Demystification of quantum entanglement

Andrei Khrennikov
School of Mathematics and Systems Engineering
University of Växjö, S-35195, Sweden

May 29, 2009

Abstract
Recently a new attempt to go beyond QM was performed in the form of so called prequantum classical statistical field theory (PCSFT). In this approach quantum systems are described by classical random fields, e.g., electron field or neutron field. Averages of quantum observables arise as approximations of averages of classical variables (functionals of “prequantum fields”) with respect to fluctuations of fields. For classical variables given by quadratic functionals of fields, quantum and prequantum averages simply coincide. In this paper we generalize PCSFT to composite quantum system. The main discovery is that, opposite to a rather common opinion, a composite system can be described by Cartesian product of state spaces (like in classical physics) and not by the tensor product of them (like in conventional QM). A natural interpretation of quantum pure state for a composite system is proposed: it is the nondiagonal block in the covariance matrix of the random field describing a composite system. The interpretation of a pure state due to Dirac and von Neumann seems to be an artifact of the conventional mathematical description of micro-systems. Entanglement was finally demystified.

1 Introduction
The question of inter-relation between classical and quantum models is of the greatest complexity and it is far from to be solved completely, see, e.g., [1]-[4] for recent debates. It is characterized by huge diversity of opinions: from Bohr’s belief [5] in completeness of QM (and
hence total impossibility to go beyond it in the classical-like framework) to Einstein’s belief [6]–[8] in incompleteness of QM (and hence a possibility to create a version of classical statistical mechanics which would reproduce quantum averages). Einstein’s views are especially important for this paper. They were discussed in detail in author’s paper “Einstein’s dream” [9]. Here I just mention that Einstein did not expect that “prequantum classical statistical mechanics” would be a plane generalization of classical mechanics for ensembles of particles. Einstein advocated the viewpoint, see especially [7], that particles should be totally excluded from “final physical theory”. Fields will be the basic entities of coming new theory. It seems that fields were interpreted by Einstein as classical fields.

Recently I made his dream come true. A new attempt to go beyond QM was performed in the form of so called prequantum classical statistical field theory (PCSFT), [10]–[15]. In this approach quantum systems are described by classical random fields, i.e., classical fields depending on a random parameter $\omega$. Operation with random fields is equivalent to operation with ensembles of fields. Thus our theory is classical statistical mechanics of fields (not particles). In the formal mathematical framework this means transition from finite dimensional phase space to infinite dimensional phase space.

Quantum averages can be interpreted as approximations of averages of functionals of classical ("prequantum") fields with respect to fields’ fluctuations. By restricting the class of classical variables to quadratic functionals of fields we reproduce quantum averages precisely. In this paper we generalize PCSFT to composite quantum systems. The main discovery is that, opposite to a rather common opinion, a composite system can be described by Cartesian product of state spaces (like in classical physics) and not by the tensor product of them (like in QM). The natural interpretation of a pure quantum state for a composite system is proposed: it is a symbolic (vector) representation of the nondiagonal block in the covariance matrix of the

---

1We remark that appealing to classical random fields is rather common in various attempts to create a kind of classical statistical mechanics which reproduces predictions of QM. We can mention stochastic electrodynamics, e.g., [16], [19], or the semiclassical model, e.g., [20]–[22]. Bohmian mechanics also contains a kind of classical field, the pilot wave. However, in this model randomness is coupled to particles and not to fields. The same can be said about Nelson’s stochastic QM [23] and its generalization due to Mark Davidson [24], [25] as well as the recent prequantum model of ‘t Hooft [26], [27], see also Thomas Elze [28], [29].
random field describing a composite system. The interpretation of a pure state due to Dirac and von Neumann seems to be an artifact of the conventional mathematical description of micro-systems.

In a few words PCSFT for composite systems can be presented in the following way. Classical (prequantum) state space of a “quantum particle” is the $L_2$-space $L_2(\mathbb{R}^3)$. Its points are interpreted as classical fields: $x \mapsto \phi(x)$. Classical statistical states represent ensembles of fields (by QM terminology ensembles of quantum particles). Mathematically (in the complete agreement with classical probability theory) a classical statistical state is represented by a random field, say $\phi(x,\omega)$. Here $\omega$ is a random parameter. A random field is a random vector belonging to classical state space $L_2(\mathbb{R}^3)$. In quantum information one can restrict considerations (at least formally) to finite dimensional state space. The correspondence between PCSFT and QM is established via identification of quantum states with classical (prequantum) statistical states: quantum density operator $\rho$ is identified with the covariance operator of corresponding prequantum random field. This approach is consistent with Einstein-Margenau-Ballentine interpretation of quantum states (including so called pure states) as describing identically prepared ensembles of quantum particles, see, e.g., Ballentine [30]. The presented prequantum classical statistical model for ensembles of quantum particles was developed in [10]–[15]. We remark that in PCSFT conventional QM appears as its approximation. Therefore PCSFT gives a possibility not only to reproduce QM-predictions (as approximations), but even to go beyond QM. However, in the present paper we do not explore the total power of PCSFT. We are “simply” concentrated on reproduction of QM-predictions.

Consider now a system composed of, e.g., two electrons, $S_i, i = 1, 2 : S = (S_1, S_2)$. PCSFT-state space of a pair of electrons is Cartesian product of state spaces of electrons composing this pair, namely, $\mathbf{H} = L_2(\mathbb{R}^3) \times L_2(\mathbb{R}^3)$. An ensemble of pairs is described by the vector random field with coordinates $\phi_i(x,\omega), i = 1, 2$. Consider the covariance operator of this random field, say $D$. It describes correlations between various degrees of freedom of this field. In particular, its nondiagonal block describes correlations between degrees of freedom of electrons (electronic fields) in pairs. Now consider the QM description of a pair of quantum particles. The state space is given by the

\footnote{We remind that in PCSFT each electron is represented by a classical random field, i.e., by a vector belonging to $L_2(\mathbb{R}^3)$.}
tensor product $H = L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3)$. Consider a pure quantum state $\Psi \in H$. The density operator corresponding to this state is $\rho = \Psi \otimes \Psi$. Consider also reduced density operators $\rho^{S_i}, i = 1, 2$. We establish correspondence between the QM and PCSFT descriptions in the following way. The canonical operator associated with $\Psi$, see section 3, represents the nondiagonal block of the covariance operator $D$. The diagonal blocks are given by combinations of reduced density operators $\rho^{S_i}$ with the covariance operator of the white noise field (the “background field”).

Direct realization of this scheme induces some mathematical difficulties related to measure theory on infinite dimensional spaces. We remark that random fields are probability distributions on state spaces. To make this paper more readable, we consider finite dimensional state spaces. Moreover, we restrict our considerations to real Hilbert spaces. All physical problems related to creation of a classical-like representation for a composite quantum system are already present in this simplified model. Finally, we notice that considerations of this paper can be generalized from pure states to general states of composite systems (given by density matrices).

2 PCSFT: single system case

Here we briefly present essentials of PCSFT for ensembles composed of single particles, see [10]–[15]. We start with the remark that both classical statistical mechanics (CSM) and QM are statistical theories which predict probabilities for various variables with respect to large (in the limit infinitely large) ensembles of physical systems. Both theories have the same aim: to predict averages of these variables. At the first sight these theories differ crucially. In particular, they are based on the use of (at the first sight) totally different mathematical tools.

In CSM averages are given by integrals (over the space of parameters characterizing systems$^3$); variables are given by functions of parameters, $\phi \rightarrow f(\phi)$, where $\phi \in M$ and $M$ is space of states. Denote by $\mu$ the probability distribution of random parameters, it is a probability measure on $M$. Average of a variable $f(\phi)$ with respect to the

$^3$In the canonical formalism these are points of phase-space.
statistical state $\mu$ is given by

$$\langle f \rangle_\mu = \int_M f(\phi) d\mu(\phi).$$  \hspace{1cm} (1)$$

By using the language of probability theory we can say that there is given a random vector $\phi(\omega)$, where $\omega$ is a random parameter, taking values in $M$. Then $\langle f \rangle_\phi = Ef(\phi(\omega)) = \langle f \rangle_\mu$. Here and everywhere below $E$ denotes classical mathematical expectation (average).

If the state space $M$ is a space of functions, e.g., $X = L_2(\mathbb{R}^3)$, then $X$-valued random vectors are called random fields. For each $\omega$, $\phi(\omega)$ is a function of $x \in \mathbb{R}^3 : \phi(x, \omega)$. We will work in the finite dimensional case, so we will not operate with random fields in our calculations. However, in discussions on physical interpretation of our theory random fields will be used.

To finish this short remark on CSM, we emphasize that if systems $S_i, i = 1, 2, ..., k$, have state spaces $M_i$, respectively, then the composite system $S = (S_i)_{i=1}^k$ has state space $M = M_1 \times \ldots \times M_k$, Cartesian product of state spaces $M_i$. Ensembles of $S$-systems are described by random vectors in $M : \phi(\omega) = (\phi_1(\omega), \ldots, \phi_k(\omega))$. A trivial, but important, remark is that in general components of $\phi(\omega)$ are not independent. There are nontrivial correlations between them. The best way to describe these correlations is to use the covariance operator (it will be defined little bit later).

In QM-formalism observables are given by self-adjoint operators and states by density operators. These operators act in Hilbert space. A special class of states, so called pure states, are represented by normalized vectors of Hilbert space. Average of an observable $\hat{A}$ with respect to a state $\rho$ is defined as

$$\langle \hat{A} \rangle_\rho = \text{Tr}\rho \hat{A}.$$  \hspace{1cm} (2)$$

If quantum systems $S_i, i = 1, 2, ..., k$, have state spaces $H_i$, respectively, then the system $S = (S_i)_{i=1}^k$ has state space $H = H_1 \otimes \ldots \otimes H_k$, the tensor product of state spaces $H_i$.

At the first sight the difference between CSM and QM is huge. Impossibility to reduce quantum average (2) to classical one (1) induced the idea on irreducible quantum randomness, see [31]. The crucial difference in the descriptions of composite systems, by Cartesian product in CSM and by the tensor product in QM, induced the idea that state space of quantum composite system has essentially (exponentially) greater capacity than state space of a classical composite.
system (since the dimension of Cartesian product increases linearly and the dimension of the tensor product exponentially). In \[10\]– \[15\] I presented the viewpoint that mentioned crucial difference between CSM and QM might be simply a mathematical artifact. In particular, it was shown that quantum average (2) can be easily reduced to classical one (1).

Take Hilbert space $M$ as space of classical states. To simplify the presentation, we assume that $M$ is a real Hilbert space with the scalar product $(\cdot, \cdot)$. Consider a probability distribution $\mu$ on $M$ having zero average $^4$ and covariance operator $D$ which is defined by a symmetric positively defined bilinear form:

$$ (Dy_1, y_2) = \int_M (y_1, \phi)(y_2, \phi)d\mu(\phi), y_1, y_2 \in M, \quad (3) $$

By scaling we obtain operator $\rho = D/\text{Tr}D$ with $\text{Tr}\rho = 1$. It can be considered as (Landau-von Neumann) density operator. By PCSFT quantum state, a density operator, is simply a symbolic representation of the covariance operator of corresponding prequantum (classical) probability distribution (random field). In general a probability distribution is not determined by its covariance operator. Thus correspondence PCSFT $\rightarrow$ QM is not one-to-one. However, if the class of prequantum probability distributions is restricted to Gaussian, then this correspondence becomes one-to-one. In PCSFT classical variables are defined as functions from (Hilbert) state space $M$ to real numbers, $f = f(\phi)$. By PCSFT quantum observable, a self-adjoint operator, is simply a symbolic representation of $f$ by means of its second derivative (Hessian), $f \rightarrow \hat{A} = \frac{1}{2}f''(0)$. This correspondence is neither one-to-one. However, by restricting the class of classical variables to quadratic forms on Hilbert state space $M, f_A(\phi) = (\hat{A}\phi, \phi)$, we make correspondence PCSFT $\rightarrow$ QM one-to-one. And finally, we present the basic equality

$$ \langle f_A \rangle_\mu (\equiv Ef_A(\phi(\omega))) = \int_M f_A(\phi)d\mu(\phi) = \text{Tr}\hat{A} = (\text{TrD}) \langle \hat{A} \rangle_\rho. \quad (4) $$

Thus quantum average $\langle \hat{A} \rangle_\rho$ can be obtained as just scaling of the classical average. We remark that scaling parameter $\text{TrD}$ is, in fact, dispersion of the probability distribution $\mu$:

$$ \sigma^2 \equiv E||\phi(\omega)||^2 = \int_M ||\phi||^2d\mu(\phi) = \text{Tr}D. \quad (5) $$

\footnote{It means that $\int_M (y, \phi)d\mu(\phi) = 0$ for any $y \in M$.}
It describes strength of deviations of a random vector \( \phi(\omega) \) from its average. The latter is zero in our model. If \( M = L_2 \) then we can consider \( \phi(\omega) \) as a random field \( \phi(x, \omega) \). Its average is the field which is equal zero everywhere. Suppose that \( \sigma^2 \) is very small, so deviations from zero are small. Since

\[
\langle \hat{A} \rangle_\rho = \frac{1}{\sigma^2} \langle f_A \rangle_\mu,
\]

quantum average can be considered as average of amplified prequantum fluctuations. Here \( \frac{1}{\sigma^2} f_A(\phi) = f_A(\hat{\phi}) \). Thus PCSFT describes averages of signals of very small magnitude. These prequantum signals are amplified by measurement devices. Averages of amplified signals are QM-averages (for quadratic variables, cf. with general PCSFT [10]–[15]).

This CSM-model for averages over ensembles consisting of single quantum particles was presented in [10]–[15]. To make it consistent with coming CSM-model for ensembles of quantum composite systems, suppose that prequantum fields (representing quantum particles) “live” not in vacuum (“empty space”), but that, like in SED, space is filled by a background random field of the white noise type. Consider averages with respect to combination of a prequantum random field for a quantum system, e.g., electron, and the background random field. Compute average with respect such \( \phi(\omega) \) and try to extract quantum average by ignoring the contribution of the background field.

Let \( \rho \) be a density operator. Set \( D = \rho \). This operator determines uniquely Gaussian prequantum random vector. In the case \( M = L_2 \) it is the prequantum random field, e.g., electronic random field. Set \( \hat{D} = D + \alpha I \), where \( I \) is the unit operator. Consider corresponding Gaussian random vector \( \phi(\omega) \). Denote by \( \mu \) its probability distribution. In the case \( M = L_2 \) it describes a mixture of the prequantum random field describing quantum particles with the background (say zero point) random field. The latter is white noise in \( L_2 \). We have

\[
Ef_A(\phi(\omega)) = \langle f_A \rangle_\mu = \text{Tr} \hat{D} \hat{A} = \text{Tr} \hat{\rho} \hat{A} + \alpha \text{Tr} \hat{A}.
\]

In this model (modification of PCSFT created in [10]–[15]) quantum

\footnote{At the moment we forget about presented above argument on scaling with respect to dispersion of the prequantum random field. It is important if one proceed beyond QM, but in this paper we just reproduce statistical predictions of QM.}
average can be obtained as a shift of classical average:

$$\langle \hat{A} \rangle^\rho = \langle f_A \rangle^\mu - \alpha \text{Tr} \hat{A}. \quad (7)$$

Thus quantum average is extracted from prequantum average by ignoring the contribution of the background field. We remark that in physical case, $M = L_2$, in general $\text{Tr} \hat{A}$ diverges. QM formalism is a way to ignore such divergences.

### 3 Operator-representation of the wave function of a composite system

Let $W$ be a real Hilbert space. We denote the space of self-adjoint operators acting in $W$ by the symbol $L_s(W)$. Since in this paper we consider only the finite dimensional real case, this space can be realized as space of all symmetric matrices.

Let $H_1$ and $H_2$ be two real (finite dimensional) Hilbert spaces. We put $H = H_1 \otimes H_2$. Any vector $\Psi \in H$ can be represented in the form

$$\Psi = \sum_{j=1}^{m} \psi_j \otimes \chi_j, \quad \psi_j \in H_1, \chi_j \in H_2, \quad (8)$$

and it determines a linear operator from $H_2$ to $H_1$

$$\hat{\Psi} \phi = \sum_{j=1}^{m} (\phi, \chi_j) \psi_j, \quad \phi \in H_2. \quad (9)$$

We remark that this operator is uniquely determined by $\Psi$, i.e., it does not depend on representation $[\Psi]$. Its adjoint operator $\Psi^*$ acts from $H_1$ to $H_2$:

$$\hat{\Psi}^* \psi = \sum_{j=1}^{m} (\psi, \psi_j) \chi_j, \quad \psi \in H_1. \quad (10)$$

Of course, $\hat{\Psi} \hat{\Psi}^* : H_1 \to H_1$ and $\hat{\Psi}^* \hat{\Psi} : H_2 \to H_2$ and these operators are self-adjoint and positively defined.

Consider operator $\rho = \Psi \otimes \Psi$. Then operators $\rho^{(1)} = \text{Tr}_{H_2} \rho = \hat{\Psi} \hat{\Psi}^*$ and $\rho^{(2)} = \text{Tr}_{H_1} \rho = \hat{\Psi}^* \hat{\Psi}$.

**Lemma 1.** Let $\Psi \in H$ of the form $[\Psi]$ be normalized by 1. Then, for any pair of operators $\hat{A_j} \in L_s(H_j), j = 1, 2$,

$$\text{Tr} \hat{\Psi} \hat{A_2} \hat{\Psi}^* \hat{A_1} = (\hat{A_1} \otimes \hat{A_2}) \Psi \equiv (\hat{A_1} \otimes \hat{A_2} \Psi, \Psi). \quad (11)$$
Proof. Set $\hat{C} = \hat{\Psi} \hat{A}_2 \hat{\Psi}^* \hat{A}_1$, it acts from $H_1$ to $H_1$. We have $\hat{C} u = \sum_k \sum_j (\hat{A}_1 u, \psi_j) (\hat{A}_2 \chi_j, \chi_k) \psi_k$. Thus, for an orthonormal basis $\{e_i\}$ in $H_1$, 

$$\text{Tr} \hat{C} = \sum_i \langle \hat{C} e_i, e_i \rangle = \sum_k \sum_j \langle \hat{A}_2 \chi_j, \chi_k \rangle \sum_i \langle \hat{A}_1 e_i, \psi_j \rangle (e_i, \psi_k)$$

$$= \sum_k \sum_j \langle \hat{A}_1 \psi_j, \psi_k \rangle (\hat{A}_2 \chi_j, \chi_k).$$

On the other hand, 

$$\langle \hat{A}_1 \otimes \hat{A}_2 \Psi, \Psi \rangle = \sum_k \sum_j \langle \hat{A}_1 \psi_j \otimes \hat{A}_2 \chi_j, \psi_k \otimes \chi_k \rangle = \sum_k \sum_j \langle \hat{A}_1 \psi_j, \psi_k \rangle (\hat{A}_2 \chi_j, \chi_k).$$

4 Interpretation of the state vector of a composite quantum system

Let the state vectors of systems $S_1$ and $S_2$ belong to Hilbert spaces $H_1$ and $H_2$, respectively. Then by axiomatics of QM, see [31], the state vector $\Psi$ of the composite system $S = (S_1, S_2)$ belongs to $H = H_1 \otimes H_2$. We remark that the interpretation of the state vector $\Psi$ of a composite system is not as straightforward as for a single system. It is known that, in general, the pure state $\Psi$ of a composite system does not determine pure states for its components. This point was discussed in detail by von Neumann, see [31]. The same problem was discussed by Schrödinger in his famous paper on Schrödinger’s cat [?]. He pointed out that in such a situation it is not natural to consider $\Psi \in H$ as the state vector of a composite system. The $\Psi$ is simply a table of correlations between systems $S_1$ and $S_2$. We also mention Naimark’s theorem by which any density operator can be realized as the reduced density operator corresponding to a pure state from a larger Hilbert space. This mathematical result looks quite confusing if one interprets $\Psi$ as the state of a composite system.

---

6Take the density operator $\rho = \Psi \otimes \Psi$ corresponding to the pure state $\Psi$. In general reduced density operators $\rho^{S_1} = \text{Tr}_{H_2} \rho$ and $\rho^{S_2} = \text{Tr}_{H_1} \rho$ do not correspond to pure states in $H_1$ and $H_2$, respectively. Here $\text{Tr}_{H_i}$ is the partial trace with respect to $H_i$. Only in the case of factorizable state $\Psi = \psi_1 \otimes \psi_2$ one can assign the pure state $\psi_1$ to $S_1$ and the pure state $\psi_2$ to $S_2$.

7Thus $\Psi$ cannot be considered as “physical state” of a concrete composite system $S$. 

9
This viewpoint matches well with the PCSFT-approach. We interpret a normalized vector \( \Psi \in H \) not as the state vector of a concrete composite system \( S = (S_1, S_2) \), but as one of blocks of the covariance operator for the prequantum random field \( \phi(\omega) = (\phi_1(\omega), \phi_2(\omega)) \) describing \( S = (S_1, S_2) \).

## 5 Coupling between PCSFT and QM averages for composite systems

Let \( \phi_1(\omega) \) and \( \phi_2(\omega) \) be two random vectors, in Hilbert spaces \( H_1 \) and \( H_2 \), respectively. Consider Cartesian product of these Hilbert spaces: \( H = H_1 \times H_2 \) (don’t mix with \( H = H_1 \otimes H_2 \)) and the random vector \( \phi(\omega) = (\phi_1(\omega), \phi_2(\omega)) \in H \) such that:

- a) its expectation \( E\phi = 0 \);
- b) its dispersion \( \sigma^2(\phi) = E||\phi||^2 < \infty \). Take its covariance operator \( D \) which is determined by the symmetric (positively defined) bilinear form: \( (Du, v) = E(u, \phi(\omega))(v, \phi(\omega)) \), where vectors \( u, v \in H \).

This operator has the block structure

\[
D = \begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{pmatrix},
\]

where \( D_{ii} : H_i \to H_i, D_{ij} : H_j \to H_i \). The operator is self-adjoint.

Hence \( D_{ii}^* = D_{ii} \), and \( D_{12}^* = D_{21} \). Here:

\[
(D_{ij}u_j, v_i) = E(u_j, \phi_j(\omega))(v_i, \phi_i(\omega)), u_i \in H_i, v_j \in H_j.
\]

Let \( \tilde{A}_i \in L_s(H_i), i = 1, 2 \). It determines the quadratic function on the Hilbert space \( H_i \):

\[
f_{\tilde{A}_i}(\phi_i) = (\tilde{A}_i \phi_i, \phi_i).
\]

We recall that we proceed in the finite dimensional real case. Thus \( H_1 = \mathbb{R}^m \) and \( H_2 = \mathbb{R}^n \), and \( H = \mathbb{R}^{n+m} \); \( \phi_1(\omega) = (\phi_{11}(\omega), \phi_{12}(\omega), ..., \phi_{1m}(\omega)) \) and \( \phi_2(\omega) = (\phi_{21}(\omega), \phi_{22}(\omega), ..., \phi_{2n}(\omega)) \). Here \( D \) is represented by \((n+m) \times (n+m)\) matrix, \( D_{11} \) by \( m \times m \) matrix and \( D_{22} \) by \( n \times n \) matrix; all these matrices are positively defined; \( D_{12} \) is represented by \( m \times n \) matrix.

Consider a Gaussian random vector. It is determined by its probability distribution on the space \( H \). In the case \( D > 0 \) it has the density:

\[
p_D(u) = \frac{1}{\sqrt{(2\pi)^n \det D}} e^{-\frac{1}{2}(Ku, u)}, \quad K = D^{-1}.
\]
The inverse matrix $K$ of the covariance matrix $D$ also has the block structure. For the variable $u = (x, y), x \in H_1, y \in H_2$, the Gaussian density can be written as

$$p_D(x, y) = \frac{1}{\sqrt{(2\pi)^n \det D}} e^{-\frac{1}{2}[(K_{11}x,x) + 2(K_{12}y,x) + (K_{22}y,y)]}$$

Finally, we remark that in the finite dimensional case quadratic functions have the form:

$$f_{A_1}(x) = \sum_{i,j=1}^{m} A_{1ij}x_ix_j, x \in H_1, f_{A_2}(y) = \sum_{i,j=1}^{m} A_{2ij}y_iy_j, y \in H_2.$$  

**Lemma 2.** For any Gaussian random vector $\phi(\omega) = (\phi_1(\omega), \phi_2(\omega))$ having zero average and any pair of operators $\hat{A}_i \in \mathcal{L}_s(H_i), i = 1, 2$, the following equality takes place:

$$\langle f_{A_1}, f_{A_2} \rangle_{\phi} \equiv E f_{A_1}(\phi_1(\omega)) f_{A_2}(\phi_2(\omega)) = (\text{Tr} D_{11} \hat{A}_1)(\text{Tr} D_{22} \hat{A}_2) + 2\text{Tr} D_{12} \hat{A}_2 D_{21} \hat{A}_1. \quad (12)$$

To proof this Lemma, we calculate the integral

$$\int_{\mathbb{R}^{n+m}} (\hat{A}_1 x, x)(\hat{A}_2 y, y)p_D(x, y)dx dy.$$

We remark that

$$\text{Tr} D_{ii} \hat{A}_i = E f_{A_i}(\phi_i(\omega)), i = 1, 2. \quad (13)$$

Thus we have

$$E f_{A_1} f_{A_2} = E f_{A_1} E f_{A_2} + 2\text{Tr} D_{12} \hat{A}_2 D_{21} \hat{A}_1. \quad (14)$$

Now take an arbitrary pure state of a composite system $S = (S_1, S_2)$, a normalized vector $\Psi \in H = H_1 \otimes H_2$.

Consider a Gaussian vector random field such that $D_{12} = \hat{\Psi}$. By Lemma 1 the last summand in the right-hand side of (14) is equal to QM-average. Hence, we obtain:

$$\frac{1}{2} E(f_{A_1} - E f_{A_1})(f_{A_2} - E f_{A_2}) = \langle \hat{A}_1 \otimes \hat{A}_2 \Psi, \Psi \rangle \equiv \langle \hat{A}_1 \otimes \hat{A}_2 \rangle_{\Psi}, \quad (15)$$

or, for covariance of two classical random vectors $f_{A_1}, f_{A_2}$, we have:

$$\frac{1}{2} \text{cov} (f_{A_1}, f_{A_2}) = \langle \hat{A}_1 \otimes \hat{A}_2 \rangle_{\Psi}. \quad (16)$$
5.1 PCSFT-interpretation of the tensor product

All previous considerations can be repeated in infinite dimensional Hilbert spaces, in particular, for $L_2$-space. In this case we consider random fields, instead of random vectors. Since the real physical situation is described by $L_2$-space, in all discussions on interpretation we will operate with random fields.

Coupling (16) between classical and quantum averages established in this section implies that the state $\Psi \in H$ of a composite system $S = (S_1, S_2)$ can be interpreted as the vector-representation of the nondiagonal block of the covariance operator $D$ of corresponding pre-quantum random field, see by (15).

By PCSFT an ensemble of composite systems $S = (S_1, S_2)$ is described by a random fields $\phi(\omega) = (\phi_1(\omega), \phi_2(\omega))$. A quantum state $\Psi$ is just a symbolic representation of $D_{12}$-block of the covariance operator of this field. We emphasize that state space of this field is the Carhesian product $H = H_2 \times H_2$ of corresponding state spaces.

6 Diagonal terms of the covariance operator

Operators $D_{ii}$ are responsible for averages of functionals $f(\phi_i(\omega))$, i.e., depending only on one of components of the vector random field $\phi(\omega)$. In particular, $E f_{\hat{A}_i}(\phi_i(\omega)) = \text{Tr} D_{ii} \hat{A}_i$.

We will construct such a random field that these averages will match those given by QM. For the latter, we have:

$$\langle \hat{A}_1 \rangle_\Psi = (\hat{A}_1 \otimes I_2 \Psi, \Psi) = \text{Tr}(\Psi \Psi^*) \hat{A}_1; \langle \hat{A}_2 \rangle_\Psi = (I_1 \otimes \hat{A}_2 \Psi, \Psi) = \text{Tr}(\hat{\Psi}^* \hat{\Psi}) \hat{A}_2,$$

where $I_i$ denotes the unit operator in $H_i, i = 1, 2$. Thus it would be natural to take

$$D_\Psi = \begin{pmatrix} \hat{\Psi}^* \hat{\Psi} & \hat{\Psi} \\ \hat{\Psi}^* & \hat{\Psi}^* \hat{\Psi} \end{pmatrix}.$$

However, in general (i.e., for an arbitrary pure state $\Psi$) this operator is not positively defined. Therefore (in general) it could not be chosen as the covariance operator of a random vector\(^8\). Let us con-

\[^8\text{One cannot construct a probability measure on Hilbert space } H \text{ with the covariance operator } D_\Psi.\]
Consider a modification of $D_\Psi$ which will be positively defined and such that quantum and prequantum averages will be coupled by a simple rule. Thus from quantum averages one can easily find prequantum averages and vice versa.

**Proposition 1.** For any pure state $\Psi \in H$, the operator

$$\tilde{D}_\Psi = \left( \begin{array}{cc} (\tilde{\Psi}^* + I_{1/4}) & \tilde{\Psi} \\ \tilde{\Psi}^* & (\Psi^* + I_{2/4}) \end{array} \right)$$

is positively defined.

**Proof.** For $\phi = (\phi_1, \phi_2) \in H$, we have:

$$\langle \tilde{\Psi} \tilde{\Psi}^* + I_{1/4} \rangle = \langle \tilde{\Psi}^* \phi_1 \rangle = \langle ||\tilde{\Psi}^* \phi_1||^2 + \frac{||\phi_1||^2}{4} + (\tilde{\Psi} \phi_2, \phi_1) + ||\tilde{\Psi} \phi_2||^2 + \frac{||\phi_2||^2}{4} \rangle \geq \langle ||\Psi \phi_1||^2 - ||\Psi^* \phi_1|| ||\phi_2|| + \frac{||\phi_2||^2}{4} \rangle \geq 0.$$ Thus operator $\tilde{D}_\Psi$ is positively defined.

Suppose now that $\phi(\omega)$ is a random vector with the covariance operator $\tilde{D}_\Psi$. Then $Ef_{A_1}(\phi_1(\omega)) = \text{Tr}[(\tilde{\Psi}^* \tilde{\Psi} + I_{1/4}) A_1] = (\tilde{A}_1 \otimes I_2 \Psi, \Psi) + \frac{1}{4} \text{Tr} \tilde{A}_1 = \langle \tilde{A}_1 \rangle_\Psi + \frac{1}{4} \text{Tr} \tilde{A}_1$, where $\langle \tilde{A}_1 \rangle_\Psi$ is quantum average of the observable $\tilde{A}_1$. Thus:

$$\langle \tilde{A}_1 \rangle_\Psi = Ef_{A_1}(\phi_1(\omega)) - \frac{1}{4} \text{Tr} \tilde{A}_1.$$  \hspace{1cm} (17)

This relation for averages together with relation (16) provides coupling between PCSFT and QM. Quantum statistical quantities can be obtained from corresponding quantities for classical random field. Thus, “irreducible quantum randomness” [31], can be, finally reduced to randomness of classical prequantum fields.

In the infinite dimensional case the shift $I/4$ corresponds to white noise. It can be considered as the ground noise (fluctuations of vacuum?), a kind of zero-point field, cf. with stochastic electrodynamics. It is reasonable to exclude this ground noise from theory giving predictions for observations. It seems that it was done in QM.

---

9 Of course, the same effect can be approached by adding $\alpha I$ for $\alpha \geq 1/4$. 

13
7 Consequences for quantum information

7.1 Demystification of entanglement

While the mathematical description of entanglement in conventional QM is well established, its physical meaning is far from to be clear. Einsteinian viewpoint \[8\] was that entanglement is a special exhibition of classical statistical correlations. The latter are determined by the common preparation procedure for, e.g., pairs of entanglement photons. Since an entangled state contains correlations for incompatible quantum observables, this viewpoint is based on the assumption that QM is not complete. It was mentioned already in EPR article \[8\] that under the assumption of completeness of QM there is no other possibility to explain entanglement than to assume action at the distance, so called “quantum nonlocality.” If correlations (with respect to incompatible observables) were not present already in the initial preparation, then they can be created only via action at the distance. The latter possibility was considered by Einstein as totally absurd, see, e.g., \[8\]. Surprisingly this possibility was taken seriously by the modern quantum community (especially its quantum information part). The explanation of the essence of entanglement as the presence of nonlocal correlations dominates in theoretical and especially experimental quantum physics. However, the PCSFT-model supports Einsteinian viewpoint.

Consider a pair of systems in the entangled state (in QM formalism): \( \Psi = (|+\rangle |-\rangle - |−\rangle |+\rangle)/\sqrt{2} \). In PCSFT \( \Psi \) is interpreted as the block of the covariance matrix for a pair of random fields. All manipulations with “entangled states” are simply local manipulations with fields inducing changes in correlations. Consider, for example, quantum teleportation. From PCSFT-viewpoint, this procedure is not about state teleportation, but about preparation of two systems (one for Alice and another for Bob) which are correlated in a special way.

7.2 Quantum version of classical wave computer?

If physical reality is really described by PCSFT-formalism, then the conventional postulates of quantum information theory should be care-
fully reconsidered. Since in PCSFT state space of a composite system is Cartesian and not tensor product of state spaces of single-systems, there is no reason to assume that a composite quantum system has exponentially larger information capacity than combination of classical random fields. It seems that either classical field version of “quantum computer” is possible or it will not provide advantages comparing with classical probabilistic computer. Of course, the problem is very complicated and it should be analyzed in more detail. Finally, we mention that PCSFT-viewpoint on entanglement, namely, as representation of correlations of classical random fields implies (at least theoretical) possibility to create entanglement of macroscopic objects, cf. [32].

The project “Beyond Quantum” has been supported by the grant Mathematical Modelling of Växjö University during the last 15 years. This paper was finally completed on the basis of visiting professorships at Tokyo University of Science (the grant Quantum Bio Informatics), March 2009, and Capital Normal University of Beijing (the grant Quantum Information and Entanglement). I would like to thank M. Ohya, N. Watanabe and S.-M. Fei for hospitality.

References

[1] Khrennikov, A.(ed): Foundations of Probability and Physics. Series PQ-QP: Quantum Probability and White Noise Analysis 13. WSP, Singapore (2001)

[2] Khrennikov, A.(ed): Quantum Theory: Reconsideration of Foundations. Ser. Math. Model. 2, Växjö University Press, Växjö (2002); electronic volume: http://www.vxu.se/msi/forskn/publications.html

[3] Adenier, G., Khrennikov, A. and Nieuwenhuizen, Th.M. (eds.): Quantum Theory: Reconsideration of Foundations-3. American Institute of Physics, Ser. Conference Proceedings 810, Melville, NY (2006)

[4] Adenier, G., Fuchs, C. and Khrennikov, A.(eds): Foundations of Probability and Physics-3. American Institute of Physics, Ser. Conference Proceedings 889, Melville, NY (2007)

[5] N. Bohr, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 48, 696-702 (1933).
[6] A. Einstein, *The Collected Papers of Albert Einstein*, Princeton Univ. Press, Princeton, 1993.

[7] A. Einstein and L. Infeld, *Evolution of Physics: The Growth of Ideas from Early Concepts to Relativity and Quanta*, Simon and Schuster, 1961.

[8] Einstein, A., Podolsky, B., and Rosen, N., Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* 47, 777–780 (1935).

[9] A. Yu. Khrennikov, Einstein’s dream. Proceedings of Conference *The nature of light: What are photons?* C. Roychoudhuri, A. F. Kracklauer, K. Creath. Proceedings of SPIE, 6664, 2007, 666409-1 – 666409-9.

[10] A. Yu. Khrennikov, “Prequantum classical statistical model with infinite dimensional phase-space,” *J. Phys. A: Math. Gen.* 38, pp. 9051-9073, 2005.

[11] A. Yu. Khrennikov, “Generalizations of quantum mechanics induced by classical statistical field theory,” *Found. Phys. Letters* 18, pp. 637-650, 2005.

[12] A. Yu. Khrennikov, “Nonlinear Schrödinger equations from prequantum classical statistical field theory,” *Physics Letters A* 357, pp. 171-176, 2006.

[13] A. Yu. Khrennikov, “Prequantum classical statistical field theory: Complex representation, Hamilton-Schrödinger equation, and interpretation of stationary states,” *Found. Phys. Lett.* 19, pp. 299-319, 2006.

[14] A. Yu. Khrennikov, “On the problem of hidden variables for quantum field theory,” *Nuovo Cimento B* 121, pp. 505-515, 2006.

[15] A. Yu. Khrennikov, Born’s rule from classical random fields, *Physics Letters A*, 372, N 44, 6588-6592 (2008).

[16] De la Pena, L. and Cetto, A. M.: The quantum dice: An introduction to stochastic electrodynamics. Kluwer, Dordrecht (1996)

[17] A. Casado, T. Marshall, E. Santos, *J. Opt. Soc. Am. B* 14, pp. 494-205, 1997.

[18] G. Brida, M. Genovese, M. Gramegna, C. Novero and E. Predazzi, *Phys. Lett A* 299, pp. 121-141, 2002.
[19] Boyer, T. H.: A brief survey of stochastic electrodynamics. In: Barut, A. O. (ed) Foundations of Radiation Theory and Quantum Electrodynamics, pp. 141-162. Plenum, New York (1980)

[20] Scully, M. O. and Zubairy, M. S.: Quantum optics. Cambridge University Press, Cambridge (1997)

[21] Louisell, W. H.: Quantum statistical properties of radiation. J. Wiley, New York (1973)

[22] Mandel, L. and Wolf, E.: Optical coherence and quantum optics. Cambridge University Press, Cambridge (1995)

[23] Nelson, E: Quantum fluctuation Princeton Univ. Press, Princeton (1985)

[24] Davidson, M.: J. Math. Phys. 20, 1865-1870 (1979)

[25] Davidson, M.: Stochastic models of quantum mechanics - a perspective. In: Adenier, G., Fuchs, C. and Khrennikov, A. (eds.) Foundations of Probability and Physics-4. American Institute of Physics, Ser. Conference Proceedings, vol. 889, pp. 106–119. Melville, NY (2007)

[26] 't Hooft, G.: Quantum mechanics and determinism. hep-th/0105105 (2001)

[27] 't Hooft, G.: Determinism beneath quantum mechanics. quant-ph/0212095 (2002)

[28] 't Hooft, G.: The free-will postulate in quantum mechanics. quant-ph/0701097 (2007)

[29] Elze, T.: The attractor and the quantum states. arXiv: 0806.3408 (2008)

[30] L. E. Ballentine, Rev. Mod. Phys., 42, 358–381 (1970).

[31] J. von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton Univ. Press, Princeton, N.J., 1955.

[32] A. Allahverdyan, A. Khrennikov, and Th. M. Nieuwenhuizen, Brownian entanglement, Phys. Rev. A, 71, 032102-1 – 032102-14 (2005).