Heisenberg model in pseudo–Euclidean spaces II

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Abstract. In the review we describe a relation between the Heisenberg spin chain model on pseudospheres and light–like cones in pseudo–Euclidean spaces and virtual billiards. A geometrical interpretation of the integrals associated to a family of confocal quadrics is given, analogous to Moser’s geometrical interpretation of the integrals of the Neumann system on the sphere.

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1. Introduction

In the paper we round up our study of geometry of discrete (contact) integrable systems with constraints starting with the Heisenberg system in pseudo–Euclidean spaces $E^{k,l}$ (see [14]) and continued with billiard system within ellipsoid [16], i.e., virtual billiard system within quadrics in $E^{k,l}$ [18].

It is well known that the Heisenberg system on a sphere can be seen as a square root of the ellipsoidal billiard [23, 28], as well that it can be seen as a Bäcklund transformation of the Neumann system [25]. For the latter, Moser gave a nice geometrical
interpretation of integrability (e.g., see [22]). We feel that it would be interesting to formulate analogous pseudo-Euclidean statements. In this sense we compiled a review paper, with some additional analysis concerning mostly the light–like case. We note that integrable discretizations are usually considered for complexified objects. Here we work within real domains. For example, the Moser–Veselov skew hodograph mapping naturally follows from the requirement that a quadratic generating function defines a symplectic mapping for real objects (see Lemma 2.1). As an example we obtain the symplectic billiard mapping for the ellipsoid recently introduced in [2] (see Example 2.1).

We consider the Heisenberg model on a pseudosphere (light–like cone)

\[ S_c^{n-1} = \{ q \in \mathbb{E}^{k,l} \mid \langle q, q \rangle = c \}, \quad c = \pm 1, 0 \]

in a pseudo–Euclidean space \((\mathbb{E}^{k,l}, \langle \cdot, \cdot \rangle)\) of signature \((k,l)\), \(k + l = n\) (see [14]). It is defined as a discrete Lagrangian system given by the action functional

\[ S[q] = \sum L(q_k, q_{k+1}), \quad L(q_k, q_{k+1}) = \langle q_k, J q_{k+1} \rangle, \]

where \(q = (q_k), k \in \mathbb{Z}\) is a sequence of points on \(S_c^{n-1}\) and \(J = \text{diag}(J_1, \ldots, J_n)\), \(\det J \neq 0\). In the Euclidean case the functional defines the energy of a spin chain of the Heisenberg model, see Veselov [28].

The equations of the stationary configuration have the form

\[ \frac{\partial L(q_k, q_{k+1})}{\partial q_k} + \frac{\partial L(q_{k-1}, q_k)}{\partial q_k} = E J q_{k+1} + E J q_k = \lambda_k E q_k, \quad k \in \mathbb{Z}, \quad k \neq 0 \]

where

\[ E = \text{diag}(\tau_1, \ldots, \tau_n), \quad \tau_i = 1, \quad i = 1, \ldots, k, \quad \tau_i = -1, \quad i = k + 1, \ldots, n. \]

The multipliers

\[ \lambda_k = 2 \langle J^{-1} q_k, q_{k-1} \rangle / \langle J^{-2} q_k, q_k \rangle \]

are determined by the constraints \(\langle q_k, q_k \rangle = c\), and they are defined outside the singular set \(\langle J^{-2} q_k, q_k \rangle = 0\).

The equations (1.1), (1.2) determine the symplectic mapping

\[ \Phi: P_c \to P_c, \quad \Phi(q_{k-1}, q_k) = (q_k, q_{k+1}) \]

with respect to the 2-form \(\Omega = \sum \tau_i J_i dQ_i \wedge dQ_i\), where

\[ P_c(q, Q): \quad \langle q, q \rangle = c, \quad \langle Q, Q \rangle = c, \quad c = \pm 1, 0, \]

\[ \langle q, J^{-1} Q \rangle \neq 0, \quad \langle Q, J^{-2} Q \rangle \neq 0, \quad \langle Q, J^{-2} Q \rangle \neq 0 \]

(see [14]). It is a completely integrable discrete Hamiltonian system. For \(J_j^2 \neq J_i^2\), the integrals can be written in the form

\[ f_i(q_{k-1}, q_k) = c \cdot \tau_i (q_{k-1})^2 + \sum_{j \neq i} \tau_i \tau_j (J q_j)_i (q_{k-1})_j - (q_{k-1})_j (J q_j)_i)^2 \]

\[ J_i^2 - J^2 \]

\[ i = 1, \ldots, n, \]

with the relation \(\sum_i f_i \equiv c^2\) among them. Furthermore, on the light–like cone, the mapping \(\Phi\) leads to an integrable contact system as well (see [14]).

\[ ^2\text{We hope that it will be clear from the context when } k \text{ denotes the discrete time, and when the signature of the metric.} \]

\[ ^3\text{Actually, the function } \langle q_k, J^{-1} q_{k+1} \rangle \text{ is the first integral [13], and so the condition } \langle q_k, J^{-1} q_{k+1} \rangle \neq 0 \text{ is invariant of the dynamics, while } \langle J^{-2} q_k, q_k \rangle \neq 0 \text{ is not. If } J^{-2} q_k, q_k = 0, \text{ by definition the flow stops. In this sense, in the codomain of } \Phi \text{ we should take the manifold defined without the assumption } \langle Q, J^{-2} Q \rangle \neq 0. \]
Outline and results of the paper. In the Euclidean case, there is a remarkable relation between the ellipsoidal billiard and the Heisenberg spin chain model established by the use of so called skew hodograph mapping (see Moser and Veselov [23]). Recently, in [15], a simple observation concerning generating functions for systems with constraints (see Theorem 2.1) is used for another interpretation of the skew-hodograph mapping. Following [15], we establish analogous relation between virtual billiards and the pseudo–Euclidean Heisenberg model, which also includes the symmetries of the system (Theorem 2.2 Section 2). As a by-product, we obtain the symplectic billiard within ellipsoid given in [2] (Example 2.1), as well as a ”big” \( n \times n \)-matrix representations of the virtual billiard flow (Theorems 2.3 Section 2).

Further, in Sections 3 and 4, as a straightforward generalization of the Euclidean case (see [25]), we consider a discrete Legendre transformation of the Heisenberg model and define the associated 1:2 symplectic correspondence on the domains \( \mathfrak{M}_{1,1} \) of the cotangent bundle of pseudospheres \( S^{n-1}_{\pm 1} \) (Theorem 3.2), i.e, the domain \( \mathfrak{M}_0 \) of the cotangent bundle of a light–like cone \( S^{n-1}_0 \) (Theorem 3.1). The small \( 2 \times 2 \)-matrix representations for the systems are also given (Theorems 3.3 4.2).

We show that the Heisenberg model on \( \mathfrak{M}_{1,1} \) is a Bäcklund transformation (Theorem 3.1) of the integrable variant of the Neumann system in pseudo–Euclidean spaces described in Theorem 5.1. On the other hand, the Heisenberg model on \( \mathfrak{M}_0 \) has a one-parameter family of invariant hypersurfaces \( \Sigma \). The restriction of the correspondence to \( \Sigma \) is a natural example of completely integrable contact system (Theorem 4.3).

Motivated by Moser’s geometric interpretation of the integrals of the Neumann system on a sphere (see [22]), in section 5 we consider the following pseudo–confocal family of quadrics in \( \mathbb{R}^{k,l} \):

\[
Q_{c,\lambda}: \quad \langle (U - \lambda I)^{-1} x, x \rangle = \sum_{i=1}^{n} \frac{U_i^2}{U_i - \lambda} = c, \quad \lambda \neq U_i, \quad i = 1, \ldots, n,
\]

where \( U_i = J_i^2 \), \( i = 1, \ldots, n \). In the light–like case, to a given trajectory \( \{q_k \mid k \in \mathbb{Z}\} \) we associate a sequence of planes

\[ \pi_k = \text{span} \{q_k, Jq_k\}, \quad k \in \mathbb{Z}. \]

Then, if \( \pi_k \) is tangent to a cone \( Q_{0,\lambda} \) from the pseudo–confocal family \( 1.4 \) for a certain \( k \), then it is tangent to \( Q_{0,\lambda^*} \) for all \( k \in \mathbb{Z} \). In the case \( c = \pm 1 \), instead of planes, to a trajectory \( \{q_k \mid k \in \mathbb{Z}\} \) we associate sequence of lines

\[ l_k = Jq_k + \text{span} \{q_k\}, \quad k \in \mathbb{Z} \]

with the same property (Theorem 5.1). Further, under the condition \( U_1 < U_2 < \cdots < U_n \), we estimate the number of (real) quadrics tangent to planes \( \pi_k \) (lines \( l_k \)) for a generic trajectory \( \{q_k \mid k \in \mathbb{Z}\} \) (Theorem 5.3).

2. Heisenberg model and billiards

2.1. Generating functions for systems with constraints. In what follows, we will use the following simple observation (see [15]). Consider \((2n - 2m)\)-dimensional submanifolds \( M \subset \mathbb{R}^{2n}(x, p) \) and \( N \subset \mathbb{R}^{2n}(X, P) \), defined by the constraints of the form

\[
\begin{align*}
M: \quad & f_i(x) = 0, \quad f_{m+i}(p, x) = 0, \quad i = 1, \ldots, m, \\
N: \quad & F_i(X) = 0, \quad F_{m+i}(P, X) = 0, \quad i = 1, \ldots, m.
\end{align*}
\]

We suppose that \( M \) and \( N \) are symplectic submanifolds with respect to the canonical symplectic forms, that is

\[
\det(f_i, f_j) \neq 0 | M, \quad \det(F_i, F_j) \neq 0 | N, \quad i, j = 1, \ldots, 2m.
\]
where $\{\cdot, \cdot\}$ are the canonical Poisson bracket (e.g., see [25]).

**Theorem 2.1.** If a graph $\Gamma_\phi$ of the diffeomorphism $\phi: M \to N$ can be given by

\[
p = \frac{\partial S(x, X)}{\partial x} + \sum_{i=1}^m \lambda_i \frac{\partial f_i}{\partial x}, \quad P = \frac{\partial S(x, X)}{\partial X} \sum_{i=1}^m \lambda_i \frac{\partial F_i}{\partial X},
\]

for certain Lagrange multipliers $\lambda_i, \Lambda_i$, then $\phi$ is symplectic. Similarly, if (2.1) defines a diffeomorphism $\phi: M \to N$, then $\phi$ is symplectic.

**2.2. Virtual billiards.** Let

\[
\mathbb{Q}^{n-1} = \{ x \in \mathbb{E}^{k,l} | \langle A^{-1}x, x \rangle = c \}, \quad c = \pm 1, 0
\]

be a $(n - 1)$-dimensional quadric, where

\[
A = \text{diag}(a_1, \ldots, a_n), \quad \det A \neq 0.
\]

A point $x \in \mathbb{Q}^{n-1}$ is **singular** if the induced metric is degenerate at $x$, i.e., if a pseudo-Euclidean normal $A^{-1}x$ at $x$ is light-like: $\langle A^{-2}x, x \rangle = 0$  

The virtual billiard mapping $\phi: (x_k, y_k) \mapsto (x_{k+1}, y_{k+1})$ is defined by:

\[
x_{k+1} = x_k + \mu_k y_k = x_k - 2 \frac{\langle A^{-1}x_k, y_k \rangle}{\langle A^{-1}y_k, y_k \rangle} y_k,
\]

\[
y_{k+1} = y_k + \nu_k A^{-1}x_{k+1} = y_k + 2 \frac{\langle A^{-1}x_{k+1}, y_k \rangle}{\langle A^{-1}y_k, x_{k+1} \rangle} A^{-1}x_{k+1},
\]

where the multipliers $\mu_k, \nu_k$ are determined from the conditions that the "impact" points $x_k$ belong to the quadric (2.2) and that the outgoing and incoming directions at $x_{j+1}$ have the same norms: $\langle y_{k+1}, y_{k+1} \rangle = \langle y_k, y_k \rangle$.

Geometrically (2.4) means that $y_k \mapsto y_{k+1}$ is the billiard reflection at $x_{k+1} \in \mathbb{Q}^{n-1}$ in the pseudo-Euclidean space $\mathbb{E}^{k,l}$, but $\mu_k$ in (2.3) can be less than zero as well. Thus, the segments $x_{k-1}x_k$ and $x_kx_{k+1}$ determined by 3 successive points of the mapping (2.3), (2.4) may be either on the same side of the tangent plane $T_{x_k}\mathbb{Q}^{n-1}$ (the usual billiard reflection at $x_k$), or on the opposite sides of $T_{x_k}\mathbb{Q}^{n-1}$. Such configurations were studied in [3, 4, 5, 12].

The system is defined outside the singular set

\[
\Sigma = \{(x, y) \in \mathbb{R}^{2n} | \langle A^{-2}x, x \rangle = 0 \lor \langle A^{-1}x, y \rangle = 0 \lor \langle A^{-1}y, y \rangle = 0 \},
\]

and if $(x_{k+1}, y_{k+1})$ is singular, the flow stops. The lines $l_k = \{ x_k + sy_k | s \in \mathbb{R} \}$ containing segments $x_kx_{k+1}$ of a given virtual billiard trajectory are of the same type: they are all either space-like ($\langle y_k, y_k \rangle > 0$), time-like ($\langle y_k, y_k \rangle < 0$) or light-like ($\langle y_k, y_k \rangle = 0$). Also, the function $\langle A^{-1}x_k, y_k \rangle$ is the first integral of the system.

Consider the submanifold of the symplectic linear space $\mathbb{R}^{2n}(x, y)$

\[
M_{c,h} = \{(x, y) \in \mathbb{R}^{2n} \setminus \Sigma | \phi_1 = \langle A^{-1}x, x \rangle = c, \phi_2 = \langle y, y \rangle = h \},
\]

where we take the symplectic form

\[
\sum_i \tau_i dy_i \wedge dx_i
\]
obtained from the canonical symplectic form on $\mathbb{R}^{2n}(x,p)$ after the identification $p = Ey$. Since $\{\phi_1, \phi_2\} = 4(A^{-1}x, y) \neq 0$ for $c,h$, it follows that $M_{c,h}$ is a symplectic submanifold of $\mathbb{R}^{2n}(x,y)$ and the mapping $\phi$ is a symplectic transformation of $M_{c,h}$ (see Theorem 2.1, [18]).

The Hamiltonian and contact integrability of the virtual billiard dynamics is described in [18]. In the case when $EA$ is positive definite, $c = +1$, this is a billiard system within ellipsoid $Q^{n-1}$ in the pseudo-Euclidean space (see [19, 5]).

For $c = 0$, the dynamics (2.3), (2.4) induces a well defined dynamics of the lines span $\{x_k\}$, i.e, the points $p_k = [x_k] \in Q^{n-2}$ of the $(n-1)$-dimensional projective space $\mathbb{P}(\mathbb{E}^{k,l})$ outside the singular set

$$\Xi = \{[x] \in \mathbb{P}(\mathbb{E}^{k,l}) \mid \langle A^{-2}x, x \rangle = 0\},$$

where $Q^{n-2}$ is the projectivization of the cone (2.2) within $\mathbb{P}(\mathbb{E}^{k,l})$. A sequence $\{p_k\}$ is a billiard trajectory within the quadric $Q^{n-2}$ in the projective space $\mathbb{P}(\mathbb{E}^{k,l})$ with respect to the metric induced from the pseudo–Euclidean space $\mathbb{E}^{k,l}$. In particular, for the signature $(n,0)$ and the condition

$$0 < a_1, a_2, \ldots, a_{n-2}, a_{n-1} < -a_n,$$

and the signature $(n-1,1)$ with the condition

$$0 < a_1, a_2, \ldots, a_{n-2}, a_{n-1} < a_n,$$

we obtain ellipsoidal billiards on the sphere and the Lobachevsky space, respectively (see [18, 15]).

### 2.3. The skew hodograph mapping and quadratic generating functions.

There is a remarkable relation between the ellipsoidal Euclidean billiards and the Heisenberg system established by the use of the so called skew hodograph mapping (see Moser and Veselov [23]). In [15], the skew-hodograph mapping is interpreted as a symplectic transformation with a quadratic generating function for a system with constraints. Here, we shall give analogous mapping for virtual billiards, which also include the symmetries of the system. Another construction, related to pluri-Lagrangian systems, that associate generating functions to the billiard system within ellipsoid is recently given in [26].

For the Euclidean case when $Q^{n-1}$ is an ellipsoid, we have the following characterisation of quadratic generating functions.

**Lemma 2.1.** A quadratic generating function $S(x,X) = \langle Bx,X \rangle$, det $B \neq 0$, defines a symplectic transformation $\psi: M_{1,1} \to M_{1,1}$ within a real domain only if $|B^T A^{1/2}| = |BA^{1/2}| = 1$, where $|\cdot|$ is the operator norm of the matrix.

The proof of Lemma 2.1 is given in the Appendix. Apart from the obvious solution $B = A^{-1/2}$ of the stated necessary conditions that leads to the skew-hodograph mapping (see [15]), we have a family of solutions related to the symmetry of the ellipsoid $Q^{n-1}$. Namely, let $R \in O(n)$ be an orthogonal matrix that commute with $A$: $\text{Ad}_R(A) = A$. Then we can take $B = RA^{-1/2} = A^{-1/2}R$.

The above construction can be considered in pseudo–Euclidean spaces as well, provided $A$ is positive definite. Recall that if some of the eigenvalues of the matrix
A are the same, we deal with virtual billiards with symmetries and the corresponding dynamics is integrable in a noncommutative sense (see [18]). The set of all symmetries
\[ \mathbf{R} \in O(k, l) : \quad \text{Ad}_\mathbf{R}(A) = A \]
(\[ \mathbf{R} \mathbf{ER}^T = E, \mathbf{RAR}^{-1} = A \]) is isomorphic to \( O(k_1, l_1) \times \cdots \times O(k_r, l_r) \). If all eigenvalues of \( A \) are distinct, the only symmetries are
\[ \mathbf{R} \in \mathbb{Z}_2^2 \subset O(k, l), \quad \text{i.e.,} \quad \mathbf{R} = \text{diag}(\pm 1, \ldots, \pm 1), \]
and the system is integrable in the usual commutative sense.

Let
\[ B = \mathbf{RA}^{-1/2}, \]
where \( \mathbf{R} \in O(k, l) \) is a symmetry of the quadric. Consider the symplectic manifold \( M_{c,c}, c = \pm 1, 0 \) and the generating function
\[ S(x, X) = \langle Bx, X \rangle = \langle \mathbf{RA}^{-1/2}x, X \rangle. \]

The equations (2.7) become
\[ \begin{align*}
E_y &= B^T EX + \lambda E A^{-1}x = \mathbf{R}^T A^{-1/2}EX + \lambda E A^{-1}x, \\
EY &= -EBx - \Lambda E A^{-1}X = -E\mathbf{RA}^{-1/2}x - \Lambda E A^{-1}X,
\end{align*} \]
where \( \langle A^{-1}x, x \rangle = c, \langle A^{-1}X, X \rangle = c \). We have four real values for \( (\lambda, \Lambda) \) given by
\[ \lambda = 0 \quad \text{or} \quad \lambda = -2\langle A^{-1}x, \mathbf{R} A^{-1/2}X \rangle/\langle A^{-1}x, x \rangle, \]
\[ \Lambda = 0 \quad \text{or} \quad \Lambda = -2\langle A^{-1}X, \mathbf{R} A^{-1/2}x \rangle/\langle A^{-2}X, X \rangle. \]

For \( \lambda = 0, \Lambda \neq 0 \), the relations (2.7), (2.8) define the symplectic mapping \( \psi_{\mathbf{R}} : M_{c,c} \to M_{c,c} \) given by
\[ \begin{align*}
X &= \mathbf{RA}^{-1/2}y, \\
Y &= -\mathbf{RA}^{-1/2}(x + \mu y), \\
\mu &= -2\langle A^{-1}x, y \rangle/\langle A^{-1}y, y \rangle.
\end{align*} \]

Let \( I \) be the identity \( n \times n \)-matrix.

**Theorem 2.2.** (i) The mapping \( \psi_{\mathbf{R}} \) commute with the virtual billiard mapping \( \phi \). In other words, let \( (x_k, y_k) \) be a solution of (2.7), (2.8) with \( \langle y_k, y_k \rangle = c \). Then \( (x'_k, y'_k) = \psi_{\mathbf{R}}(x_k, y_k) \) is a solution of (2.9), (2.10) with \( \langle y'_k, y'_k \rangle = c \). Moreover, \( \psi_{\mathbf{R}} = -\mathbf{R}^2 \circ \phi \) :
\[ \begin{align*}
x''_k &= -\mathbf{R}^2 x_{k+1}, \\
y''_k &= -\mathbf{R}^2 y_{k+1}.
\end{align*} \]

(ii) Let \( (x_k, y_k) \) be a trajectory of the mapping \( \psi = \psi_{\mathbf{I}} \). Then \( q_k = y_k \) is a solution of the Heisenberg model (1.1) on \( S^*_{c-1} \) with \( J = A^{1/2} \). Conversely, if \( J \) is positive definite and \( q_k \) is a solution of the Heisenberg system (1.1) on \( S^*_{c-1} \), then
\[ x_k = (-1)^kJ_{q_{2k}}, \quad \bar{x}_k = (-1)^kJ_{q_{2k+1}} \]
are billiard trajectories within the quadric \( \langle A^{-1}x_k, x_k \rangle = c \), where \( A = J^2 \).

**Proof.** (i) Let \( (x_k, y_k) \) be a solution of (2.3), (2.4) with \( \langle y_k, y_k \rangle = c \) and let
\[ (x'_k, y'_k) = \psi_{\mathbf{R}}(x_k, y_k) = (\mathbf{RA}^{-1/2}y_k - \mathbf{RA}^{-1/2}x_k) \]
Then with \( k \) replaced by \( k + 1 \) we obtain, respectively,
\[ \begin{align*}
x'_{k+1} &= x'_k + \mu'_k y'_k, \\
y'_{k+1} &= y'_k + \nu'_k A^{-1}x'_{k+1},
\end{align*} \]
where \( \mu'_k = -\nu_k, \nu'_k = -\mu_{k+1} \).
Further, we have
\[ x''_k = R A^{1/2} y'_k = -R A^{1/2} A^{-1/2} x_{k+1} = -R^2 x_{k+1}, \]
\[ y''_k = -R A^{-1/2} x'_{k+1} = -R A^{-1/2} A^{1/2} y_{k+1} = -R^2 y_{k+1}. \]

(ii) The second statement follows from the relations
\[ y_{k+2} = -A^{-1/2} (x_{k+1} + \mu_{k+1} y_{k+1}) = -A^{-1/2} (A^{1/2} y_k + \mu_{k+1} y_{k+1}) = -y_k - \mu_{k+1} A^{-1/2} y_{k+1}. \]

\[ \Box \]

For \( \psi = \psi_1 \), we have the following commutative diagram

\[
\begin{array}{ccc}
P_c & \xrightarrow{\Delta} & M_{c,e} \\
\phi & & \phi \\
P_c & \xrightarrow{\Delta} & M_{c,e} \\
\end{array}
\]

where \( \Delta: P_c \to M_{c,e} \) is a symplectomorphism \( x = Jq, y = Q, J = A^{1/2} \).

Also, since \( \psi^2 = -\phi \), if \( q_k \) is periodic with period \( 4N \) (respectively, \( 4N + 1, 4N + 2, 4N + 3 \)), then \( x_k, \tilde{x}_k \) are periodic with period \( 2N \) (respectively, \( 8N + 2, 4N + 2, 8N + 6 \)).

Figure 1. 6–periodic trajectory of the Heisenberg model (green lines) and the corresponding 6–periodic space–like trajectory of the virtual billiard for \( c = 1 \) (blue lines) in \( E^{1,1} \).

In the signature \( (n - 1, 1) \) the statement relates the ellipsoidal billiard on the Lobachevsky space and the Heisenberg system on the light–like cone \( S_0^{n-1} \) with the matrix \( A \) given by \( (2.6) \) (see \[15\]).
Example 2.1. As an example of a system with symmetry, consider the billiard within ellipsoid

\[ Q^{2n-1}: \quad \langle A^{-1} z, z \rangle = \frac{|z_1|^2}{a_1^2} + \cdots + \frac{|z_n|^2}{a_n^2} = 1 \]

in the Euclidean space \( \mathbb{R}^{2n} \cong \mathbb{C}^n \), \( z = (z_1, \ldots, z_n) \) (in \( \mathbb{C}^n \) we studied the reduction of symmetries of the given billiard with additional Hook’s potential). In the complex notation we have

\[ M_{1,1} = \{(z, v) \in \mathbb{C}^{2n} \mid \langle A^{-1} z, z \rangle = 1, \langle v, v \rangle = 1 \}. \]

Note that for \( a_i \neq a_j, i \neq j \), we have \( O(2)^n \) (i.e., \( U(n)^n \)) symmetry of \( Q^{2n-1} \) and the symplectic mapping \( (2.14) \), \( (2.10) \) reads

\[ Z_{k+1} = RA^{1/2}v_k, \quad \mu_k = -2\Re\langle A^{-1} z_k, \bar{v}_k \rangle / \langle A^{-1} v_k, \bar{v}_k \rangle, \]

where

\[ R = (e^{i\theta_1}, \ldots, e^{i\theta_n}). \]

In particular, for \( R = (i, \ldots, i) = iE \) we obtain \( \psi_{iE}^2 = \phi \), that is \( (2.12), (2.13) \) is exactly the square root of the billiard. This mapping coincides with the symplectic billiard mapping for the ellipsoid \( Q^{2n-1} \) introduced in \( \mathbb{C}^n \). More precisely, after setting \( k + 1 \) instead \( k \) in \( (2.12) \), we get

\[ Z_{k+2} = iA^{1/2}v_{k+1} = iA^{1/2}(-iA^{1/2}) (z_k + \mu_k v_k) = z_k - \mu_k iA^{-1/2} z_{k+1}. \]

Thus, \( \{z_k\} \) is a trajectory of a symplectic billiard within the ellipsoid \( Q^{2n-1} \) (corresponding to the ellipsoid \( \mathbb{C}^n \) where we set \( a_i \) instead of \( a_i^2 \)).

2.4. The \((n \times n)\)–matrix representation of the virtual billiards. Motivated by the Lax representation for elliptical billiards with the Hooke’s potential (Fedorov \[ \mathbb{C}^n \], see also \[ \mathbb{C}^n \]), a "small" \( 2 \times 2 \) matrix representation for the virtual billiard mapping is given in \( \mathbb{C}^n \). On the other hand, in \( \mathbb{C}^n \) we presented the following "big" \( n \times n \)–matrix representation of the Heisenberg system, a modification of the \( n \times n \) matrix representation given in \( \mathbb{C}^n \). Let

\[ F = \text{diag}(1, \ldots, 1, i, \ldots, i), \]

where the first \( k \) components are equal to 1, and the last \( n - k \) components are equal to the imaginary unit \( i \) (\( F^2 = E \)). The equations \( \mathbb{C}^n \) imply the discrete Lax representation

\[ L_{k+1}(\lambda) = A_k(\lambda)L_k(\lambda)A_k^{-1}(\lambda), \]

where

\[ L_k(\lambda) = J^2 + \lambda F_{qk-1} \wedge F_{qk} - c \cdot \lambda^2 F_{qk-1} \otimes F_{qk-1}, \]

\[ A_k(\lambda) = J - \lambda F_{qk} \otimes F_{qk-1}. \]

Note that in the light–like cone case, the \( L \)–matrix is linear in \( \lambda \). Also, if \( J_1^2 \neq J_2^2 \), the integrals obtained from the matrix representation can be written in the form \( \mathbb{C}^n \).

In the Euclidean case, the skew hodograph mapping relates \( n \times n \) matrix representations of the Heisenberg model and the elliptic billiard \( \mathbb{C}^n \). Although we have an analog of the skew hodograph mapping only for \( A > 0 \), the modification of the matrix representation for the Heisenberg model from the Euclidean to the pseudo–Euclidean spaces \( \mathbb{C}^n \) suggests the following matrix representation for virtual billiards.
Theorem 2.3. The virtual billiard mapping (2.3), (2.4) implies the discrete Lax representation
\[ L_{k+1}(\lambda) = A_k(\lambda) L_k(\lambda) A_k^{-1}(\lambda), \]
where
\[ L_k(\lambda) = A - \lambda F_{x_k} \wedge F_{y_{k-1}} - c \cdot \lambda^2 F_{y_{k-1}} \otimes F_{y_{k-1}}, \]
\[ A_k(\lambda) = A - \lambda (F_{x_k} \otimes F_{y_{k-1}} - F_{y_k} \otimes F_{x_k}) - c \cdot \lambda^2 F_{y_k} \otimes F_{y_{k-1}}. \]
The proof is given in the Appendix.

3. Bäcklund transformation of the Neumann system

3.1. Continuous limit and the Neumann system. Following Moser and Veselov [23], by taking
\[ J(\epsilon) = I + \frac{1}{2} \epsilon^2 U, \quad q_k = q(t_0 + k\epsilon) \]
for small \( \epsilon \), from (1.1) we obtain the equation for \( q(t) \)
\[ (I + \frac{1}{2} \epsilon^2 U)(2q + \epsilon^2 \ddot{q}) \approx \lambda q, \]
that is
\[ \ddot{q} \approx -Uq + \mu q, \quad \mu = (\lambda - 2)\epsilon^{-2}. \]
The last equation in the case \( c = \pm 1 \) describes the Neumann system on a pseudo-sphere. This is the Lagrangian system with the Lagrangian
\[ L(q, \dot{q}) = \frac{1}{2} \langle \dot{q}, \dot{q} \rangle - \frac{1}{2} \langle Uq, q \rangle, \]
subjected to the constraint \( \langle q, q \rangle = c, \ c = \pm 1, \) where
\[ U = \text{diag}(U_1, \ldots, U_n). \]
Indeed, the associated Euler–Lagrange equation on the tangent bundle \( T^*S_c^{n-1} \) realized by equations
\[ \langle q, q \rangle = c, \quad \langle q, \dot{q} \rangle = 0, \]
reads
\[ \ddot{q} = -Uq + \mu q, \quad \mu = -\frac{1}{c} \left( \langle \dot{q}, \dot{q} \rangle - \langle Uq, q \rangle \right). \]
In the case of the light–like cone \( c = 0 \), the Lagrangian \( L \) is degenerate, since all points of \( S_c^{n-1} \) are singular.

We will show that the cotangent bundle formulation of the Heisenberg model provides a Bäcklund transformation of the Neumann system. The construction is a straightforward generalization of the discretization of the Neumann system presented by Suris [25].

Firstly, we need a Hamiltonian formulation of the Neumann flow. Consider the realization of the cotangent bundle \( T^*S_c^{n-1} \) as a submanifold of \( \mathbb{R}^{2n}(q, p) \) endowed with the canonical symplectic form \( \omega = dp \wedge dq \):
\[ T^*S_c^{n-1}: \quad \varphi_1 = \langle q, q \rangle = c, \quad \varphi_2 = \langle q, Ep \rangle = 0. \]
This is a symplectic submanifold for \( c = \pm 1 \), since \( \{\varphi_1, \varphi_2\} = 2\langle q, q \rangle = c \neq 0 \) at \( T^*S_c^{n-1} \). Moreover, the restriction \( \omega|_{T^*S_c^{n-1}} \) coincides with the canonical symplectic form on \( T^*S_c^{n-1} \). The induced Poisson–Dirac bracket reads
\[ \{f_1, f_2\}_D = \{f_1, f_2\} - \frac{\{\varphi_1, f_1\}\{\varphi_2, f_2\} - \{\varphi_2, f_1\}\{\varphi_1, f_2\}}{\varphi_1, \varphi_2}. \]
The equation
\begin{equation}
\dot{p} = \frac{\partial L}{\partial \dot{q}} + \lambda \dot{q} = E\dot{q} + \lambda \dot{q}
\end{equation}
with constraints (3.2) and (3.4) implies \( \lambda = 0 \) and \( \dot{q} = Ep \). Thus, the Legendre transformation of \( L(q, \dot{q}) \) yields the Hamiltonian function
\begin{equation}
H(q, p) = \frac{1}{2} \langle p, p \rangle + \frac{1}{2} \langle Uq, q \rangle.
\end{equation}
The equations (3.3) are equivalent to the Hamiltonian equations with constraints
\begin{align*}
\dot{q} &= \frac{\partial H}{\partial p} - \mu \frac{\partial \varphi_1}{\partial p} - \nu \frac{\partial \varphi_2}{\partial p} = Ep - \nu q, \\
\dot{p} &= -\frac{\partial H}{\partial q} + \mu \frac{\partial \varphi_1}{\partial q} + \nu \frac{\partial \varphi_2}{\partial q} = -EUq + \mu Eq + \nu p,
\end{align*}
where the multipliers \( \mu, \nu \), determined from the conditions \( \dot{\varphi}_1 = \dot{\varphi}_2 = 0 \) are given by
\begin{equation}
\mu = -\frac{1}{\langle q, q \rangle} \left( \langle p, p \rangle - \langle Uq, q \rangle \right), \quad \nu = 0.
\end{equation}
Let
\begin{equation}
Q_\lambda(x, y) = \langle (\lambda I - U)^{-1} x, y \rangle = \sum_i \frac{\tau_i x_i y_i}{\lambda - U_i},
\end{equation}

**Theorem 3.1.** The Neumann flow (3.8), (3.9) implies the matrix representation
\begin{equation}
\frac{d}{dt} \mathcal{L}_{q,p}(\lambda) = [\mathcal{L}_{q,p}(\lambda), \mathcal{A}_{q,p}(\lambda)],
\end{equation}
with \( 2 \times 2 \) matrices depending on the parameter \( \lambda 
\begin{equation}
\mathcal{L}_{q,p}(\lambda) = \begin{pmatrix} -Q_\lambda(q, Ep) & -Q_\lambda(q, q) \\ c + Q_\lambda(p, p) & Q_\lambda(q, Ep) \end{pmatrix}, \quad \mathcal{A}_{q,p}(\lambda) = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}.
\end{equation}
The system is completely integrable. For \( U_i \neq U_j, \ i \neq j \), from the expression
\begin{equation}
\det \mathcal{L}_{q,p}(\lambda) = Q_\lambda(q, q)(c + Q_\lambda(p, p)) - Q_\lambda(q, Ep)^2 = \sum_{i=1}^n \frac{f_i(q, p)}{\lambda - U_i},
\end{equation}
we obtain a complete set of integrals
\begin{equation}
f_i(q, p) = c \cdot \tau_i q_i^2 + \sum_{j \neq i} \frac{\tau_i \tau_j (\tau_j p_j q_i - \tau_i q_j p_i)^2}{U_i - U_j},
\end{equation}
where \( \{f_i, f_j\} = 0 \), \( i, j = 1, \ldots, n \), and \( \sum_i f_i \equiv 1 \).

A "big" \( n \times n \) matrix representation and integration of equations (3.3) in the signature \( (n - 1, 1) \), i.e., of the Neumann system in the Lobachevsky space is given by Veselov (see Appendix B, [27]). A generalization of the Neumann system to the Stiefel varieties, as well as its integrable discretization, is given in [10] and [11], respectively.

### 3.2. Discrete Legendre transformation for \( c = \pm 1 \)
Following [23, 28, 25], we consider the associated discrete cotangent bundle dynamics of the Heisenberg system. Let \( \mathfrak{M}_c, c = \pm 1 \), be a domain within \( T^* S_c^{n-1} \) defined by the inequalities
\begin{equation}
D_c(q, p) = \langle J^{-2} Ep, q \rangle^2 - \langle J^{-2} q, q \rangle \langle (J^{-2} p, p) - c \rangle > 0, \quad \langle J^{-2} q, q \rangle \neq 0.
\end{equation}
Theorem 3.2. The relations $\Psi$ defined by
(3.15) $Q = EJ^{-1}p + \mu J^{-1}q$,
(3.16) $P = -EQ + \mu EQ$,
where $\mu$ is the solution of the quadratic equation
(3.17) $(J^{-2}q,q)\mu^2 + 2\mu(J^{-2}Ep,q) + (J^{-2}p,p) - c = 0$,
deﬁne 1 : 2 symplectic correspondence $\Psi: \mathcal{M}_c \to \mathcal{M}_c$ ($c = \pm 1$).

Proof. Consider a transformation of $T^*S^m_c$ defined by constraints (3.4) and the
generating function given by the discrete Lagrangian:

$$ S(q,Q) = \langle JQ,q \rangle. $$

The equations (2.1) read
(3.18) $p = EQ + \lambda Eq$,
(3.19) $P = -EJq - \LambdaEQ$,
where the Lagrange multipliers $\lambda = \Lambda = -\langle JQ,q \rangle/c$ are determined from the con-
straints $\langle Eq,p \rangle = \langle EQ,P \rangle = 0$.

Let $L_1: P_c(q,Q) \to T^*S^m_c(q,p)$, $L_2: P_c(q,Q) \to T^*S^m_c(Q,P)$ be the mappings
defined by (3.18) and (3.19), respectively. They can be seen as a discrete analogue
of the Legendre transformation (3.6). Let

$$ N_c = L_1(P_c) = L_2(P_c). $$

We have that $D_c(q,p)$ is greater then zero on $N_c$:

$$ D_c = (J^{-2}Ep,q)^2 - \langle J^{-2}q, q \rangle (\langle J^{-2}p, p \rangle - c) $$$$ = (J^{-2}q, JQ + \lambda q)^2 - (J^{-2}q, q) (\langle J^{-1}Q + \lambda J^{-2}q, JQ + \lambda q \rangle - c) $$$$ = (J^{-1}q, Q)^2 > 0,$$

since $(J^{-1}q, Q) \neq 0$ at $P_c$. Thus, $N_c$ is a subset of $M_c$.

Vice versa, assume $(q,p) \in M_c$. The relation (3.15), can be rewritten into the form
(3.17), where $\mu$ is unknown multiplier. From the constraint $(Q, Q) = c$ we get the equation
(3.18) determining $\mu$ as a two-valued function of $(q,p)$

$$ \mu(q,p) = \frac{-\langle J^{-2}Ep, q \rangle \pm \sqrt{D_c(q,p)}}{\langle J^{-2}q, q \rangle}. $$

As a result we obtain two points $Q_1$ and $Q_2$ such that $(q,p) = L_1(q,Q_1) = L_1(q,Q_2)$,
and $M_c = N_c$.

Therefore, according to Theorem 3.2, we get a two-valued symplectic transformation

$$ \Psi: M_c(q,p) \to M_c(Q,P) $$
such that $\Psi(q,p) = L_2(L_1^{-1}(q,p))$. □

Since all equations are algebraic, we have that (3.16), (3.15) is a symplectic 1:2 correspondence on $T^*S^m_c$ for complexified objects with $D_c = 0$ deﬁning the set of
branch points. Note that the discriminant $4D_c$ is the ﬁrst integral of (3.16), (3.15). It
can be veriﬁed directly. Also it follows from the Lax representation (3.23) given below.
Namely,

$$ D_c = -\det \mathcal{L}_k|_{\lambda = -c^{-2}}. $$

Recall that the commutative diagram (2.11) relates the Heisenberg system with
the virtual billiard dynamics. Now we have:
Lemma 3.1. The following diagram is commutative

\[
\begin{array}{c}
P_c(q,Q) \quad \xrightarrow{L_1} \quad M_c(q,p) \\
\Phi \downarrow \quad \downarrow L_2 \\
P_c(q,Q) \quad \xrightarrow{L_1} \quad M_c(Q,P)
\end{array}
\]

in the sense that two-valued map Ψ satisfies Ψ(q,p) = L_1(Φ(L^{-1}_1(q,p))).

Lemma 3.1 is a direct corollary of the definition of discrete Legendre transformations (3.18), (3.19) and the equation of the stationary configuration (1.1). For the completeness of the exposition the proof is included in the Appendix.

As a result, for \(c = \pm 1\), we refer to the correspondence

\[
\begin{align*}
q_{k+1} &= EJ^{-1}p_k + \mu_k J^{-1}q_k, \\
p_{k+1} &= -EJq_k + \mu_k Eq_{k+1}, & k \in \mathbb{Z}, \\
\mu_k &= \left( -\langle J^{-2}E p_k, q_k \rangle \pm \sqrt{D_c(q_k, p_k)} \right) / \langle J^{-2}q_{k+1}, q_k \rangle
\end{align*}
\]

as the Heisenberg model on \(M_c\). If \(\langle J^{-2}q_{k+1}, q_k \rangle = 0\), by definition the flow stops.

By subtracting (3.20) and (3.21), where we set \(k\) instead of \(k+1\), we obtain the equation of stationary configuration (1.1) with the Lagrange multipliers (1.2) and \(\mu_k\) related by

\[
\lambda_k = \mu_k + \mu_{k-1}, & k \in \mathbb{Z}.
\]

By using the integrals (1.3), Lemma 3.1 and the equation (3.20) we get:

Theorem 3.3. The Heisenberg system (3.20), (3.21) is completely integrable with a complete set of integrals

\[
f_i(q,p) = c \cdot \tau_i q_i^2 + \sum_{j \neq i} \frac{\tau_i \tau_j (\tau_i p_j q_i - \tau_j p_i q_j)^2}{J_i^2 - J_j^2},
\]

where \(\{f_i, f_j\} = 0\), \(i, j = 1, \ldots, n\), and \(\sum_i f_i \equiv 1\).

3.3. Bäcklund transformation. Usually, a Bäcklund transformation for a system of differential equations is a mapping which takes solution into solutions, or in the framework of integrable systems, the symplectic mapping which preserves Liouville foliation [25]. We saw that the Moser–Veselov choice \(J(\epsilon) = I + \frac{1}{2} \epsilon^2 U\) for small \(\epsilon\) approximates the Neumann dynamics (3.3). However, it does not preserve the foliation given by (3.13). Instead, as in the Euclidean case (see [25]), we take

\[
J(\epsilon) = \frac{1}{\epsilon} \sqrt{1 + \epsilon^2 U} = \frac{1}{\epsilon} I + \frac{1}{2} \epsilon U + \ldots
\]

Then, from (1.1), by taking \(q_k = q(t_0 + kc)\), \(\epsilon \approx 0\), we again obtain (3.1) with \(\lambda\) replaced by \(\epsilon \lambda\). Therefore, the Heisenberg system with \(J(\epsilon) = \frac{1}{\epsilon} \sqrt{1 + \epsilon^2 U}\) is also a discretization of the Neumann system on the pseudosphere \(S^n_{c-1}\). On the other hand

\[
1/(J_i^2(\epsilon) - J_j^2(\epsilon)) = 1/(U_i - U_j)
\]

and the integrals (3.22) reduce to the integrals (3.13). Therefore, the corresponding Heisenberg model is a Bäcklund transformation of the Neumann system. Moreover, we have the following Lax representation depending on the parameter \(\epsilon\).
Theorem 3.4. Let \( J^2(\epsilon) = U + \epsilon^{-2}I \). The Heisenberg system \((3.20), (3.21)\) on \( M_c, c = \pm 1 \), implies the matrix equations with a spectral parameter \( \lambda \)
\[
\mathcal{L}_{k+1}(\lambda) = \mathcal{M}_k(\lambda)\mathcal{L}_k(\lambda)\mathcal{M}_k(\lambda)^{-1},
\]
where
\[
\mathcal{L}_k(\lambda) = \begin{pmatrix} -Q_\lambda(q_k, E_p) & -Q_\lambda(q_k, q_k) \\ c + Q_\lambda(p_k, p_k) & Q_\lambda(q_k, E_p) \end{pmatrix},
\]
\[
\mathcal{M}_k(\lambda) = \begin{pmatrix} -\mu_k & 1 \\ \mu_k^2 + \lambda - \epsilon^2 & -\mu_k \end{pmatrix}, \quad c = \pm 1,
\]
and \( Q_\lambda \) id given by \((3.10)\).

4. Light–like cone and contact integrability.

4.1. Discrete Legendre transformation for the light–like case. Instead of \( \varphi_2 \), for a description of the cotangent bundle of the light–like cone \( S_{n-1}^0 \) we use the function \( \varphi_3 = \langle p, q \rangle \):
\[
T^*S_{n-1}^0: \quad \varphi_1 = \langle q, q \rangle = 0, \quad \varphi_3 = \langle q, p \rangle = 0.
\]

Then \( \{\varphi_1, \varphi_3\} = 2\langle EQ, q \rangle \neq 0 \), for \( q \neq 0 \). Denote the new Dirac–Poisson bracket by \( \{\cdot, \cdot\}_D \). Repeating the arguments from the previous section, by taking \( S(q, Q) = \langle q, JQ \rangle \) for a generating function, we get the discrete Legendre transformations:
\[
\mathbb{L}_1^0: P_0(q, Q) \rightarrow T^*S_{n-1}^0(q, p), \quad p = EJQ - \frac{\langle EQ, q \rangle}{\langle EQ, q \rangle} EQ,
\]
\[
\mathbb{L}_2^0: P_0(q, Q) \rightarrow T^*S_{n-1}^0(Q, P), \quad P = -EJq + \frac{\langle EQ, q \rangle}{\langle EQ, Q \rangle} EQ,
\]
and the 1:2 symplectic correspondence
\[
\Psi: \mathfrak{M}_0 \rightarrow \mathfrak{M}_0
\]
given by
\[
Q = EJ^{-1}p + \mu J^{-1}q,
\]
\[
P = -EJq + \tilde{\mu} EQ,
\]
where \( \mu = \frac{\langle J^{-2}Ep, q \rangle \pm \sqrt{D_0(q, p)}}{\langle J^{-2}q, q \rangle} \), \( \tilde{\mu} = \langle EQ, q \rangle / \langle EQ, Q \rangle \), and \( \mathfrak{M}_0 \) is a subset of \( T^*S_{n-1}^0 \) defined by the inequalities \((3.11)\) for \( c = 0 \).

Lemma 3.1 also applies, which together with the integrals \((1.3)\) implies the following statement.

Theorem 4.1. The Heisenberg system on \( \mathfrak{M}_0 \)
\[
q_{k+1} = EJ^{-1}p_k + \mu_k J^{-1}q_k,
\]
\[
p_{k+1} = -EJq_k + \tilde{\mu}_k EQ_{k+1}, \quad k \in \mathbb{Z},
\]
is completely integrable. The complete set of first integrals is
\[
f_i(q, p) = \sum_{j \neq i} \frac{\tau_i \tau_j (\tau_i p_j q_i - \tau_j q_i p_i)^2}{J_i - J_j^2},
\]
where \( \{f_i, f_j\}_D = 0, i, j = 1, \ldots, n, \) and \( \sum_i f_i \equiv 0 \).
Again, if \( \langle J_2 q_k, q_{k+1} \rangle = 0 \), by definition the flow stops. Now, the Lagrange multipliers (1.2) of stationary configuration (1.1) and the correspondence (4.1), (4.2) are related by
\[
\lambda_k = \mu_k + \tilde{\mu}_k - 1, \quad k \in \mathbb{Z},
\]
and we have an analog of Theorem 3.4.

**Theorem 4.2.** Let \( U = J^2 \). The Heisenberg system (4.1), (4.2) on \( \mathcal{M}_0 \) implies the matrix equations with a spectral parameter \( \lambda \)
\[
L_{k+1}(\lambda) = M_k(\lambda) L_k(\lambda) M_k(\lambda)^{-1},
\]
where
\[
L_k(\lambda) = \begin{pmatrix} -Q_\lambda(q_k, E p_k) & -Q_\lambda(q_k, q_k) \\ Q_\lambda(p_k, p_k) & Q_\lambda(q_k, E p_k) \end{pmatrix}, \quad M_k(\lambda) = \begin{pmatrix} -\mu_k & 1 \\ \mu_k \tilde{\mu}_k - \lambda & -\tilde{\mu}_k \end{pmatrix},
\]
and \( Q_\lambda \) id given by (3.10).

**Remark 4.1.** Obviously, the constraint \( \varphi_3 = 0 \) can be used for the Heisenberg systems on \( T^*S^{n-1} \pm 1 \) as well, but \( \varphi_2 = 0 \) is more appropriate for a continuous Neumann system (3.3). Namely, the equation
\[
p = \frac{\partial L}{\partial \dot{q}} + \lambda E q = E \dot{q} + \lambda E q
\]
with constraints \( \varphi_1 = c, \varphi_3 = 0 \) and (4.2) implies \( \lambda = -(E \dot{q}, q)/(E q, q) \) and
\[
\dot{q} = E p - \frac{(E p, q)}{(q, q)}.q.
\]

Thus, in this case, the Legendre transformation of \( L(q, \dot{q}) \) yields the Hamiltonian function
\[
H(q, p) = \frac{1}{2} \langle p, p \rangle - \frac{1}{2} \langle E p, q \rangle + \frac{1}{2} \langle U q, q \rangle,
\]
having the extra term \( \langle E p, q \rangle/2(q, q) \).

### 4.2. Contact integrability

The next statement is a cotangent variant of Theorems 2.1 and 3.3 given in [14].

**Theorem 4.3.** (i) The Heisenberg system (4.1), (4.2) satisfies the invariant relation
\[
\langle E q_k, p_k \rangle + \langle E q_{k+1}, p_{k+1} \rangle = 0.
\]

(ii) The restriction of the correspondence (4.1), (4.2) to the invariant manifold
\[
\Sigma_\kappa \subset \mathcal{M}_0: \quad \varphi_2(q, p) = (E q, p) = \pm \kappa, \quad \kappa > 0
\]
is a completely integrable discrete contact system, with respect to the contact form \( \theta = pdq|_{\Sigma_\kappa} \).

**Proof.** The statement follows from Theorems 2.1, 3.3 of [14] and Lemma 3.1. For the completeness of the exposition, we present a direct proof in the Appendix. \( \square \)
5. Geometric interpretation of the integrals

**Theorem 5.1.** (i) If a sequence of planes

\[(5.1) \quad \pi_j = \{s_1 Ep_j + s_2 q_j \mid s_1, s_2 \in \mathbb{R}\}, \quad j \in \mathbb{Z}\]

determined by a trajectory \([(q_j, p_j) \mid j \in \mathbb{Z}]\) of the Heisenberg model \((4.1), \ (4.2)\) is tangent to a cone \(Q_{0, \lambda^*}\) from the pseudo-confocal family \((1.4)\) for a certain \(j\), then it is tangent to \(Q_{0, \lambda^*}\) for all \(j \in \mathbb{Z}\).

(ii) If a sequence of lines

\[(5.2) \quad l_j = \{Ep_j + sq_j \mid s \in \mathbb{R}\}, \quad j \in \mathbb{Z}\]

determined by a trajectory \([(q_j, p_j) \mid j \in \mathbb{Z}]\) of the Heisenberg model \((5.20), \ (5.21)\) is tangent to a quadric \(Q_{c, \lambda^*}\) from the pseudo-confocal family \((1.4)\) for a certain \(j\), then it is tangent to \(Q_{c, \lambda^*}\) for all \(j \in \mathbb{Z}\).

**Proof.** (i) Let \(\pi_j, I = (i_1, \ldots, i_r), 1 \leq i_1 < i_2 < \cdots < i_r \leq n\) be the Plücker coordinates of a \(r\)-dimensional subspace \(\pi\) in \(\mathbb{R}^n\). Then \(\pi\) is tangent to a nondegenerate cone \(K = \{\sum_i b_i x_i^2 = 0\}\) if and only if \(\sum_i b_i \cdots b_{n} \pi_i^2 = 0\) (see Fedorov \[8\]). For \(r = 2\), \(\pi = \text{span} \{x, y\}\) the condition reduces to

\[(5.3) \quad \left(\sum_i b_i x_i^2\right)\left(\sum_i b_i y_i^2\right) - \left(\sum_i b_i x_i y_i\right)^2 = 0.\]

Thus, by taking \(b_i = \tau_i/(U_1 - \lambda^*)\), we get that \(\pi_j = \text{span} \{Ep_j, q_j\}\) is tangent to \(Q_{0, \lambda^*}\) if and only if

\[(5.4) \quad Q_{\lambda^*}(q_j, q_j)Q_{\lambda^*}(Ep_j, Ep_j) - Q_{\lambda^*}(q_j, Ep_j)^2 = 0.\]

On the other hand, from Theorem 4.2 we have that \((5.4)\) is the integral of the system \(Q_{\lambda^*}(q_j, q_j)Q_{\lambda^*}(Ep_j, Ep_j) = Q_{\lambda^*}(q_j, Ep_j)^2 = 0\). Therefore, if \(\pi_j\) is tangent to \(Q_{0, \lambda^*}\), it is tangent to \(Q_{c, \lambda^*}\) for all \(j \in \mathbb{Z}\).

(ii) For \(c = \pm 1\), we consider \((n + 1)\)-dimensional space \(\mathbb{R}^{n+1}(x_0, x_1, \ldots, x_n)\). The plane \(\pi_j = \text{span} \{(0, q_j), (1, Ep_j)\}\) is tangent to the cone

\[K_{c, \lambda^*} : \quad \frac{c x_0^2}{\lambda^* - U_1} + \cdots + \frac{\tau_n x_n^2}{\lambda^* - U_n} = 0\]

if and only if

\[\text{det} L_j(\lambda^*) = Q_{\lambda^*}(q_j, q_j)(c + Q_{\lambda^*}(Ep_j, Ep_j)^2 - Q_{\lambda^*}(q_j, Ep_j)^2 = 0.\]

Here \(L_j(\lambda)\) is given by Theorem 5.4 with \(\epsilon = \infty\). Thus, as in item (i), if \(\pi_j\) is tangent to \(K_{c, \lambda^*}\), it is tangent to \(K_{c, \lambda^*}\) for all \(j \in \mathbb{Z}\). Now, the statement follows from the identities

\[Q_{c, \lambda^*} \cong K_{c, \lambda^*} \cap \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 = 1\}, \quad l_j = Ep_j + sq_j \mid s \in \mathbb{R} \cong \pi_j \cap \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 = 1\}.\]

Obviously, item (ii) holds for the continuous Neumann system \((3.8), \ (3.9)\) as well, by replacing \([(q_j, p_j) \mid j \in \mathbb{Z}]\) by a trajectory \([(q(t), p(t)) \mid t \in \mathbb{R}]\). For the Euclidean case it is proved by Moser (e.g., see \[22\]). The above proof is taken from \[10\], where it is given for the Neumann systems on Stiefel varieties.

Let us assume

\[U_1 < U_2 < \cdots < U_n.\]

In the case of the Euclidean space \((k = n)\), it is well known that outside coordinates hyperplanes through \(q \in \mathbb{E}^n\) it pass exactly \(n\), i.e., \(n - 1\) quadrics from the confocal family \((1.4)\), for \(c = \pm 1\) and \(c = 0\), respectively. They define ellipsoidal coordinates,
Theorem 5.2. For \( c = 1 \) through a generic point in \( \mathbb{E}^k \) pass \( n \) quadrics from the pseudo-confocal family (5.2), for \( c = -1 \) pass \( n \) or \( n - 2 \), and for \( c = 0 \) exactly \( n - 1 \) quadrics.

Proof. For \( c = 1 \) and \( A = EU \), the confocal family (1.4) corresponds to the confocal family studied in (18). Consider the confocal family (1.4) written in the form

\[
(5.5) \quad c + Q_{\lambda}(q, q) = c + \sum_{i=1}^{k} \frac{q_i^2}{\lambda - U_i} = \sum_{i=k+1}^{n} \frac{q_i^2}{\lambda - U_i} = \frac{c\lambda^n + \ldots}{(\lambda - U_1) \ldots (\lambda - U_n)} = 0.
\]

From \( \lim_{\lambda \to U_i \pm} q_i^2/(\lambda - U_i) = \pm \infty \) we see that there exist at least \( n - 2 \) solutions \( \zeta_1 \in (U_1, U_2), \ldots, \zeta_{n-1} \in (U_{k-1}, U_k), \zeta_{k+1} \in (U_{k+1}, U_{k+2}), \ldots, \zeta_{n-1} \in (U_{n-1}, U_n) \) of (5.5) outside coordinates hyperplanes.

Next, from

\[
(5.6) \quad Q_{\lambda}(q, q) = \frac{\lambda^{n-1} \langle q, q \rangle + \ldots}{(\lambda - U_1) \ldots (\lambda - U_n)} \sim \frac{\langle q, q \rangle}{\lambda}, \quad \lambda \to \pm \infty,
\]

and

\[
\lim_{\lambda \to U_i \pm} q_i^2/(\lambda - U_1) = -\infty, \quad \lim_{\lambda \to U_i \pm} q_i^2/(\lambda - U_n) = -\infty,
\]

it follows that in the case \( c = 1 \) there are two additional solutions \( \zeta_0 \in (-\infty, U_1) \) and \( \zeta_0 \in (U_n, \infty) \).

Further, for \( c = 0 \) and \( \langle q, q \rangle < 0 \), from (5.6), we have a solution \( \zeta_0 \) within \( (-\infty, U_1) \), while for \( \langle q, q \rangle > 0 \) we have a solution \( \zeta_0 \in (U_n, \infty) \).

Theorem 5.3. For \( c = 1 \) and a generic trajectory \( \{(q_j, p_j) \mid j \in \mathbb{Z}\} \), the sequence of lines (5.2) is tangent to \( n - 1 \) quadrics from the pseudo-confocal family (1.4), while for \( c = 0 \), the sequence of planes (5.1) is tangent to \( n - 2 \) cones \( Q_{\zeta,\lambda} \). For \( c = -1 \) and a generic trajectory \( \{(q_j, p_j) \mid j \in \mathbb{Z}\} \), the sequence of lines (5.2) is, depending on the initial position, tangent to \( n - 1 \) or \( n - 3 \) quadrics.

Proof. According to Theorem 5.1, we need to estimate the number of the real zeros of the equation \( \mathcal{L}_j(\lambda) = 0 \). To simplify the notation, in what follows we will omit the index \( j \) and use \( p, q, \mathcal{L}(\lambda) \), instead of \( p_j, q_j, \mathcal{L}_j(\lambda) \).

Recall the equation (3.12) and rewrite it as

\[
(5.7) \quad Q_{\lambda}(q, q)(c + Q_{\lambda}(Ep, Ep)) - Q_{\lambda}(q, Ep)^2 = \sum_{i=1}^{n} \frac{f_i(q, p)}{\lambda - U_i} = \frac{P_c(\lambda)}{\prod_i(\lambda - U_i)},
\]

where \( f_i \) are the integrals (3.13) and \( P_c(\lambda) \) is a polynomial of degree \( n - 1 \) for \( c = \pm 1 \) and \( n - 2 \) for \( c = 0 \). Thus, the maximal number of quadrics \( Q_{\zeta,\lambda} \) is \( n - 1 \) (for \( c = \pm 1 \)), i.e., \( n - 2 \) (for \( c = 0 \)). Due to relations

\[
f_1 + \cdots + f_n = c^2, \quad U_1 f_1 + \cdots + U_n f_n = -\langle Ep, q \rangle^2 \quad \text{for} \ c = 0,
\]

the leading terms of polynomials \( P_c(\lambda) \) are given by

\[
P_{\pm 1}(\lambda) = \lambda^{n-1} + \cdots, \quad P_0(\lambda) = -\langle Ep, q \rangle^2 \lambda^{n-2} + \cdots.
\]

Firstly, let us assume \( c = \langle q, q \rangle = -1, q_1 \ldots q_n \neq 0 \). As in the proof of Theorem 5.2, there are \( n - 1 \) solutions

\[
\zeta_0 \in (-\infty, U_1), \ldots, \zeta_{n-1} \in (U_{k-1}, U_k), \zeta_{k+1} \in (U_{k+1}, U_{k+2}), \ldots, \zeta_{n-1} \in (U_{n-1}, U_n)
\]

of the equation \( Q_{\lambda}(q, q) = 0 \).
The left hand side of (5.7) is negative at the ends of all \( n - 3 \) intervals
\[
(\zeta_0, \zeta_1), (\zeta_1, \zeta_2), \ldots, (\zeta_{k-2}, \zeta_{k-1}), (\zeta_{k+1}, \zeta_{k+2}), \ldots, (\zeta_{n-2}, \zeta_{n-1}),
\]
which contain \( U_1, U_2, \ldots, U_{k-1}, U_{k+1}, \ldots, U_{n-1} \), respectively. Owing to
\[
\lim_{\lambda \rightarrow U_i \pm} \frac{f_i}{\lambda - U_i} = (\pm \text{sgn} f_i) \cdot \infty,
\]
we see that each interval in (5.9) contains a solution of \( \det L(\lambda) = 0 \).

In the case \( c = \langle q, q \rangle = 1 \), with \( U_1 \) and \( (\zeta_0, \zeta_1) \) replaced by \( U_n \) and \( (\zeta_{n-1}, \zeta_n) \), we get the existence of \( n - 3 \) solutions of \( \det L(\lambda) = 0 \). On the other side, from (5.8) we get the asymptotic expansion
\[
\sum_{i=1}^{n} \frac{f_i(q,p)}{\lambda - U_i} \sim \frac{1}{\lambda}, \quad \lambda \rightarrow \pm \infty,
\]
leading to a solution within \( (\zeta_n, \infty) \) as well. Since the polynomial \( P_1(\lambda) \) has real coefficients, degree \( n - 1 \), and \( n - 2 \) real zeros (none of the given zeros is of multiplicity greater then 1), it has an additional real zero.

For the case \( c = \langle q, q \rangle = 0 \) we proceed analogously. As in the proof of Theorem 5.2 there are \( n - 2 \) solutions
\[
\zeta_1 \in (U_1, U_2), \ldots, \zeta_{k-1} \in (U_{k-1}, U_k), \zeta_{k+1} \in (U_{k+1}, U_{k+2}), \ldots, \zeta_{n-1} \in (U_{n-1}, U_n)
\]
of the equation \( Q_\lambda(q, q) = 0 \). The left hand side of (5.7) is negative at the ends of all \( n - 4 \) intervals
\[
(\zeta_1, \zeta_2), \ldots, (\zeta_{k-2}, \zeta_{k-1}), (\zeta_{k+1}, \zeta_{k+2}), \ldots, (\zeta_{n-2}, \zeta_{n-1}),
\]
that contain \( U_2, U_3, \ldots, U_{k-1}, U_{k+2}, \ldots, U_{n-1} \). From (5.10), we obtain that each interval in (5.11) contains a solution of \( \det L(\lambda) = 0 \). Moreover, due to (5.8), we have the asymptotic expansion
\[
\sum_{i=1}^{n} \frac{f_i(q,p)}{\lambda - U_i} \sim -\frac{\langle Ep, q \rangle^2}{\lambda^2}, \quad \lambda \rightarrow \pm \infty
\]
implies that there exist \( \zeta_0 < U_1 \) and \( \zeta_n > U_n \), such that the left hand side of (5.7) is less then zero.

Therefore, the equation \( \det L(\lambda) = 0 \) has \( n - 2 \) real solutions.

\[\square\]

Remark 5.1. The signatures \((1, n-1)\) and \((n-1, 1)\) should be treated separately, however for \( c = 1, c = 0, \) and \( c = -1 \) and the signature \((1, n - 1)\) the conclusions are the same. Suppose \( c = -1 \) and \( k = n - 1 \). Now the left hand side of (5.7) is negative at the ends of intervals
\[
(\zeta_0, \zeta_1), (\zeta_1, \zeta_2), \ldots, (\zeta_{n-3}, \zeta_{n-2})
\]
that contain \( U_1, U_2, \ldots, U_{n-3}, U_{n-2} \), so we get \( n - 2 \) real solutions of \( \det L(\lambda) = 0 \). Again, since \( P_{-1}(\lambda) \) has \( n - 2 \) real zeros, it has the additional real zero: the sequence of lines (5.2) is tangent to \( n - 1 \) quadrics \( Q_{-1, \lambda} \) for a generic initial conditions.

Remark 5.2. If we assume \( c = -1 \) and that the value of the integral \( f_k \) is less then zero or the value of \( f_{k+1} \) is greater then zero, then the sequence of lines (5.2) is tangent to \( n - 1 \) quadrics \( Q_{-1, \lambda} \). Indeed, then, from (5.10), there exists \( \zeta_k \in (U_k, U_{k+1}) \) with \( \det L(\zeta_k) < 0 \). Since
\[
U_k \in (\zeta_{k-1}, \zeta_k) \quad \text{and} \quad U_{k+1} \in (\zeta_k, \zeta_{k+1}) \quad \text{for} \ k < n - 1,
\]
there exist two additional real solutions of \( \det L(\lambda) = 0 \).
Remark 5.3. Note that one can consider a symmetric Heisenberg model, i.e., Neumann system on $S^n$ as well, when some of $U_i$ are mutually equal:

$$U_1 = \cdots = U_{p_1} < U_{p_1+1} < \cdots < U_{n-p_1+1} = \cdots = U_n,$$

$\rho_1 + \cdots + \rho_r = n$. Then the set of all symmetries $R \in O(k, l)$: $Ad_R(U) = U$ is either $O(p_1) \times \cdots \times O(p_r)$, or $O(p_1) \times \cdots \times O(p_{p-1}) \times O(k_p, l_p) \times O(p_{p-1}) \times \cdots \times O(p_r)$, $k_p + l_p = p_r$.

Similarly like in the case of virtual billiard dynamics \[18\], the systems are integrable in a noncommutative sense and the phase spaces $M_c, c = \pm 1, 0$ are foliated on invariant $(N - 1)$-dimensional isotropic varieties, where

$$N = r + \mathbb{Z} \{s \mid \rho_s > 1, s = 1, \ldots, r\}.$$

Further, some additional careful analysis is needed in order to estimate the number of real caustics and their maximal number is $N - 1$ for $c = \pm 1$ and $N - 2$ for $c = 0$.

6. Appendix

Proof of Lemma 2.1. Let $S(x, X) = \langle Bx, X \rangle$, where $B$ is a nonsingular matrix. The equations (2.1) become

\[
(6.1) \quad y = B^T X + \lambda A^{-1} x, \quad Y = -Bx - \Lambda A^{-1} X, \quad \text{where} \quad \langle A^{-1} x, x \rangle = 1, \quad \langle A^{-1} X, X \rangle = 1.
\]

From the constraints $\langle y, y \rangle = 1, \langle Y, Y \rangle = 1$, we get that $\lambda$ and $\Lambda$ are solutions of the equations

\[
(6.2) \quad 1 = \langle B^T X, B^T X \rangle + 2\lambda \langle A^{-1} x, B^T X \rangle + \lambda^2 \langle A^{-2} x, x \rangle,
\]

\[
(6.3) \quad 1 = \langle B x, B x \rangle + 2\Lambda \langle A^{-1} X, B x \rangle + \Lambda^2 \langle A^{-2} X, X \rangle.
\]

One can easily see that if

$$\max_{\xi \in \mathbb{Q}^{n-1}} |B^T \xi| = \max_{\xi \in \mathbb{Q}^{n-1}} |B^T A^{1/2} \xi| > 1, \quad \max_{\xi \in \mathbb{Q}^{n-1}} |B \xi| = \max_{\xi \in \mathbb{Q}^{n-1}} |BA^{1/2} \xi| > 1,$$

then there exists $(x, X) \in \mathbb{Q}^{n-1} \times \mathbb{Q}^{n-1}$ such that the discriminant of (6.2), respectively (6.3), is less than zero. On the other hand, if $\max_{\xi \in \mathbb{Q}^{n-1}} |B^T A^{1/2} \xi| \leq 1$ and $\max_{\xi \in \mathbb{Q}^{n-1}} |BA^{1/2} \xi| \leq 1$, the discriminants are greater than zero and we have real multipliers as functions on $\mathbb{Q}^{n-1} \times \mathbb{Q}^{n-1}(x, X)$. Further, if the above relations define the mapping $\psi: M_{1, 1} \rightarrow M_{1, 1}$, we have

$$\max_{\xi \in \mathbb{Q}^{n-1}} \langle (B^T)^{-1} y - \nu(AB^T)^{-1} x, (B^T)^{-1} y - \nu(AB^T)^{-1} x \rangle = 1$$

(6.4)

$$\max_{\xi \in \mathbb{Q}^{n-1}} \langle (B^T)^{-1} y - \nu(AB^T)^{-1} x, (B^T)^{-1} y - \nu(AB^T)^{-1} x \rangle = 1$$

(6.5)

$$< Bx - \mu A^{-1} X = -Bx - \mu A^{-1} ((B^T)^{-1} y - \nu(AB^T)^{-1} x),$$

for some multipliers $\nu, \mu$, now functions on $M_{1, 1}(x, y)$. From (6.3) and the constraint $\langle A^{-1} X, X \rangle = 1$, we get

$$\nu^2 \langle A^{-1}(AB^T)^{-1} x, (AB^T)^{-1} x \rangle - 2\nu \langle (B^T) A^{-1} y, (AB^T)^{-1} x \rangle + \langle (B^T A^{1/2})^{-1} y \rangle^2 = 1.$$

Again, if

$$\max_{\xi \in \mathbb{Q}^{n-1}} |(B^T A^{1/2})^{-1} \xi| = \max_{\xi \in \mathbb{Q}^{n-1}} 1/|B^T A^{1/2} \xi| > 1,$$

there exists $(x, y) \in M$ such that the discriminant of the above quadratic equation is less then zero. Thus, in that case, (6.1) defines a dynamics for complexified objects only. Therefore, we obtain the necessary condition $|B^T A^{1/2}| = 1$. A similar analysis for $(x, y)$ to be expressed as functions of $(X, Y)$, leads to the condition $|BA^{1/2}| = 1$. \[\square\]
Proof of Theorem 2.3. We have
\[ A_k L_k = A^2 + \lambda K_1 + \lambda^2 K_2 + \lambda^3 c K_3 + \lambda^4 c^2 K_4, \]
\[ L_{k+1} A_k = A^2 + \lambda S_1 + \lambda^2 S_2 + \lambda^3 c S_3 + \lambda^4 c^2 S_4, \]
where
\[ K_1 = F_y k \otimes AF_{xk} - F_{xk} \otimes AF_{yk-1} + AF_{yk-1} \otimes F_{xk} - AF_{xk} \otimes F_{yk-1}, \]
\[ K_2 = -c F_y k \otimes AF_{yk-1} - c AF_{yk}\otimes F_{yk-1} + (y_{k-1}, x_k) F_{xk} \otimes F_{yk-1} \]
\[ - (y_{k-1}, y_{k-1}) F_{xk} \otimes F_{xk} + (x_{k}, y_{k-1}) F_{yk} \otimes F_{xk} - (x_{k}, x_{k}) F_{yk} \otimes F_{yk-1}, \]
\[ K_3 = (y_{k-1}, y_{k-1}) F_{xk} \otimes F_{yk-1} - (y_{k-1}, y_{k-1}) F_{yk} \otimes F_{xk}, \]
\[ K_4 = (y_{k-1}, y_{k-1}) F_{yk} \otimes F_{yk-1}, \]
and
\[ S_1 = F_y k \otimes AF_{xk+1} - F_{xk+1} \otimes AF_{yk} + AF_{yk} \otimes F_{xk} - AF_{xk} \otimes F_{yk-1}, \]
\[ S_2 = -c F_y k \otimes AF_{yk} - c AF_{yk} \otimes F_{yk-1} + (y_{k}, x_k) F_{xk} \otimes F_{yk-1} \]
\[ - (y_{k+1}, x_k) F_{yk} \otimes F_{yk-1} - (y_{k}, y_{k}) F_{xk+1} \otimes F_{xk} + (x_{k+1}, y_{k}) F_{yk} \otimes F_{xk}, \]
\[ S_3 = (y_{k}, y_{k}) F_{xk+1} \otimes F_{yk-1} - (y_{k+1}, y_{k}) F_{yk} \otimes F_{xk} \]
\[ + (x_{k}, y_{k}) F_{yk} \otimes F_{xk} \]
\[ + (y_{k}, y_{k}) F_{yk} \otimes F_{xk} \]
\[ S_4 = (y_{k}, y_{k}) F_{yk} \otimes F_{yk-1}. \]

It is evident that \( K_4 = S_4. \) From (2.3), (2.4) we obtain
\[ K_1 - S_1 = F_y k \otimes AF_{xk} - F_{xk} \otimes AF_{yk-1} + AF_{yk-1} \otimes F_{xk} \]
\[ - F_y k \otimes AF(x_k + \mu_k y_k) + F(x_k + \mu_k y_k) \otimes AF y_k - AF y_k \otimes F_{xk} \]
\[ = (AF y_k - 1) AF y_k \wedge F_{xk} = -\nu_{k-1} AAF^{-1} x_k \wedge F_{xk} = 0, \]
\[ K_2 - S_2 = c (F_y k \otimes (AF_{yk} - AF_{yk-1}) + (AF y_k - AF_{yk-1}) \otimes F_{yk-1}) \]
\[ + (y_{k-1}, x_k) F_{xk} \otimes F_{yk-1} - (y_{k+1}, y_k) F_{yk} \otimes F_{xk} \]
\[ + (x_k, y_k) F_{yk} \otimes F_{xk} - (x_k, y_k) F_{yk} \otimes F_{xk} \]
\[ - (y_k, x_k) F_{xk+1} \otimes F_{yk} + (x_k, y_k) F_{yk} \otimes F_{yk-1} + (x_k, y_k) F_{yk} \otimes F_{yk-1} \]
\[ + (y_k, y_k) F_{xk+1} \otimes F_{yk} + (x_k, y_k) F_{yk} \otimes F_{xk}, \]
that is,
\[ K_2 - S_2 = c (\nu_{k-1} F_{yk} \otimes F_{xk} + \nu_{k-1} F_{xk} \otimes F_{yk-1}) \]
\[ + (y_{k-1} - y_k, y_k) F_{xk} \otimes F_{yk-1} + (x_k, y_{k-1} - y_k) F_{yk} \otimes F_{xk} \]
\[ = c (\nu_{k-1} F_{yk} \otimes F_{xk} + \nu_{k-1} F_{xk} \otimes F_{yk-1}) \]
\[ + (-\nu_{k-1} A^{-1} x_k, x_k) F_{xk} \otimes F_{yk-1} + (x_k, -\nu_{k-1} A^{-1} x_k) F_{yk} \otimes F_{xk} = 0, \]
and
\[ K_3 - S_3 = (y_{k-1}, y_{k-1}) F_{xk} \otimes F_{yk-1} - (y_{k-1}, y_{k-1}) F_{yk} \otimes F_{xk} \]
\[ - (y_k, x_k) F_{xk+1} \otimes F_{yk} + (x_k, y_k) F_{yk} \otimes F_{xk-1} \]
\[ - (y_k, x_k) F_{yk} \otimes F_{yk-1} + (y_k, y_k) F_{yk} \otimes F_{xk} = 0. \]
Proof of Lemma 3.1. Let \( L_i^{-1}(q,p) = \{(q,Q_1),(q,Q_2)\}, (Q_i,P_i) = L_i(Q_i,Q_i), i.e., \)
\[
p = EJQ_i - \frac{1}{c}(JQ_i,q)EQ_i,
\]
\[
P_i = -EJq + \frac{1}{c}(JQ_i,q)EQ_i,
\]
\[
Q_i = -q + 2\frac{(J^{-1}q,Q_i)}{(J^{-2}Q_i,Q_i)}J^{-1}Q_i,
\]
\[
P_i = EJQ_i - \frac{1}{c}(JQ_i,q)EQ_i, \quad i = 1,2.
\]

Now, Lemma follows from the identity
\[
P_i = -EJq + \frac{1}{c}(JQ_i,q)EQ_i - \frac{1}{c}\langle JQ_i, Q_i \rangle EQ_i - 2\frac{(J^{-1}q,Q_i)}{(J^{-2}Q_i,Q_i)}J^{-1}Q_i.
\]
\[
= -EJq + \frac{1}{c}(JQ_i,q)EQ_i = P_i. \quad \square
\]

Proof of Theorem 4.3. (i) From (4.1) and (4.2) we have
\[
\langle E_{k+1},p_{k+1} \rangle = \langle J^{-1}p_k + \mu_k EJ^{-1}q_k, -EJq_k + \bar{\mu}_k EQ_{k+1} \rangle
\]
\[
= -\langle E_{k},p_k \rangle - \mu_k \langle q_k,q_k \rangle + \mu_k \bar{\mu}_k \langle J^{-1}q_k,q_{k+1} \rangle + \bar{\mu}_k \langle J^{-1}p_k, EQ_{k+1} \rangle
\]
\[
= -\langle E_{k},p_k \rangle + \bar{\mu}_k \langle EJ^{-1}p_k + \mu_k J^{-1}q_k, q_{k+1} \rangle
\]
\[
= -\langle E_{k},p_k \rangle + \bar{\mu}_k \langle q_k+1,q_{k+1} \rangle - \langle E_{q},p_k \rangle
\]
\[
= \langle E_{q},p_k \rangle.
\]

(ii) Consider the graph \( \Gamma_\kappa \) of the correspondence \( \Psi|_{\Sigma_\kappa} \):
\[
\Gamma_\kappa \subset \Sigma_\kappa(q,p) \times \Sigma_\kappa(Q,P) \subset \mathbb{R}^{2n}(q,p) \times \mathbb{R}^{2n}(Q,P)
\]
Note that the generating function \( S = \langle q, JQ \rangle \) of the mappings \( L_1^0, L_2^0 \) satisfies \( S = \pm \kappa |r_\kappa|, \) i.e., \( dS|_{r_\kappa} = 0. \) Therefore,
\[
PdQ - pdq = -dS + \bar{\mu} EQdQ + \mu Eqdq = \frac{\bar{\mu}}{2} d\langle Q,Q \rangle + \frac{\mu}{2} d\langle q,q \rangle = 0|_{r_\kappa}.
\]

Thus, the Heisenberg system restricted to \( \Sigma_\kappa \) preserves the 1-form \( \theta = pdq|_{\Sigma_\kappa} \):
\[
(\Psi|_{\Sigma_\kappa})^*\theta = \theta.
\]

The Hamiltonian flow of \( \varphi_2 = \langle E_p,q \rangle \) with respect to the Dirac–Poisson bracket \( \{\cdot,\cdot\}_D \) equals
\[
X = \sum_i q_i \partial / \partial q_i - p_i \partial / \partial p_i.
\]
The submanifold \( \Sigma_\kappa \) is a contact manifold with respect to \( \theta \), if and only if \( \theta(X) \neq 0 \) on \( \Sigma_\kappa \) (e.g., see [21]). We have \( \theta(X) = \varphi_2 = \pm \kappa \). Therefore, \( \Sigma_\kappa \) is a contact manifold with respect to \( \theta \) for \( \kappa \neq 0 \) with the Reeb vector field \( Z = \pm \frac{1}{\kappa} X \).

Next, since \( \{\varphi_2, f_i\}_D = 0, i = 1, \ldots , n \), where integrals \( f_i \) are given by (4.3), from \( \{f_i, f_j\}_D = 0 \), using the theorem on isoenergetic integrability (see [17]), we get that the restrictions of \( f_i|_{\Sigma_\kappa} \) commute with respect to the Jacobi bracket on \( (\Sigma_\kappa, \theta) \). We have two relations
\[
\sum_i f_i|_{\Sigma_\kappa} = 0, \quad \sum_i J^2 f_i|_{\Sigma_\kappa} + \kappa^2 = 0,
\]
and there are \( n - 2 \) independent integrals on \( \Sigma_\kappa \). Thus, the mapping \( \Psi \) is a completely integrable contact 1:2 correspondence (see [20], [18]). \( \square \)
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