Another model for the regularized big bang

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Abstract

We propose a gravitational model with a Brans–Dicke-type scalar field having, in the would-be action, a “wrong-sign” kinetic term and a quartic interaction term. In a cosmological context, we obtain, depending on the boundary conditions, either the Friedmann solution or a kink-bounce solution. The expanding-universe Friedmann solution has a big bang curvature singularity, whereas the kink-bounce solution has a nonsingular bouncing behavior of the cosmic scale factor. The bounce occurs precisely at the moment when the kink-type configuration of the scalar field goes through zero, making for a vanishing effective gravitational coupling.

PACS numbers: 04.50.Kd, 98.80.Bp, 98.80.Jk

Keywords: modified theories of gravity, big bang theory, mathematical and relativistic aspects of cosmology
I. INTRODUCTION

Recently, we have shown that the singular Friedmann solution [1–4] from Einstein’s general theory of relativity can be modified (regularized) by considering a degenerate metric with a nonzero length scale $b$ [5, 6] (cosmological aspects have been studied in Refs. [7, 8]).

The goal, here, is to look for a modified version of the theory, which gives similar results. This turns out to be surprisingly difficult, but perhaps this was to be expected, as we are trying to model an entirely new phase from which classical spacetime and the universe are supposed to emerge (see also the discussion in App. B of Ref. [6], which contains further references).

Throughout, we use the metric signature ($-$, $+$, $+$, $+$), curvature conventions following Ref. [2], and natural units with $c = \hbar = 1$.

II. MOTIVATION

The action of Einstein’s general theory of relativity reads as follows [2–4]:

$$S = \int d^4x \sqrt{-g} \mathcal{L},$$  \hspace{1cm} (2.1a)

$$\mathcal{L} = \frac{1}{16\pi G_N} R + \mathcal{L}_M,$$ \hspace{1cm} (2.1b)

where $g$ is the determinant of the metric $g_{\mu\nu}$, $R$ the Ricci curvature scalar, $G_N$ the positive Newton gravitational coupling constant, and $\mathcal{L}_M$ the Lagrange density of the standard matter (the standard-matter fields are generically denoted by $\psi$).

In Ref. [3], we proposed the following degenerate-metric Ansatz for a spatially flat universe:

$$ds^2 \equiv g_{\mu\nu}(x) dx^\mu dx^\nu = -\frac{t^2}{b^2 + t^2} dt^2 + a(t) \delta_{mn} dx^m dx^n,$$ \hspace{1cm} (2.2a)

$$b^2 > 0,$$ \hspace{1cm} (2.2b)

$$a(t) \in \mathbb{R},$$ \hspace{1cm} (2.2c)

$$t \in (-\infty, \infty),$$ \hspace{1cm} (2.2d)

$$x^m \in (-\infty, \infty),$$ \hspace{1cm} (2.2e)

where the spatial indices $m, n$ run over \{1, 2, 3\}. The metric from (2.2) is degenerate (with a vanishing determinant at $t = 0$) and describes a spacetime defect with characteristic length scale $b > 0$; see Ref. [9] for a review of this type of spacetime defect.

The standard Einstein gravitational field equation from (2.1) reads [2, 3]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G_N T_{\mu\nu}^{(M)},$$ \hspace{1cm} (2.3)
where $R_{\mu\nu}$ is the Ricci curvature tensor and $T_{\mu\nu}^{(M)}$ the energy-momentum tensor of the matter. It is straightforward to evaluate (2.3) for the metric (2.2a), as long as the curvature tensors at $t = 0$ are obtained by taking the limit $t \to 0$; see Ref. [10] for details. If we also assume the energy-momentum tensor $T_{\mu\nu}^{(M)}$ of a homogeneous perfect fluid [with energy density $\rho_M(t)$ and pressure $P_M(t)$ satisfying the standard energy conditions], then modified Friedmann equations are obtained. These modified Friedmann equations have been given as Eqs. (2.2) in Ref. [6].

A heuristic understanding [6] of these modified Friedmann equations is that they can be rewritten as the standard Friedmann equations with an additional effective energy density $\rho_{\text{defect}}$ and an additional effective pressure $P_{\text{defect}}$, both proportional to $-b^2/(b^2 + t^2)$. These effective quantities then violate the null energy condition ($\rho_{\text{defect}} + P_{\text{defect}} < 0$) and allow for a bounce solution at $t = 0$; see Ref. [11] for a general discussion on cosmic bounces and energy conditions.

But the modified Friedmann equations, as given by Eqs. (2.2) in Ref. [6], can also be written in another way:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G_{\text{eff}} \rho_M , \tag{2.4a}
\]

\[
\ddot{a} + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 = -4\pi G_{\text{eff}} P_M + \frac{b^2}{b^2 + t^2} \frac{1}{t} \dot{a} , \tag{2.4b}
\]

\[
\frac{d}{da} \left[ a^3 \rho_M(a) \right] + 3 a^2 P_M(a) = 0 , \tag{2.4c}
\]

\[
G_{\text{eff}} = \frac{t^2}{b^2 + t^2} G_N , \tag{2.4d}
\]

where the overdot stands for the derivative with respect to $t$. The above equations have the same form as the standard Friedmann equations [2], except that Newton’s gravitational coupling constant $G_N$ is replaced by the time-dependent coupling $G_{\text{eff}}$ from (2.4d) and that there is an extra term on the right-hand side of (2.4b). This extra term makes the equations (2.4) consistent for $b \neq 0$ and vanishes, formally, for $b = 0$.

The basic idea, now, is to get a Brans–Dicke-type scalar-tensor theory [12], which has a particular kink-type solution of the scalar field that gives a behavior for the effective gravitational coupling similar to (2.4d). In order to get this kink-type solution, the kinetic term in the scalar field equation must have a “wrong sign,” which matches with the previous heuristic discussion about an effective violation of the null energy condition.

### III. MODEL

The model to be presented in this section is not yet definitive. Moreover, we do not have a local four-dimensional action but only field equations. It is, of course, known that certain theories exist without having a local four-dimensional action (see Ref. [13] for a
general discussion). A model action will be given in App. A, but that model does not give
the desired cosmology. In fact, our main interest lies in the cosmological equations, to be
discussed in Sec. IVB. The (incomplete) model field equations of this section are to be
considered as a road towards the appropriate cosmological equations.

With a dimensionless real scalar field \( \eta(x) \), we define the model by its field equations,

\[
\begin{align}
\Box \eta &= \lambda b^{-4} (1 - \eta^2) \eta, \\
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= -8\pi G \eta^2 \left[ T_{\mu\nu}^{(\eta)} + T_{\mu\nu}^{(M)} \right] + X_{\mu\nu}, \\
T_{\mu\nu}^{(\eta)} &= -b^{-2} \eta_{,\mu} \eta_{,\nu} - g_{\mu\nu} \left[ \frac{\lambda}{4} b^{-4} (1 - \eta^2)^2 - \frac{1}{2} b^{-2} \eta_{,\lambda} \eta^{,\lambda} \right],
\end{align}
\]

(3.1a-c)

\[
G > 0, \quad \lambda > 0, \quad b > 0,
\]

(3.1d)

where \( G \) is a gravitational coupling constant (mass dimension \(-2\)), \( \lambda \) a dimensionless quartic
coupling constant (mass dimension 0), and \( 1/b \) a mass scale (mass dimension 1). In addition,
we use the standard notation \([2, 3]\) of a comma for the derivative and a semicolon for the
covariant derivative (for example, \( \eta_{,\mu} \equiv \partial \eta / \partial x^\mu \equiv \partial_\mu \eta \)). We keep the standard notation
\( \Box \) for the d’Alembertian, \( \Box \eta \equiv \eta_{,\mu\nu} \) in the comma/semicolon notation for derivatives.

The gravitational field equation (3.1b) contains the tensor \( X_{\mu\nu} \) which is a functional of
the metric and the scalar field, and possibly also the standard-matter fields \( \psi \),

\[
X_{\mu\nu} = X_{\mu\nu} [g_{\rho\sigma}, \eta, \partial_\rho \eta, \psi, \partial_\rho \psi, \cdots].
\]

(3.2a)

This tensor \( X_{\mu\nu} \) has the purpose of implementing energy-momentum conservation in (3.1b),

\[
\left( X_{\mu\nu} - 8\pi G \eta^2 \left[ T_{\mu\nu}^{(\eta)} + T_{\mu\nu}^{(M)} \right] \right)^{,\mu} = 0.
\]

(3.2b)

A further condition on \( X_{\mu\nu} \) is that it vanishes if the scalar field is in the vacuum configuration,

\[
X_{\mu\nu} \bigg|_{\eta(x) = \pm 1} = 0.
\]

(3.2c)

For the moment, we will just assume that there exists a proper expression for \( X_{\mu\nu} \) which
gives the cosmological equations of Sec. IVB (a suggestion for a possible term \( X_{\mu\nu} \) is given
in App. B). If such an \( X_{\mu\nu} \) does not exist, then the model needs to be extended.

Comparing the energy-momentum tensor (3.1c) to the one of a standard scalar field \( \phi \),
we see that the two derivative terms in (3.1c) have “wrong signs.” The crucial point of this
paper is that the parameter \( b \) appears already in the theory (3.1), instead of only in the
solution, as considered previously in (2.2).

Obviously, we do not consider the model field equations (3.1) to describe a realistic theory,
as we expect instabilities and nonunitarity. (Incidentally, the Pauli–Villars regulator fields
of quantum electrodynamics also have wrong signs or wrong statistics \([14, 15]\).) At best, the
model field equations (3.1) would be embedded into a consistent UV completion of Einstein’s
classical gravitation theory, perhaps related to string theory. For the moment, we only use
the model field equations (3.1) to describe the regularized big bang singularity, without
degenerate metrics but with nonstandard matter fields (here, the scalar field \( \eta \)).
IV. COSMOLOGY

A. Ansätze

We take the standard spatially flat Robertson–Walker (RW) metric \[2, 3\],
\[
\begin{align*}
\text{ds}^2 &= -dt^2 + a^2(t) \delta_{mn} dx^m dx^n, \\
ap(t) &\in \mathbb{R}, \\
t &\in (-\infty, \infty), \\
x^m &\in (-\infty, \infty),
\end{align*}
\]
(4.1a)\quad (4.1b)\quad (4.1c)\quad (4.1d)
where the cosmic time coordinate \(t\) runs over the whole real line.

For the normal-matter content, we use again a homogeneous perfect fluid, with energy density \(\rho_M\) and pressure \(P_M\) satisfying the standard energy conditions. The “wrong-sign” scalar field \(\eta\) is also taken to be homogeneous. All these fields depend on time only,
\[
\begin{align*}
\rho_M &= \rho_M(t), \\
P_M &= P_M(t), \\
\eta &= \eta(t).
\end{align*}
\]
(4.2a)\quad (4.2b)\quad (4.2c)
We will now determine the reduced field equations from these Ansätze.

B. Cosmological equations

With the metric Ansatz (4.1) and homogeneous matter fields (4.2), the scalar field equation (3.1a) and the 00 component of the gravitational field equation (3.1b) [under the assumption that \(X_{00}\) effectively vanishes, see below] reduce to two coupled ordinary differential equations (ODEs). Added to these two ODEs are the energy-conservation equation of the matter and an equation of state \(P_M = P_M(\rho_M)\) [here, we take a constant equation-of-state parameter \(w_M\)]. All in all, the reduced equations are as follows:
\[
\begin{align*}
\ddot{\eta} + 3 \frac{\dot{a}}{a} \dot{\eta} &= -\lambda \ b^{-2} \left(1 - \eta^2\right) \eta, \\
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3} \ G \ \eta^2 \left[\rho_M - \frac{1}{2} \ b^{-2} \ \dot{\eta}^2 + \frac{\lambda}{4} \ b^{-4} \ (1 - \eta^2)^2\right], \\
\dot{\rho}_M + 3 \frac{\dot{a}}{a} (\rho_M + P_M) &= 0, \\
\frac{P_M(t)}{\rho_M(t)} &= w_M = \text{const},
\end{align*}
\]
(4.3a)\quad (4.3b)\quad (4.3c)\quad (4.3d)
where the last equation gives the equation of state of the normal matter \((w_M = 0\) for nonrelativistic matter and \(w_M = 1/3\) for relativistic matter). Observe the wrong sign of the \(\dot{\eta}^2\) term in the energy density on the right-hand side of (4.3b).

For the record, we mention that the 11 component of the gravitational field equation (3.1b) with an appropriate expression for \(X_{11}\) (see below) gives

\[
\frac{\ddot{a}}{a} + \frac{1}{2} \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G \eta^2 \left[ P_M - \frac{1}{2} b^{-2} \dot{\eta}^2 - \frac{\lambda}{4} b^{-4} (1 - \eta^2)^2 \right] + \frac{\dot{a} \dot{\eta}}{a \eta},
\]

but this equation is redundant. With a given equation of state (4.3d), the ODE (4.4) can, in fact, be shown to follow from the derivative of the ODE (4.3b), by use of the ODEs (4.3a), (4.3b), and (4.3c). This redundancy of the four ODEs is, in fact, a useful check on their correctness and traces back to the contracted Bianchi identities and energy-momentum conservation.

Note that the ODEs in (4.3) essentially have the standard form [with wrong-sign kinetic terms of the scalar \(\eta\) in (4.3a) and (4.3b) and with an effective gravitational coupling \(G \eta^2\) in the first-order Friedmann equation (4.3b)], but that (4.4) has an entirely new term \((\dot{a}/a)(\dot{\eta}/\eta)\) on the right-hand side. This extra term traces back to the term \(X_{11}\) in the original field equation (3.1), if the other reduced equations are used. Equally, the term \(X_{00}\) vanishes effectively. In short, we suppose that, with the Ansätze of Sec. IV A and the reduced field equations of this subsection, we have

\[
X_{\mu\nu} \bigg|_{\text{reduced}} \overset{\circ}{=} 0, \quad \text{for } \mu \neq \nu,
\]

where the spatial index “\(m\)” on the left-hand side of (4.5b) is not summed over and where the symbol “\(\overset{\circ}{=}\)” indicates that the equality holds only “on-shell.”

The cosmological equations (4.3) and (4.4) are the main general result of this paper.

V. COSMOLOGICAL SOLUTIONS

A. Constant EOS parameter

In the following, we use a constant equation-of-state (EOS) parameter (4.3d). Then, the solution of (4.3c) is explicitly given by

\[
\rho_M(a) = \rho_{M0} a^{-3(1+w_M)},
\]

\[
\rho_{M0} \geq 0,
\]

where \(a = a(t)\) is assumed to be positive. This implies that there are essentially two dimensionless functions to be determined from the ODEs (4.3a) and (4.3b), namely \(\eta(t)\) and \(a(t)\). Different boundary conditions result, of course, in different solutions.
B. Dimensionless ODEs

At this moment, it turns out to be useful to introduce the following dimensionless quantities:

\[ g \equiv G/b^2, \quad (5.2a) \]
\[ \tau \equiv t/b, \quad (5.2b) \]
\[ r_M(\tau) = \rho_M(t) b^4, \quad (5.2c) \]

together with, by the usual abuse of notation, \( a(\tau) = a(t) \) and \( \eta(\tau) = \eta(t) \). From now on, there is no danger of misunderstanding the meaning of \( g \), as the determinant of the metric no longer appears.

Introducing dimensionless quantities, we obtain from (4.3) and (5.1) the following dimensionless cosmological equations:

\[ \ddot{\eta} + 3 \frac{\dot{a}}{a} \dot{\eta} = -\lambda \left( 1 - \eta^2 \right) \eta, \quad (5.3a) \]
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} g \eta^2 \left[ r_M - \frac{1}{2} \dot{\eta}^2 + \frac{\lambda}{4} \left( 1 - \eta^2 \right)^2 \right], \quad (5.3b) \]

\[ r_M = r_{M0} a^{-3(1+w_M)}, \quad (5.3c) \]

where the overdot now stands for the derivative with respect to \( \tau \), \( w_M \) is the constant EOS parameter, and \( r_{M0} \geq 0 \) is a constant from the boundary conditions. The dimensionless 11-component gravitational equation from (4.4) reads

\[ \frac{\ddot{a}}{a} + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 - \frac{\dot{a}}{a} \frac{\dot{\eta}}{\eta} = -4\pi g \eta^2 \left[ w_M r_M - \frac{1}{2} \dot{\eta}^2 - \frac{\lambda}{4} \left( 1 - \eta^2 \right)^2 \right], \quad (5.4) \]

with \( r_M \) given by (5.3c).

C. Scalar vacuum solution without matter

We take the following boundary conditions:

\[ \lim_{\tau \to \infty} \eta(\tau) = 1, \quad (5.5a) \]
\[ \eta(0) = 1, \quad (5.5b) \]
\[ a(0) = 1, \quad (5.5c) \]
\[ r_M(0) = 0. \quad (5.5d) \]
The solution of the ODEs (5.3) and (5.4) is then as follows:

\[ \eta(\tau) = 1, \quad a(\tau) = 1, \quad r_M(\tau) = 0, \]

which corresponds to an empty static universe with the Minkowski metric and the \( \eta \)-vacuum.

D. Scalar vacuum solution with relativistic matter

We now allow for the presence of relativistic matter with equation-of-state parameter

\[ w_M = \frac{1}{3}. \quad (5.7) \]

Other values \( w_M \in [0, 1] \) are certainly possible, but, for definiteness, we only consider \( w_M = 1/3 \) in the following. In addition, we take the boundary conditions

\[ \lim_{\tau \to \infty} \eta(\tau) = 1, \quad \eta(\tau_0) = 1, \quad a(\tau_0) = 1, \quad r_M(\tau_0) = r_{M0} > 0, \]

for a fixed finite time \( \tau_0 > 0 \). Alternative boundary conditions for the scalar field are \( \eta(\tau_0) = 1 \) and \( \dot{\eta}(\tau_0) = 0 \), but the scalar boundary conditions (5.8) are more suitable for the comparison with the boundary conditions in Sec. V E 1.

The solution of the ODEs (5.3) and (5.4) with boundary conditions (5.8) is as follows:

\[ \eta(\tau) = 1, \quad a(\tau) = \sqrt{\frac{\tau}{\tau_0}}, \quad r_M(\tau) = r_{M0} \frac{\tau_0^2}{\tau^2}, \]

which corresponds to the standard Friedmann–Lemaître–Robertson–Walker (FLRW) universe [1–4]. The solution (5.9) holds only for \( \tau > 0 \) and there is a big bang singularity at \( \tau = 0 \) with diverging curvature and energy density. Figure [1] gives a sketch, in order to prepare for the comparison with the solution of Sec. [V E].
FIG. 1. Qualitative behavior of the FLRW solution of the cosmological equations (5.3) and (5.4): the horizontal curve shows the constant scalar field $\eta(\tau) = 1$ and the rising curve shows the cosmic scale factor $a(\tau)$ for $\tau > 0$. Relativistic matter is assumed to be present and its energy density is given by $r_M(\tau) = r_M^0/a(\tau)^4$ for $r_M^0 > 0$.

E. Scalar kink solution with relativistic matter

1. Boundary conditions

In order to “tame” the big bang singularity of Sec. V D, we suggest to use a nontrivial scalar field $\eta(t)$. Specifically, we consider again relativistic matter with the equation-of-state parameter

$$w_M = \frac{1}{3},$$

(5.10)

but now take the following boundary conditions:

$$\lim_{\tau \to \infty} \eta(\tau) = 1,$$

(5.11a)

$$\eta(0) = 0,$$

(5.11b)

$$a(0) = 1,$$

(5.11c)

$$r_M(0) = r_M^0 > 0.$$  

(5.11d)

The boundary condition (5.11b) rules out the trivial solution $\eta(t) = 1$ and fixes the core of the temporal-kink solution to be at $\tau = 0$ (see Sec. V E 2 for details). The boundary condition $\lim_{\tau \to -\infty} \eta(\tau) = -1$ also rules out the trivial solution but leaves the core location of the temporal-kink solution free.
2. **Exact flat-spacetime solution**

As a start, we give an exact solution for the case of no direct gravitational interaction \(G = 0\). Using dimensionless variables, this temporal-kink solution of the ODEs (5.3a), (5.3b), and (5.4) is given by

\[
\eta_{\text{exact-sol}}^{(g=0)}(\tau) = \tanh \left( \sqrt{\lambda/2} \tau \right),
\]

(5.12a)

\[
a_{\text{exact-sol}}^{(g=0)}(\tau) = 1.
\]

(5.12b)

The homogeneous matter content is irrelevant for this static cosmology, as its gravitational interaction vanishes.

The time-dependent homogeneous scalar configuration (5.12a) has, by construction, the same mathematical structure as the static finite-energy kink soliton of the quartic scalar theory [16, 17].

The solution for \(g > 0\) that we will discuss in the rest of this section can be viewed as a deformation of the solution (5.12). By a continuity argument, we expect such a solution to exist over an interval \([-\tau_{\text{max}}, \tau_{\text{max}}]\) for small enough values of \(g > 0\), \(r_M \geq 0\), and \(\tau_{\text{max}} > 0\).

3. **Series solution**

From the ODEs (5.3), we obtain the following series solution near \(\tau = 0\):

\[
\eta_{\text{pert-sol}}(\tau) = e_1 \tau + e_3 \tau^3 + e_5 \tau^5 + \cdots,
\]

(5.13a)

\[
a_{\text{pert-sol}}(\tau) = 1 + a_2 \tau^2 + a_4 \tau^4 + \cdots,
\]

(5.13b)

\[
r_{M_{\text{pert-sol}}}(\tau) = r_M [a_{\text{pert-sol}}(\tau)]^{-4},
\]

(5.13c)

with coefficients (for general values of \(w_M\))

\[
e_1 = \sqrt{\lambda/2},
\]

(5.14a)

\[
e_3 = -\frac{\lambda}{6} \left( \sqrt{\lambda/2} + 6 \pi \sqrt{g r_M} \right),
\]

(5.14b)

\[
e_5 = \frac{\lambda^{3/2}}{120 \sqrt{2}} \left( \lambda (4 - 9 g \pi) + 10 \sqrt{3 \pi} \sqrt{\lambda g r_M} + 6 \pi \left[ 11 + 3 w_M \right] g r_M \right),
\]

(5.14c)

\[
a_2 = \sqrt{\pi/3} \sqrt{\lambda g r_M},
\]

(5.14d)

\[
a_4 = -\frac{\lambda}{72} \left( 2 \sqrt{3 \pi} \sqrt{\lambda g r_M} - 9 g \pi \left[ \lambda - 2 r_M (1 + w_M) \right] \right),
\]

(5.14e)

where the positive root for \(a_2\) has been chosen. For \(g r_M \lesssim \lambda\), we see that the above series are essentially a perturbation expansion in \(\lambda\).
Observe that, even in the absence of normal matter, the spacetime is curved:

\[
a_2 \bigg|_{rM=0} = 0, \\
a_4 \bigg|_{rM=0} = (\pi/8) g \lambda^2,
\]

(5.15a) (5.15b)

where the last coefficient does not vanish, as long as both \( g \) and \( \lambda \) are nonzero. Incidentally, the coefficients \( a_2 \) and \( a_4 \) do vanish if \( g = 0 \), consistent with the exact solution (5.12).

The kink-bounce series solution (5.13) and (5.14) is the main analytic result of this paper.

4. Approximate solution

In addition to the series solution, we have the following approximate solution for \( w_M = 1/3 \) (similar results are obtained for other values of \( w_M \)):

\[
\eta_{\text{approx-sol}}(\tau) = \tanh \left( \sqrt{\lambda/2} \tau \right), \\
a_{\text{approx-sol}}(\tau) = \sqrt{1 + 8 \pi/3 \sqrt{g rM/\lambda}} \ln \left[ \cosh \left( \sqrt{\lambda/2} \tau \right) \right], \\
r_{M,\text{approx-sol}}(\tau) = rM_0 \left[ a_{\text{approx-sol}}(\tau) \right]^{-4},
\]

(5.16a) (5.16b) (5.16c)

where we have chosen the expanding branch for \( \tau > 0 \). The configurations (5.16b) and (5.16c) provide an exact solution of the first-order Friedmann equation (5.3b) for the given scalar function (5.16a). But (5.16a) is only an approximate solution of the second-order scalar equation (5.3a), due to the uncancelled “friction” term \( 3 (\ddot{a}/a) \dot{\eta} \). Indeed, the Taylor coefficients from (5.16a) do not involve \( rM_0 \), whereas the coefficients \( e_3 \) and \( e_5 \) of the genuine solution do have a nontrivial dependence on \( rM_0 \), according to the results (5.14). Equally, the cosmic scale factor configuration (5.16b) provides only an approximative solution, as it has \( a_{\text{approx-sol}}(\tau) = 1 \) for \( rM_0 = 0 \), whereas the genuine solution still has a nontrivial time dependence of the cosmic scale factor for \( rM_0 = 0 \), according to (5.15).

For the record, the corresponding dimensionless Hubble variable \( (h \equiv \dot{a}/a) \) reads:

\[
h_{\text{approx-sol}}(\tau) = 2 \sqrt{2 \pi/3 \sqrt{g rM_0}} \tanh \left( \sqrt{\lambda/2} \tau \right),
\]

(5.17)

The cosmic scale factor and the Hubble variable near \( \tau = 0 \) are, now, given by

\[
a_{\text{approx-sol}}(\tau) = 1 + \sqrt{\pi/3} \sqrt{g \lambda rM_0} \tau^2 + O(\tau^4), \\
h_{\text{approx-sol}}(\tau) = 2 \sqrt{\pi/3 \sqrt{g \lambda rM_0}} \tau + O(\tau^3).
\]

(5.18a) (5.18b)

As \( |\tau| \to \infty \), these functions are given by

\[
a_{\text{approx-sol}}(\tau) \propto (\tau^2)^{1/4}, \\
h_{\text{approx-sol}}(\tau) \sim \frac{1}{2} \tau^{-1}.
\]

(5.19a) (5.19b)
FIG. 2. Qualitative behavior of the kink-bounce solution of the cosmological equations (5.3) and (5.4): the bottom curve shows the kink configuration of the scalar field $\eta(\tau)$ and the top curve shows the cosmic scale factor $a(\tau)$ with a bounce at $\tau = 0$. The zero of the $\eta(\tau)$ function sets the moment of the bounce [stationary point of the curve $a(\tau)$]. Relativistic matter is assumed to be present and its energy density is given by $r_M(\tau) = r_{M0}/a(\tau)^4$ for $r_{M0} > 0$.

For the numerics later on, we will use the approximate solution (5.16) for $\tau \to \infty$, which approaches the exact FLRW solution (5.9).

At this moment, we observe two different time scales in the perturbative solution (5.13), one for the scalar field $\eta$ and another for the cosmic scale factor $a$:

$$\tau_1^{(\eta)} \equiv (\lambda)^{-1/2},$$  \hspace{1cm} (5.20a)

$$\tau_1^{(a)} \equiv (\lambda g r_{M0})^{-1/4},$$  \hspace{1cm} (5.20b)

where we have assumed that $r_{M0} \lesssim \lambda$ and $g \lesssim 1$. We expect that the perturbative solution (5.13) changes to the approximate solution (5.16) around $|\tau| \sim \sqrt{2/\lambda}$ with $\eta \sim 1/2$ (see Fig. 2 for a sketch and compare with Fig. 1). Numerical results are needed to cover the intermediate range of $|\tau|$.

5. Numerical solution

We have obtained the numerical solution of the second-order ODEs (5.3a) and (5.4) over the time interval $[\tau_{\text{min}}, \tau_{\text{max}}]$, for $\tau_{\text{max}} > \tau_{\text{min}} > 0$. With boundary conditions at $\tau = \tau_{\text{min}}$ from the perturbative solution (5.13) and (5.14), we numerically integrate forward in time. With boundary conditions at $\tau = \tau_{\text{max}}$ from the approximate solution (5.16), we numerically integrate backward in time. These two numerical solutions are then matched at an appropriate intermediate time $\tau_{\text{match}} \in [\tau_{\text{min}}, \tau_{\text{max}}]$. The boundary conditions at $\tau = \tau_{\text{min}}$ and $\tau = \tau_{\text{max}}$ satisfy the first-order ODE (5.3a) and the residue of this ODE can be used to monitor the accuracy of the numerical solution at intermediate time values.
FIG. 3. Numerical solution of the cosmological ODEs (5.3) and (5.4) over the time interval \([\tau_{\text{min}}, \tau_{\text{max}}]\), with \(\tau_{\text{min}} = 1/100\) and \(\tau_{\text{max}} = 4\). The boundary conditions at \(\tau = \tau_{\text{min}}\) follow from the perturbative solution (5.13) and (5.14), while the boundary conditions at \(\tau = \tau_{\text{max}}\) follow from the approximate solution (5.16). The model parameters are \(g = 1/10\), \(\lambda = 1/2\), and \(w_M = 1/3\). The matter density at the moment of the bounce is \(r_{M0} = 1/10\).

Figure 3 shows the numerical solution for a particular choice of model parameters \(g\), \(\lambda\), and \(w_M\), and the value of the matter density \(r_{M0}\) at the moment of the bounce.

VI. DISCUSSION

In the present article, we have presented a somewhat baroque model with cosmological equations that may have a nonsingular bounce solution. The bounce solution appears if the boundary conditions allow for a kink-type solution of the “wrong-sign” scalar field. If the boundary conditions allow only for a vacuum-type solution of the scalar field, then the standard Friedmann solution is recovered, which has a big bang curvature singularity. It is rather interesting that a single model, regardless of how baroque, allows for both types of behavior (sketched in Figs. 1 and 2). The time scale \(b/c\) that describes the structure of the bounce is set by the model field equations and not by the Ansatz for the solution, as is the case for the degenerate-metric bounce [5].

Even with a different origin of the parameter \(b\), the scalar-model cosmological equations (4.3) and (4.4) are quite similar to the degenerate-metric cosmological equations (2.4). The interpretation is that these scalar-model cosmological equations provide a dynamic realization of the effective gravitational coupling that was obtained from a degenerate-metric Ansatz in general relativity.

We need to mention one important open problem, namely the study of perturbations and stability of the kink-bounce solution of Sec. V E. For the degenerate-metric bounce, these issues were studied in Ref. [8].

In closing, we have two general remarks. First, we note that the “future” boundary condition (5.11a) of the kink-bounce solution is somewhat surprising in view of the other “initial” boundary conditions (5.11b), (5.11c), and (5.11d). But two-times boundary conditions may be less amazing than they appear at first sight; see Ref. [18] for further discussion.

Second, the present article has considered a modified gravity theory with a nonstandard
matter field and a mass parameter $1/b$. Perhaps it is also possible to construct a modified geometric theory with a length parameter $b$. Such a new geometric theory, if it exists, may or may not be approximated by general relativity with degenerate metrics.

**ACKNOWLEDGMENTS**

It is a pleasure to thank E. Battista and Z.L. Wang for comments on the manuscript.

**Appendix A: Brans–Dicke model**

1. **Action and field equations**

With a dimensionless real scalar field $\eta(x)$, the Brans–Dicke action [12] is taken as follows:

$$\tilde{S} = \int d^4x \sqrt{-g} \tilde{\mathcal{L}},$$  \hspace{1cm} (A1a)

$$\tilde{\mathcal{L}} = \frac{1}{16\pi G} \frac{1}{\eta^2} R - \frac{1}{2} b^{-2} \partial_\mu \eta \partial^\mu \eta + \lambda \frac{1}{4} b^{-4} \left(1 - \eta^2\right)^2 + \mathcal{L}_M,$$  \hspace{1cm} (A1b)

$$G > 0, \quad \lambda > 0, \quad b > 0,$$  \hspace{1cm} (A1c)

where $G$ is a gravitational coupling constant and $1/b$ a mass scale. With a negative metric component $g_{00}$, we see that the overall sign of the term $(\partial_t \eta)^2$ from the scalar kinetic term in (A1b) equals the sign of the potential term, so that we have a “wrong-sign” kinetic term for $\eta$ in our model action (recall the classical-mechanics expression $L = T - V$, with $L$ the Lagrangian, $T$ the kinetic energy, and $V$ the potential energy).

The model field equations from (A1) are:

$$b^{-2} \Box \eta = \lambda b^{-4} \left(1 - \eta^2\right) \eta + \frac{1}{8\pi G} \frac{1}{\eta^3} R,$$  \hspace{1cm} (A2a)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G \eta^2 \left[T^{(\eta)}_{\mu\nu} + T^{(M)}_{\mu\nu}\right] - \eta^2 \left[\left(\eta^{-2}\right)_{;\mu;\nu} - g_{\mu\nu} \Box \left(\eta^{-2}\right)\right],$$  \hspace{1cm} (A2b)

$$T^{(\eta)}_{\mu\nu} = -b^{-2} \eta_{;\mu} \eta_{;\nu} - g_{\mu\nu} \left[\lambda \frac{1}{4} b^{-4} \left(1 - \eta^2\right)^2 - \frac{1}{2} b^{-2} \eta_{;\lambda} \eta^{;\lambda}\right],$$  \hspace{1cm} (A2c)

where we use the standard notation [2, 3] of a comma for the derivative and a semicolon for the covariant derivative.

Just as for the model of Sec. III, we do not consider the model action (A1) to describe a realistic theory, as we expect instabilities and nonunitarity.

2. **Cosmological equations**

In the cosmological context, we now make the same Ansätze as in Sec. IV A. With the metric Ansatz (4.1) and homogeneous matter fields (4.2), the scalar field equation (A2a) and
the 00 component of the gravitational field equation (A2b) reduce to two coupled ordinary differential equations (ODEs), to which are added the energy-conservation equation of the matter and an equation of state (EOS) $P = P(\rho_M)$ [here, we choose a constant equation-of-state parameter $w_M$]. Hence, the cosmological equations are as follows:

\[
\begin{align*}
\dot{b} - 2 & \left( \ddot{\eta} + 3 \frac{\dot{a}}{a} \eta \right) = -\lambda b^{-4} (1 - \eta^2) \eta + \frac{3}{4\pi G} \frac{1}{\eta^2} \left[ \frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right], \\
\left( \frac{\dot{a}}{a} \right)^2 & = \frac{8\pi}{3} G \eta^2 \left[ \rho_M - \frac{1}{2} b^{-2} \eta^2 + \frac{\lambda}{4} b^{-4} (1 - \eta^2)^2 \right] - \frac{\dot{a}}{a} \eta^2 \partial_t \left[ \eta^{-2} \right], \\
\dot{\rho}_M + 3 \frac{\dot{a}}{a} (\rho_M + P_M) & = 0, \\
P_M(t) \rho_M(t) & = w_M = \text{const},
\end{align*}
\]

where the last equation gives the equation of state of the normal matter.

The 11 component of the gravitational field equation (A2b) gives

\[
\ddot{a} + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 = -4\pi G \eta^2 \left[ P_M - \frac{1}{2} b^{-2} \eta^2 - \frac{\lambda}{4} b^{-4} (1 - \eta^2)^2 \right] + \frac{1}{2} \eta^2 \left[ \frac{\dot{a}}{a} \partial_t \left( \eta^{-2} \right) - a^{-3} \partial_t \left[ a^2 \partial_t \left( \eta^{-2} \right) \right] \right],
\]

but this equation is redundant. With a given EOS (A3d), the ODE (A4) can be shown to follow from the derivative of the ODE (A3b), together with the ODEs (A3a), (A3b), and (A3c). The redundancy of these four ODEs traces back to the contracted Bianchi identities and energy-momentum conservation, which, in turn, result from the general coordinate invariance of the underlying model (cf. Sec. 15.1 of Ref. [2] and App. E.1 Ref. [4]).

At this moment, we observe that the ODE (A3b) is really a quadratic in the Hubble parameter $H \equiv \dot{a}/a$ and we get the following roots:

\[
H_\pm = \frac{\dot{\eta}}{\eta} \pm \sqrt{\left( \frac{\dot{\eta}}{\eta} \right)^2 + \frac{8\pi}{3} G \eta^2 \left[ \rho_M - \frac{1}{2} b^{-2} \eta^2 + \frac{\lambda}{4} b^{-4} (1 - \eta^2)^2 \right]}.
\]

If the scalar field $\eta(t)$ has the kink-type structure (5.12a), the problem from (A5) is obvious: $\dot{\eta}/\eta$ diverges at $t = 0$ and a bounce-type behavior with $H(0) = 0$ requires, for $t > 0$, the minus sign in (A5), which then implies a contracting phase for larger values of $t$, as the $G \rho_M$ term in the root becomes important. Hence, removing the big bang at $t = 0$ now results in a new big bang at $t = t_{bb}$ for $t_{bb} > 0$. For $t < 0$, the plus sign in (A5) would be needed. All in all, we do not get a bounce similar to the one of the upper curve in Fig. 2.

After we completed the calculations reported in this paper, we have become aware of earlier papers in the literature, which discuss bouncing cosmology from a wrong-sign Brans-Dicke scalar (see, e.g., Ref. [19] for a research paper and Ref. [20] for a review of this and other types of models). Equation (7) of Ref. [19] has essentially the same quadratic
structure as (A3b). New, here, is the possible role of a kink-type configuration, which has certain advantages but also spells trouble if there is a linear term in the $H$ quadratic. For this reason, we have aimed at a first-order Friedmann equation of the form (4.3b).

**Appendix B: Possible nonlocal term**

Inspired by certain results from App. A, we suggest the following Ansatz for the tensor $X_{\mu\nu}$ in the model field equations (3.1):

$$\tilde{X}_{\mu\nu} = -\frac{2}{3} \eta^{-1} \left[ \eta_{,\kappa;\lambda} - g_{\kappa\lambda} \Box \eta \right] \frac{\eta^{\kappa} \eta^{\lambda}}{(\eta, \eta, \eta)^2} \left[ \eta_{,\mu} \eta_{,\nu} - g_{\mu\nu} \eta_{,\sigma} \eta^{,\sigma} \right].$$  \hspace{1cm} (B1)

This term has no direct dependence on the mass scale $1/b$ and is nonlocal due to the presence of the factor $1/(\eta, \eta, \eta)^2$.

With the spatially flat Robertson–Walker metric (4.1a) and homogeneous matter fields (4.2), the Ansatz (B1) reproduces the expressions (4.3). In fact, the reduced form of last term in square brackets in (B1) gives precisely the diagonal structure in $\mu\nu$ and a vanishing 00 component. The other terms in (B1) make that the $mm$ component matches the expression (4.3b). Remark that the limits $a(t) \to 1$ and $\eta(t) \to 1$ of the reduced $\tilde{X}_{\mu\nu}$ expressions from (4.3) define $\tilde{X}_{\mu\nu} = 0$ in the $\eta$-vacuum of Minkowski spacetime.

Turning to the energy-momentum conservation condition (3.2b), that equation can be simplified somewhat by use of the scalar field equation (3.1a) and the energy conservation condition $T^{(M):\mu}_{\nu} = 0$ of the normal matter. In terms of the contravariant tensor $X^{\mu\nu}$, the simplified energy-momentum conservation condition reads

$$X^{\mu\nu}_{,\nu} - 16\pi G \eta \eta_{,\nu} \left[ T^{\mu\nu}_{(\eta)} + T^{\mu\nu}_{(M)} \right] = 0. \hspace{1cm} (B2)$$

The question, now, is if the Ansatz (B1) satisfies the condition (B2), upon use of the field equations (3.1). We have a few partial answers in the affirmative but not a definitive general answer.

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