Spatial patterns and scale freedom in Prisoner’s Dilemma cellular automata with Pavlovian strategies

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Abstract. A cellular automaton in which cells represent agents playing the Prisoner’s Dilemma (PD) game following the simple ‘win—stay, lose—shift’ strategy is studied. Individuals with binary behaviour, such that they can either cooperate (C) or defect (D), play repeatedly with their neighbours (Von Neumann’s and Moore’s neighbourhoods). Their utilities in each round of the game are given by a rescaled pay-off matrix described by a single parameter $\tau$, which measures the ratio of 	extit{temptation to defect} to 	extit{reward for cooperation}. Depending on the region of the parameter space $\tau$, the system self-organizes—after a transient—into dynamical equilibrium states characterized by different definite fractions of C agents $\bar{c}_\infty$ (two states for the von Neumann neighbourhood and four for the Moore neighbourhood). For some ranges of $\tau$ the cluster size distributions, the power spectra $P(f)$ and the perimeter–area curves follow power law scalings. Percolation below threshold is also found for D agent clusters. We also analyse the asynchronous dynamics version of this model and compare results.

Keywords: cellular automata, game-theory (theory), applications to game theory and mathematical economics, interacting agent models

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1. Introduction

The Prisoner’s Dilemma (PD) game plays in game theory a role similar to that of the harmonic oscillator in physics. Indeed, this game, developed in the early 1950s, offers a very simple and intuitive approach to the problem of how cooperation emerges in societies of ‘selfish’ individuals, i.e. individuals who pursue exclusively their own self-benefit. It was used in a series of works by Axelrod and co-workers [1] to examine the basis of cooperation in a wide variety of contexts. Furthermore, approaches to cooperation based on the PD have shown their usefulness in political science [2]–[4], economics [5]–[11], international affairs [12]–[15], theoretical biology [16]–[18] and ecology [19, 20].

The PD game consists in two players each confronting two choices: cooperate (C) or defect (D); and each makes the choice without knowing what the other will do. The four possible outcomes for the interaction of the two agents are: (1) they both cooperate: (C, C), (2) they both defect: (D, D), (3) one of them cooperates and the other defects: (C, D) or (D, C). Depending on the case, (1)–(3), the agents get respectively: the ‘reward’ $R$, the ‘punishment’ $P$ or the ‘sucker’s pay-off’ $S$ for the agent who plays C and the ‘temptation to defect’ $T$ for the agent who plays D. These four pay-offs obey the relations

$$T > R > P > S \quad \text{and} \quad 2R > S + T.$$  

The last condition is required in order that the average utilities for each agent of a cooperative pair ($R$) are greater than the average utilities for an exploitative–exploiter pair ($(R + S)/2$). One can assign a pay-off matrix $M$ to the PD game given by

$$M = \begin{pmatrix} (R, R) & (S, T) \\ (T, S) & (P, P) \end{pmatrix},$$

which summarizes the pay-offs for row actions when confronted with column actions. Clearly it pays more to defect: if one of the two players—say $i$—defects, the other who...
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cooperates will end up with nothing. In fact, even if agent $i$ cooperates, agent $j$ should defect, because in that case s/he will get $T$ which is larger than $R$. That is, independently of what the other player does, defection $D$ yields a higher pay-off than cooperation and is the dominant strategy for rational agents. Furthermore, is the Nash equilibrium [21]—i.e. a best reply to itself—of the PD game. The dilemma is that if both defect, both do worse than if both had cooperated: both players get $P$ which is smaller than $R$. A possible way out of this dilemma is to play the game repeatedly. In this iterated Prisoner’s Dilemma (IPD), there are several strategies that outperform the dominant $[D, D]$ one-shot strategy and lead to some non-null degree of cooperation.

The attainment of cooperation in PD simulations relies on different mechanisms and factors. A popular point of view regards direct reciprocity as the crucial ingredient. A typical exponent of this viewpoint is the strategy known as tit for tat (TFT): cooperate on the first move, and then cooperate or defect exactly as your opponent did on the preceding encounter. This requires either memory of previous interactions or features (‘tags’) permitting cooperators and defectors to distinguish one another [22].

Spatial structure has also been identified as an influential factor in building cooperation. For instance, in [23] the authors neglected all strategical complexities or memories of past encounters. Instead, they show that spatial effects by themselves in a classic Darwinian setting are sufficient for the evolution of cooperation.

The problem of cooperation is approached mainly from a Darwinian evolutionary perspective: strategies that incorporate some dose of cooperative behaviour are the most successful and propagate displacing competing strategies that do not. In that sense, a central concept is that of the evolutionary stable strategy (ESS) [24,25]: a strategy which if adopted by all members of a population cannot be invaded by a mutant strategy through the operation of natural selection. The evolutionary game theory, originated as an application of the mathematical theory of games to biological issues, later spread to economics and social sciences.

In this work, we follow a different approach: there is no competition of different strategies; all the agents follow a natural strategy of ‘win—stay, lose—shift’ known as Pavlovian [26]. We do not worry about the resistance of the strategy against invasion by other strategies (such as unconditional defectors or ALL D that play D independently of what the opponent does); rather we take the Pavlovian strategy for granted. The rationale for this relies on several facts. First, the Pavlovian strategy seems to be a widespread strategy in nature [27]. Second, the Pavlovian strategy does fairly well when competing with several other strategies including generous tit-for-tat GTFT as was shown by Nowak and Sigmund [28]. Moreover, they found that in a non-spatial setting, while the Pavlovian strategy can be invaded by ALL D, a slightly stochastic variant cannot. Third, experiments with humans have shown that a great fraction of individuals do indeed use Pavlovian strategies [29].

Therefore, we address the analysis of the self-organized states that emerge when simple agents, possessing neither long term memory nor tags, play the PD game in a spatial setting using the Pavlovian strategy. We this aim we resort to a cellular automaton in

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4 Indeed the game that they considered is not exactly the PD and implies a ‘weak dilemma’ in which $D$ does not strictly dominate.

5 GTFT cooperates after the opponent has cooperated in the previous round, but it also cooperates with a non-null probability after the opponent has defected.
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which each cell is either black or white representing, respectively, a D or a C agent. Each agent plays with those belonging to her or his neighbourhood, and the total utilities s/he gets determine the update of her or his individual state.

We consider pay-off matrices implying strict dilemmas defined by equations (1) rather than weak ones in which the inequalities are relaxed (for instance \( P = S \)). To simplify things we parametrize the pay-off matrix in terms of a single parameter \( \tau \), which measures the ratio of temptation to defect to reward for cooperation.

Different self-organizations occur depending on the value of \( \tau \), the type of dynamics and the neighbourhood considered. In particular, for a range of values of \( \tau \) (that depends on the neighbourhood) we found power law behaviour that might be a signature of self-organized criticality [30].

Previously, a similar non-spatial model, in which pairs of agents were chosen at random, was analysed in [31]. Also, a mean field stochastic version was considered in [32].

This work is organized as follows. In section 2 we describe the model. In section 3 we present the results of simulations as well as analytical results obtained by using a mean field approximation that neglects all spatial correlations (details are given in the appendix at the end). Section 4 is devoted to conclusions and final remarks.

2. The model

The model is very simple: we assign to each agent, located at the cell with centre at \((x, y)\), a binary behavioural variable \( c(x, y) \) which takes the value ‘1’ for C agents and ‘0’ for D agents. This agent plays with the \( z \) agents belonging to her or his neighbourhood \( N(x, y) \) getting a pay-off \( U_1(x, y) \) with the first neighbour played, \( U_2(x, y) \) with the second one and so on\(^6\). The total utilities \( U(x, y) = U_1(x, y) + U_2(x, y) + \cdots + U_z(x, y) \) got by playing with her or his neighbourhood determine the update of her or his individual state. More technically, we have an outer totalistic cellular automaton; i.e. the state of a cell at the next time step depends only on its own state, and the sum of the states of its neighbours.

The dynamic is synchronous: all the agents update their states simultaneously at the end of each lattice sweep. In addition to this synchronous dynamics or ‘parallel updating’, we also explored, with less detail, the asynchronous dynamics or ‘sequential updating’, in which the state of an agent is updated after s/he plays.

We considered two different neighbourhoods: (a) the von Neumann neighbourhood \((z = 4)\) neighbour cells: the cell above and below, right and left from a given cell) and (b) the Moore neighbourhood \((z = 8)\) neighbour cells: the von Neumann neighbourhood + diagonals).

The pay-off matrix is parametrized in terms of a single parameter \( \tau \equiv T/R \):

\[
M = \begin{pmatrix}
(1, 1) & (-\tau, \tau) \\
(\tau, -\tau) & (-1, -1)
\end{pmatrix},
\]

with \( \tau > 1 \). The total utilities of the agent at \((x, y)\) at time \( t \), \( U(x, y, t) \), are the sums of the utilities collected by playing with each neighbour, as prescribed by the pay-off matrix.

A typical value for the population of agents is \( N_{ag} = 10,000 \) (100 \( \times \) 100 lattice)\(^7\).

\(^6\) The order in which a given agent plays with neighbours does not matter; it can be fixed or randomly chosen.

\(^7\) However, in some cases we considered \( N_{ag} \) up to 1000000 (1000 \( \times \) 1000 lattice) in order to ensure that the transients become long enough for extracting the power spectrum.

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Table 1. Utilities of a given agent depending on whether her or his state is C (row 1) or D (row 2) and the states of her or his neighbourhood (columns 2–6) for a von Neumann neighbourhood.

|   | 4C, 0D | 3C, 1D | 2C, 2D | 1C, 3D | 0C, 4D |
|---|-------|-------|-------|-------|-------|
| C | 4     | 3 − τ | 2 − 2τ| 1 − 3τ| −4τ   |
| D | 4τ    | 3τ − 1| 2τ − 2| τ − 3 | −4    |

Table 2. The same as table 1 but for a Moore neighbourhood.

|   | 8C, 0D | 7C, 1D | 6C, 2D | 5C, 3D | 4C, 4D | 3C, 5D | 2C, 6D | 1C, 7D | 0C, 8D |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| C | 8     | 7 − τ | 6 − 2τ| 5 − 3τ| 4 − 4τ| 3 − 5τ| 2 − 6τ| 1 − 7τ| −8τ   |
| D | 8τ    | 7τ − 1| 6τ − 2| 5τ − 3| 4τ − 4| 3τ − 5| 2τ − 6| τ − 7 | −8    |

The initial state at \( t = 0 \) is taken as \( c(x, y; 0) = 0 \) or 1 (D or C respectively), chosen at random for each cell \((x, y)\). Then the system evolves by iteration during \( t_f \) time steps until it reaches a stationary or dynamical equilibrium state.

The Pavlovian strategy works as follows. The agent at \((x, y)\) will change state for the next time step \( t + 1 \): \( c(x, y, t + 1) = 1 − c(x, y, t) \) (from C to D or vice versa) if \( U(x, y, t) < 0 \), and will remain the same: \( c(x, y, t + 1) = c(x, y, t) \), if \( U(x, y, t) > 0 \) (when \( U(x, y, t) = 0 \) the agent changes with probability 0.5). Once all the agents have played, their states are updated for the next time iteration.

For the von Neumann neighbourhood then, each agent plays with the four nearest neighbours. Let us analyse what is expected to happen for different values of the parameter \( \tau \). Let us focus on the agent at \((x, y)\) and her or his possible configurations (C or D) and the ones of her/his neighbourhood (number of C and D neighbours) in each case her/his corresponding utilities. These results are shown in table 1.

From table 1, since \( \tau > 1 \), we observe that the sign of the utilities \( U(x, y) \) of the agent located at site \((x, y)\)—which determines the update of her or his \( c(x, y) \)—depends on the value of \( \tau \) only for two cases: (a) if the agent plays C and her or his neighbourhood consists of three C agents and one D or (b) if the agent plays D and her or his neighbourhood consists of one C agent and three D agents. In both cases the update rule depends thus on whether \( \tau > 3 \) or \( \tau < 3 \). So, \textit{a priori}, one would expect the existence of a ‘critical’ value of the parameter \( \tau^* = 3 \) such that the results depend on whether \( \tau \) is greater or smaller than this critical value. Intuitively one can argue that since for \( \tau > 3 \) there are more favourable situations for D agents and disadvantageous ones for C agents, the mean cooperation of the system when the dynamical equilibrium is reached, \( c_\infty = (1/N_{ag}) \sum_{N_{ag}} c(x, y, t) \)—after the transient—will be smaller than when \( \tau < 3 \).

Table 2 summarizes the utilities of a player for each possible configuration of her or his neighbours for the case of a Moore neighbourhood.

A completely analogous reasoning for the Moore neighbourhood leads to three ‘critical’ values: \( \tau^*_1 = 5/3, \tau^*_2 = 3 \) and \( \tau^*_3 = 7 \). Here we would expect also that \( c_\infty \) will diminish as \( \tau \) crosses each frontier value \( \tau^*_i \) from left to right.

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Table 3. The asymptotic fraction of cooperators $c_\infty$ for a $z = 4$ von Neumann neighbourhood. Column 2: simulations. Column 3: MF approximation (see the appendix).

| $\tau$  | Simulations | MF   |
|---------|-------------|------|
| $\tau < 3$ | 0.485 ± 0.002 | 0.430 |
| $\tau \geq 3$ | 0.280 ± 0.002 | 0.342 |

Table 4. The asymptotic fraction of cooperators $c_\infty$ for a $z = 8$ Moore neighbourhood. Column 2: simulations. Column 3: MF approximation (see the appendix).

| $\tau$  | Simulations | MF   |
|---------|-------------|------|
| $1 < \tau < 5/3$ | 0.563 ± 0.002 | 0.461 |
| $5/3 \leq \tau < 3$ | 0.436 ± 0.002 | 0.420 |
| $3 \leq \tau < 7$ | 0.366 ± 0.003 | 0.386 |
| $8 \leq \tau$ | 0.320 ± 0.003 | 0.334 |

3. Results

To avoid dependence on the initial conditions, the measures correspond to averages taken over an ensemble of 100 systems with arbitrary initial conditions. In general, the results for the asymptotic regime, after a transient, become almost independent of the lattice size $L$ for $L \gtrsim 100$. Therefore in what follows, unless stated otherwise, the results correspond to simulations performed in $100 \times 100$ lattices.

As we have anticipated, we observe that the stationary state of the system changes as the parameter $\tau$ moves from one region to another (two regions in the case of a $z = 4$ von Neumann neighbourhood and four regions for a $z = 8$ Moore neighbourhood).

3.1. The asymptotic average fraction of cooperators $\bar{c}_\infty$

The asymptotic or equilibrium mean fraction of C agents $\bar{c}_\infty$ takes constant values in each of the regions delimited by the ‘critical’ $\tau^*$. Hence we have one sharp step at $c_\infty = 3$ for $z = 4$ and three sharp steps at $c_\infty = \frac{5}{3}$, 3 and 7 for $z = 8$.

It is interesting to compare the $\bar{c}_\infty$ produced by simulations with the $c_\infty^{MF}$ obtained by elementary calculations using a mean field (MF) approximation that neglects all spatial correlations (see the appendix).

In tables 3 and 4 we present $\bar{c}_\infty$ and $c_\infty^{MF}$ for $z = 4$ and 8 respectively. Clearly, as expected, the MF approximation improves with increasing $z$. In addition, divergences between spatial games and the MF approximation become maximum in the ‘cooperative’ sector of the parameter $\tau$ (leftmost region, producing $c_\infty \gtrsim 0.5$). This can be explained in terms of the particular cluster structure of that region exhibiting power law scalings (see the next subsection).

The upper bar in $\bar{c}_\infty$ denotes an average over 100 simulations with different initial conditions.

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3.2. Spatial patterns: the cluster structure

The von Neumann neighbourhood. In figure 1 we present snapshots—after the transient—of the cellular automaton for $\tau < 3$ and $\tau > 3$. These ‘cooperation maps’ illustrate the differences between the typical spatial patterns that arise in the two parameter regions divided by $\tau^* = 3$.

For $\tau < 3$ we found that:

(I) Although the asymptotic probability for D agents is $\bar{d}_\infty = 1 - \bar{c}_\infty \simeq 0.5$, which is below the percolation threshold $\rho_c \approx 0.59275$, giant spanning D clusters often occur. Percolation below threshold is a known fact in other models. In general, when there are correlations between the sites, the threshold is shifted—as happens, for instance, in the square Ising model when percolation occurs, at the critical temperature, when the concentration is also 0.5.

(II) Different quantities behave as power laws, implying thus the emergence of scale free phenomena. For instance, the size distribution of clusters of D agents exhibits power law scaling.

For $\tau > 3$ the distribution of D clusters is bimodal with a peak for very small clusters (size = 1) and a secondary peak for very large clusters. The main peak for very small clusters can be explained by the small correlation length. On the other hand, the secondary peak for very large sizes arises because the probability for a given site to be in the D state $d_\infty \equiv 1 - c_\infty$ is over the site percolation threshold and thus spanning clusters are much more abundant than when $\tau < 3$, in which case $d_\infty < \rho_c$.

Figure 2 is a plot of the log of the number of clusters of C and D agents versus the log of their size for $\tau < 3$ and $\tau > 3$, using 400 x 400 lattices. In both cases giant spanning clusters of D agents were excluded. This, in particular for $\tau > 3$, eliminates a large number of clusters belonging to the secondary peak of its bimodal distribution and explains why there are fewer ‘+’ points in figure 2(b) than in (a) (the shortage of ‘*’ points, representing C clusters, is obviously related to the fact that $c_\infty$ is smaller on the $\tau > 3$ side).

The data points for D clusters seem consistent with a power law scaling over a couple of decades, with a critical exponent of approximately $-1.79 \pm 0.02$. The graphic also shows

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(a)

Figure 2. The number of clusters of C (*) and D (+) agents versus the size of the clusters for the von Neumann neighbourhood in a 400 × 400 lattice. The clusters are summed over the last 150 lattice sweeps after the transient for: (a) τ < 3, (b) τ > 3. In both cases, giant spanning D clusters were not included.

A difference between C and D clusters: the first ones exhibit much greater deviations from an exact power law, although they also occur over a wide range of scales. This asymmetry can be traced to the difference that exists for the possible stable configurations of clusters of Cs or Ds; while the first ones need at least three C neighbours to remain C, the second ones can do well with only two C neighbours. Then the D agents can form thinner clusters than the C agents. This fact increases the probability of agents D yielding larger clusters. This also can explain why although the equilibrium probability for D agents is below the percolation threshold, giant spanning D clusters are observed.

For τ > 3 the situation changes drastically as figure 2(b) reflects; here it can be seen that the data do not fit well with a power law, either for D or for C clusters.

Remark. To check that the power law scaling is not dependent on the particular parametrization of the pay-off matrix that we are using, we measured the cluster distribution for many other pay-off matrices not described by (2). For instance, we considered this alternative parametrization of the pay-off matrix

\[
M' = \begin{pmatrix}
(1,1) & (\tau/2 - 3, \tau) \\
(\tau, \tau/2 - 3) & (-1, -1)
\end{pmatrix},
\]

for (3).

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Figure 3. Perimeter versus area for the clusters of C and D agents for $\tau = 4$. The perimeter’s values plotted are averages of perimeters of clusters of the same size, taken over the last 500 lattice sweeps after the transient.

with $3 - \tau/2 < -1 < \tau$. Again, we found power law behaviour for the leftmost region in $\tau$. Thus, it seems that this power law scaling for an entire collection of PD pay-off matrices is a robust property of the model.

Another clue about the dynamics of the clusters can be obtained by examining the relation of the perimeter to the area of the clusters. We define the perimeter of a cluster C (D) as the set of sites $(x, y)$ with behavioural variable $c(x, y) = 1$ ($c(x, y) = 0$) belonging to the cluster with at least one neighbour with the opposite behavioural variable, i.e. $c(x, y) = 0$ ($c(x, y) = 1$). The mean perimeter $P(A)$, for a given area $A$, is then given by averaging over all the perimeters of clusters with area $A$. Figure 3 shows that for $\tau < 3$ the perimeter scales linearly with the area, that is, at the fastest rate possible, implying that the clusters are highly ramified. The fraction of the area that is interior to the clusters can be easily calculated.

By fitting the points of figures 3(a) and (b) we get the following expressions for the perimeter as a function of the area, for $\tau < 3$:

$$P_C \approx 0.82A_C$$
for clusters of C agents

$$P_D \approx 0.86A_D$$
for clusters of D agents.

(4)
Then the cluster interior fraction is $F = (A - P)/A$. Thus we get that approximately

$$
F_C \simeq 0.18 \quad \text{for clusters of C agents}
$$

$$
F_D \simeq 0.16 \quad \text{for clusters of D agents.}
$$

This shows that the clusters have almost no interior, and confirms our previous observation concerning the clusters of D agents being thinner than those of C agents. This supports quantitatively the explanation of why percolation of D agents is observed but not that of C agents. The linear behaviour shown in figure 3(c), with slope approximately equal to 1, can be understood by inspection of figure 1(b), where it is clearly seen that the C agents form small ‘laddered’ clusters in which the perimeter is equal to the area.

The Moore neighbourhood. For arbitrary random initial conditions, the equilibrium cooperation maps are shown in figure 4 for $\tau$ in the different regions of interest.

As can be seen from table 4, when $\tau$ is within the interval $(1, 5)$, $\bar{c}_\infty \simeq 0.6$, which is higher than the values obtained for the von Neumann neighbourhood for any $\tau$. This implies that increasing the number of neighbours in general produces a higher fraction of
cooperators, although this higher value of $\bar{c}_\infty$ is stable for narrower domain values of $\tau$. We checked this for the case in which 12 neighbours are taken into account, achieving a value of $\bar{c}_\infty \simeq 0.8$ for $\tau \in (1, \frac{5}{2})$.

Let us analyse what happens to the clusters of Cs and Ds for the different values of $\tau$, this time for the Moore neighbourhood. The results are shown in figure 5.

In figure 5(a), corresponding to $\tau \in (1, \frac{5}{2})$ and $c_\infty \simeq 0.57$, we can observe power law behaviour for clusters of C and D agents, with the same critical exponent of approximately $-1.62 \pm 0.02$. This symmetry between Cs and Ds is broken when we take $\tau \in (\frac{5}{2}, 3)$ (figure 5(b), $c_\infty \simeq 0.44$): here we recover the kind of behaviour that we found for $\tau < 3$ in the case of the von Neumann neighbourhood (see figure 2(a)), for which the power law scaling for D agents is much clearer than that for C agents. In this case we find an exponent of approximately $-1.98 \pm 0.04$. Remarkably, criticality seems to persist, although not so clearly as in the previous cases, even for values of $\tau$ in the interval $(3, 7)$ (figure 5(c)). For $\tau > 7$, power law behaviour is completely lost, as figure 5(d) shows.

Figure 5. The number of clusters of C (*) and D (+) agents versus the size of the clusters, summed over the last 500 times after $10^4$ iterations for $z = 8$, on a logarithmic scale. The plots correspond to: (a) $\tau \in (1, \frac{5}{2})$, (b) $\tau \in (\frac{5}{2}, 3)$, (c) $\tau \in (3, 7)$, (d) $\tau > 7$. There is a percolation peak for clusters of D agents in (b), (c) and (d) since they are above the percolation threshold ($d > p_c$).
3.3. Power spectra

The power laws that we found for spatial observables might be interpreted as signatures of self-organized criticality (SOC). In order to elucidate the criticality or lack of it for the dynamics, we analysed temporal correlations. Specifically, we calculated the power spectrum $P(f)$ (i.e. the absolute value of the Fourier transform) for the time autocorrelation function $G(t)$ of the cooperative fraction $c(t)$. $G(t)$ is defined as

$$G(t) \equiv \langle c(t_0)c(t_0 + t) \rangle - \langle c(t_0) \rangle^2,$$

where the average is taken over all possible temporal origins $t_0$.

It turns out that although the transients are not very long, $P(f)$ exhibits power law behaviour, for the same range of values of $\tau$ that we found this type of behaviour for in the cluster size distributions, for almost two decades. For instance, in the case of the von Neumann neighbourhood, we have a power law power spectrum for $\tau < 3$ which is lost for $\tau > 3$ (which is consistent with the fact that the simulations have shown that for this region the system behaves periodically, with a very short period). This is shown in figure 6.

The correlation function $G(t)$ is calculated for the transient. In order to maximize this transient, an initial $c(t = 0) = 0.1$ very different from the known equilibrium value of $c_\infty \simeq 0.5$ was taken, together with a large lattice of $1000 \times 1000$. This power law scaling of $P(f)$, for the same region as we found this type of behaviour for in the cluster sizes, can be interpreted as another signature for the possible existence of critical dynamics.

3.4. Asynchronous dynamics

As we mentioned in the previous section, besides exploring the synchronous dynamics, we also performed some runs using asynchronous dynamics, in which the state of each agent is updated after s/he has played with her or his neighbourhood.

The asynchronous update produces a much less interesting situation. The power laws are lost, both for the von Neumann and Moore neighbourhoods: we find no power laws for the cluster sizes or for the power spectrum and the cooperation values decrease significantly. Still, there is a change in the mean value of the cooperation as the parameter $\tau$ goes through the critical values calculated earlier. For the von Neumann neighbourhood, for $\tau < 3$, $\bar{c}_\infty \simeq 0.34$. For $\tau > 3$, cooperation decreases to $\bar{c}_\infty \simeq 0.23$ and there is no clear pattern of behaviour. For the Moore neighbourhood, results are similar, with $\bar{c}_\infty \simeq 0.34$, $0.30$, $0.21$ and $0.13$ for $\tau \in (1, \frac{5}{3})$, $\left(\frac{5}{3}, 3\right)$, $(3, 7)$ and $\tau > 7$ respectively.

4. Conclusions

For a cellular automata, representing a system of agents playing the IPD governed by Pavlovian strategies in a simple territorial setting, we explored the steady states for different values of the parameter $\tau$, which measures the ratio of temptation to defect to reward. Both for the Von Neumann and Moore neighbourhoods we found sharp steps for $\bar{c}_\infty$ versus $\tau$ (one step in the first case and three steps in the second case).

We found power law scaling for different quantities, measuring either spatial (cluster size distributions) or temporal correlation ($P(f)$), for entire regions in parameter $\tau$ space. All this may be interpreted as consistent evidence of self-organized criticality in a spatial...
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Figure 6. The power spectrum for a $z = 4$ Von Neumann neighbourhood: (a) $\tau < 3$, (b) $\tau > 3$.

game which is not evolutionary (at least in the ordinary Darwinian sense). This result, which is qualitatively robust against changes of the pay-off matrix and the neighbourhood, is novel (as far as we know). (It is worth mentioning that the parametrization (3) allows one to study two other games besides the PD: if $-1 < \tau < 1$ ($R > T > P > S$) the game is known as a 'stag hunt' (SH), while when $4 < \tau < 8$ ($T > R > S > P$) the game is called the 'hawk–dove' (HD) game. We simulated these two games, which are popular in social sciences and biology respectively, and we found, in contrast to what happens with the PD, no power law behaviour. On the other hand, the occurrence of critical dynamics in certain spatial evolutionary games has been observed. For instance, in [33] it was shown that for a certain range of a parameter, which determines the punishment, the spatial HD game exhibits large temporal and spatial correlations and various processes governed by power laws. This is in contrast with the simplified version of the PD considered in [23], which does not exhibit complex critical dynamics of this type; rather it has periodic or chaotic dynamics. Nevertheless, for a stochastic version of this evolutionary weak dilemma, power law behaviour consistent with directed percolation has been measured [34].

We also have shown that percolation below the threshold value occurs for D agents for the case of the von Neumann neighbourhood. The asymmetry between C and D

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9 Results for SH and HD games will be published elsewhere.

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Figure 7. A Pareto optimal states configuration for a small $6 \times 6$ lattice, $z = 4$ and $\tau = 2$. Left: the $c(i,j)$ matrix. Right: the corresponding utilities $U(i,j)$: the utilities for all the agents are positive and thus they do not change their behavioural variables.

clusters, even in cases in which agents of the two types appear with equal probability, can be explained in terms of the Pavlovian strategy and the asymmetry of the pay-offs (see table 1).

A result worth remarking on is that the degree of cooperation can be increased by enlarging the neighbourhood but, simultaneously, the temptation parameter $\tau$ must be restricted to smaller values.

Another interesting general result is the effect of changing the dynamics from synchronous to asynchronous. The scale invariance that we found for the synchronous update disappears when we turn to the asynchronous update. The fact that the general qualitative behaviour of asynchronous models may differ greatly from that of the synchronous version was noted in [35].

Let us mention some interesting future extensions of the work presented here. For instance, we observed that for small lattices this simple deterministic system often reaches true equilibrium configurations, in which all the agents are happy (all get utilities above 0) and do not change their respective states—in other words, Pareto optimal states (POS), i.e. states in which none of the players can increase their pay-off without decreasing the pay-off of at least one of the other players. In figure 7 an example of such equilibrium states is presented for a small $(6 \times 6)$ lattice, $z = 4$ and $\tau = 2$.

When the lattice size grows, the system becomes unable to reach these POS. The explanation that we found for this is, as the size grows, the fraction of POS with respect to the possible configurations decreases. Additionally, it is plausible that the entirely deterministic update does not provide a path in configuration space connecting the initial state with a POS. The introduction of noise in the update rule, in some particular cases, might help in promoting ergodicity. The effect of the introduction of noise in spatial evolutionary games was analysed for example in [37] and [38]. An interesting goal is using noise to avoid entrainment in non-efficient states, i.e. implementing a sort of simulated annealing approach [36] allowing one to reach these optimal equilibria.
Another issue that seems worth exploring is the extension of the present approach, beyond the PD game, to games that are useful for modelling other different everyday situations, such as ‘stag hunt’, ‘chicken’ (see footnote 9).

Finally, after we completed this manuscript, one of the referees pointed out the study of the PD game by Posch et al [39] using ‘win—stay, lose—shift’ strategies in a non-spatial set-up. This work offers a stimulating discussion of when satisficing can become optimizing.

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Appendix: Mean field computations

An estimate of $c_\infty$ can be obtained by elementary calculations using a mean field approximation that neglects all spatial correlations.

Once the stationary state is reached, the transitions from D to C, on average, must equal those from C to D. Thus, the average probability of cooperation $c_\infty$ is obtained by equalizing the flux from C to D, $J_{CD}$, and the flux from D to C, $J_{DC}$. The possible utilities for a C player range from $R = z \times 1 = z$ to $S = -z\tau$ (see tables 1 and 2). Let us consider separately the $z = 4$ von Neumann neighbourhood and the $z = 8$ Moore neighbourhood.

- $z = 4$

We have two different situations depending on the value of $\tau$: $\tau < 3$ or $\tau \geq 3$.

- $\tau < 3$:

  In that case, the utilities $U_C$ ($U_D$) of a C (D) player are negative, and thus s/he changes from C to D (D to C) if at least two (three) neighbours play D. For a given average probability of cooperation $c$, the probabilities of a C agent facing two, three and four neighbours playing D are respectively $c^3(1-c)^2$, $c^2(1-c)^3$ and $c(1-c)^4$. Consequently, $J_{CD}$ can be written as

  $$J_{CD} \propto c^3(1-c)^2 + c^2(1-c)^3 + c(1-c)^4.$$ (7)

  On the other hand, the probabilities of a D agent facing three and four neighbours playing D are respectively $(1-c)^4c$ and $(1-c)^5$. Therefore $J_{DC}$ is given by

  $$J_{DC} \propto c(1-c)^4 + (1-c)^5.$$ (8)

  Thus the algebraic equation for $c_\infty$ is

  $$c_\infty^4 + c_\infty^3(1-c_\infty) - (1-c_\infty)^3 = 0,$$ (9)

  with only one real root in the interval $[0, 1]$: $c_{\infty}^{MF} = 0.430$.

- $\tau \geq 3$:

  In that case, the utilities $U_C$ ($U_D$) of a C (D) player are negative, and thus s/he changes from C to D (D to C) except (only) if s/he has all her or his four neighbours playing C (D). Therefore, $J_{CD}$ must be modified summing a term $c^4(1-c)$ to equation (7) and the term $c(1-c)^4$ must be suppressed from the expression (8) for $J_{DC}$. Hence, we get the following algebraic equation for $c_\infty$:

  $$c_\infty^4 + c_\infty^3(1-c_\infty) + c_\infty^2(1-c_\infty)^2 - c_\infty(1-c_\infty)^3 - (1-c_\infty)^4 = 0,$$ (10)

  with only one real root in the interval $[0, 1]$: $c_{\infty}^{MF} = 0.342$. 

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We have four different situations depending on the region in the parameter space $\tau$. The corresponding polynomials for $c_\infty$ are obtained exactly as was done for $z = 4$ and one can easily check that are given as follows:

- $1 < \tau < 5/3$:
  \[
  c_\infty^5 + c_\infty^4 (1 - c_\infty) - (1 - c_\infty)^5 = 0, \tag{11}
  \]
  with only one real root in the interval $[0, 1]$: $c_{\text{MF}}^\text{eq1} = 0.461$.

- $5/3 \leq \tau < 3$:
  \[
  c_\infty^6 + c_\infty^5 (1 - c_\infty) + c_\infty^4 (1 - c_\infty)^2 + c_\infty^3 (1 - c_\infty)^3 - (1 - c_\infty)^6 = 0, \tag{12}
  \]
  with only one real root in the interval $[0, 1]$: $c_{\text{MF}}^\text{eq2} = 0.420$.

- $3 \leq \tau < 7$:
  \[
  c_\infty^7 + c_\infty^6 (1 - c_\infty) + c_\infty^5 (1 - c_\infty)^2 + c_\infty^4 (1 - c_\infty)^3 \\
  + c_\infty^3 (1 - c_\infty)^4 + c_\infty^2 (1 - c_\infty)^5 - (1 - c_\infty)^7 = 0, \tag{13}
  \]
  with only one real root in the interval $[0, 1]$: $c_{\text{MF}}^\text{eq3} = 0.386$.

- $7 \leq \tau$:
  \[
  c_\infty^8 + c_\infty^7 (1 - c_\infty) + c_\infty^6 (1 - c_\infty)^2 + c_\infty^5 (1 - c_\infty)^3 + c_\infty^4 (1 - c_\infty)^4 \\
  + c_\infty^3 (1 - c_\infty)^5 + c_\infty^2 (1 - c_\infty)^6 + c_\infty (1 - c_\infty)^7 - (1 - c_\infty)^8 = 0, \tag{14}
  \]
  with only one real root in the interval $[0, 1]$: $c_{\text{MF}}^\text{eq4} = 0.334$.

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