Variational principle for the Wheeler-Feynman electrodynamics

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We adapt the formally-defined Fokker action into a variational principle for the electromagnetic two-body problem. We introduce properly defined boundary conditions to construct a Poincaré-invariant-action-functional of a finite orbital segment into the reals. The boundary conditions for the variational principle are an endpoint along each trajectory plus the respective segment of trajectory for the other particle inside the lightcone of each endpoint. We show that the conditions for an extremum of our functional are the mixed-type-neutral-equations with implicit state-dependent-delay of the electromagnetic-two-body problem. We put the functional on a natural Banach space and show that the functional is Fréchet-differentiable. We develop a method to calculate the second variation for $C^2$ orbital perturbations in general and in particular about circular orbits of large enough radii. We prove that our functional has a local minimum at circular orbits of large enough radii, at variance with the limiting Kepler action that has a minimum at circular orbits of arbitrary radii. Our results suggest a bifurcation at some $O(1)$ radius below which the circular orbits become saddle-point extrema. We give a precise definition for the distributional-like integrals of the Fokker action and discuss a generalization to a Sobolev space $H^2_0$ of trajectories where the equations of motion are satisfied almost everywhere. Last, we discuss the existence of solutions for the state-dependent delay equations with slightly perturbed arcs of circle as the boundary conditions and the possibility of nontrivial solenoidal orbits.

INTRODUCTION

We construct a variational principle for the electromagnetic two-body problem with finite integration limits. Unlike the Fokker action that involves an infinite integration and has a formal meaning only[1–3], our Poincaré-invariant functional maps a finite segment of trajectory into a finite real number. Our variational principle uses boundary conditions consisting of (i) the initial point $O_A$ for the trajectory of particle 1 plus the segment of trajectory of particle 2 inside the lightcone of $O_A$, and (ii) the endpoint $L_B$ for the trajectory of particle 2 plus the segment of trajectory of particle 1 inside the lightcone of $L_B$. For trajectories respecting the above boundary conditions we show that the conditions for an extremum of our functional are the two-body equations of motion of the Wheeler-Feynman electrodynamics[3]. Our first functional is the natural generalization of the Fokker action and it can not be defined for trajectories travelling faster than light (superluminal). We construct a norm for the linear space of $C^1$ orbits satisfying the above boundary conditions and show that our functional is Fréchet differentiable at subluminal orbits. In order to obtain a functional defined everywhere on a natural Banach space of $C^1$ orbits we give up the parametrization-independence and construct a second generalized functional. The electromagnetic equations of motion follow from the conditions for an extremum of our functionals in the subspace of $C^2$ orbital variations. The extremum conditions are parametrization-independent for the Fokker-like functional, while for the generalized functional the conditions require the parameter to be proper-time because of a conservation law that separates the extremal orbits in three classes. The domain of our second functional is a Banach space, and moreover along its extremal orbits the particle trajectories turn out to be of three possible types (a) subluminal trajectories travelling slower than light everywhere, (b) luminal trajectories travelling at the speed of light everywhere or (c) superluminal trajectories travelling faster than light everywhere. We calculate the second variation of the action about general orbits for $C^2$ orbital variations satisfying the above boundary conditions and in particular about the Schoenberg-Schild-circular-orbit-solutions of a large radius[9, 10]. We prove that the second variation is positive-definite about circular orbits of large enough radii, so that circular orbits are local minima of our functionals. Our results suggest a bifurcation at some $O(1)$ radius below which circular orbits become saddle-point extrema, at variance with the Kepler action for which circular orbits of arbitrary radii are minima[12]. We discuss a use of the variational principle to solve the neutral-delay equations of the electromagnetic two-body problem as a boundary-value problem with a variational integrator [4]. We discuss a generalization to a Sobolev space $H^2_0$ of trajectories where the equations of motion are valid almost everywhere and the existence of solutions with slightly perturbed circular boundaries. Last, we discuss the physics of the Fokker action and the existence of nontrivial solenoidal orbits.

The Fokker action functional is a synthetic principle of
electrodynamics discovered in the early 20th century\cite{1,2} and used in 1945 by Wheeler and Feynman\cite{3} to construct an electrodynamics of point charges. The Wheeler-Feynman electrodynamics is an alternative description of classical electromagnetism that avoids the notion of field to describe the classical laws of Gauss, Faraday, Ampère, and Biot-Savart \cite{3,5}. The theory describes point charges interacting in pairs via the half-retarded plus half-advanced solutions of Maxwell’s equations for the fields\cite{6}. Here we avoid the popular name action-at-a-distance electrodynamics because it can suggest action-at-the-same-time connecting spatially-separated points, while the Wheeler-Feynman theory involves only Einstein-local interactions along lightcones. Among the existing versions of electrodynamics of point charges\cite{7}, the selling points of the Wheeler-Feynman theory are (i) The point-charge-limit is regular, i.e., a spherical charge distribution of a small radius does not make a force on itself and its mass is not renormalized and (ii) The theory reduces to the usual Dirac electrodynamics with retarded-only interactions\cite{8} when the far fields vanish asymptotically, a condition named the absorber hypothesis in Refs. \cite{3}. The equations for two-body motion of the Wheeler-Feynman theory are state-dependent neutral-delay equations and little is known about their solutions, besides the existence of a one-parameter family of circular-orbit solutions\cite{9,10}. An existence result was proved in Ref. \cite{17} for the two-body problem with equal charges (repulsive interaction) and initial condition restricted to colinear orbits of large separations, a case where the equations of motion are no longer neutral but rather delay-only. References \cite{18,19} considered satisfying the state-dependent neutral-delay equations almost-everywhere. In Ref. \cite{20} the equations of motion were expressed as an algebraic-differential system by solving for the most advanced accelerations, an approach also used in Ref. \cite{21} to prove wellposedness and existence for $C^\infty$ initial data consisting of maximal independent past segments. The initial conditions consisting of maximal independent history segments developed in Ref. \cite{21} are different from the initial conditions used in our variational method, which combine future and past data. Last, the simpler delay-only state-dependent two-body equations with initial condition restricted to colinear orbits were studied numerically in Ref. \cite{25} for the case of repulsive interaction and in Ref. \cite{26} for the case of attractive interaction (opposite charges). This paper aims to introduce the problem for a mathematical audience. In the introduction we start from the naive and formal language of physics, posing the problem first at an intuitive level. In the later sections we make an attempt to proceed with rigor and precise definitions by presenting the results in the form of theorems.

The paper is divided as follows: In section 1 we give a crash review of Minkowski spaces and put in one place the ingredients later used to show that the equations of motion separate the orbits in three invariant classes and to construct an action defined on a Banach space. In section 2 we introduce the finite action and the boundary conditions. We construct a norm enforcing the property that perturbations with a small norm of subluminal orbits yield subluminal orbits. For such norm the functional is Fréchet-differentiable along subluminal orbits. In order to obtain a functional defined on a complete normed space we relax the parametrization independence requirement and construct a second functional that can be extended to all types of orbits of a natural Banach space. In this section we discuss the advantages of using the variational method as an alternative to the state-dependent neutral-delay equations of motion as far as numerical stability. In section 3 we give a method to calculate the second variation about arbitrary solutions for $C^2$ orbital perturbations. In particular we calculate the second variation about low-velocity-circular-orbit-solutions. We show that the quadratic form of second variation about low-velocity-circular-orbits is positive-definite if the circular radius is large enough. In this section we develop the idea of a sewing grid which appears naturally in the integration of the quadratic form with delay and is useful for the numerical analysis of state-dependent delay problems. In section 4 we put the discussions and the conclusion. We discuss the variational method as a tool to investigate solenoidal and other types of orbits of the electromagnetic two-body problem. We also discuss the variational problem with slightly perturbed circular-orbit boundary data. Last, in the appendix we review the physics of the Fokker action and the conserved momenta of Noether’s theorem. We discuss the nontrivial possibility of solenoidal orbits with both particles gyrating near the speed of light with finite and small momenta.

**PRELIMINARIES AND DEFINITIONS**

We start by explaining the natural coordinatization for Lorentz-invariant dynamics, i.e., the Lorentz four-space $L^4$ attached to an inertial frame by Einstein synchronization of clocks (the $L$ in $L^4$ stands for Lorentz). A point in $L^4$ is defined by a time $t$ and a spatial position $\vec{r}$ in the inertial frame, $x^\mu \equiv (t, \vec{r})$, henceforth called the time component $t$ and the three-vector spatial component $\vec{r}$. The index $\mu$ belongs to $(1, 2, 3, 4)$, with $\mu = 1$ denoting the time-component and $\mu = 2, 3, 4$ denoting the spatial components. From any Minkowski vector $a^\mu = (a_\mu, \vec{a})$ we define its dual vector by $a_\mu \equiv (a_\mu, -\vec{a})$. The Minkowski scalar product is a bilinear product defined as the usual scalar product on $\mathbb{R}^4$ between the first vector and the second vector’s dual (or vice-versa), i.e., $(a \cdot b) \equiv a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4$. This definition gives only a pseudo-scalar bilinear product, and the pseudo-norm $|a|^2 \equiv (a \cdot a)$ induced by the Minkowski product is sensible only for time-like vectors, i.e., when $(a \cdot a) \geq 0$.  


The Minkowski product divides the vectors of $L^4$ in three classes: (i) if $(\mathbf{a} \cdot \mathbf{a}) > 0$ the vector is called time-like (for example the four-velocity along a subluminal orbit), (ii) if $(\mathbf{a} \cdot \mathbf{a}) < 0$ the vector is called space-like (for example the four-acceleration of a subluminal orbit) and last (iii) if $(\mathbf{a} \cdot \mathbf{a}) = 0$ the vector is called a null-vector or light-like. The four-vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal if $(\mathbf{a} \cdot \mathbf{b}) = 0$. The properties of the Lorentz group and the Minkowski product are discussed in Ref. [23], of which we list a few:–(a) Two orthogonal light-like vectors are necessarily multiples of each other because $(\mathbf{a} \cdot \mathbf{a}) = (\mathbf{b} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) = 0$ implies the Cartesian product of the three-vector components satisfies $\vec{r}_a \cdot \vec{r}_a = ||\vec{r}_a||^2$ (double bars denoting the Euclidean modulus of the three-vector), (b) All vectors orthogonal to a time-like vector are space-like and form a three-dimensional space-like subspace. (c) Given a time-like four-vector $\mathbf{z}$ and an arbitrary four-vector $\mathbf{x}$ there is a unique decomposition $\mathbf{x} = \mathbf{y} + \alpha \mathbf{z}$, where $\mathbf{y}$ is space-like and $\alpha$ a real scalar, and (d) any orthogonal basis for $L^4$ must contain one time-like four-vector and 3 space-like four-vectors[23]. (e) For time-like vectors the invariant reverse-Schwartz-inequality holds for the Minkowski product, i.e., $(\mathbf{a} \cdot \mathbf{b})^2 \geq (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b})$, equality holding iff the vectors are parallel, and last (f) For a time-like and a space-like vectors, the reverse Schwartz holds without the equal-sign case, i.e., $(\mathbf{a} \cdot \mathbf{b})^2 > (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b})$ [23].

The positivity of the Minkowski product $(\mathbf{a} \cdot \mathbf{a})$ for the four-velocity in arbitrary parametrization is the physical condition that the particle travels slower than light. The four-velocity is light-like in the limit situation when the particle travels at the speed of light. The Minkowski scalar product is left invariant by Lorentz transformations and it is useful to express the equations of motion and the action functional in a form explicitly equivalent under the Lorentz group. The last three components of a Minkowski vector form a spatial three-vector usually treated differently from the first component, and the various norms used in this paper are henceforth denoted as follows: (I) the Minkowski norm is indicated with single bars, i.e., $|\mathbf{a}|$; (II) the Euclidean $\mathbb{R}^3$ norm of the spatial three-vector part is indicated by double bars, i.e., $||\vec{r}_a||$. We also use double bars to indicate the absolute value of a real number and (III) the Euclidean $\mathbb{R}^4$ norm of a four-vector is indicated by double bars with sub-index 4, i.e., $||\mathbf{a}||_4$ and last (IV) The norm defined on our functional linear space of trajectories of section 3 is denoted by $|\mathbf{b}|_{\mathcal{X}(x_1)}$.

To abbreviate the notation, we henceforth drop the 4-index label and keep only a lower index $j \in (1,2)$ to identify each particle of the two-body problem, e.g., $j = 1$ denotes electronic four-vector quantities and $j = 2$ denotes protonic four-vector quantities. For subluminal orbits it is convenient to express the equations of motion in terms of a Lorentz-invariant parameter defined by the squared-Minkowski-norm of the infinitesimal displacement vector $dx_i$, i.e.,

$$(dx_i)^2 = (dt_i)^2 - (dx_j)^2 - (dy_j)^2 - (dz_j)^2 > 0.$$  

The left-hand side of Eq. (1) is positive for subluminal orbits, zero for luminal orbits and negative for superluminal orbits. The parameter $\tau_i$ defined by Eq. (1) is called the proper-time and it is a property of each particle’s trajectory, the usual parametrization by arc-length of differential geometry.

Next we introduce the naive Fokker action in the above defined Lorentz four-space $L^4$ using a normalized unit system where the speed of light is $c = 1$ and the electron and the proton have mass and charge $m_1 = 1$ and $e_1 = -1$ and $m_2 = 1824$ and $e_2 = 1$ respectively. Let the trajectory of each particle in $L^4$ be a differentiable function $\mathbf{x}_i(\lambda_i) : \mathbb{R} \to L^4$ of a parameter $\lambda_i$ with $i = 1, 2$ indicating respectively the electron and the proton trajectories. The Fokker action[1, 2] is defined in the original literature by a formal integration along the whole trajectories as

$$S = -\int m_1 \dot{\mathbf{x}}_1 \cdot d\lambda_1 - \int m_2 \dot{\mathbf{x}}_2 \cdot d\lambda_2 + \int \delta(|\mathbf{x}_1 - \mathbf{x}_2|^2) \dot{\mathbf{x}}_1 \cdot \dot{\mathbf{x}}_2 d\lambda_1 d\lambda_2,$$

where overdot denotes derivative respect to the parameter of each trajectory. Action (2) is formally independent of the parametrizations, a geometric property easily checked by changing the parameter of each trajectory with the chain rule. The peculiar last integral of the right-hand-side of Eq. (2) involves the composition of the Dirac delta-function $\delta(x)$ with the real function

$$d(\lambda_1, \lambda_2) \equiv |\mathbf{x}_1(\lambda_1) - \mathbf{x}_2(\lambda_2)|^2,$$

where single bars stand for the Minkowski norm of the four-separation $\mathbf{x}_{12} \equiv (x_1 - x_2)$. The peculiar combination appearing in Eq. (2) comes from the Green’s function of Maxwell’s equations and becomes ill-defined along $C^1$ trajectories or in a Sobolev space. Here we give a precise definition for the right-hand-side of Eq. (2), and in Section 3 we define a consistent derivative for such operation before evaluating the second variation, thus avoiding the loose derivatives of the Dirac delta-function. Condition (1) is sufficient for the separation $d(\lambda_1, \lambda_2)$ of Eq. (3) to have precisely two zeros for each fixed $\lambda_1$ along a $C^1$ trajectory [21]. In Ref. [21] it is proved that along a subluminal orbit light captures the slower moving particle once in the past and once in the future. The integration on the last term of the right-hand-side of Eq. (2) gives a nonzero contribution at each zero $(\lambda_1, \lambda_2)$ of Eq. (3). At a given $\lambda_1$ the condition

$$|\mathbf{x}_1(\lambda_1) - \mathbf{x}_2(\lambda_2)|^2 = 0,$$

can be solved for the time-component $t_2(\lambda_2)$ of vector $\mathbf{x}_2(\lambda_2)$, yielding a retarded time and an advanced time,
each defined implicitly by
\[ t_2(\lambda_2) = t_1(\lambda_1) + ||\vec{r}_2(\lambda_2) - \vec{r}_1(\lambda_1)||. \]  

(5)

where double bars stand for the Euclidean norm of the spatial separation \( \vec{r}_2(\lambda_2) - \vec{r}_1(\lambda_1) \). Either one of the equivalent Eqs. (4) or (5) are henceforth called the lightcone condition. Equation (5) is an implicit condition for \( \lambda_2 \) because \( \lambda_2 \) appears on both sides as an unknown argument. Condition (4) is symmetric on particle trajectories, so that the lightcone condition for the pronetic trajectory is still Eq. (5), as obtained by either rearranging Eq. (5) to isolate \( t_1 \) on the left-hand-side or by exchanging the indices 1 and 2 of Eq. (5). In the following we assume the orbital parameters are restricted to the intervals \([L_{\alpha}, L_{\alpha F}]\) for \( \alpha = 1, 2 \), as defined in the next section. Let the zeros of \( d(\lambda_1, \lambda_2) \) evaluated at each zero \((\lambda_1, \lambda_2^{(j)})\) of \( d(\lambda_1, \lambda_2) \). The summation on the right-hand-side of Eq. (6) includes all the zeros of the lightcone condition inside \([L_{21}, L_{2F}]\). Once the separation \( d(\lambda_1, \lambda_2) \) is completely symmetric on particle quantities, definition (6) has a symmetric definition as follows:–

**Definition 1:** We henceforth define the integral involving the Dirac delta-function composed with \( d(\lambda_1, \lambda_2) \) by
\[
\int_{L_{21}}^{L_{2F}} \delta(d(\lambda_1, \lambda_2)) f(\lambda_1, \lambda) d\lambda \equiv \sum_j \frac{f(\lambda_1, \lambda_2^{(j)})}{||\frac{\partial d}{\partial \lambda_2}(\lambda_1, \lambda_2^{(j)})||}.
\]

(6)

where \( ||\frac{\partial d}{\partial \lambda_2}(\lambda_1, \lambda_2^{(j)})|| \) is the absolute value of the partial derivative of \( d(\lambda_1, \lambda_2) \) evaluated at each zero \((\lambda_1, \lambda_2^{(j)})\) of \( d(\lambda_1, \lambda_2) \). The summation on the right-hand-side of Eq. (6) includes all the zeros of the lightcone condition inside \([L_{21}, L_{2F}]\). Once the separation \( d(\lambda_1, \lambda_2) \) is completely symmetric on particle quantities, definition (6) has a symmetric definition as follows:–

**Definition 2:** We henceforth define the integral over \( \lambda_1 \in [L_{11}, L_{1F}] \) involving the Dirac delta-function composed with \( d(\lambda_1, \lambda_2) \) by
\[
\int_{L_{11}}^{L_{1F}} \delta(d(\lambda_1, \lambda_2)) f(\lambda, \lambda_2) d\lambda \equiv \sum_k \frac{f(\lambda_1^{(k)}, \lambda_2)}{||\frac{\partial d}{\partial \lambda_2}(\lambda_1^{(k)}, \lambda_2)||}.
\]

(7)

where \( ||\frac{\partial d}{\partial \lambda_2}(\lambda_1^{(k)}, \lambda_2)|| \) is the absolute value of the partial derivative of \( d(\lambda_1, \lambda_2) \) evaluated at each zero \((\lambda_1^{(k)}, \lambda_2)\) of \( d(\lambda_1, \lambda_2) \) for a fixed \( \lambda_2 \in [L_{21}, L_{2F}] \) and \( \lambda_1 \in [L_{11}, L_{1F}] \). For subluminal orbits the interval \([L_{kI}, L_{kF}]\) can include at the most the two zeros proved in Ref. [21], while for superluminal orbits there can be several zeros inside \([L_{kI}, L_{kF}]\), or even none. If no zero exists in the integration interval the right-hand side of either Eqs. (6) or (7) is defined to be zero. Definitions (6) and (7) are motivated by the evaluation of the respective left-hand sides of Eqs. (6) and (7) using the Dirac delta-function with a \( C^\infty \) separation \( d(\lambda_1, \lambda_2) \) and changing variables using the absolute value of the Jacobian of the local coordinate change near each zero. Here we avoid distributional operations with the Dirac delta-function and henceforth take Eqs. (6) and (7) as defining a functional of \( d(\lambda_1, \lambda_2) \), \( f(\lambda_1, \lambda_2) \) and the intervals \([L_{kI}, L_{kF}]\) into the reals. It is further useful to define the function \( A(\lambda_2) : L^4 \rightarrow L^4 \) by
\[
A_k(\lambda_2) \equiv \int_{L_{kI}}^{L_{kF}} \delta(|x_k(\lambda_2) - x_j(\lambda_2)|^2) \dot{x}_k(\lambda_2) d\lambda_2,
\]

(8)

where \((k, j) = (1, 2) \) or \((2, 1) \) and the integration on the right-hand side of (8) is defined either by Eq. (6) or Eq. (7). The vector function defined by Eq. (8) is often called the vector-potential in physics. Assuming integral (8) to exist for both \((k, j) = (1, 2) \) and \((2, 1) \), the interaction double-integral of the right-hand-side of Eq. (2) can be expressed by
\[
I = \int_{L_{ij}}^{L_{IF}} A_k(\lambda_2) \cdot \dot{x}_j d\lambda_2,
\]

(9)

with either \((k, j) = (1, 2) \) or \((2, 1) \). Using either definition (6) or (7) we can express the interaction term (9) in the two equivalent forms
\[
I = \int_{L_{ij}}^{L_{IF}} d\lambda_2 \sum_k \frac{\dot{x}_1(\lambda_1^{(k)}) \cdot \dot{x}_2(\lambda_2)}{||\frac{\partial d}{\partial \lambda_2}(\lambda_1^{(k)}, \lambda_2)||}
\]

(10)

It is instructive to check the equivalence of formulas (10) by changing the integration variable from \( \lambda_2 \) to \( \lambda_1 \) about each zero. Condition (4) defines \( \lambda_1 \) as a function of \( \lambda_2 \) by the the implicit function theorem and the Jacobian of the coordinate change transforms the first line of Eq.(10) into the second line of Eq.(10). Last, to express the Jacobian in the usual form of physics textbooks we define
\[
J_{\lambda_2}^\pm \equiv -\frac{1}{2} \frac{\partial d}{\partial \lambda_2}(\lambda_1^\pm, \lambda_2^\mp)
\]

(11)

For subluminal orbits \( \dot{x}_2(\lambda_2^\pm) \) is a time-like vector with a positive time-velocity, and once \( x_1(\lambda_1^\pm) - x_2(\lambda_2^\pm) \) is a null-vector, condition 11 defines a positive \( J_{\lambda_2}^+ \) on the retarded lightcone and a negative \( J_{\lambda_2}^- \) on the advanced cone. For superluminal orbits \( J_{\lambda_2}^\pm \) can have any sign in either lightcone, so that it is best to keep the moduli in the denominators of (6).

**ACTION WITH BOUNDS**

The guiding principle to construct an action functional is that the extremum condition should generate the electromagnetic equations of motion[1-3]. In the following we start from the naive Fokker action (2) and explain
how to restrict the integration to suitable finite segments of trajectory, using particle-time parametrization just for simplicity of the exposition. The original works [1–3] extended the integration of (2) from plus to minus infinity as a simple solution to include the needed future or past of the other particle at endpoints. There is no reason to assume such integration should converge, so that the infinite integral (2) has a formal-only meaning[15]. Moreover, the Fokker action yields the electromagnetic equations of motion[1–3] only if the condition of extremum is enforced formally with trajectory variations of compact support. Here we avoid the shortcomings of a formal-only action and give instead a finite-valued functional. The boundary conditions can be restricted to a point and a segment along each trajectory in a way that the future and the past lightcone points exist everywhere along both trajectories, as follows: Let the initial point of trajectory 1 be point \( O_A \) at \( t_1 = 0 \) and the endpoint of trajectory 2 be point \( L_B \) at \( t_2 = T_2 \) as illustrated in Fig. 3.1. The trajectory of particle 1 to be varied extends from \( O_A \) to point \( L^- \) at \( t_1 = T_1 \) where trajectory 1 intersects the past lightcone of \( L_B \) (indicated in green in Fig.1). The future history of particle 1 is needed from point \( L^- \) to point \( L^+ \) at \( t_1 = \Lambda^+_1 > T_1 \) where trajectory 1 intersects the advanced lightcone of \( L_B \) (the red portion of the upper trajectory of Fig. 3.1). The past history of particle 2 is needed from point \( O^- \) at \( t_2 = \Lambda^-_2 \) where trajectory 2 intersects the past lightcone of \( O_A \) up to point \( O^+ \) at \( t_2 = \Lambda^+_2 < T_2 \) where trajectory 2 intersects the future lightcone of \( O_A \) (also indicated in red in Fig.1). The trajectory of particle 2 to be varied goes from \( O^+ \) to \( L_B \). The combination of the initial point \( O_A \) along trajectory 1 and the final point \( L_B \) along trajectory 2, plus the respective segments of trajectory inside the lightcones of these endpoints is henceforth called exchange-of-history boundary conditions (EHBCs) as indicated in red in Fig. 3.1. Our construction is Lorentz-invariant because lightcones are Lorentz-invariant objects. The construction is unique up to a time-reversed construction using an endpoint along trajectory 1 and an initial point along trajectory 2 plus the history segments inside the respective lightcones. For \( C^1 \) orbits the EHBCs complete the trajectories in such a way that any point along each trajectory has the two lightcone roots inside the evaluation interval for either one of the interaction formulas (10).

The restrictions for the EHBCs histories are; (a) it must be possible to travel from the initial point to the final point of each trajectory at a speed lesser (or equal at the most) than light, and (b) The \textit{minimally short} condition that trajectory 1 must intersect the future lightcone of \( O^+ \) before arriving at endpoint \( L^- \) (at time \( t_1 = T_1 \)). In this way the past history of particle 2 does not interact with the future history of particle 1. Beyond that the variational method can be postulated with otherwise arbitrary histories. The advantage of solving the state-dependent delay equations using the variational method with the EHBCs is the numerical stability: Once the equations of motion are time-reversible the stable and unstable manifolds exist in pairs, so that tying both ends down with the EHBCs avoids the orbit to diverge either in the future direction along the unstable manifold or in the past direction along the stable manifold. Since the maximum spatial velocity is \( c = 1 \), the spatial position of particle 1 is bounded by a sphere of radius \( T_1 \) centered at point \( O_A \), while the spatial position of particle 2 is bounded by a sphere of radius \((T_2 - \Lambda^+_2)\) centered at \( O^+ \). Therefore the subluminal trajectories satisfying the EHBCs are spatially bounded and there are no runaway orbits satisfying the EHBCs during the optimization. The interaction formula of Eq. (10) needs the position in lightcone along the other trajectory, which is naturally approximated numerically using the trapezoidal rule with an integration grid consisting of the union of sewing chains defined as follows:—(i) A forward sewing chain is a set of consecutive points in lightcone starting from an arbitrary point on the boundary segment from \( O_- \) to \( O_+ \) (as illustrated in Fig. 4.1). The chain goes up to the corresponding point in future lightcone along trajectory 1 and back down and up until the last point along the boundary segment of trajectory 1 from \( L^- \) to \( L^+ \) and (ii) A backward sewing chain is a set of consecutive points in lightcone starting from any point on the boundary segment from \( L^- \) and \( L^+ \) of the trajectory 1 (as illustrated in Fig. 4.1). The sewing chain proceeds to the corresponding point in past lightcone along trajectory 2 and back down and up until the last backward point on the boundary segment from \( O_- \) to \( O_+ \). It is important to include in the sewing grid the forward chain starting from \( O_A \) and the backward chain starting from \( L_B \) because these chains separate boundary data from orbital data. Notice that a sewing chain starts from a point along one trajectory and ends with a point along the other trajectory, so that each chain defines the same number of points along each orbit.

For arbitrary trajectory variations satisfying the EHBCs the linearized functional variation is a sum of the linear variations along the two special cases;—(i) one fixes trajectory 2 while varying trajectory 1 arbitrarily and (ii) one fixes trajectory 1 while varying trajectory 2 arbitrarily, so that it suffices to study problems (i) and (ii). In the following we study (i) using particle-time parametrization, for which we integrate over \( t_2 \) in the double integral of action (2) with the help of Eq. (6). The half-Jacobian needed for Eq. (6) is a case of Eq. (11) with the choice of parameter \( \lambda_2 = t_2 \), i.e.,

\[
J_{t_2} \equiv [\dot{x}_1(t_1) - \dot{x}_2(t_2)] \cdot \dot{x}_2(t_2). \tag{12}
\]

The dot over \( x_2 \) in Eq. (12) denotes derivative respect to particle-time. Using (6) to integrate over \( \lambda_2 = t_2 \) inside the double-integral on the right-hand side of Eq.
(2) yields

\[ S = \int_{0}^{T_1} -m_1 \sqrt{\dot{x}_1 \cdot \dot{x}_1} dt_1 + \int_{\Lambda_2^+}^{T_2} -m_2 \sqrt{\dot{x}_2 \cdot \dot{x}_2} dt_2 + \int_{0}^{T_1} \frac{\dot{x}_1 \cdot \dot{x}_2 \cdot x_1}{2||J_t^x||} dt_1 + \int_{0}^{T_1} \frac{\dot{x}_1 \cdot \dot{x}_2 \cdot x_2}{2||J_t^x||} dt_1, \]

(13)

where the superscripts \( \pm \) on \( J_t^x \) indicate evaluation on the advanced/retarded lightcone points. Notice that action (13) is defined only for subluminal and luminal orbits. To evaluate the functional derivative of \( S \) with respect to variations of trajectory 1 we can drop the last term of the first line on the right-hand side of Eq. (13), which is independent of trajectory 1. Last, the integration over the future history of particle 1 extending from \( t_1 = T_1 \) to \( t_1 = \Lambda_1^+ \) is left invariant by a variation of trajectory 1 respecting the EHBCs, so that we can replace the upper limit of the last integral on the right-hand side of Eq. (13) by \( T_1 \), yielding an integration over \( t_1 \in [0, T_1] \) of a Lagrangian function \( L_1(x_1, \dot{x}_1) \) defined as,

\[ S_1 = \int_{0}^{T_1} L_1(x_1, \dot{x}_1) dt_1 \]

(14)

In Eq. (14) the advanced/retarded position and velocity of particle 2, indicated by \( \pm \), are evaluated with the fixed trajectory of particle 2 at the advanced/retarded points defined by the roots \( t_2^\pm (t_1) \) of Eq. (4), which are implicit functions of the updated trajectory of particle 1. Notice that even though the trajectory of particle 2 is fixed, the corresponding lightcone points move away from \( t_2^\pm \) along the fixed trajectory 2 as we vary the trajectory 1. The gradient of \( t_2^\pm \) respect to the four-position \( x_1 \) is obtained relating the differential \( dx_1 \) along trajectory 1 to the differential \( dt_2^\pm \) via the derivative of the implicit condition (4), i.e.,

\[ -2J_t^x dt_2^x + 2x_1^x \cdot dx_1 = 0, \]

(15)

where \( J_t^x \) is defined by Eq. (12) and \( x_1^x \equiv x_1(t_1) - x_2(t_2^x) \). Therefore the derivative of \( t_2^x \) with respect to \( x_1 \) along the fixed orbit of particle 2 is

\[ \frac{\partial t_2^x}{\partial x_1} = \frac{x_1^x}{J_t^x}. \]

(16)

Next we construct a linear space consisting of the \( C^1 \) orbital neighborhood of any \( C^1 \) subluminal orbit \( x_1 \) satisfying the EHBCs. Without loss of generality we operate with perturbations of trajectory 1 only, defined as type (i) in the paragraph above Eq. (12). Definition 3: For a \( C^1 \) subluminal orbit \( x_1 \) satisfying the EHBCs we define the linear space \( N^{(1)}(x_1) \) as the set of all \( C^1 \) trajectories defined by a perturbation function \( b_1 : [0, \lambda_{1F}] \rightarrow L^4 \), i.e.,

\[ u_1 \equiv x_1 + b_1, \]

(17)

\[ \dot{u}_1 \equiv \dot{x}_1 + \dot{b}_1, \]

where \( b_1 \) vanishes at the endpoints in accordance with the EHBCs, i.e.,

\[ b_1(\lambda_1 = 0) \equiv 0, \]

(18)

\[ b_1(\lambda_1 = \lambda_{1F}) \equiv 0. \]

Notice that the EHBCs forbid orbital perturbations \( b_1(\lambda_1) \) with a monotonically increasing time-component because condition (18) is impossible for a monotonically increasing time-component. The usual norm for the space of \( C^1 \) functions is given by \( || \equiv \sup ||b_1||_4 + \sup ||\dot{b}_1||_4 \) and because of conditions (18) it turns out that \( \sup ||b_1||_4 \leq |\lambda_{1F}| \sup ||\dot{b}_1||_4 \) for \( 0 < \lambda_1 < \lambda_{1F} \) as can be shown using either one of conditions (18). For example using \( b_1(\lambda_1 = \lambda_{1F}) = 0 \) we have

\[ b_1 = -\frac{\lambda_{1F}}{\lambda_1} \int_{\lambda_1}^{\lambda_{1F}} \dot{b}_1(\lambda) d\lambda, \]

(19)

so that \( \sup ||b_1||_4 \leq |\lambda_{1F}| \sup ||\dot{b}_1||_4 \) for \( 0 < \lambda_1 < \lambda_{1F} \). Therefore we can drop the sup \( ||b_1||_4 \) term of the norm and henceforth our norm is simply defined by the sup of the Euclidean \( \mathbb{R}^4 \) norm of \( \dot{b}_1 \), i.e., \( b_1|_{N(x_1)} \equiv \sup ||\dot{b}_1||_4 \).

Notice that whenever \( \dot{b}_1 = 0 \) the condition \( \sup \|\dot{b}_1\|_4 = 0 \) plus the endpoint condition (18) selects the single constant element \( b_1 = 0 \), so that \( \sup ||b_1||_4 \) defines a norm on the linear space of \( C^1 \) functions \( b_1 : [0, \lambda_{1F}] \rightarrow L^4 \) satisfying the EHBCs. The linear space \( N^{(1)}(x_1) \) can be shown to be a Banach space with this norm in the usual way. Proposition 1: Subluminal orbits \( (t_1(\lambda_1), \vec{r}_1(\lambda_1)) \) have small neighborhoods in \( N^{(1)}(x_1) \) containing only subluminal orbits. To show it we define the local Cartesian velocity respect to particle-time by

\[ \vec{v}(\lambda_1) \equiv \frac{(d\vec{r}_1/d\lambda_1)}{(dt_1/d\lambda_1)}. \]

(20)

a three-vector function of \( \lambda_1 \) with Euclidean norm lesser than one by condition (1). Along a subluminal orbit condition (1) is positive on the compact set \([0, \lambda_{1F}]\), so that \((1-||\vec{v}(\lambda_1)||^2) > 0 \) on \([0, \lambda_{1F}]\). For any orbit \( x_1(\lambda_1) \) satisfying Eq.(1) we can further define \( h_1 \equiv (dt_1/d\lambda_1) > 0 \) and express the velocity \( \dot{x}_1 \) by

\[ \dot{x}_1 = (h_1, h_1 \vec{v}_1). \]

(21)
Given a perturbation $\dot{b}_1 \equiv (\dot{b}, b\dot{b}_2)$ and a subluminal orbit $\dot{x}_1 = (h_1, h_1\dot{v}_1)$, substitution of (17) into (1) yields

$$ \dot{(x_1 + b_1)^2} = (h_1 + b)^2 - ||h_1\dot{v}_1 + b\dot{v}_2||^2 = h_1^2(1 - ||\dot{v}_1||^2) + b^2(1 - ||\dot{v}_2||^2) + 2h_1b(1 - \dot{v}_1 \cdot \dot{v}_2), $$

(22)

Since the positivity of Eq. (22) is independent of monotonic reparametrizations, in the following we use parametrization by the time-component of $x_1$, so that $h_1 = 1$. The norm $|b_1|_{N(x_1)}$ dominates the absolute value of the time-velocity perturbation $\dot{b}$ defined above Eq. (22), i.e., $|b_1|_{N(x_1)} \geq |b|$, so that one can limit $|b|$ by choosing $\delta/4 > |b_1|_{N(x_1)} \geq |b|$. Equation (22) with $\delta/4 > |b|$ and $h_1 = 1$ shows that the perturbed element is subluminal for small enough $|b_1|_{N(x_1)}$, so that subluminal orbits have small neighborhoods containing only subluminal orbits.

Next we define the Frechét derivative of action (13) about a subluminal orbit $x_1$: Let $S_1(b_1, \dot{b}_1) : N^{(1)}(x_1) \to \mathbb{R}$ be defined by substituting (17) into Eq. (14) and expanding to linear order for small $|b_1|_{N(x_1)}$. The linear expansion of $S_1(b_1, \dot{b}_1)$ in terms of $b_1$ and $\dot{b}_1$ involves integrals controlled by an $O(|b_1|^2_{N(x_1)})$ error in the above defined subluminal neighborhood $|b_1|_{N(x_1)} < \delta/4$ because the Euclidean norm $|b_1|_{4}$ is also bounded by $|\lambda_{1F}| \sup |\dot{b}_1|_4$ as explained above Eq. (19). The linear expansion of $S_1$ is already the desired Frechét derivative, i.e.,

$$ \delta S_1 = \int_0^{\lambda_{1F}} \frac{\partial L_1}{\partial x_1} \cdot b_1 + \frac{\partial L_1}{\partial \dot{x}_1} \cdot \dot{b}_1 \, d\lambda_1, $$

(23)

Even though the functional is already Frechét differentiable in $N^{(1)}(x_1)$, the electromagnetic equations require at least a $C^2$ orbit, as follows: For a $C^2$ orbit $x_1$ the second term on the right-hand side of Eq. (23) can be further integrated by parts using (18) to yield a term linear in $b_1$, so that $\delta S_1$ becomes

$$ \delta S_1 = \int G_1 \cdot b_1 \, d\lambda_1, $$

(24)

with

$$ G_1 \equiv -\frac{d}{d\lambda_1}\left(\frac{\partial L_1}{\partial \dot{x}_1}\right) + \frac{\partial L_1}{\partial x_1}, $$

(25)

where $L_1$ is defined by Eq. (14) and $G_1 \in L^1$ is defined only along any $C^2$ orbit of the natural neighborhood $N^{(2)}(x_1)$. Notice that Eq. (14) is independent of the parametrization and the expression of $L_1$ in terms of $\lambda_1$ is obtained simply by replacing $t_1$ with $\lambda_1$ in Eq. (14). Expressing Eq. (14) as a function of $\lambda_1$ and evaluating $G_1$ with Eq. (25) yields

$$ G_1 = \frac{d}{d\lambda_1}\left(m_1 \frac{\dot{x}_1}{\sqrt{\dot{x}_1 \cdot \dot{x}_1}} - \frac{\dot{x}_{2+}}{2||J_{\lambda_1}||} - \frac{\dot{x}_{2-}}{2||J_{\lambda_2}||}\right) + \frac{\partial}{\partial x_1}\left(\frac{\dot{x}_1 \cdot \dot{x}_{2+}}{2||J_{\lambda_1}||} + \frac{\dot{x}_1 \cdot \dot{x}_{2-}}{2||J_{\lambda_2}||}\right), $$

(26)

where the dot over $x_i$ denotes derivative respect to $\lambda_i$ for $i = 1, 2$ and $\lambda_2$ is the arbitrary parameter of trajectory 2. The condition for an extremum that follows from Eq. (24) is $G_1 = 0$ (plus the symmetric condition $G_2 = 0$ obtained by varying trajectory 2). Notice that $G_1$ must be zero only in the open interval $(0, \lambda_{1F})$ because the integrand of Eq. (24) vanishes at the boundaries with $b_1$. To pass from Eq. (23) to Eq. (24) the vanishing perturbations at $O_A$ and $L_B$ were enough to get rid of the boundary terms. The perturbations of velocity and acceleration are arbitrary at $O_A$ and $L_B$ because there is no prescribed orbit either before $O_A$ or after $L_B$, while the velocity and acceleration perturbations at $L^-$ and $O^+$ must vanish for a $C^2$ match with the histories. The condition $G_1 = 0$ defined by Eq. (26) yields the electromagnetic equations of motion with the Liénard-Wiechert-Lorentz force [3], as evaluated in the Appendix. The gradient for variations of trajectory 2 is obtained analogously, by discarding the integration over the past history of particle 2 and defining a sub-functional $S_2(b_2, \dot{b}_2)$ obtained from the above $S_1$ (14) by exchanging particle indices. The Banach space for arbitrary $C^2$ variations of both trajectories respecting the EHBCs is the direct product $N^{(2)}(x_1) \otimes N^{(2)}(x_2)$ with the norm given by $|b_1, b_2|_{N(x_1, x_2)} \equiv \sup |b_1|_4 + \sup |b_2|_4 + \sup |\dot{b}_1|_4 + \sup |\dot{b}_2|_4$, which is the natural physical space of orbits satisfying the EHBCs.

Action (13) is not defined for the superluminal elements of $N^{(1)}(x_1)$ (which have a large norm $|b_1|_{N(x_1)}$) because it involves taking the square-root of a negative number. The above defined norm guarantees that sufficiently small neighborhoods of subluminal orbits contain only subluminal orbits (by Proposition 1), but the set of subluminal orbits is not closed because Cauchy sequences of subluminal orbits can converge to luminal orbits. Moreover, luminal orbits can have small neighborhoods containing superluminal orbits, for which again action (13) is not even defined. In the following we relax the parametrization-invariance and construct a second Poincaré-invariant functional defined everywhere in $N^{(1)}(x_1)$ and yielding the same electromagnetic equations of motion. For superluminal trajectories the lightcone condition (4) can have an arbitrary number of zeros, and for these the double integration on the right-hand side of Eq. (2) is generalized by extending formula (6) to all zeros of (4) in the integration interval $(0, \lambda_{1F})$, which prescribes a vanishing integral in the case of no solution in the interval. Our second functional is obtained by fur-
ther generalizing the kinetic terms, i.e.,
\[
\Omega \equiv -\frac{\lambda_1^F}{2p} \frac{m_1}{0} \left( \dot{x}_1 \cdot \dot{x}_1 \right)^p d\lambda_1 - \frac{\lambda_2^F}{2p} \frac{m_2}{0} \left( \dot{x}_2 \cdot \dot{x}_2 \right)^p d\lambda_2
\]
\[
+ \frac{\lambda_3^F}{2} \frac{A(x_j) \cdot \dot{x}_j d\lambda_j.}{0}
\]
(27)

The last term of action (27) is the double integral of action (13) written in a convenient form and extended to arbitrary orbits by evaluating \(A(x_j)\) with Eq. (8) extended to all the zeros of the lightcone condition inside \([0, \lambda_1^F]\). Notice that the last term of (27) is still parametrization-independent, unlike the generalized kinetic terms of (27) that are parametrization-invariant only if \(p = 1/2\). The Euler-Lagrange condition of extremum (25) applied to action (27) yields
\[
m_i \frac{d}{d\lambda_i} \left[ (\dot{x}_i \cdot \dot{x}_i)^p \right] = \frac{4}{(2p - 1)} \frac{d}{d\lambda_i} \left[ (\dot{x}_i \cdot \dot{x}_i)^p \right] = 0.
\]
(29)

The Fokker-like action (13) has the form of the right-hand side of (6) and (7). The following proposition justifies the formal manipulation of the \(\delta\) symbol inside integration-by-parts formulas as long as the integrand vanishes at the endpoints of the integration interval. To motivate our next definition we start from formulas (6) and (7) with trajectories given by a perturbed circular orbit, i.e., \(d(\lambda_1, \lambda_2, \varepsilon) = d_0(\lambda_1, \lambda_2) + \varepsilon u(\lambda_1, \lambda_2, \varepsilon)\) with \(u(\lambda_1, \lambda_2, \varepsilon)\) given by a polynomial function of the \(b_1(\lambda_1)\) vanishing at the endpoints of \([L_{21}, L_{2F}]\) according to (18). Definition 1 yields
\[
\int_{L_{2F}}^{L_{21}} \left( \delta(d(\lambda_1, \lambda_2, \varepsilon)) f(\lambda_1, \lambda_2) d\lambda
\]
\[
\equiv \sum_{\chi_2^{(j)}} \frac{f(\lambda_1, \chi_2^{(j)})}{\|2J_{2}\|},
\]
where \(2J_{2} \equiv -\frac{df}{d\lambda_2}(\lambda_1, \chi_2^{(j)}, \varepsilon)\) and the summation of Eq. 22 is extended to all zeros \((\lambda_1, \chi_2^{(j)})\) of \(d(\lambda_1, \lambda_2, \varepsilon)\) with \(\chi_2^{(j)} \in [L_{21}, L_{2F}]\) for any fixed \(\lambda_2 \in [L_{11}, L_{1F}]\). The condition \(u(\lambda_1, \lambda_2, \varepsilon) = 0\) at the endpoints ensures that the lightcone condition is not perturbed at the endpoints, so that no zero \(\chi_2^{(j)}\) of \(d(\lambda_1, \lambda, \varepsilon)\) leaves or enters the interval \([L_{21}, L_{2F}]\) for small \(\varepsilon\). The implicit function theorem for \(d(\lambda_1, \lambda, \varepsilon) = 0\) defines \(\chi_2^{(j)}\) as a function of \(\varepsilon\) with derivative
\[
\frac{\partial \chi_2^{(j)}}{\partial \varepsilon} = -\frac{\frac{df}{d\lambda_2}(\lambda_1, \chi_2^{(j)}, \varepsilon)}{\frac{df}{d\lambda_2}(\lambda_1, \chi_2^{(j)}, \varepsilon)}.
\]
(31)

The derivative of the right-hand-side of Eq. (30) respect to \(\varepsilon\) can be expressed with the help of (31) in the form
\[
\sum_{\chi_2^{(j)}} \frac{\partial \chi_2^{(j)}}{\partial \varepsilon} \frac{df}{d\lambda_2}(\lambda_1, \lambda_2) d_\varepsilon / 2J_{2}\]
\[
\equiv \sum_{\chi_2^{(j)}} \frac{df}{d\lambda_2}(\lambda_1, \lambda_2) \frac{d_\varepsilon}{2J_{2}},
\]
(32)
where \(2J_{2} \equiv -\frac{df}{d\lambda_2}(\lambda_1, \lambda_2, \varepsilon)\) and \(d_\varepsilon \equiv \frac{df}{d\varepsilon}(\lambda_1, \lambda_2, \varepsilon)\). Equation (32) is formula (6) with \(f(\lambda_1, \lambda_2)\) replaced by \(\partial \lambda(f(\lambda_1, \lambda_2)) d_\varepsilon / 2J_{2}\), an equality that justifies the use
of a formal derivative of the delta-function symbol as follows

$$\frac{d}{d\varepsilon} \int_{L_{2}}^{L_{2}^{P}} \delta(d(\lambda_{1}, \lambda, \varepsilon)) f(\lambda_{1}, \lambda) d\lambda$$

$$= \int_{L_{2}}^{L_{2}^{P}} \delta'(d(\lambda_{1}, \lambda, \varepsilon)) f(\lambda_{1}, \lambda) d\lambda$$

$$= \int_{L_{2}}^{L_{2}^{P}} \partial_{\lambda} f(\lambda_{1}, \lambda) d_{\varepsilon}/2J_{\lambda_{1}} \delta(d(\lambda_{1}, \lambda, \varepsilon)) d\lambda$$

where again $2J_{\lambda_{2}} \equiv -\frac{\partial d}{\partial \lambda_{2}}(\lambda_{1}, \lambda_{2}, \varepsilon)$ and $d_{\varepsilon} \equiv \frac{\partial d}{\partial \varepsilon}(\lambda_{1}, \lambda_{2}, \varepsilon)$ vanishes at the integration limits. We henceforth use (33) to define the formal derivatives of the delta symbol, stressing that there is no distributional limit involved but rather the above-defined operation. Moreover, actions (27) and (13) depend on a double integral, i.e., either one of formulas (10). The derivative of the interaction $I$ defined by Eq. (30) with an $\varepsilon$-dependent $d(\lambda_{1}, \lambda_{2}, \varepsilon)$ is given by either one of formulas

$$\frac{\partial I}{\partial \varepsilon} = \int_{L_{2}}^{L_{2}^{P}} d\lambda_{2} \sum_{k} \frac{\partial_{\lambda_{1}}[\langle \dot{x}_{1} \cdot \dot{x}_{2} \rangle d_{\varepsilon}/2J_{\lambda_{1}}]}{\|2J_{\lambda_{1}}\|} \hat{\lambda}_{1}^{(k)}$$

$$\int_{L_{1}}^{L_{1}^{P}} d\lambda_{1} \sum_{j} \frac{\partial_{\lambda_{2}}[\langle \dot{x}_{1} \cdot \dot{x}_{2} \rangle d_{\varepsilon}/2J_{\lambda_{2}}]}{\|2J_{\lambda_{2}}\|} \hat{\lambda}_{2}^{(j)}$$

with $2J_{\lambda_{k}} \equiv -\frac{\partial d}{\partial \lambda_{k}}(\lambda_{1}, \lambda_{2}, \varepsilon)$, as long as $d_{\varepsilon} \equiv \frac{\partial d}{\partial \varepsilon}(\lambda_{1}, \lambda_{2}, \varepsilon) = 0$ at the integration limits. Otherwise we might have to chose the line of Eq. (34) for which $\frac{\partial d}{\partial \varepsilon}(\lambda_{1}, \lambda_{2}, \varepsilon)$ vanishes at the integration limits.

For the second variation we vary both trajectories simultaneously according to

$$x_{1} = x_{1}^{c} + \varepsilon b_{1}, \quad x_{2} = x_{2}^{c} + \varepsilon b_{2},$$

$$\dot{x}_{1} = \dot{x}_{1}^{c} + \varepsilon \dot{b}_{1}, \quad \dot{x}_{2} = \dot{x}_{2}^{c} + \varepsilon \dot{b}_{2},$$

$\varepsilon \in [0, 1]$. We require vanishing perturbations at the endpoints,

$$b_{1}(O_{A}) = b_{1}(L_{-}) = 0,$$

$$b_{2}(O_{+}) = b_{2}(L_{B}) = 0,$$

and vanishing velocity and acceleration perturbations on the history side of each trajectory, i.e., at point $O_{+}$ of trajectory 2 and at point $L^{\prime}$ of trajectory 1,

$$\dot{b}_{1}(L^{\prime}) = \ddot{b}_{2}(L^{\prime}) = 0,$$

$$\dot{b}_{2}(O_{+}) = \ddot{b}_{2}(O_{+}) = 0,$$

so that the trajectories can be continued to a $C^{2}$ trajectory $b_{1} = 0$ on the boundary segment $(L^{-}, L^{+})$ and $b_{2} = 0$ on the boundary segment $(O^{-}, O^{+})$. The quadratic integrand of the Taylor expansion involves products of variations at points connected by the lightcone condition (rather than variations at the same time as in the Kepler problem). We expand the action in a Taylor series up to the second order in $\varepsilon$ by using a directional derivative along the $C^{2}$ trajectory variation (35) of $N^{(2)}(x_{1}^{c}, x_{2}^{c})$. Once the circular orbit is an extremum, the first variation vanishes so that Taylor’s theorem gives the functional at $\varepsilon = 1$ as a sum of its value at $\varepsilon = 0$ plus the second-variation evaluated at some $\varepsilon \in [0, 1]$.

The second variation of the first term on the right-hand-side of Eq. (13), representing the kinetic energy is

$$\Delta^{(2)} K_{1} = m_{1} \varepsilon^{2} \int_{O_{A}}^{L^{+}} d\lambda_{1} \left\{ \frac{\langle \dot{x}_{1}^{c} \cdot \dot{b}_{1} - \dot{x}_{2}^{c} \cdot \dot{b}_{2} \rangle^{2}}{2 \langle \dot{x}_{1}^{c} \rangle^{3/2}} \right\}.$$  

Formula (38) is positive-definite, which is seen as follows:—If $\dot{b}_{1}$ is time-like, the positivity is given by the reverse-Schwarz inequality of time-like vectors mentioned in the introduction, while for a space-like $\dot{b}_{1}$ Eq. (38) is a sum of positive terms. Since the interaction integral is naturally expressed as a double integral times the Dirac delta-function we henceforth normalize all integrals to that form. To normalize Eq. (38) we simply add a dummy integration over $d\lambda_{2}$ multiplied by the integrating factor $2\|J_{x_{2}}\|d\lambda_{2}$ and use that $\int_{O^{-}}^{L^{+}} 2\|J_{x_{2}}\|d\lambda_{2} = 1$, yielding

$$\Delta^{(2)} K_{1} = m_{1} \varepsilon^{2} \int_{O_{A}}^{L^{+}} \int_{O_{A}}^{L^{+}} d\lambda_{1} d\lambda_{2} \left\{ \frac{\langle \dot{x}_{1}^{c} \cdot \dot{b}_{1} - \dot{x}_{2}^{c} \cdot \dot{b}_{2} \rangle^{2}}{\langle \dot{x}_{1}^{c} \rangle^{3/2}} \right\} \|J_{x_{2}}\| \delta_{D}.$$  

(39)

The symbol $\delta_{D}$ is an abbreviation for $\delta(d(\lambda_{1}, \lambda_{2}, \varepsilon))$ as of definitions (6) and (7) while upper index $c$ denotes the circular-orbit functions. Notice that the low-velocity-limit of $\|J_{x_{2}}\|$ in particle-time parametrization is the spatial separation in light-cone, $r_{12}$, as defined by Eq. (12). To abbreviate notation we henceforth indicate the double-integral over both circular orbits of any integrand $g(\lambda_{1}, \lambda_{2}, \varepsilon)$ times $\delta(d(\lambda_{1}, \lambda_{2}, \varepsilon))$ by $\int_{c} g$ . For example the kinetic term Eq. (39) is abbreviated to

$$\Delta^{(2)} K_{1} = m_{1} \varepsilon^{2} \int_{C} \int_{O_{A}}^{L^{+}} \left\{ \frac{\langle \dot{x}_{1}^{c} \cdot \dot{b}_{1} - \dot{x}_{2}^{c} \cdot \dot{b}_{2} \rangle^{2}}{\langle \dot{x}_{1}^{c} \rangle^{3/2}} \right\} \|J_{x_{2}}\|.$$  

Next we calculate the second-variation of the interaction term by substituting variation (35) into the integrand $I_{F} \equiv \delta(\|x_{1} - x_{2}\|^{2}) \dot{x}_{1} \cdot \dot{x}_{2}$ and expand in a Taylor series in $\varepsilon$ using the above define rules for the formal derivative. The separation $d(\lambda_{1}, \lambda_{2}, \varepsilon)$ is perturbed along variation
so that the formal expansion of $\delta_D$ becomes

$$
\delta_D = \delta_D(|x_i^c - x_{i2}^c|^2) + 2\epsilon(x_i^c - x_{i2}^c) \cdot (b_1 - b_2) |b_1 - b_2|^2 \delta^\prime_D + 2|x_{i2}^c \cdot (b_1 - b_2)|^2 \delta''^D + O(3),
$$

(42)

where $x_{i2}^c \equiv (x_i^c - x_{i2}^c)$. The bilinear product $\hat{x}_1 \cdot \hat{x}_2$ is perturbed to

$$
\hat{x}_1 \cdot \hat{x}_2 = \hat{x}_i^c \cdot \hat{x}_{i2}^c + 2\epsilon(\hat{x}_i^c \cdot b_1 + \hat{x}_{i2}^c \cdot b_2) + \hat{b}_1 \cdot \hat{b}_2
$$

(43)

Henceforth one or two primes over $\delta_D$ denote respectively one or two formal derivatives as defined by formulas (33) and (34). The quadratic term of the Taylor expansion of $I_F \equiv \delta(|x_i^c - x_{i2}^c|) \hat{x}_1 \cdot \hat{x}_2$ is obtained multiplying (42) by (43) and collecting the second order terms, yielding

$$
\Delta^{(2)}I_F = \epsilon^2 \hat{b}_1 \cdot \hat{b}_2 \delta_D + \epsilon^2 \hat{x}_i^c \cdot \hat{x}_{i2}^c (b_1 - b_2) |b_1 - b_2|^2 \delta^\prime_D + 2\epsilon^2 |x_{i2}^c \cdot (b_1 - b_2)| \Delta(\hat{x}_1 \cdot \hat{x}_2) \delta^\prime_D + 2\epsilon^2 \hat{x}_i^c \cdot \hat{x}_{i2}^c |b_1 - b_2|^2 \delta''^D,
$$

(44)

where $\Delta(\hat{x}_1 \cdot \hat{x}_2) \equiv (\hat{x}_i^c \cdot \hat{b}_1 + \hat{x}_{i2}^c \cdot \hat{b}_1)$ and $x_{i2}^c \equiv (x_i^c - x_{i2}^c)$. The first term on the right-hand side of Eq. (44) is already in the normalized form of Eq. (39). We henceforth drop the $\epsilon^2$ factor of the second-order expansion. The second term on the right-hand side of Eq. (44) must be split in three monomials, $\hat{x}_i^c \cdot \hat{x}_{i2}^c (b_1^2 - 2b_1 \cdot b_2 + b_2^2) \delta_D$, and the formal integration by parts to get rid of the $\delta_D$ must treat each monomial differently cause formula (33) needs a vanishing perturbation at the endpoints; For example the monomial $\hat{x}_i^c \cdot \hat{x}_{i2}^c b_1^2 \delta_D$ must be dealt with according to the first line of (34), i.e.,

$$
\int_{O_A}^{L^+} \hat{x}_i^c \cdot \hat{x}_{i2}^c b_1^2 \delta_D d\lambda_1 = \int_{O_A}^{L^+} d\lambda_1 \delta_D \frac{\partial}{\partial \lambda_1} \left( \frac{\hat{x}_i^c \cdot \hat{x}_{i2}^c b_1^2}{2||J_{\lambda_1}||} \right)
$$

(45)

since the integrand on the left-hand side of Eq. (45) vanishes with $b_1^2$ at $L^+$ and $O_A$ (the EHBCs). Notice that the monomial with $b_2^2$ does not vanish at $L^+$ and $O_A$. In that case the integration of choice would be over $d\lambda_2$. Using the above term-wise integration, the second term of the first line on the right-hand side of Eq. (44) yields

$$
\int_C \frac{\partial}{\partial \lambda_1} \left( \frac{\hat{x}_i^c \cdot \hat{x}_{i2}^c (b_1 \cdot b_2 - b_1^2)}{2||J_{\lambda_1}||} \right) + \int_C \frac{\partial}{\partial \lambda_2} \left( \frac{\hat{x}_i^c \cdot \hat{x}_{i2}^c (b_1 \cdot b_2 - b_2^2)}{2||J_{\lambda_2}||} \right).
$$

(46)

Next integrating by parts on the second line of the right-hand-side of Eq. (44) yields

$$
- \int_C \frac{\partial}{\partial \lambda_1} \left( \frac{\Delta(\hat{x}_i^c \cdot \hat{x}_{i2}^c) (x_{i2}^c \cdot b_1)}{||J_{\lambda_1}||} \right) + \int_C \frac{\partial}{\partial \lambda_2} \left( \frac{\Delta(\hat{x}_i^c \cdot \hat{x}_{i2}^c) (x_{i2}^c \cdot b_2)}{||J_{\lambda_2}||} \right),
$$

(47)

where again $\Delta(\hat{x}_1 \cdot \hat{x}_2) \equiv (\hat{x}_1 \cdot \hat{b}_2 + \hat{x}_2 \cdot \hat{b}_1)$ and $x_{i2}^c \equiv (x_i^c - x_{i2}^c)$. Last, the third line of the right-hand-side of Eq. (44) is transformed after two integrations by parts into

$$
\frac{1}{2} \int_C \frac{\partial}{\partial \lambda_1} \frac{1}{||J_{\lambda_1}||} \frac{\partial}{\partial \lambda_1} \left( \frac{\hat{x}_i^c \cdot \hat{x}_{i2}^c (x_{i2}^c \cdot b_1)^2}{||J_{\lambda_1}||} \right) + \frac{1}{2} \int_C \frac{\partial}{\partial \lambda_2} \frac{1}{||J_{\lambda_2}||} \frac{\partial}{\partial \lambda_2} \left( \frac{\hat{x}_i^c \cdot \hat{x}_{i2}^c (x_{i2}^c \cdot b_2)^2}{||J_{\lambda_2}||} \right) - \int_C \frac{\partial}{\partial \lambda_1} \frac{1}{||J_{\lambda_1}||} \frac{\partial}{\partial \lambda_2} \left( \frac{\hat{x}_i^c \cdot \hat{x}_{i2}^c (x_{i2}^c \cdot b_2)(x_{i2}^c \cdot b_1)}{||J_{\lambda_1}||} \right).
$$

(48)

Henceforth we specify the circular orbit adopting particle-time parametrization, i.e., $b_i \equiv (0, b_i)$ and $\hat{b}_i \equiv (0, \hat{b}_i)$ and $\hat{b}_1 \cdot \hat{b}_j \equiv -b_1 \cdot b_j$, where a dot between the vector parts henceforth denotes Cartesian product. The velocities along a limiting circular orbit of large radius are given by $\hat{x}_i^c = (1, \vec{v}_i^c)$ with

$$
\vec{v}_1^c = \frac{m_2}{M \sqrt{r_{12}}} \hat{v}(t_1),
$$

$$
\vec{v}_2^c = -\frac{m_1}{M \sqrt{r_{12}}} \hat{v}(t_2).
$$

(49)

In Eq. (49) $r_{12}$ is the constant separation in lightcone along the circular orbit, $\hat{v}(t)$ is the unit vector along the trajectory of particle 1 and $M \equiv m_1 + m_2$ (the Kepler orbit is discussed in Ref. [11]). The period of the circular orbit is given by Kepler’s law

$$
T = 2\pi \sqrt{\frac{M}{m_1 m_2}} r_{12}^{3/2},
$$

(50)

so that the lightcone separation $t_1 = t_2 \pm r_{12}$ is a negligible fraction of the period for large $r_{12}$, i.e., the times in lightcone are almost equal ($t_1 \simeq t_2$), the spatial positions are almost in diametral opposition and the velocities have nearly opposite directions. Using the above
circular orbit we calculate $||J_{t_1}|| = ||J_{t_2}|| \simeq r_{12}$ and 
$\partial_{t_1}||J_{t_1}|| = \partial_{t_2}||J_{t_2}|| = \dot{\hat{x}}^1 \cdot \dot{\hat{x}}^2 \simeq 1$ in the limit of a large $r_{12}$.

Theorem: The second variation about circular orbits of large enough radius is a strongly-positive quadratic form for $C^2$ trajectory variations satisfying (36) and (37).

Proof:—There are three basic types of integrals of quadratic monomials in Eqs. (46), (47) and (48), namely (a) velocity-velocity, (b) position-position and (c) position-velocity. Notice that integrals of type $\int \delta_D(\mathbf{A} \cdot \hat{b}_n)(\mathbf{B} \cdot \hat{b}_n)$ can be re-expressed as an integral of a quadratic form of position and velocity variations only using (34). In the following we inspect each type of integral, finding that (a) is strongly-positive while (b) and (c) are dominated by (a) at large enough radii, as follows:—

(a) The velocity-velocity terms of the second-variation are

$$\Delta^{(2)}V = \int \left( m_1 r_{12} \hat{b}_1^2 + m_2 r_{12} \hat{b}_2^2 + \hat{b}_1 \cdot \hat{b}_2 \right), \quad (51)$$

which is strongly positive-definite at large separations, $m_1 r_{12} \gg 1$. (b) The dominant quadratic terms in the displacements are

$$\Delta^{(2)}R = \int \left( \frac{\dot{b}_1 - \dot{b}_2}{2r_{12}^2} + \frac{3(\hat{n} \cdot \dot{b}_1 - \hat{n} \cdot \dot{b}_2)^2}{2r_{12}^2} \right)$$

$$+ \int \frac{2(\hat{n} \cdot \dot{b}_1)(\hat{n} \cdot \dot{b}_2)}{r_{12}}$$

$$+ \int \left( \frac{\dot{\hat{v}}_1 \cdot \dot{b}_1}{r_{12}^2} + \frac{\dot{\hat{v}}_2 \cdot \dot{b}_1}{r_{12}^2} + \frac{\dot{\hat{v}}_1 \cdot \dot{b}_2}{r_{12}^2} + \frac{\dot{\hat{v}}_2 \cdot \dot{b}_2}{r_{12}^2} \right)$$

where $\hat{n} \equiv (\dot{b}_1 - \dot{b}_2)/r_{12}$. Quadratic form (52) is not positive-definite, and in fact for $\dot{b}_1 = \dot{b}_2 \equiv \delta \mathbf{R}||\hat{n}$ with $\hat{n} \cdot \hat{v} = 0$ we have $\Delta^{(2)}R = -2m_1 m_2 ||\delta \mathbf{R}||^2/(M r_{12}^2)$. The first two lines on the right-hand side of Eq. (52) have a non-negative sum, while we can show using (49) that the last line is bounded, i.e.,

$$\Delta^{(2)}R \geq -\frac{m_1 m_2}{M r_{12}^2} \int \left( ||\dot{\hat{b}}_1||^2 + ||\dot{\hat{b}}_2||^2 \right). \quad (53)$$

Lemma 1:—For variations $\hat{b}_i$ vanishing at $t_i = 0$ and $t_i = T_{\phi}$ it follows from the Fourier series $\hat{b}_i = \sum \hat{a}_k \sin(\pi k T_{\phi})$ that

$$\int \frac{||\dot{\hat{b}}_i||^2}{T_{\phi}^2} \geq \frac{\pi^2}{T_{\phi}^2} \int \frac{||\hat{b}_i||^2}{C}.$$ 

where $\hat{b}_i = d\hat{b}_i/dt_i$. In Eq. (54), $T_{\phi}$ is the time for the circular rotation to travel the angle $\phi$ from $O_A$ to $L_-$, i.e., $T_{\phi} = (\phi/2\pi)T$ where $T$ is the period as defined by Eq. (50). The equal sign in (54) holds iff the first Fourier mode alone is present, i.e., $\hat{a}_k = 0$ for $k \neq 1$.

The following inequality is true for arbitrary arcs of circle $\phi < 2\pi$ but for simplicity we write it for EHBCs going a complete turn, $T_{\phi} = T$, i.e.,

$$\int \frac{r_{12} m_1 ||\dot{\hat{b}}_1||^2}{C} \geq \frac{m_1 m_2 m_3}{4M r_{12}^2} \int \frac{||\hat{b}_i||^2}{C}. \quad (55)$$

Using Eqs. (55) and (53) we can show that an arbitrary fraction $0 < f < 1$ of the kinetic term (51) dominates the quadratic form (52) for sufficiently large $r_{12}$, i.e., $f \Delta^{(2)}V \geq \Delta^{(2)}R$.

(c) The quadratic terms involving position-velocity perturbations are also dominated by the kinetic terms, as follows:—Notice that the position-velocity terms coming from (46) integrate to zero, i.e.,

$$\frac{1}{2r_{12}^2} \int \left( \hat{b}_1 \cdot \dot{b}_2 - \hat{b}_2 \cdot \dot{b}_1 \right) + \frac{1}{2r_{12}^2} \int \left( \hat{b}_2 \cdot \dot{b}_1 - \hat{b}_1 \cdot \dot{b}_2 \right), \quad (56)$$

where we used the large-radius limits $||J_{t_1}|| = ||J_{t_2}|| \simeq r_{12}$ and $\hat{q}_1 \cdot \hat{q}_2 \simeq 1$ and moved $r_{12}$ outside of the integration sign because it is constant along circular orbits. After integration over one parameter Eq. (56) reduces to the integration of an exact differential vanishing at the boundaries, so that (56) vanishes. The largest non-vanishing position-velocity terms come from (47) and (48), i.e.,

$$\Delta^{(2)}VR = \int \frac{\hat{v}_1 \cdot \dot{\hat{v}}_2}{r_{12}^2} (\hat{n} \cdot \dot{b}_1 - \hat{n} \cdot \dot{b}_2)$$

$$- \int \frac{\hat{v}_2 \cdot \dot{\hat{v}}_1}{r_{12}^2} (\hat{n} \cdot \dot{b}_1 - \hat{n} \cdot \dot{b}_2),$$

where $\hat{n} \equiv (\dot{b}_1 - \dot{b}_2)/r_{12}$ and $\Delta(\hat{v}_1 \cdot \hat{v}_2) \equiv (\hat{v}_1 \cdot \dot{b}_2 + \hat{v}_2 \cdot \dot{b}_1)$. To show that the kinetic form (51) dominates the velocity-position quadratic terms for large enough $r_{12}$ we use inequality (55) to derive Lemma 2:

$$\int \frac{r_{12} (m_1 ||\dot{\hat{b}}_1||^2 + m_2 ||\dot{\hat{b}}_2||^2)}{C} \geq \int \left( \frac{r_{12} m_1 ||\dot{\hat{b}}_1||^2}{C} + \frac{m_1 m_2 m_3}{4M r_{12}^2} \int \frac{||\hat{b}_i||^2}{C} \right) \quad (58)$$

where the last inequality is simply the completion of a binomial square. It can be verified with Eq. (49).
that the coefficients of the monomials in the integrals of (57) are dominated by $1/r_{12}^{3/2}$, so that Lemma 2 as of (58) is enough for the kinetic terms to dominate all type (b) terms. To show that the second-variation is positive-definite we divide the kinetic energy (51) in three equal parts: The first third dominates the position-squared terms (52) for large enough $r_{12}$, as explained below Eq. (55), while the second third dominates the velocity-position terms by inequality (58). The last third is a non-degenerate positive-definite quadratic form of the velocities, so that the second variation about circular orbits of large enough radii is positive-definite, proving that circular orbits are local minima. Moreover, the last third-part of Eq. (51) has all positive eigenvalues, so that the second variation is strongly positive.

**CONCLUSION AND DISCUSSIONS**

An important question is the existence of an extremizing orbit for the functional (13) with arbitrary past data for particle 2 plus arbitrary future data for particle 1, i.e., the existence result for solutions of the mixed-type neutral-delay electromagnetic equations of motion with general boundaries. There are no existence results for the electromagnetic two-body problem apart from a few obtained for a one-dimensional motion with repulsive interaction [17], a qualitatively different and simpler case where the equations are not neutral but rather delay-only. For sufficiently small $C^2$ deformations of the circular EHBCs preserving the boundary lightcones $O_A - O^+$, $O_A - O^-$ and $L^- - L_B$ and $L^+ - L_B$, the second variation can be proved positive-definite with analogous methods. Moreover, on a subset $\Theta \subset N^{(2)}(x^1, x^2)$ of orbits satisfying $M \geq \sup(||b_1||_4) + \sup(||b_2||_4)$ for some $M$ we can reconstruct the $C^2$ perturbation using (18) and the one-sided conditions (37), a formula analogous to Eq. (19). For example for $b_1(\lambda_1)$ we have

$$b_1(\lambda_1) = \int \lambda_1 \frac{d\lambda_b}{\lambda_1} \int b_1(\lambda_a) d\lambda_a, \quad (59)$$

from which it follows that $\sup(||b_1^{(k)}||) \leq \lambda_{2F}^{2-k} \sup ||\tilde{b}_1||_4$, with an analogous condition holding for $b_2$ from the other side. Conditions (37) can be used to show that the $C^2$ perturbations inside $\Theta$ are equicontinuous and uniformly bounded, so that by the Arzela-Ascoli theorem the set $\Theta$ is compact. If the second variation is positive-definite, the functional is bounded from below on the compact set $\Theta$ and assumes its minimum inside $\Theta$. We conjecture that this point of minimum is an interior point of the compact set. That granted, the minimum has a whole neighborhood inside $\Theta$, so that Eq. (24) holds for arbitrary $b_k$ and the gradients must vanish at the minimum. Conditions $G_k = 0$ with $G_k$ defined by Eq. (26) are the state-dependent neutral-delay equations of motion, so that this would be an existence result for the state-dependent neutral-delay equations. This result would be the analogous of the ”Kurzweil small delays don’t matter theorem” for global trajectories of DDE’s on compact sets[16]. The uniqueness theory also differs from the case of Ref. [21], and here one should again start from the case of slightly perturbed circular boundaries, a case where the equations of motion are approximated by neutral-delay-equations with constant advance and delay. For circular orbits of intermediate radius some inspection suggests the minimum should become a saddle in a bifurcation at a finite $O(1)$ radius in our unit system, i.e., of the order of the classical electronic radius.

The existence proof is much harder for the solenoidal orbits discussed in the appendix because of the denominators. For solenoidal orbits with a fast velocity the functional might have a maximum as suggested by the kinetic term.

A useful generalization of our second functional is for orbits defined on a Sobolev space $H^2_D$ with derivatives defined almost everywhere. For that we need to generalize the lightcone condition to arbitrary trajectories and to generalize Eq. (6) to a sum over all zeros of the lightcone condition. The fact that Eq. (6) is further integrated over the other orbital parameter to make Eq. (10) compensates for the extra zeros gained by changing the trajectories on a set of zero measure, so that the functional can be defined on $H^2_D$. This generalization could be useful in proving existence for the case of general boundaries.

Another question of interest regards the search for periodic orbits and the possibility to restrict the variational method to the family of periodic orbits satisfying the EHBCs. The reduction is possible to a sub-family of periodic orbits by identifying the spatial components of $O_A$ with those of $L^+$ for trajectory 1 and the spatial components of $O_-$ with those of $L^B$ for trajectory 2, which must be the case along a periodic orbit. The orbital variation inside the family of periodic orbits must preserve the history segment of each trajectory, as illustrated in red in Fig. 3.1, which is a sub-family of the family of periodic orbits. Last, it is possible to extremize the functionals directly in the space of $C^1$ orbits without even respecting the former sub-family conditions. The conditions for an extremum with these most general variations are no longer the electromagnetic equations of motion but rather the overdetermined equations obtained by vanishing both linear terms on the right-hand side of Eq. (23) separately.

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APPENDIX: PHYSICS OF THE FOKKER

ACTION

Here we evaluate the gradient (26) explicitly and discuss the physics of the two-body problem using proper-time parametrization for the trajectories. The velocity respect to proper-time can be expressed either in the form (21) with \( h_1 = \gamma_i \), i.e.,

\[
v_i = (\gamma_i, \gamma_i \tilde{v}_i), \tag{60}
\]
or in the form

\[
v_i = (\gamma_i, \frac{d\tilde{r}_i}{d\tau_i}), \tag{61}
\]

where

\[
\gamma_i \equiv \frac{dt_i}{d\tau_i} > 0. \tag{62}
\]

According to Eq.(1) the velocity respect to proper-time along physical orbits, \( v_i \equiv d\vec{x}_i/d\tau_i \), satisfies

\[
(v_i \cdot v_i) = 1, \tag{63}
\]

which can be solved for \( \gamma_i \) using either Eq.(61) or Eq.(60), yielding

\[
\gamma_i = \sqrt{1 + \| \frac{d\tilde{r}_i}{d\tau_i} \|^2} = (1 - \| \tilde{v}_i \|^2)^{-1/2}. \tag{64}
\]

Notice that \( \| \frac{d\tilde{r}_i}{d\tau_i} \| \) is unbounded and becomes arbitrarily large when the time-velocity approaches the speed of light. Using condition (3) to solve for the Euclidean norm of the spatial separation, we can express the separation vector \( \vec{x}_{12} \equiv (\vec{x}_1 - \vec{x}_2) \) as

\[
\vec{x}_{12} = (\mp r_{12}, r_{12} \hat{n}^\pm), \tag{65}
\]

where \( r_{12} \) is the distance in light-cone and \( \hat{n}^\pm \) is defined by

\[
\hat{n}^\pm \equiv \frac{\tilde{r}_1(t_1) - \tilde{r}_2(t_{2\pm})}{r_{12}^\pm}. \tag{66}
\]

a unitary Euclidean three-vector. Notice that the spatial distance in light-cone \( r_{12} \equiv \| \tilde{r}_2(t_{2\pm}) - \tilde{r}_1(t_1) \| \) is a different function for each light-cone. The time-component of \( \vec{x}_{12} \equiv (\vec{x}_1 - \vec{x}_2) \) is simply \( t_1 - t_2 \) and evaluated with the negative sign of Eq.(5) (for the retarded cone) yields the positive number \( t_1 - t_2 = r_{12} = \| \tilde{r}_2(t_{2\pm}) - \tilde{r}_1(t_1) \| \). The same Eq.(5) with the plus sign (for the advanced cone) yields the negative number \( t_1 - t_2 = -r_{12} = -\| \tilde{r}_2(t_{2\pm}) - \tilde{r}_1(t_1) \| \). The sign of the time-component is explicitly indicated by the plus or minus on the first entry of Eq.(65). The half-Jacobian (11) with proper-time parametrization is calculated using Eqs. (60) with index 1 replaced by 2 and definition (65), i.e.,

\[
J^\pm_{\tau_2} = (x_{12} \cdot \dot{x}_{2\pm}) = \mp \gamma_2 r_{12\pm} (1 \pm \hat{n} \cdot \tilde{v}_2), \tag{67}
\]

where overdot represents derivative respect to proper-time when the particle moves near the speed of light.

The partial derivative of \( \tau_2^\pm \) with respect to \( x_1 \) in Eq.(26) along the fixed trajectory of particle 2 is given by formula (16) with \( \partial \tau_2^\pm \) replaced by \( \partial \tau_2^\pm \). Since we are operating with proper-time we can set \( \sqrt{\vec{x}_1 \cdot \vec{x}_1} = 1 \) in the first denominator on the right-hand-side of Eq. (26), yielding

\[
G_1 = m_1 \dot{x}_1 \tag{68}
\]

\[
\frac{d}{d\tau_1} \left( \frac{\dot{x}_2 - 2J_{\tau_2}{\dot{J}_{\tau_2}}^+}{2J_{\tau_2}^+} \right) + \frac{\partial}{\partial x_1} \left( \frac{\dot{x}_1 \cdot \dot{x}_2 - 2J_{\tau_2}}{2J_{\tau_2}^+} \right),
\]

where we took out the modulus sign using that \( J_{\tau_2}^{\pm} \equiv || \tilde{r}_2^{\pm} \| \) is always positive. The derivative respect to \( \tau_1 \) on the right-hand side of Eq.(68) also acts on the arguments \( \tau_2^\pm \) since these are functions of \( \tau_1 \) by the light-cone conditions

\[
|x_1(\tau_1) - x_2(\tau_2^\pm)|^2 = 0. \tag{69}
\]

To evaluate the derivative of the retarded/advanced proper-time \( \tau_2^\pm \) with respect to \( \tau_1 \) we take the differential of the light-cone condition (69), i.e.,

\[
2(x_{12} \cdot \dot{x}_1)^\pm d\tau_1 - 2(x_{12} \cdot \dot{x}_2)^\pm d\tau_2^\pm = 0, \tag{70}
\]

which yields

\[
\frac{d\tau_2^\pm}{d\tau_1} = \left( \frac{x_{12} \cdot \dot{x}_1)^\pm}{x_{12} \cdot \dot{x}_2)^\pm} \right), \tag{71}
\]

where \( x_{12}^\pm \equiv (x_1 - x_2)^\pm \). Formula (71) is valid for both the retarded and the advanced lighcones. The same separation \( x_{12} \) appears on both numerator and denominator on the right-hand-side of Eq.(71), so that the plus or minus sign of Eq.(65) cancels out and the derivative (71) is always positive as it should be. We can use the two signs of Eq.(71) to calculate the derivative of the most retarded argument with respect to the most advanced argument by the chain rule

\[
\frac{d\tau_2^-}{d\tau_2^+} = \left( \frac{d\tau_2^-}{d\tau_1}(\frac{d\tau_1}{d\tau_2^+}) \right), \tag{72}
\]
where $\frac{dx_i}{d\tau_2} = \left(\frac{dx_i}{d\tau_1}\right)^{-1}$ and Eq.(72) is a non-negative rate because it is a product of two positive factors. The fact that the retarded and the advanced arguments have non-negative rates ensures the continuity of any piecewise-continuous solution at a breaking point.[22]. Therefore the usual mechanism for a neutral-delay equation to loose its piece-wise continuous solution at a breaking point is absent and the neutral equations of electrodynamics never loose solutions for this reason.

Using Eq.(71), the second line on the right-hand side of Eq.(68) evaluates to

$$\frac{1}{2(x_{12} \cdot \dot{x}_{2})_{\pm}^2}\frac{d(x_{12} \cdot \dot{x}_2)}{d\tau_1} \dot{x}_2 - (x_{12} \cdot \dot{x}_1) a_{2\pm}$$ (73)

where the lower index $\pm$ after the bracket indicates evaluation in the advanced/retarded light-cone respectively and $a_{2\pm} \equiv d^2x_2/d\tau_2^2$ denotes the acceleration of particle 2 respect to proper time in the advanced/retarded light-cone respectively. Last, on the third line of the right-hand side of Eq. (68) the partial derivative respect to $x_1$ acts on $x_1$ and also on quantities of particle 2 by the rule

$$\frac{\partial}{\partial x_1} = \frac{\partial \tau_2}{\partial x_1} \frac{d}{d\tau_2},$$ (74)

with $\frac{\partial x_i}{d\tau_2}$ given by Eq.(16). The manipulations are simple and the third line on the right-hand side of Eq.(68) becomes

$$\frac{\dot{x}_1 \cdot a_2}{2(x_{12} \cdot \dot{x}_2)_{\pm}} \frac{d(x_{12} \cdot \dot{x}_2)}{d\tau_1} \dot{x}_2 - (x_{12} \cdot \dot{x}_1) a_{2\pm}$$ (75)

Using Eqs.(68), (73) and (75) we can express the gradient as

$$G_1 = m_1 \ddot{x}_1 - \frac{1}{2} F_2^+ - \frac{1}{2} F_2^-,$$ (77)

where

$$F_2^\pm \equiv \frac{(x_{12} \cdot v_2)_\pm}{2\rho_2^\pm}[(x_{12} \cdot v_1) a_2 - (v_1 \cdot a_2) x_{12}] \pm$$ (78)

$$+ \frac{1}{2}\left(1 - x_{12} \cdot a_2\right)_{\pm} [(x_{12} \cdot v_1)v_2 - (v_1 \cdot v_2) x_{12}] \pm,$$

where $\rho_2^\pm \equiv \|J^\pm_{21}\|^3$ and $v_{2\pm} \equiv dx_2/d\tau_2^\pm$.

The condition $G_1 = 0$ yields a familiar Newtonian-like equation of motion with the Lorentz-force of the other particle as a semi-sum of advanced/retarded Liénard-Wiechert fields, i.e.,

$$m_1 \frac{d^v_1}{d\tau_1} = \frac{1}{2} F_2^+ (x_1, v_1, x_2(\tau_2^+), v_2(\tau_2^+), a_2(\tau_2^+))$$ (79)

$$+ \frac{1}{2} F_2^- (x_1, v_1, x_2(\tau_2^-), v_2(\tau_2^-), a_2(\tau_2^-)),$$

$$m_2 \frac{d^v_2}{d\tau_2} = \frac{1}{2} F_1^+ (x_2, v_2, x_1(\tau_1^+), v_1(\tau_1^+), a_1(\tau_1^+))$$ (80)

$$+ \frac{1}{2} F_1^- (x_2, v_2, x_1(\tau_1^-), v_1(\tau_1^-), a_1(\tau_1^-)),$$

where $x_i, v_i, a_i$ are respectively the position, velocity and acceleration of particle $i$ with respect to proper-time $\tau_i$. In Eqs. (79) and (80) the forces $F_k^\pm$ depend respectively on the other particle’s retarded/advanced position, velocity and acceleration, as well as on the object-particle’s present position and velocity. Moreover the retarded/advanced points are implicitly defined by Eq. (5), so that Eqs. (79) and (80) are neutral-delay equations of mixed-type with implicit state-dependent delay. The forces $F_k^\pm$ of Eqs. (79) and (80) are the Lorentz force of the Liénard-Wiechert fields of standard electrodynamics textbooks[6, 23]. Notice that each line of Eq. (78) is orthogonal to $v_1$, so that it follows from Eq. (79) that

$$(v_1 \cdot \dot{v}_1) = 0,$$ (81)

in agreement with Eq. (29) for $p \neq 1/2$ and for $p = 1/2$ Eq.(81) is the definition of proper-time parametrization. Condition (63) can be solved for the time-velocity with Eq. (64), thereby reducing the dynamics to the spatial components of Eqs. (79) and (80). Last, in the following we discuss the denominators of Eqs. (79) and (80). The Liénard-Wiechert force (78) involves denominators of type (67), which become singular when the other particle travels near the speed of light. This is illustrated expressing the vector-part of the equation of motion (79) using particle-time parametrization and using only the leading singular term of force (78), i.e.,

$$\frac{d}{dt}\left(\frac{m_i \ddot{v}_i}{\sqrt{1 - \ddot{v}_i^2}}\right) = -\frac{1}{r_{12}(1 \pm n \cdot \ddot{v}_i)^3} \hat{n} \times (\hat{n} \times \hat{a}_j) + \ldots$$ (82)

Notice that the left-hand-side of Eq. (82) becomes singular when particle $i$ travels near the speed of light while the right-hand side of Eq. (82) becomes singular when particle $j$ travels near the speed of light in either the past/future lightcone points, so that if the two motions synchronize a solenoidal orbit with a fast velocity could exist, as suggested in Ref. [11]. Surprisingly this non-trivial motion does not require large total momenta, as follows:— Action (13) is invariant by the Lorentz group if one also moves the boundary-condition-segments with the group element (the red segments of Fig. 3.1). Noether’s theorem [13] applies to action (13)
in a way completely analogous to the formal derivation of Schild[10], as explained in Ref. [13], yielding invariants defined by finite integrals, i.e.,

$$p^\mu = p^\mu_1(\tau_1) + p^\mu_2(\tau_2) +$$

$$-2 \int_{\tau_1}^{\tau_2} (x_1^\alpha - x_2^\alpha) \delta'(|x_1 - x_2|^2) \dot{x}_1 \cdot \dot{x}_2 d\tau_1 d\tau_2$$

$$+2 \int_{\tau_2}^{\tau_1} (x_1^\alpha - x_2^\alpha) \delta'(|x_1 - x_2|^2) \dot{x}_1 \cdot \dot{x}_2 d\tau_1 d\tau_2,$$

and

$$L^{\alpha\beta} = \left( r_2^\beta p_1^\beta - r_1^\beta p_1^\alpha \right) |_{\tau_1} + \left( r_2^\beta p_2^\beta - r_1^\beta p_2^\alpha \right) |_{\tau_2}$$

$$-2 \int_{\tau_1}^{\tau_2} (x_1^\alpha x_2^\alpha - x_1^\beta x_2^\beta) \delta'(|x_1 - x_2|^2) \dot{x}_1 \cdot \dot{x}_2 d\tau_1 d\tau_2$$

$$+2 \int_{\tau_2}^{\tau_1} (x_1^\alpha x_2^\alpha - x_1^\beta x_2^\beta) \delta'(|x_1 - x_2|^2) \dot{x}_1 \cdot \dot{x}_2 d\tau_1 d\tau_2$$

$$+ \int_{\tau_1}^{\tau_2} (\dot{x}_1^\alpha x_2^\beta - \dot{x}_1^\beta x_2^\alpha) \delta'(|x_1 - x_2|^2) \dot{x}_1 \cdot \dot{x}_2 d\tau_1 d\tau_2$$

$$- \int_{\tau_2}^{\tau_1} (\dot{x}_1^\alpha x_2^\beta - \dot{x}_1^\beta x_2^\alpha) \delta'(|x_1 - x_2|^2) \dot{x}_1 \cdot \dot{x}_2 d\tau_1 d\tau_2.$$

where

$$p_1^\mu \equiv \frac{m_1 v_1^\mu}{\sqrt{1 - |v_1|^2}}$$

$$= \frac{v_1^-}{2r_{12}(1 - \dot{n} \cdot \dot{v}_{12}^-)} - \frac{v_2^+}{2r_{12}(1 - \dot{n} \cdot \dot{v}_{12}^+)}$$

$$p_2^\mu \equiv \frac{m_2 v_2^\mu}{\sqrt{1 - |v_2|^2}}$$

$$= \frac{v_1^-}{2r_{21}(1 - \dot{n} \cdot \dot{v}_{21}^-)} - \frac{v_2^+}{2r_{21}(1 - \dot{n} \cdot \dot{v}_{21}^+)}.$$

Notice that $p_1$ and $p_2$ as defined by Eqs. (85) and (86) can be small even at fast velocities, so that a solenoidal motion with a stiff gyration near the speed of light is possible with finite and small mechanical momenta (83), as illustrated in Fig. 6.1. The solenoidal orbits of Ref. [11] were estimated to have a velocity near the speed of light, and the physical interest stems from the fact that these can be found in the physical region of small 4-momentum and angular-momentum. At present there is no numerical integrator available to integrate such non-trivial state-dependent neutral-delay equations, and we hope this work contributes to the construction of such integrator.

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