Finite horizon mean field games on networks

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Abstract
We consider finite horizon stochastic mean field games in which the state space is a network. They are described by a system coupling a backward in time Hamilton–Jacobi–Bellman equation and a forward in time Fokker–Planck equation. The value function $u$ is continuous and satisfies general Kirchhoff conditions at the vertices. The density $m$ of the distribution of states satisfies dual transmission conditions: in particular, $m$ is generally discontinuous across the vertices, and the values of $m$ on each side of the vertices satisfy some compatibility conditions. The stress is put on the case when the Hamiltonian is Lipschitz continuous. Existence, uniqueness and regularity results are proven.

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1 Introduction and main results

The theory of mean field games (MFGs for short) is more and more investigated since the pioneering works [15–17] of Lasry and Lions: it aims at studying the asymptotic behavior of differential games (Nash equilibria) as the number of agents tends to infinity. In the present work, we study stochastic MFGs on networks with finite time horizon; they are described by a system of PDEs coupling a Fokker–Planck (FP) equation for the density of the distribution of states (forward in time) and a Hamilton–Jacobi–Bellman (HJB) equation for the optimal
value of a representative agent (backward in time). This work is a continuation of [2], which was devoted to MFGs on networks in the case of an infinite time horizon. We refer to [2] for a more extended discussion on MFGs and for additional references on the analysis of the systems of PDEs that stem from the model when there is no common noise.

A network (or a graph) is a set of items, referred to as vertices (or nodes or crosspoints), with connections between them referred to as edges. In the recent years, there has been an increasing interest in the investigation of dynamical systems and differential equations on networks, in particular in connection with problems of data transmission and traffic management. The literature on optimal control in which the state variable takes its values on a network is recent: deterministic control problems and related Hamilton–Jacobi equations were studied in [1,3,11,12,21,22]. Stochastic processes on networks and related Kirchhoff conditions at the vertices were studied in [8,9].

There are two articles devoted to infinite horizon MFGs on networks, [6] and [2]. In [6], Camilli and Marchi consider a particular type of Kirchhoff condition at the vertices for the value function, which comes from an assumption which can be informally stated as follows: consider a vertex $\nu$ and assume that it is the intersection of $p$ edges $\Gamma_1, \ldots, \Gamma_p$; if, at time $\tau$, the controlled stochastic process $X_t$ associated to a given agent hits $\nu$, then the probability that $X_{\tau+}$ belongs to $\Gamma_i$ is proportional to the diffusion coefficient in $\Gamma_i$. Under this assumption, it can be seen that the density of the distribution of states is continuous at the vertices of the network. In [2], the above mentioned assumption is not made any longer, so the value function satisfies general Kirchhoff conditions and accordingly, the continuity condition for the density of the distribution of states is replaced by suitable compatibility conditions on the jumps across the vertices. In the same article, [2], one can find a complete study of the system of differential equations arising in infinite horizon MFGs on networks with at most quadratic Hamiltonians and quite general coupling costs.

Here, we study finite horizon MFGs with the same Kirchhoff conditions for the value function and transmission conditions for the density of the distribution of states as in [2]; these are obtained from the theory of stochastic control in [8,9], see Sect. 1.3 below. We focus on the case of globally Lipschitz Hamiltonians and concentrate on the difficulties induced by the Kirchhoff conditions in the time-dependent MFG system of PDEs. Therefore, this work should be seen as a first and necessary step in order to deal with more difficult situations in the finite horizon setting, for example with quadratic or subquadratic Hamiltonians. We believe that handling the latter cases (at least in the framework of weak solutions) will be possible by combining our results with techniques that can be found in [17,23,24].

After justifying the transmission conditions at the vertices for both the value function and the density, we shall prove existence and uniqueness of weak solutions of the uncoupled HJB and FP equations (in suitable space-time Sobolev spaces), then regularity results. An important step consists of obtaining regularity results for HJB equations with Kirchhoff conditions at the vertices. This will be achieved by differentiating the Bellman equation, carefully coping with the Kirchhoff conditions and using the hypothesis that the Hamiltonian is globally Lipschitz continuous. Without the latter hypothesis, the question of regularity is open to the best of our knowledge, and the main difficulty lies in obtaining gradient bounds on the solutions. This aspect is an important difference with the previous work, [2], devoted to the infinite horizon case.

General references on classical solutions of linear and nonlinear parabolic equations can be found in [14,18]. For weak solutions of HJB equations, a well known reference is [5], the techniques of which will be used in the present work. Concerning networks, Von Below [25] established the existence and uniqueness of classical solutions of linear parabolic equations with smooth coefficients and general Kirchhoff conditions. There are few works on parabolic
1.1 Networks and function spaces

1.1.1 The geometry

A bounded network \( \Gamma \) (or a bounded connected graph) is a connected subset of \( \mathbb{R}^n \) made of a finite number of bounded non-intersecting straight segments, referred to as edges, which connect nodes referred to as vertices. The finite collection of vertices and the finite set of closed edges are respectively denoted by \( \mathcal{V} := \{v_i, i \in I\} \) and \( \mathcal{E} := \{\Gamma_\alpha, \alpha \in \mathcal{A}\} \), where \( I \) and \( \mathcal{A} \) are finite sets of indices contained in \( \mathbb{N} \). We assume that for \( \alpha, \beta \in \mathcal{A} \), if \( \alpha \neq \beta \), then \( \Gamma_\alpha \cap \Gamma_\beta \) is either empty or made of a single vertex. The length of \( \Gamma_\alpha \) is denoted by \( \ell_\alpha \). Given \( v_i \in \mathcal{V} \), the set of indices of edges that are adjacent to the vertex \( v_i \) is denoted by \( \mathcal{A}_i = \{ \alpha \in \mathcal{A} : v_i \in \Gamma_\alpha \} \). A vertex \( v_i \) is named a boundary vertex if \( |\mathcal{A}_i| = 1 \), otherwise it is named a transition vertex. The set containing all the boundary vertices is named the boundary of the network and is denoted by \( \partial \Gamma \) hereafter.

The edges \( \Gamma_\alpha \in \mathcal{E} \) are oriented in an arbitrary manner. In most of what follows, we shall make the following arbitrary choice that an edge \( \Gamma_\alpha \) connecting two vertices \( v_i \) and \( v_j \), with \( i < j \) is oriented from \( v_i \) toward \( v_j \): this induces a natural parameterization \( \pi_\alpha : [0, \ell_\alpha] \rightarrow \Gamma_\alpha = [v_i, v_j] \):

\[
\pi_\alpha(y) = (\ell_\alpha - y)v_i + yv_j \quad \text{for} \quad y \in [0, \ell_\alpha].
\]

For a function \( v : \Gamma \rightarrow \mathbb{R} \) and \( \alpha \in \mathcal{A} \), we define \( v_\alpha : (0, \ell_\alpha] \rightarrow \mathbb{R} \) by

\[
v_\alpha(y) := v \circ \pi_\alpha(y), \quad \text{for all} \quad y \in (0, \ell_\alpha).
\]

The function \( v_\alpha \) is a priori defined only in \((0, \ell_\alpha)\). When it is possible, we extend it by continuity at the boundary by setting \( v_\alpha(0) := \lim_{y \to 0^+} v_\alpha(y) \) and \( v_\alpha(\ell_\alpha) := \lim_{y \to \ell_\alpha} v_\alpha(y) \). In that latter case, we can define

\[
v |_{\Gamma_\alpha} (x) = \begin{cases} v_\alpha(\pi_\alpha^{-1}(x)), & \text{if} \ x \in \Gamma_\alpha \setminus \mathcal{V}, \\ v_\alpha(0) = \lim_{y \to 0^+} v_\alpha(y), & \text{if} \ x = v_i, \\ v_\alpha(\ell_\alpha) = \lim_{y \to \ell_\alpha} v_\alpha(y), & \text{if} \ x = v_j. \end{cases}
\]  

(1.1)
Notice that \( v|_{\Gamma_a} \) does not coincide with the original function \( v \) at the vertices in general when \( v \) is not continuous.

**Remark 1.1** In what precedes, the edges have been arbitrarily oriented from the vertex with the smaller index toward the vertex with the larger one. Other choices are of course possible. In particular, by possibly dividing a single edge into two, adding thereby new artificial vertices, it is always possible to assume that for all vertices \( \nu_i \in \mathcal{V} \),

\[
\text{either } \pi_\alpha(0) = \nu_i, \text{ for all } \alpha \in \mathcal{A}_i \text{ or } \pi_\alpha(\ell_\alpha) = \nu_i, \text{ for all } \alpha \in \mathcal{A}_i. \tag{1.2}
\]

This idea was proposed by Von Below in [25]: some edges of \( \Gamma \) are cut into two by adding artificial vertices so that the new oriented network \( \tilde{\Gamma} \) has the property (1.2), see Fig. 1 for an example.

### 1.1.2 Function spaces related to the space variable

The set of continuous functions on \( \Gamma \) is denoted by \( C(\Gamma) \) and we set

\[
PC(\Gamma) = \left\{ v : \Gamma \to \mathbb{R} : \text{for all } \alpha \in \mathcal{A}, \ v_\alpha \in C(0, \ell_\alpha) \right\}.
\]

By the definition of piecewise continuous functions \( v \in PC(\Gamma) \), for all \( \alpha \in \mathcal{A} \), it is possible to define \( v|_{\Gamma_a} \) by (1.1) and we have \( v|_{\Gamma_a} \in C(\Gamma_a), v_\alpha \in C([0, \ell_\alpha]) \).

For \( m \in \mathbb{N} \), the space of \( m \)-times continuously differentiable functions on \( \Gamma \) is defined by

\[
C^m(\Gamma) := \left\{ v \in C(\Gamma) : v_\alpha \in C^m([0, \ell_\alpha]) \right\}.
\]

Notice that \( v \in C^m(\Gamma) \) is assumed to be continuous on \( \Gamma \), and that its restriction \( v|_{\Gamma_a} \) to each edge \( \Gamma_a \) belongs to \( C^m(\Gamma_a) \). The space \( C^m(\Gamma) \) is endowed with the norm \( \| v \|_{C^m(\Gamma)} := \sum_{\alpha \in \mathcal{A}} \sum_{k \leq m} \| \partial^k v_\alpha \|_{L^\infty(0, \ell_\alpha)} \). For \( \sigma \in (0, 1) \), the space \( C^{m,\sigma}(\Gamma) \), contains the functions \( v \in C^m(\Gamma) \) such that \( \partial^m v_\alpha \in C^{0,\sigma}([0, \ell_\alpha]) \) for all \( \alpha \in \mathcal{A} \); it is endowed with the norm

\[
\| v \|_{C^{m,\sigma}(\Gamma)} := \| v \|_{C^m(\Gamma)} + \sup_{\alpha \in \mathcal{A}} \sup_{y \neq z \in [0, \ell_\alpha]} \frac{|\partial^m v_\alpha(y) - \partial^m v_\alpha(z)|}{|y - z|^{\sigma}}.
\]
For a positive integer $m$ and a function $v \in C^m(\Gamma)$, we set for $k \leq m$,
\[ \partial^k v (x) = \partial^k v_{\alpha} (\pi_{\alpha}^{-1} (x)) \quad \text{if} \quad x \in \Gamma_{\alpha} \setminus \mathcal{Y} . \]
For a vertex $v$, we define $\partial_{\alpha} v (v)$ as the outward directional derivative of $v|_{\Gamma_{\alpha}}$ at $v$ as follows:
\begin{equation}
\partial_{\alpha} v (v) := \begin{cases} 
\lim_{h \to 0^+} \frac{v_{\alpha} (0) - v_{\alpha} (h)}{h}, & \text{if} \quad v = \pi_{\alpha} (0), \\
\lim_{h \to 0^+} \frac{v_{\alpha} (\ell_{\alpha}) - v_{\alpha} (\ell_{\alpha} - h)}{h}, & \text{if} \quad v = \pi_{\alpha} (\ell_{\alpha}). 
\end{cases}
\end{equation}
(1.3)
For all $i \in I$ and $\alpha \in \mathcal{A}_i$, setting
\begin{equation}
n_{i\alpha} = \begin{cases} 
1 & \text{if} \quad v_i = \pi_{\alpha} (\ell_{\alpha}), \\
-1 & \text{if} \quad v_i = \pi_{\alpha} (0), 
\end{cases}
\end{equation}
(1.4)
we have
\[ \partial_{\alpha} v (v_i) = n_{i\alpha} \partial v|_{\Gamma_{\alpha}} (v_i) = n_{i\alpha} \partial v_{\alpha} (\pi_{\alpha}^{-1} (v_i)). \]
(1.5)

**Remark 1.2** Changing the orientation of the edge does not change the value of $\partial_{\alpha} v (v)$ in (1.3).

We say that $v$ is Lebesgue-integrable on $\Gamma_{\alpha}$ if $v_{\alpha}$ is Lebesgue-integrable on $(0, \ell_{\alpha})$. In this case, for all $x_1, x_2 \in \Gamma_{\alpha}$,
\begin{equation}
\int_{[x_1, x_2]} v (x) \, dx := \int_{\pi_{\alpha}^{-1} (x_2)}^{\pi_{\alpha}^{-1} (x_1)} v_{\alpha} (y) \, dy.
\end{equation}
(1.6)
When $v$ is Lebesgue-integrable on $\Gamma_{\alpha}$ for all $\alpha \in \mathcal{A}$, we say that $v$ is Lebesgue-integrable on $\Gamma$ and we define
\[ \int_{\Gamma} v (x) \, dx := \sum_{\alpha \in \mathcal{A}} \int_0^{\ell_{\alpha}} v_{\alpha} (y) \, dy. \]
The space $L^p (\Gamma) = \{ v : v|_{\Gamma_{\alpha}} \in L^p (\Gamma_{\alpha}) \text{ for all } \alpha \in \mathcal{A}, \ p \in [1, \infty] \}$, is endowed with the norm
\[ \| v \|_{L^p (\Gamma)} := \left( \sum_{\alpha \in \mathcal{A}} \| v_{\alpha} \|_{L^p (0, \ell_{\alpha})}^p \right)^{\frac{1}{p}} \]
if $1 \leq p < \infty$, and $\max_{\alpha \in \mathcal{A}} \| v_{\alpha} \|_{L^\infty (0, \ell_{\alpha})}$ if $p = +\infty$.

We shall also need to deal with functions on $\Gamma$ whose restrictions to the edges are weakly-differentiable: we shall use the same notations for the weak derivatives.

**Definition 1.1** For any integer $s \geq 1$ and any real number $p \geq 1$, the Sobolev space $W^{s, p}_b (\Gamma)$ (the index $b$ stands for “broken”) is defined as follows
\[ W^{s, p}_b (\Gamma) := \{ v : v \rightarrow \mathbb{R} \text{ s.t. } v_{\alpha} \in W^{s, p} (0, \ell_{\alpha}) \text{ for all } \alpha \in \mathcal{A} \}, \]
and endowed with the norm
\[ \| v \|_{W^{s, p}_b (\Gamma)} = \left( \sum_{k=1}^{s} \sum_{\alpha \in \mathcal{A}} \| \partial^k v_{\alpha} \|_{L^p (0, \ell_{\alpha})}^p + \| v \|_{L^p (\Gamma)}^p \right)^{\frac{1}{p}}. \]
For $s \in \mathbb{N} \setminus \{0\}$, we also set $H^s_b (\Gamma) = W^{s, 2}_b (\Gamma)$ and $H^s (\Gamma) = C (\Gamma) \cap H^s_b (\Gamma)$.

Finally, when dealing with probability distributions in mean field games, we will often use the set $\mathcal{M}$ of probability densities, i.e., $m \in L^1 (\Gamma)$, $m \geq 0$ and $\int_{\Gamma} m (x) \, dx = 1$. 
1.1.3 Some space-time function spaces

The space of continuous real valued functions on $\Gamma \times [0, T]$ is denoted by $C(\Gamma \times [0, T])$.

Let $PC(\Gamma \times [0, T])$ be the space of the functions $v : \Gamma \times [0, T] \to \mathbb{R}$ such that

1. for all $t \in [0, T]$, $v(\cdot, t)$ belongs to $PC(\Gamma)$
2. for all $\alpha \in \mathscr{A}$, $v|_{\Gamma_\alpha \times [0, T]}$ is continuous on $\Gamma_\alpha \times [0, T]$.

For a function $v \in PC(\Gamma \times [0, T])$, $\alpha \in \mathscr{A}$, we set $v_\alpha(y, t) = v|_{\Gamma_\alpha \times [0, t]}(\pi_\alpha(y), t)$ for all $(y, t) \in [0, \ell_\alpha] \times [0, T]$.

For two nonnegative integers $m$ and $n$, let $C^{m,n}(\Gamma \times [0, T])$ be the space of continuous real valued functions $v$ on $\Gamma \times [0, T]$ such that for all $\alpha \in \mathscr{A}$, $v|_{\Gamma_\alpha \times [0, T]} \in C^{m,n}(\Gamma_\alpha \times [0, T])$.

Useful results on continuous and compact embeddings of space-time function spaces are recalled in “Appendix A”.

1.2 A class of stochastic processes on $\Gamma$

For all $\alpha \in \mathscr{A}$ and $i \in I$, consider positive constants $\mu_\alpha > 0$, $p_i > 0$ with $\sum_{\alpha \in \mathscr{A}_i} p_i = 1$ and a real valued function $a \in PC(\Gamma \times [0, T])$, such that $a|_{\Gamma_\alpha}(\cdot, t)$ belongs to $C^1(\Gamma_\alpha)$ for all $t \in [0, T]$.

Without loss of generality, we may assume that $\ell_\alpha = 1$ for all $\alpha \in \mathscr{A}$ (see Remark 1.4) and, following Remark 1.1, that (1.2) holds by possibly adding artificial nodes: if $v_i$ is such an artificial node, then $\mathcal{Z}(\mathcal{A}_i) = 2$, and we assume that $p_i = 1/2$ for $\alpha \in \mathcal{A}_i$. The diffusion parameter $\mu$ has the same value on the two sides of an artificial vertex. Similarly, the function $a$ does not have jumps across an artificial vertex.

Consider a Brownian motion $(W_t)$ defined on the real line. Following Freidlin and Sheu [8], we know that there exists a unique Markov process on $\Gamma$ with continuous sample paths that can be written $(X_t, \alpha_t)$ where $X_t \in \Gamma_{\alpha_t}$ (if $X_t = v_i$, $i \in I$, $\alpha_t$ is arbitrarily chosen as the smallest index in $\mathcal{A}_i$) such that, defining the process $x_t = \pi_{\alpha_t}(X_t)$ with values in $[0, 1]$, we have

$$- dx_t = \sqrt{2\mu_\alpha} dW_t + a_{\alpha_t}(x_t) dt + d\ell_{\alpha_t} + dh_{\alpha_t},$$

$- \ell_{\alpha_t}$ is continuous non-decreasing process (measurable with respect to the $\sigma$-field generated by $(X_t, \alpha_t)$) which increases only when $X_t = v_i$ and $x_t = 0$,

$- h_{\alpha_t}$ is continuous non-increasing process (measurable with respect to the $\sigma$-field generated by $(X_t, \alpha_t)$) which decreases only when $X_t = v_i$ and $x_t = 1$,

For all $i \in I$ and $\alpha \in \mathcal{A}_i$, $\mathbb{P}(X_{t_i} \in \Gamma_\alpha | X_t = v_i) = p_i$,

and for all function $v \in C^{2,1}(\Gamma \times [0, T])$ such that

$$v(\cdot, t) \in D := \left\{ u \in C^2(\Gamma) : \sum_{\alpha \in \mathcal{A}_i} p_i \partial_{\alpha} u(v_i) = 0, \text{ for all } i \in I \right\} \text{ for all } t \in [0, T],$$

the process

$$M_t = v(X_t, t) - \int_0^t \left( \partial_t v(X_s, s) + \mu_{\alpha_t} \partial^2 v(X_s, s) + a|_{\Gamma_{\alpha_t}}(X_s, s) \partial v(X_s, s) \right) ds$$

is a martingale, i.e.,

$$\mathbb{E}(M_t | X_s) = M_s, \quad \text{for all } 0 \leq s < t \leq T.$$
Consider \( u \in D \), the condition at boundary vertices boils down to a Neumann condition.

Remark 1.3 If \( u \in D \), the condition at boundary vertices boils down to a Neumann condition.

Remark 1.4 The assumption that all the edges have unit length is not restrictive, because we can always rescale the constants \( \mu_\alpha \) and the piecewise continuous function \( a \).

The goal of this paragraph is to justify formally the boundary value problem satisfied by the law of the stochastic process \( X_t \). At a formal level, we may assume that the latter has a density \( m(x, t) \) with respect to Lebesgue measure on \( \Gamma \) and that \( m \) is regular enough so that what follows makes sense:

\[
\mathbb{E}[u(X_t, t)] = \int_\Gamma v(x, t) m(x, t) \, dx, \quad \text{for all } v \in PC(\Gamma \times [0, T]). \tag{1.10}
\]

Using \( u \in C^{2,1}(\Gamma \times [0, T]) \) such that for all \( t \in [0, T], u(\cdot, t) \in D \). Then, from (1.8)–(1.9), we see that

\[
\mathbb{E}[u(X_t, t)] = \mathbb{E}[u(X_0, 0)] + \mathbb{E}\left[ \int_0^t \left( \partial_t u(X_s, s) + \mu_\alpha \partial^2 u(X_s, s) + a|\Gamma_a (X_s, s) \partial u(X_s, s) \right) \, ds \right]. \tag{1.11}
\]

Using (1.10) and taking the time-derivative of each member of (1.11), we obtain

\[
\int_\Gamma \partial_t (um)(x, t) \, dx = \mathbb{E}\left( \partial_t u(X_t, t) + \mu_\alpha \partial^2 u(X_t, t) + a|\Gamma_a (X_t, t) \partial u(X_t, t) \right). \tag{1.12}
\]

By integration by parts, recalling (1.2), we get

\[
0 = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_a} \left( \partial_t m(x, t) - \mu_\alpha \partial^2 m(x, t) + \partial (am)(x, t) \right) u(x, t) \, dx
\]

\[
- \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} \left[ n_{ia} a|\Gamma_a(v_i, t)m|\Gamma_a(v_i, t) - \mu_\alpha \partial_\alpha m(v_i, t) \right] u|\Gamma_a(v_i, t)
\]

\[
- \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} \mu_\alpha m|\Gamma_a(v_i, t) \partial_\alpha u(v_i, t), \tag{1.12}
\]

where \( n_{ia} \) is defined in (1.4).

We choose first, for every \( \alpha \in \mathcal{A} \), a smooth function \( u \) which is compactly supported in \((\Gamma_a \setminus \mathcal{F}) \times [0, T]\). Hence \( u|\Gamma_a(v_i, t) = 0 \) and \( \partial_\beta u(v_i, t) = 0 \) for all \( i \in I, \beta \in \mathcal{A}_i \). Notice that \( u(\cdot, t) \in D \). It follows that \( m \) satisfies

\[
\left( \partial_t m - \mu_\alpha \partial^2 m + \partial (am) \right)(x, t) = 0, \quad \text{for } x \in \Gamma_a \setminus \mathcal{F}, \ t \in (0, T), \ \alpha \in \mathcal{A}. \tag{1.14}
\]

For a smooth function \( \chi : [0, T] \to \mathbb{R} \) compactly supported in \((0, T)\), we may choose for every \( i \in I \), a smooth function \( u \) such that \( u(v_j, t) = \chi(t) \delta_{i,j} \) for all \( t \in [0, T], j \in I \) and \( \partial_\alpha u(v_j, t) = 0 \) for all \( t \in [0, T], j \in I \) and \( \alpha \in \mathcal{A}_j \), we infer a condition for \( m \) at the vertices,

\[
\sum_{\alpha \in \mathcal{A}_i} n_{ia} a|\Gamma_a(v_i, t)m|\Gamma_a(v_i, t) - \mu_\alpha \partial_\alpha m(v_i, t) = 0 \quad \text{for all } i \in I, \ t \in (0, T). \tag{1.15}
\]

This condition is called a transmission condition if \( v_i \) is a transition vertex and reduces to a Robin boundary condition when \( v_i \) is a boundary vertex.
Finally, for a smooth function $\chi : [0, T] \to \mathbb{R}$ compactly supported in $(0, T)$, for every transition vertex $v_i \in \mathcal{V} \setminus \partial \Gamma$ and $\alpha, \beta \in \mathcal{A}_i$, we choose $u$ such that $u(\cdot, t) \in D$, $\partial_{\alpha} u(v_i, t) = \chi(t)/p_{i\alpha}$, $\partial_{\beta} u(v_i) = -\chi(t)/p_{i\beta}$, $\partial_{\gamma} u(v_i) = 0$ for $\gamma \in \mathcal{A}_i \setminus \{\alpha, \beta\}$, and the directional derivatives of $u$ at the vertices $v \neq v_i$ are 0. Using such a test-function in (1.12) yields a jump condition for $m$,

$$
m|_{\Gamma_{\alpha}}(v_i, t) = \frac{m|_{\Gamma_{\beta}}(v_i, t)}{\gamma_{i\beta}}, \quad \text{for all } \alpha, \beta \in \mathcal{A}_i, v_i \in \mathcal{V}, t \in (0, T),
$$
in which

$$
\gamma_{i\alpha} = \frac{p_{i\alpha}}{\mu_{\alpha}}, \quad \text{for all } i \in I, \alpha \in \mathcal{A}_i. \tag{1.13}
$$

Summarizing, we get the following boundary value problem for $m$ [recall that the coefficients $n_{i\alpha}$ are defined in (1.4)]:

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{A}_i} \mu_{\alpha} \partial_{\alpha} m (v_i, t) - n_{i\alpha} a|_{\Gamma_{\alpha}}(v_i, t)m|_{\Gamma_{\alpha}}(v_i, t) = 0, & \quad t \in (0, T), v_i \in \mathcal{V}, \\
\partial_{\mu} m - \mu_{\alpha} \partial_{\alpha} m + \partial (ma) = 0, & \quad (x, t) \in (\Gamma_{\alpha} \setminus \mathcal{V}) \times (0, T), \alpha \in \mathcal{A}, \\
m(x, 0) = m_0(x), & \quad x \in \Gamma.
\end{aligned} \tag{1.14}
$$

1.3 Formal derivation of the MFG system on $\Gamma$

Here we aim at obtaining, at least formally, the MFG system of forward-backward partial differential equations on the network. The assumptions that we are going to make below on the optimal control problem are a little restrictive, for two reasons: first, we wish to avoid some technicalities linked to the measurability of the control process; second, the assumptions on the costs must be consistent with the assumptions that we shall make on the Hamiltonian, see Sect. 1.4.2 below. In particular, we shall impose that the Hamiltonian is globally Lipschitz continuous. More general cases, e.g. with quadratic Hamiltonians, will be the subject of a future work.

Consider a continuum of indistinguishable agents moving on the network $\Gamma$. The state of a representative agent at time $t$ is a time-continuous controlled stochastic process $X_t$ as defined in Sect. 1.2, associated with some diffusion coefficients $\mu_{\alpha}$ in the edges and some transition probabilities $p_{i\alpha}$ at the vertices. The control is the drift $a_t$. Let $m(\cdot, t)$ be the probability measure on $\Gamma$ that describes the distribution of states at time $t$.

For a representative agent, the optimal control problem is of the form

$$
v(x, t) = \inf_{a_t} \mathbb{E}_{x,t} \left[ \int_t^T (L(X_s, a_s) + \mathcal{V}[m(\cdot, t)](X_s)) \, ds + v_T(X_T) \right], \tag{1.15}
$$

where $\mathbb{E}_{x,t}$ stands for the expectation conditioned by the event $X_t = x$.

We discuss the ingredients appearing in (1.15):

- We assume that the control is in feedback form, $a_t = a(X_t, t)$, where the function $a$, defined on $\Gamma \times [0, T]$, is sufficiently regular in the edges of the network. Then, almost surely if $X_t \in \Gamma_{\alpha} \setminus \mathcal{V}$,

$$
d\pi_{\alpha}^{-1}(X_t) = a_\alpha(\pi_{\alpha}^{-1}(X_t), t) \, dt + \sqrt{2\mu_\alpha} \, dW_t.
$$
An informal way to describe the behavior of the process at the vertices is as follows: if $X_t$ hits $v_i \in V$, then it enters $\Gamma_\alpha$, $\alpha \in A$, with probability $p_{i\alpha} > 0$ ($p_{i\alpha}$ was introduced in Sect. 1.2). We assume that there is an optimal feedback law $a^*$.

- We assume that for all $\alpha \in A$, $a_\alpha$ maps $[0, \ell_\alpha] \times [0, T]$ to a compact interval $A_\alpha = [a_{\alpha}, \overline{a}_\alpha]$.
- The contribution of the control to the running cost involves the Lagrangian $L$, i.e., a real valued function defined on $\cup_{\alpha \in A} (\Gamma_\alpha \setminus V) \times A_\alpha$. If $x \in \Gamma_\alpha \setminus V$ and $a \in A_\alpha$, then $L(x, a) = L_\alpha(\pi_\alpha^{-1}(x), a)$, where $L_\alpha$ is a continuous real valued function defined on $[0, \ell_\alpha] \times A_\alpha$. We assume that $L_\alpha(x, \cdot)$ is strictly convex on $A_\alpha$.
- The contribution of the distribution of states to the running cost involves the coupling cost operator, which can either be nonlocal, i.e., $V : \mathcal{P}(\Gamma) \to \mathcal{P}^2(\Gamma)$ (where $\mathcal{P}(\Gamma)$ is the set of Borel probability measures on $\Gamma$), or local, i.e., $V[m](x) = F(m(x))$ for a continuous function $F : \mathbb{R}^+ \to \mathbb{R}$, assuming that $m$ is absolutely continuous with respect to the Lebesgue measure and identifying it with its density.
- The last term is the terminal cost $v_T$, which depends only on the state variable for simplicity.


Ito calculus as in [8,9] and the dynamic programming principle lead to the following HJB equation on $\Gamma$, more precisely the following boundary value problem

$$
\begin{align*}
- \partial_t v - \mu_\alpha \partial^2 v + H(x, \partial v) = \mathcal{V}[m(\cdot, t)](x), & \quad \text{in } (\Gamma_\alpha \setminus V) \times (0, T), \alpha \in A, \\
\sum_{\alpha \in A} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v (v_i, t) = 0, & \quad \text{if } (v_i, t) \in \mathcal{V} \times (0, T), \\
v|_{\Gamma_{\alpha}} (v_i, t) = v|_{\Gamma_{\beta}} (v_i, t) & \quad \text{for all } v_i \in \mathcal{V}, t \in (0, T) \alpha, \beta \in A_i, \\
v(x, T) = v_T(x) & \quad \text{in } \Gamma.
\end{align*}
$$

(1.16)

We refer to [15–17] for the interpretation of the value function $v$. Let us comment the different Eq. in (1.16):

1. The first equation is a HJB equation, the Hamiltonian $H$ of which is a real valued function defined on $(\cup_{\alpha \in A} \Gamma_\alpha \setminus V) \times \mathbb{R}$ given by

$$
H(x, p) = \sup_{a \in A_\alpha} \left\{ -ap - L_\alpha(\pi_\alpha^{-1}(x), a) \right\} \quad \text{for } x \in \Gamma_\alpha \setminus V \text{ and } p \in \mathbb{R}.
$$

We assume that $L$ is such that the Hamiltonians $H|_{\Gamma_\alpha \times \mathbb{R}}$ are Lipschitz continuous with respect to $p$ and $C^1$.

2. The second equation in (1.16) is a Kirchhoff transmission condition (or Neumann boundary condition if $v_i \in \partial \Gamma$); it is the consequence of the assumption on the behavior of $X_s$ at vertices. It involves the positive constants $\gamma_{i\alpha}$ defined in (1.13).

3. The third condition means that $v$ is continuous at the vertices.

4. The fourth condition is a terminal condition for the backward in time HJB equation.

If (1.16) has a smooth solution, then it provides a feedback law for the optimal control problem:

$$
a^*(x, t) = -\partial_p H(x, \partial v(x, t)).
$$
At the MFG equilibrium, $m$ is the density of the distribution of states associated with the optimal feedback law, so, according to Sect. 1.2, it satisfies (1.14), where $a$ is replaced by $a^* = -\partial_p H (x, \partial v(x, t))$. We end up with the following system:

$$
\begin{align*}
&-\partial_t v - \mu_a \partial^2 v + H(x, \partial v) = \nu [m(\cdot, t)](x), & (x, t) \in (\Gamma_\nu \setminus \nu') \times (0, T), \alpha \in A, \\
&\partial_t m - \mu_a \partial^2 m - \partial (m \partial_p H (x, \partial v)) = 0, & (x, t) \in (\Gamma_\nu \setminus \nu') \times (0, T), \alpha \in A, \\
&\sum_{\alpha \in A} \gamma_{1\alpha} \mu_a \partial v (v_i, t) = 0, & (v_i, t) \in \nu' \times (0, T),
\end{align*}
$$

$$
\begin{align*}
&\sum_{\alpha \in A} \mu_a \partial v (v_i, t) + n_{i\alpha} \partial_p H^\alpha (v_i, \partial v|_{\Gamma_\alpha} (v_i, t)) m|_{\Gamma_\alpha} (v_i, t) = 0, & (v_i, t) \in \nu' \times (0, T), \\
v|_{\Gamma_\alpha} (v_i, t) = v|_{\Gamma_\beta} (v_i, t), & m|_{\Gamma_\alpha} (v_i, t) = m|_{\Gamma_\beta} (v_i, t), & \alpha, \beta \in A, (v_i, t) \in \nu' \times (0, T), \\
v(x, T) = v_T (x), & m(x, 0) = m_0 (x), & x \in \Gamma,
\end{align*}
$$

(1.17)

where $H^\alpha := H|_{\Gamma_\alpha \times \mathbb{R}}$. At a vertex $v_i, i \in I$, the transmission conditions for both $v$ and $m$ consist of $d_{v_i} = \nu(A)$ linear relations, which is the appropriate number of relations to have a well posed problem. If $v_i \in \partial \Gamma$, there is of course only one Neumann like condition for $v$ and for $m$.

1.4 Assumptions and main results

Before giving the precise definition of solutions of the MFG system (1.17) and stating our result, we need to introduce some suitable functions spaces.

1.4.1 Function spaces related to the Kirchhoff conditions

The function spaces $V := H^1 (\Gamma)$ and

$$
W := \left\{ w : \Gamma \to \mathbb{R} : w \in H^1 (\Gamma) \quad \text{and} \quad \frac{w|_{\Gamma_\alpha} (v_i)}{\gamma_{1\alpha}} = \frac{w|_{\Gamma_\beta} (v_i)}{\gamma_{1\beta}} \quad \text{for all} \quad i \in I, \quad \alpha, \beta \in A \right\}
$$

$$
\subset H^1 (\Gamma),
$$

will be the key ingredients in order to build weak solutions of (1.17).

We will intensively use the test-function $\varphi \in W$ defined as follows:

$$
\begin{align*}
\varphi & \quad \text{is affine on} \quad (0, \ell_\alpha), \\
\varphi|_{\Gamma_\alpha} (v_i) & = \gamma_{1\alpha}, \quad \text{if} \quad \alpha \in A', \\
\varphi & = \text{constant on the edges} \quad \Gamma_\alpha \quad \text{which touch the boundary of} \quad \Gamma.
\end{align*}
$$

Note that $\varphi$ is positive and bounded. We set $\varphi = \max_\Gamma \varphi, \varphi = \min_\Gamma \varphi$.

**Remark 1.5** One can see that $v \in V \mapsto v\varphi$ is an isomorphism from $V$ onto $W$ and $w \in W \mapsto w\varphi^{-1}$ is the inverse isomorphism.

1.4.2 Running assumptions (H)

(1.17) (Diffusion coefficients) $(\mu_\alpha)_{\alpha \in A'}$ is a family of positive numbers.

(1.17) (Jump coefficients) $(\gamma_{1\alpha})_{\alpha \in A'}$ is a family of positive numbers such that $\sum_{\alpha \in A'} \gamma_{1\alpha} \mu_\alpha = 1$. 

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(Hamiltonian) The Hamiltonian $H$ is defined by the collection $H^\alpha := H|_{\Gamma \times \mathbb{R}}$, $\alpha \in \mathscr{A}$. We assume that there exists a constant $C_0 > 0$ independent of $\alpha$ such that

\[
H^\alpha \in C^1 (\Gamma \times \mathbb{R}),
\]

$H^\alpha (x, \cdot)$ is convex in $p$, for any $x \in \Gamma$, $H^\alpha (x, p) \leq C_0 (|p| + 1)$, for any $(x, p) \in \Gamma \times \mathbb{R}$; \hspace{1cm} (1.18)

\[
|\partial_p H^\alpha (x, p)| \leq C_0, \quad \text{for any } (x, p) \in \Gamma \times \mathbb{R}, \quad (1.19)
\]

\[
|\partial_x H^\alpha (x, p)| \leq C_0 (|p| + 1), \quad \text{for any } (x, p) \in \Gamma \times \mathbb{R}. \quad (1.20)
\]

(Coupling operator) We assume that $\mathcal{V}$ is a continuous map from $L^2 (\Gamma)$ to $L^2 (\Gamma)$, such that for all $m \in L^2 (\Gamma)$,

\[
\|\mathcal{V} [m]\|_{L^2 (\Gamma)} \leq C (\|m\|_{L^2 (\Gamma)} + 1).
\]

Note that such an assumption is in particular satisfied by local operators of the form $\mathcal{V} [m] (x) = F (m (x))$ where $F$ is a Lipschitz-continuous function.

(Initial and terminal data) $m_0 \in L^2 (\Gamma) \cap \mathscr{A}$ and $v_T \in H^1 (\Gamma)$.

The above set of assumptions, referred to as (H), will be the running assumptions hereafter. We will use the following notation: $\underline{\mu} := \min_{\alpha \in \mathscr{A}} \mu_\alpha > 0$ and $\overline{\mu} := \max_{\alpha \in \mathscr{A}} \mu_\alpha$.

1.4.3 Monotone coupling

We will also say that the coupling $\mathcal{V}$ is monotone, i.e., for any $m_1, m_2 \in \mathscr{M} \cap L^2 (\Gamma)$,

\[
\int_{\Gamma} (m_1 - m_2) (\mathcal{V} [m_1] - \mathcal{V} [m_2]) dx \geq 0
\]

and equality implies $\mathcal{V} [m_1] = \mathcal{V} [m_2]$.

1.4.4 Stronger assumptions on the coupling operator

We will sometimes need to strengthen the assumptions on the coupling operator, namely that $\mathcal{V}$ has the following smoothing properties:

$\mathcal{V}$ maps the topological dual of $W$ to $H^1_b (\Gamma)$, see Definition 1.1; more precisely, $\mathcal{V}$ defines a Lipschitz map from $W'$ to $H^1_b (\Gamma)$.

Note that such an assumption is not satisfied by local operators.

1.4.5 Definition of solutions and main result

Definition 1.2 (solutions of the MFG system) A weak solution of the Mean Field Games system (1.17) is a pair $(v, m)$ such that

\[
v \in L^2 (0, T; H^2 (\Gamma)) \cap C ([0, T]; V), \quad \partial_t v \in L^2 (\Gamma \times (0, T)),
\]

\[
m \in L^2 (0, T; W) \cap C ([0, T]; L^2 (\Gamma) \cap \mathscr{A}), \quad \partial_t m \in L^2 (0, T; V'),
\]

$v$ satisfies

\[
\begin{cases}
- \sum_{\alpha \in \mathscr{A}} \int_{\Gamma_\alpha} [\partial_t v (x, t) w (x) + \mu_\alpha \nu (x, t) \partial w (x) + H (x, \partial v (x, t)) w (x)] dx \\
\quad = \int_{\Gamma} \mathcal{V} [m (\cdot, t)] (x) w (x) dx, \quad \text{for all } w \in W, \text{ a.e. } t \in (0, T),
\end{cases}
\]

$v (x, T) = v_T (x)$ for a.e. $x \in \Gamma$.
and \( m \) satisfies
\[
\begin{cases}
\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left[ \partial_t m(x, t) v(x) \, dx + \mu_\alpha \partial m(x, t) \partial v(x) + \partial_\mu H(x, \partial v(x), t) m(x, t) \partial v(x) \right] \, dx \\
= 0, \quad \text{for all } v \in V, \text{ a.e. } t \in (0, T), \\
m(x, 0) = m_0(x) \quad \text{for a.e. } x \in \Gamma,
\end{cases}
\]
where \( V \) and \( W \) are introduced in Sect. 1.4.1.

We are ready to state the main result:

**Theorem 1.1** Under assumptions \( (H) \),

(i) \( (\text{Existence}) \) There exists a weak solution \((v, m)\) of (1.17).

(ii) \( (\text{Uniqueness}) \) If \( \mathcal{Y} \) is monotone (see Sect. 1.4.3), then the solution is unique.

(iii) \( (\text{Regularity}) \) If \( \mathcal{Y} \) satisfies furthermore the stronger assumptions made in Sect. 1.4.4 and if \( v_T \in C^{2,1}(\Gamma) \), then \( v \in C^{2,1}(\Gamma \times [0, T]) \).

Moreover, if for all \( \alpha \in \mathcal{A} \), \( \partial_\mu H^\alpha(x, p) \) is a Lipschitz function defined in \( \Gamma_\alpha \times \mathbb{R} \), and if \( m_0 \in W \), then \( m \in C([0, T]; W) \cap W^{1,2}(0, T; L^2(\Gamma)) \cap L^2(0, T; H^1_\beta(\Gamma)) \).

**Remark 1.6** If \( m_0 \in L^\infty(\Gamma) \), then, from the Lipschitz continuity of \( H \), the solution \( m \) of the Fokker–Planck in (1.17) belongs to \( L^\infty(\Gamma \times [0, T]) \), independently of the coupling term \( \mathcal{Y} \). This implies that the first point in Theorem 1.1 holds if \( m_0 \in L^\infty(\Gamma) \) and \( \mathcal{Y} \) is a local operator of the form \( \mathcal{Y}[m](x) = F(m(x)) \) where \( F \) is any continuous function on \( \mathbb{R} \) (the Lipschitz continuity assumption on \( \mathcal{Y} \) in Assumption \( (H) \) is not needed anymore). To prove this, one can adapt classical arguments on the global boundedness of weak solutions to parabolic equations in divergence form, see [14, V, Theorem 2.2, page 429]. The main difference is that before using Moser’s iterations, one has to test the Fokker–Planck equation by \((\varphi^{-1}m-k)_+\), where \( k \) is a suitable constant. Here \( \varphi \) is the function introduced in paragraph 1.4.1.

2 Preliminary: a modified heat equation on the network with general Kirchhoff conditions

This section contains results on the solvability of some linear boundary value problems with terminal condition, that will be useful in what follows. Consider
\[
\begin{cases}
- \partial_t v - \mu_\alpha \partial^2 v = h, & \text{in } (\Gamma_\alpha \setminus \mathcal{Y}) \times (0, T), \alpha \in \mathcal{A}, \\
v|_{\Gamma_\alpha} (v_i, t) = v|_{\Gamma_\beta} (v_i, t), & t \in (0, T), \alpha, \beta \in \mathcal{A}, v_i \in \mathcal{Y}, \\
\sum_{\alpha \in \mathcal{A}} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(v_i, t) = 0, & t \in (0, T), v_i \in \mathcal{Y}, \\
v(x, T) = v_T(x), & x \in \Gamma.
\end{cases}
\]

**Definition 2.1** If \( v_T \in L^2(\Gamma) \) and \( h \in L^2(0, T; W') \), a weak solution of (2.1) is a function \( v \in L^2(0, T; V) \cap C([0, T]; L^2(\Gamma)) \) such that \( \partial_t v \in L^2(0, T; W') \) and
\[
\begin{cases}
- \langle \partial_t v(t), w \rangle_{W', W} + \mathcal{B} (v(\cdot, t), w) = \langle h(t), w \rangle_{W', W} \quad \text{for all } w \in W \text{ and a.e. } t \in (0, T), \\
v(x, T) = v_T(x),
\end{cases}
\]

where $\mathcal{B} : V \times W \to \mathbb{R}$ is the bilinear form $\mathcal{B}(v, w) := \int_\Gamma \mu \partial v \partial w dx = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v \partial w dx$.

We use the Galerkin’s method (see [7]), i.e., we construct solutions of some finite-dimensional approximations to (2.1).

Recall that $\varphi$ has been defined in Sect. 1.4.1. We notice first that the symmetric bilinear form $\mathcal{B}(u, v) := \int_\Gamma \mu \varphi \partial u \partial v$ is such that $(u, v) \mapsto (u, v)_{L^2(\Gamma)} + \mathcal{B}(u, v)$ is an inner product in $V$ equivalent to the standard inner product in $V$, namely $(u, v)_V = (u, v)_{L^2(\Gamma)} + \int_\Gamma \partial u \partial v$. Therefore, by Fredholm’s theory, there exists a non decreasing sequence of nonnegative real numbers $(\lambda_k)_{k=1}^\infty$, that tends to $+\infty$ as $k \to \infty$ and a Hilbert basis $(v_k)_{k=1}^\infty$ of $L^2(\Gamma)$, which is also a total sequence of $V$ (and orthogonal if $V$ is endowed with the scalar product $(u, v)_{L^2(\Gamma)} + \mathcal{B}(u, v)$), such that $\mathcal{B}(v_k, v) = \lambda_k (v_k, v)_{L^2(\Gamma)}$ for all $v \in V$. Note that

$$\int_\Gamma \mu \partial v_k \partial v' \varphi dx = \begin{cases} \lambda_k & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell. \end{cases}$$

Note also that $v_k$ is a weak solution of

$$\begin{aligned} -\mu_\alpha (\varphi \partial v_k) &= \lambda_k v_k, & \text{in } \Gamma_\alpha \setminus \mathcal{V}, \alpha \in \mathcal{A}, \\ v_k|_{\Gamma_\alpha} (v_i) &= v_k|_{\Gamma_\beta} (v_i), & \alpha, \beta \in \mathcal{A}_i, \\ \sum_{\alpha \in \mathcal{A}} \gamma_{\alpha \beta} \partial_\alpha v_k (v_i) &= 0, & \gamma_{\alpha \beta} \in \mathcal{V}, \end{aligned}$$

which implies that $v_k \in C^2(\Gamma)$.

Finally, by Remark 1.5, the sequence $(\varphi v_k)_{k=1}^\infty$ is a total family in $W$ (but is not orthogonal if $W$ is endowed with the standard inner product).

**Lemma 2.1** For any positive integer $n$, there exist $n$ absolutely continuous functions $y^n_k : [0, T] \to \mathbb{R}$, $k = 1, \ldots, n$, and a function $v_n : [0, T] \to L^2(\Gamma)$ of the form

$$v_n (x, t) = \sum_{k=1}^n y^n_k (t) v_k (x),$$

(2.3)

such that

$$y^n_k (T) = \int_\Gamma v_T v_k dx, \quad \text{for } k = 1, \ldots, n,$$

(2.4)

$$-\frac{d}{dt} (v_n, v_k \varphi)_{L^2(\Gamma)} + \mathcal{B}(v_n, v_k \varphi) = \langle h(t), v_k \varphi \rangle, \quad \text{for a.a. } t \in (0, T), \quad k = 1, \ldots, n.$$  

(2.5)

**Moreover, there exists a constant $C$ depending only on $\Gamma$, $(\mu_\alpha)_{\alpha \in \mathcal{A}}$, $T$ and $\varphi$ such that**

$$\|v_n\|_{L^\infty(0, T; L^2(\Gamma))} + \|v_n\|_{L^2(0, T; V)} + \|\partial_t v_n\|_{L^2(0, T; W')} \leq C \left( \|h\|_{L^2(0, T; W')} + \|v_T\|_{L^2(\Gamma)} \right).$$

**Proof** For $n \geq 1$, the symmetric $n$ by $n$ matrix $M_n$ defined by $(M_n)_{k\ell} = \int_\Gamma v_k v_\ell \varphi dx$ is positive definite and there exist two constants $c, C$ independent of $n$ such that

$$c |\xi|^2 \leq \sum_{k, \ell=1}^n (M_n)_{k\ell} \xi_k \xi_\ell \leq C |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n.$$  

(2.6)
Looking for \( v_n \) of the form (2.3), and setting \( Y = (y_1^n, \ldots, y_n^n)^T, \hat{Y} = \left( \frac{d}{dt} y_1^n, \ldots, \frac{d}{dt} y_n^n \right)^T \), (2.5) implies that we have to solve the following system of differential equations

\[
-M_n \dot{Y} + BY = F_n, \quad Y(T) = \left( \int_{\Gamma} v_T v_1, \ldots, \int_{\Gamma} v_T v_n \right)^T,
\]

where \( B_{kk} = \mathcal{B}(v_k, v_k \varphi) \) and \( F_n(t) = (\langle h(t), v_1 \varphi \rangle, \ldots, \langle h(t), v_n \varphi \rangle)^T \). Since \( M_n \) is invertible, the system of ODEs has a unique absolutely continuous solution. The first part of the lemma is proved.

We turn to the uniform estimates for \( \{v_n\} \). Multiplying (2.5) by \( y_k^n(t) e^{\lambda t} \) for a positive constant \( \lambda \) to be chosen later, summing for \( k = 1, \ldots, n \) and using (2.3), we get

\[
- \int_{\Gamma} \partial_t v_n v_n e^{\lambda t} \varphi dx + \int_{\Gamma} \mu \partial v_n \partial \left( v_n e^{\lambda t} \varphi \right) dx = e^{\lambda t} \langle h(t), v_n \varphi \rangle_{W', W},
\]

and

\[
- \int_{\Gamma} \left[ \partial_t \left( \frac{v_n^2}{2} e^{\lambda t} \right) - \lambda \frac{v_n^2}{2} e^{\lambda t} \right] \varphi dx + \int_{\Gamma} \mu \partial v_n \partial \left( v_n e^{\lambda t} \varphi \right) dx + \int_{\Gamma} \mu \partial v_n v_n e^{\lambda t} \partial \varphi dx = e^{\lambda t} \langle h(t), v_n \varphi \rangle.
\]

Integrating both sides from \( s \) to \( T \), we obtain

\[
\int_{\Gamma} \left( \frac{v_n^2(x, s)}{2} e^{\lambda s} - \frac{v_n^2(x, T)}{2} e^{\lambda T} \right) \varphi dx + \frac{\lambda}{2} \int_s^T \int_{\Gamma} v_n^2 e^{\lambda t} \varphi dx dt + \int_{\Gamma} \mu \partial v_n v_n e^{\lambda t} \partial \varphi dx dt + \int_{\Gamma} \mu \partial v_n \partial \left( v_n e^{\lambda t} \varphi \right) dx dt = \int_s^T e^{\lambda t} \langle h(t), v_n \varphi \rangle dt
\]

\[
\leq C \int_s^T e^{\lambda t} \| h(t) \|_{W'} \| v_n(t) \|_V dt
\]

\[
\leq \frac{1}{2} \int_s^T \int_{\Gamma} \left( \| (\partial v_n)^2 + v_n^2 \|_V \right) e^{\lambda t} \varphi dx dt + \frac{C^2}{2\mu} \int_s^T \int_{\Gamma} e^{\lambda t} \| h(t) \|_{W'}^2 dt,
\]

where \( C \) is positive constant depending on \( \varphi \), from Remark 1.5. Therefore,

\[
e^{\lambda s} \int_{\Gamma} \frac{v_n^2(x, s)}{2} \varphi dx + \frac{1}{4} \int_s^T \int_{\Gamma} \mu (\partial v_n)^2 \varphi e^{\lambda t} \varphi dx dt
\]

\[
+ \left( \frac{\lambda}{2} - \frac{\mu}{2} - \frac{\| \partial \varphi \|^2_{L^\infty(\Gamma)}}{\varphi^2} \right) \int_s^T \int_{\Gamma} v_n^2 e^{\lambda t} \varphi dx dt
\]

\[
\leq e^{\lambda T} \int_{\Gamma} \frac{v_n^2(x, T)}{2} \varphi dx + \frac{C^2}{2\mu} e^{\lambda T} \int_s^T \| h(t) \|_{W'}^2 dt.
\]

Choosing \( \lambda \geq 1/2 + \overline{\mu} + 2\overline{\mu} \| \partial \varphi \|^2_{L^\infty(\Gamma)} / \varphi^2 \) and noticing that \( \int_{\Gamma} v_n^2(x, T) \varphi dx \) is bounded by \( \varphi \int_{\Gamma} v_n^2 dx \) from (2.4), it follows that

\[
\int_{\Gamma} v_n^2(x, s) \varphi dx + \int_s^T \int_{\Gamma} v_n^2 \varphi dx dt + \int_s^T \int_{\Gamma} \mu (\partial v_n)^2 \varphi dx dt \leq 2e^{\lambda T} \left( \frac{C^2}{\mu} \| h \|^2_{L^2(\partial_0, T; W')} + \overline{\varphi} \int_{\Gamma} v_n^2 dx \right). \quad (2.7)
\]
Estimate of \( v_n \) in \( L^\infty (0, T; L^2 (\Gamma)) \) and \( L^2 (0, T; V) \). From (2.7), we deduce that
\[
\|v_n\|_{L^\infty (0, T; L^2 (\Gamma))} + \|v_n\|_{L^2 (0, T; V)} \leq C \left( \|h\|_{L^2 (0, T; W')} + \|v_T\|_{L^2 (\Gamma)} \right)
\] (2.8)
for some constant \( C \) depending only on \((\mu_\alpha)_{\alpha \in \mathcal{A}}, \varphi \) and \( T \).

Estimate \( \partial_t v_n \) in \( L^2 (0, T; W') \). Consider the closed subspace \( G_1 \) of \( W \) defined by
\[
G_1 = \left\{ w \in W : \int_{\Gamma} v_k w dx = 0 \text{ for all } k \leq n \right\}.
\]
It has a finite co-dimension equal to \( n \). Consider also the \( n \)-dimensional subspace
\[
G_2 = \text{span} \{ v_1 \varphi, \ldots, v_n \varphi \}
\]
of \( W \). The invertibility of the matrix \( M_n \) introduced in the proof of Lemma 2.1 implies that \( G_1 \cap G_2 = \{0\} \). This implies that \( W = G_1 \oplus G_2 \). For \( w \in W \), we can write \( w = w_n + \hat{w}_n \), where \( w_n \in G_2 \) and \( \hat{w}_n \in G_1 \). Hence, from (2.3) and (2.5), one gets for a.e. \( t \in [0, T] \):
\[
\langle \partial_t v_n (t), w \rangle_{W^*, W} = \frac{d}{dt} \left( \int_{\Gamma} v_n w dx \right) = \frac{d}{dt} \left( \int_{\Gamma} v_n w_n dx \right) = -\langle h(t), w_n \rangle_{W^*, W} + \int_{\Gamma} \mu \partial v_n \partial w_n dx.
\]
Since there exists a constant \( C \) independent of \( n \) such that \( \|w_n\|_W \leq C \|w\|_W \), it follows that \( \|\partial_t v_n (t)\|_{W^*} \leq C \left( \|h(t)\|_W^* + \|w_n(t)\|_V \right) \) for almost every \( t \). Then (2.8) yields
\[
\|\partial_t v_n (t)\|_{L^2 (0, T; W')}^2 \leq C \left( \|h\|_{L^2 (0, T; W')}^2 + \|v_T\|_{L^2 (\Gamma)}^2 \right),
\]
for a constant \( C \) independent of \( n \). \( \square \)

**Theorem 2.1** There exists a unique solution \( u \) of (2.1), which satisfies
\[
\|u\|_{L^\infty (0, T; L^2 (\Gamma))} + \|u\|_{L^2 (0, T; V)} + \|\partial_t u\|_{L^2 (0, T; W')} \leq C \left( \|h\|_{L^2 (0, T; W')} + \|v_T\|_{L^2 (\Gamma)} \right),
\]
where \( C \) is a constant that depends only on \( \Gamma \), \((\mu_\alpha)_{\alpha \in \mathcal{A}}, T \) and \( \varphi \).

**Proof** From Lemma 2.1, the sequence \( (v_n)_{n \in \mathbb{N}} \) is bounded in \( L^2 (0, T; V) \) and the sequence \( (\partial_t v_n)_{n \in \mathbb{N}} \) is bounded in \( L^2 (0, T; W') \). Hence, up to the extraction of a subsequence, there exists a function \( u \) such that \( u \in L^2 (0, T; V) \), \( \partial_t u \in L^2 (0, T; W') \) and
\[
v_n \rightharpoonup u \text{ weakly in } L^2 (0, T; V), \quad \partial_t v_n \rightharpoonup \partial_t u \text{ weakly in } L^2 (0, T; W').
\] (2.10)
Fix an integer \( N \) and choose a function \( \overline{v} \in C^1 ([0, T]; V) \) of the form \( \overline{v}(t) = \sum_{k=1}^N d_k (t) v_k \), where \( d_1, \ldots, d_N \) are given real valued \( C^1 \) functions defined in \([0, T]\). For all \( n \geq N \), multiplying (2.5) by \( d_k (t) \), summing for \( k = 1, \ldots, n \) and integrating over \((0, T)\) leads to
\[
-\int_0^T \int_{\Gamma} \partial_t v_n \overline{v} \varphi dx dt + \int_0^T \int_{\Gamma} \mu \partial v_n \partial (\overline{v} \varphi) dx dt = \int_0^T \langle h, \overline{v} \varphi \rangle dt.
\] (2.11)
Letting \( n \to +\infty \), we obtain from (2.10) that
\[
-\int_0^T \langle \partial_t v, \overline{v} \varphi \rangle dt + \int_0^T \int_{\Gamma} \mu \partial v \partial (\overline{v} \varphi) dx dt = \int_0^T \langle h, \overline{v} \varphi \rangle dt.
\] (2.12)
Since the functions \( \overline{v} \) are dense in \( L^2 (0, T; V) \), (2.12) holds for all test function \( \overline{v} \in L^2 (0, T; V) \). Recalling the isomorphism \( \overline{v} \in V \mapsto \overline{v} \varphi \in W \) (see Remark 1.5), we obtain
that, for all \( w \in W \) and \( \psi \in C^1_c(0, T) \), 
\[ -\int_0^T (\partial_t v, w) \psi \, dt + \int_0^T \int_{\Gamma} \mu \partial v \partial w \psi \, dx \, dt = \int_0^T (h, w) \psi \, dt. \]
This implies that, for a.e. \( t \in (0, T) \),
\[ -\partial_t v, w \right) + \mathcal{B} (v, w) = \left( h, w \right) \quad \text{ for all } w \in W. \]

Using [20, Theorem 3.1] (or the same argument as in [7, pages 287-288]), we see that \( v \in C ([0, T]; L^2(\Gamma)) \), where \( L^2(\Gamma) = \{ w : \Gamma \to \mathbb{R} : \int_{\Gamma} w^2 \, dx < +\infty \} \), and since \( \varphi \) is bounded from below and above by positive numbers, \( L^2(\Gamma) = L^2(\Gamma) \) with equivalent norms. Moreover,
\[ \max_{0 \leq t \leq T} \| v(\cdot, t) \|_{L^2(\Gamma)} \leq C \left( \| \partial_t v \|_{L^2(0, T; W')} + \| v \|_{L^2(0, T; V)} \right). \]

We are now going to prove \( v(T) = v_T \). For all \( \overline{v} \in C^1 ([0, T]; V) \) such that \( \overline{v}(0) = 0 \), we deduce from (2.11) and (2.12) that
\[
-\int_0^T \int_{\Gamma} \partial_t \overline{v} v_n \varphi \, dx \, dt - \int_0^T \varphi (T) v_n (T) \varphi \, dx + \int_0^T \int_{\Gamma} \mu \partial v_n \partial (\overline{v} \varphi) \, dx \, dt = -\int_0^T \int_{\Gamma} \partial_t \overline{v} v \varphi \, dx \, dt - \int_0^T \varphi (T) v \varphi \, dx + \int_0^T \int_{\Gamma} \mu \partial v \partial (\overline{v} \varphi) \, dx \, dt.
\]

We know that \( v_n (T) \to v_T \) in \( L^2(\Gamma) \). Then, (2.10) yields \( \int_{\Gamma} \varphi (T) v_T \varphi \, dx = \int_{\Gamma} \varphi (T) v \varphi \, dx \). Since the functions of the form \( \sum_{k=1}^N d_k (T) v_k \) are dense in \( L^2(\Gamma) \), we conclude that \( v = v_T \).

In order to prove the energy estimate (2.9), we use \( ve^{\lambda t} \varphi \) as a test function in (2.2) and apply similar arguments as in the proof of Lemma 2.1 for \( \lambda \) large enough, we get (2.9).

Finally, if \( h = 0 \) and \( v_T = 0 \), by the energy estimate for \( v \) in (2.9), we deduce that \( v = 0 \). Uniqueness is proved.

**Theorem 2.2** If \( v_T \in V \) and \( h \in L^2(\Gamma \times (0, T)) \), then the unique solution \( v \) of (2.1) satisfies \( v \in L^2 (0, T; H^2(\Gamma)) \cap C ([0, T]; V) \) and \( \partial_t v \in L^2 (\Gamma \times (0, T)) \). Moreover,
\[ \| v \|_{L^\infty (0, T; V)} + \| v \|_{L^2 (0, T; H^2(\Gamma))} + \| \partial_t v \|_{L^2 (\Gamma \times (0, T))} \leq C \left( \| h \|_{L^2 (\Gamma \times (0, T))} + \| v_T \| \right), \]
(2.13)
for a positive constant \( C \) that depends only on \( \Gamma, (\mu_a)_{a \in \mathcal{A}}, T \) and \( \varphi \).

**Proof** It is enough to prove estimate (2.13) for \( v_n \).

Multiplying (2.5) by \( -\frac{d}{dt} \partial_t v_n \), summing for \( k = 1, \ldots, n \) and using (2.3) leads to
\[ \int_{\Gamma} (\partial_t v_n) \varphi \, dx - \int_{\Gamma} \mu \partial v_n \partial (\partial_t v_n \varphi) \, dx = -\int_{\Gamma} h \partial_t v_n \varphi \, dx, \]
hence \( \int_{\Gamma} (\partial_t v_n) \varphi \, dx - \int_{\Gamma} \mu \partial_t \frac{\partial v_n}{2} \varphi \, dx = -\int_{\Gamma} h \partial_t v_n \varphi \, dx. \)

Multiplying by \( e^{\lambda t} \) where \( \lambda \) will chosen later, and taking the integral from \( s \) to \( T \), we obtain
\[
\int_s^T \int_{\Gamma} (\partial_t v_n) e^{\lambda t} \varphi \, dx \, dt - \int_s^T \int_{\Gamma} \frac{\mu}{2} [((\partial v_n (T))^2) e^{\lambda T} - (\partial v_n (s))^2) e^{\lambda s}] \varphi \, dx \\
+ \lambda \int_s^T \int_{\Gamma} \frac{\mu}{2} (\partial v_n)^2 e^{\lambda t} \varphi \, dx \, dt \int_s^T \int_{\Gamma} \mu \partial v_n (T) v_n e^{\lambda t} \partial \varphi \, dx \, dt \\
= -\int_s^T \int_{\Gamma} h \partial_t v_n e^{\lambda t} \varphi \, dx \, dt.
\]
Let us deal with the term \( \int_{\Gamma} (\partial v_n (x, t))^2 \varphi dx \). From (2.4),
\[
\int_{\Gamma} \mu (\partial v_n (x, t))^2 \varphi dx \leq \sum_{k=1}^{\infty} \lambda_k \left( \int_{\Gamma} v_T v_k dx \right)^2
\]
\[
= \int_{\Gamma} \mu (\partial v_T (x))^2 \varphi dx \leq \overline{\mu} \int_{\Gamma} (\partial v_T (x))^2 \varphi dx.
\]
Then, choosing \( \lambda = 2 \overline{\mu} \| \varphi \|_{L^2(\Gamma)}^2 / (\varphi^2 \mu) \), we obtain that
\[
\int_{\Gamma} 2 \mu (\partial v_n (x, s))^2 \varphi dx + \int_{s}^{T} \int_{\Gamma} (\partial v_n)^2 \varphi dx dt \leq 2 e^{\lambda T} \varphi \left( \| h \|_{L^2((0, T))}^2 + \overline{\mu} \int_{\Gamma} (\partial v_T)^2 dx \right).
\]
Estimate of \( \partial v_n \) in \( L^\infty (0, T; L^2 (\Gamma)) \) and \( \partial v_n \) in \( L^2 (\Gamma \times (0, T)) \). Estimate (2.14) yields
\[
\| \partial v_n \|_{L^\infty (0, T; L^2 (\Gamma))} + \| \partial v_n \|_{L^2 (\Gamma \times (0, T))} \leq C \left( \| h \|_{L^2((0, T))} + \| \partial v_T \|_{L^2(\Gamma)} \right)
\]
for some constant \( C \) depending only on \( \Gamma, \mu, T \) and \( \varphi \).

Estimate of \( \partial^2 v_n \) in \( L^2 (\Gamma \times (0, T)) \). Finally, using the PDE in (2.1), we can see that \( \partial^2 v_n \) belongs to \( L^2 (\Gamma \times (0, T)) \) and is bounded by \( C \left( \| h \|_{L^2((0, T))} + \| \partial v_T \|_{V} \right) \), hence \( v_n \) is bounded in \( L^2 (0, T; H^2 (\Gamma)) \) by the same quantity. The Kirchhoff conditions (which boil down to Neumann conditions at \( \partial \Gamma \)) are therefore satisfied in a strong sense for almost all \( t \).

Using [20, Theorem 3.1] (or a similar argument as [7, pages 287-288]), we see that \( v \) in \( C([0, T]; V) \). \( \square \)

3 The Fokker–Planck equation

This paragraph is devoted to a boundary value problem including a Fokker–Planck equation

\[
\begin{align*}
\frac{\partial m}{\partial t} - \mu_\alpha \partial^2 m - \partial (bm) &= 0, & \text{in } (\Gamma_\alpha \setminus \mathcal{Y}) \times (0, T), & \alpha \in \mathcal{A}, \\
\frac{m|_{\Gamma_\alpha}}{\bar{y}_{i\alpha}} &= \frac{m|_{\Gamma_\beta}}{\bar{y}_{i\beta}}, & t \in (0, T), & \alpha, \beta \in \mathcal{A}, \; r_i \in \mathcal{Y} \setminus \partial \Gamma, \\
\sum_{\alpha \in \mathcal{A}} \mu_\alpha \partial_\alpha m (v_i, t) + n_{i\alpha} b (v_i, t) m|_{\Gamma_\alpha} (v_i, t) &= 0, & t \in (0, T), & v_i \in \mathcal{Y}, \\
m (x, 0) &= m_0 (x), & x \in \Gamma,
\end{align*}
\]

where \( b \in PC (\Gamma \times [0, T]) \) and \( m_0 \in L^2 (\Gamma) \).

**Definition 3.1** A weak solution of (3.1) is a function \( m \in L^2 (0, T; W) \cap C([0, T]; L^2 (\Gamma)) \) such that \( \partial_t m \in L^2 (0, T; V') \) and

\[
\begin{align*}
\langle \partial_t m, v \rangle_{V', V} + \mathcal{A} (m, v) &= 0 \quad \text{for all } v \in V \text{ and a.e. } t \in (0, T), \\
m (\cdot, 0) &= m_0.
\end{align*}
\]

where \( \mathcal{A} : W \times V \to \mathbb{R} \) is the bilinear form: \( \mathcal{A} (v, w) = \int_{\Gamma} \mu \partial m \partial v dx + \int_{\Gamma} bm \partial v dx \).

Using a Galerkin method as in Sect. 2, we obtain the following result, the proof of which is omitted.
Theorem 3.1 If \( b \in L^\infty(\Gamma \times (0, T)) \) and \( m_0 \in L^2(\Gamma) \), there exists a unique function \( m \in L^2(0, T; W) \cap C([0, T]; L^2(\Gamma)) \) such that \( \partial_t m \in L^2(0, T; V') \) and (3.2). Moreover, there exists a constant \( C_0 \) which depends on \((\mu_\alpha)_{\alpha \in \mathcal{A}}, \|b\|_\infty, T \) and \( \varphi \), such that

\[
\|m\|_{L^2(0,T;W)} + \|m\|_{L^\infty(0,T;L^2(\Gamma))} + \|\partial_t m\|_{L^2(0,T;V')} \leq C \|m_0\|_{L^2(\Gamma)}. 
\] (3.3)

Remark 3.1 If \( m_0 \in \mathcal{M} \), then \( m(\cdot, t) \in \mathcal{M} \) for all \( t \in [0, T] \). To see that, we first use \( v \equiv 1 \in V \) as a test-function for (3.1). Since \( \partial v = 0 \), integrating (3.2) from 0 to \( t \), we get

\[
\int_0^t \int_\Gamma \partial_t m(x, s) dx ds = 0. 
\]

This implies that \( \int_\Gamma m(x, t) dx = \int_\Gamma m_0(x) dx = 1 \), for all \( t \in (0, T] \). Setting \( m^\tau = -\int_{m < 0} m \) we can also use \( v = \varphi^{-1} m^\tau e^{-\lambda t} \) as a test-function for \( \lambda \in \mathbb{R}_+ \). Indeed, the latter function belongs to \( L^2(0, T; V) \). Taking \( \lambda \) large enough and using similar arguments as for the energy estimate (3.3) yields that \( m^\tau = 0 \), i.e., \( m \geq 0 \).

We finally state a stability result, which will be useful in the proof of the main theorem.

Lemma 3.1 Let \( m_{0e}, b_e \) be sequences of functions satisfying

\[
m_{0e} \rightarrow m_0 \text{ in } L^2(\Gamma), \quad b_e \rightarrow b \text{ in } L^2(\Gamma \times (0, T)),
\]

and for some positive number \( K \) independent of \( \varepsilon \), \( \|b\|_{L^\infty(\Gamma \times (0, T))} \leq K \). \( \|b_e\|_{L^\infty(\Gamma \times (0, T))} \leq K \).

Let \( m_e \) (respectively \( m \)) be the solution of (3.2) corresponding to the datum \( m_{0e} \) (resp. \( m_0 \)) and the coefficient \( b_e \) (resp. \( b \)). The sequence \( m_e \) converges to \( m \) in \( L^2(0, T; W) \cap L^\infty(0, T; L^2(\Gamma)) \), and the sequence \( \partial_t m_e \) converges to \( \partial_t m \) in \( L^2(0, T; V') \).

Proof Taking \((m_e - m)e^{-\lambda t}\varphi^{-1}\) as a test-function in the versions of (3.2) satisfied by \( m_e \) and \( m \), we obtain that

\[
\int_\Gamma \left[ \frac{1}{2} \partial_t \left( (m_e - m)^2 e^{-\lambda t} \right) + \frac{\lambda}{2} (m_e - m)^2 e^{-\lambda t} \right] \varphi^{-1} dx + \int_\Gamma \mu(\partial (m_e - m))^2 e^{-\lambda t} \varphi^{-1} dx
\]

\[
+ \int_\Gamma \mu (m_e - m) \partial (m_e - m) e^{-\lambda t} \varphi^{-1} dx + \int_\Gamma (b_e - b)m \partial (m_e - m) e^{-\lambda t} \varphi^{-1} dx
\]

\[
+ \int_\Gamma (b_e - b)m (m_e - m) e^{-\lambda t} \varphi^{-1} dx = 0.
\]

Using that \( \|b_e\|_\infty, \|b\|_\infty \leq K \) for all \( \epsilon \), we see that there exists a positive constant \( C \) (\( C \) will vary from one line to the other in what follows) such that

\[
\int_\Gamma \left[ \frac{1}{2} \partial_t \left( (m_e - m)^2 e^{-\lambda t} \right) + \frac{\lambda}{2} (m_e - m)^2 e^{-\lambda t} \right] \varphi^{-1} dx + \int_\Gamma \mu (\partial (m_e - m))^2 e^{-\lambda t} \varphi^{-1} dx
\]

\[
\leq C \int_\Gamma \left( |m_e - m|^2 + |m_e - m| |\partial (m_e - m)| \right)
\]

\[
+ |m| |b_e - b| (|\partial (m_e - m)| + |m_e - m|) e^{-\lambda t} \varphi^{-1} dx
\]

\[
\leq C \int_\Gamma \left( |m_e - m|^2 + |b_e - b|^2 m^2 \right) e^{-\lambda t} \varphi^{-1} dx + \int_\Gamma \frac{\mu}{2} (\partial (m_e - m))^2 e^{-\lambda t} \varphi^{-1} dx.
\]

The assumptions on \( b_e \) and \( b \) imply that \( b_e \rightarrow b \) in \( L^p(\Gamma \times (0, T)) \) for all \( 1 \leq p < \infty \). On the other hand, we know that \( m \in L^q(\Gamma \times (0, T)) \) for all \( 1 \leq q < \infty \). From the latter observation with \( p = q = 4 \), we see that the quantity \( \int_0^T \int_\Gamma (|b_e - b|^2 m^2) e^{-\lambda t} \varphi^{-1} dx dt \) tends to 0 as \( \epsilon \rightarrow 0 \) uniformly in \( \lambda > 0 \). Choosing \( \lambda \) large enough and integrating the latter inequality from 0 to \( t \in [0, T] \), we obtain:

\[
\lim_{\epsilon \rightarrow 0} \left( \|m_e - m\|_{L^2(0,T;W)} + \|m_e - m\|_{L^\infty(0,T;L^2(\Gamma))} \right) = 0.
\]
Subtracting the two versions of (3.2) and using the latter estimate also yields
\[ \lim_{\epsilon \to 0} \| \partial_t m_\epsilon - \partial_t m \|_{L^2(0,T;V')} = 0, \]
which achieves the proof. \( \square \)

4 The Hamilton–Jacobi equation

This section is devoted to the following boundary value problem including a Hamilton–Jacobi equation

\[
\begin{aligned}
&-\partial_t v - \mu \partial^2 v + H(x, \partial v) = f, \quad \text{in } (\Gamma_\alpha \setminus \mathcal{V}) \times (0, T), \quad \alpha \in \mathcal{A}, \\
v|_{\Gamma_\alpha} (v_i, t) = v|_{\Gamma_\beta} (v_i, t), \quad t \in (0, T), \quad \alpha, \beta \in \mathcal{A}, \quad v_i \in \mathcal{V}, \\
\sum_{\alpha \in \mathcal{A}_i} \gamma_{i_\alpha} \mu_\alpha \partial_\alpha v(v_i, t) = 0, \quad t \in (0, T), \quad v_i \in \mathcal{V}, \\
v(x, T) = v_T(x), \quad x \in \Gamma.
\end{aligned}
\]  

(4.1)

**Definition 4.1** For \( f \in L^2(\Gamma \times (0, T)) \) and \( v_T \in V \), a weak solution of (4.1) is a function \( v \in L^2(0, T; H^2(\Gamma)) \cap C([0, T]; V) \) such that

\[
\begin{aligned}
&\int_{\Gamma} (-\partial_t vw + \mu \partial v \partial w + H(x, \partial v) w) dx = \int_{\Gamma} f w dx \quad \text{for all } w \in W, \ a.a. \ t \in (0, T), \\
v(x, T) = v_T(x),
\end{aligned}
\]

(4.2)

We start by proving existence and uniqueness of a weak solution for (4.1). Next, further regularity for the solution will be obtained under stronger assumptions.

4.1 Existence and uniqueness for the Hamilton–Jacobi equation

**Theorem 4.1** Under the running assumptions (H), if \( f \in L^2(\Gamma \times (0, T)) \), the boundary value problem (4.1) has a unique weak solution.

Uniqueness is a direct consequence of the following proposition.

**Proposition 4.1** (Comparison principle) Under the same assumptions as in Theorem 4.1, let \( v \) and \( \hat{v} \) be respectively weak sub- and super-solution of (4.1), i.e., \( v, \hat{v} \in L^2(0, T; H^2(\Gamma)) \), \( \partial_t v, \partial_t \hat{v} \in L^2(\Gamma \times (0, T)) \) such that

\[
\begin{aligned}
&\int_{\Gamma} (-\partial_t vw + \mu \partial v \partial w + H(x, \partial v) w) dx \leq \int_{\Gamma} f w dx, \\
&\int_{\Gamma} (-\partial_t \hat{w} w + \mu \partial \hat{v} \partial w + H(x, \partial \hat{v}) w) dx \geq \int_{\Gamma} f w dx, \\
v(x, T) \leq v_T(x) \leq \hat{v}(x, T)
\end{aligned}
\]

for a.a. \( x \in \Gamma \).

Then \( v \leq \hat{v} \) in \( \Gamma \times (0, T) \).

**Proof** Setting \( \tilde{v} = v - \hat{v} \), we have, for all \( w \in W \) such that \( w \geq 0 \),

\[
\int_{\Gamma} -\partial_t \tilde{v} w + \mu \partial \tilde{v} \partial w + (H(x, \partial v) - H(x, \partial \hat{v})) w dx \leq 0,
\]

\( \square \)
for a.a \( t \in (0, T) \), and \( \overline{v} (x, T) \leq 0 \) for all \( x \in \Gamma \). Setting \( \overline{v}^+ = \overline{v} \chi_{|\overline{v}|>0} \) and \( w = \overline{v}^+ e^{\lambda t} \varphi \), we get

\[
\int_{\Gamma} \left( \frac{\overline{v}^+(T)^2}{2} - \frac{\overline{v}^+(T)^2}{2} e^{\lambda t} \right) \varphi dx + \int_{0}^{T} \int_{\Gamma} \lambda \frac{(\overline{v}^+)^2}{2} e^{\lambda t} \varphi dxdt + \int_{0}^{T} \int_{\Gamma} \mu (\partial \overline{v}^+)^2 e^{\lambda t} dxdt + \int_{0}^{T} \int_{\Gamma} H (x, \partial v) \overline{v}^+ e^{\lambda t} dxdt = 0.
\]

From (1.19), \( |H (x, \partial v) - H (x, \partial \hat{v})| \leq C_0 |\partial v| \). Hence, since \( \overline{v}^+(T) = 0 \) and \( |\partial v| \overline{v}^+ = |\partial \overline{v}^+| \overline{v}^+ \) almost everywhere, we get

\[
\int_{0}^{T} \int_{\Gamma} \left( \frac{\lambda}{2} (\overline{v}^+)^2 + \mu (\partial \overline{v}^+)^2 \right) e^{\lambda t} \varphi dxdt - \int_{0}^{T} \int_{\Gamma} (\mu |\partial \varphi| + C_0 \varphi) |\partial \overline{v}^+| \overline{v}^+ e^{\lambda t} dxdt \leq 0.
\]

For \( \lambda \) large enough, the first term in the left hand side is not smaller than the second term. This implies that \( \overline{v}^+ = 0 \).

Now we prove Theorem 4.1, following the ideas in [5]. We start with a bounded Hamiltonian.

**Proof of existence in Theorem 4.1 when \( H \) is bounded by \( C_H \).** Take \( \overline{v} \in L^2 (0, T; V) \) and \( f \in L^2 (\Gamma \times (0, T)) \). From Theorem 2.1 and Theorem 2.2 with \( h = f - H (x, \partial \overline{v}) \) and \( v_T \in V \), the following boundary value problem

\[
\begin{align*}
\begin{cases}
-\partial_t v - \mu_\alpha \partial^2 v &= f - H (x, \partial \overline{v}), & \text{in } (\Gamma \alpha \setminus \mathcal{Y}) \times (0, T), & \alpha \in \mathcal{A}, \\
v |_{\Gamma_a} (v_i, t) &= v |_{\Gamma_b} (v_i, t), & t \in (0, T), & \alpha, \beta \in \mathcal{A}_i, & v_i \in \mathcal{Y}_i, \\
\sum_{\alpha \in \mathcal{A}_i} \gamma_{\alpha} \mu_\alpha \partial_\alpha v (v_i, t) &= 0, & t \in (0, T), & v_i \in \mathcal{Y}, \\
v (x, T) &= v_T (x), & x \in \Gamma,
\end{cases}
\end{align*}
\]

has a unique weak solution \( v \in L^2 (0, T; H^2 (\Gamma)) \cap C (0, T; V) \cap W^{1,2} (0, T; L^2 (\Gamma)) \). This allows us to define the map \( \mathcal{T} : \overline{v} \in L^2 (0, T; V) \mapsto v \in L^2 (0, T; V) \). From (1.19), \( \overline{v} \mapsto H (x, \partial \overline{v}) \) is continuous from \( L^2 (0, T; V) \) into \( L^2 (\Gamma \times (0, T)) \). From Theorem 2.2, we also see that \( \mathcal{T} \) is continuous from \( L^2 (0, T; V) \) to \( L^2 (0, T; V) \). Moreover, there exists a constant \( C \) depending only on \( C_H, \Gamma, (\mu_\alpha)_{\alpha \in \mathcal{A}}, f, T, \varphi \) and \( v_T \) such that

\[
\| \partial_t v \|_{L^2 (\Gamma \times (0, T))} + \| v \|_{L^2 (0, T; H^2 (\Gamma))} \leq C.
\]

Therefore, from Aubin–Lions theorem (see Lemma A.1), we obtain that \( \mathcal{T} (L^2 (0, T; V)) \) is relatively compact in \( L^2 (0, T; V) \). By Schauder fixed point theorem, see [10, Corollary 11.2], \( \mathcal{T} \) admits a fixed point which is a weak solution of (4.1).

**Proof of existence in Theorem 4.1 in the general case.** We truncate the Hamiltonian as follows:

\[
H_n (x, p) = \begin{cases}
H (x, p) & \text{if } |p| \leq n, \\
H \left( x, \frac{p}{|p|} n \right) & \text{if } |p| > n.
\end{cases}
\]

From the previous step for bounded Hamiltonians, we see that for all \( n \), there exists a solution \( v_n \in L^2 (0, T; H^2 (\Gamma)) \cap C (0, T; V) \cap W^{1,2} (0, T; L^2 (\Gamma)) \) of (4.1), where \( H \) is replaced by \( H_n \). We propose to send \( n \) to \( +\infty \) and find a subsequence of \( \{ v_n \} \) which converges to a solution of (4.1). For that, we need some uniform estimates on \( \{ v_n \} \). As in the proof of
Proposition 4.1, using $-v e^{\lambda t}$ as a test-function, integrating from 0 to $T$ and noticing that $H$ is sublinear, see (1.18), we obtain
\[
\int_{\Gamma} \left[ \frac{v^2(x,0)}{2} - \frac{v^2(x,T)}{2} \right] e^{\lambda t} dx + \int_{0}^{T} \int_{\Gamma} \left[ \frac{\lambda}{2} v^2 e^{\lambda t} + \mu \|v_n\|^2 e^{\lambda t} + \mu \partial v_n v_n e^{\lambda t} \right] dx dt = - \int_{0}^{T} \int_{\Gamma} H_n(x, \partial v_n) v_n e^{\lambda t} dx dt + \int_{0}^{T} \int_{\Gamma} f v_n e^{\lambda t} dx dt \\
\leq C_0 \int_{0}^{T} \int_{\Gamma} (1 + |\partial v_n|) |v_n| e^{\lambda t} dx dt + \frac{1}{2} \int_{0}^{T} \int_{\Gamma} f^2 e^{\lambda t} dx dt + \frac{1}{2} \int_{0}^{T} \int_{\Gamma} v^2_n e^{\lambda t} dx dt.
\]

In the following lines, the constant $C$ above will vary from line to line and will depend only on $(\mu_n)_{\alpha \in \mathcal{A}}, C_H, T$ and $\phi$. Taking $\lambda$ large enough leads to the following estimate:
\[
\|v_n\|_{L^2(0,T;V)} \leq C \left( \|f\|_{L^2(0,T;L^2(\Gamma))} + \|v_T\|_{L^2(\Gamma)} + 1 \right),
\]
and thus, from (1.18) again, we also obtain
\[
\int_{0}^{T} \int_{\Gamma} |H_n(x, \partial v_n)|^2 dx dt \leq \int_{0}^{T} \int_{\Gamma} C_0^2 (|\partial v_n| + 1)^2 dx dt \leq \int_{0}^{T} \int_{\Gamma} 2C_0^2 (|\partial v_n|^2 + 1) dx dt \\
\leq C \left( \|f\|^2_{L^2(0,T;L^2(\Gamma))} + \|v_T\|^2_{L^2(\Gamma)} + 1 \right).
\]

Therefore, $\{H_n(x, \partial v_n) - f\}$ is uniformly bounded in $L^2(0,T;L^2(\Gamma))$. From Theorem 2.2, we obtain that $(v_n)_{n \in \mathbb{N}}$ is uniformly bounded in $L^2(0,T;H^2(\Gamma)) \cap C([0,T];V) \cap W^{1,2}(0,T;L^2(\Gamma))$. By Aubin–Lions theorem (see Lemma A.1), $(v_n)_{n}$ is relatively compact in $L^2(0,T;V)$ (and bounded in $C([0,T];V)$). Hence, up to the extraction of a subsequence, there exists $v \in L^2(0,T;V) \cap W^{1,2}(0,T;L^2(\Gamma))$ such that $v_n \rightarrow v$ strongly in $L^2(0,T;V)$ and $\partial_t v_n \rightarrow \partial_t v$ weakly in $L^2(\partial \Gamma \times (0,T))$, and $H_n(x, \partial v_n) \rightarrow H(x, \partial v)$ a.e. in $\Gamma \times (0,T)$. From Lebesgue dominated convergence theorem, (that can be applied because $H_n(x, \partial v_n) \leq H(x, \partial v_n) \leq C_0 (1 + |\partial v_n|)$), $H_n(x, \partial v_n) \rightarrow H(x, \partial v)$ in $L^2(\Gamma \times (0,T))$. Thus, it is possible to pass to the limit in the weak formulation satisfied by $v_n$ and obtain that for all $w \in W$, $\chi \in C_c(0,T)$,
\[
\int_{0}^{T} \chi(t) \left( - \int_{\Gamma} \partial_t v w dx + \int_{\Gamma} \partial v w dx + \int_{\Gamma} H(x, v) w dx \right) dt = \int_{0}^{T} \chi(t) \left( \int_{\Gamma} f w dx \right) dt.
\]

Therefore, $v$ satisfies the first line in (4.2).

Since for all $\alpha \in \mathcal{A}$, $(v_n)_{n}$ tends to $v$ in $L^2(\Gamma \times (0,T))$ strongly and in $W^{1,2}(\Gamma \times (0,T))$ weakly, $v_n |_{\Gamma \times \{t=T\}}$ converges to $v|_{\Gamma \times \{t=T\}}$ in $L^2(\Gamma \times \{t\})$ strongly. Moreover, $v_n(T) = v_T$, from Theorem 2.1. Passing to the limit in the latter identity, we get the second condition in (4.2). We have proven that $v$ is a weak solution of (4.1). $\square$

We end the section with a stability result for the Hamilton–Jacobi equation.

**Lemma 4.1** Let $(v_{T_\\epsilon})_{\epsilon}, (f_\\epsilon)_{\epsilon}$ be sequences of functions such that $v_{T_\\epsilon} \rightarrow v_T$ in $V$ and $f_\\epsilon \rightarrow f$ in $L^2(\Gamma \times (0,T))$. The sequence $(v_\\epsilon)_{\epsilon}$ of weak solutions of (4.1) with data $v_{T_\\epsilon}, f_\\epsilon, (v_\\epsilon)$ converges in $L^2(0,T;H^2(\Gamma)) \cap C([0,T];V) \cap W^{1,2}(0,T;L^2(\Gamma))$ to the weak solution $v$ of (4.1) with data $v_T, f$.

**Proof** Subtracting the two PDEs for $v_\\epsilon$ and $v$, multiplying by $(v_\\epsilon - v) e^{\lambda t} - 1$, taking the integral on $\Gamma \times (0,T)$ and using similar arguments as in the proof of Proposition 4.1, we obtain
\[
\|v_\\epsilon - v\|_{L^2(0,T;V)} \leq C \left( \|f_\\epsilon - f\|_{L^2(\Gamma \times (0,T))} + \|v_{T_\\epsilon} - v_T\|_{L^2(\Gamma)} \right), \quad \text{for } \lambda \text{ large enough}
\]
C independent of ε. This proves the convergence of \( v_\varepsilon \) to \( v \) in \( L^2(0, T; V) \). Then, the convergence in \( L^2(0, T; H^2(\Gamma)) \cap C([0, T]; V) \cap W^{1, 2}(0, T; L^2(\Gamma)) \) results from the assumption that \( H \) is Lipschitz with respect to its second argument, and from stability results for the linear boundary value problem (2.1) which are obtained with similar arguments as in the proof of Theorem 2.2.

\[ \square \]

### 4.2 Regularity for the Hamilton–Jacobi equation

In this section, we prove further regularity for the solution of (4.1). Note that viscosity solutions of HJB equations with Kirchhoff conditions were obtained in [13] and [22] under the assumption that the diffusion degenerates at the vertices, so the latter results are actually more related to first-order Hamilton–Jacobi equations. Wahbi [26] recently obtained classical solutions for parabolic HJB equations with a quadratic Hamiltonian under the restrictive assumption that the coefficients and data do not depend on time. Unfortunately, this result and the related techniques are not relevant in the context of mean field games, because the density of the state distribution is involved in the Bellman equation and makes it time-dependent. The latter aspect seems to be a significant difficulty in obtaining bounds on the gradient when the Hamiltonian is not globally Lipschitz continuous.

**Theorem 4.2** We suppose that the assumptions of Theorem 4.1 hold and that, in addition, \( v_T \in H^2(\Gamma) \) satisfies the Kirchhoff conditions given by the third equation in (4.1), \( f \in PC(\Gamma \times [0, T]) \cap L^2(0, T; H^1_0(\Omega)) \) and \( \partial_t f \in L^2(0, T; H^1_0(\Gamma)) \).

Then, the unique solution \( v \) of (4.1) satisfies \( v \in L^2(0, T; H^1(\Gamma)) \) and \( \partial_t v \in L^2(0, T; H^1(\Gamma)) \). Moreover, there exists a constant \( C \) depending only on \( \|v_T\|_{H^2(\Gamma)}, (\mu_\alpha)_{\alpha \in \mathcal{A}}, H \) and \( f \) such that

\[
\|v\|_{L^2(0, T; H^1(\Gamma))} + \|\partial_t v\|_{L^2(0, T; H^1(\Gamma))} \leq C. \tag{4.3}
\]

If, in addition, there exists \( \eta \in (0, 1) \) such that \( v_T \in C^{2+\eta}(\Gamma) \), then there exists \( \tau \in (0, 1) \) such \( v \in C^{2+\tau, 1+\frac{\tau}{2}}(\Gamma \times [0, T]) \), and \( v \) is a classical solution of (4.1).

In the proof of Theorem 4.2, the main idea is to differentiate (4.1) with respect to the space variable and to prove some regularity properties for the derived equation. Let us describe the method formally: assuming that the solution \( v \) of (4.1) is in \( C^{2-1}(\Gamma \times (0, T)) \) and taking the space-derivative of (4.1) on \( (\Gamma_\alpha \setminus \mathcal{V}) \times (0, T) \), we get

\[-\partial_t v - \mu_\alpha \partial^2 v + \partial (H(x, \partial v)) = \partial f.\]

Therefore, \( u = \partial v \) satisfies

\[-\partial_t u - \mu_\alpha \partial^2 u + \partial (H(x, u)) = \partial f, \]

with terminal condition \( u(x, T) = \partial v_T(x) \). From the Kirchhoff conditions in (4.1) and Remark 1.1, we obtain a condition for \( u \) of Dirichlet type, namely

\[
\sum_{\alpha \in \mathcal{A}} \mu_\alpha \gamma_{\alpha} n_\alpha u|_{\Gamma_\alpha}(v_i, t) = 0, \quad t \in (0, T), \quad v_i \in \mathcal{V}.
\]

At the boundary vertices of \( \Gamma \), the latter condition is of an homogeneous Dirichlet condition.

Now, by extending continuously the PDEs in (4.1) until the vertex \( v_i \) in the edges \( \Gamma_\alpha \) and \( \Gamma_\beta, \alpha, \beta \in \mathcal{A}_i \), and using the continuity condition in (4.1), one gets

\[-\mu_\alpha \partial^2 v|_{\Gamma_\alpha} + H^\alpha(v_i, \partial v|_{\Gamma_\alpha}(v_i, t)) - f|_{\Gamma_\alpha}(v_i, t) = -\mu_\beta \partial^2 v|_{\Gamma_\beta} + H^\beta(v_i, \partial v|_{\Gamma_\beta}(v_i, t)) - f|_{\Gamma_\beta}(v_i, t).
\]
This gives a second transmission condition for \( u \) at \( v_i \in \mathcal{V} \setminus \partial \Gamma \) of Robin type, namely
\[
\mu_\alpha \partial u |_{\Gamma_{\alpha}} (v_i, t) - H^\alpha (v_i, u |_{\Gamma_{\alpha}} (v_i, t)) + f |_{\Gamma_{\alpha}} (v_i, t) = \mu_\beta \partial u |_{\Gamma_{\beta}} (v_i, t) - H^\beta (v_i, u |_{\Gamma_{\beta}} (v_i, t)) + f |_{\Gamma_{\beta}} (v_i, t).
\] (4.4)

Recalling (1.5), we end up with the following nonlinear boundary value problem for \( u = \partial v \),
\[
\begin{cases}
-\partial_t u - \mu_\alpha \partial^2 u + \partial \left( H (x, u) \right) = \partial f (x, t), & (x, t) \in (\Gamma_{\alpha} \setminus \mathcal{V}) \times (0, T), \ \alpha \in \mathcal{A}, \\
\sum_{a \in \mathcal{A}_i} \gamma_{ia} \mu_\alpha n_{i\alpha} u |_{\Gamma_{\alpha}} (v_i, t) = 0, & t \in (0, T), \ \psi \in \mathcal{V}, \\
\mu_\alpha n_{i\alpha} \partial u (v_i, t) - H^\alpha (v_i, u |_{\Gamma_{\alpha}} (v_i, t)) + f |_{\Gamma_{\alpha}} (v_i, t) = \mu_\beta n_{i\beta} \partial u (v_i, t) - H^\beta (v_i, u |_{\Gamma_{\beta}} (v_i, t)) + f |_{\Gamma_{\beta}} (v_i, t), & t \in (0, T), \ \alpha, \beta \in \mathcal{A}_i, \ \psi \in \mathcal{V} \setminus \partial \Gamma, \\
u (x, T) = u_T (x), & x \in \Gamma,
\end{cases}
\] (4.5)

where \( \partial f \in L^2 (\Gamma \times (0, T)) \) and \( u_T \in F \), where \( F \) is defined in (4.6) below. Theorem 4.2 will follow by choosing \( u_T = \partial v_T \).

In order to define the weak solutions of (4.5), we need the following subspaces of \( H^1_b (\Gamma) \),
\[
F := \left\{ u \in H^1_b (\Gamma) \text{ and } \sum_{a \in \mathcal{A}_i} \gamma_{ia} \mu_\alpha n_{i\alpha} u |_{\Gamma_{\alpha}} (v_i) = 0 \text{ for all } v_i \in \mathcal{V} \right\},
\]
\[
E := \left\{ e \in H^1_b (\Gamma) \text{ and } \sum_{a \in \mathcal{A}_i} n_{i\alpha} e |_{\Gamma_{\alpha}} (v_i) = 0 \text{ for all } v_i \in \mathcal{V} \right\},
\] (4.6)

and we will use the test function \( \psi \) defined by
\[
\begin{align*}
\psi & \text{ is affine on } (0, \ell_\alpha), \\
\psi |_{\Gamma_{\alpha}} (v_i) & = \mu_\alpha \gamma_{i\alpha}, \text{ if } v_i \in \mathcal{V} \setminus \partial \Gamma, \ \alpha \in \mathcal{A}_i, \\
\psi & \text{ is constant on the edges } \Gamma_{\alpha} \text{ which touch the boundary of } \Gamma.
\end{align*}
\]

Note that \( \psi \) is positive and bounded. The map \( f \mapsto f \psi \) is an isomorphism from \( F \) onto \( E \).

**Definition 4.2** A weak solution of (4.5) is a function \( u \in L^2 (0, T; F) \) such that \( \partial_t u \in L^2 (0, T; E') \), and
\[
\begin{cases}
-\langle \partial_t u, e \rangle_{E', E} + \int_{\Gamma} \left( \mu \partial u \partial e - \partial (H (x, u)) \partial e \right) dx = - \int_{\Gamma} f \partial e dx, & \text{for all } e \in E, \ a.a \ t \in (0, T), \\
u (\cdot, T) = u_T.
\end{cases}
\] (4.7)

**Remark 4.1** Note that, if \( u \) is regular enough, then (4.7) can also be written
\[
\begin{align*}
- \langle \partial_t u, e \rangle_{E', E} & + \int_{\Gamma} \left( \mu \partial u \partial e + \partial (H (x, u)) \partial e \right) dx - \sum_{i \in I} \sum_{a \in \mathcal{A}_i} n_{i\alpha} \left[ H^\alpha (v_i, u |_{\Gamma_{\alpha}} (v_i, t)) - f |_{\Gamma_{\alpha}} (v_i, t) \right] e |_{\Gamma_{\alpha}} (v_i) \\
& = \int_{\Gamma} (\partial f) \partial e dx \quad \text{for all } e \in E, \ a.a \ t \in (0, T).
\end{align*}
\]
Remark 4.2 To explain formally the definition of weak solutions, let us use \( e \in E \) as a test-function in the PDE in (4.5). After an integration by parts, we get

\[
\int_{\Gamma} \left( -\partial_t u e + \mu \partial u \partial e + \partial (H(x,u)) e \right) dx - \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \mu_{i\alpha} \partial u |_{\Gamma^e}(v_i, t) e |_{\Gamma^e}(v_i) = \int_{\Gamma} (\partial f) e dx,
\]

where \( n_{i\alpha} \) is defined in (1.4). On the one hand, from the second transmission condition, there exists a function \( c_i : (0, T) \rightarrow \mathbb{R} \) such that \( \mu_{i\alpha} \partial u |_{\Gamma^e}(v_i, t) - H^\alpha(v_i, u |_{\Gamma^e}(v_i, t)) + f |_{\Gamma^e}(v_i, t) = c_i(t) \) for all \( \alpha \in \mathcal{A}_i \). It follows that

\[
- \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \mu_{i\alpha} \partial u |_{\Gamma^e}(v_i, t) e |_{\Gamma^e}(v_i) = - \sum_{i \in I} c_i(t) \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} e |_{\Gamma^e}(v_i) + \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \left[ -H^\alpha(v_i, u |_{\Gamma^e}(v_i, t)) + f |_{\Gamma^e}(v_i, t) \right] e |_{\Gamma^e}(v_i)
\]

\[
= \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \left[ -H^\alpha(v_i, u |_{\Gamma^e}(v_i, t)) + f |_{\Gamma^e}(v_i, t) \right] e |_{\Gamma^e}(v_i),
\]

because \( e \in E \). Then we may use the Remark 4.1 and obtain (4.7).

We start by proving the following result about (4.5) and then give the proof of Theorem 4.2.

Theorem 4.3 Under the running assumptions, if \( u_T \in F, f \in C(\Gamma \times [0, T]) \cap L^2(0, T; H^1_0(\Gamma)) \) and \( \partial_t f \in L^2(0, T; H^1_0(\Gamma)) \), then (4.5) has a unique weak solution \( u \). Moreover, there exists a constant \( C \) depending only on \( \Gamma, T, \Psi, \|u_T\|_F, \|\partial_t f\|_{L^2(\Gamma \times [0, T])} \) and \( \|\partial_t f\|_{L^2(0, T; H^1_0(\Gamma))} \) such that

\[
\|u\|_{L^2(0, T; H^1_0(\Gamma))} + \|u\|_{C([0, T]; F)} + \|\partial_t u\|_{L^2(\Gamma \times [0, T])} \leq C. \tag{4.8}
\]

Remark 4.3 Theorem 4.3 implies that for all \( \alpha \in \mathcal{A}, u(\cdot, t) \in C^1(\Gamma_\alpha) \) for a.e. \( t \). Hence, the transmission conditions for \( u \) hold in a classical sense for a.e. \( t \in [0, T] \).

As in Sect. 2, we use the Galerkin’s method to construct solutions of certain finite-dimension approximations to (4.5).

We first notice that the symmetric bilinear form \( \mathcal{B}(u, v) := \int_{\Gamma} \mu \psi^{-1} \partial u \partial v \) is such that \( (u, v) \mapsto (u, v)_L^2(\Gamma) + \mathcal{B}(u, v) \) is an inner product in \( E \) equivalent to the standard inner product in \( E \), namely \( (u, v)_E = (u, v)_L^2(\Gamma) + \int_{\Gamma} \mu \partial u \partial v \). From Fredholm’s theory, there exists a non decreasing sequence of nonnegative real numbers \( (\lambda_k)_{k=1}^\infty \), that tends to +\( \infty \) as \( k \to \infty \) and a Hilbert basis \( (e_k)_{k=1}^\infty \) of \( L^2(\Gamma) \), which is also a total sequence of \( E \) (and orthogonal if \( E \) is endowed with the scalar product \( (u, v)_{L^2(\Gamma)} + \mathcal{B}(u, v) \), such that

\( \mathcal{B}(e_k, e) = \lambda_k (e_k, e)_{L^2(\Gamma)} \) for all \( e \in E \). Note that

\[
\int_{\Gamma} \mu \partial e_k \partial e_k \psi^{-1} dx = \begin{cases} 
\lambda_k & \text{if } k = \ell, \\
0 & \text{if } k \neq \ell.
\end{cases}
\]

Note also that \( e_k \) is a weak solution of

\[
\begin{cases} 
- \mu_{i\alpha} \partial (\psi^{-1} \partial e_k) = \lambda_k e_k & \text{in } \Gamma_{\alpha} \setminus \mathcal{V}, \alpha \in \mathcal{A}, \\
\partial_{\alpha} e_k (v_i) = \frac{\partial_{\beta} e_k (v_i)}{\gamma_{i\beta}} & \text{for all } \alpha, \beta \in \mathcal{A}, \\
\sum_{\alpha \in \mathcal{A}} n_{i\alpha} e_k |_{\Gamma_{\alpha}} (v_i) = 0 & \text{if } v_i \notin \mathcal{V}
\end{cases}
\]
which implies that \( e_k|_{\Gamma_0} \in C^2(\Gamma_\omega) \) for all \( \omega \in \mathcal{A} \).

Finally, the sequence \( (f_k)_{k=1}^\infty \): \( f_k = \psi^{-1}e_k \), is a total (but not orthogonal) family in \( F \).

**Lemma 4.2** Under the assumptions made in Theorem 4.3, for any positive integer \( n \), there exist \( n \) absolutely continuous functions \( y^T_k : [0, T] \to \mathbb{R} \), \( k = 1, \ldots, n \), and a function \( u_n : [0, T] \to L^2(\Gamma) \) of the form

\[
u_n(t) = \sum_{k=1}^n y^T_k(t) f_k, \tag{4.9}\]

such that for all \( k = 1, \ldots, n \),

\[
y^T_k(T) = \int_\Gamma u_T f_k \psi^2 dx, \quad -\frac{d}{dt}(u_n, f_k \psi)_{L^2(\Gamma)} + \int_\Gamma (\mu \partial u_n - H(x, u_n)) \partial(f_k \psi) dx = -\int_\Gamma f \partial(f_k \psi) dx. \tag{4.10}\]

Moreover, there exists a constant \( C \) depending only on \( \Gamma \), \( T \), \( \psi \), \( \|u_T\|_{F} \), \( \|\partial f\|_{L^2(\Gamma \times [0, T])} \) \( \|f\| \in C(\Gamma \times [0, T]) \) and \( \|\partial_t f\|_{L^2(\Gamma \times [0, T])} \) such that

\[
u_n\|_{L^\infty(0, T; F)} + \|u_n\|_{L^2(0, T; H^1(\Gamma))} + \|\partial_t u_n\|_{L^2(\Gamma \times [0, T])} \leq C. \]

**Proof** The proof follows the same lines as the one of Lemma 2.1 but it is more technical.

Step 1: Existence. For \( n \geq 1 \), we consider the symmetric and positive definite matrix \( M_n \) defined by \( (M_n)_{k\ell} = \int_\Gamma f_k f_\ell \psi dx \), which satisfies, for some constants \( c, C \) independent of \( n \),

\[
c |\xi|^2 \leq \sum_{k, \ell=1}^n (M_n)_{k\ell} \xi_k \xi_\ell \leq C |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{4.11}\]

Looking for \( u_n \) of the form (4.9) and setting \( Y = (y^n_1, \ldots, y^n_n)^T \), \( \dot{Y} = (\frac{d}{dt}y^n_1, \ldots, \frac{d}{dt}y^n_n)^T \), (2.5) implies that we have to solve the following system of ODEs:

\[
\begin{cases}
-M_n \dot{Y}(t) + BY(t) + \mathcal{H}(Y)(t) = G(t), \quad t \in [0, T] \\
Y(T) = \left( \int_\Gamma u_T f_1 \psi^2 dx, \ldots, \int_\Gamma u_T f_n \psi^2 dx \right)^T,
\end{cases} \tag{4.12}\]

where

\[-B_{k\ell} = \int_\Gamma \mu \partial f_\ell \partial(f_k \psi) dx
-\mathcal{H}_i(Y) = -\int_\Gamma H(x, Y^T F) \partial(f_i \psi) dx \quad \text{with } F = (f_1, \ldots, f_n)^T \text{ and } Y^T F = \sum_{\ell} y^n_\ell f_\ell = u_n
-G_i(t) = -\int_\Gamma f(x, t) \partial(f_i \psi) dx \quad \text{for all } i = 1, \ldots, n.\]

Since the matrix \( M \) is invertible and the function \( \mathcal{H} \) is Lipschitz continuous by (1.19), the system (4.12) has a unique global solution.

Step 2: Uniform estimates of \( u_n \) in \( L^\infty(0, T; L^2(\Gamma)) \), \( L^2(0, T; F) \) and \( W^{1,2}(0, T; E') \). Testing (4.10) by \( y^T_k(t) f_k e^{\lambda t} \psi \) where \( \lambda \) is a positive constant to be chosen later, summing for \( k = 1, \ldots, n \) and using (4.9), we get

\[-\int_\Gamma \partial_t u_n u_n e^{\lambda t} \psi dx + \int_\Gamma (\mu \partial u_n - H(x, u_n)) \partial(u_n e^{\lambda t} \psi) dx = -\int_\Gamma f \partial(u_n \psi e^{\lambda t}) dx.\]
Since \( H \) satisfies (1.18) and \( f \) is bounded, there exists a constant \( C \) (\( C \) may vary from lines to lines hereafter) such that
\[
- \int_{\Gamma} \left[ \frac{1}{2} \left( u_n^{2} e^{\lambda t} \right) - \frac{\lambda}{2} u_n^{2} e^{\lambda t} \right] \psi dx + \int_{\Gamma} \mu \left| \partial u_n \right|^{2} e^{\lambda t} \psi dx - C \int_{\Gamma} \left| u_n \right| \left( |u_n| + |\partial u_n| \right) e^{\lambda t} dx \\
\leq C \int_{\Gamma} \left( |u_n| + |\partial u_n| \right) e^{\lambda t} dx.
\]

The desired estimate on \( u_n \) is obtained from the previous inequality in a similar way as in the proof of Lemma 2.1, by taking \( \lambda \) large enough.

**Step 3: Uniform estimates of \( u_n \) in \( L^\infty(0, T; F) \cap L^2(0, T; H^2(\Gamma)) \) and of \( \partial_t u_n \) in \( L^2(\Gamma \times (0, T)) \).**

Multiplying (4.10) by \( \partial_t y_n^\alpha (t) \int_k e^{\lambda t} \psi \) where \( \lambda \) is a positive constant to be chosen later, integrating by part the term containing \( H \) and \( f \) (all the integration by parts are justified) summing for \( k = 1, \ldots, n \) and using (4.9), we obtain that
\[
- \int_{\Gamma} \left( \partial_t u_n \right)^2 e^{\lambda t} \psi dx + \int_{\Gamma} \mu \partial u_n \partial \left( \partial_t u_n e^{\lambda t} \psi \right) dx + \int_{\Gamma} \partial \left( H (x, u_n) \right) \partial_t u_n e^{\lambda t} \psi dx \\
- \sum_{i \in I} \sum_{\alpha \in \mathcal{O}_i} n_i \alpha \left[ H^\alpha (v_i, u_n | r_{\alpha} (v_i, t)) - f | r_{\alpha} (v_i, t) \right] \partial_t u_n | r_{\alpha} (v_i, t) \psi | r_{\alpha} (v_i) e^{\lambda t} \\
= \int_{\Gamma} \partial f \partial_t u_n \psi e^{\lambda t} dx.
\]

Note that from (1.19) and (1.20),
\[
|\partial (H (x, u_n))| \leq C_0 (1 + |u_n| + |\partial u_n|)
\]
so, from Step 2, this function is bounded in \( L^2(\Gamma \times (0, T)) \) by a constant. Moreover,
\[
\int_s^T \int_{\Gamma} \partial f \partial_t u_n \psi e^{\lambda t} dx dt \leq C \left( \int_s^T \int_{\Gamma} (\partial f)^2 e^{2\lambda t} dx dt \right)^{1/2} \left( \int_s^T \int_{\Gamma} (\partial_t u_n)^2 e^{2\lambda t} dx dt \right)^{1/2},
\]
and we can also estimate the term \( \int_{\Gamma} \mu \partial u_n \partial \left( \partial_t u_n e^{\lambda t} \psi \right) dx \) as in the proof of Theorem 2.2. Therefore, the only new difficulty with respect to the proof of Theorem 2.2 consists of obtaining a bound for the term
\[
\sum_{i \in I} \sum_{\alpha \in \mathcal{O}_i} n_i \alpha \left[ H^\alpha (v_i, u_n | r_{\alpha} (v_i, t)) - f | r_{\alpha} (v_i, t) \right] \partial_t u_n | r_{\alpha} (v_i, t) \psi | r_{\alpha} (v_i) e^{\lambda t}.
\]

Let \( J_{i\alpha} (p) \) be the primitive function of \( p \mapsto H^\alpha (v_i, p) \) such that \( J_{i\alpha} (0) = 0 \):
\[
H^\alpha (v_i, u_n | r_{\alpha} (v_i, s)) \partial_t u_n | r_{\alpha} (v_i, s) = \frac{d}{dt} J_{i\alpha} (u_n | r_{\alpha} (v_i, s)).
\]

We can then write
\[
- \int_s^T \left( n_i \alpha H^\alpha (v_i, u_n | r_{\alpha} (v_i, t)) \partial_t u_n | r_{\alpha} (v_i, t) e^{\lambda t} \psi | r_{\alpha} (v_i) \right) dt \\
= n_i \alpha \psi | r_{\alpha} (v_i) \left( - J_{i\alpha} (u_n | r_{\alpha} (v_i, T)) e^{\lambda T} + J_{i\alpha} (u_n | r_{\alpha} (v_i, s)) e^{\lambda s} \right) \\
+ \lambda \int_s^T J_{i\alpha} (u_n | r_{\alpha} (v_i, t)) e^{\lambda t} dt.
\]
Since $H^\alpha(x, \cdot)$ is sublinear, see (1.18), $|\mathcal{J}_i\alpha(p)|$ is subquadratic, i.e., $|\mathcal{J}_i\alpha(p)| \leq C(1 + p^2)$, for a constant $C$ independent of $\alpha$ and $i$. This implies that

$$
\left| \int_s^T (n_i\alpha H^\alpha (v_i, u_n|\Gamma^\alpha (v_i, t)) \partial_t u_n|\Gamma^\alpha (v_i, t) e^{\lambda t} \psi|\Gamma^\alpha (v_i)) dt \right| \quad l
$$

\[
\leq C \left( e^{\lambda T} + u_n^2|\Gamma^\alpha (v_i, T) e^{\lambda T} + u_n^2|\Gamma^\alpha (v_i, s) e^{\lambda s} \right) + C\lambda \int_0^T (1 + u_n^2|\Gamma^\alpha (v_i, t)) e^{\lambda t} dt.
\]

Note that, from Step 2 and the continuous embedding of $H^1(\Gamma^\alpha)$ in $C(\Gamma^\alpha)$, $\lambda \int_0^T (1 + u_n^2|\Gamma^\alpha (v_i, t)) e^{\lambda t} dt \leq C\lambda e^{\lambda T}$. To summarize

$$
\left| \int_s^T (n_i\alpha H^\alpha (v_i, u_n|\Gamma^\alpha (v_i, t)) \partial_t u_n|\Gamma^\alpha (v_i, t) e^{\lambda t} \psi|\Gamma^\alpha (v_i)) dt \right| \quad (4.16)
$$

Similarly, using the fact that $f \in C(\Gamma \times [0, T])$ and $\partial_t f|\Gamma^\alpha (v_i, \cdot) \in L^2(0, T)$, and integrating by part, we see that

$$
\left| \int_s^T f|\Gamma^\alpha (v_i, t) \partial_t u_n|\Gamma^\alpha (v_i, t) e^{\lambda t} dt \right|
$$

\[
= \left| (f|\Gamma^\alpha u_n)|\Gamma^\alpha (v_i, T) e^{\lambda T} - (f|\Gamma^\alpha u_n)|\Gamma^\alpha (v_i, t) e^{\lambda s} 
- \int_s^T (\lambda f|\Gamma^\alpha (v_i, t) + \partial_t f|\Gamma^\alpha (v_i, t)) u_n|\Gamma^\alpha (v_i, t) e^{\lambda t} dt \right|

\leq C \left( |u_n|\Gamma^\alpha (v_i, T) e^{\lambda T} + |u_n|\Gamma^\alpha (v_i, s) e^{\lambda s} + \lambda \int_s^T |u_n|\Gamma^\alpha (v_i, t) |e^{\lambda t} dt \right)

+ \frac{1}{2} \int_s^T u_n^2|\Gamma^\alpha (v_i, t) e^{\lambda t} dt + \frac{1}{2} \int_s^T (\partial_t f|\Gamma^\alpha (v_i, t)) e^{\lambda t} dt.
\]

From Step 2 and the assumptions on $f$, the last three terms in the right hand side of the latter estimate are bounded by a constant depending on $\lambda$, but not on $n$. To summarize,

$$
\left| \int_s^T f|\Gamma^\alpha (v_i, t) \partial_t u_n|\Gamma^\alpha (v_i, t) e^{\lambda t} dt \right| \leq C \left( |u_n|\Gamma^\alpha (v_i, T) |e^{\lambda T} + |u_n|\Gamma^\alpha (v_i, s) e^{\lambda s} \right) + C(\lambda).
\]

(4.17)

To conclude from (4.16) and (4.17), we use the following estimates

$$
\left\{ \begin{array}{l}
|u_n|\Gamma^\alpha (v_i, t) \leq C \left( \int_{\Gamma^\alpha} |u_n (x, t)| dx + \int_{\Gamma^\alpha} |\partial u_n (x, t)| dx \right),

u_n^2|\Gamma^\alpha (v_i, t) \leq C \left( \int_{\Gamma^\alpha} u_n^2 (x, t) dx + \int_{\Gamma^\alpha} |u_n \partial u_n (x, t)| dx \right),
\end{array} \right. \quad (4.18)
$$

for $t = s$ and $t = T$.

Then proceeding as in the proof of Theorem 2.2 and combining (4.13), (4.14), (4.15), (4.16) and (4.17) with (4.18), we find the desired estimates by taking $\lambda$ large enough.

Let us end the proof by proving (4.18). The function $\phi = u_n|\Gamma^\alpha (s, t)$ is in $H^1(\Gamma^\alpha)$. By the continuous embedding $H^1(\Gamma^\alpha) \hookrightarrow C(\Gamma^\alpha)$, we can define $\phi$ in the pointwise sense (and even at two endpoints of any edges, see (1.1)). For all $\alpha \in \mathcal{A}$ and $x, y \in \Gamma^\alpha$, we have
\[ \phi(x) = \phi(y) + \int_{[y, x]} \partial \phi(\xi) d\xi. \]

It follows
\[
|\Gamma_\alpha| \phi(x) = \int_{\Gamma_\alpha} \phi(x) dy = \int_{\Gamma_\alpha} \phi(y) dy + \int_{\Gamma_\alpha} \int_{[y, x]} \partial \phi(\xi) d\xi dy \leq \int_{\Gamma_\alpha} |\phi(\xi)| d\xi
\]
\[
+ |\Gamma_\alpha| \int_{\Gamma_\alpha} |\partial \phi(\xi)| d\xi,
\]
which gives the first estimate setting \( x = v_i \). The second estimate is obtained in the same way replacing \( \phi \) by \( \phi^2 \) and using the fact that \( W^{1,1}(\Gamma_\alpha) \) is continuously imbedded in \( C(\Gamma_\alpha) \).

\[ \square \]

**Proof of Theorem 4.3.** From Lemma 4.2, up to the extraction of a subsequence, there exists \( u \in L^2\left(0, T; H^2_0(\Gamma)\right) \cap W^{1,2}(\Gamma \times (0, T)) \) such that
\[
\begin{cases}
  u_n \to u, & \text{in } L^2\left(0, T; F \cap H^2_0(\Gamma)\right), \\
  \partial_t u_n \to \partial_t u, & \text{in } L^2(\Gamma \times (0, T)).
\end{cases}
\]

Moreover, by Aubin–Lions Theorem (see Lemma A.1),
\[
L^2\left(0, T; F \cap H^2_0(\Gamma)\right) \cap W^{1,2}(\Gamma \times (0, T)) \xrightarrow{\text{compact}} L^2\left(0, T; F\right),
\]
so up to the extraction of a subsequence, we may assume that \( u_n \to u \) in \( L^2(0, T; F) \) and almost everywhere. Moreover, from the compactness of the trace operator from \( W^{1,2}(\Gamma_\alpha \times (0, T)) \) to \( L^2(\partial \Gamma_\alpha \times (0, T)) \), \( u_n|_{\partial \Gamma_\alpha \times (0, T)} \to u|_{\partial \Gamma_\alpha \times (0, T)} \) in \( L^2(\partial \Gamma_\alpha \times (0, T)) \) and for almost every \( t \in (0, T) \). Similarly, \( u_n|_{\partial \Gamma_\alpha \times \{t = T\}} \to u|_{\partial \Gamma_\alpha \times \{t = T\}} \) in \( L^2(\partial \Gamma_\alpha) \) and almost everywhere in \( \Gamma_\alpha \). Then, using the Lipschitz continuity of \( H \) with respect to its second argument, and similar arguments as in the proof of Theorem 2.1, we obtain the existence of a solution of (4.5) satisfying (4.8) by letting \( n \to +\infty \). Since \( H^2(\Gamma_\alpha) \subset C^{1+\sigma}(\Gamma_\alpha) \) for some \( \sigma \in (0, 1/2) \), \( u(\cdot, t) \in C^{1+\sigma}(\Gamma_\alpha) \) for all \( \alpha \in \mathscr{A} \) and a.a. \( t \). Finally, the proof of uniqueness is a consequence of the energy estimate (4.8) for \( u \).

\[ \square \]

Next, we want to prove that, if \( u \) is the solution of (4.5) and \( v \) is the solution of (4.1), then \( \partial u = v \). It means that we have to define a primitive function on the network \( \Gamma \).

**Definition 4.3** Let \( x \in \Gamma_{\alpha_0} = [v_{i_0}, v_{i_1}] \) and \( y \in \Gamma_{\alpha_m} = [v_{i_m}, v_{i_{m+1}}] \). We denote the set of paths joining from \( x \) to \( y \) by \( \mathcal{L} \). More precisely, if \( \mathcal{L} \subset [x, y] \), we can write \( \mathcal{L} \) under the form
\[
\mathcal{L} = x \to v_{i_1} \to v_{i_2} \to \ldots \to v_{i_m} \to y,
\]
with \( v_{i_k} \in \mathcal{V} \) and \([v_{i_k}, v_{i_{k+1}}] = \Gamma_{\alpha_k}\). The integral of a function \( \phi \) on \( \mathcal{L} \) is defined by
\[
\int_{\mathcal{L}} \phi(\xi) d\xi = \int_{[x, v_{i_1}]} \phi(\xi) d\xi + \sum_{k=1}^{m} \int_{[v_{i_k}, v_{i_{k+1}}]} \phi(\xi) d\xi + \int_{[v_{i_m}, y]} \phi(\xi) d\xi,
\]
recalling that the integrals on a segment are defined in (1.6).

**Lemma 4.3** Let \( u \) be the unique solution of (4.5) with \( u_T = \partial v_T \). Then for all \( x, y \in \Gamma \) and a.e. \( t \in [0, T] \), \( \int_{\mathcal{L}_1} u(\xi, t) d\xi = \int_{\mathcal{L}_2} u(\xi, t) d\xi \), for all \( \mathcal{L}_1, \mathcal{L}_2 \subset [x, y] \). This means that the integral of \( u \) from \( x \) to \( y \) does not depend on the path. Hence, for any \( \mathcal{L} \subset [x, y] \), we can define \( \int_{[x, y]} u(\xi, t) d\xi := \int_{\mathcal{L}} u(\xi, t) d\xi \).
**Proof** First, it is sufficient to prove \( \int_{\mathcal{L}} u(\zeta, t) \, d\zeta = 0 \) for all \( \mathcal{L} \in \tilde{x}x \). Secondly, if a given edge is taken twice in opposite senses, the two related contributions to the integral sum to zero. It follows that, without loss of generality, we only need to consider loops in \( \tilde{x}x \) such that all the complete edges that it contains are taken once only. It is also easy to see that we can focus on the case when \( x \in \mathcal{Y} \). To summarize, we only need to prove that \( \int_{\mathcal{L}} u(\zeta, t) \, d\zeta = 0 \) when \( v_{i_0} \in \mathcal{Y} \setminus \partial\Gamma \) and \( \mathcal{L} = v_{i_0} \to v_{i_1} \to \ldots \to v_{i_m} \to v_{i_0} \), where \( v_{i_k} \neq v_{i_l} \) for \( k \neq l \).

The following conditions

1. \( e|_{\Gamma_{i_0}} = 0 \) on each edge \( \Gamma_{i_0} \) not contained in \( \mathcal{L} \)
2. for all \( k = 0, \ldots, m-1 \), \( e|_{\Gamma_{ik}} = 1 \) if \( \Gamma_{ik} \) is the edge joining \( v_{ik} \) and \( v_{ik+1} \)
3. \( e|_{\Gamma_{im}} = 1 \) if \( \Gamma_{im} \) is the edge joining \( v_{im} \) and \( v_{i_0} \)

define a unique function \( e \in E \) which takes at most two values on \( \mathcal{L} \), namely \( \pm 1 \).

From Definition 4.3, we have

\[
\frac{d}{dt} \int_{\mathcal{L}} u(\zeta, t) \, d\zeta = \sum_{k=0}^{m} \frac{d}{dt} \int_{[v_{ik}, v_{ik+1}]} u(\zeta, t) \, d\zeta + \frac{d}{dt} \int_{[v_{im}, v_{i_0}]} u(\zeta, t) \, d\zeta
\]

Then, using Definition 4.2, Remark 4.1 and Remark 4.2 yields that

\[
\frac{d}{dt} \int_{\mathcal{L}} u(\zeta, t) \, d\zeta = \sum_{\alpha \in \mathcal{S}} \int_{\Gamma_{\alpha}} \left[ -\mu_{\alpha} \partial^2 u(\zeta, t) + \partial H(\zeta, u(\zeta, t)) - \partial f(\zeta, t) \right] \, e(\zeta) \, d\zeta
\]

Thus, for a.e. \( t \) from the regularity of \( u \) and the fact that \( e \in E \), we conclude that \( \frac{d}{dt} \int_{\mathcal{L}} u(\zeta, t) \, d\zeta = 0 \). Hence

\[
\int_{\mathcal{L}} u(\zeta, t) \, d\zeta = \int_{\mathcal{L}} u(\zeta, T) \, d\zeta = \int_{\mathcal{L}} u_T(\zeta) \, d\zeta = \int_{\mathcal{L}} \partial v_T(\zeta) \, d\zeta = 0,
\]

where the last identity comes from the continuity of \( v_T \) since \( v_T \in V \).

**Lemma 4.4** If \( u_T = \partial v_T \in F \), then the weak solution \( u \) of (4.5) satisfies \( u = \partial v \) where \( v \) is the unique solution of (4.1).

**Proof** For simplicity, we write the proof in the case when \( \partial\Gamma \neq \emptyset \). The proof is similar in the other case.

Let us fix some vertex \( v_k \in \partial\Gamma \). From standard regularity results for Hamilton–Jacobi equation with homogeneous Neumann condition, we know that that there exists \( \omega \), a closed neighborhood of \( \{v_k\} \) in \( \Gamma \) made of a single straight line segment and containing no other vertices of \( \Gamma \) than \( v_k \), such that \( v|_{[\omega \times (0, T)]} \in L^2(0, T; H^2(\omega)) \cap C([0, T]; H^2(\omega)) \cap W^{1,2}(0, T; H^1(\omega)) \). Hence, \( v \) satisfies the Hamilton–Jacobi equation at almost every point of \( \omega \times (0, T) \). Moreover the equation

\[
\partial_t v(v_k, t) + \mu \partial^2 v(v_k, t) - H(v_k, 0) + f(v_k, t) = 0
\]

(4.19)
holds for almost every \( t \in (0, T) \) and in \( L^2(0, T) \).

For every \( x \in \Gamma \) and \( t \in [0, T] \), we define

\[
\hat{v}(x, t) = v(v_k, t) + \int_{v_k}^{x} u(\zeta, t) \, d\zeta.
\] (4.20)

Note that, if \( \partial \Gamma = \emptyset \), then the proof should be modified by replacing \( v_k \) by a point \( v \in \Gamma \setminus \mathcal{V} \) and by using local regularity results for the HJB equation in (4.1).

We claim that \( \hat{v} \) is a solution of (4.1).

First, \( \hat{v}(\cdot, t) \) is continuous on \( \Gamma \). Indeed, \( \hat{v}(y, t) - \hat{v}(x, t) = \int_{xy}^{y} u(\zeta, t) \, d\zeta \). On the other hand, \( u \in C([0, T]; F) \subset L^\infty(\Gamma \times [0, T]) \). It follows that \( |\hat{v}(y, t) - \hat{v}(x, t)| \leq ||u||_{L^\infty(\Gamma \times [0, T])} \text{dist}(x, y) \) which implies that \( \hat{v}(\cdot, t) \) is continuous on \( \Gamma \).

Next, from the terminal conditions for \( u \),

\[
\hat{v}(x, T) = v(v_k, T) + \int_{v_k}^{x} u(\zeta, T) \, d\zeta = v_T(v_k) + \int_{v_k}^{x} \partial v_T(\zeta) \, d\zeta = v_T(x),
\]

where the last identity follows from the continuity of \( v_T \) on \( \Gamma \).

Let us check the Kirchhoff condition for \( \hat{v} \). Take \( v_i \in \mathcal{V} \) and \( \alpha \in \mathcal{A}_i \). From (1.5), for a.e. \( t \in (0, T) \), \( \partial_\alpha \hat{v}(v_i, t) = n_{i\alpha} \partial v|_{\Gamma_\alpha}(v_i, t) \) and from (4.20), \( \partial \hat{v}|_{\Gamma_\alpha}(v_i, t) = u|_{\Gamma_\alpha}(v_i, t) \). Since \( u(\cdot, t) \in F \), we get

\[
\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} u_{\alpha} \partial \alpha \hat{v}(v_i, t) = \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} u_{\alpha} n_{i\alpha} u|_{\Gamma_\alpha}(v_i, t) = 0,
\]

which is exactly the Kirchhoff condition for \( \hat{v} \) at \( v_i \).

There remains to prove that \( \hat{v} \) solves the Hamilton–Jacobi equation in \( \Gamma \setminus \mathcal{V} \): Take \( x \in \Gamma_\alpha \setminus \mathcal{V} \) for some \( \alpha \in \mathcal{A} \) and consider a path \( v_k \) to \( v \in \Gamma \). Let \( v_{i_{m+1}} = 0 \) be the other endpoint of \( \Gamma_\alpha \). We proceed as in the proof of Lemma 4.3: the following conditions

1. \( e|_{\Gamma_\alpha} = 0 \) on each edge \( \Gamma_\alpha \) not contained in \( L \)
2. for all \( j = 0, \ldots, m \), \( e|_{\Gamma_\alpha} = 1_{ij} - 1_{ij+1} \) if \( \Gamma_j \) is the edge joining \( v_i \) and \( v_{i+1} \)

define a unique piecewise constant function \( e \) which takes at most two values on \( L \), namely \pm 1. Note that \( e \) does not belong to \( E \) because \( e(v_k) \neq 0 \), but that \( e \) satisfies \( \sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} e|_{\Gamma_\alpha}(v_i) = 0 \) for all \( v_i \in \mathcal{V} \setminus \partial \Gamma \).

Using this function, a similar computation as in the proof of Lemma 4.3 implies that, for almost every \( t \in (0, T) \),

\[
\partial_t \hat{v}(x, t) - \partial_x v(x, t) = -u_{\alpha} \partial u|_{\Gamma_\alpha}(x, t) + H(x, u|_{\Gamma_\alpha}(x, t)) - f(x, t)
\]

Then, using (4.19) and the fact that \( \partial \hat{v} = u \), the latter identity yields that for almost every \( (x, t) \in (0, T) \times \Gamma \), \( \partial_t \hat{v}(x, t) + u_{\alpha} \partial_\alpha \hat{v}(x, t) - H(x, \partial \hat{v}(x, t)) + f(x, t) = 0 \). We have proven that \( \hat{v} \) is a solution of (4.1). Since \( v \) is the unique solution of (4.1), we conclude that \( v = \hat{v} \) and \( \partial v = u \).

We are now ready to give the proof of Theorem 4.2.

**Proof of Theorem 4.2.** Since \( \partial v = u \) by Lemma 4.4 and \( u \) satisfies (4.8) by Theorem 4.3, we obtain that \( v \in L^2(0, T; H^3(\Gamma)) \) and \( \partial_t v \in L^2(0, T; H^1(\Gamma)) \), so (4.3) holds.

Therefore, using an interpolation result combined with Sobolev embeddings, see [4] or Lemma A.2 in the Appendix, \( v \in C^{1+\sigma, \sigma/2}(\Gamma \times [0, T]) \) for some \( 0 < \sigma < 1 \).
Finally, we know that since \( f \in W^{1,2}(0, T, H^1_b(\Gamma)) \), \( f|_{\Gamma_t \times [0, T]} \in C^{\alpha, \eta}(\Gamma_t \times [0, T]) \) for all \( \eta \in (0, 1/2) \). Observe that \( v \) satisfies the (modified) heat equation with right hand side \( g = f - H(x, \partial v) \) and the same Kirchhoff conditions as in (4.1). Now, if \( f \in C^{\alpha, \eta}(\Gamma_t \times [0, T]) \) for some \( \eta \in (0, 1/2) \), then \( g = f - H(x, \partial v) \in C^{\tau, 2}(\Gamma_t \times [0, T]) \), where \( 1/2 > \tau = \min(\sigma, \eta) > 0 \). If \( v_T \in C^{2+\tau}(\Gamma) \), we can apply the main theorem in [25] (regularity for solutions of linear parabolic equations on networks) and obtain that \( v \in C^{2+\tau, 1+\tau/2}(\Gamma \times [0, T]) \), then that \( v \) is a classical solution of (4.1).

\[ \square \]

5 Existence, uniqueness and regularity for the MFG system (Proof of theorem 1.1)

**Proof of existence in Theorem 1.1.** Given \( m_0 \) and \( v_T \), let us construct the map \( \mathcal{T} : L^2(0, T; V) \to L^2(0, T; V) \) as follows.

Given \( v \in L^2(0, T; V) \), we first define \( m \) as the weak solution of (3.1) with initial data \( m_0 \) and \( b = H_p(x, \partial v) \). We know that \( m \in L^2(0, T; W) \cap C([0, T]; L^2(\Gamma)) \cap W^{1,2}(0, T; V') \).

We claim that if \( v_n \to v \) in \( L^2(0, T; V) \) then \( H_p(\cdot, \partial v_n) \) tends to \( H_p(\cdot, \partial v) \) in \( L^2(\Gamma \times (0, T)) \). To prove the claim, we argue by contradiction: assume that there exist a positive number \( \epsilon \) and a subsequence \( v_{(n)} \) such that \( \| H_p(\cdot, \partial v_{(n)}) \|_{L^2(\Gamma \times (0, T))} > \epsilon \). Then since \( \partial v_{(n)} \) tends to \( \partial v \) in \( L^2(\Gamma \times (0, T)) \), we can extract another subsequence \( v_{(n)} \) from \( v_{(n)} \) such that \( \partial v_{(n)} \) tends to \( \partial v \) almost everywhere in \( \Gamma \times (0, T) \). From the continuity of \( H_p \), we deduce that \( H_p(\cdot, \partial v_{(n)}) \) tends to \( H_p(\cdot, \partial v) \) almost everywhere in \( \Gamma \times (0, T) \).

Since there exists a positive constant \( C_0 \) such that \( \| H_p(\cdot, \partial v_{(n)}) \|_{L^2(\Gamma \times (0, T))} \leq C_0 \), we used Lemma 3.1, see We claim that if \( m_n \to m \) in \( L^2(0, T; W) \cap L^\infty(0, T; L^2(\Gamma)) \cap W^{1,2}(0, T; V') \). Hence, the map \( v \mapsto m \) is continuous from \( L^2(0, T; V) \) to \( L^2(0, T; W) \cap L^\infty(0, T; L^2(\Gamma)) \cap W^{1,2}(0, T; V') \). Moreover, the a priori estimate (3.3) holds uniformly with respect to \( v \).

Then, knowing \( m \), we construct \( \mathcal{T}(v) = \tilde{v} \) as the unique weak solution of (4.1) with

\[ f(x, t) = \mathcal{T}(m(\cdot, t))(x). \]

Note that \( m \mapsto f \) is continuous and locally bounded from \( L^2(\Gamma \times (0, T)) \) to \( L^2(\Gamma \times (0, T)) \). Then Lemma 4.1 ensures that the map \( m \mapsto \tilde{v} \) is continuous from \( L^2(\Gamma \times (0, T)) \) to \( L^2(0, T; H^2(\Gamma)) \cap L^\infty(0, T; V) \cap W^{1,2}(0, T; L^2(\Gamma)) \). From Aubin–Lions theorem, see Lemma A.1, \( m \mapsto \tilde{v} \) maps bounded sets of \( L^2(\Gamma \times (0, T)) \) to relatively compact sets of \( L^2(0, T; V) \).

Therefore, the map \( \mathcal{T} : v \mapsto \tilde{v} \) is continuous from \( L^2(0, T; V) \) to \( L^2(0, T; V) \) and has a relatively compact image. Finally, we apply Schauder fixed point theorem [10, Corollary 11.2] and conclude that the map \( \mathcal{T} \) admits a fixed point \( v \). We know that \( v \in L^2(0, T; H^2(\Gamma)) \cap L^\infty(0, T; V) \cap W^{1,2}(0, T; L^2(\Gamma)) \) and \( m \in L^2(0, T; W) \cap L^\infty(0, T; L^2(\Gamma)) \cap W^{1,2}(0, T; V'(\Gamma)) \).

Hence, there exists a weak solution \((v, m)\) to the mean field games system (1.17). \( \square \)
Proof of uniqueness in Theorem 1.1. We assume that there exist two solutions \((v_1, m_1)\) and \((v_2, m_2)\) of (1.17). We set \(\overline{v} = v_1 - v_2\) and \(\overline{m} = m_1 - m_2\) and write the system for \(\overline{v}, \overline{m}\)

\[
\begin{align*}
-\partial_t \overline{v} - \mu_x \partial^2_x \overline{v} + H (x, \partial_x v_1) - H (x, \partial_x v_2) - (\mathcal{V}[m_1] - \mathcal{V}[m_2]) &= 0, & x \in \Gamma_\alpha \setminus \mathcal{V}, \ t \in (0, T), \\
\partial_t \overline{m} - \mu_x \partial^2_x \overline{m} - \partial \left( m_1 \partial_p H (x, \partial_x m_1) - m_2 \partial_p H (x, \partial_x m_2) \right) &= 0, & x \in \Gamma_\alpha \setminus \mathcal{V}, \ t \in (0, T), \\
\overline{v}|_{\Gamma_\alpha} (v, t) &= \overline{v}|_{\Gamma_\beta} (v, t), & \overline{m}|_{\Gamma_\alpha} (v, t) = \overline{m}|_{\Gamma_\beta} (v, t), \quad \gamma_{\alpha \beta} (v, t), \quad \nu = \mathcal{V}, \quad v_1 \in \mathcal{V}, \ v_2 \in \mathcal{V}, \ t \in (0, T), \\
\sum_{\alpha \in \mathcal{A}} \mu_x \partial_x \overline{m} (v, t) &= 0, & v_1 \in \mathcal{V}, \ t \in (0, T), \\
\sum_{\alpha \in \mathcal{A}} m_1 |_{\Gamma_\alpha} (v_1) \partial_p H^\alpha (v_1, \partial_x v_1 |_{\Gamma_\alpha} (v_1, t)) - m_2 |_{\Gamma_\alpha} (v_1) \partial_p H^\alpha (v_1, \partial_x v_2 |_{\Gamma_\alpha} (v_1, t)) &= 0, \\
+ \sum_{\alpha \in \mathcal{A}} \mu_x \partial_x \overline{m} (v, t) &= 0, & v_1 \in \mathcal{V}, \ t \in (0, T), \\
\forall (x, t) = 0, & \overline{m} (x, 0) = 0.
\end{align*}
\]

Testing by \(\overline{m}\) the boundary value problem satisfied by \(\overline{u}\), testing by \(\overline{m}\) the boundary value problem satisfied by \(\overline{m}\), subtracting, we obtain

\[
\begin{align*}
\int_0^T \int_{\Gamma} (m_1 - m_2) (\mathcal{V}[m_1] - \mathcal{V}[m_2]) \, dx \, dt + \int_0^T \int_{\Gamma} \partial_t (\overline{m} \overline{v}) \, dx \, dt + \\
\sum_{\alpha \in \mathcal{A}} m_1 \left[ H (x, \partial_x v_1) - H (x, \partial_x v_1) - \partial_p H (x, \partial_x v_1) \partial \overline{v} \right] \, dx + \\
\sum_{\alpha \in \mathcal{A}} m_2 \left[ H (x, \partial_x v_1) - H (x, \partial_x v_2) + \partial_p H (x, \partial_x v_1) \partial \overline{v} \right] \, dx &= 0.
\end{align*}
\]

Since \(\mathcal{V}\) is monotone, the first term is nonnegative. Moreover,

\[
\int_0^T \int_{\Gamma} \partial_t (\overline{m} \overline{v}) \, dx \, dt = \int_{\Gamma} \overline{m} (x, T) \overline{v} (x, T) - \overline{m} (x, 0) \overline{v} (x, 0) \, dx = 0,
\]

since \(\overline{v} (x, T) = 0\) and \(\overline{m} (x, 0) = 0\). From the convexity of \(H\) and the fact that \(m_1, m_2\) are nonnegative, the last two sums are nonnegative. Therefore, all the terms are zero and thanks again to the monotonicity assumption on \(\mathcal{V}\), we obtain \(\mathcal{V}[m_1] = \mathcal{V}[m_2]\). From Theorem 4.1, we obtain that \(v_1 = v_2\). Finally, Theorem 3.1 implies that \(m_1 = m_2\).

\[\square\]

Proof of regularity in Theorem 1.1. We make the stronger assumptions written in Sect. 1.4.4 on the coupling operator \(\mathcal{V}\). We know that \(\mathcal{V}[m] \in W^{1,2} (0, T; H^1_b (\Gamma)) \cap PC (\Gamma \times [0, T])\). Assuming also that \(v_T \in V\) and \(\partial v_T \in F\), we can apply the regularity result in Theorem 4.2: \(v \in L^2 (0, T; H^3 (\Gamma)) \cap W^{1,2} (0, T; H^1 (\Gamma))\).

Moreover, since \(\mathcal{V}[m] \in W^{1,2} (0, T, H^1_b (\Gamma))\), we know that \((\mathcal{V}[m])|_{\Gamma_\alpha \times [0, T]} \in C^{\sigma, \sigma/2} (\Gamma_\alpha \times [0, T])\) for all \(0 < \sigma < 1/2\). If \(v_T \in C^{2+\eta} \cap D\) for some \(\eta \in (0, 1)\) (\(D\) is defined in (1.7)), then from Theorem 4.2, \(v \in C^{3+\tau, 1+\tau/2} (\Gamma \times [0, T])\) for some \(\tau \in (0, 1)\) and the boundary value problem for \(v\) is satisfied in a classical sense.

In turn, if for all \(\alpha \in \mathcal{A}\), \(\partial_p H^\alpha (x, p)\) is a Lipschitz function defined in \(\Gamma_\alpha \times \mathbb{R}\), and if \(m_0 \in W\), then we can use the latter regularity of \(v\) and arguments similar to those contained in the proof of Theorem 2.2 and prove that \(m \in C ([0, T]; W) \cap W^{1,2} (0, T; L^2 (\Gamma)) \cap L^2 (0, T; H^1_b (\Gamma))\).

\[\square\]

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A Some continuous and compact embeddings

Lemma A.1 (Aubin–Lions Lemma, see [19]) Let $X_0, X$ and $X_1$ be function spaces, ($X_0$ and $X_1$ are reflexive). Suppose that $X_0$ is compactly embedded in $X$ and that $X$ is continuously embedded in $X_1$. Consider some real numbers $1 < p, q < +\infty$. The space \{\(v : (0, T) \mapsto X_0 : v \in L^p(0, T; X_0), \frac{\partial}{\partial t}v \in L^q(0, T; X_1)\}\} is compactly embedded in $L^p(0, T; X)$.

Lemma A.2 (Amann, see [4]) Let $\phi : [a, b] \times [0, T] \mapsto \mathbb{R}$ such that $\phi \in L^2(0, T; H^2(a, b))$ and $\frac{\partial}{\partial t}\phi \in L^2(0, T; L^2(a, b))$. Then $\phi \in C^s(0, T; H^1(a, b))$ for some $s \in (0, 1/2)$.

This result is a consequence of the general result [4, Theorem 1.1] taking into account [4, Remark 7.4]. More precisely, we have

\[ E_1 := H^2(a, b)^{\text{compact}} \hookrightarrow E := H^1(a, b) \hookrightarrow E_0 := L^2(a, b). \]

Set $r_0 = r_1 = r = 2$, $\sigma_0 = 0$, $\sigma_1 = 2$ and $\sigma = 1$. For any $v \in (0, 1)$, we define $\sigma_v := (1 - v)\sigma_0 + \sigma_1$ and $r_v$ such that $\frac{1}{r_v} = \frac{1}{r_0} + \frac{1-v}{r_1}$, so $r_v = 2$ and $\sigma_v = 2v$. Observe that if $v \in (1/2, 1)$, then the following inequality is satisfied: $\sigma - 1/r < \sigma_v - 1/r_v < \sigma_1 - 1/r_1$. Hence, from [4, Remark 7.4],

\[ E_1 \hookrightarrow (E_0, E_1)_{\nu, 1} \hookrightarrow (E_0, E_1)_{\nu, r_v} = W^{\sigma_v, r_v}(a, b) \hookrightarrow E, \]

where $(E_0, E_1)_{\nu, 1}$, $(E_0, E_1)_{\nu, r_v}$ are interpolation spaces. This is precisely the assumption allowing to apply [4, Theorem 1.1], which gives the result of Lemma A.2.

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