A class of infinite dimensional stochastic processes with unbounded diffusion

John Karlsson* and Jörg-Uwe Löbus

Matematiska Institutionen, Linköpings Universitet, SE-581 83 Linköping, Sweden

(Received 14 March 2014; accepted 27 August 2014)

The paper studies Dirichlet forms on the classical Wiener space and the Wiener space over non-compact complete Riemannian manifolds. The diffusion operator is almost everywhere an unbounded operator on the Cameron–Martin space. In particular, it is shown that under a class of changes of the reference measure, quasi-regularity of the form is preserved. We also show that under these changes of the reference measure, derivative and divergence are closable with certain closable inverses. We first treat the case of the classical Wiener space and then we transfer the results to the Wiener space over a Riemannian manifold.

Keywords: Dirichlet form on Wiener space; Dirichlet form on Wiener space over non-compact manifold; closability; weighted Wiener measure; quasi-regularity

AMS Subject Classification: 60J60; 58J65

1. Introduction

This paper is concerned with Dirichlet forms of type

\[ \mathcal{E}(F, G) = \int \langle DF, ADG \rangle_{\mathcal{H}} \varphi \, d\nu \]

\[ = \int \left( DF, \sum_{i=1}^{\infty} \lambda_i \langle S_i, DG \rangle_{\mathcal{H}} S_i \right) \varphi \, d\nu, \tag{1} \]

where the diffusion operator \( A \) is in general \( \nu \)-a.e. unbounded, cf. [11]. This topic has received increasing interest over the past years, see, e.g. [3,18,19]. We are interested in weight functions \( \varphi \) of the form

\[ \varphi(\gamma) = \exp \left\{ \int_{0}^{1} \langle b(\gamma_s), d\gamma_s \rangle_{\mathcal{E}} - \frac{1}{2} \int_{0}^{1} |b(\gamma_s)|^2 \, ds \right\}. \]

This choice of weight functions is motivated by the geometric setting in the works [18,19] by Wang and Wu. Accordingly the weight function \( \varphi \) will be specified in Section 6, where we consider the above bilinear form in a geometric framework. In addition we study a weight function of similar type in the flat case. We present our ideas in terms of infinite dimensional processes on the classical Wiener space using a coordinate representation. This makes the subject comprehensible, in particular, to readers familiar with the finite
dimensional theory. The form is studied on the set of smooth cylindrical functions of type

\[ F, G \in Y = \{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)) : s_j \text{ is a dyadic point} \}, \]

where \( \gamma \) is a Wiener trajectory. Simultaneously we will also study the form on the more common set of cylindrical functions

\[ F, G \in Z = \{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)) : s_j \in [0,1] \}. \]

Well-definiteness of \( \mathcal{E} \) on \( Y \) is a consequence of the fact that the sum in (1) is finite. Well-definiteness of \( \mathcal{E} \) on \( Z \) requires the convergence of the sum in (1). These two different initial situations result in possibly different closures of \((\mathcal{E}, Y)\) and \((\mathcal{E}, Z)\) on \( L^2(\varphi \nu) \). Using the coordinate representation in (1) we give conditions for closability. It turns out that in the classical case, i.e. \( \lambda_1 = \lambda_2 = \cdots = 1 \), the condition \( \varphi^{-1} \in L^1(\nu) \) is sufficient for closability. However, the paper investigates forms of structure (1) with an in general \( \nu \)-a.e. unbounded diffusion operator. 

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In paper [4] the classical quasi-regular Dirichlet form on a compact Riemannian manifold has been constructed and associated with a diffusion process. The paper [11] extends the same idea in particular to the case of unbounded diffusion. There the form is

\[ \int \langle DF, ADG \rangle_{\mathbb{H}} \text{d}\nu, \]  

where operator \( A \) acts in \( L^2(\nu, \mathbb{H}) \), has a representation

\[ A\Phi(\gamma) := \sum_{i=1}^{\infty} \lambda_i \langle S_i, \Phi(\gamma) \rangle_{\mathbb{H}} S_i \]

and the constant diffusion coefficients \( \lambda_n \) are assumed to be bounded from below and possibly unbounded. It is shown that under the assumption \( \lambda_n \leq cn^{1-\epsilon} \), the form is a quasi-regular Dirichlet form and thus has an associated process.

The approach in this paper is similar to that in [18]. Again, infinite dimensional diffusion processes are investigated, where the diffusion is related to possibly unbounded operators. This article and the work by Wang and Wu [18] refer to the ideas of Lòbus [11], however study forms of type (2) with non-constant diffusion coefficients. The theory in [18] is presented in a geometric framework, i.e. on the Wiener trajectories over a complete non-compact Riemannian manifold while here we first consider the form on the classical Wiener space and later turn to a geometric setting. In addition we also take particular consideration to the special case when the form is defined pointwise, constant over the trajectory space as in [11].
The operator \((A, D(A))\) in [18] is introduced as a densely defined self-adjoint operator such that
\[
Af\Phi = fA\Phi \quad \text{for } f \in L^\infty(\nu) \text{ and } \Phi \in D(A) \text{ such that } f\Phi \in D(A)
\]
and \(A \succeq \varepsilon \text{Id}\) for some \(\varepsilon > 0\).

Here \(\nu\) is the Wiener measure on the flat trajectory space, or trajectory space over a complete non-compact Riemannian manifold. Furthermore \(\text{Id}\) denotes the identity operator. Under \((A0)\) the following is obtained. For \(f, \Phi\) as above and \(\Psi \in D(A)\) such that \(f\Psi \in D(A)\) it holds that
\[
\int_0^1 \langle (A\Phi)\cdot \Psi \rangle \, ds \, d\nu = \int_0^1 \Phi(A(f\Psi)) \, ds \, d\nu
\]
\[
= \int_0^1 \Phi(A\Psi) \, ds \, d\nu,
\]
which implies
\[
\langle A\Phi(\gamma), \Psi(\gamma) \rangle_{\mathbb{H}} = \langle \Phi(\gamma), A\Psi(\gamma) \rangle_{\mathbb{H}},
\]
for \(\nu\text{-a.e. } \gamma \in \Omega\). This observation suggests that
\[
(A\Phi)(\gamma) = A(\gamma)\Phi(\gamma),
\]
\[\text{cf. assumption (2) in Remark 1.1 of [18], that is } (A\Phi)^{1/2}(\gamma) = A(\gamma)^{1/2}\Phi(\gamma) \text{ for } \nu\text{-a.e. } \gamma \in \Omega.\]
Below we specify
\[
D(A) := \left\{ \Phi \in L^2(\nu; \mathbb{H}) : \sum_{i=1}^\infty \lambda_i(\gamma)^2 \langle S_i, \Phi(\gamma) \rangle_{\mathbb{H}}^2 \, d\nu < \infty \right\},
\]
\[
A\Phi(\gamma) := \sum_{i=1}^\infty \lambda_i(\gamma) \langle S_i, \Phi(\gamma) \rangle_{\mathbb{H}} S_i, \quad \gamma \in \Omega, \quad \Phi \in D(A).
\]
In [18] the form
\[
E_A(F, G) = \int_{0}^{1} \left\langle A^{1/2}DF, A^{1/2}DG \right\rangle_{\mathbb{H}} \, d\mu
\]
is defined where \(F, G \in \{ f \in \mathcal{F}C_0^\infty : Df \in D(A^{1/2}) \}\) and \(\mathcal{F}C_0^\infty\) denotes the set of cylindrical functions \(\{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)) : s_j \in [0,1], f \in C_0^\infty \}\). We take over this concept however in the flat case we use the cylindrical functions in \(Y\) or \(Z\) for the initial definition.

We give a necessary and sufficient condition for well-definiteness and closability of the bilinear form. In the particular case of non-decreasing eigenvalues \(0 < \lambda_1(\gamma) < \)
\( \lambda_2(\gamma), \ldots \) it reads as
\[
\sum_{p=0}^{\infty} \frac{\int \lambda_{2p} \varphi \, d\nu}{2^p} < \infty.
\]

In order to handle weights, \( \varphi \), with respect to the Wiener measure, the paper [18] introduces geometric conditions. We use
\[
\frac{f}{\varphi} \in L^1(\nu), \quad f \in Z,
\]
to verify closability of the classical form, which was first treated in [4]. To show quasi-regularity this paper and [18] both use the method of [4,11,15]. The proof in [18] involves a geometric result of [17]. We use alternative conditions on the manifold.

2. Definitions
We study the form on the space \( L^2(\varphi \nu) = L^2(\Omega, \varphi \nu) \) where \( \Omega := C_0([0,1]; \mathbb{R}^d) := \{ \gamma \in C([0,1]; \mathbb{R}^d), \gamma(0) = 0 \} \), \( \nu \) is the Wiener measure on \( \Omega \) and \( \varphi \) is a density function specified below. As stated earlier, the form is given by
\[
\mathcal{E}(F,G) = \int (DF, ADG)_{\mathbb{H}} \varphi \, d\nu, \quad F, G \in D(\mathcal{E}),
\]
where \( \mathbb{H} \) is the Cameron–Martin space, i.e. the space of all absolutely continuous \( \mathbb{R}^d \)-valued functions \( f \) on \([0,1]\), with \( f(0) = 0 \) and equipped with inner product
\[
\langle \varphi, \psi \rangle_{\mathbb{H}} := \int_{[0,1]} \langle \varphi(x), \psi(x) \rangle_{\mathbb{R}^d} \, dx.
\]
Motivated by [18], we suppose for proving quasi-regularity of the form that \( \varphi : \Omega \to [0, \infty] \) has the form
\[
\varphi(\gamma) = \exp \left\{ \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^1 |b(\gamma_s)|^2 \, ds \right\}
\]
where we choose \( b \) from two different points of view. On the one hand we are interested in closability, see Section 3. On the other hand in Section 6, these coefficients correspond to a change of the Wiener measure to a certain diffusion measure on the manifold. We define the set of all cylindrical functions
\[
Z := \{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)), \gamma \in \Omega : 0 < s_1 < \cdots < s_k = 1, f \in C_0^{\infty}( (\mathbb{R}^d)^k ), k \in \mathbb{N} \},
\]
where \( f \in C^\infty_p \) means that \( f \) and all its partial derivatives are smooth with polynomial growth. We also define

\[
Y := \left\{ F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k)), \gamma \in \Omega : \right. \\
\left. F \in Z, s_1, \ldots, s_k \in \left\{ \frac{l}{2^n} : l \in \{1, \ldots, 2^n\} \right\}, n \in \mathbb{N} \right\}.
\]

For \( F \in Z \) and \( \gamma \in \Omega \) the gradient operator \( D \) is defined by

\[
D_s F(\gamma) = \sum_{i=1}^k (s_i \wedge s)(\nabla_s f)(\gamma), \quad s \in [0, 1],
\]

where \((\nabla_s f)(\gamma) \equiv (\nabla_s f)(\gamma(s_1), \ldots, \gamma(s_k))\) denotes the gradient of the function \( f \) relative to the \( i \)th variable while holding the other variables fixed. We let \((e_j)_{j=1, \ldots, d}\) denote the standard basis in \( \mathbb{R}^d \) and

\[
H_1(t) = 1, \\
H_{2m+k}(t) = \begin{cases} 
2^{m/2} & \text{if } t \in \left[ \frac{k-1}{2^m}, \frac{2k-1}{2^m+1} \right), \\
-2^{m/2} & \text{if } t \in \left[ \frac{2k-1}{2^m+1}, \frac{k}{2^m} \right), \quad k = 1, \ldots, 2^m, \ m = 0, 1, \ldots \\
0 & \text{otherwise},
\end{cases}
\]

denote the system of the Haar functions on \([0, 1]\). We also define

\[
g_d(r-1)+j := H_r \cdot e_j, \quad r \in \mathbb{N}, \ j \in \{1, \ldots, d\}
\]

and

\[
S_n(s) := \int_0^s g_n(u) \, du, \quad s \in [0, 1], \ n \in \mathbb{N}.
\]

Let

\[
D(A) := \left\{ \Phi \in L^2(\nu; \mathbb{H}) : \sum_{i=1}^\infty \lambda_i(\gamma)^2 \langle S_i, \Phi(\gamma) \rangle_{\mathbb{H}} \, d\nu < \infty \right\},
\]

\[
A\Phi(\gamma) := \sum_{i=1}^\infty \lambda_i(\gamma) \langle S_i, \Phi(\gamma) \rangle_{\mathbb{H}} S_i, \quad \gamma \in \Omega, \ \Phi \in D(A).
\]

We can then conclude that

\[
\mathcal{E}(F, F) = \left\langle DF, ADF \right\rangle_{\mathbb{H}} \varphi \, d\nu = \left\langle DF, \sum_{i=1}^\infty \lambda_i(\gamma) S_i, ADF \right\rangle_{\mathbb{H}} S_i \varphi \, d\nu = \sum_{i=1}^\infty \lambda_i(\gamma) \langle S_i, DF(\gamma) \rangle_{\mathbb{H}}^2 \varphi(\gamma) \, d\nu < \infty, \quad F \in Y
\]

is well defined since, for \( F \in Y \), this is just a finite sum.
3. Closability of derivative, divergence and their inverses

In this section we are interested in the closability of the derivative, the divergence and their inverses with respect to a weighted Wiener measure. We use the standard definition of \((\delta, \text{Dom } \delta)\) found in e.g. [13]. For well-definiteness we assume that \(\varphi \in L^1(\nu)\). The following lemma provides a general density result when using a weighted measure.

**Lemma 3.1.** Let \(m\) be a finite Borel measure over a separable metric space \(E\) and \(\mathbb{H}\) be a separable Hilbert space.

(a) Let \(F \subset L^\infty(E, m)\) be dense in \(L^2(E, m/\varphi)\). If \(\varphi \in L^1(E, m)\) such that \(1/\varphi \in L^1(E, m)\), then \(G := \{f/\varphi : f \in F\}\) is dense in \(L^2(E, \varphi m)\).

(b) If \(F \subset L^2(E, m)\) is dense in \(L^2(E, m)\) then \(F\mathbb{H} := \{\sum_{i=1}^k f_i \cdot h_i : f_i \in F, h_i \in \mathbb{H}, k \in \mathbb{N}\}\) is dense in \(L^2(E, m; \mathbb{H})\).

(c) Let \(F\) be as in (a) and \(F\mathbb{H}\) be defined as in (b). Then \(G\mathbb{H} := \{f/\varphi : f \in F\mathbb{H}\}\) is dense in \(L^2(E, \varphi m; \mathbb{H})\).

**Proof.**

(a) The statement is equivalent to that \(\{f/\sqrt{\varphi} : f \in F\}\) is dense in \(L^2(m)\). Approximating any \(\Psi \in L^2(m)\) by \(f_n/\sqrt{\varphi}\) in \(L^2(m)\) is equivalent to approximation of \(\sqrt{\varphi} \Psi \in L^2(m/\varphi)\) by \(f_n\) in \(L^2(m/\varphi)\). The statement now follows from the density of \(F\).

(b) This is a standard result and the proof is therefore omitted.

(c) By (a), \(G \subset L^2(E, \varphi m)\) is dense in \(L^2(E, \varphi m)\). The statement now follows from (b) with \(m\) replaced by \(\varphi m\) and \(F\) replaced by \(G\).

**Remark 1.** We note that in Lemma 3.1 the condition \(F \subset L^\infty(E, m)\) dense in \(L^2(E, m/\varphi)\) can be replaced by the condition \(F \subset L^\infty(E, m)\) dense in \(L^2(E, m)\).

**Proof.** We assume \(F \subset L^\infty(E, m)\) is dense in \(L^2(E, m)\). Let \(g \in L^2(E, m/\varphi)\). Given \(\varepsilon > 0\) let

\[
E^n := \left\{ x : \frac{1}{n} < \frac{1}{\varphi(x)} < n, x \in E \right\}
\]

and \(g_n := g \cdot \chi_{E^n}\). It follows \(g_n \to g\) in \(L^2(m/\varphi)\) when \(n \to \infty\) and thus there exists \(N \in \mathbb{N}\) such that \(\|g_n - g\|_{L^2(m/\varphi)} < \varepsilon/2\) when \(n \geq N\). Since \(g_n \in L^2(m)\) we can find \(f_n \in F\) with support on \(E^n\), such that \(\|f_n - g_n\|_{L^2(m)} < \varepsilon/2n\) and thus \(\|f_n - g\|_{L^2(m/\varphi)} < \varepsilon/2\). It follows that \(\|f_n - g\|_{L^2(m/\varphi)} < \varepsilon, n \geq N\) and the statement is proved.

We now proceed by defining \(Z\mathbb{H}\) as the set of all cylindrical functions with values in \(\mathbb{H}\) of the form \(\sum_{i=1}^k F_i h_i\), where \(F_i \in \mathbb{Z}\), \(h_i \in \mathbb{H}\) and \(k \in \mathbb{N}\). We also let \(Z^b\) and \(Z^b\mathbb{H}\) denote the bounded counterparts to \(Z\) and \(Z\mathbb{H}\), respectively. We define \(Z^b\mathbb{H}\) as the set of \(\mathbb{H}\)-valued cylindrical functions of form \(\sum_{i=1}^k \phi_i h_i\), where \(h_i \in \mathbb{H}\), \(k \in \mathbb{N}\), \(\phi_i\) are polynomials in \(\int_0^1 H_i \, d\gamma\). Here \(H_i, i \in \mathbb{N}\), are defined as in (6).
Lemma 3.2. If \( x_n \in Z_{\mathbb{H}}, \mathbb{H} = L^2(T), T = [0, 1] \) and

\[
\lim_{n \to \infty} \int \psi \delta(x_n) \, d\nu = 0
\]  

for all \( \psi \) in \( Z \). Then

\[
\langle x_n, \tilde{\phi} \rangle_{L^2(\mathbb{H}^\nu)} \to 0
\]

for all \( \tilde{\phi} \in Z^0_{\mathbb{H}}. \)

**Proof.** For \( j \in \mathbb{N} \) and \( f \in L^2(T^j) \), symmetric in all \( j \) variables, define

\[
I_j(f) = j! \int_{t_0}^{t_1} \int_{t_{j-1}}^{t_j} \ldots \int_{t_1}^{t_2} f(t_1, \ldots, t_j) \, d\gamma_1, \ldots, d\gamma_j.
\]

We have for all \( n \in \mathbb{N} \) the representation

\[
x_n(t) = \sum_{j=0}^{\infty} I_j(f^{(n)}_j(\cdot, t)),
\]

for some \( f^{(n)}_j \in L^2(T^{j+1}) \), where for \( j \geq 1 \), \( f^{(n)}_j \) is symmetric in the first \( j \) variables and \( I_0(f^{(n)}_0(t)) = E[x_n(t)] \). Thus

\[
\delta(x_n) = \sum_{j=0}^{\infty} I_{j+1}(\tilde{f}^{(n)}_j),
\]

where \( \tilde{f}^{(n)}_j \) denotes the symmetrization of \( f^{(n)}_j \). For any \( \phi \in \mathcal{H}_{m+1} \), we have \( \phi = I_{m+1}(g) \) for some symmetric \( g \in L^2(T^{m+1}) \). We even assume that \( \phi \in Z \). By hypothesis (9) we have

\[
(m+1)! \langle g, \tilde{f}^{(n)}_m \rangle_{L^2(T^{m+1})} = \int I_{m+1}(g) I_{m+1}(\tilde{f}^{(n)}_m) \, d\nu
\]

\[
= \int \phi \delta(x_n) \, d\nu \to 0, \quad \text{as} \quad n \to \infty.
\]

Since \( g \in L^2(T^{m+1}) \) is symmetric we even have

\[
\langle g, f^{(n)}_m \rangle_{L^2(T^{m+1})} \to 0, \quad \text{as} \quad n \to \infty.
\]

For all symmetric polynomials \( h \in L^2(T^m; \mathbb{H}) \) we obtain

\[
\langle h, f^{(n)}_m(\cdot, t) \rangle_{L^2(T^m; \mathbb{H})} \to 0, \quad \text{as} \quad n \to \infty,
\]

where the letter \( t \) indicates the variable for the function in \( \mathbb{H} \). Using the fact that the \( m \)th Wiener chaos is isometric to the space of all symmetric functions in \( L^2(T^m) \), we get for
any $\psi \in L^2(\nu; \mathbb{H})$ such that $\psi(\cdot, t) \in \mathcal{H}_m$ for a.e. $t \in [0, 1]$, $\psi = I_m(h)$ that

$$\left\langle I_m(h), I_m(f^{(m)}(\cdot, t)) \right\rangle_{L^2(\nu; \mathbb{H})} \rightarrow 0,$$

as $n \rightarrow \infty$.

With (10) we obtain

$$\left\langle I_m(h), x_n \right\rangle_{L^2(\nu; \mathbb{H})} \rightarrow 0.$$

It follows that

$$\left\langle x_n, \tilde{\phi} \right\rangle_{L^2(\nu; \mathbb{H})} \rightarrow 0,$$

for all finite linear combinations $\tilde{\phi} = \alpha_1 \psi_1 + \cdots + \alpha_k \psi_k$, and some $k \in \mathbb{N}$, where the functions $\psi_i \in L^2(\nu; \mathbb{H})$ are in a way that $\psi_i(\cdot, t)$ is from the $i$th Wiener chaos for a.e. $t \in [0, 1]$ and $i \in \{1, \ldots, k\}$. In particular we restrict ourselves to the particular case when $\psi_i$, $i \in \{1, \ldots, k\}$, have a representation of the form

$$\psi_i = \sum_{j=1}^l \tilde{\phi}_j(\gamma) \cdot h_j(t), \quad (11)$$

with $\tilde{\phi}_j \in \mathcal{H}_m$, $h_j \in \mathbb{H}$ and some $l \in \mathbb{N}$. Moreover, since polynomials in $L^2([0, 1]^m)$ correspond to polynomials in $\mathcal{H}_m$ we even assume $\tilde{\phi}_i$ to be polynomials in $\int_0^1 H_1 \, d\gamma_1, \ldots, \int_0^1 H_m \, d\gamma$, where $H_1, \ldots, H_m$ are defined in (6). The statement now follows.

**Corollary 3.3.** Let $x_n \in Z_{\mathbb{H}}$ and $x_n \rightarrow x$ in $L^2(\varphi \nu; \mathbb{H})$ for some $x \in L^2(\varphi \nu; \mathbb{H})$, $\mathbb{H} = L^2(T), T = [0, 1]$. Assume

$$\frac{f}{\varphi} \in L^1(\nu), \quad f \in Z \quad (12)$$

and

$$\lim_{n \rightarrow \infty} \int \psi \delta(x_n) \, d\nu = 0 \quad (13)$$

for all $\psi$ in $Z$. Then $x = 0$.

**Proof.** By (12) we have

$$\lim_{n \rightarrow \infty} \int \left\langle x - x_n, \frac{\tilde{\psi}}{\varphi} \right\rangle_{\mathbb{H}} \varphi \, d\nu = 0,$$

for all $\tilde{\psi} \in Z_{\mathbb{H}}$, thus

$$\lim_{n \rightarrow \infty} \left\langle x_n, \tilde{\psi} \right\rangle_{\mathbb{H}} \, d\nu = \int \left\langle x, \tilde{\psi} \right\rangle_{\mathbb{H}} \, d\nu, \quad \tilde{\psi} \in Z_{\mathbb{H}}.$$
By Lemma 3.2 the right hand side is 0 for all \( \psi \in \mathbb{Z}_H^b \). Since \( \mathbb{Z}_H^b \) is dense in \( L^2(\nu; \mathbb{H}) \) the statement follows. \( \square \)

We emphasize that Corollary 3.3, which is based on Lemma 3.2, is crucial for the proof of Proposition 3.7. In addition we get the following proposition as an independent result.

**Proposition 3.4.** If \((12)\) holds then \( \{1/\varphi D\psi : \psi \in Z\} \) is a dense subset of \( L^2(\varphi \nu; \mathbb{H}) \).

**Proof.** We assume the contrary. Then we can find \( x \in L^2(\varphi \nu; \mathbb{H}) \) and \( x_n \in \mathbb{Z}_H, n \in \mathbb{N} \), with values in \( \mathbb{H} \), such that \( x \neq 0 \) and \( x_n \rightharpoonup x \) in \( L^2(\varphi \nu; \mathbb{H}) \) and

\[
\int \frac{1}{\varphi} \langle D\psi, x \rangle_{\mathbb{H}} \varphi \, d\nu = 0
\]

for all \( \psi \in Z \). It follows that

\[
0 = \int \langle D\psi, x \rangle_{\mathbb{H}} \varphi \, d\nu = \lim_{n \to \infty} \int \langle \frac{D\psi}{\varphi}, x_n \rangle_{\mathbb{H}} \varphi \, d\nu = \lim_{n \to \infty} \int \langle D\psi, x_n \rangle_{\mathbb{H}} \, d\nu
\]

\[
= \lim_{n \to \infty} \int \psi \delta(x_n) \, d\nu
\]

for all \( \psi \in Z \). From this we get \( x = 0 \) by Corollary 3.3, and we have a contradiction. \( \square \)

**Proposition 3.5.** Assume \( \varphi \in L^1(\nu) \) and \((12)\) holds, i.e.

\[
\frac{f}{\varphi} \in L^1(\nu), \quad f \in Z.
\]

Then \((\mathcal{D}, Z)\) defined by

\[
\mathcal{D}(f, g) := \frac{1}{2} \int \langle Df, Dg \rangle_{\mathbb{H}} \varphi \, d\nu, \quad f, g \in Z,
\]

is closable on \( L^2(\varphi \nu) \) and we denote this closure by \( \mathcal{D}^1 \). We also note that \((D, Z)\) is closable as an operator \( L^2(\varphi \nu) \supset Z \to L^2(\varphi \nu; \mathbb{H}) \).

**Proof.** We observe that under \((12)\), \( \delta(\psi)/\varphi \in L^2(\varphi \nu) \) for \( \psi \in \mathbb{Z}_H^b := \{ \sum_{i=1}^k \phi_i h_i : k \in \mathbb{N}, \phi_i \in Z \cap L^\infty(\nu), h_i \in \mathbb{H} \} \). Now for \( u_n \in Z, \psi \in Z_H \) and

\[
u_n \to 0 \text{ in } L^2(\varphi \nu), \quad D\nu_n \to f \text{ in } L^2(\varphi \nu; \mathbb{H}),
\]

we have

\[
\frac{1}{2} \int \langle D\nu_n, \psi / \varphi \rangle_{\mathbb{H}} \varphi \, d\nu = \frac{1}{2} \int \nu_n \frac{\delta(\psi)}{\varphi} \varphi \, d\nu.
\]
By the earlier observation the right hand side of (14) tends to 0. By Lemma 3.1(c), test functions of form $\psi/\varphi$, $\psi \in \mathbb{H}_1$ are dense in $L^2(\varphi \nu; \mathbb{H})$. It follows that $f = 0$ and thus $D$ is closable on $L^2(\varphi \nu)$.

\begin{proposition}
If (12) holds then $(\delta, Z_{\mathbb{H}})$ is a closable operator in $L^2(\varphi \nu; \mathbb{H}) \supset Z_{\mathbb{H}} \to L^2(\varphi \nu)$.
\end{proposition}

\begin{proof}
Let $x_n \in Z_{\mathbb{H}}$ such that

$$x_n \to 0 \quad \text{in} \quad L^2(\varphi \nu), \quad \delta(x_n) \to f \quad \text{in} \quad L^2(\varphi \nu; \mathbb{H}).$$

Then for all $y \in \{z/\varphi : z \in Z^b\}$ we have

$$\langle f, y \rangle_{L^2(\varphi \nu)} = \lim_{n \to \infty} \langle \delta(x_n), y \rangle_{L^2(\varphi \nu)} = \lim_{n \to \infty} \left( x_n, \frac{D(y \varphi)}{\varphi} \right)_{L^2(\varphi \nu; \mathbb{H})} = 0. \quad \square$$

We proceed by defining equivalence classes in $L^2(\varphi \nu)$. Let $x \in L^2(\varphi \nu)$ be the representative of $x(x) := \{x + c1 : c \in \mathbb{R}\}$, $x \in L^2(\varphi \nu)$, where 1 is the constant function taking the value 1 on $\Omega$. We also define

$$L^2(\varphi \nu) \ominus 1 := \{x(x) : x \in L^2(\varphi \nu)\}$$

as well as $\delta(z)$ which denotes the equivalence class of $\delta(z)$, $z \in Z_{\mathbb{H}}$. We remark that every $x \in L^2(\varphi \nu) \ominus 1$ has a unique representative $x_0 \equiv x_0(x)$ with $\int x_0 \varphi \, d\nu = 0$. $L^2(\varphi \nu) \ominus 1$ becomes a Hilbert space with the inner product

$$\langle x_1, x_2 \rangle_{L^2(\varphi \nu) \ominus 1} := \langle x_0(x_1), x_0(x_2) \rangle_{L^2(\varphi \nu)}, \quad x_1, x_2 \in L^2(\varphi \nu) \ominus 1.$$  

\begin{proposition}
Assume that (12) holds. Then $\delta^{-1}$ defined on $Z_{\mathbb{H}}^{-1} := \{\delta(z) : z \in Z_{\mathbb{H}}\}$ as an operator $L^2(\varphi \nu) \ominus 1 \supset Z_{\mathbb{H}}^{-1} \to L^2(\varphi \nu; \mathbb{H})$ is closable.
\end{proposition}

\begin{proof}
Let $y_n \in Z_{\mathbb{H}}^{-1}$ such that

$$y_n \to 0 \quad \text{in} \quad L^2(\varphi \nu) \ominus 1, \quad \delta^{-1}(y_n) \to f \quad \text{in} \quad L^2(\varphi \nu; \mathbb{H}).$$

Let $y_n = \delta(z_n)$ for some $z_n \in Z_{\mathbb{H}}$. Then

$$\delta(z_n) \to 0 \quad \text{in} \quad L^2(\varphi \nu) \ominus 1$$
which is equivalent to \( x_0(\delta(z_n)) \to 0 \) in \( L^2(\varphi \nu) \). Noting that \( x_0(\delta(z_n)) = \delta(z_n) + c_n1 \) for some \( c_n \in \mathbb{R} \). We have

\[
c_n = \langle \delta(z_n) + c_n1, 1 \rangle_{L^2(\nu)} = \left\langle \delta(z_n) + c_n1, \frac{1}{\varphi} \right\rangle_{L^2(\varphi \nu)} = \left\langle x_0(\delta(z_n)), \frac{1}{\varphi} \right\rangle_{L^2(\varphi \nu)} \to 0.
\]

Summarizing we get

\[
\delta(z_n) \to 0 \text{ in } L^2(\varphi \nu) \quad \text{and} \quad z_n = \delta^{-1}(\delta(z_n)) \to f \text{ in } L^2(\varphi \nu; \mathbb{H})
\]

and conclude

\[
\left\langle \psi, \delta(z_n) \right\rangle = \left\langle \frac{\psi}{\varphi}, \delta(z_n) \right\rangle_{L^2(\varphi \nu)} \to 0
\]

for all \( \psi \in Z \). By Corollary 3.3 we have \( f = 0 \). The statement follows. \( \square \)

In order to have a well-defined inverse of the gradient operator, we define equivalence classes in \( Z \),

\[
Z^e := \{ x(x) : x \in Z \}
\]

and

\[
D^e x = D^e x(x) := Dx, \quad x \in Z^e,
\]

where \( D^e \) is invertible on \( Z^e \). We denote the inverse by \( D^{-1} \) and the corresponding domain by \( Z^{-1} := \{ D^e(x) : x \in Z^e \} \).

**Proposition 3.8.** Assume (12). The operator \((D^{-1}, Z^{-1})\) as an operator \( L^2(\varphi \nu; \mathbb{H}) \supset Z^{-1} \to L^2(\varphi \nu) \Theta 1 \) is closable.

**Proof.** Let \( x_n \in Z^{-1} \) such that

\[
x_n \to 0 \text{ in } L^2(\varphi \nu; \mathbb{H}), \quad D^{-1}(x_n) \to f \text{ in } L^2(\varphi \nu) \Theta 1.
\]

For \( x_n = D^e x_n, x_n \in Z^e \), and all \( y \in \tilde{Y} := \{ x(y) : y \varphi = \delta(z), \text{ for some } z \in Z^0 \} \), we have

\[
\langle f, y \rangle_{L^2(\varphi \nu) \Theta 1} = \lim_{n \to \infty} \langle x_0(D^{-1} x_n), x_0(y) \rangle_{L^2(\varphi \nu)} = \lim_{n \to \infty} \langle x_0(x_n), x_0(y) \rangle_{L^2(\nu)} = \lim_{n \to \infty} \langle x_0(x_n), \delta(z) \rangle_{L^2(\nu)} = \lim_{n \to \infty} \langle D^e(x_n), \delta \rangle_{L^2(\nu; \mathbb{H})} = \lim_{n \to \infty} \left\langle x_n, \frac{z}{\varphi} \right\rangle_{L^2(\varphi \nu; \mathbb{H})} = 0.
\]

\[15\]
We now show that \( \mathcal{Y} \) is dense in \( L^2(\varphi \nu) \oplus 1 \). Since \( 0 = E[(f/\varphi)\delta(z)] \), \( z \in Z^b_H \) implies \( f = c \cdot \varphi \), \( c \in \mathbb{R} \), we obtain that \( \{\delta(z) + c1 : c \in \mathbb{R}, z \in Z^b_H\} \) is dense in \( L^2(\nu) \). Now Lemma 3.1 says that \( \{(\delta(z) + c1)/\varphi : c \in \mathbb{R}, z \in Z^b_H\} \) is dense in \( L^2(\varphi \nu) \). Using the fact that

\[
\int \frac{\delta(z) + c1}{\varphi} \varphi \, d\nu = 0, \quad z \in Z^b_H
\]

yields that \( c = 0 \), it follows that \( \{\delta(z)/\varphi : z \in Z^b_H\} \) is dense in \( L^2(\varphi \nu) \oplus \{c1 : c \in \mathbb{R}\} \). Thus \( \mathcal{Y} \) is dense in \( L^2(\varphi \nu) \oplus 1 \). Equation (15) shows that \( f = 0 \) and the claim follows. \( \square \)

**Remark 2.** Under condition (12), i.e.

\[
\frac{f}{\varphi} \in L^1(\nu), \quad f \in Z,
\]

operators \( D, \delta, D^{r-1} \) and \( \delta^{-1} \) are closable in the sense of Propositions 3.5–3.8.

In the following sections we are interested in weight functions of the form

\[
\varphi(y) := \exp\left\{ \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^1 |b(\gamma_s)|^2 \, ds \right\},
\]

(16)

where \( \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} \) is a stochastic Itô integral in the sense of [14]. It is natural to ask for conditions on \( b \) so that \( (f/\varphi) \in L^1(\nu), f \in Z \). Related questions have been addressed in great detail in [16] which we follow for the subsequent results.

**Proposition 3.9.** Let \( d \geq 3 \) and \( \varphi \) be given by (16) where \( b = (b^1, \ldots, b^d) \) such that \( b^j \) is for all \( j = 1, \ldots, d \) continuous of type \( b^j : \mathbb{R}^d \to \mathbb{R} \), \( \mathbb{R}^d \) being endowed with a suitable topology.

(a) Let \( b \) be spherically symmetric, i.e. \( b(x) = g(|x|), x \in \mathbb{R}^d \), for some \( g : \mathbb{R} \to \mathbb{R}^d \). Then \( (f/\varphi) \in L^1(\nu) \) for all \( f \in Z \) if and only if

\[
\int_0^1 rg(r)^2 \, dr < \infty \quad \text{and} \quad \sup_{s \geq 2} \int_{s-1}^{s+1} g(r)^2 \, dr < \infty.
\]

(b) In the case of a non-spherically symmetric \( b \),

\[
\sup_{x \in \mathbb{R}^d} \int_{\|x-y\| < 1} |b(y)|^p \, dy < \infty \quad \text{for some} \quad p > d,
\]

(18)

implies \( (1/\varphi^q) \in L^1(\nu) \) for all \( q \geq 1 \) and hence \( (f/\varphi) \in L^1(\nu) \) for all \( f \in Z \).

**Proof.** In order to follow [16] it is important to note that process \( X \) in this reference is a Brownian motion with respect to measure \( P \).
(a) **Step 1.** We show sufficiency of (17). Let

\[ Y(t) := \exp \left\{ -2 \int_0^t \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} - \int_0^t |b(\gamma_s)|^2 \, ds \right\}, \quad t \in [0, 1]. \]

Using the Cauchy–Schwartz inequality we obtain

\[
E \left[ \frac{f}{\varphi} \right] = E \left[ f(\gamma) \cdot \exp \left\{ \frac{1}{2} \int_0^1 |b(\gamma_s)|^2 \, ds - \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} \right\} \right]^2 \\
\leq E \left[ f(\gamma)^2 \cdot \exp \left\{ 2 \int_0^1 |b(\gamma_s)|^2 \, ds \right\} \right] \cdot E[Y(1)].
\]

Here the second factor is bounded by 1 since \( Y(t), t \in [0, 1] \), is a supermartingale, see [10, Chapter 3.5D]. Noting that \( f^4 \in L^1(\nu) \), it remains to show that

\[
E \left[ \exp \left\{ 4 \int_0^1 |b(\gamma_s)|^2 \, ds \right\} \right] < \infty.
\]

However, this is an immediate consequence of Theorem 2.4 in [16].

**Step 2.** We show that (17) is necessary. Let

\[
I(\gamma) := \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d}.
\]

According to the above hypotheses on \( b \), we may assume well-definiteness of \( I(\cdot) \) as a random variable. In addition, because of the spherical symmetry of \( b \) we may suppose that \( I(\gamma) = -I(-\gamma) \). Since \( 1/\varphi \in L^1(\nu) \) we have

\[
\infty > E \left[ \exp \left\{ \frac{1}{2} \int_0^1 |b(\gamma_s)|^2 \, ds - \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} \right\} \right] \\
\geq E \left[ \chi_{|I(\gamma) < 0|} \cdot \exp \left\{ \frac{1}{2} \int_0^1 |b(\gamma_s)|^2 \, ds - \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} \right\} \right] \\
\geq E \left[ \chi_{|I(\gamma) < 0|} \cdot \left\{ \frac{1}{2} \int_0^1 |b(\gamma_s)|^2 \, ds - \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} \right\} \right] \\
\geq E \left[ \chi_{|I(\gamma) > 0|} \cdot \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} \right],
\]

with \( \chi \) denoting the indicator function, and thus

\[
\left| E \left[ \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} \right] \right| < \infty.
\]  

(19)

Using Jensen’s inequality and recalling (16), it follows from \( 1/\varphi \in L^1(\nu) \) that

\[
\exp \left\{ E \left[ \frac{1}{2} \int_0^1 |b(\gamma_s)|^2 \, ds - \int_0^1 \langle b(\gamma_s), d\gamma_s \rangle_{\mathbb{R}^d} \right] \right\} < \infty.
\]  

(20)
From (20) and (19) we obtain

$$E \left[ \int_0^1 |b(\gamma_t)|^2 \, ds \right] < \infty. \quad (21)$$

Since $\gamma$ is a $d$-dimensional Wiener process and $|b(\gamma)|^2 = b(|\gamma|)^2$ we get

$$E \left[ \int_0^1 |b(\gamma_t)|^2 \, ds \right] = \int_{r=0}^\infty b(r)^2 \int_{s=0}^1 \int_{\gamma \in S_r} (2\pi s)^{-d/2} e^{-r^2/2s} \, dy \, ds \, dr, \quad (22)$$

where $S_r$ denotes the $(d - 1)$-sphere of radius $r$. Equations (21) and (22) give

$$\infty > \int_{r=0}^1 b(r)^2 \int_{s=0}^1 \frac{r^{d-1}}{s^{d/2}} e^{-r^2/2s} \, ds \, dr + \int_{r=1}^\infty b(r)^2 \int_{s=0}^1 \frac{r^{d-1}}{s^{d/2}} e^{-r^2/2s} \, ds \, dr$$

$$= C \int_{r=0}^1 b(r)^2 \, dr + C \int_{r=1}^\infty b(r)^2 \, dr \geq C \int_{r=0}^1 b(r)^2 \, dr + C \sup_{s \geq 2} \int_{s=1}^{s+1} b(r)^2 \, dr,$$

where constant $C$ results from integration with respect to $s$. The statement follows.

(b) We follow exclusively the argumentation of [16]. Our relation (18) is (1.12) in [16]. This implies condition (1.5) of [16] by the proof of Corollary 1.7 in [16]. Using Theorem 1.2 of [16] we get (1.1) of [16]. Replacing $b$ by $-b$ this yields $(1/\varphi^q) \in L^1(\nu)$ for all $q \geq 1$ according to Theorem 5.1 of [16].

4. Closability of the bilinear form

We formulate the following lemma that serves an important role in proving and formulating the closability conditions of the form.

**Lemma 4.1.** If $s_k = \sum_{i=1}^r c_i 2^{-i}$, $j \in \{1, \ldots, d\}$ and

$$i_{p,j} = i_{p,j}(c_0, c_1, \ldots) := \begin{cases} 2^{p-1}(d + j - 1) + 1 + \sum_{q=0}^{p-1} c_q 2^{p-q-1}, & p \geq 1, \\ j, & p = 0, \end{cases}$$

where $c_1, \ldots, c_{r-1} \in \{0, 1\}$, $c_r = 1$, $c_0 = c_{r+1} = 0$ and $r \geq 2$, then

$$\sum_{i=1}^\infty \lambda_i \langle S_i(s_k), e_j \rangle_{R^d}^2 = \lambda_j s_k^2 + \sum_{p=1}^r \lambda_{i_p} 2^{p-1} \left( c_p 2^{-p} + (-1)^r \sum_{q=p+1}^{r+1} 2^{-q} c_q \right)^2.$$

(b) The relation

$$\sup_{c_1, \ldots, c_{r-1} \in \{0, 1\}, \quad c_r, c_{r+1} = 0, \quad r \geq 2} \left[ \sum_{j=1}^d \sum_{p=1}^r \lambda_{i_{p,j}} 2^{p-1} \left( c_p 2^{-p} + (-1)^r \sum_{q=p+1}^{r+1} 2^{-q} c_q \right)^2 \varphi \, dv \right] < \infty \quad (23)$$
is equivalent to

\[
\sup_{c_1, c_2, \ldots \in \{0, 1\}} \sum_{j=1}^d \sum_{p=1}^\infty \frac{\lambda_{j, p} \varphi \, d\nu}{2^p} < \infty. \tag{24}
\]

**Proof.** Since \( e_j \perp S_i \) for \( i \mod d \neq j \) and \( S_i(s_k) = 0 \) for \( i > d2^r \) we get

\[
\sum_{i=1}^\infty \lambda_j \langle S_i(s_k), e_j \rangle_{\mathbb{R}^d}^2 = \lambda_j \langle S_j(s_k), e_j \rangle_{\mathbb{R}^d}^2 + \sum_{i=1}^{2^r} \lambda_{di+j} \langle S_{di+j}(s_k), e_j \rangle_{\mathbb{R}^d}^2.
\]

Now since

\[
\lambda_j \langle S_j(s_k), e_j \rangle_{\mathbb{R}^d}^2 = \lambda_j s_k^2
\]

and

\[
S_{di+j}(s_k) = 0
\]

unless \( di + j = 2^{p-1}(d + j - 1) + 1 + \sum_{q=0}^{p-1} c_q 2^{p-q-1} \) for some \( p \geq 1 \). We get

\[
\sum_{i=1}^{2^r} \lambda_{di+j} \langle S_{di+j}(s_k), e_j \rangle_{\mathbb{R}^d}^2 = \sum_{p=1}^r \lambda_{ip} \langle S_{ip}(s_k), e_j \rangle_{\mathbb{R}^d}^2,
\]

where

\[
\langle S_{ip}(s_k), e_j \rangle_{\mathbb{R}^d} = \begin{cases} 
2^{(p-1)/2} \sum_{q=p+1}^{r+1} 2^{-q} c_q & \text{if } c_p = 0 \\
2^{-(p+1)/2} - 2^{(p-1)/2} \sum_{q=p+1}^{r+1} 2^{-q} c_q & \text{if } c_p = 1 
\end{cases}
\]

\[
= 2^{(p-1)/2} \left( c_p 2^{-p} + (-1)^p \sum_{q=p+1}^{r+1} 2^{-q} c_q \right),
\]

and thus the statement follows.

(b) Assuming (23) and recalling that \( \lambda > 0 \).

\[
\sup_{c_1, \ldots, c_r, \ldots \in \{0, 1\}, \ c_r=0, \ c_{r+1}=0, \ \forall j \in \{1, \ldots, d\}, r \geq 2}
\sum_{j=1}^d \sum_{p=1}^\infty \lambda_{j, p} 2^{p-1} \left( c_p 2^{-p} + (-1)^p \sum_{q=p+1}^{r+1} 2^{-q} c_q \right) \varphi \, d\nu
\]

\[
= \sup_{c_1, c_2, \ldots \in \{0, 1\}} \sum_{j=1}^d \sum_{p=1}^\infty \lambda_{j, p} 2^{(2-p-1)} \varphi \, d\nu = \sup_{c_1, c_2, \ldots \in \{0, 1\}} \frac{1}{8} \sum_{j=1}^d \sum_{p=1}^\infty \lambda_{j, p} \varphi \, d\nu.
\]
Assuming (24) we obtain

$$
\sup_{c_1, \ldots, c_r \in [0,1], \quad c_r = 1, c_{r+1} = 0, \quad r \geq 2} \left\{ \sum_{j=1}^{d} \sum_{p=1}^{r} \lambda_{j,p} 2^{p-1} \left( c_p 2^{-p} + (-1)^p \sum_{q=p+1}^{r+1} 2^{-q} \right) \right\}^2 \varphi \, d\nu
$$

$$
\leq \sup_{c_1, c_2, \ldots \in [0,1]} \left\{ \sum_{j=1}^{d} \sum_{p=1}^{\infty} \lambda_{j,p} 2^{p-1} (2^{-p})^2 \varphi \, d\nu \right\} = \sup_{c_1, c_2, \ldots \in [0,1]} \left\{ \frac{1}{2} \sum_{j=1}^{d} \sum_{p=1}^{\infty} \int \lambda_{j,p} \varphi \, d\nu \right\}.
$$

The statement now follows.

**Remark 1.** We are most interested in investigating the degree of increase for the eigenvalues. Keeping this in mind, it makes sense to assume that the sequence of eigenvalues is non-decreasing $0 < \lambda_1(\gamma) < \lambda_2(\gamma), \ldots$ Under this condition, relation (24) simplifies to

$$
\sum_{p=0}^{\infty} \int \lambda_{i,p} \varphi \, d\nu < \infty.
$$

**Proposition 4.2.** Let $\varphi$ satisfy (12) and assume that $A \succeq \varepsilon \text{Id}$ for some $\varepsilon > 0$.

(a) The form $(\mathcal{E}, Y)$ is closable in $L^2(\varphi \nu)$. Let $(\mathcal{E}, D_Y(\mathcal{E}))$ denote the closure of $(\mathcal{E}, Y)$ in $L^2(\varphi \nu)$.

(b) If

$$
\sup_{c_1, c_2, \ldots \in [0,1]} \sum_{j=1}^{d} \sum_{p=0}^{\infty} \int \lambda_{j,p} \varphi \, d\nu < \infty,
$$

where $i_{j,p}$ and $c_1, c_2, \ldots$ are constructed as in Lemma 4.1, then $(\mathcal{E}, Z)$ is closable and we let $(\mathcal{E}, D_Z(\mathcal{E}))$ denote the closure of $(\mathcal{E}, Z)$ on $L^2(\varphi \nu)$. Furthermore $D_Z(\mathcal{E}) = D_Y(\mathcal{E})$ under (25).

(c) We have $Z \subseteq D_Y(\mathcal{E})$ if and only if (25) holds.

(d) If $(\mathcal{E}, Z)$ is closable in $L^2(\varphi \nu)$ then (25) holds.

**Proof.**

(a) We have

$$
\mathcal{E}(F, F) = \sum_{i=1}^{\infty} \left\langle \lambda_i(S_i, DF) H, \varphi \right\rangle = \left\langle A^{1/2}DF, A^{1/2}DF \right\rangle_{\mathbb{H}} \varphi \, d\nu.
$$
Suppose \(\{F_n\}_{n \geq 1} \subset Y\) such that \(F_n \to 0\) in \(L^2(\varphi_v)\) and \(\mathcal{E}(F_n - F_m, F_n - F_m) \to 0\). Since it follows that \(A^{1/2}DF_n\) is Cauchy in \(L^2(\varphi_v; H)\) we may define

\[
\psi := \lim_{n \to \infty} A^{1/2}DF_n.
\]

We also define

\[
JG := \sum_{i=1}^{\infty} \lambda_i^{-1/2} \langle S_i, G \rangle H S_i \quad G \in L^2(\varphi_v; H).
\]

Operator \(J\) is bounded on \(L^2(\varphi_v; H)\) and it follows

\[
DF_n = JA^{1/2}DF_n \to J\psi \quad \text{in} \quad L^2(\varphi_v; H).
\]

From Proposition 3.5, it is known that \((D, Z)\) is closable on \(L^2(\varphi_v)\). It follows that \(DF_n \to 0\) and thus \(J\psi = 0\). Since \(\lambda_i > 0\) and \(\lambda_i^{-1/2} > 0\), (26) gives \(\psi = 0\). Now \(A^{1/2}DF_n \to 0\) and thus \(\mathcal{E}(F_n, F_n) = \int \langle A^{1/2}DF_n, A^{1/2}DF_n \rangle_H \varphi \, dv \to 0\) as \(n \to \infty\).

(b) We show that (25) implies \(Z \subset D_Y(\mathcal{E})\). Then \(Y \subset Z \subset D_Y(\mathcal{E})\). Since \((\mathcal{E}, Y)\) is closable with closure \((\mathcal{E}, D_Y(\mathcal{E}))\), cf. (a), \((\mathcal{E}, Z)\) is then also closable and has \((\mathcal{E}, D_Y(\mathcal{E}))\) as its closure, i.e. \(D_Y(\mathcal{E}) = D_Z(\mathcal{E})\).

Let \(x^v(p)\) denote the \(v\)th coordinate of \(p \in \mathbb{R}^d\), \(v \in \{1, \ldots, d\}\). Let us demonstrate that \(F(\gamma) = x^v(\gamma(s)) \in D_Y(\mathcal{E})\) for all \(v \in \{1, \ldots, d\}\) and all \(s \in [0, 1]\). Fix \(s \in [0, 1]\), \(v \in \{1, \ldots, d\}\) and let \(s_k \to s\), where \(s_k\) is a sequence of dyadic numbers. Now let

\[
F_{v,k}(\gamma) := x^v(\gamma(s_k)) \in Y \subset D_Y(\mathcal{E}), \quad k \in \mathbb{N}.
\]

We aim to apply Lemma 4.1. By (25) we have

\[
\int \sum_{i=1}^{\infty} \lambda_i(\gamma) \langle S_i, DF(\gamma) \rangle_H^2 \varphi \, dv = \int \sum_{i=1}^{\infty} \lambda_i(\gamma) \langle S_i(s), e_v \rangle_{\mathbb{R}^d}^2 \varphi \, dv
\]

\[
\leq \sup_k \int \sum_{i=1}^{\infty} \lambda_i(\gamma) \langle S_i(s_k), e_v \rangle_{\mathbb{R}^d}^2 \varphi \, dv \leq \sum_{v=0}^{d} \int \sum_{p=0}^{\infty} \frac{\lambda_{p,v}}{2^p} \varphi \, dv < \infty.
\]

Furthermore \(F_{v,k} \to F\) in \(L^2(\varphi_v)\) and

\[
\int \sum_{i=1}^{\infty} \lambda_i(\gamma) \langle S_i, DF - F_{v,k}(\gamma) \rangle_H^2 \varphi \, dv \leq \int \sum_{i=1}^{\infty} \lambda_i(\gamma) \langle S_i, DF(\gamma) \rangle_H^2 \varphi \, dv
\]

for some sequence \(n_k \to \infty\) as \(k \to \infty\). This says \(F_{v,k} \to F\) in \(\mathcal{E}_1\)-norm and \(F \in D_Y(\mathcal{E})\), where we recall that \(||\cdot||_{\mathcal{E}_1} = \mathcal{E}(\cdot, \cdot) + ||\cdot||_{L^2(\varphi_v)}\). Indeed, the same conclusion applies to an arbitrary \(F \in Z\) with \(F(\gamma) = f(\gamma(s_1), \ldots, \gamma(s_k))\), \(s_1, \ldots, s_k \in [0, 1]\) since because of (5) we have

\[
\langle S_i, DF(\gamma) \rangle_H = \sum_{j=1}^{d} \sum_{r=1}^{k} \frac{\partial f}{\partial x_{r,j}} (\gamma(s_1), \ldots, \gamma(s_k)) \langle S_i(s_r), e_j \rangle_{\mathbb{R}^d}.
\]
We then apply (8) and (27).
(c) Recalling (b), we still have to show that \( Z \subseteq D_\gamma(\mathcal{E}) \) implies (25).

We suppose \( Z \subseteq D_\gamma(\mathcal{E}) \). Therefore \( x_\nu(\gamma(s)) \in D_\gamma(\mathcal{E}) \) for all \( \nu \in \{1, \ldots, d\} \) and all \( s \in [0, 1] \). First we check the expression of Lemma 4(a) for all dyadic points \( s_k \). To get a bound on

\[
\sum_{i=1}^{\infty} \lambda_i(S_i(s), e_i)^2_{R^d} \varphi \, d\nu,
\]

when \( s \in [0, 1] \) is no longer dyadic, we need (25). The statement follows.

(d) \((Z, \mathcal{E})\) closable in \( L^2(\varphi \nu)\) implies \( \mathcal{E}_1(F, F) < \infty \) for all \( F \in Z \), in particular for \( F(\gamma) = x^\nu(\gamma(s)) \) for all \( \nu \in \{1, \ldots, d\} \) and all \( s \in [0, 1] \). We have

\[
\mathcal{E}_1(F, F) = \|F\|^2_{L^2(\varphi \nu)} + \sum_{i=1}^{\infty} \lambda_i(S_i(s), e_i)^2_{R^d} \varphi \, d\nu < \infty.
\]

Thus \( \int \sum_{i=1}^{\infty} \lambda_i(S_i(s), e_i)^2_{R^d} \varphi \, d\nu < \infty \) for all \( \nu \in \{1, \ldots, d\} \) and all \( s \in [0, 1] \).

Now Lemma 4(a),(b) implies (25).

\[\square\]

**Proposition 4.3.** Let \( \varphi \) satisfy (12) and (25), and assume \( A \geq \epsilon \, \text{Id} \) for some \( \epsilon > 0 \). Then the form \((\mathcal{E}, D_Z(\mathcal{E}))\) is a Dirichlet form on \( L^2(\varphi \nu)\).

**Proof.** We use Proposition I.4.10 from [12]. It follows that we must show \( \mathcal{E}(1 \wedge F^+, 1 \wedge F^+) \leq \mathcal{E}(F, F) \). We know that for \( F \in Y \)

\[
\mathcal{E}(F, F) = \sum_{i=1}^{\infty} \lambda_i(S_i, DF)^2_{R^d} \varphi \, d\nu = \sum_{i=1}^{\infty} \lambda_i(\partial S_i, F)^2 \varphi \, d\nu
\]

\[
= \sum_{i=1}^{\infty} \lambda_i(\gamma) \left( \frac{d}{dt} \bigg|_{t=0} F(\gamma + t S_i) \right)^2 \varphi \, d\nu.
\]

Let \( \xi_{\epsilon} : \mathbb{R} \to [-\epsilon, 1+\epsilon] \) be non-decreasing such that \( \xi_{\epsilon}(t) = t \) for all \( t \in [0, 1] \), \( 0 \leq \xi_{\epsilon} \leq 1 \). It follows that \( \xi_{\epsilon} \circ F \to 1 \wedge F^+ \). An application of the chain rule gives

\[
\mathcal{E}(\xi_{\epsilon} \circ F, \xi_{\epsilon} \circ F) = \sum_{i=1}^{\infty} \lambda_i(\gamma) \left( \frac{d}{dt} \bigg|_{t=0} \xi_{\epsilon} \circ F(\gamma + t S_i) \right)^2 \varphi \, d\nu
\]

\[
= \sum_{i=1}^{\infty} \lambda_i(\gamma) \xi_{\epsilon}'(\gamma) \left( \frac{d}{dt} \bigg|_{t=0} F(\gamma + t S_i) \right)^2 \varphi \, d\nu
\]

\[
\leq \sum_{i=1}^{\infty} \lambda_i(\gamma) \cdot \left( \frac{d}{dt} \bigg|_{t=0} F(\gamma + t S_i) \right)^2 \varphi \, d\nu = \mathcal{E}(F, F),
\]

from which we derive \( \mathcal{E}(1 \wedge F^+, 1 \wedge F^+) \leq \mathcal{E}(F, F) \). Thus \( \mathcal{E} \) is a Dirichlet form. \( \square \)
Proposition 4.4. Let $\varphi$ satisfy (12) and (25), and assume $A \geq \varepsilon I$ for some $\varepsilon > 0$. Then the form $(\mathcal{E}, D_Z(\mathcal{E}))$ is local.

Proof. This is shown in the same manner as in Proposition 3.4 of [11] using the fact that $\nu$-a.e. implies $\varphi \nu$-a.e. \hfill \Box

5. Quasi-regularity

In this section we consider weight functions $\varphi^+$ and $\varphi^-$ of the form

$$
\varphi^+(\gamma) := \exp \left\{ \int_0^1 \langle b_t(\gamma), d\gamma_t \rangle \gamma_t - \frac{1}{2} \int_0^1 |b_t(\gamma)|^2 ds \right\}
$$

and

$$
\varphi^-(\gamma) := \exp \left\{ \int_0^1 \langle -b_t(\gamma), d\gamma_t \rangle \gamma_t - \frac{1}{2} \int_0^1 |b_t(\gamma)|^2 ds \right\},
$$

where $b_t(\gamma)$ is adapted to the natural filtration of the Wiener process. Under the condition

$$
\frac{f}{\varphi^+} \in L^1(\nu) \quad \text{and} \quad \frac{f}{\varphi^-} \in L^1(\nu),
$$

for all $f \in Z$, the closability results of Sections 3 and 4 hold for $\varphi^+$ as well as for $\varphi^-$.Lemma 5.1. Let $\varphi^+$ and $\varphi^-$ be defined as in (28) and (29). If (30) holds then we have the Novikov condition

$$
E \left[ \exp \left\{ \frac{1}{2} \int_0^1 |b_t(\gamma)|^2 ds \right\} \right] < \infty.
$$

In particular, $E[\varphi^+] = E[\varphi^-] = 1$.

Proof. We denote

$$
\alpha := \exp \left\{ \int_0^1 \langle b_t(\gamma), d\gamma_t \rangle \gamma_t \right\} \quad \text{and} \quad \beta := \exp \left\{ -\frac{1}{2} \int_0^1 |b_t(\gamma)|^2 ds \right\}.
$$

By (30) we have $E[\alpha/\beta] < \infty$ and $E[1/(\alpha\beta)] < \infty$. Thus

$$
\infty > E \left[ \frac{\chi_{\{\alpha < 1\}} (1/\alpha) + \chi_{\{\alpha \geq 1\}} \alpha}{\beta} \right] \geq E \left[ \frac{1}{\beta} \right].
$$

The claim follows. \hfill \Box

Proposition 5.2. Suppose condition (30) and the hypotheses of Proposition 4.2, namely conditions (12), (25), and that $A > \varepsilon I$ for some $\varepsilon > 0$. Then, for $\varphi = \varphi^+$ and for $\varphi = \varphi^-$,
the closure of

\[ \mathcal{E}(F, F) = \sum_{i=1}^{\infty} \int \lambda_i(S_i, DF_i) \varphi \, d\nu, \quad F \in Z \]

in \( L^2(\varphi \nu) \), is quasi-regular.

**Proof.** We prove the claim for \( \varphi = \varphi^+ \). The case \( \varphi = \varphi^- \) is similar. To simplify notation we write \( \varphi \) for \( \varphi^+ \). We follow [11], see also [4,15].

**Step 1.** For \( r \in \mathbb{N}, l \in \{0, \ldots, 2^{r-1} - 1\} \) and \( k = 2^{r-1} + l \), set \( s_k := (2l + 1)2^{-r} \). Let \( x^v(p) \) denote the \( v \)th coordinate of \( p \in \mathbb{R}^d, v \in \{1, \ldots, d\} \). Fix \( \tau \in \Omega, k = 2^{r-1} + l \) and \( v \in \{1, \ldots, d\} \). Consider the functions \( f_{v,k,\tau}(p) := x^v(p) - x^v(\tau(s_k)), p \in \mathbb{R}^d \) and

\[ F_{v,k,\tau}(\gamma) := f_{v,k,\tau}(\gamma(s_k)) = x^v(\gamma(s_k)) - x^v(\tau(s_k)), \quad \gamma \in \Omega, \]

we have \( F_{v,k,\tau} \in Y \). Using the procedure of Proposition 4.2(b) we get

\[ \mathcal{E}(F_{v,k,\tau}, F_{v,k,\tau}) \leq C_1 < \infty, \quad (31) \]

where \( C_1 \) the bound is obtained from (27).

**Step 2.** Let

\[ G_{n,\tau}(\gamma) := G_{n,\tau} := \sup_{k \in \{1, \ldots, n\}} \| F_{v,k,\tau} \|, \quad n \in \mathbb{N}, \]

then

\[ \mathcal{E}(G_{n,\tau}, G_{n,\tau}) \leq \sup_{k \in \{1, \ldots, n\}} \left( \sum_{i=1}^{\infty} \lambda_i(\gamma \langle S_i, DF_{v,k,\tau}(\gamma) \rangle) + \varphi \, d\nu \right) \]

\[ \leq C_1 < \infty, \quad n \in \mathbb{N}, \quad (32) \]

as in [11], Step 2 of the proof of Proposition 3.3 together with Lemma 3.2 of [11].

We show \( G_{n,\tau} \in L^2(\varphi \nu) \). Define

\[ (T\gamma)_t := \gamma_t - \int_0^t (b^v(\gamma))_{v=1,\ldots,d} \, ds, \]

where \( b \) is from (28) such that (30) is satisfied and \( b^v \) denotes the \( v \)th coordinate. By Lemma 5.1 and the Girsanov theorem \( (T\gamma)_t \) is a Brownian motion under \( \varphi \nu \), cf.
Theorem 3.5.1 and Corollary 3.5.13 in [10]. We have

\[
(T^{-1}\gamma)_t^\nu = \gamma_t^\nu + \int_0^t b_s^\nu(T^{-1}\gamma) \, ds = \left(\frac{1}{2} \gamma_t^\nu + \int_0^t \chi_{[b_s^\nu(T^{-1}\gamma) > 0]} b_s^\nu(T^{-1}\gamma) \, ds \right) + \left(\frac{1}{2} \gamma_t^\nu + \int_0^t \chi_{[b_s^\nu(T^{-1}\gamma) \leq 0]} b_s^\nu(T^{-1}\gamma) \, ds \right) =: M_1^\nu(t) - M_2^\nu(t),
\]

where \(M_1^\nu = M_1^\nu(\gamma)\) and \(M_2^\nu = M_2^\nu(\gamma)\) are submartingales. For the next calculation we remind the fact that \(\gamma\) is a random element while \(\tau\) is non-random and fixed.

We obtain

\[
E\left[ G_{n,\tau}^2 \varphi \right] = E \left[ \sup_{\nu \in [1, \ldots, d]} \left| x^{\nu}(\gamma_t) - x^{\nu}(\gamma_s) \right|^2 \right] \varphi \\
\leq 2E \left[ \left( \sup_{\nu \in [1, \ldots, d]} \left| x^{\nu}(\gamma_s) \right| \right)^2 \right] + 2 \sup_{\nu \in [1, \ldots, d]} \left| x^{\nu}(\tau_s) \right|^2 \\
= 2E \left[ \sup_{\nu \in [1, \ldots, d]} \left| x^{\nu}(T^{-1}\gamma_s) \right|^2 \right] + C_2(\tau) \\
\leq 4E \left[ \sum_{\nu=1}^d \sup_{s \in [0,1]} |M_1^\nu(s)|^2 + \sup_{s \in [0,1]} |M_2^\nu(s)|^2 \right] + C_2(\tau).
\]

Using Doob's inequality we get

\[
E\left[ G_{n,\tau}^2 \varphi \right] \leq 16 \sum_{\nu=1}^d E\left[ |M_1^\nu(1)|^2 \right] + 16 \sum_{\nu=1}^d E\left[ |M_2^\nu(1)|^2 \right] + C_2(\tau). \tag{33}
\]
We have

\[
E\left[|M_n^1(1)|^2\right] = E\left[\left(\frac{1}{2} \gamma_1^* + \int_0^1 \chi(b(y(T^{-1}y))b(y(T^{-1}y))\,dy\right)^2\right]
\]

\[
\leq \frac{1}{2} E[(\gamma_1^*)^2] + 2E\left[\left(\int_0^1 |b(y(T^{-1}y))\,dy\right)^2\right]
\]

\[
= \frac{1}{2} + 2E\left[\left(\int_0^1 |b(y)|\,dy\right)^2 \varphi(y)\right]
\]

\[
\leq \frac{1}{2} + 2E\left[\exp\left(\int_0^1 |b(y)|^2\,dy\right) \varphi(y)\right] = \frac{1}{2} + 2E\left[\frac{1}{\varphi}\right].
\]

By (30) this implies \(E[|M_n^1(1)|^2] < \infty\). The same calculation holds for \(E[|M_n^2(1)|^2]\) which gives by (33)

\[
E[G_{n,\tau}^2] \leq C_3(\tau), \quad n \in \mathbb{N},
\]

for some positive number \(C_3\) which depends on \(\tau\). By (32) and (34) we have

\[
E(G_{n,\tau}, G_{n,\tau}) < C_1 + C_3(\tau) < \infty, \quad n \in \mathbb{N}.
\]

**Step 3.** We recall again that for all \(\tau \in \Omega\) the sequence \((G_{n,\tau})_{n \in \mathbb{N}}\) satisfies \(G_{n,\tau} \leq G_{n+1,\tau}, n \in \mathbb{N}\). By using Lemma I.2.12 of [12] we observe that the function

\[
H_\tau(\gamma) := \sup_{s \in [0,1]} |x^v(\gamma(s)) - x^v(\tau(s))|, \quad \gamma \in \Omega,
\]

belongs to \(D(\mathcal{E})\), and that \(\mathcal{E}(H_\tau, H_\tau) \leq C_1\) as well as

\[
E_1(H_\tau, H_\tau) < C_1 + C_3(\tau) < \infty.
\]

Let \(\{\tau_k : k \in \mathbb{N}\}\) be a dense set in \(\Omega\) where we have chosen \(\tau_1 = \tau\). Set

\[
K_n := \inf_{1 \leq k \leq n} H_{\tau_k}, \quad n \in \mathbb{N}.
\]

For all \(n \in \mathbb{N}\) we have \(\mathcal{E}(K_n, K_n) \leq C_1\) using [11, Lemma 3.2] or [12, Lemma IV.4.1]. Thus

\[
E_1(K_n, K_n) \leq C_1 + C_3(\tau), \quad n \in \mathbb{N}.
\]

Using Lemma I.2.12 of [12] we obtain \(K_n \in D(\mathcal{E})\). Furthermore \(K_n, n \in \mathbb{N}\), is bounded in \((D(\mathcal{E}), \mathcal{E}^{1/2}_1)\) by (36). We apply the Banach–Saks theorem in the form of Lemma I.2.12 of [12], which states that every bounded sequence in \((D(\mathcal{E}), \mathcal{E}^{1/2}_1)\) has a subsequence whose Cesaro means, \(K_{n_k}\), converge strongly. Because \(\|K_n\|_{L^2(\gamma)}\) decreases in
\( n \in \mathbb{N} \) we get

\[
\lim_{k \to \infty} \mathcal{E}_1(K_{n_k}, K_{n_k}) = 0
\]
since \( \{\tau_l\} \) is dense and \((\mathcal{E}, D(\mathcal{E}))\) is closed. As \( K_n, n \in \mathbb{N}, \) is continuous, we may use Proposition 3.5 of [12] Chapter III, from which it follows that there exists a subsequence \( K_{n_l}, l \in \mathbb{N} \) of \( K_{n_k}, k \in \mathbb{N}, \) and an \( \mathcal{E} \)-nest \( F_m, m \in \mathbb{N}, \) such that \( K_{n_l} \) converges uniformly to zero on each \( F_m \) as \( l \to \infty. \) Since \( K_n \) is decreasing this holds even on the whole original sequence \( K_n. \) Let us follow an idea of [15], proof of Proposition 3.1. Given \( \delta > 0 \) we can find \( n \) such that \( K_n < \delta. \) We have

\[
F_m \subset \bigcup_{i=1}^{n} B(\tau_i, \delta),
\]

by (35), where \( B(x, \delta) \) denotes the ball of radius \( \delta \) centred at \( x. \) Now it follows that each \( F_m \) is totally bounded. \( F_m \) closed and totally bounded implies \( F_m \) compact. Thus \( F_m, m \in \mathbb{N}, \) forms an \( \mathcal{E} \)-nest consisting of compact sets.

Step 4. For fixed \( \tau \in \Omega, \) the system of functions \( F_{v,k,\tau}, \ v \in \{1, \ldots, d\}, k \in \mathbb{N}, \) separates the points in \( \Omega, \) Step 3 showed that there is an \( \mathcal{E} \)-nest consisting of compact sets and the form is a Dirichlet form by Proposition 4.4. Quasi-regularity now follows from its definition.

Remark 1. We observe that to obtain (31) we need condition (25). We recall that by Proposition 4.2(b) we have \((\mathcal{E}, D_Y(\mathcal{E})) = (\mathcal{E}, D_Z(\mathcal{E})).\)

Corollary 5.3. There exists a diffusion process properly associated with \((\mathcal{E}, D_Z(\mathcal{E})).\)

Proof. The result is an immediate consequence of Propositions 4.4 and 5.2 using Theorem IV.3.5 of [12].

6. Transferring the results to a geometric setting

Let \( M \) be a connected geometrically and stochastically complete manifold in the sense of [8,9]. As in our main reference, paper [18], we assume that the manifold is torsion free. We study the form

\[
\hat{\mathcal{E}}(\hat{F}, \hat{G}) = \left\langle \hat{D}\hat{F}, A\hat{D}\hat{G} \right\rangle_{\|\cdot\|_1} \hat{\varphi} \, d\hat{\nu}
= \left\langle \hat{D}\hat{F}, \sum_{i=1}^{\infty} \lambda_i \left\langle S_i, \hat{D}\hat{G} \right\rangle_{\|\cdot\|_1} S_i \right\rangle_{\|\cdot\|_1} \hat{\varphi} \, d\hat{\nu},
\]

where \( \hat{\nu} \) is the Wiener measure on the path space \( P_{m_0}(M) := \{\hat{\gamma} \in C([0,1]; M) : \hat{\gamma}(0) = m_0\}, m_0 \in M \) and \( \hat{\varphi} \) is a weight function we wish to specify. We mention that similar forms and operators have been considered in [2,8,18].
More precisely let \( P_0(\mathbb{R}^d) = \Omega := \{ \gamma \in C([0,1]; \mathbb{R}^d) : \gamma(0) = 0 \} \), \( I : P_0(\mathbb{R}^d) \to P_{m_0}(M) \) be the Itô map and \( \dot{\nu} \) be the image measure of the Wiener measure on \( P_0(\mathbb{R}^d) \) under \( I \). In order to recall the construction of the Itô map let \( O(M) \) denote the orthonormal frame bundle with respect to \( M \), \( \pi \) be the canonical projection \( O(M) \to M \) and \( H_1, \ldots, H_d \) be the canonical horizontal vector fields. Choose \( r_0 \in O(M) \) such that \( \pi(r_0) = m_0 \).

We introduce \( r_\gamma \) as the solution to the Stratonovich stochastic differential equation

\[
\begin{aligned}
\partial r_\gamma(t) &= \sum_{i=1}^d H_i(r_\gamma(t)) \partial x_i, & t \in [0,1], \\
r_\gamma(0) &= r_0,
\end{aligned}
\]

(38)

where \( \gamma = (\gamma_1, \ldots, \gamma_d) \in P_0(\mathbb{R}^d) \). This defines a.e. a mapping \( I : P_0(\mathbb{R}^d) \to P_{m_0}(M) \) by \( I(\gamma)(t) := \pi(r_\gamma(t)), \gamma \in P_0(\mathbb{R}^d), t \in [0,1] \), the Itô map. We also denote by \( T_x M \) as the tangent space of \( M \) at the point \( x \in M \).

The following paragraph provides compatibility with [18]. Let \( K \in C((0,\infty)) \) such that

\[
\text{Ric}(X,X) \geq -K(r)|X|^2, \quad X \in T_x M, \; x \in B(m_0, r), \; r > 0,
\]

(39)

where \( B(m_0, r) \) is the geodesic ball at \( m_0 \) with radius \( r \). Let \( \rho \) denote the Riemannian distance on \( M \) and \( \rho_{m_0}(x) := \rho(x, m_0), x \in M \). We assume that there are constants \( c_1, c_2, r_1 > 0 \) such that the following conditions hold,

\[
\frac{1}{2} \sqrt{(d-1)K(r)} \leq c_1 r, \quad r \geq r_1
\]

and

\[
|\text{Ric}(X,Y)|^p \leq c_2 \exp\left\{ \frac{1}{2} e^{-1-2c_1 \rho_{m_0}(x)^2} \right\}
\]

(40)

for some \( p \geq 2 \) and \( x \in M, X, Y \in T_x M, |X| = |Y| = 1 \). Following the proof of Lemma 2.2 in [18] word for word we obtain

\[
E\left[ \int_0^1 \|\text{Ric}_{r_\gamma(t)}\|^p dt \right] < \infty,
\]

(41)

where, as in [18], \( \text{Ric}_{r_\gamma(t)} : \mathbb{R}^d \to \mathbb{R}^d \) is defined by

\[
\langle \text{Ric}_{r_\gamma(t)}(a), b \rangle := \text{Ric}(r_\gamma(t)a, r_\gamma(t)b), \; a, b \in \mathbb{R}^d.
\]

In particular this provides the well-definiteness in \( L^2(\dot{\nu}) \) of the adjoint directional derivative \( \dot{B}_h^* \) below. We define

\[
\hat{Z} := \{ \hat{F}(\hat{\gamma}) = \hat{f}(\hat{\gamma}(s_1), \ldots, \hat{\gamma}(s_k)), \hat{\gamma} \in P_{m_0}(M) : 0 < s_1 < \cdots < s_k = 1, \hat{f} \in C^\infty_b(M^k), k \in \mathbb{N} \}
\]

(42)
\(\hat{Y} := \{ \hat{F}(\hat{y}) = \hat{f}(\hat{y}(s_1), \ldots, \hat{y}(s_k)), \hat{y} \in P_{m_0}(M) : \hat{F} \in \hat{Z}, s_1, \ldots, s_k \in \left\{ \frac{l}{2^n} : l \in \{1, \ldots, 2^n\} \right\}, n \in \mathbb{N} \} \). (43)

For \(\hat{F} \in \hat{Y}\) and \(\hat{f}\) as in (43) we introduce \((\nabla_{s_i} \hat{f})(\hat{y}) := \nabla_{s_i} \hat{f}(\hat{y}(s_1), \ldots, \hat{y}(s_k))\) as the gradient of function \(\hat{f}\) relative to the \(i\)th variable while keeping the other variables fixed. We note that \((\nabla_{s_i} \hat{f})(\hat{y})\) is an element of the tangent space \(T_{\hat{y}(s_i)}\) at \(\hat{y}(s_i)\). We now define

\[\hat{D}_s \hat{F}(\hat{y}) := \sum_{i=1}^{k} s \land s_i \cdot T^{-1}_{i-1}(\hat{y}(s_i))(\nabla_{s_i} \hat{f})(\hat{y}), \quad s \in [0, 1], \quad \hat{y} \in P_{m_0}(M),\]

where \(r\) is defined in (38). We also introduce

\[\hat{D}_s \hat{F}(\hat{y}) := \sum_{i=1}^{k} \chi_{[0,s]}(s)(T^{r^*}_{\hat{y}_{s_i}}(\nabla_{s_i} \hat{f})(\hat{y})), \quad s \in [0, 1], \quad \hat{y} \in P_{m_0}(M).\]

In addition as in [7] we define the damped version

\[\hat{D}_s \hat{F}(\hat{y}) := \sum_{i=1}^{k} \chi_{[0,s]}(s)\left(Q^*_{s,s} T^{r^*}_{\hat{y}_{s_i}}(\nabla_{s_i} \hat{f})(\hat{y})\right), \quad s \in [0, 1], \quad \hat{y} \in P_{m_0}(M),\] (44)

where \(Q^*\) denotes the adjoint of \(Q_{s,t} : \mathbb{R}^d \to \mathbb{R}^d\), which is in [7] defined by

\[\frac{dQ_{s,t}}{ds} = -\frac{1}{2} \text{Ric}_{r,(s)}Q_{s,t}, \quad Q_{t,t} = \text{Id}_{m_0}, \quad 0 \leq t \leq s\]

or equivalently

\[\frac{dQ_{s,t}}{dr} = \frac{1}{2} Q_{s,t} \text{Ric}_{r,(t)}, \quad Q_{s,s} = \text{Id}_{m_0}, \quad 0 \leq t \leq s,\] (45)

cf. also [5,6]. Finally for \(h \in \mathcal{H}, t \in \mathbb{R}, s \in [0, 1], \hat{y} \in P_{m_0}\), and \(\sigma\) being the solution to
the geometric flow equation (see [4,8,9])

\[ \frac{\partial}{\partial t} \sigma(t,s) = T^{\sigma^h(t)}_{s=0} r_0 h(s), \]

\[ \sigma^h(0,s)(\gamma) = \gamma(s), \]

we define the directional derivative

\[ \hat{D}_h \hat{F} := \lim_{t \to 0} \frac{\hat{F}(\sigma^{h(t)}) - \hat{F}}{t}, \quad \hat{F} \in \hat{Y}. \]

It is known that \( \hat{D}_h \hat{F} = \langle \hat{D} \hat{F}, h \rangle_{\mathbb{H}} \) \( \hat{\nu} \)-a.e. In addition we know from [2] that the adjoint \( \hat{D}_h^* \) of \( \hat{D}_h \) is given by

\[ \hat{D}_h^* = -\hat{D}_h + l_h, \]

where

\[ l_h(\gamma) = \int_0^1 \left\langle \hat{h}_t + \frac{1}{2} \text{Ric}_{r_0(t)} h_t, d\gamma_t \right\rangle \]

and \( \hat{\gamma} = I(\gamma) \). Having defined the geometric counterparts to the objects previously studied in this paper, analogous results hold in this new setting. We define \( \hat{Z}_\mathbb{H} \) to be the set of test functions of form \( \sum_{i=1}^k \hat{\phi}_i(\gamma) \cdot h_i(t) \), where \( \hat{\phi}_i \in \hat{Z}, h_i \in \mathbb{H} \) and \( k \in \mathbb{N} \).

**Lemma 6.1.** Let \( M \) be specified as above. If for \( p \) specified in (40) we have

\[ \frac{\hat{g}}{\hat{\phi}} \in L^1(\hat{\nu}), \quad \hat{g} \in L^{p/2}(\hat{\nu}), \quad (46) \]

then \( (\hat{D}, \hat{Z}) \) defined by

\[ \hat{D}(\hat{F}, \hat{G}) := \int \langle \hat{D} \hat{F}, \hat{D} \hat{G} \rangle_{\mathbb{H}} \hat{\phi} \, d\hat{\nu}, \quad \hat{F}, \hat{G} \in \hat{Z}, \]

is closable on \( L^2(\hat{\phi} \hat{\nu}). \)

**Proof.** Let \( \hat{u}_n \in \hat{Z} \),

\[ \hat{u}_n \to 0 \text{ in } L^2(\hat{\phi} \hat{\nu}) \text{ and } \hat{D} \hat{u}_n \to \hat{f} \text{ in } L^2(\hat{\phi} \hat{\nu}; \mathbb{H}). \]

Choose an arbitrary \( \hat{Z}_\mathbb{H} \ni \hat{\psi} = \sum_{i=1}^k \hat{\phi}_i \cdot h_i \). We observe that under (46)
the Burkholder–Davis–Gundy inequality in the form of \[14, \text{Chapter IV, Theorem 4.1}\], we obtain

\[
\left\langle \hat{D}\hat{u}_n, \frac{\hat{\psi}}{\hat{\phi}} \right\rangle H \hat{\phi} d\hat{\nu} = \sum_{i=1}^{k} \left\langle \hat{D}\hat{u}_n, h_i \right\rangle H \hat{\phi}_i d\hat{\nu} = \sum_{i=1}^{k} (\hat{D}_{h_i} \hat{u}_n) \hat{\phi}_i d\hat{\nu}
\]

\[
= \sum_{i=1}^{k} \int \hat{u}_n \hat{D}_{h_i} \hat{\phi}_i d\hat{\nu} = \sum_{i=1}^{k} \int \hat{u}_n \frac{\hat{D}_{h_i} \hat{\phi}_i}{\hat{\phi}} - \hat{\phi} d\hat{\nu}
\]

\[
= \sum_{i=1}^{k} \int -\hat{u}_n \frac{1}{\hat{\phi}} \left( \hat{D}_{h_i} \hat{\phi}_i - \int_{0}^{1} \left\langle \hat{h}_i(t) + \frac{1}{2} \text{Ric}_{r_i(t)} \hat{h}_i(t), d\gamma_t \right\rangle \cdot \hat{\phi}_i \right) \hat{\phi} d\hat{\nu}.
\]

This tends to 0 as \( n \to \infty \) provided that

\[
\frac{1}{\hat{\phi}} \left( \hat{D}_{h_i} \hat{\phi}_i - \int_{0}^{1} \left\langle \hat{h}_i(t) + \frac{1}{2} \text{Ric}_{r_i(t)} \hat{h}_i(t), d\gamma_t \right\rangle \cdot \hat{\phi}_i \right) \in L^2(\hat{\phi} \hat{\nu}) \tag{47}
\]

for \( i = 1, \ldots, k \). The statement will follow then. To show (47) we first observe that

\[
\frac{\hat{D}_{h_i} \hat{\phi}_i - \int_{0}^{1} \left\langle \hat{h}_i(t), d\gamma_t \right\rangle \hat{\phi}_i}{\hat{\phi}} \in L^2(\hat{\phi} \hat{\nu})
\]

by (46). We denote

\[
M_s \equiv M_s(\gamma) := \int_{0}^{s} \langle \text{Ric}_{r_i(t)} \hat{h}_i(t), d\gamma_t \rangle
\]

and note that \( M \) is a local martingale since \( P(\int_{0}^{1} |\text{Ric}_{r_i(t)} \hat{h}_i(t)|^2 dt < \infty) = 1 \) by (41). Using the Burkholder–Davis–Gundy inequality in the form of \[14, \text{Chapter IV, Theorem 4.1}\], we obtain

\[
E[|M_1|^p] \leq C_p E \left[ \langle M_1, M_1 \rangle^{p/2} \right] = C_p E \left[ \left( \int_{0}^{1} |\text{Ric}_{r_i(t)} \hat{h}_i(t)|^2 dt \right)^{p/2} \right]
\]

\[
\leq C_p E \left[ \int_{0}^{1} |\text{Ric}_{r_i(t)} \hat{h}_i(t)|^p dt \right]
\]

for some positive constant \( C_p \). With (41) we get \( E[|M_1|^p] < \infty \) and from (46) we obtain \( M_1/\hat{\phi} \in L^2(\hat{\phi} \hat{\nu}) \). We have verified (47). From [2] we know that \( \hat{Z} \) is dense in \( L^2(\hat{\nu}) \) and thus we note that \( \{ \hat{\psi}/\hat{\phi} : \psi \in \hat{Z}_{\delta_1} \} \) is dense in \( L^2(\hat{\phi} \hat{\nu}, H) \) cf. Lemma 3.1. The statement now follows.

**Proposition 6.2.** Let \( M \) be specified as above, let \( \hat{\phi} \) satisfy (46), and assume that \( A \geq \varepsilon \text{Id} \) for some \( \varepsilon > 0 \). Then Proposition 4.2(a)–(d) holds for \( \hat{E}, \hat{Y}, \hat{Z}, \hat{\phi}, \hat{\nu} \), respectively and where \( (\hat{E}, D_Y(\hat{E})) \) and \( (\hat{E}, D_Z(\hat{E})) \) are replaced by the respective closures of \( (\hat{E}, \hat{Y}) \) and \( (\hat{E}, \hat{Z}) \) on \( L^2(\hat{\phi} \hat{\nu}) \). In particular we mention that the counterpart to (25) reads now as (55).

\( \square \)
We are interested in $\hat{\psi}(\hat{\gamma})$ of the form

$$\hat{\psi}(\hat{\gamma}) = \exp \left\{ \int_0^1 \langle \hat{\psi}(\hat{\gamma}_t), d\hat{\gamma}_t \rangle_{T_{\hat{\gamma}_t}} - \frac{1}{2} \int_0^1 |\hat{\psi}(\hat{\gamma}_t)|^2_{T_{\hat{\gamma}_t}} \, dt \right\}, \quad \hat{\gamma} \in P_{m}(M).$$

as $\hat{\psi}$ is then the Radon–Nikodym derivative of the diffusion measure corresponding to the general diffusion process on $M$ with generator $(1/2)\Delta_M + \tilde{V}$, see [2], with respect to $\hat{\nu}$. We observe that

$$V := r^{-1}\hat{\psi} \circ \pi \circ \rho \equiv r^{-1}\hat{\psi} \circ I,$$

which says that $V \equiv V(\gamma)$ is an adapted $\mathbb{R}^d$-valued process. It turns out that

$$\hat{\psi}(\hat{\gamma}) = \exp \left\{ \int_0^1 \langle \tilde{V}(\gamma_t), d\gamma_t \rangle_{L^2} - \frac{1}{2} \int_0^1 |(V(\gamma_t))|^2_{L^2} \, dt \right\} =: \varphi(\gamma), \quad \hat{\gamma} = \pi \circ \rho(\gamma) = I(\gamma), \quad \gamma \in P_0(\mathbb{R}^d).$$

In other words $\varphi(\gamma) = \hat{\psi}(\hat{\gamma}) = \hat{\psi} \circ I(\gamma), \gamma \in P_0(\mathbb{R}^d)$. As in Section 5, having the corresponding conditions to (28) and (29), i.e.

$$\frac{\hat{f}}{\varphi^+} \in L^1(\hat{\nu}) \quad \text{and} \quad \frac{\hat{f}}{\varphi^-} \in L^1(\hat{\nu})$$

for all $\hat{f} \in \hat{Z}$, we get

$$\hat{E} \left[ \exp \left\{ \frac{1}{2} \int_0^1 |\tilde{V}(\gamma)_t|^2_{L^2} \, dt \right\} \right] < \infty,$$

where $\hat{E}$ denotes expectation taken with respect to $\hat{\nu}$. Thus we have the Novikov condition

$$E \left[ \exp \left\{ \frac{1}{2} \int_0^1 |(V(\gamma(t)))|^2_{L^2} \, dt \right\} \right] < \infty.$$  

In order to obtain quasi-regularity we sharpen condition (39) and instead assume the Ricci curvature to be bounded from below, i.e. there exists some $c \in \mathbb{R}$, not necessarily non-negative, such that

$$\text{Ric}(X, X) \geq c\|X\|^2, \quad X \in T_xM, \quad x \in M.$$  

(52)

We note that (52), the symmetry of the Ricci tensor, and the definition of the matrix valued function $(s, t) \mapsto Q_{s,t}$, as the solution of (45), implies that there exists $C \in \mathbb{R}$
such that
\[ \sup_{s,t \in [0,1]} \|Q_{s,t}\| \leq C. \]  
(53)

We also sharpen (41) and assume in addition
\[ E \left[ \int_0^1 \| \text{Ric}_{r(t)} \|^p \, dr \right] < \infty \]  
(54)
for some \( p > 2 \). Since \( \varphi \in L^1(\nu) \), (54) is for example satisfied if \( \text{Ric} \) is bounded on \( M \).

**Proposition 6.3.** Suppose all the above specifications on \( M \). Assume the following conditions on \( \hat{\varphi} \), respectively \( \varphi \), (46), (50) and (54). Let \( A \geq \varepsilon \text{Id} \) for some \( \varepsilon > 0 \), and
\[ \sup_{c_1,c_2,\ldots \in [0,1]} \sum_{j=1}^d \sum_{p=0}^{\infty} \int \frac{\lambda_{i,p,j} \hat{\varphi} d\hat{\varphi}}{2^p} < \infty, \]  
(55)
where \( i_{p,j} \) as in Lemma 4.1. Then
\[ \hat{E}(\hat{F},\hat{F}) = \left[ \hat{D}\hat{F}, \sum_{i=1}^{\infty} \lambda_i \langle S_i, \hat{D}\hat{F} \rangle_{\|\cdot\|} S_i \right]_{\|\cdot\|} \hat{\varphi} d\hat{\varphi}, \quad \hat{F} \in \hat{Z}, \]  
(56)
is quasi-regular in \( L^2(\hat{\varphi} \hat{\nu}) \).

**Proof.** We proceed as we did for Proposition 5.2. For \( r \in \mathbb{N}, l \in \{0, \ldots, 2^{r-1} - 1\} \) and \( k = 2^{r-1} + l \), set \( s_k := (2l + 1)2^{-r} \). Let \( x^v(p) \) denote the \( v \)-th coordinate of \( p \in M \), embedded in \( \mathbb{R}^N \), \( v \in \{1, \ldots, N\} \). We also write \( x^v(y) \equiv x^v(y_v) \). Fix \( \hat{\gamma} \in P_{m_0}(M), k = 2^{r-1} + l \) and \( v \in \{1, \ldots, N\} \). Consider the functions \( \hat{F}_{v,k}(p) := x^v(p) - x^v(\hat{\gamma}(s_k)), p \in M \) and
\[ \hat{F}_{v,k}(\hat{\gamma}) := \hat{F}_{v,k}(\hat{\gamma}(s_k)) = x^v(\hat{\gamma}(s_k)) - x^v(\hat{\gamma}(s_k)), \quad \hat{\gamma} \in P_{m_0}(M) \]
we note \( \hat{F}_{v,k} \in D(\hat{E}) \). We have
\[ \langle S_i, \hat{D}\hat{F}_{v,k}(\hat{\gamma}) \rangle_{\|\cdot\|} = \left| \langle \nabla_S x^v(\hat{\gamma}(s_k)), r_{T^{-1}(\hat{\gamma})} S_i(s_k) \rangle_{\|\cdot\|} \right| \\
= \left| \langle r_{T^{-1}(\hat{\gamma})} \nabla_S x^v(\hat{\gamma}(s_k)), S_i(s_k) \rangle_{\|\cdot\|} \right|. \]
Now applying the Cauchy–Schwarz inequality and using the fact that we can isometrically embed \( M \) in \( \mathbb{R}^N \), cf. [18], we obtain
\[ \left| \langle S_i, \hat{D}\hat{F}_{v,k}(\hat{\gamma}) \rangle_{\|\cdot\|} \right| \leq \left| \nabla_{S_i}^N x^v(\hat{\gamma}(s_k)) \right| |S_i(s_k)| \leq |S_i(s_k)|^v, \]
where $\nabla^N$ denotes the gradient in $\mathbb{R}^N$. Thus by Proposition 6.2 we get

$$
\hat{E}(\hat{F}_{v,k},\hat{F}_{v,k}) = \int_0^\infty \sum_{i=1}^\infty \lambda_i(\hat{\gamma})(\hat{D} \hat{F}_{v,k}(\hat{\gamma}),S_j)\hat{\phi} d\hat{\nu} \leq C_1 < \infty
$$

for some $C_1 > 0$ not depending on $v,k,\hat{\tau}$, cf. Lemma 4.1 and (55). Let

$$
\hat{G}_{n,\hat{\tau}} = \sup_{v \in \{1,\ldots,N\}} |\hat{F}_{v,k,\hat{\tau}}|, \ n \in \mathbb{N},
$$

then

$$
\hat{E}(\hat{G}_{n,\hat{\tau}},\hat{G}_{n,\hat{\tau}}) \leq C_1 < \infty, \ n \in \mathbb{N},
$$

as in Proposition 5.2. Our objective is to show $\hat{G}_{n,\hat{\tau}} \in L^2(\hat{\phi} \hat{\nu})$. In this way we obtain a bound on $\hat{E}_1(\hat{G}_{n,\hat{\tau}},\hat{G}_{n,\hat{\tau}})$ depending only on $\hat{\tau}$. Then we apply the method of Proposition 5.2. Define

$$
(T\gamma)_t := \gamma_t - \int_0^t ((V^w(\gamma))_{s})_{w=1,\ldots,d} ds,
$$

where $V$ is from (48) such that (50) is satisfied and $V^w$ denotes the $w$th coordinate. By Lemma 5.1 and the Girsanov theorem $T\gamma$ is a Brownian motion under $\varphi \nu$, cf. Theorem 3.5.1 in [10] together with (51) and Corollary 3.5.13 in [10]. We have

$$
(T^{-1}\gamma)_t^w = \gamma_t^w + \int_0^t (V^w(T^{-1}\gamma))_s ds = \gamma_t^w + B_t^w,
$$

where $B$ is of bounded variation. Using Proposition 2.2 of [7] we have

$$
\mathbb{E}\left[\hat{G}_{n,\hat{\tau}}^2 \hat{\phi}\right] \leq \mathbb{E}\left[\sup_{v \in \{1,\ldots,N\}} \left|\frac{x_v(\hat{\gamma}) - x_v(\hat{\tau})}{s \in [0,1]}\right|^2 \hat{\phi}\right] \leq 2\mathbb{E}\left[\sup_{v \in \{1,\ldots,N\}} \left|\int_0^1 E[\hat{D}_t[x_v(U(\gamma))]|\mathcal{F}_t] d\gamma_t\right| \varphi\right] + C_2(\hat{\tau}),
$$
where
\[
C_2(\tilde{\tau}) := 2 \left( \sup_{s \in [0,1]} \hat{E} \left[ |x_s^\nu| \right] + \sup_{s \in [0,1]} |x_s^\nu(\tilde{\tau})| \right)^2 < \infty
\]
and as before \( \gamma = I^{-1}(\tilde{\tau}) \). We have used \( x^\nu \in L^1(\tilde{\tau}) \), see, e.g., [18]. We write
\[
\alpha_s^\nu(t)(\gamma) := E [\mathcal{D}_t \{ x_s^\nu(I(\gamma)) \} | \mathcal{F}_t].
\] (57)

Taking into consideration that \( \alpha_s(t)(\gamma) = 0 \) for \( t > s \), we have
\[
\hat{E} \left[ \hat{G}_{n,x}^2 \phi \right] \leq 2E \left[ \left( \sup_{v \in \{1, \ldots, N\}} \left( \int_0^s \alpha_s^\nu(t)(\gamma) \ d\gamma \right) \varphi \right)^2 \right] + C_2(\tilde{\tau})
\]
\[
= 2E \left[ \left( \sup_{v \in \{1, \ldots, N\}} \left( \int_0^s \alpha_s^\nu(t)(T^{-1}\gamma) \ d(\gamma + B_t) \right) \right)^2 \right] + C_2(\tilde{\tau})
\] (58)
\[
\leq 4E \left[ \left( \sup_{v \in \{1, \ldots, N\}} \left( \int_0^s \alpha_s^\nu(t)(T^{-1}\gamma) \ d\gamma \right) \right)^2 \right] + C_2(\tilde{\tau}).
\]
\[+ 4E \left[ \left( \sup_{v \in \{1, \ldots, N\}} \left( \int_0^s \alpha_s^\nu(t)(T^{-1}\gamma) \ dB_t \right) \right)^2 \right] + C_2(\tilde{\tau}).
\]

By (44), (53) and (57) we observe that the integrand in \( \int_0^s \alpha_s^\nu(t)(T^{-1}\gamma) \ d\gamma \) is continuously differentiable according to the construction via (45). Therefore the integral can path-wise
be treated as a Riemann–Stieltjes integral. We obtain

\[
4E \left( \sup_{s \in [0,1]} \left( \int_0^s \alpha_s^x(t)(T^{-1}\gamma) \, dt \right)^2 \right) \\
\leq 4 \sum_{v=1}^{N} E \left( \sup_{s \in [0,1]} \left( \int_0^s \alpha_s^x(t)(T^{-1}\gamma) \, dt \right)^2 \right) \\
= 4 \sum_{v=1}^{N} E \left( \sup_{s \in [0,1]} \left( \gamma_s \alpha_s^x(s)(T^{-1}\gamma) - \int_0^s \gamma_t \frac{d}{dt} \alpha_s^x(t)(T^{-1}\gamma) \, dt \right)^2 \right) \\
\leq 4 \sum_{v=1}^{N} E \left[ 2 \sup_{s \in [0,1]} \left( \gamma_s \alpha_s^x(s)(T^{-1}\gamma) \right)^2 + 2 \sup_{s \in [0,1]} \left( \int_0^s \gamma_t \frac{d}{dt} \alpha_s^x(t)(T^{-1}\gamma) \, dt \right)^2 \right] \\
\leq 8 \sum_{v=1}^{N} E \left[ \sup_{s \in [0,1]} \left( \gamma_s \right)^2 \right] \sup_{s \in [0,1]} \left( \alpha_s^x(s)(T^{-1}\gamma) \right)_{L^\infty(\nu)} \left[ \gamma_s \right] \left[ \int_0^s \left( \frac{d}{dt} \alpha_s^x(t)(T^{-1}\gamma) \right)^2 \, dt \right] \\
+ 8 \sum_{v=1}^{N} E \left[ \sup_{s \in [0,1]} \left( \int_0^s \gamma_t^2 \, dt \int_0^s \left( \frac{d}{dt} \alpha_s^x(t)(T^{-1}\gamma) \right)^2 \, dt \right) \right] \\
\leq 8 \sum_{v=1}^{N} 4C_3 + 8 \sum_{v=1}^{N} E \left[ \left( \sup_{s \in [0,1]} \left( \int_0^s \left( \frac{d}{dt} \alpha_s^x(t)(T^{-1}\gamma) \right)^2 \, dt \right) \right)^{1/q} \right]^{1/p} \cdot E \left[ \left( \int_0^1 \gamma_t^2 \, dt \right)^{1/p} \right]^{1/q},
\]

where \( C_3 := \sup_{s \in [0,1]} \alpha_s^x(s)(T^{-1}\gamma) \) \( L^\infty(\nu)_< \infty \) according to (53). Using (45), (53) and \( |(T_{0-s} \nabla s_x^a)|_{T_{m_0}} \leq 1 \) since \( M \) is isometrically embedded, we get

\[
E \left[ \left( \sup_{s \in [0,1]} \left( \int_0^s \left( \frac{d}{dt} \alpha_s^x(t)(T^{-1}\gamma) \right)^2 \, dt \right) \right)^{1/q} \right]^{1/p} \leq E \left[ \sup_{s \in [0,1]} \left( \int_0^s E \left[ \left. \left\| \text{Ric}_{t^{-1}} \right\| \mathcal{F}_t \right|_{T_{m_0}} \right|^2 q \, dt \right)^{1/q} \right] \\
\leq C^2 E \left[ \sup_{s \in [0,1]} \left( \int_0^s E \left[ \left\| \text{Ric}_{t^{-1}} \right\| \mathcal{F}_t \right|_{T_{m_0}} \right|^2 q \, dt \right)^{1/q} \right] \\
\leq C^2 \left[ \int_0^1 \left\| \text{Ric}_{t^{-1}} \right\|_{T_{m_0}} \, dt \right]^{2q} \leq \left[ \int_0^1 \left\| \text{Ric}_{t^{-1}} \right\|^2 \, dt \right]^{1/q} < \infty,
\]
where, for the last line, we have taken into consideration (54). For the second term in (58) we have

\[
E \sup_{\gamma \in \{1, \ldots, N\}} \left[ \int_0^s \alpha_s(t)(T^{-1}\gamma) \, dB_t \right]^{2\gamma}
\]

\[
\leq \sum_{\gamma = 1}^N E \left( \sup_{s \in [0,1]} \left| \langle \alpha_s(t)(T^{-1}\gamma), V(T^{-1}\gamma) \rangle_{\mathbb{R}^d} \right|^2 \right)
\]

\[
\leq \sum_{\gamma = 1}^N \int_0^1 \sup_{s \in [0,1]} \left| \alpha_s(t)(T^{-1}\gamma) \right|^2 \, dt \cdot \int_0^1 \left| V(T^{-1}\gamma) \right|^2 \, dt.
\]

From (53) we get

\[
E \left[ \int_0^1 \sup_{s \in [0,1]} \left| \alpha_s(t)(T^{-1}\gamma) \right|^2 \, dt \right] < \infty \tag{62}
\]

and by (49) we get

\[
E \left[ \int_0^1 \left| V(T^{-1}\gamma) \right|^2 \, dt \right] = E \left[ \int_0^1 \left| V(\gamma) \right|^2 \, dt \right] \varphi
\]

\[
\leq E \exp \left\{ \int_0^1 \left| V(\gamma) \right|^2 \, dt \right\} = E \left[ \frac{1}{\varphi} \right] < \infty.
\tag{63}
\]

Plugging (62) and (63) in (61), this together with (60) and (58), shows \( \hat{G}_{n,\bar{\tau}} \in L^2(\varphi \hat{\nu}) \), the above formulated goal. Now having verified

\[
\hat{E}(\hat{G}_{n,\bar{\tau}}, \hat{G}_{n,\bar{\tau}}) \leq C_1 + \| \hat{G}_{n,\bar{\tau}} \|^2_{L^2(\varphi \hat{\nu})} < \infty,
\]

the rest of the proof can be carried out in the same way as the proof of Proposition 5.2, Steps 3 and 4. The claim follows.

References

[1] N. Bouleau, and F. Hirsch, Dirichlet Forms and Analysis on Wiener Space, De Gruyter, Berlin, 1991.

[2] M. Capitaine, E.P. Hsu, and M. Ledoux, Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces, Electron. Commun. Probab. 2 (1997), pp. 71–81.

[3] X. Chen, and B. Wu, Functional inequality on path space over a non-compact Riemannian manifold, J. Funct. Anal. 266(12) (2014), pp. 6753–6779.

[4] B. Driver, and M. R"ockner, Construction of diffusions on path and loop spaces of compact Riemannian manifolds, C.R. Acad. Sci. Paris 315 (1992), pp. 603–608.

[5] S. Fang, Stochastic anticipative integrals on a Riemannian manifold, J. Funct. Anal. 131(1) (1995), pp. 228–253.
[6] S. Fang, and P. Malliavin, Stochastic analysis on the path space of a Riemannian manifold, J. Funct. Anal. 118(1) (1993), pp. 249–274.

[7] C. Houdré, and N. Privault, A concentration inequality on Riemannian path space, in Stochastic Inequalities and Applications, Progress in Probability, Vol. 56, Birkhäuser, Basel, 2003, pp. 15–21.

[8] E.P. Hsu, Quasi-invariance of the Wiener measure on path spaces: Noncompact case, J. Funct. Anal. 193(2) (2002), pp. 278–290.

[9] E.P. Hsu, and C. Ouyang, Quasi-invariance of the Wiener measure on the path space over a complete Riemannian manifold, J. Funct. Anal. 257(5) (2009), pp. 1379–1395.

[10] I. Karatzas, and S.E. Shreve, Brownian Motion and Stochastic Calculus, 2nd ed., Springer, Berlin, 2000.

[11] J.-U. Löbus, A class of processes on the path space over a compact Riemannian manifold with unbounded diffusion, Trans. Am. Math. Soc. 356(9) (2004), pp. 3751–3767.

[12] Z.-M. Ma, and M. Röckner, Introduction to the Theory of (Non-symmetric) Dirichlet Forms, Springer, Berlin, 1992.

[13] D. Nualart, The Malliavin Calculus and Its Applications, AMS, Providence, RI, 2009.

[14] D. Revuz, and M. Yor, Continuous Martingales and Brownian Motion, Springer, Berlin, 1999.

[15] M. Röckner, and B. Schmuland, Tightness of general $c_{1,p}$ capacities on Banach space, J. Funct. Anal. 108(1) (1992), pp. 1–12.

[16] W. Stummer, The novikov and entropy conditions of multidimensional diffusion processes with singular drift, Probab. Theory Relat. Fields 97(4) (1993), pp. 515–542.

[17] F.-Y. Wang, Weak poincaré inequalities on path spaces, Int. Math. Res. Not. 2004(2), (2004), pp. 90–108.

[18] F.-Y. Wang, and B. Wu, Quasi-regular Dirichlet forms on Riemannian path and loop spaces, Forum Math. 20(6) (2008), pp. 1085–1096.

[19] F.-Y. Wang, and B. Wu, Quasi-regular dirichlet forms on free Riemannian path spaces, Infin. Dimens. Anal. Quant. Probab. Relat. Top. 12(2) (2009), pp. 251–267.