DYNAMICS OF THE CONFORMAL FACTOR IN 4D GRAVITY †

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ABSTRACT

We argue that 4D gravity is drastically modified at distances larger than the horizon scale, due to the large infrared quantum fluctuations of the conformal part of the metric. The infrared dynamics of the conformal factor is generated by an effective action, induced by the trace anomaly of matter in curved space, analogous to the Polyakov action in two dimensions. The resulting effective scalar theory is renormalizable, and possesses a non-trivial, infrared stable fixed point, characterized by an anomalous scaling dimension of the conformal factor. We argue that this theory describes a large distance scale invariant phase of 4D gravity and provides a framework for a dynamical solution of the cosmological constant problem.

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The cosmological constant, if not identically zero, is the smallest fundamental mass scale in nature. In contrast to other fine-tuning problems, an adjustment of its value once is not sufficient to explain its smallness at all epochs in the evolution of the universe. Thus, any attempt at an explanation cannot rely on an exact symmetry of the fundamental quantum theory of gravitation (such as supersymmetry), if that symmetry is ultimately broken in the low energy effective theory. Instead, the fact that the effective cosmological “constant” is dynamically dependent on the vacuum state of all quantum fields in nature implies that the physics of the Planck scale, but the low-energy, or infrared dynamics of gravity is essential to a resolution of the problem.

This conclusion seems to contradict the naive expectation that at distances larger than the Planck length gravity is very well described by the classical Einstein theory. There are indeed two main reasons that could invalidate this expectation: (i) On the one hand, earlier studies of quantum fields in curved spacetime made in evidence peculiar long distance dynamics for massless fields in rapidly expanding cosmological backgrounds, such as de Sitter space which is the natural maximally symmetric ground state of Einstein gravity with a positive cosmological term. In particular, the graviton propagator grows without bound at distances larger than the horizon length \[1,2\], while the spin zero or conformal part of the propagator provides the dominant contribution \[2\]. This suggests that classical theory is only valid at intermediate distances, larger than the Planck scale but smaller than the horizon, and that in the far infrared regime quantum gravitational effects become important.

(ii) On the other hand, the existence of trace anomaly implies that the trace of the classical equations of motion cannot in general be imposed consistently with general covariance at the quantum level. A clear example is provided by the situation in two dimensions, where the quantum trace anomaly changes discontinuously the classical theory and generates an effective non-local action for 2D gravity \[3\]. In the conformal gauge, this action becomes a local kinetic term for the conformal part of the metric, the so-called Liouville mode, whose dynamics determines the non-trivial critical behavior of 2D gravity.

We are led to study the infrared dynamics of 4D gravity, at distances larger than the horizon. This region is expected to be insensitive to the ultraviolet structure of the theory. To find the relevant effective action, one could therefore put an ultraviolet cutoff at the horizon length and integrate all high frequency modes up to that scale. At such large distances with several causally disconnected regions, all
matter interactions can be neglected, while massive matter being red-shifted away very rapidly is decoupled. The remaining massless fields could then be treated as free and classically conformally invariant. The metric can be parametrized in conformal coordinates

\[ g_{ab}(x) = e^{2\sigma(x)} \tilde{g}_{ab}(x), \]  

where the conformal factor \( e^{2\sigma(x)} \) is factorized and \( \tilde{g}_{ab}(x) \) contains the transverse spin-2 excitations. At a first approximation, one can treat \( \tilde{g}_{ab}(x) \) as a fixed fiducial metric and study the infrared dynamics of the conformal factor. Although in the classical theory the conformal part of the metric is completely constrained, the situation changes dramatically at the quantum level due to the trace anomaly. Remarkably, the \( \sigma \)-dependence of the induced non-local action turns out to be local, as in two dimensions, and an exact analysis of the infrared critical behavior of the theory can be performed [4].

Although in two dimensions the freezing of \( \tilde{g}_{ab} \) corresponds to a choice of gauge, in four dimensions it represents a severe truncation of the full theory. However, there are good reasons to believe that this could be still a good approximation in order to study the long distance dynamics: In fact, the conformal factor gives rise to the most singular infrared behavior in perturbation theory around de Sitter space. Its classical equations of motion, which fix the curvature scalar to be constant, are modified by the quantum trace anomaly. Finally, it plays a dominant role in cosmology and the truncated theory is already much richer than the “minisuperspace” truncation to a finite number of degrees of freedom. After all, the transverse spin-2 excitations could be treated as the remaining massless matter fields, which just amounts to provide their own contribution to the anomaly coefficients [5].

Apart from the cosmological constant problem, the dynamics of the conformal factor may have other interesting applications, as in inflationary cosmology, large-scale structure, and the dark matter problem. Moreover, it is an interesting non-trivial field theory in its own, which generalizes the Liouville theory in four dimensions and describes an exact treatment of 4D self-dual gravity [6].

The effective theory

In order to determine the effective action, which describes the large-distance dynamics of the conformal factor, we proceed in analogy with the two-dimensional case which we review below. The classical action of 2D gravity contains no deriva-
tives of the metric since the Einstein-Hilbert term is a total derivative:

\[ S_{cl}^{(2)} = \int \sqrt{-g} \left( \gamma R - 2\lambda \right) = 4\pi \gamma \chi - 2\lambda \int \sqrt{-g} e^{2\sigma}, \]

where \( \chi \) is the Euler number and in the second line of (2) we used the conformal gauge (1). To avoid possible confusion due to the triviality of pure gravity in two-dimensions, we consider also the presence of \( d \) matter fields. The classical theory appears then to have \( d - 2 \) dynamical degrees of freedom, in view of the equations of motion: \( T_{\text{matter}}^{ab} = \lambda g^{ab} \); two of these equations act as constraints while the third one, corresponding to the trace, determines \( \sigma \) in terms of the matter fields. At the quantum level the dynamics of the theory changes discontinuously because of the trace anomaly. In fact the trace of the classical equations of motion, which corresponds precisely to the \( \sigma \)-variation, cannot be preserved simultaneously with general coordinate invariance. For conformally invariant matter, the trace anomaly takes the form:

\[ T = \frac{c}{24\pi} R = \frac{c}{24\pi} e^{-2\sigma} \left( R - 2 \Box \sigma \right), \quad D = 2 \]

in the decomposition (1), where \( T \equiv T_a^a \). The central charge \( c \) is given by:

\[ c = d - 26 + 1, \quad d = N_S + N_F, \]

where -26 and +1 stand for the contribution of the reparametrization ghosts and \( \sigma \)-field itself, and the contribution to the matter central charge \( d \) of \( N_S \) free scalars and \( N_F \) free (Dirac) fermions is also presented. There are two ways to define a consistent quantum theory:

(i) Cancel the conformal anomaly when \( d = 26 \), in which case \( \sigma \) decouples and one is left with critical string theory. The classical counting of \( d - 2 \) degrees of freedom remains valid.

(ii) Modify the classical action (2) by adding the anomaly-induced Polyakov term [3]:

\[ S_{\text{anom}}^{(2)} = -\frac{c}{96\pi} \int \sqrt{-g} R \frac{1}{\Box} R = \frac{c}{24\pi} \int \sqrt{-g} (-\sigma \Box \sigma + \bar{R} \sigma), \]
such that

\[ T = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta \sigma} S_{\text{anom}}, \]

and in the second line of (5) a \( \sigma \)-independent term has been dropped. In the resulting Liouville action, the conformal factor has now two derivatives and acquires its own dynamics. One is left with non-critical string theory describing \( d-1 \) physical propagating modes.

Let us now consider the same line of reasoning in four dimensions. The classical action is:

\[ S_{cl}^{(4)} = \frac{1}{2\kappa} \int \sqrt{-g} (R - 2\Lambda) \]

\[ = \frac{1}{2\kappa} \int \sqrt{-\bar{g}} \left\{ e^{2\sigma} (\bar{R} + 6(\nabla \sigma)^2) - 2\Lambda e^{4\sigma} \right\}, \]

where \( \kappa = 8\pi G_N \). Although \( \sigma \) appears with two derivatives in (7), it is still non-propagating due to the reparametrization constraints. On the other hand, the metric \( \bar{g}_{ab} \) in the decomposition (1) cannot be fixed by a gauge choice as in two dimensions, since it contains the two physical transverse, trace-free, spin-2 excitations. At the quantum level, the general form of the trace anomaly for conformally invariant matter in a four dimensional curved space-time is a linear combination of the square of the Weyl tensor \( C^2 \), the Gauss-Bonnet combination \( G \), and \( \Box R \) [7]:

\[ T = bC^2 + b'(G - \frac{2}{3} \Box R) + \frac{\zeta}{3} \Box R; \quad D = 4 \]

\[ C^2 = R_{abcd} R^{abcd} - 2R_{ab} R^{ab} + \frac{1}{3} R^2, \]

\[ G = R_{abcd} R^{abcd} - 4R_{ab} R^{ab} + R^2. \]

For free fields, the anomaly coefficients \( b \) and \( b' \) are given by:

\[ b = \frac{1}{120(4\pi)^2} (N_S + 6N_F + 12N_V - 8), \]

\[ b' = -\frac{1}{360(4\pi)^2} (N_S + 11N_F + 62N_V - 28), \]

with \( N_V \) the number of vectors and the last numbers denote the \( \sigma \)-contribution [5]. Unlike \( b \) and \( b' \), the coefficient of the \( \Box R \) term in the anomaly is altered by the addition of a local \( R^2 \) term in the action, whose conformal variation is exactly \( \Box R \), so that it must be treated as an additional renormalized coupling, and the \( \zeta \) coefficient in (8) is left undetermined. In addition to the \( C^2 \), \( G \) and \( \Box R \) terms in the general form of the trace anomaly for \( D = 4 \), an \( R^2 \) term is allowed by
naive dimensional analysis. However, such a term is forbidden by the Wess-Zumino consistency condition [8], which in the present context is simply the statement that the variational relation (6) is integrable, \textit{i.e.} that there exists an effective action functional (local or non-local) $S_{\text{anom}}$, whose $\sigma$ variation is the anomaly.

In the parametrization (1), the quantity $\sqrt{-g}C^2$ is independent of $\sigma$, while the combination $\sqrt{-g}(G - \frac{2}{3}\Box R)$ becomes only linear in $\sigma$:

$$\sqrt{-g}(G - \frac{2}{3}\Box R) = \sqrt{-g}(4\Box\sigma + G - \frac{2}{3}\Box\bar{R}),$$

where $\Delta$ is the Weyl covariant fourth order operator acting on scalars:

$$\Delta = \Box^2 + 2R^{ab}\nabla_a\nabla_b - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^a R)\nabla_a.$$  

Thus, as in two dimensions, the $\sigma$ dependence of the non-local anomaly-induced action becomes local in the conformal parameterization (1):

$$S^{(4)}_{\text{anom}} = -\int \sqrt{-g}\left\{\frac{1}{8b'}[bC^2 + b'(G - \frac{2}{3}\Box R)]\frac{1}{\Delta}[bC^2 + b'(G - \frac{2}{3}\Box R)] - \frac{\zeta}{36}R^2\right\}$$

$$= \int \sqrt{-g}\{2b'\sigma\Delta\sigma + [b\bar{C}^2 + b'(\bar{G} - \frac{2}{3}\Box\bar{R})]\sigma - \frac{\zeta}{36}[\bar{R} - 6\Box\sigma - 6(\nabla\sigma)^2]^2\},$$

where in the second line of (12) a $\sigma$-independent term has been dropped.

This action plays a role similar to the Wess-Zumino action of low energy pion physics, as realized in the Skyrme model, for example. That is, it can be interpreted as an effective action at low energies, which describes all modifications to the Ward-identities due to the presence of the quantum trace anomalies. In particular, it guarantees that the total trace anomaly vanishes at any fixed point of the matter theory, provided that the coefficients $b$, $b'$ and $\zeta$ are chosen appropriately. Using in fact the integrability condition (6), one has the identity:

$$S_{\text{matter}}[g_{ab} = e^{2\sigma}\bar{g}_{ab}] = S_{\text{matter}}[\bar{g}_{ab}] + S_{\text{anom}}[\bar{g}_{ab}; \sigma].$$  

Both sides of the above equation are trivially invariant under the transformation

$$\bar{g}_{ab} \to e^{2\omega}\bar{g}_{ab}$$

$$\sigma \to \sigma - \omega,$$

which leaves the total metric unchanged. The infinitesimal variation of (13) with respect to $\omega$ then yields:

$$T_{\text{matter}} + T_\sigma = \frac{\delta S_{\text{anom}}}{\delta \sigma},$$

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implying that the total trace vanishes when one uses the $\sigma$-equation of motion.

In the induced action (12) $\sigma$ appears with four derivatives, as opposed to two in the classical theory (7), indicating the presence of an extra mode. This raises the question whether any consistent theory of gravity requires the existence of an additional scalar field with the properties of $\sigma$. The answer is not obvious because an essential difference between four and two dimensions is that the classical theory is non-renormalizable, implying that not only the trace, but all the components of the classical equations of motion are in principle inconsistent at the quantum level. Note however that in string theory, which is the only known example of quantum gravity, there is such a massless scalar mode, the dilaton, which at the lowest order couples to the Gauss-Bonnet integrand as $\sigma$ and can be used to cancel the trace anomaly. For our purposes, we are interested in the effective theory at very large distances, where the ultraviolet structure of quantum gravity is not relevant. On the other hand, anomalies correspond to non-local terms which can not be removed by local counter terms and, therefore, they are important at all scales and have to be taken into account. In particular, the quantum trace anomaly (8) modifies precisely the trace of the classical equations of motion of Einstein’s theory (7), which, for conformally invariant matter, fixe the value of the curvature scalar $R = \Lambda$. In the induced $\sigma$-action, $R$ appears with derivatives and the conformal factor aquires non-trivial dynamics which in principle could distabilize the classical vacuum.

When $\bar{g}_{ab}$ is conformally flat, the effective $\sigma$-action may be derived by a completely different method, based on symmetry argument. In fact, conformal flatness implies that the fiducial metric has a set of $D + 1$ conformal Killing vectors, denoted generically by $\xi_a$, satisfying:

$$ (L\xi)_{ab} \equiv \nabla_a \xi_b + \nabla_b \xi_a - \frac{2}{D} \bar{g}_{ab} \nabla \cdot \xi = 0. \quad (16) $$

As a result, the effective $\sigma$-theory has a residual global conformal invariance, which is a remnant of the coordinate invariance of the full theory. It arises as a combination of the coordinate transformation $x^a \rightarrow x^a + \xi^a$ and the rescaling (14) with $\omega = -\frac{2}{D} \nabla \cdot \xi$, which leaves the metric $\bar{g}_{ab}$ invariant. It is now easy to show that (2)+(5) or (7)+(12) is the most general local action for $\sigma$ in two or four dimensions containing up to two or four derivatives, respectively, which is invariant under the global conformal transformations: $\delta \sigma = \xi \cdot \nabla \sigma + \frac{1}{D} \nabla \cdot \xi$. Under this transformation, although $\sigma$ transforms inhomogeneously, the conformal factor $e^\sigma$ has scaling dimension equal to unity (as does $\partial \sigma$).
Quantization of \( \sigma \)-theory

When the conformal factor is quantized, the vanishing of the total trace at the quantum level requires to find a non-trivial critical behaviour at the infrared of the effective \( \sigma \)-theory. Remarkably, the effective action (12)+(7) is ultraviolet renormalizable and the various beta functions may be studied in ordinary flat space perturbation theory. The \( \zeta \) coupling is infinitely renormalized, and contributes an \( R^2 \) term to the anomaly, proportional to its \( \beta \)-function, \( \beta_\zeta \). Such a term is forbidden by the Wess-Zumino consistency condition, implying that this \( \beta \)-function must vanish. By simple power counting, it follows that \( \zeta \)-renormalization is not affected by the interactions of (7), and a straight-forward one-loop calculation shows that \( \zeta = 0 \) is an infrared stable perturbative fixed point [4]. This condition fixes the arbitrariness present in \( \zeta \), the coefficient of the local \( R^2 \) term, which is usually present in higher-derivative theories of gravity. The induced action (12) is then given entirely by the non-local term, in analogy with the Polyakov action (5) in two dimensions. In conformal coordinates (1), it becomes:

\[
S_{\text{anom}}^{(4)} = \int \sqrt{-g}\{ -\frac{Q^2}{(4\pi)^2}[\sigma \Delta \sigma + \frac{1}{2}(\bar{D} - \frac{2}{3}[\bar{R}])\sigma + b\overline{\sigma}^2 \sigma],
\]

where \( Q^2 = -32\pi^2 b' \). Recall that \( b' \) was the coefficient of the Gauss-Bonnet term in the trace anomaly (8), which is negative definite for all known matter fields (9) corresponding to a bounded Euclidean action. This is in contrast to the two dimensional case (4), where the matter contributes to a negative kinetic term for the Liouville theory (5). This quantity has been proposed as the four dimensional analog of two dimensional central charge, \( c \), for which the Zamolodchikov theorem applies [9], namely it decreases monotonically under the renormalization group flow from an ultraviolet to an infrared fixed point. At \( \zeta = 0 \), the contribution to the anomaly coefficients \( b \) and \( b' \) of the \( \sigma \)-field itself is coming from the quartic operator \( \Delta \) (11) and was computed in ref.[5]; the results are quoted in (9).

The effective action (17) involves four derivatives of \( \sigma \) and raises the problem of unitarity, known to plague local higher derivative theories. However, recall that the Einstein action in its covariant form (7) appears to contain a negative metric scalar degree of freedom, but that this state is removed by the reparameterization constraints. In the quartic action (17) there remains in principle one additional scalar dilaton mode not cancelled by the constraints. It is instructive to consider background metrics parametrizing a general Einstein space, \( \overline{R}_{ab} = \overline{R}_4 \overline{g}_{ab} \), in which
case the quartic operator $\Delta$ factorizes into the product of two second order operators: $\Delta = \Box (\Box - \frac{R}{6})$. This shows that the quartic action may be regarded as containing a conformally coupled scalar mode from $\Box - \frac{R}{6}$ which has a negative kinetic term, and a minimally coupled scalar from $\Box$ with a positive kinetic term. The first state is identical to the one of Einstein gravity (7) and must be cancelled by the ghosts if the theory is to be unitary, in analogy with the Liouville mode in two dimensions when $d > 25$. The operator $\Box - \frac{R}{6}$ is the precise analog of the scalar Laplacian in two dimensions which contributes to the central charge like one additional scalar degree of freedom (4). The minimally coupled scalar mode with positive kinetic term has no analog in two dimensions or the Einstein theory. It makes negative the full contribution of $\Delta$ to the central charge (9) and is responsible for the unusual behavior of this theory in the infrared, as will be discussed below [4].

We now turn to the problem of the ghost, as well as the transverse spin-2 contribution to the effective theory of $\sigma$. As already mentioned in the introduction, these graviton modes will be treated as the remaining massless matter fields, which amounts to evaluate their own contribution to the anomaly coefficients. This approximation is justified if the quantum fluctuations of transverse modes can be neglected at such large distances we consider. Note that this is not the case in the quadratic approximation of the Einstein theory around de Sitter space, where it was found that the transverse, traceless part of the graviton propagator grows logarithmically at large distances. Such a behaviour results from taking the inverse of the corresponding kinetic operator $-\Box + \frac{R}{6}$. However, when the $\sigma$-equation of motion which it is associated to the trace anomaly is not used, the kinetic operator becomes $-\Box + \frac{2}{3}R - 2\Lambda$ whose inverse has not singular infrared behaviour for generic values of $\Lambda$.

Although no particular gauge choice is preferred over any other, a convenient and geometric way to compute the ghost and spin-2 contribution to the trace anomaly is by factorizing from a general metric the diffeomorphisms and Weyl rescalings, in analogy with two dimensions [3,10]:

\[
g_{ab}(x) = \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\nu}{\partial x^b} e^{2\sigma(X)} g^{\perp}_{\mu\nu}(X),
\]

where $g^{\perp}_{\mu\nu}$ denotes the transverse trace-free part. The covariant measure on the function space of metrics is defined by means of the DeWitt supermetric on this space. The change of variables from $g_{ab}$ to $\sigma$ and $g^{\perp}_{\mu\nu}$ after dividing by the volume
of diffeomorphisms, results in a Jacobian factor in the measure:

\[ J = \det' \frac{1}{2} (L \dagger L), \tag{19} \]

where \( L \dagger \) is the Hermitian adjoint of \( L \) as defined in (16). The prime in (19) indicates that the zero modes of \( L \) must be excluded from \( J \) and treated separately.

\( L \dagger L \) is the product of two operators each of which transform covariantly under a local conformal transformation:

\[ L = e^{2\sigma} T e^{-2\sigma}, \quad L \dagger = e^{-D\sigma} T \dagger e^{(D-2)\sigma}, \tag{20} \]

under the substitution (1). Using the heat kernel definition for the determinant and these transformation properties we find:

\[ -\frac{1}{2} \delta \ln \det' (L \dagger L) = \frac{1}{2} \text{Tr}' \delta \int_\epsilon^\infty ds e^{-sL \dagger L} \]
\[ = -\frac{D+2}{2} \text{Tr}' \delta \sigma e^{-\epsilon L \dagger L} + \frac{D}{2} \text{Tr}' \delta \sigma e^{-\epsilon LL \dagger}, \tag{21} \]

where the cyclic property of the trace has been used repeatedly, and the lower limit of the proper time heat kernel has been regulated by \( \epsilon \), to be taken to zero in the end. Because of the explicit appearance of \( \text{Tr}' \) over the subspace of nonzero modes of \( L \), the upper limit of the evaluation of the integral in \( s \) vanishes and only the lower limit survives in (21). Here, an essential difference from the two dimensional case manifests itself in the appearance of the tensor operator \( LL \dagger \) whose kernel is infinite dimensional, being spanned by all transverse, traceless graviton mode fluctuations. Unlike for \( D = 2 \), where the zero modes of \( LL \dagger \) are countable by the Riemann-Roch theorem, and their conformal variations may be added explicitly to (21), in \( D = 4 \) these modes cannot be “counted” without some action over the transverse, traceless degrees of freedom. Equivalently, if we exclude these modes by restriction to the non-zero mode space of \( LL \dagger \), then the conformal variation of the ghost operator \( L \dagger L \) in (21) is necessarily non-local, and violates the Wess-Zumino consistency by itself. The ghost operator cannot yield a coordinate invariant effective action unless it is combined with the action for the physical graviton modes.

Considering the Einstein action for the spin-2 degrees of freedom, we run into several difficulties. On the one-hand, this theory is not classically conformally invariant. On the other hand, loop calculations in quantum gravity, as in non-Abelian gauge theory, make sense only if the the background field equations are satisfied. However, the field equations of the Einstein theory obscure the WZ condition, since
they imply that any $R^2$ term in the anomaly is indistinguishable from $R_{ab}R^{ab}$, on shell. Another example is the Weyl-squared action, which has the advantage of being classically conformally invariant, so that the WZ condition may be checked explicitly. Moreover, the restriction to the one-loop contribution of this action is equivalent to imposing a self-duality constraint on the graviton degrees of freedom [6]. Of course, use of this higher derivative action for the graviton modes leads to perturbative non-unitarity, about which we have nothing new to add. If we simply ignore these difficulties, and perform the calculation in the weak-field limit, one finds that actually the numerical results for these two actions are not significantly different [5]:

$$
\begin{align*}
  b^\text{grav} &= \frac{1}{(4\pi)^2} \frac{611}{120}, & b^\text{grav}' &= -\frac{1}{(4\pi)^2} \frac{1411}{360} \quad \text{(Einstein)} \\
  b^\text{grav} &= \frac{1}{(4\pi)^2} \frac{199}{30}, & b^\text{grav}' &= -\frac{1}{(4\pi)^2} \frac{87}{20} \quad \text{(Weyl)}
\end{align*}
$$

This gives a positive contribution to $Q^2$ of 7.9 or 8.7 for the Einstein or Weyl theory, respectively. In either case, it is noteworthy that the total $b$ and $-b'$ coefficients are positive, and dominated by the ghost + graviton contributions, which add with the same sign as the matter contributions. This is different from the two dimensional result that the matter and ghosts contribute to the central charge with opposite sign.

### Anomalous scaling behaviour

Once the classical action (7) is added to the induced action (17), one obtains a four-dimensional analog of the Liouville theory (5)+(2). The exponential interactions of (2) for $D = 2$, or (7) for $D = 4$, are classically conformally invariant with the conformal factor $e^\sigma$ having scaling dimension equal to unity. When $\sigma$ is quantized, there is anomalous scaling behaviour which can be determined by the requirement of vanishing $\beta$-functions for the couplings of the exponential interactions. In fact, the Liouville theory is superrenormalizable and, thus, the exact scaling dimensions can be computed in ordinary perturbation theory by analyzing only a finite set of divergent diagrams. A convenient way to do the calculation is to assume that $e^\sigma$ acquires a scaling dimension $\alpha$ and to define $\sigma = \alpha \hat{\sigma}$, so that the rescaled field $e^{\hat{\sigma}}$ has weight one.
To illustrate the idea let us consider first the two-dimensional case [11]:

\[
L_{\text{eff}}^{(2)} = -\frac{Q^2}{4\pi}[(\nabla \sigma)^2 + R\sigma] - 2\lambda e^{2\sigma},
\]

where \(Q^2 = \frac{1}{12}(25 - d)\). Ordinary power counting implies that primitive divergences arise only from tadpoles, in which the coupling \(\lambda\) appears exactly once. Substituting \(\sigma = \alpha \tilde{\sigma}\), and varying the coupling \(\lambda\) with respect to some mass scale, one finds that its \(\beta\)-function is:

\[
\beta_\lambda = (2 - 2\alpha + \frac{\alpha^2}{Q^2})\lambda,
\]

where the first two terms in the r.h.s. of (24) represent the classical contribution, while the third term is the quantum contribution from the one-loop tadpole graph. The vanishing of this beta function for \(\lambda \neq 0\) yields a quadratic relation for the anomalous scaling of \(e^{\sigma}\):

\[
\alpha = \frac{1 - \sqrt{1 - \frac{2}{Q^2}}}{\frac{2}{Q^2}}, \quad D = 2
\]

where we have chosen the negative branch of the square root, so that the classical scaling \(\alpha = 1\) is obtained in the limit \(Q^2 \to \infty\). The critical exponent (25) is real only for \(Q^2 \geq 2\) corresponding to \(d \leq 1\).

Going back to four dimensions, one has [4]:

\[
L_{\text{eff}}^{(4)} = -\frac{Q^2}{(4\pi)^2}[\sigma \Box(\Box - \frac{R}{6})\sigma + \frac{R^2}{12}\sigma] + \gamma e^{2\sigma}[\nabla^2 \sigma + \frac{R}{6}] - \lambda e^{4\sigma},
\]

where \(\gamma = \frac{3}{\kappa}, \lambda = \frac{\Lambda}{\kappa}\), and for simplicity we consider as backgrounds Einstein spaces with vanishing Weyl-squared; these include maximally symmetric spaces, as de Sitter space-time. It is easy to show that the four-dimensional Liouville-like theory (26) is superrenormalizable because of the quartic propagator. In this case, primitive divergences arise not only from tadpoles involving the \(\gamma\) or \(\lambda\) couplings, but also from graphs containing two \(\gamma\)-vertices [4]. However, the renormalization of \(\gamma\)-coupling is multiplicative and arises only from tadpoles, leading to the following \(\beta\)-function:

\[
\beta_\gamma = (2 - 2\alpha + 2\frac{\alpha^2}{Q^2})\gamma,
\]

11
which is analogous to (24) in two dimensions. Its vanishing for $\gamma \neq 0$ yields the anomalous scaling dimension of the conformal factor $e^{\sigma}$:

$$\alpha = 1 - \sqrt{1 - \frac{4}{Q^2}}, \quad D = 4. \quad (28)$$

The value $Q^2 = Q_{cr}^2 = 4$ corresponds to $d = 1$ in two dimensions, where the theory could exhibit a phase transition or qualitatively new phenomena. However, it seems from (9) and (22) that the physically relevant case in four dimensions is always $Q^2 > 4$, corresponding to $d < 1$ in two dimensions. Finally, $\lambda$ coupling is not multiplicatively renormalized, but it mixes with $\gamma$ from non-tadpole diagrams. Its $\beta$-function reads:

$$\beta_\lambda = (4 - 4\alpha + \frac{8\alpha^2}{Q^2})\lambda - \frac{8\pi^2\alpha^2}{Q^4}\gamma^2(1 + \frac{4\alpha^2}{Q^2} + \frac{6\alpha^4}{Q^4}). \quad (29)$$

Since $\alpha$ has already been determined by (28), setting $\beta_\lambda = 0$ gives a non-trivial relation for the cosmological constant $\lambda$ in Planck “units” $\gamma$, which is a function of the “central charge” $Q^2$.

There is an equivalent way of deriving the scaling relation (28) by requiring that the operator, $\sqrt{-g}R$ has conformal weight equal to four. For this, one must use that the dimension of the operator $e^{p\sigma}$ is

$$[e^{p\sigma}] = p - \frac{p^2}{2Q^2}, \quad (30)$$

where $p$ is its classical value and the second term represents the quantum contribution. Once this condition has been imposed, note that one can no longer insist that the cosmological term $e^{4\alpha\sigma}$ have the same conformal weight. Instead, there is a non-trivial mixing between the $\lambda$ and $\gamma$ couplings, so that invariance can be enforced only by the relation $\beta_\lambda = 0$ in (29).

The above calculations, we performed using the short-distance behaviour of the theory in flat space, can also be carried out in the infrared around de Sitter space yielding the same $\beta$-functions. In fact, the propagator corresponding to the quartic term, $\Box(\Box - \frac{R}{6})$ is dominated by the (non unitary) $\Box^2$ term at short distances, and by the (unitary) $-\frac{R}{6}\Box$ term at large distances. However, in de Sitter space [2],

$$\Box_{x\neq x'}^{-1} = -\frac{1}{8\pi^2} \frac{2}{s^2(x, x')} - H^2 \ln\left(\frac{H^2s^2(x, x')}{4}\right), \quad \Box = 12H^2. \quad (31)$$
where \( s(x, x') \) is the geodesic distance between the two points. Hence, the quartic propagator has the same logarithmic behavior in both limits. This is not surprising, as it is known from critical phenomena that there is a close interplay between ultraviolet and infrared behavior in systems with conformal symmetry.

We now observe that if the expectation value of the Ricci scalar is different from zero, then the global conformal symmetry must be spontaneously broken. In fact in the semi-classical limit, when the anomaly induced fluctuations are suppressed by \( \frac{1}{Q^2} \) for large \( Q^2 \), the dimension of the conformal factor \( \alpha = 1 \) and the weight of \( R \) is zero, i.e. it transforms like a scalar under global conformal transformations. However, this is not the case at the non-trivial fixed point (28), where \( \alpha \neq 1 \) and the conformal weight of \( R \) is not zero:

\[
[R] = -2\alpha - \frac{2\alpha^2}{Q^2} + 2 = 4(1 - \alpha),
\]

where we used (30). This implies that \( \langle R \rangle \) becomes an order parameter for the spontaneous breaking of global conformal symmetry, in sharp contrast to the classical situation in which \( \langle R \rangle \) can take on any value consistent with the symmetry. As a result, the cosmological “constant” problem reduces to the question of whether this symmetry remains spontaneously broken, or is restored in the quantum theory.

We may compare this case to that of spontaneous breaking of a continuous symmetry in two dimensions [12]. Consider a complex scalar field \( \phi(x) \) with a tree-level potential giving rise to symmetry breaking. When the field is quantized, the corresponding massless Goldstone boson has a propagator which grows logarithmically at large distances. This infrared divergence implies instability of the spontaneously broken vacuum due to quantum fluctuations. Because of this instability of the ordered state, the system becomes disordered and the \( U(1) \) symmetry is restored at the quantum level. Locally, there are regions of broken symmetry in which the classical description remains valid. However, as we consider regions of larger and larger size, the classical description breaks down and the average expectation value vanishes. The quantitative description of this phenomenon is given by the power law fall-off of the correlation function \( \langle \phi(x)\phi(0) \rangle \). Introducing the nonlinear polar field decomposition \( \phi = \rho e^{i\theta} \) and neglecting the fluctuations of the massive \( \rho \) field in the infrared, one finds that the angular field \( \theta \) may be treated as a free field with the propagator:

\[
\langle \theta(x)\theta(0) \rangle = -\frac{1}{4\pi \rho^2} \ln(\mu^2 x^2).
\]
In this infrared scaling limit, the correlation function for \( \phi \) has a power law behavior:

\[
\langle \phi(x)\phi^\dagger(0) \rangle \sim \rho^2 \langle e^{i\theta(x)} e^{-i\theta(0)} \rangle \\
\sim \rho^2 e^{\langle \theta(x)\theta(0) \rangle} \sim \rho^2 |x|^{-\frac{1}{2\pi\rho^2}}. \tag{34}
\]

Consider now by analogy the correlation function of Ricci scalars, \( \langle R(x)R(x') \rangle \) at two different points. Using (1) with \( \sigma \) replaced by \( \alpha \sigma \), and the \( \sigma \)-propagator from (26) and (31), we find at large distances \( |s(x,x')| \to \infty \):

\[
\langle R(x)R(x') \rangle \sim H^4 \langle e^{-2\alpha \sigma(x)} e^{-2\alpha \sigma(x')} \rangle \\
\sim H^4 e^{4\alpha^2 \langle \sigma(x)\sigma(x') \rangle} \sim H^4 |Hs(x,x')|^{-4 \alpha^2 Q^2}, \tag{35}
\]

where only the dominant infrared behavior has been retained. The result (35) states that the effective cosmological “constant” goes to zero with a definite power law behavior for large distances. In other words, there is screening in the infrared of the effective value of vacuum energy at larger and larger scales. In contrast with the 2D case (34), the value of the power is universal, depending only on \( Q^2 \) which counts the effective number of massless degrees of freedom. In particular, it depends neither on the classical value of the background curvature \( R \), nor on the Planck scale. This is essential for a scale invariant, phenomenologically acceptable solution of the cosmological constant problem.

To summarize, we believe that the effective theory of the conformal factor presented here provides a useful framework for studying the infrared behavior of gravity in four dimensions and addressing the cosmological constant problem. The anomalous scaling of the conformal factor may be the key to understanding why \( \langle R \rangle = 0 \) in the observed universe. However, many unanswered questions and open problems remain. An explicit proof of the unitarity of the \( \sigma \)-theory when one applies the diffeomorphism constraints, an explicit verification of the approximation of neglecting the contribution of transverse spin-2 modes in the infrared, a better understanding of the correlation functions and their scaling behavior at large distances, and observational implications to cosmology, large scale structure and the missing matter puzzles.

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