DEFORMATIONS OF THE LIE ALGEBRA $\mathfrak{o}(5)$ IN CHARACTERISTICS 3 AND 2

SOFIANE BOUARROUDJ$^1$, ALEXEI LEBEDEV$^2$, FRIEDRICH WAGEMANN$^3$

Abstract. The finite dimensional simple modular Lie algebras with Cartan matrix cannot be deformed if the characteristic $p$ of the ground field is equal to 0 or greater than 3. If $p = 3$, the orthogonal Lie algebra $\mathfrak{o}(5)$ is one of the two simple modular Lie algebras with Cartan matrix that have deformations (the Brown algebras $\mathfrak{br}(2; \alpha)$ are among these 10-dimensional deformations and hence are not counted separately); the 29-dimensional Brown algebra $\mathfrak{br}(3)$ is the only other simple Lie algebra with Cartan matrix that has deformations. Kostrikin and Kuznetsov described the orbits (isomorphism classes) under the action of the group $O(5)$ of automorphisms of $\mathfrak{o}(5)$ on the space $H^2(\mathfrak{o}(5); \mathfrak{o}(5))$ and produced representatives of the isomorphism classes. Here we explicitly describe global deformations of $\mathfrak{o}(5)$ and of the simple analog of this orthogonal Lie algebra in characteristic 2.

1. Introduction

In what follows fields of positive characteristic $p$ are supposed to be algebraically closed. The algebraic closedness is needed since computing cohomology we assume that all algebraic equations have solutions. This paper can be considered as an elucidation of a very interesting but too short paper [KK] and a first step in getting results analogous to those of [KK] for $p = 2$.

1.1. Setting. It is well known ([Fu, FF2]) that the finite dimensional simple Lie algebras over $\mathbb{C}$ are rigid meaning that they do not have non-trivial (even infinitesimal) deformations. Over fields of characteristic $p > 3$, the same is true not for all finite dimensional simple Lie algebras, but is true for those which possess a Cartan matrix (for a precise definition of this notion, see [BGL]), see [Ru].

For $p = 3$, Rudakov and Kostrikin ([Kos, Ru] also cited in [WK] as an example in the classification of simple modular Lie algebras with Cartan matrices) introduced a parametric family of Lie algebras $\mathfrak{L}(\varepsilon)$, which includes the 1-parameter family $\mathfrak{L}(\varepsilon, 0, 0)$. For $p = 3$, Rudakov and Kostrikin ([Kos, Ru] also cited in [WK] as an example in the classification of simple modular Lie algebras with Cartan matrices) introduced a parametric family of Lie algebras containing $\mathfrak{o}(5)$; the generic member of this family, denoted $\mathfrak{L}(\varepsilon)$, has Cartan matrix

$$\left(\begin{array}{cc} 2 & -1 \\ -2 & 1 - \varepsilon \end{array}\right), \text{ where } \varepsilon \neq 0.$$  

The Lie algebra $\mathfrak{L}(\varepsilon)$ is isomorphic to $\mathfrak{o}(5) \simeq \mathfrak{sp}(4)$ for $\varepsilon = -1$, to the Brown algebra $\mathfrak{br}(2; \alpha)$ for $\varepsilon = 1 + \frac{1}{\alpha}$, where $\varepsilon = 1$ is also possible, see [WK, BGL]. Actually, the parameter $\alpha$ is more convenient than $\varepsilon$: the value $\varepsilon = 0$ is excluded because $\mathfrak{L}(0) \simeq \mathfrak{psl}(3)$ is of dimension 8, so differs drastically from the other members of the parametric family.

It soon became clear that the deformation of $\mathfrak{o}(5)$ depends on more than one parameter: Kostrikin and Kuznetsov [KK] considered a certain 3-parameter family of Lie algebras $\mathfrak{L}(\varepsilon, \delta, \rho)$, which includes the 1-parameter family $\mathfrak{L}(\varepsilon) = \mathfrak{L}(\varepsilon, 0, 0)$. The family $\mathfrak{L}(\varepsilon, \delta, \rho)$ was...
explicitly constructed in [Kos] (we reproduce it in sec. 3.1), but the mechanism producing it remained unclear. There is, however, a construction of the same family with clear origin.

Earlier, using the lucid explicit description of the 3-parameter family \( \mathbb{T}(a,b,c) \) of irreducible 3-dimensional \( \mathfrak{sl}(2) \)-modules in characteristic 3 due to Rudakov and Shafarevich [RSh], Rudakov (cited in [Kos]) constructed the Cartan prolong \( L(a,b,c) := (\mathbb{T}(a,b,c), \mathfrak{gl}(2))_N \) which is a 3-parameter family of deformations of \( \mathfrak{o}(5) \). This construction is lucid; for explicit expressions, see [GL]; both the depth and height 1 grading of \( \mathfrak{o}(5) \) in its realization as \( \mathfrak{sp}(4) \) are preserved in \( L(a,b,c) \). Although nobody bothered to express \((\varepsilon,\delta,\rho)\) in terms of \((a,b,c)\) or the other way round, the families of deformations are isomorphic, as noted already in [Kos]. (Note that Grozman and Leites [GL] found the exceptional values of parameters \((a,b,c)\) for which the Cartan prolong \( (\mathbb{T}(a,b,c), \mathfrak{gl}(2))_N \) is of height \( h > 1 \) and simple; the Lie algebras \( L(a,b,c) \) (that exist for all values of parameters \((a,b,c)\) for which \( \mathbb{T}(a,b,c) \) is an irreducible \( \mathfrak{gl}(2) \)-module) are partial prolongs of height \( h = 1 \); for the definition of various prolongs, as well as of a shearing parameter \( N \) for \( p > 0 \), see [Shch].

Kostrikin ([Kos]) proved that (for a natural explanation of the condition in terms of reflections similar to elements of the Weyl group, see [BGL])

\[
L(\varepsilon) \simeq L(\varepsilon') \iff \varepsilon \varepsilon' = 1 \quad (\text{for } \varepsilon \neq \varepsilon')
\]

and stated the isomorphy problem of the algebras \( L(a,b,c) \) for various values of parameters.

Kostrikin and Kuznetsov ([KK]) were the first to find out that \( \dim H^2(L(\varepsilon); L(\varepsilon)) = 5 \), so the family of deformations of \( \mathfrak{o}(5) \) depends on at most 5 parameters. For an explanation of the meaning of the words “at most” here, see sec. 5.2.

Kostrikin and Kuznetsov described the orbits under the action of the group \( O(5) \) of automorphisms of \( \mathfrak{o}(5) \) (see [FG]) in the variety of Lie algebras containing \( L(-1) \simeq \mathfrak{o}(5) \), which means that Kostrikin and Kuznetsov listed all isomorphy classes of the members of the 5-parameter family of deformations of \( \mathfrak{o}(5) \). The answer is as follows:

“For \( p = 3 \), the Lie algebra \( L(-1,-1) \) and the Lie algebras \( L(\varepsilon) = L(1 + \frac{1}{\varepsilon}) \)

\[
(2) \quad \text{with the cases } \varepsilon = 1 \text{ included and } \varepsilon = 0 \text{ excluded, represent all the isomorphy classes of simple 10-dimensional Lie algebras deforming } \mathfrak{o}(5), \text{ minding (P)}.\]

(If \( p \) were equal to 0, the radius \( r \) of the sphere in the identity representation of \( O(5) \) in the 5-dimensional space \( H^2(\mathfrak{o}(5); \mathfrak{o}(5)) \) would have been a natural parameter, the extra case of \( r = 0 \) might have occurred for \( p > 0 \). The above answer from [KK] resembles this count.)

Here we answer the following natural questions arising after reading the above cited papers:

(Q1) is there a basis in the space \( H^2(\mathfrak{o}(5); \mathfrak{o}(5)) \) consisting of cocycles each of which determines a global deformation linear in the parameter?

(Q2) is there a 5-parameter family of (10-dimensional simple) Lie algebras (over a field of characteristic 3) which includes the families from (Q1) and which corresponds to an arbitrary linear combination of 5 basis cocycles from (Q1)?

The answer is “yes” to both questions, and we will construct the family via obstruction theory using Massey brackets in §3. (Actually, the question (Q1) is already answered in affirmative in [KK] but the arguments based on algebraic geometry are indirect and non-constructive, whereas here we prove this directly giving an additional verification.)

1.2. \( p = 2 \). Are there analogs of the above results for \( p = 2 \)? Leites told us that in the early 1970s, he suggested to divide the last row of the standard Cartan matrix of \( \mathfrak{o}(2n+1) \) by 2, thus making it possible to retain simplicity for \( p = 2 \). The algebra \( \mathfrak{o}(2n+1) \) itself does not, however, possess a Cartan matrix nor is it simple if \( p = 2 \); it is its derived \( \mathfrak{o}^{(1)}(2n+1) \)

\footnote{For an explicit form of these cocycles, see [BGL], where this was rediscovered.}
that does and is, see [BGL]; and it is \(o(1)(2n + 1)\) that looked as a new series of simple Lie algebras in [WK] but actually was (the derived of) the old and well-known \(o(2n + 1)\) with non-conventionally normalized Cartan matrix. (At the time [WK] was written, the term “Lie algebra \(g(A)\) with Cartan matrix \(A\)” was not properly defined and was applied not only to Lie algebras of the form \(g(A)\) but also to their subquotients and algebras of derivations which have no Cartan matrix, cf. [BGL].) The infinitesimal deformations of \(o(2n + 1)\) and \(o(1)(2n + 1)\) are calculated for small values of \(n\) in [BGL4]; here we describe how the infinitesimal deformation of \(o(1)(5)\) may be integrated to a multiparameter family of Lie algebras.

1.3. **Open problems.** (a) Obtain the analog of the result of [KK] to interpret our result for \(p = 2\): Describe the orbits of \(Sp(4) \simeq O(5) = aut(o(1)(5))\) in the 4-dimensional space \(H^{2}(o(1)(5); o(1)(5))\).

(b) Explicitly describe the non-isomorphic deforms as Lie algebras preserving geometric objects (a tensor or a distribution) both for \(p = 2\) and \(p = 3\).

(c) In [BGL4], the infinitesimal deformations of \(br(3)\) are described. Study their integrability, isomorphy classes of the deforms, and their interpretations as in (b).

2. Deformation and cohomology

2.1. **In characteristic different from 2.** Let \(k\) be a field of any characteristic \(p \neq 2\), and \(g\) a Lie algebra over \(k\). A basic reference for questions about the cohomology of Lie algebras, especially in relation to their deformation theory, is the book by Fuchs [Fu].

A multiparameter deformation of \(g\), or multiparameter family of Lie algebras containing \(g\) as a special member, is a Lie algebra \(g_{t}\), where \(t = (t_{1}, \ldots, t_{r})\), given by a Lie algebra structure on the tensor product \(g \otimes_{k} k[[t]]\) such that the Lie algebra \(g_{0}\), i.e., the one obtained when we set \(t = 0\), is isomorphic to \(g\) and such that \(t_{1}, \ldots, t_{r}\) are scalars with respect to the deformed bracket. *A posteriori* we see that in this paper we can confine ourselves to polynomials instead of formal power series in \(t\).

The bracket in the deformed Lie algebra is of the form

\[
[x, y]_{t_{1}, \ldots, t_{r}} = c_{0}(x, y) + t_{1}c_{1}(x, y) + \ldots + t_{r}c_{r}(x, y) +
\]

\[
+ t_{1}^{2}c_{11}(x, y) + t_{1}t_{2}c_{12}(x, y) \ldots + t_{r}^{2}c_{r,r}(x, y) + \ldots
\]

for any \(x, y \in g\), where \(c_{0}(x, y) := [x, y]\) is just the bracket of \(x\) and \(y\) in \(g\). By linearity, it suffices to specify the deformed bracket of elements in \(g\). The first degree conditions say that the maps \(c_{i} : g \otimes_{k} g \rightarrow g\) must be anti-symmetric and 2-cocycles (with coefficients in the adjoint module), i.e., for all \(i = 1, \ldots, r\), we have

\[
dc_{i}(x, y, z) := c_{i}([x, y], z) + c_{i}([y, z], x) + c_{i}([z, x], y) -
\]

\[
- [x, c_{i}(y, z)] - [y, c_{i}(z, x)] - [z, c_{i}(x, y)] = 0.
\]

Two (formal) 1-parameter deformations \(g_{t}\) and \(\tilde{g}_{t}\) given by the collections \(c = (c^{1}, c^{2}, \ldots)\) and \(\tilde{c} = (\tilde{c}^{1}, \tilde{c}^{2}, \ldots)\), where \(c^{i}\) and \(\tilde{c}^{i}\) are coefficients of \(t^{i}\), lead to equivalent deforms (results of deformations) (i.e., \(g_{t}\) and \(\tilde{g}_{t}\) are isomorphic as Lie algebras by an isomorphism of the form \(\tau(x) = id_{g}(x) + \sum_{i \geq 1} \tau_{i}(x)t^{i}\) for any \(x \in g\)) if and only if \(\tau\) links \(c\) and \(\tilde{c}\) by the following formulae (for all \(n > 0\)):

\[
\sum_{i+j=n} \tau_{i}(\tilde{c}^{j}(x, y)) = \sum_{i+j+k=n} c^{i}(\tau_{j}(x), \tau_{k}(y)).
\]
For the first (i.e., infinitesimal) terms, this means that two 1-parameter deformations are \( \textit{infinitesimally equivalent} \) (i.e., \( \tau = \text{id} + t\tau_1 \) and one reasons modulo \( t^2 \)) if and only if their 2-cocycles differ by a coboundary. This coboundary is nothing else than \( \tau_1 \). A similar statement is true for multiparameter deformations. In particular, if two multiparameter deformations are infinitesimally equivalent, then the corresponding infinitesimal cocycles are linearly dependent up to coboundaries.

For the sake of brevity, we shall recall properties of deformations using only 1-parameter deformations; to generalize them to the multidimensional case is routine. The Jacobi identity imposes conditions on all terms in the deformed bracket, which must be satisfied degree by degree.

Thus, the search for the most general multiparameter deformation of a given Lie algebra usually begins with the determination of the space \( H^2(\mathfrak{g}; \mathfrak{g}) \). An explicit basis given by 2-cocycles (representing the classes) determines an infinitesimal deformation. One then tries to prolong this infinitesimal deformation to all degrees. This prolongation method brings up the \textit{Massey brackets} which we will now describe, see [Fu, FL, Fia, Mill].

Let \( \mathfrak{g}_t \) be a 1-parameter deformation of a Lie algebra \( \mathfrak{g} \), given by an infinitesimal cocycle \( c = c^1 \) and higher degree terms \( c^2, c^3, \ldots \). The Jacobi identity modulo \( t^{n+1} \) reads

\[
\sum_{i+j=n, i,j \geq 0} (c^i(c^j(x,y),z) + c^j(c^i(x,y),z) + \text{cyclic}(x,y,z)) = 0,
\]

where \( \text{cyclic}(x,y,z) \) denotes the sum of all cyclic permutations of the arguments of the expression written on the left of it.

The expression (3) can be rewritten as

\[
\sum_{0 \leq i,j \leq n, i+j = n} [[c^i, c^j]](x,y,z) = 0,
\]

where the brackets \( [[\cdot, \cdot]] \) are called \textit{Nijenhuis brackets} (in differential geometry) or \textit{Massey brackets} (in deformation theory). The collection of the brackets \( [[\cdot, \cdot]] \) defines a graded Lie superalgebra structure on \( H^* (\mathfrak{g}; \mathfrak{g}) \) (for examples, see [LLS, GL1]). The whole sum may then be expressed as a \textit{Maurer—Cartan} equation:

\[
\frac{1}{2} \sum_{i+j=n, i,j > 0} [[c^i, c^j]] = dc^n
\]

because the term \( [[c^0, c^n]] \) is just the left hand side of the 2-cocycle condition on \( c^n \) in the Lie algebra \( \mathfrak{g} \) (with adjoint coefficients) for the cochain \( c^n \).

This gives a clear procedure for the prolongation of an infinitesimal deformation (expressed here for simplicity only for a 1-parameter deformation): given a first degree deformation via a cocycle \( c = c^1 \), one must compute its \textit{Massey square} \( [[c, c]] \). If \( [[c, c]] = 0 \), the infinitesimal deformation fulfills the Jacobi identity and is thus a true deformation. If \( [[c, c]] \in Z^3(\mathfrak{g}, \mathfrak{g}) \) is not a coboundary, the infinitesimal deformation is obstructed and cannot be prolonged. If \( [[c, c]] = d\alpha \) with \( \alpha \neq 0 \), then \( -\alpha t^2 \) is the second degree term of the deformation. In order to prolong to the third degree, one has to compute the next step — the Massey product \( [[c, \alpha]] \). Once again, there are the three possibilities \( [[c, \alpha]] = 0 \), \( [[c, \alpha]] = d\beta \) with \( \beta \neq 0 \) or \( [[c, \alpha]] \neq d\beta \) for any \( \beta \). If \( [[c, \alpha]] = d\beta \), then \( \beta \) gives the third degree prolongation of the deformation. In order to go up to degree 4 then, one has to be able to compensate \( [[\alpha, \alpha]] + [[c, \beta]] \) by a coboundary \( d\gamma \), and so on. One must be careful to keep track of all terms coming in to compensate low degree Massey brackets in a multiparameter deformation.
The main difficulty in this kind of obstruction calculus is that the representatives of the cohomology classes and the α-, β-, etc. cochains are not uniquely\(^2\) defined. A good choice of cochains may considerably facilitate computations.

We computed cohomology and Massey products using Grozman’s Mathematica-based package SuperLie. The formula of the following lemma was helpful in the computations. For any finite dimensional Lie algebra \(\mathfrak{g}\), all cochains with adjoint coefficients may be expressed as sums of tensor products of the form \(x \otimes \omega\), where \(x \in \mathfrak{g}\) and \(\omega \in \bigwedge^*(\mathfrak{g}^*)\). We are working with a fixed basis of \(\mathfrak{g}\) and the dual basis of \(\mathfrak{g}^*\).

**Lemma** (Grozman). For any \(c = a \otimes \omega\), where \(x \in \mathfrak{g}\) and \(\omega \in \bigwedge^*(\mathfrak{g}^*)\), let \(dc\) denote the coboundary of \(c\) in the complex with adjoint coefficients, while \(d\omega\) denotes the coboundary in the complex with trivial coefficients and \(da\) denotes the coboundary of \(a \in \mathfrak{g}\) considered as a 0-cochain in the complex with adjoint coefficients. If \(c = a \otimes \omega\), then \(dc = a \otimes d\omega + da \wedge \omega\).

**Proof.** For any \(x_1, \ldots, x_{p+1} \in \mathfrak{g}\), we have:

\[
dc(x_1, \ldots, x_{p+1}) = \sum_{1 \leq i \leq j \leq p+1} (-1)^{i+j-1} a \otimes \omega([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1}) + \\
+ \sum_{1 \leq i \leq p+1} (-1)^i [x_i, a] \otimes \omega(x_1, \ldots, \hat{x}_i, \ldots, x_{p+1}) = \\
= (a \otimes d\omega)(x_1, \ldots, x_{p+1}) + (da \wedge \omega)(x_1, \ldots, x_{p+1}).
\]

\[\square\]

2.2. **In characteristic 2.** Let now \(k\) be an algebraically closed field of characteristic 2. In this subsection \(-1 = 1\), of course; the signs are kept to make expressions look like in characteristics 0.

The vector space \(\mathfrak{g}\) is a *Lie algebra* if endowed with a bilinear map \([\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) satisfying the Jacobi identity and anti-symmetry which for \(p = 2\) means \([x, x] = 0\) for any \(x \in \mathfrak{g}\). For vector spaces, the wedge product is defined without a normalization factor:

\[a \wedge b = a \otimes b - b \otimes a.\]

Grozman communicated to us the following definition of Lie algebra cohomology in char = 2 implemented in his SuperLie:

For 1-cochains with trivial coefficients, the codifferential is defined as an operation dual to the Lie bracket:

\[d : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*.\]

For \(q\)-cochains with trivial coefficients, \(d\) is defined via the Leibniz rule. For cochains with coefficients in a module \(M\), we set

\[d(m) := - \sum_{1 \leq i \leq \dim \mathfrak{g}} [g_i, m] \otimes g_i^*,\]

\[d(m \otimes \omega) := d(m) \wedge \omega + m \otimes d(\omega)\]

for any \(m \in M\), any \(q\)-cochain \(\omega\), where \(q > 0\), and any basis \(g_i\) of \(\mathfrak{g}\), cf. Lemma 2.1.

The Massey product is defined as follows:

\[[a, b](x, y, z) := a(b(x, y), z) + b(a(x, y), z) + \text{cyclic}(x, y, z),\]

\[\text{If } \alpha \text{ is a solution to the equation } da = [[c, c]], \text{ then } \alpha + \text{cocycle is also a solution. The choice of a certain }\]
\[\alpha \text{ effects the expression of the } \beta\text{'s. The problem is how to find a "nice" } \alpha \text{ in order to have as few } \beta\text{-terms as possible and, more importantly, vanishing Massey products in degrees higher than that of } \beta. \text{ If we fail to achieve this with } \alpha, \text{ let us try to perform this with } \beta\text{'s, and so on.}\
}
if \( a \) and \( b \) are non-proportional, whereas
\[
[a, a](x, y, z) := a(a(x, y), z) + \text{cyclic}(x, y, z).
\]

### 3. Main results: \( p = 3 \)

#### 3.1. Deformations of \( \mathfrak{o}(5) \): The results known.

1) A **1-parameter** family of deformations of \( \mathfrak{o}(5) \) is given by Cartan matrices of \( L(\varepsilon) \).

Denote by \( x \)'s and \( y \)'s the Chevalley generators of \( \mathfrak{o}(5) \).

#### 3.1.1. Proposition.

The Lie algebra \( L(\varepsilon) \) can be obtained as a deformation of \( \mathfrak{o}(5) \) generated by the 2-cocycle \( c_0 \) below. The bracket is as follows:
\[
[\cdot, \cdot]_{-1-\varepsilon} = [\cdot, \cdot] - (1 + \varepsilon) c_0 + (1 + \varepsilon)^2 \alpha_0,
\]
where
\[
(4)
\]
\[
c_0 = h_1 \otimes (dx_2 \wedge dy_2) + 2 h_1 \otimes (dx_3 \wedge dy_3) + 2 h_1 \otimes (dx_4 \wedge dy_4) + h_2 \otimes (dx_4 \wedge dy_4)
+ x_1 \otimes (dh_2 \wedge dx_1) + 2 x_2 \otimes (dh_2 \wedge dx_2) + x_2 \otimes (dx_4 \wedge dy_3) + x_3 \otimes (dx_4 \wedge dy_2)
+ 2 x_4 \otimes (dh_2 \wedge dx_4) + 2 y_1 \otimes (dh_2 \wedge dy_1) + y_2 \otimes (dr_2 \wedge dy_2) + 2 y_2 \otimes (dx_3 \wedge dy_4)
+ 2 y_3 \otimes (dx_2 \wedge dy_4) + y_4 \otimes (dh_2 \wedge dy_4);
\]
\[
\alpha_0 = h_1(x_4 \wedge y_4).
\]

**Proof.** Let us denote by \( X \)'s and \( Y \)'s the Chevalley generators of \( L(\varepsilon) \). The isomorphism is given by
\[
X_i \leftrightarrow x_i, \ Y_i \leftrightarrow y_i, \ H_1 \leftrightarrow h_1, \ H_2 \leftrightarrow h_2 + (2-\varepsilon) h_1.
\]

2) The **3-parameter** family of deformations of \( \mathfrak{o}(5) \), denoted here by \( L(\varepsilon, \delta, \rho) \), was constructed by Kostrikin (see [Kos]) as follows. Consider the contact Lie algebra \( \mathfrak{k}(3; N) \), where \( N = (N_1, N_2, N_3) \in \mathbb{Z}^3 \), generated by indeterminates \( x, y \) and \( t \) forming the algebra of divided powers. As a vector space, \( \mathfrak{k}(3; N) \) is the subspace of \( \mathbb{K}[x, y, t] \) spanned by the monomials \( x^i y^j t^k \) with \( 0 \leq i < p^{N_1}, 0 \leq j < p^{N_2} \), and \( 0 \leq k < p^{N_3} \). As usual in the divided power algebra, one has
\[
w^i \cdot w^j = \binom{i+j}{i} w^{i+j}, \ \text{and} \ \partial_w w^i = w^{i-1}
\]
for \( w = x, y \) or \( t \). The contact bracket of polynomials \( f \) and \( g \) is defined by
\[
(5)\quad [f, g] = \triangle f \cdot \partial_t g - \partial_t f \cdot \triangle g + \partial_x f \cdot \partial_y g - \partial_y f \cdot \partial_x g
\]
with \( \triangle f = 2f - x \partial_x f - y \partial_y f \).

The **standard** \( \mathbb{Z} \)-grading \( \deg_{\text{Lie}} \) of \( \mathfrak{k}(3; N) \) is defined by setting \( \deg_{\text{Lie}}(f) = \deg f - 2 \), where \( \deg(x) = \deg(y) = 1 \) and \( \deg(t) = 2 \). Then a basis of \( L(\varepsilon, \delta, \rho) \) is given as follows:

| \( p \) | \( \deg \) | \( \text{deg} \) | \( \text{generator with weight} = \text{its generating function} \) |
|---|---|---|---|
| -2 | \( E_{-2a-\beta} = [E_{-a}, E_{-a-\beta}] = 1 \); |
| -1 | \( E_{-a} = x; \ E_{-a-\beta} = [E_{-\beta}, E_{-a}] = y; \) |
| 0 | \( H_\alpha = 2\varepsilon t + xy; \ H_\beta = -xy; \ E_\beta = x^2; \ E_{-\beta} = -y^2; \) |
| 1 | \( E_\alpha = -(1 + \varepsilon)xy^2 + \varepsilon yt; \ E_{\alpha+\beta} = [E_\beta, E_\alpha] = (1 + \varepsilon)x^2y + \varepsilon xt; \) |
| 2 | \( E_{2\alpha+\beta} = [E_\alpha, E_{\alpha+\beta}] = \varepsilon(1 + \varepsilon)x^2y^2 + \varepsilon^2 t^2 \). |
The brackets involving new parameters are as follows
\begin{equation}
\begin{aligned}
[E_{-2\alpha-\beta}, E_{-\alpha-\beta}] &= \delta E_{t_4}, & [E_{-2\alpha-\beta}, E_{-\alpha}] &= \rho E_{-\beta}, & [E_{-2\alpha-\beta}, E_{-\beta}] &= -\delta E_{\alpha+\beta}, \\
[E_{-2\alpha-\beta}, E_{\beta}] &= \rho E_{\alpha}, & [E_{-\alpha-\beta}, E_{-\beta}] &= -\delta E_{2\alpha+\beta}, & [E_{-\alpha}, E_{\beta}] &= -\frac{\delta}{\epsilon} E_{2\alpha+\beta}.
\end{aligned}
\end{equation}

3.2. Remark. Kostrikin and Kuznetso in [KK] write $H_{\alpha}$ as $t + xy$ instead.

3.3. Proposition. The Lie algebra $L(\varepsilon, \delta, \rho)$ can be obtained as a deformation of $\mathfrak{o}(5)$ generated by the cocycle and the following cocycles
\begin{equation}
\begin{aligned}
c_3 &= x_2 \otimes (x_1^* \wedge y_4^*) + x_4 \otimes (x_2^* \wedge y_2^*) + y_1 \otimes (y_2^* \wedge y_4^*), \\
c_6 &= x_1 \otimes (y_2^* \wedge y_4^*) + 2x_3 \otimes (y_1^* \wedge y_4^*) + x_4 \otimes (y_1^* \wedge y_3^*).
\end{aligned}
\end{equation}

The bracket is as follows:
\begin{equation}
[\cdot, \cdot]_t = [\cdot, \cdot] + t_2^2 c_0 + t_3 c_3 + t_4 c_6 + t_4^2 \alpha_0 + t_1 t_3 \alpha_3 + t_1 t_4 \alpha_6 + t_1^2 t_3 \beta_3 + t_1^2 t_4 \beta_6 + t_1^3 t_4 \gamma + t_1^4 t_4 \theta,
\end{equation}
where
\begin{equation}
\varepsilon = 2 - t_1, \quad \rho = \varepsilon(\varepsilon + 2) t_3, \quad \delta = \varepsilon(\varepsilon + 2)(2 + 2\varepsilon + \varepsilon^2) t_4.
\end{equation}

and
\begin{equation}
\alpha_0 = -x_4 \otimes (y_2^* \wedge y_3^*), \quad \alpha_3 = -x_4 \otimes (x_1^* \wedge y_2^*), \quad \beta_3 = -x_2 \otimes (x_1^* \wedge y_4^*) - y_1 \otimes (y_2^* \wedge y_4^*), \\
\beta_6 = x_4 \otimes (y_1^* \wedge y_3^*), \quad \gamma = -x_4 \otimes (y_1^* \wedge y_2^*), \quad \theta = x_3 \otimes (x_1^* \wedge y_3^*) - x_1 \otimes (y_2^* \wedge y_3^*).
\end{equation}

Proof. We can, of course, write down the whole multiplication table of $\mathfrak{o}(5)$, but to make the paper shorter we will not do it. Let us write only those constant structures for which we can deduce the values of $\rho$ and $\delta$. Indeed
\begin{equation}
[y_2, x_2] = (2 - t_1) x_3, \quad [y_2, x_1] = (t_1 t_3 - t_3) x_4, \quad [y_4, x_3] = (t_4^4 + t_4) x_1.
\end{equation}

Since $L(\varepsilon, \delta, \rho)$ was constructed in terms of $\mathfrak{o}(5)$, then, deforming the bracket, it is natural to define the isomorphisms between $L(\varepsilon, \delta, \rho)$ and $\mathfrak{o}(5)$, as follows:
\begin{equation}
x_1 \leftrightarrow E_{\beta}, \quad y_1 \leftrightarrow E_{-\beta}, \quad h_1 \leftrightarrow H_{\beta}, \quad x_2 \leftrightarrow E_{\alpha}, \quad y_2 \leftrightarrow E_{-\alpha}, \\
h_2 + (2 - \varepsilon) h_1 \leftrightarrow H_{\alpha}, \quad x_3 \leftrightarrow E_{\alpha+\beta}, \quad y_3 \leftrightarrow E_{-\alpha-\beta}, \quad x_4 \leftrightarrow E_{2\alpha+\beta}, \quad y_4 \leftrightarrow E_{-2\alpha-\beta}.
\end{equation}

3) Rudakov (cited in [Kos]) constructed a 3-parameter family of deformations of $\mathfrak{o}(5)$ as the Cartan prolong of the pair $(\mathbb{T}(a, b, c), \mathfrak{gl}(2))$. By construction, these deforms linearly depend on parameters.

3.4. The deforms of $\mathfrak{o}(5)$: General picture. Since $\dim H^2(\mathfrak{o}(5); \mathfrak{o}(5))) = 5$, we will be dealing with five parameters, denoted by $t_1, \ldots, t_5$. We denote the Chevalley generators corresponding to positive (resp. negative) roots by $x$ (resp. $y$). The Lie algebra $\mathfrak{o}(5)$ has infinitesimal deformations given by the following cocycles whose index is equal to their degree induced by the $\mathbb{Z}$-grading of $\mathfrak{o}(5)$ for which $\deg x_1 = \deg x_2 = 1$ (here $x_3 = [x_1, x_2], x_4 = [x_2, x_3]$ and similarly for the $y$'s):
\begin{equation}
\begin{aligned}
c_6 &= x_1 \otimes (y_2^* \wedge y_3^*) + 2x_3 \otimes (y_2^* \wedge y_4^*) + x_4 \otimes (y_1^* \wedge y_3^*), \\
c_3 &= x_2 \otimes (x_1^* \wedge y_3^*) + x_4 \otimes (x_1^* \wedge y_4^*) + y_1 \otimes (y_2^* \wedge y_4^*), \\
c_0 &= 2h_1 \otimes (x_2^* \wedge y_2^*) + 2h_1 \otimes (x_2^* \wedge y_4^*) + 2x_1 \otimes (y_2^* \wedge y_3^*) + y_1 \otimes (y_2^* \wedge y_3^*), \\
c_{-3} &= 2x_1 \otimes (x_2^* \wedge x_4^*) + y_2 \otimes (x_4^* \wedge y_1^*) + y_4 \otimes (x_2^* \wedge y_1^*), \\
c_{-6} &= y_1 \otimes (x_2^* \wedge x_3^*) + 2y_3 \otimes (x_1^* \wedge x_4^*) + y_4 \otimes (x_1^* \wedge x_3^*).
\end{aligned}
\end{equation}
Observe a symmetry between \(c_6\) and \(c_{-6}\), and between \(c_3\) and \(c_{-3}\): there is an involution on the Lie algebra interchanging \(x\)-generators and \(y\)-generators. One has to pay attention that there is a sign involved \((2 = -1)\) when passing from \(c_3\) to \(c_{-3}\).

### 3.5. Theorem

The Lie algebra \(\mathfrak{so}(5)\) admits a 5-parameter family of deformations denoted by \(\mathfrak{so}(5,t)\), where \(t = (t_1, t_2, t_3, t_4, t_5)\).

The deformed bracket is defined by

\[
[\cdot, \cdot]_{t_1, t_2, t_3, t_4, t_5} = [\cdot, \cdot] + t_1 c_0 + t_2 c_{-3} + t_3 c_3 + t_4 c_6 + t_5 c_{-6} + t_1 t_4 \alpha_{0,6} + t_1 t_2 \alpha_{0,-3} +
\]

\[
+ t_1 t_5 \alpha_{0,-6} + t_1 t_3 \alpha_{0,3} + t_4 t_5 \alpha_{6,-6} + t_2 t_3 \alpha_{3,-3} + t_1 t_4 t_5 \beta_{-6,0,6} + t_1 t_2 t_3 \beta_{3,0,-3},
\]

where

\[
\begin{align*}
\alpha_{0,-6} &= y_1 \otimes \left(x_3^* \wedge x_4^*\right), & \alpha_{0,6} &= x_1 \otimes \left(y_3^* \wedge y_4^*\right), \\
\alpha_{0,3} &= y_1 \otimes \left(y_2^* \wedge y_4^*\right), & \alpha_{0,-3} &= 2x_1 \otimes \left(x_2^* \wedge x_4^*\right),
\end{align*}
\]

\[
\begin{align*}
\alpha_{-3,3} &= 2h_2 \otimes \left(x_2^* \wedge y_2^*\right) + h_2 \otimes \left(x_3^* \wedge y_4^*\right) - h_2 \otimes \left(x_1^* \wedge y_1^*\right) + 2x_1 \otimes \left(x_3^* \wedge y_2^*\right) + x_2 \otimes \left(h_1^* \wedge y_2^*\right) +
\]

\[
+ 2x_2 \otimes \left(x_3^* \wedge y_4^*\right) + x_3 \otimes \left(h_1^* \wedge x_1^*\right) + 2x_4 \otimes \left(h_1^* \wedge x_2^*\right) + x_4 \otimes \left(x_2^* \wedge y_2^*\right) +
\]

\[
+ 2y_2 \otimes \left(x_3^* \wedge y_4^*\right) + 2y_3 \otimes \left(h_1^* \wedge y_2^*\right) + 2y_3 \otimes \left(x_3^* \wedge y_2^*\right) - 2y_3 \otimes \left(y_1^* \wedge y_2^*\right) - 2y_4 \otimes \left(h_1^* \wedge y_1^*\right),
\]

\[
\begin{align*}
\alpha_{-6,6} &= h_2 \otimes \left(x_3^* \wedge y_4^*\right) + 2h_2 \otimes \left(x_3^* \wedge y_4^*\right) + h_2 \otimes \left(x_4^* \wedge y_1^*\right) + 2x_1 \otimes \left(x_4^* \wedge y_2^*\right) + 2x_2 \otimes \left(h_1^* \wedge x_2^*\right) +
\]

\[
+ 2x_2 \otimes \left(x_3^* \wedge y_4^*\right) + 2x_3 \otimes \left(h_1^* \wedge x_1^*\right) + x_4 \otimes \left(h_1^* \wedge x_2^*\right) + x_4 \otimes \left(x_2^* \wedge y_1^*\right) + y_2 \otimes \left(h_1^* \wedge y_2^*\right) +
\]

\[
+ 2y_2 \otimes \left(x_3^* \wedge y_4^*\right) + y_3 \otimes \left(h_1^* \wedge y_4^*\right) + 2y_3 \otimes \left(x_2^* \wedge y_1^*\right) + y_3 \otimes \left(y_1^* \wedge y_4^*\right) + 2y_4 \otimes \left(h_1^* \wedge y_1^*\right),
\]

and

\[
\begin{align*}
\beta_{-6,0,6} &= 2h_1 \otimes \left(x_3^* \wedge y_2^*\right) + h_1 \otimes \left(x_3^* \wedge y_4^*\right) + 2h_1 \otimes \left(x_4^* \wedge y_2^*\right) + 2x_1 \otimes \left(x_3^* \wedge y_2^*\right) + 2x_2 \otimes \left(h_1^* \wedge x_2^*\right) +
\]

\[
+ 2x_3 \otimes \left(h_1^* \wedge x_1^*\right) + x_4 \otimes \left(h_1^* \wedge x_2^*\right) + y_2 \otimes \left(h_2^* \wedge y_2^*\right) + y_3 \otimes \left(h_2^* \wedge y_1^*\right) + 2y_4 \otimes \left(h_2^* \wedge y_1^*\right),
\]

\[
\begin{align*}
\beta_{3,0,-3} &= h_1 \otimes \left(x_3^* \wedge y_2^*\right) + 2h_1 \otimes \left(x_3^* \wedge y_4^*\right) + h_1 \otimes \left(x_4^* \wedge y_1^*\right) + 2x_1 \otimes \left(x_4^* \wedge y_2^*\right) + x_2 \otimes \left(h_2^* \wedge y_2^*\right) +
\]

\[
+ x_3 \otimes \left(h_2^* \wedge x_3^*\right) + 2x_4 \otimes \left(h_2^* \wedge x_1^*\right) + 2x_4 \otimes \left(x_3^* \wedge x_3^*\right) + 2y_2 \otimes \left(h_2^* \wedge y_2^*\right) + y_2 \otimes \left(x_3^* \wedge y_1^*\right) +
\]

\[
+ 2y_3 \otimes \left(h_2^* \wedge y_3^*\right) + y_3 \otimes \left(x_2^* \wedge y_2^*\right) + y_4 \otimes \left(h_2^* \wedge y_2^*\right).
\]

\[
\text{Proof.} \quad \text{The proof is a direct computation assisted by Grozman's Mathematica-based package SuperLie [Gr]. We compute the Massey brackets in each degree and check if this bracket is a coboundary. For example, we can easily get}
\]

\[
[[c_{-3}, c_{-6}]] = 0, \quad [[c_3, c_3]] = 0,
\]

\[
[[c_0, c_6]] = 2h_1 \otimes \left(y_1^* \wedge y_3^* \wedge y_4^*\right) + 2x_1 \otimes \left(y_1^* \wedge y_3^* \wedge y_4^*\right) + 2x_3 \otimes \left(x_1^* \wedge y_3^* \wedge y_4^*\right).
\]

Besides, we can show that \([[c_0, c_6]] = -2d\alpha_{0,6}\), where \(\alpha_{0,6}\) is as above. It is here that we use the formula of Lemma 2.1 in order to compute \(d\alpha_{0,6}\). Indeed, we have:

\[
d\alpha_{0,6} = d(x_1 \otimes y_3^* \wedge y_4^*) =
\]

\[
= dx_1 \otimes y_3^* \wedge y_4^* + x_1 \otimes dy_3^* \wedge y_4^* + 2x_1 \otimes y_3^* \wedge dy_4^* =
\]

\[
= x_3 \otimes x_2^* \wedge y_3^* \wedge y_4^* + h_1 \otimes y_1^* \wedge y_3^* \wedge y_4^* + x_1 \otimes h_1^* \wedge y_3^* \wedge y_4^* +
\]

\[
+ 2x_1 \otimes h_2^* \wedge y_3^* \wedge y_4^* + x_1 \otimes y_3^* \wedge y_2^* \wedge y_4^* + x_1 \otimes y_3^* \wedge h_1^* \wedge y_4^* +
\]

\[
+ x_1 \otimes y_3^* \wedge y_4^* \wedge h_2^* =
\]

\[
= x_3 \otimes x_2^* \wedge y_3^* \wedge y_4^* + h_1 \otimes y_1^* \wedge y_3^* \wedge y_4^* + x_1 \otimes y_1^* \wedge y_2^* \wedge y_4^*.
\]

In this computation we used the knowledge of the explicit form of \(dx_1\) (the coboundary of a 0-cochain with adjoint coefficients) which we extracted from the multiplication table:

\[
dx_1 = x_3 \otimes x_2^* + h_1 \otimes y_1^* + y_2 \otimes y_3^* + x_1 \otimes h_1^* + 2x_1 \otimes h_2^*,
\]
and the explicit form of $dy^*_3$ and $dy^*_4$ (the coboundary of a 1-cochain with values in the ground field):

\[
\begin{align*}
  dy^*_3 &= y^*_1 \wedge y^*_2 + y^*_4 \wedge x^*_2 + y^*_3 \wedge h^*_1, \\
  dy^*_4 &= 2y^*_4 \wedge h^*_2 + 2y^*_3 \wedge y^*_2.
\end{align*}
\]

We have chosen the cocycles so that their Massey squares are 0. As explained above, the $\alpha$’s are not unique. We hoped that we can choose them so that the $\beta$’s (corresponding to Massey products of degree three) are ALL zero. Unfortunately, this is not possible. Nevertheless, we can choose the $\alpha$’s so that a large number of the $\beta$’s vanish. Once this is done, we can deal with the $\beta$’s. Rather long computations with the remaining free parameters in degree 4 show that the $\alpha$- and $\beta$-cochains can be chosen so that all Massey brackets in degree 4 vanish. This was our choice.

\[\square\]

4. Main results: $p = 2$

4.1. Deforms of $\mathfrak{o}^{(1)}(5)$. In this subsection, $p = 2$, and hence the orthogonal Lie algebra $\mathfrak{o}(5)$ is not simple and of dimension 15. The Lie algebra $\mathfrak{o}^{(1)}(5)$, the derived of $\mathfrak{o}(5)$, is simple and of dimension 10. It is realized by means of the Cartan matrix and the generators (same with the $y$’s):

\[
\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}; \quad x_1, x_2, x_3 = [x_1, x_2], \quad x_4 = [x_1, x_3];
\]

see \cite{BGL}. From \cite{BGL4} we know that $\dim H^2(\mathfrak{o}^{(1)}(5); \mathfrak{o}^{(1)}(5)) = 4$, so we will be dealing with four parameters, denoted by $t_1, \ldots, t_4$.

The Lie algebra $\mathfrak{o}^{(1)}(5)$ has infinitesimal deformations given by the following cocycles:

\[
\begin{align*}
  c_4 &= h_1 \otimes (y^*_2 \wedge y^*_1) + x_1 \otimes (y^*_2 \wedge y^*_1) + x_2 \otimes (h^*_1 \wedge y^*_1) + x_3 \otimes (h^*_1 \wedge y^*_1) + x_4 \otimes (h^*_2 \wedge y^*_1) + y_1 \otimes (y^*_2 \wedge y^*_1), \\
  c_{-2} &= h_1 \otimes (x^*_1 \wedge y^*_1) + x_2 \otimes (h^*_1 \wedge x^*_1) + x_2 \otimes (h^*_1 \wedge x^*_1) + x_3 \otimes (x^*_1 \wedge x^*_1) + y_1 \otimes (h^*_1 \wedge y^*_1) + y_1 \otimes (h^*_1 \wedge x^*_1) + \\
  &\quad + y_1 \otimes (x^*_1 \wedge y^*_1) + y_1 \otimes (h^*_1 \wedge y^*_1) + y_4 \otimes (h^*_2 \wedge y^*_1) + y_4 \otimes (x^*_1 \wedge y^*_1), \\
  c_2 &= h_1 \otimes (x^*_2 \wedge y^*_1) + x_1 \otimes (h^*_1 \wedge y^*_1) + x_1 \otimes (h^*_1 \wedge y^*_1) + x_1 \otimes (x^*_1 \wedge y^*_1) + x_3 \otimes (x^*_1 \wedge y^*_1) + x_4 \otimes (h^*_1 \wedge x^*_1) + \\
  &\quad + x_4 \otimes (h^*_1 \wedge x^*_1) + y_2 \otimes (h^*_1 \wedge y^*_1) + y_2 \otimes (h^*_1 \wedge y^*_1) + y_2 \otimes (y^*_1 \wedge y^*_1) + y_2 \otimes (y^*_1 \wedge y^*_1),
\end{align*}
\]

and the 2-cocycle $c_{-4}$ is obtained from $c_4$ by changing $x$ by $y$ and $y$ by $x$. These cocycles $c_i$ are chosen so that $[[c_i, c_i]] = 0$ (which, fortunately, is possible) and having shortest possible expression (for esthetic reasons).

4.2. Theorem. The Lie algebra $\mathfrak{o}^{(1)}(5)$ admits a 4-parameter family of deformations denoted by $\mathfrak{o}^{(1)}(5; t)$, where $t = (t_1, t_2, t_3, t_4)$. The deformed bracket is given by the formula

\[
[\cdot, \cdot]_{t_1, t_2, t_3, t_4} = [\cdot, \cdot] + t_1 c_{-4} + t_2 c_4 + t_3 c_{-2} + t_4 c_2 + t_1 t_3 \alpha_{-1, -2} + t_1 t_4 \alpha_{-1, 2} + t_2 t_3 \alpha_{4, -2} + \\
+ t_2 t_4 \alpha_{4, 2} + t_3 t_4 \alpha_{-2, 2} + t_2 t_3 t_4 \beta_{-1, -2, 2} + t_3^2 t_4 \beta_{-2, -2, 2} + t_3 t_4^2 \beta_{-2, 2, 2} + t_3^2 t_4^2 \beta_{2, 2, 2} + t_3^3 t_4^3 \beta_{3, 2, 2}.
\]
where
\[
\begin{align*}
\alpha_{-4,-2} &= y_4 \otimes (h_2^* \wedge x_4^*), \\
\alpha_{4,-2} &= x_2 \otimes (h_2^* \wedge y_2^*) + y_1 \otimes (y_2^* \wedge y_3^*), \\
\alpha_{4,2} &= y_2 \otimes (h_2^* \wedge x_2^*), \\
\beta_{2,-2,2} &= h_1 \otimes (x_2^* \wedge y_3^*) + x_1 \otimes (x_3^* \wedge y_1^*), \\
\beta_{-2,-2,2} &= h_1 \otimes (x_4^* \wedge y_2^*) + y_1 \otimes (x_3^* \wedge y_2^*), \\
\beta_{4,-2,2} &= x_3 \otimes (h_2^* \wedge y_3^*), \\
\beta_{4,2,2} &= h_1 \otimes (x_2^* \wedge y_3^*) + h_1 \otimes (x_4^* \wedge y_4^*) + y_1 \otimes (x_3^* \wedge y_4^*) + x_1 \otimes (x_3^* \wedge y_2^*) + y_1 \otimes (x_4^* \wedge y_2^*) + y_2 \otimes (h_2^* \wedge y_2^*) + y_3 \otimes (h_2^* \wedge y_3^*) + y_4 \otimes (h_2^* \wedge y_4^*).
\end{align*}
\]

Claim. The deformation (8) of the initial bracket is trivial.

Proof. Let us consider linear operators \(A_a\) which act as follows on basic elements of \(\mathfrak{g}\):
\[
A_a f = \sqrt{1 + a} f; \quad A_a e_i = (\sqrt{1 + a})^i e_i \quad \text{for all } i = 0, \ldots, p - 1.
\]
Then
\[
[A_a x, A_a y] = A_a [x, y]_a \quad \text{for all } x, y \in \mathfrak{g}.
\]

The 2-cocycle corresponding to the infinitesimal version of the deformation (8) is proportional to
\[
z = e_0 \otimes \psi \wedge \phi_{p-1},
\]
where \((\phi_0, \ldots, \phi_{p-1}, \psi)\) is the basis of \(\mathfrak{g}^*\) dual to \((e_0, \ldots, e_{p-1}, f)\).

Claim. The element \(z\) represents a nontrivial cocycle of \(H^2(\mathfrak{g}; \mathfrak{g})\).
On deformations of $\mathfrak{g}(5)$ in characteristics 3 and 2

Proof. The algebra $\mathfrak{g}$ has a $\mathbb{Z}/p\mathbb{Z}$-grading such that

$$\deg f = 1; \quad \deg e_i = i \quad \text{for } i = 0, \ldots, p - 1.$$  

The degree of $z$ in the corresponding grading of $C^*(\mathfrak{g}; \mathfrak{g})$ is equal to 0. The subspace of $C^1(\mathfrak{g}; \mathfrak{g})$ of degree 0 is spanned by the elements

$$e_i \otimes \phi_i \quad \text{for } i = 0, \ldots, p - 1;$$  

$$e_1 \otimes \psi, \quad f \otimes \phi_1, \quad f \otimes \psi.$$  

So $B^2(\mathfrak{g}; \mathfrak{g})_0$ is the linear span of the elements (here $i \pm 1$ or $j \pm 1$ in the subscript should be understood modulo $p$)

$$d(e_i \otimes \phi_i) = e_{i+1} \otimes \phi_i \wedge \psi - e_i \otimes \phi_{i-1} \otimes \psi \quad \text{for } i = 0, \ldots, p - 1;$$  

$$d(f \otimes \phi_1) = \sum_{i \neq 1} e_{i+1} \otimes \phi_i \wedge \phi_1 - f \otimes \phi_0 \wedge \psi;$$  

$$d(f \otimes \psi) = \sum_{i = 0} e_{i+1} \otimes \phi_i \wedge \psi. \quad (10)$$

(The differential $d(e_1 \otimes \psi)$ vanishes, so does not count.) Let us consider the linear map

$$L : C^2(\mathfrak{g}; \mathfrak{g}) \to \mathbb{K}$$  

defined on the basic 2-cochains (here $i, j, k = 0, \ldots, p - 1$) as follows:

$$L(e_i \otimes \phi_j \wedge \phi_k) = 0; \quad L(e_i \otimes \phi_j \wedge \psi) = \delta_{i,j+1};$$  

$$L(f \otimes \phi_j \wedge \phi_j) = 0; \quad L(f \otimes \phi_j \wedge \psi) = 0.$$  

The value of $L$ on all the elements $\{10\}$ is equal to 0, but $L(z) = -1$. Thus $z$ is not a linear combination of the elements $\{11\}$. \hfill \Box

If the deformation $[z]$ is trivial, why can $z$ not be obtained as the differential of 1-cochain $C$ such that $C(x) = \frac{\partial A_x}{\partial a}$ (whatever such partial derivative might mean in characteristic $> 0$) for any $x \in \mathfrak{g}$? This is because in characteristic $p$ the function $\sqrt{1 + a}$ is not differentiable (again, whatever “differentiable” means here).

The situation is opposite, in a way, to the one described in [FF] for infinite dimensional Lie algebras over algebraically closed fields of characteristic 0.

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1Department of Mathematics, United Arab Emirates University, Al Ain, PO. Box: 17551; Bouarroudj.sofiane@uaeu.ac.ae, 2Nizhegorodskij Univ. RU-603950 Russia, Nizhny Novgorod, pr. Gagarina 23; yorool@mail.ru, 3Laboratoire de Mathématiques Jean Leray, UMR 6629 du CNRS, Université de Nantes, 2 rue de la Houssinière, 44322 France; wagemann@math.univ-nantes.fr