New Phantom and non-Phantom Wormhole Solutions with Generic Cosmological Constant

Y. Heydarzade*, N. Riazi‡ and H. Moradpour§

1 Department of Physics, Azarbaijan Shahid Madani University, Tabriz, 53714-161 Iran
2 Department of Physics, Shahid Beheshti University, Evin, Tehran 19839, Iran
3 Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran.

There are a number of reasons to study wormholes with generic cosmological constant $\Lambda$. Recent observations indicate that present accelerating expansion of the universe demands $\Lambda > 0$. On the other hand, some extended theories of gravitation such as supergravity and superstring theories possess vacuum states with $\Lambda < 0$. Even within the framework of general relativity, a negative cosmological constant permits black holes with horizons topologically different from the usual spherical ones. These solutions are convertible to wormhole solutions by adding some exotic matter. In this paper, the phantom and non-phantom matter wormhole solutions in the presence of cosmological constant are studied. By constructing a specific class of shape functions, mass function, energy density and pressure profiles which support such a geometry are obtained. It is shown that for having such a geometry, the wormhole throat $r_0$, the cosmological constant $\Lambda$ and the equation of state parameter $\omega$ should satisfy two specific conditions. The possibility of setting different values for the parameters of the model helps us to find exact solutions for the metric functions, mass functions and energy-momentum profiles. At last, the volume integral quantifier, which provides useful information about the total amount of energy condition violating matter is discussed briefly.

I. INTRODUCTION

In general relativity, geometrical bridges connecting two distant regions of a universe or even two different universes are in principle possible. Spacetimes containing such bridges appear as solutions of the Einstein field equations. The term “wormhole” for these bridges was used for the first time in 1957 by J. A. Wheeler [1, 2]. Many years later in 1988, the notion of traversable Lorentzian wormholes attracted the attention of physicists by the fundamental papers of Morris, Thorne and Yurtsewer [3, 4]. In these papers it was shown that such wormholes could allow humans not only to travel between universes, or distant parts of the same universe, but also to construct time machines. Also, it has been suggested that black holes and wormholes are interconvertible structures and stationary wormholes could be possible as final states of black-hole evaporation [5]. Moreover, it is shown that astrophysical accretion of ordinary matter could convert wormholes into black holes [6–8]. In the wormhole physics, it is known that these structures do not satisfy common energy conditions. The energy-momentum tensor of the matter supporting such geometries violates the null energy condition at least in the vicinity of the wormhole throat [9–11]. The matter that violates the null energy condition is usually called as exotic matter. Since the violation of the energy conditions is conventionally considered as a problematic issue, minimizing its usage seems to be useful. One may obtain that in the context of thin-shell wormholes using the cut-and-paste procedure [12, 13]. In this context, the exotic matter is concentrated at the throat of the wormhole, which is localized on the thin shell. Another approach lies within modified theories of gravity, where normal matter threading the wormhole satisfies the energy conditions, and they are the higher order curvature terms that support these exotic geometries. In the context of modified gravity, the gravitational field equation may be written as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}^{eff}$, where $T_{\mu\nu}^{eff}$ is the effective energy-momentum tensor. Then, in modified theories of gravity, the effective energy-momentum tensor involving higher order derivatives violates the null energy condition, i.e., $T_{\mu\nu}^{eff} k^\mu k^\nu < 0$ where $k^\mu$ is a null vector. This approach is widely analysed in the literature as in the fame work of $f(R)$ gravity [14, 15], curvature matter couplings [16, 17], conformal Weyl gravity [18] and braneworlds [19].

On the other hand, according to the recent discoveries in cosmology, our universe is in accelerated expansion [20–23]. A dominating dark energy component with an equation of state $p = \omega \rho$ with $\omega < -\frac{1}{3}$, is thought to be responsible for this accelerated expansion phase of universe. The specific ranges of $\omega < -1$, $\omega = -1$ and $-1 < \omega < -\frac{1}{3}$ correspond to the phantom energy, cosmological constant and quintessence matter, respectively. Then, one of the reasons to study wormholes with generic cosmological constant $\Lambda$ and specially $\Lambda > 0$ turns to the accelerated expansion of the universe. Another reason to investigate wormhole solutions with generic cosmological constant $\Lambda$ turns to supergravity

* heydarzade@azaruniv.edu
† n-riazi@sbu.ac.ir
‡ h.moradpour@riaam.ac.ir
and superstring theories which have vacuum states with $\Lambda < 0$. Also, in the framework of general relativity, a negative cosmological constant allows black hole solutions with horizons that are topologically different from the usual spherical ones. Adding some exotic matter can convert these black hole solutions to wormhole solutions \[24, 25\].

In addition, the phantom energy possess some special features such as a divergent cosmic scale factor in a finite time \[26, 27\], leading to appearance of negative entropy and temperature \[28, 29\] and predicting a new long range force \[30\]. Since the fundamental ingredient of wormhole geometries is the null energy condition violation, phantom energy can provide a means to support traversable wormhole geometries \[31–33\]. Indeed, due to the acceleration of the universe, it seems possible that macroscopic wormholes naturally grow from submicroscopic states that originally pervaded the quantum foam. Moreover, it could be imagined an absurdly advanced civilization mining the cosmic fluid for phantom energy necessary to construct and sustain a traversable wormhole \[32, 33\]. Another point is that as the phantom energy equation of state represents a spatially homogeneous cosmic fluid and is assumed not to cluster, it is also possible that inhomogeneities may arise due to gravitational instabilities. Thus, density fluctuations in the cosmological background may be the origin of phantom wormholes. It can also be considered that these structures are sustained by their own quantum fluctuations \[34–36\].

Many of the papers published on the phantom energy wormholes are not asymptotically flat \[31–33\]. The approach of these papers is to glue the interior wormhole metric to a vacuum exterior spacetime at a junction interface \[37–41\]. Recently, new asymptotically flat phantom wormhole solutions with no need to surgically pasting the interior wormhole geometry to exterior vacuum spacetime have been found in \[42\]. On the other hand, spherically symmetric and static traversable Morris-Thorne wormholes in the presence of a generic cosmological constant $\Lambda$ are analyzed in \[43\]. In that paper, two spacetimes are glued into each other and explored under matching conditions for the interior and exterior spacetimes. Another paper in this direction is \[44\] in which the cosmological constant is considered as a space variable scalar ($\Lambda = \Lambda(r)$).

In the present paper which can be considered as a follow up of \[42\], we shall study the phantom and non-phantom wormhole solutions including a cosmological constant. Our solutions include an asymptotic space time which should normally be de Sitter or anti-de Sitter. It is shown that for having such a geometry, the wormhole throat $r_0$, cosmological constant $\Lambda$ and the equation of state parameter $\omega$ must satisfy certain conditions. The organization of this paper is as follows: In section II, general geometries and constraints of Lorentzian wormholes are outlined. In section III, Einstein field equations and the metric functions are studied and the above mentioned conditions on $r_0$, $\Lambda$ and $\omega$ are obtained. In sections IV and V, some specific solutions with their mass function and energy-momentum tensor profiles are presented. At the end of section V, the volume integral quantifier, for general solutions obtained in section III, is briefly mentioned. Finally, in section VI, we present our concluding remarks. Throughout this work, units of $G = c = 1$ are used.

II. GENERAL GEOMETRY AND CONSTRAINTS OF LORENTZIAN WORMHOLES

The general static and spherically symmetric Lorentzian wormhole metric is given by

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{1 - b(r)} + r^2 d\Omega^2,$$  \hspace{1cm} (1)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. The metric functions $U(r)$ and $b(r)$ are referred to as the redshift function and shape function, respectively. The general constraints on the redshift and shape functions which build up a wormhole are as follow:

1. The wormhole throat, which connects two asymptotic regions, is located at the minimum radial coordinate $r_0$ at which $b(r_0) = 0$.

2. The shape function $b(r)$ must satisfy the so-called flaring-out condition given by

$$\frac{b(r) - rb'(r)}{2b'(r)} > 0,$$  \hspace{1cm} (2)

which at the throat of the wormhole reduces to $b'(r_0) < 1$.

3. In order to keep the proper signature of the metric, for the radial coordinates $r > r_0$, the shape function should satisfy the condition

$$1 - \frac{b(r)}{r} > 0.$$  \hspace{1cm} (3)
4. In order to have asymptotically flat geometries, the metric functions need to obey the following conditions at \( r \to \infty \):

\[
U(r) \to 1, \\
b(r) \to 0.
\]

(4)

Obviously, these conditions may be relaxed for no-asymptotically flat wormholes.

5. To ensure the absence of horizons and singularities, it is also required that \( U(r) \) be finite and nonzero throughout the spacetime.

Notice that the above constraints provide a minimum set of conditions which is mandatory for characterizing the geometry of two asymptotically flat regions connected by a bridge [45]. Of course, conditions 4 and 5 need not hold for a wormhole living in a de Sitter or anti-de Sitter background.

III. EINSTEIN FIELD EQUATIONS AND THE METRIC FUNCTIONS

We consider an anisotropic fluid for the matter content of the spacetime in the form of \( T^\mu_\nu = \text{diag}(-\rho, p_r, p_l, p_l) \) where \( \rho(r) \) represents the energy density, \( p_r(r) \) is the radial pressure and \( p_l(r) \) stands for the lateral pressure measured in the orthogonal direction to the radial direction. The Einstein equation with the cosmological constant \( \Lambda \)

\[
G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu},
\]

(5)

leads to the following equations

\[
b'(r) = (8\pi \rho(r) - \Lambda) r^2,
\]

(6)

\[
\frac{U'(r)}{U(r)} = \frac{8\pi p_r(r)r^3 + \Lambda r^3 + b(r)}{r(r-b(r))},
\]

(7)

\[
p_r(r) = p_l(r) + \frac{r}{2} \left[p'_l(r) + (\rho(r) + p_r(r)) \frac{U'(r)}{2U(r)} \right]
\]

(8)

where the prime sign denotes the derivative with respect to the radial coordinate \( r \). From equation (6), it is apparent that the total density \( \rho(r) \) is as \( \rho(r) = \rho_w(r) + \rho_\Lambda \) where \( \rho_w(r) = \frac{b'(r)}{8\pi r} \) is the density profile induced by the wormhole structure and \( \rho_\Lambda \equiv \frac{\Lambda}{8\pi} \) is the density of the cosmological constant \( \Lambda \) which can be either positive or negative representing the de Sitter and anti-de Sitter regimes, respectively.

One can define a mass function \( m(r) \) according to

\[
m(r) = \int_{r_0}^r 4\pi r^2 \rho(r) dr,
\]

(9)

This equation together with the equation (6) leads to

\[
m(r) = \frac{1}{2} \left[b(r) - r_0 + \frac{\Lambda}{3} (r^3 - r_0^3) \right],
\]

(10)

which clearly vanishes at the wormhole throat \( r = r_0 \). Indeed, the inclusion of the cosmological constant will shift the respective values of \( \rho(r) \), \( p_r(r) \) and \( p_l(r) \) and might help in minimizing the amount of energy condition violating matter which can be seen directly from equation (10). Although the corresponding mass of the cosmological constant is unbounded, the wormhole part may or may not be bounded. We will consider these two possibilities in coming next sections.

In this paper, we are interested in the wormhole solutions using the barotropic equation of state \( p_r(r) = \omega \rho(r) \). Thus, using equations (6) and (7) we obtain

\[
\frac{U'(r)}{U(r)} = \frac{r \omega b'(r) + b(r) + \Lambda (1 + \omega)r^3}{r(r-b(r))}.
\]

(11)
Before going further, it is important to point out a subtlety in considering the phantom energy equation of state in inhomogeneous spherically symmetric wormhole spacetimes. As emphasized in [31] and [32]–[33], the phantom dark energy is a homogeneously distributed fluid with an isotropic pressure. However, it can be extended to the context of inhomogeneous spacetimes by considering a negative radial pressure in the equation of state. Then, the lateral pressure can be obtained using equation (3) which is coming from the Einstein field equations. This approach is motivated by the discussion of the inhomogeneities that may appear because of gravitational instabilities and through the analysis carried out in [40]. The authors of [40] investigated a spherically symmetric time dependent wormhole solution in a cosmological context with a ghost scalar field. As a result, the radial pressure is negative through the spacetime and for large values of the radial coordinate equals to the lateral pressure, which shows the behaviour of ghost scalar field as dark energy.

We have now four equations, equations (6)–(8) and (11), with five unknown quantities \( U(r) \), \( b(r) \), \( \rho(r) \), \( p_r(r) \) and \( p_l(r) \). There are two different approaches for solving the field equations. One approach is to consider a specific distribution of the energy density threading the wormhole, like the approach of [31] and consequently finding the metric functions \( U(r) \) and \( b(r) \). The second approach involves proposing a model wormhole geometry by imposing specific choices for the shape and redshift functions and obtaining the supporting energy-momentum tensor profile [32, 33]. In this paper, since we are preliminary interested in finding asymptotically de Sitter and anti-de Sitter wormhole solutions, which are supported by phantom or non-phantom matter contents, the second approach is followed.

We consider wormholes with the shape function

\[
\frac{b(r)}{r_0} = a \left( \frac{r}{r_0} \right)^\alpha + C, \tag{12}
\]

where \( a \), \( \alpha \) and \( C \) are dimensionless constants. The first constraint imposes that \( C = 1 - a \). In order to satisfy the fourth constraint, we get \( \alpha < 1 \). Thus, the shape function takes the form

\[
b(r) = r_0 + ar_0 \left[ \left( \frac{r}{r_0} \right)^\alpha - 1 \right]. \tag{13}
\]

In order to satisfy the flaring out condition we obtain

\[
1 - a + a \left( \frac{r}{r_0} \right)^\alpha (1 - \alpha) > 0. \tag{14}
\]

Form the previously obtained result, \( \alpha < 1 \), we may divide the solutions of this condition into the following cases: i) \( 0 \leq \alpha \leq 1 \) which clearly satisfies the above condition, ii) \( \alpha > 1 \) whose exact value depends on \( \alpha \) and \( r \) and iii) \( \alpha < 0 \) which similar to the previous case, its exact value depending on \( \alpha \) and \( r \). In addition, these three classes should also satisfy the condition \( \alpha a \alpha < 1 \) coming from the flaring out condition at the throat.

Using equations (6) and (13), we can obtain the total energy density \( \rho(r) \) as

\[
\rho(r) = \frac{1}{8\pi} \left( \frac{aa}{r_0^3} \left( \frac{r}{r_0} \right)^{\alpha-3} + \Lambda \right), \tag{15}
\]

where for the case of \( \Lambda = 0 \), the profile density of [12] are recovered. Also, from equation (13), it is seen that in order to obtain de Sitter or anti-de Sitter solutions when \( r \to \infty \) we should have \( \alpha < 3 \). Then, our obtained restricted regime \( \alpha < 1 \) includes these asymptotic behaviors.

Also, the total energy density \( \rho(r) \), equation (15), should satisfy the positive energy condition

\[
\frac{aa}{r_0^3} \left( \frac{r}{r_0} \right)^{\alpha-3} + \Lambda \geq 0, \tag{16}
\]

which is valid for \( \Lambda > 0 \) with \( aa \geq 0 \). Then, with respect to the above three classes of \( a \) values and the condition \( aa \alpha < 1 \) coming from the flaring out condition at the throat, we will have \( 0 \leq aa \alpha < 1 \) and the following classes are distinguished: i) \( \Lambda > 0 \) with ranges of \( 0 \leq \alpha \leq 1 \) and \( 0 \leq \alpha < 1 \), ii) \( \Lambda > 0 \) with ranges of \( 1 < \alpha \) and \( 0 \leq \alpha < 1 \) and iii) \( \Lambda > 0 \) with ranges of \( 0 \leq \alpha \) and \( \alpha \leq 0 \).

The condition (16) can be satisfied for \( \Lambda < 0 \) with \( 0 < aa \alpha < 1 \) and \( \Lambda > 0 \) with \( aa \alpha < 0 \), but these cases are completely dependent on exact numerical value of \( \Lambda \). The case of \( aa \alpha < 0 \) is arising from the presence of a positive cosmological constant \( \Lambda \) which is not allowed in the absence of \( \Lambda \) as in [12]. In addition, when \( \Lambda < 0 \), the positive energy condition for the energy density \( \rho(r) = \rho_w(r) + \rho_\alpha \geq 0 \) can be violated. The same situation occurs in the anti-de Sitter spacetime [47].
Returning to the equation (13), we can obtain a condition for the throat of the wormhole, \( r = r_0 \), when \( \Lambda > 0 \) with the range of \( 0 \leq \alpha a < 1 \) as

\[
r_0^2 \geq \frac{\alpha a}{\Lambda},
\]

where it is trivial and does not put any restriction on the size of the throat \( r_0 \). For the case of \( \Lambda > 0 \) with \( \alpha a < 0 \), we also recover this condition but since \( \alpha a \) has negative values, it will be a nontrivial restriction on the size of wormhole throat. If we consider \( \Lambda < 0 \), we obtain

\[
r_0^2 \leq -\frac{\alpha a}{\Lambda},
\]

where is a nontrivial restriction on throat size, since for this case we should have just \( 0 < \alpha a < 1 \). Equations (17) and (18) reveal the dependence of the size of the wormhole throat on the cosmological constant \( \Lambda \) and shape function parameters \( a \) and \( \alpha \).

Due to the finiteness of \( U(r) \), using the fifth constraint and equation (6), the radial pressure at the throat should be as follows

\[
p_r(r_0) = -\frac{1}{8\pi} \left( \Lambda + \frac{1}{r_0^2} \right),\]

which together with the equation of state \( p_r(r) = \omega \rho(r) \) leads to

\[
\rho(r_0) = -\frac{1}{8\pi \omega} \left( \Lambda + \frac{1}{r_0^2} \right).
\]

On the other hand, by substituting \( r = r_0 \) in equation (12) we obtain

\[
\rho(r_0) = \frac{1}{8\pi} \left( \frac{\alpha a}{r_0^2} + \Lambda \right),
\]

where the consistency between the two above equations gives the cosmological constant as

\[
\Lambda = -\frac{1 + \alpha a \omega}{r_0^2 (1 + \omega)}.
\]

In the absence of \( \Lambda \) as in [42], for the considered shape function (13), the supporting matter with \( \omega = -\frac{1}{\alpha a} \) lies just in the phantom area. But with respect to the obtained equation (22), it is seen that non-phantom matter is also allowed. Thus, in order to obtain solutions for \( \Lambda > 0 \) with \( 0 \leq \alpha a < 1 \), we should have \(-\frac{1}{\alpha a} < \omega < -1 \) which points to a restricted phantom era with a lower bound specified by given \( \alpha \) and \( a \) values. Also, when we have \( \Lambda < 0 \) and \( 0 < \alpha a < 1 \), there are accessible solutions for both ranges \( \omega < -\frac{1}{\alpha a} < -1 \) and \( \omega > -1 \) which correspond the phantom and non-phantom regimes, respectively.

Considering the shape function given by equation (13), the ordinary differential equation for the redshift function (11) takes the following form

\[
\frac{U'(r)}{U(r)} = \frac{\Lambda (1 + \omega) r}{1 - \left( \frac{r}{r_0} \right)^\alpha} \frac{1 + \left( \frac{r}{r_0} \right)^\alpha (1 + \omega) - 1}{1 - \left( \frac{r}{r_0} \right)^{\alpha - 1}},
\]

where the results of [42] are simply recovered by substituting \( \Lambda = 0 \). On the other hand, equations (13) and (15) can be used to write the shape function as

\[
b(r) = \frac{8\pi}{\alpha} \rho(r)r^3 - \frac{\Lambda}{\alpha} r^3 + r_0 (1 - a)
\]

\[
= \frac{8\pi}{\alpha \omega} p_r(r)r^3 - \frac{\Lambda}{\alpha} r^3 + r_0 (1 - a).
\]

It is seen that this equation reveals the de Sitter or anti-de Sitter nature of the whole spacetime. In fact, the considered asymptotic flat form for the shape function, equation (13), affects the amount of energy condition violating matter and couples the wormhole throat size \( r_0 \), cosmological constant \( \Lambda \) and equation of state parameter \( \omega \) to each other.
as obtained in equations (17), (18) and (22). Consequently, one is able to substitute equation (22) into equation (11) and after using equation (9), obtain the following
\[
\frac{U'(r)}{U(r)} = \frac{8\pi p_r \left( \frac{\omega b}{\omega a} \right) r^3 + \Lambda \left( \frac{\omega - 1}{\omega a} \right) r^3 + r_0(1-a)}{r^2 \left( 1 - \frac{8\pi}{\alpha} p_r r^2 + \frac{\Lambda r^2}{\alpha} - \frac{r_0}{r} (1-a) \right)}.
\]

Unfortunately, equations (23) and (25) in general have not an exact solution. Thus, in order to deduce exact wormhole solutions for these equations, we will consider some specific choices for the parameters \(a\) and \(\alpha\) in the next sections.

Meanwhile, using equations (24) and (25) we can rewrite the lateral pressure (8) in a general form as
\[
p_t = p_r \left( \frac{\omega - 1}{2} \right) + \frac{\Lambda \omega (3 - \alpha)}{16\pi} + p_r \left( \frac{1 - \alpha a}{4(\Lambda r_0^2 + 1)} \right) \frac{8\pi p_r \left( \frac{\omega b + 1}{\omega a} \right) r^2 + \Lambda \left( \frac{\omega - 1}{\omega a} \right) r^2 + r_0(1-a)}{1 - \frac{8\pi}{\alpha} p_r r^2 + \frac{\Lambda r^2}{\alpha} - \frac{r_0}{r} (1-a)},
\]
where the first and second terms are due to the pure mass and pure cosmological constant effects, respectively, while the third term is a mixed term.

Clearly, using the shape function \(b(r)\), equation (13), we have de Sitter or anti-de Sitter behaviour as \(r \rightarrow \infty\). Thus, the radial pressure will take an asymptotically de Sitter or anti-de Sitter behaviour \(p_r \rightarrow \frac{\Lambda}{8\pi r^2}\) as \(r \rightarrow \infty\) where \(\omega\) should be \(-1\) at spatial infinity. For the lateral pressure we encounter a divergence as \(r \rightarrow \infty\) because of the first term of \(U'/U\) in equation (23) due to the presence of the cosmological constant. However, for solving this problem, one may impose an asymptotic redshift function \(U(r) \rightarrow 1\) as \(r \rightarrow \infty\), like the asymptotically flat shape function, or consider a constant redshift function in which case we obtain de Sitter or anti-de Sitter asymptotics (30). In what follows, we briefly study the validity of the second approach. Since any constant redshift function can be absorbed in the re-scaled time coordinate, we will consider \(U(r) = 1\). By this consideration, we recover the field equations as follows
\[
\frac{b'(r)}{r^2} = 8\pi \rho + \Lambda,
\]
\[
-b \frac{r^3}{r^2} = 8\pi p_r + \Lambda,
\]
\[
-b - \frac{b'}{2r^3} = 8\pi p_r + \Lambda.
\]
Clearly, using the shape function \(b(r)\), equation (13), we have de Sitter or anti-de Sitter asymptotics as \(\rho \rightarrow \frac{\Lambda}{8\pi}\), \(p_r \rightarrow -\frac{\Lambda}{8\pi}\) and \(p_t \rightarrow -\frac{\Lambda}{8\pi}\) as \(r \rightarrow \infty\). Also, for the case of constant redshift function, one may go further and consider equation of state \(p_r = \omega \rho\) and using equations (27) and (28) obtains the space varying equation of state parameter \(\omega\) as
\[
\omega(r) = -\frac{\Lambda r^3 + b}{\Lambda r^3 + rb'},
\]
which help us achieve a better understanding of the behaviour of the dominant fluid and reveals de Sitter or anti-de Sitter nature of spacetime as \(\omega \rightarrow -1\) as \(r \rightarrow \infty\). This equation leads to equation (22) at the throat of the wormhole and shows that \(\omega\) as a space varying parameter is not allowed to be \(\omega = -1\) at the throat of the wormhole. The generalization of equation (30) to the case of \(U(r) \neq \text{constant}\) is also applicable by using equations (9) and (10).

Finally, using the shape function, equation (13), we can express the third wormhole constraint mentioned in section II, by the following inequality
\[
H(x, a, \alpha) \equiv ax^{-\alpha} - x^{-1} + 1 - a < 0,
\]
where we defined \(x \equiv \frac{r_0}{r}\). In order to cover the entire spacetime, the \(x\) has the range of \(0 < x \leq 1\) where \(x = 1\) corresponds to the wormhole throat, \(r = r_0\), and \(x \rightarrow 0\) corresponds to spatial infinity. In order to check this condition, we shall plot \(H(x, a, \alpha)\) versus \(x \equiv \frac{r_0}{r}\) for some specific choices of \(a\) and \(\alpha\) at the end of the next section.
IV. SPECIFIC WORMHOLES WITH AN UNBOUNDED MASS FUNCTION

A. The case $a = 1$ and $\alpha = \frac{1}{2}$

This case is allowed for both $\Lambda > 0$ and $\Lambda < 0$. For this specific case, equation (23) can be solved:

$$U(r) = U_1 e^{\left[-\frac{\Lambda \left(3r^2 + 6r_0 r + 4r_0^2 \left(\frac{r}{r_0}\right)^{\frac{3}{2}} + 12r_0^2 \left(\frac{r}{r_0}\right)^{\frac{3}{2}} + 6r_0^2 \ln \left(\frac{r}{r_0}\right)\right)}{12\Lambda r_0^2 + 6}\right]},$$

where we can absorb the constant $U_1$ into the re-scaled time coordinate. The mass function takes the simple form

$$m(r) = \frac{r_0}{2} \left[\left(\frac{r}{r_0}\right)^{\frac{3}{2}} - 1\right] + \frac{\Lambda}{6} (r^3 - r_0^3),$$

and the pressures will be

$$p_r(r) = \frac{\omega}{8\pi} \left[\frac{1}{2r_0^2} \left(\frac{r}{r_0}\right)^{\frac{3}{2}} + \Lambda\right],$$

$$p_t(r) = -\frac{1}{4} p_r + \frac{5\Lambda\omega}{32\pi} + \frac{8p_r \left(\frac{r}{r_0}\right)}{8(\Lambda r_0^2 + 1) \left(1 - \frac{4\pi}{\omega} p_r r^2 + 2\Lambda r^2\right)} - \Lambda,$$

As mentioned at the end of the previous section, in contrast to the energy density and radial pressure leading to de Sitter or anti-de Sitter asymptotics, the lateral pressure diverges as $r \to \infty$. By considering the constant redshift function $U(r) = 1$, we can avoid the infiniteness of $p_t$. By this consideration we recover $m(r)$ in equation (33) and the pressures will be

$$p_r(r) = -\frac{1}{8\pi} \left[\frac{r_0}{r^3} \sqrt{\frac{r}{r_0}} + \Lambda\right],$$

$$p_t(r) = \frac{1}{8\pi} \left[\frac{1}{4r^2} \sqrt{\frac{r}{r_0}} - \Lambda\right],$$

which have the asymptotics as $r \to \infty$ as

$$p_r(r) \to -\frac{\Lambda}{8\pi},$$

$$p_t(r) \to -\frac{\Lambda}{8\pi},$$

B. The case $a = 1$ and $\alpha = \frac{1}{3}$

This case is also allowed for both $\Lambda > 0$ and $\Lambda < 0$. For the specific case $a = 1$ and $\alpha = \frac{1}{3}$, solving the equation (23) gives the solution

$$U(r) = U_2 e^{\left[-\frac{\Lambda \left(2r^2 + 3r_0 r \left(\frac{r}{r_0}\right)^{\frac{3}{2}} + 6r_0^2 \left(\frac{r}{r_0}\right)^{\frac{3}{2}} + 4r_0^2 \ln \left(\frac{r}{r_0}\right)\right)}{6\Lambda r_0^2 + 2}\right]},$$

where we can treat $U_2$ as previous section.

Also, for the mass function, we will have

$$m(r) = \frac{r_0}{2} \left[\left(\frac{r}{r_0}\right)^{\frac{3}{2}} - 1\right] + \frac{\Lambda}{6} (r^3 - r_0^3).$$
The pressures also will be

\[ p_r(r) = \omega \rho(r) = \frac{\omega}{8\pi} \left[ \frac{1}{3r_0^2} \left( \frac{r_0}{r} \right)^{\frac{4}{3}} + \Lambda \right], \]

\[ p_t(r) = -\frac{1}{3} p_r + \frac{\Lambda \omega}{6\pi} + \frac{4\pi p_r \left( \frac{\omega+3}{3} \right) r^2 - \Lambda r^2}{3 \left( \Lambda r_0^2 - 1 \right) \left( 1 - \frac{4\pi}{24\pi} p_r r^2 + 3\Lambda r^2 \right) p_r}. \] (39)

For this case, by considering \( U(r) = 1 \), to avoid the infiniteness of \( p_t \) as \( r \to \infty \), we recover \( m(r) \) in equation (38) and the pressures will be

\[ p_r(r) = -\frac{1}{8\pi} \left[ \frac{r_0}{r^2} \left( \frac{r}{r_0} \right)^{\frac{4}{3}} + \Lambda \right], \]

\[ p_t(r) = \frac{1}{8\pi} \left[ \frac{1}{3r_0^2} \left( \frac{r_0}{r} \right)^{\frac{4}{3}} - \Lambda \right], \] (40)

which have the asymptotics as \( r \to \infty \) as

\[ p_r(r) \to -\frac{\Lambda}{8\pi}, \]

\[ p_t(r) \to -\frac{\Lambda}{8\pi}, \] (41)

which have the asymptotics as \( r \to \infty \) as

\[ p_r(r) \to -\frac{\Lambda}{8\pi}, \]

\[ p_t(r) \to -\frac{\Lambda}{8\pi}, \] (41)

which have the asymptotics as \( r \to \infty \) as

\[ p_r(r) \to -\frac{\Lambda}{8\pi}, \]

\[ p_t(r) \to -\frac{\Lambda}{8\pi}, \] (41)

corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

C. The case \( a = \frac{1}{2} \) and \( \alpha = \frac{1}{2} \)

This case is also acceptable for \( \Lambda > 0 \) and \( \Lambda < 0 \). For this configuration, one obtains the following solution

\[ U(r) = U_3 \exp \left[ -\frac{\Lambda}{32r_0^2} \left( \frac{4r_0}{r} - \frac{r}{r_0} \right) \right], \]

\[ \times \left[ -\frac{12r^2 + 18r r_0 + 8r_0^2 r \left( \frac{r}{r_0} \right)^{\frac{4}{3}} + 30r_0^2 \left( \frac{r}{r_0} \right)^{\frac{4}{3}} + 32r_0^2 \ln \left( \frac{r}{r_0} \right) + 31r_0^2 \ln \left( 2 \left( \frac{r}{r_0} \right)^{\frac{4}{3}} + 1 \right) \right] \]

\[ \times -\frac{32r_0^2 + 8}{4 - 4 \sqrt{\frac{r_0}{r} - \frac{r_0}{r}}} \] (42)

where again, absorption of the constant \( U_3 \) would be applicable into the re-scaled time coordinate.

The mass function will be written as

\[ m(r) = \frac{r_0}{4} \left[ \left( \frac{r}{r_0} \right)^{\frac{4}{3}} - 1 \right] + \frac{\Lambda}{6} (r^3 - r_0^3), \] (43)

while the pressures are obtained as

\[ p_r(r) = \omega \rho(r) = \frac{\omega}{8\pi} \left[ \frac{1}{4r_0^2} \left( \frac{r_0}{r} \right)^{\frac{4}{3}} + \Lambda \right], \]

\[ p_t(r) = -\frac{1}{3} p_r + \frac{4 \Lambda \omega}{32\pi} + \frac{24\pi p_r \left( \frac{\omega+2}{3} \right) r^2 - 3\Lambda r^2 + 3r_0}{16 \left( \Lambda r_0^2 + 1 \right) \left( 1 - \frac{16\pi}{24\pi} p_r r^2 + 2\Lambda r^2 - \frac{16\pi}{24\pi} \right) p_r}. \] (44)

For this case, by considering \( U(r) = 1 \), to avoid the infiniteness of \( p_t \) as \( r \to \infty \), we recover \( m(r) \) in equation (43) and the pressures will be

\[ p_r(r) = -\frac{1}{8\pi} \left[ \frac{r_0}{2r^3} \left( 1 + \left( \frac{r}{r_0} \right)^{\frac{4}{3}} \right) + \Lambda \right], \]

\[ p_t(r) = \frac{1}{8\pi} \left[ \frac{r_0}{8r^3} \left( 2 + \left( \frac{r}{r_0} \right)^{\frac{4}{3}} \right) - \Lambda \right], \] (45)
which have the asymptotics as \( r \to \infty \) as

\[
    p_r(r) \to -\frac{\Lambda}{8\pi},
    \\
p_t(r) \to -\frac{\Lambda}{8\pi},
\]

(46)
corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

D. The case \( a = -1 \) and \( \alpha = \frac{1}{2} \)

This case is only allowed for \( \Lambda > 0 \). Applying the values \( a = -1 \) and \( \alpha = \frac{1}{2} \), gives a solution for the equation (23) as

\[
    U(r) = U_5 \times \exp \left[ -\frac{\Lambda}{4\pi r_0^2} \left( 3r^2 - 6r_0^2 \left( \frac{r}{r_0} \right)^{\frac{3}{2}} + 18r_0^2 r + 124r_0^3 \ln \left( \frac{r}{r_0} \right) \right) - 2 \ln \left( \frac{r}{r_0} \right) + 4 \ln \left( \left( \frac{r}{r_0} \right)^{\frac{3}{2}} + 2 \right) \right],
\]

(47)
where the constant \( U_4 \) can be absorbed similar to the previous solutions.

The mass function would be written as

\[
    m(r) = \frac{r_0}{2} \left[ 1 - \left( \frac{r}{r_0} \right)^{\frac{3}{2}} \right] + \frac{\Lambda}{6} \left( r^3 - r_0^3 \right),
\]

(48)
and the pressures will be

\[
    p_r(r) = \omega \rho(r) = \frac{\omega}{8\pi} \left[ -\frac{1}{2r_0^3} \left( \frac{r}{r_0} \right)^{\frac{3}{2}} + \Lambda \right],
    \\
p_t(r) = -\frac{1}{4} \frac{\Lambda \omega}{32\pi} + \frac{24\pi p_r \left( \frac{r}{r_0} \right)^{\frac{3}{2}} - 3 \Lambda r^2 + 6 \frac{r_0}{r} \left( \frac{r}{r_0} \right)^{\frac{3}{2}} + 2 \Lambda r^2 - 2 \frac{r_0}{r} \rho_r}{8 \left( \Lambda r_0^2 + 1 \right) \left( 1 + \frac{48}{\omega} \rho_r \right) p_r + 2 \Lambda r^2 - 2 \frac{r_0}{r} p_r}.
\]

(49)
For this case, by considering \( U(r) = 1 \), to avoid the infiniteness of \( p_t \) as \( r \to \infty \), we recover \( m(r) \) in equation (48) and the pressures will be

\[
    p_r(r) = -\frac{1}{8\pi} \left[ \frac{r_0}{r^3} \left( 2 - \left( \frac{r}{r_0} \right)^{\frac{3}{2}} \right) + \Lambda \right],
    \\
p_t(r) = \frac{1}{8\pi} \left[ \frac{r_0}{4r^3} \left( 4 - \left( \frac{r}{r_0} \right)^{\frac{3}{2}} \right) - \Lambda \right],
\]

(50)
which have the asymptotics as \( r \to \infty \) as

\[
    p_r(r) \to -\frac{\Lambda}{8\pi},
    \\
p_t(r) \to -\frac{\Lambda}{8\pi},
\]

(51)
corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

V. SPECIFIC WORMHOLES WITH A BOUNDED MASS FUNCTION

A. The case \( a = -\frac{1}{2} \) and \( \alpha = -1 \)

This case is allowed for both \( \Lambda > 0 \) and \( \Lambda < 0 \). Applying the values \( a = -\frac{1}{2} \) and \( \alpha = -1 \), gives a solution for equation (23) as

\[
    U(r) = U_4 \exp \left[ -\frac{\Lambda}{8\pi} \left( 2r^2 + 6r_0 r - 9r_0^2 \ln (2r - r_0) + 16r_0^3 \ln (r) \right) - 12 \ln (2r - r_0) + 12 \ln (r) \right] \frac{8r_0^2}{8Ar_0^2 + 4},
\]

(52)
where the constant $U_4$ can be absorbed similar to the previous solutions. The mass function would be written as

$$m(r) = \frac{r_0}{4} \left[ 1 - \frac{r_0}{r} \right] + \frac{\Lambda}{6} \left( r^3 - r_0^3 \right). \quad (53)$$

The pressures will be

$$p_r(r) = \omega \rho(r) = \frac{\omega}{\sqrt{8\pi}} \left[ \frac{1}{2r_0^4} \left( \frac{r_0}{r} \right)^4 + \Lambda \right],$$

$$p_t(r) = -p_r + \frac{\Lambda \omega}{4\pi} + \frac{8\pi \rho_r (\omega - \frac{1}{\omega})}{8 (\Lambda r_0^2 + 1)} \left( 1 + \frac{8\pi}{\omega} \rho_r r^2 - \Lambda r^2 - \frac{8\pi}{2r} \right) p_r. \quad (54)$$

For this case, by considering $U(r) = 1$, to avoid the infiniteness of $p_t$ as $r \to \infty$, we recover $m(r)$ in equation (53) and the pressures will be

$$p_r(r) = -\frac{1}{8\pi} \left[ \frac{r_0}{2r^2} (3r - r_0) + \Lambda \right],$$

$$p_t(r) = \frac{1}{8\pi} \left[ \frac{r_0}{4r^4} (3r - 2r_0) - \Lambda \right], \quad (55)$$

which have the asymptotics as $r \to \infty$ as

$$p_r(r) \to -\frac{\Lambda}{8\pi},$$

$$p_t(r) \to -\frac{\Lambda}{8\pi}, \quad (56)$$

corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

B. The case $a = \frac{1}{2}$ and $\alpha = -1$

This case is only allowed for $\Lambda > 0$. For the specific case $a = \frac{1}{2}$ and $\alpha = -1$, solving the equation (23) gives the solution

$$U(r) = U_2 \exp \left[ -\frac{\Lambda}{8} \left( 6r^2 + 6r_0 r - 7r_0^3 \ln (2r + r_0) + 16r_0^3 \ln (r) \right) - 4 \ln (2r + r_0) + 4 \ln (r) \right], \quad (57)$$

where we can treat $U_2$ as previous sections.

For the mass function, we have

$$m(r) = \frac{r_0}{4} \left[ \left( \frac{r_0}{r} \right) - 1 \right] + \frac{\Lambda}{6} \left( r^3 - r_0^3 \right). \quad (58)$$

The pressures will be

$$p_r(r) = \omega \rho(r) = \frac{\omega}{\sqrt{8\pi}} \left[ -\frac{1}{2r_0^4} \left( \frac{r_0}{r} \right)^4 + \Lambda \right],$$

$$p_t(r) = -p_r + \frac{\Lambda \omega}{4\pi} + \frac{24\pi \rho_r (\omega - \frac{1}{\omega})}{8 (\Lambda r_0^2 + 1)} \left( 1 + \frac{8\pi}{\omega} \rho_r r^2 - \Lambda r^2 - \frac{4\pi}{2r} \right) p_r. \quad (59)$$

For this case, by considering $U(r) = 1$, to avoid the infiniteness of $p_t$ as $r \to \infty$, we recover $m(r)$ in equation (58) and the pressures will be

$$p_r(r) = -\frac{1}{8\pi} \left[ \frac{r_0}{2r^2} (r + r_0) + \Lambda \right],$$

$$p_t(r) = \frac{1}{8\pi} \left[ \frac{r_0}{4r^4} (r + 2r_0) - \Lambda \right], \quad (60)$$
which have the asymptotics as \( r \to \infty \) as

\[
p_r(r) \to -\frac{\Lambda}{8\pi}, \\
p_t(r) \to -\frac{\Lambda}{8\pi},
\]

(61)
corresponding to the asymptotically de Sitter or anti-de Sitter spacetimes.

The corresponding \( H(x, a, \alpha) \) function for all of these cases are shown in the following figure, Figure 1.

![Figure 1](image)

FIG. 1: The plots depict \( H(x, a, \alpha) \) function in which the solid, dot, spacedash, dash, longdash and dashdot plots stand for the \( H(x, 1, 1/2), H(x, 1, 1/3), H(x, 1/2, 1/2), H(x, -1, 1/2), H(x, -1/2, -1) \) and \( H(x, 1/2, -1) \), respectively. The parameter \( x = r_0/r \), lying in the range \( 0 < x \leq 1 \), has been defined in order to cover the entire spacetime.

As seen from the figure, the function \( H(x, a, \alpha) \) is negative for all cases throughout the entire range of \( x \), indicating the satisfaction of the third wormhole condition.

Also, it is interesting to evaluate the "volume integral quantifier" \([49, 50]\) which provides information about the "total amount" of energy condition violating matter. This quantity is given by

\[
I_V \equiv \int \left[ \rho(r) + p_r(r) \right] dV = 2 \int_{r_0}^{\infty} \left[ \rho(r) + p_r(r) \right] 4\pi r^2 dr,
\]

(62)
which by considering the equation \([18]\) and the equation of state \( p_r(r) = \omega \rho(r) \) gives the solution as

\[
I_V = (\omega + 1) \left[ a r_0 \left( \frac{r}{r_0} \right)^\alpha + \frac{1}{3} \Lambda r^3 \right] \bigg|_{r_0}^\infty.
\]

(63)
This equation shows that in the absence of cosmological constant, in order to have a finite amount of "energy condition violating matter", the \( \alpha \) values must be negative, in which case we have

\[
I_V \to -(\omega + 1) a r_0.
\]

(64)
Therefore, as \( a \to 0 \), we have \( I_V \to 0 \) which reflects arbitrary small quantities of energy condition violating matter. In the presence of cosmological constant, it is seen that the sign of equation \([50]\) is not fixed and depends on the parameters of the model and cosmological constant.

VI. CONCLUDING REMARKS

We investigated the phantom and non-phantom wormhole solutions including a generic cosmological constant with de Sitter or anti-de Sitter asymptotics. It was shown that for constructing such a geometry, the wormhole throat
To $r_0$, cosmological constant $\Lambda$ and the equation of state parameter $\omega$ should satisfy certain relations. With respect to the sign of $\Lambda$, corresponding to de Sitter or anti-de Sitter spacetime, a new restricting condition on wormhole throat size was obtained. Also, it was shown that in the presence of cosmological constant $\Lambda$ with a general redshift function $U(r)$, the energy density profile $\rho(r)$ and the radial pressure $p_r(r)$ take an asymptotically de Sitter or anti-de Sitter behaviour as $r \to \infty$. For the lateral pressure $p_t(r)$, a divergence appears as $r \to \infty$ due to the behaviour of $U'/U$ for the redshift function due to the presence of the cosmological constant. It was briefly discussed that in order to obtain de Sitter or anti-de Sitter asymptotics for the energy density and pressures, one may consider a constant redshift function or impose an asymptotic redshift function $U \to 1$ as $r \to \infty$. It was shown that for the case of $U(r) = 1$, de Sitter or anti-de Sitter asymptotics are achievable. Using the possibility of setting different values for the parameters of the model, we found some exact solutions leading to specific metrics, mass functions and supporting energy momentum profiles. We also briefly mentioned the volume integral quantifier, which provides useful information about the total amount of energy condition violating matter. It was shown that the amount of this energy condition violation depends on parameters of the model and the value of cosmological constant which can be fixed from observations.

Acknowledgment

Y. Heydarzade acknowledges F. Parsaei and R. Monadi for checking some calculations and useful comments.

[1] J. A. Wheeler, Phys. Rev 97, 511 (1955).
[2] J. A. Wheeler, Ann. Phys 2, 604 (1957).
[3] M. S. Morris, K. S. Thorne and U. Yurtsewer, Phys. Rev. Lett 61, 1446 (1988).
[4] M. S. Morris, K. S. Thorne, Ann J. Phys 56, 395 (1988).
[5] S. A. Hayward, Int. J. Mod. Phys. D 8, 373 (1999).
[6] N. S. Kardashev, I. D. Novikov and A. A. Shatskiy, Int. J. Mod. Phys. D 16, 909 (2007).
[7] P. K. F. Kuhfittig, Schol. Res. Exch 296158 (2008).
[8] S. V. Sushkov and O. B. Zaslavskii, Phys. Rev.D 79, 067502 (2009).
[9] D. Hochberg and M. Visser, Phys. Rev. Lett 81, 746 (1998).
[10] D. Hochberg and M. Visser, arXiv:gr-qc/9901020.
[11] D. Hochberg and M. Visser, Phys. Rev. D 58, 044021 (1998).
[12] M. Visser, Phys. Rev. D 39, 3182 (1989).
[13] N. M. Garcia, F. S. N. Lobo and M. Visser, Phys. Rev. D 86, 044026 (2012).
[14] F. S. N. Lobo and M. A. Oliveira, Phys. Rev. D 80, 104012 (2009).
[15] S. N. Sajadi and N. Riazi, Prog.Theor.Phys. 126, 753-760 (2011).
[16] N. M. Garcia and F. S. N. Lobo, Phys. Rev. D 82, 104018 (2010).
[17] N. M. Garcia and F. S. N. Lobo, Class. Quant. Grav 28, 085018 (2011).
[18] F. S. N. Lobo, Class. Quant. Grav 25, 175006 (2008).
[19] F. S. N. Lobo, Phys. Rev. D 75, 064027 (2007).
[20] A. Riess et al., Astron. J 116, 1009 (1998).
[21] S. J. Perlmutter et al., Astrophys. J 517, 565 (1999).
[22] C. L. Bennett et al., Astrophys. J. Suppl. Ser 148, 1 (2003).
[23] G. Hinshaw et al., Astrophys. J. Suppl. Ser 148, 135 (2003).
[24] J. P. S. Lemos, Phys. Lett. B 352, 46 (1995).
[25] J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D 54, 3840 (1996).
[26] R. R. Caldwell, Phys. Lett. B 545, 23 (2002).
[27] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett 91, 071301 (2003).
[28] I. Brevik, S. Nojiri, S. D. Odintsov and L. Vanzo, Phys. Rev. D 70, 043520 (2004).
[29] S. Nojiri and S. D. Odintsov, Phys. Rev. D 70, 103522 (2004).
[30] L. Amendola, Phys. Rev. Lett. 93, 181102 (2004).
[31] S. V. Sushkov, Phys. Rev. D 71, 043520 (2005).
[32] F. S. N. Lobo, Phys. Rev. D 71, 084011 (2005); F. S. N. Lobo, Phys. Rev. D 71, 124022 (2005).
[33] O. B. Zaslavskii, Phys. Rev. D 72, 061303 (2005).
[34] R. Garattini and F. S. N. Lobo, Class. Quant. Grav. 24, 2401 (2007).
[35] R. Garattini, Class. Quant. Grav. 22, 1105 (2005).
[36] R. Garattini and F. S. N. Lobo, Phys. Lett. B 671, 146 (2009).
[37] J. P. S. Lemos, F. S. N. Lobo and S. Q. de Oliveira, Phys. Rev. D 68, 064004 (2003).
[38] F. S. N. Lobo, Class. Quant. Grav 21, 4811 (2004).
[39] F. S. N. Lobo and P. Crawford, Class. Quant. Grav 22, 4869-4886 (2005).
[40] J. P. S. Lemos and F. S. N. Lobo, Phys. Rev. D 69, 104007 (2004).
[41] J. P. S. Lemos and F. S. N. Lobo, Phys. Rev. D 78, 044030 (2008).
[42] F. S. N. Lobo, F. Parsaei and N. Riazi, Phys. Rev. D 87, 084030 (2013).
[43] J. P. S. Lemos, F. S. N. Lobo and S. Quinet de Oliveira, Phys. Rev. D 68, 064004 (2003).
[44] F. Rahaman, M. Kalam, M. Sarker, A. Ghosh and B. Raychaudhuri, Gen. Rel. Grav 39, 145-151 (2007).
[45] N. Dadhich, S. Kar, S. Mukherjee and M. Visser, Phys. Rev. D 65, 064004 (2002).
[46] S. V. Sushkov and S. W. Kim, Gen. Rel. Grav 36, 1671 (2004).
[47] A. Liddle, An Introduction to Modern Cosmology, John Wiley and Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex PO19 8SQ, England (2003).
[48] M. Cataldo, P. Meza and P. Minning, Phys.Rev.D 83,044050 (2011).
[49] M. Visser, S. Kar and N. Dadhich, Phys. Rev. Lett 90, 201102 (2003).
[50] S. Kar, N. Dadhich and M. Visser, Pramana 63, 859-864 (2004).