THE COMBINATORIAL PART OF THE COHOMOLOGY OF A SINGULAR VARIETY

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Abstract. We study the first step of the weight filtration on the cohomology of a proper complex algebraic variety, which we call the combinatorial part. We obtain a natural upper bound on its size, which gives rather strong information about the topology of rational singularities.

Given a possibly reducible complex algebraic variety $X$, we define the combinatorial part of the compactly supported cohomology to a subspace $KH^i_c(X) \subseteq H^i_c(X, \mathbb{Z})$ characterized by the following axioms:

(K1) These subspaces are preserved by proper pullbacks.
(K2) If $X$ is smooth and complete, $KH^0_c(X) = H^0_c(X)$ and $KH^i_c(X) = 0$ for $i > 0$.
(K3) If $U \subseteq X$ is an open immersion and $Z = X - U$, then the standard exact sequence

$$\ldots H^{i-1}_c(Z) \to H^i_c(U) \to H^i_c(X) \to \ldots$$

restricts to an exact sequence

$$\ldots KH^{i-1}_c(Z) \to KH^i_c(U) \to KH^i_c(X) \to \ldots$$

The proof of uniqueness, when $X$ is complete, given below is a simple induction. Existence will follow by identifying $KH^i_c(X)$ with the first step of the weight filtration $W_0H^i_c(X)$ of Deligne [D] and Gillet-Soulé [GS]. It will be both convenient and necessary to review the basic construction which gives a method for calculating this in terms of the underlying combinatorics of a simplicial resolution. In simple cases, such as when $X$ has simple normal crossing singularities, this can be made quite explicit. We note that in this paper varieties are reduced schemes of finite type over $\mathbb{C}$. We can extend this an arbitrary complex scheme of finite type $X$, by defining $KH^i_c(X) = KH^i_c(X_{\text{red}})$.

Work of Stepanov [S] and the second author [B] suggested a certain natural bound on the dimension of the combinatorial part of cohomology of the exceptional divisor of a singularity. The main purpose of this note is to verify this in a refined form. Given a proper map of varieties $f : X \to Y$, we show that $\dim KH^i(f^{-1}(y))$ is bounded above by $\dim(R^if_*\mathcal{O}_X)_y \otimes \mathcal{O}_y/m_y$. In particular, in accordance with a conjecture of Stepanov, the first space vanishes for a resolution of a rational singularity.

1. Uniqueness for complete varieties

As a warm up, we prove the uniqueness statement for complete varieties. For this it is convenient to replace (K3) by (K3') below.

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Lemma 1.1. Assume that $KH^i_\mathcal{L}$ satisfies the axioms (K1)-(K3). Given a complete variety $X$ with closed set $S$ and a desingularization $f : \tilde{X} \to X$ which is an isomorphism over $X - S$. Let $E = f^{-1}(S)$.

(K3') Then there is an exact sequence

$$\ldots \to KH^{i-1}(E) \to KH^i(X) \to KH^i(\tilde{X}) \oplus KH^i(S) \to \ldots$$

Proof. This follows from diagram chase on

$$\begin{array}{ccccccc}
KH^{i-1}(S) & \to & KH^i(U) & \to & KH^i(X) & \to & KH^i(S) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
KH^{i-1}(E) & \to & KH^i(U) & \to & KH^i(\tilde{X}) & \to & KH^i(E)
\end{array}$$

\[\square\]

Remark 1.2. In general, by the same argument we get a sequence

$$\ldots \to KH^{i-1}_\mathcal{L}(E) \to KH^i_\mathcal{L}(X) \to KH^i_\mathcal{L}(\tilde{X}) \oplus KH^i_\mathcal{L}(S) \to \ldots$$

Lemma 1.3. Assume that $KH^i$ satisfies the axioms (K1), (K2) and (K3'). Given a complete variety $X$ with a closed set $S$ and a desingularization $f : \tilde{X} \to X$ which is an isomorphism over $X - S$. Let $\tilde{S} \to S$ be a desingularization of $S$ and $F = \tilde{S} \times_X \tilde{X}$. Then there is an exact sequence

$$\ldots \to KH^{i-1}(F) \to KH^i(X) \to KH^i(\tilde{X}) \oplus KH^i(\tilde{S}) \to \ldots$$

Proof. Consider the diagram

$$\begin{array}{ccc}
\tilde{F} & \to & \tilde{X} \times \tilde{S} \\
\downarrow & & \downarrow f \\
F & \to & \tilde{X} \\
\downarrow & & \downarrow p \\
S & \to & X \times \tilde{S} \\
\downarrow \gamma & & \downarrow \gamma \\
S & \to & X \\
\downarrow & & \downarrow p \\
X & & \\
\end{array}$$

where the maps labelled $p$ are projections, $\gamma$ is the graph of the composition $\tilde{S} \to S \to X$, and $\tilde{F} = \tilde{S} \times_X S (\tilde{X} \times \tilde{S})$. The lefthand square containing $f$ and $id_s$ is easily seen to be Cartesian. Therefore $f$ gives an isomorphism $\tilde{F} \cong F$. Thus from the previous lemma, we obtain an exact sequence

$$\ldots \to KH^{i-1}(F) \to KH^i(X \times S) \to KH^i(\tilde{X} \times S) \oplus KH^i(\tilde{S}) \to \ldots$$

Choose base points $s_1, \ldots, s_N$ in each connected component of $S$. Define $\sigma = \frac{1}{N} \sum_j (id \times s_j)^* : H^i(X \times S) \to H^i(X)$. This gives a left inverse to $p^*$. Then
There is at most one collection of subspaces $KH^i(X) \subseteq H^i(X,\mathbb{Z})$, with $X$ complete, satisfying axioms (K1), (K2) and (K3).

Proof. We prove this by induction on $i$. First we check that $KH^0(X) = H^0(X)$. We can assume that $X$ is connected. If $p \in X$, then

$$H^0(X) = H^0(p) = KH^0(p) = KH^0(X)$$

by the axioms.

By lemma 1.3 there is an exact sequence

$$KH^{i-1}(F) \to KH^i(X) \to KH^i(\tilde{X}) \oplus KH^i(\tilde{S}) = 0$$

So $KH^i(X) = im[KH^{i-1}(F) \to H^i(X)]$. □

2. Simplicial resolutions

The general construction is based on simplicial resolutions. We start by recalling some standard material [D, GNPP, PS]. A simplicial object in a category is a diagram

$$\cdots X_2 \xrightarrow{\delta_1} X_1 \xrightarrow{\delta_0} X_0$$

with $n$ face maps $\delta_i : X_n \to X_{n-1}$ satisfying the standard relation $\delta_i \delta_j = \delta_{j-1} \delta_i$ for $i < j$; this would be more accurately called a “strict simplicial” or “semisimplicial” object since we do not insist on degeneracy maps going backwards. The basic example of a simplicial set, i.e. simplicial object in the category of sets, is given by taking $X_n$ to be the set of $n$-simplices of a simplicial complex on an ordered set of vertices. Let $\Delta^n$ be the standard $n$-simplex with faces $\delta'_i : \Delta^{n-1} \to \Delta^n$.

Given a simplicial set or more generally a simplicial topological space, we can glue the $X_n \times \Delta^n$ together by identifying $(\delta_i x, y) \sim (x, \delta'_i y)$. This leads to a topological space $|X_\bullet|$ called the geometric realization, which generalizes the usual construction of the topological space associated a simplicial complex.

Given a simplicial space, filtering $|X_\bullet|$ by skeleta $\bigcup_{n \leq N} X_n \times \Delta^n / \sim$ yields the spectral sequence

$$E_1^{pq} = H^q(X_p, A) \Rightarrow H^{p+q}(|X_\bullet|, A) \quad (1)$$

for any abelian group $A$. It is convenient to extend this. A simplicial sheaf on $X_\bullet$ is a collection of sheaves $\mathcal{F}_n$ on $X_n$ with “coface” maps $\delta^* \mathcal{F}_{n-1} \to \mathcal{F}_n$ satisfying the face relations. For example, the constant sheaves $\mathbb{Z}_{X_\bullet}$ with identities for coface maps forms a simplicial sheaf. If $X_\bullet$ is a simplicial object in the category of complex manifolds, then $\Omega^*_{X_\bullet}$ with the obvious maps, forms a simplicial sheaf. We can define cohomology by setting

$$H^i(X_\bullet, \mathcal{F}_\bullet) = Ext^i(\mathbb{Z}_{X_\bullet}, \mathcal{F}_\bullet)$$
This generalizes sheaf cohomology in the usual sense, and it can be extended to the case where $F_\bullet$ is a bounded below complex of simplicial sheaves by using a hyper $Ext$. When $F = A$ is constant, this coincides with $H^i([X_\bullet], A)$. But in general the meaning is more elusive. There is a spectral sequence
\begin{equation}
E^{pq}_1(F_\bullet) = H^q(X_p, F_p) \Rightarrow H^{p+q}(X_\bullet, F_\bullet)
\end{equation}
generalizing (1). Filtering $F_\bullet$ by the “stupid filtration” $F^{\geq n}_\bullet$ yields a different spectral sequence
\begin{equation}
E^{pq}_1 = H^q(X_\bullet, F^p_\bullet) \Rightarrow H^{p+q}(X_\bullet, F^\bullet_\bullet)
\end{equation}

**Theorem 2.1** (Deligne). If $X_\bullet$ is a simplicial object in the category of compact Kähler manifolds and holomorphic maps. The spectral sequence (1) degenerates at $E_2$ when $A = Q$.

**Remark 2.2.** The theorem follows from a more general result in [DGMS] 8.1.9. However the argument is very complicated. Fortunately, as pointed out in [DGMS], this special case follows easily from the $\partial \bar{\partial}$-lemma. Here we give a more complete argument.

**Proof.** It is enough to prove this after tensoring with $\mathbb{C}$. We can realize the spectral sequence as coming from the double $(E^\bullet(X_\bullet), d, \pm \delta)$, where $(E\bullet, d)$ is the $C^\infty$ de Rham complex, and $\delta$ is the combinatorial differential. (We are mostly going to ignore sign issues since they are not relevant here.) In fact this is a triple complex, since each $E^\bullet(-)$ is the total complex of the double complex $(E^{\bullet\bullet}(-), \partial, \bar{\partial})$.

Given a class $[\alpha] \in H^i(X_j)$ lying in the kernel of $\delta$, we have $\delta \alpha = d \beta$ for some $\beta \in E^{i-1}(X_{j+1})$ Then $d_2([\alpha])$ is represented by $\delta \beta \in E^{i-1}(X_{j+2})$. We will show this vanishes in cohomology. The ambiguity in the choice of $\beta$ will turn out to be the key point.

By the Hodge decomposition, we can assume that $\alpha$ is pure of type $(p, q)$. Therefore $\delta \alpha$ is also pure of this type. We can now apply the $\partial \bar{\partial}$-lemma [GH p 149] to write $\alpha = \partial \bar{\partial} \gamma$ where $\gamma \in E^{p-1,q-1}(X_{j+1})$. This means we have two choices for $\beta$. Taking $\beta = \partial \gamma$ shows that $d_2([\alpha])$ is represented by a form of pure type $(p-1, q)$. On the other hand, taking $\beta = -\bar{\partial} \gamma$ shows that this class is of type $(p, q-1)$. Thus $d_2([\alpha]) \in H^{p-1,q} \cap H^{p,q-1} = 0$.

By what we just proved $\delta \alpha = d \beta, \delta \beta = d \eta$, and $\delta \eta$ represents $d_3([\alpha])$. It should be clear that one can kill this and higher differentials in the exact same way.

**Corollary 2.3.** With the same assumptions as the theorem, the spectral sequence (2) degenerates at $E_2$ when $F = O_{X_\bullet}$.

(This fixes an incorrect proof in [S] 2.4.)

**Proof.** By the Hodge theorem, the spectral sequence for $F = O_{X_\bullet}$ is a direct summand of the spectral sequence for $F = \mathbb{C}$. \qed

**Theorem 2.4** (Deligne). Given any (possibly reducible) variety $X$, there exists a smooth simplicial variety $X_\bullet$, which we call a simplicial resolution, with proper morphisms $\pi_\bullet : X_\bullet \to X$ (commuting with face maps) inducing a homotopy equivalence between $|X_\bullet|$ and $X$. Given a morphism $f : X \to Y$ there exists simplicial resolutions $X_\bullet, Y_\bullet$ and a morphism $f_\bullet : X_\bullet \to Y_\bullet$ compatible with $f$. \qed
The theorem is a consequence of resolution of singularities. Proofs can be found in [D, CNPP, PS]. Note that the original construction of Deligne results in a necessarily infinite diagram, whereas the method of Guillen et. al yields a fairly economical resolution. Here are some examples.

Example 2.5. Suppose that \( X \) is an analytic space, whose irreducible components \( X^i \) are compact Kähler, and suppose that their intersections \( X^{ij} \) are all smooth. This includes the case of a divisor with simple normal crossings. Then an explicit simplicial resolution is given by taking \( X \) to be the disjoint union of \((i+1)\)-fold intersection of components of \( X \). The face map \( \delta_k \) is given by inclusions

\[
X^{i_1 \ldots i_n} \subset X^{i_1 \ldots i_k \ldots i_n} \quad (i_1 < \ldots < i_k)
\]

Remark 2.6. The above construction makes perfect sense for general \( X \), and it yields a (generally singular) simplicial variety \( X_\bullet \) with \( |X_\bullet| \) homotopic to \( X \). \( X_\bullet \) will always be dominated by a simplicial resolution.

Example 2.7. Following the method of [CNPP] we can construct a simplicial resolution of a variety \( X \) with isolated singularities as follows. Let \( f : \tilde{X} \to X \) be a resolution of singularities such that the exceptional divisor \( E = \bigcup E^i \) is a divisor with simple normal crossings. Write \( E^{ij} = E^i \cap E^j \) and \( E_n = \coprod E^{i_0 \ldots i_n} \). Let \( S_0 \subset X \) be set of singular points, \( S_1 \subset S_0 \) be the set of images of \( \bigcup E^{ij} \) and so on. Then the simplicial resolution is given by

\[
\cdots E_1 \sqcup S_2 \longrightarrow E_0 \sqcup S_1 \longrightarrow \tilde{X} \sqcup S_0
\]

where the face maps are given by inclusions \( S_i \to S_{i-1} \) on the second component. On the first component \( \delta_k \) is given by

\[
\begin{cases}
E^{i_1 \ldots i_n} \subset E^{i_1 \ldots i_k \ldots i_n} & \text{if } k \leq n \\
f : E^{i_1 \ldots i_n} \to S_{n-1} & \text{if } k = n + 1
\end{cases}
\]

Given a simplicial resolution, the spectral sequence (1) will then converge to \( H^*(X, A) \). More generally for any sheaf, there is an isomorphism

\[
H^i(X, \mathcal{F}) \cong H^i(X_\bullet, \pi_\bullet^* \mathcal{F})
\]

for any sheaf \( \mathcal{F} \) on \( X \). The last property goes by the name of cohomological descent.

Given a closed subvariety \( \iota : Z \subset X \), there exists simplicial resolutions \( Z_\bullet \to Z \), \( X_\bullet \to X \) and a morphism \( \iota_\bullet : Z_\bullet \to X_\bullet \) covering \( \iota \). Then there is a new smooth simplicial variety \( \text{cone}(\iota_\bullet) \) ([D §6.3], [CNPP IV §1.7]) whose geometric realization is homotopy equivalent to \( X/Z \). So that the spectral sequence (1) converges to \( H^*_c(X - Z, A) \). Although simplicial resolutions are far from unique, the filtration on \( H^*_c(X - Z, A) \) is the weight filtration \( W \) [GS], and this is canonically determined by \( X - Z \) alone. When \( A = \mathbb{Q} \), this part of the datum of the canonical mixed structure.

Let \( X \) be a proper variety with a possibly empty closed set \( Z \). Let \( U = X - Z \). Choose a simplicial resolution \( C_\bullet = \text{Cone}(Z_\bullet \to X_\bullet) \) as above. By convention \( W \) is an increasing filtration indexed so that

\[
W_q H^{p+q}_c(U)/W_{q-1} = E_{q+1}^p \cong E_{q+1}^{p+1} \quad \text{over } \mathbb{Q}
\]

In particular, \( W_{-1} = 0 \). The part of interest \( W_0 \), can be computed as follows. We can form a simplicial set by applying the connected component functor \( \pi_0 \) to \( C_\bullet \).
This simplicial set $|\pi_0(C_b)|$ is called the dual complex or nerve of the simplicial resolution. We have
\[ W_0H^i_c(U, \mathbb{Q}) \cong H^i(\ldots \to H^0(C_p, \mathbb{Q}) \to H^0(C_{p+1}, \mathbb{Q}) \ldots) \cong H^i(|\pi_0(C_b)|, \mathbb{Q}) \]
For integer coefficients, $W_0H^i(U, \mathbb{Z}) = \pi^+H^i(|\pi_0(C_b)|, \mathbb{Z})$ where $\pi : C_b \to \pi_0(C_b)$ is the constant map on components. So that this piece of the filtration is determined by the underlying combinatorial information encoded by the dual complex.

**Theorem 2.8.** There is a collection of subspaces $KH^i_c(X) \subseteq H^i_c(X, \mathbb{Z})$ satisfying axioms (K1)-(K3) given in the introduction. Moreover, it is uniquely characterized by axioms.

**Proof.** For existence, we note that $KH^i_c(X) = W_0H^i_c(X)$ satisfies these axioms by [GS] [3.1]. (For rational coefficients, this goes back to [D].)

So it remains to check uniqueness. We already checked this when $X$ is complete. The nonsingular case follows from this. If $X$ is nonsingular, we can choose a nonsingular compactification $\bar{X}$. Then from the axioms, we get $KH^i_c(X) = \text{im}(KH^{i-1}(\bar{X} - X))$. Then the general case now follows from the main theorem of [GN] together with remark [1.2] \qed

From the formula $K = W_0$, we can deduce further properties.

**Corollary 2.9.** $KH^i_c(X \times Y) = \bigoplus_{j+k=i} KH^j_c(X) \otimes KH^k_c(Y)$

**Proof.** [GS] thm 3]. \qed

**Corollary 2.10.** Let $\pi : \bar{X} \to X$ be a resolution of a complete variety such that the exceptional divisor $E$ has normal crossings. Let $S = \pi(E) \subset X$. Then $\dim KH^i(X)$ is the $(i-1)$st Betti number $b_{i-1}$ of the dual complex of $E$ when $i > 2 \dim(S) + 1$. If $S$ is nonsingular, then this holds for $i > 1$. When $i = 2 \dim(S) + 1$, $\dim KH^i(X) = b_{i-1} - \text{number of irreducible components of } S \text{ of maximum dimension}$.

**Proof.** This follows from lemma [1.1] the identification of $KH^i(E) = W_0H^i(E)$ and the above remarks. \qed

When $X$ is a divisor with simple normal crossings, $KH^i_c(X)$ is the cohomology of the dual complex. As remarked earlier [2.3] we can use a construction to build a simplicial variety canonically attached to $X$, for any $X$. If we apply $\pi_0$ to this simplicial variety, we get a simplicial set $\Sigma_X$ canonically attached to $X$, that we will call the nerve or dual complex. There is a canonical map $H^i(|\Sigma_X|, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ coming from the spectral sequence (1) associated to this simplicial variety. From the discussion in [2.5] and [2.6] we can see that:

**Lemma 2.11.** If $X$ is complete, the image $H^i(|\Sigma_X|, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ lies in $KH^i(X, \mathbb{Q})$. If $X$ satisfies the assumptions of example [2.5] then these subspaces coincide.

3. **Bounds on the combinatorial part**

Suppose that $X$ is a complete variety. Then in addition to the weight filtration $H^i(X, \mathbb{C})$ carries a second filtration, called the Hodge filtration induced on the abutment $H^i(X, \Omega^*_X) \cong H^i(X, \mathbb{C})$ of the spectral sequence (3) for $\Omega^*_X$. By convention $F$ is decreasing. We have $F^0 = H^1(X, \mathbb{C})$ and $F^0H^i(X, \mathbb{C})/F^1 \cong H^i(X, \mathbb{C})$. 

\[ F^0H^i(X, \mathbb{C})/F^1 \cong H^i(X, \mathbb{C}) \]
\( W \) induces the same filtration on the right as the one coming from \( \{2\} \). In particular,
\[
W_0 \text{Gr}_p^0 H^i(X, \mathbb{C}) = H^i(\ldots \to H^0(X_p, \mathcal{O}) \to H^0(X_{p+1}, \mathcal{O}) \ldots)
\]
\[
\cong H^i([\pi_0(X^\bullet)], \mathbb{C}) \cong W_0 H^i(X, \mathbb{C})
\]
This means that Hodge filtrations becomes trivial on \( W_0 H^i(X) \). So that this is a vector space and nothing more.

**Theorem 3.1.**

(a) If \( X \) is a complete variety, then there is an inclusion \( KH^i(X, \mathbb{C}) \hookrightarrow H^i(X, \mathcal{O}_X) \).

(b) If \( f: X \to Y \) a proper morphism of varieties, then there is an inclusion \( KH^i(f^{-1}(y), \mathbb{C}) \hookrightarrow (R^i f_* \mathcal{O}_X)_y \otimes \mathcal{O}_y/m_y \) for each \( y \in Y \).

**Proof.** The canonical map \( \kappa \) factors
\[
\begin{array}{ccc}
H^i(X, \mathbb{C}) & \xrightarrow{\kappa} & H^i(X, \mathcal{O}_X) \\
\downarrow \text{Gr}_p^0 H^i(X, \mathbb{C}) & & \downarrow \text{Gr}_p^0 H^i(X, \mathcal{O}_X)
\end{array}
\]
Thus
\[
W_0 H^i(X, \mathbb{C}) \subseteq \text{Gr}_p^0 H^i(X, \mathbb{C}) = \text{im}[H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X)]
\]
which implies (a).

For (b), let \( X_y \) be the reduced fibre over \( y \), and \( X^{(n)}_y \) the fibre with its \( n \)th infinitesimal structure. From (a), we have a natural inclusion \( s : W_0 H^i(X_y, \mathbb{C}) \hookrightarrow H^i(X_y, \mathcal{O}_{X_y}) \). After choosing a simplicial resolution of the fibre \( f_*: X^\bullet \to X_y \), \( s \) can be identified with the composition
\[
E_2^0(\mathbb{C}) \to E_2^0(\mathcal{O}_{X_y}) \to H^i(X_y, \mathcal{O}_{X_y})
\]
where the first map is induced by the natural map \( \mathbb{C} \to \mathcal{O} \), and the last map is the edge homomorphism. Applying the same construction to the simplicial sheaf \( f_*^* \mathcal{O}_{X^{(n)}_y} \) yields a map \( s_n \) fitting into a commutative diagram
\[
\begin{array}{ccc}
W_0 H^i(X, \mathbb{C}) & \xrightarrow{s} & H^i(X_y, \mathcal{O}_{X_y}) \\
\downarrow s_n & & \downarrow \\
H^i(X_y, \mathcal{O}_{X^{(n)}_y})
\end{array}
\]
Furthermore, these maps are compatible, thus they pass to map \( s_\infty \) to the limit. Together with the formal functions theorem \([H \text{ III 11.1]}\), this yields a commutative diagram
\[
\begin{array}{ccc}
W_0 H^i(X, \mathbb{C}) & \xrightarrow{s} & H^i(X_y, \mathcal{O}_{X_y}) \\
\downarrow s_\infty & & \downarrow s' \\
\lim_{\rightarrow} H^i(X_y, \mathcal{O}_{X^{(n)}_y}) \xrightarrow{\sim} (R^i f_* \mathcal{O}_X)_y \to (R^i f_* \mathcal{O}_X)_y \otimes \mathcal{O}_y/m_y
\end{array}
\]
Since \( s \) is injective, the map labeled \( s' \) is injective as well. \( \square \)
Remark 3.2. In item (a), we actually proved the sharper statement
\[ W_0 H^i(X, C) \hookrightarrow Gr^0_f H^i(X, C) = \text{im}[H^i(X, C) \to H^i(X, O_X)] \]

For certain classes of singularities called Du Bois singularities \([\text{PS}, \S 7.3.3]\), which include rational singularities \([\text{K}]\), \(Gr^0_f H^i(X, C) = H^i(X, O_X)\). But this is not true in general.

Corollary 3.3. Suppose that \(f : X \to Y\) is a resolution of singularities.

1. If \(Y\) has rational singularities then \(W_0 H^i(f^{-1}(y), C) = 0\) for \(i > 0\).
2. If \(Y\) has isolated normal Cohen-Macaulay singularities, \(W_0 H^i(f^{-1}(y), C) = 0\) for \(0 < i < \dim Y - 1\)

Proof. The first statement is an immediate consequence of the theorem. The second follows from the well known fact given below. We sketch the proof for lack of a suitable reference.

Proposition 3.4. If \(f : X \to Y\) is a resolution of a variety with isolated normal Cohen-Macaulay singularities, then \(R^i f_* O_X = 0\) for \(0 < i < \dim Y - 1\)

Sketch. We can assume that \(Y\) is projective. By the Kawamata-Viehweg vanishing theorem \([\text{Ka}, \text{V}]\)

\[ H^i(X, f^* L^{-1}) = 0, \quad i < \dim Y = n, \]

where \(L\) is ample. Replace \(L\) by \(L^N\), with \(N \gg 0\). Then by Serre vanishing and Serre duality (we use the CM hypothesis here)

\[ H^i(Y, L^{-1}) = H^{n-i}(Y, \omega_Y \otimes L) = 0, \quad i < n. \]

The Leray spectral sequence together with \([\text{Ka}, \text{V}]\) imply

\[ H^0(R^i f_* O_X \otimes L^{-1}) = 0, \quad i < n - 1 \]

Since the sheaves \(R^i f_* O_X\) have zero dimensional support, the proposition follows.

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