SIGNED DIFFERENTIAL POSETS AND SIGN-IMBALANCE

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Abstract. We study signed differential posets, a signed version of differential posets. These posets satisfy enumerative identities which are signed analogues of those satisfied by differential posets. Our main motivations are the sign-imbalance identities for partition shapes originally conjectured by Stanley, now proven in \cite{4,5,7}. We show that these identities result from a signed differential poset structure on Young’s lattice, and explain similar identities for Fibonacci shapes.

1. Introduction

For a poset $P$ and a field $K$, we let $KP$ denote the vector space of finite linear combinations of elements of $P$. Differential posets are posets $P$ naturally equipped with linear operators $U, D : KP \to KP$ satisfying $DU - UD = I$. Differential posets were introduced by Stanley in \cite{9} and independently discovered by Fomin \cite{1} who called them $Y$-graphs. Differential posets satisfy many enumerative properties generalizing properties of Young’s lattice $Y$. These include

\begin{align*}
\sum_{x \in P_n} f(x)^2 &= n! \quad (1) \\
\sum_{x \in P_n} f(x) &= \# \{ w \in S_n \mid w^2 = 1 \} \quad (2)
\end{align*}

where $f(x)$ denotes the number of maximal chains from the minimum element of $P$ to $x \in P$.

In this paper we study signed differential posets. A signing $(P, s, v)$ of a poset $P$ is an assignment of a sign $v(x) \in \{\pm 1\}$ to each element $x \in P$, and a sign $s(x < y)$ to each cover relation $x < y$ of $P$. Given a signed poset $P$, one defines linear operators $U, D : KP \to KP$. Signed differential posets are those signed posets which give rise to the relation $DU + UD = I$. Signed differential posets come in a number of variations, and as the most interesting example, $\beta$-signed differential poset satisfy the enumerative identities

\begin{align*}
\sum_{x \in P_n} v(x)e(x)^2 &= 0 \quad (3) \\
\sum_{x \in P_n} e(x) &= 2^{\lfloor n/2 \rfloor} \quad (4)
\end{align*}

where $e(x)$ is a signed sum of chains in $P$, defined using the signs $s(x < y)$ of the cover relations.

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Our investigations were motivated by identities involving the sign-imbalance of partition shapes, a topic studied in \cite{4, 5, 7, 11, 14}. For a poset $P$ and a labeling $\omega : P \to \{1, 2, \ldots, |P|\}$ one can define its sign-imbalance $I_{P,\omega} \in \mathbb{Z}$, as a sum of signs over all linear extensions of $P$. In the case that $P$ is a Young diagram $\lambda$, the number $I_\lambda$ is the sum of the signs $s(T) \in \{\pm 1\}$ of the reading words of the standard tableaux $T$ of shape $\lambda$. Stanley \cite{11} conjectured that the sign-imbalances $I_\lambda$ satisfy identities similar to (3) and (4), with $I_\lambda$ replacing $e(\lambda)$ (and Young’s lattice $Y$ taking the place of $P$). These (and more general) identities were proved in \cite{4, 5, 7, 8}. We show that $Y$ can be given the structure of a signed differential poset and that the sign-imbalance identities are a consequence of (3) and (4) which hold for all signed differential posets. This exhibits (3) and (4) vividly as signed analogues of (1) and (2).

In Section 2, we define $\alpha$-signed differential posets and $\beta$-signed differential posets both of which are contained in the larger class of weakly signed differential posets. In Section 3, we give identities involving signed chains and signed walks in these classes of posets. Our first main aim here is to generalize as many of the results in Stanley’s work \cite{9} as possible. Because of the potential for cancelation in enumerative problems for signed differential posets, our identities are often simpler, and enumerative constants more likely to be zero, than for differential posets.

In Section 4, we discuss our two main examples of signed differential posets: the signed Young lattice and the signed Fibonacci differential poset. The signed Fibonacci differential poset admits a simple construction using a signed analogue of the reflection extension which constructs the Fibonacci differential poset. As a consequence one can obtain in this way a large family of signed differential posets. It is curious that the underlying posets for our main examples are themselves differential posets. We have no simple explanation of this phenomenon and have not yet found a natural signed differential poset which is not also a differential poset. In Section 5, we relate signed differential posets to sign imbalance. Besides the case of partition shapes, we show that the elegant sign-imbalance identities are also satisfied for Fibonacci shapes. For the case of Fibonacci shapes, one can also define sign-imbalance as the sum of signs of reading words of Fibonacci tableaux.

There are a number of generalizations of this work which we have not included here to keep the connection with sign-imbalance transparent, but we intend to investigate these in a sequel. These include the study of the relation $DU + UD = rI$ for $r > 1$, the study of the $q$-analogue $DU - qUD = rI$ of all these commutation relations, and the extension to the “weighted” situation, for example in the sense of Fomin’s dual-graded graphs \cite{2, 3}. Some of our identities are formal consequences of Fomin’s work (and many are not), though strictly speaking Fomin disallows graphs with negatively weighted edges, thus excluding signed differential posets.

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2. Signed differential posets

2.1. Differential posets. Let $P = \cup_{n \geq 0} P_n$ be a graded poset with finitely many elements of each rank and with a minimum $\hat{0} \in P_0$. We denote the partial order on $P$ by “$<$” and we let $x \leq y$ denote a cover in $P$. Thus, $x \leq y$ if and only if $x < y$ and for any $z \in P$ satisfying $x \leq z \leq y$ we have $z \in \{x, y\}$.

Let $K$ be a field of characteristic 0. Let $KP$ denote the $K$-vector space with basis $P$ and let $\hat{KP}$ denote the $K$-vector space of arbitrary linear combinations of elements of $P$.

A linear transformation $T : \hat{KP} \to \hat{KP}$ is called continuous if it preserves infinite linear combinations. Define two continuous linear transformations $\bar{U}, \bar{D} : \hat{KP} \to \hat{KP}$ by

$$\bar{U}x = \sum_{y \triangleright x} y \quad \bar{D}x = \sum_{y \triangleright x} y.$$

We use the notation $\bar{U}$ and $\bar{D}$ here instead of $U$ and $D$ so as to save the latter notation for the signed case. The poset $P$ is called $r$-differential if $\bar{D} \bar{U} - \bar{U} \bar{D} = rI$, where $I$ is the identity transformation and $r \in \mathbb{N}$ is a positive integer. Differential posets were introduced by Stanley in [9] and independently studied from a more combinatorial perspective by Fomin [1] (for $r = 1$). Our approach imitates the linear-algebraic approach used by Stanley though many of our results can also be established in the framework of Fomin’s growth diagrams. For the purposes of the current article we will always assume that $r = 1$, and omit the mention of $r$.

If $S \subset P$, we write $S = \sum_{x \in S} x \in \hat{KP}$. The following theorem follows quickly from the definition.

**Theorem 2.1 ([9, Theorem 2.3]).** If $P$ is a differential poset, then

$$\bar{D}P = (\bar{U} + I)P.$$

The following result are special cases of enumerative properties of differential posets.

**Theorem 2.2 ([1] [9]).** Suppose $P$ is a differential poset. Then the identities (1) and (2) hold for each $n \in \mathbb{N}$.

2.2. Signed posets. Let $P = \cup_{n \in \mathbb{Z}} P_n$ be a graded poset. Denote by $E(P)$ the set of edges of its Hasse diagram, or equivalently the set $\{x \leq y \mid x, y \in P\}$ of covers in $P$. A signing of $P$ is a pair $(s, v)$ where $s : E(P) \to \{\pm 1\}$ is a labeling of the edges of the Hasse diagram of $P$ with a sign $\pm 1$, and $v : P \to \{\pm 1\}$ is a labeling of the elements of $P$ with a sign. We will call the triple $(P, s, v)$ a signed poset. If the signing is understood, then we may just say that $P$ is a signed poset.
Let \((P, s, v)\) be a signed poset where \(P\) has finitely many elements of each rank. We define two continuous linear operators \(U, D : \hat{KP} \to \hat{KP}\) by

\[
U x = \sum_{x \preceq y} s(x \preceq y) y
\]

\[
D x = \sum_{y \preceq x} s(y \preceq x) v(x) v(y) y.
\]

The function \(s^\prime : E(P) \to \{\pm 1\}\) given by \(s^\prime(x \preceq y) = s(x \preceq y) v(x) v(y)\) is called the conjugate of \(s\) by \(v\). The reader concerned with the asymmetry in the definitions of \(U\) and \(D\) above should note that the roles are swapped if \(s\) is replaced by its conjugate \(s^\prime\).

The vector space \(KP\) is naturally equipped with a symmetric bilinear inner product \(\langle.,.\rangle\) defined by \(\langle x, y \rangle = \delta_{xy}\) for \(x, y \in P\). Define a deformed inner product \(\langle x, y \rangle_v\) by \(\langle x, y \rangle_v = \delta_{xy} v(x)\). Then the operators \(U\) and \(D\) defined above are adjoint with respect to \(\langle x, y \rangle_v\).

2.3. Signed differential posets. Recall that if \(S \subset P\), we write \(S = \sum_{x \in S} x \in \hat{KP}\).

**Definition 2.3.** Let \((P, s, v)\) be a signed poset, where \(P\) is a graded poset with a minimum \(\hat{0} \in P_0\) which has finitely many elements of each rank. Then \(P\) is weakly signed differential if we have \(v(\hat{0}) = 1\) and

\[
UD + DU = I.
\]

If, in addition, we have

\[
(U + D)P = P
\]

then we say that \(P\) is a \(\alpha\)-signed differential poset. If instead we have

\[
(D - U)P = P
\]

then we say that \(P\) is a \(\beta\)-signed differential poset.

Thus a \(\alpha\) or \(\beta\)-signed differential poset is automatically weakly signed differential. In the case of differential posets or in the case of (some of) Fomin’s self-dual graphs, the analogue of the equation \((U + D)P = P\) can be deduced from the analogue of \(UD + DU = I\). However, in our case such an equation is not automatically satisfied, and indeed we have interesting examples satisfying the \(\beta\)-equation (6). Note that the relations for a \(\beta\)-signed differential poset is a relation not even formally considered by Fomin [2].

Define \(\varepsilon : \mathbb{N} \to \{0, 1\}\) by \(\varepsilon(i) = \frac{1+(-1)^{i+1}}{2}\). Alternatively, \(\varepsilon(i) = 0\) if \(i\) is even and \(\varepsilon(i) = 1\) if \(i\) is odd. The following lemma follows formally from calculations in [2], though strictly speaking Fomin only allows “positive” edges, thus excluding the relation \(UD + DU = I\).
Lemma 2.4. Let \((P, s, v)\) be a weakly signed differential poset. Then for each \(k, l \in \mathbb{N}\) the linear operators \(U, D\) satisfy the following relations:

\[
DU^k = \varepsilon(k) U^{k-1} + (-1)^k U D
\]
\[
D^k U = \varepsilon(k) D^{k-1} + (-1)^k U D^k
\]
\[
D^l U^{k+l} = U^k \prod_{s=k+1}^{k+l} \left((-1)^s U D + \varepsilon(s)\right)
\]
\[
D^k U^{l+k} = \prod_{s=k+1}^{k+l} \left((-1)^s U D + \varepsilon(s)\right) D^k.
\]

Note that the factors in the above products commute, so their order is not important.

Proof. The first and second equations follow by induction from \(DU = I - UD\). The third equation follows from the first by induction from the calculation

\[
D((-1)^s U D + \varepsilon(s)) = ((-1)^{s+1} U D + \varepsilon(s + 1)) D.
\]

The last equation follows from a similar calculation. □

Note that we may swap \(U\) and \(D\) in any identity we deduce from the identity \(UD + DU = I\). If \(w \in \{U, D\}^*\) is a word in the letters \(U\) and \(D\), we let \(\bar{w}\) be \(w\) reversed. Since \(UD + DU = I\) is \(\sim\)-invariant, we may also reverse the order of all words in the identities of Lemma 2.4.

Lemma 2.5. Let \((P, s, v)\) be a weakly signed differential poset and \(n \in \mathbb{N}\). Then

\[
D^n U^n = \begin{cases} 
U^n D^n & \text{if } n \text{ is even}, \\
U^{n-1} D^{n-1} - U^n D^n & \text{if } n \text{ is odd}.
\end{cases}
\]

Proof. We proceed by induction. The result follows from \(UD + DU = I\) when \(n = 1\). Now suppose \(n\) is even and the result has been shown for smaller \(n\). Then using Lemma 2.4 repeatedly,

\[
D^n U^n = D(U^{n-2} D^{n-2} - U^{n-1} D^{n-1}) U
\]
\[
= U^{n-2} D D^{n-2} U - (U^{n-2} - U^{n-1} D) D^{n-1} U
\]
\[
= U^n D^n.
\]

Now suppose that \(n\) is odd. We calculate using the inductive hypothesis

\[
D^n U^n = D(U^{n-1} D^{n-1}) U = U^{n-1} D^n U = U^{n-1} D^{n-1} - U^n D^n.
\]

□

3. Signed chains in signed differential posets

3.1. Enumeration for weakly signed differential posets. We collect here a few enumerative results which can be proved for all weakly signed differential posets.
Let \((P, s, v)\) be a weakly signed differential poset. For \(x \in P_r\), let \(S(x) = \{C = (\hat{0} < x_1 < x_2 < \cdots < x_r = x)\}\) denote the set of maximal chains \(C\) from \(\hat{0}\) to \(x\). Define the signed sum of chains

\[
e(x) = \sum_{C=(x_i) \in S(x)} s(\hat{0} < x_1) s(x_1 < x_2) \cdots s(x_{r-1} < x_r).
\]

The following lemma is immediate from the definitions.

**Lemma 3.1.** For \(x \in P_r\) we have

\[
e(x) = \langle U^r \hat{0}, x \rangle = v(x) \langle U^r \hat{0}, x \rangle_v = v(x) \langle D^r x, \hat{0} \rangle_v = v(x) \langle D^r x, \hat{0} \rangle.
\]

The following result is the signed analogue of (1).

**Theorem 3.2.** Let \((P, s, v)\) be a weakly signed differential poset. Then for \(n \geq 2\),

\[
\sum_{x \in P_n} v(x) e(x)^2 = 0.
\]

**Proof.** By Lemma 3.1,

\[
\langle D^n U^n \hat{0}, \hat{0} \rangle = \sum_{x \in P_n} \langle U^n \hat{0}, x \rangle \langle D^n x, \hat{0} \rangle = \sum_{x \in P_n} v(x) e(x)^2.
\]

By Lemma 2.4 we have \(D^n U^n = \prod_{k=n+1}^{2n}((-1)^k UD + \varepsilon(s))\). Since \(\varepsilon(s) = 0\) when \(s\) is even and \(UD \hat{0} = 0\) we have \(\langle D^n U^n \hat{0}, \hat{0} \rangle = 0\) for \(n \geq 2\). Alternatively, we could have used Lemma 2.5. \(\square\)

Theorem 3.2 can be generalized to show that other enumerative invariants of a weakly signed differential poset \(P\) are independent of the poset \(P\). We now give two examples of this.

Suppose that \(w \in \{U, D\}^*\) is a word in the letters \(\{U, D\}\). We say that \(w\) has rank \(\rho(w) = l\) if the difference between the number of \(U\)'s and the number of \(D\)'s in \(w\) is equal to \(l\). We say that \(w\) vanishes if there is a representation of \(w\) as a concatenation \(UDv\) for \(u, v \in \{U, D\}^*\) such that \(\rho(v)\) is even.

**Theorem 3.3.** Let \((P, s, v)\) be a weakly signed differential poset. Suppose that \(w \in \{U, D\}^*\) and \(x \in P\) satisfy \(\rho(w) = \rho(x)\). Then

\[
\langle w \hat{0}, x \rangle = \begin{cases} 0 & w \text{ vanishes,} \\ e(x) & \text{otherwise.} \end{cases}
\]

In other words, if \(e(x) \neq 0\), we have \(\frac{\langle w \hat{0}, x \rangle}{e(x)} \in \{0, 1\}\) not depending on \(P\).

**Proof.** For each word \(w \in \{U, D\}^*\) one can write, using the relation \(DU + UD = I\) only, \(w = \sum_{i,j} c_{ij}(w) U^i D^j\) for some coefficients \(c_{ij}(w) \in \mathbb{Z}\). The coefficient \(c_{ij}(w)\) is zero unless \(i - j = \rho(w)\). We explain why the \(c_{ij}(w)\) are unique later in Remark 4.3. For
now we imitate \cite{9} and give a method to calculate the \( c_{ij}(w) \) unambiguously. It is easy to see that \( c_{ij}(Uw) = c_{i-1,j}(w) \). We also have by Lemma 2.4,

\[
Dw = \sum_{i,j} c_{ij}(w) DU^i D^j
= \sum_{i,j} c_{ij}(w)(\varepsilon(i)U^{i-1} + (-1)^iU^iD)D^j.
\]

Thus \( c_{ij}(Dw) = \varepsilon(i+1)c_{i+1,j}(w) + (-1)^ic_{i,j-1} \). When \( j = 0 \), we have

\[
c_{i,0}(Uw) = c_{i-1,0}(w)
\]

\[
c_{i,0}(Dw) = \varepsilon(i+1)c_{i+1,0}(w).
\]

Thus \( c_{\rho(w),0}(w) = 0 \) or 1 depending on whether \( w \) vanishes. Since \( D^j\hat{0} = 0 \) for \( j > 0 \), we have

\[
\langle w\hat{0}, x \rangle = c_{\rho(w),0} (U^{\rho(w)}\hat{0}, x) = c_{\rho(w),0} e(x).
\]

\[\square\]

**Remark 3.4.** The coefficients \( c_{ij}(w) \) are the signed normal order coefficients. The normal order coefficients have many interpretations, for example as rook numbers on a Ferrers board. The signed normal order coefficients can be obtained as specializations of the \( q \)-normal order coefficients \cite{13}. The \( q \)-normal order coefficients correspond to the relation \( DU - qUD = I \) which we will study in a separate article.

Define the rank generating function \( F(P, t) = \sum_{x \in P} t^\rho(x) \), where \( \rho : P \to \mathbb{N} \) is the rank function of \( P \). Let \( k, n \in \mathbb{N} \) and define integers \( \kappa_{n,k} = \sum_{x \in P_n} \langle D^kU^k x, x \rangle \). Now define \( F_k(P, t) = \sum_{n \geq 0} \kappa_{n,k} t^n \). Thus \( F(P, t) = F_0(P, t) \).

In \cite{9}, Stanley proved in the case of (non-signed) differential posets that the ratios \( F_k(P, t)/F(P, t) \) were rational functions not depending on \( P \). Here we obtain the signed-analogue of this result.

**Theorem 3.5.** Let \((P, s, v)\) be a weakly signed differential poset. Then

\[
F_k(P, t) = \begin{cases} 
F(P, t) & \text{if } k = 0, \\
F(P, t)/(1 + t) & \text{if } k = 1, \\
0 & \text{if } k \geq 2.
\end{cases}
\]

**Proof.** Note that \( \kappa_{n,k} = \sum_{x \in P_n} \langle D^kU^k x, x \rangle = \sum_{x \in P_{n+k}} \langle U^kD^k x, x \rangle \). Using Lemma 2.5, we calculate,

\[
\kappa_{n,k} = \sum_{x \in P_n} \langle D^kU^k x, x \rangle
= \begin{cases} 
\sum_{x \in P_n} \langle U^kD^k x, x \rangle = \kappa_{n-k,k} & \text{if } k \text{ is even}, \\
\sum_{x \in P_n} \langle (U^{k-1}D^{k-1} - U^kD^k) x, x \rangle = \kappa_{n-k+1,k-1} - \kappa_{n-k,k} & \text{if } k \text{ is odd}.
\end{cases}
\]
Thus,
\[
F_k(P, t) = \sum_{n \geq 0} \kappa_{n,k} t^n
\]
\[
= \begin{cases} 
\sum_{n \geq 0} \kappa_{n-k,k} t^n = t^k F_k(P, t) & \text{if } k \text{ is even}, \\
\sum_{n \geq 0} (\kappa_{n-k+1,k-1} - \kappa_{n-k,k}) t^n = t^{k-1} F_{k-1}(P, t) - t^k F_k(P, t) & \text{if } k \text{ is odd}.
\end{cases}
\]
So we have \( F_k(P, t) = 0 \) if \( k > 0 \) is even and by definition \( F_0(P, t) = F(P, t) \). We also have
\[
F_k(P, t) = t^{k-1} F_{k-1}(P, t) - t^k F_k(P, t)
\]
if \( k \) is odd, giving the stated result. \( \square \)

3.2. Enumeration for \( \alpha \) and \( \beta \)-signed differential posets. In addition to the enumerative properties shared by all weakly signed differential posets, the \( \alpha \) and \( \beta \)-signed differential posets satisfy more enumerative identities.

Using the relations of a signed differential poset, it is easy to see that there are polynomials \( g^\alpha_k(z), g^\beta_k(z) \in \mathbb{Z}[z] \) such that \( D^k P = g^\alpha_k(U) P \) and \( D^k P = g^\beta_k(U) P \) in all \( \alpha \)- or \( \beta \)-signed differential posets. We will explain in Remark 4.3 why these polynomials are unique. (For now, one may think of them as defined modulo the ideal \( I = \{ f(z) \mid f(U) P = 0 \} \subset \mathbb{Z}[z] \).)

**Lemma 3.6.** Suppose \((P, s, v)\) is a \( \alpha \)-signed differential poset. Then for \( k \in \mathbb{N} \) and \( i \in \{1, 2, 3, 4\} \), we have
\[
\begin{align*}
g^\alpha_{4k+1}(z) &= z^{4k} - z^{4k+1} \\
g^\alpha_{4k+2}(z) &= -z^{4k+2} \\
g^\alpha_{4k+3}(z) &= -z^{4k+2} + z^{4k+3} \\
g^\alpha_{4k+4}(z) &= z^{4k+4}.
\end{align*}
\]

**Proof.** Using Lemma 2.4 and (5), we have
\[
DU^k P = (\varepsilon(k) U^{k-1} + (-1)^k U^k + (-1)^{k+1} U^{k+1}) P,
\]
and the result follows from induction with a case-by-case analysis. \( \square \)

**Lemma 3.7.** Suppose \((P, s, v)\) is a \( \beta \)-signed differential poset. Then for \( l \in \mathbb{N} \), we have
\[
\begin{align*}
g^\beta_{2l}(z) &= (2 - z^2)^l.
\end{align*}
\]
The polynomial \( g^\beta_{2l+1}(z) \) can be obtained from \( g^\beta_{2l}(z) \) by applying the linear transformation \( z^{2i} \mapsto z^{2i} + z^{2i+1} \) on \( \mathbb{Z}[z] \).

**Proof.** Using Lemma 2.4 and (5), we have
\[
DU^k P = (\varepsilon(k) U^{k-1} + (-1)^k U^k + (-1)^{k+1} U^{k+1}) P,
\]
for any $k \in \mathbb{N}$. The second statement of the theorem does follow easily from the first. Iterating (7) twice for $k$ even we have

$$D^2U^{2l}P = (2U^{2l} - U^{2l+2})P,$$

proving the first statement by induction. □

Our first theorem here is the signed analogue of (2).

**Theorem 3.8.** Let $(P, s, v)$ be an $a$-signed differential poset where $a \in \{\alpha, \beta\}$. Then for $n \geq 2$, we have

$$\sum_{x \in P_n} v(x)e(x) = \begin{cases} 0 & \text{if } a = \alpha, \\ 2\lfloor n/2 \rfloor & \text{if } a = \beta. \end{cases}$$

**Proof.** Let $P_n = \sum_{x \in P_n} x \in KP$. Then

$$\sum_{x \in P_n} v(x)e(x) = \langle D^n P_n, \hat{0} \rangle_v = \langle D^n P, \hat{0} \rangle_v.$$

The result thus follows from Lemmas 3.6 and 3.7, since $\langle U^j P, \hat{0} \rangle = 0$ for $j > 0$. □

Generalizing Theorem 3.8, we define for each $k, n \in \mathbb{N}$, the sum

$$\tau_{k,n} = \langle D^k P_{n+k}, P_n \rangle_v.$$

Also let $G_k(P, t) = \sum_{n \geq 0} \tau_{k,n} t^n$. If we let $G(P, t) = \sum_{x \in P} v(x) t^{\rho(x)}$ denote the $v$-weighted rank generating function of $P$, then we have $G(P, t) = G_0(P, t)$. The following result is a signed analogue of [9, Theorem 3.2].

**Theorem 3.9.** Let $(P, s, v)$ be an $a$-signed differential poset where $a \in \{\alpha, \beta\}$ and $k \in \mathbb{N}$. Then the ratio $G_k(P, t)/G(P, t)$ is a rational function of $t$ only depending on $k$ and $a$.

**Proof.** Using Lemmas 3.6 and 3.7 we can write $g_k^a(z) = a_k z^k + \cdots + a_0$. We have

$$\tau_{k,n} = \langle D^k P_{n+k}, P_n \rangle_v = \langle D^k P, P_n \rangle_v$$

$$= \langle g_k^a(U) P, P_n \rangle_v = \sum_{i=0}^k a_i \langle U^i P, P_n \rangle_v$$

$$= \sum_{i=0}^k a_i \langle D^i P, P_n \rangle_v = \sum_{i=0}^k a_i \langle D^i P_n, P_{n-i} \rangle_v.$$

Thus

$$G_k(P, t) = \sum_{n \geq 0} \tau_{k,n} t^n = \sum_{i=0}^k a_i \sum_{n \geq 0} \langle D^i P_n, P_{n-i} \rangle_v t^n$$

$$= \sum_{i=0}^k a_i \tau_{i,n-i} t^n = \sum_{i=0}^k a_i t^i G_i(P, t).$$

We may assume by induction that $G_i(P, t)$ has the form given in the theorem for $0 \leq i < k$, and so we can rearrange to write $G_k(P, t)$ as a rational function times $G(P, t)$. The constants $\{a_i\}$ do not depend on $(P, s, v)$ so we are done. □
For \( a \in \{ \alpha, \beta \} \), we denote by \( A_k^a(t) = G_k(P,t)/G(P,t) \) the rational function defined by Theorem \[3.9\]. We can calculate \( A_k^\alpha(t) \) explicitly.

**Proposition 3.10.** Let \( k \in \mathbb{N} \). Then

\[
A_k^\alpha(t) = \begin{cases} 
1 & \text{if } k = 0, \\
\frac{1}{1+t^2} & \text{if } k = 1, \\
0 & \text{if } k > 1.
\end{cases}
\]

**Proof.** The result follows immediately from the recursion in the proof of Theorem \[3.9\] and Lemma \[3.6\]. \( \square \)

In the \( \beta \)-case, the polynomials \( A_k^\beta(t) \) for even \( k \) have a simple form. The odd case appears to be considerably more complicated.

**Proposition 3.11.** Let \( l \in \mathbb{N} \). Then

\[
A_2^\beta(t) = \left( \frac{2}{1+t^2} \right)^l.
\]

**Proof.** We proceed by induction. By definition, \( A_0^\beta(t) = 1 \). Using the recursion in the proof of Theorem \[3.9\] and Lemma \[3.7\] one needs to check that \( A_{2k}^\beta \left( \frac{2}{1+t^2} \right)^k \) satisfies the equality

\[
A_{2k}^\beta = \sum_{i=0}^{k} 2^{k-i} \binom{n}{i} (-1)^i t^{2i} A_{2i}^\beta.
\]

The left hand side is equal to

\[
2^k \sum_{i=0}^{k} \binom{n}{i} \left( \frac{-t^2}{1+t^2} \right)^i = 2^k \left( 1 - \frac{t^2}{1+t^2} \right)^k,
\]

consistent with the claimed formula. \( \square \)

4. Two fundamental examples

Our two examples of signed differential posets come from signings of the two fundamental examples of differential posets. While it is possible to construct trivial examples of (weakly) signed differential posets, Theorem \[3.8\] shows that a \( \beta \)-signed differential poset must be infinite and non-trivial.

4.1. Young’s Lattice. Let \( Y \) denote Young’s lattice. Thus \( Y \) is the poset of partitions \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0) \) ordered by inclusion of Young diagrams (see for example \[12\]). We will often identify a partition with its Young diagram without comment, and will always think of Young diagrams in the English notation (top-left justified). The rank function \( \rho : Y \rightarrow \mathbb{N} \) is given by \( \rho(\lambda) = |\lambda| = \lambda_1 + \cdots + \lambda_l \). Thus the Young diagram of \( \lambda \) has \( \rho(\lambda) \) boxes. A partition \( \mu \) covers \( \lambda \) in \( Y \) if \( \mu \) and \( \lambda \) differ by a box. If \( \lambda \) is a partition then \( \lambda' \) denotes the conjugate partition, obtained by reflecting the Young diagram along the main diagonal. Recall that an outer corner of \( \lambda \) is a box that can be
added to $\lambda$ to obtain a partition, while an inner corner is a box that can be similarly removed.

Define

\[
a(\lambda) = (-1)^{\lambda_2 + \lambda_4 + \cdots}
\]

\[
a'(\lambda) = a(\lambda') = (-1)^{\lambda_2' + \lambda_4' + \cdots}
\]

We will set $a(\mu/\lambda) = a(\lambda)a(\mu)$ and similarly for $a'$. Note that $a(\mu/\lambda)$ does not depend on $\mu$ and $\lambda$, but only depends on the set of squares which lie in the difference of their Young diagrams. If $\lambda \lessdot \mu$ is a cover with a box added in the $i$-th row then

\[
a'(\mu/\lambda) = (-1)^{\lambda_i} = \begin{cases} 1 & \text{if } \mu/\lambda \text{ is a box on an odd column}, \\ -1 & \text{if } \mu/\lambda \text{ is a box on an row column}. \end{cases}
\]

Define the function $s_\alpha$ on covers $\lambda \lessdot \mu$ in $Y$ by

\[
s_\alpha(\lambda \lessdot \mu) = (-1)^{\lambda_1 + \lambda_2 + \cdots + \lambda_i}
\]

if the box $\mu/\lambda$ is on the $i$-th row. Define the function $s_\beta$ by $s_\beta(\lambda \lessdot \mu) = a(\mu/\lambda)s_\alpha(\lambda \lessdot \mu)$. Thus we obtain two signed posets $Y_\alpha = (Y, s_\alpha, a')$ and $Y_\beta = (Y, s_\beta, a')$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig.png}
\caption{The signed poset $Y_\alpha$. The shapes $\lambda$ such that $a'(\lambda) = v(\lambda) = -1$ are shaded.}
\end{figure}

The following theorem is similar to a calculation made in [11], with a different definition of $U$ and $D$.

**Theorem 4.1.** The signed posets $Y_\alpha$ and $Y_\beta$ are weakly signed differential.

**Proof.** We first prove the theorem for $Y_\alpha$. Let $\lambda$ and $\mu$ be two distinct partitions satisfying $n = |\lambda| = |\mu|$. If $\langle (UD + DU)\lambda, \mu \rangle \neq 0$ then it must be the case that $\lambda \cap \mu = \nu$ where
Figure 2. The signed poset $Y_β$. The shapes $λ$ such that $a′(λ) = v(λ) = -1$ are shaded.

$|ν| = n - 1$. Let $ρ = λ ∪ μ$. Suppose (without loss of generality) $λ/ν$ lies on the $i$-th row and $μ/ν$ lies on the $j$-th row, where $i < j$. Then

$$
\langle UD λ, μ \rangle = (-1)^{μ_1 + \cdots + μ_i}(-1)^{μ_1 + \cdots + μ_j}a′(λ/ν)
$$

$$
= (-1)^{λ_i + 1 + \cdots + λ_j}a′(λ/ν)
$$

and

$$
\langle DU λ, μ \rangle = (-1)^{λ_1 + \cdots + λ_j}(-1)^{μ_1 + \cdots + μ_j}a′(ρ/μ)
$$

$$
= (-1)^{1 + λ_i + \cdots + λ_j}a′(ρ/μ) = \langle −UD λ, μ \rangle
$$

using the fact that $μ_i = λ_i + 1$ and the equality $a′(ρ/μ) = a′(λ/ν)$.

Now we check that $\langle (UD + DU)λ, λ \rangle = 1$. We have

$$
UD λ = \left( \sum_{μ ≤ λ} a′(λ/μ) \right) λ
$$

and

$$
DU λ = \left( \sum_{μ ≥ λ} a′(ν/λ) \right) λ.
$$

Using (8) we may pair up each inner corner of $λ$ with the outer corner of $λ$ in the next column to obtain the required identity. The coefficient of 1 arises from the outer corner $ν > λ$ of $λ$ in the first column, which has coefficient $a′(ν/λ) = 1$.

Now for $Y_β$, the calculation of $\langle (UD + DU)λ, λ \rangle$ is identical, while the calculation of $\langle UDλ, μ \rangle$ and $\langle DUλ, μ \rangle$ is modified by a factor of $a(λ)a(μ)$ throughout. □

Obviously the edge labelings $s_α, s_β$ can be modified in other ways to still obtain a weakly differential poset.
Theorem 4.2. The signed poset $Y_\alpha$ is $\alpha$-signed differential and the signed poset $Y_\beta$ is $\beta$-signed differential.

Proof. After Theorem 4.1, we need only check the equations (5) and (6). For the proof.

\[ s_\beta(\lambda < \nu)a'(\nu / \lambda) - \sum_{\mu < \lambda} s_\beta(\mu < \lambda). \]

If $\nu / \lambda$ is an outer corner in row $i$ of $\lambda$ and $\lambda / \mu$ is the inner corner in row $i - 1$ then

\[ a(\lambda / \mu) = (-1)^{\lambda_1 + \cdots + \lambda_i}a(\lambda / \mu) = -a(\lambda / \mu) (-1)^{\lambda_1 + \cdots + \lambda_i}a(\nu / \lambda). \]

and using (5),

\[ a'(\nu / \lambda)s_\beta(\lambda < \nu) = (-1)^{\lambda_i}(-1)^{\lambda_1 + \cdots + \lambda_i}a(\nu / \lambda) = (-1)^{\lambda_1 + \cdots + \lambda_i}a(\nu / \lambda). \]

Since $a(\lambda / \mu) = -a(\nu / \lambda)$ the contributions of these two corners cancel out. Finally for the unique outer corner $\nu / \lambda$ in the first row we obtain a coefficient of $(-1)^{\lambda_1}a'(\nu / \lambda)a(\nu / \lambda) = 1$.

For the $\alpha$ case, we note that without the additional factors $a(\nu / \lambda)$ and $a(\lambda / \mu)$, the contributions of the paired corners $\nu / \lambda$ and $\lambda / \mu$ will still cancel out if we calculate $(U + D)P$ instead.

\[ \square \]

Remark 4.3. Theorem 4.2 allows one to show that the polynomials $g_k^\alpha(z)$, $g_k^\beta(z)$ in Lemmas 3.6 and 3.7 and the coefficients $c_{ij}(w)$ in the proof of Theorem 3.3 are uniquely defined. This follows from the fact that in $Y_\alpha$ or $Y_\beta$, the element $U^n0$ contains the partition $(n)$ with non-zero coefficient and so is non-zero. Similarly, $D^n(n)$ is a non-zero multiple of $0$. Thus one can “extract” the coefficients of $g_k^\alpha(z)$, $g_k^\beta(z)$ and $c_{ij}(w)$ one by one.

4.2. Fibonacci poset and signed reflection extensions. Let $(P, s, v)$ be a signed poset such that $P = \bigcup_{0 \leq i \leq n} P_i$. Suppose $P$ is $\alpha$- or $\beta$-signed differential up to level $n - 1$. In other words $UD + DU = I$ when restricted to $\hat{K}(\bigcup_{0 \leq i \leq n-1} P_i)$, and we have $(U + D)P$ or $(U - D)P$ equal to $\sum_{0 \leq i \leq n-1} P_i$ modulo $KP_i$.

We will now construct a signed poset $P^+ = \bigcup_{0 \leq i \leq n+1} P_i^+$ with one more level than $P$. The signed poset $(P^+, s^+, v^+)$ will satisfy $P_1^+ = P_1$ for $0 \leq i \leq n$ and also the equalities $s^+|P = s$ and $v^+|P = v$. First let $P_{n+1}^+$ consist of elements $x^+$ for each $x \in P_{n-1}$ and elements $y^*$ for each $y \in P_n$. We will add the cover relations $y^* > y$ for each $y \in P_n$ and $x^+ > y$ if $y > x$, for each $x \in P_{n-1}, y \in P_n$. We then define

\[ v^+(y^*) = v(y), \quad v^+(x^+) = -v(x) \]

and

\[ s^+(y < y^*) = 1, \quad s^+(y < x^+) = \begin{cases} -v^+(x^+)v^+(y)s(x < y) & \text{in the } \alpha \text{ case,} \\ v^+(x^+)v^+(y)s(x < y) & \text{in the } \beta \text{ case.} \end{cases} \]

Let us use the notation $P^{+t}$ defined inductively $P^{+t} = (P^{+(t-1)})^+$. The following proposition is a signed analogue of [9, Proposition 6.1].
Proposition 4.4. Suppose \((P = \bigcup_{0 \leq i \leq n} P_n, s, v)\) is \(\alpha\)- or \(\beta\)-signed differential up to level \(n - 1\). Then \((P^+, s^+, v^+ )\) is \(\alpha\)- or \(\beta\)-signed differential up to level \(n\). Thus \(\lim_{t \to \infty} P^{+t}\) is a \(\alpha\)- or \(\beta\)-signed differential poset.

Proof. The proof is a straightforward case-by-case analysis: the signed contribution of each cover cancels out with the contribution from the reflected cover. \(\Box\)

Remark 4.5. Just as in the case of differential posets, the construction described in Proposition 4.4 allows one to describe infinitely many non-isomorphic signed differential posets. They are obtained by applying the signed reflection extension to the first \(n\)-levels of the \(\alpha\)- or \(\beta\)-signed Young lattices.

Let \(Q = (\hat{0}, s, v)\) be the one element signed poset with \(v(\hat{0}) = 1\). Let \(F_\alpha = (F_\alpha, s_\alpha, v_\alpha)\) and \(F_\beta = (F_\beta, s_\beta, v_\beta)\) denote the \(\alpha\)- and \(\beta\)-signed differential posets obtained by the construction \(\lim_{t \to \infty} Q^{+t}\). Note that \(v_\alpha = v_\beta\). We call these the \((\alpha \text{ or } \beta)\) signed Fibonacci differential posets.

We now give a non recursive description of \(F_\alpha\) and \(F_\beta\). Define the Fibonacci differential poset \(F = \bigcup_{r \geq 0} F_r\) by letting \(F_r\) be the set of words in the letters \(\{1, 2\}\) such that the sum of the letters is equal to \(r\). The covering relations \(x < y\) in \(F\) are of two forms:

(a) \(x\) is obtained from \(y\) by changing a 2 to a 1, provided that the only letters to the left of this 2 are also 2’s; or

(b) \(x\) is obtained from \(y\) by deleting the first 1 occurring in \(y\).

The Fibonacci differential poset \(F\) was defined in [9] and independently in [1] where it was called the Young-Fibonacci Lattice.

The poset \(F\) is the underlying poset of the signed posets \(F_\alpha\) and \(F_\beta\). The reflection extension can be described explicitly as follows. The word \(x^+\) is obtained from \(x\) by prepending a 2. The word \(y^*\) is obtained from \(y\) by prepending a 1. If \(a(x)\) denotes the number of 2’s in \(x\) then define \(v(x) = (-1)^{a(x)}\) and

\[
\begin{align*}
  s'_\alpha(x \ll y) &= \begin{cases} 
  (-1)^{i+1} & \text{if } x \text{ is } y \text{ with the (first) 1 in the } i\text{-th place deleted}, \\
  (-1)^i & \text{if } x \text{ is } y \text{ with a 2 changed to a 1 in the } i\text{-th place.}
  \end{cases} \\
  s'_\beta(x \ll y) &= \begin{cases} 
  (-1)^{i+1} & \text{if } x \text{ is } y \text{ with the (first) 1 in the } i\text{-th place deleted}, \\
  (-1)^i & \text{if } x \text{ is } y \text{ with a 2 changed to a 1 in the } i\text{-th place.}
  \end{cases}
\end{align*}
\]

(10) \hspace{1cm} (11)

Proposition 4.6. The signed posets \((F, s'_\alpha, v)\) and \((F, s'_\beta, v)\) are identical to \((F_\alpha, s_\alpha, v_\alpha)\) and \((F_\beta, s_\beta, v_\beta)\) respectively.

Proof. The fact that the underlying posets are equal is straightforward to verify (see also [9]). The equality \(v = v_\alpha = v_\beta\) follows immediately from induction.

To show that \(s'_\alpha = s_\alpha\) and \(s'_\beta = s_\beta\) we again proceed by induction, using the following observations. First, clearly the definitions agree on covers of the form \(y \ll y^*\).

If \(x\) is obtained from \(y\) by changing a 2 to a 1 in the \(i\)-th position then \(y\) is obtained from \(x^+ = 2x\) by deleting a 1 in the \((i + 1)\)-position. In this case we have \(s_\alpha(x \ll y) = -s_\alpha(y \ll x^+)\) and \(s_\beta(x \ll y) = s_\beta(y \ll x^+)\).
If \( x \) is obtained from \( y \) by deleting a 1 in the \( i \)-position, then \( y \) is obtained from \( x^+ = 2x \) by changing a 2 to a 1 in the \( i \)-th position. In this case we have \( s_\alpha(x < y) = s_\alpha(y < x^+) \) and \( s_\beta(x < y) = -s_\beta(y < x^+) \). \( \square \)

The weighted sum of chains \( e_\alpha(x) \) and \( e_\beta(x) \) for \( F_\alpha \) and \( F_\beta \) can be calculated explicitly. We say that a word \( x \in F \) is \textit{domino-tileable} if every non-initial, maximal, consecutive subsequence of 1’s in \( x \) has even length. For example \( x = 1221121111 \) is domino-tileable but \( y = 11212 \) is not.

**Theorem 4.7.** Suppose \( x \in F \). Then

\[
e_\alpha(x) = \begin{cases} 
1 & \text{if } x \text{ is domino-tileable,} \\
0 & \text{otherwise.}
\end{cases}
\]

\[
e_\beta(x) = \begin{cases} 
v(x) & \text{if } x \text{ is domino-tileable,} \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** We proceed by induction. The result is clearly true for \( x = \hat{0} \), the empty word. Now let \( x \in F \) be an arbitrary word, and suppose \( x = 2^j \) for some word \( w \). Then in \( F \), the word \( x \) covers the set of words \( C^-(x) = \{2^i12^{j-i-1}w\} \cup \{2^jw\} \) where \( 0 \leq i \leq j - 1 \).

We use the recursive formula

\[
e(x) = \sum_{y \in C^-(x)} s(y < x) e(y).
\]

Suppose that the formula is known for all \( y < x \). If \( j = 0 \) the formula is immediate from \( s(y < 1y = x) = 1 \). For \( j \geq 1 \), the only possibly domino-tileable \( y \in C^-(x) \) are \( y_1 = 12^{j-1}w, y_2 = 2^{j-1}11w, y_3 = 2^jw \). If \( x \) is domino-tileable then only \( y_1 \) is (if \( j = 1 \), then \( y_1 = y_2 \)), and \( e(x) = s(y_1 < x) e(y_1) \), which agrees with the theorem, using equations (10) and (11).

Otherwise, \( x \) is not domino-tileable. If \( j = 1 \), then \( y_1 = y_2 \). The word \( y_1 \) is domino-tileable if and only if \( y_3 \) is and using equations (10) and (11) we see that these contributions to \( e(x) \) cancel out. Thus we may assume that \( j > 1 \). Suppose first that \( w \) is domino-tileable. In this case only \( y_2 \) and \( y_3 \) are domino-tileable and again their contributions to \( e(x) \) cancel out. If \( w \) is not domino-tileable, then none of \( y_1, y_2, y_3 \) are domino-tileable, so again \( e(x) = 0 \).

Finally we consider the case \( x = 2^j \), where \( j > 0 \). In this case \( x \) is domino-tileable but covers only a single domino-tileable word \( y = 12^{j-1} \). Again the stated result follows inductively. \( \square \)

5. **Sign-imbalance**

We indicate here how our results can be applied to sign-imbalance. If \( P \) is a poset then a bijection \( \omega : P \rightarrow [n] = \{1, 2, \ldots, n\} \) a is called a \textit{labeling} of \( P \). A linear extension of \( P \) is an order-preserving map \( f : P \rightarrow [n] \). Given a labeled poset \((P, \omega)\) and a linear extension \( f \) of \( P \), we obtain a permutation \( \pi(f) = \omega(f^{-1}(1))\omega(f^{-1}(2)) \cdots \omega(f^{-1}(n)) \in S_n \). We denote the set of linear extensions of \((P, \omega)\) by \( L(P, \omega) \). The \textit{sign-imbalance}
of \((P, \omega)\) is the sum \(I_{P,\omega} \sum_{f \in L(P,\omega)} \text{sign}(\pi(f))\). Up to sign, \(I_{P,\omega}\) only depends on \(P\). Sign-imbalance was first studied by Ruskey [6].

We first note a general basic property of the sign-imbalance of any poset \(P\) (see [11]). Let \(P\) be a finite poset with minimum element \(\hat{0}\). We say that \(P\) is domino-tileable if we can find an increasing chain of order ideals (called a domino tiling)

\[
D = (I_0 \subset I_1 \subset \cdots \subset I_r = P)
\]

where the set theoretic difference \(I_i - I_{i-1}\) for \(1 \leq i \leq r\) is a chain consisting of two elements and \(|I_0| = 1\) or \(0\) (depending on whether \(|P|\) is odd or even). Note that each domino tiling \(D\) of \(P\) gives rise to a linear extension \(f_D\) of \(P\), which is the unique linear extension satisfying \(f_D(I_{i-1}) < f_D(I_{i} - I_{i-1})\). The following Lemma follows from a sign-reversing involution argument (see [4, 11]).

**Lemma 5.1.** Let \(P\) be a finite poset with minimum element \(\hat{0}\) and \(\omega : P \to \{1, 2, \ldots, n\}\) any labeling of \(P\). If \(P\) is domino-tileable, then

\[
I_{P,\omega} = \sum_D \text{sign}(\pi(f_D))
\]

where the summation is over the domino-tilings of \(P\). If \(P\) is not domino-tileable then \(I_{P,\omega} = 0\).

5.1. **Sign-imbalance of partition shapes.** First we consider the case of Young’s lattice \(Y\). Let \(\lambda\) be a partition and \(T\) a standard Young tableau (SYT) of shape \(\lambda\) (see [12]). We will always draw our partition and tableaux in English notation. Picking the reverse of the standard labeling of the poset \(P_\lambda\) corresponding to the Young diagram of \(\lambda\), we can define the sign imbalance \(I_\lambda\) explicitly as follows. The reading word \(r(T)\) (or more precisely the reverse row reading word) is the permutation obtained from \(T\) by reading the entries of \(T\) from right to left in each row, starting with the bottom row and going up. The sign \(s(T)\) is the sign of \(r(T)\) as a permutation. Then the sign imbalance is given by

\[
I_\lambda = \sum_T s(T)
\]

where the summation is over all standard Young tableaux \(T\) with shape \(\lambda\). We omit the labeling of the poset \(P_\lambda\) in our notation.

![Tableau](https://via.placeholder.com/150)

**Figure 3.** A tableau \(T\) with shape 531, reading word \(r(T) = 496387521\) and sign \(s(T) = 1\).

**Remark 5.2.** Our reading order is the reverse of the reading order usually used to define sign-imbalances for partitions [4, 5, 7, 11]. The resulting sign-imbalances differ by a factor of \((-1)^\binom{n}{2}\), where \(n = |\lambda|\).
We can connect the sign imbalance of Young diagrams with signed differential posets as follows. For \( \lambda \in Y \) denote by \( e_\alpha(\lambda) \) and \( e_\beta(\lambda) \) the signed sums of chains in the signed posets \( Y_\alpha \) and \( Y_\beta \) respectively.

**Proposition 5.3.** Let \( \lambda \in Y \). Then \( e_\alpha(\lambda) = I_\lambda \) and \( e_\beta(\lambda) = a(\lambda)I_\lambda \).

**Proof.** A standard Young tableau \( T \) of shape \( \lambda \) is simply a maximal chain \( \emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(l)} = \lambda \) in \( Y \). For a cover \( \lambda^{(i-1)} < \lambda^{(i)} \) on the \( r \)-th row, the sum \( \lambda^{(i-1)}_1 + \cdots + \lambda^{(i-1)}_r \) is equal to the number of letters less than \( i \) appearing after \( i \) in \( r(T) \). \( \square \)

As corollaries we obtain the following theorem.

**Theorem 5.4.** Suppose \( n \geq 2 \). Then
\[
\sum_{\lambda \vdash n} a'(\lambda) I_\lambda^2 = 0, \quad \sum_{\lambda \vdash n} a'(\lambda) I_\lambda = 0, \quad \sum_{\lambda \vdash n} a(\lambda)a'(\lambda) I_\lambda = 2^{|n/2|}.
\]

Theorem 5.4 was earlier conjectured in [11] and proved independently in [4, 5, 7].

**Proof.** Using Proposition 5.3 and Theorem 4.2, the result follows immediately from applying Theorems 3.2 and 3.8 to \( Y_\alpha \) and \( Y_\beta \). \( \square \)

One can define \( I_{\lambda/\mu} \) for the Young diagrams of skew shapes in an analogous manner to \( I_\lambda \), by using a fixed reading order. For our purposes, we suppose that we have picked a reading order so that \( I_{\lambda/\mu} = (U^n, \mu, \lambda) \) in the \( \alpha \)-case and \( I_{\lambda/\mu} = a(\lambda/\mu) (U^n, \mu, \lambda) \) in the \( \beta \)-case. Here \( n = |\lambda/\mu| \). This differs from using the (reverse row) reading word by a factor of \((-1)^a\), where \( a \) is equal to the number of pairs \((s, t)\) of squares \( s \in \lambda/\mu \) and \( t \in \mu \) such that \( t \) is either higher than or on the same row and to the left of \( s \).

The following result, due originally to Sjöstrand [8], follows from Lemma 2.5.

**Theorem 5.5.** Let \( \lambda \) be a fixed partition, and \( n \in \mathbb{N} \). Then
\[
\sum_{\mu/\lambda \vdash n} a'(\mu) I_{\mu/\lambda}^2 = \begin{cases} 
\sum_{\lambda/\mu \vdash n} a'(\mu) I_{\lambda/\mu}^2 & \text{if } n \text{ is even,} \\
\sum_{\lambda/\mu \vdash n-1} a'(\nu) I_{\lambda/\mu}^2 - \sum_{\lambda/\mu \vdash n} a'(\nu) I_{\lambda/\mu}^2 & \text{if } n \text{ is odd.}
\end{cases}
\]

In Theorem 5.5 the coefficients \( a'(\mu) \) and \( a'(\nu) \) can be replaced with \( a(\mu) \) and \( a(\nu) \) by making a similar change in the definition of \( Y_\alpha \). In this form, Theorem 5.5 is exactly Theorem 4.4 of [8]. As a consequence of Theorem 3.5, we have the following result.

**Theorem 5.6.** Let
\[
F_k(t) = \sum_{|\mu/\lambda| = k} a'(\mu/\lambda) I_{\mu/\lambda}^2 t^{|\lambda|}.
\]
Then
\[
F_k(t) = \begin{cases} 
\prod_{i=1}^{\infty} \frac{1}{1-t^i} & \text{if } k = 0, \\
\frac{1}{1+t} \prod_{i=0}^{\infty} \frac{1}{1-t^i} & \text{if } k = 1, \\
0 & \text{if } k \geq 2.
\end{cases}
\]

We have used the fact that \( \sum_{\lambda \in Y} t^{|\lambda|} = \prod_i \frac{1}{1-t^i} \). We write down one more result explicitly from Proposition 3.11.
Theorem 5.7. Let \( l \in \mathbb{N} \). Define
\[
G_{2l}(t) = \sum_{|\lambda/\mu| = 2l} a(\lambda/\mu) I_{\lambda/\mu} t^{\mu}. \]

Then
\[
G_{2l}(t) = \left( \frac{2}{1 + t^2} \right)^{l} \sum_{\lambda \in \mathcal{Y}} a'(\lambda) t^{\lambda} \]
\[
= \left( \frac{2}{1 + t^2} \right)^{l} \prod_{i=0}^{\infty} \left( \frac{1}{(1 - t^{4i+1})(1 + t^{4i+2})(1 + t^{4i+3})(1 - t^{4i+4})} \right). \]

Proof. We have \( G_{2l}(t) = G_{2l}(Y_\beta, t) \) and so the first equation follows from Proposition 3.11. The explicit expression for \( \sum_{\lambda \in \mathcal{Y}} a'(\lambda) t^{\lambda} \) follows directly from the definition of \( a'(\lambda) \). \( \square \)

The constant term of the identity in Theorem 5.7 recovers the \( 2^{\lfloor n/2 \rfloor} \) identity of Theorem 5.4. One can deduce many other results concerning sign-imbalance from signed differential posets. We leave this translation of our other results, such as Proposition 3.10 to the reader. An interpretation of Theorem 3.3 would require defining the sign of an oscillating tableau. However, such a definition does not seem completely natural in the setting of sign-imbalance.

5.2. Fibonacci sign-imbalance. We now define the Fibonacci distributive lattice \( \text{Fib} \). The poset \( \text{Fib} \) has the same set elements as the Fibonacci differential poset \( F \), the set of words in the letters \( \{1, 2\} \). However the cover relations are defined differently. If \( x = x_1 x_2 \cdots x_r \) and \( y = y_1 y_2 \cdots y_l \) then \( x \prec y \) if \( r \leq l \) and \( x_i \leq y_i \) for \( 1 \leq i \leq r \). The poset \( \text{Fib} \) is a distributive lattice. It is equal to the lattice of order ideals in an infinite dual (upside-down) tree. For \( x \in \text{Fib} \), we let \( T_x \) denote the corresponding dual tree. A chain from \( \hat{0} \) to \( x \) in \( \text{Fib} \) is a linear extension of \( T_x \) which can be expressed simply as a tableau \( T \) of the form shown in Figure 4. The column lengths, read from left to right, give the word \( x \). The top row is required to be increasing, and each column is also required to be increasing. The reading word \( r(T) \) of such a Fibonacci tableau is obtained by reading the columns from bottom to top, starting with the leftmost column. This reading order defines a labeling \( \omega_x \) of \( T_x \) and the corresponding sign-imbalance is
\[
I_x = I_{T_x, \omega_x} = \sum_T s(T) \]
where \( s(T) \) is the sign of \( r(T) \) and \( T \) varies over all Fibonacci tableaux with shape \( x \).

![Figure 4. A Fibonacci tableau T with shape 212112 and reading word r(T) = 312745698.](image)
We observe that a word $x$ is domino-tileable in the notation of Section 4.2 if and only if the tree $T_x$ is domino-tileable as a poset. Also recall that we define $v(x)$ to be $(-1)^{a(x)}$ where $a(x)$ is the number of 2’s in $x$.

**Proposition 5.8.** Let $x \in \text{Fib}$. Then
\[
I_x = \begin{cases} 
 v(x) & \text{if } x \text{ is domino-tileable}, \\
0 & \text{otherwise}.
\end{cases}
\]
Thus $I_x = e(x)$ when $x$ is considered an element of $F_\beta$.

**Proof.** The proposition follows nearly immediately from Lemma 5.1. When $T_x$ is domino-tileable, it has a unique domino tiling with corresponding linear extension of the form
\[
\begin{array}{cccccccc}
1 & 2 & 4 & 6 & 7 & 8 \\
3 & 5 & 9
\end{array}
\]
Here the boxes occupied by $\{2,3\}$, $\{4,5\}$, $\{6,7\}$ and $\{8,9\}$ form the dominos. The reading word of such a linear extension has exactly as many inversions as columns of length 2, and so $I_x = v(x)$. \(\square\)

Proposition 5.8 is another example of enumerative properties agreeing for the Fibonacci differential poset $F$ and the Fibonacci distributive lattice $\text{Fib}$ (see [10]). The Fibonacci analogue of Theorem 5.4 is the following result.

**Theorem 5.9.** Suppose $n \geq 2$. Then
\[
\sum_{x \in \text{Fib}_n} v(x) I_x^2 = 0, \quad \sum_{x \in \text{Fib}_n} v(x) I_x = 2^{\lfloor n/2 \rfloor}.
\]

**Proof.** Since $I_x = e(x)$ with $x$ considered an element of $F_\beta$, the result follows from applying Theorems 3.2 and 3.8 to the $\beta$-signed differential poset $F_\beta$. \(\square\)

Theorem 5.9 is not difficult to prove directly. For example, it is easy to see that there are $2^{\lfloor n/2 \rfloor}$ domino-tileable $\{1,2\}$-words with sum $n$: this corresponds to the $\lfloor n/2 \rfloor$ choices between a ‘2’ or a ‘11’, or alternatively between a vertical domino and a horizontal domino. One can obtain many more identities for Fibonacci sign-imbalance using our results, and we leave the experimentation to the reader. We note the signed rank generating function
\[
\sum_{x \in \text{Fib}} v(x) t^{|x|} = \frac{1}{1 - t + t^2}.
\]

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