LITTLEWOOD-PALEY THEOREM, NIKOLSKII INEQUALITY, BESOV SPACES, FOURIER AND SPECTRAL MULTIPLIERS ON GRADED LIE GROUPS

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Abstract. In this paper we investigate Besov spaces on graded Lie groups. We prove a Nikolskii type inequality on graded Lie groups and as consequence we obtain embeddings of Besov spaces. We prove a version of the Littlewood-Paley theorem on graded Lie groups. The results are applied to obtain multiplier theorems for both spectral and Fourier multipliers in Besov spaces on graded Lie groups.

1. INTRODUCTION

In this paper we are interested in advancing the notions and results of harmonic analysis in the setting of graded Lie groups, building up on the fundamental book [9] of Folland and Stein, as well as on more recent developments over the decades, in particular summarised in the recent book [6] by Véronique Fischer and the second author. Indeed, as it was pointed out by Folland and Stein, the setting of homogeneous groups is ideal for the distillation of those results of harmonic analysis that depend only on the group and dilation structures of the underlying space, while the setting of graded Lie groups allows one to also use the techniques coming from the theory of partial differential operators. The difference between the classes of nilpotent, homogeneous and graded Lie groups is rather small, with the majority of nilpotent Lie groups allowing for a compatible graded structure, see [6, Chapter 3] for a detailed explanation. In particular, this setting includes the class of stratified groups ([8]) when the Rockland operator can be chosen to be the sub-Laplacian. We also mention that general Rockland operators on graded Lie groups naturally appear when one is dealing with questions concerning general partial differential operators on manifolds, as their liftings following the celebrated lifting procedure of Rothschild and Stein [17].

Summarising the research of this paper, here we obtain the following results:

• establish the Nikolskii (or the reverse Hölder) inequality in the setting of graded Lie groups in terms of its homogeneous dimension. We believe such a result to be new already on stratified groups, and even on the Heisenberg group;

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prove the Littlewood-Paley theorem on graded Lie groups for the dyadic decompositions associated to positive Rockland operators;
• investigate homogeneous and inhomogeneous Besov spaces in terms of Rockland operators and prove their embedding properties. We show that the Besov spaces in this context are also the interpolation spaces between Sobolev spaces, and prove that they are independent of a particular choice of the Rockland operator used to define them. We also prove their embedding properties with the usual (locally defined) Besov spaces on \( \mathbb{R}^n \);
• apply these results to establish multiplier theorems for spectral and Fourier multipliers in Besov spaces on graded Lie groups. More precisely, we prove a Marcinkiewicz type theorem for spectral multipliers on \( L^p \) and on Besov spaces. We also give negative results on the boundedness of invariant operators in Besov spaces. For Fourier multipliers, we show that the boundedness between \( L^p \)-spaces implies the boundedness on Besov spaces and give several applications of this result to Fourier multipliers using Hörmander-Mihlin type and other theorems in this setting.

Nikolskii-type inequalities, following the usual terminology, are, roughly speaking, inequalities between different metrics of the same function (usually trigonometric polynomials). Nikolskii [14] in 1951 proved the inequalities for \( 1 \leq p \leq q \leq \infty \):

\[
\| T_{L_1, L_2, \ldots, L_n} \|_{L^q([0, 2\pi])} \leq 2^n [(2\pi)^n L_1 L_2 \cdots L_n]^{\frac{1}{p} - \frac{1}{q}} \| T_{L_1, L_2, \ldots, L_n} \|_{L^p([0, 2\pi])},
\]

for trigonometric polynomials of the form

\[
T_{L_1, L_2, \ldots, L_n} = \sum_{k=1}^{n} \sum_{j_k=-L_k}^{L_k} c_{j_1, j_2, \ldots, j_k} e^{i(j_1 x_1 + \cdots + j_k x_k)},
\]

as well as for entire functions of exponential type. Sometimes such inequality is also called the reverse Hölder inequality in the literature.

On \( \mathbb{R}^n \), the Nikolskii inequality takes the form

\[
\| f \|_{L^q(\mathbb{R}^n)} \leq C [\mu(\text{c.h.}[\text{supp}(\hat{f})])]^{\frac{1}{p} - \frac{1}{q}} \| f \|_{L^p(\mathbb{R}^n)},
\]

for every function \( f \in L^p(\mathbb{R}^n) \) with Fourier transform \( \hat{f} \) of compact support. Here, c.h.(E) denotes the convex hull of the set \( E \). Recently, the Nikolskii inequality has been considered in the setting of Lie groups \( G \). In [16], Pesenson has obtained the Nikolskii inequality in symmetric spaces \( G/K \) of non-compact type. On the other hand, for compact homogeneous manifolds \( G/K \), in [15] the following Nikolskii inequality was obtained:

\[
\| T_L \|_{L^q(G/K)} \leq N(L)^{\frac{1}{p} - \frac{1}{q}} \| T_L \|_{L^p(G/K)},
\]

for \( 0 < p < q \leq \infty \); here, if \( 0 < p \leq 2 \), \( \rho := 1 \), and for \( 2 < p \leq \infty \), \( \rho := \lceil \frac{p}{2} \rceil + 1 \), \( N(L) \approx L^{\dim G/K} \) is the Weyl eigenvalue counting function for the elliptic pseudodifferential operator \( (I - L_{G/K})^{\frac{1}{2}} \), where \( L_{G/K} \) is the Laplacian on \( G/K \).

In this paper we prove a Nikolskii type inequality in the framework of graded Lie groups \( G \). We believe this to be new also on stratified groups, even on the Heisenberg group.
This inequality is important in mathematical analysis because it is a fundamental tool in the proof of several embeddings properties of important function spaces such as Besov spaces. The Besov spaces form scales \( B_{p,q}^r(G) \) carrying three indices \( r \in \mathbb{R}, 0 < p, q \leq \infty \), and they can be obtained by interpolation of suitable Sobolev spaces. As it was observed in \([7]\), Sobolev spaces can be defined on \( \mathbb{R}^n \), and on compact and non-compact Lie groups in various equivalent ways. In a recent work of the second author with V. Fischer, Sobolev spaces were introduced on arbitrary graded Lie groups by using positive Rockland operators (see \([7]\)). It is important to mention that Sobolev spaces on stratified Lie groups were introduced by Folland in \([8]\) by using sub-Laplacians, and it was proved (see also \([9]\)) that these spaces are different of their Euclidian counterpart defined by the Fourier transform or by using the local properties of the Laplace operators. The Folland’s Sobolev spaces coincide with those introduced in \([7]\) on graded Lie groups in the setting of stratified groups. We also refer to \([2]\) for a number of useful inequalities on graded Lie groups.

In this paper we use positive Rockland operators in order to introduce Besov spaces on graded Lie groups, and later on, we prove that our Besov spaces can be obtained by interpolation of the Sobolev spaces introduced in \([7]\). For special cases of parameters \( p, q \) and \( r \), Besov spaces were also considered by Bahouri, Gérard and Xu in \([3]\). Apart of the trivial embeddings that can be obtained on the \( q \) parameters for Besov spaces \( B_{p,q}^r(G) \), the Nikolkii inequality will be a useful tool in order to establish embeddings that involve the parameters \( r \) and \( p \).

As a substitute of the Plancherel theorem on \( L^2(G) \), in \( L^p(G) \) spaces, we prove a version of the Littlewood-Paley theorem and we will use both, our Nikolskii inequality and our Littlewood-Paley theorem in order to get boundedness of Fourier multipliers and spectral multipliers on Besov spaces. For the case of Fourier multipliers we will use the version of the Hörmander-Mihlin theorem in the nilpotent setting \([5]\).

We note that in the case of the sub-Laplacian, a wealth of results in available, to mention only a few, see e.g. Folland \([8]\) and Saka \([18]\) for Sobolev spaces and Besov spaces on stratified groups, respectively; Furioli, Melzi and Veneruso \([10]\) and Alexopoulos \([1]\) for the Littlewood-Paley theorem and Besov spaces, and for spectral multiplier theorems for the sub-Laplacian on Lie groups of polynomial growth, respectively. The novelty of this paper is that we are working with Rockland operators; these are linear invariant homogeneous hypoelliptic partial differential operators, in view of the Helffer and Nourrigat’s resolution of the Rockland conjecture in \([12]\). Such operators always exist on graded Lie groups and, in fact, the existence of such operators on nilpotent Lie groups does characterise the class of graded Lie groups, see \([6]\, Section 4.1\) for further details and references. As the literature concerning the analysis based on sub-Laplacians is immense, we do not review it here, but refer to the introduction in \([6]\) for a more extensive presentation of the subject.

This paper is organised as follows. In Section 2 we present some preliminaries on the Fourier analysis of graded Lie groups and its homogeneous structure, and we present positive Rockland operators and elements of their functional calculus. For this we follow \([6]\). In Section 3.1 we prove our version of the Nikolskii inequality for functions defined on graded Lie groups. In Section 4 we prove our version of the Littlewood-Paley theorem. In Section 5 we define Besov spaces and we prove some embeddings properties for these spaces. In Section 6 we prove that Besov spaces can
be obtained by interpolation of Sobolev spaces in the nilpotent setting. In Section 7 we show embedding properties between localisation of these Besov spaces and the usual (Euclidean) Besov spaces.

Finally, in Section 8 we study the boundedness of Fourier multipliers and spectral multipliers in Besov spaces. For the case of spectral multipliers we generalise a classical result by Marcinkiewicz, which asserts that if \( m \in BV(\mathbb{R}) \) is a bounded function which has uniformly bounded variation on every dyadic interval of \( \mathbb{R} \), then \( m \) is a multiplier on \( L^p(\mathbb{R}) \) for every \( 1 < p < \infty \). We prove that our version of the Marcinkiewicz theorem generates spectral multipliers bounded on Besov spaces. In the case of Fourier multipliers, we prove that \( \mathcal{F}(G) \)-multipliers on graded nilpotent Lie groups generate multipliers in Besov spaces \( B^{r,p,q}(G) \). As a consequence of this fact, we end Section 8 with several examples on multipliers.

2. Preliminaries

In this section, we recall some preliminaries on graded and homogeneous Lie groups \( G \). The unitary dual of these groups will be denoted by \( \hat{G} \). We also present the notion of Rockland operators and Sobolev spaces on \( G \) and on the unitary dual \( \hat{G} \) by following [5], to which we refer for further details on constructions presented in this section.

2.1. Homogeneous and graded Lie groups. Let \( G \) be a graded Lie group. This means that \( G \) is a connected and simply connected nilpotent Lie group whose Lie algebra \( \mathfrak{g} \) may be decomposed as the sum of subspaces \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s \) such that \( [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \), and \( \mathfrak{g}_{i+j} = \{0\} \) if \( i + j > s \). This implies that the group \( G \) is nilpotent because the sequence

\[
\mathfrak{g}(1) := \mathfrak{g}, \quad \mathfrak{g}(n) := [\mathfrak{g}, \mathfrak{g}(n-1)]
\]

defined inductively terminates at \( \{0\} \) in a finite number of steps. Examples of such groups are the Heisenberg group \( \mathbb{H}^n \) and more generally any stratified groups where the Lie algebra \( \mathfrak{g} \) is generated by \( \mathfrak{g}_1 \). The exponential mapping from \( \mathfrak{g} \) to \( G \) is a diffeomorphism, then, we can identify \( G \) with \( \mathbb{R}^n \) or \( \mathfrak{g}_1 \times \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s \) as manifolds. Consequently we denote by \( \mathscr{S}(G) \) the Schwartz space of functions on \( G \), by considering the identification \( G \equiv \mathbb{R}^n \). Here, \( n \) is the topological dimension of \( G \), \( n = n_1 + \cdots + n_s \), where \( n_k = \text{dim} \mathfrak{g}_k \). A family of dilations \( D_r, r > 0 \), on a Lie algebra \( \mathfrak{g} \) is a family of linear mappings from \( \mathfrak{g} \) to itself satisfying the following two conditions:

- For every \( r > 0 \), \( D_r \) is a map of the form

\[
D_r = \text{Exp}(rA)
\]

for some diagonalisable linear operator \( A \) on \( \mathfrak{g} \).

- \( \forall X, Y \in \mathfrak{g} \), and \( r > 0 \), \( [D_rX, D_rY] = D_r[X,Y] \).

We call to the eigenvalues of \( A, \nu_1, \nu_2, \cdots, \nu_n \), the dilations weights or weights of \( G \). A homogeneous Lie group is a connected simply connected Lie group whose Lie algebra \( \mathfrak{g} \) is equipped with a family of dilations \( D_r \). In such case, and with the notation above,
the homogeneous dimension of $G$ is given by
\[ Q = \text{Tr}(A) = \sum_{l=1}^{s} l \cdot \dim g_l. \]

We can transport dilations $D_r$ of the Lie algebra $g$ to the group by considering the family of maps
\[ \exp_G \circ D_r \circ \exp_G^{-1}, \quad r > 0, \]
where $\exp_G : g \to G$ is the usual exponential function associated to the Lie group $G$. We denote this family of dilations also by $D_r$ and we refer to them as dilations on the group. If we write $rx = D_r(x), \ x \in G, \ r > 0$, then a relation on the homogeneous structure of $G$ and the Haar measure $dx$ on $G$ is given by
\[ \int_G (f \circ D_r)(x) dx = r^{-Q} \int_G f(x) dx. \]

2.2. The unitary dual and the Plancherel theorem. We will always equip a graded Lie group with the Haar measure $dx$. For simplicity, we will write $L^p(G)$ for $L^p(G, dx)$. We denote by $\hat{G}$ the unitary dual of $G$, that is the set of equivalence classes of unitary, irreducible, strongly continuous representations of $G$ acting in separable Hilbert spaces. The unitary dual can be equipped with the Plancherel measure $d\mu$. So, the Fourier transform of every function $\varphi \in \mathcal{S}(G)$ at $\pi \in \hat{G}$ is defined by
\[ (\mathcal{F}_G \varphi)(\pi) \equiv \hat{\varphi}(\pi) = \int_G \varphi(x) \pi(x)^* dx, \]
and the corresponding Fourier inversion formula is given by
\[ \varphi(x) = \int_{\hat{G}} \text{Tr}(\pi(x) \hat{\varphi}(\pi)) d\mu(\pi). \]

In this case, we have the Plancherel identity
\[ \| \varphi \|_{L^2(G)} = \left( \int_{\hat{G}} \text{Tr}(\hat{\varphi}(\pi) \hat{\varphi}(\pi)^*) d\mu(\pi) \right)^{\frac{1}{2}} = \| \hat{\varphi} \|_{L^2(\hat{G})}. \]

We also denote $\| \hat{\varphi} \|^2_{HS} = \text{Tr}(\hat{\varphi}(\pi) \hat{\varphi}(\pi)^*)$ the Hilbert-Schmidt norm of operators. Roughly speaking, a Fourier multiplier is formally defined by
\[ T_\sigma u(x) = \int_{\hat{G}} \text{Tr}(\pi(x) \sigma(\pi) \hat{f}(\pi)) d\mu(\pi), \quad (2.1) \]
where the symbol $\sigma(\pi)$ is defined on the unitary dual $\hat{G}$ of $G$. For a rather comprehensive treatment of this quantization we refer to [6] and to references therein.

2.3. Homogeneous linear operators and Rockland operators. A linear operator $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ is homogeneous of degree $\nu \in \mathbb{C}$ if for every $r > 0$
\[ T(f \circ D_r) = r^\nu (Tf) \circ D_r \]
holds for every $f \in \mathcal{D}(G)$. If for every representation $\pi \in \widehat{G}$, $\pi : G \to U(H_\pi)$, we denote by $H_\pi^\infty$ the set of smooth vectors, that is, the space of elements $\nu \in H_\pi$ such that the function $x \mapsto \pi(x)\nu, \ x \in \hat{G}$ is smooth, a Rockland operator is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree $\nu = \nu_\mathcal{R}$
and such that, for every unitary irreducible non-trivial representation $\pi \in \hat{G}$, $\pi(\mathcal{R})$ is injective on $\mathcal{H}^\infty_\pi$; $\sigma_{\mathcal{R}}(\pi) = \pi(\mathcal{R})$ is the symbol associated to $\mathcal{R}$. It coincides with the infinitesimal representation of $\mathcal{R}$ as an element of the universal enveloping algebra. It can be shown that a Lie group $G$ is graded if and only if there exists a differential Rockland operator on $G$. If the Rockland operator is formally self-adjoint, then $\mathcal{R}$ and $\pi(\mathcal{R})$ admit self-adjoint extensions on $L^2(G)$ and $\mathcal{H}_\pi$ respectively. Now if we preserve the same notation for their self-adjoint extensions and we denote by $E$ and $E_{\pi}$ their spectral measures, by functional calculus we have

$$\mathcal{R} = \int_{-\infty}^{\infty} \lambda dE(\lambda), \quad \pi(\mathcal{R}) = \int_{-\infty}^{\infty} \lambda dE_{\pi}(\lambda).$$

We now recall a lemma on dilations on the unitary dual $\hat{G}$, which will be useful in our analysis of spectral multipliers. For the proof, see Lemma 4.3 of [5].

**Lemma 2.1.** For every $\pi \in \hat{G}$ let us define $D_r(\pi) = \pi(r\cdot)$ by $D_r(\pi)(x) = \pi(rx)$ for every $r > 0$ and $x \in G$. Then, if $f \in L^\infty(\mathbb{R})$ then $f(\pi(r)(\mathcal{R})) = f(r^\nu \pi(\mathcal{R}))$.

We refer to [6, Chapter 4] and references therein for an exposition of further properties of Rockland operators and their history, and to ter Elst and Robinson [4] for their spectral properties.

### 2.4. Sobolev spaces and the Hörmander-Mihlin theorem.

In order to define Sobolev spaces, we choose a positive left-invariant Rockland operator $\mathcal{R}$ of homogeneous degree $\nu > 0$. With notations above one defines Sobolev spaces as follows (c.f [6]).

**Definition 2.2.** Let $r \in \mathbb{R}$, the homogeneous Sobolev space $\dot{H}^{r,p}(G)$ consists of those $f \in \mathcal{D}'(G)$ satisfying

$$\|f\|_{\dot{H}^{r,p}(G)} := \|\mathcal{R}^\frac{r}{2} f\|_{L^p(G)} < \infty. \quad (2.3)$$

Analogously, the inhomogeneous Sobolev space $H^{r,p}(G)$ consists of those distributions $f \in \mathcal{D}'(G)$ satisfying

$$\|f\|_{H^{r,p}(G)} := \|(I + \mathcal{R})^\frac{r}{2} f\|_{L^p(G)} < \infty. \quad (2.4)$$

By using a quasi-norm $|\cdot|$ on $G$ we can introduce for every $r \geq 0$, the inhomogeneous Sobolev space of order $r$ on $\hat{G}$, $H^r(\hat{G})$ which is defined by

$$H^r(\hat{G}) = \mathcal{F}_G(L^2(G, (1 + |\cdot|^2)\tilde{\mathcal{S}}d\mathcal{S})), \quad (2.5)$$

where $\mathcal{F}_G$ is the Fourier transform on the group $G$. In a similar way, for $r \geq 0$ the homogeneous Sobolev space $\dot{H}^r(\hat{G})$ is defined by

$$\dot{H}^r(\hat{G}) = \mathcal{F}_G(L^2(G, |\cdot|^r d\mathcal{S})). \quad (2.6)$$

As usual if $r = 0$ we denote $L^2(\hat{G}) = \dot{H}^0(\hat{G}) = H^0(\hat{G})$. Characterisations of Sobolev spaces on $G$ and the unitary dual $\hat{G}$ in terms of homogeneous norms on $G$ can be found in [5] and [6], respectively.
Finally we present the Hörmander-Mihlin theorem for graded nilpotent Lie groups. This theorem will be useful in our proof of the Littlewood-Paley theorem. The formulation of such result requires a local notion of Sobolev space on the dual space $\hat{G}$. We introduce this as follows. Let $s \geq 0$, we say that the field $\sigma = \{\sigma(\pi) : \pi \in \hat{G}\}$ is locally uniformly in right-$H^s(\hat{G})$ (resp. left-$H^s(\hat{G})$) if there exists a positive Rockland operator $R$ and a function $\eta \in \mathcal{D}(G)$ satisfying

$$\|\sigma\|_{H^s_{l.u,\eta,\mathcal{R}}} := \sup_{r>0} \left\|\{\sigma(\pi^{(r)})\} \eta(\sigma(\mathcal{R}))\right\|_{H^s(\hat{G})} < \infty,$$  \hspace{1cm} (2.5)

respectively,

$$\|\sigma\|_{H^s_{l.u,\eta,\mathcal{R}}} := \sup_{r>0} \left\|\{\eta(\sigma(\mathcal{R}))\} \sigma(\pi^{(r)})\right\|_{H^s(\hat{G})} < \infty.$$  \hspace{1cm} (2.6)

Now, we state the Hörmander-Mihlin theorem on the graded nilpotent Lie group $G$. (c.f. Theorem 4.11 of [5]):

**Theorem 2.3.** Let $G$ be a graded Lie group. Let $\sigma \in L^2(\hat{G})$. If

$$\|\sigma\|_{H^s_{l.u,\eta,\mathcal{R}}} \leq \|\sigma\|_{H^s_{l.u,\eta,\mathcal{R}}} < \infty$$

with $s > \frac{Q}{2}$, then the corresponding multiplier $T_{\sigma}$ extends to a bounded operator on $L^p(G)$ for all $1 < p < \infty$. Moreover

$$\|T_{\sigma}\|_{L^p(G)} \leq C \max\{\|\sigma\|_{H^s_{l.u,\eta,\mathcal{R}}} \leq \|\sigma\|_{H^s_{l.u,\eta,\mathcal{R}}} \leq \infty\}. \hspace{1cm} (2.7)$$

In the next sections, we present our main results. We start with a formulation of the Nikolskii inequality on graded Lie groups.

### 3. Nikolskii Inequality on Graded Lie Groups

Let $G$ be a graded Lie group with a family of dilations $D_t$, $t > 0$. Let $\mathcal{R}$ be a positive Rockland operator and for every $L > 0$ let us consider the operator, defined by functional calculus, $\chi_L(\mathcal{R})$ where $\chi_L$ is the characteristic function of $[0, L]$. For $f \in \mathcal{S}(G)$ let us define the operator $T_L$ by

$$T_L f := \chi_L(\mathcal{R}) f.$$  \hspace{1cm} (3.1)

Then $T_L$ is a spectral multiplier and

$$\mathcal{F}_G(T_L f)(\pi) = \left( \int_0^L \chi_L(\lambda) dE_{\pi}(\lambda) \right) \hat{f}(\pi),$$

where $(E_{\pi}(\lambda))_{\lambda \geq 0}$ is the spectral resolution of $\pi(\mathcal{R}) := d\pi(\mathcal{R})$. We denote $E_{\pi}(L) = \int_{[0,L]} dE_{\pi}(\lambda)$. In terms of the Fourier inversion formula we have

$$T_L f(x) = \int_G \text{Tr}[\hat{\pi}(x) E_{\pi}(L) \hat{f}(\pi)] d\pi. \hspace{1cm} (3.1)$$

With notations above we present our version of the Nikolskii inequality in the following theorem.

**Theorem 3.1.** Let $G$ be a graded Lie group of homogeneous dimension $Q$, and let us consider the operator $T_L$ as in (3.1). If $1 \leq p \leq q \leq \infty$ then

$$\|T_L f\|_{L^q} \leq \|\mathcal{F}_G^{-1}[E_{\pi}(1)]\|_{L^r} L^\frac{Q}{r} \left( \frac{1}{L^\frac{Q}{r}} \right)^{\frac{1}{Q}} \|T_L f\|_{L^p}, \hspace{1cm} (3.2)$$

where $r = (1 + (1/q - 1/p))^{-1}$.
Proof. Let us define for every \( L > 0 \), the function
\[
g_L = L^{-\frac{\alpha}{p}}(T_L f) \circ D_L^{-\frac{1}{p}},
\]
i.e., \( g_L(x) = L^{-\frac{\alpha}{p}}(T_L f)(L^{-\frac{1}{p}} x), \) \( x \in G \). Denoting by \( e = e_G \) the identity element of \( G \), for every \( \pi \in \hat{G} \) we have
\[
\hat{g}_L(\pi) = \int_G L^{-\frac{\alpha}{p}}(T_L f)(L^{-\frac{1}{p}} x) \pi(x)^* dx
= \int_G (T_L f)(y) \pi(L^\frac{1}{p} y)^* dy
= \overline{T_L f}(\pi(L^\frac{1}{p})).
\]

We observe that
\[
\overline{T_L f}(\pi(L^\frac{1}{p})) = E_{\pi(L^\frac{1}{p})}(L) \hat{f}(\pi(L^\frac{1}{p})) = \chi_L(\pi(L^\frac{1}{p})(R)) \hat{f}(\pi(L^\frac{1}{p})),
\]
and by considering that
\[
\chi_L(L\pi(R)) = \int_0^\infty \chi_L(L\lambda) dE_\pi(\lambda) = \int_0^\infty \chi_1(\lambda) dE_\pi(\lambda) = E_\pi(1)
\]
we obtain that \( \hat{g}_L(\pi) = E_\pi(1) \hat{f}(\pi(L^\frac{1}{p})) \). By using that \( E_\pi(1) \) is a projection, we have
\[
\hat{g}_L(\pi) = E_\pi(1) \hat{g}_L(\pi).
\]
Hence
\[
g_L(x) = g_L * F^{-1}[E_\pi(1)](x), \ x \in G.
\]

By applying Young inequality we have
\[
\|g_L\|_{L^q} \leq \|F^{-1}[E_\pi(1)]\|_{L^r} \|g_L\|_{L^p}, \tag{3.3}
\]
provided that \( \frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1 \). We observe that the condition \( 1 \leq p \leq q \leq \infty \) implies that \( 0 \leq \frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} \leq 1 \) and consequently \( 1 \leq r \leq \infty \). Observe that for every \( a > 0 \) we have
\[
\|g_L\|_{L^a(G)} = \left( \int_G |g_L(x)|^a dx \right)^{\frac{1}{a}} = \left( \int_G L^{-\frac{Qa}{p}}|T_L f(L^{-\frac{1}{p}} x)|^a dx \right)^{\frac{1}{a}}
= \left( \int_G L^{[Q-Qa]/\nu}|T_L f(y)|^a dy \right)^{\frac{1}{a}}
= L^{\frac{Q}{\nu}(\frac{1}{a}-1)}\|T_L f\|_{L^a}.
\]

So, by the inequality (3.3), we have
\[
L^{\frac{Q}{\nu}(\frac{1}{a}-1)}\|T_L f\|_{L^q} \leq \|F^{-1}[E_\pi(1)]\|_{L^r} L^{\frac{Q}{\nu}(\frac{1}{p}-1)}\|T_L f\|_{L^p}. \tag{3.4}
\]
Thus, we obtain
\[ \| T_L f \|_{L^q} \leq \| \mathcal{F}^{-1}[E_\pi(1)] \|_{L^{p',\frac{Q}{p}-\frac{s}{q}}} \| T_L f \|_{L^p}. \] (3.5)

This completes the proof. \qed

4. Littlewood-Paley theorem on graded Lie groups

In this section we prove the Littlewood-Paley theorem by using the Hörmander-Mihlin theorem on graded Lie groups as a fundamental tool.

For versions of the Littlewood-Paley theorem for the sub-Laplacian on the Heisenberg group we can refer to Bahouri, Gérard and Xu [3], and for sub-Laplacians on groups of polynomial growth see Furioli, Melzi and Veneruso [10]. Here we prove it for general Rockland operators on graded groups.

The following lemma will be useful in our formulation of the Littlewood-Paley theorem.

**Lemma 4.1.** Let \( \sigma \in L^2(\hat{G}) \). If \( r > 0 \) and \( s \geq 0 \) then

\[ \| \sigma \circ D_r \|_{\dot{H}^s(\hat{G})} = r^{-s-\frac{Q}{p}} \| \sigma \|_{\dot{H}^s(\hat{G})}. \] (4.1)

This implies that \( \sigma \in \dot{H}^s(\hat{G}) \) if only if for every \( r > 0 \), \( \sigma \circ D_r \in \dot{H}^s(\hat{G}) \).

**Proof.** By Lemma 4.3 of [5] we have

\[
\| \sigma \circ D_r \|_{\dot{H}^s(\hat{G})} = \| |s|^{\frac{Q}{2}} \mathcal{F}_G^{-1}(\sigma \circ D_r) \|_{L^2(G)} = \| |s|^{r^{-Q}} \mathcal{F}_G^{-1}(\sigma)(r^{-1}) \|_{L^2(G)} = r^{-s-\frac{Q}{2}} \| \sigma \|_{\dot{H}^s(\hat{G})}.
\]

With the equality above, it is clear that \( \sigma \in \dot{H}^s(\hat{G}) \) if only if for every \( r > 0 \), \( \sigma \circ D_r \in \dot{H}^s(\hat{G}) \). \qed

The Littlewood-Paley theory provides a partial substitute in \( L^p \) spaces for the results derived from the Plancherel theorem. The main notion in the Littlewood-Paley theory is the concept of a dyadic decomposition. Here, the sequence \( \{ \psi_l \}_{l \in \mathbb{N}_0} \) is a dyadic decomposition, defined as follows: we choose a function \( \psi \in C_0^\infty(G) \) supported in \([1/4, 2] \), \( \psi = 1 \) on \([1/2, 1] \). Denote by \( \psi_l \) the function \( \psi_l(t) = \psi(2^{-l}t) \), \( t \in \mathbb{R} \). For some smooth compactly supported function \( \psi_0 \) we have

\[ \sum_{l \in \mathbb{N}_0} \psi_l(\lambda) = 1, \text{ for every } \lambda > 0. \] (4.2)

Now we present the Littlewood-Paley theorem in the form of the following result.

**Theorem 4.2.** Let \( 1 < p < \infty \) and let \( G \) be a graded Lie group. If \( \mathcal{R} \) is a positive Rockland operator then there exist constants \( 0 < c_p, C_p < \infty \) depending only on \( p \) such that

\[ c_p \| f \|_{L^p(G)} \leq \left\| \sum_{l=0}^{\infty} |\psi_l(\mathcal{R}) f|^2 \right\|_{L^p(G)}^{\frac{1}{2}} \leq C_p \| f \|_{L^p}, \] (4.3)

holds for every \( f \in L^p(G) \).
Proof. We observe that for \( l \geq 1 \) the spectral multiplier \( \psi_l(\mathcal{R}) \) has symbol \( \sigma_l(\pi) = \psi_l(\pi(\mathcal{R})) \). By using Lemma 2.1 we obtain

\[
\sigma_l(\pi) = \psi_l(\pi(\mathcal{R})) = \int_{-\infty}^{\infty} \psi_1(2^{-l}\lambda)dE_{\pi}(\lambda) = \psi_1(2^{-l}\pi(\mathcal{R}))
\]

\[
= \psi_1(2^{-\frac{1}{\nu}}\pi(\mathcal{R})) = \psi_1(\pi(2^{-\frac{1}{\nu}}))(\mathcal{R})
\]

\[
= \sigma_1 \circ D_{2^{-\frac{1}{\nu}}}(\pi).
\]

Taking into account Lemma 4.1 with \( r = 2^{-\frac{1}{\nu}} \) we have, for \( s \geq 0 \), the inequality

\[
\|\sigma_l\|_{H^s(\mathcal{G})} \lesssim 2^{-s^\nu + \frac{Q}{2}}. \tag{4.4}
\]

In particular, by the Hörmander-Mihlin theorem in the form of Theorem 2.3, we obtain that, for \( s > \frac{Q}{2} \),

\[
\|\psi_l(\mathcal{R})\|_{L^p(\mathcal{G})} \leq C_p 2^{-\frac{1}{p}(s-\frac{Q}{2})}, \tag{4.5}
\]

for all \( 1 < p < \infty \). Clearly \( \psi_0 \) is bounded on \( L^p(\mathcal{G}) \). Now, we will prove the right hand side of (4.3) provided that \( s > \frac{Q}{2} \):

\[
\left\| \sum_{l \in \mathbb{N}_0} |\psi_l(\mathcal{R})f|^2 \right\|_{L^p(\mathcal{G})}^{\frac{1}{2}} \leq \left\| \sum_{l \in \mathbb{N}_0} |\psi_l(\mathcal{R})f| \right\|_{L^p(\mathcal{G})} \leq \sum_{l \in \mathbb{N}_0} \|\psi_l(\mathcal{R})f\|_{L^p(\mathcal{G})}
\]

\[
\lesssim \sum_{l \in \mathbb{N}_0} 2^{-\frac{1}{p}(s-\frac{Q}{2})} \|f\|_{L^p(\mathcal{G})}
\]

\[
\equiv C_p \|f\|_{L^p(\mathcal{G})}.
\]

The proof of the left hand side is as follows. Now, let us denote by \( (E(\lambda))_{\lambda \geq 0} \) the spectral resolution associated to \( \mathcal{R} \), and for every \( \pi \in \mathcal{G} \) denote by \( (E_\pi(\lambda))_{\lambda \geq 0} \) the spectral resolution of \( \pi(\mathcal{R}) \). We observe that by duality

\[
\|f\|_{L^p(\mathcal{G})} = \sup \{ | \int_G f(x)g(x)dx | : g \in \mathcal{D}(\mathcal{G}), \|g\|_{L^{p'}} = 1 \}
\]

\[
= \sup \{ | \int_G \sum_{l \in \mathbb{N}_0} [\psi_l(\mathcal{R})f](x)g(x)dx | : g \in \mathcal{D}(\mathcal{G}), \|g\|_{L^{p'}} = 1 \}
\]

\[
= \sup \{ | \int_G \sum_{l \in \mathbb{N}} [E[2^{l-1}, 2^{l+2}]\psi_l(\mathcal{R})f](x)g(x)dx
\]

\[
+ \int_G [E[-1, 1]\psi_0(\mathcal{R})f(x)]g(x)dx | : g \in \mathcal{D}(\mathcal{G}), \|g\|_{L^{p'}} = 1 \}
\]

\[
= \sup \{ | \int_G \sum_{l \in \mathbb{N}} [\psi_l(\mathcal{R})f](x)E[2^{l-1}, 2^{l+2}]g(x)dx
\]

\[
+ \int_G [\psi_0(\mathcal{R})f(x)]E[-1, 1]g(x)dx | : g \in \mathcal{D}(\mathcal{G}), \|g\|_{L^{p'}} = 1 \}.
\]
If $E^{(l)} := E[2^{l-1}, 2^{l+2}]$ for $l \geq 1$ and $E^{(0)} := E[-1, 1]$ then

$$
\|f\|_{L^p(G)} \leq \sup \left\{ \int_G \left[ \sum_{l \in \mathbb{N}_0} \left| \psi_l(R)f(x) \right|^2 \right]^\frac{1}{2} \left[ \sum_{l \in \mathbb{N}_0} \|E^{(l)}g(x)\|^2 \right]^\frac{1}{2} \, dx : g \in \mathcal{D}(G), \|g\|_{L^{p'}} = 1 \right\}
$$

$$
\leq \sup \left\{ \left[ \sum_{l \in \mathbb{N}_0} \left| \psi_l(R)f(x) \right|^2 \right]^\frac{1}{2} \|g\|_{L^{p'}} : g \in \mathcal{D}(G), \|g\|_{L^{p'}} = 1 \right\}.
$$

By a similar argument to (4.4), the following estimate

$$
\|E^{(l)}\|_{H^s(G)} = \|\chi_{\left[ \frac{1}{2}, 1 \right]}(2^{-l}\pi(R))\|_{H^s} = \|\chi_{\left[ \frac{1}{2}, 1 \right]}\circ D_{2^{-\frac{s}{2}}}\|_{H^s} \lesssim 2^{-\frac{s}{2}}(s-\frac{Q}{2}),
$$

holds for every $s > \frac{Q}{2}$. So, the Hörmander-Mihlin Theorem gives the boundedness of the multiplier $E^{(l)}$ on $L^p(G)$ with the estimate

$$
\|E^{(l)}f\|_{L^p} \lesssim 2^{-\frac{s}{2}}(s-\frac{Q}{2})\|f\|_{L^p}.
$$

Thus, by using the Minkowski integral inequality we obtain

$$
\left[ \sum_{l \in \mathbb{N}_0} \|E^{(l)}g(x)\|^2 \right]^\frac{1}{2} \lesssim \left[ \sum_{l \in \mathbb{N}_0} 2^{-\frac{s}{2}(2s-Q)}\|g\|^2_{L^{p'}} \right]^\frac{1}{2}
$$

$$
\lesssim \|g\|_{L^{p'}}.
$$

Thus we have

$$
\|f\|_{L^p(G)} \lesssim \left[ \sum_{l \in \mathbb{N}_0} \left| \psi_l(R)f(x) \right|^2 \right]^\frac{1}{2} \|g\|_{L^{p'}}
$$

completing the proof. \(\square\)

5. Homogeneous and Inhomogeneous Besov spaces

Let $R$ be a (left-invariant) positive Rockland operator on a graded Lie group $G$. In order to define the family of Besov spaces on $G$, let us assume that $R$ is homogeneous of degree $\nu > 0$ and let us fix a dyadic decomposition of its spectrum: we choose a function $\psi \in C_0^\infty(\mathbb{R})$ supported in $[1/4, 2]$, $\psi = 1$ on $[1/2, 1]$. Denote by $\psi_t$ the function $\psi_t(t) = \psi(2^{-t}t)$, $t \in \mathbb{R}$. For some smooth compactly supported function $\psi_0$ we have

$$
\sum_{l \in \mathbb{N}_0} \psi_l(\lambda) = 1, \text{ for every } \lambda > 0. \tag{5.1}
$$

With notations above we define (left) Besov spaces associated to a (left-invariant) positive Rockland operator as follows.

**Definition 5.1.** Let $r \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. The homogeneous Besov space $\dot{B}^{r}_{p,q}(G)$ associated to $(R, (\psi_t)_t)$ consists of those $f \in \mathcal{D}'(G)$ satisfying

$$
\|f\|_{\dot{B}^{r}_{p,q}(G)} := \left( \sum_{l \in \mathbb{N}_0} 2^{lr} \|\psi_l(R)f\|_{L^p(G)}^q \right)^\frac{1}{q} < \infty, \tag{5.2}
$$
for $0 < q < \infty$ and for $q = \infty$,
\[ \|f\|_{B^{r,\infty,\psi,\mathcal{R}}_{p,\infty}(G)} := \sup_{l \in \mathbb{N}_0} 2^l \| \psi_l(\mathcal{R}) f \|_{L^p(G)} < \infty. \] (5.3)

Analogously, the inhomogeneous Besov space $B^{r}_{p,q,\psi,\mathcal{R}}(G)$ is defined as the space of distributions $f \in \mathcal{D}'(G)$ satisfying
\[ \|f\|_{B^{r}_{p,q,\psi,\mathcal{R}}(G)} := \left( \sum_{l \in \mathbb{N}_0} 2^l \| \psi_l(I + \mathcal{R}) f \|_{L^p(G)}^q \right)^{\frac{1}{q}} < \infty, \] (5.4)
if $0 < q < \infty$ and, for $q = \infty$,
\[ \|f\|_{B^{r,\infty,\psi,\mathcal{R}}_{p,\infty}(G)} := \sup_{l \in \mathbb{N}_0} 2^l \| \psi_l(I + \mathcal{R}) f \|_{L^p(G)} < \infty. \] (5.5)

Homogeneous and inhomogeneous Besov spaces do not depend on a particular choice of a positive Rockland operator $\mathcal{R}$ and of the sequence of smooth functions $\psi_l$. We will prove this fact in the following section (see Theorem 6.1). Now, we prove the following embedding properties of Besov spaces. We use the simplified notation motivated by Theorem 6.1,
\[(\dot{B}^{r,q}_{p,q}(G), \| \cdot \|_{\dot{B}^{r,q}_{p,q}(G)}) = (\dot{B}^{r,\infty,\psi,\mathcal{R}}_{p,q,\mathcal{R}}(G), \| \cdot \|_{\dot{B}^{r,\infty,\psi,\mathcal{R}}_{p,q,\mathcal{R}}(G)})\]
and
\[(B^{r,q}_{p,q}(G), \| \cdot \|_{B^{r,q}_{p,q}(G)}) = (B^{r,\infty,\psi,\mathcal{R}}_{p,q,\mathcal{R}}(G), \| \cdot \|_{B^{r,\infty,\psi,\mathcal{R}}_{p,q,\mathcal{R}}(G)}).\]

For Sobolev spaces $H^{r,p}(G)$ and $\dot{H}^{r,p}(G)$ and their properties we refer to [6, Section 4].

We also note that similar results would hold if we chose right-invariant (instead of left-invariant) Rockland operator in the definition of Besov spaces, see Remark 5.4.

**Theorem 5.2.** Let $G$ be a graded Lie group of homogeneous dimension $Q$ and let $r \in \mathbb{R}$. Then

1. $\dot{B}^{r,\infty}_{p,q,\mathcal{R}}(G) \hookrightarrow \dot{B}^{r}_{p,q,\mathcal{R}}(G) \hookrightarrow \dot{B}^{r}_{p,q,\mathcal{R}}(G) \hookrightarrow \dot{B}^{r}_{p,\infty,\psi,\mathcal{R}}(G), \varepsilon > 0, 0 < p \leq \infty, 0 < q_1 \leq q_2 \leq \infty.$

2. $\dot{B}^{r,\infty}_{p,q,\mathcal{R}}(G) \hookrightarrow \dot{B}^{r}_{p,q,\mathcal{R}}(G), \varepsilon > 0, 0 < p \leq \infty, 1 \leq q_2 < q_1 < \infty.$

3. $\dot{B}^{r}_{p_{1},q}(G) \hookrightarrow \dot{B}^{r}_{p_{2},q}(G), 1 \leq p_1 \leq p_2 \leq \infty, 0 < q < \infty, r_1 \in \mathbb{R}$ and $r_2 = r_1 - Q\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$.

4. $\dot{H}^{r}(G) = \dot{B}^{r}_{2,2}(G)$ and $\dot{B}^{r}_{p,p}(G) \hookrightarrow \dot{H}^{r,p}(G) \hookrightarrow \dot{B}^{r}_{p,2}(G), 1 < p \leq 2.$

5. $\dot{B}^{r}_{p,1}(G) \hookrightarrow L^{q}(G), 1 \leq p \leq q \leq \infty, r = Q\left(\frac{1}{p} - \frac{1}{q}\right).$
Proof. For the proof of (1) we observe that

\[
\|f\|_{\dot{B}^{r}_{p,\infty}} = \sup_{s \in \mathbb{N}_0} 2^{r \frac{s}{2}} \|\psi_s(\mathcal{R}) f\|_{L^p}
\]

\[
\leq \|\{2^{r \frac{s}{2}} \|\psi_s(\mathcal{R}) f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{\ell^2(\mathbb{N}_0)}
\]

\[
= \|f\|_{\dot{B}^{r}_{p,2}}
\]

\[
\leq \|\{2^{r \frac{s}{2}} \|\psi_s(\mathcal{R}) f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{\ell^{q_1}(\mathbb{N}_0)}
\]

\[
= \|f\|_{\dot{B}^{r}_{p,q_1}}
\]

\[
\leq \|\{2^{r \frac{s}{2} + \frac{\epsilon}{2}} \|\psi_s(\mathcal{R}^{\frac{1}{2}}) f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{\ell^{q_1}(\mathbb{N}_0)}
\]

\[
= \|f\|_{\dot{B}^{r+\frac{\epsilon}{2}}_{p,q_1}}.
\]

For the proof of (2) we use Hölder inequality as follows

\[
\|f\|_{\dot{B}^{r}_{p,2}} = \|\{2^{r \frac{s}{2}} \|\psi_s(\mathcal{R}) f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{\ell^2(\mathbb{N}_0)}
\]

\[
= \|\{2^{r \frac{s}{2} + \frac{\epsilon}{2}} \|\psi_s(\mathcal{R}) f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{\ell^2(\mathbb{N}_0)}
\]

\[
\leq \|\{2^{r \frac{s}{2} + \frac{\epsilon}{2}} \|\psi_s(\mathcal{R}) f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{\ell^{q_1}(\mathbb{N}_0)} \sum_{s \in \mathbb{N}_0} 2^{\frac{\epsilon}{2} q_2} \|\psi_s(\mathcal{R}) f\|_{L^p}^{q_2}
\]

\[
\approx \|f\|_{\dot{B}^{r+\frac{\epsilon}{2}}_{p,q_1}}.
\]

In order to proof (3) we use Nikolskii inequality from Theorem 3.1 taking into account the estimate

\[
\|\psi_s(\mathcal{R}) f\|_{L^{p_2}} \leq C2^{\frac{Q}{q_1}} \left(\frac{1}{p_1} - \frac{1}{p_2}\right) \|\psi_s(\mathcal{R}) f\|_{L^{p_1}},
\]

so that we have

\[
\left(\sum_{s \in \mathbb{N}_0} 2^{\frac{s}{2} q_2} \|\psi_s(\mathcal{R}) f\|_{L^{p_2}(G)}^{q_2}\right)^{\frac{1}{q_2}} \leq \left(\sum_{s \in \mathbb{N}_0} 2^{\frac{s}{2} \left[Q(\frac{1}{p_1} - \frac{1}{p_2})\right]} \|\psi_s(\mathcal{R}) f\|_{L^{p_1}(G)}^{q_1}\right)^{\frac{1}{q_1}}.
\]
Now we will prove (4) that is $\dot{B}_{p,\alpha}(G) \hookrightarrow \dot{H}^{r,p}(G) \hookrightarrow \dot{B}_{p,2}(G)$, for $1 < p \leq 2$. In fact, we have

\[
\|f\|_{\dot{H}^{r,p}}^p = \|\dot{R}f\|_{L^p}^p = \int_0^\infty \lambda^\frac{r}{p} dE(\lambda) f^p_{\lambda L^p} = \int_0^\infty \lambda^\frac{r}{p} dE(\lambda) f^p_{\lambda L^p} + \lambda^\frac{r}{p} dE(\lambda) f^p_{\lambda L^p} = \int_0^\infty \lambda^\frac{r}{p} dE(\lambda) f^p_{\lambda L^p} + \int_0^\infty \lambda^\frac{r}{p} dE(\lambda) f^p_{\lambda L^p}
\]

\[
= \sum_{s \in \mathbb{Z}} \int_{2^s}^{2^{s+1}} \lambda^\frac{r}{p} dE(\lambda) f^p_{\lambda L^p} \leq \sum_{s \in \mathbb{Z}} \int_{2^s}^{2^{s+1}} \lambda^\frac{r}{p} dE(\lambda) f^p_{\lambda L^p}
\]

\[
= \sum_{s \in \mathbb{Z}} 2^r \int_{2^s}^{2^{s+1}} 2^{-\frac{r}{p}} \lambda^\frac{r}{p} dE(\lambda) f^p_{\lambda L^p}
\]

\[
= \sum_{s \in \mathbb{Z}} 2^r \int_{2^s}^{2^{s+1}} \psi_s(\lambda) dE(\lambda) f^p_{\lambda L^p}
\]

\[
= \sum_{s \in \mathbb{Z}} 2^r \int_{2^s}^{2^{s+1}} \psi_s(\lambda) dE(\lambda) f^p_{\lambda L^p}
\]

For the other embedding we use the following version of the Minkowski integral inequality

\[
\left( \sum_j \left( \int f_j d\mu \right)^\frac{\alpha}{p} \right)^\frac{1}{\alpha} \leq \left( \sum_j \left( \int f_j^\alpha dx \right)^\frac{1}{\alpha} \right)^\frac{1}{\alpha},
\]

with $\alpha = \frac{2}{p}$. So, we get

\[
\|f\|_{\dot{B}_{p,2}} = \left( \sum_{s \in \mathbb{N}_0} 2^r \int_{2^s}^{2^{s+1}} \psi_s(\mathcal{R}) f^p_{\lambda L^p} \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{s \in \mathbb{N}_0} 2^r \left( \int_{G} |\psi_s(\mathcal{R}) f(x)|^p dx \right)^\frac{1}{p} \right)^{\frac{1}{2}}
\]

\[
\leq \left[ \int_{G} \left( \sum_{s \in \mathbb{N}_0} 2^r |\psi_s(\mathcal{R}) f(x)|^p dx \right)^\frac{1}{p} \right]^\frac{1}{2}
\]

\[
= \left[ \int_{G} \left( \sum_{s \in \mathbb{N}_0} 2^r |\psi_s(\mathcal{R}) f(x)|^p dx \right)^\frac{1}{p} \right]^\frac{1}{2}
\]

\[
= \left[ \int_{G} \left( \sum_{s \in \mathbb{N}_0} 2^r |\psi_s(\mathcal{R}) f(x)|^p dx \right)^\frac{1}{p} \right]^\frac{1}{2}
\]

\[
\leq \|R^{\frac{1}{p}} f\|_{L^p} = \|f\|_{\dot{H}^{r,p}},
\]

using Littlewood-Paley theorem (Theorem 4.2). We observe that in the embedding $\dot{B}_{p,\alpha}(G) \hookrightarrow \dot{H}^{r,p}(G) \hookrightarrow \dot{B}_{p,2}(G)$, if $p = 2$ then $\dot{H}^{r,2}(G) = \dot{B}_{2,2}^{r}(G)$. Now, for the proof
of (5) we use Nikolskii inequality,
\[
\|f\|_{L^q} = \| \int_G \text{Tr}[\pi(x)\tilde{f}(\pi)]d\pi \|_{L^q} \\
= \| \sum_{s\in\mathbb{N}_0} \int_G \text{Tr}[\pi(x)\psi_s[\pi(R)\tilde{f}(\pi)]d\pi \|_{L^q} \\
\leq \sum_{s\in\mathbb{N}_0} \| \int_G \text{Tr}[\pi(x)\psi_s[\pi(R)\tilde{f}(\pi)]d\pi \|_{L^q} \\
= \sum_{s\in\mathbb{N}_0} \|\psi_s(R)f\|_{L^q} \leq \sum_{s\in\mathbb{N}_0} 2^{\frac{Q}{p}(\frac{1}{p} - \frac{1}{q})}\|\psi_s(R)f\|_{L^p} \\
= \|f\|_{B^{Q(\frac{1}{p} - \frac{1}{q})}_{p,1}}.
\]

This completes the proof.

In the following theorem we present embeddings properties for inhomogeneous Besov spaces $B^r_{p,q}(G)$. The proof is similar to the homogeneous case, so we omit it.

**Theorem 5.3.** Let $G$ be a graded Lie group of homogeneous dimension $Q$ and let $r \in \mathbb{R}$. Then

1. $B^{r_1}_{p,q_1}(G) \hookrightarrow B^r_{p,q_2}(G) \hookrightarrow B^r_{p,\infty}(G)$, $\varepsilon > 0$, $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$.

2. $B^{r_1}_{p,q_1}(G) \hookrightarrow B^r_{p,q_2}(G)$, $\varepsilon > 0$, $0 < p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.

3. $B^r_{p_1,q_1}(G) \hookrightarrow B^r_{p_2,q_2}(G)$, $1 \leq p_1 \leq p_2 \leq \infty$, $0 < q < \infty$, $r_1 \in \mathbb{R}$ and $r_2 = r_1 - Q(\frac{1}{p_1} - \frac{1}{p_2})$.

4. $H^r(G) = B^r_{2,2}(G)$ and $B^r_{p,p}(G) \hookrightarrow H^{r,p}(G) \hookrightarrow B^r_{p,2}(G)$, $1 < p \leq 2$.

5. $B^r_{p,1}(G) \hookrightarrow L^2(G)$, $1 \leq p < \infty$, $r = Q(\frac{1}{p} - \frac{1}{q})$.

**Remark 5.4** (Right Besov spaces). Throughout this section we have considered Besov spaces associated to (left-invariant) positive Rockland operators. A similar formulation of homogeneous and inhomogeneous (right) Besov spaces can be obtained if we choose (right-invariant) positive Rockland operators. It can be shown that these spaces satisfy (right) versions of Theorems 5.2 and 5.3. When properties that we want to consider hold for left and right Besov spaces, we omit the prefixes left and right, nevertheless, we consider in the proofs the case of (left) Besov spaces.

### 6. Independence of Rockland Operators and Interpolation of Sobolev Spaces

In this section we prove the independence of the choice of Rockland operator and the dyadic partition $\psi_i$ in the definition of Besov spaces. For this, we show that Besov spaces can be obtained as interpolation of Sobolev spaces. If $X_0$ and $X_1$ are Banach spaces, the main notion in real interpolation theory is the $K$-functional, defined by

\[
K(f,t) = \inf \{ \|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1 \}, \quad t \geq 0.
\]
If \( 0 < \theta < 1 \) and \( 1 \leq q < \infty \), the real interpolation space \( X_{\theta,q} := (X_0, X_1)_{\theta,q} \) is defined by those vectors \( f \in X_0 + X_1 \) satisfying

\[
\|f\|_{\theta,q} = \left( \int_0^\infty (t^{-\theta} K(f,t))^{q} \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \quad \text{if } q < \infty,
\]

and for \( q = \infty \)

\[
\|f\|_{\theta,q} = \sup_{t > 0} t^{-\theta} K(f,t) < \infty.
\]

For our purposes, the following discrete form (see [13], p. 1136) will be useful

\[
\|f\|_{\theta,q} \approx \inf \left\{ \left( \sum_{k \in \mathbb{Z}} \max\{\|f_k\|_{X_0}, 2^k \|f_k\|_{X_1}\}^q \right)^{\frac{1}{q}} : f = \sum_{k \in \mathbb{Z}} 2^{k\theta} f_k \right\}.
\]

with \( 1 \leq q < \infty \).

**Theorem 6.1.** Let \( G \) be a graded Lie group, and let \( \mathcal{R} \) and \( \mathcal{R}' \) be two positive Rockland operators with homogeneity degrees \( \nu > 0 \) and \( \nu' > 0 \), respectively. If \( (\psi_l)_l \) and \( (\psi'_l)_l \) are sequences satisfying (5.1), \( 1 < p < \infty \) and \( 1 \leq q < \infty \), the spaces \( \dot{B}_{p,q,\psi,\mathcal{R}}^r(G) \) and \( \dot{B}_{p,q,\psi,\mathcal{R}'}^r(G) \) have equivalent norms, as well as the spaces \( B_{p,q,\psi,\mathcal{R}}^r(G) \) and \( B_{p,q,\psi,\mathcal{R}'}^r(G) \). We also have the following interpolation properties:

\[
B_{p,q}^r(G) = (H^{b,p}(G), H^{a,p}(G))_{\theta,q}, \quad a < r < b, \quad r = b(1 - \theta) + a\theta,
\]

and

\[
\dot{B}_{p,q}^r(G) = (\dot{H}^{b,p}(G), \dot{H}^{b,a}(G))_{\theta,q}, \quad a < r < b, \quad r = b(1 - \theta) + a\theta.
\]

**Proof.** It was proved in [6, Theorem 4.4.20], that the definition of (homogeneous and inhomogeneous) Sobolev spaces \( \dot{H}^{r,p}(G) \) and \( H^{r,p}(G) \), respectively) does not depend on the choice of Rockland operators. Hence the independence of the choice of Rockland operators and of the dyadic decomposition \( \psi_l \) in the case of Besov spaces would follow if we show that Besov spaces can be obtained by interpolation of Sobolev spaces. So, it suffices to prove (6.5) and (6.6). First we will show that for \( r > 0 \),

\[
\dot{B}_{p,q}^r = (\dot{H}^{r_1,p}, \dot{H}^{r_0,p})_{\theta,q}, \quad \text{where} \quad 0 < r_0 < r < r_1 = r_0 + \nu, \quad r = r_1 + (r_0 - r_1)\theta, \quad \text{and later we will deduce the general case from this fact.}
\]

For \( f \in \dot{H}^{r_1,p} + \dot{H}^{r_0,p} \) we write

\[
f = \sum_{l \geq 0} \psi_l(\mathcal{R}) f = \sum_{l \geq 0} 2^{l\theta} f'_l, \quad f'_l = 2^{-l\theta} \psi_l(\mathcal{R}) f.
\]

Hence

\[
\|f\|_{\theta,q}^q \lesssim \sum_{l \geq 0} \max\{\|f'_l\|_{\dot{H}^{r_1,p}}, 2^l \|f'_l\|_{\dot{H}^{r_0,p}}\}^q.
\]

Now, if \( (E_\lambda)_{\lambda \geq 0} \) denotes the spectral resolution associated to \( \mathcal{R} \), we have
\[ \| f \|_{\delta,q}^q \lesssim \sum_{l \geq 0} 2^{-\theta l q} \max \{ \| \psi_l(\mathcal{R}) f \|_{H^{r_1-p}}, 2^l \| \psi_l(\mathcal{R}) f \|_{H^{r_0-p}} \}^q \]

\[ = \sum_{l \geq 0} 2^{-\theta l q} \max \{ \| \int_{R_{l-1}}^{R_{l+1}} \psi_l(\lambda) dE \|_{H^{r_1-p}}, 2^l \| \int_{R_{l-1}}^{R_{l+1}} \psi_l(\lambda) dE \|_{H^{r_0-p}} \}^q \]

\[ = \sum_{l \geq 0} 2^{-\theta l q} \max \{ \| \int_{R_{l-1}}^{R_{l+1}} \lambda^{\frac{\nu}{q}} \psi_l(\lambda) dE \|_{L^p}, 2^l \| \int_{R_{l-1}}^{R_{l+1}} \lambda^{\frac{\nu}{q}} \psi_l(\lambda) dE \|_{L^p} \}^q \]

\[ \lesssim \sum_{l \geq 0} 2^{-\theta l q} \max \{ 2^{\frac{\nu l}{q}} \int_{R_{l-1}}^{R_{l+1}} \psi_l(\lambda) dE \|_{L^p}, 2^l \| \int_{R_{l-1}}^{R_{l+1}} \psi_l(\lambda) dE \|_{L^p} \}^q \]

\[ = \sum_{l \geq 0} 2^{-\theta l q} \max \{ 2^{\frac{\nu l}{q}} \| \psi_l(\mathcal{R}) f \|_{L^p}, 2^l \| \psi_l(\mathcal{R}) f \|_{L^p} \}^q. \]

Since

\[ \max \{ 2^{\frac{\nu l}{q}} \| \psi_l(\mathcal{R}) f \|_{L^p}, 2^l \| \psi_l(\mathcal{R}) f \|_{L^p} \} = \max \{ 2^{\frac{\nu l}{q}}, 2^{\nu l + 1} \} \| \psi_l(\mathcal{R}) f \|_{L^p}, \]

we obtain

\[ \| f \|_{\delta,q}^q \lesssim \sum_{l \geq 0} 2^{-\theta l q} \max \{ 2^{\frac{\nu l q}{q} + l q}, 2^l \} \| \psi_l(\mathcal{R}) f \|_{L^p}^q \]

\[ = \sum_{l \geq 0} \max \{ 2^{\frac{\nu l q}{q} - \theta l q}, 2^l \} \| \psi_l(\mathcal{R}) f \|_{L^p}^q \]

\[ = \sum_{l \geq 0} \max \{ 2^{-\theta l q} (2^{\nu l q/\nu} + 1), 2^{\nu l (1-\theta)} (2^{\nu l q/\nu} + 1) \} 2^{q l} \| \psi_l(\mathcal{R}) f \|_{L^p}^q. \]

Taking into account that \( r_0 - r_1 + \nu = 0 \) we have

\[ \| f \|_{\delta,q}^q \lesssim \sum_{l \geq 0} 2^{q l} \| \psi_l(\mathcal{R}) f \|_{L^p}^q \equiv \| f \|_{\mathcal{E}^{r_0}_{p,q}}^q. \]  \hspace{1cm} (6.8)\]

Now, in order to proof the converse inequality we use the following estimate on the operator norm of \( \psi_l \) for \( l \) large enough: \( \| \psi_l(\mathcal{R}) \|_{L(L^p)} \lesssim 2^{-\frac{\nu l q}{p} (s - \frac{q}{2})} \) (c.f. equation (4.5)), we observe that by the Liouville theorem (see Geller [11] or [6, Section 3.2.8]),
\( \lambda = 0 \) is not a eigenvalue of \( \mathcal{R} \). So, we have,
\[
\|f\|_{\dot{B}^s_{p,q}}^q = \sum_{l \geq 0} \frac{2^{lq}}{2^{lp}} \|\psi_l(\mathcal{R}) f\|_{L^p}^q
= \sum_{l \geq 0} \frac{2^{lq}}{2^{lp}} \|\psi_l(\mathcal{R}) \mathcal{R}^{-\frac{r}{p}} \mathcal{R}^\frac{r}{q} f\|_{L^p}^q
= \sum_{l \geq 0} \frac{2^{lq}}{2^{lp}} \| \int_{2^{l-1}}^{2^{l+1}} \psi_l(\lambda) \lambda^{-\frac{r}{p}} dE_\lambda \mathcal{R}^\frac{r}{q} f\|_{L^p}^q
\leq \sum_{l \geq 0} \frac{2^{lq} - 2^{lq} - 2^{lq} (s - \frac{q}{p})}{2^{lp}} \| \mathcal{R}^\frac{r}{q} f\|_{L^p}^q \lesssim C_s \| \mathcal{R}^\frac{r}{q} f\|_{L^p}^q.
\]

Hence
\[
\|f\|_{\dot{B}^r_{p,q}} \lesssim C_s \|f\|_{\dot{H}^{r_0,p}}.
\]

In a similar way, if we assume \( s > \frac{q}{2} + r - r_0 \) we obtain
\[
\|f\|_{\dot{B}^r_{p,q}} \lesssim \|f\|_{\dot{H}^{r_0,p}}.
\]

So, we have the embedding \( \dot{H}^{r_1,p} \hookrightarrow \dot{B}^r_{p,q} \) for \( i = 0, 1 \). Hence \( (\dot{H}^{r_1,p}, \dot{H}^{r_0,p})_{\theta,q} \hookrightarrow \dot{B}^r_{p,q} \). So we conclude that \( \|f\|_{\dot{B}^r_{p,q}} \lesssim \|f\|_{\dot{H}^{r_0,p}} \). In the case where \( r < 0 \) we observe that
\[
(I + \mathcal{R})^{\frac{|r| - r}{2}} : \dot{B}^r_{p,q} \to \dot{B}^r_{p,q} \text{ is an isomorphism and for } 0 < r_0 < |r| < r_0 + \nu = r_1 \text{ we obtain,}
\]
\[
\dot{B}^r_{p,q} = (I + \mathcal{R})^{\frac{|r| - r}{2}} (\dot{E}|^{|r|}) = ((I + \mathcal{R})^{\frac{|r| - r}{2}} \dot{H}^{r_0,p}, (I + \mathcal{R})^{\frac{|r| - r}{2}} \dot{H}^{r_1,p})_{\theta,q}
= (\dot{H}^{r_0 + |r| - r, p}, \dot{H}^{r_1 + |r| - r, p})_{\theta,q},
\]
with \( |r| = r_1 + \theta(r_0 - r_1) \). The general case where \( a < r < b \) and \( r = b(1 - \theta) + a\theta \) now follows if we consider \( r_0 = r - \frac{\nu}{2}, r_1 = r + \frac{\nu}{2} \) and by observing that
\[
r = \frac{1}{2} r_1 + \frac{1}{2} r_2 = r_1 + \frac{1}{2} (r_2 - r_1).
\]
So we get
\[
(\dot{H}^{b,p}, \dot{H}^{a,p})_{\theta,q} = (\dot{H}^{r_1,p}, \dot{H}^{r_0,p})_{\frac{1}{2},q}.
\]

Since \( (\dot{H}^{r_1,p}, \dot{H}^{r_0,p})_{\frac{1}{2},q} = \dot{B}^r_{p,q} \) we conclude the proof of the homogeneous case. An analogous proof can be adapted to the inhomogeneous case.

\[\square\]

7. Localisation of Besov spaces on graded Lie groups

In this section we prove local embedding properties of Besov spaces \( \dot{B}^r_{p,q}(G) \) with the ones defined in a local way on \( \mathbb{R}^n \). First we recall the notion of Besov spaces on...
For $x,h \in \mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n)$, let us denote
\[
\Delta^m_h f(x) := \sum_{k=0}^{m} C_k^m (-1)^{m-k} f(x + kh) \quad (7.1)
\]
and
\[
\omega^m_p(t,f) := \sup_{|h| \leq t} \| \Delta^m_h f \|_{L^p(\mathbb{R}^n)} \quad (7.2)
\]
Then, by following \cite{19} for $r > 0$ and $1 \leq p,q \leq \infty$, the Euclidean Besov space $B^r_{p,q}(\mathbb{R}^n)$ can be considered endowed with the norm
\[
\|f\|_{B^r_{p,q}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \sum_{m=0}^{n} \left( \int_0^\infty (t^{-r} \omega^m_p(t,f))^q dt \right)^{\frac{1}{q}} \quad (7.3)
\]
for $q < \infty$, and with an obvious modification in the case $q = \infty$. By considering the property $(I - \mathcal{L})^{\frac{s}{2}} (B^r_{p,q}(\mathbb{R}^n)) = B^{r-s}_{p,q}(\mathbb{R}^n)$, where $\mathcal{L}$ is the Laplace operator on $\mathbb{R}^n$, for $r < 0$, we can consider on $B^r_{p,q}(\mathbb{R}^n)$, $1 \leq p,q \leq \infty$, the norm
\[
\|f\|_{B^r_{p,q}(\mathbb{R}^n)} = \|(I - \mathcal{L})^{-\frac{s}{2}} f\|_{B^{r+s}_{p,q}(\mathbb{R}^n)}, \quad (7.4)
\]
where $s$ is a fixed real satisfying $s + r > 0$. If we denote for a graded Lie group $G$ the localisation space by
\[
B^r_{p,q}(G,loc) = \{ f \in \mathcal{D}'(G) : \phi \cdot f \in B^r_{p,q}(G), \text{ for all } \phi \in C^\infty_0(G) \} \quad (7.5)
\]
we have the following result.

**Proposition 7.1.** If $B^r_{p,q}(G,loc)$ denotes the local Besov space defined above, then for all $r \in \mathbb{R}$, $1 < p < \infty$ and $0 < q \leq \infty$ we have
\[
B^r_{\nu_1}(G,loc) \subset B^r_{p,q}(\mathbb{R}^n,loc) \subset B^r_{\nu_n}(G,loc), \quad (7.6)
\]
where $\nu_1$ and $\nu_n$ are respectively the smallest and the largest weights of the dilations.

**Proof.** It was proved in \cite[Theorem 4.4.24]{6} that the following embedding of local Sobolev spaces holds:
\[
H^{s,p}(G,loc) \subset H^s_p(\mathbb{R}^n,loc) \subset H^{s-p}(G,loc), \quad (7.7)
\]
for all $s \in \mathbb{R}$. Thus, the result now follows by using real interpolation in the sense of Theorem 6.1. \hfill \Box

### 8. Fourier multipliers and spectral multipliers

In this section we give results for the boundedness of spectral and of Fourier multipliers in Besov spaces on graded Lie groups.
8.1. **Spectral multipliers and left-invariant operators.** We also want to study in the context of nilpotent and graded Lie groups necessary conditions, under which, a spectral multiplier or a Fourier multiplier extends to a bounded operator from $B_{p,q}^r(G)$ into $B_{p,q}^{r}(G)$.

We present a generalisation of a classical result by Marcinkiewicz, as an application of our Littlewood-Paley theorem, which asserts that if $m \in \text{BV}(\mathbb{R})$ is a bounded function which has uniformly bounded variation on every dyadic interval of $\mathbb{R}$: in symbols, if $\|m\|_{\text{BV}[a,b]}$ denotes the bounded variation norm on $[a,b]$; the preceding condition on $m$ can be written as

$$\sup_{j \in \mathbb{Z}} \|m\|_{\text{BV}[2^{j-1},2^j]} < \infty,$$

then $m$ is a multiplier on $L^p(\mathbb{R})$ for every $1 < p < \infty$. Now we present our version of the Marcinkiewicz theorem.

**Theorem 8.1.** Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. If $m$ is a bounded function which has uniformly bounded variation on every dyadic interval of $\mathbb{R}$, then $m(\mathcal{R})$ extends to a bounded spectral multiplier on $L^p(G)$ for every $1 < p < \infty$. Moreover, for every $0 < q \leq \infty$, and $r \in \mathbb{R}$, $m(\mathcal{R})$ is bounded on $B_{p,q}^r(G)$.

**Proof.** Let $T_l = m \cdot \psi_l(\mathcal{R})$, where $(\psi_l)_{l \in \mathbb{N}_0}$ is a dyadic decomposition as in (5.1). If $(E_\lambda)_{\lambda > 0}$ denotes the spectral resolution associated to $\mathcal{R}$, then for every $f \in C_0(G)$ we have, for $l \geq 1$,

$$T_l f = \int_{2^{l-1}}^{2^{l+1}} m(\lambda) \psi_l(\lambda) dE_\lambda f = \int_{2^{l-1}}^{2^{l+1}} (m(2^{l-1}) + \int_{2^{l-1}}^{\lambda} dm(t)) \psi_l(\lambda) dE_\lambda f$$

$$= m(2^{l-1}) \psi_l(\mathcal{R}) f + \int_{2^{l-1}}^{2^{l+1}} \psi_{l,t}(\mathcal{R}) f dm(t),$$

where $\psi_{l,t} = \psi_t|_{[2^{l-1},2^l]}$, $2^{l-1} \leq t \leq 2^{l+1}$. But then by (4.5) every multiplier $T_l$ is bounded on $L^p(G)$ and by Minkowski’s integral inequality every operator $T_l$ is bounded on $L^p(G)$ with bound of order $2^{s - \frac{s}{p} (s - \frac{Q}{2})}$, for any $s > \frac{Q}{2}$, multiplied with a factor depending only on the $L^\infty$ norm of the function $m$ and the uniform bound of the variation of $m$ on every dyadic interval. In fact, for $s > \frac{Q}{2} + \nu$ we have

$$\|T_l f\|_{L^p(G)} \leq \|m(2^{l-1}) \psi_l(\mathcal{R}) f\|_{L^p(G)} + \int_{2^{l-1}}^{2^{l+1}} \|\psi_{l,t}(\mathcal{R}) f\|_{L^p(G)} dm(t)$$

$$\leq \|m(2^{l-1})\| \|\psi_l(\mathcal{R}) f\|_{L^p(G)} + \int_{2^{l-1}}^{2^{l+1}} \|\psi_{l,t}(\mathcal{R}) f\|_{L^p(G)} dm(t)$$

$$\leq \|m\|_{L^\infty} 2^{-\frac{s}{p} (s - \frac{Q}{2})} \|f\|_{L^p(G)} + \int_{2^{l-1}}^{2^{l+1}} 2^{-\frac{s}{p} (s - \frac{Q}{2})} \|f\|_{L^p(G)} dm(t)$$

$$\leq (\|m\|_{L^\infty} + \sup_{l \geq 0} \|m\|_{\text{BV}[2^{l-1},2^{l+1}]}) 2^{-\frac{s}{p} (s - \frac{Q}{2})} \|f\|_{L^p(G)},$$

and consequently we obtain the following estimate for the $L^p$-operator norm of every $T_l$, $l \geq 0$:

$$\|T_l\|_{L^p(G)} \leq C_s 2^{-\frac{s}{p} (s - \frac{Q}{2})}.$$  \hspace{1cm} (8.1)
Hence, by using the Littlewood-Paley theorem, we obtain
\[
\|m(R)f\|_{L^p(G)} \lesssim \left\| \left( \sum_{l \geq 0} |\psi_l(R)m(R)f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)}
\]
\[
= \left\| \left( \sum_{l \geq 0} |m(R)\psi_l(R)f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)}
\]
\[
= \left\| \left( \sum_{l \geq 0} \left| T_l f(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \leq \left\| \sum_{l \geq 0} T_l f \right\|_{L^p(G)}
\]
\[
\leq \sum_{l \in \mathbb{N}_0} \|T_l f\|_{L^p(G)} \lesssim \left[ \sum_{l \in \mathbb{N}_0} 2^{-\frac{1}{2}(q-s-\frac{1}{q})} \right] \|f\|_{L^p(G)}
\]
\[
\equiv C_p \|f\|_{L^p(G)},
\]
which proves the boundedness of \(m(R)\) on \(L^p(G)\). Now, for the boundedness on \(B^r_{p,q}(G)\), \(0 < q < \infty\), we observe that
\[
\|m(R)f\|_{B^r_{p,q}(G)} = \left( \sum_{l \in \mathbb{N}_0} 2^{\frac{1}{2}r q} \|\psi_l(R)m(R)f\|_{L^p(G)}^q \right)^{\frac{1}{q}}
\]
\[
= \left( \sum_{l \in \mathbb{N}_0} 2^{\frac{1}{2}r q} \|m(R)\psi_l(R)f\|_{L^p(G)}^q \right)^{\frac{1}{q}}
\]
\[
\leq \left( \sum_{l \in \mathbb{N}_0} 2^{\frac{1}{2}r q} \|\psi_l(R)f\|_{L^p(G)}^q \right)^{\frac{1}{q}} \|m(R)\|_{L^p(G)}
\]
\[
= \|m(R)\|_{L^p(G)} \|f\|_{B^r_{p,q}(G)}.
\]
A proof for the case where \(q = \infty\) is similar. \(\square\)

**Theorem 8.2.** Let \(G\) be a graded Lie group and let \(T\) be a linear left-invariant operator bounded from \(B^r_{p,q}(G)\) (respectively, \(B^r_{p,q}(G)\)) into \(B^\tilde{r}_{\tilde{p},\tilde{q}}(G)\) (respectively, \(B^\tilde{r}_{\tilde{p},\tilde{q}}(G)\)), for \(1 \leq p, \tilde{p} < \infty\), \(-\infty < r, \tilde{r} < \infty\), and \(0 < q, \tilde{q} \leq \infty\). If \(1 \leq \tilde{p} < p < \infty\), then \(T = 0\).

**Proof.** Let \(|\cdot|\) be a homogeneous norm on \(G\). It is known (see [5, Lemma 3.2.5]) that
\[
\lim_{|h| \to \infty} \|f + \tau_h f\|_{L^p(G)} = 2^{\frac{1}{q}} \|f\|_{L^p(G)},
\]
where \(\tau_h\) is defined by \(\tau_h f(x) = f(hx)\), \(x, h \in G\). First, we will prove the case where \(0 < \tilde{q} < \infty\) in the inhomogeneous case. By the boundedness of \(T\) we have
\[
\|Tf\|_{B^\tilde{r}_{\tilde{p},\tilde{q}}(G)} \leq \|T\| \|f\|_{B^r_{p,q}(G)},
\]
where \(\|T\| = \|T\|_{B^{(r,p,q;\tilde{r},\tilde{p},\tilde{q})}}\) is the usual operator norm. So, for every \(h \in G\) we have
\[
\|T(f + \tau_h f)\|_{B^\tilde{r}_{\tilde{p},\tilde{q}}(G)} \leq C \|f + \tau_h f\|_{B^r_{p,q}(G)}.
\]
Now, we compute both sides as $|h| \to \infty$. We observe that
\[
\|T(f + \tau_f)\|_{B^p_q(G)} = \left( \sum_{l=0}^{\infty} 2^{\frac{l}{q'}} \|\psi_l(I + R)T(f + \tau_h f)\|_{L^{\hat{p}}(G)}^q \right)^{\frac{1}{q'}}
\]
\[
= \left( \sum_{l=0}^{\infty} 2^{\frac{l}{q'}} \|\psi_l(I + R)T f + \psi_l(I + R)T \tau_h f\|_{L^{\hat{p}}(G)}^q \right)^{\frac{1}{q'}}.
\]
Because, $T$ and $\psi_l(R)$, $l \in \mathbb{N}_0$, are left-invariant, we obtain
\[
\lim_{|h| \to \infty} \|\psi_l(I + R)T f + \psi_l(I + R)T \tau_h f\|_{L^{\hat{p}}(G)} = \lim_{|h| \to \infty} \|\psi_l(I + R)T f + \tau_h \psi_l(I + R)T f\|_{L^{\hat{p}}(G)}
\]
\[
= 2^\frac{1}{q'} \|\psi_l(I + R)T f\|_{L^{\hat{p}}(G)}.
\]
Hence
\[
\lim_{|h| \to \infty} \|T(f + \tau_h f)\|_{B^p_q(G)} = 2^\frac{1}{q'} \|T f\|_{B^p_q(G)}.\]

With a similar proof we obtain
\[
\lim_{|h| \to \infty} \|f\|_{B^p_q(G)} = 2^\frac{1}{q'} \|f\|_{B^p_q(G)}.
\]

Hence
\[
2^\frac{1}{q'} \|T f\|_{B^p_q(G)} \leq 2^\frac{1}{q'} \|T\| \|f\|_{B^p_q(G)}.
\]

The last inequality implies that $\|T\| \leq 2^\frac{1}{q' - \frac{1}{2}} \|T\|$. Thus, if $p > \hat{p}$ then $T$ is the null operator. The proof for $\hat{q} = \infty$ is analogous. \qed

8.2. Fourier multipliers. Throughout this subsection we consider (right) homogeneous and inhomogeneous Besov spaces. In order to introduce our main result of this subsection we consider the following remark on the commutativity of operators with spectral measures.

**Remark 8.3.** Let $R$ be a self-adjoint operator with spectral measure $E(\lambda)_{\lambda > 0}$. Then, the spectral theorem gives $R = \int \lambda dE(\lambda)$, and by the Stone's formula we have the following integral representation for every spectral projection $E(\lambda)$, (see Theorem 7.17 of [20])
\[
E(\lambda) = \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\lambda + \delta} ([t - i\varepsilon - R]^{-1} - [t + i\varepsilon - R]^{-1}) dt. \tag{8.2}
\]

If $T$ commutes with $R$, then $T$ commutes with its resolvent operator $(z - R)^{-1}$ and hence with its spectral measure $(E(\lambda))_{\lambda > 0}$. Now, if $f$ is a bounded continuous function on $[0, \infty)$ and
\[
f(R) = \int f(\lambda) dE(\lambda), \tag{8.3}
\]
then we can write
\[
f(R) = \lim_{\|P\| \to 0^+} \sum_{i=1, \lambda_i \in P}^{\infty} [f(\lambda_i)][E_{\lambda_i} - E_{\lambda_{i-1}}], \tag{8.4}
\]
where in the limit above, \( P = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \} \) denotes a partition of \([0, \infty)\). So, if \( T \) commutes with \( R \), then it also commutes with every bounded continuous function of \( R \) defined by the functional calculus.

Now we present the following theorem on Fourier multipliers in Besov spaces where we establish a connection between \( L^p \) boundedness and Besov continuity of Fourier multipliers.

**Theorem 8.4.** Let \( G \) be a graded Lie group. Let \( \sigma = \{\sigma(\pi) : \pi \in \hat{G}\} \) be a \( \mu \)-measurable field of operators in \( L^2(\hat{G}) \). Let us assume that the corresponding operator \( T = T_\sigma \), given by

\[
T_\sigma u(x) = \int_{\hat{G}} \text{Tr}(\pi(x)\sigma(\pi)\hat{f}(\pi))d\mu(\pi),
\]

is a bounded operator from \( L^{p_1}(G) \) into \( L^{p_2}(G) \), \( 1 \leq p_i \leq \infty \). Then \( T \) is a bounded operator from the (right) Besov space \( \dot{B}^{p_1,q}_{p_2,q}(G) \) into the (right) Besov space \( \dot{B}^{p_2,q}_{p_1,q}(G) \), for all \(-\infty < r < \infty \) and \( 0 < q \leq \infty \). Moreover, \( T \) is also a bounded operator from the (right) Besov space \( B^{p_1,q}_{p_2,q}(G) \) into the (right) Besov space \( B^{p_2,q}_{p_1,q}(G) \).

**Proof.** For \( f \in \mathcal{S}(G) \) we have \( \mathcal{F}(Tf)(\pi) = \sigma(\pi)\hat{f}(\pi) = \mathcal{F}(f * (\mathcal{F}^{-1}\sigma))(\pi) \). If \( \mathcal{R} \) is a right invariant positive Rockland operator, then for every \( a \in \mathbb{C} \) (see Proposition 4.4.30 of [6])

\[
\mathcal{R}^aTf = \mathcal{R}^a(f * \mathcal{F}^{-1}\sigma) = (\mathcal{R}^a f) * \mathcal{F}^{-1}\sigma = T(\mathcal{R}^a f),
\]

in particular \( T \) commutes with \( \mathcal{R} \). Since \( T \) commutes with \( \mathcal{R} \), it commutes with its spectral measures, and with every bounded function of \( \mathcal{R} \) defined by functional calculus (see, Remark 8.3). So,

\[
\|Tf\|_{\dot{B}^{p_2,q}_{p_1,q}(G)}^q = \sum_{l \in \mathbb{N}_0} 2^{\frac{ql}{r}} \|\psi_l(\mathcal{R})Tf\|_{L^{p_2}}^q
= \sum_{l \in \mathbb{N}_0} 2^{\frac{ql}{r}} \|T\psi_l(\mathcal{R})f\|_{L^{p_2}}^q
\leq \sum_{l \in \mathbb{N}_0} 2^{\frac{ql}{r}} \|T\|_{L(L^{p_1}(G),L^{p_2}(G))}^q \|\psi_l(\mathcal{R})f\|_{L^{p_1}}^q
= \|T\|_{L(L^{p_1}(G),L^{p_2}(G))}^q \|f\|_{\dot{B}^{p_2,q}_{p_1,q}(G)}^q.
\]

Thus \( \|Tf\|_{\dot{B}^{p_2,q}_{p_1,q}(G)}^q \leq \|T\|_{L(L^{p_1}(G),L^{p_2}(G))} \|f\|_{\dot{B}^{p_2,q}_{p_1,q}(G)}^q \). The proof for the inhomogeneous case is similar. So we end the proof. \( \square \)

We end this section with applications of Theorem 8.4 to some examples for the Fourier multipliers bounded on \( L^p \) and (right) Besov spaces. For notations and terminologies we follow [6].

**Example 8.5.** Let \( T : \mathcal{S}(G) \to \mathcal{S}(G) \), \( G \) be a graded Lie group of homogeneous dimension \( Q \). If \( T \) is left-invariant and homogeneous of degree \( \nu \) with

\[
-Q < \text{Re}(\nu) < 0,
\]

(8.6)
and such that the right convolution kernel of $T$ is continuous away from the origin, then $T : L^p(G) \to L^q(G)$ is a bounded operator for $1 < p, q < \infty$ and

$$\frac{1}{q} - \frac{1}{p} = \frac{\text{Re}(\nu)}{Q}.$$  

(c.f. Proposition 3.2.8 of [6, p. 138]). By Theorem 8.4, $T$ is a bounded operator from $B^r_{p,s}(G)$ into $B^r_{q,s}(G)$ with $p$ and $q$ satisfying (8.7), $r \in \mathbb{R}$ and $0 < s \leq \infty$.

**Example 8.6.** Let $T : L^2(G) \to L^2(G)$ be a bounded and left-invariant operator. Let us assume that its distributional kernel coincides on $G \setminus \{0\}$ with a continuously differentiable function $k$ with

$$\int_{|x| \geq \frac{1}{2}} |k(x)| dx \leq A < \infty, \quad \sup_{0 < |x| \leq 1} |x|^Q |k(x)| \leq A,$$

$$\sup_{0 < |x| \leq 1} |x|^{Q + v_j} |X_j k(x)| \leq A, \quad j = 1, 2, \ldots,$$

for some homogeneous quasi-norm $| \cdot |$ on $G$ and for some $A > 0$. Then $T$ is weak type $(1,1)$ and bounded on $L^p(G)$, $1 < p < \infty$, (c.f. [6, p. 145]). By using Theorem 8.4 we obtain the boundedness of $T$ on $B^r_{p,q}(G)$, $0 < q \leq \infty$ and $r \in \mathbb{R}$.

**Example 8.7.** Let $G$ be a graded Lie group. Let $\sigma \in L^2(\hat{G})$. If

$$\|\sigma\|_{H^s, l.u., L, \eta, R}, \|\sigma\|_{H^s, l.u., R, \eta, R} < \infty$$

with $s > \frac{Q}{2}$, then the corresponding multiplier $T_\sigma$ extends to a bounded operator on $L^p(G)$ for all $1 < p < \infty$. By Theorem 2.3 we have

$$\|T_\sigma\|_{L(L^p(G))} \leq C \max\{\|\sigma\|_{H^s, l.u., L, \eta, R}, \|\sigma\|_{H^s, l.u., R, \eta, R}\}.$$  

This is the Hörmander-Mihlin Theorem presented in [5]. By Theorem 8.4, we obtain the boundedness of $T_\sigma$ on $B^r_{p,q}(G)$ and by observing the proof of such theorem we conclude that

$$\|T_\sigma\|_{L(B^r_{p,q}(G))} \leq C \max\{\|\sigma\|_{H^s, l.u., L, \eta, R}, \|\sigma\|_{H^s, l.u., R, \eta, R}\}.  \quad (8.9)$$

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