From isospin generators to BRST quantization of higher spin massless fields

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Abstract

Motivated by construction of isospin generators in particle physics (built form the SU(2) algebra), we find an equivalence between the algebra of these generators and those of the Virasoro algebra. The form of the starting generators is fixed and in order to obtain a full equivalence we introduce a new matrix product. The Cartan structure of the starting algebra is reproduced for the Virasoro-like case and a natural BRST quantization as in String Field Theory is “induced” in the Fock space of the creation/annihilation operators. Following this procedure, we find a rather trivial Lie algebra form which we obtain a gauge theory of an infinity on non-interacting massless particles of arbitrary integer spin and symmetry. Among others we find the free Maxwell field, the free (linearized) gravitational field and also the axion field with their appropriate gauge transformations.

1 Introduction

Isospin is not only important in nuclear physics for the description of the interactions between protons $p$ and neutrons $n$ in the nuclei and other particles participating in strong interactions. Most importantly, the idea of isospin invariance led Yang and Mills to the formulation of gauge theories \cite{yang_mills}. The idea of isospin, originally due to Heisenberg \cite{heisenberg}, is that in nuclear physics, if one forgets about the electromagnetic and the weak interactions,
the strong interactions are (almost) insensitive to the change of protons by neutrons and vice-versa. Hence, in this context, there is an (almost) exact symmetry and therefore the Hamiltonian of strong interactions $H_S$ has (almost) degenerate states with the same energy. The failure to be an exact symmetry being due to the slight mass difference between the proton and the neutron, $m_p = 938.3$ MeV, $m_n = 939.6$ MeV. Therefore protons and neutrons could be thought of as two states of a single “particle”, the nucleon doublet $N$,

$$N = \begin{pmatrix} p \\ n \end{pmatrix}.$$  

As we will see in section 2, one can build operators that realize the aforementioned symmetry under $p \leftrightarrow n$ exchange as:

$$T_a = \frac{1}{2} \sum_{i,j} \sum_{\alpha} a_{i,\alpha}^\dagger (\sigma_a)_{ij} a_{j,\alpha},$$

where $a_{i,\alpha}^\dagger (a_{i,\alpha})$ are creation/annihilation operators for nucleons of the species $i$ in the state $\alpha$, and $(\sigma_a)_{ij}$ are the components of the $a$-th Pauli matrix.

In the present work, however, we do not focus on isospin generators per se, rather, we note that the bilinearity in creation/annihilation operators bears close resemblance with the Virasoro generators of string theory.

On the other hand, the BRST quantization method has proven to be very useful in gauge theories (as constrained Hamiltonian systems in general) and also in string theory. Therefore we seek to find a quantum theory describing particles’ excitations from a BRST symmetry based on “generalized isospin” operators similar to those in (2) which we will define below. Since the idea of generalized operators is not restrictive to describe spin-1/2 systems we expect our theory to describe higher spin (HS) particles as well (this is clear form the group theoretic point of view in which there is no reason to consider only spins \leq 2 as candidates for elementary fields -as Nature seems to suggest-since the Lorentz group admits representations of arbitrary integer of half-integer spins). Since this is a gauge theory we will be dealing with the massless fields.

The interest in HS particles dates back to the work of Fierz and Pauli for massive case. This was in the context of finding a covariant description of

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3 If state $B$ is obtained from state $A$ by exchanging $p \leftrightarrow n$, then these states $A, B$ correspond to two (almost) degenerate eigenstates of $H_S$ with (almost) the same energy.
particles carrying arbitrary representations of the Lorentz group. The study
of the massless case was considered in [4, 5, 6]. In [7] free HS gauge fields
were studied with its underlying BRST structure and in [8] the interacting case
was obtained. Then the above was generalized to mixed symmetry massless
fields in [9, 10, 11].

The importance of HS gauge fields is not merely academic. One of the
most appealing reasons concerns string theory. The low energy limit of string
theory is relatively well understood, and for example, there are several field
theoretical descriptions of it as effective theories. On the contrary, its high
energy limit is not clear yet and neither is there a field theoretical description
in that limit. However, HS gauge field theory may provide such a description
since there is evidence supporting a relation between HS gauge fields and the
high energy limit of string field theory [12, 13, 14, 15, 16, 17, 18]. Further
developments on the BRST formulation of the problem we made on [19, 20,
21] and after many efforts, an appropriate description of the interactions of
HS massless gauge field was derived [22, 23, 24, 25]. For a recent review see
[26] and references therein.

The structure of the paper is as follows: In section 2 we review the con-
struction of isospin operators. The basics of the definition of the BRST charge
and the BRST quantization method is given in section 3. Next in section 4
we find an algebra built as “isospin operators” but with creation/annihilation
operators satisfying commutation relations as in string theory. For the
algebra thus obtained to mimic the Cartan structure of the Virasoro algebra
we must implement a new product between the matrix representatives of the
algebra. We also show how do the matrix representations look like with the
newly introduced product for the SU(2) case. Next we focus on the sim-
plest closed subalgebra of the Heisenberg algebra from which we construct
the corresponding BRST charge to initiate our quantization programme in
section 5 finding the correct equations of motion describing free photons and
gravitons. In section 6 we extend our analysis to include mixed symmetry
2\textsuperscript{nd} rank tensor fields and among the new states we find the axion of string
theory. In section 7 we extend our algebra to the Heisenberg algebra thus
allowing for higher spin gauge fields of arbitrary (integer) spin. We show
how our results can be properly understood as a limiting case of the BRST
quantization in string theory and connect with previous works on higher spin
gauge fields. In section 8 we conclude by summarizing our results and some
further comments.
2 Generalized Isospin operators

The following analysis is mostly taken from [27] and presented here for completeness and in order to fix the notation. As mentioned above the idea of isospin was based on forgetting about the electromagnetic interaction for the study of nuclear forces, therefore, interactions must be charge independent. Recalling the usual spin-1/2 case where the spin operator acts on the two \( J_3 \) components of a spin-1/2 representation, in this case, charge independence would be granted if there existed a conserved sort of “spin” operator acting on the \( \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \) doublet.

Thinking of protons (neutrons) in state \( \alpha \) as being created from the vacuum \( |0 \rangle \) by the operators \( a_{N,1/2,\alpha}^\dagger \) \((a_{N,-1/2,\alpha}) \) respectively, and annihilated by \( a_{N,1/2,\alpha} \) \((a_{N,-1/2,\alpha}) \) we can describe the states of a single proton \( p \) in state \( \alpha \) or and \( r \)-nucleon state with the nucleon \( m_i \) in the state \( \alpha_i \) as:

\[
|p,\alpha\rangle = a_{N,1/2,\alpha}^\dagger |0\rangle \quad \text{and,} \quad \sum_{\alpha} a_{N,m_1,\alpha_1}^\dagger a_{N,m_2,\alpha_2} \cdots a_{N,m_r,\alpha_r}^\dagger |0\rangle ,
\]

respectively. The label \( m_i \) telling whether the nucleon is a proton \( N = 1/2 \) or a neutron \( N = -1/2 \). The subscript \( N \) is to remind that, so far, these are creation/annihilation operators acting on nucleons \( N \). Since protons and neutrons are fermions the creation/annihilation operators satisfy anticommutation relations:

\[
\{a_{N,a,\alpha},a_{N,b,\beta}^\dagger\} = \delta_{ab}\delta_{\alpha\beta} ,
\]

\[
\{a_{N,a,\alpha},a_{N,b,\beta}^\dagger\} = \{a_{N,a,\alpha},a_{N,b,\beta}\} = 0 .
\]

Isospin symmetry had to do with \( p, n \) exchange. This can be realized by considering the following operators.

\[
T^+ = \frac{1}{\sqrt{2}} \sum_{\alpha} a_{N,1/2,\alpha}^\dagger a_{N,-1/2,\alpha}
\]

\[
(T^+)^\dagger = T^- = \frac{1}{\sqrt{2}} \sum_{\alpha} a_{N,-1/2,\alpha}^\dagger a_{N,1/2,\alpha} .
\]

One can verify that the first operator when acting on a general state of \( n \) protons in states \( \alpha_1, \ldots \alpha_n \) and \( m \) neutrons in states \( \beta_1, \ldots, \beta_m \) produces a state of \( n \) protons in the same states as before and if the one neutron in the
state, say, $\beta_i$ is in the state $\alpha$, then this neutron is exchanged by a proton in that same state. The other neutrons are left the same. If there are no neutrons in state $\alpha$ the action of $T^+$ on the general state yields 0. Similarly by considering the action of $T^-$ on the general state we can see that it exchanges a proton by a neutron in the state $\alpha$. Defining $T_3 \equiv [T^+, T^-]$ it can be checked that:

$$T_3 = \frac{1}{2} \sum_\alpha (a_{N,1/2,\alpha}^\dagger a_{N,1/2,\alpha} - a_{N,-1/2,\alpha}^\dagger a_{N,-1/2,\alpha}) ,$$  \hspace{1cm} (9)$$

$$[T^+, T^-] = T_3 ,$$ \hspace{1cm} (10)$$

$$[T_3, T^\pm] = \pm T^\pm ,$$ \hspace{1cm} (11)$$

i.e., these operators satisfy the angular momentum algebra as was to be expected and they are called the isospin generators.

The $T_3$ operator can be rewritten as (sum over repeated Latin indices understood)

$$T_3 = \frac{1}{2} \sum_\alpha a_{i,\alpha}^\dagger (\sigma_3)_{ij} a_{j,\alpha} ,$$ \hspace{1cm} (12)$$

where $(\sigma_3)_{ij}$ are the components of the third Pauli matrix. In general,

$$T_a = \frac{1}{2} \sum_\alpha a_{i,\alpha}^\dagger (\sigma_a)_{ij} a_{j,\alpha} .$$ \hspace{1cm} (13)$$

Since not only protons and nucleons are involved in strong interactions, one can generalize the above construction to the other particles involved in strong interactions. Thus we can write generalized isospin generators as:

$$\hat{T}_a = \frac{1}{\sqrt{2}} \sum a_{x,i,\alpha}^\dagger I_x [T_a^{I_x}]_{ij} a_{x,j,\alpha} ,$$ \hspace{1cm} (14)$$

where: the sum is over all particles $x$ in the state $\alpha$; $I_x$ is the isospin of particle $x$; $i, j$ take values on the $T_3$ values for the particles $x$ and $a_{x,i,\alpha}^\dagger (a_{x,i,\alpha})$ are creator/annihilator operators for the particles of type-$x$ and satisfy commutation or anticommutation relations whether the particle $x$ is bosonic or fermionic.

It is a simple task to check that the generalized operators $\hat{T}_a$ thus built satisfy the same “isospin” algebra as the isospin operators $T_a$ provided the creation/annihilation operators satisfy the usual commutation/anticommutation
relations:
\[
[a_{x,i,\alpha}, a_{y,j,\beta}^\dagger]_\pm = \delta_{xy}\delta_{ij}\delta_{\alpha\beta}, \quad (15)
\]
\[
[a_{x,i,\alpha}^\dagger, a_{y,j,\beta}^\dagger]_\pm = [a_{x,i,\alpha}, a_{y,j,\beta}]_\pm = 0. \quad (16)
\]
In the above, when both \(x\) and \(y\) are fermions, the + subscript rules and the anticommutator is used. Otherwise, the − does and the commutator is considered.

The generalized isospin generators are very similar in form to the generators of the Virasoro algebra written in terms of oscillators, namely:
\[
L_m = \begin{cases} 
\frac{1}{2} \mu^2 + \frac{1}{2} \sum_1^\infty a_{-n}^\mu a_n^\mu & \text{for } m = 0, \\
\frac{1}{2} \sum_{-\infty}^\infty a_{m-n}^\mu \sqrt{(m-n)n} a_n^\mu & \text{for } m \neq 0.
\end{cases}
\]
Where as is customary in string theory, \(a_{-m}^\mu = a_{m}^{\dagger \mu}\). In the following we will start form a very simple Lie algebra, whose generators will play the role of the spin generators above. From them we will build Virasoro-like “generalized isospin” generators and pursue a BRST quantization programme. Therefore in the next section we will review the basics of BRST symmetry.

3 The BRST charge

The BRST charge, \(\hat{Q}_{BRST}\), is a nilpotent operator that acts upon the states provided by the theory. Due to its nilpotency, this operator generates a gauge symmetry of the states, namely, if \(|\phi'\rangle = |\phi\rangle + \hat{Q}|\lambda\rangle\) then \(\hat{Q} |\phi\rangle = \hat{Q} |\phi'\rangle\) and since physical states may not depend on the gauge chosen, the physical states must be invariant under the action of the operator.

Let us remind the definition of the BRST charge and the meaning of invariance of the physical states under the gauge transformation generated by \(\hat{Q}\) in some detail so as to demonstrate the previous statements. The BRST charge (\(\hat{Q}\)) is defined in the following way:
\[
\hat{Q} \equiv c^i X_i - 1/2 \; f_{ij}^k \; c^i c^j b_k, \quad (17)
\]
where \(X_i\) are the generators of a Lie algebra with \(f_{ij}^k\) as structure constants, the variables \(c^i \; y \; b_j\) are Grassmannian fermionic fields satisfying the following anticommutation relation
\[
\{c^i, b_j\} = \delta^i_j. 
\]
The nilpotency of the charge \( \hat{Q}^2 = 0 \) is easily demonstrated from the property above, from the algebra defining the generators \([X_i, X_j] = f_{ij}^k X_k\) and from the Jacobi identity satisfied by the structure constants. The ghost number operator is defined as \( U \equiv \sum_i c_i b_i \). A certain state \( |\phi\rangle \) with ghost number \( n \) will be BRST invariant if \( \hat{Q} |\phi\rangle = 0 \). Now all there is to do for finding physical states is to give a meaning to the ghost number operator and search for states that are BRST invariant with a given ghost number. In principle one would expect physical states not to contain any ghost or antighost excitations at all so one would look for states with ghost number zero, but analyzing the ghost number operator, specifically the normal ordering problem of the fermionic fields, shows that there exists physical states with ghost number different from zero, as a result of the ordering of \( c \)'s and \( b \)'s but still with no ghost excitations.

Now suppose the state \( |\phi\rangle \) has ghost number zero. This means that \( |\phi\rangle \) must be annihilated by the \( b_k \)'s so
\[
\hat{Q} |\phi\rangle = \sum_i c_i X_i |\phi\rangle . \tag{18}
\]

But from the anticommutation relations of the fermionic fields, if \( |\phi\rangle \) is annihilated by the \( b_k \)'s it cannot be annihilated by the \( c_i \)'s. Thus if the state \( |\phi\rangle \) is physical (BRST invariant) then it must be invariant under the action of the generators \( X_i \).
\[
X_i |\phi\rangle = 0, \quad i = 1, \ldots, n. \tag{19}
\]

In string theory, Virasoro algebra is not just a coincidence. In fact it is intimately connected with basic properties of the theory, such as conformal invariance (Virasoro generators are the generators of conformal transformations of the intrinsic coordinates in the world-sheet of strings) and to the solutions to the Euler-Lagrange equations derived from the Polyakov action. This algebra, however, is infinite-dimensional and the nilpotency of the BRST charge, constructed from it, is a bit more subtle due to anomalies and other problems related to the normal ordering of the fermionic fields, all of which are solved if certain conditions are met. These problems will not concern us here, since, as we will see, we will circumvent them by restricting to a much simpler case.
4 BRST quantization with a Lie algebra different from Virasoro algebra

Virasoro algebra is certainly very important because without it quantization of string theories would be a difficult task. Our question is then the following. Is there something even more fundamental for physical theories in Virasoro algebra than in strings themselves? To answer this we focused on the fact that the generators of the Virasoro algebra can be classified à la Cartan if one forgets about the central charge. Particularly one generator \( L_0 \) that commutes with all the resting generators and that can be identified with the Hamiltonian of the theory and an infinity of raising and lowering operators \( \{L_{\pm m}\}_{m=1}^{\infty} \). Our goal then is to seek for Lie algebras whose generators can: be identified with the Virasoro generators, that exhibit the Cartan structure and that we can construct a reasonable BRST charge from them.

4.1 Construction of the Lie Algebra

Let us define creation/annihilation operators \( a^\nu_n \) that are related to the creation/annihilation operators of fields’ excitations in string theory \( \alpha^\mu_n \) as in (20) and (21).

\[
\begin{align*}
\alpha^\mu_n &= \sqrt{|n|} \ a^\mu_{n}^\dagger \quad \text{si } n < 0 \quad \text{and} \\
\alpha^\mu_n &= \sqrt{n} \ a^\mu_{n} \quad \text{si } n > 0.
\end{align*}
\]

These operators, however are not exactly the same as the usual \( \alpha^\nu_n \) of string theory. Their Poisson bracket is defined as:

\[
\{a^\mu_n, a^\nu_m\}_{P,B} = i\sigma(n)\delta_{m+n}\eta^{\mu\nu},
\]

where: \( a^0_0 = p^\mu \), \( \sigma(n) \) is the sign of \( n \) such that \( \sigma(0) = 0 \) and of course \( a^\nu_n \) are creation (annihilation) operators if \( n < 0(>0) \). Let’s define operators \( \hat{T}^i \) as

\[
\text{The Virasoro algebra is } [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}.
\]

The second term on the right side arises from the normal-ordering ambiguities of the operators that define the Virasoro generators at the moment of quantizing the commutators of the classical generators. Then our remark is perfectly true for \( L_{-1}, L_0 \) and \( L_1 \), which generates a closed subalgebra, without anomaly. Furthermore, for the anomaly to be cancelled in the general case, a particular value of the space-time dimension \( D \) must be taken since \( c = c(D) \). In the case of the bosonic string, the anomaly cancellation condition demands \( D = 26 \). As can be seen, in our case, our results will be valid in for any \( D \), since for the closed subalgebra chosen, the central charge vanishes.
follows:

\[ T^i \equiv \frac{1}{\sqrt{2}} \sum_{\mu} \sum_{n,m \neq 0} a^\mu_n T^i_{nm} a_{\mu m} \quad i \neq 0. \quad (23) \]

One can prove that if the matrices \( T^i_{nm} \) satisfy the algebra \([T^i, T^j] = f^{ij}_{k} T^k\) then the operators defined in (23) satisfy an algebra with the same structure constants as the matrices \( T^i_{nm} \), provided the creation/annihilation operators be the usual ones (and not those in (22)) and that the matrix product is also the usual one, concerning positive components for the matrices. Let’s compute the Poisson bracket between any two of our generators.

\[
\{\hat{T}^i, \hat{T}^j\}_{PB} = \frac{i}{2} \sum_{\mu,\nu} \sum_{m,n,o,p} T^i_{n,m} T^j_{o,p} \{a^\mu_n a_{\mu m}, a^\nu_o a_{\nu p}\}_{PB}
\]

\[
= \frac{i}{2} \left[ a^\mu_n a_{\mu m} \sigma(n) T^i_{n,m} T^j_{-m,p} + a^\mu_o a_{\mu m} \sigma(n) T^i_{n,m} T^j_{o,-n} + a^\mu_n a_{\mu p} \sigma(m) T^i_{n,m} T^j_{-m,p} + a^\mu_n a_{\mu o} \sigma(m) T^i_{n,m} T^j_{o,-m} \right]. \quad (24)
\]

In equations (22), a sum over repeated indices (Latin indices \( \neq 0 \)) and the identity \([AB, CD] = [A, C]DB + [A, D]B + A[B, C]D + AC[B, D] \), were used. Now if in the first term of the second expression we do \( p \rightarrow n \), \( m \rightarrow p \) and \( n \rightarrow k \), in the second term \( o \rightarrow n \), \( m \rightarrow p \) and \( n \rightarrow k \) and finally in the fourth term \( o \rightarrow p \), we get:

\[
\{\hat{T}^i, \hat{T}^j\}_{PB} = \frac{i}{2} a^\mu_n a_{\mu p} \left\{ \sigma(m) T^i_{n,m} T^j_{-m,p} + \sigma(m) T^i_{n,m} T^j_{p,-n} + \sigma(k) T^i_{k,p} T^j_{-k,n} + \sigma(k) T^i_{k,p} T^j_{n,-k} \right\}. \quad (25)
\]

The operators in (23) are manifestly symmetric for the operators in (22) are in fact commuting functions, thus \( \sqrt{2} \hat{T}^i \equiv a^\mu_n T^i_{n,m} a_{\mu m} = a^\mu_m T^i_{n,m} a_{\mu n} \) and relabeling the dummy indices \( n \leftrightarrow m \) yields \( T^i_{n,m} = T^i_{m,n} \). Therefore if we do \( k \leftrightarrow -m \) and promoting the Poisson brackets to quantum mechanical commutators, then equation (25) reads

\[
[\hat{T}^i, \hat{T}^j] = \frac{1}{2} a^\mu_n a_{\mu p} \left\{ \sigma(m) \left[ T^i_{n,m} T^j_{-m,p} - T^j_{n,m} T^i_{-m,p} \right] + \sigma(m) \left[ T^i_{n,m} T^j_{-m,p} - T^j_{n,m} T^i_{-m,p} \right] \right\}
\]

\[
= a^\mu_n a_{\mu p} \left\{ \sigma(m) \left[ T^i_{n,m} T^j_{-m,p} - T^j_{n,m} T^i_{-m,p} \right] \right\}. \quad (26)
\]
4.2 A new matrix product

Let us define the following matrix product.

\[(A \star B)_{np} \equiv \sigma(k) \ A_{n,k} B_{-k,p}, \tag{27}\]

which is perfectly associative and distributive

\[
(A \star (B + C))_{i,j} = \sigma(k) \ A_{i,k} (B_{-k,j} + C_{-k,j}) \\
= \sigma(k) \ A_{i,k} B_{-k,j} + \sigma(k) \ A_{i,k} C_{-k,j} \\
= (A \star B)_{i,j} + (A \star C)_{i,j}, \tag{28}\]

\[
((A \star B) \star C)_{i,j} = \sigma(m) \ (A \star B)_{i,m} C_{-m,j} \\
= \sigma(m) \ \sigma(k) \ A_{i,k} B_{-k,m} C_{-m,j} \\
= \sigma(k) \ A_{i,k} (\sigma(m) \ B_{-k,m} C_{-m,j}) \\
= \sigma(k) \ A_{i,k} (B \star C)_{-k,j} = (A \star (B \star C))_{i,j}. \tag{29}\]

Now with this new matrix product \(\star\), it is easy to prove that:

\[
[\hat{T}^i, \hat{T}^j]_{n,p}^* = \sum_{\mu} \sum_{n,p \neq 0} a^\mu_n \ [\hat{T}^i, \hat{T}^j]_{n,p}^* a^\mu_p . \tag{30}\]

Therefore, with the matrix product introduced, the operators \(\hat{T}^i\) are good “generalized isospin operators” with respect to the \(T_i\) ones, as sketched in section 2. With this in hand we are ready to construct an algebra that resembles the Virasoro Algebra (quadratic in creation/annihilation operators).

All we have to do is to find matrices (with components running from \(-\infty\) to \(+\infty\)) that when multiplied via the \(\star\) product can be classified à la Cartan.

4.3 Connection with Virasoro algebra

It is important to check that our generators include the Virasoro’s. To see this let’s write Virasoro’s generators in terms of the usual creation/annihilation operators, \(a^\mu_n\), according to (20) and (21).

\[
L_m = \begin{cases} 
\frac{1}{2} p^2 + \frac{1}{2} \sum_1^\infty a^\mu_n \ a^\mu_n & \text{for } m = 0, \\
\frac{1}{2} \sum_{-\infty}^\infty a^\mu_m a^\mu_n \sqrt{|(m-n)n|} \ a^\mu_n & \text{for } m \neq 0.
\end{cases}
\]
So comparing these with (23) and choosing:

\[
[T^m]_{ij} = \begin{cases} 
\frac{1}{\sqrt{2}} \{ \delta_{i+0} \delta_{j+0} + i \delta_{i+j} \} & \text{for } m = 0, \\
\frac{1}{\sqrt{2}} \sqrt{|ij|} \delta_{i+j-m} & \text{for } m \neq 0.
\end{cases}
\]

we recover Virasoro generators.

4.4 A suitable Lie algebra for this programme

Let consider first a Lie algebra of three generators. But before starting we have to study how it’s matrix representations look like with our \( \star \) product. First of all to keep in contact with the Virasoro algebra we want the generators that are to be identified with \( L_0 \) to be diagonal, or more precisely hermitian and thus diagonalizable. So our diagonal matrices are to be diagonal in the same sense as \( L_0 \) is, \( i.e. \) the entry \((0,0) \neq 0 \) because from it we generate the dynamical term \( p^2 \) so that to make sense out of \( L_0 \), the entries \((-n,n) \neq 0 \) for \( n = 1 \cdots \infty \) simply from looking at the expression for \( L_0 \) and finally el the other entries equal to zero. In the particular case of \( SU(2) \) there is only one diagonal matrix belonging to the Cartan subalgebra, \( J_3 \). So the basic elements of this algebra are:

\[
J_3 = \begin{pmatrix} 0 & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & 0 \end{pmatrix},
\]

\[
J_+ = \begin{pmatrix} r & s & t \\ s & u & v \\ t & v & w \end{pmatrix}.
\]

The last matrix of the algebra, \( J_- \), is obviously \( J_+^\dagger \). To these matrices we impose \([J_3, J_\pm]^* = \pm J_\pm \) and \([J_+, J_-]^* = J_3 \). Which sets conditions upon the constants involved. Particularly for \( J_3 \) the conditions are \( \gamma = 0 \) or \( \beta = 0 \). Let’s remember that \( J_3 \) is to be associated with \( L_0 \) so necessarily it must contain a dynamical term \( p^2 \), for it to be a proper candidate to the

\[\text{[It is interesting to see that this is an eigenvalue problem in the adjoint representation of the algebra but with the matrix product defined here. This needed be so because symmetrical matrices are closed under commutation for which our product \( \star \) proved necessary.]}\]
Hamiltonian. This term is to come from the \((0,0)\) entry of the matrix so the solution demanding \(\gamma = 0\) is discarded. Now the second solution, \(\beta = 0\), does not contain terms proportional to the fields’ excitations so this solution implies our Hamiltonian predicts the dynamics of massless particles. This solution requires \(-2sv = \gamma, \beta = 0, u, r, t, w = 0\) and \(s, v \neq 0\). So finally we are left with a much simpler algebra than SU(2) namely \([J_3, J_+ + J_-] = 0\) and \([J_+, J_-] = J_3\). Now following our definition of the generators we can explicitly write them in terms of the creation/annihilation operators of the fields’ excitations.

\[ \hat{J}_3 = \frac{1}{2} \gamma p^2 , \]

\[ \hat{J}_+ = sp_\mu a_\mu^\dagger , \]

\[ \hat{J}_- = ep_\mu a_\mu . \]

Demanding \(J_1^+ = J_\perp\) then \(s^* = e, \langle -2|s|^2 = \gamma \rangle\). That the algebra mentioned above is satisfied by these is straightforward.

5 Massless spin-1 and spin-2 fields

The construction of the BRST charge using these three generator and noting that two of the three commutators vanishes identically yields:

\[ \hat{Q} = c_0 \hat{J}_3 + c_{-1} \hat{J}_- + c_1 \hat{J}_+ - c_{-1} c_1 b_0 , \]

which is perfectly nilpotent. In this case we have no certainty about the eventual degeneracy in ghost number of the vacuum state \(|-\rangle\), but we can be certain that it is physical, so analyzing \(\hat{Q}|-\rangle = 0\), we can conclude that \(c_1|-\rangle = b_1|-\rangle = 0, c_{-1}|-\rangle \neq 0 \neq b_{-1}|-\rangle, c_0|-\rangle = |+\rangle\) and \(b_0|-\rangle = 0\). Having considered that the vacuum is annihilated by \(J_\perp J_3\).

Now the following step is to find the field equations induced by the BRST symmetry of a certain scalar wave function \(|A\rangle\), explicitly the symmetry transformation induced is:

\[ |A\rangle \longrightarrow |A'\rangle = |A\rangle + \hat{Q}|A\rangle , \]

\(^6\) It is interesting to note that the fact that our model will describe massless particles is not only a consequence of it being a (BRST) gauge theory, rather, it is also consequence of the choice above. That \(\beta \neq 0\) would lead to a gauge theory for massive HS fields is a novel feature though not clear yet and needs to be explored in the future. 

\(^7\) This algebra is a proper Lie algebra that has been studied already in ref. [28] p. 306.
where the transformation of the field $|A\rangle$ is $\delta |A\rangle = \hat{Q} |A\rangle$. Obviously the field equations would be given by the condition that $|A\rangle$ be physical i.e., $\hat{Q} |A\rangle = 0$.

A suitable gauge parameter $|\Lambda\rangle$ such that $\hat{Q} |\Lambda\rangle = \delta |A\rangle$ has the correct ghost number is:

$$|\Lambda\rangle = [b^{-1} \lambda(x) + b^{-1} a^\dagger_1 \Lambda_\mu(x)]|\rangle .$$

(36)

From now on the subscript 1 of the creation/annihilation operators from the second term will be omitted, to this extent we will always be working to first level in fields’ excitations. Computing $\hat{Q} |\Lambda\rangle$ will give us an idea of the field $|A\rangle$.

$$\hat{Q} |\Lambda\rangle = c_0 \gamma p^2 b^{-1} \lambda(x)|\rangle + c_1 s p_\mu a^\dagger b^{-1} \lambda(x)|\rangle$$

$$+ \frac{7}{2} p^2 \Lambda_\mu(x) a^\dagger c_0 b^{-1} |\rangle + s^* p_\mu \Lambda_\mu(x) c_{-1} b^{-1} |\rangle$$

$$+ \frac{s}{2} (p_\mu \Lambda_\nu(x) + p_\nu \Lambda_\mu(x)) a^\dagger a^\dagger|\rangle .$$

(37)

Identifying amongst the terms above that appear multiplying all fields, those that are scalars, vectors, tensors, etc., tells us that $|A\rangle$ should have the following form:

$$|A\rangle = \Omega(x) c_0 b^{-1} \lambda(x)|\rangle + A_\mu(x) a^\dagger b^{-1} \lambda(x)|\rangle$$

$$+ \psi_\mu(x) a^\dagger c_0 b^{-1} |\rangle + \phi(x) c_{-1} b^{-1} |\rangle$$

$$+ h_{\mu \nu}(x) a^\dagger a^\dagger|\rangle .$$

(38)

such that the variations of the auxiliary fields $\Omega(x), A_\mu(x), \phi(x), \psi_\mu(x), \eta(x), h_{\mu \nu}(x)$ be:

$$\delta \phi(x) = 0 ,$$

(39)

$$\delta \Omega(x) = 1/2 \gamma p^2 \lambda(x) ,$$

(40)

$$\delta A_\mu(x) = s p_\mu \lambda(x) .$$

(41)

and

$$\delta \psi_\nu(x) = \frac{\gamma p^2}{2} \Lambda_\nu(x) ,$$

(42)

$$\delta \eta(x) = s^* p_\nu \Lambda_\nu(x) ,$$

(43)

$$\delta h_{\mu \nu}(x) = \frac{s}{2} [p_\mu \Lambda_\nu(x) + p_\nu \Lambda_\mu(x)] .$$

(44)
As we said before the field equations are obtained imposing $\hat{Q}|A\rangle = 0$. From equations (34) and (38) and demanding that each of the different excitations of the vacuum state vanishes independently, yields:

\[ \Box A_\mu(x) - \frac{2is}{\gamma} \partial_\mu \Omega(x) = 0 \, , \]  
\[ \Box \phi(x) = 0 \, , \]  
\[ \partial_\nu A_\nu(x) + \frac{i}{s^*} \Omega(x) = 0 \, , \]  
\[ \Box \eta(x) - \frac{2is^*}{\gamma} \partial_\mu \psi_\mu(x) = 0 \, , \]  
\[ \Box h_{\mu\nu}(x) - \frac{is}{\gamma} [\partial_\mu \psi_\nu(x) + \partial_\nu \psi_\mu(x)] = 0 \, , \]  
\[ s \partial_\nu \eta(x) - 2s^* \partial_\mu h_{\mu\nu}(x) - i\psi_\nu(x) = 0 \, . \]

All of which are invariant under the transformations (39) to (44). Now replacing (47) in (45) and since $2|s|^2 = -\gamma$, we get:

\[ \partial_\nu (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)) = \partial_\nu F_{\mu\nu} = F_{\mu\nu,\nu} = 0 \, . \]

Thus we have recovered Maxwell’s equations. Furthermore solving for $h_{\mu\nu}(x)$ yields:

\[ \Box h_{\mu\nu} + h_{\lambda\lambda,\mu\nu} - (h_{\lambda\nu,\lambda\mu} + h_{\lambda\mu,\lambda\nu}) = 0 \, . \]

This last equation is nothing but Einstein’s linearized equation of the gravitational field where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, which is very easy to see if we compare with the Einstein’s field equations for $h_{\mu\nu}$ (to first order in $h$) where the matter term is present

\[ \Box h_{\mu\nu} + h_{\lambda\lambda,\mu\nu} - (h_{\lambda\nu,\lambda\mu} + h_{\lambda\mu,\lambda\nu}) = -16\pi G S_{\mu\nu} \, . \]

### 6 Mixed symmetry 2nd rank tensor field

Leaving aside the method used to construct the previous algebra and based on its simplicity the generalization to an algebra with more generators is almost immediate. However for the incorporation of mixed symmetry tensor fields it suffices to consider a 5-generator algebra $\{\hat{J}_{++}, \hat{J}_+, \hat{J}_0, \hat{J}_-, \hat{J}_{--}\}$ with
\[ \hat{J}_0 = \gamma p^2; \quad \hat{J}_{++} = sp_\mu a_2^{\mu\dagger} \text{ and so on.} \] The BRST charge \( \hat{Q} \) is constructed from the algebra (considering, obviously that we have as many ghost field operators as generators in the algebra),

\[ \hat{Q} = c_0 \hat{J}_0 + c_1 \hat{J}_+ + c_2 \hat{J}_{++} + c_{-1} \hat{J}_- + c_{-2} \hat{J}_{--} - c_{-2} c_{-1} \hat{b}_0 - c_1 c_{-1} \hat{b}_0 \] (54)

and the procedure is essentially the one used previously, we start from the following gauge field:

\[ |\Lambda\rangle = [b_{-2} \Lambda_{(1)}^\mu (x) a_1^{\mu\dagger} + b_{-1} \Lambda_{(2)}^\mu (x) a_2^{\mu\dagger}]|\rangle \] (55)

Then \( \hat{Q}|\Lambda\rangle = \delta|A\rangle \) implies \( |A\rangle \) should be:

\[ |A\rangle = [\psi_{(1)}^\mu (x) a_1^{\mu\dagger} c_{-2} + \psi_{(2)}^\mu (x) a_2^{\mu\dagger} c_{-1} + M_{12}^{(12)} (x) a_1^{\mu\dagger} a_2^{\nu\dagger} + \Omega_{(1)}^\mu (x) c_{-1} b_{-2} + \Omega_{(2)}^\mu (x) c_{-2} b_{-1}]|\rangle \] (56)

such that the variation of the fields are:

\[ \delta\psi_{(i)}^\mu (x) = -\gamma \Box \Lambda_{(i)}^\mu (x) \]
\[ \delta M_{12}^{(12)} (x) = -i \sigma (\partial_{\mu} \Lambda_{(1)}^\mu (x) + \partial_{\nu} \Lambda_{(2)}^\nu (x)) \]
\[ \delta \Omega_{(i)}^\mu (x) = -i \sigma^* \partial_{\mu} \Lambda_{(i)}^\mu (x) \] (57)

Note that \( M_{12}^{(12)} (x) \) does not have a definite symmetry. Then demanding that \( |A\rangle \) be physical, i.e. \( \hat{Q}|A\rangle = 0 \), we get:

\[ -\gamma \Box M_{12} (x) + i \sigma \psi_{(2)}^\nu (x) + i \sigma \psi_{(1)}^\mu (x) = 0 \] (58)
\[ -\gamma \Box \Omega_{(i)}^\mu (x) + i \sigma^* \psi_{(i)}^\mu (x) = 0 \] (59)
\[ \psi_{(1/2)}^\mu (x) - i \sigma^* M_{\sigma,\nu}^\mu (x) + i \sigma \Omega_{(2/1)}^\nu (x) = 0 \] (60)

Which simplify to:

\[ M_{\mu\sigma,\nu\sigma} (x) + M_{\sigma,\mu\nu} (x) - \Box M_{12} (x) - \Box^{-1} M_{\alpha\beta,\alpha\beta\nu\mu} (x) = 0 \] (61)

\[ 8[\hat{f}_+, \hat{f}_-] = \hat{J}_0 \] demands \( \gamma = -|s|^2 \).
6.1 The ‘axion’ field

If \( M_{\mu\nu}(x) = b_{\mu\nu}(x) + h_{\mu\nu}(x) \), where \( b_{\mu\nu} = 1/2(M_{\mu\nu} - M_{\nu\mu}) \), \( h_{\mu\nu} = 1/2(M_{\mu\nu} + M_{\nu\mu}) \), then eqn. (61) read:

\[
-H_{\mu\nu\sigma,\sigma}(x) - \Box h_{\mu\nu} - \Box^{-1} h_{\alpha\beta,\alpha\beta\mu\nu} + h_{\mu\sigma,\nu\sigma} + h_{\sigma\nu,\mu\sigma} = 0. \tag{62}
\]

where we have made the following definition:

\[
H_{\mu\nu\sigma}(x) \equiv b_{\mu\nu,\sigma}(x) + b_{\nu\sigma,\mu}(x) + b_{\sigma\mu,\nu}(x) \tag{63}
\]

Now, (63) has a purely symmetric and an antisymmetric part, therefore each part must vanish separately:

\[
H_{\mu\nu\sigma,\sigma}(x) = 0 \tag{64}
\]

\[
- \Box h_{\mu\nu} - \Box^{-1} h_{\alpha\beta,\alpha\beta\mu\nu} + h_{\mu\sigma,\nu\sigma} + h_{\sigma\nu,\mu\sigma} = 0 \tag{65}
\]

Now, taking the trace of the last equation then:

\[
\Box h_{\lambda\lambda} = h_{\alpha\beta,\alpha\beta} \tag{66}
\]

which if replaced in the former equations yields:

\[
- \Box h_{\mu\nu} - h_{\lambda\lambda,\mu\nu} + h_{\mu\sigma,\nu\sigma} + h_{\sigma\nu,\mu\sigma} = 0. \tag{67}
\]

Summarizing:

\[
H_{\mu\nu\sigma,\sigma}(x) = 0, \tag{68}
\]

\[
- \Box h_{\mu\nu} - h_{\lambda\lambda,\mu\nu} + h_{\mu\sigma,\nu\sigma} + h_{\sigma\nu,\mu\sigma} = 0. \tag{69}
\]

6.2 Gauge transformations of the physical fields

From (57) we see that \( \delta M^{(12)}_{\mu\nu} = -is(\partial_\mu \Lambda^{(1)}_\nu + \partial_\nu \Lambda^{(2)}_\mu) \) then we can know how the physical fields \( b_{\mu\nu}, h_{\mu\nu} \) transform since \( b_{\mu\nu} \) is the antisymmetric part of \( M^{(12)}_{\mu\nu} \) and \( h_{\mu\nu} \) its symmetric part, then

\[
\delta b_{\mu\nu} = -\frac{is}{2} (\Lambda^{(1)}_{\nu,\mu} + \Lambda^{(2)}_{\mu,\nu} - \Lambda^{(1)}_{\mu,\nu} - \Lambda^{(2)}_{\nu,\mu})
\]

\[
= -\frac{is}{2} (\Lambda^{(1)}_{\nu} - \Lambda^{(2)}_{\nu})_{,\mu} - (\Lambda^{(1)}_{\mu} - \Lambda^{(2)}_{\mu})_{,\nu}
\]

\[
= -\frac{is}{2} (\varepsilon^{(-)}_{\nu} - \varepsilon^{(-)}_{\mu}) \tag{70}
\]
Similarly

$$\delta h_{\mu \nu} = -\frac{i s}{2} (\varepsilon^{(+)}_{\nu, \mu} + \varepsilon^{(+)}_{\mu, \nu})$$  \hspace{1cm} (71)$$

Note that $h_{\mu \nu}$ transforms the same way the gravitational field under general coordinate transformation and $b_{\mu \nu}$ transforms just as the axion does, so the fact that our equations were linearized versions of the field equations of these fields was to be expected.

7 HS gauge fields and the high-energy limit of string theory

Already in the previous section we could see that our first algebraic construction was rather intricate and needed some refinement. To do this let us write Virasoro generators with their dependence on the string constant $\alpha'$ explicit:

$$L_m = \begin{cases} 
\frac{1}{4} \alpha' p^2 + \frac{1}{2} \sum_{1}^{\infty} \alpha_{-n} \alpha_{mn} & m = 0, \\
\frac{1}{2} \{ \sqrt{2 \alpha'} p_{\mu} \alpha_{\mu}^m + \sum_{-\infty}^{\infty} \alpha_{m-n} \alpha_{mn} \} & m \neq 0.
\end{cases}$$

Now let us define the generators

$$\hat{J}_0 \equiv \left[ \lim_{\alpha' \to \infty} \frac{L_0}{\alpha'} = \frac{1}{4} p^2 \right]$$ \hspace{1cm} (72)$$

$$\hat{J}_m \equiv \left[ \lim_{\alpha' \to \infty} \frac{L_m}{\sqrt{2 \alpha'}} = \frac{1}{2} p_{\mu} \alpha_{\mu}^m \right]$$ \hspace{1cm} (73)$$

So now we have made an infinite-dimensional algebra, (whose truncation provides the algebras with which we worked previously), from the formal limit $\alpha' \to \infty$ of the rescaled Virasoro generators. From this construction, the generalization of the above is immediate. Note that this rescaled and Virasoro-like generators can be obtained as our original motivation, i.e., building them like “generalized isospin operators”. The form of these would change and another formula like 4.3 would be obtained, whose specific expression is not relevant now.
Considering as gauge parameters the following:

\[ \Lambda(x) = M_\mu a_1^{\mu\dagger}, \]

\[ \Lambda^2(x) = M_\mu M_\nu a_1^{\mu\dagger} a_1^{\nu\dagger} = M_{(\mu\nu)} a_1^{\mu\dagger} a_1^{\nu\dagger}, \]

\[ \vdots \]

\[ \Lambda^n(x) = M_{\mu_1} \cdots M_{\mu_n} a_1^{\mu_1\dagger} \cdots a_1^{\mu_n\dagger} = M_{(\mu_1\cdots\mu_n)} a_1^{\mu_1\dagger} \cdots a_1^{\mu_n\dagger}. \] (74)

and defining:

\[ |\Lambda\rangle = b_{-1} e^{\Lambda(x)} |\rangle, \] (75)

and following the BRST procedure, i.e. \( \delta |A\rangle = \hat{Q} |\Lambda\rangle \) will provide a gauge invariant theory for symmetric tensor fields. Obviously the previous results are included in this last procedure. For the case of higher spin and arbitrary symmetry this generalization can also be accommodated as outlined in section 6.

As mentioned in the introduction, the connection between HS gauge fields and the high-energy (low-tension) limit of string theory is a rather old idea. So is the BRST formulation for the general theory of massless higher spin and arbitrary gauge fields. However, the approach here taken of building a BRST operator from “generalized isospin-like operators” is interesting in its own right.

\section{Conclusions}

We have been able to construct a quantum field theory for massless fields based upon the BRST symmetry induced by the “generalized isospin-like operators”. These operators were built keeping close contact with certain properties of the Virasoro algebra, which allowed us to find an extremely simple Lie algebra with which we could obtain, without many complications, some very interesting results.

Particularly uncoupled fields \( A_\mu \) and \( h_{\mu\nu} \) corresponding to the photon and graviton respectively. Besides the representations with which we worked were symmetrical ones from which, as a by-product we obtained a gauge invariant theory of symmetric tensor fields through the nilpotent operator \( \hat{Q} \) keeping close resemblance with the exterior derivative operator which provides a gauge invariant theory but for antisymmetric fields. The generalization to higher spins and arbitrary symmetry was also outlined.
Although our results (fields equations and transformation properties) are particular cases of the works on BRST formulation of massless higher spin gauge fields, these are not our main results. Instead we want to stress that our approach is not only original but also makes contact with long established results, giving support for our choice of “generalized isospin-like operators” as starting point and thus raising the question of the future relevance in other contexts or applications outside nuclear physics.

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