Brauer-Thrall type theorems for derived category

Yang Han and Chao Zhang

KLMM, ISS, AMSS, Chinese Academy of Sciences, Beijing 100190, P.R. China.
E-mail: hany@iss.ac.cn; zhangc@amss.ac.cn

Abstract

The numerical invariants (global) cohomological length, (global) cohomological width, and (global) cohomological range of complexes (algebras) are introduced. Cohomological range leads to the concepts of derived bounded algebras and strongly derived unbounded algebras naturally. The first and second Brauer-Thrall type theorems for the bounded derived category of a finite-dimensional algebra over an algebraically closed field are obtained. The first Brauer-Thrall type theorem says that derived bounded algebras are just derived finite algebras. The second Brauer-Thrall type theorem says that an algebra is either derived discrete or strongly derived unbounded, but not both. Moreover, piecewise hereditary algebras and derived discrete algebras are characterized as the algebras of finite global cohomological width and finite global cohomological length respectively.

Mathematics Subject Classification (2010): 16G60; 16E35; 16G20

Keywords: derived bounded; derived finite; derived discrete; strongly derived unbounded.

1 Introduction

Throughout this paper, $k$ is an algebraically closed field, all algebras are finite-dimensional basic connected associative $k$-algebras with identity, and all modules are finite-dimensional right modules, unless stated otherwise. One of the main topics in representation theory of algebras is to study the classification and distribution of indecomposable modules. In this aspect two famous problems are Brauer-Thrall conjectures I and II:
Brauer-Thrall conjecture I. The algebras of bounded representation type are of finite representation type.

Brauer-Thrall conjecture II. The algebras of unbounded representation type are of strongly unbounded representation type.

Here, we say an algebra is of finite representation type or representation-finite if there are only finitely many isomorphism classes of indecomposable modules. An algebra is said to be of bounded representation type if the dimensions of all indecomposable modules have a common upper bound, and of unbounded representation type otherwise. We say an algebra is of strongly unbounded representation type if there are infinitely many $d \in \mathbb{N}$ such that there exist infinitely many isomorphism classes of indecomposable modules of dimension $d$. Brauer-Thrall conjectures I and II were formulated by Jans [21]. Brauer-Thrall conjecture I was proved for finite-dimensional algebras over an arbitrary field by Roiter [26], for Artin algebras by Auslander [4]. Brauer-Thrall conjecture II was proved for finite-dimensional algebras over a perfect field by Nazarova and Roiter using matrix method [23, 27], and over an algebraically closed field by Bautista using geometric method [5]. Refer to [25] for more on Brauer-Thrall conjectures.

Since Happel [14, 15], the bounded derived categories of finite-dimensional algebras have been studied widely. The study of the classification and distribution of indecomposable objects in the bounded derived category of an algebra is still an important theme in representation theory of algebras. It is natural to consider the derived versions of Brauer-Thrall conjectures. For this, one needs to find an invariant of a complex analogous to the dimension of a module. In this aspect, Vossieck is undoubtedly a pioneer. He introduced and classified derived discrete algebras, i.e., the algebras whose bounded derived categories admit only finitely many isomorphism classes of indecomposable objects of arbitrarily given cohomology dimension vector, in his brilliant paper [29]. According to cohomology dimension vector, all algebras are divided into two disjoint classes: derived discrete algebras and derived indiscrete algebras. Namely, cohomology dimension vector is really a nice invariant of complexes for distinguishing whether an algebra is derived discrete or not. Nevertheless, cohomology dimension vector is seemingly not a perfect invariant of complexes at least in the context of derived versions of Brauer-Thrall conjectures, because it is too fine to identify an indecomposable complex with its shifts.

In this paper, we shall introduce the cohomological range of a bounded complex which is a numerical invariant up to shift and isomorphism. It leads
to the concepts of derived bounded algebras and strongly derived unbounded algebras naturally. We shall prove the following two Brauer-Thrall type theorems for derived categories:

**Theorem I.** Derived bounded algebras are just derived finite algebras.

**Theorem II.** An algebra is either derived discrete or strongly derived unbounded, but not both.

According to cohomological range and Theorems I and II, all algebras are divided into three disjoint classes: derived finite algebras, derived discrete but not derived finite algebras, and strongly derived unbounded algebras. In particular, there does not exist such an algebra that there are only finitely many cohomological ranges of each which there are infinitely many indecomposable complexes up to shift and isomorphism.

The paper is organized as follows: In Section 2, we shall introduce some numerical invariants of complexes (algebras) including (global) cohomological length, (global) cohomological width, and (global) cohomological range, and observe their behaviors under derived equivalences. Global cohomological width provides an alternative definition of strong global dimension on the level of bounded derived category, and piecewise hereditary algebras are characterized as the algebras of finite global cohomological width. Furthermore, we shall prove Theorem I. In Section 3, we shall show that strongly derived unboundedness is invariant under derived equivalences, and observe its relation with cleaving functor. Furthermore, we shall prove Theorem II for simply connected algebras, gentle algebras, and finally all algebras by using cleaving functor and covering theory. Moreover, derived discrete algebras are characterized as the algebras of finite global cohomological length.

## 2 The first Brauer-Thrall type theorem

### 2.1 Some invariants of complexes and algebras

Let $A$ be an algebra. Denote by $\text{mod}
A$ the category of all finite-dimensional right $A$-modules, and by $\text{proj}
A$ its full subcategory consisting of all finitely generated projective right $A$-modules. Denote by $C(A)$ the category of all complexes of finite-dimensional right $A$-modules, and by $C^b(A)$ and $C^{-b}(A)$ its full subcategories consisting of all bounded complexes and right bounded complexes with bounded cohomology respectively. Denote by $C^b(\text{proj}
A)$ and $C^{-b}(\text{proj}
A)$ the full subcategories of $C^b(A)$ and $C^{-b}(A)$ respectively consisting of all complexes of finitely generated projective modules. Denote
by \( K(A) \), \( K^b(\text{projA}) \) and \( K^{-b}(\text{projA}) \) the homotopy categories of \( C(A) \), \( C^b(\text{projA}) \) and \( C^{-b}(\text{projA}) \) respectively. Denote by \( D^b(A) \) the bounded derived category of \( \text{modA} \).

Now we introduce some invariants of complexes and algebras.

The **cohomological length** of a complex \( X^\bullet \in D^b(A) \) is

\[
\text{hl}(X^\bullet) := \max\{\dim H^i(X^\bullet) \mid i \in \mathbb{Z}\}.
\]

The **global cohomological length** of \( A \) is

\[
\text{gl.hl} \ A := \sup\{\text{hl}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.
\]

Obviously, the dimension of an \( A \)-module \( M \) is equal to the cohomological length of the stalk complex \( M \). Since there is a full embedding of \( \text{modA} \) into \( D^b(A) \) which sends a module to the corresponding stalk complex, by Roiter’s theorem on Brauer-Thrall conjecture I, if \( \text{gl.hl} A < \infty \) then \( A \) is representation-finite.

The **cohomological width** of a complex \( X^\bullet \in D^b(A) \) is

\[
\text{hw}(X^\bullet) := \max\{j - i + 1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\}.
\]

The **global cohomological width** of \( A \) is

\[
\text{gl.hw} \ A := \sup\{\text{hw}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.
\]

Clearly, the cohomological width of a stalk complex is 1. If \( A \) is a hereditary algebra then every indecomposable complex \( X^\bullet \in D^b(A) \) is isomorphic to a stalk complex by [135 I.5.2 Lemma]. Thus \( \text{gl.hw} A = 1 \).

The **cohomological range** of a complex \( X^\bullet \in D^b(A) \) is

\[
\text{hr}(X^\bullet) := \text{hw}(X^\bullet) \cdot \text{hl}(X^\bullet).
\]

The **global cohomological range** of \( A \) is

\[
\text{gl.hr} \ A := \sup\{\text{hr}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.
\]

The cohomological range of a complex will play a similar role to the dimension of a module. It is invariant under shift and isomorphism, since cohomological length and cohomological width are.

Next we observe the behaviors of these invariants under derived equivalences. For this we need do some preparations.
Let $\mathcal{T}$ be a triangulated $k$-category with $[1]$ the shift functor. For $T \in \mathcal{T}$, we define $\langle T \rangle_n$ inductively by

$$\langle T \rangle_0 := \{ X \in \mathcal{T} \mid X \text{ is a direct summand of } T[i] \text{ for some } i \in \mathbb{Z} \},$$

and

$$\langle T \rangle_n := \left\{ X \in \mathcal{T} \mid Y' \to X \oplus Y \to Y'' \to \text{ is a triangle in } \mathcal{T} \right\}.$$ 

Clearly, $\langle T \rangle_{n-1} \subseteq \langle T \rangle_n$ and $\langle T \rangle := \bigcup_{n \geq 0} \langle T \rangle_n$ is the smallest thick subcategory of $\mathcal{T}$ containing $T$. For $X \in \langle T \rangle$, the distance of $X$ from $T$ is

$$d(X, T) := \min \{ n \in \mathbb{N} \mid X \in \langle T \rangle_n \}.$$ 

**Lemma 1.** (See Geiss and Krause [13, Lemma 4.1]) Let $\mathcal{T}$ be a triangulated $k$-category, $T \in \mathcal{T}$ and $X \in \langle T \rangle$. Then for all $Y \in \mathcal{T}$

$$\dim \text{Hom}_\mathcal{T}(X, Y) \leq 2^{d(X, T)} \sup_{i \in \mathbb{Z}} \dim \text{Hom}_\mathcal{T}(T[i], Y).$$

**Proposition 1.** Let $A$ and $B$ be two algebras, $A^T_B$ a two-sided tilting complex, and $F = - \otimes_A^L T_B : D^b(A) \to D^b(B)$ a derived equivalence. Then there are $N_0, N_1, N_2 \in \mathbb{N}$ such that for all $X^\bullet \in D^b(A)$

1. $\text{hw}(F(X^\bullet)) \leq \text{hw}(X^\bullet) + N_0$,
2. $\text{hl}(F(X^\bullet)) \leq N_1 \cdot \text{hl}(X^\bullet)$,
3. $\text{hr}(F(X^\bullet)) \leq N_2 \cdot \text{hr}(X^\bullet)$.

**Proof.** (1) Recall that the width of a complex $X^\bullet \in C^b(A)$ is

$$w(X^\bullet) := \max \{ j - i + 1 \mid Y^j \neq 0 \neq Y^i \}.$$ 

For any $X^\bullet \in D^b(A)$, there exist a complex $\tilde{X}^\bullet \in D^b(A)$ which can be obtained from $X^\bullet$ by good truncations, such that $\text{hw}(\tilde{X}^\bullet) = w(\tilde{X}^\bullet)$ and $\tilde{X}^\bullet \cong X^\bullet$. Since $A^T_B$ is a two-sided tilting complex, there is a perfect complex $\tilde{A}^\bullet$ such that $A^\bullet \cong \tilde{A}^\bullet$. Thus, viewed as a complex of $k$-vector spaces, $F(\tilde{X}^\bullet) \cong \tilde{X}^\bullet \otimes_A^L T^\bullet \cong \text{Tot}^\oplus(\tilde{X}^\bullet \otimes_A^L T^\bullet)$. Hence $\text{hw}(F(\tilde{X}^\bullet)) = \text{hw}(F(X^\bullet)) = \text{hw}(\text{Tot}^\oplus(\tilde{X}^\bullet \otimes_A^L T^\bullet)) \leq \text{hw}(\text{Tot}^\oplus(\tilde{X}^\bullet \otimes_A^L T^\bullet)) \leq w(\tilde{X}^\bullet_A) + w(A^\bullet_T) = \text{hw}(\tilde{X}^\bullet_A) + w(A^\bullet_T).$ So $N_0 := w(A^\bullet_T)$ is as required.
(2) Let $G : D^b(B) \to D^b(A)$ be a quasi-inverse of $F$. Then $G(B) \in K^b(\text{proj}A) = \langle A \rangle$. Thus $B \cong FG(B) \in \langle F(A) \rangle$. By Lemma 1 we have

$$\dim H^i(F(X^\bullet)) = \dim \text{Hom}_{D^b(B)}(B, F(X^\bullet)[i]) \leq 2^{d(B,F(A))} \sup_{j \in \mathbb{Z}} \dim \text{Hom}_{D^b(B)}(F(A), F(X^\bullet)[j])$$

$$= 2^{d(B,F(A))} \sup_{j \in \mathbb{Z}} \dim H^j(F(X^\bullet))$$

$$= 2^{d(B,F(A))} \text{hl}(X^\bullet).$$

Thus $N_1 := 2^{d(B,F(A))}$ is as required.

(3) It follows from (1) and (2) that $\text{hl}(F(X^\bullet)) = \text{hw}(F(X^\bullet)) \cdot \text{hl}(F(X^\bullet)) \leq (\text{hw}(X^\bullet) + N_0) \cdot N_1 \cdot \text{hl}(X^\bullet) \leq (N_0 + 1)N_1 \cdot \text{hr}(X^\bullet)$. Thus $N_2 := (N_0 + 1)N_1$ is as required.

**Corollary 1.** Let two algebras $A$ and $B$ be derived equivalent. Then $\text{gl.hw} A < \infty$ (resp. $\text{gl.hl} A < \infty$, $\text{gl.hr} A < \infty$) if and only if $\text{gl.hw} B < \infty$ (resp. $\text{gl.hl} B < \infty$, $\text{gl.hr} B < \infty$).

**Proof.** By the assumption $A$ and $B$ are derived equivalent, there is a two-sided tilting complex $AT^\bullet_B$ such that $- \otimes^L_A T^\bullet_B : D^b(A) \to D^b(B)$ is a derived equivalence [24]. So the corollary follows from Proposition 1.

**2.2 Strong global dimension**

Strong global dimension was introduced by Skowroński in [28]. Happel and Zacharia characterized piecewise hereditary algebras as the algebras of finite strong global dimension [20]. Here, we adopt the definition of strong global dimension in [20], which is slightly different from that in [28].

Recall that a complex $X^\bullet = (X^i, d^i) \in C(A)$ is said to be *minimal* if $\text{Im}d^i \subseteq \text{rad}X^{i+1}$ for all $i \in \mathbb{Z}$. For any complex $P^\bullet = (P^i, d^i) \in K^b(\text{proj}A)$, there is a minimal complex $\tilde{P}^\bullet = (\tilde{P}^i, \tilde{d}^i) \in K^b(\text{proj}A)$, which is unique up to isomorphism in $C(A)$, such that $P^\bullet \cong \tilde{P}^\bullet$ in $K^b(\text{proj}A)$. The *length* of $P^\bullet$ is

$$l(P^\bullet) := \max\{j - i \mid \tilde{P}^i \neq 0 \neq \tilde{P}^j\}.$$

The *strong global dimension* of $A$ is

$$\text{s.gl.dim} A := \sup\{l(P^\bullet) \mid P^\bullet \in K^b(\text{proj}A) \text{ is indecomposable}\}.$$
Proposition 2. Let $P^* \in K^{-b}(\text{proj} A)$ be a minimal complex and $n := \min\{i \in \mathbb{Z} \mid H^i(P^*) \neq 0\}$. Then $P^*$ is indecomposable if and only if so is the brutal truncation $\sigma_{\geq j}(P^*) \in K^b(\text{proj} A)$ for all or some $j < n$.

Proof. Since $K^{-b}(\text{proj} A) \simeq D^b(A)$ is Hom-finite and Krull-Schmidt, the complex $P^* \in K^{-b}(\text{proj} A)$ is indecomposable if and only if its endomorphism algebra $\text{End}_{K(A)}(P^*)$ is local, if and only if $\text{End}_{K(A)}(P^*)/\text{radEnd}_{K(A)}(P^*) \cong k$. Hence, it suffices to show

$$\text{End}_{K(A)}(P^*)/\text{radEnd}_{K(A)}(P^*) \cong \text{End}_{K(A)}(\sigma_{\geq j}(P^*))/\text{radEnd}_{K(A)}(\sigma_{\geq j}(P^*)).$$  

Since $P^*$ is minimal, all null homotopic cochain maps in $\text{End}_{C(A)}(\sigma_{\geq j}(P^*))$ are in $\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^*))$. Thus $\text{End}_{K(A)}(\sigma_{\geq j}(P^*))/\text{radEnd}_{K(A)}(\sigma_{\geq j}(P^*)) \cong \text{End}_{C(A)}(\sigma_{\geq j}(P^*))/\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^*))$. Hence, it is enough to show

$$\text{End}_{K(A)}(P^*)/\text{radEnd}_{K(A)}(P^*) \cong \text{End}_{C(A)}(\sigma_{\geq j}(P^*))/\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^*)).$$  

Consider the composition of homomorphisms of algebras $\text{End}_{C(A)}(P^*) \xrightarrow{\psi} \text{End}_{C(A)}(\sigma_{\geq j}(P^*)) \xrightarrow{\phi} \text{End}_{C(A)}(\sigma_{\geq j}(P^*)/\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^*))$, where $\phi$ is the natural restriction and $\psi$ is the canonical projection. Since $\sigma_{\leq j-1}(P^*)$ is a minimal projective resolution of $\text{Ker}d^j$, any cochain map in $\text{End}_{C(A)}(\sigma_{\geq j}(P^*))$ can be lifted to a cochain map in $\text{End}_{C(A)}(P^*)$, i.e., $\phi$ is surjective. Thus the composition $\varphi = \psi\phi$ is surjective. Since $P^*$ is a minimal complex, $\phi$ maps all null homotopic cochain maps in $\text{End}_{C(A)}(P^*)$ into $\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^*))$. Thus $\varphi$ induced a surjective homomorphism of algebras

$$\tilde{\varphi} : \text{End}_{K(A)}(P^*) \twoheadrightarrow \text{End}_{C(A)}(\sigma_{\geq j}(P^*)/\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^*)).$$

Now it is sufficient to show that $\text{Ker}\tilde{\varphi} = \text{radEnd}_{K(A)}(P^*)$. Clearly, $\text{Ker}\tilde{\varphi} \supseteq \text{radEnd}_{K(A)}(P^*)$. Conversely, for any $f^* \in \text{Ker}\tilde{\varphi}$, we have $\phi(f^*) = \tilde{\varphi}(f^*) = 0$. Thus $\phi(f^*)$ is nilpotent, i.e., there exists $t \in \mathbb{N}$ such that $(f^*)^t = 0$ for all $i \geq j$. Since $\sigma_{\leq j-1}(P^*)$ is a minimal projective resolution of $\text{Ker}d^j$, the restriction $\sigma_{\leq j-1}(f^*) \in \text{End}_{C(A)}(\sigma_{\leq j-1}(P^*))$ of $f^*$ is a lift of the restriction of $f^j$ on $\text{Ker}d^j$. Thus $(\sigma_{\leq j-1}(f^*))^t$ is a lift of the restriction
of \((f^j)^t\) on \(\text{Ker} d^t\). Hence \((\sigma_{\leq j-1}(f^*))^t\) is null homotopic. Therefore, \(f^\bullet\) is nilpotent in \(\text{End}_{K(A)}(P^\bullet)\), and thus \(f^\bullet \in \text{radEnd}_{K(A)}(P^\bullet)\). Consequently, \(\text{Ker} \bar{\varphi} \subseteq \text{radEnd}_{K(A)}(P^\bullet)\).

\[\square\]

**Corollary 2.** \(\text{gl.dim} A \leq \text{s.gl.dim} A\).

**Proof.** We have known \(\text{gl.dim} A \leq \text{s.gl.dim} A\) if \(\text{gl.dim} A < \infty\). If \(\text{gl.dim} A = \infty\) then there is a simple \(A\)-module \(S\) of infinite projective dimension. Thus \(S\) admits an infinite minimal projective resolution. By Proposition 2 there are indecomposable complexes in \(K^b(\text{proj} A)\) of arbitrarily large length, which implies \(\text{s.gl.dim} A = \infty\).

\[\square\]

**Remark 1.** It is possible \(\text{gl.dim} A < \text{s.gl.dim} A\). Indeed, since piecewise hereditary algebras are the factors of finite-dimensional hereditary algebras [18, Theorem 1.1], all algebras of finite global dimension and with oriented cycles in their quivers are of infinite strong global dimension by [20, Theorem 3.2].

Note that the definition of the length of a perfect complex is indirected, and the strong global dimension of an algebra is defined by the lengths of the indecomposable complexes in its perfect derived category but not bounded derived category. The following result implies that the global cohomological width can provide an alternative definition of strong global dimension on the level of bounded derived category.

**Proposition 3.** Let \(A\) be an algebra. Then \(\text{gl.hw} A = \text{s.gl.dim} A\).

**Proof.** First we show \(\text{gl.hw} A \leq \text{s.gl.dim} A\). If \(\text{s.gl.dim} A = \infty\) then we have nothing to do. Now we assume \(\text{s.gl.dim} A = n < \infty\). By Corollary 2 we have \(\text{gl.dim} A < \infty\). Thus \(D^b(A) \simeq K^b(\text{proj} A)\). Hence, for any indecomposable complex \(X^\bullet \in D^b(A)\), there is a minimal complex \(P^\bullet X^\bullet \in K^b(\text{proj} A)\), such that \(X^\bullet \cong P^\bullet X^\bullet\). Since \(\text{hw}(X^\bullet) = \text{hw}(P^\bullet X^\bullet) \leq l(P^\bullet X^\bullet) + 1\), we have \(\text{gl.hw} A \leq \text{s.gl.dim} A + 1\). Actually, we must have \(\text{gl.hw} A \leq \text{s.gl.dim} A\). Assume on the contrary that \(\text{gl.hw} A = \text{s.gl.dim} A + 1\). Then there is an indecomposable minimal complex \(P^\bullet \in K^b(\text{proj} A)\) with \(\text{hw}(P^\bullet) = \text{gl.hw} A = \text{s.gl.dim} A + 1 = n + 1\) and of the form

\[
P^\bullet = 0 \longrightarrow P^{-n} \overset{d^{-n}}{\longrightarrow} P^{-n+1} \overset{d^{-n+1}}{\longrightarrow} \cdots \overset{d^{-2}}{\longrightarrow} P^{-1} \overset{d^{-1}}{\longrightarrow} P^0 \longrightarrow 0,
\]

such that \(H^0(P^\bullet) \neq 0 \neq H^{-n}(P^\bullet)\). Clearly, \(\text{Ker} d^{-n} = H^{-n}(P^\bullet) \neq 0\). It admits a minimal projective resolution

\[
Q^\bullet = 0 \longrightarrow P^{-t} \overset{d^{-t}}{\longrightarrow} \cdots \overset{d^{-2}}{\longrightarrow} P^{-n-1} \longrightarrow 0.
\]

8
Gluing $Q^\bullet$ and $P^\bullet$ together, we get a minimal complex
\[ P'^\bullet = 0 \longrightarrow P^{-t} \overset{d^{-t}}{\longrightarrow} \cdots \overset{d^{-n-2}}{\longrightarrow} P^{-n-1} \overset{\epsilon}{\longrightarrow} P^{-n} \overset{d^{-n}}{\longrightarrow} \cdots \overset{d^{-1}}{\longrightarrow} P^0 \longrightarrow 0, \]
where $\epsilon$ is the composition $P^{-n-1} \rightarrow \text{Ker} d^{-n} \hookrightarrow P^{-n}$. Since $P^\bullet = \sigma_{\geq -n}(P'^\bullet)$ is indecomposable and $H^i(P'^\bullet) = 0$ for all $i \leq -n$, by Proposition \[\ref{prop1}\] $P'^\bullet$ is indecomposable as well, which contradicts to $\text{s.gl.dim } A = n$. Hence $\text{gl.hw } A \leq \text{s.gl.dim } A$.

Next we show $\text{gl.hw } A \geq \text{s.gl.dim } A$. Assume on the contrary that $m := \text{gl.hw } A < \text{s.gl.dim } A$. Then there is an indecomposable minimal complex
\[ P^\bullet = \cdots \longrightarrow P^{-m-1} \overset{d^{-m-1}}{\longrightarrow} P^{-m} \overset{d^{-m}}{\longrightarrow} \cdots \overset{d^{-1}}{\longrightarrow} P^{-1} \overset{d^{-1}}{\longrightarrow} P^0 \longrightarrow 0 \in K^b(\text{proj } A) \]
such that $P^{-m-1} \neq 0$. Since $P^\bullet$ is minimal, we have $H^0(P^\bullet) \neq 0$. Thus $\text{gl.hw } A = m$ implies $H^i(P^\bullet) = 0$ for all $i \leq -m$. By Proposition \[\ref{prop1}\] we know $\sigma_{\geq -m}(P^\bullet)$ is indecomposable. Since $H^{-m}(\sigma_{\geq -m}(P^\bullet)) = \text{Ker} d^{-m} \neq 0$, we have $\text{hw}(\sigma_{\geq -m}(P^\bullet)) = m + 1$, which contradicts to $\text{gl.hw } A = m$. Consequently, $\text{gl.hw } A = \text{s.gl.dim } A$. \hfill \qedsymbol

Recall that an algebra $A$ is said to be piecewise hereditary if there is a triangle equivalence $D^b(A) \simeq D^b(H)$ for some hereditary abelian $k$-category $H$ (Ref. \[\ref{ref18}\]). In this case, $H$ must have a tilting object (Ref. \[\ref{ref17}\]). Thus there are exactly two classes of piecewise hereditary algebras whose derived categories are triangle equivalent to either $D^b(kQ)$ for some finite connected quiver $Q$ without oriented cycles, or $D^b(\text{coh } X)$ for some weighted projective line $X$ (Ref. \[\ref{ref16}\]).

As a corollary of Proposition \[\ref{prop3}\] piecewise hereditary algebras can be characterized as the algebras of finite global cohomological width.

**Corollary 3.** An algebra $A$ is piecewise hereditary if and only if $\text{gl.hw } A < \infty$.

**Proof.** It follows from \[\ref{cor2}\] Theorem 3.2 and Proposition \[\ref{prop3}\]. \hfill \qedsymbol

### 2.3 The first Brauer-Thrall type theorem

Recall that an algebra $A$ is said to be derived finite if up to shift and isomorphism there are only finitely many indecomposable objects in $D^b(A)$ (Ref. \[\ref{ref7}\]). We say an algebra $A$ is derived bounded if $\text{gl.hr } A < \infty$, i.e., the cohomological ranges of all indecomposable complexes have a common upper bound.

Now we can prove Theorem I.
Theorem 1. Let $A$ be an algebra. Then the following assertions are equivalent:

1. $A$ is derived bounded;
2. $A$ is derived finite;
3. $A$ is piecewise hereditary of Dynkin type.

Proof. (1) $\Rightarrow$ (3): By assumption, $\text{gl.hr} A < \infty$. Thus $\text{gl.hw} A < \infty$. It follows from Corollary 3 that $A$ is piecewise hereditary. By [16, Theorem 3.1], $D^b(A) \simeq D^b(kQ)$ for some finite connected quiver $Q$ without oriented cycles, or $D^b(A) \simeq D^b(\text{coh}\mathbb{X})$ for some weighted projective line $\mathbb{X}$. In the first case, by Corollary 1 we have $\text{gl.hr} kQ < \infty$. Hence $Q$ is a Dynkin quiver. In the second case, by [12, Theorem 3.2], $D^b(A)$ is triangle equivalent to $D^b(C)$ for a canonical algebra $C$. Since $C$ is representation-infinite, $\text{gl.hr} C = \infty$. By Corollary 1, we have $\text{gl.hr} A = \infty$, which is a contradiction.

(3) $\Rightarrow$ (2): This is well-known [15].

(2) $\Rightarrow$ (1): Trivial. \hfill $\Box$

3 The second Brauer-Thrall type theorem

3.1 Strongly derived unbounded algebras

Recall that the cohomology dimension vector of a complex $X^\bullet \in D^b(A)$ is $d(X^\bullet) := (\dim H^n(X^\bullet))_{n \in \mathbb{Z}} \in \mathbb{N}^{(\mathbb{Z})}$. An algebra $A$ is said to be derived discrete if for any $d \in \mathbb{N}^{(\mathbb{Z})}$, up to isomorphism, there are only finitely many indecomposable complexes in $D^b(A)$ of cohomology dimension vector $d$ (Ref. [29]). It is easy to see that an algebra $A$ is derived discrete if and only if for any $r \in \mathbb{N}$, up to shift and isomorphism, there exist only finitely many indecomposable complexes in $D^b(A)$ of cohomological range $r$.

We say an algebra $A$ is strongly derived unbounded if there is an infinite increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for each $r_i$, up to shift and isomorphism, there are infinitely many indecomposable complexes in $D^b(A)$ of cohomological range $r_i$. Note that all representation-infinite algebras are strongly unbounded by the Nazarova-Roiter’s theorem on Brauer-Thrall conjecture II, thus strongly derived unbounded. Moreover, it is impossible that an algebra is both derived discrete and strongly derived unbounded.

Now we show that derived equivalence preserves strongly derived unboundedness.

Proposition 4. Let two algebras $A$ and $B$ be derived equivalent. Then $A$ is strongly derived unbounded if and only if so is $B$. 

10
Proof. Let $A^\bullet T_B^\bullet$ be a two-sided tilting complex such that $F = - \otimes_A T_B^\bullet : D^b(A) \to D^b(B)$ is a derived equivalence. Assume that $A$ is strongly derived unbounded. Then there exist an infinite increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and infinitely many indecomposable complexes $\{X_{ij}^\bullet \in D^b(A) \mid i, j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\text{hr}(X_{ij}^\bullet) = r_i$ for all $j \in \mathbb{N}$. It follows from Proposition 3.2(3) that there exist two positive integers $N$ and $N'$, such that for any $X_{ij}^\bullet$,

$$\frac{1}{N'} \cdot \text{hr}(X_{ij}^\bullet) \leq \text{hr}(F(X_{ij}^\bullet)) \leq N \cdot \text{hr}(X_{ij}^\bullet).$$

In order to show that $B$ is strongly derived unbounded, we shall find inductively an infinite increasing sequence $\{r'_i\}$ and infinitely many indecomposable complexes $\{Y_{ij}^\bullet \in D^b(B) \mid i, j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\text{hr}(Y_{ij}^\bullet) = r'_i$ for all $j \in \mathbb{N}$. For $i = 1$, we have $0 < \text{hr}(F(X_{1j}^\bullet)) \leq N \cdot \text{hr}(X_{1j}^\bullet) = N \cdot r_1$. Since $F(X_{1j}^\bullet), j \in \mathbb{N}$, are also pairwise different indecomposable complexes up to shift and isomorphism, we can choose $0 < r'_1 \leq N r_1$ and infinitely many indecomposable complexes $\{Y_{ij}^\bullet \mid j \in \mathbb{N}\} \subseteq \{F(X_{1j}^\bullet) \mid j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\text{hr}(Y_{ij}^\bullet) = r'_1$ for all $j \in \mathbb{N}$. Assume that we have found $r'_i$. We choose some $r_i$ with $r_i > N' \cdot r'_i$. Since

$$r'_i < \frac{1}{N'} \cdot N' = \frac{1}{N'} \cdot \text{hr}(X_{ij}^\bullet) \leq \text{hr}(F(X_{ij}^\bullet)) \leq N \cdot \text{hr}(X_{ij}^\bullet) = N \cdot r_i,$$

we can choose $r'_i < r'_{i+1} \leq N \cdot r_i$ and infinitely many indecomposable complexes $\{Y_{i+1,j}^\bullet \mid j \in \mathbb{N}\} \subseteq \{F(X_{i+1,j}^\bullet) \mid j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\text{hr}(Y_{i+1,j}^\bullet) = r'_{i+1}$ for all $j \in \mathbb{N}$. \]

3.2 Cleaving functors

In the context of cleaving functor, bounded quiver algebras are viewed as bounded spectroids \[\] Recall that a locally bounded spectroid is a $k$-linear category $A$ satisfying:

1. different objects in $A$ are not isomorphic;
2. the endomorphism algebra $A(a, a)$ is local for all $a \in A$;
3. $\dim_k \sum_{x \in A} A(a, x) < \infty$ and $\dim_k \sum_{x \in A} A(x, a) < \infty$ for all $a \in A$.

A bounded spectroid is a locally bounded spectroid having only finitely many objects. Note that a bounded quiver algebra $A = kQ/I$ with $Q$ a finite quiver and $I$ an admissible ideal can be viewed as a bounded spectroid by taking the vertices in $Q_0$ as objects and the $k$-linear combinations of paths in $kQ/I$.
as morphisms. Conversely, a bounded spectroid $A$ admits a presentation $A \cong kQ/I$ for a finite quiver $Q$ and an admissible ideal $I$. A right $A$-module $M$ is just a covariant $k$-linear functor from $A$ to the category of $k$-vector spaces. The dimension of $M$ is $\dim M := \sum_{a \in A} \dim_k M(a)$. Denote by $\text{Mod}A$ the category of all right $A$-modules, and by $\text{mod}A$ the full subcategory of $\text{Mod}A$ consisting of all finite-dimensional $A$-modules. The indecomposable projective $A$-modules are $P_a = A(a, -)$ and indecomposable injective $A$-modules are $I_a = DA(-, a)$ for all $a \in A$, where $D = \text{Hom}_k(-, k)$. Moreover, all the concepts and notations defined for a bounded quiver algebra make sense for a bounded spectroid.

To a $k$-linear functor $F : B \to A$ between bounded spectroids, we associates a restriction functor $F_* : \text{mod}A \to \text{mod}B$, which is given by $F_*(M) = M \circ F$ and exact. The restriction functor $F_*$ admits a left adjoint functor $F^*$, called the extension functor, which sends a projective $B$-module $B(b, -)$ to a projective $A$-module $A(Fb, -)$. Moreover, $F_*$ extends naturally to a derived functor $F_* : D^b(A) \to D^b(B)$, which has a left adjoint $LF^* : D^b(B) \to D^b(A)$. Note that $LF^*$ is the left derived functor associated with $F^*$ and maps $K^b(\text{proj}B)$ into $K^b(\text{proj}A)$. We refer to [30] for the definition of derived functor.

A $k$-linear functor $F : B \to A$ between bounded spectroids is called a cleaving functor [6, 29] if it satisfies the following equivalent conditions:

1. The linear map $B(b, b') \to A(Fb, Fb')$ associated with $F$ admits a natural retraction for all $b, b' \in B$;

2. The adjunction morphism $\phi_M : M \to (F_* \circ F^*)(M)$ admits a natural retraction for all $M \in \text{mod}B$;

3. The adjunction morphism $\Phi_{X^\bullet} : X^\bullet \to (F_* \circ LF^*)(X^\bullet)$ admits a natural retraction for all $X^\bullet \in D^b(B)$.

**Proposition 5.** Let $F : B \to A$ be a cleaving functor between bounded spectroids and $\text{gl.dim}B < \infty$. Then the following assertions hold:

1. If $B$ is strongly derived unbounded then so is $A$.

2. If $\text{gl.hl}A < \infty$ (resp. $\text{gl.hw}A < \infty$, $\text{gl.hr}A < \infty$) then $\text{gl.hl}B < \infty$ (resp. $\text{gl.hw}B < \infty$, $\text{gl.hr}B < \infty$).

**Proof.** (1) Assume that there exists an increasing infinite sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and indecomposable complexes $\{X^\bullet_{ij} \in D^b(B) \mid j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\text{hr}(X^\bullet_{ij}) = r_i$ for all $j \in \mathbb{N}$. Since $F$ is a cleaving functor, $X^\bullet_{ij}$ is a direct summand of $(F_* \circ LF^*)(X^\bullet_{ij})$. Thus for any $X^\bullet_{ij}$, we can choose an indecomposable direct
summand $Y_{ij}^\bullet$ of $LF^*(X_{ij}^\bullet)$, such that $X_{ij}$ is a direct summand of $F_s(Y_{ij}^\bullet)$. Clearly, for any $i \in \mathbb{N}$, the set $\{ Y_{ij}^\bullet \mid j \in \mathbb{N}\}$ contains infinitely many elements which are pairwise different up to shift and isomorphism. To prove that $A$ is strongly derived unbounded, by the proof of Proposition 4, it is enough to show there exist $N \in \mathbb{N}$ and a function $f : \mathbb{N} \to \mathbb{N}$ such that for any $X_{ij}$ we have $\frac{1}{N} \cdot \text{hr}(X_{ij}) \leq \text{hr}(Y_{ij}^\bullet) \leq f(\text{hr}(X_{ij}))$.

For any $a \in A$, we have

$$H^m(\text{LF}^*(X_{ij}^\bullet))(a) \cong \text{Hom}_{D^b(A)}(\text{LF}^*(X_{ij}^\bullet), I_a[m]) \cong \text{Hom}_{D^b(B)}(X_{ij}^\bullet, F_s(I_a)[m]) \cong H^m(\text{RHom}_B(X_{ij}^\bullet, F_s(I_a))).$$

Since $\text{gl.dim}B < \infty$, $F_s(I_a)$ admits a minimal injective resolution

$$0 \to F_s(I_a) \to E_1^0 \to E_1^1 \to \cdots \to E_r^r \to 0.$$

There is a bounded converging spectral sequence

$$\text{Ext}_B^p(H^q(X_{ij}^\bullet), F_s(I_a)) \Rightarrow H^{p+q}(\text{RHom}_B(X_{ij}^\bullet, F_s(I_a))),$$

thus $\text{hw}(Y_{ij}^\bullet) \leq \text{hw}(\text{LF}^*(X_{ij}^\bullet)) \leq \text{hw}(X_{ij}^\bullet) + \text{gl.dim}B$, and

$$\dim H^m(Y_{ij}^\bullet) = \sum_{a \in A} \dim H^m(Y_{ij}^\bullet)(a) \leq \sum_{a \in A} \dim H^m(\text{LF}^*(X_{ij}^\bullet))(a) = \sum_{a \in A} \dim H^m(\text{RHom}_B(X_{ij}^\bullet, F_s(I_a))) \leq \sum_{a \in A} \sum_{p+q=m} \dim \text{Ext}_B^p(H^q(X_{ij}^\bullet), F_s(I_a)) \leq \sum_{a \in A} (\dim H^m(X_{ij}^\bullet) \cdot \dim F_s(I_a) + \sum_{p=1}^r \dim H^{m-p}(X_{ij}^\bullet) \cdot \dim E_a^p) \leq \sum_{a \in A} \text{hw}(X_{ij}^\bullet) \cdot \text{hl}(X_{ij}^\bullet) \cdot \max_{1 \leq p \leq r_a} \{ \dim F_s(I_a), \dim E_a^p \} \leq n_0(A) \cdot \text{hr}(X_{ij}^\bullet) \cdot \max_{a \in A, 1 \leq p \leq r_a} \{ \dim F_s(I_a), \dim E_a^p \},$$

where $n_0(A)$ denotes the number of objects in $A$.

Set $N_0 = \max_{a \in A, 1 \leq p \leq r_a} \{ \dim F_s(I_a), \dim E_a^p \}$. Then

$$\text{hr}(Y_{ij}^\bullet) = \text{hw}(Y_{ij}^\bullet) \cdot \text{hl}(Y_{ij}^\bullet) \leq (\text{hw}(X_{ij}^\bullet) + \text{gl.dim}B) \cdot n_0(A) \cdot N_0 \cdot \text{hr}(X_{ij}^\bullet) \leq n_0(A) \cdot N_0 \cdot (\text{gl.dim} B + 1) \cdot \text{hr}(X_{ij}^\bullet)^2.$$
Assume the indecomposable projective $B$-module $Q_b = B(b, -)$ and indecomposable projective $A$-module $P_a = A(a, -)$ for all $b \in B$ and $a \in A$. Then

$$
\dim H^m(X^*_{ij}) \leq \dim H^m(F_*(Y^*_{ij}))
= \sum_{b \in B} \dim \text{Hom}_{D^b(B)}(Q_b, F_*(Y^*_{ij})[m])
= \sum_{b \in B} \dim \text{Hom}_{D^b(A)}(L F^*(Q_b), Y^*_{ij}[m])
= \sum_{b \in B} \dim \text{Hom}_{D^b(A)}(F^*(Q_b), Y^*_{ij}[m])
= \sum_{b \in B} \dim \text{Hom}_{D^b(A)}(P_{F(b)}, Y^*_{ij}[m])
\leq n_0(B) \cdot \sum_{a \in A} \dim \text{Hom}_{D^b(A)}(P_a, Y^*_{ij}[m])
\leq n_0(B) \cdot \dim H^m(Y^*_{ij}),
$$

where $n_0(B)$ denotes the number of objects in $B$. Thus $\text{hl}(X^*_{ij}) \leq n_0(B) \cdot \text{hl}(Y^*_{ij})$, $\text{hw}(X^*_{ij}) \leq \text{hw}(Y^*_{ij})$, and $\text{hr}(Y^*_{ij}) \geq \frac{1}{n_0(B)} \cdot \text{hr}(X^*_{ij})$.

(2) It can be read off from the proof of (1). \qed

### 3.3 Simply connected algebras

To a tilting $A$-module $T_A$, one can associate a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\text{mod}A$, and a torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\text{modEnd}_A(T)$. The Brenner-Butler theorem in classical tilting theory establishes the equivalence between $\mathcal{F}(T)$ and $\mathcal{X}(T)$ under the restriction of functor $F = \text{Ext}^1_A(T_A, -)$, and the equivalence between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$ under the restriction of functor $G = \text{Hom}_A(T_A, -)$ (Ref. [19, Theorem (2.1)]). We say a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod}A$ splits if any indecomposable $M$ in $\text{mod}A$ is either in $\mathcal{T}$ or in $\mathcal{F}$. A tilting $A$-module $T$ is said to be separating if the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ splits. A tilting $A$-module $T$ is said to be splitting if the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ splits.

Recall that an algebra $A$ is said to be triangular if its quiver $Q_A$ has no oriented cycles. A triangular algebra $A$ is said to be simply connected if for any presentation $A \cong kQ/I$, the fundamental group $\Pi_1(Q, I)$ of $(Q, I)$ is trivial [22].

Now we prove Theorem II for simply connected algebras.

**Lemma 2.** A simply connected algebra $A$ is either derived discrete or strongly derived unbounded. Moreover, a simply connected algebra $A$ is derived discrete if and only if it is piecewise hereditary of Dynkin type, if and only if $\text{gl.hl}A < \infty$. 

14
Proof. According to Corollary 1 and Proposition 4, it is enough to show that a simply connected algebra $A$ is tilting equivalent to either a hereditary algebra or a representation-infinite algebra. If $A$ is itself hereditary or representation-infinite then we have nothing to do. If $A$ is representation-finite but not hereditary then, by [1, Theorem], there exists a separating but not splitting basic tilting $A$-module $T$. Put $A_1 = \text{End}_A(T)$. Then there are more indecomposable modules in $\text{mod} A_1$ than in $\text{mod} A$, in particular $A$ and $A_1$ are not isomorphic as algebras. Moreover, $A_1$ is still simply connected by [3, Corollary] and thus triangular. Since $A_1$ is a tilted algebra of $A$, they have the same number of simple modules [19, Corollary (3.1)]. If $A_1$ is hereditary or representation-infinite then we have nothing to do. If $A_1$ is representation-finite but not hereditary then there exists a separating but not splitting basic tilting $A_1$-module, and we can proceed as above repetitively. We claim this process must stop in finite steps, and thus $A$ is tilting equivalent to either a hereditary algebra or a representation-infinite algebra. Indeed, for any $n \in \mathbb{N}$, there are only finitely many basic representation-finite triangular algebras having $n$ simple modules up to isomorphism. We can prove this by induction on $n$. If $n = 1$, then there exists only one basic triangular algebra up to isomorphism. Assume that it is true for $n - 1$ and $B$ is a basic representation-finite triangular algebra having $n$ simple modules. Then $B$ is a one-point extension of a basic representation-finite triangular algebra with $n - 1$ simples $C$ by some $C$-module $M = \bigoplus_{i=1}^{r} M_i$ with $M_i$ being indecomposable. Since $C$ is representation-finite, we have $r \leq 3$. Thus the number of the isomorphism classes of basic representation-finite triangular algebras having $n$ simple modules is finite. Hence the tilting process above must stop in finite steps, since representation-finite simply connectedness and the number of simples are invariant under this process.

3.4 The second Brauer-Thrall type theorem

Bekkert and Merklen have classified the indecomposable objects in the derived category of a gentle algebra [7, Theorem 3]. It follows Theorem II for gentle algebras.

Lemma 3. A gentle algebra $A$ is either derived discrete or strongly derived unbounded. Moreover, $A$ is derived discrete if and only if $\text{gl.hl} A < \infty$.

Proof. It follows from [1, Theorem 4] that a gentle algebra $A$ is derived discrete if and only if $A$ does not contain generalized bands. If $A$ contains a generalized band $w$, then one can construct indecomposable complexes
of two types of generalized strings or their inverses:

\( \{ P_{w,f} \mid f = (x - \lambda)^d \in k[x], \lambda \in k^*, d > 0 \} \) which are pairwise different up to shift and isomorphism such that \( P_{w,f} \) and \( P_{w,f'} \) are of the same cohomological range or cohomological length if and only if \( \deg(f) = \deg(f') \) (Ref. [7, Definition 3]). Thus \( A \) is strongly derived unbounded and \( \text{gl.hl}A = \infty \).

If \( A = kQ/I \) does not contain generalized bands we shall prove \( \text{gl.hl}A < \infty \). By Bobiński, Geiss and Skowroński’s classification of derived discrete algebras [8, Theorem A], we know that \( A \) is derived equivalent to a gentle algebra \( \Lambda(r, n, m) \) with \( n \geq r \geq 1 \) and \( m \geq 0 \), which is given by the quiver

with the relations \( \alpha_{n-1}\alpha_0, \alpha_{n-2}\alpha_{n-1}, \ldots, \alpha_{n-r}\alpha_{n-r+1} \). According to Corollary [7] it suffices to show that \( \text{gl.hl} \Lambda(r, n, m) \leq \dim \Lambda(r, n, m) < \infty \). Note that any generalized string \( w \) of \( \Lambda(r, n, m) \) must be a sub-generalized string of following two types of generalized strings or their inverses:

1. \( (\alpha_i \cdots \alpha_{n-r})[\alpha_{n-r+1} \cdots \alpha_{n-1}(\alpha_0 \cdots \alpha_{n-r})]^p(\alpha_{n-r+1} \cdots \alpha_{n-1}(\alpha_j \cdots \alpha_{n-r}))^{-1} \), with \( -m \leq i \leq n - r, -m \leq j \leq -1 \) and \( p \geq 0 \);
2. \( (\alpha_i \cdots \alpha_{n-r})[\alpha_{n-r+1} \cdots \alpha_{n-1}(\alpha_0 \cdots \alpha_{n-r})]^p(\alpha_{n-r+1} \cdots \alpha_{n-1}(\alpha_j \cdots \alpha_{n-r}))^{-1} \), with \( -m \leq i \leq n - r, -m \leq j \leq n - r, j \neq 0 \) and \( p \geq 1 \).

By Bekkert and Merklen’s construction of the indecomposable objects in the bounded derived category of a gentle algebra [7, Definition 2 and Theorem 3], every indecomposable projective direct summand of each component of the indecomposable complex \( P^*_w \in K^b(\text{proj}\Lambda(r, n, m)) \) is multiplicity-free, and hence \( \text{gl.hl} \Lambda(r, n, m) \leq \dim \Lambda(r, n, m) < \infty \).

Let \( A^m_n \) be the bounded spectroid defined by the quiver

\[
\begin{array}{cccccccc}
 n & \alpha_{n-1} & n-1 & \alpha_{n-2} & \cdots & \alpha_2 & 2 & \alpha_1 & 1 \\
 & & & & & & & \\
 \end{array}
\]

and the admissible ideal generated by all paths of length \( m \).

**Lemma 4.** The bounded spectroid \( A^m_{3m}, m \geq 3 \), is strongly derived unbounded and \( \text{gl.hl}A^m_{3m} = \infty \).

**Proof.** Let \( B = A^m_{3m}, w_1 = \alpha_{3m-1}, w_2 = \alpha_{3m-2} \cdots \alpha_{2m}, w_3 = \alpha_{2m-1} \cdots \alpha_{m+1}, w_4 = \alpha_m \cdots \alpha_2, w_5 = \alpha_1, w'_1 = \alpha_{3m-1} \cdots \alpha_{2m+1}, w'_2 = \alpha_{2m}, w'_3 = w_3, \)
\[ w'_1 = \alpha_m \] and \[ w'_5 = \alpha_{m-1} \cdots \alpha_1. \] Then we construct a family of complexes \( \{P_{\lambda,d} | \lambda \in k, d \geq 1 \} \) by

\[
P_{\lambda,d}^\bullet := 0 \rightarrow P^d_1 \xrightarrow{\delta^0} P^d_m \oplus P^d_2 \delta^1 \rightarrow P^d_{m+1} \oplus P^d_{m+1} \delta^2 \rightarrow P^d_{2m+1} \oplus P^d_{3m-1} \delta^3 \rightarrow P^d_{3m} \rightarrow 0
\]

with the differential maps

\[
\delta^0 := \begin{pmatrix} P(w'_1)I_d & 0 \\ P(w'_5)J_{\lambda,d} & P(w'_1)I_d \end{pmatrix}, \quad \delta^1 := \begin{pmatrix} P(w'_5)I_d & 0 \\ 0 & P(w'_1)I_d \end{pmatrix}, \quad \text{for } i = 1, 2, 3,
\]

and \( \delta^i := (P(w'_5)I_d, P(w'_1)I_d) \). Here \( J_{\lambda,d} \) denotes the \( d \times d \) Jordan block with eigenvalue \( \lambda \in k \), and \( P(u) \) is the canonical map from \( P_{t(u)} \) to \( P_{s(u)} \) given by the left multiplication induced by the path \( u \). In fact, the complex \( P_{\lambda,d}^\bullet \) can be illustrated as follows

\[
\begin{array}{cccccccc}
P_1 & \xrightarrow{P(w'_1)I_d} & P_m & \xrightarrow{P(w'_1)I_d} & P_{m+1} & \xrightarrow{P(w'_1)I_d} & P_{2m} & \xrightarrow{P(w'_1)I_d} & P_{3m} \\
\downarrow{P(w_5)J_{\lambda,d}} & & \downarrow{P(w_5)J_{\lambda,d}} & & \downarrow{P(w_5)J_{\lambda,d}} & & \downarrow{P(w_5)J_{\lambda,d}} & & \downarrow{P(w_5)J_{\lambda,d}} \\
P_2 & \xrightarrow{P(w'_1)I_d} & P_{m+1} & \xrightarrow{P(w'_1)I_d} & P_{2m} & \xrightarrow{P(w'_1)I_d} & P_{3m} & & \\
& & \downarrow{P(w_5)I_d} & & \downarrow{P(w_5)I_d} & & \downarrow{P(w_5)I_d} & & \\
& & \downarrow{P(w_5)I_d} & & \downarrow{P(w_5)I_d} & & \downarrow{P(w_5)I_d} & & \\
& & & & \downarrow{P(w_5)I_d} & & \downarrow{P(w_5)I_d} & & \\
& & & & & & \downarrow{P(w_5)I_d} & & \\
& & & & & & & & \downarrow{P(w_5)I_d} \\
& & & & & & & & \\
\end{array}
\]

where \( P_1^d \) lies in the 0-th component of \( P_{\lambda,d}^\bullet \).

It is elementary to show that \( \text{End}_{\text{C}^0(B)}(P_{\lambda,d}^\bullet) \) is local, i.e., the complex \( P_{\lambda,d}^\bullet \) is indecomposable, for all \( \lambda \in k \) and \( d \geq 1 \). Moreover, the complexes \( \{P_{\lambda,d}^\bullet \ | \lambda \in k, d \geq 1 \} \) are pairwise different up to shift and isomorphism.

Now it suffices to show that \( \text{hr}(P_{\lambda,d}^\bullet) = \text{hr}(P_{\lambda,d}^\bullet) \) and \( \text{hl}(P_{\lambda,d}^\bullet) = \text{hl}(P_{\lambda,d}^\bullet) \) if and only if \( d = d' \), which implies \( B \) is strongly derived unbounded and \( \text{gl.hl.B} = \infty \). Indeed, it is clear that \( H^1(P_{\lambda,d}^\bullet) \) is independent of \( \lambda \) except \( i = 0, 1 \). Moreover, \( H^0(P_{\lambda,d}^\bullet) = 0 \) and \( \dim H^1(P_{\lambda,d}^\bullet) \) is independent of \( \lambda \) since \( \delta^0 \) is injective. Hence, \( P_{\lambda,d}^\bullet \)'s are of the same cohomological range and cohomological length for a fixed \( d \). Conversely, we have \( h\text{w}(P_{\lambda,d}^\bullet) = 5 \) due to \( H^1(P_{\lambda,d}^\bullet) \neq 0 \neq H^0(P_{\lambda,d}^\bullet) \) and \( \text{hl}(P_{\lambda,d}^\bullet) = d \cdot \text{hl}(P_{\lambda,d}^\bullet) \). Thus, \( P_{\lambda,d}^\bullet \)'s are of distinct cohomological ranges and cohomological lengths for different \( d \).

\[ \text{Lemma 5.} \] If a bounded spectroid \( A \) is not strongly derived unbounded or of finite global cohomological length then the endomorphism algebra \( A(a,a) \) is isomorphic to either \( k \) or \( k[x]/(x^2) \) for all \( a \in A \).

\[ \text{Proof.} \] If \( A \) is not strongly derived unbounded or of finite global cohomological length then \( A \) is representation-finite. Thus for any \( a \in A \), \( A(a,a) \) is uniserial, and then \( A(a,a) \cong k \) or \( A(a,a) \cong k[x]/(x^m) \) with \( m \geq 2 \). Note
that the functor $F : A^n_m \to A$ given by $F(i) = a$ and $F(\alpha_j) = x$ is a cleaving functor. If $m \geq 3$ then, by Lemma 4, $A^n_{3m}$ is strongly derived unbounded and $\text{gl.hl} A^n_{3m} = \infty$. It follows from Proposition 5 that $A$ is strongly derived unbounded and $\text{gl.hl} A = \infty$, which is a contradiction.

Now we can prove Theorem II for all algebras.

**Theorem 2.** A bounded spectroid is either derived discrete or strongly derived unbounded.

**Proof.** Assume that a bounded spectroid $A$ is not strongly derived unbounded. It follows from Lemma 5 that the endomorphism algebra $A(a, a)$ is isomorphic to either $k$ or $k[x]/(x^2)$ for all $a \in A$. Thus $A$ does not contain Riedtmann contours, and hence it is standard [6, Section 9].

If $A$ is simply connected then $A$ is derived discrete by Lemma 2. If $A$ is not simply connected then $A$ admits a Galois covering $\pi : \tilde{A} \to A$ with non-trivial free Galois group $G$ such that $\tilde{A}$ is simply connected, hence the filtered union of its simply connected convex finite full subspectroids [9, 10]. Any convex finite full subspectroid $B$ of $\tilde{A}$ is simply connected, thus $\text{gl.dim} B < \infty$. Note that the composition of the embedding functor $B \hookrightarrow \tilde{A}$ and the covering functor $\pi$ is a cleaving functor, by Proposition 5, $B$ is not strongly derived unbounded. It follows from Lemma 2 that $B$ is piecewise hereditary of Dynkin type. By the same argument as that in the proof of [29, Lemma 4.4], we obtain $B$ is piecewise hereditary of type $A$. Thus $\tilde{A}$ admits a presentation given by a gentle quiver $(Q, I)$ (Ref. [2, Theorem]), and so does $A$. Therefore, $A$ is derived discrete by Lemma 3.

Next we show that derived discrete algebras can be characterized as the algebras of finite global cohomological length.

**Proposition 6.** A bounded spectroid $A$ is derived discrete if and only if $\text{gl.hl} A < \infty$.

**Proof.** If $A$ is derived discrete then by Vossieck’s classification of derived discrete algebras [29, Theorem], $A$ is either piecewise hereditary of Dynkin type or derived equivalent to some gentle algebras without generalized bands. In the case $A$ is piecewise hereditary of Dynkin type, by Corollary 1, we have $\text{gl.hl} A < \infty$. In the other case, by Lemma 3 we have $\text{gl.hl} A < \infty$.

Conversely, it is enough to repeat the proof of Theorem 2 and replace the phrase “not strongly derived unbounded” with “of finite global cohomological length”.  

18
Remark 2. By Corollary 3 and Proposition 6, we know piecewise hereditary algebras and derived discrete algebras can be characterized as the algebras of finite global cohomological width and finite global cohomological length respectively, which provides the other proof of the first Brauer-Thrall type theorem. Indeed, an algebra $A$ satisfies $\text{gl.hr}A < \infty$ if and only if both $\text{gl.hw}A < \infty$ and $\text{gl.hl}A < \infty$, if and only if $A$ is both piecewise hereditary and derived discrete, i.e., piecewise hereditary of Dynkin type.

ACKNOWLEDGMENT. The authors thank Xiao He and Yongyun Qin for helpful discussions on this topic. The authors are sponsored by Project 11171325 NSFC.

References

[1] I. Assem, Separating splitting tilting modules and hereditary algebras, Canad. Math. Bull. 30 (1987), 177–181.

[2] I. Assem and D. Happel, Generalized tilted algebras of type $A_n$, Comm. Algebra 9 (1981), 2101–2125.

[3] I. Assem, A. Skowroński, Tilting simply connected algebras, Comm. Algebra 22(1994) 4611-4619

[4] M. Auslander, Representation theory of artin algebras II, Comm. Algebra 2 (1974), 269–310.

[5] R. Bautista, On algebras of strongly unbounded representation type, Comment. Math. Helv. 60 (1985), 392–399.

[6] R. Bautista, P. Gabriel, A. V. Roiter and L. Salmerón, Representation-finite algebras and multiplicative bases, Invent. Math. 81 (1985), 217–285.

[7] V. Bekkert and H. Merklen, Indecomposables in derived categories of gentle algebras, Alg. Rep. Theory 6 (2003), 285–302.

[8] G. Bobiński, Ch. Geiss and A. Skowroński, Classification of discrete derived categories, Cent. Eur. J. Math. 2 (2004), 19–49.

[9] O. Bretscher and P. Gabriel, The standard form of a representation-finite algebra, Bull. Soc. Math. France 111 (1983), 21–40.

[10] P. Gabriel, The universal cover of a representation-finite algebra, Lecture Notes in Math. 903 (1981), 68–105.

[11] P. Gabriel and A.V. Roiter, Representations of finite-dimensional algebras, Springer-Verlag, Berlin, Heidelberg, New York, 1997.

[12] W. Geigle and H. Lenzing, A class of weighted projective curves arising in representation theory of finite-dimensional algebras, Lecture Notes in Math. 1273 (1987), 265–297.
[13] Ch. Geiss and H. Krause, On the notion of derived tameness, J. Algebra Appl. 1 (2002), 133–157.
[14] D. Happel, On the derived category of a finite-dimensional algebra, Comment. Math. Helv. 62 (1987), 339–389.
[15] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Math. Soc. Lecture Notes Ser. 119, 1988.
[16] D. Happel, A characterization of hereditary categories with tilting object, Invent. Math. 144 (2001), 381–398.
[17] D. Happel and I. Reiten, Directing objects in hereditary categories, Contemp. Math. 229 (1998), 169–179.
[18] D. Happel, I. Reiten and S. Smalø, Piecewise hereditary algebras, Arch. Math. 66 (1996), 182–186.
[19] D. Happel and C.M. Ringel, Tilted algebras, Trans. Amer. Math. Soc. 274 (1982), 399–443.
[20] D. Happel and D. Zacharia, A homological characterization of piecewise hereditary algebras, Math. Z. 260 (2008), 177–185.
[21] J.P. Jans, On the indecomposable representations of algebras, Ann. Math. 66 (1957), 418–429.
[22] R. Martínez-Villa and J.A. de la Peña, The universal cover of a quiver with relations, J. Pure Appl. Algebra 30 (1983), 277–292.
[23] L.A. Nazarova and A.V. Roiter, Kategoriielle matrizen-probleme und die Brauer-Thrall-vermutung, Mitt. Math. Sem. Giessen Heft 115 (1975), 1–153.
[24] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. 43 (1991), 37–48.
[25] C.M. Ringel, Report on the Brauer-Thrall conjectures: Roiter’s theorem and the theorem of Nazarova and Roiter, Lecture Notes in Math. 831 (1980), 104–136.
[26] A.V. Roiter, The unboundedness of the dimensions of the indecomposable representations of algebras that have an infinite number of indecomposable representations, Izv. Akad. Nauk SSSR Ser. Math. 32 (1968), 1275–1282, English transl.: Math. USSR, Izv. 2 (1968), 1223–1230.
[27] A.V. Roiter, Matrix problems, Proc. ICM Helsinki, 1978, 319–322.
[28] A. Skowroński, On algebras with finite strong global dimension, Bull. Polish Acad. Sci. 35 (1987), 539–547.
[29] D. Vossieck, The algebras with discrete derived category, J. Algebra 243 (2001), 168–176.
[30] C.A. Weibel, An introduction to homological algebra, Cambridge Studies in Adv. Math., Vol. 38, Cambridge Univ. Press, Cambridge, 1994.