Asymptotic formula for the $M_2$-ranks of overpartitions

Helen W.J. Zhang$^{1,2}$ and Ying Zhong$^1$

$^1$School of Mathematics  
Hunan University  
Changsha 410082, P. R. China

$^2$Hunan Provincial Key Laboratory of Intelligent Information Processing and Applied Mathematics  
Changsha 410082, P. R. China

Emails: helenzhang@hnu.edu.cn, YingZhong@hnu.edu.cn

Abstract

Let $N_2(a, c, n)$ be the number of overpartitions of $n$ whose the $M_2$-rank is congruent to $a$ modulo $c$. In this paper, we obtain the asymptotic formula of $N_2(a, c, n)$ utilizing the Ingham Tauberian Theorem. As applications, we derive inequalities concerning with $N_2(a, c, n)$ including its strict concavity and log-concavity.

Mathematics Subject Classification: 05A17, 11P72, 11P82

Keywords: Overpartition, the $M_2$-rank, asymptotic formula, the Ingham Tauberian Theorem

1 Introduction

The objective of this paper is to give an asymptotic formula for the $M_2$-rank of overpartition. Based on this formula, we derived several inequalities on the $M_2$-rank of overpartition such as concavity and log-concavity.

Hardy and Ramanujan [15] initially introduced the celebrated Circle Method (now known as the Hardy-Ramanujan circle method) to obtain the asymptotic formula for integer partitions originated from finding an exact formula for it. Recall that a partition of a positive integer $n$ is a sequence of non-increasing positive integers whose sum equals $n$ and $p(n)$ is defined as the number of partitions of $n$. Hardy and Ramanujan [15] proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}}e^{\pi \sqrt{\frac{n}{3}}}, \text{ for } n \to \infty.$$  

Asymptotic formulas have been proven as powerful tools to gain inequalities of the partition function. For example, DeSalvo and Pak [10] established strong log-concavity of the $p(n)$, that is, for all $n > m > 1$,

$$p(n)^2 > p(n-m)p(n+m)$$
from the asymptotic formula above. See [2,8] for further applications.

Other statistics involving partitions have been introduced since. To interpret Ramanujan’s congruences for \(p(n)\) combinatorially, Dyson [12] introduced the rank of a partition. This is defined as the largest part of the partition minus the number of parts. Bringmann obtained the asymptotic formula of \(N(a, c, n)\) in [3] by the Hardy-Ramanujan circle method. Moreover, she confirmed the conjecture which was given by Andrews and Lewis [1],

\[
N(0, 3, n) < N(1, 3, n) \quad \text{if } n \equiv 0 \text{ or } 2 \pmod{3},
\]

\[
N(0, 3, n) > N(1, 3, n) \quad \text{if } n \equiv 1 \pmod{3}.
\]

With the help of the Ingham Tauberian Theorem, Males [20] stated another asymptotic formula of \(N(a, c, n)\), that is, for fixed \(0 \leq r < t < \infty\),

\[
N(r, t, n) \sim \frac{1}{4tn\sqrt{3}} e^{\pi \sqrt{n}}, \quad \text{as } n \to \infty.
\]

The asymptotic formula yields that when \(0 \leq r < t \leq \infty\),

\[
N(r, t, a)N(r, t, b) > N(r, t, a + b)
\]

for sufficiently large \(a\) and \(b\). Several other asymptotic formulas and inequalities for rank of a partition have also been obtained, for example [5,11,13,16].

Bringmann and Males illustrated two different approaches to studying the asymptotic behavior of the coefficients of a series: One can either use the Hardy-Ramanujan circle method [6,15] or apply the Ingham Tauberian Theorem [17,18]. In this paper, we study an asymptotic formula for the \(M_2\)-rank of overpartition in the spirit of The Ingham Tauberian Theorem.

The \(M_2\)-rank for overpartitions was introduced by Lovejoy [19]. Recall that an overpartition \(\lambda\) of a nonnegative integer \(n\) is a partition of \(n\) where the first occurrence of each distinct part may be overlined. Denote \(\overline{p}(n)\) by the number of overpartitions of \(n\). Let \(l(\lambda)\) be the largest part of \(\lambda\), \(n(\lambda)\) be the number of parts of \(\lambda\) and \(n(\lambda_o)\) be the number of odd non-overlined parts of \(\lambda\). Then the \(M_2\)-rank of an overpartition \(\lambda\) is defined as

\[
M_2\text{-rank}(\lambda) = \left\lfloor \frac{l(\lambda)}{2} \right\rfloor - n(\lambda) - n(\lambda_o) - \chi(\lambda),
\]

where \(\chi(\lambda) = 1\) if \(l(\lambda)\) is odd and non-overlined and \(\chi(\lambda) = 0\) otherwise. Denote \(\overline{N}_2(a, c, n)\) by the number of overpartitions of \(n\) whose the \(M_2\)-rank is congruent to \(a\) modulo \(c\). By investigating the asymptotic behavior of the Appell function associated with the generating function of \(\overline{N}_2(a, c, n)\), we confirm that the \(M_2\)-rank of overpartition fulfills the conditions in the Ingham Tauberian Theorem. Hence we obtain the following asymptotic formula for the \(M_2\)-rank of overpartition.

**Theorem 1.1.** If the odd integer \(c \geq 2\), then for any \(0 \leq a < c\) we have

\[
\overline{N}_2(a, c, n) \sim \frac{1}{c \overline{p}(n)} \sim \frac{1}{8cn} e^{\pi \sqrt{n}}
\]

as \(n \to \infty\).
We deduce strict concavity of $N_2(a, c, n)$ by applying Theorem 1.1.

**Theorem 1.2.** If the odd integer $c \geq 2$, then for any $0 \leq a < c$ we have

$$N_2(a, c, n_1)N_2(a, c, n_2) > N_2(a, c, n_1 + n_2)$$

for sufficiently large $n_1$ and $n_2$.

The following inequality also serves as an application of Theorem 1.1.

**Theorem 1.3.** Let $c \geq 2$ be an odd integer and $0 \leq a < c$. Then for sufficiently large $n_1$ and $n_2$ with $n_1 < n_2 + 1$, we have

$$N_2(a, c, n_1)N_2(a, c, n_2) > N_2(a, c, n_1 - 1)N_2(a, c, n_2 + 1).$$

Setting $n_1 = n_2$ in Theorem 1.3, we show that $N_2(a, c, n)$ is asymptotically log-concave.

**Corollary 1.4.** If the odd integer $c \geq 2$, then for any $0 \leq a < c$, we have

$$N_2(a, c, n)^2 > N_2(a, c, n - 1)N_2(a, c, n + 1)$$

for sufficiently large $n$.

The paper is organized as follows. In Section 2, we introduce the Ingham Tauberian Theorem [4, 17] and the monotonicity of rank of overpartition obtained by Xiong and Zang [21]. In Section 3, we obtain the asymptotic behavior of the Appell function associated with the generating function of $N_2(a, c, n)$. In Section 4, we prove our main result Theorem 1.1 and provide its applications in deriving inequalities (Theorems 1.2 and 1.3).

2 Preliminary

In this section, we recall some facts about the Appell function without proof. Let the Appell function [22]

$$\mu(u, z; \tau) := \frac{e^{\pi i u}}{\vartheta(z; \tau)} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n^2 + n) \tau} e^{2 \pi i n z},$$

where Jacobi theta function

$$\vartheta(z; \tau) := \sum_{n \in \frac{1}{2} + \mathbb{Z}} e^{\pi i n^2 \tau + 2 \pi i n (z + \frac{1}{2})},$$

with $z \in \mathbb{C}$. And $\vartheta$ satisfies

$$\vartheta(z; \tau) = \frac{i}{\sqrt{-1 \tau}} e^{\pi i z^2} \vartheta \left( \frac{z}{\tau}; -\frac{1}{\tau} \right).$$
What’s more, Zwegers showed that
\[
\mu(u, v; \tau) = -\frac{1}{\sqrt{-i\tau}} e^{\frac{\pi i (u-v)^2}{\tau}} \mu \left( \frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau} \right) + \frac{1}{2i} h(u - v; \tau),
\] (2.4)
where \( h \) is the Mordell integral
\[
h(z; \tau) := \int_{\mathbb{R}} \frac{e^{\pi i x^2 - 2\pi x z}}{\cosh(\pi x)} dx,
\] (2.5)
which satisfied the following transformation
\[
h(z; \tau) = \frac{1}{\sqrt{-i\tau}} e^{\frac{\pi i z^2}{\tau}} h \left( \frac{z}{\tau}; -\frac{1}{\tau} \right).
\] (2.6)

Recently, Zwegers [23] introduced the Appell function of higher level
\[
A_\ell(u, v; \tau) := e^{\pi i \ell u} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\ell n(n+1)/2} e^{2\pi i u v}}{1 - e^{2\pi i u} q^n},
\] (2.7)
which satisfies the following transformation formulae
\[
A_\ell(u, v; \tau) = \sum_{k=0}^{\ell-1} e^{2\pi i u k \ell} \left( v + k\tau + \frac{\ell - 1}{2}; \ell \tau \right) \mu \left( \ell u, v + k\tau + \frac{\ell - 1}{2}; \ell \tau \right).
\] (2.8)

In this paper, we shall use the following theorem [4, Theorem 1.1] due to Ingham [17] to obtain the asymptotic formula of \( N_2(a, c, n) \).

**Theorem 2.1.** Let \( f(q) := \sum_{n \geq 0} a(n) q^n \) be a power series with weakly increasing non-negative coefficients and radius of convergence equal to one. If there exist constants \( A > 0, \lambda, \alpha \in \mathbb{R} \) such that
\[
f(e^{-t}) \sim \lambda t^{\alpha} e^{A t} \quad \text{as } t \to 0^+
\]
and
\[
f(e^{-\varepsilon}) \ll |\varepsilon|^\alpha e^{A |\varepsilon|} \quad \text{as } \varepsilon \to 0,
\] (2.9)
with \( \varepsilon = x + iy \) (\( x, y \in \mathbb{R}, x > 0 \)) in each region of the form \(|y| \leq \Delta x\) for \( \Delta > 0 \), then
\[
a(n) \sim \frac{\lambda}{2} \frac{A^{\frac{\alpha}{2} + \frac{1}{4}}}{n^{\frac{\alpha}{2} + \frac{3}{4}}}
\]
as \( n \to \infty \).

**Remark 2.2.** Bringmann, Jennings-Schaffer and Mahlburg [4, Remark of Theorem 1.1] noted that if in each region \(|y| \leq \Delta x\)
\[
f(e^{-\varepsilon}) \sim \lambda \varepsilon^{\alpha} e^{A \varepsilon}
\]
then the second bound in (2.9) is automatically satisfied.
Let \( \overline{N}_2(m, n) \) denote the number of overpartitions of \( n \) with the \( M_2 \)-rank \( m \). Xiong and Zang \[21\] proved the following inequality on \( \overline{N}_2(m, n) \) conjectured by Chan and Mao \[7\]. We highlight this result to confirm the monotonicity of \( \overline{N}_2(a, c, n) \) in Theorem 2.1.

**Lemma 2.3.** For all \( m \geq 0 \) and \( n > 0 \), we have
\[
\overline{N}_2(m, n) \geq \overline{N}_2(m, n - 1).
\]

3. **Asymptotic behavior of \( A_1(\pm z, \tau; 2\tau) \)**

In this section, we investigate the asymptotic behavior of the Appell function \( A_1(\pm z, \tau; 2\tau) \) associated with the generating function of \( \overline{N}_2(a, c, n) \). This confirms the asymptotic condition in the Ingham Tauberian Theorem in our context.

**Theorem 3.1.** Let \( 0 < z < \frac{1}{2} \), and \( \tau = \frac{i\varepsilon}{2\pi} \) with \( \varepsilon = x + iy \) in each region of the form \( |y| \leq \Delta x \) for \( \Delta > 0 \) then
\[
A_1(\pm z, \tau; 2\tau) \to 0
\]
as \( \varepsilon \to 0 \).

**Proof.** Using (2.8), we have
\[
A_1(\pm z, \tau; 2\tau) = \vartheta(\tau; 2\tau) \mu(\pm z, \tau; 2\tau).
\]

Then combining (2.3) and (2.4), we find that
\[
A_1(\pm z, \tau; 2\tau) = \frac{1}{2\tau} e^{-\frac{\pi i z^2}{4\tau}} \vartheta \left( \frac{1}{2}; -\frac{1}{2\tau} \right) \mu \left( \frac{\pm z}{2\tau}, \frac{1}{2}; -\frac{1}{2\tau} \right) + \frac{1}{\sqrt{-8i\tau}} e^{-\frac{\pi i \tau}{4\tau}} \vartheta \left( \frac{1}{2}; -\frac{1}{2\tau} \right) h(\pm z - \tau; 2\tau). \tag{3.1}
\]

Let \( S_1 \) and \( S_2 \) denote the summations on the right-hand side of (3.1), respectively. We first investigate the terms from \( S_1 \). By (2.1), we have
\[
\vartheta \left( \frac{1}{2}; -\frac{1}{2\tau} \right) \mu \left( \frac{\pm z}{2\tau}, \frac{1}{2}; -\frac{1}{2\tau} \right) = e^{\pm \pi i s} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{-\frac{\pi i n^2}{4\tau}} e^{\pi i n}}{1 - e^{-\frac{\pi i n}{2\tau}} e^{\frac{\pi i n}{2\tau}}} = q_0^\mp \sum_{n \in \mathbb{Z}} \frac{q_0^{\frac{\pi^2 n}{4\tau}}} {1 - q_0^{\frac{\pi^2 n}{2\tau}}}, \tag{3.2}
\]

where \( q_0 := e^{-\frac{\pi i}{\tau}} = e^{-\frac{\pi x^2}{\varepsilon}} \).

In view of (3.2), \( S_1 \) can be rewritten as
\[
S_1 = \frac{1}{2\tau} q_0^{-\frac{1}{2} (z^2 + z)} e^{\mp \pi i z} \sum_{n \in \mathbb{Z}} \frac{q_0^{\frac{\pi^2 n}{4\tau}}} {1 - q_0^{\frac{\pi^2 n}{2\tau}}}. \tag{3.3}
\]
Noting that
\[ \left| e^{-\frac{1}{2}} \right| = e^{-\frac{1}{2} + y^2} \leq e^{-\frac{1}{2} (1+\Delta^2) x} \leq e^{-\frac{1}{2} (1+\Delta^2)|\epsilon|}, \]
so if the lowest power of \( q_0 \) is positive, \( S_1 \) tends to 0 in our asymptotic limit.

We first consider the plus sign, that is
\[ S_1^+ := \frac{1}{2\tau} q_0^{-\frac{1}{2}(z^2+z)} e^{-\pi iz} \sum_{n \in \mathbb{Z}} \frac{q_0^{n^2+n}}{1 - q_0^{n-z}}. \]
Considering only the inner sum without the prefactor, the \( n = 0 \) term is
\[ \frac{1}{1 - q_0^{-z}} = -q_0^z - q_0^{2z} - \cdots, \]
where \( z \in (0, \frac{1}{2}) \) and \( \tau \in \mathbb{H} \). For \( n \geq 1 \), it is clear that the lowest power of \( q_0 \) in the inner sum is 1. When \( n < 0 \), we have \( n - z < 0 \) and hence the term
\[ \frac{q_0^{n^2+n}}{1 - q_0^{n-z}} = -q_0^{\frac{n^2-n+2z}{2}} \sum_{j \geq 0} q_0^{(z-n)j}, \]
with the lowest order term \(-q_0^{\frac{n^2-n+2z}{2}}\). Obviously, \( \frac{n^2-n+2z}{2} \geq 1 + z \). Therefore, for \( z \in (0, \frac{1}{2}) \), the lowest power of \( q_0 \) is given by the \( n = 0 \) term, which is
\[ -\frac{1}{2\tau} q_0^{-\frac{1}{2}(z^2+z)} e^{-\pi iz} q_0^z = -\frac{1}{2\tau} q_0^{-\frac{1}{2}(z^2-z)} e^{-\pi iz}. \]
Using the facts that \( 0 < z < \frac{1}{2}, -\frac{1}{2}(z^2 - z) > 0 \), we infer that \( S_1^+ \to 0 \) as \( \varepsilon \to 0 \).

We now consider the minus sign
\[ S_1^- := \frac{1}{2\tau} q_0^{-\frac{1}{2}(z^2-z)} \epsilon^{\pi iz} \sum_{n \in \mathbb{Z}} \frac{q_0^{n^2+n}}{1 - q_0^{n+z}}. \]
Similar to the plus sign \( S_1^+ \), we consider the three cases about \( n \). Then we find that the lowest power of \( q_0 \) is also given by the \( n = 0 \) term, that is
\[ \frac{1}{2\tau} q_0^{-\frac{1}{2}(z^2-z)} e^{\pi iz}. \]
Since \(-\frac{1}{2}(z^2 - z) > 0 \), we have \( S_1^- \to 0 \).

We now verify the remaining term \( S_2 \to 0 \) as well. Applying (2.6), we find that
\[ S_2 = \frac{i}{4\tau} e^{\frac{\pi i}{2\tau} (z^2+2x\tau)} \theta \left( \frac{1}{2}; -\frac{1}{2\tau} \right) h \left( \frac{\pm z - \tau}{2\tau}; -\frac{1}{2\tau} \right). \]
On the one hand, by (2.2), we have
\[
\psi \left( \frac{1}{2}; -\frac{1}{2\tau} \right) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} e^{-\frac{\pi in^2}{2\tau} + 2\pi in} = \sum_{n \in \frac{1}{2} + \mathbb{Z}} q_0^{\frac{n^2}{\tau}} e^{2\pi in}.
\]

It is obvious that the smallest power of \(q_0\) appearing in \(e^{\frac{\pi i (z^2 + 2\tau^2)}{2\tau}} \psi \left( \frac{1}{2}; -\frac{1}{2\tau} \right)\) is given by \((n = \pm \frac{1}{2})\)
\[
q_0^{\frac{z^2}{\tau} + \frac{1}{8}} e^{\pm \pi iz} (e^{\pi i} + e^{-\pi i}).
\]

Then the power of \(q_0\) in \(e^{\frac{\pi i (z^2 + 2\tau^2)}{2\tau}} \psi \left( \frac{1}{2}; -\frac{1}{2\tau} \right)\) is positive due to \(0 < z < \frac{1}{2}\). Hence these terms give no contribution in the limiting situation.

On the other hand, with the definition of \(h(z; \tau)\) in (2.5), we find that
\[
h \left( \pm \frac{z - \tau}{2\tau}; -\frac{1}{2\tau} \right) = \int_{\mathbb{R}} e^{-\frac{\pi i x^2}{2\tau} - 2\pi i \left( \frac{\pm z}{2\tau} \frac{1}{2} \right)x} \frac{dx}{\cosh(\pi x)}
= e^{-\frac{\pi i z^2}{4\tau} \pm \pi iz} \int_{\mathbb{R} \pm iz} \frac{e^{-\frac{\pi i w^2}{4\tau} + \pi w}}{\cosh(\pi (w \mp iz))} dw,
\]

here we let \(w = x \pm iz\). Since \(0 < z < \frac{1}{2}\), the integrand has no poles. So we can shift the path of integration back to \(\mathbb{R}\),
\[
h \left( \pm \frac{z - \tau}{2\tau}; -\frac{1}{2\tau} \right) = e^{-\frac{\pi i z^2}{4\tau} \pm \pi iz} \int_{\mathbb{R}} \frac{e^{-\frac{\pi i w^2}{4\tau} + \pi w}}{\cosh(\pi (w \mp iz))} dw.
\]

Using the trivial bound
\[
\frac{1}{|\cosh(\pi (w \mp iz))|} = \frac{2}{|e^{\pi (w \mp iz)} + e^{-\pi (w \mp iz)}|} \leq \frac{2}{|e^{-\pi iz} + e^{\pi iz}|} = |\sec(\pi z)|,
\]
and the fact \(\int_{\mathbb{R}} e^{-tx^2} dx = \sqrt{\pi}\), we find that
\[
\left| h \left( \pm \frac{z - \tau}{2\tau}; -\frac{1}{2\tau} \right) \right| \leq |\sec(\pi z)| \exp \left( -\frac{\pi^2 z^2}{\varepsilon} + \varepsilon \right) \pi^{-\frac{1}{2}} e^{\frac{1}{2}},
\]

where \(\tau = \frac{i \varepsilon}{2\pi}\). Hence, we have
\[
\left| \frac{i}{4\tau} h \left( \pm \frac{z - \tau}{2\tau}; -\frac{1}{2\tau} \right) \right| \leq |\sec(\pi z)| \exp \left( -\frac{\pi^2 z^2}{\varepsilon} + \varepsilon \right) \frac{\pi^3}{2\varepsilon^2}.
\]

It is easy to see that as \(\varepsilon \to 0\) this contribution vanishes. This completes the proof. \[\blacksquare\]
4 Asymptotic formula and inequalities

In this section, we employ the Ingham Tauberian Theorem on the generating function of \( N_2(a, c, n) \) to deduce its asymptotic formula. Several inequalities concerning on \( N_2(a, c, n) \) including its strict concavity and log-concavity are obtained based on this formula.

We basically handle with the generating function of \( N_2(m, n) \). Lovejoy [19] found that

\[
R_2(\zeta, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m, n) \zeta^m q^n = \sum_{n=0}^{\infty} \frac{(-1; q)_{2n} q^n}{(\zeta q^2; q^2)_n (\zeta^{-1} q^2; q^2)_n},
\]

where

\[
(q; q)_\infty = \prod_{n=0}^{\infty} (1 - q^n) \quad \text{and} \quad (a; q)_\infty = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.
\]

In particular,

\[
R_2(1, q) = \sum_{n=0}^{\infty} \mathcal{P}(n) q^n.
\]

The following lemmas play central roles in the proof of Theorem 1.1.

**Lemma 4.1.** For integer 0 ≤ a < c and odd integer c, we have

\[
\sum_{n=0}^{\infty} N_2(a, c, n) q^n = \frac{1}{c} \left[ \sum_{n=0}^{\infty} \mathcal{P}(n) q^n + \sum_{j=1}^{c-1} \left( \zeta_c^{-a_j} + \zeta_c^{a_j} \right) R_2(\zeta_c^j, q) \right],
\]

where \( \zeta_c = e^{2\pi i/c} \) and \( q = e^{2\pi i \tau}, \tau \in \mathbb{H} \).

To verify (4.2), we need to show that

\[
\sum_{n=0}^{\infty} N_2(a, c, n) q^n = \frac{1}{c} \sum_{n=0}^{\infty} \mathcal{P}(n) q^n + \frac{1}{c} \sum_{j=1}^{c-1} \zeta_c^{-a_j} R_2(\zeta_c^j, q),
\]

where \( \zeta \) is the root of unity.

By (4.1), we notice that the right hand side of (4.3) can be expanded as

\[
\frac{1}{c} \sum_{n=0}^{\infty} \mathcal{P}(n) q^n + \frac{1}{c} \sum_{j=1}^{c-1} \zeta_c^{-a_j} R_2(\zeta_c^j, q) = \frac{1}{c} \sum_{j \equiv 0 \mod c} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m, n) \zeta_m \zeta_c^{-a_j} q^n
\]

\[
= \frac{1}{c} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m, n) \left( \sum_{j \equiv 0 \mod c} \zeta_m \zeta_c^{-a_j} \right) q^n.
\]
By the orthogonality of the roots of unity
\[ \sum_{j \mod c} \zeta_{c}^{rj} = \begin{cases} 0, & r \not\equiv 0 \mod c, \\ c, & r \equiv 0 \mod c, \end{cases} \]
we obtain
\[
\frac{1}{c} \sum_{n=0}^{\infty} p(n) q^{n} + \frac{1}{c} \sum_{j=1}^{c-1} \zeta^{-aj} R_{2} (\zeta_{c}^{j}; q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_{2}(m, n) q^{n}
\]
\[
= \sum_{n=0}^{\infty} N_{2}(a, c, n) q^{n},
\]
which gives (4.3). Since \( R_{2} (\zeta; q) = R_{2} (\zeta^{-1}; q) \), then applying (4.3), we can get (4.2). \( \square \)

Then we prove an identity on the \( M_{2} - \) rank generating function \( R_{2} (\zeta, q) \).

**Lemma 4.2.** Let \( \tau \in \mathbb{H}, z \in \mathbb{R}, q := e^{2\pi i \tau} \) and \( \zeta := e^{2\pi i z} \). We have
\[
R_{2} (\zeta, q) = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left( \zeta \frac{1}{1 - \zeta^{2} q^{n}} - \zeta^{-\frac{1}{2}} A_{1}(z, \tau; 2\tau) - \zeta^{\frac{1}{2}} A_{1}(-z, \tau; 2\tau) \right). \tag{4.4}
\]

**Proof.** One can easily derive that
\[
\frac{\zeta}{1 - \zeta^{2} q^{n}} - \frac{\zeta^{-1}}{1 - \zeta^{-1} q^{2n}} = \frac{\zeta - \zeta^{-1}}{(1 - \zeta q^{2n})(1 - \zeta^{-1} q^{2n})}. \tag{4.5}
\]
Substituting (4.5) into (4.1), we obtain that
\[
R_{2} (\zeta, q) = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^{n} q^{n^{2} + 2n} \left( \frac{\zeta}{1 - \zeta q^{2n}} - \frac{\zeta^{-1}}{1 - \zeta^{-1} q^{-2n}} \right) \frac{(1 - \zeta)(1 - \zeta^{-1})}{\zeta - \zeta^{-1}}
\]
\[
= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \frac{1 - \zeta}{1 + \zeta} \left( \sum_{n=-\infty}^{\infty} (-1)^{n} q^{n^{2} + 2n} \zeta - \sum_{n=-\infty}^{\infty} (-1)^{n} q^{n^{2} + 2n} \zeta^{-1} \right)
\]
\[
= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \frac{(1 - \zeta)}{1 + \zeta} \left\{ \zeta^{\frac{1}{2}} A_{1}(z, \tau; 2\tau) - \zeta^{-\frac{1}{2}} A_{1}(-z, \tau; 2\tau) \right\}. \tag{4.6}
\]
This completes the proof. \( \square \)

Now we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We can rewrite \( \overline{N}_{2}(a, c, n) \) as
\[
\overline{N}_{2}(a, c, n) = \sum_{k \in \mathbb{Z}} \overline{N}_{2}(a + kc, n),
\]
which is a finite sum, since for \( |a + kc| > n \) we have \( \overline{N}_{2}(a + kc, n) = 0 \).
By Lemma 2.3, we derive that for all non-negative integers \( m \) and positive integers \( n \),
\[
N_2(a, c, n) \geq N_2(a, c, n - 1).
\]

We are therefore in the situation where we may apply Theorem 2.1. By Remark 2.2, we only need to investigate the asymptotic behavior
\[
\lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} N_2(a, c, n)e^{-\varepsilon n}.
\]
Combining (4.2) and (4.4), we obtain that
\[
\sum_{n=0}^{\infty} N_2(a, c, n)q^n = \frac{1}{c} \sum_{n=0}^{\infty} p(n)q^n \left( 1 + \sum_{j=1}^{\infty} \left( \zeta_c^{-aj} + \zeta_c^{aj} \right) \frac{1 - \zeta_c^j}{1 + \zeta_c^j} \right)
\times \left( \zeta_c^{j} A_1\left( \frac{j}{c}, \tau; 2\tau \right) - \zeta_c^{-j} A_1\left( -\frac{j}{c}, \tau; 2\tau \right) \right).
\]
Let \( \tau = \frac{i\varepsilon}{2\pi} \) with \( \varepsilon = x + iy \) in each region \( |y| \leq \Delta x \) for \( \Delta > 0 \) and consider \( \varepsilon \to 0 \). Since \( 0 < \frac{j}{c} < \frac{1}{2} \), we can know that the term in the biggest brackets is asymptotically equal to 1 in this limit by Theorem 3.1. Hence we have
\[
\lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} N_2(a, c, n)e^{-\varepsilon n} \sim \frac{\varepsilon \frac{1}{2}}{2c\sqrt{\pi}} e^{\frac{\varepsilon^2}{4\pi}}
\]
in these regions of restricted angle. Then applying Theorem 2.1 we see that for \( n \to \infty \)
\[
N_2(a, c, n) \sim \frac{1}{8cn} e^{\pi\sqrt{n}}.
\]
By the following well-known fact (see, e.g., [14])
\[
\overline{p}(n) \sim \frac{1}{8n} e^{\pi\sqrt{n}}, \quad \text{for } n \to \infty, \quad (4.6)
\]
we arrive at (1.1), which completes the proof.

We prove Theorems 1.2 and 1.3 as applications of Theorem 1.1.

**Proof of Theorem 1.2.** Consider the ratio
\[
\frac{N_2(a, c, n_1)N_2(a, c, n_2)}{N_2(a, c, n_1 + n_2)}
\]
as \( n_1, n_2 \to \infty \). By Theorem 1.1 we have
\[
\frac{N_2(a, c, n_1)N_2(a, c, n_2)}{N_2(a, c, n_1 + n_2)} \sim \frac{n_1 + n_2}{8cn_1n_2} e^{(\pi\sqrt{n_1} + \pi\sqrt{n_2})} > 1
\]
as \( n_1, n_2 \to \infty \), which completes the proof.
Proof of Theorem 1.3. Similar to the proof of Theorem 1.2, we have to consider the ratio

$$\frac{N_2(a, c, n_1)N_2(a, c, n_2)}{N_2(a, c, n_1 - 1)N_2(a, c, n_2 + 1)}.$$

For $n_1 < n_2 + 1$, by Theorem 1.1, as $n_1, n_2 \to \infty$ we have

$$\frac{N_2(a, c, n_1)N_2(a, c, n_2)}{N_2(a, c, n_1 - 1)N_2(a, c, n_2 + 1)} \sim \frac{(n_1 - 1)(n_2 + 1)}{n_1n_2} e^{(\pi \sqrt{n_1} + \pi \sqrt{n_2})} e^{(\pi \sqrt{n_1 - 1} + \pi \sqrt{n_2 + 1})} > 1.$$

This completes the proof. 

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant Nos. 12001182 and 12171487), the Fundamental Research Funds for the Central Universities (Grant No. 531118010411) and Hunan Provincial Natural Science Foundation of China (Grant No. 2021JJ40037).

References

[1] G.E. Andrews and R. Lewis, The ranks and cranks of partitions moduli 2, 3, and 4, J. Number Theory 85 (1) (2000) 74–84.
[2] C. Bessenrodt and K. Ono, Maximal multiplicative properties of partitions, Ann. Comb. 20(1) (2016) 59–64.
[3] K. Bringmann, Asymptotics for rank partition functions, Trans. Amer. Math. Soc. 361 (7) (2009) 3483–3500.
[4] K. Bringmann, C. Jennings-Schaffer and K. Mahlburg, On a Tauberian theorem of Ingham and Euler-Maclaurin summation, The Ramanujan Journal (2021) https://doi.org/10.1007/s11139-020-00377-5.
[5] K. Bringmann and B. Kane, Inequalities for differences of Dyson’s rank for all odd moduli, Math. Res. Lett. 17 (5) (2010) 927–942.
[6] K. Bringmann, and K. Mahlburg, An extension of the Hardy-Ramanujan circle method and applications to partitions without sequences, Amer. J. Math. 133 (4) (2011) 1151–1178.
[7] S.H. Chan and R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions, Adv. Math. 258 (2014) 414–437.
[8] W.Y.C. Chen, D.X.Q. Jia and L.X.W. Wang, High order Turán inequalities for the partition function, Trans. Amer. Math. Soc. 372 (3) (2019) 2143–2165.
[9] S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356 (2004) 1623–1635.
[10] S. DeSalvo and I. Pak, Log-concavity of the partition function, Ramanujan J. 38 (1) (2015) 61–73.
[11] J. Dousse and M.H. Mertens, Asymptotic formulae for partition ranks, Acta Arith. 168 (1) (2015) 83–100.
[12] F.J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.
[13] K. Gomez and E. Zhu, Bounds for coefficients of the \( f(q) \) mock theta function and applications to partition ranks, J. Number Theory 226 (2021) 1–23.
[14] G.H. Hardy and S. Ramanujan, Asymptotic formulæ for the distribution of integers of various types, Proc. Lond. Math. Soc. 16 (2) (1918) 112–132.
[15] G.H. Hardy and S. Ramanujan, Asymptotic formulæ in combinatory analysis, Proc. London Math. Soc. (2) 17 (1918) 75–115.
[16] E. Hou and M. Jagadeesan, Dyson’s partition ranks and their multiplicative extensions, Ramanujan J. 45 (3) (2018) 817–839.
[17] A.E. Ingham, A Tauberian theorem for partitions, Ann. of Math. 42 (2) (1941) 1075–1090.
[18] M.-J. Jang, Asymptotic behavior of odd-even partitions, Electron. J. Combin. 24 (3) (2017) Paper No. 3.62 15 pp.
[19] J. Lovejoy, Rank and conjugation for a second Frobenius representation of an over-partition, Ann. Comb. 12 (1) (2008) 101–113.
[20] J. Males, Asymptotic equidistribution and convexity for partition ranks, Ramanujan J. 54 (2) (2021) 397–413.
[21] H. Xiong and W.J.T. Zang, Monotonicity properties for ranks of overpartitions, J. Number Theory 209 (2020) 330–346.
[22] S. Zwegers, Mock theta functions, Ph.D. Thesis, Universiteit Utrecht (2002).
[23] S. Zwegers, Multivariable Appell functions and nonholomorphic Jacobi forms, Res. Math. Sci. 6 (1) (2019) Paper No. 16, 15 pp.