RADIAL SOLUTIONS FOR A CLASS OF HÉNON TYPE SYSTEMS WITH PARTIAL INTERFERENCE WITH THE SPECTRUM

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Abstract. We investigate the existence of radial solutions for a class of Hénon type systems with nonlinearities reaching the critical growth and interacting with the spectrum of the operator with the possibility of double resonance. The proof is made using variational methods, combining Brézis and Nirenberg arguments with Ni compactness result and Rabinowitz linking theorem.

1. Introduction. Let $N > 2$ and $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$ be an unity open ball centered at the origin. In this paper we study the existence of solutions for the following system

\[
\begin{align*}
-\Delta u &= a|x|^\mu u + b|x|^\mu v + |x|^\alpha \left[ \frac{p}{p+q} |u|^{p-2} u + \xi_1 |u|^{p+q-2} \right] \\
-\Delta v &= b|x|^\mu u + c|x|^\mu v + |x|^\alpha \left[ \frac{q}{p+q} |v|^{q-2} v + \xi_2 |v|^{p+q-2} \right] \\
\quad u = v &= 0
\end{align*}
\] in $B_1,$

\] in $B_1,$

\] on $\partial B_1,$

\[ (1.1) \]

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where \( \mu \geq \alpha \geq 0, \xi_1, \xi_2 \geq 0 \) and \( p, q > 1 \) are constants such that
\[
p + q = 2\alpha := 2(N + \alpha)/(N - 2).
\]
In order to state and compare our results to the scalar case, it is convenient to rewrite system (1.1) in the vector and matrix forms such as
\[
\begin{cases}
-\Delta U = |x|^\mu AU + |x|^\alpha \nabla F(U) & \text{in } B_1, \\
U = 0 & \text{on } \partial B_1,
\end{cases}
\]
where
\[
U = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad -\Delta U = \begin{pmatrix} -\Delta u \\ -\Delta v \end{pmatrix} \in M_{2 \times 1}(\mathbb{R}),
\]
\[
A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_{2 \times 2}(\mathbb{R}) \text{ is positive definite},
\]
\[
F(U) = \frac{1}{p + q} \left( |u|^p |v|^q + \xi_1 |u|^{p+q} + \xi_2 |v|^{p+q} \right)
\]
and \( \nabla \) is the gradient operator.

Let us denote by
\[
0 = \lambda_{0,0} < \lambda_{1,0} < \lambda_{2,0} \leq \lambda_{3,0} \leq \ldots \leq \lambda_{j,0} \leq \lambda_{j+1,0} \leq \ldots, \quad \lambda_{k,0} \to \infty, \quad \text{as } k \to \infty,
\]
the sequence of eigenvalues of the problem with homogeneous Dirichlet boundary condition
\[
-\Delta u = \lambda |x|^\mu u \quad \text{in } B_1,
\]
and denote by \( \varphi_{1,0} \) the positive eigenfunction associated to \( \lambda_{1,0} \).

Let \( \mu_1, \mu_2 \) be real eigenvalues of the positive definite symmetric matrix \( A \), which will assume \( \mu_1 \leq \mu_2 \). Thus, it is verified that \( \mu_1 |U|^2 \leq (AU, U)_{\mathbb{R}^2} \leq \mu_2 |U|^2 \), for all \( U := (u, v) \in \mathbb{R}^2 \).

The interaction of these eigenvalues with the spectrum of (1.3) will play an important role in the existence’s study of the solutions.

The purpose of this work is to prove the existence of solutions of this class of gradient systems of elliptic equations on the hypothesis of an interaction of the eigenvalues \( \mu_1, \mu_2 \) of the matrix \( A \) with eigenvalues of the Laplacian operator with weight \( |x|^\mu \), which we shall denote by \( (-\Delta, |x|^\mu) \). More exactly, when the interval \([\mu_1, \mu_2]\) does not containing any eigenvalue, it could happen resonance phenomena and the case in which the interval \([\mu_1, \mu_2]\) contains an eigenvalue of the operator \((-\Delta, |x|^\mu)\) a double resonance can occur.

Problem (1.1) is an extension to systems of the scalar Hénon problem considered in [5], in which (1.1) was studied with the particular matrix
\[
A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \in M_{2 \times 2}(\mathbb{R}).
\]
In [5], under appropriate hypotheses on the parameter \( \lambda \), the authors established an existence result of at least one radial solution for the scalar problem
\[
\begin{cases}
-\Delta u = \lambda |x|^\mu u + |x|^\alpha |u|^{2\alpha - 2} u & \text{in } B_1, \\
u = 0 & \text{on } \partial B_1.
\end{cases}
\]

When \( \alpha = \mu = 0 \), the problem (1.4) belongs to the class of the so called Brézis-Nirenberg type problems [8] which have been studied by several authors in the last decades with different approaches. A version for variational systems defined on \( \Omega \),
where \( \Omega \) may be either a bounded domain or the whole \( \mathbb{R}^N \), was studied in [1] (see also [2] for systems without weight function, but in a bounded domain \( \Omega \)). When \( \alpha > 0 \) and \( \lambda = 0 \), these classes of problems are known in the literature by Hénon type problems [19] which, in view of its applications, a great deal of attention has been dispensed to the study of this type of nonlinear equations.

In the important paper [24], Ni established that the embedding \( H^1_{0,\text{rad}}(B_1) \subset L^p(B_1,|x|^\alpha) \) is compact for all \( p \in [1,2^*_\alpha) \), in order to get radial solutions. This result was extended to more general quasilinear operator in [15]. Still in the case where \( \lambda = 0 \), in [3] Badiale and Serra established multiplicity results of nonradial solutions (see [14] for extensions).

For ground state profile or concentration phenomena see for example [9, 10, 11, 12, 21, 26, 34] and references therein, and for Hénon problem involving usual Sobolev exponents we would like to cite [22, 20, 28, 29] and references. When \( \lambda > 0 \) is smaller than the first eigenvalue, in [4] it is studied by a nonhomegeneous perturbations, while in [18] it is treated some concentration phenomena for linear perturbation when \( \lambda \) is small enough.

The novelty of this paper is, up to our knowledge, the works that have been appeared in the literature up to now doesn’t treat the system (1.1) involving the Sobolev critical exponent \( 2^*_\alpha \) given by Ni and involving nonlinearities interacting with the spectrum of \( (-\Delta,|x|^{\mu}) \). In our case, the presence of the mathematical term
\[
F(U) = \frac{1}{p+q} \left( |u|^p|v|^q + \xi_1|u|^{p+q} + \xi_2|v|^{p+q} \right)
\]
includes both an uncoupled and a coupled nonlinearity.

2. Notations and preliminary stuff. We consider the subspace of \( H^1_0(B_1) \) of the radial functions given by
\[
H^1_{0,\text{rad}}(B_1) := \{ u \in H^1_0(B_1) : u(Sx) = u(x) \text{ for all } S \in O(N) \},
\]
where \( O(N) \) is space of rotations in \( \mathbb{R}^N \). Notice that \( H^1_{0,\text{rad}}(\mathbb{R}^N) \) is a Hilbert space with the inner product of \( H^1_0(B_1) \) defined by
\[
\langle u, v \rangle_{H^1_{0,\text{rad}}(B_1)} := \int_{B_1} \nabla u \nabla v \, dx,
\]
which induces the norm
\[
\|u\|_{H^1_{0,\text{rad}}(B_1)} = \left( \int_{B_1} |\nabla u|^2 \, dx \right)^{1/2}
\]
for all \( u \in H^1_{0,\text{rad}}(B_1) \). We also denote the \( L^2 \) space with weight \( |x|^{\mu} \) on \( B_1 \) by \( L^2(B_1,|x|^{\mu}) \) with the norm \( \|u\|_{L^2(B_1,|x|^{\mu})} = \left( \int_{B_1} |x|^{\mu} |u|^2 \, dx \right)^{1/2} \), where \( \mu \geq 0 \).

Since we are wanted to obtain a radial solution for the problem (1.1) with critical growth, we defined \( S_\alpha \) be the best constant for the Sobolev-Hardy embedding
\[
H^1_{0,\text{rad}}(\mathbb{R}^N) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^N, |x|^{\alpha}).
\]
The constant
\[
S_\alpha = \inf_{u \in H^1_{0,\text{rad}}(B_1) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx}{\left( \int_{\mathbb{R}^N} |x|^{\alpha} |u(x)|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha}} = \inf_{u \in H^1_{0,\text{rad}}(B_1) \setminus \{0\}} S_\alpha(u) \quad (2.3)
\]
is achieved by the family of functions
\[ U_\varepsilon(x) = \frac{[(N + \alpha)(N - 2)\varepsilon]^{(N-2)/(2+\alpha)}}{(\varepsilon + |x|^{2+\alpha})^{(N-2)/(2+\alpha)}} \] (2.4)
defined for each \( \varepsilon > 0 \). Indeed, these functions are minimizers of \( S_\alpha \) in the set of radial functions in case \( \alpha > -2 \). Furthermore, \( U_\varepsilon \) are the only positive radial solutions of
\[
\begin{aligned}
-\Delta u &= |x|^{-\alpha}u^{2^*_\alpha - 2} \quad \text{in } \mathbb{R}^N; \\
u(x) &\to 0 \quad \text{as } |x| \to \infty.
\end{aligned}
\] (2.5)
verifying the property
\[
\int_{\mathbb{R}^N} |\nabla U_\varepsilon(x)|^2 \, dx = \int_{\mathbb{R}^N} |x|^{-\alpha}|U_\varepsilon(x)|^{2^*_\alpha} \, dx = S_\alpha^{(N+\alpha)/(\alpha+2)}. \] (2.6)

For details and more general results, see [4, 15]. The imbedding \( H^1_{0,\text{rad}}(B_1) \hookrightarrow L^r(B_1, |x|^{\alpha}) \) is continuous for \( r \) in \([1, 2^*_\alpha]\) and compact for \( r \) in \([1, 2^*_\alpha]\) (see [24]).

2.1. An eigenvalue problem. For \( \lambda \in \mathbb{R} \), we consider the problem with homogeneous Dirichlet boundary condition
\[
\begin{aligned}
-\Delta u &= \lambda|x|^\mu u \quad \text{in } B_1, \\
u(x) &= 0 \quad \text{on } \partial B_1.
\end{aligned}
\] (2.7)
If (2.7) admits a weak solution \( u \in H^1_{0,\text{rad}}(B_1) \setminus \{0\} \), then \( \lambda \) is called an eigenvalue and \( u \) a \( \lambda|x|^\mu \)-eigenfunction. The set of all eigenvalues is referred as the spectrum of \(-\Delta, |x|^\mu\) in \( H^1_{0,\text{rad}}(B_1) \) and denoted by \( \sigma(-\Delta, |x|^\mu) \). Since \( K = [-\Delta]^{-1} \) is a compact operator, the problem (2.7) can be written as \( u = \lambda K(|x|^\mu u) \) with \( u \in L^2(B_1, |x|^\mu) \), hence the following results are true:

(i) Problem (2.7) admits an eigenvalue \( \lambda_{1,\mu} = \min \sigma(-\Delta, |x|^\mu) > 0 \) that can be characterized as follows
\[
\lambda_{1,\mu} = \min_{u \in H^1_{0,\text{rad}}(B_1) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx}{\int_{\mathbb{R}^N} |x|^\mu |u(x)|^2 \, dx}; \] (2.8)

(ii) There exists a nonnegative function \( \varphi_{1,\mu} \in H^1_{0,\text{rad}}(B_1) \), which is an eigenfunction corresponding to \( \lambda_{1,\mu} \), attaining the minimum in (2.8);

(iii) All \( \lambda_{1,\mu} |x|^{\mu} \) - eigenfunctions are proportional, and if \( u \) is a \( \lambda_{1,\mu} |x|^{\mu} \)-eigenfunction, then either \( u(x) > 0 \) a.e. in \( B_1 \) or \( u(x) < 0 \) a.e. in \( B_1 \);

(iv) The set of the eigenvalues of problem (2.7) consists of a sequence \( \{\lambda_{k,\mu}\} \) satisfying
\[
0 = \lambda_{0,\mu} < \lambda_{1,\mu} < \lambda_{2,\mu} \leq \lambda_{3,\mu} \leq \cdots \leq \lambda_{j,\mu} < \lambda_{j+1,\mu} \leq \cdots, \quad \lambda_{k,\mu} \to \infty, \quad \text{as } k \to \infty,
\]
and \( \lambda_{k+1,\mu} \) is characterized by
\[
\lambda_{k+1,\mu} = \min_{u \in \mathbb{P}_{k+1} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx}{\int_{\mathbb{R}^N} |x|^\mu |u(x)|^2 \, dx},
\]
where
\[
\mathbb{P}_{k+1} = \left\{ u \in H^1_{0,\text{rad}}(B_1) : \langle u, \varphi_{j,\mu} \rangle_{H^1_{0,\text{rad}}(B_1)} = \int_{B_1} \nabla u \nabla \varphi_{j,\mu} \, dx = 0, j = 1, 2, \ldots, k \right\};
\]

(v) If \( \lambda \in \sigma(-\Delta, |x|^\mu) \setminus \{\lambda_{1,\mu}\} \) and \( u \) is a \( \lambda|x|^\mu \)-eigenfunction, then \( u \) changes sign in \( B_1 \).
Denote by \( \varphi_{k,\mu} \) the eigenfunction associated to the eigenvalue \( \lambda_{k,\mu} \), for each \( k \in \mathbb{N} \). The sequence \( \{ \varphi_{k,\mu} \} \) is an orthonormal basis of \( L^2(B_1, |x|^{\rho}) \) and an orthogonal basis of \( H^1_{0,\text{rad}}(B_1) \).

**Remark 1.** For fixed \( k \in \mathbb{N} \) \((k \geq 1)\), we can assume \( \lambda_{k,\mu} < \lambda_{k+1,\mu} \), otherwise we can suppose that \( \lambda_{k,\mu} \) has multiplicity \( p \in \mathbb{N} \), that is

\[
\lambda_{k-1,\mu} < \lambda_{k,\mu} = \lambda_{k+1,\mu} = \ldots = \lambda_{k+p-1,\mu} < \lambda_{k+p,\mu},
\]

and we denote \( \lambda_{k+p,\mu} = \lambda_{k+1,\mu} \).

Now, to state our results we introduce the Hilbert space given by the product space

\[
Y(B_1) := H^1_{0,\text{rad}}(B_1) \times H^1_{0,\text{rad}}(B_1),
\]
equipped with the inner product

\[
\langle (u,v), (\varphi,\psi) \rangle_Y := \langle u, \varphi \rangle_{H^1_{0,\text{rad}}(B_1)} + \langle v, \psi \rangle_{H^1_{0,\text{rad}}(B_1)}
\]
and the norm

\[
\|(u,v)\|_Y := \left( \|u\|^2_{H^1_{0,\text{rad}}(B_1)} + \|v\|^2_{H^1_{0,\text{rad}}(B_1)} \right)^{1/2}.
\]

The space \( L^r(B_1, |x|^{\rho}) \times L^r(B_1, |x|^{\rho}) \) \((r > 1)\) is considered with the standard product norm

\[
\|(u,v)\|_{(L^r \times L^r)(B_1, |x|^{\rho})} := \left( \|u\|^2_{L^r(B_1, |x|^{\rho})} + \|v\|^2_{L^r(B_1, |x|^{\rho})} \right)^{1/2}.
\]

Besides, we recall that

\[
\mu_1 |U|^2 \leq (AU, U)_{\mathbb{R}^2} \leq \mu_2 |U|^2, \quad \text{for all } U := (u, v) \in \mathbb{R}^2,
\]
where \((\cdot, \cdot)_{\mathbb{R}^2}\) is the usual inner product in \( \mathbb{R}^2 \) and \( \mu_1 \) and \( \mu_2 \) are the eigenvalues of symmetric matrix \( A \) given above. Without loss of generality, we may assume \( \mu_1 \leq \mu_2 \).

In this paper, we consider the notation for product space \( \mathcal{S} \times \mathcal{S} := \mathcal{S}^2 \).

The following are the main results of the paper.

**Theorem 2.1.** Assume \( p + q = 2_\alpha^*, \quad \xi_1, \xi_2 > 0 \) and \( \mu \geq \alpha > 0 \) and suppose \( \lambda_{k,\mu} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,\mu} \) for some integer \( k \geq 0 \). Then \((1.1)\) admits a nontrivial radial solution if one of the following conditions hold,

1. \( N > 4 + \mu; \)
2. \( N = 4 + \mu \) and \( \mu_1 > \lambda_{1,\mu} \) with \( \mu_1 \neq \lambda_{k,\mu} \) for all \( k = 1, 2, \ldots; \)
3. \( N < 4 + \mu \) and \( k \) is an integer large enough with \( \lambda_{k,\mu} \neq \mu_1 \).

**Theorem 2.2.** Assume \( p + q = 2_\alpha^*, \quad \xi_1, \xi_2 > 0 \) and \( \mu \geq \alpha > 0 \) and suppose \( \lambda_{k-1,\mu} \leq \mu_1 < \lambda_{k,\mu} \leq \mu_2 < \lambda_{k+1,\mu} \), for some integer \( k \geq 1 \). Then \((1.1)\) admits a nontrivial radial solution, if one of the following conditions hold,

1. \( N > 4 + \mu \) and \( \mu_1 > 0; \)
2. \( N = 4 + \mu \) and \( \mu_1 \neq \lambda_{k-1,\mu} \) for all \( k = 1, 2, \ldots; \)
3. \( N < 4 + \mu \) and \( k \) is an integer large enough with \( \lambda_{k-1,\mu} \neq \mu_1 \).

**Theorem 2.3.** Suppose \( N \geq 4 + \mu, \quad p + q = 2_\alpha^*, \quad \mu \geq \alpha > 0, \quad \xi_1, \xi_2 \geq 0 \) and \( 0 < \mu_1 \leq \mu_2 < \lambda_{1,\mu} \), then \((1.1)\) admits a positive radial solution.
Remark 2 (Properties of homogeneous functions). If \( G \) is a \( C^1 \)--function and \( r \)-homogeneous with \( r \geq 1 \), that is \( G(\lambda(s,t)) = \lambda^r G(s,t), \forall (s,t) \in \mathbb{R}^2, \forall \lambda \geq 0, \) then

- there exists \( K_G > 0 \) such that
  \[
  |G(s,t)| \leq K_G (|s|^r + |t|^r), \quad s,t \in \mathbb{R},
  \]
  where \( K_G = \max\{G(s,t) : s, t \in \mathbb{R}, |s|^r + |t|^r = 1\} \) is attained in some \( (s_o,t_o) \in \mathbb{R}^2 \).
- \( (\nabla G(s,t), (s, t))_{\mathbb{R}^2} = s G_s(s,t) + t G_t(s,t) = r G(s,t), \forall (s,t) \in \mathbb{R}^2. \)
- \( G_s \) and \( G_t \) are \((r-1)\)-homogeneous.

Remark 3. The nonlinearity \( F \) is \((p+q)\)-homogeneous.

The purpose of this work is to prove the existence of solutions for a class of gradient systems of elliptic equations under the assumption on an interaction of the eigenvalues \( \mu_1 \leq \mu_2 \) of the symmetric matrix \( A \) with eigenvalues of the Laplacian operator \((-\Delta, |x|^\mu)\).

Preliminary results. In order to prove Theorem 2.3, we shall make use of the following:

Define the following minimizing problems

\[
S_{p+q}^\alpha(B_1) = \inf_{u \in H_0, rad(B_1) \setminus \{0\}} \frac{\int_{B_1} |\nabla u(x)|^2dx}{\left(\int_{B_1} |x|^{\alpha} |u(x)|^{p+q}dx\right)^{\frac{2}{p+q}}} \tag{2.10}
\]

and

\[
\tilde{S}_{p+q}^\alpha(B_1) = \inf_{u,v \in H_0, rad(B_1) \setminus \{0\}} \frac{\int_{B_1} (|\nabla u(x)|^2 + |\nabla v(x)|^2)dx}{\left(\int_{B_1} |x|^{\alpha} |u(x)|^p |v(x)|^q + \xi_1 |u(x)|^{p+q} + \xi_2 |v(x)|^{p+q}dx\right)^{\frac{2}{p+q}}} \tag{2.11}
\]

If \( p + q = 2^*_\alpha \), we denote

\[S_\alpha = S_{p+q}^\alpha(B_1) \quad \text{and} \quad \tilde{S}_\alpha = \tilde{S}_{p+q}^\alpha(B_1).\]

The proof of the next result follows arguing as was made in [2].

Lemma 2.4. Suppose \( p + q = 2^*_\alpha \). Then there exists a positive constant \( m \) such that

\[\tilde{S}_\alpha = m S_\alpha.\]  

Moreover, if \( w_\alpha \) realizes \( S_\alpha \) then \( (s_\alpha w_\alpha, t_\alpha w_\alpha) \) realizes \( \tilde{S}_\alpha \), for some \( s_\alpha, t_\alpha > 0. \)
Proposition 2.6. Setting.

Notice that (see also [4, Thm 2.3], [15, Prop 7.3]) and [6, Lemma 2].

Let \( m \) be the constant obtained by defining the function

\[
H(u, v) := (p + q)F(u, v) = |u|^p|v|^q + \xi_1|u|^{p+q} + \xi_2|v|^{p+q}.
\]

Notice that \( H(u, v)^{\frac{1}{p+q}} \) is 2-homogeneous, there exists a constant \( M > 0 \) satisfying

\[
H(u, v)^{\frac{1}{p+q}} \leq M(|u|^2 + |v|^2), \quad \text{for all } u, v \in \mathbb{R},
\]

where \( M \) is the maximum of the function \( H^{\frac{1}{p+q}} \) attained in some \((s_o, t_o)\) (with \( s_o, t_o \geq 0 \)) of the compact set \( \{(s, t) \in \mathbb{R}^2, |s|^2 + |t|^2 = 1\} \).

Let \( m = M^{-1} \), so we have that

\[
H(s_o, t_o)^{\frac{1}{p+q}} = m^{-1}(s_o^2 + t_o^2).
\]

We will make some estimates similar to Brézis-Niremberg Lemma [8, Lemma 1.2] (see also [4, Thm 2.3], [15, Prop 7.3]) and [6, Lemma 2].

Fix \( \delta > 0 \) such that \( B_{2\delta} \subset \Omega \) and \( \epsilon \in C_0^\infty(\mathbb{R}^N) \) a radial cut-off function such that \( 0 \leq \epsilon \leq 1 \) in \( \mathbb{R}^N \), \( \epsilon = 1 \) in \( B_{\delta} \) and \( \epsilon = 0 \) in \( B_{2\delta}^c = \mathbb{R}^N \setminus B_{2\delta} \), where \( B_r = B_r(0) \) is the ball centered at origin with radius \( r > 0 \).

Now define the family of nonnegative truncated functions

\[
u_{\epsilon}(x) = \eta(x)U_{\epsilon}(x), \quad x \in \mathbb{R}^N,
\]

and note that \( u_{\epsilon} \in H_{1,rad}^1(B_1) \).

Proposition 2.5. For each \( \mu, \alpha \geq 0 \) and \( \epsilon > 0 \) sufficiently small, we have

\[
\begin{align*}
a) \quad & \Vert u_{\epsilon} \Vert^2_{H_{1,rad}^1(B_1)} = S_{\alpha}^{(N+\alpha)/(2+\alpha)} + O(\epsilon^{(N-2)/(2+\alpha)}); \\
b) \quad & \Vert u_{\epsilon} \Vert^2_{L_{rad}^2(B_1, \vert x \vert^\alpha)} = S_{\alpha}^{(N+\alpha)/(2+\alpha)} + O(\epsilon^{(N+\alpha)/(2+\alpha)}); \\
c) \quad & \Vert u_{\epsilon} \Vert^2_{L_{rad}^2(B_1, \vert x \vert^\alpha)} = \begin{cases} K_1 \epsilon^{(2+\mu)/(2+\alpha)} + O(\epsilon^{(N-2)/(2+\alpha)}) & \text{if } N > 4 + \mu; \\
K_1 \epsilon^{(2+\mu)/(2+\alpha)} |\ln \epsilon| + O(\epsilon^{(2+\mu)/(2+\alpha)}) & \text{if } N = 4 + \mu; \\
K_1 \epsilon^{(N-2)/(2+\alpha)} & \text{if } N < 4 + \mu; \end{cases} \\
d) \quad & \Vert u_{\epsilon} \Vert_{L^1(B_1, \vert x \vert^\alpha)} \leq K_{2\epsilon}^{(N-2)/(2(2+\alpha))}; \\
e) \quad & \Vert u_{\epsilon} \Vert_{L_{rad}^{2-1}(B_1, \vert x \vert^\alpha)} \leq K_{3\epsilon}^{(N-2)/(2(2+\alpha))},
\end{align*}
\]

where \( K_1, K_2 \) and \( K_3 \) are positive constants.

Now consider the following minimization problem

\[
S_{\alpha, \lambda, \mu} = \inf_{v \in H_{1,rad}^1(B_1) \setminus \{0\}} S_{\alpha, \lambda, \mu}(v),
\]

where

\[
S_{\alpha, \lambda, \mu}(v) = \frac{\int_{B_1} \vert \nabla v(x) \vert^2 \, dx - \lambda \int_{B_1} \vert x \vert^\alpha |v(x)|^2 \, dx}{\left( \int_{B_1} \vert x \vert^\alpha |v(x)|^{2+\alpha} \, dx \right)^{\frac{2}{\alpha}}}. 
\]

Arguing as in [8], the following Brézis-Niremberg estimates that can be proved as in [33, Section 4.2] the first item and [32, Corollary 8] the second for the nonlocal setting.

Proposition 2.6. Let \( \alpha, \mu \geq 0 \) and \( \epsilon > 0 \) sufficiently small.

a) If \( N \geq 4 + \mu \), we have \( S_{\alpha, \lambda, \mu}(u_{\epsilon}) < S_{\alpha} \), for all \( \lambda > 0 \).

b) If \( N < 4 + \mu \), there exists \( \lambda_o > 0 \) such that for all \( \lambda > \lambda_o \), we have \( S_{\alpha, \lambda, \mu}(u_{\epsilon}) < S_{\alpha} \).
Proof. For the sake of the completeness, we give a sketch of the proof. By Proposition 2.5, we infer that

**Case 1:** $N > 4 + \mu$.

$$S_{\alpha,\lambda,\mu}(u_\varepsilon) \leq \frac{S_{\alpha}(\varepsilon^{(\alpha+1)/(2\alpha)}) + O(\varepsilon^{(N-2)/(2\alpha)}) - \lambda K_1 \varepsilon^{(2+\mu)/(2\alpha)}}{(S_{\alpha}(\varepsilon^{(\alpha+1)/(2\alpha)}) + O(\varepsilon^{(N+\alpha)/(2\alpha)}))^{1/2}}$$

$$\leq S_\alpha + O(\varepsilon^{(N-2)/(2\alpha)}) - \lambda K_1 \varepsilon^{(2+\mu)/(2\alpha)},$$

$<S_\alpha$, for $\lambda > 0$, $\varepsilon > 0$ is sufficiently small and $K_1 > 0$ a constant.

**Case 2:** $N = 4 + \mu$.

$$S_{\alpha,\lambda,\mu}(u_\varepsilon) \leq \frac{S_{\alpha}(\varepsilon^{(\alpha+1)/(2\alpha)}) + O(\varepsilon^{(2+\mu)/(2\alpha)}) - \lambda K_1 \varepsilon^{(2+\mu)/(2\alpha)}|\ln\varepsilon|}{(S_{\alpha}(\varepsilon^{(\alpha+1)/(2\alpha)}) + O(\varepsilon^{(N+\alpha)/(2\alpha)}))^{1/2}}$$

$$\leq S_\alpha + O(\varepsilon^{(2+\mu)/(2\alpha)}) - \lambda K_1 \varepsilon^{(2+\mu)/(2\alpha)}|\ln\varepsilon|,$$

$<S_\alpha$, for $\lambda > 0$, $\varepsilon > 0$ sufficiently small and $K_1 > 0$ a constant.

**Case 3:** $N < 4 + \mu$.

$$S_{\alpha,\lambda,\mu}(u_\varepsilon) \leq \frac{S_{\alpha}(\varepsilon^{(\alpha+1)/(2\alpha)}) + O(\varepsilon^{(N-2)/(2\alpha)}) - \lambda C_{\alpha}\varepsilon^{(N-2)/(2\alpha)} + O(\varepsilon^{(2+\mu)/(2\alpha)})}{(S_{\alpha}(\varepsilon^{(\alpha+1)/(2\alpha)}) + O(\varepsilon^{(N+\alpha)/(2\alpha)}))^{1/2}}$$

$$\leq S_\alpha + \varepsilon^{(N-2)/(2\alpha)}(O(1) - \lambda C_{\alpha}) + O(\varepsilon^{(2+\mu)/(2\alpha)}),$$

$<S_\alpha$, for all $\lambda > 0$ large enough ($\lambda \geq \lambda_0$, $\varepsilon > 0$ sufficiently small and $C_{\alpha} > 0$ a constant. \)

For our purposes suppose $p + q = 2\alpha$ and define the following minimization problem

$$\tilde{S}_{\alpha,\lambda}(u,v) = \inf_{u,v \in H_{rad}(B_1) \setminus \{0\}} S_{\alpha,A,\mu}(u,v),$$

where

$$S_{\alpha,A,\mu}(u,v) = \frac{\int_{B_1} |u|^2 dx + \int_{B_1} |\nabla u|^2 dx - \mu \int_{B_1} |x|^\alpha(A(u,v),(u,v)) dx}{(\int_{B_1} |x|^\alpha(\|u\|^p u + \xi_1 u|p+q + \xi_2 |u|^{p+q}) dx)^{1/2}}.$$ 

**Proposition 2.7.** Suppose $\mu_1$ given in (2.9) be positive and $\alpha, \mu \geq 0$. Then

$$\tilde{S}_{\alpha,A,\mu} < \tilde{S}_\alpha$$

if either $N \geq 4 + \mu$ or $N < 4 + \mu$ and $\mu_1$ large enough with $\mu_1 \neq \lambda_{k,\mu}$.

**Proof.** From Proposition 2.6, we have $S_{\alpha,\mu_1}(u_\varepsilon) < S_\alpha$ thanks to the fact that $\mu_1 > 0$ and provided $\varepsilon > 0$ is sufficiently small.

Let $s_\alpha, t_\alpha > 0$ obtained in Lemma 2.4. From (2.9) and (2.14), combined with the above estimate, we infer that

$$\tilde{S}_{\alpha,A,\mu} \leq S_{\alpha,A,\mu}(s_\alpha u_\varepsilon, t_\alpha u_\varepsilon)$$

$$\leq \frac{(s_{\alpha}^2 + t_{\alpha}^2)}{(s_{\alpha}^p t_{\alpha}^{p+q} + \xi_1 s_{\alpha}^{p+q} + \xi_2 t_{\alpha}^{p+q})^{2/2\alpha}}$$

$$= mS_{\alpha,\mu_1}(u_\varepsilon) < mS_{\alpha} = \tilde{S}_\alpha.$$
This concludes the proof. 

The proof of our result was inspired in [5] for scalar case. In this paper, we applied the following generalized Mountain Pass Theorem [27, thm 5.3, remark 5.5(iii)].

**Theorem 2.8.** Let $Y$ be a real Banach space with $Y = V \oplus W$, where $V$ is finite dimensional. Suppose $I \in C^1(Y, \mathbb{R})$ and

(i) There are constants $\rho, \beta > 0$ such that $I|_{\partial B_{\rho} \cap W} \geq \beta$, and

(ii) There is an $e \in \partial B_1 \cap W$ and $R_1, R_2 > \rho$ such that $I|_{\partial Q} \leq 0$, where

$$Q = (\overline{B}_{R_1} \cap V) \oplus \{re, 0 < r < R_2\}.$$  

Then $I$ possesses a $(PS)_c$ sequence where $c \geq \beta$ can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)),$$

where

$$\Gamma = \{h \in C(\overline{Q}, Y) : h = id \text{ on } \partial Q\}.$$  

**Remark 4.** Here $\partial Q$ is the boundary of $Q$ relative to space $V \oplus \text{span}\{e\}$. When $V = \{0\}$ this theorem refer to usual mountain pass theorem. We recall that if $I|_{V} \leq 0$ and $I(u) \leq 0$, $\forall u \in V \oplus \text{span}\{e\}$ with $\|u\| \geq R$, then $I$ verifies (ii) for $R$ large.

**3. Proof of the Theorem 2.1.**

**3.1. Linking geometry.** Define the following subspaces of $Y(B_1)$,

$$V_k^- = \text{span}\{(0, \varphi_{1,\mu}), (\varphi_{1,\mu}, 0), \ldots, (0, \varphi_{k,\mu}), (\varphi_{k,\mu}, 0)\}$$

and

$$W_k^+ = (V_k^-)^\perp = (\mathbb{F}_{k+1})^2,$$

for some $k \in \mathbb{N}$ and analogously to $H^1_0,\text{rad}(B_1)$, we can consider the decomposition of the product space $Y(B_1)$ by

$$Y(B_1) = V_k^- \oplus W_k^+.$$

In order to get weak solutions to system (1.2), we now define the functional $I : Y(B_1) \to \mathbb{R}$ by setting

$$I(u, v) = \frac{1}{2} \|(u, v)\|_Y^2 - \frac{1}{2} \int_{B_1} |x|^{\mu}(A(u, v), (u, v))_{\mathbb{R}^2} dx - \int_{B_1} |x|^{\alpha} F(u, v) dx,$$

whose Fréchet derivative is given by

$$I'(u, v)(\varphi, \psi) = \left((u, v), (\varphi, \psi)\right)_Y - \int_{B_1} |x|^{\mu}(A(u, v), (\varphi, \psi))_{\mathbb{R}^2} dx$$

$$- \int_{B_1} |x|^{\alpha}(\nabla F(u, v), (\varphi, \psi))_{\mathbb{R}^2} dx,$$  

(3.1)

for every $(\varphi, \psi) \in Y(B_1)$.

We shall observe that the weak solutions of problem (1.2) correspond to the critical points of the functional $I$.

Our goal now is to prove the Theorem 2.3. Therefore, under hypothesis $\lambda_{k, \mu} \leq \mu_1 < \mu_2 < \lambda_{k+1, \mu}$, for some nonnegative integer $k$, we will show that the functional $I$ has the geometric structure required by the Linking Theorem. Before, we need the following result:
Lemma 3.1. Let $\varphi_{k,\mu}$ be a sequence of eigenfunctions of $(-\Delta, |x|^\mu)$ in $H^1_{0,\text{rad}}(B_1) = \mathcal{V}_{k,\mu}^- \oplus \mathcal{W}_{k,\mu}^+$, where

\[
\mathcal{V}_{k,\mu}^- = \text{span}\{\varphi_1,\ldots, \varphi_{k,\mu}\}
\]

and

\[
\mathcal{W}_{k,\mu}^+ = (\mathcal{V}_{k,\mu}^-)^\perp = \text{span}\{\varphi_{k+1}, \varphi_{k+2}, \ldots\}H^1_{0,\text{rad}}(B_1).
\]

Then we have the following estimates:

1. $\|v\|^2_{L^2(B_1, |x|^\mu)} \geq \frac{1}{\lambda_{k,\mu}} \|v\|^2_{H^1_{0,\text{rad}}(B_1)}$ for all $v \in \mathcal{V}_{k,\mu}^-$;

2. $\|w\|^2_{L^2(B_1, |x|^\mu)} \leq \frac{1}{\lambda_{k+1,\mu}} \|w\|^2_{H^1_{0,\text{rad}}(B_1)}$ for all $w \in \mathcal{W}_{k,\mu}^+$.

Proof. Indeed, every $v \in \mathcal{V}_{k,\mu}^-$ has the form $v = \sum_{i=1}^{k} \xi_i \varphi_{i,\mu}$ with $\xi_i \in \mathbb{R}$. Then

\[
\|v\|^2_{H^1_{0,\text{rad}}(B_1)} = \int_{B_1} \nabla v \nabla v dx = \sum_{i=1}^{k} \xi_i^2 \|\varphi_{i,\mu}\|^2_{H^1_{0,\text{rad}}(B_1)} = \sum_{i=1}^{k} \xi_i^2 \lambda_{i,\mu} \int_{B_1} |x|^\mu |\varphi_{i,\mu}|^2 dx
\]

\[
\leq \sum_{i=1}^{k} \xi_i^2 \lambda_{i,\mu} \|\varphi_{i,\mu}\|^2_{L^2(B_1, |x|^\mu)} \leq \lambda_{k,\mu} \sum_{i=1}^{k} \xi_i^2 \|\varphi_{i,\mu}\|^2_{L^2(B_1, |x|^\mu)} = \lambda_{k,\mu} \|v\|^2_{L^2(B_1, |x|^\mu)}.
\]

Similarly, the statement 2. is proved using the Fourier series $w = \sum_{i=k+1}^{\infty} \xi_i \varphi_{i,\mu} \in \mathcal{W}_{k,\mu}^+$.

\[\square\]

Proposition 3.2. Suppose $p + q = 2^*_\alpha$, $\xi_1, \xi_2 > 0$ and $\lambda_{k,\mu} \leq \mu_1 \leq \mu_2 < \lambda_{k+1,\mu}$, for some integer $k \geq 0$. Then the functional $I$ satisfies:

1. There exist $\beta, \rho > 0$ such that $I(u,v) \geq \beta$ for all $(u,v) \in W_{k}^+$ with $||(u,v)||_Y = \rho$.

2. If $\mathcal{F}$ is a finite dimensional subspace of $Y(B_1)$, then there exists $R > \rho$ such that $I(u,v) \leq 0$, for all $(u,v) \in \mathcal{F}$ with $||(u,v)||_Y \geq R$.

Proof. Let $(u,v) \in W_{k}^+$. Since $|u(x)|^p |v(x)|^q \leq |u(x)|^{p+q} + |v(x)|^{p+q}$, by (2.9) and Lemma 3.1, we have

\[
I(u,v) \geq \frac{1}{2} \left(1 - \frac{\mu_2}{\lambda_{k+1,\mu}}\right) ||(u,v)||^2_Y - C ||(u,v)||^2_{\mathcal{F}^*},
\]

where $C > 0$ is a constant. This proves 1.

To prove 2., by (2.9), for all $(u,v) \in \mathcal{F}$ we have

\[
I(u,v) \leq \frac{1}{2} ||(u,v)||^2_Y - \frac{\mu_1}{2} ||(u,v)||^2_{L^2(B_1, |x|^\mu)} - \frac{1}{2\alpha} \int_{B_1} |x|^\alpha (|u(x)|^p |v(x)|^q + \xi_1 |u(x)|^{p+q} + \xi_2 |v(x)|^{p+q}) dx
\]

\[
\leq \frac{1}{2} ||(u,v)||^2_Y - \frac{1}{2\alpha} \min\{\xi_1, \xi_2\} ||(u,v)||^2_{L^2(B_1, |x|^\mu)} - \frac{1}{2\alpha} ||(u,v)||^2_Y - K ||(u,v)||^2_{\mathcal{F}^*},
\]

for some positive constant $K$, due to the fact that in any finite dimensional space all the norms are equivalent. Since $2^*_\alpha > 2$, we have that $I(u,v) \leq 0$, for all $(u,v) \in \mathcal{F}$ with $||(u,v)||_Y \geq R$. \[\square\]
Remark 5. For all \((u, v) \in V_k^-\), we have

\[
(u, v) = \left( \sum_{i=1}^{k} u_i \phi_{i, \mu}, \sum_{i=1}^{k} v_i \phi_{i, \mu} \right), \quad \int_{B_1} |u|^2 \, dx = \sum_{i=1}^{k} u_i^2 \quad \text{and} \quad \int_{B_1} |v|^2 \, dx = \sum_{i=1}^{k} v_i^2.
\]

Also

\[
\|(u, v)\|_Y^2 = \sum_{i=1}^{k} (u_i^2 + v_i^2) \| \phi_{i, \mu} \|^2_{H^1_0, rad(B_1)} = \sum_{i=1}^{k} (u_i^2 + v_i^2) \lambda_{i, \mu}.
\]

Then, using (2.9) and knowing that \(\mu_1 \geq \lambda_{i, \mu}, \forall i = 1, 2, \ldots, k\), we get

\[
I(u, v) \leq \frac{1}{2} \sum_{i=1}^{k} (u_i^2 + v_i^2) \lambda_{i, \mu} - \frac{\mu_1}{2} \sum_{i=1}^{k} (u_i^2 + v_i^2) \left( \int_{B_1} |x|^\alpha (|u(x)|^p |v(x)|^q + \xi_1 |u(x)|^{p+q} + \xi_2 |v(x)|^{p+q}) \, dx \right)
\]

\[
\leq \frac{1}{2} \sum_{i=1}^{k} (u_i^2 + v_i^2) (\lambda_{i, \mu} - \mu_1) \leq 0.
\]

Therefore, for the proof of the geometry, it is enough to use the Remark 5 and to apply Proposition 3.2 to the finite dimensional subspace \(V_k^- \oplus \text{span}(\{e\})\) containing \(Q = (V_k^- \cap B_R(0)) \oplus [0, R]E\), for some \(E \in W_k^+ \cap \partial B_1(0)\) and \(R > \rho\).

Remark 6. Notice that, by remark 5, we can choose the finite dimensional subspace \(F\) of \(Y(B_1)\) as

\[
F = \mathbb{F}_\epsilon = V_k^- \oplus \text{span}(\{z_\epsilon, 0\}),
\]

where \(V_k^- = \text{span}\{(0, \phi_{1, \mu}), (\phi_{1, \mu}, 0), (0, \phi_{2, \mu}), (\phi_{2, \mu}, 0), \ldots, (0, \phi_{k, \mu}), (\phi_{k, \mu}, 0)\}, z_\epsilon = \frac{z_\epsilon}{\|z_\epsilon\|_{H^1_0, rad(B_1)}}, \text{and} \ u_\epsilon = u - \sum_{j=1}^{k} \left( \int_{B_1} |x|^\alpha u \phi_{j, \mu} \, dx \right) \phi_{j, \mu}, \text{and} \ u_\epsilon \text{defined in (2.15)}.

From Proposition 3.2, we can apply the Theorem 2.8 for the functional \(I\) with

\[
Q = (B_R \cap V) \oplus \{r(z_\epsilon, 0) : 0 < r < R\},
\]

which the critical level is characterized as

\[
c = \inf_{h \in \Gamma} \max_{(u, v) \in Q} I(h(u, v)),
\]

where

\[
\Gamma = \{h \in C(\overline{Q}, Y) : h = \text{id on } \partial Q\}.
\]

4. Palais-Smale condition for the functional. Let \(J : E \rightarrow \mathbb{R}\) be a functional defined on a real Banach space \(E\). We recall that a sequence \((w_j)_{j \in \mathbb{N}}\) in \(E\) is said to be Palais-Smale sequence for \(J\) in \(E\) for the level \(c \in \mathbb{R}\) if

\[
(J(w_j)) \rightarrow c \quad \text{and} \quad \sup\{|\langle J'(w_j), \Psi \rangle| : \Psi \in E, \|\Psi\| = 1\} \rightarrow 0,
\]

as \(j \rightarrow +\infty\). The functional \(J\) satisfies the Palais-Smale compactness condition in \(E\) for the level \(c\) (or \((PS)_c\) condition), if every Palais-Smale sequence for \(J\) in the level \(c\) admits a (strongly) convergent subsequence in \(E\).

Lemma 4.1. Fix \(k\) an integer nonnegative. Suppose \(\lambda_{k, \mu} \leq \mu_1 \leq \mu_2 < \lambda_{k+1, \mu}\) and let \(c \in \mathbb{R}\) be such that

\[
c < \frac{(\alpha + 2)}{2(N + \alpha)} \left( \frac{\bar{S}_\alpha}{\bar{S}_\alpha} \right)^{\frac{N+\alpha}{2+\alpha}}.
\]

Then, the functional \(I\) satisfies the \((PS)_c\) condition.
Proof. Let \( U_n \in Y(B_1) = V_k^- \oplus W_k^+ \) be a \((PS)\_c\)-sequence. We have that \( U_n = U_n^- + U_n^+ \) with \( U_n^- = (u_n^-, v_n^-) \in V_k^- \) and \( U_n^+ = (u_n^+, v_n^+) \in W_k^+ \).

In order to prove Lemma 4.1, we proceed by steps.

**Step 1:** The sequence \((U_n)\) is bounded in \(Y(B_1)\).

**First case:** \( \lambda_k \mu < \mu_1 \leq \mu_2 < \lambda_{k+1} \mu \). Indeed, If \( k = 0 \), that is, \( \lambda_0 \mu = 0 \), we get for every \( n \in \mathbb{N} \)

\[
C + C\| (u_n, v_n) \|_Y \geq I(u_n, v_n) - \frac{1}{2\alpha} I'(u_n, v_n)(u_n, v_n)
\]

Using Hölder’s inequality, Young’s inequality and imbedding

\[
\begin{align*}
(\frac{\mu_1}{\lambda_{k, \mu}} - 1) \| U_n^- \|_Y^2 & \leq \int_{B_1} |x|^\mu (AU_n^-, U_n^-)_{\mathbb{R}^2} dx - \| U_n^- \|_Y^2 \\
& = - \int_{B_1} |x|^\alpha (\nabla F(U_n), U_n^-)_{\mathbb{R}^2} dx - I'(U_n)(U_n^-) \\
& \leq \int_{B_1} |x|^\alpha |\nabla F(U_n)| \| U_n^- \| dx + C \| U_n^- \|_Y. \tag{4.2}
\end{align*}
\]

By Remark 3 (iii), there is a constant \( K > 0 \) such that

\[
|\nabla F(U_n)| \leq K |u_n|^{2\alpha - 1} + |v_n|^{2\alpha - 1}.
\]

By (4.2) and estimates above, follows that

\[
\begin{align*}
(\frac{\mu_1}{\lambda_{k, \mu}} - 1) \| U_n^- \|_Y^2 & \leq K \int_{B_1} |x|^\alpha |u_n|^{2\alpha - 1} + |v_n|^{2\alpha - 1} |u_n^-| dx \\
& + K \int_{B_1} |x|^\alpha |u_n|^{2\alpha - 1} + |v_n|^{2\alpha - 1} |v_n^-| dx + C \| U_n^- \|_Y.
\end{align*}
\]

Using Hölder’s inequality, Young’s inequality and imbedding

\[
B_{0, \text{rad}}^1(B_1) \hookrightarrow L^r(B_1, |x|^{\alpha}), \quad \forall \ 1 \leq r \leq 2^\alpha,
\]

we have

\[
\begin{align*}
(\frac{\mu_1}{\lambda_{k, \mu}} - 1) \| U_n^- \|_Y^2 & \leq K \int_{B_1} |x|^\alpha |u_n|^{2\alpha - 1} + |v_n|^{2\alpha - 1} |u_n^-| dx \\
& + K \int_{B_1} |x|^\alpha |u_n|^{2\alpha - 1} + |v_n|^{2\alpha - 1} |v_n^-| dx + C \| U_n^- \|_Y \\
& \leq K \left[ \varepsilon \left( \int_{B_1} |x|^\alpha |u_n^-|^{2\alpha} dx \right)^{\frac{2}{2\alpha}} + \left( \int_{B_1} |x|^\alpha |v_n^-|^{2\alpha} dx \right)^{\frac{2}{2\alpha}} \right].
\end{align*}
\]
Therefore
\[
C\|U_n^-\|_Y^2 \leq \varepsilon C_1\|U_n^-\|_Y^2 + C_2\|U_n^-\|_Y + C_\varepsilon \left( \int_{B_1} |x|^\alpha |u_n|^2^\ast \, dx \right)^{2(2^\ast - 1)} + \left( \int_{B_1} |x|^\alpha |v_n|^2^\ast \, dx \right)^{2(2^\ast - 1)} + C\|U_n^-\|_Y.
\] (4.3)

On the other hand,
\[
I(U_n) - \frac{1}{2} I'(U_n)U_n = \frac{1}{2} \int_{B_1} |x|^\alpha (\nabla F(U_n), U_n)_{\mathbb{R}^2} \, dx - \int_{B_1} |x|^\alpha F(U_n) \, dx.
\]
Now, by Remark 2(ii) and Remark 3,
\[
\left( \frac{2^\ast}{2} - 1 \right) \int_{B_1} |x|^\alpha F(U_n) \, dx = I(U_n) - \frac{1}{2} I'(U_n)U_n \leq C + C\|U_n\|_Y
\]
and consequently, using the fact that $2^\ast > 2$, we get
\[
\int_{B_1} |x|^\alpha F(U_n) \, dx \leq C + C\|U_n\|_Y. \tag{4.4}
\]
Taking $\varepsilon > 0$ small enough in (4.3) and using (4.4), we deduce that
\[
\|U_n^-\|_Y^2 \leq C_1\|U_n^-\|_Y + C_2(1 + \|U_n\|_Y)^{2(2^\ast - 1)}. \tag{4.5}
\]
Similarly we can also estimate
\[
\|U_n^+\|_Y^2 \leq C_3\|U_n^+\|_Y + C_4(1 + \|U_n\|_Y)^{2(2^\ast - 1)}. \tag{4.6}
\]
Adding the last two estimates, we obtain
\[
\|U_n\|_Y^2 \leq C_5\|U_n\|_Y + C_5(1 + \|U_n\|_Y)^{2(2^\ast - 1)}. \tag{4.7}
\]
Since $2(2^\ast - 1) < 2$, we conclude that $(U_n)$ is bounded in $Y(B_1)$.

**Second case:** If $\lambda_{k,\mu} = \mu_1 \leq \mu_2 < \lambda_{k+1,\mu}$. We follow the notations of the previous proof.

Let $U_n \in Y(B_1)$ such that $I(U_n) \rightarrow c$ and $I'(U_n) \rightarrow 0$ in the dual space $Y(B_1)'$. Writing $Y(B_1) = V_{k-1}^- \oplus W_k^+ \oplus Z_k$, consequently we have
\[
U_n = U_n^- + U_n^+ + \beta_n Y_n := W_n + \beta_n Y_n,
\]
where $U_n^- \in V_{k-1}^-$, $U_n^+ \in W_k^+$, and $Y_n \in Z_k = \text{span}\{(\varphi_{k,\mu}, 0), (0, \varphi_{k,\mu})\}$ with $\|Y_n\|_Y = 1$. Using similar arguments as in (4.5) and (4.6), we obtain
\[
\|W_n\|_Y^2 \leq C(1 + \|U_n\|_Y)^{2\tau} + C\|W_n\|_Y, \tag{4.7}
\]
where $\tau = \frac{2^\ast - 1}{2^{2^\ast}}$. We can assume $\|U_n\|_Y \geq 1$ (if $\|U_n\|_Y \leq 1$, the sequence $(U_n)$ is bounded in $Y(B_1)$). Then, since $\|U_n\|_Y \leq \|W_n\|_Y + |\beta_n|$, from (4.7), we have
\[
\|W_n\|_Y^2 \leq C_1(\|W_n\|_Y + |\beta_n|)^{2\tau} + C\|W_n\|_Y. \tag{4.8}
\]
If $\beta_n$ is bounded, since $\tau < 1$, by (4.8) we conclude that $(W_n)$ is bounded in $Y(B_1)$ and consequently $(U_n)$ is bounded in $Y(B_1)$ as well. Otherwise, we may assume $\beta_n \to +\infty$, therefore, from (4.8), it follows that
\[
\left\| \frac{W_n}{\beta_n} \right\|_Y^2 \leq C_1 \left\{ \left( \frac{\|W_n\|_Y + |\beta_n|}{{|\beta_n|}} \right) \right\}^2 + C \frac{1}{\beta_n} \left\| \frac{W_n}{\beta_n} \right\|_Y^2 \leq C_1 \left\{ \frac{1}{|\beta_n|^{1-\tau}} \left\| \frac{W_n}{\beta_n} \right\|_Y^2 + \frac{1}{|\beta_n|^{1-\tau}} \right\}^2 + C \frac{1}{\beta_n} \left\| \frac{W_n}{\beta_n} \right\|_Y^2.
\] (4.9)

Using again the fact that $\tau < 1$, the above estimate yields that
\[
\left\| \frac{W_n}{\beta_n} \right\|_Y^2 \leq C_2 \left\| \frac{W_n}{\beta_n} \right\|_Y^{2\tau} + C_3 \left\| \frac{W_n}{\beta_n} \right\|_Y + C_4
\]
and consequently the sequence $\left\{ \frac{W_n}{\beta_n} \right\}$ is bounded in $Y(B_1)$ and by (4.9), $\left\| \frac{W_n}{\beta_n} \right\|_Y \to 0$, as $n \to \infty$.

Therefore, possibly up to a subsequence, $W_n/\beta_n \to 0$ a.e. in $B_1$ and strongly in $L^q(B_1, |x|^{\alpha}) \times L^q(B_1, |x|^{\alpha})$, $1 \leq q < 2^*_\alpha$; $Y_n \to Y_0 \in Z_k$ a.e. in $B_1$ and strongly in $Y(B_1)$ and $L^q(B_1, |x|^{\alpha}) \times L^q(B_1, |x|^{\alpha})$, $1 \leq q < 2^*_\alpha$.

Now, taking $Y_n \in Z_k$ as test function, we get
\[
I'(U_n)(Y_n) = \beta_n \left( \|Y_n\|_Y^2 - \int_{B_1} |x|^{\alpha}(\nabla F(U_n), Y_n)_{\mathbb{R}^2} dx \right)
- \int_{B_1} |x|^{\alpha}(\nabla F(U_n), Y_n)_{\mathbb{R}^2} dx.
\] (4.10)

Since $Z_k \subset V_k$ and $W_{k-1} \subset W_{k-1}$ and $\|Y_n\|_Y = 1$, there exist constants $K_1, K_2$ such that
\[
K_1 \left(1 - \frac{H_2}{\lambda_{k,\mu}} \right) \leq \|Y_n\|_Y^2 - \int_{B_1} |x|^{\alpha}(AY_n, Y_n)_{\mathbb{R}^2} dx \leq K_2 \left(1 - \frac{H_1}{\lambda_{k,\mu}} \right) = 0,
\]
and consequently
\[
\frac{1}{(\beta_n)^{2\tau-2}} \left( \|Y_n\|_Y^2 - \int_{B_1} |x|^{\alpha}(AY_n, Y_n)_{\mathbb{R}^2} dx \right) \to 0, \text{ as } n \to \infty.
\]
Using that $(U_n)$ is a $(PS)_c$-sequence, by (4.10),
\[
o(1) = \frac{1}{(\beta_n)^{2\tau-2}} I'(U_n)(Y_n) = - \frac{1}{(\beta_n)^{2\tau-1}} \int_{B_1} |x|^\alpha(\nabla F(U_n), Y_n)_{\mathbb{R}^2} dx.
\]

Now, from remark 2 (iii),
\[
\int_{B_1} |x|^\alpha(\nabla F \left( \frac{U_n}{\beta_n} \right), Y_n)_{\mathbb{R}^2} dx = \frac{1}{(\beta_n)^{2\tau-1}} \int_{B_1} |x|^\alpha(\nabla F(U_n), Y_n)_{\mathbb{R}^2} dx \to 0. \tag{4.11}
\]

On the other hand, since $U_n = W_n + \beta_n Y_n$, we have that $\frac{U_n}{\beta_n} \to Y_0$ in $L^q(B_1, |x|^{\alpha}) \times L^q(B_1, |x|^{\alpha})$ for all $1 \leq q < 2^*_\alpha$ and for a.e. in $B_1$. So, by the Dominated Convergence Theorem and by (4.11), it follows that
\[
\int_{B_1} |x|^\alpha(\nabla F \left( \frac{U_n}{\beta_n} \right), Y_n)_{\mathbb{R}^2} dx \to \int_{B_1} |x|^\alpha(\nabla F(Y_0), Y_0)_{\mathbb{R}^2} dx = 0.
\]
Hence, by the remark 2 (ii), we concluded that $\int_{B_1} |x|^\alpha F(Y_0) dx = 0$.

Finally, using the notation $Y_0 = (y_1^0, y_2^0)$, it follows that $y_1^0 = 0 = y_2^0$, contradicting $\|Y_0\|_Y = 1$. Therefore $(U_n)$ is bounded.
Step 2: Problem (1.1) admits a solution $U \in Y(B_1)$.

Since $U_n$ is bounded in $Y(B_1)$, up to a subsequence, still denoted by $U_n$, there exists $U \in Y(B_1)$ such that

$$U_n \rightharpoonup U \text{ in } Y(B_1).$$

Since $Y(B_1) \hookrightarrow L^{2n}(B_1, |x|^{\alpha}) \times L^{2n}(B_1, |x|^{\alpha})$, we have that $U_n$ is bounded in $L^{2n}(B_1, |x|^{\alpha}) \times L^{2n}(B_1, |x|^{\alpha})$, and so, up to a subsequence,

- $U_n \to U$ in $L^{2n}(B_1, |x|^{\alpha}) \times L^{2n}(B_1, |x|^{\alpha})$,
- $U_n \to U$ for a.e. $x$ in $B_1$,
- $U_n \to U$ in $L^r(B_1, |x|^{\alpha}) \times L^r(B_1, |x|^{\alpha})$, for all $r \in [1, 2\alpha)$,
- $U_n \to U$ in $L^\mu(B_1, |x|^{\alpha}) \times L^\mu(B_1, |x|^{\alpha})$, for all $r \in [1, 2\alpha)$ if $\mu \geq \alpha$.

Moreover, by Remark 2 (i), (iii), there exists a constant $K > 0$ such that

$$|\nabla F(U_n)| \leq K |u_n|^{2\alpha - 1} + |v_n|^{2\alpha - 1}. \quad (4.16)$$

We have that $|u_n|^{2\alpha - 1}$ and $|v_n|^{2\alpha - 1}$ are bounded in $L^{\frac{2\alpha}{2\alpha - 1}}(B_1, |x|^{\alpha})$ and consequently by (4.16), $|\nabla F(U_n)|$ is bounded in $L^{\frac{2\alpha}{2\alpha - 1}}(B_1, |x|^{\alpha})$. Therefore, by this and by (4.12), it follows that

$$\nabla F(U_n) \to \nabla F(U) \text{ in } L^{\frac{2\alpha}{2\alpha - 1}}(B_1, |x|^{\alpha}) \times L^{\frac{2\alpha}{2\alpha - 1}}(B_1, |x|^{\alpha}). \quad (4.17)$$

Since $(2\alpha/(2\alpha - 1))' = 2\alpha$, it is easily seen that

$$\int_{B_1} |x|^{\alpha}(\nabla F(U_n), \Theta)_{\mathbb{R}^2} dx \to \int_{B_1} |x|^{\alpha}(\nabla F(U), \Theta)_{\mathbb{R}^2} dx,$$

for all $\Theta \in L^{2n}(B_1, |x|^{\alpha}) \times L^{2n}(B_1, |x|^{\alpha})$.

In particular

$$\int_{B_1} |x|^{\alpha}(\nabla F(U_n), \Theta)_{\mathbb{R}^2} dx \to \int_{B_1} |x|^{\alpha}(\nabla F(U), \Theta)_{\mathbb{R}^2} dx, \forall \Theta \in Y(B_1), \quad (4.18)$$

as $n \to \infty$.

On the other hand, for any $\Theta \in Y(B_1)$, we have the convergence to zero of $I'(U_n)(\Theta)$; i.e.

$$\langle U_n, \Theta \rangle_Y - \int_{B_1} |x|^{\alpha}(AU_n, \Theta)_{\mathbb{R}^2} dx - \int_{B_1} |x|^{\alpha}(\nabla F(U_n), \Theta)_{\mathbb{R}^2} dx \to 0, \quad (4.19)$$

so that, passing the limit in the above expression as $n \to \infty$ and taking into account the convergences (4.15) and (4.18), we get

$$\langle U, \Theta \rangle_Y - \int_{B_1} |x|^{\alpha}(AU, \Theta)_{\mathbb{R}^2} dx - \int_{B_1} |x|^{\alpha}(\nabla F(U), \Theta)_{\mathbb{R}^2} dx = 0, \quad (4.20)$$

for all $\Theta \in Y(B_1)$ and consequently the Step 2 follows.

Step 3: The following relations hold true:

a) $I(U) = \left(\frac{\alpha}{2} - 1\right) \int_{B_1} |x|^{\alpha} F(U) dx \geq 0$.

b) $I(U_n) = I(U) + \frac{1}{2} \|U_n - U\|_Y^2 - \int_{B_1} |x|^{\alpha} F(U_n - U) dx + o(1)$.

c) $\|U_n - U\|_Y^2 = 2\alpha \int_{B_1} |x|^{\alpha} F(U_n - U) dx + o(1)$. 
Proof of a) Taking $\Theta = U \in Y(B_1)$ as a test function in (4.20), we get

$$0 = I'(U)U = ||U||^2_Y - \int_{B_1} |x|^{\alpha} (AU, U)_{\mathbb{R}^2} dx - \int_{B_1} |x|^{\alpha} (\nabla F(U), U)_{\mathbb{R}^2} dx.$$ 

Therefore, by Remark 2 (ii)

$$I(U) = \frac{1}{2} \int_{B_1} |x|^{\alpha} (\nabla F(U), U)_{\mathbb{R}^2} dx - \int_{B_1} |x|^{\alpha} F(U) dx$$

$$= \frac{2^*_n}{2} \int_{B_1} |x|^{\alpha} F(U) dx - \int_{B_1} |x|^{\alpha} F(U) dx$$

$$= \left( \frac{2^*_n}{2} - 1 \right) \int_{B_1} |x|^{\alpha} F(U) dx.$$ 

Proof of b) By Step 1 the sequence $U_n$ is bounded in $Y(B_1) \hookrightarrow L^{2^*_n}(B_1, |x|^{\alpha}) \times L^{2^*_n}(B_1, |x|^{\alpha})$, hence $U_n$ is bounded in $L^{2^*_n}(B_1, |x|^{\alpha}) \times L^{2^*_n}(B_1, |x|^{\alpha})$. Since $U_n \to U$ for a.e. in $B_1$, by the Brézis-Lieb Lemma [7, Theorem 2] (see also [16, Lemma 5]), we have

$$||U_n||_Y^2 = ||U_n - U||_Y^2 + ||U||_Y^2 + o(1), \quad (4.21)$$

$$||U_n||_{(L^{2^*_n}(B_1, |x|^{\alpha})}^2 = ||U_n - U||_{(L^{2^*_n}(B_1, |x|^{\alpha})}^2 + ||U||_{(L^{2^*_n}(B_1, |x|^{\alpha})}^2 + o(1). \quad (4.22)$$

Otherwise, by the Brézis-Lieb Lemma for homogeneous functions (Lemma 5 in [16]),

$$\int_{B_1} |x|^{\alpha} F(U_n) dx = \int_{B_1} |x|^{\alpha} F(U) dx + \int_{B_1} |x|^{\alpha} F(U_n - U) dx + o(1). \quad (4.23)$$

Therefore, using that $U_n \to U$ in $L^r(B_1, |x|^{\alpha}) \times L^r(B_1, |x|^{\alpha})$, for all $r \in [1, 2^*_n)$, by the definition of $I$, $I(4.21)$, $I(4.22)$ and $(4.23)$, we deduce that

$$I(U_n) = \frac{1}{2} ||U_n - U||_Y^2 + \frac{1}{2} ||U||_Y^2 - \frac{1}{2} \int_{B_1} |x|^{\alpha} (AU, U)_{\mathbb{R}^2} dx$$

$$- \int_{B_1} |x|^{\alpha} F(U) dx - \int_{B_1} |x|^{\alpha} F(U_n - U) dx + o(1)$$

$$= I(U) + \frac{1}{2} ||U_n - U||_Y^2 - \int_{B_1} |x|^{\alpha} F(U_n - U) dx + o(1).$$

Proof of c) By (4.12), (4.17) and Remark 2 (ii),

$$\int_{B_1} |x|^{\alpha} (\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2} dx$$

$$= \int_{B_1} |x|^{\alpha} (\nabla F(U_n), U_n - U)_{\mathbb{R}^2} dx - \int_{B_1} |x|^{\alpha} (\nabla F(U), U_n - U)_{\mathbb{R}^2} dx + o(1)$$

$$= \frac{2^*_n}{2} \int_{B_1} |x|^{\alpha} F(U_n) dx - \frac{2^*_n}{2} \int_{B_1} |x|^{\alpha} F(U) dx + o(1).$$

Therefore, using (4.23), we get

$$\int_{B_1} |x|^{\alpha} (\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2} dx = \frac{2^*_n}{2} \int_{B_1} |x|^{\alpha} F(U_n - U) dx + o(1). \quad (4.24)$$

On the other hand, by the Step 2,

$$o(1) = I'(U_n)(U_n - U) = I'(U_n)(U_n - U) - I'(U)(U_n - U)$$

$$= (U_n - U, U_n - U)_Y - \int_{B_1} |x|^{\alpha} (A(U_n - U), U_n - U)_{\mathbb{R}^2} dx$$
\[- \int_{B_1} |x|^\alpha(\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2} dx.\]

Hence, from (4.15) and (4.24), it follows that
\[
\|U_n - U\|^2_Y = 2^* \int_{B_1} |x|^\alpha F(U_n - U) dx + o(1), \quad \text{as } n \to \infty.
\]

Now, we can conclude the proof of the Lemma 4.1.

By Step 3- c), it is follows that
\[
\frac{1}{2} \|U_n - U\|^2_Y - \int_{B_1} |x|^\alpha F(U_n - U) dx = \left(\frac{1}{2} - \frac{1}{2^*}\right) \|U_n - U\|^2_Y + o(1) = \frac{(\alpha + 2)}{2(N + \alpha)} \|U_n - U\|^2_Y + o(1).
\]

Therefore, using the Step 3- b) and above equality, we see that
\[
I(U) + \frac{(\alpha + 2)}{2(N + \alpha)} \|U_n - U\|^2_Y = I(U_n) + o(1) = c + o(1), \quad \text{as } n \to \infty.
\] (4.25)

Now, by Step 1, the sequence \(\|U_n\|_Y\) is bounded in \(\mathbb{R}\). So, up to a subsequence, if necessary, we can assume that
\[
\|U_n - U\|^2_Y \to L \in [0, \infty) \quad \text{as } n \to \infty.
\] (4.26)

Again, as a consequence of Step 3- c),
\[
2^* \int_{B_1} |x|^\alpha F(U_n - U) dx \to L, \quad \text{as } n \to \infty.
\] (4.27)

Therefore, by definition of \(\tilde{S}_{\alpha}^p(B_1)\) (see (2.11)), since \(U_n - U \in Y(B_1) \setminus \{(0, 0)\}\), we have
\[
\tilde{S}_{\alpha} := \tilde{S}_{\alpha}^p(B_1) \leq \left(2^* \int_{B_1} |x|^\alpha F(U_n - U) dx\right)^{\frac{1}{2^*}}.
\]

Hence, by (4.26) and (4.27), we conclude that
\[
L \geq \tilde{S}_{\alpha}^{\frac{2^*}{\alpha}} \tilde{S}_{\alpha}
\]
and consequently, either
\[
L = 0 \quad \text{or} \quad L \geq (\tilde{S}_{\alpha})^{\frac{N+\alpha}{2(N+\alpha)}},
\]
If \(L \geq (\tilde{S}_{\alpha})^{\frac{N+\alpha}{2(N+\alpha)}},\) by (4.25), (4.26) and Step 3-a), we would get
\[
c = I(U) + \frac{(\alpha + 2)}{2(N + \alpha)} L \geq \frac{(\alpha + 2)}{2(N + \alpha)} (\tilde{S}_{\alpha})^{\frac{N+\alpha}{2(N+\alpha)}},
\]
which contradicts (4.1). Thus \(L = 0\) and therefore, by (4.26), we have
\[
\|U_n - U\|^2_Y \to 0 \quad \text{as } n \to \infty
\]
and so the assertion of Lemma 4.1 follows. \(\square\)
Let $\mathcal{F}_\varepsilon := \mathcal{V}_{k,\mu}^\varepsilon \oplus \text{span}\{z_\varepsilon\} \subset H^1_{0,\text{rad}}(B_1)$, where $\mathcal{V}_{k,\mu}^\varepsilon = \text{span}\{\varphi_{1,\mu}, \varphi_{2,\mu}, \ldots, \varphi_{k,\mu}\}$ and $z_\varepsilon = \frac{\varepsilon}{\|u\|_{H^1_{0,\text{rad}}(B_1)}}$, with $u_\varepsilon = u - \sum_{j=1}^k (\int_{B_1} |x|^\mu u_j \varphi_{j,\mu} dx) \varphi_{j,\mu}$ and $u_\varepsilon$ defined in (2.15). Note that $\mathcal{F}_\varepsilon \times \{0\} \subset \mathcal{F}_\varepsilon$. See also the Remark 6.

Define
\[
M_\varepsilon := \max_{u \in \mathcal{F}_\varepsilon} S_{\alpha,\mu,\mu}(u) = \max_{\|u\|_{L^2(B_1,|x|^\mu)} = 1} (\|u\|_{H^1_{0,\text{rad}}(B_1)}^2 - \mu_1 \|u\|_{L^2(B_1,|x|^\mu)}^2) \quad (4.28)
\]

The proof of the next result is made following the arguments as in [32, prop 12] (and in [30, prop 13]) for the nonlocal setting.

**Proposition 4.2.** Suppose $\mu \geq \alpha > 0$ and $\lambda_{k,\mu} < \mu_1 \leq \mu_2 < \lambda_{k+1,\mu}$, for some integer $k \geq 0$, then

a) $M_\varepsilon$ is achieved by $u_M \in \mathcal{F}_\varepsilon$, characterized by
\[
u_M = v + t z_\varepsilon,
\]
where $t \geq 0$ and $v = v + t \sum_{i=1}^k (\int_{B_1} |x|^\mu u_i \varphi_{i,\mu} dx) \varphi_{i,\mu}$ and $v \in \mathcal{V}_{k,\mu}^\varepsilon$.

b) The following estimate holds for $t > 0$ and $\varepsilon > 0$ small enough,
\[
M_\varepsilon \leq (\lambda_{k,\mu} - \mu_1) \|v\|_{L^2(B_1,|x|^\mu)}^2 + \mathcal{S}_{\alpha,\mu,\mu}(u_\varepsilon) \left(1 + \mathcal{O}(\varepsilon^{(N-2)/[2(2+\alpha)]}) \|v\|_{L^2(B_1,|x|^\mu)}^2\right)
\]
\[
+ \mathcal{O}(\varepsilon^{(N-2)/[2(2+\alpha)]}) \|v\|_{L^2(B_1,|x|^\mu)}^2,
\]

c) $M_\varepsilon < \mathcal{S}_{\alpha,\mu}$, provided
i) $N \geq 4 + \mu$ or
ii) $N \leq 4 + \mu$ and $k$ is an integer large enough (consequently $\mu_1$ is large).

**Proof.** a) By the Weierstrass Theorem, (4.28) is achieved by a function $u_M \in \mathcal{F}_\varepsilon$ such that
\[
M_\varepsilon = \|u_M\|_{H^1_{0,\text{rad}}(B_1)}^2 - \mu_1 \|u_M\|_{L^2(B_1,|x|^\mu)}^2 = \|u_M\|_{L^2(B_1,|x|^\mu)}^2 = 1.
\]

Therefore $u_M \neq 0$ and since $u_M \in \mathcal{F}_\varepsilon$, we have that
\[
u_M = v + t z_\varepsilon
\]
for some $\varepsilon \in \mathcal{V}_{k,\mu}^\varepsilon$. Since $z_\varepsilon = \frac{z_\varepsilon}{\|z_\varepsilon\|_{H^1_{0,\text{rad}}(B_1)}}$, we can rewrite
\[
u_M = v + t u_\varepsilon
\]
with $t = \frac{s}{\|z_\varepsilon\|_{H^1_{0,\text{rad}}(B_1)}} \geq 0$ (otherwise, if $t \leq 0$ we change the sign of $u_M$) and $v = \overline{v}$

\[
- t \sum_{i=1}^k (\int_{B_1} |x|^\mu u_i \varphi_{i,\mu} dx) \varphi_{i,\mu} \in \mathcal{V}_{k,\mu}^\varepsilon.
\]

This concludes the proof of assertion a).

b) The proof is divided in two cases:

**First case:** If $t = 0$, we have $u_M = v \in \mathcal{V}_{k,\mu}^\varepsilon$ and consequently
\[
M_\varepsilon = \|v\|_{H^1_{0,\text{rad}}(B_1)}^2 - \mu_1 \|v\|_{L^2(B_1,|x|^\mu)}^2 \leq (\lambda_{k,\mu} - \mu_1) \|v\|_{L^2(B_1,|x|^\mu)}^2 < 0.
\]

**Second case:** If $t > 0$, we have that $\overline{v}$ and $z_\varepsilon$ are orthogonal in $L^2(B_1,|x|^\mu)$. Then
\[
\|u_M\|_{L^2(B_1,|x|^\mu)}^2 = \|\overline{v}\|_{L^2(B_1,|x|^\mu)}^2 + t^2 \|z_\varepsilon\|_{L^2(B_1,|x|^\mu)}^2 \geq \|\overline{v}\|_{L^2(B_1,|x|^\mu)}^2.
\]
\[
(4.29)
\]
Let $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, so by the Hölder inequality, we can bound $u_M$ as follows

$$
\|u_M\|_{L^2(B_1, |x|^{\alpha})}^2 = \int_{B_1} |x|^\mu \left( |x|^{\frac{\mu}{2} + \frac{\mu}{q}} |u_M|^2 \right) dx = \int_{B_1} \left( |x|^{\frac{\mu}{2}} \right) \left( |x|^{\frac{\mu}{q}} |u_M|^2 \right) dx
\leq \left( \int_{B_1} |x|^{\frac{2\mu}{2\mu - q}} dx \right)^{\frac{2\mu - q}{2\mu}} \left( \int_{B_1} |x|^{\frac{2\mu}{q}} |u_M|^{2*} dx \right)^{\frac{2}{2\mu}}.
$$

Choosing $q$ such that $\frac{2\mu}{2\mu - q} = \alpha$, we have that $p = \frac{2\mu}{2\mu - 2\alpha} > 0$ (See Remark 7). So, using that $0 < \frac{2\mu}{2\mu - 2\alpha} = \frac{2\mu}{2\alpha - 2} - \frac{2\alpha - 2}{2\alpha - 2}$ and $|x| \leq 1$, we get

$$
\|u_M\|_{L^2(B_1, |x|^{\alpha})}^2 \leq \left| B_1 \right| \frac{2\alpha - 2}{2\alpha - 2} \left( \int_{B_1} |x|^{\alpha} |u_M|^{2*} dx \right)^{\frac{2}{2\alpha - 2}} = C \|u_M\|_{L^2(B_1, |x|^{\alpha})}^2.
$$

Therefore, by (4.30) and (4.29), we obtain that $\|u_M\|^2_{L^2(B_1, |x|^{\alpha})}$ and $\|\tilde{v}\|^2_{L^2(B_1, |x|^{\alpha})}$ are bounded uniformly in $\varepsilon$ by the constant $C$.

On the other hand, we claim that $t \leq \tilde{C}$, for some positive constant $\tilde{C}$. Indeed, $\varphi_{i, \mu} \in L^\infty(B_1)$, for all $i \in \mathbb{N}$, hence we have $\tilde{v} \in V_{k, \mu}^+$ and consequently since $B_1$ is bounded, $\tilde{v} \in L^\infty(B_1)$. Moreover, using the fact that in a finite dimensional space, all norms are equivalent, we get

$$
\|\tilde{v}\|_{L^\infty(B_1, |x|^{\alpha})} \leq C \|\tilde{v}\|_{L^\infty(B_1)} \leq \tilde{C}.
$$

By Proposition 2.5-d), for $\varepsilon > 0$ sufficiently small, we obtain

$$
\left| \int_{B_1} |x|^\mu u^\varepsilon \varphi_{j, \mu} dx \right| \leq \|u^\varepsilon\|_{L^1(B_1, |x|^{\alpha})} \|\varphi_{j, \mu}\|_{L^\infty(B_1)} = O(\varepsilon \frac{N-2}{2(2+\alpha)}).
$$

Therefore, using the definition of $z_\varepsilon$ and again Proposition 2.5,

$$
\|z_\varepsilon\|_{L^\infty(B_1, |x|^{\alpha})} \geq \|u^\varepsilon\|_{L^\infty(B_1, |x|^{\alpha})} - \sum_{j=1}^k \left| \int_{B_1} |x|^\mu u^\varepsilon \varphi_{j, \mu} dx \right| \|\varphi_{j, \mu}\|_{L^\infty(B_1, |x|^{\alpha})} \geq S_\alpha(N-2)/[2(2+\alpha)] + O(\varepsilon(N-2)/[2(2+\alpha)]) \geq 1 + \frac{1}{2} S_\alpha(N-2)/[2(2+\alpha)].
$$

So, since $u_M = \tilde{v} + tz_\varepsilon$ and $t > 0$, by (4.31)

$$
\frac{1}{2} S_\alpha(N-2)/[2(2+\alpha)] \leq t \leq \tilde{C}.
$$

and consequently

$$
\tilde{C}.
$$

Now, using the convexity of the map $t \mapsto t^2$, the monotonicity of the integrals, (4.32) and that $V_{k, \mu}^+$ is a finite dimensional space (all norms are equivalent), yield

$$
1 = \|u_M\|^2_{L^\infty(B_1, |x|^{\alpha})} = \int_{B_1} |x|^\alpha |u_M|^2 dx
\geq \int_{B_1} |x|^\alpha |tu^\varepsilon|^2 dx + 2^* \int_{B_1} |x|^\alpha |tu^\varepsilon|^2 \tilde{1} \tilde{v} dx
\geq \|tu^\varepsilon\|^2_{L^\infty(B_1, |x|^{\alpha})} - K \|u^\varepsilon\|_{L^\infty(B_1, |x|^{\alpha})} \|\tilde{v}\|_{L^2(B_1, |x|^{\alpha})},
$$

for some positive constant $K$. Hence

$$
\|tu^\varepsilon\|^2_{L^\infty(B_1, |x|^{\alpha})} \leq 1 + \frac{1}{K} \|u^\varepsilon\|_{L^\infty(B_1, |x|^{\alpha})} \|\tilde{v}\|_{L^2(B_1, |x|^{\alpha})}.
$$

(4.33)
From \( u_M = v + tu_u \), we get
\[
M_\varepsilon = \|u_M\|_{L^2_0(B_1)}^2 - \mu_1 \|u_M\|_{L^2(B_1)}^2 \\
\leq (\lambda_{k,\mu} - \mu_1) \|v\|_{L^2(B_1)}^2 + S_{\alpha,\mu_1,\mu}(u_u) \|tu_u\|_{L^{2,\alpha}(B_1)}^2 \\
+ 2t(t_u, v)_{H_0^1(B_1)} - 2\mu_1 t(t_u, v)_{L^2(B_1)}.
\] (4.34)

Using (4.34), from Hölder and using again the equivalence of the norms in a finite dimensional space, we conclude that
\[
M_\varepsilon \leq (\lambda_{k,\mu} - \mu_1) \|v\|_{L^2(B_1)}^2 + S_{\alpha,\mu_1,\mu}(u_u) \|tu_u\|_{L^{2,\alpha}(B_1)}^2 \\
+ c_1 \|u_u\|_{L^1(B_1)} \|\Delta v\|_{L^\infty(B_1)} + c_2 \|u_u\|_{L^1(B_1)} \|v\|_{L^\infty(B_1)}.
\] (4.35)

Since \( v \in V_{k,\mu}^- \) and all norms are equivalent on finite dimensional space, the estimate above and (4.33) yield
\[
M_\varepsilon \leq (\lambda_{k,\mu} - \mu_1) \|v\|_{L^2(B_1)}^2 \\
+ S_{\alpha,\mu_1,\mu}(u_u) \left(1 + K \|u_u\|_{L^{2,\alpha}(B_1)}^{2,\alpha,\alpha - 1} \|v\|_{L^2(B_1)}^\alpha \right)^{2,\alpha} \\
+ \mathcal{O}(\varepsilon^{(N - 2)/(2\alpha)}) \|v\|_{L^2(B_1)}^\alpha
\]
and consequently being \( 2,\alpha > 2 \), by Proposition 2.5, for \( \varepsilon > 0 \) small enough, we have
\[
M_\varepsilon \leq (\lambda_{k,\mu} - \mu_1) \|v\|_{L^2(B_1)}^2 + S_{\alpha,\mu_1,\mu}(u_u) \left(1 + \mathcal{O}(\varepsilon^{(N - 2)/(2\alpha)}) \right) \|v\|_{L^2(B_1)}^\alpha
\]
and
\[
\mathcal{O}(\varepsilon^{(N - 2)/(2\alpha)}) \|v\|_{L^2(B_1)}^\alpha.
\]

c) Since \( \lambda_{k,\mu} < \mu_1 \), from item b), we get \( M_\varepsilon \leq S_{\alpha,\mu_1,\mu}(u_u) \) for \( \varepsilon > 0 \) sufficiently small.

Now, by the Proposition 2.6, \( S_{\alpha,\mu_1,\mu}(u_u) < S_{\alpha} \) provided that either \( N \geq 4 + \mu \), or \( N < 4 + \mu \) and \( k \) is an integer large enough (hence \( \mu_1 \) is large).

\textbf{Remark 7.} Since \( \frac{N + \alpha}{N - 2} > 1 \) and \( \mu \geq \alpha > 0 \) it follows that \( \left(\frac{N + \alpha}{N - 2}\right)\mu > \alpha \) and consequently \( 2,\alpha > 2\alpha \).

The proof of the Proposition below follows arguments as in [31, prop 3.1] and it is analogous to the proof of the Proposition 4.2, so we will omit some details.

\textbf{Proposition 4.3.} Suppose \( \mu \geq \alpha > 0 \) and \( \mu_1 = \lambda_{k,\mu} \leq \mu_2 < \lambda_{k+1,\mu} \), for some integer \( k \geq 0 \).

a) \( M_\varepsilon \) is achieved by \( u_M \in I_\varepsilon \), characterized by
\[
u = v + P_k\bar{v} + t\bar{u}_u,
\]
where \( t > 0, \bar{u}_u = u_u - P_ku_u, u_u \) defined in (2.15), \( P_k \) denotes the projection operator of \( w \) on the direction \( \varphi_{k,\mu} \), that is,
\[
P_kw = \left( \int_{B_1} |x|^\mu w \varphi_{k,\mu} dx \right) \varphi_{k,\mu},
\]
and
\[
v = \sum_{i=1}^{k-1} \left( \int_{B_1} |x|^\mu (\bar{v} - tu_u) \varphi_{i,\mu} dx \right) \varphi_{i,\mu} \in V_{k-1,\mu}^- \quad \text{and} \quad \bar{v} \in V_{k,\mu}^-,
\]
where \( V_{k-1,\mu}^- := \text{span} \{ \varphi_{1,\mu}, \varphi_{2,\mu}, \ldots, \varphi_{k-1,\mu} \} \).
b) The following estimate holds for \( t > 0, \varepsilon > 0 \) small enough and some \( \sigma < \mu_1 - \lambda_{k-1,\mu} \)

\[
M \leq \left( \lambda_{k-1,\mu} - \mu_1 + \sigma \right) \|v\|_{L^2(B_1,|x|^\nu)}^2 + S_{\alpha, \lambda_{k-\mu}}(ue) \left( 1 + O(\varepsilon^{(N-2)/(2+\alpha)}) \right) + O(\varepsilon^{(N-2)/(2+\alpha)}). \tag{4.36}
\]

c) \( M < S_\alpha \), provided \( N > 4 + \mu \).

**Proof.** a) By the Weierstrass Theorem, (4.28) is achieved by a function \( u_M = \tilde{\nu} + t\varepsilon \in \mathcal{F}_\varepsilon \) that can be rewritten as

\[
u = \sum_{i=1}^{k-1} \left( \int_{B_1} |x|^\mu (\tilde{\nu} - tu) \varphi_{i,\mu} dx \right) \varphi_{i,\mu} \in V_{k-1,\mu}, \tag{4.37}
\]

and the map \( H_{0,\text{rad}} \ni z \mapsto P_k z \) denotes the projection of \( z \) on the direction \( \varphi_{k,\mu} \). Moreover, there exists a positive constant \( \bar{\kappa} \), independent of \( \varepsilon \), such that

\[
\int_{B_1} |x|^\mu \tilde{\nu} \varphi_{i,\mu} dx = \int_{B_1} |x|^\mu u \varphi_{i,\mu} dx \leq \bar{\kappa} \|v\|_{L^2(B_1,|x|^\nu)} \|u\|_{L^1(B_1,|x|^\nu)}, \tag{4.39}
\]

and

\[
|\langle \tilde{\nu}, v \rangle|_{H_{0,\text{rad}}(B_1)} \leq \bar{\kappa} \|v\|_{L^2(B_1,|x|^\nu)} \|u\|_{L^1(B_1,|x|^\nu)} \tag{4.40}
\]

for any \( \varepsilon > 0 \). Indeed, since \( u_M = \tilde{\nu} + t\varepsilon \), by the definition of \( z \) (as given in Proposition 4.2), it is easily seen that

\[
u_M = \sum_{i=1}^{k} \left( \int_{B_1} |x|^\mu \varphi_{i,\mu} dx \right) \varphi_{i,\mu} + t \left( u - \sum_{i=1}^{k} \left( \int_{B_1} |x|^\mu u \varphi_{i,\mu} dx \right) \varphi_{i,\mu} \right) \tag{4.38}
\]

\[
= \sum_{i=1}^{k-1} \left( \int_{B_1} |x|^\mu (\tilde{\nu} - tu) \varphi_{i,\mu} dx \right) \varphi_{i,\mu} + P_k \tilde{\nu} + t(u - P_k u) = v + P_k \tilde{\nu} + t\varepsilon,
\]

with \( v \) and \( \tilde{\nu} \) as in (4.37) and (4.38), respectively.

Let us start showing that (4.39) holds true. For this, note that \( v \) and \( P_k u \) are orthogonal in \( L^2(B_1,|x|^\mu) \), so that

\[
\int_{B_1} |x|^\mu \tilde{\nu} \varphi_{i,\mu} dx = \int_{B_1} |x|^\mu (u - P_k u) \varphi_{i,\mu} dx = \int_{B_1} |x|^\mu u \varphi_{i,\mu} dx,
\]

while the H"older inequality and the equivalence of the norm in a finite dimensional space give

\[
\int_{B_1} |x|^\mu \tilde{\nu} \varphi_{i,\mu} dx \leq \|u\|_{L^1(B_1,|x|^\nu)} \|v\|_{L^\infty(B_1)} \leq \bar{\kappa} \|u\|_{L^1(B_1,|x|^\nu)} \|v\|_{L^2(B_1,|x|^\nu)}
\]

for a suitable \( \bar{\kappa} > 0 \), independent of \( \varepsilon \). Thus, (4.39) is proved.
Now, let us show (4.40). At this purpose, we write

$$v = \sum_{i=1}^{k-1} v_i \varphi_{i,\mu}$$

(4.41)

for some $v_i \in \mathbb{R}$, so that $\|v\|^2_{L^2(B_1, |x|^\nu)} = \sum_{i=1}^{k-1} v_i^2$. By (4.41), the fact that $\varphi_{i,\mu}$ is an eigenfunction of $(-\Delta_i |x|^\mu)$ with eigenvalue $\lambda_{i,\mu}$, (that is, $-\Delta \varphi_{i,\mu} = \lambda_{i,\mu} |x|^\mu \varphi_{i,\mu}$) and the definition of scalar product in $H^k_0(B_1)$, we have

$$\langle \tilde{u}_\varepsilon, v \rangle_{H^k_0(B_1)} = \sum_{i=1}^{k-1} v_i \langle \tilde{u}_\varepsilon, \varphi_{i,\mu} \rangle_{H^k_0(B_1)}$$

$$= \sum_{i=1}^{k-1} \lambda_{i,\mu} v_i \langle \tilde{u}_\varepsilon, \varphi_{i,\mu} \rangle_{L^2(B_1, |x|^\nu)} = \sum_{i=1}^{k-1} \lambda_{i,\mu} v_i \langle u_\varepsilon, \varphi_{i,\mu} \rangle_{L^2(B_1, |x|^\nu)},$$

also thanks to the definition of $\tilde{u}_\varepsilon$ and the orthogonality properties of $\varphi_{i,\mu}$. So, by this and again the Hölder inequality, we get

$$\|\langle \tilde{u}_\varepsilon, v \rangle_{H^k_0(B_1)} \| \leq \sum_{i=1}^{k-1} \lambda_{i,\mu} |v_i| \|u_\varepsilon\|_{L^1(B_1, |x|^\nu)} \|\varphi_{i,\mu}\|_{L^\infty(B_1)}$$

$$\leq \tilde{\kappa} \|u_\varepsilon\|_{L^1(B_1, |x|^\nu)} \|v\|_{L^2(B_1, |x|^\nu)},$$

(4.42)

for a suitable $\tilde{\kappa} > 0$ possibly depending on $k$, but independent of $\varepsilon$. Hence, (4.40) is proved.

b) In doing this, we have to take into account that $u_M = v + P_k \tilde{v} + \tilde{u}_\varepsilon$ and that, in particular, we have to estimate three different contributions coming from $v$, $P_k \tilde{v}$ and $\tilde{u}_\varepsilon$. With respect to similar calculations carried on in Proposition 4.2, here we have to pay attention to the contribution coming from $v$, due to the resonance occurring in this case.

Let us show (4.36). Since $u_M = v + P_k \tilde{v} + \tilde{u}_\varepsilon$ we have that

$$M_\varepsilon = \|v + P_k \tilde{v} + \tilde{u}_\varepsilon\|_{H^k_{0,\text{rad}}(B_1)}^2 - \lambda_{k,\mu} \|v + P_k \tilde{v} + \tilde{u}_\varepsilon\|^2_{L^2(B_1, |x|^\nu)}$$

$$= \|v\|^2_{H^k_{0,\text{rad}}(B_1)} + \|P_k \tilde{v}\|^2_{H^k_{0,\text{rad}}(B_1)} + 2 \langle \tilde{u}_\varepsilon, v \rangle_{H^k_0(B_1)}$$

$$- \lambda_{k,\mu} \left( \|v\|^2_{L^2(B_1, |x|^\nu)} + \|P_k \tilde{v}\|^2_{L^2(B_1, |x|^\nu)} + 2 \langle \tilde{u}_\varepsilon, v \rangle_{L^2(B_1, |x|^\nu)} \right) - 2 \lambda_{k,\mu} u_\varepsilon \langle \tilde{u}_\varepsilon, v \rangle_{L^2(B_1, |x|^\nu)} + 2 \lambda_{k,\mu} t \langle \tilde{u}_\varepsilon, v \rangle_{L^2(B_1, |x|^\nu)},$$

(4.43)

thanks to the orthogonality properties of $v$, $P_k \tilde{v}$ and $\tilde{u}_\varepsilon$ and also to the definition of $\lambda_{k,\mu}$.

Now, note that by (4.38)

$$\|\tilde{u}_\varepsilon\|_{H^k_{0,\text{rad}}(B_1)}^2 - \lambda_{k,\mu} \|\tilde{u}_\varepsilon\|^2_{L^2(B_1, |x|^\nu)} = \|u_\varepsilon - P_k u_\varepsilon\|^2_{H^k_{0,\text{rad}}(B_1)} - \lambda_{k,\mu} \|u_\varepsilon - P_k u_\varepsilon\|^2_{L^2(B_1, |x|^\nu)}$$

$$= \|u_\varepsilon\|^2_{H^k_{0,\text{rad}}(B_1)} - \lambda_{k,\mu} \|u_\varepsilon\|^2_{L^2(B_1, |x|^\nu)} - 2 \langle P_k u_\varepsilon, u_\varepsilon \rangle_{L^2(B_1, |x|^\nu)}$$

$$= \|u_\varepsilon\|^2_{H^k_{0,\text{rad}}(B_1)} - \lambda_{k,\mu} \|u_\varepsilon\|^2_{L^2(B_1, |x|^\nu)},$$

(4.44)

thanks to the definition of $P_k$. 

Then, as proved in the previous Proposition we obtain (4.32), so combining (4.39), (4.40), (4.43) and (4.44), we get

\[ M_\varepsilon = \|v\|^2_{H^1_0(B_1)} + t^2\|\tilde{u}_\varepsilon\|^2_{H^1_0(B_1)} + 2t\tilde{u}_\varepsilon v + 2\lambda_{k,\mu}t\tilde{u}_\varepsilon^2v - 2\lambda_{k,\mu}t\tilde{u}_\varepsilon v + \lambda_{k,\mu}v^2 - \lambda_{k,\mu}v^2 - \lambda_{k,\mu}v^2 \]

\[ \leq \|v\|^2_{H^1_0(B_1)} + \lambda_{k,\mu}\|v\|^2_{L^2(B_1,|x|^\alpha)} + t^2\left(\|u\|^2_{H^1_0(B_1)} - \lambda_{k,\mu}\|u\|^2_{L^2(B_1,|x|^\alpha)}\right) \]

\[ + \tilde{k}\|v\|^2_{L^2(B_1,|x|^\alpha)} + \lambda_{k,\mu}\|v\|^2_{L^2(B_1,|x|^\alpha)} \]

(4.45)

provided \( \varepsilon > 0 \) is sufficiently small and for some \( \tilde{k} > 0 \), independent of \( \varepsilon \).

Since \( v \in V_{k-1,\mu} \), by (4.45), we have

\[ M_\varepsilon \leq \|v\|^2_{H^1_0(B_1)} + \lambda_{k,\mu}\|v\|^2_{L^2(B_1,|x|^\alpha)} + t^2\left(\|u\|^2_{H^1_0(B_1)} - \lambda_{k,\mu}\|u\|^2_{L^2(B_1,|x|^\alpha)}\right) \]

\[ + \tilde{k}\|v\|^2_{L^2(B_1,|x|^\alpha)} + \lambda_{k,\mu}\|v\|^2_{L^2(B_1,|x|^\alpha)} \]

(4.46)

if \( \varepsilon \) is small enough, where \( S_{\alpha,\lambda,\mu}(\cdot) \) is the function defined as

\[ H^1_0(B_1) \ni \alpha \mapsto S_{\alpha,\lambda,\mu}(\alpha) := \int_{B_1} |\nabla u|^2 + |x|^\alpha|u|^2 dx \]

(4.47)

Now, since \( u_M = tu_x + \tilde{w} \) with \( \tilde{w} := v + P_k \tilde{v} - tP_k u_x \), by the convexity\(^1\) of the map \( t \mapsto t^{2\alpha} \), by the monotonicity properties of the integrals, by (4.32) and the fact that in \( V_{k,\mu} \) all the norms are equivalent, just as in the proof of Proposition 4.2, we obtain

\[ 1 = \|u_M\|^2_{L^2(B_1,|x|^\alpha)} \geq \|tu_x\|^2_{L^2(B_1,|x|^\alpha)} + \tilde{c}\|u_x\|^2_{L^2(B_1,|x|^\alpha)} \]

for some positive constant \( \tilde{c} \), and consequently

\[ \|tu_x\|^2_{L^2(B_1,|x|^\alpha)} \leq 1 + \tilde{c}\|u_x\|^2_{L^2(B_1,|x|^\alpha)} \]

(4.48)

for \( \varepsilon \) sufficiently small. Moreover, by the Young inequality for any \( \sigma > 0 \) we have

\[ \tilde{k}\|v\|^2_{L^2(B_1,|x|^\alpha)} \leq \sigma\|v\|^2_{L^2(B_1,|x|^\alpha)} + \frac{\tilde{k}^2}{4\sigma} \|u\|^2_{L^1(B_1,|x|^\alpha)} \]

(4.49)

for any \( \varepsilon > 0 \). Hence, (4.46), (4.48) and (4.49) give, for \( \varepsilon \) small enough

\[ M_\varepsilon \leq \left(\lambda_{k-1,\mu} - \lambda_{k,\mu}\right)\|v\|^2_{L^2(B_1,|x|^\alpha)} + \sigma\|v\|^2_{L^2(B_1,|x|^\alpha)} + \frac{\tilde{k}^2}{4\sigma} \|u\|^2_{L^1(B_1,|x|^\alpha)} \]

\[ + S_{\alpha,\lambda,\mu}(u_x) \left(1 + \tilde{c}\|u_x\|^2_{L^2(B_1,|x|^\alpha)}\right) \]

\[ \leq \left(\lambda_{k-1,\mu} + \sigma\right)\|v\|^2_{L^2(B_1,|x|^\alpha)} + S_{\alpha,\lambda,\mu}(u_x) \left(1 + O(\varepsilon^{(N-2)/2(2+\alpha)})\right) \]

\[ + O(\varepsilon^{(N-2)/(2+\alpha)}). \]

(4.50)

\(^1\)If \( f \) is a differentiable convex function, then \( f(y) \geq f(x) + f'(x)(y - x) \). Here we take \( f(s) = \tilde{s}^{2\alpha}, x = tu_x \) and \( y = u_M = \tilde{w} + tu_x \).
Now, let us choose \( \sigma > 0 \) be such that \( \sigma < \lambda_{k,\mu} - \lambda_{k-1,\mu} \) (this choice is admissible since \( \lambda_{k,\mu} - \lambda_{k-1,\mu} > 0 \)). Then, (4.50) yields
\[
M_\varepsilon \leq S_{\alpha,\lambda_{k,\mu},\mu}(u_\varepsilon) \left( 1 + O(\varepsilon^{(N-2)/(2+\alpha)}) \right) + O(\varepsilon^{(N-2)/(2+\alpha)}) ,
\]
for \( \varepsilon \) small enough.

3. Now, the validity of the estimate
\[
M_\varepsilon = \|u_M\|_{H^1_0(B_1)}^2 - \lambda_{k,\mu} \|u_M\|_{L^2(B_1,|x|^\mu)}^2 < S_{\alpha}
\]
follows the same arguments as in Proposition 4.2 for the case \( N > 4 + \mu \).

4.1. End of the proof of Theorem 2.1. To complete the proof of the Theorem 2.1, we have to show that condition (4.1) is satisfied.

Proposition 4.4. The Linking critical level \( c \) of the functional \( I \) satisfies
\[
c < \frac{(\alpha + 2)}{2(N + \alpha)} \left( S_{\alpha} \right)^{\frac{N+\alpha}{4+\alpha}}.
\]

Proof. Notice that, for all \( h \in \Gamma \), we have
\[
c = \inf_{h \in \Gamma} \max_{(u,v) \in Q} I(h(u,v)) \leq \max_{(u,v) \in Q} I(h(u,v)).
\]

Let \( \mathbb{F}_\varepsilon \) as in Remark 6 with \( \varepsilon \) sufficiently small. Since \( Q \subset (\mathbb{F}_\varepsilon)^2 \), taking \( h = \text{id} \) and recalling that \( (\mathbb{F}_\varepsilon)^2 \) is a linear subspace, we obtain
\[
c = \inf_{h \in \Gamma} \max_{(u,v) \in Q} I(h(u,v)) \leq \max_{(u,v) \in Q} I((u,v)) \leq \max_{(u,v) \in \mathbb{F}_\varepsilon^2} I((u,v)) \leq \max_{(u,v) \in \mathbb{F}_\varepsilon^2} I(\eta(u,v)).
\]

We claim that
\[
\max_{(u,v) \in \mathbb{F}_\varepsilon^2} I(\eta(u,v)) < \frac{(\alpha + 2)}{2(N + \alpha)} \left( S_{\alpha} \right)^{\frac{N+\alpha}{4+\alpha}}.
\]

To verify the Claim, fixed \( U = (u,v) \in \mathbb{F}_\varepsilon^2 \) such that \( uv \neq 0 \), by (2.9), for all \( r \geq 0 \), we infer
\[
I(rU) \leq \frac{1}{2} \left( \|U\|_{L^2}^2 - \mu_1 \|U\|_{(L^2)^2(B_1,|x|^\mu)}^2 \right) - \frac{r^{2\alpha}}{2\alpha} \int_{B_1} |x|o(u(x)|p|^p v(x)|q^q + \xi_1 |u(x)|^{p+q} + \xi_2 |v(x)|^{p+q})dx
\]
\[
= Ar^2 - \frac{Br^{2\alpha}}{2\alpha} := g(r).
\]

Notice that \( r_0 = \left( \frac{A}{B} \right)^{1/(2\alpha-2)} \) is the maximum point of \( g(r) \), which maximum value is given by
\[
\max_{r \geq 0} I(rU) = \frac{(\alpha + 2)}{2(N + \alpha)} \left( \frac{A}{B^{2/\alpha}} \right)^{(N+\alpha)/(\alpha+2)}.
\]
Therefore, it is sufficient to show that

\[ \mathcal{M}_\epsilon := \max_{U \in (\mathbb{R}^+)^2} \left( \frac{\|U\|_Y^2 - \mu_1 \|U\|_{L^2(B_1,\|\cdot\|)}^2}{\left( \int_{B_1} |x|^\alpha (|u(x)|^p |v(x)|^q + \xi_1 |u(x)|^{p+q} + \xi_2 |v(x)|^{p+q}) \, dx \right)^{2/\gamma}} \right) \]

is achieved by \( u_M \in \mathcal{F}_\epsilon \).

Taking \( s_o, t_o > 0 \) as in Lemma 2.4 and \( u_M \) as in Proposition 4.3 and Proposition 4.2. Then \( \mathcal{M}_\epsilon \) is achieved by function \( U_M = (s_o u_M, t_o u_M) \). Therefore, from Proposition 4.3 and Proposition 4.2, and using (2.14), we can conclude that

\[ \widetilde{\mathcal{M}}_\epsilon = \frac{\|U_M\|_Y^2 - \mu_1 \|U_M\|_{L^2(B_1,\|\cdot\|)}^2}{\left( \int_{B_1} |x|^\alpha (|s_o u_M|^p |t_o u_M|^q + \xi_1 |s_o u_M|^{p+q} + \xi_2 |t_o u_M|^{p+q}) \, dx \right)^{2/\gamma}} \]

\[ = \frac{\left( s_o^2 + t_o^2 \right) \left( \|u_M\|_{H^1_{\text{rad}}(B_1)}^2 - \mu_1 \|u_M\|_{L^2(B_1,\|\cdot\|)}^2 \right)}{\left( \int_{B_1} |x|^\alpha |u_M|^2 \, dx \right)^{2/\gamma}} \]

\[ = m M_\epsilon < m S_\alpha = \hat{S}_\alpha, \]

if one of the following conditions hold

i) \( N > 4 + \mu \).

ii) \( N = 4 + \mu \) and \( \mu_1 \neq \lambda_{k,\mu}, \forall k \in \mathbb{N} \).

iii) \( N < 4 + \mu \) and \( k \in \mathbb{N} \) large enough (\( \mu_1 \) large) with \( \lambda_{k,\mu} \neq \mu_1 \).

This completes the proof. \( \square \)

5. **Proof of the Theorem 2.2.**

Now our goal is to get weak solutions to system (1.1) under hypothesis \( \lambda_{k-1,\mu} \leq \mu_1 < \lambda_{k,\mu} \leq \mu_2 < \lambda_{k+1,\mu} \), for some \( k \in \mathbb{N} \) \( (k \geq 1) \). We will show that the functional \( I \) has the geometric structure required by the Linking Theorem.

**Lemma 5.1.** Suppose \( p + q = 2_\alpha^* \) and \( \mu_1, \xi_1, \xi_2 > 0 \). If \( F \) be a finite dimensional subspace of \( Y(B_1) \), then there exists \( R > 0 \) large enough such that \( I(u, v) \leq 0 \), for all \( (u, v) \in F \) with \( ||(u, v)||_Y \geq R \).

**Proof.** See the Proposition 3.2 item (2). \( \square \)

**Proposition 5.2.** Assume that \( p + q = 2_\alpha^*, \xi_1, \xi_2 > 0 \), and that the following condition hold,

\[ \lambda_{k-1,\mu} \leq \mu_1 < \lambda_{k,\mu} \leq \mu_2 < \lambda_{k+1,\mu}, \text{ for some integer } k \geq 1. \] (5.1)
Then the functional $I$ satisfies:
1. There exist $\beta, \rho > 0$ such that $I(u, v) \geq \beta$ for all $(u, v) \in W^+_{k-1}$ with $\| (u, v) \|_Y = \rho$.
2. If $Q = (V^+_{k-1} \cap \overline{B}_R(0)) \oplus [0, R]E$, where $E \in W^+_{k-1} \cap \partial B_1(0)$ is a fixed vector, then $I(u, v) < 0$, for all $(u, v) \in \partial Q$ and $R > \rho$ large enough.

**Proof.** Considering the following subspaces $W^+_{k-1} = Z_k \oplus W^+_k$, where

$Z_k = \text{span}\{(\varphi_{k, \mu}, 0), (0, \varphi_{k, \mu})\}$ and $W^+_k = \text{span}\{(\varphi_{k+1, \mu}, 0), (0, \varphi_{k+1, \mu}), \ldots, \}^{d_{k, \text{rad}}}$,

we have that if $U \in W^+_{k-1}$, then $U = U^k + U$ with $U^k \in Z_k$ and $U \in W^+_k$.

Since

$$\| U \|_Y^2 = \| U^k \|_Y^2 + \| U \|_Y^2,$$

and

$$|u(x)|^p|v(x)|^q \leq |u(x)|^{p+q} + |v(x)|^{p+q},$$

by (2.9) we have

$$I(U) \geq \frac{1}{2} \left( \| U^k \|_Y^2 - \frac{\mu_2}{2} \| U \|_{(L^2)\alpha(B_1, \|x\|)}^2 + \| U \|_{(L^2)\alpha(B_1, \|x\|)}^2 \right) - \frac{C}{2} \left( \| U^k \|_Y^2 + \| U \|_Y^2 \right) \frac{z^2}{Y},$$

where $U = (u, v)$ and $C := C(\xi_1, \xi_2) > 0$ is a constant. Therefore, using that $U^k \in Z_k \subset W^+_{k-1}$ and $U \in W^+_k$, from Lemma 3.1 we obtain

$$\| U^k \|_{(L^2)\alpha(B_1, \|x\|)}^2 \leq \frac{1}{\lambda_{k, \mu}} \| U \|_Y^2$$

and

$$\| U \|_{(L^2)\alpha(B_1, \|x\|)}^2 \leq \frac{1}{\lambda_{k+1, \mu}} \| U \|_Y^2.$$

Consequently,

$$I(U) \geq \left( \frac{1}{2} \| U \|_Y^2 - \frac{\mu_2}{2} \| U \|_{(L^2)\alpha(B_1, \|x\|)}^2 \right) + \left( \frac{1}{2} \| U^k \|_Y^2 - \frac{\mu_2}{2} \| U \|_{(L^2)\alpha(B_1, \|x\|)}^2 \right) - \frac{C}{2} \left( \| U^k \|_Y^2 + \| U \|_Y^2 \right) \frac{z^2}{Y} \geq \left( 1 - \frac{\mu_2}{\lambda_{k+1, \mu}} \right) \| U \|_Y^2 + \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k, \mu}} \right) \| U^k \|_Y^2 - \frac{C}{2} \| U \|_{(L^2)\alpha(B_1, \|x\|)}^2 - C \| U \|_Y^2 \frac{z^2}{Y}. \quad (5.2)$$

Taking $\| U \|_Y = \rho$ small enough, since $\| U \|_Y^2 = \| U^k \|_Y^2 + \| U \|_Y^2$, we get that $\| U \|_Y := k(\rho)$ and $\| U \|_Y := z(\rho) = z$ are small enough. Now consider the function

$$\alpha(z) = \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k+1, \mu}} \right) z^2 + \frac{1}{2} \left( 1 - \frac{\mu_2}{\lambda_{k, \mu}} \right) (k(\rho))^2 - C \left( k(\rho) \right)^2 - C \| U \|_{Y}^2 \frac{z^2}{Y},$$

where $h(z) = \left( 1 - \frac{\mu_2}{\lambda_{k+1, \mu}} \right) z^2 - C \| U \|_{Y}^2$. By (5.1), the maximum value of $h(z)$, for $\rho$ sufficiently small, is given by

$$h := \left( \frac{1}{2\alpha C} \right)^{2\alpha-2} \left( 1 - \frac{\mu_2}{\lambda_{k+1, \mu}} \right) \left( \frac{2\alpha-2}{2\alpha} \right) > 0,$$

which is independent of $\rho$ and it is assumed at

$$\pi := \left[ \frac{1}{2\alpha C} \left( 1 - \frac{\mu_2}{\lambda_{k+1, \mu}} \right) \right]^{\frac{1}{2\alpha-2}}.$$
Therefore, it is possible to choose \( k(\rho) \) small enough, such that
\[
\alpha(z) = \frac{\lambda_1}{k(\rho)} - C k(\rho) \geq \frac{\lambda_1}{k} - (c + C) k(\rho) > 0,
\]
where \( c = \frac{1}{2} \left( \frac{\mu_0}{\lambda_{k+1}} - 1 \right) \geq 0 \) and \( C > 0 \).

Hence, by the estimate (5.2) and by the above information, for \( \|U\| \gamma = \rho \) small enough, there exists \( \beta > 0 \) such that \( I(U) \geq \beta \). This proves the statement (1).

Finally, to prove the item (2), we take \( U = (u, v) \in V_{k-1}^- \) to obtain
\[
(u, v) = \left( \sum_{i=1}^{k-1} u_i \varphi_{i, \mu}, \sum_{i=1}^{k-1} v_i \varphi_{i, \mu} \right), \quad \int_{B_1} |x|^\alpha |u|^2 dx = \sum_{i=1}^{k-1} u_i^2 \text{ and } \int_{B_1} |x|^\alpha |v|^2 dx = \sum_{i=1}^{k-1} v_i^2.
\]
Also
\[
\| (u, v) \|_Y^2 = \sum_{i=1}^{k-1} (u_i^2 + v_i^2) \lambda_{i, \mu}.
\]

Then, using (2.9), we are going to prove that \( I(U) < 0 \) on \( V_{k-1}^- \).

Let \( U = (u, v) \in V_{k-1}^- \), since \( \lambda_{k-1, \mu} \leq \mu_1 < \lambda_{k, \mu} \leq \mu_2 < \lambda_{k+1, \mu} \), we have that
\[
I(u, v) \leq \frac{1}{2} \sum_{i=1}^{k-1} (u_i^2 + v_i^2) \lambda_{i, \mu} - \frac{\mu_1}{2} \sum_{i=1}^{k-1} (u_i^2 + v_i^2) - \frac{1}{2^\alpha} \int_{B_1} |x|^\alpha |u|^p |v|^q + \xi_1 |u|^{p+q} \quad \text{and} \quad \xi_2 |v|^{p+q} dx
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{k-1} (u_i^2 + v_i^2) (\lambda_{i, \mu} - \mu_1) < 0.
\]

Now, to end the proof, it is enough to apply the Lemma 5.1 to the finite dimensional subspace \( V_{k-1}^- \oplus \text{span} \{ E \} \) containing \( Q = (V_{k-1}^- \cap \partial B(0)) \oplus \{ 0, R \} \), for some \( E \in W \cap \partial B(1) \) and \( R > \rho \).

**Proof of Theorem 2.2.** If \( \lambda_{k-1, \mu} \leq \mu_1 < \lambda_{k, \mu} \leq \mu_2 < \lambda_{k+1, \mu} \) for some integer \( k \geq 1 \), Lemma 5.1 and Proposition 5.2 ensure that the functional \( I \) satisfies the geometric structure required by the Linking Theorem. The proof of the \((PS)\) condition, in this case, is analogous to that made in Theorem 2.1, more exactly arguing as in [23, Theorem 1.3], it is sufficient to replace \( \lambda_{i, \mu} \) by \( \lambda_{k-1, \mu} \) in the Lemma 4.1 and Propositions 4.2 and 4.3.

**Proof of Theorem 2.3.** If \( 0 = \lambda_0, \mu_1 \leq \mu_2 < \lambda_{1, \mu} \), to show that the functional \( I \) satisfies the geometrical conditions of the Mountain Pass Theorem, it is enough to apply the Proposition 3.2, with \( W_k^+ = Y(B_1) \) and finite dimensional subspace \( \{ (0, 0) \} \oplus \text{span} \{ E \} \) for some \( E \in Y(B_1) \setminus \{ (0, 0) \} \) such that \( R \| E \|_Y > \rho \) with \( R > 0 \) sufficient large to ensure that \( I(RE) < 0 \). The \((PS)\) condition is guaranteed by making \( k = 0 \) in the Lemma 4.1 and Proposition 4.2. Therefore, there exists a non-trivial solution \( U \) for the problem (1.1). The positivity of the solution is obtained concluding that \( U_- = (u_-, v_-) = (0, 0) \) (by multiplying the first equation by \( u_- \), the second one by \( v_- \), where \( s_- = \min \{ s, 0 \} \) and by integrating on \( B_1 \) and applying the maximum principle.
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