Entropy Optimization of Scale-Free Networks
Robustness to Random Failures

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Abstract

Many networks are characterized by highly heterogeneous distributions of links, which are called scale-free networks and the degree distributions follow \( p(k) \sim ck^{-\alpha} \). We study the robustness of scale-free networks to random failures from the character of their heterogeneity. Entropy of the degree distribution can be an average measure of a network’s heterogeneity. Optimization of scale-free network robustness to random failures with average connectivity constant is equivalent to maximize the entropy of the degree distribution. By examining the relationship of the entropy of the degree distribution, scaling exponent and the minimal connectivity, we get the optimal design of scale-free network to random failures. We conclude that the entropy of the degree distribution is an effective measure of network’s resilience to random failures.

\textit{Key words:} scale-free networks; information theory; entropy; random failures
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1 Introduction

Many complex systems are characterized by the network of interactions among its components. It has been shown that most of the networks are highly
heterogeneous in their connectivity patterns. Heterogeneity can be easily identified by looking at the degree distribution \( p(k) \), which gives the probability of having a node with \( k \) links. Most complex networks can be described by degree distributions \( p(k) \sim c k^{-\alpha} \), with \( \alpha \in (2, 3) \). These networks include social networks (such as movie-actor networks, science citations and cooperation networks), the Internet and the World Wide Web, metabolic networks and protein interaction networks, etc [1,2,3,4,5].

Since Albert et al. raised the question of random failures and intentional attack on networks [6], enormous interest has been devoted to the study of the resilience of networks to failures of nodes or to intentional attacks [7,8,9,10,11,12]. Magoni studied a general strategy of attacks on the Internet [13]. It is important to understand how to design networks which are optimally robust against both failures and attacks.

Most of researchers use percolation theory to study this problem [7,14]. A fraction \( p \) of the nodes and their connections are removed randomly—the integrity might be compromised: for \( \alpha > 3 \), there exists a certain threshold value \( p_c \). When \( p > p_c \), the network disintegrates into smaller, disconnected parts. Below that critical threshold value, there still exists a connected cluster. For \( 2 < \alpha < 3 \), the networks are more resilient and the threshold value \( p_c \) approaches 1 [7]. Also, some researchers work on optimizing network robustness to both random failures and attacks with percolation theory [15,16,17].

A simple but essential character of scale-free network is its heterogeneous link distribution. Moreover, heterogeneity is in direct relationship with the network’s resilience to attacks. Many real-world networks are scale-free and they are robust to random failures but vulnerable to targeted attacks. Heterogeneity can be measured by entropy [18,19]. Solé et al. used entropy of the remaining degree and mutual information to study some model networks of different heterogeneity and randomness [19].

In this paper, different from percolation theory, we try an alternative point of view—the entropy of the degree distribution to describe scale-free network’s heterogeneity. To determine the optimal design of scale-free network to random failures, we maximize the robustness of scale-free network to random failures while keeping the cost constant (that is, the average number of links per node remains constant). We find that optimization of scale-free network’s robustness to random failures is equivalent to maximize the entropy of the degree distribution. By optimizing the entropy of the degree distribution, we get the scenario of optimal design of scale-free network to random failures.
In general, scale-free networks have the degree distributions \( p(k), p(k) \sim ck^{-\alpha} \), \( k = m, m + 1, \ldots, K \), where \( m \) is the minimal connectivity and \( K \) is an effective connectivity cutoff present in finite networks. Then the entropy of the degree distribution can be defined as follows:

\[
H = - \sum_{k=1}^{N-1} p(k) \log p(k) \tag{1}
\]

where \( N \) is the total number of nodes in the network and the restriction to the degree distribution is \( p(k) \sim ck^{-\alpha} \). Within the context of complex networks, the entropy of the degree distribution provides an average measure of network's heterogeneity, since it measures the diversity of the link distribution \cite{19}. Two extreme cases are the maximal value and the minimal one. The maximum value is \( H_{\text{max}} = \log (N - 1) \) obtained for \( p(k) = \frac{1}{N-1} (\forall k = 1, 2, \ldots, N - 1) \). \( H_{\text{min}} = 0 \) occurs when \( p(k) = \{0, \ldots, 1, \ldots, 0\} \).

For the power-law degree distribution of scale-free network, the impact of diversity (long tails) is obvious, increasing the uncertainty. As the scaling exponent increases or the cut-off decreases, the network becomes less heterogeneous and as a result a lower entropy is observed \cite{19}.

In this paper, we study the entropy of the degree distribution for scale-free networks. With continuous approximation, for the power law degree distribution of scale-free network, entropy \( H \) can be expressed as follows:

\[
H = - \int_{m}^{K} p(k) \log p(k) dk
= - \int_{m}^{K} c k^{-\alpha} \log (ck^{-\alpha}) dk
= - \frac{c \log c}{1 - \alpha} \cdot (K^{1-\alpha} - m^{1-\alpha}) + \frac{c\alpha}{1 - \alpha} \cdot \left[ \log K \cdot K^{1-\alpha} - \log m \cdot m^{1-\alpha} \right]
- \frac{1}{1 - \alpha} \cdot (K^{1-\alpha} - m^{1-\alpha}) \tag{2}
\]

The largest connectivity \( K \approx mN^{\frac{1}{\alpha}} \) \cite{7}, The coefficient \( c \) is obtained from
\[1 = \int_{m}^{\infty} p(k)dk = \int_{m}^{\infty} c k^{-\alpha}dk = \frac{c}{1-\alpha} \cdot (-m^{1-\alpha}) \]  

(3)

and thus

\[c = (\alpha - 1) \cdot m^{\alpha - 1} \]  

(4)

Then entropy \( H \) can be expressed as a function of scaling exponent \( \alpha \), the minimal connectivity \( m \) and network scale \( N \).

\[H(\alpha, m, N) = \left( \log \left( \frac{\alpha - 1}{m} \right) + \frac{\alpha}{1-\alpha} \right) \cdot \left( \frac{1}{N} - 1 \right) - \frac{\alpha}{\alpha - 1} \cdot \log \frac{N}{N} \]  

(5)

Fig. 1 displays the relationship between entropy \( H \) and \( \alpha \) for different \( N \) when the minimal connectivity \( m \) is defined. We know that with the increase of scaling exponent \( \alpha \), the cut-off connectivity \( K \) decreases, which makes the network less heterogeneous. As a result, a lower entropy is observed. So different scaling exponent \( \alpha \) brings the network different degree of heterogeneity.

The average connectivity \( < k > \) is obtained with continuous approximation,

\[< k > = \int_{m}^{K} k \cdot p(k)dk = \int_{m}^{K} k \cdot c k^{-\alpha}dk = \frac{c}{2-\alpha} \cdot (K^{2-\alpha} - m^{2-\alpha}) \]  

(6)
and thus

\[ < k > \approx \frac{\alpha - 1}{2 - \alpha} \cdot m \cdot (N^{\frac{2}{\alpha-1}} - 1) \]  

(7)

According to the above results, we can formulate the scale-free network’s resilience to random failures problem as the following nonlinear mixed integer programming.

\[
\begin{align*}
\text{max } H(\alpha, m, N) \\
\text{s.t. } < k > = \text{const}
\end{align*}
\]  

(8)

The problem mentioned above is different from the usual maximizing the entropy of the network problem. Note that the highest entropy for a network with a given \( < k > \) is exponential random graph [20]. To get the optimal design of scale-free networks, we restrict the degree distribution to be a power-law distribution. The problem can be rewritten as follows:

\[
\begin{align*}
\text{max } H(\alpha, m, N) \\
\text{s.t. } & \frac{\alpha - 1}{2 - \alpha} \cdot m \cdot (N^{\frac{2}{\alpha-1}} - 1) = \text{const} \\
& 2 < \alpha < 3 \\
& 1 \leq m \leq (N - 1) \\
& m \in \text{integer}
\end{align*}
\]  

(9)

By solving the mixed integer programming problem, we get the optimal value \( H \), the optimal solutions \( \alpha \) and \( m \) for different average connectivity \( < k > \). The results are shown in Fig. 2.

From Fig. 2, we can see that, in general, with the increase of average connectivity \( < k > \), entropy \( H \) as well as the minimal connectivity \( m \) increases, too. In part, the optimal minimal connectivity \( m \) keeps constant for a certain range of \( < k > \), where scaling exponent \( \alpha \) decreases with the increase of \( < k > \). As a result, the optimal network becomes much more heterogeneous than the one with bigger scaling exponent.

To prove the entropy of the degree distribution is indeed an effective measure of network robustness under node removal, we examine the relationship
Fig. 2. The optimal value $H$, the optimal solutions $\alpha$ and $m$ versus $<k>$ with $N = 1000$.

between the entropy of the degree distribution and the network threshold value $p_c$, which was presented by Cohen et al. with percolation theory [7]. The criterion to calculate the percolation critical threshold value $p_c$ of a randomly connected network is:

$$1 - p_c = \frac{1}{\kappa_0 - 1}$$  \hspace{1cm} (10)

where $\kappa_0 = \frac{\langle k^2 \rangle}{\langle k \rangle}$ is calculated from the original distribution before the random failures. With $K \approx mN^{\frac{1}{\alpha - 1}}$, $\kappa_0$ can be expressed as follows:

$$\kappa_0 \approx \frac{\alpha - 2}{3 - \alpha} \cdot m \cdot (N^{\frac{3 - \alpha}{\alpha - 1}} - 1)$$  \hspace{1cm} (11)

Fig. 3. $p_c$ and $m$ versus $<k>$ with $N = 1000$. 

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In Fig. 3, $p_c$ is calculated from the optimal value $\alpha$ and $m$ which are obtained from maximizing $H$ for different average connectivity $< k >$. The results indicate that for some constant $m$, $p_c$ increases with the increase of $< k >$.

Since both $\alpha$ and $N$ affect network's resilience to random failures, we may ask how the network's robustness changes with $N$. We just examine the relationship between $p_c$ and $\alpha$ for different $N$. The results are shown in Fig. 4. From Fig. 4, we can see that with the increase of $\alpha$, $p_c$ decreases, while the threshold value $p_c$ becomes bigger with the increase of network scale $N$, which are consistent with the results shown in Fig. 1.

To get an intuitive understanding of the relationship between the two measures, we show the results of the threshold value $p_c$ and $H$ for different average connectivity $< k >$ in Fig. 5. It can be seen that $p_c$ and $H$ vary consistently, that is, $p_c$ increases with $H$ for different average degree $< k >$. Also, with the increase of $< k >$, both $p_c$ and $H$ get improved.

![Fig. 4. $p_c$ for different $N$ versus $\alpha$ with $m = 1$.](image)

![Fig. 5. The relationship between $p_c$ and $H$ for different average connectivity $< k >$. For all curves, $\alpha \in (2, 3)$, $N \in [10^4, 9 * 10^4]$.](image)
3 Conclusions

In real-world networks one is often interested in the question of how to design a robust network while keeping the cost constant. It is well known that scale-free networks are resilient to random failures for their heterogeneous links distributions.

In this paper, different from percolation theory to evaluate the threshold value \( p_c \), we use the entropy of the degree distribution to study the resilience of scale-free networks to random failures. Heterogeneity is a simple but essential character of scale-free networks and it is in direct relationship with the network’s resilience to random failures. This character motivates us to use the entropy of the degree distribution, which provides an average measure of heterogeneity to study the effect of random failures on scale-free networks. By optimizing the entropy of the power-law degree distribution, we get the optimal value of entropy \( H \), the optimal solutions of \( \alpha \) and \( m \) for different \( \langle k \rangle \). The results indicate that when the network scale \( N \) is given, the optimal solution \( m \) is constant for a certain range of \( \langle k \rangle \), in which range the scaling exponent \( \alpha \) decreases. So a higher entropy is obtained. Of course, large average connectivity brings high cost. We then examine the relationship between threshold value \( p_c \) and \( N \) for different \( \alpha \). The results are also consistent with that of the entropy of the degree distribution. We also show the relationship of the two measures—\( p_c \) and \( H \). The results indicate that \( p_c \) and \( H \) vary consistently, that is, \( p_c \) increases with \( H \) for a certain average degree \( \langle k \rangle \). Also, both \( p_c \) and \( H \) get improved with the increase of the average degree \( \langle k \rangle \).

We just try an alternative point of view to analyze the robustness of the network from its heterogeneity. The results indicate that the entropy of the degree distribution is an effective measure of scale-free network’s resilience to random failures. Deep discussions on the resilience of scale-free networks to targeted attacks will be made in the future.

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