Abstract: New algorithms for construction of asymptotic expansions for stationary distributions of nonlinearly perturbed semi-Markov processes with finite phase spaces are presented. These algorithms are based on a special technique of sequential phase space reduction, which can be applied to processes with an arbitrary asymptotic communicative structure of phase spaces. Asymptotic expansions are given in two forms, without and with explicit bounds for remainders.

Keywords: Markov chain; semi-Markov process; nonlinear perturbation; stationary distribution; expected hitting time; Laurent asymptotic expansion

2010 Mathematics Subject Classification: Primary 60J10, 60J27, 60K15, Secondary 65C40.

1. Introduction

In this paper, we present new algorithms for construction of asymptotic expansions for stationary distributions of nonlinearly perturbed semi-Markov processes with a finite phase space.

This is Part I of the paper, where algorithms for constructing of asymptotic expansions with remainders of a standard form $o(\cdot)$ are given. In Part II, we present algorithms for construction asymptotic expansions of a more advanced form, with explicit upper bounds for remainders.

We consider models, where the phase space is one class of communicative states, for embedded Markov chains of pre-limiting perturbed semi-Markov

---

1Department of Mathematics, Stockholm University, SE-106 81 Stockholm, Sweden.
Email address: silvestrov@math.su.se

2Division of Applied Mathematics, School of Education, Culture and Communication, Mälardalen University, SE-721 23 Västerås, Sweden.
Email address: sergei.silvestrov@mdh.se
processes, while it can possess an arbitrary communicative structure, i.e., can consist of one or several closed classes of communicative states and, possibly, a class of transient states, for the limiting embedded Markov chain.

The initial perturbation conditions are formulated in the forms of Taylor and Laurent asymptotic expansions, respectively, for transition probabilities (of embedded Markov chains) and expectations of sojourn times, for perturbed semi-Markov processes. Two variants of these expansions are considered, with remainders given without and with explicit upper bounds.

The algorithms are based on special time-space screening procedures for sequential phase space reduction and algorithms for re-calculation of asymptotic expansions and upper bounds for remainders, which constitute perturbation conditions for the semi-Markov processes with reduced phase spaces.

The final asymptotic expansions for stationary distributions of non-linearly perturbed semi-Markov processes are given in the form of Taylor asymptotic expansions with remainders given, as was mentioned above, in two variants, without (in Part I) and with explicit upper bounds (in Part II).

Models of perturbed Markov chains and semi-Markov processes, in particular, for the most difficult cases of perturbed processes with absorption and so-called singularly perturbed processes, attracted attention of researchers in the mid of the 20th century.

An interest to these models has been stimulated by applications to control and queuing systems, information networks, epidemic models and models of mathematical genetics and population dynamics. As a rule, Markov-type processes with singular perturbations appear as natural tools for mathematical analysis of multi-component systems with weakly interacting components.

The first works related to asymptotical problems for the above models are, Meshalkin (1958), Simon and Ando (1961), Hanen (1963), Seneta (1967), Schweitzer (1968), and Korolyuk (1969).

The methods used for construction of asymptotic expansions for stationary distributions and related functionals such as moments of hitting times can be split in three groups.

The most widely used methods are based on analysis of generalized resolvent type inverses for transition matrices and operators for singularly perturbed Markov chains and semi-Markov processes. Mainly, models with linear, polynomial and analytical perturbations have been objects of studies. We refer here to works by Schweitzer (1968), Turbin (1972), Poliščuk and Turbin (1973), Koroljuk, Brodi and Turbin (1974), Pervozvanskii and Smirnov (1974), Courtois and Louchard (1976), Korolyuk and Turbin (1976,
Aggregation/disaggregation methods based on various modifications of Gauss elimination method and space screening procedures for perturbed Markov chains have been employed for approximation of stationary distributions for Markov chains in works by Coderch, Willsky, Sastry and Castañoñ (1983), Delebecque (1983), Gaîtsgori and Pervozvanskiï (1983), Chatelin and Miranker (1984), Courtois and Semal (1984), Seneta (1984, 1991), Cao and Stewart (1985), Vantilborgh (1985), Feinberg and Chiu (1987), Haviv (1987, 1992, 1999), Rohlichek (1987), Rohlicek and Willsky (1988), Sumita and Reiders (1988), Meyer (1989), Schweitzer (1991), Stewart and Zhang (1991), Stewart (1993, 1998, 2001), Kim and Smith (1995), Marek and Pultarová (2006), Marek, Mayer and Pultarová (2009), and Avrachenkov, Filar and Howlett (2013).

Alternatively, methods based on regenerative properties of Markov chains and semi-Markov processes, in particular, relations which link stationary probabilities and expectations of return times, have been used for getting approximations for expectations of hitting times and stationary distributions in works by Grassman, Taksar and Heyman (1985), Hassin and Haviv (1992) and Hunter (2005). Also, the above mentioned relations and methods, based on asymptotic expansions for nonlinearly perturbed regenerative processes developed in works by Silvestrov (1995, 2010, 2014), Englund and Silvestrov (1997), Gyllenberg and Silvestrov (1999, 2000, 2008), Englund (2001), Ni, Silvestrov and Malyarenko (2008), Ni (2011, 2014), Petersson (2013, 2014), and Silvestrov and Petersson (2013), have been used for getting asymptotic expansions for stationary and quasi-stationary distributions for nonlinearly perturbed Markov chains and semi-Markov processes with absorption.

A more comprehensive bibliography of works in the area can be found in the research report by Silvestrov, D. and Silvestrov, S. (2015), which is an extended preliminary version of the present paper.

In the present paper, we combine methods based on the stochastic aggre-
ation/disaggregation approach with the methods based on asymptotic expansions for perturbed regenerative processes applied to perturbed semi-Markov processes.

In the above mentioned works based on the stochastic aggregation/disaggregation approach, space screening procedures for discrete time Markov chains are used. In this case, a Markov chain with reduced phase space is constructed from the initial one as the sequence of its states at sequential moment of hitting into the reduced phase space. Times between sequential hitting of the reduced phase space are not taken into account. Such screening procedure preserves ratios of hitting frequencies for states from the reduced phase space and, thus, ratios of stationary probabilities are the same for the initial and the reduced Markov chains. This implies that the stationary probabilities for the reduced Markov chain coincide with the corresponding stationary probabilities for the initial Markov chain up to the change of the corresponding normalizing factor.

We use another more complex type of time-space screening procedures, for semi-Markov processes. In this case, a semi-Markov process with reduced phase space is constructed from the initial one as the sequence of its states at sequential moment of hitting into the reduced phase space, and times between sequential jumps of the reduced semi-Markov process are times between sequential hitting of the reduced space by the initial semi-Markov process. Such screening procedure preserves hitting times for states from the reduced phase space, i.e., these times and, thus, their expectations are the same for the initial and the reduced semi-Markov processes.

We formulate perturbation conditions in terms of asymptotic expansions for transition characteristics of perturbed semi-Markov processes. The remainders in these expansions and, thus, the transition characteristics of perturbed semi-Markov processes can be non-analytical functions of perturbation parameter. This makes a difference with the results for models with linear, polynomial and analytical perturbations.

The methods of asymptotic analysis for nonlinearly perturbed regenerative processes developed in works by Silvestrov (1995, 2010) and Gyllenberg and Silvestrov (1999, 2000, 2008) are employed. However, we use the technique of more general Laurent asymptotic expansions, instead of Taylor asymptotic expansions used in the above mentioned works, and combine these methods with the aggregation/disaggregation approach, instead of the approach based on generalized matrix inverses. This let us consider perturbed semi-Markov processes with an arbitrary asymptotic communicative
structure of the phase space.

An important novelty of our studies also is that we consider asymptotic expansions with remainders given not only in a standard form of \( o(\cdot) \), but, also, in a more advanced form, with explicit power-type upper bounds for remainders, uniform with respect to a perturbation parameter.

Semi-Markov processes are a natural generalization of Markov chains, important theoretically and essentially extending applications of Markov-type models. The asymptotic results obtained in the paper are a good illustration for this statement. In particular, they automatically yield analogous asymptotic results for nonlinearly perturbed discrete and continuous time Markov chains.

We also show how algorithms based on sequential phase space reduction can be used for getting Laurent asymptotic expansions for expected hitting times, for nonlinearly perturbed semi-Markov processes. In the context of the present paper, such expansions play an intermediate role. At the same time, they, obviously, have their own theoretical and applied values.

The method proposed in the paper can be interpreted as a stochastic analogue of the Gauss elimination method. It is based on the procedure of sequential exclusion of states from the phase space of perturbed semi-Markov processes accompanied by re-calculation of asymptotic expansions penetrating perturbation conditions for semi-Markov processes with reduced phase spaces. The corresponding algorithms are based on some kind of “operational calculus” for Laurent asymptotic expansions with remainders given in two forms, without and with explicit upper bounds. These algorithms have an universal character. They can be applied to nonlinearly perturbed semi-Markov processes with an arbitrary asymptotic communicative structure of the phase space. The algorithms are computationally effective, due to a recurrent character of the corresponding computational procedures.

Part I of the paper includes seven sections. In Section 2, we present operational rules for Laurent asymptotic expansions. In Section 3, we formulate basic perturbation conditions for Markov chains and semi-Markov processes. In Section 4, we give some basic formulas for stationary distributions for semi-Markov processes, in particular, formulas connecting stationary distributions with expectations of return times. In Section 5, we present an one-step time-space screening procedure of phase space reduction for perturbed semi-Markov processes. In Section 6, we present algorithms for re-calculation of asymptotic expansions for transition characteristics of nonlinearly perturbed semi-Markov processes with reduced phase spaces. In
Section 7, we present an algorithm for sequential reduction of phase space for semi-Markov processes and construction of Laurent asymptotic expansions for expected return times. In Section 8, we present the main result in Part I of the paper that is a new algorithm for construction of asymptotic expansions for stationary distributions of nonlinearly perturbed semi-Markov processes.

2. Laurent asymptotic expansions

In this section, we present so-called operational rules for Laurent asymptotic expansions. The corresponding proofs and comments are given in Appendix A, in Part II of the paper.

Let \( A(\varepsilon) \) be a real-valued function defined on an interval \((0, \varepsilon_0]\), for some \( 0 < \varepsilon_0 \leq 1 \), and given on this interval by a Laurent asymptotic expansion,

\[
A(\varepsilon) = a_{h_A} \varepsilon^{h_A} + \cdots + a_{k_A} \varepsilon^{k_A} + o_A(\varepsilon^{k_A}),
\]

where (a) \(-\infty < h_A \leq k_A < \infty\) are integers, (b) coefficients \(a_{h_A}, \ldots, a_{k_A}\) are real numbers, (c) function \(o_A(\varepsilon^{k_A})/\varepsilon^{k_A} \to 0\) as \(\varepsilon \to 0\).

We refer to such Laurent asymptotic expansion as a \((h_A, k_A)\)-expansion.

We say that \((h_A, k_A)\)-expansion \(A(\varepsilon)\) is pivotal if it is known that \(a_{h_A} \neq 0\).

**Lemma 1.** If function \(A(\varepsilon) = a'_{h_A} \varepsilon^{h_A} + \cdots + a'_{k_A} \varepsilon^{k_A} + o'_{A}(\varepsilon^{k_A}), \varepsilon \in (0, \varepsilon_0]\) can be represented as, respectively, \((h'_A, k'_A)\)-and \((h''_A, k''_A)\)-expansion, then the asymptotic expansion for function \(A(\varepsilon)\) can be represented in the following the most informative form \(A(\varepsilon) = a_{h_A} \varepsilon^{h_A} + \cdots + a_{k_A} \varepsilon^{k_A} + o_A(\varepsilon^{k_A}), \varepsilon \in (0, \varepsilon_0]\) of \((h_A, k_A)\)-expansion, with parameters \(h_A = h'_A \vee h''_A, k_A = k'_A \wedge k''_A\), and coefficients \(a_{h_A}, \ldots, a_{k_A}\), and remainder \(o_A(\varepsilon^{k_A})\) given by the following relations:

(i) \(a'_l = 0\), for \(h'_A \leq l < h_A\) and \(a''_l = 0\), for \(h''_A \leq l < h_A\);

(ii) \(a_l = a'_l\), for \(h_A \leq l \leq k_A = k'_A \wedge k''_A\);

(iii) \(a_l = a''_l\), for \(k_A = k'_A < l \leq k_A\) if \(k'_A < k''_A\);

(iv) \(a_l = a'_l\), for \(k_A = k''_A < l \leq k_A\) if \(k''_A < k'_A\);

(v) \(o'_A(\varepsilon^{k_A}) + \sum_{k_A \leq l \leq k_A} a'_l \varepsilon^l = o''_A(\varepsilon^{k_A}) + \sum_{k_A \leq l \leq k_A} a''_l \varepsilon^l, \varepsilon \in (0, \varepsilon_0]\) and \(o_A(\varepsilon^{k_A})\) coincides, for \(\varepsilon \in (0, \varepsilon_0]\), with \(o'_A(\varepsilon^{k_A})\) if \(k'_A < k''_A\); \(o''_A(\varepsilon^{k_A})\) if \(k'_A = k''_A\); or \(o'_A(\varepsilon^{k_A})\) if \(k'_A > k''_A\).

The asymptotical expansion \(A(\varepsilon)\) is pivotal if and only if \(a_{h_A} = a'_{h_A} = a''_{h_A} \neq 0\).
It is also useful to mention that a constant $a$ can be interpreted as function $A(\varepsilon) \equiv a$. Thus, 0 can be represented, for any integer $-\infty < h \leq k < \infty$, as the $(h,k)$-expansion, $0 = 0\varepsilon^h + \ldots + 0\varepsilon^k + o(\varepsilon^k)$, with remainder $o(\varepsilon^k) \equiv 0$. Also, 1 can be represented, for any integer $0 \leq k < \infty$, as the $(0,k)$-expansion, $1 = 1 + 0\varepsilon + \ldots + 0\varepsilon^k + o(\varepsilon^k)$, with remainder $o(\varepsilon^k) \equiv 0$.

Let us consider four Laurent asymptotic expansions, $A(\varepsilon) = a_h \varepsilon^{h_A} + \cdots + a_k \varepsilon^{k_A} + o_A(\varepsilon^{k_A})$, $B(\varepsilon) = b_h \varepsilon^{h_B} + \cdots + b_k \varepsilon^{k_B} + o_B(\varepsilon^{k_B})$, $C(\varepsilon) = c_h \varepsilon^{h_C} + \cdots + c_k \varepsilon^{k_C} + o_C(\varepsilon^{k_C})$, and $D(\varepsilon) = d_h \varepsilon^{h_D} + \cdots + d_k \varepsilon^{k_D} + o_D(\varepsilon^{k_D})$ defined for $0 < \varepsilon \leq \varepsilon_0$, for some $0 < \varepsilon_0 \leq 1$.

The following lemma presents operational rules for Laurent asymptotic expansions.

**Lemma 2.** The following operational rules take place for Laurent asymptotic expansions:

(i) If $A(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_A, k_A)$-expansion and $c$ is a constant, then $C(\varepsilon) = c A(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_C, k_C)$-expansion such that:

(a) $h_C = h_A$, $k_C = k_A$;

(b) $c_{h_C+r} = c a_{h_C+r}$,

(c) $c_{h_C} = c a_A(\varepsilon^{k_A})$.

This expansion is pivotal if and only if $c_{h_C} = c a_A(\varepsilon^{k_A}) \neq 0$.

(ii) If $A(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_A, k_A)$-expansion and $B(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_B, k_B)$-expansion, then $C(\varepsilon) = A(\varepsilon) + B(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_C, k_C)$-expansion such that:

(a) $h_C = h_A \land h_B$, $k_C = k_A \land k_B$;

(b) $c_{h_C+r} = a_{h_C+r} + b_{h_C+r}$,

(c) $c_{h_C} = a_{h_A} + b_{h_B}$.

This expansion is pivotal if and only if $c_{h_C} = a_{h_A} + b_{h_B} \neq 0$.

(iii) If $A(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_A, k_A)$-expansion and $B(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_B, k_B)$-expansion, then $C(\varepsilon) = A(\varepsilon) \cdot B(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a $(h_C, k_C)$-expansion such that:

(a) $h_C = h_A + h_B$, $k_C = (k_A + h_B) \land (k_B + h_A)$;

(b) $c_{h_C+r} = a_{h_A+r} + b_{h_B+r}$,

(c) $c_{h_C} = a_{h_A} b_{h_B} + \sum_{i+j \leq k_A, h_A \leq i \leq k_B, h_B \leq j \leq k_B} a_i b_j \varepsilon^{i+j}$.

This expansion is pivotal if and only if $c_{h_C} = a_{h_A} b_{h_B} \neq 0$.

(iv) If $B(\varepsilon), \varepsilon \in (0, \varepsilon_0]$ is a pivotal $(h_B, k_B)$-expansion, then there exists $0 < \varepsilon_0' \leq \varepsilon_0$ such that $B(\varepsilon) \neq 0, \varepsilon \in (0, \varepsilon_0']$, and $C(\varepsilon) = \frac{1}{B(\varepsilon)}, \varepsilon \in (0, \varepsilon_0']$ is a
pivotal \((h_C, k_C)\)-expansion such that:

(a) \(h_C = -h_B; k_C = k_B - 2h_B\);

(b) \(c_{hc} = b_{-1}^{-1}; c_{hc} + r = -b_{-1}^{-1}\sum_{i \leq r} b_{h_B+i}c_{hC+r-i}, r = 1, \ldots, k_C - h_C\);

(c) \(a_C(\varepsilon^{k_C}) = -\frac{b_{-1}^{-1}\sum_{h_B < h_B + i < k_B, h_C < j < k_C} b_{h_B+i}c_{hC+j} + \sum_{h_B \leq i \leq k_B} c_{hC+j} b_i\varepsilon^{i}o_B(\varepsilon^{k_B})}{b_{h_B}^{e^{k_B}} + \sum_{h_B < h_B + i < k_C} b_{h_B+i}^{e^{k_B}} + o_B(\varepsilon^{k_B})}\).

(v) If \(A(\varepsilon), \varepsilon \in (0, \varepsilon_0] \) is a \((h_A, k_A)\)-expansion, and \(B(\varepsilon), \varepsilon \in (0, \varepsilon_0] \) is a pivotal \((h_B, k_B)\)-expansion, then, there exists \(0 < \varepsilon_0 \leq \varepsilon_0 \) such that \(B(\varepsilon) \neq 0, \varepsilon \in (0, \varepsilon_0]\), and \(D(\varepsilon) = A(\varepsilon), \varepsilon \in (0, \varepsilon_0]\) is a \((h_D, k_D)\)-expansion such that:

(a) \(h_D = h_A + h_C, k_D = (h_A + h_C)^{-1} + (k_A + h_C) \wedge (k_A + h_B) \wedge (h_B - 2h_B)\);

(b) \(d_{h_D+r} = \sum_{0 \leq i \leq r} a_{h_A+i+c_{hc}+r-i}, r = 0, \ldots, k_D - h_B\);

(c) \(a_D(\varepsilon^{k_D}) = \sum_{k_D < i+j \leq k_B, h_C \leq j \leq k_C} b_i c_j \varepsilon^{i+j} + \sum_{h_B \leq i \leq k_B} b_i \varepsilon^{i}a_C(\varepsilon^{k_C}) + \sum_{h_B \leq i \leq k_B} c_{hC+j} b_i \varepsilon^{i}o_B(\varepsilon^{k_B})\),

where \(c_{hc+j}, j = 0, \ldots, k_C - h_C\) and \(a_C(\varepsilon^{k_C})\) are, respectively, the coefficients and the remainder of the \((h_C, k_C)\)-expansion \(C(\varepsilon) = \frac{A(\varepsilon)}{\varepsilon^{k_C}}\) given in the above proposition (iv), or by the following formulas,

(d) \(d_{h_D+r} = \sum_{0 \leq i \leq r} a_{h_A+i+c_{hc}+r-i}, r = 0, \ldots, k_B - h_B\);

(e) \(d_{h_B+r} = b_{h_B}^{-1}(a_{h_A+r} - \sum_{0 \leq i \leq r} b_{h_B+i}d_{h_B+r-i}), r = 0, \ldots, k_D - h_B\);

(f) \(a_D(\varepsilon^{k_D}) = \frac{\sum_{k_A < k_B + h_A - h_B + i < k_B} a_i \varepsilon^{i}o_A(\varepsilon^{k_A})}{b_{h_B}^{e^{k_B}} + \sum_{h_B < h_B + i < k_B} b_{h_B+i}^{e^{k_B}} + o_B(\varepsilon^{k_B})} - \frac{\sum_{k_A < k_B + h_A - h_B + i < k_B} b_j d_{j} \varepsilon^{j}o_B(\varepsilon^{k_B})}{b_{h_B}^{e^{k_B}} + \sum_{h_B < h_B + i < k_B} b_{h_B+i}^{e^{k_B}} + o_B(\varepsilon^{k_B})}\).

This expansion is pivotal if and only if \(d_{h_D} = a_{h_A} c_{h_C} = a_{h_A}/b_{h_B} \neq 0\).

Remark 1. By Lemma 1, the Laurent asymptotic expansions for function \(D(\varepsilon), \varepsilon \in (0, \varepsilon_0]\), given by the alternative formulas (a) – (c) and (d) – (f) in proposition (v) of Lemma 2, coincide. Also, these Laurent asymptotic expansions coincide with the expansions given by formulas (a) – (c) in propositions (iv) of Lemma 2, if \(A(\varepsilon) \equiv 1\). In this case, 1 should be interpreted as the \((0, k_B - h_B)\)-expansion, \(1 = 1 + 0\varepsilon + \ldots + 0\varepsilon^{k_B-h_B} + o(\varepsilon^{k_B-h_B})\), with remainder \(o(\varepsilon^{k_B-h_B}) \equiv 0\).

The following operational rules for multiple summation and multiplication of Laurent asymptotic expansions, used in what follows, are direct corollaries of the corresponding summation and multiplication rules given in Lemma 2.

Lemma 3. Let \(A_m(\varepsilon) = a_{kAm,m} e^{hAm} + \ldots + a_{kAm,0} e^{kAm} + o(e^{kAm}), \varepsilon \in (0, \varepsilon_0]\) be a \((h_A, k_A)\)-expansion, for \(m = 1, \ldots, N\). In this case:

(i) \(B_n(\varepsilon) = A_1(\varepsilon) + \ldots + A_n(\varepsilon), \varepsilon \in (0, \varepsilon_0]\) is, for every \(n = 1, \ldots, N\), a \((h_B, k_B)\)-expansion, where:
The following lemma summarizes some basic algebraic properties of Laurent asymptotic expansions. It is a corollary of Lemmas 1 and 2.

**Lemma 4.** The summation and multiplication operations for Laurent asymptotic expansions defined in Lemma 2 possess the following algebraic properties, which should be understood as identities for the corresponding Laurent asymptotic expansions (i.e., identities for the corresponding parameters $h, k, \text{coefficients and remainders}$) of functions represented in two alternative forms in the functional identities given below:

(i) The summation and multiplication operations for Laurent asymptotic expansions satisfy the “elimination” identities that are implied by the corresponding functional identities, $A(\varepsilon) + 0 \equiv A(\varepsilon)$, \( A(\varepsilon) \cdot 1 \equiv A(\varepsilon) \), \( A(\varepsilon) - A(\varepsilon) \equiv 0 \) and \( A(\varepsilon) \cdot A(\varepsilon)^{-1} \equiv 1 \).

(ii) The summation operation for Laurent asymptotic expansions is commutative and associative that is implied by the corresponding functional identities, \( A(\varepsilon) + B(\varepsilon) \equiv B(\varepsilon) + A(\varepsilon) \) and \( (A(\varepsilon) + B(\varepsilon)) + C(\varepsilon) \equiv A(\varepsilon) + (B(\varepsilon) + C(\varepsilon)) \).

(iii) The multiplication operation for Laurent asymptotic expansions is commutative and associative that is implied by the corresponding functional identities, \( A(\varepsilon) \cdot B(\varepsilon) \equiv B(\varepsilon) \cdot A(\varepsilon) \) and \( (A(\varepsilon) \cdot B(\varepsilon)) \cdot C(\varepsilon) \equiv A(\varepsilon) \cdot (B(\varepsilon) \cdot C(\varepsilon)) \).
(iv) The summation and multiplication operations for Laurent asymptotic expansions possess distributive property that is implied by the corresponding functional identity, \((A(\varepsilon) + B(\varepsilon)) \cdot C(\varepsilon) \equiv A(\varepsilon) \cdot C(\varepsilon) + B(\varepsilon) \cdot C(\varepsilon)\).

Remark 2. In proposition (i) of Lemma 4, 0 should be interpreted as the \((h_A, k_A)\)-expansion, 0 = 0 + 0\varepsilon^{k_A} + \ldots + 0\varepsilon^{k_A} + o(\varepsilon^{k_A})\), with remainder \(o(\varepsilon^{k_A}) \equiv 0\), and 1 as \((0, k_A - h_A)\)-expansion, 1 = 1 + \varepsilon + \ldots + \varepsilon^{k_A-h_A} + o(\varepsilon^{k_A-h_A})\), with remainder \(o(\varepsilon^{k_A-h_A}) \equiv 0\).

Remark 3. The Laurent asymptotic expansion \(A(\varepsilon)\) is assumed to be pivotal, in the elimination identity implied by functional identity \(A(\varepsilon) \cdot A(\varepsilon)^{-1} \equiv 1\), and to hold, for \(0 < \varepsilon \leq \varepsilon_0'\) such that \(A(\varepsilon) \neq 0\), \(\varepsilon \in (0, \varepsilon_0']\).

3. Nonlinearly perturbed semi-Markov processes

Let \(X = \{1, \ldots, N\}\) and \((\eta^{(e)}_n, \kappa^{(e)}_n), n = 0, 1, \ldots\) be, for every \(\varepsilon \in (0, 1]\), a Markov renewal process, i.e., a homogeneous Markov chain with the phase space \(X \times [0, \infty)\), an initial distribution \(p^{(e)} = \{p_i^{(e)} = \mathbb{P}\{\eta^{(e)}_0 = i, \kappa^{(e)}_0 = 0\} = \mathbb{P}\{\eta^{(e)}_0 = i\}, i \in X\}\) and transition probabilities,

\[
Q^{(e)}_{ij}(t) = \mathbb{P}\{\eta^{(e)}_t = \kappa^{(e)}_t = 0, \eta^{(e)}_0 = i, \kappa^{(e)}_0 = s; \kappa^{(e)}_{t-1} = i, \kappa^{(e)}_{t+1} = j\}, (i, s, (j, t) \in X \times [0, \infty). \tag{2}
\]

In this case, the random sequence \(\eta^{(e)}\) is also a homogeneous (embedded) Markov chain with the phase space \(X\) and the transition probabilities,

\[
p_{ij}(\varepsilon) = \mathbb{P}\{\eta^{(e)}_1 = j, \kappa^{(e)}_1 = 0, \kappa^{(e)}_0 = i\} = Q^{(e)}_{ij}(\infty), i, j \in X. \tag{3}
\]

The following condition plays an important role in what follows:

A: There exist sets \(Y_i \subseteq X, i \in X\) and \(\varepsilon_0 \in (0, 1]\) such that: (a) probabilities \(p_{ij}(\varepsilon) > 0, j \in Y_i, i \in X\), for \(\varepsilon \in (0, \varepsilon_0]\); (b) probabilities \(p_{ij}(\varepsilon) = 0, j \in Y_i, i \in X\), for \(\varepsilon \in (0, \varepsilon_0]\); (c) there exists, for every pair of states \(i, j \in X\), an integer \(n_{ij} \geq 1\) and a chain of states \(i = l_{ij,0}, l_{ij,1}, \ldots, l_{ij,n_{ij}} = j\) such that \(l_{ij,1} \in Y_{l_{ij,0}}, \ldots, l_{ij,n_{ij}} \in Y_{l_{ij,n_{ij}-1}}\).

We refer to sets \(Y_i, i \in X\) as transition sets. Conditions A implies that all sets \(Y_i \neq \emptyset, i \in X\).

Condition A also implies that the phase space \(X\) of Markov chain \(\eta^{(e)}\) is one class of communicative states, for every \(\varepsilon \in (0, \varepsilon_0]\).

We also assume that the following condition excluding instant transitions holds:
**B:** \( Q^{(\varepsilon)}_{ij}(0) = 0, \ i, j \in \mathbb{X}, \) for every \( \varepsilon \in (0, \varepsilon_0]. \)

Let us now introduce a semi-Markov process,

\[ \eta^{(\varepsilon)}(t) = \eta^{(\varepsilon)}_\nu(t), \ t \geq 0, \]

where \( \nu^{(\varepsilon)}(t) = \max(n \geq 0 : \zeta^{(\varepsilon)}_n \leq t) \) is a number of jumps in the time interval \([0, t], \) for \( t \geq 0, \) and \( \zeta^{(\varepsilon)}_n = \kappa^{(\varepsilon)}_1 + \cdots + \kappa^{(\varepsilon)}_n, \ n = 0, 1, \ldots, \) are sequential moments of jumps, for the semi-Markov process \( \eta^{(\varepsilon)}(t). \)

If \( Q^{(\varepsilon)}_{ij}(t) = I(t \geq 1)p_{ij}(\varepsilon), \ t \geq 0, i, j \in \mathbb{X}, \) then \( \eta^{(\varepsilon)}(t) = \eta^{(\varepsilon)}_{ij}, t \geq 0 \) is a discrete time homogeneous Markov chain embedded in continuous time.

If \( Q^{(\varepsilon)}_{ij}(t) = (1-e^{-\lambda_i(\varepsilon)t})p_{ij}(\varepsilon), \ t \geq 0, i, j \in \mathbb{X} \) (here, \( 0 < \lambda_i(\varepsilon) < \infty, i \in \mathbb{X}, \)) then \( \eta^{(\varepsilon)}(t), t \geq 0 \) is a continuous time homogeneous Markov chain.

Let us also introduce expectations of sojourn times,

\[ e_{ij}(\varepsilon) = E_i\kappa^{(\varepsilon)}_1 I(\eta^{(\varepsilon)}_1 = j) = \int_0^\infty tQ^{(\varepsilon)}_{ij}(dt), \ i, j \in \mathbb{X}. \]

Here and henceforth, notations \( P_i \) and \( E_i \) are used for conditional probabilities and expectations under condition \( \eta^{(\varepsilon)}(0) = i. \)

We also assume that the following condition holds:

**C:** \( e_{ij}(\varepsilon) < \infty, \ i, j \in \mathbb{X}, \) for \( \varepsilon \in (0, \varepsilon_0]. \)

In the case of discrete time Markov chain, \( e_{ij}(\varepsilon) = p_{ij}(\varepsilon), i, j \in \mathbb{X}. \)

In the case of continuous time Markov chain, \( e_{ij}(\varepsilon) = \lambda_i(\varepsilon)^{-1}p_{ij}(\varepsilon), i, j \in \mathbb{X}. \)

Conditions **A** (a) – (b) and **B** imply that, for every \( \varepsilon \in (0, \varepsilon_0], \) expectations \( e_{ij}(\varepsilon) > 0, \) for \( j \in \mathbb{Y}_i, i \in \mathbb{X}, \) and \( e_{ij}(\varepsilon) = 0, \) for \( j \in \mathbb{X}_i, i \in \mathbb{X}. \)

Let us assume that the following perturbation condition, based on Taylor asymptotic expansions, holds:

**D:** \( p_{ij}(\varepsilon) = \sum_{l=l_{ij}}^{l_{ij}^+} a_{ij}[l]c^l + o_{ij}(\varepsilon^{l_{ij}^+}), \ v \in (0, \varepsilon_0], \) for \( j \in \mathbb{Y}_i, i \in \mathbb{X}, \) where (a)

\[ a_{ij}[l_{ij}] > 0 \text{ and } 0 \leq l_{ij} \leq l_{ij}^+ < \infty, \] for \( j \in \mathbb{Y}_i, i \in \mathbb{X}; \) (b) \( o_{ij}(\varepsilon^{l_{ij}^+})/\varepsilon^{l_{ij}^+} \to 0 \) as \( \varepsilon \to 0, \) for \( j \in \mathbb{Y}_i, i \in \mathbb{X}. \)

We also assume that the following perturbation condition, based on Laurent asymptotic expansions, holds:
E: \( e_{ij}(\varepsilon) = \sum_{l=m_{ij}} b_{ij}[l] \varepsilon^l + o_{ij}(\varepsilon^{m_{ij}^+}) \), \( \varepsilon \in (0, \varepsilon_0) \), for \( j \in \mathbb{Y}_i, i \in \mathbb{X} \), where
(a) \( b_{ij}[m_{ij}^+] > 0 \) and \( -\infty < m_{ij}^+ \leq m_{ij}^+ < \infty \), for \( j \in \mathbb{Y}_i, i \in \mathbb{X} \); (b) \( o_{ij}(\varepsilon^{m_{ij}^+})/\varepsilon^{m_{ij}^+} \to 0 \) as \( \varepsilon \to 0 \), for \( j \in \mathbb{Y}_i, i \in \mathbb{X} \).

Conditions A, D and E, assumed to hold for some \( \varepsilon_0 \in (0, 1] \), also hold for any \( \varepsilon' \in (0, \varepsilon_0] \).

It worth to note that an actual value of parameter \( \varepsilon_0 \in (0, 1] \) is not important in propositions concerned asymptotic expansions with remainders given in form of \( o(\cdot) \).

Let us, for the moment, exclude sub-condition (a) from condition A. Conditions D and E imply that there exits \( \varepsilon_0 \in (0, \varepsilon_0) \) such that \( p_{ij}(\varepsilon) = \sum_{l=i_{ij}^+}^l a_{ij}[l] \varepsilon^l + o_{ij}(\varepsilon^{l_{ij}^+}) > 0 \) and \( e_{ij}(\varepsilon) = \sum_{l=m_{ij}} b_{ij}[l] \varepsilon^l + o_{ij}(\varepsilon^{m_{ij}^+}) > 0 \), for \( j \in \mathbb{X}, i \in \mathbb{X}, \varepsilon \in (0, \varepsilon_0] \). We can, just, decrease parameter \( \varepsilon_0 \) and to take the new \( \varepsilon_0 = \tilde{\varepsilon}_0 \). Condition A (a) holds for this new value of \( \varepsilon_0 \).

We, however, do prefer to include sub-condition (a) in condition A, in order to have a clear description for the communicative structure of the phase space \( \mathbb{X} \), in one condition. In this case, the above inequalities hold for \( \tilde{\varepsilon}_0 = \varepsilon_0 \).

Conditions D and E are consistent with condition A (a), according the above remarks.

Matrix \( \|p_{ij}(\varepsilon)\| \) is stochastic, for every \( \varepsilon \in (0, \varepsilon_0] \). This model stochasticity assumption holds by the default.

Condition D should, also, be consistent with this model stochasticity assumption.

Condition D and proposition (i) (the multiple summation rule) of Lemma 3 imply that sum \( \sum_{j \in \mathbb{Y}} p_{ij}(\varepsilon) \) can, for every subset \( \mathbb{Y} \subseteq \mathbb{Y}_i \) and \( i \in \mathbb{X} \), be represented in the form of the following Laurent asymptotic expansion,
\[
\sum_{j \in \mathbb{Y}} p_{ij}(\varepsilon) = \sum_{l=l_{ij}^-}^{l_{ij}^+} a_{i,Y}[l] \varepsilon^l + o_{i,Y}(\varepsilon^{l_{ij}^+}),
\]
(6)

where: (a) \( l_{ij}^\pm = \min_{j \in \mathbb{Y}} l_{ij}^\pm \); (b) \( a_{i,Y}[l] = \sum_{j \in \mathbb{Y}} a_{ij}[l], \ l = l_{ij}^-, \ldots, l_{ij}^+ \), where \( a_{ij}[l] = 0 \), for \( 0 \leq l < l_{ij}^- \), \( j \in \mathbb{Y} \), and (c) \( o_{i,Y}(\varepsilon^{l_{ij}^+}) = \sum_{j \in \mathbb{Y}} (\sum_{l_{ij}^+ < l \leq l_{ij}^+} a_{ij}[l] \varepsilon^l + o_{ij}(\varepsilon^{l_{ij}^+})) \).

Let us introduce the following condition, which presents additional links between the asymptotic expansions penetrating condition D, which are caused by the above model stochasticity assumption:
\( F: (a) a_{i,Y_i}[l] = \sum_{j \in Y_i} a_{ij}[l] = 1 (l = 0), \ 0 = l^-_{i,Y_i} \leq l \leq l^+_{i,Y_i}, \ i \in \mathbb{X}, \) where \( a_{ij}[l] = 0, \) for \( 0 \leq l < l^-_{ij}, \ j \in Y_i, \ i \in \mathbb{X}; \) \( b) a_{i,Y_i}(\varepsilon_{l,Y_i}) = o(\varepsilon_{l,Y_i}) = 0, \varepsilon \in (0, \varepsilon_0], \ i \in \mathbb{X}. \)

**Lemma 5.** Let conditions \( A (a) - (b) \) and \( D \) hold. In this case, condition \( F \) is equivalent to the model stochasticity assumption that matrix \( \| p_{ij}(\varepsilon) \| \) is stochastic, for every \( \varepsilon \in (0, \varepsilon_0]. \)

**Proof.** The model stochasticity assumption for matrices \( \| p_{ij}(\varepsilon) \|, \varepsilon \in (0, \varepsilon_0], \) takes, under conditions \( A (a) - (b) \), the form of the following identity, which should hold for every \( i \in \mathbb{X}, \)

\[
\sum_{j \in Y_i} p_{ij}(\varepsilon) = 1, \varepsilon \in (0, \varepsilon_0]. \tag{7}
\]

Condition \( D \) let us apply Lemma 1 to the identity given in relation \( (7) \), for every \( i \in \mathbb{X}. \) The asymptotic expansion given in relation \( (3) \), for the case \( Y = Y_i, \) and the \( (0,k) \)-expansion, \( 1 = 1 + 0\varepsilon + \cdots + 0\varepsilon^k + o(\varepsilon^k), \) with remainder \( o(\varepsilon^k) \equiv 0 \) and \( k = l^+_{i,Y_i}, \) should be used. This proves that identities given in relation \( (7) \) imply holding of condition \( F. \) The opposite implication of identities given in relation \( (7) \) by condition \( F \) is obvious. \( \square \)

Additional comments concerned the link between perturbation condition \( D \) and the model stochasticity assumption for matrices \( \| p_{ij}(\varepsilon) \|, \varepsilon \in (0, \varepsilon_0] \) are given in Appendix B, in Part II of the paper.

It is also worse to note that, under the assumption of holding condition \( A (a), \) the perturbation conditions \( D \) and \( E \) are independent.

To see this, let us take arbitrary positive functions \( p_{ij}(\varepsilon), j \in \mathbb{X}_i, i \in \mathbb{X} \) and \( e_{ij}(\varepsilon), j \in Y_i, i \in \mathbb{X} \) satisfying, respectively, conditions \( D \) and \( E, \) and, also, the corresponding stochasticity identities \( (7). \) Then, there exist semi-Markov transition probabilities \( Q^{(e)}_{ij}(t), t \geq 0, j \in Y_i, i \in \mathbb{X} \) such that \( Q^{(e)}_{ij}(\infty) = p_{ij}(\varepsilon), j \in Y_i, i \in \mathbb{X} \) and \( \int_0^{\infty} t Q^{(e)}_{ij}(dt) = e_{ij}(\varepsilon), j \in Y_i, i \in \mathbb{X}, \) for every \( \varepsilon \in (0, \varepsilon_0]. \) It is readily seen that, for example, semi-Markov transition probabilities \( Q^{(e)}_{ij}(t) = 1(t \geq e_{ij}(\varepsilon)/p_{ij}(\varepsilon))p_{ij}(\varepsilon), t \geq 0, j \in Y_i, i \in \mathbb{X} \) satisfy the above relations.

4. Semi-Markov processes with reduced phase spaces

Let us choose some state \( r \in \mathbb{X} \) and consider the reduced phase space \( \mathbb{R} = \mathbb{X} \setminus \{r\}, \) with the state \( r \) excluded from the phase space \( \mathbb{X}. \)
Let us assume that the initial distributions satisfy the following assumption,

\[ p_r^{(e)} = P \{ \eta_0^{(e)} = r \} = 0, \; \varepsilon \in (0, \varepsilon_0). \]  

(8)

Let us define the sequential moments of hitting the reduced space \( rX \) by the embedded Markov chain \( \eta^{(e)} \),

\[ r\xi_n^{(e)} = \min(k > r\xi_{n-1}^{(e)}, \eta_k^{(e)} \in rX), \; n = 1, 2, \ldots, \; r\xi_0^{(e)} = 0. \]  

(9)

Now, let us define the random sequence,

\[
(r\eta_n^{(e)}, r\kappa_n^{(e)}) = \begin{cases} 
(\eta_0^{(e)}, 0) & \text{for } n = 0, \\
(\eta_n^{(e)}, \sum_{k=r\xi_{n-1}^{(e)}+1}^{r\kappa_n^{(e)}} k^{(e)}) & \text{for } n = 1, 2, \ldots.
\end{cases}
\]

(10)

This sequence is also a Markov renewal process with a phase space \( rX \times [0, \infty) \), the initial distribution \( r\bar{p}^{(e)} = \langle rP_i^{(e)} = p_i^{(e)}, i \in rX \rangle \) (recall that \( p_r^{(e)} = 0 \)), and transition probabilities defined for \( (i, s), (j, t) \in rX \times [0, \infty) \),

\[ rQ_{ij}^{(e)}(t) = P \{ r\eta_1^{(e)} = j, r\kappa_1^{(e)} \leq t / r\eta_0^{(e)} = i, r\kappa_0^{(e)} = s \}. \]  

(11)

Respectively, one can define the transformed semi-Markov process with the reduced phase space \( rX \),

\[ r\eta^{(e)}(t) = r\eta_{r\nu^{(e)}(t)}^{(e)}, \; t \geq 0, \]

(12)

where \( r\nu^{(e)}(t) = \max(n \geq 0 : r\xi_n^{(e)} \leq t) \) is a number of jumps at time interval \( [0, t] \), for \( t \geq 0 \), and \( r\xi_n^{(e)} = r\kappa_1^{(e)} + \cdots + r\kappa_n^{(e)} \), \( n = 0, 1, \ldots \) are sequential moments of jumps, for the semi-Markov process \( r\eta^{(e)}(t) \).

The transition probabilities \( rQ_{ij}^{(e)}(t) \) are expressed via the transition probabilities \( Q_{ij}^{(e)}(t) \) by the following formula, for \( t \geq 0, \; i, j \in rX \),

\[ rQ_{ij}^{(e)}(t) = Q_{ij}^{(e)}(t) + \sum_{n=0}^{\infty} Q_{ir}^{(e)}(t) * Q_{rr}^{(e)n}(t) * Q_{rj}^{(e)}(t). \]  

(13)

Here, symbol \( * \) is used to denote the convolution of distribution functions (possibly improper), and \( Q_{rr}^{(e)n}(t) \) is the n times convolution of the distribution function \( Q_{rr}^{(e)}(t) \).
Relation (13) directly implies the following formula for transition probabilities of the reduced embedded Markov chain \( r\eta_n^{(\varepsilon)} \), for \( i, j \in rX \),

\[
r p_{ij}(\varepsilon) = r Q_{ij}^{(\varepsilon)}(\infty) = \rho_{ij}(\varepsilon) + \sum_{n=0}^{\infty} \rho_{ir}(\varepsilon) p_{rr}(\varepsilon)^n p_{rj}(\varepsilon) \]

\[
= \rho_{ij}(\varepsilon) + \rho_{ir}(\varepsilon) \frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)}. \tag{14}
\]

Note that condition A implies that probabilities \( p_{rr}(\varepsilon) \in [0, 1) \), \( r \in X \), \( \varepsilon \in (0, \varepsilon_0] \).

Let us introduce sets, \( Y_{ir}^- = \{ j \in rX : j \in Y_r \} \) if \( r \in Y_i \), or \( \emptyset \) if \( r \notin Y_i \), and \( Y_{ir}^+ = \{ j \in rX : j \in Y_i \} \) for \( i, r \in X \).

We omit the proof of the following simple lemma.

**Lemma 6.** Condition A, assumed to hold for the Markov chains \( \eta_n^{(\varepsilon)} \), also holds for the Markov chains \( r\eta_n^{(\varepsilon)} \), with the same parameter \( \varepsilon_0 \) and transition sets \( rY_i \) defined by the following relation, for \( i, r \in X \),

\[
r Y_i = \{ j \in rX : r p_{ij}(\varepsilon) > 0, \varepsilon \in (0, \varepsilon_0] \} = Y_{ir}^- \cup Y_{ir}^+. \tag{15}
\]

Let us introduce expectations,

\[
r e_{ij}(\varepsilon) = \int_0^\infty t \cdot r Q_{ij}^{(\varepsilon)}(dt), \ i, j \in rX. \tag{16}
\]

Relation (13) directly implies the following formula for expectations of sojourn times for the reduced semi-Markov process \( r\eta^{(\varepsilon)}(t) \), for \( i, j \in rX \),

\[
r e_{ij}(\varepsilon) = e_{ij}(\varepsilon) + \sum_{n=0}^{\infty} \left( e_{ir}(\varepsilon) p_{rj}(\varepsilon) + (n + 1) e_{ir}(\varepsilon) p_{ir}(\varepsilon) p_{rj}(\varepsilon) \right) \\
+ e_{rj}(\varepsilon) p_{ir}(\varepsilon)) p_{rr}(\varepsilon)^n = e_{ij}(\varepsilon) + e_{ir}(\varepsilon) \frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)} \\
+ e_{rr}(\varepsilon) \frac{p_{ir}(\varepsilon)}{1 - p_{rr}(\varepsilon)} \frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)} + e_{rj}(\varepsilon) \frac{p_{ir}(\varepsilon)}{1 - p_{rr}(\varepsilon)}. \tag{17}
\]

The following simple lemma is the direct corollary of relation (17).

**Lemma 7.** Conditions B and C, assumed to hold for the semi-Markov processes \( \eta^{(\varepsilon)}(t) \), also hold for the semi-Markov processes \( r\eta^{(\varepsilon)}(t) \).
The following theorem plays the key role in what follows.

**Theorem 1.** Let conditions A – C hold for semi-Markov processes \( \eta^{(e)}(t) \). Then, for any state \( j \in \mathcal{X} \), the first hitting times \( \tau_j^{(e)} \) and \( \tau_j^{(r_e)} \) to the state \( j \), respectively, for semi-Markov processes \( \eta^{(e)}(t) \) and \( \eta^{(r_e)}(t) \), coincide, and, thus, the expectations of hitting times \( E_{ij}(\varepsilon) = E_{i,\tau_j^{(e)}} = E_{i,\tau_j^{(r_e)}} \), for any \( i, j \in \mathcal{X} \) and \( \varepsilon \in (0, \varepsilon_0] \).

**Proof.** The first hitting times to a state \( j \in \mathcal{X} \) are connected for Markov chains \( \eta^{(e)}(\varepsilon) \) and \( \eta^{(r_e)}(\varepsilon) \) by the following relation,

\[
\nu_j^{(e)} = \min(n \geq 1 : \eta_n^{(e)} = j) = \min(n \geq 1 : r\eta_n^{(e)} = j) = r\nu_j^{(r_e)},
\]

(18)

where \( r\nu_j^{(r_e)} = \min(n \geq 1 : r\eta_n^{(r_e)} = j) \).

The above relations imply that the following relation holds for the first hitting times to a state \( j \in \mathcal{X} \), for the semi-Markov processes \( \eta^{(e)}(t) \) and \( \eta^{(r_e)}(t) \),

\[
\tau_j^{(e)} = \sum_{n=1}^{\nu_j^{(e)}} \kappa_n^{(e)} = \sum_{n=1}^{r\nu_j^{(r_e)}} \kappa_n^{(r_e)} = \sum_{n=1}^{r\nu_j^{(r_e)}} r\kappa_n^{(r_e)} = r\tau_j^{(r_e)}.
\]

(19)

The equality of expectations is an obvious corollary of relation (19). □

5. Asymptotic expansions for transition characteristics of perturbed semi-Markov processes with reduced phase spaces

As was mentioned above, condition A implies that sets \( \mathcal{Y}_{rr}^+ \neq \emptyset, r \in \mathcal{X} \) and the non-absorption probability \( \bar{p}_{rr}(\varepsilon) = 1 - p_{rr}(\varepsilon) > 0 \), for \( r \in \mathcal{X}, \varepsilon \in (0, \varepsilon_0] \). This probability satisfies the following relation, for every \( r \in \mathcal{X} \) and \( \varepsilon \in (0, \varepsilon_0] \),

\[
\bar{p}_{rr}(\varepsilon) = 1 - p_{rr}(\varepsilon) = \sum_{j \in \mathcal{Y}_{rr}^+} p_{rj}(\varepsilon).
\]

(20)

**Lemma 8.** Let conditions A and D hold. Then, the pivotal \((\bar{l}_{rr}, \bar{r}_{rr})\)-expansions for the non-absorption probabilities \( \bar{p}_{rr}(\varepsilon), r \in \mathcal{X} \) are given by the algorithm described below, in the proof of the lemma.

**Proof.** Let \( r \in \mathcal{Y}_r \). First, proposition (i) (the multiple summation rule) of Lemma 3 should be applied to the sum \( \sum_{j \in \mathcal{Y}_r^+, r_{rj}(\varepsilon)} \). Second, propositions (i) (the multiplication by constant \(-1\)) and (ii) (the summation with...
constant 1) of Lemma 2 should be applied to the asymptotic expansion for probability $p_{rr}(\varepsilon)$ given in condition $\mathbf{B}$, in order to get the asymptotic expansion for function $1 - p_{rr}(\varepsilon)$. Third, Lemma 1 should be applied to the asymptotic expansion for function $\bar{p}_{rr}(\varepsilon)$ given in two alternative forms by relation (20). Note that condition $\mathbf{F}$ holds also for the above case, where the asymptotic expansion for probability $\bar{p}_{rr}(\varepsilon)$, obtained at the second step, is replaced by the improved version of this expansion, obtained with the use of Lemma 1 at the third step. The case $r \notin \mathbb{Y}_r$ is trivial, since, in this case, probability $\bar{p}_{rr}(\varepsilon) \equiv 1$. According to Lemmas 1–3, $(\bar{I}_{rr}, \bar{I}_{rr}^+)$-expansions $\bar{p}_{rr}(\varepsilon) = \sum_{l_{rr}} a_{rr} \left[ \varepsilon^l + \bar{a}_{rr}(\varepsilon^{l_{rr}}) \right], \varepsilon \in (0, \varepsilon_0], r \in \mathbb{X}$, yielded by the above algorithm, are pivotal.

Let us now describe an algorithm for construction of asymptotic expansions for transition probabilities $\bar{p}_{ij}(\varepsilon)$ given by relation (11).

**Theorem 2.** Conditions $\mathbf{A}$ and $\mathbf{D}$, assumed to hold for the Markov chains $\eta_i^{(\varepsilon)}$, also hold for the reduced Markov chains $\bar{\eta}_i^{(\varepsilon)}$, with the same parameter $\varepsilon_0$ and the transition sets $\mathbb{Y}_i, i \in \mathbb{X}$, given by relation (15). The pivotal $(l_{ij}^{\varepsilon}, l_{ij}^{\varepsilon+})$-expansions penetrating condition $\mathbf{D}$ are given for transition probabilities $\bar{p}_{ij}(\varepsilon), j \in \mathbb{Y}_i, i \in \mathbb{X}, r \in \mathbb{X}$ by the algorithm described below, in the proof of the theorem.

**Proof.** Lemma 6 implies that condition $\mathbf{A}$ holds for the Markov chains $\bar{\eta}_i^{(\varepsilon)}$, with the same parameter $\varepsilon_0$ as for the Markov chains $\eta_i^{(\varepsilon)}$, and the transition sets $\mathbb{Y}_i, i \in \mathbb{X}$ given by relation (15).

Let us prove that condition $\mathbf{D}$ also holds for the Markov chains $\bar{\eta}_i^{(\varepsilon)}$, with the same parameter $\varepsilon_0$ and the transition sets $\mathbb{Y}_i, i \in \mathbb{X}$ given by relation (15). In order to do this, let us construct the corresponding asymptotic expansions penetrating this condition. Let $j, r \in \mathbb{Y}_i \cap \mathbb{Y}_r$. First, proposition (v) (the division rule) of Lemma 2 should be applied to the quotients $\frac{p_{rr}(\varepsilon)}{1 - p_{rr}(\varepsilon)}$. Second, proposition (iii) (the multiplication rule) of Lemma 2 should be applied to the product $p_{ir}(\varepsilon) \cdot \frac{p_{ij}(\varepsilon)}{1 - p_{ij}(\varepsilon)}$. Third, proposition (ii) (the summation rule) of Lemma 2 should be applied to sum $\bar{p}_{ij}(\varepsilon) = p_{ij}(\varepsilon) + p_{ir}(\varepsilon) \cdot \frac{p_{ij}(\varepsilon)}{1 - p_{ij}(\varepsilon)}$. The asymptotic expansions for probabilities $p_{ij}(\varepsilon), p_{ir}(\varepsilon)$, and $p_{ij}(\varepsilon)$, given in condition $\mathbf{B}$, and probability $1 - p_{rr}(\varepsilon)$, given in Lemma 8, should be used. If $j \notin \mathbb{Y}_i$ then $p_{ij}(\varepsilon) \equiv 0$; if $j \notin \mathbb{Y}_r$ then $p_{ij}(\varepsilon) \equiv 0$; if $r \notin \mathbb{Y}_i$ then $p_{ir}(\varepsilon) \equiv 0$; if $r \notin \mathbb{Y}_r$ then $1 - p_{rr}(\varepsilon) \equiv 1$. In these cases, the above algorithm is readily simplified with the use of Lemma 4. Note that parameter $\varepsilon_0$ does not change in the multiplication and summation steps as well as in the division step, since
1 − p_{rr}(ε) > 0, ε ∈ (0, ε₀]. According to Lemma 2, the (r_{ij}, r_{ij}^1)-expansions

\[ r_{p_{ij}}(ε) = \sum_{r_{ij}} r_{a_{ij}}[l]ε^l + r_{o_{ij}}(ε^{r_{ij}}), \]  

\[ ε ∈ (0, ε₀], \quad j ∈ r_{Y_i}, \quad i ∈ r_{X}, \quad r ∈ X, \]  

yielded by the above algorithm, are pivotal. □

**Remark 4.** The matrix of transition probabilities \( ||r_{p_{ij}}(ε)|| \) is stochastic, for every \( ε ∈ (0, ε₀] \). Thus, under conditions of Theorem 2, condition F holds for the asymptotic expansions of transition probabilities \( r_{p_{ij}}(ε), j ∈ r_{Y_i}, i ∈ r_{X}, \) given in this theorem.

Let us now describe an algorithm for construction of asymptotic expansions for expectations \( r_{e_{ij}}(ε) \) given by relation (17).

**Theorem 3.** Conditions A – E, assumed to hold for the semi-Markov processes \( η^{(e)}(t) \), also hold for the reduced semi-Markov processes \( r_{η^{(e)}}(t) \). Parameter \( ε₀ \), in conditions A, D and E, is the same for processes \( η^{(e)}(t) \) and \( r_{η^{(e)}}(t) \). The transition sets \( r_{Y_i}, i ∈ r_{X} \) are given for processes \( r_{η^{(e)}}(t) \) by relation (15). The pivotal \( (r_{m_{ij}}, r_{m_{ij}^1}) \)-expansions penetrating this condition. Let \( i, j, r_{Y_i}, r \) be given by relation (15). Also, conditions B and C hold for the semi-Markov processes \( r_{η^{(e)}}(t) \), by Lemma 7.

In order to prove that condition E also holds for the semi-Markov processes \( r_{η^{(e)}}(t) \), with the same parameter \( ε₀ \) as for the semi-Markov processes \( η^{(e)}(t) \), and the transition sets \( r_{Y_i}, i ∈ r_{X} \) given by relation (15), let us construct the corresponding asymptotic expansions penetrating this condition. Let \( j, r ∈ Y_i \cap Y_r \). First, proposition (v) (the division rule) of Lemma 2 should be applied to the quotients \( \frac{p_{ir}(ε)}{1−p_{rr}(ε)} \) and \( \frac{p_{ir}(ε)}{1−p_{rr}(ε)} \). Second, proposition (iii) (the multiplication rule) of Lemma 2 should be applied to the products \( e_{ir}(ε) \cdot \frac{p_{ir}(ε)}{1−p_{rr}(ε)} \) and \( e_{ir}(ε) \cdot \frac{p_{ir}(ε)}{1−p_{rr}(ε)} \), and proposition (ii) (the multiple multiplication rule) of Lemma 3 to the product \( e_{rr}(ε) \cdot \frac{p_{ir}(ε)}{1−p_{rr}(ε)} \cdot \frac{p_{rr}(ε)}{1−p_{rr}(ε)} \). Third, proposition (i) (the multiple summation rule) of Lemma 3 should be applied to sum \( r_e_{ij}(ε) = e_{ij}(ε) + e_{ir}(ε) \cdot \frac{p_{ir}(ε)}{1−p_{rr}(ε)} + e_{rr}(ε) \cdot \frac{p_{ir}(ε)}{1−p_{rr}(ε)} + e_{ir}(ε) \cdot \frac{p_{ir}(ε)}{1−p_{rr}(ε)} \). The asymptotic expansions for probabilities \( p_{ij}(ε), p_{ir}(ε) \) and \( p_{rr}(ε) \), given in condition D, probability \( 1−p_{rr}(ε) \), given in Lemma 8, and expectations \( e_{ij}(ε), e_{ir}(ε), e_{rr}(ε) \) and \( e_{ir}(ε) \), given in condition E, should be used. If \( j \notin Y_i \) then \( p_{ij}(ε) \equiv 0 \).
and \( e_{ij}(\varepsilon) \equiv 0 \); if \( j \notin \Upsilon_r \) then \( p_{rj}(\varepsilon) \equiv 0 \) and \( e_{rj}(\varepsilon) \equiv 0 \); if \( r \notin \Upsilon_i \) then \( p_{ir}(\varepsilon) \equiv 0 \) and \( e_{ir}(\varepsilon) \equiv 0 \); if \( r \notin \Upsilon_r \) then \( 1 - p_{rr}(\varepsilon) \equiv 1 \) and \( e_{rr}(\varepsilon) \equiv 0 \). In these cases, the above algorithm is readily simplified with the use of Lemma 4. As in Theorem 2, parameter \( \varepsilon_0 \) does not change in the multiplication and summation steps as well as in the division step, since \( 1 - p_{rr}(\varepsilon) > 0 \), \( \varepsilon \in (0, \varepsilon_0] \). According to Lemmas 2 and 3, the \( (r_{m_{ij}}, r_{m_{ij}^+}) \)-expansions \( \varepsilon e_{ij}(\varepsilon) = \sum \varepsilon m_{ij} r_{b_{ij}^i} \epsilon^l + \varepsilon o_{ij}(\varepsilon m_{ij}^+), \varepsilon \in (0, \varepsilon_0], j \in \Upsilon_i, i \in \Upsilon, r \in \Upsilon \), yielded by the above algorithm, are pivotal. \( \square \)

It is worth to note that, despite bulky forms, formulas for parameters and algorithms for computing coefficients in the asymptotic expansions, presented in Lemma 8 and Theorems 2 and 3, are computationally effective.

7. Sequential reduction of phase spaces for perturbed semi-Markov processes

In what follows, let \( \bar{r}_{i,N} = \langle r_{i,1}, \ldots, r_{i,N} \rangle = \langle r_{i,1}, \ldots, r_{i,N-1}, i \rangle \) be a permutation of the sequence \( \langle 1, \ldots, N \rangle \) such that \( r_{i,N} = i \), and let \( \bar{r}_{i,n} = \langle r_{i,1}, \ldots, r_{i,n} \rangle, n = 1, \ldots, N \) be the corresponding chain of growing sequences of states from space \( \Upsilon \).

Theorem 4. Let conditions A – E hold for semi-Markov processes \( \eta^{(e)}(t) \). Then, for every \( i \in \Upsilon \), the pivotal \( (M_{ii}^-, M_{ii}^+) \)-expansion for the expectation of hitting time \( E_{ii}(\varepsilon) \) is given by the algorithm based on the sequential exclusion of states \( r_{i,1}, \ldots, r_{i,N-1} \) from the phase space \( \Upsilon \) of the processes \( \eta^{(e)}(t) \). This algorithm is described below, in the proof of the theorem. The above \( (M_{ii}^-, M_{ii}^+) \)-expansion is invariant with respect to any permutation \( \bar{r}_{i,N} = \langle r_{i,1}, \ldots, r_{i,N-1}, i \rangle \) of sequence \( \langle 1, \ldots, N \rangle \).

Proof. Let us assume that \( p_{i}^{(e)} = 1 \). Denote as \( \bar{r}_{i,0} \eta^{(e)}(t) = \eta^{(e)}(t) \), the initial semi-Markov process. Let us exclude state \( r_{i,1} \) from the phase space of semi-Markov process \( \bar{r}_{i,0} \eta^{(e)}(t) \) using the time-space screening procedure described in Section 5. Let \( \bar{r}_{i,1} \eta^{(e)}(t) \) be the corresponding reduced semi-Markov process. The above procedure can be repeated. The state \( r_{i,2} \) can be excluded from the phase space of the semi-Markov process \( \bar{r}_{i,1} \eta^{(e)}(t) \). Let \( \bar{r}_{i,2} \eta^{(e)}(t) \) be the corresponding reduced semi-Markov process. By continuing the above procedure for states \( r_{i,3}, \ldots, r_{i,n} \), we construct the reduced semi-Markov process \( \bar{r}_{i,n} \eta^{(e)}(t) \).

The process \( \bar{r}_{i,n} \eta^{(e)}(t) \) has the phase space \( \bar{r}_{i,n} \Upsilon = \Upsilon \setminus \{r_{i,1}, r_{i,2}, \ldots, r_{i,n}\}. \)
The transition probabilities of the embedded Markov chain $\bar{r}_{i,n}P_{i'j'}(\varepsilon), i', j' \in \bar{r}_{i,n}X$, and the expectations of sojourn times $\bar{r}_{i,n}e_{i'j'}(\varepsilon), i', j' \in \bar{r}_{i,n}X$ are determined for the semi-Markov process $\bar{r}_{i,n}\eta^{(e)}(t)$ by the transition probabilities and the expectations of sojourn times for the process $\bar{r}_{i,n-1}\eta^{(e)}(t)$, respectively, via relations (14) and (17).

By Theorem 1, the expectation of hitting time $E_{i'j'}(\varepsilon)$ coincides for the semi-Markov processes $\bar{r}_{i,0}\eta^{(e)}(t), \bar{r}_{i,1}\eta^{(e)}(t), \ldots, \bar{r}_{i,n}\eta^{(e)}(t)$, for every $i', j' \in \bar{r}_{i,n}X$.

By Theorems 2 and 3, the semi-Markov processes $\bar{r}_{i,n}\eta^{(e)}(t)$ satisfies conditions $\textbf{B}$, $\textbf{C}$ and, also, conditions $\textbf{A}$, $\textbf{D}$ and $\textbf{E}$, with the same parameter $\varepsilon_0$ as for processes $\bar{r}_{i,n-1}\eta^{(e)}(t)$. The transition sets $\bar{r}_{i,n}Y_{i'}, i' \in \bar{r}_{i,n}X$ determined by the transition sets $\bar{r}_{i,n-1}Y_{i'}, i' \in \bar{r}_{i,n-1}X$, via relation (15) given in Lemma 6. Therefore, the pivotal $\bar{r}_{i,n}l_{i'j'}^-(\varepsilon), \bar{r}_{i,n}l_{i'j'}^+(\varepsilon)$-expansions, $\bar{r}_{i,n}P_{i'j'}(\varepsilon) = \sum_{\bar{r}_{i,n}l_{i'j'}^-} A_{i'j'}[l]e^l + \bar{r}_{i,n}a_{i'j'}(\varepsilon, \bar{r}_{i,n}l_{i'j'}^+), \varepsilon \in (0, \varepsilon_0], j' \in \bar{r}_{i,n}Y_{i'}, i' \in \bar{r}_{i,n}X$, and the pivotal $\bar{r}_{i,n}m_{i'j'}^-(\varepsilon), \bar{r}_{i,n}m_{i'j'}^+(\varepsilon)$-expansions, $\bar{r}_{i,n}e_{i'j'}(\varepsilon) = \sum_{\bar{r}_{i,n}m_{i'j'}^-} b_{i'j'}[l]e^l + \bar{r}_{i,n}b_{i'j'}(\varepsilon, \bar{r}_{i,n}m_{i'j'}^+), \varepsilon \in (0, \varepsilon_0], j' \in \bar{r}_{i,n}Y_{i'}, i' \in \bar{r}_{i,n}X$, can be constructed by applying the algorithms given in Theorems 2 and 3, respectively, to the $l_{i,n-1}l_{i'j'}^-(\varepsilon), l_{i,n-1}l_{i'j'}^+(\varepsilon)$-expansions for transition probabilities $\bar{r}_{i,n-1}P_{i'j'}(\varepsilon), j' \in \bar{r}_{i,n-1}Y_{i'}, i' \in \bar{r}_{i,n-1}X$ and to the $m_{i,n-1}m_{i'j'}^-(\varepsilon), m_{i,n-1}m_{i'j'}^+(\varepsilon)$-expansions for expectations $\bar{r}_{i,n-1}e_{i'j'}(\varepsilon), j' \in \bar{r}_{i,n-1}Y_{i'}, i' \in \bar{r}_{i,n-1}X$.

The algorithm described above has a recurrent form and should be realized sequentially for the reduced semi-Markov processes $\bar{r}_{i,1}\eta^{(e)}(t), \ldots, \bar{r}_{i,n}\eta^{(e)}(t)$ starting from the initial semi-Markov process $\bar{r}_{i,0}\eta^{(e)}(t)$.

For every $j' \in \bar{r}_{i,n}Y_{i'}, i' \in \bar{r}_{i,n}X, n = 1, \ldots, N - 1$, the asymptotic expansions for the transition probability $\bar{r}_{i,n}P_{i'j'}(\varepsilon)$ and the expectation $\bar{r}_{i,n}e_{i'j'}(\varepsilon)$, resulted by the recurrent algorithm of sequential phase space reduction described above, are invariant with respect to any permutation $\bar{r}_{i,n} = \langle r'_{i,1}, \ldots, r'_{i,n} \rangle$ of sequence $\bar{r}_{i,n} = \langle r_{i,1}, \ldots, r_{i,n} \rangle$.

Indeed, for every permutation $\bar{r}_{i,n}$ of sequence $\bar{r}_{i,n}$, the corresponding reduced semi-Markov process $\bar{r}_{i,n}\eta^{(e)}(t)$ is constructed as the sequence of states for the initial semi-Markov process $\eta^{(e)}(t)$ at sequential moment of its hitting into the same reduced phase space $\bar{r}_{i,n}X = X \setminus \{r'_{i,1}, \ldots, r'_{i,n}\} = \bar{r}_{i,n}X = X \setminus \{r_{i,1}, \ldots, r_{i,n}\}$. The times between sequential jumps of the reduced semi-Markov process $\bar{r}_{i,n}\eta^{(e)}(t)$ are the times between sequential hitting of the
above reduced phase space by the initial semi-Markov process $\eta^{(\varepsilon)}(t)$.

This implies that the transition probability $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ and the expectation $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ are, for every $j' \in \bar{r}_{i,n}^{(\varepsilon)}Y_{i'}$, $i' \in \bar{r}_{i,n}^{(\varepsilon)}X$, $n = 1, \ldots, N - 1$, invariant (as functions of $\varepsilon$) with respect to any permutation $\bar{r}_{i,n}^{(\varepsilon)}$ of the sequence $\bar{r}_{i,n}^{(\varepsilon)}$. Moreover, as follows from algorithms presented above, in Lemma 8 and Theorems 2 and 3, the transition probability $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ is a rational function of the initial transition probabilities $p_{i'j'}^{(\varepsilon)}(\varepsilon), j'' \in \bar{Y}_{i'}, i'' \in \bar{X}$, and the expectation $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ is a rational function of the initial transition probabilities $p_{i'j'}^{(\varepsilon)}(\varepsilon), j'' \in \bar{Y}_{i'}, i'' \in \bar{X}$ and the initial expectations of sojourn times $e_{i'j'}^{(\varepsilon)}(\varepsilon), j'' \in \bar{Y}_{i'}, i'' \in \bar{X}$ (quotients of sums of products for some of these probabilities and expectations), which, according the above remarks, are invariant with respect to any permutation $\bar{r}_{i,n}^{(\varepsilon)}$ of the sequence $\bar{r}_{i,n}^{(\varepsilon)}$.

By using identity arithmetical transformations (disclosure of brackets, imposition of a common factor out of the brackets, bringing a fractional expression to a common denominator, permutation of summands or multipliers, elimination of expressions with equal absolute values and opposite signs in the sums and elimination of equal expressions in quotients) the rational functions $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ and $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ can be transformed, respectively, into the rational functions $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ and $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ and wise versa. By Lemma 4, these transformations do not affect the corresponding asymptotic expansions for functions $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ and $\bar{r}_{i,n}^{(\varepsilon)}(\varepsilon)$ and, thus, these expansions are invariant with respect to any permutation $\bar{r}_{i,n}^{(\varepsilon)}$ of the sequence $\bar{r}_{i,n}^{(\varepsilon)}$.

In fact, one should only check the above invariance propositions for the case, where the permutations $\bar{r}_{i,n}^{(\varepsilon)}$ is obtained from the sequence $\bar{r}_{i,n}^{(\varepsilon)}$ by exchange of a pair of neighbor states $r_{i,k}$ and $r_{i,k+1}$, for some $1 \leq k \leq n - 1$. Then, the proof can be repeated for a pair of neighbor states for the sequence $\bar{r}_{i,n}^{(\varepsilon)}$, etc. In this way, the proof can be expanded to the case of an arbitrary permutation $\bar{r}_{i,n}^{(\varepsilon)}$ of the sequence $\bar{r}_{i,n}^{(\varepsilon)}$. The above mentioned proof of pairwise permutation invariance involves processes $\bar{r}_{i,k-1}^{(\varepsilon)}(t), \bar{r}_{i,k}^{(\varepsilon)}(t)$ and $\bar{r}_{i,k+1}^{(\varepsilon)}(t)$. It is absolutely analogous, for $1 \leq k \leq n - 1$. Taking this into account, we just show how this proof can be accomplished for the case $k = 1$.

The transition probabilities $\bar{r}_{i,2}^{(\varepsilon)}(\varepsilon)$ and $\bar{r}_{i,2}^{(\varepsilon)}(\varepsilon)$ for the sequences $\bar{r}_{i,2} = (r_1, r_2)$ and $\bar{r}_{i,2}^{(\varepsilon)} = (r_2, r_1)$ (here, $i, i', j' \neq r_1, r_2$) can be transformed into the same symmetric (with respect to $r_1, r_2$) rational function of $\varepsilon \in (0, \varepsilon_0]$, using the identity arithmetical transformations listed above,

$$\bar{r}_{i,2}^{(\varepsilon)}(\varepsilon) = r_1P_{i'j'}^{(\varepsilon)}(\varepsilon) + r_1P_{i't}^{(\varepsilon)}(\varepsilon) \frac{r_1P_{i'j'}^{(\varepsilon)}(\varepsilon)}{1 - r_1P_{i't}^{(\varepsilon)}(\varepsilon)}$$
\[ \begin{align*}
= p_i \bar{\nu} j'(\varepsilon) + p_i \bar{\nu} r_1(\varepsilon) \frac{p_{r_1 j'}(\varepsilon)}{1 - p_{r_1 r_1}(\varepsilon)} \\
+ \left( p_i \bar{\nu} r_2(\varepsilon) + p_i \bar{\nu} r_1(\varepsilon) \frac{p_{r_1 r_2}(\varepsilon)}{1 - p_{r_1 r_1}(\varepsilon)} \right) \frac{1}{1 - p_{r_2 r_2}(\varepsilon) - p_{r_2 r_1}(\varepsilon)} \frac{p_{r_2 j'}(\varepsilon) + p_{r_2 r_1}(\varepsilon) \frac{p_{r_2 j'}(\varepsilon)}{1 - p_{r_2 r_1}(\varepsilon)}}{1 - p_{r_1 r_1}(\varepsilon) - p_{r_2 r_1}(\varepsilon)} \\
= p_i \bar{\nu} j'(\varepsilon) + p_i \bar{\nu} r_1(\varepsilon) p_{r_1 j'}(\varepsilon) (1 - p_{r_2 r_2}(\varepsilon)) + p_i \bar{\nu} r_1(\varepsilon) p_{r_1 r_2}(\varepsilon) p_{r_2 j'}(\varepsilon) \\
+ p_i \bar{\nu} r_2(\varepsilon) p_{r_2 j'}(\varepsilon) (1 - p_{r_1 r_1}(\varepsilon)) + p_i \bar{\nu} r_2(\varepsilon) p_{r_2 r_1}(\varepsilon) p_{r_1 j'}(\varepsilon) \\
= r_2 p_i \bar{\nu} j'(\varepsilon) + r_2 p_i \nu r_1(\varepsilon) \frac{r_2 p_{r_1 j'}(\varepsilon)}{1 - r_2 p_{r_1 r_1}(\varepsilon)} = \nu_{i,2} p_i \nu j'(\varepsilon). \quad (21)
\end{align*} \]

Therefore, by Lemma 4, the Laurent asymptotic expansions for transition probabilities \( \bar{\nu}_{i,2} p_{i,j'}(\varepsilon) \) and \( \nu_{i,2} e_{i,j'}(\varepsilon) \), given by the recurrent algorithm of sequential phase space reduction described above, are identical.

The proof of identity for the Laurent asymptotic expansions of expectations \( \bar{\nu}_{i,2} e_{i,j'}(\varepsilon) \) and \( \nu_{i,2} e_{i,j'}(\varepsilon) \), given by the recurrent algorithm of sequential phase space reduction described above, is analogous.

Let us take \( n = N - 1 \). In this case, the semi-Markov process \( \bar{r}_{i,N-1} \eta^{(\varepsilon)}(t) \) has the phase space \( \bar{r}_{i,N-1} \mathbb{X} = \mathbb{X} \setminus \{ r_{i,1}, r_{i,2}, \ldots, r_{i,N-1} \} = \{ i \} \), which is a one-state set. The process \( \bar{r}_{i,N-1} \eta^{(\varepsilon)}(t) \) returns in state \( i \) after every jump. Its transition probability \( \bar{r}_{i,N-1} p_{ii}(\varepsilon) = 1 \) and the expectation of hitting time \( E_{ii}(\varepsilon) = \bar{r}_{i,N-1} e_{ii}(\varepsilon) \).

Thus, the above recurrent algorithm of sequential phase space reduction makes it possible to write down the following pivotal Laurent asymptotic expansion,

\[ E_{ii}(\varepsilon) = \sum_{l = M_{ii}^-}^{M_{ii}^+} B_{ii}[l] \varepsilon^l + \tilde{\nu}_{ii}(\varepsilon^{M_{ii}^+}) = \bar{r}_{i,N-1} e_{ii}(\varepsilon^{M_{ii}^+}), \varepsilon \in (0, \varepsilon_0], \quad (22) \]

where (a) \( M_{ii}^\pm = \bar{r}_{i,N-1} m_{ii}^\pm \); (b) \( B_{ii}[l] = \bar{r}_{i,N-1} b_{ii}[l], l = M_{ii}^-, \ldots, M_{ii}^+ \); (c) \( \tilde{\nu}_{ii}(\varepsilon^{M_{ii}^+}) = \bar{r}_{i,N-1} \tilde{\nu}_{ii}(\varepsilon^{M_{ii}^+}) \).

By the above remarks, the asymptotic expansion given in relation \( (22) \) is invariant with respect to the choice of sequence \( \bar{r}_{i,N-1} = \{ r_{i,1}, \ldots, r_{i,N-1} \} \). This legitimates notations (with omitted index \( \bar{r}_{i,N-1} \)) used for parameters, coefficients and remainder in the above asymptotic expansion.

The algorithm for construction of the Laurent asymptotic expansion for expectation \( E_{ii}(\varepsilon) \), given in relation \( (22) \), can be repeated for every \( i \in \mathbb{X} \). \( \square \)
Remark 5. Since matrices $\|\tilde{r}_{i,n}p_{i,j}(\varepsilon)\|$, $\varepsilon \in (0, \varepsilon_0]$, $n = 0, \ldots, N - 1$ are stochastic, the asymptotic expansions for transition probabilities $\tilde{r}_{i,n}p_{i,j}(\varepsilon)$, $j' \in \tilde{r}_{i,n}Y_i$, $i' \in \tilde{r}_{i,n}X$ satisfy condition $F$, for every $n = 0, \ldots, N - 1$.

8. Asymptotic expansions for stationary distributions of nonlinearly perturbed semi-Markov processes

In this section, we describe an algorithm for construction of asymptotic expansions for stationary distributions of nonlinearly perturbed semi-Markov processes.

The following theorem is the main new result in Part I of the present paper.

**Theorem 5.** Let conditions $A \rightarrow E$ hold for semi-Markov processes $\eta^{(c)}(t)$. Then, for every $i \in X$, the pivotal $(n^-_i, n^+_i)$-expansion for the stationary probability $\pi_i(\varepsilon)$ is given by the algorithm based on the sequential exclusion of states $r_{i,1}, \ldots, r_{i,N-1}$ from the phase space $X$ of the processes $\eta^{(c)}(t)$. This algorithm is described below, in the proof of the theorem. The above $(n^-_i, n^+_i)$-expansion is invariant with respect to any permutation $\tilde{r}_{i,N} = \langle r_{i,1}, \ldots, r_{i,N-1}, i \rangle$ of sequence $\langle 1, \ldots, N \rangle$. Relations (1) – (6), given in the proof, hold for these expansions.

**Proof.** First, condition $E$ and proposition (i) (the multiple summation rule) of Lemma 3 make it possible to write down pivotal $(m^-_i, m^+_i)$-expansions for expectations $e_i(\varepsilon), i \in X$. These expansions take the following form, for $i \in X$,

$$e_i(\varepsilon) = \sum_{j \in Y_i} e_{ij}(\varepsilon) = \sum_{l=m^-_i}^{m^+_i} b_{ij}[l] \varepsilon^l + \hat{o}_i(\varepsilon^{m^+_i}), \varepsilon \in (0, \varepsilon_0],$$

(23)

where (a) $m^+_i = \min_{j \in Y_i} m^+_i$, (b) $b_{ij}[m^-_i + l] = \sum_{j \in Y_i} b_{ij}[m^-_i + l], l = 0, \ldots, m^+_i - m^-_i$, where $b_{ij}[m^-_i + l] = 0$, for $0 \leq l < m^-_i - m^-_i, j \in Y_i$; (c) $\hat{o}_i(\varepsilon^{m^+_i})$ is given by formula (c) from proposition (i) (the multiple summation rule) of Lemma 3, which should be applied to the corresponding Laurent asymptotic expansions given in condition $E$.

Second, conditions $A \rightarrow E$, the asymptotic expansions given in relations (22) and (23), and proposition (v) (the division rule) of Lemma 2 make it possible to write down $(n^-_i, n^+_i)$-expansions for the stationary probabilities $\pi_i(\varepsilon) = \frac{e_i(\varepsilon)}{E_{ii}(\varepsilon)}, i \in X$. These expansions take the following form, for $i \in X$,
\[ \pi_i(\varepsilon) = \sum_{l=n_i^-}^{n_i^+} c_i[l] \varepsilon^l + o_i(\varepsilon^{n_i^+}), \varepsilon \in (0, \varepsilon_0], \quad (24) \]

where: (a) \( n_i^- = m_i^- - M_{ii}^- \), \( n_i^+ = (M_{ii}^+ - M_{ii}^-) \wedge (M_{ii}^+ - 2M_{ii}^- + m_i^-) \); (b) \( c_i[n_i^- + l] = B_{ii}[M_{ii}^-]^{-1}[b_i[m_i^- + l] - \sum_{1 \leq l' < l} B_{ii}[M_{ii}^- + l']c_i[n_i^- + l - l']] \), \( l = 0, \ldots, n_i^+ - n_i^- \); (c) \( o_i(\varepsilon^{n_i^+}) \) is given by formula (f) from proposition (v) (the division rule) of Lemma 2, which should be applied to the asymptotic expansions given in relations (22) and (23).

Since the asymptotic expansions given in relations (22) and (23) are pivotal, the expansions given in relation (24) are also pivotal, i.e., \( c_i[n_i^-] = b_{ii}[m_i^-]/B_{ii}[M_{ii}^-] \neq 0, i \in X \). Moreover, since \( \pi_i(\varepsilon) > 0, i \in X, \varepsilon \in (0, \varepsilon_0] \), the following relation takes place, (1) \( c_i[n_i^-] > 0, i \in X \).

By the definition, \( c_i(\varepsilon) \leq E_{ii}(\varepsilon), i \in X, \varepsilon \in (0, \varepsilon_0] \). This implies that parameters \( M_{ii}^- \leq m_i^- \), \( i \in X \), and thus, (2) \( n_i^- \geq 0, i \in X \).

Since, \( \sum_{i \in X}\pi_i(\varepsilon) = 1, \varepsilon \in (0, \varepsilon_0] \), parameters \( n_i^+, i \in X \), and coefficients \( c_i[l], l = n_i^-, \ldots, n_i^+ \), \( i \in X \), satisfy relations, (3) \( n_i^- = \min_{i \in X} n_i^- = 0 \), and, (4) \( \sum_{i \in X} c_i[l] = 1(l = 0), 0 \leq l \leq n_i^+ = \min_{i \in X} n_i^+ \). Moreover, the remainders of asymptotic expansions given in (24) satisfy identity, (5) \( \sum_{i \in X}(\sum_{n_i^- < l \leq n_i^+} c_i[l] \varepsilon^l + o_i(\varepsilon^{n_i^+})) = 0, \varepsilon \in (0, \varepsilon_0] \).

By the above remarks, (6) there exists \( \lim_{\varepsilon \to 0} \pi_i(\varepsilon) = \pi_i(0) \), which equals to \( c_i[0] > 0 \) if \( i \in X_0 \), or \( 0 \) if \( i \notin X_0 \), where \( X_0 = \{ i \in X : n_i^- = 0 \} \).

As follows from Theorem 4, the asymptotic expansion (22) for expectation \( E_{ii}(\varepsilon) \) and, thus, the asymptotic expansion (24) for stationary probability \( \pi_i(\varepsilon) \) is, for every \( i \in X \), invariant with respect to any permutation \( \bar{r}_{i,N} = \{ r_{i,1}, \ldots, r_{i,N-1}, i \} \) of sequence \( \{1, \ldots, N\} \). \( \square \)

It is appropriate to add some comments concerned two key components of the method proposed in the paper.

First of all, we would like to stress the principal role of semi-Markov setting used instead of a more traditional Markov setting. The time-space screening procedure used in the paper transforms any initial semi-Markov process to a new semi-Markov process with reduced phase space. Moreover, this procedure transforms the initial perturbation conditions, given in the form of asymptotic expansions for transition probabilities and expectations of sojourn times, to similar perturbation conditions for the reduced semi-Markov processes. However, this time-space screening procedure does not preserve Markov setting, except some trivial cases. Usually, this procedure,
applied to a discrete or continuous time Markov chain, results in a semi-
Markov process, which is not a Markov chain. This is because of the times
between sequential hitting of the reduced phase space by the initial process,
as a rule, have distributions, which differ of geometrical or exponential ones.

Also, the use of Laurent asymptotic expansions for expectations of sojourn
times for perturbed semi-Markov processes is an adequate and necessary el-
lement of the method. Expectations of sojourn times may be asymptotically
bounded (as functions of the perturbation parameter) and represented by
Taylor asymptotic expansions, for all states of the initial semi-Markov pro-
cesses. Even in this case, the exclusion of asymptotically absorbing states
from the phase space can cause appearance of states with asymptotically
unbounded expectations of sojourn times represented by Laurent asymptotic
expansions, for the reduced semi-Markov processes.

In conclusion, we would like to mention that the results presented in the
paper have a good potential for continuation of research studies (asymptotic
expansions for power and exponential moments for hitting times, asymptotic
expansions for quasi-stationary distributions, aggregated time-space screening
procedures, etc.). Some more detailed comments are given in the last
section of Part II of the paper.

References

[1] Avrachenkov, K. E. (1999). Analytic Perturbation Theory and Its Applications.
PhD Thesis, University of South Australia.

[2] Avrachenkov, K. E., Filar, J. A. and Howlett, P. G. (2013). Analytic Per-
turbation Theory and Its Applications. SIAM, Philadelphia, PA, xii+372 pp.

[3] Avrachenkov, K. E. and Haviv, M. (2003). Perturbation of null spaces with
application to the eigenvalue problem and generalized inverses. Linear Algebra Appl.,
369, 1–25.

[4] Avrachenkov, K. E. and Haviv, M. (2004). The first Laurent series coefficients
for singularly perturbed stochastic matrices. Linear Algebra Appl., 386, 243–259.

[5] Avrachenkov, K. E. and Lasserre, J. B. (1999). The fundamental matrix of
singularly perturbed Markov chains. Adv. Appl. Probab., 31, no. 3, 679–697.

[6] Bini, D. A., Latouche, G. and Meini, B. (2005). Numerical Methods for Struct-
tured Markov Chains. Numerical Mathematics and Scientific Computation, Oxford
Science Publications, Oxford University Press, New York, xii+327 pp.

[7] Cao, W. L. and Stewart, W. J. (1985). Iterative aggregation/disaggregation
techniques for nearly uncoupled Markov chains. J. Ass. Comp. Mach., 32, 702–719.
[8] Chatelin, F. and Miranker, W. L. (1984). Aggregation/disaggregation for eigenvalue problems. SIAM J. Numer. Anal., 21, no. 3, 567–582.

[9] Coderch, M., Willsky, A. S., Sastry, S. S. and Castaño, D. A. (1983). Hierarchical aggregation of singularly perturbed finite state Markov processes. Stochastics, 8, 259–289.

[10] Courtois, P. J. and Louchard, G. (1976). Approximation of eigen characteristics in nearly-completely decomposable stochastic systems. Stoch. Process. Appl., 4, 283–296.

[11] Courtois, P. J. (1977). Decomposability: Queueing and Computer System Applications. ACM Monograph Series, Academic Press, New York, xiii+201 pp.

[12] Courtois, P. J. and Semal, P. (1984). Error bounds for the analysis by decomposition of non-negative matrices. In: Iazeolla, G., Courtois, P.J. and Hordijk, A. (Eds.) Mathematical Computer Performance and Reliability. North-Holland, Amsterdam, 209–224.

[13] Craven, B. D. (2003). Perturbed Markov processes. Stoch. Models, 19, no. 2, 269–285.

[14] DeLebecque, F. (1983). A reduction process for perturbed Markov chains. SIAM J. Appl. Math., 43, 325–350.

[15] Englund, E. (2001). Nonlinearly Perturbed Renewal Equations with Applications. Doctoral dissertation, Umeå University.

[16] Englund, E. and Silvestrov, D. S. (1997). Mixed large deviation and ergodic theorems for regenerative processes with discrete time. In: Jagers, P., Kulldorff, G., Portenko, N. and Silvestrov, D. (Eds.) Proceedings of the Second Scandinavian–Ukrainian Conference in Mathematical Statistics, Vol. I, Umeå, 1997. Theory Stoch. Process., 3(19), no. 1–2, 164–176.

[17] Feinberg, B. N. and Chiu, S. S. (1987). A method to calculate steady-state distributions of large Markov chains by aggregating states. Oper. Res., 35, no. 2, 282–290.

[18] Gaïtsgory, V. G. and Pervozvanskii, A. A. (1975). Aggregation of states in a Markov chain with weak interaction. Kibernetika, no. 3, 91–98 (English translation in Cybernetics, 11, no. 3, 441–450).

[19] Grassman, W. K., Taksar, M.I. and Heyman, D. P. (1985). Regenerative analysis and steady state distributions for Markov chains. Oper. Res., 33, 1107–1116.

[20] Gyllenberg, M. and Silvestrov, D. S. (1999). Quasi-stationary phenomena for semi-Markov processes. In: Janssen, J., Limnios, N. (Eds.) Semi-Markov Models and Applications. Kluwer, Dordrecht, 33–60.

[21] Gyllenberg, M. and Silvestrov, D. S. (2000). Nonlinearly perturbed regenerative processes and pseudo-stationary phenomena for stochastic systems. Stoch. Process. Appl., 86, 1–27.
[22] Gyllenberg, M. and Silvestrov, D. S. (2008). *Quasi-Stationary Phenomena in Nonlinearly Perturbed Stochastic Systems*. De Gruyter Expositions in Mathematics, 44, Walter de Gruyter, Berlin, ix+579 pp.

[23] Hanen, A. (1963). Théorèmes limites pour une suite de chaînes de Markov. *Ann. Inst. H. Poincaré*, 18, 197–301.

[24] Hassin, R. and Haviv, M. (1992). Mean passage times and nearly uncoupled Markov chains. *SIAM J. Disc. Math.*, 5, 386–397.

[25] Haviv, M. (1986). An approximation to the stationary distribution of a nearly completely decomposable Markov chain and its error analysis. *SIAM J. Algebr. Discr. Meth.*, 7, no. 4, 589–593.

[26] Haviv, M. (1987). Aggregation/disaggregation methods for computing the stationary distribution of a Markov chain. *SIAM J. Numer. Anal.*, 24, no. 4, 952–966.

[27] Haviv, M. (1992). An aggregation/disaggregation algorithm for computing the stationary distribution of a large Markov chain. *Comm. Statist. Stoch. Models*, 8, no. 3, 565–575.

[28] Haviv, M. (1999). On censored Markov chains, best augmentations and aggregation/disaggregation procedures. Aggregation and disaggregation in operations research. *Comput. Oper. Res.*, 26, no. 10-11, 1125–1132.

[29] Haviv, M., Ritov, Y. and Rothblum, U. G. (1987). Iterative methods for approximating the subdominant modulus of an eigenvalue of a nonnegative matrix. *Linear Algebra Appl.*, 87, 61–75.

[30] Hunter, J. J. (1986). Stationary distributions of perturbed Markov chains. *Linear Algebra Appl.*, 82, 201–214.

[31] Kartashov, N. V. (1985). Asymptotic expansions and inequalities in stability theorems for general Markov chains under relatively bounded perturbations. In: *Stability Problems for Stochastic Models*. VIINISI, Moscow, 75–85 (English translation in *J. Soviet Math.*, 40, no. 4, 509–518).

[32] Kartashov, M. V. (1996). *Strong Stable Markov Chains*. VSP, Utrecht and TBiMC, Kiev, 138 pp.

[33] Kim, D. S. and Smith, R. L. (1995). An exact aggregation/disaggregation algorithm for large scale Markov chains. *Naval Res. Logist.*, 42, no. 7, 1115–1128.

[34] Kokotović, P. V., Phillips, R. G. and Javid, S. H. (1980). Singular perturbation modeling of Markov processes. In: Bensoussan, A., and Lions, J. L. (Eds.) *Analysis and Optimization of Systems: Proceedings of the Fourth International Conference on Analysis and Optimization*. Lecture Notes in Control and Information Science, 28, Springer, Berlin, 3–15.

[35] Korolyuk, V. S. (1969). On asymptotical estimate for time of a semi-Markov process being in the set of states. *Ukr. Mat. Zh.*, 21, 842–845 (English translation in *Ukr. Math. J.*, 21, 705–710).

[36] Korolyuk, V. S., Brodi, S. M. and Turbin, A. F. (1974). Semi-Markov processes and their application. *Probability Theory, Mathematical Statistics. Theoretical Cybernetics*, Vol. 11, VINTI, Moscow, 47–97.
[37] Korolyuk, V. S. and Korolyuk, V. V. (1999). Stochastic Models of Systems. Mathematics and its Applications, 469, Kluwer, Dordrecht, xii+185 pp.

[38] Koroliuk, V. S. and Limnios, N. (2005). Stochastic Systems in Merging Phase Space. World Scientific, Singapore, xv+331 pp.

[39] Korolyuk, V. S. and Turbin, A. F. (1976). Semi-Markov Processes and its Applications. Naukova Dumka, Kiev, 184 pp.

[40] Korolyuk, V. S. and Turbin, A. F. (1978). Mathematical Foundations of the State Lumping of Large Systems. Naukova Dumka, Kiev, 218 pp. (English edition: Mathematics and its Applications, 264, Kluwer, Dordrecht, 1993, x+278 pp.).

[41] Latouche, G. and Louchard, G. (1978). Return times in nearly decomposable stochastic processes. J. Appl. Probab., 15, 251–267.

[42] Marek, I., Mayer, P. and Pultarová, I. (2009). Convergence issues in the theory and practice of iterative aggregation/disaggregation methods. Electron. Trans. Numer. Anal., 35, 185–200.

[43] Marek, I. and Pultarová, I. (2006). A note on local and global convergence analysis of iterative aggregation-disaggregation methods. Linear Algebra Appl., 413, no. 2-3, 327–341.

[44] Meshalkin, L. D. (1958). Limit theorems for Markov chains with a finite number of states. Teor. Veroyatn. Primen., 3, 361–385 (English translation in Theory Probab. Appl., 3, 335–357).

[45] Meyer, C. D. (1989). Stochastic complementation, uncoupling Markov chains, and the theory of nearly reducible systems. SIAM Rev., 31, no. 2, 240–272.

[46] Ni, Y. (2011). Nonlinearly Perturbed Renewal Equations: Asymptotic Results and Applications. Doctoral dissertation, 106, Mälardalen University, Västerås.

[47] Ni, Y. (2014). Exponential asymptotical expansions for ruin probability in a classical risk process with non-polynomial perturbations. In: Silvestrov, D. and Martin-Löf, A. (Eds.) Modern Problems in Insurance Mathematics. EAA series, Springer, Cham, 67–91.

[48] Ni, Y., Silvestrov, D. and Malyarenko, A. (2008). Exponential asymptotics for nonlinearly perturbed renewal equation with non-polynomial perturbations. J. Numer. Appl. Math., 1(96), 173–197.

[49] Pervozvanskii, A. A. and Smirnov, I. N. (1974). An estimate of the steady state of a complex system with slowly varying constraints. Kibernetika, no. 4, 45–51 (English translation in Cybernetics, 10, no. 4, 603–611).

[50] Petersson, M. (2013). Quasi-stationary distributions for perturbed discrete time regenerative processes. Teor. Īmovīrn. Mat. Stat., 89, 140–155 (Also in Theor. Probab. Math. Statist., 89, 153–168).

[51] Petersson, M. (2014). Asymptotics of ruin probabilities for perturbed discrete time risk processes. In: Silvestrov, D. and Martin-Löf, A. (Eds.) Modern Problems in Insurance Mathematics. EAA series, Springer, Cham, 93–110.
[52] Poliščuk, L. I. and Turbin, A. F. (1973). Asymptotic expansions for certain characteristics of semi-Markov processes. *Teor. Veroyatn. Mat. Stat.*, **8**, 122–127 (English translation in *Theory Probab. Math. Statist.*, **8**, 121–126).

[53] Rohlichek, J. R. (1987). *Aggregation and time scale analysis of perturbed Markov systems*. Ph.D. Thesis, Massachusetts Inst. Tech., Cambridge, MA.

[54] Rohlicek, J. R. and Willsky, A. S. (1988). The reduction of perturbed Markov generators: an algorithm exposing the role of transient states. *J. Assoc. Comput. Mach.*, **35**, no. 3, 675–696.

[55] Schweitzer, P. J. (1968). Perturbation theory and finite Markov chains. *J. Appl. Probab.*, **5**, 401–413.

[56] Schweitzer, P. J. (1991). A survey of aggregation-disaggregation in large Markov chains. In: Stewart, W. J. (Ed.) *Numerical Solution of Markov Chains*. Probability: Pure and Applied, **8**, Marcel Dekker, New York, 63–88.

[57] Schweitzer, P. and Stewart, G. W. (1993). The Laurent expansion of pencils that are singular at the origin. *Linear Algebra Appl.*, **183**, 237–254.

[58] Seneta, E. (1967). Finite approximations to infinite non-negative matrices. *Proc. Cambridge Philos. Soc.*, **63**, 983–992.

[59] Seneta, E. (1984). Iterative aggregation: convergence rate. *Econom. Lett.*, **14**, no. 4, 357–361.

[60] Seneta, E. (1991). Sensitivity analysis, ergodicity coefficients, and rank-one updates for finite Markov chains. In: Stewart, W. J. (Ed.) *Numerical Solution of Markov Chains*. Probability: Pure and Applied, **8**, Marcel Dekker, New York, 121–129.

[61] Seneta, E. (2006). *Non-Negative Matrices and Markov Chains*. Springer Series in Statistics. Springer, New York, xvi+287 pp. (A revised reprint of the second (1981) edition).

[62] Silvestrov, D. S. (1995). Exponential asymptotic for perturbed renewal equations. *Teor. Imovirn. Mat. Stat.*, **52**, 143–153 (English translation in *Theory Probab. Math. Statist.*, **52**, 153–162).

[63] Silvestrov D. S. (2010). Nonlinearly Perturbed Stochastic Processes and Systems. In: Rykov, V., Balakrishnan, N. and Nikulin, M. (Eds.) *Mathematical and Statistical Models and Methods in Reliability*. Birkhäuser, Basel, 19–38.

[64] Silvestrov D. S. (2014). Improved asymptotics for ruin probabilities. In: Silvestrov, D. and Martin-Löf, A. (Eds.) *Modern Problems in Insurance Mathematics*. EAA series, Springer, Cham, 93–110.

[65] Silvestrov, D. S. and Abadov, Z. A. (1991). Uniform asymptotic expansions for exponential moments of sums of random variables defined on a Markov chain and distributions of passage times. 1. *Teor. Veroyatn. Mat. Stat.*, **45**, 108–127 (English translation in *Theory Probab. Math. Statist.*, **45**, 105–120).

[66] Silvestrov, D. S. and Abadov, Z. A. (1993). Uniform asymptotic expansions for exponential moments of sums of random variables defined on a Markov chain, and of distributions of passage times. 2. *Teor. Veroyatn. Mat. Stat.*, **48**, 175–183 (English translation in *Theory Probab. Math. Statist.*, **48**, 125–130).
[67] Silvestrov, D. S. and Petersson, M. (2013). Exponential expansions for perturbed discrete time renewal equations. In: Karagrigoriou, A., Lisnianski, A., Kleyner, A. and Frenkel, I. (Eds.) Applied Reliability Engineering and Risk Analysis. Probabilistic Models and Statistical Inference, Wiley, Chichester, 349–362.

[68] Silvestrov, D. and Silvestrov, S. (2015). Asymptotic expansions for stationary distributions of perturbed semi-Markov processes. Research Report 2015-9, Department of Mathematics, Stockholm University, 75 pp. and arXiv:1603.03891.

[69] Simon, H. A. and Ando, A. (1961). Aggregation of variables in dynamic systems. Econometrica, 29, 111–138.

[70] Stewart, G. W. (1993). Gaussian elimination, perturbation theory, and Markov chains. In: Meyer, C. D. and Plemmons, R. J. (Eds.) Linear Algebra, Markov Chains, and Queueing Models. IMA Volumes in Mathematics and its Applications, 48, Springer, New York, 59–69.

[71] Stewart, G. W. (1998). Matrix Algorithms. Vol. I. Basic Decompositions. SIAM, Philadelphia, PA, xx+458 pp.

[72] Stewart, G. W. (2001). Matrix Algorithms. Vol. II. Eigensystems. SIAM, Philadelphia, PA, xx+469 pp.

[73] Stewart, G. W. and Sun, J. G. (1990). Matrix Perturbation Theory. Computer Science and Scientific Computing, Academic Press, Boston, xvi+365 pp.

[74] Stewart, G. W. and Zhang, G. (1991). On a direct method for the solution of nearly uncoupled Markov chains. Numer. Math., 59, no. 1, 1–11.

[75] Sumita, U. and Reiders, M. (1988). A new algorithm for computing the ergodic probability vector for large Markov chains: Replacement process approach. Probab. Eng. Infor. Sci., 4, 89–116.

[76] Turbin, A. F. (1972). An application of the theory of perturbations of linear operators to the solution of certain problems that are connected with Markov chains and semi-Markov processes. Teor. Veroyatn. Mat. Stat., 6, 118–128 (English translation in Theory Probab. Math. Statist., 6, 119–130).

[77] Vantilborgh, H. (1985). Aggregation with an error of $O(\varepsilon^2)$. J. Assoc. Comput. Mach., 32, no. 1, 162–190.

[78] Yin, G. and Zhang, Q. (2003). Discrete-time singularly perturbed Markov chains. In: Yao, D. D., Zhang, H., and Zhou, X. Y. (Eds.) Stochastic Modeling and Optimization, Springer, New York, 1–42.

[79] Yin, G. G. and Zhang, Q. (2005). Discrete-Time Markov Chains. Two-time-scale methods and applications. Stochastic Modelling and Applied Probability, Springer, New York, xix+348 pp.

[80] Yin, G. G. and Zhang, Q. (2013). Continuous-Time Markov Chains and Applications. A Two-Time-Scale Approach. Second edition, Stochastic Modelling and Applied Probability, 37, Springer, New York, xxii+427 pp. (An extended variant of the first (1998) edition).