CUBIC THREEFOLDS, FANO SURFACES AND THE MONODROMY OF THE GAUSS MAP

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Abstract. The Tannakian formalism allows to attach to any subvariety of an abelian variety an algebraic group in a natural way. The arising groups are closely related to moduli questions such as the Schottky problem, but their geometric interpretation is still mysterious. We show that for the theta divisor on the intermediate Jacobian of a cubic threefold, the Tannaka group is an exceptional group of type $E_6$. This is the first known exceptional case, and it suggests a connection with the monodromy of the Gauss map.

Introduction

Let $A$ be a complex abelian variety. To any closed subvariety $Z \hookrightarrow A$ one may attach a semisimple complex algebraic group $G_Z$ in a natural way by the Tannakian formalism [17] [20], see section 1. The arising Tannaka groups are closely related to classical topics such as the Schottky problem [18], intersections of theta divisors [16], Torelli’s theorem [20] and Brill-Noether theory on algebraic curves [23]. In all these examples the Tannaka groups turn out to be classical groups, and they are simply connected unless $Z$ is a sum of two positive-dimensional subvarieties of $A$. One may thus wonder whether there exist examples in which $G_Z$ is an exceptional group, and whether its simply connected cover is always realized as the Tannaka group of some other subvariety. In this note we discuss a family of principally polarized abelian varieties of dimension five which gives a positive answer to the first question: The intermediate Jacobians of cubic threefolds [8].

Theorem 1. Let $\Theta \subset A$ be the theta divisor on the intermediate Jacobian $A$ of a smooth cubic threefold. Then

$$G_\Theta \cong E_6(\mathbb{C})/Z,$$

where $Z$ denotes the center of the simply connected complex algebraic group $E_6(\mathbb{C})$.

A more precise statement may be found in theorem 2 which in particular shows that the universal covering group $E_6(\mathbb{C})$ is realized as the Tannaka group for the Fano surface of lines on the threefold. Notice that the Weyl group $W(E_6)$ is also the symmetry group for the 27 lines on a cubic surface. The proof of theorem 2 will not use this fact explicitly, indeed it needs surprisingly few geometry and rests on representation theory, see lemma 4. But the result gives a hint towards a geometric interpretation of the abstractly defined Tannaka groups $G_Z$ which would explain their relevance to concrete applications: For any smooth subvariety $Z \hookrightarrow A$ the Weyl group $W(G_Z)$ should contain the monodromy group of a certain Gauss map.

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as a subgroup of small index as we briefly explain in section[4]. I would like to thank
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1. The Tannakian framework

Let \( D(A) = D^b_c(A, \mathbb{C}) \) be the bounded derived category of constructible sheaves
of complex vector spaces on a complex abelian variety \( A \), and \( P(A) \subset D(A) \) its
full abelian subcategory of perverse sheaves \[4\] \[15\]. The group law \( a : A \times A \to A \)
edows these with a rich structure: The convolution product

\[ \ast : P(A) \times P(A) \to D(A), \quad \delta_1 \ast \delta_2 = R\alpha_*(\delta_1 \boxtimes \delta_2) \]

leads to a Tannakian description for the category of perverse sheaves in terms of
group representations \[17\] \[20\]. There is a slight technicality here since in general
the convolution of perverse sheaves is no longer perverse. To get around this, let us
say a perverse sheaf \( \delta \in P(A) \) with hypercohomology \( H^\bullet(A, \delta) \) is
negligible if the Euler characteristic

\[ \chi(\delta) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} H^i(A, \delta) \]

vanishes. The negligible perverse sheaves have been classified in \[24\] \[21, cor. 5.2\]
and may be described explicitly in terms of perverse sheaves on non-trivial abelian
quotient varieties of \( A \). If \( N(A) \subset D(A) \) denotes the full subcategory of all sheaf
complexes whose perverse cohomology sheaves are negligible, it has been shown
in \[20\] that the triangulated quotient category \( D(A)/N(A) \) inherits a well-defined
convolution product \( \ast \) and that this latter product indeed preserves the full abelian
subcategory \( P(A) = P(A)/(N(A) \cap P(A)) \subset D(A)/N(A) \). Furthermore \( P(A) \) is a
direct limit of Tannakian categories: The subquotients of the convolution powers of
any given perverse sheaf \( \delta \in P(A) \) generate a full abelian subcategory \( \langle \delta \rangle \subset P(A) \)
which admits an equivalence

\[ \omega : \langle \delta \rangle \xrightarrow{\sim} \text{Rep}(G) \]

with the category of finite-dimensional complex linear representations of a complex
algebraic group \( G = G(\delta) \) such that \( \omega(\delta_1 \ast \delta_2) \cong \omega(\delta_1) \otimes \omega(\delta_2) \) for all \( \delta_1, \delta_2 \in \langle \delta \rangle \). It
follows from the construction of this equivalence of categories that the dimension of the arising representations is the Euler characteristic: \( \dim_{\mathbb{C}} \omega(\delta) = \chi(\delta) \).

In principle this reduces the study of perverse sheaves on \( A \) to the representation
theory of algebraic groups, though it may be hard to determine the groups \( G(\delta) \)
explicitly. The most interesting case arises when \( \delta = \delta_Z \) is taken to be the perverse
intersection cohomology sheaf supported on a closed subvariety \( Z \hookrightarrow A \). Often \( Z \)
is only determined up to a translation by some point of the abelian variety, but the
group \( G(\delta_Z) \) depends on the chosen translate. So instead of this group we consider
the semisimple group \( G_Z = [G(\delta_Z)^o, G(\delta_Z)^o] \) which is the derived group of the
connected component. This derived group remains unaffected if \( Z \) is replaced by a
translate. Let \( \omega_Z = \omega(\delta_Z)|_{G_Z} \in \text{Rep}(G_Z) \) denote its defining representation.
2. Intermediate Jacobians

Let $V \subset \mathbb{P}^4$ be a smooth cubic threefold. Clemens and Griffiths have shown [8] that the Fano surface $S$ of lines on $V$ is smooth with $\chi(\delta_S) = 27$, embeds in the intermediate Jacobian

$$J V = \text{Hom}(H^{2,1}(V), \mathbb{C})/H_3(V, \mathbb{Z}),$$

and that the latter is a principally polarized abelian variety of dimension $g = 5$ which admits as a theta divisor the image $\Theta = S - S \subset JV$ of the difference morphism $d : S \times S \to JV, (s, t) \mapsto s - t$. This theta divisor has an isolated singularity at the origin, and there its projective tangent cone is isomorphic to $V$ by [3]. One then computes $\chi(\delta_\Theta) = 78$, see section 3.

The numbers 27 and 78 also arise in representation theory as the dimensions of the smallest irreducible representations of the simply connected complex algebraic group $E_6(\mathbb{C})$. Up to duality they are attained precisely for the 27-dimensional first fundamental representation $\omega_1$ and the 78-dimensional adjoint representation $Ad$ of this group on its Lie algebra. Note that the adjoint representation factors over the quotient $E_6(\mathbb{C})/Z$ by the centre $Z = Z(E_6(\mathbb{C}))$.

**Theorem 2.** Let $A$ be the intermediate Jacobian of a smooth cubic threefold, and consider the corresponding Fano surface $S \subset A$ and the theta divisor $\Theta = S - S \subset A$ as above. Then

$$(G_S, \omega_S) \cong (E_6(\mathbb{C}), \omega_1) \quad \text{and} \quad (G_\Theta, \omega_\Theta) \cong (E_6(\mathbb{C})/Z, Ad).$$

**Proof.** We have already remarked that $\Theta \subset A$ is the image of the difference morphism $d : S \times S \to A$. The latter is generically finite, so by adjunction it follows from the decomposition theorem that $\delta_\Theta$ occurs as a direct summand of the convolution $Rd_*(\delta_S \boxtimes \delta_S) = \delta_S * \delta_{-S}$. On the Tannakian side this gives an embedding

$$\omega_\Theta \hookrightarrow \omega_S \otimes \omega_S^*$$

for the dual $\omega_S^* = \text{Hom}(\omega_S, \mathbb{C})$, so that $G_\Theta$ becomes identified with the image of the Tannaka group $G_S$ under the tensor product representation $\omega_S \otimes \omega_S^*$. The rest of the proof is an application of representation theory which may be found in lemma 4 below. \qed

**Remark 3.** A result of Collino [9] says that in the moduli space $A_5$ of principally polarized abelian fivefolds, the closure of the locus of intermediate Jacobians of smooth cubic threefolds contains the locus of Jacobians of hyperelliptic curves. See also [7]. For a degeneration of a general intermediate Jacobian into a hyperelliptic one, the Fano surface $S$ degenerates to the Brill-Noether subvariety $W_2$. The latter is by definition the image of the symmetric square of the curve in its Jacobian variety, and it has the Tannaka group $G_{W_2} \cong Sp_8(\mathbb{C})/\pm 1$ by [19, th. 6.1] [23]. By semicontinuity [13, sect. 4] this group must be a subquotient of the Tannaka group for the Fano surface, which gives a consistency check for our results and could be used for an alternative proof of theorem 2.

$$\begin{array}{ccc}
Sp_8(\mathbb{C}) & \hookrightarrow & E_6(\mathbb{C}) = G_S \\
& & \downarrow \\
G_{W_2} & = & Sp_8(\mathbb{C})/\pm 1
\end{array}$$
Lemma 4. If \( G \) is a connected semisimple complex algebraic group which admits a faithful irreducible representation \( V \in \text{Rep}(G) \) of dimension 27 such that \( V \otimes V^\vee \) contains an irreducible direct summand \( W \in \text{Rep}(G) \) of dimension 78, then up to duality

\[
G \cong E_6(\mathbb{C}), \quad V \cong \omega_1 \quad \text{and} \quad W \cong \text{Ad}.
\]

Proof. To any connected semisimple algebraic group \( G \) one may associate its universal covering \( \tilde{G} = G_1 \times \cdots \times G_n \twoheadrightarrow G \), where each \( G_i \) is simply connected and simple modulo its center. It then follows that

\[
V|_{\tilde{G}} \cong V_1 \otimes \cdots \otimes V_n
\]

is an exterior product of irreducible representations \( V_i \in \text{Rep}(G_i) \) with finite kernel and dimension \( d_i = \dim(V_i) > 1 \). Having \( d_1 \cdots d_n = \dim(V) = 27 \) forces that \( n \leq 3 \) and \( d_i \in \{3, 9, 27\} \). Now for complex simple Lie algebras of any Dynkin type, the highest weights of all irreducible representations of a given (small) dimension are easily determined via the result of [1], see table 1 where we denote by \( \omega_1, \omega_2, \ldots \) the fundamental weights. In dimensions \( d_i \in \{3, 9, 27\} \) we are left with the following representations and their duals:

| \( d_i \) | 3 | 9 | 27 |
|---|---|---|---|
| \( G_i \) | \( A_1 \) | \( A_2 \) | \( A_1 \) | \( A_2 \) | \( A_1 \) | \( A_2 \) | \( A_2 \) | \( B_4 \) | \( B_4 \) | \( B_3 \) | \( B_3 \) | \( C_4 \) | \( E_6 \) | \( G_2 \) |
| \( V_i \) | \( 2\omega_1 \) | \( \omega_1 \) | \( 8\omega_1 \) | \( \omega_1 \) | \( \omega_1 \) | \( 26\omega_1 \) | \( 2\omega_1 + 2\omega_2 \) | \( \omega_1 \) | \( \omega_1 \) | \( \omega_1 \) | \( \omega_1 \) | \( 2\omega_1 \) |

In our case each \( V_i \otimes V_i^\vee \) must by assumption have an irreducible direct summand \( W_i \) such that

\[
W|_{\tilde{G}} \cong W_1 \otimes \cdots \otimes W_n.
\]

Since \( \dim(W) = 78 \), a direct computation shows that the only possibility is \( n = 1 \) and that the universal covering group \( \tilde{G} = G_1 \) is isomorphic to \( E_6(\mathbb{C}) \). \( \square \)

3. The magic number 78

It remains to check that for the theta divisor on the intermediate Jacobian of a smooth cubic threefold we have \( \chi(\delta_3) = 78 \). This is a standard computation, but we include it since the outcome is crucial for our proof of theorem 2. We begin with a blowup formula. Let \( A \) be a complex abelian variety, and consider an effective divisor \( D \subset A \) whose singular locus \( D^{\text{sing}} = \{p_1, \ldots, p_n\} \) consists of finitely many isolated points \( p_i \) of multiplicity \( m_i \). Blowing up these finitely many points in \( A \) we obtain a Cartesian diagram

\[
\begin{array}{ccc}
E & \to & \hat{D} \\
\downarrow & & \downarrow \pi \\
D^{\text{sing}} & \to & A
\end{array}
\]

where \( E \) denotes the exceptional divisor of the blowup and where \( \hat{D} \) is the strict transform of the divisor \( D \). The restriction \( \pi : \hat{D} \to D \) is the blowup of our original divisor in the singular locus and so the scheme-theoretic intersection \( E \cap \hat{D} \subset \hat{D} \) is a Cartier divisor. In particular, if the latter is smooth, then \( \hat{D} \) is smooth.
Proposition 5. If $\hat{D}$ is smooth, then
\[
(-1)^{g-1} \chi(\hat{D}) = \deg [D]^g - \sum_{i=1}^n m_i \left[ \frac{(1 + t)(1 - t)^g}{1 - m_i t} \right]_{g-1}
\]
where $\deg : H^{2g}(A, \mathbb{Z}) \rightarrow \mathbb{Z}$ denotes the degree map and where the brackets on the right hand side indicate the coefficient of $t^{g-1}$ in the power series.

Proof. Let $E_i = \pi^{-1}(p_i) \subset \hat{A}$ denote the exceptional fibres of the blowup, and consider the fundamental cohomology classes $\eta_i = [E_i]$, $\theta = [\hat{D}]$ in $H^2(\hat{A}, \mathbb{Q})$. Since the tangent bundle to $A$ is trivial, the formula for the total Chern class of a blowup in [13, ex. 15.4.2] says that $\deg \eta_i^g = (-1)^{g-1}$ for all $i$. This last intersection number is obtained from the conormal sequence for the smooth divisor $E_i \cong \mathbb{P}^{g-1}$ with deg $c_{g-1}(E_i) = \chi(E_i) = g$. \hfill $\Box$

Corollary 6. The theta divisor $\Theta \subset A$ on the intermediate Jacobian $A = JV$ of a smooth cubic threefold $V$ has
\[
\chi(\delta_\Theta) = 78.
\]

Proof. Let $\pi : \tilde{\Theta} \rightarrow \Theta$ denote the blowup of the theta divisor in the origin. The fibre $\pi^{-1}(0)$ is isomorphic to our threefold $V$ by [13, th. 1], so base change implies that the stalk cohomology of $R\pi_*(\delta_\Theta)$ in the origin is $H^2(R\pi_*(\delta_\Theta))_0 \cong H^{1+4}(V, \mathbb{C})$ for all $i \in \mathbb{Z}$. On the other hand
\[
R\pi_*(\delta_\Theta) \cong \delta_\Theta \oplus \varepsilon
\]
by the decomposition theorem, where $\varepsilon$ is a skyscraper complex which is supported in the origin and is stable under the Lefschetz operator and its inverse. Since the stalk cohomology of $\delta_\Theta$ vanishes in all degrees $i \geq 0$, a comparison with $H^{1+4}(V, \mathbb{C})$ shows $\varepsilon \cong \delta_0[2] \oplus \delta_0 \oplus \delta_0(-2)$ and hence $\chi(\delta_\Theta) = \chi(\delta_\Theta) + 3$. Therefore our claim follows from the above proposition which in the special case $n = 1$, $m_1 = 3$ shows that $\chi(\delta_\Theta) = 5! - 39 = 81 = 78 + 3$. \hfill $\Box$

Remark 7. (a) The Euler characteristic of a perverse sheaf has a simple meaning in terms of Gauss maps. To explain this, let $A$ be any complex abelian variety, and put $\Omega = H^0(A, \Omega^1_A)$. For any closed subvariety $Z \hookrightarrow A$, the closure of the conormal bundle to the smooth locus of $Z$ is an irreducible subvariety $\Lambda_Z \hookrightarrow \mathcal{T}_A^* = A \times \Omega$ of the cotangent bundle to the abelian variety, and we define the corresponding Gauss map to be the composite
\[
\gamma_Z : \Lambda_Z \hookrightarrow \mathcal{T}_A^* = A \times \Omega \rightarrow \Omega
\]
Note that this is a generically finite map; we will denote by $\deg(\gamma_Z) \in \mathbb{N} \cup \{0\}$ its generic degree. Now any perverse sheaf $\delta \in P(A)$ defines via the Riemann-Hilbert
correspondence a characteristic cycle in the sense of [13], which is a finite formal sum
\[
CC(\delta) = \sum_{Z \rightarrow A} m_Z(\delta) \cdot \Lambda_Z
\]
with coefficients \( m_Z(\delta) \in \mathbb{Z} \). Franecki and Kapranov [12] have shown that in this situation
\[
\chi(\delta) = \sum_{Z \rightarrow A} m_Z(\delta) \cdot \deg(\gamma_Z).
\]
If \( \delta = \delta_W \) is taken to be the perverse intersection cohomology sheaf supported on a closed subvariety \( W \hookrightarrow A \), then \( m_W(\delta) = 1 \), but depending on how bad the singularities of the subvariety are, we may also have \( m_Z(\delta) \neq 0 \) for some \( Z \hookrightarrow W^{\text{reg}} \) inside the singular locus.

(b) The theta divisor \( \Theta \subset A \) on the intermediate Jacobian \( A \) of a smooth cubic threefold has an isolated singularity at the origin, so \( CC(\delta_\Theta) = \Lambda_\Theta + m \cdot \Lambda_{\{0\}} \) for some \( m \in \mathbb{Z} \). To compute the multiplicity \( m \), recall that by [8, sect. 13] we have a commutative diagram of rational maps

\[
\begin{array}{ccc}
S \times S & \xrightarrow{\Psi} & \Theta \\
\downarrow \Phi & & \downarrow \Phi \\
P^4 & \xrightarrow{\gamma_\Theta} & \Omega
\end{array}
\]

where \( \Psi \) has generic degree six and where the generic fibre of \( \Phi \) can be identified with the set of \( 27 \cdot 26 = 432 \) pairs of skew lines on the smooth cubic surface arising as the generic hyperplane section of the threefold. Note that there is a counting error in loc. cit. in the paragraph after (13.7): there all \( 27 \cdot 26 \) pairs of lines are considered, but the non-skew pairs are mapped by \( \Phi \) to a proper closed subset of \( P^4 \) and do not contribute to the generic fibre. With the corrected number we get \( \deg(\gamma_\Theta) = 432/6 = 72 \), so corollary [9] and the formula of Franecki and Kapranov imply
\[
CC(\delta_\Theta) = \Lambda_\Theta + m \cdot \Lambda_{\{0\}} \quad \text{with} \quad m = 78 - 72 = 6.
\]
The nontrivial contribution of the singular locus may be surprising since irreducible theta divisors have only mild singularities [11].

4. The monodromy of the Gauss map

Joint with the previously known cases, theorem [2] suggests that for any smooth subvariety \( Z \rightarrow A \) of a complex abelian variety \( A \), the Weyl group \( W_Z = W(G_Z) \) of the corresponding Tannaka group should be related to the monodromy of the Gauss map \( \gamma_Z : \Lambda_Z \rightarrow \Omega \) from remark [7]. By [25, 21, cor. 5.2] this Gauss map is dominant unless the subvariety \( Z \) is stable under translations by all points of some non-zero abelian subvariety, in which case \( G(\delta_Z) = \{1\} \). Discarding this trivial case, we may assume the Gauss map restricts over an open dense subset of \( \Omega \) to a finite étale cover of degree \( \deg(\gamma_Z) = \chi(\delta_Z) > 0 \). We define its monodromy group \( M_Z \) to be the Galois group of the Galois hull of this cover, which is isomorphic to the image of the monodromy representation on a general fibre of \( \gamma_Z \).
Theorem 8. We have the following monodromy and Weyl groups:

1. If $A$ is the Jacobian variety of a smooth projective curve $C \hookrightarrow A$ of genus $g$, then
   \[ M_C \cong W_C \cong \begin{cases} 
   (\pm 1)^{g-1} \rtimes S_{g-1} & \text{if } C \text{ is hyperelliptic,} \\
   S_{2g-2} & \text{otherwise.}
   \end{cases} \]

2. If $A$ is the intermediate Jacobian of a smooth cubic threefold with Fano surface $S \hookrightarrow A$, then
   \[ M_S \cong W_S \cong W(E_6). \]

3. If $A$ is a general principally polarized abelian variety of dimension $g > 1$ with theta divisor $\Theta \hookrightarrow A$, then
   \[ M_\Theta \cong (\pm 1)_0^{g} \rtimes S_r \quad \text{and} \quad W_\Theta \cong \begin{cases} 
   (\pm 1)^r \rtimes S_r & \text{if } 2 \mid g, \\
   (\pm 1)_0^{g} \rtimes S_r & \text{if } 2 \nmid g,
   \end{cases} \]
   where $r = g!/2$ and where $(\pm 1)^r = \{(\epsilon_1, \ldots, \epsilon_r) \in (\pm 1)^r \mid \epsilon_1 \cdots \epsilon_r = \pm 1\}.$

Proof. For the Weyl groups this follows from theorem [2] and [19] th. 6.1 [18], so it only remains to discuss the monodromy groups. For the Jacobian varieties in (1) we have $\Omega = H^0(C, \Omega^1_C)$ and
   \[ \Lambda_C = \{ (p, \omega) \in C \times \Omega \mid \omega(p) = 0 \} \hookrightarrow \mathcal{J}_A^* = A \times \Omega. \]

Let $\iota : C \to \mathbb{P}^* \Omega$ be the canonical map and $\overline{C} = \iota(C)$ its image. With the usual identification of points in a projective space and hyperplanes in the dual space, this gives the following factorization for the projectivized Gauss map:
   \[ \mathbb{P}\Lambda_C = \{ (p, H) \in C \times \mathbb{P}\Omega \mid \iota(p) \in H \} \xrightarrow{\alpha} \{ (\overline{\eta}, H) \in \mathbb{P}\overline{C} \times \mathbb{P}\Omega \mid \overline{\eta} \in H \} \xrightarrow{\beta} \mathbb{P}\Omega \]

By the uniform position principle [2] lemma on p. 111] the monodromy group of $\beta$ is the symmetric group $S_d$ of degree $d = \deg(\overline{C})$. In the non-hyperelliptic case we have $d = 2g - 2$ and $\alpha$ is an isomorphism, so we are done. It remains to discuss the hyperelliptic case. In that case $d = g - 1$ and $\alpha$ is the quotient by the hyperelliptic involution. So the general fibre of the Gauss map consists of $g - 1$ pairs of points that are interchanged under the hyperelliptic involution. The monodromy $M_C$ is then a subgroup of the semidirect product $(\pm 1)^{g-1} \rtimes S_{g-1}$ which surjects onto the quotient $S_{2g-2}$ via the induced permutation action on a general fibre of $\beta$. It remains to show that for any of the $g - 1$ pairs of points in a general fibre of the Gauss map, the group $M_C$ contains a permutation which interchanges the two points of this pair but fixes all the other points in the fibre. But this follows from the observation that through any branch point $\overline{\eta} \in \mathbb{P}\overline{C}$ of the double cover $C \to \mathbb{P}\overline{C} \cong \mathbb{P}^1$ there exists a hyperplane $H \in \mathbb{P}\overline{C}$ which does not meet any other branch point of this cover and over which furthermore the cover $\beta$ is unramified, meaning that this hyperplane is nowhere tangent to the curve $\overline{C}$. Hence part (1) follows.

For the intermediate Jacobian $A$ of a smooth cubic threefold $V$ in (2), recall that the Fano surface $S \subset A$ parametrizes the lines on the threefold. In fact there exists by [3] prop. 6] an embedding $V \hookrightarrow \mathbb{P}^* = \mathbb{P}T_0(A)$ such that for any $p \in S(\mathbb{C})$ the
corresponding line \( L_p \subset V \) is identified with the projective tangent space to the Fano surface at that point:

\[
\begin{array}{ccc}
L_p & \subset & V \\
\downarrow & & \downarrow \\
\mathbb{P}T_p(S) & \subset & \mathbb{P}T_p(A)
\end{array}
\]

Identifying points \( H \in \mathbb{P}\Omega \) with hyperplanes \( H \subset \mathbb{P}\Omega^* \) in the dual projective space, we obtain that

\[
\mathbb{P}\Lambda_S = \{(p, H) \in S \times \mathbb{P}\Omega \mid L_p \subset H \cap V \}.
\]

So the fibre of the Gauss map \( \gamma_S \) over a general point \( H \in \mathbb{P}\Omega \) is identified with the 27 lines on the smooth cubic surface \( H \cap V \), and the monodromy group \( M_S \) is the group of permutations of these 27 lines which is induced by variations of the hyperplane \( H \in \mathbb{P}\Omega \). By [22, VI.20] this group is \( W(E_6) \), so (2) follows.

For part (3) we take a translate of the theta divisor \( \Theta \subset A \) which is stable under the involution \( \iota = -id : A \rightarrow A \). The projectivized Gauss map factors over the quotient by this involution:

\[
\mathbb{P}\Lambda_\Theta = \Theta \xrightarrow{\alpha} \Theta/\langle \iota \rangle \xrightarrow{\beta} \mathbb{P}\Omega.
\]

From [10 sect. 1] we know that over a general point \( H \in \mathbb{P}\Omega \) in the branch locus of the generically finite map \( \beta \), the fibre \( \beta^{-1}(H) \) will consist of precisely \( r-1 \) distinct points where \( r = \deg(\beta) = g!/2 \). Furthermore \( \alpha \) is étale over the complement of the finitely many singular points of \( \Theta/\langle \iota \rangle \). So the restriction of the cover \( \beta \circ \alpha \) to a general line \( \mathbb{P}1 \hookrightarrow \mathbb{P}\Omega \) gives morphisms

\[
X = \Theta \times_{\mathbb{P}1} \mathbb{P}1 \xrightarrow{\alpha'} X/\langle \iota \rangle \xrightarrow{\beta'} \mathbb{P}1
\]

of smooth projective curves, where \( \alpha' \) is an étale double cover while \( \beta' \) is a simply branched cover. The Zariski-Lefschetz theorem for monodromy groups [14] implies that for a sufficiently general line \( \mathbb{P}1 \hookrightarrow \mathbb{P}\Omega \) the monodromy groups of the branched covers \( \alpha \circ \beta \) and \( \alpha' \circ \beta' \) coincide. Since the cover \( \alpha \circ \beta \) is irreducible, the corresponding monodromy operation is transitive. Hence the same also holds for \( \alpha' \circ \beta' \), so \( X \) is irreducible and our claim follows from [5 th. 1].

\[
\square
\]

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| $d$ | $A_2$ | $A_3$ | $A_4$ | $A_6$ | $A_8$ | $A_7$ | $B_2$ | $B_4$ | $C_2$ | $C_3$ | $C_4$ | $D_4$ | $D_5$ | $E_6$ | $F_4$ | $G_2$ |
|-----|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 2   | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    |
| 3   | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    |
| 4   | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    |
| 5   | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | -    | $\omega_2$ | -    | -    | -    | -    |
| 6   | $2\omega_1$ | $\omega_2$ | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 7   | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 8   | $\omega_1 + \omega_2$ | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 9   | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 10  | $3\omega_1$ | $2\omega_1$ | $\omega_2$ | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 11  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 12  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 13  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 14  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 15  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 16  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 17  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 18  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 19  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 20  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 21  | $5\omega_1$ | - | - | - | $2\omega_1$ | $\omega_2$ | - | $\omega_2$ | - | - | $2\omega_1$ | - | - | - | - | - |
| 22  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 23  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 24  | $3\omega_1 + \omega_2$ | - | - | $\omega_1 + \omega_4$ | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 25  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 26  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27  | $2\omega_1 + \omega_2$ | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 28  | $6\omega_1$ | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 29  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 30  | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |

Table 1. Up to duality, all irreducible representations of dimension $d \leq 30$ for the complex simple Lie algebras of type $\neq A_1$, with the exception of the defining representations of the classical Lie algebras of type $A_n$, $B_n$, $C_n$, $D_n$ with $d = n + 1$, $2n + 1$, $2n$, $2n$. We denote the representations by their highest weights, using the fundamental weights $\omega_1$, $\omega_2$, ... listed as in [6].
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