The role of Killing-Yano tensors in supersymmetric mechanics on a curved manifold

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ABSTRACT

The supersymmetric extension of charged point particle’s motion is applied to investigate symmetries of gravitational fields and electromagnetic fields. We mainly focus on the role of the Killing-Yano tensors of both usual and generalized types. Results obtained by systematic analysis strengthen the connection of the Killing-Yano tensor and superinvariants (functions commuting with the supercharge).

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I. INTRODUCTION

Recently, Gibbons, Rietdijk and van Holten \cite{1} investigated symmetries of spacetimes systematically in terms of the motion of pseudo-classical spinning point particles described by the supersymmetric extension of the usual relativistic point particle \cite{2,3}. Such a supersymmetric theory possesses a supercharge $Q$ generating the supersymmetry transformation between particle’s position $x^\mu$ and particle’s “spin” $\xi^a$, which must be introduced to forbid the negative norm state of spin due to the indefinite Lorentz metric $\eta_{ab}$. One outstanding feature of such a theory is to have an algebra like \{\{Q, Q\} \propto H\}, where $H$ is the Hamiltonian. Due to this relation and the Jacobi identity, superinvariants $J$ such as \{\{Q, J\} = 0\} are simultaneously constants of motion \{\{H, J\} = 0\}, so that superinvariants are of particular importance in supersymmetric theories. It was a big success of Gibbons et al. to have been able to show that the Killing-Yano tensor, which had long been known for relativists as rather mysterious structure, can be understood as an object generating a ‘nongeneric’ supersymmetry, i.e. supersymmetry appearing only in specific spacetimes. The corresponding supercharge $Q_f$ generated by the Killing-Yano tensor is a superinvariant rather than merely a constant of motion. The Killing-Yano tensor here is a 2-form, $f_{\mu\nu} = f_{[\mu\nu]}$, which satisfies the Penrose-Floyd equation \cite{7}

$$D_{(\mu} f_{\nu)\lambda} = 0. \tag{1}$$

It is also worth noting that the square of a Killing-Yano tensor makes the associated Killing tensor $K_{\mu\nu}$ as

$$K_{\mu\nu} = f_{\mu\lambda} f_{\nu}^{\lambda}. \tag{2}$$

It is of some interest that so-called the Carter’s constant $K_{\mu\nu} u^\mu u^\nu$ is the bosonic sector of square of $Q_f$, $\{Q_f, Q_f\}$. ($u^\mu$ is the particle’s tangent.) We may call 2-forms satisfying Eq.(1) Killing-Yano tensors of usual type, whereas we call $r$-forms satisfying similar equation

$$D_{(\mu_1} f_{\mu_2)\mu_3...\mu_{r+1}} = 0 \tag{3}$$

*Killing-Yano tensors of valence r* \cite{8,9}.
In this paper we discuss the role of these generalized Killing-Yano tensors, with the framework extended to include electromagnetic interactions.

We shall first retrace the argument in [1] with the extended framework and see the manifestation of electromagnetic interactions. One notable consequence would be the condition of the electromagnetic tensor \( F_{\mu\nu} \) to maintain the nongeneric supersymmetry. Using the Killing-Yano tensor, \( f_{\mu\nu} \), this condition will be expressed as

\[
 f^\lambda [\mu F_{\nu}]\lambda = 0. \tag{4}
\]

This has also been known in the approach using the conformal Killing spinor [10], \( \chi_{AB} \), as the condition to maintain the constant of motion, \( \chi = \chi_{AB} \lambda^A \lambda^B \), along the null geodesics generated by \( \lambda^A \bar{\lambda}^{A'} \). In 2-spinor notation, this condition is expressed as

\[
 \chi_B (A \phi_C) B = 0, \tag{5}
\]

where \( \phi_{CB} \) is the electromagnetic spinor. If the conformal Killing spinor \( \chi_{AB} \) satisfies a subsidiary condition \( \nabla_{A'C'} \chi_{C'A'} - \nabla_{AC'} \bar{\chi}_{C'A'} = 0 \), then \( \chi_{AB} \) is called the Killing spinor in strong sense [11] and coincides with the spinor version of the Killing-Yano tensor of usual type. With such \( \chi_{AB} \), Eq.(5) is equivalent to Eq.(4). It is worth noticing that the condition implies that the principal null directions (the PND) of the electromagnetic field must be aligned with those of the Killing spinor [10].

We then discuss the role of Killing-Yano tensors of valence \( r \), \( f_{\mu_1 \cdots \mu_r} \). We know that, in the usual relativistic point particle theory, Killing tensors imply constants of motion, i.e., if the spacetime admits a Killing tensor \( K_{\mu_1 \cdots \mu_r} \) of valence \( r \), then the phase space function \( K_{\mu_1 \cdots \mu_r} u^{\mu_1} \cdots u^{\mu_r} \) is constant along the geodesic [12]. What we point out in this paper is a counterpart to this in the supersymmetric theory. We will find the one-to-one correspondence between Killing-Yano tensors and superinvariants, of which forms are rather nontrivial. We also examine the brackets of such superinvariants with generic constants of motion, and thereby discuss the associated constants of motion with such superinvariants.

Although it has no obstacles in passing to quantum mechanics, we shall concentrate on classical analysis. Nevertheless, we know that Killing-Yano tensors can play a key role in the Dirac’s theory on a curved spacetime [13]. Our results may strengthen the connection
of Killing-Yano tensors with the supersymmetric classical and quantum mechanics on curved manifolds.

The plan of this paper is as follows. In sect. II we establish the canonical formulation of pseudo-classical charged spinning particles in an arbitrary background spacetime, using Grassmann-valued pseudo-Lorentz vector to describe the spin degrees of freedom. In sect. III we formulate component equations for extra symmetries, i.e. for constants of motion and superinvariants. In sect. IV we see if the nongeneric supersymmetry survives when the electromagnetic interactions are taken into account. Sections II through IV are also reviews for the treatment of the symmetries of supersymmetric point particle theory. In sect. V we establish the role of general Killing-Yano tensor of valence $r$. We in sect. VI consider the possibility of spacetimes admitting a Killing-Yano tensor to have larger symmetries. Finally, sect. VII is devoted to conclusions.

II. THE PSEUDO-CLASSICAL DESCRIPTION OF A CHARGED DIRAC PARTICLE

In this section, we establish the pseudo-classical description of our charged Dirac particle. Note first that, while usual point particle is described by its point $x^\mu$ on a Lorenzian manifold $(M, g_{\mu\nu})$, our pseudo-classical (charged) Dirac particle has also a freedom of spin which is represented by a Grassmann-valued pseudo-Lorentz vector $\xi^a$. Such descriptions was considered in Refs. [2–6], and in particular we shall employ the linearized Lagrangian treated in Ref. [3]. We thus start with the Lagrangian,

$$ L = \frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + e A_\mu \dot{x}^\mu + \frac{i}{2} \left( \xi^a \frac{D\xi^a}{d\tau} - \frac{e}{m} F_{ab} \xi^a \xi^b \right), $$

where $m$ and $e$ are, respectively, the mass and the charge of a particle, and $A_\mu(x)$ and $F_{\mu\nu}(x)$, respectively, the vector potential and the field strength of the electromagnetic field, both of which are considered as external fields, and so is the spacetime metric $g_{\mu\nu}(x)$. Greek and Latin indices refer to world and Lorentz indices, respectively, and are converted into each other by the vielbein $e_a^\mu$. The dot over $x^\mu$ represents the derivative with respect to a parameter $\tau$, while $D\xi^a/d\tau$ represents the covariant derivative with respect to $\tau$;
\[
\frac{D\xi^a}{d\tau} = \dot{\xi}^a + \omega^a_{\ b\mu} \xi^b \dot{x}^\mu, \\
\]
where \(\omega_{ab\mu}\) is the connection 1-form.

Since our Lagrangian is a gauge-fixed one, we have to add appropriate constraints. One is given by
\[
H \equiv m \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{ie}{2m} F_{ab} \xi^a \xi^b \approx -\frac{m}{2},
\]
which ensures the time-reparametrization invariance. Also, this tells that the parameter \(\tau\) is a generalization of the proper time. The other constraint is
\[
Q \equiv e_{a\mu} \dot{x}^\mu \xi^a \approx 0,
\]
which generates the supersymmetry transformation \(\dagger\). The equations of motion derived from the above Lagrangian will be invariant under the transformation generated through appropriate Poisson-Dirac bracket with the above constraints. Variable \(\xi^a\) is the superpartner of \(x^\mu\) for the supersymmetry transformation generated by \(Q\). Our Lagrangian gives, in conjunction with the constraints \(H\) and \(Q\), the pseudo-classical description of charged Dirac (spinning) particles.

Since the conjugate momenta are
\[
p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = mg_{\mu\nu} \dot{x}^\nu + \omega_\mu + e A_\mu, \quad \pi_a = \frac{\partial L}{\partial \dot{\xi}^a} = -\frac{i}{2} \xi^a
\]
with \(\omega_\mu \equiv (i/2)\omega_{ab\mu} \xi^a \xi^b\), the second class constraint for \(\pi_a\) yields the following Poisson-Dirac bracket;
\[
\{F, G\} = \frac{\partial F}{\partial \dot{x}^\mu} \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial G}{\partial \dot{x}^\mu} + i(\partial F/\partial \xi^a) \frac{\partial G}{\partial \xi^a},
\]
\(\dagger\)The full Lagrangian contains, apart from the Lagrange multipliers, a Grassmann-valued pseudo-Lorentz scaler \(\xi_5\) as a variable, which must be introduced to ‘carry’ the mass. The supercharge \(\dagger\) should have been \(Q \equiv e_{a\mu} \dot{x}^\mu \xi^a + \xi_5 \approx 0\) to recover the massive Dirac equation when quantized. However, in the present gauge, \(\xi_5\) is found to be a constant, so that the subsequent classical analysis will not be affected with \(\xi_5\) suppressed. The constancy of \(\xi_5\) will appear as the existence of the chiral charge (See Eq.(33)).
where \((\partial F/\partial \xi^a)\) is a right differentiation which will take the opposite sign to \(\frac{\partial F}{\partial \xi^a}\) when \(F\) is Grassmann-odd. With this bracket, we can check the canonical relations, \(\{x^\mu, p_\nu\} = \delta_\nu^\mu\) and \(\{\xi^a, \xi^b\} = -i\eta^{ab}\). For convenience, we introduce the gauge-covariant variable \(\Pi_\mu\) defined by
\[
\Pi_\mu = p_\mu - \omega_\mu - e A_\mu (= mg_{\mu\nu}\dot{x}^\nu).
\] (12)

With this variable, the bracket becomes
\[
\{F, G\} = (\mathcal{D}_\mu F) \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu} (\mathcal{D}_\mu G) + (R_{\mu\nu} + e F_{\mu\nu}) \frac{\partial F}{\partial \Pi_\mu} \frac{\partial G}{\partial \Pi_\nu} + i(\partial F/\partial \xi^a) \frac{\partial G}{\partial \xi^a},
\] (13)
where we have defined the spin-valued Riemann tensor
\[
R_{\mu\nu} \equiv i \frac{1}{2} R_{ab\mu\nu} \xi^a \xi^b
\] (14)
and the phase space covariant derivative operator
\[
\mathcal{D}_\mu F \equiv \frac{\partial F}{\partial x^\mu} + \Pi_\lambda \Gamma^\lambda_{\mu\nu} \frac{\partial F}{\partial \Pi_\nu} - \omega_a^\mu \Pi^a_x \xi^b \frac{\partial F}{\partial \xi^a}.
\] (15)

Now, with this bracket it is easy to see for the constraints
\[
H = \frac{1}{2m} g^{\mu\nu} \Pi_\mu \Pi_\nu + \frac{ie}{2m} F_{ab} \xi^a \xi^b \approx \frac{m}{2}
\] (16)
and
\[
Q = \frac{1}{m} e_a^\mu \Pi_\mu \xi^a \approx 0
\] (17)
that the usual supersymmetry algebra
\[
\{Q, H\} = 0, \quad \{Q, Q\} = -\frac{2i}{m} H.
\] (18)
holds.

**III. GENERALIZED KILLING EQUATIONS AND THEIR SQUARE ‘ROOTS’**

In this section we write down the equations for constants of motion and superinvariants, which will be applied in the subsequent sections.
First, for any constant of motion $J(x, \Pi, \xi)$, the bracket with $H$ vanishes, $\{H, J\} = 0$. With the bracket (13), this reduces to

$$\Pi \mu \left\{ D_\mu J - \frac{\partial J}{\partial \Pi_\nu} (R_{\mu\nu} + eF_{\mu\nu}) \right\} = eF_\mu \frac{\partial J}{\partial \Pi_\mu} + e \xi^a F_a^b \frac{\partial J}{\partial \xi^b};$$

where $F_\mu \equiv (i/2)(D_\mu F_{ab}) \xi^a \xi^b$. Following [1], let us expand $J(x, p, \Pi)$ in powers of $\Pi_\mu$;

$$J = \sum_{n=0}^{\infty} \frac{1}{n!} J^{(n)}(x, \xi) \Pi_{\mu_1} \cdots \Pi_{\mu_n}.$$

Then we have for the coefficients $J^{(n)}$ the following generalized Killing equations;

$$D_{(\mu_1 \cdots \mu_n)} J^{(n)}(x, \xi) \Pi_{\mu_1} \cdots \Pi_{\mu_n} = - (R_{\nu(\mu} + eF_{\nu(\mu}), J^{(n+1)}(x, \xi) \Pi_{\mu_1} \cdots \Pi_{\mu_n})$$

and

$$F_{\mu} J^{(1)}(x, \xi) \Pi_{\mu_1} \cdots \Pi_{\mu_n} = 0.$$

These are a direct generalization of Eq.(41) in [1], though there exist some differences of sign due to the difference of sign convention of connection 1-form $\omega_{ab\mu}$. We shall refer to Eq.(22) as the component equation of the generalized Killing equation (21).

The equation for superinvariants is derived from the equation $\{Q, J\} = 0$. Such a superinvariant $J$ is automatically a constant of motion, i.e., $\{H, J\} = 0$, as confirmed by the Jacobi identity with Eq.(18). Again, with the bracket (13), we have

$$\xi^\mu \left( D_\mu J - eF_{\mu\nu} \frac{\partial J}{\partial \Pi_\nu} \right) + i \xi^a \frac{\partial J}{\partial \xi^a} = 0.$$

Expanding $J^{(n)}$ in powers of $\xi^a$ and letting the coefficients be $f^{(m, n)}_{\mu_1 \cdots \mu_m}(x)$, i.e.,

$$J = \sum_{m,n=0}^{\infty} \frac{i^{[m]}}{m! n!} \xi^{\mu_1} \cdots \xi^{\mu_m} f^{(m, n)}(x) \Pi_{\mu_1} \cdots \Pi_{\mu_n},$$

we obtain from Eq.(23) the component equation;

$$m e_{[a} \mu \Pi_\mu f^{(m-1, n)}_{a_1 \cdots a_{m-1} \mu_1 \cdots \mu_m} - m e F_{\mu\nu} e_{[a} \mu f^{(m-1, n+1)}_{a_1 \cdots a_{m-1} \mu_1 \cdots \mu_n} - n f^{(m+1, n-1)}_{ba_1 \cdots a_{m-1} \mu_1 \cdots \mu_m} e^{b\mu_n} = 0.$$
IV. NONGENERIC SUPERSYMMETRIES

Following Ref. [1], we search for nongeneric supersymmetry with the generator of the form

$$Q_f = \xi^a f_a^\mu(x) \Pi_\mu + \frac{i}{3!} c_{abc}(x) \xi^a \xi^b \xi^c + h_a(x) \xi^a,$$

(26)

where $f_a^\mu(x), c_{abc}(x)$ and $h_a(x)$ are functions of $x^\mu$. The first term of the right side is an analogue of the supercharge $Q$ (see (17)). This charge generates the supersymmetry transformation such as

$$\delta x^\mu = i \epsilon \{ Q_f, x^\mu \} = -i \epsilon \xi^a f_a^\mu,$$

(27)

where the infinitesimal parameter $\epsilon$ of the transformation is Grassmann-odd.

We do not investigate the conditions that $Q_f$ commute with $H$, but with $Q$, since we are interested in the Killing-Yano tensor, which will be found to have close relationship with a superinvariant rather than a constant of motion.

We evaluate all nontrivial components of Eq.(25) with $J$ being $Q_f$ given by Eq.(26).

First of all, component $(m,n) = (0,1)$ gives $h_b e^{b\mu} = 0$. That is, $h_a$ must vanish. Next, look at component $(m,n) = (0,2)$, giving $f_b^{(\mu_1 e^{b\mu_2})} = 0$. Introducing $f_{\mu \nu} \equiv f_{a\mu} e^{a\nu}$, this implies that $f_{\mu \nu}$ must be antisymmetric,

$$f_{(\mu \nu)} = 0.$$

(28)

Then, look at component $(m,n) = (2,1)$, which gives

$$2 e^{[a \mu} D_\mu f_{b] \mu_1} - c_{cab} e^{b\mu_1} = 0.$$

(29)

Again, it will be useful to introduce the world-indices version of $c_{abc}$, $c_{\mu \nu \lambda} = c_{abc} e^{a \mu} e^{b \nu} e^{c \lambda}$.

Then, we observe from Eq.(29) that $D_{[\mu} f_{\nu]} \lambda$ must be skew-symmetric in accordance with the skew-symmetry of $c_{abc}$ or $c_{\mu \nu \lambda}$, so that taking Eq.(28) into account we have the Penrose-Floyd equation (1). This implies that $f_{\mu \nu}$ is the Killing-Yano tensor, as in the vacuum case. With Eqs.(1) and (28), Eq.(29) yields

$$c_{\mu \nu \lambda} = -2 D_\mu f_{\nu \lambda} (= -2 D_{[\mu} f_{\nu \lambda]}),$$

(30)
so that $c_{\mu \nu \lambda}$ is given by differentiation of $f_{\mu \nu}$ and is exact. It is easy to see that component $(m, n) = (4, 0)$, which is $D_{[\mu} c_{\nu \lambda \sigma]} = 0$, becomes trivial, since $c_{\nu \lambda \sigma}$ is exact. Finally, component $(m, n) = (2, 0)$ gives Eq. (3), the condition for the coincidence of PND’s alignment of the electromagnetic spinor and the Killing spinor.

After all, we have the final form of $Q_f$:

$$Q_f = \xi^\nu f^\mu_{\nu \Pi \mu} - \frac{i}{3} \xi^\mu \xi^\nu \xi^\lambda D_{[\mu} f_{\nu \lambda]}.$$  (31)

Difference from the vacuum case is only the definition (12) of $\Pi_\mu$ in terms of $p_\mu$, if Eq. (4) holds. This type of superinvariants exists in the Kerr-Newman spacetime [1, 11]. Although there are not so many physically interpretable spacetimes which admit a Killing-Yano tensor [8, 13], another such interesting example would be the Taub-NUT spacetime [14, 15], which admits four independent Killing-Yano tensors [15].

It is straightforward to calculate the constant of motion $K \equiv \frac{i}{2} \{Q_f, Q_f\}$, which is given by

$$K = \frac{1}{2} f^\mu_{\lambda \Pi \mu} f^{\nu \lambda} \Pi_{\mu \nu} + \frac{i}{2} \xi^\mu \xi^\nu \left\{ 2(D_\lambda f^{\sigma \mu}_{\nu}) f^{\lambda \nu \Pi \sigma} + c_{\mu \nu \lambda} f^{\sigma \lambda} \Pi_{\sigma} + e F_{\lambda \sigma} f^{\lambda \mu} f^{\sigma \nu} \right\} + \frac{1}{4} \xi^\mu \xi^\nu \xi^\lambda \xi^\rho \left\{ R_{\mu \nu \rho \omega} f^{\sigma \lambda} f^{\omega \sigma} - \frac{1}{2} c_{\mu \nu \rho} c_{\lambda \sigma} \right\}.$$  (32)

where we have used the relation $D_\mu c_{abc} = 3 f_{[c}^{\nu} R_{ab]} \mu \nu$. As expected, the bosonic sector of $K$ is the quadratic $\frac{1}{2} K_{\mu \nu \omega} u^\mu u^\nu \omega$ (with Eq. (2)).

**V. GENERALIZED KILLING-YANO TENSORS AND CORRESPONDING SUPERINVARIANTS**

We are now in a position to discuss the role of general Killing-Yano tensors. This will respond to the question of how profound the connection of the appearance of nongeneric supersymmetries and the existence of the Killing-Yano tensors is, and will give a useful tool in investigating a supersymmetric dynamical system.

What we want to note first is the Killing-Yano tensor of valence $d = \dim(M)$, which is generic and coincides with the volume form $\epsilon_{\mu_1 \cdots \mu_d}$ up to a constant factor. This object appears in two generic constants of motion, the chiral charge [16].
\( \Gamma_* \equiv \frac{i[d]}{d!} \epsilon_{a_1 \cdots a_d} \xi^{a_1} \cdots \xi^{a_d}, \) (33)

and the dual supercharge

\[ Q^* = i \{ Q, \Gamma_* \} = \frac{-i[d]}{(d-1)!} \epsilon_{a_1 \cdots a_d} e^{a_1 \mu} \Pi^\mu \xi^{a_2} \cdots \xi^{a_d}. \] (34)

(The form of these generic charges is regardless of the existence of electromagnetic interactions.) We note that the dual supercharge \( Q^* \) is superinvariant, and the form of it, Eq.(34), is similar to that of the nongeneric supercharge (31).

An analogy leads us to try to find supercharges in the form

\[ J = \frac{i[r]}{(r-1)!} f^{(r-1,1)} \mu \xi_{a_1} \cdots \xi_{a_{r-1}} + \frac{i[r+1]}{(r+1)!} f^{(r+1,1)} \xi_{a_1} \cdots \xi_{a_{r+1}}. \] (35)

It is easy to examine Eq.(25) for Eq.(35). Calculations done are completely parallel to those in the previous section. The following theorem summarizes the result.

**Theorem:** If the spacetime admits a Killing-Yano tensor of valence \( r \), \( f_{\mu_1 \cdots \mu_r} \), and the electromagnetic field \( F_{\mu \nu} \) satisfies the condition

\[ F^\nu_{\mu [a_{r-1]} \mu_{a_1} \cdots f_{a_1} \cdots a_{r-1}]} = 0, \] (36)

then the function

\[ Y_r = \xi^{a_2} \cdots \xi^{a_r} f_{\mu_1 \cdots \mu_r} \Pi^{\mu_1} - \frac{i}{r+1} \xi^{\mu_1} \cdots \xi^{\mu_{r+1}} D_{[\mu_1} f_{\mu_2 \cdots \mu_{r+1}]} \] (37)

is a superinvariant, \( \{ Q, Y_r \} = 0 \), for the bracket defined by Eq.(13). The converse also holds.

Here, \( f_{\mu_1 \cdots \mu_r} \) corresponds to \( f^{(r-1,1)} \nu g_{\nu \mu_1} e^{a_1}_{\mu_2} \cdots e^{a_{r-1}}_{\mu_r} \).

Thus, we know

\[ Q^* = \frac{-i[d]}{(d-1)!} Y_d, \] (38)

provided that \( f_{\mu_1 \cdots \mu_d} = \epsilon_{\mu_1 \cdots \mu_d} \).

Eq.(3) implies that a Killing-Yano tensor of valence 1 is a usual Killing vector. Let \( \zeta^\mu \) be a Killing vector, then it is a direct consequence of the theorem that
\[ Y_1 = \zeta^\mu \Pi_\mu - \frac{i}{2} \zeta^\mu \xi^\nu D_\mu \zeta_\nu \]  

is superinvariant, if \[ F_{\mu\nu} \zeta^\nu = 0 \] holds. However, if we consider an alternative function \[ J_\zeta = \zeta^\mu (\Pi_\mu + eA_\mu) - \frac{i}{2} \zeta^\mu \xi^\nu D_\mu \zeta_\nu, \] this is superinvariant, regardless of Eq.(40), provided that the Lie derivative of the vector potential with respect to \( \zeta^\mu \) vanishes, \( \mathcal{L}_\zeta A_\mu = 0 \). We would have got Eq.(41) as a result of trying to obtain a constant of motion (not superinvariant) associated with a Killing vector, using Eq.(21) (cf. Ref. [10]), however Eq.(41) happens to be superinvariant. This is a special feature for \( r = 1 \).

**VI. THE CONSTANTS OF MOTION ASSOCIATED WITH \( Y_r \)**

As already established, if a spacetime admits a Killing-Yano tensor of valence \( r \) and if the electromagnetic tensor satisfies Eq.(36), then \( Y_r \) is a constant of motion of a spinning particle in the spacetime. Possibly, there exist other constants of motion associated with \( Y_r \), i.e., there may exist nonvanishing brackets of \( Y_r \) with other known constants of motion. We here discuss generic feature of such constants of motion, i.e., we suppose there are no nongeneric constants of motion other than \( Y_r \) for specific value of \( r \).

It is obvious that we can construct such constants of motion first by taking brackets of \( Y_r \) with the four generic constants of motion, \( H, Q, \Gamma_*, \) and \( Q^* \). Since \( Y_r \) is a (super)invariant, we cannot use \( H \) and \( Q \) for the present purpose. Moreover, since \( Q^* \) has connection with \( \Gamma_* \) through \( Q^* = i \{ Q, \Gamma_* \} \), we do not have to discuss \( Q^* \) and \( \Gamma_* \) separately. In fact, if \( \dim(M) = d \), we have

\[
\{ Y_r, Q^* \} = i \{ Y_r, \{ Q, \Gamma_* \} \} \\
= -i(-1)^d \{ \Gamma_*, \{ Y_r, Q \} \} - i(-1)^{d(r-1)} \{ Q, \{ \Gamma_*, Y_r \} \} \\
= i \{ Q, \{ Y_r, \Gamma_* \} \} \\
( = \{ Q, Y_r^* \}),
\]  

(42)
where the second line is the consequence of the Jacobi identity and we have defined the dual of $Y_r$ as $Y^\ast_r \equiv i \{ Y_r, \Gamma_r \}$. Hence the bracket of $Y_r$ with the dual supercharge $Q^\ast$ is also the bracket of the supercharge $Q$ and the dual of $Y_r$, so that we only need to start with seeing if there exist nonvanishing duals of $Y_r$.

Since $D_\mu \Gamma_s = 0$ and $(\partial \Gamma_s / \partial \Pi_\mu) = 0$, we have

$$Y^\ast_r = (-1)^r \frac{\partial Y_r}{\partial \xi^a} \frac{\partial \Gamma_s}{\partial \xi_a}. \quad (43)$$

The numbers of the Grassmann vectors in $Y_r$ and $\Gamma_r$ are, respectively, $r - 1$ (the least number) and $d$, so that that of Eq.(43) is $r - 1 + d - 2 = r + d - 3$, which must be not greater than $d$ in order that $Y^\ast_r$ do not vanish. We can therefore have nonvanishing $Y^\ast_r$ only for $r \leq 3$. However, for $r = 1, 3$, we find by direct calculations that $Y^\ast_r$ vanishes after all. Thus, we can generate new constants of motion only for $Y^\ast_2 = Q^\ast_f$.

We can immediately calculate $Y^\ast_2 = Q^\ast_f$, which gives

$$Q^\ast_f = \frac{-i \{ 4 \}}{(d - 1)!} \epsilon_{a_1 \cdots a_d} e^{a_1 \mu} f^\nu \Pi_\nu \xi^{a_2} \cdots \xi^{a_d}. \quad (44)$$

Then we can calculate the bracket of $Q^\ast_f$ with the supercharge, for which we define $A_f$;

$$A_f \equiv m \{ Q, Q^\ast_f \} = m \{ Q_f, Q^\ast \} = e F_\mu \nu f^\mu \Gamma_s - \frac{i \{ 2 \} + 1}{(d - 2)!} \epsilon_{a_1 \cdots a_d} e^{a_1 \mu} \Pi_\mu e^{a_2 \nu} f^\lambda \Pi_\lambda \xi^{a_3} \cdots \xi^{a_d}. \quad (45)$$

This is the end of our construction — $\Gamma_s$, $Q$, $Q^\ast$, $Q_f$, $Q^\ast_f$ and $A_f$ with $K$ and $H$ constitute a closed algebra $G_2$, where $K$ is defined in Eq.(32) and $H$ is the Hamiltonian (16). Fig.1 summarizes the relation in $G_2$.

Of particular interest is the maximal abelian subalgebra, $H$, of $G_2$. We can easily find that $\Gamma_s$, $Q^\ast$, $Q^\ast_f$ and $A_f$ with $K$ and $H$ constitute such an algebra $H$ and the dimension of it is six. In the Kerr-Newman spacetime, we have another two commuting constants of motion, $J_\zeta$ and $J_\psi$, coming from the two commuting Killing vectors $\zeta$ and $\psi$, where, say, $\zeta$ is timelike and $\psi$ is the spacelike Killing vector generating closed orbits. Functions $J_\zeta$ and $J_\psi$ also commute with all elements of $H$, and with $H$ form the largest abelian algebra. This is a classical justification of the separability of the Dirac equation in the Kerr-Newman spacetime [17,18]. We can easily find that the Taub-NUT spacetime is also in the same situation.
FIG. 1. The algebra $\mathcal{G}_2$. Real lines stand for vanishing of the Poisson-Dirac brackets, whereas dashed lines for non-zeros. Characters above dashed lines are reminders of the non-vanishing brackets, e.g., the bracket of $\Gamma_*$ and $Q$ is proportional to $Q^*$. Real circles stand for vanishing of the Poisson-Dirac brackets with oneself, whereas dashed circles for non-zeros. The meaning of characters above the dashed circles is the same as for the lines. $H$ and $K$ commute with any functions listed in the figure. Note that functions $\Gamma_*$, $Q^*$, $Q_f^*$, and $A_f$ are mutually connected with real lines, so constitutes the (maximal) abelian subalgebra of $\mathcal{G}_2$. (See the last paragraph of this section.)

VII. CONCLUSIONS

We have shown that, if a spacetime admits a Killing-Yano tensor of valence $r$, the spinning particles moving on it possess the superinvariant $Y_r$ defined by (37). To hold the symmetry associated with the Killing-Yano tensor, the electromagnetic tensor must satisfy Eq.(36). The function $Y_2$, which is made from the Killing-Yano tensor of usual type, is in a particular position, since only this can have nonvanishing bracket with the chiral charge, $\Gamma_*$, for which we can find the associated other constants of motion, (44) and (45).

It should be noted that these facts do not depend on the dimension of spacetime. This enables us to apply our results to other supersymmetric systems, e.g. supersym-
metric cosmologies [19], where point particles in spacetimes are replaced by points in the minisuperspaces. Since the Lagrangians used there are not the same as the one used here, the form of Eq. (37) will vary. However, it is plausible that the Killing-Yano tensor can be a useful tool in investigating such systems.

Passing to quantum mechanics and the applications to supersymmetric cosmologies will be discussed elsewhere.

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