Signature change induces compactification *

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Abstract

It is shown – using a FRW model with $S^3 \times S^6$ as spatial sections and a positive cosmological constant – that classical signature change implies a new compactification mechanism. The internal scale factor is of the order $\Lambda^{-1/2}$, and the solutions are stable against small perturbations. In the case of compactified $S^6$, it is shown that the effective four-dimensional space-time metric has Lorentzian signature, undergoes exponential inflation in $S^3$ and is unique. Speculations concerning relations to quantum cosmology and conceivable modifications are added.

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1 Introduction

The aim of this paper is to present a mechanism that combines the issues of classical signature change and compactification. Both topics are of general interest in the recent discussion on gravity theory and cosmology, the former being comparably young, whereas the latter may be called an old tradition.

Our approach is mainly a cosmological one. Given a higher dimensional model including gravity, and assuming the spatial sections to be of topology, say $S^3 \times M$, one would like to know whether there is a dynamical mechanism that drags the size of $M$ down and keeps it in a stable way at an unobservably small ("compactified") scale. Without repeating all the advantages that the existence of internal dimensions could provide for various aspects of particle physics and field theory [1], we just mention that the Kaluza-Klein idea is crucial for several theories (like supergravity and superstring models) whose formulation requires a particular number of dimensions.

In the cosmological context, there are a lot of mechanisms at hand that prevent some of the dimensions from expanding [1]–[4]. The long-time stability of the geometrical configurations emerging is a more subtle problem, and the most popular methods to prevent internal dimensions from collapsing use particular forms of matter couplings. One famous example is the Freund-Rubin compactification in eleven-dimensional supergravity [3]. Among various other possibilities, we just mention the gravitational Casimir effect [4] and the approaches provided by quantum cosmology [5]–[6].

Our motivation lies in the search for fundamental compactifications mechanisms, i.e. models that are mainly based on gravity. Leaving apart sophisticated matter couplings, we choose the very simple model of pure gravity with a positive cosmological constant.

The second issue we have mentioned is classical signature change. In this approach, one allows the metric to be of different signature in different regions of the total manifold considered. There has been a recent discussion on the relevance and the physical nature of such models [1]–[2]. In the most important version, the signature of the metric may change from Euclidean to Lorentzian type. Usually, the Euclidean domain is considered as related to the early universe, and the existence of a physical time is a consequence of a signature change [4]. In some sense, this approach is a classical alternative to quantum cosmology, where Euclidean and
Lorentzian geometries are interlinked as well, but in a different way. We will use a specific approach to classical signature change that is effectively based on the junction conditions advocated in Ref.\footnote{1} (and that has been denoted weak signature change in Ref.\footnote{3}).

Putting these two ingredients together, we can show that a repeated sequence of signature changes can effectively compactify and stabilize internal spaces. This has been worked out in detail for a Friedmann-Robertson-Walker (FRW) model with $S^3 \times S^6$ as spatial sections. Some of the ideas contained in this paper have been developed in Ref.\footnote{11}.

Our presentation begins with an outline of the model and the choice of convenient variables in Sect.2. The compactification mechanism is discussed in Sect.3. It is based on the interplay between the causal structure of the Wheeler-DeWitt metric and the sign of the potential $W$ showing up in the equations of motion of the two scale factors. Euclidean solutions have the tendency to leave the domain $W < 0$, whereas a large portion of Lorentzian solutions cannot escape to arbitrarily large values of $W$, but recollapse. Thus, assuming actual occurrence of a signature change whenever it is possible (the condition being $W = 0$), one obtains a family of metrics whose signature type “oscillates” between Lorentzian and Euclidean, and whose location in the minisuperspace built up by the scale factors is near the curve $W = 0$ (which in turn describes compactification of either sphere). An example for such a metric that has been obtained numerically, is displayed.

In Sect.4 we assume that the effective (physical) metric is provided by a sort of coarse graining average over the true (oscillating) one. For the case of compactified $S^6$ we show that the resulting effective metric is unique, has Lorentzian signature (which is due to a small domination of the Lorentzian periods over the Euclidean ones) and describes exponential inflation of the remaining three-space $S^3$. Sect.5 is devoted to the discussion of physical problems related to quantum gravity and the necessity to exit inflation. Various conceivable modifications and generalizations are pointed out. In the last Section we comment on the alternative case of compactified $S^3$, and on the generalization to a $S^m \times S^n$ model. The structure of the equations of motion turns out to be such that only internal spaces with non-vanishing Ricci-curvature can compactify by signature change.
2 The model and its variables

We consider a FRW model with the product $S^3 \times S^6$ of round spheres as spatial sections of space-time, and a positive cosmological constant $\Lambda$. Some of the results we will achieve may be generalized to $S^m \times S^n$, and we comment on this at the end of the paper. The class of metrics is thus described by

$$ds^2 = \mp N(t)^2 dt^2 + a_1(t)^2 d\sigma_3^2 + a_2(t)^2 d\sigma_6^2,$$  \hspace{1cm} (2.1)

where $d\sigma_n^2$ is the line-element on the round unit $n$-sphere. Here and in what follows, the upper sign belongs to the Lorentzian (i.e. Pseudo-Riemannian) and the lower sign to the Euclidean (Riemannian) version. In this Section, we consider these cases independently. The ten-dimensional Einstein-Hilbert action including a positive cosmological constant and the usual boundary term that absorbs the second time-derivatives, may be written as

$$S = \mp C \int_{M_{10}} d^{10}x \sqrt{|g|} (10R + 2\Lambda) \mp 2 C \int_{\partial M_{10}} d^9x \sqrt{h} K,$$  \hspace{1cm} (2.2)

where

$$C = \frac{m_P^2}{16\pi} (\text{volume of internal space today})^{-1},$$  \hspace{1cm} (2.3)

$h_{ij}$ the metric induced by $g_{\mu\nu}$ on the boundary $\partial M_{10}$ and $K$ the trace of its extrinsic curvature. The above choice of $C$ ensures the correct gravitational constant today if $S^6$ is compactified (as internal space) at a small value of its scale factor $a_2$. The $S^3$ is then the physical (external) space we observe. (However, since we are interested here only in classical solutions to Einstein’s field equations, the constant $C$, being just a prefactor of the action, drops out anyway). Inserting (2.1) into (2.2) results into

$$S = wC \int dt \left( a_1^3 a_2^6 \frac{\dot{N}}{N} \left( -6 \frac{\dot{a}_1^2}{a_1^2} - 36 \frac{a_1 \dot{a}_2}{a_1 a_2} - 30 \frac{\dot{a}_2^2}{a_2^2} \right) \right)$$

$$\mp N \left( -6 a_1 a_2^6 - 30 a_1^3 a_2^4 + 2\Lambda a_1^3 a_2^6 \right).$$  \hspace{1cm} (2.4)

Here, $w = 32\pi^5/15$, and the volume of the $t = \text{const}$ space sections is given by

$$V = \int_{M_9} d^9x \sqrt{h} = wa_1^3 a_2^6.$$  \hspace{1cm} (2.5)
As is well known, the variation of this action with respect to the variables $N(t), a_1(t)$ and $a_2(t)$ yields the full set of Einstein’s field equations for the ansatz (2.1).

In order to diagonalize the kinetic part of the action, we change variables according to

$$N = \frac{\Lambda}{3\sqrt{10}} a_2 N,$$  \hfill (2.6)

$$u = \frac{\Lambda^2}{180} a_1^2 a_2^2,$$  \hfill (2.7)

$$v = \frac{\Lambda^3}{5400\sqrt{5}} a_1 a_2^5,$$  \hfill (2.8)

and define (using $\Lambda \equiv \ell_\Lambda^{-2}$ and $m_P \equiv \ell_P^{-1}$)

$$\Lambda_{\text{eff}} = \frac{375^{1/4} \Lambda}{2^{1/8} 6 (wC)^{1/4}} \equiv \left( \frac{1}{\pi} \left( \frac{5}{3\sqrt{2}} \right)^3 \left( \frac{\ell_P}{\ell_\Lambda} \right)^2 \left( \frac{a_2(\text{today})}{\ell_\Lambda} \right)^6 \right)^{1/4},$$  \hfill (2.9)

The action (2.4) simplifies to

$$S = \frac{1}{\Lambda_{\text{eff}}^4} \int dt \left( -\frac{\dot{u} \dot{v}}{N} \mp NW(u,v) \right).$$  \hfill (2.10)

The potential is given by

$$W(u,v) = -v - u^{3/2} + 2 u^{5/4} v^{1/2},$$  \hfill (2.11)

the origin of the three contributions being the curvature of $S^3$, the curvature of $S^6$ and the cosmological constant, respectively. The lapse variable $N$ corresponds to the choice of a particular (Lorentzian or Euclidean) ”time” parameter $t$, and the only dynamical degrees of freedom in our model are $u$ and $v$, both ranging from 0 to $\infty$. A ”point” $(u,v)$ in this minisuperspace represents a nine-geometry. Classical solutions may be displayed as trajectories (a continuous sequence of nine-geometries), and a (gauge) choice fixing the time parameter $t$ leads to parametrized curves $(u(t),v(t))$.

The equations of motion following from variation of (2.10) are

$$\frac{\dot{u} \dot{v}}{N^2} = \pm W,$$  \hfill (2.12)

$$\frac{1}{N} \frac{d}{dt} \left( \frac{\dot{u}}{N} \right) = \pm \partial_u W,$$  \hfill (2.13)

$$\frac{1}{N} \frac{d}{dt} \left( \frac{\dot{v}}{N} \right) = \pm \partial_v W.$$  \hfill (2.14)
the first being the constraint due to the absence of $\dot{N}$ in the action. It translates to the Wheeler-DeWitt equation in the framework of quantum cosmology and, once being satisfied at $t = t_0$, it is conserved in time by the two remaining equations.

The kinetic part of the action (2.10) shows that $u$ and $v$ are "lightlike" coordinates with respect to the Wheeler-DeWitt metric $ds^2_{WD} = -dudv$. The constraint equation (2.12) implies that Lorentzian trajectories that represent classical solutions are timelike ($dudv > 0$) curves in the regions of minisuperspace where $W > 0$ and spacelike ($dudv < 0$) when $W < 0$. Conversely, Euclidean trajectories are spacelike if $W > 0$ and timelike if $W < 0$.

One may perform further transformations of variables $\overline{u} = u(u), \overline{v} = v(v)$, thereby retaining the structures (2.10) and (2.12-2.14) if one sets

$$W = W \frac{du}{d\overline{u}} \frac{dv}{d\overline{v}}, \quad \overline{N} = N \frac{d\overline{u}}{du} \frac{d\overline{v}}{dv}. \quad (2.15)$$

We will use such a change of variables later on.

Sometimes one may like to ask for the dynamics in a region of minisuperspace where $a_1$ is large whereas $a_2$ is small. In such a case, the curvature contribution stemming from $S^3$ may be neglected, and $W$ is replaced by

$$W_{\text{approx}} = -u^{3/2} + 2u^{5/4}v^{1/2} \quad (2.16)$$

in (2.10) and (2.12-2.14). One would have obtained this as the correct potential, had one started with a $T^3 \times S^6$ model from the outset. In this case, one would replace $d\sigma_3^2$ in (2.1) by the metric on the flat three-torus. Letting the coordinate on the unit torus range from $0$ to $(2\pi^2)^{1/3}$, all the previous formulae remain valid if the $S^3$ curvature contributions are omitted, including (2.3) and (2.4). One may even interpret the model based on $W_{\text{approx}}$ as having $R^3 \times S^6$ as spatial sections, but in this case a proper integration over $\mathcal{M}_9$ as in (2.2) is of course not possible. One would then ignore (2.3) and (2.4), but otherwise arrive at the same equations of motion by directly using Einstein’s field equations. However this case is interpreted, the equations of motion will leave the absolute scale of $a_1$ free to an arbitrary rescaling $a_1 \rightarrow c a_1$ (i.e. $u^5 v^{-2} \rightarrow c^8 u^5 v^{-2}$), while $a_2 \rightarrow a_2$ (i.e. $v^2 u^{-1} \rightarrow v^2 u^{-1}$).

Summarizing, let us state that we will work in the framework defined by the structure of the action (2.10) and the equations of motion (2.12-2.14), possibly redefined by a transformation of the type (2.15). The potential (2.11) represents the full $S^3 \times S^6$ model, whereas (2.16) may be viewed either as an approximation in regions of large external space, or as exact version of the $T^3 \times S^6$ or $R^3 \times S^6$ model.
3 Compactification by signature change

One key feature of the model described in the previous Section is the interplay between the "causal structure" of the Wheeler-DeWitt metric and the sign of the potential $W$. The set of pairs $(u,v)$ for which $W = 0$ is a curve lying entirely in the interior of minisuperspace. Its explicit form is exhibited by solving for $v$ as a function of $u$, thus giving rise to two "branches", an "upper" one,

$$v = -u^{3/2} + 2u^2 \left( \sqrt{u} + \sqrt{u - 1} \right) \sim 4u^{5/2}$$  \hspace{1cm} (3.1)

and a "lower" one

$$v = -u^{3/2} + 2u^2 \left( \sqrt{u} - \sqrt{u - 1} \right) \sim -\frac{1}{4} u^{1/2},$$  \hspace{1cm} (3.2)

both branches being connected smoothly at $u = v = 1$. The asymptotic forms given above apply for large $u$, i.e. far away from the origin, and the second one is at the same time the exact solution of $W_{\text{approx}} = 0$. The whole curve may be parametrized as $u(\lambda) = \cosh^2(\lambda)$, $v(\lambda) = \cosh^3(\lambda) \exp(2\lambda)$, $\lambda$ taking all real values. Asymptotically, along the upper branch $a_1 \to \ell_A \sqrt{3}$ (while $a_2 \to \infty$), along the lower one $a_2 \to \ell_A \sqrt{15}$ (while $a_1 \to \infty$). The dashed curve in Fig.1 shows the location of this curve. The domain near the origin $(u,v)$ small has $W < 0$.

The curve $W = 0$ divides minisuperspace more or less naturally into a "Lorentzian" region $W > 0$ and a "Euclidean" region $W < 0$. This notation is motivated by the fact that Lorentzian trajectories may not emerge from the "regular zero-geometry" $u = v = 0$, and Euclidean trajectories inside the $W > 0$ domain must necessarily hit the zero potential curve and thus cannot evolve towards nine-geometries with arbitrarily large volume. The behaviour of the two types of trajectories is related to their role in quantum cosmology $[E]$. There, one usually constructs a path integral in the region $W < 0$ around Euclidean trajectories which describe regular ten-geometries. Near the spacelike part of $W = 0$, a family of Lorentzian trajectories that are supposed to represent the classical evolution of the universe is defined by WKB-techniques.

Euclidean trajectories corresponding to regular ten-geometries (by virtue of $[E]$) behave like $v \sim c_1 u^{5/2}$ (then $a_1(0) = \text{finite}$) or $v \sim c_2 u^{1/2}$ (then $a_2(0) = \text{finite}$) near the origin. There are two preferred solutions (the instantons) that display high symmetry: One of these is given by the piece of the curve $v = (9/16)u^{5/2}$ inside the domain $W < 0$, has $a_1 = \ell_A \sqrt{8}$ and corresponds to the ten-geometry $S^3 \times S^7$. The
other one is given by $v = (4/9)u^{1/2}$ inside $W < 0$, has $a_2 = \ell_A\sqrt{20}$ and describes $S^4 \times S^6$. Both solutions have turning points (i.e. $\dot{u} = \dot{v} = 0$ in the gauge $N = 1$) at $W = 0$.

A generic Euclidean trajectory starting from the origin is confined to satisfy $dudv > 0$ in the Euclidean region. This (and actually the stronger property $du > 0, dv > 0$) follows from the constraint equation (2.12) with the lower sign, together with the fact that $W < 0$. The trajectory will eventually hit the curve $W = 0$, thereby having either horizontal ($dv = 0$) or vertical ($du = 0$) tangent. When evolved further into the Lorentzian region $W > 0$ (but still as a Euclidean trajectory), the sign change of $W$ will enforce $dudv < 0$, and hence drive the evolution towards the curve $W = 0$ again. Re-entering the Euclidean region, the trajectory evolves with $dudv > 0$ (actually $du < 0, dv < 0$), until it hits one of the axes and thus describes a Kasner-type final singularity (i.e. $a_1 \to 0, a_2 \to \infty$ or vice versa). In the very special case of the instantons, the two intersection points between the trajectory and the curve $W = 0$ coincide and form one single turning point, the evolution leading back to the origin (according to the regularity of the ten-geometry described by these two instanton solutions).

In contrast, Lorentzian trajectories can (due to the constraint equation (2.12) with the upper sign) never emerge from the origin. Hence we restrict our attention to the Lorentzian trajectories starting at the curve $W = 0$ (with – generically – $dudv = 0$, hence horizontal or vertical tangent). We encounter two classes of behaviour: Trajectories that evolve towards arbitrarily large geometries (i.e. values of the spatial volume) and trajectories that don’t (but instead ”recollapse” towards one of the axes).

The first class (i.e. those Lorentzian trajectories that may represent a reasonable classical behavior of the universe – regardless of the space-time dimension, for the moment) fall into two sub-classes: In the generic case, one gets exponential inflation in both scale factors $a_1$ and $a_2$. However, there are two isolated solutions for which one of the two scale factors is constant. They match the two instanton solutions at $W = 0$ (where they have turning points just as these), and lie on the $W > 0$ pieces of the curves $v = (9/16)u^{5/2}$ and $v = (4/9)u^{1/2}$, respectively. Note however that these solutions, although describing compactification of either scale factor, are not stable against small perturbations.

The second class of Lorentzian trajectories (i.e. those who do not evolve towards large geometries) is usually not considered as realistic: They enter the Lorentzian
region, but re-approach the curve $W = 0$, hence leave the Lorentzian region and eventually recollapse towards one of the axes ($u = 0$ or $v = 0$) in a Kasner-type singularity (i.e. $a_1 \to 0, a_2 \to \infty$ or vice versa). One may say that a universe described by such a recollapsing trajectory is "created" with too small an amount of kinetic energy as to become arbitrarily large. However, this is only a valid statement in the framework of a model in which one does not admit a change of the metric signature, once the universe is Lorentzian. In the approach we are advocating here, those pieces of the recollapsing Lorentzian trajectories which lie inside the Lorentzian region, will play a dominant role.

In quantum cosmology, the interplay between Euclidean trajectories (representing in some sense a full quantum or tunneling state of the universe) and Lorentzian trajectories (representing a semiclassical state) is quite implicit, and there is no individual one-to-one matching. One may, as a different point of view, regard the transition from one type of evolution to the other as a classical phenomenon, thereby matching a particular Euclidean trajectory to a particular Lorentzian one. The ten-geometry resulting from such a process will then, by virtue of (2.1), undergo a classical signature change. There has been an extensive discussion in the recent literature whether such a transition is physically reasonable [7–8, 12–13]. In the context of cosmology, it is usually conceived to have happened in the early universe, but its relation to a "quantum signature change" as described by the Euclidean path integral formulation of quantum cosmology is not quite clear.

Taking this possibility serious, we are led to the question under which conditions trajectories may change their signature type without rendering the resulting ten-metric too singular. The answer is to some extent a matter of taste, and we will require continuity in $\dot{u}/N$ and $\dot{v}/N$. Then the constraint equation (2.12) implies that a classical change of signature can only happen at points $(u, v)$ for which $W = 0$.

Let us note for completeness that this is not the only reasonable choice: In the most restrictive versions of classical signature change one requires that the extrinsic curvature vanishes at the matching nine-surface [12–13]. (In Ref. [1], this scenario has been called strong signature change). The extrinsic curvature is in our model essentially given by the traces over the two factor spheres

$$
K_1 = \frac{1}{2N} \dot{h}^{ij}_{(1)} \dot{h}_{ij}^{(1)} = \frac{3}{N} \frac{\dot{a}_1}{a_1} = \frac{\sqrt{3} \Lambda}{8N} \frac{v^{1/4}}{u^{1/8}} \left(5 \frac{\dot{u}}{u} - 2 \frac{\dot{v}}{v}\right),
$$

$$
K_2 = \frac{1}{2N} \dot{h}^{ij}_{(2)} \dot{h}_{ij}^{(2)} = \frac{6}{N} \frac{\dot{a}_2}{a_2} = \frac{\sqrt{3} \Lambda}{4N} \frac{v^{1/4}}{u^{1/8}} \left(2 \frac{\dot{v}}{v} - \frac{\dot{u}}{u}\right).
$$

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Using the gauge $N = 1$ (or any gauge which ensures $N \neq 0$ when the trajectory approaches $W = 0$), the condition $K_i = 0$ together with the constraint (2.12) at $W = 0$ implies $\dot{u} = \dot{v} = 0$, hence the existence of turning points for both partners. This is however a rare event: Among the Euclidean trajectories describing regular ten-geometries only the two instanton solutions mentioned above are run into a turning point. Matching these to the corresponding Lorentzian solutions provides a saddle point approximation to the Euclidean path integral in quantum cosmology [6] and leads (semiclassically) to an (unstable) compactification of $S^3$ or (preferably) $S^6$.

There are, however, approaches that are less inspired by quantum cosmology and allow for weaker junction conditions [7]–[8]. In the spirit of these, we only require continuity of the extrinsic curvature. This is actually identical to assuming continuity of $\ddot{u}/N$ and $\ddot{v}/N$. (In Ref. [9], the scenario based on these junction conditions has been called weak signature change). To be more explicit, consider, in the gauge $N = 1$, a trajectory of either type approaching the zero potential curve. Leaving aside the rare possibility of turning points, one of the two quantities $\dot{u}, \dot{v}$ must become zero. If, e.g. $\dot{v} = 0$ (horizontal tangent), this trajectory may be matched to one of the opposite type having $\dot{v} = 0$ as well, such that $\dot{u}$ of the resulting "mixed" trajectory is continuous. There has recently been some controversy about this approach in the literature. We will justify its use after having written down the action (below equation (3.8)).

Let us add here a remark in order to avoid confusion. In Ref. [7], which is one of the most important papers on the "weaker" approach to classical signature change, a situation similar to ours (pure gravity with a cosmological constant) is considered as an example, but within a FRW model containing only a single scale factor $R$. In this case, the constraint equation is of the structure $\dot{R}^2/N^2 = \pm W(R)$ instead of (2.12). Hence, $W = 0$ implies $\dot{R}/N = 0$, which means vanishing extrinsic curvature. This statement does not carry over to the multidimensional case we are considering. As a consequence, Ellis et al precisely recover the classical metric that corresponds to the Hartle-Hawking no-boundary prescription [10], i.e. half of $S^4$ (the instanton) matched to half of the deSitter hyperboloid (representing the classical evolution), whereas we are free to admit a variety of signature change configurations, the instanton trajectories providing just a very special isolated case.
As a second assumption, we not only admit the possibility of signature change but require that it will happen whenever a trajectory approaches the curve $W = 0$. In terms of a single expression, such mixed trajectories are produced by the action

$$ S = \frac{1}{\Lambda_{\text{eff}}^4} \int dt \left( -\frac{\dot{u}\dot{v}}{N} - N|W(u,v)| \right). $$

(3.5)

Denoting $s \equiv \text{sgn}(W)$, hence $|W| \equiv sW$, the equations of motion are given by

$$ \frac{\ddot{u}}{N^2} = sW, $$

(3.6)

$$ \frac{1}{N} \frac{d}{dt} \left( \frac{\dot{u}}{N} \right) = s \partial_u W, $$

(3.7)

$$ \frac{1}{N} \frac{d}{dt} \left( \frac{\dot{v}}{N} \right) = s \partial_v W. $$

(3.8)

This form is suitable for numerical methods as well. One may of course replace $s\partial W$ by $\partial|W|$.

Here, a further remark on the junction conditions is in place. Recently, it was claimed that the original relaxation of jump conditions as advocated in Ref. [7] does not correctly take into account the distributional nature of Einstein’s field equations (see e.g. Ref. [14] and the first of Ref. [13]). Hence, in the strict sense, Einstein’s equations are not satisfied at the hypersurfaces of signature change. Thereby we denote by ”Einstein’s equations” those derived from the action (2.2) with the upper sign for both signature types. In the minisuperspace model under consideration, this would give rise to a Lagrangian proportional to $-s\dot{u}\dot{v}/N - NW$, instead of (3.3). However, there is another possibility to obtain a model for signature change, namely by inserting an additional minus sign for the Euclidean case. This is indicated by the double sign in (2.2) and leads to (3.3), hence to a Lagrangian proportional to $-\dot{u}\dot{v}/N - sNW$. The difference between these two approaches at the level of the field equations consists essentially of $\dot{s}$-terms, hence $\delta$-distributions on the hypersurface of signature change. As is clear from (3.6-3.8), such terms do not arise in the second approach. Thus we end up with a model described by the action (3.3), implying the junction conditions developed and exploited in Refs.[7]–[8].

This line of reasoning is not restricted to the FRW model we consider here, but follows a general pattern. In Ref. [11], the junction conditions demanding continuity of the extrinsic curvature were denoted as the weak ones and have been studied
in the context of a classification of possible covariant action integrals for signature change. The general justification of such a model arises from the fact that the Einstein-Hilbert action for the Euclidean part of the space-"time" manifold may in general be assumed to have either the same or the opposite sign as compared to the Lorentzian Einstein-Hilbert action,

\[ S \sim \pm \int_{\mathcal{M}_{\text{eucl}}} d^n x \sqrt{|g|} R - \int_{\mathcal{M}_{\text{lor}}} d^n x \sqrt{|g|} R. \]  

Both of these two models are thus defined by generally covariant actions, and since we do not know the "correct" sign in front of the Euclidean Ricci scalar, they are à priori of equal right. (Some subtleties like how to integrate across the hypersurface of signature change, and the possible inclusion of boundary integrals are discussed in detail in Ref. [9]). The first possibility (equal signs) leads to strong signature change, the second one (opposite signs) leads to weak signature change, and it is this second one which we choose as the underlying theory.

We will now show that the weak signature change approach as formulated above leads to a new type of compactification mechanism. Begin with a Euclidean trajectory that is supposed to describe the "birth" of the very early universe. (It need not represent a regular ten-geometry in this context, but it will at least be reasonable to let it start from the origin \( u = v = 0 \)). The trajectory eventually approaches the curve \( W = 0 \), where it is matched to a Lorentzian solution according to our prescription. As mentioned above, it may happen that this trajectory does not escape towards arbitrarily large nine-space volumes but re-approaches \( W = 0 \) and thus re-enters the Euclidean domain. Instead of recollapsing (as is usually assumed), the trajectory becomes Euclidean there by signature change, and will in turn re-appear in the Lorentzian region, where it becomes Lorentzian again. This process may continue indefinitely. In summary, it consists of assuming the trajectory to be of Lorentzian type in the Lorentzian region, and of Euclidean type in the Euclidean region. In the generic case, the resulting trajectory "oscillates" in signature type and remains always near one of the two branches (3.1–3.2) of \( W = 0 \). The corresponding ten-metric (consisting of pieces (2.1)) undergoes an infinite sequence of signature changes.

We recall that the two branches of the zero potential curve correspond asymptotically to constant values of one of the scale factors. Approximately half of all mixed trajectories oscillate around the lower branch, and thus undergo a "time" evolution \( a_1(t) \to \infty, a_2(t) \to \ell_A \sqrt{15} \). For these solutions, \( S^6 \) compactifies to the
scale set by the cosmological constant \( \ell_\Lambda \equiv \Lambda^{-1/2} \). Alternatively (though physically less desirable), \( S^3 \) may compactify as well.

Fig. 1 presents an example of a mixed trajectory describing compactification of \( S^6 \). It emerges from the origin as \( u^{-1/2} v \to 0.5 \) (corresponding to a regular ten-geometry) and has been evolved numerically in the gauge \( N = 1 \). The dashed curve is \( W = 0 \). Fig. 2 shows the dependence of the two scale factors \( a_1, a_2 \) on the "time" parameter \( t \) (the unit on the vertical axis is \( \ell_\Lambda \)). In Fig. 3, the graph of the function \( a_3(t) \) is displayed using a finer resolution. This numerical solution suggests that \( a_1(t) \) oscillates around some linearly growing average, whereas \( a_2(t) \) performs damped oscillations around its limiting value \( \ell_\Lambda \sqrt{15} \).

The stability of this compactification scheme is evident, since small perturbations in the scale factors do not alter the qualitative behaviour of mixed trajectories. This is due to the fact that the configurations we are talking about do not provide isolated solutions. In other words, given a reasonable measure on the set of all possible initial (Euclidean) trajectories that emerge from the origin, compactification of either scale factor will occur with finite probability.

However nice these pictures look, the most important questions remain to be answered: What is the (effective, i.e. physically measurable) metric, and why do we experience it as a Lorentzian rather than a Euclidean one? What is the evolution of the large scale factor in terms of a physical time coordinate? These questions can be answered indeed, and the next Section is devoted to them.

## 4 Effective space-time metric

The metric described by a mixed oscillating trajectory may be expressed in terms of the "proper time" gauge \( N = 1 \) (the corresponding coordinate being denoted \( \tau \)). It consists of pieces (2.1) with alternating signs. Let us write such a metric as

\[
ds^2 = -s(\tau)d\tau^2 + a_1(\tau)^2d\sigma_3^2 + a_2(\tau)^2d\sigma_6^2,
\]

where \( s \equiv \text{sgn}(W) = \pm 1 \). (The coordinate \( \tau \) is identical to what is called \( \sigma \) in Ref. [8].) Let the signature change occur at values \( \tau_j \) \((j = 1, 2, 3...)\) of the time parameter. We will adopt the convention that the Lorentzian intervals are given by \( \tau_{j-1} < \tau < \tau_j \) for even integers \( j \). Hence, \( \Delta \tau_j = \tau_j - \tau_{j-1} \) is a Lorentzian type parameter time interval if \( j \) is even, and a Euclidean type interval if \( j \) is odd.
Since the size of these intervals is of the order $\ell_\Lambda$ at the beginning (one usually assumes $\ell_\Lambda \approx \text{compactification radius} > o \approx \ell_P$), we expect some average over (4.1) to yield the effective physical space-time metric. The most natural measure for such an average procedure is provided by the proper "time" coordinate $\tau$ as used in (4.1) and its intervals $\Delta \tau_j$, although what follows is quite robust against the use of a different time coordinate. Hence, we assume the effective metric for large $\tau$ to be

$$ds^2_{\text{eff}} = g^0_{00}(\tau) d\tau^2 + a_1(\tau)^2 d\sigma_3^2 + a_2(\tau)^2 d\sigma_6^2$$

(4.2)

with $a_2^\text{eff}(\tau) = \ell_\Lambda \sqrt{15} \equiv (15/\Lambda)^{1/2}$ and

$$g^\text{eff}_{00}(\tau) = - \frac{\sum (-)^j \Delta \tau_j}{\sum \Delta \tau_j},$$

(4.3)

where the sum is over a large number of intervals located near $\tau$. Due to the qualitative behaviour of the mixed trajectory and the scale factors as displayed in Fig.1 and Fig.2, one would expect the Lorentzian and the Euclidean contributions to $g^\text{eff}_{00}$ to be of equal size and thus to cancel, giving $g^\text{eff}_{00} = 0$. This is however only true in the limit $\tau \to \infty$, and we will show in the following that the actual behaviour of the metric is $g^\text{eff}_{00} \sim \tau^{-2}$ for large $\tau$.

In order to apply analytic methods to some relevant order, we consider a mixed oscillating trajectory that has already evolved along the lower branch of $W = 0$ into the region with large $a_1$. (A similar computation is of course possible for the upper branch, in which case $a_1$ compactifies). As mentioned in Sect.2, the dynamics is well described by $W^\text{approx}$ from (2.16). The structure of this approximation is most easily exhibited by performing a change of variables of the type (2.15). Let us call the new variables $(x, y, \tilde{N})$ and set

$$u = x^{2/3}, \quad v = y^{1/3}.$$  

(4.4)

Furthermore, we define

$$\frac{y}{x} = \left(\frac{v}{u^{1/2}}\right)^3 \equiv \zeta \equiv \tilde{z}^6.$$  

(4.5)

The potential arising is given by

$$\tilde{W} = W^\text{approx} \frac{du\,dv}{dx\,dy} = \frac{2}{9} \left(-\zeta^{-2/3} + 2\zeta^{-1/2}\right) \equiv \tilde{W}(\zeta).$$  

(4.6)
The new lapse variable $\tilde{N}$ relates to the previous ones by
\[ N = \frac{3}{\Lambda a_2} \tilde{N} = \sqrt{\frac{3}{\Lambda}} \zeta^{-1/12} N = \frac{2}{9} \sqrt{\frac{3}{\Lambda}} x^{-1} \zeta^{-3/4} \tilde{N}, \tag{4.7} \]
and for convenience we note the transformation formulae
\begin{align*}
a_1 &= \sqrt{\frac{6}{\Lambda}} u^{5/8} v^{-1/4} = \sqrt{\frac{6}{\Lambda}} x^{1/3} \zeta^{-1/12} = \sqrt{\frac{6}{\Lambda}} x^{1/3} z^{-1/2}, \\ a_2 &= \sqrt{\frac{30}{\Lambda}} u^{-1/8} v^{1/4} = \sqrt{\frac{30}{\Lambda}} \zeta^{1/12} = \sqrt{\frac{30}{\Lambda}} z^{1/2}. \tag{4.8} \end{align*}

The sign of $\tilde{W}$ translates into $s = \text{sgn}(\zeta - \zeta_0) = \text{sgn}(z - z_0)$, with $\zeta_0 = 1/64$, $z_0 = 1/2$ representing the zero potential curve as well as the limiting value of $a_2$.

The action and the equations of motion are now of the type (3.5-3.8) with $u \to x$, $v \to y$, and $W \to \tilde{W}$ replaced. The fact that $\tilde{W}$ depends only on the ratio $y/x$ (and is thus homogenous of degree zero) corresponds to the fact that the absolute scale of $a_1$ is free to rescalings (cf. the remarks at the end of Sect.2). Due to this symmetry, the equations of motion simplify if we choose the gauge $\tilde{N} = x^{1/3} z^{3/4}$. This corresponds to $N = (2/9)(3/\Lambda)^{1/2}$, which implies
\[ \tau = \frac{2}{9} \sqrt{\frac{3}{\Lambda}} t. \tag{4.10} \]

For convenience, we will work with $t$ instead of $\tau$ ($t_j$ and $\Delta t_j$ being defined in an obvious way).

It turns out that, when written in terms of $x$ and $z$ as independent variables, the equations of motion contain $\dot{x}$ but not $x$. Setting
\[ \sigma = \frac{\dot{x}}{x}, \tag{4.11} \]
and performing some algebra, they take the form
\begin{align*}
\dot{z} &= -\frac{1}{6} z \sigma + \frac{s}{27 \sigma} (2z - 1), \\ \dot{\sigma} &= -\frac{3}{4} \sigma^2 + \frac{s}{54} \left( 6 - \frac{1}{z} \right), \\ \ddot{z} &= -\frac{\dot{z}^2}{2z} - \sigma \dot{z} + \frac{4s}{81} \left( \frac{3z}{2} - 1 \right). \tag{4.14} \end{align*}
The structure of the equations has now changed: (4.12,4.13) define a two-dimensional dynamical system, whereas (4.14) is a consequence thereof and may be omitted. In numerical computations, the use of equations (4.13,4.14) which are not singular at \( \sigma = 0 \) may turn out to be more appropriate. In this case, (4.12) gives the initial value for \( \dot{z} \) if those of \( z \) and \( \sigma \) are prescribed. The appearance of such reductions in simple cosmological models is well known [15].

Since (4.12-4.14) describe escaping trajectories as well as oscillating ones, one has to prescribe appropriate initial values at some time \( t_{\text{ini}} \). In Fig.4, an example (with \( z(0) = 0.6, \sigma(0) = 0.31 \)) is displayed. The dashed line represents \( \sigma(t) \), the solid one shows (using a magnification factor of 5 in order to keep common units on the vertical axis) the function \( 5(z(t) - 1/2) \). During the Lorentzian periods, \( \sigma \) increases, during the Euclidean periods, it decreases to zero. Its maxima decrease and converge to zero for \( t \to \infty \). \( z \) performs damped oscillations around its limiting value \( z_0 = 1/2 \). Equation (4.13) tells us that \( \sigma(t) \) tends to a zigzag curve with slope \( \pm 2/27 \).

Our main concern is the estimation of the "time" intervals \( \Delta t_j \) and the long-time behaviour of \( a_1(t) \). Let us solve the equations of motion near some value \( t = t_{\text{ini}} \) that is large enough for our approximation to hold. The most convenient choice is \( t_{\text{ini}} \equiv t_j \) for some odd integer \( j \). In other words, we place the initial time at the end of a Euclidean interval (at the beginning of a Lorentzian one). The interval \( I_j \), defined by \( t_{j-1} < t < t_{j+1} \), is understood as a "pair" of two periods of the type (eucl,lor), and it is followed by another pair \( t_{j+1} < t < t_{j+3} \) of the same type, and so on. A power series ansatz reveals that \( q \equiv \dot{z}(t_j) \) is the only free parameter (\( z = 1/2 \) and \( \sigma = 0 \) there). Within the interval \( I_j \), we may use \( s \equiv \text{sgn}(t-t_j) \). Denoting \( \bar{t} \equiv t-t_j \), the solution reads, to within the order that it necessary for the applications we have in mind,

\[
\begin{align*}
z(t) &= \frac{1}{2} + q \bar{t} + A \bar{t}^2 + B \bar{t}^3 + D \bar{t}^4 + \ldots, \\
\sigma(t) &= \frac{2s}{27} \bar{t} + \frac{sq}{27} \bar{t}^2 + E \bar{t}^3 + F \bar{t}^4 + \ldots,
\end{align*}
\]

where

\[
\begin{align*}
A &= -\frac{1}{162} (s + 81 q^2), \\
B &= \frac{1}{243} (sq + 162 q^3),
\end{align*}
\]
\[ D = \frac{1}{78732} (2 - 729 sq^2 - 91854 q^4), \]
\[ E = -\frac{5}{6561} (2 + 81 sq^2), \]
\[ F = -\frac{1}{26244} (13 q - 3240 sq^3). \]

Recall that this solutions applies only in the interval \( I_j \). Clearly, in the succeeding interval \( I_{j+2} \), which is again of the type (eucl,lor), a similar solution applies (with \( q \to q' \equiv \dot{z}(t_{j+2}) \) and \( t \to t - t_{j+2} \) replaced).

The size of the two periods that make up \( I_j \) may now be computed as the zeros of \( z(t) - 1/2 \) as given by (4.15). For the sake of notational ease, we will use the equality sign to indicate the first relevant order(s) of various quantities. It turns out that

\[
\Delta t_{\text{eucl}} = \Delta t_j = 162 q - 21870 q^3, \quad (4.17) \\
\Delta t_{\text{lor}} = \Delta t_{j+1} = 162 q + 21870 q^3. \quad (4.18)
\]

This shows a small amount of asymmetry between the Lorentzian and the Euclidean interval sizes that will be important below.

The period sizes inside the interval \( I_{j+2} \) shall be denoted as \( (\Delta t'_{\text{eucl}}, \Delta t'_{\text{lor}}) \). Using (4.13), one finds

\[ \dot{z}(t_{j\pm}) = -q \pm 324 q^3. \quad (4.19) \]

This provides a characterization of the Taylor expansions we used: The series are truncated at an order such as to reproduce (4.17)–(4.19) correctly, where it is understood that \( O(q^4) \)-contributions are neglected. Thus, to the order considered,

\[ q' = q - 648 q^3 \quad (4.20) \]

plays exactly the same role for the interval \( I_{j+2} \) as \( q \) does for the interval \( I_j \) (recall \( q' \equiv \dot{z}(t_{j+2}) \)). Repeating (4.17,4.18) for this new interval, we obtain the period sizes in terms of \( q' \):

\[
\Delta t'_{\text{eucl}} = \Delta t_{j+2} = 162 q - 126846 q^3, \quad (4.21) \\
\Delta t'_{\text{lor}} = \Delta t_{j+3} = 162 q - 83106 q^3. \quad (4.22)
\]
These results provide enough information to answer all the questions posed so far. Denoting \( q_j \equiv \dot{z}(t_j) \), equation (4.20) tells us
\[
q_{j+2} - q_j = -648 q_j^3.
\] (4.23)

For large \( j \), we can rewrite this as the differential equation \( dq/dj = -324q^3 \) and obtain to leading order
\[
q_j = \frac{1}{18\sqrt{2j}}.
\] (4.24)

This implies that the period sizes are
\[
\Delta t_j = \frac{9}{\sqrt{2j}}.
\] (4.25)

We will add a next-order correction to this expression later on. These numbers can be summed over to provide (to leading order) the value \( t_j \) corresponding to the \( j \)-th signature change,
\[
t_j = \frac{9}{\sqrt{2}} \sum_{k=1}^{j} k^{-1/2} = 9\sqrt{2j}.
\] (4.26)

Note that the deviations from (4.25) that stem from the early evolution where the approximation \( W \approx W_{\text{approx}} \) is not valid, contribute at most an additive constant to \( t_j \), and thus are irrelevant.

An immediate consequence of this is (using the zigzag limit of \( \sigma(t) \) inside each interval \( I_j \))
\[
\frac{1}{3} \int_0^{t_j} dt' \sigma(t') = \frac{1}{81} \sum_{k=1}^{j} \Delta t_k^2 = \frac{1}{2} \sum_{k=1}^{j} \frac{1}{k} = \frac{1}{2} \ln j = \frac{1}{2} \ln t^2 = \ln t.
\] (4.27)

Hence, with (4.11), we obtain the long-time behaviour \( x(t) \sim t^3 \). Inserting this into (4.8) and letting \( z \to 1/2 \), we find for large \( t \)
\[
a_1(t) = \tilde{C}t,
\] (4.28)

where \( \tilde{C} \) is a constant. This has been already anticipated in the previous Section.

The most important result, however, is contained in the equations (4.17,4.18) and (4.21,4.22). They provide a subtle pattern of small perturbations to (4.25).
Looking at the different signs showing up in these four expressions, one may recast the general behavior into the form

$$\Delta t_j = \frac{9}{\sqrt{2}j} + (-)^j \frac{33}{8\sqrt{2}j^{3/2}}. \quad (4.29)$$

This equation, along with (4.24) and (4.26), is valid for odd as well as for even integers \(j\). It displays a small predominance of the Lorentzian period sizes (\(j\) even) over the Euclidean ones (\(j\) odd).

As a last application of this machinery, we may compute the amplitudes of the oscillations. Denoting by \(\delta\) the absolute value of the maximal deviations of a quantity from its long-time limit during one period, we find to leading order

$$\delta \sigma = \frac{2}{27} \Delta t = \frac{1}{3} \sqrt{\frac{7}{j}} = \frac{6}{t},$$

$$\delta z = \frac{1}{6} \delta x = \frac{1}{3} \frac{\delta a_1}{a_1} = \frac{\delta a_2}{a_2} = \frac{81}{2} \frac{j^2}{16j} = \frac{81}{16t^2}.$$ \quad (4.31)

This shows the amount of damping that occurs to the various quantities.

Having evaluated the necessary ingredients, we return to the computation of the effective space-time metric (4.12), which amounts to perform some average procedure of the type (4.3) to (4.29). Again, we encounter some subtleties. Naively, one would expect to obtain \(g_{00}^{\text{eff}}\) as an average over two neighbouring time periods. Let us try this for a pair \((\Delta t_{j-1}, \Delta t_j)\), with \(j\) of either parity. The prescription (4.3) leads to

$$- \frac{(-)^j \Delta t_j + (-)^{j-1} \Delta t_{j-1}}{\Delta t_j + \Delta t_{j-1}} = \frac{1}{j} \left( \frac{(-)^j}{4} - \frac{11}{24} \right). \quad (4.32)$$

Hence, an average over a pair of periods of type (eucl,lor) gives a result different from an average over (lor,eucl). However, since to first order the neighbouring \(\Delta t\)'s are of equal size, we average over (4.32) for two neighbouring \(j\)'s (i.e. an even and an odd one). The result is \(-11/(24j) = -297/(4t^2)\), which is negative and falls off to zero as \(t \to \infty\). Taking into account the factor (4.10) between \(t\) and \(\tau\), the effective metric becomes

$$ds_{\text{eff}}^2 = -\frac{11}{\Lambda} \frac{dt^2}{\tau^2} + C^2 \tau^2 d\sigma_3^2 + \frac{15}{\Lambda} d\sigma_6^2.$$ \quad (4.33)
The constant $C$ may be set to any given value by a trivial rescaling of $\tau$. (If the formalis
m based on $\tilde{W}$ that we used in this Section is interpreted as the exact version of a $T^3 \times S^6$ or $R^3 \times S^6$ model, then the origin of such a freedom is clear. If, on the other hand, this formalism is considered as a large-$a_1$ approxima-
tion of the $S^3 \times S^6$ model we started with, (4.33) shows that there is enough loss of informa-
tion about the initial trajectory to allow for such a rescaling as well.) A further transfor-
mation to the effective (physical) proper time

$$\sqrt{\frac{11}{\Lambda}} \ln \left( \frac{\tau}{\tau_0} \right) = \eta, \quad (4.34)$$

with $\tau_0$ an arbitrary constant, gives

$$ds^2_{\text{eff}} = -d\eta^2 + \frac{1}{\Lambda} \exp \left( 2 \sqrt{\frac{\Lambda}{11}} \eta \right) d\sigma_3^2 + 15 \Lambda d\sigma_6^2, \quad (4.35)$$

where $\tau_0$ has been used to produce a nice prefactor of the exponential. (Note that by $\eta \rightarrow \eta + \text{const}$, this prefactor may be rescaled to any other value). The effective

four-metric is obtained by omitting the $d\sigma_6^2$-term. Thus, the small predominance of the Lorentzian over the Euclidean periods as described by (4.29) leads to an effective Lorentzian (i.e. Pseudo-Riemannian) metric displaying exponential inflation in the remaining scale factor $a_1$. This is our main result. As a particularly nice feature typical for inflation we observe that the metric is unique, hence independent of the initial values of the trajectory.

Let us finally write down the physical time values at which signature change

occurs,

$$\eta_j = \frac{1}{2} \sqrt{\frac{11}{\Lambda}} \ln j, \quad (4.36)$$

and the corresponding period sizes

$$\Delta \eta_j = \frac{1}{2j} \sqrt{\frac{11}{\Lambda}} = 81 \sqrt{\frac{11}{\Lambda}} \exp \left( -2 \sqrt{\frac{\Lambda}{11}} \eta_j \right). \quad (4.37)$$

This concludes the presentation of the signature change model of compactification.

The remaining two Sections are devoted to a discussion of open questions, specula-
tions and concluding remarks.
5 Physical problems

Although the effective metric (4.35) looks appealing, the way it has been obtained appears somewhat \textit{at hoc}. The two essential steps in our argumentation are on the one hand the assumption that signature change occurs whenever it is possible, and on the other hand the prescription (4.3) that amounts to perform some average over the true metric in order to get the effective one. A possible point of view about this procedure is that what we have done is effectively some approximation or limiting case to a different theory, possibly connected with some version of quantum gravity. One should keep in mind that our first assumption (the actual occurrence of signature change) raises the issue of the dynamical interplay between Lorentzian and Euclidean geometries, whereas the second one (the average procedure) seems to mimic some coarse graining (though at a classical level). Both of these topics are under discussion in recent quantum cosmology as well. There has even been a proposal of a quantum theory of what we called ”classical” signature change [16] (that leads to a modified Wheeler-DeWitt equation). As already mentioned in Ref. [11], mixed trajectories in superspace might give rise to semiclassical states in such a theory. The effective metric (4.35) would then probably be a prediction based on a (non-standard) theory of quantum cosmology (at least for some proper time interval after which the notion of classical signature oscillations might completely break down – see below). Regrettably, we have to leave these very fundamental questions open.

Examining the predictive power of the mechanism we presented, we observe that the metric (4.35) – although being unique and nicely describing an inflationary universe – does not provide a hint how inflation eventually would stop, and how the evolution of the large scale factor $a_1$ would become a standard (e.g. radiation or matter dominated) one. Usually, one expects this sort of problem in a theory containing a positive cosmological constant. In the standard approaches [17]–[18], $\Lambda$ mimicks the potential value $V(\phi_0)$ of some initial scalar field $\phi_0$. An exit out of inflation is provided by the actual dynamics of $\phi$: As $V(\phi)$ decreases, the cosmological constant ”decays”. Such a procedure does not seem to work in the model we are considering. The reason is that the compactification scale $a_2 \sim \ell_\Lambda \equiv \Lambda^{-1/2}$ would blow up as $\Lambda \to 0$, and the main purpose of the mechanism would be lost. In other words, in order to retain compactification for all times, the cosmological constant should be fundamental and not just a convenient way to mimick matter in the early stages of the evolution. The problem is here to exit inflation without switching off
Let us list some speculations how it might be overcome.

From the point of view of any quantum theory of gravity, the appearance of classical time periods below the Planck scale are problematic. However, this is precisely what happens in our model. The time interval between two signature changes drop down to zero quite rapidly. This is independent on whether the characteristic scale is expressed in terms of the underlying "microscopic proper time" \( \tau \)

\[
\Delta \tau_j = \frac{12 \ell_P^2}{\tau_j}
\]  

(which follows from (4.25, 4.26) and (4.11)), or in terms of the physical proper time \( \eta \), as in (4.31). Even if \( \ell_\Lambda \) exceeds \( \ell_P \) by some orders of magnitude, the period sizes \( \Delta \tau \) become Planckian at \( \tau \approx \ell_\Lambda^2/\ell_P \), the physical time intervals \( \Delta \eta \) become Planckian at \( \eta \approx \ell_\Lambda \ln(\ell_\Lambda/\ell_P) \). One thus expects quantum gravity effects to become important at some stage of the inflationary evolution (4.35). These would certainly modify the model in the sense that the classical oscillating trajectories are replaced by some prescription how to compute expectation values. It is not even clear to what extent WKB techniques would apply, because it is conceivable (especially if \( \ell_\Lambda \approx \ell_P \)) that the universe is permanently in some full (non-WKB) quantum state as far as \( a_2 \) is concerned, and only the observable \( a_1 \) behaves classical. However, one might expect the classical arguments leading to (4.28) to break down or to become strongly modified so that inflation is eventually stopped by quantum effects. This touches upon the fundamental question how physical time is "created" from a quantum state \([19]\). In other words: the way physical time was constructed in the previous Section (by averaging with respect to a "microscopic" oscillating-signature type parameter \( \tau \)) might break down as the oscillation periods fall below Planck scale. In addition, a realistic model would contain matter fields. For example, in the case of a single scalar field, the total cosmological constant could be a sum

\[
\Lambda_{\text{tot}} = \Lambda_{\text{fund}} + V(\phi)
\]  

(5.2)

of a fundamental and a "decaying" part. Only \( \Lambda_{\text{fund}} \) would survive and guarantee compactification, while the scalar field would produce density fluctuations and matter particles. Whether such a modification may be constructed, and whether the resulting effective four-metric produces an amount of inflation and structure that is compatible with observational constraints must be left for future research.

We can add a technical remark that is related to the problem of constructing semiclassical states around mixed trajectories, and that has a purely classical counterpart, too. Usually, one expects a one-dimensional family of trajectories emerging
from a common point in minisuperspace (the origin, say) to allow for a description in terms of the classical action $S$ (integrated along the trajectories) and a set of Hamilton-Jacobi equations. In the gauge $N = 1$, these are

$$\dot{u} = -\partial_{v} S, \quad \dot{v} = -\partial_{u} S. \tag{5.3}$$

However, such a formulation is only possible if the different trajectories do not intersect each other. In our model, the trajectories actually do intersect. One can show that this situation generates an infinite sequence of functions $S_k$, each describing only a subset of the family of solutions by means of (5.3) in a restricted domain of minisuperspace. In a sloppy manner, one can imagine $S_k$ to be an evaluation of (3.5) along a trajectory having undergone $k$ oscillations already. This raises the question how the procedure to assign a semiclassical state $\psi \sim \exp(iS)$ to a family of classical trajectories is modified. If, by construction, such a state is based on the set of partial actions $\{S_k\}$, one might speculate that some ”damping” effect alters the effective dynamics of the large scale factor and hence cures the problem of eternal inflation.

There is, however, a chance to obtain a classical, non-inflationary long-time behavior even in a rather standard manner, namely by introducing additional matter fields and couplings, or by incorporating a phenomenological (perfect fluid type) energy momentum tensor obeying some equation of state. It is conceivable that a perturbation of the dynamical system (4.12-4.14) along these lines alters the amplitudes of the various quantities during the oscillations. The changes required are not drastic: Retaining, for example, the relation between $t$, $\tau$ and the physical proper time $\eta$, one would need $a_1 \sim (\ln t)^{\kappa}$ as $t \to \infty$ instead of (4.28). If $0 < \kappa < 1$, it follows $a_1 \sim \eta^\kappa$, hence non-inflationary cosmic expansion. This may in turn be achieved if the amplitudes of the oscillations in $\sigma$ behave like $\delta \sigma = 6\kappa(t \ln t)^{-1}$ instead of $\delta \sigma = 6t^{-1}$, as given by (4.30). In other words, any additional coupling that increases the damping effects on $\sigma$ works against inflation.

There is another possible way out of inflation that even gives up $\Lambda$ as a fundamental constant: We have implicitly assumed that the effective four-metric is just the averaged truncated version of the full ten-metric. This is not necessarily the case. One may embed the model into a theory in which the true effective metric is given by $d\sigma^2_{\text{eff}}$ (as evaluated in the previous Section) only up to a conformal factor,

$$d\sigma^2_{\text{true}} = \Omega(\eta)^2 d\sigma^2_{\text{eff}}. \tag{5.4}$$
One often encounters such a situation in the dimensional reduction of supergravity
and superstring theories (see e.g. Ref. [20] for an article explaining the rescaling of
various ”metrics” – as appearing in supergravity – in great detail, and Ref. [21] for
a bulk of material on superstrings). The conformal factor is usually related to some
scalar field (as e.g. a dilaton). Moreover, assume \( \Lambda \) to be an effective energy of some
field, typically \( V(\phi) \), such that it decays slowly (“adiabatically”). We indicate this
by writing \( \Lambda = \Lambda(\tau) \). Using \( ds^2_{\text{eff}} \) in the form (4.33), compactification is retained
if \( \Omega(\tau) \sim \Lambda(\tau)^{-1/2} \). The type of three-space expansion is then determined by the
”constant” \( C \). Although completely arbitrary in the large-\( a_1 \) approximation we used,
it may to an even higher approximation be related to \( \Lambda(\tau) \) and the initial values of
the trajectory. As a possible way out of eternal inflation, one could try to adjust
the field contents and the couplings of the theory such that \( C \) becomes effectively a
decreasing function of \( \tau \). In the case such a scheme exists, it will certainly provide
a strong constraint on the structure of the underlying theory.

Let us conclude this Section by noting that the signature change model of com-
pactification might possibly provide a new mechanism for creating density fluctua-
tions. In the FRW ansatz (2.1) we have neglected small inhomogeneities. However,
if \( S^3 \) is slightly distorted, one would expect local versions of our dynamical variables
to oscillate in different points with slightly different amplitudes and phases. This
should in turn have some imprint on the resulting effective metric (and the matter
density in the case additional fields are present). To which extent such perturbations
are wiped out by the expansion is yet another interesting question to be pursued.

6 Concluding remarks

In Sect.4 we have worked out the effective metric for the case that the mixed tra-
jectory oscillates along the lower branch of the \( W = 0 \) curve. This led to the
compactification of \( S^6 \). One may of course reverse the situation and consider a
mixed trajectory moving near the upper branch, thus describing compactification of
\( S^3 \) (i.e. \( a_1 \to \ell_A \sqrt{3} \), while \( a_2 \to \infty \)). Neglecting – in analogy with the previous case
– the curvature contributions of \( S^6 \), amounts to omit the \( u^{3/2} \)-term in the potential
(2.11). After a change of variables of the type (2.15), namely

\[
\begin{align*}
u &= X^{1/3}, \\
v &= Y^{5/6}
\end{align*}
\]  

23
(cf. equation \((4.4)\)), we arrive at a potential

\[ W = \frac{5}{18} \left( -\xi^{-2/3} + 2\xi^{-1/4} \right) \]  

(6.2)

with \(\xi \equiv X/Y\) (cf. equation \((4.3)\)). This is very similar in structure to \((4.0)\), and without having done the computation in detail, we expect that this case yields a result analogous to \((1.3)\). Thus, in total, letting the initial (Euclidean) trajectory starting from the origin, there are three generic possibilities for the long-time evolution: compactification of either scale factor and escape into the Lorentzian region. In this last case, both scale factors expand exponentially. In addition, there are two isolated solutions (namely if the initial trajectory coincides with one of the instantons). Assuming a reasonable measure on the set of initial conditions, we expect the three generic cases to occur with comparable finite probabilities. Computing such probabilities in detail (possibly on the basis of a path integral formulation suggesting a suitable measure) would reveal whether a four-dimensional effective space-time is favoured over a seven-dimensional one.

Let us finally comment on the question of the dimensions that may be put into a signature change model of compactification from the outset. A natural generalization of the version we were dealing with in this paper is to consider a FRW model based on \(S^m \times S^n\) as spatial sections. The most important thing to notice in this context is that a "branch" of the zero potential curve is lost if the curvature of some factor space is zero. In a \(S \times S^n\) (or, more general, a \(T^m \times S^n\) or \(R^m \times S^n\)) model with \(n > 1\), only \(S^n\) can compactify by signature change. In this sense, curvature acts as an "attractive force" upon the corresponding scale factors. One might try to formulate a statement like: Signature change compactifies and stabilizes only internal spaces with non-vanishing Ricci-curvature.

Another observation in this context is that – when trying to express the action in terms of "lightlike" variables along the lines of \((2.10)\) – some ugly numerics appears. In the general case, the potential is of the structure

\[ W = \sum_{i=1}^{3} c_i u^{\alpha_i} v^{\beta_i}, \]  

(6.3)

up to transformations of the type \((2.13)\). Among all combinations \(m \leq n < 10\) we find only for \((m, n) = (2, 8), (3, 6), (5, 5), (6, 10)\) and \((7, 8)\) that \(\alpha_i\) and \(\beta_i\) are rational numbers. As an example, for \((m, n) = (2, 3)\), a simplification of the potential similar
to (2.10) produces the exponents

\[\alpha_1 = 0, \quad \alpha_2 = 1 + 2\sqrt{\frac{2}{3}}, \quad \alpha_3 = 1 + \sqrt{\frac{3}{2}},\]

\[\beta_1 = \frac{1}{2} + \sqrt{\frac{3}{2}}, \quad \beta_2 = 0, \quad \beta_3 = \frac{1}{2} \alpha_3.\]  

(6.4)

In general, one encounters the square roots of \(m(m + n - 1)/n\) and \(n(m + n - 1)/m\). Nevertheless, the overall structure of the potential is comparable to (2.11), and we expect analogous effects to arise here as well. Whether the ”beautiful” cases mentioned above are distinguished from the others, and what we can learn from structures like (6.4) about the possibility and physics of signature change induced compactification in various dimensions are again problems that deserve further study.

References

[1] D. Bailin and A. Love, ”Kaluza-Klein theories”, Rep. Prog. Phys. 50, 1087 (1987).

[2] P. G. O. Freund, ”Kaluza-Klein Cosmologies”, Nucl. Phys. B 209, 146 (1982);

A. Chodos and S. Detweiler, ”Where has the fifth dimension gone?”, Phys. Rev. D 21, 2167 (1980);

R. A. Matzner and A. Mezzacappa, ”Professor Wheeler and the Crack of Doom: Closed Cosmologies in the 5-d Kaluza-Klein Theory”, Found. Phys. 16, 227 (1986);

D. Sahdev, ”Towards a realistic Kaluza-Klein cosmology”, Phys. Lett. 137 B, 155 (1984);
R. B. Abbott, S. M. Barr and S. D. Ellis, "Kaluza-Klein cosmologies and inflation", *Phys. Rev. D* 30, 720 (1984);

E. W. Kolb, D. Lindley and D. Seckel, "More dimensions – Less entropy", *Phys. Rev. D* 30, 1205 (1984);

D. Sahdev, "Perfect fluid higher-dimensional cosmologies", *Phys. Rev. D* 30, 2495 (1984);

M. Yoshimura, "Effective action and cosmological evolution of scale factors in higher-dimensional curved space", *Phys. Rev. D* 30, 344 (1984);

K. Maeda, "Stability and attractor in a higher-dimensional cosmology: I,II", *Class. Quantum Grav.* 3, 233 (1986); *Class. Quantum Grav.* 3, 651 (1986);

O. Bertolami, Yu. A. Kubyshin and J. M. Mourão, "Stability of compactification in Einstein-Yang-Mills theories after inflation", *Phys. Rev. D* 45, 3405 (1992).

O. Bertolami, J. M. Mourão and Yu. A. Kubyshin, "On the stability of compactification after inflation", in: H. Sato and T. Nakamura (eds.), *Proceedings of the 6th Marcel Grossmann Meeting 1991*, World Scientific (Singapore, 1992), p. 625.

[3] P. G. O. Freund and M. A. Rubin, "Dynamics of dimensional reduction", *Phys. Lett.* 97 B, 233 (1980).

[4] T. Appelquist and A. Chodos, "Quantum Effects in Kaluza-Klein Theories", *Phys. Rev. Lett.* 50, 141 (1983);

T. Appelquist and A. Chodos, "Quantum dynamics of Kaluza-Klein theories", *Phys. Rev. D* 28, 772 (1983);
T. Appelquist, A. Chodos and E. Myers, "Quantum instability of dimensional reduction", *Phys. Lett.* **127** B, 51 (1983);

M. A. Rubin and B. D. Roth, "Fermions and stability in five-dimensional Kaluza-Klein theory", *Phys. Lett.* **127** B, 55 (1983);

A. Chodos and E. Myers, "Gravitational Contribution to the Casimir Energy in Kaluza-Klein Theories", *Ann. Phys. (N.Y.)* **156**, 412 (1984);

A. Chodos and E. Myers, "Gravitational Casimir energy in non-Abelian Kaluza-Klein theories", *Phys. Rev. D* **31**, 3064 (1985).

[5] X. M. Hu and Z. C. Wu, "Quantum Kaluza-Klein Cosmologies (III)", *Phys. Lett.* **155** B, 237 (1985);

S. R. Lonsdale, "Wave function of the universe for N=2 6D supergravity", *Nucl. Phys. B* **175**, 312 (1986);

J. J. Halliwell, "Classical and Quantum cosmology of the Salam-Sezgin model", *Nucl. Phys. B* **286**, 729 (1987);

J. J. Halliwell, "The quantum cosmology of Einstein-Maxwell theory in six dimensions", *Nucl. Phys. B* **266**, 228 (1986).

[6] U. Carow-Watamura, T. Inami and S. Watamura, "A quantum cosmological approach to Kaluza-Klein theory and the boundary condition of 'no boundary'", *Class. Quantum Grav.* **4**, 23 (1987);

Z. C. Wu, "Space-Time is Four-Dimensional", *Gen. Relativ. Gravit.* **17**, 1217 (1985);

Z. C. Wu, "Dimension of the Universe", *Phys. Rev. D* **31**, 3079 (1985);

X. M. Hu and Z. C. Wu, "Quantum Kaluza-Klein Cosmologies (II)", *Phys. Lett.* **149** B, 87 (1984).
[7] G. Ellis, A. Sumeruk, D. Coule and C. Hellaby, "Change of signature in classical relativity", Class. Quantum Grav. 9, 1535 (1992).

[8] G. F. R. Ellis, "Covariant Change of Signature in Classical Relativity", Gen. Relativ. Gravit. 24, 1047 (1992);

T. Dereli and R. W. Tucker, "Signature dynamics in general relativity", Class. Quantum Grav. 10, 365 (1993);

R. Kerner and J. Martin, "Change of signature and topology in a five-dimensional cosmological model", Class. Quantum Grav. 10, 2111 (1993);

T. Dray, C. A. Manogue and R. W. Tucker, "Particle Production from Signature Change", Gen. Relativ. Gravit. 23, 967 (1991);

C. Hellaby and T. Dray, "Failure of standard conservation laws at a classical change of signature", Phys. Rev. D 49, 5096 (1994).

[9] F. Embacher, "Actions for signature change", University Vienna preprint UWThPh-1995-1, also preprint gr-qc/9501004.

[10] J. B. Hartle and S. W. Hawking, "Wave function of the universe", Phys. Rev. D 28, 2960 (1983).

[11] F. Embacher, "Comments on signature change and the multi-dimensional Wheeler-DeWitt equation", Talk given at the International School-Seminar "Multidimensional Gravity and Cosmology", Yaroslavl, June 1994, to appear in the proceedings, preprint gr-qc/9409016, to appear in Gravitation and Cosmology 1 (1995).

[12] G. W. Gibbons and J. B. Hartle, "Real tunneling geometries and the large-scale topology of the universe", Phys. Rev. D 42, 2458 (1990);

[13] S. A. Hayward, "Signature change in general relativity", Class. Quantum Grav. 9, 1851 (1992);
S. A. Hayward, "On cosmological isotropy, quantum cosmology and the Weyl curvature hypothesis", *Class. Quantum Grav.* **10**, L7 (1993).

[14] M. Kossowski and M. Kriele, "Smooth and discontinuous signature type change in general relativity", *Class. Quantum Grav.* **10**, 2363 (1993);

M. Kriele and J. Martin, "Black holes, cosmological singularities and change of signature", *preprint* gr-qc/9411063.

[15] D. L. Wiltshire, "Global properties of Kaluza-Klein cosmologies", *Phys. Rev. D* **36**, 1634 (1987);

M. Szydlowski and M. Biesiada, "Inflation as a dynamical effect of higher dimensions", *Phys. Rev. D* **41**, 2487 (1990).

[16] J. Martin, "Hamiltonian quantization of general relativity with the change of signature", *Phys. Rev. D* **49**, 5086 (1994).

[17] J. J. Halliwell, "Introductory lectures on quantum cosmology", *in*: S. Coleman *et. al.* (eds.), *Quantum cosmology and baby universes*, World Scientific (Singapore, 1991), p. 159.

[18] S. W. Hawking, "The quantum state of the universe", *Nucl. Phys. B* **239**, 257 (1984).

[19] A. Ashtekar and J. Stachel (eds.), *Conceptual Problems of Quantum Gravity*, Birkhäuser (Boston, 1991).

[20] E. Cremmer and B. Julia, "The SO(8) supergravity", *Nucl. Phys. B* **159**, 141 (1979).

[21] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory, Vols. 1,2*, Cambridge University Press (Cambridge, 1987).

**Figure captions:**
**Fig. 1**
A typical mixed trajectory \((u(t), v(t))\) in minisuperspace is shown. The dashed curve is \(W = 0\) (with its two "branches"). The initial condition of the trajectory near the origin is \(u^{-1/2}v \to 0.5\). The gauge condition defining \(t\) is \(N = 1\), and the evolution has been performed numerically. The first intersection of the trajectory with the zero potential curve has actually a horizontal tangent (\(\dot{v} = 0\)) – this fact is suppressed by the low resolution of the graphics. The long-time behavior, which is essential for our purposes, is illustrated: the trajectory remains near the lower branch of the dashed curve, which implies compactification of \(S^6\). The reverse situation, i.e. a trajectory oscillating around the upper branch in much the same way, is possible as well.

**Fig. 2**
This shows the graphs of the functions \(a_1(t)\) and \(a_2(t)\) corresponding to the trajectory displayed in Fig.1. The unit on the vertical axis is \(\ell_\Lambda\). The plot demonstrates that \(a_1\) behaves effectively linear in \(t\) when the universe is already large, and that \(a_2\) converges to its limiting value \(\ell_\Lambda\sqrt{15}\).

**Fig. 3**
The graph of \(a_2(t)\) is displayed using a better resolution than in Fig.2. It shows that \(a_2\) performs damped oscillations around its limit.

**Fig. 4**
This plot shows a numerical solution of the dynamical system (4.12-4.14). The dashed line is \(\sigma(t)\), the solid line represents \(5(z(t) - 1/2)\) (the magnification factor of 5 has been introduced for convenience). The initial conditions have been chosen as \(z(0) = 0.6\) and \(\sigma(0) = 0.31\), the gauge condition defining \(t\) is \(\mathcal{N} = (2/9)(3/\Lambda)^{1/2}\). The zigzag limit of \(\sigma\) as well as the damped nature of the oscillations are well illustrated. The Lorentzian periods are those for which the solid graph has positive values (or, equivalently, during which the dashed graph increases).