Complex Numbers, Quantum Mechanics and the Beginning of Time

G. W. GIBBONS

D.A.M.T.P.
University of Cambridge
Silver Street
Cambridge CB3 9EW
U.K.

H.-J. POHLE

Department of Physics
University of California
Santa Barbara, Ca 93106

January 12, 1993

Abstract

A basic problem in quantizing a field in curved space is the decomposition of the classical modes in positive and negative frequency. The decomposition is equivalent to a choice of a complex structure in the space of classical solutions. In our construction the real tunneling geometries provide the link between this complex structure and analytic properties of the classical solutions in a Riemannian section of space. This is related to the Osterwalder-Schrader approach to Euclidean field theory.
1 Introduction

In quantum gravity are seeks to unite in a harmonious whole two distinct theo-
ries, classical general relativity and quantum mechanics. It seems reasonable to
suppose that what eventually emerges will entail a substantial revision of both
theories so that both quantum mechanics and general relativity emerge merely
as approximations to some underlying exact theory. In the context of cosmology
for example the idea of a classical spacetime in which all conventional laws of
quantum mechanics hold would expected to be a good approximation at large
times.

In fact classical relativity already brings about a radical modificatio
of
the idea of time used in non-relativistic physics while compared with clas sical
theory one of the striking new features of quantum mechanics is its use of
complex structures, particularly in the description of time-evolution. Thus it
seems reasonable to expect that it is in the way that complex structures enter
quantum mechanics where modifications due to quantum gravity will arise. One
possible viewpoint is that the complex structure of the quantum mechanical
Hilbert space only emerges late in the history of the universe.

Of course in the absence of a complete theory of quantum gravity one can
only speculate about such things unless one tries to construct approximate mod-
els in which the well known divergence difficulties of quantum gravity are ig-
nored in the hope that in the more fundamental theory the divergencies are
eliminated while at the same time the qualitative features of the model at large
scales persist. If one cannot construct approximate models exhibiting certain
features at large times, it seems unlikely that such features could arise in any
more fundamental theories. If one can, strictly speaking one can say nothing
except perhaps to confess to a feeling of optimism. This seems to be the pre-
vailing view in quantum cosmology and in that spirit we are going to investigate
a simple-minded model for the quantum creation of the universe mediated by
real tunneling geometries [1].

These arise in the WKB ansatz for the solution of the Wheeler-DeWitt equa-
tion as a special case of the general situation in which one considers "complex
paths" which are spacetimes with complex metrics. In the WKB ansatz one
uses a fixed background geometry and questions about the complex structure
of quantum mechanics are equivalent to asking how the complex structure of
the quantum mechanical Hilbert space for quantum fields around a fixed space-
time is determined, quantum fields correspond to the fluctuations around that
background.
The decomposition into positive- and negative-frequency parts is related to a natural complex structure $J$. It is $J$ that determines what we mean by complex numbers in the quantum theory of a classical system. For Lorentzian spacetime this is a problem studied by Ashtekar and Magnon [2] and we have been greatly influenced by that paper.

Real tunneling geometries are partially Lorentzian and partially Riemannian. Alternatively the Lorentzian and Riemannian portion may be regarded as different real slices of a complex spacetime $M_C$, which have a common boundary $\Sigma$. This surface $\Sigma$ acts as an initial Cauchy surface for the Lorentzian spacetime and its quantum fields ($\Sigma$ might be called the "beginning of time").

Real tunneling geometries give a privileged notation of positive and negative frequency and hence a privileged "vacuum" state brought about the tunneling process. The choice of positive frequency or of a complex structure may be thought of as a direct sum decomposition of the boundary data on $\Sigma$ into data which evolve to give a solution everywhere bounded in $M^+_R$ or everywhere bounded in $M^-_R$, where $M^+_R$ and $M^-_R$ are the two halves of the Riemannian slice separated by $\Sigma$. In this way our construction associates in a very clear way the complex structure of quantum mechanics with the beginning of time.

Additionally using the ideas of Euclidean Quantum Field Theory generalized to this particular class of curved spacetime the complex structure of the quantum mechanical Hilbert space is intimately related to Osterwalder and Schrader’s use of Reflection Positivity to construct the Quantum Mechanical Hilbert Space [3], [4], [5], [6] and [7]. In that construction one considers reflections about a spacelike hypersurface. This surface corresponds in our construction to the boundary $\Sigma$.

We shall show in section 5 that this agrees with previous work in the simplest possible case - that of the deSitter spacetime. We also apply it to the Page metric which is also a Real Tunneling Geometry but one which is not as symmetric as deSitter spacetime.

In section 6 we shall also indicate how the formalism we have developed in section 5 may be applied to other tunneling geometries. In all these sections we will restrict ourselves to Klein-Gordon fields. In section 7 we treat spinors. section 8 is a conclusion.
2 Complex Structures

In order to clarify the purpose of the work described below we shall review the relation between time and the complex numbers in quantum mechanics from the particular perspective we have adopted in this paper. Our basic viewpoint is that classically we start with real variables - for instance real valued classical fields - while quantum mechanically we deal in an essential way with complex variables. Of course we often use complex numbers in classical mechanics but this is merely as a book-keeping device and has no fundamental significance.

In conventional quantum mechanics it is a basic postulate that physical states correspond to rays in a complex Hilbert space. One way of seeing why this must be so in the standard formulation is that "observables" have a dual role. On the one hand they give the outcomes of measurements in physical states and thus correspond to quadratic forms. On the other hand they generate infinitesimal transformations in the space of physical states into themselves and thus correspond to linear maps of endomorphisms. If the space of physical state vectors is viewed as a real vector space $V$ whose $\Phi$ vectors have real components $\phi^a, a = 1, 2, \ldots, m = \dim_{\mathbb{R}}(V)$ then the observables should be regarded as second rank symmetric covariant tensors with components:

$$O_{ab} = O_{ba}, \quad \text{(2.1)}$$

in their first role quadratic forms, and once contravariant once covariant tensors $T^a{}_b$ in their second role. The vector space $V$ possesses a distinguished positive definite observable, the quantum-mechanical metric

$$g_{ab} = g_{ba}. \quad \text{(2.2)}$$

Regarding positive semi-definite observables as mixed states or density matrices one may view the metric $g_{ab}$ as the density matrix associated to complete ignorance. In any event the expectation value of the observable $O_{ab}$ in the pure state with components $\phi^a$ is

$$\langle O \rangle = \phi^a O_{ab} \phi^b / \phi^a g_{ab} \phi^b. \quad \text{(2.3)}$$

Clearly state vectors which differ by a real multiple give rise to the same expectation values for all observables and hence the physical states would seem on the face of it to correspond to the real projective space $P(V) \equiv P_m(\mathbb{R})$. Clearly every observable in the first sense can be diagonalized with respect to the positive definite metric $g_{ab}$ over the reals (using orthogonal transformations) and thus has real eigen-values.
In their second role as infinitesimal transformations observables generate rotations of $V$ into itself which preserve the metric $g_{ab}$. Thus they satisfy:

$$T_{ab} \equiv g_{ac} T^c_b = -T_{ba} \quad (2.4)$$

In fact every linear bijection of the convex cone of positive semi-definite observables preserving the metric $g_{ab}$ and the trace $g^{ab}O_{ab}$ is induced by an orthogonal transformation. Thus the assumption that physical transformations are orthogonal transformations seems to be well founded, though one might wish to relax the condition the map of density matrices to density matrices be bijective.

How can one link the antisymmetric tensor $T_{ab}$ to the symmetric tensor $O_{ab}$? One way (and this is the way it is done in standard quantum mechanics) is to invoke the existence on $V$ of a symplectic form, i.e. a fundamental anti-symmetric tensor:

$$\Omega_{ab} = -\Omega_{ba} \quad (2.5)$$

which is non-degenerate in the sense that it provides an isomorphism between $V$ and its dual space $V^*$. This imposes the constraint on the real dimension $m$ of the vector space $V$ that it be even. If we demand that in addition to preserving the metric $g_{ab}$ physical transformations preserve the symplectic form it follows that

$$O[T^b_c]_{ac} \equiv \Omega_{ab} T^b_c = O_{ca} \quad (2.6)$$

Since the metric $g_{ab}$ is really just another observable it should be the case that there is an infinitesimal transformation associated to it. Let us call this $J^a_{\ b}$. It is defined by:

$$g_{ac} = \Omega_{ab} J^b_c \quad (2.7)$$

In standard quantum mechanics the tensor $J^a_{\ b}$ coincides with what is usually known as a complex structure. However in order that this be so $J^a_{\ c}$ must satisfy the condition that:

$$J^a_{\ c} J^c_{\ b} = -\delta^a_b \quad (2.8)$$

where $\delta^a_b$ is the Kronecker delta. This condition imposes a compatibility condition on the metric $g_{ab}$ and the symplectic form $\Omega_{ab}$ which may be expressed in various ways. For example $J^a_{\ b}$ is not only an infinitesimal isometry of the metric $g_{ab}$ but a finite one as well:

$$g_{ab} J^a_{\ c} J^b_{\ d} = g_{cd}. \quad (2.9)$$
Similarly it is not only an infinitesimal symplectic transformation but a finite one as well:

$$\Omega_{ab} J^a \cdot c J^b \cdot d = \Omega_{cd}. \quad (2.10)$$

From a physical point of view the simplest justification of the compatibility conditions would seem to be one of economy. If it were not true then successive powers of the endomorphism $J^a \cdot b$ would, when contracted with the metric or the symplectic form, produce a further fundamental symmetric, or antisymmetric tensors with no obvious physical interpretation. In any event given that the compatibility conditions hold it now follows that observables must be hermitean, in the sense that:

$$O_{ab} J^a \cdot c J^b \cdot d = O_{cd} \quad (2.11)$$

It is at this stage that the quantum mechanical phases as opposed to $\pm 1$ factors enters the formalism. One now has that the state vector $\phi^a$ and the state vector $\exp(\alpha J^a \cdot b) \phi^b$ give the same expectation values for all (hermitean) observables $O_{ab}$ and real phases $\alpha$. Thus if one is interested in the question of how these phases enter into quantum mechanics it is precisely when we introduce the complex structure.

It is important to realize that given a metric $g_{ab}$ or a symplectic form $\Omega_{ab}$ there is no unique compatible complex structure associated to it. In fact it is not uncommon to impose more than one complex structure on the same real vector space. For example one may start with the 4-dimensional space of real Majorana spinors (using a signature for the metric in which $\gamma_2 = -1$, and $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$) and regard it as a 2-dimensional complex vector space by choosing $\gamma_5$ as the complex structure, in which case one arrives at Weyl spinors, or one may choose $\gamma_0$ as the complex structure in which case one arrives at (non-relativistic) Pauli spinors.

One way in which complex structures arise naturally is when one has a one parameter subgroup $R(t)$ of the orthogonal group $SO(m)$ acting on the real vector space $V$. For a generic rotation $R(t)$ there will be $m/2$ mutually orthogonal real 2-planes such that the rotation is through an angle $\lambda_i t$ in the $i$’th 2-plane. The quantities $\lambda_i$ are the skew eigen-values of the infinitesimal generator $T^a \cdot b$ of the rotation $R(t) = \exp(tT^a \cdot b)$. The rotation $R(t)$ commutes with any one of the $2^{m/2}$ possible complex structures defined by rotations of $\pm \frac{\pi}{2}$ in these orthogonal 2-planes. With respect to any of these complex structures the rotation $R(t)$ is a $U(1)$ subgroup of $U(\mathbb{H})$. There is an associated conjugation operator with each such choice. Now if the rotation $R(t)$ where thought of as time translations, the eigen-values $\lambda$ being thought of as energies, then the complex structure with all
positive signs is precisely the ”i” of quantum mechanics and the associated conjugation operator is the time reversal operator. In this way one may say that a particular quantum mechanical Hamiltonian picks out or fixes the complex structure. Note that by an appropriate choice of basis we could always choose the complex structure so as to arrange that the energy eigen-values were all positive. However if the complex structure is given ahead of time then of course the signs of the energy eigenvalues are determined.

Let us now consider the situation in which there is an additional one parameter subgroup of rotations $R'(t)$, for example that associated with some conserved charge. In the standard formalism as we have described it above $R'(t)$ must commute with the ”i” of quantum mechanics, i.e. it must be a $U(1)$ subgroup of $SO(m)$ with respect to the unitary structure determined by $J$. Now $R'(t)$ determines its own complex structure, call it $J'$, and its own conjugation operator, call it $C$. The operators $J$ and $J'$ commute. The operators $-$ and $J$ anti-commute. Similarly the operators $J'$ and $C$ anti-commute. However $C$ commutes with $J$, that is the charge conjugation operator $C$ is a linear operator with respect to the usual complex structure of quantum mechanics, unlike the time reversal operator. We could of course introduce a complex notation associated to the complex structure $J'$ but this would be confusing because we would have between the ”i” of quantum mechanics and this new $i'$, they are after all not the same operator. In classical field theory where the ”i” of quantum mechanics does not enter one frequently does use a complex notation with the usual ”i”, when dealing with gauge transformations for example, and no confusion arises but in quantum mechanics this is not possible.

We restrict ourselves to Klein-Gordon fields, classically defined by the equation

$$(-\nabla_g^2 + m^2) \Phi = 0$$  \hspace{1cm} (2.12)

on a manifold $M$ with Lorentz metric $g$. Let $V$ denote the vector-space of all well-behaved real-valued solutions of equation (2.12). On $V$ we can define an antisymmetric bilinear form $\Omega$ by means of the integral

$$\Omega(\Phi_1, \Phi_2) = \int_\sigma (\Phi_2 \nabla_\mu \Phi_1 - \Phi_1 \nabla_\mu \Phi_2) d\Sigma^\mu$$  \hspace{1cm} (2.13)

over a surface $\sigma$, where $\Phi_1$ and $\Phi_2$ are in $V$.

The one-particle Hilbert space $\mathcal{H}$ is a copy of $V$. But to represent quantum states of a particle, $\mathcal{H}$ must have, in addition, the structure of a complex Hilbert space. So we have to introduce on $V$ a hermitean inner product together with a
complex structure $J$, which is compatible with the symplectic structure $\Omega$, i.e. $J^2 = -1$ and

\[ \Omega(J\Phi_1, J\Phi_2) = \Omega(\Phi_1, \Phi_2). \quad (2.14) \]

If $\Omega(., J.)$ is non-degenerate then it automatically follows that

\[ \langle \Phi_1, \Phi_2 \rangle = \Omega(\Phi_1, J\Phi_2) + i\Omega(\Phi_1, \Phi_2) \quad (2.15) \]

provides a Hilbertian norm on $V$.

Given any complex structure $J$, regardless whether or not it is compatible with a symplectic form, one may introduce the standard complex notation in which the operator $J$ becomes multiplication by "$i$". Formally this proceeds as follows. One first complexifies the real vector space $V$ by tensoring with an algebra which is isomorphic to the complex numbers and which we denote for the time being by $C$ with imaginary unit denoted as usual by $i$ to obtain a complex vector space,

\[ V_C = V \times_R C. \quad (2.16) \]

One extends the action of $J$ to $V$ in a $C -$linear fashion. One then has a preferred direct sum decomposition of $V_C$:

\[ V_C = \mathcal{H} \oplus \overline{\mathcal{H}}, \quad (2.17) \]

where

\[ \mathcal{H} = \frac{1}{2}(1 - iJ)V_C \quad (2.18) \]

and

\[ \overline{\mathcal{H}} = \frac{1}{2}(1 + iJ)V_C. \quad (2.19) \]

It is easy to see that the action of $J$ on any element of $\mathcal{H}$ is just given by multiplication with $i$. Associated to the complex structure $J$ is a complex conjugation operator (written as $\overline{\cdot}$) which anti-commutes with the operator $J$. Any real vector of $V$ can now be decomposed as $\Phi = \Phi^+ + \Phi^-$. The vector $\Phi^-$ is the complex conjugate of $\Phi^+$. Although $\Phi$ and $J\Phi$ are real, $\Phi^+$ and $\Phi^-$ are complex. Thus physically $\mathcal{H}$ has as many "degrees of freedom" as the original vector space.

The quantum mechanical Hilbert space is then given as the space of analytic functions, where analyticity is defined with respect to $J$ and the scalar product in $\mathcal{H}$ is

\[ \langle \Phi_1, \Phi_2 \rangle = i\Omega(\Phi_1, \overline{\Phi_2}). \quad (2.20) \]
Of course it has to be shown in every special case, that the product (2.21) is positive definite.

In a static spacetime, with \( \sigma \) in equation (2.13) to be a Cauchy hypersurface orthogonal to the time-like Killing field, the procedure above decomposes every real solution \( \Phi \) of the Klein-Gordon equation into positive and negative frequency parts \( \Phi = \Phi^+ + \Phi^- \). This decomposition provides a preferred complex structure which is given by \( J\Phi = i\Phi^+ + (-i)\Phi^- \).

Finally, to obtain a description of quantum fields in curved spacetime we have to construct the Hilbert space of states \( \mathcal{F} \), as a sum of symmetrized tensor products:

\[
\mathcal{F} = C \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus ... \tag{2.21}
\]

The summand \( C \) in (2.21) is a vacuum or ground state.

On the level of the bosonic Fock space we have to associate with each pair of solutions \( \Phi_k^+, \Phi_k^- \) of \( \mathcal{H} \) or \( \mathcal{H} \) resp. creation and destruction operators \( a_k, a_k^\dagger \) and to impose the canonical quantization relation

\[
[ a_k, a_{k'}^\dagger ] = \delta_{kk'} \tag{2.22}
\]

and

\[
[ a_k, a_{k'} ] = 0, \quad [ a_k^\dagger, a_{k'}^\dagger ] = 0. \tag{2.23}
\]

Now we can associate with each element \( \Phi \) of \( V \) a field-operator \( \Phi \) by means of the relation

\[
\Phi = \sum_k (a_k \Phi_k^- + a_k^\dagger \Phi_k^+), \tag{2.24}
\]

where the modes are normalized with respect to (2.20). The vacuum state is defined by

\[
a_k |0> = 0. \tag{2.25}
\]

Now we see that the definition of a complex structure \( J \) is crucial for the definition of the field-operator and the vacuum state. The Wightman function is given by the vacuum expectation-value of two field-operators:

\[
W(x, y) =< 0 | \Phi(x) \Phi(y) |0> = \sum_k \Phi_k^-(x) \Phi_k^+(y) \tag{2.26}
\]

The basic example we have in mind is that of a simple harmonic oscillator. Quantum Field Theory, from that point of view, is just a collection of many such oscillators.
The oscillator position $y$ satisfies:

$$\frac{d^2 y}{dt^2} + \nu^2 y = 0 \quad (2.27)$$

with $\nu$ real and positive. A basis for the real 2-dimensional space $V$ of classical solutions is provided by

$$e_1 = \frac{1}{\sqrt{\nu}} \cos \nu t, \quad e_2 = \frac{1}{\sqrt{\nu}} \sin \nu t. \quad (2.28)$$

Their Wronskian is independent of time

$$\Omega(e_1, e_2) = e_1 \dot{e}_2 - e_2 \dot{e}_1 = 1 \quad (2.29)$$

and gives the symplectic form on the real 2-dimensional vector space $V$. A complex structure on $\mathcal{H}$ is induced from the complex structure $J$ on the space of real solutions $V$ given by:

$$Je_1 = e_2, \quad Je_2 = -e_1 \quad (2.30)$$

so that

$$J \left( \frac{1}{\sqrt{\nu}} e^{\mp i \nu t} \right) = (\pm i) \frac{1}{\sqrt{\nu}} e^{\mp i \nu t}. \quad (2.31)$$

The norm of a quantum mechanical state $\Phi = \phi_1 e_1 + \phi_2 e_2$ is $\|\Phi\| = \phi_1^2 + \phi_2^2$. The usual procedure in second quantization (in the Heisenberg picture) is to regard $y$ as an operator $\hat{y}$

$$\hat{y} = a \frac{1}{\sqrt{\nu}} e^{-i \nu t} + a^\dagger \frac{1}{\sqrt{\nu}} e^{i \nu t} \quad (2.32)$$

and impose the canonical quantization relation

$$[a, a^\dagger] = 1 \quad (2.33)$$

In the Bargmann-Fock representation we consider wave functions which are anti-holomorphic functions of $a$, $\Phi(\overline{a})$. One has

$$a^\dagger \rightarrow \overline{a}, \quad a \rightarrow \frac{\partial}{\partial \overline{a}}. \quad (2.34)$$

The inner product is

$$< \Phi_1 | \Phi_2 > = \int da d\overline{a} \frac{1}{2\pi i} e^{-a \overline{a}} \Phi_1(\overline{a}) \Phi_2(\overline{a}). \quad (2.35)$$
We may identify $\mathcal{H}$ with the space of anti-holomorphic functions of $a$. The ground state correspond to the constant function. The first excited state, corresponding to $\mathcal{H}$ is given by $\Phi_1 = \pi$. The n'th exited state has wave functions $\Phi_n = \pi^n / \sqrt{n!}$ and corresponds to $(\mathcal{H} \otimes \ldots \otimes \mathcal{H})_S$, the n-fold symmetric product. The action of $J$ on $\Phi$ is
\[
J\Phi(\pi) = i\Phi(\pi).
\]

### 3 Globally Static Metrics

In this section we reanalyse the complex structure in a globally static space-time and its connection with analytic properties of the analytic continuation of positive and negative frequency modes in a corresponding Riemannian space.

In a globally static Lorentzian spacetime $M_L$ for which the metric may be written as:
\[
ds^2 = -v(x)dt^2 + g_{ij}(x)dx^i dx^j, \quad v > 0,
\]
a basis for the space $V$ of real solutions of the Klein-Gordon equation is given by
\[
\Phi_c(x, t; \omega) = \cos(\omega t) \chi_\omega(x), \quad \omega > 0
\]
and
\[
\Phi_s(x, t; \omega) = \sin(\omega t) \chi_\omega(x), \quad \omega > 0
\]
where $\chi_\omega(x)$ satisfies the spatial equation:
\[
-\frac{1}{\sqrt{v}} \nabla_i (\sqrt{v} \nabla^i \chi_\omega) + \frac{\omega^2}{v} \chi_\omega + m^2 \chi_\omega = 0.
\]

We can endow the space $V$ of real valued solutions of the Klein-Gordon equation with the symplectic form
\[
\Omega(\Phi, \Phi') = \int_\Sigma (\Phi \partial_\mu \Phi' - \Phi' \partial_\mu \Phi) d\Sigma^\mu
\]
the integral being taken over a Cauchy surface $\Sigma$. To convert $V$ to the one-particle Hilbert space $\mathcal{H}$ we introduce the complex structure $J$ with
\[
J\Phi_c = \Phi_s \quad \text{and} \quad J\Phi_s = -\Phi_c,
\]
and perform the preferred direct sum decomposition (2.17). We obtain the space $\mathcal{H}$ with the basis
\[ \Phi^+(x, t; \omega) = \exp(-i\omega t) \chi_\omega(x), \quad \omega > 0. \]  
(3.7)

This is the "obvious" definition of positive frequency. If we decompose a solution \( \Phi \) of \( V \) in its positive and negative frequency part we find that our complex structure is given by

\[ J\Phi = i\Phi^+ + (-i)\Phi^- . \]  
(3.8)

Setting \( t = i\tau \), \( \tau \) real, we pass to a metric of a Riemannian space \( M_R \). We regard \( M_L \) and \( M_R \) as real slices of a complex manifold \( M_C \) with the common boundary \( \Sigma \).

We see that a superposition \( \Phi \) of the analytic continuation of purely positive frequency functions satisfies the elliptic Klein-Gordon equation

\[ -\frac{1}{\sqrt{v}} \nabla_i(\sqrt{v}\nabla^i\Phi) - \frac{1}{v} \frac{\partial^2 \Phi}{\partial \tau^2} + m^2 \Phi = 0 \]  
(3.9)

for all negative values of \( \tau \), i.e. it is everywhere bounded on \( M^-_R \), defined by \( \tau < 0 \). In other words \( \mathcal{H} \) analytically continues to the space \( H^+ \) of solutions of the homogeneous Klein-Gordon equation bounded in \( M^-_R \) similarly \( \overline{\mathcal{H}} \), i.e. the negative frequency solutions, analytically continue to solutions in \( H^- \) of the homogeneous Klein-Gordon equation which are bounded on \( M^+_R \) i.e. if \( \tau > 0 \). The reflection map \( \theta : \tau \rightarrow -\tau \) maps the two spaces \( H^+ \) and \( H^- \) into one another:

\[ \theta H^\pm = H^\mp \]  
(3.10)

Now we can give a more geometrical motivation for the complex structure introduced in (3.6): Instead of referring to the space of solutions, either of the Lorentzian or the Riemannian Klein-Gordon equation we may refer instead to their values and the values of their normal derivatives ( either \( \frac{\partial \Phi}{\partial t} \) or \( \frac{\partial \Phi}{\partial \tau} \) on the boundary surface \( \Sigma \), i.e. at \( \tau = 0 \). In the Lorentzian case this boundary data is the (complex-valued) Cauchy data \( V_C \). In the Riemannian case the boundary data provide a real vector space of pairs ( \( \Phi|_{\Sigma}, \frac{\partial \Phi}{\partial \tau}|_{\Sigma} \) ) The choice of positive frequency or of a complex structure may thus be thought of as a direct sum decomposition of the boundary data on \( \Sigma \) into data which evolve to give a solution everywhere bounded in \( M^-_R \) (elements of \( H^+ \)) or everywhere bounded in \( M^+_R \) (elements of \( H^- \)). In this way our construction associates in a very clear way the complex structure of quantum mechanics with the beginning of time.
Note that while $H$ and $\overline{H}$ may be thought of as complex conjugate of one another, $H^+$ and $H^-$ are both real vector spaces. Note also that the decomposition of the real boundary data into $H^+ \oplus H^-$ is non-local with respect to the boundary $\Sigma$. In a quite recent paper Ashtekar, Tate and Uggla showed, that complex structures can be constructed from Killing vectors. So the complex structure for globally static spacetime is very related to the time-like Killing vector in these geometries.

4 Real Tunneling Geometries

The aim of this section is to extend the theory developed in section 3 to the more general case when the Lorentzian spacetime, which we call $2M_L$, admits a moment of time symmetry $\Sigma$. Thus $2M_L$ can be decomposed as

$$2M_L = M^+_L \cup M^-_L$$

(4.1)

where time reversal $T$ is an isometry of $g_L$ such that

$$TM^\pm_L = M^\mp_L$$

(4.2)

and

$$\partial M^+_L = \Sigma = \partial M^-_L$$

(4.3)

where

$$\partial \Sigma = \Sigma$$

(4.4)

We assume (following) that in the compactification $M_C$ of $2M_L$ there is another real section $2M_R$ with Riemannian metric $g_R$ and a reflection map $\theta$. The conditions (4.2) - (4.4) continue to hold but with $L$ replaced by $R$ and $T$ replaced by $\theta$. The real tunneling geometry is the manifold: $M^-_R \cup M^+_L$ with the metric $g_R$ on $M^-_R$ and $g_L$ on $M^+_L$. The surface $\Sigma$ thus represents the "beginning of time" in the model. In [1] it was further assumed that $M^-_R$ was compact and hence the boundary $\Sigma$ had to be compact as well. In our work we shall not necessarily be making that assumption. However we shall assume that $\Sigma$ is the only boundary of $M^-_R$. This assumption would rule out the globally static example discussed in section 3 for which $M^-_R = \Sigma \times (-\infty, 0]$, $\Sigma$ being the spatial cross section and $(-\infty, 0]$ corresponds to $-\infty < \tau \leq 0$. Demanding that various fields vanish at $\tau \to -\infty$ we have the effect of rendering irrelevant the non-compactness of $M^-_R$ and in fact our results will cover that more general case as well. All that
is really needed for that the only boundary terms we need to consider are those on \( \Sigma \).

We now proceed very much as in section \( \overline{3} \). We define the spaces \( H^+ \) and \( H^- \) of real solutions of the equation

\[
(-\nabla^2_g + m^2) \Phi_R = 0 \tag{4.5}
\]

which are everywhere bounded on \( M^-_R \) or \( M^+_R \) respectively. We will identify these spaces with the boundary data \((\Phi_R, \frac{\partial \Phi_R}{\partial \tau})\) where the normal derivative \( \frac{\partial \Phi_R}{\partial \tau} \) is outward with respect to \( M^-_R \) and inward with respect to \( M^+_R \). The map \( \theta \) extends to a map from

\[
\theta : H^\pm \rightarrow H^\mp \tag{4.6}
\]

by pull back, that is for all \( \Phi_R \in H^\pm \) we define

\[
\theta \Phi_R(x) = \Phi_R(\theta x) = \Phi_{R\theta}(x). \tag{4.7}
\]

Now we perform an analytic continuation of solutions in \( H^+ \) on \( M^-_R \) to get complex solutions in \( M^+_R \). These solutions define a vector space \( \mathcal{H} \) and we consider these solutions as to be of ”positive” frequency. In a similar way we obtain the space \( \overline{\mathcal{H}} \) of negative frequency from the solutions bounded in \( M^+_R \). The spaces \( \mathcal{H} \) and \( \overline{\mathcal{H}} \) are a direct decomposition of the vector space \( V \) of solutions in \( M_L \). So any solution \( \Phi \) in \( M_L \) can be decomposed in a positive and negative component \((\Phi = \Phi^+ + \Phi^-)\). This gives the possibility to define a complex structure \( J \) with \( J\Phi = i\Phi^+ + (-i)\Phi^- \) on \( \Sigma \), which is compatible with the natural symplectic form

\[
\Omega(\Phi, \Phi') = \int_{\Sigma} (\Phi \partial_t \Phi' - \Phi' \partial_t \Phi) d\Sigma^\mu_L \tag{4.8}
\]

on \( M_L \).

To make \( \mathcal{H} \) to a one-particle Hilbert space it remains to construct a scalar product. Following (2.20) we define

\[
\langle \Phi_1, \Phi_2 \rangle = i \int_{\Sigma} (\overline{\Phi_2} \partial_t \Phi_1 - \Phi_1 \partial_t \overline{\Phi_2}) d\Sigma_L. \tag{4.9}
\]

It remains to show, that this product is positive definite for functions in \( \mathcal{H} \). Therefore we consider \( \Phi \) on \( \Sigma \) and its derivatives as the boundary values of a function \( \Phi_R \) in \( H^+ \) and \( \overline{\Phi} \) and its derivative as boundary values of a function \( \Phi_{R\theta} \) in \( H^- \). Thus on \( \Sigma \) we have for the modes
\[
\frac{\partial \Phi}{\partial t} = -i \frac{\partial \Phi_R}{\partial \tau} \text{ and } \frac{\partial \Phi^*}{\partial t} = -i \frac{\partial \Phi_{R\theta}}{\partial \tau}.
\] (4.10)

If we choose the coordinate \(x^0\) in \(M^+_L\) and \(M^-_R\) to be orthogonal to \(\Sigma\) it will be easy to see that the measures of integration, which are the dual forms with respect to \(M^+_L\) or \(M^-_R\) of the volume 3-form on \(\Sigma\), are equal. We get

\[
d\Sigma_L = (-g^0_0) \sqrt{-g_L} d^3x = g^0_0 \sqrt{g_R} d^3x = d\Sigma_R.
\] (4.11)

This gives

\[
\langle \Phi, \Phi \rangle = \int_{\Sigma} (\Phi_{R\theta} \partial_{\tau} \Phi_R - \Phi_R \partial_{\tau} \Phi_{R\theta}) d\Sigma_R.
\] (4.12)

From Gauss’ formula the integration over \(\Sigma\) can be replaced by an integration over \(M^-_R\):

\[
\langle \Phi, \Phi \rangle = \int_{M^-_R} (\Phi_{R\theta} \nabla^2 \Phi_R - \Phi_R \nabla^2 \Phi_{R\theta}) dV_R.
\] (4.13)

The function \(\Phi_R\) is not bounded in \(M^+_R\). It satisfies the modified Klein-Gordon equation

\[
(-\nabla^2_{g_R} + m^2) \Phi_R = j.
\] (4.14)

The source \(j\) has support only in \(M^+_R\). Similarly the function \(\Phi_{R\theta}\) has a source \(j_\theta\) in \(M^-_R\) and we can write

\[
\langle \Phi, \Phi \rangle = \int_{M^-_R} \Phi_{R\theta} j_\theta dV_R.
\] (4.15)

Let \(G(x, y)\) be the unique inverse of the Klein-Gordon operator \(-\nabla^2_{g_R} + m^2\) on \(2M_R\), then \(\Phi_R(x)\) can be written as:

\[
\Phi_R(x) = \int_{M^-_R} G(x, y) j(y) dV_Ry
\] (4.16)

and we have

\[
\langle \Phi, \Phi \rangle = \int_{M^+_R} \int_{M^-_R} j(x) G(x, y) j(y) dV_Rx \ dV_Ry.
\] (4.17)

Because \(j\) vanishes in \(M^-_R\) and \(j_\theta\) in \(M^+_R\), we may extend the integral to \(2M_R \times 2M_R\) and we can write also:

\[
\langle \Phi, \Phi \rangle = \int_{2M_R \times 2M_R} j(x) G(x, \theta y) j(y) dV_Rx \ dV_Ry.
\] (4.18)
We have a suitable scalar product \([4.9]\), if the Green’s function \(G(x, y)\) is such that the expression \([4.17]\) is non-negative for all possible sources \(j\) e.g. those satisfying the requirement of Reflection Positivity will give a satisfactory physical inner product. The requirement of reflection positivity is quite stringent. It is not sufficient for example that \(G(x, y)\) be pointwise positive. If one considers a Gaussian in flat one dimensional Euclidean space,

\[
G(x, y) = \exp\left(-\frac{(x-y)^2}{2}\right)
\]

(4.19)

and take the source to be

\[
j(x) = x \exp\left(-\frac{(x - 1)^2}{2}\right),
\]

(4.20)

the integral turns out to be

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j(x)G(x, y)j(-y)\,dx\,dy = -\frac{4\pi}{3^{5/2}e^{2/3}}.
\]

(4.21)

It can be seen not to satisfy positivity.

For the simple case of a free theory one may show directly that the expression \([4.17]\) is positive as follows. We define the function

\[
G_D(x, y) = G(x, y) - G(x, \theta y).
\]

(4.22)

While \(G\) is the Green’s function on \(2M_R\), \(G_D\) is the Green’s function on \(2M_R\) restricted to functions which vanish on \(\Sigma\). So as operators it follows that \(G_D \leq G\) and we can see that the expression \([4.18]\) rewritten as

\[
\langle \Phi, \Phi \rangle = \int_{2M_R \times 2M_R} j(x)G(x, y)j(y)\,dV_x\,dV_y
\]

\[
- \int_{2M_R \times 2M_R} j(x)G_D(x, y)j(y)\,dV_x\,dV_y
\]

(4.23)

is indeed positive, provided the \(j\)'s are in a class of functions which guarantees the finiteness of the integrals.

Just as in Lorentzian space we can define suitable field-operators in the Riemannian section \(2M_R\). We set

\[
\Phi_R = \sum_k \left( a_k \Phi_{Rk}^- + a_k^\dagger \Phi_{Rk}^+ \right),
\]

(4.24)

where the normalized modes \(\Phi_{Rk}^\pm\) are elements of \(H^\pm\). The vacuum expectation value of the product of two field-operators gives just the Riemannian Green’s
function:
\[ G(x, y) = \langle 0 | \Phi_R(x) \Phi_R(x) | 0 \rangle = \sum_k \Phi_{Rk}^-(x) \Phi_{Rk}^+(y) \] (4.25)

The connection between quantum field theory in Minkowski space and properties of corresponding Green’s functions in Euclidean space is well investigated. Let us compare our results with Osterwalder and Schrader’s approach to Euclidean quantum field theory. In their paper they gave necessary and sufficient conditions under which Euclidean Green’s functions have analytic continuations whose boundary values define a unique set of Wightman distributions. These conditions were

(E0) Temperedness
(E1) Euclidean covariance
(E2) Reflection positivity
(E3) Symmetry
(E4) Cluster property.

Caused by the curvature and possible finiteness of the Riemannian space \(2M_R\) in our approach we can not expect something analog to the conditions of Euclidean covariance (E1) and the cluster property (E4), but these conditions where anyway only included to guarantee similar properties for the corresponding Wightman functions. In our approach we assumed that the integrals in (4.23) exist, which is in analogy to (E0). The way, we constructed the Euclidean Green’s functions makes them symmetric (E3) and guarantees the reflection positivity (E2). Then, because of the fact, that the functions \(\Phi_R^\pm\) in \(2M_R\) are a complex continuation of the functions \(\Phi^\pm\), the corresponding Wightman function (2.26) is a continuation of the Riemannian Green’s function. In this sense our paper presents a generalization of the results of Osterwalder and Schrader in the case of tunneling geometries.

5 Examples

5.1 DeSitter space

The simplest case of a tunneling geometry is deSitter space. The complexified space can be considered as the surface given by
\[ (z^1)^2 + (z^2)^2 + (z^3)^2 + (z_4)^2 + (z_5)^2 = 1. \] (5.1)

The Riemannian real part \(2M_R\) is given by the involution
\[ J_R : (z^1, z^2, z^3, z_4, z_5) \rightarrow (\overline{z^1}, \overline{z^2}, \overline{z^3}, \overline{z_4}, \overline{z_5}), \] (5.2)
which is a four sphere $S_3$. We may consider $M_R^-$ as the lower and $M_R^+$ as the upper half sphere. The Lorentzian real part $2M_L$ is given by the involution

$$J_R : (z^1, z^2, z^3, z^4, z^5) \rightarrow (\overline{z}^1, \overline{z}^2, \overline{z}^3, \overline{z}^4, -\overline{z}^5). \quad (5.3)$$

The intersection $\Sigma = 2M_R \cap 2M_L$ of both spaces is a three sphere.

With suitable coordinates the metric of the Riemannian section can be cast in the form

$$ds^2 = -\frac{1}{\cosh^2 \tau} (d\tau^2 + d\Omega_3^2) \quad (5.4)$$

where $d\Omega_3$ is the metric of the sphere $S_3$ and $\tau \in (-\infty, \infty)$. The boundary $\Sigma$ is given by $\tau = 0$. In these coordinates the Klein-Gordon equation (4.5) takes the form

$$-\cosh^4 \tau \frac{\partial}{\partial \tau} \left( \frac{1}{\cosh^2 \tau} \frac{\partial \Phi}{\partial \tau} \right) - \cosh^2 \tau \Delta_3 \Phi_L + m^2 \Phi_L = 0 \quad (5.5)$$

and we assume $m^2 > 9/4$.

By a separation of variables we find the following set of linearly independent solutions:

$$\Phi_{R \; pq\tau}(\tau, \Omega; m) = y^\pm_{R \; pq}(\tau; m) \mathcal{Y}_{pq\tau}(\Omega) \quad (5.6)$$

$$p = 0, 1, 2, \cdots \; ; \; q = 0, 1, \cdots, p \; ; \; r = -q, \cdots, q.$$ 

The index $R$ indicates that we have a mode in Riemannian space and the $\mathcal{Y}_{pq\tau}(\Omega)$ are ortho-normal real-valued surface harmonics of degree $p$ on $S_3$ obeying

$$-\Delta_3 \mathcal{Y}_{pq\tau}(\Omega) = p(p + 2) \mathcal{Y}_{pq\tau}(\Omega). \quad (5.7)$$

The real valued functions $y^\pm_{R \; pq}(\tau; m)$ may be expressed in terms of a hypergeometric function

$$y^\pm_{R \; pq}(\tau; m) = \frac{1}{p!} \Gamma(p + 1/2 + i\gamma) \Gamma(p + 1/2 - i\gamma) \left( \cosh \tau \exp(\pm p\tau) \right)^{1/2} \text{F}_1 \left( 1/2 + i\gamma, 1/2 - i\gamma, p + 1; \exp(\pm \tau)/(2 \cosh \tau) \right) \quad (5.8)$$

where $\gamma = (m^2 - 9/4)^{1/2}$. According to our definition the $\Phi_{R \; pq}(\tau; m)$ describe positive frequency modes, which are bounded for $\tau < 0$. The relation $\theta \Phi_{R \; pq} = \Phi_{R \; pq}$ holds true.

By rotating back the time axis we get the positive frequency modes

$$\Phi_{L \; pq\tau}(t; m) = \Phi_{R \; pq}(i\tau; m) \quad (5.9)$$

and their span gives the set of positive frequency functions in the Lorentzian section of deSitter space. With this unique definition of positive frequency
we can expand the quantized Klein-Gordon field $\Phi$ in $2M_L$ in a Fock space representation

$$\Phi_L(t, \Omega; m) = \sum_{pqr} (a_{pqr} \Phi_{L,pqr}^- (m) + a_{pqr}^+ \Phi_{L,pqr}^+ (m)) \quad (5.10)$$

and the vacuum state would then be uniquely specified by $a_{pqr} |0\rangle = 0$. We shall now elaborate this example further because we think it contains many features which can be generalized.

In analogy to formula (5.10) we define in Riemannian space:

$$\Phi_R(\tau, \Omega; m) = \sum_{pqr} (a_{pqr} \Phi_{R,pqr}^- (m) + a_{pqr}^+ \Phi_{R,pqr}^+ (m)) \quad (5.11)$$

In equation (5.11) the operators $a_{pqr}$ and $a_{pqr}^+$ are the same as those in (5.10) and they act on the same Hilbert space, which consists of suitable boundary data on $\Sigma$.

A general one-particle state is given by the superposition

$$|1\rangle = \sum_{pqr} \alpha_{pqr} a_{pqr}^+ |0\rangle. \quad (5.12)$$

This can also be expressed in terms of the field operators in Lorentzian space as

$$|1\rangle = \int_{M_L^+} dV_L \alpha_L(t, \Omega) \Phi_L(t, \Omega; m) |0\rangle$$

$$= \sum_{pqr} \int_{M_L^+} dV_L \alpha_L(t, \Omega) \Phi_{L,pqr}^+ (t, \Omega; m) a_{pqr}^+ |0\rangle. \quad (5.13)$$

and in Riemannian space

$$|1\rangle = \int_{M_R^+} dV_R \Phi_R(\tau, \Omega; m) |0\rangle$$

$$= \sum_{pqr} \int_{M_R^+} dV_R \alpha_R(\tau, \Omega) \Phi_{R,pqr}^+ (\tau, \Omega; m) a_{pqr}^+ |0\rangle. \quad (5.14)$$

What is the relation between the functions $\alpha_L(t, \Omega)$ and $\alpha_R(\tau, \Omega)$? Caused by the orthogonality of the Lorentzian modes of different mass

$$\int_{M_L^+} dV_L \Phi_{L,pqr}^-(t, \Omega; m) \Phi_{L,p'q'r'}^+ (t, \Omega; m') = \delta_{pqr,p'q'r'}/2$$

$$\times \left\{ \begin{array}{ll} \delta(m^2 - m'^2) & \text{if } m, m' > 3/2 \\ \infty & \text{if } m, m' < 3/2 \end{array} \right.$$
we can rewrite the Riemannian expression (5.14) as

$$|1\rangle = \sum_{pqr} \int_{M_R} dV_R \alpha_R(\tau, \Omega) \sum_{p'q'r'} \int_{m'^2 > 9/4} dm'^2 \int_{M_R} dV'_L \Phi_{R_{p'q'r'}}(\tau, \Omega; m') \Phi_{L_{pqr}}(t', \Omega'; m') a_{pqr}^\dagger |0\rangle. \quad (5.15)$$

By introducing the Green’s function

$$G(\tau', \Omega'; \tau, \Omega; m') = \langle 0| \Phi_R(\tau', \Omega'; m') \Phi_R(\tau, \Omega; m') |0\rangle = \sum_{p'q'r'} \Phi_{R_{p'q'r'}}(\tau', \Omega'; m') \Phi_{L_{pqr}}(t', \Omega'; m') \quad (5.16)$$

and analytic continuation we can rewrite our one-particle state as

$$|1\rangle = \sum_{pqr} \int_{M_L} dV_L \int_{m'^2 > 9/4} dm'^2 \int_{M_R} dV'_R \Phi(\tau', \Omega') \alpha_R(\tau', \Omega') a_{pqr}^\dagger G(-it, \Omega; \tau'; \Omega'; m') |0\rangle. \quad (5.17)$$

A comparison with (5.13) showes that we can express the Lorentzian function $\alpha_L$ in terms of the Riemannian function $\alpha_R$ by means of

$$\alpha_L(t, \Omega) = \int_{m'^2 > 9/4} dm'^2 \int_{M_R} dV'_R \Phi(\tau', \Omega'; m') \alpha_R(\tau', \Omega'). \quad (5.18)$$

We see that we can construct a one-particle state from the field operators $\Phi_R$ defined in the Riemannian sector as well as from the field operators in the Lorentzian sector.

5.2 Page Metric

The Page metric is a solution of the Einstein equation with the cosmological constant $\Lambda$. It belongs to the Bianchi IX type solutions and is invariant under a group homomorphic to $U(1) \times SU(2)$. The line element can be expressed in terms of the coordinates $\eta, \psi, \theta, \phi$ as

$$ds^2 = a^2 b^2 c^2 d\eta^2 + a^2 d\sigma_1^2 + b^2 d\sigma_2^2 + c^2 d\sigma_3^2 \quad (5.19)$$

Here the $\sigma_1, \sigma_2, \sigma_3$ are left-invariant one forms on $SU(2)$ such that

$$\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi$$
$$\sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi$$
$$\sigma_3 = d\psi + \cos \theta d\phi$$
The functions $a, b, c$ are given by the relations

\begin{align*}
a^2 &= b^2 = \frac{a_0^2}{\lambda} (1 - \nu^2 \tau^2) \quad (5.20) \\
c^2 &= \frac{c_0^2}{\lambda} \frac{\Delta(\tau)}{(1 - \nu^2 \tau^2)} (1 - \tau^2) \quad (5.21) \\
d\eta &= \frac{\lambda}{a_0^2 c_0 \Delta(\tau)} (1 - \tau^2) \quad (5.22)
\end{align*}

with $(-1 < \tau < 1)$. The constant $\nu$ is the solution $\nu \approx 0.28 \ldots$ of the equation

\[ \nu^4 + 4\nu^3 - 6\nu^2 + 12\nu - 3 = 0 \quad (5.23) \]

and we used the definitions

\begin{align*}
\Delta(\tau) &= 3 - \nu^2 - \nu^2 (1 + \nu^2) \tau^2 \\
\lambda &= \frac{\Lambda}{3(1 + \nu^2)} \quad (5.24)
\end{align*}

and

\begin{align*}
a_0^2 &= \frac{1}{3 + 6\nu^2 - \nu^4} \\
c_0^2 &= \frac{1}{(3 + \nu^2)^2}
\end{align*}

For real $\tau$ the equations (5.20), (5.21) and (5.23) give the compact Riemannian section $2M_R$. The reflexion map is given by $\theta : \tau \rightarrow -\tau$ and thus the surface $\tau = 0$ is our nucleation surface $\Sigma$, which is a squashed three-sphere. To obtain the Lorentzian section we have to take $\tau$ to be pure imaginary. We get an ever expanding universe in which $a$ and $c$ grow exponentially as $\exp(\Lambda \tau/3)$.

With the help of the vector fields

\begin{align*}
\xi_1 &= -\cot \theta \cos \psi \frac{\partial}{\partial \psi} - \sin \psi \frac{\partial}{\partial \theta} + \cos \psi \frac{\partial}{\partial \phi} \\
\xi_2 &= -\cot \theta \sin \psi \frac{\partial}{\partial \psi} + \cos \psi \frac{\partial}{\partial \theta} + \sin \psi \frac{\partial}{\partial \phi} \\
\xi_3 &= \frac{\partial}{\partial \psi}
\end{align*}

the Klein-Gordon equation can be written as

\[- \frac{1}{a^2 b^2 c^2} \frac{\partial^2}{\partial \eta^2} \Phi - \left( \frac{\xi_1^2}{a^2} + \frac{\xi_2^2}{b^2} + \frac{\xi_3^2}{c^2} \right) \Phi + m^2 \Phi = 0 \quad (5.25)\]
A separation of variables is possible and we can use the following ansatz for the linearly independent solutions

\[ \Phi_R^{(i)}(\tau, \psi, \theta, \phi) = y_{Rpq}(\tau) D_{qr}^{(i)p}(\psi, \theta, \phi), \quad (5.26) \]

where \( i = 1, 2 \). The functions \( D_{qr}^{(1)p} \) are the real and the imaginary part of the Wigner functions \( D_{pq}^p = D_{pq}^{(1)p} + iD_{pq}^{(2)p} \). They satisfy the relations

\[ \xi_3 D_{pq}^p = iqD_{pq}^p, \quad (5.27) \]
\[ (\xi_3^2 + \xi_2^2 + \xi_1^2) D_{pq}^p = -p(p+1)D_{pq}^p \quad (5.28) \]

and

\[ \sum_p \overline{D}_{qr}^p D_{qr}^p = \sum_p \overline{D}_{qr}^q D_{qr}^q = \delta_{rr'} \quad (5.29) \]

The ansatz (5.26) together with the relations (5.27) and (5.28) lead to the equation

\[ -\frac{1}{a^4c^2} \partial^2_{\eta^2} y_{Rpq} + \left( \frac{p(p+1)}{a^2} + q^2 \left( \frac{1}{c^2} - \frac{1}{a^2} \right) \right) y_{Rpq} + m^2 y_{Rpq} = 0 \quad (5.30) \]

With the substitution \( \tau^2 = z \) we get

\[ -y_{Rpq}'' - \left( \frac{1}{2z} + \frac{\Delta'}{\Delta} - \frac{1}{1-z} \right) y_{Rpq}' + \frac{1-\nu^2z}{4z(1-z)\Delta} \]
\[ \times \left( \frac{p(p+1)}{a^2(1-\nu^2z)} + q^2 \left( \frac{1-\nu^2z}{c^2(1-z)\Delta} - \frac{1}{a^2(1-\nu^2z)} \right) + \frac{m^2}{\lambda} \right) y_{Rpq} = 0 \quad (5.31) \]

The 2-surface ”bolt” \( z = 1 \) which closes up the space corresponds to a regular singular point of the differential equation (5.32). With the ansatz

\[ y_{Rpq} = (1-z)^k u(z) \quad (5.32) \]

where \( u \) is to be considered analytic and different from zero at \( z = 1 \), we get the indicial equation

\[ k^2 - 2k - \frac{(1-\nu^2)^2}{4c^2\Delta^2(1)}q^2 = 0 \quad (5.33) \]

which has the solution

\[ k = 1 \pm \sqrt{1 + \frac{(1-\nu^2)^2}{4c^2\Delta^2(1)}q^2} \approx 1 \pm \sqrt{1 + 3.07q^2}. \quad (5.34) \]

We see that there is at least one solution which is regular at \( z = 1 \). Because equation (5.32) has no other singularity in the circle \( |1-z| < 0 \) and our space
$2M_R$ is covered by the interval $0 < z < 1$ it remains to consider the point $z = 0$. According to Fuchs’s theory this point also represents a regular singular point of the differential equation and we use an ansatz similar to (5.32), in which we assume that the function $v(z)$ is analytic and different from zero at $z = 0$

$$y_{R pqr} = (z)^{l}v(z)$$

and we get

$$l(l - \frac{1}{2}) = 0$$

(5.36)

From the properties of the function $v$ we see, that there is no singularity in the general solution of equation (5.32) at $z = 0$. Together with what we found about the behaviour of the solutions around $z = 1$ and transforming back to the parameter $\tau$ we conclude that there exist of solution $y_{R pqr}$ which are everywhere regular in the interval $0 > \tau > -1$. These can be used to construct the positive frequency modes via the ansatz (5.26).

### 6 Spinors

In this section we shall treat the spinorial case with what may seem to the reader to be positively painful pedantry. However in view of the selection rule found in [1] in the purely Lorentzian theory we feel that this is justified since there appears to be a genuine difference between the Lorentzian and the Riemannian theory (For example in the Lorentzian theory there is no Lorentz -cobordism of the $S^3$ admitting an SL(2, C) spinor structure. On the other hand one can clearly consider $S^3$ as the boundary of the 4-ball, $B^4$ and put a Riemannian metric on it). In calculations of the wave function of the universe there appears to be no obstacle to including spinors and indeed this has been done by Halliwell and D’Eath [19]. This contrast between the Riemannian and the Lorentzian theory is rather puzzling. We shall in fact find, in accordance with the results of Halliwell and D’Eath that there is no apparent difficulty here.

The treatment sketched below is related to that of [10], [11] and [12] but we mentioned in section 2 our strategy is to start with Majorana spinors and then pass to Dirac spinors just as for spin zero particles we started with a real scalar field and then passed to the charged case. Thus we shall use conventions for the $\gamma$ -matrices $\gamma^\mu$ in Minkowski space-time as follows:

$$(\gamma^{0}_L)^2 = -1, \quad (\gamma^{i}_L)^2 = 1.$$  

(6.1)
The matrices $\gamma^\mu$ may be taken to be real with $\gamma^0$ anti-symmetric and $\gamma^i$ symmetric. The Dirac equation is thus

$$ (\gamma^\mu_L D_\mu - m)\psi = 0. \quad (6.2) $$

where $\psi$ is a real four-component spinor. Given a Cauchy surface $\Sigma$ the restriction of $\psi$ to $\Sigma$ gives four real functions which constitute the real vector space $V$ of cauchy data for the Dirac equation. The real vector space $V$ admits an invariant positive definite inner product:

$$ \int_\Sigma \bar{\psi} \gamma^\mu_L \psi d\Sigma_\mu = \int_\Sigma \psi^i \psi \sqrt{h} d^3 x, \quad (6.3) $$

where $\bar{\psi}$ is the Dirac adjoint, which coincides with the Majorana adjoint in our case. The problem of quantization in this case is to endow $V$ with a complex structure (or equivalently a symplectic structure) compatible with the positive definite inner product. As before we complexify $V$ and consider complex-valued (i.e. Dirac) spinors and seek an orthogonal direct sum decomposition

$$ V_C = H_1 \oplus \overline{H}_1. \quad (6.4) $$

The Cauchy data may also be regarded as complex valued boundary data for the Riemannian Dirac equation:

$$ (\gamma^\mu_R D_\mu - m)\psi = 0, \quad (6.5) $$

now,

$$ (\gamma^0_R)^2 = 1. \quad (6.6) $$

In fact choosing:

$$ \gamma^0_R = i \gamma^0_L \quad (6.7) $$

one has that the $\gamma^\mu_R$ are hermitean. The Dirac adjoint in $M_R$ may thus be taken as Hermitian conjugation. The conserved current $J^\mu$ on $M_R$ is:

$$ \int_\Sigma \psi^i \gamma^\mu_R \psi d\Sigma_\mu = \int_\Sigma \psi^i i \gamma^0_R \sqrt{h} d^3 x \quad (6.8) $$

It is important to realize that this expression is not the same as the analytic continuation of the Lorentzian inner product. Thus it cannot be regarded as the quantum-mechanical metric. This will be defined below. One can now decompose
\(V_C\) into data \(\in H^+\) which is non-singular everywhere on \(M^-\) and \(\in H^+\) which is everywhere non-singular on \(M^+\). We can extend \(\theta\) to a map
\[
\theta : H^+ \to H^\pm
\] (6.9)
by defining:
\[
\psi_\theta = \gamma^0 R \psi(\theta x)
\] (6.10)
and the inner product to be:
\[
\int_\Sigma \psi^\dagger_\theta \gamma^0 R \psi \sqrt{\hbar} d^3x.
\] (6.11)
Clearly this expression (i.e. (6.11)) unlike that in (6.8) coincides with the usual Lorentzian inner product when restricted to the appropriate set of Cauchy data.

Let us now turn to the case of Dirac spinors. It is here that the notation can become confusing and the idea of doubling enters. We must start with classical solutions of the Dirac equation. These are already "complex valued". The associated complex conjugation operator is of course what one calls "charge conjugation". (In the Majorana representation that we are using there is no need for an explicit charge conjugation matrix). However this charge conjugation operator when extended to the quantum theory is expected to be a linear rather than an antilinear operator. It follows that strictly speaking that Dirac spinors do not take their values in the usual complex numbers of quantum mechanics but in a commutative field (in the algebraic sense of those words) which is of course isomorphic but not naturally so in the mathematical sense and not physically equivalent. We could adopt the notation suggested earlier and introduce a new imaginary unit \(i'\). If the Dirac spinors corresponded to electrons in QED for example the new unit \(i'\) corresponds to an electromagnetic gauge transformation of \(\pi/2\). Now let us turn to the quantum theory. We must further complexify the space of classical solutions of the Dirac equation by taking the tensor product with the usual quantum mechanical complex numbers. We then impose on this extended space a choice of positive and negative frequency, i.e. a complex structure, in the same way that did for Majorana spinors using the reflection map \(\theta\). Using the notation \(i\) for this complex structure we have that \(i\) and \(i'\) commute.

The problem becomes more confusing when we want to consider Weyl spinors. As pointed out earlier, in Lorentzian spacetime, pointwise, we may obtain Weyl spinors by using \(\gamma_5\) as a complex structure. This means that we need to act on the Dirac spinors with the projection operator:
\[
\frac{1}{2}(1 - i'\gamma_5)
\]
to obtain Weyl spinors satisfying:

\[ \gamma_5 \psi = i \psi. \]

Now since \( \gamma_5 \) does not commute with \( \gamma_0 \), the two complex structures, corresponding to the distinction between two different chiralities and between particle and antiparticle respectively do not commute.

7 Conclusion

In section 2 we summarized the relation between the direct sum decomposition of the space of real classical solutions of the Klein-Gordon system with the choice of a complex structure. Then we focused our attention to the question how the complex structure might be related with geometrical properties of spacetime. If we follow the ideas of the no-boundary proposal and the "tunneling of the universe from nothing" then we have to assume a close physical relation between Lorentzian and Riemannian sections of space. So the analytic continuation of a Lorentzian spacetime to a Riemannian section is not only formal and we followed the ideas of Euclidean field theory. We found that we could relate analytic properties of the classical solutions in a Riemannian section of space with a preferred complex structure on the nucleation surface. So in a more popular way we can say, that in our construction with the beginning of time a preferred decomposition of matter in particles and anti-particles is given, which is strictly related with non-local properties of wave-mechanics "before the creation of the universe".

References

[1] G. W. Gibbons, J. B. Hartle, Phys. Rev. D 42, 2458 (1990)

[2] A. Ashtekar, A. Magnon, Proc. R. Soc. Lond. A 346, 375 (1975)

[3] K. Osterwalder, R. Schrader, Comm. Math. Phys. 31, 83 (1973)

[4] A. Uhlmann, "On the Quantization in Curved Spacetime." in Proceedings of the 1979 Serpukhov International Workshop on High Energy Physics; Czech. J. Phys. B31 1249 (1981), B32 573 (1982); Abstracts of Contributed Papers to GR9 (1980)
[5] G. W. Gibbons, "Quantization about Classical Background Metrics" in Proceedings of the 9th G.R.G. Conference ed. E. Schmutzer, Deutscher Verlag der Wissenschaft (1981)

[6] J. Glimm, A. Jaffe, "Quantum Physics", Springer New York (1981)

[7] G. F. De Angelis, D. de Falco and G. Di Genova, Comm. Math. Phys. 103, 297 (1985)

[8] A. Ashtekar, R. S. Tate, C. Uggla, Syracuse preprint SU-GP 92/2-5 (1992)

[9] G. W. Gibbons, S. W. Hawking, "Selection rules for topology change", Comm. Math. Phys. (in press)

[10] K. Osterwalder, R. Schrader, Phys. Rev. Lett. 29, 1423 (1972)

[11] J. Fröhlich, K. Osterwalder, E. Seiler, Ann. Phys. (N.Y.) 118, 461 (1983)

[12] H. Nicolai, Nucl. Phys. B140, 294 (1978)

[13] E. A. Tagirov, Ann. Phys. (N.Y.) 76, 561 (1973)

[14] G. W. Gibbons, C. N. Pope, Comm. Math. Phys. 61, 239 (1978)

[15] D. N. Page, Phys. Lett 78B, 249 (1978)

[16] B. L. Hu, Phys. Rev. D 8, 1048 (1973)

[17] A. S. Dawydov, "Quantum Mechanics", Neo Press, Ann Arbor (1966)

[18] E. L. Ince, "Ordinary Differential Equations", Dover Publications, New York (1944)

[19] P. D. D'Eath, J. J. Halliwell, Phys. Rev. D 35, 1100 (1987)