S1. Supplementary Material

S1.1. Algorithms

Algorithm 1 Standard Empirical Mode Decomposition

1: procedure EMD($x(t)$, STOPPAGE CRITERIA, NUMBER OF IMFS) \[\triangleright\]
2: $x(t)$ is the input signal
3: $\tilde{x}(t) \leftarrow x(t)$
4: $e_{\text{max}}(t), e_{\text{min}}(t) \leftarrow$ form upper and lower envelopes from all local maxima and minima, respectively, using cubic splines
5: $m(t) \leftarrow \frac{e_{\text{min}}(t) + e_{\text{max}}(t)}{2}$
6: $d(t) \leftarrow \tilde{x}(t) - m(t)$
7: $\tilde{x}(t) \leftarrow d(t)$ go to line 3; \textbf{repeat} until $d(t)$ becomes an IMF \[\triangleright\]
8: \textbf{repeat} lines 3:7 until desired number of IMFs obtained
Algorithm 2 Bivariate Empirical Mode Decomposition

1: procedure BEMD($x(t)$, STOPPAGE CRITERIA, NUMBER OF IMFs) ⊢
   
   $x(t)$ is the input signal

2: \hspace{1em} $\tilde{x}(t) \leftarrow x(t)$

3: \hspace{1em} for $1 \leq k \leq N$ do

4: \hspace{2em} Project the complex-valued signal $x(t)$ on direction $\varphi_k$: $p_{\varphi_k}(t) = \text{Re}\left(e^{-i\varphi_k} x(t)\right)$

5: \hspace{2em} Extract the maxima of $p_{\varphi_k}(t)$: $\{t^k_{j}, p_{j}^k\}$

6: \hspace{2em} Interpolate the set $\{(t^k_{j}, p_{j}^k)\}$ to obtain the "tangent" in direction $\varphi_k$: $e_{\varphi_k}'(t)$

7: \hspace{1em} end for

8: \hspace{1em} Compute the mean of all tangents: $m(t) = \frac{1}{N} \sum_{k=1}^{N} e_{\varphi_k}'(t)$

9: \hspace{1em} $d(t) \leftarrow \tilde{x}(t) - m(t)$

10: \hspace{1em} $\tilde{x}(t) \leftarrow d(t)$ go to line 3; repeat until $d(t)$ becomes an IMF ⊢ STOPPAGE CRITERION for a bivariate IMF

11: \hspace{1em} repeat lines 3:10 until desired number of IMFs obtained

Algorithm 3 Multivariate Empirical Mode Decomposition

1: procedure MEMD($x(t)$, STOPPAGE CRITERIA, NUMBER OF IMFs) ⊢
   
   $x(t)$ is the input signal

2: \hspace{1em} $\tilde{x}(t) \leftarrow x(t)$

3: \hspace{1em} Choose the point set based on the Hammersley sequence for sampling on an $(n - 1)$–sphere (Rehman and Mandic, 2010)

4: \hspace{2em} for $1 \leq k \leq N$ do

5: \hspace{3em} Compute a projection, $p_{\varphi_k}(t)_{t=1}^{T}$ of input signal $x(t)_{t=1}^{T}$ along the direction vector $v(t)_{t=1}^{T}$: $p_{\varphi_k}(t) = x(t)v(t)_{t=1}^{T}$

6: \hspace{3em} Find time instants $\{t^k_{j}\}$ corresponding to the maxima of $p_{\varphi_k}(t)$

7: \hspace{3em} Interpolate the set $\{(t^k_{j}, x(t^k_{j}))\}$ to obtain the multivariate envelope $e_{\varphi_k}(t)$

8: \hspace{2em} end for

9: \hspace{1em} Compute the mean $m(t)$ of the envelope as: $m(t) = \frac{1}{N} \sum_{k=1}^{N} e_{\varphi_k}(t)$

10: \hspace{1em} $d(t) \leftarrow \tilde{x}(t) - m(t)$

11: \hspace{1em} $\tilde{x}(t) \leftarrow d(t)$ go to line 3; repeat until $d(t)$ satisfies the ⊢ STOPPAGE CRITERION for a multivariate IMF

12: \hspace{1em} repeat line 2:11 until desired number of IMFs obtained
Algorithm 4 Multivariate Variational Mode Decomposition

1: Initialize: \( \{\hat{u}_{k,n}^1\}, \{\omega_k^1\}, \lambda_n^1, m \leftarrow 0 \)
2: repeat
3: \( m \leftarrow m + 1 \)
4: \( \text{for } k = 1 : K \text{ do} \)
5: \( \text{for } n = 1 : N \text{ do} \)
6: \( \text{Update mode } \hat{u}_{k,n}^m(\omega) \leftarrow \frac{\hat{x}_n(\omega) - \sum_{i \neq k} \hat{u}_{i,n}(\omega) + \frac{\lambda_n^m(\omega)}{2}}{1 + 2\alpha(\omega - \omega_k^m)^2} \)
7: \( \text{end for} \)
8: \( \text{end for} \)
9: \( \text{for } k = 1 : K \text{ do} \)
10: \( \text{Update center frequency } \omega_k: \omega_k^{m+1} \leftarrow \frac{\sum_{n=0}^{\infty} \int_{0}^{\infty} \omega |\hat{u}_{k,n}^{m+1}(\omega)|^2 \, d\omega}{\sum_{n=0}^{\infty} \int_{0}^{\infty} |\hat{u}_{k,n}^{m+1}(\omega)|^2 \, d\omega} \)
11: \( \text{end for} \)
12: \( \text{for } n = 1 : N \text{ do} \)
13: \( \text{Update } \lambda_n \text{ for all } \omega \geq 0: \lambda_n^{m+1}(\omega) = \lambda_n^m(\omega) + \tau \left( \hat{x}_n(\omega) - \sum_k \hat{u}_{k,n}^{m+1}(\omega) \right) \)
14: \( \text{end for} \)
15: until Convergence \( \sum_k \sum_n \frac{||\hat{u}_{k,n}^{m+1} - \hat{u}_{k,n}^m||_2^2}{||\hat{u}_{k,n}^m||_2^2} < \epsilon \)
S1.2. Simulations with time-varying amplitudes

In Simulations 1-3, the amplitude was assumed to be constant throughout time. However, Bedrosian’s theorem comes into play when the goal is to separate two different time varying signals that are multiplied with one another. To investigate the performance of each combination of MD technique and PS measure in this setting we performed a follow-up simulation where both the amplitude and phase were allowed to be time varying.

Let us revisit the set-up of Simulation 3, where the phase relationship varies according to a sigmoid function. However, let us extend the simulation by allowing the amplitudes to vary according to three different scenarios outlined below. Throughout, we express the signals of interest as follows:

\[
x(t) = A_x(t) \cos(\omega_0 t) + \varepsilon_x(t) \\
y(t) = A_y(t) \cos \left( \omega_0 t + \frac{a}{1 + \exp(b(t - t_0))} \right) + \varepsilon_y(t)
\]  

(14)

**Simulation A**: Let the amplitude of the signal \( x(t) \) vary according to an exponentially decaying function,

\[
A_x(t) = \alpha + e^{\beta t}
\]  

(15)

where \( \alpha = 1 \) and \( \beta = 200 \).

Figure S1 shows the results of the simulation. In this setting, Bedrosian’s theorem still holds and the proposed MD phase synchronization approaches successfully address the problem and recover the relationship between the phase of the two signals. It should be noted that MVMD still outperforms the other two approaches in a manner consistent with the results of the previous simulation with constant amplitude; see Simulation 3.

**Simulation B**: Let the amplitude of the signal \( x(t) \) vary according to a linear-frequency chirp function \( \sin(\vartheta) \), in which the instantaneous frequency
Figure S1: Results of Simulation A. A comparison of BEMD, na-MEMD, and MVMD-based PS when the phase shift corresponds to a sigmoid function. Here the amplitude of \( x(t) \) varies according to an exponentially decaying function while the amplitude of \( y(t) \) remains constant. Results are based on an IMF extracted with a central frequency of 0.05 Hz. Results are shown for tWPS using circular-circular correlation (CIRC) and IPS using cosine of relative phase (CRP). In each panel, the mean and 95% interval for each measure are shown at each time point. For the tWPS measures results are shown for three different window lengths (30, 60, and 120).

\( f(t) \) changes linearly with respect to time \( f(t) = \gamma t + f_0 \). The amplitude \( A_x \) is expressed as follows:

\[
A_x(t) = \eta + \sin \left( \vartheta_0 + 2\pi \left( \frac{\gamma}{2} t^2 + f_0 t \right) \right)
\]  

(16)
where $\eta$ is the DC component (offset) of the time varying amplitude, $\vartheta_0$ is the phase at time 0, $f_0$ is the starting frequency at which the frequency starts to change, and $\gamma$ denotes the chirp rate. Here $\gamma = (f_1 - f_0)/T$ where $f_1$ is the final frequency and $T$ is the time between the initial and final frequencies. We set $\eta = 1$, $T = 334$, $\vartheta_0 = 0$, $f_0 = 0.01$, and $f_1 = 2f = 2 \times 0.05 = 0.1$ Hz. The function and its Short-time Fourier transform (STFT) are shown in Figure S2.

![Short-Time Fourier Transform](image)

![Amplitude of signal x(t)](image)

Figure S2: (Top) The short-time Fourier transform (STFT) of the amplitude created based on the linear frequency chirp function. The plot illustrates the linearly increasing frequency, and the horizontal yellow line corresponds to the DC component of the amplitude. (Bottom) The amplitude $A_x$ used in Simulation B.

Figure S3 shows the results of the simulation. They are consistent with the previous simulation results with constant amplitude with regards to recovering the phase relationship between the two signals; see Simulation 3.
However, we can observe that the mean of the estimated PS measures contain increased fluctuations for BEMD and na-MEMED caused by mode mixing.

**Simulation C:** Let the amplitude of the signal $x(t)$ vary according to a linear frequency chirp function and the amplitude of $y(t)$ vary according to an...
exponential (also referred to as geometric) chirp function where $f(t) = f_0 \xi^t$. Here we can write $A(t) = \sin \left( \theta_0 + 2\pi f_0 \left( \frac{\xi^{t-1}}{1-\ln \xi} \right) \right)$, where $A$, $\theta_0$, $f_0$, and $\xi$ denote the amplitude, the initial phase at time $0$, starting frequency at $t = 0$ and exponentially increasing frequency rate, $\xi = \left( \frac{f_1}{f_0} \right)^{\frac{1}{t_1}}$, respectively. Thus, the amplitudes for this simulation can be expressed as follows:

$$A_x(t) = \eta + \sin \left( \theta_0 + 2\pi \left( \gamma t^2 + f_0 t \right) \right)$$

$$A_y(t) = \alpha + \sin \left( \theta_0 + 2\pi f_0 \left( \frac{\xi^t - 1}{\ln \xi} \right) \right)$$

(17)

We keep the parameters for the linear frequency chirp function the same as in Simulation B (see Figure S2). The parameters for the exponential chirp function are set to $\theta_0 = 0$, $\alpha = 3$, $f_0 = 0.01$ Hz, $f_1 = 5 \times f_0 = 0.25$ Hz, and $\xi = 1.0048$. Figure S4 shows the STFT of the exponential chirp amplitude.

The results of the simulation shows that the performance of the various MD-based phase synchronization approaches are consistent with those obtained in the constant amplitude case. It should be noted that both BEMD and na-MEMD suffer from issues related to mode-mixing, which manifests itself in fluctuations throughout the time course (Figure S5). The performance of MVMD appears to be negligibly affected, and it outperforms the other MD approaches.
Figure S4: (Top) The short-time Fourier transform (STFT) of the amplitude created based on the exponential frequency chirp function. The plot illustrates the exponentially increasing frequency, and the horizontal yellow line corresponds to the DC component of the amplitude. (Bottom) The amplitude $A_y$ used in Simulation C.
Figure S5: Results of Simulation C. A comparison of BEMD, na-MEMD, and MVMD-based PS when the phase shift corresponds to a sigmoid function. Here the amplitude of $x(t)$ varies according to a linear frequency chirp function, while the amplitude of $y(t)$ varies according to an exponential chirp function. Results are based on an IMF extracted with a central frequency of 0.05 Hz. Results are shown for tWPS using circular-circular correlation (CIRC) and IPS using cosine of relative phase (CRP). In each panel, the mean and 95% interval for each measure are shown at each time point. For the tWPS measures results are shown for three different window lengths (30, 60, and 120).
In Simulation 6, we evaluated the ability of the different combinations of MD techniques and PS measures to overcome increasing noise levels. Throughout, we assumed the noise level was the same in both signals being studied. Here we investigate the case where the noise added to the two signals have different variances. We find, perhaps not unexpectedly, that the width of the confidence interval is primarily driven by the noise in the signal with the larger variance. Figure S6 shows the results of the simulation. The first three columns show how the width of the confidence intervals increase as the noise level increase, and correspond to the results seen in Simulation 6. The fourth column corresponds to the case where $\text{Var}(\varepsilon_x) = 1$ and $\text{Var}(\varepsilon_y) = 4$. Clearly, the results are closer to the case where $\text{Var}(\varepsilon_x) = \text{Var}(\varepsilon_y) = 4$ (second column), than the case where $\text{Var}(\varepsilon_x) = \text{Var}(\varepsilon_y) = 1$ (first column).

Figure S6: The first three columns illustrate how the width of the confidence intervals increase as the noise level increase (which is the same for the two signals). The fourth column corresponds to the case where $\text{Var}(\varepsilon_x) = 1$ and $\text{Var}(\varepsilon_y) = 4$. Clearly, the results are closer to the case where $\text{Var}(\varepsilon_x) = \text{Var}(\varepsilon_y) = 4$ (second column), than the case where $\text{Var}(\varepsilon_x) = \text{Var}(\varepsilon_y) = 1$ (first column).
**S1.4. Measuring Simulations - MSE**

For each of Simulations 1-4 we calculated the mean square error (MSE) between the true and estimated values for each combination of MD technique and PS measure. They are shown in the tables below.

| PS Measure | Simulation 1 | Simulation 2 | Simulation 3 |
|------------|--------------|--------------|--------------|
|            | BEMD         | NA-MEMD      | MVMD         |
| CIRC, w= 30 | 0.2022       | 0.2007       | 0.1028       |
| CIRC, w=60  | 0.1009       | 0.0995       | 0.0556       |
| CIRC, w=120 | 0.0508       | 0.0497       | 0.0279       |
| CRP         | 0.5003       | 0.5001       | 0.4971       |

**Table S1:** The mean square error (MSE) between the true and estimated values for each combination of MD technique and PS measure for Simulation 1.

| PS Measure | Simulation 2 | Simulation 3 |
|------------|--------------|--------------|
|            | BEMD         | NA-MEMD      | MVMD         |
| CIRC, w= 30 | 0.2918       | 0.2850       | 0.1588       |
| CIRC, w=60  | 0.3236       | 0.3170       | 0.2113       |
| CIRC, w=120 | 0.4031       | 0.4003       | 0.3303       |
| CRP         | 0.3037       | 0.2947       | 0.0765       |

**Table S2:** The mean square error (MSE) between the true and estimated values for each combination of MD technique and PS measure for Simulation 2.

| PS Measure | Simulation 3 |
|------------|--------------|
|            | BEMD         | NA-MEMD      | MVMD         |
| CIRC, w= 30 | 0.2230       | 0.2154       | 0.1476       |
| CIRC, w=60  | 0.1951       | 0.1945       | 0.1381       |
| CIRC, w=120 | 0.1964       | 0.1942       | 0.1546       |
| CRP         | 0.2823       | 0.2753       | 0.0689       |

**Table S3:** The mean square error (MSE) between the true and estimated values for each combination of MD technique and PS measure for Simulation 3.
Table S4: The mean square error (MSE) between the true and estimated values for each combination of MD technique and PS measure for Simulation 4.

| PS Measure   | MSE Simulation 4 |   |   |
|--------------|------------------|---|---|
|              | BEMD             | NA-MEMD | MVMD |
| CIRC, w= 30  | 0.3895           | 0.2943  | 0.1537 |
| CIRC, w=60   | 0.3219           | 0.2410  | 0.1495 |
| CIRC, w = 120| 0.2683           | 0.2139  | 0.1594 |
| CRP          | 0.5972           | 0.3926  | 0.0889 |