IMPROVEMENT ON THE BLOW-UP FOR A WEAKLY COUPLED WAVE EQUATIONS WITH SCALE-INvariant DAMPING AND MASS AND TIME DERIVATIVE NONLINEARITY

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Abstract. An improvement of [18] on the blow-up region and the lifespan estimate of a weakly coupled system of wave equations with damping and mass in the scale-invariant case and with time-derivative nonlinearity is obtained in this article. Indeed, thanks to a better understanding of the dynamics of the solutions, we give here a better characterization of the blow-up region. Furthermore, the techniques used in this article may be extended to other systems and interestingly they simplify the proof of the blow-up result in [3] which is concerned with the single wave equation in the same context as in the present work.

1. Introduction

The weakly coupled system of semilinear wave equations in the presence of damping and mass terms in scale-invariant case with time derivative nonlinearity reads as follows:

\[
\begin{aligned}
  &u_{tt} - \Delta u + \frac{\mu_1}{1 + t}u_t + \frac{\nu_1^2}{(1 + t)^2}u = |\partial_t v|^p, \quad (x, t) \in \mathbb{R}^N \times [0, \infty), \\
  &v_{tt} - \Delta v + \frac{\mu_2}{1 + t}v_t + \frac{\nu_2^2}{(1 + t)^2}v = |\partial_t u|^q, \quad (x, t) \in \mathbb{R}^N \times [0, \infty), \\
  &u(x, 0) = \varepsilon f_1(x), \quad v(x, 0) = \varepsilon f_2(x), \quad x \in \mathbb{R}^N, \\
  &u_t(x, 0) = \varepsilon g_1(x), \quad v_t(x, 0) = \varepsilon g_2(x), \quad x \in \mathbb{R}^N,
\end{aligned}
\]  

(1.1)

where \(\mu_1, \mu_2, \nu_1^2\) and \(\nu_2^2\) are nonnegative constants. The positive parameter \(\varepsilon\) characterizes the size of the initial data, and \(f_1, f_2, g_1\) and \(g_2\) are positive functions assumed to be compactly supported on \(B_{\mathbb{R}^N}(0, R), R > 0\).

Naturally, along this article, we suppose that \(p, q > 1\).

First, we recall the Glassey exponent \(p_G\) which is given by

\[
p_G = p_G(N) := 1 + \frac{2}{N - 1}.
\]

(1.2)

It is well-known that the aforementioned critical value, \(p_G\), characterizes the threshold between the global existence \((p > p_G)\) and the nonexistence \((p \leq p_G)\) regions; see e.g.
In this paragraph, we recall some results related to the massless case for a single equation. We start by mentioning the blow-up result for the solution of a single equation inherited from (1.1) without mass term. Indeed, Lai and Takamura showed in [13] an upper bound estimate of the lifespan. Later, in [18], Palmieri and Tu enhanced this result by extending the blow-up region for \( p \) in the case of a single equation (with mass term). More precisely, they obtain a blow-up result for \( p \in (1, p_G(N + \sigma(\mu, 0))) \) where

\[
\sigma = \sigma(\mu, \nu) := \begin{cases} 
\mu + 1 - \sqrt{\delta} & \text{if } \delta \in [0, 1), \\
\mu & \text{if } \delta \geq 1,
\end{cases}
\]

and \( \delta \) is given by (1.5) below. Note that the result in [18] was recently improved in [6] by extending the upper bound for \( p \) from \( p_G(N + \sigma(\mu, 0)) \) to \( p_G(N + \mu) \).

Recently, in [3], the case of the scale-invariant damped equation with mass and time-derivative nonlinearity was studied; this reads

\[
\begin{cases} 
\frac{u_{tt} - \Delta u}{1+t} + \frac{\mu}{u_t} + \frac{\nu^2}{(1+t)^2} u = |u_t|^p, & \text{in } \mathbb{R}^N \times [0, \infty), \\
u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbb{R}^N.
\end{cases}
\]

Let us introduce the following quantity, that we assume to be positive,

\[
\delta = \delta(\mu, \nu) := (\mu - 1)^2 - 4\nu^2.
\]

Furthermore, we define

\[
\sigma = \sigma(\mu, \nu) := \begin{cases} 
\mu + 1 - \sqrt{\delta} & \text{if } \delta \in [0, 1), \\
\mu & \text{if } \delta \geq 1.
\end{cases}
\]

Using a functional approach, a blow-up result is proven in [3] for (1.4), improving thus the one obtained in [18]. In fact, the blow-up interval, \( p \in (1, p_G(N + \sigma)) \) (\( \sigma \) is given by (1.6)), is ameliorated in comparison with [18], to show the blow-up inside the region \( p \in (1, p_G(N + \mu)] \), with \( \delta \in (0, 1) \). However, for \( \delta \geq 1 \), the two results, in [3] and [18], are the same. In relationship with these works, we also mention the articles [6, 13] where the massless case is investigated.
Now, letting $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$ in (1.1) we find the following coupled system:

\[
\begin{align*}
&u_{tt} - \Delta u = |\partial_t v|^p, \\
&v_{tt} - \Delta v = |\partial_t u|^q, \\
&u(x,0) = \varepsilon f_1(x), \quad v(x,0) = \varepsilon f_2(x), \quad x \in \mathbb{R}^N, \\
&u_t(x,0) = \varepsilon g_1(x), \quad v_t(x,0) = \varepsilon g_2(x), \quad x \in \mathbb{R}^N.
\end{align*}
\]

(1.7)

For the global existence of solutions to (1.7), we refer the reader to [11]. However, the blow-up of (1.7) has been the subject of several works; see e.g. [1, 9, 15, 16]. More precisely, the critical (in the sense of interface between blow-up and global existence) curve for $p, q$ is given by

\[
\Upsilon(N, p, q) := \max(\Lambda(N, p, q), \Lambda(N, q, p)) = 0,
\]

where

\[
\Lambda(N, p, q) := \frac{p+1}{pq-1} - \frac{N-1}{2}.
\]

Under some assumptions, the solution $(u, v)$ of (1.7) blows up in finite time $T(\varepsilon)$ for small initial data (of size $\varepsilon$), namely

\[
T(\varepsilon) \leq \begin{cases} 
C\varepsilon^{-T(N,p,q)} & \text{if } \Upsilon(N, p, q) > 0, \\
\exp(C\varepsilon^{-(pq-1)}) & \text{if } \Upsilon(N, p, q) = 0, \ p \neq q, \\
\exp(C\varepsilon^{-(p-1)}) & \text{if } \Upsilon(N, p, q) = 0, \ p = q.
\end{cases}
\]

(1.10)

In the context of the present work, Palmieri and Tu [18] proved a blow-up result for the system (1.1). More precisely, the authors in [18] proved that there is blow-up for the system (1.1) for $p, q$ satisfying

\[
\Omega(N, \sigma_1, \sigma_2, p, q) := \max(\Lambda(N + \sigma_1, p, q), \Lambda(N + \sigma_2, q, p)) \geq 0,
\]

where $\Omega$ is given by (1.9) and $\sigma_i = \sigma(\mu_i, \nu_i), i = 1, 2$ is given by (1.6).

Indeed, for small initial data (of size $\varepsilon$), the solution $(u, v)$ of (1.1) blows up in finite time $T(\varepsilon)$ that is bounded as

\[
T(\varepsilon) \leq \begin{cases} 
C\varepsilon^{-\Omega(N,\sigma_1,\sigma_2,p,q)} & \text{if } \Omega(N, \sigma_1, \sigma_2, p, q) > 0, \\
\exp(C\varepsilon^{-(pq-1)}) & \text{if } \Omega(N, \sigma_1, \sigma_2, p, q) = 0, \\
\exp(C\varepsilon^{-\min\left(\frac{2q-1}{pq-1}, \frac{p-1}{pq-1}\right)}) & \text{if } \Lambda(N + \sigma_1, p, q) = \Lambda(N + \sigma_2, p, q) = 0.
\end{cases}
\]

(1.12)

Concerning the case of coupled equations, namely (1.1) without mass terms ($\nu_1 = \nu_2 = 0$), an improvement was obtained in [5] for the blow-up results and the lifespan. The results in [5] are ameliorating the ones in [18].

In this article, we refine the results in [18] when at least one of the coefficients $\delta_i, i = 1, 2$, is in $[0, 1)$. In fact, for positive values of $\delta_i (i = 1, 2)$, we extend the results obtained
in [18] to show that the new blow-up region does not depend on the mass parameters \( \nu_i, i = 1, 2 \) (and hence nor on \( \sigma_i, i = 1, 2 \)). A similar observation was concluded for the case of a one equation that we studied in [3], and the aim here is to extend this examination to the case of the coupled system (1.1). But, although the obtaining of similar results for damped coupled systems with mass terms is predictable, the situation is somehow more delicate. Indeed, the techniques used in [3] are now longer effective for the system (1.1). To overcome this difficulty, we first write the linear problem associated with (1.1) which reduces in this case to the following single equation:

\[
w_{tt}^L - \Delta w^L + \frac{\mu_i}{1+t}w_t^L + \frac{\nu_i^2}{(1+t)^2}w^L = 0,
\]

where \( w \) stands for \( u^L \) or \( v^L \), the solutions of the linear problem associated with (1.1). Clearly, the equation (1.13) is invariant under the following transformation:

\[
\tilde{w}^L(x,t) = w^L(\alpha x, \alpha(1+t) - 1), \quad \alpha > 0.
\]

Taking advantage of the aforementioned invariance properties, we refine our choice for a functional family that is indexed by a positive parameter \( \eta \). This judicious choice implies a better description of the dynamics of the solution of (1.1). More precisely, we obtain, for \( \eta \) large enough, the coercivity of the functional that will be introduced later on to show the blow-up results.

Thanks to the above observations, we finally enhance the result on the blow-up region, defined by (1.11) and obtained in [18], by showing that the critical curve is characterized by

\[
\Omega(N, \mu_1, \mu_2, p, q) = \max(\Lambda(N + \mu_1, p, q), \Lambda(N + \mu_2, q, p)) \geq 0,
\]

where \( \Lambda \) is given by (1.9).

The outline of this article is presented as follows. First, we introduce in Section 2 the weak formulation of (1.1) in the energy space. Then, in the same section, we state our main result. Some technical lemmas are proven in Section 3. Finally, we show the proof of the main result in Section 4.

2. MAIN RESULT

The aim of this section is to state our main result for which we will write the equivalent of the system (1.1) in the corresponding energy space. More precisely, the weak formulation associated with (1.1) reads as follows:

**Definition 2.1.** We say that \((u, v)\) is an energy solution of (1.1) on \([0, T)\) if

\[
\left\{
\begin{array}{l}
u, v \in C([0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n)), \\
u_t \in L^q_{loc}((0, T) \times \mathbb{R}^n), \quad v_t \in L^p_{loc}((0, T) \times \mathbb{R}^n)
\end{array}
\right.
\]

\]
satisfies, for all $\Phi, \tilde{\Phi} \in C_0^\infty(\mathbb{R}^N \times [0, T])$ and all $t \in [0, T)$, the following equations:

\[
\begin{align*}
\int_{\mathbb{R}^N} u_t(x,t)\Phi(x,t)dx - \int_{\mathbb{R}^N} u_t(x,0)\Phi(x,0)dx - \int_0^t \int_{\mathbb{R}^N} u_t(x,s)\Phi_t(x,s)dxds \\
+ \int_0^t \int_{\mathbb{R}^N} \nabla u(x,s) \cdot \nabla \Phi(x,s)dxds &+ \int_0^t \int_{\mathbb{R}^N} \frac{\mu_1}{1+s}u(x,s)\Phi(x,s)dxds \\
+ \int_0^t \int_{\mathbb{R}^N} \frac{\nu_1^2}{(1+s)^2} u(x,s)\Phi(x,s)dxds = \int_0^t \int_{\mathbb{R}^N} |v_t(x,s)|^p\Phi(x,s)dxds,
\end{align*}
\]

(2.1)

and

\[
\begin{align*}
\int_{\mathbb{R}^N} v_t(x,t)\tilde{\Phi}(x,t)dx - \int_{\mathbb{R}^N} v_t(x,0)\tilde{\Phi}(x,0)dx - \int_0^t \int_{\mathbb{R}^N} v_t(x,s)\tilde{\Phi}_t(x,s)dxds \\
+ \int_0^t \int_{\mathbb{R}^N} \nabla v(x,s) \cdot \nabla \tilde{\Phi}(x,s)dxds &+ \int_0^t \int_{\mathbb{R}^N} \frac{\mu_2}{1+s}v(x,s)\tilde{\Phi}(x,s)dxds \\
+ \int_0^t \int_{\mathbb{R}^N} \frac{\nu_2^2}{(1+s)^2} v(x,s)\tilde{\Phi}(x,s)dxds = \int_0^t \int_{\mathbb{R}^N} |u_t(x,s)|^q\tilde{\Phi}(x,s)dxds,
\end{align*}
\]

(2.2)

together with the conditions $u(x,0) = \varepsilon f_1(x)$ and $v(x,0) = \varepsilon f_2(x)$ being satisfied in $H^1(\mathbb{R}^N)$.

After some elementary computations, (2.1) and (2.2) can be written, respectively, in the following way

\[
\begin{align*}
\int_{\mathbb{R}^N} [u_t(x,t)\Phi(x,t) - u(x,t)\Phi_t(x,t) + \frac{\mu_1}{1+t}u(x,t)\Phi(x,t)]dx \\
\int_0^t \int_{\mathbb{R}^N} u(x,s) \left[ \Phi_t(x,s) - \Delta \Phi(x,s) - \frac{\partial}{\partial s} \left( \frac{\mu_1}{1+s} \Phi(x,s) \right) + \frac{\nu_1^2}{(1+s)^2} \Phi(x,s) \right]dxds \\
= \int_0^t \int_{\mathbb{R}^N} |u_t(x,s)|^p\Phi(x,s)dxds + \varepsilon \int_{\mathbb{R}^N} [-f_1(x)\Phi_t(x,0) + (\mu_1f_1(x) + g_1(x))\Phi(x,0)]dx,
\end{align*}
\]

and

\[
\begin{align*}
\int_{\mathbb{R}^N} [v_t(x,t)\tilde{\Phi}(x,t) - v(x,t)\tilde{\Phi}_t(x,t) + \frac{\mu_2}{1+t}v(x,t)\tilde{\Phi}(x,t)]dx \\
\int_0^t \int_{\mathbb{R}^N} v(x,s) \left[ \tilde{\Phi}_t(x,s) - \Delta \tilde{\Phi}(x,s) - \frac{\partial}{\partial s} \left( \frac{\mu_2}{1+s} \tilde{\Phi}(x,s) \right) + \frac{\nu_2^2}{(1+s)^2} \tilde{\Phi}(x,s) \right]dxds \\
= \int_0^t \int_{\mathbb{R}^N} |v_t(x,s)|^q\tilde{\Phi}(x,s)dxds + \varepsilon \int_{\mathbb{R}^N} [-f_2(x)\tilde{\Phi}_t(x,0) + (\mu_2f_2(x) + g_2(x))\tilde{\Phi}(x,0)]dx.
\end{align*}
\]

(2.3)

(2.4)

Remark 2.1. Since $f$ and $g$ are supported on $B_{\mathbb{R}^N}(0, R)$, one can see that $\text{supp}(u), \text{supp}(v) \subset \{(x,t) \in \mathbb{R}^N \times [0, \infty) : |x| \leq t + R\}$. Consequently, one can choose any test function $\Phi$ which is not necessarily compactly supported.

In the following, we state the main result of this article.
Theorem 2.2. Let \( p, q > 1 \). For \( i = 1, 2 \), let \( \mu_i, \nu_i^2 > 0 \), with \( \delta_i := \delta(\mu_i, \nu_i) > 0 \) (see (1.5)), such that

\[
\Omega(N, \mu_1, \mu_2, p, q) \geq 0,
\]

where \( \Omega \) is defined by (1.11).

Assume that \( f_1, f_2 \in H^1(\mathbb{R}^N) \) and \( g_1, g_2 \in L^2(\mathbb{R}^N) \) are non-negative functions which are compactly supported on \( B_{\mathbb{R}^N}(0, R) \), and do not vanish everywhere. Furthermore, we suppose that

\[
\frac{\mu_i - 1 - \sqrt{\delta_i}}{2} f_i(x) + g_i(x) \geq 0, \quad i = 1, 2.
\]

Let \((u, v)\) be an energy solution of (2.1)-(2.2) on \([0, T_\varepsilon)\) such that \( \text{supp}(u), \text{supp}(v) \subset \{(x, t) \in \mathbb{R}^N \times [1, \infty) : |x| \leq t+R\} \). Then, there exists a constant \( \varepsilon_0 = \varepsilon_0(f_1, f_2, g_1, g_2, N, R, p, q, \mu_1, \mu_2) > 0 \) such that \( T_\varepsilon \) verifies

\[
T(\varepsilon) \leq \begin{cases} 
C\varepsilon^{-\Omega(N, \mu_1, \mu_2, p, q)} & \text{if } \Omega(N, \mu_1, \mu_2, p, q) > 0, \\
\exp(C\varepsilon^{-(pq-1)}) & \text{if } \Omega(N, \mu_1, \mu_2, p, q) = 0, \\
\exp(C\varepsilon^{-\min\left(\frac{pq-1}{pq+1}, \frac{pq-1}{q+1}\right)}) & \text{if } \Lambda(N + \mu_1, p, q) = \Lambda(N + \mu_2, q, p) = 0.
\end{cases}
\]

where \( C \) is a positive constant independent of \( \varepsilon \) and \( 0 < \varepsilon \leq \varepsilon_0 \).

Remark 2.2. The blow-up result in Theorem 2.2 exhibits the new region obtained for the critical curve for \( p, q \) as a shift of the dimension by \( \mu_1, \mu_2 \). We believe that this new blow-up region delimitation coincides with the critical one. Of course, a rigorous confirmation should be proved by a global existence result.

Remark 2.3. The result in Theorem 2.2 holds true for the following generalized system:

\[
\begin{cases}
\begin{align*}
u_{tt} - \Delta u + \left( b_1(t) + \frac{\mu_1}{1+t} \right) u_t + \left( c_1(t) + \frac{\nu_1^2}{(1+t)^2} \right) u &= |\partial_x v|^p, & (x, t) \in \mathbb{R}^N \times [0, \infty), \\
\nu_{tt} - \Delta v + \left( b_2(t) + \frac{\mu_2}{1+t} \right) v_t + \left( c_2(t) + \frac{\nu_2^2}{(1+t)^2} \right) v &= |\partial_x u|^q, & (x, t) \in \mathbb{R}^N \times [0, \infty),
\end{align*}
\end{cases}
\]

where \( b_i(t), (1+t)c_i(t) \) belong to \( L^1(0, \infty) \) and \( \mu_i, \nu_i \) are such that \( \delta_i > 0; \ i = 1, 2 \).

The proof of the generalized damping case (2.8) can be performed by mimicking the one of Theorem 2.2 with the necessary modifications.

Remark 2.4. The techniques used in this article can be of course adapted in other contexts. More precisely, the case of a single equation corresponding to (1.1) can be simplified by taking into account the invariance property (1.14) which is related to the introduction of the parameter \( \eta \). Furthermore, one can use the aforementioned techniques to study systems like (1.1) with mixed nonlinearities, this will be the subject of a forthcoming work which will somehow constitute an extension of our previous works [4, 5, 6], see also [13, 17].
3. Some auxiliary results

We first introduce a positive test function which is defined as

\[(3.1) \quad \psi_\eta^i(x,t) := \rho_\eta^i(t)\phi_\eta(x), \ i = 1, 2, \ \forall \ \eta > 0,\]

where

\[(3.2) \quad \phi_\eta(x) := \begin{cases} \int_{S^{N-1}} e^{\eta x} \omega d\omega & \text{for } N \geq 2, \\ e^{\eta x} + e^{-\eta x} & \text{for } N = 1. \end{cases}\]

Note that the function \(\phi_\eta(x)\) is introduced in [25] and \(\rho_\eta^i(t), [15, 19, 21, 22]\), is solution

\[(3.3) \quad \frac{d^2 \rho_\eta^i(t)}{dt^2} - \frac{d}{dt} \left( \frac{\mu_i}{1 + t} \rho_\eta^i(t) \right) + \left( \frac{\nu_i^2}{(1 + t)^2} - \eta^2 \right) \rho_\eta^i(t) = 0, \ i = 1, 2.\]

From the literature, it is well-known that the expression of \(\rho_\eta^i(t)\) is given by (see the Appendix for more details),

\[(3.4) \quad \rho_\eta^i(t) = (\eta(t + 1))^{\frac{\mu_i + 1}{2}} K_{\frac{\nu_i}{\mu_i}}(\eta(t + 1)), \ i = 1, 2,\]

where

\[K_{\nu}(t) = \int_0^\infty \exp(-t \cosh \zeta) \cosh(\nu \zeta) d\zeta, \ \nu \in \mathbb{R}.\]

Furthermore, we have that the function \(\phi_\eta(x)\) satisfies

\[\Delta \phi_\eta = \eta^2 \phi_\eta.\]

One can easily see that the function \(\psi_\eta^i(x,t)\) fulfills the following conjugate equation:

\[(3.5) \quad \partial_t^2 \psi_\eta^i(x,t) - \Delta \psi_\eta^i(x,t) - \frac{\partial}{\partial t} \left( \frac{\mu_i}{1 + t} \psi_\eta^i(x,t) \right) + \frac{\nu_i^2}{(1 + t)^2} \psi_\eta^i(x,t) = 0.\]

In what follows and through this article, the constant \(C\) stands for any generic positive number which may depend on the data \((p, q, \mu_i, N, R, f_i, g_i)_{i=1,2}\) but not on \(\varepsilon\) and whose value may change from line to line. However, in some occurrences and when it is necessary, we will precise the dependence of the constant \(C\) on the parameters involved in this work.

Now, we state without proof the following lemma which gives a useful estimate for the function \(\psi_\eta^i(x,t)\).

**Lemma 3.1 ([25]).** Let \(r > 1\). Then, there exists a constant \(C = C(\eta, N, R, r) > 0\) such that

\[(3.6) \quad \int_{|x| \leq t + R} (\phi_\eta(x))^r \leq Ce^{rt}(1 + t)^{\frac{(2-r)(N-1)}{2}}, \ \forall \ t \geq 0.\]
In order to show the blow-up result later on, the following functionals are introduced here.

\begin{align}
F_1^\eta(t) := e^{-\eta t} \int_{\mathbb{R}^N} u(x,t)\phi^\eta(x)dx, \quad F_2^\eta(t) := e^{-\eta t} \int_{\mathbb{R}^N} v(x,t)\phi^\eta(x)dx,
\end{align}

and

\begin{align}
\tilde{F}_1^\eta(t) := e^{-\eta t} \int_{\mathbb{R}^N} \partial_t u(x,t)\phi^\eta(x)dx, \quad \tilde{F}_2^\eta(t) := e^{-\eta t} \int_{\mathbb{R}^N} \partial_t v(x,t)\phi^\eta(x)dx,
\end{align}

where \( \eta \) is a positive constant that will be determined later on. We also define the multiplier \( m_i(t) \) for \( i = 1, 2 \) as follows:

\begin{align}
m_i(t) = (1 + t)^{\mu_i}, \quad i = 1, 2.
\end{align}

Hence, the next two lemmas give the first lower bounds for \( F_i^\eta(t) \) and \( \tilde{F}_i^\eta(t), \quad i=1,2, \) respectively.

**Lemma 3.2.** Assume that the assumption in Theorem 2.2 holds. Then, we have

\begin{align}
F_i^\eta(t) & \geq 0, \quad \text{for all } t \in [0, T), \quad i = 1, 2,
\end{align}

for all \( \eta \geq \eta_0 \) where \( \eta_0 \) is given by

\begin{align}
\eta_0 := 1 + \max(|\nu_1|, |\nu_2|).
\end{align}

**Proof.** Let \( t \in [0, T) \). We first employ Definition 2.1, perform an integration by parts in space in the fourth term in the left-hand side of (2.1) and then choose \( \psi_1^\eta(x,t) \) as a test function\(^1\), we obtain that

\begin{align}
m_1(t) \int_{\mathbb{R}^N} u_t(x,t)\psi_1^\eta(x,t)dx - \varepsilon \int_{\mathbb{R}^N} g_1(x)\psi_1^\eta(x,0)dx
\end{align}

\begin{align}
+ \int_0^t m_1(s) \int_{\mathbb{R}^N} \left\{ u_t(x,s)\psi_1^\eta(x,s) - \eta^2 u(x,s)\psi_1^\eta(x,s) \right\} dx ds
\end{align}

\begin{align}
+ \int_0^t \int_{\mathbb{R}^N} \frac{\nu_1^2 m_1(s)}{(1 + s)^2} u(x,s)\psi_1^\eta(x,s)dx ds
\end{align}

\begin{align}
= \int_0^t m_1(s) \int_{\mathbb{R}^N} |v_t(x,s)|^p \psi_1^\eta(x,s)dx ds,
\end{align}

where \( m_1(t) \) is defined by (3.9).

Using the fact that

\begin{align}
\int_0^t m_1(s) \frac{dF_1^\eta(s)}{ds}ds = - \int_0^t m'_1(s)F_1^\eta(s)ds + m_1(t)F_1^\eta(t) - F_1^\eta(0),
\end{align}

\(^1\)Note that it is possible to consider here not compactly supported test functions thanks to the support property of \( u \). Indeed, it is sufficient to replace \( \psi_1^\eta(x,t) \) by \( \psi_1^\eta(x,t)\chi(x,t) \) where \( \chi \) is compactly supported such that \( \chi(x,t) \equiv 1 \) on \( \text{supp}(u) \).
and the definition of $F_1^\eta$, the equation (3.12) gives
\begin{equation}
(3.13)
\frac{dF_1^\eta}{dt}(t) + (1 + \eta)F_1^\eta(t) = \int_0^t \left\{ \frac{\mu_1}{1 + s} - \frac{\nu_1^2}{(1 + s)^2} - \eta + \eta^2 \right\} m_1(s)F_1^\eta(s)ds \\
+ \int_0^t m_1(s) \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1^\eta(x, s)dx ds,
\end{equation}
where
\begin{equation}
C_0^\eta(f_1, g_1) := \int_{\mathbb{R}^N} \{f_1(x) + g_1(x)\} \phi^\eta(x)dx.
\end{equation}

Multiplying (3.13) by $e^{(1+\eta)t}/m_1(t)$, we deduce after integrating over $[0, t]$ that
\begin{equation}
(3.14)
F_1^\eta(t) \geq F_1^\eta(0)e^{-(1+\eta)t} + \varepsilon C_0^\eta(f_1, g_1) \int_0^t \frac{e^{(1+\eta)(s-t)}}{m_1(s)}ds \\
+ \int_0^t \frac{\mu_1 e^{(1+\eta)(s-t)}}{m_1(s)} \int_0^s \left\{ \frac{\mu_1}{1 + \eta} - \frac{\nu_1^2}{(1 + \tau)^2} - \eta + \eta^2 \right\} m_1(\tau)F_1^\eta(\tau)d\tau ds.
\end{equation}
Since $\eta \geq \eta_0$, then we have $-\nu_1^2 - \eta + \eta^2 > 0$.
Thanks to (3.14) and the information that $F_1^\eta(0) > 0$, we deduce that $F_1^\eta(0) > 0, \forall t \geq 0$; see [12, Sec. 3].
Similarly, one can prove that $F_2^\eta(t)$ is bounded by zero from below thanks to the fact that $\eta \geq \eta_0$.
This ends the proof of Lemma 3.2. $\square$

The next step consists in proving the positivity of the functional $\tilde{F}_i^\eta(t)$ which is subject of the following lemma.

**Lemma 3.3.** Under the assumption as in Theorem 2.2, it holds that
\begin{equation}
(3.15)
\tilde{F}_i^\eta(t) \geq 0, \quad \text{for all } t \in [0, T), \quad \eta \geq \eta_0, \quad i = 1, 2,
\end{equation}
where $\eta_0$ is given by (3.11).

**Proof.** Let $t \in [0, T)$. Using the definition of $F_1^\eta$ and $\tilde{F}_1^\eta$, given respectively by (3.20) and (3.21), and the fact that
\begin{equation}
(3.16)
\frac{dF_1^\eta}{dt}(t) + \eta F_1^\eta(t) = \tilde{F}_1^\eta(t),
\end{equation}
the equation (3.13) yields
\begin{equation}
(3.17)
m_1(t)(\tilde{F}_1^\eta(t) + F_1^\eta(t)) - \varepsilon C_0^\eta(f_1, g_1) = \int_0^t \left\{ \frac{\mu_1}{1 + s} - \frac{\nu_1^2}{(1 + s)^2} - \eta + \eta^2 \right\} m_1(s)F_1^\eta(s)ds \\
+ \int_0^t m_1(s) \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1^\eta(x, s)dx ds.
\end{equation}
Differentiating the equation (3.17) in time and using (3.16), we obtain

\[
\frac{d}{dt} \left\{ \tilde{F}_1^\eta(t)m_1(t) \right\} + m_1(t)\tilde{F}_1^\eta(t) = \left\{ -\frac{\nu_1^2}{(1+t)^2} + \eta^2 \right\} m_1(t)F_1^\eta(t) \\
+ m_1(t) \int_{\mathbb{R}^N} |v_t(x,t)|^p \psi_1^\eta(x,t) dx.
\]

(3.18)

Thanks to the fact that \( \eta \geq \eta_0 \), we can ignore the right-hand side in (3.18) which is now positive. Then, the identity (3.18) yields

\[
\frac{d}{dt} \left\{ \tilde{F}_1^\eta(t)m_1(t)e^t \right\} \geq 0.
\]

(3.19)

An analogous estimate to (3.19) can be derived for \( F_2^\eta(t) \) as well. Finally, since \( \tilde{F}_i^\eta(0) > 0, i = 1, 2 \), we conclude the proof of Lemma 3.3. \( \square \)

**Remark 3.1.** One can note that the lower bounds in Lemmas 3.2 and 3.3 are not optimal. However, the aforementioned results are sufficient to prove our main result since we only need the positivity of \( F_i^\eta(t) \) and \( \tilde{F}_i^\eta(t), i = 1, 2 \), for all \( t > 1 \). To enhance these lower bounds, we will instead introduce new functionals as we will see in the next lemmas.

Although the computations in the rest of this article can be carried out for all \( \eta \geq \eta_0 \), we choose from now to set \( \eta = \eta_0 \) and ignore the dependence on \( \eta \) for all the functions (already introduced or which will be later on) that will be subsequently used. For example, the function \( \psi_i^\eta \) will be simply denoted by \( \psi_i \), and the same for all the other functions (including the constants) unless otherwise specified.

The following functionals can be now introduced and will be used subsequently in the proof of the blow-up criteria later on,

\[
G_1(t) := \int_{\mathbb{R}^N} u(x,t)\psi_1(x,t) dx, \quad G_2(t) := \int_{\mathbb{R}^N} v(x,t)\psi_2(x,t) dx,
\]

and

\[
\tilde{G}_1(t) := \int_{\mathbb{R}^N} \partial_t u(x,t)\psi_1(x,t) dx, \quad \tilde{G}_2(t) := \int_{\mathbb{R}^N} \partial_t v(x,t)\psi_2(x,t) dx.
\]

(3.20) \quad (3.21)

The aim of the next two lemmas is to prove that the functions \( \varepsilon^{-1}G_i(t) \) and \( \varepsilon^{-1}\tilde{G}_i(t) \) are coercive. Indeed, as we will see later on in (3.22) and (3.35) below, we will improve the lower bounds already obtained for the functionals \( F_i(t) \) and \( \tilde{F}_i(t) \). This improvement will be useful in the proof of the main result of this article.

Although the techniques used here are somehow close to the ones in our previous work [6] (which studies the one single equation corresponding to (1.1)), but, the situation is slightly different for the system (1.1). So, we will include all the details about the proofs of the next two lemmas. However, we will only show the proofs for the solution \( u \), and for \( v \) the computations follow similarly.
Lemma 3.4. Assume that the assumptions in Theorem 2.2 hold. Let \((u, v)\) be an energy solution of (2.1)-(2.2). Then, for \(i = 1, 2\), there exists \(T_0 = T_0(\mu_i, \nu_i, \eta_0) > 1\) such that we have

\[
G_i(t) \geq C_{G_i} \varepsilon, \quad \text{for all } t \in [T_0, T),
\]

where \(C_{G_i}\) is a positive constant which depends on \(f_i, g_i, N, R, \eta_0\) and \(\mu_i, \nu_i\).

Proof. Let \(t \in [0, T)\). Replacing \(\Phi\) by \(\psi_1\) in (2.3) and employing (3.5) yield

\[
\int_{\mathbb{R}^N} \left[ u_t(x, t)\psi_1(x, t) - u(x, t)\frac{\partial \psi_1(x, t)}{\partial t} + \frac{\mu_1}{1 + t} u(x, t)\psi_1(x, t) \right] dx
\]

\[
= \int_0^t \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1(x, s) dx ds + \varepsilon C_1(f_1, g_1),
\]

where

\[
C_1(f_1, g_1) := \int_{\mathbb{R}^N} \left[ (\mu_1 \rho_1(0) - \rho'_1(0)) f_1(x) + \rho_1(0) g_1(x) \right] \phi(x) dx.
\]

Now, using the definition of \(\rho_1\), given by (3.4), and (5.2), we deduce that

\[
\mu_1 \rho_1(0) - \rho'_1(0) = \frac{\mu_1 - 1 - \sqrt{\delta_1}}{2} K_{\frac{N}{2}}(1) + K_{\frac{N}{2} + 1}(1).
\]

Therefore, we obtain that

\[
C_1(f_1, g_1) = K_{\frac{N}{2}}(1) \int_{\mathbb{R}^N} \left[ \left( \frac{\mu_1 - 1 - \sqrt{\delta_1}}{2} f_1(x) + g_1(x) \right) \phi(x) dx \right] + K_{\frac{N}{2} + 1}(1) \int_{\mathbb{R}^N} f_1(x) \phi(x) dx.
\]

Thanks to the hypotheses in Theorem 2.2, namely the positivity of the initial data and (2.6), the constant \(C_1(f_1, g_1)\) is positive.

Recall (3.20) and (3.1), the equation (3.23) gives

\[
G_1'(t) + \Gamma_1(t) G_1(t) = \int_0^t \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1(x, s) dx ds + \varepsilon C_1(f_1, g_1),
\]

where

\[
\Gamma_1(t) := \frac{\mu_1}{1 + t} - 2 \frac{\rho'_1(t)}{\rho_1(t)}.
\]

Now, we multiply (3.27) by \(\frac{(1 + t)^{\mu_1}}{\rho_1^2(t)}\) and integrate over \((0, t)\), we deduce that

\[
G_1(t) \geq \frac{G_1(0)}{\rho_1^2(0)} \frac{\rho_1^2(t)}{(1 + t)^{\mu_1}} + \varepsilon C_1(f_1, g_1) \frac{\rho_1^2(t)}{(1 + t)^{\mu_1}} \int_0^t (1 + s)^{\mu_1} ds.
\]
Observing that $G_1(0) = \varepsilon K_{1/2}(\eta_0) \int_{\mathbb{R}^N} f_1(x)\phi(x)dx > 0$ and employing (3.4), the estimate (3.29) yields

$$G_1(t) \geq \varepsilon C_1(f_1,g_1)(1 + t)K^2_{1/2}(\eta_0(t + 1)) \int_{t/2}^t \frac{1}{(1 + s)K^2_{1/2}(\eta_0(s + 1))}ds.$$  

(3.30)

Thanks to (5.3), we deduce the existence of $T^1_0 = T^1_0(\mu_1, \nu_1, \eta_0) > 1$ such that, for all $t \geq T^1_0$,

$$\left\{ \begin{array}{l}
(1 + t)K^2_{1/2}(\eta_0(t + 1)) > \frac{\pi}{4\eta_0}e^{-2\eta_0(t+1)}, \\
\text{and} \\
(1 + t)^{-1}K^{-2}_{1/2}(\eta_0(t + 1)) > \frac{\eta_0}{\pi}e^{2\eta_0(t+1)}.
\end{array} \right.$$  

(3.31)

Combining (3.30) and (3.31), we infer that

$$G_1(t) \geq \frac{\varepsilon}{4\eta_0}C_1(f_1,g_1)e^{-2\eta_0t} \int_{t/2}^t e^{2\eta_0s}ds \geq \frac{\varepsilon}{8\eta_0}C_1(f_1,g_1)e^{-2\eta_0t}(e^{2\eta_0t} - e^{\eta_0t}), \forall t \geq T^1_0.$$  

(3.32)

Finally, using $e^{2\eta_0t} > 2e^{\eta_0t}, \forall t \geq 1$, we conclude that

$$G_1(t) \geq \frac{\varepsilon}{16\eta_0}C_1(f_1,g_1), \forall t \geq T^1_0.$$  

(3.33)

Similarly, we have an analogous estimate to (3.33) for $G_2(t)$, namely

$$G_2(t) \geq \frac{\varepsilon}{16\eta_0}C_2(f_2,g_2), \forall t \geq T^2_0.$$  

(3.34)

Hence, it suffices to set $T_0 = \max(T^1_0, T^2_0)$ to achieve the proof of Lemma 3.4. \hfill \Box

In the following, we will prove a lower bound for the functional $\tilde{G}_i(t)$, defined by (3.21). This will be the subject of the next lemma.

**Lemma 3.5.** Suppose that the assumptions in Theorem 2.2 are fulfilled. Let $(u,v)$ be an energy solution of (2.1)-(2.2). Then, for $i = 1, 2$, there exists $T_i = T_i(\mu_i, \nu_i, \eta_0) > 1$ such that

$$\tilde{G}_i(t) \geq C_{\tilde{G}_i}\varepsilon, \text{ for all } t \in [T_i, T),$$  

(3.35)

where $C_{\tilde{G}_i}$ is a positive constant which depends on $f_i, g_i, N, R, \mu_i$ and $\eta_0$.

**Proof.** The proof of the lemma will be carried out for $u$. The same conclusion can be similarly perfomed for $v$.

Let $t \in [0, T)$. Thanks to the definition of $G_1$ and $\tilde{G}_1$, given by (3.20) and (3.21), respectively, and using (3.1) together with the following identity

$$\frac{d\tilde{G}_1}{dt}(t) - \frac{\rho'_1(t)}{\rho_1(t)}G_1(t) = \tilde{G}_1(t),$$  

(3.36)
the equation (3.27) gives
\begin{equation}
\tilde{G}_1(t) + \left( \frac{\mu_1}{1 + t} - \frac{\rho_1'(t)}{\rho_1(t)} \right) G_1(t)
= \int_0^t \int_{\mathbb{R}^N} |v_1(x, s)|^p \psi_1(x, s) dx \, ds + \varepsilon C_1(f_1, g_1).
\end{equation}

Now, taking the time-derivative of the equation (3.37), we infer that
\begin{equation}
\frac{d\tilde{G}_1}{dt}(t) + \left( \frac{\mu_1}{1 + t} - \frac{\rho_1'(t)}{\rho_1(t)} \right) \frac{dG_1}{dt}(t) - \left( \frac{\mu_1}{(1 + t)^2} + \frac{\rho_1''(t)\rho_1(t) - (\rho_1'(t))^2}{\rho_1(t)} \right) G_1(t)
= \int_{\mathbb{R}^N} |v_1(x, t)|^p \psi_1(x, t) dx.
\end{equation}

Employing (3.3) and (3.36), the equation (3.38) yields
\begin{equation}
\frac{d\tilde{G}_1}{dt}(t) + \left( \frac{\mu_1}{1 + t} - \frac{\rho_1'(t)}{\rho_1(t)} \right) \tilde{G}_1 - \left( \eta_0^2 - \frac{\nu_1^2}{(1 + t)^2} \right) G_1(t) = \int_{\mathbb{R}^N} |v_1(x, t)|^p \psi_1(x, t) dx.
\end{equation}

Using (3.28), the definition of \( \Gamma_1(t) \), we conclude that
\begin{equation}
\frac{d\tilde{G}_1}{dt}(t) + \frac{3\Gamma_1(t)}{4} \tilde{G}_1(t) \geq \Sigma_1^1(t) + \Sigma_1^2(t) + \Sigma_1^3(t),
\end{equation}
where
\begin{equation}
\Sigma_1^1(t) := \left( \frac{-\rho_1'(t)}{2\rho_1(t)} - \frac{\mu_1}{4(1 + t)} \right) \left( \tilde{G}_1(t) + \left( \frac{\mu_1}{1 + t} - \frac{\rho_1'(t)}{\rho_1(t)} \right) G_1(t) \right),
\end{equation}
\begin{equation}
\Sigma_1^2(t) := \left( \eta_0^2 - \frac{\nu_1^2}{(1 + t)^2} \right) + \left( \frac{\rho_1'(t)}{2\rho_1(t)} + \frac{\mu_1}{4(1 + t)} \right) \left( \frac{\mu_1}{1 + t} - \frac{\rho_1'(t)}{\rho_1(t)} \right) G_1(t),
\end{equation}
and
\begin{equation}
\Sigma_1^3(t) := \int_{\mathbb{R}^N} |v_1(x, t)|^p \psi_1(x, t) dx.
\end{equation}

Combining the use of (3.37) and (5.4), we have the existence of \( T_1^1 = T_1^1(\mu_1) \geq T_0 \) such that
\begin{equation}
\Sigma_1^1(t) \geq \frac{\eta_0^2}{4} C_1(f_1, g_1) + \frac{\eta_0}{4} \int_0^t \int_{\mathbb{R}^N} |v_1(x, s)|^p \psi_1(x, s) dx \, ds, \quad \forall \, t \geq T_1^1.
\end{equation}

Employing Lemma 3.2 and (5.4), one can obtain the existence of a time \( \bar{T}_1^1 = \bar{T}_1^1(\mu_1) \geq T_1^1 \) for which we have
\begin{equation}
\Sigma_1^2(t) \geq 0, \quad \forall \, t \geq \bar{T}_1^1.
\end{equation}
Gathering all the above results, namely (3.40), (3.43), (3.44) and (3.45), we end up with the following estimate

\[
\frac{d\tilde{G}_1}{dt}(t) + \frac{3\Gamma_1(t)}{4}\tilde{G}_1(t) \geq \frac{\eta_0\varepsilon}{4} C_1(f_1, g_1) + \frac{\eta_0}{4} \int_0^t \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1(x, s)dxds
+ \int_{\mathbb{R}^N} |v_t(x, t)|^p \psi_1(x, t)dx, \quad \forall \ t \geq \tilde{T}_1^1.
\]

At this level, we can eliminate the nonlinear terms\(^2\) and we write

\[
\frac{d\tilde{G}_1}{dt}(t) + \frac{3\Gamma_1(t)}{4}\tilde{G}_1(t) \geq \frac{\eta_0\varepsilon}{4} C_1(f_1, g_1), \quad \forall \ t \geq \tilde{T}_1^1.
\]

Integrating (3.47) over \((\tilde{T}_1^1, t)\) after multiplication by \(\frac{t^{3\mu_1/4}}{\rho_1(t)}\), we obtain

\[
\tilde{G}_1(t) \geq \tilde{G}_1(\tilde{T}_1^1) \left(1 + \tilde{T}_1^1\right)^{3\mu_1/4} \rho_1^{3/2}(t) \int_{\tilde{T}_1^1}^t \left(1 + s\right)^{3\mu_1/4} \frac{1}{t^{3\mu_1/4}} ds, \quad \forall \ t \geq \tilde{T}_1^1.
\]

Recall that \(\tilde{G}_1(t) = \rho_1(t)e^t \tilde{F}^{\eta_0}(t)\), where \(\tilde{F}^{\eta_0}(t)\) is given by (3.8), and using Lemma 3.3 we deduce that \(\tilde{G}_1(t) \geq 0\) for all \(t \in [0, T]\).

Hence, the fact that \(\tilde{G}_1(t)\) is nonnegative together with the definition of \(\rho_1(t)\), given by (3.4), yield

\[
\tilde{G}_1(\tilde{T}_1^1) \left(1 + \tilde{T}_1^1\right)^{3\mu_1/4} \rho_1^{3/2}(\tilde{T}_1^1) \geq 0, \quad \forall \ t \in [0, T).
\]

Using (3.4), (3.31) and (3.49), the estimate (3.48) implies that

\[
\tilde{G}_1(t) \geq C \varepsilon e^{-3t/2} \int_{t/2}^t e^{3s/2}ds, \quad \text{for all} \ t \geq T_1^1 := 2\tilde{T}_1^1.
\]

Consequently, we see that

\[
\tilde{G}_1(t) \geq C \varepsilon, \quad \forall \ t \geq T_1^1.
\]

Note that, similarly for \(v\), we obtain the existence of \(T_1^2 = T_1^2(\mu_2) > 1\). Finally, by setting \(T_1 = \max(T_1^1, T_1^2)\), we conclude the proof of Lemma 3.5. \(\square\)

4. PROOF OF THEOREM 2.2.

In this section we will prove Theorem 2.2. For that purpose, we will make use of the results obtained in Section 3. In fact, thanks to the invariance of the linear problem

\(^2\)In fact, for a subsequent use in the proof of the main result, we choose here to keep the nonlinear terms up to this step in our computations. Otherwise, omitting the nonlinear terms can be done earlier in the proof of this lemma.
associated with (1.1) and the coercive properties of $\tilde{G}_i(t)$, as stated in Lemma 3.5, we will show the blow-up result of (1.1). Note that the techniques used in our previous works [3, 5] cannot be entirely followed here. By introducing some new functionals $L_1(t)$ and $L_2(t)$ (see (4.1) and (4.2) below), which verify two integral inequalities similar to the ones in [5], we improve the blow-up result in [18] for the solution of (1.1).

Let

$$L_1(t) := \frac{1}{8} \int_{T_2}^{t} \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1(x, s) dx ds + \frac{C_3 \varepsilon}{8},$$

and

$$L_2(t) := \frac{1}{8} \int_{T_2}^{t} \int_{\mathbb{R}^N} |u_t(x, s)|^q \psi_2(x, s) dx ds + \frac{C_3 \varepsilon}{8},$$

where $C_3 = \min(C_1(f_1, g_1)/4, C_2(f_2, g_2)/4, 8C_{\tilde{G}_1}, 8C_{\tilde{G}_2})$ (see Lemma 3.5 for the constants $C_{\tilde{G}_1}$ and $C_{\tilde{G}_2}$) and $T_2 := T_2(\mu_1, \mu_2) > T_1$ is a positive time such that $\frac{\eta_0}{4} - \frac{3\Gamma_1(t)}{32} > 0$ and $\Gamma_i(t) > 0$, for $i=1,2$, for all $t \geq T_2$; see (3.28) and (5.4).

Now, we introduce

$$\mathcal{F}_i(t) := \tilde{G}_i(t) - L_i(t), \quad \forall \ i = 1, 2.$$ 

Hence, thanks to (3.46), we see that $\mathcal{F}_1$ satisfies

$$\mathcal{F}_1'(t) + \frac{3\Gamma_1(t)}{4} \mathcal{F}_1(t) \geq \left( \frac{\eta_0}{4} - \frac{3\Gamma_1(t)}{32} \right) \int_{T_2}^{t} \int_{\mathbb{R}^N} |v_t(x, s)|^p \psi_1(x, s) dx ds$$

$$+ \frac{7}{8} \int_{\mathbb{R}^N} |v_t(x, t)|^p \psi_1(x, t) dx + C_3 \left( \frac{\eta_0}{4} - \frac{3\Gamma_1(t)}{32} \right) \varepsilon$$

$$\geq 0, \quad \forall \ t \geq T_2.$$

After multiplying (4.3) by $\frac{\rho_1(t)^{3/4}}{\rho_1^{3/4}(t)}$ and integrating over $(T_2, t)$, we obtain

$$\mathcal{F}_1(t) \geq \mathcal{F}_1(T_2) \frac{(T_2)^{3/4} \rho_1^{3/4}(t)}{\rho_1^{3/4}(t)} \frac{T_2^{3/4}}{(T_2)^{3/4}}, \quad \forall \ t \geq T_2,$$

where $\rho_1(t)$ is defined by (3.4).

Using Lemma 3.5 and $C_3 = \min(C_1(f_1, g_1)/4, C_2(f_2, g_2)/4, 8C_{\tilde{G}_1}, 8C_{\tilde{G}_2}) \leq 8C_{\tilde{G}_1}$, one can see that $\mathcal{F}_1(T_2) = \tilde{G}_1(T_2) - \frac{C_3 \varepsilon}{8} \geq C_{\tilde{G}_1} \varepsilon - \frac{C_3 \varepsilon}{8} \geq 0$.

Consequently, we infer that

$$\tilde{G}_1(t) \geq L_1(t), \quad \forall \ t \geq T_2.$$

In a similar way, we have an analogous lower bound for $\tilde{G}_2(t)$, that is

$$\tilde{G}_2(t) \geq L_2(t), \quad \forall \ t \geq T_2.$$
Employing the Hölder’s inequality together with the estimates (3.6) and (3.35), a lower bound for the nonlinear term can written as
\begin{align}
\int_{\mathbb{R}^N} |v_t(x,t)|^p \psi_1(x,t) dx & \geq (\mathcal{G}_2(t))^p \left( \int_{|x| \leq t + R} \left( \psi_2(x,t) \right)^{\frac{p}{2}} \left( \psi_1(x,t) \right)^{-\frac{1}{2}} dx \right)^{-(p-1)} \\
& \geq C(\mathcal{G}_2(t))^p \rho_1(t) \rho_2^{-p}(t) e^{-(p-1)t} t^{-\frac{(N-1)(p-1)}{2}}.
\end{align}

From (3.4) and (3.31), observe that
\begin{align}
\rho_1(t)e^t \leq Ct^\frac{N}{2}, \quad \forall \ t \geq T_0/2.
\end{align}

Note that similar estimate holds for \(\rho_2(t)\).

Combining (4.8) (and the equivalent estimate for \(\rho_2(t)\)) and (4.7), we deduce that
\begin{align}
\int_{\mathbb{R}^N} |v_t(x,t)|^p \psi_1(x,t) dx \geq C t^{-\frac{(N-1)(p-1) + \frac{\mu_1}{2} - \frac{\mu_2}{2} p}{2}} (\mathcal{G}_2(t))^p, \quad \forall \ t \geq T_2.
\end{align}

Now, recall the definition of \(L_1(t)\), given by (4.1), and injecting (4.6) in (4.9), we conclude that
\begin{align}
L'_1(t) \geq C t^{-\frac{(N-1)(p-1) + \frac{\mu_1}{2} - \frac{\mu_2}{2} p}{2}} (L_2(t))^p, \quad \forall \ t \geq T_2.
\end{align}

Likewise, we have
\begin{align}
L'_2(t) \geq C t^{-\frac{(N-1)(q-1) + \frac{\mu_1}{2} - \frac{\mu_2}{2} q}{2}} (L_1(t))^q, \quad \forall \ t \geq T_2.
\end{align}

A straightforward integration of (4.10) and (4.11) on \((T_2, t)\) yields, respectively,
\begin{align}
L_1(t) & \geq \frac{C_3 \varepsilon}{8} + C \int_{T_2}^t (1 + s)^{-\frac{(N-1)(p-1) + \frac{\mu_1}{2} - \frac{\mu_2}{2} p}{2}} (L_2(s))^p ds, \quad \forall \ t \geq T_2,
\end{align}

and
\begin{align}
L_2(t) & \geq \frac{C_3 \varepsilon}{8} + C \int_{T_2}^t (1 + s)^{-\frac{(N-1)(q-1) + \frac{\mu_1}{2} - \frac{\mu_2}{2} q}{2}} (L_1(s))^q ds, \quad \forall \ t \geq T_2.
\end{align}

Observe that \(\frac{1}{T_2}(T_2 + s) \leq 1 + s \leq T_2 + s\) for all \(s \in (T_2, t)\), because \(T_2 > 1\), we infer that
\begin{align}
L_1(t) & \geq \frac{C_3 \varepsilon}{8} + C \int_{T_2}^t (T_2 + s)^{-\frac{(N-1)(p-1) + \frac{\mu_1}{2} - \frac{\mu_2}{2} p}{2}} (L_2(s))^p ds, \quad \forall \ t \geq T_2,
\end{align}

and
\begin{align}
L_2(t) & \geq \frac{C_3 \varepsilon}{8} + C \int_{T_2}^t (T_2 + s)^{-\frac{(N-1)(q-1) + \frac{\mu_1}{2} - \frac{\mu_2}{2} q}{2}} (L_1(s))^q ds, \quad \forall \ t \geq T_2.
\end{align}

At this level, the remaining part of the proof is the same as the one in [18, Sections 4.2 and 4.3]. More precisely, here (4.14) (resp. (4.15) corresponds to (25) (resp. (26)) in
Nevertheless, in the present work the shift of the dimension $N$ is with $\mu_i$ instead of $\sigma(\mu_i)$ in [18], where $\sigma(\mu_i)$ is defined by (1.6).

This achieves the proof of Theorem 2.2.

5. APPENDIX

The aim of this appendix is to recall some properties of the function $\rho^\eta_i(t)$, for $i = 1, 2$, the solution of (3.3). Mainly, we will use the computations in [22]. Hence, we can write the expression of $\rho^\eta_i(t)$ as follows:

$$\rho^\eta_i(t) = (\eta(t + 1))^{\mu_i+1} K_{\sqrt{\gamma_i}}(\eta(t + 1)), \quad i = 1, 2,$$

where

$$K_{\nu}(t) = \int_0^\infty \exp(-t \cosh \zeta) \cosh(\nu \zeta) d\zeta, \quad \nu \in \mathbb{R}.$$

From the proof of [22, Lemma 2.1], one can see that

$$\frac{1}{\rho^\eta_i(t)} \frac{d\rho^\eta_i(t)}{dt} = \frac{\mu_i + 1 + \sqrt{\gamma_i}}{2(t + 1)} - \eta \frac{K_{\sqrt{\gamma_i}+1}(\eta(t + 1))}{K_{\sqrt{\gamma_i}}(\eta(t + 1))}, \quad i = 1, 2.$$

On the other hand, the function $K_{\nu}(t), \nu \in \mathbb{R}$, satisfies ([2])

$$K_{\nu}(t) = \sqrt{\frac{\pi}{2t}} e^{-t}(1 + O(t^{-1})), \quad \text{as } t \to \infty.$$

A combination of (5.2) and (5.3) yields

$$\frac{1}{\rho^\eta_i(t)} \frac{d\rho^\eta_i(t)}{dt} = -\eta + O(t^{-1}), \quad \text{as } t \to \infty, \quad i = 1, 2.$$

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