transformation \[ 31 \] for the Dirac equation. The unitary-transformed BdG Hamiltonian is \( \mathcal{H}_{k,\eta} = S_\eta \mathcal{H}_k S_\eta^{-1} \), with

\[
S_\eta = \exp \left[ \frac{\eta}{2} s^3 \otimes \left( \frac{i\Delta}{|\Delta|} \tau^+ + \text{h.c.} \right) \right].
\]

We take so small angle \( \eta \) that \( \eta \sim \mathcal{O}(\alpha/\hbar) \). In other words, we partially diagonalize the Hamiltonian, with the restricted rotation \( \mathcal{S} \). We choose a free part of \( \mathcal{H}_{k,\eta} \), as seen in Eq. (A1) in Appendix A. Then, the 2nd-order Brillouin-Wigner approach \[ 22 \] leads to an effective Hamiltonian,

\[
\mathcal{H}_{k,\eta}^{\text{eff}} = s^0 + s^3 \otimes \left[ \frac{s^0 + s^3}{2} \mathcal{H}_k^{\text{eff}} + \frac{s^0 - s^3}{2} (h_{-k}^{\text{eff}})^* \right. \\
\left. + (i\Delta_k^{\text{eff}} \tau^+ + \text{h.c.}) + \mathcal{O}(\alpha^2/\hbar^2) \right].
\]

FIG. 6. (Color online) (a) Phase diagram with respect to a topological invariant. The red solid line denotes the topological phase transition in the bulk, defined by Eq. (6). The blue dotted line denotes the topological phase transition on the line-potential, defined by Eq. (7). (b) Schematic diagram of the different phases. The phase III only has the bound states at the boundary.
FIG. 7. (Color online) Spin-resolved local density of states at energy $E = 9.8290 \times 10^{-2}t$ in the system with a point-type potential ($V = 100t$). See Fig. 1(b). The Zeeman magnetic field is $h = 2t$. The other parameters are the same as in Fig. 2. (a) Up spin component and (b) down spin component of the wave functions. We note that in this figure we focus on the area around the potential center.

with $h_k^{\text{eff}} = \varepsilon_k - h - (\alpha^2/h)|\ell_k|^2 + (|\Delta|^2/h)$ and

$$\Delta_k^{\text{eff}} = -\frac{2\alpha}{h}\Delta(\ell_{2,k} + i\ell_{1,k}) - \frac{2i\eta|\Delta|}{h}\Delta. \quad (10)$$

The detail calculations are shown in Appendix A but we concisely illustrate how effective gap function (10) is produced from the original $s$-wave mean-filed term. In Eq. (3), the spin-orbit couplings (the second line) do not commute with the pairing potential term (the third line). Thus, in the perturbation analysis, the cross terms between them may exist, and they lead to the effective superconducting gaps. When $\eta \to 0$ (i.e., no unitary transformation is performed), the chiral $p$-wave low-energy behavior is survived, whereas the $s$-wave contribution disappears. This result is consistent with the low-energy theory in the semiconductor-superconductor junction systems. 19 Therefore, the full gap feature emerges as a higher-order correction in the low-energy effective Hamiltonian. When a magnetic field is high, the $p$-wave component is predominant. The occurrence of the MGS via the potential terms is consistent with this statement.

A similar argument is possible for a 3D topological superconductor. The low-energy effective theory is attainable, taking large-mass (nonrelativistic) limit. 24 This assertion is reasonable, since the Zeeman magnetic field in the 2D topological superconductor is regarded as the mass term. Moreover, we mention that, taking massless (ultrarelativistic) limit 24,25 (i.e. a weak magnetic field in our system), the superconductivity is robust against non-magnetic impurities, like $s$-wave superconductivity.

FIG. 8. (Color online) Magnetic-field dependence of the lowest absolute eigenvalue for point-type potential, with heights $V = 5t$ and $V = 100t$. The dashed line ($V = 0t$) is obtained by exact bulk energy spectrum (6). The other parameters are the same as in Fig. 2.

IV. SUMMARY

We studied the MGS in a two-dimensional time-reversal-symmetry-broken topological $s$-wave superconductor with spatial inhomogeneity. Two kinds of the potential functions, line-type potential and point-type potential were examined. We showed that the mid-gap states are locally bound in the vicinity of the potential region. We obtained a smooth connection of the MGS into the zero-energy states, changing the potential into a hard-wall potential. To obtain better understanding for the results, we derived a low-energy effective theory.
We claim that the violation of Anderson’s theorem in our system has the same mechanism as the cases of other unconventional superconductivity, such as p-wave.

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**Appendix A: Derivation of an effective Hamiltonian**

We show the derivation of Eq. (A1). We start with the unitary-transformed BdG Hamiltonian $H_{k,\eta} = \Sigma \eta H_{k,\eta}^{-1}$, where $\Sigma \eta$ is defined by Eq. (A2). We take so small angle $\eta$ that $\eta \sim O(\alpha/h)$. After straightforward calculations, we find that $H_{k,\eta} = H_{k,\eta}^{(0)} + \nu_{k,\eta} + O(\eta^2)$, with

$$H_{k,\eta}^{(0)} = \frac{s^0 + s^3}{2} \otimes [(\varepsilon_k - \hbar)\tau^3 + (i\Delta_{k,\eta}^{(+)})\tau^+ + \text{h.c.}]$$

$$+ \frac{s^0 - s^3}{2} \otimes [(\varepsilon_k + \hbar)\tau^3 + (i\Delta_{k,\eta}^{-} )\tau^+ + \text{h.c.}],$$

$$\nu_{k,\eta} = s^3 \otimes (\alpha\ell_{1,k} + i\Delta_{k,\eta}^{(1)})\tau^0$$

$$+ s^2 \otimes [\alpha\ell_{2,k}\tau^3 + (i\Delta_{k,\eta}^{(2)})\tau^+ + \text{h.c.}],$$

(A1)

The four additional gap functions are $\Delta_{k,\eta}^{(+)} = \mp i\eta(\varepsilon_k \mp \hbar)(\Delta^{(+)} / |\Delta|)$, $\Delta_{k,\eta}^{(1)} = -i\eta|\Delta|$, and $\Delta_{k,\eta}^{(2)} = [1 - (\eta\alpha\ell_{1,k} / |\Delta|)]\Delta$. We set the free part for the perturbation approach as $H_{k,\eta}^{(0)}$.

Now, we derive the low-energy effective Hamiltonian. The 2nd-order Brillouin-Wigner approach leads to an effective Hamiltonian

$$H_{k,\eta}^{\text{eff}} = \mathcal{P} H_{k,\eta}^{(0)} \mathcal{P} + \sum_{m=1}^{2} (\mathcal{P} \nu_{k,\eta} Q) R_{k,\eta,m}(Q \nu_{k,\eta} \mathcal{P}),$$

(A3)

with $\mathcal{P} = (1/2)(s^0 + s^3) \otimes \tau^0$, $Q = s^0 \otimes \tau^0 - \mathcal{P}$ and $R_{k,\eta,m} = Q[E_{k,\eta,m}^{(0)} - Q H_{k,\eta}^{(0)} Q^{-1}]$. The non-perturbative energy $E_{k,\eta}^{(0)}$ is related to the eigenvalues of $\mathcal{P} H_{k,\eta}^{(0)} \mathcal{P}$: $\mathcal{P} H_{k,\eta}^{(0)} \mathcal{P} = (1/2)(s^0 + s^3) \otimes \text{diag}\{E_{k,\eta,1}^{(0)}, E_{k,\eta,2}^{(0)}\}$. We approximate $R_{k,\eta,m} \approx R_{k_F,\eta,m}$, with the Fermi momentum $k_F$ given by $\epsilon_{k_F} = \hbar + O(\alpha^2 / h^2)$.

Taking the leading term, we find that $R_{k,\eta,m} \approx (1/2)(s^0 - s^3) \otimes (-1/2)\eta^2$. Therefore, performing straightforward algebraic calculations about the Pauli matrices, we obtain Eq. (A1). In the effective gap $\Delta_{k,\eta}^{\text{eff}}$, the chiral p-wave component (the first term) comes from $\Delta_{k,\eta}^{(+)}$. In contrast, the s-wave component (the second term) is attributable to $\Delta_{k,\eta}^{(2)}$. We remark that the contribution from $\Delta_{k,\eta}^{(+)}$ in Eq. (A1) is negligible, since on the Fermi surface $\Delta_{k,\eta}^{(+)} \sim \eta O(\alpha^2 / h^2) \sim O(\alpha^3 / h^3)$. The contribution from $\Delta_{k,\eta}^{(-)}$ is irrelevant to the low-energy sector, owing to the spin-up projector $\mathcal{P}$.

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