All solutions to the relaxed commutant lifting problem

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Abstract

A new description is given of all solutions to the relaxed commutant lifting problem. The method of proof is also different from earlier ones, and uses only an operator-valued version of a classical lemma on harmonic majorants.

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0 Introduction

In this paper we give a new, more refined, description of all solutions to the relaxed commutant lifting problem. Let us first recall the formulation of this problem. The starting point is a data set \( \{A, T', U', R, Q\} \) consisting of five Hilbert space operators. The operator \( A \) is a contraction mapping \( \mathcal{H} \) into \( \mathcal{H}' \), the operator \( U' \) on \( \mathcal{K}' \) is a minimal isometric lifting of the contraction \( T' \) on \( \mathcal{K}' \), and \( R \) and \( Q \) are operators from \( \mathcal{H}_0 \) to \( \mathcal{H} \), satisfying the following constraints:

\[
T'AR = AQ \quad \text{and} \quad R^*R \leq Q^*Q. \tag{0.1}
\]

Given this data set the relaxed commutant lifting problem (RCL problem) is to find all contractions \( B \) from \( \mathcal{H} \) to \( \mathcal{K}' \) such that

\[
\Pi_{\mathcal{H}'}B = A \quad \text{and} \quad U'BR = BQ. \tag{0.2}
\]

Here \( \Pi_{\mathcal{H}'} \) is the orthogonal projection from \( \mathcal{K}' \) onto \( \mathcal{H}' \).

The RCL problem has been introduced in \([9]\), and in the paper \([9]\) also an explicit construction for a particular solution is given. By choosing \( \mathcal{H}_0 = \mathcal{H} \) with \( R \) the identity operator on \( \mathcal{H} \) and \( Q = T \) an isometry on \( \mathcal{H} \), one sees that the solution of the RCL problem in \([9]\) contains the classical Sz-Nagy-Foias commutant lifting theorem \([15]\) as a special case. Also a number of recent generalizations of the commutant lifting theorem can be seen as special cases of the solution to the RCL problem presented in \([9]\). This includes the Treil-Volberg version \([17]\), which appears when one takes \( R = I \), and the weighted commutant lifting theorem from \([5]\). Finally, \([9]\) also shows that the solution of the RCL problem allows one to solve relaxed versions of most metric constrained interpolation problems from \([10]\), and their \( H^2 \) versions.

In \([12]\) a Redheffer type description is given of all solutions to the RCL problem by using the theory of isometric realizations and Arocena’s coupling method from \([2]\) and \([3]\), see also Section VII.8 in \([7]\). A choice sequence approach for the description of all solutions, also using the coupling

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framework, can be found in [13]. In the present paper we give a more refined and more explicit description of all solutions than the one appearing in [12]. Furthermore, our proof will be rather elementary and uses only an operator-valued version of a classical result on harmonic majorants. Our approach is even interesting in the classical commutant lifting setting, and provides a new proof for Theorem XIII.3.4 in [7] (see the final part of Section 1).

The paper consists of three sections not counting the present introduction. In the first section we introduce the necessary terminology, state our two main theorems, and specify our results for the commutant lifting setting. The second section contains preliminary material on positive real operator-valued functions and presents an operator-valued version of a classical result on least harmonic majorants (cf., [6], page 28). The proofs of our two main theorems are given in the third section.

We conclude with a few words about notation and terminology. Throughout capital calligraphic letters denote Hilbert spaces. The Hilbert space direct sum of \( \mathcal{U} \) and \( \mathcal{V} \) is denoted by

\[
\mathcal{U} \oplus \mathcal{V} \quad \text{or by} \quad \begin{bmatrix} \mathcal{U} \\ \mathcal{V} \end{bmatrix}.
\]

The term \textit{operator} stands for a bounded linear transformation acting between Hilbert spaces. The set of all operators from \( \mathcal{U} \) into \( \mathcal{V} \) is denoted by \( \mathbf{L}(\mathcal{U}, \mathcal{V}) \). The identity operator on the space \( \mathcal{U} \) is denoted by \( I_\mathcal{U} \) or just by \( I \), when the underlying space is clear from the context. As usual, given a contraction \( A \) from \( \mathcal{U} \) into \( \mathcal{V} \), we write \( D_A \) for the defect operator \((I_\mathcal{U} - A^*A)^{1/2} \) and \( \mathcal{D}_A \) for the closure of the range of \( D_A \). For the definition of an isometric lifting and a review of its properties we refer to Section IV.1 in [10]. By definition, a \( \mathbf{L}(\mathcal{U}, \mathcal{V}) \)-valued \textit{Schur class function} is a function which is analytic on the open unit disk \( \mathbb{D} \) and whose values are contractions from \( \mathcal{U} \) to \( \mathcal{V} \). The class of these functions is denoted by \( \mathbf{S}(\mathcal{U}, \mathcal{V}) \) and is called a \textit{Schur class}. Notice that a function \( F \) belongs to the Schur class \( \mathbf{S}(\mathcal{E}, \mathcal{Y}_1 \oplus \mathcal{Y}_2) \) if and only if \( F \) admits a matrix representation of the form

\[
F(\lambda) = \begin{bmatrix} F_1(\lambda) \\ F_2(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{D}, \tag{0.3}
\]

where \( F_1 \) is in \( \mathbf{S}(\mathcal{E}, \mathcal{Y}_1) \) and \( F_2 \) is in \( \mathbf{S}(\mathcal{E}, \mathcal{Y}_2) \) such that \( F_1(\lambda)^*F_1(\lambda) + F_2(\lambda)^*F_2(\lambda) \leq I \) for all \( \lambda \in \mathbb{D} \).

For convenience a function \( F \) that is represented as in (0.3) will be denoted by \( F = \text{col} [F_1, F_2] \). By \( H^2(\mathcal{U}) \) we denote the Hardy space of all \( \mathcal{U} \)-valued analytic functions \( f \) on \( \mathbb{D} \) such that \( \sum_{\nu=0}^{\infty} \|f_\nu\|^2 < \infty \), where \( f_0, f_1, f_2, \ldots \) are the Taylor coefficients of \( f \) at zero. Finally, \( S_\mathcal{U} \) denotes the unilateral shift on \( H^2(\mathcal{U}) \) and \( E_\mathcal{U} \) is the canonical embedding of \( \mathcal{U} \) onto the space of constant functions in \( H^2(\mathcal{U}) \) defined by \( (E_\mathcal{U}v)(\lambda) \equiv v \) for all \( v \in \mathcal{U} \). We simply write \( S \) and \( E \) if the underlying space is clear from the context.

1 Main theorems

Let \( \{A, T', U', R, Q\} \) be a fixed data set. In the sequel we say that \( B \) is a \textit{solution to the RCL problem} for the data set \( \{A, T', U', R, Q\} \) if \( B \) is a contraction from \( \mathcal{H} \) into \( \mathcal{K}' \) satisfying (1.2).

Without loss of generality we shall assume that \( U' \) is the Sz.-Nagy-Schäffer minimal isometric lifting of \( T' \), that is,

\[
U' = \begin{bmatrix} T' & 0 \\ ED_{T'} & S \end{bmatrix} \quad \text{on} \quad \mathcal{K}' = \begin{bmatrix} \mathcal{H}' \\ H^2(D_{T'}) \end{bmatrix}. \tag{1.1}
\]

Here \( S \) is the unilateral shift on \( H^2(D_{T'}) \) and \( E \) is the canonical embedding of \( D_{T'} \) into \( H^2(D_{T'}) \) defined by \( (Ed)(\lambda) \equiv d \) for all \( d \in D_{T'} \).

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Since we assume that $\mathcal{K}' = \mathcal{H}' \oplus H^2(D_{T'})$, an operator $B$ from $\mathcal{H}$ into $\mathcal{K}'$ is a contraction satisfying $\Pi_{\mathcal{H}'} B = A$, as in the first identity of (1.2), if and only if $B$ can be represented in the form

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix}: \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(D_{T'}) \end{bmatrix},$$

where $\Gamma$ is a contraction from $\mathcal{D}_A$ into $H^2(D_{T'})$. Moreover, $B$ and $\Gamma$ determine each other uniquely. Using this representation of $B$ and the fact that $U'$ is given by (1.1), the constraint $U'BR = BQ$ in (1.2) is equivalent to

$$ED_{T'}AR + STD_AR = \Gamma D_A Q.$$

Therefore, with $U'$ as in (1.1), the RCL problem for $\{A, T', U', R, Q\}$ is equivalent to the problem of finding all contractions $\Gamma$ from $\mathcal{D}_A$ into $H^2(D_{T'})$ such that (1.3) holds.

To state our two main theorems we need some additional notation. Observe that, because of (0.1), for each $h \in \mathcal{H}_0$ we have

$$\|D_AQh\|^2 = \|Qh\|^2 - \|AQh\|^2 \geq \|Rh\|^2 - \|T'ARh\|^2 = \|ARh\|^2 - \|T'ARh\|^2 + \|Rh\|^2 - \|ARh\|^2 = \|D_{T'}ARh\|^2 + \|D_ARh\|^2.$$

Hence the identity

$$\omega D_AQh = \begin{bmatrix} D_{T'}ARh \\ D_ARh \end{bmatrix}, \quad h \in \mathcal{H}_0,$$  \hspace{1cm} (1.5)

uniquely defines a contraction $\omega$ from $\mathcal{F} = \overline{D_AQH}$ into $\mathcal{D}_{T'} \oplus \mathcal{D}_A$. Let $\omega_1$ be the contraction mapping $\mathcal{F}$ into $\mathcal{D}_{T'}$, determined by the first component of $\omega$ and $\omega_2$ be the contraction mapping $\mathcal{F}$ into $\mathcal{D}_A$ determined by the second component of $\omega$, that is,

$$\omega_1D_AQh = D_{T'}ARh \quad \text{and} \quad \omega_2D_AQh = D_ARh, \quad \text{for all } h \in \mathcal{H}_0.$$

Notice that we have equality in (1.3) if and only if $R^*R = Q^*Q$. In other words, $\omega$ is an isometry if and only if $R^*R = Q^*Q$, which happens in many applications. In particular, $\omega$ is an isometry in the setting of the commutant lifting problem. The equation in (1.3) can equivalently be represented in terms of $\omega_1$ and $\omega_2$ as

$$E\omega_1 + ST\omega_2 = \Gamma|\mathcal{F}.$$  \hspace{1cm} (1.6)

In the sequel we shall call a pair of operator-valued functions $\{F, G\}$ a Schur pair associated with the data set $\{A, T', U', R, Q\}$ if $\text{col } [F, G]$ is in $\mathbf{S}(D_A, D_{T'} \ominus D_A)$ and $\text{col } [F, G](\lambda)|\mathcal{F} = \omega$ for all $\lambda \in \mathbb{D}$. In other words, $\{F, G\}$ is a Schur pair if both $F$ and $G$ are analytic operator-valued functions, where $F: \mathbb{D} \to \mathbf{L}(D_A, D_{T'})$ and $G: \mathbb{D} \to \mathbf{L}(D_A, D_A)$, such that

$$F(\lambda)^*F(\lambda) + G(\lambda)^*G(\lambda) \leq I, \quad F(\lambda)|\mathcal{F} = \omega_1, \quad G(\lambda)|\mathcal{F} = \omega_2, \quad \text{for all } \lambda \in \mathbb{D}. $$  \hspace{1cm} (1.7)

We can now state the first main theorem.

**Theorem 1.1** Consider the data set $\{A, T', U', R, Q\}$ with $U'$ being given by (1.1). Then all solutions to the corresponding RCL problem are given by

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix}: \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(D_{T'}) \end{bmatrix},$$  \hspace{1cm} (1.8)
where \( \Gamma \) is a contraction from \( \mathcal{D}_A \) into \( H^2(\mathcal{D}_{T'}) \) given by
\[
(\Gamma d)(\lambda) = F(\lambda)(I - \lambda G(\lambda))^{-1}d, \quad d \in \mathcal{D}_A, \lambda \in \mathbb{D},
\]
with \( \{F, G\} \) an arbitrary Schur pair associated with the given data set.

The mapping \( \{F, G\} \mapsto B \) from the set of Schur pairs to the solutions of the RCL problem described in Theorem 1.1 is onto but not necessarily one to one. In other words, in general there can be many Schur pairs associated with a specified solution \( B \) via \( (1.8) \) and \( (1.9) \). However, in the classical commutant lifting setting the mapping \( \{F, G\} \mapsto B \) is onto and one to one, see \( \mathbb{R} \) and the final part of this section. To describe the non-uniqueness we need some additional notation.

Let \( B \) in \( (1.8) \) be a fixed solution to the RCL problem for the data set \( \{A, T', U', R, Q\} \) with \( U' \) being given by \( (1.1) \), and let \( \Gamma \) be the contraction from \( \mathcal{D}_A \) into \( H^2(\mathcal{D}_{T'}) \) determined by \( B \) via \( (1.8) \). Then \( \Gamma \) satisfies \( (1.3) \). This implies that there exists a contraction \( \Omega \) mapping \( \mathcal{F}_T = \overline{\mathcal{D}_T F} \) into \( \mathcal{D}_T \) satisfying
\[
\Omega D_T D_A Qh = D_T D_A Rh, \quad h \in \mathcal{H}_0.
\]
To see this we use \( (1.3) \) and \( (1.5) \) to show that for all \( h \in \mathcal{H}_0 \), we have
\[
||D_T D_A Qh||^2 = ||D_A Qh||^2 - ||\Gamma D_A Qh||^2 = ||D_A Qh||^2 - ||E D_T A Rh||^2 - ||ST D_A Rh||^2
\]
\[
= ||D_A Qh||^2 - ||D_T A Rh||^2 - ||\Gamma D_A Rh||^2
\]
\[
= ||D_T D_A Rh||^2 + ||D_A Qh||^2 - ||D_T A Rh||^2 - ||\Gamma D_A Rh||^2
\]
\[
= ||D_T D_A Rh||^2 + ||D_A Qh||^2 - ||\omega D_A Qh||^2
\]
\[
\geq ||D_T D_A Rh||^2.
\]
Thus \( ||D_T D_A Qh|| \geq ||D_T D_A Rh|| \) for all \( h \in \mathcal{H}_0 \). So the relation \( \Omega D_T D_A Q = D_T D_A R \) uniquely defines a contraction from \( \mathcal{F}_T = \overline{\mathcal{D}_T F} \) into \( \mathcal{D}_T \). By employing the definition of \( \omega \) observe that for all \( f \in \mathcal{F} \) we have \( \Omega D_T f = D_T \omega f \). From the calculation leading to \( (1.11) \) we also see that \( \Omega \) is an isometry if and only if \( \omega \) is an isometry, and as we saw the latter happens if and only if \( R^* R = Q^* Q \). In particular, \( \Omega \) is an isometry in the setting of the commutant lifting theorem.

Now for \( \Gamma \) and \( \Omega \) as in the previous paragraph, let \( \mathbf{S}_\Omega(\mathcal{D}_T, \mathcal{D}_F) \) be the subset of the Schur class \( \mathbf{S}(\mathcal{D}_T, \mathcal{D}_F) \) defined by
\[
\mathbf{S}_\Omega(\mathcal{D}_T, \mathcal{D}_F) = \{ C \in \mathbf{S}(\mathcal{D}_T, \mathcal{D}_F) : C(\lambda)|\mathcal{F}_T = \Omega \text{ for all } \lambda \in \mathbb{D} \}.
\]
Notice that \( \mathbf{S}_\Omega(\mathcal{D}_T, \mathcal{D}_F) \) is not empty. For example, it contains the function \( C \) given by \( C(\lambda) = \Omega \Pi_{\mathcal{F}_R} \) for all \( \lambda \in \mathbb{D} \). Here \( \Pi_{\mathcal{F}_R} \) is the orthogonal projection from \( \mathcal{D}_T \) onto \( \mathcal{F}_T \). We claim that for the given contraction \( \Gamma \), the set of all Schur pairs \( \{F, G\} \) associated with the data set \( \{A, T', U', R, Q\} \) and satisfying \( (1.3) \) is parameterized by the set \( \mathbf{S}_\Omega(\mathcal{D}_T, \mathcal{D}_F) \). To make this precise, we first define a mapping \( J_T \) from \( \mathbf{S}(\mathcal{D}_T, \mathcal{D}_F) \) into \( \mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A) \) as follows
\[
J_T C = \begin{bmatrix} F \\ G \end{bmatrix}, \quad \begin{bmatrix} F(\lambda) \\ G(\lambda) \end{bmatrix} = \begin{bmatrix} 2\Theta(\lambda) (W(\lambda) + I)^{-1} \\ \lambda^{-1}(W(\lambda) - I) (W(\lambda) + I)^{-1} \end{bmatrix},
\]
where
\[
\Theta(\lambda)d = (\Gamma d)(\lambda), \quad d \in \mathcal{D}_A,
\]
\[
W(\lambda) = \Gamma^*(I + \lambda S^*) (I - \lambda S^*)^{-1} \Gamma + D_T(I + \lambda C(\lambda))(I - \lambda C(\lambda))^{-1} D_T, \quad \lambda \in \mathbb{D}.
\]
Here \( S \) is the unilateral shift on \( H^2(\mathcal{D}_{T'}) \) and \( E \) is the canonical embedding of \( \mathcal{D}_{T'} \) onto the set of constant function in \( H^2(\mathcal{D}_{T'}) \). We are now ready to state the second main theorem.
Theorem 1.2 Let $B$ in (1.12) be a solution to the RCL problem for the data set $\{A,T',U',R,Q\}$ with $U'$ being given by (1.11), and let $\Gamma$ be the contraction determined by $B$ via (1.2). Then the mapping $J_{1}$ from $S(\mathcal{D}_{\Gamma}, \mathcal{D}_{\Gamma})$ into $S(\mathcal{D}_{A}, \mathcal{D}_{A})$ defined in (1.13) maps $S_{\Omega}(\mathcal{D}_{\Gamma}, \mathcal{D}_{\Gamma})$ in a one to one way onto the set of all Schur pairs $\{F,G\}$ associated with the given data set such that (1.8) and (1.9) hold.

To give some further insight in the set $S_{\Omega}(\mathcal{D}_{\Gamma}, \mathcal{D}_{\Gamma})$ appearing in Theorem 1.2 put $\mathcal{G}_{\Gamma} = \mathcal{D}_{\Gamma} \ominus \mathcal{F}_{\Gamma}$, and let $\Pi_{\mathcal{F}_{\Gamma}}$ and $\Pi_{\mathcal{G}_{\Gamma}}$ be the orthogonal projections from $\mathcal{D}_{\Gamma}$ onto $\mathcal{F}_{\Gamma}$ and $\mathcal{G}_{\Gamma}$, respectively. Using Corollary XXVII.5.3 in [13] it follows that $S_{\Omega}(\mathcal{D}_{\Gamma}, \mathcal{D}_{\Gamma})$ if and only if

$$C(\lambda) = \Omega \Pi_{\mathcal{F}_{\Gamma}} + D_{\Omega} C_{1}(\lambda) \Pi_{\mathcal{G}_{\Gamma}}, \quad \lambda \in \mathbb{D},$$

(1.15)

for some function $C_{1}$ in the Schur class $S(\mathcal{G}_{\Gamma}, \mathcal{D}_{\Omega})$. Moreover, $C_{1}$ and $C$ in (1.15) determine each other uniquely. Hence, instead of $S_{\Omega}(\mathcal{D}_{\Gamma}, \mathcal{D}_{\Gamma})$, we can say, in Theorem 1.2 that the set of all Schur pairs $\{F,G\}$ satisfying (1.9) correspond to $S(\mathcal{G}_{\Gamma}, \mathcal{D}_{\Omega})$ in a one to one way.

A similar remark applies to the set of Schur pairs appearing in Theorem 1.1. To see this, notice that a pair of functions $\{F,G\}$ is a Schur pair associated to the data set $\{A,T',U',R,Q\}$ if and only if

$$\text{col} \{F,G\} \in \{H \in S(\mathcal{D}_{A}, \mathcal{D}_{T'} \oplus \mathcal{D}_{A}) : H(\lambda)|\mathcal{F} = \omega \text{ for all } \lambda \in \mathbb{D}\}.$$

Therefore, the set of Schur pairs associated to the given data set is in one to one correspondence to $S(\mathcal{G}, \mathcal{D}_{\omega})$, where $\mathcal{G} = \mathcal{D}_{A} \ominus \mathcal{F}$.

We conclude this section with the commutant lifting theorem as given by Theorem XIII.3.4 in [7]. We show how this result can be derived from Theorems 1.1 and 1.2.

Theorem 1.3 Let $\{A,T',U',R,Q\}$ be a data set with $U'$ being given by (1.11), $\mathcal{H}_{0} = \mathcal{H}$, $R = I_{\mathcal{H}}$ and $Q$ an isometry on $\mathcal{H}$. Then all solutions to the corresponding RCL problem are given by

$$B = \begin{bmatrix} A \\ \Gamma D_{A} \end{bmatrix} : \mathcal{H} \rightarrow \begin{bmatrix} \mathcal{H}' \\ H^{2}(\mathcal{D}_{T'}) \end{bmatrix},$$

(1.16)

where $\Gamma$ is a contraction from $\mathcal{D}_{A}$ into $H^{2}(\mathcal{D}_{T'})$ given by

$$(\Gamma d)(\lambda) = F(\lambda)(I - \lambda G(\lambda))^{-1} d, \quad d \in \mathcal{D}_{A}, \lambda \in \mathbb{D},$$

(1.17)

with $\{F,G\}$ an arbitrary Schur pair associated with the given data set. The solution $B$ and the Schur pair $\{F,G\}$ in (1.16) and (1.17) determine each other uniquely. Finally, there exists only one solution to the given RCL problem if and only if $\mathcal{F} = \mathcal{D}_{A}$ or $\omega \mathcal{F} = \mathcal{D}_{T'} \oplus \mathcal{D}_{A}$.

Proof. The representation of all solutions follows immediately from Theorem 1.1. Obviously, the Schur pair $\{F,G\}$ in (1.17) determines $B$ uniquely. To prove the converse implication, let $B$ be a solution to the corresponding RCL problem for the given data set, and let $\Gamma$ be the contraction from $\mathcal{D}_{A}$ into $H^{2}(\mathcal{D}_{T'})$ given by (1.16). By Theorem 1.1 it suffices to show that the set $S_{\Omega}(\mathcal{D}_{\Gamma}, \mathcal{D}_{\Gamma})$ consists of one element only. Recall that in the commutant lifting setting $R^{*}R = Q^{*}Q$, and hence, as has been remarked in the paragraph preceding (1.12), in this case the operator $\Omega$ is an isometry. Moreover, from the definition of $\Omega$ we obtain that

$$\text{Im} \Omega = \mathcal{D}_{\Gamma} D_{A} R H_{0} = \mathcal{D}_{\Gamma} D_{A} \mathcal{H} = \mathcal{D}_{\Gamma}.$$

Thus $\Omega$ is a unitary operator from $\mathcal{F}_{\Gamma}$ onto $\mathcal{D}_{\Gamma}$, and hence $D_{\Omega*} = \{0\}$. But then the remark made in the first paragraph after Theorem 1.2 shows that $S_{\Omega}(\mathcal{D}_{\Gamma}, \mathcal{D}_{\Gamma})$ is a singleton.
Now that we know that every solution uniquely corresponds to a Schur pair, we see that there is only one solution if and only if there is only one corresponding Schur pair. From the remark made in the second paragraph after Theorem 1.2, we see that the latter happens if and only if \( S(\mathcal{G}, \mathcal{D}_\omega^*) \) consists of the zero element only. In other words, there exists a unique solution if and only if \( \mathcal{G} = \{0\} \) or \( \mathcal{D}_\omega^* = \{0\} \). From \( \mathcal{G} = \mathcal{D}_A \cap \mathcal{F} \) it follows that \( \mathcal{G} = \{0\} \) if and only if \( \mathcal{F} = \mathcal{D}_A \). Since in the commutant lifting setting the operator \( \omega \) is an isometry (see the paragraph containing (1.6)), we have \( \text{Im} \omega = \ker \omega^* = (\mathcal{D}_T^* \oplus \mathcal{D}_A) \ominus \mathcal{D}_\omega^* \). Hence \( \mathcal{D}_\omega^* = \{0\} \) is equivalent to \( \omega \mathcal{F} = \mathcal{D}_T^* \oplus \mathcal{D}_A \).

For the commutant lifting setting representations of all solutions by formulas of the type (1.17) date back to [4], see also [5]. The proofs of Theorems 1.1 and 1.2 will be given in the third section.

### 2 Operator-valued positive real functions and harmonic majorants

Let \( \Theta \) be a \( \mathbf{L}(\mathcal{E}, \mathcal{Y}) \)-valued analytic function on \( \mathbb{D} \), where \( \mathcal{E} \) and \( \mathcal{Y} \) are Hilbert spaces. We say that \( \Theta \) belongs to \( H^2(\mathbf{L}(\mathcal{E}, \mathcal{Y})) \) if for each \( a \in \mathcal{E} \) the function \( \Theta(\cdot) a \) belongs to \( H^2(\mathcal{Y}) \). The latter condition is equivalent to the requirement that \( \sum_{\nu=0}^{\infty} \| \Theta_\nu a \|^2 < \infty \) for all \( a \in \mathcal{E} \). Here and in the sequel \( \Theta_0, \Theta_1, \Theta_2, \ldots \) are the Taylor coefficients of \( \Theta \) at zero. If \( \Theta \) is in \( H^2(\mathbf{L}(\mathcal{E}, \mathcal{Y})) \), then \( \Theta \) uniquely defines an operator \( \Gamma \) from \( \mathcal{E} \) into \( H^2(\mathcal{Y}) \) by

\[
(\Gamma a)(\lambda) = \Theta(\lambda) a, \quad a \in \mathcal{E}, \lambda \in \mathbb{D}.
\]

In this case, we say that \( \Gamma \) is the **operator associated** with \( \Theta \). On the other hand, if \( \Gamma \) is an operator mapping \( \mathcal{E} \) into \( H^2(\mathcal{Y}) \), then the relation \( \Theta(\lambda) a = (\Gamma a)(\lambda) \) for \( a \in \mathcal{E} \) and \( \lambda \in \mathbb{D} \) uniquely defines a function \( \Theta \) in \( H^2(\mathbf{L}(\mathcal{E}, \mathcal{Y})) \). In this case, we say with a slight abuse of terminology that \( \Theta \) is the **symbol** of \( \Gamma \).

As before, let \( \Theta \) be a function in \( H^2(\mathbf{L}(\mathcal{E}, \mathcal{Y})) \), and let \( \Gamma \) be the operator associated with \( \Theta \). Throughout this section \( S \) is the block forward shift on \( H^2(\mathcal{Y}) \), and \( E \) the canonical embedding from \( \mathcal{Y} \) onto the constant functions in \( H^2(\mathcal{Y}) \), that is, \( (Ey)(\lambda) \equiv y \) on \( \mathbb{D} \). In this case, \( \Theta_n = E^* (S^*)^n \Gamma \) for all non-negative integers \( n \). Hence \( \Theta \) admits a state space realization of the following form:

\[
\Theta(\lambda) = E^* (I - \lambda S^*)^{-1} \Gamma, \quad \lambda \in \mathbb{D}.
\]

With \( \Theta \) as above we associate the \( \mathbf{L}(\mathcal{E}, \mathcal{E}) \)-valued function

\[
V(\lambda) = \Gamma^* \Gamma + 2\lambda \Gamma^* (I - \lambda S^*)^{-1} S^* \Gamma, \quad \lambda \in \mathbb{D},
\]

where \( \Gamma \) is the operator associated with \( \Theta \) via (2.2). An easy computation shows that \( V \) can also be written as

\[
V(\lambda) = \Gamma^* (I + \lambda S^*) (I - \lambda S^*)^{-1} \Gamma, \quad \lambda \in \mathbb{D}.
\]

Obviously, \( V \) is analytic on \( \mathbb{D} \). Using \( EE^* = I - SS^* \), we see from (2.2) and (2.3) that the Taylor coefficients \( \{V_n\}_0^\infty \) of \( V \) at zero are given by

\[
V_0 = \Gamma^* \Gamma = \sum_{\nu=0}^{\infty} \Theta_\nu^* \Theta_\nu \quad \text{and} \quad V_n = 2\Gamma^* S^{*n} \Gamma = 2 \sum_{\nu=0}^{\infty} \Theta_\nu^* \Theta_{\nu+n}, \quad \text{for all } n \geq 1.
\]

The results below show that \( V \) is positive real, and therefore we shall refer to \( V \) as the **positive real function defined by** \( \Theta \).

Recall that a \( \mathbf{L}(\mathcal{E}, \mathcal{E}) \)-valued function \( W \) is **positive real** if \( W \) is analytic on \( \mathbb{D} \) and

\[
\Re W(\lambda) = \frac{1}{2} (W(\lambda)^* + W(\lambda)) \geq 0, \quad \lambda \in \mathbb{D}.
\]
It is known (see, e.g., [11], Section 1.2) that a \( L(\mathcal{E}, \mathcal{E}) \)-valued function \( W \) which is analytic at zero, \( W(\lambda) = \sum_{\nu=0}^{\infty} \lambda^{\nu} W_\nu \), say, is positive real if and only if for each \( n \) the \( n \times n \) Toeplitz operator matrix \( T_{\mathcal{R}W,n} \) given by
\[
T_{\mathcal{R}W,n} = \frac{1}{2} \begin{bmatrix}
W_0^* + W_0 & W_1^* & \cdots & W_{n-1}^* \\
W_1 & W_0^* + W_0 & \cdots & W_{n-2}^* \\
\vdots & \vdots & \ddots & \vdots \\
W_{n-1} & W_{n-2} & \cdots & W_0^* + W_0
\end{bmatrix},
\]
defines a non-negative operator on \( \mathcal{E}^n \).

Our aim in this section is to prove the following theorem which can be viewed as an operator valued version of a classical result on harmonic majorants, cf., Section 2.6 in [6].

**Theorem 2.1** Let \( \Theta \) be a function in \( H^2(\mathcal{L}(\mathcal{E}, \mathcal{Y})) \) such that the associated operator \( \Gamma \) is a contraction from \( \mathcal{E} \) into \( H^2(\mathcal{Y}) \). The set of all positive real functions \( W \) with values in \( \mathcal{L}(\mathcal{E}, \mathcal{E}) \) satisfying
\[
\Theta(\lambda)^* \Theta(\lambda) \leq \Re W(\lambda) \quad \text{for all } \lambda \in \mathbb{D} \text{ and } W(0) = I
\]
is parameterized by \( S(D_\Gamma, D_\Gamma) \). More precisely, all positive real functions \( W \) on \( \mathbb{D} \) satisfying (2.6) are given by
\[
W(\lambda) = V(\lambda) + D_\Gamma (I + \lambda C(\lambda)) (I - \lambda C(\lambda))^{-1} D_\Gamma, \quad \lambda \in \mathbb{D},
\]
where \( V \) on \( \mathbb{D} \) is given by (2.3), and \( C \) is an arbitrary function in \( S(D_\Gamma, D_\Gamma) \). Moreover, \( W \) and \( C \) in (2.7) determine each other uniquely. Finally, there is only one positive real function \( W \) satisfying (2.6) if and only if \( \Gamma \) is an isometry. In this case \( W = V \) is the only function satisfying (2.6).

In order to prove the above theorem it will be convenient to first prove a lemma and to review some theory concerning the Cayley transform of operator-valued functions.

**Lemma 2.2** Let \( \Theta \in H^2(\mathcal{L}(\mathcal{E}, \mathcal{Y})) \), and \( V \) be the \( \mathcal{L}(\mathcal{E}, \mathcal{E}) \)-valued function defined by (2.3). Then \( V \) is positive real. More precisely,
\[
\Theta(\lambda)^* \Theta(\lambda) \leq \Re V(\lambda), \quad \lambda \in \mathbb{D}.
\]

Furthermore, if \( W \) is any \( \mathcal{L}(\mathcal{E}, \mathcal{E}) \)-valued positive real function such that \( \Theta(\lambda)^* \Theta(\lambda) \leq \Re W(\lambda) \) for all \( \lambda \in \mathbb{D} \), then \( W - V \) is positive real.

To give some further insight in (2.8), let us consider the scalar case, that is, \( \mathcal{E} \) and \( \mathcal{Y} \) are equal to \( \mathbb{C} \). In that case formula (2.3) can be rewritten as
\[
V(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\omega} + \lambda}{e^{i\omega} - \lambda} |\theta(e^{i\omega})|^2 d\omega, \quad \lambda \in \mathbb{D},
\]
and the above lemma is well known (see the proof of Theorem 2.12 in [6]). In fact, in the scalar case \( \Re V \) is known as the least harmonic majorant of \( \theta(\cdot) \).

**Proof of Lemma 2.2** We split the proof into three parts. In the first part we prove (2.8).

**Part 1.** Take \( \lambda \in \mathbb{D} \). For convenience set \( \Phi(\lambda) = (I - \lambda S^*)^{-1} \). Using (2.2) and (2.3), we have
\[
\Theta(\lambda) = E \Phi(\lambda) \Gamma \quad \text{and} \quad V(\lambda) = \Gamma^* \Gamma + 2\lambda \Gamma^* \Phi(\lambda) S^* \Gamma.
\]
Note that $\Phi(\lambda) = I + \lambda \Phi(\lambda) S^*$. Since $E^* E + S S^* = I$, we obtain

\[
\Theta(\lambda) \Theta(\lambda) = \Gamma^* \Phi(\lambda) + E^* \Phi(\lambda) \Gamma = \Gamma^* \Phi(\lambda) (I - SS^*) \Phi(\lambda) \Gamma \\
= \Gamma^* \Phi(\lambda) (\Phi(\lambda) + \Gamma^* \Phi(\lambda)) - \Gamma^* \Phi(\lambda) S S^* \Phi(\lambda) \Gamma \\
= \Gamma^* [I + \lambda S \Phi(\lambda)] [I + \lambda \Phi(\lambda) S^*] \Gamma - \Gamma^* \Phi(\lambda) S S^* \Phi(\lambda) \Gamma \\
= \Gamma^* \Gamma + \lambda \Gamma^* S \Phi(\lambda) \Gamma + \lambda \Gamma^* \Phi(\lambda) S^* \Gamma \\
+ |\lambda|^2 \Gamma^* \Phi(\lambda) S S^* \Phi(\lambda) \Gamma - \Gamma^* \Phi(\lambda) S S^* \Phi(\lambda) \Gamma \\
= \frac{1}{2} \left( V(\lambda) + V(\lambda)^* + (1 - |\lambda|^2) \Gamma^* (I - \lambda S^{-1}) S^* (I - \lambda S^{-1})^{-1} \Gamma \right).
\]

The last term is non-negative. Thus (2.8) holds. In particular, $V$ is positive real.

\textbf{Part 2.} Fix $0 < r < 1$, and set $\Theta \bar{\Theta}(z) = \Theta(r z)$ for each $z \in \mathbb{D}$. Notice that $\Theta \bar{\Theta}$ is analytic in open neighborhood of $\mathbb{D}$, the closure of the open unit disc $\mathbb{D}$. Let $\Gamma$ be the operator from $\mathcal{E}$ into $H^2(\mathcal{Y})$ associated with $\bar{\Theta}$, that is, $(\Gamma a)(z) = \Theta(z) a$ for $a \in \mathcal{E}$ and $z \in \mathbb{D}$. Thus $\Gamma = \Lambda_r \Gamma$, where $\Lambda_r$ is the operator on $H^2(\mathcal{Y})$ defined by

\[
(A_r h)(z) = h(r z), \quad h \in H^2(\mathcal{Y}), \ z \in \mathbb{D}.
\]

Note that $\Lambda_r$ is bounded and $\lim_{r \to 1} \Lambda_r = I$ with pointwise convergence. Let $\bar{V}$ be the positive real function defined by $\bar{\Theta}$. Thus

\[
\bar{V}(\lambda) = \Gamma^* \Lambda_r^2 \Gamma + 2 \lambda \Gamma^* \Lambda_r (I - \lambda S)^{-1} S^* \Lambda_r \Gamma, \quad \lambda \in \mathbb{D}.
\]

Since $\Lambda_r S = r S \Lambda_r$, we have $\Lambda_r (I - \lambda S)^{-1} = (I - \lambda r S)^{-1} \Lambda_r$ for each $\lambda \in \mathbb{D}$. Taking adjoints and replacing $\lambda$ by $\bar{\lambda}$ we see that $(I - \lambda S^*)^{-1} S^* \Lambda_r = r \Lambda_r (I - \lambda r S^*)^{-1} S^*$ and hence $\bar{V}$ is also analytic on an open neighborhood of $\mathbb{D}$.

From the first part of the proof we know that for each $\lambda \in \mathbb{D}$ we have

\[
\Re \bar{V}(\lambda) - \Theta(\lambda)^* \bar{\Theta}(\lambda) = (I - |\lambda|^2) \Gamma^* \Lambda_r (I - \bar{\lambda} S)^{-1} S^* (I - \lambda S)^{-1} \Lambda_r \Gamma \\
= (I - |\lambda|^2) \Gamma^* (I - \bar{\lambda} r S)^{-1} \Lambda_r S S^* \Lambda_r (I - \lambda r S^*)^{-1} \Gamma.
\]

Since all functions involved are analytic on an open neighborhood of $\mathbb{D}$, we conclude that

\[
\Theta(e^{i \omega})^* \bar{\Theta}(e^{i \omega}) = \Re \bar{V}(e^{i \omega}), \quad 0 \leq \omega \leq 2\pi.
\]

Let $W$ be a positive real function with values in $L(\mathcal{E}, \mathcal{E})$ such that $\Theta(\lambda)^* \Theta(\lambda) \leq \Re \bar{W}(\lambda)$ for all $\lambda \in \mathbb{D}$. Set $\bar{W}(\lambda) = W(r \lambda)$ for each $\lambda \in \mathbb{D}$. Then $\Theta(\lambda)^* \Theta(\lambda) \leq \Re \bar{W}(\lambda)$ for all $\lambda \in \mathbb{D}$. Again $\bar{W}$ is analytic on an open neighborhood of $\mathbb{D}$, and thus, by continuity, $\Theta(e^{i \omega})^* \bar{\Theta}(e^{i \omega}) \leq \Re \bar{W}(e^{i \omega})$ for each $0 \leq \omega \leq 2\pi$. But then we can use the result of the previous paragraph to show that

\[
\Re \bar{V}(e^{i \omega}) \leq \Re \bar{W}(e^{i \omega}), \quad 0 \leq \omega \leq 2\pi.
\]

(2.9)

Next we show that the latter inequality implies that $\bar{W} - \bar{V}$ is positive real. To accomplish this, let $L_{\Re \bar{V}}$ and $L_{\Re \bar{W}}$ be the block Laurent operators on $\ell^2(\mathcal{Y})$ defined by $\Re V$ and $\Re W$, respectively. Since $\Re \bar{V}$ and $\Re \bar{W}$ are both continuous on the unit circle $\mathbb{T}$, these operators are well defined and bounded. Furthermore, the inequality (2.8) implies that $L_{\Re \bar{V}} \leq L_{\Re \bar{W}}$. Taking the compression to $\ell^2_{+}(\mathcal{Y})$ this implies that $T_{\Re \bar{V}} \leq T_{\Re \bar{W}}$, where $T_{\Re \bar{V}}$ and $T_{\Re \bar{W}}$ are the block Toeplitz operators on...
$\ell^2(Y)$ defined by $R\tilde{V}$ and $R\tilde{W}$, respectively. Next, taking an $n$-th section of these block Toeplitz operators, we obtain that $T_{R\tilde{V},n} \leq T_{R\tilde{W},n}$ for all integers $n \geq 0$. This implies (see the paragraph before Lemma 2.2) that $\tilde{W} - \tilde{V}$ is positive real.

**Part 3.** We continue to use the notation introduced in the preceding part, but now we make the dependence on the parameter $r$ explicit. Thus for $\tilde{V}$ we write $V\langle r \rangle$, and for $\tilde{W}$ we write $W\langle r \rangle$. Define

$$\Delta = W - V, \quad \Delta\langle r \rangle = W\langle r \rangle - V\langle r \rangle, \quad \text{for each } 0 < r < 1.$$

The result of the previous part shows that $\Delta\langle r \rangle$ is positive real for each $0 < r < 1$. Furthermore, for $r \uparrow 1$ the $n$-th Taylor coefficient of $\Delta\langle r \rangle$ converges pointwise (i.e., in the strong operator topology) to the $n$-th Taylor coefficient of $\Delta$. Here $n$ is an arbitrary non-negative integer. Hence for each $n = 0, 1, 2, \ldots$ we see that $T_{R\Delta\langle r \rangle,n} x$ converges to $T_{R\Delta,n} x$ for each $x \in E^n$ as $r \uparrow 1$. Since the operators $T_{R\Delta\langle r \rangle,n}$ are non-negative, the same holds true for $T_{R\Delta,n}$. This shows that $\Delta = W - V$ is positive real. $\square$

**Positive real functions and the Cayley transform.** For $C$ in $S(E, E)$ consider the map

$$C \mapsto K, \quad \text{where } K(\lambda) = (I + \lambda C(\lambda)) (I - \lambda C(\lambda))^{-1} \quad \text{for all } \lambda \in \mathbb{D}. \tag{2.10}$$

Since $C(\lambda)$ is contractive for each $\lambda \in \mathbb{D}$, the function $K$ is well defined by (2.10). The map $C \mapsto K$ in (2.10) establishes a one to one correspondence between the Schur class $S(E, E)$ and the set of all positive real functions $K$ satisfying $K(0) = I$. Indeed, if $K$ is defined by (2.10) for some $C \in S(E, E)$, then $K$ is analytic in $\mathbb{D}$ and $K(0) = I$ while

$$R K(\lambda) = (I - \lambda C(\lambda))^{-*} \left( I - |\lambda|^2 C(\lambda)^* C(\lambda) \right) (I - \lambda C(\lambda))^{-1}, \quad \lambda \in \mathbb{D}. \tag{2.11}$$

It follows that $R K(\lambda) > 0$ for each $\lambda \in \mathbb{D}$, and hence $K$ is positive real. Conversely, for a positive real function $K$ satisfying $K(0) = I$, the function $C$ given by

$$C(\lambda) = \frac{1}{\lambda} (K(\lambda) - I) (I + K(\lambda))^{-1}, \quad 0 \neq \lambda \in \mathbb{D}, \tag{2.12}$$

is well defined and belongs to $S(E, E)$.

If $C$ belongs to $S(E, E)$, then we call $K$ defined by (2.10) the Cayley transform of $C$. If $K$ is positive real with $K(0) = I$, then $C$ defined by (2.12) will be called the inverse Cayley transform of $K$.

**Proof of Theorem 2.1.** Let $C$ be a function in $S(D_\Gamma, D_\Gamma)$, and define $W$ by (2.7). Then

$$W(\lambda) = V(\lambda) + D_\Gamma K(\lambda) D_\Gamma \quad \text{for each } \lambda \in \mathbb{D}, \quad \text{where } K \text{ on } \mathbb{D} \text{ is the Cayley transform of } C.$$

Note that $V(0) = \Gamma^* \Gamma$. Hence $W(0) = V(0) + I - \Gamma^* \Gamma = I$. By consulting Lemma 2.2 we have

$$R W(\lambda) = R V(\lambda) + D_\Gamma (R K(\lambda)) D_\Gamma \geq R V(\lambda) \geq \Theta(\lambda)^* \Theta(\lambda) \geq 0, \quad \lambda \in \mathbb{D}.$$

Therefore $W$ is a positive real function satisfying (2.4).

Conversely, assume that $W$ is a positive real function satisfying (2.4). According to Lemma 2.2 we have that $R W(\lambda) \geq R V(\lambda)$ for all $\lambda$ in $\mathbb{D}$. Hence, the function $\Delta = W - V$ is a positive real function on $\mathbb{D}$ that satisfies $\Delta(0) = W(0) - V(0) = I - \Gamma^* \Gamma = D_\Gamma^2$. We claim that $\Delta$ admits a unique factorization of the form $\Delta(\lambda) = D_\Gamma K(\lambda) D_\Gamma$, where $K$ is a positive real function with values in $L(D_\Gamma, D_\Gamma)$ and $K(0) = I$. To see this let $\{\Delta_n\}_{n=0}^\infty$ be the Taylor coefficients of $\Delta$ at the origin. Since $T_{R\Delta,n}$ is a positive Toeplitz matrix and $\Delta(0) = D_\Gamma^2$, we see that

$$\begin{bmatrix} 2D_\Gamma^2 & \Delta_n^* \\ \Delta_n & 2D_\Gamma^2 \end{bmatrix} \geq 0, \quad n = 0, 1, 2, \ldots \tag{2.13}$$

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Recall (see Theorem XVI.1.1. in [7]) that a $2 \times 2$ operator matrix
\[
\begin{bmatrix}
A & B^* \\
B & A
\end{bmatrix}
\]
on $\mathcal{E} \oplus \mathcal{E}$ if and only if $B = A^{1/2} \Phi A^{1/2}$ for some contraction $\Phi$ on $\overline{A\mathcal{E}}$. In this case, $B$ and $\Phi$ uniquely determine each other. So from (2.13) we see that there exists a unique operator $K_n$ on $\mathcal{D}_T$ such that $\Delta_n = D_T K_n D_T$ for all integers $n \geq 0$, and $K_0 = I$. Let $T_{\mathbb{R} K, n}$ the $n \times n$ block Toeplitz operator matrix obtained by replacing $W_j$ by $K_j$ in (2.12). Notice $D_n^* T_{\mathbb{R} K, n} D_n = T_{\mathbb{R} \Delta, n}$, where $D_n$ is the diagonal operator matrix $\text{diag}(D_T)^n$ acting on $\oplus^n \mathcal{D}_T$. Since $T_{\mathbb{R} \Delta, n}$ is positive, and $D_n$ is onto a dense set in $\oplus^n \mathcal{D}_T$, it follows that $T_{\mathbb{R} K, n}$ is positive for each integer $n \geq 0$. Hence $K(\lambda) = \sum_{n=0}^{\infty} \lambda^n K_n$ is a positive real function. Therefore $\Delta(\lambda) = D_T K(\lambda) D_T$ where $K$ is a positive real function satisfying $K(0) = I$, which proves our claim.

Let $C$ on $\mathbb{D}$ be the inverse Cayley transform of $K$. Then $C$ is a function in $\mathcal{S}(\mathcal{D}_T, \mathcal{D}_T)$, and we have
\[
K(\lambda) = (I + \lambda C(\lambda)) (I - \lambda C(\lambda))^{-1}, \quad \lambda \in \mathbb{D}.
\]
Hence $W$ is given by (2.4) with $C \in \mathcal{S}(\mathcal{D}_T, \mathcal{D}_T)$ being the inverse Cayley transform of the positive real function $K$ uniquely determined by $\Delta(\lambda) = D_T K(\lambda) D_T$. Recall that the inverse Cayley transform is a bijective mapping from the set of positive real functions $K$ with $K(0) = I$ onto $\mathcal{S}(\mathcal{D}_T, \mathcal{D}_T)$. Thus $K$ and $C$ uniquely determine each other.

Moreover, since $\Delta$ and $K$ determine each other uniquely and the Cayley transform is bijective, we obtain that $C$ and $W$ in (2.4) determine each other uniquely. \qed

\section{3 Proofs of the main theorems}

In this section we prove Theorems 1.1 and 1.2. Throughout this section $\{A, T', U', R, Q\}$ is a fixed data set with $U'$ being given by (1.1). As mentioned in Section 1, an operator $B$ from $\mathcal{H}$ into $\mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$ is a solution to the corresponding RCL problem if and only if $B$ admits a representation of the form
\[
B = \begin{bmatrix}
A \\
\Gamma D_A
\end{bmatrix} : \mathcal{H} \rightarrow \begin{bmatrix}
\mathcal{H}' \\
H^2(\mathcal{D}_{T'})
\end{bmatrix},
\quad (3.1)
\]
with $\Gamma$ a contraction from $\mathcal{D}_A$ into $H^2(\mathcal{D}_{T'})$ satisfying
\[
E \omega_1 + S \Gamma \omega_2 = \Gamma |\mathcal{F}.
\quad (3.2)
\]
Here $S$ denotes the unilateral shift on $H^2(\mathcal{D}_{T'})$ and $E$ is the canonical embedding of $\mathcal{D}_{T'}$ onto the space of constant functions in $H^2(\mathcal{D}_{T'})$ defined by $(E d)(\lambda) \equiv d$ for all $d \in \mathcal{D}_{T'}$.

As a first step towards the proofs of Theorem 1.1 and 1.2 it will be convenient first to consider the case when the space $\mathcal{F}$ in (3.2) consists of the zero element only. In that case the only constraint on the operator $\Gamma$ in (3.1) is that it has to be a contraction. It follows that for $\mathcal{F} = \{0\}$ our two main theorems reduce to the following result.

\textbf{Theorem 3.1} Let $\Gamma$ be an operator from $\mathcal{E}$ into $H^2(\mathcal{V})$. Then $\Gamma$ is a contraction if and only if $\Gamma$ admits a representation of the form
\[
(\Gamma e)(\lambda) = F(\lambda)(I - \lambda G(\lambda))^{-1} e, \quad e \in \mathcal{E}, \lambda \in \mathbb{D},
\quad (3.3)
\]
where \( \text{col } [F, G] \) is any function in \( S(\mathcal{E}, \mathcal{Y} \oplus \mathcal{E}) \). Moreover, if \( \Gamma \) is a contraction, then there is a one to one correspondence between \( S(D_T, D_T) \) and the set of all Schur class functions \( \text{col } [F, G] \) in \( S(\mathcal{E}, \mathcal{Y} \oplus \mathcal{E}) \), that satisfy (3.3). To be precise, let \( J_T \) be the map from \( S(D_T, D_T) \) into \( S(\mathcal{E}, \mathcal{Y} \oplus \mathcal{E}) \) defined by

\[
J_T C = \begin{bmatrix} F \\ G \end{bmatrix} \quad (C \in S(D_T, D_T)), \quad \text{where} \quad \begin{bmatrix} F(\lambda) \\ G(\lambda) \end{bmatrix} = \begin{bmatrix} 2\Theta(\lambda)(W(\lambda) + I)^{-1} \\ \lambda^{-1}(W(\lambda) - I)(W(\lambda) + I)^{-1} \end{bmatrix} (3.4)
\]

with \( \Theta \) the symbol of \( \Gamma \), see (2.1), and

\[
W(\lambda) = \Gamma^*(I + \lambda S^*)(I - \lambda S^*)^{-1} \Gamma + D_T(I + \lambda C(\lambda))(I - \lambda C(\lambda))^{-1} D_T, \quad \lambda \in \mathbb{D}. \quad (3.5)
\]

Then \( J_T \) is a one to one mapping from \( S(D_T, D_T) \) onto the set of all functions \( \text{col } [F, G] \) in \( S(\mathcal{E}, \mathcal{Y} \oplus \mathcal{E}) \) that satisfy (3.3). In particular, the representation in (3.3) is unique if and only if \( \Gamma \) is an isometry.

In a somewhat different, less explicit form, Theorem 3.1 appears in the introduction of [12], see Corollaries 0.3 and 0.4 in [12]. These corollaries were obtained as immediate consequences of the description of all solutions to the relaxed commutant lifting problem given in [12]. In the present paper we follow a different direction: we first prove Theorem 3.1 and then derive Theorems 1.1 and 1.2 as further refinements of Theorem 3.1.

Theorem 3.1 has other partial predecessors in the literature. For example, when \( \mathcal{E} = \mathcal{Y} = \mathbb{C} \) and \( \Gamma \) is an isometry, the representation (3.3) immediately follows from the description of all solutions to the relaxed commutant lifting problem given in [18], page 490. When \( \mathcal{E} = \mathbb{C}^q \) and \( \mathcal{Y} = \mathbb{C}^p \) the first statement in Theorem 3.1 is Theorem 2.2 in [1]. The second and third part of Theorem 3.1 seem to be new, even in the scalar case.

**Proof of Theorem 3.1** Let \( \Theta \) be the symbol of \( \Gamma \), see (2.1). Take for \( C \) any function in \( S(D_T, D_T) \), and define functions \( F \) and \( G \) by (3.4) and (3.5). Then \( F \) is a \( \mathcal{L}(\mathcal{E}, \mathcal{Y}) \)-valued function and \( G \) is a \( \mathcal{L}(\mathcal{E}, \mathcal{E}) \)-valued function. From Theorem 2.1 we obtain that \( W \) in (3.5) is a positive real function satisfying (2.6). Note that \( G \) is the inverse Cayley transform of \( W \). Hence \( G \) is a function in \( S(\mathcal{E}, \mathcal{E}) \). Moreover, for each \( \lambda \in \mathbb{D} \) we have

\[
I - \lambda G(\lambda) = I - (W(\lambda) - I)(W(\lambda) + I)^{-1} = (W(\lambda) + I) - (W(\lambda) - I))(W(\lambda) + I)^{-1} = 2(W(\lambda) + I)^{-1}. \quad (3.6)
\]

Therefore, \( F \) is given by \( F(\lambda) = \Theta(\lambda)(I - \lambda G(\lambda)) \), \( \lambda \in \mathbb{D} \). In particular, \( F \) is analytic on \( \mathbb{D} \) and, since \( G \in S(\mathcal{E}, \mathcal{E}) \), we obtain that \( \Theta(\lambda) = F(\lambda)(I - \lambda G(\lambda))^{-1} \) for all \( \lambda \in \mathbb{D} \). Then the definition of \( \Theta \) shows that (2.6) is satisfied. Since \( G \) is the inverse Cayley transform of \( W \), the function \( W \) must be the Cayley transform of \( G \). Hence, using (2.11) with \( G \) in place of \( C \), the real part of \( W \) is given by

\[
\Re W(\lambda) = (I - \lambda G(\lambda))^{-*}(I - |\lambda|^2 G(\lambda)^* G(\lambda))(I - \lambda G(\lambda))^{-1}, \quad \lambda \in \mathbb{D}.
\]

Then for each \( \lambda \in \mathbb{D} \) we have

\[
(I - \lambda G(\lambda))^{-*} F(\lambda)^* F(\lambda) (I - \lambda G(\lambda))^{-1} = \Theta(\lambda)^* \Theta(\lambda) \leq \Re W(\lambda)
\]

\[
= (I - \lambda G(\lambda))^{-*} (I - |\lambda|^2 G(\lambda)^* G(\lambda))(I - \lambda G(\lambda))^{-1}.
\]

Thus \( F(\lambda)^* F(\lambda) + |\lambda|^2 G(\lambda)^* G(\lambda) \leq I \) for all \( \lambda \in \mathbb{D} \). In other words, \( \text{col } [F, \lambda G] \) is in \( S(\mathcal{E}, \mathcal{Y} \oplus \mathcal{E}) \). Using the maximum principle for analytic functions from \( \mathcal{E} \) to \( \mathcal{Y} \oplus \mathcal{E} \) we see that \( \text{col } [F, G] \) is in \( S(\mathcal{E}, \mathcal{Y} \oplus \mathcal{E}) \).
Note that $C$ and $W$ uniquely determine each other, by Theorem 2.4 and $W$ and $G$ determine each other uniquely because $G$ is the inverse Cayley transform of $W$. Hence $C$ and $G$ determine each other uniquely. In other words, the map $J_T$ is one to one.

To prove the surjectivity, let us assume that $\text{col } [F,G]$ is in $S(\mathcal{E}, \mathcal{Y} \oplus \mathcal{E})$ and satisfies (3.3). Then $G$ is a function in $S(\mathcal{E}, \mathcal{E})$. Let $W$ be the Cayley transform of $G$. Then $W$ is positive real and $W(0) = I$. Moreover, for each $\lambda$ in $\mathbb{D}$ we have

$$\Theta(\lambda)^* \Theta(\lambda) = (I - \lambda G(\lambda))^{-1} F(\lambda)^* F(\lambda) (I - \lambda G(\lambda))^{-1} \leq (I - \lambda G(\lambda))^{-1} (I - G(\lambda)^* G(\lambda)) (I - \lambda G(\lambda))^{-1} \leq (I - \lambda G(\lambda))^{-1} (I - |\lambda|^2 G(\lambda)^* G(\lambda)) (I - \lambda G(\lambda))^{-1} = Y W(\lambda) \lambda.$$ 

Thus $W$ is a $\mathbb{L}(\mathcal{E}, \mathcal{E})$-valued positive real function that satisfies (2.4), and we can apply Theorem 2.1 to show that $W$ is given by (3.5) for some function $C$ in $S(\mathcal{D}_T, \mathcal{D}_T)$. Since $W$ is the Cayley transform of $G$, we have

$$G(\lambda) = \lambda^{-1}(W(\lambda) - I)(W(\lambda) + I)^{-1}, \quad \lambda \in \mathbb{D}.$$ 

Furthermore, (3.3) and (3.6) yield $F(\lambda) = 2\Theta(\lambda)(W(\lambda) + I)^{-1}$ for all $\lambda \in \mathbb{D}$. We see that $\text{col } [F,G]$ is equal to $J_T C$.

The final statement about uniqueness is trivial, because $\Gamma$ is an isometry if and only if $D_T$ is a zero operator. \hfill \Box

Note that for the case when $\mathcal{E} = \mathcal{D}_A$ and $\mathcal{Y} = \mathcal{D}_T'$, the map $J_T$ in Theorem 3.1 is precisely the map $J_T$ in (1.18).

Next, in order to deal with the constraint in (3.2) and to prove the main theorems, we first prove the following result.

**Proposition 3.2** Consider the data set \{A, T', U', R, Q\} with $U'$ being given by (1.1). Let $\Gamma$ be a contraction from $\mathcal{D}_A$ onto $H^2(\mathcal{D}_T)$, and let $C$ be a function in $S(\mathcal{D}_T, \mathcal{D}_T)$. Define functions $F$ and $G$ by $\text{col } [F,G] = J_TC$ using (1.16) and (1.14). Then $\{F,G\}$ is a Schur pair associated with the given data set if and only if $\Gamma$ satisfies (3.2) and $C$ belongs to $S_{\Omega}(\mathcal{D}_T, \mathcal{D}_T)$.

**Proof.** Let $\Theta$ be the symbol of $\Gamma$, that is, $\Theta(\lambda)d = (\Gamma d)(\lambda)$ for all $d \in \mathcal{D}_A$ and all $\lambda \in \mathbb{D}$. Observe that $W$ in (1.14) can be rewritten as

$$W(\lambda) = \Gamma^*(I - \lambda S^*)^{-1} (I + \lambda S^*) \Gamma = D_T(I - \lambda C(\lambda))^{-1} (I + \lambda C(\lambda)) D_T, \quad \lambda \in \mathbb{D}.$$ 

Since $\Gamma^* \Gamma + D_T^2 = I$, we obtain

$$W(\lambda) - I = 2\lambda \Gamma^*(I - \lambda S^*)^{-1} S^* \Gamma + 2\lambda D_T(I - \lambda C(\lambda))^{-1} C(\lambda) D_T, \quad \lambda \in \mathbb{D}, \quad (3.7)$$

$$W(\lambda) + I = 2\Gamma^*(I - \lambda S^*)^{-1} \Gamma + 2 D_T(I - \lambda C(\lambda))^{-1} D_T, \quad \lambda \in \mathbb{D}. \quad (3.8)$$

We divide the remaining part of the proof into two parts.

**Part 1.** First, assuming that $\Gamma$ satisfies (3.2), we show that $G(\lambda)F = \omega_2$ for all $\lambda \in \mathbb{D}$ if and only if $C$ belongs to $S_{\Omega}(\mathcal{D}_T, \mathcal{D}_T)$. So assume that $\Gamma$ satisfies (3.2). Using (3.7) and (3.8) we see that for $f \in F$ and $\lambda \in \mathbb{D}$ we have

$$\lambda^{-1}(W(\lambda) - I) f = 2\Gamma^*(I - \lambda S^*)^{-1} S^* \Gamma f + 2D_T(I - \lambda C(\lambda))^{-1} C(\lambda) D_T f$$

$$= 2\Gamma^*(I - \lambda S^*)^{-1} S^* (E \omega_1 f + S \omega_2 f) + 2 D_T(I - \lambda C(\lambda))^{-1} C(\lambda) D_T f$$

$$= 2\Gamma^*(I - \lambda S^*)^{-1} \omega_2 f + 2 D_T(I - \lambda C(\lambda))^{-1} C(\lambda) D_T f$$

$$= (W(\lambda) + I) \omega_2 f - 2 D_T(I - \lambda C(\lambda))^{-1} D_T \omega_2 f + 2 D_T(I - \lambda C(\lambda))^{-1} C(\lambda) D_T f$$

$$= (W(\lambda) + I) \omega_2 f + 2 D_T(I - \lambda C(\lambda))^{-1} C(\lambda) D_T f - D_T \omega_2 f.$$
Since $G$ is defined as the inverse Cayley transform of $W$, we obtain for $f \in \mathcal{F}$ and $\lambda \in \mathbb{D}$ that
\[
G(\lambda)f = \lambda^{-1}(W(\lambda) - I)(W(\lambda) + I)^{-1}f = (W(\lambda) + I)^{-1}\lambda^{-1}(W(\lambda) - I)f = \omega_f + 2(W(\lambda) + I)^{-1}D_T(I - \lambda C(\lambda))^{-1}(C(\lambda)D_T f - D_T \omega_f).
\] (3.9)

If, in addition, $C \in S_\Omega(D_T, D_T)$, then $C(\lambda)D_T f = D_T \omega_f$ for all $f \in \mathcal{F}$ and all $\lambda \in \mathbb{D}$. In this case, the last term in (3.9) vanishes. In other words, $G(\lambda)|\mathcal{F} = \omega_f$ for all $\lambda \in \mathbb{D}$.

Conversely, if $G(\lambda)|\mathcal{F} = \omega_f$ for all $\lambda \in \mathbb{D}$, then (3.9) shows that
\[
(W(\lambda) + I)^{-1}D_T(I - \lambda C(\lambda))^{-1}(C(\lambda)D_T f - D_T \omega_f) = 0, \quad f \in \mathcal{F}, \lambda \in \mathbb{D}.
\]

Since $I - \lambda C(\lambda)$ is an invertible operator on $D_T$ for all $\lambda \in \mathbb{D}$ and $D_T|\mathcal{F}$ is one to one, this implies that $C(\lambda)D_T|\mathcal{F} = D_T \omega_f$ for all $\lambda \in \mathbb{D}$. Therefore $C$ is in $S_\Omega(D_T, D_T)$. This verifies our claim.

Part 2. In this part we prove our proposition. First assume that $\Gamma$ satisfies (3.2) and $C$ is in $S_\Omega(D_T, D_T)$. Then we obtain the first part shows that $G(\lambda)|\mathcal{F} = \omega_f$ for all $\lambda \in \mathbb{D}$. Since col $[F, G] = J_T C$, and $I - \lambda G(\lambda) = 2(I + W(\lambda))^{-1}$, we have $F(\lambda) = \Theta(\lambda)(I - \lambda G(\lambda))$ for all $\lambda \in \mathbb{D}$. Thus
\[
F(\lambda)f = \Theta(\lambda)f - \lambda \Theta(\lambda)G(\lambda)f = (\Gamma f)(\lambda) - \lambda \Theta(\lambda)\omega_f f = \omega_f + \lambda(\Gamma \omega_f)(\lambda) - \lambda(\Gamma \omega_f)(\lambda) = \omega_f, \quad f \in \mathcal{F}, \lambda \in \mathbb{D}.
\]

This proves that $\{F, G\}$ is a Schur pair.

Conversely, assume that $\{F, G\}$ is a Schur pair associated with the given data set. Since $F(\lambda)|\mathcal{F} = \omega_f$ and $F(\lambda) = \Theta(\lambda)(I - \lambda G(\lambda))$ for all $\lambda \in \mathbb{D}$, we obtain for all $f \in \mathcal{F}$ and all $\lambda \in \mathbb{D}$ that
\[
\omega_f f = F(\lambda)f = \Theta(\lambda)f - \lambda \Theta(\lambda)G(\lambda)f = (\Gamma f)(\lambda) - \lambda \Theta(\lambda)\omega_f f = (\Gamma f)(\lambda) - \lambda(\Gamma \omega_f)(\lambda).
\]

In other words, $\Gamma$ satisfies the constraint in (3.2). Using this along with $G(\lambda)|\mathcal{F} = \omega_f$ for all $\lambda \in \mathbb{D}$, the result of the first part shows that $C$ is in $S_\Omega(D_T, D_T)$. □

Proof of Theorem 3.1 Let $\{F, G\}$ be a Schur pair associated with the given data set. Then Theorem 3.1 and Proposition 3.2 show that $\Gamma$ given by (1.9) is a contraction from $D_A$ into $H^2(D_{T'})$ satisfying (1.6). Hence $B$ given by (1.8) is a solution to the RCL problem.

Conversely, assume that $B$ is a solution to the RCL problem. Then $B$ admits a matrix representation of the form (1.8), where $\Gamma$ is a contraction from $D_A$ into $H^2(D_{T'})$ satisfying (1.6). Recall that the set $S_\Omega(D_T, D_T)$ is not empty. Let $C$ be any function in $S_\Omega(D_T, D_T)$. Then we obtain from Proposition 3.2 that the pair of functions $\{F, G\}$ given by col $[F, G] = J_T C$ form a Schur pair associated with the given data set. Moreover, Theorem 3.1 shows that $\Gamma$ satisfies (1.9). □

Proof Theorem 3.2 Assume that $B$ is a solution to the RCL problem. Recall that $B$ admits a matrix representation of the form (1.8), where $\Gamma$ is a contraction from $D_A$ into $H^2(D_{T'})$ satisfying the constraint in (1.6). Then Proposition 3.2 implies that $J_T$ maps $S_\Omega(D_T, D_T)$ onto the set of Schur pairs $\{F, G\}$ such that (1.9) holds. According to Theorem 3.1 the map $J_T$ is one to one. □

As one may expect from the proof of Theorem 3.1 under appropriate additional conditions on the data set $\{A, T', U', R, Q\}$, the formula describing all solutions in Theorem 3.1 will yield a proper parametrization, that is, the relation between the Schur pair $\{F, G\}$ and the solution $B$ is one to one. We plan to come back to this question and the related question of uniqueness of the solution in a future publication.
References

[1] D. Alpay, V. Bolotnikov, and Y. Peretz, On the tangential interpolation problem for $H_2$ functions. *Trans. Amer. Math. Soc.* 347 (1995), 675–686.

[2] R. Arocena, Generalized Toeplitz kernels and dilations of intertwining operators, *Integral Equations and Operator Theory*, 6 (1983), 759–778.

[3] R. Arocena, Unitary extensions of isometries and contractive intertwining dilations, in: *The Gohberg Anniversary Collection II*, OT 41, Birkhäuser Verlag Basel, 1989, pp. 13–23.

[4] Gr. Arsene, Z. Ceausescu, and C. Foias, On intertwining dilations VIII, *J. Operator Theory* 4 (1980), 55–91.

[5] A. Biswas, C. Foias, and A. E. Frazho, Weighted Commutant Lifting, *Acta Sci. Math. (Szeged)*, 65 (1999), 657-686.

[6] P.L. Duren, *Theory of $H^p$ Spaces*, Academic Press INC., New York-London, 1970.

[7] C. Foias and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, OT 44, Birkhäuser-Verlag, Basel, 1990.

[8] C. Foias and A. E. Frazho, Constructing the Schur contraction in the commutant lifting theorem, *Acta Sci. Math. (Szeged)*, 61 (1995), 425-442.

[9] C. Foias, A.E. Frazho, and M.A. Kaashoek, Relaxation of metric constrained interpolation and a new lifting theorem, *Integral Equations and Operator Theory*, 42 (2002), 253–310.

[10] C. Foias, A.E. Frazho, I. Gohberg, and M. A. Kaashoek, *Metric Constrained Interpolation, Commutant Lifting and Systems*, Operator Theory: Advances and Applications, 100, Birkhäuser-Verlag, 1998.

[11] A.E. Frazho and M.A. Kaashoek, A Naimark dilation perspective of Nevanlinna-Pick interpolation, *Integral Equations and Operator theory*, 42 (2002), 253–310.

[12] A.E. Frazho, S. ter Horst, and M.A. Kaashoek, Coupling and relaxed commutant lifting, *Integral Equations and Operator theory*, to appear.

[13] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of Linear Operators*, Vol. II, Operator Theory: Advances and Applications, 63, Birkhäuser-Verlag, Basel, 1993.

[14] W.S. Li and D. Timotin, The relaxed intertwining lifting in the coupling approach, *Integral Equations and Operator theory*, to appear.

[15] B. Sz.-Nagy and C. Foias, Dilation des commutants d’opérateurs, *C. R. Acad. Sci. Paris, série A*, 266 (1968), 493-495.

[16] D. Sarason, Exposed points in $H^1$, I, in: *The Gohberg anniversary collection, Vol. II*, OT 41, Birkhäuser Verlag, Basel, 1989, pp. 485–496.

[17] S. Treil and A. Volberg, A fixed point approach to Nehari’s problem and its applications, *Operator Theory: Advances and Applications*, 71, Birkhäuser-Verlag, Basel, 1994, pp.165-186.
