ALGEBRAIC EXTENSIONS OF GLOBAL FIELDS WITH ONE-DIMENSIONAL LOCAL CLASS FIELD THEORY

I.D. CHIPCHAKOV

Abstract. Let $E$ be an algebraic extension of a global field $E_0$ with a nontrivial Brauer group $\text{Br}(E)$, and let $P(E)$ be the set of those prime numbers $p$, for which $E$ does not equal its maximal $p$-extension $E_p$. This paper shows that $E$ admits one-dimensional local class field theory if and only if there exists a system $V(E) = \{v(p) : p \in P(E)\}$ of (nontrivial) absolute values, such that $E_p \otimes_{E} E_{v(p)}$ is a field, where $E_{v(p)}$ is the completion of $E$ with respect to $v(p)$. When this occurs, we determine by $V(E)$ the norm groups of finite extensions of $E$, and the structure of $\text{Br}(E)$. It is also proved that if $P$ is a nonempty set of prime numbers and $\{w(p) : p \in P\}$ is a system of absolute values of $E_0$, then one can find a field $K$ algebraic over $E_0$ with such a theory, so that $P(K) = P$ and the element $\kappa(p) \in V(K)$ extends $w(p)$, for each $p \in P$.

Introduction

The purpose of this paper is to characterize the fields pointed out in the title as well as to describe the norm group $N(R/E)$ of an arbitrary finite extension $R$ of such a field $E$, and to determine the structure of the Brauer group $\text{Br}(E)$. When $\text{Br}(E) \neq \{0\}$, our research proves the validity of the Hasse principle for norm form polynomials of $R/E$. It shows that $N(R/E) = N(\Phi(R)/E)$, for some finite abelian extension $\Phi(R)$ of $E$ (by \cite{10}, Theorem 1.2, this is not necessarily true when $E$ is an arbitrary field admitting one-dimensional local class field theory (abbr, LCFT)). The field $\Phi(R)$ is uniquely determined up-to an $E$-isomorphism by the local behaviour of $R/E$ at the elements of a certain characteristic system $\{v(p) : p \in P(E)\}$ of absolute values of $E$, indexed by the set $P(E)$ of those prime numbers $p$, for which $E$ is properly included in its maximal $p$-extension $E_p$ in a separable closure $E_{\text{sep}}$ of $E$ (see Theorems 2.1 and 2.3). The extension $\Phi(R)/E$ is subject to the classical local reciprocity law, i.e. the quotient group of the multiplicative group $E^*$ by $N(\Phi(R)/E)$ is isomorphic to the Galois group $G(\Phi(R)/E)$. At the same time, this paper proves that if $E_0$ is a global field and $P$ is a set of prime numbers, then the class $\Omega_P(E_0)$ of algebraic extensions $\Sigma$ of $E_0$, which admit LCFT and satisfy the equality $P(\Sigma) = P$, is nonempty. It shows that each $K \in \Omega_P(E_0)$ includes as a subfield a unique minimal element $R(K)$ of $\Omega_P(E_0)$ (see Theorem 2.2 and Corollary 4.4). The established properties of the elements of $\Omega_P(E_0)$ extend the arithmetic basis of LCFT and thereby allow one to determine the scope

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of validity of the norm limitation theorem in LCFT (cf. [7], Sect. 4, and [10], Theorem 1.1 (ii) and Remark 3.3).

Note that a field $E$ is said to admit LCFT, if its finite abelian extensions in $E_{sep}$ are uniquely determined by their norm groups, and for each pair $(M_1, M_2)$ of such extensions, the norm group over $E$ of the compositum $M_1 M_2$ is equal to the intersection $N(M_1/E) \cap N(M_2/E)$ and $N(M_1 \cap M_2/E) = N(M_1/E)N(M_2/E)$. Clearly, this is the case if and only if $E$ admits local $p$-class field theory (i.e. finite abelian extensions of $E$ in $E(p)$ have the same properties), for each $p \in P(E)$. By [9], Theorem 3.1, $E$ admits local $p$-class field theory, for a given $p \in P(E)$, if the $p$-component $\text{Br}(E)_p$ of $\text{Br}(E)$ is nontrivial, and $E$ is a $p$-quasilocal field, i.e. its cyclic extensions of degree $p$ are embeddable as $E$-subalgebras in each central division $E$-algebra of Schur index $p$. By [9], Remark 3.4 (ii), the converse holds when $E$ is an algebraic extension of a global field $E_0$. Then $E$ admits LCFT if and only if it is a strictly primarily quasilocal field (a strictly PQL-field), i.e. $E$ is $p$-quasilocal with $\text{Br}(E)_p \neq \{0\}$, for every $p \in P(E)$. As noted above, the main results of the present paper are obtained by characterizing several types of such fields in terms of valuation theory and thereafter by a local-to-global approach.

Throughout the paper, algebras are assumed to be associative with a unit, simple algebras are supposed to be finite-dimensional over their centres, Brauer groups of fields are viewed as additively written, and Galois groups are regarded as profinite with respect to the Krull topology. The set of prime numbers is denoted by $\mathbb{P}$, and for any abelian torsion group $T$, $T_p$ stands for the $p$-component of $T$, for each $p \in \mathbb{P}$. Absolute values of fields are supposed to be nontrivial, and for each algebra $A$, the considered subalgebras of $A$ contain its unit. As usual, a field $E$ is called formally real, if $-1$ is not presentable over $E$ as a finite sum of squares; we say that $E$ is a nonreal field, otherwise. In what follows, $M(E)$ denotes a system of representatives of the equivalence classes of absolute values of $E$, and for each $n \in \mathbb{N}$, $E^{*n} = \{a^n : a \in E^*\}$. For any field extension $\Phi/E$, $I(\Phi/E)$, $\text{Br}(\Phi/E)$ and $\rho_{\Phi/E}$ denote the set of intermediate fields of $\Phi/E$, the relative Brauer group of $\Phi/E$, and the scalar extension map of $\text{Br}(E)$ into $\text{Br}(\Phi)$, respectively. When $\Phi/E$ is finite, we write $P(\Phi/E)$ for the set of all $p \in \mathbb{P}$ dividing the degree $[\Phi : E]$. The class of central simple $E$-algebras is denoted by $s(E)$, $d(E)$ stands for the class of division algebras $\Delta \in s(E)$, and for each $A \in s(E)$, $[A]$ is the equivalence class of $A$ in $\text{Br}(E)$. By a group formation, we mean a nonempty class $\chi$ of finite groups satisfying the following conditions:

- $\chi$ is closed under the formation of subgroups and homomorphic images;

If $G$ is a finite group, and $H_1, H_2$ are normal subgroups of $G$, such that $G/H_j \in \chi$; $j = 1, 2$, then $G/(H_1 \cap H_2) \in \chi$.

The group formation $\chi$ is called abelian closed, if it is closed under the formation of subgroups, finite direct products and group extensions with abelian kernels (for examples of such formations, see, e.g., [10], Remark 6.1). When this occurs, $\chi$ is saturated, i.e. a finite group $G$ lies in $\chi$, provided that $G/\Phi(G) \in \chi$, where $\Phi(G)$ is the Frattini subgroup of $G$. Our terminology in valuation theory, simple algebras, Brauer groups and abstract abelian groups is standard (as used, e.g., in [3; 22; 11; 27; 25] and [16]), as well
as the one concerning profinite groups, Galois cohomology, field extensions and Galois theory (cf. [28; 20; 21 and 22]).

The paper is organized as follows: Section 1 includes preliminaries on algebraic extensions of a global field $E_0$, used in the sequel. The main results of the paper are stated in Section 2, and proved in Sections 3, 4 and 5. Section 3 presents a characterization of the $p$-quasilocal fields in the class of algebraic extensions of $E_0$, and a proof of the local reciprocity law for strictly PQL-fields algebraic over $E_0$. In Section 4, we characterize the minimal algebraic strictly PQL-extensions of $E_0$, and we describe the Brauer groups of strictly PQL-fields algebraic over $E_0$. In Section 5 we prove the Hasse norm theorem for a finite Galois extension $M/E$, where $E \in I(E_0/E_0)$ and the group $p\text{Br}(E) = \{b \in \text{Br}(E) : pb = 0\}$ is finite, for every $p \in P(E)$. This, applied to the case where $E$ is strictly PQL, allows us to establish the claimed properties of norm groups of $E$.

1. Preliminaries on the local behaviour of algebraic extensions of global fields

Let $E_0$ be a global field, $\overline{E}_0$ an algebraic closure of $E_0$ and $A$ be a finite dimensional $E$-algebra, for some $E \in I(\overline{E}_0/E_0)$. Suppose that $B$ is a basis of $A$, $\Sigma$ is a finite subset of $A$, $\Sigma_1(B)$ is the set of structural constants of $A$, and $\Sigma_2(B)$ is the one of coordinates of elements of $\Sigma$, relative to $B$. Then the extension $E_1$ of $E_0$ generated by the union $\Sigma_1(B) \cup \Sigma_2(B)$ is finite, and the following statements are true (cf. [P, Sect. 9.2, Proposition c]):

(1.1) (i) The subring $A_1$ of $A$ generated by $E_1 \cup B$ is an $E_1$-subalgebra of $A$, such that the $E$-algebras $A_1 \otimes_E E$ and $A$ are isomorphic and $\Sigma \subseteq A_1$;
(ii) If $A/E$ is a Galois extension and $\Sigma$ contains the roots in $A$ of the minimal polynomial over $E_0$ of a given primitive element of $A/E$, then $A_1/E_1$ is Galois with $G(A_1/E_1) \cong G(A/E)$ (canonically);
(iii) If $A \in d(E)$, then $\Sigma$ can be chosen so that $A_1 \in d(E_1)$, $\exp(A_1) = \exp(A)$ and $\text{ind}(A_1) = \text{ind}(A)$.

Statement (1.1) enables one to generalize some known properties of central division algebras over global fields and local fields, as follows:

(1.2) If $E_0$ is a global or locally compact field and $E \in I(\overline{E}_0/E_0)$, then every $D \in d(E)$ is cyclic with $\text{ind}(D) = \exp(D)$; in addition, if $E_0$ is locally compact, then $\rho_{E_0/E}$ is surjective.

An absolute value $v$ of a field $E$ is said to be $\chi$-Henselian with respect to a group formation $\chi$, if $v$ is nonarchimedean and uniquely extendable to an absolute value of the compositum $E(\chi)$ of finite Galois extensions of $E$ in $E_{\text{sep}}$ with Galois groups belonging to $\chi$ (or, equivalently, if $v$ extends uniquely on each finite Galois extension $M$ of $E$ with $G(M/E) \in \chi$). It follows from Galois theory and general properties of valuation prolongations (cf. [3], Ch. VI) that $E$ is $\chi$-Henselian if and only if it is $\bar{\chi}$-Henselian, where $\bar{\chi}$ is the minimal saturated group formation including $\chi$. This allows us to restrict our considerations of $\chi$-Henselity to the special case where $\chi$ is saturated. When $\chi$ is the formation of $p$-groups, for a given $p \in P$, the valuation $v$ is called $p$-Henselian. We say that $v$ is Henselian, if it
has a unique prolongation $v_F$ on each $F \in I(E/E)$, i.e. if $v$ is Henselian with respect to the formation of all finite groups. It is known \cite{12, 29}, that in this case, every finite dimensional division $E$-algebra has a valuation extending $v$ (we also denote it by $v$). The following statements describe some basic relations between central division algebras over a Henselian valued field $(E, v)$, and the corresponding algebras over the completion $E_v$ (see, for example, \cite{8}):

(1.3) (i) The completion $D_v$ of each finite dimensional division $E$-algebra $D$ with a centre $Z(D) \in I(E_{\text{sep}}/E)$ is $E_v$-isomorphic to $D \otimes_E E_v$;
(ii) The mapping $p_{E/E_v}$ is an isomorphism;
(iii) A finite separable extension $L$ of $E$ embeds in an algebra $U \in d(E)$ if and only if $L_v$ embeds in $U_v$ over $E_v$.

When $v$ is Henselian, finite extensions of $E_v$ in $E_{v, \text{sep}}$ can be characterized as follows (cf. \cite{4}, Ch. VI, Sect. 8, No 2, Cor. 2, and \cite{8}):

(1.4) Every finite separable extension $L$ of $E_v$ is $E_v$-isomorphic to $\bar{L}_v$, where $\bar{L}$ is the separable closure of $E$ in $L$. In order that $L/E_v$ is a Galois extension it is necessary and sufficient that so is $\bar{L}/E$; when this holds, $\mathcal{G}(L/E_v) \cong \mathcal{G}(\bar{L}/E)$.

The presented results will usually be applied to the case of an algebraic extension $E$ of a local field $E_0$. It is well-known that the natural absolute value, say $v_0$, of $E_0$ is Henselian, whence its unique prolongation $v$ on $E$ preserves this property. Also, it follows from (1.1) and the classical local class field theory (cf. \cite{4}, Ch. VI, Sect. 1) that finite extensions of $E$ and central division $E$-algebras are related as follows:

(1.5) (i) A finite extension $L$ of $E$ embeds in an algebra $\Delta \in d(E)$ if and only if $[L: E]$ divides $\text{ind}(\Delta)$; $[\Delta] \in \text{Br}(L/E)$ if and only if $\text{ind}(\Delta) | [L: E]$;
(ii) The mapping $p_{E_0/E}$ is surjective;
(iii) The fields $E$ and $E_v$ admit local $p$-class field theory, for a given $p \in P(E)$, if and only if $\text{Br}(E)_p \neq \{0\}$; in order that $\text{Br}(E)_p = \{0\}$ it is necessary and sufficient that $E$ contains as a subfield an extension $E_n$ of $E$ of degree divisible by $p^n$, for each $n \in \mathbb{N}$;
(iv) If $\text{Br}(E)_p \neq \{0\}$, then there exists $K_p \in I(E/E_0)$, such that $[K_p: E_0] \in \mathbb{N}$ and $p \not| [L: K_p]$, for any $L \in I(E/K_p)$ with $[L: K_p] \in \mathbb{N}$; also, $\rho_{K_p/E}$ maps $\text{Br}(K_p)_p$ bijectively on $\text{Br}(E)_p$; in particular, $\text{Br}(E)_p$ is isomorphic to the quasicyclic $p$-group $\mathbb{Z}(p^\infty)$, i.e. to the $p$-component of the quotient group $\mathbb{Q}/\mathbb{Z}$ of the additive group of rational numbers by the subgroup of integers.

The following lemma is essentially a special case of the Grunwald-Wang theorem \cite{30}, (see also \cite{1}, Ch. 10, Theorem 5, and \cite{24}).

**Lemma 1.1.** Assume that $E$ is an algebraic extension of a global field $E_0$, \{$v_1, \ldots, v_l$\} is a finite subset of $M(E)$, $\bar{F}_1, \ldots, \bar{F}_l$ are cyclic extensions of $E_{v_1}, \ldots, E_{v_l}$, respectively, of degrees not divisible by 4, and $n = \text{l.c.m.}\{[\bar{F}_i: E_{v_i}] : i = 1, \ldots, l\}$. Then there is a cyclic extension $F/E$ with $[F: E] = n$ and $F_{v_i} \cong \bar{F}_i$ over $E_v$ whenever $i \in \{1, \ldots, l\}$, $v'_i \in M(F)$ and $v'_i$ extends $v_i$. 

Proof. It follows from Krasner’s lemma (cf. [23], Ch. II, Proposition 3) that each \( \widetilde{F}_i/E_{\alpha_i}, i \leq t \), has a primitive element say \( \alpha_i \) that is algebraic over the closure of \( E_0 \) in \( E_{\alpha_i} \). Hence, by (1.1) (ii), \( I(E/E_0) \) contains a finite extension \( L \) of \( E_0 \), whose absolute values \( w_1, \ldots, w_t \) induced by \( v_1, \ldots, v_t \), respectively, are pairwise nonequivalent, and for each \( i \leq t \), there exists an \( E_{\alpha_i} \)-isomorphism \( \widetilde{F}_i \cong \mathcal{F}_i \otimes_{L_{w_i}} E_{\alpha_i} \), for some cyclic extension \( \mathcal{F}_i \) of \( L_{w_i} \) in \( \widetilde{F}_i \) (with \( [\mathcal{F}_i : L_{w_i}] = [\widetilde{F}_i : E_{\alpha_i}] \)). By Grunwald-Wang’s theorem, \( L \) has a cyclic extension \( \Phi \) of degree \( n \) with \( \Phi_{w_i'} \cong \mathcal{F}_i \) over \( L_{w_i} \), whenever \( i \in \{1, \ldots, t\} \), \( w_i' \in M(\Phi) \) and \( w_i' \) extends \( v \). It is now easy to see that \( \Phi \otimes_L E := F \) is a field with the properties required by Lemma 1.1.

The generalization of the Grunwald-Wang theorem for independent absolute values of arbitrary fields, contained in [LR], enables one in conjunction with Galois theory to deduce the following result:

(1.6) If \( E \) is a field and \( v_1, v_2 \) are \( \Omega \)-Henselian absolute values of \( E \), where \( \Omega \) is an abelian closed group formation containing finite \( p \)-groups whenever \( p \in \mathbb{P} \) and there exists \( \Lambda_p \in \Omega \) of order divisible by \( p \), then \( v_1 \) and \( v_2 \) are equivalent unless \( E(\Omega) = E \).

Let now \( E_0 \) be a global field, \( E \in I(E_0/E_0), v \in M(E), v_0 \) the absolute value of \( E_0 \) induced by \( v \), \( E_v \) a completion of \( E \) with respect to \( v \), \( \bar{v} \) the absolute value of \( E_v \) continuously extending \( v \), \( E_0' \) the closure of \( E_0 \) in \( E_v \), \( E' = E.E_0' \), \( v' \) the absolute value of \( E' \) induced by \( \bar{v} \), and \( p \) a prime number.

Applying (1.4) and (1.5), one obtains the following:

(1.7) (i) \( E'/E_0' \) is an algebraic extension and \( E_0' \) is a completion of \( E_0 \) with respect to \( v_0 \); in particular, \( E_0' \) is isomorphic to \( \mathbb{R}, \mathbb{C} \) or a local field;

(ii) If \( v \) is archimedean, then \( E_v \) is isomorphic to \( \mathbb{R} \) or \( \mathbb{C} \); hence, \( Br(E') = \{0\} \) unless \( p = 2 \) and \( E_v \cong \mathbb{R} \);

(iii) If \( v \) is nonarchimedean, then \( v' \) is Henselian; it induces on \( E_0' \) the continuous prolongation of \( v_0 \);

(iv) The absolute Galois groups \( G_{E'} \) and \( G_{E_v} \) are continuously isomorphic, and \( \rho_{E'/E_v} \) is an isomorphism.

Next we present (slightly generalized) a part of the classical Brauer-Hasse-Noether-Albert theorem (see [31], Ch. XIII, Sects. 3 and 6).

**Proposition 1.2.** Let \( E_0 \) be a global field and \( E \) algebraic extension of \( E_0 \). Then \( \rho_{E/E_0} \) is surjective, for each \( v \in M(E) \), and the homomorphism of \( Br(E) \) into the direct sum \( \bigoplus_{v \in M(E)} Br(E_v) \) mapping, for each \( \sigma \in s(E) \), \( [A] \) into the sequence \( [A \otimes_E E_v] : v \in M(E) \), is injective.

Proof. Our latter assertion is a special case of [31], Lemma 3.3. For the proof of the former one, fix an absolute value \( v \in M(E) \) as well as an algebra \( \Delta \in d(E_v) \), and denote by \( E_v' \) and \( E' \) the fields from \( I(E_v/E_0) \) defined in (1.7). By (1.7) (iv), there exists an \( E_{\alpha_i} \)-isomorphism \( \Delta \cong \Delta \otimes_{E_v} E_{v_0} \), for some \( \Delta \in d(E') \), uniquely determined up-to an \( E' \)-isomorphism, and by (1.2), \( [\Delta] = [\Delta_0 \otimes_{E_0'} E'] \), in \( Br(E') \), for a suitably chosen \( \Delta \in d(E_0') \). Using consecutively (1.7) (i) and the description of \( Br(E_0) \) by class field theory
(see, e.g., [31], Ch. XIII, Sect. 6), one obtains the existence of an algebra \( \Delta_0 \in d(E_0) \), such that \([\Delta_0 \otimes_{E_0} E'_0] = [\hat{\Delta}_0] \) \((\mathrm{Br}(E_0))\). These observations and the general behaviour of tensor products under scalar extensions (cf. [27], Sect. 9.4, Corollary a) indicate that \([\Delta] = [(\Delta_0 \otimes_{E_0} E) \otimes_{E} E_v] \), in \(\mathrm{Br}(E_v)\), which proves the former assertion of Proposition 1.2. \qed

Applying (1.7) (ii), Proposition 1.2 and the latter part of (1.5) (iii), one proves the following result of Fein and Schacher [13], Sect. 2, Theorem 4):

(1.8) If \(E_0\) is a global or local field, \(E \in I(\mathbb{F}_0/E_0)\) and \(\mathrm{Br}(E)_p = \{0\}\), for some \(p \in \mathbb{P}\), then \(\mathrm{Br}(E_1)_p = \{0\}\), for every \(E_1 \in I(\mathbb{F}_0/E)\).

**Lemma 1.3.** Let \(F\) be a field, \(w \in M(F)\), \(M/F\) a Galois extension, and \(P_w(M)\) the set of prolongations of \(w\) on \(M\). Then the action of \(G(M/F)\) on \(P_w(M)\) by the rule \(v \rightarrow v \circ \varphi\): \(\varphi \in G(M/F), v \in P_w(M)\), is transitive.

**Proof.** The assertion is well-known in case \([M:F] \in \mathbb{N}\) (cf. [22], Ch. IX, Proposition 11), so we assume that \([M:F] = \infty\). Fix prolongations \(v_1\) and \(v_2\) of \(w\) on \(M\), denote by \(\Sigma(M/F)\) the set of finite Galois extensions of \(F\) in \(M\), and for each \(L \in \Sigma(M/F)\), let \(v_1(L)\) and \(v_2(L)\) be the absolute values of \(L\) induced by \(v_1\) and \(v_2\), respectively. We show that the sets \(P(L) = \{\sigma \in G(M/F) : \sigma(v_1)\text{ extends }v_2(L)\}\) are closed, for all \(L \in \Sigma(M/F)\), and their intersection \(P\) is nonempty. It follows from Galois theory that \(P(L)\) is a coset of a subgroup \(U_L\) of \(G(M/F)\) including \(G(M/L)\), for each \(L \in \Sigma(M/F)\). Observing also that \(P(L_1...L_s) \subseteq \cap_{j=1}^{s} P(L_j)\) whenever \(s \in \mathbb{N}\) and \(L_1,...,L_s \in \Sigma(M/F)\), one obtains from the compactness of \(G(M/F)\) that \(P \neq \emptyset\). Since \(g(v_1) = v_2\), for every \(g \in P\), this proves Lemma 1.3. \qed

## 2. Statements of the main results

It is clear from Galois theory and the general description of relative Brauer groups of cyclic extensions [P, Sect. 15.1, Proposition b] that a field \(E\) with \(\mathrm{Br}(E) = \{0\}\) is strictly PQL if and only if \(P(E) = \phi\), i.e. \(E\) does not possess proper abelian extensions. The following result characterizes the strictly PQL-fields \(E\) with \(P(E) \neq \phi\) in the set \(I(\mathbb{F}_0/E_0)\), for a global field \(E_0\):

**Theorem 2.1.** Let \(E_0\) be a global field and \(E\) an extension of \(E_0\) in \(\overline{E_0}\), such that \(P(E) \neq \phi\). Then the following conditions are equivalent:

(i) \(E\) is a strictly PQL-field;

(ii) For each \(p \in P(E)\), \(\mathrm{Br}(E)_p \neq \{0\}\) and there exists an absolute value \(v(p)\) of \(E\), such that the tensor product \(E(p) \otimes_{E} E_{v(p)}\) is a field.

When these conditions hold, \(v(p)\) is uniquely determined, up-to an equivalence, for each \(p \in P(E)\). Moreover, \(\rho_{E/E_{v(p)}}\) maps \(\mathrm{Br}(E)_p\) bijectively on \(\mathrm{Br}(E_{v(p)})_p\), and \(E(p) \otimes_{E} E_{v(p)} \cong E_{v(p)}(p)\) as \(E_{v(p)}\)-algebras.

**Definition 1.** Under the hypotheses of Theorem 2.1, a system of absolute values of \(E\) is called characteristic, if it is subject to the restrictions in (ii).
Theorem 2.1, Proposition 1.2 and statements (1.7) (iv) and (1.8) indicate that if $E$ is an algebraic extension of a global field $E_0$, $F \in I(E/E_0)$, and for each $p \in P(E)$, $w(p)$ is the absolute value of $F$ induced by $v(p)$, then the groups $\text{Br}(F)_p$ and $\text{Br}(F_{w(p)})_p$ are nontrivial. Therefore, our research concentrates on the study of the following class of fields:

**Definition 2.** Let $E_0$ be a global field, $F$ an extension of $E_0$ in $\overline{E_0}$ with $\text{Br}(F) \neq \{0\}$, $P$ a nonempty set of prime numbers $p$, for which $\text{Br}(F)_p \neq \{0\}$ and $\{w(p): p \in P\}$ a system of absolute values of $F$, such that $\text{Br}(F_{w(p)})_p \neq \{0\}$. Denote by $\Omega(F, P, W)$ the set of all $E \in I(\overline{E_0}/F)$ with the following properties:

(i) $E$ is a strictly PQL-field and $P(E) = P$;

(ii) $E$ possesses a characteristic system $\{v(p): p \in P\}$ (of absolute values), such that $v(p)$ is a prolongation of $w(p)$, for each $p \in P$.

Our main results about $\Omega(F, P, W)$ can be stated as follows:

**Theorem 2.2.** With assumptions and notations being as above, $\Omega(F, P, W)$ is a nonempty set, for which the following assertions hold true:

(i) Every $E \in \Omega(F, P, W)$ possesses a unique subfield $R(E)$ that is a minimal element of $\Omega(F, P, W)$;

(ii) The Galois closure over $F$ of any minimal element $E$ of $\Omega(F, P, W)$ has a prosolvable Galois group; furthermore, $E$ is presentable as a union $\bigcup_{n=1}^{\infty}K_n$, where $K_1 = E$ and $K_{n+1}$ is an intermediate field of the maximal extension of $K_n$ in $E_{\text{sep}}$ with a pronilpotent Galois group;

(iii) If $E$ is a minimal element of $\Omega(F, P, W)$, $p \in P$ and $F_{w(p)}$ is the closure of $F$ in $E_{v(p)}$, then the degrees of finite extensions of $F_{w(p)}$ in $E_{v(p)}$ are not divisible by $p$;

(iv) For each linearly ordered subset $\Lambda$ of $\Omega(F, P, W)$, the intersection $\cap_{E \in \Lambda}E$ lies in $\Omega(F, P, W)$.

Let $E_0$ be a global field, $E \in I(\overline{E_0}/E_0)$ and $R$ a finite extension of $E$. An element $\alpha \in E^*$ is called a local norm of $R/E$, if $\alpha \in N(R_{v}/E_{v})$, for every absolute value $v$ of $E$, and each prolongation $v'$ of $v$ on $R$. Denote by $N_{\text{loc}}(R/E)$ the multiplicative group of local norms of $R/E$. As it turns out, when $E$ is a strictly PQL-field, $N_{\text{loc}}(R/E)$ is a subgroup of $N(R/E)$ and both are determined by the local behaviour of $R/E$.

**Theorem 2.3.** Let $E_0$ be a global field, $E \in I(\overline{E_0}/E_0)$ a field admitting LCFT with $P(E) \neq \phi$, and $V = \{v(p): p \in P(E)\}$ a characteristic system of $E$. Also, let $R$ be a finite separable extension of $E$ in $\overline{E_0}$, $M$ a normal closure of $R$ in $\overline{E_0}$ over $E$, and $\Pi(M/E) = P(M/E) \cap \Pi(E)$. Then there exist abelian finite extensions $F_1$ and $F_2$ of $E$ in $\overline{E_0}$, and a real number $\varepsilon > 0$, for which:

(i) $N(F_1/E) = N(R/E)$, $N(F_2/E) = N_{\text{loc}}(R/E)$, $F_1 \subseteq F_2$ and $[F_1 : E]$ is not divisible by any prime number out of $\Pi(M/E)$; the groups $E^*/N(R/E)$ and $E^*/N_{\text{loc}}(R/E)$ are isomorphic to $\mathcal{G}(F_1/E)$ and $\mathcal{G}(F_2/E)$, respectively;

(ii) $F_1 = F_2$, provided that $\mathcal{G}(M/E)$ is nilpotent or $\Pi(M/E) = \phi$; in the latter case, $F_1 = F_2 = E$;
Remark 2.5. It is well-known that if $E$ is an algebraic extension of a global field of characteristic $q > 0$, and $E^q = \{ e^q : e \in E \}$, then $[E : E^q]$ is equal to 1 or $q$. This implies that if $R/E$ is a finite extension and $R_0$ is the separable closure of $E$ in $R$, then $N(R/R_0) = R_0^*$ and $N(R/E) = N(R_0/E)$, i.e. $N(R/E)$ can be fully described by applying Theorem 2.3. One also sees that norm forms of $R/E$ are subject to Hasse’s principle.

The following statement shows the validity the Hasse norm principle for finite Galois extensions of strictly PQL-fields.

Proposition 2.6. Assume that $E_0$ is a global field, $E$ is an algebraic strictly PQL-extension of $E_0$, $P(E) \neq \{ \phi \}$ and $V = \{ v(p) : p \in P(E) \}$ is a characteristic system of $E$. Also, let $M$ be a finite Galois extension of $E$ in $\overline{E_0}$, and let $\Pi(M/E) = P(M/E) \cap P(E)$. Then there exists a finite abelian extension $\tilde{M}$ of $E$ satisfying the following conditions:

(i) $N(\tilde{M}/E) = N(M/E) = N_{loc}(M/E)$;
(ii) $[\tilde{M} : E] = [M : E]$; in particular, $\tilde{M} = E$ in case $\Pi(M/E) = \{ \phi \}$;
(iii) For each $p \in P(M/E)$ and any absolute value $v(p)' \in \tilde{M}$ extending $v(p)$, the maximal abelian $p$-extension of $E_{v(p)}$ in $M_{v(p)'}$ is $E_{v(p)}$-isomorphic to $\tilde{M}_p \otimes_E E_{v(p)}$, where $\tilde{M}_p = \tilde{M} \cap E_p$.

The field $\tilde{M}$ is uniquely determined by $M$, up-to an $E$-isomorphism.

Proof. This follows from Theorem 2.3, Proposition 2.4 and Lemma 1.3. \qed

Under the hypotheses of Proposition 2.6, one obtains from Theorem 2.1 and Galois theory that $M_0 \subseteq \tilde{M}$, where $M_0$ is the maximal abelian extension of $E$ in $M$. In addition, it becomes clear that $\tilde{M} = M_0$, provided that $G(M/E)$ is nilpotent. This is not necessarily true when $G(M/E) \cong G$, for any given nonnilpotent finite group $G$ (cf. [7], Sect. 4). Also, Proposition 2.6 shows that the index $[E^* : N(M/E)]$ divides the order $o(G)$ of $G$ and is divisible by $|G : [G,G]|$, $[G,G]$ being the commutator subgroup of $G$. The concluding result of this Section characterizes those $G$, for which $M$ can be
chosen so that \( G(M/E) \cong G \) and \( |E^* : N(M/E)| = o(G) \). It solves for such \( G \) the inverse problem concerning the admissible values of \( |E^* : N(M/E)| \).

**Proposition 2.7.** In the setting of Proposition 2.6, if \( [M : E] = [\widehat{M} : E] \), then the Sylow \( p \)-subgroups of \( G(M/E) \) are abelian, for every \( p \in \mathbb{P} \).

Conversely, let \( G \) be a finite group with \( o(G) = m \) and abelian Sylow subgroups, and let \( n \in \mathbb{N} \) divide \( m \) and be divisible by \( [G : [G, G]] \). Then there exist subfields \( E_n \) and \( M_n \) of \( \overline{\mathbb{Q}} \), such that \( E_n \) is a strictly PQL-field, \( M_n/E_n \) is a Galois extension with \( G(M_n/E_n) \cong G \), and \( |E_n^* : N(M_n/E_n)| = n \).

**Proof.** Our former assertion can be deduced from Proposition 2.6 and Galois theory. The proof of the latter one relies on the existence of subfields \( E_m \) and \( M_m \) of \( \overline{\mathbb{Q}} \) satisfying the following conditions (see [7], Sect. 4):

\[(2.1) \ (i) \ E_m \text{ is a strictly PQL-field, } P(E_m) := P \text{ equals the set of all prime numbers, and the characteristic system } \{v_m(p) : p \in P\} \text{ has the property that } v_m(p) \text{ induces on } \mathbb{Q} \text{ the natural } p\text{-adic absolute value, for each } p \in P; \]
\[(ii) \ M_m/E_m \text{ is a Galois extension with } G(M_m/E_m) \cong G, \text{ and for each } p \in P, \ M_{m,v_m(p)}/E_{m,v_m(p)} \text{ is Galois with } G(M_{m,v_m(p)}/E_{m,v_m(p)}) \text{ isomorphic to a Sylow } p\text{-subgroup of } G, \text{ where } v_m(p)^t \text{ is a prolongation of } v_m(p) \text{ on } M.\]

Let \( P(G) \) be the set of prime divisors of \( m \), and let \( G_p \) be a Sylow \( p \)-subgroup of \( G \), for each \( p \in P(G) \). It is clear from Proposition 2.6, (2.1) (ii) and the commutativity of the groups \( G_p : p \in P(G) \), that \( E_m \) and \( M_m \) have the properties required by Proposition 2.7. In particular, \([M_m : E_m] = m\), where \( M_m \) is the abelian extension of \( E_m \) associated with \( M_m/E_m \), by Proposition 2.6. We show that \( \widehat{M}_m \) possesses a subfield \( \Phi_m \), such that \( \Phi_m \cap M_m,ab = E_m \) and \( \Phi_m M_m,ab = M_m \). Identifying as we can (cf. [20], Ch. 7, Corollary 5.3) the character groups \( C(G) \) and \( C(G_p) \) with the cohomology groups \( H^1(G, \mathbb{Q}/\mathbb{Z}) \) and \( H^1(G_p, \mathbb{Q}/\mathbb{Z}) \), respectively, one obtains from [28], Ch. I, Proposition 9) that the composition \( \text{cor}_p \circ \text{res}_p : C(G) \to C(G_p) \) and the corestriction \( \text{cor}_p : C(G_p) \to C(G) \) induces an automorphism of the \( p \)-component \( C(G)_p \) of \( C(G) \). This implies \( \text{res}_p \), maps \( C(G)_p \) bijectively on a pure subgroup of \( C(G_p) \). As pure subgroups of \( C(G_p) \) are direct summands in \( C(G_p) \) (cf. [16], Theorem 24.5), the existence of \( \Phi_m \) follows now from Galois theory. It is not difficult to see that \( \Phi_m \) contains as a subfield an abelian extension \( F_n \) of \( E_m \) of degree \( m/n \), for each \( n \in \mathbb{N} \) subject to the restrictions of Proposition 2.7. Observing that \( F_n \cap M_m = E_m \), one obtains from Galois theory that \( (M_n F_m)/F_n \) is a Galois extension with \( G((M_n F_m)/F_n) \cong G \). Fix an absolute value \( w_n(p) \) of \( F_n \) extending \( v_m(p) \), for each \( p \in P \), and put \( W_n = \{w_n(p) : p \in P\} \). Since, by (2.1) (i) and Theorem 2.1, \( \text{Br}(E_n) \cong \mathbb{Q}/\mathbb{Z} \), we have \( \text{Br}(F_n,w_n(p))_p \neq \{0\}, p \in P \), i.e. \( \Omega(F_n, P, W_n) \) is well-defined. Applying Theorem 2.2 (iii), one concludes that if \( M_n = M_m E_n \), for some minimal element \( E_n \) of \( \Omega(F_n, P, W_n) \), then \( M_n/E_n \) has the property required by Proposition 2.7. \( \square \)
3. **Algebraic p-quasilocal extensions of global fields and Galois groups of their maximal p-extensions**

In this Section we characterize algebraic extensions of global fields admitting local p-class field theory, for a given prime number p, and thereby prove Theorem 2.1. Our argument is presented by the following three lemmas.

**Lemma 3.1.** Let E be an algebraic extension of a global field $E_0$, such that $E(p) \otimes_E E_{v_1}$ is a field, for some $p \in \mathbb{P}$ and $v_1 \in M(E)$, and let $v_2 \in M(E)$ be nonequivalent to $v_1$. Then:

(i) $E_{v_2}(p) = E_{v_2}$ and $E(p) \otimes_E E_{v_1}$ is $E_{v_2}$-isomorphic to $E_{v_1}(p)$;

(ii) $Br(E_{v_2})_p = \{0\}$ and $\rho_{E/E_{v_1}}$ maps $Br(E)_p$ bijectively on $Br(E_{v_1})_p$.

**Proof.** It is clear from Galois theory (cf. [21], Proposition 2.11) that $E(p)$ embeds in $E_{v_1}(p)$ as an $E$-subalgebra. Identifying $E(p)$ with its $E$-isomorphic copy in $E_{v_1}(p)$, one gets from the condition on $v_1$ that $E(p) \cap E_{v_1} \cong E$. Note also that every extension $L$ of $E_{v_1}$ in $E_{v_1}(p)$ of degree $p^n$ equals $L E_{v_1}$, for some $L \in I(E(p)/E)$ with $[L : E] = p^n$. If $n = 1$, this is implied by Lemma 1.1, so we assume that $n \geq 2$. It follows from the subnormality of proper subgroups of finite $p$-groups (cf. [22], Ch. I, Sect. 6) and Galois theory that $L$ contains as a subfield a cyclic extension $L_0$ of $E_{v_1}$ of degree $p$. As $L_0(p) = E(p)$ and $E(p) \otimes_{L_0} L_0, v'$ is a field, $v'$ being a prolongation of $v$ on $L_0$ (unique when $v$ is nonarchimedean), this enables one to complete the proof of our assertion, arguing by induction on $n$. The obtained result shows that $E_{v_1}(p) = E(p).E_{v_1}$ and there is an $E_{v_1}$-isomorphism $E_{v_1}(p) \cong E(p) \otimes_E E_{v_1}$. Using again Galois theory, one also concludes that $G(E(p)/E) \cong G(E_{v_1}(p)/E_{v_1})$. It is now clear from Lemma 1.1 and the condition on $E(p)$ and $E_{v_1}$ that $p \notin P(E_{v_1})$. Hence, by (1.2), $Br(E_{v_2})_p = \{0\}$, so Lemma 3.1 (ii) reduces to a consequence of Proposition 1.2. □

**Lemma 3.2.** An algebraic extension $E$ of a global field $E_0$ admits local p-class field theory, for a given $p \in P(E)$, if and only if $Br(E)_p \neq \{0\}$ and there exists an absolute value $v$ of $E$ such that $E(p) \otimes_E E_v$ is a field. When this occurs, $v$ is unique, up-to an equivalence.

**Proof.** It is clear from Lemma 3.1 that $Br(E)_p = \{0\}$ in case $E$ has nonequivalent absolute values $v_1$ and $v_2$ for which $E(p) \otimes_E E_{v_1}$ and $E(p) \otimes_E E_{v_2}$ are fields. This proves the concluding statement of Lemma 3.2. Suppose now that $E$ admits local p-class field theory. This means that $E$ is p-quasilocal with $Br(E)_p \neq \{0\}$. Hence, by Merkurjev’s theorem [25], Sect. 4, Theorem 2, there exists $D \in d(E)$ of index $p$, by Proposition 1.2, $D \otimes_E E_v \in d(E_v)$, for some absolute value $v$ of $E$. We show that $E(p) \otimes_E E_v$ is a field by assuming the opposite. In view of Galois theory and the subnormality of proper subgroups of finite $p$-groups, then $E_v$ contains as a subfield a cyclic extension $L$ of $E$ of degree $p$. Since the algebras $D \otimes_E E_v$ and $(D \otimes_E L) \otimes_L E_v$ are isomorphic over $E_v$ (cf. [27], Sect. 9.4, Corollary a), this leads to the conclusion that $[D] \notin Br(L/E)$. The obtained result contradicts the assumption that $E$ is p-quasilocal, and thereby, proves that $E(p) \otimes_E E_v$ is a field.
Conversely, assume that $Br(E)_p \neq \{0\}$ and $E(p) \otimes_E E_v$ is a field, for some absolute value $v$ of $E$. By (1.2), then we have $E(p) \neq E$. Let $\overline{E}$ be an algebraic closure of $E_v$, $M$ a cyclic extension of $E$ in $\overline{E}_v$ of degree $p$, and $E'$ the intermediate field of $E_v/E$ defined as in (1.7). We show that $[\Delta] \in Br(M/E)$, for each $\Delta \in d(E)$ of index $p$. It follows from Galois theory and the assumptions on $v$ and $M$ that $ME_v/E_v$ and $ME'/E'$ are cyclic field extensions of degree $p$. This means that $v$ is uniquely extendable to an absolute value $v_M$ of $M$, and $ME_v$ is a completion of $M$ with respect to $v_M$ (cf. [4], Ch. II, Theorem 10.2). Applying now (1.2), Lemma 3.1 and Proposition 1.2, one concludes that our assertion will be proved, if we show that $[\Delta \otimes_E M] \in Br(ME_v/M)$. Since the algebras $(\Delta \otimes_E M) \otimes_M (ME_v)$, $\Delta \otimes_E (ME_v)$ and $(\Delta \otimes_E (ME')) \otimes_M (ME_v)$ are isomorphic over $ME_v$, it is sufficient to establish that $[\Delta] \in Br(ME'/E)$. This property of $\Delta$ follows from the fact that $E'$ is quasilocal, the equality $[(ME')': E'] = p$ and the existence of an $(ME')$-isomorphism $\Delta \otimes_E (ME') \cong (\Delta \otimes_E E') \otimes_{E'} (ME')$, so the proof of Lemma 3.2 is complete. □

**Lemma 3.3.** Let $E_0$ be a global field, $E$ an algebraic extension of $E_0$ admitting local $p$-class field theory, for some $p \in P(E)$, and let $v \in M(E)$ be chosen so that $E(p) \otimes_E E_v$ is a field. Then:

(i) $Br(E)_p$ is isomorphic to the group $\mathbb{Z}(p^\infty)$ unless $p = 2$ and $E$ is a formally real field; when this is the case, $v$ is nonarchimedean;

(ii) $Br(E)_2$ is of order 2 in case $E_v$ is isomorphic to $\mathbb{R}$.

**Proof.** By [6], I, Lemma 3.6, a 2-quasilocal field $E$ is formally real if and only if $Br(E)_2$ is of order 2. This, combined with (1.5) (iv), (1.7) (i), (iv) and Lemma 3.1, proves our assertion. □

Our next result shows that an algebraic extension $E$ of a global field $E_0$ satisfying the condition $Br(E) \neq \{0\}$ is a field with local class field theory in the sense of [26] if and only if finite extensions of $E$ are strictly PQL-fields.

**Corollary 3.4.** Let $E$ be an algebraic extension of a global field $E_0$ and $\Pi(E)$ the set of those prime numbers $p$ for which $G_E$ is of nonzero cohomological $p$-dimension. Then finite extensions of $E$ are strictly PQL-fields if and only if the following assertions hold true:

(i) $E_{sep} \otimes_E E_v$ is a field, for some $v \in M(E)$;

(ii) $Br(E)_p \neq \{0\}$, for every $p \in \Pi(E)$.

**Proof.** Statements (1.8) and the concluding observations in [9], Sect. 1, imply finite extensions of $E$ are strictly PQL-fields and only if $E$ is quasilocal and $cd_p(G_E) \neq 1$, for any $p \in \mathbb{P}$. In view of [3], Proposition 3.1, this means that if $E$ is formally real, then the Sylow pro-$p$-subgroups of $G_E$ are isomorphic to $\mathbb{Z}_p^2$, for each $p \in \Pi(E) \setminus \{2\}$. The obtained result in fact proves that $E$ is real closed, since, by [18], Theorem 2.3, the algebraicity of $E/Q$ guarantees that abelian subgroups of $G_E$ are procyclic. Our argument also relies on the fact [6], I, Lemma 3.6, that $E$ is formally real and 2-quasilocal if and only if $|E(2): E| = 2$, and when this occurs, $Br(E)_2$ is of order 2.
Lemma 3.1 and the structure of Br(\mathcal{E}) extensions of local fields (cf. [15], Ch. IV, Sect. 1). When follows from (1.7) (iv) and the solvability of the Galois groups of finite Galois rings the equivalent conditions of Corollary 3.4, then this case, by (1.2), \text{Br}(\mathcal{E})_p \cong \mathbb{Z}(p^\infty), for each p \in \Pi(\mathcal{E}). Condition (i) is equivalent to the one that \nu is Henselian. The prolongation \nu_F is also Henselian, whence \text{F}_{\text{sep}} \otimes \text{F}_{\nu_F} is a field as well. As the exponent of \text{Br}(\mathcal{F}/\mathcal{E}) divides \[ F : \mathcal{E} \], we have \text{Br}(\mathcal{F})_p \neq \{0\}, for each p \in \Pi(\mathcal{E}). It is now easy to see from Lemma 3.2 that \mathcal{F} is a strictly PQL-field.

Step 2. Assume that finite extensions of \mathcal{E} are strictly PQL-fields, and fix some p \in \Pi(\mathcal{E}). Clearly, if M/\mathcal{E} is a finite Galois extension with p | [M : \mathcal{E}], and \mathcal{M}_p is the fixed field of some Sylow p-subgroup of \text{G}(\mathcal{M}/\mathcal{E}), then p \in P(M_p). Combining Lemmas 3.2 and 3.3 with (1.8), one concludes that \text{Br}(\mathcal{M}_p) \cong \text{Br}(\mathcal{E})_p \cong \mathbb{Z}(p^\infty). In particular, p \in P(\mathcal{E}), whence E(p) \otimes \mathcal{E}_{\nu(p)} is a field, for some \nu(p) \in M(\mathcal{E}). Let now \mathcal{F} be a finite Galois extension of \mathcal{E} in \text{E}_{\text{sep}} and \nu(p)' an absolute value of \mathcal{F} extending \nu(p). In view of Lemma 1.3, then \text{Br}(\mathcal{F}_{\nu(p)}/\mathcal{E}_{\nu(p)}) is of exponent dividing \[ F : \mathcal{E} \], so it follows from Lemma 3.1 and the structure of \text{Br}(\mathcal{E})_p that \text{Br}(\mathcal{F}_{\nu(p)'})_p \neq \{0\} and the prolongation \nu(p)' is unique. This means that \text{E}_{\text{sep}} \otimes \mathcal{E}_{\nu(p)} is a field (cf. [4], Ch. II, Theorem 10.2), which completes the proof of Corollary 3.4. □

Remark 3.5. Lemma 3.3 indicates that if \mathcal{E} is an algebraic strictly PQL-extension of a global field \mathcal{E}_0, then finite subgroups of \text{Br}(\mathcal{E}) are cyclic. This implies that the local reciprocity law for arbitrary strictly PQL-fields (cf. [9], Theorem 2) has the same form as in the case of local fields. In other words, when R/\mathcal{E} is a finite abelian extension, \text{E}^* / N(R/\mathcal{E}) \cong \text{G}(R/\mathcal{E}).

Note finally that if \mathcal{E} is an algebraic extension of a global field \mathcal{E}_0 satisfying the equivalent conditions of Corollary 3.4, then \mathcal{G}_E is prosolvable. This follows from (1.7) (iv) and the solvability of the Galois groups of finite Galois extensions of local fields (cf. [15], Ch. IV, Sect. 1). When \mathcal{E} \in I(\mathcal{E}_0/\mathcal{E}_0) is merely strictly PQL, \mathcal{G}_E is rarely prosolvable, as can be seen, for example, in the formally real case, from Theorem 2.2 and the following result.

Corollary 3.6. Let \mathcal{E} be a formally real algebraic extension of \mathbb{Q} admitting LCF, and let p \in P(\mathcal{E}). Then there exists a finite Galois extension of \mathcal{E} with a Galois group isomorphic to the symmetric group \text{Sym}_p.

Proof. It is clearly sufficient to consider only the special case of p \neq 2. In this case, by (1.2), \mathcal{E}(p) \neq \mathcal{E}, i.e. there exists a cyclic extension L of \mathcal{E} of degree p. Fix \nu(p) and \nu(2) in M(\mathcal{E}) so that \mathcal{E}(p) \otimes \mathcal{E}_{\nu(p)} and \mathcal{E}(2) \otimes \mathcal{E}_{\nu(2)} are fields. By Lemmas 3.2 and 3.3, \nu(p) and \nu(2) are nonequivalent, so it follows from the (weak) approximation theorem (cf. [23], Ch. II, Theorem 1) and Krasner's lemma that there exists a polynomial \text{f}_p \in \mathcal{E}[X] of degree
\(p\) with the following properties: (i) the root field of \(f_p(X)\) over \(E_{v(p)}\) is \(E_{v(p)}^*\) isomorphic to \(L \otimes_{E} E_{v(p)}\); (ii) \(f_p(X)\) has exactly \(p - 2\) zeroes in \(E_{v(2)}\). This implies that the Galois group \(G_p\) of \(f_p(X)\) over \(E\) is isomorphic to a transitive subgroup of \(\text{Sym}_p\) with a transposition, which ensures that \(G_p \cong \text{Sym}_p\). \(\Box\)

4. Minimal algebraic strictly PQL-extensions of global fields

Let \(E_0\) be a global field. Theorem 2.1 shows that a field \(E \in \mathcal{I}(\overline{E}_0/E_0)\) with \(P(E) \neq \emptyset\) is strictly PQL if and only if it possesses a system \(V(E) = \{v(p) \in M(E) : p \in P(E)\}\), such that \(v(p_i)\) is \(p\)-Henselian, for each \(p \in P(E)\), unless \(p = 2 \in P(E)\) and \(v(2)\) is archimedean. Therefore, it is convenient to describe a number of properties of the considered fields in the language of valuation theory. Our objective in the present Section is to demonstrate this by proving the existence of some families of algebraic strictly PQL-extensions of \(E_0\), in a form adequate to the needs of LCFT. For the purpose, we need the following definition.

**Definition 3.** Let \(F\) be an algebraic extension of a global field \(E_0\) with \(\text{Br}(F) \neq \{0\}\), \(P\) a subset of \(\{p \in \mathbb{P} : \text{Br}(F)_p \neq \{0\}\}\), \(P \neq \emptyset\), and \(W = \{w(p) \in M(F) : p \in P\}\) a system chosen so that \(\text{Br}(F_{w(p)})_p \neq \{0\}\), for each \(p \in P\). Assume that \(\Pi = \mathbb{P}\setminus P\), \(P(I) = P_i : i \in I\), is a partition of \(P\), i.e. a sequence of nonempty subsets of \(P\), such that \(\bigcup_{i \in I} P_i = P\) and \(P_j \cap P_{j'} = \emptyset\), for each pair of distinct indices \(j, j'\), \(\chi = \{\chi_i : i \in I\}\) is a system of group classes. The partition \(P(I)\) is said to be compatible with \(W\), if the set \(W_i = \{w(p_i) : p_i \in P_i\}\) consists of equivalent absolute values; the system \(\chi\) is called admissible by the pair \((W, P(I))\), if the following conditions hold:

- (c) \(\chi_i\) consists of \(P_i \cup \Pi_i\)-groups, where \(\Pi_i \subseteq \Pi\), for each \(i \in I\);
- (cc) \(\chi_i\) is a saturated group formation including all finite \(p_i\)-groups, for each \(i \in I\) and \(p_i \in P_i\), unless \(P_i = \{2\}\) and \(w(2)\) is archimedean; if \(P_i = \{2\}\) and \(w(2)\) is archimedean, then \(\chi_i\) equals the class of groups of orders \(\leq 2\).

When \(F, W, P, P(I)\) and \(\chi\) are fixed as in Definition 4.1, we denote by \(\Omega_\chi(F, P, W)\) the set of strictly PQL-fields \(K \in \mathcal{I}(\overline{E}_0/F)\) with \(P(K) = P\) and a characteristic system \(V(K) = \{v(p) : p \in P\}\), such that \(v(p)\) extends \(w(p)\) whenever \(p \in P\), the absolute values \(v(p_i), p_i \in P_i\), are equivalent, for each \(i \in I\), and \(v(p_i)\) is \(\chi_i\)-Henselian, in case \(w(p_i)\) is nonarchimedean. Our main results concerning this set can be stated as follows:

**Theorem 4.1.** Let \(E_0\) be a global field and \(F\) an extension of \(E_0\) in \(\overline{E}_0\), such that \(\text{Br}(F) \neq \{0\}\). Assume that \(P, \Pi\), and \(W = \{w(p) : p \in P\}\) are defined as above, \(P(I) = P_i : i \in I\), is a partition of \(P\) compatible with \(W\), and \(\chi = \{\chi_i : i \in I\}\) is a system of group classes admissible by \((W, P(I))\). Then the set \(\Omega_\chi(F, P, W)\) has the following properties:

(i) \(\Omega_\chi(F, P, W)\) is nonempty and satisfies the conditions of Zorn’s lemma with respect to the partial ordering inverse to inclusion;

(ii) If \(\chi_i, i \in I\), are abelian closed unless \(P_i = \{2\}\) and \(w(2)\) is Archimedean, then every \(E \in \Omega_\chi(F, P, W)\) possesses a unique subfield \(R(E)\), which is a minimal element of \(\Omega_\chi(F, P, W)\);
(iii) If \( K \in \Omega_\chi(F, P, W) \) is minimal, \( V(K) = \{ \kappa(p): \ p \in P \} \) is a characteristic system of \( K \), and \( F_{w(p)} \) is the closure of \( F \) in \( K_{\kappa(p)} \), then the degrees of finite extensions of \( F_{w(p)} \) in \( K_{\kappa(p)} \) are not divisible by \( p \).

Proof. We first show that \( \Omega_\chi(F, P, W) \) contains a minimal element \( K \) with the properties required by Theorem 4.1 (iii). Denote by \( R(p) \) the maximal \( p \)-extension of \( R \) in \( \mathcal{E}_0 \), provided that \( R \in I(F/F) \) and \( p \in \mathbb{P} \). Fix an algebraic closure \( \overline{F}_{w(p)} \) of \( F_{w(p)} \) as well as an embedding \( \theta_i \) of \( F \) in \( \overline{F}_{w(p)} \) as an \( F \)-subalgebra, for each \( i \in I \), and consider the tower \( \{ K_n: \ n \in \mathbb{N} \} \) of extensions of \( F \) in \( \overline{F} \), defined inductively as follows:

\[
(4.1) \ K_1 = F, \text{ and for each } n \in \mathbb{N}, K_{n+1} \text{ is the compositum of the fields } K_n, \text{ and put } \kappa(p_i)(\alpha) = \overline{w}(p_i)(\theta_i(\alpha)), \text{ for each } \alpha \in K, \text{ in case } p_i \in P_i \text{ and } i \in I, \text{ where } \overline{w}(p_i) \text{ is the unique absolute value of } \overline{F}_{w(p_i)} \text{ extending the continuous prolongation of } w(p_i) \text{ on } F_{w(p_i)}.
\]

Also, let \( \kappa_i(p) \) be the absolute value of \( K_i \) induced by \( \kappa(p) \), for each pair \( (p, \nu) \in (P \times \mathbb{N}) \). We show that \( K \) is a minimal element of \( \Omega_\chi(F, P, W) \) with a characteristic system \( V(K) = \{ \kappa(p): p \in P \} \). It follows from (1.1) (ii) that every finite Galois extension \( L \) of \( K \) in \( \overline{F} \) is isomorphic over \( K \) to \( L_n \otimes K_n \), for some index \( n \) depending on \( L \), and a suitably chosen finite Galois extension \( L_n \) of \( K_n \) in \( L \). Since \( \mathcal{G}(L_n/K_n) \cong \mathcal{G}(L/K) \), this ensures that \( L_n \cap K = L_n \cap K_{n+1} = K_n \). It is therefore clear that if \( p \in P \) and \( \mathcal{G}(L_n/K_n) \in \chi^{(i)} \), for some \( i \in I \), then \( \kappa_i(p) \) and \( \kappa(p) \) have unique prolongations on \( L_n \) and \( L \), respectively. This means that if \( w(p) \) is nonarchimedean, then \( \kappa(p) \) is \( \chi^{(i)} \)-Henselian. A similar argument also shows that \( K(p) = K \), in case \( p \in \Pi \). Now fix an index \( i \in I \), and a prime \( p_i \in P_i \), and for each \( R \in I(\mathcal{E}_0/F) \), let \( w_R(p_i) \) be the absolute value of \( \theta_i(R).F_{w(p_i)} \) induced by \( \overline{w}(p_i) \), and \( \tilde{R}_{p_i} \), the completion of \( \theta_i(R).F_{w(p_i)} \) with respect to \( w_R(p_i) \). Then \( \tilde{w}R(p_i) \) is Henselian, which means that \( \theta_i(\tilde{R}).F_{w(p_i)} \) is separably closed in \( \tilde{R}_{p_i} \). In view of (4.1) and Definition 4.1, this result, applied to the case of \( \tilde{R} = K \), implies that \( p_i \) does not divide the degrees of finite extensions of \( F_{w(p_i)} \) in \( K_{\kappa(p)} \). Hence, by (1.3) (ii) and the behaviour of Schur indices of central simple algebras under finite extensions of their centres (cf. [P, Sect. 13.4]), we have \( \text{Br}(K_{\kappa(p_i)})_p \neq \{ 0 \} \). Observing also that there exists an \( F \)-isomorphism \( K_{\kappa(p_i)} \cong K_{\kappa(p_i)} \), acting on \( K \) as \( \theta_i \), one concludes that \( \text{Br}(K_{\kappa(p_i)})_p \neq \{ 0 \} \) and \( p_i \) does not divide the degrees of finite extensions of \( F_{w(p_i)} \) in \( K_{\kappa(p_i)} \). This, combined with (1.2) and Proposition 1.2, yields consecutively \( \text{Br}(K)_{P_i} \neq \{ 0 \} \) and \( K_{P_i} \neq K \); one also sees that if \( P_i = \{ 2 \} \) and \( w(2) \) is archimedean, then \( K \) is formally real and \( K_{\kappa(2)} \cong \mathbb{R} \). The assumptions of Theorem 4.1 and the obtained results indicate that \( K \) is a minimal element of \( \Omega_\chi(F, P, W) \). Let \( M \) be a proper extension of \( F \) in \( K \). Then there exists an index \( m \in \mathbb{N} \), such that \( K_m \subseteq M \) and \( K_{m+1} \nsubseteq M \).
This means that $K_m(\pi) \not\subseteq M$, for some $\pi \in \Pi$, or $K_{m,j} \not\subseteq M$, for some $j \in I$.

In the former case, this means that $MK_m(\pi)$ contains as a subfield a cyclic extension of $M$ of degree $\pi$, whence $\pi \in P(M)$. In the latter one, it turns out that $M$ admits a proper finite extension $M'$ in $MK_{m,j}$. Let $M_1$ be the normal closure of $M'$ in $\mathfrak{E}_0$ over $M$. Then $M_1$ is a subfield of $MK_m(\chi_i)$, so it follows from Galois theory that $G(M_1/M) \subseteq \chi_i$. On the other hand, (4.1), the inequality $M' \neq M$ and the inclusions $M \subseteq M' \subseteq MK_{m,j}$ indicate that the absolute value of $M$ induced by $\kappa(p_j)$, for a given $p_j \in \mathcal{P}_j$, has at least 2 different prolongations on $M'$ and on $M_1$. Applying now Lemma 3.1, one sees that $M \not\subseteq \Omega_{\chi}(F, P, W)$, i.e. $K$ is minimal in $\Omega_{\chi}(F, P, W)$, as claimed.

In the rest of the proof of Theorem 4.1 we identify the completions $E_{v(p_i)}$, $p_i \in P_i$, for each $i \in I$. Our considerations rely on the following lemma.

**Lemma 4.2.** Let $E_0$ be a global field, $E$ and $T$ extensions of $E_0$ in $\mathfrak{E}_0$, such that $T \subseteq E$, and let $v \in M(E)$ and $u$ be the absolute value of $T$ induced by $v$. Assume that $E$ and $T$ admit local $p$-class field theory, for some $p \in P(E)$, and $E(p) \otimes_E E_v$ is a field. Then $T(p) \otimes_T T_u$ is a field and $p \in P(T)$.

**Proof.** Since $T_u$ is isomorphic to the closure of $T$ in $E_v$, this can be deduced from (1.8) and Lemmas 3.1 (ii) and 3.2. □

We turn to the proof of Theorem 4.1 (i). Fix an element $E$ of $\Omega_{\chi}(F, P, W)$, a characteristic system $V(E) = \{v(p): p \in P\}$ of $E$, and a nonempty linearly ordered subset $\Lambda$ of $I(E/F) \cap \Omega_{\chi}(F, P, W)$. Denote by $L$ the intersection of the fields from $\Lambda$, and by $v_L(p)$ the absolute value of $L$ induced by $v(p)$, for each $p \in P$. We show that $L \in \Omega_{\chi}(F, P, W)$ and $V(L) = \{v_L(p): p \in P\}$ is a characteristic system for $L$. Note first that $P(L) = P$. Indeed, (1.2), (1.8) and Theorem 2.1 indicate that $Br(L)_p \neq \{0\}$ and $L(p) \neq L$, for every $p \in P$. Assuming that $\bar{p} \in \mathbb{F} \setminus P$, one obtains from Galois theory that $L(\bar{p}).R/R$ is a $\bar{p}$-extension, which yields $L(\bar{p}).R = R$, for each $R \in \Lambda$, and so proves that $L(\bar{p}) = L$. Suppose now that $i \in I$, $p_i \in P_i$, $R \in \Lambda$, $v_R(p_i)$ is the absolute value of $R$ induced by $v(p_i)$, and $\Sigma_i$ is an algebraic closure of $E_{v(p_i)}$.

Identifying $\mathfrak{E}_0$ with its $L$-isomorphic copy in $\Sigma_i$, and $L_{v_L}(p)$ with the closures in $E_{v(p_i)}$ of $L$ and $R$, respectively, one obtains from Galois theory, general properties of tensor products (cf. [P, Sect. 9.2, Proposition c]), the choice of $v_R(p_i)$, and Lemma 4.2 that $R(\lambda_i) \cap R_{v_R(p_i)} = R$. Since $L(\lambda_i).R \subseteq R(\chi_i)$, this implies that $L(\lambda_i) \cap L_{v_L}(p_i) \subseteq R$. As $R$ is an arbitrary element of $\Lambda$, the obtained result proves that $L(\lambda_i) \cap L_{v_L}(p_i) = L$. This means that $L(\lambda_i) \otimes_L L_{v_L}(p_i)$ is a field, so Theorem 4.1 (i) is proved.

Our objective now is to prove Theorem 4.1 (iii). Let $L$ be a minimal element of $\Omega_{\chi}(F, P, W)$ and $V(L) = \{\lambda(p): p \in P\}$. It is clearly sufficient to show that $L$ can be viewed as an extension of $F$ defined in accordance with (4.1). Identifying, for each $i \in I$ and $p_i \in P_i$, $L$ with its canonically isomorphic copy in $L_{\lambda(p_i)}$, and $F_{v_L}(p_i)$ with the closure of $F$ in $L_{\lambda(p_i)}$, fix an embedding $j_i$ of $L$ in $\Sigma_i$ as an $E$-subalgebra. Also, let $K$ be the union of the extensions $K_n$: $n \in \mathbb{N}$, of $F$ in $\mathfrak{E}$ associated with the maps $j_i$: $i \in I$, as in (4.1). Proceeding by induction on $n$, and arguing as in the proof of Theorem 4.1 (i), one obtains that $K_n \subseteq E$, for every $n \in \mathbb{N}$, i.e. $K \subseteq E$.  


Since $K$ is a minimal element of $\Omega_\chi(F,P,W)$, this yields $K = E$, and so completes the proof of Theorem 4.1 (iii).

It remains for us to prove Theorem 4.1 (ii). In view of Theorem 4.1 (i), it suffices to show that if $E \in \Omega_\chi(F,P,W)$, then $I_\chi(E/F)$ contains a unique minimal element. Assuming the opposite, take a field $E \in \Omega_\chi(F,P,W)$ so that $I_\chi(E/F)$ contains two different elements, say, $K$ and $L$. As shown in the proof of Theorem 4.1 (iii), $K = \bigcup_{n=1}^{\infty} K_n$ and $L = \bigcup_{m=1}^{\infty} L_m$, the unions being defined as in (4.1). As $K \neq L$, this means that $K_n = L_n$ and $K_{n+1} \neq L_{n+1}$, for some $n \in \mathbb{N}$. Specifically, $K_{n,i} \neq L_{n,i}$, for some $i \in I$. In view of (1.6), this means that $R_i(\chi_i) = R_i$, where $R_i = K_{n,i}L_{n,i}$. Hence, by Definition 3 and the assumptions of Theorem 4.1 (ii), $p_i \notin P(R_i)$, for any $p_i \in P_i$. It is therefore clear from (1.8) that $\text{Br}(R'_i)_{p_i} = \{0\}$, for every $R'_i \in I(\overline{E}_0/R_i)$ and each $p_i \in P_i$. Since $E \in I(\overline{E}_0/R_i)$ and $\text{Br}(E)_{p_i} = \{0\}$ when $p_i$ runs across $P_i$, this is a contradiction proving our assertion about $I_\chi(E/F)$, the concluding step towards the proof of Theorem 4.1.

Proof of Theorem 2.2. Statements (i), (iii) and (iv) are obtained by applying Theorem 4.1 to the special case in which $I = P$ and, for each $p \in P$, $\chi_p$ is the formation of finite $p$-groups unless $2 \in P$ and $w(2)$ is archimedean. Since the class of solvable groups is closed under taking subgroups, quotient groups and group extensions, this enables one to deduce Theorem 2.2 (ii) from Galois theory and (4.1).

Corollary 4.3. Let $E_0$ be a global field, $F$ an algebraic extension of $E_0$, such that $\text{Br}(F) \neq \{0\}$, and $P$ a nonempty subset of $\mathbb{P}$, for which $\text{Br}(F)_{p} \neq \{0\}$, $p \in P$. Then the set $\Omega_P(\overline{E}_0/F)$ of strictly PQL-fields $\Sigma \in I(\overline{E}_0/F)$ with $P(\Sigma) = P$ is nonempty. Moreover, every $K \in \Omega_P(\overline{E}_0/F)$ possesses a unique subfield $R(K)$ that is a minimal element of $\Omega_P(\overline{E}_0/F)$.

Proof. It is easily obtained from (1.8) and Lemma 4.2 that $\Omega_P(\overline{E}_0/F)$ equals the union of the sets $\Omega(F,P,W)$, taken over all $W$ admissible by Definition 1. Observing also that $I(E/F) \cap \Omega_P(\overline{E}_0/F) \subseteq \Omega(F,P,W)$, for each $E \in \Omega(F,P,W)$, one proves our assertions by applying Theorem 4.1 to the set $\Omega_\chi(F,P,W)$ considered in the proof of Theorem 2.2, for an arbitrary system $W = \{w(p) \in M(F): p \in P\}$, such that $\text{Br}(F_{w(p)})_{p} \neq \{0\}$, $p \in P$. \hfill $\Box$

Corollary 4.4. Under the hypotheses of Theorem 4.1, suppose that $F = E_0$, $2 \in P$, $w(2)$ is Archimedean, $K$ is a minimal element of $\Omega_\chi(F,P,W)$, and $n$ is an integer $\geq 5$. Then $I(\overline{E}_0/K)$ contains Galois extensions $M_k, L_k$, $k \in \mathbb{N}$, of $K$, such that $G(M_k/K) \cong \text{Sym}_n$ and $G(L_k/K) \cong \text{Alt}_n$, for every $k \in \mathbb{N}$.

Proof. Let $d_1$ and $d_2$ be squarefree integers, such that $d_1 < 0 < d_2$ and $d_2 \in E_0^{\ast 2}$ (the existence of $d_2$ is implied Lemma 1.1). It is known (cf. [17]) that there exists a set $\{A_k, B_k: k \in \mathbb{N}\}$ of Galois extensions of $\mathbb{Q}$ in $\mathbb{P}$ with $G(A_k/\mathbb{Q}) \cong G(B_k/\mathbb{Q}) \cong \text{Sym}_n$, $\sqrt{d_1} \in A_k$ and $\sqrt{d_2} \in B_k$, for every $k \in \mathbb{N}$. Note also that the Galois closure $K'$ of $K$ in $\mathbb{P}$ over $F$ has a prosolvable Galois group. Since, by the proof of Theorem 4.1, $K$ is obtained from $F$ in accordance with (4.1), and the class of solvable groups is closed under
taking subgroups, quotient groups and group extensions, the prosolvability of \( G(K'/F) \) can be deduced from Galois theory and the solvability of finite groups of odd order [FT]. Hence, by Galois theory and the simplicity of \( \text{Alt}_n \), \( n \geq 5 \), when \( k \) is sufficiently large, the fields \( A_k K \) and \( B_k K \) are Galois extensions of \( K \) with \( G(A_k K/K) \cong \text{Sym}_n \) and \( G(B_k K/K) \cong \text{Alt}_n \). □

Remark 4.5. Corollary 4.4 retains validity, if \( F = F_0 \), \( 2 \in P \) and, in terms of Definition 3, \( P_j = \{2\} \) and \( \Pi_j \) contains at most one element, for some \( j \in I \); in particular, this holds when \( \mathbb{P} \setminus P \) contains at most one element and \( \omega(2) \) is nonequivalent to \( w(p) \), for any \( p \in \mathbb{P} \setminus \{2\} \). Indeed, in this case, \( G(K'/F) \) is prosolvable, \( K' \) being again the Galois closure of \( K \) in \( \overline{\mathbb{Q}}_0 \) over \( F \) (as follows from the solvability of finite groups of odd or biprimary order). Omitting the details, note that the extensions \( A_k \) and \( B_k \) of \( F \) can be constructed as root fields of polynomials in \( F[X] \) of degree \( n \) with a suitably chosen local behaviour (ensured by applying the approximation theorem).

**Proposition 4.6.** In the setting of Theorem 4.1, let \( F \) be a global field. Then the minimal elements of \( \Omega_{\chi}(F,P,W) \) form a single \( F \)-isomorphism class if and only if some of the following assertions holds true:

(i) \( P = P_1 \);

(ii) \( P = \mathbb{P} \) and for each \( i \in I \), \( p_i \in P_1 \), the system \( W_i = \{w(q_i) \colon q_i \in P_1\} \) contains all \( w(p) \in W \) that are equivalent to \( w(p_i) \).

**Proof.** We first show that if none of conditions (i) and (ii) holds, then \( \Omega_{\chi}(F,P,W) \) contains a pair of nonisomorphic fields (over \( F \)). Our argument relies on two observations the latter of which is a special case of Lemma 1.1:

(4.2) (i) If \( E_1 \) and \( E_2 \) are fields lying in \( \Omega_{\chi}(F,P,W) \), with characteristic systems \( V(E_\pi) = \{v_\pi(p) \colon p \in P\} \), \( u = 1,2 \), and if there exists an \( F \)-isomorphism \( \varphi \colon E_1 \rightarrow E_2 \), then the absolute values \( v_1(p) \) and \( v_2(p) \circ \varphi \) of \( E_1 \) are equivalent, for each \( p \in P \);

(ii) If \( \pi \in \mathbb{P} \), \( p \in P \) and \( S \) is a finite subset of \( W \), then there exists a cyclic extension \( \Phi_{\pi} \) of \( F \) in \( F_{\text{sep}} \), such that \( [\Phi_{\pi} : F] = \pi \) and each \( s \in S \) has \( \pi \) distinct prolongations on \( \Phi_{\pi} \).

Suppose first that \( \pi \notin P \) and there exist \( p_1, p_2 \in P \), for which \( w(p_1) \) is not equivalent to \( w(p_2) \). Assume also that \( S = \{w(p_1), w(p_2)\} \), fix a generator \( \psi_{\pi} \) of \( G(\Phi_{\pi}/F) \) and prolongations \( w(p_1)' \) and \( w(p_2)' \) on \( \Phi_{\pi} \) of \( w(p_1) \) and \( w(p_2) \), respectively. By Lemma 1.3, the set \( S_u \) of absolute values of \( \Phi_{\pi} \) extending \( w(p_u) \) equals \( \{w(p_u)' \circ \psi_{\pi}^j \colon j = 0,1, \ldots, \pi - 1\} \), for each index \( u \).

It is therefore clear that \( \Phi_{\pi} \in I(E/F) \), for any \( E \in \Omega_{\chi}(F,P,W) \). Put \( W_u = \{w_u(p) \colon p \in P\} \), where \( w_u(p) \) is a prolongation of \( w(p) \) on \( \Phi_{\pi} \), for each \( p \in P \), \( w_1(p_u) = w(p_u)' \) and \( w_2(p_u) = w(p_u)' \circ \psi_{\pi}^{\pi - 1} \colon u = 1,2 \). Applying (4.1) and Lemma 1.3, one obtains that \( \Omega_{\chi}(F,P,w) \) contains minimal elements \( E_1 \) and \( E_2 \) possessing characteristic systems \( V(E_1) \) and \( V(E_2) \), respectively, such that \( V(E_\pi) = \{v_{\pi}(p) \colon p \in P\} \) and \( v_{\pi}(p) \) extends \( w_u(p) \) whenever \( p \in P \) and \( u = 1,2 \). Hence, by (4.2) (i) and the choice of \( w_u(p_1) \) and \( w_u(p_2) \), the fields \( E_1 \) and \( E_2 \) are not isomorphic over \( F \).

Suppose now that there exist different indices \( j, m \in I \), for which \( \{w(p_j') \colon p_j' \in P_j \cup P_m \} \) consists of equivalent absolute values. Assume also that \( \pi \in P_j \).
and \( S = \{ w(\pi) \} \), and fix some \( p \in P_m \). It is easily verified that the field \( \Phi_\pi \) considered in (4.2) (ii) is included in all fields from \( \Omega_\chi(F,P,W) \). Arguing as in the case of \( \pi \notin P \), one obtains that \( \Omega_\chi(F,P,W) \) contains minimal elements \( E_1 \) and \( E_2 \), such that the system \( \{ v_1(p') \in V(E_1): p' \in P_1 \cup P_m \} \) consists of equivalent absolute values, whereas \( \{ v_2(p') \in V(E_2): p' \in P_1 \cup P_m \} \) does not possess this property. It is therefore clear from (4.2) (i) that \( E_1 \) and \( E_2 \) are not \( F \)-isomorphic, which completes the proof of the left-to-right implication of Proposition 4.6.

Conversely, let some of conditions (i), (ii) be in force, take minimal elements \( K \) and \( L \) of \( \Omega_\chi(F,P,W) \), as well as characteristic systems \( V(K) = \{ \kappa(p): p \in P \} \) and \( V(L) = \{ \lambda(p): p \in P \} \), and for each \( n \in \mathbb{N} \), denote by \( K_n \) and \( L_n \), the fields in \( I(K/F) \) and \( I(L/F) \), respectively, defined in accordance with (4.1). Also, let \( \kappa_n(p) \) and \( \lambda_n(p) \) be the absolute values of \( K_n \) and \( L_n \), induced by \( \kappa(p) \) and \( \lambda(p) \), respectively, for any index \( n \). Proceeding by induction on \( n \) (and taking the inductive step by a repeated application of Lemma 1.3), one proves the existence of a sequence \( \psi = \{ \psi_n: n \in \mathbb{N} \} \) of \( F \)-isomorphisms \( \psi_n: K_n \to L_n \), such that \( \psi_{n+1} \) extends \( \psi_n \), and \( \kappa_n(p) \) equals the composition \( \lambda_n(p) \circ \psi_n \), for each pair \( (n, p) \in \mathbb{N} \times P \). The sequence \( \psi \) gives rise to an \( F \)-isomorphism \( K \cong L \), so Proposition 4.6 is proved. \( \square \)

Proposition 4.6 shows that if \( F \) is a global field, then the minimal elements of \( \Omega(F,P,W) \) form an \( F \)-isomorphism class if and only if \( F \) contains only one element or \( P = \mathbb{P} \) and \( W \) consists of pairwise nonequivalent absolute values. The concluding result of this Section determines the structure of the Brauer groups of algebraic strictly PQL-extensions of global fields, and gives a classification of abelian torsion groups realizable as such Brauer groups.

**Proposition 4.7.** Let \( E_0 \) be a global field, \( T \) be a nontrivial divisible abelian torsion group, \( P(T) \) the set of those \( p \in \mathbb{P} \) for which \( T_p \neq \{ 0 \} \), and \( T_0 \) the direct sum \( \oplus_{p \in P(T)} \mathbb{Z}(p^{\infty}) \), where \( P_0(T) = P(T) \setminus \{ 2 \} \). Then \( T \cong \text{Br}(E(T)) \), for some strictly PQL-field \( E(T) \in I(T_0/E_0) \), if and only if \( T \) is isomorphic to one of the following groups:

(i) the direct sum of \( T_0 \) by a group of order 2; when this occurs, \( E(T) \) is formally real;

(ii) \( \oplus_{p \in P(T)} \mathbb{Z}(p^{\infty}) \); in this case, \( E(T) \) is nonreal and can be chosen so that its finite extensions are strictly PQL.

**Proof.** Let \( E \) be an extension of \( E_0 \) in \( T_0 \). It is clear from (1.2) that \( \text{Br}(E)_q = \{ 0 \} \), for every \( q \in \mathbb{P} \setminus P(E) \). This, combined with Lemma 3.3 and Theorems 2.1 and 2.2, proves all assertions of Proposition 4.7 except the one that if \( T \) is of type (ii), then \( E(T) \) can be chosen so that its finite extensions are strictly PQL. At the same time, Theorem 4.1, applied to a saturated group formation \( \chi \) (including the class of finite \( p \)-groups, for every \( p \in P(T) \)), the partition \( P = P_1 = P(T) \), and a nonarchimedean absolute value \( \omega_1 \) of \( E_0 \), indicates that there exists a \( \chi \)-Henselian algebraic strictly PQL-extension \( E \) of \( E_0 \) with \( P(E) = P(T) \). Hence, by Lemma 3.3, \( \text{Br}(E) \cong T \) if and only if \( T \) is of type (ii). By Corollary 3.4, when \( \chi \) is the formation of all finite groups, finite extensions of \( E \) are strictly PQL, so the obtained result completes the proof of Proposition 4.7. \( \square \)
5. HASSE NORM PRINCIPLE

In this Section we prove the validity of the Hasse norm principle for arbitrary finite Galois extensions of the fields pointed out at the end of the Introduction. By Theorem 2.1, this applies to algebraic strictly PQL-extensions of global fields, which is used for proving Theorem 2.3 and Proposition 2.4.

**Theorem 5.1.** Assume that $E_0$ is a global field and $E$ is an extension of $E_0$ in $\mathcal{F}_0$, such that the set $S_p(E) = \{v \in M(E) : \text{Br}(E_v)_p \neq \{0\}\}$ is finite, for each $p \in \mathcal{F}$. Let $M/E$ be a finite Galois extension. Then:

(i) $N(M/E)$ contains every $\lambda \in E^*$ satisfying the inequalities $v(\lambda - 1) < \varepsilon$: $v \in \cup_{p \in P(M/E)} S_p(E)$, for some real number $\varepsilon > 0$ depending on $M/E$;

(ii) $N(M/E) = N_{\text{loc}}(M/E)$; an element $c \in E^*$ lies in $N(M/E)$ if and only if $c \in N(M_{v'}/E_0^*)$ whenever $v \in \cup_{p \in P(M/E)} S_p(E)$ and $v'$ is an absolute value of $M$ extending $v$.

**Proof.** We first show that $N(M/E) \subseteq N_{\text{loc}}(M/E)$, i.e. of $N(M_{v'}/E_0^*)$, for $v \in E$, and an arbitrary prolongation $v'$ of $v$ on $M$. Identifying $E_v$ with the topological closure of $E$ in $M_{v'}$, one obtains without difficulty that $M_{v'}/E_v$ is a Galois extension with $G(M_{v'}/E_0^*) \cong G(M/(M \cap E_0^*))$. Observing also that $N_{E_0^*}(\mu) = N_{F_0^*}(\mu)$, for every $\mu \in M^*$, where $F = M \cap E_0^*$, one deduces the required inclusion from the following lemma proved in [5], II.

**Lemma 5.2.** Assume that $E$, $M$ and $F$ are fields, such that $M/E$ is a finite Galois extension and $F \in I(M/E)$. For each $p \in P(M/E)$, let $L_p$ be the fixed field of some Sylow $p$-subgroup $G_p$ of $G(M/E)$. Then $N(M/E) \subseteq N(M/F)$ and $N(M/E) = \cap_{p \in P(M/E)} N(M/L_p)$.

Note that the class of fields satisfying the conditions of Theorem 5.1 is closed under the formation of finite extensions. This follows from (1.8) and the fact that absolute values of fields have finitely many prolongations on their finite extensions. Hence, by Lemma 5.2, it suffices to establish the rest of Theorem 5.1 in the special case where $M \subseteq E(p)$.

**Lemma 5.3.** Under the hypotheses of Theorem 5.1, let $M/E$ be a finite $p$-extension, for some $p \in \mathcal{F}$. Then:

(i) An element $c \in E^*$ lies in $N(M/E)$, provided that $c \in N(M_{v'}/E_0^*)$ whenever $v$ runs across $S_p(E)$ and $v'$ is a prolongation of $v$ on $M$;

(ii) There exists a real number $\varepsilon > 0$, such that $N(M/E)$ contains every $\lambda \in E^*$ satisfying $v(\lambda - 1) < \varepsilon$: $v \in S_p(E)$.

**Proof.** For each $v \in S_p(E)$ and any absolute value $v'$ of $M$ extending $v$, denote by $\bar{v}$ and $\bar{v}'$ their continuous prolongations on $E_v$ and $M_{v'}$ respectively. Also, let $\eta_{v'}$ be a primitive element of $M_{v'}/E_v$ taken so that $\bar{v}'(\eta_{v'} - 1) < 1$, and let $h_{v'}(X)$ be the minimal polynomial of $\eta_{v'}$ over $E_{v'}$. It follows from Krasner’s lemma that $M_{v'}/E_v$ contains as a primitive element a root of the polynomial $h_{v'}(X) + (\lambda - 1)a_{v'}$, provided that $a_{v'}$ is the free term of $h_{v'}(X)$ and $\lambda$ is any element of $E_v^*$ for which $v(\lambda - 1)$ is sufficiently small. As
$S_p(E)$ is finite, this enables one to deduce Lemma 5.3 (ii) from Lemma 5.3 (i) and \[118, I, Lemma 4.2 (ii).

We prove Lemma 5.3 (i). Suppose that $[M : E] = p^n$, for some $n \in \mathbb{N}$, fix a field $F \in I(M/E)$ so that $[F : E] = p$, and denote by $\varphi$ some generator of $G(F/E)$. Assume that $c \in N_{\text{loc}}(M/E)$. Considering the cyclic $E$-algebra $(F/E, \varphi, c)$, one obtains from Proposition 1.2 and \[27, Sect. 15.1, Proposition b,\] that $c \in N(F/E)$, which proves Lemma 5.3 in case $n = 1$.

Let now $n \geq 2$ and $\alpha \in F^*$ be an element of norm $N_E^F(\alpha) = c$. We show that there exists $\beta \in F^*$, such that $\alpha(\varphi(\beta)\beta^{-1}) \in N(M/F)$. Denote by $S(F)$ the set of absolute values of $F$ extending elements of $M(E)$ and by $\tilde{S}_p(F)$ the subset of $S(F)$ consisting of prolongations of elements of $S_p(E)$. It is clear from (1.8) that $S_p(F) \subseteq \tilde{S}_p(F)$ and equality holds in case $E$ is a nonreal field. Proceeding by induction on $n$, one concludes that it suffices to establish the existence of $\beta$, under the hypothesis that $N(M/F)$ has the properties required by Lemma 5.3. Fix an element $\omega$ of $S_p(F)$ as well as a prolongation $\omega'$ of $\omega$ on $M$, and denote by $v$ the absolute value of $E$ induced by $\omega$. Suppose first that $\omega$ is the unique prolongation of $v$ on $F$. Then $F_\omega$ is $E_\omega$-isomorphic to $F \otimes_E E_\omega$. This means that $F_\omega/E_\omega$ is a cyclic extension of degree $p$, or equivalently, that $\varphi$ extends uniquely to an $E_\omega$-automorphism $\varphi_\omega$ of $F_\omega$. As $c \in N(R_{\omega'}/E_\omega)$, $N_E^F(\alpha) = c$ and $E$ is dense in $E_\omega$, the obtained result, combined with Hilbert’s Theorem 90 and the inductive hypothesis, proves that $\alpha \varphi(\beta_{\omega'})\beta^{-1}_{\omega'} \in N(R_{\omega'}/F_\omega)$, for some $\beta_{\omega'} \in F^*$.

Suppose finally that $v$ is not uniquely extendable on $F$, and fix a real number $\varepsilon > 0$. Then, by Lemma 1.3, the compositions $\omega \circ \varphi^i := \omega_i$, $i = 0, ..., p-1$, are the prolongations of $v$ on $F$ (and are pairwise nonequivalent). Note also that $E$ is a dense subfield of $F_{\omega_0}, ..., F_{\omega_{p-1}}$, so one can find elements $c_0, ..., c_{p-1}$ of $E^*$ so that $\omega_i(\alpha - c_i) < \varepsilon$, for each index $i$. Clearly, $E^*$ contains elements $\mu_0, ..., \mu_{p-1}$ satisfying the equalities $\mu_j \mu_{j+1}^{-1} = c_j$: $j = 0, ..., p - 2$, and by the approximation theorem, there exists $\beta_\omega \in F^*$, such that $\omega_j(\beta_\omega - \mu_j) < \varepsilon$, for every $j \in \{0, ..., p - 1\}$. Therefore, the product $\alpha_\omega := \alpha \varphi^{(p-1)}(\beta_\omega)\beta_\omega^{-1}$ satisfies the inequalities $\omega_j(\alpha_\omega - c) < \varepsilon$. When $\varepsilon$ is sufficiently small, this ensures that $\alpha_\omega \in N(M_{\omega'}/E_\omega)$, provided that $i \in \{0, 1, ..., p-1\}$ and $v_i$ is an absolute value of $M$ extending $\omega_i$ (cf. \[24, Ch. II, Proposition 2\]). Summing up the obtained results and using again the approximation theorem, one concludes that there exists $\beta \in F^*$ such that $\alpha \varphi(\beta)\beta^{-1} \in N(M_{\omega'}/F_\omega)$, for each $\omega \in S_p(F)$, and any prolongation $\omega'$ of $\omega$ on $M$. Now the inductive hypothesis leads to the conclusion that $\alpha \varphi(\beta)\beta^{-1} \in N(M/F)$ and $c \in N(M/E)$, which completes the proof of Lemma 5.3 and Theorem 5.1.

\[\square\]

**Corollary 5.4.** Assume that $E_0$, $E$ and $S_p(E)$, $p \in \mathbb{P}$, satisfy the conditions of Theorem 5.1, $R$ is a finite extension of $E$ in $E_{\text{sep}}$, $M$ is the normal closure of $R$ in $E_{\text{sep}}$ over $E$, $f$ is a norm form of $R/E$, and $c$ is an element of $E^*$. Then $c \in N(R/E)$ if and only if $c$ is presentable by $f$ over $E_{\text{sep}}$, for each $v \in M(E)$. This occurs if and only if $c$ is presentable by $f$ over $E_v$, for each $v$ lying in the union $\Delta(R/E)$ of the sets $S_p(E)$, $p \in P(M/E)$.
Proof. The assertion that $c \in N(R/E)$ is obviously equivalent to the one that $c$ is presentable by $f$ over $F$. Since $N(M/E) \subseteq N(R/E)$ (cf. [22], Ch. VIII, Sect. 5), Lemmas 5.2 and 5.3 indicate that $N(R/E)$ includes as a subgroup the set $\Sigma_{\varepsilon} = \{\lambda \in E^*: v(\lambda - 1) < \varepsilon, v \in \Delta(R/E)\}$, for some $\varepsilon > 0$ depending on $M/E$. Suppose now that $c$ is presentable by $f$ over $E_v$, for each $v \in \Delta(R/E)$, and fix a positive number $\varepsilon'$. The polynomial $f$ maps $E_v$ continuously into itself, for each $v \in \Delta(R/E)$, so the approximation theorem implies the existence of an element $c_{\varepsilon'} \in N(R/E)$ such that $v(c - c_{\varepsilon'}) < \varepsilon'$, $v \in \Delta(R/E)$. It is therefore clear that $c c_{\varepsilon'}^{-1} \in N(R/E)$ when $\varepsilon'$ is sufficiently small, so Corollary 5.4 is proved.

Corollary 5.5. In the setting of Corollary 5.4, $N_{loc}(R/E) \leq N(R/E)$.

Proof. Our argument relies on the known fact $N_{E_v}^{R}(\rho) = \prod_{j=1}^{t} N_{E_{v,j}}^{R_{E_v}}(\rho)$, for each $\rho \in R^*$, $v \in M(E)$, where $\omega_j$: $j = 1, ..., s$, are the prolongations of $v$ on $R$ (cf. [22], Ch. XII, Proposition 10). Also, it follows from the approximation theorem that if $c_j \in E^* \cap N(R_{E_v}/E_v)$: $j = 1, ..., t$, and $\varepsilon$ is a real positive number, then there is $\rho_v \in R^*$, such that $v(c_j - N_{E_v}^{R_{E_v}}(\rho_v)) < \varepsilon$, for each index $j$. This implies that if $c \in N_{loc}(R/E)$, then one can find an element $\alpha_v \in R^*$ so that $v(N_{E_v}^{R}(\alpha_v) - c) < \varepsilon$, for every $v \in \Delta(R/E)$. Therefore, Lemmas 5.2 and 5.3 yield $N_{E_v}^{R}(\alpha_v)c^{-1} \in N(M/E)$, in case $\varepsilon$ is sufficiently small. Hence, $N_{loc}(R/E) \subseteq N(R/E)$, as claimed.

To prove Theorem 2.3 we also need the following lemma.

Lemma 5.6. Let $E$ be an algebraic extension of a global field $E_0$ and $R$ a finite separable extension of $E_0$, for some $v \in M(E)$. Also, let $E'$ be the subfield of $E_v$ defined in (1.7), $R'$ the separable closure of $E'$ in $R$, and $R_0$ the maximal abelian extension of $E_v$ in $R$. Then $N(R/E_v) = N(R_0/E_v)$ and $N(R'/E') = E'^* \cap N(R/E_v)$.

Proof. It follows from (1.7) and [6], Theorem 8.1, that $E'$ is separably closed in $E_v$. Therefore, for each finite extension $L$ of $E_v$ in $E_v,sep$, the norm map $N_{E_v}^{L'}$ is a prolongation of $N_{E_v}^{J'}$, $L'$ being the separable closure of $E'$ in $L$. Considering $R$ with its unique absolute value extending the continuous prolongation $\bar{v}$ of $v$ on $E_v$, one obtains as in the proof of Lemma 5.3 (ii) that $N(R/E_v)$ includes the set $\{\lambda \in E_v^*: \bar{v}(\lambda - 1) < \varepsilon\}$, for some real number $\varepsilon > 0$. These observations show that $E_{v}^* = E'^*N(R/E_v)$ and $N(R'/E') = E'^* \cap N(R/E_v)$. Let $\Sigma$ be the normal closure of $R$ in $E_v,sep$ over $E_v$, and let $F'$ be the algebraic closure of $E'$ in $F$, for each $F' \in I(\Sigma/E_v)$. It is clear from (1.3) and (1.4) that $\Sigma/E_v$ is a Galois extension and the mapping of $I(\Sigma/E_v)$ into $I(\Sigma'/E')$ by the rule $F \rightarrow F'$ is bijective. Applying (1.4), (1.7) (ii) and Galois theory to $\Sigma'/E'$, one also obtains that $R_0$ is the maximal abelian extension of $E'$ in $R'$. Since $E'$ is quasilocal and $\rho_{E'/L'}$ is surjective, for every finite extension $L'$ of $E'$ (the latter follows from (1.5) (iv) and the Albert-Hochschild theorem, see [28], Ch. II, 2.2), this enables one to deduce from [10], Theorem 1.1 (i), that $N(R'/E') = N(R_0/E')$. Observing now
that \( N(R_0/E_v) = N(R_0/E^\prime)N(R/E_v) = N(R/E^\prime)N(R/E_v) \), one concludes that \( N(R_0/E_v) \subseteq N(R/E_v) \). The inclusion \( N(R/E_v) \subseteq N(R_0/E_v) \) follows at once from the transitivity of norm maps in towers of finite separable extensions (cf. [22], Ch. VIII, Sect. 5), so Lemma 5.6 is proved.

**Proof of Theorem 2.3 and Proposition 2.4.** In view of Theorem 2.1 and Lemma 3.1, \( E \) satisfies the conditions of Theorem 5.1. This reduces Proposition 2.4 to a special case of Corollary 5.4, and enables one to deduce from Corollary 5.5 that \( N_{\text{loc}}(R/E) \subseteq N(R/E) \). It follows from Lemma 3.3 and the former part of (1.5) (iii) that \( E_{v(p)} \) admits local \( p \)-class field theory, for every \( p \in P(E) \). Since \( N^R_{E}(\rho) = \prod \rho_{\pi(p)} \in \Sigma_{\rho} N_{E_{\pi(p)}}^{R}(\rho) \), for each \( \rho \in \mathbb{R}^* \), these observations, combined with Theorem 2.1 and Lemma 5.6, prove that \( N(R/E) = N(F_1/E) \) and \( N_{\text{loc}}(R/E) = N(F_2/E) \), where \( F_1 \in I(\mathcal{T}_{0}/E) \), \( j = 1, 2 \), are determined by Theorem 2.3 (iii) and (iv). They also show that \( F_1 \subseteq F_2 \) and \( P(F_2/E) \subseteq P(M/E) \). Now Theorem 2.3 (ii) can be deduced from Burnside-Wielandt’s characterization of nilpotent finite groups (cf. [19], Ch. 6, Sect. 2), Galois theory and Lemma 3.1 (i). As the latter part of Theorem 2.3 (i) is contained in Remark 3.6, Theorem 2.3 is proved.

We conclude this Section with an example of a field extension \( R/E \) satisfying the conditions of Theorem 2.3, for which \( N_{\text{loc}}(R/E) \neq N(R/E) \).

**Example.** Let \( w(p) \) the \( p \)-adic absolute value of \( \mathbb{Q} \), for each \( p \in \mathcal{P} \), \( W = \{ w(p) : p \in \mathcal{P} \} \). \( E \) a minimal element of \( \Omega(\mathbb{Q}, \mathcal{P}, W) \), \( V = \{ v(p) : p \in \mathcal{P} \} \) a characteristic system of \( E \), and \( R = E(\alpha) \), where \( \alpha \in \mathbb{Q} \) is a root of the polynomial \( f(X) = X^4 + 8X^2 + 256X - 20 \). Denote by \( g(X) = X^3 - 16X^2 + 144X + 256^2 \) the resolution polynomial of \( f(X) \), and put \( \mathbb{Q}_2 = \mathbb{Q}_{\text{sep}} \). It is easily verified that \( f(X) \) has roots in \( \mathbb{Q}_3 \) and in the subfields \( \mathbb{Q}_2(\sqrt{2}) \) and \( \mathbb{Q}_2(\sqrt{-10}) \) of \( \mathbb{Q}_2 \), but has no roots in \( \mathbb{Q}_2 \) (cf. [24], Ch. II, Proposition 2). It is similarly obtained that \( g(X) \) is irreducible over \( \mathbb{Q}_3 \) and the discriminant \( d(g) = d(f) \) lies in \( \mathbb{Q}_3^2 \). Observing also that \(-5d(f) \in \mathbb{Q}_2^2 \), whereas \(-5 \notin \mathbb{Q}_2^2 \), one proves that \( f(X) \) and \( g(X) \) are irreducible over \( \mathbb{Q} \) with Galois groups \( G_f \cong \text{Sym}_4 \) and \( G_g \cong \text{Sym}_3 \). These results, combined with Theorem 2.2 (iii), imply that \( g(X) \) and \( f(X) \) remain irreducible over \( E \) with the same Galois groups. They also show that \( v(p) \) has two nonequivalent prolongations \( v_1(p) \) and \( v_2(p) \) on \( R \), for \( p = 2, 3 \). Hence, by Theorem 2.3, \( N(R/E) = E^* \) and \( N_{\text{loc}}(R/E) \) is a subgroup of \( E^* \), such that \( E^*/N_{\text{loc}}(R/E) \) is noncyclic of order 12. Moreover, it becomes clear that an integer \( \beta \) not divisible by 3 lies in \( N_{\text{loc}}(R/E) \) if and only if \( \beta \) or \(-2\beta \in \mathbb{Q}_2^2 \). In particular, \(-20 \notin N_{\text{loc}}(R/E) \).

**References**

[1] E. Artin and J. Tate, *Class Field Theory*, Benjamin, New York-Amsterdam, 1968.

[2] N. Bourbaki, *Topologie Generale*, 3me Ed. Chap. I: Structures topologiques, Chap. II: Structures uniformes, Hermann, Paris, 1961.

[3] N. Bourbaki, *Algebres Commutatives*, Chap. V: Entiers. Chap. VI: Valuations, Hermann, Paris, 1964.

[4] J.W.S. Cassels and A. Fröhlich (Eds.), *Algebraic Number Theory*, Academic Press, London-New York, 1967.
[5] I.D. Chipchakov, *On the Galois cohomological dimensions of Henselian valued stable fields*, Comm. Algebra 30 (2002), 1549-1574.

[6] I.D. Chipchakov, *On the residue fields of Henselian valued stable fields*, I, J. Algebra 319 (2008), 16-49; II, C.R. Acad. Bulg. Sci. 61 (2008), 1229-1238.

[7] I.D. Chipchakov, *On nilpotent Galois groups and the scope of the norm limitation theorem in one-dimensional abstract local class field theory*, In: Proc. of ICTAMI 05, Alba Iulia, Romania, 15.9, 2005: Acta Univ. Apulensis No. 10 (2005), 149-167.

[8] I.D. Chipchakov, *On the Brauer groups of closed subfields of generalized Tate fields*, Preprint.

[9] I.D. Chipchakov, *Primarily quasilocal fields and 1-dimensional abstract local class field theory*, Preprint, arXiv:math/0506515v7 [math.RA], June 24, 2009.

[10] I.D. Chipchakov, *On the Brauer groups of quasilocal fields and the norm groups of their finite Galois extensions*, Preprint, arXiv:math/0707.4245v6 [math.RA], Feb. 6, 2009.

[11] I. Efrat, *A Hasse principle for function fields over PAC-fields*, Isr. J. Math. 122 (2001), 43-60.

[12] Yu.L. Ershov, *Co-Henselian extensions and Henselization of division algebras*, Algebra i Logika 27 (1988), 649-658 (Russian: English transl. in Algebra i Logika 27 (1988)).

[13] B. Fein and M. Schacher, *Brauer groups of fields algebraic over Q*, J. Algebra 43 (1976), 328-337.

[14] W. Feit and J. Thompson, *Solvability of groups of odd order*, Pac. J. Math. 13 (1963), 775-1029.

[15] I.B. Fesenko and S.V. Vostokov, *Local Fields and Their Extensions*, Translations of Mathematical Monographs, vol. 121, Amer. Math. Soc., Providence, RI, 2002.

[16] L. Fuchs, *Infinite Abelian Groups*, Academic Press, New York and London, 1970.

[17] T.A. Gekht, *G-closed fields and imbeddings of quadratic number fields*, J. Number Theory 8 (1976), 58-72.

[18] W.-D. Geyer, *Unendliche algebraische Zahlkörper, über denen jede Gleichung auflösbar von Beschränkter Stufe ist*, J. Number Theory 1 (1969), 346-374.

[19] M.I. Kargapolov and Yu.I. Merzlyakov, *Fundamentals of Group Theory*, 3rd Ed. Nauka, Moscow, 1982.

[20] G. Karpilovsky, *Topics in Field Theory*, North-Holland Math. Stud. 155, Amsterdam, 1989.

[21] H. Koch, *Galoische Theorie der p-Erweiterungen*, Springer, New York, 1970.

[22] S. Lang, *Algebra*, Addison-Wesley, Reading, MA, 1965.

[23] S. Lang, *Algebraic Number Theory*, Addison-Wesley, Reading MA, 1970.

[24] F. Lorenz and P. Roquette, *The theorem of Grunwald-Wang in the setting of valuation theory*, F.-V. Kuhlmann (ed.) et. al., Valuation Theory and Its Applications, vol. II (Saskatoon, SK, 1999), 175-212, Fields Inst. Comm. 33, Amer. Math. Soc., Providence, RI, 2003.

[25] A.S. Merkurjev, *Brauer groups of fields*, Comm. Algebra 11 (1983), No 22, 2611-2624.

[26] J. Neukirch and R. Perlis, *Fields with local class field theory*, J. Algebra 42 (1976), 531-536.

[27] R. Pierce, *Associative Algebras*, Springer-Verlag, New York, 1982.

[28] J.-P. Serre, *Cohomologie Galoisienne*, Lect. Notes in Math. 5, Springer-Verlag, Berlin, 1965.

[29] A.R. Wadsworth, *Extending valuations to finite dimensional division algebras*, Proc. Am. Math. Soc. 98 (1986), 20-22.

[30] Sh. Wang, *On Grunwald’s theorem*, Ann. Math. 51 (1950), 471-484.

[31] A. Weil, *Basic Number Theory*, Springer-Verlag, Berlin, 1967.

E-mail address: chipchak@math.bas.bg

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., bl. 8, 1113, Sofia, Bulgaria