A new error in variables model for solving positive definite linear system using orthogonal matrix decompositions

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Abstract The need to estimate a positive definite solution to an overdetermined linear system of equations with multiple right hand side vectors arises in several process control contexts. The coefficient and the right hand side matrices are respectively named data and target matrices. A number of optimization methods were proposed for solving such problems, in which the data matrix is unrealistically assumed to be error free. Here, considering error in measured data and target matrices, we present an approach to solve a positive definite constrained linear system of equations based on the use of a newly defined error function. To minimize the defined error function, we derive necessary and sufficient optimality conditions and outline a direct algorithm to compute the solution. We provide a comparison of our proposed approach and two existing methods, the interior point method and a method based on quadratic programming. Two important characteristics of our proposed method as compared to the existing methods are computing the solution directly and considering error both in data and target matrices. Moreover, numerical test results show that the new approach leads to smaller standard deviations of error entries and smaller effective rank as desired by control problems. Furthermore, in a comparative study, using the Dolan-Moré performance profiles, we show the approach to be more efficient.

Keywords Error in variables models · Positive definiteness constraints · Overdetermined linear system of equations · Multiple right hand side vectors

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1 Introduction

Computing a symmetric positive definite solution of an overdetermined linear system of equations arises in a number of physical problems such as estimating the mass inertia matrix in the design of controllers for solid structures and robots; see, e.g., [1, 4, 9, 14, 17]. Modeling a deformable structure also leads to such a mathematical problem; e.g., see [26]. The problem turns into finding an optimal solution of the system

$$DX \simeq T,$$

where $D, T \in \mathbb{R}^{m \times n}$, with $m \geq n$, are given and a symmetric positive definite matrix $X \in \mathbb{R}^{n \times n}$ is to be computed as a solution of (1). In some special applications, the data matrix $D$ has a simple structure, which may be taken into consideration for efficiently organized computations. Estimation of the covariance matrix and computation of the correlation matrix in finance are two such examples where the data matrices are respectively block diagonal and the identity matrix; e.g., see [32].

A number of least squares formulations have been proposed for physical problems, which may be classified as ordinary and error in variables (EIV) models. Also, single or multiple right hand side least squares may arise. With a single right hand side, we have an overdetermined linear system of equations $Dx \simeq t$, where $D \in \mathbb{R}^{m \times n}$, $t \in \mathbb{R}^{m \times 1}$, with $m \geq n$, are known and the vector $x \in \mathbb{R}^{n \times 1}$ is to be computed. In an ordinary least squares formulation, the error is only attributed to $t$. So, to minimize the corresponding error, the following mathematical problem is devised:

$$\min \|\Delta t\| \quad \text{s.t.} \quad Dx = t + \Delta t.$$  \hfill (2)

There are a number of methods for solving (2), identified as direct and iterative methods. A well known direct method is based on using the QR factorization of the matrix $D$ [28]. An iterative method has also been introduced in [7] for solving (2) using the GMRES algorithm.

In an EIV model, however, errors in both $D$ and $t$ are considered; e.g., see [3]. Total least squares formulation is a well-known EIV model, where the goal is to solve the following mathematical problem (e.g., see [6] and [16]):

$$\min \|[(\Delta D, \Delta t)]\| \quad \text{s.t.} \quad (D + \Delta D)x = t + \Delta t.$$  \hfill (3)

We note that $\|\cdot\|$ in (2) and (3) respectively denote the vector 2-norm and the matrix Frobenius norm. Both direct [25] and iterative [12] methods have been presented for solving (3). Moreover, the scaled total least squares formulation has been considered to unify both ordinary and total least squares formulation; e.g., see [25]. In a scaled total least squares formulation, the mathematical problem

$$\min \|[(\Delta D, \Delta t)]\| \quad \text{s.t.} \quad (D + \Delta D)x = \lambda t + \Delta t$$  \hfill (4)
is to be solved for an arbitrary scalar $\lambda$. Zhou [19] has studied the effect of perturbation and gave an error analysis of such a formulation.

A least squares problem with multiple right hand side vectors can also be formulated as an overdetermined system of equations $DX \simeq T$, where $D \in \mathbb{R}^{m \times n}$, $T \in \mathbb{R}^{m \times k}$, with $m \geq n$, are given and the matrix $X \in \mathbb{R}^{n \times k}$ is to be computed. With ordinary and total least squares formulations, the respective mathematical problems are:

$$\min \| \Delta T \|
$$

$$\text{s.t. } DX = T + \Delta T \quad X \in \mathbb{R}^{n \times k}$$

and

$$\min \| [\Delta D, \Delta T] \|
$$

$$\text{s.t. } (D + \Delta D)X = T + \Delta T \quad X \in \mathbb{R}^{n \times k}.$$  \hspace{1cm} (5)

Common methods for solving (5) are similar to the ones for (2); see, e.g., [7, 28]. Solving (6) is possible by using the method described in [8], based on the SVD factorization of the matrix $[D, T]$. Connections between ordinary least squares and total least squares formulations have been discussed in [11].

Here, we consider a newly defined EIV model for solving a positive definite linear problem. Our goal is to compute a symmetric positive definite solution $X \in \mathbb{R}^{n \times n}$ to the overdetermined system of equations $DX \simeq T$, where both matrices $D$ and $T$ may contain errors. We refer to this problem as positive definite linear system of equations. No EIV model, even the well-known total least squares formulation, is considered for solving the positive definite linear system of equations in the literature. Several approaches have been proposed for this problem, commonly considering the ordinary least squares formulation and minimizing the error $\| \Delta T \|_F$ over all $n \times n$ symmetric positive definite matrices, where $\| \cdot \|_F$ is the Frobenious norm; see e.g. [10, 24]. Larson [13] discussed a method for solving a positive definite least squares problem considering the corresponding normal system of equations. He considered both symmetric and positive definite least squares problems. Krislock [26] proposed an interior point method for solving a variety of least squares problems with positive semi-definite constraints. Woodgate [18] described a new algorithm for solving a similar problem in which a symmetric positive semi-definite matrix $P$ is computed to minimize $\| F - PG \|$, with known $F$ and $G$. Hu [10] presented a quadratic programming approach to handle the positive definite constraint. In her method, the upper and lower bounds for the entries of the target matrix can be given as extra constraints. In real measurements, however, both the data and target matrices may contain errors; hence, the total least squares formulation appears to be appropriate.

The rest of our work is organized as follows. In Section 2, we define a new error function and discuss some of its characteristics. A method for solving the resulting optimization problem with the assumption that $D$ has full column rank is presented in Section 3. In Section 4, we generalize the method to the case of data matrix having an arbitrary rank. In Section 5, a detailed discussion is made on computational
complexity of both methods. Computational results and comparisons with available methods are given in Section 6. Section 7 gives our concluding remarks.

2 Problem formulation

Consider a single equation \( ax \simeq b \), where \( a, b \in \mathbb{R}^n \) and \( x \in \mathbb{R}^+ \). As shown in Fig. 1, errors in the \( i \)th entry of \( b \) and \( a \) are respectively equal to \( | a_i x - b_i | \) and \( | a_i - \frac{b_i}{x} | \); e.g., see [25].

In [25], \( \sum_{i=1}^{n} L_i \) was considered as a value to represent errors in both \( a \) and \( b \). As shown in Fig. 1, \( L_i \) is the height of the triangle ABC which turns to be equal to \( L_i = \frac{|b_i - a_i x|}{\sqrt{1 + x^2}} \). Here, to represent the errors in both \( a \) and \( b \), we define the area error to be

\[
\sum_{i=1}^{n} |b_i - a_i x| \left| a_i - \frac{b_i}{x} \right| , \tag{7}
\]

which is equal to

\[
\sum_{i=1}^{n} (b_i - a_i x) \left( a_i - \frac{b_i}{x} \right) ,
\]

for \( x \in \mathbb{R}^+ \).

Considering the problem of finding a symmetric and positive definite solution to the overdetermined system of linear equations \( DX \simeq T \), in which both \( D \) and \( T \) include error, the values \( DX \) and \( TX^{-1} \) are predicted values for \( T \) and \( D \) from the model \( DX \simeq T \); hence, vectors \( \Delta T_j = (DX - T)_j \) and \( \Delta D_j = (D - TX^{-1})_j \) are the entries of errors in the \( j \)th column of \( T \) and \( D \), respectively. Extending the error formulation (7), the value

\[
E = \sum_{j=1}^{n} (DX_j - T_j)^T (D_j - (TX^{-1})_j )
\]

\[\text{Fig. 1 Error formulation for a single equation}\]
seems to be an appropriate measure of error. We also have
\[
E = \sum_{j=1}^{n} \sum_{i=1}^{m} (DX - T)_{ij} (D - TX^{-1})_{ij} = \text{tr}((DX - T)^T (D - TX^{-1})),
\]
with \( \text{tr}(.) \) standing for trace of a matrix. Therefore, the problem can be formulated as
\[
\min_{X > 0} \text{tr}((DX - T)^T (D - TX^{-1})),
\]
where \( X \) is symmetric and by \( X > 0 \), we mean \( X \) is positive definite. Problem (9) poses a newly defined EIV model for solving the positive definite linear system of equations.

In Lemma 2, we represent an equivalent formulation for the error, \( E \). First, consider to a well-known property of positive definite matrices.

**Note 1** A matrix \( X \in \mathbb{R}^{n \times n} \) is positive definite if and only if there exists a nonsingular matrix \( Y \in \mathbb{R}^{n \times n} \) such that \( X = YY^T \).

The following results about the trace operator are also well-known; e.g., see [22].

**Lemma 1** For an nonsingulartible matrix \( P \in \mathbb{R}^{n \times n} \) and arbitrary matrices \( Y \in \mathbb{R}^{n \times n}, A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{m \times n} \) we have
1. \( \text{tr}(Y) = \text{tr}(P^{-1}YP) \).
2. \( \text{tr}(AB) = \text{tr}(BA) \).

**Lemma 2** The error \( E \), defined by (8), is equal to
\[
E = \|DY - TY^{-T}\|_F^2, \tag{10}
\]
where \( X = YY^T \) and \( \| . \|_F \) denotes the Frobenius norm of a matrix.

**Proof** Substituting \( X = YY^T \) in (8) and using Lemma 1, we get
\[
E = \text{tr}((DX - T)^T (D - TX^{-1})) = \text{tr}((DX - T)^T (DX - T)X^{-1}) \\
= \text{tr}((DX - T)^T (DX - T)Y^{-T}Y^{-1}) = \text{tr}(Y^{-1}(DX - T)^T (DX - T)Y^{-T}) \\
= \text{tr}((DXY^{-T} - TY^{-T})^T (DXY^{-T} - TY^{-T})) \\
= \text{tr}((DY - TY^{-T})^T (DY - TY^{-T})) \\
= \|DY - TY^{-T}\|_F^2.
\]

Considering this new formulation for \( E \), it can be concluded that by use of our newly defined EIV model, computing a symmetric and positive definite solution to the over-determined system of equations \( DX \simeq T \) is equivalent to computing a nonsingular matrix \( Y \in \mathbb{R}^{n \times n} \) to be the solution of
\[
\min \|DY - TY^{-T}\|_F^2.
\]
and letting $X = YY^T$. A similar result is obtained by considering the overdetermined system $DX \simeq T$ with $X = YY^T$ and multiplying both sides by $Y^{-T}$. We have,

$$DYY^T \simeq T,$$

or equivalently,

$$DY \simeq TY^{-T}. \quad (11)$$

Now, to assign a solution to (11), it makes sense to minimize the norm of residual. Thus, to compute $X = YY^T$, it is sufficient to let $Y$ to be the solution of

$$\min \|DY - TY^{-T}\|_F^2.$$

**Note 2** An appropriate characteristic of the error formulation proposed by (8) is that for a symmetric and positive definite matrix $X$, the value of $E$ is nonnegative and it is equal to zero if and only if $DX = T$.

### 3 Mathematical solution: full rank data matrix

Here, we are to develop an algorithm for solving (9) with the assumption that $D$ has full column rank. Using Lemma 1, with $X$ being symmetric, we have

$$\text{tr}((DX - T)^T (D - TX^{-1})) = \text{tr}(D^T DX + X^{-1} T^T T) - 2 \text{tr}(T^T D).$$

So, (9) can be written as

$$\min \text{tr}(AX + X^{-1} B), \quad (12)$$

where $A = D^T D$ and $B = T^T T$ and the symmetric and positive definite matrix $X$ is to be computed.

To explain our method for solving (12), we present the following theorems.

**Theorem 1** The solution $X^*$ for problem (12) satisfies

$$X^* AX^* = B.$$

**Proof** Let $\Phi(X) = \text{tr}(AX + X^{-1} B)$. Assuming $d(\Phi(X))$ to be the first order differential of $\Phi$ [30], the first order necessary conditions for (12) [33] is obtained to be

$$d(\Phi(X)) = \text{tr}(A - X^{-1} BX^{-1}) = 0,$$

which is implied by

$$X^* AX^* = B, \quad (13)$$

where $X^*$ is symmetric and positive definite.

The following theorem helps us to check whether the first order necessary conditions defined in Theorem 1 are sufficient for optimality.
Theorem 2 (Sufficient optimality conditions) [31] Consider the optimization problem

$$\min f(X)$$

$$s.t. g(X) = 0.$$  \hfill (14)

Suppose that $L(X, \lambda) = f(X) - \lambda g(X)$ is the corresponding Lagrangian and $d^2(L)$ is its Hessian matrix. If the matrices $X^*$ and $\lambda^*$ satisfy the KKT necessary conditions and $s^T d^2(L(X^*, \lambda^*)) s$ is positive for each feasible direction $s$ from $X^*$, then $X^*$ is a strict local solution for (14). Also, if $f(X)$ is strictly convex and $\{X | g(X) = 0\}$ is convex, then $X^*$ is the unique global solution.

Corollary 1 We note that for each $X^*$ satisfying the first order necessary conditions of (12), the sufficient optimality conditions described in Theorem 2 are satisfied. Moreover, in the following lemma we show that $\Phi_1(X) = tr(AX + X^{-1}B)$ is strictly convex on the cone of symmetric positive definite matrices. Hence, we confirm that the symmetric positive definite matrix satisfying the KKT necessary conditions mentioned in Theorem 1 is the unique global solution of (12).

Lemma 3 $\Phi(X) = tr(AX + X^{-1}B)$ is strictly convex on the cone of symmetric positive definite matrices.

Proof Let $\Phi_1(X) = tr(AX)$ and $\Phi_2(X) = tr(X^{-1}B)$. Then, $\Phi_1(X)$ being an affine function is convex on $\mathbb{R}^{n \times n}$. We show that $\Phi_2(X) = tr(X^{-1}B) = tr\left((B^{-1}X)^{-1}\right) = tr\left((B^{-\frac{1}{2}}XB^{-\frac{1}{2}})^{-1}\right)$ is strictly convex on the cone of symmetric positive definite matrices to conclude that $\Phi(X) = \Phi_1(X) + \Phi_2(X)$ is strictly convex. To this end, we first prove the strict convexity of $tr(X^{-1})$ on the cone of symmetric positive definite matrices. We know that a function $tr(f(X))$ is strictly convex on the cone of symmetric positive definite matrices if and only if $f(t)$ is strictly convex on $\mathbb{R}^+$; see, e.g., [21]. By letting $f(t) = \frac{1}{t}$, the strict convexity of $tr(X^{-1})$ on the cone of symmetric positive definite matrices is established. Now, from the definition of strict convexity it is implied that for arbitrary symmetric positive definite matrices $P$ and $Q$ we have

$$tr\left((\lambda P + (1 - \lambda)Q)^{-1}\right) < \lambda tr(P^{-1}) + (1 - \lambda) tr(Q^{-1}).$$  \hfill (15)

Letting $P = B^{-\frac{1}{2}}XB^{-\frac{1}{2}}$ and $Q = B^{-\frac{1}{2}}YB^{-\frac{1}{2}}$, from (15) we get

$$tr\left((\lambda B^{-\frac{1}{2}}XB^{-\frac{1}{2}} + (1 - \lambda)B^{-\frac{1}{2}}YB^{-\frac{1}{2}})^{-1}\right)$$

$$= \Phi_2(\lambda X + (1 - \lambda)Y)$$

$$< \lambda tr\left((B^{-\frac{1}{2}}XB^{-\frac{1}{2}})^{-1}\right) + (1 - \lambda) tr\left((B^{-\frac{1}{2}}YB^{-\frac{1}{2}})^{-1}\right)$$

$$= \lambda \Phi_2(X) + (1 - \lambda)\Phi_2(Y).$$

This establishes the strict convexity of $\Phi_2$ on the cone of symmetric positive definite matrices. \hfill \Box
3.1 Computing the positive definite matrix satisfying KKT conditions

As mentioned in Theorem 1, the KKT conditions lead to the nonlinear matrix equation

\[ XAX = B. \] (16)

Note that (16) is an special case of the continuous time Riccati equation (CARE), [23]

\[ A^T XE + E^T XA - (E^T XB + S)R^{-1}(B^T XE + S^T) + Q = 0, \] (17)

with \( R = 0, \) \( E = \frac{A}{2} \) and \( Q = -B. \) There is a MATLAB routine to solve CARE for arbitrary values of \( A, E, B, S, R \) and \( Q. \) To use the routine, it is sufficient to type the command

\[ X = \text{care}(A, B, Q, R, S, E), \]

for the input arguments as in (17). Higham [23] developed an effective method for computing the positive definite solution to this special CARE when \( A \) and \( B \) are symmetric and positive definite using well-known decompositions. Lancaster and Rodman [29] also discussed solving different types of algebraic Riccati equations. Moreover, they derived a perturbation analysis for these matrix equations.

**Note 3** (QR decomposition) The QR decomposition [28] of a matrix \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n, \) is a decomposition of the form \( A = QR, \) where \( R \) is an \( m \times n \) upper triangular matrix and \( Q \) satisfies \( QQ^T = Q^T Q = I. \) Moreover, if \( A \) has full column rank, then \( R \) also has full column rank.

**Note 4** (Cholesky decomposition) A Cholesky decomposition [28] of a symmetric positive definite matrix \( A \in \mathbb{R}^{n \times n} \) is a decomposition of the form \( A = RT R, \) where \( R, \) known as the Cholesky factor of \( A, \) is an \( n \times n \) nonsingular upper triangular matrix.

**Note 5** (Spectral decomposition) [28] All eigenvalues of a symmetric matrix, \( A \in \mathbb{R}^{n \times n}, \) are real and there exists an orthonormal matrix with columns representing the corresponding eigenvectors. Thus, there exist an orthonormal matrix \( U \) with columns equal to the eigenvectors of \( A \) and a diagonal matrix \( D \) containing the eigenvalues such that \( A = UD U^T. \) Also, if \( A \) is positive definite, then all of its eigenvalues are positive, and so we can set \( D = S^2. \) Thus, spectral decomposition for a symmetric positive definite matrix \( A \) is a decomposition of the form \( A = US^2 U^T, \) where \( U^T U = UU^T = I \) and \( S \) is a diagonal matrix.

**Theorem 3** [23] Assume \( D, T \in \mathbb{R}^{m \times n} \) with \( m \geq n \) are known and \( \text{rank}(T) = n. \) Let \( D = QR \) be the QR factorization of \( D. \) Let \( A = DT D \) and \( B = TT T. \) Define the matrix \( \tilde{Q} = RB R^T \) and compute its spectral decomposition, that is, \( \tilde{Q} = RB R^T = US^2 U^T. \) Then, (12) has a unique solution, given by

\[ X^* = R^{-1} U \tilde{S} U^T R^{-T} \]
Proof Based on Theorem 2 and the afterwards discussion, it is sufficient to show that \( X^* \) satisfies the necessary optimality conditions, \( X^*AX^* = B \). Note that from \( D = QR \), we have
\[
A = D^T D = R^T Q^T Q R = R^T R.
\]
Substituting \( X^* \), we have
\[
X^*AX^* = R^{-1}U \tilde{S}U^T R^{-T} R^T R R^{-1}U \tilde{S}U^T R^{-T}
= R^{-1}U \tilde{S}^2 U^T R^{-T} = R^{-1}RR^T R^{-T} = B.
\]
\[\square\]

Note 6 To compute \( R \), it is also possible to first compute \( A = D^T D \) and then calculate the Cholesky decomposition for \( A \). However, because of more stability, in Theorem 3 the QR decomposition of \( D \) is used.

We are now ready to outline the steps of our proposed algorithm.

Algorithm 1 Solving the EIV model for positive definite linear system using QR decomposition

\[
\text{PROCEDURE PDEIV-QR}(D, T)
1: \text{Compute the QR decomposition for } D \text{ and let } D = QR.
2: \text{Let } \tilde{Q} = RBR^T, \text{ where } B = T^T T \text{ and compute the spectral decomposition of } \tilde{Q}, \text{ that is, } \tilde{Q} = U \tilde{S}^2 U^T.
3: \text{Set } X^* = R^{-1}U \tilde{S}U^T R^{-T}.
4: \text{Set } E = \text{tr}((DX^* - T)^T(D - TX^*-1)).
\]

Note that Algorithm 1 computes the solution of (9) directly.

The following theorem shows that by use of spectral decomposition of \( A \) a method similar to the one introduced in [23] is in hand for solving the continuous time Riccati equation.

Theorem 4 Let \( A = D^T D \) and \( B = T^T T \) with \( D, T \in \mathbb{R}^{m \times n} \), \( m \geq n \) and \( \text{rank}(D) = n \). Let the spectral decomposition of \( A \) be \( A = US^2U^T \). Define the matrix \( \tilde{Q} = SU^T BUS \) and compute its spectral decomposition, \( \tilde{Q} = U \tilde{S}^2 U^T \). Then, the unique minimizer of (12) is
\[
X^* = US^{-1}U \tilde{S}U^T S^{-1}U^T.
\]

Proof Similar to the proof of Theorem 3, it is sufficient to show that the mentioned \( X^* \) satisfies \( X^*AX^* = B \). Substituting \( X^* \), we have
\[
X^*AX^* = US^{-1}U \tilde{S}U^T S^{-1}U^T US^2U^T US^{-1}U \tilde{S}U^T S^{-1}U^T
= US^{-1}U \tilde{S}^2 U^T S^{-1}U^T = US^{-1}SU^T BUSS^{-1}U^T = B.
\]
\[\square\]
Next, based on Theorem 4, we outline an algorithm for solving (9).

**Algorithm 2** Solving the EIV model for positive definite linear system using spectral decomposition

**PROCEDURE** PDEIV-Spec\((D, T)\)

1. Let \(A = D^T D\) and compute its spectral decomposition: \(A = US^2U^T\).
2. Let \(\tilde{Q} = SU^T BUS\), where \(B = T^T T\) and compute the spectral decomposition of \(\tilde{Q}\), that is, \(\tilde{Q} = \tilde{U} \tilde{S}^2 \tilde{U}^T\).
3. Set \(X^* = US^{-1} \tilde{U} \tilde{S} \tilde{U}^T S^{-1} UT\).
4. Set \(E = \text{tr}((DX^* - T)(D - TX^* - 1))\).

In Section 4 we generalize our proposed method for solving positive definite linear system of equations when the data matrix is rank deficient.

## 4 Mathematical solution: rank deficient data matrix

Since the data matrix \(D\) is usually produced from experimental measurements, we may have \(\text{rank}(D) < n\). Here, we are to generalize Algorithm 1 for solving (9), assuming that \(\text{rank}(D) = r < n\). In Section 4.1 we outline two algorithms to compute the general solution of (9). It will be shown that, in general, (9) may not have a unique solution. Hence, in Section 4.2 we discuss how to find a particular solution of (9) having desirable characteristics for control problems.

### 4.1 General solution

Based on Theorems 1 and 2, a symmetric positive definite matrix \(X^*\) is a solution of (9) if and only if

\[
X^*AX^* = B. \tag{18}
\]

Therefore, in the following, we discuss how to find a symmetric positive definite matrix \(X^*\) satisfying (18).

First we note that in case \(D\) and \(T\) are rank deficient, there might be no solution for (18), and if there is any, it is not necessarily a unique solution; see, e.g., [23]. Higham [23] considered to \(X = B^\frac{1}{2} (B^\frac{1}{2} AB^\frac{1}{2})^{-\frac{1}{2}} B^\frac{1}{2}\) as a solution of (18), which is symmetric and positive semidefinite. However, we are interested in finding a symmetric positive definite solution to (18). Hence, in the following, first the necessary and sufficient conditions on \(A\) and \(B\) to guarantee the existence of positive definite solution to (18) are discussed. We then outline two algorithms to compute such a solution.

Let the spectral decomposition of \(A\) be \(A = U \begin{pmatrix} S^2 & 0 \\ 0 & 0 \end{pmatrix} U^T\), where \(S^2 \in \mathbb{R}^{r \times r}\) is a diagonal matrix having the positive eigenvalues of \(A\) as its diagonal entries.
Substituting the decomposition in (18), we get

\[ X^*U \begin{pmatrix} S^2 & 0 \\ 0 & 0 \end{pmatrix} U^T X^* = B. \]  

(19)

Since \( U \) is orthonormal, (19) can be written as

\[ U^T X^*U \begin{pmatrix} S^2 & 0 \\ 0 & 0 \end{pmatrix} U^T X^*U = U^T BU. \]

Then, letting \( \tilde{X} = U^T XU \) and \( \tilde{B} = U^T BU \), we have

\[ \tilde{X} \begin{pmatrix} S^2 & 0 \\ 0 & 0 \end{pmatrix} \tilde{X} = \tilde{B}. \]

(20)

Thus, the matrix \( X = U \tilde{X} U^T \) is a solution of (9) if and only if \( \tilde{X} \) is symmetric positive definite and satisfies (20). Substituting the block form \( \tilde{X} = \begin{pmatrix} \tilde{X}_{rr} & \tilde{X}_{r,n-r} \\ \tilde{X}_{n-r,r} & \tilde{X}_{n-r,n-r} \end{pmatrix} \), \( \tilde{X}_{rr} \in \mathbb{R}^{r \times r} \), \( \tilde{X}_{r,n-r} = \tilde{X}_{n-r,r} \in \mathbb{R}^{r \times (n-r)} \) and \( \tilde{X}_{n-r,n-r} \in \mathbb{R}^{(n-r) \times (n-r)} \), in (20) leads to

\[ \begin{pmatrix} \tilde{X}_{rr} S^2 \tilde{X}_{rr} & \tilde{X}_{rr} S^2 \tilde{X}_{r,n-r} \\ \tilde{X}_{n-r,r} S^2 \tilde{X}_{rr} & \tilde{X}_{n-r,r} S^2 \tilde{X}_{r,n-r} \end{pmatrix} = \tilde{B} = \begin{pmatrix} \tilde{B}_{rr} & \tilde{B}_{r,n-r} \\ \tilde{B}_{n-r,r} & \tilde{B}_{n-r,n-r} \end{pmatrix}, \]

which is satisfied if and only if

\[ \tilde{X}_{rr} S^2 \tilde{X}_{rr} = \tilde{B}_{rr}, \]  

(21a)

\[ \tilde{X}_{rr} S^2 \tilde{X}_{r,n-r} = \tilde{B}_{r,n-r}, \]  

(21b)

\[ \tilde{X}_{n-r,r} S^2 \tilde{X}_{r,n-r} = \tilde{B}_{n-r,n-r}. \]  

(21c)

Before discussing how to compute \( \tilde{X} \), we show that if (18) has a symmetric and positive definite solution, then \( \tilde{B}_{rr} \) must be nonsingular. The matrix \( \tilde{X}_{rr} \) as a main minor of the positive definite matrix \( \tilde{X} \) is nonsingular. \( S \) is also supposed to be nonsingular. Hence, it can be concluded from (21a) that \( \tilde{B}_{rr} \) is nonsingular.

Let \( \bar{D} = S \) and suppose \( \bar{T} \) satisfies \( \bar{T}^T \bar{T} = \bar{B}_{rr} \). Consider problem (9) corresponding to the data and target matrices \( \bar{D} \) and \( \bar{T} \) as follows:

\[ \min_{\tilde{X} > 0} \text{tr}((\bar{D} \tilde{X} - \bar{T})^T(\bar{D} - \bar{T} \tilde{X}^{-1})). \]

(22)

We know from Theorems 1 and 2 that the necessary and sufficient optimality conditions for the unique solution of problem (22) implies (21a). Thus, \( \tilde{X}_{rr} \) can be computed using Algorithm 1 for the input arguments \( \bar{D} \) and \( \bar{T} \). Substituting the computed \( \tilde{X}_{rr} \) in (21b), the linear system of equations

\[ \tilde{X}_{rr} S^2 \tilde{X}_{r,n-r} = \tilde{B}_{r,n-r} \]  

arises, where \( \tilde{X}_{rr}, S^2 \in \mathbb{R}^{r \times r} \) are known and \( \tilde{X}_{r,n-r} \in \mathbb{R}^{r \times (n-r)} \) is to be computed. Since \( \tilde{X}_{rr} \) is positive definite and \( S^2 \) is nonsingular, the coefficient matrix of the linear system (23) is nonsingular and \( \tilde{X}_{r,n-r} \) can be uniquely computed.

It is clear that since \( \tilde{X} \) is symmetric, \( \tilde{X}_{n-r,r} \) is the same as \( \tilde{X}_{r,n-r}^T \). Now, we check whether the computed \( \tilde{X}_{n-r,r} \) and \( \tilde{X}_{r,n-r} \) satisfy (21c). Inconsistency of (22) means that there is no symmetric positive definite matrix satisfying (21a–21c), and if so,
(9) has no solution. Thus, in solving a specific positive definite system with rank deficient data and target matrices using the presented EIV model, a straightforward method to investigate the existence of solution is to check whether (21c) holds for the given data and target matrices. On the other hand, for numerical results, it is necessary to generate meaningful test problems. Hence, in the following two lemmas, we investigate the necessary and sufficient conditions for satisfaction of (21c).

**Lemma 4** Let the spectral decomposition of \( A \) be determined as

\[
A = U \begin{pmatrix} S^2 & 0 \\ 0 & 0 \end{pmatrix} U^T,
\]

where \( S^2 \in \mathbb{R}^{r \times r} \) and \( \text{rank}(A) = \text{rank}(B) = r \). The necessary and sufficient condition for satisfaction of (21c) is that the columns of

\[
BU_r(U_r^T BU_r)^{-1}U_r^T B - B
\]

belong to the null space of \( U_{n-r}^T \).

**Proof** From (21a), we have

\[
\tilde{X}_{rr}^{-1} S^{-2} \tilde{X}_{rr}^{-1} = \tilde{B}_{rr}^{-1},
\]

and from (21c), we get

\[
\tilde{X}_{r,n-r} = S^{-2} \tilde{X}_{rr}^{-1} \tilde{B}_{r,n-r},
\]

\[
\tilde{X}_{n-r,r} = \tilde{B}_{n-r,r} \tilde{X}_{rr}^{-1} S^{-2}.
\]

Manipulating (21c) with (24) and (25), we get

\[
\tilde{B}_{n-r,r} \tilde{B}_{rr}^{-1} \tilde{B}_{r,n-r} = \tilde{B}_{n-r,n-r}.
\]

Considering the block form \( U = (U_r U_{n-r}) \), where \( U_r \in \mathbb{R}^{n \times r} \) and \( U_{n-r} \in \mathbb{R}^{n \times (n-r)} \), we have

\[
\tilde{B} = U^T BU = \begin{pmatrix} U_r^T \\ U_{n-r}^T \end{pmatrix} B \begin{pmatrix} U_r \\ U_{n-r} \end{pmatrix}
\]

\[
= \begin{pmatrix} U_r^T BU_r & U_r^T BU_{n-r} \\ U_{n-r}^T BU_r & U_{n-r}^T BU_{n-r} \end{pmatrix}.
\]

Rewriting (26) results in

\[
U_{n-r}^T BU_r(U_r^T BU_r)^{-1}U_r^T BU_{n-r} = U_{n-r}^T BU_{n-r},
\]

which is equivalent to (e.g., see [20])

\[
BU_r(U_r^T BU_r)^{-1}U_r^T B = B + Z,
\]

where the columns of \( Z \in \mathbb{R}^{n \times n} \) are in the null space of \( U_{n-r}^T \). Thus, (18) has a positive definite solution if and only if the columns of \( BU_r(U_r^T BU_r)^{-1}U_r^T B - B \) belong to \( \text{Null}(U_{n-r}^T) \). \( \square \)
Note 7 For real problems with arbitrary values of $D$ and $T$, the necessary and sufficient condition given in Lemma 4 may not be satisfied, in general. Hence, we are to propose a threshold to determine if

$$F = U_{n-r}^T \left( BU_r(U_r^T BU_r)^{-1} U_r^T B - B \right)$$

is close enough to zero. In the following, we show that if $\|F\| < \delta$, for a sufficiently small scalar $\delta$, then $X_{r(n-r)}$ computed from (21b) is a proper approximation for the solution of (21c). Substituting $F$ in (27), we have

$$\tilde{B}_{n-r,r} \tilde{B}_{r,n-r}^{-1} \tilde{B}_{r,n-r} - \tilde{B}_{n-r,n-r} = FU_{n-r},$$

and

$$\tilde{X}_{n-r,r} S^2 \tilde{X}_{r,n-r} - \tilde{B}_{n-r,n-r} = FU_{n-r}.$$ Let $X^*$ satisfy (21c), that is,

$$X^*_{n-r,r} S^2 X^*_{r,n-r} - \tilde{B}_{n-r,n-r} = 0.$$ Then, we have

$$\tilde{X}_{n-r,r} S^2 \tilde{X}_{r,n-r} - X^*_{n-r,r} S^2 X^*_{r,n-r} = FU_{n-r}. \quad (29)$$ Letting $\tilde{Y} = S \tilde{X}_{r,n-r}$ and $Y^* = SX^*_{r,n-r}$, (29), we get

$$\tilde{Y}^T \tilde{Y} = Y^* Y^* + (FU_{n-r})_{ij}, \quad (30)$$

where $\tilde{y}_i$ and $y^*_i$ are the $i$th column of $\tilde{Y}$ and $Y^*$ respectively. Now, since the 2 norm of each column of $U_{n-r}$ is equal to one, every entry of $U_{n-r}$ is less than or equal to one. Moreover, under the assumption $\|F\| < \delta$, none of the entries of $F$ are greater than $\delta$. Hence, we have

$$|(FU_{n-r})_{ij}| = |f_i^T u_j| \leq |f_i + \cdots + f_{i(n-r)}| < (n-r)\delta, \quad (31)$$

where $f_i^T$ and $u_j$ are the $i$th row of $F$ and the $j$th column of $U_{n-r}$ respectively. Now, (30) together with (31) gives

$$|\tilde{y}_i^T \tilde{y}_j - y^*_i y^*_j| < (n-r)\delta.$$ Hence, there is a constant $c_{ij}$ such that

$$|\tilde{y}_{ij} - y^*_{ij}| < c_{ij}, \quad (32)$$

where $\tilde{y}_{ij}$ and $y^*_{ij}$ are the $(i,j)$th entry of $\tilde{Y}$ and $Y^*$ respectively. Letting $S = \text{diag}(s_1, \ldots, s_r)$, from (32) we get

$$|s_i||(|\tilde{X}_{n-r,r})_{ij} - (X^*_{n-r,r})_{ij}| \leq c_{ij},$$

for $i = 1, \ldots, r$ and $j = 1, \ldots, n-r$ and

$$\|\tilde{X}_{r,n-r} - X^*_{r,n-r}\| \leq C.$$ Hence, assuming

$$\tilde{X}_{r,r} = X^*_{r,r},$$
we have $\|\tilde{X} - X^*\| < \alpha$ which means that if $FU_{n-r}$ is close enough to zero, then the computed solution from the approximate satisfaction of (21c) would be close enough to the exact solution.

In the following lemma, we give a sufficient condition which guarantees the existence of a solution for (18). We later use this result to generate consistent test problems in Section 6.

**Lemma 5** Let the spectral decomposition of $B$ be $B = V \left( \begin{array}{cc} \sum^2 & 0 \\ 0 & 0 \end{array} \right) V^T$, where \( \sum^2 \in \mathbb{R}^{r \times r} \) and \( \text{rank}(A) = \text{rank}(B) = r \). A sufficient condition for satisfaction of (21c) is that

$$V = U \left( \begin{array}{cc} Q & 0 \\ 0 & P \end{array} \right),$$

where $Q \in \mathbb{R}^{r \times r}$ and $P \in \mathbb{R}^{(n-r) \times (n-r)}$ satisfy $QQ^T = Q^T Q = I$ and $PP^T = P^T P = I$.

**Proof** A possible choice for $Z$ in Lemma 5 is zero, for which (28) is equivalent to

$$U_r(U_r^T BU_r)^{-1}U_r^T = B^+ + W,$$

where the columns of $W \in \mathbb{R}^{n \times n}$ belong to the null space of $B$. To obtain a simplified sufficient condition for existence of a positive definite solution to (18), we let $W = 0$. Multiplying (34) by $U_r^T$ and $U_r$ respectively on the left and right, and substituting the spectral decomposition of $B$, we get

$$\left( U_r^T V_r \sum^2 V_r^T U_r \right)^{-1} = U_r^T B^+ U_r = U_r^T V_r \sum^2 V_r^T U_r.$$

Letting $M = U_r^T V_r$, we get

$$\left( M \sum^2 M^T \right)^{-1} = M \sum^2 M^T.$$

Since $M$ has full rank, we get

$$M^{-T} \sum^2 M^{-1} = M \sum^2 M^T.$$

Now, since $\sum^2$ is nonsingular, (36) holds if and only if

$$M^T M = I.$$ (37)

This leads to

$$\left(U_r^T V_r\right)^T U_r^T V_r = V_r^T U_r U_r^T V_r = I.$$ (38)

Since $U$ is orthonormal, we have $UU^T = U_r U_r^T + U_{n-r} U_{n-r}^T = I$. Hence, we get

$$U_r U_r^T = I - U_{n-r} U_{n-r}^T.$$ (39)

Substituting (39) in (38), we get

$$V_r^T (I - U_{n-r} U_{n-r}^T) V_r = I - V_r^T U_{n-r} U_{n-r}^T V_r = I.$$
which is satisfied if and only if \( U_{n-r}^T V_r = 0 \). Since the columns of \( U_r \) form an orthogonal basis for the null space of \( U_{n-r}^T \) [28], it can be concluded that each column of \( V_r \) is a linear combination of the columns of \( U_r \). Thus,

\[
V_r = U_r Q
\]

is a necessary condition for (37) to be satisfied, and since both \( U_r \) and \( V_r \) have orthogonal columns, \( Q \in \mathbb{R}^{r \times r} \) satisfies \( QQ^T = Q^T Q = I \). On the other hand, we know from the definition of the spectral decomposition that \( V V^T = U U^T = I \). Thus,

\[
V_r V_r^T + V_{n-r} V_{n-r}^T = I,
U_r U_r^T + U_{n-r} U_{n-r}^T = I.
\]

(41)

Manipulating (40) with (41), we get

\[
V_{n-r} V_{n-r}^T = U_{n-r} U_{n-r}^T,
\]

(42)

which holds if and only if there exists a matrix \( P \in \mathbb{R}^{(n-r) \times (n-r)} \) such that \( PP^T = P^T P = I \) and

\[
V_{n-r} = P U_{n-r}.
\]

(43)

It can be concluded from (40) and (43) that \( V = U \left( \begin{array}{cc} Q & 0 \\ 0 & P \end{array} \right) \), where \( QQ^T = Q^T Q = I \) and \( PP^T = P^T P = I \).

**Corollary 2** The matrices \( P \) and \( Q \) defined in Lemma 4 can set to be rotation matrices [28] to satisfy

\[
P P^T = P^T P = I,
Q Q^T = Q^T Q = I.
\]

Thus, to compute a target matrix, \( T \), satisfying Lemma 4, it is sufficient to first compute \( V \) from (33) with \( Q \in \mathbb{R}^{r \times r} \) and \( P \in \mathbb{R}^{(n-r) \times (n-r)} \) arbitrary rotation matrices and \( U \) as defined in Lemma 4 and then set \( T = \bar{U} \left( \begin{array}{cc} \sum & 0 \\ 0 & 0 \end{array} \right) V^T \), where \( \bar{U} \in \mathbb{R}^{m \times m} \) and \( \sum \in \mathbb{R}^{r \times r} \) are arbitrary orthonormal and diagonal matrices.

Thus, problem (9) has a solution if and only if the data and target matrices satisfy Lemma 4. In this case, \( \tilde{X}_{rr}, \tilde{X}_{r,n-r} \) and its transpose, \( \tilde{X}_{n-r,r} \), are respectively computed from (21a) and (21b). Hence, the only remaining step is to compute \( \tilde{X}_{n-r,n-r} \) so that \( \tilde{X} \) is symmetric and positive definite.

We know that \( \tilde{X} \) is symmetric positive definite if and only if there exists a nonsingular lower triangular matrix \( L \in \mathbb{R}^{n \times n} \) so that

\[
\tilde{X} = LL^T,
\]

(44)

where \( L \) is lower triangular and nonsingular. Considering the block forms

\[
\tilde{X} = \begin{pmatrix} \tilde{X}_{rr} & \tilde{X}_{r,n-r} \\ \tilde{X}_{n-r,r} & \tilde{X}_{n-r,n-r} \end{pmatrix}
\]
and

\[
L = \begin{pmatrix}
L_{rr} & 0 \\
L_{n-r,r} & L_{n-r,n-r}
\end{pmatrix},
\]

where \(L_{n-r,r}\) is an \((n-r) \times r\) matrix and \(L_{rr} \in \mathbb{R}^{r \times r}\) and \(L_{n-r,n-r} \in \mathbb{R}^{(n-r) \times (n-r)}\) are nonsingular lower triangular matrices, we get

\[
\begin{pmatrix}
\tilde{X}_{rr} & \tilde{X}_{r,n-r} \\
\tilde{X}_{n-r,r} & \tilde{X}_{n-r,n-r}
\end{pmatrix} = \begin{pmatrix}
L_{rr} & 0 \\
L_{n-r,r} & L_{n-r,n-r}
\end{pmatrix} \begin{pmatrix}
L_{rr}^T & 0 \\
0 & L_{n-r,n-r}^T
\end{pmatrix}.
\]

Thus,

\[
\tilde{X}_{rr} = L_{rr}L_{rr}^T, \quad \tilde{X}_{r,n-r} = L_{rr}L_{n-r,r}^T, \quad \tilde{X}_{n-r,r} = L_{n-r,r}L_{rr}^T, \quad \tilde{X}_{n-r,n-r} = L_{n-r,r}L_{n-r,n-r}^T + L_{n-r,n-r}L_{n-r,n-r}^T.
\]

Therefore, to compute a symmetric positive definite \(\tilde{X}\), (46a–46d) must be satisfied. Let \(\tilde{X}_{rr} = \tilde{L} \tilde{L}^T\) be the Cholesky decomposition of \(\tilde{X}_{rr}\). \(L_{rr} = \tilde{L}\) satisfies (46a). Substituting \(L_{rr}\) in (46b), \(L_{n-r,r}^T\) is computed uniquely by solving the resulting linear system. Since (46c) is transpose of (46b), it does not give any additional information. Finally, to compute a matrix \(\tilde{X}_{n-r,n-r}\) to satisfy (46d), it is sufficient to choose an arbitrary lower triangular nonsingular matrix \(L_{n-r,n-r}\) and substitute it in (46d). The resulting \(\tilde{X}_{n-r,n-r}\) gives a symmetric positive definite \(\tilde{X}\) as follows:

\[
\tilde{X} = \begin{pmatrix}
\tilde{X}_{rr} & \tilde{X}_{r,n-r} \\
\tilde{X}_{n-r,r} & \tilde{X}_{n-r,n-r}
\end{pmatrix}.
\]

Now, based on the above discussion, we outline the steps of our algorithm for solving (9) in the case \(\text{rank}(D) = r < n\).

Next, we show how to use the complete orthogonal decomposition of the data matrix \(D\) instead of the spectral decomposition of \(A\).

**Note 8 (Complete Orthogonal Decomposition)** [28] Let \(A \in \mathbb{R}^{m \times n}\) be an arbitrary matrix with \(\text{rank}(A) = r\). There exist \(R \in \mathbb{R}^{r \times r}\), \(U \in \mathbb{R}^{m \times m}\) and \(V \in \mathbb{R}^{n \times n}\) so that \(R \in \mathbb{R}^{r \times r}\) is upper triangular, \(UU^T = U^TU = I\), \(VV^T = V^TV = I\) and \(A = U \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^T\).

Next, Algorithm 4 is presented using the complete orthogonal decomposition of \(D\).

In the following, we discuss finding a particular solution of (9) having proper characteristics.

### 4.2 Particular solution

Based on Algorithms 3 and 4, in the case of rank deficient data matrix, problem (9) has infinitely many solutions. These solutions are generated by having different choices of \(L_{n-r,n-r} \in \mathbb{R}^{(n-r) \times (n-r)}\), an arbitrary nonsingular lower triangular matrix. Here, we describe how to find a particular solution \(X\) having desired characteristics.
Algorithm 3 Solving the EIV model for positive definite linear system with rank deficient data and target matrices using spectral decomposition

**PROCEDURE PDEIV-RD-Spec(D, T, δ)**

1: δ as the upper bounds for absolute error is taken to be close to the machine (or user’s) zero.
2: Let $A = D^T D$ and compute its spectral decomposition:

$$A = U \begin{pmatrix} S^2 & 0 \\ 0 & 0 \end{pmatrix} U^T.$$

3: Let $B = T^T T$ and $\tilde{B} = U^T B U$.
4: Compute $\text{rank}(D) = r$ and let

$$\tilde{B}_{rr} = \tilde{B}(1 : r, 1 : r),$$
$$\tilde{B}_{r,n-r} = \tilde{B}(1 : r, r + 1 : n),$$
$$\tilde{B}_{n-r,n-r} = \tilde{B}(r + 1 : n, r + 1 : n).$$

5: Let $\tilde{D} = S$, assume $\tilde{T}$ satisfies $\tilde{B}_{rr} = \tilde{T}^T \tilde{T}$.
6: Perform Algorithm 1 with input parameters $D = \tilde{D}$ and $T = \tilde{T}$, and let $\tilde{X}_{rr} = X^*.$
7: Solve the linear system (21b) to compute $\tilde{X}_{r,n-r}$ and let $\tilde{X}_{n-r,r} = \tilde{X}_{r,n-r}^T$.
8: if $\|U_{n-r}^T (BU_r(U_r^T BU_r)^{-1}U_r^T B - B)\| \geq \delta$ then stop ((9) has no solution)
9: else
10: Let the Cholesky decomposition of $\tilde{X}_{rr}$ be $\tilde{X}_{rr} = \tilde{L} \tilde{L}^T$ and set $L_{rr} = \tilde{L}$.
11: Solve the lower triangular system (46b) to compute $L_{n-r,r}$.
12: Let $L_{n-r,n-r} \in \mathbb{R}^{(n-r) \times (n-r)}$ be an arbitrary nonsingular lower triangular matrix and compute $\tilde{X}_{n-r,n-r}$ using (46d).
13: Let $\tilde{X} = \begin{pmatrix} \tilde{X}_{rr} & \tilde{X}_{r,n-r} \\ \tilde{X}_{n-r,r} & \tilde{X}_{n-r,n-r} \end{pmatrix}$ and $X^* = U \tilde{X} U^T.$
14: Compute $E = \text{tr}((DX^* - T)(D - T X^*^{-1})).$
15: end if.

for control problems. Effective rank and condition number, defined in the next two definitions, are two important characteristics.

**Definition 1** (Effective Rank [34]) The effective rank of a matrix $X \in \mathbb{R}^{n \times n}$ is defined to be

$$r(X) = \frac{\text{tr}(X)}{\|X\|_2}.$$

**Note 9** For $X$, a symmetric positive definite matrix, by using the spectral decomposition $X = US^2U^T$, the effective rank of $X$ is

$$r(X) = \frac{s_1^2 + \ldots + s_n^2}{s_1^2},$$

where $s_i^2$ is the $i$th diagonal entry of $S^2$. 
Algorithm 4 Solving the EIV model for positive definite linear system with rank-deficient data and target matrices using complete orthogonal decomposition

**PROCEDURE** PDEIV-RD-COD($D$, $T$, $\delta$)

1. $\delta$ as the upper bounds for absolute error is taken to be close to the machine (or user’s) zero.
2. Compute the complete orthogonal decomposition of $D$, that is, $D = U\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^T$.
3. Let $A = D^T D = V_r R^T R V_r^T$, $B = T^T T$ and $\tilde{B} = V^T B V$, where $V_r$ consists of the first $r$ columns of $V$.
4. Compute $\text{rank}(D) = r$ and let $\tilde{B}_{rr} = \tilde{B}(1 : r, 1 : r)$, $\tilde{B}_{r,n-r} = \tilde{B}(1 : r, r+1 : n)$, $\tilde{B}_{n-r,n-r} = \tilde{B}(r+1 : n, r+1 : n)$.
5. Let $\tilde{D} = R$, assume $\tilde{T}$ satisfies $\tilde{B}_{rr} = \tilde{T}^T \tilde{T}$.
6. Perform Algorithm 1 with input parameters $D = \tilde{D}$ and $T = \tilde{T}$, and let $\tilde{X}_{rr} = X^*$.
7. Solve the linear system (21b) to compute $\tilde{X}_{r,n-r}$ and let $\tilde{X}_{n-r,r} = \tilde{X}_{n-r,r}^T$.
8. **if** $\|U_{n-r}^T (BU_r(U_r^T BU_r)^{-1}U_r^T B - B)\| \geq \delta$ **then** stop ($\text{(9)}$ has no solution)
9. **else**
10. Let the Cholesky decomposition of $\tilde{X}_{rr}$ be $\tilde{X}_{rr} = \tilde{L} \tilde{L}^T$ and set $L_{rr} = \tilde{L}$.
11. Solve the lower triangular system (46b) to compute $L_{n-r,r}$.
12. Let $L_{n-r,n-r} \in \mathbb{R}^{(n-r) \times (n-r)}$ be an arbitrary nonsingular lower triangular matrix and compute $\tilde{X}_{n-r,n-r}$ using (46d).
13. Let $\tilde{X} = \begin{pmatrix} \tilde{X}_{rr} & \tilde{X}_{r,n-r} \\ \tilde{X}_{n-r,r} & \tilde{X}_{n-r,n-r} \end{pmatrix}$ and $X^* = V \tilde{X} V^T$.
14. Compute $E = \text{tr}((DX^* - T)(D - TX^*-1))$.
15. **end if.**

**Definition 2** (Condition Number [28]) Assume that $X \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. With the spectral decomposition $X = US^2U^T$, the condition number of $X$ is defined to be

$$\kappa(X) = \frac{s_1^2}{s_n^2}.$$

We will later make use of common constraints on condition number and effective rank of the particular solution of (9), as significant features for control problems.

**Proposition 1** As proper characteristics for control problems, it is appropriate for a solution $X$ of (9) to satisfy the following conditions:

1. $r(X)$ be as low as possible, [34] and
2. $\kappa(X) < K$ [2].

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Note 10 Considering the definition $\tilde{X} = U^T X U$, it can be concluded that $X$ and $\tilde{X}$ have the same effective ranks and condition numbers; see, e.g., [24]. Thus, in the following we discuss on $r(\tilde{X})$ and $\kappa(\tilde{X})$ instead of $r(X)$ and $\kappa(X)$ in Proposition 1.

We know from (45) that

$$X = \begin{pmatrix} L_{rr} L_{rr}^T & L_{rr} L_{n-r,r}^T \\ L_{n-r,r} L_{rr}^T & L_{n-r,r} L_{n-r,r}^T + \mu^2 I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & L_{n-r,n-r} L_{n-r,n-r}^T - \mu^2 I \end{pmatrix}.$$

Defining

$$F = \begin{pmatrix} L_{rr} L_{rr}^T & L_{rr} L_{n-r,r}^T \\ L_{n-r,r} L_{rr}^T & L_{n-r,r} L_{n-r,r}^T + \mu^2 I \end{pmatrix},$$

and

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & L_{n-r,n-r} L_{n-r,n-r}^T - \mu^2 I \end{pmatrix},$$

with $\mu$ being an small positive scalar, we get $X = F + Y$. Note that $F$ is positive definite. Also, assuming that all eigenvalues of $L_{n-r,n-r} L_{n-r,n-r}^T$ are greater than or equal to $\mu^2$, $Y$ is positive semi-definite. In Lemma 6 below, we review some properties of eigenvalues to simplify the specified conditions in Proposition 1.

**Lemma 6** [35] Let $A$ and $B$ be two $n \times n$ symmetric positive semi-definite matrices. The following inequalities hold for eigenvalues of $A$, $B$ and $A + B$, where $\lambda_i(.)$ denotes the $i$th largest eigenvalue of a matrix:

1. $\lambda_1(A + B) \geq \lambda_1(B)$,
2. $\lambda_1(A + B) \leq \lambda_1(A) + \lambda_1(B)$,
3. $\lambda_n(A + B) \geq \lambda_n(A) + \lambda_n(B)$.

Using Lemma 6, we get

$$r(X) = \frac{\lambda_1(F) + \ldots + \lambda_n(F) + \lambda_1(Y) + \ldots + \lambda_{n-r}(Y)}{\lambda_1(X)} \leq \frac{\lambda_1(F) + \ldots + \lambda_n(F) + \lambda_1(Y) + \ldots + \lambda_{n-r}(Y)}{\lambda_1(Y)},$$

and

$$\kappa(X) = \frac{\lambda_1(X)}{\lambda_n(X)} \leq \frac{\lambda_1(F) + \lambda_1(Y)}{\lambda_n(F) + \lambda_n(Y)},$$

where $\lambda_n(Y) = 0$, and since $F$ is nonsingular, $\lambda_n(F) \neq 0$.

Considering (47) and (48), since $F$ and thus $\lambda_i(F)$ are fixed, the sufficient condition to satisfy condition (1) in Proposition 1 is to set $\lambda_1(Y)$ as large as possible and choose $\lambda_2(Y), \lambda_3(Y), \ldots$ and $\lambda_{n-r}(Y)$ to be small positive values to decrease the value of $r(X)$. The largest possible value for $\lambda_1(Y)$ to satisfy condition (2) in Proposition 1 is $\lambda_1(Y) = K \lambda_n(F) - \lambda_1(F)$, which must be greater than or equal to zero; otherwise, condition (2) in Proposition 1 can not be satisfied for the assumed value of $K$. 
Thus, to compute a particular solution of (9) satisfying Proposition 1, it is sufficient to let \( \tilde{X}_{n-r,n-r} \) have a spectral decomposition of the form \( \tilde{X}_{n-r,n-r} = W\Sigma^2W^T \), with \( \sigma_i^2 = K\lambda_n(F) - \lambda_1(F) \) and \( \sigma_i^2, i = 2, \ldots, n-r \), having small positive values. \( K\lambda_n(F) - \lambda_1(F) \) being negative means that condition (2) in Proposition 1 can not be satisfied for the assumed value of the upper bound, \( K \).

In Section 5, we will compare the computational complexity of PDEIV-RD-Spec and PDEIV-RD-COD. Also, based on the reported numerical results in Section 6, we make a comparison of the required computing times by the algorithms.

## 5 Computational complexity

Here, we study the computational complexity of our algorithms for solving the positive definite linear system of equations using our proposed EIV model.

### 5.1 Full column rank data matrix case

The computational complexity of PDEIV-QR presented in Section 3 for the case of full column rank data matrix is same the as solving an \( n \times n \) Riccati equation and is not more than \( N_{PDEIV-QR} = \frac{19}{3}n^3 \); e.g., see [23]. The computational complexity of PDEIV-Spec is given in Table 1; for details on the indicated computational complexities, see [27].

Comparing the resulting complexities of \( N_{PDEIV-QR} \) and \( N_{PDEIV-Spec} \), it can readily be concluded that, independent of the matrix size, the computational complexity of PDEIV-QR is lower than that of PDEIV-Spec.

| Computation                                      | Time complexity |
|--------------------------------------------------|-----------------|
| \( A = D^TD \)                                   | \( \frac{1}{2}mn^2 \) |
| Spectral decomposition for \( A \in \mathbb{R}^{n \times n} \) | \( 4n^3 \) |
| \( B = T^TT \)                                   | \( \frac{1}{2}mn^2 \) |
| \( SU^T \)                                       | \( n^2 \) |
| \( \tilde{Q} = SU^TBU^T \)                       | \( \frac{1}{2}n^3 \) |
| Spectral decomposition of \( \tilde{Q} \in \mathbb{R}^{n \times n*} \) | \( 4n^3 \) |
| \( S^{-1} \)                                     | \( n \) |
| \( US^{-1} \)                                    | \( n^2 \) |
| \( \tilde{U}\tilde{S}\tilde{U}^T \)            | \( \frac{n^3}{2} + n^2 \) |
| \( X^* = US^{-1}\tilde{U}\tilde{S}\tilde{U}^TS^{-1}U^T \) | \( \frac{3}{2}n^3 \) |
| Total time complexity \( N_{PDEIV-Spec} \)       | \( mn^2 + \frac{23}{2}n^3 \) |

*The expected complexity for computing the spectral decomposition of an \( n \times n \) matrix using the divide and conquer algorithm is approximately \( 4n^3 \); e.g., see [27].

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Table 2  Needed computations in PDEIV-RD-Spec and the corresponding computational complexities

| Computation                                                                 | Time complexity                  |
|---------------------------------------------------------------------------|----------------------------------|
| $A = D^TD$                                                                | $\frac{1}{4}mn^2$                |
| Spectral decomposition for $A \in \mathbb{R}^{n \times n}$               | $4n^3$                           |
| $B = T^TT$                                                                | $\frac{1}{2}mn^2$                |
| $\tilde{B} = U^TBU$                                                      | $\frac{3}{2}n^3$                |
| $N_{PDEIV-QR}(r \times r)$ diagonal data matrix                           | $4r^3 + 3r^2$                    |
| Solving the linear system (21c)                                           | $(\frac{n^3}{3} + n^2)(n - r)$   |
| Spectral decomposition for $B \in \mathbb{R}^{n \times n}$              | $2n^3$                           |
| Cholesky decomposition for $\tilde{X}_{rr}$                             | $\frac{r^3}{6}$                 |
| Solving the lower triangular system (46b)                                 | $(\frac{n^3}{3}) (n - r)$        |
| Computing $\tilde{X}_{n-r,n-r}$ from (46d)                               | $r(n - r)^2 + \frac{(n-r)^3}{2}$ |
| Total time complexity                                                     | $N_{PDEIV-RD-Spec}^*$            |

* $N_{PDEIV-RD-Spec} = mn^2 + \frac{15}{2}n^3 + \frac{25}{6}r^3 + \left(\frac{n^3}{3} + n^2\right)(n-r) + \left(\frac{n^3}{3}\right)(n-r) + r(n-r)^2 + \frac{(n-r)^3}{2}$

5.2 Rank deficient data matrix case

The computational complexities of PDEIV-RD-Spec and PDEIV-RD-COD presented in Section 4 for the case of rank deficient data matrix are respectively provided in Tables 2 and 3.

Table 3  Needed computations in PDEIV-RD-COD and the corresponding computational complexities

| Computation                                                                 | Time complexity                  |
|---------------------------------------------------------------------------|----------------------------------|
| Complete orthogonal decomposition for $D \in \mathbb{R}^{m \times n}$     | $4mn^2 - \frac{4}{3}n^3$         |
| $A = D^TD = V_rR^TR_v$                                                   | $nr^2 + \frac{r^3}{3}$           |
| $B = T^TT$                                                                | $\frac{1}{2}mn^2$                |
| $\tilde{B} = V^TBV$                                                      | $\frac{3}{2}n^3$                |
| $N_{PDEIV-QR}(r \times r)$                                              | $\frac{19}{3}r^3$               |
| Solving the linear system (21c)                                           | $(\frac{n^3}{3} + n^2)(n - r)$   |
| Spectral decomposition for $B \in \mathbb{R}^{n \times n}$              | $2n^3$                           |
| Cholesky decomposition for $\tilde{X}_{rr}$                             | $\frac{r^3}{6}$                 |
| Solving the lower triangular system (46b)                                 | $(\frac{n^3}{3}) (n - r)$        |
| Computing $\tilde{X}_{n-r,n-r}$ from (46d)                               | $r(n - r)^2 + \frac{(n-r)^3}{2}$ |
| Total time complexity                                                     | $N_{PDEIV-RD-COD}^*$             |

* $N_{PDEIV-RD-COD} = mn^2 + nr^2 + \frac{13}{6}n^3 + 7r^3 + \left(\frac{n^3}{3} + n^2\right)(n-r) + \left(\frac{n^3}{3}\right)(n-r) + r(n-r)^2 + \frac{(n-r)^3}{2}$
Considering the results for $N_{PDEIV-RD-Spec}$ and $N_{PDEIV-RD-COD}$ in Tables 2 and 3, we have

\[
N_{PDEIV-RD-Spec} - N_{PDEIV-RD-COD} = \left( mn^2 + \frac{15}{2} n^3 + \frac{25}{6} r^3 \right) - \left( \frac{9}{2} mn^2 + nr^2 + \frac{13}{6} n^3 + \frac{49}{6} r^3 \right)
\]

\[
= -4mn^2 + \frac{16}{3} n^3 - nr^2 - \frac{17}{6} r^3.
\]

We can see that if $4mn^2 + nr^2 + \frac{17}{6} r^3 > \frac{16}{3} n^3$, then PDEIV-RD-Spec has a lower computational complexity; otherwise, the computational complexity of PDEIV-RD-COD is lower.

Thus, based on the above study, the computational complexity of PDEIV-QR is lower than that of PDEIV-Spec, for all matrix sizes. But, for the case of rank deficient data matrix, depending on the matrix size and rank, one of the algorithms PDEIV-RD-Spec and PDEIV-RD-COD may have a lower computational complexity.

6 Numerical results

Here, some numerical results are reported. We made use of MATLAB 2012b in a Windows 7 machine with a 3.2 GHz CPU and a 4 GB RAM. We generated random test problems with random data and target matrices. These random matrices were produced using the rand command in MATLAB. The command rand($m$, $n$) generates an $m \times n$ matrix with uniformly distributed random entries in the interval $[0, 1]$.

The random test problems were classified into problems with full column rank data matrix and problems with rank deficient data matrix.

In Section 6.1, we report the numerical results corresponding to full column rank data matrices. For a given matrix size, we generated 50 random test problems and reported the average time and the average error, $E$ values in Table 4. To study the effect of using QR or spectral decompositions in our proposed approach, we constructed the Dolan-Moré performance profile.

The Dolan-Moré performance profile was introduced in [5] to compare the performance of different algorithms on solving a given problem. Here, we used the new version of this performance profile which is derivative free [15].

Table 4 Average time and error values for PDEIV-QR and PDEIV-Spec

| $m$ | $n$ | Time (PDEIV-QR) | E (PDEIV-QR) | Time (PDEIV-Spec) | E (PDEIV-Spec) |
|-----|-----|----------------|-------------|-------------------|---------------|
| 100 | 10  | 0.0021         | 1.6191E+002 | 0.0014           | 1.6191E+002  |
| 100 | 50  | 0.0017         | 7.2274E+002 | 0.0022           | 7.2274E+002  |
| 100 | 100 | 0.0058         | 1.2388E+003 | 0.0072           | 1.2388E+003  |
| 1000| 100 | 0.0089         | 1.6258E+004 | 0.0104           | 1.6258E+004  |
| 1000| 200 | 0.0434         | 3.1684E+004 | 0.0505           | 3.1684E+004  |
The Dolan-Moré performance profile can be generated for different parameters. Since a desired feature in estimation of mass inertia matrix is that the standard deviation value of the resulting error matrix in $T$ be as low as possible, we compare the required times and the standard deviation values in PDEIV-QR and PDEIV-Spec; hence, we present the Dolan-Moré performance profiles for these parameters. It can be concluded from the generated performance profiles in Figs. 2 and 3 that the required time by PDEIV-QR is lower than that of PDEIV-Spec.

Also, to confirm the efficiency of our proposed approach in solving the positive definite linear systems, we reported the numerical results obtained by the interior
point method (IntP), discussed in [26], and the method based on quadratic programming proposed by Hu in [10] (HuM). We then constructed the corresponding Dolan-Moré performance profiles in Figs. 4 and 5. These performance profiles show that our proposed approach is able to compute a solution with a smaller value of standard deviation in less computing time. However, considering the ordinary formulation in IntP and HuM, the more complicated optimization problem,

$$\min \|DX - T\|_F^2,$$

$$X > 0,$$

Fig. 4 The Dolan-Moré performance profile (comparing the required time by PDEIV-QR, PDEIV-Spec, IntP and HuM)

Fig. 5 The Dolan-Moré performance profile (comparing the standard deviation values for PDEIV-QR, PDEIV-Spec, IntP and HuM)
is needed to be solved; hence, a lower computing time in our proposed approach is not surprising.

In Section 6.2, the numerical results for test problems with rank deficient data matrices are reported. In such test problems, generating an $m \times n$ random data matrix with column rank $r$ is necessary. Hence, we first used the command

$$R = \text{rand}(m, n)$$

to generate a full column rank $m \times n$ random matrix, and then set the data matrix $D$ to be equal to its singular value decomposition (SVD) of rank $r$,

$$[U, S, V] = \text{svd}(R)$$

$$D = U(:, 1 : r) \ast S(1 : r, 1 : r) \ast (V')(1 : r, :).$$

Also, the target matrix $T$ was computed from Corollary 2.

For a given matrix size and rank, we generated 50 test problems. Similar to Section 6.1, we report the average required time and average value of error, $E$ in Table 5. We also studied the effect of using complete orthogonal decomposition and spectral decomposition in the proposed approach. To compare the efficiency of these decompositions, we constructed the Dolan-Moré performance profiles of required times and standard deviation values for the numerical results produced by PDEIV-RD-Spec and PDEIV-RD-COD in Figs. 6 and 7. Our proposed approach was also compared with the other available methods based on the Dolan-Moré performance profiles as presented in Figs. 8 and 9. Also, we computed the particular solution of (9), choosing appropriate values for eigenvalues of the matrix $Y$ based on the discussion at the end of Section 4 to satisfy conditions given in Proposition 1. We presented the Dolan-Moré performance profiles of effective rank and condition number in Figs. 10 and 11 confirming the efficiency of our proposed algorithm in generating solutions with lower values of effective rank and condition number.

Numerical results also confirmed the effectiveness of Algorithms 1 through 4 in producing more accurate solutions with lower standard deviation values in lower times.

### Table 5: Average time and error values for PDEIV-RD-Spec and PDEIV-RD-COD

| $m$ | $n$ | $r$ | Time (Spec) | E (Spec) | Time (COD) | E (COD) |
|-----|-----|-----|-------------|----------|------------|--------|
| 100 | 10  | 5   | 3.6377E−004 | 1.8733E+002 | 6.3001E−004 | 1.8733E+002 |
| 100 | 50  | 20  | 1.4125E−003 | 2.0468E+003 | 1.6243E−003 | 2.0468E+003 |
| 100 | 50  | 50  | 5.1234E−003 | 3.9126E+003 | 5.9146E−003 | 3.9126E+003 |
| 1000| 100 | 50  | 6.3142E−003 | 2.0047E+004 | 1.2843E−002 | 1.6258E+004 |
| 1000| 200 | 100 | 3.0763E−002 | 5.8443E+004 | 4.3702E−002 | 5.8443E+004 |
6.1 Full column rank data matrix

In Table 4, the average error value, \( E = tr(DX^* - T)^T (D - TX^*-1) \), and the average required times (in seconds) are reported for PDEIV-QR and PDEIV-Spec. The first two columns of this table contain the matrix size, the third to sixth columns give the time and error for PDEIV-QR and the time and error for PDEIV-Spec, respectively.
The reported results in Table 4 show that PDEIV-QR is faster in computing the solution. Also, the Dolan-Moré performance profile for the required times by these algorithms given in Fig. 2 confirms this result.

However, based on the Dolan-Moré performance profile for the standard deviation value showed in Fig. 3, there is no significant difference between the standard deviation values generated by the two algorithms.

In the following, we compare our proposed approach with the available methods. The Dolan-Moré performance profiles for the times and standard deviation values shown in Figs. 4 and 5 confirm the efficiency of our proposed approach in computing

---

**Fig. 8** The Dolan-Moré performance profile (comparing the required time by PDEIV-RD-Spec, PDEIV-RD-COD, IntP and HuM)

**Fig. 9** The Dolan-Moré performance profile (comparing the standard deviation values for PDEIV-RD-Spec, PDEIV-RD-COD, IntP and HuM)
a solution with lower value of standard deviation of error in lower time compare to IntP and HuM. 1000 random test problems with data and target matrices size less than $1000 \times 200$ are generated to construct the Dolan-Moré performance profiles.

**Note 11** There are three important points about the value of error bound, TOL, in the iterative methods IntP and HuM:

![Fig. 10](image1.png) **Fig. 10** The Dolan-Moré performance profile (comparing the values of effective rank for PDEIV-RD-Spec, IntP and HuM)

![Fig. 11](image2.png) **Fig. 11** The Dolan-Moré performance profile (comparing the values of condition number for PDEIV-RD-Spec, IntP and HuM)
Both IntP and HuM as iterative methods need a TOL, the maximum value of possible error, as an input.

Smaller values of TOL result in larger computing times without any considerable change in standard deviation values.

For all matrix sizes, except for $1000 \times 200$, the value of TOL was taken to be $10^{-6}$. In $1000 \times 200$ case, the value of TOL was set to 0.001 considering the fact that with a smaller value of TOL, an out of memory error occurred for both algorithms IntP and HuM.

### 6.2 Rank deficient data matrix

Here, we report the obtained numerical results, similar to Section 6.1, for test problems with rank deficient data matrix. In Table 5 and Figs. 6 and 7, we see numerical results obtained by PDEIV-RD-Spec and PDEIV-RD-COD. In Table 5, the average error value and the required time for PDEIV-RD-Spec and PDEIV-RD-COD are reported.

In Figs. 6 and 7, the Dolan-Moré performance profiles for time and standard deviation values of PDEIV-RD-Spec and PDEIV-RD-COD are shown.

These results show that PDEIV-RD-Spec computes the solution faster, but there is no significant difference in the obtained standard deviations.

To compare our proposed approach with IntP and HuM, the Dolan-Moré profiles corresponding to required times and standard deviations for IntP, HuM, PDEIV-RD-Spec and PDEIV-RD-COD are represented in Figs. 8 and 9. To construct these profiles, 1000 random test problems with data and target matrices size less than $1000 \times 100$ are generated. The Dolan-Moré profiles confirm that PDEIV-RD-Spec and PDEIV-RD-COD compute solutions with lower values of standard deviation in lower times.

The Dolan-Moré performance profiles for effective rank and condition number presented in Figs. 10 and 11 confirm the efficiency of our proposed algorithm in generating solutions with lower values of effective rank and condition number.

Considering the numerical results reported in this section, for the data matrix $D$ having full column rank, we observe:

1. Required time by PDEIV-QR is lower than that of PDEIV-Spec.
2. Required time and standard deviation in error entries for PDEIV-QR and PDEIV-Spec are considerably lower than those of IntP and HuM.

And, if the data matrix is rank deficient, we observe:

1. Required time by PDEIV-RD-Spec is lower than that of PDEIV-RD-COD.
2. Required time and standard deviation values for both PDEIV-RD-Spec and PDEIV-RD-COD are considerably lower than those of IntP and HuM, and the standard deviation values for PDEIV-RD-Spec is lower than those of the other three methods.
3. PDEIV-RD-Spec can generate particular solutions with considerably lower values of effective rank and condition number than IntP and HuM.
7 Concluding remarks

We first proposed a new error in variables (EIV) model to solve positive definite systems. We then established the necessary and sufficient conditions of optimality for the proposed EIV model and outlined an algorithm to solve a positive definite linear system, offering three main desirable features. First, consideration of our proposed EIV model in both the data and target matrices admits a more realistic problem formulation. Second, our proposed algorithm computes the exact solution directly, and, as shown by our numerical results on randomly generated test problems, is faster than two other existing methods, IntP and HuM. The generated Dolan-More performance profiles also confirm the efficiency of our proposed algorithm in computing a solution faster than the other methods. The lower computing time is mainly due to fact that the generated optimization problem by our newly defined EIV model is not as complicated as the one considered by the other methods. Numerical results showed lower standard deviation of the error in the target matrix and lower values of effective rank and condition number, as desired by control problems. The new approach lead us to the development of algorithms for the EIV model of positive definite linear system using QR decomposition (PDEIV-QR) and the EIV model of positive definite linear system using spectral decomposition (PDEIV-Spec) for the case of data matrix having full column rank, and the EIV model of positive definite linear system with rank deficient data and target matrices using spectral decomposition (PDEIV-RD-Spec) and the EIV model of positive definite linear system with rank deficient data and target matrices using complete orthogonal decomposition (PDEIV-RD-COD) for the case of rank deficient data matrix. The numerical results also showed PDEIV-QR, using the QR decomposition, to compute the solution faster in the case of full column rank data matrix. However, PDEIV-RD-Spec showed to be more efficient when the data matrix is rank deficient.

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References

1. Alizadeh, F., Pierre, J., Heaberly, A., Overton, M.L.: Primal-dual interior point methods for semidefinite programming: convergence rates, stability and numerical result. SIAM J. Optim. 8, 746–768 (1998)
2. Aubry, A., Maio, A.D., Pallotta, L., Farina, A.: Maximum likelihood estimation of a structured covariance matrix with a condition number constraint. IEEE Trans. On Signal Processing 60(6), 3004–3021 (2012)
3. Cheng, C.L., Kukush, A., Mastronardi, N., Paige, C., Van Huffel, S.: Total least squares and errors-in-variables modeling. Comput. Stat. Data Anal. 52, 1076–1079 (2007)
4. Deng, Y., Boley, D.: On the optimal approximation for the symmetric procrustes problems of the matrix equation AXB = C. Proceedings of the International Conference on Computational and Mathematical Methods in Science and Engineering, Chicago, pp. 159–168 (2007)
5. Dolan, E.D., Moré, J.J.: Benchmarking optimization software with performance profiles. Math. Program. 91, 201–213 (2012)
6. Golub, G.H., Van Loan, C.F.: An analysis of the total least squares problem. SIAM J. Numer. Anal. 17, 883–893 (1980)
7. Hayami, K., Yin, J.F., Ito, T.: GMRES method for least squares problems. SIAM J. Matrix Anal. Appl. 31(5), 2400–2430 (2010)
8. Hnetynková, I., Plesinger, M., Sima, D.M., Strakos, Z., Van Huffel, S.: The total least squares problem in $AX \approx B$, A new classification with the relationship to the classical works. SIAM J. Matrix Anal. Appl. 32(3), 748–770 (2011)
9. Hu, H., Olkin, I.: A numerical procedure for finding the positive definite matrix closest to a patterned matrix. Stat. Probabil. Lett. 12, 511–515 (1991)
10. Hu, H.: Positive definite constrained least-squares estimation of matrices. Linear Algebra Appl. 229, 167–174 (1995)
11. Van Huffel, S., Vandewalle, J.: Algebraic connections between the least squares and total least squares problems. Numer. Math. 55, 431–449 (1989)
12. Kang, B., Jung, S., Park, P.: A new iterative method for solving total least squares problem. Proceeding of the 8th Asian Control Conference (ASCC). Kaohsiung (2011)
13. Larson, H.J.: Least squares estimation of the components of a symmetric matrix. Technometrics 8(2), 360–362 (1966)
14. McInroy, J., Hamann, J.C.: Design and control of flexure jointed hexapods. IEEE Trans. Robot. Autom. 16(4), 372–381 (2000)
15. Moré, J.J., Wild, S.M.: Benchmarking derivative-free optimization algorithms. SIAM J. Optim. 20, 172–191 (2009)
16. Paige, C.C., Strakos, Z.: Scaled total least squares fundamentals. Numer. Math. 91, 117–146 (2000)
17. Poignet, P., Gautier, M.: Comparison of weighted least squares and extended kalman filtering methods for dynamic identification of robots. Proceedings of the IEEE Conference on Robotics and Automation, San Francisco, pp. 3622–3627 (2000)
18. Woodgate, K.G.: Least-squares solution of $F = PG$ over positive semidefinite symmetric $P$. Linear Algebra Appl. 245, 171–190 (1996)
19. Zhou, L., Lin, L., Wei, Y., Qiao, S.: Perturbation analysis and condition numbers of scaled total least squares problems. Numer. Algorithms 51, 381–399 (2009)
20. Banerjee, S., Roy, A.: Quadratic Forms, Linear Algebra and Matrix Analysis for Statistics. Chapman & Hall/CRC Texts in Statistical Sciences, pp. 441–442 (2014)
21. Carlen, E.: Trace inequalities and quantum entropy: an introductory course. Contemporary Math. AMS 529, 73–140 (2009)
22. Gill, P.E., Murray, W., Wright, M.H.: Numerical Linear Algebra and Optimization. Addison Wesley (1991)
23. Higham, N.J.: Functions of Matrices: Theory and Computation. SIAM, Philadelphia (2008)
24. Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge University Press (1991)
25. Van Huffel, S., Vandewalle, J.: The Total Least Squares Problem: Computational Aspects and Analysis. SIAM, Philadelphia (1991)
26. Krislock, N.G.: Numerical Solution of Semidefinite Constrained Least Squares Problems, M. Sc. Thesis, University of British Colombia (2003)
27. Demmel, J.W. Applied Numerical Linear Algebra, 3rd edn. SIAM, Philadelphia (1996)
28. Golub, G.H., Van Loan, C.F. Matrix Computation, 4th edn. JHU Press (2012)
29. Lancaster, P., Rodman, L.: Algebraic Riccati Equations. Clarendon Press (1995)
30. Magnus, J.R., Neudecker, H. Matrix Differential Calculus with Applications in Statistics and Econometrics, 2nd edn. Wiley (1999)
31. Nocedal, J., Wright, S.J.: Numerical Optimization. Springer, New York (1999)
32. Higham, N.J. Computing the nearest correlation matrix (A problem from nance), MIMS EPrint: 2006.70. http://eprints.ma.man.ac.uk/ (2006). Accessed 26 June 2012
33. Petersen, K.B., Pedersen, M.S. The matrix cookbook. http://orion.uwaterloo.ca/hwolkowi/matrixcookbook.pdf (2008). Accessed 11 January 2013
34. Vershynin, R. Introduction to the non-asymptotic analysis of random matrices. http://arxiv.org/pdf/1011.3027v7.pdf (2011). Accessed 01 February 2013
35. American Mathematical Society, Eigenvalues and sums of Hermitian matrices. http://www.ams.org/bookstore/pspdf/gsm-132-prev.pdf (2009). Accessed 18 March 2013