A surface-aware projection basis for quasigeostrophic flow

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Recent studies indicate that altimetric observations of the ocean’s mesoscale eddy field reflect the combined influence of surface buoyancy and interior potential vorticity anomalies. The former have a surface-trapped structure, while the latter have a more grave form. To assess the relative importance of each contribution to the signal, it is useful to project the observed field onto a set of modes that separates their influence in a natural way. However, the surface-trapped dynamics are not well-represented by standard baroclinic modes; moreover, they are dependent on horizontal scale.

Here we derive a modal decomposition that results from the simultaneous diagonalization of the energy and a generalization of potential enstrophy that includes contributions from the surface buoyancy fields. This approach yields a family of orthonormal bases that depend on two parameters: the standard baroclinic modes are recovered in a limiting case, while other choices provide modes that represent surface and interior dynamics in an efficient way. For constant stratification, these modes consist of symmetric and antisymmetric exponential modes that capture the surface dynamics, and a series of oscillating modes that represent the interior dynamics. Motivated by the ocean, where shears are concentrated near the upper surface, we also consider the special case of a quiescent lower surface. In this case, the interior modes are independent of wavenumber, and there is a single exponential surface mode that replaces the barotropic mode. We demonstrate the use and effectiveness of these modes by projecting the energy in a set of simulations of baroclinic turbulence.
1. Introduction

Because direct observations of the ocean’s interior are sparse, satellite altimetry plays a crucial role in determining its time-dependent, three-dimensional velocity structure. This indirect measurement process assumes that sea surface height variations are dominated by currents with low-mode vertical structure, a result of the stiffening action of rotation and ensuing barotropization. Observations provide some support for this assumption, at least on lateral scales of order the first internal deformation scale and above. For example, using currentmeter records in conjunction with satellite observations, Wunsch (1997) argues that the bulk of the ocean’s eddy kinetic energy resides in the barotropic and first baroclinic modes. In addition, a number of studies show a strong correlation between the lateral size of eddies and the first internal deformation scale (e.g. Stammer 1997; Chelton et al. 2011).

However, recent theoretical developments, supported by simulation and improved analysis of satellite altimetry, suggest that surface signals are not well-correlated with low-mode vertical structure, especially for submesoscale motions. In particular, Lapeyre and Klein (2006) argue that surface buoyancy and upper-ocean potential vorticity are anti-correlated for eddying flow, and that the three-dimensional velocity field may be obtained, assuming quasigeostrophy, from knowledge of the surface buoyancy field alone. The dynamics at the upper surface in this view are closely related to the surface quasigeostrophic (SQG) model (Blumen 1982; Held et al. 1995), and imply a vertical structure with a surface-trapped component that is not well represented by standard baroclinic modes. This view is supported by results from simulations (LaCasce and Mahadevan 2006; Klein et al. 2008), as well as recent analyses of satellite altimetry (e.g. Isern-Fontanet et al. 2006; Le Traon et al. 2008).

Finally, in an atmospheric context, Tulloch and Smith (2009) have shown that lateral surface buoyancy gradients may interact with interior mean potential vorticity gradients to excite baroclinically unstable modes that generate SQG-like dynamics near the upper surface. In simulations, the resulting kinetic energy spectrum near the surface exhibits a steep $-3$ slope just below the deformation scale, and a flatter $-5/3$ slope at smaller scales — translated to
the oceanic context, this implies an energetic submesoscale dominated by the surface mode.

One of the most widely used tools in oceanography is the projection of the vertical structure of observed or simulated currents on simple bases of functions. The above observations and modeling results lead one to seek projection bases that faithfully represent both the low-mode interior structure and the surface dynamics. The standard basis of baroclinic modes, consisting of the eigenfunctions \( \phi(z) \) of the operator \( \partial_z [f^2/N^2(z) \partial_z \phi] \), with homogenous boundary conditions \( \partial_z \phi|_{z=0} = \partial_z \phi|_{z=-H} = 0 \), fails in this respect. By construction, it is a complete basis in which to expand the streamfunction \( \psi \) of flows provided they satisfy the same homogeneous boundary conditions, which imply zero surface and bottom buoyancy. But for realistic flows with non-zero surface buoyancy \( b = f \partial_z \psi|_{z=0} \), expansion in baroclinic modes leads to a non-uniform convergence near \( z = 0 \), and a very large set of modes is required to capture the near-surface behaviour.

As noted by Lapeyre and Klein (2006), in quasigeostrophic theory, the dynamical contribution of the surface buoyancy can be separated from that of the interior potential vorticity: taking advantage of the linearity of the inversion of the quasigeostrophic potential vorticity (PV)

\[
q = \nabla^2 \psi + \partial_z \left( \frac{f^2}{N^2} \partial_z \psi \right)
\]

the streamfunction may be decomposed into interior and surface parts, \( \psi = \psi^{\text{int}} + \psi^{\text{surf}} \) (assuming zero buoyancy at the bottom), where \( \psi^{\text{int}} \) satisfies (1) with boundary condition \( \partial_z \psi^{\text{int}}|_{z=0} = 0 \) while \( \psi^{\text{surf}} \) satisfies the zero-PV condition \( \nabla^2 \psi^{\text{surf}} + \partial_z (f^2/N^2 \partial_z \psi^{\text{surf}}) = 0 \) with \( \partial_z \psi^{\text{surf}}|_{z=0} = b/f \). The vertical structure of the interior contribution can be expanded in the standard baroclinic modes. By contrast, the surface contribution — the only one retained in SQG theory — has a vertical structure determined by the zero PV condition which couples horizontal and vertical dependence, reducing to \( \exp(\kappa N z/f) \), where \( \kappa \) is the horizontal wavenumber, in the case of constant \( N \) and for \( z \gg H \).

It is intuitively clear that an effective projection basis should somehow combine modes similar to the baroclinic modes with modes that, like the exponential modes of SQG theory,
capture the dynamical contribution of the surface buoyancy. A systematic method to obtain such a basis has remained elusive, however. Tulloch and Smith (2009) proposed a heuristic model based on a barotropic and first baroclinic mode, appended by exponential modes for each surface. Similarly, Lapeyre (2009) attempted to represent the full dynamics of the upper ocean with a truncated set of standard baroclinic modes appended by an exponential surface mode. However, these hybrid modes do not diagonalize the energy, since the surface and interior modes are not orthogonal. Moreover, because the surface modes depend on wavenumber while the interior modes do not, the energetic overlap varies with horizontal scale, increasing with increasing scale. These difficulties stem from the fact that the addition of the exponential mode makes the basis functions linearly dependent in a certain sense, leading to an overcomplete frame rather than a basis. A consequence is that the modal decomposition is non-unique. Lapeyre (2009) defined a unique basis by requiring that it minimizes a certain functional, but the results remained inconclusive. An alternative basis, involving modes satisfying the Dirichlet condition $\psi |_{z=0} = 0$ together with the barotropic mode, has recently been proposed by Scott and Furnival (2012) but this too suffers from a lack of orthogonality.

In this paper, we take a different approach and propose a new modal basis (or rather a family of bases) that diagonalizes the energy and effectively captures surface-intensified motion driven by buoyancy. Our approach relies on the observation that there are infinitely many possible (complete) bases onto which the flow may be projected which diagonalize the energy. As we show, a useful basis is obtained by demanding that it simultaneously diagonalizes both the energy and another quadratic invariant that generalizes potential enstrophy to include the variances of the surface and bottom buoyancy fields. The relative weight of the potential enstrophy and buoyancy variances in this invariant provide two parameters that determine the basis uniquely.

The eigenvalue problem that arises is similar to the standard vertical mode problem, but retains a dependence on horizontal wavenumber, and the eigenvalue appears in both the
eigenvalue equation and its boundary conditions. In a limiting case, the standard baroclinic
modes are recovered — for constant \(N\) and \(-H \leq z \leq 0\), these are \(\psi_n \propto \cos(n\pi z/H)\), \(n = 0, 1, \ldots\). Another limiting case, motivated by the ocean where shears are concentrated near
the upper surface but are weak at depth, leads to the simple basis

\[
\psi_0 \propto \cosh \left[ N\kappa(z + H)/f \right] , \quad \psi_n \propto \sin \left[ (n - 1/2)\pi z/H \right] , \quad n = 1, 2, \ldots
\]

which includes the exponential mode of SQG theory.

The paper is organized as follows. In section 2 we construct a generalized eigenvalue
problem that defines the new basis. In section 3, we derive analytical solutions and general
results for two special cases: constant \(N\), for expository purposes, and an ocean-like case,
in which the lower boundary is assumed quiescent, leading to \([2]\). These modes are tested
in section 4 on fields generated from a set of high-resolution quasigeostrophic simulations of
baroclinic turbulence. Finally, we discuss and conclude in section 5.

2. Surface-aware basis

Throughout the paper, we assume a horizontally-periodic domain bounded vertically by
rigid surfaces at \(z = z^-\) and \(z = z^+\), with total depth \(H = z^+ - z^-\). The horizontal
periodicity allows us to Fourier transform the equations in the horizontal plane, resulting in
separable dynamics and ordinary differential equations for the vertical structure. (In more
general domains, the Fourier series can be replaced by an expansion in eigenfunctions of
the horizontal Laplacian, and the results obtained here should hold essentially unchanged.)
The complex amplitudes of the quasigeostrophic potential vorticity (PV) \(q = q_{kl}(z)\), surface
buoyancies (SBs) \(b_{kl}^\pm\) and streamfunction \(\psi = \psi_{kl}(z)\) are then related by

\[
\left( \frac{f^2}{N^2} \psi' \right)' - \kappa^2 \psi = q, \quad z^- < z < z^+ \quad \text{(3a)}
\]

\[
\frac{f^2}{N^2 H} \psi' = b^\pm, \quad z = z^\pm \quad \text{(3b)}
\]
where $\kappa = (k^2 + l^2)^{1/2}$ is the wavenumber magnitude, a prime indicates a $z$ derivative, $f$ is the Coriolis frequency and $N = N(z)$ is the buoyancy frequency. We include the non-standard factor $f^2/(N^2 H)$ in our definition of the SBs so that the SBs and PV have the same dimension (inverse time), and because it ultimately yields a more natural eigenvalue problem. We have omitted the wavenumber subscript on $q$, $b^\pm$ and $\psi$ and continue to do so onward, except where confusion may occur.

The quasigeostrophic equation set has four quadratic invariants: energy, potential enstrophy, and the buoyancy variance at each surface. At each wavenumber $\kappa$, these are

$$E_\kappa = \frac{1}{2H} \int_{z^-}^{z^+} \left( \frac{f^2}{N^2} |\psi'|^2 + \kappa^2 |\psi|^2 \right) dz$$

$$Z_\kappa = \frac{1}{2H} \int_{z^-}^{z^+} |q|^2 dz$$

$$B^\pm_\kappa = \frac{1}{2} |b^\pm|^2.$$  

Summing each quantity over $(k, l)$ gives the total invariant.

We seek to define a complete basis that diagonalizes the energy. This can be done in infinitely many ways. Our strategy is based on the following principles: (i) we regard the energy as a functional, not of the streamfunction, but of the PV and of the SBs; (ii) we exploit standard results on the simultaneous diagonalization of quadratic forms. Principle (i) is grounded in the quasigeostrophic model, which makes it explicit that PV and SBs, taken together, make up the set of dynamical variables. Thus, the contribution of the SBs to the dynamics is recognized; as a result, the bases we obtain naturally represent data with non-zero surface buoyancies. Regarding (ii), we recall a classical result from linear algebra: whereas there are infinitely many bases diagonalizing a quadratic form $x^T A x$, where $A$ is a symmetric positive definite matrix, only one of these bases also diagonalizes another quadratic form $x^T B x$ (e.g. Horn and Johnson 1990). This is simply found by solving the generalized eigenvalue problem $B x = \lambda A x$. An analogous result applies to linear operators (see, e.g. Goldstein 1980). Similarly, here we can define a unique basis by insisting that it diagonalizes another quadratic form in addition to the energy $E_\kappa$. A natural choice for this
is a ‘generalized potential enstrophy’ that combines the remaining invariants into a single quantity,

\[ P_\kappa \equiv Z_\kappa + \alpha_+ B^+ + \alpha_- B^- \]  (4)

where \( \alpha_\pm > 0 \) are (nondimensional) undetermined weights, the choice of which will be discussed later. This approach yields a unique basis for fixed \( \alpha_\pm \).

To proceed, we require four objects: a vector structure that combines the SBs and interior PV, an inner product that operates on this vector, and two operators (analogous to the matrices \( A \) and \( B \) above) that give the energy and generalized potential enstrophy in terms of the inner product. These are defined as follows:

**Vector.** We define the ‘generalized potential vorticity vector’

\[ Q \equiv \begin{pmatrix} b^+ \\ q(z) \\ b^- \end{pmatrix}. \]  (5)

**Inner product.** The specific choice of inner product is unimportant for the final results; we make what appears to be the simplest choice, namely

\[ \langle Q_1, Q_2 \rangle = \frac{1}{H} \int_{z^-}^{z^+} \bar{\bar{q_1}} q_2 \, dz + \bar{\bar{b_1}} b_2^+ + \bar{\bar{b_1}} b_2^-. \]  (6)

where the overbar denotes a complex conjugate.

**Operators.** With the definitions (5) and (6), it is a simple matter to find the linear operators \( \mathcal{E} \) and \( \mathcal{P} \) such that

\[ E_\kappa = \frac{1}{2} \langle Q, \mathcal{E} Q \rangle \quad \text{and} \quad P_\kappa = \frac{1}{2} \langle Q, \mathcal{P} Q \rangle. \]  (7)

\(^1\)Notice that our \( Q \) bears a resemblance to the generalized potential vorticity of [Bretherton 1966], which in our notation is written

\[ Q_B = \left( \frac{f^2}{N^2} \psi' \right)' - \kappa^2 \psi - \frac{f^2}{N^2} \psi' \delta(z - z^+) + \frac{f^2}{N^2} \psi' \delta(z - z^-). \]

Our notation makes it plain that the PV and SBs are independent, a point that the use of \( Q_B \) might obscure.
These are given by
\[
\mathcal{E}Q = \begin{pmatrix}
\psi(z^+)
\-\psi(z)
\-\psi(z^-)
\end{pmatrix}
\quad \text{and} \quad
\mathcal{P}Q = \begin{pmatrix}
\alpha_+ b^+
q(z)
\alpha_- b^-
\end{pmatrix},
\] (8)
where the streamfunction $\psi$ is the solution of (3), given $q$ and $b^\pm$. The first of these expressions is obtained after an integration by parts; the second is immediate. These two operators are positive definite and self-adjoint (see Appendix 5 for details).

The basis we seek is now given by the eigenfunctions $\xi_n$ of the generalized eigenvalue problem
\[
\mathcal{P}\xi_n = \mu_n^2 \mathcal{E}\xi_n,
\] (9)
where the eigenvalues $\mu_n^2$ are positive for all $n$. To obtain an explicit form for (9), we define the components of $\xi_n = [\xi_n^+, \xi_n(z), \xi_n^-]^T$ analogous to those of $Q$, and the scalar streamfunctions $\phi_n(z)$ such that $\mathcal{E}\xi_n = [\phi_n(z^+), -\phi_n(z), -\phi_n(z^-)]^T$. In terms of these, the eigenvalue problem reads
\[
\begin{pmatrix}
\alpha_+ \xi_n^+
\xi_n(z)
\alpha_- \xi_n^-
\end{pmatrix}
= \mu_n^2 \begin{pmatrix}
\phi_n(z^+)
-\phi_n(z)
-\phi_n(z^-)
\end{pmatrix},
\] (10)
In view of (3), this implies that the $\phi_n$ satisfy
\[
\left(\frac{f^2}{N^2} \phi_n'\right)' - \kappa^2 \phi_n = -\mu_n^2 \phi_n \quad \text{and} \quad \frac{f^2}{N^2 H} \phi_n' = \pm \frac{\mu_n^2}{\alpha_\pm} \phi_n \ \text{at} \ z = z^\pm.
\] (11)
This eigenvalue problem is a key result of the paper. Its eigenfunctions $\phi_n$, which are purely real, give the form of the streamfunction corresponding to the basis eigenvectors $\xi_n$. The three components of these eigenvectors may be derived from the $\phi_n$ using (10), although, as shown below, this is not necessary to project data onto the modes $\xi_n$.

By construction, the eigenfunctions are orthogonal for the products $\langle \cdot, \mathcal{E} \cdot \rangle$ and $\langle \cdot, \mathcal{P} \cdot \rangle$. The choice of normalization for the eigenvectors $\xi_n$ is inessential, but it is convenient to fix
the energy of each mode to be unity, that is, to take

$$\langle \xi_m, E \xi_n \rangle = \frac{1}{H} \int_{z^-}^{z^+} \left( \frac{f^2}{N^2} \phi_m' \phi_n' + \kappa^2 \phi_m \phi_n \right) \, dz = \delta_{mn}. \quad (12)$$

The expression in terms of $\phi_m$ and $\phi_n$ is found by using (10) and (11) to eliminate $\xi_m$, $\xi_n$ and the eigenvalues, then integrating by parts, which removes boundary terms. Correspondingly,

$$\langle \xi_m, P \xi_n \rangle = \frac{\mu_n^2}{H} \int_{z^-}^{z^+} \left( \frac{f^2}{N^2} \phi_m' \phi_n' + \kappa^2 \phi_m \phi_n \right) \, dz = \mu_n^2 \delta_{mn} \quad (13)$$

and

$$\langle P^{-1} E \xi_m, E \xi_n \rangle = \frac{1}{H} \int_{z^-}^{z^+} \phi_m \phi_n \, dz + \frac{\phi_m(z^+) \phi_n(z^+)}{\alpha_+} + \frac{\phi_m(z^-) \phi_n(z^-)}{\alpha_-} = \mu_n^{-2} \delta_{mn}. \quad (14)$$

The latter relation (14) has the advantage of involving only the undifferentiated streamfunctions, while the first relation (12) is independent of the eigenvalues and $\alpha_{\pm}$.

The basis of eigenfunctions can be used to expand data: given $Q$ or $\psi$, we can write

$$Q = \sum_n a_n \xi_n \quad \text{and} \quad \psi = \sum_n a_n \phi_n, \quad (15)$$

where the $a_n$ are amplitude coefficients that can be found using one of the orthogonality relations (12) or (13); for instance

$$a_n = \langle \xi_n, E Q \rangle = \frac{1}{H} \int_{z^-}^{z^+} \left( \frac{f^2}{N^2} \phi_n' \psi' + \kappa^2 \phi_n \psi \right) \, dz.$$  

The energy and generalized potential enstrophy are then simply

$$E_\kappa = \frac{1}{2} \sum_n |a_n|^2 \quad \text{and} \quad P_\kappa = \frac{1}{2} \sum_n \mu_n^2 |a_n|^2, \quad (16)$$

respectively.

Note that, even though the eigenvalue problem (11) is not of the standard Sturm–Liouville form, because of the presence of the eigenvalue $\mu_n^2$ in the boundary conditions, the basis of eigenvectors can be shown to be complete in the sense that it provides a representation of arbitrary vectors $Q$ that converges as the number of modes tends to $\infty$. This is discussed further in Appendix 5.
Lastly, note that our choice of orthogonality conditions implies slightly unfamiliar dimensions for the eigenfunctions. Because \([q], [b^\pm] \sim [T^{-1}]\) and \([\mu] \sim [L^{-1}]\) (where \(T\) is time, \(L\) is length, and braces mean “dimensions of”), \([q]\) implies that \([\xi] \sim [L^{-2}][\phi]\). The orthogonality condition \((\text{12})\) demands \([\phi] \sim [L]\) and therefore \([\xi] \sim [L^{-1}]\). In the next section, the problem will be analyzed in an appropriate nondimensional form.

3. Structure of the surface-aware modes and special cases

The approach described above provides a family of bases parameterized by the values of \(\alpha_+\) and \(\alpha_-\). In principle, different values can be chosen for different wavenumbers \(\kappa\); here, however, we restrict attention to choices of \(\alpha_\pm\) that are independent of \(\kappa\). To clarify some general properties of the new modes, we first recast the eigenvalue problem in nondimensional form with the substitutions \(z \mapsto Hz\), \(\kappa \mapsto f/(N_0H)\kappa\) and \(\mu \mapsto f/(N_0H)\mu\), where \(N_0\) is a typical value of \(N\); thus the wavenumber and eigenvalue are scaled by the approximate deformation length, \(N_0H/f\). The non-dimensional eigenvalue problem \((\text{11})\) then becomes

\[
(s\phi_n')' = -\lambda_n^2\phi_n \quad \text{and} \quad s\phi_n' = \pm\frac{\lambda_n^2 + \kappa^2}{\alpha_\pm}\phi_n \quad \text{at} \quad z = 0, -1, \quad \text{where} \quad s = \frac{N_0^2}{N^2(z)} \quad (17)
\]

and we have defined an alternative eigenvalue \(\lambda_n\) such that

\[
\mu_n^2 = \kappa^2 + \lambda_n^2. \quad (18)
\]

Written in terms of \(\lambda_n\), the eigenvalue equation takes the form of the standard vertical mode equation, but with more complicated boundary conditions.

Analysis of the new eigenvalue problem \((\text{17})\) is complicated by its dependence on three independent parameters: \(\kappa\), \(\alpha_+\) and \(\alpha_-\). Moreover, for each choice of parameters, there is an infinite set of eigenvalues. Since the problem depends on the two weights \(\alpha_\pm\) in a nearly
equivalent way, we proceed first by setting the weights equal and defining $\alpha \equiv \alpha_+ = \alpha_-$ (a case in which the weights differ will be considered in a later subsection). The nature of the eigenproblem is then largely determined by the size of the boundary condition coefficient $\mu_n^2/\alpha$: when $\mu_n^2/\alpha \to 0$, the boundary conditions revert to the standard case $\phi_n' = 0$ at the top and bottom, while when $\mu_n^2/\alpha \to \infty$, the boundary conditions become $\phi_n = 0$ at the top and bottom. However, more subtle possibilities arise as well, because unlike the standard vertical mode problem, $\lambda_n$ may be imaginary (although $\mu_n$ is always real). When $\lambda_n$ is real, the modes are oscillatory, but when it is imaginary, the modes are evanescent — these can be interpreted either as surface modes or as extensions of the barotropic mode.

This interpretation is suggested by examining the eigenvalue problem in two limiting regimes:

$\kappa^2 \ll \alpha$: modes with real $\lambda$ satisfy the simplified boundary condition $(s\phi'_n) = \pm\lambda^2_n\phi_n/\alpha$ at $z = 0, -1$ which further reduces to $\phi_n' = 0$ for $\alpha \gg 1$, corresponding to the standard baroclinic modes. These are complemented by a barotropic mode for which the first approximation $\lambda = 0$ can be refined to the purely imaginary $\lambda = i\kappa \sqrt{2/\alpha}$.

$\kappa^2 \gg \alpha$. In this case, almost all modes have $\mu_n^2 = \kappa^2 + \lambda_n^2 \gg \alpha$ and hence satisfy the simplified boundary conditions $\phi_n = 0$ at $z = 0, -1$. There are two additional modes, however, for which $\mu_n^2 = O(\alpha)$ and hence $\lambda \sim i\kappa$. These solve

$$(s\phi'_n)' - \kappa^2\phi_n \simeq 0 \quad \text{with} \quad s\phi'_n = \pm\frac{\mu_n^2}{\alpha_{\pm}}\phi_n \quad \text{at} \quad z = 0, -1,$$

and can be recognized as surface modes, with zero interior PV.

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$^2$This approximation is not uniform in $n$ but breaks down for highly oscillatory modes, with $\lambda_n = O(\alpha)$, which satisfy $\phi' = O(\alpha) \neq 0$ at $z = 0, -1$ and thus differ from the standard high-$n$ baroclinic modes.
a. **Analytical solutions for constant N**

In the special case of constant stratification, or \( s = 1 \), the eigenvalue problem (17) can be solved in closed form. Writing the solutions as

\[
\phi_n = A \cos(\lambda_n z) + B \sin(\lambda_n z),
\]

where \( A \) and \( B \) are integration constants, and imposing the boundary conditions leads to an algebraic equation for \( \lambda_n \), which may be either real or imaginary. For \( \lambda_n^2 > 0 \), the characteristic equation (dropping the subscript \( n \)) is

\[
\tan \lambda = \frac{(\alpha_+ + \alpha_-)\lambda(\lambda^2 + \kappa^2)}{(\lambda^2 + \kappa^2)^2 - \alpha_+\alpha_-\lambda^2}.
\]

(20)

For \( \lambda^2 < 0 \) we define \( \tilde{\lambda} = i\lambda \) and obtain

\[
\tanh \tilde{\lambda} = \frac{(\alpha_+ + \alpha_-)(\kappa^2 - \tilde{\lambda}^2)}{(\kappa^2 - \lambda^2)^2 + \alpha_+\alpha_-\lambda^2}.
\]

(21)

Equations (20) and (21) are suitable for a graphical analysis. Fig. 1 shows that there are infinitely many solutions to (20) (top panel) and one or two solutions to (21) depending on \( \alpha_\pm \) (bottom panel; in both cases we set \( \alpha \equiv \alpha_+ = \alpha_- \)). An important parameter is the ratio of the slopes of the right- and left-hand sides of (20) and (21) at \( \lambda = 0 \), which in both cases is

\[
\frac{\alpha_+ + \alpha_-}{\kappa^2} \equiv \tilde{\kappa}^{-2}
\]

When \( \tilde{\kappa} < 1 \) there is only one solution to (21), and there is a solution of (20) with \( \lambda < \pi/2 \).

On the other hand, if \( \tilde{\kappa} > 1 \), there are two solutions to (21) (note that the maximum of the right-hand side of (21) is 1), and there may or may not be a solution of (20) for \( \lambda < \pi/2 \).

The solution to (21) gives either a generalization of the barotropic mode, in the case of a single solution, or two modes that capture the vertical structure of the surface modes. Setting  \[3\] Note also that if \( \alpha_+\alpha_- > 4\kappa^2 \), the denominator of the right-hand side of (20) goes to 0, but stays finite otherwise: the existence of a 0 in the denominator determines whether there is a solution to (20) with \( \lambda < \pi/2 \) in the case \( \tilde{\kappa}^{-2} > 1 \).
\[ \alpha \equiv \alpha_+ = \alpha_- \], these solutions are plotted as functions of \( \tilde{\kappa} \) in Fig. [2]. There are two solutions when \( \tilde{\kappa} > 1 \), but only one otherwise. The limiting solutions discussed in the previous section can be derived explicitly. In the limit \( \tilde{\kappa}^2 = \kappa^2/(2\alpha) \ll 1 \), the single solution of (21) is given by \( \tilde{\lambda} \sim \kappa \sqrt{2/\alpha} \), with eigenfunction \( \phi \propto 1 \), which can be interpreted as the barotropic mode.

For \( \tilde{\kappa}^2 \gg 1 \), the two solutions can be identified as surface intensified modes, one symmetric and the other antisymmetric about the center of the domain, explicitly given by

\[
\phi_0 \propto \cosh \left[ \kappa \left( z + \frac{1}{2} \right) \right] \quad \text{and} \quad \phi_1 \propto \sinh \left[ \kappa \left( z + \frac{1}{2} \right) \right],
\]

with eigenvalues \( \mu_0/\alpha = \kappa \tanh \kappa \) and \( \mu_1/\alpha = \kappa \coth \kappa \). For \( \kappa \gg 1 \), the eigenvalues are nearly identical, so that linear combinations of the eigenfunctions will also satisfy the eigenvalue problem — in particular, one can construct separate upper-surface and lower-surface modes.

For real \( \lambda \), the right-hand side of (20) tends to zero for both large and small \( \kappa \), leading to eigenvalues \( \lambda_n = n\pi, n = 1, 2 \ldots \). The eigenfunctions, however, differ in the two cases: for \( \tilde{\kappa} \ll 1 \), they have the standard form \( \phi_n \propto \cos(n\pi z) \), but for \( \tilde{\kappa} \gg 1 \), they are \( \phi_n \propto \sin(n\pi z) \).

The first four modes, for \( \alpha = 1 \) and a range of \( \kappa \) are plotted in Fig. [3].

b. An oceanic special case

Here we consider a case that is potentially the most relevant to the ocean, where shears near the surface may lead to surface-intensified modes, while the quiescent abyss may be more naturally represented by the standard boundary condition, \( \phi' = 0 \) at the bottom. The relevant limits for this case are \( \alpha_+ \ll 1 \) and \( \alpha_- \to \infty \), in which case the eigenvalue problem reduces to

\[
(s\phi'_n) - \lambda_n^2 \phi_n = 0, \quad \phi_n|_{z=0} = 0, \quad \phi'_n|_{z=-1} = 0, \quad (22a)
\]

\[
(s\phi'_0) - \kappa^2 \phi_0 = 0, \quad s\phi'_0|_{z=0} = \frac{\mu_0^2}{\alpha_+} \phi_0, \quad \phi'_0|_{z=-1} = 0. \quad (22b)
\]

to leading order in \( \alpha_+ \). The solutions \( \phi_n, n = 1, 2 \ldots \) to (22a) describe interior modes, while \( \phi_0 \) is the solution to (22b) with \( \mu_0^2/\alpha_+ = O(1) \) and represents a zero PV, surface-intensified mode.
Note that the structure of the interior modes, like that of the standard baroclinic modes, is independent of $\kappa$; the normalization of the mode energy that we have chosen however leads to $\kappa$-dependent normalization factors. Since we concentrate on the leading-order approximation to the eigenvalue problem as $\alpha_+ \rightarrow 0$, all the modes, including the surface-intensified one, are independent of $\alpha_+$ and so are the normalisation factors (because the energy does not involve $\alpha_+$). Only the eigenvalue $\mu_0^2$ depends (linearly) on $\alpha_+$, although the approximation $\mu_0^2 = 0$ can be made to conclude, in particular, that the surface-intensified mode has a generalized enstrophy which vanishes to leading order.

Recently, Scott and Furnival (2012) proposed to use the eigenfunctions of (22a), forming what they term a Dirichlet basis, in conjunction with the barotropic mode. While this set of functions, like that obtained by adding a surface mode to the standard baroclinic basis (Lapeyre 2009), does not diagonalize the energy, it is remarkable that this is achieved by the complete set of solutions of (22a) and (22b), that is, by the Dirichlet basis plus a surface mode.

For constant $N$ (or $s = 1$), the solutions to (22) may be computed explicitly; they are

$$\phi_0 = A \cosh [\kappa(z + 1)], \quad A \equiv \sqrt{\frac{2}{\kappa \sinh(2\kappa)}} \quad (23a)$$

$$\phi_n = B \sin \left[ \left( n - \frac{1}{2} \right) \pi z \right], \quad B \equiv \sqrt{\frac{2}{\pi^2(n - 1/2)^2 + \kappa^2}} \quad (23b)$$

with eigenvalues $\mu_0^2 = \alpha_+ \kappa \tanh \kappa$ (corresponding to $\tilde{\lambda} \simeq \kappa - (\alpha_+/2) \tanh \kappa$) and $\lambda_n = (n - 1/2)\pi$ with $n = 1, 2 \ldots$. Their dimensional form was given by (2) in the introduction.

Again, note that the dependence on $\kappa$ of the coefficient for the interior modes is due to the normalization choice, but is irrelevant for the projection of data.

4. Use of new basis for the projection of simulated data

As a demonstration, we use the new basis to project the energy in three simulated turbulent flows, each generated by baroclinic instability of a fixed mean state in a horizontally-
periodic quasigeostrophic model. The numerical model is spectral in the horizontal, and
finite-difference in the vertical — it is the same as used in, for example, Smith and Ferrari
(2009). Energy is dissipated by linear bottom drag, and enstrophy is removed by a highly
scale-selective exponential cutoff filter (Smith et al. 2002). In all cases, the model resolution
is $512 \times 512 \times 100$.

We analyze results from three simulations. These first two are based on highly idealized
flows, and will be used to demonstrate the fundamental structure of the basis, and how
the partition of energy depends on both the nature of the flow, and on the choice of the
nondimensional weights $\alpha_\pm$. The third simulation is based on a more realistic, ocean-like
mean state, and is designed to explore the oceanic special case considered at the end of
the last section. To project the simulated data onto the new basis, one must consider the
generalized matrix eigenvalue problem that results from the particular vertical discretization
used in the model. The details of the construction of the basis in this discretization are given
explicitly Appendix B.

a. Idealized ‘interior’ and ‘surface’ baroclinic instability simulations

Both idealized flows have constant stratification $s = 1$, a ratio of domain scale to deforma-
tion scale equal to 4 and $\beta = 0$, but mean states that generate different types of baroclinic
instability. The first simulation, is forced by an ‘interior instability,’ with a mean flow that
projects onto the first (standard) baroclinic mode, $U(z) = \cos \pi z$. Flows of this type are
unstable due to a sign change of the mean interior PV gradient, but have no mean SB gra-
dients, since $B_y^\pm \propto U_z|_{z=0.\pm1} = 0$ — we refer to this simulation as BC1. The second flow
is forced by an Eady mean state, with a linear mean shear $U(z) = z$, so the instability is
driven by mean SB gradients $B_y^\pm = 1$, resulting in energy generation near the two surfaces.

The simulations are run to statistically steady state, and snapshots of the steady-state
prognostic fields of each are used to compute horizontal (total) energy spectra. The upper
panels of Fig. 4 display the horizontal spectra for the BC1 (left) and Eady (middle) simu-
lations for a few vertical levels $z$ (the right-hand column plots will be discussed in the next subsection). It is immediately apparent that the energy in the BC1 simulation is spread rather evenly over depth; by contrast, the energy in the Eady simulation is largely concentrated at the two surfaces. The panels in the middle row of Fig. 4 show the first few modes of the energy projected onto the standard basis, $\phi_n(z) \propto \cos(n\pi z)$, $n = 1, 2, ...$ (the baroclinic modes) and $\phi_0 \propto 1$ (the barotropic mode). Consistent with the $z$-dependence of the energy in the upper panel, the energy in BC1 is largely captured by the barotropic and first baroclinic modes. By contrast, the energy in the Eady case seems to be distributed evenly across the barotropic and a large number of baroclinic modes, effectively demonstrating the failure of the standard modes to provide any insight into the energy partition in a case with large energy near the surfaces.

The bottom panels of Fig. 4 display the energy spectra for the first few modes in the projection onto the new basis (BC1, left panel; Eady, middle panel). Anticipating that the BC1 simulation is best represented by the standard baroclinic basis (recovered from the generalized basis in the limit $\alpha_\pm \gg 1$), while the Eady simulation is best represented on the generalized basis in the limit $\alpha_\pm \ll 1$, we chose $\alpha_\pm = 10^6$ for the former and $\alpha_\pm = 10^{-4}$ for the latter. As is apparent, the generalized basis with the appropriate weights more efficiently captures the surface energy in the Eady simulation much better than the standard basis.

To quantify the choice of $\alpha_\pm$, we consider the projection of energy in both the BC1 and Eady simulations with the generalized basis using weights ranging from $\alpha_\pm = 10^{-3}$ to $10^3$ (always holding $\alpha = \alpha_+ = \alpha_-$) and ask, for what weights is the energy captured by the least number of modes? A simple diagnostic for this, the ratio of the energy contained in the first two modes to the total energy as a function of $\alpha$, is shown in in Fig. 5. The results indicate that extreme values of $\alpha$ are best suited for the BC1 ($\alpha \to \infty$) and Eady ($\alpha \to 0$) simulations, thus confirming our choice for Fig. 4. In the next section we examine a third simulation where the interior and surface contributions are more balanced, so that intermediate values of $\alpha_\pm$ may be expected to be relevant.
b. A semi-realistic oceanic simulation

The third simulation is driven by a mean state typical of the mid-latitude ocean. It uses an exponential mean stratification $N^2 = N_0^2 \exp(z/h)$, so that $s = \exp(-z/h)$, with $h = 0.2$, intended to represent the pycnocline. The mean shear is $U(z) = h(z+1-h) \exp(z/h) + g(z) + C$, where $g(z)$ is the first standard baroclinic eigenfunction of the operator $(sg')' = -\lambda^2 g$, with $g' = 0$ at $z = 0, -1$, so that $U$ is surface-intensified with $U'(0) = 1$ and $U'(-1) = 0$. The constant $C$ is set to ensure $\int_{-1}^{0} U(z) \, dz = 0$. Both $U(z)$ and $N(z)$ are plotted in the top panel of Fig. 6. Note that $U$ is baroclinically unstable due to both an internal sign change of the mean PV gradient, and to the interaction of the mean interior PV gradient $Q_y$ with the mean upper SB gradient $B_y^+$. Consistent with the assumptions of the ocean modes, the lower SB gradient $B_y^- = 0$. The ratio of the domain scale to the first baroclinic deformation radius (as determined by $\lambda^{-1}$) is 5. The nondimensional Coriolis gradient $\beta U_0 L_D^{-2} = 1.2$, and energy is dissipated by a linear drag $rL_0 U_0^{-1} = 0.4$. The steady-state turbulent flow has a complicated vertical structure, as evidenced by the vertical slice of the PV shown in Fig. 7.

The energy spectra for the flow are shown in the right panels of Fig. 4, just as for the BC1 and Eady cases. The energy spectra by vertical level again indicates a very surface-intensified flow, but this time, the flow falls off from a $-5/3$ spectral slope to a more energetic interior than was the case for the Eady simulation. Projection onto the standard vertical modes (middle right panel) indicates a peak in the barotropic mode, but otherwise energy is spread evenly over a large number of baroclinic modes. Projection onto a generalized basis is shown in the bottom right panel. For this simulation with no buoyancy activity at the bottom, it is natural to use a basis with $\alpha_- \to \infty$. The maximum in the ratio of the energy in modes 1 and 2 to total energy shown in Fig. 5 suggests that the value $\alpha = \alpha_+ = 2$ is appropriate. The first few modes of the corresponding basis are shown in the bottom panels of Fig 6. This is the basis chosen for Fig. 4 and indicates that the projection is very effective, with most of the energy captured by the surface and modified first baroclinic modes. An alternative basis is the ‘oceanic’ basis of section 6 which takes $\alpha_+ \ll 1$. The spectra obtained with this
basis (not shown) are essentially identical to those obtained for $\alpha_+ = 2$. This suggests that the results are insensitive to the precise value of $\alpha_+$ and that ‘oceanic’ basis may be a good default choice to analyse typical ocean data.

5. Conclusion

This paper presents a family of basis functions designed for the projection of three-dimensional ocean velocity data. The bases diagonalize both the quasigeostrophic energy and a generalization of the quasigeostrophic potential enstrophy that includes contributions from the buoyancy variances at the upper and lower surfaces. The family of bases is parameterized by the weights $\alpha_{\pm}$ assigned to the surface buoyancy variances — the standard baroclinic modes are recovered in the limit $\alpha_\pm \to \infty$, but the modes obtained in the opposite limit allow for efficient representation of the surface buoyancy variances. The bases should prove advantageous in a number of applications, from projection of observations to the derivation of highly truncated theoretical models. Their main drawback compared to the standard basis of baroclinic modes is the dependence of the modes on the wavenumber $\kappa$ which implies a lack of separation between the horizontal vertical structure in physical space. This drawback is unavoidable if some of the modes are to reflect the SQG contribution; it is minimised for the ‘oceanic’ basis obtained for $\alpha_+ \to 0$, $\alpha_- \to \infty$ since all but one modes have a $\kappa$-independent structure.

The limit $\alpha_- \to \infty$ would seem a natural choice of generalized basis for typical ocean conditions takes because of the relative lack of buoyancy activity at the bottom. Regarding $\alpha_+$, an optimal value can in principle be chosen by inspecting the spectra for a range of values or by using a diagnostic such as that of Fig. 5. However, some simpler rules of thumb would be desirable. Intuitively, one might expect that the optimal values of $\alpha_{\pm}$ are those that balance the contributions of the enstrophy $Z_\kappa$ and of the surface-buoyancy variance $B_\kappa^+$ in the generalized enstrophy $P_\kappa = Z_\kappa + \alpha_+ B_\kappa^+$. Some support for this intuition is provided
by Fig. 8 which shows $Z_\kappa$, $B_\kappa$ and their ratio as a function of $\kappa$ for the ocean simulation. The figure shows a ratio $Z_\kappa/B^+_\kappa$ that is around 5 for a broad range of $\kappa$, roughly consistent with the value $\alpha_+ = 2$ indicated by Fig. 5. There is, however, a peak around $\kappa = 4$ and a substantial increase for $\kappa \gtrsim 20$, which suggest that better results could be obtained by allowing $\alpha_+$ to depend on $\kappa$. We have not explored this intriguing possibility here.

As an alternative to the ratio $Z_\kappa/B^+_\kappa$, it would be useful to relate more directly the value of the weights $\alpha_\pm$ most appropriate to project a flow on the large-scale characteristics of the flow. Since for flows driven by instabilities, $Z_\kappa$ and $B^\pm_\kappa$ are related to the large-scale PV and surface-buoyancy gradients $Q_y$ and $B^\pm_y$, it is plausible that the ratio $Q_y/B^\pm_y$ can be used as a guide for the choice of the weights.

The advent of higher-resolution satellite observations, expected when the Surface Water Ocean Topography satellite becomes operational (Fu and Ferrari 2008), will improve our understanding of upper-ocean submesoscale dynamics only to the extent that we can connect surface observations with the three-dimensional structure of the flow below the surface. The basis derived and demonstrated here may prove a useful tool in this goal.
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Derivation details

Here we prove a few relevant facts about the eigenvectors and eigenvalues of (9). First, we show that the operator $\mathcal{E}$ is self-adjoint, e.g. $\langle \xi_m, \mathcal{E} \xi_n \rangle = \langle \mathcal{E} \xi_m, \xi_n \rangle$. Expanding the left-hand side and integrating by parts, we find

$$\langle \xi_m, \mathcal{E} \xi_n \rangle = \frac{1}{H} \int_{z^-}^{z^+} -\bar{\xi}_m \phi_n \, dz + \bar{\xi}_m^+ \phi_n(z^+) - \bar{\xi}_m^- \phi_n(z^-),$$

$$= \frac{1}{H} \int_{z^-}^{z^+} -\phi_n \left( \frac{f^2}{N^2} \bar{\phi}_m' \right)' + \kappa^2 \bar{\phi}_m \phi_n \, dz$$

$$+ \frac{f^2}{HN^2(z^+)} \bar{\phi}_m'(z^+) \phi_n(z^+) - \frac{f^2}{HN^2(z^-)} \bar{\phi}_m'(z^-) \phi_n(z^-),$$

$$= \frac{1}{H} \int_{z^-}^{z^+} \frac{f^2}{N^2} \bar{\phi}_m' \bar{\phi}_m + \kappa^2 \bar{\phi}_m \phi_n \, dz,$$

$$= \langle \mathcal{E} \xi_m, \xi_n \rangle$$

since the expression on the penultimate line is clearly symmetric. The self-adjointness of $\mathcal{P}$ as well as the positive definiteness is obvious.

To establish the completeness of the basis of the eigenvector $\xi_n$, we rewrite the eigenvalue problem in the standard form $\mathcal{A} \xi_n = \mu_n^{-2} \xi_n$, where $\mathcal{A} = \mathcal{P}^{-1} \mathcal{E}$ is positive definite and self-adjoint. This operator is compact when acting on the Hilbert space of vectors $\mathcal{Q}$ with bounded norm $\langle \mathcal{Q}, \mathcal{Q} \rangle$. This is because it is essentially an integral operator with continuous kernel — the Green’s function of the operator $(s\phi')' - \kappa^2 \phi$ (e.g. Debnath and Mikusiński 1998, section 4.8). The Hilbert-Schmidt theorem (Debnath and Mikusiński 1998, section 4.10) then applies to guarantee that every vector $\mathcal{Q}$ has a unique convergent expansion in terms of the $\xi_n$. 
Here we construct the discrete version of the eigenvalue problem. Assuming a constant discrete coordinate \( z_j \) on \( J \) grid points, with \( z_1 = 0 \) at the top, \( z_J = -H \) at the bottom, and a constant finite difference \( \Delta z = z_j - z_{j+1} \), the mean stratification is \( N_0^2 = (g/\rho_0)\Delta \bar{\rho}/\Delta z \), where \( \Delta \bar{\rho} = \rho_j - \rho_1 \) is the average background density jump between levels, \( \rho_j = \rho(z_j) \) is the background density, and \( \rho_0 \) is the average density. The parameter \( s = N_0^2/N^2 \) is discretized as \( s_j = s(z_{j+1/2}) \equiv \Delta \bar{\rho}/(\rho_{j+1} - \rho_j) \), thus \( s_j \) is offset by a half space from \( \rho_j \). In this discretization, the SBs and PV are

\[
 b^+ = \frac{f^2}{N_0^2 H} s \psi'|_{z=0} \quad \rightarrow \quad L_D^{-2} \frac{s_1}{\delta}(\psi_1 - \psi_2)
\]

\[
 b^- = \frac{f^2}{N_0^2 H} s \psi'|_{z=-1} \quad \rightarrow \quad L_D^{-2} \frac{s_{J-1}}{\delta}(\psi_{J-1} - \psi_J)
\]

\[
 q = \left( \frac{f^2}{N_0^2} s \psi' \right)' - \kappa^2 \psi \quad \rightarrow \quad L_D^{-2} \frac{1}{\delta^2} \left[ s_{j-1} \psi_{j-1} - (s_{j-1} + s_j) \psi_j + s_j \psi_{j+1} \right] - \kappa^2 \psi_j,
\]

where \( \delta \equiv \Delta z/H \) and \( L_D \equiv N_0 H/f \). Nondimensionalizing \( \kappa \mapsto [L_D^{-1}] \kappa, \psi \mapsto [L_D^2 T^{-1}] \psi \) and \( (q, b^\pm) \mapsto [T^{-1}] (q, b^\pm) \) (for some timescale \( T \)), the discrete PV/SBs and streamfunction are related as

\[
 Q = A \psi,
\]
where

\[
A = \frac{1}{\delta^2} \begin{pmatrix}
\delta s_1 & -\delta s_1 & 0 & \cdots & 0 \\
 s_1 & -(s_1 + s_2 + \delta^2 \kappa^2) & s_2 & 0 & \cdots & 0 \\
 0 & \cdots & 0 & s_{J-2} & -(s_{J-2} + s_{J-1} + \delta^2 \kappa^2) & s_{J-1} \\
 0 & \cdots & 0 & \delta s_{J-1} & -\delta s_{J-1} & \cdots & 0 \\
\end{pmatrix}.
\]

Defining the operators

\[
\begin{align*}
B &= \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \delta & \cdots & 0 \\
0 & \cdots & \delta & 0 \\
0 & \cdots & 0 & 1 \\
\end{pmatrix}, \\
F &= \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
0 & \cdots & -1 & 0 \\
0 & \cdots & 0 & -1 \\
\end{pmatrix},
\end{align*}
\]

one sees that B plays the part of the inner product, e.g. \(\langle \xi_1, \xi_2 \rangle \rightarrow \xi_1^T B \xi_2\) and F accomplishes the awkward sign changes in the definition of the operator \(\mathcal{E}\). The energy in wavenumber \(\kappa\) is

\[
E_{\kappa} = \frac{\delta}{2} \left[ \sum_{j=1}^{J-1} s_j \left| \frac{\psi_j - \psi_{j-1}}{\delta} \right|^2 + \kappa^2 \sum_{j=2}^{J-1} |\psi_j|^2 \right] = \frac{1}{2} \psi^* FBA \psi.
\]

For consistency with the theoretical development in section 2, we may also write the energy in terms of the vector \(Q = A\psi\),

\[
E_{\kappa} = \frac{1}{2} Q^* B A^{-1} Q = \frac{1}{2} Q^* B \mathcal{E} Q
\]

where the symmetry of F and B were used, and \(\mathcal{E} \equiv FA^{-1}\) is defined to make the discrete version of the energy operator defined in (8) perfectly clear.

Similarly, the generalized enstrophy in wavenumber \(\kappa\) is

\[
P_{\kappa} = \frac{1}{2} Q^* B \mathcal{P} Q
\]
where we define

\[
P = \begin{pmatrix}
\alpha_+ & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
0 & \ldots & 1 & 0 \\
0 & \ldots & 0 & \alpha_-
\end{pmatrix}
\]

to make clear the analogy with the generalized enstrophy operator defined in (8).

Now note that \( B\mathcal{E} \) and \( B\mathcal{P} \) are both symmetric (the former can be verified by checking that \( FBA \) is symmetric), so we can simultaneously diagonalize the two quadratic forms \( E_\kappa \) and \( P_\kappa \) by solving the generalized eigenvalue problem \( B\mathcal{P}\xi_j = \mu_j^2 B\mathcal{E}\xi_j \) or, in matrix form

\[
(B\mathcal{P})X = (B\mathcal{E})XM^2
\]

where \( X \) is the matrix with columns \( \xi_j \) and \( M^2 \) has \( \mu_j^2 \) along its diagonal and zeros elsewhere.

Solutions to this generalized eigenvalue problem obey the orthogonality relations

\[
X^\top B\mathcal{E}X = I \quad \text{and} \quad X^\top B\mathcal{P}X = M^2,
\]

which are analogous to (12) and (13), respectively.

In practice, it is more convenient to define a streamfunction eigenfunction \( \phi \) such that \( A\phi = \xi_\kappa \), so that the generalized eigenvalue problem can be rewritten as \( F\mathcal{P}A\phi_j = \mu_j^2 \phi_j \), or in matrix form

\[
F\mathcal{P}A\Phi = \Phi M^2
\]

where \( \Phi \) has \( \phi_j \) as its columns. In this case, the orthogonality relations become

\[
\Phi^\top FBA\Phi = I \quad \text{and} \quad \Phi^\top PBA^2\Phi = M^2,
\]

where we’ve used the fact that \( F^2 = I \). Finally, writing \( B\mathcal{P}^{-1}F\mathcal{P}^{-1}A \) as \( \Phi^{-1}(A^{-1}P^{-1}F)\Phi = M^2 \) and using the first relation in (B5), we have the equivalent of (14),

\[
\Phi^{-1}B\mathcal{P}^{-1}\Phi = M^{-2}
\]
The expansion in the basis of eigenvectors $\phi_n$ of discrete data is readily expressed in terms of the matrix $\Phi$. Denoting by $\psi$ the column vector of the streamfunction data (Fourier transformed in the horizontal) $\psi(z_j)$, the expansion reads

$$
\psi = \Phi a, \quad (B7)
$$

where $a = (a_1, \ldots, a_J)^T$ is the column vector of the mode amplitudes. These amplitudes are obtained from the data using the relation

$$
a = \Phi^T FBA \psi,
$$

which is deduced from (B5) and (B7). The total energy at a given wavenumber $\kappa$, $E_\kappa = \frac{1}{2} \psi^* FBA \psi = \frac{1}{2} |a|^2$, where $*$ denotes the complex (conjugate) transpose, is clearly the sum of the individual contributions $|a_n|^2 / 2$ of each mode. Similarly, the generalized enstrophy,

$$
P_\kappa = \frac{1}{2} Q^* B P Q = \frac{1}{2} \psi^* PBA^2 \psi = \frac{1}{2} a^* M^2 a,
$$

is the sum of the contributions $\mu_n^2 |a_n|^2 / 2$. 

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Fig. 4. Energy spectra for the BC1 (left), Eady (middle) and Ocean (right) simulations. Top panels: spectra for selected vertical levels (see legend). Middle: spectra from fields projected onto standard vertical modes (modes 1, 2 and 3–10 are shown). Bottom: spectra from fields projected onto new modes, with $\alpha_+ = \alpha_- = 10^6$ for the BC1 case, $\alpha_+ = \alpha_- = 10^{-4}$ for the Eady case and $\alpha_+ = 2, \alpha_- = 10^6$ for the Ocean case.
Fig. 5. Ratio of the energy content of the first two modes to the total energy as a function of $\alpha = \alpha_+ = \alpha_-$ for the BC1 and Eady simulations, and as a function of $\alpha = \alpha^+$ (with $\alpha_- \to \infty$) for the Ocean simulation.
Fig. 6. Left: $N^2(z)$ and $U(z)$ for the Ocean simulation. Middle: the surface mode $\phi_0(z)$ with $\alpha_- \to \infty$ and $\alpha_+ \ll 1$ (solid) and $\alpha_+ = 2$ (dashed), for a range of wavenumbers $\kappa$ (see legend). The $\kappa = .1$ lines are on top of each other. Right: The first three interior modes with $\alpha_+ \ll 1$ and $\alpha_- \to \infty$. 
Fig. 7. Vertical slice of PV snapshot from the Ocean simulation. The flow has a complicated structure in the upper ocean, masking a more uniform flow at depth.
Fig. 8. Enstrophy $Z_\kappa$ and surface buoyancy variance $B^+_{\kappa}$ as functions of wavenumber $\kappa$ for the Ocean simulation (lines with slopes -1 and -5/3 are included for reference). The ratio $Z_\kappa/B^+_{\kappa}$, also shown, can be used to guide the choice of the weight $\alpha_+$ for an effective projection basis.