Bounded Imaginary Powers of Differential Operators on Manifolds with Conical Singularities

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Abstract. We study the minimal and maximal closed extension of a differential operator $A$ on a manifold $B$ with conical singularities, when $A$ acts as an unbounded operator on weighted $L_p$-spaces over $B$, $1 < p < \infty$.

Under suitable ellipticity assumptions we can define a family of complex powers $A^z$, $z \in \mathbb{C}$. We also obtain sufficient information on the resolvent of $A$ to show the boundedness of the purely imaginary powers.

Examples concern unique solvability and maximal regularity for the solution of the Cauchy problem for the Laplacian on conical manifolds as well as certain quasilinear diffusion equations.

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1. Introduction

Seeley’s classical paper [21], published in 1967, showed in a striking way how pseudodifferential techniques could be applied to analyze complex powers of elliptic (pseudo-)differential operators on closed manifolds. Replacing the resolvent in the Dunford integral by a parameter-dependent parametrix, he obtained a representation of the powers that was precise enough to deduce a wealth of information on eigenvalue asymptotics, zeta functions, and index theory. Seeley also extended his results to differential boundary value problems. In 1971 he showed the boundedness of the purely imaginary powers on $L_p$-spaces, [23].

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At that time the principal motivation for these studies was the description of interpolation spaces. Additional interest in the behavior of imaginary powers came from Dore and Venni’s 1987 article [8], in which they showed how the boundedness of imaginary powers can be used to derive results on maximal regularity for evolution equations.

Meanwhile, bounded imaginary powers or even the existence of a bounded $H^\infty$ calculus [14] have been established in many situations, e.g. in abstract settings [15], for classes of differential operators on $\mathbb{R}^n$ and smooth manifolds [3], boundary value problems on bounded and certain unbounded domains in $\mathbb{R}^n$, [7, 16, 26], as well as for operators in Boutet de Monvel’s calculus [25].

We shall focus here on the case of a manifold with conical singularities. This is a Hausdorff space, $B$, that is a smooth manifold outside a finite number of singular points, while, close to each of these, it has the structure of a cone with smooth, closed cross-section. Blowing up $B$ near its singular points, we obtain a manifold $\mathbb{B}$ with boundary $\partial \mathbb{B} = X$.

Near the boundary, we fix a splitting of coordinates $(t, x) \in [0, 1] \times X$. Rather than on $B$, the analysis will be performed on $\mathbb{B}$ (respectively the interior of $\mathbb{B}$). We consider so-called cone or Fuchs-type differential operators, i.e., operators which close to the boundary are of the form

\begin{equation}
A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j,
\end{equation}

where each $a_j \in C^\infty(\mathbb{R}_+, \text{Diff}^{n-j}(X))$ is a smooth family of differential operators on the cross-section. Such an $A$ acts as an unbounded operator $A : C^\infty_{\text{comp}}(\text{int} \mathbb{B}) \subset \mathcal{H}_p^{0, \gamma}(\mathbb{B}) \to \mathcal{H}_p^{0, \gamma}(\mathbb{B})$, where the space $\mathcal{H}_p^{0, \gamma}(\mathbb{B})$ away from the boundary coincides with $L_p(\mathbb{B})$ and near the boundary with

$$
t^{-\frac{n+1}{p}} L_p(0, 1 \times X, \frac{dt}{t} \frac{dx}{x}), \quad n = \text{dim } X.
$$

Here, $1 < p < \infty$, and $\gamma$ is an arbitrary real number. Justified by the fact that a change to polar coordinates shows the equivalence $L_p(\mathbb{R}^{n+1}) = t^{-\frac{n+1}{p}} L_p(\mathbb{R}_+ \times \mathbb{S}^n, \frac{dt}{t} d\varphi)$ we define the space $L_p(B)$ as $H^{0, \gamma_p}_p(B)$ for $\gamma_p = (n + 1)(\frac{1}{2} - \frac{1}{p})$.

Let $\Lambda = \Lambda_{\Delta}(\theta)$ denote a closed sector in the complex plane, symmetric about the negative real half-axis and of aperture $2(\pi - \theta)$ for some $0 < \theta < \pi$. We find conditions (Definition [4, 3]) on $A$, which depend on $\gamma \in \mathbb{R}$ but not on $1 < p < \infty$, that ensure the following:

i) the closure $A_{\text{min}}$ of $A$ has no spectrum in $\Lambda_{\Delta} \cap \{ |\lambda| > R \}$;

ii) the resolvent satisfies the uniform estimate $\| (\lambda - A)^{-1} \|_{\mathcal{L}(H^{0, \gamma}_p(B))} \leq c_p |\lambda|^{-1}$.

Moreover, we obtain very precise information on the structure of the resolvent. For this and i) see Theorem [1, 6]. ii) is shown in Proposition [7]. We also give conditions (Remark [3]) implying that the maximal extension $A_{\text{max}}$ satisfies statements analogous to i) and ii). The symmetry about the negative real axis, which here plays the role of a ray of minimal growth in the sense of Agmon [1], [23], is not essential. The case of an arbitrary symmetry axis $\{ t e^{i\varphi} | t \geq 0 \}$ can be reduced to our situation, replacing $A$ by $e^{-i\varphi} A$.

Since $A_{\text{min}}$ (respectively $A_{\text{max}}$) in the above case has compact resolvent, any “keyhole” region $\Lambda$, consisting of the sector $\Lambda_{\Delta}$ and an arbitrarily small ball around zero, only contains finitely many elements of the spectrum. Assuming that zero is the only spectral point in the keyhole (or,
alternatively, shrinking the angle of the sector and possibly rotating $A$ a little), we define complex powers $A_{\min}^z$ (respectively $A_{\max}^z$) for all complex $z$ with negative real part. This is done in terms of a Dunford integral, integrating the resolvent against $\lambda^z$ along the boundary of the keyhole. Using the specific structure of the resolvent, we show (Theorem 5.1) that

$$\|A_{\min}^z\|_{\mathcal{L}(\mathcal{H}^p)} \leq C_p e^{\Im z|\theta|} \quad \text{uniformly for all } z \text{ with } |\Re z| \text{ sufficiently small}$$

as well as the analogous estimate for $A_{\max}^z$ (Theorem 6.3). Consequently, the purely imaginary powers $A_{\min}^iy$ (respectively $A_{\max}^iy$), $y \in \mathbb{R}$, exist as suitable limits and satisfy an estimate as in iii).

It should be noted that both the construction of the complex powers and the boundedness of the imaginary powers only rely on the information about the resolvent provided by Theorem 4.6. Our conclusions therefore carry over to all situations where the resolvent has this structure.

The key to the above described results is, similar to Seeley’s classical concept, to view $\lambda - A$ as an element of a calculus of parameter-dependent pseudodifferential operators on $\mathbb{B}$, and to express $(\lambda - A)^{-1}$ within this calculus. In our context, the appropriate calculus is Schulze’s parameter-dependent cone algebra, cf. for example [19, 20]. The conditions we impose on $A$ are, more or less, ellipticity conditions on $\lambda - A$ within this calculus. We require three associated objects not to have spectrum in the sector $\Lambda_\Delta$. The first is the usual homogeneous principal symbol of $A$, defined on the cotangent bundle over the interior of $\mathbb{B}$. The second is the so-called rescaled symbol, which reflects the behavior of the principal symbol near the boundary. The third is the so-called model cone operator $\hat{A}$, which acts as an unbounded operator in Sobolev spaces on the infinite cylinder $\mathbb{R}_+ \times X$. It is induced by freezing the coefficients of $A$ at the boundary, i.e., using the notation from (1.3), $\hat{A} = t^{-\mu} \sum_{j=0}^\mu a_j(0)(-t\partial_t)^j$.

In order to separate the more general functional-analytic issues from the specific difficulties related to conical singularities, we give a review of several basic facts about complex powers of unbounded operators on a Banach space in Section 4, while in Section 5 we briefly discuss Fuchs-type operators. Sections 4, 5 and 6 are devoted to the proof of the results stated above.

In Section 6 we treat an example and show how our work can be combined with that of Dore and Venni to obtain results on unique solvability and maximal regularity for the non-homogeneous Cauchy problem in $L_p(B)$:

$$\dot{u}(\tau) - \Delta u(\tau) = f(\tau), \quad u(0) = 0.$$  

Here, $\Delta$ is the Laplace-Beltrami operator for a Riemannian metric with a conical degeneracy, and we assume $\dim \mathbb{B} > 4$. Based on this observation, we consider in Section 7 quasilinear diffusion equations of the form

$$\dot{u}(\tau) - \text{div}(a(t^c u) \text{grad } u)(\tau) = f(u, \tau) + g(\tau), \quad u(0) = u_0,$$

in weighted $L_p$-spaces on $\mathbb{B}$, and show unique solvability with the help of an abstract result by Clément and Li [20]. Here, $a$, the diffusion coefficient, is a smooth positive function; we assume that it depends on $t^c u$, where $t$ is a smooth function on $\mathbb{B}$, coinciding with the distance to the boundary near $\partial \mathbb{B}$, and $c$ is a positive constant.

It is clear that physically relevant applications would require our understanding of the Laplacian on lower-dimensional manifolds and of boundary value problems. Both topics are presently under investigation. As their analysis, however, is considerably more complicated, it makes sense to focus first on the present situation, where the ideas and techniques can be explained more easily.

An appendix relates the structure of the resolvent as we use it to that given in earlier work by Schulze and that of Gil [5]. Moreover, we collect a few definitions and notions in Section 11.

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Let us recall some well-known facts on complex powers of a closed, densely defined operator

\[ A : \mathcal{D}(A) \subset F \rightarrow F \]

in a Banach space \( F \), cf. for example [23]. We denote by \( \Lambda = \Lambda(\delta, \theta) \) the keyhole region

\[ \Lambda(\delta, \theta) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \delta \text{ or } |\arg \lambda| \geq \theta \} \]

with \( \delta > 0 \) and \( 0 < \theta < \pi \). We assume that

1. The spectrum of \( A \) has empty intersection with \( \Lambda \setminus \{0\} \).
2. \( \|(\lambda - A)^{-1}\|_{L(F)} \) is uniformly bounded for large \( \lambda \in \Lambda \).

Remark 2.1. Under conditions (A1), (A2), and (A3), the operators \( A^z \) should be viewed with a little care, since \( A^{-1} \) in general is not the inverse of \( A \), which is not required to exist.

Remark 2.2. Under conditions (A1), (A2), and (A3), the limits

\[ A^{iy} f = \lim_{\|z\| \to \infty} A^z f \]

exist for any real number \( y \) and any \( f \in F \), and thus define operators \( A^{iy} \in L(F) \). Furthermore

\[ A^{iy} f = A^{-1+iy} A f \]

for \( f \in \mathcal{D}(A) \). In particular, if we set

\[ E_0 = \frac{1}{2\pi i} \int_{|\lambda| = \delta} (\lambda - A)^{-1} \, d\lambda, \]

then \( E_0 \) is a projection in \( F \) and \( A^0 = 1 - E_0 \).

Remark 2.2 could be rephrased as follows: Under conditions (A1), (A2), and (A3), the operators

\[ T^z := A^z + E_0, \quad z \in \mathbb{H}, \]

form an analytic semi-group (with \( \lim_{z \to 0} T^z f = f \) for any \( f \in F \)) and there exist constants \( c \geq 1 \) and \( \omega \geq 0 \) such that

\[ \|T^z\|_{L(F)} \leq c e^{\omega |z|}, \quad z \in \mathbb{H}. \]

Moreover, \( (1 - E_0) A + E_0 : \mathcal{D}(A) \to F \) is an isomorphism, whose inverse is \( T^{-1} \).

In concrete situations the problem is to analyze whether an operator \( A \) satisfies conditions (A1), (A2), (A3), and then to find the best possible constant \( \omega \). Fundamental works on this topic are due to Seeley [21, 22, 23], where he gives criteria ensuring that a differential operator on a compact manifold (with boundary) has these properties. The main object of the present paper is to give such criteria for differential operators on manifolds with conical singularities.
3. Cone differential operators

We consider a differential operator $A : C^\infty(B, E) \to C^\infty(B, E)$ acting on sections of a vector bundle $E$ over $B$. We may assume that $E$ respects the product structure near the boundary $\partial B = X$, i.e. is the pull-back of a vector bundle $E_X$ over $X$ under the canonical projection $[0, 1] \times X \to X$. The operator $A$ is called a cone differential or Fuchs-type operator, cf. [8, 12] if, near the boundary it is of the form

\begin{equation}
A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j,
\end{equation}

where $a_j \in C^\infty(\mathbb{R}_+, \text{Diff}_{\mu-j}(X; E_X, E_X))$ are functions, smooth up to the boundary, with values in the differential operators on $X$. In order to keep the notation simple, we shall not indicate the bundles and write $C^\infty(B)$, $\text{Diff}_{\mu-j}(X)$, etc.

We can rewrite (3.1) as

\begin{equation}
A = t^{-\mu} \text{op}_M^{\gamma+\mu-\frac{\sigma}{2}}(f), \quad f(t, z) = \sum_{j=0}^{\mu} a_j(t)z^j,
\end{equation}

where the Mellin pseudodifferential operator is defined by

\begin{equation}
\text{op}_M^{\gamma+\mu-\frac{\sigma}{2}}(f)(t) = \int_{\Re z = \frac{\mu+1}{2} - \gamma - \mu} t^{-z} f(t, z)(Mu)(z)\, dz, \quad u \in C^\infty_{\text{comp}}([0, 1] \times X).
\end{equation}

Here, $\gamma \in \mathbb{R}$ is arbitrary, and (3.3) is independent of the choice of $\gamma$. We keep $\gamma$ in the notation, since we shall consider extensions of $A$ to different weighted Sobolev spaces, the weight being given by $\gamma$.

**Definition 3.1.** For $s \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$, and $1 < p < \infty$ we introduce $\mathcal{H}^{s, \gamma}_p(B)$ as the space of all functions $u \in H^{s, \text{loc}}(\text{int } B)$ such that

\[
t^{\frac{1}{2} + \gamma}(t\partial_t)^k \partial_x^{\alpha} (\omega u)(t, x) \in L_p(\mathbb{R}_+ \times X, \frac{dt}{t}dx) \quad \forall \ k + |\alpha| \leq s
\]

for some cut-off function $\omega \in C^\infty_{\text{comp}}([0, 1])$.

Recall that a cut-off function is a function $\omega \in C^\infty_{\text{comp}}(\text{int } B)$ such that $\omega \equiv 1$ near $t = 0$. The definition of the Banach spaces $\mathcal{H}^{s, \gamma}_p(B)$ naturally extends to real $s$ as well as to spaces of vector bundles over $B$. For more details see Section 11.

For any $s$, $\gamma$, and $p$, the operator $A$ induces continuous mappings

\[
A : \mathcal{H}^{s+\mu, \gamma+\mu}_p(B) \longrightarrow \mathcal{H}^{s, \gamma}_p(B).
\]

With $A$ we associate three symbols. The first is the usual homogeneous principal symbol $\sigma^\mu_\psi(A) \in C^\infty(T^*(\text{int } B) \setminus 0)$, taking values in the corresponding bundle homomorphisms. In local coordinates near the boundary

\begin{equation}
\sigma^\mu_\psi(A)(t, x, \tau, \xi) = t^{-\mu} \sum_{j=0}^{\mu} \sigma^\mu_\psi(a_j)(t, x, \xi)(-i\tau)^j
\end{equation}

Dropping the factor $t^{-\mu}$, replacing $t\tau$ by $\tau$, and inserting $t = 0$, we obtain

\[
\tilde{\sigma}^\mu_\psi(A)(x, \tau, \xi) := \sum_{j=0}^{\mu} \sigma^\mu_\psi(a_j)(0, x, \xi)(-i\tau)^j,
\]

which yields the rescaled symbol of $A$,

\begin{equation}
\tilde{\sigma}^\mu_\psi(A) \in C^\infty((T^*X \times \mathbb{R}) \setminus 0).
\end{equation}
The third one is the conormal symbol
\begin{equation}
\sigma^\mu(A)(z) = f(0, z) = \sum_{j=0}^\mu a_j(0)z^j,
\end{equation}
a function of \( z \in \mathbb{C} \) with values in the differential operators on \( X \).

Remark 3.2. The operator \( A \) is called elliptic with respect to the weight \( \gamma + \mu \), if both the homogeneous principal symbol and the rescaled symbol are invertible on their respective domains, and
\begin{equation}
\sigma^\mu(A)(z) : H^s(X) \to H^{s-\mu}(X), \quad \text{Re } z = \frac{n+1}{2} - \gamma - \mu,
\end{equation}
is an isomorphism for all \( z \) on that line.

It can be shown, \[18\], Theorem 3.13, that \( A \) is elliptic if and only if the operators \( A : \mathcal{H}^{s+\mu,\gamma+\mu}_p(\mathbb{B}) \to \mathcal{H}^s_p(\mathbb{B}) \) are Fredholm for any \( s \) and \( p \).

We shall consider \( A \) as the operator
\begin{equation}
A : \mathcal{D}(A) = \mathcal{H}^{\mu,\gamma+\mu}_p(\mathbb{B}) \subset \mathcal{H}^{\mu,\gamma}_p(\mathbb{B}) \to \mathcal{H}^{0,\gamma}_p(\mathbb{B}).
\end{equation}

Remark 3.3. a) In case \( A \) is elliptic with respect to \( \gamma + \mu \), \[3.8\] is the closure of \( A \) considered on the domain \( \mathcal{C}_{\text{comp}}(\mathbb{B}) \).

b) By the spectral invariance of the cone algebra, \[18\], Theorem 3.14, the spectrum of \( A \) is independent of \( 1 < p < \infty \).

With \( A \) we associate the model cone operator, which acts in Sobolev spaces on the infinite cylinder
\begin{equation}
X^\wedge := \mathbb{R}_+ \times X.
\end{equation}
Let \( (t, x) \) denote cylindrical coordinates on \( X^\wedge \). Then \( H^s_{\text{cone}}(X^\wedge) \) is the space of all distributions \( u \) whose push-forward under conical coordinates \( (t, tx) \) belongs to \( H^s_p(\mathbb{R}^{1+n}) \) (for details see \[17\], Section 4.2).

Definition 3.4. For \( s, \gamma \in \mathbb{R} \) and \( 1 < p < \infty \) the spaces \( K^{\mu,\gamma}_p(X^\wedge) \) consist of all distributions \( u \in H^s_{p,\text{loc}}(X^\wedge) \) satisfying, for some cut-off function \( \omega \in \mathcal{C}_\infty(\mathbb{R}[0,1]) \),
\[\omega u \in H^{\mu,\gamma}_p(\mathbb{B}), \quad (1 - \omega)u \in H^s_{p,\text{cone}}(X^\wedge).\]

Freezing the coefficients of \( A \) at \( t = 0 \), we obtain the model cone operator \( \hat{A} \),
\begin{equation}
\hat{A} = t^{-\mu} \sum_{j=0}^\mu a_j(0)(-\partial_t)^j : K^{\mu,\gamma+\mu}_p(X^\wedge) \to K^{0,\gamma}_p(X^\wedge).
\end{equation}

Remark 3.5. a) If \( A \) is elliptic with respect to the weight \( \gamma + \mu \) and satisfies the ellipticity condition (E1) introduced below, then it can be shown that \[3.10\] is the closure of \( \hat{A} \) considered on the domain \( \mathcal{C}_\infty(\mathbb{B}) \).

b) If we set \( \tilde{a}(\lambda) = \lambda - \hat{A} \) with \( \hat{A} \) from \[3.10\], then \( \tilde{a}(\lambda) \) corresponds to the so-called principal edge symbol of \( \lambda - A \), if we view \( \lambda - A \) as a constant coefficient edge symbol in the framework of Schulze’s theory of pseudodifferential operators on manifolds with edges, cf. for example \[8\].

It is worth mentioning that \( \tilde{a}(\lambda) = \lambda - \hat{A} \) is a homogeneous function in a specific way. Namely if we define for \( \varrho > 0 \)
\begin{equation}
\kappa_\varrho : \mathcal{C}_\infty(\mathbb{B}) \to \mathcal{C}_\infty(\mathbb{B}), \quad (\kappa_\varrho u)(t, x) = \varrho^{\frac{n+1}{2}}u(\varrho^2 t, x),
\end{equation}
then these operators extend by continuity to isomorphisms in $\mathcal{L}(K_p^{s,\gamma}(X^\land))$ and

$$
\tilde{a}(g^\mu \lambda) = g^\mu \kappa_\varrho \alpha(\lambda) \kappa_\varrho^{-1}.
$$

In particular, $\text{spec}(\tilde{A})$ is a closed conic subset of the complex plane.

Near the boundary of $\mathcal{B}$ we can write $\lambda - A = t^{-\mu} \text{op}_M^{\gamma + \mu - \frac{D}{2}}(h)(\lambda)$ with the parameter-dependent Mellin symbol

$$
h(t, z, \lambda) = \tilde{h}(t, z, t^\mu \lambda), \quad \tilde{h}(t, z, \lambda) = \lambda - f(t, z),
$$

and $f$ from (3.2).

4. The resolvent of cone differential operators

To describe the structure of the resolvent we recall some elements from the theory of parameter-dependent cone pseudodifferential operators, starting with the smoothing remainders of the calculus. To this end we introduce a family of Fréchet spaces of smooth functions on $\text{int} \mathcal{B}$ and $X^\land$, respectively.

**Definition 4.1.** For $\gamma \in \mathbb{R}$ we let $C^{\infty,\gamma}(\mathcal{B})$ denote the space of all $u \in C^{\infty}(\text{int} \mathcal{B})$ such that

$$
\sup_{0 < t < 1} t^{\frac{n+1}{2} - \gamma} ||| \log^l t (t \partial_t)^{k}(u)(t, \cdot) ||| < \infty \quad \forall k, l \in \mathbb{N}_0
$$

for any semi-norm $||| \cdot |||$ of $C^{\infty}(X)$. Similarly, $S^0_0(X^\land)$ is the space of all $u \in C^{\infty}(X^\land)$ which are rapidly decreasing as $t \to \infty$ and satisfy (4.1).

We shall say that an operator $G$ has a kernel $k$ with respect to the $H^{0,0}_2(\mathcal{B})$-scalar product if

$$(Gu)(y) = \langle k(y, \cdot), \mathfrak{w}\rangle_{H^{0,0}_2(\mathcal{B})} = \int_{\mathcal{B}} k(y, y') u(y') t(y')^n dy', \quad u \in C^{\text{comp}}(\text{int} \mathcal{B}),$$

where $t$ denotes a boundary defining function on $\mathcal{B}$ and $dy'$ refers to a density on $2\mathcal{B}$, the double of $\mathcal{B}$. We shall use the analogous notion for operators on $X^\land$, based on the scalar product of $K^{0,0}_2(X^\land) = L_2(X^\land, t^n dt dx)$.

**Definition 4.2.** An operator-family $G = G(\lambda)$, $\lambda \in \Lambda$, belongs to $C^{\infty}_G(\mathcal{B}; \Lambda, \gamma)$, $\gamma \in \mathbb{R}$, if there exists an $\varepsilon = \varepsilon(G) > 0$ such that $G(\lambda)$ has a kernel $k(\lambda) = k(\lambda, \cdot, \cdot)$ with respect to the $H^{0,0}_2(\mathcal{B})$-scalar product and

$$
k(\lambda, y, y') \in \mathcal{S}(\Lambda, C^{\infty,\gamma + \varepsilon}(\mathcal{B}) \widehat{\otimes}_{\pi} C^{\infty,\gamma + \varepsilon}(\mathcal{B})),$$

cf. (10.3): $\widehat{\otimes}_{\pi}$ denotes the completed projective tensor product. $C^{\infty}_G(\mathcal{B}; \Lambda, \gamma)$ is the residual class of the calculus. For every choice of $s$, $p$, and $\lambda$, the operator $G(\lambda)$ maps $H^{s,\gamma}_p(\mathcal{B})$ into $C^{\infty,\gamma}(\mathcal{B})$. For the description of the resolvent we shall need another class of operator-families. For each fixed $\lambda$, they are smoothing over $X^\land$, yet they have a finite order in $\lambda$.

**Definition 4.3.** Let $\gamma, \mu \in \mathbb{R}$ and $d > 0$. We define $R^{\mu,d}_G(X^\land; \Lambda, \gamma)$ as the space of all operator-families $G = G(\lambda)$ that have a kernel with respect to the $K^{0,0}_2(X^\land)$-scalar product of the form

$$
k(\lambda, t, x, t', x') = [\lambda]^{\frac{n+1}{2}} \hat{k}(\lambda, |\lambda|^d t, x, |\lambda|^d t', x'),
$$

where $[\cdot]$ is a smoothed norm-function (i.e., $[\cdot]$ is smooth, positive on $\mathbb{C}$ and $[\lambda] = |\lambda|$ for large $\lambda$) and for some $\varepsilon = \varepsilon(G) > 0$

$$
\hat{k}(\lambda, t, x, t', x') \in S^\pi_s(\Lambda) \widehat{\otimes}_{\pi} S^{\gamma + \varepsilon}(X^\land_{(t, x)}) \widehat{\otimes}_{\pi} S^{\gamma - \varepsilon}(X^\land_{(t', x')}).
$$
In this case, $G(\lambda)$ maps $K_p^s(\mathcal{X})$ into $S^\gamma_p(\mathcal{X})$ for any $s$ and $p$. See also the Appendix for more information on such operator-families. Trivially, a symbol $a \in S^\gamma_p(\Lambda)$ satisfies the estimate
\begin{equation}
|a(\lambda)| \leq c (1 + |\lambda|)^{\frac{\gamma}{2}}, \quad \lambda \in \Lambda.
\end{equation}

Recall from Section 3 that if $A$ is a cone differential operator, then $\lambda - A$ can be written in terms of Mellin symbols taking values in the differential operators on $X$, cf. (3.13). In that case the Mellin symbol is a polynomial in $z$. A general Mellin symbol is an entire function with values in the pseudodifferential operators on $X$; more precisely:

**Definition 4.4.** For $\mu \in \mathbb{R}$ and $d > 0$ let $M^\mu_d(\Lambda)$ denote the space of all functions $\tilde{g}(z; \lambda)$, which are holomorphic in $z \in \mathbb{C}$ with values in $L^\mu_d(\Lambda)$, and for which
\[ \tilde{g}(\beta, \lambda) := \tilde{g}(\beta + i\tau, \lambda) \in L^\mu_d(\Lambda, \mathbb{R} \times \Lambda) \]
is locally bounded as a function of $\beta \in \mathbb{R}$. This is a Fréchet space in a canonical way.

Let us now state the ellipticity assumptions on $A$, which ensure the existence of its resolvent in a keyhole region:

**Definition 4.5.** We call $A$ elliptic with respect to the weight $\gamma + \mu$ and the sector $\Lambda_\Delta$, cf. (10.2), if the following two conditions are satisfied:

1. **(E1)** Both the homogeneous principal symbol $\sigma^\mu_\psi(A)$ and the rescaled symbol $\tilde{\sigma}^\mu_\psi(A)$, cf. (3.5), have no spectrum in $\Lambda_\Delta$, pointwise on $T^*\!(\text{int\,} \mathbb{B}) \setminus 0$ and $(T^*\!X \times \mathbb{R}) \setminus 0$, respectively.
2. **(E2)** the model cone operator $\tilde{A}$, acting as in (3.10), has no spectrum in $\Lambda_\Delta \setminus \{0\}$.

If conditions (E1) and (E2) are satisfied, they automatically hold for a slightly larger keyhole region (by closedness of the spectrum, compactness of $\mathbb{B}$, and the homogeneity of the rescaled symbol, the homogeneous principal symbol, as well as $\tilde{\sigma}(\lambda)$, cf. (3.12)). Moreover, one can show that condition (E2) implies that (3.3) is a family of isomorphisms.

We would like to point out that, although the above conditions seem to be quite strong, it follows from more general considerations that they are essentially necessary.

Under conditions (E1) and (E2) we can now describe the resolvent of $A$:

**Theorem 4.6.** If $A$ is elliptic with respect to $\Lambda_\Delta = \Lambda_\Delta(\theta)$ and $\gamma + \mu$, then $A$ has no spectrum in $\Lambda_\Delta \cap \{ |\lambda| > R \}$ for some $R > 0$, and for large $\lambda \in \Lambda_\Delta$
\begin{equation}
(\lambda - A)^{-1} = \sigma \left\{ \mu^\lambda \text{op}_{\tilde{\mathcal{M}}}(g)(\lambda) + G(\lambda) \right\} + (1 - \sigma)I_{\psi,\gamma}^\Lambda(\lambda),
\end{equation}
where $\sigma, \sigma_0, \sigma_1 \in C_\infty(0, 1]$ are cut-off functions satisfying $\sigma_1 \sigma = \sigma_1, \sigma \sigma_0 = \sigma$, and there exists a $\delta > 0$ such that, for $\Lambda = \Lambda(\delta, \theta)$,
\begin{itemize}
\item[i)] $g(t, z, \lambda) = \tilde{g}(t, z, t^\mu \lambda)$ with $\tilde{g} \in C_\infty(\mathbb{B}, M^\mu_d\!(\Lambda))$,
\item[ii)] $I_{\psi,\gamma}^\Lambda(\lambda) \in L^{-\mu,\psi}(\text{int\,} \mathbb{B}; \Lambda)$, cf. (10.4),
\item[iii)] $G(\lambda) \in R_G^{-\mu,\psi}(\mathcal{X}; \Lambda, \gamma)$, and $G_\infty(\lambda) \in C_{\text{comp}}(\mathbb{B}; \Lambda, \gamma)$.
\end{itemize}

In view of the fact that $A$ has compact resolvent (recall that the embeddings $\mathcal{H}_p^s(\mathbb{B}) \hookrightarrow \mathcal{H}_p^r(\mathbb{B})$ are compact provided $s > r$ and $\gamma > \theta$), only finitely many points of the spectrum of $A$ will lie in $\Lambda$. Thus, after possibly rotating $A$ a little and shrinking the keyhole $\Lambda$, we can assume that $A$ has no spectrum in $\Lambda$, except perhaps $0$.

Theorem 4.6 follows from the parametrix construction in the parameter-dependent cone algebra given in [1], Theorems 3.2, 3.4, cf. also [8], Section 9.3.3, Theorem 6. An important observation we can draw from this theorem is a norm estimate of the resolvent:
PROPOSITION 4.7. Under the assumptions of Theorem 4.6 there exists a constant \( c_p \geq 0 \) such that for all sufficiently large \( \lambda \in \Lambda \)

\[
\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \leq c_p|\lambda|^{-1}.
\]

PROOF. We first reduce the case of arbitrary \( \gamma \) to the special case \( \gamma = \gamma_p = (n+1)(\frac{1}{2} - \frac{1}{p}) \).

To this end let \( b \in C^\infty(\text{int}\, \mathbb{B}) \) be a positive function such that \( b(t,x) = t^\nu, \nu = \gamma_p - \gamma \), for all \( (t,x) \in [0,1] \times X \). Multiplication by \( b \) induces isomorphisms \( \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{0,\gamma_p}(\mathbb{B}) \) with inverse induced by \( b^{-1} \). Therefore,

\[
\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \sim \|b(\lambda - A)^{-1}b^{-1}\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma_p}(\mathbb{B}))}.
\]

But for large \( |\lambda| \),

\[
b(\lambda - A)^{-1}b^{-1} = \sigma \left\{ t^{\mu}\text{op}_M^{-\frac{n}{2}}(T^{-\nu}\gamma)(\lambda) + t^{\nu}G(\lambda)t^{-\nu} + (1 - \sigma)bP(\lambda)b^{-1}(1 - \sigma_1) + bG_\infty(\lambda)b^{-1},
\]

where \( T^{-\nu}g(t,z,\lambda) = g(t,z - \nu, \lambda) \). Since \( T^{-\nu}g \) and \( bP(\lambda)b^{-1} \) are of the same quality as \( g \) and \( P(\lambda) \), respectively, and \( t^{\nu}G(\lambda)t^{-\nu} \in \mathcal{R}_{\infty}^{\mu,\mu}(X^\times; \Lambda, \gamma_p) \) and \( bG_\infty(\lambda)b^{-1} \in C_\infty^\infty(\mathbb{B}; \Lambda, \gamma_p) \), we can assume from the very beginning that \( \gamma = \gamma_p \).

The term \( G_\infty(\lambda) \) certainly behaves in the right way, since it is rapidly decreasing in \( \lambda \). Also the term \( (1 - \sigma)P(\lambda)(1 - \sigma_1) \) is good by the standard Calderon-Vaillancourt theorem. The two remaining terms \( t^{\mu}\text{op}_M(\gamma)(\lambda) \) and \( G(\lambda) \) we shall consider in the spaces

\[
\mathcal{K}^{0,\gamma_p}_{p}(X^\times) = L_p((\mathbb{R}_+ \times X, t^n\, dt\, dx).
\]

If \( \kappa_\tau \) is the group action from (3.11), then \( \|\kappa_\tau\|_{\mathcal{L}(\mathcal{K}^{0,\gamma_p}_{p}(X^\times))} = |\gamma|^n \) for all \( \gamma > 0 \). Hence for an arbitrary operator \( T \in \mathcal{L}(\mathcal{K}^{0,\gamma_p}_{p}(X^\times)) \) we have

\[
\|T\|_{\mathcal{L}(\mathcal{K}^{0,\gamma_p}_{p}(X^\times))} = \|\kappa_\tau^{-1}T\kappa_\tau\|_{\mathcal{L}(\mathcal{K}^{0,\gamma_p}_{p}(X^\times))}.
\]

Now let \( G(\lambda) \) have a kernel \( k(\lambda) \) as described in Definition 4.2 (with \( \mu \) replaced by \(-\mu \) and \( d = \mu \)). Then the operator-norm of \( G(\lambda) \) is the same as that of \( \kappa_\gamma^{1/\mu} \, G(\lambda) \, \kappa_\gamma^{1/\mu} \), which has the kernel \( \tilde{k}(\lambda, t,x, t', x') \). But this kernel is \( O(|\lambda|^{-1}) \) in \( \lambda \), cf. (4.2). To treat the last term we can pass to local coordinates, i.e. we assume \( X = \mathbb{R}^n \) and

\[
\tilde{g}_{\frac{1}{2}-\gamma_p}(t,x,\tau,\xi,\lambda) \in S^{-\mu,\mu}((\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda)\).
\]

By a tensor product argument we can assume that \( \tilde{g} \) is independent of \( (t,x) \). Conjugating with \( \kappa_\gamma^{1/\mu} \), we have to show that

\[
\text{op}_M^{-\frac{n}{2}}(g')(\lambda), \quad g_{\frac{1}{2}-\gamma_p}(t,\tau,\xi,\lambda) = t^\mu \tilde{g}_{\frac{1}{2}-\gamma_p}(\tau,\xi, t^\lambda),(\frac{1}{\lambda})
\]

is uniformly bounded in \( L_p((\mathbb{R}_+ \times \mathbb{R}^n, t^n\, dt\, dx) \) for large \( |\lambda| \). Since then \( \frac{1}{|\lambda|} \) is bounded away from zero and infinity, a simple calculation shows that

\[
|\langle \partial^k_{\tau}^{\mu,\rho} \partial^\rho_{\xi} \tilde{g}_{\frac{1}{2}-\gamma_p}(t,\tau,\xi,\lambda)| \leq c_{k,\lambda}(1 + |\tau| + |\xi|)^{-k-|\alpha|}
\]

uniformly in \( (t,\tau,\xi,\lambda) \), i.e., \( g'(\lambda) \in MS^0((\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda)\times \mathbb{R}^n) \) uniformly in \( \lambda \). Then the result follows, see the end of Section 10. \( \square \)

There are certain relations between \( A \) and \( g \) from i) respectively \( P(\lambda) \) from ii), which we are going to study now.

Let \( U \subset \mathbb{R}^n \) be a coordinate neighborhood for \( X \), whose closure is contained in another coordinate neighborhood. Condition (E1) ensures that the local symbol

\[
(4.4) \quad \tilde{q}(\mu)(t,x,\tau,\xi) := t^\mu \sigma_\mu^{\omega}(A)(t,x,t^{-1}\tau,\xi),
\]
The aim of this section is to show that a cone differential operator $A$ satisfies condition (A3). More precisely we shall show:

LEMMA 4.8. There exists a zero excision function $\chi$ on $\mathbb{R}$, such that for any $\varphi \in C^\infty_{\text{comp}}(U)$ and any $\sigma \in C^\infty_{\text{comp}}([0,1])$

$$\varphi(x)\sigma(t)\chi(|\tau,\xi|^2 + |\lambda|^2)(\lambda - \hat{q}^{(\mu)}(t,x,\tau,\xi))^{-1} \in S^{-\mu,\mu}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda).$$

PROOF. Let $a$ denote the symbol in question. For shortness let us write $y = (t,x)$ and $\eta = (\tau,\xi)$. Since the eigenvalues of $\hat{q}^{(\mu)}(y,\eta)$ are proportional to $|\eta|^\mu$ (uniformly for $y \in [0,1] \times U$) and do not lie in $\Lambda$, there exists a constant $c > 0$ such that $(\lambda - \hat{q}^{(\mu)}(y,\eta))^{-1}$ is a smooth function in

$$\{(y,\eta,\lambda) \in [0,1] \times U \times \mathbb{R}^{1+n} : \lambda \in \Lambda \}$$

Thus, if we choose $\chi$ in such a way that $\chi(|\eta|^2 + |\lambda|^2)$ vanishes for $|\eta| \leq (\delta/c)^\mu$ and $|\lambda| \leq \delta$, then $a$ is smooth on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}$. For $\Lambda = \Lambda(\delta,\theta)$. To verify that $a$ is a symbol, it suffices to show that

$$|a(y,\eta,\lambda)| \leq c(1 + |\eta|^2 + |\lambda|^2)^{\frac{\mu}{\delta}}$$

uniformly in $y \in [0,1] \times U$ and $\eta,\lambda \in \mathbb{R}^{1+n} \times \Lambda$. Since $a$ is anisotropic homogeneous of order $(-\mu,\mu)$ for large $(\eta,\lambda) \in \mathbb{R}^{1+n} \times \Lambda$, estimate $4.5$ holds on $\mathbb{R}^{1+n} \times \Lambda$. It also holds for $|\lambda| \leq \delta$ and $|\eta|$ sufficiently large, since then $|(\lambda - \hat{q}^{(\mu)}(y,\eta))^{-1}|$ is $O(|\eta|^{-\mu})$ due to the above described behavior of the eigenvalues. For $|\lambda|$ and $|\eta|$ simultaneously small, estimate $4.5$ holds anyway. \qed

For every $\beta \in \mathbb{R}$, we can associate with $\hat{g}$ from Theorem 4.4 a local symbol

$$\hat{g}_\beta = \hat{g}_\beta(t,x,\tau,\xi,\lambda) \in S^{-\mu,\mu}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda).$$

It is a consequence of the above mentioned parametrix construction in the cone calculus that the principal symbol of $\hat{g}$ is determined by the inverted principal symbol of $\lambda - A$. With the notation from Lemma 4.8 we indeed have

$$\varphi(x)\sigma(t)\left\{\hat{g}_\beta - \chi(|\tau,\xi|^2 + |\lambda|^2)(\lambda - \hat{q}^{(\mu)})^{-1}\right\} \in S^{-\mu-1,\mu}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda).$$

Similarly, the local symbols of $P(\lambda)$ from Theorem 4.4 ii) can be approximated modulo $S^{-\mu-1,\mu}$ in terms of the inverted local principal symbol of $\lambda - A$.

5. Complex powers of cone differential operators

The aim of this section is to show that a cone differential operator $A$ satisfying (E1), (E2), also satisfies condition (A3). More precisely we shall show:

THEOREM 5.1. Let $A$ be elliptic with respect to $\gamma + \mu$ and $\Lambda$, having no spectrum in the keyhole $\Lambda = \Lambda(\delta,\theta)$, except perhaps 0. Then one can define $A^\gamma$ as in (2.1) and there exists a constant $c_\mu \geq 0$ such that for all $z \in \mathbb{H}$ with $|\text{Re} z|$ sufficiently small

$$\|A^\gamma\|_{\mathcal{L}(\mathcal{H}^0_\gamma^{\beta,\gamma}(\mathcal{B}))} \leq c_\mu e^{\theta|\text{Im} z|}.$$ 

Let us first give a short outline of the proof. In view of Theorem 4.4 we can replace in (2.1) the resolvent $(\lambda - A)^{-1}$ by the right-hand side of (4.3). Then one obtains four integrals (corresponding to the four terms on the right-hand side of (4.3), each of which has to be estimated as in (1.1).

For the one associated with $G_\infty(\lambda)$ this is certainly possible, since $G_\infty(\lambda)$ is rapidly decreasing in $\lambda$ and therefore

$$\int c_\lambda^\gamma G_\infty(\lambda) d\lambda \|_{\mathcal{L}(\mathcal{H}^0_\gamma^{\beta,\gamma}(\mathcal{B}))} \leq c'_\mu \delta|\text{Re} z| e^{\theta|\text{Im} z|}.$$ 

Obviously $\delta|\text{Re} z|$ is uniformly bounded for small $|\text{Re} z|$.
For the integral connected with \((1 - \sigma)P(\lambda)(1 - \sigma_1)\) one can proceed exactly as in the proof of Theorem 1 of [23], since this term is localized away from the boundary, and there, \(H_p^{0,\gamma}(\mathbb{B})\) coincides with the usual \(L_p\)-spaces (note also the remark after formula (1.6)).

Hence it remains to consider the expressions

\[
\sigma \int_C \lambda^\varphi G(\lambda) \, d\lambda \sigma_0, \quad \sigma \int_C \lambda^\varphi \mu op_M(\gamma) \, d\lambda \sigma_0.
\]

We shall start with the analysis of the first term. Letting

\[
H_p^{0,\gamma}(X^\gamma) = L_p(X^\gamma, t_0^{\frac{n+\gamma}{n}} \mu \, dt \, dx) = t^\gamma L_p(X^\gamma, t_0^{\frac{n+\gamma}{n}} \mu \, dt \, dx),
\]

it is obvious from Definition 3.1 that multiplication with any cut-off function \(\sigma \in C_0^\infty([0,1])\) induces continuous operators \(H_p^{0,\gamma}(\mathbb{B}) \to H_p^{0,\gamma}(X^\gamma)\) and \(H_p^{0,\gamma}(X^\gamma) \to H_p^{0,\gamma}(\mathbb{B})\). Estimating the first term in (5.2) thus reduces to the following proposition:

**Proposition 5.2.** Let \(G(\lambda) \in R_G^{\mu,\omega}(X^\gamma; \lambda, \gamma)\) and \(G_z = \int_C \lambda^\varphi G(\lambda) \, d\lambda \) for \(z \in \mathbb{H}\). Then \(G_z \in L(H_p^{\gamma}(X^\gamma))\), and there exists a constant \(c_p \geq 0\) such that for \(|Re z|\) sufficiently small

\[
\|G_z\|_{L(H_p^{\gamma}(X^\gamma))} \leq c_p e^{\epsilon |z|}.\]

**Proof.** By conjugation with \(t^\gamma\) we can assume that \(\gamma = 0\) (cf. the proof of Proposition 4.7). If we split the integral into three terms according to the decomposition of \(C\) in (10.1), the integral over \(C_2\) can be estimated in the desired way, since \(|G(\lambda)\|\) is bounded on \(C_2\). By symmetry, \(C_1\) and \(C_3\) can be treated in the same way. So we shall assume for the rest of the proof that

\[
C(t) = C_1(t) = te^{i\theta}, \quad -\infty < t \leq 1,
\]

(for notational convenience we replace \(\delta\) by \(1\)). Also for convenience we suppress the \(x\)-variables from the notation. We shall frequently make use of the fact that, substituting \(\lambda = \rho^\mu e^{i\theta}\), we have

\[
\int_C f(\lambda) \, d\lambda = \mu e^{i\theta} \int_1^\infty f(\rho^\mu e^{i\theta}) \rho^{\mu-1} \, d\rho.
\]

According to Definition 13.3, \(G(\lambda)\) for \(|\lambda| \geq 1\) is an integral operator (with respect to the scalar product in \(H_2^{0,0}(X^\gamma)\)) with kernel

\[
k(\lambda, t, s) = |\lambda|^{n+\gamma} \hat{k}(\lambda, |\lambda|^{\frac{1}{n}} t, |\lambda|^{\frac{1}{n}} s),
\]

where, for some \(\varepsilon > 0\),

\[
\hat{k}(\lambda, t, s) \in S^{-1}(\Lambda) \otimes_\pi S_0^0(\mathcal{X}^\gamma) \otimes_\pi S_0^0(\mathcal{X}^\gamma)
\]

(the fact that \(\hat{k}\) is classical will not play a role for the following calculations). Then \(G_z\) is an integral operator with kernel

\[
k_z(t, s) = \int_C \lambda^\varphi k(\lambda, t, s) \, d\lambda.
\]

Writing \(\hat{k}(\lambda, t, s) = (\hat{\chi}(t) + (1 - \hat{\chi})(t))\hat{k}(\lambda, t, s)(\hat{\chi}(s) + (1 - \hat{\chi})(s))\) with the characteristic function \(\hat{\chi}\) of \([0,1]\), the proposition will be true, if we can show that in any of the four cases

\[
\begin{align*}
(5.4) & \quad k_z(t, s) = e^{-\|Im z\|} \int_C \lambda^\varphi \hat{\chi}(|\lambda|^{\frac{1}{n}} t) k(\lambda, t, s) \hat{\chi}(|\lambda|^{\frac{1}{n}} s) \, d\lambda \\
(5.5) & \quad k_z(t, s) = e^{-\|Im z\|} \int_C \lambda^\varphi \hat{\chi}(|\lambda|^{\frac{1}{n}} t) k(\lambda, t, s)(1 - \hat{\chi})(|\lambda|^{\frac{1}{n}} s) \, d\lambda \\
(5.6) & \quad k_z(t, s) = e^{-\|Im z\|} \int_C \lambda^\varphi (1 - \hat{\chi})(|\lambda|^{\frac{1}{n}} t) k(\lambda, t, s) \hat{\chi}(|\lambda|^{\frac{1}{n}} s) \, d\lambda \\
(5.7) & \quad k_z(t, s) = e^{-\|Im z\|} \int_C \lambda^\varphi (1 - \hat{\chi})(|\lambda|^{\frac{1}{n}} t) k(\lambda, t, s)(1 - \hat{\chi})(|\lambda|^{\frac{1}{n}} s) \, d\lambda
\end{align*}
\]
the associated integral operators are bounded in $\mathcal{H}^{0,0}_p(X^\wedge)$, uniformly in $-\alpha \leq \text{Re } z < 0$ for some $\alpha > 0$. The cases (5.3) and (5.4) are equivalent by symmetry (i.e. passing to the adjoint). The proofs of all cases (5.4), (5.5), and (5.7) rely on the following Hardy inequalities:

\begin{align}
(5.8) & \quad \int_0^\infty \left( \int_0^t g(s) \, ds \right)^p t^{-1-r} \, dt \leq \left( \frac{p}{r} \right)^p \int_0^\infty g(t)^p t^{p-1-r} \, dt \\
(5.9) & \quad \int_0^\infty \left( \int_t^\infty g(s) \, ds \right)^p t^{-1+r} \, dt \leq \left( \frac{p}{r} \right)^p \int_0^\infty g(t)^p t^{p-1+r} \, dt
\end{align}

for any non-negative function $g$ on $\mathbb{R}_+$ and $r > 0$ (cf. [27], Lemma 3.14, page 196). To begin with case (5.4) we use the fact that, for some fixed $\epsilon > 0$,

$$|\hat{k}(\lambda, t, s)| \leq c|\lambda|^{-1} |t - \frac{n+1}{2} + \epsilon| s^{-\frac{n+1}{2} + \epsilon}$$

uniformly in $\lambda \in C$ and $t, s > 0$, to obtain

$$|k_z(t, s)| \leq c t^{-\frac{n+1}{2}+\epsilon} s^{-\frac{n+1}{2} + \epsilon} \int_1^\infty g^{\mu \text{Re } z - 1 + 2 \epsilon} \tilde{\chi}(gt) \tilde{\chi}(gs) \, d\varrho$$

$$= c \frac{|\mu \text{Re } z + 2\epsilon|}{\mu \text{Re } z + 2\epsilon} \left( \text{min} (\frac{1}{t}, \frac{1}{s})^{\mu \text{Re } z + 2\epsilon} - 1 \right) \tilde{\chi}(t) \tilde{\chi}(s) t^{-\frac{n+1}{2} + \epsilon} s^{-\frac{n+1}{2} + \epsilon}.$$

Since $\mu \text{Re } z$ is negative, the factor $\text{min} (\frac{1}{t}, \frac{1}{s})^{\mu \text{Re } z + 2\epsilon}$ is uniformly bounded by 1 for $0 < s, t \leq 1$. If $-\frac{\epsilon}{\mu} = -\alpha \leq \text{Re } z < 0$ the factor $c$ can be estimated from above by a constant uniformly in $0 < s, t \leq 1$. Furthermore the kernel function $\tilde{\chi}(t) \tilde{\chi}(s) t^{-\frac{n+1}{2} + \epsilon} s^{-\frac{n+1}{2} + \epsilon}$ belongs to $\mathcal{H}^{0,0}_p(X^\wedge)$ and thus induces a continuous operator in $\mathcal{H}^{0,0}_p(X^\wedge)$, it remains to consider the kernel $t^{-\frac{n+1}{2} + \epsilon} s^{-\frac{n+1}{2} + \epsilon} \text{min} (\frac{1}{t}, \frac{1}{s})^{2\epsilon}$. Because this kernel is symmetric in $s$ and $t$, indeed it suffices to treat

$$k(t, s) = \begin{cases} t^{-\frac{n+1}{2} + \epsilon} s^{-\frac{n+1}{2} + \epsilon} & : s \leq t \\ 0 & : s > t \end{cases}.$$

If $G$ denotes the associated integral operator, then

$$\|G u\|_{\mathcal{H}^{0,0}_p(X^\wedge)}^p \leq \int_0^\infty \left( \int_0^\infty k(t, s)|u(s)|^p s^p \, ds \right)^p t^{\frac{n+1}{2} - p - 1} \, dt = \int_0^\infty \left( \int_0^t s^{\frac{n+1}{2} + \epsilon} |u(s)|^p \, ds \right)^p t^{1-p} \, dt$$

$$\leq \left( \frac{p}{ps} \right)^p \int_0^\infty |u(t)|^p t^{\frac{n+1}{2} - p - 1} \, dt = \left( \frac{1}{p} \right)^p \|u\|^p_{\mathcal{H}^{0,0}_p(X^\wedge)}$$

by Hardy’s inequality (5.8). This finishes case (5.4). For case (5.3) observe that

$$|\hat{k}(\lambda, t, s)(1 - \tilde{\chi})(s)| \leq c_N |\lambda|^{-1} s^{-\frac{n+1}{2} + \epsilon} s^{-N}$$

for any $N \in \mathbb{N}$ uniformly in $\lambda \in C$ and $s, t > 0$. Then

$$|k_z(t, s)| \leq c_N t^{-\frac{n+1}{2} + \epsilon} s^{-N} \int_1^\infty g^{\mu \text{Re } z + \frac{n+1}{2} + \epsilon - N} \tilde{\chi}(gt)(1 - \tilde{\chi})(gs) \, d\varrho.$$

This expression equals zero if $s \leq t$ and for $s > t$ we can estimate

$$|k_z(t, s)| \leq c_N t^{-\frac{n+1}{2} + \epsilon} s^{-N} \int_{1/t}^{1/t} g^{\mu \text{Re } z + \frac{n+1}{2} + \epsilon - N} \, d\varrho = \frac{c_N}{\mu \text{Re } z + \frac{n+1}{2} + \epsilon - N} (k^1_z(t, s) - k^2_z(t, s))$$

with kernel functions $k^1_z$ and $k^2_z$ given by

$$k^1_z(t, s) = \tilde{\chi}(t) \begin{cases} 0 & : s \leq t \\ \left( \frac{1}{s} \right)^N t^{-n-1-\mu \text{Re } z} & : s > t \end{cases}$$

$$k^2_z(t, s) = \tilde{\chi}(t) \begin{cases} 0 & : s \leq t \\ \left( \frac{1}{ts} \right)^{-\frac{n+1}{2}} \left( \frac{1}{s} \right)^\epsilon t^{-\mu \text{Re } z} & : s > t \end{cases}.$$

In order to check the uniform boundedness of the integral operator $K^1_z$ associated with $k^1_z$, $-\alpha \leq \text{Re } z < 0$, on $\mathcal{H}^{0,0}_p(X^\wedge)$ we observe that

$$\mathcal{H}^{0,0}_p(X^\wedge) = \ell^p L_p(X^\wedge, t^n \, dt \, dx)$$
with \( \beta = p(n+1)(\frac{1}{2} - \frac{1}{p}) \). The boundedness of \( K^1_z \) is equivalent to the boundedness of \( t^\beta K^1_z t^{-\beta} \) on \( L_\mu(X^\times, t^n dt dx) \). To show it, we employ Schur’s lemma: if \( N \) is sufficiently large, then
\[
\int_0^\infty t^\beta K^1_z(t,s) s^{-\beta} s^n ds = \tilde{\chi}(t) t^{\beta+N-n-1-\mu \text{Re} z} \int_t^\infty s^{-N+n-\beta} ds = \frac{1}{N-n+\beta-t^\mu \text{Re} z} \tilde{\chi}(t) \leq 1
\] and
\[
\int_0^\infty t^\beta K^1_z(t,s) s^{-\beta} s^n ds = \frac{1}{N-n+\beta-t^\mu \text{Re} z} \min(1,s)^{\beta+N-\mu \text{Re} z} \leq 1.
\]
To handle \( k^2_z \) first observe that we can drop the factor \( s^{-\mu \text{Re} z} \), since this is uniformly bounded by \( 1 \) in \( s \leq 1 \) and \( \text{Re} z < 0 \), and if \( s \geq 1 \) and \( -\frac{z}{\mu} \leq \text{Re} z < 0 \), then \( (\frac{z}{\mu})^{\epsilon/2} s^{-\mu \text{Re} z} \leq 1 \) for \( 0 \leq t \leq 1 \) (for \( t \geq 1 \) anyway \( k^2_z(t,s) = 0 \)). Thus we can assume that
\[
k^2_z(t,s) = k^2(t,s) = \tilde{\chi}(t) \left\{ \begin{array}{ll}
0 & : s \leq t \\
(s t)^{-n+1} (\frac{1}{n+1})^\epsilon t^{-n+1+\epsilon} s^{-\frac{n+1}{2}+\epsilon} & : s > t \end{array} \right.
\]
But then Hardy’s inequality \( (5.9) \) shows that the integral operator associated with the kernel \( k^2_z \) is continuous in \( \mathcal{H}_p^0(X^\times) \) with operator norm bounded by \( \frac{1}{2} \). This finishes case \( (5.5) \). For the final case \( (5.7) \) we use that
\[
| (1 - \tilde{\chi}(t)) k_0(\lambda, t, s)(1 - \tilde{\chi}(s)) | \leq c_N |\lambda|^{-1} t^{-N} s^{-N}
\]
for any \( \lambda \in \mathbb{C} \) and \( s,t > 0 \). Then
\[
| k_z(t,s) | \leq c_N t^{-N} s^{-N} \int_1^{\infty} \mu \text{Re} z + n - 2N (1 - \tilde{\chi})(\mu t)(1 - \tilde{\chi})(\mu s) d\mu
\]
\[
= -\mu \text{Re} z + n - 2N t^{-N} s^{-N} \max (1, \frac{1}{t}, \frac{1}{s}) \mu \text{Re} z + n + 1 - 2N.
\]
The factor in front is obviously uniformly bounded in \( \text{Re} z < 0 \) for \( N \) sufficiently large. Since \( \mu \text{Re} z \) is negative,
\[
| (1 - \tilde{\chi}(t)) k_z(t,s)(1 - \tilde{\chi}(s)) | \leq c (1 - \tilde{\chi}(t)) t^{-N} s^{-N} (1 - \tilde{\chi}(s)),
\]
\[
| \tilde{\chi}(t) k_z(t,s)(1 - \tilde{\chi}(s)) | \leq c \tilde{\chi}(t) t^{N-n-1} s^{-N-n} (1 - \tilde{\chi}(s)),
\]
\[
| (1 - \tilde{\chi}(t)) k_z(t,s) \tilde{\chi}(s) | \leq c (1 - \tilde{\chi}(t)) t^{N-n} s^{-N-n} \tilde{\chi}(s).
\]
All these kernel functions belong to \( \mathcal{H}_p^0(X^\times) \otimes \mathcal{H}_{p'}^0(X^\times) \) for sufficiently large \( N \) and thus induce continuous operators in \( \mathcal{H}_p^0(X^\times) \). Hence it remains to investigate \( \tilde{\chi}(t) k_z(t,s) \tilde{\chi}(s) \) and by symmetry even
\[
k_z(t,s) = \left\{ \begin{array}{ll}
0 & : s \leq t \\
(s t)^{-n+1} (\frac{1}{n+1})^\epsilon t^{-n+1+\epsilon} s^{-\frac{n+1}{2}+\epsilon} & : s > t \end{array} \right.
\]
Again Hardy’s inequality \( (5.9) \) shows that the associated operator is \( \mathcal{H}^{0,0}(X^\times) \)-continuous. \( \square \)

We consider now the second term in \( (5.4) \). Using a partition of unity on \( X \) with any two functions supported in a single coordinate neighborhood, we can assume \( X = \mathbb{R}^n \) and use local symbols compactly supported in \( x \). To complete the proof of theorem \( (5.1) \) we make use of the decomposition \( (4.6) \) and of the fact (see Section \( (10) \)) that operators defined by means of symbols \( a \in MS^0(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \Gamma_{\text{sol}}^+ - \gamma, \mathbb{R}^n; \Lambda) \) are bounded on \( \mathcal{H}_p^\gamma(\mathbb{R}^+ \times \mathbb{R}^n) \), with norm estimated in terms of the seminorms associated to \( a \). We treat the homogeneous principal symbol of \( g \) and the lower order part separately.

**Lemma 5.3.** Let \( b \in S^{-\mu-1,\mu}(\mathbb{R}^+ \times \mathbb{R}^n \times \Gamma_{\text{sol}}^+ - \gamma, \mathbb{R}^n; \Lambda) \) be compactly supported in \( t \) and
\[
b_z(t,x,\frac{n+1}{2} - \gamma + i\tau, \xi) = t^\mu \int_C \lambda^2 b(t,x,\frac{n+1}{2} - \gamma + i\tau, \xi, t^\mu \lambda) d\lambda.
\]
For Re \( z < 0 \) this defines a symbol \( b_z \in MS^0(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{n+1}{2} - \gamma} \times \mathbb{R}^n) \), and the symbol estimates of \( e^{-\theta|\text{Im} z|} b_z \) are uniform in \(-1 \leq \text{Re} z < 0\). Consequently,

\[
\|o_{\mathcal{M}}^\gamma \tilde{\Phi}(b_z)\|_{L(H^\gamma_p(\mathbb{R}_+ \times \mathbb{R}^n))} \leq c_p e^{\theta|\text{Im} z|}
\]

uniformly in \(-1 \leq \text{Re} z < 0\).

**Proof.** Without loss of generality, we can set \( \gamma = \frac{n+1}{2} \). We have to show that

\[
|\partial^k_x (t \partial_t)^k \partial^2_x b_z(t, x, i\tau, \xi)| e^{-\theta|\text{Im} z|} \langle \tau, \xi \rangle^{|\alpha|} \leq c e^{\theta|\text{Im} z|}
\]

uniformly in \( t > 0 \), \( x \in \mathbb{R}^n \), \( \tau \in \mathbb{R} \) and \(-1 \leq \text{Re} z < 0\). The totally characteristic derivatives in \( t \) can be handled very simply, observing that \( t \partial_t t^\mu = \mu t^\mu \),

\[
t \partial_t \left( \tilde{b}(t, x, i\tau, \xi, t^\mu \lambda) \right) = (t \partial_t \tilde{b})(t, x, i\tau, \xi, t^\mu \lambda) + \mu (\lambda \partial_\lambda \tilde{b})(t, x, i\tau, \xi, t^\mu \lambda)
\]

and both symbols \( t \partial_t \tilde{b} \) and \( \lambda \partial_\lambda \tilde{b} \) are of the same type as \( \tilde{b} \). Since the derivatives with respect to \( x \), \( \tau \) and \( \xi \) can be taken under the integral sign, it suffices to assume \( \tilde{b} \in S^{\mu - 1 - k, \mu}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_0 \times \mathbb{R}^n; \Lambda) \) and to show that

\[
|b_z(t, x, i\tau, \xi)| \leq c e^{\theta|\text{Im} z|} \langle \tau, \xi \rangle^{-k}
\]

uniformly in \( t > 0 \), \( x \in \mathbb{R}^n \), \( \tau \in \mathbb{R} \) and \(-1 \leq \text{Re} z < 0\). By hypothesis, we have

\[
|b_z(t, x, i\tau, \xi)| \leq c t^\mu \int_C |\lambda|^z (1 + \tau^2 + |\xi|^2 + |t^\mu \lambda|^{2/\mu})^{-\mu - 1 - k/2} \, d\lambda,
\]

and on \( C \) we can estimate \( |\lambda|^z \) from above by \( \delta \text{Re} z e^{\theta|\text{Im} z|} \). The transformation \( \varrho = t^\mu \lambda \) yields

\[
|b_z(t, x, i\tau, \xi)| \leq c \delta \text{Re} z e^{\theta|\text{Im} z|} \langle \tau, \xi \rangle^{-k} \int_{t^\mu C \setminus C(1 + |\varrho|)^{-1 - 1/k} \, d\varrho,
\]

where \( t^\mu C \) means the path \( C(t^\mu \delta, \theta) \). Since the support of \( \tilde{b} \) is compact, we may assume without loss of generality that \( 0 < t \leq 1 \). Then we obtain the estimate

\[
\int_{t^\mu C \setminus C} (1 + |\varrho|)^{-1 - 1/k} \, d\varrho \leq 2\pi \delta + \int_{C(1 + |\varrho|)^{-1 - 1/k} \, d\varrho < +\infty,
\]

and the statement follows, since \( \delta \text{Re} z \) is uniformly bounded in \(-1 \leq \text{Re} z < 0\).

**Proposition 5.4.** Let \( \tilde{g} = \tilde{g}(t, x, \zeta, \lambda) \) be a local symbol associated to the Mellin symbol \( \check{g} \) of \((\lambda - A)^{-1}\) of Theorem 4.8(i) and let

\[
g_z(t, x, \frac{n+1}{2} - \gamma + i\tau, \xi) = \sigma(t) t^\mu \int_C |\lambda|^{z\check{g}(t, x, \frac{n+1}{2} - \gamma + i\tau, \xi, t^\mu \lambda)} \, d\lambda
\]

with some cut-off function \( \sigma \in C^\infty_{\text{comp}}([0,1]) \). For \( \text{Re} z < 0 \) this defines a symbol \( g_z \in MS^0(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{n+1}{2} - \gamma} \times \mathbb{R}^n) \), and the symbol estimates of \( e^{-\theta|\text{Im} z|} g_z \) are uniform in \(-1 \leq \text{Re} z < 0\). In particular,

\[
\|o_{\mathcal{M}}^\gamma \tilde{\Phi}(g_z)\|_{L(H^\gamma_p(\mathbb{R}_+ \times \mathbb{R}^n))} \leq c_p e^{\theta|\text{Im} z|}
\]

uniformly in \(-1 \leq \text{Re} z < 0\).

**Proof.** Without loss of generality let \( \gamma = \frac{n+1}{2} \). We shall also suppress \( \sigma \) from the notation and instead assume that \( 0 < t \leq 1 \). We can also assume \( x \) confined to a compact subset of \( \mathbb{R}^n \). By 4.8,

\[
\tilde{g}(t, x, i\tau, \xi, \lambda) = \chi(|\tau, \xi|^2 + |\lambda|^2) (\lambda - \check{g}(t, x, i\tau, \xi))^{-1} \mod S^{\mu - 1, \mu}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda),
\]
where \( \tilde{q}^{(\mu)} \) denotes a local symbol of \( A \) as defined in \((4.4)\). In view of Lemma 5.3, we therefore may assume that

\[
g_z(t, x, \tau, \xi) = t^\mu \int_C \lambda^\tau \chi(|\tau, \xi|^2 + t^2 |\lambda|^2)(t^\mu \lambda - \tilde{q}^{(\mu)}(t, x, \tau, \xi))^{-1} d\lambda
\]

where we have used the substitution \( \theta = t^{\mu} \lambda \). We have to estimate this expression as in \((5.10)\). The factor \( t^{\mu-\tau} \) behaves correctly, since \( (t^{\partial_t})^k t^{\mu-\tau} = (-\mu \tau)^k t^{\mu-\tau} \) is uniformly bounded in \( 0 < t \leq 1 \) and \(-1 \leq \text{Re} \ z < 0\). For \((\tau, \xi) \neq 0\) we have

\[
\text{spec}(\tilde{q}^{(\mu)}(t, x, \tau, \xi)) \subset \{ \lambda \in \mathbb{C} \mid c_1 |\tau, \xi|^{\mu} \leq |\lambda| \leq c_2 |\tau, \xi|^{\mu} \text{ and } |\arg \lambda| < \theta \}
\]

with suitable constants \( c_1 \) and \( c_2 \). Thus for large enough \(|\tau, \xi|\) we have \( \chi(|\tau, \xi|^2 + t^{-2\mu}|q|^2) = 1 \) and the spectrum of \( \tilde{q}^{(\mu)}(t, x, \tau, \xi) \) is located to the right of the path \( C \). By Cauchy’s theorem we can then replace the path \( t^{\mu}C \) by \( C \), and obtain for large \(|\tau, \xi|\)

\[
\int_C \tilde{q}^2(\theta - \tilde{q}^{(\mu)}(t, x, \tau, \xi))^{-1} d\theta = 2\pi i \tilde{q}^{(\mu)}(t, x, \tau, \xi) z.
\]

Then we can estimate (as in \((23), (2.9)\))

\[
|\partial^l _t (t\partial_t) g_z(t, x, \tau, \xi)| \leq p(|z|) e^{\text{Im} z} |\tau, \xi|^{\mu} |\tau, \xi|^{-l-|\alpha|} \leq p(|z|) e^{\text{Im} z} |\tau, \xi|^{-l-|\alpha|}
\]

with a polynomial \( p \). However, since we can replace \( \theta \) by \( \theta - \varepsilon \) for some \( \varepsilon > 0 \) (as noted in the comments on conditions (E1) and (E2)), this yields the uniform symbol estimates of \( g_z \) for large \(|\tau, \xi|\).

For small \(|\tau, \xi|\), we now shall show that

\[
g_z(t, x, \tau, \xi) = t^{-\mu} \int_{\Upsilon(t)} \theta^2 a(t, x, \tau, \xi; \theta) d\theta,
\]

where we have set

\[
a(t, x, \tau, \xi; \theta) = \chi(|\tau, \xi|^2 + |\theta|^2)(\theta - \tilde{q}^{(\mu)}(t, x, \tau, \xi))^{-1} \in S^{-\mu}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda)
\]

and \( \Upsilon(t) \) is the path given in the following picture (with \( r_0 > 0 \) to be chosen appropriately):
In fact, the difference of both sides from (5.11) equals
\begin{equation}
\alpha(r) = \int_{C(r,0)} \varphi^* a(t, x, \tau, \xi; \varrho) \, d\varrho \quad \text{for } r = r_0.
\end{equation}
Since, for small \(|\tau, \xi|\), the spectrum of \(\varphi^{(\mu)}(t, x, i\tau, \xi)\) is contained in some ball of finite radius, \(a(t, x, \tau, \xi; \varrho)\) is holomorphic in \(\varrho\) for \(|\varrho| \geq r_0\), if \(r_0\) is chosen large enough. Thus \(\alpha(r) = \alpha(r_0)\) for all \(r \geq r_0\), by Cauchy’s theorem. For any fixed \(z\) and \((t, x, \tau, \xi)\) the integrand in (5.12) is \(O(|\varrho|^{-1+\Re z})\) for \(|\varrho| \to \infty\) and, on the radial part of \(C(r)\), the integrand is \(O(t^\Re z)\). Hence, \(\alpha(r) = \lim_{r \to +\infty} \alpha(r) = 0\), and (5.11) holds.

To estimate the right-hand side of (5.11), we split the integral into four parts, which we briefly analyze separately. First of all, observe that \(|\varphi^*|\) can be estimated from above by \(e^{\theta|\Im z|}(\mu \delta)^{\Re z}\) on the whole path. This and the fact that \(a(t, x, \tau, \xi; \varrho) \in S^{-\mu-\mu}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda)\) are enough to get the desired estimates for the terms obtained integrating along the two arcs \(A_1A_2\) and \(A_3A_4\), since they can be treated with essentially the same technique we used to prove Lemma 3.3. The term obtained integrating along \(A_2A_3\) is
\[ b(t, x, \tau, \xi) = \int_{\rho_\delta}^{r_0} (se^{i\theta})^* a(t, x, \tau, \xi; se^{i\theta}) e^{i\theta} \, ds. \]
The derivatives with respect to \(t, \xi\) and \(\tau\) can be taken under the integral sign, so that we could again start with a symbol \(a \in S^{-\mu-\mu}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda)\) and prove that, for any \(l\),
\[ |(t\partial_t)^l b(t, x, \tau, \xi)| \leq \varrho e^{\theta|\Im z|}(\tau, \xi)^{-k} \]
uniformly in \(-1 < \Re z < 0\). This is true for \(l = 0\), as one can easily check. For \(l = 1\) we get
\[ t\partial_t b(t, x, \tau, \xi) = \int_{\rho_\delta}^{r_0} (se^{i\theta})^* (t\partial_t a(t, x, \tau, \xi; se^{i\theta}) e^{i\theta} \, ds - \mu(t\delta e^{i\theta})^{z+1} a(t, x, \tau, \xi; t\delta e^{i\theta}), \]
and this also satisfies the desired estimate. In fact, the first term is of the same kind as \(b\), while for the second it suffices to use the definition of \(S^{-\mu-\mu}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{1+n}; \Lambda)\) and the fact that \(0 < t \leq 1\). The result for arbitrary \(l\) can be proved by induction and, obviously, the contribution obtained integrating along \(A_4A_1\) behaves in a completely similar way. This yields the desired symbol estimates for small \(|\tau, \xi|\) and finishes the proof.

**Remark 5.5.** Let us point out that the proof of Theorem 5.1 only makes use of assumption (E1), the structure of the resolvent (1.11), and the fact that \(\text{spec } A \cap \Lambda = \emptyset\). It does not use that the domain \(D(A)\) equals \(H^{\mu, \gamma, \mu}_{\alpha, \beta}(\mathbb{B})\). Therefore, Theorem 5.1 holds true for \(A\) considered on other domains, as long as (E1), (4.6), and spec \(A \cap \Lambda = \emptyset\) are satisfied.

Complex powers of Fuchs-type differential operators have been studied recently also by Loya [13]. He applies Melrose’s b-calculus and focuses on the analytic properties of the kernels in the spirit of Seeley [21].

In the next section we shall investigate the possible closed extensions of \(A\), and use the previous remark to obtain an analogue of Theorem 5.1 for the maximal extension of \(A\).

### 6. Closed extensions of cone differential operators

Let \(A\) be a cone differential operator, which is elliptic with respect to \(\gamma + \mu\) in the sense of Remark 6.2. If we consider \(A\) as the unbounded operator in \(H^{0, \gamma}_{\alpha, \beta}(\mathbb{B})\) with domain \(C_{\text{comp}}(\text{int } \mathbb{B})\), its closure \(A_{\text{min}} = A_{\text{min}}^{\gamma, \mu}\) is given by
\[ D(A_{\text{min}}) = H^{0, \gamma + \mu}_{\alpha, \beta}(\mathbb{B}), \]
and the maximal closed extension \(A_{\text{max}} = A_{\text{max}}^{\gamma, \mu}\) by
\[ D(A_{\text{max}}) = \{ u \in H^{0, \gamma}_{\alpha, \beta} \mathbb{B} \mid Au \in H^{0, \gamma}_{\alpha, \beta}(\mathbb{B}) \}. \]
Note that in \( [38] \) we simply wrote \( A \) instead of \( A_{\min} \). Taking into account the duality of \( \mathcal{H}^0_p(\mathbb{B}) \) and \( \mathcal{H}^0_{-\gamma}(\mathbb{B}) \), the following lemma is valid:

**Lemma 6.1.** If \( A^t \) is the formal adjoint of \( A \) with respect to the scalar product of \( \mathcal{H}^0_{-\gamma}(\mathbb{B}) \), then

\[
(A_{\min}^\gamma)^* = (A^t)^{-\gamma}_{\max}, \quad (A^\gamma_{\max})^* = (A^t)^{-\gamma}_{\min}.
\]

We shall write this more shortly as \( A_{\min}^* = A^t_{\max} \) and \( A_{\max}^* = A^t_{\min} \).

A proof of the above statements in case \( p = 2 \) and \( \gamma = 0 \) is given in \( [12] \). The argument in the general case is analogous. As a simple consequence,

\[
(\lambda - A_{\max})^{-1} = [(\lambda - A_{\min})^{-1}]^*
\]

whenever one of both sides exists. Since the structure of the resolvent of \( A = A_{\min} \) as given in Theorem 6.4 is invariant under passing to the adjoint, we obtain the following theorem:

**Theorem 6.2.** If \( A_{\min}^t \) is elliptic with respect to \( \Lambda_\Delta \) and \( -\gamma + \mu \), then \( A_{\max} \) has no spectrum in \( \Lambda_\Delta \cap \{|\lambda| > R\} \) for some \( R > 0 \), and for large \( \lambda \in \Lambda_\Delta \)

\[
(\lambda - A_{\max})^{-1} = \sigma \left\{ t^{\mu_p} \mathcal{M}_\Lambda^\gamma(g)(\lambda) + G(\lambda) \right\} \sigma_0 + (1 - \sigma)P(\lambda) (1 - \sigma_1) + G_\infty(\lambda),
\]

where \( \sigma, \sigma_0, \sigma_1 \in C^\infty([0, 1]) \) are cut-off functions satisfying \( \sigma_1 \sigma = \sigma, \sigma \sigma_0 = \sigma \), and

1. \( g(t, z, \lambda) = \tilde{g}(t, z, t^\mu \lambda) \) with \( \tilde{g} \in C^\infty(\mathbb{R}^+; \mathbb{M}^{-\mu, \mu}_G(X; \Lambda)) \),
2. \( P(\lambda) \in L^{-\mu, \mu}(\text{int} \mathbb{B}; \Lambda) \),
3. \( G(\lambda) \in R^{-\mu, \mu}_G(\mathbb{X}; \Lambda, \gamma), \) and \( G_\infty(\lambda) \in C^{-\infty}_G(\mathbb{B}; \Lambda, \gamma) \).

Proceeding exactly as in Proposition 4.7 and Section 5, we can prove a norm estimate for the complex powers of \( A_{\max}^t \):

**Theorem 6.3.** Let \( A_{\min}^t \) be elliptic with respect to \( -\gamma + \mu \) and \( \Lambda_\Delta \), having no spectrum in the keyhole \( \Lambda = \Lambda(\delta, \theta) \) except perhaps 0. Then one can define \( A_{\max}^t \) as in (2.1) and there exists a constant \( c_p \geq 0 \) such that for all \( z \in \mathbb{H} \) with \( |\text{Re} \ z| \) sufficiently small

\[
\|A_{\max}^t\|_{L(\mathcal{H}^0_p(\gamma)(\mathbb{B}))} \leq c_p e^{\theta |\text{Im} \ z|}.
\]

Of course, it is desirable to express the ellipticity assumptions made on \( A_{\min}^t \) in the previous two theorems purely in terms of \( A_{\max}^t \). This can be done as follows.

**Remark 6.4.** If \( \hat{A} \) is the model cone operator on \( \mathbb{X}^\wedge \) associated with \( A \) and \( \hat{A}_{\max} = \hat{A}_{\max}^{\gamma, \mu} \) is the closed operator given by

\[
\mathcal{D}(\hat{A}_{\max}) = \left\{ u \in \mathcal{K}^0_p(\mathbb{X}^\wedge) \mid \hat{A}u \in \mathcal{K}^0_p(\mathbb{X}^\wedge) \right\},
\]

then \( A_{\min}^t \) is elliptic with respect to \( -\gamma + \mu \) and \( \Lambda_\Delta \) if and only if \( A \) satisfies condition (E1) and (E2') \( \hat{A}_{\max} \) has no spectrum in \( \Lambda_\Delta \setminus \{0\} \).

The previous remark holds true, since similar to Lemma 6.1 \( \hat{A}_{\min}^* = \hat{A}^t_{\max} \) and \( \hat{A}_{\max}^* = \hat{A}_{\min}^t \).

It can be shown that \( \mathcal{D}(A_{\max}) \) differs from \( \mathcal{D}(A_{\min}) \) by a finite dimensional space (for the case \( p = 2 \) see 12)

\[
\mathcal{D}(A_{\max}) = \mathcal{D}(A_{\min}) \oplus V, \quad \dim V < \infty.
\]

More precisely, the dimension of \( V \) only depends on the conormal symbol of \( A \),

\[
\dim V = \sum_{-2 < \text{Re} \ z - \frac{\gamma}{p} + \gamma < 0} M(\sigma^\mu_M(A), z), \quad \text{(6.1)}
\]
where $M(h, z)$ denotes the multiplicity in $z$ in the sense of \[10\] of a function $h$, which is holomorphic in a punctured neighborhood of $z$. Moreover, $V$ consists of smooth functions of the form

$$\omega(t) \sum_{j=0}^{N} \sum_{k=0}^{k_j} c_{jk}(x)t^{-p_j} (\log t)^k, \quad c_{jk} \in C^\infty(X);$$

the coefficients $c_{jk}$, the exponents $p_j \in \mathbb{C}$ ($\frac{n+1}{2} - \gamma - \mu \leq \text{Re} p_j < \frac{n+1}{2} - \gamma$), and $k_j, N \in \mathbb{N}_0$ are determined by $A$. In particular, the only closed extensions of $A$ are the operators $A_W$ given by

$$D(A_W) = D(A_{\min}) \oplus W, \quad W \leq V.$$ 

In this notation, $A_{\min} = A_{(0)}$ and $A_{\max} = A_V$. Correspondingly,

$$D(\hat{A}_{\max}) = D(\hat{A}_{\min}) \oplus \hat{V}, \quad \dim \hat{V} = \dim V,$$

and all closed extensions $\hat{A}_W$ are given by

$$D(\hat{A}_W) = D(\hat{A}_{\min}) \oplus \hat{W}, \quad \hat{W} \leq \hat{V}.$$ 

**Remark 6.5.** If the conormal symbol $\sigma^p_M(A)(z)$ is invertible for all $\frac{n+1}{2} - \gamma - \mu < \text{Re} z < \frac{n+1}{2} - \gamma$, then $\dim V = \dim \hat{V} = 0$ by (6.1), and both $A$ and $\hat{A}$ have only one closed extension in $\mathcal{H}_p^0(\mathbb{B})$ and $\mathcal{K}_p^0(\mathbb{B}^\gamma)$, respectively.

**7. Example: The Cauchy problem for Laplacians**

Let $g(t)$ be a family of metrics on $X$, depending smoothly on a parameter $t \in \mathbb{R}_+$, and $\Delta_X(t)$ the corresponding Laplacian on $X$. If we equip $\text{int} \mathbb{B}$ with a metric that coincides with $dt^2 + t^2 g(t)$ near $t = 0$, the associated Laplacian $\Delta$ is near the boundary given by

$$t^{-2} \left\{ (t\partial_t)^2 + (n-1 + t G^{-1}(t) (\partial_t G(t)) t \partial_t + \Delta_X(t) \right\},$$

where $G = (\det(g_{ij}))^{\frac{1}{2}}$ and $n = \dim X$. Hence $\Delta$ is a cone differential operator in the sense of (3.1). We shall prove the following theorem:

**Theorem 7.1.** Let $\Delta$ be the Laplacian on $\text{int} \mathbb{B}$ in the above sense, $1 < p < \infty$ such that

\begin{equation}
2 \max(p, p') - 1 < n = \dim \partial \mathbb{B}.
\end{equation}

If $\gamma_p = (n+1)(\frac{1}{2} - \frac{1}{p'})$, then $\Delta$ defined on $C^\infty_\text{comp}(\text{int} \mathbb{B})$ has for any $1 < q < \infty$ a unique closed extension $\Delta_{p,q}$ in $\mathcal{H}_q^{0,\gamma_p}(\mathbb{B})$, which is given by

$$D(\Delta_{p,q}) = \mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B}).$$

Moreover, $-\Delta_{p,q}$ is elliptic with respect to $\gamma_p + 2$ and any sector $\Lambda_{\Delta} \subset \mathbb{C} \setminus \mathbb{R}_+.$

**Proof.** Let us set $A = -\Delta$. The rescaled symbol of $A$ is

$$\hat{\sigma}_q^2(A)(x, \tau, \xi) = \tau^2 + |\xi|^2,$$

where $|\xi|$ refers to the metric $g(0)$ on $X$. Hence $A$ satisfies the ellipticity condition (E1) for any $\Lambda_{\Delta}$ in question. The conormal symbol of $A$, cf. (3.6) and (3.7), is

$$\hat{\sigma}_Q^2(A)(z) = -z^2 + (n-1)z - \Delta_X(0) : H^s(X) \longrightarrow H^{s-2}(X).$$

If $0 = \lambda_0 \geq \lambda_1 \geq \ldots$ are the eigenvalues of $\Delta_X(0)$, then $\sigma_{Q}^2(A)(z)$ is not bijective if and only if

$$z \in \left\{ \frac{n+1}{2} \pm \left( \frac{n-1}{2} - \lambda_j \right)^{\frac{1}{2}} | j \in \mathbb{N}_0 \right\}.$$ 

Note that, in particular, $\sigma_{Q}^2(A)(z)$ is invertible for all $z$ with $0 < \text{Re} z < n - 1$, and thus by condition (7.1) for all $z$ with $\frac{n+1}{2} - \gamma_p - 2 < \text{Re} z \leq \frac{n+1}{2} - \gamma_p$. This shows that $A$ is elliptic with respect to $\gamma_p + 2$ in the sense of Remark 3.2 and has only one closed extension

$$A_{p,q} : \mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B}) \subset \mathcal{H}_q^{0,\gamma_p}(\mathbb{B}) \longrightarrow \mathcal{H}_q^{0,\gamma_p}(\mathbb{B})$$.
by Remark 6.3. The model cone operator is
\[ \hat{A} = -t^{-2} \left\{ (t\partial_t)^2 + (n-1)(t\partial_t) + \Delta_X(0) \right\}, \]
i.e. \( \hat{A} \) is the Laplacian on \( X^\gamma \) with respect to the metric \( dt^2 + t^2g(0) \). As before, \( \hat{A} \) has a unique closed extension
\[ \hat{A}_{p,q} : \mathcal{K}^{2,\gamma_p+2}(X^\gamma) \subset \mathcal{K}^{0,\gamma_p}(X^\gamma) \to \mathcal{K}^{0,\gamma_p}(X^\gamma). \]
Since \( \hat{A} \) is symmetric and non-negative, \( \hat{A}_{2,2} \) is self-adjoint and \( \text{spec}(\hat{A}_{2,2}) \subset \mathbb{R}_+ \). Let us show that
\[ \text{spec}(\hat{A}_{p,q}) \subset \mathbb{R}_+ \quad \forall 1 < q < \infty, \ p \text{ satisfying } (7.1). \]
By Corollary 3.15 of [18] (in the version for operators in the cone algebra \( C^m(X^\gamma; (\gamma, \gamma - \mu, \Theta)) \)) on \( X^\gamma \), which is introduced in Section 8.2.5 of [3], the spectrum of \( \hat{A}_{p,q} \) is independent of \( 1 < q < \infty \). Thus we can set \( q = 2 \) and write \( \hat{A}_p = \hat{A}_{p,2} \). We can assume \( p \geq 2 \), by passing to the adjoint (i.e., \( \hat{A}_p^* = \hat{A}_{p'} \) and \( -\gamma_p = \gamma_{p'} \)). Then \( \ker(\lambda - \hat{A}_p) \subset \ker(\lambda - \hat{A}_2) = \{0\} \), since \( \mathcal{K}^{2,\gamma_p+2}(X^\gamma) \subset \mathcal{K}^{2,2}(X^\gamma) \) in view of \( \gamma_p \geq \gamma_2 = 0 \). The fact that \( \sigma_{\text{clos}}^2(\hat{A}(z)) \) is invertible for \( 0 < \text{Re} \ z \leq \frac{\omega - \lambda - \gamma_p - 2}{2} \) implies that
\[ \ker(\lambda - \hat{A}_p^*) \subset \mathcal{K}^{2,2}(X^\gamma) = \mathcal{K}^{2,\gamma_p+2}(X^\gamma), \quad \lambda \notin \mathbb{R}_+; \]
we shall give the argument below. As a consequence, we have for the adjoint
\[ \ker(\lambda - \hat{A}_p^*)^* = \ker(\lambda - \hat{A}_{p'}) \subset \ker(\lambda - \hat{A}_2), \quad \lambda \notin \mathbb{R}_+, \]
\( \lambda - \hat{A}_{p'} \) is bijective for \( \lambda \notin \mathbb{R}_+ \).

In order to see (7.2) set \( \gamma^1 = \min(\gamma_{p'} + 2, 0) \). The invertibility of the conormal symbol implies that \( \lambda - \hat{A} \) is elliptic with respect to \( \gamma^1 + 2 \). Moreover, the minimal and maximal extensions of \( \lambda - \hat{A} \), considered as unbounded operators in \( \mathcal{K}^{2,\gamma^1}(X^\gamma) \), coincide and their domain is \( \mathcal{K}^{2,\gamma^1+2}(X^\gamma) \). In particular, \( \mathcal{N} = \ker(\lambda - \hat{A} : \mathcal{K}^{2,\gamma^1+2}(X^\gamma) \to \mathcal{K}^{2,\gamma^1+2}(X^\gamma)) \) is a subset of the maximal domain, thus it is included in \( \mathcal{K}^{2,\gamma^1+2}(X^\gamma) \). Iterating this process, we see that \( \mathcal{N} \subset \mathcal{K}^{2,\gamma^j+2}(X^\gamma) \) for all \( \gamma^j := \min(\gamma^{j-1} + 2, 0) = \min(\gamma_{p'} + 2j, 0) \). Choosing \( j \) large enough we get (7.2).

As a consequence of Theorem 7.2, we get the following result on the maximal regularity for solutions of the Cauchy problem for the Laplacian:

**Theorem 7.2.** Let \( \Delta \) be the Laplacian on \( \text{int} \ B \) as described above, \( 1 < p < \infty \), and \( 2 \max(p, p') < \dim B \). Then the Cauchy problem
\[ (7.3) \quad \hat{u}(\tau) - \Delta u(\tau) = f(\tau), \quad 0 < \tau < T; \quad u(0) = 0, \]
has a unique solution
\[ u \in W^1_r \left( [0, T], \mathcal{H}^{0,\gamma_p}(B) \right) \cap L_r \left( [0, T], \mathcal{H}^{2,\gamma_p+2}(B) \right) \]
for every
\[ f \in L_r \left( [0, T], \mathcal{H}^{0,\gamma_p}(B) \right), \quad 1 < q, r < \infty. \]
Furthermore, \( u, \ u' \), and \( \Delta u \) depend continuously on \( f \).

In fact, in Theorem 7.1 above, we have shown that \( -\Delta \) is elliptic with respect to \( \gamma_p + 2 \) and any sector \( A_\delta \) not containing \( \mathbb{R}_+ \). Moreover, the problem (7.3) is equivalent to \( \hat{u}(\tau) - (\Delta - c)v(\tau) = e^{i\tau} f(\tau), \)
\( v(0) = 0 \), and, for sufficiently large \( c \), the operator \( \Delta + c \) satisfies the assumptions of Theorem 5.1 for any fixed \( 0 < \theta < \frac{\pi}{2} \) and \( \delta > 0 \). Then
\[ \|(-\Delta + c)^{iy}\|_{\mathcal{L}(\mathcal{H}^{0,\gamma_p}(B))} \leq c_{p,q} e^{\delta|y|} \quad \forall y \in \mathbb{R}, \]
and Theorem 7.2 immediately follows from Theorem 3.2 of [1].
Remark 7.3. a) An interpolation result of Amann [2], Theorem III.4.10.2, shows that
\[ W^1_r((0,T], H^0_q(\mathbb{B})) \cap L_r((0,T], H^2_q(\mathbb{B})) \to C([0,T], (H^0_q(\mathbb{B}), H^2_q(\mathbb{B}))_{\frac{1}{4},r}). \]
We will be interested in the special case, where \( r = q \). Here, we know from [3], Corollary 5.5, that
\[ (H^0_q(\mathbb{B}), H^2_q(\mathbb{B}))_{\frac{1}{4},q} \to H^s(\mathbb{B}) \quad \text{for any} \quad s < \frac{1}{q}, \delta < \gamma + \frac{1}{p}. \]
In particular, we deduce that the solution \( u \) in Theorem 7.2 is a continuous function on \([0,T]\) with values in \( H^s(\mathbb{B}) \).
b) It follows from [2], Theorem III.4.10.7 and Remark III.4.10.9(c), that Theorem 7.2 also holds for initial values \( u(0) = u_0 \in (H^0_q(\mathbb{B}), H^2_q(\mathbb{B}))_{\frac{1}{4},r} \).

Condition (7.3) implies that \( \dim \mathbb{B} > 4 \). Its strength lies in the fact that it ensures that \( \Delta \) is essentially self-adjoint on \( L_2(\mathbb{B}) \) with domain \( H^{2,2}_0(\mathbb{B}) \). It was shown by Cheeger that essentially self-adjointness also holds for \( \dim \mathbb{B} = 4 \). In that case, however, the domain is larger than \( H^{2,2}_0(\mathbb{B}) \).
For \( \dim \mathbb{B} < 4 \), the Laplacian is not essentially self-adjoint. We then have \( \Delta_{\min} \subset \Delta_F \subset \Delta_{\max} \), where \( \Delta_F \) is Friedrich’s extension. Hence the resolvent set on \( L_2(\mathbb{B}) \) of both \( \Delta_{\min} \) and \( \Delta_{\max} \) is empty, and Theorem 7.2 will certainly not be true for the minimal or the maximal extension.

8. Application: A quasilinear diffusion equation

As explained for example in the introduction of [2], diffusion processes are governed by a quasilinear equation of the form
\[ \dot{u}(\tau) - \text{div} \, D(u) \text{grad} \, u(\tau) = f(\tau, u), \quad 0 < \tau < T. \]
We now want to illustrate how the above analysis of the Laplacian allows us to solve problems of this kind for certain choices of the ‘diffusion matrix’ \( D \) and the nonlinearity \( f \).
To this end we shall make use of results obtained in Section 5 of [3] and a theorem of Clément and Li, which reads as follows.

Theorem 8.1. (Clément & Li) Let \( E_0 \) and \( E_1 \) be Banach spaces, \( E_1 \) densely and continuously embedded in \( E_0 \). For fixed \( 1 < q < \infty \) denote by \( E = (E_1, E_0)_{1/q,q} \) the real interpolation space.

For the quasilinear equation
\[ \partial_\tau u(\tau) - \hat{A}(u(\tau)) = f(\tau, u) + g(\tau) \quad \text{on} \quad [0,T], \quad u(0) = u_0 \in E, \]
to have a unique solution \( u \in W^1_q([0,T], E_0) \cap L_q([0,T], E_1) \) for some \( 0 < T_1 \leq T \), it is sufficient that there exists an open neighborhood \( U \subset E \) of \( u_0 \) such that

(H1) \( \hat{A} : U \to \mathcal{L}(E_1, E_0) \) is Lipschitz continuous and \( \hat{A}(u_0) \) is of maximal regularity with respect to \( E_0, E_1, \) and \( q \),
(H2) \( \hat{f} : [0,T] \times U \to E_0 \) is Lipschitz continuous,
(H3) \( g \in L_q([0,T], E_0) \).

In (H1), maximal regularity means that the Cauchy problem \( \partial_\tau v - \hat{A}(u_0)v = h, \quad v(0) = v_0 \), has a unique solution \( v \in W^1_q([0,T], E_0) \cap L_q([0,T], E_1) \) for every \( h \in L_q([0,T], E_0), \quad v_0 \in E, \) with \( v, \partial_\tau v, \) and \( \hat{A}(u_0)v \) depending continuously on \( h \) and \( v_0 \).
In order to apply this theorem to our situation, we fix a boundary defining function \( t \), which we also use as a coordinate in a neighborhood of the boundary, and choose a Riemannian metric \( h_{cone} \) on \( \mathbb{B} \) with a conical degeneracy at the boundary. We let \( \text{div} \) and \( \text{grad} \) denote the divergence
and gradient, respectively, with respect to \( h_{\text{cone}} \). More explicitly, writing \( h_{\text{cone}} = dt^2 + t^2g(t) \) in a neighborhood of \( \partial B \) with a smooth family \( g(\cdot) \) of metrics on the cross-section, we have

\[
\text{grad } u = t^{-2} \left( t^2 \partial_t u \partial_t + \sum_{i,j=1}^n g^{ij} \partial_{x_i} u \partial_{x_j} \right).
\]

Next, we let \( a \in C^\infty(\mathbb{R}^2) \) denote an arbitrary smooth, positive function. We shall consider the case where the diffusion matrix is a scalar multiple of the identity on \( TB \) of the form \( D(u) = a(t^nu)I_{TB} \) with the above boundary defining function \( t \) and an arbitrary positive constant \( c \). Here, we identify the complex values of \( u \) with elements of \( \mathbb{R}^2 \). Instead of being constant, \( c \) might be a smooth, real-valued function on \( B \) which is positive and constant at the boundary.

We let \( E_0 = \mathcal{H}^{\alpha,\gamma}(B) \) and \( E_1 = \mathcal{H}^{\alpha,\gamma+2}(B) \) with \( p \) and \( q \) to be determined later on. Moreover, we define the second order differential operator

\[
A(u) = \text{div}(a(t^nu) \text{ grad })
\]

Note that the point evaluation is defined for \( u \in E = (E_1, E_0)_{1/q,q} \) by the Sobolev embedding theorem provided \( q > (n + 3)/2 \). We are going to show the following theorem:

**Theorem 8.2.** Assume that \( \dim B > 4 \) and let \( T > 0 \). Then there exists a suitable choice of numbers \( 1 < p, q < \infty \) and \( 0 < T_1 \leq T \) such that the equation

\[
\partial_\tau u(\tau) - A(u)u(\tau) = f(\tau, u) + g(\tau), \quad u(0) = u_0 \in E,
\]

has a unique solution

\[
u \in W^1_q([0,T], E_0) \cap L_q([0,T], E_1)
\]

for every initial value \( u_0 \in C^\text{comp}(\text{int } B) \), every \( f \in \text{Lip}([0,T] \times U, E_0) \), and every \( g \in L_q([0,T], E_0) \).

As we showed in [3], Corollary 5.11, examples of functions \( f \) satisfying the above assumption (for suitable \( p, q \)) include \( f(u) = |u|^\alpha, \alpha \in \mathbb{R} \), or \( f(u) = u^\alpha, \alpha \geq 1 \) (and hence their linear combinations).

A specific example here is the time-dependent Ginzburg-Landau equation \( \partial_\tau u - \Delta u = u - u^3 \) for initial data \( u(0) = u_0 \in C^\text{comp}(\text{int } B) \).

We deduce Theorem 8.2 from 8.1. As a preparation we rewrite equation (8.1) as

\[
\partial_\tau u - \hat{A}(u)u = f(u) + g, \quad u(0) = u_0 \in E
\]

with \( \hat{A}(u) = a(t^nu) \Delta \) and

\[
f(\tau, u) = f(\tau, u) - (\partial_1 a)(t^nu) \text{ grad } (t^nu \text{ Re } u), \text{ grad } u - i(\partial_2 a)(t^nu) \text{ grad } (t^nu \text{ Im } u), \text{ grad } u.
\]

Here, \( \langle \cdot, \cdot \rangle \) is the complexified pointwise scalar product on \( TB \) given by \( h_{\text{cone}} \) and \( \Delta \) is the associated Laplacian. For large \( q \), solving (8.3) in \( W^1_q([0,T], E_0) \cap L_q([0,T], E_1) \) is equivalent to solving (8.2) in that space. In fact, for \( q > n + 3 \) the Sobolev embedding theorem implies that \( E \subseteq C^1(\text{int } B) \). Hence the solutions to both equations will be functions continuous in \( \tau \) and continuously differentiable in \( (t, x) \) so that \( \hat{A}(u)u + f(\tau, u) \) and \( \hat{A}(u)u + \hat{A}(u)u + f(\tau, u) \) coincide as distributions on \( \text{int } B \) for each \( \tau \).

In [3], Theorem 5.7, we already have shown that \( \hat{A} \) satisfies condition (H1), provided \( p < \frac{n+1}{2} \) is close to \( \frac{n+1}{2} \), and \( q \) is large. It is therefore sufficient to show the Lipschitz continuity of the map

\[
u \rightarrow (\partial_1 a)(t^nu) \text{ grad } (t^nu \text{ Re } u), \text{ grad } u : U \rightarrow E_0
\]

with \( p \) and \( q \) subject to the above condition; the other term in (8.4) can be treated in the same way.

**Lemma 8.3.** Let \( c > 0 \). For \( p < \frac{n+1}{2} \) sufficiently close to \( \frac{n+1}{2} \) and \( q \) sufficiently large, the mapping

\[
u \rightarrow (\partial_1 a)(t^nu) : U \rightarrow L^\infty(B) \text{ is Lipschitz continuous on bounded subsets of } E.
\]
Proposition 8.4. Let $D_1, D_2$ be as above, $1 < q < \infty$, and $s, \delta, \gamma \in \mathbb{R}$ be such that $s > 1 - \frac{n+1}{q}$ and $\delta - 1 + c/2 > (n + 1 + 2\gamma)/4$. Then the map

$$u \mapsto (D_1 u)(D_2 u) : \mathcal{H}_q^{s,\delta}(\mathbb{B}) \rightarrow \mathcal{H}_q^{\delta,\gamma}(\mathbb{B})$$

is Lipschitz continuous on bounded sets.

Proof. The proof is similar to that of [2], Theorem 5.15. For the convenience of the reader we provide the details. Choose a cut-off function $\omega \in C_0^\infty([0,1])$ and $\psi \in C_0^\infty(\text{int } \mathbb{B})$ with $\omega^2 + \psi^2 = 1$. Then $(D_1 u)(D_2 u) = (\omega D_1 u)(\omega D_2 u) + (\psi D_1 u)(\psi D_2 u).$ We first focus on the analysis near the boundary, i.e. we show that

$$u \mapsto \omega D_1 u)(\omega D_2 u) : \mathcal{H}_q^{s,\delta}(\mathbb{B}) \rightarrow \mathcal{H}_q^{\delta,\gamma}(\mathbb{B})$$

is Lipschitz continuous on bounded sets. We abbreviate $u_j = D_j u$ and $v_j = D_j v$, $j = 1, 2$, for $u, v \in \mathcal{H}_q^{s,\delta}(\mathbb{B})$. Since $u_j$ and $v_j$ have their support near $t = 0$, we have according to (10.5)

$$\|u_1 u_2 - v_1 v_2\|_{\mathcal{H}_q^{s,\gamma}(\mathbb{B})} = \|S_\gamma(u_1 u_2) - S_\gamma(v_1 v_2)\|_{L_q(\mathbb{R} \times X)} = \|S_\gamma u_1(S_\gamma u_2) - (S_\gamma v_1)(S_\gamma v_2)\|_{L_q(\mathbb{R} \times X)} \leq \|S_\gamma u_1\|_{L_{2q}(\mathbb{R} \times X)}\|S_\gamma v_2\|_{L_{2q}(\mathbb{R} \times X)} + \|S_\gamma v_2\|_{L_{2q}(\mathbb{R} \times X)}\|S_\gamma u_1 - v_1\|_{L_{2q}(\mathbb{R} \times X)}.$$

Here, we chose $\gamma = (n+1+2\gamma)/4$, and we employed Hölder’s inequality. Next we use the embedding $\mathcal{H}_q^{s-1}(\mathbb{R}^{n+1}) \rightarrow L_{2q}(\mathbb{R}^{n+1}),$ valid for $s - 1 - \frac{2}{q} \geq -\frac{n+1}{2q},$ cf. [28] 2.8.1 Remark 2. Since we assumed that $s - 1 > \frac{n+1}{q}$ we deduce that

$$\|S_\gamma u_1\|_{L_{2q}(\mathbb{R} \times X)} \leq \|S_\gamma v_2\|_{L_{2q}(\mathbb{R} \times X)} \leq C \|u_1\|_{\mathcal{H}_q^{s-1}(\mathbb{R}^{n+1})} \leq C \|u\|_{\mathcal{H}_q^{s-1+c/2}(\mathbb{B})} \leq C \|u\|_{\mathcal{H}_q^{s,\delta}(\mathbb{B})}.$$

The second estimate results from the continuity of $D_1$; for the third we used that $\gamma \leq \delta - 1 + c/2$. In the same way we estimate $\|S_\gamma v_2\|_{L_{2q}(\mathbb{R} \times X)}$ and finally

$$\|S_\gamma (u_1 - v_1)\|_{L_{2q}(\mathbb{R} \times X)} \leq \|u - v\|_{\mathcal{H}_q^{s,\delta}(\mathbb{B})}, \quad j = 1, 2.$$
plus the fact that the norms of \( u_1, v_2 \) and \( u_j - v_j \) can be estimated by the norms of \( u, v \), and \( u - v \) in \( \mathcal{H}_q^{r,\gamma}(\mathbb{B}) \) completes the argument. \( \square \)

**Corollary 8.5.** Let \( D_1, D_2 \) as above. Then there exist \( 1 < p, q < \infty \) such that the map
\[
  u \mapsto (D_1u)(D_2u) : E \to E_0 = \mathcal{H}_q^{0,\gamma_0}(\mathbb{B})
\]
is Lipschitz continuous on bounded sets.

**Proof.** We have \( E \to \mathcal{H}_q^{s,\gamma}(\mathbb{B}) \) for any \( s < 2/q' \) and \( \delta < \gamma_0 + 2/q' \). Choosing \( p < (n+1)/2 \) close to \((n+1)/2 \) and \( q \) large, we have
\[
  \gamma_0 + \frac{2}{q'} - 1 + \frac{c}{2} > \frac{n + 1 + 2\gamma_0}{4} \quad \text{and} \quad \frac{2}{q'} - 1 > \frac{n + 1}{q}.
\]
We can then pick \( s, \delta \) with \( \gamma_0 + \frac{2}{q'} - 1 + \frac{c}{2} > \delta - 1 + \frac{s}{q'} > \frac{n + 1 + 2\gamma_0}{4} \) and \( \frac{2}{q'} - 1 > s - 1 > \frac{n + 1}{q} \) and apply Lemma 8.5. \( \square \)

9. Appendix: Smoothing Mellin symbols and Green symbols

The structure of the resolvent (respectively parametrix) of a differential operator \( A \) as given in Theorem 8.4 at the first glance does not coincide with those which you find for example in [8] or [9]. This is mainly due to the fact that we consider \( A \) as an unbounded operator in \( \mathcal{H}_p^{r,\gamma}(\mathbb{B}) \) whose resolvent acts continuously in \( \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \), and do not consider \( A \) as a bounded operator acting from \( \mathcal{H}_p^{s,\mu,\gamma+\mu}(\mathbb{B}) \) to \( \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \). We shall use this appendix to clarify this point.

Let us begin with a discussion of so-called Green symbols. Let us set
\[
  \mathcal{K}^{s,\gamma}(X^\nu) = (t^\gamma K_2^{s,\gamma}(X^\nu)
\]
for real \( \nu \), cf. Definition 3.4. These are Hilbert spaces, and \( \mathcal{K}^{s,-\gamma}(X^\nu)_{-\nu} \) can be identified with the dual space of \( \mathcal{K}^{s,\gamma}(X^\nu) \) via the scalar-product in \( \mathcal{K}^{0,0}(X^\nu) \). The operators \( \kappa_\delta \) defined in (3.14) extend by continuity to operators in \( \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\nu)) \).

For \( \mu \in \mathbb{R} \) and \( d > 0 \) we let
\[
  S^{\mu,d}(\Lambda; \mathcal{K}^{s,\gamma}(X^\nu), \mathcal{K}^{s',\gamma'}(X^{\nu'}))
\]
denote the space of all smooth functions \( a \in C^\infty(\Lambda, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\nu), \mathcal{K}^{s',\gamma'}(X^{\nu'}))) \) satisfying
\[
  \| \kappa_{(\lambda)}^{-1/d} \{ \partial_\nu^\alpha a(\lambda) \} \kappa_{(\lambda)}^{1/d} \| \leq c_\alpha \| \lambda \|^{|\alpha|} \]
uniformly for \( \lambda \in \Lambda \) and all multiindices \( \alpha \).

We call a smooth function \( b \in C^\infty(\Lambda, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\nu), \mathcal{K}^{s',\gamma'}(X^{\nu'}))) \) twisted homogeneous of degree \((\mu, d)\) if it fulfills
\[
  b(q^d \lambda) = q^\mu \kappa_\delta b(\lambda) \kappa_\delta^{-1}
\]
for all \( \lambda \) and \( q > 0 \). Note that multiplying \( b \) with a 0-exception function (supported sufficiently far away from zero) yields a symbol in \( S^{\mu,d}(\Lambda; \mathcal{K}^{s,\gamma}(X^\nu), \mathcal{K}^{s',\gamma'}(X^{\nu'})) \). The space
\[
  S_d^{\mu,d}(\Lambda; \mathcal{K}^{s,\gamma}(X^\nu), \mathcal{K}^{s',\gamma'}(X^{\nu'}))
\]
then consists of all symbols from \( S^{\mu,d}(\Lambda; \mathcal{K}^{s,\gamma}(X^\nu), \mathcal{K}^{s',\gamma'}(X^{\nu'})) \), that have asymptotic expansions \( a \sim \sum_{j=0}^\infty a^{(\mu-j,d)} \) with functions \( a^{(\mu-j,d)} \) that are twisted homogeneous of degree \((\mu - j, d)\).

**Definition 9.1.** Let \( \gamma, \gamma' \in \mathbb{R} \). If \( g \in S^{\mu,d}(\Lambda; \mathcal{K}^{0,\gamma}(X^\nu), \mathcal{K}^{0,\gamma'}(X^{\nu'})) \), we can form the adjoint symbol \( g^* \in S^{\mu,d}(\Lambda; \mathcal{K}^{0,-\gamma}(X^\nu), \mathcal{K}^{0,-\gamma'}(X^{\nu'})) \) by taking pointwise the adjoint with respect to the
$K^{0,0}(X^\lor)$-scalar product. We then call $g$ a Green symbol if additionally there exists an $\varepsilon = \varepsilon(g) > 0$, such that

$$g \in s, s', \nu, \nu' \cap S_{cl}^{\mu, \nu}(\Lambda; K^{s, \gamma}(X^\lor)^{\nu}, K^{s', \gamma'+\varepsilon}(X^\lor)^{\nu'})$$

$$g^* \in s, s', \nu, \nu' \cap S_{cl}^{\mu, \nu}(\Lambda; K^{s, -\gamma'}(X^\lor)^{\nu}, K^{s', -\gamma'+\varepsilon}(X^\lor)^{\nu'})$$

The entity of all such Green symbols we shall denote by

$$R_G^{\mu, \nu}(X^\lor; \Lambda, (\gamma, \gamma'))$$

It is a trivial fact that if $\gamma' \geq \gamma''$, then

$$R_G^{\mu, \nu}(X^\lor; \Lambda, (\gamma, \gamma')) \subset R_G^{\mu, \nu}(X^\lor; \Lambda, (\gamma, \gamma''))$$

Moreover it can be shown, cf. [24], that in case $\gamma = \gamma'$ both Definitions 9.1 and 4.3 yield the same symbols respectively operator-families. In other words, Green symbols can either be characterized by their mapping properties in Sobolev spaces or by the structure of their kernels.

Let us now return to the resolvent, cf. Theorem 4.6. If you compare with [5], you will find that there ‘our’ term $G(\lambda)$ is replaced by a term of the form $G_0(\lambda) + M(\lambda)$, where

$$G_0(\lambda) \in R_G^{-\mu, \nu}(X^\lor; \Lambda, (\gamma, \gamma + \mu))$$

is a Green symbol and

$$M(\lambda) = \omega(t[\lambda]^{\frac{1}{d}}) t^\mu \text{op}_M^{-\gamma - \frac{\varepsilon}{d}}(h_0(t[\lambda]^{\frac{1}{d}}))$$

for some cut-off functions $\omega, \omega_0 \in C_\text{comp}^\infty(\mathbb{R}_+)$, and a meromorphic Mellin symbol

$$h \in M_{\mathbb{R}_+}^{-\infty}(X).$$

The last notation roughly means that $h$ is a meromorphic function on the complex plane with values in $L^{-\infty}(X)$, the smoothing pseudodifferential operators on $X$, having only finitely many poles in any vertical strip $|\text{Re } z| \leq \beta$, $\beta > 0$, and the Laurent coefficients of the principal part of $h$ at any such pole are finite rank operators. For more details see [8], Section 8.1.2. By the above observation,

$$G_0(\lambda) \in R_G^{-\mu, \nu}(X^\lor; \Lambda, \gamma).$$

The same is also true for $M$, since it is easy to see that

$$M \in S_{cl}^{\mu, \nu}(\Lambda; K^{s, \gamma}(X^\lor)^{\nu}, K^{s', \gamma+\mu}(X^\lor)^{\nu'})$$

for all $s, s', \nu, \nu'$ (note that $M$ is twisted homogeneous for large $|\lambda|$). Observe that $\mu \geq 1$ since we are dealing with differential operators of positive order. The adjoint symbol is given by

$$M^*(\lambda) = \omega_0(t[\lambda]^{\frac{1}{d}}) t^\mu \text{op}_M^{-\gamma - \frac{\varepsilon}{d}}(h^*)(\omega(t[\lambda]^{\frac{1}{d}}) + G_1(\lambda)$$

where $h^*(z) = h(n + 1 - \mu - \overline{z})^*$ and

$$G_1(\lambda) = t^\mu \omega_0(t[\lambda]^{\frac{1}{d}}) \left\{ \text{op}_M^{-\gamma - \frac{\varepsilon}{d}}(h^*) - \text{op}_M^{-\gamma - \frac{\varepsilon}{d}}(h^*) \right\} \omega(t[\lambda]^{\frac{1}{d}}).$$

Here, $\gamma_+ = \gamma$ if $h^*$ has no pole on the line $\text{Re } z = \frac{n+1}{2} + \gamma$, otherwise $\gamma_+ > \gamma$ sufficiently close to $\gamma$. However, it is known, cf. [8], Section 8.1.2, Theorem 6, that then $G_1 \in R_G^{-\mu, \nu}(X^\lor; \Lambda, (\gamma, -\gamma))$ and

$$M^* - G_1 \in S_{cl}^{\mu, \nu}(\Lambda; K^{s, -\gamma}(X^\lor)^{\nu}, K^{s', -\gamma+\mu}(X^\lor)^{\nu'})$$

for all $s, s', \nu, \nu'$. All together this shows that $M \in R_G^{-\mu, \nu}(X^\lor; \Lambda, \gamma)$ and hence justifies the description of the resolvent we have given in Theorem 4.6.
10. Notation

For $0 \neq \lambda \in \mathbb{C}$ we let $\arg \lambda$ be the unique number $-\pi \leq \arg \lambda < \pi$ such that $\lambda = |\lambda| e^{i \arg \lambda}$. For $z \in \mathbb{C}$ we then set

$$\lambda^z = |\lambda|^z e^{iz \arg \lambda}.$$

For fixed $z$ this is a holomorphic function in $\lambda \in \mathbb{C} \setminus \{\lambda \in \mathbb{R} \mid \lambda \leq 0\}$.

For $\delta > 0$ and $0 < \theta < \pi$ we let $\Lambda = \Lambda(\delta, \theta)$ denote the closed keyhole region

$$\Lambda(\delta, \theta) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \delta \text{ or } |\arg \lambda| \geq \theta \}$$

and $C = C(\delta, \theta)$ its parametrized boundary, $C = C_1 \cup C_2 \cup C_3$, with

$$(10.1) \quad C_1(t) = te^{i\theta}, \quad -\infty < t \leq \delta; \quad C_2(t) = \delta e^{-it}, \quad -\theta \leq t \leq \theta; \quad C_3(t) = te^{-i\theta}, \quad \delta \leq t < \infty.$$

We let $\Lambda_{\Delta} = \Lambda_{\Delta}(\theta)$ denote the closed sector contained in $\Lambda$,

$$\Lambda_{\Delta}(\theta) = \{ \lambda \in \mathbb{C} \mid |\arg \lambda| \geq \theta \} \cup \{0\}$$

and, similar to (10.1), $C_{\Delta}$ its parametrized boundary.

We now recall various spaces of pseudodifferential symbols and operators we shall use throughout this paper. In the following we let $\mu, d \in \mathbb{R}$ and $d$ positive.

We call a function smooth on $\Lambda$, if it is the restriction to $\Lambda$ of a function which is smooth in an open neighborhood of $\Lambda$. If $E$ is a Fréchet space, then

$$S(\Lambda, E)$$

consists of all $u \in C^\infty(\Lambda, E)$ satisfying

$$\sup_{\lambda \in \Lambda} \|\partial_\lambda^\gamma u(\lambda)\| |\lambda|^N < \infty$$

for any multi-index $\gamma \in \mathbb{N}_0^2$, any $N \in \mathbb{N}$, and any continuous semi-norm $\| \cdot \|$ of $E$. The space of symbols of order $\mu$ and anisotropy $d$,

$$S^{\mu, d}(\mathbb{R}^m_y \times \mathbb{R}^n_\eta \times \Lambda),$$

consists of all functions (possibly matrix-valued) $a \in C^\infty(\mathbb{R}^m_y \times \mathbb{R}^n_\eta \times \Lambda)$, which fulfill the estimates

$$|\partial_y^\beta \partial_\eta^\gamma \partial_\lambda^\alpha a(y, \eta, \lambda)| \leq c_{\alpha\beta\gamma} \langle \eta, \lambda \rangle^{\mu - |\alpha| - d|\gamma|}, \quad \langle \eta, \lambda \rangle_d = (1 + |\eta|^2 + |\lambda|^2)^{\frac{d}{2}},$$

where $c_{\alpha\beta\gamma}$ are certain constants and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}^n$. The space of symbols $S^{\mu, d}(\mathbb{R}^m_y \times \mathbb{R}^n_\eta \times \Lambda)$ is a Fréchet algebra under the usual product of symbols.
for any multi-indices $\alpha$, $\beta$, and $\gamma$. Further we set

$$S^{\mu,d}(\mathbb{R}^+ \times \mathbb{R}^{m-1} \times \mathbb{R}^n; \Lambda) = S^{\mu,d}(\mathbb{R}^m \times \mathbb{R}^n; \Lambda)_{|\mathbb{R}^+ \times \mathbb{R}^{m-1} \times \mathbb{R}^n \times \Lambda}.$$  

For a compact manifold $X$, $\dim X = n$, the space

$$L^{\mu,d}(X; \Lambda)$$

of parameter-dependent pseudodifferential operators of order $\mu$ and anisotropy $d$ (acting on sections of a vector bundle) consists of all operator-families, which are obtained as a sum (according to a covering of $X$ by coordinate neighborhoods) of local operators with symbols from $S^{\mu,d}(\mathbb{R}^n \times \mathbb{R}^n; \Lambda)$ and a smoothing remainder from $L^{-\infty}(X; \Lambda) := \mathcal{S}(\Lambda, L^{-\infty}(X))$. In the last definition, $L^{-\infty}(X)$ is the usual space of smoothing operators on $X$, i.e. the space of all integral operators having a smooth kernel.

If $\gamma \in \mathbb{R}$ and $\Gamma_\gamma$ denotes the vertical line in the complex plane

$$\Gamma_\gamma = \{ z \in \mathbb{C} \mid \text{Re} z = \gamma \},$$

the space of symbols

$$MS^\mu(\mathbb{R}^+ \times \mathbb{R}^n \times \Gamma_\gamma)$$

consists of all functions $a \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^n \times \Gamma_\gamma)$ which satisfy the estimates

$$|\partial_t (t \partial_t)^k \partial_x^\gamma \partial_z^\alpha a(t, x, z) - 1 + \gamma + i \tau, \xi| \leq c_{k\alpha} |\tau, \xi|^{\mu - |\alpha|}, \quad \langle \tau, \xi \rangle = (1 + \tau^2 + |\xi|^2)^{1/2}.$$  

The associated (Fourier-Mellin) pseudodifferential operator is

$$[op_M^{\gamma - \frac{d}{2}}(a)](t, x) = \int_{\text{Re} z = \frac{\mu - \gamma}{\mu}} z^{-1} op(a)(t, z) (\mathcal{M}u)(z, x) dz,$$

where $op$ is the standard Fourier pseudodifferential operator on $\mathbb{R}^n$, and $\mathcal{M}$ the Mellin transform

$$(\mathcal{M}v)(z) = \int_0^\infty t^z v(t) \frac{dt}{t}.$$  

The operator $op_M^{\gamma - \frac{d}{2}}(a)$ induces continuous mappings on the associated Mellin Sobolev spaces

$$op_M^{\gamma - \frac{d}{2}}(a) : \mathcal{H}_q^s(\mathbb{R}^+ \times \mathbb{R}^n) \rightarrow \mathcal{H}_q^{s - \gamma}(\mathbb{R}^+ \times \mathbb{R}^n),$$

$s \in \mathbb{R}$, $1 < q < \infty$, where

$$\mathcal{H}_q^{s,\gamma}(\mathbb{R}^+ \times \mathbb{R}^n) := \{ u \mid (Su)(t, x) := e^{i(\frac{d}{2} - \gamma)t} u(e^{-t}, x) \in H^s_q(\mathbb{R}^{1+n}) \}.$$  

The continuity is due to the fact that a Mellin pseudodifferential operator on $\mathbb{R}^+ \times \mathbb{R}^n$ transforms, under conjugation by $S$, to a usual pseudodifferential operator on $\mathbb{R}^{1+n}$, and then the Calderón-Vaillancourt theorem applies.

Using local coordinates, we obtain the spaces $\mathcal{H}_q^{s,\gamma}(X)$ for an arbitrary closed manifold $X$ with the corresponding map $S_\gamma : \mathcal{H}_q^{s,\gamma}(X) \rightarrow \mathcal{H}_q^s(\mathbb{R}^n \times X).$  

The space $\mathcal{H}_q^{s,\gamma}(\mathbb{B})$ consists of all $u \in H_q^{s,\gamma}(X)$ such that $\omega u \in \mathcal{H}_q^{s,\gamma}(X)$ for a cut-off function $\omega$.

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