New Subquadratic Approximation Algorithms for the Girth

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Abstract

We consider the problem of approximating the girth, \( g \), of an unweighted and undirected graph \( G = (V, E) \) with \( n \) nodes and \( m \) edges. A seminal result of Itai and Rodeh [SICOMP’78] gave an additive 1-approximation in \( O(n^2) \) time, and the main open question is thus how well we can do in subquadratic time.

In this paper we present two main results. The first is a \( 1 + \varepsilon, O(1) \)-approximation in truly subquadratic time. Specifically, for any \( k \geq 2 \) our algorithm returns a cycle of length \( 2^{g/2 + 2 \left( \frac{2}{2^{k-1}} \right)} \) in \( \tilde{O}(n^{2-1/k}) \) time. This generalizes the results of Lingas and Lundell [IPL’09] who showed it for the special case of \( k = 2 \) and Roditty and Vassilevska Williams [SODA’12] who showed it for \( k = 3 \). Our second result is to present an \( O(1) \)-approximation running in \( O(n^{1+\varepsilon}) \) time for any \( \varepsilon > 0 \). Prior to this work the fastest constant-factor approximation was the \( \tilde{O}(n^{3/2}) \) time 8/3-approximation of Lingas and Lundell [IPL’09] using the algorithm corresponding to the special case \( k = 2 \) of our first result.

1 Introduction

In this paper we consider the basic graph theoretical problem of computing the shortest cycle of an unweighted and undirected graph. The length of this cycle, \( g \), is also known as the girth of a graph.

Computing the girth of a graph has been studied since the 1970s. In a seminal paper from 1978, Itai and Rodeh [5] showed that the girth of an \( n \)-node, \( m \)-edge graph can be computed in \( O(n^\omega) \) time using fast matrix multiplication, where \( \omega < 2.373 \) is the matrix multiplication constant [4]. They also observed that running \( n \) breadth first searches gives an \( O(mn) \) time combinatorial algorithm for finding the girth. Here combinatorial means an algorithm that does not employ Strassen-like cancellation tricks and arithmetic operations over some field. Furthermore, it was shown by Vassilevska Williams and Williams [11] that any combinatorial algorithm computing the girth in \( O(n^{3-\varepsilon}) \) for any \( \varepsilon > 0 \) would imply a truly subcubic algorithm (i.e. \( O(n^3) \) for some \( \varepsilon > 0 \)) for combinatorial boolean matrix multiplication. Obtaining such an algorithm is widely conjectured to be impossible.

This seeming barrier, combined with fast matrix multiplication being deemed impractical, motivates the study of approximation algorithms for the girth, \( g \). In the paper of Itai and Rodeh [5] they also presented an algorithm computing a cycle of length at most \( g + 1 \) in \( O(n^2) \) time using a simple BFS approach. In some sense this is an optimal approximation algorithm, as the input may indeed be as large as \( \Theta(n^2) \) and we cannot hope to get a better approximation. However, we also know due to a classic result by Bondy and Simonovits [2] that any undirected

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graph with more than $200n^{3/2}$ edges contains a 4-cycle. Furthermore, we can find such a cycle in $O(n)$ expected time using the subroutine of Yuster and Zwick [12] giving an $O(n)$ time algorithm returning a cycle of length at most $g + 1$ for graphs with many edges (since $g \geq 3$). It thus remains interesting to obtain approximation algorithms with subquadratic running time.

This problem was initially studied by Lingas and Lundell [6] and later by Roditty and Vassilevska Williams [10]. In this paper we say that an algorithm is a $(c_1, c_2)$-approximation to the girth for $c_1 \geq 1, c_2 \geq 0$, if the algorithm returns a cycle with length $\hat{g}$ such that $g \leq \hat{g} \leq c_1 \cdot g + c_2$. We say that a $(c_1, 0)$-approximation is a multiplicative $c_1$-approximation (or just $c_1$-approximation), and that a $(1, c_2)$-approximation is an additive $c_2$-approximation (or just $+c_2$-approximation). Lingas and Lundell [6] initialized the study of subquadratic girth approximation algorithms by giving a Monte Carlo $(2, 2)$-approximation in expected time $O(n^{3/2}\sqrt{\log n})$. It is worth noting that this algorithm represents an (at worst) $8/3$-approximation. They stated as an open question whether a subquadratic time algorithm giving a multiplicative approximation factor of two or better exists. This question was answered by Roditty and Vassilevska Williams [10] who presented several subquadratic approximation algorithms for the girth. In particular they gave a $2$-approximation in $O(n^{5/3}\log n)$ time. In fact, they showed a more general result returning a cycle of length at most $2 \left\lceil g/2 \right\rceil + 2 \left\lceil g/4 \right\rceil$ (although they state it quite differently)\footnote{For a graph with girth $g = 4c - z$ for some $c \geq 1$ and $z \in \{0, 1, 2, 3\}$ their algorithm returns a cycle of length $6c - z$ for even $g$ and $6c - z + 1$ for odd $g$. It can be verified by inspection that this is indeed $2\left\lceil g/2 \right\rceil + 2 \left\lceil g/4 \right\rceil$.} in $O(n^{5/3}\log n)$. This can be seen as an “almost-but-not-quite” $3/2$-approximation. They also conjecture that obtaining a $(2 - \varepsilon)$-approximation requires essentially quadratic time. Complementing this conjecture, they present a randomized algorithm which beats this barrier for triangle-free graphs, giving an $8/5$-approximation in $O(n^{1.968})$ which can be improved to $O(n^{1.942})$ using the result of [3] as a lemma. Finally, in the same paper, Roditty and Vassilevska Williams present an additive $3$-approximation (additive $2$ for even $g$) in time $O(n^{3/2}/m \log^2 n)$.

1.1 Our contribution

In this paper we address the following two questions: “How good an approximation can we give for the girth in subquadratic time?” and “What is the fastest running time of any constant-factor approximation algorithm to the girth?”.

The conjecture of Roditty and Vassilevska Williams [10] suggests that we cannot hope to obtain a $(2 - \varepsilon)$-approximation faster than $\Omega(n^2)$. However, we show that if we allow a small additive error, we can an arbitrarily good multiplicative approximation in subquadratic time – that is, a $(1 + \varepsilon, O(1))$-approximation – for any $\varepsilon > 0$. Specifically, we show the following theorem.

**Theorem 1.** Let $G$ be a graph with $n$ nodes and let $k \geq 2$ be any integer. Then there exists an algorithm that runs in time $O(n^{2-1/k}(\log n)^{1-1/k})$ and finds a cycle with length at most $2 \left\lceil \frac{g}{2} \right\rceil + 2 \left\lceil \frac{g}{2^{k-1}} \right\rceil$, where $g$ is the girth of $G$.

Theorem 1 generalizes the result of Roditty and Vassilevska Williams [10, Thm. 1.3], who showed it for the special case of $k = 3$ (with an additional $\log^{1/3} n$ factor) and the result of Lingas and Lundell [6, Thm. 1] who showed a Monte Carlo version for the special case of $k = 2$. As a corollary, we also get an improvement on the result for triangle-free graphs [10, 3] improving the running time from $O(n^{1.942})$ to $O(n^{9/8}\log^{4/5} n)$ and generalizing to families of graph with girth $g > 2\ell - 1$ for any positive integer $\ell$.

**Corollary 1.** Let $\ell$ be a positive integer and $k = 2\ell - \left\lceil \frac{\ell}{3} \right\rceil + 2$. There exists an algorithm that given a graph $G$ with $n$ nodes and girth $> 2\ell - 1$ runs in time $O(n^{2-1/k})$ and gives a $(1 + \frac{3}{2^{\ell+1}})$-approximation of the girth.
For completeness we show how Theorem 1 directly implies Corollary 1 above in Appendix A.

Complementing Theorem 1 we also show that for any constant \( \varepsilon > 0 \) it is possible to obtain a constant-factor approximation in \( O(n^{1+\varepsilon}) \) expected time. Specifically, we show the following theorem.

**Theorem 2.** Let \( G \) be a graph with \( n \) nodes and \( k \) an integer \( \geq 2 \). There exists an algorithm giving a \( 2^k \)-approximation to the girth of \( G \) with probability \( 1 - \frac{1}{n} \) running in expected time \( O(n^{1+1/k}k \log n) \).

Our algorithms are quite simple and rely on sampling and a “stop-early” BFS procedure similar to previous work \([3, 6, 10]\). Using techniques of \([10]\) it is possible to derandomize the sampling in Theorem 1, however this procedure is too slow and does not work for Theorem 2, hence this theorem is the only available in a randomized variant.

**1.2 Related work**

The problem of approximating the girth has also been considered in other settings. For undirected graphs with weights in \( \{1, \ldots, M\} \) Lingas and Lundell \([4]\) gave a 2-approximation of the minimum weight cycle running in \( O(n^2 \log n (\log n + \log M)) \). Roditty and Tov \([8]\) improved the approximation factor to \( 4/3 \) while maintaining the running time, and also gave a \( O(\frac{1}{\varepsilon}n^2 \log n \log \log n) \) time \( (4/3 + \varepsilon) \)-approximation for graphs with non-negative real weights. Furthermore, it was shown by Roditty and Vassilevska Williams \([9]\) that the problem (also for directed graphs) reduces to finding a minimum weight triangle in an undirected graph with \( O(n) \) nodes and weights in \( \{1, \ldots, O(M)\} \). For directed graphs a recent paper by Pachocki, et al. \([12]\) gave a \( O(k \log n) \)-approximation in \( O(mn^{1/k} \log^3 n) \) for any \( k \geq 1 \) and an additive \( O(n^\alpha) \)-approximation in \( O(mn^{1-\alpha}) \) for any \( \alpha \in (0, 1) \).

Closely related to the problem of finding the girth of a graph is the problem of finding a cycle of a fixed length \( k \). For undirected graphs, Yuster and Zwick \([1]\) showed that this can be done for even \( k \) in \( O(f(k) \cdot n^2) \). Alon, Yuster and Zwick \([1]\) showed that for directed and undirected graphs this can be done in \( O(f(k) \cdot m^{2-2/k}) \) if \( k \) is even and \( O(f(k) \cdot m^{2-2/(k+1)}) \) if \( k \) is odd. And for undirected graphs when \( k \) is even they give an algorithm with running time \( O(m^{2-((k+1)/4)^{-1}}/(k/2+1)) \). For even \( k \) this was improved \([3]\) to \( O(f(k) \cdot m^{2-4/(k+2)}) \). This problem of finding an even \( k \)-cycle was used as a subroutine by Roditty and Vassilevska Williams \([10]\) in some of their algorithms for approximating the girth.

## 2 Preliminaries

We will assume that all graphs \( G = (V, E) \) in the paper are undirected, unweighted, connected and contain at least one cycle. If \( u \) is some node in the graph \( G \) and \( \ell \) is an integer, we denote by \( B_G(u, \ell) \) the ball of radius \( \ell \) around \( u \) in \( G \), i.e. the set of all nodes of distance at most \( \ell \) from \( u \) in \( G \). We will sometimes denote this simply \( B(u, \ell) \) when \( G \) is clear from the context. We let \( \Gamma(u) \) denote the neighbourhood of \( u \), i.e. \( \Gamma(u) = B(u, 1) \setminus \{u\} \). For a set of nodes \( S \subseteq V \), let \( B(S, \ell) = \bigcup_{u \in S} B(u, \ell) \). We denote by \( C_\ell \) the simple cycle of length \( \ell \) and let \( \log x \) be the natural logarithm of \( x \).

We will need the following lemma, which was used in a slightly weaker version by Roditty and Vassilevska Williams \([10]\).

**Lemma 1.** Let \( A_1, \ldots, A_k \) be sets over a universe \( U \) of \( n \) elements such that for every \( i \leq k \) we have \( |A_i| = x \) for some positive \( x \). Then we can find a set \( S \subseteq U \) with \( |S| \leq \frac{n}{x} \log k \) in \( O(kx + n) \) time such that for each \( 1 \leq i \leq k \) we have \( S \cap A_i \neq \emptyset \).
Proof. Let \( t(u) = |\{i \mid u \in A_i\}| \). We start by computing this quantity for each \( u \in U \) in \( O(kx + n) \) time by simply traversing all the sets \( A_i \). We now keep \( k \) linked lists \( L_1, \ldots, L_k \), where \( L_i \) contains the elements \( u \) such that \( t(u) = i \). In addition, we keep a table \( T \) where \( T[u] \) contains a pointer to the node in \( L_1, \ldots, L_k \) containing \( u \).

The algorithm now works as follows. Let \( S = \emptyset \). Repeatedly find an element \( u \) with maximum \( t(u) \), add it to \( S \), and for each remaining \( A_i \) such that \( u \in A_i \), we remove \( A_i \) and for each \( v \in A_i \) update \( t(v) \) to \( t(v) - 1 \) and move it from \( L_{t(v)} \) to \( L_{t(v)-1} \). We note that we may update an element \( v \) several times, and that if \( t(v) \) becomes 0 we simply ignore \( v \) for the remainder of the algorithm.

Since the maximum value \( t(u) \) can never decrease we can keep track of the maximum value in total \( O(k) \) time over all iterations. Furthermore, all updates take \( O(kx) \) time using the lists \( L_1, \ldots, L_k \) and our table \( T \). The analysis of the size of \( S \) now follows exactly as in \cite{10}.

Similar to previous papers \cite{3, 10, 6} we will use a procedure called **BFS-Cycle**\((G, u)\), which is simply the algorithm which runs a BFS from \( u \) in \( G \) until a node \( v \) is visited twice. In this case the algorithm returns the simple cycle containing \( v \) in the BFS tree including the last edge visited. We will need the following well-known lemma from the literature.

**Lemma 2. \cite{3, 6}** Let \( G = (V, E) \) and let \( u \in V \) be any vertex. Then running **BFS-Cycle**\((G, u)\) takes \( O(n) \) time. Furthermore, if \( v \) is a vertex at distance \( \ell \) from \( u \) and \( v \) is contained in a simple cycle of length \( k \), then **BFS-Cycle**\((G, u)\) returns a cycle of length at most \( 2\lfloor k/2 \rfloor + 2\ell \).

Sometimes we will need to run a restricted version of **BFS-Cycle** which stops after visiting a certain number of nodes (or if a cycle was found before). Let this procedure be denoted by **BFS-Cycle**\((G, u, y)\), where \( y \) is the bound on the number of visited nodes. It is clear that **BFS-Cycle**\((G, u, y)\) takes at most \( O(y) \) time if we assume an adjacency list representation.

We will need the following algorithm, which samples a set \( S \) and runs **BFS-Cycle** from each \( u \in S \).

**Definition 1.** Let \( A(G, x, y) \) be an algorithm that takes an \( n \)-node graph \( G = (V, E) \) and two positive numbers \( x, y \). The algorithm creates a set \( S \subseteq V \) of nodes by sampling each node of \( V \) independently with probability \( \min\{\frac{x}{\ell}, 1\} \). Clearly \( S \) has expected size \( x \). The algorithm then runs **BFS-Cycle**\((G, u, y)\) for each \( u \in S \) and returns the smallest cycle found or nothing if no cycle is found.

We will also need the following deterministic variant, where the set \( S \) is picked using Lemma \cite{1}

**Definition 2.** Let \( A_{det}(G, x, y) \) be the same algorithm as \( A(G, x, y) \) with the following modification to how the set \( S \) is picked. Fix \( \tilde{x} = \frac{n \log n}{x} \) and denote the nodes of \( G \) by \( u_1, \ldots, u_n \). The algorithm first creates the sets \( A_1, \ldots, A_n \), where \( A_i \) is the set containing the \( \tilde{x} \) closest nodes to \( u_i \) (breaking ties arbitrarily, eg. by the order in the adjacency lists). It then creates the set \( S \subseteq V \) by running the algorithm of Lemma \cite{1} on \( A_1, \ldots, A_n \).

It is easy to see that \( A(G, x, y) \) runs in \( O(n + xy) \) expected time and that \( A_{det}(G, x, y) \) runs in \( O\left(\frac{n^2 \log n}{x} + xy\right) \) time.

### 3 Subquadratic girth approximations

In this section we present our algorithm for obtaining subquadratic approximations of the girth. In particular we will prove Theorem \cite{1}.
We will now show how to obtain a constant factor approximation randomized in near-linear time. We can now use Lemma 2 to conclude that the call to BFS-Cycle produces a cycle in \( G, u, \frac{n \log n}{x} \) keeping track of the smallest cycle seen so far. The output of our algorithm is the smallest cycle produced by any of the above steps.

It remains to prove that the algorithm returns a cycle of length at most \( 2 \lfloor g/2 \rfloor + 2 \left[ \frac{g}{2(k-1)} \right] \). Let \( C \) be a cycle of length \( g \) in \( G \) and let \( t = \min_{u \in C} \{ r(u) \} \). We know that the set \( S \) produced by \( A_{det}(G, \frac{n \log n}{x}, n) \) contains a node with distance at most \( t + 1 \) to \( C \) and therefore, by Lemma 2, this algorithm returns a cycle with length at most \( 2 \lfloor g/2 \rfloor + 2(t + 1) \). It follows that the call to \( A_{det}(G, \frac{n \log n}{x}, n) \) suffices for \( t < \left[ \frac{g}{2(k-1)} \right] \). Now, assume that \( t \geq t_0 \), where \( t_0 = \left[ \frac{g}{2(k-1)} \right] \) and let \( u \) be the last node of \( C \) that was added to \( H \). Let \( H_u \) denote the graph \( H \) after adding \( u \). It follows by the definition of \( u \) that the cycle \( C \) is contained in \( H_u \). Furthermore, for each \( v \in H_u \) we have \( r(v) \geq r(u) \geq t_0 \) and thus \( |B_{H_u}(v, t_0)| \leq x^{t_0} \) for each \( v \in H_u \) and since \( (k-1)t_0 \geq g/2 \) by assumption, this implies that \( |B_{H_u}(u, [g/2])| \leq x^{k-1} \). We can now use Lemma 2 to conclude that the call to BFS-Cycle returns a cycle of length at most \( 2 \lfloor g/2 \rfloor \).

For the running time of the algorithm observe first that we can find \( r(u) \) for each \( u \in O(x) \) time using a BFS. Moreover, since we only add nodes \( u_i \) to \( H \) when \( r(u_i) > 0 \) it follows that we only consider at most \( O(nx) \) edges for addition to \( H \). The running time is now bounded by \( O(n \cdot \frac{n \log n}{x} + nx^{k-1}) \), which is minimized when \( x = (n \log n)^{1/k} \) giving a total running time of \( O(n^{2-1/k}(\log n)^{1-1/k}) \). \( \square \)

## 4 Constant approximation in near-linear time

We will now show how to obtain a constant factor approximation randomized in near-linear time.

**Proof of Theorem 2.**\( \square \) The algorithm is very straight-forward: For each \( i = 1, \ldots, k \) we run \( A(G, n^{1+(1-i)/k} \log n, n^{r/k}) \) and return the minimal cycle found or nothing if no cycle is reported.

We now analyze this procedure. Let \( g \) be the girth of \( G \) and \( C \) a cycle in \( G \) with length \( g \). Let \( r \) be the smallest non-negative integer such that

\[
|B(C, \lfloor g/2 \rfloor \cdot (2^r - 1))| \leq n^{r/k}.
\]

Clearly such an \( r \) exists, as we may pick \( r = k \). Also observe that \( r > 0 \) since, in particular, the ball with \( r = 0 \) contains the cycle \( C \) itself. We will show that the \( r \)th iteration of the algorithm gives a sufficiently small cycle. Consider the algorithm \( A(G, n^{1+(1-r)/k} \log n, n^{r/k}) \) and let \( S \) be the set of nodes sampled by this algorithm. We will show that there exists a node \( u \in S \) close to \( C \), such that \( C \) is contained in the tree explored by BFS-Cycle\( (G, u, n^{r/k}) \). Consider the slightly smaller ball \( B(C, \lfloor g/2 \rfloor \cdot (2^{r-1} - 1)) \). By the minimality of \( r \), the number of nodes in \( S \) belonging to this ball in expectation is at least

\[
|B(C, \lfloor g/2 \rfloor \cdot (2^{r-1} - 1))| \cdot n^{(1-r)/k} \log n \geq n^{(r-1)/k} \cdot n^{(1-r)/k} \log n = \log n.
\]

Therefore the probability that no node from \( B(C, \lfloor g/2 \rfloor \cdot (2^{r-1} - 1)) \) is sampled is at most \( \frac{1}{n} \). We thus assume that such a sampled node, \( u \), exists. We now argue that the BFS search starting in \( u \) gives the desired cycle. This is illustrated in Figure 4. Since \( u \) is contained in
$C \leq \lceil g/2 \rceil \cdot (2^{r-1} - 1)$

$\leq \lceil g/2 \rceil \cdot (2^{r-1} - 1)$

Figure 1: Illustration of the proof of Theorem 2. $u$ is a sampled node, and running BFS-Cycle from $u$ visiting at most $n^{r/k}$ gives the desired approximation.

\[ B(C, [g/2] \cdot (2^{r-1} - 1)) \]

we know that $B(u, [g/2] \cdot 2^{r-1}) \subseteq B(C, [g/2] \cdot (2^{r-1}))$ and thus, by the definition of $r$, we know that $|B(u, [g/2] \cdot 2^{r-1})| \leq n^{r/k}$.

Furthermore, this ball around $u$ also contains the cycle $C$ and thus $A(G, n^{1+(1-r)/k} \log n, n^{r/k})$ returns a cycle of length at most $2 \cdot [g/2] \cdot 2^{r-1}$. Since $r \leq k$ the length of the cycle returned is at most $2^k [g/2]$.

Since we invoke that algorithm $A$ exactly $k$ times and each invocation takes $O(n^{1+1/k} \log n)$ in expectation the running time follows.

We note that we cannot employ the algorithm $A_{det}$ instead in the algorithm above, as the task of creating a ball around each node $u \in V$ in order to employ Lemma 1 takes too long. This is the main bottle-neck in obtaining a deterministic variant of Theorem 2.

5 Conclusion and open problems

In this paper we have studied the problem of obtaining subquadratic approximation algorithms for the girth of an undirected and unweighted graph. We have shown how to obtain a multiplicative $(1 + \varepsilon)$-approximation with small additive error in subquadratic time, and an $O(1)$-approximation in $O(n^{1+\varepsilon})$ time for any $\varepsilon > 0$. It remains as the main open question whether one can obtain a multiplicative $(2 - \varepsilon)$-approximation in $O(n^{2-\varepsilon})$ time or perhaps show that obtaining such an approximation requires $n^{2-o(1)}$ using the framework of Hardness in P. Another interesting question is whether one can improve on our Theorem 2 and obtain a multiplicative $O(1)$-approximation in $n^{1+o(1)}$ time or an additive $O(1)$-approximation in $O(n^{2-\varepsilon})$ time. Finally, it is an interesting question whether one can improve on Corollary 1 and obtain a multiplicative $(1 + \varepsilon)$-approximation in subquadratic time for graphs with girth $> 2\ell - 1$.

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Appendix

A Omitted proofs

Proof of Corollary 1. We use Theorem 1 with \(k = 2\ell - \left\lceil \frac{\ell}{3} \right\rceil + 2\). In order to prove that we get a \(1 + \frac{3}{2\ell + 1}\)-approximation we just need to prove that

\[
2 \left\lceil \frac{g}{2} \right\rceil + 2 \left\lceil \frac{g}{2(k-1)} \right\rceil \leq 1 + \frac{3}{2\ell + 1},
\]

(1)

whenever \(g \geq 2\ell\). For \(g = 2\ell\) (1) holds, so assume that \(g \geq 2\ell + 1\).

For \(2\ell + 1 \leq g \leq 2(k-1)\) we note that \(2 \left\lceil \frac{g}{2} \right\rceil \leq g + 1\) and \(2 \left\lceil \frac{g}{2(k-1)} \right\rceil = 2\). Therefore:

\[
2 \left\lceil \frac{g}{2} \right\rceil + 2 \left\lceil \frac{g}{2(k-1)} \right\rceil \leq \frac{g + 3}{g} \leq 1 + \frac{3}{2\ell + 1},
\]

and (1) holds. So assume that \(g > 2(k-1)\). Then we can write \(g = 2(k-1)q + r\) for some positive integer \(q\) and some \(r \in \{1, 2, \ldots, 2(k-1)\}\). We again use that \(2 \left\lceil \frac{g}{2} \right\rceil \leq g + 1\) and see that \(\left\lceil \frac{g}{2(k-1)} \right\rceil = q + 1\). So we get that:

\[
2 \left\lceil \frac{g}{2} \right\rceil + 2 \left\lceil \frac{g}{2(k-1)} \right\rceil \leq \frac{g + 1 + 2q + 2}{g} = 1 + \frac{3 + 2q}{g} \leq 1 + \frac{3 + 2q}{2(k-1)q + 1}. \tag{2}
\]

The right hand side of (2) is maximized when \(q = 1\), and therefore we get:

\[
2 \left\lceil \frac{g}{2} \right\rceil + 2 \left\lceil \frac{g}{2(k-1)} \right\rceil \leq 1 + \frac{5}{2k - 1},
\]

and therefore we just need to prove that \(\frac{5}{2k-1} \leq \frac{3}{2\ell + 1}\). This can be rewritten as

\[
k \geq \frac{5\ell + 4}{3}.
\]

But we have that \(\left\lceil \frac{\ell}{3} \right\rceil \leq \frac{\ell + 2}{3}\) and therefore

\[
k = 2\ell - \left\lceil \frac{\ell}{3} \right\rceil + 2 \geq 2\ell - \frac{\ell + 2}{3} + 2 = \frac{5\ell + 4}{3},
\]

as desired. \(\square\)