Ehrenfest scheme for $P$-$V$ criticality of the $d$-dimensional AdS black holes surrounded by perfect fluid in Rastall theory

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Abstract

We discuss the $d$-dimensional anti-de-Sitter (AdS) black holes surrounded by perfect fluid in the Rastall theory of gravity characterized by its mass ($M$), the field structure parameter ($N_s$), the Rastall parameter ($\psi$), and the cosmological constant ($\Lambda$). We derive the quantities like the mass, and the Hawking temperature of the black holes and demonstrate the effects caused by the various parameters. We investigate the thermodynamics for $P$-$V$ criticality and phase transition in the extended phase space of the AdS black holes by considering the cosmological constant as pressure and its conjugate quantity as thermodynamic volume. We complete the analogy of this system with the liquid-gas system and study its critical point, which occurs at the point of divergence of the specific heat at constant pressure ($C_P$), volume expansion coefficient ($\alpha$) and isothermal compressibility coefficient ($\kappa_T$). Using these expressions we calculate the Ehrenfest equations and carry out an analytical check at the critical points of $P$-$V$ criticality and show that the results resemble with the nature of liquid-gas phase transition at the critical point. These considerations lead us to understand the comparison of AdS black holes and liquid-gas systems which perfectly matches with the behavior of van der Waals (vdWs) gas.

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I. INTRODUCTION

The Einstein’s general relativity (GR) has successfully described many predictions within its regime and has been the most popular and successful theory of gravity. In the framework of GR, the geometry and matter fields are coupled minimally which results in the covariant conservation relation of the energy-momentum tensor (EMT). It has been shown that when the non-minimal coupling of the geometry and matter fields occurs, they get affected by their mutual changes [1–5], and hence the covariant conservation of the matter EMT may be violated [6, 7]. The idea of the covariant conservation based on spacetime symmetries has been implemented only in the Minkowski flat or weak field regime of gravity. However, in the strong domain of gravity the actual nature of the spacetime geometry and the covariant conservation relation, is still debated. Taking advantage of this fact, Rastall [6, 7] proposed a phenomenological model where the covariant conservation of the EMT would be non-vanishing and has the form: $T^\mu_\nu;_\mu = \lambda R^\nu$, where $R$ is the Ricciscalar, and $\lambda$ is the Rastall coupling parameter. Here, the parameter $\lambda$ measures the potential deviations of Rastall theory from GR and shows a tendency of curvature-matter coupling in a non-minimal way. Interestingly, all eletrovacuum solutions to Einstein’s GR are also the solutions in Rastall gravity and asymptotically they approach the Mankowski spacetime. However, all the non-vacuum solutions in Rastall gravity contain Rastall parameter and are significantly different from corresponding solutions in GR, and thereby making the Rastall gravity aesthetically rich [8].

In the recent years, the Rastall theory has attracted great attention and a rich diversified research dedicated to it are available in literature including on some phenomenological results, related to both astrophysical [8–12] and cosmological consequences [13–19]. The static, cylindrically symmetric black hole solutions to the Rastall gravity coupled to the $U(1)$ Abelian-Higgs model that takes into account the quantum effects in curved spacetime has been characterized in a phenomenological way [20]. Nevertheless, recently many works on the various black hole solutions have been investigated based on Rastall theory. Among them include the spherically symmetric black hole solutions of Rastall gravity [8, 12], the rotating black holes [21, 22], their shadow properties [23] and also the thermodynamics and other theoretical aspects of black holes [24–26]. Besides, some works comparing the Rastall theory with standard GR have been also discussed [27, 28]. The essence of a perfect dark
energy fluid called the quintessence has been included in the black hole solution. Following Kiselev [29], the black hole solutions surrounded by the quintessence field have been generalized to higher dimensions in GR [30] as well as in other theories of gravity [31]. The quintessence black hole solutions have been also extended to Rastall theory [8]. Recently, a \( d \)-dimensional solution to Rastall theory in presence of perfect fluid has been obtained [32] and extended to the charged anti-de Sitter (AdS) spaces [33].

The purpose of this paper is an analytical study of the extended phase space thermodynamics of the \( d \)-dimensional AdS black holes in Rastall theory and also their \( P-V \) criticality. The idea of AdS black holes dated back to the pioneering work due to Hawking and Page [34], describing a first-order phase transition between the Schwarzschild AdS black holes and the thermal AdS spaces familiarly known as Hawking-Page phase transition. Since then, people have found continued interests in the study of the phase transition of AdS black holes. The black holes in AdS spacetimes seem to be a real thermodynamic system as it contains the pressure, volume and temperature which have been important in the study of the thermodynamic phenomena and the rich phase structure, such as van der Waals (vdWs) phase transition, \( P-V \) criticality, reentrant phase transitions, triple point, isolated critical point, and superfluidity [35–45]. These considerations enhance us to study the AdS black holes as an extended thermodynamic system. On the other hand, the \( d \)-dimensional charged AdS black holes also exhibit a vdWs phase transition [38–45]. In the reduced parameter space, the critical phenomena were also found to be charge-independent [38–45]. The critical phenomena of the AdS black holes have also been extended for the AdS black holes in some modified gravity theories, e.g., massive gravity theory [46], power-law Maxwell field [47], Born-Infeld theory [48], and also to the regular AdS black holes [49]. Therefore, it will be interesting to study the thermodynamics phase transition and the \( P-V \) criticality of the \( d \)-dimensional AdS black holes in the presence of perfect fluid in Rastall gravity. The effect of the Rastall parameter in the thermodynamic quantities will be rather important since it contributes significantly to them.

Our paper is organized as follows. In the next section, we briefly discuss the \( d \)-dimensional AdS black holes in the presence of a perfect fluid in the Rastall theory. In Sec. III, we evaluate the thermodynamic quantities of the black holes and find the expressions for thermodynamic pressure and volume as thermodynamic variables. In Sec. IV, we have used the classical Ehrenfest equations to analytically check the AdS black holes at the critical points.
of $P$-$V$ criticality.

II. $d$-DIMENSIONAL ADS BLACK HOLES SPACETIME SURROUNDED BY PERFECT FLUID

The non-conservation of EMT in the strong gravity regime as proposed by Rastall [6, 7] leads to the choice that $T_{\mu\nu;\mu} = \lambda R^{\nu}$, where $R$ is the Ricciscalar and $\lambda$ is the Rastall parameter. This assumption leads to modify the Einstein’s field equations which can be written as

$$G_{\mu\nu} = \kappa \left( T_{\mu\nu} - \lambda g_{\mu\nu} R \right)$$

(1)

where $\kappa$ is a coupling constant related to the Newton’s gravitational constant. If one includes the negative cosmological constant $\Lambda$, the field equations (1) can be recast as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \left( T_{\mu\nu} - \lambda g_{\mu\nu} R \right).$$

(2)

The trace of the Eq. (2) leads to have the following form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \tilde{T}_{\mu\nu},$$

(3)

where $\tilde{T}_{\mu\nu}$ is the effective EMT which has the form

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{\left( \psi T - \tilde{\Lambda}d \right)}{\left( \psi - 1/2 \right)d + 1} g_{\mu\nu}$$

(4)

in which $d$ is the spacetime dimension, $\tilde{\Lambda} = \Lambda/\kappa$ and $T_{\mu\nu}$ is the EMT for surrounding quintessence field and $T$ is its trace. Here and henceforth we consider $\psi = \kappa \lambda$. The EMT $T_{\mu\nu}$ in $d$-dimensional spacetime [30, 31] of the surrounding quintessence field is given as

$$T^t_t = T^r_r = -\rho_s,$$

(5)

and

$$T^{\theta_1\theta_1} = T^{\theta_2\theta_2} = \ldots = T^{\theta_{d-2}\theta_{d-2}}$$

$$= \frac{\rho_s}{d-2} \left[ (d-1) \omega_s + 1 \right]$$

(6)
With these expressions of $T^{\mu\nu}$, the components of effective EMT $\tilde{T}^{\mu\nu}$ read as

$$
\tilde{T}^t_t = \tilde{T}^r_r = -\frac{\left[ (2\psi - 1)(d - 1)^2 (1 + \omega_s) - (d - 2)^2 \right] \rho_s - 2\bar{\Lambda}d(d - 2)}{(d - 2)((2\psi - 1)d + 2)},
$$

$$
T^{\theta_1\theta_1} = T^{\theta_2\theta_2} = \ldots = T^{\theta_{d-2}\theta_{d-2}},
$$

$$
= \frac{\left[ (2\psi - 1)(d - 1)(1 + \omega_s) + 1 \right] \rho_s - 2\bar{\Lambda}d(d - 2)}{(d - 2)((2\psi - 1)d + 2)}.
$$

(7)

The black holes solutions in a $d$-dimensional static spherically symmetric spacetime surrounded by a perfect fluid have been analyzed [32] and then extended to charged AdS spacetimes [33]. However, we briefly discuss the $d$-dimensional static spherically symmetric black holes surrounded by perfect fluid in AdS spacetime [32, 33]. The $d$-dimensional spherically symmetric Schwarzschild-Tangherlini-like line element reads

$$
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2_{d-2},
$$

(8)

where $d\Omega^2_{d-2}$ is the line element of the $(d - 2)$-dimensional unit sphere given by

$$
d\Omega^2_{d-2} = d\theta_1^2 + \sum_{i=2}^{d-2} \left[ \prod_{j=2}^{i} \sin^2 \theta_{j-1} \right] d\theta_i^2.
$$

(9)

The field equations (3) together with the metric (8) lead to the following independent equations:

$$
\frac{d - 2}{2r^2} \left[ rf' + (d - 3)(f - 1) \right] + \Lambda = \kappa \tilde{T}^t_t
$$

(10)

$$
\frac{1}{2r^2} \left[ r^2f'' + 2(d - 3)rf' + (d - 3)(d - 4)(f - 1) \right] + \Lambda = \kappa \tilde{T}^{\theta_{d-2}\theta_{d-2}}
$$

(11)

From Eqs. (7)-(11) we get a master differential equation to solve $f$:

$$
\left[ 1 - 2\psi \frac{(d - 1)(1 + \omega_s)}{d - 2} \right] r^2f'' + [(d - 1)\omega_s + (2d - 5) - 4\psi(d - 1)(1 + \omega_s)] rf' + (d - 2) [(d - 1)\omega_s + d - 3 - 2\psi(d - 1)(1 + \omega_s)] (f - 1)
$$

$$
= \frac{2(d - 1)(1 + \omega_s)}{(d - 2)} \Lambda r^2
$$

(12)
Eq. (12) admits the solution \[ f(r) = 1 - \frac{m}{r^{d-3}} - \frac{N_s}{r^\xi} + \frac{r^2}{l^2}, \] (13)

where

\[ \xi = \frac{(d-3) + (d-1) \omega_s - 2\psi (d-1) (1 + \omega_s)}{1 - 2\psi \left(\frac{d-1}{d-2}\right) (1 + \omega_s)} \] (14)

and

\[ \frac{1}{l^2} = -\frac{2\Lambda}{(d-1)(d(1-2\psi) - 2)}, \] (15)

where \( l \) is the curvature radius. The energy density \( \rho_s \) can be obtained when one put \( f(r) \) from Eq. (13) in Eq. (11), which lead to have the following form

\[ \rho_s(r) = -\frac{1}{2r} \frac{W_s N_s}{r^{\frac{(d-1)\omega_s - 2\psi (d-1)(1 + \omega_s)}{1 - 2\psi \left(\frac{d-1}{d-2}\right) (1 + \omega_s)}}}, \] (16)

where

\[ W_s = \frac{[(d-1)\omega_s - 2\psi \left(\frac{d-1}{d-2}\right) (1 + \omega_s)] [2\psi d - (d - 2)]}{[1 - 2\psi \left(\frac{d-1}{d-2}\right) (1 + \omega_s)]^2}, \] (17)

where \( m \) and \( N_s \) are two integration constants, respectively, related to the the black hole mass and density of the surrounding perfect field. The parameter \( m \) related to the Arnowitt-Deser-Misner (ADM) mass \( M \) of the black hole as

\[ M = \frac{(d-2)}{16\pi} \Omega_{d-2} m, \quad \Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d}{2}\right)} \] (18)

The spacetime (8) depends not only on the dimension \( d \), but also on the state parameter \( \omega_s \) of the perfect fluid. In the case when \( d = 4 \) and \( 1/l^2 = 0 \), the metric solution (13) reproduces [8] and for \( \psi = 0 \), the metric function (13) reduces to the \( d \)-dimensional AdS black holes in quintessence background, which in addition for \( 1/l^2 = 0 \) reproduces the \( d \)-dimensional quintessence black holes [30]. In the limit \( \omega_s = -1 \) and \( 1/l^2 = 0 \), the spacetime (8) for \( N_s > 0 \) becomes

\[ ds^2 = -\left[1 - \frac{m}{r^{d-3}} - N_s r^2\right] dt^2 + \frac{dr^2}{1 - \frac{m}{r^{d-3}} - N_s r^2} + r^2 d\Omega_{d-2}^2, \] (19)
which corresponds to the $d$-dimensional Schwarzschild-Tangherlini-de Sitter black holes [50]. Interestingly, Eq. (19) does not contain the factor $\psi$ and thus the Rastall and Einstein theories merge for $\omega_s = -1$. For $N_s < 0$ and $\omega_s = -1$, the metric (8) becomes

$$ds^2 = -\left[1 - \frac{m}{r^{d-3}} - \Lambda_{\text{eff}} r^2 \right] dt^2 + \frac{dr^2}{1 - \frac{m}{r^{d-3}} - \Lambda_{\text{eff}} r^2} + r^2 d\Omega_{d-2}^2,$$

where $\Lambda_{\text{eff}} = (N_s + 1/l^2)$ and the metric reduces to the $d$-dimensional Schwarzschild-Tangherlini-AdS black holes.

III. THERMODYNAMICS

In this section, we would like to briefly discuss the thermodynamics of the $d$-dimensional AdS black holes surrounded by perfect fluid. The metric element (8) is invariant under the transformation $t \rightarrow -t$ and correspondingly we have a time-like Killing vector of the form $\xi^\mu = \delta^\mu_t$. The Killing vector $\xi^\mu$ is a null generator of the event horizon, i.e., $\xi^\mu \xi_\mu = 0$, which in turn gives $g_{tt}|_{r=r_+} = g^{rr}|_{r=r_+} = f(r_+) = 0$. This relation is used to determine the event horizon radius which has a complicated structure and cannot be solved analytically. The ADM mass $M$ is obtained by solving $f(r_+) = 0$ in terms of the event horizon $r = r_+$, which reads

$$M = \frac{(d-2)\Omega_{d-2}}{16\pi} \left[1 - \frac{N_s}{r_+^{d-3}} + \frac{r_+^{d-3}}{l^2} \right] r_+^{d-3}.$$

The surface gravity is defined as $\kappa = \sqrt{-\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu/2}$ and is related to the temperature $T$ through $T = \kappa/2\pi$. Thus the temperature of the black holes in $d$-dimensional spacetime reads

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi} \left[\frac{(d-3)}{r_+} + (d-1)\frac{r_+}{l^2} + (3 - d + \xi)\frac{N_s}{r_+^{d-3}} \right].$$

The entropy can be calculated as

$$S = \int T^{-1} \left(\frac{\partial M}{\partial r_+}\right) dr_+ = \frac{\Omega_{d-2} r_+^{d-2}}{4}.$$

In extended phase space one can identify the cosmological constant as thermodynamic pressure which for the metric (13) reads

$$P = -\frac{\Lambda}{8\pi} - \frac{(d-1)(d(1-2\psi) - 2)}{16\pi l^2}.$$
The expression for mass (21) in terms of pressure (24) can now be expressed as

\[
M = \frac{(d - 2)\Omega d-2}{16\pi} \left[1 - \frac{N_s}{r_+^d} + \frac{16\pi P}{(d-1)(d(1 - 2\psi) - 2)} r_+^2 \right] r_+^{d-3} \tag{25}
\]

We observe that the entropy, Eq. (23) does not contain the contribution from perfect fluid as well as the Rastall parameter \(\psi\). However, the thermodynamic pressure, Eq. (24) is independent of the parameter \(N_s\) but not of the Rastall parameter \(\psi\) and hence affects the thermodynamics of the black holes in the extended phase space.

It is suggested that the first law of black hole mechanics must include the variation of the cosmological constant as thermodynamic variable when written for AdS black holes. This treatment leads us to consider the cosmological constant as pressure and its conjugate quantity as thermodynamic volume in the extended phase space of black hole thermodynamics. The thermodynamics in such an extended phase space have been studied in various theories of gravity. To confront the idea of phase transition of the black hole system with the usual classical van der Waals system such treatments have been provided to investigate the critical phenomena and related pathologies [35–45, 51–57]. Therefore, the first law of black hole thermodynamics in the extended phase space reads

\[
dM = TdS + VdP + \Theta_s dN_s, \tag{26}
\]

where \(S\) is the entropy and \(P\) is related to the cosmological constant of the black holes defined in Eqs. (23) and (24). Here we treat \(\Theta_s\) as a generalized force conjugate to the surrounding perfect fluid structure parameter \(N_s\) and is introduced to make the the first law to be consistent with the Smarr-Gibbs-Duhem relation. Thus on using Eqs. (25) and (26), we obtain \(\Theta_s\) having the following form [30]

\[
\Theta_s = \left(\frac{\partial M}{\partial N_s}\right)_{S,P,Q} = -\frac{(d - 2) (2\pi)^{d/2}}{16\pi \Gamma^{d-1/2}} \frac{1}{r_+^d}. \tag{27}
\]

The thermodynamic volume term from Eq. (26) reads

\[
V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,N_s} \tag{28}
\]

Using Eqs. (24) and (25), the Eq. (28) is obtained as

\[
V = \frac{(d - 2)\Omega d-2r_+^{d-1}}{(d - 1)(d(1 - 2\psi) - 2)} \tag{29}
\]
Therefore, the corresponding Smarr-Gibbs-Duhem formula \([38, 39]\) when using the Euler’s theorem is obtained as

\[
\frac{d - 3}{d - 2} M = TS - \frac{2}{d - 2} VP + \frac{\xi}{d - 2} \Theta_s N_s. \tag{30}
\]

It is clear that the mass depends on the state parameter \(\omega_s\) of the perfect fluid as it is contained in \(\xi\). In the limit \(\omega_s \to -1\), the last term in the right hand side of Eq. (30) becomes \(-\frac{2}{d-2} \Theta_s N_s\). Setting \(N_s = 1/l^2\), we have \(\Theta_l = \frac{\partial M}{\partial N_s} = -\frac{2}{d-2} \Theta_s\) and \(-\frac{2}{d-2} \Theta_s N_s = \frac{1}{d-2} \Theta_l l\). The fact that if \(\Lambda\) and hence \(P\) is treated truly as a constant, the second last term in (26) is essentially zero. However, it is not zero always and can have physical consequences. It is of common belief that in the inflationary models, \(\Lambda\) can be identified as a variable quantity \([51, 54]\).

**IV. P-V CRITICALITY AND ANALYTICAL CHECK OF CLASSICAL EHRENFEUST EQUATIONS IN THE EXTENDED PHASE SPACE**

Next, we study the critical phenomena of the black hole under thermodynamic equilibrium. The Hawking temperature (22) when expressed in terms of pressure (24) is written as

\[
T = \frac{1}{4\pi} \left( \frac{(d - 3)}{r_+} + \frac{16\pi r_+ P}{(d(1 - 2\psi) - 2)} + (3 - d + \xi) \frac{N_s}{r_+^{1+\xi}} \right). \tag{31}
\]

Before expressing the Eq. (31) for pressure we need to introduce the thermodynamic volume of the vdW-like fluid. In this way one can, in general, express the equation of state relating its pressure to other thermodynamic quantities e.g. temperature, volume and other related parameters specifying the AdS black holes in arbitrary dimensions in the presence of perfect fluid in Rastall theory such that \(P = P(T, V, Q, N_s)\). One can thus write the fluid volume as discussed in \([35, 36]\) in the following form

\[
v = \frac{4\ell_P^{d-2}}{d - 2} r_+ = \frac{4\ell_P^{d-2}}{d - 2} \left( \frac{(d - 1)(d(1 - 2\psi) - 2)V}{(d - 2)\Omega_{d-2}} \right)^{\frac{1}{d-1}}. \tag{32}
\]

In geometric units \(\ell_P = 1\) and we have

\[
r_+ = \frac{d - 2}{4} v. \tag{33}
\]
Thus replacing \( r_+ \) as defined in Eq.(33) we obtain the equation of state from Eq. (31) as follows

\[
P = \frac{(d-1-2\psi) - 2) T}{d-2} \frac{1}{v} \left( \frac{d}{d-3}(d-1-2\psi) - 2 \right) - \frac{(3-d+\xi)(d(1-2\psi) - 2)N_s}{16\pi(d-2)^2 v^2} \tag{34}
\]

In the limit \( \psi \to 0 \), the equation of state (34) reduces to \( d \)-dimensional AdS black holes in quintessence background in Einstein’s GR. When both \( N_s \) and \( \psi \) tend to zero the equation of state (34) reduces to Schwarzschild-Tangherlini-AdS black holes in higher dimensions [36].

The critical points are determined as

\[
\left( \frac{\partial P}{\partial v} \right)_{T=T_c} = \left( \frac{\partial^2 P}{\partial^2 v} \right)_{T=T_c} = 0. \tag{35}
\]

The coupled equations (35) can be solved analytically. Therefore, the critical quantities now read

\[
v_c = \left[ \frac{(3-d+\xi)(1+\xi)(2+\xi)N_s}{2(d-3)} \right]^{\frac{1}{\xi}} \tag{36}
\]

\[
T_c = \frac{2(d-3)\xi}{\pi(d-2)(1+\xi)v_c} \tag{37}
\]

\[
P_c = \frac{(d(1-2\psi) - 2)(d-3)\xi}{\pi(d-2)^2(2+\xi)v_c^2} \tag{38}
\]

The subscript “c” denotes the values of the physical quantities at the critical points. The Eqs. (36), (37) and (38) lead to the following universal ratio

\[
\rho_c = \frac{P_cv_c}{T_c} = \frac{d(1-2\psi) - 2}{2(d-2)} \left( \frac{1+\xi}{2+\xi} \right) \tag{39}
\]

This ratio is independent of the surrounding field parameter \( N_s \) but is dependent on the Rastall coupling constant \( \psi \). In Einstein’s GR, i.e., when \( \psi \to 0 \), the ratio recovers the \( d \)-dimensional AdS black holes in quintessence background. When \( \psi = 0 \) and \( \omega_s = \frac{d-3}{d-1} \), the universal ratio corresponds to the \( d \)-dimensional Reissner-Nordström black holes [36] for \( N_s = -q^2 \), which reads

\[
\rho_c = \frac{2d-5}{4d-8} \tag{40}
\]

We recover the ratio \( \rho_c = 3/8 \) in \( d = 4 \), a universal feature of the vdW fluid. It is seen that the critical parameters, Eqs. (36)-(38), depend on the fluid parameter and therefore
reflecting in this way the presence of some exotic matter fields, e.g., the dark energy on
the critical behavior of the vdW fluid. If we put $\psi = 0$, and $N_s = -q^2$, all these critical
quantities reduce to those of $d$-dimensional RN-AdS black holes [36].
Next we derive another important thermodynamic quantity in equilibrium physics, the Gibbs
free energy of the thermodynamic system. In the extended phase space, the enthalpy of the
thermodynamic system in interpreted as mass of the black holes. Thus treating mass as
enthalpy one can obtain the Gibbs free energy, $G = H - TS = M - TS$ [53] in the following
form

$$G = \frac{\Omega_{d-2}}{16\pi} \left[ r_+^{d-3} - \frac{16\pi r_+^{d-1} P}{(d-1)(d(1-2\psi)-2)} - \frac{(1+\xi)N_s}{r_+^{\xi+3-d}} \right].$$

(41)

A. Analytical study of classical Ehrenfest equations

To obtain a distinct second order phase transition at the critical points, the Ehrenfest
equations have been successfully applied for the vdW like fluid in the extended phase space
of black hole thermodynamics for various solutions in Einstein’s GR and other modified
theories of gravity [35, 55–57]. The distinct second order phase transition of the vdWs fluid
are described by the Ehrenfest equations as follows

$$\left( \frac{\partial P}{\partial T} \right)_S = \frac{C_{P_2} - C_{P_1}}{TV(\alpha_{P_2} - \alpha_{P_1})} = \frac{\Delta C_P}{TV\Delta \alpha}$$

(42)

$$\left( \frac{\partial P}{\partial T} \right)_V = \frac{\alpha_2 - \alpha_1}{\kappa_{T_2} - \kappa_{T_1}} = \frac{\Delta \alpha}{\Delta \kappa_T}$$

(43)

where $\alpha = \frac{1}{V} (\frac{\partial V}{\partial T})_P$ and $\kappa_T = -\frac{1}{V} (\frac{\partial V}{\partial P})_T$, respectively, are the isobaric volume expansion
coefficient and the isothermal compressibility coefficient.

At the critical points of $P$-$V$ criticality in the extended phase space thermodynamics, we
analytically study the Ehrenfest schemes given by Eqs. (42) and (43). The temperature
when expressed in terms of entropy (31) can be obtained as

$$T = \frac{1}{4\pi} \left[ (d-3) \left( \frac{4S}{\Omega_{d-2}} \right)^{\frac{1}{d-2}} + \frac{16\pi P}{(d(1-2\psi)-2)} \left( \frac{4S}{\Omega_{d-2}} \right)^{\frac{1}{d-2}} + (3-d+\xi)N_s \left( \frac{4S}{\Omega_{d-2}} \right)^{-\frac{1+\xi}{d-2}} \right]$$

(44)
Utilizing the Eqs. (23), (31) and (44), we calculate the specific heat at constant pressure, the volume expansion coefficient and the isothermal compressibility of the black holes as

\[ C_P = T \left( \frac{\partial S}{\partial T} \right)_P \]

\[ (d - 2)S \left[ 1 + \frac{16\pi P}{(d(1-2\psi)-2)(d-3)} \left( \frac{4S}{\Omega_{d-2}} \right)^{\frac{2}{d-2}} + \frac{(3-d+\xi)}{d-3} N_s \left( \frac{4S}{\Omega_{d-2}} \right)^{-\frac{\xi}{d-2}} \right] \]

\[ - 1 + \frac{16\pi P}{(d(1-2\psi)-2)(d-3)} \left( 4S \right)^{\frac{2}{d-2}} - \frac{(1+\xi)(3-d+\xi)}{d-3} N_s \left( \frac{4S}{\Omega_{d-2}} \right)^{-\frac{\xi}{d-2}} \]

(45)

\[ \alpha = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P \]

\[ = \frac{4\pi \left( \frac{d-1}{d-3} \right) \left( \frac{4S}{\Omega_{d-2}} \right)^{\frac{2}{d-2}}}{-1 + \frac{16\pi P}{(d(1-2\psi)-2)(d-3)} \left( 4S \right)^{\frac{2}{d-2}} - \frac{(1+\xi)(3-d+\xi)}{d-3} N_s \left( \frac{4S}{\Omega_{d-2}} \right)^{-\frac{\xi}{d-2}} \] \]

\[ \kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T \]

\[ = - \frac{16\pi}{(d(1-2\psi)-2)(d-3)} \left( \frac{4S}{\Omega_{d-2}} \right)^{\frac{2}{d-2}} - \frac{(1+\xi)(3-d+\xi)}{d-3} N_s \left( \frac{4S}{\Omega_{d-2}} \right)^{-\frac{\xi}{d-2}} \]

(46)

\[ \text{In deriving the Eq. (47), we use the following cyclic thermodynamic identity of a } (P,V,T) \text{ system such that} \]

\[ \left( \frac{\partial V}{\partial P} \right)_T \left( \frac{\partial P}{\partial T} \right)_V \left( \frac{\partial T}{\partial V} \right)_P = -1 \]

(48)

It is to be noted that the denominators of \( C_P \), \( \alpha \) and \( \kappa_T \) share the same factor, and thereby indicating that they diverge at the same point critical points and therefore satisfy the condition

\[ -1 + \frac{16\pi P}{(d(1-2\psi)-2)(d-3)} \left( \frac{4S_c}{\Omega_{d-2}} \right)^{\frac{2}{d-2}} - \frac{(1+\xi)(3-d+\xi)}{d-3} N_s \left( \frac{4S_c}{\Omega_{d-2}} \right)^{-\frac{\xi}{d-2}} = 0 \]

(49)

However, using Eqs. (23), (33) and (36), we obtain

\[ S_c = \frac{\Omega_{d-2}}{4} \left[ \frac{(3-d+\xi)(1+\xi)(2+\xi)N_s}{2(d-3)} \right]^{\frac{d-2}{2}} \]

(50)
\[ P_c = \frac{(d(1-2\psi)-2)(d-3)\xi}{\pi(d-2)^2(2+\xi)} \left[ \frac{(3-d+\xi)(1+\xi)(2+\xi)N_s}{2(d-3)} \right]^{\frac{\pi^2}{2}} \]  

Thus the Eq. (49) shows that \( C_P, \alpha \) and \( \kappa_T \) are discontinuous at the critical points \( S_c \) and \( P_c \).

Next, we move on to check the validity of the Ehrenfest equations (42)-(43) at the critical point. The definition of the volume expansion coefficient can be rearranged as

\[ V\alpha = \left( \frac{\partial V}{\partial T} \right)_P = \left( \frac{\partial V}{\partial S} \right)_T \left( \frac{\partial S}{\partial T} \right)_P = \left( \frac{\partial V}{\partial S} \right)_P \frac{C_P}{T}, \]  

hence using Eqs. (23) and (29), the R.H.S of the equation (42) can be converted to

\[ \frac{\Delta C_P}{TV\Delta \alpha} = \left[ \frac{d - 2}{4} - \frac{\psi d}{2} \right] \left( \frac{S_c}{\Omega_{d-2}} \right)^{-\frac{1}{2}} \]  

Utilising Eq. (44), the L.H.S of Eq. (42) can be derived as

\[ \left[ \left( \frac{\partial P}{\partial T} \right)_S \right]_c = \left[ \frac{d - 2}{4} - \frac{\psi d}{2} \right] \left( \frac{S_c}{\Omega_{d-2}} \right)^{-\frac{1}{2}} \]  

From Eqs. (53) and (54) we conclude that the first equation of the Ehrenfest equations is valid at the critical point.

Next we calculate the second Ehrenfest equation defined in Eq. (43) and check its validity. Using Eqs. (23), (32) and (44), we have L.H.S of Eq. (43) at the critical point as

\[ \left[ \left( \frac{\partial P}{\partial T} \right)_V \right] = \left[ \frac{d - 2}{4} - \frac{\psi d}{2} \right] \left( \frac{S_c}{\Omega_{d-2}} \right)^{-\frac{1}{2}} \]  

Using the defining relations of \( \kappa_T \) and \( \alpha \), we have

\[ V\kappa_T = -\left( \frac{\partial V}{\partial P} \right)_T = \left( \frac{\partial T}{\partial \Omega} \right)_V \left( \frac{\partial V}{\partial T} \right)_P = \left( \frac{\partial T}{\partial P} \right)_V V\alpha, \]  

such that the R.H.S of Eq. (43) can be written

\[ \frac{\Delta \alpha}{\Delta \kappa_T} = \left[ \left( \frac{\partial P}{\partial T} \right)_V \right] _c = \left[ \frac{d - 2}{4} - \frac{\psi d}{2} \right] \left( \frac{S_c}{\Omega_{d-2}} \right)^{-\frac{1}{2}} \]  

In calculating the Eq. (57), once again we have used the cyclic rule of the \( P-V-T \) system defined in Eq. (48). Therefore, the second Ehrenfest equation is also valid at the critical points as is confirmed by Eq. (57). Therefore, both the Ehrenfest equations are found to be valid at the critical point of the \( P-V \) criticality of the \( d \)-dimensional Rastall AdS black hole.
surrounded by perfect fluid.

Using Eqs. (53) and (57), the Prigogine-Defay (PD) ratio is derived as

$$\Pi = \frac{\Delta C_P \Delta \kappa T}{TV (\Delta \alpha)^2} = 1.$$  \hspace{1cm} (58)

Since PD ratio identically equals to unity, it is proved that the $d$-dimensional AdS black holes in the perfect fluid background in Rastall theory has a distinct second order phase transition. The PD ratio was first introduced in [58] and later it was extensively investigated in [59]. The PD ratio can be used to measure the potential deviation from for the system which does not show the similar behavior as those of vdWs fluid [60, 61]. For the vdWs system, the PD ratio equals to unity while for a glassy phase transition, it lies between 2 to 5 [60–63]. The Eq. (58) together with the Ehrenfest ensures the validity Eqs. (42) and (43) confirms the second-order phase transition at the critical points. Therefore, for a $d$-dimensional AdS black holes in Rastall gravity, the phase transition is no longer an exception and follows the same features of a vdWs like liquid-gas system.

V. CONCLUSIONS

In this paper, we briefly discussed the $d$-dimensional AdS black holes in the perfect fluid background in Rastall gravity. As limiting cases, when $\omega_s = -1$ and $1/l^2 \neq 0$, the metric (13) reduces to the $d$-dimensional Schwarzschild-Tangherlini (A)dS black holes. We have extended the first law in AdS spaces which includes $\Theta_s dN_s$ term in addition to $VdP$ term. We have studied the thermodynamics of this $d$-dimensional black hole spacetime and defines a quantity $\Theta_s$ conjugate to $N_s$ in order to be consistent with the Smarr-Gibbs-Duhem relation.

Next, we have investigated the extended phase space thermodynamics of $d$-dimensional AdS black holes in perfect fluid background. The definitions of the thermodynamic pressure and volume term includes the Rastall parameter $\psi$ and hence modify the thermodynamic quantities in AdS spacetime. We have employed the classical Ehrenfest schemes to study the essence of the phase transition at the critical points of $P-V$ criticality. The consideration of cosmological constant as thermodynamic pressure in AdS spaces and its conjugate quantity as thermodynamic volume led us to analyze the classical Ehrenfest equations by calculating heat capacity at constant pressure $C_P$, and the isobaric volume expansion coefficient $\alpha$, and
the isothermal compressibility $\kappa_T$. The effect of Rastall parameter $\psi$ is also demonstrated in the derived thermodynamic quantities in extended phase space of black hole thermodynamics and found that they diverge exactly at the same critical points. The PD ratio is also calculated and found that it exactly equals to unity. In the limit $\psi \to 0$, our results reduce to the expressions calculated in Einstein’s GR. Therefore, the universal character of the vdW gas in Rastall gravity help us to understand the relations of the AdS black holes and liquid-gas system.

Our results may be important in the context of AdS/CFT correspondence. An extension to $d$-dimensional charged AdS black holes and their phase analysis may be a natural addition to the work. A rotating version of the black holes and their $P$-$V$ criticality investigation will be physically motivated.

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