Limit state of a rectangular reinforced section using elastoplastic material strain diagrams

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Abstract. The paper considers the methodology of constructing the strength area of a rectangular reinforced section. The notion of "section strength area" is used in the calculations of structures according to the limit equilibrium. The peculiarity of the strength region is that inside the strength region the section works in the elastic stage, and at its boundary it transits to the limit state with the possibility of unrestricted plastic deformation. The equations describing the boundary of the section's strength region are often called yield conditions. In this paper, it is assumed that the material of which the section is made and the reinforcing material deform according to the ideal elastoplastic body law. Thus, the deformation diagrams of the materials are described by the Prandtl diagram. The material, of which the section is made, has different tensile and compressive yield strengths. The reinforcing material has the same tensile and compressive yield strengths. In deriving the equations describing the boundary of the section strength area, it was assumed that the bending moment and the longitudinal force applied in the center of the rectangle act in the section. Given that the section may have asymmetrical reinforcement, different equations are used to describe the upper and lower boundaries of the strength area. In order to construct the strength area, it is necessary to solve the optimization problem of finding the extreme value of the moment taking into account the constraints (equations and inequalities) for a given value of the longitudinal force. The analysis of results obtained in this way for a symmetrically reinforced cross-section made it possible to propose a simpler technique for constructing the strength area of a rectangular reinforced cross-section without solving the optimization problem.

1. Analysis of recent research
The problem of finding the destructive load for a structure was first formulated by Galileo Galilei. But only in the first half of the twentieth century was such a problem solved in the works of Gvozdev A. A. [1], Rzhanitsyn A. R. [2], Chiras A. A. [3], Pikovsky A. A. [4] and other researchers. In most of the works it was considered that the longitudinal force has little effect on the value of ultimate load and only bending moment was taken into account in the calculation. This is true for beam systems and, to some extent, for frame systems. As for the arch systems, the failure to consider the longitudinal forces leads to errors in determining the ultimate load [3], [5], [6].

The strength areas of unreinforced cross-sections of different shapes are presented in [3], [4], [5], [6]. In [7] the strength area of reinforced concrete rectangular section with symmetrical reinforcement is presented, in [8] the construction of strength area for rectangular section with single reinforcement is shown.

The equations describing the boundaries of the strength domain were used to compose an extreme problem in order to find the ultimate load. In [5], [6] the direct method [9] was proposed to find the ultimate load. This method has some advantages over other methods and makes it easy to compose a program for calculating the ultimate load.

2. The purpose of this work is to develop a methodology for constructing the strength domain of a reinforced rectangular section made of materials whose physical properties are described by a Prandtl diagram.
3. Materials and methods
A rectangular reinforced section made of elastoplastic materials is considered. The strength area of the section is constructed by solving an extreme problem using standard programs.

4. Research results
Consider an asymmetrically reinforced rectangular cross-section in the limiting state. In this study, it is assumed that the physical characteristics of the material of which the section is made and of the reinforcing material correspond to the characteristics of an ideal elastic-plastic body (Figure 1).

![Figure 1. Deformation diagrams: a) – sectional material, b) – reinforcement.](image)

The section is in equilibrium under the action of boundary forces $M_b$, $N_b$ and forces arising in the reinforcement and the material of the section (Figure 2). For this section, it is required to obtain the dependences describing the boundary of the strength area and to plot the strength area in the coordinates "longitudinal force - bending moment".

![Figure 2. Scheme of forces in a rectangular cross-section under the action of moment and longitudinal force.](image)

Write equilibrium equations:

$$\sum X = -b\sigma_y \left(h/2 - y\right) + b\sigma_y \left(h/2 + y\right) - N_s' + N_s - N_b = 0; \quad (1)$$

$$\sum m_o = M_b - 0.5b\sigma_y \left(h/2 - y\right)^2 - 0.5b\sigma_y \left(h/2 + y\right)^2 - N_s' \left(h/2 - y - a'\right)$$

$$- N_o \left(h/2 + y - a\right) + N_o y = 0. \quad (2)$$

From equation (1) we obtain the expressions for $y$ and for $N_b$:

$$y = \left[0.5h \left(\sigma_{y1} - \sigma_{y2}\right) + \frac{1}{b} \left(N_s' - N_s + N_b\right)\right] \left(\sigma_{y1} + \sigma_{y2}\right)^{-1}; \quad (3)$$
Using equation (2), we write down the expression for the boundary moment:

\[ M_b = 0.5b\sigma_y (h/2 - y)^2 + 0.5b\sigma_y (h/2 + y)^2 + N'_s (h/2 - y - a') + N_s (h/2 + y - a) - N_b y. \]  

(5)

The presented dependences (3) - (5) are used to construct the upper boundary of the section strength area. To construct the lower boundary of the section strength area, the dependences were obtained:

\[ y = \left[ 0.5h \left( \sigma_{cy} - \sigma_{oy} \right) + \frac{1}{b} \left( N'_s - N_s + N_b \right) \right] \left( \sigma_{cy} + \sigma_{oy} \right)^{-1}; \]  

(6)

\[ \sigma_s \leq \sigma_{oy}; \]

\[ \sigma'_s \leq \sigma_{oy}, \]

(7)

Where \( M_b \) is determined in accordance with (8).

In expressions (3) - (8) the unknowns are \( N_i, \sigma_s, \sigma'_s \). To determine the boundary torque \( M_b \) at a given value of the longitudinal force \( N_b \) we form an optimization problem (9), (10), where the target function is the moment \( M_b \), variable of design - stresses in the reinforcement \( \sigma_s \) and \( \sigma'_s \). Since the upper and lower bounds of the strength area are described by different expressions, we use the solution of problem (9) to construct the upper bound, and the solution of problem (10) to construct the lower bound.

Before solving problems (9), (10) it is necessary to determine the limits of change in the longitudinal force \( N_b \).

The rightmost point of the strength region corresponds to the state when the entire section is stretched. Since the reinforcement is asymmetrical, a moment must be applied to the section to achieve this state (Figure 3).
Write equilibrium equations:

\[ \sum X = bh\sigma_y + N_s - N'_s - N_{bs} = 0; \]
\[ \sum m_y = M_{bs} - 0.5bh^2\sigma_y - N_s(h-a) + N'_s a' + 0.5hN_{bs} = 0. \]

From the equilibrium equations we find:

\[ N_{bs} = bh\sigma_y + N_s - N'_s; \]
\[ M_{bs} = 0.5bh^2\sigma_y + N_s(h-a) - N'_s a' - 0.5hN_{bs} . \]

At the limit state \( N_s = A_s\sigma_{sy} \); \( N'_s = -A'_s\sigma_{sy} \). Because of this:

\[ N_{bs} = bh\sigma_y + \sigma_{sy}(A_s + A'_s); \]
\[ M_{bs} = 0.5bh^2\sigma_y + \sigma_{sy}[A_s(h-a) + A'_s(h-a')] - 0.5hN_{bs}. \] (11)

(12)

The leftmost point of the strength region corresponds to the state when the entire section is compressed. (Figure 4).

Write equilibrium equations:

\[ \sum X = -bh\sigma_y + N_s - N'_s - N_{bc} = 0; \]
\[ \sum m_y = M_{bc} - 0.5bh^2\sigma_y - N_s a - N'_s(h-a') - 0.5hN_{bc} = 0. \]

From the equilibrium equations we find:

\[ N_{bc} = -bh\sigma_y + N_s - N'_s; \]
\[ M_{bc} = 0.5bh^2\sigma_y + \sigma_{sy}[A_s(h-a) + A'_s(h-a')] + 0.5hN_{bc}. \]

At the limit state \( N_s = -A_s\sigma_{sy} \); \( N'_s = A'_s\sigma_{sy} \). Because of this:

\[ N_{bc} = -bh\sigma_y - \sigma_{sy}(A_s + A'_s); \]
\[ M_{bc} = 0.5bh^2\sigma_y + \sigma_{sy}[A_s(h-a) + A'_s(h-a')] - 0.5hN_{bc}. \] (13)

(14)
The upper and lower extremum points of the boundary of the strength area are defined at \( y = 0 \). From expressions (4) and (7) we find:

\[
N_b = -0.5bh\left(\sigma_{cy} - \sigma_{sy}\right) - N_s' + N_s'.
\]

From expressions (5) and (8) we find:

\[
M_{b,\text{max}} = 0.125bh^2\left(\sigma_{cy} + \sigma_{sy}\right) + N_s\left(h/2 - a\right) + N_s'(h/2 - a').
\]

The coordinates of the point of the upper extremum are determined at \( \sigma_s = \sigma_{sy} \); \( \sigma_s' = \sigma_{sy} \):

\[
N_b = -0.5bh\left(\sigma_{cy} - \sigma_{sy}\right) - \sigma_{sy}\left(A_s - A_s'\right);
\]

\[
M_{b,\text{max}} = 0.125bh^2\left(\sigma_{cy} + \sigma_{sy}\right) + \sigma_{sy}\left[A_s\left(h/2 - a\right) + A_s'\left(h/2 - a'\right)\right].
\]

The coordinates of the point of the lower extremum are determined at \( \sigma_s = -\sigma_{sy} \); \( \sigma_s' = -\sigma_{sy} \):

\[
N_b = -0.5bh\left(\sigma_{cy} - \sigma_{sy}\right) + \sigma_{sy}\left(A_s - A_s'\right);
\]

\[
M_{b,\text{min}} = 0.125bh^2\left(\sigma_{cy} + \sigma_{sy}\right) - \sigma_{sy}\left[A_s\left(h/2 - a\right) + A_s'\left(h/2 - a'\right)\right].
\]

4.1. Example 1

Construction of the Strength Area for a Rectangular Symmetrically Reinforced Section.

Consider a rectangular section with the characteristics: \( b \times h = 0.2 \times 1.0 \text{m} \), \( E_c = 2.3 \times 10^7 \text{kN/m}^2 \), \( \sigma_{cy} = 14500 \text{kN/m}^2 \), \( \sigma_{sy} = 1300 \text{kN/m}^2 \), \( E_{sy} = 2.1 \times 10^8 \text{kN/m}^2 \), \( \sigma_{sy} = 365000 \text{kN/m}^2 \), \( A' = A_s = 6.283 \text{cm}^2 \) (2020), \( a' = a = 0.03 \text{m} \).

Limits of variation of longitudinal force:

\[
N_{lc} = -\left(bh\sigma_{cy} + 2A_s\sigma_{sy}\right) = -3358.67 \text{kN} , \quad N_{ls} = bh\sigma_{sy} + 2A_s\sigma_{sy} = 718.67 \text{kN}.
\]

Coordinates of the extreme points of the section strength area are determined using the dependencies (15) – (18):

\[
N_b = -0.5bh\left(\sigma_{cy} - \sigma_{sy}\right) = -1320.0 \text{kN} ;
\]

\[
M_{b,\text{max}} = 0.125bh^2\left(\sigma_{cy} + \sigma_{sy}\right) + A_s\sigma_{sy}\left(h - a - a'\right) = 610.576 \text{kNm} ;
\]

\[
M_{b,\text{min}} = 0.125bh^2\left(\sigma_{cy} + \sigma_{sy}\right) - A_s\sigma_{sy}\left(h - a - a'\right) = -610.576 \text{kNm}.
\]

To solve optimization problems (9) and (10) we use EXCEL. The boundary of the strength area is shown in Figure 1. Moreover, the areas EG, BD, EP, RD are described by linear functions, and the areas GJB and PQR are described by quadratic functions.
Figure 5. Strength area of a rectangular section with symmetrical reinforcement.

Plots of stresses in the reinforcement and coordinates of the bending center of the section at $M_b > 0$ are presented in Figures 6, 7. It should be noted that these graphs are broken lines, the break points of which correspond to the same values of $N_b$.

Analyzing the graphs shown in Figures 6, 7, we can note their characteristic features, namely:

- on segment $EF$ $\sigma_x = -\sigma_{xy}$, $\sigma'_x = \sigma_{xy}$, $-(h/2 - a) \leq y \leq h/2$;
- on segment $FG$ $-\sigma_{xy} < \sigma_x < \sigma_{xy}$, $\sigma'_x = \sigma_{xy}$, $y = -(h/2 - a)$;
- on segment $GB$ $\sigma_x = \sigma'_x = -\sigma_{xy}$, $-(h/2-a) < y < (h/2-a')$;
- on segment $BC$ $\sigma_x = \sigma_{xy}$, $-\sigma_{xy} < \sigma'_x < \sigma_{xy}$, $y = h/2 - a'$;
- on segment $CD$ $\sigma_x = \sigma_{xy}$, $\sigma'_x = -\sigma_{xy}$, $(h/2-a') \leq y \leq h/2$.

Figure 6. Stresses in the reinforcement for the upper boundary of the section strength area with symmetrical reinforcement – $A_x$.

The coordinates of the break points of the graphs (Figures 6, 7), calculated by (4), (5) are presented in Table 1.

For the case where \( M_b < 0 \) (the lower boundary of the section strength area) the graphs of force changes in the reinforcement and the graph of bending center coordinate changes will be symmetrical to the graphs presented in Figures 6, 7 relative to the horizontal axis.

![Figure 7. Change in the coordinate of the bending center of a section with symmetrical reinforcement for the upper boundary of the section strength area.](image)

**Table 1.** The values of the limit forces at the break points of the graphs in Figures 6 and 7.

| Point | \( y \) | \( \sigma_x \) | \( \sigma_y \) | \( N_b \), kN | \( M_b \), kNm |
|-------|-------|-------|-------|---------|---------|
| E     | \(-h/2\) | \(-\sigma_{sy}\) | \(\sigma_{sy}\) | -3358.67 | 0       |
| F     | \(-(h/2-a)\) | \(-\sigma_{sy}\) | \(\sigma_{sy}\) | -3263.873 | 45.978  |
| G     | \(-(h/2-a)\) | \(\sigma_{sy}\) | \(\sigma_{sy}\) | -2805.200 | 261.554 |
| J     | 0      | \(\sigma_{sy}\) | \(\sigma_{sy}\) | -1320.0   | 610.576 |
| B     | \(h/2-a'\) | \(\sigma_{sy}\) | \(\sigma_{sy}\) | 165.200   | 261.554 |
| C     | \(h/2-a'\) | \(\sigma_{sy}\) | \(-\sigma_{sy}\) | 623.873   | 45.978  |
| D     | \(h/2\) | \(\sigma_{sy}\) | \(-\sigma_{sy}\) | 718.67    | 0       |

Taking into account that the coordinate of the bending center and the stresses in the reinforcement vary linearly, the boundary of the section strength area can also be constructed without solving problems (9), (10). The order of constructing the upper boundary of the section strength area is as follows:

– at a given value of the longitudinal force, we calculate the values of \( y, \sigma_x, \sigma'_x \);
– using expression (5), we calculate the boundary moment.

The lower boundary of the section strength area with symmetrical reinforcement is constructed by symmetrically mapping the upper boundary relative to the horizontal axis.

**4.2. Example 2**

Construction of the Strength Area for a Rectangular Asymmetrically Reinforced Section.
Consider the section from example 1 with reinforcement $A_s = 12.322 \text{cm}^2$ (2028), $A'_s = 6.283 \text{cm}^2$ (2020).

Dependencies (3) - (5) are used to construct the upper limit of the section strength area. First of all, we determine the coordinates of characteristic points on the diagrams of stresses change in reinforcement and change of coordinate of bending center of section (Table 2), (Figure 8, 9).

Table 2. The values of the limit forces at the break points of the graphs in Figures 8 and 9.

| Point | $y$          | $\sigma_s$ | $\sigma'_s$ | $N_b$, kN | $M_b$, kNm |
|-------|--------------|------------|-------------|-----------|------------|
| E     | $-h/2$       | $-\sigma_{sy}$ | $\sigma_{sy}$ | -3578.835 | -103.477   |
| F     | $-(h/2 - a)$ | $-\sigma_{sy}$ | $\sigma_{sy}$ | -3484.04  | -57.4985   |
| G     | $-(h/2 - a)$ | $\sigma_{sy}$ | $\sigma_{sy}$ | -2585.04  | 365.0306   |
| J     | 0            | $\sigma_{sy}$ | $\sigma_{sy}$ | -1099.84  | 714.0526   |
| B     | $h/2 - a'$   | $\sigma_{sy}$ | $\sigma_{sy}$ | 385.3628  | 365.0306   |
| C     | $h/2 - a'$   | $\sigma_{sy}$ | $-\sigma_{sy}$ | 844.0353  | 149.4545   |
| D     | $h/2$        | $\sigma_{sy}$ | $-\sigma_{sy}$ | 938.835   | 103.477    |

Figure 8. Stresses in the reinforcement for the upper boundary of the strength area of an asymmetrically reinforced section – $A_s$, $A'_s$. 
Figure 9. Change in the coordinate of the bending center of an asymmetrically reinforced section for the upper boundary of the strength region.

To construct the lower boundary of the section strength area, using dependencies (6) - (8) we calculate the coordinates of characteristic points on the graphs $\sigma$, $\sigma'$, $y$ (Table 3) and construct the corresponding graphs (Figures 10, 11).

Table 3. The values of the limit forces at the break points of the graphs shown in Figures 10 and 11.

| Point | $y$ | $\sigma$ | $\sigma'$ | $N_b$, kN | $M_b$, kNm |
|-------|-----|----------|-----------|----------|-----------|
| K     | $h/2$ | $-\sigma_y$ | $\sigma_y$ | -3578.835 | -103.477  |
| L     | $h/2 - a$ | $-\sigma_y$ | $\sigma_y$ | -3484.035 | -149.455  |
| P     | $h/2 - a$ | $-\sigma_y$ | $-\sigma_y$ | -3025.363 | -365.031  |
| Q     | 0 | $-\sigma_y$ | $-\sigma_y$ | -1540.163 | -714.053  |
| R     | $-(h/2 - a')$ | $-\sigma_y$ | $-\sigma_y$ | -54.963 | -365.031  |
| S     | $-(h/2 - a')$ | $\sigma_y$ | $-\sigma_y$ | 844.035 | 57.499 |
| T     | $-h/2$ | $\sigma_y$ | $-\sigma_y$ | 938.835 | 103.477 |
Figure 10. Stresses in the reinforcement for the lower boundary of the strength area with asymmetrical reinforcement – — $\sigma_s$, — $\sigma_{s'}$.

Having the graphs shown in Figures 8 - 11, you can easily build the boundary of the section strength area (Figure 12). For this purpose, using the available graphs (Figures 8 - 11) and Table 2, 3 for a given value of $N_b$, we determine $\sigma_s$, $\sigma_{s'}$, $y$ and use the appropriate formulas to calculate $M_b$.

Figure 11. Change in coordinate of bending center of section with asymmetrical reinforcement for the lower boundary of the strength area.
Similarly to the symmetrical reinforcement at the EG, KP, BD, RT sections, the boundary of the strength area is described by linear functions, and the upper and lower parts by quadratic functions.

The presence of equations describing the boundary of the section strength area allows us to solve the problem of finding the ultimate load for the structure using the method of limiting equilibrium [6].

5. Conclusions.
1. The boundary of the strength area of a rectangular reinforced section is described by linear and quadratic functions.
2. Extreme values of the boundary moments $M_{b,\text{max}}, M_{b,\text{min}}$ and the corresponding values of longitudinal forces $N_b$ both for symmetrical and asymmetrical reinforcement are determined by $y = 0$, i.e., when the bending center of the rectangular section is on the symmetry axis of the rectangle.
3. The reinforced section reaches the limit state when the whole section and the reinforcement (or one of the reinforcements) enter the plastic state.
4. Linear graphs of changes in the coordinate of the bending center of the section and stresses in the reinforcement allow us to construct the boundary of the area of strength of a rectangular reinforced section without solving optimization problems (9), (10).

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