New Solutions for Nonlinear Schrodinger Equations using Adomian Decomposition and Homotopy Perturbation Methods

Norhasimah Mahiddin1,2*

1Centre of Foundation Studies for Agriculture Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
2Institute For Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia; shimamahiddin@gmail.com

Abstract

Coupled non-linear Schrodinger equation (CNLS) with two mode propagations in an optical fibre was studied. Using Painleve analysis, we are successfully solving the focusing Manakov system, bright and dark solitons using Adomian Decomposition Method (ADM) and Homotopy Perturbation Method (HPM). Both methods are fast convergent, do not need linearization and eliminates massive computation work. Its hows that HPM does not involve the Adomian polynomials when dealing with non-linear problems.

Keywords: Adomian Decomposition, Homotopy Perturbation, Schrodinger, Soliton

1. Introduction

There is growing interest in doing research of optical soliton pulses in fibres due to their capability of propagating long distances. Many mathematical models are based on the Coupled Non-Linear Schrodinger equations (CNLS). With the exception in a limited number of these models, problems are existed to find their exact analytical solutions. As a result, they have to be solved by using numerical method such as finite difference schemes. However, this method required discretization of space-time variables.

Approximate analytical solutions such as Back transformation, Hirota's bilinear method, Darboux transformation, tanh method, homogeneous balance method, sine-cosine method, variational iteration method and others were shown to be effective, easier and accurate to solve a large class of non-linear problems with approximations converging rapidly to accurate solutions.

In this paper, we present the approximate analytical solutions of CNLS solved by two analytical methods i.e.: ADM and HPM. Both methods provide direct scheme problem solving without resort for linearization and discretization. Results from both methods are then compared and reveal their capability, effectiveness and convenience.

2. Problem Formulation

Interaction of two optical modes \(q_1\) and \(q_2\) for shorter wave lengths in a fibre is governed by CNLS equations:

\[
\begin{align*}
iq_{1t} + c_1q_{1tt} + 2\left(\alpha |q_1|^2 + \beta |q_2|^2\right)q_1 &= 0 \\
iq_{2t} + c_2q_{2tt} + 2\left(\alpha |q_1|^2 + \beta |q_2|^2\right)q_2 &= 0
\end{align*}
\]

where, \(c_1, c_2, \alpha, \beta\) and \(\gamma\) are real parameters. The parameter \(\beta\) is the cross-coupling coefficient, \(c_1\) and \(c_2\) are the signum functions depending on the signs of the group velocity dispersion (GVD), that is \(c = +1\) (anomalous GVD) and \(c = -1\) (normal GVD).

Equations (1) possesses Painleve (P) property for the two specific parametric choices:

\[
\begin{align*}
c_1 &= c_2 \ ; \alpha = \beta = \gamma \\
c_1 &= -c_2 \ ; \alpha = -\beta = \gamma
\end{align*}
\]

and integrable.

The P-property for Equations (1) for parametric (2) and (3) implies that these solutions of Equation (1) must...
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In this paper, we obtain the approximate analytical solutions for bright and dark soliton for the parametric (1).

2.1 Bright Solitons
Manakov proposed integrable model for Equation (1) under the parametric restriction (2), namely

\[
\alpha = \beta = \gamma = \mu , c_1 = c_2 = +1 \quad \text{corresponds to the anomalous GVD region where bright solitons existed.}
\]

Hence, Equation (1) becomes

\[
iq_1 + c_1q_{1tt} + 2(\alpha |q_1|^2 + \beta |q_2|^2)q_1 = 0 \quad (5a)
\]

\[
iq_2 + c_2q_{2tt} + 2(\alpha |q_1|^2 + \beta |q_2|^2)q_2 = 0 \quad (5b)
\]

The approximate solutions of (5)-(6) are obtained by integrating each equation once with respect to and using the initial condition.

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For bright dark

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For bright dark (±) soliton

\[
q_1(x, t) = q_1(t) \pm i \int_0^x \left[ \frac{\partial^2 q_1}{\partial t^2} + 2 \mu (|q_1|^2 + |q_2|^2)q_1 \right] dx \quad (7a)
\]

\[
q_2(x, t) = q_2(t) \pm i \int_0^x \left[ \frac{\partial^2 q_2}{\partial t^2} + 2 \mu (|q_1|^2 + |q_2|^2)q_2 \right] dx \quad (7b)
\]

In (7), we assume \( q_1(t) \) and \( q_2(t) \) are bounded for all \( t \) in \( J = [0, T] \) (\( T \in \mathbb{R} \)) and

\[|x - x'| \leq m'^2 \text{ for all } 0 \leq x', x' \leq T.\]

The terms \( \frac{\partial^4 q_1}{\partial t^4}, \frac{\partial^2 q_2}{\partial t^2}, m_1(q_1, q_2) = 2 \mu (|q_1|^2 + |q_2|^2)q_1 \) and \( m_2(q_1, q_2) = 2 \mu (|q_1|^2 + |q_2|^2)q_2 \) are Lipschitz continuous with

\[
\left| \frac{\partial^2 q_1}{\partial t^2} - \frac{\partial^2 q_1^*}{\partial t^2} \right| \leq L_1 |q_1 - q_1^*| ,
\]

\[
\left| m_1(q_1, q_2) - m_1^*(q_1, q_2) \right| \leq L_2 \left[ |q_1|^2 + |q_2|^2 \right] |q_1 - q_1^*| + |q_2 - q_2^*| \quad (8)
\]

3. Mathematical Methods

3.1 Adomian Decomposition Methods
ADM is applied in Eqs (5a-b). For bright solitons:

\[
L_x q_1 = i q_{1tt} + 2 \mu (|q_1|^2 + |q_2|^2)q_1 \quad (9a)
\]

\[
L_x q_2 = i q_{2tt} + 2 \mu (|q_1|^2 + |q_2|^2)q_2 \quad (9b)
\]

where, \( L_x = \frac{\partial}{\partial x} \) is integrable differential operator with

\[
L_x^{-1} = \int_0^t dt.
\]

Operating on both sides of (9a-b) with the integral operator \( L_x^{-1} \) leads to

\[
q_1(x, t) = q_1(0, t) + i L_x^{-1} q_{1tt} + i L_x^{-1} \left\{ M_1(q_1, q_2) \right\} \quad (10a)
\]

\[
q_2(x, t) = q_2(0, t) + i L_x^{-1} q_{2tt} + i L_x^{-1} \left\{ M_2(q_1, q_2) \right\} \quad (10b)
\]

where,

\[
M_1(q_1, q_2) = 2 \mu (|q_1|^2 + |q_2|^2)q_1
\]

\[
M_2(q_1, q_2) = 2 \mu (|q_1|^2 + |q_2|^2)q_2
\]

are the nonlinear terms. The solutions of \( q_1(x, t) \)
and \( q_2(x,t) \) can be decomposed by an infinite series as follows \[14\]:

\[
q_1(x,t) = \sum_{i=0}^{\infty} q_{1i}(x,t) \quad (11a)
\]

\[
q_2(x,t) = \sum_{i=0}^{\infty} q_{2i}(x,t) \quad (11b)
\]

where, \( q_{1i}(x,t) \) and \( q_{2i}(x,t) \) are the components of \( q_1(x,t) \) and \( q_2(x,t) \) which elegantly be determined. The non-linear term \( M(x,t) \) can be decomposed by the following infinite series:

\[
M_k(q_1,q_2) = \sum_{i=0}^{\infty} A_{kn}, \quad k = 1,2
\]

where, \( A_{kn} \) is called Adomian's polynomial and defined by

\[
A_{kn} = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} M_k \left( \sum_{i=0}^{\infty} \lambda_i q_{1i}, \sum_{i=0}^{\infty} \lambda_i q_{2i} \right) \right], \quad i = 0
\]

From the above consideration, ADM defines the components of \( q_1(x,t) \) and \( q_2(x,t) \) for \( i \geq 0 \) through the following recursive relationships.

For \( q_1(x,t) \):

\[
q_{10}(x,t) = q_1(0,t)
\]

\[
q_{1m+1}(x,t) = i \int [q_{1m} + m_1 (q_1,q_2)] dx, \quad n \geq 0
\]

For \( q_2(x,t) \):

\[
q_{20}(x,t) = q_2(0,t)
\]

\[
q_{2m+1}(x,t) = i \int [q_{2m} + m_2 (q_1,q_2)] dx, \quad n \geq 0
\]

### 3.2 Homotopy Perturbation Method (HPM)

Eqs. (5a-b) solved with the HPM method by constructing the following homotopy:

\[
H_i(q_1,q_2,p) = (1-p) \left[ \frac{\partial q_{10}}{\partial x} - \frac{\partial q_{10}}{\partial x} + p \left[ \frac{\partial q_{10}}{\partial x} - \frac{i \partial^2 q_{10}}{\partial t^2} - 2\mu \left[ |q_1|^2 + |q_2|^2 \right] q_1 + \frac{\partial q_{10}}{\partial x} \right] = 0 \right]
\]

\[
H_i(q_1,q_2,p) = (1-p) \left[ \frac{\partial q_{20}}{\partial x} - \frac{\partial q_{20}}{\partial x} + p \left[ \frac{\partial q_{20}}{\partial x} - \frac{i \partial^2 q_{20}}{\partial t^2} - 2\mu \left[ |q_1|^2 + |q_2|^2 \right] q_1 + \frac{\partial q_{20}}{\partial x} \right] = 0 \right]
\]

Or

\[
\left\{ \frac{\partial q_1}{\partial x} - \frac{\partial q_1}{\partial x} + p \left[ -i \frac{\partial^2 q_1}{\partial t^2} - 2\mu \left[ |q_1|^2 + |q_2|^2 \right] q_1 + \frac{\partial q_1}{\partial x} \right] = 0 \right\}
\]

\[
\left\{ \frac{\partial q_2}{\partial x} - \frac{\partial q_2}{\partial x} + p \left[ -i \frac{\partial^2 q_2}{\partial t^2} - 2\mu \left[ |q_1|^2 + |q_2|^2 \right] q_2 + \frac{\partial q_2}{\partial x} \right] = 0 \right\}
\]

In HPM, the solutions of (16a-b) are expressed as power series in \( p \):

\[
q_1(x,t) = q_{10}(x,t) + p q_{11}(x,t) + p^2 q_{12}(x,t) + p^3 q_{13}(x,t) + \ldots \quad (17a)
\]

\[
q_2(x,t) = q_{20}(x,t) + p q_{21}(x,t) + p^2 q_{22}(x,t) + p^3 q_{23}(x,t) + \ldots \quad (17b)
\]

where, \( p \in [0,1] \) and \( q_{10} \) and \( q_{20} \) are the embedding parameter and arbitrary initial approximation, respectively. This will satisfy the given initial condition. As \( p \) approaches to 1, we obtained

\[
q_1(x,t) = \lim_{p \to 1} q_1 = q_{10} + q_{11} + q_{12} + q_{13} + \ldots = \sum_{i=0}^{\infty} q_{1i}
\]

\[
q_2(x,t) = \lim_{p \to 1} q_2 = q_{20} + q_{21} + q_{22} + q_{23} + \ldots = \sum_{i=0}^{\infty} q_{2i}
\]

Substituting (17) into (16a),

\[
\left\{ \frac{\partial q_{10}}{\partial x} - \frac{\partial q_{10}}{\partial x} + p \left[ -i \frac{\partial^2 q_{10}}{\partial t^2} - 2\mu \left[ |q_1|^2 + |q_2|^2 \right] q_1 + \frac{\partial q_{10}}{\partial x} \right] = 0 \right\}
\]

Substituting (17) into (16b),

\[
\left\{ \frac{\partial q_{20}}{\partial x} - \frac{\partial q_{20}}{\partial x} + p \left[ -i \frac{\partial^2 q_{20}}{\partial t^2} - 2\mu \left[ |q_1|^2 + |q_2|^2 \right] q_2 + \frac{\partial q_{20}}{\partial x} \right] = 0 \right\}
\]
EQUATING THE COEFFICIENTS OF THE TERMS IN (18a) WITH THE IDENTICAL POWERS OF \( p \), we obtained the following:

\[
p^0 : \frac{\partial q_{10}}{\partial x} - \frac{\partial q_{10}}{\partial x} = 0
\]

\[
p^1 : \frac{\partial q_{11}}{\partial x} - i\frac{\partial^2 q_{11}}{\partial t^2} - 2i\mu|q_{10}|^2 q_{10} - 2i\mu|q_{20}|^2 q_{20} + \frac{\partial q_{10}}{\partial x} = 0
\]

\[
p^2 : \frac{\partial q_{12}}{\partial x} + \frac{\partial^2 q_{12}}{\partial t^2} - 2i\mu|q_{20}|^2 q_{20} - 2i\mu|q_{10}|^2 q_{20} + \frac{\partial q_{20}}{\partial x} = 0
\]

\[
p^3 : \frac{\partial q_{13}}{\partial x} + \frac{\partial^2 q_{13}}{\partial t^2} - 2i\mu|q_{20}|^2 q_{20} - 2i\mu|q_{10}|^2 q_{20} + \frac{\partial q_{20}}{\partial x} = 0
\]

\[\vdots \]

\[p^n \]

Likewise, equating the coefficients of the terms in (18b) with the identical powers of \( p \), we obtained the following:

\[
p^0 : \frac{\partial q_{20}}{\partial x} - \frac{\partial q_{20}}{\partial x} = 0
\]

\[
p^1 : \frac{\partial q_{21}}{\partial x} + i\frac{\partial^2 q_{21}}{\partial t^2} - 2i\mu|q_{20}|^2 q_{20} - 2i\mu|q_{10}|^2 q_{20} + \frac{\partial q_{20}}{\partial x} = 0
\]

\[
p^2 : \frac{\partial q_{22}}{\partial x} + \frac{\partial^2 q_{22}}{\partial t^2} - 2i\mu|q_{20}|^2 q_{20} - 2i\mu|q_{20}|^2 q_{20} + \frac{\partial q_{20}}{\partial x} = 0
\]

\[
p^3 : \frac{\partial q_{23}}{\partial x} + \frac{\partial^2 q_{23}}{\partial t^2} - 2i\mu|q_{20}|^2 q_{20} - 2i\mu|q_{20}|^2 q_{20} + \frac{\partial q_{20}}{\partial x} = 0
\]

\[\vdots \]

\[p^n \]

4. Existence and Convergence of ADM and HPM

Theorem 1: Let \( 0 < \alpha < 1 \), then (5)-(6) have a unique solution.

Proof: Let \( q_1 \) and \( q_1^* \) be two different solutions of (7), then

\[
|q_1 - q_1^*| = \left| \int_0^t \left[ \frac{\partial q_{10}}{\partial x} - \frac{\partial q_{10}}{\partial x} \right] \right| \leq \left| \int_0^t \left[ \frac{\partial q_{10}}{\partial x} - \frac{\partial q_{10}}{\partial x} \right] \right|
\]

\[
\leq T(mL_1 + m' L_2) \left| q_1 - q_1^* \right| + \left| q_2 - q_2^* \right| \left( \left| q_1 - q_1^* \right| + \left| q_2 - q_2^* \right| \right)
\]

\[= \alpha \left| q_1 - q_1^* \right|
\]

From which we get \( (1 - \alpha) \left| q_1 - q_1^* \right| \leq 0 \). Since \( 0 < \alpha < 1 \), \( \left| q_1 - q_1^* \right| = 0 \), implies \( q_1 = q_1^* \) and completes the proof.

The proof for unique solution of \( q_2 \) is similar.

5. Results and Discussion

Let the initial approximation:

\[ q_{10}(x,t) = a_0(1 - \varepsilon \cos \sqrt{2\alpha} x) \] and \( q_{20}(x,t) = b_0(1 - \varepsilon \cos \sqrt{2\alpha} x) \)

For bright solution:

\[ q_1(x,t) = a_1(1 - \varepsilon \cos \sqrt{2\alpha} x) + a_2(1 + \varepsilon \cos \sqrt{2\alpha} x) \]

For dark solution:

\[ q_1(x,t) = a_1(1 - \varepsilon \cos \sqrt{2\alpha} x) + a_2(1 + \varepsilon \cos \sqrt{2\alpha} x) \]

For HPM:

\[ q_1(x,t) = a_1(1 - \varepsilon \cos \sqrt{2\alpha} x) + a_2(1 + \varepsilon \cos \sqrt{2\alpha} x) \]

For bright solution:

\[ q_2(x,t) = a_1(1 - \varepsilon \cos \sqrt{2\alpha} x) + a_2(1 + \varepsilon \cos \sqrt{2\alpha} x) \]

For dark solution:

\[ q_2(x,t) = a_1(1 - \varepsilon \cos \sqrt{2\alpha} x) + a_2(1 + \varepsilon \cos \sqrt{2\alpha} x) \]
6. Conclusion

Both methods (ADM and HPM) are simple, easy without any linearization and produce the same results.

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