The product of Gamma Subacts of Multiplication Gamma Act.

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Abstract. In this paper, we introduce the notion of prime gamma subact and study some of its properties. Also, we define the concept of product gamma subacts of gamma act, particularly product gamma subacts of multiplication gamma act then we obtain some related results. Using our concept to introduce new mathematical system which is called the semigroup associative with multiplication gamma act and its basic properties are discussed. Also, by definition of product gamma subacts, we introduce the concept of gamma nilpotent. Finally, the concepts of (meet principal and weak meet principle ) gamma subact are introduced to study the product of multiplication gamma ideals of gamma semigroups. Relation among the multiplication, meet principal and weak meet principle gamma subacts are investigated.

Keywords. Multiplication gamma act, prime gamma subact, gamma nilpotent, gamma presentation, meet principal gamma subact, weak meet principal.

1. Introduction
In 1981 the concept of gamma semigroup was introduced by M.K. Sen [1]. Let S and Γ be nonempty sets. If there is a mapping $S \times \Gamma \times S \rightarrow S$ defined by $(s, \alpha, s_2) \mapsto s \alpha s_2$, satisfying, $s_1(\alpha(s_2)s_3) = (s_1\alpha s_2)s_3$ for all $s_1, s_2, s_3$ in S and $\alpha, \beta$ in Γ then S is called Γ-semigroup. A Γ-semigroup S, is called commutative if $srt = tars$ for all $s,t \in S$ and $\alpha \in \Gamma$. An element $a$ of Γ-semigroup S is said to be identity if $aas = s =sass$ for all $s \in S$ and $\alpha \in \Gamma$. A Γ-semigroup with identity is called Γ-monoid. A nonempty subset A is of a Γ-semigroup S called an Γ-iddeal or two sided Γ-ideal of S (left and right) if $S \alpha A \subseteq A$ and $\alpha S \subseteq A$. An element $s \in S$ is called idempotent if $sas = s$ for all $\alpha \in \Gamma, (s \Gamma s = s)$. Then, S is said to be a strongly idempotent if for all element in S is an idempotent. A Γ-ideal A of a Γ-semigroup S is said to be globally idempotent if $\Gamma A = A$. For any subsets $A, B$ of $S$, then $A \Gamma B = \{ a \alpha b : a \in A, b \in B$ and $\alpha \in \Gamma \}$. A Γ-ideal $P$ of $S$ is said to be a prime provided that for any two Γ-ideals $A, B$ of $S$, $A \Gamma B \subseteq P$, either $A \subseteq P$ or $B \subseteq P$[2]. Let $A$ be a Γ-ideal of $S$, then the intersection of all prime Γ-ideals of $S$ containing $A$ is called Γ-radical of $A$ and it's denoted by $rad_{\Gamma}(A)$. Let $A$ be ideal of Γ-semigroup $S$, If $a \in rad_{\Gamma}(A)$, then $(a \Gamma)^{n-1} a \subseteq A$ for some positive integer $n$ [1]. Recently, M.S. Abbas and Abdulqader Faris introduced and studied in [3] the concept of gamma acts which is a generalization of gamma semigroups. A nonempty set $M$ is called left gamma act over $S$ (denoted by $S_l$-act) if there is a mapping $S \times M \rightarrow M$ defined by $(s, \alpha, m) \mapsto s \alpha m$, satisfying $(s_1\alpha s_2)\beta m = s_1\alpha (s_2\beta m)$ for all $s_1, s_2 \in S$, $\alpha, \beta \in \Gamma$ and $m \in M$. Similarily, one can define a right gamma acts. If $S$ is a commutative Γ-monoid, then every left $S_l$-act is right $S_r$-act. A non-empty subset $N$ of a left $S_l$-act $M$ is called gamma subact (denoted by $S_l$-
subact) if for all \( s \in S, \alpha \in \Gamma \) and \( n \in N \) implies that \( s \alpha n \in N \). An element \( \theta \in M \) is called a zero of \( M \) if \( s\theta = \theta \) and if \( S \theta \) is a \( \Gamma \)-semigroup with zero then \( 0m = \theta \) for all \( m \in M, s \in S \) and \( \alpha \in \Gamma \). Let \( N \) be an \( S_r \)-subact of \( S_r \)-act. Then, \([N:M] = \{ s \in S \mid s \alpha n \in N \text{ for all } \alpha \in \Gamma \text{ and } m \in M \}\). Let \( M \) be an \( S_r \)-act and \( X \) a nonempty subset of \( M \). Then \( [X]_M = \bigcup \{ S \alpha x \mid s \in S \text{ and } \alpha \in \Gamma \}\), if \( M = [X]_M \) then \( X \) is said to be a generating set of \( M \). Also, if \( |X| < \infty \), then \( M \) is finitely generated and \( M \) is a cyclic if \( M = \{ [u]_M \} \) for some \( u \in M \). Let \( M \) be an \( S_r \)-act. Then, \( M \) is a simple \( S_r \)-act, if it contain no gamma subact other than \( M \). A \( \Gamma \)-semigroup \( S \) is said to be simple if \( S \) is \( S_r \)-act. In this paper we introduce the concept of prime gamma subact and we give some notions about it. Also, we investigate the notion of product gamma subact of multiplication gamma act and its basic properties are discussed. In particular, we show that if two gamma subacts are finitely generated then so is their product. The concept of \( \Gamma \)-presentation is introduced to study the relation between it and multiplication gamma act. Moreover, we introduce the definition of meet principal gamma subact to study the product of multiplication \( \Gamma \)-ideals of \( \Gamma \)-Semigroup. Throughout this paper \( S \) will be denote a commutative \( \Gamma \)-semigroup.

2. Preliminaries.

In this section we will introduce the definitions and results which are needed in the next section.

**Definition 2.1.**[4] An \( S_r \)-act \( M \) is called a multiplication gamma act if for every \( S_r \)-subact \( N \) of \( M \) there exists a \( \Gamma \)-ideal \( A \) of \( S, \) such that \( N = A \Gamma M \). A \( \Gamma \)-ideal \( A \) of \( S, \) is multiplication if \( A \) is \( S_r \)-subact of \( S, \). A \( \Gamma \)-monoid \( S, \) is called multiplication if all its \( \Gamma \)-ideals are multiplication.

**Proposition 2.2.**[4] Let \( N \) be a \( S_r \)-subact of \( M \). Then \( M \) is multiplication \( S_r \)-act if and only if \( N = [N:M] \Gamma M \).

**Proposition 2.3.**[4] Let \( S \) be a \( \Gamma \)-semigroup and \( \{ A_i \}_{i \in I} \) be a family of \( \Gamma \)-ideals of \( S, \) \( S \) is an \( S_r \)-act, then \( \bigcup_{i \in I} (A_i \Gamma M) = ( \bigcup_{i \in I} A_i ) \Gamma M \).

**Theorem 2.4.**[4] Let \( S \) be a \( \Gamma \)-monoid and \( M \) a faithful \( S_r \)-act. Then \( M \) is a multiplication if and only if

\[
\bigcap_{i \in I} (A_i \Gamma M) = ( \bigcap_{i \in I} A_i ) \Gamma M \text{ for any nonempty collection of } \Gamma \text{-ideals } A_i, \ (i \in I) \text{ of } S, \text{ with } ( \bigcap_{i \in I} A_i ) \neq \emptyset, \text{ and for any } S_r \text{-subact } N \text{ of } M \text{ and } \Gamma \text{-ideal } A \text{ of } S \text{ such that } N \subseteq A \Gamma M \text{ there exists an } \Gamma \text{-ideal } B \subseteq A \text{ and } N \subseteq B \Gamma M.
\]

**Lemma 2.5.** Let \( S \) be a \( \Gamma \)-monoid and \( M \) an \( S_r \)-act. Then \( K \) is a multiplication \( S_r \)-subact of \( M \) if and only if \( N(K) = [N:K] \Gamma K \) for every \( S_r \)-subact \( N \) of \( M \).

**Proof.** (\( \Leftarrow \)) Let \( x \in N \cap K, \) then \( x \in K, \) since \( K \) is a multiplication \( S_r \)-subact of \( M, \) there exists an \( \Gamma \)-ideal \( A \) in \( S \) such that, \( Sx = A \Gamma K \subseteq N. \) It follows that \( A \subseteq [N:K] \) and hence \( A \Gamma K \subseteq [N:K] \Gamma K \). Therefore, \( x \in [N:K] \Gamma K \). For the other inclusion, let \( y \in [N:K] \Gamma K \). Then \( y = sk \alpha k \) for some \( s \in [N:K], \alpha \in \Gamma \) and \( k \in K. \) If \( s \Gamma K \subseteq N, \) \( k \in K \). Since \( [N:K] \Gamma K \subseteq S \Gamma K \subseteq K \) then \( y \in K. \) From that, we conclude that \( y \in N \cap K. \) Hence, \( N \cap K = [N:K] \Gamma K \). \( (\Rightarrow) \) Let \( L \) be a \( S_r \)-subact of \( K. \) Then by hypothesis, \( L = L \Gamma K = [L:K] \Gamma K \). Thus by Proposition 2.2, \( K \) is multiplication.

Now, we introduce the following definition.

**Definition 2.5.** A proper \( S_r \)-subact \( N \) of \( M \) is prime if for any \( m \in M \) and \( s \in S, \) the set inclusion \( s \Gamma m \subseteq N \) implies either \( m \in N \) or \( s \in (N:M) \).

**Proposition 2.6.** Let \( S \) be a \( \Gamma \)-monoid, \( M \) is a multiplication \( S_r \)-act. Then for \( S_r \)-subact \( N \) of \( M \), the following statements are equivalent

- \( N \) is prime \( S_r \)-subact of \( M, \)
- \([N:M]\) is prime \( \Gamma \)-ideal of \( \Gamma \)-semigroup \( S, \)
- There exists a prime \( \Gamma \)-ideal \( P \) of \( S, \) which is maximal with the property \( P \Gamma M = N. \)

**Proof.** (i) \( \Rightarrow \) (ii) Consider the inclusion \( A \Gamma B \subseteq [N:M], \) for two \( \Gamma \)-ideals \( A, B, \) of \( S, \) Assume \( A \subset [N:M], \) then there exists \( \alpha \in [N:M] \) such that \( \alpha x \in N, \) where \( x \in M \) and \( \alpha \in \Gamma. \) Let \( b \in B. \) Then \( b \Gamma (\alpha x) = (b \Gamma \alpha) x = (a \Gamma b) x \subseteq N. \) Since \( N \) is a prime \( S_r \)-subact of \( M \) and \( a \alpha x \in N, \) then \( b \Gamma M \subseteq N, \) that is, \( b \in [N:M]\) prime \( \Gamma \)-ideal.
(ii) $\Rightarrow$ (iii) Consider the family, $\mathcal{T} = \{ P : N = \Gamma M \text{ and } P \text{ is an } \Gamma\text{-ideal of } S \}$. Since $M$ is multiplication $S_1$-act, then $(\mathcal{T}, \subseteq)$ is a nonempty partial order set. Let $\{ P_i \} \subseteq \mathcal{T}$ be a chain. By Proposition 2.3, $\bigcup_{i \in I} P_i \in \mathcal{T}$ is an upper bound of $\{ P_i \} \subseteq \mathcal{T}$. Zorn's Lemma implies that $\mathcal{G}$ has a maximal element such as $P$ (say). Now, we will show that, $P$ is prime $\Gamma$-ideal of $S$. Let $x, y \in S$ and $x \Gamma S \Gamma y \subseteq P$. Thus $x \Gamma S \Gamma y \subseteq P \cap M \subseteq N$, and hence $x \Gamma S \Gamma y \subseteq [N : M]$, by statement (2), $x \in [N : M]$ or $y \in [N : M]$. Since $M$ is multiplication, then by Proposition 2.2, $N = [N : M] \Gamma M$, it follows that $[N : M] \subseteq T$. Clearly $P \subseteq [N : M]$ and $P = [N : M]$. Thus $P$ is prime $\Gamma$-ideal of $S$.

(iii) $\Rightarrow$ (i) Let $P$ be a prime $\Gamma$-ideal of $S$, which is maximal with the property $P \Gamma M \subseteq N$. It is clear that, $P = [N : M]$. Let $x \in S$ and $m \in M$ such that $x \Gamma S \Gamma m \subseteq N$. Since $M$ is a multiplication, then there exists an ideal $A$ of $S$, such that $S \Gamma m = \Gamma A M$, it follows that $x \Gamma S \Gamma m \subseteq N$. Thus $x \Gamma A \subseteq [N : M]$. Since $[N : M] = P$ and $P$ is prime $\Gamma$-ideal of $S$, then $x \in [N : M]$. Since $x \in [N : M]$, $[N : M]$ is a prime $\Gamma$-ideal of $S$, then $x \in [N : M]$ or $A \subseteq [N : M]$, i.e. $x \in [N : M]$. Then any nonempty subset of $N$ is the intersection of all prime $S_1$-subacts of $M$ containing $N$ and denoted by $\text{rad}_{M}(N)$.

Proposition 2.8. Let $S$ be a $\Gamma$-monoid and $N$ a proper $\Gamma$-subact of a faithful multiplication $S_1$-act $M$. Then $\text{rad}_{M}(N) = \sqrt{(N : M)} \Gamma M$. Proof: Let $\mathcal{P} = \{ P : P \text{ is prime } \Gamma$-ideal of $S$; $[N : M] \subseteq P \}$. If $B = \sqrt{(N : M)}$ then $B = \bigcap_{P \in \mathcal{P}} P$ and hence, by Theorem 2.4, $B \Gamma M = \bigcap_{P \in \mathcal{P}} (P \cap M \Gamma M)$. Let $P \in \mathcal{P}$. If $M \subseteq \Gamma M$ then $N = [N : M] \Gamma M \subseteq P \cap M$. If $M \neq \Gamma M$ then $N = [N : M] \Gamma M \subseteq P \cap M$ by Proposition 2.6, $P \cap M$ is prime $\Gamma$-subact of $M$. Thus $\text{rad}_{M}(N) \subseteq P \cap M$. Therefore, $\text{rad}_{M}(N) \subseteq B \Gamma M$. Conversely, let $K$ be a prime $\Gamma$-subact of $M$ containing $N$. By Proposition 2.6, there is a prime $\Gamma$-ideal $P$ of $S$, such that $K = P \cap M$ and since $[N : M] \Gamma M = K \cap M$ then $[N : M] \subseteq P$. Hence, $\sqrt{(N : M)} \subseteq P$ and so $\sqrt{(N : M)} \Gamma M = K$. Therefore, $\text{rad}_{M}(N) \subseteq P$.

A $\Gamma$-semigroup $S$, is said to be completely globally idempotent if every $\Gamma$-ideal of $S$, is globally idempotent. The proof of following proposition is clear.

Proposition 2.9. Let $S$ be a $\Gamma$-semigroup, if $A$ and $B$ are globally idempotent $\Gamma$-ideals of $S$, then $\text{A} \cap \text{B}$ is globally idempotent $\Gamma$-ideal.

Proposition 2.10. Let $S$ be completely globally idempotent $\Gamma$-semigroup. Then $S$ is a multiplication.

Proof: Let $A$ and $B$ be $\Gamma$-ideals of $S$, such that $B \subseteq A$. Then $B = \text{B} \cap \text{B} \subseteq \text{B} \cap \text{A} \subseteq \text{B}$. Hence $B = \text{B} \cap \text{A}$. Therefore, $S$ is a multiplication.

3. The product of $S_1$-subacts in multiplication $S_1$-acts.

In this section the product of gamma subacts have been studied and a number of results giving their structures are obtained. We start with a result generalizing definition of the product gamma subact.

Definition 3.1. Let $M$ be an $S_1$-act and $N_1, N_2$ be $S_1$-subacts of $M$. Then the product of $N_1$ and $N_2$ is defined as: $N_1 \times N_2 = \{ [N_1 : M] \Gamma [N_2 : M] \Gamma M \}$. It’s clear that $N_1 \times N_2$ is an $S_1$-subact of $M$.

Example 3.2. (1) Let $S = \{ 5n + 4 : n \in \mathbb{N} \}$, $\Gamma = \{ 5n + 1 : n \in \mathbb{N} \}$ and $M = \{ 5n : n \in \mathbb{N} \}$. Under the usual addition of natural numbers, $S$ is a $\Gamma$-semigroup and $M$ is a $\Gamma$-act. If $N_1 = \{ 5n : n \leq 2, n \in \mathbb{N} \}$ and $N_2 = \{ 5n : n \geq 3, n \in \mathbb{N} \}$, then $N_1 \times N_2$ are $S_1$-subacts of $M$ such that $[N_1 : M] = \{ 5n + 4 : n \geq 1, n \in \mathbb{N} \}$ and $[N_2 : M] = \{ 5n + 4 : n \geq 2, n \in \mathbb{N} \}$. Thus, $N_1 \times N_2 = \{ [N_1 : M] \Gamma [N_2 : M] \Gamma M \} = \{ 5n + 10 : n \geq 3, n \in \mathbb{N} \}$.

Definition 3.3. Let $M$ be an $S_1$-act and $N$ an $S_1$-subact of $M$ such that $N = \Gamma M$ for some $\Gamma$-ideal $A$ of $S$. Then $A$ is called as a $\Gamma$-presentation of $N$.

It is clear that, if $S$-act $M$ is a multiplication, if and only if every $S_1$-subact of $M$ has a $\Gamma$-presentation (By Proposition 2.2). In particular, for every $S_1$-subact $N$ of a multiplication $S_1$-act $M$, $[N : M]$ is a $\Gamma$-presentation for $N$. Note that, it is possible that for a $S_1$-subact $N$, no such $\Gamma$-presentation exists. As shows in following example.

Let $S = \mathbb{Z}$, $\Gamma = \mathbb{N}$. Then, $S$ is a $\Gamma$-semigroup by the mapping $(z, n, z_2) \mapsto (z, n, z_2)$ where $z, n, z_2$ being the usual multiplication. Now, let $M = \{ 1, 2, 3, 4, 5, 6 \}$, and define the mapping $S \times \Gamma \times M \mapsto M$ by $(z, n, m) \mapsto 2$. Then any nonempty subset of $M$ that contains 2 is $S_1$-subact of $M$, in particular if $N = \{ 1, 2 \}$ is $S_1$-subact of $M$ then $N \neq \{ 2 \}$. Hence, $\Gamma M$ for each $\Gamma$-ideals $A$ of $S$. 


Now, we introduce particularly the product of two $S^r$-subacts when the gamma act is a multiplication, and we give some of its properties.

**Definition 3.4.** Let $M$ be a multiplication $S^r$-act and $N_1, N_2$ be $S^r$-subacts of $M$. If $N_1 = \Lambda M$ and $N_2 = \Gamma M$ for some $\Gamma$-ideals $A$ and $B$ of $S$, then the product of $N_1$ and $N_2$ is denoted by $N_1 \cdot N_2$ and is defined by $N_1 \cdot N_2 = (A \cdot B)\Gamma M$. Clearly, $N_1 \cdot N_2$ is an $S^r$-subact of $M$.

**Theorem 3.5.** Let $S$ be a $\Gamma$-semigroup and $N_1, N_2$ be $S^r$-subacts of multiplication $S^r$-act $M$. Then the product of $N_1$ and $N_2$ is independent of $\Gamma$-presentations of $N_1$ and $N_2$. **Proof:** Suppose $N_1 = A_1 \Gamma M = A_2 \Gamma M$ and $N_2 = B_1 \Gamma M = B_2 \Gamma M$ for $\Gamma$-ideals $A_1$ and $B_1$ of $S$, $i = 1, 2$. Now, $(A_1 \Gamma B_1)\Gamma M = A_1 \Gamma (B_1 \Gamma M) = B_2 \Gamma (A_1 \Gamma M) = (A_2 \Gamma B_2)\Gamma M$.

**Corollary 3.6.** Let $N_1$ and $N_2$ be $S^r$-subacts of a multiplication $S^r$-act $M$. Then the $\Gamma$-ideals $[N_1 : M][N_2 : M]$ and $[N_1 \cdot N_2 : M]$ are $\Gamma$-presentations of $N_1 \cdot N_2$ and they are equal. **Proof:** Since $M$ is multiplication $S^r$-act, then $N_1 = [N_1 : M]\Gamma M$ and $N_2 = [N_2 : M]\Gamma M$. Thus $N_1 \cdot N_2 = [N_1 : M][N_2 : M]\Gamma M$. Hence $[N_1 : M][N_2 : M]$ presentation of $N_1 \cdot N_2$. On the other hand, since $N_1 \cdot N_2$ is an $S^r$-subact of $M$ then $N_1 \cdot N_2 = [N_1 \cdot N_2 : M]\Gamma M$. Thus, $[N_1 \cdot N_2 : M]$ presentation of $N_1 \cdot N_2$.

**Proposition 3.7.** Let $N = \Lambda M$ be a $S^r$-subact of multiplication $S^r$-act $M$, for some $\Gamma$-ideal $A$ of $S$. If $A$ is globally idempotent, then $A$ is $\Gamma$-presentation of $N \subseteq N$. **Proof:** Clear.

If $N_1$ and $N_2$ finitely generated $S^r$-subacts such that $N_1$ generated $n$ and $N_2$ generated by $m$ then the following Theorem shows that $N_1 \cdot N_2$ generated by $nm$.

**Theorem 3.8.** Let $S$ be a $\Gamma$-semigroup and $N_1$ and $N_2$ finitely generated $S^r$-subacts of multiplication $S^r$-act $M$ then $N_1 \cdot N_2$ is also finitely generated. **Proof:** Since $M$ is multiplication $S^r$-act, then there exists $\Gamma$-ideals $A$ and $B$ of $S$, such that $N_1 = \Lambda M$ and $N_2 = \Gamma M$. By hypothesis,

$$\Lambda M = \bigcup_{i=1}^{n} \mathcal{S}(a_i a_i m_i) \text{ and } \Gamma M = \bigcup_{j=1}^{k} \mathcal{S}(b_j b_j m_j') \text{ where } a_i \in A, b_j \in B$$

,$$a_i, b_j \in \Gamma \text{ and } m_i, m_j' \in M \text{ for all } i, j.$$

We will prove that,

$$(\Lambda \Gamma M) = \bigcup_{j=1}^{k} \mathcal{S}(a_i a_i b_j b_j m_j m_j') = \bigcup_{j=1}^{k} \mathcal{S}(a_i a_i b_j b_j m_j m_j'), \text{ for each } i = 1 \ldots n, j = 1 \ldots k.$$

Clearly, $\bigcup_{j=1}^{k} \mathcal{S}(a_i a_i b_j b_j a_i a_i m_i) = (\Lambda \Gamma M)$.

For the other inclusion, let $aab\beta m \in (\Lambda \Gamma M)$ where $a \in A, b \in B, a, b, \beta \in \Gamma$ and $m \in M$. Then, there is $c, t \in S$ and $a', b' \in \Gamma$ such that $aab\beta m = a(a' b'b' m', b' m') = \beta a b a'(ab' m')$.

Therefore,

$$aab\beta m \in \bigcup_{j=1}^{k} \mathcal{S}(a_i b_j b_j a_i a_i m_i).$$

Similarly, we have $N_1 \cdot N_2 = (\Lambda M) = \bigcup_{j=1}^{k} \mathcal{S}(a_i b_j b_j a_i a_i m_i)$. In following we characterize prime $S^r$-subact in multiplication $S^r$-act by product $S^r$-subacts.

**Theorem 3.9.** Let $S$ be a $\Gamma$-monoid and $P$ a proper $S^r$-subact of a multiplication $S^r$-act $M$. Then $P$ is prime if and only if $N_1 \cdot N_2 \subseteq P$ then either $N_1 \subseteq P$ or $N_2 \subseteq P$ for each $S^r$-subact $N_1$ and $N_2$ of $M$. **Proof:** \((\Rightarrow)\) Let $P$ be prime and $N_1 \cdot N_2 \subseteq P$, but neither $N_1 \subseteq P$ nor $N_2 \subseteq P$ for some $S^r$-subacts $N_1$ and $N_2$ of $M$. Since $M$ is a multiplication $S^r$-act, then $N_1 = \Lambda M$ and $N_2 = \Gamma M$ for some $\Gamma$-ideals $A$ and $B$ of $S$. So, there is $a \in A, b \in B$, and $a, b, \alpha, \beta \in \Gamma$ such that $aab\beta m \in N_1 \setminus P$ and $b b' m' \in N_2 \setminus P$. Since, $(\Lambda \Gamma M) \subseteq P$. Thus, $aa(b b' m') \in P$. Since $P$ is prime $S^r$-subact of $M$, then either $a \in [P : M]$ that is, $aa \in P$ or $b b' m' \in P$ which is a contradiction.

\((\Leftarrow)\) Let $S^r S^r x \subseteq P$ for some $x \in S$ and $x \in M \cap P$. We will show that $s T M \subseteq P$. Suppose that $m \in M$. Since $M$ is multiplication $S^r$-act, then $S^r T M = \Lambda M$ and $S^r (s a m) = \Gamma M$ for some $\Gamma$-ideals $A$ and $B$ of $S$. Now, $S^r x \subseteq P$ or $s T m \subseteq P$, but $x \in P$ so $aa m \notin P$. Hence, $P$ is prime $S^r$-subact.
Let $M$ be a multiplication $S_I$-act and $m, m' \in M$. Then there are $\Gamma$-ideals $A, B$ such that $S \Gamma m = A \Gamma M$ and $S \Gamma m' = B \Gamma M$. So, we mean by $m \ast m'$, the product of $S \Gamma m$ and $S \Gamma m'$, which equal to $(A \Gamma B) \Gamma M$. As consequence of Theorem 3.9, we give the following corollary:

**Corollary 3.10.** Let $S$ be a $\Gamma$-monoid and $P$ a proper $S_I$-subact of a multiplication $S_I$-act $M$. Then $P$ is prime if and only if $m \ast m' \in P$ then either $m \in P$ or $m' \in P$ for each $m, m' \in M$. **Proof.** Clear.

Now, we introduce the concept of $\Gamma$-nilpotent $S_I$-subacts of multiplication $S_I$-act and study some properties about it.

**Definition 3.11.** For subsets $A, B$ of a $\Gamma$-semigroup $S$, the product set of $A$ and $B$ relative to $S$ denoted by $A \cdot B$ is defined as $A \cdot \Gamma : B = \{a \cdot \alpha \ast b \mid a \in A, b \in B, \alpha \in \Gamma \}$. And for $A \subseteq S$, the product set $A \cdot \Gamma : S$ relative to $S$ is defined as $A^2 = A \cdot \Gamma : A \cdot \Gamma : S$. More general $A^k = A \cdot \Gamma : A \cdot \Gamma : \ldots : \Gamma : A$, where $k$ positive integer. A $\Gamma$-ideal $A$ is called $\Gamma$-nilpotent if $A^k = \emptyset$ for some positive integer $k$.

**Definition 3.12.** Let $M$ be a multiplication $S_I$-act and $N$ a $S_I$-subact of $M$. Then, $N$ is called $\Gamma$-nilpotent if $N^k = \emptyset$ for some positive integer $k$, where $N^k$ means the product of $N$, $k$ times. An element $m$ of $M$ is called $\Gamma$-nilpotent if $S \Gamma m$ is $\Gamma$-nilpotent, i.e $m^k = \emptyset$ for some positive integer $k$. The set of all $\Gamma$-nilpotent elements of $M$ is denoted by $N_{\Gamma M}$.

**Proposition 3.13.** Let $M$ be a multiplication $S_I$-act. Then, $N_{\Gamma M}$ is a $S_I$-subact of $M$. **Proof:** Let $s \in S, \alpha \in \Gamma$ and $m \in N_{\Gamma M}$. Then, there is positive integer $k$ such that $m^k \ast A^k \Gamma M = \emptyset$ for some $\Gamma$-ideal $A$ of $S$.

**Theorem 3.14.** Let $N$ be a $S_I$-subact of a multiplication $S_I$-act $M$. Then $rad_f(N) = \{m \in M \mid m^k \subseteq N \text{ for some } k \geq 0\}$. **Proof:** Suppose $N = \{m \in M \mid m^k \subseteq N \text{ for some } k \geq 0\}$. First, we show that $N$ is $S_I$-subact of $M$. Let $s \in S, \alpha \in \Gamma$ and $m \in N$. Then, $m^k \subseteq N$ that is $(s \Gamma m_1)^k \subseteq N$, for some positive integer $k$. Now, $(s \Gamma (s \Gamma m_1))^k \subseteq (s \Gamma m_1)^k \subseteq N$, that is $(s \Gamma m_1)^k \subseteq N$ and hence $N$ is $S_I$-subact of $M$. Now, we will show that $rad_f(N) = \Delta$. Suppose $m \in N$, then $S \Gamma m = \Gamma M$ where $A$ is a $\Gamma$-ideal of $S$. So, $m^k = A^k \Gamma M \subseteq N$ for some $k \geq 1$ and hence by Proposition 2.8, we have $rad_f(m^k) = rad_f(A^k \Gamma M) = \sqrt{\Gamma} \Gamma M \subseteq rad_f(N)$, since $rad_f(A^k \Gamma M) \subseteq rad_f(N)$. Therefore, $m \in S \Gamma m = \Gamma M \subseteq \Gamma \Gamma M$ and hence $N \subseteq rad_f(N)$. For the other inclusion, let $m \in rad_f(N) = \sqrt{(N:M) \Gamma M}$. Then, $m = s \Gamma am'$ where $s \in \sqrt{(N:M)}, \alpha \in \Gamma$ and $m' \in M$. Then there exist a positive integer $n$ such that $(s \Gamma)^{n-1} s \subseteq (N:M)$. So, $(s \Gamma)^{n-1} s \Gamma am' = s \Gamma a \Gamma am' = s \Gamma a \Gamma am' \ast \Gamma am' \ast \ldots \ast \Gamma am' = (s \Gamma am')^n$. So, we have $(s \Gamma am')^n \subseteq (N:M) \Gamma M \subseteq N$. Therefore, $rad_f(N) = \Delta$.

The proof of the following corollary is immediately by Definition 2.7 and Theorem 2.14.

**Corollary 3.15.** Let $M$ be a multiplication $S_I$-act. Then $N_{\Gamma M}$ is the intersection of all prime $S_I$-subacts of $M$.

**Corollary 3.16.** Let $M$ be a faithful multiplication $S_I$-act. Then $N_{\Gamma M} = \sqrt{\Gamma} M$. **Proof.** Clear.

**Theorem 3.17.** Let $S$ be a $\Gamma$-semigroup with zero and $M$ a multiplication $S_I$-act. An $S_I$-subact $N$ of $M$ is $\Gamma$-nilpotent if and only if for every $\Gamma$-presentation $A$ of $N$, $A^k \subseteq [\theta:M]$ for some positive integer $k$. **Proof:** $(\Rightarrow)$Let $A$ be a $\Gamma$-presentation $\Gamma$-ideal of $N$. If $N$ is $\Gamma$-nilpotent, then $N^k = \emptyset$ for some positive integer $k$, that is, $N^k = A^k \Gamma M = \emptyset$. Thus, $A^k \subseteq [\theta:M]$. $(\Leftarrow)$ Suppose that $A^k \subseteq [\theta:M]$ for some presentation $\Gamma$-ideal $A$ of $N$. Then there is positive integer $k$, $N^k = A^k \Gamma M \subseteq [\theta:M] \Gamma M = \emptyset$. Hence, $N$ is $\Gamma$-nilpotent.

**Corollary 3.18.** Let $M$ be a faithful multiplication $S_I$-act and $N$ a $S_I$-subact of $M$. Then, $N$ is $\Gamma$-nilpotent if and only if every presentation $\Gamma$-ideal of $N$ is a $\Gamma$-nilpotent.

If $M$ is a multiplication $S_I$-act we can use the product Previously defined between $S_I$-subacts of $M$ to introduce new mathematical system as the following definition.
Definition 3.19. Let $S$ be a $\Gamma$-semigroup and $M$ a multiplication $S_\Gamma$-act. Consider the family denoted by $S(M)$ of all cyclic $S_\Gamma$-subacts of $M$. Define the mapping, $\sigma : S(M) \times S(M) \to S(M)$ written by $\sigma (m, m') \mapsto m \ast m'$ for all $m, m' \in S(M)$. Theorem 3.8 guarantee that $\sigma (m, m') \in S(M)$. Also, by Theorem 3.5, $\sigma$ is well-defined.

Verification: Let $m_1, m_2, m_3 \in M$. Then, $S\Gamma m_i, i = 1, 2, 3$ are cyclic $S_\Gamma$-subacts of $M$. Thus, there is $\Gamma$-ideals $A_1, A_2$ and $A_3$ of $S$, such that $S\Gamma m_1 = A_1\Gamma M$, $S\Gamma m_2 = A_2\Gamma M$, $S\Gamma m_3 = A_3\Gamma M$. Now, $m_1 \ast (m_2 \ast m_3) = A_1\Gamma M \ast (A_2\Gamma M \ast A_3\Gamma M) = A_1\Gamma M \ast (A_2\Gamma A_3)\Gamma M = (A_1\Gamma A_3)\Gamma M = (A_1\Gamma A_2)\Gamma M = (A_1\Gamma M) \ast (A_2\Gamma M) \ast (A_3\Gamma M) = (m_1 \ast m_2) \ast m_3$. Hence, $(S(M)$ is called the semigroup associative with $M$.

Example 3.20. Let $S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \Gamma = \{\{a, b\}\}$ and $M = S$. Then $M$ is an $S_\Gamma$-act under the mapping: $S \times \Gamma \times M \to M$ defined by $(A, B, C) \mapsto A \cap B \cap C$. $M$ is cyclic $S_\Gamma$-act generated by the set $\{a, b\}$. So, $M$ is multiplication. Here, $A_1 = \{\emptyset\}, A_2 = \{\emptyset, \{a\}\}, A_3 = \{\emptyset, \{b\}\}$ and $A_4 = \{\emptyset, \{a\}, \{b\}\}$ are the $\Gamma$-ideals of $S$. Here, $(m_1 \ast m_2) \ast m_3 = (m_1 \ast m_2) \ast m_3$, for all $m_i \in M, i = 1, 2, 3$. Therefore, $(S(M)$ is semigroup.

Recall that [5], if $C$ and $D$ are categories of acts, and $F : C \to D$ assignment of a unique object $F(A) \in D$ to each object $A \in C$ and unique morphism $F(f)$ in $D$ to each morphism $f : A \to B$ in $C$. Consider the following conditions: (1) $F(id_{A}) = id_{F(A)}$ for $A \in C$, we say that $F$ preserves identities. (2) $F(f_2 \circ f_1) = F(f_1) \circ F(f_2)$ for $f_1 \circ f_2 \in Mor_C(A_1, A_2)$ and $A_1, A_2 \in C$ we say that $F$ preserves a composition. If $F$ satisfies (1) and (2), then $F$ is called covariant functor.

In [6] mention the concept of the category of gamma acts which denoted by $S_\Gamma$-ACT as following: Let $S$ be $\Gamma$-semigroup. The objects of $S_\Gamma$-ACT consist of all left $S_\Gamma$-acts over the same $\Gamma$-semigroup $S$. The morphisms of $S_\Gamma$-ACT is the $S_\Gamma$-homomorphisms between the objects of $S_\Gamma$-ACT.

Now, consider $C = \{M_1, M_2, M_3, \ldots\}$ be a family of multiplication $S_\Gamma$-acts then $C$ is a subcategory of the category of gamma acts, because the objects of $C$ belong to the category of gamma acts and they are both equal in morphism, i.e (S_\Gamma-homomorphism of S_\Gamma-acts).

Definition 3.21. Let $C = \{M_1, M_2, M_3, \ldots\}$ and $D = \{S(M_1), S(M_2), S(M_3), \ldots\}$ be a families of multiplication $S_\Gamma$-acts and semigroups associative with $M_i$ for all $i = 1, 2, 3, \ldots$ resp. Define a mapping: $F : C \to D$ by:

$$F(M_1) = S(M_1), \text{ for all } M_1 \in C.$$  

$$F(f) \in Mor_D(S(M_1), S(M_2)) \text{ for all } f \in Mor_C(M_1, M_2).$$

Remark 3.22. $F$ is covariant functor. Since, (1) $F(id_{M}) (m) = id_{F(M)} (m) = m$. Hence, $F(id_{M}) (m) = id_{F(M)}$, i.e $F$ is preserves identities. (2) $F(f_2 \circ f_1)(m) = (F(f_2) \circ F(f_1))(m)$. On other hand $F(f_1)F(f_2)(m) = F(f_1)F(f_2)(m) = f_1(f_2(m))$. Hence, $F(f_2 \circ f_1) = F(f_2) \circ F(f_1)$ for all $f_1, f_2 \in Mor_C(M_1, M_2).$ Thus, we started with multiplication gamma act and we moved to subcategory of the category of monoid. The following Proposition guarantee that the convers is true.

Proposition 3.23. Let $S$ be a monoid. Then $S(S) = S$. Proof. Since $S$ is a monoid, thus $S$ is $S_\Gamma$-act. Let $s \in S$. Then, $S\Gamma s \in S(S)$ and hence, $\Gamma s \in S(S)$ which implies that $s \in S(S)$. Therefore, $S \subseteq S(S)$. Similarly, $S(S) \subseteq S$. Therefore, $S(S) = S$.

Now, we introduce the concept of meet principal $S_\Gamma$-subact to study the product of multiplication $\Gamma$-ideals of $S_\Gamma$-Semigroup.

Definition 3.24. Let $S$ be a $\Gamma$-monoid. An $S_\Gamma$-subact $N$ of $S_\Gamma$-act $M$ is said to be meet principal (weak meet principal) if $\forall \Gamma N \cap K = (\forall \Gamma (K : N)) \cap (\forall K = [K : N])$, for all $\Gamma$-ideal $A$ of $S$ and $S_\Gamma$-subact $K$ of $M$.

It's clear that meet principal (multiplication) $S_\Gamma$-acts is weak meet principal.

Theorem 3.25. Let $S$ be a $\Gamma$-monoid. If $M$ is meet principal $S_\Gamma$-act then $M$ is a multiplication. The converse is true, if $M$ is faithful.

Proof. Let $B$ be an $S_\Gamma$-subact of $M$. Then, $B = S\Gamma M \cap B = (S \cap (B : M)) \Gamma M = (B : M) \Gamma M$. On other hand, let $M$ be a faithful multiplication $S_\Gamma$-act, $A$ is a $\Gamma$-ideal of $S$ and $B$ is $S_\Gamma$-subact of $M$. Then, by Theorem 2.4, $A \Gamma M \cap B = A \Gamma M \cap (B : M) \Gamma M = (A \cap (B : M)) \Gamma M$. Therefore, $M$ is meet principal $S_\Gamma$-act.
**Corollary 3.26.** Let $S$ be a $\Gamma$-monoid. If $A$ and $B$ are faithful multiplication $\Gamma$-ideals of $S$, then $A\Gamma B$ is a multiplication $\Gamma$-ideal of $S$. **Proof.** Let $C$, $D$ be any $\Gamma$-ideals of $S$. Then, by Definition 3.24, and Theorem 3.25, $C\Gamma(A\Gamma B)\cap D = (C\Gamma A)\Gamma B \cap D = [C\Gamma A \cap (D:\ B)]\Gamma B = [C \cap (D:\ A))\Gamma A] \Gamma B = [C \cap (D:\ A\Gamma B)]\Gamma (A\Gamma B)$. Thus, by Theorem 3.25, $A\Gamma B$ is meet principal $\Gamma$-ideal. Hence, $A\Gamma B$ is a multiplication $\Gamma$-ideal of $S$.

**Theorem 3.27.** Let $S$ be $\Gamma$-monoid and $M$ be $S$-$\Gamma$-act. Consider the following conditions for a $S$-$\Gamma$-subact $N$ of $M$:

- $N$ is meet principal.
- $N$ is weak meet principal.
- $N$ is multiplication.

Then $i \implies ii \implies iii \implies i$, if $N$ is faithful.

**Proof:** Follows from Lemma 2.5, and Corollary 3.26.

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