§0. Introduction.

In [19], Strominger, Yau and Zaslow made a surprising conjecture about pairs of mirror manifolds, which, if true, should at last provide a true geometric understanding of mirror symmetry. Simply put, string theory suggests that if $X$ and $\tilde{X}$ are mirror pairs of $n$-dimensional Calabi-Yau manifolds, then on $X$ there should exist a special Lagrangian $n$-torus fibration $f : X \to B$, (with some singular fibres) such that $\tilde{X}$ is obtained by finding some suitable compactification of the dual of this fibration. More precisely, if $B_0 \subseteq B$ is the largest set such that $f_0 = f|_{f^{-1}(B_0)}$ is smooth, then $\tilde{X}$ should be a compactification of the fibration $\tilde{f}_0 : R^1 f_0^* (\mathbb{R}/\mathbb{Z}) \to B_0$.

As yet, not a great deal is known about whether this conjecture is true. As was observed in [19], K3 surfaces do have special Lagrangian two-torus fibrations. In two dimensions it is easy to construct such fibrations because K3 surfaces are hyperKähler, having an $S^2$ of complex structures compatible with a given Ricci-flat metric. Submanifolds that are special Lagrangian in one complex structure are holomorphic in another one of these complex structures. Thus one simply looks for elliptic fibrations in a different complex structure from the original one. Of course, dualizing an elliptic fibration over a curve has no effect on the topology, which fits with the fact that the mirror of a K3 surface is a K3 surface. Thus one does not see any of the more subtle behaviour of mirror symmetry at first.

Using the Borcea-Voisin construction of mirror symmetry between Calabi-Yau threefolds of the form $K3 \times E/(\iota,-1)$, Pelham Wilson and I gave an example of the SYZ construction in the three-dimensional case. We were able to confirm the conjecture that the known mirror was indeed a compactification of the dual of the three-torus fibration one obtained by using a two-torus fibration on the K3 surface and a circle fibration on

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the elliptic curve. Unfortunately the metric is degenerate, and this is still not a complete example.

Any complete clarification of the SYZ construction will require a great deal more. In particular, there is still not a single example of a special Lagrangian three-torus fibration on a Calabi-Yau threefold with a non-degenerate Ricci-flat metric. We need an understanding of special Lagrangian submanifolds which will allow us to construct such fibrations. Next, one has to have sufficient knowledge about the singular fibres to understand how to compactify the dual fibrations. One needs to understand how to put a complex and Kähler structure on the mirror, thus defining the mirror map between complex and Kähler moduli space. Finally, one needs to understand how this process allows one to count holomorphic curves.

This paper will not treat most of these issues. Instead, we will assume that we can construct torus fibrations on Calabi-Yau manifolds and that their mirrors are obtained by dualizing these fibrations. Our goal will be to discuss what the expected topological properties of these fibrations and Calabi-Yau manifolds must be. In doing so we will have to make some assumptions about the nature of special Lagrangian torus fibrations. The assumptions about the existence and properties of these fibrations make this paper a rather speculative one. In order to ground these assumptions in reality, we refer to the known examples of special Lagrangian fibrations for K3 surfaces and Borcea-Voisin examples. We also give, in §1, a local example of a special Lagrangian fibration. This example has singular fibres that have not previously appeared. Hopefully, these examples can be taken as evidence for the assumptions we make.

In §2, we consider the Leray spectral sequence for a special Lagrangian torus fibration \( f : X \to B \) and deduce some elementary consequences about dualizing from this. In §3, we back up and investigate some of the implications of the existence of a large complex structure limit point in the complex moduli space of a Calabi-Yau manifold. This allows us to formulate a key conjecture about the nature of the monodromy about a branch of the discriminant locus in complex structure moduli space passing through a large complex structure limit point. This conjecture, Conjecture 3.7, states that such monodromy is given by translation of the \( T^n \) fibration by a section. This is precisely a generalization of the Dehn twist of an elliptic curve.

§4 is devoted to working out the consequences of the conjecture. By computing the action of translation by a section on cohomology, we are able to conclude that the limiting values of the \( (1, n-1) \) Yukawa couplings agree with the topological couplings of the mirror. In addition, if \( n = 3 \), the monodromy weight filtration and the Leray filtration coincide,
as conjectured in [11] and [18].

§5 is even more speculative. We use the earlier results of the paper to give, in the threefold case, an isomorphism between $H^{even}(\bar{X}, Q)$ and $H^{odd}(X, Q)$. A priori there are many such isomorphisms, and any such choice needs justification as to its naturality. In this case, we are motivated by Kontsevich’s homological mirror symmetry conjecture. Hopefully, we provide a natural choice for the mirror map on the level of cohomology. One byproduct of this effort is that we provide a description of $H^{odd}(X, Q)$ in which the monodromy transformations about boundary divisors passing through the large complex structure limit point have a very simple description.

A note on notation: If $X$ is a compact manifold of dimension $n$, we denote by $\cup$ the cup product in $H^*(X, G)$ for a group $G$. If $X$ has an orientation, giving a canonical isomorphism $H^n(X, \mathbb{Z}) \cong \mathbb{Z}$, then for $\alpha_i \in H^{d_i}(X, \mathbb{Z})$, $\sum_{i=1}^{m} d_i = n$, the intersection number $\alpha_1 \cdot \ldots \cdot \alpha_m$ is the image of $\alpha_1 \cup \ldots \cup \alpha_m$ under the canonical isomorphism. The reason for being didactic about distinguishing cup products and intersection numbers is that in the situation studied in this paper, $\bar{X}$ will not have a canonical choice of orientation until we place a complex structure on $\bar{X}$.

We also note the convention here that for a compact oriented submanifold $M$ of dimension $p$ of an oriented manifold $X$, the cohomology class $[M]$ of $M$ is the class such that

$$\int_M \alpha = \int_X \alpha \cup [M]$$

for all $p$-forms $\alpha$ on $X$.

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§1. Some Special Lagrangian Fibrations.

Let $X$ be a Riemannian manifold, and $\alpha$ a $p$-form on $X$. Recall from [12] that $\alpha$ is a calibration if $d\alpha = 0$ and for each $p$-plane $\xi$ in $T_{X,x}$, $\|\alpha|_\xi\| \leq \|\text{Vol}(\xi)\|$, with equality holding for at least one $\xi$, where $\text{Vol}(\xi)$ is the volume form on $\xi$ induced by the metric. An oriented $p$-dimensional submanifold $M \subseteq X$ is a calibrated submanifold if $\alpha|_M = \text{Vol}(M)$, the volume form on $M$ induced by the metric on $X$. Any calibrated submanifold is necessarily minimal.
One can extend the notion of a calibrated submanifold to singular subsets. The proper context to do this, as pointed out in [12], is that of rectifiable $p$-currents, and one then obtains natural compactness statements for spaces of calibrated currents. As this is quite technical, we will not go into this here. Rather, given a subset $M \subseteq X$ with $M_0 \subseteq M$ a smooth $p$ dimensional submanifold and with $M - M_0$ of Hausdorff dimension less than $p$, we say $M$ is calibrated if $M_0$ is.

We recall Proposition-Definition 1.1. [12] Let $X$ be an $n$-dimensional complex manifold with a nowhere-vanishing holomorphic $n$-form $\Omega$, and a Riemannian metric $g$ with associated Kähler form $\omega$ ($g(X,Y) = \omega(JX,Y)$) such that

$$\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n \Omega \wedge \bar{\Omega}.$$  

Then $\text{Re} \Omega$ is a calibration, called the special Lagrangian calibration. Furthermore, modulo orientation, a submanifold $M \subseteq X$ of real dimension $n$ is special Lagrangian if and only if $\omega|_M = 0$ and $\text{Im} \Omega|_M = 0$.

This proposition was not stated in this generality in [12], but merely for the standard holomorphic $n$-form and symplectic form on $\mathbb{C}^n$ (i.e. $\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i$, $\Omega = e^{i\theta} dz_1 \wedge \cdots \wedge dz_n$ for some fixed $\theta$). However, the given normalization condition assures that each point, there is a basis of tangent vectors in which $\omega$ and $\Omega$ can be written as the above standard symplectic and holomorphic forms. We emphasize also that the condition that $X$ be a Kähler manifold (i.e. $d\omega = 0$) is unnecessary. This suggests an analogy between the study of pseudo-holomorphic curves on symplectic manifolds with an almost complex structure, and special Lagrangian submanifolds on a complex manifold with an almost symplectic structure.

Another reason for emphasizing that the Kähler condition is unnecessary is that I will now give some examples of special Lagrangian torus fibrations. To date, I have been unable to construct Kähler metrics on these examples in which the fibrations are special Lagrangian, even in the two-dimensional case where such metrics are known to exist [9].

Example 1.2. Let

$$X = \mathbb{C}^n - \{1 + \prod_{j=1}^n z_j = 0\}$$

with $z_1, \ldots, z_n$ coordinates on $\mathbb{C}^n$,

$$\Omega = \frac{1}{i^n(1 + \prod z_j)} dz_1 \wedge \cdots \wedge dz_n$$
and
\[\omega = \frac{1}{|1 + \prod z_j|^{2/n}} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.\]

Let \( f : X \to \mathbb{R}^n \) be given by \( f = (f_1, \ldots, f_n) \), with
\[f_1(z_1, \ldots, z_n) = |z_1|^2 - |z_2|^2\]
\[\vdots\]
\[f_{n-1}(z_1, \ldots, z_n) = |z_1|^2 - |z_n|^2\]
\[f_n(z_1, \ldots, z_n) = \log |1 + \prod z_j|\]

This is a globalization of the example of [12], III.3.A. To check that the fibres of \( f \) are Lagrangian with respect to \( \omega \), one can check just as well that they are Lagrangian with respect to \( \omega' = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \), the standard symplectic form on \( \mathbb{C}^n \). To do this, it is enough to check that \( \{f_i, f_j\} = 0 \) for all \( i \) and \( j \), which is easily done. To check that \( \text{Im} \Omega|_{f^{-1}(x)} = 0 \), note that, as in [12], III.2.C, the columns of the complex matrix \((i\partial f_k/\partial \bar{z}_j)_{j,k}\) at a point \( z = (z_1, \ldots, z_n) \) span the tangent space to \( f^{-1}(f(z)) \), and thus \( \text{Im} \Omega|_{f^{-1}(f(z))} = 0 \) if and only if
\[\text{Im} \frac{1}{i^n(1 + \prod z_j)} \det(i\partial f_k/\partial \bar{z}_j) = 0.\]

Again, this is easily checked.

Note that \( X \) has a diagonal \( T^{n-1} \) action given by
\[(z_1, \ldots, z_n) \mapsto (e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)\]
with \( \theta_1 + \cdots + \theta_n = 0 \). This action acts on the fibres of \( f \), and is useful for understanding these fibres. Indeed, let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and consider \( z = (z_1, \ldots, z_n) \in f^{-1}(x) \).

Then \(|1 + \prod z_j| = e^{x_n}\), so \( \prod z_j \) lies on a circle in \( \mathbb{C} \) of radius \( e^{x_n} \), centered at \(-1\). It is then easy to check that
\[\{z' \in f^{-1}(x) | \prod z'_j = \prod z_j\} = T^{n-1} \cdot z,\]

the orbit of \( z \) under the \( T^{n-1} \) action, and that for each \( c \in \mathbb{C} \) with \(|1 + c| = e^{x_n}\), there exists a \( z \in f^{-1}(x) \) with \( \prod z_j = c \). Since the orbit of \( z \) under \( T^{n-1} \) is homeomorphic to \( T^{n-1} \) unless \( z \in \bigcup_{1 \leq i < j \leq n} P_{ij} \) where
\[P_{ij} = \{(z_1, \ldots, z_n) \in X | z_i = z_j = 0\},\]
we see that any fibre $f^{-1}(x)$ disjoint from $\bigcup P_{ij}$ is an $n$-torus. Furthermore, the discriminant locus of $f$ is precisely $\Delta = f(\bigcup P_{ij})$. We have

$$f(P_{ij}) = \{(x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^n | x_{j-1} = 0 \text{ and } x_k \leq 0 \text{ for all } k\}$$

and for $i, j > 1$,

$$f(P_{ij}) = \{(x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^n | x_{i-1} = x_{j-1} \text{ and } x_{i-1}, x_{j-1} \geq x_k \text{ for all } k\}.$$
\[ \gamma_n \mapsto \gamma_n \pm \delta, \text{ the sign depending on chosen orientation of } \gamma_n \text{ and } \partial \Delta. \text{ This gives a complete description of the monodromy.} \]

For example, in the \( n = 3 \) case, the monodromy about the three branches of the discriminant locus have matrices in the basis \( \gamma_1, \gamma_2, \gamma_3 \) with suitable choice of orientations of \( \gamma_n \) and \( \partial \Delta \)

\[
T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.
\]

**Remark 1.3.** We note that the discriminant locus of the above fibration is codimension 2. Using the theory of volume minimizing manifolds, one can prove that if \( f : X \to B \) is a \( C^\infty \) special Lagrangian torus fibration with \( B \) smooth, and if the fibres of \( f \) are reduced, \( \dim_X X = n \leq 6 \), then the discriminant locus \( \Delta \subseteq B \) has Hausdorff dimension \( \leq n - 2 \). This is in distinction with the general case of a Lagrangian torus fibration, where one might have a codimension one discriminant locus, as well as fibres of real dimension \( < n \). This will be discussed in [10].

**Remark 1.4.** An obvious question: what is the dual of the special Lagrangian torus fibration constructed above? To date, I have not been able to construct a dual to this example. One can make a natural guess for what the dual singular fibres are however. I would conjecture that if there is a natural choice for a dual special Lagrangian \( T^n \) fibration, then the fibre dual to a fibre homeomorphic to \( T^{n-l} \times ((S^1 \times T^{l-1})/\{pt\} \times T^{l-1}) \) could be described as follows: thinking of a torus \( T^i \) as \([0, 1]^i\) with opposite sides identified, let \( V^i = T^i - (0, 1)^i \). Then the conjectural dual fibre is

\[
T^{n-l} \times ((T^{l-1} \times S^1)/\sim)
\]

where \( (t_1, s_1), (t_2, s_2) \in T^{l-1} \times S^1 \) are equivalent if either \( (t_1, s_1) = (t_2, s_2) \), or \( t_1 = t_2 \) and \( t_1, t_2 \in V^i \).

§2. Topological Mirror Symmetry and the Leray Spectral Sequence.

In this section \( X \) will denote a Calabi-Yau \( n \)-fold, and let \( B \) be a real \( n \) dimensional manifold with \( f : X \to B \) a special Lagrangian fibration whose general fibres are \( n \)-tori. In this section, \( f \) only need be \( C^0 \), since we will only be concerned about topology. We will assume in all that follows that \( B \) is a compact manifold without boundary.\footnote{This assumption is not necessarily well-justified: perhaps \( B \) might be singular at singular fibres of \( f \).} Presumably
being special Lagrangian places some strong additional conditions on the nature of the singular fibres of $f$. Since we do not know yet what these singular fibres might be, we will make some assumptions about the fibration $f$.

Let $\Delta \subseteq B$ be the discriminant locus of $f$, and let $B_0 = B - \Delta$, $f_0 = f|_{f^{-1}(B_0)}$, $j : B_0 \hookrightarrow B$ the inclusion. Note that $B_0$ is an open subset of $B$ by the results of [15].

**Definition 2.1.** Let $G$ be an abelian group (in cases of interest $G = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$). We say $f$ is $G$-simple if

$$j_*R^qf_0_*G = R^qf_*G$$

for all $q$. If $G = \mathbb{Q}$, we just say $f$ is simple, instead of $\mathbb{Q}$-simple.

This means essentially that the cohomology of the singular fibres is determined by monodromy about $\Delta$. In particular, note the fact that $j_*R^n f_0_* \mathbb{Q} = R^n f_* \mathbb{Q}$ implies that all fibres are irreducible. For example, an elliptic fibration is simple if and only if all fibres are irreducible. We will show in §3 that we expect such irreducibility for a special Lagrangian torus fibration if $X$ is sufficiently general in complex moduli. In most of the paper, we will only be concerned about $\mathbb{Q}$-simplicity, but near the end we will need $\mathbb{Z}$-simplicity. The difference is a matter of torsion.

**Example 2.2.** It is easy to check $\mathbb{Z}$-simplicity for the fibration of Example 1.2. Indeed, it is enough to check that $H^i(X_0, \mathbb{Z}) = H^i(X_b, \mathbb{Z})^G$, where $b \in \mathbb{R}^n$, $b \notin \Delta$, and $G$ is the group generated by all monodromy transformations $T_{ij}$, $1 \leq i < j \leq n$. Let $X_0^\#$ be the smooth part of $X_0$; then $X_0^\# \cong T^{n-1} \times \mathbb{R}$ and $X_0 = X_0^\# \cup \{0\}$, $0$ the origin in $\mathbb{C}^n$. Then the exact sequence

$$\cdots \to H^i_c(X_0^\#, \mathbb{Z}) \to H^i(X_0, \mathbb{Z}) \to H^i(\{0\}, \mathbb{Z}) \to \cdots$$

shows that

$$H^i(X_0, \mathbb{Z}) \cong \begin{cases} H^0(\{0\}, \mathbb{Z}) & i = 0 \\ H^i_c(X_0^\#, \mathbb{Z}) \cong H^{n-i}(X_0^\#, \mathbb{Z})^\vee \cong H^{n-i}(T^{n-1}, \mathbb{Z})^\vee \cong H^{i-1}(T^{n-1}, \mathbb{Z}) & i > 0. \end{cases}$$

On the other hand, if $\gamma_1^*, \ldots, \gamma_n^*$ is the basis of $H^1(X_b, \mathbb{Z})$ dual to the basis $\gamma_1, \ldots, \gamma_n$ of $H_1(X_b, \mathbb{Z})$ given in §1, then $H^i(X_b, \mathbb{Z}) \cong \bigwedge^i H^1(X_b, \mathbb{Z})$ and $H^i(X_b, \mathbb{Z})^G$ is easily seen to be generated by

$$\{\gamma_1^* \wedge \gamma_n^* | I \subseteq \{1, \ldots, n-1\}, \#I = i - 1\}.$$ 

This gives equality between $H^i(X_0, \mathbb{Z})$ and $H^i(X_b, \mathbb{Z})^G$. 

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Definition 2.3. If $f : X \to B$ is simple, an $n$-torus fibration $\tilde{f} : \tilde{X} \to B$ is called a permissible dual of $f$ if $\tilde{f}$ is simple and $\tilde{f}^{-1}(B_0) \to B_0$ is isomorphic to $R^1f_{0*}(R/Z) \to B_0$.

It is not clear if permissible duals exist or are unique. The reason for working with this simplicity condition is that the cohomology of $\tilde{X}$ is then closely related to the cohomology of $X$, something we need for mirror symmetry to hold. First, recall that a special Lagrangian submanifold $M$ is oriented so that $\Omega|_M$ is the volume form on $M$. Thus the fibres of the special Lagrangian fibration $f : X \to B$ come along with a canonical orientation, and hence we obtain a natural isomorphism $R^n f_{0*}Q \cong Q_{B_0}$. Having fixed this isomorphism, Poincaré duality gives a perfect pairing

$$R^q f_{0*}Q \times R^{n-q} f_{0*}Q \to R^n f_{0*}Q \cong Q_{B_0}$$

which yields

$$R^q f_{0*}Q \cong (R^{n-q} f_{0*}Q)^\vee.$$  

Now

$$(R^{n-q} f_{0*}Q)^\vee \cong R^{n-q} \tilde{f}_{0*}Q,$$

so applying $j_*$ and the assumption that $f$ and $\tilde{f}$ are simple yields

$$R^q f_*Q \cong R^{n-q} \tilde{f}_*Q.$$  

In particular,

(2.1) $$H^p(B, R^q f_*Q) \cong H^p(B, R^{n-q} \tilde{f}_*Q).$$

Note that if furthermore $f$ is $\mathbb{Z}$-simple, we similarly obtain

(2.2) $$H^p(B, R^q f_*\mathbb{Z}) \cong H^p(B, R^{n-q} \tilde{f}_*\mathbb{Z}).$$

We now use the Leray spectral sequence for $f$ and $\tilde{f}$. In general, $f_*Q = Q$, and by simplicity, $R^n f_*Q \cong Q$. If $n = 2$, $f : X \to B$ is an elliptic fibration, and if $X$ is simply connected, $B = S^2$, and the Leray spectral sequence takes the form

$$
\begin{array}{ccc}
Q & 0 & Q \\
0 & H^1(B, R^1 f_*Q) & 0 \\
Q & 0 & Q
\end{array}
$$

It is standard that $H^0(B, R^1 f_*Q) = 0$ for a non-trivial elliptic fibration (see [5]).

For $n = 3$, we have
Lemma 2.4. If $n = 3$, $f : X \to B$ simple with permissible dual $\tilde{f} : \tilde{X} \to B$, and $X, \tilde{X}$ simply connected, then the Leray spectral sequences for $f$ and $\tilde{f}$ degenerate at the $E_2$ term. In addition, one obtains (not natural!) isomorphisms

\[
H^{even}(X, \mathbb{Q}) \cong H^{odd}(\tilde{X}, \mathbb{Q}),
\]

\[
H^{odd}(X, \mathbb{Q}) \cong H^{even}(\tilde{X}, \mathbb{Q}),
\]

and

\[
h^{1,1}(X) = h^{1,2}(\tilde{X}),
\]

\[
h^{1,2}(X) = h^{1,1}(\tilde{X}).
\]

Proof. Since $X$ is simply connected, so is $B$, so $H^1(B, \mathbb{Q}) = H^2(B, \mathbb{Q}) = 0$. Thus we also have $H^i(B, R^3f_*\mathbb{Q}) = 0$ for $i = 1, 2$. In addition, since $H^1(X, \mathbb{Q}) = H^5(X, \mathbb{Q}) = 0$, we must have $H^0(B, R^1f_*\mathbb{Q}) = 0$ and $H^3(B, R^2f_*\mathbb{Q}) = 0$. The same holds true for $\tilde{f}$, and using (2.1), we obtain the following $E_2$ term for $X$:

\[
\begin{array}{ccc}
\mathbb{Q} & 0 & 0 \\
0 & H^1(B, R^2f_*\mathbb{Q}) & H^2(B, R^2f_*\mathbb{Q}) \\
0 & H^1(B, R^1f_*\mathbb{Q}) & H^2(B, R^1f_*\mathbb{Q}) \\
\mathbb{Q} & 0 & 0 \\
\end{array}
\]

The only possible non-zero differentials then occur in the following exact sequences:

\[
H^3(X, \mathbb{Q}) \xrightarrow{\varphi} H^0(B, R^3f_*\mathbb{Q}) \xrightarrow{d_2} H^2(B, R^2f_*\mathbb{Q})
\]

and

\[
H^1(B, R^1f_*\mathbb{Q}) \xrightarrow{d_2} H^3(B, f_*\mathbb{Q}) \xrightarrow{f^*} H^3(X, \mathbb{Q}).
\]

Now if $\sigma \in H^3(X, \mathbb{Q})$ is a class such that $\sigma \cdot [T^3] \neq 0$, then $\varphi(\sigma) \neq 0$, so $\varphi_1$ is surjective. If $[p]$ is the class of a point of $B$, then $f^*([p]) = [T^3] \neq 0$. Thus $f^*$ is injective. Thus in both cases, $d_2$ is zero, and the spectral sequence degenerates. The remainder of the theorem then follows immediately from (2.1), observing that the isomorphism in cohomology is not natural since it involves choosing a splitting of the Leray filtration on $H^3(X, \mathbb{Q})$, and also that $\dim H^1(B, R^2f_*\mathbb{Q}) = \dim H^2(B, R^1f_*\mathbb{Q})$ by (2.1) and the fact that $\dim H^2(\tilde{X}, \mathbb{Q}) = \dim H^4(\tilde{X}, \mathbb{Q})$, so that $h^{1,2} = \frac{1}{2} \dim H^3(X, \mathbb{Q}) - 1 = \dim H^1(B, R^2f_*\mathbb{Q})$. •

Remark 2.5. It is not clear what to conjecture in higher dimensions. Demanding that the spectral sequences degenerate might be too strong a condition. The ideal outcome would be that the spectral sequence does degenerate. A weaker condition which would be useful to know is that if $H^2(X, \mathcal{O}_X) = 0$, then $H^1(B, R^1f_*\mathbb{Q}) \cong H^2(X, \mathbb{Q})$; this would be especially useful in light of the discussion of monodromy in §3.
One can make a few elementary observations.

(1) If $f$ has a section $\sigma_0 : B \to X$, then the maps $d_2 : H^{i-2}(B, R^1 f_* G) \to H^i(B, G)$ are zero for any coefficient group $G$: indeed, the map $f^* : H^i(B, G) \to H^i(X, G)$ is injective, as $\sigma_0^*$ is a left inverse of $f^*$.

(2) If $H^2(X, \mathcal{O}_X) = 0$, then every element of $H^2(X, \mathbb{Z})$ is represented by a divisor. On the other hand, if $\alpha \in H^2(B, \mathbb{Z})$ is non-zero and $\beta \in H^{n-2}(B, \mathbb{Z})$ with $\alpha \cup \beta \neq 0$, then $f^*(\alpha \cup \beta) \cup [\Omega] \neq 0$, since this is proportional to the volume of a fibre. But for any divisor $D$ on $X$, $\Omega|_{\text{Supp}(D)} = 0$, so $D \cup f^* \beta \cup [\Omega] = 0$. This contradicts $f^* \alpha \in H^2(X, \mathbb{Z})$ being representable by a divisor. Thus $H^2(B, \mathbb{Z})$ must be a torsion group if $H^2(X, \mathcal{O}_X) = 0$.

Note though that if $X$ is a holomorphic symplectic four-fold, we expect $H^2(B, \mathbb{Z}) \neq 0$. For example, if $X$ is the symmetric square of a K3 surface with a special Lagrangian $T^2$-fibration over $\mathbb{C}P^1$, then $X$ has a special Lagrangian $T^4$-fibration over $\mathbb{C}P^2$.

§3. Large Complex Structure Limits and Monodromy.

We will now study some of the geometry of Calabi-Yau manifolds near large complex structure limit points of the moduli space. More precisely, let $\bar{S}$ be some suitable compactification of a $h^{1,n-1}$-dimensional parameter space $S$ of Calabi-Yau $n$-folds, $\bar{\mathcal{X}} \to \bar{S}$ a compactification of the family $\mathcal{X} \to S$ of Calabi-Yau manifolds. Assume $\bar{S} - S = B_1 \cup \cdots \cup B_{h^{1,n-1}}$ is a divisor with simple normal crossings. We recall from [17] the definition of a maximally unipotent point (= large complex structure limit point) of $\bar{S}$.

**Definition 3.1.** Let $p \in \bar{S}$ be a point which is contained in boundary divisors $B_1, \cdots, B_s$, $s = h^{1,n-1} = \dim \bar{S}$. We say $p$ is maximally unipotent, or a large complex structure limit point, if

(1) The monodromy transformations $T_j$ around $B_j$ near $p$ are all unipotent.

(2) Let $N_j = \log T_j$, $N = \sum a_j N_j$ for some $a_j > 0$, $0 \subseteq W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq H^n(X, \mathbb{Q})$ the induced weight filtration. Then $\dim W_0 = \dim W_1 = 1$ and $\dim W_2 = 1 + s$.

(3) Let $g_0, \ldots, g_r$ be a basis of $W_2$ such that $g_0$ spans $W_0$, and define $m_{jk}$ by $N_j g_k = m_{jk} g_0$ for $1 \leq j, k \leq s$. Then the matrix $M := (m_{jk})$ is invertible.

We note that in case $n = 3$, this implies that the Hodge diamond of the mixed Hodge
structure associated with this degeneration looks like
\[
\begin{array}{cccc}
1 & & & W_6/W_5 \\
0 & 0 & & W_5/W_4 \\
0 & h^{1,2} & 0 & W_4/W_3 \\
0 & 0 & 0 & W_3/W_2 \\
0 & h^{1,2} & 0 & W_2/W_1 \\
0 & 0 & & W_1/W_0 \\
1 & & & W_0
\end{array}
\]
and \(W_0\) is the only subspace of \(H^3(X, \mathbb{Q})\) invariant under all \(T_1, \ldots, T_s\).

In [17], Morrison proposes a general mirror symmetry conjecture, which, imprecisely, states that associated to such a large complex structure limit point, there should exist a mirror Calabi-Yau manifold \(\tilde{X}\) whose Kähler moduli space is locally isomorphic to \(S\), and such that the Gauss-Manin connection on \(S\) agrees with a connection induced by the Gromov-Witten invariants on the Kähler moduli space of \(\tilde{X}\). As we will not be discussing the counting of rational curves in this paper, we will ignore this latter aspect. However, we need to make some of this more precise.

Recall a framing of \(\tilde{X}\) is a cone \(\Sigma = \mathbb{R}_+ e_1 + \cdots + \mathbb{R}_+ e_r \subseteq H^2(\tilde{X}, \mathbb{R})\) generated by a basis \(e_1, \ldots, e_r\) of \(H^2(\tilde{X}, \mathbb{Z})\). \(\Sigma\) should be a subcone of the Kähler cone of \(\tilde{X}\). This gives a part of the complexified Kähler moduli space by
\[
\mathcal{M}_A(\Sigma) = (H^2(\tilde{X}, \mathbb{R}) + i\Sigma)/H^2(\tilde{X}, \mathbb{Z}).
\]

The map
\[
(q_1, \ldots, q_r) \in \mathbb{C}^r \mapsto \sum \left( \frac{1}{2\pi i} \log q_j \right) e_j \in \mathcal{M}_A(\Sigma)
\]
gives a description of \(\mathcal{M}_A(\Sigma)\) as
\[
\mathcal{M}_A(\Sigma) = \{(q_1, \ldots, q_r) \in \mathbb{C}^r | 0 < |q_i| < 1\}
\]
and thus has a natural compactification
\[
\overline{\mathcal{M}_A(\Sigma)} = \{(q_1, \ldots, q_r) \in \mathbb{C}^r | 0 \leq |q_i| < 1\}.
\]

The mirror conjecture then posits the existence of an isomorphism between an open neighborhood \(U\) of the large complex structure limit point \(p \in \bar{S}\) and \(\overline{\mathcal{M}_A(\Sigma)}\), for some framing \(\Sigma\), taking \(U \cap S\) to \(\mathcal{M}_A(\Sigma)\). (More generally, one might need a multi-valued map between these two spaces.) This isomorphism should identify the connections involved.
We also recall the nilpotent orbit theorem. Choose \( q \in S \) near \( p \), \( X = X_q \). Then \( H^n(X, \mathbb{C}) \) comes along with a limiting Hodge filtration
\[
H^n(X, \mathbb{C}) = F^n_{\text{lim}} \supseteq \cdots \supseteq F^0_{\text{lim}} \supseteq 0.
\]
Let \( \Omega_{\text{lim}} \in H^n(X, \mathbb{C}) \) span \( F^n_{\text{lim}} \), which is one-dimensional. The nilpotent orbit of \( \Omega_{\text{lim}} \) is, for local coordinates \( q_1, \ldots, q_s \) on \( \bar{S} \) and coordinates \( t_j = \frac{\log q_j}{2\pi i} \) on the universal cover of \( S \),
\[
\Omega_{\text{nil}}(t_1, \ldots, t_s) = \exp \left( \sum \frac{\log q_j}{2\pi i} N_j \right) \Omega_{\text{lim}} = \exp \left( \sum t_j N_j \right) \Omega_{\text{lim}}.
\]
The nilpotent orbit theorem states that \( \Omega_{\text{nil}} \) is asymptotic to the actual periods. More precisely, let \( \beta_0 \) be an integral generator of \( W_0 \), and normalize \( \Omega_{\text{lim}} \) so that \( [\Omega_{\text{lim}}] \cdot \beta_0 = 1 \). Denote by \( \Omega(t_1, \ldots, t_s) \) the cohomology class of the holomorphic \( n \)-form on \( X \), normalized so that \( \Omega(t) \cdot \beta_0 = 1 \), for \( t = (t_1, \ldots, t_s) \) on the universal cover of \( S \). Then \( \Omega_{\text{nil}}(t) \) and \( \Omega(t) \) are asymptotic as \( \text{Im} t \to \infty \). Thus in particular the Yukawa couplings, for \( \frac{\partial}{\partial t_{i_1}}, \ldots, \frac{\partial}{\partial t_{i_n}} \in H^1(T_X) \),
\[
\left\langle \frac{\partial}{\partial t_{i_1}}, \ldots, \frac{\partial}{\partial t_{i_n}} \right\rangle := \int_X \Omega(t) \wedge \frac{\partial}{\partial t_{i_1}} \cdots \frac{\partial}{\partial t_{i_n}} \Omega(t)
\]
can be approximated, for \( \text{Im} t \) large, by
\[
\left\langle \frac{\partial}{\partial t_{i_1}}, \ldots, \frac{\partial}{\partial t_{i_n}} \right\rangle_{\text{nil}} := \int_X \Omega_{\text{nil}}(t) \wedge \frac{\partial}{\partial t_{i_1}} \cdots \frac{\partial}{\partial t_{i_n}} \Omega_{\text{nil}}(t) = \int_X \Omega_{\text{nil}}(t) \wedge N_{i_1} \cdots N_{i_n} \Omega_{\text{lim}}.
\]
Now \( N_{i_1} \cdots N_{i_n} \Omega_{\text{lim}} \in W_0 \), from which we conclude that
\[
N_{i_1} \cdots N_{i_n} \alpha_0 = \left\langle \frac{\partial}{\partial t_{i_1}}, \ldots, \frac{\partial}{\partial t_{i_n}} \right\rangle_{\text{nil}} \beta_0,
\]
where \( \alpha_0 \) is any cohomology class such that \( \alpha_0 \cdot \beta_0 = 1 \). This is the well-known fact that the asymptotic behaviour of the Yukawa coupling is determined by the action of the monodromy.

What information does the Strominger-Yau-Zaslow conjecture give about this form of the mirror symmetry conjecture? We state here a refined subconjecture of the conjecture given in [11].

Conjecture 3.2. Let \( \mathcal{X} \to S \) be a family of simply connected Calabi-Yau threefolds, with compactification \( \tilde{\mathcal{X}} \to \tilde{S} \), and let \( p \in \tilde{S} \) be a large complex structure limit point. Then for some open neighbourhood \( U \subseteq \tilde{S} \) of \( p \) and a dense set \( V \subseteq U \cap S \), for \( q \in V \), \( X = X_q \).
and “general choice”* of Kähler form $\omega$ on $X$ corresponding to a Ricci flat metric, there is a $C^\infty$ special Lagrangian 3-torus fibration $f : X \to B$ with the following properties:

1. $B$ is homeomorphic to $S^3$ and $f$ has a topological section $\sigma_0$.†
2. $f$ is simple and has a simply connected permissible dual $\bar{f} : \bar{X} \to B$.
3. The Leray spectral sequence for $f : X \to B$ degenerates at the $E_2$ level and looks like

$$
\begin{array}{cccc}
\mathbb{Q}\sigma_0 & 0 & 0 & \mathbb{Q} \\
0 & E^{1,2}_2 & H^4(X, \mathbb{Q}) & 0 \\
0 & H^2(X, \mathbb{Q}) & E^{2,1}_2 & 0 \\
\mathbb{Q} & 0 & 0 & \mathbb{Q}[T^3]
\end{array}
$$

with $\dim E^{1,2}_2 = \dim E^{2,1}_2 = h^{1,2}(X)$, and the induced filtration on $H^3(X, \mathbb{Q})$ coincides with the weight filtration of the mixed Hodge structure associated with the large complex structure limit point $p$.

We will go into more detail on a number of these issues, first making two simple observations.

**Observation 3.3.** Given the hypotheses of the conjecture, then for some open neighborhood $U$ of $p \in \bar{S}$, and $q \in U \cap S$, $\mathcal{X}_q$ contains an open subset which is fibred in $n$-tori. Furthermore, these $n$-tori represent a cohomology class which spans $W_0$ over $\mathbb{Q}$.

**Proof.** Let $\Delta \to \bar{S}$ be a mapping of a 1-dimensional disk into $\bar{S}$ with $0 \in \Delta$ going to $p \in S$, and with $\Delta$ meeting $B_1, \ldots, B_s$ transversally at $p$. Then the family $\mathcal{X}_\Delta \to \Delta$ has a semistable reduction $\mathcal{Y} \to \Delta'$, with $\mathcal{Y}_0$ a variety with simple normal crossings. Let $\Gamma$ be the dual graph of $\mathcal{Y}_0$. $\Gamma$ is a simplicial complex with one vertex $P_i$ for each component $Y_i$ of $\mathcal{Y}_0$, such that the simplex $\langle P_{i_0}, \ldots, P_{i_k} \rangle$ belongs to $\Gamma$ if and only if $Y_{i_0} \cap \cdots \cap Y_{i_k} \neq \phi$. Then ([16], pg. 109) $W_0 \cong H^n(|\Gamma|, \mathbb{Q})$. Since $\dim W_0 = 1$ by hypothesis, we must have an $n$-simplex in $\Gamma$, i.e. $\mathcal{Y}_0$ has an $n+1$-tuple point $q$, locally isomorphic to $x_0 \cdots x_n = 0$, and the fibration $\mathcal{Y} \to \Delta'$ is given locally in a neighborhood of this point by $x_0 \cdots x_n = t$, $t$ the coordinate on $\Delta'$. We note that for a given $t$, $U_t = \{ (x_0, \ldots, x_n) \in \mathbb{C}^{n+1} | x_0 \cdots x_n = t \}$ has a $T^n$-fibration as follows. Let

$$
R_t = \{ (x_0, \ldots, x_n) \in \mathbb{R}_+^{n+1} | x_0 \cdots x_n = |t| \}.
$$

Then define a map $U_t \to R_t$ by $(x_0, \ldots, x_n) \mapsto (|x_0|, \ldots, |x_n|)$. The fibres are clearly $n$-tori. Thus, for $t$ sufficiently close to 0, one can find a subset $R'_t \subseteq R_t$ and $U'_t = T^n \times R'_t \subseteq U_t$.

---

* I don’t know what this means, but I suspect $\omega$ should not be too close to the walls of the Kähler cone.

† See Remark 3.5 on this latter point.
with $U'_t$ homeomorphic to a subset of $Y_t$. It follows from the construction of the Picard-\Lefschetz transformation in [4], §7 that the cohomology class of this $T^n$ is a generator of $W_0$. 

The difficult and most crucial part of the program to understand mirror symmetry from the point of view of the SYZ construction is to show that there exists a special Lagrangian $T^n$ in this cohomology class, and that it then gives rise to a fibration. We will not consider this problem at all in this paper, but rather assume that such exists.

**Observation 3.4.** Suppose $n = 3$. Let $\beta_0 \in H^3(X, \mathbb{Z})$ be an integral generator of $W_0$. Then there is a dense set $V \subseteq S$ (the complement of a countable number of closed real codimension 1 subsets) with the following property: for any $q \in V$, $\Omega$ a holomorphic 3-form on $X_q$ normalized so that $\Omega.\beta_0 = 1$, $\alpha \in H^3(X, \mathbb{Z})$, we have $\text{Im } \Omega.\alpha = 0$ if and only if $\alpha \in W_0$. In particular, if $X_q$ has a special Lagrangian torus fibration $f : X_q \to B$ with a section, $q \in V$, and the general fibre of $f$ has cohomology class in $W_0$, then all fibres of $f$ are irreducible.

**Proof.** For a given $\alpha \in H^3(X, \mathbb{Z})$, the set

$$V_\alpha = \{ \beta \in H^3(X, \mathbb{C}) | \text{Im } \beta.\alpha = 0 \}$$

is a real hyperplane in $H^3(X, \mathbb{C})$. Let $D \subseteq H^3(X, \mathbb{C})$ be the image of the (multi-valued) period map $S \to H^3(X, \mathbb{C})$ given by $q \mapsto \Omega_q \in H^3(X, \mathbb{C})$ with $\Omega_q$ normalized as above. Then $D$ is analytic, by analyticity of the period mapping, and so either $D \subseteq V_\alpha$ or $V_\alpha$ intersects $D$ in a real codimension one subset. Thus if we show $D \subseteq V_\alpha$ if and only if $\alpha \in W_0$, the result follows by removing the countable number of real hypersurfaces from $S$ corresponding to $V_\alpha \cap D$, for $\alpha \in H^3(X, \mathbb{Z})$.

Fix $\alpha \in H^3(X, \mathbb{Z})$. If $\alpha \in W_0$, clearly $D \subseteq V_\alpha$. Conversely, suppose there is a point $q \in S$ with $\Omega := \Omega_q \in V_\alpha$. Define

$$\alpha' = \alpha - (\text{Re } \Omega.\alpha)\beta_0 \in H^3(X, \mathbb{R}).$$

If $\alpha' = 0$, then $\alpha \in W_0$. So, assume $\alpha' \neq 0$. Since $\text{Im } \Omega.\alpha = 0$, we see $\Omega.\alpha' = 0$. In addition,

$$(H^{3,0}(X_q) \oplus H^{2,1}(X_q))^\perp = H^{3,0}(X_q) \oplus H^{2,1}(X_q),$$

which intersects $H^3(X, \mathbb{R})$ only in zero, so there must exist a $\gamma \in H^{2,1}(X_q)$ such that $\gamma.\alpha' \neq 0$. By multiplying $\gamma$ by a phase factor, we can insure that $\text{Im } \gamma.\alpha' \neq 0$. Applying the Bogomolov-Tian-Todorov theorem, we can construct a curve $q(t)$ in $S$, $t$ a real variable,
with \( q(t) = q \), and \( \Omega_{q(t)} = C(t)(\Omega + t\gamma + O(t^2)) \) where \( C(t) \) is the normalization factor required to ensure that \( \Omega_{q(t)}\beta_0 = 1 \). In fact,

\[
\Omega_{q(t)}\beta_0 = C(t)(1 + t(\gamma, \beta_0) + O(t^2)),
\]

so

\[
C(t) = 1 - t(\gamma, \beta_0) + O(t^2)
\]

and

\[
\Omega_{q(t)} = \Omega + t(\gamma - (\gamma, \beta_0)\Omega) + O(t^2).
\]

Let

\[
\alpha'(t) = \alpha - (\text{Re} \, \Omega_{q(t)} \cdot \alpha)\beta_0.
\]

If \( \text{Im} \, \Omega_{q(t)} \cdot \alpha = 0 \), then \( \Omega_{q(t)} \cdot \alpha'(t) = 0 \). But calculating, we see

\[
\Omega_{q(t)} \cdot \alpha'(t) = [\Omega + t(\gamma - (\gamma, \beta_0)\Omega) + O(t^2)].[\alpha' - t(\text{Re} \, (\gamma - (\gamma, \beta_0)\Omega) \cdot \alpha)\beta_0 + O(t^2)]
\]

\[
= t(- \text{Re} \, (\gamma \cdot \alpha) + (\text{Re} \, (\gamma, \beta_0)) (\Omega \cdot \alpha) + \gamma \cdot \alpha') + O(t^2)
\]

\[
= t(\text{Im} \, \gamma \cdot \alpha') + O(t^2)
\]

which is non-zero for small \( t, t \neq 0 \). We conclude \( \Omega_{q(t)} \not\in V_\alpha \) for these \( t \). Thus \( D \subseteq V_\alpha \) if and only if \( \alpha \in W_0 \).

Finally, suppose \( q \in V \) and \( f : X_q \rightarrow B \) is a special Lagrangian fibration with a section, with fibre \([T^3] \in W_0 \). Since \( f \) has a section \( \sigma_0 \), \( \sigma_0 \cdot [T^3] = \pm 1 \), so \([T^3] \) must be primitive. Thus if a fibre of \( f \) is reducible, i.e. can be written as the sum of two special Lagrangian currents representing cohomology classes \( \alpha_1 \) and \( \alpha_2 \), then \([T^3] = \alpha_1 + \alpha_2 \) and \( \Omega \cdot \alpha_1, \Omega \cdot \alpha_2 > 0 \) implies \( \alpha_1, \alpha_2 \not\in W_0 \). But since \( q \in V \), we then have \( \text{Im} \, \Omega \cdot \alpha_1, \text{Im} \, \Omega \cdot \alpha_2 \neq 0 \), a contradiction.

\textbf{Remark 3.5.} We do not expect that every special Lagrangian \( T^n \)-fibration has a section. In fact, in the K3 case, this only occurs when the mirror of a family of \( M \)-polarized K3 surfaces is constructed using a \( U(1) \subseteq M^\perp \). (Recall \( U(n) \) is the lattice with Cartan matrix \( \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} \)). More generally, one can construct a mirror family given a primitively embedded \( U(n) \subseteq M^\perp \) in which case the special Lagrangian fibration arising only has a topological \( n \)-section. In this case, the mirror of the mirror is not the original K3 surfaces. See [6], [11] for more details. One would expect similar phenomena to arise in higher dimensions. We may not have noticed such phenomena yet since most examples studied are toric, where the mirror of the mirror brings us back to the original family.
Nevertheless, we will continue with the assumption of the existence of a section, as that is the most important case given known examples. If one wanted to extend the discussion below to a fibration \( f : X \to B \) without a section, one would want to consider it as a torsor over the double dual \( \tilde{f} : \tilde{X} \to B \).

We next consider \( C^\infty \) sections of the fibration \( f : X \to B \). Since no such section of \( f \) can intersect the critical locus of \( f \), let \( X^\# \) be the complement of the critical locus of \( f \). We have

**Theorem 3.6.** Let \( X \) be a Calabi-Yau \( n \)-fold, \( B \) a smooth real \( n \)-dimensional manifold, with \( f : X \to B \) a \( C^\infty \) special Lagrangian torus fibration such that \( R^n f_* \mathbb{Q} = \mathbb{Q}_B \). Suppose furthermore that \( f \) has a \( C^\infty \) section \( \sigma_0 \). Then \( X^\# \) has the structure of a fibre space of groups with \( \sigma_0 \) the zero section. In fact there is an exact sequence of sheaves of abelian groups

\[
0 \to R^{n-1} f_* \mathbb{Z} \to T_B^* \to X^\# \to 0.
\]

Given a section \( \sigma \in \Gamma(U, X^\#) \), one obtains a \( C^\infty \) diffeomorphism \( T_\sigma : f^{-1}(U) \cap X^\# \to f^{-1}(U) \cap X^\# \) given by \( x \mapsto x + \sigma(f(x)) \), and this diffeomorphism extends to a diffeomorphism \( T_\sigma : f^{-1}(U) \to f^{-1}(U) \).

The proof of this theorem will be given in [10]. It is just a variant of action-angle coordinates.

Thus we have an exact sequence

\[
0 \to H^0(B, R^{n-1} f_* \mathbb{Z}) \to H^0(B, T_B^*) \to H^0(B, X^#) \xrightarrow{\delta} H^1(B, R^{n-1} f_* \mathbb{Z}) \to 0.
\]

Exactness on the right follows since \( T_B^* \) is a fine sheaf. Any section of \( f \) coming from \( H^0(B, T_B^*) \) is homotopic to zero; hence \( H^1(B, R^{n-1} f_* \mathbb{Z}) \) classifies sections of \( f \) modulo \( f \) homotopic to zero.

Let \( \sigma : B \to X \) be a section of \( f \). We denote by \( T_\sigma : X \to X \) translation by \( \sigma \) as in Theorem 3.6, and denote by \( [\sigma] \in H^1(B, R^{n-1} f_* \mathbb{Z}) \) the image of the section \( \sigma \) under the map \( \delta \). Explicitly, if \( \{U_i\} \) is an open covering of \( B \) such that \( \sigma : B \to X \) is represented by \( \sigma_i \in \Gamma(U_i, T_B^*) \), then \( [\sigma] \) is represented by the \( \check{\text{Čech}} \) cocycle \((U_{ij}, \sigma_j - \sigma_i)\), with \( \sigma_j - \sigma_i \in \Gamma(U_{ij}, R^{n-1} f_* \mathbb{Z}) \). \( T_\sigma \) induces a map \( T_\sigma^* \) on the cohomology of \( X \) which only depends on \( [\sigma] \in H^1(B, R^{n-1} f_* \mathbb{Z}) \).

The key conjecture of this paper connects this group structure on \( X \) to the monodromy action around the boundary divisors passing through a large complex structure limit point.
Let $\mathcal{X} \to S$ be a family of Calabi-Yau manifolds as earlier, $t \in S$ with $\mathcal{X}_t = X$, and $\tilde{S}$ a compactification of $S$ with $p \in \tilde{S}$ a large complex structure limit point. As above, write $\tilde{S} - S = B_1 \cup \cdots \cup B_s$, a divisor with simple normal crossings, $s = h^{1,n-1}$. Given a loop around $B_i$, based at $t \in U$, one obtains a diffeomorphism $T_i : X \to X$ which is determined up to isotopy, and $T_1, \ldots, T_s$ induce the monodromy transformations on cohomology. On the other hand, given a section $\sigma : B \to X$, one obtains a diffeomorphism $T_{\sigma} : X \to X$ by translation. We can ask what the relationship between these two sets of diffeomorphisms are.

Conjecture 3.7. There exist sections $\sigma_1, \ldots, \sigma_s$ of $f : X \to B$ such that $T_i$ and $T_{\sigma_i}$ are isotopic. Furthermore, $[\sigma_1], \ldots, [\sigma_s]$ form a basis for $H^1(B, R^{n-1}f_*\mathbb{Q})$.

We can also add that in this case, the framing of $\check{X}$ should be given by the cone generated by the images of $[\sigma_1], \ldots, [\sigma_s]$ in $H^1(B, R^1\check{f}_*\mathbb{Q})$ under the isomorphism (2.1).

I cannot prove this conjecture yet, not even knowing the existence or precise properties of these torus fibrations, but it is a reasonable conjecture, and I will spend some time showing how natural it is. The initial motivation for this conjecture is that the fibre class $[T^n]$ is the unique cycle invariant under all monodromy transformations $T_1, \ldots, T_s$, so one might hope one can find that the monodromy diffeomorphisms act on the fibres of the special Lagrangian fibration. In addition, this conjecture fits very much with the spirit of the construction of the Picard-Lefschetz transformation in [4, §7]. Finally, as the group of sections mod homotopy is $H^1(B, R^{n-1}f_*\mathbb{Z})$, if $f$ and $\check{f}$ are $\mathbb{Z}$-simple this is isomorphic to $H^1(B, R^1\check{f}_*\mathbb{Z})$. Under the best circumstances (e.g. $n = 3$ and $X, \check{X}$ are simply connected), this is isomorphic to $H^2(\check{X}, \mathbb{Z})$. Now $H^2(\check{X}, \mathbb{Z})$ is the monodromy group of the Kähler moduli space for $\check{X}$; thus our conjecture gives a very natural way of connecting this group, the integral shifts in the $B$-field, with the complex structure monodromy group of $X$.

Example 3.8. Let $\mathcal{E} \to \Delta$ be a degenerating family of elliptic curves with $\mathcal{E}_0$ a fibre of type $I_1$, $E = \mathcal{E}_t$ a smooth elliptic curve. As is well-known, the monodromy diffeomorphism induced by a loop around $0 \in \Delta$ is described as a Dehn twist. Now we have a natural $T^1$-fibration $f : E \to S^1$, and the group of sections mod homotopy is isomorphic to $H^1(S^1, f_*\mathbb{Z}) = \mathbb{Z}$. The Dehn twist is the same as translating by a section generating this group. Thus the above proposed form for the monodromy should be thought of as a generalization of the Dehn twist.

Example 3.9. Let us consider Conjecture 3.7 for K3 surfaces. If we are interested only in continuous ($C^0$) isotopy as opposed to smooth isotopy, then [7], Theorem 10.1 tells us that a homeomorphism of a K3 surface is determined up to isotopy by its action
on cohomology. Thus to check Conjecture 3.7, it is enough to determine that the relevant monodromy transformations on \( H^2(X, \mathbb{Z}) \) coincide with the action of translation by sections.

More precisely, let \( L \) denote the K3 lattice, \( M \subseteq L \) a primitive sublattice of signature \((1, t)\), and suppose \( T = M^\perp \) decomposes as \( U(1) \oplus \tilde{M} \). Mirror symmetry exchanges the complex and Kähler moduli of \( M \)-polarized K3 surfaces: see [1], [6]. If \( E, E' \) are generators of the \( U(1) \), we showed in [11] that an \( M \)-polarized K3 surface \( X \) has a special Lagrangian \( T^2 \)-fibration \( f : X \to S^2 \) with fibre of cohomology class \( E \). This \( T^2 \) fibration is just an elliptic fibration in a different complex structure on \( X \).

It follows from [6], Proposition 6.2 that the monodromy group at the large complex structure limit point of the complex moduli space of \( M \)-polarized K3 surfaces is isomorphic to \( \tilde{M} \), acting, for \( D \in \tilde{M} \), by

\[
T_D(E) = E \\
T_D(\alpha) = \alpha - (D.\alpha)E \quad \text{for } \alpha \in U(1)^\perp \\
T_D(E') = E' + D - \frac{D^2}{2}E.
\]

Let \( \sigma_0 : S^2 \to X \) be the chosen zero section. We refer to the cohomology class of the image of \( \sigma_0 \) also as \( \sigma_0 \), so with the correct orientation, \( \sigma_0 = E' - E \). There is a natural map from the group of sections

\[
\Gamma(B, X^\#) \to H^1(B, R^1 f_* \mathbb{Z})
\]

given by taking \( \sigma \in \Gamma(B, X^\#) \) to the cohomology class \( \sigma - \sigma_0 \). (Note: a priori, this is not the map \( \delta \) defined above, though Theorem 4.1 proves that this map coincides with \( \delta \) when the dimension of \( X \) is even.)

We also have a natural choice of splitting for the Leray filtration of \( H^2(X, \mathbb{Z}) \), i.e. by writing

\[
L = \mathbb{Z}E' \oplus U(1)^\perp \oplus \mathbb{Z}E.
\]

Note for future reference that \( E' = \sigma_0 + \frac{c_2(X)}{24} E \). This gives us an isomorphism between \( H^1(B, R^1 f_* \mathbb{Z}) \) and \( U(1)^\perp \), and hence this latter group is also isomorphic to the group of sections of \( f \) modulo homotopy. Explicitly, a section \( \sigma \) is taken to \( \sigma - \sigma_0 - (E', (\sigma - \sigma_0))E \in U(1)^\perp \). Conversely, if \( D \in U(1)^\perp \), we denote the cohomology class of the corresponding section by \( \sigma_D \). In particular, if we write \( \oplus \) for the group law in sections mod homotopy, \( \sigma_{D_1} \oplus \sigma_{D_2} = \sigma_{D_1 + D_2} \). Note that since \( \sigma_D^2 = -2 \) for any section \( \sigma_D \), we in fact have

\[
\sigma_D = E' + D - \left( \frac{D^2}{2} + 1 \right) E.
\]
Now let $T_{\sigma_D} : X \to X$ denote translation by $\sigma_D$. Clearly

$$T^*_D(E) = E.$$ 

Also, as $T^*_D(\sigma_0) = \sigma_D$, we see that

$$T^*_D(E' - E) = E' + D - \left(\frac{D^2}{2} + 1\right) E$$

and thus

$$T^*_D(E') = E' + D - \frac{D^2}{2} E.$$ 

Finally, for $\alpha \in U(1)^\perp$,

$$T^*_D(\alpha) = T^*_D(\sigma_\alpha - E' + (1 + \alpha^2/2)E)$$

$$= \sigma_{\alpha + D} - (E' + D - \frac{D^2}{2} E) + \left(\frac{\alpha^2}{2} + 1\right) E$$

$$= \alpha - (\alpha D) E.$$ 

Thus we see that $T^*_D$ and $T_D$ coincide, for any $D \in \tilde{M}$ (in fact for any $D \in U(1)^\perp$).

§5 is partially motivated by the desire to find a similar splitting for the Leray filtration in the three-dimensional case which will then allow one to write precisely the action of monodromy on $H^3$.

§4. The Monodromy Action on the Leray Filtration.

In this section we assume $f : X \to B$ is a simple $C^\infty$ special Lagrangian torus fibration with section $\sigma_0$, with permissible dual $\tilde{f} : \tilde{X} \to B$. Suppose $\sigma : B \to X$ is an arbitrary $C^\infty$ section. We would like, at least on the level of the Leray filtration, to describe the action of $T^*_\sigma - I : H^i(X, \mathbb{Q}) \to H^i(X, \mathbb{Q})$.

To do so, first observe there are natural pairings given by the composition of cup products

$$(4.1) \quad H^p(B, R^q f_* \mathbb{Q}) \otimes H^{p'}(B, R^{q'} f_* \mathbb{Q}) \to H^{p+p'}(B, R^q f_* \mathbb{Q} \otimes R^{q'} f_* \mathbb{Q})$$

which are compatible with the intersection pairing of the total cohomology

$$H^p(X, \mathbb{Q}) \otimes H^{p'}(X, \mathbb{Q}) \to H^{p+p'}(X, \mathbb{Q}).$$

(See [3], IV 6.8.)
We obtain from this pairing, using the isomorphism (2.1), an action of cohomology of \( X \) on the cohomology of \( X \):

\[
H^p(B, R^q f_* Q) \otimes H^{p'}(B, R^q f'_* Q) \cong H^p(B, R^{n-q} \hat{f}_* Q) \otimes H^{p'}(B, R^q \hat{f}'_* R)
\]

(4.2)

We will denote this pairing by \( \langle \cdot, \cdot \rangle \).

Given a section \( \sigma : B \to X \), we obtain as in §3 a cohomology class \([\sigma] \in H^1(B, R^{n-1} f_* Z)\), hence also in \( H^1(B, R^{n-1} f_* Q)\). Via (2.1), this gives an element \( D \in H^1(B, R^1 \hat{f}_* Q)\). If \( f \) is \( Z\)-simple, then by (2.2) one obtains in addition \( D \in H^1(B, R^1 \hat{f}_* Z)\).

Our main theorem relates the action of \( T_\sigma \) on cohomology with the action \( \langle \cdot, D \rangle \) of \( D \) on the cohomology of \( X \).

**Theorem 4.1.** Let \( H^p(X, Q) = F_p \supseteq F_{p-1} \supseteq \cdots \supseteq F_0 \supseteq 0 \) be the Leray filtration with \( F_i/F_{i-1} \cong E_{\infty}^{p-i, i} \). Let \( \sigma : B \to X \) be a section, \([\sigma] \in H^1(B, R^{n-1} f_* Z)\) the corresponding cohomology class, and \( D \in H^1(B, R^1 \hat{f}_* Q)\) the image of \([\sigma]\) under the isomorphism (2.1). Then \((T^*_\sigma - I)(F_i) \subseteq F_{i-1}\) and the induced map

\[
T^*_\sigma - I : F_i/F_{i-1} \to F_{i-1}/F_{i-2}
\]

is induced by

\[
\langle \cdot, (-1)^i D \rangle : H^{p-i}(B, R^i f_* Q) \to H^{p-i+1}(B, R^{i-1} f_* Q).
\]

In what follows, we will be making use of singular cohomology. We denote by \( \Delta_q \) the standard simplex: the convex hull of \( P_0 = (0, \ldots, 0), P_1 = (1, 0, \ldots, 0), \ldots, P_q = (0, \ldots, 0, 1) \) in \( R^q \) with coordinates \( s_1, \ldots, s_q \) and standard orientation \( \partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_q \). Orient \( \Delta_q \times [0, 1] \) by \( \partial/\partial s_1 \wedge \cdots \wedge \partial/\partial s_q \wedge \partial/\partial t \). We choose a standard triangulation of \( \Delta_q \times [0, 1] \). For example, if we let \( P_0, \ldots, P_q \) be the vertices of \( \Delta_q \times \{0\} \) and \( P'_0, \ldots, P'_q \) be the vertices of \( \Delta_q \times \{1\} \), then

\[
\Delta_q \times [0, 1] = \bigcup_{i=0}^q \langle P_0, \ldots, P_i, P'_i, \ldots, P'_q \rangle.
\]

This extends to a triangulation of \( K \times [0, 1] \) for any chain complex \( K \), and one can check one obtains the formula

\[
\partial(\Delta_q \times [0, 1]) = (\partial \Delta_q) \times [0, 1] + (-1)^q \Delta_q \times \{1\} - (-1)^q \Delta_q \times \{0\}
\]

as chain complexes.

Before getting into the details of the proof, let us give a lemma which lies at the heart of the proof and which gives some hint as to why the formula given in Theorem 4.1 is correct.
Lemma 4.2. Let $T^n = V/\Lambda$ for a real $n$-dimensional vector space $V$, a lattice $\Lambda$, and a fixed isomorphism $\bigwedge^n V \cong \mathbb{R}$. Fix a base-point $p_0 \in T^n$. Define a pairing

$$\langle \cdot, \cdot \rangle'': H^q_{\text{sing}}(T^n, \mathbb{R}) \times \pi_1(T^n, p_0) \to H^{q-1}_{\text{sing}}(T^n, \mathbb{R})$$

as follows: if $\alpha$ is a singular cocycle on $T^n$ and $\gamma : [0,1] \to T^n$ is a loop, then define $\langle \alpha, \gamma \rangle''$ to be the following $q-1$-cocycle: if $\Delta : \Delta_{q-1} \to T^n$ is a singular simplex, then $\langle \alpha, \gamma \rangle''(\Delta) = \alpha(\Delta \gamma)$. Here $\Delta \gamma$ is the singular $q$-chain $\Delta \gamma : \Delta_{q-1} \times [0,1] \to T^n$ defined by $\Delta \gamma(s,t) = \Delta(s) + \gamma(t)$. Then if we identify $H^q(T^n, \mathbb{R})$ with $\bigwedge^q V^\vee$ and $\pi_1(T^n, p_0)$ with $\Lambda$, there is a commutative diagram

$$\begin{array}{ccc}
H^q_{\text{sing}}(T^n, \mathbb{R}) \times \pi_1(T^n, p_0) & \langle \cdot, \cdot \rangle'' & H^{q-1}_{\text{sing}}(T^n, \mathbb{R}) \\
\downarrow \cong & \downarrow \cong & \\
\bigwedge^q V^\vee \times \Lambda & \mapsto & \bigwedge^{q-1} V^\vee \\
\downarrow \cong & \downarrow \cong & \\
\bigwedge^{n-q} V \times \Lambda & (-1)^{n-q} \wedge & \bigwedge^{n-q+1} V \\
\end{array}$$

The middle row is given by $(v^\ast, \lambda) \mapsto \iota(\lambda)v^\ast$. Here contraction is defined via the convention $(\iota(\lambda)v^\ast)(v_1, \ldots, v_{q-1}) = v^\ast(v_1, \ldots, v_{q-1}, \lambda)$. The isomorphism connecting the middle and bottom rows are given via the perfect pairings

$$\bigwedge^q V \times \bigwedge^{n-q} V \to \bigwedge^n V \cong \mathbb{R}$$

and

$$\bigwedge^{q-1} V \times \bigwedge^{n-q+1} V \to \bigwedge^n V \cong \mathbb{R}.$$

Proof. One checks that the cohomology class of $\langle \alpha, \gamma \rangle''$ only depends on the cohomology class of $\alpha$ and the homotopy class of $\gamma$, so that $\langle \cdot, \cdot \rangle''$ is well-defined.

One natural way to define the isomorphism $\bigwedge^q V^\vee \cong H^q(T^n, \mathbb{R})$ is by considering $\alpha \in \bigwedge^q V^\vee$ as defining a $q$-form on $V$, hence a translation invariant $q$-form on $T^n$, and then defining the singular cocycle $\alpha$ by $\Delta \mapsto \int_{\Delta} \alpha$. Thus to understand $\langle \cdot, \cdot \rangle$, choose $\alpha \in \bigwedge^q V^\vee$, $\lambda \in \Lambda$, and take $\gamma(t) = t\lambda + p_0$. If $\Delta : \Delta_{q-1} \to T^n$ is a singular $q-1$ simplex, then $\Delta_{\gamma} : \Delta_{q-1} \times [0,1] \to T^n$ is given by $\Delta_{\gamma}(s,t) = \Delta(s) + \gamma(t)$. Then

$$\Delta_{\gamma\ast}(\partial/\partial s_i) = \Delta_\ast(\partial/\partial s_i), \Delta_{\gamma\ast}(\partial/\partial t) = \lambda.$$
Thus, if $L_q$ denotes the $q$-dimensional Lebesgue measure on $\mathbb{R}^q$,

$$
\langle \alpha, \gamma \rangle''(\Delta) = \alpha(\Delta_{\gamma})
$$

$$
= \int_{\Delta_{\gamma}} \alpha
$$

$$
= \int_{\Delta_{q-1} \times [0,1]} (\Delta_{\gamma}^* \alpha)(\partial/\partial s_1, \ldots, \partial/\partial s_{q-1}, \partial/\partial t) dL^q
$$

$$
= \int_{\Delta_{q-1} \times [0,1]} \alpha(\Delta_{\gamma s} \partial/\partial s_1, \ldots, \Delta_{\gamma s} \partial/\partial s_{q-1}, \Delta_{\gamma s} \partial/\partial t) dL^q
$$

$$
= \int_{\Delta_{q-1}} \int_0^1 \alpha(\Delta_{s} \partial/\partial s_1, \ldots, \Delta_{s} \partial/\partial s_{q-1}, \lambda) dt dL^{q-1}
$$

$$
= \int_{\Delta_{q-1}} \Delta^*(\iota(\lambda) \alpha)(\partial/\partial s_1, \ldots, \partial/\partial s_{q-1}) dL^{q-1}
$$

$$
= \int_{\Delta} \iota(\lambda) \alpha.
$$

Thus $\langle \alpha, \gamma \rangle'' = \iota(\gamma) \alpha$.

The commutativity of the bottom square is standard multilinear algebra. •

**Proof of Theorem 4.1.** To see that $(T_{\sigma}^* - I)(F_i) \subseteq F_{i-1}$, note that since $T_{\sigma}$ acts on the fibres of $f$ by translation, $T_{\sigma}$ induces the identity $T_{\sigma}^* : R^i f_* Q \to R^i f_* Q$ and hence $T_{\sigma}^* : H^{p-i}(B, R^i f_* Q) \to H^{p-i}(B, R^i f_* Q)$ is the identity. Thus in particular, the map $T_{\sigma}^* - I : F_i/F_{i-1} \to F_i/F_{i-1}$ is zero.

To prove the second statement, we have to be more explicit about the Leray spectral sequence. First, we will use singular cohomology on $X$. Let $G$ be the coefficient group $(G = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \ldots)$. We let $S^p(U, G)$ be the space of $G$-valued $p$-cochains on $U$, and let $S^p(X, G)$ be the sheaf associated to the presheaf $U \mapsto S^p(U, G)$. Note that the presheaf $S^p(\cdot, G)$ satisfies the sheaf gluing axiom. Then, as $X$ is a manifold,

$$
0 \to G \to S^*(X, G)
$$

is a flabby resolution of $G$, with coboundary maps $d : S^p(X, G) \to S^{p+1}(X, G)$ the usual singular coboundary map: $(d\alpha)(\Delta) = \alpha(\partial \Delta)$ for a singular simplex $\Delta : \Delta_{p+1} \to X$. (See [3], pg. 26 for details about these sheaves.)

Next, we will use Čech cohomology to compute cohomology of sheaves on $B$. Life is simplest if there is an open covering $U = \{ U_i \}$ of $B$ for which $H^i(U/B, R^j f_* Q) = H^i(B, R^j f_* Q)$, for all $i$ and $j$, in which case we denote by $C^*(U/B, F)$ the Čech complex of $F$ with differential $d'$. If we cannot find such an open covering, instead consider open
covariances $\mathcal{U}$ of $B$ indexed by the set $B$, with $b \in U_b$, and take

$$C^p(B, \mathcal{F}) = \lim_{\mathcal{U}} C^p(\mathcal{U}/B, \mathcal{F})$$

where the limit is over all such coverings (see [3], pg. 28). Then the complex $C^*(B, \mathcal{F})$ computes $\hat{H}^p(B, \mathcal{F})$. For simplicity of notation, however, we will assume we are in the case where we do have a nice open covering, but everything carries over to this more general case.

We now obtain a double complex

$$C^{p,q} = C^p(\mathcal{U}/B, f_*S^q(X, \mathbb{Q}))$$

(or $C^{p,q} = C^p(B, f_*S^q(X, \mathbb{Q}))$ in the more general case) with horizontal boundary maps $d' : C^{p,q} \to C^{p+1,q}$ and vertical boundary maps $(-1)^pd : C^{p,q} \to C^{p,q+1}$. (The $(-1)^p$ is required to ensure anticommutativity of the vertical and horizontal boundary maps.) The spectral sequence arising from this double complex is the Leray spectral sequence for $f$. Let $(\text{Tot}^\cdot(C^\cdot), d_{tot})$ be the total complex of this double complex.

Now let $\alpha \in F_i(\text{Tot}^p(C^\cdot)) = \bigoplus_{j=0}^i C^{i-j,j}$, $\alpha = (\alpha_{p-i,i}, \ldots, \alpha_{p,0})$, $\alpha_{j,k} \in C^{j,k}$, and suppose $\alpha$ represents a class in $F_i(H^p(X, \mathbb{Q}))$, so $d_{tot}\alpha = 0$. In particular, $d\alpha_{p-i,i} = 0$. We wish to understand the action of $T^*_\sigma - I$ on $\alpha$. Write the Čech cochain $\alpha_{p-j,j}$ as

$$\alpha_{p-j,j} = (U_{i_0\ldots i_{p-j}}, \alpha_{i_0\ldots i_{p-j}}),$$

for $j \leq i$, $\alpha_{i_0\ldots i_{p-j}}$ a singular $j$-cochain on $f^{-1}(U_{i_0\ldots i_{p-j}})$. Choose also, on each $U_i$, a representative $\sigma_i \in \Gamma(U_i, T^*_B)$ of the section $\sigma \in \Gamma(B, X^\#)$. The class $(T^*_\sigma - I)(\alpha)$ is represented by $\alpha' = (\alpha'_{p-j,j})_{0 \leq j \leq i}$ with

$$\alpha'_{p-j,j} = (U_{i_0\ldots i_{p-j}}, T^*_\sigma \alpha_{i_0\ldots i_{p-j}} - \alpha_{i_0\ldots i_{p-j}})$$

where $T^*_\sigma$ denotes translation of cochains: $(T^*_\sigma \beta)(\Delta) = \beta(T_{\sigma} \circ \Delta)$.

For an open set $U \subseteq U_k$ and a $q$-cochain $\beta$ on $f^{-1}(U)$, denote by $\beta_{T^*_\sigma}$ the $q-1$-cochain given, for a singular $q-1$-simplex $\Delta : \Delta_{q-1} \to f^{-1}(U)$, by $\beta_{T^*_\sigma}(\Delta) = \beta(\Delta_{T^*_\sigma})$, where $\Delta_{T^*_\sigma} : \Delta_{q-1} \times [0,1] \to f^{-1}(U)$ is defined by $\Delta_{T^*_\sigma}(s,t) = \Delta(s) + t\sigma(f(\Delta(s)))$. It follows from (4.3) that

$$d(\beta_{T^*_\sigma}) = (d\beta)_{T^*_\sigma} - (-1)^q T^*_\sigma \beta + (-1)^q \beta.$$

Thus, if $\delta = (U_{i_0\ldots i_{p-i}}, (-1)^p(\alpha_{i_0\ldots i_{p-i}})_{T^*_\sigma}) \in C^{p-i,i-1}$, then $(-1)^{p-i}d\delta + \alpha'_{p-i,i} = 0$. It follows that $\alpha'' := \alpha' + d_{tot}\delta = \alpha' - \alpha'_{p-i,i} + d'\delta \in F_{i-1}(\text{Tot}^p(C^\cdot))$ is cohomologous to $\alpha' = (T^*_\sigma - I)\alpha$ in $H^p(X, \mathbb{R})$. 

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Recall also that $d_{\text{tot}}\alpha = 0$, so that $d'\alpha_{p-i,i} + (-1)^{p-i+1}d\alpha_{p-i+1,i-1} = 0$, i.e. on $U_{i_0 \ldots i_{p-i+1}}$, 

$$
\sum_{j=0}^{p-i+1} (-1)^j \alpha_{i_0 \ldots i_j \ldots i_{p-i+1}} + (-1)^{p-i+1}d\alpha_{i_0 \ldots i_{p-i+1}} = 0.
$$

Applying $T_{\sigma_{p-i+1}}$ to this and multiplying by $(-1)^p$, we obtain the equality 

$$
\sum_{j=0}^{p-i} (-1)^{p+j}(\alpha_{i_0 \ldots i_j \ldots i_{p-i+1}})T_{\sigma_{i_{p-i+1}}} - (-1)^i(\alpha_{i_0 \ldots i_{p-i}})T_{\sigma_{i_{p-i+1}}} - (-1)^i(d\alpha_{i_0 \ldots i_{p-i+1}})T_{\sigma_{i_{p-i+1}}} = 0.
$$

We thus obtain 

$$
\alpha''_{p-1,i,i-1} = \alpha'_{p-i+1,i-1} + d'\delta
$$

$$
= (U_{i_0 \ldots i_{p-i+1}}, (T_{\sigma}^* - I)\alpha_{i_0 \ldots i_{p-i+1}}
+ \sum_{j=0}^{p-i} (-1)^{p+j}(\alpha_{i_0 \ldots i_j \ldots i_{p-i+1}})T_{\sigma_{i_{p-i+1}}} - (-1)^i(\alpha_{i_0 \ldots i_{p-i}})T_{\sigma_{i_{p-i}}}
+ (-1)^i(\alpha_{i_0 \ldots i_{p-i}})T_{\sigma_{i_{p-i+1}}} + (-1)^i(d\alpha_{i_0 \ldots i_{p-i+1}})T_{\sigma_{i_{p-i+1}}}).
$$

Taking $\delta' = (U_{i_0 \ldots i_{p-i+1}}, (-1)^p(\alpha_{i_0 \ldots i_{p-i+1}})T_{\sigma_{i_{p-i+1}}}) \in C^{p-i-1,i-2}$, we see that if we set 

$$
\beta = \alpha'' + d_{\text{tot}}\delta',
$$

then 

$$
\beta_{p-i+1,i-1} = (U_{i_0 \ldots i_{p-i+1}}, (-1)^i(\alpha_{i_0 \ldots i_{p-i}})T_{\sigma_{i_{p-i+1}}} - (-1)^i(\alpha_{i_0 \ldots i_{p-i}})T_{\sigma_{i_{p-i}}})
$$

and $\beta$ still represents $(T_{\sigma}^* - I)\alpha$, and $\beta_{p-i+1,i-1}$ represents a class in $H^{p-i+1}(B, R^{i-1}f_*Q)$.  

Recall now how to define the map 

$$
\langle \cdot, D \rangle : H^p(B, R^q f_*Q) \to H^{p+1}(B, R^{q-1}f_*Q)
$$

in terms of Čech cohomology ([3], Pg. 194). First, define the pairing $\langle \cdot, \cdot \rangle'$ as the composition 

$$
\langle \cdot, \cdot \rangle' : R^q f_*Q \otimes R^1 \tilde{f}_*Q \xrightarrow{\cong} R^{q+1} \tilde{f}_*Q \otimes R^1 \tilde{f}_*Q \xrightarrow{\cup} R^{q-1} \tilde{f}_*Q \xrightarrow{\cong} R^q f_*Q.
$$
Now the element $[\sigma] \in H^1(B, R^{n-1} f_* \mathbb{Z})$ is represented by the Čech cocycle $(U_{i_0i_1}, \sigma_{i_1} - \sigma_{i_0})$. We denote by $D_{i_0i_1}$ the image of $\sigma_{i_1} - \sigma_{i_0}$ in $H^0(U_{i_0i_1}, R^1 f_* \mathbb{Q})$ under the isomorphism (2.1), so $D$ is represented by the Čech cocycle $(U_{i_0i_1}, D_{i_0i_1})$.

Given $\alpha = (U_{i_0...i_p}, \alpha_{i_0...i_p})$ a representative for an element of $H^p(B, R^q f_* \mathbb{Q})$, $(\alpha, D)$ is represented by

$$ (U_{i_0...i_{p+1}}, (-1)^n-q \langle \alpha_{i_0...i_p}, D_{i_p,i_{p+1}} \rangle '). $$

The sign $(-1)^{n-q}$ comes from the sign in (4.1). The theorem will then follow from the following claim:

**Claim.** If $U \subseteq U_i \cap U_j$ is an open set and $\alpha$ is a $q$-cocycle in $f^{-1}(U)$ representing an element of $\Gamma(U, R^q f_* \mathbb{Q})$, then $(-1)^{n-q} \langle \alpha, D_{ij} \rangle'$ is represented by the $q-1$ cocycle $\alpha T_{\sigma_j} - \alpha T_{\sigma_i}$.

**Proof.** We have two maps $\delta_1, \delta_2 : \Gamma(U, R^q f_* \mathbb{Q}) \to \Gamma(U, R^{q-1} f_* \mathbb{Q})$ defined by $\delta_1(\alpha) = (-1)^{n-q} \langle \alpha, D_{ij} \rangle'$ and $\delta_2(\alpha) = \alpha T_{\sigma_j} - \alpha T_{\sigma_i}$. Both maps are compatible with restriction to open subsets $V \subseteq U$, and by the assumption of simplicity, are then completely determined by their restrictions to $\Gamma(B_0 \cap U, R^q f_* \mathbb{Q})$, which in turn is determined by the restrictions to $\Gamma(V_i, R^q f_* \mathbb{R})$ for a covering $\{V_i\}$. Thus we can assume $U$ is contractible, then $\Gamma(U, R^q f_* \mathbb{Q}) \cong H^q(f^{-1}(U), \mathbb{Q}) \cong H^q(X_b, \mathbb{Q})$ for any $b \in U$. One then define

$$ \delta'_1, \delta'_2 : H^q(X_b, \mathbb{Q}) \to H^{q-1}(X_b, \mathbb{Q}) $$

by $\delta'_1(\alpha) = (-1)^{n-q} \langle \alpha, D_{ij} \rangle' \sigma_j$ and $\delta'_2(\alpha) = \alpha T_{\sigma_j} - \alpha T_{\sigma_i} \sigma_j$. Clearly $\delta'_1$ and $\delta'_2$ coincide with $\delta_1$ and $\delta_2$ under the above isomorphisms.

Write $X_b = V/\Lambda$, so that we are in the situation of Lemma 4.2.

Let $\gamma : I \to X_b$ be the loop based at $\sigma_i(b)$ given by

$$ \gamma(t) = \begin{cases} (1 - 2t)\sigma_i(b) & 0 \leq t \leq 1/2; \\ (2t - 1)\sigma_j(b) & 1/2 \leq t \leq 1. \end{cases} $$

Then the homology class of $\gamma$ in $H_1(X_b, \mathbb{Z})$ coincides with $\sigma_j(b) - \sigma_i(b) \in H^{n-1}(T^n, \mathbb{Z})$ via Poincaré duality, and then by using the third row of the diagram of Lemma 4.2,

$$ \delta'_1(\alpha) = \langle \alpha, \gamma \rangle' \sigma_j. $$

In addition,

$$ (\alpha T_{\sigma_j} - \alpha T_{\sigma_i})(\Delta) = \alpha(\Delta T_{\sigma_j} - \Delta T_{\sigma_i}) $$

$$ = \alpha(\Delta \gamma) $$

$$ = \langle \alpha, \gamma \rangle' \sigma_j. $$

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so by the definition of $\langle \cdot, \cdot \rangle''$ in Lemma 4.2,

$$\delta'_2(\alpha) = \langle \alpha, \gamma \rangle''.$$  

This proves the claim.  

Now we see that in $H^{p-i+1}(B, R^{i-1}f_*Q)$,

$$\beta_{p-i+1,i-1} = \langle U_{i_0\ldots i_{p-i+1}}, (\alpha_{i_0\ldots i_{p-i}}) T_{\sigma_{i_{p-i}+1}} - (\alpha_{i_0\ldots i_{p-1}}) T_{\sigma_{i_{p-1}}} \rangle $$

$$= (U_{i_0\ldots i_{p-i+1}}, (\alpha_{i_0\ldots i_{p-i}}) D_{i_{p-i+1}})$$

by the claim,

$$= \langle \alpha, (\alpha_{i_0\ldots i_{p-1}}) D \rangle$$

by (4.4).

This is the desired result.  

We now encounter another problem whose solution must be deferred to a future paper.  

We would like to compare certain intersection numbers on $X$ with intersection numbers on $\tilde{X}$.  To actually obtain numbers, we need to orient $\tilde{X}$.  Now $X$ is a complex manifold and hence comes with a canonical orientation.  Once we understand how to put a complex structure on $\tilde{X}$, $\tilde{X}$ will also come with a canonical orientation, which is the orientation we want to use.  But until then, we have to make an assumption about what this orientation is.  So we have

**Convention 4.3.** Orient $\tilde{X}$ as follows: $(-1)^n[T^n] \in f^*H^n(B, f_*Q) \subseteq H^n(X, Q)$ corresponds to an element $\varphi \in H^n(B, R^n f_*Z) \cong H^{2n}(\tilde{X}, Q)$ via (2.1).  $\varphi$ determines an orientation on $\tilde{X}$, so that $\int_{\tilde{X}} \varphi > 0$.  This is the orientation we will use.

We will justify this choice in [10].  Having chosen an orientation on $\tilde{X}$, we can define $D_1, \ldots, D_n = \int_{\tilde{X}} D_1 \cup \cdots \cup D_n$ for $D_1, \ldots, D_n \in H^2(\tilde{X}, Q)$.  Note that by Remark 2.5, $H^1(B, R^1 f_*Z)$ is a subquotient of $H^2(\tilde{X}, Z)$, so if $D_1, \ldots, D_n$ are actually just elements of $H^1(B, R^1 f_*Z)$, we can lift them to elements $D_1, \ldots, D_n$ on $H^2(\tilde{X}, Z)$, and $D_1 \cup \cdots \cup D_n$ is independent of the lifting.

For the remainder of this section, we will assume that $f : X \to B$ is in fact $\mathbb{Z}$-simple, thus allowing us to apply the isomorphisms (2.2).  We actually only need (2.2) for $q = n-1$.

**Definition 4.4.**  For $D \in H^1(B, R^1 f_*Z)$, let $\sigma_D$ denote a section of $f : X \to B$ such that $[\sigma_D] \in H^1(B, R^{n-1} f_*Z)$ corresponds to $D$ under the isomorphism (2.2).  We orient $\sigma_D(B) \subseteq X$ so that the cohomology class of $\sigma_D(B)$, which we also write as $\sigma_D$, satisfies $\sigma_D[T^n] = 1$.  Note this cohomology class only depends on $D$ and not the particular choice of section.  We will also write $T_D := T_{\sigma_D}$.
Corollary 4.5. Let $\alpha_0 \in H^n(X, \mathbb{Q})$ be any cohomology class which represents $1 \in H^0(B, R^n f_* \mathbb{Q})$ (so that $\alpha_0[T^n] = 1$). Then

$$\alpha_0(T_{D_1} - I) \cdots (T_{D_n} - I)\alpha_0 = (-1)^{n(n-1)/2}D_1 \cdots D_n.$$ 

Proof. By Theorem 4.1, $(T_{D_1} - I) \cdots (T_{D_n} - I)\alpha_0 \in F_0 \cong H^n(B, \mathbb{Q})$ via the map $f^*: H^n(B, \mathbb{Q}) \to H^n(X, \mathbb{Q})$. Furthermore,

$$(T_{D_1} - I) \cdots (T_{D_n} - I)\alpha_0 = (-1)^{n+1/2} \langle \cdots \langle \alpha_0, D_1 \rangle, D_2 \rangle, \ldots, D_n \rangle.$$ 

Under the isomorphism (2.1), $\alpha_0$ coincides with $1 \in H^0(B, \tilde{f}_* \mathbb{Q}) = H^0(\tilde{X}, \mathbb{Q})$, and thus

$$\langle \cdots \langle \alpha_0, D_1 \rangle, D_2 \rangle, \ldots, D_n \rangle$$

coincides with $D_1 \cup \cdots \cup D_n \in H^{2n}(\tilde{X}, \mathbb{Q})$. Now using Convention 4.3 to orient $\tilde{X}$, if $1 \in H^{2n}(\tilde{X}, \mathbb{Q})$ is chosen so that $\int_{\tilde{X}} 1 = 1$, we see that

$$(D_1 \cdots D_n)1 = D_1 \cup \cdots \cup D_n$$

and $(D_1 \cdots D_n)1$ then coincides with $(-1)^{n}(D_1 \cdots D_n)[T^n]$ under the isomorphism (2.1). Thus

$$(T_{D_1} - I) \cdots (T_{D_n} - I)\alpha_0 = (-1)^n(-1)^{n+1/2}D_1 \cdots D_n[T^n].$$ 

Since $\alpha_0[T^n] = 1$, the result follows. $ullet$

Part (2) of the following Corollary, along with Conjecture 3.7, implies part (3) of Conjecture 3.2. Part (1) shows that Conjecture 3.7 implies the $(1,n-1)$ Yukawa couplings near the large complex structure limit point have the expected limiting values.

Corollary 4.6. Suppose that $X$ is a member of a family of Calabi-Yau manifolds $\mathcal{X} \to S$, $p \in \overline{S}$ a large complex structure limit point. Suppose that $X$ possesses a $\mathbb{Z}$-simple $C^\infty$ special Lagrangian fibration such that Conjecture 3.7 holds. Then

1. The asymptotic values of the $(1,n-1)$-Yukawa coupling coincide, up to a sign, with the $(1,1)$-topological coupling of the mirror.

2. If $n = 3$, the weight filtration on $H^3(X, \mathbb{Q})$ associated to the large complex structure limit point coincides with the Leray filtration, i.e. $W_{2i} = W_{2i+1} = F_i$.

Proof. Suppose the monodromy about boundary divisors $B_1, \ldots, B_s$ are $T_1, \ldots, T_s$, which by Conjecture 3.7 are induced by translation by sections $\sigma_1, \ldots, \sigma_s$ corresponding to $D_1, \ldots, D_s \in H^1(B, R^1 f_* \mathbb{Z})$ on $\tilde{X}$. Then if $\alpha_0$ is as in Corollary 4.5, we have

$$N_{i_1} \cdots N_{i_n} \alpha_0 = \pm(D_{i_1} \cdots D_{i_n})[T^n].$$

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Note that this is not always zero: if it was, then for $N = \sum a_i N_i$, $a_i > 0$, $N^n = 0$ on $H^n(X, \mathbb{Q})$. This is not the case, since $p \in \tilde{S}$ is a large complex structure limit point. Thus using the definition of the monodromy weight filtration, we see $\alpha_0 \in W_{2n}$, $\alpha_0 \notin W_{2n-1}$ and $[T^n] \in W_0$. Since $\alpha_0, [T^n] = 1$, it follows from (3.1) that

$$\left\langle \frac{\partial}{\partial t_{i_1}}, \ldots, \frac{\partial}{\partial t_{i_n}} \right\rangle_{nil} = \pm D_{i_1} \cdots D_{i_n},$$

proving (1). (2) follows from the definition of the weight filtration and Theorem 4.1. •

Remark 4.7. In higher dimensions, one needs some additional hypotheses to ensure that the Leray and weight filtrations coincide. One such condition is the following. Suppose the Leray spectral sequences for $f$ and $\tilde{f}$ degenerate, and a monodromy operator $T : X \to X$ is induced by translation by a section $\sigma$ corresponding to a divisor $D$. Then if $\{W_i\}$ is the weight filtration on $H^p(X, \mathbb{Q})$ induced by $T$, then $W_{2i} = W_{2i+1} = F_i(H^p(X, \mathbb{Q}))$ for all $i, p$ if

$$\cup D^{n-j} : F_i(H^j(\tilde{X}, \mathbb{Q})) \to F_{n+i-j}(H^{2n-j}(\tilde{X}, \mathbb{Q}))$$

is an isomorphism for all $i, j$. One can think of this as a compatibility between the spectral sequence for $\tilde{f}$ and Hard Lefschetz.

Corollary 4.8. Let $D \in H^1(B, R^1 f_* \mathbb{Z})$. Then $(-1)^{n(n-1)/2} \sigma_0 \sigma_t D$ is a polynomial of degree $n$ in $t$, and this polynomial is even if $n$ is even and odd if $n$ is odd, with leading term $D^n t^n / n!$. In particular, if $n = 2$ and $X$ is a K3 surface, then

$$-\sigma_0 \sigma_D = \frac{D^2}{2} + 2$$

and if $n = 3$ then there exists a class $C \in H^4(\tilde{X}, \mathbb{Q})$ such that

$$-\sigma_0 \sigma_D = \frac{D^3}{6} + C.D.$$

If $X$ and $\tilde{X}$ are simply connected, then

$$C \equiv \frac{c_2(\tilde{X})}{12} \mod \mathbb{Z}.$$

Proof. If $f : \mathbb{Z} \to \mathbb{Z}$ is a function, denote by $\Delta f$ the difference function of $f$, i.e. $(\Delta f)(t) = f(t+1) - f(t)$. Note that if $f(t) = \sigma_0 \sigma_{tD}$, then

$$(\Delta f)(t) = \sigma_0 \sigma_{(t+1)D} - \sigma_0 \sigma_{tD}$$

$$= \sigma_0 (T_D - I) \sigma_{tD},$$
and inductively,
\[(\Delta^i f)(t) = \sigma_0. (T_D - I)^i \sigma_t D.\]

Now since \(\sigma_0\) and \(\sigma_t D\) are cohomology classes representing 1 in \(H^0(B; R^n \hat{f}_* \mathbb{Q}) \cong \mathbb{Q}\), it follows from Corollary 4.5 that \(\sigma_0. (T_D - I)^n \sigma_t D = (-1)^{(n-1)/2} D^n\). Thus \((\Delta^n f)(t) = (-1)^{(n-1)/2} D^n\) for all \(t\), so \(f\) is a polynomial of degree \(n\) in \(t\) with
\[f(t) = (-1)^{(n-1)/2} D^n t^n/n! + O(t^{n-1}).\]

Also, we note that
\[\sigma_0. \sigma_t D = (-1)^n \sigma_t D. \sigma_0 = (-1)^n (T_t D(\sigma_0)). \sigma_0 = (-1)^n \sigma_0. T_{-t} D(\sigma_0) = (-1)^n \sigma_0. T_{-t} D(\sigma_0)\]
so \(f\) is even or odd according to the parity of \(n\).

If \(X\) is a K3 surface, then \(f(0) = \sigma_0. \sigma_0 = -2\), which proves the formula in this case. If \(n = 3\), then
\[f(t) = -D^3 t^3/6 - t \varphi(D)\]
for some function \(\varphi : H^1(B, R^1 \hat{f}_* \mathbb{Z}) \to \mathbb{Q}\). Using similar difference equation techniques, one can compute \(\sigma_0. (\sigma_{t_1 D_1 + t_2 D_2} - \sigma_{t_1 D_1} - \sigma_{t_2 D_2})\) and find that \(\varphi\) is linear. We omit the details. Once we know \(\varphi\) is linear, we know by Poincaré duality that there exists a \(C \in H^4(\hat{X}, \mathbb{Q})\) such that \(\varphi(D) = C.D\).

Finally, if \(X\) and \(\hat{X}\) are simply connected and \(n = 3\), then by the results of §2, \(H^{2i}(\hat{X}, \mathbb{Q}) = H^i(B, R^1 \hat{f}_* \mathbb{Q})\), and then \(C\) is completely determined by \(\varphi\). It then follows from Riemann-Roch and integrality of \(f(t)\) that \(C \equiv c_2/12 \mod \mathbb{Z}\).

This leads to a conjecture which is inspired by Kontsevich’s homological mirror symmetry conjecture [14].

**Conjecture 4.9.** (Mirror Riemann-Roch) If \(D \in H^1(B, R^1 \hat{f}_* \mathbb{Z})\) represents the first Chern class of a line bundle \(\mathcal{O}_X(D)\), then
\[(-1)^{(n-1)/2} \sigma_0. \sigma_D = \chi(\mathcal{O}_X(D)).\]

This is true from Corollary 4.8 if \(n = 2\). At the time of completing this paper, I can prove this conjecture for \(n = 3\) with some additional assumptions on the nature of special Lagrangian fibrations, which I do not believe should be necessary. This does not
seem satisfactory, so I have decided to postpone a proof until the future. Nevertheless, we
shall assume Conjecture 4.9 is true for $n = 3$ in the next section. The point is that for
$n = 3$, as we will see in §5, knowing Conjecture 4.9 gives us a complete description of the
intersection pairing in $H^3(X, \mathbb{Z})$ if we note in addition that in the Leray filtration,

$$H^3(X, \mathbb{Q}) = F_3 \supseteq F_2 \supseteq F_1 \supseteq F_0 \supseteq 0,$$

for any $\alpha, \beta \in F_1, \alpha.\beta = 0$.

§5. The Mirror Map on Cohomology for Threefolds.

Suppose now that $f : X \to B$ is a $\mathbb{Z}$-simple $C^\infty$ special Lagrangian $T^3$-fibration, with
$\tilde{f} : \tilde{X} \to B$ $\mathbb{Z}$-simple, $X$, $\tilde{X}$ simply connected. Our goal is to speculate on what the SYZ
construction might tell us about a mirror map between $H^{odd}(X, \mathbb{Q})$ and $H^{even}(\tilde{X}, \mathbb{Q})$ in
the three-dimensional case. As we saw in Lemma 2.4, there do exist isomorphisms between
these groups, but we have not given a natural one. The difficulty is that a priori there is not
a natural splitting of the Leray (or weight) filtration on $H^{odd}(X, \mathbb{Q})$; once one determines a
splitting compatible with the Leray filtration, one would obtain an isomorphism compatible
with the isomorphisms (2.1). This is exactly what we did in the two-dimensional case in
Example 3.9 in order to determine the action of translation by a section on cohomology.

One benefit of having a natural isomorphism, as we shall see, is that the action of
monodromy on $H^3(X, \mathbb{Q})$ then gives an action on $H^{even}(\tilde{X}, \mathbb{Q})$, and this action can then
be completely described in a natural way. This gives us more information than Theorem
4.1, which only yields information about the monodromy action on graded pieces of the
Leray filtration. Some extra information must be put in to get a complete description of
monodromy: this extra ingredient is mirror Riemann-Roch (Conjecture 4.9) which we will
assume for $n = 3$.

What an appropriate mirror map is certainly depends on the point of view. We will
present one method of producing an isomorphism here, with motivation being Kontsevich’s
homological mirror symmetry conjecture [14].

In [14], Kontsevich suggests that there should be an isomorphism between $\mathcal{D}^b(\tilde{X})$, the
bounded derived category of coherent sheaves on $\tilde{X}$, and a mysterious category involving
$X$ which is a derived version of Fukaya’s $A^\infty$ category [8], possibly with extra structure.
For convenience, I’ll call it $\mathcal{D}^b(X)$. We don’t have a definition for this category, and will
make no attempt to describe it here, other than to mention that the objects of this category
should be related to Lagrangian submanifolds of $X$. One should expect a map

$$\psi : \mathcal{D}^b(X) \to H^3(X, \mathbb{Z})$$
taking an object of $\mathcal{D}^b(X)$ to the cohomology class of its underlying Lagrangian submanifold. If $E$ and $F$ are objects of $\mathcal{D}^b(X)$, there should be an Euler characteristic

$$\chi(E, F) = \sum_i (-1)^i \dim \text{Ext}_i^{\mathcal{D}^b(X)}(E, F),$$

where these Exts are Floer-type groups, and the conjectures of [14] suggest that

$$\chi(E, F) = \psi(E).\psi(F).$$

On the other hand, consider the following skew-symmetric pairing $(\cdot, \cdot)$ on $H^{even}(\tilde{X}, \mathbb{Q})$. If $\alpha = (\alpha_0, \alpha_2, \alpha_4, \alpha_6), \beta = (\beta_0, \beta_2, \beta_4, \beta_6) \in \bigoplus_i H^{2i}(\tilde{X}, \mathbb{Q})$, we put

$$(\alpha, \beta) = \alpha_0.\beta_6 - \alpha_2.\beta_4 + \alpha_4.\beta_2 - \alpha_6.\beta_0,$$

with orientation on $\tilde{X}$ defined as in Convention 4.3. If $\mathcal{E} \in \text{Coh}(\tilde{X})$, then the Mukai vector of $\mathcal{E}$ is $v(\mathcal{E}) = \text{ch}(\mathcal{E})\sqrt{\text{Td}(\tilde{X})} = \text{ch}(\mathcal{E})(1 + c_2/24) \in H^{even}(\tilde{X}, \mathbb{Q})$. (Note that to talk about $\text{Coh}(\tilde{X})$ assumes a complex structure on $\tilde{X}$; we will just assume here that this complex structure yields the same orientation on $\tilde{X}$ as does Convention 4.3.) Then

$$\chi(\mathcal{E}, \mathcal{F}) := \sum_{i=0}^3 (-1)^i \dim \text{Ext}_i^{\mathcal{E}, \mathcal{F}}(\mathcal{E}, \mathcal{F}) = (v(\mathcal{E}), v(\mathcal{F})).$$

We now might conjecture the existence of a commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^b(X) & \xrightarrow{\psi} & \mathcal{H}^3(X, \mathbb{Q}) \\
\downarrow^{\phi_1} & & \uparrow^{\phi_2} \\
\mathcal{D}^b(\tilde{X}) & \xrightarrow{v} & \mathcal{H}^{even}(\tilde{X}, \mathbb{Q})
\end{array}$$

I am going to try to guess what properties $\phi_2$ should have in order to make the diagram commutative. This is of course imprecise since we have no idea $\phi_1$ and $\psi$ are, but we can make educated guesses.

We want the following properties for the map $\phi_2$:

(5.1) $\phi_2$ should be a symplectic isomorphism: $\phi_2(\alpha) . \phi_2(\beta) = (\alpha, \beta)$ for $\alpha, \beta \in \mathcal{H}^{even}(\tilde{X}, \mathbb{Q})$.

(5.2) Both $\mathcal{H}^3(X, \mathbb{Q})$ and $\mathcal{H}^{even}(\tilde{X}, \mathbb{Q})$ are filtered, $\mathcal{H}^3(X, \mathbb{Q})$ by the Leray filtration and $\mathcal{H}^{even}(X, \mathbb{Q})$ by

$$\tilde{F}_i = \bigoplus_{j=3-i}^3 H^{2j}(\tilde{X}, \mathbb{Q}).$$
Here

\[ F_i/F_{i-1} \cong H^{3-i}(B, R^i f_* \mathbb{Q}) \]

and

\[ \tilde{F}_i/\tilde{F}_{i-1} \cong H^{3-i}(B, R^{3-i} \tilde{f}_* \mathbb{Q}), \]

which are isomorphic via (2.1). The map \( \phi_2 \) should respect these isomorphisms. We shall see in a moment that this is not quite correct, however.

These two properties do not completely determine \( \phi_2 \). There is some additional information one can obtain following Kontsevich’s philosophy that a monodromy transformation on \( X \) should yield an automorphism of \( \mathcal{D}^b(X) \), and hence one of \( \tilde{X} \). Tensoring with \( \mathcal{O}_X(D) \) induces an automorphism \( t_D : \mathcal{D}^b(\tilde{X}) \to \mathcal{D}^b(\tilde{X}) \), and clearly \( v(t_D(\mathcal{E})) = e^D v(\mathcal{E}) \).

We would expect this to correspond to the action of \( T_D \) on \( H^3(X, \mathbb{Q}) \), and so obtain the formula

(5.3) \[ T_D(\phi_2(\alpha)) = \phi_2(e^D \alpha). \]

Another canonical automorphism of \( \mathcal{D}^b(\tilde{X}) \) given by Kontsevich is constructed as follows*. Let \( p_1, p_2 : \tilde{X} \times \tilde{X} \to \tilde{X} \) be the two projections, \( \Delta \subseteq \tilde{X} \times \tilde{X} \) the diagonal. Then

\[ \mathcal{E} \mapsto p_2^*(\mathcal{I}_\Delta \otimes p_1^* \mathcal{E})[1] \]

yields an automorphism of \( \mathcal{D}^b(\tilde{X}) \), with \( v(p_2^*(\mathcal{I}_\Delta \otimes p_1^* \mathcal{E})[1]) = v(\mathcal{E}) + (v(\mathcal{E}), v(\mathcal{O}_\tilde{X}))v(\mathcal{O}_\tilde{X}) \).

(The shift by 1 has the effect of negation of Chern character of a complex). This induces an action

\[ \tilde{\gamma} : \alpha \mapsto \alpha + (\alpha, v(\mathcal{O}_\tilde{X}))v(\mathcal{O}_\tilde{X}) \]

on \( H_{\text{even}}(\tilde{X}, \mathbb{Q}) \). We expect that this will correspond to the Picard-Lefschetz transformation associated with \( \sigma_0 \) on \( X \), namely for \( \beta \in H^3(X, \mathbb{Q}) \)

\[ \gamma : \beta \mapsto \beta + (\beta, \sigma_0)\sigma_0. \]

Thus we should expect \( \phi_2(v(\mathcal{O}_\tilde{X})) = \sigma_0 \). Combining this with (5.3), we obtain

(5.4) \[ \phi_2(v(\mathcal{O}_X(D))) = \sigma_D. \]

* Somewhat earlier, H. Kim suggested in [13] a similar construction on vector bundles (which agrees with Kontsevich’s construction in some cases), and he suggested that this construction should be mirror to a Picard-Lefschetz type transformation.
Unfortunately conditions (5.1)-(5.4) are incompatible. One reason is that the alternating sign appearing in Theorem 4.1 makes (5.2) and (5.3) incompatible. So we have to change the sign in some of the isomorphisms of (2.1) to make things work. The theorem below gives choices of sign which work. It is difficult at this point to justify the choice of sign conventions; perhaps in the future a different choice of sign will seem more natural. Other than the choice of signs, everything in this theorem fits with (5.1)-(5.4).

**Theorem 5.1.** There exists a unique isomorphism $\phi_2 : H^\text{even}(\bar{X}, \mathbb{Q}) \to H^3(X, \mathbb{Q})$ of filtered vector spaces with the following properties:

(5.5) $\phi_2(v(O_{\bar{X}}(D))) = \sigma_D$,

(5.6) $\phi_2 : F_i/F_{i-1} \to \tilde{F}_i/\tilde{F}_{i-1}$ is the duality isomorphism $H^{3-i}(B, R^i f_\ast \mathbb{Q}) \cong H^{3-i}(B, R^{3-i} \tilde{f}_\ast \mathbb{Q})$ of (2.1) for $i = 2$ and 3, but the negative of (2.1) for $i = 0$ and 1.

(5.7) $(\alpha, \beta) = \phi_2(\alpha)\cdot \phi_2(\beta)$ for all $\alpha, \beta \in H^\text{even}(\bar{X}, \mathbb{Q})$

Furthermore, under this isomorphism,

(5.8) $\phi_2(e^D \alpha) = T_D \phi_2(\alpha)$

and

(5.9) $\phi_2(\tilde{\gamma}(\alpha)) = \gamma(\phi_2(\alpha))$.

**Proof.** First, since $\phi_2(\tilde{F}_0) = F_0$ and $\phi_2|_{\tilde{F}_0} : \tilde{F}_0 \to F_0$ coincides with the negative of the isomorphism (2.1), we must have $\phi_2(0, 0, 0, 1) = [T^3]$ by Convention 4.3.

Now for $C \in H^4(\bar{X}, \mathbb{Q})$, $\left(\sqrt{Td(\bar{X})}, C\right) = 0$, so

$$0 = \phi_2 \left(\sqrt{Td(\bar{X})}\right) \cdot \phi_2(C) = \phi_2(v(O_{\bar{X}})) \cdot \phi_2(C) = \sigma_0 \cdot \phi_2(C)$$

by (5.5) and (5.7). In conjunction with (5.6), this uniquely determines $\phi_2(C)$ with $\phi_2$ giving an isomorphism between $H^4(\bar{X}, \mathbb{Q})$ and $\sigma_0^\perp \cap F_1$. We have now determined $\phi_2 : \tilde{F}_1 \to F_1$ completely. Thus

$$\sigma_0 = \phi_2(v(O_{\bar{X}}))$$

$$= \phi_2(1, 0, c_2/24, 0)$$

$$= \phi_2(1, 0, 0, 0) + \phi_2(c_2/24)$$

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so $\phi_2(1,0,0,0) = \sigma_0 - \phi_2(c_2/24)$. Furthermore, by (5.5), we must have

$$\sigma_{-D} = \phi_2(v(O_X(D))) = \phi_2 \left( 1, D, \frac{D^2}{2} + \frac{c_2}{24}, \frac{D^3}{6} + \frac{D.c_2}{24} \right)$$
$$= \left[ \sigma_0 - \phi_2 \left( \frac{c_2}{24} \right) \right] + \phi_2(D)$$
$$+ \phi_2 \left( \frac{D^2}{2} + \frac{c_2}{24} \right) + \left( \frac{D^3}{6} + \frac{D.c_2}{24} \right) [T^3],$$

from which we conclude we must have

$$\phi_2(D) = \sigma_{-D} - \sigma_0 - \phi_2 \left( \frac{D^2}{2} \right) - \left( \frac{D^3}{6} + \frac{D.c_2}{24} \right) [T^3].$$

To finish the proof we must check (5.6)-(5.9). Checking (5.6) and (5.9) is easy. To check (5.7), we need to show that $\phi_2(1,0,0,0).\phi_2(D) = 0$, $\phi_2(D).\phi_2(E) = 0$, $\phi_2(D).\phi_2(C) = -D.C$ for $C \in H^4(\tilde{X}, Q)$, and $\phi_2(D).[T^3] = 0$. The last of these is obvious. To perform the other calculations, we need the following:

$$(\sigma_{-D} - \sigma_0).\phi_2(C) = T_{-D}(\sigma_0).\phi_2(C) - \sigma_0.\phi_2(C)$$
$$= \sigma_0.(T_D(\phi_2(C)) - \phi_2(C))$$
$$= \sigma_0.(T_D - I)\phi_2(C)$$
$$= \sigma_0.(-D.C)[T^3] \text{ by Theorem 4.1 and (5.6)}$$
$$= -D.C.$$

Thus

$$\phi_2(D).\phi_2(C) = (\sigma_{-D} - \sigma_0).\phi_2(C)$$
$$= -D.C$$

as desired. Also, using the above calculations and mirror Riemann-Roch,

$$\phi_2(1,0,0,0).\phi_2(D) = (\sigma_0 - \phi_2(c_2/24)).(\sigma_{-D} - \sigma_0 - (D^3/6 + D.c_2/24)[T^3])$$
$$= \sigma_0.\sigma_{-D} + (\sigma_{-D} - \sigma_0).\phi_2(c_2/24) - (D^3/6 + D.c_2/24)\sigma_0.[T^3]$$
$$= D^3/6 + D.c_2/12 - D.c_2/24 - D^3/6 - D.c_2/24$$
$$= 0$$

* Compare this with the expression $E' = \sigma_0 + (c_2/24)E$ of Example 3.9.
and
\[
\phi_2(D) \phi_2(E) = (\sigma_D - \sigma_0 - \phi_2(D^2/2)).(\sigma_E - \sigma_0 - \phi_2(E^2/2))
\]
\[
= (\sigma_D - \sigma_0).(\sigma_E - \sigma_0) + DE^2/2 - D^2E/2
\]
\[
= T_D(\sigma_0).(\sigma_E - \sigma_0) - \sigma_0.(\sigma_E - \sigma_0) + DE^2/2 - D^2E/2
\]
\[
= \sigma_0 T_D(\sigma_E - \sigma_0) - \sigma_0.\sigma_E + DE^2/2 - D^2E/2
\]
\[
= \sigma_0. (\sigma_D - \sigma_D - \sigma_E) + DE^2/2 - D^2E/2
\]
\[
= - \frac{(D - E)^3}{6} + \frac{D^3}{6} + \frac{(-E)^3}{6} - (D - E - D + E).c_2/12 + DE^2/2 - D^2E/2
\]
\[
= 0
\]
as desired.

We need to check that (5.8) holds. We do this for $\alpha \in H^{2i}(\tilde{X}, Q)$ for each $i$.

For $i = 3$, the result is clear.

For $i = 2$, $C \in H^4(\tilde{X}, Q)$,
\[
T_D(\phi_2(C)) = \phi_2(C) + (T_D - I)(\phi_2(C)),
\]
and since $(T_D - I)\phi_2(C) \in F_0$, Theorem 4.1 tells us that $(T_D - I)\phi_2(C) = (D.C)[T^3]$.

On the other hand, $\phi_2(e^D C) = \phi_2(C) + (D.C)[T^3]$, so (5.8) holds for $C \in H^4(\tilde{X}, Q)$.

For $E \in H^2(\tilde{X}, Q)$,
\[
T_D(\phi_2(E)) = T_D \left( \sigma_E - \sigma_0 - \phi_2 \left( \frac{E^2}{2} \right) - \left( \frac{E^3}{6} + \frac{E.c_2}{24} \right) \right) [T^3]
\]
\[
= \sigma_D - \sigma_D - \phi_2 \left( e^D \frac{E^2}{2} \right) - \left( \frac{E^3}{6} + \frac{E.c_2}{24} \right) [T^3]
\]
and
\[
\phi_2(e^D E) = \phi_2(0, E, E.D, E.D^2/2)
\]
\[
= \left[ \sigma_E - \sigma_0 - \phi_2 \left( \frac{E^2}{2} \right) - \left( \frac{E^3}{6} + \frac{E.c_2}{24} \right) \right] [T^3]
\]
\[
+ \phi_2(E.D) + \frac{E.D^2}{2} [T^3].
\]

Thus
\[
T_D(\phi_2(E)) - \phi_2(e^D E) = (\sigma_D - \sigma_E) - (\sigma_D - \sigma_0) - \phi_2(E.D) - \left( \frac{DE^2}{2} + \frac{ED^2}{2} \right) [T^3]
\]
\[
= (T_E - I)(T_D - I)\sigma_0 - \phi_2(E.D) - \left( \frac{DE^2}{2} + \frac{ED^2}{2} \right) [T^3].
\]
By Theorem 4.1, this is in fact in $F_0$, so to test to see if this is zero, we just need to show that $\sigma_0(T_D(\phi_2(E)) - \phi_2(e^D E)) = 0$. This is straightforward to check using mirror Riemann-Roch.

Finally, we check that

$$T_D(\phi_2(v(\mathcal{O}_X))) = T_D(\sigma_0) = \sigma_D = \phi_2(e^D v(\mathcal{O}_X)).$$

Thus (5.8) holds in general. •

Remark 5.2. Ultimately, I expect this mirror map should be understood in terms of a type of Fourier-Mukai transform using a Poincaré bundle on $X \times_B \bar{X}$. A version of this was done for K3 surfaces in [2]. While it is not difficult to construct a Poincaré bundle on $X \times_B \bar{X}$ in general, this will merely be a $C^\infty$ complex line bundle, and it is not clear how to extract information in the derived categories from this non-algebraic situation.

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