A minimal representation of the orthosymplectic Lie supergroup

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Outline

Introduction

Construction
Classification

Goal
Classification of all possible representations of a given group/algebra.
Classification

Goal
Classification of all irreducible representations of a given group/algebra.
Goal
Classification of all unitary irreducible representations of a given Lie group.
Connected compact groups

Figure: Élie Cartan  CC BY-SA 2.5, MFO

Figure: Hermann Weyl  CC BY-SA 3.0, ETH-Bibliothek
Semisimple groups

**Figure:** Harish-Chandra  CC BY-SA 4.0, Pratham Cbh
The orbit method

The orbit method (or geometric quantization)
Gives a connection between
- the unitary irreducible representations of $G$
- the coadjoint orbits of $g^*$. 

Figure: Alexandre Kirillov
Minimal representations

Minimal representation: hand-waving definition
The representation associated to the minimal nilpotent coadjoint orbit via the orbit method.

Special properties

- Very small: lowest possible Gelfand-Kirillov dimension.
- Difficult from orbit method point of view.
A unitary representation of a simple real Lie group $G$ is called \textit{minimal} if the annihilator ideal of the derived representation of the universal enveloping algebra of $\text{Lie}(G)_\mathbb{C}$ is the Joseph ideal.

\textbf{Definition (Joseph ideal)}

The Joseph ideal is the unique completely prime, two-sided ideal in the universal enveloping algebra such that the associated variety is the closure of the minimal nilpotent coadjoint orbit.

\cite{GanSavin2005}
Minimal representations: an example

The metaplectic representation

Unitary irreducible representation of $Mp(2n, \mathbb{R})$, a double cover of $Sp(2n, \mathbb{R})$, on $L^2_{even}(\mathbb{R}^n)$. On algebra level it is given by

$$d\mu \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} = -\pi i \sum_{i,j=1}^{n} C_{ij}y_iy_j$$

for $C \in \text{Sym}(n, \mathbb{R})$

$$d\mu \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} = -\frac{1}{2} \text{tr}(A) - \sum_{i,j=1}^{n} A_{ij}y_j\partial_i$$

for $A \in \text{M}(n, \mathbb{R})$

$$d\mu \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \frac{1}{4\pi i} \sum_{i,j=1}^{n} B_{ij}\partial_i\partial_j$$

for $B \in \text{Sym}(n, \mathbb{R})$. 
Other prominent example is given by the minimal representation of $O(p, q)$.

There exists a unified construction of minimal representation using Jordan algebras developed in [HKM].

[HKM] J. Hilgert, T. Kobayashi, J. Möllers. Minimal representations via Bessel operators. J. Math. Soc. Japan 66 (2014), no. 2, 349–414.
Supersymmetry

- Introduced in the 70s.

- Treat bosons and fermions at the same footing.

- Add ‘odd stuff’ to the ordinary (even) ‘stuff’.
Super vector space

**Definition**
A super vector space is a $\mathbb{Z}_2$ graded vector space, i.e.

$$V = V_0 \oplus V_1.$$ 

The elements in $V_0 \cup V_1$ are called homogeneous.

We define parity for homogeneous elements as

$$|u| = i \quad \text{if} \quad u \in V_i.$$
Definition of a Lie superalgebra

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a super vector space with a bilinear product $[\ , \ ]$ which

- is a graded product

$$[g_i, g_j] \subset g_{i+j}, \text{ for } i, j \in \mathbb{Z}_2$$

- is super anti-commutative

$$[X, Y] = -(-1)^{|X||Y|} [Y, X]$$

- satisfies the super Jacobi identity

$$(-1)^{|X||Z|} [X, [Y, Z]] + (-1)^{|Y||X|} [Y, [Z, X]] + (-1)^{|Z||Y|} [Z, [X, Y]] = 0.$$
The orthosymplectic Lie superalgebra

Consists of the $(p + q + 2n) \times (p + q + 2n)$ matrices for which

$$X^{st} \Omega + \Omega X = 0$$

with

$$\Omega = \begin{pmatrix} I_p & -I_q \\ -I_q & -I_n \end{pmatrix}.$$ 

Bracket: $[X, Y] = XY - (-1)^{|X||Y|} YX$. 
The orthosymplectic Lie superalgebra \( \mathfrak{osp}(1, 0|2) \)

Defining equation

\[
\begin{pmatrix}
  a & d & g \\
  -b & e & f \\
  -c & h & i
\end{pmatrix}
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & -1 \\
  0 & 1 & 0
\end{pmatrix}
+ \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & -1 \\
  0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{pmatrix} = 0.
\]

So \( \mathfrak{osp}(1, 0|2n) = \left\{ X = \begin{pmatrix}
  0 & b & c \\
  c & e & f \\
  -b & h & -e
\end{pmatrix} \mid b, c, e, f, h \in \mathbb{R} \right\} \).

Even part:

\[
X_0 = \begin{pmatrix}
  0 & e & f \\
  e & f & 0 \\
  h & -e & 0
\end{pmatrix}
\]

Odd part:

\[
X_1 = \begin{pmatrix}
  c & b & c \\
  b & c & 0 \\
  -b & h & 0
\end{pmatrix}
\]
Goal

Construct minimal representations for Lie supergroups.

Focus on the example $\text{OSp}(p, q|2n)$.

Approach

Generalize the unified construction of minimal representation using Jordan algebras developed in [HKM].

[HKM] J. Hilgert, T. Kobayashi, J. Möllers. Minimal representations via Bessel operators. J. Math. Soc. Japan 66 (2014), no. 2, 349–414.
How to construct minimal representations for simple Lie groups?

- Start from a simple Jordan algebra.

- Associate some Lie algebras/groups:
  - structure algebra/group
  - the *Tits-Kantor-Koecher Lie algebra* / conformal group.

- Construct a representation from this TKK Lie algebra on the Jordan algebra.

→ Representation is still too big.
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How to construct minimal representations for simple Lie groups?

- Study the orbits of the Jordan algebra under the action of the structure group.
- Show that this representation restricts to the minimal orbit.
- Infinitesimally unitary representation with respect to some $L^2$ measure.
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Minimal representations for Lie supergroups: what do we need?

- Jordan superalgebras ✓ [Ka]
- Structure algebra and TKK algebras ✓ [BC1]
- Representation on the Jordan superalgebra ✓ [BC2]

[Ka] V. G. Kac. Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras. Comm. Algebra 5 (1977), no. 13, 1375-1400.
Minimal representation for \( \mathfrak{osp}(p,q|2n) \)

- Construction
- The super case

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[BC1] S. Barbier, K. Coulembier. On structure and TKK algebras for Jordan superalgebras. Comm. Algebra 46 (2018), no 2, 684-704.
Structure algebra and TKK

The spin factor Jordan superalgebra

\[ J := \mathbb{R} e \oplus \mathbb{R}^{p+q-3|2n} \]

The structure algebra

\[ \text{str}(J) = \mathfrak{osp}(p - 1, q - 1|2n) \oplus \mathbb{R} L_e \]

The Tits-Kantor-Koecher construction

\[ \text{TKK}(J) = J \oplus \text{str}(J) \oplus J = \mathfrak{osp}(p, q|2n) \]
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[BC2] S. Barbier, K. Coulembier. Polynomial Realisations of Lie (Super)Algebras and Bessel Operators. International Mathematics Research Notices 2017, no. 10, 3148-3179.
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These steps were done in general. For the next steps we restrict to \(\mathfrak{osp}(p, q|2n)\).
Minimal representations for $\mathfrak{osp}(p, q|2n)$: what do we need?

- Jordan superalgebras ✓
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- Representation on the Jordan superalgebra ✓
- Minimal orbit and restriction to this orbit ✓ [BF]
- Integration to group level ✓ [BF]
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[BF] S. Barbier and J. Frahm, A minimal representation of the orthosymplectic Lie superalgebra, 45 pages, arXiv:1710.07271.
Harish-Chandra supermodules

\[ G = (G_0, g, \sigma) \] a Lie supergroup,
\( G_0 \) is connected and real reductive,
\( K_0 \) is a maximal compact subgroup of \( G_0 \).

Definition (Harish-Chandra supermodule)

A super vector space \( V \) is a Harish-Chandra supermodule if \( V \)

- is a locally finite \( K_0 \)-representation
- it has a compatible \( g \)-module structure
- finitely generated over \( U(g) \)
- \( K_0 \)-multiplicity finite.

A. Alldridge. Fréchet Globalisations of Harish-Chandra Supermodules. Int Math Res Notices 2017, no. 17, 5137-5181.
A Harish-Chandra supermodule

Set $\mu = \max(p - 2n, q) - 3$, and $\nu = \min(p - 2n, q) - 3$

$$g = \mathfrak{osp}(p, q|2n), \quad \mathfrak{t} = \mathfrak{osp}(p|2n) \oplus \mathfrak{so}(q).$$

Define

$$W = U(g)\tilde{K}_{\nu/2}(|X|)$$

with $\tilde{K}_{\nu/2}(|X|)$ the modified Bessel function of the third kind.

**Theorem**

If $p + q$ is even and $p - 2n > 0$, then $W$ is a Harish-Chandra supermodule with $\mathfrak{t}$-decomposition $W = \bigoplus_j W_j$

$$W_j \cong \mathcal{H}^{\frac{\mu-\nu}{2} + j}(\mathbb{R}^{p|2n}) \otimes \mathcal{H}^j(\mathbb{R}^q) \quad \text{if } p - 2n \leq q,$$

$$W_j \cong \mathcal{H}^j(\mathbb{R}^{p|2n}) \otimes \mathcal{H}^{\frac{\mu-\nu}{2} + j}(\mathbb{R}^q) \quad \text{if } p - 2n \geq q.$$
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[BF] S. Barbier and J. Frahm, A minimal representation of the orthosymplectic Lie superalgebra, 45 pages, arXiv:1710.07271.
Properties of the minimal representation

- Gelfand-Kirillov dimension: $p + q - 3$.
- The annihilator ideal is the Joseph ideal constructed in [CSS] if $p + q - 2n - 2 > 0$.
- There exists non-degenerate superhermitian, sesquilinear form for which the representation is skew-symmetric.

K. Coulembier, P. Somberg, V. Souček. Joseph ideals and harmonic analysis for $\mathfrak{osp}(m|2n)$. Int. Math. Res. Not. IMRN (2014), no. 15, 4