We present the first study of a dynamical quantum game. Each agent has a ‘memory’ of her performance over the previous m timesteps, and her strategy can evolve in time. The game exhibits distinct regimes of optimality. For small m the classical game performs better, while for intermediate m the relative performance depends on whether the source of qubits is ‘corrupt’. For large m, the quantum players dramatically outperform the classical players by ‘freezing’ the game into high-performing attractors in which evolution ceases.

The new field of quantum games is attracting significant interest [1,2]. We recently conjectured [2] that novel features should arise for quantum games with N ≥ 3 players. Benjamin and Hayden [6] subsequently created a Prisoner’s Dilemma-like game for N = 3 with a high-payoff ‘coherent quantum equilibrium’ (CQE). Johnson [7] showed that this quantum advantage can become a disadvantage when the game’s external qubit source is corrupted by a noisy ‘demon’. So far, all studies have been restricted to static games.

Here we introduce an iterated version of the game, where players (agents) may modify their strategies based on information from the past – i.e. ‘they learn’ from their mistakes [3]. This evolutionary game produces highly non-trivial dynamics in both quantum and classical regimes, and represents the first step toward understanding iterated games employing temporal quantum coherence. Agents are provided with the minimum resources necessary for adaptability - specifically, each agent possesses a measure of her past success through a parameter $s_m$ whose value reflects the payouts from recent rounds of the game. The fixed ‘memory’ parameter m effectively governs the number of rounds upon which $s$ depends (see later Eq. (1)), and turns out to be fundamental for deciding the relative superiority of the quantum and classical games. For small memory m, the classical game is superior (in the sense that the average payout per player is higher). For intermediate m, relative superiority is determined by the reliability of the external qubit source, while for large m the quantum game is radically superior due to evolutionary ‘freezing’ into a high-paying attractor.
players from ‘flip’/‘don’t flip’ to a set of quantum operations, and (c) introducing ‘gates’ to entangle the qubits - specifically the entangling gate is $J = \frac{1}{\sqrt{2}} (I \otimes 3 + iF \otimes 3)$ where $F = \sigma_z$. Without entanglement, the quantum game is trivially equivalent to the classical variant [1] - therefore we can make a correspondence between the quantum and classical games simply by removing the $J$ gates. The lower inset in Fig.1 shows the flow of information in the quantum game. In general the actions of the players in the quantum game will result in a final state which is a superposition. Measurement then collapses this state to one of the classical outcomes, and the payoff is then be determined from the table (see Fig. 1) as in the classical game. Among the conclusions of Ref. [1] was the discovery that the quantum players could escape the DSE that traps the classical players, and hence outperform the classical players by a considerable factor. In Ref. [2] it was shown that this quantum advantage depends on the reliability of the source of qubits: if the source is believed to be generating qubits in the $|0\rangle$ state, but is in fact generating qubits in state $|1\rangle$, then the classical game players will out-perform the quantum ones.

Following the previous literature on iterated classical multi-player games [3], we limit the classical moves to three options: ‘definitely flip’, ‘definitely do not flip’ and ‘flip with probability 1/2’. These options are denoted by setting $p = 0, 1/2, 1$ respectively. This simplification from the full (continuous) range of physically possible $p$ values clarifies the analysis while retaining the basic structure of the game. There are now $3^3 = 27$ possible profiles or ‘configurations’ $(p_1, p_2, p_3)$. These yield ten ‘classes’ each containing $C \geq 1$ configurations which are equivalent under interchange of player label [3]. The upper table in Fig. 2 shows the average payoffs for each configuration class for the classical game. Given that the input is $0$, the dominant strategy equilibrium corresponds to all players choosing $p = 0$, i.e. class (iv) in the table. Hence although the continuous-parameter $p$-space has been discretized to only three values, this description includes the fundamental dominant strategy equilibrium.

To maintain a correspondence between the quantum and classical games, we also permit our quantum players just three different moves: $I, \tilde{\sigma}_z$, and $\sqrt{\frac{1}{2}} (\tilde{\sigma}_x + \tilde{\sigma}_z)$. It was earlier shown that these moves can give rise to superior quantum performance for the static quantum game with reliable qubit source [3]. Moreover when the entangling $J$ gates are removed, making the game classical, these three moves correctly correspond to our allowed classical moves $p = 1, 0, 1/2$ (respectively). Therefore we label the three quantum moves with the same $p$ notation. A fully comprehensive study of the quantum game would also examine the case where the players are permitted a larger set of moves, e.g. one that is closed under composition. However the aim here is to make a first direct move-for-move comparison between the iterated quantum and classical games, and this necessitates restricting the larger set of quantum options. The resulting table (see Fig. 2) provides a simple quantum analog of the classical case.

### Classical

| Class | $p=0$ | $p=1/2$ | $p=1$ | Average |
|-------|-------|---------|-------|---------|
| i)    | -     | aaaa[2][1] | -     | (1/2)[1/2] |
| ii)   | aa[21/4][17/4] | aa[3/4][1] | -     | (9/4)[5/4] |
| iii)  | a[11/2][9/2] | a[3/2][1] | -     | (3/2)[17/6] |
| iv)   | aaaa[2][0] | -     | aaaa[2][1] | (2/0) |
| v)    | a[1][1] | -     | aaaa[0][2] | (0/2) |
| vi)   | a[9/4][9/4] | -     | a[1][1] | (17/3)[17/3] |
| vii)  | a[5][4]/2 | a[0][0] | a[4][5]/2 | (13/3)[13/3] |
| viii) | -     | aaaa[1][4]/3 | aaaa[17/4][21]/4 | (5/4)[9/4] |
| ix)   | -     | aaaa[1][2][3][2] | aaaa[17/4][21]/4 | (17/4)[9/4] |
| x)    | -     | aaaa[1][2][3][1] | aaaa[9/2][11]/2 | (17/4)[25/6] |

### Quantum

| Class | $p=0$ | $p=1/2$ | $p=1$ | Average |
|-------|-------|---------|-------|---------|
| i)    | -     | aaaa[15/4][18]/4 | -     | (15/4)[18/4] |
| ii)   | aaaa[9/4][19]/4 | aaaa[15/4][18]/4 | -     | (15/4)[18/4] |
| iii)  | aaaa[7/2][9]/2 | aaaa[3/2][1] | -     | (11/4)[19/6] |
| iv)   | aaaa[2][0] | -     | aaaa[2][1] | (2/0) |
| v)    | aaaa[1][1] | -     | aaaa[0][0] | (0/2) |
| vi)   | aaaa[9][0] | aaaa[1][1] | aaaa[9][0] | (17/3)[19/3] |
| vii)  | aaaa[9][0] | aaaa[1][1] | aaaa[9][0] | (17/3)[19/3] |
| viii) | aaaa[5][4]/2 | aaaa[5][4]/2 | aaaa[5][4]/2 | (17/3)[19/3] |
| ix)   | -     | aaaa[19/4][15]/4 | aaaa[19/4][15]/4 | (19/4)[15/4] |
| x)    | -     | aaaa[19/4][15/4] | aaaa[19/4][15/4] | (19/4)[15/4] |

### Example

$$ F = \frac{1}{\sqrt{2}} (I \otimes 3 + iF \otimes 3) $$

$$ J = \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z) $$

$\Delta t$ denotes the move interval, and $m$ is the number of moves since the last update.

[FIG. 2. Average payoffs for classical game (upper) and quantum game (lower). Agents are denoted by ‘a’. $p$ corresponds to the probability of not flipping the input qubit - for the quantum case, $p$ is an operator (see text). Average payoffs for input qubit $0$ are shown as $(.)$; those for input $1$ are shown as $[.]$; those for 50 : 50 mixture of input qubits are shown without parentheses. Final column gives the average payoff per player for a given input qubit.]

We assume that in both quantum and classical games, players are unable to communicate between themselves and hence cannot coordinate which player picks which strategy. In the quantum game, this is more critical since the CQE (i.e. the Nash equilibrium given by class (viii) in the lower table of Fig. 2) involves players using different $p$’s. We will find that the evolutionary aspect of our iterated game allows the quantum players to find cooperative solutions even in the absence of communication, provided that the memory $m$ is not too small.

The agents are allowed to evolve (‘mutate’) their strategies based on past success, according to the following prescription. After timestep $t - 1$, each player holds a ‘moving average’ $\$ \_m,t{-1}$ of her payoffs which is then updated at timestep $t$:

$$ \$ \_m,t = \frac{m - 1}{m} \$ \_m,t{-1} + \frac{1}{m} \Delta t $$

where $\$ \_m,t$ is the updated moving average and $\Delta t$ is the payoff to the player at turn $t$. The information about previous outcomes therefore has a ‘half life’ since the contribution of a given round’s payoff falls exponentially.
with successive rounds. When the player’s moving average falls below a certain threshold $d$, she mutates - she chooses one of the other two allowed strategies with a 50 : 50 chance. After mutation, the moving average $s_m$ continues to be updated, but the player enters a ‘trial period’ of $Y$ turns before considering mutating again. For simplicity, we set $Y = m$ for the remainder of this paper. We also consider fixed external qubit source orientations, although each of these restrictions can be relaxed. We choose a mutation threshold $d = 3.0$ in order to be above the 2 point maximum possible payoff per player in the classical static game (see payoff table in Fig. 1). We performed runs with $10^7$ timesteps - this is sufficient for the resulting statistics to be stable to within a few percent.

Figure 1 shows the results for the average payoff per turn for the three agents combined. The classical dominant strategy equilibrium (2, 2, 2) corresponds to a value 6, while the CQE (5, 9, 5) and its permutations correspond to a value 19. Figure 3 shows the corresponding average lifetime of the agents before mutation. Figure 1 shows that for memory $m < 8$, the quantum game players are worse off than the classical game players regardless of the input qubit source orientation (0 or 1). (The particular orientation of the fixed source doesn’t affect the results for the classical game.) The classical game players are also less prone to mutation, as seen from Figure 3. For $8 < m < 25$, the quantum game players do better than the classical players if the source is 0 (as they believe it is) but do worse if the source is 1 (i.e. the source is ‘corrupt’ [7]). The same is qualitatively true for the lifetimes (Fig. 3) but for a smaller range $8 < m < 19$. For $m > 25$, the quantum game players achieve better average payoffs than the classical game players irrespective of the source orientation, but those with the reliable (i.e. non-corrupt) source do significantly better. They eventually ‘freeze’ into a cooperative state, as explained later.

Figure 1 reveals an interesting effect within the classical game: for $m > 1$ the classical game players do better than the classical dominant strategy equilibrium value of $2 + 2 + 2 = 6$. If we had set the threshold $d < 2$, the players would quickly ‘freeze’ into the configuration $p = 0, 0, 0$, which guarantees 2 points per player per turn. By adopting a higher satisfaction limit, i.e. by being more ‘greedy’, the classical players introduce instability into the game – although this implies that in any given round at least one player is typically worse off, in the long term the instability actually benefits all the players.

The explanation for the interesting large $m$ behavior lies in the cycles through which the game moves on short and long timescales. In general, the behavior of both classical and quantum systems for larger $m$ is dominated by a small number of possible attractors which trap the system for a certain period, before it jumps into a brief period of turbulence and then ends up in another attractor.

**FIG. 4.** Schematic game dynamics showing the attractors which dominate both the classical and quantum games in the large $m$ limit. (a) The classical game, and (b) the quantum game with a reliable source ($s = 0$). Ellipses represent specific strategy profiles and are labeled by the $p$ values adopted by the three players. The circle labeled $T$ represents the set of all other profiles, which are visited only during brief turbulent periods in the game’s dynamics. Arrows indicate the probability that one profile follows another over a period of $m$ rounds. Data are collected over $10^7$ rounds with $m = 30$. In the classical game the profile ($p_1 = 0, p_2 = 0, p_3 = 0$), corresponding to the dominant strategy equilibrium, occurs in each of the three attractor cycles.

Figure 4 shows the structure of the attractors in the classical and quantum cases. In the classical game, the attractors correspond to two players fixing their strategy as ‘definitely flip’ ($p = 0$). While they maintain these choices, the third player can never achieve satisfactory payouts, hence the short timescale dynamics correspond to this third player jumping around the three possible $p$ values, constantly mutating. Eventually one of the other two players will encounter a run of losses and mutate her strategy. The game then enters a (poorly performing) turbulent period before settling back into either the previous attractor or one of the 2 permutation-equivalent cycles. In the quantum case, however, these three short cycles are replaced by six attractors, each a single config-

**FIG. 3.** Average time agents maintain a given strategy before mutating, as a function of the memory $m$. Inset: Results for a larger y-scale, illustrating the ‘freezing’ of the dynamics in the quantum game with a reliable qubit source ($s = 0$).
uration such as $p = 0, 1, 1/2$. Because all such attractors yield satisfactory expected payoff to all players, the system will remain in an attractor until one player has an exceptionally long run of losses. Figure 3 indicates that the probability of this occurrence falls exponentially with $m$ - i.e. the system 'freezes' into one configuration profile.

In summary, we have presented the first results for an evolutionary quantum game. The dynamics are non-trivial, even in the present case where the game's history is assumed to be classical in that a measurement is taken at each timestep. The next logical step is to allow for a superposition (or 'multiverse') of histories over $R > 1$ turns. Because the coherence in the present game need only be maintained for $R = 1$ turns, the present game could be implemented as an algorithm in an elementary quantum computer containing 3 or more qubits - such an experiment could be performed with existing NMR technologies. More generally, it is not inconceivable that such 'games' are already being played at some microscopic level in the physical world - indeed, it has recently been shown that nature plays classical dominant-strategy games using clones of a virus that infects bacteria \[^{[10]}\].

---

1. J. Eisert, M. Wilkens, M. Lewenstein, Phys. Rev. Lett. 83, 3077 (1999). See also J. Eisert and M. Wilkens, preprint quant-ph/0004076.
2. D.A. Meyer, Phys. Rev. Lett. 82, 1052 (1999); preprint quant-ph/0004092. See also L. Marinatto and T. Weber, Phys. Lett. A 272, 291 (2000) and the Comment by S.C. Benjamin quant-ph/0008127.
3. S.C. Benjamin and P.M. Hayden, quant-ph/0003036.
4. J. Du, H. Li, X. Xu, M. Shi, X. Zhou and R. Han, quant-ph/0010092.
5. N.F. Johnson and S.C. Benjamin (unpublished).
6. S.C. Benjamin and P.M. Hayden, quant-ph/0007035.
7. N.F. Johnson, Phys. Rev. A (Rapid Comm.) 63, 020302 (2001).
8. N.F. Johnson, P.M. Hui, R. Jonson and T.S. Lo, Phys. Rev. Lett. 82, 3360 (1999).
9. N.F. Johnson, D.J.T. Leonard, T.S. Lo and P.M. Hui, Physica A 283, 568 (2000).
10. P.E. Turner and L. Chao, Nature 398, 441 (1999); M.A. Nowak and K. Sigmund, Nature 398, 367 (1999).