ON CREMONA TRANSFORMATIONS OF $\mathbb{P}^3$ WHICH FACTORIZE IN A MINIMAL FORM

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ABSTRACT. We consider Cremona transformations of the dimension 3 complex projective space which factorize as a product of $\ell \leq 2$ elementary links of type II, without small contractions, connecting two Fano 3-folds. We show there are exactly twelve classes of such transformations. Moreover, we give a geometric description of the elements in each of these classes.

1. Introduction

Denote by $\mathbb{P}^n = \mathbb{P}^n_\mathbb{C}$ the dimension $n$ projective space over the field $\mathbb{C}$ of complex numbers. A Cremona transformation of $\mathbb{P}^n$ is a birational map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$; the set $\text{Cr}(\mathbb{P}^n)$ of Cremona transformations is the Cremona group of $\mathbb{P}^n$. Since the beginning, from the Luigi Cremona works in the 1860’s until now-days, the Cremona transformations in the plane have been object of interest and the knowledge one has about that is considered to be satisfactory (for a survey see [Hu], [SCE] or [Alb]; for some recent works see for example [Gi82], [Is85], [Gi94], [BaBe00], [dFe04], [Be07], [Bl107], [Bl207], [Do09], [DoIs09], [BlPaVu09], [CaLa], [CeDe]). For higher dimension there has also been a lot of contemporary research on the subject (see for example [ST68], [ST69], [ST70], [ESB89], [CK89], [CK91], [Pa97], [Pa00], [RS01], [PRV01], [PR05], [GSP06], [AGSP08], [Bl11], [Do11]), though the results obtained remain sporadic and, in general, there are no substantial advances with respect to the pioneering works in the knowledge either about the structure of arbitrary Cremona transformations themselves or about the structure of the group $\text{Cr}(\mathbb{P}^n)$, even for $n = 3$.

On the other hand, in [Co95] A. Corti proved that the so-called Sarkisov program, consisting of an algorithm to decompose a birational map between two Mori fiber spaces as product of elementary links, works in dimension 3 in the context of terminal singularities. This deep result about the structure of birational maps straightforward gives neither information about the structure of Cremona transformations nor about the structure of $\text{Cr}(\mathbb{P}^3)$. In fact, consider the directed graph whose vertexes end edges are, respectively, the (isomorphism classes of) Mori fiber spaces and the elementary links connecting two Mori fiber spaces. For a Cremona transformation the Sarkisov program establish a strategy to...
construct a non trivial path in that graph which starts and ends at $\mathbb{P}^3$; moreover, that path is in general not unique.

In [Ka87] S. Katz rediscover the interesting classical result that general cubo-cubic Cremona transformations (i.e. defined by the maximal minors of a $4 \times 3$-matrix of general linear forms) are the unique birational maps $\varphi : \mathbb{P}^3 \dasharrow \mathbb{P}^3$ not defined exactly along a smooth curve; in particular he shows such a $\varphi$ may be realized by blowing up that curve and then contracting an irreducible divisor onto a smooth curve which is, in fact, projectively isomorphic to the first one; in other words, $\varphi$ is a special kind of elementary link of type II in the sense of the Sarkisov program (see [Co95], [Ma Chap. 14] or [Do11 Chap. 5]). More generally, a **Special Cremona transformation** is a birational map $\phi : \mathbb{P}^n \dasharrow \mathbb{P}^n$ whose base locus scheme is a smooth and connected subvariety $V \subset \mathbb{P}^n$; in [CrKa89] and [ESB89] Special Cremona transformations have been classified for $\dim V \leq 2$ and $\text{codim} V = 2$; all these transformations correspond to special elementary links of type II.

The study of special Cremona transformations was initiated, at least systematically, by Semple and Tyrrell in [ST68], [ST69] and [ST70]. This type of Cremona transformation is extremely rare and it is exactly by this reason, following the philosophy of [ST70], we think they become interesting.

Returning to the dimension three case, we reconsider the arguments in [Ka87] in the context of the Sarkisov program and notice that the main result therein may be reinterpreted by saying that general cubo-cubic Cremona transformations are, in fact, the unique links connecting $\mathbb{P}^3$ to itself if we assume those links arise into the smooth varieties category and do not contain any small contraction (that is, logarithmic flips in general); this condition, i.e. smoothness and absence of small contractions, is a “specialness” condition on links of type II. One also may read this result as saying that general cubo-cubic Cremona transformations are the shortest non trivial way to go from $\mathbb{P}^3$ to $\mathbb{P}^3$ in the subgraph of the one mentioned above, whose vertexes are (isomorphism classes of) smooth varieties with Picard number 1; more generally, the links without small contractions are such shortest ways to connect two (maybe coincident) such vertexes.

In order to search other interesting Cremona transformations, and taking into account the considerations above, it seems us to be natural to ask for birational maps of $\mathbb{P}^3$ factorizing as a product of elementary links of type II which are “special” (i.e. in the smooth varieties category and without small contractions).

In this paper we consider that problem in the following simplest case: the number of spacial links is $\leq 2$. We think this special case merits to be study for several reasons.

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1Semple and Tyrrell start [ST70] by writing: **Any Cremona transformation defined by a homaloidal system of primals with a single irreducible non-singular base variety is a rare enough phenomenon to merit special study.**
Indeed, first of all, the 3-folds associated to the vertexes above (the Fano 3-folds) have historically revealed very interesting properties from which one can expect the same is true for the Cremona transformations they constitute; on the other hand, this kind of transformation is extremely rare since, as we will see, it only arises for very particular degrees; finally, it shows another difference between the situation we find in \( \mathbb{P}^3 \) and that of \( \mathbb{P}^2 \), where such a transformation may not exist as follows from a classical result of Max Noether ([Alb, Lemma 2.6.1]).

We prove that, besides cubo-cubic transformations, which are already special links of type II, there are exactly eleven types of Cremona transformation of \( \mathbb{P}^3 \) which factorize as a product of two special links (Theorem 14 and Corollary 15); as we expected, these transformations are very rare, they are defined by polynomials of degrees 2, 3, 4 or 6. And moreover, for each type which does not factorize by way of a cubo-cubic Cremona transformation, we give a geometric description of transformations therein (Theorems 16 and 18).

In order to get our results we need to describe all type II links, under the above specialness assumption, connecting \( \mathbb{P}^3 \) to a projective 3-folds of Picard number 1, that is, to a rational Fano 3-fold (Theorem 7). We will use the classification of Fano 3-folds of Picard number 1 obtained by V. Iskovskih in [Is77] and [Is78] and these papers will be our main references for Fano 3-folds properties; for the rationality of Fano 3-folds we also consult [Be77], [IsSh79], [Is80], [Is83] and [Vo88].

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2. Special Links of type II from \( \mathbb{P}^3 \) to a Fano 3-fold

In this section we classify links of type II connecting \( \mathbb{P}^3 \) to a Fano 3-fold which are special in the sense that they arise in the smooth varieties category and do not contain small contractions.

Let \( Z \) be a dimension 3 smooth projective variety; as always \( K_Z \) denote the canonical class of \( Z \). Let \( p : Z \to \mathbb{P}^3 \) be a \( K_Z \)-negative extremal contraction whose center is an irreducible curve \( \Gamma \subset \mathbb{P}^3 \). By the Mori’s classification of (divisorial) extremal contractions ([Mo82 Thm. 3.3 and Cor. 3.4]) we deduce \( \Gamma \) is a smooth curve and \( p \) is the blow-up of it.

The closed cone of curves \( \overline{\text{NE}}(Z) \) has dimension 2 and one of its edges is the negative ray corresponding to \( p \). We assume that the other edge of that cone is also a \( K_Z \)-negative extremal ray; denote by \( q : Z \to X \) the extremal contraction associated to it, which may not be a flip since \( Z \) is smooth. We fix an embedding \( X \subset \mathbb{P}^N \) which is supposed to be non-degenerate (the case \( X = \mathbb{P}^3 \) is not excluded).
When \( \dim X = 3 \) it is normal, \( \mathbb{Q} \)-factorial, and has at most terminal singularities. A general codimension 2 linear space in \( \mathbb{P}^N \) intersects \( X \) along a smooth irreducible curve, say of genus \( g \); we call it the sectional genus by curves of \( X \). When \( \dim X = 2 \), then \( X \) is a smooth surface and a general fiber of \( q : Z \to X \) is a smooth rational curve. In the case where \( \dim X = 1 \) this curve is smooth and a general fiber of \( q \) is a del Pezzo surface (\cite{Mo82} Thm. 3.5).

On the other hand, the rational map \( \chi := q \circ p^{-1} : \mathbb{P}^3 \to X \) is defined by a linear system \( \Lambda_\chi \) of degree \( n \geq 1 \) surfaces containing \( \Gamma \), a subsystem of \( |\chi^*\mathcal{O}_{\mathbb{P}^N}(1)| \); we denote by \( m \geq 1 \) the multiplicity of a general member of that linear system at a general point of \( \Gamma \).

The following result was inspired by \cite{Ka87} (see also \cite{Pa10} Lemma 3).

**Proposition 1.** Suppose \( \dim X > 1 \) and let \( d := \deg \Gamma, \ d_0 := (\dim X - 2) \deg X \) and \( g_0 = (\dim X - 2)g \). Then

\[
\begin{align*}
(n^2 - m^2d)(4m - n) &= 2m(d_0 + 1 - g_0) - d_0 & (1) \\
n^2 > m^2d & & (2) \\
m^2|(n^3 - d_0), & & m|(n^2(2n) + 1 - g_0). & (3)
\end{align*}
\]

**Proof.** Take a general divisor \( H \in |p^*\mathcal{O}_{\mathbb{P}^3}(1)| \) and let us denote \( E := p^{-1}(\Gamma) \). By construction, \( q \) is defined by the complete linear system \( |nH - mE| \). A general element in \( |nH - mE| \) corresponds by \( p \) to a degree \( n \) surface in \( \mathbb{P}^3 \) passing through \( \Gamma \) with multiplicity \( m \) from which it follows the inequality \( (2) \).

The projection formula implies

\[
H^3 = 1, \quad H^2 \cdot E = 0, \quad H \cdot E^2 = -d;
\]

Therefore, we have

\[
(nH - mE)^3 = n^3 - 3nm^2d - m^3E^3 = d_0,
\]

since it corresponds to intersect \( p(Z) = X \subset \mathbb{P}^N \) with a general codimension 3 linear space. Then \( m^2 \) divides \( n^3 - d_0 \) as stated in \( (3) \).

On the other hand, since \( q : Z \to q(Z) = X \subset \mathbb{P}^N \) is a morphism, \( \dim X > 1 \) and \( Z \) is smooth the Bertini’s Theorem implies that the strict transform of a general codimension 2 linear space in \( \mathbb{P}^N \), by \( q \), is a smooth complete intersection of two smooth members of \( |nH - mE| \); of course, when \( \dim X = 2 \) that complete intersection is the disjoint union
of deg $X$ smooth rational curves. The adjunction formula yields
\[
2g_0 - 2 = (nH - mE)^2 \cdot [2(nH - mE) + p^*(K) + E] \\
= (n^2H^2 - 2nmH \cdot E + m^2E^2) \cdot [(2n - 4)H - (2m - 1)E] \\
= 2 \left[ n^2(n - 2) - nm(n - 2)d \right] - \\
m^2(2m - 1)E^3; \\
\]
in particular, we obtain the second divisibility condition in (3).

Finally, by eliminating $(2m - 1)m^3E^3$ between (5) and (4), we obtain the relation (1).

A dimension 3 projective variety $X$ is said to be a Fano 3-fold if it is smooth and its anti-canonical class $-K_X$ is ample. In [Is77] and [Is78] V. Iskovskih gives an essentially complete classification of Fano 3-folds with Picard number 1 (see also [Is83]); following Iskovskih the index of such a Fano 3-fold is the positive integer $r$ such that $-K_X \cong H^r$, where $H_X$ generates the Picard group of $X$. On the other hand, we are interested in those Fano 3-folds which are rational and will only pay attention to these varieties, though only once we will need to consider a Fano 3-fold whose rationality was even not established according to [IsSh79] and (specially) [Is83].

Let us recall what an elementary link of type II is; we give the more general definition though we will use it in a special setup.

Let $X_1, X_2$ be normal projective dimension 3 varieties which are $\mathbb{Q}$-factorial and have at most terminal singularities. Suppose $X_i$ admits a Mori fiber structure, that is a fibration $X_i \to Y_i$, $\dim Y_i < \dim X_i$, which is an extremal $K_{X_i}$-negative contraction, for $i = 1, 2$. A link of type II connecting $X_1$ to $X_2$ is a birational map $\chi : X_1 \dasharrow X_2$ which admits a decomposition
\[
X_1 \xrightarrow{\tau_1} Z_1 \dasharrow Z_2 \xrightarrow{\tau_2} X_2
\]
where $\tau_i$ is a divisorial extremal contraction with respect to $K_{Z_i}$, $Z_i$ is also a normal projective (dimension 3) variety, $\mathbb{Q}$-factorial and having at most terminal singularities, $i = 1, 2$, and $Z_1 \dasharrow Z_2$ a sequence of logarithmic flips (small contractions); in this work we will always deal with the case $Z_1 = Z_2$, that is, there are no small contractions, and where $X_1 = \mathbb{P}^3$ and $X = X_2$ is smooth. Since $X$ has Picard number 1 the Mori fiber structure is nothing but $X \to \text{Spec}(\mathbb{C})$ and we know $X$ is a Fano 3-fold, that is, its anti-canonical class $-K_X$ is ample. We say $\chi$ is nontrivial if it is not defined somewhere.

**Definition 1.** A link of type II is said to be special if it arises into the smooth varieties category and does not contain small contractions.

**Remark 2.** If $\chi = q \circ p^{-1} : \mathbb{P}^3 \dasharrow X$ is a special link of type II, the center of $p$ must be a curve; indeed, otherwise $p$ is the blow-up of a point ([Mo82 Thm. 3.3]) and then $q$ is a Mori Fiber Space.
In the following series of examples we give a list of special links of type \( \text{II} \) connecting \( \mathbb{P}^3 \) to a Fano rational 3-fold. For a curve \( \Gamma \subset \mathbb{P}^3 \) we denote by \( \mathcal{J}_\Gamma \subset \mathcal{O}_{\mathbb{P}^3} \) its ideal sheaf.

**Examples 3.**

\((L.1)\). Let \( \Gamma \subset \mathbb{P}^3 \) be a smooth quintic curve of genus 2. Then \( h^0(\mathcal{J}_\Gamma(3)) = 6 \). If \( \phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5 \) is the rational map defined by the choice of a basis of \( H^0(\mathcal{J}_\Gamma(3)) \), it is birational onto its image; this image is exactly the famous quadratic complex of lines, that is, a complete intersection of smooth hyperquadrics (see for example [GH, Chap. 6]).

\((L.2)\). Let \( \Gamma \subset \mathbb{P}^3 \) be a smooth rational curve of degree 4. Using a parameterization \( \eta : \mathbb{P}^1 \to \Gamma \) by polynomials of degree 4 we obtain \( h^0(\mathcal{J}_\Gamma(3)) = 7 \) and \( h^0(\mathcal{J}_\Gamma(2)) = 1 \); take \( S \) and \( Q \) irreducible cubic and quadric containing \( \Gamma \), respectively. Hence \( S \cap Q = \Gamma \cup C \) where \( \text{deg } C = 2 \). If \( C \) were in a plane, then \( \Gamma \) should be arithmetically Cohen-Macaulay and would contradict formulae in [PS74, Prop. 3.1], since \( \Gamma \) has arithmetic genus 0. Therefore \( C \) is the union of two skew lines.

We obtain a rational map \( \phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^6 \) defined by cubic homogeneous polynomials. By restriction to a general plane we easily conclude that the blow-up \( p : Z \to \mathbb{P}^3 \) of \( \mathbb{P}^3 \) along \( \Gamma \) resolves the indeterminacies of \( \phi \); denote by \( q : Z \to \mathbb{P}^6 \) the morphism obtained by composing \( \phi \) with that blow-up. If \( H \subset X \) is the pullback of a general plane in \( \mathbb{P}^3 \), then \( q^*\mathcal{O}_{\mathbb{P}^6}(1) = \mathcal{O}_Z(3H - E) \) where \( E \) is the exceptional divisor produced by blowing up \( \Gamma \). Since \( p_*(E^3) = 2 - 2g(\Gamma) + K_{\mathbb{P}^3} \cdot \Gamma = -14 \), from the equation \((4)\) we obtain \( (3H - E)^3 = 5 \). Therefore \( q(Z) \subset \mathbb{P}^6 \) is a degree 5 threefold.

Finally, fix a point \( x \in \Gamma \) and denote by \( W_x \) the strict transform in \( Z \) of a general plane passing through \( x \); by construction the restriction of \( q \) to \( W_x \) is the anticanonical embedding of a del Pezzo surface of degree 5. We conclude that \( X := q(Z) \) is smooth.

\((L.3)\). Let \( \Gamma \subset \mathbb{P}^3 \) be a smooth conic. Then \( h^0(\mathcal{J}_\Gamma(2)) = 5 \), and we obtain a rational map \( \phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4 \) defined by quadratic homogeneous polynomials. This is a link of type \( \text{II} \) obtained by blowing up the conic \( C \) and contracting the (strict transform of) chords of \( \Gamma \).

\((L.4)\). Let \( \Gamma \subset \mathbb{P}^3 \) be a smooth elliptic curve of degree 5. Then \( h^0(\mathcal{O}_\Gamma(3)) = 15 \). It follows that \( r := h^0(\mathcal{J}_\Gamma(3)) \geq 5 \); denote by \( \phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r-1} \) the rational map defined by the cubic forms vanishing on \( \Gamma \). If \( p : Z \to \mathbb{P}^3 \) is the blow-up of \( \mathbb{P}^3 \) along \( \Gamma \), by restricting \( \phi \) to planes, we observe that \( q = \phi \circ p \) is a morphism whose image, say \( X \), is a 3-fold. Take a general divisor \( H \in |p^*\mathcal{O}_{\mathbb{P}^3}(1)| \) and let us denote \( E := p^{-1}(\Gamma) \). By construction, \( q \) is defined by the complete linear system \( |3H - E| \).

This time \( H \cdot E^2 = -5, E^3 = -20 \); hence \( (3H - E)^3 = 2 \). Since \( q(Z) \) is nondegenerate by construction, we deduce \( q \) is birational onto its image which is, in fact, a dimension 3 variety of degree 2, that is, \( r - 1 = 4 \) and \( q(Z) \subset \mathbb{P}^4 \) is a hyperquadric \( Q \).
Projecting from a point of $\Gamma$ to $\mathbb{P}^2$, we see that through such a point, there pass two 3-secant lines of $\Gamma$; then the union of the strict transforms, in $Z$, of these 3-secant lines defines a (divisor) surface $F$. Moreover, since

$$K_Z = -4H + E = -3H_Z + F,$$

where $H_Z \in |3H - E|$, we deduce, on one side, $q$ is an extremal contraction which contracts $F$, and on the other side, $F \in |5H - 2E|$, that is, $p(F)$ is a degree 5 surface passing through $\Gamma$ with multiplicity 2.

Finally, the computations $F^2 \cdot H_Z = -5$ and $F^3 = -15$ imply $q(F)$ is an elliptic quintic curve on $Q$. Moreover, this quintic is not contained in a dimension 3 linear space: indeed, otherwise there would be an effective divisor in the class $H_Z - F = -2H + E$, which is impossible since, for example, $H^2 \cdot (-2H + E) = -7$.

(L.5). Let $\Gamma \subset \mathbb{P}^3$ be a smooth genus 3 curve of degree 6 which is arithmetically Cohen-Macaulay. That is, the sheaf ideal $J_\Gamma$ is generated by the maximal minors of a $4 \times 3$ matrix of linear forms. These four homogeneous polynomials define a birational map $\chi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ whose inverse is defined also by cubic polynomials. This map is a link of type II obtained by blowing up the curve $\Gamma$ and then by contracting the (strict transform of) the trisecant lines to $\Gamma$ (see [Ka87]).

Remark 4. In case (L.2) above the unique quadric containing $\Gamma$ is smooth: in fact, our argument above shows that quadric contains two skew lines.

Remark 5. Note that in the case where $d_0 = 2g_0 - 2$, the equation (1) becomes

$$(n^2 - m^2d)(4m - n) = (m - 1)d_0. \quad (6)$$

In this case $(m, n, d) = (1, n, n^2)$ always gives a solution for it. On the other hand, there is no birational map $\chi : \mathbb{P}^3 \dashrightarrow X$ satisfying such a solution: indeed, this solution corresponds to a pencil.

A divisorial extremal $K_Z$-negative contraction $p : Z \rightarrow Y$ is (also) called an extraction; denote by $E$ the exceptional divisor associated to $p$.

Consider now a birational map $\chi : \mathbb{P}^3 \dashrightarrow X \subset \mathbb{P}^N$ and let $\mathcal{H}$ be the linear system of strict transforms, by $\chi$, of hyperplane sections of $X$, and call that the homaloidal transform of the linear system of these; $\mathcal{H}$ is a linear system of surfaces, say of degree $n$. If $p : Z \rightarrow \mathbb{P}^3$ is an extraction, the homaloidal transform $\mathcal{H}_Z := p^{-1}_* \mathcal{H}$ of $\mathcal{H}$ satisfies

$$\mathcal{H}_Z = p^* \mathcal{H} - m_E E$$

for a integer $m_E \geq 0$. The ramification formula for $p$ is

$$K_Z = p^* K_{\mathbb{P}^3} + a_E E,$$
where \( a_E > 0 \) is the discrepancy of \( E \) which is known to be a rational number. If \( \chi \) is not a morphism, the Noether-Fano-Iskovskih Criterion ([Ma Prop. 13-1-3]) implies \( m_E/a_E > n/4 \).

In the following Lemma and Theorem we need to solve the arithmetic equation (1) under the conditions (2) and (3) where \( d_0 \) and \( g_0 \) take some fixed values. For this purpose we use Maxima Algebra System, a free computer algebra system (see details in [Pa11]); nevertheless we try there to bound the values of solutions in such a way that the reader may, if preferred, do all computations by hand instead (maybe a long and tedious work).

**Lemma 6.** Let \( X \) be a rational Fano 3-fold of Picard number 1 and index \( r = 1 \). Then there are no non trivial special links of type II connecting \( \mathbb{P}^3 \) to \( X \).

**Proof.** Suppose there exists a non trivial special link of type II, say \( \chi = q \circ p^{-1} : \mathbb{P}^3 \dashrightarrow X \). We will show this gives a contradiction.

First of all we observe that \( p : Z \to \mathbb{P}^3 \) is the blow-up of an irreducible smooth curve, say \( \Gamma \), of degree \( d \) (Remark [2]); we keep the integers \( m \) and \( n \) as in Proposition [1]. Therefore the discrepancy associated to \( p \) is \( a_E = 1 \). Moreover, since the birational map \( \chi \) is not a morphism then \( n < 4m \). Since \( \chi \) is birational we get \( m < n \) (inequality (2)); hence the equation (6) implies \( m > 1 \).

Second, since \( X \) is rational and following the Iskovskih Classification of Fano 3-folds (see [Is83 Thm. 1]) we may suppose either \( g_0 = 6 \) and \( X = X_{10} \subseteq \mathbb{P}^9 \) has degree 10 or \( X = X_{2g_0-2} \subseteq \mathbb{P}^{g_0+1} \) has degree \( d_0 = 2g_0 - 2 \in \{10, 12, 16, 18, 22\} \); note that the cases of degree 10 are known to be non-rational only for general varieties.

In the sequel we then assume that the center of \( p \) is a smooth and irreducible curve \( \Gamma \) of degree \( d \geq 1 \).

Take two general degree \( n \) surfaces \( S, S' \) in the linear system \( \Lambda_\chi \) defining \( \chi = q \circ p^{-1} \). Then, thinking of \( S \cap S' \) as a 1-cycle we have \( S \cap S' = m^2\Gamma + T \) where \( T \) is a curve of degree \( n^2 - m^2d \) which is the strict transform by \( \chi \) of a general codimension 2 linear section of \( X \); in particular \( T \) has (geometric) genus \( g_0 \).

a) Case \( d_0 = 10 \) and \( g_0 = 6 \). From (3) we obtain \( m^2|(n^3 - 10) \) and \( m|(n^3 - 2n^2 - 5) \). Hence \( m \) divides

\[
135 = (2n^2 + 4n - 11)(n^3 - 10) - (2n^2 + 8n + 5)(n^3 - 2n^2 - 5).
\]

Moreover, \( m < n < 4m \) by (2). The solutions \((m, n, d)\) for the equation (6) are

\((3, 7, 5), (3, 10, 10)\).
The first case is excluded for otherwise the curve $T$ above should be a quartic curve of genus 6. In the second one, suppose that solution gives a birational morphism $q : Z \to X_{10} \subset \mathbb{P}^7$. From Equation (1) we deduce
\[ 10 = (10H - 3E)^3 \]
\[ = -1700 - 27E^3 \]
which implies $E^3$ is not an integer number: contradiction.

c) Case $d_0 = 12, g_0 = 7$. Then $m^2|(n^3 - 12)$ and $m|(n^3 - 2n^2 - 6)$, and $m$ divides
\[ 156 = (2n^2 + 4n - 10)(n^3 - 12) - (2n^2 + 8n + 6)(n^3 - 2n^2 - 6). \]
There are no solutions for the equation (6).

d) Case $d_0 = 16, g_0 = 9$. Then $m$ divides
\[ 96 = (n^2 + 2n - 4)(n^3 - 16) - (n^2 + 4n + 4)(n^3 - 2n^2 - 8). \]
The corresponding solutions for the equation (6) are $(2, 4, 3), (2, 6, 7)$.

Now we exclude the case $(2, 6, 7)$: suppose that solution gives a birational morphism $q : Z \to X_{20} \subset \mathbb{P}^{10}$, denote by $F$ the exceptional divisor associated to $q$. Comparing the ramification formulae for $p$ and $q$, respectively, we obtain
\[ -4H + E = -6H + 2E + F. \]
We deduce $p(F) \supset \Gamma$ is a quadric surface. By restricting to a general plan we get all sextic surfaces singular along $\Gamma$ must contain that quadric: contradiction.

e) Case $d_0 = 18, g_0 = 10$. Then $m$ divides
\[ 414 = (4n^2 + 8n - 14)(n^3 - 18) - (4n^2 + 16n + 18)(n^3 - 2n^2 - 9). \]
There are no solutions for the equation (6) in this case.

f) Finally we deal with the case $d_0 = 22, g_0 = 12$. Then $m$ divides
\[ 464 = (4n^2 + 8n - 10)(n^3 - 22) - (4n^2 + 16n + 22)(n^3 - 2n^2 - 11). \]
We obtain the solution $(7, 16, 5)$. This solution may also be excluded: indeed, suppose that solution gives a birational morphism $q : Z \to X_{22} \subset \mathbb{P}^{13}$. Since $H \cdot E^2 = -5$ we deduce
\[
\deg X_{22} = 22 = (16H - 7E)^3 \\
= -7664 - 7^3E^3
\]
which implies $E^3$ is not an integer number: contradiction.
Theorem 7. Let $X$ be a smooth Fano 3-fold with Picard number 1 and index $r$. Let $\chi = q \circ p^{-1} : \mathbb{P}^3 \to X$ be a non-trivial special link of type II. Then, $r \geq 2$ and we have exactly one of the following statements:

a) $r = 2$, $X$ is the complete intersection of two hyperquadrics in $\mathbb{P}^5$ and $\chi$ is as in (L.1).

b) $r = 2$, $X$ is a quintic in $\mathbb{P}^6$ and $\chi$ is as in (L.2).

c) $r = 3$, $X$ is a hyperquadric in $\mathbb{P}^4$ and $\chi$ is as in (L.3) or (L.4).

d) $r = 4$, $X = \mathbb{P}^3$ and $\chi$ is as in (L.5).

Proof. From Lemma 6 we get $r \geq 2$ and from the Iskovskih Classification of Fano 3-folds we get that the unique cases we need to consider are those as in the statement of Theorem 7. We keep all notations, $m, n, d$, etc., as before.

a) From (1) we know $m^2 | (n^3 - 4)$ and $m | (n^3 - 2n^2)$. Therefore $m$ divides $8 = (n^3 - 4)(n^2 - 2) - (n^3 - 2n^2)(n^2 + 2n + 2)$.

On the other hand, since $m < n < 4m$ we deduce $m \leq 2$. A straightforward computation yields $(m, n, d) = (1, 3, 5)$ as the unique solution of (1).

Take general cubic surfaces $S, S'$ in the linear system $\Lambda_\chi$. Then $S \cap S' = \Gamma \cup T$ where $T$ is a quartic curve whose geometric genus is $g_0 = 1$. Since $T$ is not contained into a plane and all non-degenerate singular quartic curve is rational it follows $T$ is smooth; in particular $T$ is arithmetically Cohen-Macaulay. By liaison ([PS74, Prop. 3.1(vi)]) we deduce $\Gamma$ is a an arithmetically Cohen-Macaulay curve of arithmetic genus 2.

b) An easy computation using the adjunction formula yields $g_0 = 1$ in this case. From (1) we obtain $m^2 | (n^3 - 5)$ and $m | (n^3 - 2n^2)$. Hence $m$ divides $15 = (2n^2 - 3)(n^3 - 5) - (2n^2 + 4n + 5)(n^3 - 2n^2)$; as we know $m < n < 4m$, and then $m > 1$ is easily seen to be not possible. Solving the equation (1) we get the unique solution $(1, 3, 4)$.

Now we know $\Gamma$ is a non-degenerate smooth quartic, it suffices to show that curve is not elliptic. In fact, such an elliptic curve is a complete intersection of quadrics which implies $h^0(J_\Gamma(3)) = 8$, and then $X$ is the projection of a dimension three variety $X' \subset \mathbb{P}^7$, from a point $p \notin X'$. The birationality of $\chi$ implies that projection is an isomorphism, which is not possible because $X$ is projectively normal (see [Is77, Prop. 4.4(iii)]).
c) Then $d_0 = 2$ and $g_0 = 0$. As before we have $m$ divides
\[
5 = (6n^2 - 4n - 7)(n^3 - 2) - (6n^2 + 8n + 9)(n^3 - 2n^2 + 1);
\]
we obtain the solutions $(1, 2, 2)$ and $(1, 3, 5)$ and, as before, we see these are the required solutions.

d) Now $d_0 = 1$ and $g_0 = 0$. In this case $m$ divides
\[
2(n - 1) = (n - 2)(n^3 - 1) + n(n^3 - 2n^2 + 1).
\]
Since $n < 4m$ we need to solve (equation (1))
\[
(n^2 - m^2d)(4m - n) = 4m - 1,
\]
where $2(n - 1) = mk$ for $k = 1, \ldots, 7$. We get $(1, 3, 6)$ is the unique solution. The result follows form [Ka87] in this case. □

3. Associated Cremona Transformations and Main Theorem

In this section we classify all Cremona transformations which are either a special link of type II or may be factorized as $\chi^{-1}_i \chi_1$, where $\chi_i : \mathbb{P}^3 \dasharrow X$ is a special link of type II onto a smooth 3-fold $X \not\cong \mathbb{P}^3$ ($i = 1, 2$). Since $X$ must be a Fano 3-fold, by Theorem $\square$ we already know $\chi_i$ is necessarily as in examples (L.1), (L.2), (L.3) or (L.4).

In order to establish the kind of transformation we deal with we need first to understand the geometry of links involved in Theorem $\square$ This is the subject of the following paragraph.

3.1. Geometric description of links.

3.1.1. Cases (L.1), (L.3) and (L.5). The links of type II as in case (L.1) are described in [AGSP08, §4]. Analogously, the links of type II as in cases (L.3) and (L.5) are well known (the former case is very classical and easy to describe and for the last one see [Ka87], [SR, Chap. VIII, §4] or [Hu, Chap. HIV, §11]). For the convenience of the reader we include here a slight description of the situation for these three cases:

**Case (L.1):** The Fano 3-fold $X$ is a complete intersection of two hyperquadrics in $\mathbb{P}^5$. There is a dimension 2 family of lines on $X$ where two such lines may intersect themselves or not. If $L \subset X$ is a line on $X$, we have:

(a) there are two possibilities for the normal bundle $\mathcal{N}_L X$ of $L$ in $X$: it is isomorphic to either $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

(b) a general projection $\pi_L : X \dasharrow \mathbb{P}^3$ from $X$ with center $L$ is a birational map which defines a special link of type II connecting $X$ to $\mathbb{P}^3$; this link is obtained by blowing up the line and then contracting the strict transform of lines on $X$ passing through a point
of $L$; these lines contract onto a smooth quintic curve of genus $2$ which is contained in a unique quadric: it is smooth or not, depending on the two possibilities for $\mathcal{N}_L X$ described above, respectively. Moreover, the inverse map $\chi := \chi^{-1}_L : \mathbb{P}^3 \dasharrow X \subset \mathbb{P}^5$ is as in the first part of Examples 3.

**Case (L.3):** Here $X$ is a smooth hyperquadric in $\mathbb{P}^4$. The projection $\pi_x : X \dasharrow \mathbb{P}^3$ from $X$ with center a point $x \in X$ is a birational map which defines a special link. This link is obtained by blowing up $x$ and then contracting the strict transform of lines on $X$ passing through $x$; these lines contract onto a smooth conic. Moreover, the inverse map $\chi := \chi^{-1}_x : \mathbb{P}^3 \dasharrow X \subset \mathbb{P}^4$ is as in the third part of Examples 3.

**Case (L.5):** The link in this case is exactly the one of the last part of Examples 3. This is obtained by blowing up a smooth sextic curve of genus $3$, say $\Gamma$, which is not contained into a quadric, and then contracting the $3$-secant lines to $\Gamma$ onto a curve projectively equivalent to $\Gamma$.

3.1.2. **Case (L.2).** Let $\Gamma \subset \mathbb{P}^3$ be a smooth rational quartic curve. We first note that through a point $y \in \Gamma$ there passes a unique trisecant line to $\Gamma$: in fact, by projecting $\Gamma$ from $y$ to a general plane we get a (rational) singular cubic curve. Then the unique smooth quadric $Q$ containing $\Gamma$ (Remark 4) is none other than the trisecant variety $\text{Sec}_3(\Gamma)$; moreover, $Q$ is the exceptional set of the extremal contraction $q : Z \to X = X_5 \subset \mathbb{P}^6$ (we keep notations from §2) and the numerical equivalence class of the strict transform of a trisecant line to $\Gamma$ generates the corresponding (negative) extremal ray. Therefore the center of $q : Z \to X$ is necessarily a smooth rational curve, say $C$ ([Mo82, Thm. 3.3 and Cor. 3.4]); moreover, $C$ is a conic in this case, because the image $q(Q)$ spans a dimension $2$ linear space in $\mathbb{P}^6$.

Take a general plane $\Pi \subset \mathbb{P}^3$; all trisecant lines to $\Gamma$ intersect $\Pi$ at a point and $\Pi \cap Q$ is a smooth conic $C'$. The strict transform $p^{-1}_*\Pi$ of $\Pi$ by $p : Z \to \mathbb{P}^3$ is an abstract del Pezzo surface $S$ of degree $5$. Since $p^{-1}_*\Pi = p^*(\Pi)$ and $(3H - E)^2 \cdot p^*(\Pi) = 5$ we deduce that $q(S)$ is a (isomorphic to $S$) hyperplane section of $X$ and $q(C') = C$. We conclude that the inverse link $\chi^{-1} : X \dasharrow \mathbb{P}^3$ is the restriction to $X$ of a (linear) projection of $\mathbb{P}^6$ from the plane containing $C$.

Moreover, let $\ell \subset \mathbb{P}^3$ be a general line, we may suppose $\ell$ in the general plane $\Pi$ considered above. Then, the strict transform $\gamma := \chi_*(\ell)$ of $\ell$ is a twisted cubic which is $2$-secant to $C$.

Conversely, we show all conics on $X$ are as above. Let $C \in C_2$ be a (smooth) conic and denote by $P \subset \mathbb{P}^6$ the plane containing it. We know $X$ does not contain planes ([Is78 Prop. 5.3]), so $P \not\subset X$. Since $X$ is cut out by hyperquadrics ([Is77 Thm. 4.2]) we infer $X \cap P = C$ is a theoretical-scheme intersection. That also implies the points of $\mathbb{P}(\mathcal{N}_C P)$ corresponding to normal direction of $C$ in $P$ does not correspond to normal directions of
C in X. If \( \sigma : Z \rightarrow X \) is the blow-up of \( X \) along \( C \) and \( \pi : X \rightarrow \mathbb{P}^3 \) is the restriction to \( X \) of a general projection of \( \mathbb{P}^6 \) from \( P \), then we deduce a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\tau} & \mathbb{P}^3 \\
\downarrow{\pi} & & \downarrow{\pi} \\
X & \xrightarrow{\sigma} & X
\end{array}
\]

(7)

where \( \tau \) is a morphism; in particular, a general hyperplane section \( S \) of \( X \) containing \( C \) is smooth, that is, it is a del Pezzo surface of degree 5. We conclude \( C \) is realized as the strict transform of (see Lemma 8):

(a) a line passing through only one of \( p_1, p_2, p_3, p_4 \).

(b) a conic passing through \( p_1, p_2, p_3, p_4 \).

Let \( H_Z \) be the pullback by \( \sigma \) of a general hyperplane and set \( F := \sigma^{-1}(C) \). A straightforward computation as in the beginning of \( \S 2 \) shows \( (H_Z - F)^3 = 1 \) from which it follows \( \tau \), and then \( \pi \), is birational. Moreover, \( \pi \) contracts all lines in \( X \) intersecting \( C \). We may realize such a line, say \( L \subset S \subset X \), by thinking of \( S \) as the blow-up \( f : S \rightarrow \mathbb{P}^2 \) in four points \( p_1, p_2, p_3, p_4 \in \mathbb{P}^2 \), as follows:

(a') If \( C \) is as in (a), then \( L \) is either \( f^{-1}(p_i) \) or the strict transform of a line passing through \( p_j, p_k \), \( i \not\in \{j, k\} \).

(b') If \( C \) is as in (b), then \( L \) is \( f^{-1}(p_j) \) for a \( j \not\in \{1, 2, 3, 4\} \).

In both cases above we deduce \( \pi \) contracts four \((-1)\)-curves on each general surface \( S \) containing \( C \), these lines are the unique ones intersecting \( C \). Furthermore, we have

\[
K_Z \cdot \sigma^{-1}_*(L) = (\sigma^*K_X + F) \cdot \sigma^{-1}_*(L) = -2
\]

from which we conclude \( \tau \) is an extraction and \( \tau \circ \sigma^{-1} \) is a link of type II inverse of that of Examples 3, part (L.2); in other words, \( \tau = p \) and \( \sigma = q \) relatively to notations therein.

Now we describe the family \( C_2 = C_2(X) \) of all irreducible conics on \( X \), the centers of all possible contractions \( q = \sigma : Z \rightarrow X \) as above. As we saw before, a general hyperplane section of \( X \) containing a fixed conic \( C \in C_2 \) is a (smooth) del Pezzo surface of degree 4 relative to that hyperplane.

**Lemma 8.** The family \( C_2 \) is a pure dimension 4 quasi-projective variety such that \( N_C X \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \) for all \( C \in C_2 \). Moreover, if \( S \subset X \) is a smooth hyperplane section containing a conic \( C \in C_2 \), \( f : S \rightarrow \mathbb{P}^2 \) the blow-up of four points, then \( f(C) \) is as in (a) or (b) above.

**Proof.** The quasi-projective structure on \( C_2 \) follows from the general theory on Hilbert schemes. The assertion about the dimension of \( C \) follows from the second part of the Lemma.
Write \( C \sim aL - \sum_{i=1}^{4} b_iE_i, \) \( b_i \geq 0, \) as a divisor class in \( \text{Pic}(S) \), where \( L \) is the pullback of a general line in \( \mathbb{P}^2 \) and \( E_i \)’s the exceptional divisors; hence \( i \neq j \) implies \( b_i + b_j \leq a. \)

On the other hand, since \( K_S \cdot S C = -2, \) by the adjunction formula on \( S \) we get \( C \cdot S C = 0, \) where the subindex “\( S \)“ in the dot means the intersection number is taken on \( S \). We deduce

\[
3a - \sum_{i=1}^{4} b_i = 2, \quad a^2 - \sum_{i=1}^{4} b_i^2 = 0.
\]

Hence \( a \leq 2 \) from which it follows \( f(C) \) is as we wanted.

Finally, let us prove the assertion about the normal sheaf of \( C \) in \( X \). By relating normal bundles on \( C, S \) and \( X \) we obtain an exact sequence of bundles on \( C \):

\[
0 \longrightarrow \mathcal{N}_C S \longrightarrow \mathcal{N}_C X \longrightarrow \mathcal{N}_S X|_C \longrightarrow 0. \tag{8}
\]

Since \( \mathcal{N}_C S \cong \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^3}(2), \) we deduce \( \mathcal{N}_C X \cong \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \) for integer numbers \( a, b \) such that \( a + b = 2, \ a \leq b. \) From the cohomology long sequence associated to \( (8) \) it follows \( (a, b) \in \{(−1, 3), (1, 1), (0, 2)\}. \)

On the other hand, the contraction \( p = \tau : Z \rightarrow \mathbb{P}^3 \) induces a birational morphism from the exceptional divisor \( F \) of \( q \) onto a smooth quadric (Remark \([4]\)). Since \( (a, b) = (−1, 3) \) or \( (a, b) = (0, 2) \) would imply \( F \) is a Nagata-Hirzebruch surface \( \mathbb{F}_2, \) in which case that morphism can not exist, we conclude \( (a, b) = (1, 1). \)

Finally, we have

**Lemma 9.** Let \( C, C' \in \mathcal{C}_2 \) be two distinct conics. Then \( C \) and \( C' \) are not coplanar, intersect transversally and \( |C \cap C'| \in \{0, 1\}. \) Moreover, the two possibilities for \( |C \cap C'| \) occur.

**Proof.** Let \( P \) and \( P' \) be the planes where \( C \) and \( C' \) lie. We know \( P \cap X = C, \) \( P' \cap X = C'. \) Hence

\[
P \cap C' = P' \cap C = C \cap C',
\]

and \( |C \cap C'| \leq 2. \)

Suppose \( C \) and \( C' \) intersect at two points (counting multiplicities) and consider \( \sigma : Z \rightarrow X, \) the blow-up of \( X \) along \( C, \) \( F := \sigma^{-1}(C) \) and let \( \mathcal{H}_Z \) be the pullback of a general hyperplane section of \( X; \) \( \tau : Z \rightarrow \mathbb{P}^3 \) is the related extraction which is the blow-up of a rational quartic curve \( \Gamma \subset \mathbb{P}^3; \) as usual \( E := \tau^{-1}(\Gamma). \) If \( D \) denotes the strict transform of \( C' \) by \( \sigma, \) then

\[
K_Z \cdot D = -2, \quad (\mathcal{H}_Z - F) \cdot D = 0.
\]

The second equality implies \( \tau \) contracts \( D \) and the first one implies \( D \equiv 2E_y \) where \( E_y = \tau^{-1}(y) \) for a point \( y \in \Gamma. \) This is not possible because \( \tau|_E : E \rightarrow \Gamma \) is a projective fibration and \( D \) is irreducible. Then \( |C \cap C'| \leq 1. \)
Finally, on a (degree 5) del Pezzo surface $S \subset \mathbb{P}^5$ there are pairs of disjoint conics and pairs of intersecting conics.

3.1.3. **Case (L.4).** Here $X$ is also a smooth hyperquadric in $\mathbb{P}^4$. We keep notations as in Examples 3 As we saw $C_5 := q(F)$ is a smooth quintic curve of genus 1, not contained in a dimension 3 linear space, and $F \in |5H - 2E|$. By construction, a general plane $\Pi \subset \mathbb{P}^3$ is transformed by the link $\chi : \mathbb{P}^3 \dashrightarrow X$ onto the blow-up of $\Pi$ along the five points $\Pi \cap \Gamma = \{p_1, \ldots, p_5\}$, which is a del Pezzo surface of degree 4. Therefore $C_5$ is the image of a quintic curve in $\Pi$ with double points $p_1, \ldots, p_5$, that is $C_5 = \chi_*(p(F) \cap \Pi)$, and the inverse link $\chi^{-1} : X \dashrightarrow \mathbb{P}^3$ is defined by the linear system $|2H_X - C_5| := q_*|2H_Z - F|$, a linear system of del Pezzo quartic surfaces containing $C_5$. Notice that for a general line $\ell \subset \mathbb{P}^3$, the strict transform $\chi_*(\ell)$, which one may suppose to be contained in $\Pi$, is a twisted cubic 5-secant to $C_5$.

Consider a degree 4 del Pezzo surface $S \subset \mathbb{P}^4$, let $f : S \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at $p_1, \ldots, p_5$, let $C \subset S$ be and a non degenerate elliptic quintic curve; denote by $E_i := f^{-1}(p_i) \subset S$ the corresponding exceptional divisors, $i = 1, \ldots, 5$.

Suppose for a moment $C \cdot S E_1 \geq 3$. By projecting $X$ from the line $E_1$ to $\mathbb{P}^3$ we get $C$ is birationally equivalent to a conic: impossible. Then $E_i \cdot S C \leq 2$ for all $i$. Write $C \sim aL - \sum_{i=1}^{5} b_i E_i$, $b_i \geq 0$, as a divisor class in $\text{Pic}(S)$, where $L$ is the pullback of a general line in $\mathbb{P}^2$; hence $b_i \leq 2$.

On the other hand, since $K_S \cdot S C = -5$, by the adjunction formula on $S$ we get $C \cdot S C = 5$. We deduce

$$3a - \sum_{i=1}^{5} b_i = 5, \quad a^2 - \sum_{i=1}^{5} b_i^2 = 5.$$  

Since $C$ is not rational (then $a \geq 3$) we conclude $3 \leq a \leq 5$, and $C$ is the strict transform by $f$ of one of the following:

(a) a quintic curve with double points at $p_1, \ldots, p_5$

(b) a quartic curve with double points at $p_i, p_j$ and passing through all $p_k$ with $k \neq i, j$.

(c) a smooth cubic curve passing through four of $p_1, \ldots, p_5$.

A first consequence of the description above is that every elliptic quintic curve $C$ in $X \subset \mathbb{P}^4$ which is not contained in a 3-space admit (five) chords: indeed, in the case

(a) they are the (-1)-curves $f^{-1}(p_i)$'s, in case (b) they are $f^{-1}(p_i), f^{-1}(p_j)$ and the strict transform of lines passing through two of the remaining points, etc.

Second, such a quintic curve is the base locus scheme of a special link, inverse to a link as in case (L.4): indeed, denote by $\phi : X \dashrightarrow \mathbb{P}^4$ the rational map defined by the linear system $|2H_X - C|$ and by $\sigma : Z \to X$ the blow-up of $X$ along $C$. By restricting $\phi$
to (general) dimension 3 linear spaces intersecting \( C \) transversely, we deduce \( s = 3 \) and \( \tau := \phi \circ \sigma \) is a morphism (the linear system of quadrics in \( \mathbb{P}^3 \) passing through five points in general position has dimension 4, contains smooth members and its indeterminacy may be resolved by blowing up these points). In particular, a general member in \( |2\mathcal{H}_X - C| \) is smooth.

By using the first consequence we know there exists a chord \( L \subset X \) of \( C \). Its strict transform \( \sigma^{-1}(L) \) is contracted by \( \tau \) and satisfies

\[
K_Z \cdot \sigma^{-1}(L) = (-3\mathcal{H}_X + F) \cdot \sigma^{-1}(L) = -1;
\]

that is, \( \tau \) is an extraction. By Theorem 7 it suffices to show that the center of \( \tau \) is a curve.

Let \( E \) be the exceptional divisor of \( \tau \) (recall that \( \tau \) is the blow-up of its center). By writing \( K_Z = \tau^*K_{\mathbb{P}^3} + aE = -4H + aE \), for \( a > 0 \), we deduce \( H \cdot a^2E^2 = -5 \) from which it follows \( a = 1 \) and the center of \( \tau \) is a quintic curve.

Now we describe the family \( \mathcal{C}_5 = \mathcal{C}_5(X) \) of genus 1 smooth quintic curves on \( X \) which are not contained in a three dimensional linear space.

**Lemma 10.** The family \( \mathcal{C}_5 \) is a pure dimension 15 quasi-projective variety such that \( \mathcal{N}_C X \) is indecomposable, extension of a bundle of degree 5 by a bundle of degree 10, for all \( C \in \mathcal{C}_5 \). Moreover, if \( S \subset X \) is a degree 4 del Pezzo surface containing \( C \), \( f : S \to \mathbb{P}^2 \) the blow-up of five points, then \( f(C) \) is as in (a), (b) or (c) above.

**Proof.** As before, the quasi-projective structure on \( \mathcal{C}_5 \) follows from the general theory on Hilbert schemes. Notice that all linear system of curves as in (a), (b) or (c) above has dimension 5.

Fix \( C \in \mathcal{C}_5 \). Since the set of hyperquadric sections of \( X \) constitute a linear system of dimension 13 on \( X \), whose elements containing \( C \) define a linear subsystem of dimension 3, we get the assertion relative to the dimension.

On the other hand, by relating the normal bundle on \( C \), \( S \) and \( X \) we obtain an exact sequence of bundles on \( C \):

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{N}_CS & \longrightarrow & \mathcal{N}_CX & \longrightarrow & \mathcal{N}_S|_C \longrightarrow 0.
\end{array}
\]

We already saw \( \deg(\mathcal{N}_C S) = C \cdot S = 5 \) and, since \( K_S \cdot S = -5 \), then \( \deg(\mathcal{N}_S|_C) = 10 \).

Moreover, since \( S \) is a complete intersection, the sequence above splits if and only if \( C \) is the complete intersection of \( S \) with a hypersurface \( V \subset \mathbb{P}^4 \) ([HaHu]): it can not happen since \( C \) is not degenerated in \( \mathbb{P}^4 \) and \( \deg(C) = 5 \). We deduce that \( \mathcal{N}_C X \) is indecomposable which completes the proof.

\( \square \)
Corollary 11. Let \( C_5 \in C_5 \). There are no 3-secant lines to \( C_5 \) and \( X \) contains a dimension 1 family of 2-secant lines to \( C_5 \).

Proof. On one side it is clear \( C \) does not admit 3-secant lines, since by projecting \( C \) from one of its points we get a quartic curve with geometric genus 1, then a smooth quartic. On the other side, in the three possibilities for the curve \( f(C) \) there is always \((-1)\) curves in \( S \) which are 2-secant to \( C \).

\[ \square \]

Finally, we have:

Corollary 12. Let \( C, C' \in C_5 \) be two genus 1 quintic curves in \( X \). Then \( C \) and \( C' \) may be contained or not in the same hyperquadric section of \( X \). Moreover, we have:

(i) in the first case, \( |C \cap C'| \leq 7 \) and it may take all values 5, 6 and 7.

(ii) in the second case, \( |C \cap C'| \leq 10 \) and it may take all values between 0 and 5.

Proof. The first assertion follows directly from Lemma 10 and its proof.

To prove (i) we only need to consider the different possibilities to intersect curves as in (a), (b) or (c).

To prove (ii), suppose \( C \) is contained in a (degree 4) del Pezzo surface \( S \) and let \( Q \subset \mathbb{P}^4 \) be a hyperquadRIC such that \( S' := Q \cap X \) is a del Pezzo surface distinct from \( S \). We distinguish two cases:

(1) \( S' \) does not contain \( C \). A linear system on \( S' \) constituted by strict transform of curves as in (a), (b) or (c) defines an immersion \( \eta : S' \to \mathbb{P}^5 \). Since \( S' \) intersects \( C \) in at most 10 points, then \( \eta(C \cap S') \) consists of at most 10 points, being the maximum if and only if \( S' \) is transversal to \( C \). It suffices to chose a hyperplane in \( \mathbb{P}^5 \) passing through 0, 1, 2, 3, 4 or 5 of these points, we obtain a curve \( C' \) as we wanted.

(2) \( S' \) contains \( C \). Then \( S' \cap S \) is a degree 8 curve in \( X \subset \mathbb{P}^4 \), one of its component is \( C \). Then \( S' \cap S = C \cup D \) where \( D \) is curve of degree 3 (taking into account multiplicities) and arithmetic genus 0. As in the beginning of § 3.1.3 we deduce \( D \) intersects \( C \) along a zero scheme of length 5; in particular, a quintic \( C' \subset S' \), different from \( C \), may intersect \( C \) in at most 5 points.

\[ \square \]

3.2. Pure Special type II Cremona transformations. A Cremona transformation of \( \mathbb{P}^3 \) is a birational map \( \varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3 \). When \( \varphi \) and its inverse are defined by homogeneous polynomials of degrees \( d \) and \( e \), respectively, we say \( \varphi \) has bidegree \((d, e)\). In this case, if \( S, S' \) are general surfaces of degree \( d \) in the linear system defining \( \varphi \), then
$S \cap S' = \Gamma_1 \cup C$ where $\Gamma_1$ contains the base locus scheme $\text{Base}(\varphi)$ of $\varphi$ and $C$ is a rational irreducible curve of degree $e$. Note that the theoretical-scheme structure of $\Gamma_1$ depends on $S$ and $S'$, but $\deg \Gamma_1 = d^2 - e$, then $\Gamma_1$ defines a unique class in the Chow group of $\mathbb{P}^3$.

We introduce the 1-cycle class associated to a Cremona transformation $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ as the class of 1-cycles defined as above; we denote it by $\text{Cyc}(\phi)$.

A transformation with bidegree $(d, e) \in \{(2, 2), (3, 3)\}$ will be called general if its base locus scheme is smooth (we could have called it “special”, but the base locus may be disconnected which disagrees with what one find in the literature). Finally, on one side, a bidegree $(3, 3)$ transformation $\varphi$ is said to be de Jonquières if, up to linear change of coordinates in the domain and the target, we may write

$$\varphi = (g : kx : ky : kz), \ g, k \in \mathbb{C}[w, x, y, z] \text{ homogeneous},$$

where $g$ is irreducible and vanishes at $(1 : 0 : 0 : 0)$ with order 2, and $k(1 : 0 : 0 : 0) = 0$; in particular $(1 : 0 : 0 : 0)$ is an embedded point in $\text{Base}(\varphi)$ and the strict transform of a general line by $\varphi$ is a plane cubic curve. On the other side, $\varphi$ is said to be determinantal if it is defined by the maximal minors of a $4 \times 3$ matrix of linear forms; of course, a general bidegree $(3, 3)$ transformation is determinantal.

Remark 13. When $\varphi$ is as in (9), for general lines $\ell \subset \mathbb{P}^3$, the plane curve $\varphi^{-1}(\ell)$ spans a general plane passing through $(1 : 0 : 0 : 0)$.

We will say a link as in the case $(L.i)$, see Examples 3 is a link in the class $(L.i)$, $1 = 1, 2, 3, 4, 5$.

**Theorem 14.** Let $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation not defined somewhere which is a product of $\ell$ special links of type II, with $\ell \leq 2$. Then, exactly one of the following assertions hold:

(a) $\ell = 1$ and $\varphi$ is a general Cremona transformation of bidegree $(3, 3)$.

(b) $\ell = 2$ and $\varphi$ is a product $\chi_2^{-1} \circ \chi_1$ where $\chi_1, \chi_2$ are links of type II in the class $(L.i)$, with $i = 1, 2, 3, 4, 5$.

(c) $\ell = 2$ and $\varphi$ is a product $\chi_2^{-1} \circ \chi_1$ where $\chi_1$ is a link of type II in the class $(L.5)$ and $\chi_2$ is a link of type II in the class $(L.i)$, with $i = 1, 2, 3, 4$.

(d) $\ell = 2$ and $\varphi$ is a product $\chi_2^{-1} \circ \chi_1$ where $\chi_1$ and $\chi_2$ are links of type II either in classes $(L.3)$ and $(L.4)$, respectively, or conversely.

**Proof.** If $\ell = 1$, we know $\varphi$ is a general cubo-cubic transformation by Theorem 7. In the sequel we assume $\ell = 2$.

By hypothesis $\varphi = \chi_2^{-1} \circ \chi_1$ where $\chi_i : \mathbb{P}^3 \dashrightarrow X$ is a special link of type II onto a smooth 3-fold $X$, $i = 1, 2$. As we know by Remark 2 each $\chi_i$ is not defined along an
irreducible curve, say $\Gamma_i \subset \mathbb{P}^3$, and then $\chi_i = q_i \circ p_i^{-1}$ where $p_i : Z_i \to \mathbb{P}^3$ is the blow-up of $\Gamma_i$. By construction $\chi_i$ is not trivial, that is, it is not an isomorphism. Theorem \ref{thm:blow-up} implies the result.

For simplicity, a Cremona transformation as in Theorem \ref{thm:blow-up} is said to be \textit{Pure Special type II}.

**Corollary 15.** There exist twelve classes of Pure Special type II Cremona transformations with $\ell \leq 2$.

In the theorems below we will not include an explicit description of Pure Special type II Cremona transformations containing links in the class $(L.5)$, where it is clear the situation we will find there but to do that may be longer than interesting.

**Theorem 16.** Let $\varphi = \chi_2^{-1} \circ \chi_1 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Pure Special type II Cremona transformation. Suppose $\chi_1, \chi_2$ are both in the same class $(L.i)$, $i = 1, \ldots, 4$. Then, exactly one of the following assertions hold:

(a) In the class $(L.1)$ case $\varphi$ is a bidegree $(3,3)$ transformation whose base locus scheme contains a smooth genus 2 quintic curve $\Gamma$ and it holds one of the following:

i) $\text{Base}(\varphi) = \Gamma \cup L$ where $L$ is a 2-secant line to $\Gamma$ if and only if $\text{Base}(\chi_1^{-1}) \cap \text{Base}(\chi_2^{-1}) = \emptyset$; in this case $\varphi$ is determinantal.

ii) $\text{Base}(\varphi) = \Gamma \cup L$ with an embedded point where $L$ is a trisecant line to $\Gamma$ if and only if $\text{Base}(\chi_1^{-1}) \cap \text{Base}(\chi_2^{-1}) \neq \emptyset$; in this case $\varphi$ is de Jonquières.

(b) In the class $(L.2)$ case $\varphi$ is a bidegree $(3,3)$ transformation whose base locus scheme contains a smooth rational quartic curve $\Gamma$ and it holds one of the following:

i) $\text{Base}(\varphi) = \Gamma \cup D$ where $D$ is a rank 1 or 3 conic which is 4-secant to $\Gamma$ if and only if $\text{Base}(\chi_1^{-1}) \cap \text{Base}(\chi_2^{-1}) = \emptyset$; in this case $\varphi$ is determinantal.

ii) $\text{Base}(\varphi) = \Gamma \cup D$ where $D = L \cup L'$ is a rank 2 conic which is 4-secant to $\Gamma$, with $L$ being 3-secant, if and only if $\text{Base}(\chi_1^{-1}) \cap \text{Base}(\chi_2^{-1}) \neq \emptyset$; in this case $\varphi$ is determinantal.

(c) In the class $(L.3)$ case $\varphi$ is a general bidegree $(2,2)$ transformation whose base locus scheme is the union of a smooth conic and a point not lying on its plane.

(d) In the class $(L.4)$ case $\varphi$ is a bidegree $(6,6)$ Cremona transformation whose base locus scheme contains an elliptic quintic curve $\Gamma$ and another curve $D$ such that $\text{Cyc}(\varphi) = n\Gamma + D$, with $n \geq 4$, and it holds one of the following:

i) the curves $\text{Base}(\chi_1^{-1})$ and $\text{Base}(\chi_2^{-1})$ intersect at $m$ (taking into account multiplicities) points, with $0 \leq m \leq 10$, and $D$ is supported on the union of an irreducible curve $C_{10-m}$ of degree $10 - m$, which is birationally equivalent to $\text{Base}(\chi_2^{-1})$, and
m lines 3-secant to $\Gamma$; in this case $n = 4$, $C_{10-m}$ is smooth in $\mathbb{P}^3 \setminus \Gamma$ and it is $(25 - 3m)$-secant to $\Gamma$.

(ii) the curves $\text{Base}(\chi_1)$ and $\text{Base}(\chi_2)$ intersect at $m$ (taking into account multiplicities) points, with $m \in \{0, 5\}$, and $D$ is supported on the union of $m$ lines 3-secant to $\Gamma$; in this case $n = m = 5$ or $n = 6$ and $m = 0$.

Proof. The case (a) follows from [AGSP08] and the case (c) is essentially trivial. Let us consider the cases (b) and (d).

Case (b). We keep all notations from §2.1, $\chi_1 = q \circ p^{-1}$ and $\Gamma := \text{Base}(\chi_1)$. Then, for each $i = 1, 2$, the birational map $\chi_i = \pi_i : X \dashrightarrow \mathbb{P}^3$ is the restriction to $X$ of a general projection from a plane, $\text{Base}(\pi_1) = C \in C_2$, $\text{Base}(\pi_2) = C' \in C_2$.

The strict transform $S \subset X$ of a general plane by $\pi_2$ is a del Pezzo surface of degree 5 containing the conic $C' = \text{Base}(\pi_2)$. As we saw in §3.1 the surface $S$ is a hyperplane section of $X$ from which we deduce it intersects $C = \text{Base}(\pi_1)$ at two points. Therefore the strict transform $(\chi_1)_*^{-1}(S) = (\pi_1)_*(S)$ is a cubic surface. By symmetry we get $\varphi$ has bidegree $(3, 3)$.

Furthermore, let $\ell \subset \mathbb{P}^3$ be a general line. As we saw, the strict transform $\gamma := (\pi_2^{-1})_* (\ell)$ of $\ell$ is a twisted cubic 2-secant to $C'$. By construction $\gamma \cup C'$ is a hyperplane section of $X$, general among those containing the plane $P' \supset C'$.

Suppose $C \cap C' = \emptyset$. By genericity on $\ell$, we may suppose that the linear 3-space $\langle \gamma \rangle$ spanned by $\gamma$ does not intersect the plane $P$ of $C$. We deduce $\pi_1$ restricts isomorphically on $\gamma$ and then $\pi_1(\gamma)$ is a twisted cubic curve in $\mathbb{P}^3$.

Take irreducible cubic surfaces, say $W, W' \subset \mathbb{P}^3$, such that $W \cap W' = \Gamma \cup D_2 \cup \pi_1(\gamma)$, where $D_2 := \pi_1(C')$ is a conic which has rank 3 (i.e. it is smooth) if $P \cap P' = \emptyset$ and has rank 1 otherwise. Since a twisted cubic curve is arithmetically Cohen-Macaulay of genus 0, by liaison $\Gamma \cup D_2$ is an arithmetically Cohen-Macaulay curve of degree 6 and arithmetic genus 3, whose ideal is generated by 4 independent cubic forms which are the maximal minors of a $4 \times 3$ matrix of linear forms ([PS74 §3]). Hence $\varphi$ is determinantal. By adjunction on $S$, we have that arithmetic genus 3 for $\Gamma \cup D_2$ implies $D_2$ is 4-secant to $\Gamma$.

Now suppose $C \cap C'$ consists of a unique point, say $x \in X$; by Lemma 9 there is no other possibility. Then $P \cap P' = \{x\}$ from which it follows $P'$ contracts (via $\pi_1$) to a line $L'$. We deduce this line is a chord of $\varphi_*^{-1}(\ell) = (\pi_1)_*(\gamma)$ and this curve is a twisted curve, which as before implies $\varphi$ is determinantal: in fact, otherwise it would be de Jonquières and $\varphi_*^{-1}(\ell)$ should contradict what we said in Remark 13.
Furthermore, the map $\pi_2 \circ q$ in not defined along the line (fiber) $F_x = q^{-1}(x)$ which corresponds via $p$ to a line intersecting $L$; that line is 3-secant to $\Gamma$ since $H = H_Z - F$ and $H_Z = 3H - E$, $H_Z \in \{q^*\mathcal{O}_X(1)\}$, implies $E = 2H_Z - 3F$, then $E \cdot F_x = 3$.

If $D_2 = L \cup L'$ we note $D_2$ is 4-secant to $\Gamma$, as before, which completes the proof of the statement (b).

**Case** (d). Now $C, C' \in \mathcal{C}_5$ are elliptic quintic curves on the hyperquadric $X \subset \mathbb{P}^4$. The reduced structure on $\text{Base}(\varphi)$ is then $\Gamma \cup p(q^{-1}(C'))$, where $q \circ p^{-1} = \chi_1$. As we saw the (classes of) exceptional divisors $E$ and $F$ of $p$ and $q$, respectively, satisfy

$$H_Z := q^*(H_X) = 3H - E, F = 5H - 2E,$$

or equivalently

$$H = 2H_Z - F, E = 5H_Z - 3F. \tag{11}$$

Then $\varphi$ is defined by a linear subsystem of $p_*q^*(\{2H_X\}) = p_*(\{6H - 2E\})$, from which it follows its linear system consists of degree 6 surfaces passing through $\Gamma$ with multiplicity at least 2; in particular, $\text{Cyc}(\varphi) = n\Gamma + D$ with $n \geq 4$ and, by symmetry, $\text{bideg}(\varphi) = (6, 6)$.

On the other hand, notice that (by definition) $m := q_*^{-1}(C') \cdot F$.

Suppose $C'$ is not contained in $q(E)$, that is $q_*^{-1}(C') \not\subset E$. The restriction of $p$ to $q_*^{-1}(C')$ induces a birational morphism which is injective on $Z \setminus E$. Then $C_{10-m} := p(q_*^{-1}(C'))$ is a curve of degree $10 - m$, smooth on $\mathbb{P}^3 \setminus \Gamma$, which intersects $\Gamma$ at a zero scheme of length $s = q_*^{-1}(C') \cdot E$. We deduce

$$s = q_*^{-1}(C') \cdot (5H_Z - 3F) = 25 - 3m.$$ 

Take $S, S'$ general elements in the linear system defining $\varphi$. The scheme theoretical intersection $S \cap S'$ defines a 1-cycle of degree 36 supported on

$$\Gamma \cup p(q_*^{-1}(C')) \cup p(q^{-1}(C \cap C')).$$

By taking into account multiplicities, the last part in the union above consists of $m$ lines 3-secant to $\Gamma$. By computing degrees we obtain

$$36 - 6 = n \deg \Gamma + 10,$$

that is, $n = 4$.

Now suppose $C' \subset q(E)$. Since $E$ is a projective lines bundle over $\Gamma$ we deduce $p(q_*^{-1}(C')) = \Gamma$, then $n > 4$. The last assertions follow easily. □

**Remark 17.** (1) Transformations as in (a)i correspond to type $T_{33}^{(3)}$ in table [SR, pag. 185]. Analogously, transformations as in (b)i correspond to type $T_{33}^{(4)}$ in [SR, pag. 185] together with degeneration of it.

(2) We do not know if the case (ii) in part (d) of the theorem above may effectively occur.
Theorem 18. Let \( \varphi = \chi_2^{-1} \circ \chi_1 : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \) be a Pure Special type II Cremona transformation. Suppose \( \chi_1 \) and \( \chi_2 \) are links of type II either in classes \( (L.3) \) and \( (L.4) \), respectively, or conversely. Then, exactly one of the following assertions hold:

(a) In the first case the base locus scheme of \( \varphi \) contains a smooth conic \( \Gamma \) and it holds one of the following:

(i) \( \text{Base}(\varphi) = \Gamma \cup D_5 \) where \( D_5 \) is a genus 1 quintic curve, with at most a double point, which is 5-secant to \( \Gamma \) if and only if \( \text{Base}(\chi_1) \cap \text{Base}(\chi_2^{-1}) = \emptyset \); in this case \( \text{bideg}(\varphi) = (4, 3) \) and \( \text{Cyc}(\varphi) = 4\Gamma + D_5 \).

(ii) \( \text{Base}(\varphi) = \Gamma \cup D_4 \) where \( D_4 \) is an elliptic quartic curve 3-secant to \( \Gamma \) if and only if \( \text{Base}(\chi_1^{-1}) \cap \text{Base}(\chi_2^{-1}) \neq \emptyset \); in this case \( \text{bideg}(\varphi) = (3, 3) \) and \( \text{Cyc}(\varphi) = \Gamma + D_4 \).

(b) In the second case the base locus scheme of \( \varphi \) contains an elliptic quintic curve \( \Gamma \) and it holds one of the following:

(i) \( \text{Base}(\varphi) = \Gamma \cup \{o'\} \) where \( o' \) is either an infinitely near point or an isolated point if and only if \( \text{Base}(\chi_1^{-1}) \cap \text{Base}(\chi_2^{-1}) = \emptyset \); in this case \( \text{bideg}(\varphi) = (3, 4) \) and \( \text{Cyc}(\varphi) = \Gamma \).

(ii) \( \text{Base}(\varphi) = \Gamma \cup L \) where \( L \) is a line 3-secant to \( \Gamma \) if and only if \( \text{Base}(\chi_1^{-1}) \cap \text{Base}(\chi_2^{-1}) \neq \emptyset \); in which case \( \text{bideg}(\varphi) = (3, 3) \) and \( \text{Cyc}(\varphi) = \Gamma + L \).

Proof. Recall the description given in §3.1.3. The linear system defining the special link of class \( (L.4) \) consists of the hyperquadric sections of \( X \) containing a quintic elliptic curve \( C_5 \in \mathcal{C}_5 \); we denote by \( |2H_X - C_5| \) this linear system. On the other hand, the special link of class \( (L.3) \) is the projection of \( X \subset \mathbb{P}^4 \) from a point \( o \in X \).

We also keep notations as in the previous theorem. The divisors \( E, F, H, \mathcal{H}_Z \) always refer to the morphism \( p \) and \( q \) where \( \chi_1 = q \circ p^{-1} \).

(a) In this case we have the equalities of divisor classes

\[
H = \mathcal{H}_Z - F, \quad \mathcal{H}_Z = 2H - E
\]

where \( q(F) = \{o\} \) and \( p(E) = \Gamma \).

If \( o \notin C_5 \), then \( C_5 \) projects birationally to a quintic curve \( D_5 \). Notice that \( D_5 \) has at most one double point, and this occurs only when \( o \) belongs to a 2-secant line to \( C_5 \) (see Corollary 11). This curve intersects \( \Gamma \) in a 0-scheme of length \( s := q_s^{-1}(C_5) \cdot E \). Since \( E = \mathcal{H}_Z - 2F \) and \( F \cap q_s^{-1}(C_5) = \emptyset \), we get \( s = 5 \). Moreover, \( p_s q_s^*|2H_X - C_5| \subset p_s^*|2H_Z| = |4H - 2E| \), from which we deduce \( \varphi \) is defined by a linear system of quartic surfaces containing \( \Gamma \) with multiplicity \( \geq 2 \); in particular \( \text{deg}(\varphi) = 4 \) and \( \text{Cyc}(\varphi) = n\Gamma + D_4 \) with \( n \geq 4 \). Furthermore, since a twisted cubic on \( X \) which does not pass through \( o \) is projected to a cubic curve we also obtain \( \text{deg}(\varphi^{-1}) = 3 \), that is, \( \text{bideg}(\varphi) = (4, 3) \).
Finally, the 1-cycle above has degree $16 - 3 = 2n + 5$, then $n = 4$, which completes the proof of (i).

Now suppose $o \in C_5$; in this case $C_5$ projects to an elliptic quartic curve $D_4$. Hence $F \cdot q_*^{-1}(C_5) = 1$ and $q^{-1}(C_5) = q_*^{-1}(C_5) \cup F$. Since $p$ (defined by $|\mathcal{H}_Z - F|$) restricts to $F \simeq \mathbb{P}^2$ as a linear immersion and $E = \mathcal{H}_Z - 2F$, we deduce $D_4 := p_*(q_*^{-1}(C_5))$ is an elliptic quartic curve 3-secant to $\Gamma$ and the rest of the assertion (ii) follows by arguing as before.

(b) The assertion relative to bidegrees follows from (a).

To prove the remaining part, first suppose $o \notin C_5$. If $o \in q(E)$, then $o' := q^{-1}(o) \in E$ and $p(o') \in \Gamma$. For a general line $\ell \subset \mathbb{P}^3$, the strict transform $(\chi_2^{-1} \circ q)_*^{-1}(\ell)$ is a curve passing through $o'$, then $\varphi \circ p$ in not defined at $o'$; we deduce $o'$ is an infinitely near point lying over $\Gamma$.

If $o \notin q(E)$, then $o' := p(q^{-1}(o)) \notin \Gamma$ defines an isolated point of $\text{Base}(\varphi)$.

Finally, we suppose $o \in C_5$. In this case the fiber $F_o := q^{-1}(o) \subset F$ is a smooth rational curve. From (11) we get

$$H \cdot F_o = 1, \quad E \cdot F_o = 3.$$  

Then $p(F_o)$ is a line 3-secant to $\Gamma$. \hfill \Box

Remark 19. With a small additional effort we may prove that the bidegree $(3,3)$ Cremona transformations appearing in the theorem are determinantal. In fact, a union of curves as in $\text{Base}(\varphi)$, in cases (a)ii and (b)ii, both are sextic curves of arithmetic genus 3 and one may proves such curves are Arithmetic Cohen-Macaulay (or use [Pa97]). The maps in (a)ii and (b)ii correspond, respectively, to the cases $T_{33}^{(6)}$ and $T_{33}^{(2)}$ in table [SR, pag. 185].

From the Theorems above it follows (to compare with Remarks 17 and 19):

**Corollary 20.** The transformations of bidegree $(3,3)$ of types $T_{33}^{(1)}$, $T_{33}^{(5)}$, $T_{33}^{(7)}$ and $T_{33}^{(8)}$ in table [SR, pag. 185] are not Pure Special type II.

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