Remarks on non-compact gradient Ricci solitons

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Abstract In this paper we show how techniques coming from stochastic analysis, such as stochastic completeness (in the form of the weak maximum principle at infinity), parabolicity and $L^p$-Liouville type results for the weighted Laplacian associated to the potential may be used to obtain triviality, rigidity results, and scalar curvature estimates for gradient Ricci solitons under $L^p$ conditions on the relevant quantities.

Keywords Ricci solitons · Triviality · Scalar curvature · Maximum principles · Liouville-type theorems

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0 Introduction

Let $(M, (\cdot, \cdot))$ be a Riemannian manifold. A Ricci soliton structure on $M$ is the choice of a smooth vector field $X$ (if any) satisfying the soliton equation

$$\text{Ric} + \frac{1}{2} L_X (\cdot, \cdot) = \lambda (\cdot, \cdot),$$

(1)

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for some constant $\lambda \in \mathbb{R}$. Here, Ric denotes the Ricci curvature of $M$ and $L_X$ stands for the Lie derivative in the direction $X$. The Ricci soliton $(M, \langle \cdot, \cdot \rangle, X)$ is said to be shrinking, steady or expansive according to whether the coefficient $\lambda$ appearing in Eq. (1) satisfies $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$.

In the special case where $X = \nabla f$ for some smooth function $f : M \to \mathbb{R}$, we say that $(M, \langle \cdot, \cdot \rangle, \nabla f)$ is a gradient Ricci soliton with potential $f$. In this situation, the soliton equation reads

$$\text{Ric} + \text{Hess} (f) = \lambda \langle \cdot, \cdot \rangle. \quad (2)$$

Clearly, Eqs. (1) and (2) can be considered as perturbations of the Einstein equation

$$\text{Ric} = \lambda \langle \cdot, \cdot \rangle,$$

and reduce to this latter in case $X$ is a Killing vector field. In particular, if $X = 0$, we call the underlying Einstein manifold a trivial Ricci soliton. It is easy to show that Einstein, gradient Ricci solitons are either trivial or Ricci flat; see, e.g., the proof of Theorem 1.3 in [18].

In this note we will focus our attention on geodesically complete, gradient Ricci Solitons. Here are some typical examples, [11].

Example The standard Euclidean space $(\mathbb{R}^m, \langle \cdot, \cdot \rangle, \nabla f)$ with

$$f (x) = \frac{1}{2} A |x|^2 + \langle x, B \rangle + C,$$

for arbitrary $A \in \mathbb{R}$, $B \in \mathbb{R}^m$ and $C \in \mathbb{R}$. Note that $f$ is the essentially unique solution of the equation $\text{Hess} (f) = A \langle \cdot, \cdot \rangle$ on $\mathbb{R}^m$. This follows by integrating on $[0, |x|]$ the equation

$$\frac{d^2}{ds^2} (f (sv)) = A,$$

with $v \in \mathbb{R}^m$ such that $|v| = 1$. In fact, a kind of converse also holds; [9,11,20]. In the Appendix we will provide a straight-forward proof.

Theorem 1 Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold. Suppose that there exists a smooth function $f : M \to \mathbb{R}$ satisfying $\text{Hess} (f) = \lambda \langle \cdot, \cdot \rangle$, for some constant $\lambda \neq 0$. Then $M$ is isometric to $\mathbb{R}^m$.

Example The Riemannian product

$$\left( \mathbb{R}^m \times N^k, \langle \cdot, \cdot \rangle + \langle \cdot, \cdot \rangle_{N^k}, \nabla f \right)$$

where $(N^k, \langle \cdot, \cdot \rangle)$ is any $k$-dimensional Einstein manifold with Ricci curvature $\lambda \neq 0$, and $f (t, x) : \mathbb{R}^m \times N^k \to \mathbb{R}$ is defined by

$$f (x, p) = \frac{\lambda}{2} |x|^2_{\mathbb{R}^m} + \langle x, B \rangle_{\mathbb{R}^m} + C,$$

with $C \in \mathbb{R}$ and $B \in \mathbb{R}^m$.

As generalizations of Einstein manifolds, Ricci solitons enjoy some rigidity properties, which can take the form of classification (metric rigidity), or alternatively, triviality of the soliton structure (soliton rigidity). For instances of the former, see e.g. the recent far-reaching paper [24] and references therein.

As for the latter, it has been known for some time that compact, expanding Ricci solitons are necessarily trivial, [3]. Our first main result, Theorem 2 below, extends this conclusion.

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