EXISTENCE OF TRACIAL STATES ON REDUCED GROUP C*-ALGEBRAS

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Abstract. Let $G$ be a locally compact group. It is not always the case that its reduced C*-algebra $C^*_r(G)$ admits a tracial state. We exhibit closely related necessary and sufficient conditions for the existence of such. We gain a complete answer when $G$ compactly generated. In particular for $G$ almost connected, or more generally when $C^*_r(G)$ is nuclear, the existence of a trace is equivalent to amenability. We exhibit two examples of classes of totally disconnected groups for which $C^*_r(G)$ does not admit a tracial state.

1. Introduction and background

If $G$ is a discrete group, then it is well-known that its reduced C*-algebra $C^*_r(G)$ admits a tracial state. The condition of when this trace on $C^*_r(G)$ is unique is linked to the simplicity $C^*_r(G)$. Indeed, Kalantar and Kennedy [15] showed that the trace on $C^*_r(G)$ is unique when $C^*_r(G)$ is simple and a complete characterization of when $C^*_r(G)$ admits a unique tracial state for a discrete group $G$ was solved by Breuillard, Kalantar, Kennedy and Ozawa [5] and Haagerup [13] by using techniques developed for studying when $C^*_r(G)$ is simple.

The study of simplicity for $C^*_r(G)$ has recently migrated to the case when $G$ is a totally disconnected group due to significant work of Raum [21, 22] and Suzuki [23]. In his example, Suzuki showed that that if $C^*_r(G)$ admits a tracial state, then it must be unique.

We focus our attention on the following question.

Question 1.1. What conditions on a locally compact group $G$ characterize when $C^*_r(G)$ admits a tracial state.

Ng [19] considered this question from the perspective of finding a condition which may be added to the assumption that $C^*_r(G)$ is nuclear to determine that $G$ is amenable. He showed that this extra

Date: July 20, 2017.

1991 Mathematics Subject Classification. Primary 22D25; Secondary 22D05, 43A30, 43A07, 46L05, 46L35.

The first two authors were partially supported by an NSERC Discovery Grants.
assumption is the existence of a tracial state. We give an alternate proof to Ng’s result in Section 3. He used a theory of strictly amenable representations of C*-algebras; we use only one aspect of the theory of amenable traces.

More generally we answer this Question 4.1 fully when $G$ is compactly generated, in particular when $G$ is almost connected or when $G$ has property (T). This is given in Section 2.

1.1. Background. The \textit{reduced C*-algebra} of $G$ is the C*-algebra generated by the left regular representation on the $L^2$-space with respect to a left Haar measure: $\lambda_G : G \to B(L^2(G))$ ($\lambda_G$ will be denoted $\lambda$ when the group is unambiguous), $\lambda(s)g(t) = g(s^{-1}t)$ for $s$ in $G$, $g$ in $L^1(G)$ and a.e. $t$ in $G$. We have an integrated form $\lambda : L^1(G) \to B(L^2(G))$,

$$\lambda(f)g(t) = f * g(t) = \int_G f(s)g(s^{-1}t)\,dt,$$

where this integral may be understood in the weak operator topology. Then $C_r^*(G) = \overline{\lambda(L^1(G))}$, the norm closure in $B(L^2(G))$.

We shall use the duality $C^*_r(G)^* = B_r(G)$ established by Eymard [9]. Let us briefly recall the definition of the \textit{reduced Fourier-\-Stieltjes algebra} $B_r(G)$. By Godement [12] the cone $\text{PL}^2(G)$ of square-integrable positive definite functions consists of certain elements of the form $\langle \lambda(g), g \rangle = \check{g} * \hat{g}$ where $g \in L^2(G)$, and $\check{g}(t) = g(y^{-1})$ for a.e. $t$ in $G$. Furthermore, for any $u$ in span$\text{PL}^2(G) \subset L^\infty(G)$ we have

$$\sup \left\{ \left| \int_G f(s)u(s)\,ds \right| : f \in L^1(G), \|\lambda(f)\| \leq 1 \right\} < \infty$$

so span$\text{PL}^2(G)$ forms a subspace of $C^*_r(G)^*$, which is, in fact weak* dense; we denote this space $B_r(G)$. The space may also be realized as the space of all matrix coefficients $\langle \pi(\cdot)\xi, \eta \rangle$ where $\pi : G \to B(\mathcal{H})$ is a continuous unitary representation which is weakly contained in $\lambda$, and $\xi, \eta \in \mathcal{H}$. In particular, the norm closure $A(G)$ of span$\text{PL}^2(G)$ in $B_r(G)$ is called the \textit{Fourier algebra}, and may be realized all matrix coefficients of the left regular representation: functions of the form $\langle \lambda(g), h \rangle = \check{h} * \hat{g}$ where $g, h \in L^2(G)$. We let

$$S_r(G) = \{ u \in B_r(G) : u \text{ is positive definite and } u(e) = 1 \}$$

which is the state space of $C^*_r(G)$, i.e. each $u$ corresponds to a positive norm one functional.

A state is called \textit{tracial} if $\int_G f * f'(s)u(s)\,ds = \int_G f' * f(s)u(s)\,ds$ for each $f, f'$ in $L^1(G)$. By left invariance of the measure, we see that this is equivalent to having $u(ts) = u(st)$ for each $s, t$ in $G$. Hence we consider the set of tracial states, given by

$$\text{TS}_r(G) = \{ u \in S_r(G) : u(ts^{-1}) = u(s) \text{ for all } s, t \text{ in } G \}.$$
Our goal is to study when this set is non-empty.

The following observations are well-known.

**Lemma 1.2.** (i) Let $H$ be a closed subgroup of $G$. If there is $u$ in $\mathcal{B}_r(G)$ for which the restriction $u|_H = 1$, then $H$ is amenable.

(ii) Let $N$ be an amenable normal subgroup of $G$. Then $\mathcal{A}(G/N)$ is isomorphic to a subalgebra of elements in $\mathcal{B}_r(G)$.

**Proof.**

(i) The restriction theorem ([14, 1.7]) tells us that $\mathcal{A}(G)|_H = \mathcal{A}(H)$. Hence weak* density of Fourier algebras in reduced Fourier algebras provides that $\mathcal{B}_r(G)|_H \subseteq \mathcal{B}_r(H)$. In particular, we see that $1 = u|_H \in \mathcal{B}_r(H)$, which tells us that $H$ is amenable.

(ii) Let $\lambda_{G:N} : G \to \mathcal{B}(L^2(G/N))$ denote the left quasi-regular representation of $G$, $\lambda_{G:N}(s)g(tN) = g(s^{-1}tN)$ for a.e. $tN$ in $G/N$. Then amenability of $N$ and Fell’s continuity of induction [11] gives weak containment relation

$$\lambda_{G:N} = \text{ind}_{N}^G 1_{N} \prec \text{ind}_{N}^G \lambda_{N} = \lambda_{G}$$

and it follows that $\mathcal{A}(G/N)$, which is identifiable with the space of matrix coefficients of $\lambda_{G:N}$, embeds naturally as a closed subspace of $\mathcal{B}_r(G)$. (This is in the nature of the functorial results of Arsac [1].) We also remark that his result is also observed in [17]. □

### 2. Main result

We say that $G$ is an *invariant neighbourhood group*, or [IN]-group, if there exists a relatively compact neighbourhood $W$ of $e$ for which $xWx^{-1} = W$ for all $x$ in $G$; and a *small invariant neighbourhood group*, or [SIN]-group if $G$ admits a neighbourhood basis of invariant neighbourhoods. We observe that the class of [SIN]-groups contains discrete groups, abelian groups and compact groups. Both classes are closed under products. The class of [IN]-groups is closed under extension when the normal subgroup is compact. See the discussion in the book of Palmer [20] 12.6.15 & 12.6.16. Moreover, [20] 12.1.31] shows that any [IN]-group $G$ admits a compact normal subgroup $K$ by which $G/K$ is [SIN] – we write $[\text{IN}]= [\text{K}^{\text{SIN}}]$. Any almost connected [SIN] group is of the form $V \rtimes K$ where $V$ is a vector group and $K$ has a finite action of $V$, and is hence amenable, hence so too is any almost connected [IN]-group.

**Theorem 2.1.** Consider the following conditions on $G$:

(i) $C^*_r(G)$ admits a tracial state, i.e. $\mathcal{TS}_r(G) \neq \emptyset$;

(ii) $G$ admits a normal amenable closed subgroup $N$ for which $G/N$ is an [IN]-group — we say $G$ is of class $[\text{Am}^{\text{IN}}]$;
(ii’) $G$ admits a normal amenable closed subgroup $N$ for which each compactly generated open subgroup of $G/N$ is an [SIN]-group.

Then (ii) $\Rightarrow$ (i) $\Rightarrow$ (ii’). In particular, if $G$ is compactly generated, then (i) $\iff$ (ii).

See Remark 2.5 below to see that (ii’) does not imply (i), in general. Also, comments above show that $[\text{Am}^\text{IN}] = [\text{Am}^\text{SIN}]$.

Proof. (ii) $\Rightarrow$ (i). Lemma 1.2 (ii) shows that $A(G/N)$ identifies as a subalgebra of $B_r(G)$. Let $W$ be a compact neighbourhood of $G/N$ which is invariant under inner automorphisms. We let for $sN$ in $G/N$

$$u(sN) = \langle \lambda_{G/N}(sN)1_{W}, 1_{W} \rangle = m((sN)W \cap W)$$

which defines an element of $\text{PL}^2(G/N) \subseteq A(G/N)$, where $m$ denotes the Haar measure. If $t \in G$ we have

$$u(tst^{-1}N) = m((tst^{-1}N)W \cap W) = m(tN[(sN)W \cap W]t^{-1}N)$$

since $(tN)W(t^{-1}N) = W$. Hence $s \mapsto \frac{1}{m(W)}u(sN)$ defines an element of $TS_r(G)$.

(i) $\Rightarrow$ (ii’). Let $u \in TS_r(G)$. We may replace $u$ by $|u|^2 = uu^*$ in $TS_r(G)$, so we may assume that $0 \leq u \leq 1$. Any set $S = u^{-1}(B)$, where $\emptyset \neq B \subseteq [0,1]$ is necessarily satisfies $sSs^{-1} = S$ for all $s$ in $G$.

Let $N = u^{-1}(\{1\})$. It is well-known that $N$ is a subgroup. Indeed, write $u = \langle \pi(\cdot)\xi, \xi \rangle$ where $\pi : G \to B(\mathcal{H})$ is a unitary representation and $\|\xi\| = 1$ in $\mathcal{H}$, and uniform convexity provides that $N = \{s \in G : \pi(s)\xi = \xi\}$. Hence $N$ is a normal subgroup with $u$ constant on cosets of $N$. Lemma 1.2 (i) tells us that $N$ is amenable.

For ease of notation, let us now replace $G$ by $G/N$ and assume that

$$(2.1) u^{-1}(\{1\}) = \{e\}.$$

Let $H$ be a compactly generated open subgroup of $G$, so there is a symmetric compact neighbourhood $K$ of $e$ for which $H = \bigcup_{n=1}^{\infty} K^n$. Let $0 < \varepsilon < 1$ be such that

$$u^{-1}((1-\varepsilon,1]) \cap K^3 \subseteq (K^3)^o \text{ (interior)}.$$ 

If $K$ is open, $\varepsilon$ may be chosen arbitrarily; if not, $(2.1)$ provides $\varepsilon$ so $u(s) < 1 - \varepsilon$ for $s$ in $\partial(K^3)$ (boundary). For natural numbers $n$ for which $n > 1/\varepsilon$ we let

$$V_n = u^{-1}((1 - \frac{1}{n}, 1])$$ and $W_n = V_n \cap K^3$.

If $U$ is an open neighbourhood of $e$, let $n$ be so $u(s) < 1 - \frac{1}{n}$ for $s$ in $K^3 \setminus U$ — $(2.1)$ provides that we may find $\frac{1}{n} < \min_{x \in K^3 \setminus U} [1 - u(s)]$ — and we see that $W_n \subseteq U$. Hence $\{W_n\}^\infty_{n=[1/\varepsilon]}$ is a neighbourhood base
for $e$. In particular, for sufficiently large $n$, say $n \geq n_0 \geq \lceil 1/\varepsilon \rceil$, we have $W_n \subseteq K$. Then for such $n$, the symmetry of $K$ provides that
\[ \bigcup_{s \in K} sW_n s^{-1} \subseteq KW_n K \subseteq K^3. \]
But also
\[ \bigcup_{s \in G} sW_n s^{-1} \subseteq \bigcup_{s \in G} sV_n s^{-1} = V_n \]
so we conclude that
\[ \bigcup_{s \in K} sW_n s^{-1} \subseteq V_n \cap K^3 = W_n \]
whence $sW_n s^{-1} = W_n$ for each $s$ in $K$. It is immediate that $sW_n s^{-1} = W_n$ for each $s$ in $H = \bigcup_{n=1}^{\infty} K^n$. Hence $\{W_n\}_{n=n_0}$ shows that $H$ is a $[\text{SIN}]$-group.

For separable groups, the following is proved by Ng [19], using much more indirect techniques.

**Corollary 2.2.** If $G$ is almost connected, then $C^*_r(G)$ admits a tracial state if and only if $G$ is amenable.

**Proof.** This follows immediately from the theorem, above, and the fact that almost connected groups are compactly generated and that almost connected $[\text{IN}]$-groups are amenable, thanks to the structure results indicated at the beginning of this section.

To put Theorem 2.1 in context, we shall use the following observation.

**Lemma 2.3.** Let $G$ be a totally disconnected $[\text{IN}]$-group. Then $G$ admits an open compact normal subgroup.

**Proof.** As noted at the beginning of this section, there is a compact normal subgroup $K$ for which $G/K$ is a $[\text{SIN}]$-group. Then $G/K$ is totally disconnected and there is an open compact subgroup $M$, and an invariant neighbourhood $W$, so $K \subseteq W \subseteq M$. We again use total disconnectivity of $G/K$ to find an open subgroup $L$ with $K \subseteq L \subseteq W$. The the subgroup generated by $\bigcup_{x \in G} xLx^{-1}$ is open and normal, but contained in $M$ and hence compact.

Actually, the above lemma establishes that a totally disconnected $[\text{SIN}]$-group is *pro-discrete*, i.e. admits arbitrarily small compact open normal subgroups.
Corollary 2.4. The following three classes of locally compact groups coincide 
(a) $[AM^\text{IN}]$ 
(b) $[AM^\text{SIN}]$ 
(c) groups admitting an open normal amenable subgroup.

Proof. Clearly (c) implies (b) implies (a). Let us see that (a) implies (c). If $G$ is in $[AM^\text{IN}]$, and $N$ is a normal amenable subgroup for which $G/N$ is $[IN]$. Then the connected component of the identity $(G/N)_e$ is a connected $[IN]$-group, hence amenable, and $(G/N)/(G/N)_e$, is a totally disconnected $[IN]$-group. Thus Lemma 2.3 provides an open normal compact subgroup of $(G/N)/(G/N)_e$, whose pre-image with respect to the quotient map $q : G \rightarrow (G/N)/(G/N)_e$ is the desired open normal amenable subgroup. □

Remark 2.5. Condition (ii') does not guarantee the existence of a tracial state on $C^*_r(G)$.

Suzuki [23] considers totally disconnected groups of the form $G = \bigoplus_{n \in \mathbb{N}} (\Gamma_n \rtimes F_n)$ where each pair $(\Gamma_n, F_n)$ consists of a discrete group and a finite group acting on it for which $C^*_r(\Gamma_n \rtimes F_n)$ admits unique tracial state. Here $F = \prod_{n \in \mathbb{N}} F_n$ is equipped with product topology, and the action of $F$ on $\Gamma = \bigoplus_{n \in \mathbb{N}} \Gamma_n$ is componentwise. Each compactly generated open subgroup of $G$ is contained in one of the form $H = \Lambda \rtimes K$ where $\Lambda \subseteq \bigoplus_{j=1}^m \Gamma_{n_j}$ and $K$ is an open subgroup of $F$. Then $K \cap \prod_{n \in \mathbb{N} \setminus \{n_1, \ldots, n_m\}} F_n$ is an open, compact normal subgroup of $H$, thereby showing that $H$ is an $[IN]$-group. Thus (ii') of Theorem 2.1 holds.

The sequences $K_n = \prod_{k=n+1}^\infty F_n$ and $L_n = \bigoplus_{k=1}^n \Gamma_k \rtimes F_k$ satisfy that each $K_n \lhd L_n$, and with each $C^*_r(L_n/K_n) \cong \bigotimes_{\text{min}, k=1}^n C^*_r(\Gamma_k \rtimes F_k)$ admitting a unique tracial state. On $\lambda(\xi_{K_n})C^*_r(L_n) \cong C^*_r(L_n/K_n)$, where $\xi_{K_n} = \frac{1}{m(K_n)} 1_{K_n}$, this state is given by

$$u_n(x) = \frac{1}{m(K_n)} \langle \lambda(x)1_{K_n}, 1_{K_n} \rangle = \frac{m(xK_n \cap K_n)}{m(K_n)} \quad \text{for} \quad x \in L_n.$$ 

Now suppose there is $u$ in $TS_r(G)$. Then $\xi_{K_n} * u * \xi_{K_n}$ is a sequence of positive definite elements converging in $B_r(G)$-norm to $u$, hence converging uniformly. Hence we have uniform on compact convergence

$$u = \lim_n \xi_{K_n} * u * \xi_{K_n}|_{L_n} = \lim_n u_n.$$ 

But on any fixed $L_{n_0}$, the latter sequence converges pointwise to $1_{\{e\}}$, which is not continuous as $G$ is not discrete. Thus $TS_r(G) = \emptyset$. 

Remark 2.6. It curious to note how some conditions in Theorem 2.1 relate to *inner amenability*, the property that \(L^\infty(G)\) admits a conjugation-invariant mean.

(i) If \(G\) admits the property that each compactly generated open subgroup \(H\) is \([IN]\), then \(G\) is inner amenable.

Indeed, for each such \(H\) let \(W_H\) be a relatively compact invariant neighbourhood. Any cluster point in \(L^\infty(G)^*\) of the net of elements \(\xi_H = \frac{1}{m(W_H)}1_{W_H}\), indexed over increasing \(H\), is a conjugation invariant mean.

Moreover, if we can arrange that \(W_H \searrow \{e\}\), i.e. eventually enter any neighbourhood of \(e\), then \((\xi_H)_H\) shows that \(G\) has the quasi-small invariant neighbourhood property (QSIN) of Losert and Rindler [18]. Notice that this occurs in Suzuki’s example, above.

(ii) It is not the case that \([Am^{IN}]\)-groups are generally inner amenable. In particular, the class of inner amenable groups is not closed under extension.

Breuillard and Gelander [4] provide a dense free group \(F_6\) of rank 6 in \(S = \text{SL}_2(\mathbb{R})\). Let \(G = \mathbb{R}^2 \rtimes F_6\). If \(M\) were a conjugation-invariant mean on \(L^\infty(G)\), then it would restrict to one on \(\ell^\infty(F_6) \cong L^\infty(G/\mathbb{R}^2)\).

An observation of Effros [8] (see, also [18]) shows that the only such mean on \(\ell^\infty(F_6)\) is evaluation at \(e\). Hence \(M\) is concentrated on the identity component \(L^\infty(\mathbb{R}^2)\), hence giving an \(F_6\)-invariant mean on the latter.

A standard procedure, akin to that of building nets satisfying Reiter’s property \((P_1)\), allows us to construct a net \((f_\alpha)\) of probabilities in \(L^1(\mathbb{R}^2)\) for which
\[
\|f_\alpha \circ \sigma - f_\alpha\|_1 \to 0 \quad \text{for each } \sigma \text{ in } F_6.
\]

Since the action of \(\text{SL}_2(\mathbb{R})\) is continuous on \(\mathbb{R}^2\), it is continuous on \(L^1(\mathbb{R}^2)\), and hence, by density of \(F_6\) in \(S\), (2.2) holds for \(\sigma\) in \(S\).

Since the nilpotent part \(N\) of the Iwasawa decomposition \(S = KAN\) is the stabilizer of \(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\) in \(S\), we obtain an isomorphism of \(S\)-spaces
\[
k(\theta)a(t)N \mapsto k(\theta)a(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^t \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} : S/N \to \mathbb{R}^2 \setminus \{0\}
\]

where \(k(\theta)\) is the usual rotation by angle \(\theta\) and \(a(t)\) diagonal matrix with entries \(e^t\) and \(e^{-t}\). The usual formula for the Haar integral in \(S\) gives invariant integral \(\int_\mathbb{R} \int_0^{2\pi} f(k(\theta)a(t)N)e^{2t} d\theta dt\) on \(S/N\), while the Jacobian of \((\theta, t) \mapsto e^t \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}\) in \(\mathbb{R}^2 \setminus \{0\}\) is \(e^{2t}\). Hence the identification (2.3) identifies the \(S\)-invariant measure on \(S/N\) with the Lebesgue measure on \(\mathbb{R}^2 \setminus \{0\}\).
Thus the net \((f_\alpha) \subset L^1(\mathbb{R}^2)\), above, induces a net of means \((m_\alpha)\) on the space of left uniformly continuous functions \(LUC(S/N)\) which is asymptotically invariant for left translation. Any weak* cluster point of \((m_\alpha)\) must be an invariant mean, and hence \(G/N\) is an amenable homogeneous space in the sense of Eymard [10]. Since \(N\) is itself amenable, [10, §3 1°] would imply that \(S\) is amenable, which is absurd. Hence a mean \(M\), as above, cannot exist.

We wish to thank Jason Crann for pointing out an error in an earlier version of Remark 2.6. This also addresses an error in [17, Rem. 2.4].

3. Nuclearity and amenable traces

We wish to give another perspective to the question of existence of a trace. For unital C*-algebras the following is well-known.

**Lemma 3.1.** Let \(A\) be a non-unital C*-algebra and \(\tau\) a tracial state on \(A\). Then the map
\[
a \otimes b^\text{op} \mapsto \tau(ab)
\]
extends to a state on the maximal tensor product \(A \otimes_{\text{max}} A^\text{op}\).

**Proof.** We sketch a minor variant of the Gelfand-Neimark-Segal (GNS) construction. We let \(A_1\) denote the C*-unitization of \(A\), to which any state canonically extends. We complete the space \(A_1/N\) where \(N = \{a \in A : \tau(a^*a) = \tau(aa^*) = 0\}\) with inner product \(\langle a + N, b + N \rangle = \tau(b^*a) = \tau(ab^*)\), to get \(\mathcal{H}\), and let \(\pi : A \to B(\mathcal{H})\) and \(\pi^\text{op} : A^\text{op} \to B(\mathcal{H})\) be the respective left and right regular representations on \(A_1/N\), i.e. on \(\mathcal{H}\). Then \(\pi\) and \(\pi^\text{op}\) have commuting ranges, and we see that
\[
\tau(ab) = \langle \pi(a)\pi^\text{op}(b^\text{op})(1 + N), 1 + N \rangle
\]
which gives the result. □

In the spirit of Brown [6, 3.1.6] (really Kirchberg [10, Prop. 3.2]) we shall say that a tracial state \(\tau\) on \(A\) is amenable (or liftable) provided that (3.1) extends to a state on the minimal tensor product \(A \otimes_{\text{min}} A^\text{op}\).

The following is well-known for discrete groups.

**Theorem 3.2.** For a locally compact group \(G\), \(C^*_r(G)\) admits an amenable tracial state if and only if \(G\) is amenable.

**Proof.** If \(G\) is amenable, then it is well known that \(C^*_r(G)\) is nuclear (see for example [3, IV.3.5.2]) and hence any trace is amenable. As \(G\) is amenable \(1_G \in TS_r(G)\).

To see necessity let us first recall the well-known fact that \(C^*_r(G)\) is symmetric. The map \(f \mapsto \hat{f}\) on \(L^1(G)\), where \(\hat{f}(t) = \frac{1}{\Delta(t)}f(t^{-1})\) for
a.e. \( t \), is isometric and anti-multiplicative. The map \( \lambda(f) \mapsto \lambda(\tilde{f})^{op} \) extends to an isomorphism between \( C^*_r(G) \) and \( C^*_r(G)^{op} \). Indeed, for \( \xi, \eta \) in \( L^2(G) \) we have

\[
\langle \lambda(\tilde{f})\xi, \eta \rangle = \langle \xi, \lambda(\tilde{f})\eta \rangle = \langle \lambda(f)\bar{\eta}, \bar{\xi} \rangle
\]

and we take the supremum of the absolute value of this quantity over all choices of \( \|\xi\|_2 = \|\eta\|_2 = 1 \) to see that

\[
\|\lambda(\tilde{f})\| = \|\lambda(f)\|.
\]

Hence if \( C^*_r(G) \) admits an amenable tracial state \( \tau \), with associated \( u \) in \( TS_r(G) \), then

\[
\lambda(f) \otimes \lambda(g) \mapsto \lambda(f) \otimes \lambda(\tilde{g})^{op} \mapsto \tau(\lambda(f)\lambda(\tilde{g})) = \int_G f \ast \tilde{g} u
\]

defines an element of the dual of \( C^*_r(G \times G) \cong C^*_r(G) \otimes_{\min} C^*_r(G) \). We note that

\[
\int_G f \ast \tilde{g} u = \int_G \int_G f(t)\tilde{g}(t^{-1}s) dt u(s) ds
\]

\[
= \int_G \int_G f(t)\frac{g(s^{-1})}{\Delta(s)} u(ts) ds dt = \int_G \int_G f(t)g(s)u(ts^{-1}) ds dt
\]

so \( (s, t) \mapsto u(ts^{-1}) \) defines an element of \( B_r(G \times G) \). But then Lemma 1.2 (i) provides that \( 1 \in B_r(G \times G)|_D \subseteq B_r(D) \), where \( D = \{(s, s) : s \in G \} \cong G \), so \( G \) is amenable. \( \square \)

The following is part of the main result of Ng [19]; we use exactly his observations for sufficiency, but offer a different proof for necessity.

**Corollary 3.3.** A locally compact group is amenable if and only if \( C^*_r(G) \) is nuclear and admits a tracial state.

**Proof.** Necessity is proved in the sufficiency condition of the theorem above. It is evident that any tracial state on a nuclear C*-algebra is amenable, which gives sufficiency. \( \square \)

**Example 3.4.** Let \( G \) be an algebraic linear group which is semisimple, e.g. \( G = SL_n, n \geq 2 \), so \( G(\mathbb{Q}_p) \) is non-amenable. However, \( G(\mathbb{Q}_p) \) is type I by a result of Bernstein [2] and hence \( C^*_r(G(\mathbb{Q}_p)) \) is nuclear (see, for example, [3 IV.3.31 & IV.3.35]). Thus thanks to the last corollary, \( C^*_r(G(\mathbb{Q}_p)) \) admits no tracial state.

The field \( \mathbb{Q}_p \) of \( p \)-adic numbers is topologically singly generated as a ring: \( \mathbb{Q}_p = \langle 1/p \rangle \). Hence \( G(\mathbb{Q}) \) is compactly generated. Hence it follows Theorem 2.1 that this group is not in the class \([\text{Am}^{IN}]\).
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