Racks and orbits of dressing transformations

A. A. Balinsky *

Technion-Israel Institute of Technology
Department of Mathematics
32000 Haifa, ISRAEL

Abstract

New algebraic structure on the orbits of dressing transformations of the quasitriangular Poisson Lie groups is provided. This gives the topological interpretation of the link invariants associated with the Weinstein–Xu classical solutions of the quantum Yang-Baxter equation. Some applications to the three-dimensional topological quantum field theories are discussed.

*e-mail: balin@leeor.technion.ac.il
Introduction.

Three-dimensional topological quantum field theories and especially Chern–Simons type theory (see [1, 2, 3, 4]) have been attracting interest of mathematicians and physicists. Many of them give us the new invariants of links and 3-manifolds. The article [2] gives an excellent account of main developments in knot theory which followed upon the discovery of the Jones polynomials [13] in 1984. But from our point of view, we nevertheless are far from the understanding of the topological meaning of the new invariants.

In the very deep paper [8] A. Weinstein and P. Xu defined a broad class of knot and links invariants using a kind of the classical solutions of the quantum Yang–Baxter equation. In [8] the classical analogue is developed for part of the standard construction in which generalized Jones invariants are produced from representations of quantum groups. As far as I know their article is the first attempt to understand the topological meaning of the quantum group invariants on the quasi-classical level. It was established in [11] that in the case of factorizable Poisson Lie group $G$ the Weinstein-Xu link invariant coincide with the space of link group representations in $G$. The general case of an arbitrary quasitriangular Poisson Lie group is connected with Joyce’s theory of knot quandles or fundamental racks.

Any codimension two link has a fundamental rack which contains more information than the fundamental group and which is a complete invariant for irreducible links in any 3-manifold.

Rack provides complete algebraic and topological framework in which to study links and knots in 3-manifolds. The finding rack structures inside of the quantum link invariants and of the three-dimensional topological quantum field theories looks like the finding hidden symmetries in the integrable equations. It was done in [9] for the topological quantum field theories associated to finite groups. I think that this is a very perspective area for the investigation and that the concept of rack gives us the powerful tools for description of the topological quantum field theories.

Our main goal in this paper is to give the rack structure for Poisson Lie group. This is the main ingredient in our interpretation of the Weinstein-Xu link invariant in the general case of an arbitrary quasitriangular Poisson Lie group.

1 Racks and Quandles

In this section we state some properties of the racks and quandles which will be used in this paper. For more details on this subject, see [5, 6, 7]. To simplify reading, we keep the notations of [5] wherever possible, on one hand, but give all necessary definitions, on the other.

Recall that a set with product is a pair $(\Delta, \ast)$ where $\Delta$ is a set and $\ast$ is a map $\Delta \times \Delta \to \Delta$. The value of this map for $(a, b)$ will be denoted by $a^b$ or by $a \ast b$. The reasons for writing the operation exponentially are explained in [5]. A morphism of sets with product $(\Delta, \ast) \to (\Delta', \ast')$ is a map $\phi : \Delta \to \Delta'$ such that $\phi(a \ast b) = \phi(a) \ast' \phi(b)$. For any set with product $(\Delta, \ast)$ and $b \in \Delta$ the right translation
$r_b$ is the map $r_b : \Delta \rightarrow \Delta$ defined by $r_b(a) = a \ast b = a^b$.

**Definition 1.1 ([5])** A rack is a non-empty set $\Delta$ with a product satisfying the following two axioms:

- Given $a, b \in \Delta$ there is a unique $c \in \Delta$ such that $a = c^b$.
- Given $a, b, c \in \Delta$ the formula
  \[ a^{bc} = a^{cb} \]
  holds.

Here $a^{bc}$ means $(a^b)^c$ and $a^{bc}$ means $a^{(bc)}$.

In other words, a set with product $(\Delta, \ast)$ is a rack, iff all right translations are automorphisms:

- $\forall a, b \in \Delta \exists! c \in \Delta \ a = c \ast b$
- $\forall a, b, c \in \Delta \ (a \ast c) \ast (b \ast c) = (a \ast b) \ast c$.

One can find many examples of automorphic sets in [4, 7].

The rack axioms are the algebraic distillation of two of the Reidemeister moves (the second and third moves).

**Definition 1.2 ([8, 14])** A map $R : S \times S \rightarrow S \times S$, where $S$ is any set, is called a solution to the set-theoretic quantum Yang–Baxter equation if

\[ R_{13} R_{23} R_{12} = R_{12} R_{23} R_{13}, \]

where $R_{ij} : S \times S \times S \rightarrow S \times S \times S$ is $R$ on the $i^{th}$ and $j^{th}$ factors of the cartesian product and $Id$ on the third one.

The following fact is crucial in the using of racks in low-dimensional topology.

**Lemma 1.3** If $\Delta$ is an rack then the map

\[ R : \Delta \times \Delta \rightarrow \Delta \times \Delta \quad (a, b) \mapsto (a, b^a) \]

is a solution to the set-theoretic Yang-Baxter equation.

Given a rack $\Delta$, we can get an action of the braid group $B_n$ on $(\Delta)^n$. More precisely, suppose that $R : \Delta \times \Delta \rightarrow \Delta \times \Delta$ is a solution to the set-theoretic Yang-Baxter equation from the Lemma above. Let $\hat{R} = R \circ \sigma$ with $\sigma : \Delta \times \Delta \rightarrow \Delta \times \Delta$ being the exchange of components, and let $\hat{R}_i(n)$ be the endomorphism of the cartesian power $\Delta^n$ defined by:

\[ \hat{R}_i(n)((x_1, \ldots, x_n)) = (x_1, \ldots, x_{i-1}, \hat{R}(x_i, x_{i+1}), x_{i+2}, \ldots, x_n). \]

Then by the assignment of $\hat{R}_i(n)$ to the $i^{th}$ generator $b_i$ of the braid group $B_n$ we obtain an action of $B_n$ on $\Delta^n$ for each $n$. 2
In what follows all the examples will be satisfied the identity
\[ a^a = a \text{ for all } a \in \Delta, \]
which we call the quandle condition. This condition is quarantine the first Reide-
meister move. We shall call a rack satisfying the quandle condition a quandle rack
or quandle. The term quandle is due to Joyce [6].

Finally, we recall a definition of Freyd and Yetter [15] (see also [5]).

**Definition 1.4** A (right) crossed \( G \)-set for a group \( G \) is a set, \( X \), with a right
action of the group \( G \), which we write \((x,g) \mapsto x \cdot g\) where \( x,x \cdot g \in X \) and \( g \in G \)
and a function \( \delta : X \rightarrow G \) satisfying the augmentation identity:
\[ \delta(a \cdot g) = g^{-1}(\delta(a))g \text{ for all } a \in X, g \in G, \]
which is precisely the same as saying that \( \delta \) is \( G \)-map when \( G \) is regarded as a right
\( G \)-set under right conjugation.

Given a crossed \( G \)-set \( X \), we can define an operation of \( X \) on itself by defining \( a^b \)
to be \( a \cdot \delta(b) \). One can easily check that the operation \((a,b) \mapsto a^b\) gives us the rack
structure on \( X \), which we call the augmented rack with augmentation \( \delta \). For more
details on the theory of augmented rack see [3].

## 2 Quasitriangular Poisson Lie Groups

Let \( G \) be a Poisson Lie group. This means that \( G \) is a Lie group equipped with a
Poisson structure \( \pi \) such that the multiplication in \( G \) viewed as map \( G \times G \rightarrow G \)
is a Poisson mapping, where \( G \times G \) carries the product Poisson structure. The theory
of Poisson Lie groups is a quasiclassical version of the theory of qua ntum groups.

One can easily check that the Poisson structure \( \pi \) must vanish at the identity \( e \in G \), so that its linearization \( d_e \pi : G \rightarrow \mathfrak{g} \wedge \mathfrak{g} \) at \( e \) is well defined ( here \( \mathfrak{g} \) is the
Lie algebra of \( G \) ). It turns out that this linear homomorphism is a 1– cocycle with
respect to the adjoint action. Moreover, the dual homomorphism \( \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^* \)
satisfies the Jacobi identity; i.e. \( \mathfrak{g}^* \) becomes a Lie algebra. Such a pair \((\mathfrak{g}, \mathfrak{g}^*)\)
is called a Lie bialgebra [16]. Each Lie bialgebra corresponds to a unique connected,
simply connected Poisson Lie group. It is easy to show that the pair \((\mathfrak{g}^*, \mathfrak{g})\) is a Lie
bialgebra as soon as \((\mathfrak{g}, \mathfrak{g}^*)\) is one. The Poisson Lie group \((\mathfrak{g}^*, \pi^*)\) corresponding to
\((\mathfrak{g}^*, \mathfrak{g})\) will be called dual to \((G, \pi)\). Thus ( connected, simply connected ) Poisson
Lie groups come in dual pairs.

\( \mathfrak{g} \) and \( \mathfrak{g}^\ast \) may be put as Lie subalgebras into the greater Lie algebra \( \hat{\mathfrak{g}} \) which is
called the double Lie algebra. A vector space \( \hat{\mathfrak{g}} \) equals \( \mathfrak{g} \oplus \mathfrak{g}^\ast \), with Lie bracket
\[ [X + \xi, Y + \eta] = [X, Y] + [\xi, \eta] + ad^\ast \xi \eta - ad^\ast \eta \xi + ad^\ast \xi Y - ad^\ast \eta Y \]
Here $X,Y \in \mathcal{G}$, $\xi, \eta \in \mathcal{G}^*$ and $ad^*$ denotes the coadjoint representations of $\mathcal{G}$ on $\mathcal{G}^*$ and of $\mathcal{G}^*$ on $\mathcal{G} = (\mathcal{G}^*)^*$. We use $[,]$ to denote both the bracket on $\mathcal{G}$ and $\mathcal{G}^*$.

With respect to $ad$–invariant non-degenerate canonical bilinear form

$$(X + \xi, Y + \eta) = \langle X, \eta \rangle + \langle Y, \xi \rangle$$

$\mathcal{G}$ and $\mathcal{G}^*$ form maximal isotropic subspaces of $\tilde{\mathcal{G}}$.

The simply connected group $\tilde{G}$ corresponding to $\tilde{\mathcal{G}}$ is the classical Drinfeld double of the Poisson Lie group $(G, \pi)$.

Conversely, any Lie algebra $\tilde{G}$ with a non–degenerate symmetric $ad$–invariant bilinear form and a pair of maximal isotropic subalgebras (a Manin triple) gives a pair of dual Lie bialgebras by identifying one of the subalgebras with the dual of the other by means of this bilinear form.

Let $r = \sum a_i \otimes b^i$ be an element of $\mathcal{G} \otimes \mathcal{G}$; we say that $r$ satisfies the classical Yang–Baxter equation if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$ 

Here, for instance, $[r_{12}, r_{13}] = \sum_{i,j} [a_i, a_j] \otimes b^i \otimes b^j$. A quasitriangular Lie bialgebra is a pair $(\mathcal{G}, r)$, where $\mathcal{G}$ is Lie bialgebra, $r \in \mathcal{G} \otimes \mathcal{G}$, the coboundary of $r$ is the cobracket $d_e \pi : \mathcal{G} \to \mathcal{G} \wedge \mathcal{G}$ and $r$ satisfies the classical Yang–Baxter equation.

Let us associate with $r$ a linear operator

$$r_+ : \mathcal{G}^* \to \mathcal{G}, \quad \xi \mapsto \langle r, \xi \otimes id \rangle.$$

Its adjoint is given by

$$-r_- = r_+^* : \mathcal{G}^* \to \mathcal{G}, \quad \xi \mapsto \langle r, id \otimes \xi \rangle = \langle P(r), \xi \otimes id \rangle,$$

where $P$ is the permutation operator in $\mathcal{G} \times \mathcal{G}$, $P(X \otimes Y) = Y \otimes X$.

The Lie bracket $[,]$ in $\mathcal{G}^*$ is given by

$$[\xi, \eta] = ad^*_{r_+(\xi)} \eta - ad^*_{r_-(\eta)} \xi.$$

**Lemma 2.1** ([8], [17]) For any quasitriangular Lie bialgebra $(\mathcal{G}, r)$, the linear maps

$$r_+, r_- : \mathcal{G}^* \to \mathcal{G},$$

defined above, are both Lie algebra homomorphisms.

We now turn our attention to groups.

**Definition 2.2** ([8]) A Poisson Lie group $G$ is called quasitriangular if its corresponding Lie bialgebra $(\mathcal{G}, \mathcal{G}^*)$ is quasitriangular and if the Lie algebra homomorphisms $r_+$ and $r_-$ from $\mathcal{G}^*$ to $\mathcal{G}$ lift to Lie group homomorphisms $R_+$ and $R_-$ from $G^*$ to $G$. 

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It turns out that if $G$ is quasitriangular, the maps $\phi$ and $\psi$ from $G^*$ to $G$ are Poisson morphisms, where $\phi(x) = R_+(x^{-1})$, $\psi(x) = R_-(x^{-1})$, for any $x \in G^*$. For every Poisson Lie group $G$ there are naturally defined left and right “dressing” actions of $G$ on $G^*$, whose orbits are exactly the symplectic leaves of $G^*$. When $G$ has the zero Poisson structure, its dual Poisson Lie group is simply $G^*$ with the abelian Lie group structure and ordinary Lie–Poisson bracket. The left and right dressing actions in this case are simply the left and right coadjoint actions of $G$ on $G^*$.

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3 Poisson Lie Rack

Let $G$ be a quasitriangular Poisson Lie group with Lie bialgebra $\mathcal{G}$, and let $G^*$ be its simply connected dual. We can lift the Lie algebra homomorphisms $r_\pm : \mathcal{G}^* \to \mathcal{G}$ to the group homomorphisms $R_\pm : G^* \to G$, and define the map $J : G^* \to G$ by $J(x) = R_+(x)(R_-(x))^{-1}$. The group $G$ is factorizable if $J$ is a global diffeomorphism. In this case for each element $x \in G$ we have a factorization $x = x_+ x_-^1$, where $x_\pm = R_\pm(J^{-1}(x))$.

The following Theorem gives us an augmented rack structure for $G^*$.

**Theorem 3.1** Let $G$ be a quasitriangular Poisson Lie group and let $G^*$ be its simply connected dual. Then $G^*$ has a structure of crossed $G$-set with (right) action

$$(x, g) \mapsto \lambda_g^{-1} x$$

and augmentation $\delta$:

$$\delta(x) = \phi(x^{-1})\psi(x).$$

This Theorem is the generalization for quasitriangular Poisson Lie group of the well-known fact that the left dressing action of $G$ on $G^*$, for factorizable Poisson Lie group $G$, coincides with the conjugation action $Ad_x$ under the identification of $G$ with $G^*$ by $J$.

**Proof.** It follows from Lemma 2.1 that the Lie algebra homomorphisms $r_\pm$ naturally extend to Lie algebra homomorphisms $f_\pm$ from the double Lie algebra $\tilde{G}$ onto $\mathcal{G}$ defined by: $f_\pm(X + \xi) = X + r_\pm\xi$. By $F_\pm$ we denote the Lie group homomorphism from the classical Drinfeld double $\tilde{G}$ of the Poisson Lie group $(G, \pi)$. For any
\[ d = gu = \bar{u}\bar{g} \in \bar{G} \text{ with } g, \bar{g} \in G \text{ and } u, \bar{u} \in G^*, \text{ we have} \]
\[ F_+ = g\phi(u^{-1}) = \phi(\bar{u}^{-1})\bar{g}, \]
and
\[ F_- = g\psi(u^{-1}) = \psi(\bar{u}^{-1})\bar{g}. \]
This implies that
\[ \phi(\bar{u}^{-1})\psi(\bar{u}) = g\phi(u^{-1})\psi(u)g^{-1}. \]
Finally, a straightforward calculation based on the identity \( \lambda_gu = \bar{u} \) shows that \( G^* \) has a structure of crossed \( G \)-set. Q.E.D.

It follows from Weinstein-Xu result (Lemma 8.5 from [8]) that the rack from Theorem 3.1 is the quandle.

**Definition 3.2** Let \( G \) be a quasitriangular Poisson Lie group and let \( G^* \) be its simply connected dual. The **Poisson Lie quandle** is \( G^* \) with the following rack operation:
\[ b^a = \lambda_{\psi(a^{-1})\phi(a)}b. \]
Symplectic Poisson Lie quandles are the symplectic leaves of \( G^* \) (orbits of “dressing” actions of \( G \) on \( G^* \)) with the same rack operation.

For any rack \( \Delta \) we have braid group \( B_n \) action on \( \Delta^n \). Recall that two braids give rise to equivalent links if and only if they are equivalent under Markov moves. There are two types of Markov moves: one is conjugation \( A \rightarrow BAB^{-1} \); the other is by increasing the number of strings in braid by a simple twist: \( A \rightarrow Ab_n^\pm \), for \( A \in B_n \), where \( b_n \) is \( n^{th} \) generator of \( B_{n+1} \). After an elementary calculation we get

**Lemma 3.3** Suppose that \( \Delta \) is a quandle. If \( A \in B_n \) and \( B \in B_m \) define equivalent links, then the fixed point sets of \( A \) on \( \Delta^n \) and of \( B \) on \( \Delta^m \) are isomorphic.

This implies that for the Poisson Lie quandle \( G^* \) we have link invariant as the fixed point set of the corresponding braid action on \( (G^*)^n \). These invariant is the space of the representations of the fundamental augmented rack of a link into Poisson Lie quandle (see Proposition 7.6 from [5]).

It turns out that this space of the representations of the fundamental augmented rack of a link into Poisson Lie quandle equal to the Weinstein-Xu link invariant, associated with the quasitriangular Poisson Lie group \( G \) [8]. This will be proofed in the next paper.

**Problem 1.** Find for the Poisson Lie quandle the analogue of the “exchange” and of the compatibility conditions for quantum R-matrix (see Theorem 5.4 from [8]).

It is well-known (S. Majid [20]) that a vector-space with a basis having the structure of crossed \( G \)-set is equivalent to a representation of the quantum double of the group algebra \( C[G] \) with the usual Hopf algebra structure.

**Problem 2.** Find the quantum analogue of the rack operation on \( G^* \).
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