Are Maxwell’s equations Lorentz-covariant?

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Abstract. The statement that Maxwell’s electrodynamics in vacuum is already covariant under Lorentz transformations is commonplace in the literature. We analyse the actual meaning of that statement and demonstrate that Maxwell’s equations are perfectly fit to be Lorentz-covariant; they become Lorentz-covariant if we construct to be so, by postulating certain transformation properties of field functions. In Aristotelian terms, the covariance is a plain potentiality, but not necessarily entelechy.

1. Introduction

Lorentz-covariance of Maxwell’s equations is certainly the key link between classical electrodynamics and special relativity. While there is a clear consensus in the literature that ‘the electrodynamic foundation of Maxwell–Lorentz’s theory is in agreement with the principle of relativity,’ and thus that Maxwell’s equations are Lorentz-covariant, the true meaning of that statement appears to be somewhat elusive. Generally, it is demonstrated that Maxwell’s equations are Lorentz-covariant if and only if the electric and magnetic fields and charge and current densities appearing in them transform according to some specific transformation laws. As is well known, this can be done basically in two ways: either transforming directly Maxwell’s equations (‘steep and difficult mountaineer’s path’) as Einstein originally did [1, 2, 3, 4], or employing the powerful and elegant, almost dazzling, tensorial approach in Minkowski space-time. Neither way is very transparent to the student.

On the other hand, the student of relativity encounters frequently some potentially confusing locutions on Lorentz-covariance of Maxwell’s equations which, in the long run, might lead the student to think that ‘requirement of form–invariance is automatically fulfilled for Maxwell’s fundamental equations of electrodynamics in vacuo.’ For example, in his classic book, Møller [5] states: ‘we saw that it is necessary to change the fundamental equations of mechanics in order to bring them into accordance with the principle of relativity. This is not so with the equations of electrodynamics in vacuum, the Maxwell equations, which, as we shall see, are already covariant under Lorentz transformations [...]’. In the same vein, Rindler [6] writes: ‘Having examined and relativistically modified Newtonian particle mechanics, it would be natural to look next with the same intentions at Maxwell’s electrodynamics, at first in vacuum. But
that theory turns out to be already “special-relativistic”. In other words, its basic laws, as summarized by the four Maxwell equations plus Lorentz’s force law, are form-invariant under Lorentz transformations, i.e. under transformations from one inertial frame to another.’ Similarly, Mario Bunge [7] asserts that relativistic electrodynamics ‘is not a new theory but a reformulation of CEM [classical electromagnetism], which was relativistic without knowing it.’ Also, in his fine book [8], Ugarov affirms: ‘It is remarkable that the system of Maxwell’s equations formulated fifty years prior to the advent of the special theory of relativity proved to be covariant with respect to the Lorentz transformation, i.e. it retains its appearance, with the accuracy of variables’ designations, under the Lorentz transformation. This signifies that the system of Maxwell’s equations retains its appearance in any inertial frame of reference, and the principle of relativity holds automatically.’ As the last characteristic example, I quote from a recent book by Christodoulides [9]: ‘It is obvious that electromagnetic theory, as expressed by Maxwell’s equations, is a relativistic theory, whose equations needed no modification in order to become compatible with the Theory of Relativity, at least as these apply to the vacuum.’

Recently, I pointed out that the above statements should be taken *cum grano salis*: Lorentz-covariance of Maxwell’s equations is *not* fulfilled automatically [10]. I noted that, for example, the so-called source-free Maxwell’s equations, \( \text{curl} \mathbf{E} = -\partial \mathbf{B} / \partial t \) and \( \text{div} \mathbf{B} = 0 \), are Lorentz-covariant if one defines \( \mathbf{E}' \) and \( \mathbf{B}' \) via \( \mathbf{E} \) and \( \mathbf{B} \) as given by the well-known transformation rules. A complete but succinct discussion of the issue is given in [11]. However, taking into account a possible relevance of the issue for the student of relativity, it is perhaps worthwhile to discuss in some detail what the above-mentioned authors actually meant by ‘Maxwell’s equations are already covariant under Lorentz transformations.’

2. Lorentz-covariance of Maxwell’s equations

2.1. Mathematical prelude

The problem of Lorentz-covariance of Maxwell’s equations is basically a mathematical question. In this Subsection, for the convenience of the reader, we recall some familiar results in the simple way, from vectorial perspective.

We begin by writing a set of coupled partial differential equations

\[
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t},
\]

\[
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t},
\]

\[
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t},
\]

\[
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0,
\]
where \( E_i = E_i(x, y, z, t) \) and \( B_i = B_i(x, y, z, t) \), \( i \) stands for subscripts \( x, y, z \), are functions of the mutually independent variables \( x, y, z \), and \( t \). Introduce another set of the mutually independent variables \( x', y', z' \), and \( t' \), and let them be the following functions of \( x, y, z \), and \( t \):

\[
x' = \gamma(x - vt) \quad ; \quad y' = y \quad ; \quad z' = z \quad ; \quad t' = \gamma(t - vx/c^2) , \tag{5}
\]

where \( \gamma \equiv (1 - v^2/c^2)^{-1/2} \), \( c \equiv \sqrt{1/\varepsilon_0 \mu_0} \), \( \varepsilon_0 \) and \( \mu_0 \) are positive constants, and \( v \) is a nonnegative constant satisfying \( 0 \leq v < c \).

As is well known, expressing unprimed by primed variables in equations (1)-(4), employing the standard procedure which involves the chain rule for differentiation, after some manipulations one obtains that the following primed equations apply (a detailed derivation is found, e.g., in [2], Section 8.2):

\[
\begin{align*}
\frac{\partial \mathcal{E}_x'}{\partial y'} - \frac{\partial \mathcal{E}_y'}{\partial z'} &= - \frac{\partial \mathcal{B}_x'}{\partial t'} , \tag{6} \\
\frac{\partial \mathcal{E}_x'}{\partial z'} - \frac{\partial \mathcal{E}_z'}{\partial x'} &= - \frac{\partial \mathcal{B}_y'}{\partial t'} , \tag{7} \\
\frac{\partial \mathcal{E}_y'}{\partial x'} - \frac{\partial \mathcal{E}_x'}{\partial y'} &= - \frac{\partial \mathcal{B}_z'}{\partial t'} , \tag{8} \\
\frac{\partial \mathcal{B}_x'}{\partial x'} + \frac{\partial \mathcal{B}_y'}{\partial y'} + \frac{\partial \mathcal{B}_z'}{\partial z'} & = 0 , \tag{9}
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{E}_x' & \equiv E_x \\
\mathcal{E}_y' & \equiv \gamma(E_y - vB_z) \\
\mathcal{E}_z' & \equiv \gamma(E_z + vB_y) \\
\mathcal{B}_x' & \equiv B_x \\
\mathcal{B}_y' & \equiv \gamma(B_y + \frac{v}{c}E_z) \\
\mathcal{B}_z' & \equiv \gamma(B_z - \frac{v}{c}E_y) \quad \left\{ \begin{array}{l}
\end{array} \right. \tag{10}
\end{align*}
\]

In equations (10) \( \mathcal{E}_i' = \mathcal{E}_i'(x', y', z', t') \) and \( E_i = E_i[\gamma(x' + vt'), y', z', \gamma(t' + vx'/c^2)] \) and analogously for \( \mathcal{B}_i' \) and \( B_i \). Obviously, equations (6)-(9) have the same form as equations (1)-(4). Thus, transforming equations (1)-(4) by transformation of variables (5), one reveals that those equations imply that, in the primed variables, equations (6)-(9) of the same form apply under the proviso that \( \mathcal{E}_i' \) and \( \mathcal{B}_i' \) therein be given by identities (10). Consequently, if \( E_i \) and \( B_i \) satisfy unprimed equations (1)-(4), one knows that \( \mathcal{E}_i' \) and \( \mathcal{B}_i' \) determined by identities (10) satisfy primed equations (6)-(9).

Note that, from equations (10) and (5), \textit{mutatis mutandis}, one obtains the following inverse identities:

\[
\begin{align*}
E_x' & \equiv \mathcal{E}_x' \\
E_y' & \equiv \gamma(\mathcal{E}_y' + v\mathcal{B}_z') \\
E_z' & \equiv \gamma(\mathcal{E}_z' - v\mathcal{B}_y') \\
B_x' & \equiv \mathcal{B}_x' \\
B_y' & \equiv \gamma(\mathcal{B}_y' - \frac{v}{c}\mathcal{E}_z') \\
B_z' & \equiv \gamma(\mathcal{B}_z' + \frac{v}{c}\mathcal{E}_y') \quad \left\{ \begin{array}{l}
\end{array} \right. \tag{11}
\end{align*}
\]

which of course are obtained quickly by interchanging primed and unprimed quantities and replacing \( v \) by \(-v\) in (10).
Assume now that functions \( E_i \) and \( B_i \), in addition to equations (1)-(4), must also satisfy another set of equations:

\[
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 \rho u_x + \frac{1}{c^2} \frac{\partial E_x}{\partial t},
\]

\[
\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 \rho u_y + \frac{1}{c^2} \frac{\partial E_y}{\partial t},
\]

\[
\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 \rho u_z + \frac{1}{c^2} \frac{\partial E_z}{\partial t},
\]

\[
\rho = \epsilon_0 \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \equiv \rho_E ,
\]

where \( \rho(x, y, z, t) \) is the 'charge density' (without attaching any physical meaning to it), and \( u_i = u_i(x, y, z, t) \) are Cartesian components of the velocity field of the 'charge.' The only constraint imposed by equations (12)-(15) on \( \rho \) and \( u_i \) is the 'equation of continuity,'

\[
\frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} + \frac{\partial \rho}{\partial t} = 0 ,
\]

which is a necessary condition for the validity of equations (12)-(15). Thus, the equation of continuity may apply even if equations (12)-(15) do not apply [12].

Introduce symbol

\[
\rho'_E \equiv \epsilon_0 \left( \frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} \right) ,
\]

where \( E'_i \) are given by identities (10). Employing the standard procedure, one finds that \( \rho'_E \) transforms according to equation

\[
\rho'_E = \gamma \left[ \rho_E - \epsilon_0 v \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{1}{c^2} \frac{\partial E_x}{\partial t} \right) \right] ,
\]

wherefrom using equations (12) and (15) one obtains

\[
\rho'_E = \gamma \left( \rho - \frac{v}{c^2} \rho u_x \right) ,
\]

Making use of the familiar transformations for velocity field components,

\[
u'_x = \frac{u_x - v}{1 - u_x v/c^2} , \quad u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)} , \quad u'_z = \frac{u_z}{\gamma(1 - u_x v/c^2)} ,
\]

and their inverse, from eq. (19) one gets

\[
\rho = \gamma \left( \rho'_E + \frac{v}{c^2} \rho'_E u'_x \right) .
\]

Transforming eq. (12) through eqs. (11) yields directly

\[
\frac{\partial B'_z}{\partial y'} - \frac{\partial B'_y}{\partial z'} = \mu_0 \left( \frac{\rho u_x}{\gamma} - \rho'_E v \right) + \frac{1}{c^2} \frac{\partial E'_x}{\partial t'} ,
\]
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which using eq. (19) and the first formula (20) gives

\[
\frac{\partial B'_y}{\partial y'} - \frac{\partial B'_z}{\partial z'} = \mu_0 \left[ \gamma \left( \rho - \frac{v}{c^2} \rho u_x \right) \right] u'_x + \frac{1}{c^2} \frac{\partial E'_x}{\partial t'},
\]

(23)

Transforming in the same way eqs. (13)-(15), and using formulas (20) and eq. (23), one gets

\[
\frac{\partial B'_x}{\partial z'} - \frac{\partial B'_y}{\partial x'} = \mu_0 \left[ \gamma \left( \rho - \frac{v}{c^2} \rho u_x \right) \right] u'_y + \frac{1}{c^2} \frac{\partial E'_y}{\partial t'},
\]

(24)

\[
\frac{\partial B'_y}{\partial x'} - \frac{\partial B'_x}{\partial y'} = \mu_0 \left[ \gamma \left( \rho - \frac{v}{c^2} \rho u_x \right) \right] u'_z + \frac{1}{c^2} \frac{\partial E'_z}{\partial t'},
\]

(25)

\[
\gamma \left( \rho - \frac{v}{c^2} \rho u_x \right) = \rho'_E,
\]

(26)

respectively. Eq. (26) is identical with eq. (19), as it should be.

Equations (1)-(4) and (12)-(15) can obviously be recast into the compact form

\[
\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot B = 0, \quad \nabla \times B = \mu_0 \rho u + \epsilon_0 \mu_0 \frac{\partial E}{\partial t}, \quad \nabla \cdot E = \frac{\rho}{\epsilon_0},
\]

(27)

and the transformed equations (6)-(9) and (23)-(26) can be recast into

\[
\nabla' \times E' = -\frac{\partial B'}{\partial t'}, \quad \nabla' \cdot B' = 0, \quad \nabla' \times B' = \mu_0 \rho' u' + \epsilon_0 \mu_0 \frac{\partial E'}{\partial t'}, \quad \nabla' \cdot E' = \frac{\rho'}{\epsilon_0},
\]

(28)

where \(\rho' \equiv \gamma \left( \rho - \frac{v}{c^2} \rho u_x \right)\).

Thus, transforming equations (27) by transformation of variables (5), one obtains that, in primed variables, equations (28) of the same form apply under the proviso that \(E'\) and \(B'\) therein be defined by identities (10). Consequently, if \(E\) and \(B\) satisfy unprimed equations (27), one knows that \(E'\) and \(B'\) defined by identities (10) satisfy primed equations (28). This is all one can extract from the unprimed Maxwell’s equations (27), transforming them by the Lorentz transformation (5).

2.2. Are Maxwell’s equations Lorentz-covariant?

Comparing equations (27) and (28), the following conclusion is readily reached: in order that Maxwell’s equations which apply in unprimed variables, hold also in primed variables, (that is, in standard parlance, in order that Maxwell’s equations be Lorentz-covariant), it suffices to define \(E'\) and \(B'\) (by which we mean counterparts of \(E\) and \(B\) in primed variables) by equations

\[
E' \overset{d}{=} E', \quad B' \overset{d}{=} B',
\]

(29)

and the ‘charge density’ in primed variables, \(\rho'\), by equation

\[
\rho' \overset{d}{=} \gamma \left( \rho - \frac{v}{c^2} \rho u_x \right).
\]

(30)
Since transformation properties of $E$, $B$ and $\rho$ are not known beforehand, $E'$, $B'$ and $\rho'$ have to be defined by equations (29) and (30), i.e. through transformation laws. Introducing those definitions is an inconspicuous but indispensable step, a conditio sine qua non for Lorentz-covariance of Maxwell’s equations in the strict sense of the word. \[11\]

Now we are armed with all the facts necessary to answer our query, are Maxwell’s equations Lorentz-covariant. That is, do Maxwell’s equations retain their form under transformation of variables (5)? The correct answer appears to be: the Maxwell equations are ready-made to be Lorentz-covariant, but they are actually Lorentz-covariant only if we construct to be so (cf, e.g., \[13, 14\]). As was demonstrated above, what exactly is sufficient to be postulated for the covariance emanates from the equations themselves. In this sense, and in this sense only, one can speak about ‘a miracle [that] Maxwell, fully unaware of relativity, had nevertheless written his equations in a relativistically covariant form straight away’ \[15\]. However, the covariance is not fulfilled automatically; there is no covariance without postulating specific transformation properties of the quantities appearing in the equations (with the exception of course of purely geometric quantities, such as velocity and acceleration, which are already defined in both unprimed and primed coordinates, and whose transformation properties follow from the definitions). In Aristotelian terms, Lorentz-covariance is contained in Maxwell’s equations as a plain potentiality, but not as entelechy. One should keep this in mind.

Einstein’s original demonstration that ‘the electrodynamic foundation of Lorentz’s theory of the electrodynamics of moving bodies agrees with the principle of relativity,’ \[1\] is basically mathematics disguised as physics. Einstein postulates that Maxwell’s equations conform to the principle of relativity and thus that both eqs. (27) and equations

$$\nabla' \times E' = -\frac{\partial B'}{\partial t'}, \nabla' \cdot B' = 0, \nabla' \times B' = \mu_0 \rho' u' + \epsilon_0 \mu_0 \frac{\partial E'}{\partial t'}, \nabla' \cdot E' = \frac{\rho'}{\epsilon_0},$$

(31)

hold; basically, he thus postulates that Maxwell’s equations are Lorentz–covariant. Comparing (mutually equivalent) equations (28) and (31), he deduces the necessary conditions for the covariance (eqs. (29) and (30) regarded as transformation laws).

3. Concluding comments

Take now that the Lorentz transformation (5) has its received physical meaning, i.e., assume that it relates space and time coordinates of an event in a given inertial frame $S$ with the space and time coordinates of the same event in an inertial frame $S'$ which is in a standard configuration with $S$. As is well known, assuming the validity of Maxwell’s

\[\dagger\] While it appears that a multiplicative factor $\Psi(v)$ could be included in eqs. (29) and (30), a simple analysis demonstrates that $\Psi(v)$ must equal one \[1\].

\[\ddagger\] Incidentally, recall that the equation of continuity (16) itself is ready-made to be not only Lorentz-covariant but also Galilei-covariant (cf, e.g., \[16, 17\]). Whichever covariance is preferred on physical grounds, the remaining one then becomes a purely mathematical property.
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In the given frame $S$, and also taking that $E'$, $B'$ and $\rho'$ are defined by equations (29) and (30) (achieving thus Lorentz-covariance of Maxwell’s equations), would ensure the validity of Maxwell’s equations in any reference frame $S'$ in uniform translation with respect to $S$, if the special theory of relativity is valid. In this context, definitions (29) and (30) express the electric and magnetic fields and charge density in $S'$, and thus, basically, represent a fundamental physical assumption. However, as Bartocci and Mamone Capria pointed out, the plain possibility of achieving Lorentz-covariance of Maxwell’s equations can be regarded as nothing more than an interesting mathematical property devoid of any physical contents [18]. It is perhaps instructive to recognize that the formal covariance can be employed as a handy tool, quite outside the relativistic framework [10].

To summarize, in the context of physics, if Maxwell’s equations describe physical fields in an inertial frame $S$, and Lorentz transformations relate space and time coordinates of the same event as observed in two inertial frames $S$ and $S'$ in relative motion, Lorentz-covariance of Maxwell’s equations expresses a fundamental physical assumption that the same (primed!) Maxwell’s equations describe the physical fields also in the $S'$ frame. On the other hand, from the mathematical side, what is latent in Maxwell’s equations is, first, that they are ready-made to be Lorentz-covariant, and, second, the precise ‘recipe’ how to achieve that they actually be Lorentz-covariant. Shortly, Maxwell’s equations are Lorentz-covariant if we construct to be so, but they need not be. However, it was indeed a miracle that Maxwell had written his equations in a form perfectly fit to be Lorentz-covariant. From this perspective, Heinrich Hertz’s feeling that Maxwell’s equations ‘give back to us more than was originally put into them,’ proved prophetic.

Finally, note that, as is well known, analysis of Maxwell’s equations can be often made much easier in terms of potentials. For the sake of completeness, a brief discussion of Lorentz-covariance of Maxwell’s equations from the perspective of potentials, skipping the familiar details, is given in Appendix.

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Appendix

First of all, recall that Maxwell’s equations imply the equation of continuity (16):

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z} = 0,$$

(A.1)

Thus, Rindler’s formulation that Maxwell’s equations ‘fit perfectly into the scheme of special relativity’ [19], should perhaps be amended as ‘can fit perfectly into the scheme of special relativity.’
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which is a necessary condition for the validity of Maxwell’s equations and thus it may be valid even if Maxwell’s equations do not apply. Recall also that Maxwell’s equations possess, inter alia, a nice property that they allow themselves to be considerably simplified mathematically by expressing \( E_i \) and \( B_i \) in terms of potentials \( \Phi \) and \( A_i \) introduced by

\[
\begin{align*}
E_x &= -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t}, \\
E_y &= -\frac{\partial \Phi}{\partial y} - \frac{\partial A_y}{\partial t}, \\
E_z &= -\frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{\partial t}, \\
B_x &= \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}, \\
B_y &= \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}, \\
B_z &= \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}
\end{align*}
\]

(A.2)

Assuming that the potentials satisfy the Lorenz gauge condition,

\[
\frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0,
\]

(A.3)

it follows that the potentials then should satisfy the inhomogeneous d’Alembert type equations:

\[
\Box \Phi = -\frac{\rho}{\epsilon_0}, \quad \Box A_x = -\mu_0 \rho u_x, \quad \Box A_y = -\mu_0 \rho u_y, \quad \Box A_z = -\mu_0 \rho u_z,
\]

(A.4)

(A.5)

where

\[
\Box \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2},
\]

(A.6)

Now transform the continuity equation (A.1) replacing unprimed by primed variables according to equations (5), employing formulae for changing partial differential coefficients

\[
\frac{\partial}{\partial t'} = \gamma \left( \frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'} \right), \quad \frac{\partial}{\partial x'} = \gamma \left( \frac{\partial}{\partial x'} - \frac{v}{c^2} \frac{\partial}{\partial t'} \right), \quad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z'}.
\]

(A.7)

One obtains

\[
\frac{\partial}{\partial t'} \gamma (\rho - \frac{v}{c^2} \rho u_x) + \frac{\partial}{\partial x'} \gamma (\rho u_x - \rho v) + \frac{\partial}{\partial y'} (\rho u_y) + \frac{\partial}{\partial z'} (\rho u_z) = 0.
\]

(A.8)

Using equations (20), one has

\[
\frac{\partial}{\partial t'} \gamma (\rho - \frac{v}{c^2} \rho u_x) + \frac{\partial}{\partial x'} \gamma \rho u'_x (1 - \frac{u_x v}{c^2}) + \frac{\partial}{\partial y'} (\rho u_y) + \frac{\partial}{\partial z'} (\rho u_z) = 0.
\]

(A.9)

Inspecting the last equation, the following conclusion is readily reached: in order that equation of continuity (A.1) implies equation of the same form and content in primed variables, it suffices to define the charge density in primed coordinates, \( \rho' \), by

\[
\rho' = \gamma \left( \rho - \frac{v}{c^2} \rho u_x \right).
\]

(A.10)

With that definition, equation (A.9) obviously reduces to

\[
\frac{\partial \rho'}{\partial t'} + \frac{\partial (\rho' u'_x)}{\partial x'} + \frac{\partial (\rho' u'_y)}{\partial y'} + \frac{\partial (\rho' u'_z)}{\partial z'} = 0,
\]

(A.11)
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which is identical with equation (A.1), except for primes. Functions \( \rho' u'_i \) are Cartesian components of the convection current density in the \( S' \) frame, as \( \rho u_i \) are in the \( S \) frame. Clearly, \( \rho c, \rho u_x, \rho u_y \) and \( \rho u_z \) transform according to the rules

\[
\rho' c = \gamma (\rho c - \frac{v}{c} \rho u_x), \quad \rho' u'_x = \gamma (\rho u_x - \frac{v}{c} \rho c), \quad \rho' u'_y = \rho u_y, \quad \rho' u'_z = \rho u_z, \quad (A.12)
\]

under the Lorentz transformation (5).

Now transform in the same way another simple equation, the Lorenz gauge condition (A.3). One obtains automatically

\[
\frac{1}{c^2} \frac{\partial}{\partial t'} \gamma (\Phi - v A_x) + \frac{\partial}{\partial x'} \gamma (A_x - \frac{v}{c^2} \Phi) + \frac{\partial}{\partial y'} A_y + \frac{\partial}{\partial z'} A_z = 0. \quad (A.13)
\]

Obviously, in order that equation (A.3) be Lorentz-covariant, it suffices to define functions of primed variables \( \Phi', A'_x, A'_y \) and \( A'_z \) by equations

\[
\Phi' = \gamma (\Phi - v A_x), \quad A'_x = \gamma (A_x - \frac{v}{c^2} \Phi), \quad A'_y = A_y, \quad A'_z = A_z, \quad (A.14)
\]

the result that Poincaré reached a long time ago by a different path, postulating charge invariance and invariance of the inhomogeneous d’Alembert type equations for the potentials \([20]\). Since the primed functions are defined by equations (A.14), it follows that \( \Phi, A_x, A_y \) and \( A_z \) a fortiori transform according to equations (A.14) under the Lorentz transformation (5). For convenience, recast the transformation rules into

\[
\frac{\Phi'}{c} = \gamma \left( \frac{\Phi}{c} - \frac{v}{c} A_x \right), \quad A'_x = \gamma \left( A_x - \frac{v}{c^2} \Phi \right), \quad A'_y = A_y, \quad A'_z = A_z. \quad (A.15)
\]

A glance at equations (A.12) reveals that, with definition (A.10) of \( \rho' \) density, \( \rho c, \rho u_x, \rho u_y \) and \( \rho u_z \) become contravariant components of a 4-vector of Minkowski space-time; equation (A.15) shows that the analogous conclusion applies to \( \Phi/c, A_x, A_y \) and \( A_z \). Thus, 4-current density \( J^\mu \) and 4-potential \( A^\mu \) are constructed.

Now we arrived at the familiar, wide and well trodden path, and no need to go further. Namely, as is well known, in the tensorial notation, with 4-vectors \( J^\mu \) and \( A^\mu \), Lorentz-covariance of Maxwell’s equations is an obvious fact, offered as on a plate. What is perhaps less obvious is that, instead of simply asserting that \( \Phi \) and \( \mathbf{A} \) together constitute a 4-vector, it would be more correct to specify that now \( \Phi \) and \( \mathbf{A} \) together constitute a 4-vector per definitionem, namely, we constructed to be so.\[\footnote{As Rindler ([6], p 155) notes, ‘[...] we can construct a tensor by specifying its components arbitrarily in one coordinate system, say \( \{x^i\} \), and then using the transformation law [expressing the familiar informal definition of tensors] to define its components in all other systems, or, in the case of a qualified tensor, in all those systems which are mutually connected by transformations belonging to the chosen subgroup.’}]

Thus, in the language of potentials and 4-tensors, our main conclusion is reached in a simpler and more transparent way. Maxwell’s equations are perfectly fit to be Lorentz-covariant; they become Lorentz-covariant only if we define the primed potentials and
charge density so that \((\Phi/c, A)\) and \((pc, \rho u)\) be 4-vectors of Minkowski space-time. Deciding that \((\Phi/c, A)\) and \((pc, \rho u)\) be contravariant components of 4-vectors ensures Lorentz-covariance of Maxwell’s equations, enabling us to recast those equations in an explicitly Lorentz-covariant form.

Thus, we can agree with Sommerfeld’s [21] simile that ‘the true mathematical structure of these entities [\(\Phi\) and \(A\)] will appear only now [in the language of 4-tensors], as in a mountain landscape when the fog lifts,’ only in the framework of the interpretation given above. Sommerfeld’s claim that ‘by reducing the Maxwell equations to the four-vector \([A^\mu, J^\mu\) and the operators \(\partial_\mu, \Box = -\partial_\mu \partial^\mu\)] we have demonstrated at the same time their general validity, independent of the coordinate system’ is, strictly speaking, incorrect. Basically, we have assumed a four-vector character of \((\Phi/c, A)\) and \((pc, \rho u)\) and thus we have constructed that ‘the Maxwell equations satisfy the relativity postulate from the very beginning.’ This constructional aspect of Lorentz-covariance of Maxwell’s equations, clearly enunciated by Einstein [22] and by Bergmann [23], seems to be understated in the literature.

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