INTRODUCTION TO THE DECLINATION FUNCTION FOR GERRYMANDERS

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ABSTRACT. The declination is introduced in [War17b] as a new quantitative method for identifying possible partisan gerrymanders by analyzing vote distributions. In this expository note we explain and motivate the definition of the declination. We end by computing its values on several recent elections.

There are two main methods used for mathematically identifying partisan gerrymanders. The first is to define functions that identify oddly shaped districts under the assumption that unusual shapes are likely due to gerrymandering. The second is to consider how votes between the two parties are distributed among the districts. The declination, which we introduce in [War17b] takes this second approach. (Note that simulated district plans are frequently used in conjunction with either or both of these methods.) The purpose of this exposition is to provide an approachable introduction to the declination, a method for measuring the degree of a gerrymander. The declination treats asymmetry in the vote distribution as indicative of gerrymandering. We refer the reader to [War17b] for references to some of the significant other mathematical work on gerrymandering as well as a more comprehensive analysis of elections using the declination.

1. BRIEF INTRODUCTION

We first briefly introduce the declination measure and mention some of its strengths and weaknesses. The definition is motivated, and terms are explained more fully, in the next section.

Plot the democratic vote fractions in each electoral district in increasing order. Place three points on the diagram:

- A point $F$ at the center of mass of the points corresponding to republican districts. The $y$-value (i.e., democratic vote fraction) of this point is the average of the $y$-values for these districts. The $x$-value is centered horizontally on the republican districts.
- A similar point $H$ corresponding to the democratic districts.
- A point $G$ whose $y$-value is at one-half and which horizontally resides at the transition between the republican districts and the democratic districts.

Draw the line segments $FG$ and $GH$. Compute the angle between them (in radians). Then multiply by $2/\pi \approx 0.64$ so that the resulting value is between $-1$ and $1$. This is the declination, $\delta$. Positive values indicate asymmetry that favors Republicans while negative values indicate asymmetry that favors Democrats. The name declination is in analogue to the angle between true north and magnetic north.

The following Python function computes the declination when given the democratic vote fraction in each district as a list of numbers (each between 0 and 1). The code to compute the declination and two variants introduced in [War17b], along with R code to compute the same, can be found at the author’s website [War17a].
**Figure 1.** Example illustrating the three points $F, G$ and $H$ arising in the definition of the declination. Data is from the 2014 North Carolina election for the US House.

```python
import math
import numpy as np

def declination(vals):
    """ Compute the declination of an election. """
    Rwin = sorted(filter(lambda x: x <= 0.5, vals))
    Dwin = sorted(filter(lambda x: x > 0.5, vals))

    # Undefined if each party does not win at least one seat
    if len(Rwin) < 1 or len(Dwin) < 1:
        return False
    theta = np.arctan((1-2*np.mean(Rwin))*len(vals)/len(Rwin))
    gamma = np.arctan((2*np.mean(Dwin)-1)*len(vals)/len(Dwin))

    # Convert to range [-1,1]
    # A little extra precision just in case.
    return 2.0*(gamma-theta)/3.1415926535

**Strengths of the declination**

1. Is a measure of partisan symmetry that does not assume any particular seats-votes proportionality.
2. Is a geometric angle that can be easily visualized on top of a plot of the votes among the various districts.
3. When scaled by half the number of districts, corresponds to the number of seats won by one side that are allocable to the asymmetry.
(4) Can easily be used in conjunction with simulations to account for external sources of asymmetry such as geographic clustering.
(5) Provably increases in absolute value in response to packing and cracking.
(6) Continues to work even when one party is dominant statewide.
(7) Is insensitive to incumbency gerrymandering (when done by both parties) as well as the degree of competitiveness of the election.

Weaknesses of the declination

(1) Is not defined when one party sweeps all seats.
(2) Is noisy when there are very few seats or when one party wins almost all of the seats (say, greater than 90%).

2. Motivation

To begin, suppose we have a state with ten electoral districts and two major parties: the Democrats and the Republicans. The support of each party will vary from district to district. Assuming the parties are equally popular, there are probably a few districts in which the Democrats are dominant; others in which the Republicans are dominant; and a few where the races are likely to be competitive. If we write down the fraction of Democrats voters in each district we get a sequence of ten numbers, each between 0 and 1. A number close to zero means the Democrats are a small minority in that district while a number close to 1 indicates they are overwhelming favorites. In Fig. 2 we have plotted the results for a few hypothetical elections for which the parties are equally matched in the state overall. Each dot corresponds to a single district. We have chosen to sort the districts in increasing order of democratic vote. Doing so makes it easier to see what is going on, but there is nothing magical about this ordering.

In Fig. 2A, we have the situation sketched in the previous paragraph: the first three districts are dominated by the Republicans, the last three by the Democrats, and the four in the middle are dominated by neither party. In Fig. 2B, we have a similar scenario, except now there is much less variation from district to district. The election results depicted are what you might expect from sprinkling voters from either party down on the landscape at random. By chance there will be some areas with a few more voters of a given party, but overall the distribution of voters is relatively homogeneous. In Fig. 2C there is a significant amount of variation from district to district. For whatever reason, the Democrats and Republicans are each clustered in five of the ten districts.

In Figs. 3A and B we illustrate elections in which the Republicans and the Democrats are the majority party, respectively. The distributions still seem intuitively equitable. In Fig. 3A, the
Democrats have 40% of the statewide vote and win three of ten seats. While one might suppose that the Democrats should win four seats, there isn’t anything obviously unfair about how the votes are distributed. The only unusual aspect is that none of the districts are particularly competitive. The Democrats’ votes are a little more efficiently distributed than those of the Republicans in that the Democrats win about 60% of the vote in the districts they win and the Republicans win about 68% of the vote in the districts they win. In Fig. 3B, the Democrats have 65% of the vote and win eight of the ten seats. In this case, there are some competitive districts. In Fig. 3A we had two types of districts — Democrat-dominated and Republican-dominated. Here we have a spectrum of districts, so it is harder to see if he distribution is equitable. We can say, at the least, that there is a reasonably continuous spectrum of districts ranging from narrow Republican majorities to Democrat dominated.

So far we have illustrated elections that are not, on their face, obviously unfair to one of the parties. So how does one party get an advantage? Partisan gerrymanders that advantage the Republicans at the expense of the Democrats are created by “packing and cracking” the Democrat voters. The most efficient way to win seats is through narrow victories and, to the extent necessitated by overall support, overwhelming defeats. When the Democrats win a district with an overwhelming majority, it likely means there was another district that could have been won by the Democrats had the voters been distributed more evenly among the two districts. Likewise, two narrow losses by the Democrats could likely have been one win and one loss had the votes been distributed less evenly.

In Fig. 4B we have displayed what happens to the election of Fig. 2A when extra Democrats are packed into districts they were already going to win. The Democrats now only win three districts; the ones they do win are won overwhelmingly. The two additional districts the Republicans pick
up are narrow wins, but wins nonetheless. In terms of the plot of the vote-fractions, we see that the dots for the districts the Democrats still win are further away from the 50% line while the dots for the two districts that changed hands are just below 50%. In Fig. 4C we show an instance of cracking the election from Fig. 2A — votes are taken from districts the Democrats should have won and distributed to other districts that they still have no hope of winning. Once again, the Republican victories are, on average, narrower than the Democrat victories.

As an initial attempt to determine whether the distribution of votes is fair, we could compare the average democratic vote in the districts the Democrats win to the average republican vote in the districts the Republicans win (this is closely related to the lopsided means test suggested by Wang [Wan16]). This works pretty well when the parties are evenly matched statewide. Suppose the statewide average is 50%. If the Democrats win less than half of the districts, their average vote in those districts is necessarily higher than the average vote for the Republicans in the districts the Republicans win. For example, in Fig. 5A, the Democrats win only three districts and do so with an average winning margin in these three districts of 20%. In contrast, the Republicans are able to win the complementary seven districts with an average winning margin of about 8%.

But simply comparing the two averages doesn’t work as well when the parties are not as closely matched in the state as a whole. Consider Fig. 3B (repeated as Fig. 5B). The Democrats have 65% of the overall vote. It’s not surprising at all that the Republicans only win two seats. There has to be a fair amount of geographic heterogeneity for there to be even two districts in which the Republicans are the majority. Likewise, it’s not surprising that the average Democrat vote in the districts they win is far above 50% while the Republican vote is barely above 50% in the two districts they do win. By the logic of the above paragraph, a simple comparison of average winning margins would indicate that the Democrats are grossly disadvantaged by the district plan since their average winning margin is so much greater. But this is misleading. If anything, they’ve won more districts then we might think they should (65% of the vote but 80% of the seats).

The declination addresses this by incorporating the fraction of seats won into the comparisons of the averages. If a party only wins one or two seats, we’d expect these wins to be relatively narrow. If the party wins a lot of seats, we’d expect the average margin to be relatively high. In light of this, the declination computes the ratio of “average winning margin” to “fraction of seats won” for each party. There’s no particular assumption about an appropriate ratio; the ratio will depend on how the supporters of each party are distributed geographically. If the populations are distributed relatively evenly across the state (lots of mixing of members of the two parties), the ratio will be quite low. If there is little mixing, the ratios are likely to be higher. Regardless, the underlying
assumption of the declination is that the ratios of the two parties should be comparable. If not, then the parties are being treated differently and one is getting an advantage from how the votes are distributed.

For each party we thus have a right triangle: Its base is proportional to (i.e., half of) the fraction of districts won and its height is equal to the average margin of victory in the districts that are won. This triangle has a hypotenuse of a given slope. The declination compares these slopes by computing the angle between lines of those slopes. We refer the reader to [War17b] for an trigonometric expression of the definition. For now it suffices to know that the declination ranges between $-1$ and $1$ with larger (absolute) values more indicative of partisan asymmetry.
As is suggested from the data in [War17b], for an election with 10 districts,
• values above around 0.3 could be considered indicative of likely gerrymandering (barring inherent geographic advantages) and
• multiplying the value of the declination by 5 estimates the number of seats switched due to gerrymandering.

Figs. 6–9 repeat Figs. 2–5 but with the declination shown (written as δ). For each election, we have plotted the hypotenuses of the two relevant triangles. The larger the difference between the slopes, the larger (in absolute value) is the value of the declination and the more unfair is the election.

In Fig. 10 we see that the 2012 district plans for North Carolina and Pennsylvania are advantageous to the Republicans while the Arizona plan looks relatively neutral. (As described in [War17b], we impute the vote fractions for uncontested races.) The heuristic for identifying the number of seats that have changed parties is to multiply the declination by half the number of districts. (See [BW17] for a more sophisticated analysis of how the declination relates to the number of seats switched.) For North Carolina, this leads to an estimate of 2.9 seats and for Pennsylvania, to an estimate of 4.8 seats.

As stated initially, the aim of this note is to give some intuition for the motivation for the definition of the declination from a slightly different perspective from that provided in our research articles [War17b, BW17]. Our article [War17b] in particular provides more in-depth analysis including
• a theorem formally relating the declination with “packing and cracking”,
• a discussion of a variant of the declination that is more appropriate when one wishes to compare the declination for elections with different numbers of districts,
• many more examples, and
• a comparison to other quantitative measures of gerrymandering such as the efficiency gap [McG14, MS15] and the mean-median difference [Wan16, MB15],
• a discussion of confounding factors such as possible “self-packing” of Democrats and the Voting Rights Act of 1965.

Of course, to be useful as part of a manageable standard for identifying candidate gerrymanders (beyond the scope of both this note and [War17b]), further validation analysis of the declination must be done.

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