Einstein-Maxwell-scalar black holes with massive and self-interacting scalar hair

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Abstract

Recently, spontaneous scalarization of charged black holes (BHs) has attracted a lot of attention and motivated several studies of Einstein-Maxwell-scalar models. These studies have, however, only considered a massless and non-self-interacting scalar field. In this work a more realistic treatment of the problem is considered by studying the effects of scalar field mass and self-interacting terms on spontaneous scalarization in EMS models, as well as on the string theory motivated dilatonic BH. We assess the domains of existence of BH solutions, thermodynamic preference, scalar field radial profiles and perturbative stability of the scalarized solutions, focusing on spherical perturbations. The mass term is found to alter the threshold for the onset of scalarization and the self-interacting term suggests a stabilizing effect on the BH solutions, mimicking the behavior observed in extended-Scalar-Tensor-Gauss-Bonnet models.

1 Introduction

Einstein-Maxwell-scalar (EMS) models are generically described by the action (in units with $8\pi G = c = 1$)

$$S = \int d^4x\sqrt{-g} \left[ R - 2\partial_\mu \phi \partial^\mu \phi - f(\phi)F_{\mu\nu}F^{\mu\nu} - U(\phi) \right].$$

This action describes a real scalar field $\phi$ with a potential $U(\phi)$ minimally coupled to Einstein’s gravity ($R$ is the Ricci scalar) and non-minimally coupled to Maxwell’s electromagnetism ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual Maxwell tensor) through a function $f(\phi)$. Non-minimal couplings between the electromagnetic and a scalar field, and the corresponding BH solutions, have long been considered in the context of, e.g., Kaluza-Klein theory and supergravity [1, 2]. More recently, EMS models have been shown to allow the phenomenon of spontaneous scalarization of asymptotically flat, charged black holes (BHs) [3–6]. Spontaneous scalarization, first considered in a BH context in extended-Scalar-Tensor-Gauss-Bonnet (eSTGB) models [7–12], is a strong gravity phase transition. It occurs when two phases (classes of solutions) co-exist, one of them becoming dynamically preferred. In EMS models, for certain choices of the coupling function $f(\phi)$, the standard electrovacuum (scalar-free) Reissner-Nordström (RN) BH solves the equations of motion, along with a new class of BHs that allow a non-trivial equilibrium scalar-field configuration (scalar-hair). For sufficiently large BH charge to mass ratio, however, the RN BH becomes unstable against scalar perturbations and the formation of these hairy BHs is conjectured to be the endpoint of the instability [3, 4].

Studies up to now have focused on the simple case of massless and non-self-interacting scalar field [3, 6, 13–23] with the exception of [24], which studied the case of a configuration of the EMS model with a massive scalar field ($U(\phi) \sim \mu^2 \phi^2$) for only a specific BH charge to mass ratio. A more realistic treatment of the problem requires a full analysis of spontaneous scalarization in EMS models with a massive (and even self-interacting) scalar field, which is the aim of this work. We extend previous studies to inquire the effects of a massive and self-interacting scalar-field ($U(\phi) \sim \mu^2 \phi^2 + \lambda \phi^4$) for a generic EMS BH for several coupling functions. Certain sectors of extended scalar-tensor theories arise naturally in string theory and at low energies supersymmetry is broken which leads to a massive scalar field. The inclusion of scalar field mass suppresses the scalar field at the length scale of the order of the Compton wavelength, which may help in reconciling the theory with observations for a much broader range of the coupling parameters and functions [8]. Such study of spontaneous scalarization with a massive and self-interacting scalar field was performed in [7] in the context of eSTGB models leading to two main conclusions: (i) a mass term for the scalar field alters the threshold for the onset of scalarization; (ii) the quartic self-coupling is sufficient to produce scalarized solutions that are stable against radial perturbations, without the need to resort to (exotic) higher-order terms in the Gauss-Bonnet coupling function. In this work we inquire if a parallelism between the results for eSTGB and...
EMS models emerge. Scalar field self-interactions have been further studied, e.g., in the context of Kerr BHs with synchronized hair [25,26].

This paper is organized as follows. In Section 2 we present the model, obtain the field equations for the ansatz that describes the class of solutions of interest and provide some details on the construction of the solutions and, following [5], propose a classification of the BH solutions, based on the behaviour of the coupling function. Then we describe how the emergence of the scalarised solutions is computed, in linear theory, comparing our results with the ones from [24] and discuss the effective potential for spherical perturbations, a simple tool that can establish perturbative stability and diagnose possible instabilities. Section 3 contains the bulk of our results and the analysis of the two main examples considered (dilaton and scalarized cases). This includes obtaining the domains of existence, studying the thermodynamic preference, the scalar field radial profiles and the effective potential for spherical perturbations. Final remarks are presented in Section 4.

2 The model

The action of the family of EMS models we wish to consider is given by [1,1] We focus on the case where the scalar field is massive and self-interacts

\[ U(\phi) / 2 = \mu^2 \phi^2 + \lambda \phi^4 , \]  

(2.2)

where \( \mu \) is the scalar field mass and the self-coupling is positive, \( \lambda > 0 \). The equations of motion that follow from (1.1) are

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{1}{2} \left[ T_{\mu \nu}^{(\phi)} + T_{\mu \nu}^{(EM)} \right] , \]  

(2.3)

\[ \partial_\mu \left( \sqrt{-g} f(\phi) F^{\mu \nu} \right) = 0 , \]  

(2.4)

\[ \square \phi = \frac{\dot{f}(\phi) F_{\mu \nu} F^{\mu \nu} + U(\phi)}{4} , \]  

(2.5)

where the dot denotes differentiation with respect to the scalar field, i.e., \( \dot{f}(\phi) \equiv df / d\phi \) and the energy-momentum tensor is given by

\[ T_{\mu \nu}^{(\phi)} / 4 = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu \nu} \left[ \partial_\alpha \phi \partial^\alpha \phi + \frac{1}{2} U(\phi) \right] , \quad T_{\mu \nu}^{(EM)} / 4 = f(\phi) \left( F_{\mu \alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} \right) . \]  

(2.6)

A generic, static and spherically symmetric line element used to describe both scalar-free and scalarized solutions is

\[ ds^2 = -N(r)e^{-2\delta(r)} dt^2 + \frac{dr^2}{N(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) . \]  

(2.7)

Spherical symmetry requires the scalar field \( \phi(r) \) to have a radial dependence only, and an electromagnetic 4-potential ansatz of the following type,

\[ A = V(r) dt . \]  

(2.8)

Functions \( N, \delta, V, \phi \) have radial dependence only; for ease of notation this dependence will be omitted henceforth and a radial derivative will be denoted by a prime. With this ansatz, the equations of motion (2.3, 2.4, 2.5) reduce to

\[ V' = \frac{Q}{r^2 \dot{f}(\phi)} e^{-\delta} , \quad \left( e^{-\delta} r^2 N\phi' \right)' = -\frac{1}{2} \delta r^2 \dot{f}(\phi) V^2 + \frac{e^{-\delta}}{4} r^2 U(\phi) , \]  

(2.9)

\[ \delta' = -r \phi^2 , \quad N' = \frac{1 - N(1 - r \delta)}{r} - \frac{1}{2} r U(\phi) - \frac{Q^2}{r^3 \dot{f}(\phi)} , \]  

(2.10)

where \( Q \) is an integration constant interpreted as the electric charge measured at infinity. To solve this set of coupled, non-linear ordinary differential equations, we have to implement suitable boundary conditions for the desired functions
and corresponding derivatives. We assume the existence of an event horizon at \( r = r_H > 0 \) and that the solution possesses a power series expansion in \((r - r_H)\)

\[
N(r) = N_1 (r - r_H) + \ldots , \quad \delta (r) = \delta_0 + \delta_1 (r - r_H) + \ldots , \\
\phi (r) = \phi_0 + \phi_1 (r - r_H) + \ldots , \quad V (r) = v_1 (r - r_H) + \ldots .
\]  

Plugging these expansions in the field equations, the lower order coefficients are determined to be

\[
N_1 = - \frac{Q^2 - r_H^2 f(0)}{r_H f'(0)} - \frac{1}{2} r_H U'(0), \quad \delta_1 = - \frac{\phi_1}{r_H}, \quad \phi_1 = - \frac{2Q^2 f(0) - r_H^2 f'(0) U'(0)}{2Q^2 r_H f(0) - 4r_H f'(0)^2 U'(0)}.
\]  

One observes that only two of the six parameters introduced in the expansions (2.11) are independent, which we choose to be \(\phi_0\) and \(\delta_0\), the remaining being derived from these ones. The solutions in the vicinity of the horizon are determined by these two parameters, together with \((r_H, Q, \mu, \lambda)\). Some physical horizon quantities, such as the Hawking temperature \(T_H\), the horizon area \(A_H\), the energy density \(\rho(r_H)\) and the Kretschmann scalar \(K(r_H)\), are then determined by these parameters as follows:

\[
T_H = \frac{1}{4 \pi} N_1 e^{-\delta_0}, \quad A_H = 4\pi r_H^2, \quad \rho (r_H) = \frac{2Q^2}{r_H f'(0)} + U(\phi_0), \quad K(r_H) = \frac{4}{r_H f'(0)^2} (5Q^4 - 6r_H^2 Q^2 f(\phi_0) + 3f(\phi_0)^2 r_H^4) + 4U(\phi_0) \rho(\phi_0)^2 r_H^2 - 2 r_H^5.
\]  

To obtain the boundary conditions at spatial infinity one performs an asymptotic approximation of the solution in the far field. Then the equations of motion yield:

\[
N(r) = 1 - \frac{2M}{r} + \frac{Q^2 + \Lambda^2}{r^2} + \ldots , \quad \phi (r) = \frac{Q}{r} e^{-\mu r} + \ldots , \quad V (r) = \Phi + \frac{Q}{r} + \ldots ,
\]  

which introduce three new parameters: the scalar charge \(Q\), the electrostatic potential difference between the horizon and infinity \(\Phi\), and the ADM mass \(M\). From these asymptotic expansions one collects a set of 9 independent parameters, therefore: \((r_H, Q, \mu, \lambda, \phi_0, \delta_0, Q, \Phi, M)\). As we shall see below, the full integration of the field equations relates these parameters, and, for each choice of the coupling functions, the solutions of interest actually form a family of solutions with only 4 (continuous) parameters, typically taken to be the global charges \((M, Q)\) and the self-coupling parameters \((\mu, \lambda)\). For later use we gather the following results and definitions:

\[
q \equiv \frac{Q}{M}, \quad a_H \equiv \frac{A_H}{16\pi M^2}, \quad t_H \equiv 8\pi M T_H, \quad \hat{\mu} \equiv Q \mu, \quad \hat{\lambda} \equiv Q^2 \lambda,
\]  

where \(\hat{\mu}\) and \(\hat{\lambda}\) are dimensionless self-coupling parameters, \(q\) is the reduced charge, \(a_H\) is the reduced horizon area and \(t_H\) the reduced BH temperature. These reduced quantities are convenient because they are invariant under the scaling symmetry

\[
r \to \eta r, \quad \xi \to \eta \xi, \quad \mu \to \mu / \eta, \quad \lambda \to \lambda / \eta^2,
\]  

where \(\eta\) is a constant and \(\xi\) represents any of the global charges of the model, while \(f(\phi)\) remains unchanged.

### 2.1 Classification of EMS models

Depending on the choice of coupling \(f(\phi)\), the RN BH may or may not be a solution of the equations of motion of the model, which can be better seen from the scalar field equation of motion (2.5). This leads to two classes of EMS models [5].
2.1.1 Class I - Models without a scalar-free solution

In this class of EMS models $\phi(r) = 0$ does not solve the field equations and so, the RN BH is not a solution. From the scalar field equation of motion (2.5)

$$\dot{f}(0) \neq 0 .$$

(2.17)

Such representative coupling is the standard dilatonic coupling (studied in the massless, non-self-interacting case in [1,2,5,27]).

$$f(\phi) = e^{\alpha \phi}$$

(2.18)

in which case we refer to $\phi$ as a dilaton field. Three reference values for the dilaton coupling constant $\alpha$ are [5]: $\alpha = 0$ (Einstein-Maxwell theory), $\alpha = 2$ (low energy strings), $\alpha = 2\sqrt{3}$ (Kaluza-Klein theory). Massive dilaton studies have been conducted in [28,29].

2.1.2 Class II - Models with a scalar-free solution

In this class of EMS models $\phi(r) = 0$ solves the field equations and so, the RN BH is a solution. From the scalar field equation of motion (2.5).

$$\dot{f}(0) = 0 .$$

(2.19)

The RN solution, however, is (in general) not unique. These EMS models may contain a new set of BH solutions, with a non-trivial scalar field profile - the scalarized BHs. As discussed in [5], such new set of BH solutions may exist in models where the RN BH is (class IIA) or is not (class IIB) unstable. In this work we are interested in the first (IIA), for which spontaneous scalarization occurs. The second case is studied in great detail in [13]. The spontaneously scalarized (hereby dubbed "scalarized") BHs bifurcate from RN BHs, and reduce to the latter for $\phi = 0$. This bifurcation moreover, may be associated to a tachyonic instability, against scalar perturbations $\delta \phi$, of the RN BH. These obey

$$(\square - \mu_{eff}^2)\delta \phi = 0 ,$$

(2.20)

with $\mu_{eff}^2 < 0$ given by

$$\mu_{eff}^2 = -\frac{\dot{f}(0)Q^2}{2r^2} + \mu^2,$$

(2.21)

being unaffected by the self-coupling. Such representative coupling that will be studied in greater detail later is

$$f(\phi) = e^{\alpha \phi^2}.$$

(2.22)

The coupling constant $\alpha$ is taken as a positive. This coupling was studied in the massless, non-self-interacting case in detail in [3,5].

2.1.2.1 Bifurcation of solutions: the existence line

Let us now consider the onset of spontaneous scalarization. We assume that the model under consideration admits the RN BH of Einstein-Maxwell theory as the scalar-free solution, that is (2.7)-(2.8) with

$$\delta = 0 , \quad N(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} , \quad V(r) = \frac{Q}{r} .$$

(2.23)

The scalarization phenomenon is assessed by considering scalar perturbations of the RN solution within the considered model. Following [3,6], we take a spherical harmonics decomposition of the scalar field perturbation:

$$\delta \phi = \sum_{r,m} Y_{r,m}(\theta, \phi)U_r(r) .$$

(2.24)
With this ansatz, the scalar field equation of motion simplifies to

\[
\frac{e^{\delta}}{r^2} \left( \frac{r^2 N e^{\delta} U_\ell}{r^2} \right)' - \left[ \frac{\ell (\ell + 1)}{r^2} + \mu_{\text{eff}}^2 \right] U_\ell = 0.
\]  

(2.25)

Once the coupling functions are fully fixed, solving (2.25) is an eigenvalue problem: for a given \( \ell \), requiring an asymptotically vanishing, regular at the horizon, smooth scalar field, a discrete set of BHs solutions are selected, \textit{i.e.} a discrete set of RN solutions, each with a certain reduced charge \( q \). These are the \textit{bifurcation points} from the scalar-free solution. They are labelled by an integer \( n \in \mathbb{N}_0; n = 0 \) is the fundamental mode, whereas \( n \geq 1 \) are excited states (overtones). The RN solutions with a smaller (larger) \( q \) than that of the bifurcation point are stable (unstable) against the corresponding scalar perturbation. In particular, the first bifurcation point, \textit{i.e.}, the one with the smallest \( q \), which corresponds to the mode \( \ell = 0 \) and \( n = 0 \), marks the onset of the scalarization instability. Only RN BHs with \( q \) smaller than the first bifurcation point are stable against any sort of scalar perturbation.

At each bifurcation point, a new family of (fully non-linear) scalarized BH solutions emerges from the RN family, as static solutions of the equations of motion of the full model. In this paper we shall consider only the first bifurcation point and the corresponding new family of spherically symmetric scalarized BHs that bifurcate from the RN family. The existence line (the set of bifurcating points from the scalar-free solution) is altered by the scalar-field mass because it has a suppressing effect for the tachyonic instability, altering the threshold for the onset of scalarization as seen in Fig. 1 where the existence lines are plotted for a sample of \( \hat{\mu} \) values. Besides shifting the minimum value of \( \alpha \) to higher values, for each constant \( \alpha \) line, a higher value of \( \hat{\mu} \) implies a higher charge to mass ratio \( q \) for bifurcation.

Figure 1: Set of bifurcating points - the existence lines - of scalarized solutions in the \((\alpha, q)\) plane for a sample of \( \hat{\mu} \) values. Higher \( \hat{\mu} \) values lead to higher required \( q \) for bifurcation, altering the threshold for the onset of scalarization.

Our values for the bifurcation points match with great precision the ones obtained in Table I of [24], for a study of scalarized BHs of the same model considered in this work, with a specific value of \( q = 0.7 \) and a scalar-field mass term \( \mu^2 = \alpha/\beta \), where \( \beta \) is a mass-like parameter. In ref. [24] the results for the minimum values of \( \alpha \) for bifurcation for a RN BH with \( q = 0.7 \) are presented in such a way that they depend on the parameter \( \beta \), which is not scale invariant. Table I presents the same results but with the minimum values of \( \alpha \) for bifurcation depending on \( \hat{\mu} \), a scale invariant quantity, so that these results can be applied to any system in consideration.

| \( \hat{\mu} \) | 0.0  | 0.0228 | 0.0365 | 0.05 | 0.10  | 0.365 | 0.50  |
|---------------|------|--------|--------|------|-------|-------|-------|
| \( \alpha \)   | 8.019| 8.493  | 8.82   | 9.18 | 10.69 | 21.80 | 29.52 |

Table 1: Bifurcation points \((n = 0 \text{ mode})\) for several values of \( \hat{\mu} \) for a RN BH with \( q = 0.7 \). The results match the ones from [24].
2.2 Effective potential for spherical perturbations

Let us also introduce a diagnostic analysis of perturbative stability, against spherical perturbations, that shall be applied to the solutions derived and discussed in the next section. Following a standard technique, see e.g. [6], we consider spherically symmetric, linear perturbations of an equilibrium solution, keeping the metric ansatz, but allowing the functions $N, \delta, \phi, V$ to depend on $r$ as well as on $t$:

$$ds^2 = -\tilde{N}(r,t)e^{-2\tilde{\delta}(r)}dt^2 + \frac{dr^2}{N(r,t)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad A = \tilde{V}(r,t)dt, \quad \phi = \tilde{\phi}(r,t). \quad (2.26)$$

The time dependence enters as a Fourier mode with frequency $\Omega$, for each of these functions:

$$\tilde{N}(r,t) = N(r) + \varepsilon N_1(r)e^{-i\Omega t}, \quad \tilde{\delta}(r,t) = \delta(r) + \varepsilon \delta_1(r)e^{-i\Omega t}, \quad \tilde{V}(r,t) = V(r) + \varepsilon V_1(r)e^{-i\Omega t}. \quad (2.27)$$

From the linearized field equations around the background solution, the metric perturbations and $V_1(r)$ can be expressed in terms of the scalar field perturbation,

$$N_1 = -2rN\phi'\phi_1, \quad \delta_1' = -2r\phi'\phi_1', \quad V_1' = -V' \left( \delta_1 + \frac{\dot{\phi}}{f(\phi)} \phi_1' \right), \quad (2.28)$$

thus yielding a single perturbation equation for $\phi_1$. Introducing a new variable $\Psi(r) = r\phi_1$, the scalar-field equation of motion may be written as

$$\left( Ne^{-\delta} \right)^2 \Psi'' + Ne^{-\delta} \left( Ne^{-\delta} \right)' \Psi' + (\Omega^2 - U_\Omega) \Psi = 0, \quad (2.29)$$

which, by introducing the 'tortoise' coordinate $x$ as $dx/dr = e^\delta/N$, can be written in the standard one-dimensional Schrödinger-like form:

$$-\frac{d^2}{dx^2} \Psi + U_\Omega \Psi = \Omega^2 \Psi. \quad (2.30)$$

The effective potential that describes spherical perturbations $U_\Omega$ is defined as:

$$U_\Omega = U_{SI} + U_0, \quad (2.31)$$

with

$$U_{SI} = \frac{e^{-2\delta}}{2} \left\{ \mu^2 \left[ 1 + 4r\phi \phi' + \phi^2 (-1 + 2r^2 \phi'^2) \right] + \lambda \phi^2 \left[ 6 + 8r\phi \phi' + \phi^2 (-1 + 2r^2 \phi'^2) \right] \right\},$$

$$U_0 = \frac{e^{-2\delta}}{r^2} \left\{ 1 - N - 2r^2 \phi^2 - \frac{Q^2}{r^2 f(\phi)} \left( 1 - 2r^2 \phi'^2 + \frac{\dot{\phi}}{f(\phi)} \right) + 2r \phi^2 \left( \frac{\dot{\phi}}{f(\phi)} \right)^2 \right\}. \quad (2.32)$$

An unstable mode would have $\Omega^2 < 0$, which for the asymptotic boundary conditions of our model is a bound state. It follows from a standard result in quantum mechanics (see e.g. [31]), however, that eq. (2.30) has no bound states if $U_\Omega$ is everywhere larger than the lowest of its two asymptotic values, i.e., if it is positive in our case.\(^3\) Thus an everywhere positive effective potential proofs mode stability against spherical perturbations.

We remark that the existence of a region of negative potential is a necessary but not sufficient condition for instabilities to be present. In fact, for the fundamental, spherically symmetric scalarized solutions in [34] with $\hat{\mu} = 0$ and $\hat{\lambda} = 0$, this region occurs for some solutions, which are, nonetheless, stable [16].

\(^3\)A simple proof is as follows. Write Eq. (2.30) in the equivalent form

$$\frac{d}{dx} \left( \Psi \frac{d\Psi}{dx} \right) = \left( \frac{d\Psi}{dx} \right)^2 + (U_\Omega - \Omega^2)\Psi^2. \quad (2.33)$$

After integrating from the horizon to infinity it follows that

$$\int_{-\infty}^{\infty} dx \left( \frac{d\Psi}{dx} \right)^2 + U_\Omega \Psi^2 = \Omega^2 \int_{-\infty}^{\infty} dx \Psi^2 \quad (2.34)$$

which for $U_\Omega > 0$ implies $\Omega^2 > 0$. 

6
3  Numerical results for the BH solutions

In this section we present, for both the dilatonic and scalarized couplings, the main results, namely the effects of the mass term and self-interactions on the domains of existence, radial function profiles, thermodynamic preference and effective potential for spherical perturbations of the BH solutions.

3.1 Domain of Existence

For the scalarized case, the domain of existence was obtained in the massless, non-self-interacting case in [3–5], being bounded by an existence line and (in the absence of a magnetic charge) by a critical line, at which BH solutions are singular - numerics suggest a divergence of the Kretschmann scalar at the horizon and that $A_H \to 0$. The existence line will change depending on the scalar-field mass and is independent of the self-coupling $\lambda$ as previously observed in Fig. 1. The domain of existence for the scalarized coupling is presented in Fig. 2. As seen in Fig. 2 (left panel), the scalar field mass term has a narrowing effect on the domain of existence of scalarized solutions, because higher reduced charge $q$ is required for bifurcation and because overcharging is restricted - the critical set occurs for smaller $q$ values as compared to the massless case. On the other hand, the self-interaction effects widen the domain of existence - c.f. Fig. 2(right panel). The existence line remains unchanged, while the critical set occurs for higher charge to mass ratios, suggesting a stabilizing effect.

Concerning the dilaton case, the domain of existence was obtained in the massless, non-self-interacting case in [5], being bounded (in the absence of a magnetic charge) by a critical line. Note that there is no existence line since the model does now allow a scalar-free solution. Similarly to the scalarized case, as seen in Fig. 3(left panel), the scalar field mass term has a narrowing effect on the domain of existence of dilatonic solutions - the critical set occurs for smaller $q$ values as compared to the massless case. On the other hand, the self-interaction effects widen the domain of existence - c.f. Fig. 3(right panel).

3.2 Thermodynamic preference

Concerning thermodynamic preference, since the model under consideration is General Relativity minimally coupled to some matter, the Bekenstein-Hawking BH entropy formula holds. Thus, the entropy analysis reduces to the analysis of the horizon area. It is convenient to use the reduced event horizon area $a_H$. In the region where the RN BHs and scalarized BHs co-exist - the non-uniqueness region -, for the same $q$, the scalarized solutions are always entropically preferred as seen in Fig. 4(left). Also, for the same coupling strength $\alpha$, charge to mass ratio $q$, and $\lambda = 0$, massless solutions are generically entropically preferred over the massive ones (exceptions occur for solutions close to...
Figure 3: Domain of existence of dilaton solutions in the \((\alpha, q)\) plane. Left: \(\hat{\lambda} = 0\) with \(\hat{\mu} = 0\) and \(\hat{\mu} = 0.1\) cases. The mass term has a narrowing effect on the domain of existence. Right: \(\hat{\mu} = 0\) with \(\hat{\lambda} = 0\) and \(\hat{\lambda} = 0.01\) cases. The self-interaction effects widen the domain of existence.

bifurcation), while the solutions with non-zero self-coupling \(\lambda\) are always entropically preferred over solutions with \(\lambda = 0\) (considering \(\mu = 0\)). For the dilatonic coupling, conclusions are qualitatively similar to the scalarized case (c.f. Fig. 4, right panel). For the same coupling strength \(\alpha\), for the same \(q\), massless solutions are generically entropically preferred over the massive ones (for \(\lambda = 0\)), while the solutions with non-zero self-coupling \(\lambda\) are always entropically preferred over solutions with zero self-coupling \(\lambda = 0\).

Figure 4: \(a_H\) vs. \(q\). (Left) The black line represents scalar-free RN BHs, while the red, blue and green lines are sequences of (numerical data points representing) scalarized BHs for \(\alpha = 10\). (Right) The red, blue and green lines represent sequences of (numerical data points representing) dilaton BHs for \(\alpha = 5\).

3.3 Scalar field radial profiles

The radial profiles for the scalar field were also studied - c.f. Fig. 5. The mass term, as expected, has a suppressing effect on the scalar field (as seen, for instance, in the scalar field value at the event horizon). Also, the mass term leads to a scalar field radial profile more concentrated in the neighborhood of the event horizon and a faster decay. This is expected since the decay is approximately exponential with the scalar field mass. No relevant discrepancies were observed in the radial profiles between the solutions with and without self-coupling \(\lambda\). This can be understood from the scalar field equation of motion (2.5), where the term proportional to \(\lambda\) can be neglected as rough approximation.
it is not dominant near the horizon, and it is cubic on an asymptotically vanishing scalar field.

\[ \lambda = 0 \]

\[ q \approx 0.727 \]

\[ \alpha = 10 \]

\[ \mu = 0 \]

\[ \mu = 0.1 \]

\[ \log(\frac{r}{r_H}) \]

\[ \phi(r) \]

\[ U_\Omega \]

3.4 Effective potential for spherical perturbations

The effective potential for spherical perturbations was computed (c.f. Fig. 6) for both scalarized (left panel) and dilatonic (right panel) solutions. In the scalarized case the effective potential reveals that the self-interacting, massless solutions generically yield an everywhere positive effective potential with vanishing asymptotic values, thus being free of instabilities. On the other hand solutions with a mass term for the scalar field generically yield a negative region in the effective potential, thus instabilities cannot be excluded \textit{a priori}. However, in [24] it was shown that these negative regions do not correspond to instabilities (for the some specific nodeless solutions), suggesting that these solutions are perturbatively stable. Another possible way to establish stability of the solutions in the presence of such negative potential regions would be via the S-deformation method [13, 32]. An interesting feature of these massive solutions is the asymptotic value of the effective potential \( U_\Omega \rightarrow \mu^2/2 \) as \( r \rightarrow \infty \) as can easily be observed in Eq. (2.32). For the dilatonic case all solutions generically yield an everywhere positive effective potential with zero as the lowest of the asymptotic values, thus being free of instabilities.

\[ \alpha = 10 \]

\[ q = 0.727 \]

\[ U_\Omega \]

\[ \log(\frac{r}{r_H}) \]

3.4 Effective potential for spherical perturbations

Figure 5: (Left) Scalar field radial profiles for the scalarized case with \( \hat{\mu} = 0 \) and \( \hat{\mu} = 0.1 \) while \( q \approx 0.727, \alpha = 10 \) and \( \hat{\lambda} = 0 \). (Right) Scalar field radial profiles for the dilatonic case with \( \hat{\mu} = 0 \) and \( \hat{\mu} = 0.1 \) while \( q \approx 0.747, \alpha = 5 \) and \( \hat{\lambda} = 0 \). The mass term leads to a scalar field radial profile more concentrated in the neighborhood of the event horizon, while being unaffected by the self-interaction term.

Figure 6: (Left) Effective potential for several scalarized solutions. (Right) Effective potential for several dilatonic solutions.

\[ \lambda = 0 \]

\[ \mu = 0 \]

\[ \mu = 0.1 \]

\[ \alpha = 5 \]

\[ q = 0.747 \]
4 Conclusions

In this work we have studied the impact of scalar field mass and self-interactions on two paradigmatic EMS models - the dilaton and scalarized cases. Depending on the choice of the coupling $f(\phi)$, the model can accommodate BHs with scalar-hair and, may or may not accommodate the standard RN BH of electrovacuum. In the first case, the RN BH may become unstable against scalar perturbations and spontaneously develop scalar-hair.

It was found that the presence of a mass term alters the threshold for the onset of scalarization (which is independent of the self-interaction $\lambda$) as was already observed for eSTGB in [7]. The results obtained for the bifurcation points of the $n = 0$ mode of scalarized BHs agree with the ones obtained in [24] for the specific case of $q = 0.7$, and are presented in a scale-invariant form.

The domain of existence is bounded by critical lines where BH solutions are singular for both dilaton and scalarized BHs. The presence of a scalar field mass term narrows the domain of existence as the critical set occurs for smaller BH charge to mass ratios, while the scalar-field self-interaction has the opposite effect. For both couplings it was generically found that BHs with massless scalar-hair are thermodynamically preferred over the massive counterparts, and on the other hand, thermodynamically unfavored when comparing to solutions with self-interacting scalar hair. We observed that the scalar field self-coupling $\lambda$ has no measurable effect on the scalar field radial profiles, while the mass term causes an exponential decay. In what concerns the effective potential for spherical perturbations, for the dilaton coupling all solutions generically yield an everywhere positive effective potential with zero as the lowest of the asymptotic values, thus being free of instabilities. The same behavior is observed in the scalarized case for the massless, self-interacting case. In the massive scalar field case, the potential generically yields a negative region, that does not correspond to an instability as studied in [24]. Our analysis then suggests that the self-coupling $\lambda$ has a possible stabilizing effect on the BH solutions, mimicking the behaviour observed for eSTGB models in [7], completing a parallelism between the two models.

A closing remark: a study of dyonic solutions (magnetically charged BH solutions) was performed in a similar fashion as in section 3 leading to similar global conclusions. However, as in the massless, non-self-interacting case the domain of existence is bounded by an extremal set, at which BHs are extremal. This extremal set is obtained, for the same $\alpha$, for smaller values of $q$ if a mass term is considered in spite of the massless case, and for higher values of $q$ if a non-zero self-coupling is considered (similarly to what is observed in the purely electric case for the critical set).

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