Tetrahedralization of a Hexahedral Complex

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Abstract

Two important classes of three-dimensional elements in computational meshes are hexahedra and tetrahedra. While several efficient methods exist that convert a hexahedral element to a tetrahedral elements, the existing algorithm for tetrahedralization of a hexahedral complex is the marching tetrahedron algorithm which limits pre-selection of face divisions. We generalize a procedure for tetrahedralizing triangular prisms to tetrahedralizing cubes, and combine it with certain heuristics to design an algorithm that can triangulate any hexahedra.

\textbf{Keywords:} Finite element, Tetrahedra, Hexahedra, Triangulation

1. Introduction

Tetrahedralization of hexahedra has many uses: rendering engines may only process tetrahedra, discretization methods may require tetrahedra, and some geometric algorithms are only phrased over tetrahedra. It would be advantageous to convert a hexahedral mesh into as few tetrahedra as possible. However, the common algorithms described below cannot guarantee a conforming division of an arbitrary hexahedral complex, due to non-matching face divisions. We detail an algorithm to divide all hexahedra into five or six tetrahedra, except in an exceptional, degenerate case in which we use twelve tetrahedra.

2. Background

2.1. Marching Tetrahedron (MT): Limitations

In marching tetrahedra (Doi and Koide, 1991), each cube is split into six irregular tetrahedra by cutting the cube in half three times, cutting diagonally through each of the three pairs of opposing faces. In this way, the tetrahedra all share one of the main diagonals of the cube.

An obvious limitation of this algorithm is that the cuts are predetermined: that is, we cannot arbitrarily select cuts on any on the faces. We remedy this by augmenting the algorithm by introducing two types of division: \textit{prism decomposition} and \textit{five irregular tetrahedra}. Moreover, our algorithm allows the user to preselect any of the face cuts and can also be run in parallel across an entire mesh.

2.2. Prism Decomposition (PD)

We start by discussing the prism decomposition procedure we employ in the first three of our cases. (Erleben et al., 2005) provide an algorithm for triangulating a prism by choosing face cuts carefully. In particular, it first defines rising (R) and falling (F) cuts:

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For any of the cuts that are not degenerate, which we specify below, a canonical division into three tetrahedra is possible:

2.3. Degenerate Cases

Further, (Erleben et al., 2005) claim that the only the degenerate cases are FFF and RRR as presented below:

Indeed, the configurations of these face divisions can characterize the impossibility of triangulating the prism.

**Observation 2.1.** For a tetrahedral decomposition of a prism, at least two (exterior) face cuts must meet at some vertex.

**Proof.** Assume, for the sake of contradiction, that none of the face cuts meet at any vertex. Note that each vertex in a tetrahedra has three incident triangles, and consequently three incident edges. Each of the prism vertices must be present in at least two tetrahedra which share at most one face. This means that at least five faces, and thus five edges, must be incident to a vertex in a tetrahedra. However, if only at most one face cut meets a vertex, then combined with the three previous incident edges, the total number of incident edges can never exceed four, contradicting the fact that a vertex must have five incident edges in a triangulation.

(Erleben et al., 2005) “fixes” this issue by looking at the neighboring prisms and changing their configurations to transform these into the non-degenerate cases. Unfortunately, this may not be possible for some global mesh configurations.
3. General Hex-to-Tet: A General Algorithm for Tetrahedralizing a Hexahedral Complex

3.1. Generalizing Prism Decomposition to Cubes

A trivial observation that any cube can be divided into two prisms by simply cutting across a diagonal plane allows us to partially reduce arbitrary tetrahedron decomposition of a cube to a decomposition of prisms:

The above allows us to separately triangulate each prism with the main diagonal cut (red) serving as a face cut for both prisms. Recall that our main goal was to allow the user to arbitrarily select the face cuts across the six faces; our only freedom being able to choose the main diagonal. We claim that this procedure always works for cases where up to three cuts have been predetermined. For more than three predetermined cuts, the procedure works if the cuts are lined up accordingly.

For up to three predetermined cuts, we can use the prism decomposition method without running into the degenerate cases from Subsection 2.3 as we will always have at least two outside face cuts to choose from. We can choose these cuts in such a way that, along with the main diagonal, these cuts make sure that we get two cuts that meet at a vertex in each of the prisms, utilizing Observation 2.1. We will specify this case-by-case below:

3.1.1. Zero or One Predetermined Cut

We can simply run the MT algorithm here. Or, we can choose an arbitrary main diagonal along with the face cuts in each of the prisms so that PD can be performed.

3.1.2. Two Predetermined Cuts

If the two cuts are not opposite to one another, then we can again run MT. This is also possible if the two opposite cuts are either both falling or both rising.

Crucially, even in the case where the two cuts have opposite orientations, we can choose the main diagonal so that its ends meet at the endpoint of both predetermined cuts as follows:

Figure 6: Cutting with the opposite faces in the same orientation (blue) yields FFR for both prisms.

3.1.3. Three Predetermined Cuts

None of the cuts are opposite of one another. In this case, we should be able to select the opposite cuts for each of the three predetermined cuts. Hence, we can apply MT to get the canonical division.
A pair with opposite face cuts with same orientation. This is again trivial: we can simply use MT with one of the pairs having already determined.

A pair with opposite face cuts with different orientation.

1. Choose the uncut pair and decompose the cube into prism by cutting this pair to form the diagonal plane.
2. On whichever prism the third predetermined cut falls, the main diagonal is cut to avoid the prism degenerate case(s).
3. The second prism has an uncut external face, which is again used to avoid the prism degenerate case(s).

![Diagram of cube with cuts](image)

2. The red diagonal is chosen to configure the prism to FFR

1. Choose a dividing plane.

3. The blue cut is then picked to avoid the degenerate cases.

![Diagram of cube with cuts](image)

Figure 7: Two in three predetermined cuts with different orientation.

3.1.4. Four and Five Predetermined Cuts

Again, we will handle the easy cases first. Also, note that at least one of the pairs of cuts must be opposite to one another here (indeed, two of the pairs are opposite of another in the five case.)

At least one pair opposite of each other with the same orientation. This case is easy as we can choose such a pair to create the diagonal plane for decomposing the cube into prisms. If two of the remaining cuts lie on the same prism, then we use the diagonal cut to avoid the degenerate case which leaves us with two (or one) remaining cuts in another prism. If only one cut is present in any one of the prisms, then again we can easily avoid RRR/FFF cases.

Only one pair opposite of each other with different orientation. We can choose one of the adjacent predetermined cuts and cut across its opposite face with the same orientation. This allows us to choose a diagonal plane with one of the resulting prisms containing two of the predetermined cuts. Here, we can again use the main diagonal cut to escape the degenerate case, while the remaining prism has an extra uncut external face.
The red diagonal is chosen to configure the prism to FFR. The green cut is then picked to avoid the degenerate cases. The side opposite to the green cut is chosen so that two of the predetermined cuts get isolated in one of the prisms. Both pairs are opposite of one another with different orientation.

**Meet at a vertex:** If the pairs meet at a vertex while being in different orientation to their opposite cuts, then we can simply decompose into prisms using the remaining uncut pair (we can cut in the same orientation as the remaining determined cut in the five case), and the resulting prisms are obviously non-degenerate as the external faces meet at a vertex as shown below:

**None meet at a vertex** This is the degenerate case, where prism decomposition fails. Geometrically, it is equivalent to having a FFF/RRR case as in Observation 2.1 for cubes. Note that the remaining one/two cuts cannot save this from being degenerate:
Observation 3.1. If two of the opposite pairs are in FFFF/RRRR configuration, then prism decomposition fails.

3.1.5. Six Predetermined Cuts

Again, any two opposite cuts with the same orientation implies that prism decomposition works. Otherwise, if at least two opposite pairs have different orientation, then when all pairs meet at (some) vertices to one another, we have the following five tetrahedral decomposition:

Again, if any cut is isolated, then the cube cannot be triangulated as Observation 3.1 comes into play.

3.2. Solving the Degenerate Cases

As we outlined above in Observation 3.1, the cubes fail to be triangulated in the usual way only if one or more cut(s) are isolated. We conjecture that this cannot be resolved using one of the procedures above, and one of the following methods must be followed:

3.2.1. Flipping Neighboring Cubes

Recall that we had several degrees of freedom when choosing one of the cuts when decomposing and later when avoiding the prism degenerate cases. Indeed, the procedure above is invariant with respect to opposite face cuts of the same orientation. That is, we have

Theorem 1. Changing the orientation of a pair of opposite face cuts with the same orientation still yields a valid prism decomposition.
Proof. We want to show that triangulation is invariant under flipping opposite pairs with the same orientation. However, this is essentially changing the diagonal plane that yields those prisms.

Assume however that after flipping the orientation, one of the new prism acquires a degenerate configuration (RRR/FFF). In order for this to happen, we must be constrained to cut the main diagonal in some orientation $R$ (respectively $F$) for one of the prisms with other cuts having orientations $FF$ (respectively $RR$). The main diagonal will then have orientation $F$ (respectively $R$), for the other prism, yielding $FFF$ (respectively $RRR$). But, this implies that we started with a degenerate $FFFF/RRRR$ exterior face cuts, which must be impossible as this would not have yielded a valid decomposition before flipping.

In light of Theorem 1, in order to avoid the degenerate case for four predetermined case, note that changing the orientation of one of the external face cuts suffices.

Figure 13. Flipping opposite face cuts of adjacent cubes. This is an instance when flipping fails as all four adjacent cubes’ opposite faces have cuts with different orientations.

3.2.2. Steiner Points

Our last resort is introducing new vertices, called Steiner points, which presents an easy solution to the above problem as any number of predetermined cuts can be triangulated to form 12 tetrahedra (De Berg et al., 2000): All eight original vertices are connected to the Steiner point to decompose the cube into six pyramids. Any of the face cuts now yields two tetrahedra.
1. Connect each of the vertices with the steiner point.

2. This yields six pyramids, which is then cut across to get tetrahedra using the predetermined cuts (red).

*Figure 14.* Steiner points allows decomposition into twelve tetrahedra at the expense of an additional vertex.

### 3.3. The Main Algorithm

Below we present a more succinct version of the algorithm that covers all the cases above:

**Algorithm 1** Hex-to-Tet (A hexahedral mesh $M$)

1. while there exists a hexahedron $H$ that is unmarked do
2. $N \leftarrow$ number of cut exterior faces of $H$
3. if $N \geq 4$ with two pairs of opposite face cuts with different orientation then
4. if the pairs meet at a vertex then ▶ This is the five irregular tetrahedral decomposition.
5. while there exists an uncut face do
6. Cut the face so that the cut meets at the predetermined cut(s) at some vertex
7. end while
8. else
9. Degenerate-Case ($H$)
10. end if
11. else
12. Prism-Decomposition ($H$)
13. end if
14. mark $H$
15. end while
16. return “Done”

**Algorithm 2** Prism-Decomposition (Hexahedron $H$)

1. if there does not exist a pair of opposite cuts with the same orientation then
2. Cut one of the uncut pairs in this manner
3. end if
4. Cut across such pair to create a diagonal plane and two prisms.
5. if there exists a prism with two of the exterior faces cut then
6. Use the middle diagonal to avoid RRR/FFF
7. end if
8. Cut the remaining uncut faces of the prism to get valid decompositions
Algorithm 3 Degenerate-Case (Hexahedron $H$)

1: Get the adjacent cubes of the four faces with each pair having different opposite orientation
2: if any of the four cuts form an opposite cut pair with the same orientation in their adjacent cubes then
3:    Flip the cut, along with the opposite face cut in the adjacent cube $\triangleright$ The triangulation is preserved in the neighboring cube.
4: else
5:    Introduce a Steiner point $P$ in the middle of the cube.
6:    Connect each of the vertices of the cube with $P$ using six new interior edges
7:     while there exists an uncut exterior face do
8:        Cut the face so that the cut is in the same orientation as its opposite face $\triangleright$ This is to make sure that there is greater chance of flipping the cuts for adjacent cubes.
9:    end while
10: end if

4. Conclusion

In this paper we presented a triangulation algorithm for hexahedra; contrary to the existing algorithm (marching tetrahedra), our algorithm does not depend on a predefined set of face cuts. Our algorithm identifies the number of predetermined face divisions and uses an extension of prism decomposition algorithm and several other techniques to decomposition the hexahedra into tetrahedra. We have also shown that using a predefined set of face cuts restricts the triangulation in certain instances and the use of Steiner points is warranted to completely solve the triangulation problem for hexahedra.

In future work, we plan to implement the algorithm above in the PETSc (Balay et al., 2022a,b) libraries in order to convert meshes with tensor product cells to simplicial cells as part of its DMPlex mesh capabilities (Knepley and Karpeev, 2009; Lange et al., 2016; Knepley et al., 2017).

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References

Balay, S., Abhyankar, S., Adams, M.F., Benson, S., Brown, J., Brune, P., Buschelman, K., Constantinescu, E., Dalcin, L., Dener, A., Eijkhout, V., Gropp, W.D., Hapla, V., Isaac, T., Jolivet, P., Karpeev, D., Kaushik, D., Knepley, M.G., Kong, F., Kruger, S., May, D.A., McInnes, L.C., Mills, R.T., Mitchell, L., Munson, T., Roman, J.E., Rupp, K., Sanan, P., Sarich, J., Smith, B.F., Zampini, S., Zhang, H., Zhang, J., 2022a. PETSc/TAO Users Manual. Technical Report ANL-21/39 - Revision 3.17. Argonne National Laboratory.

Balay, S., Abhyankar, S., Adams, M.F., Benson, S., Brown, J., Brune, P., Buschelman, K., Constantinescu, E.M., Dalcin, L., Dener, A., Eijkhout, V., Gropp, W.D., Hapla, V., Isaac, T., Jolivet, P., Karpeev, D., Kaushik, D., Knepley, M.G., Kong, F., Kruger, S., May, D.A., McInnes, L.C., Mills, R.T., Mitchell, L., Munson, T., Roman, J.E., Rupp, K., Sanan, P., Sarich, J., Smith, B.F., Zampini, S., Zhang, H., Zhang, J., 2022b. PETSc Web page. URL: https://petsc.org/.

De Berg, M.T., Van Kreveld, M., Overmars, M., Schwarzkopf, O., 2000. Computational geometry: algorithms and applications. Springer Science & Business Media.

Doi, A., Koide, A., 1991. An efficient method of triangulating equi-valued surfaces by using tetrahedral cells. IEICE TRANSACTIONS on Information and Systems 74, 214–224.

Erlleben, K., Dohlmann, H., Sporring, J., 2005. The adaptive thin shell tetrahedral mesh.

Knepley, M.G., Karpeev, D.A., 2009. Mesh algorithms for PDE with Sieve I: Mesh distribution. Scientific Programming 17, 215–230. URL: http://arxiv.org/abs/0908.4427, doi:10.3233/SPR-2009-0249.

Knepley, M.G., Lange, M., Gorman, G.J., 2017. Unstructured overlapping mesh distribution in parallel. arXiv:1506.06194.

Lange, M., Mitchell, L., Knepley, M.G., Gorman, G.J., 2016. Efficient mesh management in Firedrake using PETSc-DMF. Parallel and Distributed Computing and Applications, Springer International Publishing, pp. 100–111, doi:10.1007/978-3-319-29141-7_4.