Nitsche’s Finite Element Method for Model Coupling in Elasticity

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Abstract
We develop a Nitsche finite element method for a model of Euler–Bernoulli beams with axial stiffness embedded in a two–dimensional elastic bulk domain. The beams have their own displacement fields, and the elastic subdomains created by the beam network are triangulated independently and are coupled to the beams weakly by use of Nitsche’s method in the framework of hybridization.

1 Introduction
In this paper we continue our work on coupling of elastic models[1, 2, 3, 4]. Unlike previous coupling models[3, 5], we take as our starting point the hybridized approach of Burman et al.[6], where an auxiliary interface variable is introduced in the Nitsche framework. The hybridized formulation conveniently supports solution and preconditioning based on substructuring where the bulk variables are eliminated resulting in a system for the hybrid variable. Furthermore, the hybrid variable may be used to model interfaces with mechanical properties such as bending and membrane stiffness. We consider in particular embedded interfaces made up by beams-trusses embedded in a two dimensional elastic membrane with both strong and cohesive coupling between the beam-truss interface and the membrane. Using the hybridized Nitsche framework we easily derive a weak formulation, which directly leads to a finite element method by replacing the function spaces with conforming finite dimensional finite element spaces. The cohesive formulation is designed in such a way that we may let the stiffness in the coupling tend to infinity without loss of stability or convergence. We focus our attention on fitted meshes, that are not required to match on the interface, but the approach can directly be extended to cut finite element formulations.

The outline of the paper is as follows: In Section 2 we introduce the hybridized formulation for an elastic interface problem, in Section 3 we consider
interfaces with bending and membrane stiffness as well as strong and cohesive coupling to the elastic problem. In Section 4 we present some numerical examples illustrating the method and we observe optimal order convergence properties.

2 The Elastic Interface Problem

We begin by extending the hybridized Nitsche method proposed by Burman et al. [6] to the case of linearized elasticity. Let \( \Omega \) denote a bounded domain in \( \mathbb{R}^2 \). For ease of presentation, we consider only the case where \( \Omega \) is divided into two non-overlapping subdomains \( \Omega_1 \) and \( \Omega_2 \), \( \Omega_1 \cup \Omega_2 \), with interface \( \Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2 \); the case of several domains is a straightforward extension. We further assume that the subdomains are polygonal so that \( \Gamma \) is piecewise linear. In each subdomain, we assume plane stress linearized elasticity with homogeneous Dirichlet boundary conditions, i.e., we seek displacement fields \( (u_1, u_2, u_\Gamma) \) that are zero on \( \partial \Omega \). The problem takes the form: find \( u_i : \Omega_i \rightarrow \mathbb{R}^2 \) and \( u_\Gamma : \Gamma \rightarrow \mathbb{R}^2 \) such that

\[
\begin{align*}
\sigma(u_i) &= 2\mu \varepsilon(u_i) + \lambda \nabla \cdot u_i I \quad \text{in } \Omega_i \quad (2.1) \\
-\sigma(u_i) \cdot \nabla &= f_i \quad \text{in } \Omega_i \quad (2.2) \\
\llbracket \sigma(u) \cdot n \rrbracket &= 0 \quad \text{on } \Gamma \quad (2.3) \\
u_i - u_\Gamma &= 0 \quad \text{on } \Gamma \cap \partial \Omega_i \quad (2.4)
\end{align*}
\]

Here we used the notation

\[
\llbracket \sigma(v) \cdot n \rrbracket := \sigma(v_1) \cdot n_1 + \sigma(v_2) \cdot n_2
\]

where \( n_i \) denotes the outward pointing normal to \( \Omega_i \), \( \sigma(u) \) is the stress tensor, \( \varepsilon(u) = [\varepsilon_{ij}(u)]_{i,j=1}^2 \) is the strain tensor with components

\[
\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

\( I = [\delta_{ij}]_{i,j=1}^2 \) with \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \), and \( \lambda \) and \( \mu \) are the Lamé parameters in plane stress, so in terms of Young’s modulus, \( E \), and Poisson’s ratio, \( \nu \), we have

\[
\lambda = \frac{E \nu}{1 - \nu^2}, \quad \mu = \frac{E}{2(1 + \nu)}.
\]

Now, multiplying (2.2) by test functions \( v_i \), \( v_i = 0 \) on \( \partial \Omega \setminus \Gamma \), integrating
by parts over \( \Omega_i \), and using (2.4) we find

\[
\sum_i (f_i, v_i)_\Omega = \sum_i (-\sigma(u_i) \cdot \nabla, v_i)_{\Omega_i} = \sum_i (\sigma(u_i), \varepsilon(v_i))_{\Omega_i} - \sum_i (\sigma(u_i) \cdot n_i, v_i)_{\partial\Omega_i} = \sum_i (\sigma(u_i), \varepsilon(v_i))_{\Omega_i} - \sum_i (\sigma(u_i) \cdot n_i, v_i)_\Gamma - \sum_i (u_i - u_\Gamma, \sigma(v_i) \cdot n_i)_\Gamma + (\gamma_i(u_i - u_\Gamma, v_i - v_\Gamma)_\Gamma
\]

(2.8)

where \( \gamma_i \in \mathbb{R}^+ \) are arbitrary. Using the interface condition (2.3), in the form

\[
(J \sigma(u), n_K, u_\Gamma) = 0
\]

we finally obtain

\[
\sum_i (f_i, v_i)_\Omega = \sum_i (\sigma(u_i), \varepsilon(v_i))_{\Omega_i} - \sum_i (\sigma(u_i) \cdot n_i, v_i - v_\Gamma)_\Gamma - \sum_i (u_i - u_\Gamma, \sigma(v_i) \cdot n_i)_\Gamma + (\gamma_i(u_i - u_\Gamma, v_i - v_\Gamma)_\Gamma
\]

(2.12)

To formulate a finite element method we let \( V_h^1, V_h^2, V_h^\Gamma \) be conforming finite element spaces, such that \( u_h^i \in V_h^i \) and \( u_\Gamma^h \in V_h^\Gamma \). We then obtain the hybridized Nitsche method from Burman et al.[6], extended to linear elasticity: find \( (u_1^h, u_2^h, u_\Gamma^h) \in V_h^1 \oplus V_h^2 \oplus V_h^\Gamma \) such that

\[
\sum_i (f_i, v_i)_\Omega = \sum_i (\sigma(u_i^h), \varepsilon(v_i))_{\Omega_i} - \sum_i (\sigma(u_i^h) \cdot n_i, v_i - v_\Gamma)_\Gamma - \sum_i (u_i^h - u_\Gamma^h, \sigma(v_i) \cdot n_i)_\Gamma + \sum_i (\gamma_i(u_i^h - u_\Gamma^h, v_i - v_\Gamma)_\Gamma
\]

(2.14)

for all \( (v_1, v_2, v_\Gamma) \in V_h^1 \oplus V_h^2 \oplus V_h^\Gamma \).

**Remark 2.1** With local meshsize \( h_i \) on \( \Omega_i \) and the choice

\[
\gamma_i = \gamma_{0,i} h_i^{-1}
\]

(2.17)

it is possible to show that the bilinear form is coercive on the finite element space, provided the parameters \( \gamma_{0,i} \) are taken large enough, which together with Galerkin orthogonality and approximation properties of the finite element spaces leads to optimal order a priori error estimates. We refer to Burman et al.[6] for details.
3 Interfaces with Bending and Membrane Stiffness

3.1 Strong Coupling

We now add bending and membrane stiffness of the interface to our functional in the vein of model coupling in creeping flow [7]. We assume that we are given and arclength parameter \( s \) and a unit tangent vector \( t \) along \( \Gamma \), which creates an orthonormal system with the unit normal \( n \) to \( \Gamma \). For definiteness we assume that \( n = n_1 = -n_2 \), and \( t = t_1 = -t_2 \). We further split \( u_\Gamma \) into a normal and a tangential part

\[
u_\Gamma = u_n n + u_t t
\]

with \( u_t = t \cdot u_\Gamma \) and \( u_n = n \cdot u_\Gamma \).

The equilibrium equations on \( \Gamma \) are then assumed as follows

\[
\frac{d^2}{ds^2} \left( EI \frac{d^2 u_n}{ds^2} \right) = f_n - n \cdot \sigma \cdot n
\]

and

\[
- \frac{d}{ds} \left( EA \frac{du_t}{ds} \right) = f_t - t \cdot \sigma \cdot n
\]

where \( f_n \) and \( f_t \) are given external loads. Here \( EI \) denotes bending stiffness (with \( I \) the second moment of inertia) and \( EA \) axial stiffness (with \( A \) the cross section area), both possibly varying with position. These two equilibrium equations now replace the interface equilibrium (2.3) which no longer holds.

Again, multiplying (2.2) by test functions \( v_i, v_i = 0 \) on \( \partial \Omega \setminus \Gamma \), integrating by parts over \( \Omega_i \), and using (2.4) we find

\[
\sum_i (f_i, v_i)_{\Omega_i} = \sum_i - (\sigma(u_i) \cdot \nabla, v_i)_{\Omega_i}
\]

\[
= \sum_i (\sigma(u_i), \varepsilon(v_i))_{\Omega_i} - \sum_i (\sigma(u_i) \cdot n_i, v_i)_{\partial \Omega_i}
\]

\[
= \sum_i (\sigma(u_i), \varepsilon(v_i))_{\Omega_i} - \sum_i (\sigma(u_i) \cdot n_i, v_i)_{\Gamma}
\]

\[
- \sum_i (u_i - u_\Gamma, \sigma(v_i) \cdot n_i)_{\Gamma} + (\gamma_i(u_i - u_\Gamma), v_i)_{\Gamma}
\]

Writing \( v_\Gamma = v_n n + v_t t \) we see that

\[
\sum_i (\sigma(u_i) \cdot n_i, v_\Gamma)_{\Gamma} = ([\sigma \cdot n], v_\Gamma)_{\Gamma}
\]

\[
= (t \cdot [\sigma \cdot n], v_t)_{\Gamma} + (n \cdot [\sigma \cdot n], v_n)_{\Gamma}
\]

\[
= \left( f_t + \frac{d}{ds} \left( EA \frac{du_t}{ds} \right), v_t \right)_{\Gamma}
\]

\[
+ \left( f_n - \frac{d^2}{ds^2} \left( EI \frac{d^2 u_n}{ds^2} \right), v_n \right)_{\Gamma}
\]
Our Nitsche method thus takes the form: find \((u_1^h, u_2^h, u_n^h, u_t^h) \in V_1^h \oplus V_2^h \oplus V_n^h \oplus V_t^h\) such that

\[
\begin{align*}
&\sum_i (\sigma(u_i^h), \varepsilon(v_i))_\Omega - \sum_i (\sigma(u_i^h) \cdot n_i, v_i - v_n n - v_t t)_{\Gamma} \\
&\quad - \sum_i (u_i^h - u_n^h n - u_t^h t, \sigma(v_i) \cdot n_i)_{\Gamma} \\
&\quad + \sum_i (\gamma_i (u_i^h - u_n^h n - u_t^h t), v_i - v_n n - v_t t)_{\Gamma} \\
&\quad + \left( \frac{EA}{ds} \frac{du_i^h}{ds}, \frac{dv_i}{ds} \right)_{\Gamma} + \left( \frac{EI}{ds^2} \frac{d^2 u_i^h}{ds^2}, \frac{d^2 v_n}{ds^2} \right)_{\Gamma} \\
&= \sum_i (f_i, v_i)_\Omega + (f_t, v_t)_{\Gamma} + (f_n, v_n)_{\Gamma}
\end{align*}
\] (3.12)

for all \((v_1, v_2, v_n, v_t) \in V_1^h \oplus V_2^h \oplus V_n^h \oplus V_t^h\).

- In this setting, \(V_n^h\) must be a space of \(C^1(\Gamma)\)–continuous polynomials, whereas \(V_t^h\) can be \(C^0(\Gamma)\)–continuous. Thus it is reasonable to choose different discretizations for \(u_n^h\) and \(u_t^h\).

- For an interface that have a corner or bifurcates in a points we can not use \((u_n^h, u_t^h)\) as global degrees of freedom since these are not continuous if the segments meet at an angle. Thus we must transform the variables back to Cartesian coordinates. We give details for our chosen discretization below.

For brevity let us define

\[
A(u, v) := \sum_i (\sigma(u_i), \varepsilon(v_i))_\Omega + \left( \frac{EA}{ds} \frac{du_i}{ds}, \frac{dv_i}{ds} \right)_{\Gamma} + \left( \frac{EI}{ds^2} \frac{d^2 u_n}{ds^2}, \frac{d^2 v_n}{ds^2} \right)_{\Gamma}
\] (3.17)

\[
L(v) := \sum_i (f_i, v_i)_\Omega + (f_t, v_t)_{\Gamma} + (f_n, v_n)_{\Gamma}
\] (3.18)

The finite element method takes the form: find \(u^h := (u_1^h, u_2^h, u_n^h, u_t^h) \in V^h := V_1^h \oplus V_2^h \oplus V_n^h \oplus V_t^h\) such that

\[
\begin{align*}
L(v) &= A(u^h, v) - \sum_i (\sigma(u_i^h) \cdot n_i, v_i - v_n n - v_t t)_{\Gamma} \\
&\quad - \sum_i (u_i^h - u_n^h n - u_t^h t, \sigma(v_i) \cdot n_i)_{\Gamma} \\
&\quad + \sum_i (\gamma_i (u_i^h - u_n^h n - u_t^h t), v_i - v_n n - v_t t)_{\Gamma}, \quad \forall v \in V^h
\end{align*}
\] (3.19)

where we recall that

\[
u_t^h = u_n^h n + u_t^h t, \quad v_{\Gamma} = v_n n + v_t t
\] (3.22)
3.2 Cohesive Coupling

To model a weaker coupling we proceed in the spirit of Juntunen and Stenberg [8] and Hansbo and Hansbo [9]. The cohesive model, replacing the strong condition \( u_i = u_\Gamma \) on \( \Gamma \), is given by

\[
\sigma(u_i) \cdot n_i + S_i(u_i - u_\Gamma) = 0 \quad \text{on } \partial \Omega_i \cap \Gamma \tag{3.23}
\]

where \( S_i \) are coupling stiffness matrices, assumed to be of the form

\[
S_i = \frac{1}{\alpha_i} n \otimes n + \frac{1}{\beta_i} t \otimes t \tag{3.24}
\]

where \( 1/\alpha_i \) and \( 1/\beta_i \) are stiffness parameters normal and tangential to the interface, respectively. We are interested in the case where \( \alpha_i \) and \( \beta_i \) can be arbitrarily small, so we write (3.23) as

\[
C_i \sigma(u_i) \cdot n_i + u_i - u_\Gamma = 0 \tag{3.25}
\]

where \( C_i = S_i^{-1} = \alpha_i n \otimes n + \beta_i t \otimes t \). We then find

\[
L(v) = A(u, v) - \sum_i (\sigma(u_i) \cdot n_i, v_i - v_\Gamma)_\Gamma \tag{3.26}
\]

\[
= A(u, v) + \sum_i (\sigma(u_i) \cdot n_i, C_i \sigma(v_i) \cdot n_i)_\Gamma \tag{3.27}
\]

\[
- \sum_i (\sigma(u_i) \cdot n_i, C_i \sigma(v_i) \cdot n_i + v_i - v_\Gamma)_\Gamma \tag{3.28}
\]

\[
= A(u, v) + \sum_i (\sigma(u_i) \cdot n_i, C_i \sigma(v_i) \cdot n_i)_\Gamma \tag{3.29}
\]

\[
- \sum_i (\sigma(u_i) \cdot n_i, C_i \sigma(v_i) \cdot n_i + v_i - v_\Gamma)_\Gamma \tag{3.30}
\]

\[
- \sum_i (C_i \sigma(u_i) \cdot n_i + u_i - u_\Gamma, \sigma(v_i) \cdot n_i)_\Gamma \tag{3.31}
\]

\[
+ \sum_i (C_i \sigma(u_i) \cdot n_i + u_i - u_\Gamma, \tau_i(C_i \sigma(u_i) \cdot n_i + u_i - u_\Gamma))_\Gamma \tag{3.32}
\]

where the last two terms are zero due to the interface condition and the resulting form on the right hand side is symmetric. Furthermore, \( \tau_i \) is a stabilization matrix of the form

\[
\tau_i = \tau^i_n n \otimes n + \tau^i_t t \otimes t, \quad \tau^i_n = \frac{1}{h_i/\gamma_{0,i} + \alpha_i}, \quad \tau^i_t = \frac{1}{h_i/\gamma_{0,i} + \beta_i} \tag{3.33}
\]

where \( \gamma_{0,i} \) is sufficiently large (cf. Remark 2.1).

3.3 One-Sided Cohesive Coupling

It is clear that the cohesive model is unphysical in that the beams are allowed to penetrate the domains. Thus we need to enforce strong continuity of contact
type in such situations. We then enforce the contact constraints by way of
\( n_i \cdot (u_i - u_n n) \leq 0 \). With \( \sigma_n := n \cdot \sigma \cdot n \), \( \sigma_i := t \cdot \sigma \cdot n \), \( [v_n^i] := n_i \cdot (v_i - v_n n) \), and \( [v_t^i] := t_i \cdot (v_i - v_n t) \) our contact conditions on \( \Gamma \) can then be formulated, following Burman and Hansbo\[10]:

\[
\beta_i \sigma_i(u_i) + [u_t^i] = 0 \quad \text{on} \quad \partial \Omega_i \cap \Gamma \\
[u_n^i] \leq 0 \quad \text{on} \quad \partial \Omega_i \cap \Gamma,
\]

\[
\sigma_n(u_i) + \alpha_i^{-1}[u_n^i] \leq 0 \quad \text{on} \quad \partial \Omega_i \cap \Gamma
\]

\[
(\sigma_n(u_i) + \alpha_i^{-1}[u_n^i])[u_n^i] = 0 \quad \text{on} \quad \partial \Omega_i \cap \Gamma
\]

where we recognise \((3.35)-(3.37)\) as the Kuhn–Tucker conditions. We begin by rewriting \((3.32)\) in normal and tangential components, writing:

\[
L(v) = A(u, v) + \sum_i (\beta_i \sigma_i(u_i), \sigma_i(v_i))_\Gamma
\]

\[
- \sum_i (\sigma_i(u_i), \beta_i \sigma_i(v_i) + [v_t^i])_\Gamma
\]

\[
- \sum_i (\beta_i \sigma_i(u_i) + [u_t^i], \sigma_i(v_i))_\Gamma + \sum_i (\beta_i \sigma_i(v_i) + [v_t^i], \tau_i(\beta_i \sigma_i(u_i) + [u_t^i]))_\Gamma
\]

\[
- \sum_i (\sigma_n(u_i), [v_n^i])_\Gamma
\]

where we did not introduce the normal component of the cohesive law.

We now turn to the alternative formulation of the Kuhn–Tucker conditions due to Rockafellar\[11\], introduced in a Nitsche formulation for contact analysis by Chouly and Hild\[12\]. We recognize that

\[
p_i := \sigma_n(u_i) + \alpha_i^{-1}[u_n^i]
\]

act as multipliers (cf. Burman and Hansbo\[10\], Section 5.1), and the Kuhn–Tucker conditions are then equivalent to the relation

\[
p_i = -\frac{1}{\epsilon_i}([u_n^i] - \epsilon_i p_i)_+
\]

where \([x]_+ = \max(x, 0)\) and \(\epsilon_i > 0\) but arbitrary. We then write

\[
(\sigma_n(u_i), [v_n^i])_\Gamma = (\sigma_n(u_i) + \alpha_i^{-1}[u_n^i], [v_n^i])_\Gamma - (\alpha_i^{-1}[u_n^i], [v_n^i])_\Gamma
\]

\[
= (\sigma_n(u_i) + \alpha_i^{-1}[u_n^i], [v_n^i] - \epsilon_i(\sigma_n(v_i) + \alpha_i^{-1}[v_n^i]))_\Gamma
\]

\[
+ (\epsilon_i(\sigma_n(u_i) + \alpha_i^{-1}[u_n^i]), \sigma_n(v_i) + \alpha_i^{-1}[v_n^i]))_\Gamma
\]

\[
- (\alpha_i^{-1}[u_n^i], [v_n^i])_\Gamma
\]
Thus we have that
\[
- (\sigma_n(u_i), \|v_n^i\|)_\Gamma = (\alpha_i^{-1}\|u_n^i\|, \|v_n^i\|)_\Gamma \\
+ \left( \epsilon_i^{-1}\left[\|u_n^i\| - \epsilon_i(\sigma_n(u_i) + \alpha_i^{-1}\|u_n^i\|)\right] + \|v_n^i\| - \epsilon_i(\sigma_n(v_i) + \alpha_i^{-1}\|v_n^i\|)\right)_\Gamma \\
- (\epsilon_i(\sigma_n(u_i)) + \alpha_i^{-1}\|u_n^i\|), \sigma_n(v_i) + \alpha_i^{-1}\|v_n^i\|)_\Gamma
\]

To obtain well conditioned systems for small \( \alpha_i \), we examine the case when \( p_i = 0 \) and \( p_i \neq 0 \).

- If \( p_i \neq 0 \) we are in contact and
  \[
  - (\sigma_n(u_i), \|v_n^i\|)_\Gamma = - (\alpha_i^{-1}\|u_n^i\|, \|v_n^i\|)_\Gamma + (\epsilon_i^{-1}\|u_n^i\|, \|v_n^i\|)_\Gamma \\
  - (\sigma_n(u_i), \|v_n^i\|)_\Gamma - (\|u_n^i\|, \sigma_n(v_i))_\Gamma
  \]
  which, with \( \epsilon_i^{-1} = \gamma_{0,i}/h_i + \alpha_i^{-1} \) gives a scheme of the type (2.16).

- If \( p_i = 0 \) we have the cohesive law and
  \[
  - (\sigma_n(u_i), \|v_n^i\|)_\Gamma = (\alpha_i^{-1}\|u_n^i\|, \|v_n^i\|)_\Gamma \\
  - (\epsilon_i(\sigma_n(u_i) + \alpha_i^{-1}\|u_n^i\|), \sigma_n(v_i))_\Gamma \\
  = - (\alpha_i \sigma_n(u_i) + \|u_n^i\|, \sigma_n(v_i))_\Gamma \\
  - (\sigma_n(u_i), \alpha_i \sigma_n(v_i))_\Gamma + (\alpha_i \sigma_n(u_i), \sigma_n(v_i))_\Gamma \\
  + (\alpha_i - \epsilon_i)(\sigma_n(u_i) + \alpha_i^{-1}\|u_n^i\|), \sigma_n(v_i) + \alpha_i^{-1}\|v_n^i\|)_\Gamma
  \]
  and using again \( \epsilon_i^{-1} = \gamma_{0,i}/h_i + \alpha_i^{-1} \) we find the standard method for cohesive laws with penalty parameter \( (\alpha_i - \epsilon_i)/\alpha_i^2 = 1/(h_i/\gamma_{0,i} + \alpha_i) = \tau_i^n \).

Thus the parameter \( \epsilon_i \) does not change in contact and out of contact and the schemes become insensitive to small \( \alpha_i \).

4 Numerical examples

In this Section we illustrate the properties of the model and method by presenting some basic numerical examples. In all cases we used Young’s modulus \( E = 10^6 \) and Poisson’s ratio \( \nu = 1/3 \) in the bulk, and \( \gamma_{0,i} = 20(\lambda + \mu) \) as a Nitsche penalty parameter.

We consider a macro domain \( \Omega = (0, 2) \times (0, 1) \) split into 5 subdomains \( \Omega_1 \) to \( \Omega_5 \) separated by 6 line segments \( AB, BC, BD, CE, DE, \) and \( EF \), where \( A = (0, 1/2), B = (1 - 1/\sqrt{2}, 1/2), C = (1, 1), D = (1, 0), E = (1 + 1/\sqrt{2}, 1/2), \) \( F = (2, 1/2) \), see Fig. 1.

We use one cubic \( C^1 \) polynomial in each segment for the approximation of the interface normal displacement \( u_n \), and one linear polynomial in each segment for the tangential displacement \( u_t \). Continuity at the endpoints of the segments
is achieved by a transformation to Cartesian coordinates of the nodal variables by use of

\[
\begin{bmatrix}
  u_x \\
  u_y \\
  \theta
\end{bmatrix}
= \begin{bmatrix}
  n_x & t_x & 0 \\
  n_y & t_y & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  u_n \\
  u_t \\
  \theta
\end{bmatrix}
\]  

(4.1)

where \( \theta := u_n'(s) \) is the “rotation” degree of freedom in the \( C^1 \) approximation.

The boundary conditions are: Dirichlet boundary conditions \((u_x, u_y) = (0, 0)\) at \( x = 0 \) and zero Neumann conditions elsewhere. These are also imposed on the interface variables. We further impose zero rotation for the interface variable \( u_n \) at \( x = 0 \) in the case \( EI > 0 \). The approximation in the domains is a \( P^1-C^0 \) approximation (constant strain triangle). In the case of cohesion, we use the same constants \( \alpha_i =: \alpha \) and \( \beta_i =: \beta \) for all interfaces.

4.1 Bending of a Cantilever Structure

We consider constant loads \( f_i = (0, -2 \times 10^4), i = 1, \ldots, 5 \), and show the computational results for the standard hybrid method \((EI = EA = \alpha = \beta = 0)\) in Fig. 2. We next show the effect of increasing bending stiffness on the interface, with \( EI = 10^4 \) and \( EI = 10^5 \) in Fig. 3. The stiffening effect is noticeable.

Finally, in Fig. 4 we show the effect of normal compliance at a fixed bending stiffness \( EI = 10^4 \). We show the results for \( \alpha = 10^{-6} \) and \( \alpha = 10^{-5} \). The contact algorithm is invoked to avoid domain penetrations.

4.2 Stretching

The loads in this example are \( f_i = (10^5, 0), i = 1, \ldots, 5 \), to give a stretch of the domain. In Fig. 5 we show the result for the standard hybridized method. We next show the effect of adding membrane and bending stiffness on the interface, with \( EA = 10^6 \) and \( EI = 0 \) and with \( EA = 10^6, EI = 10^4 \) in Fig. 6. We then compare the effect of tangential cohesion, \( \beta = 10^{-5} \), with that of normal cohesion, \( \alpha = 10^{-5} \), in Fig. 7. Finally, in Fig. 8 we show the effect of having both tangential and normal cohesion.

5 Concluding Remarks

We have introduced a hybridized Nitsche method for linearized elasticity which uses an auxiliary interface displacement, modelled independently of the domains. This allows for easy modeling of different stiffness models at the interface. We have focused here on Euler–Bernoulli beam bending and membrane stiffness, but other models can be easily accommodated; the only requirement is that continuity of displacements between the interface field and the domain fields can be represented. We have also suggested weaker couplings between the interface and domains in the form of a cohesive interface law, with no-penetration fulfilled. This leads to a nonlinear contact problem which fits straightforwardly in the general framework of Nitsche’s method. Some numerical examples are provided.
to show how different parameter choices affect the solution in bending and in stretching of a plane elasticity problem.

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Figure 1: Cantilever domain.
Figure 2: Deformations for the hybrid method without interface stiffness.

Figure 3: Deformations with interface bending stiffness, $EI = 10^4$ (left) and $EI = 10^5$ (right).
Figure 4: Deformations with interface bending stiffness and normal cohesion, $EI = 10^4$; $\alpha = 10^{-6}$ (left) and $\alpha = 10^{-5}$ (right).

Figure 5: Stretch deformations for the hybrid method without interface stiffness.
Figure 6: Deformations with interface membrane stiffness and with combined membrane/bending stiffness, $EA = 10^6$; $EI = 0$ (left) and $EI = 10^4$ (right).

Figure 7: Deformations with interface stiffness and with cohesion, $EA = 10^6$, $EI = 10^4$; $\beta = 10^{-5}$, $\alpha = 0$ (left) and $\beta = 0$, $\alpha = 10^{-5}$ (right).
Figure 8: Deformations with interface stiffness and with cohesion, $EA = 10^6$, $EI = 10^4$, $\beta = 10^{-5}$, $\alpha = 10^{-5}$. 