DC Semidefinite programming and cone constrained DC optimization I: theory

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Abstract
In this two-part study, we discuss possible extensions of the main ideas and methods of constrained DC optimization to the case of nonlinear semidefinite programming problems and more general nonlinear cone constrained optimization problems. In the first paper, we analyse two different approaches to the definition of DC matrix-valued functions (namely, order-theoretic and componentwise), study some properties of convex and DC matrix-valued mappings and demonstrate how to compute DC decompositions of some nonlinear semidefinite constraints appearing in applications. We also compute a DC decomposition of the maximal eigenvalue of a DC matrix-valued function. This DC decomposition can be used to reformulate DC semidefinite constraints as DC inequality constrains. Finally, we study local optimality conditions for general cone constrained DC optimization problems.

Keywords DC optimization · Semidefinite programming · DC decomposition · Cone constrained optimization

Mathematics Subject Classification 90C22 · 90C26

1 Introduction
Starting with the pioneering works of Hiriart-Urruty [21, 22], Pham Dinh and Souad [49], Strekalovsky [54], Tuy [63], and many others in the 1980s, DC (Difference of Convex functions) programming has been an active area of research in nonlinear nonconvex optimization. One of the main features of DC optimization problems is
the fact that one can derive constructive global optimality conditions [11, 23, 55, 65, 73] and develop deterministic global optimization methods [13, 24, 37, 57, 58, 64] for this class of problems. Local search methods for minimizing DC functions have also attracted a considerable attention of researchers (see [15, 26, 56, 61] and the references therein).

Perhaps, the most efficient and well-known numerical method for DC optimization problems is the so-called DCA, originally presented by Pham Dinh and Souad [49] and later on thoroughly investigated in the works of Le Thi and Pham Dinh et al. [36, 38, 39, 47, 48] (a particular version of the DCA is sometimes called the concave-convex/convex-concave procedure [32, 72]). Some closely related local search methods were studied in the works of de Oliveira et al. [9, 10, 66–69]. For a detailed survey on DC programming, DCA, and their applications see [34, 35, 46]. A comprehensive literature review of the DCA, the convex-concave procedure, and other related optimization methods can be found in [40].

Cone constrained optimization is one the central areas of constrained optimization, since it provides a unified setting for many different problems appearing in applications. Standard equality and inequality constrained problems, semidefinite programming problems [30, 53, 60], second order cone programming problems [2], semi-infinite programming problems [16, 50], and many other particular problems (see, e.g. [3, 6, 43]) can be formulated as general cone constrained optimization problems.

A detailed theoretical analysis of smooth and nonsmooth cone constrained optimization problems was presented in [5, 14, 28, 42, 62, 74]. Optimization methods for solving various convex cone constrained optimization problems can be found in [3, 6, 43], while algorithms for solving various classes of smooth nonconvex cone constrained optimization problems were developed, e.g. in [7, 29, 30, 53, 70, 71] (see also the references therein).

Despite the abundance of publications on cone constrained optimization and (usually inequality) constrained DC optimization problems, very little attention has been paid to extensions of the main results and methods of DC optimization to the case of problems with cone constraints. Even in the comprehensive survey paper [35], only unconstrained and inequality constrained DC optimization problems are discussed.

The convex-concave procedure and the penalty convex-concave procedure for solving cone constrained DC optimization problems were proposed by Lipp and Boyd [40], where an application of these methods to multi-matrix principal component analysis was presented. However, to the best of the author’s knowledge, a convergence analysis of these methods remains an open problem. An application of the DCA to bilinear and quadratic matrix inequality feasibility problems was considered by Niu and Dinh [44]. Finally, optimality conditions for DC semi-infinite programming problems were studied in the recent paper [8].

The main goal of this paper is to fill in the gap and extend some of the main results and methods of inequality constrained DC optimization (such as the DCA) to the case of DC optimization problems with DC cone constraints, particularly, DC semidefinite programming problems. The motivation behind this extension is connected to the fact that the DC optimization approach allows one to develop general
methods for solving nonsmooth cone constrained optimization problems, as well as extend global DC optimization methods to the case of such problems. Furthermore, the nonlocal nature of the DCA (the method uses global majorants of the objective function and constraints) in some cases allows this algorithm to find better local solutions than traditional optimization methods. This peculiarity makes the DCA a potentially appealing alternative to existing methods for solving cone constrained optimization problems (see [40] for some promising results of numerical experiments).

In the first part of our study, we present a detailed discussion of two different approaches to the definition of DC matrix-valued mappings: order-theoretic and componentwise. We obtain several useful properties of convex and DC matrix-valued functions, prove that any DC (in the order-theoretic sense) matrix-valued map is necessarily componentwise DC, and demonstrate how one can compute DC decompositions of several nonlinear matrix-valued functions appearing in applications. We also construct a DC decomposition of the maximal eigenvalue of a componentwise DC matrix-valued mapping. This result allows one to easily extend all ideas and methods of inequality constrained DC optimization to the case of DC optimization problems with componentwise DC semidefinite constraints. Finally, we also derive local optimality conditions for general cone constrained DC optimization problems in several different forms.

The second part of our study contains a detailed convergence analysis of the algorithms for solving cone constrained DC optimization problems proposed in [40], thus providing a theoretical foundation for applications of the methods from [40]. Namely, we prove a global convergence of the DCA from [40] to a critical point of the problem under consideration and present a comprehensive analysis of a penalized version of this method. We obtain sufficient conditions for the exactness of the penalty subproblem, establish a global convergence of the exact penalty DCA to generalized critical points, and provide two types of sufficient conditions for a convergence of the exact penalty DCA to a feasible and critical point of a cone constrained DC optimization problem from an infeasible starting point. We also discuss why the exact penalty DCA might be superior to the non-penalized version of this method in the case when the feasible starting point is known.

The paper is organized as follows. Order-theoretic and componentwise approaches to DC matrix valued functions are studied in Sect. 2, while a DC structure of the maximal eigenvalue of a nonlinear matrix-valued mapping is discussed in Sect. 3. Finally, Sect. 4 is devoted to the derivation of local optimality conditions for general cone constrained DC optimization problems.

2 Two approaches to DC matrix-valued functions

Denote by $\mathbb{S}^\ell$ the space of all real symmetric matrices of order $\ell \in \mathbb{N}$, and let $\preceq$ be the Löwner partial order on $\mathbb{S}^\ell$, i.e. $A \preceq B$ for some matrices $A, B \in \mathbb{S}^\ell$ if and only if the matrix $B - A$ is positive semidefinite. Nonlinear semidefinite optimization is concerned with problems of minimizing functions subject to constraints of the form $F(x) \preceq 0$, where $F : \mathbb{R}^d \to \mathbb{S}^\ell$ is a given nonlinear mapping.
To extend the main ideas and results of DC optimization to the case of nonlinear semidefinite programming problems, first one must introduce a suitable definition of a DC matrix-valued mapping $F$. There are two possible approaches to this definition: order-theoretic and componentwise. Let us discuss and compare these approaches.

Recall that the matrix-valued function $F$ is called convex (see, e.g. [5, Sect. 5.3.2] and [6, Sect. 3.6.2]), if

$$F(\alpha x_1 + (1 - \alpha)x_2) \preceq \alpha F(x_1) + (1 - \alpha)F(x_2) \quad \forall x_1, x_2 \in \mathbb{R}^d, \alpha \in [0, 1].$$

Therefore it is natural to call the function $F$ DC (Difference-of-Convex), if there exist convex mappings $G, H : \mathbb{R}^d \to \mathcal{S}$ such that $F = G - H$. Any such representation of the function $F$ (or, equivalently, any such pair of functions $(G, H)$) is called a DC decomposition of $F$.

The definition of matrix-valued DC mapping given above has several disadvantages. Firstly, the convexity of matrix-valued functions is much harder to verify than the convexity of real-valued functions. Many matrix-valued mappings that might seem to be convex judging by the experience with the real-valued case are, in actuality, nonconvex. In particular, the convexity of each component $F_{ij} \quad (\cdot)$ of $F$ is not sufficient to ensure the matrix convexity of $F$.

**Example 1** Let $d = 1, \ell = 2$, and $F(x) = \begin{pmatrix} 1 & x^2 \\ x^2 & 1 \end{pmatrix}$. Then for $x_1 = 1$ and $x_2 = -1$ one has

$$\alpha F(x_1) + (1 - \alpha)F(x_2) - F(\alpha x_1 + (1 - \alpha)x_2) = \begin{pmatrix} 0 & 1 - (2\alpha - 1)^2 \\ 1 - (2\alpha - 1)^2 & 0 \end{pmatrix}.$$

This matrix is not positive semidefinite for any $\alpha \in (0, 1)$, which implies that the map $F$ is nonconvex.

Secondly, recall that the set $\mathcal{S}$ equipped with the Löwen partial order is not a vector lattice, since by Kadison’s theorem [27] the least upper bound (the supremum) of two matrices in the Löwen order exists if and only if these matrices are comparable. Therefore, many standard results and techniques from convex analysis do not admit a direct extension to the case of matrix convexity (cf. the general theory of convex vector-valued maps [31, 45, 59], in which the assumption on the completeness of partial order is often indispensable). For example, in most cases the supremum of two convex matrix-valued functions is not correctly defined.

Nevertheless, there are some similarities between matrix-valued DC mappings and their real-valued counterparts. In particular, one can construct a DC decomposition of a twice continuously differentiable matrix-valued map with bounded Hessian in the same way one can construct DC decomposition of a twice continuously differentiable real-valued function.

Let $I_{\ell}$ be the identity matrix of order $\ell$. Denote by $| \cdot |$ the Euclidean norm, by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^k$, and by $\|A\|_F = \sqrt{\text{Tr} (A^T A)}$ the Frobenius norm of a real matrix $A$, where $\text{Tr} (\cdot)$ is the trace of a square matrix.
Theorem 1 Let a map $F : \mathbb{R}^d \to \mathbb{S}^\ell$ be twice continuously differentiable and suppose that there exists $M > 0$ such that $\|\nabla^2 F_{ij}(x)\|_F \leq M$ for all $i, j \in \{1, \ldots, \ell\}$. Then the mapping $F$ is DC and for any $\mu \geq \ell M$ both pairs $(G_k, H_k), k \in \{1, 2\}$, with

$$G_1(x) = F(x) + \frac{\mu}{2} |x|^2 I_\ell, \quad H_1(x) = \frac{\mu}{2} |x|^2 I_\ell,$$

and

$$G_2(x) = \frac{\mu}{2} |x|^2 I_\ell, \quad H_2(x) = \frac{\mu}{2} |x|^2 I_\ell - F(x) \quad \forall x \in \mathbb{R}^d,$$

are DC decompositions of $F$.

Proof Observe that by the definitions of matrix convexity and the Löewner partial order, a mapping $G : \mathbb{R}^d \to \mathbb{S}^\ell$ is convex if and only if for any $z \in \mathbb{R}^\ell$ one has

$$\langle z, (\alpha G(x_1) + (1 - \alpha)G(x_2) - G(\alpha x_1 + (1 - \alpha)x_2))z \rangle \geq 0$$

for all $x_1, x_2 \in \mathbb{R}^d$ and $\alpha \in [0, 1]$ or, equivalently,

$$\langle z, G(\alpha x_1 + (1 - \alpha)x_2)z \rangle \leq \alpha \langle z, G(x_1)z \rangle + (1 - \alpha)\langle z, G(x_2)z \rangle.$$

Therefore, a map $G : \mathbb{R}^d \to \mathbb{S}^\ell$ is convex if and only if for any $z \in \mathbb{R}^\ell$ the real-valued function $G_z(\cdot) = \langle z, G(\cdot)z \rangle$ is convex. Consequently, in the case when $G$ is twice continuously differentiable, this function is convex if and only if for any $z$ the Hessian of the function $G_z$ is positive semidefinite, i.e. for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^\ell$ the matrix

$$\nabla^2 G_z(x) = \sum_{i,j=1}^\ell z_i z_j \nabla^2 G_{ij}(x)$$

is positive semidefinite.

Let us now turn to the proof of the theorem. Denote $G(x) = F(x) + \frac{\mu}{2} |x|^2 I_\ell$ and $H(x) = \frac{\mu}{2} |x|^2 I_\ell$. Let us check that the mappings $G$ and $H$ are convex, provided $\mu \geq \ell M$. Then one can conclude that $F$ is a DC function and the pair $(G_1, H_1)$ from the formulation of the theorem is a DC decomposition of $F$. The fact that the pair $(G_2, H_2)$ is also a DC decomposition of $F$ can be proved in a similar way.

Indeed, for any $x \in \mathbb{R}^d$ and $v, z \in \mathbb{R}^\ell$ one has

$$\langle v, \nabla^2 G_z(x)v \rangle = \sum_{i,j=1}^\ell z_i z_j \langle v, \nabla^2 F_{ij}(x)v \rangle + \mu \sum_{i=1}^\ell z_i^2 |v_i|^2 \quad (1)$$

Let us estimate the first term on the right-hand side of this equality. Indeed, applying the obvious inequality $2|z_i z_j| \leq z_i^2 + z_j^2$, one gets
By the Cauchy–Bunyakovsky–Schwarz inequality and the fact that the Frobenius norm is compatible with the Euclidean norm one has

\[
\sum_{i,j=1}^{\ell} z_i z_j \langle v, \nabla^2 F_{ij}(x)v \rangle \geq - \sum_{i,j=1}^{\ell} |z_i z_j| |\langle v, \nabla^2 F_{ij}(x)v \rangle| \geq - \frac{1}{2} \sum_{i,j=1}^{\ell} (z_i^2 + z_j^2) |\langle v, \nabla^2 F_{ij}(x)v \rangle|.
\]

which implies that

\[
\sum_{i,j=1}^{\ell} z_i z_j \langle v, \nabla^2 F_{ij}(x)v \rangle \geq - \frac{|v|^2}{2} \sum_{i,j=1}^{\ell} (z_i^2 + z_j^2) \|\nabla^2 F_{ij}(x)\|_F
\]

\[
= - \frac{|v|^2}{2} \sum_{i=1}^{\ell} 2z_i^2 \left( \sum_{j=1}^{\ell} \|\nabla^2 F_{ij}(x)\|_F \right),
\]

where the last equality follows from the fact that the matrix \( F(x) \) is by definition symmetric. Combining this inequality with (1), one finally obtains that

\[
\langle v, \nabla^2 G_{\ell}(x)v \rangle \geq |v|^2 \sum_{i=1}^{\ell} \left( \mu - \sum_{j=1}^{\ell} \|\nabla^2 F_{ij}(x)\|_F \right) z_i^2.
\]

Hence for any \( x \in \mathbb{R}^d, \ z \in \mathbb{R}^\ell, \) and \( \mu \geq \ell M \) one has

\[
\langle v, \nabla^2 G_{\ell}(x)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^\ell,
\]

that is, the Hessian \( \nabla^2 G_{\ell}(x) \) is positive semidefinite. Thus, one can conclude that the matrix-valued mapping \( G(x) = F(x) + \frac{\mu}{2} |x|^2 I_\ell \) is convex. The convexity of \( H \) can be readily verified directly.

The difficulties connected with the use of matrix convexity motivate us to consider a different approach to the definition of DC matrix-valued mappings.

**Definition 1** A function \( F : \mathbb{R}^d \to \mathbb{S}^\ell \) is called componentwise convex, if each component \( F_{ij}(\cdot), \ i,j \in \{1, \ldots, \ell\}, \) is convex. The function \( F \) is called componentwise DC, if there exist componentwise convex functions \( G, H : \mathbb{R}^d \to \mathbb{S}^\ell \) such that \( F = G - H. \) Any such representation of \( F \) (or, equivalently, any such pair of functions \((G, H)\)) is called a componentwise DC decomposition of \( F. \)

Many properties of real-valued DC functions can be easily extended to the case of componentwise DC matrix-valued mappings. For example, a linear combination of componentwise DC mappings is obviously componentwise DC. With the
use of the well-known results of Hartman [19], one can easily see that the Hadamard and the Kronecker products of componentwise DC matrix-valued mappings are componentwise DC, etc.

Let us point out some connections between convex/DC and componentwise convex/DC matrix-valued mappings. As Example 1 demonstrates, componentwise convex matrix-valued functions need not be convex. On the other hand, from the fact that for any convex matrix-valued map $F$, the real-valued function $\langle z, F(\cdot)z \rangle$ is convex for all $z \in \mathbb{R}^\ell$, it follows that all diagonal components $F_{ii}(\cdot)$ of a convex matrix-valued map $F$ must be convex (put $z = e_i$ for every vector $e_i$ from the canonical basis of $\mathbb{R}^\ell$). However, non-diagonal components of $F$ need not be convex.

**Example 2** Let $d = 1$, $\ell = 2$, and $F(x) = \begin{pmatrix} 0.5x^2 \sin x \\ \sin x \ 0.5x^2 \end{pmatrix}$. Then for all $z \in \mathbb{R}^2$ and $x \in \mathbb{R}$ one has

$$\sum_{i,j=1}^{2} z_i z_j \nabla^2 F_{ij}(x) = z_1^2 - 2\sin x z_1 z_2 + z_2^2 \geq z_1^2 - 2|z_1||z_2| + z_2^2$$

$$= (|z_1| - |z_2|)^2 \geq 0.$$ 

Consequently, the function $F$ is convex by [5, Proposition 5.72, part (ii)], despite the fact that non-diagonal elements of $F$ are nonconvex.

Although non-diagonal elements of a convex matrix-valued mapping $F$ might be nonconvex, they cannot be too ‘wild’, e.g. discontinuous. Namely, the following result holds true.

**Theorem 2** Let a map $F : \mathbb{R}^d \to \mathbb{S}^\ell$ be convex. Then for all $i, j \in \{1, \ldots, \ell\}$, $i \neq j$, the function $F_{ij}$ is DC and, therefore, Lipschitz continuous on any bounded set and twice differentiable almost everywhere.

**Proof** We prove the theorem by induction in $\ell$. The case $\ell = 1$ is trivial. Let us prove the case $\ell = 2$ in order to highlight the main idea of the proof.

As was noted above, the function $\langle z, F(\cdot)z \rangle$ is convex for all $z \in \mathbb{R}^\ell$, which, in particular, implies that the functions $F_{11}(\cdot)$ and $F_{22}(\cdot)$ are convex. For the vector $z = (1, 1)^T$ one obtains that the function

$$F_z(x) = \langle z, F(x)z \rangle = F_{11}(x) + 2F_{12}(x) + F_{22}(x), \quad x \in \mathbb{R}^d$$

is convex as well. Therefore the function

$$F_{12}(x) = F_{21}(x) = \frac{1}{2}F_z(x) - \frac{1}{2}(F_{11}(x) + F_{22}(x))$$

is DC, which completes the proof of the case $\ell = 2$.

**Inductive step** Suppose that the theorem is valid for some $\ell \in \mathbb{N}$. Let us prove it for $\ell + 1$. The function $F_z(\cdot) = \langle z, F(\cdot)z \rangle$ is convex for all $z \in \mathbb{R}^{\ell+1}$. Putting
\( z = (z_1, \ldots, z_\ell, 0)^T \) and \( z = (0, z_2, \ldots, z_{\ell+1})^T \) for any \( z_i \in \mathbb{R}, \ i \in \{1, \ldots, \ell + 1\} \) one obtains that the matrix-valued mappings

\[
G(x) = \begin{pmatrix} F_{11}(x) & \cdots & F_{1\ell}(x) \\ \vdots & \ddots & \vdots \\ F_{\ell1}(x) & \cdots & F_{\ell\ell}(x) \end{pmatrix}, \quad H(x) = \begin{pmatrix} F_{22}(x) & \cdots & F_{2(\ell+1)}(x) \\ \vdots & \ddots & \vdots \\ F_{(\ell+1)2}(x) & \cdots & F_{(\ell+1)(\ell+1)}(x) \end{pmatrix}
\]

are convex. Therefore, by the induction hypothesis all functions \( F_{ij}, i, j \in \{1, \ldots, \ell + 1\} \) are DC, except for \( F_{1(\ell+1)} \) (or, equivalently, \( F_{(\ell+1)1} \), since \( F(x) \) is by definition a symmetric matrix).

For \( z = (1, \ldots, 1)^T \) one gets that the function

\[
F_z(x) = \sum_{i,j=1}^{\ell+1} F_{ij}(x), \quad x \in \mathbb{R}^d
\]

is convex, which obviously implies that the function \( F_{1(\ell+1)} \) is DC.

Finally, taking into account the fact that finite-valued convex functions are Lipschitz continuous on bounded sets [52, Thm. 10.4] and twice differentiable almost everywhere by the Busemann–Feller–Aleksandrov theorem (see, e.g. [4]), one can conclude that for all \( i, j \in \{1, \ldots, \ell\} \) the functions \( F_{ij} \) are Lipschitz continuous on bounded sets and twice differentiable almost everywhere.

As simple corollaries to the previous theorem we obtain straightforward extensions of some well-known results for real-valued convex function to the matrix-valued case.

**Corollary 1** Let a map \( F : \mathbb{R}^d \to \mathbb{S}^\ell \) be convex. Then \( F \) is Lipschitz continuous on bounded sets, i.e. for any bounded set \( K \subset \mathbb{R}^d \) there exists \( L > 0 \) such that \( \|F(x_1) - F(x_2)\|_F \leq L|x_1 - x_2| \) for all \( x_1, x_2 \in K \).

**Corollary 2** (Busemann–Feller–Aleksandrov theorem for matrix-valued functions)

Let a map \( F : \mathbb{R}^d \to \mathbb{S}^\ell \) be convex. Then \( F \) is twice differentiable almost everywhere.

**Remark 1** Note that the statement of Theorem 2 is obviously true for locally convex (i.e. convex in a neighbourhood of every point) matrix-valued mappings defined on not necessarily convex sets. Therefore, the previous corollary remains true in this case as well. Namely, every locally convex map \( F : U \to \mathbb{S}^\ell \) defined on an open set \( U \subset \mathbb{R}^d \) is twice differentiable almost everywhere on \( U \).

Since the difference of two real-valued DC functions is a DC function, Theorem 2 also allows one to point out a direct connection between DC and componentwise DC matrix-valued mappings.

**Corollary 3** Any DC map \( F : \mathbb{R}^d \to \mathbb{S}^\ell \) is componentwise DC.
Since the definition of DC function provides a lot of flexibility (namely, there are infinitely many DC decompositions of a given function), it seems reasonable to assume that despite some drawbacks of matrix convexity the class of matrix-valued DC mappings is sufficiently rich. In particular, one might ask whether the class of matrix valued DC functions coincides with the class of componentwise DC functions or there are some componentwise DC mappings that are not DC (a characterization of such functions would provide a deep insight into the structure of DC matrix-valued mappings). Another interesting question is whether the matrix DC property is preserved under standard operations, such as the Hadamard/Kronecker product and inversion. Arguing in the same way as in the proof of Theorem 1, one can easily check that for twice continuously differentiable matrix-valued mappings the answer to this question is positive, provided one considers locally DC functions. However, it is unclear whether the classes of locally and globally DC mappings coincide in the matrix-valued case (for componentwise DC functions this statement is true due to the celebrated result of Hartman [19]).

In the end of this section, let us present several simple examples of DC semidefinite constraints appearing in applications and their DC decompositions. These examples, in particular, demonstrate some benefits of using matrix-valued DC mappings in comparison with componentwise DC mappings.

**Example 3 (Quadratic/Bilinear Constraints)** Suppose that

\[
F(x) = C + \sum_{i=1}^{d} x_i B_i + \sum_{i,j=1}^{d} x_i x_j A_{ij}
\]

for some matrices \(C, B_i, A_{ij} \in S^c\). In particular, one can suppose that the map \(F(x)\) is bilinear/biaffine, that is,

\[
F(x, y) = A_{00} + \sum_{i=1}^{d} x_i A_{i0} + \sum_{j=1}^{m} y_j A_{0j} + \sum_{i=1}^{d} \sum_{j=1}^{m} x_i y_j A_{ij}, \quad \forall x \in \mathbb{R}^d, \; y \in \mathbb{R}^m
\]

for some matrices \(A_{ij} \in S^c\). Such nonlinear matrix constraints appear in problems of simultaneous stabilisation of single-input single-output linear systems by one fixed controller of a given order [20, 53], robust gain-scheduling and some decentralized control problems [17, 18], problems of maximizing the minimal eigenfrequency of a given structure [53], etc.

By Theorem 1 the map \(F\) of the form (2) is DC and for any \(\mu \geq \ell M\), where

\[
M^2 = \max_{s,k \in \{1, \ldots, c\}} \sum_{i,j=1}^{d} [A_{ij}]_{sk}^2
\]

the pair

\[
G(x) = C + \sum_{i=1}^{d} x_i B_i + \sum_{i,j=1}^{d} x_i x_j A_{ij} + \frac{\mu}{2} |x|^2 I_c, \quad H(x) = \frac{\mu}{2} |x|^2 I_c
\]
is a DC decomposition of $F$. Note that to compute a componentwise DC decomposition of $F$ one would have to compute DC decompositions of $\ell^2$ quadratic functions of the form

$$\sum_{i,j=1}^{d} [A_{ij}]_{sk} x_i x_j, \quad s, k \in \{1, \ldots, \ell\}.$$ 

Moreover, in the general case the mapping $H$ (the concave part) from a componentwise DC decomposition of $F$ would not be diagonal.

It should be noted that a different DC decomposition of the mapping $F$ can be constructed. Namely, as was shown in [5, Example 5.74], a matrix-valued map $F$ of the form (2) is convex, if the $d \times d$ block matrix $A = (A_{ij})_{i,j=1}^{d}$ is positive semidefinite (note that replacing, if necessary, $A$ with $0.5(A + A^T)$, one can suppose that the block matrix $A$ is symmetric). Therefore, if a decomposition $A = A_+ + A_-$ of the matrix $A$ onto positive semidefinite and negative semidefinite parts is known, one can define

$$G(x) = C + \sum_{i=1}^{d} x_i B_i + \sum_{i,j=1}^{d} x_i x_j (A_+)_{ij}, \quad H(x) = -\sum_{i,j=1}^{d} x_i x_j (A_-)_{ij}$$

Such DC decomposition can be used, if the block matrix $A$ has a relatively simple structure, e.g. when only the diagonal blocks $A_{ii}$ are nonzero.

Example 4 (Bilinear/Biaffine Matrix Constraints) Consider the map

$$R(X_1, X_2, X_3) = \begin{bmatrix} X_1 \\ X_3(A + BX_2 C)X_3 \end{bmatrix}$$

for all $X_1, X_3 \in \mathbb{S}^\ell$, $X_2 \in \mathbb{R}^{m \times m}$, and for some matrices $A \in \mathbb{R}^{\ell \times \ell}$, $B \in \mathbb{R}^{\ell \times m}$, and $C \in \mathbb{R}^{m \times \ell}$. Nonlinear semidefinite constraints involving such mappings $R$ (or similar ones) appear, e.g. in optimal $H_2/H_\infty$-static output feedback problems [33, 53].

To apply the results presented in this section to the mapping $R$, define $d = 0.5\ell(\ell + 1) + m^2 + 0.5\ell(\ell + 1)$ (here we used the fact that a matrix $X \in \mathbb{S}^\ell$ is defined by $\ell(\ell + 1)/2$ variables). For any $x \in \mathbb{R}^d$ let $(X_1, X_2, X_3)$ be the corresponding triplet of matrices from $\mathbb{S}^\ell \times \mathbb{R}^{m \times m} \times \mathbb{S}^\ell$, and let $F(x) = R(X_1, X_2, X_3)$.

By Theorem 1 the map $F$ is DC and for any $\mu \geq \ell M$, where

$$M^2 = \max_{i \in \{1, \ldots, \ell\}} \sum_{k_i=1}^{m} \sum_{k_2=1}^{\ell} \sum_{k_3=1}^{\ell} (B_{ik_1} C_{k_2 k_3})^2,$$

the pair

$$G(x) = F(x) + \frac{\mu}{2} (\|X_2\|_F^2 + \|X_3\|_F^2)I_{2\ell}, \quad H(x) = \frac{\mu}{2} (\|X_2\|_F^2 + \|X_3\|_F^2)I_{2\ell}$$

is a DC decomposition of $F$. 

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Example 5 (The Stiefel Manifold/Orthogonality Constraint) Let \( d = m \times \ell \) for some \( m \in \mathbb{N} \), i.e. \( x \) is a real matrix of order \( m \times \ell \), which we denote by \( X \). Consider the equality constraint
\[
X^T X = I_\ell,
\]
which is known as the Stiefel manifold or orthogonality constraint appearing in many applications [1, 12, 40, 41].

Following Lipp and Boyd [40], we rewrite equality constraint (3) as two matrix inequality constraints:
\[
G(X) = X^T X - I_\ell \preceq 0, \quad H(X) = I_\ell - X^T X \succeq 0.
\]

Let, as above, \( G_z(X) = \langle z, G(X)z \rangle \). Observe that for any \( X_1, X_2 \in \mathbb{R}^{m \times \ell} \) and \( \alpha \in [0, 1] \) one has
\[
\alpha G_z(X_1) + (1 - \alpha)G_z(X_2) - G_z(\alpha X_1 + (1 - \alpha)X_2) \\
= (\alpha - \alpha^2)\langle z, X_1^T X_1 z \rangle + ((1 - \alpha) - (1 - \alpha)^2)\langle z, X_2^T X_2 z \rangle \\
- \alpha(1 - \alpha)\langle z, (X_1^T X_2 + X_2^T X_1)z \rangle \\
= \alpha(1 - \alpha)\left( |X_1 z|^2 + |X_2 z|^2 - 2\langle X_1 z, X_2 z \rangle \right) \\
= \alpha(1 - \alpha)|X_1 z - X_2 z|^2 \geq 0.
\]

Consequently, the function \( G_z \) is convex for any \( z \in \mathbb{R}^\ell \), which implies that the functions \( G \) and \(-H\) are matrix convex. Thus, equality constraint (3) can be rewritten as two DC semidefinite constraints. It should be noted that although this transformation is degenerate (we rewrite an equality constraint as two inequality constraints), numerical experiments reported in [40] demonstrate the effectiveness of an optimization method based on such transformation.

3 DC Structure of the maximal eigenvalue function

Since there is no obvious connection between componentwise convexity and the Löewner partial order/matrix convexity, componentwise DC matrix-valued mappings cannot be utilised directly in the abstract setting of nonlinear semidefinite programming problems. Instead, it is natural to apply componentwise DC property to a reformulation of such problems, in which the constraint \( F(x) \preceq 0 \) is replaced by the equivalent inequality constraint \( \lambda_{\max}(F(x)) \leq 0 \), where \( \lambda_{\max}(A) \) is the maximal eigenvalue of a symmetric matrix \( A \).

Our aim is to show that for componentwise DC mappings \( F \) the inequality constraint \( \lambda_{\max}(F(x)) \leq 0 \) is also DC, and one can compute a DC decomposition of the maximal eigenvalue function \( \lambda_{\max}(F(\cdot)) \), if a componentwise DC decomposition
of the map $F$ is known. With the use of this result one can easily extend standard results and algorithms from the theory of DC constrained DC optimization problems to the case of DC semidefinite programming problems.

**Theorem 3** Let $F : \mathbb{R}^d \to \mathbb{S}^\ell$ be a componentwise DC mapping and $F_{ij} = G_{ij} - H_{ij}$ be a DC decomposition of each component of $F$, $i, j \in \{1, \ldots, \ell\}$. Then the function $\lambda_{\max}(F(\cdot))$ is DC and the pair $(g, h)$ with

$$
g(x) = \max_{|v| \leq 1} \sum_{i,j=1}^{\ell} \left( (v_i v_j + 1)G_{ij}(x) + (1 - v_i v_j)H_{ij}(x) \right),
$$

$$
h(x) = \sum_{i,j=1}^{\ell} \left( G_{ij}(x) + H_{ij}(x) \right)
$$

for all $x \in \mathbb{R}^d$ is a DC decomposition of the function $\lambda_{\max}(F(\cdot))$.

**Proof** Fix any $x \in \mathbb{R}^d$. As is well-known and easy to check, the following equality holds true:

$$
\lambda_{\max}(F(x)) = \max_{|v| \leq 1} \langle v, F(x)v \rangle = \max_{|v| \leq 1} \sum_{i,j=1}^{\ell} v_i v_j F_{ij}(x).
$$

Adding and subtracting $G_{ij}(x) + H_{ij}(x)$ for all $i, j \in \{1, \ldots, \ell\}$ and taking into account the equality $F_{ij}(x) = G_{ij}(x) - H_{ij}(x)$, one obtains that

$$
\lambda_{\max}(F(x)) = \max_{|v| \leq 1} \sum_{i,j=1}^{\ell} \left( (v_i v_j + 1)G_{ij}(x) + (1 - v_i v_j)H_{ij}(x) \right)
$$

$$
- \sum_{i,j=1}^{\ell} \left( G_{ij}(x) + H_{ij}(x) \right) =: g(x) - h(x).
$$

The function $h$ is obviously convex as the sum of convex functions. Moreover, note that $v_i v_j + 1 \geq 0$ and $1 - v_i v_j \geq 0$ for all $|v| \leq 1$. Therefore, the function $g$ is also convex as the maximum of the family of convex functions

$$
(v_i v_j + 1)G_{ij}(x) + (1 - v_i v_j)H_{ij}(x), \quad |v| \leq 1.
$$

Thus, the function $\lambda_{\max}(F(\cdot))$ is DC and the pair $(g, h)$ defined in (4) is a DC decomposition of this function. \hfill \Box

**Remark 2** Let us make an almost trivial, yet useful observation. By definition $g(x) = \lambda_{\max}(F(x)) + h(x)$. Therefore, there is no need to directly compute the maximum in the definition of $g$ in order to compute $g(x)$. One simply has to find the maximal eigenvalue of the matrix $F(x)$ and then add $h(x)$. 

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For the sake of completeness, let us point out explicit formulae for the subdifferentials of the convex functions $g$ and $h$ from the theorem above. To this end, for any matrix $A \in \mathbb{S}^c$ denote by $\lambda_{\max}(A)$ the eigenspace of $\lambda_{\max}(A)$.

**Proposition 1** Under the assumptions of Theorem 3 for any $x \in \mathbb{R}^d$ one has

$$\partial g(x) = \text{co} \left\{ \sum_{i,j=1}^\ell (v_i v_j + 1) \partial G_{ij}(x) + (1 - v_i v_j) \partial H_{ij}(x) \left| v \in \mathcal{E}_{\max}(F(x)), \quad |v| = 1 \right. \right\}$$

and $\partial h(x) = \sum_{i,j=1}^\ell (\partial G_{ij}(x) + \partial H_{ij}(x))$, where ‘co’ stands for the convex hull.

**Proof** The expression for $\partial h(x)$ follows directly from the standard rules of subdifferential calculus. Let us prove the equality for $\partial g(x)$.

Indeed, fix any $x \in \mathbb{R}^\ell$ and denote by $V(F(x))$ the set of all those $v \in \mathbb{R}^\ell$ with $|v| \leq 1$ for which the maximum in the definition of $g(x)$ is attained. Clearly, $V(F(x))$ is a compact set. With the use of the theorem on the subdifferential of the supremum of an infinite family of convex functions (see, e.g. [25, Thm. 4.2.3]) one obtains that

$$\partial g(x) = \text{co} \left\{ \sum_{i,j=1}^\ell (v_i v_j + 1) \partial G_{ij}(x) + (1 - v_i v_j) \partial H_{ij}(x) \left| v \in V(F(x)) \right. \right\}.$$

Note that this convex hull is closed as the convex hull of a compact set. Therefore, it remains to show that $v \in V(F(x))$ if and only if $v \in \mathcal{E}_{\max}(F(x))$ and $|v| = 1$.

Observe that for any $v \in \mathbb{R}^\ell$ one has

$$\sum_{i,j=1}^\ell (v_i v_j + 1) G_{ij}(x) + (1 - v_i v_j) H_{ij}(x)$$

$$= \sum_{i,j=1}^\ell v_i v_j (G_{ij}(x) - H_{ij}(x)) + h(x) = \langle v, F(x)v \rangle + h(x).$$

Therefore, the maximum over all $v \in \mathbb{R}^\ell$ with $|v| \leq 1$ of the left-hand side of this equality (which is equal to $g(x)$) is attained at exactly the same $v$ as the maximum over all $v \in \mathbb{R}^\ell$ with $|v| \leq 1$ of the right-hand side of this equality (which is equal to $\lambda_{\max}(F(x)) + h(x)$). Consequently, one has

$$V(F(x)) = \left\{ v \in \mathbb{R}^\ell \left| |v| \leq 1, \quad \langle v, F(x)v \rangle = \lambda_{\max}(F(x)) \right. \right\}.$$

With the use of the spectral decomposition of the matrix $F(x)$ one can easily verify that $\lambda_{\max}(F(x)) = \langle v, F(x)v \rangle$ for some $|v| \leq 1$ if and only if $|v| = 1$ and $v$ is an eigenvector of the matrix $F(x)$ corresponding to its maximal eigenvalue (i.e. $v \in \mathcal{E}_{\max}(F(x))$), which implies the required result. \qed

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Thus, if an eigenvector $v$ with $|v| = 1$ of the matrix $F(x)$ corresponding to the maximal eigenvalue $\lambda_{\text{max}}(F(x))$ is computed, one can easily compute subgradients of DC components of the function $\lambda_{\text{max}}(F(\cdot))$ at the point $x$ with the use of subgradients of the functions $G_{ij}$ and $H_{ij}$.

**Remark 3** Let us note once again that one can rewrite nonlinear semidefinite programming problem

$$\minimize \ f_0(x) \ \text{subject to} \ F(x) \leq 0, \ x \in Q. \quad (5)$$

where $Q$ is a closed convex set, as the following equivalent inequality constrained problem

$$\minimize \ f_0(x) \ \text{subject to} \ \lambda_{\text{max}}(F(x)) \leq 0, \ x \in Q. \quad (5)$$

In the case when the function $f_0$ is DC and the map $F$ is componentwise DC, one can easily extend all existing results and methods for inequality constrained DC optimization problems to the case of problem (5) with use of Theorem 3 and Proposition 1. For the sake of shortness, we leave the tedious task of explicitly reformulating existing results and methods in terms of problem (5) to the interested reader.

### 4 Cone constrained DC optimization

In the previous section, we pointed out how methods and results of DC optimization can be applied to nonlinear semidefinite optimization problems with componentwise DC constraints. Let us now show how one can extend standard results from DC optimization to the case when the semidefinite constraint is DC in the order-theoretic sense. Since such extension does not rely on any particular properties of semidefinite problems (i.e. any properties of matrix-valued mappings, the Löwner partial order, etc.) or the finite dimensional nature of the problem, following Lipp and Boyd [40], below we study optimality conditions for DC semidefinite programming problems in the more general setting of DC cone constrained problems of the form

$$\minimize \ f_0(x) = g_0(x) - h_0(x), \ \text{subject to} \ F(x) = G(x) - H(x) \preceq_K 0, \ x \in Q. \quad (P)$$

Here $g_0, h_0$ are real-valued closed convex functions defined on $\mathbb{R}^d$, $K$ is a proper cone in a real Banach space $Y$ (that is, $K$ is a closed convex cone such that $K \cap (-K) = \{0\}$), $\preceq_K$ is the partial order induced by the cone $K$, i.e. $x \preceq_K y$ if and only if $y - x \in K$, the mappings $G, H : \mathbb{R}^d \to Y$ are convex with respect to the cone $K$ (or $K$-convex), that is,

$$G(\alpha x_1 + (1 - \alpha)x_2) \preceq_K \alpha G(x_1) + (1 - \alpha)G(x_2) \ \forall \alpha \in [0, 1], \ x_1, x_2 \in \mathbb{R}^d$$

and the same inequality holds for $H$, and, finally, $Q \subseteq \mathbb{R}^d$ is a closed convex set. Note that the constraint $F(x) \preceq_K 0$ can be rewritten as $F(x) \in -K$. 

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Thus, the problem \((P)\) is a cone constrained DC optimization problem that consists in minimizing the DC objective function \(f_0\) subject to the generalized inequality (or cone) constraint that is DC with respect to the cone \(K\). In the case when \(Y = \mathbb{S}_d^+\) and \(K\) is the cone of positive semidefinite matrices, the problem \((P)\) becomes a standard nonlinear semidefinite programming problem.

### 4.1 Some properties of convex mappings

Before we proceed to the study of cone constrained DC optimization problems, let us first present two well-known auxiliary results on convex mappings and convex multifunctions, whose formulations are tailored to our specific setting. For the sake of completeness, we provide detailed proofs of these results.

We start with the following well-known characterisation of \(K\)-convex mappings in terms of their derivatives.

**Lemma 1** Let \(X\) be a real Banach space. A Gâteaux differentiable mapping \(\Phi : X \to Y\) is \(K\)-convex if and only if
\[
\Phi(x_1) - \Phi(x_2) \succeq_K \Phi'(x_2)(x_1 - x_2) \quad \forall x_1, x_2 \in \mathbb{R}^d,
\]
where \(\Phi'(x)\) is the Gâteaux derivative of \(\Phi\) at \(x\).

**Proof** Let \(\Phi\) be convex. Then by definition
\[
\alpha \Phi(x_1) + (1 - \alpha) \Phi(x_2) - \Phi(\alpha x_1 + (1 - \alpha)x_2) \in K \quad \forall \alpha \in [0, 1], x_1, x_2 \in X.
\]
Since \(K\) is a cone, for any \(\alpha \in (0, 1)\) one has
\[
\Phi(x_1) - \Phi(x_2) - \frac{1}{\alpha} \left( \Phi(x_2 + \alpha(x_1 - x_2)) - \Phi(x_2) \right) \in K.
\]
Passing to the limit as \(\alpha \to +0\) and taking into account the fact that the cone \(K\) is closed, one obtains that
\[
\Phi(x_1) - \Phi(x_2) - \Phi'(x_2)(x_1 - x_2) \in K \quad \forall x_1, x_2 \in X
\]
or, equivalently, condition (6) holds true.

Conversely, if condition (6) holds true then for all \(x_1, x_2 \in X\) and for any \(\alpha \in [0, 1]\) one has
\[
\Phi(x_1) - \Phi(x(\alpha)) - (1 - \alpha) \Phi'(x(\alpha))(x_1 - x_2) \in K, \\
\Phi(x_2) - \Phi(x(\alpha)) - \alpha \Phi'(x(\alpha))(x_2 - x_1) \in K.
\]

where \(x(\alpha) = \alpha x_1 + (1 - \alpha)x_2\). Multiplying the first expression by \(\alpha\) and the second expression by \(1 - \alpha\) and bearing in mind the fact that a convex cone is closed under addition, one obtains that
\[
\alpha \Phi(x_1) + (1 - \alpha) \Phi(x_2) - \Phi(x(\alpha)) \in K \quad \forall \alpha \in [0, 1], x_1, x_2 \in X,
\]
that is, $\Phi$ is $K$-convex.

Let us also present a lemma on solutions of perturbed convex generalized equations, based on some well-known results on metric regularity of convex multifunctions (see, e.g. [51]). For any metric space $(X, \rho)$ and all $x \in X$ denote $B(x, r) = \{ x' \in X \mid \rho(x', x) \leq r \}$. If $X$ is a normed space, then $B_X = B(0, 1)$.

**Lemma 2** Let $X$ be a real Banach space and $Z$ be a metric space. Suppose that $M_z : X \rightrightarrows Y$, $z \in Z$, is a family of closed convex multifunctions such that for some $z_0 \in Z$ and $x \in X$ one has $0 \in \text{int} \ M_{z_0}(X)$ and $0 \in M_{z_0}(x)$. Suppose also that the function $z \mapsto \text{dist} (0, M_z(x))$ is continuous at $z_0$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$
M_{z_0}(x + B_X) \subseteq M_{z_0}(x) + \varepsilon B_Y \quad \forall z \in B(z_0, \delta).
$$

Then there exist a neighbourhood $U$ of $z_0$ and a mapping $\xi : U \to X$ such that $0 \in M_{z_0}(\xi(z))$ for all $z \in U$, $\xi(z_0) = \bar{x}$, and $\xi(z) \to \bar{x}$ as $z \to z_0$.

**Proof** Since $0 \in \text{int} \ M_{z_0}(X)$ and $0 \in M_{z_0}(x)$, by [51, Thm. 1] there exists $\eta > 0$ such that $\eta B_Y \subseteq M_{z_0}(x + B_X)$. By our assumption there exists $\delta > 0$ such that

$$
\eta B_Y \subseteq M_{z_0}(x + B_X) \subseteq M_{z_0}(x) + \varepsilon B_Y \quad \forall z \in B(z_0, \delta),
$$

which by [51, Thm. 2] implies that for all $x \in X$ and $z \in B(z_0, \delta)$ one has

$$
\text{dist} (x, M_{z_0}^{-1}(0)) \leq \frac{2}{\eta} (1 + \|x - \bar{x}\|) \text{dist} (0, M_{z_0}(x)).
$$

Putting $x = \bar{x}$ one gets that for any $z \in B(z_0, \delta)$ there exists $\bar{\xi}(z) \in M_{z_0}^{-1}(0)$ such that $\|\bar{x} - \bar{\xi}(z)\| \leq (4/\eta) \text{dist} (0, M_{z_0}(\bar{x}))$. Note that $\bar{\xi}(z_0) = \bar{x}$, since $0 \in M_{z_0}(\bar{x})$. Moreover, from the fact that the function $z \mapsto \text{dist} (0, M_z(x))$ is continuous at $z_0$ it follows that $\bar{\xi}(z) \to \bar{x}$ as $z \to z_0$, which completes the proof.

**Remark 4** Roughly speaking, the previous lemma states that if $0 \in \text{int} \ M_{z_0}(X)$ and $0 \in M_{z_0}(x)$, then under certain semicontinuity assumptions for any $z$ in a neighborhood of $z_0$ there exists a solution $\bar{\xi}(z)$ of the generalized equation $0 \in M_{z_0}(x)$ continuously depending on $z$ and such that $\bar{\xi}(z_0) = \bar{x}$.

**Corollary 4** Let $X$ be a real Banach space, $W \subseteq X$ be a closed convex set, $E \subseteq Y$ be a proper cone, and $\Phi, \Psi : X \to Y$ be $E$-convex mappings. Suppose that $\Phi$ is continuous on $W$, $\Psi$ is continuously Fréchet differentiable on $W$, and the following constraint qualification holds true

$$
0 \in \text{int} \left\{ \Phi(x) - \Psi(x_0) - D\Psi(x_0)(x - x_0) + E \mid x \in W \right\}
$$

for some $x_0 \in W$ such that $\Phi(x_0) = \Psi(x_0) \leq_E 0$, where $D\Psi(x_0)$ is the Fréchet derivative of $\Psi$ at $x_0$. Then for any $\bar{x} \in W$ such that
\[ \Phi(x) - \Psi(x) - D\Psi(x)(x-x) \leq 0 \]

there exists a neighbourhood \( \mathcal{U} \) of \( x \) and a mapping \( \xi : \mathcal{U} \cap W \to W \) such that
\[ \Phi(\xi(z)) - \Psi(z) - D\Psi(z)(\xi(z) - z) \leq 0 \quad \forall z \in \mathcal{U} \cap W \]
\( \xi(x) = x \), and \( \xi(z) \to x \) as \( z \to x \).

**Proof** For any \( z \in X \) introduce the \( E \)-convex function \( \Phi_z : X \to Y \) defined as
\[ \Phi_z(x) = \Phi(x) - \Psi(z) - D\Psi(z)(x-z) \]
and the set-valued mapping
\[
M_z(x) = \begin{cases} 
\Phi_z(x) + E, & \text{if } x \in W, \\
\emptyset, & \text{if } x \not\in W.
\end{cases}
\]

(8)

The multifunction \( M_z \) is closed due to the facts that the mapping \( \Phi_z(\cdot) \) is continuous and the sets \( W \) and \( E \) are closed. Moreover, this multifunction is convex.

Indeed, by the convexity of \( \Phi \) for any \( x_1, x_2 \in W \) and all \( \alpha \in [0, 1] \) one has
\[ \alpha\Phi_z(x_1) + (1 - \alpha)\Phi_z(x_2) \in \Phi_z(\alpha x_1 + (1 - \alpha)x_2) + E, \]
which due to the convexity of the cone \( E \) implies that
\[ \alpha M_z(x_1) + (1 - \alpha)M_z(x_2) \subseteq \Phi_z(\alpha x_1 + (1 - \alpha)x_2) + E + \alpha E + (1 - \alpha)E \]
\[ \subseteq M_z(\alpha x_1 + (1 - \alpha)x_2) \]
for all \( x_1, x_2 \in W \) and \( \alpha \in [0, 1] \), that is, the graph of \( M_z \) is convex.

Our aim is to apply Lemma 2 with \( Z = W \) and \( z_* = x_* \). Indeed, by definition \( 0 \in M_{z_*}(\bar{x}) \), while condition (7) implies that \( 0 \in \text{int} M_{z_*}(X) \).

From the fact that \( \Psi \) is continuously Fréchet differentiable on \( W \) it follows that for any \( \varepsilon > 0 \) there exists \( \delta < \min\{1, \varepsilon/3(1 + \|D\Psi(z_*)\|)\} \) such that
\[ \|\Psi(z) - \Psi(z_*)\| < \frac{\varepsilon}{3}, \quad \|D\Psi(z) - D\Psi(z_*)\| < \frac{\varepsilon}{3(2 + \|ar{x}\| + \|z_*\|)} \]
for all \( z \in B(z_*, \delta) \cap W \). Choose any \( y \in M_{z_*}(\bar{x} + B_X) \). By definition there exist \( x \in (\bar{x} + B_X) \cap W \) and \( v \in E \) such that \( y = \Phi_{z_*}(x) + v \). Observe that
\[ \|\Phi_{z_*}(x) + v - y\| = \|\Phi_{z_*}(x) - \Phi_{z_*}(x)\| \]
\[ \leq \|\Psi(z) - \Psi(z_*)\| + \|D\Psi(z) - D\Psi(z_*)\| \|x - z\| \]
\[ + \|D\Psi(z_*)\| \|z - z_*\| < \varepsilon \]
for all \( z \in B(z_*, \delta) \cap W \), which implies that
\[ M_{z_*}(\bar{x} + B_X) \subseteq M_{z_*}(\bar{x} + B_X) + \varepsilon B_Y \quad \forall z \in B(z_*, \delta) \cap W. \]

Thus, it remains to show that the restriction of the function \( \text{dist} (0, M_{z_*}(\bar{x})) \) to \( W \) is continuous.
By definition \( \text{dist} (0, M_\xi (\bar{x})) = \text{dist} (\Phi_\xi (\bar{x}), -E) \) (see (8)). With the use of the fact that \( \Psi \) is continuously Fréchet differentiable one obtains that for any \( \varepsilon > 0 \) there exists \( r < \min \{ 1, \varepsilon / 3 \} \) such that

\[
\| \Psi (z) - \Psi (z_*) \| < \frac{\varepsilon}{3}, \quad \| D\Psi (z) - D\Psi (z_*) \| < \frac{\varepsilon}{3(\| \bar{x} \| + \| z_* \| + 1)}
\]

for all \( z \in B(z_*, r) \cap W \). Therefore for any such \( z \) one has

\[
\| \Phi_\xi (\bar{x}) - \Phi_\xi (\bar{x}) \| \leq \| \Psi (z) - \Psi (z_*) \|
\]

\[
+ \| D\Psi (z) - D\Psi (z_*) \| \| \bar{x} - z \| + \| D\Psi (z_*) \| \| z - z_* \| < \varepsilon,
\]

which implies that for any \( z \in B(x_*, r) \cap W \) the following inequality holds true:

\[
\text{dist} (0, M_\xi (\bar{x})) = \text{dist} (\Phi_\xi (\bar{x}), -E) \leq \| \Phi_\xi (\bar{x}) - \Phi_\xi (\bar{x}) \| < \varepsilon
\]

(here we used the fact that \( \Phi_\xi (\bar{x}) \in -E \)). Thus, all assumptions of Lemma 2 with \( Z = W \) and \( z_* = x_* \) are valid, and by this lemma there exists a required mapping \( \xi (z) \).

\[\square\]

4.2 Optimality conditions

Let us extend well-known local optimality conditions for constrained DC optimization problems to the case of the problem \((P)\). To the best of the author’s knowledge, standard subdifferential calculus cannot be extended to the case of convex matrix-valued mappings and many other \( K \)-convex vector-valued maps, which makes it very difficult to deal with subdifferentials of such functions. Therefore, below we suppose that the mapping \( H \) (the \( K \)-concave part of \( F \)) is continuously differentiable, but do not impose any smoothness assumptions on the objective function \( f_0 \).

**Theorem 4** Let \( x_* \) be a locally optimal solution of the problem \((P)\) and the mapping \( H \) be Fréchet differentiable at \( x_* \). Then for any \( v \in \partial h_0 (x_*) \) the point \( x_* \) is a globally optimal solution of the following convex programming problem:

\[
\begin{align*}
\text{minimize} \quad & g_0 (x) - h_0 (x_*) - \langle v, x - x_* \rangle \\
\text{subject to} \quad & G(x) - H(x_*) - DH(x_*) (x - x_*) \leq_K 0, \quad x \in Q,
\end{align*}
\]

(9)

where \( DH(x_*) \) is the Fréchet derivative of \( H \) at \( x_* \).

**Proof** Denote by \( \omega_v (x) = g_0 (x) - h_0 (x_*) - \langle v, x - x_* \rangle \), \( x \in \mathbb{R}^d \), the objective function of problem (9). This function is convex. Moreover, taking into account the fact that by the definition of subgradient \( h_0 (x) \geq h_0 (x_*) + \langle v, x - x_* \rangle \), one obtains that \( \omega_v (x) \geq f_0 (x) \) for all \( x \in \mathbb{R}^d \) and \( \omega_v (x_*) = f_0 (x_*) \).

By contradiction, suppose that there exists \( v \in \partial h_0 (x_*) \) such that the point \( x_* \) is not a globally optimal solution of problem (9), i.e. there exists a feasible point \( x \) of this problem such that \( \omega_v (x) < \omega_v (x_*) \). Define \( x(\alpha) = \alpha x + (1 - \alpha) x_* \). Then

\[\Box\]
\[ f_0(x(\alpha)) \leq \omega_\nu(x(\alpha)) \leq a \omega_\nu(x) + (1 - \alpha) \omega_\nu(x_*), \quad \forall \alpha \in (0, 1] \]  

for all \( \alpha \in (0, 1] \), thanks to the convexity of \( \omega_\nu \).

Let us check that \( x(\alpha) \) is a feasible point of the problem (P) for all \( \alpha \in [0, 1] \). Then with the use of (10) one can conclude that \( x_* \) is not a locally optimal solution of the problem (P), which contradicts the assumption of the theorem.

Indeed, by Lemma 1 one has \( H(x(\alpha)) - H(x_*) - DH(x_*)(x(\alpha) - x_*) \in K \) for all \( \alpha \in [0, 1] \). Adding and subtracting \( G(x(\alpha)) \), one obtains that

\[ -F(x(\alpha)) + G(x(\alpha)) - H(x_*) - DH(x_*)(x(\alpha) - x_*) \in K \quad \forall \alpha \in [0, 1] \]

or, equivalently,

\[ F(x(\alpha)) \leq_K G(x(\alpha)) - H(x_*) - DH(x_*)(x(\alpha) - x_*) \quad \forall \alpha \in [0, 1]. \]

Hence taking into account the fact that the point \( x(\alpha) \) is feasible for problem (9) due to the convexity of this problem, one can conclude that \( F(x(\alpha)) \leq_K 0 \). Thus, \( x(\alpha) \) is a feasible point of the problem (P) and the proof is complete.

Let us reformulate optimality conditions from the previous theorem. Denote by \( \Omega(x_*) \) the feasible region of problem (9) and for any convex set \( V \subseteq \mathbb{R}^d \) and \( x \in V \) denote by \( N_\nu(x) = \{ v \in \mathbb{R}^d \mid \langle v, z - x \rangle \leq 0 \forall z \in V \} \) the normal cone to \( V \) at \( x \).

**Corollary 5** Let \( x_* \) be a locally optimal solution of the problem (P) and the map \( H \) be Fréchet differentiable at \( x_* \). Then

\[ \partial h_0(x_*) \subseteq \partial g_0(x_*) + N_{\Omega(x_*)}(x_*). \]

**Proof** Fix any \( v \in \partial h_0(x_*) \). By Theorem 4 the point \( x_* \) is a globally optimal solution of the convex problem (9). Applying standard necessary and sufficient optimality conditions for a convex function on a convex set (see, e.g. [25, Thm. 1.1.2’]), one obtains that \( 0 \in \partial \omega_\nu(x_*) + N_{\Omega(x_*)}(x_*), \) where, as above, \( \omega_\nu(x) = g_0(x) - h_0(x_*) - \langle v, x - x_* \rangle \) is the objective function of problem (9). Since \( \partial \omega(x_*) = \partial g_0(x_*) - \nu \), one gets that \( v \in \partial g_0(x_*) + N_{\Omega(x_*)}(x_*), \) which implies the desired result.

In the case when a natural constraint qualification (namely, Slater’s condition for problem (9)) holds at \( x_* \), one can show that optimality conditions from Theorem 4 coincide with standard optimality conditions for cone constrained optimization problems (see, e.g. [5]). To this end, denote by \( Y^* \) the topological dual space of \( Y \) and by \( \langle \cdot, \cdot \rangle \) the canonical duality pairing between \( Y \) and \( Y^* \), that is, \( \langle y^*, y \rangle = y^*(y) \) for any \( y^* \in Y^* \) and \( y \in Y \).

Let \( K^* = \{ y^* \in Y^* \mid \langle y^*, y \rangle \geq 0 \forall y \in K \} \) be the dual cone of \( K \) and for any \( \lambda \in Y^* \) define \( L(x, \lambda) = f_0(x) + \langle \lambda, F(x) \rangle \).
Corollary 6  Let $x_*$ be a locally optimal solution of the problem (P) and the mappings $G$ and $H$ be Fréchet differentiable at $x_*$. Suppose also that the following constraint qualification holds true:

$$0 \in \text{int} \left\{ G(x) - H(x_*) - DH(x_*)(x - x_*) + K \mid x \in Q \right\}$$

(if $K$ has nonempty interior, it is sufficient to suppose that there exists $x \in Q$ such that $G(x) - H(x_*) - DH(x_*)(x - x_*) \in -\text{int } K$). Then for any $v \in \partial h_0(x_*)$ there exists a multiplier $\lambda_* \in K^*$ such that $\langle \lambda_, F(x_*) \rangle = 0$ and

$$v \in \partial g_0(x_*) + D\left( \langle \lambda_, F(\cdot) \rangle \right)(x_*) + N_Q(x_*) .$$

In particular, if both $g_0$ and $h_0$ are differentiable at $x_*$, then there exists $\lambda_* \in K^*$ such that $\langle \lambda_, F(x_*) \rangle = 0$ and $\langle D_x L(x_*, \lambda_*) , x - x_* \rangle \geq 0$ for all $x \in Q$.

Proof  Rewriting problem (9) as the convex cone constrained problem

$$\begin{align*}
\text{minimize} & \quad g_0(x) - h_0(x_*) - \langle v, x - x_* \rangle \\
\text{subject to} & \quad G(x) - H(x_*) - DH(x_*)(x - x_*) \in -K, \quad x \in Q
\end{align*}$$

and applying standard necessary and sufficient optimality conditions for convex cone constrained optimization problems (see, for example, [5, Thm. 3.6 and Prp. 2.106]), we arrive at the required result.

Remark 5  In the case of semidefinite programs, i.e. when $Y = S^r$ and $K$ is the cone of positive semidefinite matrices, the dual cone $K^*$ coincides with $K$ (if we identify the dual of $S^r$ with the space $S^r$ itself), and thus the multiplier $\lambda_*$ from the previous corollary is a positive semidefinite matrix. In addition, the constraint qualification from the corollary takes the form: there exists $x \in Q$ such that the matrix $G(x) - H(x_*) - DH(x_*)(x - x_*)$ is negative definite.

5 Conclusions

In this paper, we developed a general theory of DC semidefinite programming problems. To this end, we studied two definition of DC matrix-valued mappings (abstract and componentwise) and their interconnections. We proved that any DC matrix-valued map is componentwise DC and demonstrated how one can compute a DC decomposition of several nonlinear semidefinite constraints appearing in applications. We also constructed a DC decomposition of the maximal eigenvalue function, which allows one to apply standard results and methods of inequality constrained DC optimization to problems with smooth and nonsmooth componentwise DC semidefinite constraints. In the case of general cone constrained DC optimization problems, we obtained local optimality conditions.
The second part of our study is devoted to a detailed convergence analysis of the DCA and its exact penalty version for cone constrained DC optimization problems proposed in [40] (see also [38, 48]).

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