ORTHOGONAL SYMMETRIC MATRICES
AND JOINS OF GRAPHS

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Abstract. We introduce a notion of compatibility for multiplicity matrices. This gives rise to a necessary condition for the join of two (possibly disconnected) graphs $G$ and $H$ to be the pattern of an orthogonal symmetric matrix, or equivalently, for the minimum number of distinct eigenvalues $q$ of $G \lor H$ to be equal to two. Under additional hypotheses, we show that this necessary condition is also sufficient. As an application, we prove that $q(G \lor H)$ is either two or three when $G$ and $H$ are unions of complete graphs, and we characterise when each case occurs.

1. Introduction

1.1. Background and related work. Orthogonal matrices, ubiquitous in matrix theory and central in applications, have been widely studied. However, some basic questions on their combinatorial structure remain open [8,20]. In this paper we advance our understanding of the zero-nonzero patterns of symmetric orthogonal matrices, drawing motivation from the Inverse Eigenvalue Problem for Graphs, a more general problem on the interplay between spectral and structural properties of a matrix.

Let $G$ be a simple graph with vertex set $V(G) = \{1, \ldots, n\}$ and edge set $E(G)$, and consider $S(G)$, the set of all real symmetric $n \times n$ matrices $A = (a_{ij})$ such that, for $i \neq j$, $a_{ij} \neq 0$ if and only if $\{i, j\} \in E(G)$, with no restriction on the diagonal entries of $A$. The Inverse Eigenvalue Problem for Graphs (IEP-G) seeks to characterise all possible sets of eigenvalues of matrices in $S(G)$. The IEP-G is unsolved for all except a few families of graphs, and it has motivated the study of several related parameters. One widely studied parameter is the minimum number of distinct eigenvalues of a graph, namely $q(G) = \min\{q(A) : A \in S(G)\}$, where $q(A)$ denotes the number of distinct eigenvalues of a square matrix $A$. For a graph $G$ the set $S(G)$ contains an orthogonal symmetric matrix (or $G$ is is realisable by an orthogonal symmetric matrix) if and only if $q(G) = 2$.

The study of the minimum number of distinct eigenvalues of a graph was initiated by Leal-Duarte and Johnson in [14], where they proved
that if $T$ is a tree, then $q(T)$ cannot be smaller than $\text{diam}(T) + 1$; this bound was later improved for infinitely many trees (see e.g., [9]). Determining $q$ for many graphs with cycles seems to be a difficult problem. It is well known that $q(G) = n$ if and only if $G$ is a path [10], and graphs with $q(G) = n - 1$ were characterised in [5] using strong spectral properties. At the other extreme, while it is clear that $q(G) = 1$ if and only if $E(G) = \emptyset$, there is no known characterisation of graphs with $q(G) = 2$, i.e., the graphs that are realisable by an orthogonal symmetric matrix.

Several families of graphs are known to have a realisation with an orthogonal symmetric matrix (see, e.g., [3, 6, 13, 16]), and several necessary conditions were determined by Adm et. al. [2, 3]. The same authors also proved that $q(G \lor G) = 2$ for any connected graph $G$. Later, this result was generalized by Monfared and Shader [18], who proved that $q(G \lor H) = 2$ for any connected graphs $G$ and $H$ with the same number of vertices. Recently, joins of disconnected graphs were investigated in [1, 2], where particular attention was given to joins of unions of complete graphs.

1.2. Overview. Results in this paper contribute to the IEP-G, to the study of combinatorial structure of orthogonal matrices, and also shed light on certain completion problems for orthogonal matrices.

The main framework is developed in Section 2, where we present a necessary condition for the join of two (possibly disconnected) graphs, $G \lor H$, to have $q(G \lor H) = 2$, as well as a closely related sufficient condition. Section 3 contains remarks, extensions of previous results and initial examples derived from results in Section 2. Finally, in Section 4 we apply our results to determine $q(G \lor H)$ whenever $G$ and $H$ are unions of complete graphs. As a preview of our results, we state the main theorem of Section 4 below.

**Theorem 1.1.** Let $G = G_1 \cup \cdots \cup G_k$ and $H = H_1 \cup \cdots \cup H_\ell$ where $G_1, \ldots, G_k$ and $H_1, \ldots, H_\ell$ are connected graphs and $k \leq \ell$. By $\text{iso}(G)$ we denote the set of isolated vertices of $G$. If at least one of the following three conditions holds, then $G \lor H$ is not realisable by an orthogonal symmetric matrix (i.e., $q(G \lor H) \neq 2$).

(a) $|G| < \ell$;
(b) $\text{iso}(G) \neq \emptyset$ and $k + \ell > |G| + |\text{iso}(G)|$;
(c) $\text{iso}(G) = \emptyset$ and $\text{iso}(H) \neq \emptyset$, and $k \leq \ell < 2k$ and $|H| < 2k$ and $|G| < k + \ell$.

Moreover, if the connected components of $G$ and $H$ are all complete graphs, then

$$q(G \lor H) = \begin{cases} 2 & \text{if none of conditions (a), (b), (c) is satisfied,} \\ 3 & \text{if at least one of (a), (b), (c) is satisfied.} \end{cases}$$
1.3. Notation. We denote by $N = \{1, 2, \ldots\}$ and $N_0 = \{0\} \cup N$ the sets of positive and nonnegative integers. Moreover, let $[n] = \{1, 2, \ldots, n\}$ and $m + [n] = \{m + 1, m + 2, \ldots, m + n\}$.

For any set $S$, we denote by $S^{m \times n}$ the set of $m \times n$ matrices with entries in $S$, and let $S^n$ denote $S^{1 \times m}$, depending on the context. For a matrix $X \in \mathbb{R}^{m \times n}$, the transpose of $X$ is denoted by $X^\top$, and we write $X > 0$ if every entry of $X$ is greater than 0. Similarly, for matrices $X = (x_{ij})$ and $Y = (y_{ij})$ in $\mathbb{R}^{m \times n}$, we write $X \geq Y$ if $x_{ij} \geq y_{ij}$ for all $i \in [m], j \in [n]$. We say that a matrix $X$ is nowhere-zero if no entry of $X$ is zero. We write $X \neq 0$ if at least one entry of $X$ is nonzero.

The following notation is used for special vectors and matrices: $1_m$ is the column vector of ones in $\mathbb{R}^{m \times 1}$, $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)^\top$ is the vector with the 1 in the $i$th entry and zeros elsewhere, $I_k$ is the $k \times k$ identity matrix, $0_{m \times n}$ is the zero $m \times n$ matrix, and we also write $0_m := 0_{m \times m}$ and $0_m := 0_{m \times 1}$. (We allow any of $k, m, n$ to be zero, in which case the corresponding matrix is empty.)

If $X \in \mathbb{R}^{m \times n}$ and $R \subseteq [m], C \subseteq [n]$, then $X[R, C]$ is the submatrix of $X$ with rows $R$ and columns $C$. Let $\bigoplus_{i \in [r]} A_i$ denote the direct sum of matrices $A_1, \ldots, A_r$, where, for technical reasons, we allow for the possibility that $A_i$ is an empty matrix. We denote the diagonal matrix with diagonal entries $\Lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r$ occurring with multiplicities $v = (v_1, \ldots, v_r)^\top \in \mathbb{N}_0^r$ by $D_{\Lambda, v} := \bigoplus_{i \in [r]} \lambda_i I_{v_i}$. Since $\Lambda$ lists the eigenvalues of this matrix (ignoring multiplicities), we sometimes refer to $\Lambda$ as an eigenvalue list. In the case $v = 1_r$ (i.e., all multiplicities are equal to one), we abbreviate this diagonal matrix as $\text{diag}(\lambda_1, \ldots, \lambda_r)$.

Let $\sigma(A)$ denote the multiset of eigenvalues of a square matrix $A$, counted with algebraic multiplicities, and let $q(A)$ denote the number of distinct eigenvalues of $A$. By $\text{mult}(\lambda, A) \in \mathbb{N}_0$ we denote the multiplicity of a real number $\lambda$ in $\sigma(A)$.

In this paper, all graphs are simple undirected graphs with a nonempty vertex sets. For a graph $G = (V(G), E(G))$, the order of $G$ is denoted by $|G| := |V(G)|$. A connected component of $G$ is a maximal subgraph of $G$ in which any two vertices are connected via a path. The set of isolated vertices of $G$, i.e., the set of vertices of degree zero, is denoted by $\text{iso}(G)$. The join $G \vee H$ of two graphs $G$ and $H$ is the disjoint graph union $G \cup H$ together with all the possible edges joining the vertices in $G$ to the vertices in $H$. We abbreviate the disjoint graph union of $k$ copies of the same graph $G$ by $kG := G \cup \cdots \cup G$. We write $P_n$, $C_n$ and $K_n$ for the path, the cycle and the complete graph on $n$ vertices, respectively, and we denote the complete bipartite graph on two disjoint sets of cardinalities $m$ and $n$ by $K_{m,n} := mK_1 \vee nK_1$.

Recall that for a graph $G$ of order $n$, with $V(G)$ identified with $[n]$, we write

$$S(G) := \{A = (a_{ij}) \in \mathbb{R}^{n \times n} : a_{ij} \neq 0 \iff \{i, j\} \in E(G) \text{ for } i \neq j\}.$$
and \(q(G) := \min\{q(A) : A \in S(G)\}\). For \(A \in S(G)\), the matrix \(A[H]\) is the principal submatrix of \(A\) whose rows and columns are the vertices of a subgraph \(H \subseteq G\).

2. Compatible multiplicity matrices

Below we introduce the notion of compatible multiplicity matrices, which will give a necessary condition for \(q(G \lor H) = 2\).

**Definition 2.1.** Let \(G\) be a connected graph, \(n := |G|\) and \(r \in \mathbb{N}\). We call a vector \(v = (v_1 \ v_2 \ \ldots \ v_r)\) \(\in \mathbb{N}_0^r\) a multiplicity vector for \(G\) if \(v\) is an ordered multiplicity list that can be realised by a matrix in \(S(G)\). In other words, \(\sum_{i=1}^r v_i = n\) and there is an eigenvalue list \(\Lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r\) with \(\lambda_i < \lambda_{i+1}\) for \(i \in [r-1]\), and an orthogonal matrix \(U \in \mathbb{R}^{n \times n}\), so that \(U \Lambda_v U^\top \in S(G)\).

**Definition 2.2.** Let \(G\) be a graph and \(r, k, \ell \in \mathbb{N}\). We say that a matrix \(V \in \mathbb{N}_0^{r \times k}\) is a multiplicity matrix for \(G\) if \(G\) has \(k\) connected components \(G_1, \ldots, G_k\), and for \(1 \leq i \leq k\), the \(i\)th column of \(V\) is a multiplicity vector for \(G_i\).

We note for future reference that \(|G| = 1^\top v 1_k\) whenever \(V\) is an \(r \times k\) multiplicity matrix for a graph \(G\).

**Definition 2.3.** For a matrix \(X\) with at least 3 rows, we write \(\tilde{X}\) for the matrix obtained by deleting the first row and the last row of \(X\). Let \(r, k, \ell \in \mathbb{N}\) with \(r \geq 3\). Two matrices \(V \in \mathbb{N}_0^{r \times k}\) and \(W \in \mathbb{N}_0^{r \times \ell}\) are said to be compatible if \(\tilde{V} 1_k = \tilde{W} 1_\ell\) and \(\tilde{V}^\top \tilde{W} > 0\). We say that two graphs \(G, H\) have compatible multiplicity matrices if there exist compatible matrices \(V, W\) where \(V\) is a multiplicity matrix for \(G\) and \(W\) is a multiplicity matrix for \(H\).

In Theorem 2.5 we will show that compatibility of multiplicity matrices is a necessary condition for the join of two graphs to have \(q = 2\). For this, we use the following lemma on the Sylvester equation (see, e.g., [12]), whose simple proof we omit.

**Lemma 2.4.** Suppose \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\) and \(C \in \mathbb{R}^{m \times m}\) satisfy the equation \(AB - BC = 0\). If \(v\) is an eigenvector of \(C\) corresponding to the eigenvalue \(\lambda\), then \(Bv\) is either equal to zero, or is an eigenvector of \(A\) corresponding to the same eigenvalue, \(\lambda\). In particular, if \(B\) has trivial kernel, then \(\sigma(C) \subseteq \sigma(A)\) and hence \(m \leq n\).

**Theorem 2.5.** Let \(G\) and \(H\) be two graphs. If \(q(G \lor H) = 2\), then \(G\) and \(H\) have compatible multiplicity matrices.

**Proof.** Decompose \(G\) and \(H\) into their connected components as \(G = \bigcup_{i \in [k]} G_i\) and \(H = \bigcup_{j \in [\ell]} H_j\). Since \(q(G \lor H) = 2\), there is an orthogonal symmetric matrix

\[
X := \begin{pmatrix} A & B \\ B^\top & -C \end{pmatrix} \in S(G \lor H)
\]
where $A := \bigoplus_{i \in [k]} A_i$ with $A_i \in S(G_i)$, $C := \bigoplus_{j \in [\ell]} C_j$ with $C_j \in S(H_j)$, and $B$ is a nowhere-zero matrix. Since $X$ is orthogonal and symmetric, we have $X^2 = I$.

Let us have a closer look at eigenvalues of $A$ and $C$. First we note that if $Y \in \{A_1, \ldots, A_k, C_1, \ldots, C_\ell\}$, then $Y$ is a principal submatrix of the orthogonal symmetric matrix $X$, so every eigenvalue of $Y$ is in the closed interval $[-1, 1]$. Moreover, at least one eigenvalue of $Y$ is in the open interval $(-1, 1)$; for otherwise, $Y$ is orthogonal and from $X^2 = I$ it follows easily that some row or column of $B$ must be zero, contrary to hypothesis.

Therefore $A \oplus C$ has at least one eigenvalue in the interval $(-1, 1)$. Write the distinct eigenvalues of $A \oplus C$ in $(-1, 1)$ as $\lambda_2 < \cdots < \lambda_{r-1}$, where $r \in \mathbb{N}$ with $r \geq 3$, and define $\lambda_1 = -1$ and $\lambda_r = 1$. Let $V$ be the multiplicity matrix for $G$ realised by $A$, i.e., $V = (v_{si})_{s,i} \in \mathbb{N}^{r \times k}$ where $v_{si} := \text{mult}(\lambda_s, A_i)$ for $s \in [r]$ and $i \in [k]$. Similarly, let $W = (w_{sj})_{s,j} \in \mathbb{N}^{r \times \ell}$ be the multiplicity matrix for $H$ realised by $C$, with $w_{sj} := \text{mult}(\lambda_s, C_j)$. We proceed to show that $V$ and $W$ are compatible.

Let $U_A := \bigoplus_{i \in [k]} U_{A_i}$ and $U_C := \bigoplus_{j \in [\ell]} U_{C_j}$ where $U_A$ and $U_C$ are orthogonal matrices which diagonalise $A_i$ and $C_i$, respectively. Consider the orthogonal symmetric matrix $X' := (U_A \oplus U_C)^\top X (U_A \oplus U_C)$. Since $X'$ is orthogonal, each of its rows and columns is a unit vector. In particular, any row or column of $X'$ with $\pm 1$ on the diagonal has every other entry equal to zero. Deleting all such rows and columns, we obtain an orthogonal symmetric matrix $X''$ of the form $X'' := \begin{pmatrix} D_G & F \\ F^\top & -D_H \end{pmatrix}$, where $D_G$ and $D_H$ are diagonal. Note that the diagonal entries of $D_G$ and $D_H$ are precisely the numbers $\lambda_s$ for $s = 2, \ldots, r-1$, where $\text{mult}(\lambda_s, D_G) = (\tilde{V}1_k)_s$, and $\text{mult}(\lambda_s, D_H) = (\tilde{W}1_\ell)_s$. Since $-1 < \lambda_s < 1$ for $s = 2, \ldots, r-1$, the matrix $F^\top F = I - D_H^{-1}$ is invertible, and hence $F$ has trivial kernel. The identity $(X'')^2 = I$ also yields $D_G F - F D_H = 0$, so by Lemma 2.4 we have $\sigma(D_H) \subseteq \sigma(D_G)$. By symmetry, $F^\top$ also has trivial kernel, allowing us to conclude $\sigma(D_H) = \sigma(D_G)$. Hence, $\tilde{V}1_k = \tilde{W}1_\ell$.

It only remains to show that $\tilde{V}^\top \tilde{W} > 0$. Let $D_{G_i}$ be the restriction of $D_G$ to the vertices in $G_i$ which survived the earlier deletion, and define $D_{H_j}$ similarly. By our earlier observations, these are all non-empty matrices. Let $F = (F_{ij})$ be the block partition of $F$ compatible with $\bigoplus_{i \in [k]} D_{G_i}$ and $\bigoplus_{j \in [\ell]} D_{H_j}$. From $(X'')^2 = I$, we conclude that $D_{G_i} F_{ij} - F_{ij} D_{H_j} = 0$. If $\sigma(D_{G_i}) \cap \sigma(D_{H_j}) = \emptyset$, then Lemma 2.3 shows that the Sylvester equation $D_{G_i} Y - Y D_{H_j} = 0$ has only the trivial solution $Y = 0$, so $F_{ij} = 0$. Tracing back, this implies that the $(i, j)$th block of $B$ is also zero, which is a contradiction. Therefore $D_{G_i}$ and $D_{H_j}$ have a common eigenvalue $\lambda_s$ for at least one $s \in \{2, \ldots, r-1\}$, for all pairs $i$ and $j$. By construction, $A_i$ and $C_j$ have the same property, so $(\tilde{V}^\top \tilde{W})_{ij} = \sum_{s=2}^{r-1} \text{mult}(\lambda_s, A_i) \text{mult}(\lambda_s, C_j) > 0$. 

\hfill \Box
Example 2.6. It is straightforward to see that $2K_2$ and $3K_1$ do not have compatible multiplicity matrices, hence $q(2K_2 \vee 3K_1) \neq 2$. This gives an explicit counterexample to the claim in [1, Lemma 3.4], which was later retracted in [2]. In contrast, we will see in Section 3 that $q(2K_2 \vee 2K_1) = q(2K_2 \vee 4K_1) = 2$.

Remark 2.7. The proof of Theorem 2.5 shows that, if $G$ is connected, then $G$ has compatible multiplicity matrices.

As an application, note that if $V, W \in \mathbb{N}_0^{r \times m}$ are compatible multiplicity matrices for $G = mK_1$ and $H = mK_1$, respectively, then we must have $V = W = e_i^\top I_m$ for some $i \in \{2, \ldots, r - 1\}$. Hence, if $q(D_1 B B^\top D_2) = 2$ where $D_1, D_2, B$ are $m \times m$ matrices with $D_1, D_2$ diagonal and $B$ nowhere-zero, then we must have $D_1 = aI_m$ and $D_2 = bI_m$ for some $a, b \in \mathbb{R}$.

The authors have not been able to determine whether the converse to Theorem 2.5 is valid in full generality. In Theorem 2.11 below a partial converse is given, which requires additional hypotheses. For this purpose, we introduce the following terminology.

Definition 2.8. Let $v \in \mathbb{N}_0^r$ be a multiplicity vector for a connected graph $G$, and recall the notation of Definition 2.1. We say that $v$ is *sane* for $G$ (standing for *spectrally arbitrary with nowhere-zero eigenbases*), if, for any eigenvalue list $\Lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r$ with $\lambda_i < \lambda_{i+1}$ for $i \in [r-1]$, there is a nowhere-zero orthogonal matrix $U$ so that $UD_{\Lambda,v}U^\top \in S(G)$. If, for any finite set $\mathcal{Y} \subseteq \mathbb{R}^n \setminus \{0\}$ and any $\Lambda$, an orthogonal matrix $U$ can be chosen as above with the additional property that $UY$ is nowhere-zero for all $y \in \mathcal{Y}$, then we say that $v$ is *generically realisable* for $G$.

Definition 2.9. Let $V$ be a multiplicity matrix for a graph $G$. If every column of $V$ is sane/generically realisable for the corresponding connected component of $G$, then we say that $V$ is *sane/generically realisable* for $G$.

Definition 2.10. If every multiplicity matrix for a graph $G$ is sane for $G$, then we say that $G$ is *sane*. If every multiplicity matrix for a graph $G$ is generically realisable for $G$, then we say that $G$ is *generically realisable*.

Remark 2.11. Clearly, for a given graph $G$, any generically realisable multiplicity vector is sane, and any sane multiplicity vector is spectrally arbitrary. In fact, in some situations those concepts agree, but as we will see in Section 3, there are cases where they are different.

Sane multiplicity vectors were considered in the literature before, although not under that name. For example, it is well known that if
For \( \mathbf{v} \in \mathbb{N}_0^r \) and \( n := \mathbf{1}_r^T \mathbf{v} \geq 2 \), then \( \mathbf{v} \) is spectrally arbitrary for \( K_n \) if it has at least two positive coordinates, and that any such \( \mathbf{v} \) is sane for \( K_n \) (see, e.g., [19] Theorem 4.5). Furthermore, any multiplicity vector \( \mathbf{v} \in \{0,1\}^r \) (i.e., with all its elements equal to 0 or 1) is sane for every connected graph \( G \) of order \( 1_r^T \mathbf{v} \), by [17] Theorem 4.2.

In fact, \( K_n \) turns out to be generically realisable, as we will see in Proposition 4.4 below. In [15], we will prove that \( G \) is also an irreducible algebraic variety.

For \( q \in \mathbb{N} \), we write \( SO(q) \) for the set of \( q \times q \) special orthogonal matrices, i.e., all \( q \times q \) orthogonal matrices with determinant 1, considered as an algebraic subvariety of \( \mathbb{R}^{q^2} \). For \( \mathbf{p} = (p_1, \ldots, p_r)^T \in \mathbb{N}^r \), we write \( SO(\mathbf{p}) := SO(p_1) \times \cdots \times SO(p_r) \). We will use the elementary fact from algebraic geometry that the algebraic variety \( SO(q) \) is irreducible for any \( q \in \mathbb{N} \) (see, e.g., [11]). Since \( SO(\mathbf{p}) \) is a product of such varieties, it is also an irreducible algebraic variety.

**Lemma 2.12.** Let \( r \in \mathbb{N} \), \( \mathbf{p} = (p_1, \ldots, p_r)^T \in \mathbb{N}^r \) and for \( s \in [r] \), let \( A_s \in \mathbb{R}^{p_s \times p_s} \) with rank \( A_s = 1 \). If \( \sum_{r=1}^r \text{tr}(A_s X_s) = 0 \) for all \( (X_1, \ldots, X_r) \in SO(\mathbf{p}) \), then \( r \geq 2 \) and \( p_s = 1 \) for each \( s \in [r] \).

**Proof.** For \( s \in [r] \), write \( A_s = a_s \mathbf{v}_s \mathbf{w}_s^T \) where \( \mathbf{v}_s, \mathbf{w}_s \in \mathbb{R}^{p_s} \) are unit vectors and \( a_s \in \mathbb{R} \setminus \{0\} \). Suppose for a contradiction that \( p_s > 1 \) for some \( s \in [r] \). Then \( SO(p_s) \) acts transitively on the unit vectors in \( \mathbb{R}^{p_s} \), so there exist \( X_s, X'_s \in SO(p_s) \) with \( X_s \mathbf{v}_s = \mathbf{w}_s = -X'_s \mathbf{v}_s \). For \( t \in [r] \setminus \{s\} \), choose any \( X_t \in SO(p_t) \) and set \( X'_t = X_t \). We have \( \sum_{r=1}^r \text{tr}(A_s X_s) = 0 = \sum_{r=1}^r \text{tr}(A_s X'_s) \), and cancelling common terms gives \( \text{tr}(A_s X_s) = \text{tr}(A_s X'_s) \). This implies that \( a_s = 0 \), a contradiction. We conclude that \( p_s = 1 \) for each \( s \in [r] \).

In particular, \( SO(p_1) = \{1\} \), hence \( \text{tr}(A_1) = \pm a_1 \neq 0 \), so \( r \neq 1 \). \( \square \)

In the next result we identify a particular situation that Lemma 2.12 will eventually be applied to.

**Lemma 2.13.** Let \( k, m, n \in \mathbb{N} \). For \( s \in [k] \), let \( p_s \in \mathbb{N} \) and \( \emptyset \neq \mathcal{R}_s, \mathcal{C}_s \subseteq [p_s] \) where \( m' := \sum_{s \in [k]} |\mathcal{R}_s| \leq m \) and \( n' := \sum_{s \in [k]} |\mathcal{C}_s| \leq n \). Let \( F : \times_{s=1}^k \mathbb{R}^{p_s \times p_s} \to \mathbb{R}^{n \times m} \) be given by

\[
F(X_1, \ldots, X_k) := \begin{pmatrix}
\Theta_{\mathcal{R}_s, \mathcal{C}_s} |X_s| & 0_{m' \times (n-n')}
v_{(m-m') \times n'} & 0_{m \times (m-m')}
\end{pmatrix},
\]

Fix \( a \in [m] \), \( b \in [n] \) and invertible nowhere-zero matrices \( S \in \mathbb{R}^{n \times n} \) and \( T \in \mathbb{R}^{m \times m} \). Then there exist matrices \( A_s \in \mathbb{R}^{p_s \times p_s} \) with rank \( A_s = 1 \) for \( s \in [k] \) so that \( (S^T F(X_1, \ldots, X_k) T)_{ab} = \sum_{s \in [k]} \text{tr}(A_s X_s) \) for any \( X_1 \in \mathbb{R}^{p_1 \times p_1}, \ldots, X_k \in \mathbb{R}^{p_k \times p_k} \).

**Proof.** Let us write \( F(\mathcal{X}) = F(X_1, \ldots, X_k) \). Define \( \mathcal{E}_s := (\sum_{s \in \mathcal{R}_s} |\mathcal{R}_s|) + |\mathcal{R}_s| \subseteq [m] \) and \( \mathcal{F}_s := (\sum_{s \in \mathcal{C}_s} |\mathcal{C}_s|) + |\mathcal{C}_s| \subseteq [n] \), so that for \( s \in [k] \) we have \( X_s[\mathcal{R}_s, \mathcal{C}_s] = F(\mathcal{X})[\mathcal{E}_s, \mathcal{F}_s] \). Let \( \chi_{\mathcal{E}_s, \mathcal{F}_s} \in \mathbb{R}^{m \times n} \) denote the matrix
with ones on $E_s \times F_s$ and zeros everywhere else, and let $\circ$ denote the Hadamard product. The block structure in the definition of $F(X)$ implies that $F(X) = F(X) \circ \sum_{s \in \{k\}} \lambda \chi_{E_s \times F_s}$. Observe that the matrix $Y := (Te_a)(Se_b)^\top \in \mathbb{R}^{n \times m}$ is rank one and nowhere-zero, so any submatrix of $Y$ shares these properties. Define $A_s \in \mathbb{R}^{ps \times ps}$ by setting $A_s[C_s, R_s] := Y[F_s, \varepsilon_s]$ and defining all other entries to be zero (i.e., $(A_s)_{\alpha, \beta} = 0$ if $(\alpha, \beta) \notin C_s \times R_s$). Then rank $A_s = 1$, and we have $tr(A_s) = tr(A_n[C_s, R_s]X_s[R_s, C_s]) = tr(Y[F_s, \varepsilon_s]F(X)[E_s, F_s]) = tr(Y(F(X) \circ \chi_{E_s \times F_s})$, so

$$\sum_{s \in \{k\}} tr(A_s)X_s = \left(Y \left(F(X) \circ \sum_{s \in \{k\}} \chi_{E_s \times F_s}\right)\right) = tr(YF(X)) = tr(Te_a(Se_b)^\top F(X)) = e_b^\top S^\top F(X)Te_a = (S^\top F(X)T)_{ab}. \square$$

**Theorem 2.14.** Suppose that $G$ and $H$ have compatible multiplicity matrices $V = (v_{ij}) \in \mathbb{N}_0^{r \times k}$ and $W = (w_{ij}) \in \mathbb{N}_0^{r \times k}$. Let $p = (p_2, \ldots, p_{r-1}) := \tilde{V}_k 1_k = \tilde{W}_1 1_1$. Then $q(G \cup H) \leq 2$ if at least one of the following conditions is satisfied:

(a) Whenever $(\tilde{V}^\top \tilde{W})_{ij} \geq 2$, there exists $s \in \{2, \ldots, r-1\}$ with $v_{si}w_{sj} \neq 0$ and $p_s \geq 2$.

(b) $V$ is a generically realisable multiplicity matrix for $G$.

(c) $W$ is a generically realisable multiplicity matrix for $H$.

**Proof.** Let $G_1, \ldots, G_k$ and $H_1, \ldots, H_\ell$ be the connected components of $G$ and $H$, and let $\Lambda = (-1, \lambda_2, \ldots, \lambda_{r-1}, 1)$ where $-1 < \lambda_2 < \ldots < \lambda_{r-1} < 1$. Write $v_i$ for $i$th column of $V$ and let $D_G \in \mathbb{R}^{[G] \times [G]}$ be the diagonal matrix $D_G = \oplus_{i \in \{k\}} D_{G_i}$, where $D_{G_i} = D_{\lambda_i, v_i}$. Similarly, let $w_j$ be the $j$th column of $W$ and let $D_H \in \mathbb{R}^{[H] \times [H]}$ be the diagonal matrix $D_H = \oplus_{j \in \{k\}} D_{H_j}$, where $D_{H_j} = D_{\lambda_j, w_j}$. Note that for $s \in \{2, \ldots, r-1\}$, we have $p_s = \text{mult}(\lambda_s, D_G) = \text{mult}(\lambda_s, D_H)$.

Let $Z = (Z_i, 1_{Z_i}) \in SO(p)$. The freedom in the choice of $Z$ will be needed at a later stage. For each $s \in \{2, \ldots, r-1\}$, let $U_s \in \mathbb{R}^{2ps \times 2ps}$ be the orthogonal symmetric matrix

$$U_s := \begin{pmatrix}
\lambda_s I_{p_s} & (1 - \lambda_s^2)^{1/2}Z_s \\
(1 - \lambda_s^2)^{1/2}Z_s^\top & -\lambda_s I_{p_s}
\end{pmatrix}.$$

Note that $U_s$ depends on our choice of $Z$. Set $n_1 := e_1^\top V 1_k + e_i^\top W 1_\ell$ to be the sum of the entries in the first row of $V$ and the last row of $W$, and $n_r := e_1^\top V 1_k + e_i^\top W 1_\ell$ to be the sum of the entries in the first row of $W$ and the last row of $V$. Consider the orthogonal matrix $U_0 := (\oplus_{s \in \{r\}} U_s) \oplus (-I_n) \oplus I_n$. By construction, the diagonal of $U_0$ is a permutation of the diagonal of $D_G \oplus (-D_H)$. It follows that $U_0$ is permutation similar to an orthogonal matrix $U := \begin{pmatrix}
D_G & B(Z) \\
B(Z)^\top & -D_H
\end{pmatrix}$ under a permutation which maps the top left block of each $U_s$ to a
submatrix of $D_G$, and the bottom right block to a submatrix of $-D_H$, where $B(Z)$ is a $|G| \times |H|$ matrix which depends on $Z$. We partition $B(Z)$ as a $k \times \ell$ block matrix $B(Z) = (B_{ij}(Z))_{i \in [k], j \in [\ell]}$ with block-partition compatible with those of $D_G = \bigoplus_{i=1}^k D_G$, and $D_H = \bigoplus_{j=1}^\ell D_H$, so that each $B_{ij}(Z)$ is an $|G_i| \times |H_j|$ matrix. For $(i, j) \in [k] \times [\ell]$, let $Q(i, j) = \{ s : 2 \leq s \leq r - 1, v_{si} w_{sj} \neq 0 \}$. Since $V$ and $W$ are compatible, $(\hat{V}^r \hat{W}^r)_{ij} \neq 0$ and so $Q(i, j) \neq \emptyset$ for every $i, j$. For $2 \leq s \leq r - 1$, note that $\lambda_s$ is a diagonal value of both $D_G$ and $D_H$, if and only if $s \in Q(i, j)$. By construction, the rows and columns of $B_{ij}(Z)$ may be permuted to obtain the $|G_i| \times |H_j|$ matrix

$$B_{ij}(Z) := \begin{pmatrix} \bigoplus_{s \in Q(i, j)} (1 + \lambda_s^2) Z_s^{1/2} \mathbf{R}_{si}, \mathbf{C}_{sj} & 0 \\ 0 & 0 \end{pmatrix}$$

where for each $s \in Q(i, j)$ both $\mathbf{R}_{si} : i \in [k]$ and $\mathbf{C}_{sj} : j \in [\ell]$ are partitions of $[p_s]$, with $|\mathbf{R}_{si}| = \text{mult}(\lambda_s, D_{G_i}) = v_{si}$ and $|\mathbf{C}_{sj}| = \text{mult}(-\lambda_s, -D_{H_j}) = w_{sj}$, and where the zeros in the matrix $B_{ij}(Z)$ above represent the (possibly empty) zero matrices which pad the direct sum out to $|G_i|$ rows and $|H_j|$ columns.

Since $V$ and $W$ are sane multiplicity matrices for $G$ and $H$, respectively, there exist nowhere-zero orthogonal matrices $S_i, T_j$ for $i \in [k], j \in [\ell]$, such that $S_i D_{G_i} S_i^T \in S(G_i)$ and $T_j D_{H_j} T_j^T \in S(H_j)$. Let $S := (S_1, \ldots, S_k)$ and $T := (T_1, \ldots, T_\ell)$, and note that $S$ and $T$ can typically be chosen in several different ways. The matrix $R := (\bigoplus_{i=1}^k S_i) \oplus (\bigoplus_{j=1}^\ell T_j)$ is orthogonal and thus $R U R^T = \begin{pmatrix} A_G & C(Z) \\ C(Z)^T & A_H \end{pmatrix}$ is an orthogonal symmetric matrix with $A_G \in S(G)$ and $A_H \in S(H)$. Hence, provided the matrix $C(Z) = (\bigoplus_{i=1}^k S_i) B(Z) (\bigoplus_{j=1}^\ell T_j)^T$ has no zero elements we will have $R U R^T \in S(G \vee H)$ and so $q(G \vee H) \leq 2$, as required. To analyse when this happens, we will now refer back to the freedom we have in choosing $Z$, $S$ and $T$.

Note that the matrix $C(Z)$ inherits the block-partition of $B(Z)$, i.e., $C(Z)$ is a $k \times \ell$ block matrix with $|G_i| \times |H_j|$ blocks $C_{ij}(Z) = S_i B_{ij}(Z) T_j^T$. We will first fix $S$ and $T$ arbitrarily and show that hypothesis [\ref{thm:orthogonal}] of the theorem guarantees that $C(Z)$ is nowhere-zero for some appropriate choice of $Z$.

For $i \in [k], j \in [\ell], a \in [|G_i|]$ and $b \in [|H_j|]$, consider the the linear functionals $L_{ab}(i, j) : SO(p) \to \mathbb{R}$ given by $L_{ab}(i, j)(Z) := (C_{ij}(Z))_{ab}$. If $q(G \vee H) > 2$, then $C(Z)$ has at least one entry equal to zero for any choice of $Z \in SO(p)$, so $SO(p) \subseteq \bigcup_{i,j,a,b} L_{ab}(i, j)^{-1}(0)$. Since $SO(p)$ is an irreducible algebraic variety, there exist $i_0, j_0, a_0, b_0$ so that for $L_0 := L_{a_0,b_0}(i_0, j_0)$ we have $SO(p) \subseteq L_{a_0,b_0}(i_0, j_0)^{-1}(0)$, i.e., $L_0(Z) = 0$ for $Z \in SO(p)$. On the other hand, $L_0(Z) = (S_{i_0} \Pi_1 B_{i_0,j_0}^{(i)}(Z) \Pi_2 T_{j_0}^T)_{a_0 b_0}$ where $\Pi_1, \Pi_2$ are permutation matrices with $\Pi_1 B_{i_0,j_0}^{(i)}(Z) \Pi_2 = B_{i_0,j_0}^{(i)}(Z)$. By equation (1) and Lemma \ref{lem:orthogonal}, there exist matrices $A_s \in \mathbb{R}^{p_s \times p_s}$ with
rank \( A_s = 1 \) for \( s \in Q(i_0,j_0) \) so that \( 0 = L_0(Z) = \sum_{s \in Q(i_0,j_0)} \text{tr}(A_s Z_s) \) for every \( Z \in SO(p) \). By Lemma 2.12 we have \((\tilde{V}^T \tilde{W})_{i_0,j_0} \geq |Q(i_0,j_0)| \geq 2\), and \( p_s = 1 \) for every \( s \in Q(i_0,j_0) \). On the other hand, for \( s \in \{2, \ldots, r - 1\} \setminus Q(i_0,j_0) \), we have \((\tilde{V}^T \tilde{W})_{i_0,j_0} = 0\). This is inconsistent with hypothesis \( [a] \). Hence, when \( [a] \) holds, we can find a suitable \( Z \) for which the orthogonal matrix \( RUR^T \) lies in \( S(G \vee H) \), so \( q(G \vee H) = 2 \) in this case.

Now suppose that hypothesis \( [b] \) holds. Fix any invertible nowhere-zero \( |G_i| \times |G_i| \) matrices \( \tilde{S}_i \) for \( i \in [k] \), and consider the linear functionals 

\[ \tilde{L}_{ab}(i,j) : SO(p) \to \mathbb{R} \]

given by \( \tilde{L}_{ab}(i,j)(Z) := (\tilde{S}_i B_{ij}(Z) T_j^T)_{ab} \). Suppose that for any \( Z \in SO(p) \), there exist \( i, j, b \) so that the \( b \)-th column of \( B_{ij}(Z) T_j^T \) is zero. Then \( SO(p) \subseteq \bigcup_{i,j,b} \tilde{L}_{ab}(i,j)^{-1}(0) \), and by the irreducibility of the algebraic variety \( SO(p) \) there exist some fixed \( i_0,j_0,b_0 \) so that \( SO(p) \subseteq \bigcap_{i,j,b} \tilde{L}_{ab}(i_0,j_0)^{-1}(0) \). In other words, for every \( Z \in SO(p) \), the \( b_0 \)-th column of \( \tilde{S}_{i_0} B_{i_0,j_0}(Z) T_{j_0}^T \) is zero, so by the invertibility of \( \tilde{S}_{i_0} \), the \( b_0 \)-th column of \( B_{i_0,j_0}(Z) T_{j_0}^T \) is zero. Now (taking \( a = 1 \)) observe that \( \tilde{L} := \tilde{L}_{1b_0}(i_0,j_0) \) vanishes at every \( Z \in SO(p) \). Since \( \tilde{S}_{i_0} \) is nowhere-zero, Lemma 2.13 implies that \( \tilde{L} \) may be written in the form 

\[ \tilde{L}(Z) = \sum_{s \in Q} \text{tr}(A_s Z_s), \]

where \( \emptyset \neq Q := Q(i_0,j_0) \) and \( \text{rank} A_s = 1 \) for each \( s \in Q \). By Lemma 2.12 for each \( s \in Q \) we have \( p_s = 1 \). So for every \( Z \in SO(p) \) and \( s \in Q \), we have \( Z_s \in SO(1) \), i.e., \( Z_s = 1 \). This implies that 

\[ B_{i_0,j_0}'(Z) = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \]

where \( D = \text{diag}((1 - \lambda_s^2)^{1/2} : s \in Q) \) is an invertible diagonal matrix and the 0s represent (possibly empty) zero matrices. (In particular, \( B_{i_0,j_0}'(Z) \) is independent of \( Z \).) This implies that the kernel of \( B_{i_0,j_0}'(Z) \) does not contain a nowhere-zero vector, and the same is therefore true of the permutation equivalent matrix \( B_{i_0,j_0}(Z) \). Since \( T_{j_0} \) is nowhere-zero, it follows that the \( b_0 \)-th column of \( B_{i_0,j_0}(Z) T_{j_0}^T \) is not zero, a contradiction.

Therefore it is possible to choose \( Z \in SO(p) \) so that \( B_{ij}(Z) T_j^T \) has no column equal to zero for all pairs \( i \) and \( j \). Since \( V \) is generically realisable for \( G \), we can find a \( k \)-tuple \( S = (S_1, \ldots, S_k) \) of nowhere-zero orthogonal matrices such that for every \( i \in [k] \), we have \( S_i D G, S_i^T \in S(G_i) \) and for every \( j \in [\ell] \), the matrix \( C_{ij}(Z) = S_i B_{ij}(Z) T_j^T \) is nowhere-zero. Hence, \( q(G \vee H) \leq 2 \) in this case. The argument for hypothesis \([c]\) is of course symmetric.

\[ \square \]

Theorems 2.5 and 2.14 yield:

**Corollary 2.15.** If \( G \) is generically realisable and \( H \) is sane, then \( q(G \vee H) = 2 \) if and only if \( G \) and \( H \) have a pair of compatible multiplicity matrices.
3. Remarks, applications, examples

Our principal application of Theorem 2.14 in this paper is to determine the minimum number of distinct eigenvalues of the join of two graphs, each of which is a union of complete graphs. Before turning to that in Section 4, we give some other consequences and related discussion.

3.1. Construction of orthogonal symmetric matrices. As is apparent from the proof of Theorem 2.14, several things would have to align for two graphs $G$ and $H$ with compatible multiplicity matrices to have $q(G \lor H) > 2$, and this won’t happen in any sufficiently generic situation. In other words, once two graphs have compatible multiplicity matrices, they will typically have $q(G \lor H)$ equal to 2. This potentially gives us a wealth of examples, including situations in which we cannot verify the technical hypotheses of Theorem 2.14.

In Pseudo-algorithm 3.1, we concisely summarise the construction used repeatedly in the previous section. This is illustrated by the example involving cycles below.

**Algorithm 3.1** Finding an orthogonal symmetric matrix in $S(G \lor H)$

Suppose $G$ and $H$ are graphs with $k$ and $\ell$ components, respectively, and $A_G \in S(G)$ and $A_H \in S(H)$ have compatible multiplicity matrices $V \in \mathbb{N}_{0}^{r \times k}$ and $W \in \mathbb{N}_{0}^{r \times \ell}$, with respect to $\Lambda = (-1, \lambda_2, \ldots, \lambda_{r-1}, 1)$ where $\lambda_i \in (-1, 1)$ for $i = 2, \ldots, r-1$.

Step 1. Choose orthogonal matrices $S, T$ with $S^\top A_G S = D_{\lambda, V_1k}$ and $T^\top A_H T = D_{\lambda, W_1\ell}$. Choose $Z_i \in O(p_i)$ for $i = 2, \ldots, r-1$, where $p_i = e_i^\top V_1k = e_i^\top W_1\ell$.

Step 2. Let $C := S \left( 0_{e_i^\top V_1k \times e_i^\top W_1\ell} \oplus (\oplus_{s=2}^{r-1} (1 - \lambda_s^2)^{1/2} Z_s) \oplus 0_{e_i^\top V_1k \times e_i^\top W_1\ell} \right) T^\top$.

Step 3. If $C$ is nowhere-zero, return $X := \begin{pmatrix} A_G & C \\ C^\top & -A_H \end{pmatrix}$. Otherwise, go to Step 1, making a different choice of orthogonal matrices $S$, $T$, and $Z_i$, if possible.

Of course, we have no guarantee that this procedure will terminate; this is why the technical hypotheses of Theorem 2.14 were imposed.

**Example 3.1.** Let $G = C_8$ and $H = C_4$. To construct an orthogonal symmetric matrix $X \in S(G \lor H)$, let $v = (2, 2, 2, 2)^\top$ and $w = (0, 2, 2, 0)^\top$. These are compatible multiplicity vectors for $G$ and $H$, respectively; for example,

$$A_G := \frac{1}{\sqrt{2+\sqrt{2}}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in S(G), \quad A_H := \frac{1}{2+\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in S(H)$$
have these multiplicities with respect to the eigenvalue list \( \Lambda = (-1, -\lambda, \lambda, 1) \), where \( \lambda = \sqrt{2} - 1 \). We execute Pseudo-algorithm 3.1 using a computer algebra system. With a little trial and error, it turns out that we can do this in exact arithmetic with \( Z_2 = I_2 \) and \( Z_3 = \frac{1}{3} (\frac{3}{4} - 3) \), to obtain the orthogonal symmetric matrix \( X = \begin{pmatrix} A_G & C \\ C^T & -A_H \end{pmatrix} \in S(G \vee H) \), where for \( \alpha(r) := \sqrt{3 + \frac{r^2}{2}} \), we have

\[
C := \sqrt{\frac{\lambda}{10}} \begin{pmatrix}
\frac{3\sqrt{2}}{2} & -1 & \frac{6\sqrt{2}}{2} & 3 \\
-\alpha(1) & 3\alpha(-7) & -\alpha(-1) & 3\alpha(7) \\
-9 & -\sqrt{2} & -3 & -2\sqrt{2} \\
3\alpha(7) & -3\alpha(1) & -\alpha(-7) & -3\alpha(-1) \\
-\alpha(-1) & 3\alpha(7) & \alpha(1) & -3\alpha(-7) \\
-3 & -2\sqrt{2} & 9 & \sqrt{2} \\
-\alpha(-7) & -3\alpha(-1) & -\alpha(7) & 3\alpha(1)
\end{pmatrix}
\approx \begin{pmatrix}
0.27 & -0.064 & 0.55 & 0.19 \\
-0.15 & 0.043 & -0.13 & 0.61 \\
-0.58 & -0.091 & -0.19 & -0.18 \\
0.20 & -0.46 & -0.014 & -0.49 \\
0.55 & 0.19 & -0.27 & 0.064 \\
-0.13 & 0.61 & 0.15 & -0.043 \\
-0.19 & -0.18 & 0.58 & 0.091 \\
-0.014 & -0.40 & -0.20 & 0.46
\end{pmatrix}.
\]

3.2. Some families of graphs realisable by an orthogonal symmetric matrix. In [15] Theorem 5.2] the authors proved that \( q(G \vee H) = 2 \) if \( G \) and \( H \) are connected graphs with \( |G| = |H| \). We can use Theorem 2.14(a) to extend their result to more than one connected component. (In fact, the statement below is also valid for \( k = 1 \) [15].)

**Theorem 3.2.** If \( G \) and \( H \) are graphs each having \( k \geq 2 \) connected components, and there is \( n \in \mathbb{N} \) so that the order of every one of these connected components is either \( n \), \( n + 1 \) or \( n + 2 \), then \( q(G \vee H) = 2 \).

**Proof.** Let \( G_1, \ldots, G_k \) and \( H_1, \ldots, H_k \) be the connected components of \( G \) and \( H \), respectively. Consider the matrices

\[
V := \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^k \end{pmatrix} \in \mathbb{N}_0^{(n+2) \times k} \quad \text{and} \quad W := \begin{pmatrix} w^1 \\ w^2 \\ \vdots \\ w^k \end{pmatrix} \in \mathbb{N}_0^{(n+2) \times k},
\]

where \( E := 1_n 1_k^T \in \mathbb{R}^{n \times k} \) is the matrix full of ones and \( v = (v_i)_{i \in [k]} \), \( v' = (v_i')_{i \in [k]} \), \( w = (w_i)_{i \in [k]} \) and \( w' = (w_i')_{i \in [k]} \) are all 0-1 vectors in \( \{0, 1\}^k \), such that \( v_i + v_i' = |G_i| - n \) and \( w_i + w_i' = |H_i| - n \) for \( i \in [k] \). Note that \( V \) and \( W \) are sane multiplicity matrices for \( G \) and \( H \), respectively, by Remark 2.11. Moreover, \( V \) and \( W \) are compatible since \( \bar{V} = \bar{W} \) and \( \bar{V}^T \bar{W} = n 1_k 1_k^T > 0 \). Since \( k \geq 2 \), Theorem 2.14(a) implies that \( q(G \vee H) = 2 \).

It is easy to see that a complete graph is generically realisable (see Proposition 4.1 below for details). This fact together with Theorem 2.14(c) allows us to extend [11] Lemmas 3.13 and 3.14, where the authors proved that if \( G \) is a connected graph of order \( \ell \) or \( \ell + 1 \), then \( q(G \vee \ell K_1) = 2 \).

**Theorem 3.3.** If \( G \) is a connected graph of order \( m \in \{\ell, \ell + 1, \ell + 2\} \) and \( n_1, \ldots, n_\ell \in \mathbb{N} \), then \( q(G \vee \bigcup_{j \in [\ell]} K_{n_j}) = 2 \).
Proof. Let us define

\[ V := \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_r \end{pmatrix} \in \mathbb{N}_0^{r+2} \quad \text{and} \quad W := \begin{pmatrix} n^r \\ I_r \\ 0 \end{pmatrix} \in \mathbb{N}_0^{(r+2)\times r}, \]

where \( \varepsilon', \varepsilon'' \in \{0, 1\} \) and \( n = (n_1 - 1 \ \ldots \ \ n_r - 1)^\top \in \mathbb{N}_0^r \). By Remark 2.11, \( V \) is a sane multiplicity matrix for \( G \), and by Proposition 4.1, \( W \) is a generically realisable multiplicity matrix for \( \bigcup_{\ell \in [\ell]} K_{n_\ell} \). Since \( \tilde{V} = \tilde{W} 1_\ell = 1_\ell \) and \( \tilde{V}^\top \tilde{W} = 1_\ell^\top \mathbb{Z} \), \( V \) and \( W \) are compatible. It follows by Theorem 2.11(e) that \( \eta(G \cup \bigcup_{\ell \in [\ell]} K_{n_\ell}) = 2 \).

3.3. Generic realisability and sanity. In the remainder of this section, we explore the relationship between the sane and the generically realisable multiplicity vectors. First, we show that provided its minimum multiplicity is greater than one, any sane multiplicity vector is automatically generically realisable.

Proposition 3.4. If \( \mathbf{v} \in \mathbb{N}_0^n \) is a sane multiplicity vector for \( G \) with no entry equal to 1, then \( \mathbf{v} \) is also generically realisable for \( G \).

Proof. Generic realisability for \( G \) is not changed by inserting or deleting zeros from \( \mathbf{v} = (v_1 \ v_2 \ \ldots \ v_r)^\top \), so we may assume that \( v_i \geq 2 \) for all \( i \in [r] \). Let \( \Lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r \) with \( \lambda_i < \lambda_{i+1} \) for \( i \in [r-1] \). Since \( \mathbf{v} \) is sane for \( G \), there is a nowhere-zero orthogonal matrix \( U \in \mathbb{R}^{n \times n} \) where \( n = |G| \) with \( UD_{\Lambda}U^\top \in S(G) \). Note that for \( \mathcal{X} = (X_1, \ldots, X_r) \in SO(\mathbf{v}) \), we have \( (\oplus_{i=1}^r X_i)D_{\Lambda, U}(\oplus_{i=1}^r X_i^\top) = D_{\Lambda, U} \), hence \( U \) can be replaced by \( U(\oplus_{i=1}^r X_i) \).

Let \( \mathcal{Y} \subseteq \mathbb{R}^n \setminus \{0\} \) be finite. For every \( \mathbf{y} \in \mathcal{Y} \), and \( j \in [n] \), we define linear functionals:

\[
L(j, \mathbf{y}) : SO(\mathbf{v}) \rightarrow \mathbb{R}
\]

\[
L(j, \mathbf{y})(\mathcal{X}) := (U(\oplus_{i=1}^r X_i)\mathbf{y})_j.
\]

If \( \mathbf{v} \) is not generically realisable, then there exists \( U \) as above, and some finite set of nonzero vectors \( \mathcal{Y} \) so that:

\[
SO(\mathbf{v}) \subseteq \bigcup_{j \in [n], \mathbf{y} \in \mathcal{Y}} L(j, \mathbf{y})^{-1}(0).
\]

Irreducibility of the algebraic variety \( SO(\mathbf{v}) \) guarantees the existence of a fixed \( j_0 \in [n] \) and \( \mathbf{y}_0 \in \mathcal{Y} \) so that \( SO(\mathbf{v}) \subseteq L(j_0, \mathbf{y}_0)^{-1}(0) \). For \( \mathcal{X} = (X_1, \ldots, X_r) \in SO(\mathbf{v}) \), we can write:

\[
L(j_0, \mathbf{y}_0)(\mathcal{X}) = \text{tr} \left((\mathbf{y}_0 \mathbf{e}_{j_0}^\top U)(\oplus_{i=1}^r X_i)\right).
\]

Let \( R_i := [0_{n_1 \times (v_1 + \ldots + v_{i-1})} I_{v_i} 0_{n_i \times (v_{i+1} + \ldots + v_r)}] \), \( \mathbf{x}_i := R_i \mathbf{y}_0 \), \( \mathbf{u}_i := R_i U^\top \mathbf{e}_{j_0} \), and \( A_i := \mathbf{x}_i \mathbf{u}_i^\top \). Since \( U \) is nowhere-zero, we have \( \mathbf{u}_i \neq 0 \) for every \( i \in [r] \), and \( \mathbf{y}_0 \neq 0 \), so the set \( Q = \{i \in [r] : \mathbf{x}_i \neq 0\} \) is non-empty, and \( L(j_0, \mathbf{y}_0)(\mathcal{X}) = \sum_{i \in Q} \text{tr}(A_i X_i) \) with rank \( A_i = 1 \) for all \( i \in Q \). Since \( v_i \geq 2 \) for all \( i \), this contradicts Lemma 2.12. \( \square \)
Our next task is to find an example of a sane multiplicity vector which is not generically realisable. First, a simple lemma.

**Lemma 3.5.** If \( \mathcal{V} \) is a subspace of \( \mathbb{R}^n \) which contains a nowhere-zero vector, then there is a nowhere-zero orthonormal basis for \( \mathcal{V} \).

**Proof.** The case \( \dim \mathcal{V} = 1 \) is trivial. Assume inductively that \( r = \dim \mathcal{V} \geq 2 \) and \( \mathbf{b}_1, \ldots, \mathbf{b}_r \) is an orthonormal basis of \( \mathcal{V} \) for which \( \mathbf{b}_1, \ldots, \mathbf{b}_{r-1} \) are nowhere-zero. For \( 0 < t < 1 \), replace \( \mathbf{b}_{r-1} \) with \( \mathbf{b}'_{r-1} = \sqrt{1-t^2} \mathbf{b}_{r-1} + t \mathbf{b}_r \) and replace \( \mathbf{b}_r \) with \( \mathbf{b}'_r = t \mathbf{b}_{r-1} + \sqrt{1-t^2} \mathbf{b}_r \). Since \( \mathbf{b}_{r-1} \) is nowhere-zero, for sufficiently small \( t > 0 \) the vectors \( \mathbf{b}'_{r-1} \) and \( \mathbf{b}'_r \) are nowhere-zero. \( \square \)

**Example 3.6.** We claim that \( \mathbf{v} = \begin{pmatrix} 1 & m & 1 \end{pmatrix}^\top \in \mathbb{N}_0^3, \, m \geq 3 \), is sane, but not generically realisable for \( K_2 \vee mK_1 \), showing that the two concepts are indeed different.

In fact, \( \mathbf{v} \) is not generically realisable for \( K_2 \vee mK_1 \) with respect to any list \( \Lambda \) of three distinct eigenvalues. To see this, it suffices by translation and scaling to take \( \Lambda = (-1, 0, \lambda) \) where \( \lambda > 0 \). So, let \( X = \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix} \in S(K_2 \vee mK_1) \) be a rank two matrix with simple eigenvalues \(-1, \lambda, \) where \( A \in S(K_2) \) and \( D \) is diagonal. Since rank \( X = 2 \) and \( m > 2 \), at least one diagonal entry of \( D \) is zero. Further, \( B \) is nowhere-zero and rank \( X = 2 \), and this in turn implies that \( D = 0 \). Considering the spectral decomposition \( X = \lambda \mathbf{x} \mathbf{x}^\top - \mathbf{w} \mathbf{w}^\top \) where \( \mathbf{x}, \mathbf{w} \) are orthogonal unit eigenvectors with eigenvalues \( \lambda, -1 \), we write

\[
\mathbf{x} = \begin{pmatrix} \sqrt{1-s^2} \mathbf{x}_1 \\ sx_2 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} \sqrt{1-t^2} \mathbf{w}_1 \\ tw_2 \end{pmatrix}
\]

where \( 0 \leq s, t \leq 1 \) and \( \mathbf{x}_i, \mathbf{w}_i \) are unit vectors with \( \mathbf{x}_1, \mathbf{w}_1 \in \mathbb{R}^2 \) and \( \mathbf{x}_2, \mathbf{w}_2 \in \mathbb{R}^m \). Now \( D = \lambda s^2 \mathbf{x}_2 \mathbf{x}_2^\top - t^2 \mathbf{w}_2 \mathbf{w}_2^\top = 0 \), so \( t = \sqrt{\lambda} s = 0 \), \( s \leq \lambda^{-1/2} \) and \( \mathbf{w}_2 = c \mathbf{x}_2 \) for some \( c \in \{-1, 1\} \), so we have

\[
\mathbf{w} = \begin{pmatrix} \sqrt{1-\lambda s^2} \mathbf{w}_1 \\ c \sqrt{\lambda s} \mathbf{x}_2 \end{pmatrix}.
\]

To see that \( \mathbf{v} \) is not generically realisable for \( K_2 \vee mK_1 \), consider the set \( \mathcal{Y}' := \{ \mathbf{y}_{-1}, \mathbf{y}_1 \} \), where \( \mathbf{y}_k := e_1 + k \sqrt{\lambda} e_{m+2} \) for \( k \in \{-1, 1\} \). Any orthogonal matrix \( U \) with \( U^\top UX = D \) for \( \mathbf{y}_{-1}, \mathbf{y}_1 \) has first and last columns equal to \( a \mathbf{w} \) and \( b \mathbf{x} \), respectively, for some \( a, b \in \{-1, 1\} \). Removing the first two entries of the vector \( U \mathbf{y}_k = a \mathbf{w} + bk \sqrt{\lambda} \mathbf{x} \) leaves \( (ac + bk) \sqrt{\lambda} s \mathbf{x}_2 = 0_m \) for \( k = -abc \), so \( U \mathbf{y}_{-abc} \) is not nowhere-zero. This proves that \( \mathbf{v} \) is not generically realisable for \( K_2 \vee mK_1 \).

To prove that \( \mathbf{v} \) is sane for \( K_2 \vee mK_1 \), it suffices for each \( \lambda > 0 \) to construct a matrix in \( S(K_2 \vee mK_1) \) with a nowhere-zero orthonormal eigenbasis, and eigenvalues \( \lambda \) and \(-1 \) with multiplicity 1, and 0 with multiplicity \( m \). To deal with the \( \lambda \neq 1 \) case, consider the unit vectors...
that the spectra of matrices in orthonormal eigenbasis, by straightforward calculation.

The rank two matrix \( X := \lambda x x^\top - w w^\top \) is then given by

\[
X = \begin{pmatrix}
(\lambda - 1)x_1 x_1^\top & \sqrt{\lambda} x_1 x_2^\top \\
\sqrt{\lambda} x_2 x_1^\top & 0_m
\end{pmatrix} \in S(K_2 \vee mK_1).
\]

It is easy to find a nowhere-zero vector orthogonal to both \( x \) and \( w \), so by Lemma 3.5 the kernel of \( X \) has a nowhere-zero orthonormal basis. Appending \( w \) and \( x \) gives a nowhere-zero orthonormal eigenbasis for \( X \), as required.

The final case, when \( \lambda = 1 \), can be resolved by the matrix \( X := \begin{pmatrix} A & B \\ B^\top & 0_m \end{pmatrix} \) where \( A = \frac{1}{20} \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix} \) and \( B = \frac{\sqrt{2}}{4 \sqrt{m}} \begin{pmatrix} 1_m \\ 2 \cdot 1_m \end{pmatrix} \).

We can check that \( X \) has rank 2, nonzero eigenvalues 1 and \(-1\), and a nowhere-zero orthonormal eigenbasis, by straightforward calculation.

4. JOINS OF UNIONS OF COMPLETE GRAPHS

Proposition 4.1 together with the solution to the IEP-G for complete graphs, emphasises that the spectra of matrices in \( S(K_n) \) are the least constrained. We exploit this fact to determine \( q \) of joins of unions of complete graphs, and derive some conditions for \( q \neq 2 \) in a more general setting.

**Proposition 4.1.** For \( n \in \mathbb{N} \), the complete graph \( K_n \) is generically realisable.

**Proof.** This is trivial for \( n = 1 \). Let \( n \geq 2 \) and recall (Remark 2.11) that any multiplicity list with sum \( n \) and at least two nonzero entries is spectrally arbitrary for \( K_n \). Hence, if \( D \) is any \( n \times n \) diagonal matrix with \( D \neq a I_n \), \( a \in \mathbb{R} \), then there is an orthogonal matrix \( U \) so that \( A := U D U^\top \in S(K_n) \). For any finite set \( \mathcal{Y} \subseteq \mathbb{R}^n \setminus \{0\} \) we can find an orthogonal \( V \) arbitrarily close to \( I_n \) so that \( V U \mathcal{Y} \) is nowhere-zero for every \( y \in \mathcal{Y} \). Since \( A \in S(K_n) \), for \( V \) sufficiently close to \( I_n \), we also have \( V A V^\top \in S(K_n) \). Hence, the multiplicity list of \( D \) is generically realisable for \( K_n \). \( \square \)

We now introduce some notation. Assume throughout this section that \( k, \ell \geq 1 \) are positive integers. Let \( \mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k \) denote a \( k \)-tuple of natural numbers. We abbreviate a tuple such as \( (m_1, \ldots, m_s, a, \ldots, a) \), with \( a \) appearing \( t \) times, as \( (m_1, \ldots, m_s, a^t) \). We define \( K_\mathbf{m} := \bigcup_{i \in [k]} K_{m_i} \), and \( |\mathbf{m}| := \sum_{i \in [k]} m_i = |K_\mathbf{m}| \). Clearly, \( k \leq |\mathbf{m}| \). Recall that \( \text{iso}(K_\mathbf{m}) \) denotes the set of isolated vertices of the graph \( K_\mathbf{m} \), and let \( \iota(\mathbf{m}) = |\{i \in [k] : m_i = 1\}| = |\text{iso}(K_\mathbf{m})| \). Note that \( |\mathbf{m}| + \iota(\mathbf{m}) \geq 2k \). In particular, if \( \iota(\mathbf{m}) = 0 \), then \( |\mathbf{m}| \geq 2k \).
Next, we wish to prove that the number of distinct eigenvalues of a join of two unions of complete graphs can only be equal to 2 or to 3.

To see this, we will use the following simple lemma, which follows from [2] Theorem 3 or [16] Lemma 2.9.

**Lemma 4.2.** Let $G$ be a graph with at least one edge, and let $v \in V(G)$. Let $H$ be the graph obtained from $G$ by replacing $v$ with a complete graph $K_n$ and adding edges from every neighbour of $v$ in $G$ to every vertex of $K_n$. Then $q(H) \leq q(G)$.

**Corollary 4.3.** Let $m, m' \in \mathbb{N}^k$, $n, n' \in \mathbb{N}^\ell$, where $n' \geq n$ and $m' \geq m$. If $q(K_m \vee K_n) = 2$, then $q(K_{m'} \vee K_{n'}) = 2$.

**Proof.** The graph $K_{m'} \vee K_{n'}$ may be obtained from $K_m \vee K_n$ by choosing a set of $k+\ell$ vertices (one from each component of $K_m$ and one from each component of $K_n$) and applying the procedure described in Lemma 4.2 successively to each one. 

**Corollary 4.4.** If $m \in \mathbb{N}^k$ and $n \in \mathbb{N}^\ell$, then

$$q(K_m \vee K_n) = \begin{cases} 2 & \text{if } K_m \text{ and } K_n \text{ have compatible multiplicity matrices,} \\ 3 & \text{otherwise.} \end{cases}$$

**Proof.** Since $K_m \vee K_n$ has at least one edge, $q(K_m \vee K_n) \geq 2$. Moreover, $q(kK_1 \vee \ell K_1) = q(K_{k,\ell}) \in \{2, 3\}$, by [3] Corollary 6.5. Applying Lemma 4.2 a total of $k + \ell$ times, we obtain $2 \leq q(K_m \vee K_n) \leq q(kK_1 \vee \ell K_1) \leq 3$, so $q(K_m \vee K_n) \in \{2, 3\}$.

By Proposition 4.1, $K_m$ and $K_n$ are generically realisable (and, hence, they are also sane). By Corollary 2.15, $q(K_m \vee K_n) = 2$ if and only if $K_m$ and $K_n$ have compatible multiplicity matrices. 

**Remark 4.5.** Corollary 4.3 can also be deduced easily from Corollary 1.4 by considering multiplicity matrices. Given $m, m', n, n'$ as in Corollary 4.3, suppose $V \in \mathbb{N}_0^{r \times k}$ and $W \in \mathbb{N}_0^{r \times \ell}$ are compatible multiplicity matrices for $K_m$ and $K_n$, respectively. Then $V' := V + e_i(m' - m)^\top$ and $W' := W + e_i(n' - n)^\top$ are multiplicity matrices for $K_{m'}$ and $K_{n'}$, respectively, and $\tilde{V} = \tilde{V}'$ and $\tilde{W} = \tilde{W}'$, so $V'$ and $W'$ are compatible. Hence, if $q(K_m \vee K_n) = 2$, then $q(K_{m'} \vee K_{n'}) = 2$.

The following easy observations about multiplicity matrices are worth recording at this point.

**Proposition 4.6.** Let $k, r \in \mathbb{N}$ with $r \geq 3$.

(a) For $m \in \mathbb{N}^k$, a matrix $V \in \mathbb{N}_0^{r \times k}$ is a multiplicity matrix for $K_m$ if and only if, for every $i \in [k]$, we have $1_i^\top V e_i = m_i$, and if $m_i \geq 2$, then $V e_i$ has at least two nonzero entries.

(b) If $V$ is a multiplicity matrix for a graph $G = \bigcup_{i=1}^k G_i$, where each $G_i$ is connected, then $V$ is a multiplicity matrix for $K_m$ where $m := (|G_1|, \ldots, |G_k|)$. 

(c) If $V$ and $W$ are compatible matrices, then $\tilde{V}$ and $\tilde{W}$ have at least one nonzero entry in each column.

Corollary 4.7. Let $G = \bigcup_{i=1}^{k} G_i$, $H = \bigcup_{j=1}^{l} H_j$, $m = (m_1, \ldots, m_k)$ and $n = (n_1, \ldots, n_l)$, where $G_i$ and $H_j$ are connected graphs with $|G_i| = m_i$ and $|H_j| = n_j$. If $q(K_m \vee K_n) = 3$, then $q(G \vee H) \geq 3$.

Proof. Since $G \vee H$ has at least one edge, $q(G \vee H) \geq 2$. If $q(G \vee H) = 2$, then by Theorem 2.5 there is a pair $V, W$ of compatible multiplicity matrices for $G$ and $H$. By Proposition 4.6, these are multiplicity matrices for $K_m$ and $K_n$, respectively. Hence, by Corollary 4.4, we have $q(K_m \vee K_n) = 2$, a contradiction. $\square$

Remark 4.8. In contrast with complete graphs, paths $P_n$ are realisable by matrices $A \in S(P_n)$ with the most constrained spectrum. We will explore this in our follow up paper [15], where it will be shown that $q(\bigcup_{i=1}^{k} P_{m_i} \vee \bigcup_{j=1}^{l} P_{n_j}) = 2$ implies $q(G \vee H) = 2$, where $G$ and $H$ are as in Corollary 4.7.

The following straightforward lemma is used several times in our arguments below. The proof is left to the reader.

Lemma 4.9. If $s, t \in \mathbb{N}$ and $c \in \mathbb{N}^s$, then there exists $b \in \mathbb{N}^s$ with $|b| = t$ and $b \preceq c$ if and only if $s \leq t \leq |c|$.

Now that we have established the underlying framework, we are left with the combinatorial question of deciding under what conditions on $m$ and $n$ compatible multiplicity matrices for $K_m$ and $K_n$ exist. Our first result in this direction gives a sufficient condition.

Proposition 4.10. For $m \in \mathbb{N}^k$, $n \in \mathbb{N}^\ell$, the following conditions are equivalent:

(a) $k + \ell \leq \min\{|m| + \iota(m), |n| + \iota(n)\}$.

(b) $K_m$ and $K_n$ have a compatible pair of multiplicity matrices in $\mathbb{N}_0^{3 \times k}$ and $\mathbb{N}_0^{3 \times \ell}$, respectively.

Moreover, if either (a) or (b) is valid, then $q(K_m \vee K_n) = 2$.

Proof. Suppose (b) holds and $V \in \mathbb{N}_0^{3 \times k}$, $W \in \mathbb{N}_0^{3 \times \ell}$ are compatible multiplicity matrices for $K_m$, $K_n$, respectively. Let $p := \tilde{V}1_k = \tilde{W}1_\ell \in \mathbb{N}$. Since $\tilde{V}^T \tilde{W} > 0$, every entry of the row vectors $\tilde{V}$ and $\tilde{W}$ is at least 1, so $p \geq \max\{k, \ell\}$. Every column of $V$ corresponds to a connected component $K_{m_i}$ of $K_m$ and is of the form $(a_i, b_i, c_i)^T$ where $b_i \geq 1$ and $a_i + b_i + c_i = m_i$. If $m_i = 1$ for some $i$, then the corresponding column is necessarily equal to $(0, 1, 0)^T$. For columns with $m_i > 1$ we have $a_i + c_i \geq 1$ by Proposition 1.6, so $1 \leq b_i \leq m_i - 1$. Summing up, we obtain

$$p = \sum_{i=1}^{k} b_i \leq \iota(m) + \sum_{i=m_i > 1} (m_i - 1) = |m| + \iota(m) - k,$$

and similarly $p \leq |n| + \iota(n) - \ell$. Since $\max\{k, \ell\} \leq p$, we get (a).

Now suppose the inequality (a) holds, and without loss of generality assume $k \leq \ell$. Then $k \leq \ell \leq |m| - k + \iota(m) = \sum_{i=1}^{k} \max\{1, m_i - 1\}$,
and there exist $t_1, \ldots, t_k \in \mathbb{N}$ with $t_i \leq \max\{1, m_i - 1\}$ for $i \in [k]$ and $\sum_{i \in [k]} t_i = \ell$, by Lemma 4.9. Now

$$V := \begin{pmatrix} m_1 - t_1 & \ldots & m_k - t_k \\ t_1 & \ldots & t_k \\ 0 & \ldots & 0 \end{pmatrix} \in \mathbb{N}_0^{3 \times k} \text{ and } W := \begin{pmatrix} n_1 - 1 & \ldots & n_\ell - 1 \\ 1 & \ldots & 1 \\ 0 & \ldots & 0 \end{pmatrix} \in \mathbb{N}_0^{3 \times \ell}$$

are compatible multiplicity matrices for $K_m$ and $K_n$, respectively. Hence (b) holds.

The final claim is immediate from Corollary 4.4. □

The next result follows from [3, Theorem 4.4]. We present an alternative proof using our methods.

**Proposition 4.11.** Let $m \in \mathbb{N}^k$, $n \in \mathbb{N}^\ell$. If $|m| < \ell$, then $q(K_m \vee K_n) = 3$.

**Proof.** Otherwise, by Corollary 4.4 there exist compatible multiplicity matrices $V \in \mathbb{N}_0^{r \times k}$ and $W \in \mathbb{N}_0^{r \times \ell}$ for $K_m$ and $K_n$, respectively. Then $\overline{V^T W} > 0$, which implies that $\overline{W}$ has no zero columns, so $1_{r-2} \overline{W}^T \geq 1_\ell^T$. Since $\overline{V}^T 1_k = \overline{W}^T 1_\ell$, it follows that $\ell = 1_\ell^T 1_\ell < 1_{r-2} \overline{W}^T 1_\ell = 1_{r-2} \overline{V}^T 1_k \leq 1_\ell^T 1_k = |K_m| = |m|$, a contradiction. □

Note that there is a gap between the sufficient conditions for $q = 2$ given in Proposition 4.10 and the necessary conditions for $q = 2$ that follow from Proposition 4.11. It turns out that isolated vertices play an important role in the complete solution, and we consider different cases that can occur below.

### 4.1. No isolated vertices.

First we examine the case when at least one of $K_m$ and $K_n$ has no isolated vertices. In particular, when neither one of these graphs has an isolated vertex we will see in Proposition 4.13 that the sufficient condition in Proposition 4.11 for $q(K_m \vee K_n)$ to be 3 is also necessary.

**Lemma 4.12.** Let $k \leq \ell$, $m \in \mathbb{N}^k$ and $n \in \mathbb{N}^\ell$ with $\nu(m) = 0$. If $|n| = 2k$, then $q(K_m \vee K_n) = 2$.

**Proof.** By permuting the entries of $n$ if necessary, we may assume that $n = (n_1, \ldots, n_t, 1^t(n))$, where $t := \ell - \nu(n)$ and $n_i \geq 2$ for $i \in [t]$. Write $\nu(n) = 2a + b$ for $b \in \{0, 1\}$. From $2k = |n| = \nu(n) + \sum_{i \in [t]} n_i$, we get $k = a + \frac{1}{2}(b + \sum_{i \in [t]} n_i) \geq a + \frac{1}{2}b + t$, and hence $k \geq a + b + t$.

Denote $R := (r_1, r_2) = (k-t-a-b, k-t-a)$, $C := (c_1, \ldots, c_t) = (n_1-2, \ldots, n_t-2)$, and note that $r_1 + r_2 = \sum_{j \in [t]} c_j$. Hence, by [7, Theorem 2.1.2], there exists a matrix $Y = [y_{ij}] \in \mathbb{N}_0^{p \times q}$ with row sums equal to $R$ and column sums equal to $C$. Since $\nu(m) = 0$, we have $m_i \geq 2$ for $i \in [k]$. Define
compatible matrices \( V \in \mathbb{N}_0^{4 \times k} \) and \( W \in \mathbb{N}_0^{4 \times \ell} \) as follows:

\[
V := \begin{pmatrix}
  m_1 - 2 & \ldots & m_k - 2 \\
  1 & \ldots & 1 \\
  1 & \ldots & 1 \\
  0 & \ldots & 0
\end{pmatrix}, \quad W := \begin{pmatrix}
  0 & \ldots & 0 & 0 & 0 \\
  y_{11} + 1 & \ldots & y_{1\ell} + 1 & 1_{a+b}^T & 0 \\
  y_{21} + 1 & \ldots & y_{2\ell} + 1 & 0 & 1_a^T \\
  0 & \ldots & 0 & 0 & 0
\end{pmatrix}.
\]

By Corollary \ref{cor:4.6}, \( V \) and \( W \) are multiplicity matrices for \( K_m \) and \( K_n \), respectively, so \( q(K_m \lor K_n) = 2 \) by Corollary \ref{cor:4.4}.

**Proposition 4.13.** Let \( k \leq \ell \), \( m \in \mathbb{N}^k \) and \( n \in \mathbb{N}^\ell \) with \( \iota(m) = 0 \). If \( 2k \leq \ell \leq |m| \) or \( \ell \leq 2k \leq |n| \), then \( q(K_m \lor K_n) = 2 \).

**Proof.** Suppose first that \( 2k \leq \ell \leq |m| \). Then \( k \leq \ell - k \leq |m| - k \), and since \( \iota(m) = 0 \), we have \( m_i \geq 2 \) for every \( i \in [k] \). By Lemma \ref{lem:4.9} there exist \( r_i \in \lceil m_i - 1 \rceil \) so that \( \ell - k = \sum_{i=1}^{k} r_i \). The matrices

\[
\begin{pmatrix}
m_1 - r_1 - 1 & \ldots & m_k - r_k - 1 \\
1 & \ldots & 1 \\
r_1 & \ldots & r_k \\
0 & \ldots & 0
\end{pmatrix}
\text{ and } \begin{pmatrix}
n_1 - 1 & \ldots & n_{k+1} - 1 & \ldots & n_{\ell} - 1 \\
1 & \ldots & 1 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 1 & \ldots & 1 \\
0 & \ldots & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

are compatible multiplicity matrices for \( K_m \) and \( K_n \), respectively, hence \( q(K_m \lor K_n) = 2 \) by Corollary \ref{cor:4.4}.

Suppose instead that \( \ell \leq 2k \leq |n| \). By Lemma \ref{lem:4.9} there is a vector \( p \in \mathbb{N}^\ell \) with \( p \leq n \) and \( |p| = 2k \). By Lemma \ref{lem:4.12} we have \( q(K_m \lor K_p) = 2 \) and it follows by Corollary \ref{cor:4.3} that \( q(K_m \lor K_n) = 2 \).

Assuming \( \iota(m) = \iota(n) = 0 \), we get the first complete resolution of \( q(K_m \lor K_n) \).

**Proposition 4.14.** Let \( m \in \mathbb{N}^k \), \( n \in \mathbb{N}^\ell \). If \( \iota(m) = \iota(n) = 0 \), then

\[
q(K_m \lor K_n) = \begin{cases}
2 & \text{if } \max\{k, \ell\} \leq \min\{|m|, |n|\}, \\
3 & \text{otherwise}.
\end{cases}
\]

**Proof.** By symmetry we may assume without loss of generality that \( k \leq \ell \). Under this assumption, we must prove that \( q(K_m \lor K_n) = 2 \) if and only if \( \ell \leq |m| \). If \( |m| < \ell \), then \( q(K_m \lor K_n) = 3 \) by Proposition \ref{prop:4.11} and if \( 2k \leq \ell \leq |m| \), then \( q(K_m \lor K_n) = 2 \) by Proposition \ref{prop:4.13}.

The only remaining case is when \( \ell \leq |m| \) and \( k \leq \ell < 2k \). Then \( 1 \leq 2k - \ell \leq k \leq \ell \) and \( \iota(m) = \iota(n) = 0 \) implies that \( m_i, n_j \geq 2 \) for all \( i \in [k] \) and \( j \in [\ell] \). We define matrices \( V \in \mathbb{N}_0^{4 \times k} \) and \( W \in \mathbb{N}_0^{4 \times \ell} \) as follows:

\[
V := \begin{pmatrix}
m_1 - 2 & \ldots & m_k - 2 \\
1 & \ldots & 1 \\
1 & \ldots & 1 \\
0 & \ldots & 0
\end{pmatrix}.
\]
Since $V,W$ are compatible multiplicity matrices for $K_m$ and $K_n$, respectively, we conclude that $q(K_m \vee K_n) = 2$ by Corollary 4.4.

4.2. Unions of complete graphs with isolated vertices. When both $K_m$ and $K_n$ have isolated vertices, we have the following characterisation.

**Proposition 4.15.** Let $m \in \mathbb{N}^k$, $n \in \mathbb{N}^\ell$. If $\iota(m) > 0$ and $\iota(n) > 0$, then the following conditions are equivalent:

(a) $q(K_m \vee K_n) = 2$;
(b) $K_m$ and $K_n$ have a compatible pair of multiplicity matrices;
(c) $K_m$ and $K_n$ have a compatible pair of multiplicity matrices in $N_{0}^{3k}$ and $N_{0}^{3\ell}$, respectively;
(d) $k + \ell \leq \min\{m + \iota(m), n + \iota(n)\}$.

**Proof.** (a) and (b) are equivalent by Corollary 4.4. (c) and (d) are equivalent by Proposition 4.10 and plainly imply (a) and (b).

Suppose now (b) holds, and $V \in \mathbb{R}^{r \times k}$ and $W \in \mathbb{R}^{r \times \ell}$ are compatible multiplicity matrices for $K_m$ and $K_n$, respectively, for some $r \geq 3$. Since $\iota(m) > 0$ and $\iota(n) > 0$, the condition $\overline{V}^\top \overline{W} > 0$ implies that for some $i \in [r-2]$, $i$th row of both $\overline{V}$ and $\overline{W}$ are nowhere-zero. Define $V' \in N_{0}^{3k}$ and $W' \in N_{0}^{3\ell}$ as follows:

$$V' := \begin{pmatrix} 1_i^\top V - e_i^\top \overline{V} \\ e_i^\top \overline{V} \\ 0_k^\top \end{pmatrix}, \quad W' := \begin{pmatrix} 1_i^\top W - e_i^\top \overline{W} \\ e_i^\top \overline{W} \\ 0_\ell^\top \end{pmatrix}.$$  

Note that $V'$ has the same column sums as $V$, and if the $j$th column of $V$ has more than one non-zero entry, then the $j$th column of $V'$ has (precisely) two non-zero entries, so by Proposition 4.6, $\overline{V'}$ is a multiplicity matrix for $K_m$; similarly, $W'$ is a multiplicity matrix for $K_n$. Since $\overline{V'} = e_i^\top \overline{V}$ and $\overline{W'} = e_i^\top \overline{W}$, it follows that $V'$ and $W'$ are compatible multiplicity matrices for $K_m$ and $K_n$, so (b) implies (c).

Suppose now $k \leq \ell$ and $\iota(n) = 0$. Under this assumption we have $|n| \geq 2\ell \geq k + \ell$. Hence the sufficient condition (c) in Proposition 4.10 for $q(K_m \vee K_n) = 2$ is equivalent to $k + \ell \leq |m| + \iota(m)$. We will show that this condition is also necessary for $q(K_m \vee K_n) = 2$ in the case $\iota(m) > 0$.

**Lemma 4.16.** If $r \geq 3$ and $V \in N_{0}^{r \times k}$ is a multiplicity matrix for a graph $G$ with $k$ connected components and $m = 1_i^\top V$ is the vector of
connected component sizes of $G$, then for any $s \in [r - 2]$, we have $e_s^1 \nabla \mathbf{1}_k \leq \iota(\mathbf{m}) + |\mathbf{m}| - k$.

Proof. We have

$$e_s^1 \nabla \mathbf{1}_k = \sum_{m_s = 1} e_s^1 \nabla e_i + \sum_{m_s > 1} e_s^1 \nabla e_i \leq \iota(\mathbf{m}) + \sum (m_i - 1) = \iota(\mathbf{m}) + |\mathbf{m}| - k.$$ 

$\Box$

**Proposition 4.17.** Let $k \leq \ell$, $\mathbf{m} \in \mathbb{N}^k$ and $\mathbf{n} \in \mathbb{N}^\ell$. If $\iota(\mathbf{m}) > 0$ and $\iota(\mathbf{n}) = 0$, then

$$q(K_\mathbf{m} \vee K_\mathbf{n}) = \begin{cases} 2 & \text{if } k + \ell \leq |\mathbf{m}| + \iota(\mathbf{m}), \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Since $k \leq \ell$ and $\iota(\mathbf{n}) = 0$, we have $|\mathbf{n}| \geq 2\ell \geq k + \ell$ and thus the sufficiency of the condition $k + \ell \leq |\mathbf{m}| + \iota(\mathbf{m})$ follows by Proposition 4.10.

Conversely, if $q(K_\mathbf{m} \vee K_\mathbf{n}) = 2$, then by Corollary 4.4 there exist compatible multiplicity matrices $V \in \mathbb{N}_{r \times k}$ and $W \in \mathbb{N}_{r \times \ell}$ for $K_\mathbf{m}$ and $K_\mathbf{n}$, respectively. Since $\iota(\mathbf{m}) > 0$, some column of $\nabla V$ is equal to $e_s$, where $s \in [r - 2]$. Since $\nabla^\top W > 0$, this implies that $e_s^1 \nabla W > 0$, so $e_s^1 \nabla W \geq 1^r_\ell$. By Lemma 4.16,

$$\ell = 1^r_\ell \leq e_s^1 \nabla^\top \mathbf{1}_\ell = e_s^1 \nabla \mathbf{1}_k \leq \iota(\mathbf{m}) + |\mathbf{m}| - k,$$

so $k + \ell \leq \iota(\mathbf{m}) + |\mathbf{m}|$. $\Box$

**Proposition 4.18.** Let $k \leq \ell$, $\mathbf{m} \in \mathbb{N}^k$ and $\mathbf{n} \in \mathbb{N}^\ell$. If $\iota(\mathbf{m}) = 0$ and $\iota(\mathbf{n}) > 0$, then

$$q(K_\mathbf{m} \vee K_\mathbf{n}) = \begin{cases} 2 & \text{if } 2k \leq \ell \leq |\mathbf{m}|, \ or \ \ell \leq 2k \leq |\mathbf{n}|, \ or \ k + \ell \leq |\mathbf{m}|, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. If $2k \leq \ell \leq |\mathbf{m}|$ or $\ell \leq 2k \leq |\mathbf{n}|$ or $k + \ell \leq |\mathbf{m}|$, then $q(K_\mathbf{m} \vee K_\mathbf{n}) = 2$ by Propositions 4.10 and 4.13. If $\ell > |\mathbf{m}|$, then $q(K_\mathbf{m} \vee K_\mathbf{n}) = 3$ by Proposition 4.11.

If none of the above relations holds, then in particular, we have $|\mathbf{n}| < 2k$ and $|\mathbf{m}| < k + \ell$. If $q(K_\mathbf{m} \vee K_\mathbf{n}) = 2$, then by Corollary 4.4 there exist $r \geq 3$ and compatible multiplicity matrices $V \in \mathbb{N}_{r \times k}$ and $W \in \mathbb{N}_{r \times \ell}$ for $K_\mathbf{m}$ and $K_\mathbf{n}$, respectively. Since $\iota(\mathbf{n}) > 0$, we can assume without loss of generality that $n_\ell = 1$ and that $\nabla^\top W e_\ell = e_1 \in \mathbb{N}^{r-2}$. The condition $\nabla^\top W > 0$ now implies that $e_1^\top \nabla \geq 1^r_\ell$. Lemma 4.16 together with $\iota(\mathbf{m}) = 0$ gives $e_1^\top \nabla \mathbf{1}_\ell = e_1^\top \nabla \mathbf{1}_k \leq |\mathbf{m}| - k < \ell$. If $e_1^\top \nabla \geq 1^r_\ell$, then we obtain an immediate contradiction. So the first row of $\nabla$ contains at least one zero entry. Hence, each of the $k$ columns of $\nabla$ necessarily has at least one nonzero entry in a row other than row 1, in addition to the nonzero entry in row 1, to satisfy $\nabla^\top W > 0$. So $2k \leq 1^r_{r-2} \nabla \mathbf{1}_k = 1^r_{r-2} \nabla \mathbf{1}_\ell \leq 1_r^\top W \mathbf{1}_\ell = |\mathbf{n}|$, a contradiction. This proves $q(K_\mathbf{m} \vee K_\mathbf{n}) = 3$. $\Box$
We remark that by the constructions in Subsections 4.1 and 4.2 whenever \( q(K_m \vee K_n) = 2 \), there exist compatible multiplicity matrices for \( K_m \) and \( K_n \) with at most four rows.

4.3. Complete result. We summarise the different cases that we have considered in a theorem that completely resolves \( q(K_m \vee K_n) \), and follows immediately from Propositions 4.14, 4.15, 4.17 and 4.18.

**Theorem 4.19.** Let \( k \leq \ell, \ m \in \mathbb{N}^k \) and \( n \in \mathbb{N}^\ell \). We have \( q(K_m \vee K_n) = 2 \) if and only if one of the following is true:

(a) \( \iota(m) = 0, \ i(n) = 0 \) and \( \ell \leq |m| \),
(b) \( \iota(m) > 0 \) and \( k + \ell \leq |m| + \iota(m) \), or
(c) \( \iota(m) = 0, \ i(n) > 0 \) and either \( k + \ell \leq |m| \), or \( 2k \leq \ell \leq |m| \), or \( \ell \leq 2k \leq |n| \).

Otherwise, \( q(K_m \vee K_n) = 3 \).

**Theorem 4.14** is a straightforward consequence of Theorem 4.19 and Corollary 4.7. Note that Theorem 4.19 allows us to observe properties of \( q(K_m \vee K_n) \). For example, for any two fixed numbers of connected components \( k \) and \( \ell \), we can always choose \( m \in \mathbb{N}^k \) and \( n \in \mathbb{N}^\ell \), with \(|m|\) and \(|n|\) sufficiently large to achieve \( q(K_m \vee K_n) = 2 \).

4.4. Multiplicities. Let \( X \) be an \( n \times n \) orthogonal symmetric matrix, and let \( i_+(X) \) and \( i_-(X) \) denote the multiplicities of \( 1 \) and \(-1\) as eigenvalues of \( X \), respectively. The study of \( i_+(X) \) can be motivated by the fact that the map \( X \mapsto P(X) = \frac{1}{2}(I + X) \) is a bijection between the \( n \times n \) orthogonal symmetric matrices and the orthogonal projections onto subspaces of \( \mathbb{R}^n \), and rank \( P(X) \) is \( i_+(X) \). Moreover, for any graph \( G \), we have \( X \in S(G) \) if and only if \( P(X) \in S(G) \).

Let \( G \) and \( H \) be graphs such that \( q(G \vee H) = 2 \). Let \( X \in S(G \vee H) \) be an orthogonal matrix with the corresponding compatible multiplicity matrices \( V, W \) guaranteed by Theorem 2.5. Let \( \mu = 1_{r\leftrightarrow}^\top V 1_k + 1_{r\leftrightarrow}^\top W 1_\ell \). By the proof of Theorem 2.5, we have \( i_+(X) \geq \mu \) and \( n - i_+(X) = i_-(X) \geq \mu \), implying \( \mu \leq i_+(X) \leq n - \mu \).

To consider the special case \( G = K_m \) and \( H = K_n \), let \( \mu(m,n) \) denote the minimum value of \( 1_{r\leftrightarrow}^\top V 1_k + 1_{r\leftrightarrow}^\top W 1_\ell \) over all compatible multiplicity matrices \( V \) and \( W \) for \( K_m \) and \( K_n \). By the previous paragraph, we have \( \mu(m,n) \leq i_+(X) \leq |m| + |n| - \mu(m,n) \) for any orthogonal \( X \in S(K_m \vee K_n) \). Moreover, if \( V_0 \in \mathbb{N}_0^{r\leftrightarrow k} \) and \( W_0 \in \mathbb{N}_0^{r\leftrightarrow \ell} \) are compatible multiplicity matrices for \( K_m \) and \( K_n \), then so are any matrices \( V \in \mathbb{N}_0^{r\leftrightarrow k} \) and \( W \in \mathbb{N}_0^{r\leftrightarrow \ell} \) satisfying \( \tilde{V} = V_0, \tilde{W} = W_0, (e_1 + e_\tau)\top V = (e_1 + e_\tau)\top V_0 \) and \( (e_1 + e_\tau)\top W = (e_1 + e_\tau)\top W_0 \). By the construction of Theorem 2.14, this implies that for every integer \( i \) with \( \mu(m,n) \leq i \leq |m| + |n| - \mu(m,n) \), we have \( i = i_+(X) \) for some orthogonal matrix \( X \in S(K_m \vee K_n) \).

Let \( k \leq \ell, \ m \in \mathbb{N}^k \) and \( n \in \mathbb{N}^\ell \), then:

\[
\ell \leq \mu(m,n) \leq \max\{\ell, 2k\}.
\]
The first inequality is clear, and the second one follows by examining the constructions of multiplicity matrices in the proof of Proposition 4.10 and in the proofs in Subsections 4.1 and 4.2.

Lemma 4.20. Let \( k, \ell \in \mathbb{N} \), \( k \leq \ell \), \( m \in \mathbb{N}^k \), \( n \in \mathbb{N}^\ell \). If \( k + \ell > |m| + \iota(m) \), then \( \mu(m, n) \geq 2k \).

Proof. Suppose \( \mu(m, n) < 2k \) and there exist \( V, W \) as above with \( \mathbf{1}_{\ell - 2}^\top \mathbf{V} \mathbf{1}_k < 2k \). Some column of \( \mathbf{V} \) then has sum less than 2, so that column has exactly one non-zero entry, say in row \( s \). Since \( \mathbf{V}^\top \mathbf{W} > 0 \), this implies that row \( s \) of \( \mathbf{W} \) is nowhere-zero. By Lemma 4.16 and the compatibility of \( V \) and \( W \), we have \( |m| + \iota(m) - k \geq e_i^\top \mathbf{V} \mathbf{1}_k = e_i^\top \mathbf{W} \mathbf{1}_\ell \geq \ell \), a contradiction. So \( \mu(m, n) \geq 2k \). □

In the next theorem we show that \( \mu(m, n) \) always reaches one of the two upper bounds given in (2).

Theorem 4.21. Let \( k \leq \ell \), \( m \in \mathbb{N}^k \) and \( n \in \mathbb{N}^\ell \). If \( q(K_m \vee K_n) = 2 \), then

\[
\mu(m, n) = \begin{cases} 2k & \text{if } |m| + \iota(m) - k < \ell < 2k, \\ \ell & \text{otherwise.} \end{cases}
\]

Consequently, there exists a matrix \( A \in S(K_m \vee K_n) \) with \( q(A) = 2 \) whose eigenvalues have multiplicities \( \{t, |m| + |n| - t\} \) if and only if \( \mu(m, n) \leq t \leq |m| + |n| - \mu(m, n) \).

Proof. If \( |m| + \iota(m) - k < \ell < 2k \), then we have \( 2k \leq \mu(m, n) \leq \max\{\ell, 2k\} = 2k \) by Lemma 4.20 and (2), so \( \mu(m, n) = 2k \). On the other hand, if \( \ell \geq 2k \), then \( \mu(m, n) = \ell \) by (2), and if \( |m| + \iota(m) - k \geq \ell \), then the matrix \( W \) constructed in the the proof of Proposition 4.10 shows that \( \mu(m, n) \leq \ell \), so \( \mu(m, n) = \ell \) by (2). The final claim regarding multiplicities follows immediately from the discussion before Lemma 4.20. □

Note that both cases in Theorem 4.21 occur, for example when \( k = 2 \) and \( \ell = 3 \) we have \( \mu((K_3 \cup K_2) \vee (K_2 \cup 2K_1)) = \ell \) and \( \mu(2K_2 \vee (K_2 \cup 2K_1)) = 2k \).

4.5. Examples. We conclude this work with two examples, that illustrate the strength of the general result.

Example 4.22. For \( m \in \mathbb{N} \) and \( n \in \mathbb{N}^\ell \), we have

\[
q(K_m \vee K_n) = \begin{cases} 2 & \text{if } \ell \leq m, \\ 3 & \text{if } \ell > m \end{cases}
\]

so for \( \ell \leq m \), we have \( \mu((m), K_n) = \ell \) by Theorem 4.21. Considering conditions (a), (b) and (c) in Theorem 4.19, we get the necessary and sufficient conditions for \( q = 2 \) as follows: (a) covers the case when \( m \geq 2 \) and \( \iota(n) = 0 \), and demands \( \ell \leq m \), (b) applies when \( m = 1 \) and in this case we get \( \ell = 1 \), finally, (c) gives three conditions for the case \( m \geq 2 \).
and \( \iota(n) > 0: 1 + \ell \leq m, 2 \leq \ell \leq m \) or \( \ell \leq 2 \leq |n| \), that together reduce to \( \ell \leq m \).

Throughout this section we were noting the effect that the number of connected components and isolated points have on \( q \). The example below exposes this behaviour on a specific example.

**Example 4.23.** Let \( a, b, s \in \mathbb{N} \). Then:

\[
q(aK_s \lor bK_1) = \begin{cases} 
2 & \text{if } s = 2 \text{ and } b \in \{a, 2a\}, \text{ or } s \neq 2 \text{ and } a \leq b \leq sa, \\
3 & \text{otherwise}
\end{cases}
\]

and in the cases that \( q(aK_s \lor bK_1) = 2 \), we have \( \mu((s^a), (1^b)) = b \) by Theorem 4.21.

To see how the formula for \( q(aK_s \lor bK_1) \) follows from conditions (a) \( [b] \) and (c) \( [c] \) in Theorem 4.19 we split into various cases. If \( s = 1 \), then condition (b) gives \( q = 2 \) if and only if \( a = b \), as required. If \( a \leq b \), then condition (b) gives \( q = 2 \) if and only if \( a = b \), as required. In the remaining case, when \( b > a \) and \( s > 1 \), the necessary and sufficient conditions for \( q = 2 \) are listed in item (c). In the notation of this example, \( q = 2 \) precisely when at least one of the following conditions holds:

\[
a + b \leq as, \text{ or } 2a \leq b \leq as, \text{ or } b = 2a.
\]

If \( s = 2 \), it is apparent that \( b = 2a \) is the only solution. For \( s \geq 3 \), we get that \( q = 2 \) if any only if \( b \in \{a + 1, a + 2, \ldots, sa\} \).

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