ON SOME HADAMARD-TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper, we prove some new inequalities of Hadamard-type for convex functions on the co-ordinates.

1. INTRODUCTION
Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a < b$. The following double inequality;

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is well known in the literature as Hadamard’s inequality. Both inequalities hold in the reversed direction if $f$ is concave.

In [1], Dragomir defined convex functions on the co-ordinates as following:

Definition 1. Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on $\Delta$ if the following inequality holds,

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda f(x, y) + (1 - \lambda)f(z, w)$$

for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

In [1], Dragomir established the following inequalities of Hadamard’s type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$. 

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Theorem 1. Suppose that \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is convex on the co-ordinates on \( \Delta \). Then one has the inequalities:

\[
\begin{align*}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq & \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\
\leq & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dxdy \\
\leq & \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_c^d f(x, d) dx \\
&+ \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\
\leq & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}.
\end{align*}
\]

The above inequalities are sharp.

Similar results can be found in [1]-[7].

The main purpose of this paper is to prove some new inequalities of Hadamard-type for convex functions on the co-ordinates.

2. MAIN RESULTS

To prove our main result, we need the following Lemma.

Lemma 1. Let \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) be a twice partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \). If \( \frac{\partial^2 f}{\partial t \partial s} \in L(\Delta) \), then the following equality holds:

\[
\begin{align*}
&\quad A + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dudv \\
= & \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (t-1)(s-1) \frac{\partial^2 f}{\partial t \partial s} (tx+(1-t)a, sy+(1-s)c) dsdt \\
&+ \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (t-1)(1-s) \frac{\partial^2 f}{\partial t \partial s} (tx+(1-t)a, sy+(1-s)d) dsdt \\
&+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (1-t)(s-1) \frac{\partial^2 f}{\partial t \partial s} (tx+(1-t)b, sy+(1-s)c) dsdt \\
&+ \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 (1-t)(1-s) \frac{\partial^2 f}{\partial t \partial s} (tx+(1-t)b, sy+(1-s)d) dsdt.
\end{align*}
\]
where

\[ A = \frac{(x-a)(y-c)f(a,c)+(x-a)(d-y)f(a,d)}{(b-a)(d-c)} \]

\[ + \frac{(b-x)(y-c)f(b,c)+(b-x)(d-y)f(b,d)}{(b-a)(d-c)} \]

\[ - \frac{(x-a)}{(b-a)(d-c)} \int_{c}^{d} f(a,v) \, dv - \frac{(b-x)}{(b-a)(d-c)} \int_{c}^{d} f(b,v) \, dv \]

\[ - \frac{(d-y)}{(b-a)(d-c)} \int_{a}^{b} f(u,d) \, du - \frac{(y-c)}{(b-a)(d-c)} \int_{a}^{b} f(u,c) \, du \]

Proof. It suffices to note that

\[ I = \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_{0}^{1} \int_{0}^{1} \left( t-1 \right) (s-1) \frac{\partial^2 f}{\partial t \partial s} \left( t^2 + (1-t) a, sy + (1-s) c \right) ds \, dt \]

\[ + \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \int_{0}^{1} \int_{0}^{1} \left( t-1 \right) (s-1) \frac{\partial^2 f}{\partial t \partial s} \left( t^2 + (1-t) a, sy + (1-s) d \right) ds \, dt \]

\[ + \frac{(b-x)(y-c)}{(b-a)(d-c)} \int_{0}^{1} \int_{0}^{1} \left( 1-t \right) (s-1) \frac{\partial^2 f}{\partial t \partial s} \left( t^2 + (1-t) b, sy + (1-s) c \right) ds \, dt \]

\[ + \frac{(b-x)(d-y)}{(b-a)(d-c)} \int_{0}^{1} \int_{0}^{1} \left( 1-t \right) (1-s) \frac{\partial^2 f}{\partial t \partial s} \left( t^2 + (1-t) b, sy + (1-s) d \right) ds \, dt. \]

Integrating by parts, we get

\[ I_1 = \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_{0}^{1} (s-1) \left[ \frac{t-1}{x-a} \frac{\partial f}{\partial s} (tx + (1-t) a, sy + (1-s) c) \right]_{0}^{1} ds \]

\[ - \frac{1}{x-a} \int_{0}^{1} \frac{\partial f}{\partial s} (tx + (1-t) a, sy + (1-s) c) \, dt \] ds

\[ = \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_{0}^{1} (s-1) \left[ \frac{1}{x-a} \frac{\partial f}{\partial s} (a, sy + (1-s) c) \right] ds \]

\[ - \frac{1}{x-a} \int_{0}^{1} \frac{\partial f}{\partial s} (tx + (1-t) a, sy + (1-s) c) \, dt \] ds
By integrating again and by changing of the variables $u = tx + (1 - t)a$, $v = sy + (1 - s)c$, we obtain

$$I_1 = \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1 (s-1) \left[ \frac{t-1}{x-a} \frac{\partial f}{\partial s} (tx + (1-t)a, sy + (1-s)c) \right]_0^1$$
$$- \frac{1}{x-a} \int_0^1 \frac{\partial f}{\partial s} (tx + (1-t)a, sy + (1-s)c) dt \right] ds$$
$$= \frac{1}{(x-a)(y-c)} f(a,c) - \frac{1}{(x-a)(y-c)^2} \int_c^y f(a,v) dv$$
$$- \frac{1}{(x-a)^2(y-c)^2} \int_a^x f(u,c) du + \frac{1}{(x-a)^2(y-c)^2} \int_a^c \int_{y}^x f(u,v) dudv.$$

By a similar argument, we have

$$I_2 = \frac{1}{(x-a)(d-y)} f(a,d) - \frac{1}{(x-a)(d-y)^2} \int_y^d f(a,v) dv$$
$$- \frac{1}{(x-a)^2(d-y)} \int_a^x f(u,d) du$$
$$+ \frac{1}{(x-a)^2(d-y)^2} \int_a^y \int_{d}^x f(u,v) dudv,$$

$$I_3 = \frac{1}{(b-x)(y-c)} f(b,c) - \frac{1}{(b-x)(y-c)^2} \int_c^y f(b,v) dv$$
$$- \frac{1}{(b-x)^2(y-c)} \int_x^b f(u,c) du$$
$$+ \frac{1}{(b-x)^2(y-c)^2} \int_x^y \int_c^b f(u,v) dudv.$$
and

\[ I_4 = \frac{1}{(b-x)(d-y)} f(b,d) - \frac{1}{(b-x)(d-y)^2} \int_y^d f(b,v) \, dv \]

\[ - \frac{1}{(b-x)^2(d-y)} \int_x^b f(u,d) \, du \]

\[ + \frac{1}{(b-x)^2(d-y)^2} \int_x^b \int_y^d f(u,v) \, dudv. \]

Therefore, we obtain

\[ I_1 + I_2 + I_3 + I_4 = \frac{1}{(b-a)(d-c)} \times \left[ A - (x-a) \left[ \int_y^d f(a,v) \, dv + \int_c^y f(a,v) \, dv \right] - (b-x) \left[ \int_c^d f(b,v) \, dv + \int_y^d f(b,v) \, dv \right] \right. \]

\[ - (d-y) \left[ \int_a^x f(u,d) \, du + \int_x^b f(u,d) \, du \right] - (y-c) \left[ \int_x^y f(u,c) \, du + \int_a^y f(u,c) \, du \right] \]

\[ + \int_x^a \int_y^c f(u,v) \, dudv + \int_x^b \int_y^d f(u,v) \, dudv + \int_x^d \int_y^a f(u,v) \, dudv + \int_x^d \int_y^a f(u,v) \, dudv \]

\[ = \frac{1}{(b-a)(d-c)} \left[ A - (x-a) \int_c^d f(a,v) \, dv - (b-x) \int_c^d f(b,v) \, dv \right. \]

\[ - (d-y) \int_a^b f(u,d) \, du - (y-c) \int_a^b f(u,c) \, du + \int_x^d \int_a^c f(u,v) \, dudv. \]

Which completes the proof. \[ \Box \]

**Theorem 2.** Let \( f : \Delta = [a,b] \times [c,d] \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a,b] \times [c,d] \) and \( \frac{\partial^2 f}{\partial s \partial t} \in L(\Delta) \). If \( \left| \frac{\partial^2 f}{\partial s \partial t} \right| \) is a convex function on the co-ordinates
on \( \Delta \), then the following inequality holds:

\[
\begin{align*}
&\left| A + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) \, du \, dv \right| \\
\leq & \frac{1}{9(b-a)(d-c)} \left[ \left( \frac{(x-a)^2 + (b-x)^2}{4} \left( \frac{(y-c)^2 + (d-y)^2}{4} \right) \right) \left| \frac{\partial^2 f}{\partial t \partial s} (x,y) \right| \\
&+ \frac{(x-a)^2 (y-c)^2}{2} \left| \frac{\partial^2 f}{\partial t \partial s} (a,c) \right| + \frac{(x-a)^2 (d-y)^2}{2} \left| \frac{\partial^2 f}{\partial t \partial s} (a,d) \right| \\
&+ \frac{(b-x)^2 (y-c)^2}{2} \left| \frac{\partial^2 f}{\partial t \partial s} (b,c) \right| + \frac{(b-x)^2 (d-y)^2}{2} \left| \frac{\partial^2 f}{\partial t \partial s} (b,d) \right| \right].
\end{align*}
\]

**Proof.** From Lemma \( \mathbb{L} \) and using the property of modulus, we have

\[
\begin{align*}
&\left| A + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) \, du \, dv \right| \\
\leq & \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 |(t-1)(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)a, sy + (1-s)c) \right| \, ds \, dt \\
&+ \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 |(t-1)(1-s)| \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)a, sy + (1-s)d) \right| \, ds \, dt \\
&+ \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 |(1-t)(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)b, sy + (1-s)c) \right| \, ds \, dt \\
&+ \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 |(1-t)(1-s)| \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)b, sy + (1-s)d) \right| \, ds \, dt.
\end{align*}
\]
Since $|\partial^2 f/\partial t \partial s|$ is co-ordinated convex, we can write

\[
\left| \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(u,v) \, du \, dv \right| \\
\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_{0}^{1} |(s-1)| \\
\times \left[ \int_{0}^{1} (t-1) t \left| \frac{\partial^2 f}{\partial t \partial s} (x, sy + (1-s) c) \right| dt \\
+ \int_{0}^{1} (t-1)(1-t) \left| \frac{\partial^2 f}{\partial t \partial s} (a, sy + (1-s) c) \right| dt \right] ds \\
+ \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \int_{0}^{1} |(1-s)| \\
\times \left[ \int_{0}^{1} (t-1) t \left| \frac{\partial^2 f}{\partial t \partial s} (x, sy + (1-s) d) \right| dt \\
+ \int_{0}^{1} (t-1)(1-t) \left| \frac{\partial^2 f}{\partial t \partial s} (a, sy + (1-s) d) \right| dt \right] ds \\
+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \int_{0}^{1} |(s-1)| \\
\times \left[ \int_{0}^{1} (t-1) t \left| \frac{\partial^2 f}{\partial t \partial s} (x, sy + (1-s) c) \right| dt \\
+ \int_{0}^{1} (t-1)(1-t) \left| \frac{\partial^2 f}{\partial t \partial s} (b, sy + (1-s) c) \right| dt \right] ds \\
+ \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \int_{0}^{1} |(1-s)| \\
\times \left[ \int_{0}^{1} (1-t) t \left| \frac{\partial^2 f}{\partial t \partial s} (x, sy + (1-s) d) \right| dt \\
+ \int_{0}^{1} (1-t)(1-t) \left| \frac{\partial^2 f}{\partial t \partial s} (b, sy + (1-s) d) \right| dt \right] ds.
\]
By computing these integrals, we obtain

\[ A + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) \, du \, dv \]

\[ \leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 |s-1| \left[ -\frac{1}{6} \frac{\partial^2 f}{\partial t \partial s} (x, sy + (1-s) c) \right] \, ds \]

\[ + \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 |1-s| \left[ -\frac{1}{6} \frac{\partial^2 f}{\partial t \partial s} (x, sy + (1-s) d) \right] \, ds \]

\[ + \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 |s-1| \left[ -\frac{1}{6} \frac{\partial^2 f}{\partial t \partial s} (b, sy + (1-s) c) \right] \, ds \]

\[ + \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 |1-s| \left[ -\frac{1}{6} \frac{\partial^2 f}{\partial t \partial s} (b, sy + (1-s) d) \right] \, ds. \]

Using co-ordinated convexity of \( \frac{\partial^2 f}{\partial t \partial s} \) again and computing all integrals, we obtain

\[ A + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) \, du \, dv \]

\[ \leq \frac{1}{9(b-a)(d-c)} \left[ \left( \frac{(x-a)^2 + (b-x)^2}{2} \right) \left( \frac{(y-c)^2 + (d-y)^2}{2} \right) \right] \left( \frac{\partial^2 f}{\partial t \partial s} (x,y) \right) \]

\[ + \frac{(x-a)^2(y-c)^2}{2} \left( \frac{\partial^2 f}{\partial t \partial s} (a,y) \right) \]

\[ + \frac{(b-x)^2(y-c)^2}{2} \left( \frac{\partial^2 f}{\partial t \partial s} (b,y) \right) \]

\[ + \frac{(y-c)^2(x-a)^2}{2} \left( \frac{\partial^2 f}{\partial t \partial s} (x,c) \right) \]

\[ + \frac{(d-y)^2(x-a)^2}{2} \left( \frac{\partial^2 f}{\partial t \partial s} (x,d) \right) \]

\[ + (x-a)^2(y-c)^2 \left( \frac{\partial^2 f}{\partial t \partial s} (a,c) \right) + (x-a)^2(d-y)^2 \left( \frac{\partial^2 f}{\partial t \partial s} (a,d) \right) \]

\[ + (b-x)^2(y-c)^2 \left( \frac{\partial^2 f}{\partial t \partial s} (b,c) \right) + (b-x)^2(d-y)^2 \left( \frac{\partial^2 f}{\partial t \partial s} (b,d) \right) \]

Which completes the proof. \( \square \)
Corollary 1. (1) Under the assumptions of Theorem 2, if we choose $x = a$, $y = c$, we obtain the following inequality:

$$\left| \frac{f(b, d)}{(b - a)(d - c)} - \frac{1}{d - c} \int_c^d f(b, v) \, dv \right|$$

$$\leq \frac{1}{2(b - a)(d - c)} \left[ \frac{\partial^2 f}{\partial t \partial s} \left( \frac{a + b + c + d}{2}, \frac{a + c + d}{2} \right) \right] + \frac{1}{18(b - a)(d - c)} \left( \frac{\partial^2 f}{\partial t \partial s} (a, c) + \frac{\partial^2 f}{\partial t \partial s} (b, c) \right) + \frac{1}{18(b - a)(d - c)} \left( \frac{\partial^2 f}{\partial t \partial s} (a, d) + \frac{\partial^2 f}{\partial t \partial s} (b, c) \right).$$

(2) Under the assumptions of Theorem 2, if we choose $x = b$, $y = d$, we obtain the following inequality:

$$\left| \frac{f(a, c)}{(b - a)(d - c)} - \frac{1}{d - c} \int_c^d f(a, v) \, dv \right|$$

$$\leq \frac{1}{36(b - a)(d - c)} \left[ \frac{\partial^2 f}{\partial t \partial s} (b, d) + \frac{1}{9(b - a)(d - c)} \left( \frac{\partial^2 f}{\partial t \partial s} (a, c) + 1 \right) \right] + \frac{1}{18(b - a)(d - c)} \left( \frac{\partial^2 f}{\partial t \partial s} (a, d) + \frac{1}{18(b - a)(d - c)} \left( \frac{\partial^2 f}{\partial t \partial s} (b, c) \right) \right).$$

(3) Under the assumptions of Theorem 2, if we choose $x = \frac{x + b}{2}$, $y = \frac{y + d}{2}$, we obtain the following inequality:

$$\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4(b - a)(d - c)} - \frac{1}{d - c} \int_c^d f(a, v) \, dv - \frac{1}{d - c} \int_c^d f(b, v) \, dv \right|$$

$$\leq \frac{1}{144(b - a)(d - c)} \left[ \frac{\partial^2 f}{\partial t \partial s} \left( \frac{a + b + c + d}{2}, \frac{a + c + d}{2} \right) \right] + \frac{1}{18(b - a)(d - c)} \left( \frac{\partial^2 f}{\partial t \partial s} (a, c) + \frac{\partial^2 f}{\partial t \partial s} (b, c) \right) + \frac{1}{18(b - a)(d - c)} \left( \frac{\partial^2 f}{\partial t \partial s} (a, d) + \frac{\partial^2 f}{\partial t \partial s} (b, c) \right).$$

Theorem 3. Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$ and $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$. If $\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|^q_{L(\Delta)}, q > 1$, is a convex function on the
coordinates on $\Delta$, then the following inequality holds:

\[
A + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(u, v) \, du \, dv \\
\leq \frac{1}{2^\frac{p}{q} \times \left( (x - a)^2 (y - c)^2 \right)} \left( \frac{\partial_x^2 f}{\partial t \partial s} (x, y)^q + \frac{\partial_x^2 f}{\partial t \partial s} (x, c)^q + \frac{\partial_x^2 f}{\partial t \partial s} (a, y)^q + \frac{\partial_x^2 f}{\partial t \partial s} (a, c)^q \right)^{\frac{1}{q}}
\]

where $p^{-1} + q^{-1} = 1$.

**Proof.** From Lemma [1] we have

\[
A + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(u, v) \, du \, dv \\
\leq \frac{(x - a)^2 (y - c)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 \left| (t - 1) (s - 1) \left| \frac{\partial_x^2 f}{\partial t \partial s} (tx + (1 - t) a, sy + (1 - s) c) \right| ds \right.
\]

\[
\left. + \frac{(x - a)^2 (d - y)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 \left| (t - 1) (1 - s) \left| \frac{\partial_x^2 f}{\partial t \partial s} (tx + (1 - t) a, sy + (1 - s) d) \right| ds \right. \]

\[
\left. + \frac{(b - x)^2 (y - c)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 \left| (1 - t) (s - 1) \left| \frac{\partial_x^2 f}{\partial t \partial s} (tx + (1 - t) b, sy + (1 - s) c) \right| ds \right. \]

\[
\left. + \frac{(b - x)^2 (d - y)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 \left| (1 - t) (1 - s) \left| \frac{\partial_x^2 f}{\partial t \partial s} (tx + (1 - t) b, sy + (1 - s) d) \right| ds \right. .
\]
By using the well known H"older inequality for double integrals, then one has:

\[
\left| A + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) \, du \, dv \right| \\
\leq \left( \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(t-1)(s-1)|^p \, ds \, dt \right) \right)^{\frac{1}{p}} \\
\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) a, sy + (1-s) c) \right|^q \, ds \, dt \right)^{\frac{1}{q}} \\
+ \left( \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(t-1)(s-1)|^p \, ds \, dt \right) \right)^{\frac{1}{p}} \\
\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) a, sy + (1-s) c) \right|^q \, ds \, dt \right)^{\frac{1}{q}} \\
+ \left( \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(1-t)(s-1)|^p \, ds \, dt \right) \right)^{\frac{1}{p}} \\
\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) b, sy + (1-s) c) \right|^q \, ds \, dt \right)^{\frac{1}{q}} \\
+ \left( \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(1-t)(1-s)|^p \, ds \, dt \right) \right)^{\frac{1}{p}} \\
\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) b, sy + (1-s) c) \right|^q \, ds \, dt \right)^{\frac{1}{q}}. 
\]

Since \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q \) is convex function on the co-ordinates on \( \Delta \), we know that for \( t \in [0,1] \)

\[
\left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) a, sy + (1-s) c) \right|^q \\
\leq t \left| \frac{\partial^2 f}{\partial t \partial s} (x, sy + (1-s) c) \right|^q + (1-t) \left| \frac{\partial^2 f}{\partial t \partial s} (a, sy + (1-s) c) \right|^q \\
\leq t \left( s \left| \frac{\partial^2 f}{\partial t \partial s} (x, y) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s} (x, c) \right|^q \right) + (1-t) \left( s \left| \frac{\partial^2 f}{\partial t \partial s} (a, y) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q \right)
\]
and by using the fact that
\[ \left( \int_0^1 \int_0^1 |(t - 1) (s - 1)|^p \, ds \, dt \right)^{\frac{1}{p}} = \frac{1}{(p + 1)^{\frac{1}{p}}} \]
we get
\[
\int_0^1 \int_0^1 \left( \frac{\partial^2 f}{\partial t \partial s} (tx + (1 - t) a, sy + (1 - s) c) \right)^q \, ds \, dt \leq \int_0^1 \int_0^1 \left( t^q + (1 - t)^q \right) \left( \frac{\partial^2 f}{\partial t \partial s} (x, y) \right)^q \, ds \, dt
\]
\[\leq \int_0^1 \int_0^1 \left( \frac{\partial^2 f}{\partial t \partial s} (x, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (x, c) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (a, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (a, c) \right)^q \, ds \, dt
\]
and similarly, we get
\[
\int_0^1 \int_0^1 \left( \frac{\partial^2 f}{\partial t \partial s} (tx + (1 - t) a, sy + (1 - s) d) \right)^q \, ds \, dt \leq \int_0^1 \int_0^1 \left( \frac{\partial^2 f}{\partial t \partial s} (x, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (x, d) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (a, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (a, d) \right)^q \, ds \, dt
\]
\[\leq \int_0^1 \int_0^1 \left( \frac{\partial^2 f}{\partial t \partial s} (x, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (x, c) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (a, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (a, c) \right)^q \, ds \, dt
\]
\[
\int_0^1 \int_0^1 \left( \frac{\partial^2 f}{\partial t \partial s} (tx + (1 - t) b, sy + (1 - s) c) \right)^q \, ds \, dt \leq \int_0^1 \int_0^1 \left( \frac{\partial^2 f}{\partial t \partial s} (x, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (x, c) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (b, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (b, c) \right)^q \, ds \, dt
\]
\[\leq \int_0^1 \int_0^1 \left( \frac{\partial^2 f}{\partial t \partial s} (x, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (x, d) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (b, y) \right)^q + \left( \frac{\partial^2 f}{\partial t \partial s} (b, d) \right)^q \, ds \, dt
\]
Then by using the inequalities (2.1), (2.4) in (??), we obtain

\[
\left| A + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) \, du \, dv \right| \\
\leq \frac{1}{(p+1)^{\frac{1}{2}}} \times \\
\left\{ \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \left( \frac{\partial^2 f}{\partial t \partial s}(x,y) \right)^q + \frac{\partial^2 f}{\partial t \partial s}(x,c)^q + \frac{\partial^2 f}{\partial t \partial s}(a,y)^q + \frac{\partial^2 f}{\partial t \partial s}(a,c)^q \right\}^{\frac{1}{q}}
\]

\[
+ \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \left( \frac{\partial^2 f}{\partial t \partial s}(x,y) \right)^q + \frac{\partial^2 f}{\partial t \partial s}(x,d)^q + \frac{\partial^2 f}{\partial t \partial s}(a,y)^q + \frac{\partial^2 f}{\partial t \partial s}(a,d)^q \right\}^{\frac{1}{q}}
\]

\[
+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \left( \frac{\partial^2 f}{\partial t \partial s}(x,y) \right)^q + \frac{\partial^2 f}{\partial t \partial s}(x,c)^q + \frac{\partial^2 f}{\partial t \partial s}(b,y)^q + \frac{\partial^2 f}{\partial t \partial s}(b,c)^q \right\}^{\frac{1}{q}}
\]

\[
+ \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \left( \frac{\partial^2 f}{\partial t \partial s}(x,y) \right)^q + \frac{\partial^2 f}{\partial t \partial s}(x,d)^q + \frac{\partial^2 f}{\partial t \partial s}(b,y)^q + \frac{\partial^2 f}{\partial t \partial s}(b,d)^q \right\}^{\frac{1}{q}}
\}

which completes the proof. \( \square \)

**Corollary 2.** (1) Under the assumptions of Theorem 5 if we choose \( x = a, y = c \), or \( x = b, y = d \), we obtain the following inequality:

\[
\frac{1}{(b-a)(d-c)} \left| f(b,d) - (b-a) \int_c^d f(b,v) \, dv - (d-c) \int_a^b f(u,d) \, du + \int_a^b \int_c^d f(u,v) \, du \, dv \right| \\
\leq \frac{1}{(b-a)(d-c) \left( p+1 \right)^{\frac{1}{2}}} \left( \frac{\partial^2 f}{\partial t \partial s}(a,c)^q + \frac{\partial^2 f}{\partial t \partial s}(a,d)^q + \frac{\partial^2 f}{\partial t \partial s}(b,c)^q + \frac{\partial^2 f}{\partial t \partial s}(b,d)^q \right)^{\frac{1}{q}}
\]

(2) Under the assumptions of Theorem 5 if we choose \( x = b, y = d \), we obtain the following inequality:

\[
\frac{1}{(b-a)(d-c)} \left| f(a,c) - (b-a) \int_c^d f(a,v) \, dv - (d-c) \int_a^b f(u,d) \, du + \int_a^b \int_c^d f(u,v) \, du \, dv \right| \\
\leq \frac{1}{(b-a)(d-c) \left( p+1 \right)^{\frac{1}{2}}} \left( \frac{\partial^2 f}{\partial t \partial s}(b,d)^q + \frac{\partial^2 f}{\partial t \partial s}(b,c)^q + \frac{\partial^2 f}{\partial t \partial s}(a,d)^q + \frac{\partial^2 f}{\partial t \partial s}(a,c)^q \right)^{\frac{1}{q}}
\]

(3) Under the assumptions of Theorem 5 if we choose \( x = a, y = d \), we obtain the following inequality:

\[
\frac{1}{(b-a)(d-c)} \left| f(b,c) - (b-a) \int_c^d f(b,v) \, dv - (d-c) \int_a^b f(u,c) \, du + \int_a^b \int_c^d f(u,v) \, du \, dv \right| \\
\leq \frac{1}{(b-a)(d-c) \left( p+1 \right)^{\frac{1}{2}}} \left( \frac{\partial^2 f}{\partial t \partial s}(a,d)^q + \frac{\partial^2 f}{\partial t \partial s}(a,c)^q + \frac{\partial^2 f}{\partial t \partial s}(b,d)^q + \frac{\partial^2 f}{\partial t \partial s}(b,c)^q \right)^{\frac{1}{q}}
\]
(4) Under the assumptions of Theorem 3, if we choose \(x = b, y = c\), we obtain the following inequality:

\[
\frac{1}{(b-a)(d-c)} \left| f(a,d) - (b-a) \int_{c}^{d} f(a,v) dv - (d-c) \int_{a}^{b} f(u,d) du + \int_{a}^{b} \int_{c}^{d} f(u,v) dvdu \right|
\]

\[
\leq \frac{1}{(b-a)(d-c)} \left( \frac{1}{2(d-c)} \int_{c}^{d} f(a,v) dv - \frac{1}{2(d-c)} \int_{c}^{d} f(b,v) dv \right)
\]

\[
\int_{a}^{b} f(u,d) du - \frac{1}{2(b-a)} \int_{a}^{b} f(u,c) du + \frac{1}{4(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(u,v) dvdu
\]

\[
\leq \frac{1}{(b-a)(d-c) (p+1)^{2} 2^{p}} \left\{ \left( \frac{\partial^{2} f}{\partial t \partial s} \left( \frac{a+b+c+d}{2}, \frac{a+b}{2}, \frac{a+b+c+d}{2}, \frac{a+b}{2} \right) \right)^{q} + \left( \frac{\partial^{2} f}{\partial t \partial s} \left( \frac{a+b+c+d}{2}, \frac{a+b}{2}, \frac{a+b+c+d}{2}, \frac{a+b}{2} \right) \right)^{q} + \left( \frac{\partial^{2} f}{\partial t \partial s} \left( \frac{a+b+c+d}{2}, \frac{a+b}{2}, \frac{a+b+c+d}{2}, \frac{a+b}{2} \right) \right)^{q} + \left( \frac{\partial^{2} f}{\partial t \partial s} \left( \frac{a+b+c+d}{2}, \frac{a+b}{2}, \frac{a+b+c+d}{2}, \frac{a+b}{2} \right) \right)^{q} \right\}.
\]

(5) Under the assumptions of Theorem 3, if we choose \(x = \frac{a+b}{2}, y = \frac{a+b}{2}\), we obtain the following inequality:

\[
\left( \frac{\partial^{2} f}{\partial t \partial s} \left( \frac{a+b+c+d}{2}, \frac{a+b}{2}, \frac{a+b+c+d}{2}, \frac{a+b}{2} \right) \right)^{q} + \left( \frac{\partial^{2} f}{\partial t \partial s} \left( \frac{a+b+c+d}{2}, \frac{a+b}{2}, \frac{a+b+c+d}{2}, \frac{a+b}{2} \right) \right)^{q} + \left( \frac{\partial^{2} f}{\partial t \partial s} \left( \frac{a+b+c+d}{2}, \frac{a+b}{2}, \frac{a+b+c+d}{2}, \frac{a+b}{2} \right) \right)^{q} + \left( \frac{\partial^{2} f}{\partial t \partial s} \left( \frac{a+b+c+d}{2}, \frac{a+b}{2}, \frac{a+b+c+d}{2}, \frac{a+b}{2} \right) \right)^{q} \right\}.
\]

**Theorem 4.** Let \(f : \Delta = [a,b] \times [c,d] \to \mathbb{R}\) be a partial differentiable mapping on \(\Delta = [a,b] \times [c,d]\) and \(\frac{\partial^{2} f}{\partial t \partial s} \in L(\Delta)\). If \(\left| \frac{\partial^{2} f}{\partial t \partial s} \right|^{q} \geq 1\), is a convex function on the
co-ordinates on $\Delta$, then the following inequality holds:

\[
A + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(u, v) 
\leq \left( \frac{1}{4} \right)^{1 - \frac{1}{q}} \left\{ K \left( \frac{1}{36} \frac{\partial^2 f}{\partial t \partial s}(x, y) \right)^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s}(x, c) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s}(a, y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q \right\}^{\frac{1}{q}} \\
+ L \left( \frac{1}{36} \frac{\partial^2 f}{\partial t \partial s}(x, y) \right)^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s}(x, d) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s}(a, y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q \right\}^{\frac{1}{q}}
\]

where

\[
K = \frac{(x - a)^2 (y - c)^2}{(b - a)(d - c)} \\
L = \frac{(x - a)^2 (d - y)^2}{(b - a)(d - c)} \\
M = \frac{(b - x)^2 (y - c)^2}{(b - a)(d - c)} \\
N = \frac{(b - x)^2 (d - y)^2}{(b - a)(d - c)}
\]

Proof. From Lemma 1 we have

\[
A + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(u, v) \\
\leq \frac{(x - a)^2 (y - c)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 |(t - 1)(s - 1)| \left| \frac{\partial^2 f}{\partial t \partial s}(tx + (1 - t)a, sy + (1 - s)c) \right| dsdt \\
+ \frac{(x - a)^2 (d - y)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 |(t - 1)(1 - s)| \left| \frac{\partial^2 f}{\partial t \partial s}(tx + (1 - t)a, sy + (1 - s)d) \right| dsdt \\
+ \frac{(b - x)^2 (y - c)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 |(1 - t)(s - 1)| \left| \frac{\partial^2 f}{\partial t \partial s}(tx + (1 - t)b, sy + (1 - s)c) \right| dsdt \\
+ \frac{(b - x)^2 (d - y)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 |(1 - t)(1 - s)| \left| \frac{\partial^2 f}{\partial t \partial s}(tx + (1 - t)b, sy + (1 - s)d) \right| dsdt.
\]
By using the well known Power mean inequality for double integrals, then one has:

\[
(2.5) \quad \left| A + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) \, du \, dv \right| \\
\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(t-1)(s-1)| \, ds \, dt \right)^{1-\frac{1}{q}} \\
\times \left( \int_0^1 \int_0^1 |(t-1)(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)a, sy + (1-s)c) \right|^q \, ds \, dt \right)^{\frac{1}{q}} \\
+ \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(t-1)(s-1)| \, ds \, dt \right)^{1-\frac{1}{q}} \\
\times \left( \int_0^1 \int_0^1 |(t-1)(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)a, sy + (1-s)d) \right|^q \, ds \, dt \right)^{\frac{1}{q}} \\
+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(t-1)(s-1)| \, ds \, dt \right)^{1-\frac{1}{q}} \\
\times \left( \int_0^1 \int_0^1 |(t-1)(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)b, sy + (1-s)c) \right|^q \, ds \, dt \right)^{\frac{1}{q}} \\
+ \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(t-1)(s-1)| \, ds \, dt \right)^{1-\frac{1}{q}} \\
\times \left( \int_0^1 \int_0^1 |(t-1)(s-1)| \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)b, sy + (1-s)d) \right|^q \, ds \, dt \right)^{\frac{1}{q}}.
\]

Since \( \left| \frac{\partial^2 f}{\partial t \partial s} \right|^q \) is convex function on the co-ordinates on \( \Delta \), we know that for \( t, s \in [0, 1] \)

\[
\left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)a, sy + (1-s)c) \right|^q \\
\leq t \left| \frac{\partial^2 f}{\partial t \partial s} (x, sy + (1-s)c) \right|^q + (1-t) \left| \frac{\partial^2 f}{\partial t \partial s} (a, sy + (1-s)c) \right|^q \\
\leq t \left( s \left| \frac{\partial^2 f}{\partial t \partial s} (x, y) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s} (x, c) \right|^q \right) + (1-t) \left( s \left| \frac{\partial^2 f}{\partial t \partial s} (a, y) \right|^q + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q \right)
\]
and by using the fact that

\[
\left( \int_0^1 \int_0^1 |(t-1)(s-1)| \, ds \, dt \right)^{1-\frac{1}{q}} = \left( \frac{1}{4} \right)^{1-\frac{1}{q}}
\]

we get

\[
\left( \int_0^1 \int_0^1 |(t-1)(s-1)| \left\{ ts \left| \frac{\partial^2 f}{\partial t \partial s} (x, y) \right|^q + t(1-s) \left| \frac{\partial^2 f}{\partial t \partial s} (x, c) \right|^q \right\} \, ds \, dt \right)^{\frac{1}{q}}
\]

\[
\leq \left( \int_0^1 \int_0^1 |(t-1)(s-1)| \left\{ t \left( \frac{\partial^2 f}{\partial t \partial s} (a, y) \right|^q + (1-t) \left( 1-s \right) \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q \right\} \, ds \, dt \right)^{\frac{1}{q}}
\]

\[
= \left( \frac{1}{36} \left| \frac{\partial^2 f}{\partial t \partial s} (x, y) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (x, c) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (a, y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right|^q \right)^{\frac{1}{q}}
\]

and similarly, we get

\[
\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)a, sy + (1-s)c) \right|^q \, ds \, dt \right)^{\frac{1}{q}}
\]

\[
\leq \left( \frac{1}{36} \left| \frac{\partial^2 f}{\partial t \partial s} (x, y) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (x, d) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (a, y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right|^q \right)^{\frac{1}{q}},
\]

\[
\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)b, sy + (1-s)c) \right|^q \, ds \, dt \right)^{\frac{1}{q}}
\]

\[
\leq \left( \frac{1}{36} \left| \frac{\partial^2 f}{\partial t \partial s} (x, y) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (x, c) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (b, y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right|^q \right)^{\frac{1}{q}},
\]

\[
\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t)b, sy + (1-s)d) \right|^q \, ds \, dt \right)^{\frac{1}{q}}
\]

\[
\leq \left( \frac{1}{36} \left| \frac{\partial^2 f}{\partial t \partial s} (x, y) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (x, d) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (b, y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right|^q \right)^{\frac{1}{q}}.
\]
Then by using the inequalities (2.6)-(2.9) in (2.3), we obtain

\[
\left| A + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(u,v)\, du \, dv \right| \\
\leq \left( \frac{1}{4} \right)^{1 - \frac{1}{q}} \left\{ K \left( \frac{1}{36} \left| \frac{\partial^2 f}{\partial t \partial s} (x,y) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (x,c) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (a,y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s} (a,c) \right|^q \right)^{\frac{1}{q}} \\
+ L \left( \frac{1}{36} \left| \frac{\partial^2 f}{\partial t \partial s} (x,y) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (x,d) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (a,y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s} (a,d) \right|^q \right)^{\frac{1}{q}} \\
+ M \left( \frac{1}{36} \left| \frac{\partial^2 f}{\partial t \partial s} (x,y) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (x,c) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (b,y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s} (b,c) \right|^q \right)^{\frac{1}{q}} \\
+ N \left( \frac{1}{36} \left| \frac{\partial^2 f}{\partial t \partial s} (x,y) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (x,d) \right|^q + \frac{1}{18} \left| \frac{\partial^2 f}{\partial t \partial s} (b,y) \right|^q + \frac{1}{9} \left| \frac{\partial^2 f}{\partial t \partial s} (b,d) \right|^q \right)^{\frac{1}{q}} \right\}
\]

which completes the proof. □

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