Decays of the Meson $B_c$ to a $P$–Wave Charmonium State $\chi_c$ or $h_c$

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Abstract

The semileptonic decays, $B_c \rightarrow \chi_c(h_c) + \ell + \nu_\ell$, and the two-body nonleptonic decays, $B_c \rightarrow \chi_c(h_c) + h$, (here $\chi_c$ and $h_c$ denote $(c\bar{c}[^3P_J])$ and $(c\bar{c}[^1P_1])$ respectively, and $h$ indicates a meson) were computed. All of the form factors appearing in the relevant weak-current matrix elements with $B_c$ as its initial state and a $P$-wave charmonium state as its final state for the decays were precisely formulated in terms of two independent overlapping-integrations of the wave-functions of $B_c$ and the $P$-wave charmonium and with proper kinematics factors being ‘accompanied’. We found that the decays are quite sizable, so they may be accessible in Run-II at Tevatron and in the foreseen future at LHC, particularly, when BTeV and LHCB, the special detectors for B-physics, are borne in mind. In addition, we also pointed out that the decays $B_c \rightarrow h_c + \cdots$ may potentially be used as a fresh window to look for the $h_c$ charmonium state, and the cascade decays, $B_c \rightarrow \chi_c[^3P_{1,2}] + l + \nu_l$ ($B_c \rightarrow \chi_c[^3P_{1,2}] + h$) with one of the radiative decays $\chi_c[^3P_{1,2}] \rightarrow J/\psi + \gamma$ being followed accordingly, may affect the observations of $B_c$ meson through the decays $B_c \rightarrow J/\psi + l + \nu_l$ ($B_c \rightarrow J/\psi + h$) substantially.

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I. INTRODUCTION

The meson $B_c$, being a unique meson, contains two different heavy flavors. It decays by one of the two heavy flavors through weak interaction and it happens that the two have a comparable possibility each other in magnitude, or by the two heavy flavor annihilation, hence, its decay-channels which have a sizable branching ratio, are manifested much richer than those of the mesons $B^\pm, B^0, B_s, D^\pm, D^0, D_s$ etc. Therefore one may study the two heavy flavors $b,c$ simultaneously with the meson $B_c$ alone, as long as its different weak decay channels can be distinguished from each other well. Of all the mesons, in studying two heavy flavor $b,c$ simultaneously, $B_c$ is unique.

The meson $B_c$ is just discovered very recently. The first positive observation was successful in CDF at Tevatron, Fermilab through the semi-leptonic decays $B_c \rightarrow J/\psi + l + \nu_l$.

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and the mass $m_{B_c} = 6.40 \pm 0.39 \pm 0.13$ GeV, the lifetime $\tau_{B_c} = 0.46^{+0.18}_{-0.16} \pm 0.03$ ps etc were obtained [1].

Before the observation of CDF, $B_c$-meson production [2, 3], spectroscopy [4, 5] and various decays [6–10] had been widely computed. Now the further experimental studies of the meson are planned at Tevatron (in Run II) and at LHC etc. Particularly, in addition to CDF, D0, ATLAS and CMS, the detectors BTeV and LHCB are specially designed for B-physics, numerous $B^\pm$ events (more than $10^8 \sim 10^{10}$ per year) at these two colliders are expected to be recorded [3, 4], so a lot of interesting decay channels of $B_c$ will be well-studied experimentally, and certain rare processes will become accessible. Therefore, further extensive theoretical studies of this meson are freshly motivated.

The semileptonic decays, $B_c \to \chi_c(h_c) + l + \nu_l$, and the two-body nonleptonic decays, $B_c \to \chi_c(h_c) + h$, i.e. the decays of the meson $B_c$ to a P-wave chamonium state are certainly interesting, but still missing in literature, thus we devote this paper to report our the latest computations on them, although the semileptonic decays were reported shortly [10]. Why the decays interest people, let us outline the reasons below.

First of all, people would like to know how sizable the decays will be, especially, to know if accessible in Run-II of Tevatron and/or in LHC. Especially the cascade decays of $B_c \to \chi_c + \cdots$ and $\chi_c \to J/\psi + \gamma$ looks quite like as a signal for the observation of the meson $B_c$ through the decays $B_c \to J/\psi + \cdots$, because the photon may be missed in detectors. In addition, two of the P-wave charmonia have a branching ratio about a few tenth for the radiative decays $\chi_c[^3P_1] \to J/\psi + \gamma (Br = 27.3\%)$ and $\chi_c[^3P_2] \to J/\psi + \gamma (Br = 13.5\%)$, so indeed the cascade decays may potentially contribute a substantial background for the observation $B_c$ meson through $B_c \to J/\psi + \cdots$. Therefore, even only from the point of view to estimate the background for the observation on $B_c$ meson, to see how great the concerned decays is very interesting.

If one would like to see $CP$ violations in $B_c$ decays, for example, to see $CP$ violation in the decays $B_c \to h + h_1 + h_2$ ($h, h_1, h_2$ denote various possible mesons), as emphasized in Ref. [10], one knows that the interference of the direct decays with a cascade one through a resonance, e.g., $\chi_c[^3P_0]$, i.e., $B_c \to \chi_c[^3P_0] + h$ and $\chi_c[^3P_0] \to h_1 + h_2$, may enhance the visible $CP$ violation effects substantially. Thus to see the advantage of this method for the purpose quantitatively, the knowledge on the decay $B_c \to \chi_c[^3P_0] + h$ is necessary.

QCD-inspired potential model works very well for nonrelativistic double-heavy systems. The systems $(c\bar{b})$ and $(c\bar{b})$ in forming bound states, except the reduce mass, are similar to the well-studied systems $(b\bar{b})$ and $(c\bar{c})$, so it is believed that with potential model the static properties of the systems $(c\bar{b})$ and $(c\bar{b})$ can be predicted very well as those of bottomium $(b\bar{b})$ and charmonium $(c\bar{c})$. In general, to apply the wave functions to computing the relevant decay matrix elements is attracting, since the potential model will have further tests. Thus with the wave functions of $B_c$ (the ground state of the system of $(c\bar{b})$) and $\chi_c(h_c)$ (the P-wave states of $(c\bar{c})$) obtained by potential model, we have applied the wave functions to compute the decays $B_c \to \chi_c(h_c) + \cdots$.

Since the mass of $B_c$ ($m_{B_c}$) is much greater than those of the $P$-wave charmonia ($m_{\chi_c}$ and $m_{h_c}$), so the momentum recoil in the concerned decays can be a great (even relativistic). If one tries to apply the Schrödinger wave functions of nonrelativistic binding systems to computing the decay processes with such a great (even relativistic) recoil momentum, one cannot carry out the computation of the decay matrix element successfully just as done in atomic and nuclear decays by taking a suitable ‘reference frame’ and then a simple ‘boosting’, since the recoil in an atomic or nuclear process is much smaller than that in the present concerned decays. The great momentum recoil obviously means the velocity between the two CMS of $B_c$ meson and the charmonium state is huge, and the potential wave functions of the parent and the daughter states, given just in each CMS respectively, cannot be applied directly just by choosing a suitable reference frame and simple boosting the wave functions to the same frame. Thus when applying the wave functions to calculation of the decays (e.g.
the semileptonic decays and most two-body nonleptonic decays here) with such a great (even relativistic) momentum recoil, special handling is needed. To deal with the momentum recoil properly, an approach for the decays from a nonrelativistic $S$-wave state to another $S$-wave one, the so-called generalized instantaneous approximation, was proposed in Ref. [8]. Since it is straightforward to extend from a nonrelativistic $S$-wave state to another $S$-wave one, to the present case, that the decays are from a nonrelativistic $S$-wave state to a $P$-wave one for the approach, hence here we do so. The key points of the approach may be outlined as the three steps: firstly, to 'extend' the potential model, which is based on Schrödinger equation, to the one on Bethe-Salpeter (B.S.) equation even for the non-relativistic binding systems; then, according to Mandelstam method [12] to formulate the (weak) current matrix element (an elementary factor for the relevant decays) sandwiched by the B.S. wave functions of the two bound-state, so that the current matrix element is written in a fully relativistic formulation; finally, by making the so-called 'generalized instantaneous approximation' on the fully relativistic matrix element i.e. to integrate out the ‘time’ component of the relative momentum in the Mandelstam formulation by a contour integration, and as the final result, the current matrix element turns out back to be formulated in terms of proper operators sandwiched by the Schrödinger wave functions of the ‘original’ potential model. Since the weak current matrix (by means of the Mandelstam method) was formulated relativistically, so we can be sure that the final formulation takes the recoil effects into account properly and no new free parameter is added at all. Besides the great recoil effects are treated properly, one additional advantage of the approach is that it has a more solid ground on quantum field theory than that on the ‘original’ potential models, because the B.S. wave functions and the Mandelstam formulation have a more solid ‘ground’ on quantum field theories and they are used as a starting point to make the generalized instantaneous approximation.

On the other hand, B.S. equation is four-dimensional in space-time to describe a bound state problem, and there are a few problems still, such as, how to determine the QCD-inspired four-dimensional interaction kernel of the equation properly, and what is the physics meaning of the excitation in its relative-time ‘freedom’ of the two components etc. In addition, the B.S. equation is harder than a Schrödinger one to solve, even when the four-dimensional kernel is fixed. Whereas with the generalized instantaneous approximation, the current matrix elements are reduced into certain proper operators sandwiched by the potential model Schrödinger wave functions finally, therefore, the approach, in the meantime to circle the difficulty about treating the great momentum recoil effects properly, has also kept some of the advantages of potential model, such as to avoid the difficulty to solve the B.S. equations etc.

Finally we should note here that in our calculating the two-body nonleptonic decays of $B_c$ to the $P$-wave $\chi_c$ and $h_c$ states, the so-called factorization assumption and the effective Lagrangian for four fermions in which the ‘short-distance’ QCD corrections have been taken into account with OPE (operator product expansion) and RGM (the renormalization group method), as done by most authors, are adopted.

The paper is organized as follows: To follow the Introduction in Section-II, the exclusive semileptonic differential decay rates, the matrix elements and form factors etc are described. In Section III, the adopted approach, the so-called generalized instantaneous approximation, to compute the form factors is illustrated precisely. In Section IV, the two-body non-leptonic decays of $B_c$ are formulated with necessary description. Finally in Section V, numerical results and discussions are presented. The dependence of the current matrix elements on the form factors, and the dependence of the form factors on $\xi_1$ and $\xi_2$, the integrations of the wave function overlapping, are put in Appendix.

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1For the binding systems, $B_c$ and $\chi_c(h_c)$, to do the extension is just by means of the original instantaneous approximation proposed by Salpeter, that can be found in many text book on quantum field theory e.g. the book [13] to ‘build’ the relation between the Schrödinger equations and the relevant B.S. ones.
II. THE EXCLUSIVE SEMILEPTONIC DECAYS AND RELEVANT CURRENT MATRIX ELEMENTS

The $T-$matrix element for the semileptonic decays $B_c \to X_{c\ell} + \ell^+ + \nu\ell$:

$$T = \frac{G_F}{\sqrt{2}} V_{ij} \bar{u}_\ell \gamma_\mu (1 - \gamma_5) v_\ell < X_{c\ell}(p', \epsilon) | J_\mu | B_c(p) >,$$  \hspace{1cm} (1)

where $X_{c\ell}$ denotes $\chi_c$ and $h_c$, $V_{ij}$ is the Cabibbo-Kobayashi-Maskawa(CKM) matrix element and $J_\mu$ is the charged current responsible for the decays, $p$, $p'$ are the momenta of initial state $B_c$ and final state $X_{c\ell}$. Thus we have:

$$\sum |T|^2 = \frac{G_F^2}{2} |V_{ij}|^2 l^{\mu\nu} h_{\mu\nu},$$  \hspace{1cm} (2)

where $h_{\mu\nu}$ is the hadronic tensor and $l^{\mu\nu}$ the leptonic tensor. The later $l_{\mu\nu}$ is easy to compute whereas in general the former $h_{\mu\nu}$ can be written as:

$$h_{\mu\nu} = -\alpha g_{\mu\nu} + \beta_{++}(p + p')_\mu (p + p')_\nu + \beta_{+-}(p - p')_\mu (p + p')_\nu + \beta_{-+}(p + p')_\mu (p - p')_\nu + \beta_{--}(p - p')_\mu (p - p')_\nu + i\gamma \epsilon_{\mu\nu\rho\sigma} (p + p')^\rho (p - p')^\sigma,$$  \hspace{1cm} (3)

and by a straightforward calculation, the differential decay-rate is obtained accordingly:

$$\frac{d^3\Gamma}{dx dy} = |V_{ij}|^2 \frac{G_F^2 M^5}{32\pi^3} \left\{ \alpha \left( \frac{y - m_{l}^2}{M^2} \right) + 2\beta_{++} \left[ 2x(1 - \frac{M^2}{M^2} + y) - 4x^2 - y \right] + \frac{m_{l}^2}{4M^2}(8x + \frac{4M^2 - m_{l}^2}{M^2} - 3y) \right\} + 4(\beta_{++} + \beta_{--}) \frac{m_{l}^2}{M^2} (2 - 4x + y - \frac{2M^2 - m_{l}^2}{M^2}) + 4\beta_{-+} \frac{m_{l}^2}{M^2}(y - \frac{m_{l}^2}{M^2}) - \gamma \left[ y(1 - \frac{M^2}{M^2} - 4x + y) + \frac{m_{l}^2}{M^2}(1 - \frac{M^2}{M^2} + y) \right],$$  \hspace{1cm} (4)

where $x \equiv E_\ell/M$ and $y \equiv (p - p')^2/M^2$, $M$ is the mass of $B_c$ meson, $M'$ is the mass of final state $X_{c\ell}$. The coefficient functions $\alpha$, $\beta_{++}$, $\gamma$ can be formulated in terms of form factors. Note here that we have kept the mass of the lepton $m_l$ precisely that is different from those by N. Isgur et al [1] and by B. Grinstein et al [14], so the formula here can be applied not only to the cases of $e$ and $\mu$ semileptonic decays but also to those of $\tau$-semileptonic decays.

1. If $X_{c\ell}$ is $h_c([P_1])$ state: the vector current matrix element

$$< X_{c\ell}(p', \epsilon) | V_\mu | B_c(p) > \equiv r_\epsilon^*_\mu + s_+(\epsilon^\ast \cdot p)(p + p')_\mu + s_-(\epsilon^\ast \cdot p)(p - p')_\mu,$$  \hspace{1cm} (5)

and the axial vector current matrix element

$$< X_{c\ell}(p', \epsilon) | A_\mu | B_c(p) > \equiv iv\epsilon_{\mu\nu\rho\sigma} \epsilon^{\ast\nu}(p + p')^\rho (p - p')^\sigma,$$  \hspace{1cm} (6)

where $p$ and $p'$ are the momenta of $B_c$ and $h_c$ respectively, $\epsilon$ is the polarization vector of $h_c$.

2. If $X_{c\ell}$ is $\chi_c([3P_0])$ state: the vector matrix element vanishes, and the axial vector current

$$< X_{c\ell}(p', \epsilon) | A_\mu | B_c(p) > \equiv iv\epsilon_{\mu\nu\rho\sigma} \epsilon^{\ast\nu}(p + p')^\rho (p - p')^\sigma,$$  \hspace{1cm} (6)
\[< X_{\chi}(p')|A_\mu|B_\chi(p) = u_+(p+p')\mu + u_-(p-p')\mu. \quad (7)\]

3. If \(X_{\chi}\) is \(\chi_c([^3P_1])\) state:
\[< X_{\chi}(p',\epsilon)|V_\mu|B_\chi(p) = i\epsilon^\mu + c_+(\epsilon^* \cdot p)(p+p')\mu + c_-(\epsilon^* \cdot p)(p-p')\mu, \quad (8)\]
and
\[< X_{\chi}(p',\epsilon)|A_\mu|B_\chi(p) = i\epsilon_{\mu\nu\rho\sigma}^*\epsilon^{*\nu}(p+p')\rho(p-p')^\sigma. \quad (9)\]

4. If \(X_{\chi}\) is \(\chi_c([^3P_2])\) state:
\[< X_{\chi}(p',\epsilon)|V_\mu|B_\chi(p) = ih + \epsilon_{\mu\rho\sigma\nu}\epsilon^{*\nu}(p+p')\rho(p-p')^\sigma, \quad (10)\]
and
\[< X_{\chi}(p',\epsilon)|A_\mu|B_\chi(p) = k\epsilon^\mu p' + b_+(\epsilon^* p^\rho p^\sigma)(p+p')\mu + b_-(\epsilon^* p^\rho p^\sigma)(p-p')\mu. \quad (11)\]

The form factors \(r, s_+, s_-, v, u_+, u_-, l, c_+, c_-, k, b_+, b_-\) and \(h_{++}\) are functions of the momentum transfer \(t = (p-p')^2\) and can be calculated precisely. In Ref. [8] we proposed an approach, the generalized instantaneous approximation, to compute those form factors for the decays of \(B_\chi\) to an \(S\)-wave charmonium state \(J/\psi\) or \(h_c\). Now we are computing the form factors \(r, s_+, s_-, \cdots\) appearing in the decays of \(B_\chi\) to a \(P\)-wave charmonium state, in fact, the approach may be used directly, thus the approach is adopted in the present calculations here.

### III. THE SO-CALLED GENERALIZED INSTANTANEOUS APPROACH TO THE WEAK CURRENT MATRIX ELEMENTS

To calculate these form factors, the approach developed in Ref. [8] is adopted. Let us outline the approach here for convenience. According to the Mandelstam formalism [12] when the considered weak (electromagnetic) current matrix element involves only one hadron in the initial state and one in final state respectively, then it may be written down in terms of Bethe-Salpeter (B.S.) wave functions which describe the hadrons as bound states exactly:

\[p^\mu = i \int \frac{d^4q}{(2\pi)^4} Tr \left[ \Gamma_{\mu
u}(q')\Gamma^\mu\chi_{\nu}(p_2 + m_2) \right], \quad (12)\]

where \(\chi_{\nu}(q), \chi_{\nu'}(q')\) are the B.S. wave functions of the initial and final states with the corresponding momenta \(p, p'\). Throughout the paper we use \(p_1, p_2\) denote the momenta of the quarks in the initial meson \(B_\chi\), and \(p_1', p_2'\) denote the momenta of the quarks in the final meson \(\chi_c\) or \(h_c\). For convenience let us introduce further definition of the relative momentum \(q\) (or \(q'\)):

\[p_1 = \alpha_1 p + q, \quad \alpha_1 = \frac{m_1}{m_1 + m_2}; \]
\[p_2 = \alpha_2 p - q, \quad \alpha_2 = \frac{m_2}{m_1 + m_2}. \]

\(p_1, p_2, m_1\) and \(m_2\) are the momenta and masses for the quark and antiquark respectively. Note that the matrix element of the current Eq. (12) now is fully relativistic, thus it can be used as the start ‘point’ to take into account the recoil effects in the decays no matter how great the recoil moment is in the considered decay. To prepare in applying the generalized instantaneous approach for the matrix element, we need to ‘convert’ the potential model onto the B.S. equation ‘ground’.
A. The Potential Model and B.S. Equation

In general, the B.S. equation for the corresponding wave function $\chi_p(q)$:

$$(p_1 - m_1)\chi_p(q)(p_2 + m_2) = i \int \frac{d^4k}{(2\pi)^4} V(p, k, q)\chi_p(k), \quad (13)$$

where $V(p, k, q)$ is the kernel between the quarks in the bound state, may describe the relevant quark-antiquark bound state well. Accordingly the B.S. wave function $\chi_p(q)$ should satisfy the normalization condition:

$$\int d^4k (2\pi)^4 Tr \left[ \chi_p(k) \frac{\partial}{\partial p_0} \left( S_1^{-1}(p_1)S_2^{-1}(p_2)\delta^4(q - q') + V(p, q, q') \right) \chi_p(q') \right] = 2ip_0, \quad (14)$$

where $S_1(p_1)$ and $S_2(p_2)$ are the propagators of the relevant particles with masses $m_1$ and $m_2$ respectively.

As pointed out in introduction, the B.S. equation in four dimension should be reduced to a one in three dimension i.e. the time-like component momentum should be integrated out (the instantaneous approximation) with a contour integration as proposed by Salpeter, especially when the kernel has the property as follows

$$V(p, k, q) \approx V\left( |\vec{k} - \vec{q}| \right),$$

to do it is very easy. When one make a contour integration of the ‘time’ component of the relative momentum on the whole B.S. equation, then the B.S. equation is deduced straightforwardly into a three-dimensional equation which just is a Schrödinger equation in momentum representation. Since the start point of the common potential model is a Schrödinger equation, thus we may convert the potential model onto a ground based on the B.S. equation in the way with instantaneous approach.

To treat the possible great recoil effects in the decays, furthermore we need to convert the instantaneous approximation to a covariant way too, i.e. to divide the relative momentum $q$ into two parts, $q_\parallel$ and $q_\perp$, a parallel (time-like) part and an orthogonal one to $p$, respectively:

$$q^\mu = q^\mu_\parallel + q^\mu_\perp,$$

where $q^\mu_\parallel \equiv (p \cdot q/M_p^2)p^\mu$ and $q^\mu_\perp \equiv q^\mu - q^\mu_\parallel$. Correspondingly, we have two Lorentz invariant variables:

$$q_p = \frac{p \cdot q}{M_p}, \quad q_{pT} = \sqrt{q_p^2 - q^2} = \sqrt{-q_{p\perp}^2}.$$

In the rest frame of the initial meson, i.e., $\vec{p} = 0$, they turn back to the usual component $q_0$ and $|\vec{q}|$, respectively.

Now the volume element of the relative momentum $k$ can be written in an invariant form:

$$d^4k = dk_p k_{pT}^2 dk_{pT} ds d\phi, \quad (15)$$

where $\phi$ is the azimuthal angle, $s = (k_p q_p - k \cdot q)/(k_{pT} q_{pT})$. Now the interaction kernel can be rewritten as:

$$V(\vec{k} - \vec{q}) = V(k_\perp, s, q_{p\perp}). \quad (16)$$

Defining:

$$\varphi_p(q^\mu_\perp) \equiv i \int \frac{dq_p}{2\pi} \chi_p(q^\mu_\parallel, q^\mu_\perp),$$
The B.S. equation now can be rewritten as:
\[
\chi_p(q_p||, q_p\perp) = S_1(p_1)\eta(q_p\perp)S_2(p_2)
\]
and the propagators can be decomposed as
\[
S_i(p_i) = \frac{\Lambda_{ip}^+(q_p\perp)}{J(i)q_p + \alpha_i M - \omega_{ip} + i\epsilon} + \frac{\Lambda_{ip}^-(q_p\perp)}{J(i)q_p + \alpha_i M + \omega_{ip} - i\epsilon},
\]
with
\[
\omega_{ip} = \sqrt{m_i^2 + q_{pT}^2}, \Lambda_{ip}^+(q_p\perp) = \frac{1}{2\omega_{ip}} \left[ \frac{\hat{p}_\perp}{M} \omega_{ip} \pm J(i)(m_i + q_{p\perp}) \right],
\]
where \(i = 1, 2\) and \(J(i) = (-1)^{i+1}\). Here \(\Lambda_{ip}^{\pm}(q_p\perp)\) satisfies the relations
\[
\Lambda_{ip}^+(q_p\perp) + \Lambda_{ip}^-(q_p\perp) = \frac{\hat{p}_\perp}{M}, \Lambda_{ip}^+(q_p\perp) \frac{\hat{p}_\perp}{M} \Lambda_{ip}^+(q_p\perp) = \Lambda_{ip}^+(q_p\perp), \Lambda_{ip}^-(q_p\perp) \frac{\hat{p}_\perp}{M} \Lambda_{ip}^-(q_p\perp) = 0.
\]
Due to these equations, \(\Lambda^{\pm}\) may be referred as the \(p\)–projection operators, while in the rest frame of corresponding meson, they turn to the energy projection operator.

We define \(\varphi_p^{\pm}(q_p\perp)\) as
\[
\varphi_p^{\pm}(q_p\perp) \equiv \Lambda_{1p}^{\pm}(q_p\perp) \frac{\hat{p}_\perp}{M} \varphi_p(q_p\perp) \frac{\hat{p}_\perp}{M} \Lambda_{2p}^{\pm}(q_p\perp),
\]
where the upper index \(C\) denotes the charge conjugation. In our notation, \(\Lambda_{2p}^{\pm}(q_p\perp) \equiv \Lambda_{2p}^{\mp}(q_p\perp)\). Integrating over \(q_p\) on both sides of Eq.(18), we obtain:
\[
(M - \omega_{1p} - \omega_{2p})\varphi_p^{++}(q_p\perp) = \Lambda_{1p}^+(q_p\perp)\eta_p(q_p\perp)\Lambda_{2p}^{-C}(q_p\perp);
\]
\[
(M + \omega_{1p} + \omega_{2p})\varphi_p^{--}(q_p\perp) = \Lambda_{1p}^-(q_p\perp)\eta_p(q_p\perp)\Lambda_{2p}^{+C}(q_p\perp);
\]
\[
\varphi_p^{+-}(q_p\perp) = \varphi_p^{-+}(q_p\perp) = 0.
\]

The normalization condition of Eq.(14) now becomes
\[
\int \frac{q_{pT}^2 dq_{pT}}{(2\pi)^2} tr \left[ \varphi^{++} \frac{\hat{p}_\perp}{M} \varphi^{++} \frac{\hat{p}_\perp}{M} - \varphi^{--} \frac{\hat{p}_\perp}{M} \varphi^{--} \frac{\hat{p}_\perp}{M} \right] = 2P_0.
\]

From these equations, one may see that in the weak binding case to compare with the factor \((M - \omega_{1p} - \omega_{2p})\), the factor \((M + \omega_{1p} + \omega_{2p})\) is large, so the negative energy components of the wave functions \(\varphi^{--}\) are small. In the present case, for the heavy quarkonium and \(B_c\) meson, it is just the case, so we ignore the negative energy components of the wave functions safely at the lowest order approximation.

Neglecting the negative energy components of the wave functions, the B.S. equation contains the positive component
\[
\varphi_p^{++}(q_p\perp) \equiv \Lambda_{1p}^+(q_p\perp) \frac{\hat{p}_\perp}{M} \varphi_p(q_p\perp) \frac{\hat{p}_\perp}{M} \Lambda_{2p}^{-C}(q_p\perp)
\]
only, and the normalization condition becomes:

\[
\int \frac{q^2 dq_T}{(2\pi)^2} tr \left[ \varphi^+ \frac{\not{p}}{M} \varphi^+ \frac{\not{p}}{M} \right] = 2P_0
\]

Now let us consider the wave function \( \varphi^+ \) appearing in the above equations. We know that the total angular momentum of a meson is composed from orbital one and spin, furthermore there are two ways i.e. S-L coupling or j-j coupling to compose the total angular momentum. Here to consider \( P \)-wave states of charmonium, we adopt the way of S-L coupling, i.e. we let the spins of the two quarks couple into a total spin, which can be either singlet or triplet, then the total spin couple to the relative orbital angular momentum, and finally we obtain the total angular momentum. In this way, the reduced B.S. wave function \( \varphi_P \) can be written approximately as:

\[
\varphi_{1S_0}(\vec{q}) = \frac{P + M}{2\sqrt{2}M} \gamma_5 \psi_{n00}(\vec{q}), \quad (24)
\]

for \( ^1S_0 \) state, and

\[
\varphi_{3S_1}^\lambda(\vec{q}) = \frac{P + M}{2\sqrt{2}M} \epsilon^\lambda \psi_{n00}(\vec{q}), \quad (25)
\]

for \( ^3S_1 \) state, where \( \epsilon^\lambda \) is the polarization of this state. For the \( P \)-wave \((c\bar{c})\) wave functions:

\[
\varphi_{1P_0}(\vec{q}) = \frac{P + M}{2\sqrt{2}M} \gamma_5 \psi_{n1M_z}(\vec{q}), \quad (26)
\]

for \( ^1P_0 \), i.e. \( h_c \) state, and

\[
\varphi_{3P_0J_z}^J(\vec{q}) = \frac{P + M}{2\sqrt{2}M} \epsilon^J(S) \psi_{n1M_z}(\vec{q}) < 1S_z, LM_z | JJ_z > , \quad (27)
\]

for \( ^3P_J(J=0, 1, 2) \) i.e. \( \chi_c \) states, where \( \epsilon \) is the polarization vector of total spin, \( < 1S_z, LM_z | JJ_z > \) is Clebsch-Gordon coefficients which couple \( L, S \) to the total angular momentum \( J \). \( \psi_{n00} \) and \( \psi_{n1M_z} \) are the full B.S. wave functions.

**B. The Radius B.S. Equation in Momentum Space**

To solve the B.S. equation, the key problem is about its radial component. If we ignore the negative energy contributions, the reduced B.S. equation Eq.(18) in the rest frame of the meson center mass system can be written as:

\[
\varphi_P(\vec{q}) = \frac{\Lambda^+_1(\vec{q}) \int \frac{d\vec{k}}{(2\pi)^3} V(\vec{k}, \vec{q}) \varphi_P(\vec{k}) \Lambda^+_2(\vec{q})}{M - \omega_1 - \omega_2}.
\]

In the frame, the energy projection operator:

\[
\Lambda^+_1 = \frac{1}{2\omega_1} (\omega_1 \gamma_0 + \vec{\gamma} \cdot \vec{q} + m_1),
\]

\[
\Lambda^+_2 = \frac{1}{2\omega_1} (\omega_2 \gamma_0 - \vec{\gamma} \cdot \vec{q} - m_2),
\]
where the kernel $V$ acts on $\varphi(\hat{q})$ as:

$$V(\hat{q})\varphi(\hat{q}) = V_s(\hat{q})\varphi(\hat{q}) + V_v(\hat{q})\gamma_\mu\varphi(\hat{q})\gamma^\mu,$$  \hspace{1cm} \text{(29)}

i.e. to correspond to the potential model more precisely, the interaction kernel can be formally divided into the corresponding non-perturbative QCD ‘linear’ one, $V_s$ (in scalar nature) and the corresponding gluon exchange one, $V_v$ (in vector nature).

When substituting Eqs.(24,26), the wave functions in the meson center mass system, to the reduced B.S. equation Eq.(28), the equation for a spin singlet state $V$ nature) and the corresponding gluon exchange one, $\phi$ where the kernel

$$\phi_{S=0}(\hat{q}) = \frac{1}{4\omega_1\omega_2(M - \omega_1 - \omega_2)} \left\{ m_1m_2 \int \left[4V_v(\hat{q}, \hat{k}) - 4V_s(\hat{q}, \hat{k})\right] \phi_{S=0}(\hat{k})d\hat{k}\right\}, \hspace{1cm} \text{(30)}$$

where the $\phi_{S=0}(\hat{q})$ is $\phi_{n00}(1S_0)$ or $\phi_{n1M_z}(1P_1)$. Since square of the relative momentum $\hat{q}^2$ is small to compare with quark mass squared in the ‘double heavy’ meson, as a lowest order approximation, we have ignored such higher terms and use $\omega_1 = m_1, \omega_2 = m_2$ in numerator.

Now let us factorize out the radial component of the wave function and its relevant B.S. equation in momentum space from the angular ones:

$$\psi_{nLM_z}(\hat{q}) = \phi_{nL}(\hat{q}) Y_{LM_z}(\theta, \phi),$$

where $n$ is the principal quantum number, $L$ is the orbital angular momentum and $M_z$ is the projection of the third component of $L$, $\phi_{nL}(\hat{q})$ is the radial wave function and $Y_{LM_z}(\theta, \phi)$ is the spherical harmonic function. For the spin singlet states, multiplying $Y_{LM_z}^*(\hat{q})$ to two sides of the reduced B.S. equation and sum over $M_z$ by using the formula,

$$\frac{4\pi}{2L + 1} \sum_{M_z} Y_{LM_z}(\hat{q})Y_{LM_z}^*(\hat{k}) = P_L(cos\theta),$$

where $\theta$ is the angular between the unit vector $\hat{q}$ and $\hat{k}$, the radial reduced B.S. equation for $1S_0$ state is obtained:

$$\phi_{n0}(\hat{q}) = \frac{1}{4\omega_1\omega_2(M - \omega_1 - \omega_2)} \left\{ m_1m_2 \int \left[4V_v(\hat{q}, \hat{k}) - 4V_s(\hat{q}, \hat{k})\right] \phi_{n0}(\hat{k})d\hat{k}\right\}. \hspace{1cm} \text{(31)}$$

Whereas for $1P_1$ state:

$$\phi_{n1}(\hat{q}) = \frac{1}{4\omega_1\omega_2(M - \omega_1 - \omega_2)} \left\{ m_1m_2 \int \left[4V_v(\hat{q}, \hat{k}) - 4V_s(\hat{q}, \hat{k})\right] \phi_{n1}(\hat{k})cos\theta d\hat{k}\right\}, \hspace{1cm} \text{(32)}$$

where $\phi_{n0}(\hat{q})$ and $\phi_{n1}(\hat{q})$ are the radial parts of the wave functions.

Similarly, for the spin triplet states $S = 1$ we have:

$$\sum_{lm} <1S_z,LM_z|JJ_z> \phi_{S=1}(\hat{q}) = \sum_{lm} <1S_z,LM_z|JJ_z> \frac{1}{4\omega_1\omega_2(M - \omega_1 - \omega_2)} \times \left\{ m_1m_2 \int \left[4V_v(\hat{q}, \hat{k}) - 4V_s(\hat{q}, \hat{k})\right] \phi_{S=1}(\hat{k})d\hat{k}\right\}. \hspace{1cm} \text{(33)}$$

where the $\phi_{S=1}(\hat{q})$ is $\phi_{n00}(3S_1)$ or $\phi_{n1M_z}(3P_J)$. Then the equation for $3S_1$ is:
\[ \phi_{n0}(|\vec{q}|) = \frac{1}{4\omega_1\omega_2(M - \omega_1 - \omega_2)} \left\{ m_1m_2 \int \left[ 4V_v(\vec{q}, \vec{k}) - 4V_s(\vec{q}, \vec{k}) \right] \phi_{n0}(|\vec{k}|)d\vec{k} \right\}; \quad (34) \]

and for \( ^3P_J \):

\[ \phi_{n1}(|\vec{q}|) = \frac{1}{4\omega_1\omega_2(M - \omega_1 - \omega_2)} \left\{ m_1m_2 \int \left[ 4V_s(\vec{q}, \vec{k}) - 4V_v(\vec{q}, \vec{k}) \right] \phi_{n1}(|\vec{k}|) \cos \theta d\vec{k} \right\}. \quad (35) \]

The normalization of \( \phi_{nL} \) now is read:

\[ \int \frac{q^2_Tdq_T}{(2\pi)^2} \left[ \frac{m_1m_2}{\omega_1\omega_2} \phi_{nL}^2(|q_T|) \right] = 2M. \]

Under the present further approximation, the three triplet \( P \)-wave states \( ^3P_J \) and the singlet \( ^1P_1 \) as well, are degenerated. The reason is that we have ignored the ‘splitting’ interactions at all.

**C. The Generalized Instantaneous Approximation**

After neglecting the negative energy component and the ‘treatment’ above, the weak current matrix elements become as follows:

\[ l^\mu = i \int \frac{d^4q}{(2\pi)^4} Tr \left[ \eta(q_{\perp}') \frac{\Lambda^+_{1\perp}(q_{\perp}')}{q_{\perp}' + \alpha_1 M' - \omega_1 + i\epsilon} \Gamma^\mu \frac{\Lambda^+_2(q_{\perp})}{q_{\perp} + \alpha_2 M - \omega_1 + i\epsilon} \times \eta(q_{\perp}) \right], \quad (36) \]

The generalized instantaneous approximation, being an extension for the original one on the B.S. equations suggested by Salpeter, with the Cauchy’s theorem performs a contour integration about the time-like component \( q_T \) in complex plan on the whole current matrix elements precisely. As the final result, the matrix elements turn out to become a three dimensional integration about the space-like components \( q_{\perp} \).

If we choose the contour along the lower half plane, after completing the contour integration, the current matrix elements become as follows:

\[ l^\mu = \int \frac{d^4q_{\perp}}{(2\pi)^4} Tr \left[ \eta(q_{\perp}') \Lambda^+_{1\perp}(q_{\perp}') \Gamma^\mu \frac{\Lambda^+_2(q_{\perp})}{M' - \omega_1 - \omega_2} \right], \]

This matrix elements can also be written in the frame where the momentum \( q_{\perp}' \) is the integral argument by means of a suitable Jacobi transformation, i.e.

\[ l^\mu(r) = \int \frac{q_{\perp}'^2dq_{\perp}'ds}{(2\pi)^2} Tr \left[ \eta^++(q_{\perp}') \Gamma_\mu \eta^++(q_{\perp}) \frac{P'}{M'} \right]. \quad (37) \]

The above formula with the argument \( q_{\perp}' \) as the integral argument is more convenient, especially, in the cases when we calculate the matrix elements involving a \( P \)-wave state in the final state.

After performing the calculations on the matrix elements \( l^\mu \) precisely, the dependence of the matrix elements on the overlapping integrations of the initial and the final state wave functions becomes transparent. So is all the form factors too.

Since there is the so-called spin symmetry for heavy mesons, all of the form factors for their decays may attributed to one ‘universal’ function i.e. the Isgur-Wise function [13].
Therefore for the double heavy meson $B_c$ to a $S$-wave charmonium, at the limiting $m_b >> m_c >> \Lambda_{QCD}$ i.e. turning to the case of the heavy mesons, the form factors are attributed to the Isgur-Wise function, and the Isgur-Wise function is related to an overlapping integration of the wave functions of $B_c$ and the $S$-wave charmonium with certain kinematics factors precisely [3][6]. Now at the present case of $B_c$ to a $P$-wave charmonium, not only due to the spin-symmetry but also due to the great recoil in the decays, alternatively there are two independent and ‘universal’ functions, essentially, just two overlapping integrations of the wave functions of the initial and final bound states, $\xi_1$ and $\xi_2$, and all of the form factors are described by the two general functions with proper kinematics factors precisely.

Since in the present case the initial state is of an $S$-wave and the final state is of a $P$-wave, so the matrix elements must be related to two kinds of terms: one is to the integration which does not depend on the relative momentum $q_{\perp}$ at all, and the one just on $q_{\perp}$ linearly. Namely all the form factors appearing in the decays depend on two universal functions $\xi_1$ and $\xi_2$ only:

$$\epsilon^\lambda(L) \cdot \epsilon_0 \xi_1 \equiv \int \frac{d^3 q_{\perp}^*}{(2\pi)^3} \psi_{n1M_\pm}(q_{\perp}^*)\psi_{n00}(q_{\parallel}^T),$$

$$\epsilon_0^\alpha(L) \xi_2 \equiv \int \frac{d^3 q_{\perp}^*}{(2\pi)^3} \psi_{n1M_\pm}(q_{\perp}^*)\psi_{n00}(q_{\parallel}^T)q_{\perp}^\alpha,$$  \hspace{0.3cm} (38)

where

$$\epsilon_0 \equiv \frac{p - \frac{p'p}{M^2}p'}{\sqrt{(p'p')^2 - M^2}}$$

describes the polarization vector along recoil momentum $p'$, $\epsilon^\lambda(L)$ is the polarization vector of the orbital angular momentum.

We should note that for the decays from an $S$-wave state to a $P$-state, the function $\xi_1$ generated in the present approach is special. Since $\xi_1$ has more direct roots to the momentum recoil, so it cannot be obtained by boosting the final state wave function as done in the cases with a small recoil. The reason is that $\xi_1$ approaches to zero when the momentum recoil vanishes. Whereas, the function $\xi_2$, as in the cases with a small recoil, can essentially involve recoil effects just by ‘boosting’ the final state wave function.

Substituting the B.S. wave functions Eq.(24) and Eqs.(26-27) into the equation of current matrix elements and using Eq.(38), the precise formula for the form factors i.e. the precise dependence of the form factors on $\xi_1$ and $\xi_2$, can be obtained and we put them in the appendices and the curves of $\xi_1$ and $\xi_2$ obtained by numerical calculations are shown in a figure. With the functions $\xi_1, \xi_2$ and the form factors, the decay rates of the semileptonic decays and the spectrum of the charged lepton for the decays can be obtained by straightforward numerical calculations.

Note that in our calculation on the form factors, we have used the relations:

$$\sum_{\lambda,\lambda'} \epsilon_\mu^\lambda(S)\epsilon_\nu^{\lambda'}(L) < 1\lambda; 1\lambda'|00 > \equiv \sqrt{\frac{1}{3}}(g_{\mu\nu} - \frac{p'_{\mu}p'_{\nu}}{M^2}),$$

$$\sum_{\lambda,\lambda'} \epsilon_\mu^\lambda(S)\epsilon_\nu^{\lambda'}(L) < 1\lambda; 1\lambda'|1\lambda'' > \equiv \sqrt{\frac{i}{2M}}\epsilon_{\mu\nu\alpha\beta}p'^\alpha\epsilon^{\beta}_{\lambda'}(J),$$

$$\sum_{\lambda,\lambda'} \epsilon_\mu^\lambda(S)\epsilon_\nu^{\lambda'}(L) < 1\lambda; 1\lambda'|2\lambda'' > \equiv \epsilon^{\lambda''}_{\mu\nu}(J),$$  \hspace{0.3cm} (39)
where $< 1S_z; 1L_z | JJ_z >$ as previous are C.-G. coefficients. The polarization vector $\epsilon_{\mu}^\lambda(J), J = 1$ and the tensor $\epsilon_{\mu\nu}^\lambda(J), J = 2$ have the projection properties:

$$
\sum_{\lambda} \epsilon_{\mu}^\lambda(J) \epsilon_{\nu}^\lambda(J) = \left( \frac{p_\mu p'_\nu}{M^2} - g_{\mu\nu} \right) \equiv P_{\mu\nu},
$$

$$
\sum_{\lambda} \epsilon_{\mu\nu}^\lambda(J) \epsilon_{\alpha\beta}^\lambda(J) = \frac{1}{2} \left( P_{\mu\alpha} P'_{\nu\beta} + P'_{\mu\beta} P_{\nu\alpha} \right) - \frac{1}{3} P'_{\mu\nu} P'_{\alpha\beta},
$$

(40)

IV. THE TWO-BODY NON-LEPTONIC DECAYS

In this section we outline how the two-body non-leptonic decays $B_c \to \chi_c(h_c) + h$ (here $h$ denotes a meson) are calculated. We adopt the factorization assumption on the decay amplitudes which is widely adopted in estimation of the non-leptonic decays for various mesons. With the assumption, the weak current matrix elements appear in the calculations precisely and they are related to the form factors just obtained in the previous section. For the non-leptonic decay modes $B_c \to \chi_c(h_c) + h$ (caused by the decay $b \to c$), the following effective Lagrangian $L_{eff}$ (QCD corrections are involved) is responsible:

$$
L_{eff} = \frac{G_F}{\sqrt{2}} \left\{ V_{cb} [c_1(\mu) Q_{1}^{cb} + c_2(\mu) Q_{2}^{cb}] + h.c. \right\} + \text{penguin operators}.
$$

(41)

$G_F$ is the Fermi constant, $V_{ij}$ are CKM matrix elements and $c_i(\mu)$ are scale-dependent Wilson coefficients. The four-quark operators $Q_1^{cb}$ and $Q_2^{cb}$ (CKM favoured only) are:

$$
Q_1^{cb} = \left[ V_{ud}^* (\bar{d}u)_{V-A} + V_{us}^* (\bar{s}u)_{V-A} + V_{cd}^* (\bar{d}c)_{V-A} + V_{cs}^* (\bar{s}c)_{V-A} \right] (\bar{c}b)_{V-A} + \left[ V_{ud}^* (\bar{c}u)_{V-A} + V_{us}^* (\bar{c}s)_{V-A} + V_{cd}^* (\bar{c}d)_{V-A} + V_{cs}^* (\bar{c}s)_{V-A} \right] (\bar{c}b)_{V-A} + V_{c2}(\bar{c}c)_{V-A} (\bar{s}b),
$$

(42)

where $(\bar{q}_1 q_2)_{V-A}$ denotes $\bar{q}_1 \gamma_\mu (1 - \gamma_5) q_2$.

Because at this moment we restrict ourselves to consider the decays in which the coefficients of ‘penguin’ operators in the effective Lagrangian are small in comparison with the two main ones $c_1$ and $c_2$, so the contribution from penguin terms is neglected in the calculations, although in the Ref. [17] it is pointed out that in total decay width the penguin may have interference with the main ones and can course an increase about %3 ~ 4. Moreover, at this stage we also restrict ourselves only to consider the decay modes where the weak annihilation contribution is small due to precise reasons e.g. the helicity suppression etc[3], namely we neither take into account the contribution from the weak annihilation here.

Precisely by means of the factorization assumption, the decay amplitudes for the non-leptonic decays can be formulated into the three factors: the so-called leptonic decay constants, which are defined by the matrix elements: $< 0 | A_\mu | M(p) > = i f M p_\mu$ (or $< 0 | V_\mu | V(p, \epsilon) > = f_V M_V \epsilon_\mu$); the weak current matrix elements $< \chi_c | V_\mu (A_\mu) | B_c >$, which are those as the semileptonic decays; and the relevant coefficients in the combinations: $a_1 = c_1 + \kappa c_2$ and $a_2 = c_2 + \kappa c_1$, here $\kappa = 1/N_c$ and $N_c$ is number of color. The coefficients in

\[2\] We will consider the contribution from penguin and weak annihilation carefully elsewhere.
the combination \( a_1, a_2 \) is due to the weak currents being ‘Fierz-reordered’. In the numerical calculation later on, we will choose \( a_1 = c_1 \) and \( a_2 = c_2 \), i.e., we take \( \kappa = 0 \) in the spirit of the large \( N_c \) limit, and QCD correction coefficients \( c_1 \) and \( c_2 \) are computed at the energy scale of \( m_h \).

Therefore with the relations between the currents and form factors obtained as in the semileptonic decays, finally the factorized amplitudes for the nonleptonic decays can be formulated in terms of the form factors and the decay constants by definitions: \(<0 A_\mu| M(p) >= i f_M p_\mu\) and \(<0 V_\mu| V(p, e) >= f_V M_V e_\mu\). Thus the decay widths for the two-body nonleptonic decays can be computed straightforward.

V. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we present the numerical results.

In the numerical calculations, based on potential models the parameters are chosen as follows:

\[
\begin{align*}
\lambda &= 0.24 \text{ GeV}^2, \\
\alpha &= 0.06 \text{ GeV}, \\
\Lambda_{QCD} &= 0.18 \text{ GeV}, \\
\alpha &= e = 2.7183, \\
V_0 &= -0.93 \text{ GeV}, \\
V_{bc} &= 0.04, \\
m_1 &= 1.846 \text{ GeV}, \\
m_2 &= 5.243 \text{ GeV}.
\end{align*}
\]

With this set of parameters, we obtain the masses:

\[
M_{B_c} = 6.33 \text{ GeV}, \\
M' = 3.50 \text{ GeV}
\]

and corresponding radial wave-functions of \( B_c \) meson and \( P \)-wave charmonium \( \chi_{c, h} \), numerically. Here in the present evaluations, we only carry out the lowest order ones without considering the splitting caused by \( L - S \) and \( S - S \) couplings, in which all the bound states \( ^3P_0(J = 0, 1, 2) \) and \( ^1P_1 \) are degenerated.

To see the behaviors of the universal function \( \xi_1(t_m - t) \) and \( \xi_2(t_m - t) \) i.e. the two overlapping integrations of the wave functions of initial and final states, we plot them explicitly in Fig.1, where \( t_m = (M - M')^2, t = (P - P')^2 \).

The lepton energy spectra for the decays \( B_c \to \chi_c(e(\mu) + \nu) \), for which the mass of charged lepton can be ignored, are shown in Fig.2, and those for the decays \( B_c \to \chi_c(\tau + \nu) \), for which the mass of charged lepton \( \tau \) cannot be ignored, are shown in Fig.3, where \( \vec{p}_l \) is the momentum of lepton. The difference between Fig.2 and Fig.3 is due to the sizable mass of \( \tau \)-lepton. For the semileptonic decays, we put the corresponding widths in Table I.

As for the non-leptonic two-body decays \( B_c \to \chi_c(h_c) + h \), we only evaluate some typical channels, whose widths are relatively larger, and put results in Table II. In the numerical calculations, we have chosen \( a_1 = c_1 \) and \( a_2 = c_2 \), i.e., \( \kappa = 0 \), and \( c_1 \) and \( c_2 \) are computed at the energy scale of \( m_h \). The values of the decay constants: \( f_{\pi^+} = 0.131 \text{ GeV}, f_{\rho^+} = 0.208 \text{ GeV}, f_{a_1} = 0.229 \text{ GeV}, f_{K^+} = 0.159 \text{ GeV}, f_{K^{*+}} = 0.214 \text{ GeV}, f_{D_s} = 0.213 \text{ GeV}, f_{D_s^*} = 0.242 \text{ GeV}, f_{D^+} = 0.209 \text{ GeV}, f_{D^{*+}} = 0.237 \text{ GeV} \) are adopted by fitting decays of \( B \) and \( D \) mesons.

If comparing the results in Table 1 with the decays of \( B_c \) to \( S \)-wave charmonium states \( J/\psi \) and \( \eta_c \), e.g., \( \Gamma(B_c \to J/\psi + l + \nu) \sim 25 \cdot 10^{-15} \text{GeV} \), one can realize the semileptonic decays of \( B_c \) to the \( P \)-wave charmonium states in magnitude are about tenth of the decay \( B_c \to J/\psi + l + \nu_l \). As for the two-body nonleptonic decays, due to the difference in momentum recoil and the fact that the recoil momentum is fixed in a given specific decay, the \( P \)-wave decay \( B_c \to \chi_c(h_c) + h \) can be greater than twentieth of the one, \( B_c \to J/\psi(h_c) + h \), to an \( S \)-wave state.

The first observation of \( B_c \) by CDF group is through the semileptonic decay \( B_c \to J/\psi + l + \nu_l \), hence, we can conclude that most of the decays concerned here are accessible in Run-II of Tevatron and in LHC, especially, when the particular detector for \( B \) physics BTeV and LHCb at the two colliders are concerned. It is because that Tevatron and LHC will have more than 20 time events of \( B_c \) meson than Run-I and have much better detectors.

Since the decays \( B_c \to \chi_c[^3P_{1,2}] + l + \nu_l \) have such a quite sizable branching ratio, so the cascade decays i.e. the decays with an according one of the radiative decays \( \chi_c[^3P_{1,2}] \to J/\psi + \gamma \) followed may affect the observation through the semileptonic decays \( B_c \to J/\psi + l + \nu_l \)
as done by CDF group substantially, especially, when the efficiency of detecting a photons for the detector is not great enough.

We also would like to point out here that with sizable branching ratio, the decays \( B_c \rightarrow h_c + l + \nu_l \) and/or \( B_c \rightarrow h_c + h \) potentially can open a fresh ‘window’ to observe the charmonium state \( h_c[^1P_1] \), especially, to note that the charmonium state \( h_c[^1P_1] \) has not been well-established experimentally yet.

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**APPENDIX A:**

In this appendix, we present the form factors and formulas for \( \alpha, \beta_{++} \) and \( \gamma \) which are required in the calculations on the exclusive semileptonic decays of \( B_c \) to \( X_{cc} \), which denotes one of \( ^1P_1, ^3P_0, ^3P_1 \) and \( ^3P_2 \) states as indicated precisely in each case below.

For convenience, we introduce the parameters below:

\[
\omega_{20} \equiv \omega_2 \frac{p \cdot p'}{M M'}, \\
\omega_{10} \equiv \sqrt{\omega_{20}^2 - m_2^2 + m_1^2}, \\
nep = \sqrt{\frac{(p \cdot p')^2}{M'^2} - M^2}.
\]

1. **\( B_c \) Meson to Charmonium \( h_c[^1P_1] \)**

The matrix elements for the vector and axial currents:

\[
< X(p', \epsilon) | V_\mu | B_c(p) > \equiv r \epsilon_\mu^* + s_+ (\epsilon^* \cdot p)(p + p')_{\mu} + s_- (\epsilon^* \cdot p)(p - p')_{\mu} \\
< X(p', \epsilon) | A_\mu | B_c(p) > \equiv iv \epsilon_{\mu \rho \sigma} \epsilon^{\nu}(p + p')^\rho (p - p')^\sigma.
\]

Where

\[
r = \frac{(m'_1 - m_2)(m_1 + \omega_{10} - m_2 - \omega_{20})\xi_2}{8m'_1\omega_{10}\omega_{20}} - \frac{(m'_1 + m_2)(m_1 + \omega_{10} + m_2 + \omega_{20})\xi_2(p \cdot p')}{8M'Mm'_1\omega_{10}\omega_{20}},
\]

\[
s_+ = \frac{m_2[M(m_2 + \omega_{20} - m_1 - \omega_{10}) - M'(m_1 + \omega_{10} + m_2 + \omega_{20})]\xi_2}{8M'M^2\omega_{10}\omega_{20}^2} + \frac{m_2[M(m_2 + \omega_{20} - m_1 - \omega_{10}) - M'(m_1 + \omega_{10} + m_2 + \omega_{20})]\xi_1}{8M'M\omega_{10}\omega_{20}nep} + \frac{m_2[M(m_2\omega_{20} + \omega_{20}^2 - m_1\omega_{20} - \omega_{10}^2) - M'(m_1\omega_{20} - \omega_{10}^2 + m_2\omega_{20} + \omega_{20}^2)]\xi_2}{8M'M^2\omega_{10}\omega_{20}}.
\]
\[ s_\pm = \frac{m_2[-M(m_2 + \omega_20 - m_1 - \omega_10) - M'(m_1 + \omega_10 + m_2 + \omega_20)]\xi_2}{8M'M^2\omega_10\omega_{20}} + \frac{m_2[-M(m_2 + \omega_20 - m_1 - \omega_10) - M'(m_1 + \omega_10 + m_2 + \omega_20)]\xi_1}{8M'M\omega_{10}\omega_{20}n e p} \]

\[ s_- = \frac{m_2[-M(m_2 + \omega_20 - m_1 - \omega_10) - M'(m_1 + \omega_10 + m_2 + \omega_20)]\xi_2}{8M'M^2\omega_10\omega_{20}^2} + \frac{m_2[-M(m_2 + \omega_20 - m_1 - \omega_10) - M'(m_1 + \omega_10 + m_2 + \omega_20)]\xi_1}{8M'M\omega_{10}\omega_{20}} \]

\[ v = -\frac{(m'_1 + m_2)(m_1 + m_2 + \omega_10 + \omega_20)\xi_2}{16M'Mm'_1\omega_{10}\omega_{20}} \]

The dependence of \( \alpha, \beta_{++} \) and \( \gamma \) on the above form factors:

\[ \alpha = r^2 + 4M^2 \frac{p'}{p} v^2, \quad (A5) \]

\[ \beta_{++} = \frac{r^2}{4M^2} - M^2 y v^2 + \frac{1}{2} \left[ \frac{M^2}{M^2} (1 - y) - 1 \right] r s_+ + M^2 \frac{p'}{p} s_+^2, \quad (A6) \]

\[ \beta_{+-} = -\frac{r^2}{4M^2} + (M^2 - M'^2) v^2 + \frac{1}{4} \left[ -\frac{M^2}{M'^2} (1 - y) - 3 \right] r s_+ + \frac{1}{4} \left[ \frac{M^2}{M'^2} (1 - y) - 1 \right] r s_- + M^2 \frac{p'}{p} s_+ s_-, \quad (A7) \]

\[ \beta_{--} = \beta_{+-}, \quad (A8) \]

\[ \beta_{-+} = \frac{r^2}{4M^2} + \left[ M^2 y - 2(M^2 + M'^2) \right] v^2 + \frac{1}{2} \left[ -\frac{M^2}{M'^2} (1 - y) - 3 \right] r s_- + M^2 \frac{p'}{p} s_-^2, \quad (A9) \]

\[ \gamma = 2rv. \quad (A10) \]
The dependence of $\alpha$, $\beta_{++}$ and $\gamma$ on the above form factors:
\( \alpha = 0, \) \hspace{1cm} (A13)

\[ \beta_{++} = u_+^2, \quad \beta_{+-} = u_+u_-, \] \hspace{1cm} (A14)

\[ \beta_{-+} = u_-u_+, \quad \beta_{--} = u_-^2, \] \hspace{1cm} (A15)

\[ \gamma = 0. \] \hspace{1cm} (A16)

### 3. \( B_c \) Meson to Charmonium \( \chi_c[^3P_1] \)

The matrix elements for the vector and axial current currents:

\[<X(p', \epsilon)|V_\mu|B_c(p)> \equiv l\epsilon^* \mu + c_+(\epsilon^* \cdot p)(p + p')_\mu + c_-(\epsilon^* \cdot p)(p - p')_\mu,\]

\[<X(p', \epsilon)|A_\mu|B_c(p)> \equiv i\epsilon_{\mu\nu\rho\sigma}(p + p')^\nu(p - p')^\rho.\]

Where

\[ l = \frac{(m_1 + \omega_{10} + m_2 + \omega_{20})\xi_1[(p' \cdot p')^2 - M^2 M'^2]m_2}{4\sqrt{2}MM'^2\epsilon_0\omega_{10}\omega_{20}} - \frac{(m_1 + \omega_{10} - m_2 - \omega_{20})\xi_1(p' \cdot p')}{4\sqrt{2}MM'_1m_2\omega_{10}} - \frac{(m_1 + \omega_{10} - m_2 - \omega_{20})\xi_2(p' \cdot p')}{4\sqrt{2}MM'_1\omega_{10}\omega_{20}} \]

\[ + \frac{\xi_2 m_2[(p' \cdot p')^2 - M^2 M'^2]}{4\sqrt{2}MM'^2\epsilon_0\omega_{10}\omega_{20}} \left[ \frac{(\omega_{10} + m_1 + \omega_{20} + m_2)}{\omega_{20}} + \frac{(m_1\omega_{20} + m_2\omega_{20} + \omega_{20}^2 - \omega_{10}^2)}{\omega_{10}} \right]. \hspace{1cm} (A16)\]
The dependence of the matrix element of the vector and axial currents:

\[ q = \frac{(\omega_{10} + m_1 + \omega_{20} + m_2)\xi_1}{8\sqrt{2M'\omega_{10}\omega_{20}}np} - \frac{m_2(-\omega_{10} - m_1 + \omega_{20} + m_2)\xi_1}{8\sqrt{2M'\omega_{10}\omega_{20}}np} \]

\[ + \frac{(\omega_{20}^2 + m_1\omega_{20} + m_2\omega_{20} - \omega_{10}^2)\xi_2}{8\sqrt{2M'M\omega_{10}^2}} + \frac{(m_2 + \omega_{20})\xi_2}{8\sqrt{2M'Mm_2\omega_{10}}} \]

\[ - \frac{\xi_2m_2}{8\sqrt{2MM'\omega_{10}\omega_{20}}} \left[ \frac{(-\omega_{10} - m_1 + \omega_{20} + m_2)}{\omega_{20}} + \frac{(-m_1\omega_{20} + m_2\omega_{20} + \omega_{20}^2 - \omega_{10}^2)}{\omega_{10}^2} \right]. \]

The dependence of \( \alpha, \beta_{++} \) and \( \gamma \) on the above form factors:

\[ \alpha = l^2 + 4M^2 p^2 q^2, \] \hspace{1cm} (A17)

\[ \beta_{++} = \frac{l^2}{4M^2} - M^2yq^2 + \frac{1}{2} \left[ \frac{M^2}{M^2}(1-y) - 1 \right] lc_+ + M^2 \frac{p^2}{M^2} c_+^2, \] \hspace{1cm} (A18)

\[ \beta_{+-} = -\frac{l^2}{4M^2} + (M^2 - M^2)q^2 + \frac{1}{4} \left[ -\frac{M^2}{M^2}(1-y) - 3 \right] lc_+ \]

\[ + \frac{1}{4} \left[ \frac{M^2}{M^2}(1-y) - 1 \right] lc_- + M^2 \frac{p^2}{M^2} c_+ c_-, \] \hspace{1cm} (A19)

\[ \beta_{-+} = \beta_{+-} \] \hspace{1cm} (A20)

\[ \beta_{--} = \frac{l^2}{4M^2} + \left[ M^2y - 2(M^2 + M^2) \right] q^2 + \frac{1}{2} \left[ -\frac{M^2}{M^2}(1-y) - 3 \right] lc_- + M^2 \frac{p^2}{M^2} c_-^2, \] \hspace{1cm} (A21)

\[ \gamma = 2lq. \] \hspace{1cm} (A22)

### 4. \( B_c \) Meson to Charmonium \( \chi_c^{[3P_2]} \)

The matrix element of the vector and axial currents:

\[ < X(p', \epsilon)|V_{\mu}|B_c(p) > \equiv ih_{+\epsilon} \epsilon_{\rho\sigma\alpha} \epsilon_{s\alpha} p_{\alpha}(p + p')^\rho(p - p')^\sigma, \]

\[ < X(p', \epsilon)|A_\mu|B_c(p) > \equiv k\epsilon_{\mu\rho} p^\rho + b_+(\epsilon_{\rho\sigma\rho} p^\rho) p_{\sigma}(p + p')_{\mu} + b_-(\epsilon_{\rho\sigma} p^\rho p^\sigma)(p - p')_{\mu}. \]

Where

\[ k = -\frac{(m_1 + \omega_{10} + m_2 + \omega_{20})\xi_1}{4\omega_{10}np} + \frac{m_2(-m_1 - \omega_{10} + m_2 + \omega_{20})\xi_1}{4\omega_{10}\omega_{20}np} \]
\[\xi_2 m_2 \frac{4 M \omega_1}{4 M \omega_1} + \xi_2 m_2 \left[ \frac{(-m_1 \omega_20 + m_2 \omega_20 + \omega_20 - \omega_10)}{\omega_20} \right] \]

\[b_+ = \frac{m_2(m_1 + \omega_1 + m_2 + \omega_20) \xi_1}{8 M' M \omega_1 \omega_20} + \xi_2 \frac{8 M' M \omega_1 \omega_20 \text{ne}p}{8 M' M \omega_1 \omega_20} \]

\[b_- = -\frac{m_2(m_1 + \omega_1 + m_2 + \omega_20) \xi_1}{8 M' M \omega_1 \omega_20} - \frac{8 M' M \omega_1 \omega_20 \text{ne}p}{8 M' M \omega_1 \omega_20} \]

\[h_{++} = -\frac{M^2 p' \cdot k b_+}{3 M^2} \left[ \frac{M^2}{M^2(1 - y) - 1} \right], \quad (A24)\]

\[h_{+-} = \frac{M^2 p' \cdot k b_+}{6 M^2} \left[ -\frac{M^2}{M^2(1 - y) - 3} \right] - \frac{M^2 p' \cdot k b_-}{6 M^2} \left[ -\frac{M^2}{M^2(1 - y) + 1} \right], \quad (A25)\]

\[\beta_{++} = \beta_{+-} = \beta_{-+} = \beta_{-+} = \beta_{+-} = \beta_{-+} = \beta_{++} = \beta_{+-} = \beta_{-+} \]

\[\beta_{+-} = \beta_{-+} \]

\[\beta_{-+} = \beta_{+-} \]

\[\beta_{++} = \beta_{+-} \]

\[\beta_{+-} = \beta_{-+} \]

\[\beta_{-+} = \beta_{++} \]
\[
\frac{k^2}{24} \left( 2 + \frac{M^2}{Mr^2} (2 - y) + \frac{4M^2 \vec{p}'^2}{M'^4} \right) - \frac{M^2 \vec{p}'^2}{3M'^2} k b_+ \left[ \frac{M^2}{M'^2} (1 - y) + 3 \right],
\]
(A27)

\[
\gamma = \frac{M^2 \vec{p}'^2}{M'^2} k h.
\]
(A28)
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TABLES

TABLE I. The semileptonic decay widths (in the unit $10^{-15}$ GeV)

| Channel | $\Gamma(B_c \rightarrow ^1P_1\ell\nu_\ell)$ | $\Gamma(B_c \rightarrow ^3P_0\ell\nu_\ell)$ | $\Gamma(B_c \rightarrow ^3P_1\ell\nu_\ell)$ | $\Gamma(B_c \rightarrow ^3P_2\ell\nu_\ell)$ |
|---------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| $\epsilon(\mu)$ | 2.509                             | 1.686                             | 2.206                             | 2.732                             |
| $\tau$    | 0.356                             | 0.249                             | 0.346                             | 0.422                             |

TABLE II. Two-body non-leptonic $B_c^+$ decay widths in unit $10^{-15}$ GeV

| Channel       | $\Gamma$       | $\Gamma(a_1 = 1.132)$ | $\Gamma(a_1 = 1.132)$ |
|---------------|----------------|-----------------------|-----------------------|
| $^1P_1\pi^+$  | $a_1^2$ 0.569  | 0.729                 | 1.40                  |
| $^3P_0\pi^+$  | $a_1^2$ 0.317  | 0.407                 | 0.806                 |
| $^3P_1\pi^+$  | $a_1^2$ 0.0815 | 0.104                 | 0.331                 |
| $^3P_2\pi^+$  | $a_1^2$ 0.277  | 0.355                 | 0.579                 |
| $^1P_1A_1$    | $a_1^2$ 1.71   | 2.19                  | 4.26$x$10^{-3}        |
| $^3P_0A_1$    | $a_1^2$ 1.03   | 1.33                  | 2.35$x$10^{-3}        |
| $^3P_1A_1$    | $a_1^2$ 0.671  | 0.859                 | 0.583$x$10^{-3}       |
| $^3P_2A_1$    | $a_1^2$ 1.05   | 1.34                  | 1.99$x$10^{-3}        |
| $^1P_1K^*$    | $a_1^2$ 7.63$x$10^{-3} | 9.78$x$10^{-3} | 4.26$x$10^{-3}       |
| $^3P_0K^*$    | $a_1^2$ 4.43$x$10^{-3} | 5.68$x$10^{-3} | 2.35$x$10^{-3}       |
| $^3P_1K^*$    | $a_1^2$ 2.05$x$10^{-3} | 2.63$x$10^{-3} | 0.583$x$10^{-3}      |
| $^3P_2K^*$    | $a_1^2$ 3.48$x$10^{-3} | 4.47$x$10^{-3} | 1.99$x$10^{-3}       |
| $^1P_1D_*$    | $a_2^2$ 1.99   | 2.56                  | 2.32                  |
| $^3P_0D_*$    | $a_2^2$ 1.48   | 1.89                  | 1.18                  |
| $^3P_1D_*$    | $a_2^2$ 2.21   | 2.83                  | 0.149                 |
| $^3P_2D_*$    | $a_2^2$ 2.68   | 3.44                  | 0.507                 |
| $^1P_1D^{**}$ | $a_1^2$ 0.788  | 0.101                 | 0.0868                |
| $^3P_0D^{**}$ | $a_1^2$ 0.0567 | 0.0726                | 0.0443                |
| $^3P_1D^{**}$ | $a_1^2$ 0.0767 | 0.0983                | 0.00610               |
| $^3P_2D^{**}$ | $a_1^2$ 0.0972 | 0.124                 | 0.0209                |
FIG. 1. The universal functions $\xi_1$ and $\xi_2$ vs. $t_m - t$. They are the overlapping-integrations of the wave functions for $\chi_c(h_c)$ and $B_c$ with the definition as in Eq.(38). The solid line is of $\xi_1$, the dashed one is of $\xi_2$.

FIG. 2. The energy spectrum of the charged lepton for the decays $B_c \rightarrow \chi_c + e(\mu) + \nu$, where the solid line is the result of $h_c[^1P_1]$ state, dotted-blank-dashed line is of $\chi_c[^3P_0]$, dashed line is of $\chi_c[^3P_1]$, dotted-dashed line is $\chi_c[^3P_2]$. 

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FIG. 3. The energy spectrum of the charged lepton for the decays $B_c \to \chi_c + \tau + \nu_\tau$, where the solid line is the result of $h_c[1^1P_1]$ state, dotted-blank-dashed line is of $\chi_c[3^3P_0]$, dashed line is of $\chi_c[3^3P_1]$, dotted-dashed line is $\chi_c[3^3P_2]$. 

\[ \frac{1}{\Gamma_i} \frac{d\Gamma_i}{d|\vec{p}|} \text{GeV}^{-1} \]