Generalized String Functions Of N=1 Space-Time Supersymmetric String Vacua

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Abstract

We present a dual formulation of the construction of $N = 2$ nonlinear $\sigma$-model type conformal field theories with $c = 9$ which are mainly used as internal sectors of Calabi-Yau heterotic string compactifications. The supercurrents $G_{\pm}(z)$ and the higher components of the spectral flow superfields $X_{\pm}(z) + \theta_{\pm}Y_{\pm}(z)$ turn out in each case to be tensor products of two simple currents of a $c = 1$ and a non-supersymmetric parafermionic $c = 8$ CFT. The characters of the latter model can be regarded as string functions of the coset $\frac{\sigma}{U(1)}$. In particular, for the $(1)^9$ Gepner model we discuss these string functions in detail, realizing this model contains a broken $E_8$ gauge symmetry of which only the abelian subalgebra remains a symmetry of the spectrum. As an example, we construct a simple model leading to a string theory with a massless spectrum of 36 $E_6$ generations and an extended gauge symmetry $E_6 \times SU(3)^4$.

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1. Introduction

Since the realization of the importance of Calabi-Yau manifolds and $N = 2$ world-sheet conformal field theories (CFT) for the construction of phenomenologically interesting four-dimensional $N = 1$ space-time supersymmetric heterotic string models [13], a large set of consistent models has been investigated [2,3,5,9,10,14,16,17], including many three generation cases [1,6,11,21,22]. In 1988 D. Gepner [10] suggested a quite interesting construction of string vacua which could in principle be performed for every $N = 2$ CFT with central charge $c = 9$. The main ingredient is an automorphism of the $N = 2$ Virasoro algebra, the so-called spectral flow, which naturally provides one with the space-time supersymmetry generator and a GSO projection on the world-sheet. The authors of [8] mainly focused their attention on CFTs describing $N = 2$ nonlinear $\sigma$-models on Calabi-Yau spaces. These $\sigma$-models could be used as the internal sector of heterotic string models leading to a minimal gauge group $E_6$. The characters of a $\sigma$-model CFT are given by flow invariant orbits of integer $U(1)$ charge of some arbitrary rational $N = 2$ CFT. Their actual idea to use directly the extension of the $N = 2$ Virasoro algebra by the local spectral flow operators $X^\pm(z) + \theta^\pm Y^\pm(z)$ of superconformal dimension $(H,Q) = (3/2, \pm3)$, a nonlinear $\mathcal{W}$-algebra, however, failed because of the non-rationality of this algebra [18]. In this letter we will show that using only a rational subalgebra of the entire spectral flow $\mathcal{W}$-algebra, one arrives at a dual formulation of $\sigma$-model CFTs, in which the roles of the spectral flow $X^\pm(z)$ and the supercurrent $G^\pm(z)$ are interchanged. The $c = 9$ theory is split into two parts of central charge $c = 1$ and $c = 8$, respectively. The former one contains the $U(1)$ current and the original spectral flow operators $X^\pm(z)$, whereas the latter part is non-supersymmetric. The characters of this $c = 8$ CFT can be regarded as generalized string functions of the $\sigma$-model with respect to the $U(1)$ current. After describing this in more detail in sect. 2, we calculate the string functions for the $(1)^9$ Gepner model, revealing a hidden $E_8$ structure which can be used to reconstruct this model in the dual approach using only $E_8$ $\Theta$-functions at level $k = 3$. Finally, in sect. 4 we constructively apply the dual formalism to a model where the $c = 8$ theory is given by four copies of the $SU(3)_1$ Kac-Moody algebra. The model turns out to have 36 $E_6$ generations and an enlarged gauge group $E_6 \times SU(3)^4$. Since our construction is quite general, all methods used for building $N = 1$ space-time supersymmetric string models should fit into this scheme, e.g. the latter fairly simple model appears in a rather complicated covariant lattice construction [17].

2. Dual construction of $N = 2$ $\sigma$-model partition functions

In this section we describe a general formalism to construct $N = 2$ supersymmetric $\sigma$-models of Calabi-Yau manifolds. \textsuperscript{†} It is partially a review of the work of Eguchi, Ooguri, Taormina, Yang [8], however, with more emphasis on rational modular properties of the characters involved. First, we recall the important role played by the spectral flow algebra, which leads to a quite general form of that part of the $\sigma$-model characters which depends

\textsuperscript{†} Recently, it has been pointed out that in order to allow also mirror manifolds of rigid models with $n_{27} = 0$ one has to consider more general classes of manifolds, so-called special Fano varieties [7,23,24].
explicitly on the $U(1)$ charge $Q$. Secondly, knowing the $\sigma$-model partition function in the $NS$ sector, we show how to construct a consistent $N = 1$ space-time supersymmetric string model from it.

Suppose, there is a given rational $N = 2$ super CFT with central charge $c = 9$ and characters $\chi^\sigma_i$ which form a finite-dimensional representation of the subgroup $M'$ of the modular group generated by $\{T^2, S\}$. The continuous automorphism of the $N = 2$ Virasoro algebra

$$L'_n = L_n + \eta J_n + \frac{c}{6} \eta^2 \delta_{n,0} \quad (G^\pm)'_n = G^\pm_{n+\eta} \quad J'_n = J_n + \frac{c}{3} \eta \delta_{n,0}$$

is called the spectral flow. For $\eta = 1$ its action on the representations of the $N = 2$ CFT arranges them into a finite number of spectral flow invariant orbits. It can be shown [10] that for $c = 3d, d \in \mathbb{N}$ all orbits with integer $U(1)$ charge form a subrepresentation of $M'$ and restriction of one modular invariant partition function onto these orbits yields another one. It has been pointed out in [8] that these new invariants are the partition functions of $N = 2$ nonlinear $\sigma$-models on Calabi-Yau manifolds. Implementing these as the internal part of a heterotic string model yields vacua with phenomenologically desired $N = 1$ space-time supersymmetry. We remark that the construction of new modular invariants outlined above can also be considered as modding out the spectral flow simple current $Y^{+2}(z)$ with superconformal weights $(H, Q) = (2, 2)$ using the Schellekens-Yankielowicz technique [19,20]. In this case, the monodromy charge is equal to the $U(1)$ charge. Furthermore, the extension of the $N = 2$ Virasoro algebra by the local, holomorphic and (anti)chiral super fields $X^{\pm3}_3(z) + \theta^\pm Y^{\pm2}_3(z)$ yields a nonlinear $\mathcal{W}$-algebra. Thus, one can also consider the spectral flow invariant orbits as general reducible representations of this $\mathcal{SW}(1, (3/2)^{\pm3})$ algebra. The main obstacle for using this extended algebra for the direct construction of new models is its non-rationality. Although only a finite number of $U(1)$ charges $Q$ are allowed, there are no restrictions on the values of the conformal dimensions $H$. The idea is to use not the complete spectral flow algebra, but only a rational subalgebra.

The $\mathcal{SW}(1, (3/2)^{\pm3})$ algebra contains two $N = 2$ Virasoro algebras, the obvious one with central charge $c = 9$ and another one with $c = 1$, the generators of which are

$$\left\{ \frac{1}{3} J, \frac{1}{6} J^2, \frac{1}{3} X^+, \frac{1}{3} X^- \right\}. \quad (2.2)$$

Since this algebra admits only three representations in the $NS$ sector with weights $(H, Q) \in \{(0, 0), (1/6, \pm1/3)\}$, the flow invariant orbits $\chi^\sigma_i$ can be expanded into three terms:

$$\chi^\sigma_i = \chi^\sigma_i(\tau, \theta) = \chi^{N=2,c=1}_i(\tau, 3\theta) S_i(\tau) + \chi^{N=2,c=1}_i(\tau, 3\theta) T_i(\tau) + \chi^{N=2,c=1}_i(\tau, 3\theta) U_i(\tau) \quad (2.3)$$

where the index $i$ runs over all flow invariant orbits with integer $U(1)$ charge and we have expressed the $c = 1$ characters in terms of $SU(2)$ $\Theta$-functions

$$\Theta_{n,m}(q, \theta) = \sum_{j \in \mathbb{Z} + \frac{n}{2m}} q^{mj^2} e^{2\pi i \theta j}. \quad (2.4)$$

† Note that the involved states need not to be (anti)chiral.
Thus, the series $S_i(\tau), T_i(\tau), U_i(\tau)$ can be interpreted as generalized string functions corresponding to the coset

$$\frac{\sigma - \text{model}}{U(1)},$$

where the $U(1)$ current $j(z) = \sqrt{3}i\partial \phi(z)$ is compactified on a circle such that the vertex operator $X^{\pm 3}(z) = \sqrt{6} : e^{\pm i\sqrt{3} \phi(z)} :$ is well-defined. However, by factorizing an $N = 2 \sigma$-model as

$$\sigma - \text{model} = (\text{compact } U(1)) \otimes (\text{non-supersymmetric } c = 8 \text{ CFT})$$

we have interchanged in some sense the roles of the two spin-3/2 fields $G^{\pm}(z)$ and $X^{\pm}(z)$. In particular, the superpartners $G^{\pm}(z)$ and $Y^{\pm 2}(z)$ are now contained in the second and third term of (2.3) with suitable fields $\Gamma^{\pm}(z)$ from the non-supersymmetric $c = 8$ CFT:

$$G^{\pm}(z) = \sqrt{6} : e^{\pm i\frac{1}{\sqrt{3}} \phi(z)} : \otimes \Gamma^{\pm}_{H=4/3}(z), \quad Y^{\pm 2}(z) = \pm 6 : e^{\pm i\frac{2}{\sqrt{3}} \phi(z)} : \otimes \Gamma^{\pm}_{H=4/3}(z)$$

whereas the total $c = 9$ energy-momentum tensor remains in the first term

$$L_{\text{tot}} = \frac{1}{6} : j j : + L_{c=8}.$$  

These observations thus reveal a different way of constructing $N = 2 \sigma$-models. One considers (2.3,2.6,2.7) as the starting point and adds a suitable rational $c = 8$ non-supersymmetric CFT containing a simple current of dimension $H = 4/3$. That the existence of such a current is sufficient for the entire model to contain the $SV(1,(3/2)^{\pm 3})$ spectral flow algebra can be seen by the following argument. Due to the general formula for a simple current of order $N [19,20]$

$$H = \frac{r(N - 1)}{2N} \mod \mathbb{Z}, \quad r \in \mathbb{Z}$$

the order of the simple current $\Gamma^{+}$ must be divisible by 3. Thus, we define $[\Gamma^{-}] = [\Gamma^{+}]^{N-1}$. Because of $[\Gamma^{+}] \times [\Gamma^{-}] = [1]$ the conformal dimension of $\Gamma^{-}$ has to be $H = 4/3$, as well. This implies the OPE to have the following parafermionic like form:

$$\Gamma^{+}(z) \Gamma^{-}(w) = \frac{1}{(z-w)^{\frac{3}{2}}} + \frac{1}{3} L_{c=8}(w) + \ldots$$

Note that in general in (2.10) there can also appear spin-1 currents. However, these can be canceled by choosing the symmetric combination $\frac{1}{2}(\Gamma^{+} + \Gamma^{-})$, which corresponds to modding out the simple currents $\Gamma^{+}$ and $\Gamma^{-}$ separately and then adding the partition functions. We will come back to this point in sect. 3. Analogous to the parafermionic case, (2.10) implies that $G^{+}(z) G^{-}(w)$ satisfies the $N = 2$ Virasoro relation. The associativity of the OPE and the primarity of $G^{\pm}(z)$ then implies that the OPEs $G^{\pm}(z) G^{\pm}(w)$ have vanishing singular parts. The OPE $G^{\pm}(z) X^{\pm 3}(w)$ shows that $Y^{\pm 2}(z)$ is the superpartner of $X^{\pm 3}(z)$, so that the entire $SV(1,(3/2)^{\pm 3})$ algebra is satisfied.
Before continuing the study of these string functions, we briefly review the construction of heterotic strings, when a $\sigma$-model partition function has already been chosen [8]. Assume, given a $\sigma$-model partition function in the $NS$ sector:

$$Z^{NS} = \sum_{ij} N_{ij} \chi^\sigma_{i,j}^{NS}(\tau, \theta) \chi^{\sigma,NS}_{i,j}(\tau, \bar{\theta})$$

(2.11)

where the characters can be expanded as in (2.3). Then the other sectors can be calculated easily:

$$\chi_i^\sim(\tau, \theta) = \frac{\Theta_{0,6} - \Theta_{6,6}}{\eta} S_i(q) + \frac{\Theta_{4,6} - \Theta_{2,6}}{\eta} T_i(q) + \frac{-\Theta_{-2,6} + \Theta_{4,6}}{\eta} U_i(q)$$

$$\chi_i^R(\tau, \theta) = \frac{-\Theta_{-3,6} + \Theta_{3,6}}{\eta} S_i(q) + \frac{\Theta_{1,6} + \Theta_{5,6}}{\eta} T_i(q) + \frac{\Theta_{-5,6} + \Theta_{1,6}}{\eta} U_i(q)$$

(2.12)

The sectors with well-defined $U(1)$ charge parity are

$$NS_i^+ = \frac{\Theta_{0,6}}{\eta} S_i + \frac{\Theta_{-4,6}}{\eta} T_i + \frac{\Theta_{4,6}}{\eta} U_i \quad (Q \in 2\mathbb{Z})$$

$$NS_i^- = \frac{\Theta_{6,6}}{\eta} S_i + \frac{\Theta_{2,6}}{\eta} T_i + \frac{-\Theta_{-2,6}}{\eta} U_i \quad (Q \in 2\mathbb{Z} + 1)$$

$$R_i^+ = \frac{\Theta_{3,6}}{\eta} S_i + \frac{\Theta_{-1,6}}{\eta} T_i + \frac{\Theta_{5,6}}{\eta} U_i \quad (Q \in 2\mathbb{Z} - 1/2)$$

$$R_i^- = \frac{\Theta_{-3,6}}{\eta} S_i + \frac{\Theta_{1,6}}{\eta} T_i + \frac{\Theta_{-5,6}}{\eta} U_i \quad (Q \in 2\mathbb{Z} + 1/2).$$

(2.13)

In order to obtain a heterotic string model we have to combine the internal part with the four-dimensional flat space-time part, their fermionic partners on the left-moving side and the gauge group $E_8 \otimes SO(10)$ on the right-moving side. How this can be done while preserving modular invariance has been shown by Gepner [10]. In light cone gauge one first couples the internal part to the two left-moving fermions which form a $SO(2)$ Kac-Moody algebra admitting the following four integrable representations: $(\chi_0)_{Q=0}^H$, $(\chi_v)^H_{Q=1/2}$, $(\chi_5)^H_{Q=1/2}$ and $(\chi_c)^H_{Q=1/2}$ where $Q$ is defined modulo $2\mathbb{Z}$. Finally, after performing a supersymmetric orbit construction using the spectral flow operator with $\eta = 1/2$ in the internal part, which corresponds to the Ramond ground state $(H, Q) = (3/8, 3/2)$, and the spinor representation in the external part, one obtains

$$\chi_{L,i}(\tau, \theta) = \chi_v^{SO(2)} NS_i^+ + \chi_0^{SO(2)} NS_i^- - \chi_c^{SO(2)} R_i^- - \chi_s^{SO(2)} R_i^+. \quad (2.14)$$

In the right-moving sector using the bosonic string map, one arrives at

$$\chi_{R,i}(\tau) = \left(\chi_0^{SO(10)} NS_i^+ + \chi_v^{SO(10)} NS_i^- + \chi_c^{SO(10)} R_i^+ + \chi_s^{SO(10)} R_i^-\right) \chi_0^{E_8}. \quad (2.15)$$

Using the explicit expressions for the $SO(10)$ characters, the splitting (2.3) of $\sigma$-model characters and the following expressions for the $E_6$ characters

$$\chi_0^{E_6} = \frac{1}{\eta^6} \{ (\theta_{0,6} + \theta_{6,6}) \theta_3^2 + (\theta_{0,6} - \theta_{6,6}) \theta_4^2 + (\theta_{-3,6} + \theta_{3,6}) \theta_2^5 \}$$

$$\chi_2^{E_6} = \frac{1}{\eta^6} \{ (\theta_{-5,6} + \theta_{1,6}) \theta_2^5 + (\theta_{-2,6} + \theta_{4,6}) \theta_3^5 + (\theta_{-2,6} + \theta_{4,6}) \theta_2^5 \}$$

$$\chi_2^{E_6} = \frac{1}{\eta^6} \{ (\theta_{1,6} + \theta_{5,6}) \theta_2^5 + (\theta_{-4,6} + \theta_{2,6}) \theta_3^5 + (\theta_{-4,6} + \theta_{2,6}) \theta_2^5 \}$$

(2.16)
one can write the two orbits in the following way:
\[
\chi_{L,i}(\tau, \theta) = \chi_i^{SO(2)} S_i + \chi_0^{SO(2)} N S_i^+ - \chi_c^{SO(2)} R_i^{-} - \chi_s^{SO(2)} R_i^{+}
\]
\[
\chi_{R,i}(\tau) = \chi_0^{E_8} S_i + \chi_2^{E_8} U_i + \chi_7^{E_8} T_i.
\]
Since both the orbit construction and the bosonic map preserve modular invariance, the partition function of the heterotic string can be written as
\[
Z \sim \frac{1}{|Im(\tau)||\eta|^4} \sum_{i,j} N_{i,j} \chi_{L,i}(\tau, \theta) \chi_{R,j}(\tau)
\]
where the constant is fixed by the requirement that the vacuum only appears once in the partition function. Thus, knowing the \(\sigma\)-model is sufficient to determine the heterotic string. Especially, the massless spectrum can be read off directly from (2.17,2.18) and all information is contained in the string functions \(S_i, T_i, U_i\), which appear in orbits of conformal dimension \(H = 0\) and \(H = 1/2\), respectively.

3. String functions of a Gepner model with 84 generations

Although this model has exhaustively been investigated in the past, we want to illuminate it from our dual point of view again. We will show that this model contains in a certain sense a broken \(E_8\) gauge symmetry of which only the abelian part \(U(1)^8\) appears in the massless spectrum. The appearance of an \(E_8\) symmetry in the massive spectrum was first observed in [25], where only the vacuum character had been investigated. We will show that the whole partition function can be reconstructed using affine \(E_8\) \(\Theta\)-functions at level \(k = 3\).

For the internal \(c = 9\) theory, one chooses a tensor product of nine copies of the \(c = 1\) unitary model of the \(N = 2\) super Virasoro algebra. In the \(NS\) sector there exist three representations labeled by \(l = m = 0\), \(l = m = 1\), \(l = -m = 1\) with conformal weights \((H, Q) = (0, 0)\) and \((1/6, \pm 1/3)\), respectively. We denote them by \(A,B,C\). Since the spectral flow cyclically permutes these three, the only flow invariant orbit containing the vacuum is
\[
\chi_0^\sigma = A^9 + B^9 + C^9.
\]
Some inspection enables one to write this character as
\[
\chi_0^\sigma = \frac{\Theta_0 + \Theta_{0,6/9}}{\eta} \frac{1}{\eta^k} \Theta_{0, k=3}(\tau, 0) + \ldots
\]
In table 1 we list all other flow invariant orbits and their combinatorial multiplicities giving the range of the index \(i\), e.g. 1680 = \((9\choose3)\).
On the level of characters for \( j \) fixed all \( \chi_i^j \) are identical. Thus, due to the philosophy of [8] after really identifying them one can calculate the \( S \)-matrix:

\[
S = \frac{1}{27\sqrt{3}} \begin{pmatrix}
1 & 1680 & 84 & 84 & 72 & 630 & 756 \\
\frac{1}{3} & -7 & 1 & 1 & -3 & -6 & 9 \\
1 & 20 & \kappa & \kappa^* & 18 & -45 & 27 \\
1 & 20 & \kappa^* & \kappa & 18 & -45 & 27 \\
1 & -70 & 21 & 21 & 27 & 0 & 0 \\
1 & -16 & -6 & -6 & 0 & 27 & 0 \\
1 & 20 & 3 & 3 & 0 & 0 & -27
\end{pmatrix} \quad (3.3)
\]

with \( \kappa = -21/2 + i 27\sqrt{3}/2 \). The factor \( 1/3 \) is due to the smaller orbit length of \( \chi_i^2 \). Thus, the following left-right symmetric combination is a modular invariant partition function (5040 = 3 · 1680):

\[
Z_{NS} \sim |\chi_0|^2 + 5040 |\chi_1|^2 + 84 |\chi_2|^2 + 84 |\chi_3|^2 + 72 |\chi_4|^2 + 630 |\chi_5|^2 + 756 |\chi_6|^2, \quad (3.4)
\]

leading to \( n_{27} = 84 \) generations and \( n_{27} = 0 \) antigenerations. Furthermore, in the massless spectrum there appear \( n_1 = 252 \) spin-zero gauge singlets and eight \( U(1)^8 \) gauge particles. Apparently from (3.3), the term \( |\chi_2 - \chi_3|^2 \) is invariant by itself, so that as suggested in [8] multiples of it can be added to (3.4) yielding string models with a wide range of Euler numbers \( \chi/2 \in \{-84, -82, \ldots, 82, 84\} \). There are two consistent arguments which show that only the model (3.4) and its mirror (i.e. \( n_{27} = 0, n_{27} = 84 \))

\[
Z_{M}^{NS} \sim Z_{NS} - 84 |\chi_2 - \chi_3|^2, \quad (3.5)
\]

really give consistent string models. Firstly, via a slightly modified Verlinde formula reflecting \( S \) not to be symmetric we have calculated the fusion rules associated to the \( S \)-matrix (3.3) which shows that only the two left-right combinations following from (3.4) and (3.5) satisfy a closed operator algebra on the tree level. Secondly, the modular invariance of all these partition functions is an accident resulting from the information-losing identification of all the characters in table 1. Treating them differently, the term added in (3.5) is no longer modular invariant, but all combinations which are antisymmetric under a \( U(1) \) flip \( P : j(z) \to -j(z) \) still form a subrepresentation of the modular group. Thus, the analogue of (3.5) is

\[
Z_{M}^{NS} \sim Z_{NS} - \sum_{i,j} |\chi_i^j - \chi_{P(i)}^{P(j)}|^2 \quad (3.6)
\]

which again can be shown to give only the mirror (3.5), for instead of coupling states with identical \( U(1) \) charges, (3.6) couples a state with charge \( Q \) to a state with charge \( -Q \). This way of constructing the mirror of a Gepner-type string model has been suggested in [12] and will be generalized in [4]. Now, we will show that this model can be rewritten as

\[
\sigma-\text{model of } (1)^9 = \text{compact } U(1) \otimes \text{compact } U(1)^8 \text{ on } 3M(E_8). \quad (3.7)
\]

For the root lattice \( M(E_8) \) we choose the following representation:

\[
M(E_8) = \left\{ (x_1, \ldots, x_8) \mid \left( \text{all } x_i \in \mathbb{Z} \lor \text{all } x_i \in \mathbb{Z} + \frac{1}{2} \right) \land \sum x_i \in 2\mathbb{Z} \right\}. \quad (3.8)
\]
It is known that for a simple Lie algebra $G$ all $\Theta$-functions

\[ \Theta_{\lambda,k}^G(\tau, \theta) = \sum_{\gamma \in M + \frac{1}{k}} q^{\frac{|\gamma|^2}{2}} e^{2\pi i \theta \gamma} \]  \tag{3.9} \]

form a finite-dimensional representation of the modular group if the weights are reduced to the finite set $\lambda \in 3M^*/M$ with $M^*$ to be the dual lattice. In the case of simply-laced Lie algebras the dual lattice is identical to the weight lattice. In table 2 we list all weights contained in $3M(E_8)/M(E_8)$ calculating also their conformal dimensions $H$ and ground state degeneracies.

| $H$ | $\lambda$ | # | # total | deg. |
|-----|-----------|---|---------|------|
| 0   | $(0^8)$   | 1 |         | 1    |
| $\frac{1}{3}$ | $(\pm 1, \pm 1, 0^9)$ | 112 |         |      |
|      | $(\pm \frac{1}{2}, \sum x_i \in 2\mathbb{Z})$ | 128 | 240     | 1    |
| $\frac{2}{3}$ | $(\pm 1^4, 0^4)$ | 1120 |         |      |
|      | $(\pm 2, 0^7)$ | 16 |         |      |
|      | $(\pm \frac{3}{2}, \pm \frac{1}{2}) \land \sum x_i \in 2\mathbb{Z}$ | 1024 | 2160 | 1    |
| 1   | $(\pm 1^6, 0^2)$ | 1792 |         |      |
|      | $(+2, +1^2, 0^5)$ | 168 |         |      |
|      | $(-2, -1^2, 0^5)$ | 168 |         |      |
|      | $(+2, -1^2, 0^5)/\mathbb{Z}_3$ | 56 |         |      |
|      | $(-2, +1^2, 0^5)/\mathbb{Z}_3$ | 56 | 2240 | 3    |
| $\frac{4}{3}$ | $(\pm 1^8) \land \sum x_i \in 4\mathbb{Z} - 2$ | 128 |         |      |
|      | $(\pm \frac{3}{2}, \pm \frac{1}{2}) \land \sum x_i \in 2\mathbb{Z}$ | 896 |         |      |
|      | $(-\frac{3}{2}, \pm \frac{1}{2}) \land \sum x_i \in 2\mathbb{Z}$ | 896 | 1920 | 9    |

Table 2: all $E_8$ weights

It turns out that all $\Theta$-functions with the same dimension are identical as $q$-series, i.e. setting $\theta = 0$ in (3.9). Identifying due to [8] all these $\Theta$-functions we can calculate the $S$-matrix for the remaining five characters with dimensions $H \in \{0, 1/3, 2/3, 1, 4/3\}$:

\[ S = \frac{1}{81} \begin{pmatrix} 1 & 240 & 2160 & 2240 & 1920 \\ 1 & 69 & 54 & -28 & -96 \\ 1 & 6 & -27 & -28 & 48 \\ 1 & -3 & -27 & 53 & -24 \\ 1 & -12 & 54 & -28 & -15 \end{pmatrix}, \quad \tag{3.10} \]

where not surprisingly the multiplicities appear in the first row. Note that due to the identification this $S$-matrix is no longer symmetric, but $S^2 = 1$ is still true. Now, we tensor this model with the universal $c = 1$ part and mod out the simple current

\[ Y^{+2}(z) = 6 : e^{\frac{i}{\sqrt{3}} \phi(z)} : \otimes : e^{\frac{i}{\sqrt{3}} \beta \Phi} : \tag{3.11} \]
with $\beta = (-1, 1, 1, 1, 1, 1, 1, 1)$ and $\Phi$ containing the eight free bosons, yielding a free field realization of the $c = 8$ part. The fusion rules for a system of $\Theta$-functions are quite simple:

$$[\Theta_{\mu, k}] \times [\Theta_{\nu, k}] = [\Theta_{\mu + \nu, k}],$$

(3.12)

so that the monodromy charge $Q(\nu)$ of some field $[1] \otimes [\Theta_{\nu, k}]$ concerning to the simple current (3.11) is

$$Q(\nu) = |\nu|^2 + |\beta|^2 - |\nu + \beta|^2 \mod \mathbb{Z}.$$  

(3.13)

Forming orbits and projecting them onto those with integer monodromy charge yields exactly the characters of table 2 including the correct conformal weights and multiplicities.

As announced in the previous section, the eight spin-one currents also appear in the OPE $G^+(z)G^-(w)$, so that we are led to choose the following symmetrized form of the $\sigma$-model orbits:

$$\chi_j^i = \frac{\Theta_{0, 0} + \Theta_{0, 0}}{\eta} \frac{1}{\eta^2} \left( \Theta_{E_8, k=3}(\tau, 0) + \Theta_{E_8, k=3}(-\tau, 0) \right)$$

$$+ \frac{\Theta_{-2, 0} + \Theta_{2, 0}}{\eta} \frac{1}{\eta^2} \left( \Theta_{E_8, k=3}(\tau, 0) + \Theta_{E_8, k=3}(-\tau, 0) \right)$$

(3.14)

For brevity, we present in table 3 only those $E_8$ weights $\lambda$ leading to the 84 massless orbits

| $\lambda^i$ | deg. |
|------------|------|
| (0; (-1)^5, 0) | 7 |
| (1; (-1)^2, 0^2) | 21 |
| (2; 1^2, 0^3) | 21 |
| (0; -2, 1^2, 0^4)/\mathbb{Z}_3 | 35 |

(3.15)

where the first entry should be regarded as fixed in every row. For the 84 antigenerations of the mirror model one has only to exchange plus and minus signs in table 3. This concludes the reconstruction of the $(1)^9$ Gepner model in terms of $E_8$ $\Theta$-functions showing that this simple model already has very intriguing string functions.

4. A simple toy model

We will choose for the non-supersymmetric $c = 8$ theory the tensor product of four copies of the Kac-Moody algebra $SU(3)_1$, which in particular satisfies the condition of containing a simple current of dimension $H = 4/3$. There exist only three representations of $SU(3)_1$ denoted by $\chi_0, \chi_3, \chi_3^*$ satisfying the following fusion rules:

$$[\chi_3] \times [\chi_0] = [\chi_3], \quad [\chi_3] \times [\chi_3] = [\chi_3^*], \quad [\chi_3] \times [\chi_3^*] = [\chi_0].$$

(4.1)

Thus, $[\chi_3]^4$ really is the desired simple current of $(SU(3)_1)^4$. Modding out the simple current $Y^+(z)$ in

$$\sigma-\text{model} = \text{compact } U(1) \otimes (SU(3)_1)^4$$

(4.2)
the orbits listed in table 4 survive.

| orbit $\chi_i$ | $(H, Q)$ | comb. factor |
|---------------|----------|--------------|
| $\chi_0$ = $A\chi_0^4 + B\chi_3^3 + C\chi_3^3$, | $(0, 0)$ | 1 |
| $\chi_1$ = $A\chi_0\chi_3^3 + B\chi_3\chi_3^3 + C\chi_3^3\chi_0^3$ | $(\frac{1}{2}, -1)$ | 4 |
| $\chi_2$ = $A\chi_0\chi_3^3 + B\chi_3\chi_3^3 + C\chi_3^3\chi_3^3$ | $(\frac{1}{2}, +1)$ | 4 |
| $\chi_3$ = $A\chi_0^2\chi_3\chi_3^3 + B\chi_3^2\chi_3^3\chi_0 + C\chi_3^2\chi_0\chi_3$ | $(\frac{2}{3}, 0)$ | 12 |
| $\chi_4$ = $A\chi_0^2\chi_3\chi_3^3 + B\chi_3^2\chi_0^2 + C\chi_0^2\chi_0^2$ | $(\frac{5}{6}, \pm 1)$ | 6 |

Table 4: orbits and multiplicities for the toy model

Now $A, B, C$ denote the three characters of the $c = 1$ part $(2, 3)$, e.g. $A = \frac{\Theta_{0,6}(\tau, \theta) + \Theta_{6,6}(\tau, \theta)}{\eta(\tau)}$. Actually, we know that in order to have a good field interpretation one has to symmetrize the simple current as $\frac{1}{2}(\chi_3 + \chi_3^*)$. This does not effect, however, the characters in table 4 as $(\tau, \theta)$ series, so that we immediately obtain the diagonal partition function, which always guarantees consistency on the tree level, as well:

$$Z^{NS} \sim |\chi_0|^2 + 4|\chi_1|^2 + 4|\chi_2|^2 + 12|\chi_3|^2 + 6|\chi_4|^2.$$ (4.3)

Using this $\sigma$-model as the internal part of a heterotic string model one obtains a vacuum with $n_{27} = 36$, $n_{27}^D = 0$, $n_1 = 324$ and an obviously enlarged gauge group $E_6 \times SU(3)^4$ in the right-moving sector. The massless fermions $\psi_n^{ab}$ carry three indices, two of them concerning to this entire gauge group: $a$ for the fundamental 27-dimensional representation of $E_6$ and $b$ for the fundamental three-dimensional representation of in each case one of the four $SU(3)$ factors. With respect to the remaining three $SU(3)$s, the fermions are singlets. Due to the degeneracy of $\chi_3(q) = 3q^\frac{3}{2} + \ldots$ the third index $n \in \{1, 2, 3\}$ just counts the particles in the left-moving sector. We have found a model with this massless spectrum and extended gauge group in [17] in the context of the covariant lattice approach, from which it is absolutely not obvious that it can be written in such a simple way. In addition, the number of (anti)generations occurs as an orbifold of the $(1)^9$ Gepner model [15].

5. Conclusion and outlook

In this letter we have outlined an alternative way of constructing $N = 2$ Calabi-Yau $\sigma$-model partition functions. We have both reconstructed one already known model realizing a broken $E_8$ symmetry and used the dual approach as a constructive one. In principle every $c = 8$ non-supersymmetric CFT containing a simple current of dimension $H = 4/3$ can be used as string functions in this formalism. Whether or not new three generation models can be achieved remains to be seen. Furthermore, it should be possible to transform other methods like the orbifold construction into our approach. Since the $c = 1$ part is universal for all $\sigma$-models, one might hope this construction scheme to be more suitable for the investigation of marginal deformations of the CFT, reflecting deformations of the complex and Kähler structure of the underlying Calabi-Yau manifold.

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