The scaling behavior of logarithmic fidelity in quantum phase transition in LMG model

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In this paper, we explore the differences between classical logarithmic fidelity and quantum fidelity. The classical logarithmic fidelity is found to be always extensive while the quantum one manifests distinct size dependence in different phases. Illustrated by the anisotropic Lipkin-Meshkov-Glick model, we found numerically and analytically that the logarithmic fidelity scales like $N$ in the symmetry-broken phase and scales like $N^0$ in the polarized phase. The singular behavior around the critical point is also investigated.

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I. INTRODUCTION

Fidelity1 is an information concept used to measure the similarity of an input and its corresponding output (classical or quantum) states of a channel. In quantum information theory, the fidelity concept is hardly related to the size dependence because information science usually focuses on quantum states (or information) carried by few-body systems. Therefore, none of previous works on the fidelity in information science, as far as we know, paid attention to the scaling behavior of the fidelity until recent application of fidelity in quantum phase transitions(QPTs)2.

Up to now, a lot of works 3 connecting QPTs and the fidelity have been done. Traditionally, a QPT occurs when there is a significant change of the system’s physical quantity. But of the quantum ground state concern, the fidelity measures the similarity between two quantum states differed by a certain fixed value of a driving parameter. When these two observations come together, the QPT can be observed as long as the fidelity drops to zero. It can be understood as the state of the system undergoes structural change from one phase to another, such that states from two phases are orthogonal to each other.4-6 Such an interesting idea means the QPTs can be characterized by solely the quantum state itself, without a priori knowledge to the symmetry and order of the system.

Since then, various fidelity-related measures have been put forward, including the fidelity per site 7, the fidelity susceptibility 8,9, thermal-state fidelity 8,9,15,18, operator fidelity 10, and density-functional fidelity 11, which can be applied under different circumstances. Remarkably, researches have been focusing on the scaling behavior of the above fidelity measures, as the second-order QPTs occur only in the thermodynamics limit 12,13. While the idea of critical exponents in physical quantities has been well-established 14, the above findings have also successfully related the critical exponent of the fidelity to those of the physical quantities, making the fidelity more physically meaningful.

However, it seems to us that the logarithmic fidelity, is presumably regarded an extensive quantity in relevant studies. Such an idea is true for thermal state. As we will show below, the logarithmic fidelity of two thermal states is proportional to the Helmholtz free energy, which is an extensive quantity for a thermodynamic system. While dramatically the logarithmic fidelity of two quantum states have distinct size dependence in different phases, which is believed to be caused by various quantum correlations. To explore the difference between the thermal-state logarithmic fidelity and the ground-state logarithmic fidelity is the key motivation of the present work.

In this paper, we show that the scaling dependence of the logarithmic fidelity may not always be universal for a thermodynamic system. The thermal-state logarithmic fidelity is extensive due to the extensibility of the Helmholtz free energy. While the quantum ground-state logarithmic fidelity might be either intensive, extensive, or superextensive, depending on the quantum adiabatic dimension of the ground state. We take the anisotropic Lipkin-Meshkov-Glick model (LMG model) 21 as an example and show that the logarithmic fidelity scales like $N$ in the model’s symmetry-broken phase and $N^0$ in the polarized phase.

This paper is organized as follows: In Sec. II, we introduce the definition of fidelity and logarithmic fidelity for both thermal state and quantum state, then explain why fidelity has different scaling behavior for both cases. In Sec. III, we take the LMG model as an example, and explicitly show that the logarithmic fidelity scales like $N$ in the symmetry broken phase and $N^0$ in the polarized phase. In Sec. IV, we study numerically the critical properties of the fidelity per site around the transition point. Finally, our conclusions are given in Sec. V.

II. THERMAL-STATE FIDELITY AND QUANTUM-STATE FIDELITY

Let a quantum-many system be characterized by a pure state $|\Psi(h)\rangle$ labeled the continuous variable $h$. In describing a QPT, the continuous variable is the driving parameter that induces the QPT, very often it is the interaction strength, or external field strength. The project between two states of different values of the variable $h$ and $h'$ can be measured by the fidelity, which is defined as

$$F(h, h') = |\langle\Psi(h)|\Psi(h')\rangle|.$$  (1)

The fidelity is zero if two states are orthogonal, one if identical.
In Ref. [6], the authors proposed the logarithm of the fidelity, namely the fidelity per site to describe QPTs. It is given by

\[ d(h, h') = \lim_{N \to \infty} F(h, h')^{\frac{1}{N}}, \]  

(2)

\[ \ln d(h, h') = \lim_{N \to \infty} \frac{1}{N} \ln F(h, h'), \]  

(3)

in which \( N \) is the system size. Usually, fidelity scale like \( d^N \) for large scale \( N \). Fidelity per site give a measurement of fidelity that is independent of the system size.

The extension of fidelity to a mixed-state or matrix-product state makes use of the density matrix \( \rho(h) \) of the system, it is defined as

\[ F(h, h') = \text{tr} \sqrt{\rho^{1/2}(h) \rho(h') \rho^{1/2}(h)}. \]  

(4)

If two mixed states are diagonal in the same set of basis, the fidelity is the trace of the product of the density matrices \( F(h, h') = \text{tr} \rho(h) \rho(h') \). Such a definition allows description of classical phase transitions, as long as the thermal state of the system is obtained.

At finite temperatures, a thermal state is described by the density matrix

\[ \rho(h) = \frac{1}{Z} \sum_n e^{-\beta E_n} |\Psi_n\rangle \langle \Psi_n| \]  

(5)

where \( \beta = T^{-1} \) denotes the inverse temperature, and the Boltzmann constant \( k \) is set to be one, \( Z \) is the partition function of the system, i.e. \( Z(\beta) = \sum_n e^{-\beta E_n} \), and \( |\Psi_n\rangle \) is the energy eigenstate of the system’s Hamiltonian. The thermal-state fidelity in the parameter space of temperature (and thus \( \beta \)) is of the form

\[ F = \frac{Z(\beta_a, \beta_b)}{\sqrt{Z(\beta_1) Z(\beta_2)}}, \]  

(6)

with different temperature \( T_i = \beta_i^{-1} \).

Let us take the logarithm of the thermal-state fidelity,

\[ \ln F = \ln Z\left(\frac{\beta_1 + \beta_2}{2}\right) - \frac{1}{2} \ln Z(\beta_1) - \frac{1}{2} \ln Z(\beta_2). \]  

(7)

Notice that

\[ G = -\frac{1}{\beta} \ln Z \]

is the Helmholtz free energy, which is an extensive quantity, thus logarithmic fidelity for thermal state should also be an extensive quantity. An other useful quantum quantity, the fidelity susceptibility \( \chi_F \) was also derived to be equal to \( C_T/4\beta^2 \) [8,9], that is proportional to \( N \).

On the other hand, the quantum-state logarithmic fidelity leads by the fidelity susceptibility [27],

\[ \ln F(h_1, h_2) = -\frac{(h_1 - h_2)^2 \chi_F}{2} + \cdots, \]

if \( h_1 \) and \( h_2 \) are close to each other. The ground-state fidelity susceptibility has its own quantum adiabatic dimension in different phases, while the thermal-state logarithmic is lead by extensive term. Therefore, we speculate that the quantum-state logarithmic fidelity might not be always extensive. In the following section, we are going to use the anisotropic LMG model as an example to demonstrate the size dependence of logarithmic fidelity in both phases.

### III. THE LOGARITHMIC FIDELITY IN THE ANISOTROPIC LMG MODEL

The Hamiltonian of the LMG model reads

\[ H_{\text{LMG}} = -\frac{1}{N} \sum_{i<j} \left( \sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y \right) - h \sum_j \sigma_j^z \]

\[ = -\frac{1}{N} (1 + \gamma) (S^2 - S_z^2 - N/2) - 2hS_z \]

\[ = -\frac{1}{2N} (1 - \gamma) (S_z^2 + S_z^2), \]

(8)

where \( \sigma_x (x, y, z) \) are the usual Pauli matrices, \( S_x = \sum_j \sigma_j^x/2 \) the collective operator, \( \gamma \leq 1 \) denotes the anisotropy parameter, and \( h \) is the external magnetic field. In its isotropic case \( (\gamma = 1) \), it undergoes a first order QPT (ground state level crossing) at \( |h| = 1 \). In our discussion, anisotropic case is concerned. In anisotropic case \( (\gamma \neq 1) \), a second order QPT at \( h = 1 \). The ground state of the system falls on \( S^2 = \frac{N}{2} \left( \frac{N}{2} + 1 \right) \).

The LMG model can be used to describe the Bose-Einstein condensate and Josephson junctions. The spectrum of the LMG model is recently reviewed by some sophisticated analytic method, including the continuous unitary transformation [22] and the spin coherent state formalism [23,24], these enriched the understanding of the model, and extended the study to a wide range of parameter values.

To study the behavior of logarithmic fidelity of the LMG model, we are going to discuss the polarized phase and symmetry-broken phase seperately.

#### A. Polarized phase \((h > 1)\)

In polarized phase, the usual treatment to obtain the solution of groundstate is by firstly, mapping the collective spin operator \( S \) into bosonic operators \( a \) and \( a^\dagger \) by Holstein-Primakoff transformation. Then \( a \) and \( a^\dagger \) are mapped into another pair of bosonic operator \( b \) and \( b^\dagger \) by standard Bogoliubov transformation, i.e.,

\[ b = \cosh \theta \frac{a}{2} - \sinh \theta \frac{a^\dagger}{2} \]

\[ b^\dagger = \cosh \theta \frac{a^\dagger}{2} - \sinh \theta \frac{a}{2}. \]

(9)

After adjusting the parameter \( \theta \), the Hamiltonian is diagonalized to the following form while \( \theta = \tanh^{-1} \left( \frac{1 - \gamma}{2\beta - 1 - \gamma} \right) \).
\[ H = \hbar(N + 1) + 2\sqrt{(h - 1)(h - \gamma)}\left(b^\dagger b + \frac{1}{2}\right) \]  
\[ \text{(10)} \]

The eigenstates are \(|n\rangle_b\), therefore the groundstate is \(|0\rangle_b\). We are interested in the fidelity \(F(h, h') = |b(0(h)[0(h')]_b\|

As its independence of system size, it is meaningless to measure \(\ln F\) for its logarithmic fidelity always tends to zero while system size increases. Therefore, for the groundstate we can start with the ground state

\[ b|\psi_0\rangle_b = 0 \]

such that the eigenstates are \(|n\rangle_b\), and \(b^\dagger b|n\rangle_b = n|n\rangle_b\). Therefore, we can start with the ground state

\[ b|\psi_0\rangle_b = 0 \]

where the relation between \(a\) and \(b\) is given by Bogoliubov transformation that

\[ b = \cosh(\theta/2)a - \sinh(\theta/2)b \]

where \(\theta\) is a function of \(h\). It is possible to express the ground state as the combination of \(|m\rangle_a\), which is the eigenstates of \(a^\dagger a\), where \(a^\dagger a|m\rangle_a = m|m\rangle_a\), i.e.,

\[ |\psi_0\rangle_b = \sum_{m=0}^{\infty} c_m m|m\rangle_a, \]

where \(c_m\) is the coefficient and it is then solved and normalized by \(|\langle \Psi_0|\Psi_0\rangle_b = 1\). The result shows,

\[ \begin{align*}
    c_{2k} &= (1 - \tanh^2(\theta/2))^{1/4} \tanh(\theta/2) \sqrt{2k'/\varphi(k')}, \\
    c_{2k+1} &= 0
\end{align*} \]

where \(k = 0, 1, 2, 3, \ldots\), with

\[ \tanh[\theta(h > 1)] = \frac{1 - \gamma}{2h - 1 - \gamma}. \]

The analytical form of fidelity between \(h\) and \(h'\) can be obtained as

\[ F(h, h') = \frac{(1 - \tanh^2(\theta'/2))^{1/4}(1 - \tanh^2(\theta/2))^{1/4}}{\sqrt{(1 - \tanh(\theta/2)\tanh(\theta'/2))}}, \]

where \(\theta = \theta(h)\) and \(\theta' = \theta(h')\). Fidelity here is an intensive value, by taking Log, the quantum logarithmic fidelity is

\[ \ln F(h, h') = \frac{1}{4}\ln\left(1 - \tanh^2(\theta/2)\right) + \frac{1}{4}\ln\left(1 - \tanh^2(\theta'/2)\right) - \frac{1}{2}\ln\left(1 - \tanh(\theta/2)\tanh(\theta'/2)\right). \]

Therefore, agreeing with the numerical result, the logarithmic fidelity is independent of the system size in the polarized phase. With the expression of fidelity, we recover the fidelity susceptibility in Ref. [25] by taking the second derivative of fidelity between \(h\) and \(h + \delta h\) with respect to \(\delta h\) for \(\delta h\) approaching 0. It gives the result of

\[ \frac{d^2F(h, h + \delta h)}{d\delta h^2} \bigg|_{\delta h \to 0} = \frac{(1 - \gamma)^2}{32(h - 1)^2(h - \gamma)^2}. \]

B. Symmetry-broken phase \((h < 1)\)

In the following, we are going to derive the \(N\) dependence of logarithmic fidelity for \(h < 1\). In semi-classical
FIG. 2: (Color online) The dependence of logarithm of fidelity to the system size of the LMG model in the symmetric-broken phase. Solid lines represent the analytical solution and dots represent the numerical results.

FIG. 3: (Color online) The logarithmic fidelity as a function of h for various systems sizes. Here the fixed point set at h = 1.1.

c1 = \frac{\Omega}{\Lambda} \cdot \frac{\sqrt{N}}{2} \Theta_i, \quad (16)

\Omega = \frac{1}{2} \left( \cosh \frac{\theta_i}{2} - \sin \frac{\theta_i}{2} \right) \sin(\alpha_1 - \alpha_2), \quad (17)

\Lambda = \frac{1}{2} \left( \cosh(\alpha_1 - \alpha_2) + 1 \right) \cosh \frac{\theta_i}{2} - \frac{1}{2} \left( \cosh(\alpha_1 - \alpha_2) - 1 \right) \sin \frac{\theta_i}{2}, \quad (18)

\Theta = \frac{1}{2} \left( \cosh(\alpha_1 - \alpha_2) - 1 \right) \cosh \frac{\theta_i}{2} - \frac{1}{2} \left( \cosh(\alpha_1 - \alpha_2) + 1 \right) \sin \frac{\theta_i}{2}, \quad (19)

and,

\tanh \left[ \theta_i(h < 1) \right] = \frac{h_i^2 - \gamma}{2 - h_i^2 - \gamma},

for \( i = 1, 2 \).

Now, we experienced in LMG model that, logarithm of fidelity is of different dependence of system size in two phases. In other words, the logarithmic fidelity is extensive in broken phase but intensive in symmetric phase. As we obtained in the first page that the logarithm of fidelity in thermal states is very close to zero and exhibits itself intensively. However extensive behavior emerges once \( h \) is smaller than the...
The scaling function in the following form:

\[ F(h, h') = \delta(h, h'). \]

This phenomenon is known as the Anderson orthogonal catastrophe. A tiny change of \( h \) makes a entirely orthogonal states.

The first derivative of logarithmic fidelity in broken phase shows a minimum occurs as quasi-critical point, \( h_{\text{min}} \), which approaches one as

\[ h_{\text{min}} = 1 - 1.0459 N^{-0.6097}, \]

as shown in the small graph of Fig. 6. It simply projects to \( h_c = 1 \) as the critical point of LMG model. In Fig. 6 we examined the critical exponent of \( \frac{d}{dh} \ln F(h, h') \). We analyzed the scaling function in the following form:

\[ 1 - \exp \left( \frac{d}{dh} \ln F \right) = f \left[ N^\nu (h - h_{\text{min}}) \right], \]

the parameter \( \nu \) is the best adjusted when curves of different size overlap. It gives \( \nu \sim 0.65 \), which is close to the accepted value \( \nu = \frac{1}{4} \).

The minima of first derivative of logarithmic fidelity shows slow divergence. It is less than \( N \ln N \) but greater than \( \ln N \).

V. SUMMARY

We first reviewed the classical logarithmic fidelity in phase transitions, and showed the logarithmic fidelity is actually the Helmholtz Free Energy, which is an extensive quantity. Then
we analyzed the LMG model numerically, and discovered the quantum logarithmic fidelity scales as $N$ in the broken phase and $N^0$ in the symmetric phase. Such a change is not observed in classical systems, and we believe it is a quantum effect.

This finding leaves a question to the definition of fidelity per site, where “per site” may occasionally not hold. Recent papers concerning the infinitesimal change of parameter in fidelity, taking its leading term, the fidelity susceptibility shows similar behavior. In Ref. [25], the authors gave analytic calculation of the fidelity susceptibility and showed the different critical exponent in two phases of the LMG model. They also confirmed with the scaling ansatz in which the critical exponent is related to the scaling dependence of the system. On the other hand, for the topological phase transition of the Kitaev model, the fidelity susceptibility also exhibits different scaling dependence in two phases [26]. The difference is even more interesting, the fidelity susceptibility scales like $L^2$ in one phase and $L^2 \ln L$ in another, it changes from an extensive quantity to a superextensive quantity. In Ref. [27], the extra $L \ln L$ dependence is considered as the characteristic in the topological QPT.

VI. ACKNOWLEDGEMENT

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