Canonical approach to the closed string non-commutativity

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Abstract

We consider the closed string moving in the weakly curved background and its totally T-dualized background. Using T-duality transformation laws, we find the structure of the Poisson brackets in the T-dual space corresponding to the fundamental Poisson brackets in the original theory. From this structure we obtain that the commutative original theory is equivalent to the non-commutative T-dual theory, whose Poisson brackets are proportional to the background fluxes times winding and momenta numbers. The non-commutative theory of the present article is more non-geometrical then T-folds and in the case of three space-time dimensions corresponds to the nongeometric space-time with $R$-flux.

1 Introduction

It is well known that the open string endpoints, attached to $Dp$-brane, are non-commutative [1, 2]. The non-commutativity is implied by the fact that, on the solution of boundary conditions the initial coordinate is given as a linear combination of the effective coordinate and the effective momentum, which have the nonzero Poisson bracket (PB). In the constant background case, the coefficient in front of momenta is proportional to the Kalb-Ramond field $B_{\mu\nu}$, whose presence is crucial in gaining the non-commutativity.

The closed string does not have endpoints and in the flat space the boundary conditions are satisfied automatically. But, to understand the closed string non-commutativity, we are going to use the similar explanation as in the open string case. We will express the closed
string coordinates in terms of the coordinates and momenta of some other space. The relation between different spaces will be established using the T-duality transformations.

The T-dualization along isometry directions, and the construction of T-dual theory was first realized through Buscher procedure [3]. The procedure is in fact a localization of the translation invariance symmetry, in which beside the covariantization of derivatives one adds the Lagrangian multiplier term to the action which insures the physical equivalence of the initial and the T-dual theory.

In the flat space, T-duality relates $\sigma$-derivatives of the coordinates of the original theory with the momenta of its T-dual theory, and vice versa. As the momenta of the original theory are taken to be commutative it follows that the coordinates commute as well. So, in the flat space there is no non-commutativity of the closed string T-dual coordinates. This is in agreement with the fact that T-duality is canonical transformation in the flat space, and the fact that PB’s are invariant under such transformations.

The closed string non-commutativity was first observed in the paper [4], and investigated further in [5, 6, 7], where it was found that the commutators of the coordinates are proportional to the flux and the winding number.

Let us briefly describe the result of Ref. [5] following its notation. After the $T_1$-dualization along coordinate $X_1$, one obtains the twisted torus with coordinates $Y_a, (a = 1, 2, 3)$ and $f$-flux. After additional $T_2$-dualization along $X^2 = Y^2$ one obtains the non-geometric background with coordinates $Z^a$ and $Q$-flux. Using the standard Buscher prescription one can not perform $T_3$-dualization along the coordinate $X^3 = Y^3 = Z^3$ because the Kalb-Ramond field $B_{ab}$ depends on $Z^3$. But it is argued in Refs. [8, 5] that $T_3$-dualization leads to the non-geometric background with R-flux configuration and coordinates $W^a$ presented in the T-duality chain

$$H_{abc}, X^a \xrightarrow{T_1} f^a_{b c}, Y^a \xrightarrow{T_2} Q^{ab}_{c}, Z^a \xrightarrow{T_3} R^{abc}, W^a. \quad (1.1)$$

In the paper [5], the non-commutativity of the non-geometric background ($Z^a$ with $Q$-flux) has been obtained using its $T_2$-duality connection $Z^a = Z^b(Y^a)$ with geometric background (twisted torus with $Y^a$ and $f$-flux).

In our paper [9], we performed generalized Buscher’s T-dualization procedure along all the coordinate directions. It corresponds to $T = T_1 \circ T_2 \circ \cdots \circ T_D$ -duality relation $y_\mu = y_\mu(x^\nu)$ connecting the beginning and the end of the T-duality chain

$$H_{\mu \nu \rho}, x^\mu \xrightarrow{T_1} (f_1)_{\mu \nu \rho}, x^\mu \xrightarrow{T_2} (f_2)_{\mu \nu \rho}, x^\mu \xrightarrow{T_3} \cdots \xrightarrow{T_D} (f_D)_{\mu \nu \rho}, x^\mu = y_\mu. \quad (1.2)$$

where $(f_i)_{\mu \nu \rho}$ and $x^\mu_i, (i = 1, 2, \cdots, D)$ are fluxes and the coordinates of the corresponding configuration. In $D$-dimensional space-time it is possible to perform T-duality along any subset of coordinates. For simplicity, in the present article we will T-dualize all the directions. The general case will be published separately.
We considered the bosonic string moving in the background with the constant metric $G_{\mu\nu} = \text{const}$ and the linear Kalb-Ramond field $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho$, where the field strength of the Kalb-Ramond field $B_{\mu\nu\rho}$ is infinitesimally small (for more details see the introductory part of Section 2). The obtained T-dual theory is of the same form as the initial theory, so that the T-dual string moves in the T-dual background but in the doubled space given by the coordinates $y_\mu, \tilde{y}_\mu$. The dual coordinates satisfy the following conditions $\dot{y}_\mu = \tilde{y}_\mu', y_\mu' = \dot{\tilde{y}}_\mu$. The improvement, in comparison to the standard Buscher procedure, is the covariantization of the coordinates $x_\mu$. In fact, because $x_\mu$ is gauge dependent, it is replaced by the gauge invariant expression $\Delta x_\mu^{\text{inv}} = \int d\xi D_\alpha x_\mu$. As pointed out in [8], the T-dual background of the present paper, is of the “new class that is even more nongeometrical than T-folds”. Unlike the T-folds, this background is not standard manifold even locally. In our formulation, this stems from the fact that the argument of the background fields $\Delta x_\mu^{\text{inv}}$ is the line integral.

In the canonical formalism, the T-dual variables can be expressed in terms of the original ones in the simple form $y_\mu' \equiv \frac{1}{\kappa} \pi_\mu - \beta_\mu^0 [x]$ and $\pi_\mu' \equiv \kappa x_\mu + \kappa^2 \theta_0^\nu \beta_\nu^0 [x]$. The infinitesimal expression $\beta_\mu^0$ is the improvement in comparison to the flat background case. Because the coordinates and momenta of the original theory do not commute, $\beta_\mu^0$ is the source of the closed string noncommutativity.

We will follow the main idea of Ref. [5], using the T-duality transformation laws between the T-dual backgrounds in order to study the non-commutativity of the coordinates. In the paper [5], the $T_2$-duality connects coordinates $Z^a = Z^a(Y^a)$ of the nongeometric background ($Z^a$ with $Q$-flux) and the geometric background (twisted torus with $Y^a$ and $f$-flux). We performed T-dualization procedure along all the coordinates, and obtained the T-duality transformation $y_\mu = y_\mu(x^\mu)$ of the locally nongeometric background (the end of the chain (1.2) with $y_\mu$ and $fD$-flux) and the geometric background (torus with $H$-flux in the beginning of the chain (1.2)). In both approaches it was assumed that the geometric backgrounds (described by $Y^a$ in [5] and by $X^a$ in our paper) have the standard commutation relations. The PB between $y_\mu$’s is proportional to the flux $B_{\mu\nu\rho}$ and the winding number $N_\mu$ of the initial theory. In addition, we obtain the complete algebra of the T-dual coordinates and momenta in terms of the fluxes.

For $D = 3$, the case of the present article corresponds to T-duality $T = T_1 \circ T_2 \circ T_3$ which connects the coordinates $W^a = W^a(X^a)$ of the nongeometric background ($W^a$ with $R$-flux) and the geometric background (torus with $X^a$ and $H$-flux). In comparison to Ref. [5], this procedure contains one T-dualization more, $T_3$-dualization along the coordinate $X^3 = Y^3 = Z^3$, which can not be done using the standard Buscher prescription because the Kalb-Ramond field $B_{ab}$ depends on $Z^3$. So, in terms of Ref. [5], we obtained the non-commutativity of the nongeometric background, with $R$-flux configuration. This background does not look like the conventional space even locally.
At the end we give three appendices. In the first one we derive in detail the expression for the dual momentum $\star \pi^\mu$, while in the second one we make a list of fluxes used in the paper. The third appendix contains the mathematical details about transition from PB $\{\Delta X, \Delta Y\}$ to PB $\{X, Y\}$.

## 2 Bosonic string in the weakly curved background and its T-dual picture

Let us consider the closed string moving in the $D$-dimensional space-time, in the coordinate $x^\mu(\tau, \sigma)$, $\mu = 0, \cdots, D - 1$ dependent background, described by the action

$$S[x] = \kappa \int_{\Sigma} d^2 \xi \, \partial_+ x^\mu \Pi_{+\mu\nu}[x] \partial_- x^\nu.$$  \hspace{1cm} (2.1)

We suppose that all the coordinates are compact with the radii $R_\mu$. The background is defined by the space-time metric $G_{\mu\nu}$ and the antisymmetric Kalb-Ramond field $B_{\mu\nu}$

$$\Pi_{\pm\mu\nu}[x] = B_{\mu\nu}[x] \pm \frac{1}{2} G_{\mu\nu}[x].$$  \hspace{1cm} (2.2)

The light-cone coordinates are

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma), \quad \partial_\pm = \partial_\tau \pm \partial_\sigma,$$  \hspace{1cm} (2.3)

and the action is given in the conformal gauge (the world-sheet metric is taken to be $g_{\alpha\beta} = e^{2F} \eta_{\alpha\beta}$).

The world-sheet conformal invariance is required, as a condition of having a consistent theory on the quantum level \[10, 11\]. This results in the following space-time equations for the background fields

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}^{\rho\sigma} = 0, \quad D_\rho B_{\mu\nu} = 0,$$  \hspace{1cm} (2.4)

in the lowest order in slope parameter $\alpha'$ and for the constant dilaton field $\Phi = const.$ Here

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$  \hspace{1cm} (2.5)

is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and $D_\mu$ are Ricci tensor and the covariant derivative with respect to the space-time metric.

We will consider the weakly curved background \[5, 9, 12, 13\], defined by

$$G_{\mu\nu}[x] = const, \quad B_{\mu\nu}[x] = b_{\mu\nu} + h_{\mu\nu}[x] = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho, \quad b_{\mu\nu}, B_{\mu\nu\rho} = const.$$  \hspace{1cm} (2.6)
Here, the constant $B_{\mu \nu \rho}$ is infinitesimally small which, according to [4, 5, 7], means that we will assume that $D$ dimensional torus is so large that for any $\mu, \nu, \rho$

\[
\frac{B_{\mu \nu \rho}}{R_\mu R_\nu R_\rho} \ll 1, \tag{2.7}
\]

where $R_\mu (\mu = 0, 1, \ldots D - 1)$ are the radii of the torus. For simplicity we will take $R_0 = R_1 = \cdots = R_{D-1}$ and rescale background fields according to App.A of Ref. [5]. The background (2.6) is the solution of (2.4) in the first order in $B_{\mu \nu \rho}$ approximation of the closed string theory (2.1).

### 2.1 T-dual bosonic string

The T-dualization of the closed string theory in the weakly curved background was a subject of investigation in [9]. There we presented the T-dualization procedure performed along all the coordinates, in a background which depends on these coordinates. Here we will give a short overview of the most important results.

The T-dual picture of the theory is given by

\[
* S[y] = \kappa \int d^2 \xi \partial_+ y_\mu \, *\Pi^\mu_\nu [\Delta V[y]] \, \partial_- y_\nu = \frac{\kappa^2}{2} \int d^2 \xi \, \partial_+ y_\mu \Theta^\mu_\nu [\Delta V[y]] \, \partial_- y_\nu, \tag{2.8}
\]

with

\[
\Theta^\mu_\nu \equiv -\frac{2}{\kappa} (G_E^{-1} \Pi \gamma G_E^{-1})^\mu_\nu = \theta^\mu_\nu \pm \frac{1}{\kappa} (G_E^{-1})^\mu_\nu, \quad G_E_{\mu \nu} \equiv G_{\mu \nu} - 4 (B G^{-1} B)_{\mu \nu}. \tag{2.9}
\]

The dual background fields defined in analogy with (2.2) as $*\Pi^\mu_\nu = *B^\mu_\nu \pm \frac{1}{2} *G^\mu_\nu$, have the form

\[
*G^\mu_\nu [\Delta V[y]] = (G_E^{-1})^\mu_\nu [\Delta V[y]], \quad *B^\mu_\nu [\Delta V[y]] = \frac{\kappa}{2} \theta^\mu_\nu [\Delta V[y]]. \tag{2.10}
\]

Using the terminology introduced in the open string case, they are equal to the inverse of the effective metric $G_E^{\mu_\nu}$ and proportional to the non-commutativity parameter $\theta^\mu_\nu$. Their argument is given by

\[
\Delta V^\mu[y] = -\kappa \theta_0^\mu_\nu \Delta y_\nu + (g^{-1})^\mu_\nu \Delta \tilde{y}_\nu, \tag{2.11}
\]

where

\[
\Delta y_\mu = \int_P (d \tau \tilde{y}_\mu + d \sigma y'_\mu) = y_\mu(\xi) - y_\mu(\xi_0), \quad \Delta \tilde{y}_\mu = \int_P (d \tau y'_\mu + d \sigma \tilde{y}_\mu), \tag{2.12}
\]

and

\[
g_{\mu \nu} = G_{\mu \nu} - 4 (b G^{-1} b)_{\mu \nu}, \quad \theta_0^\mu_\nu = -\frac{2}{\kappa} (g^{-1} b G^{-1})^\mu_\nu, \tag{2.13}
\]

are constant finite parts of the effective metric and the non-commutativity parameter. The variable $\Delta \tilde{y}_\mu$ is path independent on the zeroth order equation of motion. The T-dual theory is defined in the doubled space, defined by two coordinates $y_\mu$ and $\tilde{y}_\mu$, related by expressions $\tilde{y}_\mu = \tilde{y}_\mu, \ y'_\mu = \tilde{y}'_\mu$. 

5
2.2 Transformation laws

The T-duality transformation connecting the variables of the closed string theory in the weakly curved background and its T-dualized string theory is [9]

\[ \partial_{\pm} x^\mu \cong -\kappa \Theta^\mu_{\pm} [\Delta V] \left[ \partial_{\pm} y_\nu \mp 2 \beta_{\nu}^\mp [V] \right], \tag{2.14} \]

with

\[ \beta^\pm [x] = \frac{1}{2} (\beta^0_{\mu} \pm \beta^1_{\mu}) = \mp \frac{1}{2} h_{\mu \nu} [x] \partial_{\pm} x^\nu, \beta^0_{\mu} [x] = h_{\mu \nu} [x] x^\nu, \beta^1_{\mu} [x] = -h_{\mu \nu} [x] \dot{x}^\nu. \tag{2.15} \]

From (2.14) we can find the transformation law for \( \dot{x}^\mu \) and \( x'^{\mu} \)

\[ \dot{x}^\mu \cong -\kappa \theta^{\mu \nu} [\Delta V] \dot{y}_\nu + (G^{-1}_E)^{\mu \nu} [\Delta V] y'_\nu + (g^{-1})^{\mu \nu} \beta^0_{\nu} [V] + \kappa \theta^{\mu \nu} \beta^1_{\nu} [V]. \tag{2.16a} \]
\[ x'^{\mu} \cong (G^{-1}_E)^{\mu \nu} [\Delta V] \dot{y}_\nu - \kappa \theta^{\mu \nu} [\Delta V] y'_\nu - \kappa \theta^{\mu \nu} \beta^0_{\nu} [V] - (g^{-1})^{\mu \nu} \beta^1_{\nu} [V]. \tag{2.16b} \]

Using the expression for the canonical momentum of the original theory

\[ \pi_{\mu} = \frac{\delta S}{\delta \dot{x}^\mu} = \kappa \left[ G_{\mu \nu} \dot{x}^\nu - 2B_{\mu \nu} [x] x^\nu \right], \tag{2.17} \]

and T-dual canonical momentum

\[ ^* \pi^\mu = \frac{\delta S}{\delta \dot{y}_\mu} = \kappa (G^{-1}_E)^{\mu \nu} [\Delta V [y]] \dot{y}_\nu - \kappa^2 \theta^{\mu \nu} [\Delta V [y]] y'_\nu - \kappa (g^{-1})^{\mu \nu} \beta^1_{\nu} [V [y]], \tag{2.18} \]

derived in App. A we can rewrite the above transformations in the canonical form

\[ x'^{\mu} \cong \frac{1}{\kappa} \pi^\mu - \kappa \theta^{\mu \nu} \beta^0_{\nu} [V], \tag{2.19a} \]
\[ \pi_{\mu} \cong \kappa y'_\mu + \kappa^2 \beta^{0 \mu} [V], \tag{2.19b} \]

with \( \beta^{0 \mu} [V] \) defined in (2.15). It is shown in Ref. [9] that the T-dual of the T-dual action is the original one. The corresponding T-dual transformation of the variables law is the inverse of (2.14)

\[ \partial_{\pm} y_\mu \cong -2 \Pi_{\pm \nu} [\Delta x] \partial_{\pm} x^\nu \mp 2 \beta^{\pm \nu} [x], \tag{2.20} \]

and so the transformation laws for \( \dot{y}_\mu \) and \( y'_\mu \) are equal to

\[ \dot{y}_\mu \cong -2B_{\mu \nu} [x] \dot{x}^\nu + G_{\mu \nu} x^\nu + \beta^1_{\mu} [x], \tag{2.21a} \]
\[ y'_\mu \cong G_{\mu \nu} \dot{x}^\nu - 2B_{\mu \nu} [x] x^\nu - \beta^{0 \mu} [x]. \tag{2.21b} \]

Using (2.17) and (2.18) we obtain the canonical form of the T-dual transformations

\[ y'_\mu \cong \frac{1}{\kappa} \pi_{\mu} - \beta^0_{\mu} [x], \tag{2.22a} \]
\[ ^* \pi^\mu \cong \kappa x'^{\mu} + \kappa^2 \theta^{\mu \nu} \beta^0_{\nu} [x]. \tag{2.22b} \]
In the zeroth order one has $x^{(0)\mu} \equiv V^\mu$, and it is easy to see that (2.22) is inverse of (2.19).

Because the T-dual theory is defined in the doubled space, we will need the canonical expression for $\tilde{y}_\mu' = \hat{y}_\mu$. Using (2.21a) and (2.21b), we obtain

$$\tilde{y}_\mu' \equiv -\frac{2}{\kappa} \left( B[\Delta x] + \frac{1}{2} h[x] \right)_{\mu\nu} (G^{-1})^{\nu\rho} \pi_\rho + \left( G^{E}[\Delta x] - 2h[x]G^{-1}b \right)_{\mu\nu} x^\nu. \tag{2.23}$$

### 3 Non-commutativity relations between canonical variables

We want to establish the relation between the Poisson structures of the original and T-dual theory. The initial theory is the geometric one, described by the canonical variables $x^\mu$ and $\pi_\mu$. So, we choose the standard form of the PB’s in the original space, which are

$$\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu_\nu \delta(\sigma - \bar{\sigma}), \quad \{x^\mu(\sigma), x^\nu(\bar{\sigma})\} = 0, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar{\sigma})\} = 0. \tag{3.1}$$

The T-dual theory is the nongeometric one, defined in the doubled space, with two coordinates $y_\mu$ and $\tilde{y}_\mu$, connected by relations $\hat{y}_\mu = \tilde{y}_\mu'$, $y_\mu' = \hat{y}_\mu$. Using the T-duality transformation laws, we search for the corresponding Poisson structure in T-dual theory and express them in the form of (3.1) with $K$ and $L$ equal.

Using the transformation laws (2.21a) and (2.21b), we obtain

$$\Delta Y_{\mu}(\sigma, \sigma_0) = \int_{\sigma_0}^\sigma d\eta Y'_{\mu}(\eta) = Y_{\mu}(\sigma) - Y_{\mu}(\sigma_0), \tag{3.2}$$

$Y_\mu = y_\mu, \tilde{y}_\mu$ and calculating the equal time commutators. The fact that T-dual coordinates under T-duality transform to both coordinate and momenta dependent expressions, enables noncommutativity. The relation of the form

$$\{X'_{\mu}(\sigma), Y'_\nu(\bar{\sigma})\} \equiv K'_{\mu\nu}(\sigma)\delta(\sigma - \bar{\sigma}) + L_{\mu\nu}(\sigma)\delta'(\sigma - \bar{\sigma}), \tag{3.3}$$

implies the following relation (derived in the App. C) between coordinates

$$\{X_{\mu}(\tau, \sigma), Y_{\nu}(\tau, \bar{\sigma})\} \equiv - [K_{\mu\nu}(\sigma) - K_{\mu\nu}(\bar{\sigma}) + L_{\mu\nu}(\bar{\sigma})] \theta(\sigma - \bar{\sigma}), \tag{3.4}$$

where $\theta(\sigma)$ is the step function defined in (C.6).

In the flat space the coordinate dependent part of the Kalb-Ramond field is absent $h_{\mu\nu} = 0$, and consequently $\beta_{0\mu} = 0$. So, from (2.22a) and (2.22b) follows $y'_\mu \equiv \frac{1}{\kappa} \pi_\mu$ and $\star\pi^\mu \equiv \kappa x'^\mu$. Therefore, the PB of the canonical variables of the T-dual theory remain the standard ones, the same as in the original theory. So, the nontrivial infinitesimal expression $\beta_{0\mu}$, which exists only in the coordinate dependent backgrounds, is the source of the closed string non-commutativity.

Using the transformation laws (2.22a) and (2.22b), we can calculate PB’s $\{y'_\mu, y'_\nu\}$, $\{y'_\mu(\sigma), y'_\nu(\bar{\sigma})\}$ and $\{\tilde{y}'_\mu(\sigma), y'_\nu(\bar{\sigma})\}$ and express them in the form of (3.3) with $K$ and $L$ equal.
1. \( \{y'_\mu, y'_\nu\} \)

\[
K_{\mu\nu}[x] = \frac{3}{\kappa} h_{\mu\nu}[x] = \frac{1}{\kappa} B_{\mu\nu\rho} x^\rho, \quad L_{\mu\nu} = 0, \tag{3.5}
\]

2. \( \{\tilde{y}'_\mu, \tilde{y}'_\nu\} \)

\[
K_{\mu\nu}[x, \tilde{x}] = \frac{3}{\kappa} h_{\mu\nu}[\tilde{x}] - \frac{6}{\kappa} \left[ h[x] G^{-1} b + b G^{-1} h[x] \right]_{\mu\nu}, \quad L_{\mu\nu}[x] = \frac{1}{\kappa} g_{\mu\nu} - \frac{6}{\kappa} \left[ h[x] G^{-1} b + b G^{-1} h[x] \right]_{\mu\nu}, \tag{3.6}
\]

with

\[
\tilde{x}^\mu = \frac{1}{\kappa} (G^{-1})^{\mu\nu} \pi_{\nu} + 2 G^{-1} B^{\mu\nu} x^\nu. \tag{3.7}
\]

Using (2.6) and (B.2) expressions (3.6) can be rewritten in terms of the fluxes

\[
K_{\mu\nu}[x, \tilde{x}] = \frac{1}{\kappa} B_{\mu\nu\rho} \tilde{x}^\rho - \frac{3}{2\kappa} \Gamma^{E}_{\rho,\mu\nu} x^\rho,
\]

\[
L_{\mu\nu}[x] = \frac{1}{\kappa} g_{\mu\nu} - \frac{3}{2\kappa} \Gamma^{E}_{\rho,\mu\nu} x^\rho, \tag{3.8}
\]

3. \( \{\tilde{y}'_\mu, \tilde{y}'_\nu\} \)

\[
K_{\mu\nu}[x] = \frac{3}{\kappa} h_{\mu\nu}[x] + \frac{24}{\kappa} \left[ b h[x] \right]_{\mu\nu} + \frac{6}{\kappa} \left[ h[\tilde{x}] b - bh[\tilde{x}] \right]_{\mu\nu}, \quad L_{\mu\nu} = 0. \tag{3.9}
\]

In terms of fluxes it becomes

\[
K_{\mu\nu} = -\frac{1}{\kappa} \left[ B_{\mu\nu\rho} - 6 g_{\mu\alpha} Q_{\rho\beta}^\alpha g_{\beta\nu} \right] x^\rho + \left[ -\frac{3}{2\kappa} (\Gamma^{E}_{\mu,\nu\rho} - 2\Gamma^{E}_{\mu,\rho\nu}) + \frac{4}{\kappa} B_{\mu\nu\sigma} (G^{-1} b)^\sigma_{\rho} \right] \tilde{x}^\rho, \tag{3.10}
\]

where \( \Gamma^{E}_{\nu,\mu\rho} \) and \( Q_{\mu\nu\rho} \) are defined in (B.1) and (B.5).

For the above values of \( K \) and \( L \), the relation (3.4) gives

\[
\{y_\mu (\sigma), y_\nu (\bar{\sigma})\} \cong -\frac{1}{\kappa} B_{\mu\rho\nu} x^\rho (\sigma) - x^\rho (\bar{\sigma}) \theta (\sigma - \bar{\sigma}), \tag{3.11}
\]

\[
\{y_\mu (\sigma), \tilde{y}_\nu (\bar{\sigma})\} \cong -\left\{ -\frac{1}{\kappa} B_{\mu\rho\nu} \tilde{x}^\rho (\sigma) - \tilde{x}^\rho (\bar{\sigma}) \right\} - \frac{3}{2\kappa} \Gamma^{E}_{\rho,\mu\nu} x^\rho (\sigma) - x^\rho (\bar{\sigma}) + \frac{1}{\kappa} g_{\mu\nu} - \frac{3}{2\kappa} \Gamma^{E}_{\rho,\mu\nu} x^\rho (\sigma) - x^\rho (\bar{\sigma}) \theta (\sigma - \bar{\sigma}), \tag{3.12}
\]

\[
\{\tilde{y}_\mu (\sigma), \tilde{y}_\nu (\bar{\sigma})\} \cong -\left\{ -\frac{1}{\kappa} B_{\mu\rho\nu} - 6 g_{\mu\alpha} Q_{\rho\beta}^\alpha g_{\beta\nu} \right\} x^\rho (\sigma) - x^\rho (\bar{\sigma}) + \left[ -\frac{3}{2\kappa} (\Gamma^{E}_{\mu,\nu\rho} - \Gamma^{E}_{\mu,\rho\nu}) + \frac{4}{\kappa} B_{\mu\nu\sigma} (G^{-1} b)^\sigma_{\rho} \right] \tilde{x}^\rho (\sigma) - \tilde{x}^\rho (\bar{\sigma}) \theta (\sigma - \bar{\sigma}). \tag{3.13}
\]

After two-dimensional reparametrization, the \( \sigma \) dependent part takes the form

\[
[X^\mu (f (\sigma)) - X^\mu (f (\bar{\sigma}))] \theta [f (\sigma) - f (\bar{\sigma})],
\]
where \( f(\sigma) \) is monotonically increasing function with properties \( f(0) = 0 \) and \( f(2\pi) = 2\pi \). Therefore, the PB between different points is not reparametrization invariant. For fixed points, it can be fit to be arbitrary small, by the appropriate choice of function \( f(\sigma) \). So, only PB’s at the same point are physically significant.

Taking \( \sigma = \bar{\sigma} \) we obtain that all PB’s vanish, and consequently, coordinates commute. But, taking \( \sigma = \bar{\sigma} + 2\pi \), in the non-commutativity relation between the dual coordinates \( \bar{y}'s \) \((3.11)\), we obtain the closed string non-commutativity relation

\[
\{y_\mu(\sigma + 2\pi), y_\nu(\sigma)\} \cong -\frac{2\pi}{\kappa} B_{\mu\nu\rho} N^\rho. \tag{3.14}
\]

Here, \( N^\mu = \frac{1}{2\pi} [x^\mu(\sigma + 2\pi) - x^\mu(\sigma)] \) is winding number of the original coordinates. In sec. 4 we will compare this relation with the result of Ref.\([5, 7]\).

Similarly, from \((3.12)\) and \((3.13)\), we obtain

\[
\{y_\mu(\sigma + 2\pi), \bar{y}_\nu(\sigma)\} + \{\bar{y}_\mu(\sigma), y_\nu(\sigma + 2\pi)\} \cong -\frac{4\pi}{\kappa^2} B_{\mu\nu\rho} p^\rho + \frac{\pi}{\kappa} \left( 3 \Gamma^E_{\mu,\nu\lambda} - 8 B_{\mu\nu\lambda} b^\lambda \right) N^\rho, \tag{3.15}
\]

and

\[
\{\bar{y}_\mu(\sigma + 2\pi), \bar{y}_\nu(\sigma)\} \cong \frac{2\pi}{\kappa} \left[ -B_{\mu\nu\rho} - 6 g_{\mu\alpha} Q^{\alpha\beta} g_{\beta\nu} + 2 \bar{B}_{\mu\nu\lambda} g_{\lambda\rho} + 3 \left( \Gamma^E_{\mu,\nu\lambda} - \Gamma^E_{\nu,\mu\lambda} \right) b^\lambda \right] N^\rho + \frac{\pi}{\kappa^2} \left[ 3 \left( \Gamma^E_{\mu,\nu\rho} - \Gamma^E_{\nu,\mu\rho} \right) p^\rho - 8 B_{\mu\nu\lambda} b^\lambda \right] p^\rho. \tag{3.16}
\]

Using \((3.17)\) and integrating from \( \sigma \) to \( \sigma + 2\pi \) we have

\[
\frac{1}{2\pi} [\tilde{x}^\mu(\sigma + 2\pi) - \tilde{x}^\mu(\sigma)] = \frac{1}{\kappa} (G^{-1})^{\mu\nu} p_\nu + 2 (G^{-1})^{\mu\rho} b_{\rho\lambda} N^\lambda, \tag{3.17}
\]

where

\[
p_\mu = \frac{1}{2\pi} \int_{\sigma}^{\sigma + 2\pi} d\eta \pi(\eta) \tag{3.18}
\]

To complete the algebra, using the expressions \((2.22)\) and \((2.23)\) and after one \( \sigma \) integration, we find that the algebra of \( y_\mu \), \( \bar{y}_\mu \) and \( \pi^\mu \) is of the following form

\[
\{y_\mu(\sigma), \pi^\nu(\bar{\sigma})\} \cong \delta_\mu^\nu \delta(\sigma - \bar{\sigma}) + \kappa h_{\mu\rho} [x(\sigma)] \theta^\rho_0 \delta(\sigma - \bar{\sigma}) + \kappa h_{\mu\rho} [x'(\bar{\sigma})] \theta^\rho_0 \theta(\sigma - \bar{\sigma}), \tag{3.19}
\]

\[
\{\bar{y}_\mu(\sigma), \pi^\nu(\bar{\sigma})\} \cong \left[ -2b G^{-1} - 3h[x(\sigma)] G^{-1} - 2 \kappa h [x(\sigma)] \theta_0^\nu \delta(\sigma - \bar{\sigma}) \right] \mu \delta(\sigma - \bar{\sigma}) - \left[ 3h [x'(\bar{\sigma})] G^{-1} + 2 \kappa h [x'(\bar{\sigma})] \theta_0^\nu \theta(\sigma - \bar{\sigma}) \right] \mu \theta(\sigma - \bar{\sigma}), \tag{3.20}
\]

\[
\{\pi^\mu(\sigma), \pi^\nu(\bar{\sigma})\} \cong 0. \tag{3.21}
\]

Note that at the zeroth order one has \( \{y_\mu(\sigma), \pi^\nu(\bar{\sigma})\} = \delta_\mu^\nu \delta(\sigma - \bar{\sigma}) \) and \( \{\bar{y}_\mu(\sigma), \pi^\nu(\bar{\sigma})\} = -2b_\mu^\nu \delta(\sigma - \bar{\sigma}) \), so both doubled space variables \( y_\mu \) and \( \bar{y}_\mu \) have nontrivial PB with \( \pi^\mu \).
4 Comparison with the previous results

Let us mention that the case considered in the present paper is different from that of Ref. [5]. In Ref. [5], the non-commutativity relations in the nongeometric background with Q-flux where established, which are given in terms of winding numbers on the twisted torus $N^3 = \frac{1}{2\pi} \left( Y_3^3(\sigma + 2\pi) - Y_3^3(\sigma) \right)$. In the present article, the non-commutativity of the nongeometric background, which is not standard even locally and for $D = 3$ turns to R-flux background, was obtained in terms of the winding numbers on the torus with $H$-flux $N^\mu = \frac{1}{2\pi} \left( X^\mu(\sigma + 2\pi) - X^\mu(\sigma) \right)$.

4.1 The brief overview of the results of Ref.[5]

Before comparing the results of our paper with those of Ref. [5] let us shortly reexpress result of Ref.[5] using its notation. From the last identification in Eqs.(2.17) and the first relation in (2.25) of Ref.[5] it follows that

$$Y_1^H = Y_0^2Y_0^3 + \ldots.$$  \hfill (4.1)

Using expression for $G_{ab}(Y_3)$ for twisted torus (Table 1) of Ref. [5] we can find

$$\pi_1 = \dot{Y}_1 - HY_0^3\dot{Y}_0, \quad \pi_2 = \dot{Y}_2 - HY_0^3\dot{Y}_1,$$  \hfill (4.2)

and consequently

$$\pi_{01} = \dot{Y}_0^1, \quad \pi_{H2} = \dot{Y}_H^2 - Y_0^3\dot{Y}_0^1 = \dot{Y}_H^2 - Y_0^3\pi_{01}. \hfill (4.3)$$

The $T_2$-duality along $Y^2$, from the twisted torus to the nongeometric background produces

$$Z^1 \cong Y^1 = Y_0^1 + HY_0^2Y_3^3, \quad Z^2 \equiv \dot{Y}_0^2 - HY_0^3\dot{Y}_1 = \dot{Y}_0^2 - Y_0^3\pi_{01}.$$  \hfill (4.4)

So, we find the PB

$$\{Z^1(\sigma), Z^2(\dot{\sigma})\} \cong \{Y^1(\sigma), \pi_2(\dot{\sigma})\} = H \left[ Y_0^3(\sigma) - Y_0^3(\dot{\sigma}) \right] \delta_{2\pi}(\sigma - \dot{\sigma}). \hfill (4.5)$$

Note that $\delta_{2\pi}(\sigma - \dot{\sigma})$ is $2\pi$ periodic $\delta$-function, $\delta_{2\pi}(\alpha) = \sum_{n \in \mathbb{Z}} \delta(\alpha - 2\pi n)$, so the periodic parts in bracket in front of $\delta$-function disappear and we obtain

$$\{Z^1(\sigma), Z^2(\dot{\sigma})\} = HN^3(\sigma - \dot{\sigma})\delta_{2\pi}(\sigma - \dot{\sigma}). \hfill (4.6)$$

Here $N^3$ is winding number of $Y_0^3$ which has a general form

$$Y_0^3(\sigma) = N^3\sigma + Y_0^{3_{\text{periodic}}}(\sigma). \hfill (4.7)$$
The expression $\alpha \delta_{2\pi}(\alpha)$ is zero for $\alpha = 0$ but it is different from zero for $\alpha = 2n\pi \ (n \in \mathbb{Z}, n \neq 0)$.

The integration over $\bar{\sigma}$, from $\bar{\sigma}_0$ to $\bar{\sigma}$, produces

$$\{Z^1(\sigma), Z^2(\bar{\sigma})\} - \{Z^1(\sigma), Z^2(\bar{\sigma}_0)\} = -\frac{1}{2\pi} H N^3 \left[ F(\sigma - \bar{\sigma}) - F(\sigma - \bar{\sigma}_0) \right], \quad (4.8)$$

where

$$2\pi \int_{\alpha_0}^{\alpha} d\eta \delta_{2\pi}(\eta) = F(\alpha) - F(\alpha_0), \quad (4.9)$$

and

$$F(\alpha) = \sum_{n \neq 0} \frac{1}{n^2} e^{-ina} + i\alpha \sum_{n \neq 0} \frac{1}{n} e^{-ina} + \frac{\alpha^2}{2}. \quad (4.10)$$

The function $F(\alpha)$ is even $F(-\alpha) = F(\alpha)$ and $F(0) = \frac{\pi^2}{3}$.

So, the result for PB itself

$$\{Z^1(\sigma), Z^2(\bar{\sigma})\} = -\frac{1}{2\pi} H N^3 \left[ F(\sigma - \bar{\sigma}) + C \right], \quad (4.11)$$

is in fact the equation (4.41) of Ref.[5] up to some integration constant $C$. The undetermined constant $C$ corresponds to the contribution of the zero modes of the undetermined commutators, because one started with $\sigma$-derivative of the coordinate $Z^2$. The choice of Ref.[5] in subsection 4.4.2 is $C = 0$ which produces the expression (4.41) of Ref.[5] and the noncommutativity at the same point $\sigma = \bar{\sigma}$

$$\{Z^1(\sigma), Z^2(\sigma)\} = -\frac{1}{2\pi} H N^3 F(0) = -\frac{\pi}{6} H N^3. \quad (4.12)$$

As it was pointed out in Ref.[5], "other reasonings could as well be pursued". Following the line of our paper one can require that coordinates are commutative at the same point $(\sigma = \bar{\sigma})$ which produces

$$C = -F(0) = -\frac{\pi^2}{3}. \quad (4.13)$$

So, with this choice one has

$$\{Z^1(\sigma), Z^2(\bar{\sigma})\} = H N^3 \left[ F(\sigma - \bar{\sigma}) - \frac{\pi^2}{3} \right], \quad (4.14)$$

and obtains the non-commutativity for $\sigma = 2\pi + \bar{\sigma}$

$$\{Z^1(\sigma + 2\pi), Z^2(\sigma)\} = \pi H N^3. \quad (4.15)$$
4.2 Similarities and differences

Although we analyzed the different cases, let us compare some general features of the results considered. In both approaches the commutators are infinitesimally small and they close on some winding numbers. Note that in general, we can connect any geometric background with every nongeometric background from the chain of T-duality (1.2). Using the T-duality transformations we can calculate the noncommutativity of the coordinates of the nongeometric background in terms of the winding numbers of the geometrical background.

For arbitrary $\sigma$ and $\bar{\sigma}$, $\sigma$-dependence is different. In Ref. [5], up to the integration constant $C$ it is equal to

$$F(\sigma - \bar{\sigma}) + C,$$

and in the present article, up to the integration constant $C_1$, it is

$$[x^\mu(\sigma) - x^\mu(\bar{\sigma})] \theta(\sigma - \bar{\sigma}) + C_1.$$

The constants appear because in both approaches we started with the sigma derivatives of the coordinates. In the papers considered, the values of the constants are taken to be $C = 0$ and $C_1 = 0$. For these choices, the noncommutativity appears for $\sigma = \bar{\sigma}$ in the Ref. [5] and for $\sigma = \bar{\sigma} + 2\pi$ in the present article. For the other choice $C = -F(0) = -\frac{\pi^2}{3}$ and $C_1 = 0$, in both cases the coordinate commute at the same point $\sigma = \bar{\sigma}$ and have nontrivial PB for $\sigma = \bar{\sigma} + 2\pi$.

The main difference between two approaches is the origin of noncommutativity. The nontrivial boundary conditions given in Eq. (2.25) of Ref. [5] are the source of noncommutativity in that article. Because Ref. [5] does not consider $T_3$-dualization, $\beta_\mu^0$-functions (introduced in Eq. (2.15)) are zero and there is no noncommutativity of this kind. On the other hand, in the case considered in this paper, just these $\beta_\mu^0$ functions are the sources of the noncommutativity, even in the absence of the nontrivial boundary conditions of Ref. [5]. For complete noncommutativity relations one should take into account both kinds of noncommutativity.

5 Concluding remarks

In the present article we derived the closed string non-commutativity relations. We considered the theory describing the string moving in the weakly curved background. Its T-dual theory is obtained performing the T-dualization procedure along all the coordinates [9]. The T-dual transformation laws have the central role in our approach. These laws connect the world-sheet derivatives of the coordinates and momenta in the original and the T-dual theory. The zero orders are transformation laws of the constant background and they do not lead to the noncommutativity. The term $\beta_\mu^0$, which is infinitesimally small and bilinear in coordinates $x^\mu$, plays the key role in obtaining the noncommutativity relations.
In the original space we choose the standard Poisson brackets. The T-dual coordinates \( y^\mu \) has two terms: one linear in the original momenta and the other bilinear in the original coordinates. This explains the nontrivial PB \( \{ y^\mu, y^\nu \} \) which is linear in coordinate. Note that in the case of open string moving in the flat background coordinate is linear function in both effective momenta and coordinates. So, the corresponding PB is constant.

The T-dual momenta \( \pi^\mu \) are bilinear expressions in original coordinates. So, PB of the T-dual momenta vanishes \( (3.21) \), but PB between T-dual coordinates and momenta \( (3.19) \) obtained additional term linear in coordinates.

In the doubled space there exists the additional coordinate \( \tilde{y}^\mu \). It consists of the term linear in original momenta, but with the coefficient linear in original coordinate and the other terms bilinear in original coordinates. So, it produces the nontrivial PB with all variables \( (y^\mu, \tilde{y}^\mu, \pi^\mu) \), \( (3.12) \), \( (3.13) \) and \( (5.20) \).

The general structure of the non-commutativity relations is

\[
\{ Y^\mu(\sigma), Y^\nu(\bar{\sigma}) \} = \{ F^\mu_{\rho\nu} [x^\rho(\sigma) - x^\rho(\bar{\sigma})] + \tilde{F}^\mu_{\rho\nu} [\tilde{x}^\rho(\sigma) - \tilde{x}^\rho(\bar{\sigma})] \} \theta(\sigma - \bar{\sigma}),
\]

where \( Y^\mu = (y^\mu, \tilde{y}^\mu) \) and \( F^\mu_{\rho\nu} \) and \( \tilde{F}^\mu_{\rho\nu} \) are the constant and infinitesimally small fluxes. At the same points, for \( \sigma = \bar{\sigma} \) all PB's are zero. In the important particular case for \( \sigma = \bar{\sigma} + 2\pi \) we get

\[
\{ Y^\mu(\sigma + 2\pi), Y^\nu(\sigma) \} = 2\pi \left[ (F^\mu_{\rho\nu} + 2\tilde{F}^\mu_{\rho\nu} b^\rho_{\nu}) N^\nu + \frac{1}{\kappa} \tilde{F}^\mu_{\rho\nu} p^\nu \right],
\]

where \( N^\mu \) and \( p^\mu \) are winding numbers and momenta of the original theory. We can rewrite it in the form

\[
\{ Y^\mu(\sigma + 2\pi), Y^\nu(\sigma) \} = \oint_{C_\rho} F^\mu_{\rho\nu} dx^\rho + \oint_{\tilde{C}_\rho} \tilde{F}^\mu_{\rho\nu} d\tilde{x}^\rho,
\]

where \( C_\rho \) and \( \tilde{C}_\rho \) are cycles around which the closed string is wrapped. Note the ”wrapping” of auxiliary coordinate \( \tilde{x}^\mu \) is in accordance with \( (3.17) \) and represents linear combination of momenta \( p^\mu \) and winding numbers \( N^\mu \). This generalizes the conjecture of Ref.[14] between the closed string noncommutativity and fluxes.

In terms of Ref.[5] for the three dimensional torus \( x^\mu \rightarrow X^a, (a = 1, 2, 3) \) our case corresponds to the non-commutativity of the nongeometric background with \( W^a \) coordinates and \( R \)-fluxes obtained after the successive performance of all three T-dualizations along all three coordinates. It relates \( W^a \) with \( X^a \) coordinates of torus with \( H \)-flux, and so the PB closes on the winding number of the \( X^a \)-coordinates. We hope that these results will contribute to the better understanding of the most strange, uncommon R-flux configurations where the noncommutativity appears as a consequence of the nontrivial \( \beta^0_{\mu} \) functions. Note that Ref.[5] uses \( T_2 \)-duality (performed along \( Y^2 \)) and the relation...
\[ Z^\alpha = Z^\alpha(Y^\alpha) \] to obtain the non-commutativity of the nongeometric background with \( Q \)-flux in terms of the winding of \( Y^\alpha \)-coordinates. There the noncommutativity originates from the nontrivial boundary conditions. To obtain the general structure of the closed string noncommutativity for arbitrary background of the chain (1.2) one should find its T-duality transformations with all other backgrounds of the chain and calculate both kind of the noncommutativity originating from nontrivial boundary conditions as well as from nontrivial \( \beta_0^{\mu} \) functions.

The term of the action with the constant part of the Kalb-Ramond field \( b_{\mu\nu} \) is topological. So, it does not contribute to the equations of motion. In the open string case it contributes to the boundary conditions and it is a source of the open string noncommutativity. In the closed string case it is absent from boundary conditions as well. Classically, we can gauge it away and Kalb-Ramond field becomes infinitesimally small. But, if \( b_{\mu\nu} = 0 \) one loses topological contributions. In order to investigate the global structure of the theory with holonomies of the world sheet gauge fields in quantum theory we should preserve such term.

Putting \( b_{\mu\nu} = 0 \) the noncommutativity relations (3.14), (3.15) and (3.16) get the simpler form

\[
\begin{align*}
\{ y_\mu(\sigma + 2\pi), y_\nu(\sigma) \} & = -\frac{2\pi}{\kappa} B_{\mu\nu\rho} N^\rho, \\
\{ y_\mu(\sigma + 2\pi), \tilde{y}_\nu(\sigma) \} & = \frac{1}{\kappa} G_{\mu\nu} \left( -\frac{2\pi}{\kappa^2} B_{\mu\nu\rho} p^\rho \right), \\
\{ \tilde{y}_\mu(\sigma + 2\pi), \tilde{y}_\nu(\sigma) \} & = -\frac{6\pi}{\kappa} B_{\mu\nu\rho} N^\rho.
\end{align*}
\]

A The momentum in the T-dual theory

Let us here calculate the T-dual momentum given in (2.18). The T-dual theory depends on two variables \( y_\mu, \tilde{y}_\mu \) which are connected by the relations \( \dot{y}_\mu = \tilde{y}'_\mu, \dot{\tilde{y}}_\mu = \tilde{y}_\mu \). So, to obtain the momentum canonically conjugated to \( y_\mu \), we should vary the action with respect to both \( \dot{y}_\mu \) and \( \tilde{y}'_\mu \).

First, let us calculate the contribution from the background fields argument. With the help of the relation

\[ \Theta^\mu\nu_-[x] = \Theta^\mu\nu_0 - 2\kappa \Theta^\mu\rho_0 h_{\rho\sigma}[x] \Theta^\sigma\nu_0, \]

we can rewrite the T-dual action (2.8) as

\[
\begin{align*}
* S[y] & = * S_0 - \kappa^3 \int d^2\xi \partial_+ y_\mu \Theta^\mu\rho_0 h_{\rho\sigma}[\Delta V[y]] \Theta^\sigma\nu_0 \partial_- y_\nu, \\
* S_0 & = \frac{\kappa^2}{2} \int d^2\xi \partial_+ y_\mu \Theta^\mu\nu_0 \partial_- y_\nu.
\end{align*}
\]
Using the expression
\[ \partial_\pm V^\mu = -\kappa \Theta_{\pm \mu}^\nu \partial_\nu y_\mu^{(0)}, \] (A.3)
we obtain
\[ \star S[y] = \star S_0 + \kappa \int d^2 \xi \partial_+ V^\mu h_{\mu \nu} [\Delta V] \partial_- V^\nu = \star S_0 + \kappa \int d^2 \xi \Delta V^\mu h_{\mu \nu} [\partial_- V] \partial_+ V^\nu. \] (A.4)

Because of the relation
\[ h_{\mu \nu} [\partial_- V] \partial_+ V^\nu = \partial_0 \beta_0^\mu \sigma[V] + \partial_1 \beta_1^\mu \sigma[V], \] (A.5)
the action becomes
\[ \star S[y] = \star S_0 + \kappa \int d^2 \xi \left[ \Delta y_\mu \theta_0^\mu + \Delta \tilde{y}_\mu (g^{-1})^\mu \nu \partial_0 \beta_0^\nu \sigma[V] + \partial_1 \beta_1^\nu \sigma[V] \right] \] (A.6)

So, the contribution to the T-dual momentum, coming from the T-dual background fields argument is obtained from (A.6), integrating over \( \sigma \) by parts in \( \Delta \tilde{y}_\mu (g^{-1})^\mu \nu \partial_1 \beta_1^\nu \). Using \( \tilde{y}_\mu' = \dot{y}_\mu \) we obtain
\[ \Delta \star \pi^\mu = -\kappa (g^{-1})^\mu \nu \beta_1^\nu \sigma[V]. \] (A.7)

Therefore, the total T-dual momentum is
\[ \star \pi^\mu = \kappa (G^{-1})^\mu \nu \Delta V[y] \dot{y}^\nu - \kappa^2 \theta^\mu \nu \Delta [V[y]] y_\nu' - \kappa (g^{-1})^\mu \nu \beta_1^\nu \sigma[V[y]]. \] (A.8)

### B Fluxes

The field strength of the original Kalb-Ramond field, is given by (2.5). The original metric \( G_{\mu \nu} \) is constant, and therefore the corresponding Christoffel connection is zero. The effective metric \( G^E_{\mu \nu} \) is linear in coordinate and the corresponding Christoffel connection
\[ \Gamma^E_{\mu, \nu \rho} = \frac{1}{2} \left( \partial_\rho G^E_{\mu \sigma} + \partial_\sigma G^E_{\mu \rho} - \partial_\mu G^E_{\nu \rho} \right) = -\frac{4}{3} \left( B_{\mu \sigma \nu} (G^{-1} b)^\sigma_{\rho} + B_{\mu \rho \sigma} (G^{-1} b)^\sigma_{\nu} \right), \] (B.1)
is the infinitesimally small constant. It will be used in the following forms
\[ \Gamma^E_{\mu, \nu \rho} x^\mu = 4 \left( h[x] G^{-1} b + b G^{-1} h[x] \right)_{\nu \rho}, \] (B.2)
and
\[ (\Gamma^E_{\mu, \nu \rho} - \Gamma^E_{\nu, \mu \rho}) x^\rho = 8 h_{\mu \nu} [b x] - 4 \left( h[x] G^{-1} b - b G^{-1} h[x] \right)_{\nu \mu}. \] (B.3)

We can express the dual Kalb-Ramond field \([9]\) as
\[ \star B^{\mu \nu} \Delta V = \star b^{\mu \nu} + Q^{\mu \nu} \Delta V^\rho, \] (B.4)
where \( b^{\mu\nu} = \frac{\kappa}{2} \theta_{0}^{\mu\nu} \) and
\[
Q^{\mu\nu}_{\rho} = -\frac{1}{3} \left[ (g^{-1})^{\mu\sigma} (g^{-1})^{\nu\tau} - \kappa^{2} \theta_{0}^{\mu\sigma} \theta_{0}^{\nu\tau} \right] B_{\sigma\tau}\rho.
\] (B.5)

This will be used as
\[
Q^{\mu\nu}_{\rho} x^\rho = -(g^{-1})^{\mu\rho} \left[ h(x) + 4bG^{-1}h(x)G^{-1}b \right] (g^{-1})^{\nu\rho}.
\] (B.6)

C PB’s between pure coordinates

Starting with the PB of the \( \sigma \) derivatives of the coordinates
\[\{X'_{\mu}(\sigma), Y'_{\nu}(\bar{\sigma})\} \cong K'_{\mu\nu}(\sigma) \delta(\sigma - \bar{\sigma}) + L_{\mu\nu}(\sigma) \delta' (\sigma - \bar{\sigma}),\] (C.1)

let us find the expression for the PB between coordinates \( \{X_{\mu}(\sigma), Y_{\nu}(\bar{\sigma})\} \). From (C.1) it follows that \( \Delta X_{\mu}(\sigma, \sigma_0) \) and \( \Delta Y_{\mu}(\sigma, \sigma_0) \) defined by
\[
\Delta X_{\mu}(\sigma, \sigma_0) = \int_{\sigma_0}^{\sigma} d\eta X'_{\mu}(\eta) = X_{\mu}(\sigma) - X_{\mu}(\sigma_0),
\]
\[
\Delta Y_{\mu}(\sigma, \sigma_0) = \int_{\sigma_0}^{\sigma} d\eta Y'_{\mu}(\eta) = Y_{\mu}(\sigma) - Y_{\mu}(\sigma_0),
\] (C.2)

satisfy
\[
\{\Delta X_{\mu}(\sigma, \sigma_0), \Delta Y_{\nu}(\bar{\sigma}, \bar{\sigma}_0)\} \cong \int_{\sigma_0}^{\sigma} d\eta \int_{\bar{\sigma}_0}^{\bar{\sigma}} d\bar{\eta} \left[ K'_{\mu\nu}(\eta) \delta(\eta - \bar{\eta}) + L_{\mu\nu}(\eta) \delta'(\eta - \bar{\eta}) \right].
\] (C.3)

Integrating over \( \bar{\eta} \) and using
\[
\int_{\sigma_0}^{\sigma} d\eta f(\eta) \delta(\eta - \bar{\eta}) = f(\bar{\sigma}) [\theta(\sigma - \bar{\sigma}) - \theta(\sigma_0 - \bar{\sigma})],
\] (C.4)

we obtain
\[
\{\Delta X_{\mu}(\sigma, \sigma_0), \Delta Y_{\nu}(\bar{\sigma}, \bar{\sigma}_0)\} \cong \int_{\sigma_0}^{\sigma} d\eta \left[ K'_{\mu\nu}(\eta) \left[ \theta(\eta - \bar{\sigma}_0) - \theta(\eta - \bar{\sigma}) \right] + L_{\mu\nu}(\eta) \left[ \delta(\eta - \bar{\sigma}_0) - \delta(\eta - \bar{\sigma}) \right] \right],
\] (C.5)

where the function \( \theta(\sigma) \) is defined as
\[
\theta(\sigma) = \int_{0}^{\sigma} d\eta \delta(\eta) = \frac{1}{2\pi} \left( \sigma + 2 \sum_{n \geq 1} \frac{1}{n} \sin n\sigma \right) = \begin{cases} 0 & \text{if } \sigma = 0 \\ 1/2 & \text{if } 0 < \sigma < 2\pi, \quad \sigma \in [0.2\pi]. \end{cases}
\] (C.6)
Integrating by parts over $\eta$ and using (C.4) we get
\[
\{\Delta X_\mu(\sigma, \sigma_0), \Delta Y_\nu(\bar{\sigma}, \bar{\sigma}_0)\} \cong \\
K_{\mu\nu}(\sigma)[\theta(\sigma - \bar{\sigma}_0) - \theta(\sigma - \bar{\sigma})] - K_{\mu\nu}(\sigma_0)[\theta(\sigma_0 - \bar{\sigma}_0) - \theta(\sigma_0 - \bar{\sigma})] \\
- K_{\mu\bar{\nu}}(\bar{\sigma}_0)[\theta(\sigma - \bar{\sigma}_0) - \theta(\sigma_0 - \bar{\sigma}_0)] + K_{\mu\bar{\nu}}(\bar{\sigma})[\theta(\sigma - \bar{\sigma}) - \theta(\sigma_0 - \bar{\sigma})] \\
+ L_{\mu\nu}(\bar{\sigma}_0)[\theta(\sigma - \bar{\sigma}_0) - \theta(\sigma_0 - \bar{\sigma}_0)] - L_{\mu\nu}(\bar{\sigma})[\theta(\sigma - \bar{\sigma}) - \theta(\sigma_0 - \bar{\sigma})]. \quad \text{(C.7)}
\]
Relation
\[
\{X_\mu(\tau, \sigma), Y_\nu(\tau, \bar{\sigma})\} \cong -[K_{\mu\nu}(\sigma) - K_{\mu\nu}^{\prime}(\bar{\sigma}) + L_{\mu\nu}(\bar{\sigma})] \theta(\sigma - \bar{\sigma}), \quad \text{(C.8)}
\]
solves (C.7), up to additive constant.

For $X_\mu = Y_\mu$, the antisymmetry of the left hand side under the replacement $\mu \leftrightarrow \nu$ and $\sigma \leftrightarrow \bar{\sigma}$, produces conditions $L_{\mu\nu} = L_{\nu\mu}$ and $K_{\mu\nu} + K_{\nu\mu} = L_{\mu\nu}$.

References

[1] F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari, JHEP 02 (1999) 016; C. S. Chu and P. M. Ho, Nucl. Phys. B550 (1999) 151; N. Seiberg and E. Witten, JHEP 09 (1999) 032; F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari Nucl. Phys. B576 (2000) 578; C. S. Chu and P. M. Ho, Nucl. Phys. B568 (2000) 447; T. Lee, Phys. Rev. D62 (2000) 024022.

[2] B. Sazdović, Eur. Phys. J. C44 (2005) 599; B. Nikolić and B. Sazdović, Phys. Rev. D 74 (2006) 045024; B. Nikolić and B. Sazdović, Phys. Rev. D 75 (2007) 085011; B. Nikolić and B. Sazdović, Adv. Theor. Math. Phys. 14 (2010) 1, Lj. Davidovic and B. Sazdovic, Phys. Rev. D 83 (2011) 066014; JHEP 08 (2011) 112.

[3] T. H. Buscher, Phys. Lett. B194 (1987) 59; T. H. Buscher, Phys. Lett. B201 (1988) 466.

[4] D. Luest, JHEP 12 (2010) 084.

[5] D. Andriot, M. Larfors, D. Luest, P. Patalong, JHEP 06 (2013) 021.

[6] D. Andriot, O. Hohm, M. Larfors, D. Luest, P. Patalong, Phys. Rev. Lett. 108 (2012) 261602.

[7] D. Luest, arXiv:1205.0100 [hep-th]; R. Blumenhagen, A. Deser, D. Luest, E. Plauschinn, and F. Rennecke, J.Phys. A44 (2011) 385401; C. Condeescu, I. Florakis, and D. Luest, JHEP 04 (2012) 121.

[8] J. Shelton, W. Taylor, and B. Wecht, JHEP 10 (2005) 085; A. Dabholkar and C. Hull, JHEP 05 (2006) 009.
[9] Lj. Davidović and B. Sazdović, *arXiv:1205.1991* [hep-th].

[10] E. S. Fradkin and A. A. Tseytlin, *Phys. Lett. B* 158 (1985) 316; *Nucl. Phys. B* 261 (1985) 1.

[11] K. Becker, M. Becker and J. Schwarz *String Theory and M-Theory: A Modern Introduction*, Cambridge University Press 2006; B. Zwiebach, *A First Course in String Theory*, Cambridge University Press, 2004.

[12] V. Schomerus, *Class. Quant. Grav.* 19 (2002) 5781; L. Cornalba and R. Schiappa, *Commun. Math. Phys.* 225 (2002) 33.

[13] Lj. Davidović and B. Sazdović, *Phys. Rev. D* 83 (2011) 066014; *JHEP* 08 (2011) 112; *EPJ C* 72 No. 11 (2012) 2199.

[14] D. Andriot, O. Hohm, M. Larfors, D. Luest, and P. Patalong, *Fort. Phys.*** 60** (2012).