Deviations From Newton’s Law in Supersymmetric Large Extra Dimensions

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Abstract: Deviations from Newton’s Inverse-Squared Law at the micron length scale are smoking-gun signals for models containing Supersymmetric Large Extra Dimensions (SLEDs), which have been proposed as approaches for resolving the Cosmological Constant Problem. Just like their non-supersymmetric counterparts, SLED models predict gravity to deviate from the inverse-square law because of the advent of new dimensions at sub-millimeter scales. However SLED models differ from their non-supersymmetric counterparts in three important ways: (i) the size of the extra dimensions is fixed by the observed value of the Dark Energy density, making it impossible to shorten the range over which new deviations from Newton’s law must be seen; (ii) supersymmetry predicts there to be more fields in the extra dimensions than just gravity, implying different types of couplings to matter and the possibility of repulsive as well as attractive interactions; and (iii) the same mechanism which is purported to keep the cosmological constant naturally small also keeps the extra-dimensional moduli effectively massless, leading to deviations from General Relativity in the far infrared of the scalar-tensor form. We here explore the deviations from Newton’s Law which are predicted over micron distances, and show the ways in which they differ and resemble those in the non-supersymmetric case.
1. Introduction

The recent discovery of a non-vanishing Dark Energy density [1] throws into sharp relief the general theoretical befuddlement about how to predict reliably the gravitational response to the energy of the vacuum [2]. In a nutshell, the difficulty arises because in four dimensions gravity responds to the energy density of the vacuum, $\rho$, as if it were a cosmological constant. But $\rho$ generically receives its largest contributions from the quantum zero-point energy of the shortest-wavelength modes, for instance with modes having wavelength $\ell$ contributing in four dimensions an amount of order $\delta \rho \sim \ell^{-4}$. As such, the theoretical prediction is sensitive to the most microscopic of a theory’s details, and since any theory describing the known particle types
has $\ell^{-1}$ larger than 100 GeV, the theoretical prediction is generically many orders of magnitude larger than the observed value:\(^1\)

$$\rho_{\text{obs}} = \frac{\hbar c}{a^4} \quad \text{with} \quad \frac{\hbar c}{a} \sim 3 \times 10^{-3} \text{ eV}. \quad (1.1)$$

1.1 Naturalness Issues

Until the observational result was found to be nonzero, the theoretical goal had been to identify a theory which is both experimentally successful and technically natural in that sense that the contributions to the vacuum energy for some reason cancel once summed over the relevant particle content. Now that we know the result is nonzero the bar has been raised, and a technically natural solution would instead require a cancellation only for those modes having scales $\ell < a$. Although this kind of naturalness is not absolutely required by fundamental principles, it amounts to the requirement that the small hierarchy can be understood within an effective theory of microscopic physics, regardless of the scales for which this theory is formulated. Technical naturalness is a conservative requirement because we know that it applies for all of the many other hierarchies of scale for which we have solid evidence, in all branches of physics.

Yet it is hard to modify physics at scales above $\ell \sim a$ to get a small enough vacuum energy without also running into conflict with the many experimental measurements which are available for such scales. More recently, despair at making progress along these lines has led many to abandon the criterion of technical naturalness altogether and to turn to anthropic ideas along the lines proposed by Weinberg more than ten years ago [3, 4]. This point of view has gained additional momentum from the observation that anthropic reasoning might be required to make predictions within string theory, given the enormous number of vacua which arise there [5]. This line of argument has emboldened others to abandon using technical naturalness as a criterion for understanding the electro-weak hierarchy [6].

On the other hand, potential progress has also been made in obtaining a small gravitational response to the vacuum energy without abandoning technical naturalness, within the framework of Supersymmetric Large Extra Dimensions (SLED) [7, 8, 9]. The idea behind this proposal is that it is indeed possible to modify physics at scales $\ell \lesssim a$ in such a way as only to modify gravitational interactions without changing non-gravitational physics. As a result, although the vacuum energy may

\(^1\)From here on we adopt units for which $\hbar = c = 1$. 

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be large it may be possible to modify how this vacuum energy gravitates for scales \( \ell < a \) without also ruining the agreement between theory and the vast number of non-gravitational experiments. The framework within which this can be done is that of large extra dimensions \([10]\), according to which two internal dimensions are taken to be as large as they can possibly be without coming into conflict with observations. Interestingly enough, the largest radius which is possible for such extra dimensions turns out to be \( r \sim a \), as imposed by tests of Newton’s Law at short distances \([11, 12]\).

It is proposed that within such models supersymmetry in the extra dimensions can play the crucial role of cancelling the contributions to the vacuum energy from all modes having wavelengths \( \ell \lesssim r \sim a \). There is currently considerable activity devoted to understanding what the gravitational response to the vacuum energy would look like within this framework, in order to see whether this kind of modification really provides a technically natural description of the dark energy \([7, 8, 9, 13, 14]\).

### 1.2 The Smoking Gun

A particularly attractive feature of the SLED framework is that it is very predictive, and so eminently falsifiable. This predictiveness ultimately comes because technical naturalness requires the proposal to modify physics at length scales \( \ell \lesssim a \) (or energies above \( a^{-1} \sim 10^{-3} \text{ eV} \)) — a range of scales to which we have a great deal of experimental access. In particular SLED models add to the observational implications for the Large Hadron Collider (LHC) of the earlier large-extra-dimensional models \([15]\), as well as predicting new features to do with supersymmetry \([16, 17, 18]\). As such, SLED models provide a unique framework which links the technical naturalness of the vacuum energy with observational implications at high-energy accelerators.

Given that its raison d’être is modifying the gravitational response of the vacuum, it is the modifications to gravity which provide the most definitive signatures for the SLED proposal. These come in two types, corresponding to deviations from General Relativity over both sub-millimeter and cosmological distance scales. Over very large distances the model behaves like a scalar-tensor theory, with the light scalar (or scalars) describing the dynamics of the moduli of the extra dimensions. Unlike for non-supersymmetric models having large extra dimensions, within SLED models there is an understanding of why these modes remain essentially massless in a technically natural way, which is a consequence of the same mechanism which is purported to protect the vacuum energy in these models \([19]\). This leads to im-
lications for the time-dependence of the Dark Energy density [19, 20], as well as model-dependent implications for long-distance tests of gravity.

However, it is tests of Newton’s Law over micron length scales which ultimately provide the smoking gun for SLED models. This is because SLED models rely for their success on the existence of two large extra dimensions, and on the coincidence between the size, \( r \), of the extra dimensions and the observed scale, \( a \), in the observed Dark Energy density. This means that Newton’s law must be violated at length scales, \( \lambda \), which are of order the Kaluza Klein (KK) masses for the large extra dimensions: \( \lambda \sim r/2\pi \sim a/2\pi \), which works out to be around a micron in size [17]. Since the size of these dimensions cannot be shrunk without making the observed vacuum energy too large, there is no way to escape the implication that Newton’s law must change once tested over these distances.

Since these deviations are predicted over distances which are not too far from experimental reach, we can hope that the theory can be definitively tested within the not-to-distant future. For these purposes it is important to have precise predictions for what kinds of deviations should be expected. It is the purpose of this paper to provide a first calculation of the short-distance behaviour of force laws within the SLED picture, working within the framework of the simplest (toroidal) compactifications. In particular, we compute the forces which should be expected between bodies which are separated in the visible dimensions but are not displaced in the extra dimensions, due to the mediation of the various bosons which propagate within the extra dimensions. As might be expected, there are more of these bosons than arise in non-supersymmetric models due to the additional particle content which supersymmetry in the bulk requires. Some of these additional bosons can mediate spin-dependent forces, and the detection of such could provide a way to distinguish between the supersymmetric from non-supersymmetric options.

We organize our results as follows. The next section, §2, briefly describes the bosonic field content of 6D supergravity theories, and then describes the quantization of their linearized fluctuations about flat space. The exchange of these fluctuations is then used in §3 to compute the interaction between slowly-moving, localized classical sources, and identify their interaction potential energy. We also identify what kinds of charges such sources can carry in the static limit, and follow how the interaction energy depends on these charges. Finally, we close in §4 with a concluding discussion.
2. Forces Mediated by 6D Bosons

In this section we describe the bosonic fields which make up the bulk sector in models with supersymmetric large extra dimensions. We then identify how these bulk particles can couple to brane degrees of freedom, and compute the forces which are obtained when they are exchanged by two sources which are both localized on the same brane.

2.1 6D Supergravity

The field content of practical interest for SLED models is the bosonic sector of 6D supergravity coupled to various forms of super-matter. In (2,0) supersymmetry the bosonic field content of the supergravity multiplet consists of the metric, $g_{MN}$, and a 2-form gauge potential which is subject to a six-dimensional self-duality condition. This is normally combined with a tensor multiplet, whose bosonic content contains another 2-form potential having the opposite self-duality property, and so which combines with the previous one into an unconstrained potential, $B_{MN}$. The other boson in the tensor multiplet is a scalar field, $\phi$, known as the dilaton. The most common types of (2,0) matter multiplets are gauge multiplets — whose bosonic parts consist of the gauge potentials, $A^a_M$, of some gauge group — or hypermultiplets — whose bosonic part consists of scalar fields, $\Phi^a$. These latter fields parameterize a coset space, $G/H$, and so nominally describe the Goldstone modes for the symmetry breaking pattern $G \rightarrow H$. For some supergravities a scalar potential, $v(\Phi)$, can exist for these fields, in which case the symmetry $G$ is explicitly broken and the fields $\Phi^a$ become instead pseudo-Goldstone particles. Alternatively, if a subgroup of $G$ is gauged, then some of these hyper-scalars can be eaten by the gauge fields through the usual Anderson-Higgs-Kibble mechanism.

For a broad class of supergravities these bosons are governed by the following action\(^2\)

$$S = \int d^6 x \sqrt{-g} \left[-\frac{1}{2\kappa^2} g^{MN} \left( R_{MN} + \partial_M \phi \partial_N \phi + G_{ab}(\Phi) \partial_M \Phi^a \partial_N \Phi^b \right) - \frac{1}{12} e^{-2\phi} G_{MNP} G^{MNP} - \frac{1}{4} e^{-\phi} F^\alpha_{MN} F^\alpha_{MN} \right] , \quad (2.1)$$

\(^2\)Our metric is ‘mostly plus’ and we adopt Weinberg’s curvature conventions \(^2\), which differ from those of Misner, Thorne and Wheeler \(^2\) only in an overall change of sign of the Riemann tensor.
where $\kappa = M^{-2}$ with $M$ being the 6D Planck mass, which is of order 10 TeV [25]. The quantity $G_{ab}(\Phi)$ represents a $G$-invariant metric on the coset space $G/H$, and we here focus our attention on the case where no scalar potential exists. Supersymmetry puts a number of restrictions on the form the target-space metric, $G_{ab}$ can take, but these do not play a role in what follows. Finally, $F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + c^a_{\beta\gamma} A_M^\beta A_N^\gamma$ is the usual gauge-covariant field strength, and the field strength, $G_{MNP}$, for the 2-form potential, $B_{MN}$, is given by

$$G_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN} + \bar{c} \kappa \Omega_{MNP}, \quad (2.2)$$

where $\Omega_{MNP}$ is the Chern-Simons form, given for abelian gauge groups by $\Omega_{MNP} = F_{MN} A_P + F_{NP} A_M + F_{PM} A_N$. $\bar{c}$ here is a supergravity-specific constant, which can vanish for some of the extra-dimensional tensors appearing in the bulk supergravity.

The equations of motion obtained from the action, (2.1), are given by Einstein’s equation,

$$\frac{1}{\kappa^2} R_{MN} = \partial_M \phi \partial_N \phi + G_{ab}(\Phi) \partial_M \Phi^a \partial_N \Phi^b + \frac{1}{2} e^{-2\phi} G_{MAB} G_{N}^{AB} - e^{-\phi} F_{MA} F_{NB}$$

$$- \frac{1}{4} g_{MN} \left[ \frac{1}{3} e^{-2\phi} G_{ABC} G^{ABC} - \frac{1}{2} e^{-\phi} F_{AB} F^{AB} \right], \quad (2.3)$$

supplemented by the other field equations

$$0 = \frac{1}{\kappa^2} \Box \phi + \frac{1}{6} e^{-2\phi} \Omega_{MNP} G^{MNP} + \frac{1}{4} e^{-\phi} F_{MN} F^{MN} \quad (2.4)$$

$$0 = \Box \Phi^a + \Gamma^a_{bc}(\Phi) g^{MN} \partial_M \Phi^b \partial_N \Phi^c \quad (2.5)$$

$$0 = \nabla_P \left( e^{-2\phi} G_P^{MN} \right) \quad (2.6)$$

$$0 = \nabla_N \left( e^{-\phi} F^{NM} \right) + \kappa e^{-2\phi} G^{MAB} F_{AB} \quad . \quad (2.7)$$

Here the $\Gamma^a_{bc}(\Phi)$ represent the Christoffel symbols built in the usual way from the target-space metric, $G_{ab}(\Phi)$.

As noted earlier, these expressions do not include a potential for the scalar fields, as would be appropriate for ungauged 6D supergravity. Even for ungauged supergravity it could also happen that a scalar potential is generated for fields like $\phi$ if some of the gauge field strengths, $F_{MN}$ or $G_{MNP}$, are nonzero in the background. We do not consider here potentials coming from either sources since the success of the SLED mechanism relies on these potentials not being generated with a size which is large enough to be relevant to our present focus: the forces between macroscopic objects over sub-millimeter distances.
2.2 Toroidal Compactification

The simplest solution to these equations is the trivial one, for which the scalars are constants while all gauge field-strengths vanish: \( F^a_{MN} = 0 \) and \( G_{MNP} = 0 \). Since these assumptions ensure there is no matter stress-energy, the simplest metric configuration is flat space: \( g_{MN} = \eta_{MN} \). We compactify to 4 dimensions by taking two of these flat directions \((x^m, m = 4, 5)\) to fill out a compact 2-torus, \( T_2 \), of volume \( A \), while the other four dimensions \((x^\mu, \mu = 0, \ldots, 3)\) are large.

For these purposes we define \( T_2 \) by identifying points on the flat plane according to
\[
(x^4, x^5) \sim (x^4 + n_2 r_2 \cos \theta + n_1 r_1; x^5 + n_2 r_2 \sin \theta),
\]
where \( n_{1,2} \) are integers and \( \theta, r_1 \) and \( r_2 \) are the three real moduli of the 2-torus. Equivalently, in terms of the complex coordinate \( z = x^4 + i x^5 \) this is
\[
z \equiv z + (n_2 \tau + n_1) r_1,
\]
with the complex quantity \( \tau \) defined by \( \tau = (r_2/r_1) e^{i \theta} = \tau_1 + i \tau_2 \), and as defined \( \tau \) takes values in the upper half-plane, \( \tau_2 > 0 \). In terms of these quantities, the volume of the 2-torus becomes \( A = r_1 r_2 \sin \theta \).

To set notation, we now dimensionally reduce a free 6D scalar field on such a background. Consider therefore a 6D scalar field \( \psi \) whose 6D field equation is\(^3\)
\[
(-\Box_6 + m^2) \psi,
\]
propagating on a spacetime compactified to 4 dimensions on the above 2-torus. We assume the scalar satisfies the following boundary conditions
\[
\psi(x^\mu, x^4 + n_2 r_2 \cos \theta + n_1 r_1; x^5 + n_2 r_2 \sin \theta) = e^{2 \pi i (n_1 \rho_1 + n_2 \rho_2)} \psi(x^\mu, x^4, x^5),
\]
with \( 0 \leq \rho_{1,2} < 1 \) being two real quantities. The choices \( \rho_{1,2} = 0, \frac{1}{2} \) correspond to taking periodic or anti-periodic boundary conditions along the torus’ two cycles, although more general values of \( \rho_i \) are also possible.

The scalar field \( \phi \) may be expanded in terms of the eigenfunctions of the 2D Laplacian, \( \Box_2 = \partial^2_4 + \partial^2_5 \), corresponding to
\[
\psi(x, y) = \sum_p \psi_p(x) u_p(y),
\]
where we denote the 4D coordinates, \( x^\mu \), collectively by \( x \) and the 2D coordinates, \( x^m \), as \( y \) (with \( y^1 = x^4 \) and \( y^2 = x^5 \)). The mode functions are given by plane

\(^3\)The subscript ‘6’ here emphasizes that the d’Alembertian appearing here is the 6-dimensional one.
waves, \( u_p(y) = \mathcal{A}^{-1/2} e^{ip \cdot y} \), where the above boundary conditions imply the allowed extra-dimensional momenta are given by

\[
p \cdot y = 2\pi \left[ \frac{(n_1 + \rho_1)(y_1 - y_2 \cot \theta)}{r_1} + \frac{(n_2 + \rho_2)y_2 \csc \theta}{r_2} \right]. \tag{2.12}
\]

These modes satisfy \( \Box_2 u_p = -p^2 u_p \) with

\[
p^2 = \left( \frac{2\pi}{\mathcal{A} \tau_2} \right)^2 \left| n_2 + \rho_2 - \tau(n_1 + \rho_1) \right|^2. \tag{2.13}
\]

Viewed from a 4D perspective, each of the KK modes, \( \psi_p \), is a 4D scalar field satisfying the field equation

\[
(\Box_4 + p^2 + m^2) \psi_p = 0, \tag{2.14}
\]

and so whose 4D mass is related to the 6D mass, \( m \), of the full 6D field \( \psi \), by \( M_p^2 = p^2 + m^2 \). Similar decompositions also hold for free higher-spin fields compactified on a 2-torus, as is described in more detail (as needed) below.

### 2.3 Linear Fluctuations

The above free-field expressions are useful for the present purposes because they may be applied to the dynamics of small fluctuations about the flat vacuum solution described above. The free-field results describe the leading (linear) parts of these fluctuations. In particular, we wish to explore the potential energy which is set up by various sources through the exchange of small fluctuations in the bosonic fields which arise in 6D supergravity, described earlier.

To this end we expand the action, (2.1), about the toroidal background configuration described above, working to quadratic order in the deviations of the fields \( \phi \), \( \Phi^a \), \( A_M \), \( B_{MN} \) and \( g_{MN} \) about this background. For an appropriate choice for the various gauge-averaging terms (more about which below) this leads to a quadratic action within which fields of differing spins do not mix:

\[
S_{\text{quad}} = S_\phi + S_\Phi + S_A + S_B + S_h. \tag{2.15}
\]

Each term in this action is described in more detail in the following sections.

**The Dilaton**

In the absence of background gauge field strengths, \( F_{MN} = G_{MNP} = 0 \), there is no potential for the dilaton and so any constant value \( \phi_0 \) provides an equally good
background value. Dividing the dilaton into background and fluctuation according to
\[ \phi = \phi_0 + \kappa \hat{\phi}, \]  
(2.16)
and expanding the action to quadratic order in the fluctuation, \( \hat{\phi} \), then leads to
\[ S_\phi = -\frac{1}{2} \int d^6x \eta^{MN} \left( \partial_M \hat{\phi} \partial_N \hat{\phi} \right). \]  
(2.17)

Since this has the same form as the example discussed above of dimensional reduction of a free scalar field on \( T_2 \) (in the special case \( m = 0 \)), the results of this earlier section may be taken over in whole cloth. In particular we decompose \( \hat{\phi} \) in terms of toroidal modes as in eq. (2.11), leading to the tower of KK masses given in eq. (2.13). The momentum-space propagator for such a 6D field is then
\[ D^\phi(k^\mu, p) = \frac{1}{k^M k_M} = \frac{1}{k^2 + p^2}, \]  
(2.18)
where \( k^M = \{ k^\mu, p \} \), with \( k^\mu \) denoting the (time-like) 4-momentum in the four visible directions (with 4D contraction \( k^2 = k^\mu k_\mu \)) and \( p \) representing the (Euclidean) 2-momentum in the compact two dimensions (with 2D contraction \( p^2 = p \cdot p \)).

The dilaton can couple directly to brane matter, depending on the nature of the microscopic physics of the brane. For instance for \( D \)-branes in string theory, branes typically couple (in the absence of background gauge and Kalb-Ramond fields) to the dilaton with strength
\[ S_b = \int d^d \xi \sqrt{-\gamma} e^{\lambda \phi} T, \]  
(2.19)
where \( T \) is the brane tension, \( d \) is the spacetime dimension of the brane, \( \lambda \) is a constant whose value depends on the kind of brane involved, and \( \xi^\mu \) are coordinates on the brane world-sheet which we take to sweep out the surface \( x^M(\xi) \) in spacetime. Also, \( \gamma = \det \gamma_{\mu\nu} \) where \( \gamma_{\mu\nu} = g_{MN} \partial_\mu x^M \partial_\nu x^N \) denotes the induced metric on the brane world-sheet. For some branes it can happen that \( \lambda \) vanishes, as it does for instance in the case of \( D3 \)-branes in the 10D Einstein frame.

To linear order in the fluctuation \( \hat{\phi} \) we parameterize the dilaton interaction with the form
\[ S_{\text{int}} = \kappa \int d^d \xi \sqrt{-\gamma} \hat{\phi} \rho, \]  
(2.20)
where \( \rho \) denotes the local source density. For instance for a point particle of mass \( m \) on a 3-brane one might have \( \rho = \lambda [T + m \delta^3(\xi)] \), for some mass \( m \). In general, such a coupling to the brane tension, \( T \), can ruin the existence of the flat background
solution of interest here due to the back-reaction which the presence of a configuration of branes induces on the intervening 6D bulk fields. Indeed, an understanding of how the bulk adjusts is central to the issue of whether 6D supergravity provides a naturally small dark-energy density. In what follows we imagine that this back-reaction can be negligible to the linearized order to which we work, and focus on the implications for experiments of the dilaton couplings to localized particles.

Goldstone Modes

The discussion of the hyper-scalars, $\Phi^a$, follows a similar route, once the fields are expanded into background plus fluctuation according to

$$\Phi^a = \varphi^a + \kappa \hat{\Phi}^a. \quad (2.21)$$

The main difference is in this case the nature of the couplings to brane-bound particles which might be expected.

In general, for constant background fields, $\partial_M \varphi^a = 0$, it is always possible\(^4\) to rescale the fluctuations $\hat{\Phi}^a$ such that $G_{ab}(\varphi) = \delta_{ab}$, in which case the quadratic action for $\hat{\Phi}^a$ becomes equivalent to $N$ copies of the dilaton action, $S_\Phi$:

$$S_\Phi = -\frac{1}{2} \int d^6x \delta_{ab} \eta^{MN} \partial_M \hat{\Phi}^a \partial_N \hat{\Phi}^b. \quad (2.22)$$

This then leads to the following momentum-space propagator for $\hat{\Phi}^a$

$$D_{ab}^\Phi(k^\mu, p) = \frac{\delta_{ab}}{k^2 + p^2}, \quad (2.23)$$

where $k^2$ and $p^2$ are defined as before.

Because $\Phi^a$ arises as a Goldstone boson its couplings are typically derivative couplings, which precludes there being a dilaton-like coupling of the form (2.19) directly to brane matter. Instead, we expect the linearized coupling to have the general form

$$S_{\text{int}} = \kappa \int d^d\xi \sqrt{-\gamma} j_a^M \partial_M \hat{\Phi}^a, \quad (2.24)$$

where $j_a^M$ denotes the contribution of background brane fields to the conserved current of the symmetry for which $\Phi^a$ is the Goldstone boson.

\(^4\)We assume here the target space manifold to have Euclidean signature.
2.4 Gauge Fields

To quadratic order the expansion of the gauge-field part of the supergravity action about $A^\alpha_M = 0$ is equivalent to that for a collection of non-interacting (abelian) gauge potentials,

$$S_A = -\int d^6x \left[ \frac{1}{4} F^{\alpha}_M F^M_N + \frac{1}{2\xi_1}(\partial^M A^\alpha_M)^2 \right],$$

where a convenient covariant gauge-fixing Lagrangian has been added to remove the gauge freedom $\delta A^\alpha_M = \partial^M \epsilon^\alpha$. The gauge and Lorentz indices are raised and lowered using the metrics $\delta_{\alpha\beta}$ and $\eta_{MN}$, respectively.

This then leads to the following momentum-space propagator for $A^\alpha_M$

$$D^\alpha_{MN}(k^\mu, p) = \frac{\delta^\alpha_{\beta}}{k^2 + p^2} \left[ \eta_{MN} + (\xi_1 - 1) \frac{k_M k_N}{k^2 + p^2} \right].$$

The coupling of any such a 6D gauge boson to a brane again depends on the microscopic details. If so, then we expect the linearized coupling to have the general form

$$S_{\text{int}} = \int d^4\xi \sqrt{-\gamma} J^M_{\alpha} A^\alpha_M,$$

where we take the current, $J^M_{\alpha}$, to include the appropriate 6D gauge coupling, $g$.

One should also be alive to more complicated possibilities, however, such as if an abelian bulk gauge field, $F^{\alpha}_MN$, were to mix with an abelian brane gauge field, $V^\alpha_{\mu\nu}$, as in

$$S'_{\text{int}} = k_{\alpha\beta} \int d^4\xi \sqrt{-\gamma} F^\alpha_{\mu\nu} V^\beta_{\mu\nu},$$

where $k_{\alpha\beta}$ represent a matrix of mixing parameters. Phenomenological constraints can be quite strong for such interactions, because if they involve the electromagnetic field they can produce effects which are easily detected even if very small [24].

2.5 2-Form Gauge Potentials

A field which is ubiquitous to higher-dimensional supergravity theories is the skew-symmetric gauge potential, $B_{MN}$, since this field is usually related to the graviton by supersymmetry. The quadratic action for this field in the absence of background field strengths is

$$S_B = -\int d^6x \left[ \frac{1}{12} G^{MNP} G^MNP + \frac{1}{2\xi_a}(\partial_M B^{MN})^2 \right]$$

$$= -\int d^6x \left[ \frac{1}{4} \partial_M B_{NP} \partial^M B^{NP} + \frac{1}{2} \left( \frac{1}{\xi_a} - 1 \right) (\partial_M B^{MN})^2 \right].$$
where $G_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN}$ and the second term is the gauge-fixing Lagrangian which is added to remove the gauge freedom $\delta B_{MN} = \partial_M \Lambda_N - \partial_N \Lambda_M$, for arbitrary $\Lambda_M$. The corresponding momentum-space propagator for $B_{MN}$ becomes

$$D_B^{MN,PQ}(k^\mu, p) = \frac{1}{k^2 + p^2} \left[ (\eta_{MP} \eta_{NQ} - \eta_{MQ} \eta_{NP}) \right. $$

$$+ \left. \left( \frac{\xi_a - 1}{k^2 + p^2} \right) (\eta_{MP} k_Q k_P - \eta_{NQ} k_M k_P + \eta_{NQ} k_M k_P - \eta_{MQ} k_N k_P) \right] .$$

The field $B_{MN}$ can couple to branes, although these couplings often vanish in the absence of background field strengths in the bulk or on the brane. An example is the $B_{MN}$ coupling to $D$-branes, which is obtained (in the absence of Ramond-Ramond fields) by substituting $G_{MN} \to g_{MN} + \kappa B_{MN} + \kappa \alpha' V_{MN}$ inside $\gamma_{\mu\nu}$ in eq. (2.19), where $\alpha'$ is a dimensionful constant and as before $V_{MN} = \partial_M V_N - \partial_N V_M$ is the field strength of a brane gauge field.

Other types of antisymmetric tensors can also arise, which are not related to the extra-dimensional metric by supersymmetry. Examples of these are the Ramond-Ramond fields which arise within Type IIB supergravity models in 10 dimensions. (Higher-rank fields can also arise, although we do not pursue the implications of these further here.)

In general, for slowly varying fields we expect a linearized coupling to have the general form

$$S_{\text{int}} = \kappa \int d^4 x \sqrt{-\gamma} J^{MN} B_{MN} ,$$

for some model-dependent current, $J^{MN} = - J^{NM}$.

### 2.6 Gravity

Finally, we follow standard practice and expand the metric about the background using

$$g_{MN} = \eta_{MN} + 2 \kappa \dot{h}_{MN} .$$

Expanding the Einstein-Hilbert action to quadratic order in $\dot{h}_{MN}$ then leads to

$$S = - \int d^6 x \left[ - \frac{1}{2} \partial_M \dot{h} \partial^M \dot{h} + \frac{1}{2} \partial_N \dot{h}_{PQ} \partial^N \dot{h}^{PQ} + \partial_P \dot{h}^{PN} \partial_N \dot{h} - \partial_P \dot{h}^{PN} \partial_Q \dot{h}_N^Q \right. $$

$$- \frac{1}{\xi_2} \left( \partial^M \dot{h}_{MN} - \frac{1}{2} \partial_N \dot{h} \right)^2 ,$$

(2.33)
where \( \hat{h} = \eta^{MN}h_{MN} \). With this choice the propagator in momentum space (in 6 dimensions) is

\[
D^b_{MN,PQ}(k^\mu, p) = \frac{1}{k^2 + p^2} \left[ \frac{1}{2} (\eta^{MP}\eta^{NQ} + \eta^{MQ}\eta^{NP}) - \frac{1}{4} \eta^{MN}\eta^{PQ} \right. \\
\left. + \frac{\xi_2 - 1}{2(k^2 + p^2)} (\eta^{MP}k_Nk_Q + \eta^{MQ}k_P + \eta^{NP}k_Mk_Q + \eta^{NQ}k_Mk_P) \right].
\] (2.34)

The interaction between these fields and matter is the usual gravitation one, with \( h_{MN} \) coupling to the linearized stress tensor:

\[
S_{\text{int}} = -\kappa \int d^d x \, T^{MN} \hat{h}_{MN}.
\] (2.35)

### 3. Interactions Amongst Non-relativistic Sources

In this section we compute the interaction energy which is generated by the linearized tree-level exchange of 6D bulk fields between two classical charge distributions. For practical applications we focus on the case where the sources sit on the same brane, and so are separated only in the 3 visible spatial directions and not separated at all in the 2 compact dimensions.

Consider now the exchange of bulk fields from classical sources described by the couplings of eq. (2.20), (2.24), (2.27), (2.31) and (2.35). Such an exchange leads to the momentum-space interaction potential

\[
V(k) = \kappa^2 \left[ \rho_1(k)D^0(k)\rho_2(-k) + k^M j^{ab}_1 k_j^{(a)}(-k) + \frac{1}{2} (\eta^{MP}\eta^{NQ} + \eta^{MQ}\eta^{NP}) \right. \]
\[
\left. + \frac{\xi_2 - 1}{2(k^2 + p^2)} (\eta^{MP}k_Nk_Q + \eta^{MQ}k_P + \eta^{NP}k_Mk_Q + \eta^{NQ}k_Mk_P) \right].
\] (3.1)

where \( k^M = \{k^\mu, p\} \) represents the 6-momentum transfer which occurs due to the bulk-field exchange. Although all indices written here are 6-dimensional in practice the macroscopic sources on the brane have currents which are parallel to the branes, which we take also to be the large 4 dimensions \( x^\mu \).

For precision tests of gravity our interest is in the elastic scattering of macroscopic non-relativistic sources, and so the momentum transfer which is relevant in the large four dimensions is \( k^\mu = (0, k) \). We also keep the extra-dimensional momentum transfer, \( p \), arbitrary because here we know the brane states are localized at the brane positions, and so the brane-bulk couplings in themselves break momentum conservation in these directions.
3.1 Point Sources

In order to apply the above expression to real measurements of forces we must ask what the source currents, $\rho$, $j^a_M$, $J^a_M$, $J_{MN}$ and $T_{MN}$, would be for a brane-bound source. Although we might expect the answer in general to be model-dependent, a considerable amount can be learned from symmetries in the case when the interaction is non-relativistic and elastic and the source is labelled simply by its 4-momentum and spin. (For instance, in what follows we apply these results to the interactions of electrons or protons or any other non-relativistic object whose internal structure is not disturbed by the bulk-field exchange.) In this case currents like $J_\mu = \langle p, s| \hat{J}_\mu |p, s\rangle$ depend only on the particle 4-momentum $p^\mu$ and the spin pseudo-vector $s^\mu$, where $p^\mu p_\mu = -m^2$, $p^\mu s_\mu = 0$ and $s^\mu s_\mu = +1$, and so parity and Lorentz invariance determines their form up to an overall scalar normalization.\(^5\)

For instance, in such a case the single-particle matrix element of a 4-dimensional vector, axial vector, skew-tensor, skew axial tensor and symmetric tensor current must have the form

$$
\begin{align*}
J_\mu &= q p_\mu, & J^A_\mu &= q_A s_\mu, \\
J_{\mu\nu} &= c \epsilon_{\mu\nu\lambda\rho} p^\lambda s^\rho, & J^A_{\mu\nu} &= c_A (p_\mu s_\nu - p_\nu s_\mu) \\
\text{and} & & T_{\mu\nu} &= A p_\mu p_\nu + B s_\mu s_\nu, & (3.2)
\end{align*}
$$

where $q$, $q_A$, $c$, $c_A$, $A$ and $B$ are unknown constants whose form is not determined by 4D Lorentz invariance.

Furthermore, for static interaction energies our interest is in negligible velocities, so we may specialize to the rest frame for which $p^\mu \simeq (m, 0)$ and $s^\mu = (0, s)$, where $s$ is the unit vector pointing in the direction in which the particle’s spin angular momentum is measured. In this case the $m$ dependence of the various constants may be determined on dimensional grounds, leaving only a dimensionless number undetermined. Additionally, since all the currents $J^A_\mu$, $J_{\mu\nu}$ and $J^A_{\mu\nu}$ give the same kind of contributions to the interaction potential, we only keep $J^A_{\mu\nu}$ for simplicity.

Using these arguments we find that the quantities which appear in eq. (3.1) are approximately constant for the range of momentum transfers of interest, $\lambda(k) \approx \lambda(0)$

\(^5\)We impose parity invariance on the matrix elements since the sources of interest are bound states of parity-conserving interactions like electromagnetism, the strong force or gravity.
etc., and thus
\[ \lambda(0) = am, \]
\[ j_\alpha^\mu(0) = Q_\alpha p^\mu / m \simeq Q_\alpha \delta_0^\mu, \]
\[ J^\alpha_\mu(0) = g_\alpha q_\alpha p^\mu / m \simeq g_\alpha q_\alpha \delta_0^\mu, \]
\[ J^\mu_\nu(0) = \frac{c}{2} (p^\mu s^\nu - p^\nu s^\mu) = \frac{c}{2} ms^i (\delta_0^\mu \delta_0^\nu - \delta_0^\nu \delta_0^\mu), \]
\[ T^{\mu\nu}(0) = m \left( \delta_0^\mu \delta_0^\nu + b s^i s^j \delta_0^\mu \delta_0^\nu \right). \] (3.3)

Now \( a, Q_\alpha, q_\alpha, b \) and \( c \) are dimensionless constants, \( g_\alpha \) is the 6D gauge coupling constant (having dimensions of inverse mass) and we have used the definition of mass to fix the normalization of \( m \) in the coefficient \( A \) defined in eq. (3.2).

Inserting these expressions into the interaction potential of eq. (3.1) (and specializing for simplicity to a single gauge charge, \( q \)) then gives
\[ V(k^\mu, p) = \left\{ \kappa^2 m_1 m_2 \left( X + \frac{3}{4} \right) - g^2 q_1 q_2 \right\} \frac{1}{k^2 + p^2}, \] (3.4)

where
\[ X = a_1 a_2 - c_1 c_2 (s_1 \cdot s_2) + \frac{1}{4} (b_1 + b_2) + b_1 b_2 \left[ (s_1 \cdot s_2)^2 - \frac{1}{4} \right], \] (3.5)
and \( k^\mu = (k^0, k) \). In this expression the first term \( (a_1 a_2) \) arises due to dilaton exchange, the last term \( (q_1 q_2) \) from the exchange of bulk gauge fields, the spin-dependent term \( (c_1 c_2) \) from the exchange of the Kalb-Ramond 2-form potential, and the \( (b_1, b_2) \) terms plus the penultimate term \( (3/4) \) follows from 6D graviton exchange. Derivative couplings preclude Goldstone boson exchange from contributing in the static limit.

Taking the limit \( k^0 \to 0 \) corresponding to elastic (static) interactions and transforming to position space we then find
\[ V(r, y) = \frac{1}{\mathcal{A}} \int \frac{d^3 k}{(2\pi)^3} \sum_p V(k, p) e^{i k \cdot r + i p \cdot y} \]
\[ = -\frac{1}{\mathcal{A}} \left\{ \kappa^2 m_1 m_2 \left( X + \frac{3}{4} \right) - g^2 q_1 q_2 \right\} \sum_p \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i k \cdot r + i p \cdot y}}{k^2 + p^2} \]
\[ = -\frac{1}{4\pi r \mathcal{A}} \left\{ \kappa^2 m_1 m_2 \left( X + \frac{3}{4} \right) - g^2 q_1 q_2 \right\} \sum_p e^{-|p|r + i p \cdot y}, \] (3.6)

where \( r = |r| \). Taking two sources which share the same brane we take \( y = 0 \), and see that the resulting \( V(r) \) becomes the expected sum over Yukawa potentials, one for each KK mode.
For sufficiently large $r$ only the zero mode $p = 0$ contributes, giving the standard $1/r$ result

$$V(r) \rightarrow -\kappa^2 m_1 m_2 \left( X + \frac{3}{4} \right) \frac{g_1^2 q_1 q_2}{4 \pi r A}, \quad (3.7)$$

and from this we read off the four-dimensional Newton constant, $G_N$, by focussing on the $b$-independent gravitational term (last term in the parenthesis):

$$G_N = \frac{3 \kappa^2}{16 \pi A}. \quad (3.8)$$

Similarly, the effective gauge coupling constant, $e$, is given by

$$\alpha = \frac{e^2}{4 \pi} = \frac{g^2}{4 \pi A}. \quad (3.9)$$

Notice that although eq. (3.8) pre-multiplies just the gravitational contribution (so defined because it includes only the contribution of the massless modes of the extra-dimensional metric) it differs from what would have been obtained — namely $G_N = \kappa^2/(8 \pi A)$ — using only the 4D graviton. This difference arises because of the contribution in eq. (3.7) of other 6D metric polarizations besides the 4D graviton.

We see here another reflection of the fact that long-distance gravity also deviates from General Relativity in these extra-dimensional models, because of the existence within them of massless KK metric modes which can mediate long-range interactions. Although these same modes also exist in non-supersymmetric models with large extra dimensions, it is important to notice that in the non-supersymmetric case quantum corrections lift their masses to the generic KK mass scale, thereby removing any such long-distance signature. The same need not be true within SLED, since in this case the suppression of the quantum contributions to the vacuum energy also suppress the corrections to the light-scalar masses, leaving these modes naturally light enough to have observable effects over long distances [19]. In this sense long-distance modifications of General Relativity potentially distinguish supersymmetric from non-supersymmetric large-extra-dimensional models, although the strength of the couplings of these modes depends on the details of the extra dimensions and how they have evolved over cosmological timescales.

Although we find in the present instance more than one such massless KK mode, this is an artefact of our having compactified from 6D to 4D on a torus. Generic compactifications need have only two light scalar fields, corresponding to the dilaton arising from the approximate scale invariance of 6D supergravities together with its axionic partner under 4D supersymmetry. It is these fields which are ultimately
responsible for the time-dependence of the Dark Energy density which SLED models predict [19, 20].

Another torus-specific feature of the above results is the fact that the KK spectrum is identical for bulk fields having different spins. For general extra-dimensional geometries these spectra can differ, leading to a more complicated interplay of the strengths of different fields as a function of source separation.

With these definitions we finally obtain the following interaction potential energy for two point sources situated at the same point in the extra dimensions but separated by a distance $r$ in the large 3 space dimensions.

$$V(r) = \left\{ -\frac{G_N m_1 m_2}{r} \left( 1 + \frac{4X}{3} \right) \frac{\alpha q_1 q_2}{r} \right\} \sum_p e^{-|p|r}.$$  \hspace{1cm} (3.10)

For general separations we perform the sum in eq. (3.10) numerically, with results plotted as a function of $r$ in fig. (1). As is clear from this figure, for large $r$ the interaction potential varies as $1/r$, as appropriate for 3 spatial dimensions, while for small $r$ the sum over many KK modes combines to give a result which varies as $1/r^3$, as is appropriate for forces in 5 spatial dimensions. The various curves in this figure show how the results depend on two of the moduli of the torus: the ratio $r_1/r_2$ of the lengths of its two sides, as well as the ‘twist’ angle $\theta$. For very asymmetric tori, with one direction much longer than the other (i.e. small $r_1/r_2$ or small $\theta$), the figure also shows that there can be an intermediate regime for which there are effectively 4 spatial dimensions, and so for which the force law varies as $1/r^2$.

As the figures make clear, the crossover between the $1/r$ and $1/r^2$ or $1/r^3$ power-law behaviour occurs over quite a short range of $r$. This complicates the interpretation of experiments, for which deviations to Newton’s Law are often parameterized using a single Yukawa-type exponential,

$$V(r) \sim \frac{1}{r} \left( 1 + a e^{-br} \right),$$  \hspace{1cm} (3.11)

with the parameters $(a$ and $b)$ found by fitting to the data. Naively, one might expect this form to provide a good description, with the parameters found corresponding to the properties of the first KK mode. In fact, as shown in fig. (2), the rapid crossover to a new power law implies the best fit is usually quite different from what would be inferred from the first KK mode, and also depends strongly on the interval of $r$’s used in the fit. This emphasizes that some care is required when inferring the form of the interaction potential directly from observational data.
Figure 1: The potential in eq. (3.10) calculated numerically for different choices of the moduli $\theta$ and $\eta = r_1/r_2$ (where $r_2$ is defined to be the larger of the two toroidal radii), with $V_0/(2\pi) = -(G_N m_1 m_2/r_1)\left(1 + \frac{4\lambda}{3}\right) + (\alpha q_1 q_2/r_1)$. The large- and small-$r$ regimes where $V(r) \sim 1/r$ and $1/r^3$ respectively is clearly seen, as is an intermediate regime for which $V(r) \sim 1/r^2$ for very asymmetric torii ($\eta = 0.01$ (left figure) and $\theta = 0.01$ (right figure)).

Figure 2: The exact potential for $\eta = 1$, $\theta = \pi/2$ (solid line) is fitted to an expression with a single exponential, of the form $(1 + ae^{-br})/r$. The best fit (dashed line), obtained from a least squares fit to the logarithm of $V/V_0$, is $(1 + 29.2481 e^{-1.68385r})/r$, quite far from the "correct" expression $(1 + 4e^{-r})/r$ from the first KK mode (dotted line). Here $r$ is written in units of $r_1/(2\pi)$, and $V_0$ is defined in fig. (1).

A final feature of these expressions which is worth mention is the overall sign of the deviation from Newton’s Law. Non-supersymmetric 6D models involving only graviton exchange generically predict that the strength of the interaction potential increases relative to Newton’s law as $r$ decreases, and this prediction is robust because it follows only from the attractive nature of graviton exchange.
The same need not be true in supersymmetric models because although scalar and graviton exchange generate attractive forces between identical particles, vector exchange leads to repulsion. In order to find a weaker force than gravity as \( r \) decreases it is therefore necessary to have the lowest vector KK mode dominate over the relevant range of distances, which could in principle be arranged by ensuring that the lightest KK mode correspond to a bulk vector field. The above figures show that this does not occur when these are compactified on a torus, however, because in this case all bulk fields have identical KK spectra and so the vectors do not dominate the scalars and gravitons. However this is an toroidal artefact, and a detection of an initially negative residual could carry considerable information about the shape of the extra dimensions.

3.2 Extended Sources

It is useful to generalize the result of eq. (3.10), from point sources to extended sources. This is a necessary complication because once the interaction potential between point sources is no longer proportional to \( 1/r \) it is not true that the potential due to a spherical source distribution is the same as that of a point charge.

In the present instance the interaction energy of two sources having mass densities \( \rho_1 \) and \( \rho_2 \) whose centres are displaced by a separation \( r \) is calculated by integrating eq. (3.10), to give

\[
V(r) = -G \sum_p \int_{V_1} d^3r_1 \int_{V_2} d^3r_2 \rho_1(r_1)\rho_2(r_2) e^{-p|r_2+r_2-r_1|} \frac{1}{|r_2-r_1|}. \tag{3.12}
\]

Here

\[
G = G_N \left( 1 + \frac{4}{3} \langle X \rangle \right) - \alpha \omega_1 \omega_2 \tag{3.13}
\]

where the \( \omega_i \) denote the charge-to-mass ratios for the two source distribution, which is assumed to be constant. Similarly \( \langle X \rangle \) denotes the relevant average over the source densities.

The density integrals can be calculated explicitly for simple matter distributions, some of which we now perform for illustrative purposes.

Spherical sources

If we assume that the two sources are spheres with constant density, \( \rho_1 \) and \( \rho_2 \), and radii, \( R_1 \) and \( R_2 \), so \( M_i = 4\pi \rho_i R_i^3/3 \). We further assume \( r \geq R_1 + R_2 \) so that the
two sources do not physically overlap. With these choices the interaction potential energy becomes

\[
V(r) = -G \rho_1 \rho_2 \sum_p \int_{r_1 \leq R_1} d^3 r_1 \int_{r_2 \leq R_2} d^3 r_2 \frac{e^{-|p| r + r_2 - r_1|}}{|r + r_2 - r_1|}
\]

\[
= -G M_1 M_2 \frac{r}{r} \sum_p B(|p| R_1) B(|p| R_2) e^{-|p|r},
\]

(3.14)

where [21]

\[
B(x) = \frac{3 (x \cosh x - \sinh x)}{x^3} \rightarrow 1 + \frac{x^2}{10} + O(x^4) \quad \text{if } x \ll 1
\]

\[
- \frac{3 e^x}{2 x^2} \left( 1 - \frac{1}{x} \right) + O(e^{-x}) \quad \text{if } x \gg 1. \quad (3.15)
\]

For each KK mode the effect of distributing the matter over a sphere is to multiply each factor of mass, \( M_i \), by the function \( B(|p| R_i) \). Since \( \lim_{x \to 0} B(x) = 1 \) the contribution of any KK mode whose wavelength, \( \lambda = 1/|p| \), is longer than the sphere’s radius is not modified from that of a point source, and so in particular this is true for the KK zero modes. Since these wavelengths are at their largest shorter than a micron lengths, this means that for the spheres of practical interest the macroscopic effects encoded in \( B \) are important for all nonzero KK modes. For these it is the opposite limit, \( |p| R \gg 1 \), which is relevant, in which case

\[
V(r) \approx -G M_1 M_2 \frac{r}{r} \left[ 1 + \frac{9}{4 R_1^2 R_2^2} \sum_{p \neq 0} \frac{1}{p^4} e^{-|p|(r - R_1 - R_2)} \right].
\]

(3.16)

The sum over \( p \) is exponentially small unless the spheres are separated by micron-sized distances of order the longest KK wavelength, \( r - R_1 - R_2 \lesssim \lambda \sim \max[1/|p|] \), and for separations this large it is \( O(\lambda^4/R_1 R_2^2) \).

The sum in eq. (3.14) is performed numerically and compared to the point source potential in fig. (3), for the simplest possible torus with \( r_1 = r_2 \) and \( \theta = \pi/2 \). The ratio between the two potentials is always larger than 1 since \( B(x) \geq 1 \). We can find an upper bound to this ratio when the two spheres of equal radius touch (\( r = 2R \)) by replacing the sums with integrals:

\[
\sum_p B^2(|p| R) e^{-|p| R} \sum_p e^{-|p| R} \lesssim \int_0^{\infty} 4x B^2(x) e^{-2x} dx \approx 1.635 \, 532.
\]

(3.17)

This estimate is independent of the moduli of the torus.
\[ R_1 = R_2 = R \]

\[ \frac{V_{\text{sphere}}(r)}{V_{\text{point}}(r)} \]

\[ r = 2R \]

\[ 10^{-3} \quad 10^{-2} \quad 1 \quad 10^1 \]

\[ 1 \quad 1.1 \quad 1.2 \quad 1.3 \quad 1.4 \quad 1.5 \]

\[ R = 0 \]
\[ R = 0.1 \]
\[ R = 0.2 \]
\[ R = 0.4 \]
\[ R = 1 \]
\[ R = 1.5 \]
\[ R = 2 \]

\[ \frac{\Delta V(z)}{z} \]

\[ 10^{-9} \quad 10^{-10} \quad 10^{-11} \quad 10^{-12} \quad 10^{-13} \]

\[ r = 2R \]

\[ 10^{-2} \quad 10^{-1} \quad 1 \]

\[ R = 0 \]
\[ R = 0.1 \]
\[ R = 0.4 \]
\[ R = 1 \]
\[ R = 2 \]

**Figure 3:** The figure to the left shows the ratio between the potential of spherical sources, eq. (3.14), and the corresponding point source potential. The radius is, when going from left to right and then down, \( R = 0.05, 0.1, 0.2, 0.4, 1, 1.5 \) and 2. The figure to the right shows the relative contribution of the KK sum in (3.18), for the potential between the Earth and a small sphere. The size of the torus is here chosen as \( r_1 = 10^{-5} \) m. For both figures all lengths are written in units of \( r_1/(2\pi) \), and the moduli of the torus are chosen as \( r_1 = r_2 \) and \( \theta = \pi/2 \).

**Semi-Infinite Bulk Sources**

Of important practical interest is the case where a very small spherical object interacts with a very large one (say, the Earth). The potential energy in this case may be obtained from the above result for spheres by taking the limit where the radius of one of the spheres becomes very large. We are led in this way to the interaction energy between a sphere of mass \( M \) and radius \( R \), whose centre is displaced a distance \( z \) from the edge of the semi-infinite source.

Denoting the density and radius of the large sphere by \( \rho \) and \( L \), we define its mass-per-unit-area as \( \sigma = M/A = (4\pi \rho L^3/3)/(4\pi L^2) = \rho L/3 \). Dropping an overall additive constant and neglecting sub-dominant powers of \( z/L \) we find the interaction energy to be

\[
V(z) = M \left[ gz - 2\pi \mathcal{G} \rho \sum_{p \neq 0} B(|p| R) \frac{1}{p^2} e^{-|p|z} \right], \tag{3.18}
\]

where \( g = 4\pi \mathcal{G} \sigma \). The first term in this result describes the constant (Galilean) acceleration towards the large source due to the exchange of massless KK modes, and since this acceleration ‘sees’ the entire source its strength depends on the mass/area ratio \( \sigma \).
By contrast, the second term represents an attraction of the smaller sphere towards the nearest piece of the larger sphere due to the exchange of massive KK states, which is only sensitive to the amount of matter within a range \( \lambda \) of the edge of the small sphere due to the finite range of these interactions. Because of this finite range this part of the interaction is controlled by \( \rho \) rather than \( \sigma \), and is exponentially small unless the sphere’s edge is within \( \lambda \) (i.e. closer than a micron or so) from the edge of the semi-infinite source (see fig. 3). Because this attraction is only to nearby source material, it is also sensitive to any local deviations of the large source from spherical symmetry.

The apparent divergence in eq. (3.18) as \( z \to 0 \) (due to the divergence of the sum) is illusory so long as \( R \) is nonzero. This is because the geometry requires \( z \geq R \), with the sphere touching the slab only when \( z = R \). The result is finite in this limit, as may also be seen explicitly from its parent formula, eq. (3.16). Eq. (3.18) would diverge in the limit \( R \to 0 \), however, since in this case \( B = 1 \) and \( z \) can approach zero. This represents a divergence in the potential of a single spherical which only arises at the source’s edge, and as such it may be absorbed into the coefficient of an effective interaction which is localized on the boundary of the source.

In the limit of practical interest, \(|p| R \gg 1\), we may use in eq. (3.18) the asymptotic form for \( B(x) \) to obtain the simple formula

\[
V(z) = M \left[ gz - \frac{3 \pi G \rho}{R^2} \sum_{p \neq 0} \frac{1}{p^4} e^{-|p|d} \right],
\]  

where \( d = z - R \) denotes the separation between the nearest edges of the two sources.

4. Conclusions

The calculations in this paper allow the following conclusions to be drawn concerning the implications of SLED models for tests of gravity at short distances.

- SLED models predict deviations from the inverse-square law at micron distance scales, giving instead a position-dependence which describes a crossover from \( 1/r^2 \) to \( 1/r^4 \) over a fairly short range of distances. These predictions are not well-described by perturbing Newton’s Law with a single Yukawa-type exponential potential. This situation is similar to what is obtained in non-supersymmetric models with sub-millimeter extra dimensions, with the important difference that within SLED the scale of the crossover is related to the
observed Dark Energy density and so cannot be adjusted to smaller values if no deviations from Newton’s Law are observed. It also differs from the Yukawa-type deviations which are likely to be produced by recent string models having a string scale in the TeV region with six large dimensions [26].

• The precise shape of the deviations from Newton’s Law in the crossover region are likely to depend on the details of the Kaluza-Klein spectrum, and so also on the precise shape of the extra-dimensional geometry. This is already visible in the calculations presented here, through the dependence of the forces on the various shape moduli of the extra-dimensional torus. Calculations are underway to determine more precisely how this shape depends on these details for more general geometries than the tori considered here.

• SLED models also differ from their non-supersymmetric counterparts in the number and kinds of fields which supersymmetry dictates must populate the extra dimensions. In particular, since bulk vector exchange mediates a repulsive force, it is in principle possible to obtain an interaction energy which is weaker than Newton’s law at short distances if the lightest KK mode should be for the vector fields rather than the tensors or scalars. So far as we know this same possibility cannot arise within the purely-gravitational extra-dimensional models considered heretofore. This type of short-distance repulsive interaction turns out never to dominate for the simple toroidal compactifications considered here, but could be possible for more complicated extra-dimensional geometries.

• The presence of several bulk fields in SLED models can also generically introduce a composition dependence to the observed inter-particle force. Unfortunately, the detailed nature of this composition-dependence is difficult to predict without knowing more of the details of our extra-dimensional surroundings.

All of these conclusions underline the fact that the SLED proposal is unique in that it provides a very precise framework within which a fundamental connection is made between the observed Dark Energy density and other kinds of observable physics. In it deviations of gravity at sub-millimeter distances are also tied to observable signals at the LHC and possibly to scalar-tensor deviations from General Relativity over both astrophysical and cosmological distance scales. We hope that experimenters take up the challenge of searching for these phenomena.
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