An eigenvalue localization set for tensors and its applications

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Abstract
A new eigenvalue localization set for tensors is given and proved to be tighter than those presented by Li et al. (Linear Algebra Appl. 481:36-53, 2015) and Huang et al. (J. Inequal. Appl. 2016:254, 2016). As an application of this set, new bounds for the minimum eigenvalue of \( \mathcal{M} \)-tensors are established and proved to be sharper than some known results. Compared with the results obtained by Huang et al., the advantage of our results is that, without considering the selection of nonempty proper subsets \( S \) of \( N = \{1, 2, \ldots, n\} \), we can obtain a tighter eigenvalue localization set for tensors and sharper bounds for the minimum eigenvalue of \( \mathcal{M} \)-tensors. Finally, numerical examples are given to verify the theoretical results.

MSC: 15A18; 15A69; 65F10; 65F15

Keywords: \( \mathcal{M} \)-tensors; nonnegative tensors; minimum eigenvalue; localization set

1 Introduction
For a positive integer \( n, n \geq 2 \), \( N \) denotes the set \( \{1, 2, \ldots, n\} \). \( \mathbb{C} \) (respectively, \( \mathbb{R} \)) denotes the set of all complex (respectively, real) numbers. We call \( A = (a_{i_1 \ldots i_m}) \) a complex (real) tensor of order \( m \) dimension \( n \), denoted by \( \mathbb{C}^{[m,n]}(\mathbb{R}^{[m,n]}) \), if

\[
a_{i_1 \ldots i_m} \in \mathbb{C} (\mathbb{R}),
\]

where \( i_j \in N \) for \( j = 1, 2, \ldots, m \). \( A \) is called reducible if there exists a nonempty proper index subset \( \mathbb{I} \subset N \) such that

\[
a_{i_1 i_2 \ldots i_m} = 0, \quad \forall i_1 \in \mathbb{I}, \forall i_2, \ldots, i_m \notin \mathbb{I}.
\]

If \( A \) is not reducible, then we call \( A \) irreducible [3].

Given a tensor \( A = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]} \), if there are \( \lambda \in \mathbb{C} \) and \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C} \setminus \{0\} \) such that

\[
Ax^{m-1} = \lambda x^{[m-1]},
\]

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then \( \lambda \) is called an eigenvalue of \( A \) and \( x \) an eigenvector of \( A \) associated with \( \lambda \), where \( Ax^{m-1} \) is an \( n \) dimension vector whose \( i \)th component is

\[
(Ax^{m-1})_i = \sum_{i_2, \ldots, i_m \in \mathbb{N}} a_{i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}
\]

and

\[
x^{m-1} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T.
\]

If \( \lambda \) and \( x \) are all real, then \( \lambda \) is called an \( H \)-eigenvalue of \( A \) and \( x \) an \( H \)-eigenvector of \( A \) associated with \( \lambda \); see [4, 5]. Moreover, the spectral radius \( \rho(A) \) of \( A \) is defined as

\[
\rho(A) = \max \{ |\lambda| : \lambda \in \sigma(A) \},
\]

where \( \sigma(A) \) is the spectrum of \( A \), that is, \( \sigma(A) = \{ \lambda : \lambda \) is an eigenvalue of \( A \} \); see [3, 6].

A real tensor \( A \) is called an \( M \)-tensor if there exist a nonnegative tensor \( B \) and a positive number \( \alpha > \rho(B) \) such that \( A = \alpha I - B \), where \( I \) is called the unit tensor with its entries

\[
\delta_{i_1 \cdots i_m} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_m, \\ 0 & \text{otherwise}. \end{cases}
\]

Denote by \( \tau(A) \) the minimal value of the real part of all eigenvalues of an \( M \)-tensor \( A \). Then \( \tau(A) > 0 \) is an eigenvalue of \( A \) with a nonnegative eigenvector. If \( A \) is irreducible, then \( \tau(A) \) is the unique eigenvalue with a positive eigenvector [7–9].

Recently, many people have focused on locating eigenvalues of tensors and using obtained eigenvalue inclusion theorems to determine the positive definiteness of an even-order real symmetric tensor or to give the lower and upper bounds for the spectral radius of nonnegative tensors and the minimum eigenvalue of \( M \)-tensors. For details, see [1, 2, 10–14].

In 2015, Li et al. [1] proposed the following Brauer-type eigenvalue localization set for tensors.

**Theorem 1** ([1], Theorem 6) *Let \( A = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]} \). Then*

\[
\sigma(A) \subseteq \Delta(A) = \bigcup_{i \in \mathbb{N}, j \neq i} \Delta^j_i(A),
\]

*where*

\[
\Delta^j_i(A) = \{ z \in \mathbb{C} : |(z - a_{i_1 \cdots i})(z - a_{j_1 \cdots j}) - a_{i_1 \cdots i}a_{j_1 \cdots j}| \leq |z - a_{j_1 \cdots j}|r^j_i(A) + |a_{i_1 \cdots i}|r^j_i(A) \},
\]

\[
r^j_i(A) = \sum_{\delta_{i_2 \cdots i_m} = 0} |a_{i_2 \cdots i_m}|, \quad r^j_i(A) = \sum_{\delta_{i_2 \cdots i_m} = 0} |a_{i_2 \cdots i_m}| = r_i(A) - |a_{i_1 \cdots i}|.
\]

To reduce computations, Huang et al. [2] presented an \( S \)-type eigenvalue localization set by breaking \( \mathbb{N} \) into disjoint subsets \( S \) and \( \tilde{S} \), where \( \tilde{S} \) is the complement of \( S \) in \( \mathbb{N} \).
Theorem 2 ([2], Theorem 3.1) Let \( A = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]} \), \( S \) be a nonempty proper subset of \( N \), \( \bar{S} \) be the complement of \( S \) in \( N \). Then
\[
\sigma(A) \subseteq \Delta^S(A) = \left( \bigcup_{i \in S, j \in S} \Delta_i^j(A) \right) \cup \left( \bigcup_{i \in S, j \in \bar{S}} \Delta_i^j(A) \right).
\]

Based on Theorem 2, Huang et al. [2] obtained the following lower and upper bounds for the minimum eigenvalue of \( M \)-tensors.

Theorem 3 ([2], Theorem 3.6) Let \( A = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]} \) be an \( M \)-tensor, \( S \) be a nonempty proper subset of \( N \), \( \bar{S} \) be the complement of \( S \) in \( N \). Then
\[
\min \left\{ \min_{i \in S} \max_{j \in S} L_{ij}(A), \min_{i \in S} \max_{j \in \bar{S}} L_{ij}(A) \right\} \leq \tau(A) \leq \max \left\{ \max_{i \in S} \min_{j \in S} L_{ij}(A), \max_{i \in S} \min_{j \in \bar{S}} L_{ij}(A) \right\},
\]
where
\[
L_{ij}(A) = \frac{1}{2} \left\{ a_{i_1 \cdots i} + a_{j_1 \cdots j} - r_j^i(A) - \left[ (a_{i_1 \cdots i} - a_{j_1 \cdots j} - r_j^i(A))^2 - 4a_{i_1 \cdots j} \right] \right\}^{\frac{1}{2}}.
\]

The main aim of this paper is to give a new eigenvalue inclusion set for tensors and prove that this set is tighter than those in Theorems 1 and 2 without considering the selection of \( S \). And then we use this set to obtain new lower and upper bounds for the minimum eigenvalue of \( M \)-tensors and prove that new bounds are sharper than those in Theorem 3.

2 Main results

Now, we give a new eigenvalue inclusion set for tensors and establish the comparison between this set with those in Theorems 1 and 2.

Theorem 4 Let \( A = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]} \). Then
\[
\sigma(A) \subseteq \Delta^C(A) = \bigcup_{i \in N, j \notin i} \Delta_i^j(A).
\]

Proof For any \( \lambda \in \sigma(A) \), let \( x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n \setminus \{0\} \) be an associated eigenvector, i.e.,
\[
Ax^{m-1} = \lambda x^{[m-1]}.
\]
Let \( |x_p| = \max \{|x_i| : i \in N\} \). Then \( |x_p| > 0 \). For any \( j \in N, j \neq p \), then from (1) we have
\[
\lambda x_p^{m-1} = \sum_{\begin{smallmatrix} i_2, \ldots, i_m \in N, \delta_{i_2 \cdots i_m} = 0 \end{smallmatrix}} a_{p_1 \cdots i_m} x_{i_2} \cdots x_{i_m} + a_{p-\cdots-p} x_p^{m-1} + a_{p-\cdots-p} x_p^{m-1}
\]
and
\[
\lambda x_j^{m-1} = \sum_{\begin{smallmatrix} i_2, \ldots, i_m \in N, \delta_{i_2 \cdots i_m} = 0 \end{smallmatrix}} a_{p_1 \cdots i_m} x_{i_2} \cdots x_{i_m} + a_{p-\cdots-p} x_p^{m-1} + a_{p-\cdots-p} x_p^{m-1},
\]
equivalently,

\[
(\lambda - a_{j_{p-j}}) x_{j}^{m-1} - a_{j_{p-j}} x_{p}^{m-1} = \sum_{\delta_{j_{p-j}} = 0} \sum_{\delta_{j_{p-j}} = 0} a_{j_{p-j}} x_{j_{p-j}} 
\]

and

\[
(\lambda - a_{j_{p-j}}) x_{j}^{m-1} - a_{j_{p-j}} x_{p}^{m-1} = \sum_{\delta_{j_{p-j}} = 0} \sum_{\delta_{j_{p-j}} = 0} a_{j_{p-j}} x_{j_{p-j}} 
\]

Solving \( x_{p}^{m-1} \) from (2) and (3), we get

\[
(\lambda - a_{j_{p-j}}) (\lambda - a_{j_{p-j}} - a_{j_{p-j}} a_{j_{p-j}} p) x_{p}^{m-1} 
\]

\[
= (\lambda - a_{j_{p-j}}) \sum_{\delta_{j_{p-j}} = 0} \sum_{\delta_{j_{p-j}} = 0} a_{j_{p-j}} x_{j_{p-j}} + a_{j_{p-j}} \sum_{\delta_{j_{p-j}} = 0} \sum_{\delta_{j_{p-j}} = 0} a_{j_{p-j}} x_{j_{p-j}} 
\]

Taking absolute values and using the triangle inequality yields

\[
| (\lambda - a_{j_{p-j}}) | x_{p}^{m-1} | \leq | (\lambda - a_{j_{p-j}}) | x_{p}^{m-1} + | a_{j_{p-j}} | x_{p}^{m-1} 
\]

Furthermore, by \( |x_{p}| > 0 \), we have

\[
| (\lambda - a_{j_{p-j}}) | x_{p}^{m-1} | \leq | (\lambda - a_{j_{p-j}}) | r_{j_{p-j}}^{A} + | a_{j_{p-j}} | r_{j_{p-j}}^{A} 
\]

which implies that \( \lambda \in \Delta_{p}^{A}(\mathcal{A}) \). From the arbitrariness of \( j \), we have \( \lambda \in \bigcap_{j \in N} \Delta_{p}^{A}(\mathcal{A}) \). Furthermore, we have \( \lambda \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \Delta_{j_{p-j}}^{A}(\mathcal{A}) \). The conclusion follows. \( \square \)

Next, a comparison theorem is given for Theorems 1, 2 and 4.

**Theorem 5** Let \( A = (a_{i_{1}, \ldots, i_{m}}) \in \mathbb{C}^{[m,n]} \), \( S \) be a nonempty proper subset of \( N \). Then

\[
\Delta^{0}(A) \subseteq \Delta^{S}(A) \subseteq \Delta(A).
\]

**Proof** By Theorem 3.2 in [2], \( \Delta^{S}(A) \subseteq \Delta(A) \). Here, only \( \Delta^{0}(A) \subseteq \Delta^{S}(A) \) is proved. Let \( z \in \Delta^{0}(A) \), then there exists some \( i_{0} \in N \) such that \( z \in \Delta_{i_{0}}^{0}(A), \forall j \in N, j \neq i_{0} \). Let \( \tilde{S} \) be the complement of \( S \) in \( N \). If \( i_{0} \in S \), then taking \( j \in \tilde{S} \), obviously, \( z \in \bigcup_{i_{0} \in S, j \in \tilde{S}} \Delta_{i_{0}}^{A}(A) \subseteq \Delta^{S}(A) \). If \( i_{0} \in \tilde{S} \), then taking \( j \in S \), obviously, \( z \in \bigcup_{i_{0} \in \tilde{S}, j \in S} \Delta_{i_{0}}^{A}(A) \subseteq \Delta^{S}(A) \). The conclusion follows. \( \square \)

**Remark 1** Theorem 5 shows that the set \( \Delta^{0}(A) \) in Theorem 4 is tighter than those in Theorems 1 and 2, that is, \( \Delta^{0}(A) \) can capture all eigenvalues of \( A \) more precisely than \( \Delta(A) \) and \( \Delta^{S}(A) \).
In the following, we give new lower and upper bounds for the minimum eigenvalue of $M$-tensors.

**Theorem 6** Let $A = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible $M$-tensor. Then

$$\min_{i \in \mathbb{N}} \max_{j \neq i} L_{ij}(A) \leq \tau(A) \leq \max_{i \in \mathbb{N}} \min_{j \neq i} L_{ij}(A).$$

**Proof** Let $x = (x_1, x_2, \ldots, x_n)^T$ be an associated positive eigenvector of $A$ corresponding to $\tau(A)$, i.e.,

$$Ax^{m-1} = \tau(A)x^{m-1}. \quad (4)$$

(I) Let $x_q = \min \{x_i : i \in \mathbb{N} \}$. For any $j \in \mathbb{N}, j \neq q$, we have by (4) that

$$\tau(A)x_q^{m-1} = \sum_{\delta \phi_{i_2 \cdots i_m} = 0, \delta j_{i_2 \cdots i_m} = 0} a_{q_2 \cdots i_m}x_{i_2} \cdots x_{i_m} + a_{q_j \cdots q_{m-1}} + a_{q_j \cdots x_j^{m-1}},$$

and

$$\tau(A)x_j^{m-1} = \sum_{\delta q_{i_2 \cdots i_m} = 0, \delta j_{i_2 \cdots i_m} = 0} a_{q_2 \cdots i_m}x_{i_2} \cdots x_{i_m} + a_{j_{i_2 \cdots j}^{m-1}} + a_{j_{i_2 \cdots q_{m-1}}}.$$

equivalently,

$$\left(\tau(A) - a_{q_j \cdots j}x_j^{m-1} - a_{q_{i_2 \cdots i_m}}x_{i_2} \cdots x_{i_m}\right) = \sum_{\delta q_{i_2 \cdots i_m} = 0, \delta j_{i_2 \cdots i_m} = 0} a_{q_2 \cdots i_m}x_{i_2} \cdots x_{i_m}\quad (5)$$

and

$$\left(\tau(A) - a_{j_{i_2 \cdots j}}x_j^{m-1} - a_{q_{i_2 \cdots q_{m-1}}x_q}\right) = \sum_{\delta q_{i_2 \cdots i_m} = 0, \delta j_{i_2 \cdots i_m} = 0} a_{q_2 \cdots i_m}x_{i_2} \cdots x_{i_m}\quad (6)$$

Solving $x_q^{m-1}$ by (5) and (6), we get

$$\left(\left(\tau(A) - a_{q_j \cdots q}\right)\left(\tau(A) - a_{j_{i_2 \cdots j}} - a_{q_{i_2 \cdots q}x_q}\right)x_q^{m-1}\right)$$

$$= \left(\tau(A) - a_{j_{i_2 \cdots j}}\right)\sum\left(\delta q_{i_2 \cdots i_m} = 0, \delta j_{i_2 \cdots i_m} = 0\right) a_{q_2 \cdots i_m}x_{i_2} \cdots x_{i_m} + a_{q_{i_2 \cdots q}}\sum\left(\delta q_{i_2 \cdots i_m} = 0, \delta j_{i_2 \cdots i_m} = 0\right) a_{q_2 \cdots i_m}x_{i_2} \cdots x_{i_m}\quad (5)$$

From Theorem 2.1 in [9], we have $\tau(A) \leq \min_{i \in \mathbb{N}} a_{i_{i_2 \cdots i_m}}$ and

$$\left(\left(a_{q_j \cdots q} - \tau(A)\right)\left(a_{j_{i_2 \cdots j}} - \tau(A)\right) - a_{q_{i_2 \cdots q}x_q}\right)x_q^{m-1}\right)$$

$$= \left(a_{j_{i_2 \cdots j}} - \tau(A)\right)\sum\left(\delta q_{i_2 \cdots i_m} = 0, \delta j_{i_2 \cdots i_m} = 0\right) |a_{q_2 \cdots i_m}|x_{i_2} \cdots x_{i_m} + |a_{q_{i_2 \cdots q}}|\sum\left(\delta q_{i_2 \cdots i_m} = 0, \delta j_{i_2 \cdots i_m} = 0\right) |a_{q_2 \cdots i_m}|x_{i_2} \cdots x_{i_m}.$$
Hence,
\[
((a_{q-j} - \tau(A))(a_{j-j} - \tau(A)) - |a_{q-j}| |a_{j-j}|)x_p^{m-1} \\
\geq (a_{j-j} - \tau(A)) \sum_{\delta_{i2-\imath_m}=0} |a_{q_{i2-\imath_m}}| |a_{j-j}| \sum_{\delta_{j2-\imath_m}=0} |a_{j_{i2-\imath_m}}| \sum_{\delta_{q_{i2-\imath_m}=0}} |a_{q_{j_{i2}}}| |a_{j_{q_{i2}}}| |a_{j_{j_{i2}}}|
\]

From \(x_q > 0\), we have
\[
(a_{q-j} - \tau(A))(a_{j-j} - \tau(A)) - |a_{q-j}| |a_{j-j}| \\
\geq (a_{j-j} - \tau(A)) \sum_{\delta_{i2-\imath_m}=0} |a_{q_{i2-\imath_m}}| \sum_{\delta_{j2-\imath_m}=0} |a_{j_{i2-\imath_m}}| \\
= (a_{j-j} - \tau(A))r_q^j(A) + |a_{q-j}| r_j^j(A),
\]
equivalently,
\[
(a_{q-j} - \tau(A))(a_{j-j} - \tau(A)) - (a_{j-j} - \tau(A))r_q^j(A) - |a_{q-j}| r_j^j(A) \geq 0,
\]
that is,
\[
\tau(A)^2 - (a_{q-j} + a_{j-j} - r_q^j(A))\tau(A) + a_{q-j} a_{j-j} - a_{q-j} r_q^j(A) + a_{j-j} r_j^j(A) \geq 0.
\]
Solving for \(\tau(A)\) gives
\[
\tau(A) \leq \frac{1}{2} \left\{a_{q-j} + a_{j-j} - r_q^j(A) - \left[(a_{q-j} - a_{j-j} - r_q^j(A))^2 - 4a_{q-j} r_j^j(A)\right]^{1/2}\right\} = L_q^j(A).
\]
For the arbitrariness of \(j\), we have \(\tau(A) \leq \min_{j \neq q} L_q^j(A)\). Furthermore, we have
\[
\tau(A) \leq \max_{i \in N} \min_{j \neq i} L_q^j(A).
\]
(II) Let \(x_p = \max\{x_i : i \in N\}\). Similar to (I), we have
\[
\tau(A) \geq \min_{i \in N} \max_{j \neq i} L_q^j(A).
\]
The conclusion follows from (I) and (II).

Similar to the proof of Theorem 3.6 in [2], we can extend the results of Theorem 6 to a more general case.

**Theorem 7** Let \(A = (a_{i1-\imath_m}) \in \mathbb{R}^{[m,n]}\) be an \(M\)-tensor. Then
\[
\min_{i \in N} \max_{j \neq i} L_q^j(A) \leq \tau(A) \leq \max_{i \in N} \min_{j \neq i} L_q^j(A).
\]
By Theorems 3, 6 and 7 in [13], the following comparison theorem is obtained easily.
Theorem 8 Let $A = (a_{i_1\ldots i_m}) \in \mathbb{R}^{[m,n]}$ be an $\mathcal{M}$-tensor, $S$ be a nonempty proper subset of $N$, $\bar{S}$ be the complement of $S$ in $N$. Then

$$\min_{i \in N} R_i(A) \leq \min_{j \neq i} L_{ij}(A) \leq \min \left\{ \min_{i \in S} \max_{j \in \bar{S}} L_{ij}(A), \min_{i \in \bar{S}} \max_{j \in S} L_{ij}(A) \right\} \leq \min_{i \in N} \max_{j \neq i} L_{ij}(A)$$

where $R_i(A) = \sum_{i_2,\ldots,i_m \in N} a_{ii_2\ldots i_m}$.

Remark 2 Theorem 8 shows that the bounds in Theorem 7 are sharper than those in Theorem 3, Theorem 2.1 of [9] and Theorem 4 of [13] without considering the selection of $S$, which is also the advantage of our results.

3 Numerical examples

In this section, two numerical examples are given to verify the theoretical results.

Example 1 Let $A = (a_{ijk}) \in \mathbb{R}^{[3,4]}$ be an irreducible $\mathcal{M}$-tensor with elements defined as follows:

$$A(:,:,1) = \begin{pmatrix} 62 & -3 & -4 & -2 \\ -4 & -2 & -1 \\ -3 & -1 & -3 & -3 \\ -3 & -3 & -2 & -2 \end{pmatrix}, \quad A(:,:,2) = \begin{pmatrix} 0 & -4 & -3 & -3 \\ -1 & 28 & -2 & -2 \\ -1 & -2 & -2 & -4 \\ -2 & -2 & -3 & -1 \end{pmatrix},$$

$$A(:,:,3) = \begin{pmatrix} -2 & -1 & -2 & -2 \\ -1 & -1 & -2 \\ -2 & -4 & 63 & -4 \\ -4 & -4 & -2 & -2 \end{pmatrix}, \quad A(:,:,4) = \begin{pmatrix} -4 & -2 & -2 & -1 \\ -1 & -2 & -3 & -1 \\ -2 & -3 & -3 & -2 \\ -2 & -2 & -4 & 61 \end{pmatrix}.$$

By Theorem 2.1 in [9], we have

$$2 = \min_{i \in N} R_i(A) \leq \tau(A) \leq \min \{ \max_{i \in N} R_i(A), \min_{i \in N} a_{i-1} \} = 28.$$

By Theorem 4 in [13], we have

$$\tau(A) \geq \min_{j \neq i} L_{ij}(A) = 2.3521.$$

By Theorem 3, we have

if $S = \{1\}$, $\bar{S} = \{2,3,4\}$, \quad $3.6685 \leq \tau(A) \leq 24.2948$;

if $S = \{2\}$, $\bar{S} = \{1,3,4\}$, \quad $3.6685 \leq \tau(A) \leq 19.7199$;

if $S = \{3\}$, $\bar{S} = \{1,2,4\}$, \quad $2.3569 \leq \tau(A) \leq 27.7850$;

if $S = \{4\}$, $\bar{S} = \{1,2,3\}$, \quad $2.3521 \leq \tau(A) \leq 27.8536$;

if $S = \{1,2\}$, $\bar{S} = \{3,4\}$, \quad $2.3569 \leq \tau(A) \leq 27.7850$;

if $S = \{1,3\}$, $\bar{S} = \{2,4\}$, \quad $3.6685 \leq \tau(A) \leq 23.0477$;
if \( S = \{1, 4\}, \tilde{S} = \{2, 3\}, \) \( 3.6685 \leq \tau(A) \leq 23.9488. \)

By Theorem 7, we have

\[ 3.6685 \leq \tau(A) \leq 19.7199. \]

In fact, \( \tau(A) = 14.4049. \) Hence, this example verifies Theorem 8 and Remark 2, that is, the bounds in Theorem 7 are sharper than those in Theorem 3, Theorem 2.1 of [9] and Theorem 4 of [13] without considering the selection of \( S. \)

**Example 2** Let \( A = (a_{ijk}) \in \mathbb{R}^{[4,2]} \) be an \( M \)-tensor with elements defined as follows:

\[ a_{1111} = 6, \quad a_{1222} = -1, \quad a_{2111} = -2, \quad a_{2222} = 5, \]

other \( a_{ijk} = 0. \) By Theorem 7, we have

\[ 4 \leq \tau(A) \leq 4. \]

In fact, \( \tau(A) = 4. \)

**4 Conclusions**

In this paper, we give a new eigenvalue inclusion set for tensors and prove that this set is tighter than those in [1, 2]. As an application, we obtain new lower and upper bounds for the minimum eigenvalue of \( M \)-tensors and prove that the new bounds are sharper than those in [2, 9, 13]. Compared with the results in [2], the advantage of our results is that, without considering the selection of \( S, \) we can obtain a tighter eigenvalue localization set for tensors and sharper bounds for the minimum eigenvalue of \( M \)-tensors.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors contributed equally to this work. All authors read and approved the final manuscript.

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**References**

1. Li, CQ, Chen, Z, Li, YT: A new eigenvalue inclusion set for tensors and its applications. Linear Algebra Appl. 481, 36-53 (2015)
2. Huang, ZG, Wang, LG, Xu, Z, Cui, JJ: A new \( S \)-type eigenvalue inclusion set for tensors and its applications. J. Inequal. Appl. 2016, 254 (2016)
3. Chang, KQ, Zhang, T, Pearson, K: Perron-Frobenius theorem for nonnegative tensors. Commun. Math. Sci. 6, 507-520 (2008)
4. Qi, LQ: Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. 40, 1302-1324 (2005)
5. Lim, LH: Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing. CAMSAP, vol. 05, pp. 129-132 (2005)
6. Yang, YN, Yang, QZ: Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM J. Matrix Anal. Appl. 31, 2517-2530 (2010)
7. Ding, WY, Qi, LQ, Wei, YM: \( M \)-tensors and nonsingular \( M \)-tensors. Linear Algebra Appl. 439, 3264-3278 (2013)
8. Zhang, LF, Qi, LQ, Zhou, GL: \( M \)-tensors and some applications. SIAM J. Matrix Anal. Appl. 35, 437-452 (2014)
9. He, J, Huang, TZ: Inequalities for \( M \)-tensors. J. Inequal. Appl. 2014, 114 (2014)
10. Li, CQ, Li, YT, Kong, X: New eigenvalue inclusion sets for tensors. Numer. Linear Algebra Appl. 21, 39-50 (2014)
11. Li, CQ, Li, YT: An eigenvalue localization set for tensor with applications to determine the positive (semi-)definiteness of tensors. Linear Multilinear Algebra 64(4), 587-601 (2016)
12. Li, CQ, Jiao, AQ, Li, YT: An S-type eigenvalue location set for tensors. Linear Algebra Appl. 493, 469-483 (2016)
13. Zhao, JX, Sang, CL: Two new lower bounds for the minimum eigenvalue of $\mathcal{M}$-tensors. J. Inequal. Appl. 2016, 268 (2016)
14. He, J: Bounds for the largest eigenvalue of nonnegative tensors. J. Comput. Anal. Appl. 20(7), 1290-1301 (2016)