Abstract

We provide an overview of iterated function systems (IFS), where randomly chosen state-to-state maps are iteratively applied to a state. We aim to summarize the state of art and, where possible, identify fundamental challenges and opportunities for further research.

1 Introduction

The fundamental mathematical tool used in this study is discrete-time Markov chains, which are generated by random repetition of functions and exhibit long-term statistical behaviour. Allow us to encourage readers through an example from (Chakraborty, 2016, Section 2.1, Chapter 1); similar examples could be found in (Barnsley, 1993, Chapter 3). Assume a particle is located at the origin, i.e., at \((0,0)\) of a two-dimensional Euclidean plane. Consider a die with four sides denoted by the integers 1, 2, 3, 4. The particle moves on the plane according to rules as shown in Table 1: Initially, the particle is at \((0,0)\); now, imagine 2 shows up after throwing the die. From the rules described in the table, the particle will move to \((0.6x + 0.19, 0.6y + 0.5)\) where \((x,y)\) is its current coordinate. For us, \((x,y) = (0,0) = X_0\) the initial position of the particle, and since 2 has appeared, the next position or co-ordinate of the particle is

\[
(0.6 \times 0 + 0.19, 0.6 \times 0 + 0.5) = (0.19, 0.6) = X_1.
\]

We mark the point \((0.25,0.4)\) with a pen. We throw the die again, and depending on the number, and the particle moves to a different point from \((0.25, 0.4)\). If we repeat these steps for, say, 20,000 times (Of course, that requires the help of a computer) and mark those points on the plane, we get a picture similar to Figure 1.
These points visited by the particle are insufficient to cover all points in the leaf and produce a fully formed naturally shaped leaf devoid of holes. After several iterations, a two-dimensional maple leaf is the particle’s sole visited or so-called accessible position. One may demonstrate that with probability one, the sequence of points $X_0, X_1, X_2, \ldots$, referred to as a sample path or trajectory, would ultimately pass through any portion of the Maple leaf. The exciting aspect of the game is that even though the particle’s travel was random in each step, the movement’s long-term behaviour took on the form of a Maple leaf. The outcome will be the same regardless of how many individuals play the game concurrently or how many times. Particular emphasis should be placed on the terms “with probability one”, which will be explained subsequently. The trajectory will approach the same point if the dice consistently display the same face. However, for a fair dice, this occurrence has zero chance. The act of continually tossing the dice is referred to as “iteration”, and the rule dictating how the particle should advance to the next step from the present position is known as a “function”.

It could be easily seen that the four rules described in the table above can be seen as the following functions $w_1, w_2, w_3, w_4$ acting on $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ as follows:

\[
\begin{align*}
w_1 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.04 \end{bmatrix} \\
w_2 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.19 \\ 0.5 \end{bmatrix} \\
w_3 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0.466 & -0.466 \\ 0.466 & 0.466 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.266 \\ 0.067 \end{bmatrix} \\
w_4 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0.466 & 0.466 \\ -0.466 & 0.466 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.456 \\ 0.434 \end{bmatrix}
\end{align*}
\]
The chaotic game seen above illustrates the “random repetition of functions” (where the randomness appeared due to the throwing of the die).

1.1 Some Motivating Applications of IFS

Example 1.1. Non-linear time series in Statistics: A first-order autoregression model is defined in non-linear time series analysis as

\[ X_k = \phi X_{k-1} + Z_k, \quad k \in \mathbb{Z}, \]

is critical in time series analysis because it represents a strong (i.i.d.) white noise sequence. Establishing the requirements for a stationary solution to the first-order autoregressive equation is simple. A nearly comparable process is provided by

\[ Y_k = A_k Y_{k-1} + B_k, \quad k \in \mathbb{Z}, \tag{1} \]

where \((A_k, B_k)\) is a series of independently generated random vectors in \(\mathbb{R}^2\). These models are critical in non-linear time series modelling. The presence of strict-sense and weak-sense stable solutions to the equation (1) is a natural issue. Under certain weak circumstances, (Douc et al., 2014, Theorem 4.1) established the existence of strict-sense stationery. The theorem has been further generalized to functional autoregressive process using IFS techniques (Douc et al., 2014, Theorem 4.40, 4.41) and based on these results, a weak-sense stationary solution to (1) has been shown in (Douc et al., 2014, Theorem 4.47) under the assumptions (Douc et al., 2014, Assumption A4.43-A4.45).

Example 1.2. IFS in finance: Uniquely ergodic IFS with constant probability are valuable alternatives to the standard binomial models (see Van der Hoek and Elliott (2006)) of stock price development that are used to estimate one-period option pricing (Swishchuk and Islam, 2013, Section 3.8). A generalised binomial model Cox et al. (1979) is found using IFS with state-dependent probability for call option prices, bond valuation, stock price evolution, and interest rates. See (Swishchuk and Islam, 2013, Section 4.6.1, 4.6.4) and (Bahsoun et al., 2005b).

Example 1.3. Stability of congestion management, e.g., in Transmission Control Protocol: The distribution of limited resources among several agents in different scientific fields and applications is one of the most challenging problems. For example, the bandwidth available for communication on computer networks is limited. Each connection can simultaneously support several data flows, each seeking to optimize its portion of the available bandwidth. A circumstance like that occurs when many electric cars are charged concurrently at the same charging station and compete for the same electricity or charge rate. A scalable and robust solution to this challenge is required. Additionally, a decentralized technique is appropriate when there is limited interaction between users and maintaining anonymity is critical. The method additive increase and multiplicative decrease (AIMD) has been quite effective in resolving the issue (Chiu and Jain, 1989). In actuality, a more complex and challenging scenario occurs. To address such scenarios, the stochastic linear AIMD method and the stochastic additive increase and non-linear decrease (AINLD) algorithm developed in (Corless et al., 2016, Chapter 6, 7, 8) have been extensively investigated utilizing IFS and their ergodic features. For example, in (Corless et al., 2016, Chapter 7) the AIMD with state-dependent transition probabilities has been modelled as

\[ w(k+1) = A(k)w(k) \quad k = 0, 1, 2, \ldots, \tag{2} \]

where \(w(k) \in S_n\) is vector-valued random variable and defined as

\[ S_n = \{w(k) = (w_1(k), \ldots, w_n(k)) \in \mathbb{R}^n : \sum_{i=1}^{n} w_i(k) = c\}, \]
where for each $i$, $w_i(k)$ is the amount of share consumed by the $i^{th}$ user and $k = 0, 1, 2, \ldots$ represents discrete time instants, $c$ represent the overall capacity of the resource accessible to the whole system, and $c = 1$ is assumed for the purpose of easy computation. AIMD matrix $A(k)$ is given by

$$A(k) = \begin{bmatrix} \beta_1(k) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_n(k) \end{bmatrix} + \frac{1}{\sum_{j=1}^{n} \alpha_j} \begin{bmatrix} \alpha_1(k) \\ \vdots \\ \alpha_n(k) \end{bmatrix} \begin{bmatrix} (1 - \beta_1(k)) & \cdots & (1 - \beta_n(k)) \end{bmatrix}$$

where $\alpha_i(k) > 0$ is the rate of growth of the agent $i$ after the $k^{th}$ capacity event (a time when the resource reaches its capacity constraint, i.e., $\sum_{i=1}^{n} w_i(k) = c = 1$) and $\beta_i(k)$ denote multiplicative decrease quantity at the $k^{th}$ capacity event depending on the response of the $i^{th}$ user at the capacity event, $0 \leq \beta_i(k) \leq 1$.

It can be shown that the AIMD matrices are non-negative, column stochastic. The set of matrices arising through (2) is denoted by finite set of indices $\mathcal{S}$ and $\mathscr{A} = \{A_\sigma : \sigma \in \mathcal{S}\}$. The technique is stochastic because the index $\sigma(k)$ is created randomly for each event $k$; that is, $\sigma(k)$ is a random variable. Thus, $A(k) = A_{\sigma(k)}$.

The equation (2) may be used to represent the dynamical system of interest in the networks of AIMD flows. We refer to a state-dependent AIMD model in the paper (Corless et al., 2016, Chapter 7).

Consider $\{p_j : j \in \mathcal{S}\}$ as a collection of probability functions from the simplex $S_n$ into the closed interval $[0, 1]$ that fulfil

$$\sum_{j \in \mathcal{S}} p_j(w) = 1 \quad \text{for all } w \in S_n.$$ 

Here, $p_j(w)$ denotes the probability that the matrix $A_j$ occurs when the state or the share vector is $w$, i.e.,

$$P(\sigma(k) = j \mid x(k) = x) = p_j(x) \quad k = 0, 1, 2, \ldots$$

or,

$$P\left(A(k) = A_j \mid w(k) = w\right) = p_j(w) \quad k = 0, 1, 2, \ldots$$

Take note that (3) and (4) create a stochastic AIMD algorithm, and they are an IFS with the state (or place) dependent probabilities, and more precisely, a Markov chain on $S_n$, whose state transition probabilities are provided by

$$P(w, \mathscr{A}) = P\left(w(k + 1) \in \mathscr{A} \mid w(k) = w\right) = \sum_{j : A_j w \in \mathscr{A}} p_j(w) \quad \text{for all } w \in S_n, \text{ and for any event } \mathscr{A}. \quad (5)$$

Having created AIMD’s mathematical framework, we may question how the AIMD algorithm distributes resources efficiently among the participants. Can the algorithm ensure equal shares for all network agents? Several other relevant research questions are highlighted in (Corless et al., 2016, Chapter 1, Section 1.4).

To know more about this direction please see Dumas et al. (2002); Wirth et al. (2006b); Shorten et al. (2007); Wirth et al. (2006a); King and Shorten (2006); Shorten et al. (2007); Corless and Shorten (2012a,b); Fioravanti et al. (2017, 2019); Crisostomi et al. (2016); Corless et al. (2016); Epperlein and Mareček (2017); Wirth et al. (2019); Leśniak et al. (2021); Ghosh et al. (2022).

**Example 1.4.** Closed-loop models within sharing economy: Imagine if $\mathcal{S}_1, \ldots, \mathcal{S}_N$ an $N$ number of agent-based system with with a controller $\mathcal{C}$, filter $\mathcal{F}$, in a closed-loop feedback as in Figure 2. At time $k$, $\mathcal{C}$ generates a signal $\pi(k)$. As a result, agents adjust the way they utilize the resource. Due to unpredictability
in the agent’s reaction to the control signal, randomized control signal, or random perturbations of the control signal, for all , the th agent’s state at time is treated as a random variable, and we note that has no access to or but only able to get access to the error signal , which indicates a difference between the value that is produced by the filter and the desired output value . Smart grid and smart city applications may be shown with this simple setup. In this closed-loop feedback system, a stochastic difference equation governs the state of each agent:

where are taken as general response functions of agents in the network, are valued independent identically distributed (i.i.d) discrete random variables. This representation is precisely an IFS system.

The controller and filter take all conceivable initial state distributions into account to guarantee that the system’s long-term behaviour is acceptable for all agents involved. For each stable linear controller and filter, we seek conditions where the feedback loop controls total resource consumption and converges to a unique invariant measure. The above concept and strategy have been investigated in a variety of other engineering fields, including smart-city and smart-grid control (Fioravanti et al., 2017, 2019), congestion control (Shorten et al., 2005), resource allocation (Epperlein and Mareček, 2017), load aggregation and operation of virtual power plants (Marecek et al., 2021), social sensing platforms (Ghosh et al., 2022), and model predictive control (Kungurtsev et al., 2021).

Example 1.5. Opinion dynamics: There has long been a focus of research at the junction of economics and dynamical systems; for example, ’s pioneering work utilizing IFS Acemoglu et al. (2010); Acemoglu et al. (2013), and the references therein. The Friedkin-Johnsen model Friedkin and Johnsen (2011) accurately depicts opinion dynamics characterized by heterogeneity. ’s 2019 showed that the Friedkin-Johnsen model is a specific example of their expanded Rescorla-Wagner model Rescorla (1972). Surprisingly, this extended model with random time-varying topology is exactly a recurrent IFS, and the suggested model’s internal states exhibit both convergence and ergodicity.

Example 1.6. Stochastic dynamic decision model: Following (Hinderer, 1970, Chapter 1), let a set be called the state space, and let be another set called action space. Let be a sequence of maps from certain sets to ( factors), according as or , into the class of all non-empty subsets of such that

is called the set of histories at time , the set of all admissible histories at time , and the set of all admissible actions at time under admissible history . Denoting we
can write $H_{k+1} = S_k \times \mathcal{X}, k \in \mathbb{N}$. Let $p_0$ be a probability distribution on $\mathcal{X}$ called the initial distribution and $p_k$ is a transition probability function from $S_k$ to $\mathcal{X}$, i.e. for any $(h,a) \in S_k$, $p_k(h,a,\cdot)$ is a probability distribution on $\mathcal{X}$, called transition law from time $k$ to $k+1$. And, finally, let $r_k$ is an extended real-valued function on $S_k$ called reward during the time interval $(k,k+1)$. A collection $\{\mathcal{X}, \mathcal{A}, W, (p_k)_{k \in \mathbb{N}}, (r_k)_{k \in \mathbb{N}}\}$ with the above properties is called a stochastic dynamic decision model. Choosing a policy and an optimality criterion is necessary to define a dynamic optimisation problem under the suitable assumption on $r_k$. A deterministic admissible policy is $f = (f_k)_{k \in \mathbb{N}}$ of maps $f_k : \mathcal{X}^k \rightarrow \mathcal{A}$ such that

$$f_k(y) \in w_k \left( h_k^f(x) \right), x \in \mathcal{X}^k, k \in \mathbb{N},$$

where

$$h_k^f(x) := \begin{cases} (x_1), & \text{if } k = 1 \\ (x_1, f_1(x_1), x_2, f_2(x_1, x_2), \ldots, f_{k-1}(x_1, \ldots, x_{k-1}), x_k), & \text{if } k \geq 2. \end{cases}$$

denotes the history at time $k$ obtained when the sequence $x = (x_1, \ldots, x_k)$ of states occurred and actions were chosen according to $f$. The application of a policy $f$ generates a stochastic process, the decision process determined by $f$, is as follows: Starting at some $k = 1$ at some point $x_1$, selected from $\mathcal{X}$ according to the initial probability distribution $p_0$, one takes action $f_1(x_1)$ whereupon one gets the reward $r_1(x_1, f_1(x_1))$, and the system moves to some $x_2 \in \mathcal{X}$ selected according to $p_1(x_1, f_1(x_1), \cdot)$. Then the action $f_2(x_1, x_2)$ is taken, giving the $r_2(x_1, f_1(x_1), x_2, f_2(x_2))$ and the system moves to some $x_3 \in \mathcal{X}$ selected according to $p_2(x_1, f_1(x_1), x_2, f_2(x_1, x_2), \cdot)$ and so on. Thus for a given policy, the sequence of states $\{x_k\}_{k \in \mathbb{N}}$ then appears equivalent to the IFS.

Example 1.7. In DNA replication model: The DNA replication kinetic equation is shown in (Gaspard, 2017, Equation 2). New DNA strands are created by molecular machinery utilizing templates from previous generations. To correctly solve replication kinetic equations, one may use iterative function systems that run along the template sequence and provide copy sequence statistics and kinetic and thermodynamic parameters. DNA polymerase local velocity distributions with fractal and continuous sequence heterogeneity are studied using this approach, as well as the transition between linear and sub-linear copy growth rates.

1.2 Organization of the Survey and Past Surveys on IFS

Our survey is organized as follows: Sections 3–6 survey key research areas within the current IFS literature. Section 7 of the survey focuses on structural results, which can be seen as the geometry of the unique invariant measure. Finally, Section 8 highlights several possible research directions. In Figure 3, 4 and 5, we have broken down the key areas to ease navigating the survey. In the appendices, we provide pointers to key papers within applications of IFS.

We should also like to point to the excellent and well-known surveys of Diaconis and Freedman (1999), and Iosifescu (2009), and the lesser-known surveys and reviews of Kajiser (1981); Stenflo (2012b, 1998b); Letac et al. (1986); Chamayou and Letac (1991); Carlsson (2005a); Bhattacharya and Majumdar (2003); Athreya (2003b, 2016); Fuh (2004); Majumdar (2009); Kunze et al. (2011); Denker and Waymire (2016). For a survey on a random iteration of quadratic functions, see Athreya and Bhattacharya (2001). The sheer fact that our survey appears more than two decades after Diaconis and Freedman (1999) makes it possible to cover more of the recent developments and thus be more comprehensive.

2 Introduction and Historical Background of IFS

This part will concentrate on IFS concepts and conclusions critical for understanding how the theory might be used for various practical uses. The objective of this part is twofold: to present a brief overview of IFS
theory, concentrating on a few instances relevant to image generation, and to summarise numerous additional conclusions of general interest relevant to this thesis to keep this paper self-contained as feasible. After briefly discussing some essential facts about metric spaces, contractive transformations, and affine transformations, the rest of this section will explore the definitions and a few fundamental features of IFS. We refer Barnsley (1993); Edgar (2007); Peitgen et al. (2006) for a more in-depth discussion.

The term IFS has become well-known and popular in pure and applied science, particularly mathematics, probability, quantum physics, mathematical biology, computer science, electrical engineering, and economics, over the last seventy years. Numerous IFS theories derive from the theory of random systems with complete connections. According to Iosifescu and Grigorescu (1990)’s book, the first explicit definition of the idea of dependency with complete connection was created by Onicescu (1935), who used a discrete state space. The idea of random systems with complete connection was initially established in 1963 by Iosifescu (1963). A few applications of random systems with complete connection have been recognised in the extensively investigated field of theory of learning models in Norman (1972); Isaac (1962); Burton and Keller (1993). In this topic, Karlin et al. (1953) wrote a seminal study in which he studied a learning model that can be thought of as an IFS with place-dependent probability on an interval of the real line. The proof of (Karlin et al., 1953, Theorem 36) had some gap which later led to some exciting counterexamples in the IFS with a finite number of maps with place dependent probabilities, see for example Stenflo (2001b); Keane (1972); Barnsley and Elton (1988); Kaijser (1994, 1981) for further comments on Karlin’s work Karlin et al. (1953).

Barnsley and Demko (1985) first named the word IFS and has garnered much attention since then. Their study sparked widespread interest in field applications involving the representation of real-world images utilising IFS’s two-dimensional transformation. It was well-known that IFS could accurately depict a complicated object with just a few carefully selected parameters and that this system could be coupled to make more complex pictures.

2.1 Mathematical Preliminaries on IFS

Before we go any further, let us review some fundamental concepts, notations, and nomenclature. Throughout this article, we assume $\mathcal{X} \neq \emptyset$, and $\rho : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ is a metric on it, we shall denote $(\mathcal{X}, \rho)$ the metric space consisting of the set $\mathcal{X}$ endowed with the metric $\rho$. We will often be assumed to be complete and separable, i.e., a Polish space. The requirement that the metric space $(\mathcal{X}, \rho)$ is to be complete will be apparent upon the statement of the contraction mapping theorem and specification of its role in developing the theory of IFS.

A $\sigma$-algebra of $\mathcal{X}$ is a collection of subsets of $\mathcal{X}$ that is closed by complements, countable unions, and countable intersections operation includes $\mathcal{X}$. The smallest $\sigma$-algebra, which contains all the open sets of $\mathcal{X}$, is called the Borel $\sigma$-algebra. $\mathcal{B}(\mathcal{X})$ is the Borel sigma-algebra on $\mathcal{X}$, and $\mathcal{A}$ is always an event in $\mathcal{B}(\mathcal{X})$. $C(\mathcal{X}, \mathbb{R})$ is the Banach space that corresponds to all real-valued continuous functions operating on $\mathcal{X} \rightarrow \mathbb{R}$ endowed with the supremum norm $\|\cdot\|_{\infty}$. $C_{b}(\mathcal{X}, \mathbb{R})$ is structurally identical to $C(\mathcal{X}, \mathbb{R})$, except its functions are continuous and bounded. $\mathcal{M}(\mathcal{X})$ is the real vector space that contains all signed finite Borel measures on $\mathcal{X}$ that include $\mathcal{M}_{1}(\mathcal{X})$, which is the space that contains all positive measures. $\mathcal{M}_{b}(\mathcal{X})$ represents space of all probability measures on $\mathcal{X}$ included in $\mathcal{M}(\mathcal{X})$. $\mathcal{B}_{\text{hm}}(\mathcal{X}, \mathbb{R})$ denotes the space of all bounded, measurable, real-valued functions on $\mathcal{X}$ and $w: \mathcal{X} \rightarrow \mathbb{R}$ is always continuous, Lipschitz, and measurable and in addition to possibly having other properties which will be mentioned accordingly.

An appropriate probability triplet $(\mathcal{X}, \mathcal{E}, \mathbb{P})$, where $\mathbb{P}$ is the probability measure, the set $\mathcal{X} \neq \emptyset$, and $\mathcal{E}$ is a collection of events, is considered throughout the article. $\mathbb{P}_{x}$ denote the probability measure in a context where the initial condition is taken as $x$, more precisely, when an event $\mathcal{A}$ has probability $\mathbb{P}_{x}(\mathcal{A}) = 1$, we say that the event occurs almost surely $\mathbb{P}_{x}$ or in short a.s $\mathbb{P}_{x}$. The expectation operator is always denoted by $E$ and $E_{\mu}$ when taken with respect to a probability measure $\mu$. 


Let \([m] := \{1, 2, \ldots, m\}\) be a finite set of natural numbers from 1 to \(m < \infty\), \(\mathbb{R}, \mathbb{N}, \mathbb{Z}\) denote the set of real, natural numbers and integers respectively. \(\mathbb{R}^n\) be the \(n\) fold Cartesian product of \(\mathbb{R}\), by \(\mathbb{R}^+\) we mean the set \(\{x \in \mathbb{R} : x > 0\}\). Throughout the article a point \(x \in \mathbb{R}^n\) will be represented by a \(n \times 1\) column vector \((x_1, x_2, \ldots, x_n)^\top\), where \(\top\) is usual transpose notation. A familiar example of a complete metric space is given by \((\mathbb{R}^n, \rho)\), where \(\rho\) is the Euclidean metric on \(\mathbb{R}^n\) defined by for any two points \(x = (x_1, x_2, \ldots, x_n)^\top, y = (y_1, y_2, \ldots, y_n)^\top \in \mathbb{R}^n:\nabla
\[
\rho (x, y) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}.
\]

Given a metric space \((\mathcal{X}, \rho), (\mathcal{H} (\mathcal{X}), \rho_h)\) be another metric space formed by the collection of nonempty, compact subsets of \(\mathcal{X}\) which is denoted by \(\mathcal{H} (\mathcal{X})\) and \(\rho_h\) be the Hausdorff metric to be defined momentarily.

First define the distance from \(x\) to \(Y\) as:
\[
\rho (x, Y) = \min_{y \in Y} \{\rho(x, y)\} \quad \text{for } X, Y \in \mathcal{H} (\mathcal{X}).
\]

Since \(y \mapsto \rho(x, y)\) is a continuous map and \(Y \neq \emptyset\) and compact, the minimum is attained. Next, define \(\rho (X, Y)\), the distance between two compact sets in \(\mathcal{H} (\mathcal{X})\):
\[
\rho (X, Y) = \max_{x \in X} \{\rho (x, Y)\} \quad \text{for } X, Y \in \mathcal{H} (\mathcal{X}).
\]

For the same reason as above, the maximum is also attained. One should notice that, in general, \(\rho (X, Y)\) and \(\rho (Y, X)\) may not be same, i.e. \(\rho (\cdot, \cdot)\) is not symmetric on \(\mathcal{H} (\mathcal{X}) \times \mathcal{H} (\mathcal{X})\), and hence is not a metric. (Barnsley, 1993, Section 2.6, Exercise 6.7).

Now we are in a position to define the Hausdorff metric \(\rho_h\):
\[
\rho_h (X, Y) = \max \{\rho (X, Y), \rho (Y, X)\} \quad \text{for } X, Y \in \mathcal{H} (\mathcal{X}).
\]

We refer (Barnsley, 1993, Section 2.6, Exercise 6.15) for the fact that \((\mathcal{H} (\mathcal{X}), \rho_h)\) is indeed a metric space.

Given a non-empty set \(\mathcal{X}\), by a self-transformation or self map on it, we mean a map \(w\) with domain and range is \(\mathcal{X}\), i.e. \(w : \mathcal{X} \to \mathcal{X}\). A very important class of transformations of one metric space into itself or another, which we shall use frequently, consists of the collection of contractive transformations. Given two metric spaces \((\mathcal{X}, \rho_1)\) and \((\mathcal{Y}, \rho_2)\), a transformation \(T : \mathcal{X} \to \mathcal{Y}\) is said to be a contraction if and only if there exists a real number \(r \in [0, 1]\) such that for all \(x_1, x_2 \in \mathcal{X}\):
\[
\rho_2 (T(x_1), T(x_2)) \leq r \rho_1 (x_1, x_2).
\]

\(r\) is called a contractivity factor for \(T\). \(T\) is called a strict contraction if \(r < 1\). Thus when \(T\) is a self-transformation on \(\mathcal{X}\), i.e. \(T : \mathcal{X} \to \mathcal{X}\), then it acts on pairs of points in \(\mathcal{X}\) by bringing them closer together, their distance is reduced by a factor at least \(r\). Let \(M_n (\mathbb{R})\) denotes the set of all \(n \times n\) matrices with entries from \(\mathbb{R}\), \(T\) is called an affine transformation on \(\mathbb{R}^n\) if for some \(A \in M_n (\mathbb{R})\) and a vector \(b \in \mathbb{R}^n\), \(T\) is of the form
\[
T : \mathbb{R}^n \to \mathbb{R}^n \quad \text{and } T(x) = Ax + b.
\]

The next theorem describes a contractive transformation on a complete metric space to itself and known contraction mapping theorem. This result confirms the existence of a single fixed point for the contraction map \(T\) and proposes a mechanism for calculating a fixed point.

**Theorem 1.** (Barnsley, 1993, Section 3.6, Theorem 1) Let \(T : \mathcal{X} \to \mathcal{X}\) be a strict contraction on a complete metric space \((\mathcal{X}, \rho)\). Then, there exists a unique point \(x^* \in \mathcal{X}\) such that \(T(x^*) = x^*\). Furthermore, for any \(x \in \mathcal{X}\), we have
\[
\lim_{n \to \infty} T^n (x) = x^*,
\]
where \(T^n\) denotes the \(n\)-fold composition of \(T\) with itself.
Consider a metric space $(X, \rho)$ and a finite set of strictly contractive transformations $w_k : X \to X$, $1 \leq k \leq m$, with respective contractivity factors $r_k$. Define a transformation $w : \mathcal{H}(X) \to \mathcal{H}(X)$, where $\mathcal{H}(X)$ is the collection of nonempty, compact subsets of $X$, by:

$$w(X) = \bigcup_{k=1}^{m} w_k(X), \text{ for any } X \in \mathcal{H}(X).$$

(13)

Since $w$ is a strict contraction with contractivity factor $r^* = \max_{1 \leq k \leq m} \{r_k\}$ (Barnsley, 1993, Section 3.7, Lemma 5), and $(\mathcal{H}(X), \rho_n)$ is a complete metric space (Barnsley, 1993, Section 2.7, Theorem 1), we conclude that $w$ possesses a unique fixed point by Theorem 1. Thus if we denote the fixed point by $X^*$, then the map $w$ satisfy the following self-covering condition

$$w(X^*) = \bigcup_{k=1}^{m} w_k(X^*).$$

(14)

We are now able to define what is called hyperbolic IFS formally.

**Definition 2.1** (Hyperbolic IFS; Hutchinson (1981)). A hyperbolic IFS $\{w; w_1, w_2, \ldots, w_m\}$ consists of a complete metric space $(X, \rho)$ and a finite set of strictly contractive transformations $w_k : X \to X$ with contractivity factors $r_k$, for $k = 1, 2, \ldots, m$. The maximum $r^* = \max_{1 \leq k \leq m} \{r_k\}$ is called a contractivity factor of the IFS and the unique fixed point $X^* \in \mathcal{H}(X)$ of the transformation defined in (13) is called the attractor of the IFS.

**Definition 2.2** (Modulus of uniform continuity). Let $w : X \to \mathbb{R}$ be a uniformly continuous function. The modulus of uniform continuity is the function $\phi : [0, \infty) \to [0, \infty)$ defined by

$$\phi(t) := \sup_{x,y \in X, \rho(x,y) \leq t} |w(x) - w(y)|.$$  

(15)

**Definition 2.3** (Dini continuous). A function $w : X \to \mathbb{R}$ is said to be Dini continuous if it has modulus of continuity $\phi(t)$ such that

$$\frac{\phi(t)}{t}$$

is integrable over $(0, \delta)$ for some $\delta > 0$

(16)

and we say that $\phi$ satisfies the Dini’s condition.

Note that Lipschitz and Holder continuous functions are Dini continuous and that Dini continuity is a stronger condition than continuity.

**Definition 2.4** (Total-variation (TV) distance; Proposition 4.2 in Levin et al. (2006)). Let $\nu_1, \nu_2 \in \mathcal{M}_n(X)$, then throughout the article, the total-variation distance between $\nu_1$ and $\nu_2$ will be denoted by $d_{tv}(\nu_1, \nu_2)$, and is expressed as

$$d_{tv}(\nu_1, \nu_2) = \sup_{\mathcal{A} \in \mathcal{F}(X)} |\nu_1(\mathcal{A}) - \nu_2(\mathcal{A})|. $$

(17)

If $\mathcal{X}$ is finite with cardinality $n < \infty$, one can show the above expression is equivalent to the following:

$$d_{tv}(\nu_1, \nu_2) = \frac{1}{2} \sum_{i=1}^{n} |\nu_1(i) - \nu_2(i)|. $$

(18)
**Definition 2.5** (Prohorov metric; Billingsley (2013)). Let \( v_1, v_2 \in \mathcal{M}_p(\mathcal{X}) \) then the Prohorov distance between \( v_1 \) and \( v_2 \) is defined as

\[
d_{pr}(v_1, v_2) = \inf \{ \delta > 0 : v_1(\mathcal{X}) < v_2(\mathcal{X}_\delta) + \delta \text{ and } v_2(\mathcal{X}) < v_1(\mathcal{X}_\delta) + \delta \}
\] (19)

for all compact subsets \( \mathcal{X} \) of \( \mathcal{X} \) and \( \mathcal{X}_\delta := \{ x \in \mathcal{X} : \rho(x, \mathcal{X}) < \delta \} \). Dudley (1976) is a standard reference for further discussions on Prohorov metric.

**Definition 2.6** (Kolmogorov-Smirnov distance). For \( \mathcal{X} = \mathbb{R} \), Kolmogorov-Smirnov distance between \( v_1, v_2 \in \mathcal{M}_p(\mathcal{X}) \) is denoted by \( d_{ks} \) and is defined by

\[
d_{ks}(v_1, v_2) = \sup_{x \in \mathcal{X}} |v_1((-\infty,x]) - v_2((-\infty,x])|
\] (20)

In, other words if \( F_{v_1} \) and \( F_{v_2} \) denotes the distribution function corresponding to the probability measure \( v_1 \) and \( v_2 \) respectively then (20) is equivalent to the following expression:

\[
d_{ks}(v_1, v_2) = \sup_{x \in \mathcal{X}} |F_{v_1}(x) - F_{v_2}(x)|
\] (21)

The following metric is due to Kantorovich and Rubinstein Evans (1999); Svetlozar and Rüschendorf (1998); Villani (2009), also known as Wasserstein-1 distance.

**Definition 2.7** (Wasserstein-1 distance; Remark 6.5, p. 95 in Villani (2009)). Let \( \mathcal{W}_1 \) denote the space of all Lipschitz maps with Lipschitz constant 1, i.e

\[
\mathcal{W}_1 = \{ w \in \mathcal{C}(\mathcal{X}, \mathbb{R}) : |w(x) - w(y)| \leq \rho(x, y) \ \forall x, y \in \mathcal{X} \}.
\]

For \( v_1, v_2 \in \mathcal{M}_p(\mathcal{X}) \), \( d_{ws}(v_1, v_2) \) denotes the Wasserstein-1 distance between \( v_1, v_2 \in \mathcal{M}_p(\mathcal{X}) \) and is given by:

\[
d_{ws}(v_1, v_2) = \sup_{w \in \mathcal{W}_1} \left[ \int_{\mathcal{X}} wdv_1 - \int_{\mathcal{X}} wdv_2 \right].
\] (22)

For more on different kinds of metrics on the space of probability measures on metric space and their usefulness, see (Parthasarathy, 1967; Rachev, 1991; Chakraborty and Rao, 1998; Gibbs and Su, 2002; Gibbs, 2000; Kravchenko, 2006; Mostafaei and Kordnourie, 2011).

**Remark 2.1.** The Monge-Kantorovich metric is another name for the metric mentioned above (Kunze et al., 2012, Definition B.28). In the context of mass transportation, this metric was developed. Please see Hanin (1992, 1999); Kravchenko (2006); Vershik (2006) for more general findings and a historical timeline. Convergence in this metric is related to weak convergence, as seen from the definition of \( d_{ws} \). Convergence of probability measures in a compact metric space in \( d_{ws} \) and weak convergence are equivalent (Kunze et al., 2012, Proposition B.29). When \( \mathcal{X} \) is compact, the space \( (\mathcal{M}_p, d_{ws}) \) is complete Hanin (1992, 1999); Kravchenko (2006); Weaver (2018). Furthermore, this distance is difficult to calculate since finding an optimal \( w \in \mathcal{W}_1 \) to maximize the difference in (22) is generally not straightforward.

**Definition 2.8** (Markov Chain on Metric Space). Let \( (\Omega, \mathcal{E}, \mathbb{P}) \) be a probability space. A Markov chain on a metric space \( \mathcal{X} \) is a sequence of \( \mathcal{X} \) valued random variables \( \{X_k\}_{k \geq 0} \) whose interdependence satisfies the Markov property, i.e,

\[
\mathbb{P} [X_{k+1} \in \mathcal{A} | X_0, X_1, \ldots, X_k] = \mathbb{P} [X_{k+1} \in \mathcal{A} | X_k = x] \quad \text{for all } \mathcal{A} \in \mathcal{B}(\mathcal{X}).
\] (23)
Definition 2.9 (Stochastic Kernel or Transition Probability Function). For each $x \in \mathcal{X}$ and $\mathcal{A} \in \mathcal{B}(\mathcal{X})$, let
\[ \nu(x, \mathcal{A}) := \mathbb{P} [X_{k+1} \in \mathcal{A} | X_k = x] \tag{24} \]
is called a stochastic kernel on $\mathcal{X}$ or a transition probability function of the chain $\{X_k\}$, which means

(a) $\forall x \in \mathcal{X}$, $\nu(x, \cdot)$ is a probability measure on $\mathcal{X}$, hence a function from $\mathcal{X} \times \mathcal{B}(\mathcal{X})$ to $[0, 1]$, so if we fix some $x \in \mathcal{X}$, $\nu(x, \mathcal{A}) = \sum_{y \in \mathcal{A}} p_{x,y}$. Thus, $\nu(x, \mathcal{A})$ is simply the probability of reaching some state in $\mathcal{X}$ given that the current state is $x$. When $\mathcal{X}$ is discrete or, more precisely, finite, the transition kernel is simply a transition matrix $\nu$ with entries $p_{x,y}$ as follows:
\[ p_{x,y} = \mathbb{P} [X_n = y | X_{n-1} = x] \quad x, y \in \mathcal{X}. \]

(b) $\forall \mathcal{A} \in \mathcal{B}(\mathcal{X})$, $\nu(\cdot, \mathcal{A})$ is a measurable function from $\mathcal{X} \times \mathcal{B}(\mathcal{X})$ to $\mathcal{X}$.

One can introduce the 1-step transition probability as
\[ \mathbb{P} [X_1 \in \mathcal{A} | X_0 = x] = \nu(x, \mathcal{A}). \tag{25} \]
One can also introduce the $k$-step transition probability as
\[ \mathbb{P} [X_{k+1} \in \mathcal{A} | X_0 = x] = \nu^k(x, \mathcal{A}). \tag{26} \]

Definition 2.10 (Feller Chain; Onésimo Hernández and Lasserre (2003)). Let $\{X_k\}_{k \geq 0}$ be a Markov chain on a compact metric space $\mathcal{X}$, and then the chain is said to be a Feller chain if for every continuous function $\phi : \mathcal{X} \to \mathbb{R}$, the function
\[ x \mapsto \mathbb{E} [\phi(X_1) | X_0 = x] \tag{27} \]
is continuous.

Definition 2.11 (Weak-Feller Chain; Onésimo Hernández and Lasserre (2003)). Let $\{X_k\}_{k \geq 0}$ be a Markov chain on a metric space $\mathcal{X}$ with transition probability function $\nu$. Then the chain is said to satisfy the Weak-Feller property if every sequence $x_n \in \mathcal{X}$ such that $x_n \to x \in \mathcal{X}$, $\nu f(x_n) \to \nu f(x)$ whenever $f$ is bounded and continuous function on $\mathcal{X}$.

Definition 2.12. A Markov chain is said to have weak Feller property if its transition probability kernel (see Definition 2.9) maps a continuous function to another continuous function, i.e., it leaves $C(\mathcal{X})$ invariant.

Definition 2.13. Let $\mathcal{A} \subseteq \mathcal{X}$, the Dirac measure on a point $x \in \mathcal{X}$ is generally denoted by $\delta_x$ and defined as
\[ \delta_x(\mathcal{A}) = \begin{cases} 1 & x \in \mathcal{A} \\ 0 & x \notin \mathcal{A} \end{cases}. \]

Remark 2.2. Any discrete-time Markov chain can be generated by an IFS with probabilities see (Kifer, 2012, Section 1.1) or (Bhattacharya and Waymire, 2009, Page 228), although such representation is not unique, see, Stenflo Stenflo (1998a).
3 IFS on Compact State Space

3.1 State Independent IFS with Finite Number of Self Maps

**Definition 3.1** (IFS with state-independent probability with finite number of maps). Let \((\mathcal{X}, \rho)\) be a compact metric space. Let \(\{w_i\}_{i=1}^m\) be continuous self-transformations on \(\mathcal{X}\) and \(\mu\) be a probability measure on \([m]\). Let \(\sigma_0, \sigma_1, \ldots\) be i.i.d discrete random variables taking values in \([m]\) and

\[
P(\sigma_t = j) = \mu(j) \quad j \in [m] \text{ and } i = 0, 1, 2, \ldots, m.
\]  

(28)

An **IFS with state-independent probability** is a Markov process that is realized by the recursion

\[
X_{k+1} := w_{\sigma_k}(X_k), \quad k = 0, 1, 2, \ldots,
\]  

(29)

where \(X_0 \sim \nu \in \mathcal{M}_f(\mathcal{X})\) is assumed to be independent of \(\sigma_k \forall k\).

We describe below how the iteration progress in Algorithm 1.

**Algorithm 1**: State-Independent IFS Iteration

- **Input**: Initial value \(X_0\), \(m\) functions \(\{w_k\}_{k=1}^m\), probability measure \(\mu\) over \([m]\)
- **Output**: A sequence of random variables \(X_k\) with Markov property (see Definition 2.8)

Step 1: Set \(n = 0\)
Step 2: Repeat the following:

Step 2.1: Draw \((\sigma_n)\) from \(\mu\)
Step 2.2: \(X_{n+1} = w_{\sigma_n}(X_n)\)
Step 3: Return \(\{X_k\}_{k \in \mathbb{N}}\)

**Definition 3.2** (Markov operator). To describe the evolution of the Markov process described by (29), we define an operator, known as **Markov operator** \(P\),

\[
P : C(\mathcal{X}, \mathbb{R}) \rightarrow C(\mathcal{X}, \mathbb{R}), \quad P(w)(x) = \mathbb{E}[w(X_{k+1}) \mid X_k] = \sum_{k \in \mathbb{N} \cup \{0\}, \sigma_k \in [m]} \mu(\sigma_k)(w \circ w_{\sigma_k})(x)
\]  

(30)
To find the dual of the operator (30), suppose the elements or points in the metric space $\mathcal{X}$ are distributed according to a measure $\nu_0$. For each point, we throw a dice independently with $m$ sides for which the side $i$ falls with probability $\mu(i)$, $i = 1, 2, \ldots, m$. If $i$ falls for the point at position $x_0$, we move the point to $w_i(x_0)$, in other words, if $\sigma_0, \sigma_1, \ldots$ is a sequence of i.i.d discrete random variable taking values in $[m]$, starting from some $x_0 \in \mathcal{X}$, then the point $x_0$ moves to $w_{\sigma_i}(x_0)$ with probability $\mu(\sigma_i)$ as stated in (28). The new amount of elements in a set $\mathcal{A}$ come from the set $w_{\sigma_i}^{-1}(\mathcal{A})$ with probability $\mu(\sigma_i)$. The elements transported to $\mathcal{A}$ by $w_{\sigma_i}$ have then mass $\mu(\sigma_i) \nu_0(w_{\sigma_i}^{-1}(\mathcal{A}))$. Hence the new distribution of the elements in $\mathcal{X}$ is

$$\nu_1(\mathcal{A}) = \mu(\sigma_0) \nu_0(w_{\sigma_0}^{-1}(\mathcal{A})) + \mu(\sigma_1) \nu_0(w_{\sigma_1}^{-1}(\mathcal{A})) + \cdots + \mu(\sigma_m) \nu_0(w_{\sigma_m}^{-1}(\mathcal{A}))$$  \hspace{1cm} (31)

Thus one can consider for the IFS with maps $w_1, w_2, \ldots, w_m$ and the probabilities $\mu(1), \ldots, \mu(m)$, an operator of the evolution of densities of elements under the action of the IFS as defined below.

**Definition 3.3.** For an IFS as in Definition 3.1, the dual operator of the Markov operator defined (30) is given by

$$P^* : \mathcal{M}(\mathcal{X}) \rightarrow \mathcal{M}(\mathcal{X}), P(\nu) = \sum_{k \in \mathbb{N}_0, (0)} \int_{\mathcal{X}} \mu(\sigma_k) \nu(w_{\sigma_k}^{-1}(\mathcal{A}))$$  \hspace{1cm} (32)

A probability measure $\nu$ on $\mathcal{X}$ is called invariant under $P^*$, if $P^*\nu = \nu$.

For the dynamical systems with only one transformation $w : \mathcal{X} \rightarrow \mathcal{X}$, the operator $P^*$ is known as the *Perron-Frobenius operator*, see (Lasota and Mackey, 1998, Section 3.2). The operator defined in the Definition (32) is also known as the *Foais operator* (Lasota and Mackey, 1998, Section 12.4).

### 3.2 Ergodic Properties of the Associated Markov Processes

It may be critical in practice to determine if an IFS is uniquely ergodic or not, both theoretically and practically, since this will facilitate the conclusion of simulation results for the process. If we simulate an ergodic weak-Feller chain on a compact space, the sampled points’ distributions will converge to the invariant measure. To establish the ergodic property of the related Markov process created by an IFS, Breiman (1960) established the following version of the ergodic theorem noted in (Corless et al., 2016, Chapter 6, Theorem 6.7).

**Theorem 2 (Breiman (1960)).** Let $\{X_t\}$ be a Markov chain on a compact metric space $(\mathcal{X},\rho)$ with a unique-invariant distribution $\nu^*$, suppose also that the chain is Feller (see Definition 2.10), then for every initial condition $X_0 = x_0 \in \mathcal{X}$ and any continuous function $w : \mathcal{X} \rightarrow \mathbb{R}$, the sampled points will satisfy the following limiting behavior

$$\lim_{k \rightarrow \infty} \frac{1}{k+1} \sum_{i=0}^{k} w(X_i) = \int w(x)\nu^*(dx) \text{ almost surely } \mathbb{P}_{x_0}.$$  \hspace{1cm} (33)

Using several *contractivity condition* is a critical strategy for establishing the unique ergodicity of an IFS. There are two distinct ways to characterise these contractions. One may consider contractivity in terms of individual trajectories. If a process is contractive, the trajectories of individuals beginning from distinct positions should approach each other over time. Thus, there will be one random trajectory in the long run, and the behaviour will be determined only by the original distribution. Another method for demonstrating contractivity is to use the weak convergence of several probability metrics, as described in (Carlsson, 2005a, Section 2.2). Contractivity conditions assure that the trajectories can meet regardless of their starting points.
To see the evolution of the IFS the kernel (see Definition 2.9) of the corresponding Markov chain should be obtained. Given an initial condition $x_0 \in \mathcal{X}$, and a sequence $\{\sigma_k\}$ in $[m]$, one generates $\{x_k\}_{k \geq 0}$ through the following recurrence relation

$$x_{k+1} = w_{\sigma_k}(x_k), \quad k = 0, 1, 2, \ldots$$ (34)

The semantics of the definition is captured in Algorithm 2.

**Algorithm 2:** State-dependent IFS Iteration

- **Input:** Initial value $x_0$, $m$ functions $\{w_k\}_{k=1}^m$, $m$ probability functions $\{p_k\}_{k=1}^m$
- **Output:** A sequence of random variables $X_k$ with Markov property (see Definition 2.8)

1. Set $n = 0$
2. Repeat the following:
   1. Calculate $p(X_n) = (p_1(X_n), p_2(X_n), \ldots, p_m(X_n))$
   2. Select $w_{\sigma_{n+1}} \in \{1, \ldots, m\}$ according to $p(X_n)$
   3. Apply $w_{\sigma_{n+1}}$ to the state: $X_{n+1} = w_{\sigma_{n+1}}(X_n)$
3. Return $\{X_k\}_{k \in \mathbb{N}}$

To see the evolution of the IFS the kernel (see Definition 2.9) of the corresponding Markov chain should be obtained. Given a state $x \in \mathcal{X}$ at time $k$, the set of possible states at the next time $k + 1$ is the finite set

$$\{w_\sigma(x) : \sigma \in [m]\}$$

and the occurrence of $w_\sigma$ has the probability $p_\sigma(x)$. Hence, for any event $\mathcal{A}$,

$$\mathbb{P}(X_{k+1} \in \mathcal{A} \mid X_k = x) = \sum_{\sigma : w_\sigma(x) \in \mathcal{A}} p_\sigma(x).$$

The kernel for this Markov process is given by

$$\nu(x, \mathcal{A}) = \sum_{\sigma : w_\sigma(x) \in \mathcal{A}} p_\sigma(x) = \sum_{\sigma : x \in (w_\sigma)^{-1}(\mathcal{A})} p_\sigma(x) = \sum_{\sigma \in [m]} p_\sigma(x) 1_{(w_\sigma)^{-1}(\mathcal{A})}(x).$$ (35)
This Markov chain’s kernel is hence a Dirac-measure's sum (see Definition 2.13)

\[ \nu(x, \cdot) = \sum_{\sigma \in [m]} p_\sigma(x) \delta_{w_\sigma(x)}. \]  

(36)

To describe evolution of the dynamics (34), let us define the Markov operator \( P \) (see Definition 3.2), note that, for any function \( g : \mathcal{X} \to \mathbb{R} \),

\[ \mathbb{E} [g(X_{k+1}) | X_k = x] = \sum_{\sigma \in [m]} p_\sigma(x) g(w_\sigma(x)). \]  

(37)

Hence,

\[ (Pg)(x) = \sum_{\sigma \in [m]} p_\sigma(x) (g \circ w_\sigma)(x). \]  

(38)

The Markov operator’s Feller property is crucial, which says \( (Pg) \in C_b(\mathcal{X}) \) whenever \( g \in C_b(\mathcal{X}) \). It is clear that the Markov operator defined in (38) is indeed has Feller property see (Hairer, 2006b, Theorem 4.22). Since \( \mathcal{X} \) is compact, all functions in \( C(\mathcal{X}, \mathbb{R}) \) are bounded, and \( P : C(\mathcal{X}, \mathbb{R}) \to C(\mathcal{X}, \mathbb{R}) \), by Riesz representation theorem (originally due to Riesz (1909) or see (Royden and Fitzpatrick, 1988, Chapter 6, Section 4) for a proof), we can conclude that the dual space of \( C(\mathcal{X}, \mathbb{R}) \) is is the space of all signed or complex measures on \( \mathcal{X} \) with the total-variation norm. The dual map \( P^\star \) is given by

\[ P^\star \nu(\mathcal{A}) = \sum_{i=1}^m (p_i \circ w_i^{-1}(\mathcal{A})) \, d(\nu \circ w_i^{-1}(\mathcal{A})) \quad \text{for } \mathcal{A} \in \mathcal{B}(\mathcal{X}). \]  

(39)

where we define

\[ (\nu \circ w_i^{-1})(\mathcal{A}) = \nu(w_i^{-1}(\mathcal{A})) \quad \text{for all events } \mathcal{A} \in \mathcal{B}(\mathcal{X}). \]  

(40)

and it is not required that \( w_i \) are invertible maps, but we interpret

\[ w_i^{-1}(\mathcal{A}) = \{ x \in \mathcal{X} : w_i(x) \in \mathcal{A} \} \]  

as the pre-image of \( \mathcal{A} \) under the map \( w_i \).

The theory of Markov operators began in 1906 when Markov demonstrated that stochastic matrices might be used to study the asymptotic features of certain stochastic processes Markov (1906). Positive linear operators on \( \mathbb{R}^n \) are defined by these matrices. Markov’s concepts have been expanded in several areas. Feller developed, in particular, the theory of Markov operators operating on Borel measures defined on certain topological spaces. In Nummelin (1984); Revuz (1975); Çınlar (1974); Ethier and Kurtz (1986); Foguel (1966b,a,c, 1969) one finds some historical notes and a plethora of literature.

**Definition 3.5** (Contraction on average in 1 step), (Chiu, 2015, Section 2.2) We say that an IFS contracts on average on the metric space \( (\mathcal{X}, \rho) \) after 1 step if there exists an \( 0 < r < 1 \) such that

\[ \sup_{x,y,z \in \mathcal{X}} \sum_i p_i(x) \frac{\rho(w_i(y),w_i(z))}{\rho(y,z)} \leq r. \]  

(42)

This can be generalised to define IFS that contract on average after, say, \( k \) step where \( k \in \mathbb{N} \). Let

\[ \mathcal{J} := \{ \sigma = (\sigma_k)_{k=1}^\infty : \sigma_k \in [m] \} \]
represents the collection of all conceivable sequences of maps. Let $\circ$ denote the usual composition operation of functions, for $x_0 \in \mathcal{X}$, $\sigma \in \mathcal{I}$, $k \in \mathbb{N}$, define
\[
F_k(x, \sigma) := \begin{cases} x_0, & \text{if } k = 0 \\ (w_{\sigma_k} \circ \ldots \circ w_{\sigma_1})(x_0), & \text{if } k \geq 1. \end{cases}
\] (43)
For $x_0 \in \mathcal{X}$, we define $\mu_{x_0}$ on $\mathcal{I}$ for some $\sigma = (\sigma_1, \ldots, \sigma_k)$ as follows:
\[
\mu_{x_0}(\sigma) = p_{\sigma_1}(x) \times p_{\sigma_2}(w_{\sigma_1}(x)) \times \ldots \times p_{\sigma_k}(w_{\sigma_{k-1}} \circ \ldots \circ w_{\sigma_1})(x_0),
\] (44)
to quantify the probability that we apply the sequence of maps $w_{\sigma_1}, \ldots, w_{\sigma_k}$ given that we have started from the point $x_0$. It is easy to notice that $(F_k(x_0, \cdot))_{k \geq 1}$ is a Markov chain with transitional probability $\mu_{x_0}$.
Let $\mathcal{M}_k : [m]^k \to \mathbb{R}$ be a function on $\mathcal{I}$ depending only on the first $k$ coordinates. Then expectation with respect to $\mu_{x_0}$ is defined as (Chiu, 2015, Section 2.2)
\[
\mathbb{E}_{x_0} [\mathcal{M}_k] := \int \mathcal{M}_k(\sigma) \, d\mu_{x_0}(\sigma) = \sum_{\sigma_1, \ldots, \sigma_k \in [m]} \mu_{x_0}(\sigma) \cdot \mathcal{M}_k(\sigma).
\] (45)

**Definition 3.6** (Contraction on average in $k$ steps). (Chiu, 2015, Section 2.2) We say an IFS is *contract on average after $k$ steps* if there exists $r \in (0, 1)$ such that
\[
\sup_{x,y \in \mathcal{X}} \sum_{\sigma_1, \ldots, \sigma_k} \mu_{x_0}(\sigma) \frac{\rho(F_k(y, \sigma), F_k(z, \sigma))}{\rho(y, z)} \leq r.
\] (46)
If for a $\sigma = (\sigma_1, \ldots, \sigma_k)$, we denote $w_k(\sigma) = (w_{\sigma_k} \circ \ldots \circ w_{\sigma_1})(x)$ and assume that each map is Lipschitz and define the Lipschitz norm $\|\cdot\|_{\text{Lip}}$ as follows:
\[
\|w\|_{\text{Lip}} = \sup_{x,y \in \mathcal{X}} \frac{\rho(w(x), w(y))}{\rho(x, y)}
\] (47)
then we can re-write (46) in a compact form as follows (Chiu, 2015, Section 2.2):
\[
\sup_{x \in \mathcal{X}} \mathbb{E}_x \left[ \|w_k(\cdot)\|_{\text{Lip}} \right] \leq r.
\] (48)

In Hermer et al. (2019), several other contraction conditions were introduced to achieve convergence of random iteration of functions. To introduce them, we first need to define the following:
\[
\text{Fix } w_i = \{x \in \mathcal{X} : w_i(x) = x\}
\] (49)

**Definition 3.7.** (Quasi-nonexpansive mappings.) $w_i$ is called a quasi-nonexpansive mapping if
for all $x \not\in \text{Fix } w_i$ and, for all $y \in \text{Fix } w_i$ implies $\rho(w_i(x), y) \leq \rho(x, y)$.
(50)

**Definition 3.8.** (Paracontraction.) $w_i$ is called paracontraction if it is continuous and
for all $x \not\in \text{Fix } w_i$ and, for all $y \in \text{Fix } w_i$ implies $\rho(w_i(x), y) < \rho(x, y)$.
(51)

**Definition 3.9.** (Nonexpansive mappings.) $w_i$ is called nonexpansive if
for all $x, y \in \mathcal{X}$ implies $\rho(w_i(x), w_i(y)) < \rho(x, y)$.
(52)

For the following definition, the space $\mathcal{X}$ needs to be a normed-linear space.

**Definition 3.10.** (Averaged mappings on a normed linear space.) A mapping $w_i : \mathcal{X} \to \mathcal{X}$ is called averaged mapping if there exists an $\alpha \in (0, 1)$ such that
\[
\text{for all } x, y \in \mathcal{X} \Rightarrow \|w_i(x) - w_i(y)\|^2 + \frac{1 - \alpha}{\alpha} \| (x - w_i(x)) - (y - w_i(y)) \|^2 \leq \|x - y\|^2.
\] (53)
3.4 Ergodic Properties of the Associated Markov Processes

To ensure that the Markov process given by the IFS stated in the Definition 3.4 possesses a unique invariant measure, we will need to place some mild restrictions on the probability functions \( p_i(x) : X \to [0,1] \). Throughout the article, we will assume that each map \( w_i \) of the IFS is Lipschitz continuous and that the probabilities \( p_i(x) \) satisfies the Dini continuity as stated in the Definition 2.3. Research has focused heavily on IFS models with state-dependent probability (e.g., Stenflo (2012b, 2002b) and references there). For an IFS with state-dependent probabilities, the convergence of a chaotic game to an invariant measure was shown in Barnsley et al. (1988, 1989a). See also, Gwóźdź-Lukawska and Jachymski (2005); Andres et al. (2005) for extension of the Barnsley-Hutchinson type results to an infinite number of self-maps. Let us state a significant result on ergodicity of IFS on compact metric space due to Barnsley et al. (1988):

**Theorem 3** (Barnsley et al. (1988)). Suppose \((X, \rho)\) be a compact metric space and that \(\{w_i : X \to X\}_{i=1}^N\) be an IFS that contracts on average with corresponding Dini continuous probabilities \(\{p_i : X \to [0,1]\}_{i=1}^m\), then there exists a unique invariant Borel probability measure \(\nu^*\) for the IFS i.e. for all event \(A \in B(X)\) we have

\[
\int v(x, \mathcal{A}) \nu^* (dx) = \nu^* (\mathcal{A}).
\]

3.5 State-Dependent IFS with Infinite Number of Maps

**Definition 3.11** (Chiu (2015)). Let \(\{w_j\}_{j \in \mathbb{N}}\) be a collection of Borel measurable transformations on \(X\) along with associated continuous probability functions \(\{p_i(x)\}_{i \in \mathbb{N}}, p_i : X \to [0,1]\), where for all \(x \in X, \sum_{i \in \mathbb{N}} p_i(x) = 1\). Then similar to the Definition 3.4 if the iteration process starts from a point \(x_0\), a map \(w_{\sigma_1}(x)\) is chosen with probability \(p_{\sigma_1}(x_0)\) and if the process is repeated up to \(k > 1\) times, then a random orbit \((w_{\sigma_k} \circ w_{\sigma_{k-1}} \circ \cdots \circ w_{\sigma_1})(x_0)\) is obtained

with probability \(p_{\sigma_k}(w_{\sigma_{k-1}} \circ w_{\sigma_{k-2}} \circ \cdots \circ w_{\sigma_1}(x_0)) \times \cdots \times p_{\sigma_2}(w_{\sigma_1}(x_0)) \times p_{\sigma_1}(x_0)\).

And for a given starting point \(x_0 \in X\) and \(A \in B(X)\) the transition probability from \(x_0\) to \(A\) is given by

\[
v(x_0, A) = \sum_{k \in \mathbb{N}} p_{\sigma_k}(x) \mathbf{1}_A (w_{\sigma_k}(x_0))\]

Under some regularity assumptions on the transformations and the associated probability functions, Peigné (1993) has shown the existence of a unique, invariant measure of the Markov chain arising from the IFS defined in the Definition 3.11.

4 IFS on Polish Spaces

4.1 State-Independent IFS with Infinite Number of Self Maps

According to Dumitrut (2013), IFS with an infinite number of self-maps was first introduced due to Wicks (2006) in 1991. See Lesnian (2004); Gwóźdź-Lukawska and Jachymski (2005); Dan and Alexandru (2009); Mihail and Miculescu (2009, 2010); Mihail (2010); Secelean (2013); Mauldin (1995); Hans (2000); Hyong-Chol et al. (2005); Mendivil (1998) for their theoretical development. Some applications of IFS with an infinite number of self-maps can be found in Diaconis and Freedman (1999) and the references there. Here, we try to summarise some of their basic properties:
Definition 4.1 (Diaconis and Freedman (1999)). (State independent IFS with infinite number of self-maps.) Let \( \mu \) be a probability distribution on a countably infinite index set \( \Theta \) and, \( \{ w_\sigma : \sigma \in \Theta \} \) be a family of self-maps on the metric space \( X \). One way to characterize a state-independent IFS that has a countably infinite number of self-maps is as follows: If the starting location is \( X_0 = x_0 \in X \), then the movement occurs by selecting a \( \sigma \) at random from \( \mu \), and then continuing to \( w_\sigma(x) \). \( \mu \) is independent of any \( x \in X \). In the same way, as was covered in section (29), this process will be described inductively by the realization equation.

\[
X_{k+1} = w_{\sigma_k}(X_k)
\]

(56)

where \( \sigma_1, \sigma_2, \ldots \) are drawn independently from the \( \mu \) distribution.

Based on the above definition, the forward and the backward iteration (see, Chamayou and Letac (1991); Letac et al. (1986); Diaconis and Freedman (1999)) of the mappings are given by the compositions of iteration of the mappings as follows respectively:

\[
X_k (\cdot) := (w_{\sigma_k} \circ w_{\sigma_{k-1}} \circ \cdots \circ w_{\sigma_1}) (\cdot), \quad X_0 (\cdot) = X_0,
\]

(57)

\[
Y_k (\cdot) := (w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_k}) (\cdot), \quad Y_0 (\cdot) = Y_0.
\]

(58)

Both (57) and (58) have the same distribution for every \( k \), but their characteristics are different. For any fixed \( x \) (57) is ergodic with some regularity assumptions on the maps, while (58) converges almost surely and may not even have a Markov property (Diaconis and Freedman, 1999, Theorem 5.1). The following result is due to Letac et al. (1986), which gives a simple way to prove that (57) is ergodic and obtains its stationary distribution. Let \( \mathcal{D}(\mathcal{Y}) \) denote the distribution of a random variable \( \mathcal{Y} \) defined below.

Theorem 4 (Letac et al. (1986)). If \( \lim_{k \to \infty} Y_k(x) = \mathcal{Y} < \infty \) almost surely and independent of \( x \), then the Markov chain \( \{X_n(x)\}_{n \geq 0} \) is ergodic with unique stationary distribution \( \mathcal{L}(\mathcal{Y}) \).

The main theorem in Diaconis and Freedman (1999) and the immediate consequences were already established in several articles; for example, see Arnold and Crauel (1992); Dubins and Freedman (1966); Barnsley et al. (1988, 1989a); Duflo (2013); Elton (1990); Hutchinson (1981).

The map \( (\sigma,x) \mapsto f_\sigma(x) \) is measurable with respect to product sigma-field on \( X \times \Phi \). This representation is always viable because \( X \) is Polish, cf. Kifer (2012); Borovkov and Foss (1992). In Diaconis and Freedman (1999) it is assumed that if \( w_\sigma \) are all Lipschitz, so that there exists a \( l_\sigma \) satisfying

\[
\rho(w_\sigma(x), w_\sigma(y)) \leq l_\sigma \rho(x,y) \quad \forall x,y \in X
\]

(59)

and these are contracting on average in the sense that

\[
\mathbb{E}_\mu [\ln (l_\sigma)] < 0,
\]

(60)

and further if there exists some \( x_0 \in X \) and a \( \beta > 0 \) such that

\[
\mathbb{E}_\mu \left[l_\sigma^\beta \right] < +\infty,
\]

(61)

and there exists \( \alpha > 0 \)

\[
\mathbb{E}_\mu \left[\rho^{\alpha}(w_\sigma(x_0), x_0)\right] < +\infty
\]

(62)

then the Markov chain arising from the random iteration of functions whose realization equation is given by (56) has a unique invariant probability distribution \( \nu^* \). Moreover under any initial condition \( X_0 = x \), the distribution of \( \nu_k \to \nu^* \) in distribution in Prohorov metric where \( X_k \sim \nu_k \). We now state the main result in Diaconis and Freedman (1999):
Theorem 5 (Diaconis and Freedman (1999)). Let \((\mathcal{X}, \rho)\) be a complete, separable metric space. Let 
\(\{w_{\sigma} : \sigma \in \Theta\}\) be a family of Lipschitz function on \(\mathcal{X} \rightarrow \mathcal{X}\) and \(\mu\) be a probability distribution on \(\Theta\) and let \(\nu_k(x, dy)\) denote the transition probability kernel of the chain \(X_k\). Suppose further that conditions stated in (60), (61) and (62) holds, then

(a) the induced Markov chain has a unique invariant probability distribution \(\pi\),

(b) there is a positive, finite constant \(a_x\) and \(r \in (0, 1)\) such that
\[
\overline{d}(\nu_k(x, \cdot), \pi) \leq a_x r^k \quad \forall k = 1, 2, \ldots, \text{ and } \forall x \in \mathcal{X},
\]

(c) the constant \(r\) does not depend on \(x\) or \(k\); the constant \(a_x\) does not depend on \(k\) and
\[
a_x < a + b \rho(x, x_0) \text{ for some } 0 < a, b < +\infty.
\]

Iosifescu (2003) has shown the same exact conclusion using contraction characteristics of the two linear operators associated with the Markov chain. Wu and Woodroofe (2000) and Jarner and Tweedie (2001) exhibit these chains’ additional stability features. See Abrams et al. (2003) for an extension of some results from Diaconis and Freedman (1999) with the maps with Lipschitz number one.

In Jarner and Tweedie (2001) the following assumptions have been made on the complete separable metric space \((\mathcal{X}, \rho)\):

Assumption 4.1. (a) The metric is bounded and the diameter is finite:
\[
R = \rho(x, y) < \infty, \quad (63)
\]

(b) Maps are contracting on average:
\[
E[\rho(X_1(x), X_1(y))] \leq \rho(x, y) \quad \forall x, y \in \mathcal{X}, \quad (64)
\]

(c) Local contraction: Let \(\tau_C(x) = \inf\{n \geq 1 : X_n(x) \in C\}\). There exists \(r \in (0, 1)\) and a region \(C\) in \(\mathcal{X}\), satisfy the following:
\[
E[\rho(X_{\tau_C(x)}(x) \lor \tau_C(y)(x), X_{\tau_C(x) \lor \tau_C(y)}(y))] \leq r \rho(x, y) \quad \forall x, y \in \mathcal{X}, \quad (65)
\]

(d) Drift condition: There exists \(H : \mathcal{X} \rightarrow \mathbb{R}\) with \(H \geq 1\), \(H\) is bounded in the region \(C\) with
\[
D = \sup_{x \in C} H(x) < +\infty, \quad (66)
\]

and there exists constant \(\lambda \in (0, 1)\) and \(b < +\infty\) such that
\[
\nu H(x) = \int H(x) \nu(x, dy) \leq \lambda H(x) + b 1_C(x), \quad (67)
\]

where \(\nu_k(x, \cdot)\) is the distribution of \(X_k(x)\).

Now we state the main results from Jarner and Tweedie (2000):

Theorem 6 (Jarner and Tweedie (2000)). Under the assumptions made in Assumption 4.1, the IFS (56) possesses an unique invariant measure \(\pi\) and regardless of any initial distribution
\[
X_k \rightarrow \pi \quad \text{in distribution}, \quad (68)
\]

further there exists \(\gamma < 1, \kappa < \infty\) and for all \(x \in \mathcal{X}\), \(n \geq 1\) we have,
\[
d_{pr}(\nu_k(x, \cdot), \pi) \leq \gamma^k \kappa [H(x)]^{\frac{1}{2}}. \quad (69)
\]
It is clear from the above result that the chain exhibit stability properties as those found in Diaconis and Freedman (1999). Further, under the following assumptions:

**Assumption 4.2.** (a) There exists \( x_0 \in \mathcal{X} \) and a \( \alpha \in (0,1] \) such that

\[
\mathbb{E} (\rho^\alpha (x_0, X_1 (x_0))) = \int_\Theta \rho^\alpha (x_0, X_1 (x_0)) \mu(d\theta) < \infty \tag{70}
\]

(b) with \( x_0 \) and \( \alpha \) chosen above, further, there exists \( r(\alpha) = r \in (0, 1) \) and \( c(\alpha) = c > 0 \) such that

\[
\mathbb{E} (\rho^\alpha (X_k(x_0), X_k (x))) \leq cr^k\rho^\alpha (x_0, x) \quad \forall x_0, x \in \mathcal{X} \text{ and } \forall k \in \mathbb{N}. \tag{71}
\]

Wu and Shao (2004) has showed that (58) converges almost surely as \( n \to \infty \).

If \( X \subseteq \mathbb{R}^n \) is a convex subset, then Steinsaltz (1999) has added a local contractibility requirement that essentially amounts to the presence of a drift function \( g : \mathcal{X} \to (0, \infty) \) and a \( r \in (0, 1) \) such that

\[
\mathbb{E} \left( \limsup_{y \to x} \frac{\rho(X_k(y), X_k(x))}{\rho(y, x)} \right) \leq g(x)r^k \quad \forall x \in \mathcal{X} \text{ and } \forall k \in \mathbb{N}. \tag{72}
\]

Further to it, he proved that under (70) in conjunction with a suitable assumption on \( g \), which is a generalization of (70) a more general contraction condition than (71) does hold, but not (71) itself. Unifying results from Steinsaltz (1999) and Wu and Shao (2004), a more general contraction condition was introduced in Herkenrath and Iosifescu (2007) as follows:

(a) There exists \( x_0 \in \mathcal{X}, \alpha \in (0, 1], r \in (0, 1) \), and measurable functions \( g_k : \mathcal{X} \to [0, \infty), k \in \mathbb{N} \), such that both

\[
r_1 := \limsup_{k \to \infty} \delta_k^\frac{1}{k} \quad \text{and} \quad r_2(x) := \limsup_{k \to \infty} g_k^\frac{1}{k}(x) \quad \text{strictly exceed } r \quad \forall x \in \mathcal{X}, \text{ where}
\]

\[
\delta_k := \mathbb{E} (g_k (X_1 (x_0))) \rho^\alpha (x_0, X_1 (x_0)), \quad n \in \mathbb{N}. \tag{73}
\]

(b) For all \( x \in \mathcal{X} \) and for any \( n \in \mathbb{N} \), one has,

\[
\mathbb{E} (\rho^\alpha (X_k(x), X_k (x_0))) \leq r_k^k g_k(x)\rho^\alpha (x, x_0) \tag{74}
\]

Clearly, \( r_1 \) and \( r_2(x) \) are radius of convergence of the power series \( \sum_{n \in \mathbb{N}} \delta_k t^k \) and \( \sum_{k \in \mathbb{N}} g_k(x)t^k, \ t \in \mathbb{R} \), respectively.

## 5 Ergodic Theorems and Law of Large Number for IFS

The principal objective of this part is to provide an overview of ergodic theory as a method for investigating statistical patterns in IFS. First, we attempt to use an illustrative example from a deterministic dynamical system, and then we try to use a random example. Finally, we try to understand how the patterns rely on the system’s initial state. Assume we have modelled a highly complex phenomenon using a Markov chain formed by an IFS, \( \{X_k\} \), developing in the state space \( \mathcal{X} \). Thus, the system’s state at time instant \( k \) is \( X_k \in \mathcal{X} \). The description of the system at any point in time may include many parameters, each of which must be represented by an element of \( \mathcal{X} \). Experimental investigation of such systems may be arduous since the system’s behaviour may be so complex that following all the specifics of its development is relatively tricky or implausible. Most of the time, it is not easy to speculate on the trajectory such a system will take.
This may be the case sometimes because of computation difficulties and the system’s inherent unpredictability and sensitivity to initial inputs.

A statistical technique is one approach that may be used when attempting to characterize the characteristics of a system. In most cases, the first step in conducting an experimental study of a system is to gather statistical data about the system by making several observations or taking multiple measurements. A measurement is a technique to associate any state \( x \in \mathcal{X} \) of the system in consideration with a real number \( w(x) \in \mathbb{R} \), i.e., it is a map \( w : \mathcal{X} \rightarrow \mathbb{R} \). A measurement \( w(X_k) \) of the system \( \{X_k\}_{k \geq 0} \) at time \( k \) yields the value \( w(X_k) \). Because a single measurement like this discloses nothing about the system, it makes sense to do the measurement many times and then take the average of the results obtained over different periods. The outcome of this technique if the measurements are carried out at periods \( 1, 2, \ldots, k \) is

\[
\frac{w(X_1) + \ldots + w(X_n)}{k}. \tag{75}
\]

This time-averaged data may be used as approximations to a desired true value. Additionally, one hopes that when additional observations are averaged, the resultant estimate approaches the ideal real value. In other words, gathering increasing amounts of data is only beneficial if a variant of the law of large numbers is true, i.e., there is

\[
\lim_{k \to \infty} \frac{w(X_1) + \ldots + w(X_n)}{k} \rightarrow \bar{x}. \tag{76}
\]

Additionally, the infinite time horizon average should be independent of the system’s initial state, or the limiting value should be challenging to grasp. Consequently, the following logical problems arise: Is there any statistical regularity in the system? How does the initial condition affect the system’s statistical properties? Is the system capable of remembering its initial condition, or does it ultimately forget? Next, we try to address these questions in the context of IFS.

5.1 A Motivating Example from Deterministic Dynamics

On \( \mathbb{R} \), a basic discrete linear dynamical system (without randomness) may be described as the following transformation map for some constant \( a \in (0, 1) \):

\[
w_1 : \mathbb{R} \rightarrow \mathbb{R}, \quad w_1(x) = ax \tag{77}
\]

If we apply the map \( k \) times, we get \( X_k(x) = aX_{k-1} = w_k^k(x) = a^kx \to 0 \) as \( k \to \infty \), indicating that 0 is a stable, attractive fixed point of the dynamics regardless of where we begin. The convergence toward this single-point attractor is exponential. Its attractive domain is identical to \( \mathbb{R} \). Thus, the system rapidly loses memory because of the map’s contraction and intrinsic stability. Regardless of the original state, the sequence of points \( X_k \to 0 \) as \( k \to \infty \), i.e., the system’s initial condition is eventually forgotten. On the other hand, one may illustrate a non-stable (see Bakhtin (2015)) system with the state space \( \mathcal{X} = [0, 1] \) and the transformation

\[
w_2 : [0, 1) \rightarrow [0, 1), \quad w_2(x) = \{2x\} \text{ i.e fractional part of the real number } 2x, \ x \in [0, 1). \tag{78}
\]

As a result, we can show that stability at the trajectory level does not hold in the classical fashion by demonstrating it using these two above cases. However, a definition of stability more focused on system statistics might still be legitimate. To put it another way, if \( f \) is an integrable and Borel measurable function on \([0, 1)\), then it is true that for any \( x \in [0, 1) \) with respect to the Lebesgue measure on \([0, 1)\),

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} f(w_2^{i-1}(x)) = \int_{[0,1)} f(x)dx. \tag{79}
\]
Given that the right-hand side reflects the average of \( f \) with respect to the Lebesgue measure on \([0,1]\), one may conclude that the Lebesgue measure accurately characterises the attributes of this dynamic system or uniform distribution on \([0,1]\) in the long run.

Indeed, one interpretation of (79) is that for practically every initial condition \( x \), the distribution of \( \phi(x) \) with assigning equal mass of \( \frac{1}{k} \) unit to each of \( x, w_2(x), \ldots, w_{k-1}(x) \) converges to the Lebesgue measure. Additionally, the convergence in (79) might be seen as a law of large numbers and more precisely Lebesgue measure is unique in this system since it defines the limits in (79) and invariant under \( w_2 \). It is ergodic; it cannot be reduced to two nontrivial invariant measures.

5.2 A Motivating Example from Random Dynamics

Similar concerns, as stated just before, arise naturally when considering random maps rather than deterministic maps. In general, the long-term behaviour of chaotic deterministic dynamical system trajectories is uncertain (Boyarsky and Gora, 2012, Chapter 2). As a result, it is appropriate to use statistical methods to characterize the system’s behaviour in aggregate. Random maps may also form naturally due to random perturbations of deterministic dynamics.

For example, recall that the previous dynamics stated in (77) where the map was \( w_1 : x \mapsto ax \) for which 0 was a stable equilibrium point. Let us introduce random noise into this system to destroy the equilibrium. Assume we have a sequence of independent, identically normally distributed random variables \( \{\sigma_k(\omega)\}_{k \in \mathbb{Z}} \sim N(0, \nu^2) \) specified on \((X, \mathcal{F}, \mathbb{P})\) with \( \omega \) as usual represent a sample point. Define random maps for \( k \in \mathbb{Z} \) as follows:

\[
w_{k,\omega}(x) = ax + \sigma_k(\omega)
\]

A natural analogue of the forward orbit from the example (77) is a a stochastic process \( \{X_k\}_{k \geq 0} \) with starting from \( X_0 = x_0 \in \mathbb{R} \) such that

\[
X_{k+1} = aX_k + \sigma_k
\]

Stability analysis, in this case, is not straightforward as it was for (77). No fixed equilibrium point serves all maps \( \{w_{k,\omega}\}_{k \in \mathbb{Z}} \) at once. It is possible that the solution of \( w_{k,\omega}(x) = x \) for some values of \( k \) doesn’t make any sense to all other different values of \( k \in \mathbb{Z} \). Nevertheless, this system can produce an ergodic result analogous to (79). Let \( \mu \sim N(0, \frac{\sigma^2}{1+\sigma^2}) \). Then for any integrable function \( f \) with respect to \( \mu \), for almost every \((x_0, \omega) \in \mathbb{R} \times \Omega \) with respect to \( \mu \times \mathbb{P} \)

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} f(X_{i-1}) = \int_{\mathbb{R}} f(x) \nu(dx)
\]

This finding explains that \( \mu \) is a unique invariant measure for the Markov operator associated with the system. This example illustrates a scenario without deterministic stability, yet statistical stability exists.

5.3 Ergodic Theorems for IFS on Compact Space

For dynamical systems, Birkhoff’s Ergodic Theorem (Silva, 2008, Chapter 5, Theorem 5.1.1) is well-known: Let

\[
T : (\mathcal{X}, \mathcal{B}(\mathcal{X}), \nu) \to (\mathcal{X}, \mathcal{B}(\mathcal{X}), \nu)
\]

be an ergodic dynamical system, further, suppose \( w \in L^1(\mathcal{X}, \mathcal{B}(\mathcal{X}), \nu) \), then for almost every \( x \)

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} w(T^i(x)) = \int w d\nu.
\]
The point-wise Ergodic theorem or strong law of large number for IFS is due to Elton (1987). The result was later sharpened in (Peigné, 1993, Proposition 7.1) under some hypothesis.

**Theorem 7** (Elton (1987)). Consider an IFS described in Definition 3.4 with associated probabilities \( p_i \) that satisfy the mild technical condition in Definition 2.3. Let \( \nu^* \) denote the unique invariant probability measure of the IFS. Then for almost all \( x \in X \) and for all \( w \in C(X, \mathbb{R}) \)

\[
\lim_{k \to \infty} \frac{1}{k+1} \sum_{i=0}^{k} w (w_{\sigma_i} \circ \cdots \circ w_{\sigma_0} (x)) = \int w d\nu^*
\]  

(84)

See also Forte and Mendivil (1998), and Tricot and Riedi (1999) for short proofs of the above result. All these results hold for the state space of IFS as a closed subset of \( \mathbb{R}^n \).

5.4 Ergodic Theorems for IFS on Polish Space

In Stenflo (2001a) an weak ergodic theorem including rate of convergence for Markov chains arising from IFS with a countably infinite number of functions under a stochastically contractive and boundedness condition (Stenflo, 2001a, Condition C & D, Theorem 2) and the method of proving the results were inspired from work introduced in Letac et al. (1986); Burton and Rosler (1995); Loskot and Rudnicki (1995); Silvestrov and Stenflo (1998); Janoska (1995). Recently, in (Hermer et al., 2022, Section 2.4.3.4.3.7) under some greater regularity assumptions on non-expansive transformations (such as \( \alpha \)-firmly non-expansive), the ergodic and convergence aspects of random function iteration have been extensively explored. In (Jaroszewska, 2008, section 2) the concept of generalized contraction of a transformation map was introduced. Subsequently, the asymptotic behaviour of IFS with a finite or countably infinite number of maps with state-dependent probabilities consisting of at least one generalized contraction was studied to prove their asymptotic stability property. For more on ergodic properties of IFS on Polish space we refer Lasota and Mackey (1998); Szarek (1997, 2000a,b, 2003a,b).

6 Central Limit Theorem (CLT) and Invariance Principle for IFS

Limit theory for stochastic dynamical systems arising from the IFS defined as in the Definition 3.1 see, e.g in Borovkov (1998); Bhattacharya and Majumdar (2007); Meyn and Tweedie (1993). Suppose an IFS of the form 3.1 have a stationary distribution \( \nu^* \), let \( w \in L^2(\nu^*) \) be square integrable function on \( X \) with mean zero and let

\[
\mathcal{S}_k(w) = \sum_{i=1}^{k} w(X_i),
\]  

(85)

then Wu and Woodroofe (2000) studied central limit theorems for IFS i.e, under what conditions on \( w \) and \( w_{\sigma_k} \), is \( \frac{\mathcal{S}_k(w)}{\sqrt{k}} \) asymptotically normal as \( k \to \infty \). By decomposing (85) into a martingale and a stochastically bounded component as in Gordin and Lifsic (1978), Benda (1998) proved, that the above result is true if (62) holds for \( a = 2 \), \( w \) is Lipschitz continuous and \( E_{\mu} [I_0^2] < 1 \). Even for a discontinuous \( w \) and more relaxed conditions on the \( l_\sigma \), and (62), Wu and Woodroofe (2000); Wu and Shao (2004) showed that the central limit theorems hold. If \( w_1, \ldots, w_m \) are Lipschitz self-maps on \( X \) with \( l_1, \ldots, l_m \) Lipschitz constants respectively, then Loskot and Rudnicki (1995) has derived central limit theorem for the random variables \( w(X_k) \) for any Lipschitz function \( w : X \to \mathbb{R} \) for the IFS defined as in Definition 3.1 with the additional condition on the
Lipschitz constants, and the probabilities have been taken as follows:

\[ \mu(\sigma_i)^2 \sigma_i < 1. \] (86)

For the IFS in Definition 3.3, from Barnsley and Denko (1985); Hutchinson (1981); Elton and Piccioni (1992) one can conclude that there exists an unique compact set \( \mathcal{K} \) that is invariant for the IFS in the sense that

\[ \mathcal{K} = \bigcup_{i=1}^{N} w_i(\mathcal{K}). \]

Such chains allow greater flexibility for generating a uniform stationary probability distribution on \( \mathcal{K} \). These kinds of Markov chains are seen in the thermodynamic formalism of statistical mechanics.

Existence and uniqueness of possessing a unique stationary distribution are not usual for these chains Berger et al. (2018); Bramson and Kalikow (1993); Lacroix (2000); Stenflo (2001b, 2002b), but some additional regularity conditions are needed, see, e.g., Harris et al. (1955); Johansson and Öberg (2003); Stenflo (2002b,a). In Peigné (1993), a CLT has been proved by the spectral decomposition technique of the associated Markov operator. For details on the spectral decomposition of the Markov operator see, e.g., Hennion (1993); C Ionescu (1948); Tulcea (1959). On a non-compact metric space, a CLT for IFS with Lipschitz maps has been proved in Hennion and Herve (2004); for more literature on these, we refer Andreas and Stenflo (2005); Herkenrath and Iosifescu (2007); Horbacz (2016); Häggström and Rosenthal (2007); Häggström (2005).

6.1 CLT for IFS with State-Independent Probabilities with Countably Infinite Number of Self Maps on Polish Space

Let us formulate some background to demonstrate the law of large numbers for the chain established in Kapica and Ślęczka (2018). Let \( P \) be a Markov operator acting on the space of measures on \( \mathcal{X} \) such that

\[ P\nu_n = \nu_{n+1} \text{ i.e } P( \text{ law of } X_n ) = ( \text{ law of } X_{n+1} ), \]

where \( \nu_n \) is the distribution of \( X_n \). The following assumptions were made to establish LLN for time-homogeneous Markov chain \( X_n \).

Assumption 6.1. (a) There exists a probability measure \( \nu_\ast \in \mathcal{M}(\mathcal{X}) \), a continuous function \( g : \mathcal{X} \to \mathbb{R} \), and a \( q \in (0,1) \) such that

\[ \| P^n \delta_x - \nu_\ast \|_{FM} \leq g(x)q^n. \] (87)

Theorem 8 (Kapica and Ślęczka (2018)). Let \( P : \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{X}) \) be a Markov operator which satisfies (87). Let \( X_n \) be a time-homogeneous Markov chain taking values in \( \mathcal{X} \), with transition operator \( P \) and initial distribution \( \nu \). Further to this, there exists a \( \nu_\ast \in \mathcal{M}(\mathcal{X}) \) such that \( \lim_{n \to \infty} P^n \nu = \nu_\ast \) for every starting distribution \( \nu \). Then for every Lipschitz and bounded function \( f : \mathcal{X} \to \mathbb{R} \) the sequence \( f(X_n) \) satisfies LLN i.e

\[ \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) \xrightarrow{n \to \infty} \int_X f(x)\nu_\ast(dx). \] (88)

In addition, with the (Kapica, 2016, Assumption B1-B4),

Theorem 9 (Kapica and Ślęczka (2018)). If \( X_n \) be a homogeneous Markov chain on a polish space \( (\mathcal{X}, \rho) \) with transition operator \( P \) and initial distribution \( \mathcal{L}(X_0) = \nu \), and let \( g : \mathcal{X} \to \mathbb{R} \) be any bounded, Lipschitz function then

\[ \mathbb{E}g(X_n) \xrightarrow{n \to \infty} \int_{\mathcal{X}} g(x)\nu_\ast(dx). \] (89)
where $\nu_*$ is invariant probability distribution for $P$ and moreover the sequence $g(X_n)$ satisfies LLN i.e

$$\frac{1}{n} \sum_{k=0}^{n-1} g(X_k) \to \int g(x) \nu_*(dx) \quad \text{in probability } \mathbb{P}. \quad (90)$$

Horbacz and Ślęczka (2016) demonstrates strong law of large number for random dynamical systems with randomly chosen jumps has been established, and they have shown their result holds for the IFS with state or place-dependent probabilities with a finite number of self-maps on a Polish space. For more results on the central limit on the Polish space theorem, see Kaijser (1979); Kwiecinska and Slomczynski (2000); Marta (1997).

### 6.2 Invariance Principle of IFS

Dynamical systems and probability theory are concerned with the limiting behaviour of successive observations of random variables. CLT examines deviations from the average behaviour of sums, for example, whereas the strong law of large numbers (SLLN) (ergodic theorem) specifies the average behaviour of sums. One extension of the previous two forms of the average behaviour of a sequence of random variables is the well-known notion of *almost sure invariance principle:*

**Definition 6.1** (Strassen (1964); Strassen et al. (1967)). Let $\{X_i\}_{i \geq 0}$ be i.i.d random variables with finite $(2+\delta)$ moments, define $\mathcal{S}_n = \sum_{i=0}^{n-1} X_i$. Then there exists a Brownian motion and a probability space $\Omega$ on which $B$ and $\mathcal{S}_n$ can be re-defined such that $\mathcal{S}_n = B(n) + o(n^{\frac{1}{2}})$.

This instantly enables one to infer a variety of standard statistical features for $\mathcal{S}_n$, such as the SLLN and different modifications of the CLT, assuming that they hold for the Brownian motion $B$. One such result is the following:

**Theorem 10** (Walkden (2007), Almost sure invariance principle). Suppose $f : \mathcal{X} \to \mathbb{R}$ is a bounded Lipschitz function with $\nu(f) = 0$ and $\sigma^2(f) = \sigma^2 > 0$. Fix $x \in \mathcal{X}$, then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a one-dimensional Brownian motion $W : \Omega \to C([0,\infty),\mathbb{R})$ such that the random variable $\omega \mapsto W(\omega)(t)$ has variance $\sigma^2 t$ and sequence of random variables $\phi_{n,x} : \Omega \to \mathbb{R}$, $\psi_{n,x} : \Omega \to \mathbb{R}$ with the following properties:

(a) for all $\epsilon > 0$ we have,

$$f^n(x, \cdot) = \phi_{n,x}(\cdot) + O(\frac{1}{n^{\frac{1}{2}+\epsilon}}) \quad \mu_x \text{ a.e } \quad (91)$$

(b) the sequence $\{\phi_{n,x}(\cdot)\}$ and $\{\psi_{n,x}(\cdot)\}$ are equal in distribution.

(c) for all $\epsilon > 0$, we have,

$$\psi_{n,x}(\cdot) = W(\cdot)(n) + O(\frac{1}{n^{\frac{1}{2}+\epsilon}}) \quad \mathbb{P} \text{ a.e. } \quad (92)$$

where by a.e we mean *almost everywhere.*

### 7 Geometry of the Unique Invariant Measure

In the penultimate section of the survey, we consider structural results, which can be seen as the geometry of the unique invariant measure. While there is surely much more to be understood, the current results provide valuable insights within IFS.
Geometry of Unique Invariant Measure

7.1 A Special Case: I.I.D Random Iteration of Quadratic and Logistics Maps

For a parameter $\sigma \in (0, 4)$, consider the quadratic family of self-maps on $[0, 1]$ as follows:

$$\mathcal{O} = \{ w_{\sigma}(x) := \sigma x(1-x), 0 \leq \sigma \leq 4, 0 \leq x \leq 1 \}.$$  \hfill (93)

And the realization scheme is as follows:

$$X_{k+1} = w_{\sigma_k}(X_k), \quad k = 0, 1, 2, 3, \ldots; \sigma_1, \sigma_2, \ldots \text{ are i.i.d.} \quad \forall k \geq 1, w_{\sigma_k} \in \mathcal{O}.$$  \hfill (94)

Given a pair of parameter values $(\alpha, \beta)$ with $\alpha < \beta$, and a number $r \in (0, 1)$, random dynamical systems generated by considering i.i.d sequence of maps $\{w_k : k \geq 1, k \in \mathbb{N}\}$ with $\mathbb{P}[w_1 = w_{\alpha}] = r$ and $\mathbb{P}[w_1 = w_{\beta}] = 1-r$ from the set $\mathcal{O}$ and similar ones has been studied extensively in last seventy years, see Collet and Eckmann (2009); Devaney (2008). This dynamical system family is one of the most influential, despite its apparent simplicity. For results on a few other chaotic families, see, e.g., May (1976); De Melo and Van Strien (2012); Day et al. (1994). In the economics literature of dynamic optimization model (93) and (94) arise significantly, see, e.g. Majumdar and Mitra (1994, 2000). In Bhattacharya and Rao (1993) for specific values of $\alpha, \beta$, uniqueness and other properties of the Markov chain arising from such random iteration have been studied. Several interesting results on the uniqueness and support of the invariant measure were obtained, primarily motivated by Dubins and Freedman (1966) on random monotone maps on an interval.

**Theorem 11** (Bhattacharya and Rao (1993)).

(a) If $0 \leq \alpha < \beta \leq 1$, then $\delta_0$ is the unique invariant probability measure in $[0, 1]$.

(b) If $1 < \alpha < \beta \leq 2$, then there exists an unique invariant non-atomic probability measure $\pi$ on $(0, 1)$.

(c) If $2 < \alpha < \beta \leq 1 + \sqrt{5}$ and $\alpha \in \left[\frac{8}{\beta(4-\beta)}, \beta\right)$ then there exists an unique invariant non-atomic probability measure $\pi$ on $(0, 1)$.

Much is known about the behaviour of the dynamical system generated by the self-maps for a fixed $\sigma \in [0, 4]$, defined in the set (93) which are also known as logistics maps in the literature of dynamical systems and chaos, see Graczyk and Swiatek (1997); Arrowsmith et al. (1990); Klebaner (1997).

In Athreya and Dai (2000), the following recursive relation

$$X_{k+1} = \sigma_{k+1}X_k(1-X_k), \quad k = 0, 1, 2, 3, \ldots.$$  \hfill (95)

has been studied with great depth and importance, and necessary and sufficient conditions have been established for the existence and uniqueness of non-trivial invariant probability measure for (95), in Steinsaltz.
Consider a slightly different version of the chain (95) as follows:

\[ X_{k+1} = \sigma_{k+1} X_k w_{k+1}(X_k) \quad k = 0, 1, 2, 3, \ldots, \tag{96} \]

under some assumptions on \( \sigma_k \) and \( w_k \) for \( k = 0, 1, 2, \ldots, \) (Athreya, 2003a, Section 4) investigate conditions for the existence and uniqueness of non-trivial invariant probability measures and their support for (96). This class of maps has been widely utilized in the research on ecology and economics to simulate population and growth models. See Majumdar (2009) for additional expansion and related results on the random iteration of monotone maps. In Bhattacharya and Majumdar (2001), a convergence rate in Kolmogorov distance of the chain defined in (4.1) has shown under some mild assumption on the chain. The utility of these specific models of IFS has been extensively documented in the literature of dynamic economics see, e.g., Bhattacharya and Waymire (2009); Tong (1990), and economic growth and survival under uncertainty, see Majumdar et al. (1989); Majumdar and Radner (1992); Stokey et al. (1989); Tong (1990); Yahav (1975). In Bhattacharya et al. (2004), positive Harris recurrence and the existence and uniqueness of invariant probability measure have been established for the chain (94). A significant fact to notice that the state-space of \( f X g \) is \( \delta(0) \), since the domain of the logistic maps was restricted in \( \delta(0) \) to avoid the trivial invariant probability measure \( \delta f 0 \).

7.2 Decomposition of Measure for IFS

**Definition 7.1.** A probability measure \( \nu_1 \) is *absolutely continuous with respect to another probability measure* \( \nu_2 \) if

\[ \nu_2(A) = 0 \Rightarrow \nu_1(A) = 0 \quad \text{for all } A \in \mathcal{B}(X) \tag{97} \]

i.e. if every null set of \( \nu_2 \) is also a null set of \( \nu_1 \). Two probability measures are equivalent if they are mutually absolutely continuous or have the same null sets.

The following theorem is adapted from Dubins and Freedman (1966), and it is noted in Barnsley et al. (1988, 1989a) that it holds for a large class of IFS. Consider \( \mathcal{M}_f \) be the set of all finite non-negative Borel measures on \( X \), if any \( \nu \in \mathcal{M}_f \) can be uniquely decomposed into sum of two measures \( \nu_1 \) and \( \nu_2 \), i.e, \( \nu = \nu_1 + \nu_2 \) then we express \( \mathcal{M}_f = \mathcal{M}_f^1 \oplus \mathcal{M}_f^2 \).

**Theorem 12.** (Extension of Dubins and Freedman’s result to Polish space) Let \( \mathcal{M}_f \) denote the set of all of the non-negative finite measures on some polish space \( X \), let \( T \) be a map on \( \mathcal{M}_f = \mathcal{M}_f^1 \oplus \mathcal{M}_f^2 \) which satisfies the following:

\[ a) \ T(\nu_1 + \nu_2) = T \nu_1 + T \nu_2 \quad \forall \nu_1, \nu_2 \in \mathcal{M}_f. \]

\[ b) \ T \mathcal{M}_f^1 \subseteq \mathcal{M}_f^1. \]

\[ c) \ (T \nu)(X) = \nu(X) \quad \forall \nu \in \mathcal{M}_f. \]

Then if \( \nu \) is a fixed point of \( T \) i.e \( T \nu = \nu \) with the fact that \( \nu = \nu_1 + \nu_2 \) for some unique \( \nu_1 \in \mathcal{M}_f^1 \) \( \nu_2 \in \mathcal{M}_f^2 \) then \( \nu_1 \) and \( \nu_2 \) are also fixed point of \( T \) i.e \( T \nu_1 = \nu_1 \) and \( T \nu_2 = \nu_2 \).
Proof. Fix a \( \nu \in \mathcal{M}_f \) such that \( T\nu = \nu \), one can write \( \nu = \nu_1 + \nu_2 \) and since \( T \) is linear as a map, we have by condition a) of the statement

\[
T\nu = T(\nu_1 + \nu_2) = T\nu_1 + T\nu_2.
\]  

(98)

Now, we use the condition b) of the statement, and we get

\[
Tv_1 \in \mathcal{M}_f^1.
\]

(99)

Now, for \( Tv_2 \) we get unique \( \mu_1 \in \mathcal{M}_f^1, \mu_2 \in \mathcal{M}_f^2 \) such that

\[
Tv_2 = \mu_1 + \mu_2.
\]

(100)

So (98) can be re-written in the light of (100)

\[
Tv = T(\nu_1 + \nu_2) = Tv_1 + Tv_2 = Tv_1 + \mu_1 + \mu_2 = (Tv_1 + \mu_1) + \mu_2,
\]

(101)

where \( (Tv_1 + \mu_1) \in \mathcal{M}_f^1 \) and \( \mu_2 \in \mathcal{M}_f^2 \). Since the decomposition is unique and using the condition c) of the statement, one can write

\[
Tv_1 + \mu_1 = v_1.
\]

(102)

Now, applying the condition c) again one get

\[
(Tv_1 + \mu_1)(\mathcal{X}) = (Tv_1)(\mathcal{X}) + \mu_1(\mathcal{X}) = v_1(\mathcal{X}) + \mu_1(\mathcal{X}).
\]

(103)

But we know \( (Tv_1)(\mathcal{X}) = v_1(\mathcal{X}) \), thus we have \( \mu_1(\mathcal{X}) = 0 \), which is to say \( \mu_1 = 0 \). Hence,

\[
Tv_1 = v_1
\]

\[
Tv_2 = v_2
\]

\[\Box\]

Since excluding singular measures with respect to the Lebesgue measure would suffice to prove ergodicity, we give a necessary and sufficient condition for this, in the case of weak Feller Markov chains on a Polish state-space which is a straightforward generalization of (Carlsson, 2005b, Theorem 3).

Theorem 13. Let \( X_t(x) \) be a weak-Feller Markov chain on a complete separable metric space \( \mathcal{X} \). There is an invariant measure \( \nu \) with a positive singular part if and only if there exists a starting point \( x_0 \in \mathcal{X} \) and a compact subset \( \mathcal{K} \subseteq \mathcal{X} \) with zero Lebesgue measure such that for all \( \epsilon > 0 \):

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} P(X_t(x_0) \in \mathcal{K}) > 1 - \epsilon.
\]

(104)

Proof. Let

\[
\nu_0 = \delta_{x_0}, \quad \nu_k = \frac{1}{k} \sum_{i=0}^{k-1} T^* \nu_0
\]

(105)

Then (104) gives

\[
\lim_{k \to \infty} \nu_k (\mathcal{K}) > 0
\]

(106)
Now choose a weakly convergent sub-sequence \( k_i \) such that for some \( \tilde{\nu} \),
\[
\lim_{i \to \infty} \nu_{k_i} \to \tilde{\nu} \quad \text{weakly}
\] (107)

Finally due to Portmanteau theorem Billingsley (2013),
\[
\tilde{\nu}(\mathcal{X}) \geq \limsup_{i \to \infty} \nu_{k_i}(\mathcal{X}) > 0.
\] (108)

Now, let \( \mathcal{X} \) be a compact set of Lebesgue measure zero in \( \mathcal{X} \) such that for some small \( \epsilon > 0 \)
\[
\nu(\mathcal{X}) > 1 - \epsilon,
\]
where \( \nu \) has a positive singular part. The existence of such a set can be shown as follows: Let \( \mu_n \) be the Lebesgue measure on \( \mathbb{R}^n \). If we consider singular measures on \( \mathcal{X} = \mathbb{R}^n \), then for \( n = 1 \), we can take Cantor measure Falconer (1985); Mattila (1995) \( \nu \), that is a singular probability measure whose support is the Cantor set \( C \). It is known that \( C \) is compact, \( \mu_1(C) = 0 \), but \( \nu(C) = 1 \). For \( n > 1 \), let \( x \in \mathbb{R}^n \) be a generic non-zero vector.

Denote \( L = \{cx : c \in \mathbb{R}\} \), be the line generated by \( x \), and notice that it is a copy of \( \mathbb{R} \), with \( \mu_n(L) = 0 \). Consider the singular measure \( \nu \) that behaves like \( \mu_1 \) on \( L \), meaning that
\[
\nu(\mathcal{V}) = \mu_1(\mathcal{V} \cap L)
\]
for every measurable set \( V \subset \mathbb{R}^n \). This is a singular measure. Since \( L \) is closed in \( \mathbb{R}^n \), every compact \( \mathcal{X} \subset L \) is also a compact for \( \mathbb{R}^n \). One can easily find a compact (for example, \([-1,1] \)) inside \( L \) such that
\[
\mu(\mathcal{X}) > 0, \quad \mu_k(\mathcal{X}) \leq \mu_k(L) = 0.
\]

Now, from (Yosida, 1995, Theorem 6 (ii)), for all \( w \in \mathcal{L}^1(\nu) \)
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{E}[w(X_k(x))] = w^*(x)
\] (109)
exists \( \nu \)-almost everywhere, moreover
\[
\int_{\mathcal{X}} w d\nu = \int_{\mathcal{X}} w^* d\nu.
\] (110)

For more on the decomposition of measures related to Markov operators, we refer to (Worm, 2010, Chapter 5 - Chapter 8), Zaharopol (2005, 2009), and Buzzi (2000) for the Lasota-York type map.

### 7.3 Topological aspects of IFS

**Definition 7.2.** (Skorokhod (1987)) A discrete-time Markov chain which start from \( X_0 = x \in \mathcal{X} \), denoted by \( \{X_k(x)\} \) on a metric space \( (\mathcal{X}, \rho) \) is said to be **topologically recurrent**, if for any open set \( \mathcal{Y} \) in \( \mathcal{X} \) and any starting point \( x \in \mathcal{X} \), we have,
\[
\mathbb{P} \left( \sum_{n=1}^{\infty} 1_{\mathcal{Y}}(X_k(x)) = +\infty \right) = 1,
\] (111)
where \( 1_{\mathcal{Y}}(X_k(x)) \) is defined as follows:
\[
1_{\mathcal{Y}}(X_k(x)) := \begin{cases} 
1 & \text{if } X_k \in \mathcal{Y}, \\
0 & \text{if } X_k \in \mathcal{Y}^c = (\mathcal{X} \setminus \mathcal{Y})
\end{cases}
\]
The expression $\sum_{n=1}^{\infty} 1_{\mathcal{U}}(X_n(x))$ is infinite i.e diverges iff the set $\{k \mid X_k \in \mathcal{U}\}$ is infinite. Therefore, the notion of topological recurrence of a Markov chain means that for each starting point $x \in \mathcal{X}$ and for each open set $\mathcal{U}$ in $\mathcal{X}$, the Markov chain $X_k$ will visit the open set $\mathcal{U}$ an infinite number of times.

**Lemma 7.1.** (Skorokhod (1987)) Suppose $\{X_k\}$ is a Feller Markov chain in a compact metric space $(\mathcal{X}, \rho)$, then $\{X_k\}$ is topologically recurrent if and only if

$$P(X_k(x) \in \mathcal{U}) > 0 \text{ for any open set } \mathcal{U} \text{ in } \mathcal{X}.$$  \hspace{1cm} (112)

**Definition 7.3.** (Usachev, 1972, Definition 2.4) Suppose $X_n$ is a Markov chain with transitional probability kernel $\nu(x, \cdot)$, then a non-empty measurable set $B$ in $\mathcal{X}$ is called stochastically closed iff $\nu(x, B) = 1 \text{ for all } x \in B$.

In other words, stochastically closed means that with probability 1, the Markov chain stays in the set once it steps into that set, so this is an analogue of absorbing set and stochastic version of forward invariance. No more than one topologically and stochastically closed set exists if a Markov chain is topologically recurrent. For an IFS, topological recurrence does not depend on the fact whether the system is contracting or expanding (Carlsson, 2005b, Remark 2). For topological recurrence and its relation with ergodicity of homogeneous Markov chain on locally compact and compact metric space, we refer to articles by Skorokhod (1987, 2007); Harris (1956); Blank (2018). In Samuel and Tetenov (2017); Mihail (2012) concept of topologically contractive IFS was defined, and recently their existence of invariant probability measure was studied in Leśniak et al. (2020).

### 7.4 Conformal IFS and Teichmüller Theory for IFS

**Conformal IFS** was introduced in Mauldin and Urbański (1996). By using the Perron-Frobenius operator, symbolic dynamics on shift space with an infinite alphabet, part of the article’s main result was to show the existence of a unique invariant probability measure for conformal IFS with the countable family of self-maps is equivalent of the existence of a conformal measure. For more development of conformal IFS we refer Mauldin and Urbański (1999); Mauldin et al. (2001); Lindsay and Mauldin (2002); Urbański (2002); Käenmäki (2003); Roy and Urbański (2005); Lau et al. (2009); Jaerisch and Kesseböhmer (2011); Deng and Ngai (2011); Seuret and Wang (2015); Rempe-Gillen and Urbański (2016); Mihaiescu and Urbański (2016); Das and Simmons (2021); Spaulding (2022). D. Sullivan in Sullivan (1982) proposed a measurable version of Mostow’s rigidity theorem. If two finite Kleinian groups are conjugate under a nonsingular Borel map $F$, then $F$ agrees practically everywhere with conformal conjugacy. Since this theorem’s introduction, measurable rigidity has received much attention, and several generalizations and modifications have been produced. Hanus and Urbański (1998) proved that two non essentially affine, conformal iterated function systems are conformally comparable if their conformal measures coincide up to generator permutation. The paper Kesseböhmer and Stratmann (2006) study rigidity and flexibility for conformal-iterated function systems. They demonstrate that two non-essentially affine systems are conformally identical if and only if there is at least one level set in each Lyapunov spectrum where the Gibbs measures coincide. They compared this result to an affine case, concluded that the essentially affine systems are less rigid than non-essentially affine systems, and studied their flexibility. Teichmüller theory is rich and active research area in mathematics Hubbard (2016). For the first time, Hille and Snigireva (2012) has introduced Teichmüller theory for families of conformal iterated function systems.

### 8 Opportunities for Further Research

In the final section, we sketch out several directions for further research and pointers to existing works along these directions.
8.1 Non-asymptotic Study of Mixing Rates in IFS

It is widely understood that the mixing rate of the IFS is geometric, asymptotically, when there exists a unique invariant measure Ślęczka (2011) since the corresponding Markov operator is contractive in the Wasserstein norm Lasota (1995). Hairer (2002, 2006a); Hairer et al. (2011) have shown how to simplify the analysis using coupling arguments.

Very recently, Leśniak et al. (2022) showed that the exponent of the rate of convergence could be bounded from below in terms of dimensions of a box enclosing the attractor and from above by various terms, including the Lipschitz constants of the IFS. Related, detailed analyses seem very desirable.

Within applied probability, detailed analyses of subgeometric rates of convergence under less restrictive assumptions are now appearing Durmus et al. (2016). These often extend the coupling technique of Hairer (2002). It would be interesting to extend these to iterated function systems. See (Godland, 2022, Chapter 7) for the first step in this direction.

8.2 Continuous-Time IFS and IFS on Time Scales

Continuous-time IFS with place-dependent probabilities were studied already by Lasota and Mackey (1999). They were interested in the biological interpretation and asymptotic stability of IFS and applied stability criteria to describe the stability of cell division structures. The results of Lasota and Mackey were further generalized and extended by Szarek (1999); Horbacz and Szarek (2001), who showed in a general Polish space that such a system is asymptotically stable, and its stationary distribution is singular. A very clear treatment is presented by Wojewódka (2013). Applications in biology (Alkurdi et al., 2013, e.g.) often rely on these tools, as detailed in Appendix A.3.

It would be of considerable interest to link the work on continuous-time IFS more closely with the work on stochastic partial differential equations on one hand and with switched systems Liberzon and Morse (1999); Zappavigna et al. (2010); Shorten et al. (2007) on the other. In the same spirit, one could develop connections to analysis on time scales, utilizing the tools of Pötzsche et al. (2003). See Ghosh et al. (2019) for the first small step in this direction.

8.3 IFS with Discontinuous Probability Functions

Investigating IFS with discontinuous probability functions would be of considerable interest in many practical applications. Jaroszewska (2013) considers an IFS with discontinuous but piecewise constant probability functions was studied and used the Schauder–Tychonov fixed point theorem approach, they proved that the system possesses an invariant probability measure. Marecek et al. (2022) suggested the use of the machinery
of Filippov (2013). (Fioravanti et al., 2019, Theorem 18) suggested that one could base an alternative view of IFS with discontinuous probabilities on the work of Werner (2004, 2005e,f,d,a,b,c, 2006), although the details remain to be worked out.

8.4 Testable Conditions for the Existence and Non-Existence of a Unique Invariant Measure

In many practical applications, the self-transformations are available only in an oracular fashion, i.e., upon providing index \( j \in [m] \) and \( X \), one has access to \( w_j(X) \). In such applications, it may be hard to estimate the moduli of continuity of \( w_j, j \in [m] \), to guarantee the contraction on average. Still, a testable condition for a unique invariant measure’s (non-)existence would be of considerable interest. Perhaps, Malliavin calculus Malliavin (1978); Nualart (2006) could provide an avenue towards deriving such conditions. While one could draw connections between Malliavin calculus and IFS following (Gravereaux, 1988; Viard, 1994, e.g.), much remains to be done in this respect.

8.5 Approximating and Shaping of Invariant Measure for IFS

A great deal of interest has been shown in data compression in approximating measures and functions using IFS and other similar approaches. A substantial compression factor may be achieved by representing a target measure or function using a minimal number of IFS parameters, as has been shown in the case of image processing, see, e.g., Barnsley and Jacquin (1988); Jacquin (1990a,b,c); Jacquin et al. (1992); Jacquin (1994). Forte and Vrscay (1995) state that the inverse problem of measure construction using IFS is as follows: given a target probability measure \( \nu \in \mathcal{M}_p(\mathcal{P}) \), can we construct using a Markov operator \( P \) an IFS consisting of \( m \) functions with constant probabilities \( \{p_i\}_{i=1}^m \) that has an invariant probability measure of \( \nu^* \) that is close to the desired \( \nu \) value, i.e \( d_h(\nu^*, \nu) < \epsilon \)? Definition of Hutchinson metric \( d_h \) can be found in (Barnsley, 1993, Definition 5.2, Chapter IX) is the same as that in (22). A solution to the aforesaid problem by using IFS and moment matching process was proposed by Barnsley and Demko (1985); Barnsley et al. (1986) and (Onésimo Hernández and Lasserre, 2003, Chapter 12) discusses the problem from optimization point of view. We recommend that the reader consult Widder et al. (1945); Stieltjes (1895); Baker and Gammel (1970); Bessis and Demko (1991) for further information on the research, the connection between moments and measures and some significance of this problem. An extension of the same problem to the IFS with state-dependent probabilities has been done, and when \( \mathcal{P} = [0, 1] \), the solution of the above problem has been achieved in La Torre et al. (2018). For more on IFS, the inverse problem of measure and moments, we refer Elton and Yan (1989); Diaconis and Shahshahani (1986); Abenda (1990); Abenda et al. (1992); Cabrelli et al. (1992); Vrscay (1991); Vrscay and Weil (1991); Vrscay and Rokhlin (1989); Kunze et al. (2012). The inverse problem for continuous probability measure on \( \mathbb{R} \) has been successfully addressed by Stenflo (2012a) and the construction of asymptotically stable IFS has been described in Jaroszewska (2002); Guzik (2016a).

Let \( w \) be a self-map on \( (0, \infty) \) and \( \{\sigma_k\}_{k \geq 0} \) be i.i.d \( (0, \infty) \) valued random variables. Chamayou and Letac (1991) studied several explicit invariant probability measures for the Markov chain of the form:

\[
X_{k+1} = \sigma_k(wX_k).
\]

When \( w \) is a Mobius transformation, its outcome is a particular product of random \( 2 \times 2 \) matrices and random continuing fractions when generic exponential families on \( (0, \infty) \) are considered. Invariant measures of dynamical systems have been researched extensively in the literature. It has been shown that certain algorithms converge to the invariant measure (within the error margins of specific metrics) for certain systems. Asymptotic convergence estimate rates are sometimes provided, see, e.g., Ding et al. (1993); Froyland (2007); Murray (2010); Bose and Murray (2001); Ding and Zhou (1994); Dellnitz and Junge (1999, 2002), however.
there are a few results that provide an explicit (rigorous) error bound, see, e.g., Bahsoun and Bose (2011); Ippei (2011); Liverani (2001); Keane et al. (1998); Pollicott and Jenkinson (2000) and more recent work by Galatolo and Nisoli (2014); Galatolo et al. (2009, 2016). When the maps are multi-dimensional, Ding and Zhou (1994, 1995) has results on approximating physical invariant measure, which was further generalized by Froyland et al. (1995); Froyland (1995, 1997a,b, 1998) in a series of papers.

When there is a continuous unique invariant measure for a discrete-time dynamical system on a compact metric space arising from IFS, Hunt and Miller (1992); Hunt (1996b,a) produces techniques for approximating these invariant measures and their convergence rate through a method earlier proposed by Ulam (1960). For the method description, see (van Wyk and Ding, 2002, section 11.3). We refer to several other excellent references for approximating the invariant measure of dynamical systems and invariant measure we refer Miller (1994); Ding and Wang (2019); Diamond et al. (1995); Froyland (1999); Froyland and Aihara (2000); Imkeller and Kloeden (2003); Boyarsky and Góra (2012); Bandt and P. (2017). Under some assumptions, in Strichartz et al. (1995), invariant measures associated with IFS with finitely many maps on a compact subset of \(\mathbb{R}\) were well approximated. These results were further generalized, and some algorithms were also developed for approximating invariant measures for IFS on higher dimensions in Öberg (2005, 2006). Góra and Boyarsky (2003) has numerical studies on absolutely continuous invariant measures for state-dependent IFS on \(\mathbb{R}^d\), and these results were further extended in Bahsoun and Bose (2011); Bahsoun et al. (2005a). An interesting observation in Góra et al. (2006) is that there may be a situation where absolutely continuous invariant measures can not be observed experimentally. For more results on the absolutely continuous invariant measure, we refer Boyarsky (1990); Islam and Chandler (2015) and the references there.

8.6 Set-valued IFS

IFS over the space of set-valued self-mappings or multi-functions was re-introduced in Torre et al. (2007); La Torre and Mendivil (2008); Kunze et al. (2008); Colapinto and Torre (2008) which could be traced back due to Hutchinson (1981). Fractals and multi-valued fractals are essential in biology, quantum mechanics, computer graphics, dynamical systems, astronomy, and geophysics, see El Naschie (1994); Iovane (2006b,a); Fedeli (2005); Shi and Chen (2004). IFS over multi-functions have substantial repercussions in applied sciences. In Llorens-Fuster et al. (2009), the existence and uniqueness of self-similar mixed-iterated function sets are provided, and the well-posedness of the self-similarity issue for various iterated multi-function systems is also explored. Well-posedness is intimately connected to approximating a fixed point equation, which is vital for constructing fractals using pre-fractals. Due to the wide range of influential application research on set-valued IFS is still active Nia (2015); Guzik (2016b); Georgescu et al. (2020); Goyal and Prasad (2021); Pandey et al. (2022); Russell Luke et al. (2018).

8.7 Non-homogeneous IFS

A significant portion of the IFS literature assumes time-homogeneous probabilities on drawing functions. In addition, convergence results are obtained within the assumption of certain average cotractivity. Stenflo (1998b) and later Mendivil (2015) have convergence results based on time-dependent and time-varying probabilities. These time-varying assumptions on probabilities have been missed in the IFS literature. The results radically different from those obtained in the conventional time-homogeneous contractive IFS. In Lu and Mukherjea (1997), a necessary condition for the existence of a \(\sigma\)-finite invariant measure for Markov chains with random transitions was discovered, along with probabilities demonstrating the significance of attractors for particular IFS to their invariant measures. In Ghosh et al. (2019), the notion of time-varying piecewise IFS was presented in Polish space.
8.8 Large Deviations for IFS

Recently large deviations (Feller, 2008, Feller XVI.7) probabilities and their applications have received a considerable amount of attention; for example, in probability and statistics Varadhan (1984); Frank (2000); Deuschel and Stroock (2001); Dupuis and Ellis (2011), information theory and queuing theory Shwartz and Weiss (1995), statistical mechanics Touchette (2009); Oono (1989), DNA analysis, communications and control Amir and Ofer (2010); Freidlin et al. (2012). In a dynamical system, the theory of large deviations allows one to find the likelihood that the ergodic average $\frac{1}{n} \sum_{i=0}^{n-1} f(T^k(x))$ deviate from the mean $\int f d\mu = 0$.

(Hennion and Hervé, 2001, Chapter VIII) have obtained such results using transfer operators, and Pollicott (2001) did the same for IFSs that contract on average. However, it should be mentioned that Pollicott assumes, without complete proof, that the perturbed transfer operators are analytic perturbations of the unperturbed transfer operator, as noted in (Hennion and Herve, 2004, Page-1940). Broise (1996) looked at such results for piecewise expanding maps and was able to say that, like many large deviations results, the probability that ergodic averages deviate from the mean decays exponentially fast. Motivated from the works like (Broise, 1996; Hennion and Herve, 2004; Hennion and Hervé, 2001, Chapter 10), and Santos and Walkden (2013); Bezuglyi and Jorgensen (2018), (Chiu, 2015, Chapter 3) has several results for large deviations for IFS that contract on average.

9 In Place of Conclusions

The survey outlined the development of a relatively simple notion of contraction on average into substantial literature on iterated function systems. Indeed, our survey alone covers over 400 papers, several books, and a steady stream of PhD dissertations in Applied Probability and related areas of Mathematics: Giles (1990); Daryl (1991); Hepting (1991); Vines (1993); Doughty (1995); Stenflo (1998a); Hanus (2000); Dai (2001); Kapica (2003); Carlsson (2005a); Doan (2009); Worm (2010); Jozsef (2013); Shaun (2015); Chiu (2015); Davison (2015); Pena (2016); Wojewódka (2016); Masoumeh (2017); Godland (2022). We suggest the breadth of the literature utilizing iterated function systems in several application domains within the appendices.

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A Some Applications of IFS

A.1 Physics, Mathematics and Engineering

The book of Tong (1990) contains a wide range of practical examples of the use of IFS in Statistics. See Stenflo (2002b) for a survey of place-dependent IFS and connections to the Ruelle-Perron-Frobenius theorem of statistical mechanics, Furstenburg and Kesten (1960); Guivarc’h and Raugi (1985); Mukherjee (1992); Alsmeyer and Löwe (2013); Kaijser (1978) for connections to the theory of random walk in non-negative matrices, theory of random matrices and the products of random matrices, Iosifescu and Grigorescu (1990) for connections to the theory of continued fractions, Propf and Wilson (1996); Athreya and Stenflo (2003); Roberts and Rosenthal (2004) for a connection with Markov chain Monte-Carlo algorithms, Falconer (1985, 2004); Barnsley (1993) for more on connections to the theory to fractals, Mandelbrot and Mandelbrot (1982); Peters (1994); Kifer (1995); Peters (1996); Bratteli and Jørgensen (1999); Jorgensen (2006); Fisher (2012); Gleick (2011); Peitgen et al. (2006); Krzyzszttof (2015) for connections to representation theory, Davison (2015, 2018) in operator theory, Dutkay and Jorgensen (2007b,a); Mirek (2011); Jorgensen et al. (2011) in harmonic analysis, Berger (1992) for the theory of wavelets, Keller Keller (1998) and Pesin (2008) for connections to the theory of deterministic dynamical systems. For a number research articles on IFS on graphs see Werner (2004, 2005e,f,d,a,b,c, 2006), on stochastic fixed point theory see Hermer et al. (2019, 2020). Articles which treat some particular type of IFS to mention a few such as: random walk on $\mathbb{R}^n$ generated by affine maps by Berger and Soner (1988), random walk on $(\mathbb{R}^n)^n$ with elastic collisions on the axis Leguesdron (1989), Peigne Peigne (1992), Mauldin and Urbański (2000) discusses parabolic IFS and Vince (2013) explored several interesting results on Mobiis IFS, Fayolles, Malyshiev, and Iasenogorodski Fayolle et al. (1999), concerning quantum information theory El Naschie (1994); Słomczyński (1997); Loziński et al. (2003); Jadczyk (2004); Baraviera et al. (2009); Jurgens and Crutchfield (2020), in mathematical finance Swishchuk and Islam (2011, 2013), to other branches of physics Decrouez et al. (2009); Fatchina et al. (2018); Davison (2018). In Control of switched systems Branicky (1998), signal modeling Vines (1993); Freeland and Durrani (1991), improve the gain and efficiency of the antenna Ezhumalai et al. (2021), to model discrete sequence David and Moson (1992), in power domain and domain theory Edalat (1996, 1995), electrical circuits Chen and Ueta (2002), and in TCP Niclas Carlsson (2005).

A.2 Computer Science and Computer Graphics

IFS was used to investigate stochastic learning automata, see, e.g., Narendra and Thathachar (1974, 2012); Lakshminivarah (2012); Norman (1968, 1972); Karlin et al. (1953). IFS on the space of input weights was used to develop an associative reinforcement learning theory in neural networks Bressloff and Stark (1991, 1992); Bressloff (1992). A recurrent neural network is an artificial neural network in which node connections generate a temporally directed graph Tino and Dorffner (1998). A recurrent neural network is an artificial neural network whose node connections produce a temporal graph. In Kolen (1994), IFS is used to characterize the behaviour of recurrent networks in symbolic processing.

Stark (1991) analyzed various methods for producing images using IFS and demonstrated how they might be described as neural networks with one neuron per image pixel. Such networks can quickly produce complex graphics for real-time animation. IFS application is known in pattern recognition also, see, e.g. Román-Roldán et al. (1994). Fosson (2011) devised a low-complexity, recursive approach for linear deconvolution systems using binary input using IFS.

IFS encodes and creates fractals; Barnsley showed how to compress images. Many articles address how IFS with affine maps may be used to draw and compress images, see, e.g., Hutchinson (1981); Dubins and Freedman (1966); Diaconis and Shahshahani (1986); Barnsley (1993); Barnsley and Sloan (1990); Kouzani (2008); Peruggia (1993); Hepting et al. (1991); Hepting (1991); Berger (1989). Many works in the literature
on fractals have developed in the past few decades. It has a very deep-rooted connection with the IFS see Williams (1971); Abraham and Ueda (2000) historical remark on fractals, and for other development and applications, see Crilly et al. (1991); Barnsley et al. (1989b, 2008); Natoli (2012); Sprott (1994); Jones (2001); Garg et al. (2014); Mitra et al. (2000); Alexander et al. (2012), on computer graphics Barnsley (1993); Barnsley and Jacquin (1988); Davoine and Chassery (2009); Nikiel (2007); Kunze et al. (2012); John (2007).

A.3 Biological Science

Andrzej Lasota and his group have made seminal contributions in a series of articles Lasota and Yorke (1973); Lasota and Mackey (1984); Tyrcha (1988); Lasota et al. (1992); Lasota and Yorke (1994); Lasota (1995); Lasota and Mackey (1998, 1999) in modelling cell division processes using techniques from IFS and Markov operators taking values in Polish state space. Using the IFS model, they studied general intracellular biochemistry. Each cell has \( d \) molecules whose dynamic evolution governs its life history. They showed that cell division might create functionally and physically stable cells despite unstable molecular dynamics. Fast cell multiplication stabilizes cellular populations according to their stability. For more on the application of IFS and, in general, random dynamical systems in the field of Biological science, see Tyson and Hannsgen (1986); Doebeli (1995); Trappe (1996); Lankhorst (1996); Swishchuk (1997); Anatoliy and Wu (2003); Joseph and Sasikumar (2006); Almeida and Vinga (2009); Almeida (2014); Cuzzocrea et al. (2017); Gaspard (2017); Anatoliy and Wu (2003); Ashlock and Golden (2003); Hoskins (1995); Lasota and Mackey (1999); Wojewódka (2013); Rudnicki (2000); Rudnicki et al. (2002); Wojewódka (2013); Hille et al. (2016); Rudnicki and Tyran-Kamińska (2015); Czapła et al. (2019, 2020).