We consider a straight three dimensional quantum layer with singular potential, supported on a straight wire which is localized perpendicularly to the walls and connects them. We prove that an infinite number of embedded eigenvalues appear in this system. Furthermore, we show that after introducing a small surface impurity to the layer, the embedded eigenvalues turn to the second sheet resolvent poles, which state resonances. We discuss the asymptotics of the imaginary component of the resolvent pole with respect to the surface area.

Keywords: embedded eigenvalues, resonances, delta potential

(Some figures may appear in colour only in the online journal)

1. Introduction

The paper belongs to the line of research often called Schrödinger operators with delta potentials\(^1\). The analysis of these types of potential is motivated by mesoscopic physics systems with semiconductor structures designed in such a way that they can be mathematically modelled by the Dirac delta supported on sets of lower dimensions. The support of delta potential imitates the geometry of the semiconductor material; for example, it can take the form of one dimensional sets (wires) or surfaces with specific geometrical properties. A particle is confined in the semiconductor structure; however, the model admits a possibility of tunneling. Therefore these types of system are called in literature \textit{leaky quantum graphs} or \textit{wires}. One of the most appealing problems in this area is the question of how the geometry of a wire affects the spectrum—see [1] and [12, chapter 10]. The aim of the present paper is to discuss how the surface perturbation leads to resonances.

\(^1\) In the following we will use the notations ‘delta interaction’ and ‘delta potential’ equivalently.
We consider a non-relativistic three dimensional model of a quantum particle confined between two infinite unpenetrable parallel walls which form a straight quantum layer defined by $\Omega := \{(x, y, z) \in \mathbb{R}^2 \times [0, \pi]\}$. In the case of absence of any additional potential, the Hamiltonian of such a system is given by the negative Laplacian $-\Delta : D(\Delta) \rightarrow L^2(\Omega)$ with the domain $D(\Delta) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, i.e. with the Dirichlet boundary conditions on $\partial \Omega$. The spectrum of $-\Delta$ is determined by $\sigma_{\text{ess}}(-\Delta) = [1, \infty)$; however, it is useful to keep in mind that the energies in the $x_3$-direction are quantized and given by $(k^2)_{k=1}^\infty$.

At the first stage we introduce a straight wire $I$ which connects the walls $\partial \Omega$, being at the same time perpendicular to them. We assume the presence of interaction localized on $I$ and characterized by the coupling constant $\alpha \in \mathbb{R}$.

The symbolic Hamiltonian of such a system can be formally written

$$-\Delta + \delta_{\alpha,I},$$

where $\delta_{\alpha,I}$ represents delta potential supported on $I$. Since the interaction support in this model has the co-dimension larger then one it is called strongly singular potential. The proper mathematical definition of Hamiltonian can be formulated in the terms of boundary conditions.

More precisely, we define $H_\alpha$ as a self adjoint extension of $-\Delta|_{C^\infty_0(\Omega \setminus I)}$ determined by means of appropriate boundary conditions satisfied on $I$ by functions from the domain $D(H_\alpha)$. The coupling constant $\alpha$ is involved in the boundary conditions mentioned; however, it is worth saying at this point that $\alpha$ does not contribute additively to the structure of the Hamiltonian.

To describe spectral properties of $H_\alpha$ we can rely on radial symmetry of the system, and consider a two dimensional system with point interaction governed by the Hamiltonian $H_\alpha^{(1)}$. The spectrum of $H_\alpha^{(1)}$ consists of the positive half line and one discrete negative eigenvalue

$$\xi_\alpha = -4e^{2(-2\pi\alpha + \psi(1))},$$

where $-\psi(1) = 0.577...$ determines the Euler–Mascheroni constant. This reflects the spectral structure of $H_\alpha$; specifically, for each $l \in \mathbb{N}$ the number

$$\epsilon_l = \xi_\alpha + l^2,$$

gives rise to an eigenvalue of $H_\alpha$. Note that an infinite number of $\epsilon_l$ exist above the threshold of the essential spectrum and, consequently, the Hamiltonian $H_\alpha$ admits an infinite number of embedded eigenvalues.

In the second stage we introduce to the layer an attractive interaction supported on a finite $C^\infty$ surface $\Sigma \subset \Omega$ separated from the wire $I$ by some distance—see figure 1. Suppose that $\beta \neq 0$ is a real number. The Hamiltonian $H_{\alpha,\beta}$ which governs this system can be symbolically written as

$$-\Delta + \delta_{\alpha,I} - \beta \delta_{\Sigma},$$

where $\delta_{\Sigma}$ stands for the Dirac delta supported on $\Sigma$; this term represents weakly singular potential. Again, a proper mathematical definition of $H_{\alpha,\beta}$ can be formulated as a self adjoint extension of

$$H_\alpha^{(1)}|_{\{f \in D(H_\alpha); C^\infty(\Omega \setminus I), f = 0 \text{ on } \Sigma\}}.$$

This extension is defined by means of the appropriate boundary conditions on $\Sigma$ discussed in section 3.

The aim of this paper is to analyse how the presence of surface interaction supported on $\Sigma$ affects the embedded eigenvalues. The existence of embedded eigenvalues is a direct consequence of the symmetry. By introducing additional interaction on $\Sigma$ we break this symmetry;
however, if the perturbation is small then we may expect that the system preserves a ‘spectral memory’ on original eigenvalues. In section 5 we show, for example, that if the area $|\Sigma| \to 0$ then the embedded eigenvalues $\epsilon_l$ turn to complex poles of the resolvent of $H_{\alpha,\beta}$. These poles are given by $z_l = \epsilon_l + o(|\Sigma|)$, with $\Im z_l < 0$; the latter confirms that $z_l$ is localized on the second sheet continuation. We derive the explicit formula for the lowest order of the imaginary component of $z_l$ and show that it admits the following asymptotics

$$\Im z_l = \mathcal{O}(|\Sigma|^2).$$

The poles of resolvent state resonances in the system governed by $H_{\alpha,\beta}$ and $\Im z_l$ are related to the width of the resonance given by $-2\Im z_l$. Finally, let us mention that various types of resonator in waveguides and layers have already been analyzed. For example, in [21] the authors study resonances induced by twisting of waveguides, which is responsible for breaking symmetry. A planar waveguide with narrows playing the role of resonators has been studied in [4]. On the other hand, straight Dirichlet or Neumann waveguides with windows or barriers inducing resonances have been analyzed in [6, 15, 16]. Furthermore, resonances in curved waveguides with finite branches have been described in [10]. It is also worth mentioning that quantum waveguides with electric and magnetic fields have been considered—see [3, 8].

On the other hand, various types of resonators induced by delta potential in two or three dimensional systems have been analyzed. Let us mention the results of [11, 18, 19], which describe resonances in terms of breaking symmetry parameters, or by means of tunnelling effects. In [20] the authors consider a straight two dimensional waveguide with a semitransparent perpendicular barrier modelled by delta potential. It was shown that on slightly changing the slope of the barrier the embedded eigenvalues turn to resonances; the widths of these

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Layer with wire and surface impurity.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Construction of the second sheet continuation.}
\end{figure}
resonances can be expressed in the terms of the barrier slope. The present paper is, in a sense, an extension of [20]. However, the strongly singular character of the delta interaction supported on \( I \) means that even an infinitesimal change of the slope of \( I \) cannot be understood as a small perturbation. Therefore the resolvent poles are not interesting from the physical point of view, since they rapidly escape far away from the real line. In the present model, delta potential localized on \( \Sigma \) plays the role of small perturbation which leads to resonances.

Finally, it is worth mentioning that the spectral properties of quantum waveguides and layers with delta interaction have been studied, for example, in [13, 14]. The results of [13] concern weakly singular potentials and in [14] the authors consider strongly singular interaction. In the present paper we combine both types of delta interaction and analyze how they affect each other.

General notations:

- \( \mathbb{C} \) stands for the complex plane and \( \mathbb{C}_{\pm} \) for the upper (respectively, lower) half-plane.
- \( \| \cdot \| \), \( (\cdot, \cdot) \) denote the norm and the scalar product in \( L^2(\Omega) \) and \( (\cdot, \cdot)_{\Sigma} \) defines the scalar product in \( L^2(\Sigma) \).
- Suppose that \( A \) stands for a self-adjoint operator. We denote by \( \sigma_{\text{ess}}(A) \), \( \sigma_\rho(A) \) and \( \rho(A) \) respectively the essential spectrum, the point spectrum and the resolvent set of \( A \).
- The notation \( C \) stands for a constant, the value of which can vary from line to line.

2. Parallel walls connected by wire inducing embedded eigenvalues

2.1. Free particle in layer

Let \( \Omega \subset \mathbb{R}^3 \) stand for a layer defined by \( \Omega := \{ x = (x_1, x_3) : x \in \mathbb{R}^2, x_3 \in [0, \pi] \} \) and in the following we assume the convention \( x = (x_1, x_2) \in \mathbb{R}^2 \).

The ‘free’ Hamiltonian is determined by

\[
H = -\Delta : D(H) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \to L^2(\Omega),
\]

and admits the following decomposition:

\[
H = -\Delta^{(2)} \otimes I + I \otimes -\Delta^{(1)} \quad \text{on} \quad L^2(\mathbb{R}^2) \otimes L^2(0, \pi),
\]

where \( \Delta^{(2)} : D(\Delta^{(2)}) = W^{2,2}(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \) stands for the two-dimensional Laplacian and \( \Delta^{(1)} : D(\Delta^{(1)}) = W^{2,2}(0, \pi) \cap W^{1,2}_0(0, \pi) \to L^2(0, \pi) \) determines a one-dimensional Laplacian with the Dirichlet boundary conditions.

To define the resolvent of \( H \) it is useful to note that the sequence \( \{ \chi_n \}_{n=1}^\infty \) given by

\[
\chi_n(x_3) := \frac{\sqrt{2}}{\pi} \sin(nx_3), \quad n \in \mathbb{N}
\]

forms an orthonormal basis in \( L^2(0, \pi) \). Suppose that \( z \in \mathbb{C} \setminus [1, \infty) \). Then \( R(z) := (-\Delta - z)^{-1} \) defines an integral operator with the kernel

\[
\mathcal{G}(z; \xi, \xi', x_3, x_3') := \frac{1}{2\pi} \sum_{n=1}^\infty K_0(\kappa_n(z)|\xi - \xi'|)\chi_n(x_3)\chi_n(x_3'),
\]

where \( K_0(\cdot) \) denotes the Macdonald function—see [2]—and

\[
\kappa_n(z) := -i\sqrt{z - n^2}, \quad 3\sqrt{z - n^2} > 0.
\]

In the following we will also use the abbreviation \( \mathcal{G}(z) \) for (2.3). The threshold of spectrum of \( H \) is determined by the lowest discrete transversal energy, i.e. 1. Moreover, it is purely absolutely continuous, and consequently takes the form.
\[ \sigma(H) = [1, \infty). \]

2.2. Layer with perpendicular wire: embedded eigenvalues phenomena

We introduce a wire defined by the straight segment of width \( \pi \) and perpendicular to the walls. The presence of the wire will be modelled by delta interaction supported on \( I \subset \Omega \), where \( I := (0,0) \times [0, \pi] \).

In view of the radial symmetry the operator with delta interaction on \( I \) admits a natural decomposition on \( L^2(\Omega) = L^2(\mathbb{R}^2) \otimes L^2(0, \pi) \) and acts in the subspace \( L^2(\mathbb{R}^2) \) as the Schrödinger operator with one point interaction. Therefore, the delta potential can be determined by appropriate boundary conditions—see [1, chapter 1.5]—which can be implemented separately in each sector of the transversal energy. For this aim we decompose a function \( \psi \in L^2(\Omega) \) onto \( \psi(x) = \sum_{n=1}^{\infty} \psi_n(x) \chi_n(x) \), where \( \psi_n(x) := \int_{0}^{\pi} \psi(x, \xi) \chi_n(\xi) \, d\xi \).

(D1) We say that a function \( \psi \) belongs to the set \( D' \subset W^{2,2}_{\text{loc}}(\Omega \setminus I) \cap L^2(\Omega) \) if \( \Delta \psi \in L^2(\Omega) \), \( \psi|_{\partial \Omega} = 0 \) and the following limits

\[ \Xi_n(\psi) := - \lim_{|\xi| \to 0} \frac{1}{\ln |\xi|} \psi_n(x), \quad \Omega_n(\psi) := \lim_{|\xi| \to 0} (\psi_n(x) - \Xi_n(\psi) \ln |\xi|) \]

are finite.

(D2) For \( \alpha \in \mathbb{R} \), we define the set

\[ D(H_\alpha) := \{ \psi \in D' : 2\pi \alpha \Xi_n(\psi) = \Omega_n(\psi) \text{ for any } n \in \mathbb{N} \} \]  

and the operator \( H_\alpha : D(H_\alpha) \to L^2(\Omega) \) which acts as follows:

\[ H_\alpha \psi(x) = -\Delta \psi(x), \quad \text{for } x \in \Omega \setminus I. \]

The resulting operator \( H_\alpha : D(H_\alpha) \to L^2(\Omega) \) coincides

\[ -\Delta^{(2)}_\alpha \otimes I + I \otimes -\Delta^{(1)} \text{ on } L^2(\mathbb{R}^2) \otimes L^2(0, \pi), \]

where \( \Delta^{(2)}_\alpha : D(\Delta^{(2)}_\alpha) \to L^2(\mathbb{R}^2) \) stands for the two-dimensional Laplacian with point interaction, see [1, chapter 1.5] with the domain \( D(\Delta^{(2)}_\alpha) \). Consequently, \( H_\alpha \) is self adjoint; its spectral properties will be discussed in the next section.

2.3. Resolvent of \( H_\alpha \)

Suppose that \( z \in \mathbb{C}_+ \). We use the standard notation \( R_\alpha(z) \) for the resolvent operator, i.e. \( R_\alpha(z) := (H_\alpha - z)^{-1} \). To figure out the explicit resolvent formula we introduce

\[ \omega_n(z;x) := \frac{1}{2\pi} K_0(k_n(z)|x|) \chi_n(x), \quad n \in \mathbb{N}; \]

in the following we will use also the abbreviation \( \omega_n(z) = \omega_n(z; \cdot) \).

The following theorem states the desired result.

**Theorem 2.1.** The essential spectrum of \( H_\alpha \) is given by

\[ \sigma_{\text{ess}}(H_\alpha) = [1, \infty). \]
Furthermore, let
\[
\Gamma_n(z) := \frac{1}{2\pi} \left( 2\pi\alpha + s_n(z) \right), \quad \text{where} \quad s_n(z) := -\psi(1) + \ln \frac{\sqrt{z-(\pi^2)}}{2i}.
\]

(2.8)

Suppose that \( z \in \mathbb{C} \setminus [1, \infty) \) and \( \Gamma_n(z) \neq 0 \). Then \( z \in \rho(H_\alpha) \) and operator \( R_\alpha(z) \) admits the Krein-like form:
\[
R_\alpha(z) = R(z) + \sum_{n=1}^{\infty} \Gamma_n(z)^{-1}(\omega_n(z), \cdot)\omega_n(z).
\]

(2.9)

**Proof.** Our first aim is to show that (2.9) defines the resolvent of \( H_\alpha \). Operator \( H_\alpha \) is defined as the self-adjoint extension of \( -\Delta|\mathcal{C}_\infty^0(\Omega \setminus I) \). Suppose that \( f \in C_0^\infty(\Omega \setminus I) \). Then \( g := (-\Delta - z) f \in C_0^\infty(\Omega \setminus I) \). Employing the fact that \( \omega_n(z) = G(z) \ast (\delta \chi_n) \), where \( G(z) \) is the kernel defined by (2.3) and \( \delta = \delta(z) \), we conclude that \( (\omega_n(z), g) = (\delta \chi_n, f)_{-1,1} = 0 \), where \((\cdot, \cdot)_{-1,1} \) states the duality between \( W_0^{1,2}(\Omega) \) and \( W^{1,2}(\Omega) \). This, consequently, implies \( \Gamma_n(z)(-\Delta - z) f = R(z)(-\Delta - z) f = f \) in view of (2.9) which means that \( R_\alpha(z) \) defines the resolvent of a self-adjoint extension of \( -\Delta|\mathcal{C}_\infty^0(\Omega \setminus I) \). To complete the proof we have to show that any function \( g = R_\alpha(z) f \) satisfies boundary conditions (2.5). In fact, \( g \) admits the unique decomposition \( g = g_1 + g_2 \), where \( g_1 := R(z) f \) and \( g_2 = \sum_{n=1}^{\infty} \Gamma_n(z)^{-1}(\omega_n(z), f)\omega_n(z) \). Therefore, a nontrivial contribution to \( \Xi_n(g) \) comes from \( g_2 \) since \( g_1 \in W^{2,2}(\Omega) \). Employing the asymptotic behaviour of the Macdonald function—see [2],
\[
K_\alpha(\rho) = \ln \frac{1}{\rho} + \psi(1) + O(\rho),
\]
we get \( \Xi_n(g) = \frac{1}{2\pi} \Gamma_n(z)^{-1}(\omega_n(z), f) \) and
\[
\Omega_n(g) = (1 - \frac{1}{2\pi} \Gamma_n(z)^{-1}s_n(z))(\omega_n(z), f) = \alpha \Gamma_n(z)^{-1}(\omega_n(z), f).
\]

Using (2.8) one obtains (2.5). This completes the proof of (2.9). The stability of the essential spectrum can be concluded in an analogous way to that used in [7, theorem 3.1]. The key step is to show that \( R(z) - R_\alpha(z) \) is compact. The statement can be proved relying on compactness of the trace map \( S : W^{2,2}(\Omega) \rightarrow L^2(I) \), which follows from the boundedness of the trace map—see [22, chapter 1, theorem 8.3] and the compactness theorem—see [22, chapter 1, theorem 16.1]. This implies, in view of boundedness of \( R(z) : L^2(\Omega) \rightarrow W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \), that \( SR(z) : L^2(\Omega) \rightarrow L^2(I) \) is compact. Employing the resolvent formula—see [23]—and the fact that the remaining operators contributing to \( R(z) - R_\alpha(z) \) are bounded, we conclude that \( R(z) - R_\alpha(z) \) is compact.

**Remark 2.2.** The spectral analysis developed in this work is mainly based on the resolvent properties. In the following we will use the results of [5, 7, 23, 24], where strongly as well as weakly singular potentials were considered.

In the following theorem we state the existence of eigenvalues of \( H_\alpha \).

**Theorem 2.3.** Let \( A_\alpha := \{ n \in \mathbb{N} : \xi_n + n^2 < 1 \} \). Each \( \epsilon_n := \xi_n + n^2 \), where \( n \in A_\alpha \) defines the discrete eigenvalue of \( H_\alpha \) with the corresponding eigenfunction \( \omega_n := \omega(n) \). In

\[\text{Analogously as in the previous discussion we assume } \Im \sqrt{z - n^2} > 0. \quad \text{The logarithmic function } z \mapsto \ln z \text{ is defined in the cut plane } -\pi < \arg z < \pi \text{ and admits continuation to the entire logarithm Riemann surface.}\]
particular, this means that for any \( \alpha \) operator \( H_\alpha \) has at least one eigenvalue \( \epsilon_1 \) below the threshold of the essential spectrum.

Operator \( H_\alpha \) has an infinite number of embedded eigenvalues. More precisely, for any \( n \in \mathbb{N} \setminus \mathcal{A}_\alpha \) the number \( \epsilon_n := \xi_\alpha + n^2 \) determines the embedded eigenvalue. In particular, there exists \( \tilde{n} \in \mathbb{N} \setminus \mathcal{A}_\alpha \) such that \( \epsilon_n \in ((n-1)^2, n^2) \) for any \( n > \tilde{n} \).

**Proof.** The proof is based on the Birman–Schwinger argument which, in view of (2.9), reads

\[
z \in \sigma_p(H_\alpha) \iff \exists n \in \mathbb{N} : \Gamma_n(z) = 0,\]

—see [24, Theorem 2.2]. Note that, given \( n \in \mathbb{N} \) the function \( z \mapsto \Gamma_n(z), z \in \{ C : \Im \sqrt{z - n^2} > 0 \} \)
has the unique zero at \( z = \xi_\alpha + n^2 \), i.e.

\[\Gamma_n(\xi_\alpha + n^2) = 0.\]

Finally, it follows, for example, from [24, Theorem 3.4] that the corresponding eigenfunctions takes the form \( \mathcal{G}(\epsilon) * \chi_\alpha \delta \). This completes the proof. 

### 3. Surface impurity

We define a finite smooth parameterized surface \( \Sigma \subset \Omega \), being a graph of the map \( U \ni q = (q_1, q_2) \mapsto x(q) \in \Omega \). The surface element can be calculated by means of the standard formula \( d\Sigma = |\partial_1 x(q) \times \partial_2 x(q)| |dq| \). Additionally, we assume that \( \Sigma \cap I = \emptyset \). Furthermore, let \( n : \Sigma \to \mathbb{R}^3 \) stand for the unit normal vector (with an arbitrary orientation), and \( \partial_n \) denote the normal derivative defined by vector \( n \). Relying on the Sobolev theorem we state that the trace map \( W^{1,2}(\Omega) \ni \psi \mapsto \psi|_\Sigma \in L^2(\Sigma) \) constitutes a bounded operator; we set the notation \( (\cdot, \cdot) \) for the scalar product in \( L^2(\Sigma) \). Given \( \beta \in \mathbb{R} \setminus \{ 0 \} \) we define the following boundary conditions: suppose that \( \psi \in C(\Omega) \cap C^4(\Omega \setminus \Sigma) \) satisfies

\[
\partial_n^+ \psi|_\Sigma = \partial_n^- \psi|_\Sigma = -\beta \psi|_\Sigma, \tag{3.11}
\]

where the partial derivatives contributing to the above expression are defined as the positive (resp., negative) limits on \( \Sigma \) and signs are understood with respect to the direction of \( n \).

\((D_3)\) We say that a function \( \psi \) belongs to the set \( \tilde{D} \subset W^{1,2}_{\text{loc}}(\Omega \setminus (I \cup \Sigma)) \) if \( \Delta \psi \in L^2(\Omega) \), \( \psi|_{\partial I} = 0 \) and the limiting equations (2.5) and (3.11) are satisfied.

\((D_4)\) Define operator which for \( f \in \tilde{D} \) acts as \( -\Delta f(x) \) if \( x \in \Omega \setminus (I \cup \Sigma) \) and let \( H_{\alpha, \beta} : D(H_{\alpha, \beta}) \to L^2(\Omega) \) stand for its closure.

To figure out the resolvent of \( H_{\alpha, \beta} \) we define the operator acting from to \( L^2(\Omega) \) to \( L^2(\Sigma) \) as \( R_{\alpha, \Sigma}(z) f = (R_{\alpha, \Sigma}(z) f)|_\Sigma \). Furthermore, we introduce the operator from \( L^2(\Sigma) \) to \( L^2(\Omega) \) defined by \( R_{\alpha, \Sigma}(z) f = G_{\alpha} * f, \delta \), where \( G_{\alpha} \) stands for kernel of (2.9). Finally, we define \( R_{\alpha, \Sigma}(z) : L^2(\Sigma) \to L^2(\Sigma) \) by \( R_{\alpha, \Sigma}(z) f = (R_{\alpha, \Sigma}(z) f)|_\Sigma \). In view of (2.9) the latter takes the following form

\[
R_{\alpha, \Sigma}(z) = R_{\Sigma}(z) + \sum_{n=1}^{\infty} \Gamma_n(z)^{-1}(w_n(z), \cdot)_\Sigma w_n(z), \tag{3.12}
\]

where \( w_n(z) := \omega_n(z)|_\Sigma \) and \( R_{\Sigma}(z) : L^2(\Sigma) \to L^2(\Sigma) \) stands for the bilateral embedding of \( R(z) \).
Following the strategy developed in [23], we define the set $Z \subset \rho(H_\alpha)$ such that $z$ belongs to $Z$ if the operators 

$$(I - \beta R_\alpha,\Sigma\Sigma(z))^{-1}, \quad \text{and} \quad (I - \beta R_\alpha,\Sigma\Sigma(z))^{-1}$$

acting from $L^2(\Sigma)$ to $L^2(\Sigma)$ exist and are bounded. Our aim is to show that $Z \neq \emptyset$. (3.13)

Therefore we auxiliarily define the quadratic below-bounded form

$$\int_\Omega |\psi|^2 dx - \beta \int_\Sigma |\psi|^2 \Sigma^2 d\Sigma, \quad \psi \in W^{1,2}_0(\Omega).$$

Let $H_\beta$ stand for the operator associated to the above form in the sense of the first representation theorem—see [17, chapter VI]. Following the arguments from [7] we conclude that $I - \beta R_\Sigma(z) : L^2(\Sigma) \to L^2(\Sigma)$ defines the Birman–Schwinger operator for $H_\beta$. Using theorem 2.2 of [24] one obtains

$$z \in \rho(H_\beta) \Leftrightarrow 0 \in \rho(I - \beta R_\Sigma(z)).$$

In the following we are interested in negative spectral parameters, and thus we assume $z = -\lambda$ where $\lambda > 0$. Since the spectrum of $H_\beta$ is lower bounded we conclude

$$0 \in \rho(I - \beta R_\Sigma(-\lambda)),$$

for large enough $\lambda$. (3.14)

The next step is to find a bound for the second component contributing to (3.12). In fact, it can be majorized by

$$\sum_{n=1}^{\infty} |\Gamma_n(-\lambda)|^{-1} \|w_n(-\lambda)\|_\Sigma^2 \leq C \sum_{n=1}^{\infty} \|w_n(-\lambda)\|_\Sigma^2,$$

where we applied the uniform bound $|\Gamma_n(-\lambda)|^{-1} \leq C$, see (2.8). Using the large argument expansion, see [2],

$$K_0(z) \sim \sqrt{\frac{\pi}{2\xi}} e^{-\xi},$$

we get the estimate

$$|w_n(-\lambda, x)| \leq C \frac{1}{\lambda^{1/4}} e^{-\tau_{\text{min}}(\xi^2 + \lambda)^{1/2}} \quad \text{for} \quad \lambda \to \infty,$$

where $\tau_{\text{min}} = \min_{x \in \Sigma} |\xi|$. This implies that the norm of the second component of (3.12) behaves as $o(\lambda^{-1})$. Combining this result with (3.14) we conclude that $0 \in \rho(I - \beta R_\alpha,\Sigma\Sigma(-\lambda))$ for sufficiently large $\lambda$, which shows that (3.13) holds.

To realize the strategy of [23] we observe that the embedding operator $\tau^* : L^2(\Sigma) \to W^{-1,2}(\Omega)$ acting as $\tau^* f = f + \delta$ is bounded and, moreover,

$$\text{Ran} \tau^* \cap L^2(\Omega) = \{0\}.$$ (3.16)

Suppose that $z \in Z$. Using (3.16) together with theorem 2.1 of [23] we conclude that the expression

$$R_{\alpha,\beta}(z) = R_{\alpha}(z) + R_{\alpha,\Sigma}(z)(I - \beta R_{\alpha,\Sigma}(z))^{-1}R_{\alpha,\Sigma}(z).$$ (3.17)

defines the resolvent of the self adjoint operator.
Theorem 3.1. We have
\[ R_{α,β}(z) = (H_{α,β} - z)^{-1}. \]

Proof. To show the statement we repeat the strategy applied in the proof of theorem 2.1. Operator \( H_{α,β} \) is defined as the self-adjoint extension of \(-\Delta\chi_0^2(Ω(q,1,3))\) determined by imposing boundary conditions (2.5) and (3.11). The idea is to show that any function from the domain \( D(H_{α,β}) \) satisfies (2.5) and (3.11). Since the proof can be done by the mimicking the arguments from the proof of theorem 2.1, we omit further details.

Furthermore, repeating the arguments from the proof of theorem 2.1, we state that
\[ \sigma_{\text{ess}}(H_{α,β}) = [1, \infty). \]

Notation. In the following we will be interested in the spectral asymptotic for small \( |Σ| \). Therefore, we introduce an appropriate scaling with respect to a point \( x_0 \in Σ \). Specifically, for small positive parameter \( δ \) we define \( Σ_δ \) as the graph of \( U \ni q \mapsto x_δ(q) \in Ω \), where
\[ x_δ(q) := δx(q) - δx_0 + x_0. \]

For example, a sphere of radius \( R \) originated at \( x_0 \) turns to the sphere of radius \( δR \) after scaling. Note that equivalence \( |\partial_q x_δ(q) \times \partial_q x_δ(q)| = δ^2|\partial_q x(q) \times \partial_q x(q)| \) implies the scaling of the surface area \( |Σ_δ| = δ^2 |Σ| \).

4. Preliminary results for the analysis of poles

The Birman–Schwinger argument relates the eigenvalues of \( H_{α,β} \) and zeros of \( I - βR_{α,ΣΣ}(z) \) determined by the condition \( \ker(I - βR_{α,ΣΣ}(z)) \neq \{0\} \). To recover resonances we show that \( R_{α,ΣΣ}(z) \) has a second sheet continuation \( R_{α,ΣΣ}^{II}(z) \) and the statement
\[ \ker(I - βR_{α,ΣΣ}^{II}(z)) \neq \{0\} \]
holds for certain \( z \in \mathbb{C}_- \).

4.1. Analytic continuation of \( R_{α,ΣΣ}(z) \)

We start with the analysis of the first component of \( R_{α,ΣΣ}(z) \) determined by \( R_{ΣΣ}(z) \)—see (3.12). Since \( R_{ΣΣ}(z) \) is defined by means of the embedding of kernel \( G(z) \)—see (2.3)—the following lemma will be useful for further discussion.

Lemma 4.1. For any \( k \in \mathbb{N} \) the function \( G(z) \) admits the second sheet continuation \( G^{II}(z) \) through \( J_k := (k^2, (k + 1)^2) \) to an open set \( Π_k \subset \mathbb{C}_- \) and \( ∂Π_k \cap \mathbb{R} = J_k \). Moreover, \( G^{II}(z) \) takes the form
\[ G^{II}(z; x, x', x_3, x'_3) = \frac{1}{2\pi} \sum_{n=1}^{∞} Z_{0}(i\sqrt{z - n^2}) \chi_n(x_3) \chi_n(x'_3), \]
where
\[ Z_{0}(i\sqrt{z - n^2}) = \begin{cases} K_0(-i\sqrt{z - n^2}), & \text{for } n > k, \\ K_0(-i\sqrt{z - n^2}) + i\pi I_0(i\sqrt{z - n^2}), & \text{for } n \leq k, \end{cases} \]
and $I_0(\cdot)$ standardly denotes the Bessel function.

**Proof.** The proof is based on edge-of-the-wedge theorem, i.e. our aim is to establish the convergence

$$G(\lambda + i0) = G^{II}(\lambda - i0),$$

for $\lambda \in J_k$. In fact, it suffices to show that the analogous formula holds for $Z_0$ and $z = \lambda \pm i0$.

Assume first that $n > k$. Then $\sqrt{\lambda - n^2} \mp i0 = \sqrt{\lambda - n^2}$ since $\Im \sqrt{\lambda - n^2} > 0$. Furthermore, the function $K_0(\cdot)$ is analytic in the upper half-plane—consequently, we have $K_0(-i\sqrt{\lambda - n^2} \pm i0) = K_0(\sqrt{n^2 - \lambda})$.

Assume now that $n \leq k$. Then $\sqrt{\lambda - n^2} \pm i0 = \pm \sqrt{\lambda - n^2} \in \mathbb{R}$ which implies

$$K_0(-i\sqrt{\lambda - n^2} + i0) = K_0(-i\sqrt{\lambda - n^2} \rho). \quad (4.20)$$

On the other hand, using the analytic continuation formulae

$$K_0(ze^{i\pi}) = K_0(z) - i m \pi I_0(z) \quad \text{and} \quad I_0(ze^{i\pi}) = I_0(z),$$

for $m \in \mathbb{N}$, we get

$$Z_0(\sqrt{\lambda - n^2} - i0) = K_0(i\sqrt{\lambda - n^2} \rho) + i \pi I_0(-i\sqrt{\lambda - n^2} \rho)$$

This completes the proof. ■

The above lemma provides the second sheet continuation of $R(z)$, as well as $R_{\Sigma \Sigma}(z)$; the latter is defined as the bilateral embedding of $R^{II}(z)$ to $L^2(\Sigma)$.

**Remark 4.2.** Note that for each $k \in \mathbb{N}$ the analytic continuation of $G(\cdot)$ through $J_k$ leads to different branches. Therefore, we have to keep in mind that the analytic continuation of $G(\cdot)$ is $k$-dependent.

In the next lemma we show that the operator $R^{II}_{\Sigma \Sigma, k}(z)$ is bounded and derive the operator norm asymptotics if $\delta \to 0$.

**Lemma 4.3.** Assume that $k \in \mathbb{N}$ and $\lambda \in J_k$. Let $z = \lambda - i\varepsilon$, where $\varepsilon$ is a small positive number. Operator $R^{II}_{\Sigma \Sigma, k}(z)$ is bounded and its norm admits the asymptotics

$$\|R^{II}_{\Sigma \Sigma, k}(z)\| = o(1), \quad (4.21)$$

where the error term is understood with respect to $\varepsilon$.

**Proof.** To estimate the kernel of $R^{II}_{\Sigma \Sigma, k}(z)$ we use (4.19), i.e.

$$G^{II}_{\Sigma \Sigma, k}(z; \rho, x_3, x'_3) = \frac{1}{2\pi} \left( \sum_{n=1}^{\infty} K_0(\kappa_n(z) \rho) \chi_n(x_3) \chi_n(x'_3) \right)$$

$$+ \sum_{n \leq k} I_0(-\kappa_n(z) \rho) \chi_n(x_3) \chi_n(x'_3), \quad (4.22)$$

$$+ \sum_{n \leq k} I_0(-\kappa_n(z) \rho) \chi_n(x_3) \chi_n(x'_3) \quad (4.23)$$
where
\[ \rho = |x - x'|, \quad \text{and} \quad x, x' \in \Sigma_\delta. \]

First, we consider (4.23). The expression
\[ |I_0(-\kappa_n(z)\rho)\chi_n(x)\chi_n(x')| \]
is bounded. Therefore the operator defined by the kernel (4.23) is also bounded and the corresponding operator norm in \( L^2(\Sigma_\delta) \) behaves as \( \|\Sigma_\delta\|^2 = O(\delta^4) \).

The analysis of the term (4.22) is more involving, because it consists of an infinite number of components. The asymptotics:
\[ K_0(\kappa_n(z)\rho) - K_0(n\rho) = \ln \sqrt{1 - \frac{z}{n^2}(1 + O(\rho))} \]
implies
\[ \sum_{n=1}^{\infty} |(K_0(\kappa_n(z)\rho) - K_0(n\rho))\chi_n(x)\chi_n(x')| = C + O(\rho); \quad (4.24) \]
recall that \( \kappa_n(z) \) is defined by (2.4). To estimate \( \sum_{n=1}^{\infty} K_0(n\rho)\cos(na) \) we borrow the idea from [14] and use [25, chapter 10, II, 5.9.1.4.] to get
\[ \sum_{n=1}^{\infty} K_0(n\rho)\cos(na) = \frac{\pi}{2\sqrt{\rho^2 + a^2}} + \frac{1}{2} \left( \ln \frac{\rho}{4\pi} - \psi(1) \right) \]
\[ + \frac{\pi}{2} \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{(2n\pi + a)^2 + \rho^2}} - \frac{1}{2n\pi} \right) \]
\[ + \frac{\pi}{2} \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{(2n\pi - a)^2 + \rho^2}} - \frac{1}{2n\pi} \right). \quad (4.25) \]
\[ (4.26) \]
\[ (4.27) \]
\[ (4.28) \]

For \( x, x' \in \Sigma_\delta \) the terms (4.26) and (4.27) can be majorized by \( C \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right) \), i.e. by a uniform constant. Consequently, using the above estimates together with the equivalence \( \sin a \sin b = \frac{1}{2} \left( \cos(a - b) - \cos(a + b) \right) \) we get, after straightforward calculations,
\[ \left| \sum_{n=1}^{\infty} K_0(\kappa_n(z)\rho)\chi_n(x)\chi_n(x') \right| \leq C \left( \frac{1}{|x - x'|} + \ln \frac{|x - x'|}{\rho} \right); \quad (4.28) \]
the singular terms in the above estimates come from (4.25). Let us analyze the left-hand side of (4.28). First, we consider the component \( \mathcal{P}(x, x') := \frac{1}{|x - x'|} \) which gives
\[ \int_{\Sigma_\delta} \mathcal{P}(x, x')d\Sigma_\delta = (\mathcal{P} * \delta_{\Sigma_\delta})(x) = \int_{\Sigma_\delta} \frac{1}{|x - x'|} d\Sigma_\delta. \]

To conclude the desired convergence we employ the concept of generalized Kato measure. Specifically, since the Dirac delta on \( \Sigma_\delta \) defines Kato measure we obtain
\[
\sup_{x \in \Sigma_d} \int_{\Sigma_d} \mathcal{P}(x, x') d\Sigma_d = o(1),
\]
where the right-hand side asymptotics is understood in the sense of convergence with respect to \(\varepsilon\). Employing the Schur argument we conclude that the norm of the integral operator with the kernel \(\mathcal{P}(x, x')\) acting from \(L^2(\Sigma_d)\) to \(L^2(\Sigma_d)\) behaves as \(o(1)\). The term \(\ln|\lambda - \lambda'|\) contributing to (4.28) can be estimated analogously.

To recover the second sheet continuation of \(R_{\alpha, \Sigma} \Sigma \cdot \cdot\) it remains to construct the analytic extensions of \(\omega_0(z)\) and \(\Gamma_\varepsilon(z)\), see (3.12).

**Lemma 4.4.** Given \(n \in \mathbb{N}\) the functions \(\omega_0(z)\) and \(\Gamma_\varepsilon(z)\) admit the second sheet continuations \(\omega_n^{II}(z)\) and \(\Gamma_n^{II}(z)\) to \(\Pi_k\) through \(J_k = (k^2, (k+1)^2)\), \(k \in \mathbb{N}\) defined by

\[
\omega_n^{II}(z; x) := \frac{1}{2\pi} Z_0(i \sqrt{|z - n^2|^2}) \chi_n(x),
\]
where \(Z_0\) is determined by (4.19), and

\[
\Gamma_n^{II}(z) = \begin{cases} 
\frac{1}{2\pi} \left(2\pi \alpha - \psi(1) + \ln \frac{\sqrt{n^2 - 2\varepsilon}}{2n}\right), & \text{for } n > k \\
\frac{1}{2\pi} \left(2\pi \alpha - \psi(1) + \ln \frac{\sqrt{n^2 - 2\varepsilon} - \pi i}{2n}\right), & \text{for } n \leq k.
\end{cases}
\]

**Proof.** The construction of \(\omega_n^{II}(z)\) can be obtained by mimicking the arguments from the proof of lemma 4.1.

To get \(\Gamma_n^{II}(z)\) we first assume \(k < n\) and \(z = \lambda \pm i\varepsilon, \lambda \in (k^2, (k+1)^2)\). Then

\[
\ln \frac{\sqrt{\lambda - n^2 \pm i0}}{i} = \ln \sqrt{n^2 - \lambda} \quad \text{and, consequently,} \quad \Gamma_n(\lambda + i0) = \Gamma_n^{II}(\lambda - i0).
\]

Assumenow that \(n \leq k\). Then we have \(\lambda - n^2 > 0\) and

\[
\ln \frac{\sqrt{\lambda - n^2 \pm i0}}{i} = \ln \sqrt{\lambda - n^2 \pm i0} + \frac{\pi i}{2},
\]
which implies

\[
\Gamma_n(\lambda + i0) = \Gamma_n^{II}(\lambda - i0) = \frac{1}{2\pi} \left(2\pi \alpha - \psi(1) + \ln \sqrt{\lambda - n^2 - \frac{\pi i}{2}}\right).
\]

This, in view of edge-of-the-wedge theorem, completes the proof.

Henceforth, we assume that \(\varepsilon_\ell \neq k^2\) for any \(k, n \in \mathbb{N}\). Suppose \(z = \lambda - i\varepsilon\), where \(\varepsilon\) is a small non-negative number and \(\lambda \in J_k\). At most one eigenvalue \(\varepsilon_\ell\) can exist in the interval \(J_k\). Assuming that \(z \in (\Pi_k \cup J_k) \setminus \varepsilon_\ell\) we define the analytic functions \(z \mapsto \Gamma_n^{II}(z)^{-1}\) for \(n \in \mathbb{N}\). Then the second sheet continuation of the resolvent takes the form

\[
R_{\alpha, \Sigma}^{II}(z) = R_{\Sigma}^{II}(z) + \sum_{n=1}^{\infty} \Gamma_n^{II}(z)^{-1}(w_n^{II}(z), \cdot)_{\Sigma} w_n^{II}(z)
\]
for \(z \in (\Pi_k \cup J_k) \setminus \varepsilon_\ell\).

**Notation.** In the following we will avoid the superscript \(II\), keeping in mind that all quantities depending on \(z\) are defined for second sheet continuation if \(\Im z < 0\), which admits infinitely many branches \(\Pi_k, k \in \mathbb{N}\).

Assume that \(\varepsilon_\ell \in J_k\). Having in mind the latter purposes we define
for \( z \in (\Pi_k \cup J_k) \setminus \epsilon_l \). The following lemma states the operator norm asymptotics.

**Lemma 4.5.** Operator \( A_l(z) : L^2(\Sigma_\delta) \to L^2(\Sigma_\delta) \) is bounded and the operator norm satisfies

\[
\|A_l(z)\| \leq C|\Sigma_\delta|.
\]  
(4.34)

**Proof.** Suppose that \( z = \lambda - i \epsilon \). We derive the estimates

\[
|\langle A_l(z)f, f \rangle_{\Sigma_\delta}| \leq \left( \sum_{n \neq l} |\Gamma_n(z)^{-1}w_n(z)||^2_{\Sigma_\delta} \right) \|f\|_{\Sigma_\delta}^2
\leq C \left( \sum_{n \neq l} \|w_n(z)\|_{\Sigma_\delta}^2 \right) \|f\|_{\Sigma_\delta}^2;
\]  
(4.35)

to obtain (4.35) we use (4.30). Now our aim is to show

\[
\sum_{n \neq l} \|w_n(z)\|_{\Sigma_\delta}^2 \leq C|\Sigma_\delta|.
\]  
(4.36)

To find a bound for the left-hand side of (4.36) we analyse first the behaviour of \( w_n(z) \) for large \( n \) and \( z \in (\Pi_k \cup J_k) \setminus \epsilon_l \). With this aim we employ (4.29) and (4.19). Note that for \( n > k \) function \( w_n(z) \) admits the representation:

\[
w_n(z, x) = \frac{1}{2\pi} K_0(\kappa_n(z)|x|) \chi_n(x_3),
\]

where \( x \in \Sigma_\delta \). Using again the large argument expansion (3.15) and the fact that \( \Re(-i\sqrt{z} - n^2) \sim n \) we get the estimate

\[
|w_n(z, x)| \leq Ce^{-r_{\min}n},
\]

where \( r_{\min} = \min_{x \in \Sigma_\delta} |x_3|. \) This implies

\[
\sum_{n > k, n \neq l} \|w_n(z)\|_{\Sigma_\delta}^2 \leq C|\Sigma_\delta| = O(\delta^2).
\]  
(4.37)

On the other hand for \( n \leq k \) function \( w_n(z) \) consists of \( K_0 \) and \( I_0 \), see (4.19). Both functions are continuous on \( \Sigma_\delta \) and therefore \( \|w_n\|_{\Sigma_\delta} \leq C|\Sigma_\delta| \). Since the number of such components is finite, in view of (4.37), we come to (4.36) which completes the proof. \( \blacksquare \)

### 5. Complex poles of resolvent

Assume that \( \epsilon_l \in J_k \). Suppose that \( \delta \) is sufficiently small. It follows from lemma 4.3 and 4.5 that the operators \( I - \beta R_{\Sigma_\delta}(z) \) and \( I - \beta A_l(z) \) acting in \( L^2(\Sigma_\delta) \) are invertible for \( z \in (\Pi_k \cup J_k) \setminus \epsilon_l \) and it makes sense to introduce auxiliary notation

\[
G_{\Sigma_\delta}(z) := (I - \beta R_{\Sigma_\delta}(z))^{-1}.
\]
Since the norm of $G_{\Sigma_\delta}(z)A_\delta(z) : L^2(\Sigma_\delta) \to L^2(\Sigma_\delta)$ tends to 0 if $\delta \to 0$ therefore the operator $I + \beta G_{\Sigma_\delta}(z)A_\delta(z)$ is invertible as well.

The following theorem ‘transfers’ the analysis of resonances from the operator equation to the complex valued function equation.

**Theorem 5.1.** Suppose $\epsilon_l \in J_k$ and assume that $z \in (\Pi_k \cup J_k) \setminus \epsilon_l$. Then the condition

$$\ker(I - \beta R_{\alpha,\Sigma_\delta}(z)) \neq \{0\}$$

(5.38)

is equivalent to

$$\Gamma_l(z) + \beta(w_l(z), T_l(z)w_l(z))_{\Sigma_\delta} = 0,$$

(5.39)

where

$$T_l(z) := (I - \beta G_{\Sigma_\delta}(z)A_l(z))^{-1}G_{\Sigma_\delta}(z).$$

**Proof.** The strategy of the proof is partially based on an idea borrowed from [9]. The following equivalences

$$I - \beta R_{\alpha,\Sigma_\delta}(z) = (I - \beta R_{\Sigma_\delta}(z)) (I - \beta G_{\Sigma_\delta}(z)A_l(z))$$

$$- \beta \Gamma_l(z)^{-1}(w_l(z), \cdot)_{\Sigma_\delta} G_{\Sigma_\delta}(z)w_l(z))$$

$$= (I - \beta R_{\Sigma_\delta}(z))(I - \beta G_{\Sigma_\delta}(z)A_l(z))$$

$$\times [I - \beta \Gamma_l(z)^{-1}(w_l(z), \cdot)_{\Sigma_\delta} T_l(z)w_l(z)],$$

show that (5.38) is equivalent to

$$\ker [I - \beta \Gamma_l(z)^{-1}(w_l(z), \cdot)_{\Sigma_\delta} T_l(z)w_l(z)] \neq \{0\}.$$

The above condition is formulated for a rank one operator and, consequently, it is equivalent to (5.39).

Theorem 5.1 shows that the problem of complex poles of resolvent $R_{\alpha,\beta}(z)$ can be shifted to the problem of the roots analysis of

$$\eta(z, \delta) = 0, \quad \text{where} \quad \eta(z, \delta) := \Gamma_l(z) - \beta \sigma_l(z, \delta),$$

(5.40)

and

$$\delta(z, \delta) := (w_l(z), T_l(z)w_l(z))_{\Sigma_\delta}. $$

The further discussion is devoted to figuring out roots of (5.40). In the following we apply the expansion $(1 + A)^{-1} = (1 - A + A^2 - A^3 ...)$ valid if $\|A\| < 1$. Taking $-\beta R_{\Sigma_\delta}(z)$ as $A$ we get

$$G_{\Sigma_\delta}(z) = (I - \beta R_{\Sigma_\delta}(z))^{-1} = I + \hat{R}(z), \quad \hat{R}(z) := \sum_{n=1}^{\infty} (\beta R_{\Sigma_\delta}(z))^n. $$

(5.41)

Expanding the analogous sum for $-\beta G_{\Sigma_\delta}(z)A_l(z)$ one obtains

$$(I - \beta G_{\Sigma_\delta}(z)A_l(z))^{-1} = I + \beta A_l(z) + \beta \hat{R}(z)A_l(z) + ...$$

(5.42)

In view of lemmas 4.5 and 4.3 the norm of $R_{\Sigma_\delta}(z)A_l(z)$ behaves as $o(1)\|A_l(z)\|_{\Sigma_\delta}$ for $\delta \to 0$ and the same asymptotics holds for the operator norm of $\hat{R}(z)A_l(z).$ The further terms
in (5.42) are of smaller order with respect to \( \delta \). Consequently, applying again (5.42) we conclude that \( T_l(z) \) admits the following expansion

\[
T_l(z) = I + \beta A_l(z) + \hat{R}(z) + \ldots
\]  

Using the above statements we can formulate the main result.

**Theorem 5.2.** Suppose that \( \epsilon_l \in J_k \) and consider the function \( \eta_l(z, \delta) : \Pi_k \cup J_k \times [0, \delta_0) \to \mathbb{C} \), where \( \delta_0 > 0 \), defined by (5.40). Then the equation

\[
\eta_l(z, \delta) = 0
\]  

possesses a solution which is determined by the function \( \delta \mapsto z_l(\delta) \in \mathbb{C} \) with the following asymptotics

\[
z_l(\delta) = \epsilon_l + \mu_l(\delta), \quad |\mu_l(\delta)| = o(1).
\]  

Moreover, the lowest order term of \( \mu_l(\cdot) \) takes the form

\[
\mu_l(\delta) = 4\pi \alpha \beta \left\{ \| \omega_l(\epsilon_l) \|^2_{\Sigma_{J_k}} \right\}
\]

\[
+ \beta \sum_{n \neq l} \Gamma_n(\epsilon_l)^{-1} |(\omega_l(\epsilon_l), \omega_n(\epsilon_l))_{\Sigma_{J_k}}|^2
\]

\[
+ (\omega_l(\epsilon_l), \hat{R}(\epsilon_l) \omega_l(\epsilon_l))_{\Sigma_{J_k}}. 
\]

**Proof.** Note that \( z \mapsto \eta_l(z, \delta) \)---see (5.40)---is analytic and \( \eta_l(\epsilon_l, 0) = 0 \). Using (4.30) one obtains

\[
\left. \frac{d\Gamma_l(z)}{dz} \right|_{z=\epsilon_l} = \frac{1}{4\pi \alpha} < 0, \quad n \in \mathbb{N}.
\]

Combining this with

\[
\left. \frac{\partial \eta_l(z, \delta)}{\partial z} \right|_{z=\epsilon_l, \delta=0} = 0,
\]

we get \( \frac{\partial \eta_l(z, \delta)}{\partial \delta} \bigg|_{\delta=0} = \frac{1}{4\pi \alpha} \neq 0 \). In view of the implicit function theorem we conclude that the equation (5.40) admits a unique solution which a continuous function of \( \delta \mapsto z_l(\delta) \) and \( z_l(\delta) = \epsilon_l + o(1) \). To reconstruct asymptotics of \( z(\cdot) \) first we expand \( \Gamma_l(z) \) into the Taylor sum

\[
\Gamma_l(z) = \frac{1}{4\pi \alpha} (z - \epsilon_l) + \mathcal{O}((z - \epsilon_l)^2).
\]

Then the spectral equation (5.40) reads

\[
z = \epsilon_l + 4\pi \alpha \beta z_l(z, \delta) + \mathcal{O}((z - \epsilon_l)^2).
\]

Now we expand \( z_l(z, \delta) \). Using (5.43) and (4.33) we reconstruct its first order term which reads
\[
\left\{ \| w_l(\epsilon_l) \|_{\Sigma_\delta}^2 + \beta \sum_{n \neq l} \Gamma_n(\epsilon_l)^{-1} |(w_l(\epsilon_l), w_n(\epsilon_l))_{\Sigma_\delta}|^2 \\
+ (w_l(\epsilon_l), R_l(\epsilon_l)w_l(\epsilon_l))_{\Sigma_\delta} \right\}.
\]

Applying the asymptotics \( z_l(\epsilon_l) = \epsilon_l + o(\delta) \) and the fact that \( \vartheta_l(\cdot, \cdot) \) is analytic with respect to complex variables we get the formula for \( \mu(\cdot) \).

5.1. Analysis of imaginary part of the pole

Since the imaginary component of resonance pole has a physical meaning we dedicate a special discussion to this problem. The information on the lowest order term of the pole imaginary component is contained in (5.47) and (5.48). On the other hand, note that only the components subscripted by \( n \leq k \) admit a non-zero imaginary parts. Therefore

\[
\Im \left( 4\pi\xi_\alpha \beta \left( \sum_{n \leq k} \Gamma_n(\epsilon_l)^{-1} |(w_l(\epsilon_l), w_n(\epsilon_l))_{\Sigma_\delta}|^2 \\
+ (w_l(\epsilon_l), R_l(\epsilon_l)w_l(\epsilon_l))_{\Sigma_\delta} \right) \right).
\]

(5.49)

and

\[
\Im \left( 4\pi\xi_\alpha \beta \left( \frac{1}{\imath_{l,n} + (1/2)^2} (\imath_{l,n} + \frac{1}{2}) \right) \right).
\]

(5.50)

determines the lowest order term of \( \Im(\mu(\delta)) \).

Sign and asymptotics of \( \Im(\mu(\delta)) \) with respect to \( \Sigma_\delta \). Recall that \( \epsilon_l \in J_k \). First we analyse (5.49) and with this aim we define

\[
\imath_{l,n} := \frac{1}{2\pi} \left( 2\pi \alpha + \ln \sqrt{\epsilon_l - n^2} - \psi(1) \right),
\]

for \( n \leq k \). Relying on (4.31) we get

\[
\Gamma_l(\epsilon_l)^{-1} = \frac{1}{\imath_{l,n} + (1/2)^2} \left( \imath_{l,n} + \frac{1}{2} \right)
\]

if \( n \leq k \). Consequently, formula (5.49) is equivalent to

\[
\Im 4\pi\xi_\alpha \beta^2 \sum_{n \leq k} \frac{1}{2 \imath_{l,n} + (1/2)^2} |(w_l(\epsilon_l), w_n(\epsilon_l))_{\Sigma_\delta}|^2.
\]

The above expression is negative because \( \xi_\alpha < 0 \). Moreover, since both \( w_l(\epsilon_l) \) and \( w_l(\epsilon_n) \) are continuous in \( \Omega \setminus I \) we have \(|(w_l(\epsilon_l), w_n(\epsilon_l))_{\Sigma_\delta}|^2 \sim |\Sigma_\delta|^2 \). This means that (5.49) behaves as \( \mathcal{O}(|\Sigma_\delta|^2) \). To recover the asymptotics of (5.50) we restrict ourselves to the lowest order term of \( R(z) \)—see (5.43)—namely

\[
\psi_l := \Im 4\pi\xi_\alpha \beta^2 (w_l(\epsilon_l), R_{\Sigma_\delta}(\epsilon_l)w_l(\epsilon_l))_{\Sigma_\delta}.
\]

Using analytic continuation formulae (4.20) and employing the small argument expansion—see [2],

\[
K_0(z) \sim -\ln z,
\]

16
where \(-\pi < \arg z < \pi\) states the plane cut for the logarithmic function, one gets

\[ v_l \sim \Im \xi_\alpha \beta^2 \sum_{n \leq k} \left( \int_{\Sigma_\delta} w_l(\epsilon_l) \chi_n \right)^2 = O(|\Sigma_\delta|^2). \]

One can easily see that \(v_l \leq 0\). Summing up the above discussion we can formulate the following conclusion.

**Proposition 5.3.** \(The \) resonance pole takes the form \(z_l(\delta) = \epsilon_l + \mu(\delta)\) \text{ with the lowest order of } \(\Im \mu(\delta)\) \text{ given by}

\[ \pi \xi_\alpha \beta^2 \sum_{n \leq k} \left( \frac{2}{i_n + (1/2)} |(w_l(\epsilon_l), w_n(\epsilon_l))_{\Sigma_\delta}|^2 + \left( \int_{\Sigma_\delta} w_l(\epsilon_l) \chi_n \right)^2 \right). \]

It follows from the above formula that \(\Im \mu(\delta) \leq 0\) and the asymptotics

\[ \Im \mu(\delta) = O(|\Sigma_\delta|^2) \]

holds. Moreover, the lowest order of \(\Im \mu(\delta)\) is independent of sign of \(\beta\).

Note that for the special geometrical cases the embedded eigenvalues can survive after introducing \(\Sigma_\delta\) since the “perturbed” eigenfunctions are not affected by the presence of \(\Sigma_\delta\). Let us consider

\[ \Pi_l := \{ x \in \Omega : x = \left( \frac{x, \pi}{l} \right), \quad l \in \mathbb{N} \} \]

and assume that \(\Sigma_\delta \subset \Pi_l\). Then \(w_m(z) = 0\) for each \(m \in \mathbb{N}\) and, consequently, \(\vartheta_m(z, \delta) = 0\) —see (5.40). This implies the following statement.

**Proposition 5.4.** \(Suppose \) that \(\Sigma \subset \Pi_l\). \(Then\) for all \(m \in \mathbb{N}\) the numbers \(\epsilon_{ml}\) remain the embedded eigenvalues of \(H_{\alpha,\beta}\).

### Acknowledgments

The author thanks the referees for reading the paper carefully, removing errors and recommending various improvements in exposition.

The work was supported by the project DEC-2013/11/B/ST1/03067 of the Polish National Science Centre.

### References

[1] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 2004 *Solvable Models in Quantum Mechanics* 2nd printing (Providence, RI: American Mathematical Society) (with appendix by ed P Exner)

[2] Abramowitz M and Stegun I 1972 *Handbook of Mathematical Functions* (Mansfield Centre, CT: Martino Publishing)

[3] Baskin L M, Flamenkovskii B A and Sarafanov O V 2013 Effect of magnetic field on resonant tunneling in 3D waveguides of variable cross-section *J. Math. Sci.* 196 469–89

[4] Baskin L M, Kabardov M, Neittaanmäki P, Flamenkovskii B A and Sarafanov O V 2013 Asymptotic and numerical study of resonant tunneling in two-dimensional quantum waveguides of variable cross section *Comput. Math. Math. Phys.* 53 16641683
[5] Behrndt J, Exner P, Holzmann M and Lotoreichik V 2017 Approximation of Schrödinger operators with delta-interactions supported on hypersurfaces Math. Nach. 290 1215–48
[6] Borisov D, Exner P and Golovina A 2013 Tunneling resonances in systems without a classical trapping J. Math. Phys. 54 012102
[7] Brasche JF, Exner P, Kuperin Y A and Šeba P 1994 Schrödinger operators with singular interactions J. Math. Anal. Appl. 184 112–39
[8] Briet P and Gharsalli M 2013 Stark resonances in 2-dimensional curved quantum waveguides Rep. Math. Phys. 76 317–38
[9] Chuburin Y P 2005 Perturbation theory of resonances and embedded eigenvalues of the Schrödinger operator for a crystal film Teor. Mat. Fiz. 143 417–30
[10] Delitsyn AL, Nguyen B-T and Grebenkov DS 2012 Trapped modes in finite quantum waveguides Eur. Phys. J. B 85 1–12
[11] Exner P and Kondej S 2004 Schrödinger operators with singular interactions: a model of tunneling resonances J. Phys. A: Math. Gen. 37 8255–77
[12] Exner P and Kovařík H 2015 Quantum Waveguides (New York: Springer)
[13] Exner P and Krejčiřík D 1999 Quantum waveguides with a lateral semitransparent barrier: spectral and scattering properties J. Phys. A: Math. Gen. 32 4475–94
[14] Exner P and Nemcová K 2002 Quantum mechanics of layers with a finite number of point perturbation J. Math. Phys. 43 1152–84
[15] Frolov S V and Popov I Y 2000 Resonances for laterally coupled quantum waveguides J. Math. Phys. 41 4391–405
[16] Frolov S V and Popov I Y 2003 Three laterally coupled quantum waveguides: breaking of symmetry and resonance asymptotics J. Phys. A: Math. Gen. 36 1655–70
[17] Kato T 1980 Perturbation Theory for Linear Operators (New York: Springer)
[18] Kondej S 2012 Resonances induced by broken symmetry in a system with a singular potential Ann. Henri Poincaré 13 1451–67
[19] Kondej S and Krejčiřík D 2013 Spectral analysis of a quantum system with a double line singular interaction Publ. RIMS Kyoto Univ. 49 831–59
[20] Kondej S and Leotfski W 2014 Mathematical and theoretical J. Phys. A: Math. Theor. 47 1416–38
[21] Kovařík H and Sacchetti A 2007 Resonances in twisted quantum waveguides J. Phys. A: Math. Theor. 40 8371
[22] Lions J L and Magenes E 1972 Non-Homogeneous Boundary Value Problems and Applications vol I (Heidelberg: Springer)
[23] Posilicano A 2001 A krein-like formula for singular perturbations of self-adjoint operators and applications J. Funct. Anal. 183 109–47
[24] Posilicano A 2004 Boundary triples and Weyls function for singular perturbations of self-adjoint operators Meth. Funct. Anal. Topol. 10 57–63
[25] Prudnikov Y A P and Marichev O I 1981–1983 Integraly i rady, I. Elementarnye funkcii, II. Specialnye funkcii, III (Moskva: Nauka)