Simplifying Triangulations

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Abstract
We give a new algorithm to simplify a given triangulation with respect to a given curve. The simplification uses flips together with powers of Dehn twists in order to complete in polynomial time in the bit-size of the curve.

Keywords Triangulations of surfaces · Flip graphs · Dehn twists

Mathematics Subject Classification 57M20

1 Introduction

Fix an (orientable) punctured surface \( S \) and let \( \zeta = \zeta(S) := -3\chi(S) \). We will assume that \( S \) is sufficiently complex that \( \zeta \geq 3 \) and so \( S \) can be decomposed into an (ideal) triangulation consisting of exactly \( \zeta \) edges. Throughout, we consider these triangulations up to isotopy of the surface relative to the punctures.

A triangulation \( \mathcal{T} \) provides a coordinate system that can be used to efficiently describe isotopy classes of simple closed curves on \( S \). The isotopy class \( \gamma \) is encoded by its edge coordinate \( (\iota(\gamma, e_1), \ldots, \iota(\gamma, e_{\zeta})) \) which consists of its geometric intersection numbers with the edges \( e_1, \ldots, e_{\zeta} \) of \( \mathcal{T} \) [9, p. 28]. This can be found from the number of intersections between an \( e_i \) and a representative of \( \gamma \) that is in normal position [7], that is, where \( \gamma \) enters and exits each triangle of \( \mathcal{T} \) through different sides.

Now on a generic triangulation \( \gamma \) appears extremely complicated and so these intersection numbers are very large, however there is always a triangulation \( \mathcal{T} \) in which \( \gamma \) intersects each edge of \( \mathcal{T} \) at most twice. Such a triangulation, which we refer to as \( \gamma \)-simple, is extremely useful for performing calculations with. For example, if \( \gamma \)
is given via its edge coordinate on a $\gamma$-simple triangulation then it is straightforward
to determine its topological type.

The aim of this paper is to show that two basic moves can be used to rapidly
transform any given triangulation into a $\gamma$-simple one:

**Theorem 3.7** Let $D := 80 \zeta B (10B + 1)^C$ where $B := 52 \zeta$ and $C := 2 \zeta$. If $\iota(\gamma, \mathcal{T}) > D$ then there is a triangulation $\mathcal{T}'$ that differs from $\mathcal{T}$ by either:

- an edge flip (see Sect. 2), or
- a small power of a Dehn twist (see Sect. 3) along a short curve,

such that $\iota(\gamma, \mathcal{T}') \leq (1 - 1/D) \iota(\gamma, \mathcal{T})$.

Using this, as there is a flip or twist that reduces $\iota(\gamma, \mathcal{T})$ by a definite fraction, we
can convert $\mathcal{T}$ to a $\gamma$-simple one in only $O(\|\gamma\|_\mathcal{T})$ such moves where $\|\gamma\|_\mathcal{T} := \log \iota(\gamma, \mathcal{T})$. Furthermore, since there are only $\zeta$ edges and $2\zeta$ short curves, we can
efficiently find the move to apply at each stage by simply trying each possible move
in turn.

This result mimics several similar simplification results in other models of curves
on surfaces. For example, in:

- interval permutations by Agol et al. [1, Sect. 4],
- the street complex by Erickson and Nayyeri [4],
- straight line programs by Schaefer et al. [11], and
- Dynnikov coordinates on a punctured disk [3].

However, the edge coordinates provided by a triangulation can be used to describe
all isotopy classes of curves on $S$. This means that problems involving multiple
curves can also be tackled. For example, unlike in the other data structures, in
$O(\text{poly}(\|\gamma\|_\mathcal{T} + \|\delta\|_\mathcal{T}))$ time we can determine whether $\gamma$ and $\delta$ fill $S$. That is, whether
every essential simple closed curve on $S$ must intersect either $\gamma$ or $\delta$.

To do this we repeatedly use Theorem 3.7 to construct a sequence of at most
$O(\|\gamma\|_\mathcal{T})$ moves from $\mathcal{T}$ to a $\gamma$-short triangulation $\mathcal{T}'$ (Corollary 3.8). We apply
this sequence of moves to $\gamma$ and $\delta$ to obtain their edge coordinates with respect to $\mathcal{T}'$.
Although this sequence may make the coordinates of $\delta$ more complicated, direct
computation shows that $\|\delta\|_{\mathcal{T}'} \leq \|\delta\|_\mathcal{T} + \|\gamma\|_{\mathcal{T}}^2$. In the $\mathcal{T}'$ coordinate system, since
the edge coordinate of $\gamma$ is so simple, the complementary components of $\gamma \cup \delta$ can be
seen directly and computed in $O(\text{poly} \|\delta\|_{\mathcal{T}'})$ time. Since $\gamma$ and $\delta$ fill $S$ if and only if
all of these complementary components are disks or once-punctured disks, this allows
us to efficiently determine whether $\gamma$ and $\delta$ fill $S$.

Finally, while we will focus on the case where the surface $S$ is fixed, we note
that these simplification processes’ dependence on $S$ is entirely contained within the
$D = D(S)$ term of Theorem 3.7. Hence it, and so determining whether $\gamma$ and $\delta$ fill $S$,
is fixed parameter tractable in $\chi(S)$ [6, Defn. 2.2].

**2 Flips**

The first basic operation that we will consider in order to produce a simpler triangulation
with respect to $\gamma$ is a *flip*. We say that an edge of $\mathcal{T}$ is *flippable* if it is contained
in two distinct triangles. If \( e \) is such an edge then we may flip it to obtain a new triangulation \( \mathcal{T}(e) \) as shown in Fig. 1.

The number of intersections between \( \gamma \) and the new edge \( f \) is exactly determined by the number of intersections between \( \gamma \) and the neighbouring edges of \( e \).

**Proposition 2.1** ([9, p. 30]) Suppose that \( \gamma \) is a curve and \( e \) is a flippable edge of a triangulation \( \mathcal{T} \) as shown in Fig. 1. Then

\[
\iota(\gamma, f) = \max(\iota(\gamma, a) + \iota(\gamma, c), \iota(\gamma, b) + \iota(\gamma, d)) - \iota(\gamma, e).
\]

2.1 The Flip Graph

The flip graph \( G = G(S) \) is the graph with a vertex for each triangulation of \( S \) where two vertices are connected via an edge of length 1 if they differ by a flip. The flip graph is connected, see [8] and [9, p. 36], and so we may use a sequence of flips to convert \( \mathcal{T} \) into a \( \gamma \)-simple triangulation. To help us find such a sequence we recall the following lemma:

**Lemma 2.2** ([2, Lem. 2.4.3]) If \( \mathcal{T} \) is not a \( \gamma \)-short triangulation then there is an edge of \( \mathcal{T} \) which can be flipped in order to reduce the intersection number.

Similar results are also known for other measures of the complexity of \( \gamma \), see [10] and [9, p. 39]. We may use Lemma 2.2 repeatedly to monotonically reduce \( \iota(\gamma, \mathcal{T}) \) until we reach a \( \gamma \)-simple triangulation and so deduce

**Corollary 2.3** For each \( \mathcal{T} \in G \) and curve \( \gamma \) we can build a path in \( G \) from \( \mathcal{T} \) to a \( \gamma \)-simple triangulation whose length is \( O(\iota(\gamma, \mathcal{T})) \).

Unfortunately there are cases in which at least \( O(\iota(\gamma, \mathcal{T})) \) flips are required to reach a \( \gamma \)-simple triangulation. For example, on the triangulation of the once punctured torus shown in Fig. 2 the curve of slope \( k \) has geometric intersection number \( 2k \) while the nearest \( \gamma \)-simple triangulation is \( k - 3 \) flips away.

Such examples can be constructed by performing large powers of Dehn twists. In the next section we will show that in fact these twists are the only way to create such an obstruction.
This triangulation is far from a $\gamma$-simple one in $G$

3 Twists

To deal with triangulations which need a large number of flips in order to simplify them, we introduce a second type of move: the Dehn twist $T^k_\delta$ [5, Chap. 3]. This move cuts the surface open along the curve $\delta$ and rotates one of the boundary components $k$ times to the right (or $|k|$ times to the left if $k$ is negative) before regluing the boundary components together. We will show that if flips cannot decrease $\iota(\gamma, \mathcal{T})$ by a definite fraction, then a power of a Dehn twist can.

To do this, suppose that $\gamma$ is a fixed curve and that $\mathcal{T}$ is a fixed triangulation. We set some additional notation for terms that we will use throughout this section:

- Let $r$ denote the maximum amount that $\iota(\gamma, \mathcal{T})$ can be reduced by through a single flip, that is,

$$ r := \max_{e \in E(\mathcal{T})} (\iota(\gamma, \mathcal{T}) - \iota(\gamma, \mathcal{T}(e))). $$

- Let $e_{\text{max}}$ be an edge of $\mathcal{T}$ which meets $\gamma$ the most, that is, such that

$$ \iota(\gamma, e_{\text{max}}) = \max_{e \in E(\mathcal{T})} \iota(\gamma, e). $$

- Let $M := \iota(\gamma, e_{\text{max}})$.
- Let $P$ be the set of points of intersection between a representative of $\mathcal{T}$ and a normal representative of $\gamma$.

3.1 Insulation

Definition 3.1 Suppose that $p \in P$ lies on the edge $e$ of $\mathcal{T}$. Then $p$ is $k$-insulated if each component of $e - p$ contains at least $k$ other points in $P$. That is, if looking along $e$ there are at least $k$ points in $P$ on either side of $p$. For example, see Fig. 3.
We say that \( p, p' \in P \) are adjacent if they appear consecutively along \( \gamma \). For example, again see Fig. 3. The key property of insulation is that it only decays slightly when we move to an adjacent point.

**Lemma 3.2** Suppose that \( p \in P \) and that \( p' \in P \) is an adjacent point. If \( p \) is \( k \)-insulated then \( p' \) is \((5k - (2M + r - 2))\)-insulated.

**Proof** For convenience we will use the notation \( x := \iota(\gamma, x) \) here. We begin by considering the case in which \( p \) lies on a flippable edge \( e \). Without loss of generality, following the notation of Fig. 1, we may assume that \( b + d \geq a + c \). Recall that by Proposition 2.1 we have that

\[
b + d - e = f \geq e - r,
\]

where, as described above, \( r \) is the maximal amount that a single flip can reduce \( \iota(\gamma, T) \) by. Rearranging this, since \( b, d \leq M \), we obtain

\[
2e - (r + M) \leq b, d \leq M. \tag{3.1}
\]

Let \( W, X, Y, \) and \( Z \) be the number of times that \( \gamma \) runs around each of the corners of the square about \( e \), as shown in Fig. 3. Using this notation and inequality (3.1) we then have that

\[
-r - 2(M - e) \leq b - d = (X + Y) - (Z + W) \leq r + 2(M - e), \tag{3.2}
\]

\[
-r \leq f - e = (X + Z) - (W + Y) \leq 2(M - e). \tag{3.3}
\]

By adding and subtracting inequalities (3.2) and (3.3) we discover that

\[
|X - W| \leq r + 2(M - e) \quad \text{and} \quad |Y - Z| \leq r + 2(M - e).
\]

Now let \( e' \) be the edge containing \( p' \). All of the points of \( P \) on one component of \( e - p \) are adjacent to points on one of the components of \( e' - p' \). However, as \( X \approx W \) and \( Y \approx Z \), the other component of \( e' - p' \) can only have \( r + 2(M - e) \) fewer points of \( P \) than the other component of \( e - p \). Hence \( p' \) must be \((k - r - 2M + 2e)\)-insulated. Finally note that as \( p \) is \( k \)-insulated we must have that \( e \geq 2k + 1 \). Therefore \( p' \) is \((5k - (2M + r - 2))\)-insulated as required.
Fig. 4 If \( p \) lies on a non-flippable edge \( e \) then \( p' \) is at least as insulated as \( p \).

Now suppose that \( p \) lies on a non-flippable edge \( e \). In this case \( p' \) must lie on the bounding edge \( e' \), as shown in Fig. 4. It follows immediately that if \( p \) is \( k \)-insulated then \( p' \) is \( k \)-insulated too. As \( k \geq 5k - (2M + r - 2) \), the result also holds in this case. \( \square \)

For convenience, let \( A := (2M + r - 2)/4 \). The next corollary follows from solving the difference equation

\[
k_{i+1} = 5k_i - 4A.
\]

**Corollary 3.3** Suppose that \( p_0, p_1, \ldots \) is a sequence of pairwise adjacent points. If \( p_0 \) is \( k \)-insulated then \( p_i \) is \((A - (A - k)5^i)\)-insulated.

**3.2 Blocks**

We focus on the intersections between \( \gamma \) and \( e_{\text{max}} \), so define \( Q := P \cap e_{\text{max}} \). To enable the following definition, we fix a coorientation of \( e_{\text{max}} \), that is, a choice of unit normal vector field to \( e_{\text{max}} \).

**Definition 3.4** The **chain** of \( q \in Q \) is the sequence of pairwise adjacent points \( p_0, p_1, \ldots \) starting with \( p_0 := q \) and emanating from \( e_{\text{max}} \) in the direction of its coorientation.

If \( p_0, p_1, \ldots \) is the chain of \( q \in Q \), then we refer to the edge that \( p_i \) lies on, together with the coorientation with which the chain meets that edge at \( p_i \), as the **type** of \( p_i \).

We partition the points of \( Q \) into subsets \( Q_1, Q_2, \ldots, Q_t \) called **blocks**. The block that \( q \in Q \) is contained in is determined by the sequence of types of the points in its chain up to the first repetition of type. Of course, as there are only \( 2\xi \) different types, the block of \( q \) is determined by at most the first \( 2\xi + 1 \) points of its chain. We also note that the points in each block are consecutive along \( e_{\text{max}} \).

**Lemma 3.5** There are at most \( C := 2\xi \) blocks.

**Proof** For a point \( q \in Q \), let us refer to the smallest initial sequence of points in its chain that contains two points on \( e_{\text{max}} \) of the same type, as the **initial chain** of \( q \). We
say that two points in \( Q \) are equivalent if the sequences of types of the points in their initial chains are equal. The number of blocks is bounded above by the number of such equivalence classes and we will proceed by showing that there are at most \( 2\zeta \) classes.

Let \( \Gamma \) denote the union of all of the segments of \( \gamma \) connecting between all of the points in the initial chains of all \( q \in Q \). We retract all of the endpoints of \( \Gamma \) on \( e_{\text{max}} \) down to a single point to obtain, after identifying any parallel components, a train track \( \tau \) on \( S \). This train track has a single switch and an Euler characteristic argument shows that it has at most \( \zeta \) branches.

An initial chain gives a train-route in \( \tau \) that runs over at most two branches. Furthermore, two initial chains are equivalent if and only if their train-routes run over the same sequence of branches. However, in such a train track there are at most \( 2\zeta \) such train-routes that start by leaving the switch in the direction of the coorientation of \( e_{\text{max}} \). Hence the claimed bound on the number of equivalence classes, and so the number of blocks, holds.

\[ \Box \]

Again for convenience, let \( B := 5^{2\zeta} \).

**Proposition 3.6** Suppose that \( q \in Q \) is \( k \)-insulated and lies in the block \( Q_n \). If \( Q_n \) contains more than

\[ M - 2(A - AB + Bk) \]

points, then the lexicographically smallest pair \((i, j)\) such that \( p_i \) and \( p_j \) in the chain of \( q \) have the same type is of the form \((0, j)\).

**Proof** We will show the contrapositive. Suppose that \((i, j)\) is the lexicographically smallest pair such that \( p_i \) and \( p_j \) have the same type and that \( i > 0 \). Let \( e \) be the edge containing \( p_i \) and \( p_j \).

Suppose that there are \( d \) points of \( P \) between \( p_i \) and \( p_j \) along \( e \). Note that, as shown in Fig. 5, there are at most \( d + 1 \) points in \( Q_n \). However, by Corollary 3.3, we have that \( p_i \) and \( p_j \) are both \((A - (A - k)B)\)-insulated and so

\[ |Q_n| \leq d + 1 \leq M - 2(A - AB + Bk) \]

as required.

\[ \Box \]

We can now prove the main theorem.

**Theorem 3.7** Let \( D := 80\zeta B(10B + 1)^C \) where \( B := 5^{2\zeta} \) and \( C := 2\zeta \). Suppose that \( \iota(\gamma, \mathcal{T}) > D \), then there is a triangulation \( \mathcal{T}' \) that differs from \( \mathcal{T} \) by either:

- an edge flip (see Sect. 2), or
- a small power of a Dehn twist (see Sect. 3) along a short curve,

such that \( \iota(\gamma, \mathcal{T}') \leq (1 - 1/D)\iota(\gamma, \mathcal{T}) \).

**Proof** Let \( r := \iota(\gamma, \mathcal{T})/D \). We may assume that no flip reduces \( \iota(\gamma, \mathcal{T}) \) by more than \( r \) as otherwise we are done by performing that flip.

Without loss of generality we may also assume that the blocks \( Q_1, \ldots, Q_t \) are ordered by insulation. That is,
Fig. 5 A block must be narrow if it meets itself but not on \( e_{\max} \)

- \( Q_1 \) is the block containing \( q_1 \), a point of highest insulation in \( Q \),
- \( Q_2 \) is the block containing \( q_2 \), a point of highest insulation in \( Q \) not in \( Q_1 \),
- \( Q_3 \) is the block containing \( q_3 \), a point of highest insulation in \( Q \) not in \( Q_1 \cup Q_2 \),
  
- \( Q_t \) is the block containing \( q_t \), a point of highest insulation in \( Q \) not in \( Q_1 \cup Q_2 \cup \ldots \cup Q_{t-1} \).

Now let \( B_n := 40rB(10B + 1)^{n-1} \). If every \( Q_n \) contains at most \( B_n \) points then

\[
M \leq 40rB(10B + 1)^C.
\]

This cannot happen as it would mean that

\[
\iota(\gamma, \mathcal{T}) \leq \zeta M \leq 40\zeta rB(10B + 1)^C = \frac{rD}{2} = \frac{\iota(\gamma, \mathcal{T})}{2}.
\]

Therefore there is a smallest \( n \) such that \( Q_n \) contains more that \( B_n \) points.

Now note that \( q_n \), the point in \( Q_n \) with maximal insulation that we chose above, is \( k \)-insulated where

\[
k := \left\lfloor \frac{M - 1}{2} \right\rfloor - (B_1 + \cdots + B_{n-1}) \geq \left\lfloor \frac{M - 1}{2} \right\rfloor - \frac{B_n}{10B}.
\]

Let \( p_0, p_1, \ldots \) be the chain of \( q_n \) and let \((i, j)\) be the lexicographically smallest pair such that \( p_i \) and \( p_j \) have the same type. Now as

\[
M - 2(A - AB + Bk) \leq 2rB + \frac{B_n}{5} \leq \frac{B_n}{4},
\]
we have that \( i = 0 \) by Proposition 3.6. Therefore \( p_0 \) and \( p_j \) both lie on \( e_{\text{max}} \) and there are at most \( B_n/4 \) points between them along \( e_{\text{max}} \).

Hence we let \( \delta \) be the loop which runs parallel to \( \gamma \) from \( p_0 \) round to \( p_j \) and then connects back to \( p_0 \) by following along \( e_{\text{max}} \), as shown in Fig. 6. As these points are so close and the block is so wide, the block and its image after it has been pushed along \( \gamma \) share at least \( 3B_n/4 \) points in \( Q \). It follows that by performing at most \( B_n/2 \) Dehn twists along \( \delta \) we can reduce \( \iota(\gamma, \mathcal{T}) \) by at least \( B_n/2 > r \) as required.

We note that the constructive nature of the proof of Theorem 3.7 means that the curve \( \delta \) can easily be found. Simply trace along \( \gamma \) starting at the \( t \) different \( q_i \) for at most \( 2\zeta \) steps and look for one that gets back to \( e_{\text{max}} \) close to where it started. Additionally this construction guarantees that the required power is small (at most \( \iota(\gamma, \mathcal{T}) \)) and that \( \delta \) is short (\( \iota(\delta, \mathcal{T}) \leq 2\zeta \)).

### 3.3 The Flip–Twist Graph

To take into account the addition of the twist move, we introduce a modified version of the flip graph \( G \) with additional edges. The flip–twist graph \( \mathcal{G} = \mathcal{G}(S) \) is the graph with a vertex for each triangulation of \( S \) where \( \mathcal{T} \) and \( \mathcal{T}' \) are connected via:

- an edge of length 1 if they differ by a flip, and
- an edge of length \( \log (\iota(\delta, \mathcal{T}) + k) \) if they differ by \( T^{k}_\delta \).

These edge lengths are proportional to the computation complexity of performing these operations.

**Corollary 3.8** For each \( \mathcal{T} \in \mathcal{G} \) and curve \( \gamma \) we can build a path in \( \mathcal{G} \) from \( \mathcal{T} \) to a \( \gamma \)-simple triangulation whose length is \( O(\log(\iota(\gamma, \mathcal{T}))^2) \).
By applying Theorem 3.7 at most \( \log \frac{\tau(\gamma, T)}{\log D} \) times, we can construct a path in \( \mathcal{G} \) from \( T \) to a triangulation \( T_0 \) such that \( \tau(\gamma, T_0) \leq D \). Since any curve that we twist about during this process is short, this path has length at most

\[
\frac{\log (2 \zeta + \tau(\gamma, T)) \cdot \log \tau(\gamma, T)}{\log D}.
\]

By Corollary 2.3 we can then construct a sequence of at most \( D \in O(1) \) flips from \( T_0 \) to a \( \gamma \)-short triangulation. The result then follows from concatenating these two paths together.

Again the curves of slope \( k \) on the once-punctured torus from the end of Sect. 2 attain this logarithmic distance bound in \( \mathcal{G} \). The curve of slope \( k \) on the once-punctured torus, as shown in Fig. 2, has geometric intersection number \( 2k \) and the distance to the nearest \( \gamma \)-simple triangulation in \( \mathcal{G} \) is \( \log (k + 2) \).

4 Multicurves and Multiarcs

We finish by highlighting that a version of Theorem 3.7 holds if \( \gamma \) is a multicurve or even a multiarc. However, in the argument of Theorem 3.7 it is possible that the wide block that we find returns exactly to itself, that is, \( p_j = p_0 \). In this case \( \delta \) is disjoint from \( \gamma \) and so performing Dehn twists about it has no effect.

If this situation occurs then one can remove the entire block, which is a simple closed curve meeting each edge at most twice with high multiplicity. Again this reduces the intersection number by at least \( m \) as required. Repeating this process allows us to extract the isotopy classes of \( \gamma \) along with their multiplicities. We can then analyse each in turn in order to compute, for example, the topological types present.

Furthermore, in the multiarc case it may be necessary to repeat the analysis of the chains of \( e_{\max} \) with its other coorientation. This is because a block may terminate into a puncture before we can follow its chain for the required \( 2 \zeta \) steps. However, if it terminates in both directions then it is a short arc and we can remove it to simplify the situation before repeating the argument. Additionally, in each direction there may now be as many as \( 2 \zeta + 6 \) blocks to check and so the constant \( C \) must be slightly increased.

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