ANALYSIS AND OPTIMAL CONTROL OF SOME QUASILINEAR PARABOLIC EQUATIONS

Eduardo Casas*
Departamento de Matemática Aplicada y Ciencias de la Computación
E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria
39005 Santander, Spain

Konstantinos Chrysafinos
Department of Mathematics, School of Applied Mathematics and Physical Sciences
National Technical University of Athens, Zografou Campus
15780 Athens, Greece

Dedicated to Prof. Jiongmin Yong on the occasion of his 60th birthday

Abstract. In this paper, we consider optimal control problems associated with a class of quasilinear parabolic equations, where the coefficients of the elliptic part of the operator depend on the state function. We prove existence, uniqueness and regularity for the solution of the state equation. Then, we analyze the control problem. The goal is to get first and second order optimality conditions. To this aim we prove the necessary differentiability properties of the relation control-to-state and of the cost functional.

1. Introduction. In this paper, we analyze the following optimal control problem

\[
\begin{align*}
\min_{\alpha \leq u(x,t) \leq \beta} J(u)
\end{align*}
\]

with

\[
J(u) = \frac{1}{2} \int_{Q} (y_u(x,t) - y_d(x,t))^2 \, dx \, dt + \frac{\nu}{2} \int_{Q} u^2(x,t) \, dx \, dt,
\]

\(y_u\) being the solution of the following quasilinear partial differential equation

\[
\begin{align*}
\frac{\partial y}{\partial t} - \text{div}_x \left[ a(x,t,y(x,t)) \nabla_x y \right] + a_0(x,t,y(x,t)) &= u \quad \text{in} \ Q = \Omega \times (0,T), \\
y(x,t) &= 0 \quad \text{on} \ \Sigma = \Gamma \times (0,T), \\
y(x,0) &= y_0(x) \quad \text{in} \ \Omega.
\end{align*}
\]

Above \(\Omega \subset \mathbb{R}^n, 1 \leq n \leq 3,\) is a bounded open set with a \(C^{1,1}\) boundary \(\Gamma\) ([33, Definition 1.2.1.1]), \(T > 0, -\infty < \alpha < \beta < +\infty,\) and \(\nu > 0\) are given numbers.

Two main parts are considered in the paper. The first one concerns the analysis of the state equation and the second one, the analysis of the associated control problem. Under suitable assumptions, we prove existence, uniqueness and regularity of

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* Corresponding author: Eduardo Casas.
the solution of (1). Then, in the second part, additional differentiability assumptions on the functions \( a \) and \( a_0 \) of the state equation are imposed to prove first and second order optimality conditions for (P). In this paper, the main emphasis is to establish the theoretical results needed to carry out the numerical analysis of the control problem. Indeed, the regularity of the optimal controls deduced from the first order optimality conditions, and the sufficient second order optimality conditions are the key tools to prove error estimates for the numerical approximation of the control problem. The numerical analysis will be done in a forthcoming paper.

There are many papers and books devoted to the control of linear and semilinear parabolic equations. Let us mention the books [30], [35], [39], [40], [46]. We also cite some papers dealing with first and/or second order optimality conditions of semilinear parabolic control problems: [2], [3], [4], [6], [7], [11], [17], [19], [20], [21], [25], [37], [44].

However, just a few papers are devoted to the control of quasilinear equations of \( p \)-Laplace type: [10], [14], [15], [16], [18], [26], [41]. However, control problems where the nonlinearity is in the state, not in the gradient, have not been extensively studied. These equations are not of monotone type. In the elliptic case, we mention [12], [13], [21], [22]. We only know two papers for this type of nonlinearity in the parabolic case: [5] and [31]. In [31], the nonlinearity is quite general, but the author imposes strong assumptions on the corresponding functions, which are not required in our formulation. An important difference of our problem with the one studied in [5] is that we do not assume the boundedness of the nonlinear function \( a(x,t,y) \). Additionally, we prove more regularity for the optimal control and state, which is essential to prove error estimates for the numerical approximations.

Before finishing this section, let us introduce some notation. Given \( 1 \leq p, q \leq \infty \), we denote
\[
W^{1,0}_{p,q} = L^p(0,T;W^{1,q}_0(\Omega)) \cap W^{1,p}(0,T;W^{-1,q}_0(\Omega))
\]
and
\[
W^{2,1}_{p,q}(Q) = L^p(0,T;W^{2,q}_0(\Omega) \cap W^{1,1}_0(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)).
\]
We take \( \| \cdot \| = \| \cdot \|_X + \| \cdot \|_Y \) as the norm in the spaces \( X = X_1 \cap X_2 \). In the case that \( p = q \), we set \( W^{2,1}_{p,1}(Q) = W^{2,1}_p(Q) \) and \( W^{1,0}_{p,0}(Q) = W^{1,0}_p(Q) \), respectively. Moreover, if \( p = q = 2 \) we denote \( H^{2,1}(Q) = W^{2,1}_2(Q) \) and \( W(0,T) = W^{1,0}_2(Q) \). We also consider the Besov spaces (\( 1 < q < \infty \) and \( 1 \leq p \leq \infty \))
\[
W_{q,p}(\Omega) = (W^{-1,q}_0(\Omega),W^{1,q}_0(\Omega))_{1-\frac{1}{p},p}
\]
and
\[
B_{q,p}(\Omega) = (L^q(\Omega),W^{2,q}_0(\Omega) \cap W^{1,q}_0(\Omega))_{1-\frac{1}{p},p},
\]
where \((X,Y)_{1-\frac{1}{p},p}\) denotes the real interpolation of the Banach spaces \( X \) and \( Y \); see, for instance, [45]. We recall that for \( p = q \geq 2 \), the following identities hold [8, §14.2]
\[
W^{1-\frac{2}{p},p}_0(\Omega) = W_{p,p}(\Omega) = (W^{-1,p}(\Omega),W^{1,p}_0(\Omega))_{1-\frac{1}{p},p}
\]
and
\[
W^{2-\frac{4}{p},p}_0(\Omega) \cap W^{1,p}_0(\Omega) = B_{p,p}(\Omega) = (L^p(\Omega),W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega))_{1-\frac{1}{p},p}.
\]
From the continuous embedding \( W^{1,0}_p(Q) \subset C([0,T],W^{-1,p}(\Omega)) \), we get that \( y(0) \) is well defined for every \( y \in W^{1,0}_p(Q) \). Moreover, it is known that, for \( p \geq 2 \),
the mapping
\[ W_p^{1,0}(Q) \rightarrow W_0^{1-\frac{2}{pq}}(\Omega) \]
is continuous and surjective; see [1, Theorem III/4.10.2].

2. Analysis of the state equation. Throughout this section we make the following assumptions.

**Assumption 1.** The function \( a : \bar{Q} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and
\( \exists \Lambda > 0 \) such that \( a(x,t,y) \geq \Lambda \) \( \forall (x,t,y) \in \bar{Q} \times \mathbb{R} \); \( \forall M > 0 \exists L_M : \forall x_1, x_2 \in \bar{\Omega}, \forall t \in \mathbb{R}, \) and \( \forall |y_1|, |y_2| \leq M \)
\[ |a(x_2, t, y_2) - a(x_1, t, y_1)| \leq L_M(|x_2 - x_1| + |y_2 - y_1|). \]

**Assumption 2.** We suppose that \( a_0 : Q \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function,
monotone nondecreasing with respect to \( y \), and satisfying
\( \forall M > 0 \exists C_M : \forall |y_1|, |y_2| \leq M \) and for a.a. \((x,t) \in Q\)
\[ |a_0(x,t,y_2) - a_0(x,t,y_1)| \leq L_{0,M}|y_2 - y_1|. \]

Then we establish our first results on existence and uniqueness of a solution of (1).

**Theorem 2.1.** Let us assume that \( u, a_0(\cdot, \cdot, 0) \in L^p(0,T;W^{-1,q}(\Omega)) \) and \( y_0 \in W_{q,p}(\Omega) \cap C_0(\Omega) \) with \( p, q \in [2, +\infty) \) satisfying that \( \frac{1}{p} + \frac{n}{q} < 1 \). Then, (1) has a unique solution \( y \in C(\bar{Q}) \cap W^{1,0}(Q) \). Moreover, there exists a constant \( M_{p,q} \) depending on \( \|y_0\|_{C_0(\Omega)}, \|u\|_{L^p(0,T;W^{-1,q}(\Omega))} \), \( p \) and \( q \) such that
\[ \|y\|_{C(\bar{Q})} + \|y\|_{W^{1,0}(Q)} \leq M_{p,q}(\|u\|_{L^p(0,T;W^{-1,q}(\Omega))} + \|y_0\|_{W_{q,p}(\Omega)} + \|y_0\|_{C_0(\Omega)}). \]

Before proving this theorem we establish the following lemma.

**Lemma 2.2.** Let us assume that \( f \in L^p(0,T;W^{-1,q}(\Omega)) \) and \( y_0 \in W_{q,p}(\Omega) \) with \( p, q \in [2, +\infty) \). Given a function \( b \in C(\bar{Q}) \) with \( b(x,t) \geq \Lambda > 0 \) \( \forall (x,t) \in Q \), we consider the problem
\[ \left\{ \begin{array}{l}
\frac{\partial y}{\partial t} - \text{div}(b(x,t)\nabla y) = f \quad \text{in } Q, \\
y(x,t) = 0 \quad \text{on } \Sigma, \quad y(x,0) = y_0(x) \quad \text{in } \Omega.
\end{array} \right. \]

This problem has a unique solution \( y \in W^{1,0}(Q) \). Moreover, there exists a constant \( C_{p,q} \) depending on \( p \) and \( q \) such that
\[ \|y\|_{W^{1,0}(Q)} \leq C_{p,q}(\|f\|_{L^p(0,T;W^{-1,q}(\Omega))} + \|y_0\|_{W_{q,p}(\Omega)}). \]

**Proof.** This lemma follows from [43, Theorem 2.5] with \( X = W_0^{1,q}(\Omega), \) \( X_1 = W^{-1,q}(\Omega), \) \( A(t)v = \text{div}(b(x,t)\nabla v), \) \( f \in L^p(0,T;X_1) \) and \( y_0 \in (X,X_1)_{1-\frac{1}{p},p} = W_{q,p}. \) To this end we have to check the assumptions (H1) and (H2) of this theorem. To check (H1), first we observe that the continuity of \( b \) implies that \( A \in C([0,T],\mathcal{L}(X,X_1)). \) Moreover, fixed \( t \in [0,T], \) the operator \( -A(t) \) is an isomorphism between \( W^{1,q}(\Omega) \) and \( W^{-1,q}(\Omega) \) for every \( 1 < q < +\infty. \) This is a consequence of the continuity and strict positivity of \( b, \) and the \( C^{1,1} \) regularity of \( \Gamma; \)
see [32, Chap. 4, p. 73] or [42, pp. 156-157]. Therefore, $A(t)$ generates an analytic semigroup on $X$ for every $t \in [0, T]$; see [34, Theorem 5.4 and Remark 5.2-i)]. This proves (H1). Hypothesis (H2) is a consequence of the maximum parabolic regularity of the operator $-A(t)$ in $W^{-1,q}(\Omega)$ for every $t \in [0, T]$; see [34, Theorem 5.4] again.

\[ \text{Proof of Theorem 2.1.} \] First we define an approximating problem. For $M > 0$ fixed we define the functions

\[ a_M(x, t, y) = a(x, t, \text{Proj}_{[-M, +M]}(y)), \]
\[ a_0M(x, t, y) = a_0(x, t, \text{Proj}_{[-M, +M]}(y)), \]

where $\text{Proj}_{[-M, +M]}(s)$ is the projection of the real number $s$ on the interval $[-M, +M]$. Now we consider the approximate problem

\begin{align*}
\begin{cases}
\frac{\partial y}{\partial t} - \text{div}_x [a_M(x, t, y)\nabla_x y] + a_0M(x, t, y) = f & \text{in } Q, \\
y(x, t) = 0 & \text{on } \Sigma, \\
y(x, 0) = y_0(x) & \text{in } \Omega.
\end{cases}
\end{align*}

(11)

To prove the existence of a solution for this problem we use the Schauder’s fixed point theorem as follows. Let $F : L^2(Q) \rightarrow L^2(Q)$ be the functional associating with each element $w \in L^2(Q)$ the unique solution $F(w) = y_w \in W(0, T)$ of the linear problem

\begin{align*}
\begin{cases}
\frac{\partial y}{\partial t} - \text{div}_x [a_M(x, t, w(x, t))\nabla_x y] + a_0M(x, t, w(x, t)) = u & \text{in } Q, \\
y(x, t) = 0 & \text{on } \Sigma, \\
y(x, 0) = y_0(x) & \text{in } \Omega.
\end{cases}
\end{align*}

(12)

Recall that $a(\cdot, \cdot, 0), u \in L^p(0, T; W^{-1,q}(\Omega)) \subset L^2(0, T; H^{-1}(\Omega))$. Moreover, using (5)–(7), we deduce the existence of $L_{0, M} > 0$ and $L_M > 0$ such that for $(x, t) \in Q$

\[ |a_0M(x, t, w(x, t))| \leq |a_0(x, t, 0)| + L_{0, M} \in L^p(0, T; W^{-1,q}(\Omega)), \]
\[ a_M(x, t, w(x, t)) \geq L, \quad \text{and} \quad |a_M(x, t, w(x, t))| \leq |a(x, t, 0)| + L_M M \leq C \]

Furthermore, since $q, p \geq 2$ we have $y_0 \in W_{q,p}(\Omega) \subset L^2(\Omega)$. Indeed, from the continuous embeddings $W_0^{-1,q}(\Omega) \subset H_0^1(\Omega)$ and $W^{-1,q}(\Omega) \subset H^{-1}(\Omega)$ for $q \geq 2$ and the properties of the real interpolation of Banach spaces [45, §1.3.3]

\[ W_{q,p}(\Omega) = (W^{-1,q}(\Omega), W^{-1,q}(\Omega))_{1-p \cdot p} = (W_0^{-1,q}(\Omega), W^{-1,q}(\Omega))_{\frac{1}{2}, p} \subset (W_0^{-1,q}(\Omega), W^{-1,q}(\Omega))_{\frac{1}{2}, 2} \subset (H_0^1(\Omega), H^{-1}(\Omega))_{\frac{1}{2}, 2} = L^2(\Omega). \]

Hence, (12) has a unique solution $y_w$ in $W(0, T)$, and the following inequality holds

\[ \|y_w\|_{L^\infty(0, T; L^2(\Omega))} + \|\nabla x y_w\|_{L^2(Q)} \leq C_1 \left( \|u + a_0(\cdot, \cdot, 0) + L_{0, M} M\|_{L^2(0, T; H^{-1}(\Omega))} + \|y_0\|_{L^2(\Omega)} \right), \]

where the constant $C_1$ depends only on $\Omega$ and $L$; see, for instance, [38, Chap. 7]. Since the embedding $W(0, T) \subset L^2(Q)$ is compact, we apply the Schauder’s fixed point theorem to deduce the existence of at least one fixed point $y_M$ of the functional $F$. Obviously this fixed point is a solution of (11).

Now, arguing as in [38, Chap. 3, §7] and using (5), the monotonicity of $a_0$, and the fact that $u - a_0(\cdot, \cdot, 0) \in L^p(0, T; W^{-1,q}(\Omega))$ with \( \frac{1}{p} + \frac{n}{q} < 1 \), we deduce the existence of a constant $C > 0$, depending on $\Lambda, \Omega, \|y_0\|_{L^\infty(Q)}$, and $\|u - a_0(\cdot, \cdot, 0)\|_{L^p(0, T; L^q(\Omega))}$,
but independent of $M$, such that $\|y_M\|_{L^\infty(Q)} \leq C$ for every $M > 0$. Hence we have $a_M(x, t, y_M(x, t)) = a(x, t, y_M(x, t))$ and $a_{0M}(x, t, y_M(x, t)) = a_0(x, t, y_M(x, t))$ for every $M \geq C$. Therefore, $y_M$ is a solution of (1) $\forall M \geq C$.

So far we have proved the existence of a solution $y$ of (1) with $y \in L^\infty(Q) \cap W(0, T)$. Now, the continuity $y \in C(\bar{Q})$ follows from [38, §3.10] or [28]. Finally the regularity of $y$ and the estimate (8) is an immediate consequence of Lemma 2.2. Indeed, it is enough to take $b(x, t) = a(x, t, y(x, t))$ and replace $f(x, t)$ by $u(x, t) - a_0(x, t, y(x, t))$. Notice that (7) implies that

$$\|a_0(x, t, y(x, t))\| \leq |a_0(x, t, 0)| + L_{0,M} |y(x, t)| \leq |a_0(x, t, 0)| + L_{0,M} \|y\|_{C(\bar{Q})}$$

for $M = \|y\|_{C(\bar{Q})}$ and $(x, t) \in Q$. Therefore, $a_0(\cdot, \cdot, y) \in L^p(0, T; W^{-1,q}(\Omega))$ and its norm in this space is estimated by $\|a_0(\cdot, \cdot, 0)\|_{L^p(0, T; W^{-1,q}(\Omega))} + \|y\|_{C(\bar{Q})}$, which is estimated by $\|y_0\|_{C(\bar{Q})} + \|a_0(\cdot, \cdot, y)\|_{L^p(0, T; W^{-1,q}(\Omega))} + \|u\|_{L^p(0, T; W^{-1,q}(\Omega))}$.

Finally, we prove the uniqueness. Let us assume that $y_1$ and $y_2$ are solutions of (1) belonging to $C(\bar{Q}) \cap W^{0,1}_{p,q}(Q)$ and take $y = y_2 - y_1$. Subtracting the equations satisfied by $y_2$ and $y_1$ we get

$$\begin{cases}
\frac{\partial y}{\partial t} - \text{div}_x [a(x, t, y_2(x, t))\nabla x y] + a_0(x, t, y_2) - a_0(x, t, y_1) \\
y(x, t) = 0 \text{ on } \Sigma, \quad y(x, 0) = 0 \text{ in } \Omega.
\end{cases}$$

Multiplying this equation by $y$ and integrating in $\Omega \times (0, t)$ we get with $M = \max\{\|y_2\|_{\infty}, \|y_1\|_{\infty}\}$ and using the monotonicity of $a_0$ with respect to $y$

$$\begin{align*}
\frac{1}{2} \|y(t)\|_{L^2(\Omega)}^2 &+ \Lambda \int_0^t \int_\Omega |\nabla x y|^2 \, dx \, ds \\
&\leq \frac{1}{2} \|y(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega a(x, t, y_2)|\nabla x y|^2 \, dx \, ds \\
&+ \int_0^t \int_\Omega [a_0(x, t, y_2) - a_0(x, t, y_1)] y \, dx \, ds \\
&= \int_0^t \int_\Omega [a(x, t, y_2) - a(x, t, y_1)] \nabla x y_1 \nabla x y \, dx \, ds \\
&\leq \int_0^t \int_\Omega L_M |\nabla x y_1| |\nabla x y||y| \, dx \, ds \\
&\leq \frac{L_M^2}{2\Lambda} \int_0^t \int_\Omega |\nabla y_1|^2 \, dx \, ds + \frac{\Lambda}{2} \int_0^t \int_\Omega |\nabla x y|^2 \, dx \, ds.
\end{align*}$$

From here we get

$$\|y(t)\|_{L^2(\Omega)}^2 + \Lambda \int_0^t \int_\Omega |\nabla x y|^2 \, dx \, ds \leq \frac{L_M^2}{\Lambda} \int_0^t \int_\Omega |\nabla y_1|^2 \, dx \, ds \quad \forall t \in [0, T].$$

With Gronwall’s inequality we infer that $y = 0$, which proves the uniqueness. \(\square\)

**Remark 1.** Theorem 2.1 and Lemma 2.2 are still valid if we only assume Lipschitz regularity of $\Gamma$ and $2 < q < q_*$ for some $q_*>4$ if $n = 2$, $q_*>3$ if $n = 3$, and arbitrarily big $q_*>\infty$ if $n = 1$. Indeed, observe that the $C^{1,1}$ regularity of $\Gamma$ was only used in the proof of Lemma 2.2 to claim that the operator $-A(t) = -\text{div}_x [b(x, t)\nabla x v]$ defines an isomorphism between $W^{0,1}_0(\Omega)$ and $W^{-1,q}(\Omega)$ for every $t \in [0, T]$, which is still certain in the range of $q$ indicated; see [36] and [29, Lemma 6.2]. Finally, if
Ω is a convex polygonal or polyhedral domain, then \( A(t) : W_p^{1,q}(\Omega) \to W^{-1,q}(\Omega) \) is an isomorphism for every \( t \in [0,T] \) and every \( 1 < q < \infty \); [27]. Therefore, Theorem 2.1 and Lemma 2.2 hold under this assumption as well.

Some additional regularity of the solution of (1) is established.

**Theorem 2.3.** Let us assume that \( u, a_0(\cdot, \cdot, 0) \in L^{2p}(0,T; L^q(\Omega)) \) with \( 2 \leq p, q < \infty \) and \( \frac{1}{p} + \frac{n}{q} < 2 \), and \( y_0 \in C_0(\Omega) \cap B_{q,p} \cap W_{2q,2p} \), then (1) has a unique solution \( y \in C(Q) \cap W_{p,q}^{2,1}(Q) \). Moreover, there exists a constant \( M_{p,q} \) depending on \( y_0 \) and \( u \) such that

\[
\|y\|_{C(Q)} + \|y\|_{W_{p,q}^{2,1}(Q)} \\
\leq M_{p,q}(\|u\|_{L^{2p}(0,T; L^q(\Omega))} + \|y_0\|_{C_0(\Omega)} + \|y_0\|_{B_{q,p}} + \|y_0\|_{W_{q,p}}).
\] (13)

To prove the above theorem we will use the following lemmas.

**Lemma 2.4** ([43]). Let \( b \in C(Q) \) satisfy \( b(x,t) \geq \Lambda > 0 \ \forall (x,t) \in Q \). We assume that \( f \in L^p(0,T; L^q(\Omega)) \) and \( y_0 \in B_{q,p} \) with \( 1 < p, q < \infty \). Then the equation

\[
\begin{cases}
\frac{\partial y}{\partial t} - b(x,t)\Delta_x y = f & \text{in } Q, \\
y(x,t) = 0 & \text{on } \Sigma, \\
y(x,0) = y_0(x) & \text{in } \Omega
\end{cases}
\] (14)

has a unique solution \( y \in W_{p,q}^{2,1}(Q) \) and the following inequality holds

\[
\|y\|_{W_{p,q}^{2,1}(Q)} \leq M_{p,q}(\|f\|_{L^{p}(0,T; L^q(\Omega))} + \|y_0\|_{B_{q,p}}).
\] (15)

**Lemma 2.5.** Assume that \( p, q \in [2,\infty), y_0 \in B_{q,p} \cap W_{2q,2p}, f \in L^{2p}(0,T; L^q(\Omega)), \) and \( b \in C(Q) \) satisfies \( b(x,t) \geq \Lambda > 0 \ \forall (x,t) \in Q \) and \( \nabla_x b \in L^{2p}(0,T; L^{2q}(\Omega)) \). Then, there exists a unique solution \( y \in W_{p,q}^{2,1}(Q) \) of (9) and its norm is estimated by the norms of \( \nabla_x b, f \) and \( y_0 \) in their corresponding spaces.

**Proof.** Observe that \( L^q(\Omega) \subset W^{-1,2q}(\Omega) \) due to the fact that \( q \geq 2 \geq \frac{q}{p} \). Then, from the assumptions \( f \in L^{2p}(0,T; L^q(\Omega)) \subset L^{2p}(0,T; W^{-1,2q}(\Omega)) \) and \( y_0 \in W_{2q,2p} \), and Lemma 2.2 we infer that (9) has a unique solution \( y \in W_{p,q}^{2,1}(Q) \). Now, we rewrite (9) in the form

\[
\begin{cases}
\frac{\partial y}{\partial t} - b(x,t)\Delta_x y = \tilde{f} & \text{in } Q, \\
y(x,t) = 0 & \text{on } \Sigma, \\
y(x,0) = y_0(x) & \text{in } \Omega
\end{cases}
\] (16)

where \( \tilde{f} = f - \nabla_x b \nabla_x y \). We apply Lemma 2.4 to deduce the \( W_{p,q}^{2,1}(Q) \) regularity of \( y \). Since \( y_0 \in B_{q,p} \), we only need to check that \( \nabla_x b \nabla_x y \in L^{p}(0,T; L^q(\Omega)) \), but this is an immediate consequence of the fact that \( \nabla_x b \nabla_x y \in L^{2p}(0,T; L^{2q}(\Omega)) \).

**Proof of Theorem 2.3.** Since \( u \in L^{2p}(0,T; L^q(\Omega)) \subset L^{2p}(0,T; W^{-1,2q}(\Omega)) \) with \( \frac{1}{2p} + \frac{n}{2q} < 1 \), and \( y_0 \in C_0(\Omega) \cap W_{2q,2p} \), we deduce from Theorem 2.1 the existence of a unique solution \( y \in W_{2p,2q}^{1,0}(Q) \cap C(Q) \) of (1). Now, taking \( b(x,t) = a(x,t, y(x,t)) \) we have

\[
\nabla_x b = \nabla_x a + \nabla_y a \nabla_x y.
\]

From Assumption 1, \( y \in C(Q) \), and the fact that \( \nabla_x y \in L^{2p}(0,T; L^{2q}(\Omega)) \), we deduce that \( \nabla_x b \in L^{2p}(0,T; L^{2q}(\Omega)) \). Thus, we are under the assumptions of Lemma 2.5 and, hence, \( y \in W_{p,q}^{2,1}(Q) \) and the estimate (13) holds. \( \square \)
Remark 2. In the case $n = 2$, if we assume that $\Omega$ is convex, then Theorem 2.3 holds for $p = q = 2$. This can be proved using the above argument and the fact that $A(t)$ is an isomorphism between $W_0^{1,\lambda}(\Omega)$ and $W^{-1,\lambda}(\Omega)$ just because of the Lipschitz regularity of $\Gamma$; see Remark 1. The convexity is used to deduce that $\Delta : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is an isomorphism; see [33, Chapter 3]. If $n = 2$ or 3 and $\Omega$ is a convex polygonal or polyhedral domain, then the $W^{2,1}(\Omega)$ regularity is still valid for some $q > 2$. Indeed, for these domains the operator $\Delta$ is an isomorphism between $W^{2,q}(\Omega)$ and $L^q(\Omega)$ with $2 \leq q < q^*$ for some $q^* > 2$, and between $W^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ for every $1 < p < \infty$; see [33, Chapter 4] and [27].

Corollary 1. If $p > n$, $u \in L^{2p}(0,T;L^p(\Omega))$, and $y_0 \in W^{2-\frac{2}{p},p}(\Omega) \cap W_0^{1,p}(\Omega)$, then the equation (1) has a unique solution $y \in W^{2,1}_p(\Omega) \subset C(\bar{Q})$ and there exists a constant $M_p > 0$ depending on $\|y_0\|_{C_0(\Omega)}$ such that

$$
\|y\|_{W^{2,1}_p(\Omega)} \leq M_p \left( \|u\|_{L^{2p}(0,T;L^p(\Omega))} + \|y_0\|_{W^{2-\frac{2}{p},p}(\Omega)} \right). 
$$

This corollary follows from Theorem 2.3 taking $p = q$ and noting that $B_{p,p} = W^{2-\frac{2}{p},p}(\Omega) \cap W_0^{1,p}(\Omega)$, cf. (3), and $W_{2p,2p} = W_0^{1-\frac{2}{p},2p}(\Omega)$, cf. (2). Moreover the embedding $W^{2-\frac{2}{p},p}(\Omega) \cap W_0^{1,p}(\Omega) \subset W_0^{1-\frac{2}{p},2p}(\Omega)$ holds due to $p > n$; see [33, §1.4.4].

3. Analysis of the control problem. In this section, besides Assumptions 1 and 2, we make the following hypotheses.

Assumption 3. The initial state satisfies $y_0 \in W_0^{1-\frac{2}{p},p}(\bar{\Omega})$ with $\bar{p} > n + 2$. For the target in the cost functional we assume that $y_d \in L^\infty(Q)$.

Assumption 4. The function $a : Q \times \mathbb{R} \to \mathbb{R}$ is of class $C^2$ with respect to the last variable and the following properties are fulfilled

$$
\forall M > 0 \exists C_M > 0 \text{ such that }
\begin{cases}
\frac{\partial^j a}{\partial y^j}(x,t,y) \leq C_M \text{ for a.a. } (x,t) \in Q, \forall |y| \leq M, \text{ and } j = 0,1,2; \\
\forall \rho > 0 \text{ and } \forall M > 0 \exists \varepsilon_{M,\rho} > 0 \text{ such that for a.a. } (x,t) \in Q
\end{cases}
$$

$$
\begin{align}
\frac{\partial^2 a}{\partial y^2}(x,t,y_2) - \frac{\partial^2 a}{\partial y^2}(x,t,y_1) \leq \rho \forall |y_1| \leq M \text{ with } |y_2 - y_1| \leq \varepsilon_{M,\rho}.
\end{align}
$$

Assumption 5. $a_0 : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class $C^2$ with respect to the last variable satisfying for almost all $(x,t) \in Q$

$$
\begin{cases}
\forall M > 0 \exists C_M > 0 \text{ such that }
\frac{\partial^j a_0}{\partial y^j}(x,t,y) \leq C_M \forall |y| \leq M, \text{ and } j = 1,2; \\
\forall \rho > 0 \text{ and } \forall M > 0 \exists \varepsilon_{M,\rho} > 0 \text{ such that }
\end{cases}
$$

$$
\begin{align}
\frac{\partial^2 a_0}{\partial y^2}(x,t,y_2) - \frac{\partial^2 a_0}{\partial y^2}(x,t,y_1) \leq \rho \forall |y_1| \leq M \text{ with } |y_2 - y_1| \leq \varepsilon_{M,\rho}.
\end{align}
$$

We define the spaces

$$
U = L^{\bar{p}}(0,T;W^{-1,\frac{2}{p}}(\Omega)) \text{ and } Y = C(\bar{Q}) \cap W_0^{1,0}(\bar{Q}).
$$
The set of admissible for the control problem (P) is denoted by
\[ U_{ad} = \{ u \in L^\infty(Q) : \alpha \leq u(x,t) \leq \beta \text{ for a.a. } x \in Q \}. \]

It is immediate that the space of controls \( L^\infty(Q) \) is continuously embedded in \( U \). The following theorem establishes the existence of a unique solution for every \( u \in U \).

**Theorem 3.1.** For every \( u \in U \) the equation (1) has a unique solution \( y_u \in Y \). In addition, if \( u \in L^\infty(Q) \), \( a_0(\cdot,\cdot,0) \in L^{2\bar p}(0,T;L^\bar p(\Omega)) \), and \( y_0 \in W^{2-\frac{2}{\bar p}}(\Omega) \cap W_0^{1,\bar p}(\Omega) \), then \( y_u \in W^{2,1}_p(Q) \). Moreover, there exist constants \( C_p \) depending on \( \|y_0\|_{C_0(\Omega)} \) and \( \|u\|_{L^p(0,T;W^{-1,\bar p}(\Omega))} \), and \( M_{\alpha,\beta} \) such that
\[
\|y_u\|_Y \leq C_p \left( \|u\|_{L^p(0,T;W^{-1,\bar p}(\Omega))} + \|y_0\|_{W_0^{2,\bar p}(\Omega)} \right) \quad \forall u \in U, \tag{24}
\]
\[
\|y_u\|_{W^{2,1}_p(Q)} \leq M_{\alpha,\beta} \quad \forall u \in U_{ad}. \tag{25}
\]

**Proof.** We apply Theorem 2.1 with \( p = q = \bar p \), the fact that \( \bar p > n + 2 \), and Assumption \( \gamma \) we infer that \( y_0 \in W^{1,\bar p}(\Omega) \subset C_0(\Omega) \). Therefore, we obtain the existence and uniqueness of \( y_u \in Y \), as well as the estimate \( (24) \). The \( W^{2,1}_p(Q) \) regularity is an immediate consequence of Corollary 1 and the fact \( \square \)

Assuming more regularity of \( a_0(\cdot,\cdot,0) \), \( y_0 \) and \( \Gamma \), we can apply Theorem 2.3 to deduce additional regularity for \( y_u \).

Let us consider the mapping \( G : U \rightarrow Y \) given by \( G(u) = y_u \) solution of (1). Then, we have the following differentiability properties.

**Theorem 3.2.** The mapping \( G \) is of class \( C^2 \). Moreover, given \( u, v, v_1, v_2 \in U \), the derivatives \( z_v = G'(u)v \) and \( z_{v_1v_2} = G''(u)(v_1,v_2) \) are solutions in \( Y \) of the equations
\[
\begin{aligned}
\frac{\partial z}{\partial t} - \text{div}_x \left[ a(x,t,y_u)\nabla_x z + \frac{\partial a_0}{\partial y}(x,t,y_u)\nabla_x y_u z \right] + \frac{\partial a_0}{\partial y}(x,t,y_u)z = v & \quad \text{in } Q, \\
z(x,t) = 0 & \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega,
\end{aligned} \tag{26}
\]
and
\[
\begin{aligned}
\frac{\partial z}{\partial t} - \text{div}_x \left[ a(x,t,y_u)\nabla_x z + \frac{\partial a_0}{\partial y}(x,t,y_u)\nabla_x y_u z \right] + \frac{\partial a_0}{\partial y}(x,t,y_u)z &= \text{div}_x \left[ \frac{\partial a_0}{\partial y}(x,t,y_u)(z_{v_1}\nabla_x z_{v_2} + z_{v_2}\nabla_x z_{v_1}) + \frac{\partial^2 a_0}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2}\nabla_x y_u \right] \\
- \frac{\partial^2 a_0}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2} &= \quad \text{in } Q, \\
z(x,t) = 0 & \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega.
\end{aligned} \tag{27}
\]
respectively, where \( z_{v_i} = G'(u)v_i, \ i = 1, 2 \).

**Proof.** We apply the implicit function theorem. To this end we define the mapping
\[
F : Y \times U \rightarrow U \times W^{1,\frac{2}{\bar p}}(\Omega) \\
F(y,u) = \left( \frac{\partial y}{\partial t} - \text{div}_x \left[ a(x,t,y)\nabla_x y \right] + a_0(x,t,y) - u, y(0)-y_0 \right).
\]
Looking at the definition of $Y$ and using the assumptions along with (4) it is easy to check that $F$ is well defined. Moreover, $F$ is obviously of class $C^2$ and

$$\frac{\partial F}{\partial y}(y,u)z = \left( \frac{\partial}{\partial t} - \operatorname{div}_x \left[ a(x,t,y) \nabla_x z + \frac{\partial a}{\partial y}(x,t,y)z \nabla_x y \right] + \frac{\partial a_0}{\partial y}(x,t,y)z, z(0) \right)$$

$$= \left( - \operatorname{div}_x \left[ \frac{\partial a}{\partial y}(x,t,y)(z_1 \nabla_x z_2 + z_2 \nabla_x z_1) + \frac{\partial^2 a_0}{\partial y^2}(x,t,y)z_1z_2 \nabla_x y \right] + \frac{\partial^2 a_0}{\partial y^2}(x,t,y)z_1z_2, 0 \right).$$

Since we have that $F(y_u, u) = (0, 0)$, the theorem follows from the implicit function theorem if we prove that $\frac{\partial F}{\partial y}(y_u, u) : Y \rightarrow U \times W_0^{1-\frac{2}{p}}(\Omega)$ is an isomorphism. This is equivalent to the existence, uniqueness and continuous dependence of the solution $z \in Y$ of

$$\left\{ \begin{array}{l}
\frac{\partial z}{\partial t} - \operatorname{div}_x \left[ a(x,t,y) \nabla_x z + \frac{\partial a}{\partial y}(x,t,y) \nabla_x y u z \right] + \frac{\partial a_0}{\partial y}(x,t,y)z = v \text{ in } Q,

z(x,t) = 0 \text{ on } \Sigma, \quad z(x,0) = z_0(x) \text{ in } \Omega,
\end{array} \right.$$  

with respect to $(v, z_0) \in U \times W_0^{1-\frac{2}{p}}(\Omega)$. Since $\bar{p} > n + 2$, then $W_0^{1-\frac{2}{\bar{p}}}(\Omega) \subset C_0(\Omega)$ and the assumptions (1.1)-(1.6) and (7.1)-(7.2) of [38, Chapter 3] hold. Hence, the existence and uniqueness of a solution $z \in L^2(0,T; H_0^1(\Omega)) \cap C(\bar{Q})$ follows. Finally, the regularity in $W_0^{1,0}(Q)$ follows from Lemma 2.2. We only need to move the term $-\operatorname{div}_x \left[ \frac{\partial a}{\partial y}(x,t,y) \nabla_x y u z \right]$ to the right hand side of the equation. To treat this term we observe that $\frac{\partial a_0}{\partial y}(x,t,y) \nabla_x y u z \in L^p(Q)$ and hence $\operatorname{div}_x \left[ \frac{\partial a}{\partial y}(x,t,y) \nabla_x y u z \right] \in L^p(0,T; W^{-1,\bar{p}}(\Omega)).$

In the next theorem, we consider the extension of the forms $G'(u)$ and $G''(u)$ to $L^2(Q)$ and $L^2(\Omega \times Q)$, respectively.

**Theorem 3.3.** Suppose that $a_0(\cdot, \cdot, 0) \in \operatorname{Lip}(0,T; L^p(\Omega))$ and $y_0 \in W_0^{2-\frac{2}{\bar{p}}}(\bar{Q}) \cap W_0^{1,0}(\bar{Q})$. Then, for every $u \in L^\infty(\bar{Q})$ and $v \in L^2(\bar{Q})$, (26) has a unique solution $z_0 \in W(0,T)$, and the linear mapping $G'(u) : L^2(\bar{Q}) \rightarrow W(0,T)$ is continuous.

**Proof.** From Theorem 3.1 we obtain that $y_u \in W_0^{2,1}(\bar{Q})$. From the interpolation inequalities of Gagliardo-Nirenberg, see [9, page 313], we get

$$\|\nabla_x y_u\|_{L^p(\Omega)} \leq C \|y_u\|_{W_0^{2,1}(\bar{Q})}^{1/2} \|y_u\|_{L^\infty(\Omega)}^{1/2}.$$  

From here we infer

$$\|\nabla_x y_u\|_{L^p(\Omega)} \leq C \|y_u\|_{W_0^{2,1}(\bar{Q})}^{1/2} \|y_u\|_{L^\infty(\Omega)}^{1/2}.$$  

Since $\bar{p} > n + 2 \geq 3$, we deduce that $W_0^{2,1}(\bar{Q}) \subset C(\bar{Q})$ and, consequently, the above inequality implies that $\nabla_x y_u \in L^3(\bar{Q})$. Therefore, using again [38, Chap. 3] we deduce that (26) as a unique solution $z_0 \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)).$ Finally, the regularity $z_0$ in $W(0,T)$ follows from (26). Indeed, the only delicate point is to prove that $\operatorname{div}_x \left[ \frac{\partial a}{\partial y}(x,t,y) \nabla_x y u z \right] \in L^2(0,T; H^{-1}(\Omega)).$ From (18)}
Now we have the following identities hold Theorem 3.4.

Using an interpolation inequality in the $L^p$ spaces, see [9, page 93], and a Sobolev embedding we obtain

\[
\|z\|_{L^\gamma(\Omega)} \leq \|z\|^\frac{1}{2}_{L^2(\Omega)} \|z\|^\frac{1}{2}_{L^\gamma(\Omega)} \leq C \|z\|^\frac{1}{2}_{L^2(\Omega)} \|z\|^\frac{1}{2}_{H^\delta(\Omega)}.
\]

Now we have

\[
\|z\|_{L^\gamma(\Omega)} \leq \|z\|^\frac{1}{2}_{L^0(0,T;L^2(\Omega))} \left( \int_0^T \|z\|^\frac{3}{2}_{H^\delta(\Omega)} \right)^{1/3} 
\leq C \|z\|^\frac{1}{2}_{L^\gamma(0,T;L^2(\Omega))} \|z\|^\frac{1}{2}_{L^0(0,T;H^\delta(\Omega))} \leq C \|z\|_{W(0,T)} \leq C \|v\|_{L^2(Q)}.
\]

Hence, we conclude the proof with (28).

As a consequence of theorems 3.2 and 3.3 we get the differentiability of the cost functional $J$.

**Theorem 3.4.** The functional $J : L^\infty(Q) \rightarrow \mathbb{R}$ is of class $C^2$ and $\forall u, v, v_1, v_2 \in U$ the following identities hold

\[
J'(u)v = \int_Q \left( \varphi_u + \nu v \right) v dx dt,
\]

\[
J''(u)(v_1, v_2) = \int_Q \left( \left[ 1 - \varphi_u \frac{\partial a_0}{\partial y^2}(x, t, y_u) \right] z_{v_1} z_{v_2} + \nu v_1 v_2 \right) dx dt
\]

\[
- \int_Q \nabla \varphi_u \left[ \frac{\partial a}{\partial y}(x, t, y_u)(z_{v_1} \nabla z_{v_2} + \nabla z_{v_1} z_{v_2}) + \frac{\partial^2 a}{\partial y^2}(x, t, y_u) z_{v_1} z_{v_2} \nabla y_u \right] dx dt \tag{30}
\]

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$, and $\varphi_u \in W^{2,1}_p(Q)$ for every $2 \leq p < \infty$ is the solution of

\[
\begin{cases}
- \frac{\partial \varphi}{\partial t} - \Delta_y x \left[ a(x, t, y_u) \nabla_x \varphi \right] + \frac{\partial a}{\partial y}(x, t, y_u) \nabla_x y_u \nabla_x \varphi + \frac{\partial a_0}{\partial y}(x, t, y_u) \varphi \\
= y_u - y_d \quad \text{in } Q, \\
\varphi(x, t) = 0 \quad \text{on } \Sigma, \quad \varphi(x, T) = 0 \quad \text{in } \Omega.
\end{cases} \tag{31}
\]

Moreover, if $a_0(\cdot, \cdot, 0) \in L^2(0, T; L^p(\Omega))$ and $y_0 \in W^{2-\frac{2}{p}}(\Omega) \cap W^{1,\frac{p}{2}}(\Omega)$, then for every $u \in L^\infty(Q)$ there exist continuous extensions $J' : L^2(Q) \rightarrow \mathbb{R}$ and $J'' : L^2(Q) \times L^2(Q) \rightarrow \mathbb{R}$ given by the expressions (29) and (30), respectively.

**Proof.** The differentiability of $J$ follows from Theorem 3.2 and the chain rule. The only thing that we have to prove is the existence and uniqueness of the solution of (43). From our assumptions, the regularity of $y_u \in Y$ established in Theorem 3.1 and the fact that $y_d \in L^\infty(Q)$, the existence and uniqueness of a solution $\varphi_u \in L^2(0, T; H_0^1(\Omega)) \cap C(\overline{Q})$ follows from [38, Chap. 3].

Now we observe that the equation (43) can be written in the form

\[
\begin{cases}
- \frac{\partial \varphi}{\partial t} - a(x, t, y_u) \Delta_x \varphi = f \quad \text{in } Q, \\
\varphi(x, t) = 0 \quad \text{on } \Sigma, \quad \varphi(x, T) = 0 \quad \text{in } \Omega.
\end{cases} \tag{32}
\]
using the generalized H"older’s inequality, and arguing as at the end of the proof of 
that (30) is a bilinear continuous form in 
over, from (18) and (19) we have that 
use the interpolation inequality of Gagliardo-Nirenberg [9, p. 313]
\[
\|\nabla_x z\|_{L^{2r}(\Omega)} \leq C \|z\|_{L_{\infty}(\Omega)}^{1/2} \|z\|_{W^{2, r}(\Omega)}^{1/2} \quad \forall z \in L^\infty(\Omega) \cap W^{2, r}(\Omega).
\]
Then, \(\forall z \in L^\infty(\Omega) \cap L^r(0, T; W^{2, r}(\Omega))\) we have
\[
\|\nabla_x z\|_{L^{2r}(\Omega)} \leq C \|z\|_{L^\infty(\Omega)}^{1/2} \|z\|_{L^r(0, T; W^{2, r}(\Omega))}^{1/2}
\]
(33)
Taking \(r = 2\) in (33) and using that \(\varphi_u \in L^\infty(\Omega) \cap L^2(0, T; H^2(\Omega))\), we deduce that \(\nabla_x \varphi_u \in L^4(\Omega)\), therefore \(f \in L^4(\Omega)\) and the regularity \(\varphi_u \in L^4(0, T; W^{2, 4}(\Omega))\) follows from [43] again. Again, we use (33) with \(r = 4\) to deduce that \(\nabla_x \varphi_u \in L^8(\Omega)\) and, hence, \(f \in L^8(\Omega)\) and \(\varphi_u \in W^{2, 1}(\Omega)\). Repeating this argument we conclude that \(\varphi_u \in W^{2, 1}_p(\Omega)\) \(\forall p \in [2, \infty)\).

It is obvious that (29) defines a linear and continuous form in \(L^2(\Omega)\). Let us prove that (30) is a bilinear continuous form in \(L^2(\Omega)\). From Theorem 3.3 we know that \(z_{v_1}, z_{v_2} \in W(0, T)\) and the mapping \(v \rightarrow z_v\) is continuous from \(L^2(\Omega)\) to \(W(0, T)\). The \(W^{2, 1}_p(\Omega)\) regularity of \(\varphi_u\) for \(p\) arbitrarily big implies that \(\varphi_u \in L^\infty(\Omega)\). Moreover, from (18) and (19) we have that \(\frac{\partial a(x, t, y_a)}{\partial y} \in L^2(\Omega)\) and \(\frac{\partial^2 a(x, t, y_a)}{\partial y^2} \in L^2(\Omega)\). Therefore, to conclude the proof it is enough to estimate the terms \(\nabla_x \varphi_u \nabla_x z_{v_2}\) and \(\nabla_x \varphi_u \nabla_y z_{v_1} z_{v_2}\) in \(L^1(\Omega)\) in terms of \(\|v_1\|_{L^2(\Omega)}\|v_2\|_{L^2(\Omega)}\). This is obtained by using the generalized H"older’s inequality, and arguing as at the end of the proof of Theorem 3.3 to deduce the estimates in \(L^3(\Omega)\), as follows
\[
\|\nabla_x \varphi_u \nabla_x z_{v_1} z_{v_2}\|_{L^1(\Omega)} \leq \|\nabla_x \varphi_u\|_{L^6(\Omega)} \|\nabla_x z_{v_1}\|_{L^2(\Omega)} \|z_{v_2}\|_{L^3(\Omega)} \\
\leq C \|v_1\|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)},
\]
(34)
\[
\|\nabla_x \varphi_u \nabla_y z_{v_1} z_{v_2}\|_{L^1(\Omega)} \leq \|\nabla_x \varphi_u\|_{L^6(\Omega)} \|\nabla_y z_{v_1}\|_{L^2(\Omega)} \|z_{v_2}\|_{L^3(\Omega)} \|z_{v_2}\|_{L^3(\Omega)} \\
\leq C \|v_1\|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)}.
\]
(35)

\[\square\]

Remark 3. The \(C^{1,1}\) regularity can be relaxed for Theorems 3.1-3.4. Indeed, let us assume that \(\Omega\) is a convex polygonal or polyhedral domain. Of course, the comments of Remarks 1 and 2 apply to deduce the regularity result \(y_u \in W^{1, \beta}(\Omega)\) established in Theorem 3.1. From Remark 1 we also deduce that Theorem 3.2 is still valid. Theorem 3.3 is valid as well. We can argue in the same way, we only have to change the estimate (28). Thus, in dimension \(n = 3\), recalling that \(\bar{p} > n + 2 = 5\), we have
\[
\|\nabla_x y_u z\|_{L^2(\Omega)} \leq \|\nabla_x y_u\|_{L^3(\Omega)} \|z\|_{L^{10/3}(\Omega)}.
\]
Now using again an interpolation inequality in the \(L^p\) spaces, [9, page p3], we get
\[
\|z\|_{L^{10/3}(\Omega)} \leq \|z\|_{L^2(\Omega)}^{2/5} \|z\|_{L^6(\Omega)}^{3/5} \leq C \|z\|_{L^2(\Omega)}^{2/5} \|z\|_{H^{1/3}_3(\Omega)}^{3/5}.
\]
Combining the above inequalities we deduce
\[
\|\nabla_x y_u z\|_{L^2(\Omega)} \leq C \|z\|_{L^2(\Omega)}^{2/5} \left( \int_0^T \|\nabla_x y_u\|_{L^6(\Omega)} \|z\|_{H^{1/3}_3(\Omega)}^{6/5} dt \right)^{1/2} \\
\leq C \|z\|_{L^2(\Omega)} \|\nabla_x y_u\|_{L^5(\Omega)} \|z\|_{L^2(\Omega)} \|H^{1/3}_3(\Omega) \|z\|_{L^5(\Omega)} \|v\|_{L^2(\Omega)}.
\]
For \( n = 2 \), we use that \( \bar{p} > n + 2 = 4 \) and we argue as follows

\[
\| \nabla xyu \|_{L^2(\Omega)} \leq \| \nabla xyu \|_{L^4(\Omega)} \| z \|_{L^4(\Omega)}.
\]

Using the Galiardo-Nirenberg inequality

\[
\| z \|_{L^4(\Omega)} \leq C \| z \|_{L^4(\Omega)}^{1/2} \| z \|_{H^1_0(\Omega)}^{1/2},
\]

we infer

\[
\| \nabla xyu \|_{L^2(\Omega)} \leq C \| z \|_{L^4(\Omega)}^{1/2} \left( \int_0^T \| \nabla xyu \|_{L^4(\Omega)} \| z \|_{H^1_0(\Omega)} \, dt \right)^{1/2}
\]
\[
\leq C \| z \|_{L^4(\Omega)} \| \nabla xyu \|_{L^4(\Omega)} \| z \|_{H^1_0(\Omega)} \leq C \| \nabla xyu \|_{L^4(\Omega)} \| v \|_{L^2(\Omega)}.
\]

Finally, Theorem 3.4 remains partially valid. Since \( \bar{p} > n + 2 \), we can apply again [38, Chap. 3] to deduce the existence of a unique solution \( \varphi_u \in L^2(0, T; H^1_0(\Omega)) \cap C(\bar{Q}) \) of (43). The \( H^{2,1}(\Omega) \) regularity of \( \varphi_u \) follows from [43] and [33, Chapter 3] applied to equation (32). Additionally, there exists \( p^* > 2 \) such that the \( W^{2,1}_p(\Omega) \) regularity of \( \varphi_u \) is valid for \( 2 \leq p < p^* \), cf. [43] and [27]. Moreover, using (33) with \( r = 2 \), we deduce that \( \nabla \varphi_u \in L^4(\Omega) \). It is obvious that (29) defines a linear and continuous form \( J'(u) : L^2(\Omega) \rightarrow \mathbb{R} \). Additionally, if \( n = 2 \) then (30) defines a bilinear and continuous form in \( L^2(\Omega) \) if \( n = 2 \). To check this it is enough to replace the inequalities (34) and (35) by

\[
\| \nabla \varphi_u \|_{L^2(\Omega)} \| \nabla z v_1, z v_2 \|_{L^4(\Omega)} \leq C \| v_1 \|_{L^2(\Omega)} \| v_2 \|_{L^2(\Omega)},
\]

\[
\| \nabla \varphi_u \|_{L^2(\Omega)} \| \nabla x z yu \|_{L^4(\Omega)} \leq C \| v_1 \|_{L^2(\Omega)} \| v_2 \|_{L^2(\Omega)},
\]

In the three-dimensional case, we can prove the extension of the bilinear form \( J''(u) \) to \( L^2(\Omega) \) under an extra regularity of \( y_u \). Instead of taking \( \bar{p} > n + 2 \) in the Assumptions 3-5, we assume that \( \bar{p} \geq 12 \). Then, from Theorem 3.1 we infer that \( y_u \in W^{1,0}_p(\Omega) \), hence, \( \nabla xyu \in L^{1,2}(\Omega) \). Now, given \( v \in L^2(\Omega) \), we prove that the solution \( z v \) of (26) belongs to \( L^2(0, T; W^{1,0}_p(\Omega)) \). Indeed, from Hölder’s inequality

\[
\| \nabla xyu \|_{L^2(\Omega)} \leq \| \nabla xyu \|_{L^2(\Omega)} \| z \|_{L^4(\Omega)},
\]

and Galiardo’s inequality

\[
\| z \|_{L^4(\Omega)} \leq C \| z \|_{L^2(\Omega)}^{1/4} \| z \|_{H^1_0(\Omega)}^{3/4},
\]

we infer

\[
\| \nabla xyu \|_{L^2(\Omega)} \| z \|_{L^4(\Omega)} \leq C \| v \|_{L^2(\Omega)}.
\]

This implies that

\[
\| \text{div}_x \left[ \frac{\partial a}{\partial y} (x, t, y_u) \nabla xyu \right] \|_{L^2(0, T; W^{-1,3}_0(\Omega))} \leq C \| v \|_{L^2(\Omega)}.
\]

Furthermore, we have that \( v \in L^2(\Omega) \) implies \( v \in L^2(0, T; W^{1,0}_p(\Omega)) \). Then, we apply Lemma 2.2 with \( p = 2, q = 3, b(x, t) = a(x, t, y_u(x, t)) \), and

\[
f = \text{div}_x \left[ \frac{\partial a}{\partial y} (x, t, y_u) \nabla xyu \right] - \frac{\partial a_0}{\partial y} (x, t, y_u) z + v \in L^2(0, T; W^{-1,3}_0(\Omega)),
\]
to conclude that \( z_\varepsilon \in L^2(0, T; W^{1,3}_0(\Omega)) \). Now, using again a Hölder’s inequality
\[
\| \nabla \varphi u \nabla x z_{\varepsilon} \|_{L^3(\Omega)} \leq \| \nabla \varphi u \|_{L^4(\Omega)} \| \nabla x z_{\varepsilon} \|_{L^3(\Omega)} \| z_{\varepsilon} \|_{L^{12/5}(\Omega)},
\]
an interpolation inequality
\[
\| z_{\varepsilon} \|_{L^{12/5}(\Omega)} \leq C \| z_{\varepsilon} \|_{L^4(\Omega)}^{3/4} \| z_{\varepsilon} \|_{H^1(u)}^{1/4},
\]
we obtain
\[
\| \nabla \varphi u \nabla x z_{\varepsilon} \|_{L^1(Q)} \leq C \| \nabla \varphi u \|_{L^4(Q)} \| \nabla x z_{\varepsilon} \|_{L^2(\Omega)} \| z_{\varepsilon} \|_{L^{12/5}(\Omega)} \| z_{\varepsilon} \|_{L^4(\Omega)}^{3/4} \| z_{\varepsilon} \|_{L^{12/5}(\Omega)}^{1/4} \leq C \| u \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)}.
\]
Finally, Hölder’s inequality implies
\[
\| \nabla \varphi u \nabla x y_{\varepsilon} z_{\varepsilon} \|_{L^1(Q)} \leq \| \nabla \varphi u \|_{L^1(Q)} \| \nabla x y_{\varepsilon} \|_{L^2(\Omega)} \| z_{\varepsilon} \|_{L^1(\Omega)} \| z_{\varepsilon} \|_{L^1(\Omega)} \leq C \| u \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)}.
\]
Thus, we conclude the continuity of \( J''(u) \) on \( L^2(Q) \times L^2(Q) \).

We finish the paper by establishing the first and second order optimality conditions. In what follows, we say that \( \bar{u} \in U_{ad} \) is a local solution of (P) if there exists a ball \( B_\varepsilon(\bar{u}) \subset L^2(Q) \) such that
\[
J(\bar{u}) \leq J(u) \quad \forall u \in U_{ad} \cap B_\varepsilon(\bar{u}).
\]
As usual, we say that \( \bar{u} \in U_{ad} \) is a solution of (P) if the above inequality holds for every \( u \in U_{ad} \).

**Theorem 3.5.** (P) has at least one solution. Moreover, for any local solution \( \bar{u} \) of (P) there exist \( \bar{y} \in W^{2,1}_p(Q) \) and \( \bar{\varphi} \in W^{2,1}_p(Q) \) \( \forall p < \infty \) such that the following optimality system holds
\[
\begin{cases}
\frac{\partial \bar{y}}{\partial t} - \text{div}_x \left[ a(x, t, \bar{y}(x, t)) \nabla_x \bar{y} \right] + a_0(x, t, \bar{y}(x, t)) = \bar{u} & \text{in } Q, \\
\bar{y}(x, t) = 0 & \text{on } \Sigma, \\
\frac{\partial \bar{\varphi}}{\partial t} - \text{div}_x \left[ a(x, t, \bar{y}(x, t)) \nabla_x \bar{\varphi} \right] + \frac{\partial a_0}{\partial y} (x, t, \bar{y}) \nabla_x \bar{y} \nabla_x \bar{\varphi} + \frac{\partial a_0}{\partial y_1} (x, t, \bar{y}) \bar{\varphi} = \bar{y} - y_d & \text{in } Q, \\
\bar{\varphi}(x, t) = 0 & \text{on } \Sigma, \\
\int_Q (\bar{u} + \nu \bar{u})(u - \bar{u}) \, dx \, dt \geq 0 & \forall u \in U_{ad}.
\end{cases}
\]
Moreover, the regularity \( \bar{u} \in W^{1,p}(Q) \) \( \forall p < \infty \) is fulfilled.

The proof of this theorem is standard. The regularity result of \( \bar{u} \) follows from the regularity of \( \bar{\varphi} \) and the projection formula
\[
\bar{u}(x, t) = \text{Proj}_{[\alpha, \beta]} \left( - \frac{1}{\nu} \bar{\varphi}(x, t) \right).
\]
For the second order analysis we introduce the cone of critical directions
\[
C_{\bar{u}} = \{ v \in L^2(Q) : J''(\bar{u})v = 0 \text{ and } (40) \text{ holds} \}
\]
\[
v(x, t) \begin{cases}
\geq 0 & \text{if } \bar{u}(x, t) = \alpha, \\
\leq 0 & \text{if } \bar{u}(x, t) = \beta.
\end{cases}
\]
Remark 4. Let us observe that given \( v \in L^2(Q) \) satisfying (40), then \( v \in C_u \) if and only if \((\bar{\varphi} + \nu \bar{u}) v = 0 \) a.e. in \( Q \). Indeed, the projection formula (39) and (40) imply that \((\bar{\varphi}(x,t) + \nu \bar{u}(x,t)) v(x,t) \geq 0 \) \( \forall (x,t) \in Q \). Therefore, if \( J'(\bar{u})v = 0 \), we infer from (29) that \((\bar{\varphi} + \nu \bar{u})v = 0 \) a.e. in \( Q \).

Theorem 3.6. If \( \bar{u} \) is local minimum of (P), then \( J''(\bar{u})v^2 \geq 0 \) \( \forall v \in C_u \). Reciprocally, if \( \bar{u} \in U_{ad} \) satisfies (24) and \( J''(\bar{u})v^2 > 0 \) \( \forall v \in C_u \setminus \{0\} \), then there exist \( \delta > 0 \) and \( \varepsilon > 0 \) such that

\[
J(\bar{u}) + \frac{\delta}{2} \| u - \bar{u} \|_{L^2(Q)}^2 \leq J(u) \quad \forall u \in U_{ad} \cap B_\varepsilon(\bar{u}).
\]

This theorem follows from the abstract results proved in [23]. Indeed, it is easy to check that the assumptions (A1) and (A2) of [23] hold for \( A = U_\infty = L_\infty(Q) \), \( K = U_{ad} \), \( U_2 = L^2(Q) \) and \( \Lambda = \nu \). To check these assumptions it is enough to recall the expression of \( J'' \) given in (30) and to take into account that the weak convergence \( v_k \rightharpoonup v \) in \( L^2(Q) \) implies the weak convergence \( z_{v_k} \rightharpoonup z_v \) in \( W^0(0,T) \), hence the strong convergence \( z_{v_k} \rightarrow z_v \) in \( L^2(Q) \).

Remark 5. It is known that the condition \( J''(\bar{u})v^2 > 0 \forall v \in C_u \setminus \{0\} \) is equivalent to the existence of \( \kappa > 0 \) such that

\[
J''(\bar{u})v^2 \geq \kappa \| v \|_{L^2(Q)}^2 \quad \forall v \in C_u; \tag{42}
\]

see, for instance, [24].

Remark 6. In the case of a convex polygonal or polyhedral domain \( \Omega \), the optimality system (36)-(38) is satisfied as well. However, we have less regularity for \((\bar{u}, \bar{y}, \bar{\varphi})\). Namely, we only have \( \bar{y} \in W^{1,0}_p(Q) \cap C(\bar{Q}) \), \( \bar{\varphi} \in H^{2,1}(Q) \cap C(\bar{Q}) \), and \( \bar{u} \in H^1(Q) \cap C(\bar{Q}) \). Concerning the second order conditions established in Theorem 3.6, we can assure its validity only in dimension \( n = 2 \) (of course in dimension \( n = 1 \) as well). The reader is referred to Remark 3 for details.

Remark 7. A more general control problem associated with the state equation (1) can be considered, namely

\[
(P) \quad \min_{u \in U_{ad}} J(u)
\]

with

\[
J(u) = \frac{1}{2} \int_Q L(x,t,y_u(x,t)) \, dx \, dt + \frac{\nu}{2} \int_0^T \int_\omega u^2(x,t) \, dx \, dt,
\]

where \( \omega \) is an open subset of \( \Omega \),

\[
U_{ad} = \{ u \in L^\infty(\omega \times (0,T)) : \alpha \leq u(x,t) \leq \beta \text{ for a.a. } x \in \omega \times (0,T) \},
\]

and \( L : Q \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function of class \( C^2 \) with respect to the last variable. Additionally we make the following assumptions:

\[
L(\cdot, \cdot, 0) \in L^\infty(Q),
\]

\[
\forall M > 0 \quad \exists C_M > 0 : \left| \frac{\partial L}{\partial y^j}(x,t,y) \right| \leq C_M \text{ for } j = 1, 2,
\]

\[
\forall M > 0 \text{ and } \forall \varepsilon > 0 \quad \exists \eta > 0 : |y_2 - y_1| \leq \eta \Rightarrow \left| \frac{\partial^2 L}{\partial y^j y^k}(x,t,y_2) - \frac{\partial^2 L}{\partial y^j y^k}(x,t,y_1) \right| \leq \varepsilon,
\]

for a.a. \( (x,t) \in Q \) and every \( y, y_i \in \mathbb{R} \) with \( |y|, |y_i| \leq M, i = 1, 2 \).

All the results established in sections 2 and 3 are valid for this control problem. Indeed, since \( L^\infty(\omega \times (0,T)) \) can be considered a subspace of \( L^\infty(Q) \) formed
by functions vanishing outside $\omega \times (0, T)$, the results proved in section 2 remain unchanged. In section 3, the spaces $L^2(Q)$, $L^\infty(Q)$ or $W^{1,p}(Q)$ referred to the controls have to be changed by $L^2(\omega \times (0, T))$, $L^2(\omega \times (0, T))$ and $W^{1,p}(\omega \times (0, T))$, respectively. The identities (29) and (30) must be replaced by

$$J'(u)v = \int_0^T \int_\omega (\varphi_u + \nu u) v \, dx \, dt,$$

and

$$J''(u)(v_1, v_2) = \int_Q \left[ 1 - \varphi_u \frac{\partial^2 a_0(x, t, y, u)}{\partial y^2} \right] z_{v_1} z_{v_2} \, dx \, dt + \nu \int_0^T \int_\omega v_1 v_2 \, dx \, dt$$

$$- \int_Q \nabla \varphi_u \left[ \frac{\partial a}{\partial y}(x, t, y, u) (z_{v_1} \nabla z_{v_2} + \nabla z_{v_1} z_{v_2}) + \frac{\partial^2 a}{\partial y^2}(x, t, y, u) z_{v_1} z_{v_2} \nabla y u \right] \, dx \, dt,$$

where $\varphi_u \in W^{2,1}_p(Q)$ for every $2 \leq p < \infty$ is the adjoint state solution of

$$\begin{array}{ll}
- \frac{\partial \varphi}{\partial t} - \text{div}_x \left[ a(x, t, y, u) \nabla \varphi \right] + \frac{\partial a}{\partial y}(x, t, y, u) \nabla y u \nabla \varphi + \frac{\partial a_0}{\partial y}(x, t, y, u) \varphi = \frac{\partial L}{\partial y}(x, t, y, u) \text{ in } Q, \\
\varphi(x, t) = 0 \text{ on } \Sigma, \quad \varphi(x, T) = 0 \text{ in } \Omega.
\end{array}$$

(43)

These changes lead to the obvious modifications in (37) and (38). Finally, Theorem 3.6 is also valid and it follows from the abstract results proved in [23].

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E-mail address: eduardo.casas@unican.es
E-mail address: Chrysafinos@math.ntua.gr