ON THE CONNECTIVITY OF THE HYPERBOLICITY REGION
OF IRREDUCIBLE POLYNOMIALS

MARIO KUMMER

Abstract. We give an elementary proof for the fact that an irreducible hyperbolic polynomial has only one pair of hyperbolicity cones.

1. Introduction

A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is hyperbolic with respect to $e \in \mathbb{R}^{n+1}$ if the univariate polynomial $h(te + v)$ has only real roots for all $v \in \mathbb{R}^n$ and if $h(e) \neq 0$. The notion of hyperbolicity is transferred to the projective zero locus of $h$ in the obvious way. Hyperbolic polynomials were first studied in the context of partial differential equations but later they appeared also in many other areas like combinatorics and optimization. For a given homogeneous polynomial $h$ one can consider the set $\Lambda_h$ of all points with respect to which $h$ is hyperbolic. Lars Garding [Gar59] showed that $\Lambda_h$ is the union of several connected components of $\{a \in \mathbb{R}^n : h(a) \neq 0\}$ all of which are convex cones. Interest in these convex cones comes, among others, from convex optimization since these are the feasible sets of hyperbolic programs [Güll97]. It follows directly from the definitions that we have $-\Lambda_h = \Lambda_h$. In particular, connected components of $\Lambda_h$ come in pairs. Under the assumption that $\nabla h(a) \neq 0$ for all $0 \neq a \in \mathbb{R}^n$ there is at most one such pair by [HV07, Thm. 5.2]. In general, $\Lambda_h$ can have many connected components. For example a product of linear forms is hyperbolic with respect to any point where it is not zero. Another example is the quartic polynomial $p = (x_1^2 + x_2^2 - 2 \cdot x_1^2) \cdot (2 \cdot x_1^2 - x_2^2 - x_3^2)$ which has no linear factor. One easily checks that $\Lambda_p$ has four connected components. It seems to be consensus among experts that if $h$ is irreducible, then the set $\Lambda_h$ should have at most one pair of connected components. However, we are not aware of any proof for that statement in the literature. This note is intended to give a reference (with proof) for that statement. We were originally motivated by the questions considered in [JT17].

The above example of the quartic polynomial $p$ without a linear factor suggests that there might be no easy proof for this statement only relying on convex geometric arguments since all connected components of $\Lambda_p$ are strictly convex. Our proof will rely on elementary properties of real algebraic curves and the theorem of Bertini.

2. Real projective curves

Let $X$ be a smooth projective and geometrically irreducible curve defined over the reals. It is a well-known fact going back to Felix Klein [Kle63, §21] that the set $X(\mathbb{C}) \setminus X(\mathbb{R})$ has either one or two connected components. If it has two connected components, say $X_+$ and $X_-$, then $X$ is called a separating curve. In that case, an orientation on $X_+$ induces an orientation on $X(\mathbb{R})$. Up to global reversing, this orientation does not depend on the choice of the component and the orientation on it. It is called the complex orientation on $X(\mathbb{R})$ and was introduced by V. A. Rokhlin [Roh78, §2.1] in order to study the topology of real plane curves.
Let \( f : X \to \mathbb{P}^1 \) be a surjective morphism. We fix one of the two orientations on \( \mathbb{P}^1(\mathbb{R}) \). On every point \( p \in X(\mathbb{R}) \) where \( f \) is unramified the fixed orientation on \( \mathbb{P}^1(\mathbb{R}) \) induces an orientation of \( X(\mathbb{R}) \) locally around \( p \). Again, up to global reversing, this orientation does not depend on the choice of the orientation on \( \mathbb{P}^1(\mathbb{R}) \). We call it the orientation induced by \( f \) on \( X(\mathbb{R}) \).

Now assume that \( f \) has the property \( f^{-1}(\mathbb{P}^1(\mathbb{R})) = X(\mathbb{R}) \). Then, since \( \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \) is not connected, the set \( X(\mathbb{C}) \setminus X(\mathbb{R}) \) is also not connected. It consists therefore of two connected components \( X_+ \) and \( X_- \) which are the preimages under \( f \) of the upper and lower half-planes respectively, cf. [Roh78 §3.6]. We call \( f \) a separating morphism because it is a certificate for \( X \) being separating. Note that in that case \( f \) is unramified on all of \( X(\mathbb{R}) \) (see e.g. [Gab06 Rem. 3.2]) and the orientation induced by \( f \) is the same as the complex orientation.

3. Plane hyperbolic curves

Now let \( Y \) be a possibly singular projective and geometrically irreducible curve defined over the reals. As above we call a morphism \( f : Y \to \mathbb{P}^1 \) separating if \( f^{-1}(\mathbb{P}^1(\mathbb{R})) = Y(\mathbb{R}) \). Again this induces an orientation on the smooth points of \( Y(\mathbb{R}) \). On the other hand, the desingularization map \( g : \tilde{Y} \to Y \) together with the complex orientation on \( \tilde{Y}(\mathbb{R}) \) also induces an orientation on the smooth points of \( Y(\mathbb{R}) \). Up to global reversing, these two orientations are the same. Indeed, \( g \) is an isomorphism on the smooth points of \( Y \) and the morphism \( f \circ g \) is also separating. This gives us the following lemma.

**Lemma.** Two separating morphisms \( f_1, f_2 \) from \( Y \) induce the same orientations on the smooth points of \( Y(\mathbb{R}) \) — up to global reversing.

Let \( C \subseteq \mathbb{P}^2 \) be an irreducible projective plane curve defined over the reals. For every point \( e \in \mathbb{P}^2(\mathbb{R}) \) we consider the linear projection \( \pi_e : C \setminus \{ e \} \to L \) where \( L \subseteq \mathbb{P}^1 \) is a line not containing \( e \). Let \( p \in C(\mathbb{R}) \) and \( e_1, e_2 \in \mathbb{P}^2(\mathbb{R}) \) be two points not lying on the tangent space \( T \) of \( C \) at \( p \). Let \( L \subseteq \mathbb{P}^2 \) be a line not containing \( e_1 \) and \( e_2 \) and fix an orientation on \( L(\mathbb{R}) \). We observe that the orientation on \( C(\mathbb{R}) \) at \( p \) induced by \( \pi_{e_1} \) is the same as the one induced by \( \pi_{e_2} \) if and only if \( e_1 \) and \( e_2 \) are in the same connected component of \( \mathbb{P}^2(\mathbb{R}) \setminus (L \cup T) \). Now assume that \( C \) is hyperbolic, i.e. the set

\[ \Lambda_C = \{ e \in \mathbb{P}^2(\mathbb{R}) : C \text{ is hyperbolic with respect to } e \} \]

is not empty. Then for each \( e \in \Lambda_C \), the projection \( \pi_e : C \to \mathbb{P}^1 \) is separating. Since a separating morphism is not ramified at smooth real points, this means that \( \Lambda_C \) does not intersect any line that is tangent to \( C \) at a smooth real point.

**Proposition.** In the above situation the set \( \Lambda_C \) is connected.

**Proof.** Assume for the sake of a contradiction that \( \Lambda \) is not connected. Let \( \Lambda_1 \) and \( \Lambda_2 \) be two connected components of \( \Lambda \) and let \( e_i \in \Lambda_i \) for \( i = 1, 2 \). Let \( p_1 \) be a smooth point of \( C(\mathbb{R}) \) which is in the closure of \( \Lambda_1 \) and let \( T_1 \) be the tangent of \( C \) at \( p_1 \). Then by the above discussion we have \( \Lambda \subseteq \mathbb{P}^2(\mathbb{R}) \setminus T_1 \cong \mathbb{R}^2 \). Thus, \( \Lambda_1 \) and \( \Lambda_2 \) are two disjoint convex open subsets of \( \mathbb{R}^2 \). We can find a line \( G \) through \( e_1 \) that intersects \( \Lambda_2 \) and which intersects \( C \) transversally and only in smooth points. The linear polynomial that defines a tangent \( T_2 \) to \( C \) at one of the two intersection points of \( G \) with the boundary of \( \Lambda_2 \), lets denote it it by \( p_2 \), will separate \( e_1 \) from \( \Lambda_2 \). Thus \( e_1 \) and \( e_2 \) are in distinct connected components of \( \mathbb{P}^2(\mathbb{R}) \setminus (T_1 \cup T_2) \). Now we compare the orientations at \( p_1 \) and \( p_2 \) induced by the two morphisms \( \pi_{e_i} : C \to T_i \). By the above discussion we see that these induce different orientations at \( p_2 \) but the same at \( p_1 \). But this is a contradiction to the conclusion of the lemma. \( \square \)
4. The general case

Now the general case follows from a simple application of the theorem of Bertini.

**Theorem.** Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be irreducible and hyperbolic. If $n > 2$, the set $\Lambda_h$ of all points with respect to which $h$ is hyperbolic has two connected components.

**Proof.** Let $V \subseteq \mathbb{P}^{n-1}$ be the hypersurface defined by $h$. Assume for the sake of a contradiction that

$$\Lambda_V = \{ e \in \mathbb{P}^{n-1}(\mathbb{R}) : V \text{ is hyperbolic with respect to } e \}$$

is not connected. Then we can find a real linear subspace $E \subseteq \mathbb{P}^{n-1}$ of dimension two that intersects two different connected components of $\Lambda_V$ such that $E \cap V$ is irreducible [Jou83, Cor. 6.11]. But this contradicts the proposition which implies that $\Lambda_V \cap E$ is connected. \hfill \Box

**Acknowledgements.** I would like to thank Daniel Plaumann, who encouraged me to write this note, and Eli Shamovich for some comments.

**References**

[Gab06] Alexandre Gabard. Sur la représentation conforme des surfaces de Riemann à bord et une caractérisation des courbes séparantes. *Comment. Math. Helv.*, 81(4):945–964, 2006.

[Gär59] Lars Garding. An inequality for hyperbolic polynomials. *J. Math. Mech.*, 8:957–965, 1959.

[Gül97] Osman Güler. Hyperbolic polynomials and interior point methods for convex programming. *Math. Oper. Res.*, 22(2):350–377, 1997.

[HV07] J. William Helton and Victor Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, 60(5):654–674, 2007.

[Jou83] Jean-Pierre Jouanolou. Théorèmes de Bertini et applications, volume 42 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1983.

[JT17] Thorsten Jörgens and Thorsten Theobald. Hyperbolicity cones and imaginary projections. *arXiv preprint arXiv:1703.04988*, 2017.

[Kle63] Felix Klein. *On Riemann’s theory of algebraic functions and their integrals. A supplement to the usual treatises*. Translated from the German by Frances Hardcastle. Dover Publications, Inc., New York, 1963.

[Roh78] V. A. Rohlin. Complex topological characteristics of real algebraic curves. *Uspekhi Mat. Nauk*, 33(5(203)):77–89, 237, 1978.

Max-Planck-Institute for Mathematics in the Sciences, Leipzig, Germany

E-mail address: kummer@mis.mpg.de