Almost uniform convergence in the Wiener–Wintner ergodic theorem

by

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Abstract. We extend almost everywhere convergence in the Wiener–Wintner ergodic theorem to a generally stronger almost uniform convergence and, in the case of infinite measure, we present a universal space for which this convergence holds. We then extend this result to the case with Besicovitch weights.

1. Introduction. Let $(\Omega, \mu)$ be a measure space. Denote by $\mathcal{L}^0$ the algebra of almost everywhere (a.e.) finite complex-valued measurable functions on $(\Omega, \mu)$, and let $\mathcal{L}^p \subset \mathcal{L}^0$, $1 \leq p \leq \infty$, stand for the $L^p$-space on $(\Omega, \mu)$ equipped with the standard norm $\| \cdot \|_p$.

**Definition 1.1.** A sequence $\{f_n\} \subset \mathcal{L}^0$ is said to converge almost uniformly (a.u.) if there is $\hat{f} \in \mathcal{L}^0$ such that, given $\varepsilon > 0$, there exists $\Omega' \subset \Omega$ satisfying

$$
\mu(\Omega \setminus \Omega') \leq \varepsilon \quad \text{and} \quad \lim_{n \to \infty} \| (\hat{f} - f_n) \chi_{\Omega'} \|_{\infty} = 0.
$$

It is clear that $\{f_n\} \subset \mathcal{L}^0$ converges a.u. if and only if for every $\varepsilon > 0$ there exists $\Omega' \subset \Omega$ such that

$$
\mu(\Omega \setminus \Omega') \leq \varepsilon \quad \text{and} \quad \{f_n \chi_{\Omega'}\} \text{ converges in } \mathcal{L}^\infty,
$$

that is, this sequence converges uniformly.

In view of Egorov’s theorem, a.u. convergence and a.e. convergence coincide when $\mu(\Omega) < \infty$. But in the case $\mu(\Omega) = \infty$, a.u. convergence is generally stronger. For example, if $\Omega$ is the interval $[0, \infty)$ equipped with Lebesgue measure and $f_n = \chi_{[0,n]}$, $n \in \mathbb{N}$, then $f_n(\omega) \to 1$ for all $\omega \in \Omega$, whereas $\{f_n\}$ clearly does not converge a.u.
It is worth mentioning here that, in the noncommutative case, the notion of a.u. convergence (in Egorov’s sense) was introduced by Lance [Lan76] to prove an individual ergodic theorem for an automorphism $\alpha$ on a von Neumann algebra equipped with a faithful $\alpha$-invariant normal state. In the case of a semifinite von Neumann algebra $\mathcal{M}$, a similar bilaterally a.u. (b.a.u.) convergence was considered by Yeadon [Ye76] to establish an individual ergodic theorem in the noncommutative $L^1$-space $L^1(\mathcal{M})$. More recently, Junge and Xu [JX07] discussed a.u. and b.a.u. convergences of ergodic averages acting in the noncommutative spaces $L^p(\mathcal{M})$, $1 < p < \infty$. We note that if $\mathcal{M} = L^\infty(\Omega)$, both a.u. and b.a.u. convergences coincide with the one given in Definition 1.1.

Let $T : \Omega \to \Omega$ be a measure preserving transformation (m.p.t.). Given $\lambda \in \mathbb{C}_1 = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$ and $f \in L^0$, denote

$$M_n(T, \lambda)(f) = \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k f \circ T^k \quad \text{and} \quad M_n(T)(f) = \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k.$$

**Definition 1.2.** We write $f \in a.e.WW(\Omega, T)$, resp. $f \in a.u.WW(\Omega, T)$, if

$\exists \Omega_f \subset \Omega$ with $\mu(\Omega \setminus \Omega_f) = 0$ such that

$\{M_n(T, \lambda)(f)(\omega)\}$ converges for any $\omega \in \Omega_f$ and $\lambda \in \mathbb{C}_1$, resp.

$\forall \varepsilon > 0 \exists \Omega' = \Omega_{f,\varepsilon}$ with $\mu(\Omega \setminus \Omega') \leq \varepsilon$ such that

$\{M_n(T, \lambda)(f)\chi_{\Omega'}\}$ converges uniformly for any $\lambda \in \mathbb{C}_1$.

Clearly, we have $a.u.WW \subset a.e.WW$, but even if $\mu(\Omega) < \infty$, Egorov’s theorem does not entail the opposite inclusion. As an example, consider the set $\Omega = [0, 1]$ with Lebesgue measure $\nu$. Given $t \in [0, 1]$, define

$$f_{n,t}(\omega) = \begin{cases} \frac{1}{n(\omega-t)} & \text{if } \omega \neq t, \\ 0 & \text{if } \omega = t, \end{cases}$$

for $n = 1, 2, \ldots$. Then $\lim_{n \to \infty} f_{n,t}(\omega) = 0$ for all $\omega \in \Omega' = \Omega$ and $t \in [0, 1]$ (cf. first half of Definition 1.2). Now, pick $\varepsilon \in (0, 1)$ and let $E$ be any subset of $\Omega$ such that $\nu(\Omega \setminus E) \leq \varepsilon$. Since $\nu(E) > 0$, the set $E$ is infinite, thus has an accumulation point $t_0 \in \Omega$, and it is easy to see that the sequence $f_{n,t_0\chi_E}$ does not converge uniformly. Therefore, the property

$\forall \varepsilon > 0 \exists \Omega' \subset \Omega, \mu(\Omega \setminus \Omega') \leq \varepsilon$, \quad \text{such that}$

$\{f_{n,t\chi_{\Omega'}}\}$ converges uniformly $\forall t \in [0, 1]$ fails to hold (cf. second half of Definition 1.2).

The celebrated Wiener–Wintner theorem [WW41] asserts that $L^1 \subset a.e.WW$ provided $\mu(\Omega) < \infty$. It is shown in [As03, Theorem 2.10] that for a uniquely ergodic system, that is, when $\mu$ is the only invariant measure
for $T$, and a continuous function the convergence in the Wiener–Wintner theorem is uniform in $\Omega$; a related result was previously obtained in [Ro94]. For a review of results on uniform convergence in Wiener–Wintner-type ergodic theorems for uniquely ergodic systems and continuous functions, see [EZ13].

Furthermore, Assani’s extension of Bourgain’s return times theorem [As03, Theorem 5.1] entails that if $(\Omega, \mu)$ is $\sigma$-finite, then $L^p \subset a.e. \text{WW}$, $1 \leq p < \infty$.

Now, let $(\Omega, \mu)$ be a $\sigma$-finite measure space, and let $T : \Omega \to \Omega$ be a m.p.t. The main goal of this article is to show that if $T$ is ergodic, then $\mathcal{R}_\mu \subset a.u. \text{WW}(\Omega, T)$. Here,

$$\mathcal{R}_\mu = \{ f \in L^1 + L^\infty : \mu\{|f| > \lambda\} < \infty \text{ for all } \lambda > 0 \},$$

which coincides with $L^1$ if $\mu(\Omega) < \infty$, is a universal space, relative to a.u. convergence of the averages $M_n(T, \lambda)$, that contains not only every space $L^p$ for $1 \leq p < \infty$ but also classical function Banach spaces $X$ on $(\Omega, \mu)$ such as Orlicz, Lorentz, and Marcinkiewicz spaces if $\chi_\Omega \notin X$. Thus, by relaxing uniform convergence to almost uniform convergence, we gain convergence for a much wider class of functions than the class of continuous functions and without the assumption of finiteness of measure. Then we further generalize this result by expanding the family $\{\{\lambda^k\} : \lambda \in C_1\}$ to the class of all bounded Besicovitch sequences.

2. Preliminaries. In what follows, we reduce the problem to showing that $L^1 \subset a.u. \text{WW}$, which in turn can be derived from the case $\mu(\Omega) < \infty$ with the help of Hopf decomposition. As in the classical case, Corollary 2.2 below further reduces the problem to finding a set $\mathcal{D} \subset a.u. \text{WW}$ that is dense in $L^1$. To this end, we take the path of “simple inequality” as outlined in [As03] and employ Egorov’s theorem and a form of van der Corput’s inequality to show that $\mathcal{D} = L^2 \subset a.u. \text{WW}$.

For $\bar{b} = \{b_k\}_{k=0}^\infty \subset C$, denote

$$M_n(T, \bar{b})(f) = \frac{1}{n} \sum_{k=0}^{n-1} b_k f \circ T^k.$$

Let $\mathcal{B}$ be a subset of the set of bounded sequences $\bar{b} = \{b_k\}_{k=0}^\infty \subset C$.

**Proposition 2.1.** Let $(\Omega, \mu)$ be $\sigma$-finite, and let $1 \leq p < \infty$. Then the set

$$C_p(\mathcal{B}) = \{ f \in L^p : \forall \varepsilon > 0 \exists \Omega' \subset \Omega \text{ with } \mu(\Omega \setminus \Omega') \leq \varepsilon \text{ such that } \{M_n(T, \bar{b})(f)\chi_{\Omega'}\} \text{ converges uniformly } \forall \bar{b} \in \mathcal{B} \}$$

is closed in $L^p$. 
\begin{proof}

Given \( l \in \mathbb{N} \), denote
\[
\mathcal{B}_l = \{ \{ b_k \} \in \mathcal{B} : |b_k| \leq l \ \forall k \}.
\]

Let \( \{ f_k \} \subset \mathcal{C}_p(\mathcal{B}) \) and \( f \in \mathcal{L}^p \) be such that \( \| f - f_k \|_p \to 0 \). Given \( \varepsilon > 0 \) and \( \delta > 0 \), the maximal ergodic inequality
\[
\mu \left\{ \sup_n M_n(T)(|g|) > \frac{\varepsilon}{2l+1} \right\} \leq \left( \frac{2\|g\|_p}{t} \right)^p \quad \forall g \in \mathcal{L}^p, \ t > 0
\]
(see, for example, [CL19]) entails that, given \( l \in \mathbb{N} \), there exists \( f_{k_l} \) for which
\[
(2.1) \quad \mu \left\{ \sup_n M_n(T)(|f - f_{k_l}|) > \frac{\delta}{3l} \right\} \leq \frac{\varepsilon}{2l+1}.
\]

Next, as
\[
|M_n(T, \bar{b})(f - f_{k_l})| \leq lM_n(T)(|f - f_{k_l}|) \quad \forall \bar{b} \in \mathcal{B}_l,
\]
inequality \((2.1)\) implies that
\[
\mu \left\{ \sup_n |M_n(T, \bar{b})(f - f_{k_l})| > \frac{\delta}{3} \right\} \leq \frac{\varepsilon}{2l+1} \quad \forall \bar{b} \in \mathcal{B}_l.
\]

Therefore, with
\[
\Omega_{l,1} = \left\{ \sup_n |M_n(T, \bar{b})(f - f_{k_l})| \leq \frac{\delta}{3} \right\},
\]
we have \( \mu(\Omega \setminus \Omega_{l,1}) \leq \varepsilon/2l+1 \) and
\[
\|M_n(T, \bar{b})(f - f_{k_l})\chi_{\Omega_{l,1}}\|_\infty \leq \delta/3 \quad \forall n \in \mathbb{N}, \ \bar{b} \in \mathcal{B}_l.
\]

Now, letting \( \Omega_1 = \bigcap_{l=1}^{\infty} \Omega_{l,1} \), we obtain \( \mu(\Omega \setminus \Omega_1) \leq \varepsilon/2 \) and
\[
\|M_n(T, \bar{b})(f - f_{k_l})\chi_{\Omega_1}\|_\infty \leq \delta/3 \quad \forall n, l \in \mathbb{N}, \ \bar{b} \in \mathcal{B}.
\]

Furthermore, as \( f_{k_1} \in \mathcal{C}(\mathcal{B}) \), there exists \( \Omega_2 \subset \Omega \) with \( \mu(\Omega \setminus \Omega_2) \leq \varepsilon/2 \) such that \( \{M_n(T, \bar{b})(f_{k_1})\chi_{\Omega_2}\} \) converges uniformly for all \( \bar{b} \in \mathcal{B} \). Thus, for each \( \bar{b} \in \mathcal{B} \), there is a number \( N = N(\bar{b}) \) such that
\[
\|(M_m(T, \bar{b})(f_{k_1}) - M_n(T, \bar{b})(f_{k_1}))\chi_{\Omega_2}\|_\infty \leq \frac{\delta}{3} \quad \forall m, n \geq N.
\]

Now, setting \( \Omega' = \Omega_1 \cap \Omega_2 \), we have \( \mu(\Omega \setminus \Omega') \leq \varepsilon \) and, for each \( \bar{b} \in \mathcal{B} \) and all \( m, n \geq N(\bar{b}) \),
\[
\|(M_m(T, \bar{b})(f) - M_n(T, \bar{b})(f))\chi_{\Omega'}\|_\infty \\
\leq \|M_m(T, \bar{b})(f - f_{k_1})\chi_{\Omega_1}\|_\infty \\
+ \|(M_m(T, \bar{b})(f_{k_1}) - M_n(T, \bar{b})(f_{k_1}))\chi_{\Omega_2}\|_\infty \\
+ \|M_n(T, \bar{b})(f - f_{k_1})\chi_{\Omega_1}\|_\infty \leq \delta,
\]
implying that \( \{M_n(T, \bar{b})(f)\chi_{\Omega'}\} \) converges uniformly for all \( \bar{b} \in \mathcal{B} \), hence \( f \in \mathcal{C}_p(\mathcal{B}) \).  \( \blacksquare \)
Corollary 2.2. Let $(\Omega, \mu)$ be $\sigma$-finite. Then $L^p \cap a.u.WW$ is closed in $L^p$ for each $1 \leq p < \infty$.

Proof. Apply Proposition 2.1 to $B = \{ b = \{ \lambda^k \} : \lambda \in \mathbb{C}_1 \}$. ■

Next, let $K$ be the $\| \cdot \|_2$-closure of the linear span of the set

$$K = \{ f \in L^2 : f \circ T = \lambda f \text{ for some } \lambda \in \mathbb{C}_1 \}.$$

Proposition 2.3. $K \subset a.u.WW(\Omega, T)$.

Proof. By Corollary 2.2, it is sufficient to show that $\sum_{j=1}^m z_j f_j \in a.u.WW$ whenever $z_j \in \mathbb{C}$ and $f_j \in K$ for all $1 \leq j \leq m$. This, in turn, will easily follow from $K \subset a.u.WW$.

So, pick $f \in K$ and $\varepsilon > 0$. Then there exists $\Omega'$ such that $\mu(\Omega \setminus \Omega') \leq \varepsilon$ and $f \chi_{\Omega'} \in L^\infty$. In addition, given $\lambda \in \mathbb{C}_1$, we have

$$M_n(T, \lambda)(f) \chi_{\Omega'} = \frac{1}{n} \sum_{k=0}^{n-1} (\lambda \lambda f)^k \cdot f \chi_{\Omega'}.$$ 

Therefore, since the sequence $\{ \frac{1}{n} \sum_{k=0}^{n-1} (\lambda \lambda f)^k \}$ converges in $\mathbb{C}$, we conclude that the averages $M_n(T, \lambda)(f) \chi_{\Omega'}$ converge uniformly for any $\lambda \in \mathbb{C}_1$, hence $f \in a.u.WW$. ■

3. The case of finite measure. Let $(\Omega, \mu)$ be a finite measure space, and let $T$ be a m.p.t. If $Uf = f \circ T$, $f \in L^0$, then $U : L^2 \to L^2$ is a surjective linear isometry with $U^* = U^{-1}$. Given $f, g \in L^2$, denote $(f, g) = \int_{\Omega} f \overline{g} d\mu$, an inner product in the Hilbert space $L^2$.

If $f \in L^2$ and $l \in \mathbb{Z}$, define

$$\gamma_f(l) = \begin{cases} (f, U^{-l} f) & \text{if } l < 0, \\ (f, U^l f) & \text{if } l \geq 0. \end{cases}$$

It is easily verified that the sequence $\{ \gamma(l) \}_{-\infty}^{\infty}$ is positive definite, that is, for any $z_0, \ldots, z_m \in \mathbb{C}$,

$$\sum_{i,j=0}^m \gamma(i - j) z_i \overline{z}_j \geq 0.$$ 

Therefore, the Herglotz–Bochner theorem implies that there exists a positive Borel measure $\sigma_f$ on $\mathbb{C}_1$ such that

$$\int_{\Omega} \tilde{f} \cdot (f \circ T^l) d\mu = (f, U^l f) = \gamma_f(l) = \hat{\sigma_f}(l) = \int_{\mathbb{C}_1} e^{2\pi il \lambda} d\sigma_f(\lambda), \quad l = 1, 2, \ldots.$$

Let now $K^\perp$ be the orthogonal complement of $K$ in $L^2$. It is known that if $f \in K^\perp$, then the measure $\sigma_f$ is continuous, that is, $\sigma_f\{ \lambda \} = 0$ for every $\lambda \in \mathbb{C}_1$ (see, for example, [As03, p. 27]). Let us provide an independent proof of this claim. We will need the following.
Proposition 3.1. $U(K^\perp) \subset K^\perp$.

Proof. Since $U^* = U^{-1}$, it follows that $U^* f = \lambda_f^{-1} f$ for all $0 \neq f \in K$. Thus, given $g \in K^\perp$ and $f \in K$, we have 

$$(U g, f) = (g, U^* f) = \lambda_f^{-1} (g, f) = 0,$$

hence $U g \in K^\perp$. ■

Proposition 3.2. If $f \in K^\perp$, then $\sigma_f$ is a continuous measure.

Proof. It is known [Katz76, p. 42] that 

$$\sigma_f \{ \lambda \} = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} e^{2\pi i \lambda} \check{\sigma}_f(\lambda).$$

Therefore,

$$\sigma_f \{ \lambda \} = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} e^{2\pi i \lambda} \int_{\Omega} \check{f}(f \circ T^l) d\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} e^{2\pi i \lambda} f \circ T^l d\mu,$$

thus it is sufficient to verify that

$$(3.1) \quad \lim_{n \to \infty} \left\| \frac{1}{n} \sum_{l=1}^{n} e^{2\pi i \lambda} f \circ T^l \right\|_2 = 0.$$

The mean ergodic theorem for $\tilde{U} : L^2 \to L^2$ given by $\tilde{U} f = e^{2\pi i \lambda} U f$ implies that there is $\check{f} \in L^2$ such that

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{l=1}^{n} e^{2\pi i \lambda} f \circ T^l - \check{f} \right\|_2 = 0.$$

By Proposition 3.1, $f \circ T^l \in K^\perp$ for each $l$, so $\check{f} \in K^\perp$. Also,

$$\check{f} \circ T = \| \cdot \|_2 - \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} e^{2\pi i \lambda} f \circ T^{l+1} = e^{-2\pi i \lambda} \check{f},$$

so that $\check{f} \in K$. Therefore $\check{f} = 0$, and (3.1) follows. ■

Now we shall turn our attention to a case of van der Corput’s Fundamental Inequality. Let $n \geq 1$ and $0 \leq m \leq n - 1$ be integers, and let $f_0, \ldots, f_{n-1+m} \in L^0$ be such that $f_n = \cdots = f_{n-1+m} = 0$. Then, replacing in the inequality in [KN74, Ch. 1, Lemma 3.1] $N$ by $n+1$ and $H$ by $m+1$, we obtain

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f_k \right|^2 \leq \frac{n+m-1}{n(m+1)} \frac{1}{n} \sum_{k=0}^{n-1} |f_k|^2 + \frac{2(n+m-1)}{n(m+1)} \sum_{l=1}^{m} \frac{m+1-l}{m+1} \Re \frac{1}{n} \sum_{k=0}^{n-1} f_k f_{k+l}.$$
If \( f_0, \ldots, f_{n-1} \in L^\infty \), then the above inequality entails
\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} f_k \right\|_\infty^2 \leq \frac{n + m - 1}{n(m + 1)} \left\| \frac{1}{n} \sum_{k=0}^{n-1} |f_k|^2 \right\|_\infty + \frac{2(n + m - 1)}{n(m + 1)} \sum_{l=1}^{m} \frac{m + 1 - l}{m + 1} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \tilde{f}_k f_{k+l} \right\|_\infty,
\]
which in turn implies that
\[
(3.2) \quad \left\| \frac{1}{n} \sum_{k=0}^{n-1} f_k \right\|_\infty^2 < \frac{2}{m + 1} \left\| \frac{1}{n} \sum_{k=0}^{n-1} |f_k|^2 \right\|_\infty + \frac{4}{m + 1} \sum_{l=1}^{m} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \tilde{f}_k f_{k+l} \right\|_\infty.
\]

To prove the Wiener–Wintner theorem for a.u. convergence, we will establish an almost uniform (in \( \omega \)) enhancement—given by equation (3.3) below—of J. Bourgain’s uniform (in \( \lambda \)) Wiener–Wintner theorem stating that
\[
\lim_{n \to \infty} \sup_{\lambda \in C} \left| M_n(T, \lambda)(f) \right| = 0
\]
for every \( f \in K^\perp \) and almost all \( \omega \in \Omega \) (see, for example, [As03, Theorem 2.4]):

**Theorem 3.3.** If a m.p.t. \( T \) is ergodic, then \( L^1 \subset a.u. \text{WW}(\Omega, T) \).

**Proof.** By Corollary 2.2 as \( L^2 \) is dense in \( L^1 \), \( L^2 = K \oplus K^\perp \), and, by Proposition 2.3, \( K \subset a.u. \text{WW} \), it remains to show that \( K^\perp \subset a.u. \text{WW} \).

Let \( f \in K^\perp \) and \( \varepsilon > 0 \). By the pointwise ergodic theorem, since \( T \) is ergodic, we have

\[
M_n(T)(|f|^2) \to \|f\|_2^2 \text{ a.e. and } M_n(T)(\tilde{f} \cdot (f \circ T^l)) \to \tilde{\sigma}_f(l) \text{ a.e. } \forall l = 1, 2, \ldots.
\]

Applying Egorov’s theorem repeatedly, we can construct \( \Omega' = \Omega_{f, \varepsilon} \subset \Omega \) such that \( \mu(\Omega \setminus \Omega') \leq \varepsilon \) and
\[
(3.3) \quad \|M_n(T)(|f|^2)\chi_{\Omega'}\|_\infty \to \|f\|_2^2 \text{ and } \|M_n(T)(\tilde{f} \cdot (f \circ T^l))\chi_{\Omega'}\|_\infty \to \tilde{\sigma}_f(l)
\]
for all \( l = 1, 2, \ldots \).

If \( \lambda \in C \) and \( f_k = \lambda^k f \circ T^k\chi_{\Omega'} \), then a simple calculation yields
\[
\tilde{f}_k f_{k+l} = \lambda^l (\tilde{f} \cdot (f \circ T^l)) \circ T^k \chi_{\Omega'}, \text{ hence } |f_k|^2 = |f|^2 \circ T^k \chi_{\Omega'}, \forall k, l = 0, 1, 2, \ldots.
\]

Therefore, inequality (3.2) implies that
\[
\begin{align*}
\sup_{\lambda \in C} \left\| M_n(T, \lambda)(f) \chi_{\Omega'} \right\|_\infty^2 &< \frac{2}{m + 1} \left\| M_n(T)(|f|^2) \chi_{\Omega'} \right\|_\infty + \frac{4}{m + 1} \sum_{l=1}^{m} \left\| M_n(T)(\tilde{f} \cdot (f \circ T^l)) \chi_{\Omega'} \right\|_\infty.
\end{align*}
\]
Thus, for a fixed $m$, in view of (3.3), we obtain
\[
\lim sup_{n} \sup_{\lambda \in \mathbb{C}_1} \|M_n(T, \lambda)(f)\chi_{\Omega'}\|_{\infty}^2 \leq \frac{2}{m+1}\|f\|_{\infty}^2 + \frac{4}{m+1} \sum_{l=1}^{m}|\hat{\sigma}_f(l)|.
\]

Since, by Proposition 3.2 the measure $\sigma_f$ is continuous, Wiener’s criterion of continuity of a positive finite Borel measure [Katz76, p. 42] yields
\[
\lim_{m \to \infty} \frac{1}{m+1} \sum_{l=1}^{m}|\hat{\sigma}_f(l)|^2 = 0, \quad \text{hence} \quad \lim_{m \to \infty} \frac{1}{m+1} \sum_{l=1}^{m}|\hat{\sigma}_f(l)| = 0,
\]
and we conclude that
\[
(3.4) \quad \lim_{n} \sup_{\lambda \in \mathbb{C}_1} \|M_n(T, \lambda)(f)\chi_{\Omega'}\|_{\infty} = 0,
\]
so $f \in a.u. \mathcal{WW}(\Omega, T)$.

Remark 3.4. Convergence (3.4) can be derived directly from Bourgain’s result as follows: Given $f \in \mathcal{K}^\perp$, the function
\[g_n(\omega) = \sup_{\lambda \in \mathbb{C}_1} |M_n(T, \lambda)(f)(\omega)|\]
converges to zero a.e. on $\Omega$. Fix $\varepsilon > 0$. Since $\mu(\Omega) < \infty$, Egorov’s theorem implies that there exists $\Omega' \subset \Omega$ with $\mu(\Omega \setminus \Omega') \leq \varepsilon$ such that
\[
\lim_{n} \|g_n\chi_{\Omega'}\|_{\infty} = 0.
\]
Therefore, we can write
\[
0 = \lim_{n} \left\| \sup_{\lambda \in \mathbb{C}_1} |M_n(T, \lambda)(f)\chi_{\Omega'}| \right\|_{\infty} = \lim_{n} \sup_{\omega \in \Omega'} \sup_{\lambda \in \mathbb{C}_1} |M_n(T, \lambda)(f)(\omega)|
\]
\[
= \lim_{n} \sup_{\lambda \in \mathbb{C}_1} \sup_{\omega \in \Omega'} |M_n(T, \lambda)(f)| = \lim_{n} \sup_{\lambda \in \mathbb{C}_1} \|M_n(T, \lambda)(f)\chi_{\Omega'}\|_{\infty}.
\]

Remark 3.5. In the classical case, a simple application of the ergodic decomposition theorem yields convergence of the averages $M_n(T, \lambda)(f)$, with $f \in L^1$, on a set of full measure for all $\lambda \in \mathbb{C}_1$ without the assumption of ergodicity of the m.p.t. $T$ (see, for example, [As03 Theorem 2.12]). Unfortunately, this does not seem to be the case with a.u. convergence. So, the question whether Theorem 3.3 remains valid for a non-ergodic m.p.t. $T$ remains open.

4. The case of infinite measure. Assume now that $(\Omega, \mu)$ is $\sigma$-finite and $T: \Omega \to \Omega$ is an ergodic m.p.t. In the next theorem, we employ the idea of the proof of [As03 Theorem 5.1].
Theorem 4.1. \( \mathcal{L}^1(\Omega) \subset a.u. WW(\Omega, T) \).

Proof. Fix \( f \in \mathcal{L}^1 \) and \( \varepsilon > 0 \). Let \( \Omega = C \cup D \) be the Hopf decomposition, where \( C \) is the conservative part and \( D = \Omega \setminus C \) the dissipative part of \( \Omega \). Then, by [Kr85, §3.1, Theorem 1.6] or [Pe83 §3.7, Theorem 7.4], we have

\[
n M_n(T)(|f|)(\omega) = \sum_{k=0}^{n-1} |f|(T^k \omega) < \infty
\]

for almost all \( \omega \in D \). Moreover, since, by [CL19, Theorem 3.1], the sequence \( \{M_n(T)(|f|)\} \) converges a.u., there is \( \Omega_1 \subset D \) such that

\[
\mu(D \setminus \Omega_1) \leq \varepsilon/3 \quad \text{and} \quad \{M_n(T)(|f|)\chi_{\Omega_1}\} \text{ converges uniformly.}
\]

Then, as \( M_n(T)(|f|) \to 0 \) a.e. on \( D \), it follows that \( M_n(T)(|f|)\chi_{\Omega_1} \to 0 \) uniformly. Therefore, in view of

\[
|M_n(T, \lambda)(f)\chi_{\Omega_1}| \leq M_n(T)(|f|)\chi_{\Omega_1} \quad \forall \lambda \in \mathbb{C}_1,
\]

we conclude that

\[
(4.1) \quad M_n(T, \lambda)(f)\chi_{\Omega_1} \to 0 \quad \text{uniformly} \quad \forall \lambda \in \mathbb{C}_1.
\]

Next, since \((\Omega, \mu)\) is \( \sigma \)-finite, applying an exhaustion argument, one can construct \( p \in \mathcal{L}^1_+ \) such that \( p \circ T = p \) and \( \tilde{C} = \{p > 0\} \) is the maximal modulo \( \mu \) subset of \( C \) on which there exists a finite \( T \)-invariant measure (see [Kr85, pp. 131, 132]). Moreover, by [Kr85, Lemma 3.11, Theorem 3.12], \( \tilde{C} \) and \( C \setminus \tilde{C} \) are \( U \)-absorbing (equivalently, \( T \)-absorbing) and \( M_n(T)(|f|) \to 0 \) a.e. on \( C \setminus \tilde{C} \). Hence, as above, there exists a set \( \Omega_2 \subset C \setminus \tilde{C} \) such that

\[
\mu((C \setminus \tilde{C}) \setminus \Omega_2) \leq \varepsilon/3 \quad \text{and} \quad M_n(T)(|f|)\chi_{\Omega_2} \to 0 \quad \text{uniformly,}
\]

implying that

\[
(4.2) \quad M_n(T, \lambda)(f)\chi_{\Omega_2} \to 0 \quad \text{uniformly} \quad \forall \lambda \in \mathbb{C}_1.
\]

If we define \( \mu' = p \cdot \mu \sim \mu \), then \( \mu' \) is a \( U \)-invariant (equivalently, \( T \)-invariant, that is, \( p \circ T = p \)), finite measure on \( \tilde{C} \). It follows that \( T \) is a m.p.t. on the finite measure space \((\tilde{C}, \mu')\):

\[
\mu'(T^{-1} A) = \int_{T^{-1} A} p \, d\mu = \int_A p \circ T \, d\mu = \int_A p \, d\mu = \mu'(A).
\]

In addition, as \( \tilde{C} \) is \( T \)-absorbing, ergodicity of \( T \) and \( \mu' \sim \mu \) entail that \( T : \tilde{C} \to \tilde{C} \) is an ergodic m.p.t. Also, since \( f \in \mathcal{L}^1(\tilde{C}, \mu) \), we have \( fp^{-1} \in \mathcal{L}^1(\tilde{C}, \mu') \). Therefore, by Theorem [3.3], there exists \( \Omega_3 \subset C \) such that

\[
\mu(\tilde{C} \setminus \Omega_3) = \mu'(\tilde{C} \setminus \Omega_3) \leq \varepsilon/3
\]
and the averages
\[ M_n(T, \lambda)(f p^{-1}) \chi_{\Omega_3} = \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k (f p^{-1}) \circ T^k \chi_{\Omega_3} \]
converge uniformly for all \( \lambda \in \mathbb{C}_1 \). But \( (f p^{-1}) \circ T^k \chi_{\Omega_3} = p^{-1}(f \circ T^k) \chi_{\Omega_3} \), and we conclude that the sequence
\[
\{ M_n(T, \lambda)(f) \chi_{\Omega_3} = p M_n(T, \lambda)(f p^{-1}) \chi_{\Omega_3} \}
\]
converges uniformly \( \forall \lambda \in \mathbb{C}_1 \).

Now, with \( \Omega' = \Omega_1 \cup \Omega_2 \cup \Omega_3 \), in view of (4.1)–(4.3), we obtain
\[ \mu(\Omega \setminus \Omega') \leq \varepsilon \] and \( \{ M_n(T, \lambda)(f) \chi_{\Omega'} \} \) converges uniformly \( \forall \lambda \in \mathbb{C}_1 \).

Therefore \( f \in a.u.WW(\Omega, T) \), and the proof is complete. \( \blacksquare \)

Recall that
\[ R_\mu = \{ f \in L^1 + L^\infty : \mu\{|f| > \lambda\} < \infty \text{ for all } \lambda > 0 \}. \]

**Theorem 4.2.** \( R_\mu \subset a.u.WW(\Omega, T) \).

**Proof.** Pick \( f \in R_\mu \) and fix \( \varepsilon, \delta > 0 \). By \cite{CCLi19} Proposition 2.1, there exist \( g \in L^1 \) and \( h \in L^\infty \) such that
\[ \|h\|_\infty \leq \delta/3 \text{ and } f = g + h. \]

As \( g \in L^1 \), Theorem 4.1 entails that there exists \( \Omega' \subset \Omega \) such that for each \( \lambda \in \mathbb{C}_1 \) there is a number \( N = N(\lambda) \) satisfying
\[ \mu(\Omega \setminus \Omega') \leq \varepsilon \] and \( \|(M_m(T, \lambda)(g) - M_n(T, \lambda)(g)) \chi_{\Omega'}\|_\infty \leq \delta/3 \forall m, n \geq N. \)

Then, given \( \lambda \in \mathbb{C}_1 \) and \( m, n \geq N(\lambda) \), we have
\[ \|(M_n(T, \lambda)(f) - M_n(T, \lambda)(f)) \chi_{\Omega'}\|_\infty \]
\[ \leq \|(M_m(T, \lambda)(g) - M_n(T, \lambda)(g)) \chi_{\Omega'}\|_\infty \]
\[ + \|M_m(T, \lambda)(h)\|_\infty + \|M_n(T, \lambda)(h)\|_\infty \leq \delta/3 + 2\|h\|_\infty \leq \delta, \]

implying that \( f \in a.u.WW. \) \( \blacksquare \)

As \( L^p \subset R_\mu \) for any \( 1 \leq p < \infty \), Theorem 4.2 yields the following.

**Corollary 4.3.** \( L^p \subset a.u.WW(\Omega, T) \) for all \( 1 \leq p < \infty \).

**5. A Wiener–Wintner-type ergodic theorem with Besicovitch weights.** The goal of this section is to show that Theorem 4.2 remains valid if one expands the set \( \{\lambda^k\}_{k=0}^{\infty} : \lambda \in \mathbb{C}_1 \} \) to the family of all bounded Besicovitch sequences.

A function \( P : \mathbb{Z} \to \mathbb{C} \) is called a *trigonometric polynomial* if \( P(k) = \sum_{j=1}^{s} z_j \lambda_j^k \), \( k \in \mathbb{Z} \), for some \( s \in \mathbb{N} \), \( \{z_j\}_{j=1}^{s} \subset \mathbb{C} \), and \( \{\lambda_j\}_{j=1}^{s} \subset \mathbb{C}_1 \). A sequence \( \{b_k\}_{k=0}^{\infty} \subset \mathbb{C} \) is called a *bounded Besicovitch sequence* if
(i) $|b_k| \leq C < \infty$ for all $k$ and some $C > 0$;
(ii) for every $\varepsilon > 0$ there exists a trigonometric polynomial $P$ such that

$$\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |b_k - P(k)| < \varepsilon.$$ 

Now, by linearity, Corollary 4.3 implies the following.

**Proposition 5.1.** Let $f \in \mathcal{L}^1$. Then for every $\varepsilon > 0$ there exists $\Omega' \subset \Omega$ with $\mu(\Omega \setminus \Omega') \leq \varepsilon$ such that the sequence

$$M_n(T, P)(f)\chi_{\Omega'} = \frac{1}{n} \sum_{k=0}^{n-1} P(k)f \circ T^k \chi_{\Omega'}, \quad n = 1, 2, \ldots,$$

converges uniformly for any trigonometric polynomial $P = P(k)$.

Let us denote by $\mathcal{B}$ the set of Besicovitch sequences. The next theorem is an extension of Theorem 4.1.

**Theorem 5.2.** If $f \in \mathcal{L}^1$, then for any $\varepsilon > 0$ there exists $\Omega' \subset \Omega$ with $\mu(\Omega \setminus \Omega') \leq \varepsilon$ such that the sequence $\{M_n(T, \overline{b})(f)\chi_{\Omega'}\}$ converges uniformly for every $\overline{b} \in \mathcal{B}$.

**Proof.** In view of Proposition 2.1 it is sufficient to show that the convergence holds for any $f \in \mathcal{L}^1 \cap \mathcal{L}^\infty$. So, pick $0 \neq f \in \mathcal{L}^1 \cap \mathcal{L}^\infty$ and let $\varepsilon, \delta > 0$. By Proposition 5.1 there exists $\Omega' \subset \Omega$ with $\mu(\Omega \setminus \Omega') \leq \varepsilon$ such that for any trigonometric polynomial $P = P(k)$ there is $N_1 = N_1(P)$ satisfying

$$\|(M_m(T, P)(f) - M_n(T, P)(f))\chi_{\Omega'}\|_{\infty} \leq \delta/3 \quad \forall m, n \geq N_1.$$ 

Let $\overline{b} = \{b_k\} \in \mathcal{B}$, and let $P = P(k)$ be a trigonometric polynomial such that

$$\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |b_k - P(k)| < \frac{\delta}{3\|f\|_{\infty}}.$$ 

Then there exists $N_2$ such that $\frac{1}{n} \sum_{k=0}^{n-1} |b_k - P(k)| < \frac{\delta}{3\|f\|_{\infty}}$ whenever $n \geq N_2$. Now, if $m, n \geq \max\{N_1, N_2\}$, it follows that

$$\|(M_m(T, \overline{b})(f) - M_n(T, \overline{b})(f))\chi_{\Omega'}\|_{\infty} \\
\leq \|M_m(T, \overline{b})(f) - M_m(T, P)(f)\|_{\infty} + \|M_n(T, \overline{b})(f) - M_n(T, P)(f)\|_{\infty} \\
+ \|(M_m(T, P)(f) - M_n(T, P)(f))\chi_{\Omega'}\|_{\infty} \\
\leq 2\|f\|_{\infty} \frac{1}{n} \sum_{k=0}^{n-1} |b_k - P(k)| + \delta/3 < \delta,$$

so $\{M_n(T, \overline{b})(f)\chi_{\Omega'}\}$ converges uniformly for each $\overline{b} \in \mathcal{B}$. ■

As in Theorem 4.2 we derive the following.

**Theorem 5.3.** The conclusion of Theorem 5.2 holds for all $f \in \mathcal{R}_\mu$. 

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*Almost uniform convergence in the Wiener–Wintner ergodic theorem*
Corollary 5.4. Given $1 \leq p < \infty$, the conclusion of Theorem 5.2 holds for all $f \in L^p$.

Note that when $\mu(\Omega) = \infty$, there are functions in $\mathcal{R}_\mu$ that do not belong to any of the spaces $L^p$, $1 \leq p < \infty$, but lie in classical function Banach spaces $X$ such as Orlicz, Lorentz, or Marcinkiewicz spaces with $1 = \chi_\Omega \notin X$. If $1 \notin X$, then $X \subset R_\mu$ by [CCLi19, Proposition 6.1], hence the conclusions of Theorems 4.2 and 5.3 hold for any $f \in X$. For conditions that imply $1 \notin X$ when $X$ is an Orlicz, Lorentz, or Marcinkiewicz space, that is, for applications of Theorems 4.2 and 5.3 to these spaces, see [CL19, Section 5].

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