A class of cubic hypersurfaces and quaternionic Kähler manifolds of co-homogeneity one

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Abstract

We classify all complete projective special real manifolds with reducible cubic potential, obtaining four series. For two of the series the manifolds are homogeneous, for the two others the respective automorphism group acts with co-homogeneity one. Complete projective special real manifolds give rise to complete quaternionic Kähler manifolds via the supergravity q-map, which is the composition of the supergravity c-map and r-map. We develop curvature formulas for manifolds in the image of the q-map. Applying the q-map to one of the above series of projective special real manifolds, we obtain a series of complete quaternionic Kähler manifolds, which are shown to be inhomogeneous (of co-homogeneity one) based on our curvature formulas.

Keywords: projective special real manifolds, projective special Kähler manifolds, quaternionic Kähler manifolds, co-homogeneity one

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Contents

1 Classification of complete projective special real manifolds with reducible cubic potential

1.1 Classification of non-degenerate reducible polynomials
1.2 Classification of hyperbolic reducible polynomials and complete projective special real manifolds

2 Curvature formulas for the q-map

2.1 Conical affine and projective special Kähler geometry

2.2 The supergravity c-map

2.3 The supergravity r-map

2.4 Curvature formulas for the supergravity r-map

2.5 Levi-Civita connection for quaternionic Kähler manifolds obtained by the q-map

2.6 Curvature tensor for quaternionic Kähler manifolds obtained by the q-map

2.7 Pointwise norm of the curvature tensor for quaternionic Kähler manifolds obtained by the q-map

2.8 Example: A series of inhomogeneous complete quaternionic Kähler manifolds

A Automorphisms of manifolds in the image of the r- and c-map

Introduction

In this paper we are concerned with hypersurfaces \( \mathcal{H} \subset \mathbb{R}^{n+1} \) contained in the level set \( \{ h = 1 \} \) of a homogeneous cubic polynomial \( h \). The hypersurface is equipped with the symmetric tensor field \( g_\mathcal{H} \) on \( \mathcal{H} \) induced by \( -\frac{1}{3} \partial^2 h \). We require that \( g_\mathcal{H} \) is a Riemannian metric. Then \( (\mathcal{H}, g_\mathcal{H}) \) is called a projective special real manifold, see Definition 5. \( h \) is called its cubic potential and \( g_\mathcal{H} \) is called the projective special real metric. The polynomials \( h \) which admit such a hypersurface are called hyperbolic, cf. Definition 4.

Projective special real manifolds occur in the physics literature as the scalar manifolds of 5-dimensional supergravity coupled to vector multiplets, see [GST]. These manifolds are related to projective special Kähler manifolds [F, ACD], reviewed in Section 2.1, by a construction known as the \( r \)-map [DV], which is induced by the dimensional reduction of the supergravity theory from 5 to 4 space-time dimensions.

Similarly, projective special Kähler manifolds are related to quaternionic Kähler manifolds of negative scalar curvature by the \( c \)-map, which is induced by dimensional reduction to 3 dimensions [FS]. As shown in [ACM, ACDM], the quaternionic Kähler property of the \( c \)-map metric can be proven by showing that it is part of a one-parameter family of met-
The key fact is that the cotangent bundle of any conical affine special Kähler manifold admits an indefinite hyper-Kähler metric (known as the rigid c-map metric) and a circle action of the type required for the correspondence to yield a family of positive definite quaternionic Kähler metrics. These constructions are recovered in [MS] based on Swann’s twist construction and elementary deformation. See also [APP], where it was first shown that the c-map metric and the rigid c-map metric are related by the QK/HK-correspondence.

It is known [CHM] that the r- and c-map preserve the completeness of the underlying Riemannian metrics. It follows that the same is true for their composition, the q-map. In this way the study of the completeness of quaternionic Kähler manifolds obtained by the q-map is reduced to the study of the completeness of the initial projective special real manifold. Complete projective special real manifolds are characterized by the following theorem, to be used later on.

**Theorem 1** ([CNS, Thm. 2.5]). A projective special real manifold $\mathcal{H} \subset \mathbb{R}^{n+1}$ is complete with respect to the metric $g_\mathcal{H}$ if and only if $\mathcal{H}$ is closed as a subset of $\mathbb{R}^{n+1}$.

It follows from Theorem 1 that the classification of complete projective special real manifolds is equivalent to the solution of the following two problems:

(i) Classification of all hyperbolic homogeneous cubic polynomials $h$, up to linear transformations.

(ii) For each such polynomial determine all locally strictly convex components of the level set $\{h = 1\}$, up to linear transformations.

While it is certainly possible to solve these problems in low dimensions, see [CDL] for the solution up to polynomials in 3 variables, we do not expect a simple solution valid in all dimensions. A very rough idea about problem (i) is obtained by observing that the dimension of the space of homogeneous cubic polynomials grows cubically whereas the dimension of the general linear group grows only quadratically with the number of variables. Notice that the hyperbolic polynomials form an open subset in the space of homogeneous cubic polynomials in a given number of variables. An interesting class of projective special real manifolds is provided by considering those with reducible cubic potentials $h$, that is $h$ is a product of polynomials of lower degree. The motivation to consider this class is that the polynomial $h$ is preserved by a large group of linear

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1The same is true for the generalized r-map [CHM], where the cubic polynomial is replaced by a more general homogeneous function $h$. However, the resulting Kähler manifolds are in general no longer projective special Kähler and therefore not interesting for our present purposes.
transformations. In virtue of the general results about symmetries in Appendix A, this will eventually give rise to quaternionic Kähler manifolds with a large but not always transitive group of isometries. Applying the q-map to the complete manifolds in this class we obtain a class of complete quaternionic Kähler manifolds, as follows from the general result [CHM, Thm. 6]. In this way one obtains, in particular, the series of symmetric spaces

\[
\frac{\text{SO}(4,m)}{\text{SO}(4) \times \text{SO}(m)}, \quad m \geq 3,
\]  

(0.1)
as well as the series of homogeneous non-symmetric spaces \(T(p), p \geq 1\), of rank 3, see [DV, C]. One of the results of this paper is that one also obtains a series of complete quaternionic Kähler manifolds that are not locally homogeneous, see Theorem 22. In fact, we show that there are precisely four series of complete projective special real manifolds with reducible cubic potential. More precisely, by solving the above problems (i) and (ii) under the assumption that \(h\) is reducible we will obtain the following result.

**Theorem 2.** Every complete projective special real manifold \(H \subset \{h = 1\} \subset \mathbb{R}^{n+1}\) of dimension \(n \geq 2\) for which \(h\) is reducible is linearly equivalent to exactly one of the following complete projective special real manifolds:

\[ a) \{x_{n+1}(\sum_{i=1}^{n-1} x_i^2 - x_n^2) = 1, \quad x_{n+1} < 0, x_n > 0\}, \]

\[ b) \{(x_1 + x_{n+1})(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 + x_{n+1} < 0\}, \]

\[ c) \{x_1(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 < 0, x_{n+1} > 0\}, \]

\[ d) \{x_1(x_1^2 - \sum_{i=2}^{n+1} x_i^2) = 1, \quad x_1 > 0\}. \]

Notice that in the case \(n = 2\) the result follows from [CDL, Thm. 1] and that the above list is also valid in the case \(n = 1\) but then the curves a) and b) are linearly equivalent, as well as c) and d), see [CHM, Cor. 4].

Under the q-map the series a) with \(n \geq 1\) corresponds to the series (0.1) of symmetric quaternionic Kähler manifolds with \(m = n + 2\). Similarly, b) corresponds to the series \(T(p)\) of homogeneous quaternionic Kähler manifolds with \(p = n - 1 \geq 0\), where only the first member \(T(0) = \frac{\text{SO}(4,3)}{\text{SO}(4) \times \text{SO}(3)}\) of the series is symmetric. The quaternionic Kähler manifolds obtained from the series c) and d) admit a Lie group acting isometrically with co-homogeneity one. For d) we will prove the following stronger result.

**Theorem 3.** The quaternionic Kähler manifolds associated with the projective special real manifolds \(\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1(x_1^2 - \sum_{i=2}^{n+1} x_i^2) = 1, \quad x_1 > 0\}, \quad n \geq 1\), are complete of negative scalar curvature and the isometry group acts with co-homogeneity one.
Notice that the theorem provides examples of complete quaternionic Kähler manifolds in all dimensions $\geq 12$ for which the isometry group acts with co-homogeneity one, see [DS, PV] for results excluding the existence of such manifolds in the case of positive scalar curvature, and [P] for some examples of negative scalar curvature in dimension 4. Recall also that the quaternionic hyperbolic space admits deformations by complete quaternionic Kähler manifolds [L], but the isometry groups of these are not known.

The claim that the quaternionic Kähler manifolds in Theorem 3 and similarly the ones obtained from the series c) in Theorem 2 admit a subgroup of the isometry group acting with an orbit of codimension one follows from the fact that the automorphism group of the initial projective special real manifolds acts with an orbit of codimension one. In fact, the orthogonal group $O(n)$ in the variables $x_2, \ldots, x_{n+1}$ acts by automorphisms of the projective special real manifold. Moreover, every automorphism of a projective special real manifold extends to an isometry of the corresponding quaternionic Kähler manifold under the q-map. In addition, the r-map as well as the c-map each produce a freely acting additional solvable Lie group of automorphisms, see [DV, DVV, CHM]. The dimensions of the latter solvable groups coincide with the number of extra dimensions created by the r- and c-map, respectively. Therefore the co-homogeneity does not increase under these constructions, see Appendix A for details.

The main difficulty is to prove that the quaternionic Kähler manifolds of Theorem 3 are not of co-homogeneity zero, this is the content of Theorem 22. The proof proceeds by computing the point-wise norm of the curvature tensor and showing that for each of these manifolds it is a non-constant rational function depending only on one coordinate $x$ out of a system of $4n + 8$ global coordinates. It relies on general curvature formulas for quaternionic Kähler manifolds obtained by the q-map, which constitute another important result of this paper, see Theorem 20 and Corollary 21. Incidentally, we expect that the isometry groups of the quaternionic Kähler manifolds corresponding to the remaining series c) in Theorem 2 do likewise have co-homogeneity precisely one. The corresponding curvature calculations are more involved in that case.

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1 Classification of complete projective special real manifolds with reducible cubic potential

In this section we will classify all complete projective special real manifolds with reducible cubic potential up to linear transformations. After giving some basic definitions we will first classify up to equivalence all non-degenerate reducible homogeneous cubic polynomials in Section 1.1 and among these all hyperbolic ones in Section 1.2. In the same section we determine, for each of the resulting hyperbolic polynomials $h$, those connected components (up to linear transformations) of the level sets $\{h = 1\}$ which contain a hyperbolic point, see Definition 4. In particular we determine all such components which are locally strictly convex or, equivalently, consist solely of hyperbolic points. As a consequence of Theorem 1 these components give precisely all complete projective special real manifolds with reducible cubic potential (up to linear transformations).

Definition 4. Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a homogeneous cubic polynomial.

1. The polynomial $h$ is called non-degenerate if there exists $p \in \mathbb{R}^{n+1}$, such that $\det \partial^2 h_p \neq 0$.

2. The polynomial $h$ is called hyperbolic if there exists a hyperbolic point $p \in \mathbb{R}^{n+1}$, that is a point such that $h(p) > 0$ and $\partial^2 h_p$ is of signature $(1, n)$.

Two homogeneous cubic polynomials are called equivalent if they are related by a linear transformation.

Notice that the notions of non-degeneracy and hyperbolicity are invariant under linear transformations and that $\det \partial^2 h_p \neq 0$ implies $h(p) \neq 0$ if the tensor $\partial^2 h_p$ is non-degenerate on the hyperplane $\ker dh_p$.

Definition 5. A hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ is called a projective special real manifold if there exists a homogeneous cubic polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$, such that

(i) $\mathcal{H} \subset \{x \in \mathbb{R}^{n+1} \mid h(x) = 1\}$ and

(ii) $g_{\mathcal{H}} := -\frac{1}{3} \partial^2 h \mid_{T_{\mathcal{H}} \times T_{\mathcal{H}}} > 0$.

The hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ is endowed with the Riemannian metric $g_{\mathcal{H}}$ which is called the projective special real metric or centroaffine metric, see [CNS] for an explanation of this terminology. Two projective special real manifolds are called isomorphic if there is a linear transformation inducing a bijection between them.

2 For practical reasons, we prefer to compute $-\frac{1}{2} \partial^2 h$ instead of $-\frac{4}{3} \partial^2 h$ below.
Remark 6. Using the homogeneity of $h$ one sees that for every projective special real manifold $\mathcal{H}$ the symmetric tensor $\partial^2 h_p$ is of signature $(1,n)$ for all $p \in \mathcal{H}$ and that $\mathcal{H}$ is perpendicular to the position vector $p$ with respect to $\partial^2 h_p$. In fact, $\partial^2 h_p(p,p) = 6h(p) = 6$ and $\partial^2 h_p(p,v) = 3dh_p v = 0$ for all $p \in \mathcal{H}, v \in T_p \mathcal{H}$. In particular, $h$ is hyperbolic. Notice also that a linear transformation mapping a projective special real manifold $\mathcal{H} \subset \mathbb{R}^{n+1}$ to another projective special real manifold $\mathcal{H}' \subset \mathbb{R}^{n+1}$ is automatically an isometry with respect to the centroaffine metrics. In particular, isomorphic projective special real manifolds are isometric.

In order to avoid special cases in low dimensions, and since the case $n \leq 2$ has already been studied [CDL], we will always assume that $n \geq 3$ in the following classifications.

1.1 Classification of non-degenerate reducible polynomials

For $m \in \mathbb{N}$ and $k \in \{0, \ldots, m\}$, we introduce the following quadratic polynomials on $\mathbb{R}^m$:

$$Q_k^m := \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{m} x_i^2.$$ 

Proposition 7. Any non-degenerate reducible homogeneous cubic polynomial $h$ on $\mathbb{R}^{n+1}$, $n \geq 3$, is equivalent to precisely one of the following:

I) $x_{n+1} Q_{\frac{n}{2}}^n$, $\frac{n}{2} \leq k \leq n$,

II) $x_1 Q_k^{n+1}$, $1 \leq k \leq n + 1$,

III) $(x_1 + x_{n+1}) Q_k^{n+1}$, $\frac{n+1}{2} \leq k \leq n$.

Proof. Let $h = LQ$ be a non-zero reducible cubic polynomial on $\mathbb{R}^{n+1}$, where $L$ is a linear and $Q$ a quadratic factor. Up to a linear transformation, we can assume that $Q = Q_k^m$, $1 \leq m \leq n + 1$, $\frac{m}{2} \leq k \leq m$. In the following, let

$$L := \sum_{j=1}^{n+1} a_j x_j.$$ 

Next we examine for which choices of $Q_k^m$ and $L$ the polynomial $h = LQ_k^m$ is non-degenerate. Notice that $m = n$ or $m = n + 1$, since otherwise $0 \neq \ker dL \cap \ker \partial^2 Q \subset \ker \partial^2 h_p$ for all $p \in \mathbb{R}^{n+1}$. 


In the case \( m = n \) the non-degeneracy of \( h \) clearly implies that \( a_{n+1} \neq 0 \) and without loss of generality we can assume that \( L = x_{n+1} \). We compute

\[
\partial^2 h = 2 \begin{pmatrix}
  x_{n+1} & \cdots & x_{n+1} & x_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  -x_{n+1} & \cdots & -x_{k+1} & -x_k \\
  x_k & \cdots & x_1 & x_{n+1} \\
  x_1 & \cdots & x_k & -x_{k+1} \\
  \vdots & \ddots & \vdots & \vdots \\
  -x_{n+1} & \cdots & -x_n & 0
\end{pmatrix},
\]

where the remaining entries are zero. The determinant is given by

\[
\det \partial^2 h = 2^{n+1}(-1)^{n-k+1}x_{n+1}^{n-2}h,
\]

which shows that \( h = x_{n+1}Q^k \) is non-degenerate for all \( \frac{n-1}{2} \leq k \leq n \). These are precisely the polynomials listed in I).

It remains to check the case \( m = n + 1 \), that is, \( h = LQ^{n+1} \), \( \frac{n+1}{2} \leq k \leq n + 1 \). Using the transitive action of the pseudo-orthogonal group of the quadratic form \( Q^{n+1} \) on each pseudo-sphere and on the cone of non-zero light-like vectors we can assume up to a positive rescaling that \( L = x_1 \) (L space-like), \( L = x_{n+1} \) (L time-like), or \( L = x_1 + x_{n+1} \) (L light-like), where the latter two cases need only to be considered for \( k \leq n \). Since \( x_{n+1} \) is space-like with respect to \( -Q^{n+1} \) for \( \frac{n+1}{2} \leq k \leq n \) and \( -Q^{n+1} \) is equivalent to \( Q^{n+1-1-k} \), \( 1 \leq n + 1 - k \leq \frac{n+1}{2} \), we are left with the two cases II) and III).

In case II), \( h = x_1Q^{n+1} \) with \( 1 \leq k \leq n + 1 \) and

\[
\partial^2 h = 2 \begin{pmatrix}
  3x_1 & x_2 & \cdots & x_k & -x_{k+1} & \cdots & -x_{n+1} \\
  x_2 & x_1 & \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \ddots & x_k & x_1 & \ddots & \vdots & \vdots \\
  -x_{k+1} & \cdots & -x_1 & x_{n+1} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  -x_{n+1} & \cdots & -x_1 & 0
\end{pmatrix},
\]

We obtain

\[
\det \partial^2 h = (-1)^{n+1-k}2^{n+1}x_1^{n-2}(4x_1^3 - h),
\]

which, for all \( 1 \leq k \leq n + 1 \), is not the zero polynomial. Hence, all polynomials listed in II) are non-degenerate.

In case III), that is \( h = (x_1 + x_{n+1})Q^{n+1} \), \( \frac{n+1}{2} \leq k \leq n \), it is convenient to change the coordinates the following way:

\[
x_1 + x_{n+1} = \xi, \\
x_1 - x_{n+1} = \eta.
\]
$h$ is now of the form

$$h = \xi \left( \xi \eta + \sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2 \right).$$

In the coordinates $(\xi, \eta, x_2, \ldots, x_n)$ we have

$$\partial^2 h = 2 \begin{pmatrix}
\eta & \xi & x_2 & \ldots & x_k & -x_{k+1} & \ldots & -x_n \\
\xi & 0 & x_2 & \ldots & \xi & \ldots & \xi \\
x_2 & \xi & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \ldots & \xi & \ldots & \ldots & \ldots \\
x_k & \ldots & \xi & \ldots & \ldots & \ldots & \ldots \\
-x_{k+1} & \ldots & \ldots & \ldots & -\xi & \ldots & \ldots \\
\vdots & \vdots & \ldots & \ldots & \ldots & \ldots & \ldots \\
-x_n & \ldots & \ldots & \ldots & \ldots & \ldots & -\xi 
\end{pmatrix}.$$

It is now easy to see that

$$\det \partial^2 h = (-1)^{n+1-k} \xi^{n+1}.$$

We conclude that all polynomials considered in III) are non-degenerate. \qed

### 1.2 Classification of hyperbolic reducible polynomials and complete projective special real manifolds

Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a hyperbolic homogeneous cubic polynomial. We consider the open subset $\mathcal{H}(h)$ of the hypersurface $\{h = 1\}$ consisting of the hyperbolic points of $h$:

$$\mathcal{H}(h) = \{ p \in \mathbb{R}^{n+1} \mid h(p) = 1, -\partial^2 h_p \text{ has Lorentzian signature } (n, 1) \}.$$

**Proposition 8.** Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$, $n \geq 3$, be a reducible hyperbolic homogeneous cubic polynomial and let $(x_1, \ldots, x_{n+1})$ denote the standard coordinates of $\mathbb{R}^{n+1}$. Then $h$ is equivalent to one of the following polynomials and the corresponding hypersurface $\mathcal{H}(h)$ endowed with the Riemannian metric $-\frac{1}{2} \partial^2 h|_{T\mathcal{H}(h) \times T\mathcal{H}(h)}$ has the following properties:

a) $h = x_{n+1} \left( \sum_{i=1}^{n-1} x_i^2 - x_n^2 \right)$, $\mathcal{H}(h) = \{ h = 1, \ x_{n+1} < 0 \}$ has two connected components, both closed and isomorphic.

b) $h = (x_1 + x_{n+1}) \left( \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 \right)$, $\mathcal{H}(h) = \{ h = 1, \ x_1 + x_{n+1} < 0 \}$ has one connected component and it is closed.

c) $h = x_1 \left( \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 \right)$, $\mathcal{H}(h) = \{ h = 1, \ x_1 < 0 \}$ has two connected components, both closed and isomorphic.
d) \( h = x_1 \left( x_1^2 - \sum_{i=2}^{n+1} x_i^2 \right) \), \( \mathcal{H}(h) = \{ h = 1, \ x_1 > 0 \} \) has one connected component and it is closed.

e) \( h = x_1 \left( x_1^2 + x_2^2 - \sum_{i=3}^{n+1} x_i^2 \right) \), \( \mathcal{H}(h) = \{ h = 1 \} \cap \{ \frac{1}{x_1^2} > x_1 > 0 \} \) has two connected components. They are isomorphic and not closed.

In particular, the closed connected components of the respective \( \mathcal{H}(h) \) are complete projective special real manifolds.

**Proof.** In Proposition\(^7\) we have listed all non-degenerate cubic homogeneous polynomials up to equivalence. It remains to determine which ones are hyperbolic and to analyse the properties of the connected components of \( \mathcal{H}(h) \). In the following we treat each of the cases I-III) of Proposition\(^7\)

I) Recall that the family I) of Proposition \(^7\) contains the polynomials \( h = x_{n+1}Q^n_k \), \( \frac{n}{2} \leq k \leq n \), with

\[
-\frac{1}{2} \partial^2 h = -x_{n+1} \left( \sum_{i=1}^{k} dx_i^2 - \sum_{i=k+1}^{n} dx_i^2 \right) - 2 \left( \sum_{i=1}^{k} x_i dx_i - \sum_{i=k+1}^{n} x_i dx_i \right) dx_{n+1}.
\]

To check that a point \( p \in \mathbb{R}^{n+1} \) is hyperbolic it suffices to construct an orthogonal basis of \( T_p \mathbb{R}^{n+1} \) with respect to \(-\frac{1}{2} \partial^2 h\) and to check that the Gram matrix has Lorentzian signature. Note that the vectors \( \{ \partial_{x_1}, \ldots, \partial_{x_n} \} \) are orthogonal at each point:

\[
-\frac{1}{2} \partial^2 h(\partial_{x_i}, \partial_{x_j}) = \begin{cases} 
\delta_{ij} x_{n+1}, & 1 \leq i, j \leq k, \\
\delta_{ij} x_{n+1}, & k+1 \leq i, j \leq n, \\
0, & \text{otherwise.}
\end{cases}
\]

Now the restrictions \( n \geq 3, k \geq \frac{n}{2} \), allow us to limit the possibility of hyperbolic points to the cases \( k = n - 1 \) and \( k = n \) and we obtain the requirement \( x_{n+1} < 0 \). Otherwise we would have at least two time-like vectors in an orthogonal basis of the form \( (v, \partial_{x_1}, \ldots, \partial_{x_n}) \).

For \( v = \sum_{i=1}^{n+1} v_i \partial_{x_i} \) to be orthogonal to \( \partial_{x_i} \) for all \( 1 \leq i \leq n \) it has to fulfil

\[
x_{n+1} v_i + x_i v_{n+1} = 0 \ \forall 1 \leq i \leq n.
\]

Hence, \( v_i = -\frac{x_i v_{n+1}}{x_{n+1}} \) for \( 1 \leq i \leq n \) and \( v = v_{n+1} \left( -\sum_{i=1}^{n} \frac{x_i}{x_{n+1}} \partial_{x_i} + \partial_{x_{n+1}} \right) \). Since \( x_{n+1} < 0 \), we might choose \( v = \sum_{i=1}^{n} x_i \partial_{x_i} - x_{n+1} \partial_{x_{n+1}} \) and obtain

\[
-\frac{1}{2} \partial^2 h(v, v) = x_{n+1} \left( \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2 \right) = h.
\]

Hyperbolic points need to fulfil \( h(p) > 0 \) by definition, which implies \(-\frac{1}{2} \partial^2 h(v, v) > 0\). Hence, \( h = x_{n+1}Q^n_k, \ \frac{n}{2} \leq k \leq n \), is hyperbolic if and only if \( k = n - 1 \), that is
$h = x_{n+1} \left( \sum_{i=1}^{n-1} x_i^2 - x_n^2 \right)$ is the polynomial $a$ of this proposition. The hypersurface $\mathcal{H}(h)$ consists of the connected components

$$\mathcal{H}_1 := \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid h(x_1, \ldots, x_{n+1}) = 1, \ x_n < 0, x_{n+1} < 0 \right\}$$

and

$$\mathcal{H}_2 := \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid h(x_1, \ldots, x_{n+1}) = 1, \ x_n > 0, x_{n+1} < 0 \right\} .$$

One can easily verify that $\mathcal{H}_1$ and $\mathcal{H}_2$ are both closed in $\mathbb{R}^{n+1}$ and related by the involution $(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, -x_n, x_{n+1})$.

II) The family II) of Proposition\footnote{footnote text} contains polynomials of the form $h = x_1 Q_{k+1}^{n-1}$, $1 \leq k \leq n+1$. We will construct an orthogonal basis for each $p \in \{ h > 0 \}$, $p = (x_1, \ldots, x_{n+1})$, with respect to

$$-\frac{1}{2} \partial^2 h = x_1 \left( -3dx_1^2 - \sum_{i=2}^{k} dx_i^2 + \sum_{i=k+1}^{n+1} dx_i^2 \right) - 2dx_1 \left( \sum_{i=2}^{k} x_idx_i - \sum_{i=k+1}^{n+1} x_idx_i \right).$$

We define

$$v = x_1 \partial_{x_1} - \sum_{i=2}^{n+1} x_i \partial_{x_i}.$$ 

Then one can check, for $x_1 \neq 0$, that $(v, \partial_{x_2}, \ldots, \partial_{x_{n+1}})$ is an orthogonal basis with respect to $-\frac{1}{2} \partial^2 h$ and that

$$-\frac{1}{2} \partial^2 h(v, v) = -4x_1^3 + h.$$ 

Thus, the possible values for $k$ that do not exclude the possibility for $h$ to be hyperbolic, the respective requirements for the possibly hyperbolic points, and the corresponding polynomials are (recall $n \geq 3$):

A) $k = 1$, $x_1 > 0$, $h < 4x_1^3 \left( -\frac{1}{2} \partial^2 h(v, v) < 0 \right)$; $h = x_1 \left( x_1^2 - \sum_{i=2}^{n+1} x_i^2 \right)$,

B) $k = 2$, $x_1 > 0$, $h > 4x_1^3 \left( -\frac{1}{2} \partial^2 h(v, v) > 0 \right)$; $h = x_1 \left( x_1^2 + x_2^2 - \sum_{i=3}^{n+1} x_i^2 \right)$,

C) $k = n$, $x_1 < 0$, $h > 4x_1^3 \left( -\frac{1}{2} \partial^2 h(v, v) > 0 \right)$; $h = x_1 \left( \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 \right)$,

D) $k = n + 1$, $x_1 < 0$, $h < 4x_1^3 \left( -\frac{1}{2} \partial^2 h(v, v) < 0 \right)$; $h = x_1 \left( \sum_{i=1}^{n+1} x_i^2 \right)$.

The polynomials in A), B), and C) are, in fact, hyperbolic, as seen by specifying a hyperbolic point:

A) $p_A = (1, 0, \ldots, 0)$, $h(p_A) = 1$,

B) $p_B = (1, 2, 0, \ldots, 0)$, $h(p_B) = 5$,

C) $p_C = (-1, 0, \ldots, 0, 2)$, $h(p_C) = 3$. 

11
These three series of polynomials are, corresponding to the above order A), B), and C), the first three cases d), e), and c) of this proposition. The polynomials in D) are not hyperbolic, since the specified conditions are not compatible with $h > 0$. We will now describe the sets $\mathcal{H}(h)$.

In case A), the set of hyperbolic points of $\mathbb{R}^{n+1}$ with respect to $h$ was described by the inequalities $x_1 > 0$ and $h < 4x_1^2$. The second inequality follows from the first since $Q_{1}^{n+1} \leq x_1^2$. This shows that $\mathcal{H}(h) = \{h = 1, x_1 > 0\}$, which has one connected component. To see this consider for fixed $h$ there is a unique $x_1(u) \in (\rho, \infty)$ such that $h(x_1(u), u) = 1$. We obtain a bijection

$$\mathbb{R}^n \to \mathcal{H}(h), \ u \mapsto (x_1(u), u),$$

which is a diffeomorphism by the implicit function theorem. In particular, $\mathcal{H}(h)$ is connected. This implies that it is a connected component of $\{h = 1\}$ and, thus, closed in $\mathbb{R}^{n+1}$.

In case B), the requirement for hyperbolicity on $\{h = x_1 \left(x_2^2 + x_2 - \sum_{i=3}^{n+1} x_i^2\right) = 1\}$ is $\frac{1}{\sqrt{4}} > x_1 > 0$, which implies $x_2 \neq 0$. Observe that

$$h = 1 \iff x_2^2 = \frac{1}{x_1}(1 - x_1^3) + \sum_{i=3}^{n+1} x_i^2.$$  

Hence, $\mathcal{H}(h) = \{h = 1\} \cap \{\frac{1}{\sqrt{4}} > x_1 > 0\}$ has two connected components, namely $\{h = 1\} \cap \{\frac{1}{\sqrt{4}} > x_1 > 0\} \cap \{x_2 > 0\}$ and $\{h = 1\} \cap \{\frac{1}{\sqrt{4}} > x_1 > 0\} \cap \{x_2 < 0\}$. They are related by the involution $x_2 \mapsto -x_2$, which preserves the polynomial $h$. The two components of $\mathcal{H}(h)$ are not closed in $\mathbb{R}^{n+1}$, since its boundary is given by

$$\partial \mathcal{H}(h) = \left\{h = 1, x_1 = \frac{1}{\sqrt{4}}, \det \partial^2 h = 0\right\} = \left\{h = 1, x_1 = \frac{1}{\sqrt{4}}\right\} = \left\{x_2^2 - \sum_{i=3}^{n+1} x_i^2 = \frac{3}{4^2}\right\}.$$  

In case C), the requirement $x_1 < 0$ automatically implies the second requirement $h > 4x_1^2$ on $\{h = x_1 \left(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2\right) = 1\}$ and, hence, $\mathcal{H}(h) = \{h = 1, x_1 < 0\}$. Note that $\{h = 1\} \cap \{x_1 = 0\} = \emptyset$ implies that the connected components of $\mathcal{H}(h)$ are also connected components of $\{h = 1\}$, and thus are closed. $x_1 < 0$ and $h = x_1 \left(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2\right) = 1$ implies $\sum_{i=1}^{n} x_i^2 - x_{n+1}^2 < 0$, which implies $x_{n+1} \neq 0$. Hence, the connected components of $\mathcal{H}(h)$ are given by the two graphs $\{h = 1, x_1 < 0, x_{n+1} > 0\}$ and $\{h = 1, x_1 < 0, x_{n+1} < 0\}$. They are related by the involution $x_{n+1} \mapsto -x_{n+1}$.
III) Recall that each \( h = (x_1 + x_{n+1})Q_k^{n+1} \) contained in family III) of Proposition 7 is equivalent to \( h = \xi (\xi \eta + \sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2) \). In these coordinates

\[
-\frac{1}{2} \partial^2 h = -\eta d\xi^2 - 2\xi d\eta d\xi + \left( -2 \sum_{i=2}^{k} x_i dx_i + 2 \sum_{i=k+1}^{n} x_i dx_i \right) d\xi \\
\quad + \xi \left( -\sum_{i=2}^{k} dx_i^2 + \sum_{i=k+1}^{n} dx_i^2 \right)
\]

The set \( \{ h = \xi (\xi \eta + \sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2) = 1 \} \) consists of exactly two connected components:

\[
\mathcal{H}_1 := \left\{ (\xi, \eta, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \eta = \frac{1 - \xi \left( \sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2 \right)}{\xi^2}, \xi > 0 \right\}
\]

and

\[
\mathcal{H}_2 := \left\{ (\xi, \eta, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \eta = \frac{1 - \xi \left( \sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2 \right)}{\xi^2}, \xi < 0 \right\}
\]

In order to determine which of the polynomials in this family are hyperbolic, we will pull back \(-\frac{1}{2} \partial^2 h\) to \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively. We will use that \( h \) is hyperbolic if and only if the pullback is Riemannian at least at one point contained in \( \{ h = 1 \} \). We first determine the differential of \( \eta = \eta(\xi, x_2, \ldots, x_n) \):

\[
d\eta = \frac{-2 + \xi \left( \sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2 \right)}{\xi^3} d\xi + \frac{-2 \sum_{i=2}^{k} x_i dx_i + 2 \sum_{i=k+1}^{n} x_i dx_i}{\xi}.
\]

Hence, the pullback of \(-\frac{1}{2} \partial^2 h\) to \( \mathcal{H}_j \) which we denote by \( g_j, j \in \{1, 2\} \), is of the following form:

\[
g_j = \frac{3 - \xi \left( \sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2 \right)}{\xi^2} d\xi^2 + 2 \left( \sum_{i=2}^{k} x_i dx_i - \sum_{i=k+1}^{n} x_i dx_i \right) d\xi \\
\quad + \xi \left( -\sum_{i=2}^{k} dx_i^2 + \sum_{i=k+1}^{n} dx_i^2 \right).
\]

For each \( \frac{n+1}{2} \leq k \leq n \) there exists exactly one \( \tilde{k} \) with \( 1 \leq \tilde{k} \leq \frac{n+1}{2} \), such that \( \mathcal{H}_1 \) corresponding to \( h = (x_1 + x_{n+1})Q_k^{n+1} \) is isometric to \( \mathcal{H}_2 \) corresponding to \( \tilde{h} = (x_1 + x_{n+1})Q_{\tilde{k}}^{n+1} \), namely \( \tilde{k} = n - (k - 1) \). In the coordinates \( (\xi, \eta, x_2, \ldots, x_n) \) the corresponding isometry is given by \( \xi \mapsto -\xi, x_\ell \mapsto x_\ell - (\ell - 2) \) for \( 2 \leq \ell \leq n \). Hence, we can reduce our analysis to \( \mathcal{H}_1 \), that is \( \xi > 0 \), but need to increase the range for \( k \) to \( 1 \leq k \leq n \).
Returning to the study of $g_1$, we obtain

\[ g_1(\partial_{x_i}, \partial_{x_j}) = \begin{cases} -\delta^i_j \xi, & 2 \leq i, j \leq k, \\ \delta^i_j \xi, & k + 1 \leq i, j \leq n. \end{cases} \]

For $g_1$ to be Riemannian, this implies that $k = 1$. Hence, the only possibly hyperbolic polynomial is $h = \xi (\xi \eta - \sum_{k=2}^n x_i^2)$ and the corresponding metric $g_1$ reads

\[ g_1 = \frac{3}{\xi^2} d\xi^2 + \frac{1}{\xi} \sum_{i=2}^n (x_i d\xi - \xi dx_i)^2, \]

which is indeed Riemannian at all points of $H_1$. Hence, the only hyperbolic polynomial of the form $h = (x_1 + x_{n+1})Q_k^{n+1}$, $\frac{n+1}{2} \leq k \leq n$, is given by

\[ h = (x_1 + x_{n+1}) \left( \sum_{i=1}^n x_i^2 - x_{n+1}^2 \right). \]

The corresponding $H(h) = \{ h = 1, x_1 + x_{n+1} < 0 \}$ has a single connected component. It is closed in $\mathbb{R}^{n+1}$, since $\{ h = 1 \} \cap \{ x_1 + x_{n+1} = 0 \} = \emptyset$ implies that $H(h)$ is also a connected component of $\{ h = 1 \}$. This polynomial is the polynomial $b)$ of this proposition. \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ © : \]
ii) \((\nabla_X J)Y = (\nabla_Y J)X\) for all \(X, Y \in \Gamma(TM)\),

iii) \(\nabla \xi = D \xi = \text{Id}\), where \(D\) is the Levi-Civita connection,

iv) \(g_M\) is positive definite on \(\mathcal{D} = \text{span}\{\xi, J \xi\}\) and negative definite on \(\mathcal{D}^\perp\).

Let \((M, J, g_M, \nabla, \xi)\) be a conical affine special Kähler manifold of complex dimension \(n + 1\). Then \(\xi\) and \(J \xi\) are commuting holomorphic vector fields that are homothetic and Killing respectively \([CM]\). We assume that the holomorphic Killing vector field \(J \xi\) induces a free \(S^1\)-action and that the holomorphic homothety \(\xi\) induces a free \(\mathbb{R}^{>0}\)-action on \(M\). Then \((M, g_M)\) is a metric cone over \((S, g_S)\), where \(S := \{p \in M|g_M(\xi(p), \xi(p)) = 1\}\), \(g_S := g_M|_S\); and \(-g_S\) induces a Riemannian metric \(\bar{g}_M\) on \(\bar{M} := S/S_{J \xi}\). \((\bar{M}, -\bar{g}_M)\) is obtained from \((M, J, g)\) via a Kähler reduction with respect to \(J \xi\) and, hence, \(\bar{g}_M\) is a Kähler metric (see e.g. \([CHM]\)). The corresponding Kähler form \(\omega_{\bar{M}}\) is obtained from \(\omega_M\) by symplectic reduction. This determines the complex structure \(J_{\bar{M}}\). We will denote by \(\pi\) the projection \(M \to \bar{M}\). For future use let us mention that the metrics on \(M\) and \(\bar{M}\) are explicitly related by

\[g_M|_{V \times V} = -g_M(\xi, \xi)p^*g_M|_{V \times V}, \quad V = (\ker d\pi)^\perp \subset TM. \tag{2.1}\]

**Definition 10.** The Kähler manifold \((\bar{M}, g_{\bar{M}}, J_{\bar{M}})\) is called a **projective special Kähler manifold**.

Locally, there exist so-called conical special holomorphic coordinates \(z = (z^I) = (z^0, \ldots, z^n) : U \xrightarrow{\sim} \bar{U} \subset \mathbb{C}^{n+1}\) such that the geometric data on the domain \(U \subset M\) is encoded in a holomorphic function \(F : \bar{U} \to \mathbb{C}\) that is homogeneous of degree 2 \([ACD, CM]\). Namely, we have \([CM]\)

\[g_M|_U = \sum_{I,J} N_{IJ} dz^I d\bar{z}^J, \quad N_{IJ}(z, \bar{z}) := 2\text{Im} F_{IJ}(z) := 2\text{Im} \frac{\partial^2 F(z)}{\partial z^I \partial \bar{z}^J} (I, J = 0, \ldots, n)\]

and \(\xi|_U = \sum z^I \frac{\partial}{\partial z^I} + \bar{z}^J \frac{\partial}{\partial \bar{z}^J} \). The Kähler potential for \(g_M|_U\) is given by \(\rho|_U = g_M(\xi, \xi)|_U = \sum z^I N_{IJ} \bar{z}^J\).

The \(\mathbb{C}^*\)-invariant functions \(X^\mu := z^\mu, \mu = 1, \ldots, n\), define a local holomorphic coordinate system on \(\bar{M}\). The Kähler potential for \(g_{\bar{M}}\) is \(\mathcal{K} := -\log \sum_{I,J=0}^n X^I N_{IJ}(X) \bar{X}^J\), where \(X := (X^0, \ldots, X^n)\) with \(X^0 := 1\). Note that for every function \(f_U(z)\) on \(U\), we define a function \(f_{\bar{U}}(X)\) on the corresponding subset \(\bar{U} \subset M\) by \(f_{\bar{U}}(X) := f_U(1, X^1, \ldots, X^n)\). In most cases, we will suppress the subscripts \(U\) and \(\bar{U}\) and use the same notation for corresponding functions on \(U\) and \(\bar{U}\).
2.2 The supergravity c-map

Let \((\bar{M}, g_{\bar{M}})\) be a projective special Kähler manifold of complex dimension \(n\) which is globally defined by a single holomorphic function \(F\). The supergravity c-map \([FS]\) associates with \((\bar{M}, g_{\bar{M}})\) a quaternionic Kähler manifold \((\bar{N}, g_{\bar{N}})\) of dimension \(4n + 4\). Following the conventions of \([CHM]\), we have \(\bar{N} = \bar{M} \times \mathbb{R} > 0 \times \mathbb{R}^{2n+3}\) and

\[
g_{\bar{N}} = g_{\bar{M}} + g_G,
\]

\[
g_G = \frac{1}{4\rho^2} dp^2 + \frac{1}{4\rho^2} (d\bar{\phi} + \sum (\zeta^I d\bar{\zeta}_I - \bar{\zeta}_I d\zeta^I))^2 + \frac{1}{2\rho} \sum J_{IJ}(m) d\zeta^I d\zeta^J
\]

\[
+ \frac{1}{2\rho} \sum J^{IJ}(m) (d\bar{\zeta}_I + R_{IK}(m) d\zeta^K)(d\bar{\zeta}_J + R_{JL}(m) d\zeta^L),
\]

where \((\rho, \bar{\phi}, \zeta^I, \bar{\zeta}_I)\), \(I = 0, 1, \ldots, n\), are standard coordinates on \(\mathbb{R} > 0 \times \mathbb{R}^{2n+3}\). The real-valued matrices \(J(m) := (J_{IJ}(m))\) and \(R(m) := (R_{IJ}(m))\) depend only on \(m \in \bar{M}\) and \(J(m)\) is invertible with the inverse \(J^{-1}(m) := (J^{-1}_{IJ}(m))\). More precisely,

\[
N_{IJ} := R_{IJ} + iJ_{IJ} := F_{IJ} + i \sum K N_{IK} z^K \sum L N_{LJ} z^L \sum I J N_{IJ} z^I z^J, \quad N_{IJ} := 2 \text{Im} F_{IJ},
\]

where \(F\) is the holomorphic prepotential with respect to some system of special holomorphic coordinates \(z^I\) on the underlying conical special Kähler manifold \(M \to \bar{M}\). Notice that the expressions are homogeneous of degree zero and, hence, well defined functions on \(\bar{M}\). It is shown in \([CHM]\ Cor. 5\) that the matrix \(J(m)\) is positive definite and hence invertible and that the metric \(g_{\bar{N}}\) does not depend on the choice of special coordinates \([CHM]\ Thm. 9\). It is also shown that \((\bar{N}, g_{\bar{N}})\) is complete if and only if \((\bar{M}, g_{\bar{M}})\) is complete \([CHM]\ Thm. 5\).

Using \((p_a)_{a=1,\ldots,2n+2} := (\zeta^I, \zeta^J)_{IJ=0,\ldots,n}\) and \((\hat{H}^{ab}) := \begin{pmatrix} J^{-1} & -J^{-1}R \\ R J^{-1} & J + R J^{-1} R \end{pmatrix}\), we can combine the last two terms of \(g_G\) into \(\frac{1}{2\rho} \sum dp_a \hat{H}^{ab} dp_b\), i.e. the quaternionic Kähler metric is given by

\[
g_{FS} := g_{\bar{N}} = g_{\bar{M}} + \frac{1}{4\rho^2} dp^2 + \frac{1}{4\rho^2} (d\bar{\phi} + \sum (\zeta^I d\bar{\zeta}_I - \bar{\zeta}_I d\zeta^I))^2 + \frac{1}{2\rho} \sum dp_a \hat{H}^{ab} dp_b. \quad (2.2)
\]

2.3 The supergravity r-map

Let \((\mathcal{H} := \{x \in U \mid h(x) = 1\}, g_{\mathcal{H}} := -\partial^2 h|_{\mathcal{H}}\) be a projective special real manifold defined by a real homogeneous cubic polynomial \(h\) and an \(\mathbb{R} > 0\)-invariant domain \(U \subset \mathbb{R}^n \setminus \{0\}\). Let \(\bar{M} := \mathbb{R}^n + iU \subset \mathbb{C}^n\) be endowed with the standard complex structure \(J_{\bar{M}}\) induced from \(\mathbb{C}^n\) and with holomorphic coordinates \((X^\mu = y^\mu + i\bar{y}^\mu)_{\mu=1,\ldots,n} \in \mathbb{R}^n + iU\). We define a Kähler metric

\[
g_{\bar{M}} = \sum_{\mu,\nu=1}^n \frac{\partial^2 \mathcal{K}}{\partial X^\mu \partial X^\nu} dX^\mu d\bar{X}^\nu
\]

16
on $\bar{M}$ with Kähler potential

$$K(X, \bar{X}) := -\log 8h(x),$$

where $x = (\text{Im } X^1, \ldots, \text{Im } X^n) \in U$.

**Definition 11.** The correspondence $(\mathcal{H}, g_M) \mapsto (\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ is called the supergravity r-map.

**Remark 12.** Note that any manifold $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ in the image of the supergravity r-map is a projective special Kähler manifold (see Section 2.1). The corresponding conical affine special Kähler manifold is the trivial $\mathbb{C}^*$-bundle

$$M := \{z = z^0 \cdot (1, X) \in \mathbb{C}^{n+1} \mid z^0 \in \mathbb{C}^*, X \in \bar{M} = \mathbb{R}^n + iU\} \to \bar{M}$$
edowed with the standard complex structure $J$ and the metric $g_M$ defined by the holomorphic function

$$F : M \to \mathbb{C}, \quad F(z^0, \ldots, z^n) = \frac{h(z^1, \ldots, z^n)}{z^0}.$$

Note that in general, the flat connection $\nabla$ on $M$ is not the standard one induced from $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$. The homothetic vector field $\xi$ is given by $\xi = \sum_{I=0}^{n-1} (z^I \frac{\partial}{\partial z^I} + \bar{z}^I \frac{\partial}{\partial \bar{z}^I})$. To check that $g_{\bar{M}}$ is the corresponding projective special Kähler metric, one uses the fact that

$$8|z^0|^2 h(x) = \sum_{I, J=0}^n z^I N_{I,J}(z, \bar{z}) \bar{z}^J,$$

where as above, $x = (\text{Im } X^1, \ldots, \text{Im } X^n) = (\text{Im } \frac{z^1}{z^0}, \ldots, \text{Im } \frac{z^n}{z^0}) \in U$ (see [CHM]).

### 2.4 Curvature formulas for the supergravity r-map

Under the assumptions of Section 2.3, let $(e^a_{\mu})_{a, \mu=1,\ldots,n}$ be a real $n \times n$ matrix-valued function on some open subset in $\bar{M}$ such that $\sum_{a=1}^n e^0_{\mu} \bar{e}^a_{\nu} = \sum_{a=1}^n e^a_{\mu} e^a_{\nu} = K_{\mu\bar{\nu}}$, where

$$K_{\mu\bar{\nu}} = -\frac{\partial^2 \log h(x)}{\partial X^\mu \partial X^{\bar{\nu}}} = -\frac{h_{\mu\nu}(x)}{4h(x)} + \frac{h_{\mu}(x) h_{\nu}(x)}{4h^2(x)}. \quad (2.3)$$

Here, subscripts of the cubic polynomial $h$ denote derivatives with respect to the standard coordinates on $U$, e.g. $h_{\mu}(x) = \frac{\partial h(x)}{\partial x^\mu}$. The holomorphic one-forms

$$\sigma^a := \sum_{\mu=1}^n e^a_{\mu} dX^\mu \quad (2.4)$$

$\tilde{\nabla}$ is defined by $x^I = \text{Re } z^I$ and $y_I = \text{Re } F_I(z)$ being flat for $I = 0, \ldots, n$ (see [ACD]).
We denote the coefficients of the local Levi-Civita connection one-form associated to the Kähler metric \( g_M \) by \( \Gamma^\rho_{\sigma\mu} \). Then \( \sigma^a = 2g_M(\sigma_a, \cdot) \) and \( \sigma^a(a_b) = \delta^a_b', \sigma^a(\bar{a}_b) = \bar{\delta}^a_b = 0 \). Note that \( g_M(\sigma_a, \bar{a}_b) = \bar{\delta}^a_b \) which implies \( \langle \sigma_a, \sigma_b \rangle = \delta_{ab} \) for the corresponding sesqui-linear form \( \langle \cdot, \cdot \rangle = g_M - i\omega_M \) on \((T\bar{M}, J)\). This explains why we call the frame \((\sigma_a)\) and the dual coframe \((\sigma^a)\) unitary.

Note that the inverse of the matrix-valued function \((K_{\mu\nu})_{\mu,\nu=1,...,n}\) (see equation (2.3)) is given by

\[
K^{\nu\rho} = -4h(x)h^{\nu\rho}(x) + 2x^\nu x^\rho, \tag{2.5}
\]

where \((h^{\mu\nu})_{\mu,\nu=1,...,n} = (h_{\mu\nu})^{-1}_{\mu,\nu=1,...,n}\). We will usually write \(K^{\rho\nu}\) instead of \(K^{\nu\rho}\).

Note that in this section, \(\nabla\) denotes the Levi-Civita connection of the projective special Kähler metric \(g_M\). The expressions for the Christoffel symbols

\[
\Gamma^\rho_{\sigma\mu} := dX^\rho(\nabla_{\partial_X^\sigma} \partial_X^\mu) = \sum_{\kappa=1}^n K^\rho_{\kappa\mu} \partial_X^\sigma K^\kappa_{\mu\bar{\nu}}
\]

\[
= -\frac{i}{2h} \left( h \sum_{\kappa=1}^n h^{\kappa\mu} h^{\rho\kappa} - h_{\sigma\rho} \delta^\rho_{\mu} - h_{\mu\rho} \delta^\rho_{\sigma} + \frac{1}{2} x^\rho h_{\mu\sigma} \right)
\]

and the coefficients

\[
R^\rho_{\sigma\mu\nu} := dX^\rho \left( R(\partial_X^\nu, \partial_X^\sigma) \partial_X^\mu \right) = -\partial_X^\nu \Gamma^\rho_{\sigma\mu} = -\frac{i}{2} \partial_X^\nu \Gamma^\rho_{\sigma\mu}
\]

\[
= -\frac{1}{4h^2} \left[ \frac{1}{2} x^\rho (hh_{\mu\sigma} - h_{\mu\sigma} h_{\nu}) + h_{\mu\nu} h_{\rho\sigma} + h_{\rho\nu} h_{\mu\sigma} - h_{\sigma\rho} \delta^\rho_{\mu} - \frac{1}{2} h_{\mu\rho} \delta^\rho_{\sigma} \right] - h^2 \sum_{\alpha, \beta, \gamma = 1}^n K^{\alpha\beta} h_{\alpha\beta\gamma} h_{\gamma\mu\sigma}
\]

\[
= -\delta^\rho_{\kappa\mu} - \delta^\rho_{\mu\kappa} + \epsilon^{2X} \sum_{\alpha, \beta, \gamma = 1}^n K^{\alpha\beta} h_{\alpha\nu\beta} K^{\beta\gamma} h_{\gamma\mu\sigma} \tag{2.6}
\]

of the Riemann curvature tensor

\[
R(X, Y) Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad (X, Y, Z \in \mathfrak{X}(\bar{M}))
\]

have been calculated for instance in [CDL, Theorem 3].

We denote the coefficients of the local Levi-Civita connection one-form associated to the unitary local coframe \((\sigma^a)_{a=1,...,n}\) by \(\omega^a_b\), i.e. \(\nabla. \sigma^a = -\sum_{b=1}^n \omega^a_b \sigma^b\). Compatibility with
the metric and torsion-freeness translate into the conditions that the complex one-form valued matrix \((\omega^a_b)_{a,b=1,\ldots,n}\) is anti-Hermitian and satisfies \(d\sigma^a + \sum_{b=1}^{n} \omega^a_b \wedge \sigma^b = 0\) for \(a = 1, \ldots, n\). These are fulfilled by the following general formula that holds for all Kähler manifolds:

\[
\omega^a_b = \sum_{\mu=1}^{n} (e^a_\mu \bar{\partial} e^\mu_b - \bar{e}^a_\mu \partial e^\mu_a).
\] (2.7)

This formula is found by observing that the \((0, 1)\)-component of \(\omega^a_b\) is uniquely determined by solving the \((1, 1)\)-projection of the equation \(d\sigma^a + \sum_{b=1}^{n} \omega^a_b \wedge \sigma^b = 0\) and using the skew-Hermiticity to compute the \((1, 0)\)-component of \(\omega^a_b\). By the existence of the Levi-Civita connection the \((2, 0)\)-projection of the equation \(d\sigma^a + \sum_{b=1}^{n} \omega^a_b \wedge \sigma^b = 0\) is then automatically satisfied. In terms of the local connection one-form, the curvature tensor of a Kähler manifold is given by

\[
R(X, Y)\sigma_c = \sum_{d=1}^{n} (d\omega^d_c + \sum_{c'=1}^{n} \omega^d_{c'} \wedge \omega^c_{c'})(X, Y)\sigma_d =: \sum_{d=1}^{n} \tilde{R}^d_c(X, Y)\sigma_d.
\]

Using equation (2.6) and \(\mathcal{K}^{\mu\nu} = \sum_{c=1}^{n} e^\mu_c \bar{e}^\nu_c\), one gets the following proposition (see [D, Prop. 7.2.1]):

**Proposition 13.** In terms of the unitary local coframe \((\sigma^a)_{a=1,\ldots,n}\), the Riemann curvature tensor of a projective special Kähler manifold in the image of the supergravity r-map reads

\[
\tilde{R}^a_b = -\delta^a_b \sum_{c=1}^{n} \sigma^c \wedge \bar{\sigma}^c - \sigma^a \wedge \bar{\sigma}^b + e^{2\chi} \sum_{c,e,d=1}^{n} \tilde{h}_{ade} \tilde{h}_{ceb} \sigma^c \wedge \bar{\sigma}^d,
\]

where \(\tilde{h}_{ade} := \sum_{\mu,\nu,\sigma=1}^{n} e^\mu_a e^\nu_b e^\sigma_c h_{\mu\nu\sigma}\) for \(a, b, c = 1, \ldots, n\).

**Proof.** Using \(\mathcal{K}^{\mu\nu} = e^\mu_c \bar{e}^\nu_c\), \(\mathcal{K}^{\mu\nu} = e^\mu_c \bar{e}^\nu_c\) and the fact that \((e^a_\mu)_{a=1,\ldots,n} = (\bar{e}^\nu_b)^{-1}_{\nu, b=1,\ldots,n}\), we find

\[
\tilde{R}^a_b(\sigma_c, \bar{\sigma}_d) = \sigma^a(R(\sigma_c, \bar{\sigma}_d)\sigma_b)
\]

\[
= e^a_\rho dX^\rho(R(\partial_{X^\nu}, \partial_{\bar{X}^\sigma})\partial_{X^\nu}) e^\nu_d \bar{e}^\sigma_b
\]

\[
= e^a_\rho (-\delta^a_b \mathcal{K}^{\mu\nu} - \delta^a_b \mathcal{K}^{\sigma\nu} + e^{2\chi} \mathcal{K}^{\rho\sigma} h_{\alpha\nu\beta} \mathcal{K}^{\beta\gamma} h_{\gamma\mu\sigma}) e^\mu_d \bar{e}^\nu_b
\]

\[
= -\delta^a_b \delta^b_{de} - \delta^a_b \delta^b_{bd} + e^{2\chi} \tilde{h}_{ade} \tilde{h}_{ceb}.
\]

\[\square\]

\(^4\text{Note that for arbitrary Kähler manifolds, the functions } e^a_\mu \text{ cannot in general be chosen to be real.}\)
2.5 Levi-Civita connection for quaternionic Kähler manifolds obtained by the q-map

In this and the following section, we will introduce the quaternionic vielbein formalism, which was used in [FS] to determine the Levi-Civita connection and the Riemann curvature tensor of manifolds in the image of the supergravity c-map. The formulas in this formalism arise from well-known formulas in the differential geometry literature expressed in terms of local frames in the complex vector bundles $E$ and $H$ whose tensor product is identified with the complexified tangent bundle of a quaternionic Kähler manifold in Salamon’s $E$-$H$ formalism [S] (see e.g. [D, Ch. 7] for detailed explanations of the relation between the formulas used in the physics, respectively mathematics literature). The $q$-map is the composition of the supergravity r- and c-map. It assigns a quaternionic Kähler manifold of dimension $4m = 4(n + 1)$ to any projective special real manifold of dimension $n - 1$.

We apply the quaternionic vielbein formalism to quaternionic Kähler manifolds in the image of the q-map and derive formulas for the Levi-Civita connection and the Riemann curvature tensor of these manifolds, expressed in terms of the cubic polynomial $h$, which defines the initial projective special real manifold. Up to changing conventions and fixing inaccuracies, these results can also be obtained by restricting the formulas in [FS] for the c-map to the case of the q-map. The Riemann curvature tensor of a quaternionic Kähler manifold is determined by its trace-free part, the quaternionic Weyl tensor. The latter can be expressed in terms of a certain symmetric quartic tensor field $Ω ∈ \Gamma(S^4E^*)$ in the complex vector bundle $E$. In addition to the above-mentioned results, we derive a formula expressing this quartic tensor field in terms of the cubic polynomial $h$ for manifolds in the image of the q-map. This result is used in Subsection 2.7 to give a general formula for the squared pointwise norm of the Riemann curvature tensor of any quaternionic Kähler manifold in the image of the q-map.

We will restrict ourselves to manifolds in the image of the q-map, which is the composition of the supergravity r- and c-map, i.e. we consider the Ferrara-Sabharwal metric (2.2) defined on $\bar{N} = М × \mathbb{R}^0 × \mathbb{R}^{2n+3}$ for a projective special Kähler manifold $(\bar{M} = \mathbb{R}^n + iU, g_{\bar{M}}, J_{\bar{M}})$ in the image of the supergravity r-map, which is defined by a real homogeneous cubic polynomial $h$. On $\bar{N}$, we define the following complex-valued one-forms:

$$\beta^0 := i e^{X/2} \frac{1}{\sqrt{\rho}} \sum_{I=0}^n X^I A_I, \quad \beta^a := \sum_{I=0}^n P^a_I dX^I = \sigma^a,$$

$$\alpha^0 := -\frac{1}{2\rho} \left( d\rho - i(d\bar{\phi} + \sum_{I=0}^n (\zeta^I d\bar{\zeta}_I - \bar{\zeta}^I d\zeta_I)) \right), \quad \alpha^a := i \frac{1}{\sqrt{\rho}} e^{-X/2} \sum_{I,J=0}^n P^a_I N^{IJ} A_J$$

for $a = 1, \ldots, n$, where $X^0 = 1$, $(N^{IJ})$ is the inverse of the matrix $(N_{IJ})$, $P^a_I$ are the
components of the complex $n \times (n + 1)$ matrix-valued function
\[
(P^a_I)_{a=1,\ldots,n, I=0,\ldots,n} = (P^a_0, P^a_\mu)_{a,\mu=1,\ldots,n} := \left( -\sum_{\nu=1}^n e^a_\nu X^\nu, e^a_\mu \right)_{a,\mu=1,\ldots,n},
\]
and $A_I = d\tilde{\zeta}_I + \sum_{J=0}^n F_{IJ}(X)d\zeta_J$ for $I = 0, \ldots, n$. Note that the matrix $(P^a_I)$ represents the linear map $d\pi_p : T_pM \to T_{\pi(p)}\tilde{M}$ for $p = (X^0, X^1, \ldots, X^n)^t$ with $X^0 = 1$ in the coordinate basis $\left( \frac{\partial}{\partial X^I} \right)$ of $T_pM$ and the unitary basis $(\sigma_a)$ of $T_{\pi(p)}\tilde{M}$. In terms of these one-forms, the Ferrara-Sabharwal metric reads (see e.g. [D, Lemma 7.3.1])
\[
g_{FS} = \sum_{A=0}^n (\beta^A \bar{\beta}^A + \alpha^A \bar{\alpha}^A).
\]
The equations
\[
J_1^* \alpha^A = i\alpha^A, \quad J_1^* \beta^A = i\beta^A, \quad J_2^* \alpha^A = \bar{\beta}^A,
\]
for $A = 0, \ldots, n$ and $J_1 J_2 = J_3$ define an almost hyper-complex structure $(J_1, J_2, J_3)$ on $\tilde{N}$. $J_1$, $J_2$ and $J_3$ span a quaternionic structure $Q$ on $\tilde{N}$ that is compatible (skew-symmetric and parallel) with the quaternionic Kähler metric $g_{FS}$. In fact, the quaternionic Kähler property was proven in [FS] by computing the Levi-Civita connection. This calculation is reviewed below, see equations (2.15)-(2.16) and Proposition 16 and amounts to showing that $Q$ is parallel. Alternatively, it follows from the fact that the Ferrara-Sabharwal metric can be obtained by the geometric construction described in [ACDM] involving the HK/QK-correspondence. Note that $J_1$ defines an integrable complex structure on $\tilde{N}$.

We will now prepare for the calculation of the Levi-Civita connection of the Ferrara-Sabharwal metric. Direct calculation gives the following expressions for the exterior derivatives of the above one-forms (see [D, Prop. 7.3.3]):

**Proposition 14.**
\[
d\beta^0 = \frac{1}{2} \left( \alpha^0 + \bar{\alpha}^0 - id^c K \right) \land \beta^0 + \sum_{b=1}^n \alpha^b \land \bar{\beta}^b,
\]
\[
d\bar{\beta}^a = -\sum_{b=1}^n \omega^a_{\bar{b}} \land \beta^b,
\]
\[
d\alpha^0 = -\bar{\alpha}^0 \land \bar{\alpha}^0 + \beta^0 \land \bar{\beta}^0 - \sum_{b=1}^n \alpha^b \land \bar{\alpha}^b,
\]
\[
d\alpha^a = \frac{1}{2} \left( \alpha^0 + \bar{\alpha}^0 - id^c K \right) \land \alpha^a + \beta^0 \land \bar{\beta}^a - \sum_{b=1}^n \omega^a_{\bar{b}} \land \alpha^b - i e^c K \sum_{b,c=1}^n \tilde{h}_{abc} \bar{\alpha}^b \land \beta^c,
\]
\[\footnote{This can either be shown by direct calculation (see [CLST]) or deduced from the fact that all quaternionic Kähler manifolds obtained from the HK/QK correspondence admit a globally defined compatible integrable complex structure (see [D, Rem. 5.5.5]).}
where \( d^c = i(\bar{\partial} - \partial) \), \( \tilde{h}_{abc} = \sum_{\mu, \nu, \sigma}^n \epsilon_a^\mu \epsilon_b^\nu \epsilon_c^\sigma h_{\mu \nu \sigma} \) for \( a, b, c = 1, \ldots, n \), and \( (\omega^a)_b = 1, \ldots, n \) is the (pullback to \( \tilde{N} \) of the) local connection one-form of the Levi-Civita connection on \( \tilde{M} \) with respect to the local unitary coframe \((\sigma^a)_b = 1, \ldots, n \) on \( \tilde{M} \).

To calculate the exterior derivatives in Proposition 14 we have used the following explicit formula for the local Levi-Civita connection one-form of a projective special Kähler manifold:

**Proposition 15.** The local connection one-form for the Levi-Civita connection with respect to the unitary coframe \((\sigma^a)_b = 1, \ldots, n \) can be written as

\[
\omega^a_b = e^{-\mathcal{X}} (\bar{\partial} P^a_I N^{IJ} \bar{P}^b_J - P^a_I N^{IJ} (\partial \bar{P}^b_J))
\]

where \( P^a_I \) are defined in equation (2.22).

**Proof.** Note that \(-e^{-\mathcal{X}} g_M(d\pi^a, d\pi^\dagger)|_{V \times V} = g_M|_{V \times V} \), see (2.1) and \( e^{-\mathcal{X}} = g_M(\xi, \xi)|_{X^o = 1} \).

Dualizing yields the equation

\[
-e^{-\mathcal{X}} g_M^{-1} = g_M^{-1}(d\pi^a, d\pi^\dagger),
\]

which in components reads

\[
-e^{-\mathcal{X}} \delta^{ab} = \sum_{I, J = 0}^n P^a_I N^{IJ} \bar{P}^b_J \quad (a, b = 1, \ldots, n). \tag{2.11}
\]

Multiplication of equation (2.11) by \(-e^{-\mathcal{X}} e^\mu_b \) gives

\[
e^\mu_b = e^{-\mathcal{X}} (\bar{X}^\mu N^{0J} - N^{\mu J}) \bar{P}^b_J.
\]

This equation shows that

\[
-e^\mu_b \bar{\partial} e^a_\mu = e^{-\mathcal{X}} ((\bar{\partial} e^a_\mu) N^{\mu J} - C^\mu (\bar{\partial} e^a_\mu) N^{0J}) \bar{P}^b_J = e^{-\mathcal{X}} (\bar{\partial} P^a_I) N^{IJ} \bar{P}^b_J.
\]

Using the above equation one then finds

\[
\omega^a_b \equiv -e^\mu_b \bar{\partial} e^a_\mu + e^\mu_b \bar{\partial} e^b_\mu = e^{-\mathcal{X}} ((\bar{\partial} P^a_I) N^{IJ} \bar{P}^b_J - P^a_I N^{IJ} (\partial \bar{P}^b_J)).
\]

Adding \( 0 \equiv \delta^a_b \partial \mathcal{X} + e^{-\mathcal{X}} \partial (P^a_I N^{IJ} \bar{P}^b_J) \) to the above equation gives

\[
\omega^a_b = \delta^a_b \partial \mathcal{X} + e^{-\mathcal{X}} (\bar{\partial} \bar{P}^a_I N^{IJ} \bar{P}^b_J - P^a_I (\partial \bar{P}^b_J)) \equiv \delta^a_b \partial \mathcal{X} + e^{-\mathcal{X}} d(P^a_I N^{IJ} \bar{P}^b_J + i e^{-\mathcal{X}} P^a_I N^{IK} \bar{dF}_{KL}(\bar{X}) N^{LJ} \bar{P}^b_J).
\]

\[\square\]
The components $\bar{\theta}_\alpha$ of the local $\text{Sp}(1)$-connection one-form of a quaternionic Kähler manifold (with Levi-Civita connection $\nabla$) with respect to a local oriented orthonormal frame $(J_1, J_2, J_3)$ in the quaternionic structure are defined by
\[
\nabla J_\alpha = 2(\bar{\theta}_\beta(\cdot)J_\gamma - \bar{\theta}_\gamma(\cdot)J_\beta)
\]
for any cyclic permutation $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$. The one-forms $\bar{\theta}_\alpha$ are related to the fundamental two-forms $\omega_\alpha = g(J_\alpha, \cdot)$ by the following well known structure equations for quaternionic Kähler manifolds
\[
\nu \frac{\omega_\alpha}{2} = d\bar{\theta}_\alpha - 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma,
\]
where $\nu := \frac{\text{scal}}{4m(m+2)} (\dim \mathcal{N} = 4m = 4(n+1))$ is the reduced scalar curvature. For manifolds in the image of the supergravity c-map, we have $\nu = -2$ and
\[
\bar{\theta}_1 = -\frac{1}{4\rho}(d\tilde{\phi} + \rho \, d^c \mathcal{K} - \sum_{l=0}^{n} (\zeta_l d\zeta^l - \zeta^l d\zeta_l)) = -\frac{1}{2} \operatorname{Im} \alpha^0 - \frac{1}{4} d^c \mathcal{K},
\]
\[
\bar{\theta}_2 + i\bar{\theta}_3 = i \frac{1}{\sqrt{\rho}} e^{X/2} \sum_{l=0}^{n} X^l A_l = \beta^0.
\]
These formulas follow from the general expression for the $\text{Sp}(1)$-connection of a quaternionic Kähler manifold obtained from the HK/QK-correspondence, see [D, Thm. 4.1.2], after specialization to the case of the supergravity c-map, see [D, Rem. 5.5.3 and 5.5.4] and [ACDM].

We combine the one-forms defined in equation (2.8) into the following quaternionic vielbein, which is a $(4n + 4) \times (4n + 4)$ matrix of complex-valued one-forms:
\[
(f^{a\Gamma})_{a=1,2;\Gamma=1,\ldots,2n+2} = 
\begin{pmatrix}
    f^{1A} & f^{1\bar{A}} \\
    f^{2A} & f^{2\bar{A}}
\end{pmatrix}_{A=0,\ldots,n} := 
\begin{pmatrix}
    \beta^A & \bar{\alpha}^A \\
    -\bar{\beta}^A & \alpha^A
\end{pmatrix}_{A=0,\ldots,n}.
\]
Let $\beta_A$, $\alpha_A$ be complex-valued vector fields on $\mathcal{N}$ such that $\beta^A = 2g(\beta_A, \cdot)$ and $\alpha^A = 2g(\alpha_A, \cdot)$ for $A = 0, \ldots, n$. These vector-fields are combined into the following local frame in $T^C \mathcal{N}$, which is dual to $(f^{a\Gamma})$:
\[
(f_{a\Gamma})_{a=1,2;\Gamma=1,\ldots,2n+2} = 
\begin{pmatrix}
    f_{1A} & f_{1\bar{A}} \\
    f_{2A} & f_{2\bar{A}}
\end{pmatrix}_{A=0,\ldots,n} := 
\begin{pmatrix}
    \beta_A & \alpha_A \\
    -\bar{\beta}_A & \bar{\alpha}_A
\end{pmatrix}_{A=0,\ldots,n}.
\]
Recall that the skew-symmetric almost complex structures $J_1, J_2, J_3$ spanning the quaternionic structure $Q$ are of standard form, see (2.10), in the coframe $(f^{a\Gamma})$. Note that in our case, namely for manifolds in the image of the q-map, the frame $(f_{a\Gamma})$ is globally defined and thus establishes a global isomorphism $T^C \mathcal{N} \cong H \otimes E$, where $H$ and $E$ are trivial complex vector bundles. More specifically, $(f_{a\Gamma})$ corresponds to a tensor product of the
form \((f_{\alpha \Gamma}) = (h_{\alpha} \otimes E_{\Gamma})\), where \((h_{\alpha})\) is a frame of \(H\) and \((E_{\Gamma})\) is a frame of \(E\). To prove that \(Q\) is parallel and therefore that \(g_{FS}\) is quaternionic Kähler, it is sufficient to check that the Levi-Civita connection with respect to frame \((f_{\alpha \Gamma})\) has the following form:

\[
f^{\alpha \Gamma}(\nabla_X f_{\beta \Delta}) = p^{\alpha \beta}(X) \delta^{\Gamma}_{\Delta} + \delta^{\alpha}_{\beta} \Theta^{\Gamma}_{\Delta}(X) \tag{2.15}
\]

for \(\alpha, \beta = 1, 2\) and \(\Gamma, \Delta = 1, \ldots, 2n + 2\), where \(p = (p^{\alpha \beta})\) is a one-form with values in \(\mathfrak{sp}(1) = \mathfrak{su}(2)\), i.e. \(p^i := p^i = -p\), and \(\Theta = (\Theta^{\Gamma}_{\Delta})\) is a one-form with values in \(\mathfrak{sp}(n + 1) \subset \mathfrak{su}(2n + 2)\). The latter means that

\[
\Theta = \begin{pmatrix} q & t \\ \bar{q} & \bar{t} \end{pmatrix}, \tag{2.16}
\]

where \(q, t\) are complex 1-form-valued \((n + 1) \times (n + 1)\) matrices that are anti-Hermitian, respectively symmetric \((q^\dagger = \bar{q}, t^t = t)\).

**Proposition 16.** The \(\text{Sp}(1)\)-part of the Levi-Civita connection of a quaternionic Kähler manifold in the image of the \(q\)-map is given by

\[
p = \begin{pmatrix} -i\bar{\theta}_1 & -\bar{\theta}_2 - i\bar{\theta}_3 \\ \bar{\theta}_2 - i\bar{\theta}_3 & i\bar{\theta}_1 \end{pmatrix},
\]

see equation (2.13), and the \(\text{Sp}(n + 1)\)-part is given by

\[
\Theta = \begin{pmatrix} q^A_B & t^A_B \\ -\bar{t}^A_B & \bar{q}^A_B \end{pmatrix}_{A,B=0,\ldots,n},
\]

where

\[
q = (q^A_B)_{A,B=0,\ldots,n} = \begin{pmatrix} \frac{i}{4}d^cK + \frac{3}{4}r^0 - \alpha^0 \\ \bar{\alpha}^a \\ \omega^a_{\beta} + \frac{1}{4}(-id^cK + (\bar{\alpha}^0 - \alpha^0))\delta^a_{\beta} \end{pmatrix}_{a,b=1,\ldots,n}
\]

and

\[
t = (t^A_B)_{A,B=0,\ldots,n} = \begin{pmatrix} 0 & 0 \\ 0 & i\epsilon^c \sum_{c=1}^n \bar{h}_{abc} \alpha^c \end{pmatrix}_{a,b=1,\ldots,n}.
\]

**Proof.** The vanishing of torsion is the following system of equations for the components of the connection one-form:

\[
0 = d\beta^A + p^1_1 \wedge \beta^A - p^1_2 \wedge \bar{\alpha}^A + \sum_{B=0}^n (q^A_B \wedge \beta^B + t^A_B \wedge \alpha^B),
\]

\[
0 = d\alpha^A + p^1_1 \wedge \alpha^A + p^1_2 \wedge \bar{\beta}^A + \sum_{B=0}^n (-\bar{t}^A_B \wedge \beta^B + \bar{q}^A_B \wedge \alpha^B)
\]

for \(A = 0, \ldots, n\). This is straightforward to solve using Proposition 14. □
2.6 Curvature tensor for quaternionic Kähler manifolds obtained by the q-map

We consider a manifold in the image of the q-map and use the notation introduced in the last section. In terms of the local frame of the type \((2.14)\), the Riemann curvature tensor of a quaternionic Kähler manifold reads

\[
f^\alpha\Gamma(R(X,Y)f_\beta\Delta) = \tilde{R}_H^\alpha{}_{\beta}(X,Y)\delta^\Gamma_\Delta + \delta^\alpha_\beta \tilde{R}_E^\Gamma_\Delta(X,Y),
\]

where

\[
\tilde{R}_H = dp + p \wedge p
\]

\[
= \begin{pmatrix}
- id\bar{\theta}_1 + 2i\bar{\theta}_2 \wedge \bar{\theta}_3 & -(d\bar{\theta}_2 + id\bar{\theta}_3) + 2i\bar{\theta}_1 \wedge (\bar{\theta}_2 + i\bar{\theta}_3) \\
(d\bar{\theta}_2 - id\bar{\theta}_3) + 2i\bar{\theta}_1 \wedge (\bar{\theta}_2 - i\bar{\theta}_3) & id\bar{\theta}_1 - 2i\bar{\theta}_2 \wedge \bar{\theta}_3
\end{pmatrix}
\]

\[
\overset{2.12}{} = \nu \left( \begin{array}{cc}
-i\omega_1 & -\omega_2 - i\omega_3 \\
\omega_2 - i\omega_3 & i\omega_1
\end{array} \right)
\]

and

\[
\tilde{R}_E = d\Theta + \Theta \wedge \Theta. \tag{2.17}
\]

We write the Sp\((n)\)-part of the curvature tensor as

\[
\tilde{R}_E = \begin{pmatrix} r & s \\ -s & -r \end{pmatrix},
\]

where \(r, s\) are complex two-form valued \((n + 1) \times (n + 1)\) matrices that fulfill \(r^\dagger = -r\), \(s^t = s\). In terms of this splitting, equations \((2.16)\) and \((2.17)\) read

\[
r^A_B = dq^A_B + \sum_{C=0}^{n}(q^A_C \wedge q^C_B - t^A_C \wedge \bar{t}^C_B)
\]

\[
s^A_B = dt^A_B + \sum_{C=0}^{n}(q^A_C \wedge t^C_B + t^A_C \wedge \bar{q}^C_B),
\]

for \(A, B = 0, \ldots, n\).

Recall that in the \(E-H\)-formalism, the complexified quaternionic Kähler metric on the complexified tangent bundle \(T^C\bar{N} \cong H \otimes E\) can be written in the form \(g_{FS}^C = \omega_H \otimes \omega_E\), where \(\omega_H\) and \(\omega_E\) are non-degenerate skew-symmetric two-forms. The two-forms are represented by matrices \((\epsilon_{\alpha\beta})_{\alpha,\beta=1,2}\) and \((\frac{1}{2}C_{\Gamma\Delta})_{\Gamma,\Delta=1,\ldots,2n+2}\), where

\[
C_{\Gamma\Delta} = 2\omega_E(E_\Gamma, E_\Delta), \quad \epsilon_{\alpha\beta} = \omega_H(h_\alpha, h_\beta).
\]

We have that \(C_{\bar{A}\bar{B}} = -C_{\bar{A}B} = \delta_{AB}, C_{AB} = C_{\bar{A}\bar{B}} = 0\ (A, B = 0, \ldots, n), \) and \(\epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0\).
Proposition 17. The $\text{Sp}(n+1)$-part \(\pounds_{\Xi}^E\) of the curvature tensor can be expressed as

\[
\tilde{R}^E_{E} = \sum_{\alpha, \beta=1}^{2n+2} \frac{\nu}{4} \epsilon_{\alpha \beta} C_{\Xi \Delta} f^{\alpha \Lambda} \wedge f^\beta \Delta + \sum_{\alpha, \beta, \gamma=1}^{2n+2} C^{\Lambda \Delta} \Omega_{\Lambda \Xi \Gamma \Delta} \epsilon_{\alpha \beta} f^{\alpha \Gamma} \wedge f^\beta \Delta, \tag{2.18}
\]

where $C_{\Gamma \Delta} = -C^{\Gamma \Delta}$, and $\Omega_{\Lambda \Xi \Gamma \Delta}$ are complex-valued functions on $\tilde{N}$ that are symmetric in all four indices.

Proof. Recall that the curvature tensor of every $4m$-dimensional quaternionic Kähler manifold can be decomposed as

\[
R = \nu R_{\mathbb{H}^{P_m}} + W, \tag{2.19}
\]

where

\[
R_{\mathbb{H}^{P_m}}(X, Y)Z = \frac{1}{4}[g(Y, Z)X - g(X, Z)Y] - \frac{1}{2} \sum_{i=1}^{3} \omega_i(X, Y)J_iZ
\]

\[
+ \frac{1}{4} \sum_{i=1}^{3} [\omega_i(Y, Z)J_iX - \omega_i(X, Z)J_iY] \tag{2.20}
\]

is the curvature tensor of the quaternionic projective space, $\nu$ is the reduced scalar curvature defined above, and $W$ is a curvature tensor of type $\text{Sp}(m)$, which is related to an element in $\Omega \in \Gamma(S^4E^*)$ by the following formula:

\[
W(he, h'e')(h''e'') = -\omega_H(h, h') h'' \omega^{-1}_E(\Omega(e, e', e'', \cdot)), \tag{2.21}
\]

$h, h', h'' \in \Gamma(H), e, e', e'' \in \Gamma(E)$. Writing the formulas (2.20) and (2.21) in terms of our chosen frames with $m = n + 1$, we obtain

\[
f^{\delta \Lambda}(R_{\mathbb{H}^{P_{n+1}}}^E(f_{a \Gamma}, f_{b \Delta})f_{\gamma \Xi}) = -\frac{1}{4} \epsilon_{\alpha \beta} \delta^\delta_{\gamma} (C_{\Gamma \Xi \delta \Delta} + C_{\Delta \Xi \delta \Gamma}), \tag{2.22}
\]

where $R_{\mathbb{H}^{P_{n+1}}}^E$ is the $\text{Sp}(n+1)$-part of $R_{\mathbb{H}^{P_{n+1}}}$, and

\[
f^{\delta \Lambda}(W(f_{a \Gamma}, f_{b \Delta})f_{\gamma \Xi}) = -2 \delta^\delta_{\gamma} \epsilon_{\alpha \beta} \sum_{\Lambda'=1}^{2n} \Omega_{\Gamma \Delta \Xi \Lambda'} C^{\Lambda' \Lambda}. \tag{2.23}
\]

Equations (2.22) and (2.23) now imply

\[
\tilde{R}^E_{E}(f_{a \Gamma}, f_{b \Delta}) = \frac{\nu}{4} \epsilon_{\alpha \beta} C_{\Xi \Delta} \delta^\delta_{\gamma} - \frac{\nu}{4} \epsilon_{\beta \alpha} C_{\Xi \Gamma} \delta^\delta_{\Delta} - 2 \epsilon_{\alpha \beta} \sum_{\Lambda'=1}^{2n} \Omega_{\Gamma \Delta \Xi \Lambda'} C^{\Lambda' \Lambda}. \tag{2.18}
\]

The above equation is equivalent to (2.18). \qed
Using the expressions for the local Levi-Civita connection one-form given in Proposition [16] one obtains the following result (see [D, Prop. 7.3.5]) by inserting the Sp(n + 1)-part Θ of the Levi-Civita connection into the formula (2.17) for the Sp(n + 1)-part of the curvature:

**Proposition 18.** The Sp(n + 1)-part of the curvature two-form for any quaternionic Kähler manifold in the image of the q-map is given by \( \tilde{R}_E^{\Gamma, \Delta} = \left( \frac{r^A_B}{s^A_B} \right)_{A, B = 0, \ldots, n} \)

with

\[
\begin{bmatrix}
\frac{1}{2} (\alpha^0 \wedge \bar{\alpha}^0 - \beta^0 \wedge \bar{\beta}^0) \\
\sum_{C=0}^{n} (\alpha^C \wedge \bar{\alpha}^C - \beta^C \wedge \bar{\beta}^C) \\
\frac{1}{2} \delta^a_b \sum_{C=0}^{n} (\alpha^C \wedge \bar{\alpha}^C - \beta^C \wedge \bar{\beta}^C) \\
\end{bmatrix} \cdot \frac{1}{2} \delta^a_b \sum_{C=0}^{n} (\alpha^C \wedge \bar{\alpha}^C - \beta^C \wedge \bar{\beta}^C)
\]

and

\[
s = (s^A_B)
\]

\[
\begin{bmatrix}
0 \\
\alpha^0 \wedge \bar{\alpha}^0 + \bar{\beta}^0 \wedge \beta^0 + ie^X \bar{h}_{abc} \alpha^c \wedge \bar{\beta}^d \\
0 \\
\end{bmatrix}
\]

where

\[
S_{abcd} := -\frac{1}{2} e^{2X} \left( (\bar{h}_{bcf} \bar{h}_{fad} - 4 \bar{h}_{bc} \bar{h}_{ad}) + (\bar{h}_{acf} \bar{h}_{fbd} - 4 \bar{h}_{ac} \bar{h}_{bd}) + (\bar{h}_{abf} \bar{h}_{fde} \bar{h}_{ced} \alpha^d \wedge \bar{\beta}^e) - 2S_{abcd} \alpha^c \wedge \bar{\beta}^d \right)_{a, b = 1, \ldots, n}
\]

**Remark 19.** Note that the vanishing of the symmetric quartic tensor field

\[
S_{abcd} \sigma^a \otimes \sigma^b \otimes \sigma^c \otimes \sigma^d
\]

\[
= \frac{-1}{2 \, 4^3 h^2} \left( 3h_{\tau(\mu \nu \rho \sigma)} - 12h_{(\mu \nu \rho \sigma)} + 16h_{(\mu \nu \rho \sigma)} \right) \, dX^\mu \otimes \, dX^\nu \otimes \, dX^\sigma \otimes \, dX^\rho
\]

\[
= \frac{-1}{2 \, 4^3 h^2} \left( -12h_{(\mu \nu \rho \sigma) \tau \rho} - 12h_{(\mu \nu \rho \sigma) \tau} - 12h_{(\mu \nu \rho \sigma) \rho} + 16h_{(\mu \nu \rho \sigma)} \right) \, dX^\mu \otimes \, dX^\nu \otimes \, dX^\sigma \otimes \, dX^\rho
\]

\[
=: S_{\mu \nu \rho \sigma} \, dX^\mu \otimes \, dX^\nu \otimes \, dX^\sigma \otimes \, dX^\rho
\]

on the projective special Kähler manifold \((\tilde{M}, g_M, J_M)\) is a necessary and sufficient condition for \((\tilde{M}, g_M)\) to be symmetric [CV].

---

6 All repeated lower case indices are summed over 1, \ldots, n.

7 All repeated indices are summed over 1, \ldots, n. Note that the symmetrization denoted by \((\ldots)\) over four indices includes a factor of \(1/4!\).
Careful comparison of the expressions given in the above proposition with equation \(2.18\) leads to the following expression for the quartic symmetric tensor field determining the Riemann curvature tensor of a quaternionic Kähler manifold:

**Theorem 20.** [\(\text{D, Th. 7.3.7}\)]

For manifolds in the image of the q-map, the non-vanishing components of the quartic symmetric tensor field defined in equation \(2.18\) are given by

\[
\begin{align*}
\Omega &\equiv 0 = 1, \\
\Omega_{ab0} &\equiv 0 = \frac{1}{4} \delta_{bd}, \\
\Omega_{ab0} &\equiv 0 = \frac{1}{4} \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} - \frac{1}{2} \epsilon^{2\xi} \sum_{j=1}^{n} \tilde{h}_{abf} \tilde{h}_{jcd}, \\
\Omega_{abcd} &\equiv 0 = \Omega_{ada} = S_{abcd} \\
and symmetrization thereof, where \(a, b, c, d = 1, \ldots, n\).
\end{align*}
\]

2.7 Pointwise norm of the curvature tensor for quaternionic Kähler manifolds obtained by the q-map

In this section, we give a general formula for the curvature invariant \(S_W := \frac{1}{16} \|W\|_2^2 \in C^\infty(\tilde{N})\) for all quaternionic Kähler manifolds \(\tilde{N} = \tilde{M} \times \mathbb{R}^{n+1} \times \mathbb{R}^{2n+3}\) in the image of the q-map, where \(W\) is the quaternionic Weyl tensor, see equation \(2.19\). We express \(S_W\) as a linear combination of three curvature invariants on the corresponding projective special Kähler manifold \(\tilde{M}\). Its relation to the squared pointwise norm of the Riemann curvature tensor \(R\) is given by

\[
\|R\|_2^2 = 80(n+1)^2 + 16(n+1) + 64 S_W. \tag{2.26}
\]

This follows from the orthogonality of the decomposition \(R = \nu R_{H\mathbb{P}^{n+1}} + W\), the fact that the reduced scalar curvature is \(\nu = -2\) for quaternionic Kähler manifolds obtained via the supergravity c-map, and the following formula for the squared pointwise norm of \(R_{H\mathbb{P}^{n+1}}\):

\[
\|R_{H\mathbb{P}^{n+1}}\|_2^2 = 20n^2 + 44n + 24 = 20(n+1)^2 + 4(n+1).
\]

The above formula is obtained from equation \(2.20\).

The scalar curvature of a projective special Kähler manifold \(\tilde{M}\) in the image of the supergravity r-map is given by (see Theorem 3 in [\(\text{CDL}\) in the special case \(D = 3\)]

\[
scal_{\tilde{M}} = -2n^2 + n - 2h \sum_{\alpha,\beta,\gamma=1}^{n} \sum_{\alpha',\beta',\gamma' = 1}^{n} h_{\alpha\beta\gamma} h^{\alpha\alpha'} h^{\beta\beta'} h^{\gamma\gamma'} h_{\alpha'\beta'\gamma'} \\
= -2n(n+1) + \frac{1}{32h^2} \sum_{\alpha,\beta,\gamma=1}^{n} \sum_{\alpha',\beta',\gamma' = 1}^{n} h_{\alpha\beta\gamma} K^{\alpha\alpha'} K^{\beta\beta'} K^{\gamma\gamma'} h_{\alpha'\beta'\gamma'}. \tag{2.27}
\]

\(^{8}\)Note that compared to [\(\text{CDL}\) we scaled the projective special Kähler metric \(g_{\tilde{M}}\) by a factor of \(\frac{1}{2}\), which leads to a scaling of the scalar curvature \(scal_{\tilde{M}}\) by a factor of 2.
The squared pointwise norm of the Riemann tensor of a projective special Kähler manifold $\tilde{M}$ in the image of the $r$-map is

$$\|R_\tilde{M}\|^2 = 16 \sum_{\mu,\nu,\rho,\sigma=1}^n R_{\mu\nu\rho\sigma} K^{\mu\nu} K^{\rho\sigma},$$

$$= -32 \text{scal}_\tilde{M} - 32n(n + 1)$$

$$+ \frac{1}{44\tilde{h}^4} \sum_{\mu,\nu,\rho,\sigma=1}^n \sum_{\mu',\nu',\rho',\sigma'=1}^n B_{\rho\sigma\mu\nu} K^\rho K^{\rho'} K^{\sigma\sigma'} K^{\mu\nu} B_{\rho'\sigma'\mu'\nu'}$$

where

$$R_{\mu\nu\rho\sigma} = \sum_{\alpha=1}^n K_{\alpha \mu} R^{\alpha}_{\nu\rho\sigma} = -K_{\mu \nu} K_{\rho \sigma} + e^{2\chi} \sum_{\beta,\gamma=1}^n h_{\mu\beta} K^{\beta\gamma} h_{\gamma\sigma\nu}$$

and

$$B_{\mu\nu\rho\sigma} := \sum_{\kappa,\kappa'=1}^n h_{\mu\kappa} K^{\kappa\kappa'} h_{\kappa'\rho\sigma}. $$

The third real-valued function on $\tilde{M}$ relevant for this discussion is

$$\sum_{a,b,c,d=1}^{2n+2} (S_{abcd})^2 = \sum_{\mu,\nu,\rho,\sigma=1}^n \sum_{\mu',\nu',\rho',\sigma'=1}^n S_{\mu\nu\rho\sigma} K^{\mu\nu} K^{\rho\sigma},$$

where the respective components are defined in equations (2.24) and (2.25).

Using the quartic tensor field introduced in (2.18), we define the following function on $\tilde{N}$:

$$S_W := \sum_{\Gamma,\Gamma',\Gamma'',\Gamma'''=1}^{2n+2} \Omega_{\Gamma\Gamma'\Gamma''\Gamma'''} C^{\Gamma\Delta} C^{\Gamma'\Delta'} C^{\Gamma''\Delta''} C^{\Gamma'''\Delta'''} \Omega_{\Delta\Delta'\Delta''\Delta'''}.$$

Using the formulas for $\Omega$ given in Theorem 20, we find the following expression for $S_W$:

$$S_W = 2\Omega_{ABCD} \Omega_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} - 8\Omega_{ABCD} \Omega_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} + 6\Omega_{ABCD} \Omega_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}}$$

$$= 2\sum S_{abcd} S_{abcd} + 2n(n + 1) + \text{scal}_\tilde{M} + \frac{3}{2} (n + 1) + 6(\frac{1}{4} \|R_\tilde{M}\|^2 + \frac{1}{4} \text{scal}_\tilde{M} + \frac{n^2 + n}{8})$$

$$= 2\sum S_{abcd} S_{abcd} + \frac{1}{4} (11n + 6)(n + 1) + \frac{3}{32} \|R_\tilde{M}\|^2 + \frac{5}{2} \text{scal}_\tilde{M}. \quad (2.28)$$

Together with equation (2.26), we obtain the following corollary:

**Corollary 21.** The squared pointwise norm of the Riemann curvature tensor for any quaternionic Kähler manifold in the image of the $q$-map, defined by a cubic polynomial $h$ in $n$ variables, is

$$\|R\|^2 = 64(n + 1)(4n + 3) + 160 \text{scal}_\tilde{M} + 6\|R_\tilde{M}\|^2 + 128 \sum_{a,b,c,d=1}^n (S_{abcd})^2.$$
2.8 Example: A series of inhomogeneous complete quaternionic Kähler manifolds

For \( n \in \mathbb{N} \), we consider the following series of projective special real manifolds:

\[
\mathcal{H} = \{ h = 1, \; x > 0 \} \subset \mathbb{R}^n, \; h := x \left( x^2 - \sum_{i=1}^{n-1} y_i^2 \right). \tag{2.29}
\]

Note that this corresponds to case d) of Theorem 2, up to a shift in \( n \). The coefficients \( \mathcal{K}_{\mu\nu} \) of the inverse metric of the corresponding project special Kähler manifold \( \bar{M} \) obtained by the r-map are given by equation (2.5) in terms of the matrix

\[
(h_{\mu\nu}) = \frac{-1}{12x^2 + 4 \sum_{i=1}^{n-1} y_i^2} \begin{pmatrix} -2x & 2y \ 
2y & 6x \cdot 1 \end{pmatrix},
\]

where \( y^t = (y_1, \ldots, y_{n-1}) \). The scalar curvature of the corresponding projective special Kähler manifold \( \bar{M} \) in the image of the supergravity r-map can be calculated using equation (2.27) and reads

\[
\text{scal}_M = -n \cdot (2n - 1) + 3h \cdot \frac{n - 2}{h - 4x^3} + \frac{36x^3h^2}{(h - 4x^3)^3}.
\]

Furthermore, we find

\[
\|R_M\|^2 = \frac{16}{(h - 4x^3)^6} \left( h^8(n(3n - 8) + 9) - 4h^5(n(17n - 46) + 57)x^3 + 4h^4(n(161n - 382) + 537)x^6 - 64h^3(n(51n - 97) + 99)x^9 + 128h^2(n(73n - 107) + 78)x^{12} - 2048h(n(7n - 8) + 3)x^{15} + 1024n(9n - 8)x^{18} \right)
\]

and

\[
\sum_{a,b,c,d=1}^{n} (S_{abcd})^2 = \sum_{\mu,\nu,\sigma,\rho=1}^{n} \sum_{\mu',\nu',\sigma',\rho'=1}^{n} S_{\mu\nu\sigma\rho} \mathcal{K}^{\mu\mu'} \mathcal{K}^{\nu\nu'} \mathcal{K}^{\sigma\sigma'} \mathcal{K}^{\rho\rho'} S_{\mu'\nu'\sigma'\rho'}
\]

\[
= \frac{3x^6}{(h - 4x^3)^6} \left( h^4(n(n + 16) + 207) - 16h^3(n - 2)(n + 9)x^3 + 96h^2 \left( n^2 + n - 6 \right)x^6 - 256h(n - 2)nx^9 + 256(n - 2)n^2x^{12} \right).
\]

Using equation (2.28), the function \( S_W \) is calculated to be

\[
S_W = \frac{3}{2(h - 4x^3)^6} \left( h^6(n + 1) - 4h^5(n + 1)(5n - 2)x^3 + 8h^4(n(21n + 37) + 112)x^6 - 256h^3(n(3n + 10) - 11)x^9 + 256h^2(n(8n + 33) - 20)x^{12} - 1024h(n(3n + 11) + 2)x^{15} + 2048(n + 1)(n + 2)x^{18} \right) + \frac{3n}{4}(n + 1).
\]
One can now check that the above function is non-constant for $n > 1$. This can be seen as follows. Restricting the function to the hypersurface $(\mathbb{R}^n + i\mathcal{H}) \times \mathbb{R}^3 \times \mathbb{R}^2 \subset (\mathbb{R}^n + iU) \times \mathbb{R}^3 \times \mathbb{R}^2 = \tilde{N}$, we obtain a rational function of the real variable $x \geq 1$. It is now easy to check for all $n > 1$ that the numerator is not proportional to the denominator. Due to equation (2.26), also the squared pointwise norm of the Riemann curvature tensor is non-constant. This shows that the quaternionic Kähler metrics obtained from the series of polynomials in equation (2.29) are not locally-homogeneous for $n > 1$. In total, we have the following:

**Theorem 22.** For $n > 1$, the series of manifolds obtained from the complete projective special real manifolds in equation (2.29) via the q-map consists of complete quaternionic Kähler manifolds that are not locally homogeneous.

Note that this implies Theorem 3, since the quaternionic Kähler manifolds $\tilde{N}$ associated with the series (2.29) admit a group of co-homogeneity one as discussed after the aforementioned theorem and in Appendix A, see Example 29.

### A Automorphisms of manifolds in the image of the r- and c-map

#### The r-map

Let $\mathcal{H} \subset \mathbb{R}^n$ be a (connected) projective special real manifold with cubic polynomial $h$ and $\text{Aut}(\mathcal{H}) \subset \text{GL}(n)$ its automorphism group, which consists of linear transformations that preserve the hypersurface $\mathcal{H}$. The supergravity r-map associates to $\mathcal{H}$ a projective special Kähler domain $\tilde{M}$. This means that there exists a holomorphic function $F$ homogeneous of degree two defined on some $\mathbb{C}^*$-invariant domain $M_F \subset \mathbb{C}^{n+1} \setminus \{0\}$ such that $\tilde{M}$ is the image of the Lagrangian cone

$$
M = \left\{ \left( z^0, \ldots, z^n, w_0, \ldots, w_n \right)^t \in M_F \times \mathbb{C}^{n+1} \subset V \left| w_I = \frac{\partial F}{\partial z^I}, \ I = 0, \ldots, n \right. \right\}
$$

under the canonical projection $V \setminus \{0\} \to P(V)$, where $V = \mathbb{C}^{2n+2}$ is endowed with the canonical symplectic structure $\sum dz^I \wedge dw_I$. As part the definition of a projective special Kähler domain, one does also require $\sum z^I N_{IJ} \bar{z}^J > 0$ for all $z \in M_F$ and that the real symmetric matrix $(N_{IJ}) := (2 \text{Im} F_{IJ})$ has signature $(1, n)$ for all $z \in M_F$. Note that such a manifold $M$ is called a **conical affine special Kähler domain**. We define the automorphism group of $M$ as

$$
\text{Aut}(M) := \left\{ A \in \text{Sp} \left( \mathbb{R}^{2n+2} \right) \subset \text{Sp} \left( \mathbb{C}^{2n+2} \right) \left| AM \subset M \right. \right\}.
$$
The elements of $\operatorname{Aut}(M)$ preserve the affine special Kähler structure on $M$ induced by the embedding $M \subset V$ and, hence, also the projective special Kähler metric and the complex structure on $M$. We denote by $\operatorname{Aut}(\bar{M})$ the group of holomorphic isometries of $\bar{M}$ induced by $\operatorname{Aut}(M)$. Recall (see Remark 12) that for a conical affine special Kähler domain defined by the r-map the function $F$ takes the form $F(z^0, \ldots, z^n) = \frac{h(z^1, \ldots, z^n)}{z^0}$ and $M_F = \{ z_0(1, p) \mid z_0 \in \mathbb{C}^*, p \in \mathbb{R}^n + iU \}$, where $U = \mathbb{R}^{>0} \cdot \mathcal{H} \subset \mathbb{R}^n$ is the open cone generated by the hypersurface $\mathcal{H}$.

Next we consider the subgroup

$$\operatorname{Aff}_H(\mathbb{R}^n) := (\mathbb{R}^{>0} \times \operatorname{Aut}(\mathcal{H})) \ltimes \mathbb{R}^n \subset \operatorname{Aff}(\mathbb{R}^n)$$

and construct an embedding $\varphi_h : \operatorname{Aff}_H(\mathbb{R}^n) \rightarrow \operatorname{Sp}(\mathbb{R}^{2n+2})$ as follows. The restriction of $\varphi_h$ to the subgroup $\operatorname{Aut}(\mathcal{H}) \subset \operatorname{Aff}_H(\mathbb{R}^n)$ is defined by the canonical inclusions

$$\operatorname{Aut}(\mathcal{H}) \subset \operatorname{GL}(n, \mathbb{R}) \subset \operatorname{GL}(n+1, \mathbb{R}) \subset \operatorname{Sp}(\mathbb{R}^{2n+2}).$$

Note that under these inclusions $\operatorname{GL}(n, \mathbb{R})$ acts trivially on the coordinates $z^0$ and $w_0$. When restricted to the $\mathbb{R}^{>0}$-factor, $\varphi_h$ is given by the inclusion

$$\mathbb{R}^{>0} \ni \lambda \mapsto \left( \begin{array}{cc} \lambda^{-\frac{2}{n}} & 0 \\ 0 & \lambda^{-\frac{2}{n}} \cdot 1 \end{array} \right) \in \operatorname{GL}(n+1, \mathbb{R}) \subset \operatorname{Sp}(\mathbb{R}^{2n+2}).$$

Finally, we define the homomorphism $\varphi_h|_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \operatorname{Sp}(\mathbb{R}^{2n+2})$ by

$$\varphi_h(v) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ v & 1 & 0 \\ -H(v, v, v) & -3H(v, v, \cdot) & 1 - v^t \\ 3H(v, v, \cdot)^t & 6H_v & 0 & 1 \end{array} \right), \quad (A.1)$$

where $H \in S^3(\mathbb{R}^n)^*$ is the cubic tensor defined by $H(v, v, v) = h(v)$, $v \in \mathbb{R}^n$, and $H_v : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $z \mapsto H(v, z, \cdot)^t$.

**Proposition 23.** The above prescription defines an embedding

$$\varphi_h : \operatorname{Aff}_H(\mathbb{R}^n) \rightarrow \operatorname{Aut}(M) \subset \operatorname{Sp}(\mathbb{R}^{2n+2}).$$

The induced homomorphism $\bar{\varphi}_h : \operatorname{Aff}_H(\mathbb{R}^n) \rightarrow \operatorname{Aut}(\bar{M})$ is also an embedding.

**Proof.** It is straightforward to check that the matrix $A = \varphi_h(v)$ (A.1) is symplectic, which shows that $\varphi_h$ maps into $\operatorname{Sp}(\mathbb{R}^{2n+2})$. Similarly, one can easily verify that $\varphi_h$ is a group homomorphism. The fact that the group $\varphi_h(\operatorname{Aff}_H(\mathbb{R}^n)) \subset \operatorname{Sp}(\mathbb{R}^{2n+2})$ preserves the Lagrangian cone $M$ can be proven by checking that

$$\frac{\partial F}{\partial z} \bigg|_{z'} = w',$$

where $\begin{pmatrix} z' \\ w' \end{pmatrix} := A \begin{pmatrix} z \\ w \end{pmatrix}$. \hfill $\square$

32
Corollary 24. Let $\mathcal{H}$ be a projective special real manifold on which $\text{Aut}(\mathcal{H})$ acts with co-homogeneity $k \in \mathbb{N}_0$. Then the group $\varphi_h (\text{Aff}_\mathcal{H}(\mathbb{R}^n)) \subset \text{Aut}(\mathcal{M})$ acts with co-homogeneity $k$ on the corresponding projective special Kähler domain $\tilde{M}$ obtained by the $r$-map.

A similar result holds for Lie subgroups $L \subset \text{Aut}(\mathcal{H})$.

Example 25. The projective special real manifolds in equation (2.29) have $\text{Aut}(\mathcal{H}) = \text{O}(n-1)$. Thus $\text{Aff}_\mathcal{H}(\mathbb{R}^n) \cong (\mathbb{R}^+ \times \text{O}(n-1)) \ltimes \mathbb{R}^n$ acts with co-homogeneity one by automorphisms of the corresponding projective special Kähler domains $\tilde{M}$ obtained by the $r$-map.

The c-map

Let $\tilde{M}$ be a projective special Kähler domain of real dimension $2n$ and denote by $M \to \tilde{M}$ the corresponding conical affine special Kähler domain. The c-map associates with $\tilde{M}$ a quaternionic Kähler manifold $\tilde{N} = \tilde{M} \times G$, where $G$ is the solvable Iwasawa subgroup of $\text{SU}(1,n+2)$, which is of dimension $2n+4$. The quaternionic Kähler metric is of the form $g_N = g_M + g_G$, where $g_G$ is a family of left-invariant metrics on $G$ varying with $p \in \tilde{M}$ [CHM]. This implies the inclusion $G \subset \text{Isom}(\tilde{N})$. Moreover, the symplectic group $\text{Sp}(\mathbb{R}^{2n+2})$ is a subgroup of $\text{Aut}(G)$, as can be easily seen from the structure of $G$ as solvable extension of the $(2n+3)$-dimensional Heisenberg group. So $\text{Aut}(M) \subset \text{Sp}(\mathbb{R}^{2n+2})$ acts naturally on the trivial bundle $\tilde{N} = \tilde{M} \times G \to \tilde{M}$ mapping fibres to fibres and covering the action of $\text{Aut}(\tilde{M})$ on the base manifold.

Proposition 26. The subgroup $\text{Aut}(M) \ltimes G \subset \text{Sp}(\mathbb{R}^{2n+2}) \ltimes G$ acts by isometries on $\tilde{N}$.

Proof. This follows from [CHM, Lemma 4] by considering automorphisms of conical affine special Kähler domains rather than isomorphism between different projective special Kähler domains. $\square$

Corollary 27. Let $M$ be a conical affine special Kähler domain and $\tilde{M}$ the corresponding projective special Kähler domain. If a Lie subgroup $L \subset \text{Aut}(M)$ acts with co-homogeneity $k \in \mathbb{N}_0$ on $\tilde{M}$ then $L \ltimes G$ acts isometrically and with co-homogeneity $k$ on the corresponding quaternionic Kähler manifold $\tilde{N}$ obtained by the c-map.

The q-map

For any quaternionic Kähler manifold $\tilde{N}$ in the image of the q-map. We define

$$\text{Isom}_\mathcal{H}(\tilde{N}) := \varphi_h (\text{Aff}_\mathcal{H}(\mathbb{R}^n)) \ltimes G \subset \text{Aut}(M) \ltimes G \subset \text{Isom}(\tilde{N}),$$
where $\mathcal{H}$ denotes the underlying projective special real manifold and $M$ the corresponding conical affine special Kähler domain.

**Corollary 28.** If $\text{Aut}(\mathcal{H})$ acts with cohomogeneity $k \in \mathbb{N}_0$ on $\mathcal{H}$ then $\text{Isom}_{\mathfrak{h}}(\tilde{N})$ acts with cohomogeneity $k$ on $\tilde{N}$. As a consequence, $\text{Isom}(\tilde{N})$ has co-homogeneity $\leq k$.

**Example 29.** Consider the quaternionic Kähler manifolds $\tilde{N}$ associated with the projective special real manifolds in equation (2.29) by the q-map. Then $\text{Isom}_{\mathfrak{h}}(\tilde{N})$ acts with co-homogeneity one by isometries on $\tilde{N}$. Note that the maximal compact subgroup of $\text{Isom}_{\mathfrak{h}}(\tilde{N})$ is $O(n-1)$ and that the maximal connected subgroup $\text{Isom}_{\mathfrak{h}}^0(\tilde{N})$ has a Levi decomposition of the form

$$\text{Isom}_{\mathfrak{h}}^0(\tilde{N}) = \text{SO}(n-1) \rtimes ((\mathbb{R}^+ \times \mathbb{R}^n) \rtimes G),$$

where the semi-direct decomposition $(\mathbb{R}^+ \times \mathbb{R}^n) \rtimes G$ of the radical is defined by the embedding $\varphi_h$.

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