Persistent spin oscillations in a spin-orbit-coupled superconductor

Amit Agarwal,¹ Marco Polini,² Rosario Fazio,¹ and G. Vignale³
¹NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, I-56126 Pisa, Italy
²NEST, Istituto Nanoscienze-CNR and Scuola Normale Superiore, I-56126 Pisa, Italy
³Department of Physics and Astronomy, University of Missouri, Columbia, Missouri 65211, USA

Quasi-two-dimensional superconductors with tunable spin-orbit coupling are very interesting systems with properties that are also potentially useful for applications. In this Letter we demonstrate that these systems exhibit undamped collective spin oscillations that can be excited by the application of a supercurrent. We propose to use these collective excitations to realize persistent spin oscillators operating in the frequency range of 10 GHz – 1 THz.

PACS numbers: 74.20.-z,73.20.Mf,71.45.-d,71.70.Ej

Introduction. — Spin-orbit-coupled two-dimensional (2D) electron gases (EGs) are the focus of great interest in the field of semiconductor spintronics. This interest has been largely fueled by the hope to realize the visionary Datta-Das “spin transistor” in which the on/off state is achieved by purely-electrical control of the electron’s spin in a spin-orbit-coupled semiconductor channel placed between ferromagnetic leads. Research in spin-orbit-coupled 2DEGs has been recently revitalized by theoretical and experimental studies of the spin Hall effect, in which a current traversing the sample generates a spin-current in the orthogonal direction.

The study of the interplay between spin-orbit coupling (SOC) and superconductivity in 2D systems, stemming from the seminal works of Edelstein and Gor’kov and Rashba, has also gained impetus. There is a large variety of systems in which SOC and superconductivity coexist: two examples of great current interest are i) 2DEGs in InAs or GaAs semiconductor heterostructures that are proximized by ordinary s-wave superconducting leads — a class of systems which plays a key role in the quest for Majorana fermions — and ii) 2DEGs that form at interfaces between complex oxides such as LaAlO₃ and SrTiO₃, which display tunable SOC and superconductivity.

Motivated by this body of experimental and theoretical literature, we investigate the collective spin dynamics of an archetypical 2DEG model Hamiltonian with Rashba SOC and s-wave pairing in the presence of repulsive electron-electron (e-e) interactions. In the absence of superconductivity a Rashba 2DEG exhibits spin oscillations, which, at long wavelength and for weak repulsive interactions, have a frequency \( \alpha k_F \). When the strength of SOC and \( k_F \) the 2D Fermi wavenumber in the absence of SOC. These oscillations, however, are damped and quickly decay due to the emission of (double) electron-hole pairs, which, in the normal phase, are present at arbitrarily low energies. In this Letter we demonstrate that in a Gor’kov-Rashba superconductor (GRSC), collective spin oscillations continue to exist in a wide range of parameters, and are undamped because they lie inside the superconducting gap where no other excitation exists. Fig. [1] shows schematically the nature of the spin oscillations in a GRSC. At variance with the Cooper pairs of a standard s-wave semiconductor, the pairs of a GRSC are in a mixture of singlet and triplet states. It is this feature that enables the pairs to respond to an oscillating magnetic field applied, say, in the \( j_y \) direction. In the course of the oscillation the spins of a pair tilt in opposite directions, in a pair-breaking motion that creates a net spin polarization along the \( j_y \) axis. The spin polarization produces an exchange field, which, if the electron-electron interaction is sufficiently strong, sustains oscillations of the appropriate frequency in the absence of an external field. The essential point is that these oscillations are undamped as long as their frequency falls below the quasiparticle gap: they will therefore display an extraordinarily long lifetime.

In order to excite these long-lived spin modes one could in principle apply a short magnetic pulse, but there is also a purely-electrical method. Namely, a supercurrent pulse applied, say, in the \( j_x \) direction, will generate, via

![FIG. 1: (color online) a) Response of a Cooper pair in the \( \lambda = + \) chirality subband of a Gor’kov-Rashba superconductor subject to an oscillating magnetic field in the \( j_y \) direction. The solid circle is the Fermi surface and the black dot is the origin of momentum space. The arrows labeled by “1”, “2”, and “0” describe the orientation of the spins under the action of a magnetic field that points up, down, or vanishes. Spontaneous oscillations are sustained, in the absence of a magnetic field, by the internal exchange field. b) A supercurrent boosts the Fermi surface in the \( j_x \) direction (solid line) and creates a magnetic field in the \( j_y \) direction. As a result, spins begin to oscillate around the new equilibrium orientation, indicated by the thick red arrows.](image-url)
the Edelstein effect\cite{20} an effective magnetic field pulse in the $\hat{y}$ direction, and this should be sufficient to start the spin oscillations. This excitation mechanism is illustrated in Fig. [1]). The Fourier spectrum of the supercurrent pulse must not contain frequencies of the order of (or larger than) twice the superconducting gap to avoid the creation of quasiparticle excitations. We suggest that the new collective spin mode can be used to realize “persistent spin oscillators” operating in the frequency range of 10 GHz – 1 THz (for superconductors with a critical temperature in the range $10^{-1} – 10$ K). The microscopic mechanism responsible for the only in a thin shell of momentum space around the Fermi surface. The microscopic mechanism responsible for the new collective spin mode can be used to realize “persistence” of the spin degrees-of-freedom only. The first step is to decouple the two quartic terms, $\hat{H}_0$ and $\hat{H}_{e-e}$, by means of a suitable Hubbard-Stratonovich (HS) transformation (see e.g. Refs. \cite{16,17}). For the pairing term $\hat{H}_p$ we introduce the complex HS field $\Delta_0(r,\tau)$ which describes the superconducting order parameter.\cite{16,17} We do the decoupling in the chiral basis: this allows us to work with Cooper pairs that are protected by time-reversal symmetry.\cite{18} Transforming back to the real-spin basis we get spin-triplet pairing in addition to the regular spin-singlet pairing.

It is useful to rewrite $\hat{H}_{e-e}$ as \cite{19}

$$\hat{H}_{e-e} = \frac{V}{4} \int d^2r \left\{ \rho^2(r) - \left[ \sum_{a=1}^{3} \frac{d}{z \rho_a(r)} \right]^2 \right\} ,$$

where $\rho(r) = \sum_i \rho_i(r)$ is the total density operator, $\Delta_0(r,\tau) = \sum_{i,j} \psi_i^+(r) \sigma_{ij} \psi_j(r)$ is the usual spin-density operator, and $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ is an arbitrary unit vector in 3D space. To decouple $\hat{H}_{e-e}$ by means of HS transformation we introduce four real HS fields,\cite{19} $\phi(r, \tau)$ and $M(r, \tau)$, which are conjugate to density fluctuations and spin fluctuations, respectively.

The notation is considerably simplified by defining a four-component spinor $\Psi^\dagger(r,\tau) = [\psi_i^\dagger(r) \psi_i^\dagger(r)]$ in real-spin space. The exact microscopic action corresponding to $\hat{H}$ after the HS transformation can now be expressed in a compact form as (the variables $r, \tau$ will be suppressed from now on when needed for brevity)

$$S = \int_0^\beta d\tau \int d^2r \left[ \frac{|\Delta_0|^2}{g} + \frac{\phi^2 + M \cdot M}{V} \right]$$

$$+ \left\{ \frac{(-G^{-1} + \Sigma_0)}{2} \right\} \psi \right\} ,$$

where $\beta = (k_B T)^{-1}$, $\Sigma_0(r, \tau) = i \phi(r^3 \otimes \mathbb{I}_s)$, and $\Psi$ is the Grassmann variable corresponding to the fermionic field $\psi^\dagger$. Here $G^{-1}$ is the Green’s function of the problem defined by $\hat{H}_0 + \hat{H}_p$ and is a 4 x 4 matrix given by

$$G^{-1} = -\partial_i \mathbb{I}_s \otimes \mathbb{I}_s + \tau^3 \otimes h + \alpha \left\{ \Gamma \times (-i \nabla) \cdot \hat{z} \right\}$$

$$+ \frac{\tau^1 + i \tau^2}{2} \otimes \Delta + \frac{\tau^1 - i \tau^2}{2} \otimes \Delta.$$  

The Pauli matrices $\tau^a$ act in the $2 \times 2$ Nambu-Gor’kov space and $\mathbb{I}_s \otimes \mathbb{I}_s$ is the identity matrix in real-spin (Nambu-Gor’kov) space, $\Gamma = (\Gamma^1, \Gamma^2, \Gamma^3) \equiv (\tau^3 \otimes \sigma^3, \tau^1 \otimes \sigma^1, \tau^2 \otimes \sigma^2, \tau^3 \otimes \sigma^3)$ and $\Delta$ is a $2 \times 2$ matrix whose diagonal (off-diagonal) elements are related to the triplet (singlet) order parameter [see Eq. (A5)].

At low energies, fluctuations of the amplitude of the order parameter $\Delta_0(r,\tau)$ do not play any role while phase fluctuations give rise to the Bogoliubov-Anderson model.\cite{20} To this end, we write $\Delta_0(r, \tau) = \Delta e^{i \theta(r, \tau)}$, with $\Delta$ real. The amplitude $\Delta$ is fixed by the saddle-point equation $\delta S/\delta \Delta = 0$, which yields the BCS equation [see Eq. (A1)].

We now derive an effective low-energy action corresponding to the full Hamiltonian $\hat{H}$ in terms of spin degrees-of-freedom only. The first step is to decouple the two quartic terms, $\hat{H}_0$ and $\hat{H}_{e-e}$, by means of a suitable Hubbard-Stratonovich (HS) transformation (see e.g. Refs. \cite{16,17}). For the pairing term $\hat{H}_p$ we introduce the complex HS field $\Delta_0(r,\tau)$, which describes the superconducting order parameter.\cite{16,17} We do the decoupling in the chiral basis: this allows us to work with Cooper pairs that are protected by time-reversal symmetry.\cite{18} Transforming back to the real-spin basis we get spin-triplet pairing in addition to the regular spin-singlet pairing.

It is useful to rewrite $\hat{H}_{e-e}$ as \cite{19}
The role of the phase field \( \theta(r, \tau) \) can be made explicit in the action \( S \) by performing the following gauge transformation \( \hat{\varphi}_i(r, \tau) = \hat{\psi}_i(r, \tau) e^{i \theta(r, \tau)/2} \) to new fermionic fields \( \hat{\varphi}_i(r, \tau) \). Writing the action \( S \) in terms of the new fermionic fields generates new self-energies in the round brackets in the second line of Eq. (4): 

\[
\Sigma_1(r, \tau) = \left[ i \left( \frac{1}{2} \partial_\tau \theta + \phi \right) + \frac{(\nabla_\tau \theta)^2}{8m} \right] r^3 \otimes \mathbb{1}_\sigma \\
- \frac{i}{2m} \left[ \nabla_\tau^2 \theta - \frac{1}{2} (\nabla_\tau \theta) \cdot \nabla_\tau \right] \mathbb{1}_\tau \otimes \mathbb{1}_\sigma \ ,
\]

(6)

\[
\Sigma_2(r, \tau) = M \Gamma \quad \text{and} \quad \Sigma_3(r, \tau) = \frac{ \alpha }{ 2 } [ \Gamma \times (\nabla_\tau \theta)] \cdot \hat{\mathbf{z}} \ .
\]

(7)

The fermionic part of the action can be integrated out (since it corresponds to a Gaussian functional integral for the partition function) leaving us with the following effective action 

\[
S_{\text{eff}} = \int_0^\beta d\tau \int d^2r \left[ \frac{\Delta^2}{g} + \frac{\phi^2}{2 \tau} + \frac{M \cdot M}{V} \right] \\
- \frac{1}{2} \text{Tr} \left[ \ln (-G_0^{-1} + \Sigma) \right] \ ,
\]

(8)

where the symbol “Tr” means a trace over all degrees of freedom (including space and imaginary time).

To make further progress we need to expand the last term in \( S_{\text{eff}} \) in powers of \( \Sigma \). We keep terms up to second order in the Fourier components of the fields \( \phi, \theta, M \). A remarkable simplification occurs in the \( q \to 0 \) limit where the action reduces to the sum of independent quadratic terms (see Appendix B). Density and supercurrent oscillations on one hand and spin oscillations on the other hand decouple. As usual, the frequencies of collective modes are determined by the isolated poles of appropriate susceptibilities. For short range interactions, the density/current modes disperse linearly in \( q \) and their frequency vanishes at \( q = 0 \) as expected for a regular Goldstone mode. The spin modes, on the other hand, have a finite frequency, which increases with increasing \( \Delta \) [consistent with the fact that the resistance of Cooper pairs to the twisting motion described in Fig. 1 increases with increasing \( \Delta \)], but remains less than \( 2\Delta \), ensuring long lifetime.

Collective spin oscillations. — In the \( q \to 0 \) limit all the mixed response functions vanish (see Appendix B) and the frequency of the collective spin mode \( \omega_{\|} (\omega_{\perp}) \) at \( q = 0 \) is given by the solution of the equation

\[
2V^{-1} - \chi_{\| (\perp)}(0, \omega) = 0
\]

(9)

with respect to \( \omega \). In passing, we note that Eq. (9) can also be obtained diagrammatically from a vertex equation obtained by summing up ladder diagrams (see Appendix C). In Eq. (9), \( \chi_{\|} = \chi_{\sigma^1 \sigma^1} = \chi_{\sigma^2 \sigma^2} \) and \( \chi_{\perp} = \chi_{\sigma^3 \sigma^3} \) are the in-plane and out-of-plane dynamical spin susceptibilities of the GRSC described by \( \mathcal{H}_0 + \mathcal{H}_p \), respectively. These are obtained from the analytical continuation, \( i\nu_m \to \omega + i\theta^+ \), of the corresponding expressions in imaginary frequency:

\[
\chi_{\sigma^\pm}(0, i\nu_m) = -\frac{1}{2\beta A} \sum_{k,n} \text{Tr} \left[ \Gamma^a G_0(k, i\epsilon_n + i\nu_m/2) \right] \\
	imes \Gamma^b G_0(k, i\epsilon_n - i\nu_m/2) \ ,
\]

(10)

where “Tr” implies a trace over spin and Nambu-Gor’kov indices and \( \nu_m (\epsilon_n) \) is a bosonic (fermionic) Matsubara frequency. After analytic continuation we find, at \( T = 0 \),

\[
\chi_{\parallel}(0, \omega) = -\frac{1}{8\pi} \int_0^\infty dk \left( 1 - \frac{\xi_{\|} \xi_{\perp} + \Delta^2}{E_+ E_-} \right) \\
\times \left( \frac{1}{\omega + i\theta^+ - \mathcal{E}} - \frac{1}{\omega + i\theta^+ + \mathcal{E}} \right)
\]

(11)

and \( \chi_{\perp}(0, \omega) = 2\chi_{\parallel}(0, \omega) \left| \mathcal{E} \equiv E_+(k) + E_-(k) \right| \) and \( E_\pm^2(k) = \xi_{\|}^2(k) + \Delta^2 \). Due to the relation between out-of-plane and in-plane spin response functions, we will discuss only collective in-plane excitations.

We calculate \( \chi_{\parallel}(0, \omega) \) numerically from Eq. (11) and plot its real and imaginary parts in Fig. 2. In the limit \( \Delta = 0 \) (i.e., absence of superconductivity) – see panel a) – the imaginary part is non-zero only in the interval of frequencies between \( 2\alpha k_F, + \) and \( 2\alpha k_F, - \) \([k_F, \pm\) being the minority (majority) Fermi wave vectors for the two Rashba bands \( \xi_{\|}(k) \) and the real part exhibits (logarithmic) singularities at these boundaries (see Appendix D). When this result is inserted in Eq. (9), one finds a collective spin mode, which is undamped within this approximation. In a more refined theory (beyond Gaussian
fluctuations), however, low-energy double electron-hole excitations damp this mode. We now show that, at odds with the normal phase, in the superconducting state the mode lies (for a wide range of parameters) within the superconducting gap and thus cannot be damped by these excitations.

In panels b) - c) we plot $\chi_{\parallel}(0, \omega)$ for finite $\Delta$. In the superconducting state $\Re \chi_{\parallel}(0, \omega)$ exhibits a divergence at $\omega_1 = \min_k [E_+(k) + E_-(k)]$. In panel d) we plot $E_+(k) + E_-(k)$ as a function of $k$. In the region $0 < \omega < \omega_2$, $\Im \chi_{\parallel}(0, \omega)$ is identically zero and, since $\Re \chi_{\parallel}(0, \omega)$ diverges for $\omega \to \omega_1$, there is always an in-plane collective spin mode with frequency $\omega_1 \approx \omega_1$ for weak repulsive interactions $V$. Our results for the frequency of the in-plane collective mode $\omega_1$ as a function of $V$ and $\alpha$ (for a fixed value of $\Delta$) are summarized in Fig. 3. Note that there is a wide range of parameters such that $\omega_1$ lies within the superconducting gap, $0 < \omega_1 < 2\Delta$. We also have checked that, as expected, $\omega_1$ increases with $\Delta$.

In summary, we have shown that quasi-two-dimensional superconductors with tunable spin-orbit coupling exhibit undamped collective spin oscillations that can be excited by the application of a magnetic field or a supercurrent. The concerted action of spin-orbit coupling and electron-electron interaction is essential to the establishment of these collective oscillations. Since the frequency $\omega_1$ of these oscillations is of the order of the superconducting gap $\Delta$ we expect that our findings might enable the realization of long-lived spin oscillators operating in the frequency range of 10 GHz - 1 THz.

Acknowledgments

We acknowledge financial support by the EU FP7 Programme under Grant Agreement No. 215368-SEMiSPINNET (A.A. and M.P.), No. 234970-NANOCTM and No. 248629-SOLID (R.F.), and by the NSF under Grant No. DMR-0705460 (G.V.).

Appendix A: The Green’s function of a Gor’kov-Rashba superconductor

Let us first consider the Green’s function $G_{\epsilon}$ of a Gor’kov-Rashba superconductor in the so-called “chiral” basis. We remind the reader that we define “Gor’kov-Rashba superconductor” the system defined by the Hamiltonian $\mathcal{H}_0 + \mathcal{H}_p$, where the Rashba $\mathcal{H}_p$ and pairing $\mathcal{H}_p$ Hamiltonians have been defined in the main text.

We start by defining the four-spinor $\hat{\Psi}_n(k) = [\hat{\psi}_\lambda^\dagger(k) \hat{\psi}_\lambda^\dagger(-k) \hat{\psi}_\lambda^\dagger(-k) \hat{\psi}_\lambda^\dagger(k)]$ in momentum space, where $\hat{\psi}_\lambda$ and $\hat{\psi}_\lambda^\dagger$ are field operators corresponding to the eigenstates of the Rashba Hamiltonian $\mathcal{H}_0$ introduced in Eq. (1) of the main text. In this “chiral” basis the Matsubara Green’s function corresponding to the Gor’kov-Rashba Hamiltonian $\mathcal{H}_0 + \mathcal{H}_p$ [i.e. $G_{\epsilon}(k, \tau) = -\langle T_\tau [\hat{\Psi}_n(k, \tau) \hat{\Psi}_n^\dagger(k, 0)]\rangle$] is a $4 \times 4$ block-diagonal matrix (in Fourier transform with respect to imaginary time $\tau$):

$$G_{\epsilon}(k, i\epsilon_n) = \begin{pmatrix} G_+(k, i\epsilon_n) & 0 \\ 0 & G_-(k, i\epsilon_n) \end{pmatrix}, (A1)$$

where $G_{\pm}(k, i\epsilon_n)$ are the following $2 \times 2$ matrices:

$$G_+(k, i\epsilon_n) = -i\epsilon_n + \xi_0(k) + \lambda \Delta \cdot k/k \mathbf{c}^2_0 + E^2_0(k). (A2)$$

Here $E^2_0(k) \equiv \xi_0^2(k) + \Delta^2$ and $\epsilon_n$ is a fermionic Matsubara frequency. The gap $\Delta$ of the Gor’kov-Rashba superconducting state is given by

$$\Delta = \frac{g}{2} \sum_{k, \lambda} \lambda e^{i\phi_k} (\hat{\psi}_\lambda(-k) \hat{\psi}_\lambda(k)) = \frac{1}{2 A} \sum_{k, \lambda} \tanh \left( \frac{\beta E_0(k)}{2} \right), (A3)$$

where $g > 0$ is the pairing constant (see main text) and $\phi_k$ is the angle between $k$ and the $\hat{x}$ axis. We emphasize that, following Ref. [3], we have assumed that pairing occurs only between time-reversed partners within each Rashba spin-orbit-split band: $\langle \hat{\psi}_\lambda(-k) \hat{\psi}_\lambda(k) \rangle = 0$ if $\lambda = \lambda$. The gap $\Delta$ is fixed by the saddle-point equation $\delta S/\delta \Delta = 0$, [$S$ is given in Eq. (4) of the main text] which yields the mean-field BCS equation

$$1 = \frac{g}{2 A} \sum_{k, \lambda} \tanh \left( \frac{\beta E_0(k)}{2} \right), (A4)$$
The Green’s function $G_0$ introduced in Eq. (5) of the main text is explicitly given by the following $4 \times 4$ matrix:

\[
G_0^{-1}(r, \tau) = -\begin{pmatrix}
\partial_r - \nabla_r^2/2m & \alpha(\partial_x - i\partial_y) \\
-\alpha(\partial_x + i\partial_y) & \partial_r - \nabla_r^2/2m
\end{pmatrix}
\begin{pmatrix}
\Delta_{11} & \Delta_{12} \\
-\Delta_{12} & \Delta_{22}
\end{pmatrix}
\begin{pmatrix}
\partial_r + \nabla_r^2/2m & \alpha(\partial_x + i\partial_y) \\
-\alpha(\partial_x - i\partial_y) & \partial_r + \nabla_r^2/2m
\end{pmatrix},
\]

where $\Delta_{11} = \langle \hat{\psi}_1(r, \tau) \hat{\psi}_1(r, \tau) \rangle$, $\Delta_{12} = \langle \hat{\psi}_1(r, \tau) \hat{\psi}_2(r, \tau) \rangle$, $\Delta_{21} = \langle \hat{\psi}_2(r, \tau) \hat{\psi}_1(r, \tau) \rangle$, and $\Delta_{22} = \langle \hat{\psi}_2(r, \tau) \hat{\psi}_2(r, \tau) \rangle$. The Green’s function $G_0$ in the real-spin basis can be related to the triplet order parameter (which, of course, arises only because of the presence of spin-orbit coupling) and $\Delta_{12} = \langle \hat{\psi}_1(r, \tau) \hat{\psi}_2(r, \tau) \rangle$ is the singlet order parameter.

The 4 × 4 Green’s function $G_0$ in the real-spin basis can be related to the Green’s function $G_c$ in the chiral basis: we find (in Fourier transform with respect to space and imaginary time)

\[
\begin{align*}
G_0^{11}(k, \epsilon_n) &= G_s(k, \epsilon_n) \mathbb{1}_\sigma + G_s(k, \epsilon_n) \left[ (\hat{k} \times \sigma) \cdot \hat{z} \right], \\
G_0^{12}(k, \epsilon_n) &= F_s(k, \epsilon_n) \sigma^2 + F_s(k, \epsilon_n) \left[ (\hat{k} \times \sigma) \cdot \hat{z} \right] \sigma^2, \\
G_0^{21}(k, \epsilon_n) &= F_s(k, \epsilon_n) \sigma^2 - F_s(k, \epsilon_n) \left[ (\hat{k} \times \sigma) \cdot \hat{z} \right] \sigma^2, \\
G_0^{22}(k, \epsilon_n) &= \tilde{G}_s(k, \epsilon_n) \mathbb{1}_\sigma - \tilde{G}_s(k, \epsilon_n) \left[ (\hat{k} \times \sigma) \cdot \hat{z} \right] \mathbb{1}_\sigma.
\end{align*}
\]

Here $\hat{k} = k/|k|$ and we have defined

\[
\begin{align*}
2G_{s/a}(k, \epsilon_n) &= G_{s/a}^{11}(k, \epsilon_n) \pm \alpha \Delta \left[ (\hat{k} \times \sigma) \cdot \hat{z} \right], \\
2\tilde{G}_{s/a}(k, \epsilon_n) &= \tilde{G}_{s/a}^{11}(k, \epsilon_n) \pm \alpha \Delta \left[ (\hat{k} \times \sigma) \cdot \hat{z} \right],
\end{align*}
\]

where $\Delta = \Delta_{11}$. Using Eqs. (B1)-(B2) in the definition of the action $S$ given in Eq. (4) of the main text we find:

\[
S = \int_0^\beta d\tau \int d^2r \left[ \frac{\Delta^2}{g} + \frac{\mathcal{P}^2 + M \cdot M}{V} + \Phi \left( -G_0^{-1} + \Sigma_1 + \Sigma_2 + \Sigma_3 \right) \Phi \right],
\]

where $\Phi(r, \tau) = [\varphi_1^\dagger \varphi_2^\dagger \varphi_3^\dagger \varphi_4^\dagger]$, $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ are given by Eqs. (6) - (8) of the main text. Now the fermionic fields ($\Phi$) can be integrated out of the partition function, which is of the form $Z = Z_0 \int D[\Phi] \exp(\int B \Phi)$, by performing a Gaussian integral over the Grassman variables. We use the following relations for an arbitrary matrix $B$,

\[
\int D[\Phi] \exp \left( \frac{1}{2} \Phi B \Phi \right) = \sqrt{\text{Det}[B]}
\]

Using Eq. (B4) in Eq. (B3) immediately gives the effective action $S_{\text{eff}}$ reported in Eq. (8) of the main text.

To expand $S_{\text{eff}}$ up to second order in $\Sigma$, we use the identity

\[
\text{Tr} \left[ \ln \left( -G_0^{-1} + \Sigma \right) \right] = \text{Tr} \left[ \ln \left( -G_0^{-1} \right) \right] - \Sigma \sum_{n=1}^{\infty} \frac{1}{n} \left( G_0 \Sigma \right)^n.
\]

Appendix B: The effective action, Gaussian fluctuations, and spin collective excitations in the long-wavelength limit

To include the phase fluctuations of the order parameter in our study (ignoring the fluctuations in the modulus of the order parameter), we perform a gauge transformation to new fermionic fields

\[
\hat{\varphi}_i^\dagger(r, \tau) = \hat{\psi}_i^\dagger(r, \tau) e^{-i\theta(r, \tau)/2},
\]

and

\[
\hat{\psi}_i(r, \tau) = \hat{\varphi}_i(r, \tau) e^{i\theta(r, \tau)/2}.
\]
Using Eq. (8) in the main text and the identity above we find, up to second order in \(\Sigma\),

\[
S_{\text{eff}} \approx \int d\tau dr \left[ \frac{\Delta^2}{g} + \frac{q^2}{V} + \frac{M \cdot M}{V} \right] - \frac{1}{2} \text{Tr} \left[ \ln (\Sigma G_0^{-1}) \right] + \frac{1}{2} \text{Tr}[G_0 \Sigma] + \frac{1}{4} \text{Tr}[G_0 \Sigma G_0 \Sigma] + \mathcal{O}(\Sigma^3),
\]

(B6)

where the trace “\(\text{Tr}\)” is taken over all degrees of freedom including space and imaginary time. Note that in the last term of the previous equation we have to keep only terms up to second order in the fluctuating fields \(\theta\), \(\phi\), and \(M\) (Gaussian fluctuations).

We can rewrite \(S_{\text{eff}}\) as

\[
S_{\text{eff}} \approx \int d\tau dr \frac{\Delta^2}{g} - \text{Tr} \left[ \ln (\Sigma G_0^{-1}) \right] + S_{\text{fluct}}^{(2)},
\]

(B7)

where \(S_{\text{fluct}}^{(2)}\) is the second-order contribution to \(S_{\text{eff}}\) due to the fluctuating fields. Using Fourier transforms with respect to space and imaginary time and defining \(\Pi(q, i\nu_m) = [\theta(q, i\nu_m) \phi(q, i\nu_m) M(q, i\nu_m)]\) we find

\[
S_{\text{fluct}}^{(2)} = \sum_{q, m} \Pi(q, i\nu_m) \left\{ N(q, i\nu_m) \Pi^\dagger(-q, -i\nu_m) \right\},
\]

(B8)

where \(N\) is a \(5 \times 5\) matrix whose elements are given by various dynamical response functions and \(\nu_m\) is a bosonic Matsubara frequency. Different elements of the matrix \(N(q, i\nu_m)\) are given by

\[
N_{11}(q, i\nu_m) = \frac{\nu_m^2}{8} \chi_{\rho\rho}(q, i\nu_m) + \frac{1}{8} \sum_{i,j} q_i q_j \left[ \delta_{ij} \frac{\rho_{mf}}{m} \right] - \chi_{j'j}(q, i\nu_m) \left[ \frac{i}{8} \nu_m \sum_{i} q_i \left[ \chi_{j'j}(q, i\nu_m) + \alpha^2 q_i^2 \chi_{\phi\phi}(q, i\nu_m) \right] + \alpha \frac{1}{8} \chi_{\phi\phi}(q, i\nu_m) \right] - \frac{i}{8} \sum_{i} q_i \left[ \chi_{\phi\phi}(q, i\nu_m) + \chi_{j'j}(q, i\nu_m) \right] \right\},
\]

N_{12}(q, i\nu_m) = -\frac{i}{4} \nu_m \chi_{\rho\rho}(q, i\nu_m)
\]

(B11)

\[
N_{13}(q, i\nu_m) = -\frac{1}{4} \sum_{i} q_i \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \nu_m \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \sum_{i} q_i \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \nu_m \chi_{\rho\rho}(q, i\nu_m),
\]

(B12)

where \(a, b \in [3, 4, 5]\),

\[
N_{1a}(q, i\nu_m) = -\frac{i}{4} \nu_m \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \sum_{i} q_i \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \nu_m \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \sum_{i} q_i \chi_{\rho\rho}(q, i\nu_m),
\]

(B13)

\[
N_{2a}(q, i\nu_m) = -\frac{i}{4} \nu_m \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \sum_{i} q_i \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \nu_m \chi_{\rho\rho}(q, i\nu_m),
\]

and finally,

\[
N_{a2}(q, i\nu_m) = -\frac{i}{4} \nu_m \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \sum_{i} q_i \chi_{\rho\rho}(q, i\nu_m) + \frac{i}{4} \nu_m \chi_{\rho\rho}(q, i\nu_m).
\]

(B17)

The quantity \(\rho_{mf}\) which appears in Eq. \(\text{B9}\) is given by

\[
\rho_{mf} = \frac{1}{2\beta} \sum_{k, n} \text{Tr}[G_0(k, \epsilon_n) \tau^3 \otimes 1_\sigma],
\]

(B18)

and physically corresponds to the total electron density (superfluid and normal component).

The response functions \(\chi\) that appear in Eqs. \(\text{B9}\)-\(\text{B17}\) are dynamical susceptibilities of the Gor'kov-Rashba superconducting state in the absence of electron-electron interactions:
\[
\chi_{ij}^{c}(q, i\nu_{m}) = -\frac{1}{2\beta} \sum_{k,n} \frac{k_{i}}{m} \text{Tr} \left[ G_{0}(k^{+}, \epsilon_{n}^{+}) \tau^{3} \otimes \mathds{1}_{\sigma} G_{0}(k^{-}, \epsilon_{n}^{-}) \mathds{1}_{\tau} \otimes \mathds{1}_{\sigma} \right],
\]
\[
\chi_{\sigma\sigma'}^{c}(q, i\nu_{m}) = -\frac{1}{2\beta} \sum_{k,n} \text{Tr} \left[ G_{0}(k^{+}, \epsilon_{n}^{+}) \Gamma^{j} G_{0}(k^{-}, \epsilon_{n}^{-}) \Gamma^{j} \right],
\]
\[
\chi_{\sigma\rho}^{c}(q, i\nu_{m}) = -\frac{1}{2\beta} \sum_{k,n} \frac{k_{j}}{m} \text{Tr} \left[ G_{0}(k^{+}, \epsilon_{n}^{+}) \mathds{1}_{\tau} \otimes \mathds{1}_{\sigma} G_{0}(k^{-}, \epsilon_{n}^{-}) \mathds{1}_{\rho} \right],
\]
\[
\chi_{\phi\phi}^{c}(q, i\nu_{m}) = -\frac{1}{2\beta} \sum_{k,n} \text{Tr} \left[ G_{0}(k^{+}, \epsilon_{n}^{+}) \Gamma_{\phi} G_{0}(k^{-}, \epsilon_{n}^{-}) \Gamma_{\phi} \right],
\]
\[
\chi_{\phi\rho}^{c}(q, i\nu_{m}) = -\frac{1}{2\beta} \sum_{k,n} \text{Tr} \left[ G_{0}(k^{+}, \epsilon_{n}^{+}) \mathds{1}_{\tau} \otimes \mathds{1}_{\sigma} G_{0}(k^{-}, \epsilon_{n}^{-}) \Gamma_{\phi} \right],
\]
\[
\chi_{\phi\rho}^{c}(q, i\nu_{m}) = -\frac{1}{2\beta} \sum_{k,n} \frac{k_{i}}{m} \text{Tr} \left[ G_{0}(k^{+}, \epsilon_{n}^{+}) \mathds{1}_{\tau} \otimes \mathds{1}_{\sigma} G_{0}(k^{-}, \epsilon_{n}^{-}) \Gamma_{\phi} \right].
\]
the correction to the bare dynamical response function due to electron-electron interactions, while the second term originates from the coupling of spin fluctuations to phase fluctuations. The collective spin modes of the system can be found by solving
\[
\det[Q(q, \nu_m \to \omega + i0^+)]=0. \quad (B32)
\]

As mentioned in the main text, in this work we are interested in finding the frequency of the collective spin modes in the long-wavelength limit \( q \to 0 \). In this limit Eq. (B32) simplifies considerably and the effective action decouples into separate terms corresponding to supercurrent/density oscillations and spin oscillations, respectively. More specifically in the \( q \to 0 \) limit we have,
\[
\chi^{\sigma \rho}(0, \nu_m) = \chi^{\sigma \rho}(0, \nu_m) = 0. \quad (B33)
\]

Moreover,
\[
\chi^{\sigma \sigma}(q, \nu_m) = \chi^{\sigma \rho}(q, \nu_m) = 0 \quad (B34)
\]

for every finite \( q \). In the limit \( q \to 0 \) Eqs. (B33)-(B34) give
\[
N_{13} = N_{14} = N_{31} = N_{41} = N_{51} = N_{23} = N_{24} = N_{25} = N_{32} = N_{42} = 0, \quad (B35)
\]

and
\[
N_{34} = N_{35} = N_{43} = N_{45} = N_{53} = N_{55} = 0. \quad (B36)
\]

Using Eq. (B35) in Eq. (B31) we obtain
\[
Q_{cd}(0, \nu_m) = N_{c+2 \sigma d+2}(0, \nu_m). \quad (B37)
\]

In other words, phase/density fluctuations do not couple to spin fluctuations in the long-wavelength limit. Using Eq. (B36) in Eq. (B31) we obtain \( Q_{cd} = 0 \) for \( c \neq d \), i.e., all the off-diagonal components of the matrix \( Q \) are zero in the long wavelength limit. Eq. (B32) thus reduces to
\[
\det[Q(0, \nu_m \to \omega + i0^+)]=\left[\frac{2}{V} - \chi^{\sigma \sigma}(0, \omega)\right] \times \left[\frac{2}{V} - \chi^{\sigma \sigma}(0, \omega)\right] = 0, \quad (B38)
\]

with \( \chi^{\sigma \sigma}(0, \omega) = \chi^{\sigma \sigma}(0, \omega) \).

Note that we can also obtain Eq. (B38) directly from \( S_{\text{fluct}}^{(2)} \). Equation (B35) implies that the matrix \( N \) in Eq. (B8) has a block diagonal form comprising an upper 2x2 block corresponding to \( \theta - \phi \) fields and a lower 3x3 block corresponding to the \( M \) fields. Thus the \( q = 0 \) component of the action \( S_{\text{fluct}}^{(2)} \) can be expressed as a product of a “phase only” action and a “spin only” action, i.e.
\[
S_{\text{fluct}}^{(2)}|_{q=0} = S_{\theta,\phi}^{(2)}|_{q=0} \times S_{M}^{(2)}|_{q=0}. \quad (B39)
\]

Moreover, Eq. (B36) implies that the lower 3x3 block of the \( N \) matrix in Eq. (B8) is diagonal in the long-wavelength limit. We thus obtain
\[
S_{M}^{(2)}|_{q=0} = \frac{1}{2} \sum_{\nu_m,i} M_{i}(0, \nu_m) \left[\frac{2}{V} - \chi^{\sigma \sigma}(0, \nu_m)\right] \times M_{i}(0, \nu_m), \quad (B40)
\]

which gives the same condition for the existence of collective spin modes as Eq. (B38).

Appendix C: The ladder sum and the vertex equation

In this Section we show that the equation
\[
\frac{2}{V} - \chi^{\sigma \sigma}(0, \omega) = \frac{2}{V} - \chi^{\sigma \sigma}(0, \omega) = 0 \quad (C1)
\]

for the collective spin excitations that we found in the previous section, Eq. (B38), can also be obtained diagrammatically.

In the (conserving) ladder approximation\(^{18,19}\) the dynamical spin response function \( \tilde{\chi}^{\sigma \sigma} \) in the presence of electron-electron interactions is given by
\[
\tilde{\chi}^{\sigma \sigma}(q, \nu_m) = -\frac{1}{2\beta} \sum_{k,n} \text{Tr}\left[\Gamma^2 G_0(k^+, i\epsilon^+_n)\Lambda(q, \nu_m) \times G_0(k^-, i\epsilon^-_n)\right]. \quad (C2)
\]

The vertex function \( \Lambda(q, \nu_m) \) is a “dressed” version of the bare vertex \( \Gamma^2 = \mathbb{1}_\tau \otimes \sigma^2 \) and it accounts for the interplay between electron-electron interactions and the external electromagnetic field. Similar equations hold for \( \tilde{\chi}^{\sigma \sigma} \) and \( \tilde{\chi}^{\sigma \sigma} \).

The vertex \( \Lambda \) is a 4x4 matrix and satisfies the following equation (see Fig. 4):
\[
\Lambda(q, \nu_m) = \Gamma^2 - V \tau^3 \otimes \mathbb{1}_\sigma \left\{ \frac{1}{\beta} \sum_{k,n} G_0(k^+, i\epsilon^+_n) \times \Lambda(q, \nu_m) \times G_0(k^-, i\epsilon^-_n) \right\} \tau^3 \otimes \mathbb{1}_\sigma. \quad (C3)
\]

In the \( q \to 0 \) limit, after some lengthy but straightforward algebra, Eq. (C3) yields
\[
\Lambda(0, \nu_m) = \frac{\mathbb{1}_\tau \otimes \sigma^2}{1 - \frac{V}{2} \chi^{\sigma \sigma}(0, \nu_m)}. \quad (C4)
\]

In the ladder approximation, the interacting in-plane spin-susceptibility in the long-wavelength limit is thus given by
\[
\tilde{\chi}^{\sigma \sigma}(0, \nu_m) = \frac{\chi^{\sigma \sigma}(0, \nu_m)}{1 - \frac{V}{2} \chi^{\sigma \sigma}(0, \nu_m)}. \quad (C5)
\]
Here or more explicitly by the bare-bubble level and the spinor basis with the help of the transformation

\[ \chi(\sigma_4) = \begin{pmatrix} 1 & i \sigma_1 \\ i \sigma_1 & 1 \end{pmatrix} \]

The response function of interest is the in-plane spin-spin correlation function \( \chi(\sigma_4) \). In the following we will use the identity

\[ \sigma_4 = \sum_k \chi(\sigma_4) q_k / 2 \]

will denote the identity and the

\[ q_k = \sum_{\kappa} \chi(\sigma_4) q_{\kappa} \]

reproduces the condition given above in Eq. (C1).

\[ \chi(\sigma_4) \]

is given by

\[ \chi(\sigma_4) = \frac{1}{4\pi} \int_k k_F \cdot dk \left[ \frac{1}{\omega + 2aki + i0^+} - \frac{1}{\omega - 2aki + i0^+} \right] \]

(D1)

where \( k_F \cdot \tau \) is the Fermi wave-vector of the minority (majority) Rashba band. Performing the integration in Eq. (D1), we find that the real and imaginary parts of \( \chi(\sigma_4) \) are given by

\[ \Re \chi(\sigma_4) = \Re \left( \frac{\omega}{8ma^2} \ln \left| \frac{\omega + 2aki \tau_+ \omega - 2aki \tau_-}{\omega - 2aki \tau_+ \omega + 2aki \tau_-} \right| \right) \]

(D2)

and

\[ \Im \chi(\sigma_4) = \Im \left( \frac{\omega}{16a^2} \Theta(\omega - 2aki \tau_+ \omega + 2aki \tau_-) \right) \]

(D3)

Eqs. (D2) and (D3) agree with Eqs. (7) and (10) in Ref. 29 (after setting to zero the Dresselhaus spin-orbit coupling constant in their results).

**Appendix D: Normal Rashba Gas**

In this Section we report explicit expressions for the real and imaginary parts of the in-plane dynamical spin susceptibility \( \chi(\sigma_4) \) of a normal (non-superconducting) Rashba 2DEG in the absence of electron-electron interactions [see panel a) of Fig. 1 in the main text].

At zero temperature and in the absence of superconductivity we find:

\[ \chi(\sigma_4) = \frac{1}{4\pi} \int_k k_F \cdot dk \left[ \frac{1}{\omega + 2aki + i0^+} - \frac{1}{\omega - 2aki + i0^+} \right] \]

(D1)

Since collective modes are isolated poles in the dynamical response function \( \chi(\sigma_4) \) (located infinitesimally below the real-frequency axis), Eq. (C5) reproduces the condition given above in Eq. (C1).

---

* Electronic address: amit.agarwal@sns.it

1. I. Žutić, J. Fabian, and S. Das Sarma, Rev. Mod. Phys. 76, 323 (2004); J. Fabian et al., Acta Physica Slovaca 57, 565 (2007).
2. S. Datta and B. Das, Appl. Phys. Lett. 56, 665 (1990).
3. J.E. Hirsch, Phys. Rev. Lett. 83, 1834 (1999); S. Mukerjee, N. Nagaosa, and S.-C. Zhang, Science 301, 1348 (2003); J. Sinova et al., Phys. Rev. Lett. 92, 126603 (2004).
4. Y.K. Kato et al., Science 306, 1910 (2004); J. Wunderlich et al., Phys. Rev. Lett. 94, 047204 (2005); J. Wunderlich et al., Science 330, 1801 (2010).
5. V.M. Edelstein, Phys. Rev. Lett. 75, 2004 (1995) and Phys. Rev. B 67, 020505 (2003).
6. L.P. Gor’kov and E.I. Rashba, Phys. Rev. Lett. 87, 037004 (2001).
7. S.K. Yip, Phys. Rev. B 65, 144508 (2002); K.V. Samokhin, E.S. Zijlstra, and S.K. Bose, ibid. 69, 094514 (2004); P.A. Frigeri et al., Phys. Rev. Lett. 92, 097001 (2004); P.A. Frigeri, D.F. Agterberg, and M. Sigrist, New J. Phys. 6, 115 (2004); O. Dimitrova and M.V. Feigel’man, Phys. Rev. B 76, 014522 (2007); E. Cappelluti, C. Grimaldi, and F. Marsiglio, Phys. Rev. Lett. 98, 167002 (2007); H. Kontani, J. Goryo, and D.S. Hirashima, ibid. 102, 086602 (2009).
8. See e.g. F. Girottto et al., Phys. Rev. Lett. 87, 216808 (2001) and references therein to earlier experimental work.
9. See e.g. J. Alicea, Phys. Rev. B 81, 125318 (2010).
10. F. Wilczek, Nature Phys. 5, 614 (2009).
11. A. Ohmoto and H.Y. Hwang, Nature 427, 423 (2004).
12. A.D. Caviglia et al., Phys. Rev. Lett. 104, 126803 (2010).
13. A.D. Caviglia et al., Nature 456, 624 (2008).
14. The quality factor of the oscillations limited by radiative damping is estimated to be \( Q = c/(\omega r_0) \) where \( r_0 = c^2/(mc^2) \sim 3 \times 10^{-15} \) m is the classical electron radius, \( c \) is the speed of light. This gives a value of \( Q \) in excess of 10^16.
15. P. Morel and P.W. Anderson, Phys. Rev. 125, 263 (1962).
16. R.E. Prange and V. Korenman, Phys. Rev. B 19, 4691 (1979).
17. See e.g. L. Benfatto, A. Toschi, and S. Caprara, Phys. Rev. B 69, 184510 (2004).
18. J.R. Schrieffer, *Theory of Superconductivity* (Advanced
Book Classic, Oxford 1964).  

19 Y. Nambu, Phys. Rev. 117, 648 (1960).  

20 C. López-Bastidas, J.A. Maytorena, and F. Mireles, Phys. Status Solidi (c) 4, 4229 (2007).