Non-Markovianity and memory effects in quantum open systems

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Although a number of measures for quantum non-Markovianity have been proposed recently, it is still an open question whether these measures directly characterize the memory effect of the environment, i.e., the dependence of a quantum state on its past in a time evolution. In this paper, we present a criterion and propose a measure for non-Markovianity with clear physical interpretations of the memory effect. The non-Markovianity is defined by the inequality \( T(t_2, t_0) \neq T(t_2, t_1)T(t_1, t_0) \) in terms of memoryless dynamical map \( T \) introduced in this paper. This definition is conceptually distinct from that based on divisibility used by Rivas et al (Phys. Rev. Lett 105, 050403 (2010)), whose violation is manifested by non-complete positivity of the dynamical map. We demonstrate via a typical quantum process that without Markovian approximation, nonzero memory effects (non-Markovianity) always exist even if the non-Markovianity is zero by the other non-Markovianity measures.

I. INTRODUCTION

The Markov approximation is widely used to study open quantum systems, leading to a reduced dynamics where the future state of the system depends only on its present state. Mathematically, a quantum Markovian process can be described by a master equation in the Lindblad form, or equivalently by completely positive divisible maps\textsuperscript{1,2}. Strong-coupling, finite-size environments or small time scales might lead to the failure of the Markov approximation. Memory effects then become important, and the dynamics in this case is said to be non-Markovian. It was found that the non-Markovianity is usually associated with the occurrence of revivals, non-exponential relaxation, or negative decay rates in the dynamics. In recent years, the features of the non-Markovian quantum process attracted attention in both theoretical and experimental studies\textsuperscript{3–16}. A number of quantitative measures have been proposed to quantify non-Markovianity, such as the quantum channels\textsuperscript{17},\textsuperscript{18} the non-monotonic behaviors of distinguishability\textsuperscript{18–26}, entanglement\textsuperscript{22}, Fisher information\textsuperscript{23}, correlation\textsuperscript{24}, the volume of states\textsuperscript{25}, capacity\textsuperscript{26}, the breakdown of divisibility\textsuperscript{22,27}, the negative fidelity difference\textsuperscript{28}, the nonzero quantum discord\textsuperscript{29}, the negative decay rates\textsuperscript{30}, and the notion of non-Markovian degree\textsuperscript{31}.

Among them, the measure defined by Breuer, Laine and Piilo\textsuperscript{15} is closely related to the memory effect: Non-Markovianity in this measure manifests itself as a reverse flow of information from the environment to the system. This back-flow of information might be a sufficient condition for the memory effects. It is worth addressing that the essence of the non-Markovianity is whether the future state depends on its past state or not.

To that extent, there are no straightforward witnesses for non-Markovianity to date, although a great deal of effort has been made to understand the non-Markovianity. The reason that previous measures\textsuperscript{17–31} might not capture directly the memory effect is the physics behind the definitions. The measure in Ref.\textsuperscript{17} investigates whether a quantum channel is consistent with a Markovian evolution. The measure proposed by Rivas, Huelga, and Plenio uses divisibility as a measure of non-Markovianity which characterizes the non-complete positivity of the dynamical map\textsuperscript{22}. The noncomplete positivity is improved in\textsuperscript{27}, but it still could not capture the memory effects. The other non-Markovianity measures in Refs.\textsuperscript{18–26,28–31} are based on features different from those mentioned before, however these features are intrinsically related to the non-complete positivity of dynamical maps in quantum evolutions. Therefore, the measures so far do not directly reflect how the future state of a quantum system depends on its past. This stimulates us to consider the following questions: How do we directly quantify this memory effect? Is non-exponential but monotonic relaxation a Markovian process or a non-Markovian one? What is the essential difference between a time-dependent Markovian equation and a non-Markovian time-local equation? In this paper, we will try to study these questions by quantifying the non-Markovianity directly based on the memory effect.

Generally speaking, the conventional dynamical map \( \varepsilon(t_2, t_1) \) is ambiguous unless the initial time of the evolution is fixed. We define \( T(t_2, t_1) \) (for details, see below) as the dynamical map transferring state from \( \rho(t_1) \) to \( \rho(t_2) \), where \( t_1 \) (arbitrary) is the initial time of the evolution. The initial time means the state of the whole system is in a product state at this point in time. We will show the difference between \( T(t_2, t_1) \) and \( \varepsilon(t_2, t_1) \) in latter discussions. By this definition, \( T \) is always completely positive (CP) and trace preserving (TP) for arbitrary \( \rho(t_1) \) in quantum processes. The Markovian divisibility condition can be expressed in terms of dynamical map \( T \) as \( T(t_2, t_0) = T(t_2, t_1)T(t_1, t_0) \) because the dynamical map in a Markovian process does not depend on the ini-
tial time of evolution. In the non-Markovian regime, the
divisible condition in terms of $T$ is conceptually distinct
from the conventional one since $T(t_2, t_1)$ is no longer a
dynamical map connecting $T(t_1, t_0)$ and $T(t_2, t_0)$. Instead,$T(t_2, t_1)$ and $T(t_2, t_0)$ correspond to two evolutions with
different initial times in a quantum process. Our main
result is that the non-Markovianity in terms of $T$ is man-
ifested by inequality $T(t_2, t_0) \neq T(t_2, t_1)T(t_1, t_0)$ which
has a physical explanation that the evolution depends on
its history. This is distinct from the non-Markovianity
measure defined via completely positive divisibility dis-
cussed in Refs. 22, 27, 31 for the dynamical map.

With those notations, the non-Markovianity is then
defined as the maximum distance between $T(t_2, t_0)$ and
$T(t_2, t_1)T(t_1, t_0)$ over $t_1$ and $t_2$. It reflects the degree of
memory effects in a quantum process. We calculate the
non-Markovianity of a typical quantum process where a
qubit decays into vacuum with different environmental
spectra. The results show that a non-zero memory effect
(non-Markovianity) always exists in this process, even if
it is Markovian by the previous measures. Our measure
applies to a quantum process that can describe evolu-
tions starting from an arbitrary time $t_I$, rather than an
evolution with a fixed initial time.

The paper is organized as follows. In Sec.II, we review
the dynamics of open quantum systems and the concept
of universal dynamical map. In Sec.III, we define two
types of dynamical maps, $A$ and $T$, with different physi-
cal meanings. The non-Markovian criterion and measure
is introduced in Sec.IV in terms of $T$. In Sec.V, we cal-
culated and discuss the non-Markovianity of an exactly
solvable model. Finally, we summarize in Sec.VI.

II. DYNAMICS OF OPEN QUANTUM SYSTEMS

A natural way to model an open quantum system is to
regard it as arising from an interaction between the
system and the environment (denoted by $S$ and $E$),
which together form a closed quantum system. The
reduced density matrix for the system is obtained by
tracing out the environmental degrees of freedom, i.e.,
$\rho_S = \text{Tr}_E(\rho_{SE})$. The total Hamiltonian for the system
and the environment can be written as,

$$H = H_S + H_E + H_{SE},$$

where $H_S$, $H_E$ and $H_{SE}$ represents the Hamiltonian of
the system, the environment and the coupling, respecti-
vously. Consider the total density matrix at $t_1$, $\rho_{SE}(t_1)$,
which undergoes a unitary evolution. The system density
matrix at $t_2$ ($t_2 \geq t_1$) is given by

$$\rho_S(t_2) = \text{Tr}_E[U(t_2, t_1)\rho_{SE}(t_1)U(t_2, t_1)^\dagger],$$

where $U(t_2, t_1) = e^{-\frac{i}{\hbar}H(t_2 - t_1)}$. In a general case where
the total Hamiltonian is time-dependent, $U(t_2, t_1)$ can
be expressed as $U(t_2, t_1) = Te^{-\frac{i}{\hbar}\int_{t_1}^{t_2} H(\tau)d\tau}$ with $T$ the
chronological operator. Eq.(2) can be understood as a
map $\varepsilon(t_2, t_1)$ connecting $\rho_S(t_1)$ and $\rho_S(t_2)$, namely,

$$\rho_S(t_2) = \varepsilon(t_2, t_1)\rho_S(t_1).$$

In general, such a map depends not only on the total
evolution operator $U(t_2, t_1)$ and properties of the
environment, but also on the system state because $\rho_{SE}(t_1)$
may contain correlations between the system and the
environment $[2]$. Given a fixed $\rho_E(t_1)$ of the environment
and the correlations between $S$ and $E$, not all $\rho_S(t_1)$ are
allowed due to the positivity requirement of $\rho_{SE}(t_1)$. For
example, if the correlation is non-zero, $\rho_S(t_1)$ cannot be
a pure state $[2]$. Moreover, it is well known that $\varepsilon(t_2, t_1)$
may not be CP.

A dynamical map which is independent of the state it acts
upon is called a universal dynamical map (UDM). It
describes a plausible evolution for any states $[2]$. Be-
ting TP and CP, a UDM can be expressed in the Kraus
representation,

$$\rho_S(t_2) = \varepsilon(t_2, t_1)\rho_S(t_1) = \sum_\alpha K_\alpha(t_2, t_1)\rho_S(t_1)K_\alpha^\dagger(t_2, t_1),$$

where $\sum_\alpha K_\alpha^\dagger(t_2, t_1)K_\alpha(t_2, t_1) = 1$. It turns out that a
dynamical map is a UDM if and only if it is induced from the
initial condition $\rho_{SE}(t_1) = \rho_S(t_1) \otimes \rho_E(t_1)$, where
$\rho_E(t_1)$ is fixed (independent of the system state) for arbit-
ary $\rho_S(t_1)$ $[2]$.

Notice that in a Markovian or non-Markovian quan-
tum process, an evolution could start not only at $t = 0$, but
also at the other times, say, $t = t_I (t_I \geq 0)$ where we
refer to $t_I$ as the initial time of the evolution. Starting at
$t_I$ implies that the total state of the system and the envi-
ronment satisfies conditions at $t_I$. In this paper, we focus
on quantum processes where the evolutions start with an
initial product state of the system and the environment,
i.e., $\rho_{SE}(t_I) = \rho_S(t_I) \otimes \rho_E(t_I)$ where $\rho_E(t_I)$ is fixed and
$\rho_S(t_I)$ is arbitrary. In other words, we study models
describing a system in arbitrary states interacting with the
environment in a fixed initial state. This consideration
is reasonable as it means that the system and the envi-
ronment are independent at the beginning $t_I$ and then
start their evolution. Typically, $\rho_E(t_I) = \rho_E$ is sta-
tionary under a time-independent environment Hamiltonian
$H_E$, i.e., $\rho_{SE}(t_I) = \rho_S(t_I) \otimes \rho_E$ for any $t_I$. For example,
the equilibrium state or the vacuum state of the environ-
ment is often chosen as an initial environment state to
study the reduced system dynamics. For a general (non-
stationary) initial environment state, it can be given by

$$\rho_E(t_I) = \mathcal{T} e^{-\frac{i}{\hbar}\int_{t_I}^{t_1} H_E(\tau)d\tau} \rho_E(0),$$

where we have assumed that before an evolution starts,
initial state of the environment undergoes free evolution
driven by $H_E[\mathcal{T} \rho_E(t)]$.

With the factorized initial state, the dynamical maps
starting from any time $t_I$ are UDMs characterized by the
fixed initial environment state and the global unitary evolution (total Hamiltonian), rather than the initial system state. Consequently, the quantum processes are supposed to have fixed non-Markovianity as a function of $\rho_E(t_1)$ and $H(t)$, regardless of the system states. On the other hand, the quantum process is universal because it can describe the evolution of any initial system state. In the case of $\rho_E(t_1) = \rho_E$ and $H(t) = H$, which is the usual assumption used to investigate the dynamics of an open quantum system, we expect a time-independent value of non-Markovianity of the quantum process. Otherwise, the non-Markovianity may depend on the time interval that we are interested in since the quantum process is not time-homogenous.

### III. DYNAMICAL MAPS WITH AND WITHOUT MEMORY

The reduced dynamics of an open quantum system can be represented by dynamical maps. In particular, quantum non-Markovian behaviors are often discussed with the help of the dynamical map $\varepsilon(t_2,t_1)$ that transforms the state from $\rho_S(t_1)$ to $\rho_S(t_2)$. However, in general the physical meaning and properties of $\varepsilon(t_2,t_1)$ are not clear unless we know the total density matrix $\rho_{SE}(t_1)$, or, from another point of view, the initial time of the evolution. For instance, let $t_0 \leq t_1 \leq t_2$; when the evolution starts at $t_1$, which means $\rho_{SE}(t_1) = \rho_S(t_1) \otimes \rho_E$, the map $\varepsilon(t_2,t_1)$ is a UDM. In contrast, when the evolution starts at a fixed time $t_0$, the total density matrix at $t_1$ is $\rho_{SE}(t_1) = U(t_1,t_0)\rho_S(t_0) \otimes \rho_E U(t_1,t_0)^\dagger$, and then $\varepsilon(t_2,t_1)$ can be understood as an intermediate map that is in general not a UDM. Obviously, $\varepsilon(t_2,t_1)$ has different meanings and properties for different initial times of evolution. This is crucial for understanding non-Markovian characters since it tells us that the evolution of the system after $t_1$ [determined by $\varepsilon(t_2,t_1)$] is related to its history (from $t_0$ to $t_1$). To clarify this point, we introduce two types of dynamical maps denoted by $\Lambda$ and $T$, respectively. We will describe the two dynamical maps in the following.

#### A. Type I: dynamical map with memory

$\Lambda(t_2,t_1)$ is defined as the quantum dynamical map transferring the state from $\rho_S(t_1)$ to $\rho_S(t_2)$ when the evolution starts at a prescribed time $t_0$ ($t_0 \leq t_1 \leq t_2$), i.e., $t_1 = t_0$. When $t_1 > t_0$, $\Lambda(t_2,t_1)$ can be understood as a continuous map. The physical meaning of $\Lambda(t_2,t_1)$ can be expressed as

$$\rho_S(t_2) = \Lambda(t_2,t_1)\rho_S(t_1) = Tr_{E}[U(t_2,t_1)\rho_{SE}(t_1)U(t_2,t_1)^\dagger]$$

where $\rho_{SE}(t_1) = U(t_1,t_0)\rho_S(t_0) \otimes \rho_E U(t_1,t_0)^\dagger$ and $\rho_S(t_0) = \rho_S(t_1)$ defined by Eq. [5]. $\rho_{SE}(t_1)$ is in general not a product state with fixed environment state due to the system-environment interaction, this means $\Lambda(t_2,t_1)$ is not a UDM in general. The history of the system from $t_0$ to $t_1$ is encoded in $\rho_{SE}(t_1)$. We will refer to this type of map as dynamical map with memory. This interpretation of the dynamical map is frequently used to investigate the non-Markovianity of quantum dynamics.

In particular, when the starting time of the map is the initial time of the evolution, i.e., $t_1 = t_0$, $\Lambda(t_2,t_0)$ is a UDM represented by,

$$\rho_S(t_2) = \Lambda(t_2,t_0)\rho_S(t_0) = Tr_{E}[U(t_2,t_0)\rho_S(t_0) \otimes \rho_E(t_0)U(t_2,t_0)^\dagger]$$

which can map any system state to another physical state.

#### B. Type II: memoryless dynamical map

Next, we define $T(t_2,t_1)$ as the quantum dynamical map transferring the quantum system from state $\rho(t_1)$ to $\rho(t_2)$, where $t_1$ is the initial time of the evolution, i.e., $t_1 = t_1$. The initial total state is $\rho_{SE}(t_1) = \rho_S(t_1) \otimes \rho_E(t_1)$ with fixed $\rho_E(t_1)$ defined by Eq. [5] and arbitrary $\rho_S(t_1)$. Then, $T(t_2,t_1)$ can be expressed as,

$$\rho_S(t_2) = T(t_2,t_1)\rho_S(t_1) = Tr_{E}[U(t_2,t_1)\rho_S(t_1) \otimes \rho_E(t_1)U(t_2,t_1)^\dagger]$$

$T(t_2,t_1)$ is always a UDM in contrast to $\Lambda(t_2,t_1)$. As $\rho_E(t_1) = \rho_E(t_1)$ is independent of the system’s history, the map $T$ is memoryless. We remind the reader that in a quantum process, $T(t_2,t_1)$ and $T(t_2',t_1')$ ($t_1 \neq t_1'$) correspond to different evolutions.

In particular, when the total Hamiltonian $H$ is time-independent and the initial environmental state is stationary, we have

$$\rho_S(t_2) = T(t_2,t_1)\rho_S(t_1) = Tr_{E}[e^{-\frac{i}{\hbar}H(t_2-t_1)}\rho_S(t_1) \otimes \rho_Ee^{\frac{i}{\hbar}H(t_2-t_1)}]$$

One can easily observe that the map $T(t_2,t_1)$ is time-homogeneous, i.e., $T(t_2,t_1) = T(t_2-t_1,0)$. This feature was discussed by Chruściński and Kossakowski in Ref. [3], leading to the existence of an initial time $t_1$ in a non-Markovian time-local master equation,

$$\dot{\rho}_S = \mathcal{L}(t-t_1)\rho_S$$

such that $T(t_2,t_1) = T e^{\int_{t_1}^{t_2} \mathcal{L}(\tau-t_1)d\tau} = T e^{\int_{0}^{t_2-t_1} \mathcal{L}(\tau)d\tau} = T(t_2-t_1,0)$. Eq. [10] provide a full description of the quantum non-Markovian process where evolutions starting at any time $t_1$ are included. The initial time $t_1$ is usually ignored in literature provided that one only considers evolutions starting from $t = 0$, leading to $\dot{\rho} = \mathcal{L}(t)\rho$ which may have the same form of a time-dependent Markovian equation with non-negative decay rates. However, the
existence of the initial time $t_1$ is the essential feature of a non-Markovian time-local equation compared with a time-dependent Markovian equation. It provides an evident signature of the memory effect since the dynamical map $\epsilon(t_2, t_1) = T e^{iH(t_2-t_1)\tau} \rho_S(t_1) T^\dagger e^{-iH(t_2-t_1)\tau}$ sending $\rho_S(t_1)$ to $\rho_S(t_2)$ relies on the initial time $t_1$ ($t_2 \geq t_1 \geq t_2$), i.e., the history of the system, while in a time-dependent Markovian equation, $\epsilon(t_2, t_1) = T e^{iH(t_2-t_1)\tau} \rho_S(t_1) T^\dagger$ is uniquely defined.

When $t_1$ is the initial time of the evolution $t_0$, we define $\Lambda(t_2, t_0) = T(t_2, t_0)$ by definition. The key difference between the two types of dynamical maps $\Lambda(t_2, t_1)$ and $T(t_2, t_1)$ exists when $t_1$ is not the initial time. When $t_1 \neq t_0$, in general $\Lambda(t_2, t_1) \neq T(t_2, t_1)$ since $\rho_{SE}(t_1)$ is in general not $\rho_S(t_1) \otimes \rho_E(t_1)$ with fixed $\rho_E(t_1)$. Given a quantum process characterized by the total Hamiltonian and the initial environment state, the map $T$ is well-defined and straightforward to calculate compared with $\Lambda$. With these notations, the non-Markovianity can be defined by the use of the memoryless dynamical map in the following section.

**IV. NON-MARKOVIANITY BASED ON MEMORY EFFECTS**

In this section, we give a condition as well as a measure for quantum non-Markovinity quantifying the degree of the memory effect. Let us start from the quantum Markov process. A quantum process is called Markovian if the corresponding dynamical map, which we denote by $\Lambda_M$ in this paper, is divisible, i.e.,

$$\Lambda_M(t + \tau) = \Lambda_M(\tau)\Lambda_M(t)$$

(11)

where all maps are UDMs (CP and TP) defining a one-parameter semigroups. Equivalently, the quantum Markovian process can be described by a master equation in Lindblad form, \cite{32}

$$\dot{\rho}_S = \mathcal{L}\rho_S$$

$$= -i[H, \rho_S] + \sum_\alpha \gamma_\alpha(V_\alpha \rho_S V_\alpha^\dagger - \frac{1}{2} V_\alpha^\dagger V_\alpha \rho_S$$

$$- \frac{1}{2} \rho_S V_\alpha^\dagger V_\alpha),$$

(12)

with $\gamma_\alpha \geq 0$. The evolution of the system is given by $\rho_S(t_2) = \Lambda_M(t_2 - t_1)\rho_S(t_1)$ where $\Lambda_M(t) = e^{\mathcal{L}t}$. More generally, when the generator $\mathcal{L}$ varies with time due to the change of external conditions, the process is called time-dependent Markovian which satisfies a time-dependent master equation in Lindblad form,

$$\dot{\rho}_S = \mathcal{L}(t)\rho_S$$

$$= -i[H(t), \rho_S] + \sum_\alpha \gamma_\alpha(t)[V_\alpha(t) \rho_S V_\alpha^\dagger(t) - \frac{1}{2} V_\alpha^\dagger(t) V_\alpha(t) \rho_S - \frac{1}{2} \rho_S V_\alpha^\dagger(t) V_\alpha(t)]$$

(13)

with $\gamma_\alpha(t) \geq 0$. The dynamical map becomes $\Lambda_M(t_2, t_1) = T e^{iH(t_2-t_1)\tau} \rho_S(t_1) T^\dagger e^{-iH(t_2-t_1)\tau}$ which is inhomogeneous but still a UDM, and the evolution of the system can be described by $\rho_S(t_2) = \Lambda_M(t_2, t_1)\rho_S(t_1)$. The divisibility condition can then be written as

$$\Lambda_M(t_2, t_0) = \Lambda_M(t_2, t_1)\Lambda_M(t_1, t_0)$$

(14)

where each dynamical map is a UDM. We will discuss in the following the time-dependent Markovian process without loss of generality.

In a quantum (time-dependent) Markovian process, the dynamical map transferring $\rho_S(t_1)$ to $\rho_S(t_2)$ is uniquely given by the UDM $\Lambda_M(t_2, t_1)$ regardless of the system’s history before $t_1$. We remark that the map $\Lambda_M(t_2, t_1)$ is a UDM if and only if it is induced from a product state, i.e.

$$\rho_S(t_2) = \Lambda_M(t_2, t_1)\rho_S(t_1)$$

$$= \text{Tr}_E[U(t_2, t_1)\rho_S(t_1) \otimes \rho_E(t_1) U(t_2, t_1)^\dagger]$$

(15)

where $\rho_E(t_1)$ is a fixed state and $\rho_S(t_1)$ is arbitrary. The evolution of the system at any time $t_1$ in a (time-dependent) Markovian process can be understood by examining Eq. (15). It is interpreted as the collisional model as reviewed in Ref. \cite{33}. Particularly, when the total Hamiltonian and the initial environmental state are time-independent, the map $\Lambda_M(t_2, t_1)$ can be understood as

$$\rho_S(t_2) = \Lambda_M(t_2, t_1)\rho_S(t_1)$$

$$= \Lambda_M(t_2 - t_1)\rho_S(t_1)$$

$$= \text{Tr}_E[e^{-iH(t_2-t_1)} \rho_S(t_1) \otimes \rho_E e^{iH(t_2-t_1)}]$$

(16)

which depends only on the difference $t_2 - t_1$, corresponding to a homogeneous Markovian process described by Eq. (11) or Eq. (12).

In a real Markovian quantum process, the total state of the system and the environment may not remain perfectly factorized as in Eq. (15), and $\epsilon(t_2, t_1)$ is not a UDM in general. From this point of view, the exact dynamics of an open quantum system is never perfectly Markovian \cite{2}. However, if $\rho_{SE}(t) \approx \rho_S(t) \otimes \rho_E(t)$ with fixed $\rho_E(t)$ (independent of the system state) where the correlation between the system and environment does not affect so much the system’s dynamics, a Markovian model is still a good approximate description.

An important observation is that the evolution from any time $t_1$ in a quantum (time-dependent) Markovian process can be interpreted as Eq. (15) where the total state $\rho_{SE}(t_1)$ remains a product state and the environment state is fixed for arbitrary $\rho_S(t_1)$. No matter the initial time of an evolution is $t_1$ or $t_0$ (or any time before $t_1$), the dynamical map transferring $\rho_S(t_1)$ to $\rho_S(t_2)$ is always given by $\Lambda_M(t_2, t_1)$. Therefore, the Markovian dynamical map $\Lambda_M$ can be interpreted as both $\Lambda$ and $T$ according to our definitions in Sec. III, i.e.,

$$\Lambda_M(t_2, t_1) = \Lambda(t_2, t_1) = T(t_2, t_1)$$

(17)
for any $t_0 \leq t_1 \leq t_2$. Consequently, in a Markovian process, the divisibility condition Eq. (14) can be expressed both in terms of $A$ and $T$, respectively,

$$A(t_2, t_0) = A(t_2, t_1)A(t_1, t_0), \quad (18)$$

$$T(t_2, t_0) = T(t_2, t_1)T(t_1, t_0). \quad (19)$$

Any violation of the above two divisibility conditions will be a manifestation of non-Markovianity. Interestingly, the violations of Eq. (18) and Eq. (19) have different physical interpretations which can lead to different criteria and measures for quantum non-Markovianity. The violation of Eq. (18) is usually manifested by the non-complete positivity of the intermediate map $A(t_2, t_1)$ in evolutions starting at $t_0$, which is the ultimate reason behind the behaviors of different quantities (such as the trace distance, entanglement, correlation and so on) in Refs. [18, 26, 28, 29]. The violation itself also accounts for the measures in Refs. [22, 27, 30, 31]. In contrast, the dynamical maps in Eq. (19) are all UDMs, i.e., CP and TP, and the violation of the second divisibility condition Eq. (19) is manifested by the inequality

$$T(t_2, t_0) \neq T(t_2, t_1)T(t_1, t_0). \quad (20)$$

A quantum process is non-Markovian if there exists $t_2 \geq t_1 \geq t_0$ such that the inequality Eq. (20) holds. This criterion is conceptually different from others. It has a clear physical interpretation in terms of memory effects, which we will explain in the following.

Consider the left-hand and right-hand sides of Eq. (20) acting on an arbitrary state $\rho_S(t_0)$, respectively, as shown in Fig. 1. On the left-hand side of Eq. (20), $\rho_S(t_0)$ is mapped to $\rho_S(t_2)$ by $T(t_2, t_0)$ in evolution $A$, while on the right-hand side, $\rho_S(t_0)$ is firstly mapped to $\rho_S(t_1)$ by $T(t_1, t_0)$ in evolution $B$. Then, as an initial state, $\rho_S(t_1)$ is mapped to $\rho_S(t_2)$ by $T(t_2, t_1)$ in evolution $C$, which starts at $t_1$ with $\rho_S(t_1) = \rho_S(t_1) \otimes \rho_E(t_1)$. If $T(t_2, t_0) \neq T(t_2, t_1)T(t_1, t_0)$, there exists $\rho_S(t_0)$ such that $\rho_S(t_2) \neq \rho_S(t_2)$. We remind the reader that in both evolutions $A$ and $C$, the system state at $t_1$ is $\rho_S(t_1)$, but with different histories: in evolution $A$, $\rho_S(t_1)$ has a history from $t_0$ to $t_1$ which is taken into account by $\rho_S(t_1) = U(t_1, t_0)\rho_S(t_0) \otimes \rho_E(t_0)U(t_1, t_0)^\dagger$. Nevertheless, in evolution $C$, $\rho_S(t_1)$ serves as an initial state without any history before $t_1$. Therefore, the fact that $\rho_S(t_2) \neq \rho_S(t_2)$ is a direct manifestation of the memory effect where the future evolution (after $t_1$) of the system depends on its history (from $t_0$ to $t_1$). This is the fundamental property of non-Markovianity. If $T(t_2, t_0) = T(t_2, t_1)T(t_1, t_0)$ provided $t_2 > t_1 > t_0$, then $\rho_S(t_2) = \rho_S(t_2)$ for any $\rho_S(t_0)$, we say that the future state of the system depends only on its present state, i.e., the process is Markovian.

From the environment side, the physics of Eq. (20) can be interpreted as follows. At the end of evolution $B$, the total state of the system and the environment is $\rho_S(t_1) = U(t_1, t_0)\rho_S(t_0) \otimes \rho_E(t_0)U(t_1, t_0)^\dagger$. Then, at the beginning of evolution $C$, the environment is initialized by $T(t_2, t_1)$ such that $\rho_S(t_1) \otimes \rho_E(t_1)$, where $\rho_E(t_1)$ is the initial environmental state at $t_1$ defined in Eq. (19). The terminology initialize means the system information acquired by the environment in the time interval $[t_0, t_1]$ is erased at time $t_1$. Note that the initialization never happens in evolution $A$. After $t_1$, if the future system states in evolutions $A$ and $B$ are different, that means the environment ”remembers” the history of the system [encoded in the $\rho_S(t_1)$] and the future system is affected by this kind of memory.

We define the non-Markovianity as the maximum distance between $T(t_2, t_0)$ and $T(t_2, t_1)T(t_1, t_0)$ over all $t_1$ and $t_2$ (We may investigate a quantum process with a fixed $t_0$).

$$N_M = \max_{t_1, t_2} D(T(t_2, t_0), T(t_2, t_1)T(t_1, t_0)) \quad (21)$$

Here $D$ denotes some distance measurement between the two quantum dynamical maps $T(t_2, t_1)$ and $T(t_2, t_1)T(t_1, t_0)$. (Here the concatenation of the two UDMs $T(t_2, t_1)$ and $T(t_1, t_0)$ is also a UDM). This definition allows us to quantify non-Markovianity directly through dynamical maps without optimization of quantum states [18]. It can be understood as the maximal deviation of the divisibility condition Eq. (19). When $N_M \to 0$, a quantum process loses its memory effects and becomes Markovian.

To choose a distance measure between two dynamical maps for our non-Markovianity, we remark that a dynamical map $A$ is isomorphic to its Choi–Jamiolkowski matrix $\psi_A$ defined as $\rho_A = \Lambda \otimes I(|\psi\rangle\langle\psi|)$. Here $I$ is the identity map, and $|\psi\rangle = \sum^n_{i=1} |i\rangle|\langle i|$ is a maximally entangled state of the system and an ancillary system. The Choi–Jamiolkowski matrix has been used in Ref. [22] to quantify the non-complete positivity and measure the non-Markovianity. Meanwhile, it is often used for measuring the fidelity or distance between two quantum channels (typically a general channel and a unitary one) $[34, 37]$. Here we can easily measure the distance between two dynamical maps $A_1$ and $A_2$ through the trace distance of their Choi–Jamiolkowski matrices, i.e.,

$$D(A_1, A_2) = \frac{1}{2} \|\rho_{A_1} - \rho_{A_2}\| \quad (22)$$

where $\|A\| = \text{Tr}(\sqrt{AA^\dagger})$ is the trace norm of an operator $A$. The good properties of the trace distance can be taken advantage of for measuring the distance of dynamical maps $[34]$. Finally, from Eq. (21) and Eq. (22), the
non-Markovianity is given by
\[ N_M = \max_{t_1,t_2} \frac{1}{2} \| \rho_{T(t_2,t_0)} - \rho_{T(t_2,t_1)T(t_1,t_0)} \| \]  
(23)
which naturally gives a finite value of non-Markovianity satisfying 0 \leq N_M \leq 1 for any quantum process without normalization.

Given a theoretically described quantum process, the dynamical map \( T(t_2,t_1) \) is always CP and describes a physically plausible evolution. The determination of \( T(t_2,t_1) \) (or its Choi-Jamiolkowski matrix) is straightforward as long as the evolution starting from \( t_1 \) is known, regardless of how the process is described. In experiment, \( T(t_2,t_1) \) \( \rho_{T(t_2,t_1)} \) could be determined through quantum process tomography (quantum state tomography) [35]. When the complete information about \( T(t_2,t_1) \) is unavailable, \( \rho_{S(t_2)} \neq \rho_{S(t_2)} \) can be used as a sufficient condition of non-Markovianity which is easy to verify both theoretically and experimentally. Moreover, for any observable \( A \), \( \text{Tr}[\rho_{T(t_2)}] \neq \text{Tr}[\rho_{T(t_2)}] \) is sufficient for non-Markovianity.

V. EXAMPLE

In this section, we illustrate how our measure can be calculated with a typical quantum process. The model describes a two-level system (denoted by S) decaying into its environment (denoted by E), which is initially in the vacuum state. This model is exactly solvable and extensively discussed to study the non-Markovian behaviors. The total time-independent Hamiltonian is written as
\[ H = H_S + H_E + H_{SE} \]  
(24)
where \( H_S = \omega_0 \sigma^+ \sigma^- \), \( H_E = \sum_k \omega_k b_k^\dagger b_k \) are the free Hamiltonians of S and E, and \( H_{SE} = \sigma^+ \sum_k g_k b_k + \sigma^- \sum_k g_k^* b_k^\dagger \) denotes the coupling between the qubit and the environmental modes. In the case that \( \rho_{SE}(0) = \rho_S(0) \otimes \rho_E \) where \( \rho_S(0) \) is arbitrary and \( \rho_E = |0\rangle_E \langle 0|_E \) is the vacuum state, the evolution starting at 0 (in the interaction picture) can be expressed in terms of dynamical map \( T \) as
\[ \rho_S(t) = T(t,0) \rho_S(0) = \begin{pmatrix} |c(t)|^2 & c(t) \rho_{S11}(0) & c(t) \rho_{S12}(0) \\ c(t)^* \rho_{S21}(0) & |c(t)|^2 & \rho_{S22}(0) \\ 1 - |c(t)|^2 & \rho_{S21}(0) & \end{pmatrix} \]  
(25)
where the function \( c(t) \) satisfies
\[ \dot{c}(t) = -\int_0^t dt' f(t-t')c(t') \]  
(26)
with the correlation function \( f(t-t_1) = \int d\omega J(\omega) e^{i(\omega_0-\omega)(t-t_1)} \)

Since the total Hamiltonian is time-independent and the initial environmental state \( \rho_E \) is invariant under the total Hamiltonian, the map \( T \) is time homogeneous according to the discussion in Sec. 2, i.e., \( T(t_2,t_1) = T(t_2-t_1,0) \). Thus, the evolution starting at \( t_1 \) \( \rho_{SE}(t_1) = \rho_S(t_1) \otimes \rho_E \) can be easily obtained as
\[ \rho_S(t_2) = T(t_2,t_1) \rho_S(t_1) = T(t_2 - t_1,0) \rho_S(t_1). \]  
(27)
Alternatively, the evolution starting at 0 can be described by the following master equation
\[ \dot{\rho}_S = \mathcal{L}(t) \rho_S \]  
(28)
where \( S(t) = -2 \text{Im} \frac{\dot{c}(t)}{c(t)} \) and \( \gamma(t) = -2 \text{Re} \frac{\dot{c}(t)}{c(t)} \). According to Eq. (28), \( T(0,0) \) is given by \( T(0,0) = T e^{\int_0^0 \mathcal{L}(\tau)d\tau} \). We stress here that Eq. (28) can only describe evolutions starting at 0 rather than an arbitrary time \( t_f \). Instead, the master equation describing the full quantum process contains the initial time \( t_f \) [3].

Assume the spectral density of the bath are Lorentzian and of the form \( J_L(\omega) = \frac{1}{2\pi} \frac{\gamma_0}{\omega_0^2 - \omega^2 + \lambda^2} \) where \( \lambda \) is connected to the environment correlation time \( \tau_E \) by \( \tau_E = 1/\lambda \) and \( \gamma_0 \) determines the time scale of the system by \( \tau_S = 1/\gamma_0 \). The solution of Eq. (26) is given by \( c(t) = e^{-\lambda t/2} \left[ \cos(\frac{\lambda t}{2}) + i \frac{\gamma_0}{\lambda} \sin(\frac{\lambda t}{2}) \right] \) with \( d = \sqrt{\lambda^2 - 2\gamma_0^2} \). Since \( c(t) \) is real, we have \( S(t-t_1) = 0 \) in Eq. (29). Typically, the exact dynamics of an open quantum system is not Markovian as discussed above. In this example, when the environment correlation time \( \tau_E \) is much smaller than the system characteristic time \( \tau_S \), i.e., \( \tau_E \ll \tau_S \), a Markovian model can be a good approximation [1, 32]. Thus the non-Markovianity of this quantum process can be characterized by the parameter \( R = \frac{\tau_S}{\tau_E} = \frac{\tau_S}{\tau_E} \). When \( R < \frac{1}{2} \), the process can be described by Eq. (29) with \( \gamma(t-t_1) \geq 0 \). Interestingly, when we only consider evolutions starting at 0, Eq. (29) reduces to Eq. (28) which has the form of a time-dependent Markovian equation Eq. (13) (but with different meanings because \( T(t_2,t_1) \) is undefined in Eq. (28) for \( t_1 > t_0 \)). Therefore, the process with \( R < \frac{1}{2} \) is called Markovian by previously proposed measures that use evolutions starting at a fixed time. It seems counter-intuitive that even though \( \tau_E \) is comparable to \( \tau_S \), for example, \( R = \frac{\tau_S}{\tau_E} = 0.49 \), the process is still called Markovian. Also, other intuitively non-Markovian models might be called Markovian by previous measures [33]. In contrast, we will show that
The non-Markovianity $N_M$ for this quantum process is calculated as follows. From Eq. (23) and Eq. (27), we obtain the Choi-Jamiolkowski matrix of $T(t_2, t_0)$

$$
\rho_{T(t_2,t_0)} = T(t_2, t_0) \otimes \mathbb{I}(\psi \langle \psi |)
$$  

where $\rho_{T(t_2,t_0)} = \frac{1}{2} \left( \begin{array}{cccc} |c(\tau_{20})|^2 & 0 & 0 & c(\tau_{20}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - |c(\tau_{20})|^2 & 0 \\ c(\tau_{20})^* & 0 & 0 & 1 \end{array} \right)
$$

We first demonstrate that when $R < 0.5$, the evolution in the quantum process depends on its history proving the process is non-Markovian. Consider that the first evolution starts at 0 where the system is initially in its excited state, i.e., $\rho_{SE}(0) = |e\rangle\langle e| \otimes \rho_E$. Then, the system density matrix at $t_1$ is given by $\rho_S(t_1) = T(t_1, 0)\rho_{SE}(0)$. Now we assume the second evolution starts at $t_1$ with initial state $\rho'_S(t_1) = \rho_S(t_1) \otimes \rho_E = \rho_S(t_1) \otimes \rho_E$. At a further time $t_2$, we have $\rho_S(t_2) = T(t_2, 0)\rho_S(0)$ in the first evolution and $\rho'_S(t_2) = T(t_2, t_1)\rho'_S(t_1)$ in the second evolution. The result is visualized in Fig. 2 by evaluating the evolutions of the excited-state population $\rho_{ee}$ with $R = 0.4$. From the fact that $\rho'_S(t_2) \neq \rho_S(t_2)$, we conclude that the future states (after $t_1$) of the system are relevant to its history (from $t_0$ to $t_1$) in the process. The exited-state population $\rho_{ee}$ decays monotonically and non-exponentially in this case. Although the decay rate $\gamma(t - t_1)$ is non-negative and the revival of $\rho_{ee}$ does not occur, the environment is affected by the system’s history and then has an influence on the future evolution of the system. Thus the dashed line and the solid line in Fig. 2 do not overlap. This phenomenon never happens in a (time-dependent) Markovian process described by Eq. (13), where the dynamical map transferring $\rho_{EE}(t_2)$ to $\rho_{EE}(t_2)$ is uniquely given by $\Lambda_{ME}(t_2, t_1) = T e^{\frac{\Gamma}{2} \int_{t_1}^{t_2} C(r) dr}$ regardless of the initial time.

We calculate the non-Markovianity by numerically optimizing the two time differences $\tau_{10}$ and $\tau_{21}$. $N_M$ is plotted as a function of $\log R$ in Fig. 3 where $R$ varies from $10^{-4}$ to $10^4$. The result demonstrates that the non-Markovianity is non-zero for all $R > 0$. It monotonically decreases with $R$ and tends to 0 as $R \to 0$. It is observed that when $R < 10^{-2}$, the non-Markovianity is already very small ($N_M < 0.006$), implying that the quantum process is approaching Markovian and almost memoryless. Indeed, when $N_M$ is small, the dashed and solid lines in Fig. 3 will be very close and almost exponential.
Then, a Markovian master equation can well describe evolutions starting from any time, i.e., the full quantum process. In this model, the non-exponential relaxation is a sign of non-Markovianity.

Before closing this section, we consider the Ohmic spectral density with an exponential cutoff $J_0(\omega) = \omega\alpha e^{-\frac{\omega}{\omega_c}}$, where $\alpha$ is a dimensionless coupling strength and $\omega_c$ is the cutoff frequency. In this case, the full dynamics is still described by Eqs. (27) or (29), whereas the function $c(t-t_f)$ is complex and $S(t-t_f)$ is non-zero in general. For complex $c(t)$, the non-Markovianity has the following form,

$$\nonumber N_M = \max_{t_1 \leq t \leq t_2} \left[ \frac{1}{8} \left| M' - \sqrt{M'^2 + 4(N' - 2K)} \right| + \frac{1}{8} \left| M' + \sqrt{M'^2 + 4(N' - 2K)} \right| \right]$$

where $M' = |c(\tau_{10})|^2 - |c(\tau_{10})c(\tau_{21})|^2$, $N' = |c(\tau_{20})|^2 + |c(\tau_{10})c(\tau_{21})|^2$, and $K = \text{Re} [c(\tau_{20})^* c(\tau_{10})^* c(\tau_{21})]$.

The exact value of $c(t-t_f)$ for the Ohmic spectrum can be calculated using the analytic expressions in Ref. [9], that is,

$$\nonumber c(t) = e^{i\omega_0 t} \left\{ Z e^{-i\omega t} + \frac{2}{\pi} \int_0^\infty d\omega \frac{\omega J_0(\omega) e^{-i\omega t}}{4[\omega - \omega_0 - i\Sigma(\omega)] + [J_0(\omega)]^2} \right\}$$

where $\Sigma(\omega) = \alpha \omega_c \left[ \sqrt{\alpha \omega_c} e^{-\frac{\omega}{\omega_c}} / \sqrt{\pi} E_i\left(\frac{\omega}{\omega_c}\right) - 1 \right]$. When the condition $\alpha\omega_c > \omega_0$ holds, $Z = [1 - \Sigma'(\omega')]^{-1} \neq 0$ corresponding to a dissipationless process ($c(\infty) \neq 0$) due to the zero spectral density for negative frequencies, otherwise, $Z = 0$. Here $\omega'$ is the solution of $\omega' = \omega_0 - \Sigma'(\omega) < 0$. The global phase $e^{i\omega_0 t}$ in Eq. (34) is added compared with the expression in Ref. [9] since we are working in the interaction picture for consistency with the Lorentzian case. In fact, the non-Markovianity does not depend on the picture we choose. The non-Markovianity for different $\alpha$ (from $10^{-3}$ to $10^2$) is calculated with $\omega_0/\omega_c = 1$, as shown in Fig. 5. Although the system dynamics for the Ohmic spectrum is different from that for the Lorentzian one, especially in the strong-coupling regime, the behavior of $N_M$ as a function of the coupling strength is similar. The non-Markovianity (memory effect) is non-zero for all $\alpha > 0$ even for parameters leading to $\gamma(t-t_f) > 0$ in Eq. (29). When the coupling is weak, $\log(N_M)$ and $\log R$ (or $\log \alpha$) are in a linear relationship in both cases, which indicates that the quantum process asymptotically becomes Markovian with the decreasing of the coupling strength.

**VI. CONCLUSIONS AND DISCUSSIONS**

We present a criterion and propose a measure for non-Markovianity of quantum processes. The measure directly quantifies the degree of memory effects, i.e., how much the future state of a system depends on its past. To construct the measure, we introduce a universal dynamical map (UDM) $T(t_2,t_1)$ which corresponds to an evolution starting from $t_1$. In contrast to $A(t_2,t_1)$, $T(t_2,t_1)$ is well defined and always CP as well as easy to calculate in a quantum process. The Markovian divisibility can be expressed in terms of $T$ in Eq. (19) and the violation of it is simply manifested by the inequality Eq. (20), which has a clear physical interpretation as the memory effects. We define non-Markovianity as the maximal violation of Eq. (19) for all times. Unlike the previous proposed measures which focus on evolutions starting from a fixed time, our measure applies to a quantum process where evolutions can start at an arbitrary time.

One important result of our work is that a quantum process may have a memory effect even if it is called Markovian by other measures. Thus, the previously proposed criteria is not equivalent to the memory effect. Besides, we demonstrate that in a non-Markovian process, the dynamics may be still described by a time-dependent master equation in Lindblad-like form with non-negative decay rates. However, the non-Markovian master equation contains the initial time $t_1$, which is essentially different from a time-dependent Markovian equation [3]. When only describing evolutions starting from $0$ ($t_1 = 0$), a non-Markovian master equation may have the same form of a time-dependent Markovian master equation, e.g., Eq. (29) with $R < \frac{1}{2}$. By observing evolutions starting from different times as in Fig 2, memory effects can be revealed. Thus the negative decay rates in a master equation are not necessary to describe memory effects (non-Markovianity). And non-exponential but monotonic relaxation may occur in both non-Markovian processes and time-dependent Markovian processes.

Our measure is in units of trace distance that satisfies $0 \leq N_M \leq 1$ for any quantum process without normalization. It is easy to calculate regardless of the description of the quantum process. The optimization for quantum states or the knowledge of the environmental state is not required. When the full information of the dynamical map $T$ is unavailable, the condition $\rho_S(t_2) \neq \rho'_S(t_2)$
\[ \text{Tr}[\rho(t_2)] \neq \text{Tr}[\rho'(t_2)] \] can be used as a witness of non-Markovianity, which is easy to be examined both theoretically and experimentally.

Now we consider the renormalized spectral density with zero negative frequency components in the example \( J'_L(\omega) = J_L(\omega) \theta(\omega) \) where \( \theta(\omega) \) is the step function. By numerical simulation, we find that the dynamics and the non-Markovianity for \( J_L(\omega) \) are different from those for \( J'_L(\omega) \). The influence caused by a negative component strongly depends on \( \frac{1}{\lambda \omega_0} \). When \( \frac{1}{\lambda \omega_0} \) are small, \( J'_L(\omega) \) and \( J_L(\omega) \) leads to almost the same dynamics and non-Markovianity for both strong coupling (large \( R \)) and weak coupling. When \( \frac{1}{\lambda \omega_0} \) are large, the negative frequency of the Lorentzian spectrum alters the dynamics and the non-Markovianity significantly for all coupling strengths (even for weak couplings). The reason is that \( J'_L(\omega) \) is a symmetric function with \( x = \omega_0 \) the axis of symmetry and \( \lambda \) the peak width. Thus, \( \frac{1}{\lambda \omega_0} \) determines the weight of the component of the negative frequency which in turn alters the correlation function and the dynamics evidently, whereas the non-Markovinity for \( J'_L(\omega) \) is still non-zero for all \( R > 0 \) and tends to 0 as \( R \to 0 \).

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