KOSZULITY OF ALGEBRAS WITH NON-PURE RESOLUTIONS

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Abstract. We discuss certain homological properties of graded algebras whose trivial modules admit non-pure resolutions. Such algebras include both of Artin-Schelter regular algebras of types $(12221)$ and $(13431)$. Under certain conditions, a module with non-pure resolution is decomposed to form an extension by two modules with pure resolutions.

Introduction

Consider a (connected) graded algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$. Let $V \subset A$ be a minimal graded vector space which generates $A$, and $R \subset T(V)$ the minimal graded vector space which generates the relations of $A$. Then $A \cong T(V)/(R)$ where $(R)$ is the ideal generated by $R$, and the start of a minimal graded free resolution of the trivial left $A$-module $A_k$ is

$$\cdots \to A \otimes R \to A \otimes V \to A \to k \to 0.$$ 

More information is lurking in the subsequent terms of the resolution, and one may expect to understand for what they are standing by examining certain simple resolutions.

Perhaps the linear resolutions are the most suitable expecting. A well-known such example is Koszul algebras, introduced by Priddy in [12]. There are some generalizations on Koszul algebras. For example, $N$-Koszul algebras, motivated by the cubic Artin-Schelter regular algebras (AS-regular algebras, for short), were introduced by Berger in [1], recent developing on Calabi-Yau algebras offered a new class of $N$-Koszul algebras ([2, 3]). Almost Koszul algebras, grew out of finding periodic resolutions for the trivial extension algebras of the path algebras of Dynkin quivers, were defined in [4]. Several progresses have been given since then, such as stacked monomial algebras in [8] where the authors pointed out that all monomial algebras with pure resolutions are stacked monomial algebras, and piecewise-Koszul algebras in [10]. Note that all the objects above bear with pure resolutions, purity becomes a powerful homological tool. Our starting idea for writing this article is trying to understand the resolutions in some non-pure case.
There is an alternate generalization of the notion by using the duality of Koszul algebras. Denote $E(A)$ the Ext-algebra of $A$, Cassidy and Shelton recently introduced the notion of $K_2$ algebra in [5], for a connected graded algebra, by defining its Ext-algebra $E(A) = (E^1(A), E^2(A))$ as an algebra. Both Koszul algebras and $N$-Koszul algebras fall into the class of $K_2$ algebras. On the other hand, a piecewise-Koszul algebra, determined by a pair of parameters $(p, N)$ ($2 \leq p \leq N$), is $K_2$ if and only if the period $p = 2$ or $p = N$. Another remarkable example of $K_2$ algebras is the AS-regular algebras of type $(13431)$, and yet the AS-regular algebras of type $(12221)$ are not $K_2$.

The AS-regular algebras have been studied in many recent papers. In particular, four families of non-Koszul regular algebras were constructed in [9] by using the theory of $A_\infty$-algebras. In that paper, the AS-regular algebras of global dimension 4 generated in degree 1 are classified into the following three types according to the number of generators:

$$(12221) \quad 0 \rightarrow A(-7) \rightarrow A(-6)^{\oplus 2} \rightarrow A(-4) \oplus A(-3) \rightarrow A(-1)^{\oplus 2} \rightarrow A \rightarrow k \rightarrow 0,$$

$$(13431) \quad 0 \rightarrow A(-5) \rightarrow A(-4)^{\oplus 3} \rightarrow A(-3)^{\oplus 2} \oplus A(-2)^{\oplus 2} \rightarrow A(-1)^{\oplus 3} \rightarrow A \rightarrow k \rightarrow 0,$$

$$(14641) \quad 0 \rightarrow A(-4) \rightarrow A(-3)^{\oplus 4} \rightarrow A(-2)^{\oplus 6} \rightarrow A(-1)^{\oplus 4} \rightarrow A \rightarrow k \rightarrow 0.$$

The algebras of type $(14641)$ are Koszul. Our goal in this article is to introduce a new class of Koszul-type algebras, called bi-Koszul algebras, which includes both AS-regular algebras of types $(12221)$ and $(13431)$ as its objects. Such algebras are defined as non-pure analogue of the piecewise-Koszul algebras (see Definition 2.1).

We can connect the theory of the bi-Koszul algebras and their Koszul duality with the following, to be proved in Section 2.

**Theorem 0.1.** $A$ is a bi-Koszul algebra if and only if its Koszul dual $E(A)$ begins with $E^1(A) = E^1_1(A)$, $E^2(A) = E^2_1(A) \oplus E^2_{d+1}(A)$, $E^3(A) = E^3_{2d}(A)$, and $E^{3n}(A) = (E^3(A))^n$, $E^{3n+1}(A) = E^1(A)E^{3n}(A)$, $E^{3n+2}(A) \cong E^3(A)E^{3n}(A) \oplus E^2_{2nd+d+1}((J_{3^n}(k)))$ as $E^0(A)$-modules.

We consider a strongly bi-Koszul algebra $A$ with its Ext-algebra $E(A)$ being generated by $E^i(A)$ ($i = 0, 1, 2, 3$), and give a construction of such algebras.

The notion of (strongly) bi-Koszul modules is introduced in Section 3. One of main homological properties of bi-Koszul modules is (see the theorems 5.2 and 5.6):

**Theorem 0.2.** Let $A$ be a bi-Koszul algebra and $M \in Gr_0(A)$. We have:

1. If $E(M)$ is generated by $E^0(M)$, then $M$ is a bi-Koszul module.
2. If $M$ is a strongly bi-Koszul module, then $E(M)$ is generated by $E^0(M)$.

The resolutions of bi-Koszul modules are non-pure, which results an obstruction in the study of homological properties. A natural question is: what non-pure resolutions can be decomposed into pure resolutions. We discuss the question for a kind of decomposable modules in Section 4.
Theorem 0.3. Let $A$ be a connected noetherian graded algebra, and $M$ a decomposable bi-Koszul module (see Section 4). Then the resolution $Q$ of $M$ can be decomposed into a direct sum of two pure resolutions. Moreover, there exists a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ such that both $M'$ and $M''$ are $\delta$-Koszul modules.

1. Preliminaries

Throughout this article, $F$ is a fixed field. We always assume that $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is an associated graded $F$-algebra, where $k$ is a semisimple Artin algebra over $F$, $A_1$ is a finitely generated $F$-module, and $A$ is generated in degree $0$ and $1$. The graded Jacobson radical of $A$, denoted by $J$, is $J = A_1 \oplus A_2 \oplus \cdots$. Let $Gr(A)$ denote the category of graded left $A$-modules, and $gr(A)$ its full subcategory of finitely generated $A$-modules. The morphisms in these categories, denoted by $\text{Hom}_{Gr(A)}(M, N)$, are graded $A$-module maps of degree zero. We denote $Gr_0(A)$ and $gr_0(A)$ the full subcategories of $Gr(A)$ and $gr(A)$ whose objects are generated in degree $0$, respectively.

Let $M \in Gr(A)$, we denote the $n^{th}$ shift of $M$ by $M(n)$ where $M(n)_j = M_{j-n}$. In particular, for a graded algebra $A$, we use the notation established in $[5]$, setting

$$A(m_1, \ldots, m_s) := A(m_1) \oplus \cdots \oplus A(m_s).$$

We write $\mathcal{E}xt_A^*$ the derived functor of the graded $\text{Hom}_A^*$ functor

$$\text{Hom}_A^*(M, N) := \bigoplus_n \text{Hom}_{Gr(A)}(M, N(n)),$$

and denote

$$E(A) := \mathcal{E}xt_A^*(k, k), \quad E(M) := \mathcal{E}xt_A^*(M, k),$$

the Koszul dual (or Ext-algebra) of the algebra $A$ and the Koszul dual of the module $M \in Gr(A)$, respectively. $E(M)$ is a left $E(A)$-module by the Yoneda product as a bigraded space with the $(i, j)^{th}$ component $E_i^j(M) := \mathcal{E}xt_A^j(M, k)_j$. Similarly, $E(A)$ is a bigraded algebra with $E^0(A) = k$.

All modules bounded below in $Gr(A)$ have minimal graded projective resolutions. Thus, we fix

$$P : \ldots \rightarrow P_n \xrightarrow{d_n} \ldots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} k \rightarrow 0$$

a minimal graded projective resolution of $k$ over $A$, where $\ker d_n \subseteq JP_n$ for all $n \geq 0$. And assume

$$Q : \ldots \rightarrow Q_n \xrightarrow{\partial_n} \ldots \rightarrow Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\partial_0} M \rightarrow 0$$

is a minimal graded projective resolution of $M \in Gr(A)$, where $\ker \partial_n \subseteq JQ_n$ for all $n \geq 0$. Denote $\Omega^n(M) = \ker \partial_{n-1}$ the $n^{th}$ syzygy of $M$.

We say a graded module $N$ supported in $S$ if $N_i = 0$ for all $i \notin S$. Now, for each $n \geq 0$, $Q_n$ is graded by internal (or second) degrees, we write $Q_n = \oplus_i Q_{n,i}$. Let $Q_n$ be supported in $\{j \mid j \geq i\}$, then the following are clear:
(1) \( E^n(M) = \text{Hom}_A(Q_n, k) = \bigoplus_{j \geq 1} \text{Hom}_{Gr(A)}(Q_n, k(j)) \);

(2) there exists an integer \( s \geq i \) such that for any \( j \geq s \), \( Q_{n,j} = A_{j-s}Q_n,k \) if and only if \( \text{Hom}_{Gr(A)}(Q_n, k(j)) = 0 \).

From these facts, we have

**Lemma 1.1.** \( Q_n \) is generated in degrees \( i_1, i_2, \cdots, i_l \) if and only if \( E^n(M) \) is supported in \( \{i_1, i_2, \cdots, i_l\} \). \( \square \)

Since \( E^n(M) \) is bigraded for any \( M \in Gr(A) \), we say \( E^n(M) \) is *pure* in the sense that it is supported in a single second degree for any \( n \geq 0 \). A *pure resolution* means every projective module in a minimal projective resolution is generated in one degree; if not, we call it a *non-pure resolution*. The following result is well-known for pure parts in the resolution.

**Lemma 1.2** ([7] Proposition 3.6). Suppose that \( P_n \) is generated in single degree \( d_n \) for \( n = i, j, i + j \) satisfying \( d_i + d_j = d_{i+j} \). Then

\[
E^{i+j}(A) = E^i(A)E^j(A) = E^j(A)E^i(A)
\]

The Koszulity of algebras with pure resolutions has been studied in many papers (see, for example, [1, 4, 6, 7, 8, 10, 12]). We refer to [4] for the notion of the almost Koszul algebras, to [6] for the \( \delta \)-Koszul algebras, to [10] for the piecewise-Koszul algebras. As mentioned in the introductory section, the latter is related to a pair of integers \( (p, N) \), the parameter \( p \) shows the periodicity of generators’ degree distribution in the resolution, and the other one is related to the jumping degree. The definition agrees with the classical Koszul algebra when the period equals the jumping degree, and goes back to the \( N \)-Koszul algebra when the period \( p = 2 \). One may imply, on the other hand, that an almost Koszul algebra must be a piecewise-Koszul algebra, moreover, the two concepts become consistent if and only if \( A = A_0 \oplus A_1 \oplus \cdots \oplus A_q \) and \( N \geq p + q - 1 \). A criterion theorem for a graded algebra \( A \) to be piecewise-Koszul is that \( E(A) \) is generated by \( E^0(A), E^1(A), E^p(A) \), and \( E^p(A) = E^p_N(A) \).

By comparison, within our knowledge, we have less clue, except the recent paper [5], for the study of Koszulity of algebras whose trivial modules admitting non-pure resolutions.

### 2. Bi-Koszul Algebras

In this section, we give the definition of (strongly) bi-Koszul algebras and discuss their homological properties.

To meet the non-pure situation, we first introduce a *resolution map*

\[
\Delta : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}
\]
by

\[ \Delta(n) = \begin{cases} 
\frac{2}{3}(2d, 2d), & \text{if } n \equiv 0 \pmod{3}, \\
\frac{n-1}{3}(2d, 2d) + (1, 1), & \text{if } n \equiv 1 \pmod{3}, \\
\frac{n-2}{3}(2d, 2d) + (d, d+1), & \text{if } n \equiv 2 \pmod{3},
\end{cases} \]

where \( d \geq 2 \) is a fixed integer, and \( \mathbb{N} \times \mathbb{N} \) is a \( \mathbb{Z} \)-module with natural operations.

It is clear that the resolution map has a period of three; that is, for all \( n \geq 0 \),

\[ \Delta(n + 3) = \Delta(n) + \Delta(3). \]

For simplification, we use \( \Delta(n) \) to express both of its image \((x, y)\) and of the set \( \{x, y\} \), so \( \Delta(0) = \{0\} \), \( \Delta(1) = \{1\} \) and \( \Delta(2) = \{d, \ d + 1\} \).

**Definition 2.1.** A graded algebra \( A = k \oplus A_1 \oplus A_2 \oplus \cdots \) is called a bi-Koszul algebra if the trivial left \( A \)-module \( k \) admits a minimal graded projective resolution

\[ P : \cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow k \rightarrow 0, \]

such that each \( P_n \) is generated in degrees \( \Delta(n) \) for all \( n \geq 0 \).

Here are some examples of the bi-Koszul algebras.

**Example 2.2.** The AS-regular algebras of global dimension 4 of type \((13431)\) and \((12221)\) are the bi-Koszul algebras with taking \( d = 2 \) and \( d = 3 \), respectively.

**Example 2.3.** Let \( \Gamma \) be the following quiver:

\[
\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet \xrightarrow{\delta} \bullet \xrightarrow{\pi} \bullet \]

Set \( A = \mathbb{F}\Gamma/R \), where \( R \) is the ideal generated by the relations \( \alpha \beta, \beta \gamma \delta, \) and \( \delta \pi \). One can easily check that \( A \) is a bi-Koszul algebra.

One of effective approaches for studying the algebra \( A \) is to examine by its Ext-algebra \( E(A) \). The resolution \( P \) of a bi-Koszul algebra \( A \) contains pure and non-pure parts. It is easy to deal with the pure part by noting Lemma 1.2 and the periodicity of the resolution map \( \Delta \).

**Proposition 2.4.** Let \( A \) be a bi-Koszul algebra. Then

\[ E^{3n}(A) = E^3(A) \underbrace{E^3(A) \cdots E^3(A)}_n \]

and

\[ E^{3n+1}(A) = E^1(A)E^{3n}(A) = E^{3n}(A)E^1(A). \]

**Proof.** It is clear. \Box

Suppose \( A \) is a bi-Koszul algebra, we write

\[ E^{[P]}(A) := \bigoplus_{n \geq 0} E^{3n}(A) \bigoplus (\bigoplus_{n \geq 0} E^{3n+1}(A)), \]

and

\[ E^{[N]}(A) := \bigoplus_{n \geq 0} E^{3n+2}(A) \]
respectively, then $E(A)$ is decomposed into pure and non-pure parts as $E(A) = E^{[P]}(A) \oplus E^{[N]}(A)$.

**Corollary 2.5.** Assume $d > 2$. We have

1. $E^{[P]}(A)$ is a subalgebra of $E(A)$ which is generated by $E^0(A)$, $E^1(A)$ and $E^3(A)$.
2. $E^{[N]}(A)$ is a module over $E^{[P]}(A)$.

**Proof.** By noting the second degrees, one can easily check that $E^{3s+1}(A)E^{3t+2}(A) = E^{3s+2}(A)E^{3t+1}(A) = E^{3s+1}(A)E^{3t+1}(A) = 0$ for any integers $s, t \geq 0$. The result then follows from Lemma 1.1 and Proposition 2.4. \qed

For non-pure part of the resolution, there is an obstruction because of its non-purity. We quote two lemmas firstly.

**Lemma 2.6 [7] Lemma 3.2.** Let $M \in Gr(A)$ supported in $\{j \mid j \geq 0\}$ with the minimal graded projective resolution $Q$. Let $n \geq 1$, assume that $P_n$ is supported in $\{j \mid j \geq s\}$. Then $Q_n$ is supported in $\{j \mid j \geq s\}$. \qed

**Lemma 2.7.** Let $M \in Gr_0(A)$. For the natural induced map

$$E^n(M/JM) \xrightarrow{f} E^n(M) \xrightarrow{g} E^n(JM),$$

we have $\text{Im}(f) = E^n(A)E^0(M)$, the Yoneda product of $E^n(A)$ and $E^0(M)$.

**Proof.** The result is followed from the commutative diagram below:

$$
\begin{array}{ccc}
E^n(A) \otimes E^0(M/JM) & \rightarrow & E^n(A)E^0(M/JM) \\
\downarrow \cong & & \downarrow \cong \\
E^n(A) \otimes E^0(M) & \rightarrow & E^n(A)E^0(M) \\
\end{array}
\xrightarrow{f}
\begin{array}{ccc}
E^n(M/JM) & \rightarrow & E^n(M) \\
\downarrow f & & \downarrow \\
E^n(M) & \rightarrow & E^n(M)
\end{array}
$$

which can be found in [7]. \qed

**Proposition 2.8.** Let $A$ be a bi-Koszul algebra. Let $M \in Gr_0(A)$ with the minimal graded projective resolution $Q$. Assume that $Q_n$ is generated in degrees $\Delta(n)$. Then there exist $k$-module isomorphisms, for all $n \geq 0$

$$E^{3n+2}(M) \cong E^{3n+2}(A)E^0(M) \oplus E^{3n+2}_{2n+d+1}(JM).$$

**Proof.** Since $JM(-1) \in Gr_0(A)$, $E^n(JM)$ is supported in $\{j \mid j \geq \min \Delta(n) + 1\}$. Consider the graded short exact sequence

$$0 \rightarrow JM \rightarrow M \rightarrow M/JM \rightarrow 0.$$

Applying $\text{Ext}^*_A(-, k)$, we obtain a long exact sequence

$$0 \rightarrow E^0(M/JM) \rightarrow E^0(M) \rightarrow E^0(JM) \rightarrow E^1(M/JM) \rightarrow E^1(M) \rightarrow E^1(JM) \rightarrow E^2(M/JM) \rightarrow E^2(M) \rightarrow E^2(JM) \rightarrow E^3(M/JM) \rightarrow E^3(M) \rightarrow E^3(JM) \rightarrow E^4(M/JM) \rightarrow E^4(M) \rightarrow E^4(JM) \rightarrow E^5(M/JM) \rightarrow E^5(M) \rightarrow E^5(JM) \rightarrow \cdots.$$
Comparing the second degrees of all modules in the long exact sequence above, we have a series of exact sequences:

\[
0 \to \mathcal{E}^0(M/JM) \to \mathcal{E}^0(M) \to 0,
\]

\[
0 \to \mathcal{E}^1_0(JM) \to \mathcal{E}^1_1(M/JM) \to \mathcal{E}^1_1(M) \to 0,
\]

\[
0 \to \mathcal{E}^2_1(JM) \to \mathcal{E}^2_2(M/JM) \to \mathcal{E}^2_3(M) \to 0,
\]

\[
0 \to \mathcal{E}^2_{d+1}(JM) \to \mathcal{E}^2_{d+1}(M/JM) \to \mathcal{E}^2_{d+1}(M) \to 0,
\]

\[
0 \to \mathcal{E}^3_{d+1}(JM) \to \mathcal{E}^3_{d+1}(M/JM) \to \mathcal{E}^3_{d+1}(M) \to 0,
\]

\[
0 \to \mathcal{E}^3_{3d+1}(JM) \to \mathcal{E}^3_{3d+1}(M/JM) \to \mathcal{E}^3_{3d+1}(M) \to 0,
\]

\[
0 \to \mathcal{E}^5_{3d+1}(JM) \to \mathcal{E}^5_{3d+1}(M/JM) \to \mathcal{E}^5_{3d+1}(M) \to 0,
\]

\[
0 \to \mathcal{E}^5_{4d}(JM) \to \mathcal{E}^5_{4d}(M/JM) \to \mathcal{E}^5_{4d}(M) \to 0,
\]

\[
\cdots \cdots \cdots .
\]

Hence, by Lemma 2.7 there are exact sequences:

\[
0 \to \mathcal{E}^{3n+2}_{2nd+d}(A)\mathcal{E}^0(M) \to \mathcal{E}^{3n+2}_{2nd+d}(M) \to 0,
\]

and

\[
0 \to \mathcal{E}^{3n+2}_{2nd+d+1}(A)\mathcal{E}^0(M) \to \mathcal{E}^{3n+2}_{2nd+d+1}(M) \to \mathcal{E}^{3n+2}_{2nd+d+1}(JM) \to 0,
\]

for any \( n \geq 0 \). Note that the second short exact sequence is split as \( k \)-modules, and both \( \mathcal{E}^{3n+2}(A) \) and \( \mathcal{E}^{3n+2}(M) \) are supported only in \( \Delta(3n+2) \), so we complete the proof. \( \square \)

We may now prove Theorem 0.1 which we restate:

**Theorem 2.9.** The following statements are equivalent:

1. \( A \) is a bi-Koszul algebra;
2. \( E(A) \) begins with \( E^1(A) = E^1_1(A) \), \( E^2(A) = E^2_2(A) \oplus E^2_{d+1}(A) \), \( E^3(A) = E^3_{2d}(A) \), and for each \( n \geq 1 \),
   - (a) \( E^{3n}(A) = E^3(A) E^3(A) \cdots E^3(A) \),
   - (b) \( E^{3n+1}(A) = E^1(A) E^{3n}(A) \),
   - (c) \( E^{3n+2}(A) \cong E^2(A) E^{3n}(A) \oplus E^2_{2nd+d+1}(J\Omega^{3n}(k)) \) as \( k \)-modules.

**Proof.** Suppose we have (2), it is easy to see that \( P_1 \) is \( \Delta(1) \)-generated, \( P_2 \) is \( \Delta(2) \)-generated and \( P_3 \) is \( \Delta(3) \)-generated. For all \( n \geq 1 \), \( P_{3n} \) and \( P_{3n+1} \) are generated in degree \( \Delta(3n) \) and \( \Delta(3n+1) \) from the conditions (a) and (b), respectively. On the other hand, by the condition (c) and Proposition 2.8 we get

\[
E^{3n+2}(A) = E^2(\Omega^{3n}(k)) \cong E^2(A) E^0(\Omega^{3n}(k)) \oplus E^2_{2nd+d+1}(J\Omega^{3n}(k)).
\]

\( E^2(A) \) is supported in \( \{d,d+1\} \), and \( \Omega^{3n}(k) \) is \( 2nd \)-generated, so \( E^{3n+2}(A) \) is supported in \( \{2nd+d,2nd+d+1\} \); that is, \( P_{3n+2} \) is \( \Delta(3n+2) \)-generated. This proves that \( A \) is a bi-Koszul algebra.
Conversely, due to Lemma 1.1 and Proposition 2.4, we need only to show that the last isomorphism is true for each \( n \geq 1 \).

Since \( E^{3n+2}(A) = E^2(\Omega^{3n}(k)) \) and \( \Omega^{3n}(k) \) is 2\text{d}-generated, we have the following minimal resolution of \( \Omega^{3n}(k)(-2nd) \in Gr_0(A) \):

\[
\cdots \to P_{3n+2}(-2nd) \to P_{3n+1}(-2nd) \to P_{3n}(-2nd) \to \Omega^{3n}(k)(-2nd) \to 0.
\]

Denote \( M = \Omega^{3n}(k)(-2nd) \). By Proposition 2.8 we know that

\[
E^2(M) \cong E^2(A)E^0(M) \oplus E^2_{2nd+1}(JM).
\]

Therefore,

\[
E^{3n+2}(A) \cong E^2(A)E^{3n}(A) \oplus E^2_{2nd+d+1}(J\Omega^{3n}(k)).
\]

\[\square\]

Remark 2.1. The obstruction \( E^2_{2nd+d+1}(J\Omega^{3n}(k)) \) in Theorem 2.9 arises from the bigger degree in \( \Delta(3n + 2) \). It is vanished in the previous examples.

Definition 2.10. We call a bi-Koszul algebra \( A \) strongly if the obstruction is vanished.

Obviously, a strongly bi-Koszul algebra has nice homological properties, for example, the Koszul dual \( E(A) \) of such an algebra is generated by \( E^0(A), E^1(A), E^2(A) \) and \( E^3(A) \). The following gives a criterion for a bi-Koszul algebra to be strongly by syzygies.

Proposition 2.11. Suppose that \( A \) is a bi-Koszul algebra. Then \( A \) is strongly if and only if \( \Omega^2(J\Omega^{3n}(k)) \) and \( \Omega^{3n+3}(k) \) are generated in the same degrees for all \( n \geq 1 \).

Proof. Assume \( \Omega^{3n+3}(k) \) and \( \Omega^2(J\Omega^{3n}(k)) \) are generated in the same degrees. Note that now \( \Omega^{3n+3}(k) \) is \((2nd + 2d)\) generated, so does \( \Omega^2(J\Omega^{3n}(k)) \). That means \( E^2_{2nd+d+1}(J\Omega^{3n}(k)) = 0 \). So \( A \) is strongly bi-Koszul.

Conversely, from the proofs of Proposition 2.8 and Theorem 2.9 the assumption of \( E^2_{2nd+d+1}(J\Omega^{3n}(k)) = 0 \) concludes that \( E^2(J\Omega^{3n}(k)) \) is supported in \( \{2nd + 2d\} \). Since \( \Omega^{3n+3}(k) \) is \((2nd + 2d)\) generated, we get \( \Omega^2(J\Omega^{3n}(k)) \) and \( \Omega^{3n+3}(k) \) are generated in the same degrees.

\[\square\]

We will give another criterion for a bi-Koszul algebra being strongly next section, and will provide a method of constructing a strongly bi-Koszul algebra in the last section.

3. Bi-Koszul Modules

We mentioned a kind of modules whose degree distribution of generators in the resolution obeys some resolution map \( \Delta \) in Proposition 2.8. In this section, we turn to study such modules which are defined over the bi-Koszul algebras.
Definition 3.1. Let $A$ be a bi-Koszul algebra. Assume $M \in \text{Gr}(A)$, we call $M$ a bi-Koszul module if it has a minimal graded projective resolution of the form

$$Q: \cdots \to Q_n \xrightarrow{\partial_n} \cdots \to Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\partial_0} M \to 0,$$

in which each $Q_n$ is generated in degrees $\Delta(n)$.

Remark 3.1. If $M$ is a bi-Koszul module, then $M \in \text{Gr}_0(A)$ and for each $n \geq 0$, $\Omega^2(M)(-2nd)$ is also a bi-Koszul module.

Theorem 3.2. Let $A$ be a bi-Koszul algebra, $M \in \text{Gr}_0(A)$. Then we have:

1. if $E(M)$ is generated in degree 0 as an $E(A)$-module, then $M$ is a bi-Koszul module;
2. if $M$ is a bi-Koszul module, then $E(M)$ is generated in degree 0 if and only if $\Omega^2(JM)(-2d)$ is also a bi-Koszul module.

Proof. (1) By the assumption, for each $n \geq 0$, $E^n(M) = E^n(A)E^0(M)$, so

$$E^n(M) = E^n(A)E^0(M) = E^n_{A(n)}(M).$$

Since $A$ is a bi-Koszul algebra, the result follows by Lemma 1.1.

(2) When $M$ is a bi-Koszul module, the proof of Proposition 2.8 shows

$$E(JM) = E_0^0(JM) \oplus E_1^1(JM) \oplus E_2^2(JM) \oplus E_3^3(JM) \oplus E_4^4(JM) \oplus E_5^5(JM) \oplus E_6^6(JM) \oplus \cdots.$$ 

By Lemma 2.7, for all $n \geq 0$, $E^n(M) = E^n(A)E^0(M)$ if and only if the induced map $E(M/JM) \to E(M)$ is an epimorphism, which is equivalent to $E^n_{A(n)}(JM) = E^n_3(JM) = \cdots = 0$. So we have $E^n(M) = E^n(A)E^0(M)$ (for all $n \geq 0$) if and only if $\Omega^2(JM)(-2d)$ is also a bi-Koszul module.  

We discuss some conditions under which $\Omega^2(JM)(-2d)$ becomes a bi-Koszul module.

Firstly, we consider a kind of special bi-Koszul modules.

Definition 3.3. Let $A$ be a bi-Koszul algebra, $M \in \text{Gr}_0(A)$ a bi-Koszul module. We call $M$ strongly if, for all $n \geq 0$, $J\Omega^{3n+2}(M) = J\Omega^{3n+2}(M/JM) \cap \Omega^{3n+2}(M)$.

The following two lemmas are well-known.

Lemma 3.4. For any graded algebra $A$ with Jacobson radical $J$. Let $0 \to K \to M \to N \to 0$ be a short exact sequence in $\text{Gr}(A)$ such that $JK = K \cap JM$, then
there exists a commutative diagram:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \Omega(K) & \Omega(M) & \Omega(N) & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & Q & R & L & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & K & M & N & 0 \\
\end{array}
\]

where \( Q \to K, R \to M \) and \( L \to N \) are projective covers.

\[\Box\]

Lemma 3.5. For any graded algebra \( A \), let \( 0 \to K \to M \to N \to 0 \) be a short exact sequence in \( \text{Gr}(A) \). If \( K, M \) and \( N \) are generated in one same degree, then \( JK = K \cap JM \).

\[\Box\]

Theorem 3.6. Let \( A \) be a bi-Koszul algebra. Assume that \( M \in \text{Gr}_0(A) \) is a strongly bi-Koszul module. Then \( \Omega^2(JM)(-2d) \) is a bi-Koszul module. Hence, \( E(M) \) is generated in degree 0 as an \( E(A) \)-module.

Proof. Clearly, \( M/JM \) is also a bi-Koszul module over \( A \). For the short exact sequence

\[0 \to JM \to M \to M/JM \to 0,
\]

we have the commutative diagram as follows:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 \to \Omega^1(M) & \Omega^1(M/JM) & JM \to 0 \\
\downarrow & \downarrow \\
Q_0 \to Q_0 \\
\downarrow & \downarrow \\
0 \to JM & M & M/JM \to 0 \\
\downarrow & 0.
\end{array}
\]

Note that \( M \) is strongly bi-Koszul, starting from the short exact sequence

\[0 \to \Omega^1(M) \to \Omega^1(M/JM) \to JM \to 0,
\]

we may get projective covers sheaf by sheaf from the lemmas 3.4 and 3.5. Hence, for all \( n \geq 1 \),

\[0 \to \Omega^n(M) \to \Omega^n(M/JM) \to \Omega^{n-1}(JM) \to 0
\]

are exact, where \( \Omega^n(M), \Omega^n(M/JM) \) and \( \Omega^{n-1}(JM) \) are all generated in the same degrees. So \( \Omega^2(JM)(-2d) \) is a bi-Koszul module by comparing with degrees. \(\Box\)

The following result is evident from the proof of Theorem 3.6.
Corollary 3.7. Under the condition of Theorem 3.6, we have, for all \( n \geq 1 \),
\[
0 \to E^{n-1}(JM) \to E^n(M/JM) \to E^n(M) \to 0
\]
are exact, where \( E^{n-1}(JM) \), \( E^n(M/JM) \) and \( E^n(M) \) are all supported in the same degrees.

The following proposition gives another criterion for a bi-Koszul algebra to be strongly by the notion of bi-Koszul modules.

Proposition 3.8. Let \( A \) be a bi-Koszul algebra. Assume, for all \( n \geq 1 \), \( \Omega^{3n}(k)(-2nd) \) are strongly bi-Koszul modules. Then \( A \) is a strongly bi-Koszul algebra.

Proof. By Proposition 2.4, we know that
\[
E^{3n}(A) = E^3(A)E^3(A) \cdots E^3(A)
\]
and
\[
E^{3n+1}(A) = E^1(A)E^{3n}(A).
\]
Denote \( M = \Omega^{3n}(k)(-2nd) \), which is a strongly bi-Koszul module. So by Theorem 3.6, we get
\[
E^2(M) = E^2(A)E^0(M);
\]
that is,
\[
E^{3n+2}(A) = E^2(A)E^{3n}(A).
\]
Thus, we complete the proof. \( \square \)

We end this section with a result about the relations between bi-Koszul modules and Koszul modules.

Denote \( E^{[0]}(M) = \oplus_{n \geq 0} E^{3n}(M) \), and \( E^{[0]}(A) = \oplus_{n \geq 0} E^{3n}(A) \). Clearly, \( E^{[0]}(M) \) is a left \( E^{[0]}(A) \)-module. Denote \( M_0 := M \) and \( M_n := \Omega^2(JM_{n-1})(-2d) \) for \( n \geq 1 \).

Proposition 3.9. Let \( A \) be a bi-Koszul algebra. Assume that all \( M_n \ (n \geq 0) \) are strongly bi-Koszul modules. Then \( E^{[0]}(M) \) is a Koszul module over \( E^{[0]}(A) \).

Proof. By Corollary 3.7, there exist short exact sequences:
\[
0 \to E^{3n-1}(JM) \to E^{3n}(M/JM) \to E^{3n}(M) \to 0
\]
for all \( n \geq 0 \) (we have denoted \( E^{-1}(JM) = 0 \)). Hence
\[
0 \to \bigoplus_{n \geq 1} E^{3n-3}(\Omega^2JM) \to \bigoplus_{n \geq 0} E^{3n}(M/JM) \to \bigoplus_{n \geq 0} E^{3n}(M) \to 0;
\]
that is,
\[
0 \to \bigoplus_{n \geq 0} E^{3n}(\Omega^2JM)(-2d)(-1) \to \bigoplus_{n \geq 0} E^{3n}(M/JM) \to \bigoplus_{n \geq 0} E^{3n}(M) \to 0.
\]
Since \( \Omega^2JM \) is a strongly bi-Koszul module, \( \bigoplus_{n \geq 0} E^{3n}(\Omega^2JM)(-2d)(-1) = \Omega^1(E^{[0]}(M)) \) is 1-generated by Theorem 3.6. Continuing this fashion, we see that \( E^{[0]}(M) \) has a linear resolution. The result now follows. \( \square \)
4. Decomposition of Resolutions

In this section, we discuss another topic that whether a minimal resolution of a bi-Koszul module can be decomposed into two pure resolutions. We end the article by providing a method of constructing the strongly bi-Koszul algebras.

From now on, we assume that the graded algebra $A$ is noetherian and connected, here “connected” means $k$ itself is a field. All modules are considered finitely generated graded free $A$-modules with their homogeneous bases. Denote $|x|$ the degree of a homogeneous element $x \in M$ for $M \in \text{gr}(A)$.

Using the notation recalled in the preliminary section, we write graded free $A$-modules as

$$M = A(-m_1, \ldots, -m_s), \quad N = A(-n_1, \ldots, -n_t)$$

with all $m_i, n_j \in \mathbb{N}$ satisfying $m_1 \leq \cdots \leq m_s$, and $n_1 \leq \cdots \leq n_t$. We choose homogeneous $A$-bases $\zeta$ and $\xi$ for $M$ and $N$ respectively, where

$$\zeta = \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix}, \quad \xi = \begin{pmatrix} y_1 \\ \vdots \\ y_t \end{pmatrix},$$

with $|x_i| = m_i$ and $|y_j| = n_j$.

For $f \in \text{Hom}_{\text{Gr}(A)}(M, N)$, set $F = (a_{ij}) \in M_s \times t(A)$ and say that $F$ is a matrix representation of $f$ (with respect to the bases $(\zeta, \xi)$) in case $f(\zeta) = F\xi$. In this case, we simply say that $(f, F)$ forms a match.

**Remark 4.1.**

1. If $f \neq 0$, one can compute out that $|a_{ij}| = m_i - n_j$ if $m_i \geq n_j$; and $a_{ij} = 0$ if $m_i < n_j$.

2. There exists a nonzero entry in each row of $F$, all nonzero entries in $F$ are homogeneous.

**Lemma 4.1.** Let $(f, F)$ be a match with respect to $(\zeta, \xi)$, assume $F = \begin{pmatrix} F' & 0 \\ U & F'' \end{pmatrix}$ with both $F'$ and $F''$ being nonzero. Then there exist a commutative diagram:

$$
\begin{array}{cccccc}
0 & \to & M' & \to & M & \to & M'' & \to & 0 \\
\downarrow f' & & \downarrow f & & \downarrow f'' & & \\
0 & \to & N' & \to & N & \to & N'' & \to & 0
\end{array}
$$

such that each row is split exact, and all $(f', F'), (f'', F''), (\lambda, U)$ are matches, where $\lambda \in \text{Hom}_{\text{Gr}(A)}(M'', N')$.

**Proof.** Let $F' \in M_{s' \times t'}(A)$. Set $\zeta = \begin{pmatrix} \zeta' \\ \zeta'' \end{pmatrix}$ where $\zeta'$ has $s'$ rows, and $\xi = \begin{pmatrix} \xi' \\ \xi'' \end{pmatrix}$ where $\xi'$ has $t'$ rows. For $x \in M$ with $x = X\zeta$, write $X = (X', X'')$ where $X'$ has $s'$ columns. Since $f(x) = f(X\zeta) = XF\xi$, we have

$$f(X'\zeta' + X''\zeta'') = X'F'\xi' + X''U\xi' + X''F''\xi''.$$
Set $M', M'', N', N''$ to be free modules generated by $\zeta', \zeta'', \xi', \xi''$, respectively, and $f' = f|_{M'}, f'' = P|_{M''} \circ f|_{M''}$, $\lambda = P|_{N'} \circ f|_{M''}$, here $P|_{N'} : N \to N'$ and $P|_{N''} : N \to N''$ are projective maps. Then the result follows.

**Definition 4.2.** Let notation be assumed as in the lemma above, suppose that for any $X''$, there exists $X'$ such that $X''U = X'F'$, then we call the matrix $F$ admissible.

**Remark 4.2.**

(1) $F = \begin{pmatrix} F' & 0 \\ 0 & F'' \end{pmatrix}$ is admissible.

(2) If a match $(f, F)$ with $F$ admissible, then $\text{Im} \lambda \subseteq \text{Im} f'$ from the proof of Lemma 4.1.

**Lemma 4.3.** Assume that $M \xrightarrow{f} N \xrightarrow{g} P$ is an exact sequence, and $(f, F), (g, G)$ are matches such that both $F = \begin{pmatrix} F' & 0 \\ U & F'' \end{pmatrix}$ and $G = \begin{pmatrix} G' & 0 \\ V & G'' \end{pmatrix}$ are admissible. Then, we have the following commutative diagram:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
M' \xrightarrow{f'} & N' \xrightarrow{g'} & P' \\
\downarrow & \downarrow & \downarrow \\
M \rightarrow & N \rightarrow & P \\
\downarrow & \downarrow & \downarrow \\
M'' \xrightarrow{f''} & N'' \xrightarrow{g''} & P'' \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

such that all rows are exact, and all columns are split exact.

**Proof.** By Lemma 4.1 and the assumption, we obtain $M', N', P', M'', N''$ and $P''$ such that three columns are split exact and the diagram is commutative. Moreover, there exist $\lambda_1 \in \text{Hom}_{Gr(A)}(M'', N''), \lambda_2 \in \text{Hom}_{Gr(A)}(N'', P')$ such that
\[
f(x) = f'(x') + \lambda_1(x'') + f''(x''), \quad g(y) = g'(y') + \lambda_2(y'') + g''(y'')
\]

satisfying $\text{Im} \lambda_1 \subseteq \text{Im} f'$ and $\text{Im} \lambda_2 \subseteq \text{Im} g'$. Now we prove the first row is exact.

It is clear that $\ker g' = \ker g \cap N'$. We show that $\text{Im} f' = \text{Im} f \cap N'$. It is obvious that $\text{Im} f' \subseteq \text{Im} f \cap N'$. For any $y \in \text{Im} f \cap N'$, there exists $x = x' + x'' \in N$ such that $y = f(x) = f'(x') + \lambda_1(x'') + f''(x'')$ with $f'(x') + \lambda_1(x'') \in N'$ and $f''(x'') \in N''$. So $f(x) = f'(x') + \lambda_1(x'')$ and $f''(x'') = 0$. Since $\text{Im} \lambda_1 \subseteq \text{Im} f'$, there exists $z \in M'$ such that $\lambda_1(x'') = f'(z)$. Thus, $y = f(x) = f'(x' + z)$. Therefore, $y \in \text{Im} f'$. So we have $\ker g' = \ker g \cap N' = \text{Im} f \cap N' = \text{Im} f'$.

Next, we show that the third row is exact.

We prove $\text{Im} f'' = \text{Im} f \cap N''$ firstly. In fact, $\text{Im} f \cap N'' \subseteq \text{Im} f''$ is obvious. Conversely, for any $y \in \text{Im} f''$, there exists $x \in M''$ such that $f''(x) = y$, we get $f(x) = \lambda_1(x) + f''(x)$. Since $\text{Im} \lambda_1 \subseteq \text{Im} f'$, there exists $z \in M'$ such that
$f'(z) = \lambda_1(x)$. So $f(x - z) = \lambda_1(x) + f''(x) - f'(z) = f''(x) = y$. Therefore, $y \in \text{Im} f \cap N''$. This proves $\text{Im} f'' = \text{Im} f \cap N''$.

On the other hand, we show that $g'' = \text{ker} g \cap N''$. It is obvious that $\ker g \cap N'' \subseteq \ker g''$. For any $y \in \ker g''$, we have $y \in N''$ and $g''(y) = 0$. Since $\text{Im} \lambda_2 \subseteq \text{Im} f'$, there exists $z \in N'$ such that $g'(z) = \lambda_2(y)$. We get $g(z - y) = g'(z) - \lambda_2(y) - g''(y) = 0$. Since $g = \text{Im} f$, there exists $v = v' + v'' \in M$ such that $f(v) = f'(v') + \lambda_1(v'') + f''(v'') = z - y$. Clearly, $f''(v') + \lambda_1(v'') = z$ and $f''(v'') = -y$. Since $\text{Im} \lambda_1 \subseteq \text{Im} f'$, there exists $w \in M'$ such that $f'(w) = \lambda_1(v'')$. Thus, $f'(v' + w) = z$. We get $\lambda_2(y) = g'(z) = g'f'(v' + w) = 0$ by the exactness of the first row. So $g(y) = \lambda_2(y) + g''(y) = 0$. Therefore, $y \in \ker g \cap N''$.

Hence, we get $\ker g'' = \ker g \cap N'' = \text{Im} f \cap N'' = \text{Im} f''$. \hfill $\Box$

With preparations above, we turn our attention to a kind of bi-Koszul modules. Let $M$ be a bi-Koszul module over $A$ with the minimal free resolution:

$$Q : \cdots \rightarrow Q_{3n+2} \xrightarrow{\partial_{3n+2}} Q_{3n+1} \xrightarrow{\partial_{3n+1}} Q_{3n} \xrightarrow{\partial_n} \cdots \rightarrow Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\delta_1} Q_0 \xrightarrow{\varepsilon} M \rightarrow 0$$

where $Q_{3n+2} = A(-2nd - d)p_{3n+2} \oplus A(-2nd - d - 1)q_{3n+2}$.

We say that $M$ is decomposable in case, for each $m \geq 1$, the matrix representation $F_m$ of $\partial_m$ with the form

$$\begin{pmatrix} F'_m & 0 \\ U_m & F''_m \end{pmatrix}$$

satisfying

1. $F_m$ is admissible (if the resolution is ending at $Q_m$ with $m \neq 3n + 2$, $F_m = (F'_m \ 0)$ is permitted),
2. $F_{m+1}F_m$ is consistent according to the block multiplication, and
3. the number of rows of $F'_{3n+2}$ is equal to $p_{3n+2}$.

**Theorem 4.4.** Let $A$ be a connected noetherian graded algebra, and $M$ a decomposable bi-Koszul module defined as above. Then the resolution $Q$ can be decomposed into a direct sum of two pure resolutions. Moreover, there exists a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ such that $M'$ and $M''$ are $\delta$-Koszul modules.

**Proof.** With the assumption, applying the lemmas 4.1 and 4.3 to each exact sequence $Q_{m+1} \xrightarrow{\partial_{m+1}} Q_m \xrightarrow{\partial_m} Q_{m-1}$ produces two exact sequences

$$\cdots \rightarrow Q'_m \xrightarrow{\partial'_m} \cdots \rightarrow Q'_2 \xrightarrow{\partial'_2} Q'_1 \xrightarrow{\partial'_1} Q'_0,$$

and

$$\cdots \rightarrow Q''_m \xrightarrow{\partial''_m} \cdots \rightarrow Q''_2 \xrightarrow{\partial''_2} Q''_1 \xrightarrow{\partial''_1} Q''_0,$$

where both $Q'_m$ and $Q''_m$ are pure, moreover, $Q_m = Q'_m \oplus Q''_m$. Denote that $M' = \varepsilon(Q'_0)$, $M'' = \pi\varepsilon(Q''_0)$,
where $\pi$ is the canonical surjective morphism from $M$ to $M/M'$. We have the commutative diagram:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
Q_1' & Q_0' & M' \to 0 \\
\downarrow & \downarrow & \downarrow \\
Q_1 & Q_0 & M \to 0 \\
\downarrow & \downarrow & \downarrow \\
Q_1'' & Q_0'' & M'' \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

with $\varepsilon' = \varepsilon|_{Q_0'}$ and $\varepsilon'' = \pi \varepsilon|_{Q_0''}$. It is clear that $\varepsilon''$ is an epimorphism. The first and third rows are exact by the similar proof of Lemma 4.3.

If the resolution $Q$ is ending at $Q_m$ with $m \neq 3n + 2$, and $F_m$ has the form $(F'_m, 0)$, we can get the similar proof by taking $Q'_m = Q_m, Q''_m = 0$. □

Remark 4.3. We may describe $M'$ and $M''$ above more precisely by their resolutions:

\[Q' : \cdots \to Q'_m \xrightarrow{\partial'_m} \cdots \to Q'_2 \xrightarrow{\partial'_2} Q'_1 \xrightarrow{\partial'_1} Q'_0 \xrightarrow{\varepsilon'} M' \to 0\]

and

\[Q'' : \cdots \to Q''_m \xrightarrow{\partial''_m} \cdots \to Q''_2 \xrightarrow{\partial''_2} Q''_1 \xrightarrow{\partial''_1} Q''_0 \xrightarrow{\varepsilon''} M'' \to 0\]

where $Q'_m$ is generated in degree

\[\delta'(m) = \begin{cases} 2nd, & \text{if } m = 3n, \\ 2nd + 1, & \text{if } m = 3n + 1, \\ 2nd + d, & \text{if } m = 3n + 2, \end{cases}\]

and $Q''_m$ is generated in degree

\[\delta''(m) = \begin{cases} 2nd, & \text{if } m = 3n, \\ 2nd + 1, & \text{if } m = 3n + 1, \\ 2nd + d + 1, & \text{if } m = 3n + 2. \end{cases}\]

Both of them are one kind of $\delta$-Koszul modules introduced in [6], but here the notion is defined over an arbitrary graded algebra.

Corollary 4.5. In the case of Theorem 4.4, we get, for all $m \geq 1$, $E^m(M) = E^m(M') \oplus E^m(M'')$, where both $E^m(M')$ and $E^m(M'')$ are pure.

Example 4.6. Set

\[A = \frac{k(x, y, z, w)}{(yx, z^2y, wz)}\]

The trivial module $Ak$ has the minimal resolution:

\[0 \to A(-5) \to A(-4)^2 \to A(-2)^2 \oplus A(-3) \to A(-1)^4 \to A \to k \to 0,\]
so $A$ is a bi-Koszul algebra. Consider $M := \frac{A \oplus A}{((x, 0), (0, y))}$, it is a bi-Koszul module because it has the minimal resolution:

$$0 \to A(-5) \xrightarrow{M_4} A(-4)^2 \xrightarrow{M_3} A(-2) \oplus A(-3) \xrightarrow{M_2} A(-1)^2 \xrightarrow{M_1} A^2 \to M \to 0.$$ 

where $M_4 = \begin{pmatrix} w & 0 \\ z^2 & 0 \\ 0 & w \end{pmatrix}$, $M_3 = \begin{pmatrix} y & 0 \\ 0 & z^2 \\ z^2 & 0 \end{pmatrix}$ and $M_1 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$.

So by Theorem 4.4 we have the minimal resolutions:

$$0 \to A(-5) \to A(-4) \to A(-2) \to A(-1) \to A \to A/(x) \to 0,$$

and

$$0 \to A(-4) \to A(-3) \to A(-1) \to A \to A/(y) \to 0.$$

We end this article by constructing a strongly bi-Koszul algebra based on the notion of free products and direct sums, as well as the existence of δ-Koszul algebra mentioned above.

For the details of the free product $A \sqcup A'$ and direct sum $A \cap A'$, we refer to [11], Chapter 3, from which the following results will be used in our construction.

1. $E(A \sqcup A') = E(A) \cap E(A')$,
2. $(E(A) \cap E(A'))_i = \begin{cases} k, & \text{if } i = 0, \\ E^i(A) \oplus E^i(A'), & \text{if } i > 0, \end{cases}$
3. $E^{>0}(A) \cdot E^{>0}(A') = E^{>0}(A') \cdot E^{>0}(A) = 0.$

Now suppose that $A$ and $A'$ are two δ-Koszul algebras, the trivial modules $A^k$ and $A^{-k}$ have their minimal resolutions whose degree distributions obey the resolution maps $\delta'$ and $\delta''$ mentioned in Remark 4.3. We know that $E^i(A)E^{3n}(A) = E^{i+3n}(A)$

and $E^i(A')E^{3n}(A') = E^{i+3n}(A')$ for $i = 0, 1, 2$ and $n \geq 0$ by Lemma 4.2.

Denote $B = A \sqcup A'$, we have $E^{3n+i}(B) = E^{3n+i}(A) \oplus E^{3n+i}(A')$. On the other hand, $E^i(B)E^{3n}(B) = (E^i(A) \oplus E^i(A'))(E^{3n}(A) \oplus E^{3n}(A')) = E^i(A)E^{3n}(A) \oplus E^i(A')E^{3n}(A')$ for $i = 0, 1, 2$ and $n \geq 0$. Hence, $B$ is a bi-Koszul algebra and $E^i(B)E^{3n}(B) = E^{3n+i}(B)$ for $i = 0, 1, 2$ and $n \geq 0$. So $B$ is a strongly bi-Koszul algebra.

The above shows that one may get a strongly bi-Koszul algebra by free products of some graded algebras with pure resolutions. Unfortunately, the constructing cannot produce any non-strongly bi-Koszul algebra.

We leave the following question.

Question. Is there a bi-Koszul algebra that is not strongly? Or equivalently, must the bi-Koszul algebras be strongly?

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