Extendible cardinals and the mantle

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Received: 24 March 2018 / Accepted: 12 April 2018 / Published online: 18 April 2018 © Springer-Verlag GmbH Germany, part of Springer Nature 2018

Abstract The mantle is the intersection of all ground models of $V$. We show that if there exists an extendible cardinal then the mantle is the smallest ground model of $V$.

Keywords Extendible cardinal · Mantle · Set-theoretic geology

Mathematics Subject Classification 03E45 · 03E55

1 Introduction

Let us say that an inner model $W$ of $\text{ZFC}$ is a ground if the universe $V$ is a set-forcing extension of $W$. The set-theoretic geology, initiated by Fuchs et al. [1], is a study of the structure of all grounds of $V$. An important object in the set-theoretic geology is the mantle:

Definition 1.1 The mantle $M$ is the intersection of all grounds of $V$.

It is known that the mantle is a transitive, definable, forcing-invariant model of $\text{ZFC}$ [1,7]. The mantle itself needs not to be a ground of $V$, but if the mantle is a ground of $V$ then it is the smallest ground. In [1], they asked the following question: Under what circumstances is the mantle also a ground model of the universe? For this question, Usuba answered if some very large cardinal exists then the mantle must be a ground:

This research was supported by JSPS KAKENHI grant Nos. 18K03403 and 18K03404.

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**Fact 1.2 ([7])** Suppose there exists a hyper-huge cardinal. Then the mantle is a ground of $V$. More precisely, there is a poset $\mathbb{P} \in \mathbb{M}$ and a $(\mathbb{M}, \mathbb{P})$-generic $G$ such that $|\mathbb{P}| < \kappa$ and $V = \mathbb{M}[G]$.

An uncountable cardinal $\kappa$ is **hyper-huge** if for every cardinal $\lambda \geq \kappa$, there is an elementary embedding $j : V \rightarrow M$ for some inner model $M$ such that the critical point of $j$ is $\kappa$, $\lambda < j(\kappa)$, and $M$ is closed under $j(\lambda)$-sequences.

In this paper we prove that the hyper-huge cardinal assumption can be weakened to the extendible cardinal assumption. Recall that an uncountable cardinal $\kappa$ is **extendible** if for every ordinal $\alpha \geq \kappa$, there exists $\beta > \alpha$ and an elementary embedding $j : V_\alpha \rightarrow V_\beta$ such that the critical point of $j$ is $\kappa$ and $\alpha < j(\kappa)$. Every hyper-huge cardinal is an extendible cardinal limit of extendible cardinals.

**Theorem 1.3** Suppose there exists an extendible cardinal. Then the mantle is a ground of $V$. In fact if $\kappa$ is extendible then the $\kappa$-mantle of $V$ is its smallest ground (The $\kappa$-mantle will be defined in Definition 2.6 below).

### 2 Some materials

Let’s recall some basic definitions and facts about the set-theoretic geology. See [1] for more information.

**Fact 2.1** ([1,5]) There is a formula $\varphi(x, y)$ of set-theory such that:

1. For every $r$, the class $W_r = \{x \mid \varphi(x, r)\}$ is a ground of $V$ with $r \in W_r$.
2. For every ground $W$ of $V$, there is $r$ with $W = W_r$.

It turned out that all grounds are downward set-directed:

**Fact 2.2** ([7]) Let $\{W_r \mid r \in V\}$ be the collection of all grounds of $V$ defined as in Fact 2.1. For every set $X$, there is $r$ such that $W_r \subseteq W_s$ for every $s \in X$.

A key of the definability of grounds as in Fact 2.1 is the covering and the approximation properties introduced by Hamkins [2]:

**Definition 2.3** ([2]). Let $M \subseteq V$ be a transitive model of ZFC containing all ordinals. Let $\kappa$ be a cardinal.

1. $M$ satisfies the $\kappa$-covering property for $V$ if for every set $x$ of ordinals, if $|x| < \kappa$ then there is $y \in M$ with $x \subseteq y$ and $|y| < \kappa$.
2. $M$ satisfies the $\kappa$-approximation property for $V$ if for every set $A$ of ordinals, if $A \cap x \in M$ for every set $x \in M$ with size $< \kappa$, then $A \in M$.

**Fact 2.4** (Hamkins, see Laver [4]) Let $\kappa$ be a regular uncountable cardinal. Let $M$, $N$ be transitive models of ZFC containing all ordinals. If $M$ and $N$ satisfy the $\kappa$-covering and the $\kappa$-approximation properties for $V$, $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N$, and $\kappa^+ = (\kappa^+)^M = (\kappa^+)^N$, then $M = N$.

**Fact 2.5** ([2]) Let $\kappa$ be a regular uncountable cardinal, and $\mathbb{P}$ a poset of size $< \kappa$. Let $G$ be $(V, \mathbb{P})$-generic. Then, in $V[G]$, $V$ satisfies the $\kappa$-covering and the $\kappa$-approximation properties for $V[G]$. 
Let us make some definition and observations.

**Definition 2.6** Let $\kappa$ be a cardinal. A ground $W$ of $V$ is a $\kappa$-ground if there is a poset $\mathbb{P} \in W$ of size $< \kappa$ and a $(W, \mathbb{P})$-generic $G$ such that $V = W[G]$. The $\kappa$-mantle is the intersection of all $\kappa$-grounds.

The $\kappa$-mantle is a definable, transitive, and extensional class. It is trivially consistent that the $\kappa$-mantle is a model of ZFC, and we can prove that if $\kappa$ is strong limit, then the $\kappa$-mantle must be a model of ZF. A sketch of the proof is as follows. First we show that all $\kappa$-grounds are downward-directed. For any two $\kappa$-grounds $W_0$ and $W_1$, since $\kappa$ is a limit cardinal, there is a regular cardinal $\lambda < \kappa$ such that $W_0$ and $W_1$ are $\lambda$-grounds. Then $V$ is a $\lambda$-c.c. forcing extension of both $W_0$ and $W_1$. By the proof of Fact 2.2 (see [7]), we can find a ground $W \subseteq W_0 \cap W_1$ of $V$ such that $V$ is a $\lambda^{++}$-c.c. forcing extension of $W$. Then, we can find a poset $\mathbb{P} \in W$ of size $\leq 2^{2^{\lambda^{++}}}$ and a $(W, \mathbb{P})$-generic filter $G$ with $V = W[G]$ (e.g., see Appendix in Sargsyan-Schindler [6]), so $W$ is a $\kappa$-ground of $V$ as well. The downward-directedness of the $\kappa$-grounds implies that the $\kappa$-mantle is absolute between all $\kappa$-grounds, so the $\kappa$-mantle of $V$ is definable in all $\kappa$-grounds. Now, by Lemma 21 in [1], the $\kappa$-mantle is a model of ZF.

However we do not know whether the $\kappa$-mantle is always a model of ZFC.

**Question 2.7** For a given cardinal $\kappa$, is the $\kappa$-mantle always a model of ZFC?

By Facts 2.4 and 2.5, if $W$ is a $\kappa$-ground of $V$, then $W$ is completely determined by the set $P = \mathcal{P}(\kappa) \cap W$, that is, $W$ is a unique ground $W'$ with $\mathcal{P}(\kappa) \cap W' = P$, $\kappa^{+} = (\kappa^{+})^{W'}$, and $W'$ satisfies the $\kappa$-covering and the $\kappa$-approximation properties. This means that there are at most $2^{2^{\kappa}}$ many $\kappa$-grounds of $V$, hence there is a set $X$ of size $\leq 2^{2^{\kappa}}$ such that the collection $\{W_r \mid r \in X\}$ is the $\kappa$-grounds. We have the following by the combination of this observation and Fact 2.2:

**Lemma 2.8** Let $\kappa$ be a cardinal and $\bar{W}$ the $\kappa$-mantle of $V$. Then there is a ground $W$ such that $W \subseteq \bar{W}$.

For a class $C \subseteq V$ and an ordinal $\alpha$, let $C_\alpha = C \cap V_\alpha$, the set of all elements of $C$ with rank $< \alpha$.

**Lemma 2.9** Let $\kappa$ be a cardinal and $\bar{W}$ the $\kappa$-mantle of $V$. For an inaccessible $\theta > \kappa$, let $\bar{W}^V_\theta$ be the $\kappa$-mantle of $V_\theta$, that is, $\bar{W}^V_\theta$ is the intersection of all $\kappa$-grounds of $V_\theta$.

1. If $\theta$ is inaccessible $\kappa$ and $W \subseteq V$ is a $\kappa$-ground of $V$, then $W_\theta$ is a $\kappa$-ground of $V_\theta$.

2. $\bar{W}^V_\theta \subseteq \bar{W}_\theta$ for every inaccessible $\theta > \kappa$.

3. Suppose there are proper class many inaccessible cardinals. Then there is $\alpha > \kappa$ such that for every inaccessible cardinal $\theta > \alpha$, we have $\bar{W}^V_\theta = \bar{W}_\theta$.

**Proof** (1) is easy, and (2) easily follows from (1).

(3) Suppose not. Then, by (2), the family $C = \{\theta > \kappa \mid \theta$ is inaccessible, $\bar{W}^V_\theta \not\subseteq \bar{W}_\theta\}$ forms a proper class. For $\theta \in C$, there is a $\kappa$-ground $M^\theta$ of $V_\theta$ with $\bar{W}^V_\theta \not\subseteq M^\theta$. Fix a poset $\mathbb{P}^\theta \in (M^\theta)_\kappa$ and an $(M^\theta, \mathbb{P}^\theta)$-generic $G^\theta$ such that $V_\theta = M^\theta[G^\theta]$. By
Fact 2.5, $M^\theta$ has the $\kappa$-covering and the $\kappa$-approximation properties for $V_\theta$, and $\kappa^+ = (\kappa^+)_{V_\theta} = \langle (\kappa^+)_{M^\theta} \rangle_{V_\theta}$. Since $C$ is a proper class, there are a poset $\mathbb{P} \in V_\kappa$, a filter $G \subseteq \mathbb{P}$, and $P \subseteq \mathcal{P}(\kappa)$ such that the family $C' = \{ \theta \in C \mid M^\theta \cap \mathcal{P}(\kappa) = P, \mathbb{P}^\theta = \mathbb{P}, G^\theta = G \}$ forms a proper class. Take $\theta_0, \theta_1$ from $C'$ with $\theta_0 < \theta_1$. Then $(M^\theta_{\theta_0})_{\theta_0}$ is a model of ZFC, and $(M^\theta_{\theta_1})_{\theta_0}$ has the $\kappa$-covering and the $\kappa$-approximation properties for $V_{\theta_0}$. By applying Fact 2.4, we have $(M^\theta_{\theta_1})_{\theta_0} = M^\theta_{\theta_0}$. Hence the sequence $\langle M^\theta \mid \theta \in C' \rangle$ is coherent, that is, $(M^\theta_{\theta_1})_{\theta_0} = M^\theta_{\theta_0}$ for every $\theta_0 < \theta_1$ from $C'$. Then $M = \bigcup_{\theta \in C'} M^\theta$ is transitive, closed under the G"odel’s operations, and almost universal, hence $M$ is a model of ZFC (see e.g. Theorem 13.9 in Jech [3]). Moreover $M[G] = V$ because $M^\theta[G] = V_\theta$ for every $\theta \in C'$, so $M$ is a $\kappa$-ground of $V$. Therefore we have $\overline{W} \subseteq M$, and $\overline{W}_\theta \subseteq M_\theta = M^\theta$ for every $\theta \in C'$, this is a contradiction. □

3 The proof

We start the proof of Theorem 1.3.

Proof Let $\overline{W}$ be the $\kappa$-mantle of $V$. We prove that $\overline{W}$ is the mantle of $V$, this provides Theorem 1.3; By Lemma 2.8, there is a ground $W$ with $W \subseteq \overline{W}$. Clearly $\mathbb{M} \subseteq W \subseteq \overline{W} = \mathbb{M}$, hence $\mathbb{M} = \overline{W} = W$ is a ground of $V$.

If not, by Lemma 2.8, there is a ground $W$ of $V$ with $W \subseteq \overline{W}$. Fix a large inaccessible cardinal $\lambda > \kappa$ such that $W$ is a $\lambda$-ground of $V$ and $W_\lambda \subseteq \overline{W}_\lambda$. $W_\lambda$ and $V_\lambda$ are transitive models of ZFC. By Lemma 2.9, we can find an inaccessible $\theta > \lambda$ such that $\overline{W}^\theta = \overline{W}_\theta$, where $\overline{W}^\theta$ is the $\kappa$-mantle of $V_\theta$.

Take an elementary embedding $j : V_{\theta + 1} \rightarrow V_{j(\theta) + 1}$ such that the critical point of $j$ is $\kappa$ and $\theta < j(\kappa)$. $j(\theta)$ is inaccessible, so $V_{j(\theta)}$ and $W_{j(\theta)}$ are transitive models of ZFC. By the elementarity of $j$, the set $j(\overline{W}^\theta)$ is the $\kappa$-mantle of $j(V_\theta) = V_{j(\theta)}$.

By Lemma 2.9, $W_{j(\theta)}$ is a $\lambda$-ground of $V_{j(\theta)}$, hence $j(\overline{W}^\theta) \subseteq W_{j(\theta)}$.

Fix a sequence $S = \langle S_\alpha \mid \alpha < \lambda \rangle \subseteq W$ of pairwise disjoint sets such that each $S_\alpha$ is a stationary subset of $\lambda \cap \operatorname{Cof}(\omega)^W$ in $W$. Since $V$ is a $\lambda$-c.c. forcing extension of $W$, each $S_\alpha$ is stationary in $\lambda \cap \operatorname{Cof}(\omega)^V$ in $V$ as well. Let $\langle S^*_\alpha \mid \alpha < j(\lambda) \rangle = j(S) \in V_{j(\theta)}$. By a well-known argument by Solovay, we have that $j^{\ast\lambda} = \{ \alpha < \sup(j^{\ast\lambda}) \mid S^*_\alpha \cap \sup(j^{\ast\lambda}) \}$ is stationary in $\sup(j^{\ast\lambda})$ (e.g., see Theorem 14 in Woodin et al.[8]).

Claim 3.1 $j^{\ast\lambda} \in W_{j(\theta)}$.

Proof of the claim Since $S \in W_\theta \subseteq W_\theta = \overline{W}^\theta$, we have $j(S) \in j(\overline{W}^\theta) \subseteq W_{j(\theta)}$. $V_{j(\theta)}$ is a $\lambda$-c.c. forcing extension of $W_{j(\theta)}$. Hence for each set $S \subseteq \sup(j^{\ast\lambda})$ with $S \in W_{j(\theta)}$, the stationarity of $S$ is absolute between $V_{j(\theta)}$ and $W_{j(\theta)}$. This means that for every $\alpha < \sup(j^{\ast\lambda})$, we have that $\alpha \in j^{\ast\lambda}$ if and only if $S^*_\alpha \cap \sup(j^{\ast\lambda})$ is stationary in $\sup(j^{\ast\lambda})$ in $W_{j(\theta)}$. Thus we have $j^{\ast\lambda} \in W_{j(\theta)}$.

Finally we claim that $W_\lambda = \overline{W}_\lambda$, which yields the contradiction. The inclusion $W_\lambda \subseteq \overline{W}_\lambda$ is trivial. For the converse, we shall prove $\overline{W}_\alpha \subseteq W_\alpha$ by induction on $\alpha < \lambda$. Since the critical point of $j$ is $\kappa$, we have $j(\overline{W}^\theta_\kappa) = (\overline{W}^\theta_\kappa) = W_\kappa$. Since $j(\overline{W}^\theta_{\theta_0}) \subseteq W_{j(\theta)}$, we have $W_\kappa \subseteq W_\kappa$. Take $\alpha$ with $\kappa \leq \alpha < \lambda$, and suppose $\overline{W}_\alpha \subseteq W_\alpha$.
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To show that $W_{\alpha+1} \subseteq W$, take $X \in W_{\alpha+1}$. Since $W$ is transitive, we have $X \subseteq W_{\alpha}$. $X \in W_{\alpha+1} = (W^V)_{\alpha+1} \subseteq W^V$, hence $j(X) \in j(W^V)$. We know $W_{\alpha} = W_{\alpha} \in W_{\lambda}$ and $W_{\lambda}$ is a model of ZFC, hence there is $\gamma \in W_{\lambda}$ and a bijection $f: \gamma \to W_{\alpha}$ with $f \in W_{\lambda}$.

We conclude this paper by asking the following natural question:

**Question 3.2** Let $\kappa$ be an extendible cardinal. Is there a poset $P \in M$ of size $< \kappa$ and a $(M, P)$-generic $G$ with $V = M[G]$?

This question is equivalent to the destructibility of extendible cardinals by non-small forcings:

**Question 3.3** Let $\kappa$ be a cardinal, and $P$ a poset such that for every $p \in P$, the suborder $\{ q \in P \mid q \leq p \}$ is not forcing equivalent to a poset of size $< \kappa$. Does $P$ necessarily force that “$\kappa$ is not extendible”?

The referee pointed out that this question might be related to the following result. See Sargsyan and Schindler [6] for the definitions.

**Fact 3.4** ([6]) Let $M_{SW}$ be the least iterable inner model with a strong cardinal above a Woodin cardinal. If $\kappa$ is a strong cardinal of $M_{SW}$, then the $\kappa$-mantle of $M_{SW}$ is the smallest ground of $M_{SW}$ via some $\kappa^+$.c.c. poset of size $\kappa^+$, while $M_{SW}$ cannot be a forcing extension of its $\kappa$-mantle via a poset of size $< \kappa$.

**References**

1. Fuchs, G., Hamkins, J.D., Reitz, J.: Set-theoretic geology. Ann. Pure Appl. Logic 166(4), 464–501 (2015)
2. Hamkins, J.D.: Extensions with the approximation and cover properties have no new large cardinals. Fund. Math. 180(3), 257–277 (2003)
3. Jech, T.: Set Theory. The Third Millennium Edition, Revised and Expanded. Springer, Berlin (2003)
4. Laver, R.: Certain very large cardinals are not created in small forcing extensions. Ann. Pure Appl. Log. 149(1–3), 1–6 (2007)
5. Reitz, J.: The ground axiom. J. Symb. Log. 72(4), 1299–1317 (2007)
6. Sargsyan, G., Schindler, R.: Varsovian models I. J. Symb. Log. (To appear)
7. Usuba, T.: The downward directed grounds hypothesis and very large cardinals. J. Math. Log. 17, 1750009 (2017)
8. Woodin, W.H., Davis, J., Rodriguez, D.: The HOD dichotomy. arXiv:1605.00613 (unpublished)