FEASIBLE BASES FOR A POLYTOPE RELATED TO THE HAMILTON CYCLE PROBLEM

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Abstract. We study a certain polytope depending on a graph G and a parameter $\beta \in (0, 1)$ which arises from embedding the Hamiltonian cycle problem in a discounted Markov decision process. Eshragh et al. [11] conjectured a lower bound on the proportion of feasible bases corresponding to Hamiltonian cycles in the set of all feasible bases. We make progress towards a proof of the conjecture by proving results about the structure of feasible bases. In particular, we prove three main results: (1) the set of feasible bases is independent of the parameter $\beta$ when the parameter is close to 1, (2) the polytope can be interpreted as a generalized network flow polytope and (3) we deduce a combinatorial interpretation of the feasible bases. We also provide a full characterization for a special class of feasible bases, and we apply this to provide some computational support for the conjecture.

1. Introduction

The Hamilton Cycle Problem (HCP) is one of the classical problems in combinatorics. Given a graph $G$, the problem is to decide if $G$ contains a cycle that visits each node exactly once. Cycles that pass through every node of a graph exactly once are called Hamilton cycles. If a graph contains at least one Hamilton cycle, then it is called Hamiltonian. Otherwise, it is non-Hamiltonian. The HCP is NP-complete even for planar graphs with maximum degree three for undirected graphs, and maximum degree two for directed graphs [16, 18], so it is unlikely that there is an exact algorithm which terminates in polynomial time and solves the problem in general.

The Traveling Salesman Problem (TSP) asks for a Hamilton cycle of minimum weight in an arc-weighted graph, and therefore the HCP is a special case of the TSP where all the weights are either zero or one. Due to its importance in many applications and its rich mathematical structure the TSP has attracted the attention of many researchers. In particular, it has been one of the major driving forces for the development of polyhedral techniques in combinatorial optimization (see [2] for a nice overview). The underlying idea is to identify a subset of the arc set of a graph on $n$ vertices with its characteristic vector in $\mathbb{R}^{n(n-1)/2}$. Then the TSP is asking for the minimum of a linear function over the convex hull of the set of Hamilton cycles, and this convex hull is known as the traveling salesman polytope.

In 1994, Filar and Krass [14] proposed a new approach to the HCP, based on the theory of Markov Decision Processes (MDPs). An MDP comprises a state space, an action space, transition probabilities between states (which depend on the actions taken by the decision maker) and a reward function. In the basic setting, the decision maker takes an action, receives a reward from the environment, and the environment changes its state. Next, the decision maker identifies the state of the environment, takes a further action, obtains a reward, and so forth. The state transitions are probabilistic, and depend solely on the state and the action taken by the decision maker. The reward obtained by the decision maker depends on the action taken, and on the current state of the environment. The decision maker’s actions in each environmental state are prescribed by a policy. The model introduced by Filar and Krass [14] initiated a new line of research and has attracted growing attention (see, for instance, [3, 4, 6, 7, 8, 9, 13, 15, 17]). In particular, Feinberg [13] investigated the relationship between the HCP and discounted MDPs. In discounted MDPs, a discount factor $\beta \in (0, 1)$, which represents the difference in importance between future and present rewards, is used.

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to discount rewards. Feinberg presented a polytope, which we shall refer to as $F_\beta(G)$, constructed from an input graph $G$. He showed that the Hamilton cycles of $G$ correspond to certain extreme points of the polytope $F_\beta(G)$, called Hamiltonian extreme points. Ejov et al. [8] described some geometric properties of $F_\beta(G)$ and Eshragh et al. [12] transformed $F_\beta(G)$ to a combinatorially equivalent polytope $H_\beta(G)$ to avoid certain numerical issues. Moreover, they constructed a new polytope $WH_\beta(G)$ by adding new linear constraints, called wedge constraints, to the polytope $H_\beta(G)$.

In 2011, Eshragh and Filar [10] partitioned the extreme points of $H_\beta(G)$ into five types, consisting of Hamiltonian extreme points and four types of non-Hamiltonian extreme points. They showed that for a discount factor $\beta$ sufficiently close to one, the wedge constraints cut off the non-Hamiltonian extreme points of types 2, 3 and 4, while preserving the Hamiltonian extreme points. In addition, they proposed to use a random walk on the extreme points (or on the feasible bases) of the polytopes $H_\beta(G)$ and $WH_\beta(G)$ to search for a Hamilton cycle, and they showed that for a Hamiltonian graph $G$, the random walk algorithm detects a Hamiltonian extreme point with probability one in a finite number of iterations.

For a more precise analysis of the efficiency of this random walk approach it is necessary to understand the combinatorial structure of the polytopes. More precisely, it is required to analyze the prevalence of Hamiltonian extreme points within the set of all extreme points, as well as the mixing properties of the random walk. This was the motivation for the work of Eshragh et al. [11] who established results on the combinatorial structure of the polytope $H_\beta(G)$. They characterized feasible bases of the polytope $H_\beta(G)$ for a general input graph $G$, and determined the expected numbers of different types of feasible bases when the underlying graph is random. They showed that for a random graph, the number of feasible bases corresponding to Hamiltonian extreme points of $H_\beta(G)$ is exponentially small compared to the total number of feasible bases. Moreover, they demonstrated that the wedge constraints eliminate a large number of non-Hamiltonian feasible bases, and they provided computational evidence for the efficiency of the random walk on the feasible bases of $WH_\beta(G)$. Based on their computational and theoretical results, they conjectured that for a random graph on $n$ nodes the ratio between the number of feasible bases corresponding to Hamilton cycles and the total number of feasible bases is asymptotically bounded below by $c/n^k$ for some positive constants $c$ and $k$.

In this paper, we continue this line of research by studying the structure of the polytope $WH_\beta(G)$. In particular, we are interested in characterizing feasible bases for this polytope. In Section 2, we introduce some notation and provide relevant background from the literature. Section 3 is about general results on feasible bases for $WH_\beta(G)$. In particular, we prove that there is a constant $\alpha^* = \alpha^*(n) < 1$ such that for all graphs $G$ on $n$ vertices, the set of feasible bases for $WH_\beta(G)$ does not depend on $\beta$ as long as $\alpha^* \leq \beta < 1$. Then we establish a close relationship between $WH_\beta(G)$ and a generalized network flow polytope for a graph that is obtained from $G$ by splitting each node into two nodes. The third result of Section 3 is a proof that any feasible basis contains a set of $n$ arcs forming a collection of node-disjoint cycles, such that in the corresponding basic feasible solution, the values of the variables corresponding to these $n$ arcs tend to 1 as $\beta \rightarrow 1$, while the values of the remaining basic variables tend to 0. In Section 4, we use the relation to the generalized network flow polytope to give a complete characterization of feasible bases of $WH_\beta(K_n)$ which correspond to a Hamilton cycle in the generalized network flow setting.

2. Notation and background

Consider a digraph $G = (V, E)$ without loops and parallel arcs, where $V = [n] = \{1, 2, \ldots, n\}$ is the set of nodes and $E$ is the set of arcs. Throughout this paper, $G$ refers to such a digraph on $n$ nodes, unless otherwise stated. For each node $i \in V$, the in-neighborhood $N^-(i)$, and the out-neighborhood $N^+(i)$ are the sets

$$N^-(i) = \{j \in V : (j, i) \in E\}, \quad N^+(i) = \{j \in V : (i, j) \in E\}.$$ 

Furthermore, we use the following notations to denote the total inflow and total outflow for node $i$, respectively:

$$\Phi(i) = \sum_{j \in N^-(i)} x_{ji}, \quad \Psi(i) = \sum_{j \in N^+(i)} x_{ij}.$$ 

As indicated earlier, Feinberg [13] defined a polytope depending on the graph $G$, and showed that finding a Hamilton cycle is equivalent to finding an extreme point of this polytope whose support corresponds to a
Hamilton cycle. The support of an extreme point is defined to be the set of its non-zero coordinates. More precisely, he proved the following theorem.

**Theorem 2.1** (Feinberg [13]). Consider a digraph $G = (V, E)$, a parameter $\beta$ with $0 < \beta < 1$, and let $F_\beta(G) \subseteq \mathbb{R}^{|E|}$ be the polytope defined by the constraints

$$\sum_{j \in \mathcal{N}^+(i)} y_{ij} - \beta \sum_{j \in \mathcal{N}^-(i)} y_{ji} = 0 \quad \text{for all } i \in V \setminus \{1\},$$

$$\sum_{j \in \mathcal{N}^+(i)} y_{ij} - \beta \sum_{j \in \mathcal{N}^-(i)} y_{ji} = 1, \quad \text{for all } i \in V \setminus \{1\},$$

$$\sum_{j \in \mathcal{N}^+(1)} y_{1j} = 1 - \beta^n, \quad y_{ij} \geq 0 \quad \text{for all } (i, j) \in E.$$ (2.2)

The graph $G$ is Hamiltonian if and only if there exists an extreme point of $F_\beta(G)$ which has exactly $n$ positive coordinates tracing out a Hamilton cycle in $G$.

Eshragh et al. [12] modified $F_\beta(G)$ by a coordinate transformation $x_{ij} = (1 - \beta^n)y_{ij}$ for all $(i, j) \in E$. The resulting polytope $H_\beta(G) \subseteq \mathbb{R}^{|E|}$ is defined by the constraints

$$\sum_{j \in \mathcal{N}^+(i)} x_{ij} - \beta \sum_{j \in \mathcal{N}^-(i)} x_{ji} = 1 - \beta^n, \quad \text{for all } i \in V \setminus \{1\},$$

$$\sum_{j \in \mathcal{N}^+(i)} x_{ij} - \beta \sum_{j \in \mathcal{N}^-(i)} x_{ji} = 0 \quad \text{for all } i \in V \setminus \{1\},$$

$$\sum_{j \in \mathcal{N}^+(1)} x_{1j} = 1, \quad x_{ij} \geq 0 \quad \text{for all } (i, j) \in E.$$ (2.6)

Since values of $\beta$ close to one were shown to be important in [10, 12], this transformation eliminates numerical instability in (2.3). The following definition is motivated directly from Theorem 2.1.

**Definition 1.** Let $x$ be an extreme point of the polytope $H_\beta(G)$. If the positive coordinates of $x$ trace out a Hamilton cycle in the graph $G$, $x$ is called a Hamiltonian extreme point. Otherwise, it is called a non-Hamiltonian extreme point.

Eshragh et al. [12] observed that, if $x$ is the Hamiltonian extreme point corresponding to a Hamilton cycle $C$ then its components are given by $x_{ij} = \beta^k$ if $(i, j)$ is the $(k - 1)$-th arc in $C$ starting from node 1, and $x_{ij} = 0$ if $(i, j)$ is not contained in $C$. In particular, the Hamiltonian extreme points satisfy the $2(n - 1)$ wedge constraints

$$\beta^{n-1} \leq \sum_{j \in \mathcal{N}^+(i)} x_{ij} \leq \beta \quad \text{for all } i \in V \setminus \{1\}.$$ (2.9)
which cut off some non-Hamiltonian extreme points. Adding the wedge constraints and introducing slack
variables $y_i$, we obtain the polytope $\mathcal{WH}_\beta(G)$ described by the following constraints:

$$\sum_{j \in N^+(i)} x_{ij} - \beta \sum_{j \in N^-(i)} x_{ji} = 1 - \beta^n, \quad (2.10)$$

$$\sum_{j \in N^+(i)} x_{ij} - \sum_{j \in N^-(i)} x_{ji} = 0 \quad \text{for all } i \in V \setminus \{1\}, \quad (2.11)$$

$$\sum_{j \in N^+(i)} x_{ij} = 1, \quad (2.12)$$

$$\sum_{j \in N^+(i)} x_{ij} - y_i = \beta^{n-1} \quad \text{for all } i \in V \setminus \{1\}, \quad (2.13)$$

$$0 \leq y_i \leq \beta - \beta^{n-1} \quad \text{for all } i \in V \setminus \{1\}, \quad (2.14)$$

$$x_{ij} \geq 0 \quad \text{for all } (i, j) \in E. \quad (2.15)$$

A basis for this polytope can be specified by a triple $(B, L, U)$, where $B$ is the set of basic variables, and the
sets $L$ and $U$ are non-basic $y$-variables at their lower and upper bounds, respectively. In other words, $L \cup U$ is
the partition of the set $\{i \in V \setminus \{1\} : y_i$ is a non-basic variable$\}$, such that $y_i = 0$ for $i \in L$ and $y_i = \beta - \beta^{n-1}$
for $i \in U$. By a slight abuse of notation, we will simply call the triple $(B, L, U)$ a basis. The set $B$ can be
identified with the union $A \cup Y$, where $A \subseteq E$ is the set of arcs $(i, j) \in E$ such that $x_{ij}$ is a basic variable, and
$Y \subseteq V \setminus \{1\}$ is the set of nodes $i$ such that $y_i$ is a basic variable. A basis $(B, L, U)$ can then be interpreted
as a node-colored digraph on the node set $V$: the arc set is $A$ and the color classes are $\{1\}, Y, L$ and $U$. If the
unique solution of the system of equations (2.10)–(2.13) corresponding to $(B, L, U)$ satisfy the lower and upper bound constrains (2.14)–(2.15), then the basis $(B, L, U)$ is feasible, otherwise it is infeasible.

Remark 1. For the extreme point $(x, y)$ corresponding to a basis $(B, L, U)$, the total in and outflows of the
nodes in $L \cup U$ are as follows

$$\Phi(i) = \sum_{j \in N^-(i)} x_{ji} = \begin{cases} \beta^{n-2} & \text{for } i \in L, \\ 1 & \text{for } i \in U, \end{cases} \quad (2.16)$$

$$\Psi(i) = \sum_{j \in N^-(i)} x_{ji} = \begin{cases} \beta^{n-1} & \text{for } i \in L, \\ \beta & \text{for } i \in U, \end{cases} \quad (2.17)$$

Eshragh et al. [12] introduced the concept of quasi-Hamiltonian extreme points (bases) to search for Hamilton
cycles among the extreme points (or the feasible bases) of the polytope $\mathcal{WH}_\beta(G)$.

Definition 2. An extreme point $(x, y)$ of $\mathcal{WH}_\beta(G)$ is called quasi-Hamiltonian if any walk $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_n \rightarrow i_{n+1}$, where $i_{k+1} \in \arg \max_{j \in N^+(i_k)} \{x_{ikj}\}$ for $k = 1, \ldots, n$, is a Hamilton cycle in $G$. A feasible basis
corresponding to a quasi-Hamiltonian extreme point is called quasi-Hamiltonian basis.

For a positive integer $n$ and a probability $p$, $0 < p < 1$ (which may depend on $n$), let $G_{n,p}$ be the graph on $n$
vertices obtained by adding each arc $(i, j)$ independently with probability $p$, and then adding the arcs of a
randomly chosen Hamilton cycle (this is very similar to the random graph model studied in [5]). Motivated
by a random walk approach to the HCP for sparse random graphs, the following conjecture was made in [11].

Conjecture 1. There exist positive constants $c$, $\delta$ and $k$ such that for all $\beta \in (1 - e^{-cn}, 1)$, with high probability,
the expected proportion of feasible bases of $\mathcal{WH}_\beta(G_{n,p})$ that are quasi-Hamiltonian is at least $\delta/n^k$.

Proving this conjecture would be a first step towards a random walk based algorithm for the HCP on $G_{n,p}$
for very small $p$.

3. General results

In this section we establish general results about feasible bases of $\mathcal{WH}_\beta(G)$. In Subsection 3.1, we show
that the set of feasible bases for $\mathcal{WH}_\beta(G)$ does not depend on the parameter $\beta$ for values of $\beta$ sufficiently close
to one. Then, in Subsection 3.2, we demonstrate that $WH_β(G)$ can be interpreted as a generalized network flow polytope for a certain network on $2n$ nodes obtained by splitting each node of $G$ into two nodes. The polyhedral theory of the generalized network flow problem implies that the bases in this setting correspond to subgraphs in which every connected component is an augmented tree, that is, a graph obtained from adding a single arc to a tree, thus creating a unique cycle. We prove that the feasible bases in the generalized network flow interpretation of $WH_β(G)$ are always connected, that is, a feasible basis corresponds to a spanning augmented tree. Finally, in Subsection 3.3, we show that in the original graph $G$ every feasible basis contains a spanning collection of node-disjoint cycles, such that the values of the variables corresponding to the arcs of these cycles are close to one, while the values of the remaining variables are close to zero.

3.1. Ultimate bases. Let $Γ(G, β)$ be the set of feasible bases for $WH_β(G)$. We first show that there exists $α^* ∈ (0, 1)$ such that $Γ(G, β_1) = Γ(G, β_2)$ whenever $α^* < β_1 ≤ β_2 < 1$, that is, the set $Γ(G, β)$ does not depend on the parameter $β$, as long as it is sufficiently close to 1. Consider a triple $(B, L, U)$ with $|B| = 2n$, where $B = A ∪ Y$, $A ⊆ E$ and $Y ⊆ V \ \{1\}$, and $L ∪ U$ is a partition of the set $V \ \{(1) ∪ Y\}$. Let $M_B(β)$ be the submatrix of the constraint matrix of the polytope $WH_β(G)$ corresponding to $B$, set $y_i = 0$ for $i ∈ L$ and $y_i = β - β^{n-1}$ for $i ∈ U$, and let $b(β)$ be the vector that is obtained from the right hand sides of (2.10) through (2.13), where for $i ∈ U$ the $β^{n-1}$ in (2.13) is replaced by $β^{n-1} + (β - β^{n-1}) = β$. The triple $(B, L, U)$ is a feasible basis for $WH_β(G)$ if and only if the following two conditions are satisfied:

1. $M_B(β)$ is invertible, that is, $det(M_B(β)) ≠ 0$, and
2. the $y$-components of the vector $(x, y) = M_B(β)^{-1}b(β)$ satisfy (2.14), and the $x$-components satisfy (2.15).

We refer to these conditions as independence and feasibility, respectively. In the next two lemmas we show that these two properties do not depend on $β$ for $β$ sufficiently close to 1. The determinant $det(M_B(β))$ is a polynomial in $β$. If this is the zero polynomial, then the independence condition is not satisfied for any $β$. On the other hand, if the determinant is not the zero polynomial, then $B$ is independent for all $β$ sufficiently close to 1. For the formal argument we let $B$ be the set of all sets of 2n variables such that the corresponding columns are independent for some $β$, that is,

\[ B = \{B : B = A ∪ Y, A ⊆ E, Y ⊆ V \ \{1\}, |B| = 2n, \ det(M_B(β)) ≠ 0\}. \]

Since $B$ is finite, there is an $α ∈ [0, 1)$ such that none of the polynomials $det M_B(β)$, $B ∈ B$, has a root in the open interval $(α, 1)$. This implies the following lemma.

**Lemma 3.1.** There exists $α ∈ (0, 1)$ such that for all $β > α$, and for all $B = A ∪ Y$, $A ⊆ E$, $Y ⊆ V \ \{1\}$, with $|B| = 2n$, the matrix $M_B(β)$ is invertible for every $B ∈ B$.

**Proof.** Define a function $H : B → [0, 1]$ as follows:

\[ H(B) = \begin{cases} \max\{β : 0 ≤ β < 1 \ det(M_B(β)) = 0\} & \text{if } \ det(M_B(β)) = 0 \text{ for some } β ∈ (0, 1), \\ 0 & \text{otherwise}. \end{cases} \]

Then $α = \max\{H(B) : B ∈ B\}$ has the claimed property. It follows immediately from the construction that $0 ≤ α < 1$, and that for every $B ∈ B$ and every $β$ with $α < β < 1$, $det(M_B(β)) ≠ 0$. □

The next step is to show that feasibility is also independent of $β$ for values sufficiently close to 1.

**Lemma 3.2.** There exists $α^* ∈ (α, 1)$ such that for all $β ≥ α^*$, and for all $B ∈ B$, the vector $M_B(β)^{-1}b(β)$ satisfies (2.14) and (2.15) if and only if $M_B(α^*)^{-1}b(α^*)$ satisfies (2.14) and (2.15).

**Proof.** For $B = A ∪ Y ∈ B$, and let $β ∈ (α, 1)$, let $(x^B(β), y^B(β)) = M_B^{-1}(β)b(β)$. The components of this vector are rational functions of $β$ with denominator $det(M_B(β))$, say

\[ x^B_{ij}(β) = \frac{f^B_{ij}(β)}{det(M_B(β))} \text{ for } (i, j) ∈ A, \quad y^B_i(β) = \frac{g^B_i(β)}{det(M_B(β))} \text{ for } i ∈ Y. \]

Let $h^B_i(β) = β - β^{n-1} - g^B_i(β)/det(M_B(β))$. We define

\[ T^B_{ij}(β) = \max\{β : 0 < β < 1, det(M_B(β))f^B_{ij}(β) = 0\} \]

for $i ∈ Y$, $j ∈ A$. We set $T^B_{ij}(β) = 0$ when $f^B_{ij}(β) = 0$. If $T^B_{ij}(β) = α^*$, then $y^B_i(β) = 0$, otherwise, this is the case when $β ∈ (α^*, 1)$. This implies the following lemma.

**Lemma 3.3.** There exists $α^* ∈ (α, 1)$ such that for all $β ≥ α^*$, and for all $B ∈ B$, the vector $M_B(β)^{-1}b(β)$ satisfies (2.14) and (2.15) if and only if $M_B(α^*)^{-1}b(α^*)$ satisfies (2.14) and (2.15).
if \( \det(M_B(\beta))f_B^P(\beta) \) is not identically zero, but vanishes for some \( \beta \in (0, 1) \), and \( T_{ij}(B) = 0 \) otherwise. Similarly, we define
\[
K_i(B) = \max\{\beta : 0 < \beta < 1, h_i^B(\beta)g_i^P(\beta) = 0\}
\]
if \( h_i^B(\beta)g_i^P(\beta) \) is not identically zero, but vanishes for some \( \beta \in (0, 1) \), and \( K_i(B) = 0 \) otherwise. Let
\[
\tau = \max_B \max_{(i,j) \in A} T_{ij}(B), \quad \kappa = \max_B \max_{v \in V} K_i(B),
\]
and \( \gamma = \max\{\tau, \kappa\} \). Let \( \alpha^* = \gamma + (1 - \gamma)/2 \in (\alpha, 1) \). It follows from Lemma 3.1 and \( \alpha^* \in (\alpha, 1) \), that \( M_B(\beta) \) is invertible for every \( B \in B \) and every \( \beta \in [\alpha^*, 1) \). Furthermore, as \( \alpha^* > \max\{\tau, \kappa\} \), none of the functions \( f_B^P(\cdot), g_B^P(\cdot), h_i^B(\cdot) \) and \( \det(A_B(\cdot)) \) changes sign on the interval \([\alpha^*, 1]\), and this implies that \( \alpha^* \) has the claimed property.

Combining Lemmas 3.1 and 3.2 we obtain the following result which justifies the subsequent definition.

**Proposition 1.** There exists \( \alpha^* \geq \alpha \) such that for all \( \beta \in [\alpha^*, 1) \), \( \Gamma(G, \beta) = \Gamma(G, \alpha^*) \).

**Definition 3.** We call the elements of \( \Gamma(G, \beta) \) for \( \beta \geq \alpha^* \) ultimate bases for \( G \), and we denote the set of ultimate bases for \( G \) by \( \Gamma(G) \).

Henceforth, we consider ultimate bases of \( WH_\beta(G) \) and we set \( \beta = 1 - \delta \) where \( \delta \) tends to zero. We omit the argument \( \beta \) whenever there is no danger of confusion.

### 3.2. A generalized network flow formulation

In this subsection, we show that the polytope \( WH_\beta(G) \) can be interpreted as a generalized network flow (GNF) polytope and we establish results about feasible bases of \( WH_\beta(G) \) in the GNF setting. We first recall some definitions and results pertaining to the GNF polytope, see [1, Chapter 15] for details. Let \( X = (V(X), E(X)) \) be a digraph with capacities \( c_{v,w} \) and positive rational multipliers \( \mu_{vw} > 0 \) for each arc \((v, w) \in E(X)\), and demands/supply \( b_v \) for each node \( v \in V(X) \). The GNF polytope for this data is defined by the following constraints:

\[
\sum_{w \in N^+(v)} x_{vw} - \sum_{w \in N^-(v)} \mu_{wv}x_{wv} = b_v \quad \text{for } v \in V(X),
\]
\[
0 \leq x_{vw} \leq c_{vw} \quad \text{for } (v, w) \in E(X).
\]

A possible interpretation is that for every unit of flow that enters arc \((v, w)\) in node \( v \), \( \mu_{vw} \) units arrive at node \( w \). The polytope \( WH_\beta(G) \) given by (2.10) through (2.15) is a GNF polytope for the digraph \( G' \) obtained from \( G \) by splitting each node \( i \) into two nodes \( v_i \) and \( w_i \), replacing arcs \((i, j)\) by \((v_i, w_i), \) and adding arcs \((v_i, w_i)\) for \( i = 2, 3, \ldots, n \). Then \( y_i \) is the flow on arc \((v_i, w_i), \) the multipliers are equal to \( \beta \) for all arcs of the form \((w_i, v_i), \) and equal to 1 for all arcs of the form \((v_i, w_i), \) More precisely, the digraph \( G' = (V', E') \) has node set \( V' \) and arc set \( E' \) given by
\[
V' = \{v_i : i \in V\} \cup \{w_i : i \in V\}, \quad E' = \{(v_i, w_j) : (i,j) \in E\} \cup \{(v_i, w_i) : i \in V \setminus \{1\}\}.
\]

We denote the two parts in the partition of \( E' \) as \( E_1' \) and \( E_2' \), that is, \( E_1' = \{(w_i, v_j) : (i,j) \in E\} \) and \( E_2' = \{(v_i, w_i) : i \in V \setminus \{1\}\} \). This construction of \( G' \) is illustrated in Figure 1. The constraints (2.10) to (2.14) determine the supply/demand vector \( b \) as follows:

1. Constraint (2.12) says that \( w_1 \) has a supply of 1, that is, \( b_{w_1} = 1 \).
2. Together with (2.10), this implies that \( v_1 \) has a demand of \( \beta^{n-1} \), that is, \( b_{v_1} = -\beta^{n-1} \).
3. By constraint (2.13), node \( w_i \) for \( 2 \leq i \leq n \) has a supply of \( \beta^{n-1} \), that is \( b_{w_i} = \beta^{n-1} \).
4. From (2.13) and (2.11), it follows that, for \( i = 2, 3, \ldots, n \),
\[
y_i - \beta \sum_{j \in N^-(i)} x_{ji} = -\beta^{n-1},
\]

hence node \( v_i \) has demand \( \beta^{n-1} \), that is, \( b_{v_i} = -\beta^{n-1} \).
Figure 1. A digraph $G$, and the corresponding digraph $G'$ such that $WH_\beta(G)$ is a GNF polytope for $G'$. The sets $E'_1$ and $E'_2$ are indicated by solid and dashed lines, respectively.

Figure 2. Flow conservation for a node $i \in V \setminus \{1\}$. The two ovals represent the set $\{w_j : i \in N^-(i)\}$, and $\{v_j : i \in N^+(i)\}$, respectively. Flow conservation in $w_i$ corresponds to (2.13), and flow conservation in $v_i$ is (2.11) (substituting $y_i + \beta^{n-1}$ for $\sum_{j \in N^+(i)} x_{ij}$).

The correspondence between the GNF in $G'$ and the model (2.10) through (2.15) is illustrated in Figure 2. To summarize, the vector $b$ is given by

$$b_v = \begin{cases} 
1 & \text{for } v = w_1, \\
\beta^{n-1} & \text{for } v \in \{w_2, w_3, \ldots, w_n\}, \\
-\beta^{n-1} & \text{for } v \in \{v_1, v_2, \ldots, v_n\}.
\end{cases}$$

The interpretation of $WH_\beta(G)$ as a GNF polytope allows us to apply the known results about the structure of bases for GNF polytopes to $WH_\beta(G)$ and identify bases of $WH_\beta(G)$ with subgraphs of $G'$. For this purpose, we introduce some terminology (following [1, Section 15.3]), and then state the characterization of bases of GNF polytopes in Theorem 3.1. For a cycle $C$ (not necessarily directed) with a given orientation, let $C^-$ and $C^+$ denote the sets of forward and backward arcs in $C$. The cycle multiplier is

$$\mu(C) = \frac{\prod_{(v,w) \in C} \mu_{vw}}{\prod_{(v,w) \in C^+} \mu_{vw}}.$$ If one unit of flow is sent along $C$, starting at some node $s$, then $\mu(C)$ units return to this node. A cycle $C$ is called a breakeven cycle if $\mu(C) = 1$.

Definition 4. An augmented tree is a connected graph with exactly one cycle, called the extra cycle. An augmented forest is a collection of node-disjoint augmented trees.

Definition 5. An augmented tree is called a good augmented tree if its extra cycle is not a breakeven cycle. An augmented forest as a good augmented forest if each of its components is a good augmented tree.

In our specific setting it turns out that the breakeven condition is equivalent to having the same number of forward and backward arcs.

Definition 6. An oriented cycle is called balanced if it has the same number of forward and backward arcs.
Proposition 2. A cycle $C$ in $G'$ is breakeven if and only if it is balanced.

Proof. Let $C$ be a cycle in $G'$, and let $C_1$ and $C_2$ denote the set of arcs of $C$ in $E_1'$ and $E_2'$, respectively. As the multipliers of the arcs in $C_1$ are all equal to $\beta$ and the multipliers of the arcs in $C_2$ are all equal to 1, we have $\mu(C) = \beta^d$, where $d$ is the difference between the number of forward arcs and the number of backward arcs in $C_1$. As a consequence, $C$ is a breakeven cycle if and only if $C_1$ contains the same number of forward and backward arcs.

Let $e_1, e_2, \ldots, e_k$ be the elements of $C_2$ in the order in which they are traversed by $C$. Let $P_1, \ldots, P_k$ be the paths into which $C$ is cut by deleting the arcs in $C_2$. More precisely, $P_i$ is the path from $e_i$ to $e_{i+1}$ for $i = 1, \ldots, k-1$, and $P_k$ is the path from $e_k$ to $e_1$. Let $d_i$ be the difference between the number of forward and backward arcs on $P_i$. If $P_i$ starts and ends at a forward arc then $d_i = 1$, if it starts and ends at a backward arc, then $d_i = -1$, and otherwise $d_i = 0$. Now we conclude, that $C$ is a breakeven cycle if and only if $C_2$ contains the same number of forward and backward arcs if and only if $C$ is balanced. $\square$

From Proposition 2 we deduce that an augmented forest in $G'$ is good if and only if none of its extra cycles is balanced. This is illustrated in Figure 3 where (b) and (c) depict an augmented tree and an augmented forest, respectively, for the digraph $G'$ corresponding to the digraph $G$ in (a). The tree in Figure 3 (b) is not a good augmented tree because the extra cycle is balanced. The forest in Figure 3 (c) is a good augmented forest as the extra cycles are not balanced.

Theorem 3.1 (Section 15.5 in [1]). Let $X$ be a directed graph. For a partition $E(X) = B \cup L \cup U$, the triple $(B, L, U)$ is a basis for a GNF polytope with underlying digraph $X$ if and only if $B$ is a good augmented forest which spans all the nodes of $X$.

We denote the GNF interpretation of the polytope $\mathcal{W}_\beta(G)$ by $\mathcal{P}_\beta(G')$. In Theorem 3.2 we strengthen Theorem 3.1 for the polytope $\mathcal{P}_\beta(G')$ by showing that in order to obtain a feasible basis, the augmented forest has to be connected, that is, it has to be a good augmented tree.

Theorem 3.2. If $(B', L', U')$ is a feasible basis for $\mathcal{P}_\beta(G')$, then $B'$ is a good augmented tree.

This will be proved by showing that the assumption of more than one connected components leads to a contradiction. For this purpose we need some preliminary results which are stated in Lemmas 3.3–3.6. For the rest of this subsection, we assume that $(B', L', U')$ is a basis for $\mathcal{P}_\beta(G')$. In particular, $B'$ is a good augmented forest, and we assume it has connected components $T_1, \ldots, T_m$. We shall use $T_k$ to denote both the arc set and
the node set of the good augmented tree $T_k$. The intended meaning will be clear from the context. Without loss of generality, $w_1 \in T_1$. For convenience, we set

$$V_k = \{ i \in V : v_i \in T_k \text{ and } w_i \in T_k \},$$

$$V_k^- = \{ i \in V : v_i \not\in T_k \text{ and } w_i \in T_k \},$$

$$V_k^+ = \{ i \in V : v_i \in T_k \text{ and } w_i \not\in T_k \}.$$

Thus, $V_k$ correspond to the arcs in $E_2'$ which have both endpoints in $T_k$, $V_k^-$ corresponds to the arcs in $E_2'$ which enter $T_k$, and $V_k^+$ corresponds to the arcs in $E_2'$ which leave $T_k$. The next lemma states that every component $T_k$ must contain both $v_i$ and $w_i$ for some $i \neq 1$.

**Lemma 3.3.** For every $k \in \{1, \ldots, m\}$, the extra cycle in $T_k$ contains an arc from $E_2'$. In particular, $V_1 \setminus \{1\} \neq \emptyset$ and $V_k \neq \emptyset$, for $k \in \{2, \ldots, m\}$.

**Proof.** Fix $k \in \{1, \ldots, m\}$, let $C$ be the extra cycle in $T_k$, and assume that every arc in $C$ is in $E_1'$. Thus, in $C$ the nodes $w_i$ only have outgoing arcs and the nodes $v_i$ only have incoming arcs. This implies that $C$ is balanced which is a contradiction. $\square$

In the next lemma we analyze the interaction between components. The arcs in $E_1'$ can carry flow only if they correspond to basic variables. As a consequence, the flow transfer between connected components is only through arcs in $E_2'$, that is, arcs of the form $(v_i, w_i)$.

**Lemma 3.4.**

$$\begin{align*}
(1-\beta) \sum_{j \in V_k} \Phi(j) &= \beta \sum_{j \in V_k^-} \Phi(j) - \sum_{j \in V_k^+} \Phi(j) \quad k = 2, \ldots, m, \\
(1-\beta) \sum_{j \in V_k} \Phi(j) &= 1 - \beta^n + \beta \sum_{j \in V_k^-} \Phi(j) - \sum_{j \in V_k^+} \Phi(j). 
\end{align*}$$

**Proof.** Let $k \in \{2, \ldots, m\}$. Summing (2.11) over all $i \in V_k \cup V_k^-$ and then swapping the indices $i$ and $j$, and splitting the sum into the terms for $j \in V_k$ and the terms for $j \in V_k^-$, we obtain

$$\sum_{i \in V_k \cup V_k^-} \sum_{j \in N^+(i)} x_{w_i, v_j} = \beta \sum_{i \in V_k \cup V_k^-} \sum_{j \in N^-(i)} x_{w_i, v_j} = \beta \sum_{j \in V_k \cup V_k^-} \sum_{i \in N^-(j)} x_{w_i, v_j} = \beta \left( \sum_{j \in V_k} \Phi(j) + \sum_{j \in V_k^-} \Phi(j) \right).$$

All the nonzero terms on the left hand side have $j \in V_k \cup V_k^+$, so we can rearrange the left hand side as follows:

$$\sum_{i \in V_k \cup V_k^-} \sum_{j \in N^+(i)} x_{w_i, v_j} = \sum_{i \in V_k \cup V_k^-} \sum_{j \in N^+(i) \cap V_k} x_{w_i, v_j} + \sum_{j \in V_k \cup V_k^-} \sum_{i \in N^+(i) \cap V_k^+} x_{w_i, v_j} + \sum_{i \in V_k \cup V_k^-} \sum_{j \in N^-(j)} x_{w_i, v_j} = \sum_{j \in V_k} \Phi(j) + \sum_{j \in V_k^-} \Phi(j),$$

where for the second equality we used that for $j \in V_k \cup V_k^+$ and $i \in N^-(j)$, the variable $x_{w_i, v_j}$ can be nonzero only if $i \in V_k \cup V_k^-$. Therefore,

$$\sum_{j \in V_k} \Phi(j) + \sum_{j \in V_k^+} \Phi(j) = \beta \left( \sum_{j \in V_k} \Phi(j) + \sum_{j \in V_k^-} \Phi(j) \right).$$
Lemma 3.6. For \( k = 1 \), we proceed similarly. We start by summing constraints (2.10) and (2.11) over all \( i \in V_1 \cup V'_1 \):

\[
\sum_{i \in V_1 \cup V'_1} \sum_{j \in N^-(i)} x_{w_i v_j} = 1 - \beta^n + \beta \sum_{i \in V_1 \cup V'_1} \sum_{j \in N^-(i)} x_{w_j v_i} = 1 - \beta^n + \beta \sum_{i \in V_1 \cup V'_1} \sum_{j \in N^+(j)} x_{w_i v_j} = 1 - \beta^n + \beta \left( \sum_{j \in V_1} \Phi(j) + \sum_{j \in V'_1} \Phi(j) \right).
\]

As above, the left hand side is equal to \( \sum_{j \in V_1} \Phi(j) + \sum_{j \in V'_1} \Phi(j) \), and therefore,

\[
\sum_{j \in V_1} \Phi(j) + \sum_{j \in V'_1} \Phi(j) = 1 - \beta^n + \beta \left( \sum_{j \in V_1} \Phi(j) + \sum_{j \in V'_1} \Phi(j) \right),
\]

which is equivalent to (3.2).

Next we show that every component must have the same number of incoming and outgoing arcs in \( E'_2 \).

Lemma 3.5. For every \( k \in \{1, \ldots, m\} \), \( |V^-_k| = |V'^+_k| \).

Proof. For every \( k \in \{1, \ldots, m\} \), the left hand side of (3.1) and (3.2), respectively, is

\[
(1 - \beta) \sum_{j \in V_k} \Phi(j) = \delta |V_k| (1 - O(\delta)) = O(\delta).
\]

For \( k = 2, \ldots, m \), the right hand side of (3.1) is

\[
\beta \sum_{j \in V^-_k} \Phi(j) - \sum_{j \in V'^+_k} \Phi(j) = (1 - \delta) |V^-_k| (1 - O(\delta)) - |V'^+_k| (1 - O(\delta)) = |V^-_k| - |V'^+_k| + O(\delta),
\]

and the right side of (3.2) is

\[
1 - \beta^n + \beta \sum_{j \in V^-_k} \Phi(j) - \sum_{j \in V'^+_k} \Phi(j) = (1 - \delta) |V^-_k| (1 - O(\delta)) - |V'^+_k| (1 - O(\delta)) + O(\delta) = |V^-_k| - |V'^+_k| + O(\delta).
\]

In both cases, we conclude that \( |V^-_k| - |V'^+_k| = O(\delta) \), and the claim follows because the left hand side is an integer.

For \( i \in V^-_1 \cup V'^+_1 \), we have \((v_i, w_i) \notin B'\). As a consequence, \( i \in L' \cup L' \) which implies that \( \Phi(i) \in \{1, \beta^{n-2}\} \).

For \( k \in \{1, \ldots, m\} \), set

\[
r_k = |V^-_k| = |V'^+_k|, \quad s_k = |\{i \in V^-_k : \Phi(i) = 1\}|, \quad q_k = |\{i \in V'^+_k : \Phi(i) = 1\}|.
\]

and for \( k \in \{2, \ldots, m\} \), \( z_k = |\{i \in V'^+_k : \Phi(i) = \beta^{n-1}\}| \), or equivalently, \( z_k = 1 \) if \( 1 \in V'^+_k \), and \( z_k = 0 \) otherwise.

Lemma 3.6.

\[
(n - 2)(s_k - q_k) + z_k - r_k \geq 1 \quad k = 2, \ldots, m, \quad (3.3)
\]
\[
(n - 2)(s_1 - q_1) + n - r_1 - (z_2 + \cdots + z_m) \geq 1. \quad (3.4)
\]

Proof. We start by bounding the left hand sides of the identities (3.1) and (3.2) from below:

\[
\delta \sum_{j \in V_k} \Phi(j) \geq \delta |V_k| \beta^{n-1} = \delta |V_k| (1 - O(\delta)) \geq \delta + O(\delta^2), \quad (3.5)
\]
where the second inequality follows from Lemma 3.3. Next we express the sums on the right hand sides of (3.1) and (3.2) in terms of the quantities $r_k$, $s_k$, $q_k$ and $z_k$. First consider the case $k \in \{2, \ldots, m\}$. Then

\[
\sum_{j \in V_k^-} \Phi(j) = s_k + (r_k - s_k)\beta^{n-2} = r_k - (r_k - s_k)(n-2)\delta + O(\delta^2),
\]

\[
\sum_{j \in V_k^+} \Phi(j) = q_k + (r_k - q_k - z_k)\beta^{n-2} + z_k\beta^{n-1} = r_k - [(r_k - q_k)(n-2) + z_k] \delta + O(\delta^2).
\]

The right hand side of (3.1) is

\[
(1 - \delta) \sum_{j \in V_k^-} \Phi(j) - \sum_{j \in V_k^+} \Phi(j) = r_k - (r_k - s_k)(n-2)\delta - r_k \delta - r_k + [(r_k - q_k)(n-2) + z_k] \delta + O(\delta^2) = [(s_k - q_k)(n-2) - r_k + z_k] \delta + O(\delta^2),
\]

and with (3.5) we obtain (3.3). For $k = 1$,

\[
\sum_{j \in V_1^-} \Phi(j) = s_1 + (r_1 - s_1 - (z_2 + \cdots + z_m))\beta^{n-2} + (z_2 + \cdots + z_m)\beta^{n-1}
\]

\[
= r_1 - [(r_1 - s_1)(n-2) + (z_2 + \cdots + z_m)] \delta + O(\delta^2),
\]

\[
\sum_{j \in V_1^+} \Phi(j) = q_1 + (r_1 - q_1)\beta^{n-2} = r_1 - (r_1 - q_1)(n-2)\delta + O(\delta^2).
\]

The right hand side of (3.2) is

\[
1 - \beta^n + (1 - \delta) \sum_{j \in V_1^-} \Phi(j) - \sum_{j \in V_1^+} \Phi(j) = n \delta + r_1 - [(r_1 - s_1)(n-2) + (z_2 + \cdots + z_m)] \delta - r_1 \delta - r_1 + (r_1 - q_1)(n-2) \delta + O(\delta^2)
\]

\[
= [n + (s_1 - q_1)(n-2) - r_1 - (z_2 + \cdots + z_m)] \delta + O(\delta^2).
\]

and with (3.5) we obtain (3.4).

We are now ready to finish the proof of Theorem 3.2.

Proof of Theorem 3.2. It follows from (3.3) that $r_k \geq 1$, for $k = 2, \ldots, m$, because $r_k = 0$ implies $s_k = q_k = z_k = 0$ which forces the left hand side of (3.3) to be zero. Then (3.3) implies $(n-2)(s_k - q_k) \geq 1 + r_k - z_k \geq 1$, hence $s_k \geq q_k + 1$. On the other hand, by definition we have

\[
s_1 + \cdots + s_m = q_1 + \cdots + q_m = \left\{ i \in V \setminus \bigcup_{k=1}^m V_k : \Phi(i) = 1 \right\}.
\]

Therefore, $q_1 - s_1 = (s_2 + \cdots + s_m) - (q_2 + \cdots + q_m) \geq m - 1 \geq 1$, and (3.4) implies

\[
n - 2 \leq (n-2)(q_1 - s_1) \leq n - r_1 - (z_2 + \cdots + z_m) - 1,
\]

which simplifies to $r_1 + z_2 + \cdots + z_m \leq 1$. If $r_1 = 0$, then (3.2) becomes

\[
\sum_{j \in V_1} \Phi(j) = \frac{1 - \beta^n}{1 - \beta} = 1 + \beta + \beta^2 + \cdots + \beta^{n-1} = n + O(\delta).
\]

But $\Phi(j) = 1 - O(\delta)$ for every $j \in V$, so the left hand side is $|V_1| + O(\delta)$, and this implies $|V_1| = n$ which contradicts the assumption that $B'$ is not connected. Therefore, $r_1 = 1$ and $z_2 + \cdots + z_m = 0$. (3.4) becomes $(n-2)(q_1 - s_1 + 1) \geq 0$, and together with $s_1 - q_1 \leq 1 - m$, we obtain $m = 2$ and $q_1 = s_1 + 1$. Furthermore,
\(q_1 \leq r_1\), this implies \(q_1 = 1\) and \(s_1 = 0\), and then \(\Phi(j) = 1\) for the unique node \(j \in V_1^+\), and \(\Phi(j') = \beta^{n-2}\) for the unique node \(j' \in V_1^-\). Now (3.2) becomes

\[
(1 - \beta) \sum_{j \in V_1} \Phi(j) = 1 - \beta^n + \beta^{n-1} - 1 = (1 - \beta)\beta^{n-1}.
\]

From \(z_2 + \cdots + z_m = 0\), we have \(1 \in V_1\), and with \(\Phi(1) = \beta^{n-1}\), this implies \(\sum_{j \in V_1 \setminus \{1\}} \Phi(j) = 0\). But then \(V_1 = \{1\}\), which contradicts Lemma 3.3.

3.3. Thick and thin arcs. In this subsection, we show that for a feasible basis of the polytope \(WH_3(G)\), the values of the basic variables either tend to one or zero as \(\beta \to 1\). More precisely, we show that there are exactly \(n\) arcs corresponding to basic variables that tend to 1, and that these \(n\) arcs form a collection of node-disjoint cycles which we call thick cycles. Furthermore, we provide upper bounds for the number of non-basic \(y\)-variables which are at their lower bounds and the number of non-basic \(y\)-variables which are at their upper bounds.

Definition 7. Let \((B, L, U) \in \Gamma(G)\), where \(B = A \cup Y\) and let \(x = A_0^{-1} b\). The arc \((i, j) \in A\) is called thick with respect to \(B\), if \(x_{ij} = 1 - O(\delta)\). The arc \((i, j) \in A\) is called thin with respect to \(B\) if \(x_{ij} = O(\delta)\). Otherwise, the arc \((i, j) \in A\) is called intermediate with respect to \(B\).

We start by expressing the sum of all flow variables in terms of \(\delta\).

Lemma 3.7. For every point \(x \in WH_3(G)\),

\[
\sum_{(i, j) \in E} x_{ij} = \sum_{i=1}^{n} \Phi(i) = \sum_{i=1}^{n} \Psi(i) = \left(\frac{n}{1}\right) - \left(\frac{n}{2}\right) \delta + \left(\frac{n}{3}\right) \delta^2 - \cdots + (-1)^{n-1} \left(\frac{n}{n}\right) \delta^{n-1}.
\]

Proof. The first two equalities follow from the observation that \(\sum_{i=1}^{n} \Phi(i)\) and \(\sum_{i=1}^{n} \Psi(i)\) are two different ways of computing the sum of all the \(x_{ij}\), first by grouping the arcs according to their end node, and second by grouping them according to their start node. With \(\Psi(1) = 1\), \(\Phi(1) = \beta^{n-1}\) and \(\Psi(i) = \beta \Phi(i)\) for all \(i \in V \setminus \{1\}\), we obtain

\[
\sum_{i=1}^{n} \Psi(i) - 1 = \sum_{i=2}^{n} \Psi(i) = \beta \sum_{i=1}^{n-1} \Phi(i) = \beta \left(\sum_{i=1}^{n} \Phi(i) - \beta^{n-1}\right) = \beta \left(\sum_{i=1}^{n} \Psi(i) - \beta^{n-1}\right),
\]

which implies

\[
\sum_{i=1}^{n} \Psi(i) = \frac{1 - \beta^n}{1 - \beta} = \sum_{k=0}^{n-1} \beta^k = \sum_{k=0}^{n-1} (1 - \delta)^k = \sum_{k=0}^{n-1} \sum_{l=0}^{k} \binom{k}{l} (-\delta)^l = \sum_{l=0}^{n-1} \left(-1\right)^l \sum_{k=l}^{n-1} \binom{k}{l} \delta^l = \sum_{l=0}^{n-1} \left(-1\right)^l \left(\frac{n}{l+1}\right) \delta^l.
\]

\(\square\)

Theorem 3.3. Let \((B, L, U) \in \Gamma(G)\), where \(B = A \cup Y\) and let \(A_1, A_2\) denote the sets of thick and thin arcs, respectively. Then

(i) \(A = A_1 \cup A_2\),
(ii) \(A_1\) forms a spanning collection of node-disjoint directed cycles,
(iii) for every \(i \in V \setminus \{1\}\), the digraph with arc set \(A_1 \cup A_2\) contains a directed path from node 1 to node \(i\),
(iv) \(|L| \leq (n - 1)/2\) and \(|U| \leq (n - 1)/2\).

Proof.

(i) For the sake of contradiction, assume that there exists an intermediate arc \((i, j) \in A\). It follows from (2.10)-(2.13) that, for every \(i \in V\), the total outflow and the total inflow tend to one as \(\beta \to 1\). Thus, there is at least one more intermediate arc leaving node \(i\), say \((i, \ell)\). Similarly, nodes \(j\) and \(\ell\) have another intermediate incoming arc. This argument shows, that every node with an outgoing intermediate arc has at least two outgoing intermediate arcs, and every node with an incoming intermediate arc has at least two incoming intermediate arcs. As a consequence \(A\) contains a cycle \(C\) which alternates
between forward and backward arcs, which implies that the corresponding cycle \( C' \) in \( B' \) is balanced. This contradicts the assumption that \( B' \) is a basis for \( \mathcal{P}(G') \).

(ii) Using (2.13), (2.14), and (2.12), for every \( i \in V \),

\[
\left| \{ j \in N^+(i) : (i, j) \in A_1 \} \right| - O(\delta) = \sum_{j \in N^+(i)} x_{ij} - \sum_{j \in N^+(i)} x_{ij} = 1 - O(\delta).
\]

Thus, there is at most one thick arc leaving node \( i \). If there is a node \( i \in V \) with no thick leaving arc, then using part (i), the lower bound from the wedge constraints (2.9) and (2.10), we have

\[
1 - O(\delta) \leq \sum_{j \in N^+(i)} x_{ij} - \sum_{j \in N^+(i)} x_{ij} = O(\delta),
\]

which is a contradiction. Hence, in the subgraph \( X = (V, A_1) \) every node has exactly one leaving arc. A similar argument shows that every node has exactly one entering thick arc, and therefore, \( A_1 \) is a spanning collection of node-disjoint directed cycles.

(iii) For the sake of contradiction, let us assume that there is a node \( i^* \in V \) such that there is no directed path from node 1 to \( i^* \). Let \( V^* \subseteq V \) be the set of nodes \( j \) such that there exists a directed path from \( j \) to \( i^* \). Since \( 1 \notin V^* \), (2.11) implies

\[
\sum_{\ell \in N^+(j)} x_{\ell j} = \beta \sum_{\ell \in N^-} x_{\ell j},
\]

for all \( j \in V^* \). Summing over all \( j \in V^* \), we obtain

\[
\sum_{j \in V^*} \sum_{\ell \in N^+(j)} x_{\ell j} = \beta \sum_{j \in V^+} \sum_{\ell \in N^-} x_{\ell j}.
\]

The sum on the left hand side is over all arcs starting in \( V^* \) and we split it into the sum over the arcs with both nodes in \( V^* \) and the arcs starting in \( V^* \) and ending in \( V \setminus V^* \). The sum on the right hand side is over all arcs ending in \( V^* \), and by the definition of \( V^* \) these arcs also start in \( V^* \). This gives

\[
\sum_{(j, \ell) \in E} x_{\ell j} + \sum_{(j, \ell) \in E} x_{\ell j} = \beta \sum_{(j, \ell) \in E} x_{\ell j} = \beta \sum_{(j, \ell) \in E} x_{\ell j},
\]

which simplifies to

\[
\sum_{(j, \ell) \in E} x_{\ell j} = -\delta \sum_{(j, \ell) \in E} x_{\ell j}.
\]

With \( \delta > 0 \) and the non-negativity of the \( x_{\ell j} \), it follows that

\[
\sum_{(j, \ell) \in E} x_{\ell j} = \sum_{(j, \ell) \in E} x_{\ell j} = 0,
\]

and this implies that \( x_{\ell j} = 0 \) for all \( j \in V^* \), \( \ell \in N^+(j) \). In particular,

\[
\sum_{\ell \in N^+(j^*)} x_{\ell j^*} = 0 < \beta^{n-1},
\]

which contradicts the wedge constraint (2.9).
(iv) We use Lemma 3.7, together with \( \Psi(i) = \beta = 1 - \delta \) for \( i \in U \), and \( \Psi(i) = \beta^{n-1} = 1 - (n-1)\delta + O(\delta^2) \) for \( i \in L \):

\[
\sum_{i \in Y} \Psi(i) = \sum_{i=1}^{n} \Psi(i) - \Psi(1) - |U|(1 - \delta) - |L|(1 - (n - 1)\delta) + O(\delta^2)
\]

\[
= n - \binom{n}{2} \delta - 1 - |U| - |L| + |U|\delta + (n - 1)|L|\delta + O(\delta^2)
\]

\[
= |Y| - \left( \frac{n}{2} \right) - |U| - (n - 1)|L| \delta + O(\delta^2).
\]

On the other hand, from \(|Y| = n - 1 - |U| - |L|\) and \(\beta^{n-1} \leq \Psi(i) \leq \beta\) for every \( i \in Y \), we obtain

\[(n - 1 - |U| - |L|)(1 - (n - 1)\delta) - O(\delta^2) \leq \sum_{i \in Y} \Psi(i) \leq (n - 1 - |U| - |L|)(1 - \delta),\]

and therefore,

\[n - 1 - |U| - |L| \leq \binom{n}{2} - |U| - (n - 1)|L| \leq (n - 1 - |U| - |L|).
\]

The first of these inequalities simplifies to \(|L| \leq (n - 1)/2\), and the second one to \(|U| \leq (n - 1)/2\). \(\Box\)

Combining Theorems 3.2 and 3.3 we establish the following results on the structure of feasible bases of \( WH_3(G) \) in the GNF setting.

**Theorem 3.4.** Let \((B, L, U) \in \Gamma(G)\) where \( B = Y \cup A_1 \cup A_2 \) and let \( B' = Y' \cup A'_1 \cup A'_2 \) be the corresponding arc sets in \( G' \), that is,

\[Y' = \{(v_i, w_i) : i \in Y\}, \quad A'_1 = \{(w_i, v_j) : (i, j) \in A_1\}, \quad A'_2 = \{(w_i, v_j) : (i, j) \in A_2\}.
\]

(i) \( B' \) is a good augmented tree spanning all nodes of \( G' \).

(ii) \( A'_1 \) is the unique perfect matching in \( G' \).

(iii) \( Y' \cup A'_1 \) is a collection of \( n - |Y| \) node-disjoint paths.

**Proof.** Item (i) follows from Theorem 3.2. Theorem 3.3 (ii) implies that the arcs in \( A'_1 \) form a perfect matching in \( G' \). With Lemma 3.3, there is at least one arc of the form \((v_i, w_i)\) in the extra cycle of \( B' \). As a consequence, \( B' - Y' \) is a forest. Since a forest has at most one perfect matching, \( A'_1 \) is the unique perfect matching. The last statement can be verified by adding the arcs in \( Y' \) one-by-one to \( A'_1 \). Clearly, we start with \( n \) node-disjoint paths, each of them being a single arc. And then in each step the new arc from \( Y' \) connects the endpoints of two paths, thereby reducing the number of paths by 1. \(\Box\)

The relation between the characterizations of feasible bases for the polytope \( WH_3(G) \) provided in Theorems 3.3 and 3.4 is illustrated in Figure 4.

4. A CLASS OF FEASIBLE BASES

For the approach to the Hamiltonian Cycle Problem based on sampling extreme points of \( WH_3(G) \), an essential ingredient is a lower bound for the fraction of Hamiltonian extreme points in the set of all extreme points. In order to derive such an estimate it will be useful to understand the structure of the feasible bases corresponding to a fixed subgraph of \( G \) or \( G' \). By Theorem 3.3, a subgraph of \( G \) which corresponds to a feasible basis contains a spanning collection of node-disjoint directed cycles. In \( G' \), this corresponds to a perfect matching between the sets \( \{w_i : i \in V\} \) and \( \{v_i : i \in V\} \), and Theorem 3.4 provides the additional information that the subgraph corresponding to the basic variables is a good augmented spanning tree.

In this section we make a further step by focusing on the feasible bases corresponding to a fixed (isomorphism class of) good augmented tree in \( G' \). More precisely, we characterize the feasible bases whose corresponding good augmented tree is a Hamilton cycle in \( G' \). This does not imply that the basis is Hamiltonian in the sense that it corresponds to a Hamilton cycle in \( G \) (see Figure 5a for an illustration). From Section 3.2 we know that at least one \( y \)-variable needs to be in the basis, that is, the good augmented tree must contain at least one arc of the form \((v_i, w_i)\). We restrict our attention further to the case that the basis contains exactly one
FEASIBLE BASES FOR A POLYTOPE RELATED TO THE HAMILTON CYCLE PROBLEM

(a) An example graph where the black filling indicates node 1.

(b) A feasible basis as a subgraph of $G$.

(c) The same basis as a subgraph of $G'$.  

Figure 4. A feasible basis $(B, L, U)$ for the graph in (a) is illustrated in (b). Here $B$ contains the $x$-variables for the shown arcs and the $y$-variables for the empty nodes, while $L$ is the set with the grey node as its only element, and $U$ is the empty set. The basic feasible solution for this basis is indicated by the arc and node labels, and the thin and thick arcs are indicated by the strength of the lines. The good augmented tree in $G'$ corresponding to this basis is shown in (c). In particular, the thick arcs form a spanning collection of node-disjoint cycles in (b), and a perfect matching in (c).

$y$-variable, say $y_s$. Starting with the thick arcs $(w_i, v_j)$, and adding the arc $(v_s, w_s)$ corresponding to the basic $y$-variable, we obtain $n - 2$ isolated arcs and one path of length 3 which consists of the thick arc into $v_s$, the arc $(v_s, w_s)$ and the thick arc leaving $w_s$. In order to obtain a Hamilton cycle, the thin arcs have to connect these components in such a way that for every $i \in V \setminus \{s\}$, the node $w_i$ has exactly one outgoing arc, and node $v_i$ has exactly one incoming arc. In $G$, this corresponds to every node $i \in V \setminus \{s\}$ having exactly one thick and one thin outgoing arc, and exactly one thick and one thin incoming arc, while the only arcs incident with node $s$ are the thick arcs into and out of $s$. Up to relabeling the variables, we can then assume that the thick arc into node $s$ is the arc $(s - 1, s)$, and that the thick arc out of node $s$ is the $s$-th thick arc if we traverse the Hamilton cycle in $G'$ starting with the thick arc out of $w_1$. More precisely, up to relabeling the variables, we may assume that there is a fixed-point free permutation $\pi$ of $V$ such that the following conditions are satisfied:

- The set of thick arcs is $\{(i, \pi(i)) : i \in V\}$.
- The set of thin arcs is $\{(i + 1, \pi(i)) : i \in [n] \setminus \{s\}\}$.
- $\pi(i) = i + 1$ if and only if $i = s - 1$.

In the last condition above, and throughout this section, whenever $i \in V$, $i \pm 1$ refers to the node $i \pm 1$ (mod $n$). For a basis $(B, L, U)$, the set $B$ of basic variables is determined by the permutation $\pi$, and we still have to choose a partition $V \setminus \{1, s\} = L \cup U$ of the set of non-basic $y$-variables such that $y_i = 0$ for $i \in L$ and $y_i = \beta - \beta^{n-1} = (n - 2)\delta - (n-1)\delta^2 + O(\delta^3)$ for $i \in U$. In Figure 5, we illustrate how a basis can be completely specified by a picture, using different types of nodes to indicate if $i \in L$, $i \in U$ or $y_i$ is a basic variable. The general structure of the bases considered in this section is illustrated in Figure 6.

Our aim is to characterize the 4-tuples $(s, \pi, L, U)$ of a node $s \in \{2, 3, \ldots, n\}$, a fixed-point free permutation $\pi$ with $\pi(i) = i + 1$ if and only if $i = s - 1$, and a partition $[n] \setminus \{1, s\} = L \cup U$, such that the basis given by the triple $(B, L, U)$ is feasible, where $B = \{s\} \cup \{(i, \pi(i)) : i \in [n]\} \cup \{(i, \pi(i-1)) : i \in [n] \setminus \{s\}\}$. The cycles of the permutation $\pi$ correspond to the thick cycles in $B$. In particular, the basis $(B, L, U)$ is quasi-Hamiltonian if and only if the permutation $\pi$ is a single cycle.

In order to simplify the notation, we introduce new variables. Let $\xi_i = x_{\pi(i)}$, $i \in [n]$ be the flow on the thick arc leaving node $i$, and let $\eta_i = x_{i+1, \pi(i)}$, $i \in [n] \setminus \{s - 1\}$ be the flow on the thin arc entering node $\pi(i)$. This is illustrated in Figure 7. For all $i \in [n] \setminus \{s\}$, $\xi_i + \eta_{i-1} = \Psi(i)$, and $\xi_{\pi^{-1}(i)} + \eta_{\pi^{-1}(i)} = \Phi(i)$, and
substituting the values of $\Phi(i)$ and $\Psi(i)$ leads to the following representation of the system (2.10)--(2.13):

\[
\begin{align*}
\xi_s - \beta \xi_{s-1} &= 0, \\
\xi_{\pi^{-1}(i)} + \eta_{\pi^{-1}(i)} &= \beta^{n-1}, \\
\xi_{\pi^{-1}(i)} + \eta_{\pi^{-1}(i)} &= \beta^{n-2} & i \in L, \\
\xi_{\pi^{-1}(i)} + \eta_{\pi^{-1}(i)} &= 1 & i \in U, \\
\xi_1 + \eta_n &= 1, \\
\xi_i + \eta_{i-1} &= \beta^{n-1} & i \in L, \\
\xi_i + \eta_{i-1} &= \beta & i \in U, \\
\xi_s - y_s &= \beta^{n-1}.
\end{align*}
\]

Figure 5. A quasi-Hamiltonian feasible basis (right) and a non-quasi-Hamiltonian (left) feasible basis for $n = 6$ vertices. In the top pictures the bases are shown as subgraphs of $G$, and node types indicate the sets $L$ (empty squares), $U$ (filled squares), basic $y$-variable (filled circle), and node 1 (empty circle). The pictures on the bottom illustrate that the corresponding good augmented tree is indeed a Hamilton cycle in $G'$.

Figure 6. The structure of the bases considered in this section. The dashed arc indicates the $y$-variable.
Lemma 4.1. The quadruple \((s, \pi, L, U)\) encodes a feasible basis if and only if the unique solution of system (4.1)-(4.8) satisfies \(\eta_i \geq 0\) for all \(i \in [n] \setminus \{s - 1\}\), and \(\beta^{n-1} \leq \xi_s \leq \beta\).

Proof. If the quadruple \((s, \pi, L, U)\) encodes a feasible basis, then \(\eta_i \geq 0\) for all \(i \in [n] \setminus \{s - 1\}\), and \(0 \leq y_s \leq \beta - \beta^{n-1}\), or equivalently, \(\beta^{n-1} \leq \xi_s \leq \beta\). For the converse, let \((\xi, \eta)\) be the solution of the system (4.1)-(4.8) for \((s, \pi, L, U)\), and assume that \(\eta_i > 0\) for all \(i \in [n] \setminus \{s - 1\}\), and \(\beta^{n-1} \leq \xi_s \leq \beta\). To verify feasibility, we only need to show that \(\xi_i > 0\) for all \(i \in [n]\). For \(i = s\), this is an immediate consequence of the assumption \(\beta^{n-1} \leq \xi_s \leq \beta\) which implies \(\xi_s = 1 + O(\delta)\). Using that the right hand sides of equations (4.2) to (4.7) are all \(1 + O(\delta)\), we obtain, for every \(i \in [n] \setminus \{s\}\)

\[
\xi_i = -\eta_{i-1} + 1 + O(\delta) = -(-\xi_{i-1} + 1 + O(\delta)) + 1 + O(\delta) = \xi_{i-1} + O(\delta),
\]

and by induction \(\xi_i = 1 + O(\delta) > 0\) for all \(i \in [n]\). \(\Box\)

Lemma (4.1) implies that in order to characterize the feasible bases, we need to derive necessary and sufficient conditions for \(\eta_i \geq 0\) for all \(i \in [n] \setminus \{s - 1\}\) and \(\beta^{n-1} \leq \xi_s \leq \beta\). We start by finding necessary conditions. Expressing \(\xi_s\) in terms of \(|L|\) and \(|U|\) (Lemma 4.2) allows us to specify \(|L|\) and \(|U|\) (Lemma 4.3), and deduce that for every feasible basis \(\pi(s) \in U\) (Lemma 4.4). As a consequence, we find asymptotic expressions for \(\eta_s\) (Lemma 4.5). Eliminating the \(\xi\)-variables from the system (4.1)-(4.8) leads to a recursion for the \(\eta_i\), where the base case is given by the value \(\eta_s\) determined in Lemma 4.5. The asymptotic expression obtained from solving this recursion is given in Lemma 4.6. Assuming that \(L\) and \(U\) have the right cardinalities and \(\pi(s) \in U\), the second order term of \(\eta_s\) is positive whenever the first order term vanishes (Lemma 4.7).

Theorem 4.1 combines the necessary conditions obtained so far into a characterization of feasible \((s, \pi, L, U)\) by three properties: (i) the cardinality condition for \(L\) and \(U\), (ii) \(\pi(s) \in U\), and (iii) the non-negativity of the linear terms in the asymptotic expansions of the variables \(\eta_i\).

Lemma 4.2. Let \((\xi, \eta)\) be the solution of the system (4.1)-(4.8) for \((s, \pi, L, U)\). Then

\[
\xi_s = 1 - \left[\frac{n}{2}\right] \delta + \left[\frac{n}{3}\right] \delta^2 + O(\delta^3).
\]

Proof. We have

\[
n - \left(\frac{n}{2}\right) \delta + \left(\frac{n}{3}\right) \delta^2 + O(\delta^3) = \sum_{i=1}^{n} \Psi(i) = 1 + \xi_s + |L| \beta^{n-1} + |U| \beta
\]

\[
= 1 + |L| + |U| - (|U| + (n - 1) |L|) \delta + |L| \left(\frac{n - 1}{2}\right) \delta^2 + \xi_s + O(\delta^3)
\]

\[
= n - 1 - (|U| + (n - 1) |L|) \delta + |L| \left(\frac{n - 1}{2}\right) \delta^2 + \xi_s + O(\delta^3),
\]

where the first equation comes from Lemma 3.7, the second one follows from \(\Psi(1) = 1\), \(\Psi(s) = \xi_s\), \(\Psi(i) = \beta^{n-1}\) for \(i \in L\) and \(\Psi(i) = \beta\) for \(i \in U\), the third one from substituting \(1 - \delta\) for \(\beta\), and the last one from \([n] = \{1, s\} \cup L \cup U\). The claim follows by solving for \(\xi_s\). \(\Box\)

Lemma 4.3. If \((s, \pi, L, U)\) encodes a feasible basis, then \(|U| = [(n - 2)/2]\) and \(|L| = [(n - 2)/2]\).
Proof. We start with $|L| + |U| = n - 2$. If $n$ is even, then Theorem 3.3(iv) implies $|L| \leq (n - 2)/2$ and $|U| \leq (n - 2)/2$, and the claim follows. For odd $n$, Theorem 3.3(iv) implies only $|L|, |U| \leq (n - 1)/2$, and therefore $\{L, |U|\} = \{(n - 1)/2, (n - 3)/2\}$. For the sake of contradiction, assume $|U| = (n - 1)/2$ and $|L| = (n - 3)/2$. By Lemma 4.2,

$$\xi_s = 1 - \left[ \binom{n}{2} - \frac{n - 1}{2} - \frac{(n - 1)(n - 3)}{2} \right] \delta + \left[ \binom{n}{3} - \frac{n - 1}{2} - \frac{n - 3}{2} \right] \delta^2 + O(\delta^3)$$

$$= 1 - (n - 1)\delta - \frac{1}{2} \binom{n - 1}{3} \delta^2 + O(\delta^3) < \beta^{n - 1},$$

which is the required contradiction.

Lemma 4.4. If $(s, \pi, L, U)$ encodes a feasible basis then $\pi(s) \in U$.

Proof. Suppose $\pi(s) \notin L \cup \{1\}$. From Lemmas 4.2 and 4.3, we have that $\xi_s = 1 - (n/2)\delta + O(\delta^2)$ if $n$ is even, and $\xi_s = 1 - \delta + O(\delta^2)$ if $n$ is odd. From (4.2) and (4.3), it follows that

$$\eta_s \leq \beta^{n - 2} - \xi_s \leq -(n - 2)\delta - 1 + (n/2)\delta + O(\delta^2) = (2 - n/2)\delta + O(\delta^2) < 0,$$

which is the required contradiction.

In the next lemma, we use Lemma 4.2 to determine the asymptotics for the variables $\xi_s$ and $\eta_s$ assuming the necessary conditions from Lemma 4.3 and 4.4.

Lemma 4.5. Suppose that $|L| = \lfloor (n - 2)/2 \rfloor$ and $|U| = \lfloor (n - 2)/2 \rfloor$ and $\pi(s) \in U$.

(i) If $n$ is even then $\eta_s = 1 - \xi_s = (n/2)\delta + \frac{(n - 1)(n - 2)(n - 6)}{12} \delta^2 + O(\delta^3)$.

(ii) If $n$ is odd then $\eta_s = 1 - \xi_s = \delta + \frac{(n - 1)(n - 2)(n - 3)}{12} \delta^2 + O(\delta^3)$.

Proof. We apply Lemma 4.2. For even $n$,

$$\xi_s = 1 - \left[ \binom{n}{2} - \frac{n - 2}{2} - \frac{(n - 1)(n - 2)}{2} \right] \delta + \left[ \binom{n}{3} - \frac{n - 1}{2} - \frac{n - 2}{2} \right] \delta^2 + O(\delta^3)$$

$$= 1 - (n/2)\delta - \frac{(n - 1)(n - 2)(n - 6)}{12} \delta^2 + O(\delta^3),$$

and for odd $n$,

$$\xi_s = 1 - \left[ \binom{n}{2} - \frac{n - 3}{2} - \frac{(n - 1)^2}{2} \right] \delta + \left[ \binom{n}{3} - \frac{n - 1}{2} \right] \delta^2 + O(\delta^3)$$

$$= 1 - \delta - \frac{(n - 1)(n - 2)(n - 3)}{12} \delta^2 + O(\delta^3).$$

In both cases, the claim follows since $\eta_s = 1 - \xi_s$ by (4.4) for $i = \pi(s) \in U$.

We set $R = \{1\}$, and partition the set $[n] \setminus \{s, s - 1\}$ into nine sets $W_{PQ}$ with $P, Q \in \{L, U, R\}$, where $W_{PQ} = \{i \in [n] \setminus \{s\} : i \in P, \pi(i) \in Q\}$. In particular, $W_{RR} = \emptyset$. For every $i \in [n] \setminus \{s, s - 1\}$, we have $\eta_i + \xi_i = \Phi(\pi(i))$ and $\eta_{i+1} + \xi_i = \Psi(i)$. Eliminating the $\xi$-variables, we obtain the recursion

$$\eta_i = \eta_{i+1} + \gamma_i \quad \quad \quad \quad i \in [n] \setminus \{s - 1, s\},$$

(4.9)
If Lemma 4.6.

For equality case.

By assumption

Let

The following lemma is obtained from Lemma 4.5 and repeated application of (4.9).

Lemma 4.6. If \(|L| = \lfloor(n - 2)/2\rfloor, \ |U| = \lfloor(n - 2)/2\rfloor and \(\pi(s) \in U\), then for all \(i \in [n] \setminus \{s, s-1\}\),

\[
\eta_i = \left(\alpha_{s_1} + \sum_{j \in I(i)} \alpha_{j_1}\right) \delta + \left(\alpha_{s_2} + \sum_{j \in I(i)} \alpha_{j_2}\right) \delta^2 + O(\delta^3).
\]

We want to derive necessary and sufficient conditions for \(\eta_i \geq 0\). Clearly, it is necessary that \(\alpha_{s_1} + \sum_{j \in I(i)} \alpha_{j_1} \geq 0\), and if the inequality is strict then this is also sufficient. The next lemma deals with the equality case.

Lemma 4.7. Let \(n \geq 5\) and suppose \(|L| = \lfloor(n - 2)/2\rfloor, \ |U| = \lfloor(n - 2)/2\rfloor and \(\pi(s) \in U\). For all \(i \in [n] \setminus \{s, s-1\}\),

\[
\alpha_{s_1} + \sum_{j \in I(i)} \alpha_{j_1} = 0 \implies \alpha_{s_2} + \sum_{j \in I(i)} \alpha_{j_2} > 0.
\]

Proof. By assumption

\[
\alpha_{s_1} + N_{UU}(i) + N_{LL}(i) - (n - 3)N_{UL}(i) + (n - 1)N_{LU}(i) - (n - 2)N_{UR}(i) - (n - 2)N_{RL}(i) = 0,
\]

and thus

\[
(n - 3)N_{UL}(i) - (n - 1)N_{LU}(i) = \alpha_{s_1} + N_{UU}(i) + N_{LL}(i) - (n - 2)N_{UR}(i) - (n - 2)N_{RL}(i). \quad (4.12)
\]
For the second order term, we obtain
\[
\frac{2}{n-2} \left( \alpha_{s_2} + \sum_{j \in I(i)} \alpha_{j_2} \right) = \frac{2\alpha_{s_2}}{n-2} + (n-3)N_{UL}(i) - (n-1)N_{LU}(i) \\
+ (n-1)N_{UR}(i) + (n-3)N_{RL}(i) - 2N_{LL}(i) \\
\geq \frac{2\alpha_{s_2}}{n-2} + \alpha_{s_1} + N_{LU}(i) + N_{UR}(i) - N_{RL}(i) - N_{LL}(i) \\
\geq \frac{2\alpha_{s_2}}{n-2} + \alpha_{s_1} - (N_{RL}(i) + N_{LL}(i)) > \frac{2\alpha_{s_2}}{n-2} + \alpha_{s_1} - |L|.
\]

Substituting the values for \( \alpha_{s_1} \) and \( \alpha_{s_2} \) we obtain for even \( n \),
\[
\frac{2}{n-2} \left( \alpha_{s_2} + \sum_{j \in I(i)} \alpha_{j_2} \right) \geq \frac{1}{6}(n-1)(n-6) + \frac{n}{2} - \frac{n-2}{2} = \frac{1}{6}(n-1)(n-6) + 1 = \frac{(n-4)(n-3)}{6} > 0,
\]
and for odd \( n \),
\[
\frac{2}{n-2} \left( \alpha_{s_2} + \sum_{j \in I(i)} \alpha_{j_2} \right) \geq \frac{1}{6}(n-1)(n-3) + 1 - \frac{n-1}{2} = \frac{1}{6}(n-1)(n-3) - \frac{n-3}{2} = \frac{(n-4)(n-3)}{6} > 0. \quad \square
\]

**Theorem 4.1.** For \( n \geq 5 \), the basis encoded by \((s, \pi, L, U)\) is feasible if and only if the following conditions are satisfied:

(i) \(|U| = [(n-2)/2]\) and \(|L| = [(n-2)/2]\),

(ii) \(\pi(s) \in U\),

(iii) For every \( i \in [n] \setminus \{s, s-1\} \), \( \alpha_{s_1} + \sum_{j \in I(i)} \alpha_{j_1} \geq 0 \).

**Proof.** Suppose that the basis encoded by \((s, \pi, L, U)\) is feasible. Conditions (i) and (ii) follow from Lemmas 4.3 and 4.4, respectively, and Condition (iii) from Lemmas 4.3 and 4.6 together with \( \eta_i \geq 0 \). Conversely, suppose that the three conditions in the theorem are satisfied. By Lemma 4.1, it is sufficient to verify \( \beta^{n-1} \leq \xi_s \leq \beta \) and \( \eta_i \geq 0 \) for all \( i \in [n] \setminus \{s-1\} \). The first of these conditions follows immediately from Lemma 4.5, and the second one from (iii) together with Lemmas 4.6 and 4.7. \( \square \)

Next, we simplify Theorem 4.1(iii) and provide another characterization of feasible bases in Theorems 4.2 and 4.3. To do this, by using Lemmas 4.3 and 4.4 we deduce a restriction for the possible values of \( s-1 \) (Lemma 4.8). Furthermore, in Lemmas 4.9 and 4.10, we deduce conditions on the number of different types of arcs in various parts of the good augmented tree, where the type of an arc is determined by where it starts (in \( L, U \), or \( R \)) and where it ends (in \( L, U \) or \( R \)).

**Lemma 4.8.** If \((s, \pi, L, U)\) encodes a feasible basis, then \( s-1 \in R \) if \( n \) is odd, and \( s-1 \in R \cup U \) if \( n \) is even.

**Proof.** By Lemmas 4.3 and 4.4, \( \xi_s = 1 - \alpha_{s_1} \delta + O(\delta^2) \), and then (4.1) implies \( \xi_{s-1} = 1 - (\alpha_{s_1} - 1) \delta + O(\delta^2) \). If \( s-1 \in L \) then (4.6) implies
\[
\eta_{s-2} = \beta^{n-1} - \xi_{s-1} = 1 - (n-1)\delta - 1 + (\alpha_{s_1} - 1) \delta + O(\delta^2) = (\alpha_{s_1} - n) \delta + O(\delta^2) < 0.
\]
If \( n \) is odd then \( \alpha_{s_1} = 1 \), and for \( s-1 \in U \), (4.7) implies
\[
\eta_{s-2} = \beta - \xi_{s-1} = 1 - \delta - 1 + O(\delta^2) = -\delta + O(\delta^2) < 0. \quad \square
\]

**Lemma 4.9.** If \( \pi(s) \in U \) and \( s-1 \in R \cup U \) then \( |W_{UL}| + |W_{UR}| + |W_{RL}| = |W_{LU}| + 1 \).

**Proof.** We start with \( U = (U \cap \pi^{-1}(U)) \cup (U \cap \pi^{-1}(L)) \cup (U \cap \pi^{-1}(R)) \cup \{U \cap \{s-1\}\} \) which implies
\[
|U| = |W_{UU}| + |W_{UL}| + |W_{UR}| + \begin{cases}
1 & \text{if } s-1 \in U, \\
0 & \text{if } s-1 \in R.
\end{cases} \quad (4.13)
\]
On the other hand, $U = (U \cap \pi(U)) \cup (U \cap \pi(L)) \cup (U \cap \pi(R)) \cup (U \cap \pi(s))$, hence

$$|U| = |W_{LU}| + |W_{UR}| + |W_{RL}| + 1. \quad (4.14)$$

From (4.13) and (4.14), we obtain

$$|W_{UL}| + |W_{UR}| = |W_{LU}| + |W_{RL}| + \begin{cases} 1 & \text{if } s - 1 \in R, \\ 0 & \text{if } s - 1 \in U. \end{cases} \quad (4.15)$$

Now

$$|W_{UL}| + |W_{UR}| + |W_{RL}| \overset{(4.15)}{=} |W_{LU}| + |W_{RL}| + \begin{cases} 1 & \text{if } s - 1 \in R, \\ 0 & \text{if } s - 1 \in U = |W_{LU}| + 1, \end{cases}$$

where the last equality comes from the observation that $W_{RU} \cup W_{RL} = \emptyset$ if $s - 1 = 1$ and $W_{RU} \cup W_{RL} = \{1\}$ if $s - 1 \in U \cup L$. \hfill $\square$

**Remark 2.** Lemma 4.9 implies that if $s - 1 \in R \cup U$ and $\pi(s) \in U$, then there exists a unique index $i^* = i^*(\pi, L, U) \in [n] \setminus \{s, s - 1\}$ such that $N_{UL}(i^*) + N_{UR}(i^*) + N_{RL}(i^*) = N_{LU}(i^*) + 1$ and $N_{UL}(i) + N_{UR}(i) + N_{RL}(i) \leq N_{LU}(i)$ for all $i \in I(i^* - 1)$.

**Lemma 4.10.** Suppose the quadruple $(s, \pi, L, U)$ encodes a feasible basis, and let $i^* = i^*(\pi, L, U)$. Then

(i) If $n$ is odd, then $i^* = n$.

(ii) If $n$ is even, then

- $N_{UL}(i) + N_{UR}(i) + N_{RL}(i) \leq N_{LU}(i) + 1$ for all $i \in [n] \setminus \{s, s - 1\}$, and
- $|I(i^*) \setminus \{1, \pi^{-1}(1)\}| \geq (n - 4)/2$.

**Proof.** Theorem 4.1(iii) together with (4.11) implies

$$0 \leq \sum_{j \in I(i^*)} \alpha_{j1} = \alpha_{s1} + N_{UU}(i^*) + N_{LL}(i^*) + (3 - n)N_{UL}(i^*) + (n - 1)N_{LU}(i^*)$$

$$+ (2 - n)(N_{UR}(i^*) + N_{RL}(i^*)),$$

hence

$$0 = \alpha_{s1} + N_{UU}(i^*) + N_{LL}(i^*) + N_{UL}(i^*) + N_{LU}(i^*) + (n - 2)[N_{LU}(i^*) - N_{UL}(i^*) - N_{UR}(i^*) - N_{RL}(i^*)]$$

$$= \alpha_{s1} + N_{UU}(i^*) + N_{LL}(i^*) + N_{UL}(i^*) + N_{LU}(i^*) + N_{LU}(i^*) - (n - 2)$$

$$= \alpha_{s1} + |I(i^*) \setminus \{1, \pi^{-1}(1)\}| - (n - 2),$$

and therefore,

$$|I(i^*) \setminus \{1, \pi^{-1}(1)\}| \geq n - 2 - \alpha_{s1} = \begin{cases} n - 3 & \text{if } n \text{ is odd}, \\ (n - 4)/2 & \text{if } n \text{ is even}. \end{cases} \quad (4.16)$$

If $n$ is odd then $|I(i) \setminus \{1, \pi^{-1}(1)\}| \leq |I(i)| \leq i - 2$ for all $i \in [n] \setminus \{s, s - 1\} = \{3, 4, \ldots, n\}$, and together with $\pi^{-1}(1) \in I(n - 1)$, we deduce $i^* = n$. Now let $n$ be even, and assume that there exists $i \in I(s - 2) \setminus I(i^*)$ such that $N_{UL}(i) + N_{UR}(i) + N_{RL}(i) \geq N_{LU}(i) + 2$. As before we apply Theorem 4.1(iii) together with (4.11) and $\alpha_{s1} = n/2:

$$0 \leq n/2 + N_{UU}(i) + N_{LL}(i) + N_{UL}(i) + N_{LU}(i) + (n - 2)[N_{LU}(i) - N_{UL}(i) - N_{UR}(i) - N_{RL}(i)]$$

$$\leq n/2 + N_{UU}(i) + N_{LL}(i) + N_{UL}(i) + N_{LU}(i) - 2(n - 2) \leq -3n/2 + 4 + (n - 3) = 1 - n/2 < 0,$$

which is the required contradiction. \hfill $\square$

In the next two theorems we show that the necessary conditions from Lemmas 4.3, 4.4, 4.8, and 4.10 are also sufficient.

**Theorem 4.2.** Let $n = 2k + 1$. The quadruple $(s, \pi, L, U)$ encodes a feasible basis if and only if the following conditions are satisfied.

(i) $|U| = k - 1$ and $|L| = k$.

(ii) $\pi(s) \in U,$
by checking the feasibility of the three bases represented in Figure 8 and

\[ N_{LU}(i) - N_{UL}(i) - N_{UR}(i) \geq 0 \quad \text{for all } i \in [n] \setminus \{1, 2, n\}, \]

\[ N_{LU}(n) - N_{UL}(n) - N_{UR}(n) = -1. \]  

Now

\[ \alpha_{i} + \sum_{j \in I(i)} \alpha_{ij} = 1 + N_{UU}(i) + N_{LL}(i) + (3 - n)N_{UL}(i) + (n - 1)N_{LU}(i) + (2 - n)N_{UR}(i) \]

\[ = 1 + N_{UU}(i) + N_{LL}(i) + N_{UL}(i) + N_{LU}(i) + (n - 2) [N_{LU}(n) - N_{UL}(n) - N_{UR}(n)] \]

\[ \geq 1 + N_{UU}(i) + N_{LL}(i) + N_{UL}(i) + N_{LU}(i) \geq 0, \]

for all \( i \in [n] \setminus \{1, 2, n\} \), and

\[ \sum_{j \in I(n)} \alpha_{ij} = 1 + N_{UU}(n) + N_{LL}(n) + N_{UL}(n) + N_{LU}(n) + (n - 2) [N_{LU}(n) - N_{UL}(n) - N_{UR}(n)] \]

\[ \geq 3 - n + N_{UU}(n) + N_{LL}(n) + N_{UL}(n) + N_{LU}(n) = 3 - n + |I(n) \setminus \{\pi^{-1}(1), 2\}| = 0. \]

where the last equality is a consequence of \( |I(n) \setminus \{\pi^{-1}(1), s\}| = n - 3. \)

**Theorem 4.3.** Let \( n = 2k \) and let \( i^* = i^*(\pi, L, U) \). The quadruple \((s, \pi, L, U)\) encodes a feasible basis if and only if the following conditions are satisfied.

(i) \( |U| = |L| = k - 1 \),

(ii) \( \pi(s) \in U \),

(iii) \( s - 1 \in R \cup U \),

(iv) \( |I(i^*) \setminus \{1, \pi^{-1}(1)\}| \geq (n - 4)/2 \),

(v) \( N_{UL}(i) + N_{UR}(i) + N_{RL}(i) \leq N_{LU}(i) + 1 \) for all \( i \in I(s - 2) \setminus I(i^*) \).

Proof. If the basis corresponding to \((s, \pi, L, U)\) is feasible then conditions (i) to (iv) are implied by Lemmas 4.3, 4.8 and 4.4, 4.10, and 4.10. For the converse, suppose that the conditions are satisfied. From Theorem 4.3 together with (i) and (ii) it follows that it is sufficient to verify \( \alpha_{i} + \sum_{j \in I(i)} \alpha_{ij} \geq 0 \) for all \( i \in [n] \setminus \{s, s - 1\} \). From Lemma 4.9 with (ii) and (iii), \( i^* \) is well defined. Using (v) and the definition of \( i^* \)

\[ N_{LU}(i) - N_{UL}(i) - N_{UR}(i) - N_{RL}(i) \geq 0 \quad \text{for all } i \in I(i^* - 1), \]

\[ N_{LU}(i) - N_{UL}(i) - N_{UR}(i) - N_{RL}(i) \geq -1 \quad \text{for all } i \in I(s - 2) \setminus I(i^* - 1). \]  

Now

\[ \alpha_{i} + \sum_{j \in I(i)} \alpha_{ij} = n/2 + N_{UU}(i) + N_{LL}(i) + (3 - n)N_{UL}(i) + (n - 1)N_{LU}(i) + (2 - n)N_{UR}(i) + N_{RL}(i) \]

\[ \geq n/2 + N_{UU}(i) + N_{LL}(i) + N_{UL}(i) + N_{LU}(i) + (n - 2) [N_{LU}(i) - N_{UL}(i) - N_{UR}(i) - N_{RL}(i)] \]

\[ \geq n/2 + 2 - n + N_{UL}(i) + N_{LL}(i) + N_{UL}(i) + N_{LU}(i) \quad \text{for all } i \in I(i^* - 1), \]

\[ \geq n/2 + 2 - n + N_{UL}(i) + N_{LL}(i) + N_{UL}(i) + N_{LU}(i) \quad \text{for all } i \in I(s - 2) \setminus I(i^* - 1). \]

We illustrate Theorems 4.1, 4.2, and 4.3 by checking the feasibility of the three bases represented in Figure 8. It is immediately clear that all three examples satisfy the first two conditions in Theorem 4.1: the number of empty squares is at least the number of filled squares and at most the number of filled squares plus one, and the arc leaving the filled circle ends in a filled square.
Example 1 (Figure 8(a)). We have $5 \in W_{UL}$, $6 \in W_{UL}$, $1 \in W_{RL}$, $2 \in W_{UR}$, and therefore
\[ \alpha_{21} + \alpha_{51} + \alpha_{61} + \alpha_{11} = 3 + 5 - 3 - 4 - 4 = -3 < 0, \]
which implies that the basis is infeasible. The fact that a negative partial sum occurs in the very end corresponds to the fact that the third condition in Theorem 4.3 is violated as $3 = s - 1 \in L$.

Example 2 (Figure 8(b)). Using $3 \in W_{UL}$, $4 \in W_{UL}$, $5 \in W_{UL}$ and $7 \in W_{UL}$, we obtain that all partial sums in $\alpha_{21} + \alpha_{31} + \cdots + \alpha_{71} = 1 + 6 + 1 - 4 + 0 - 4 = 0$ are non-negative, and the basis is feasible. We get the same result by verifying the conditions in Theorem 4.2:
(i) There are $k - 1 = 2$ filled squares and $k = 3$ empty squares.
(ii) The arc leaving the filled circle ends in a filled square.
(iii) There is a thick arc from node 1 to node 2.
(iv) With $f(i) = N_{UL}(i) + N_{UR}(i) + N_{RL}(i) - N_{LU}(i)$, we have $(f(3), f(4), f(5), f(6), f(7)) = (-1, -1, 0, 0, 1)$, hence $i^* = 7$.

Example 3 (Figure 8(c)). Using $6 \in W_{UL}$, $7 \in W_{UL}$, $8 \in W_{UL}$, $1 \in W_{RL}$, $2 \in W_{UR}$ and $3 \in W_{UR}$, we obtain that all partial sums in $\alpha_{21} + \alpha_{61} + \alpha_{71} = 4 + 7 + 1 - 5 - 6 + 7 - 6 = 2$ are non-negative, so the basis is feasible. We get the same result by verifying the conditions in Theorem 4.3:
(i) There are $k - 1 = 2$ filled squares and $k = 1$ empty squares.
(ii) The thick arc leaving the filled circle ends in a filled square.
(iii) The thick arc entering the filled circle starts in a filled square or the empty circle.

With $f(i) = N_{UL}(i) + N_{UR}(i) + N_{RL}(i) - N_{LU}(i)$, we have $(f(6), f(7), f(8), f(1), f(2), f(3)) = (-1, -1, 0, 1, 0, 1)$, hence $i^* = 1$, and then
(iv) $|I(1) \setminus \{1, \pi_i^{-1}(1)\}| = |\{6, 7, 8\}| = 3 \geq (8 - 4)/2 = 2$, and
(v) $f(i) \leq 1$ for all $i \in \{6, 7, 8, 1, 2, 3\}$.

For small $n$, we can use the characterizations given in Theorems 4.1, 4.2 and 4.3 to count feasible bases of the type considered in this section. For $n = 5$, this can be done by hand: By Theorem 4.2, $s = 2$ and $\pi(2) \in U$, in particular $\pi(2) \neq 1$. Together with $\pi(i) \neq i$ for all $i$, and $\pi(i) \neq i + 1$ for all $i \neq 1$, this leaves only two permutations: $\pi_1 = (12543)$ and $\pi_2 = (124)(35)$. Then $U$ is fixed because $|U| = 1$ and $\pi(2) \in U$, hence $U = \{5\}$ for $\pi_1$ and $U = \{4\}$ for $\pi_2$. For $\pi_1$ this implies $3 \in W_{LR}$, $4 \in W_{UL}$ and $5 \in W_{UL}$, and the basis is feasible because the partial sums of $1 + 0 + 1 - 2$ are all non-negative. For $\pi_2$, $3 \in W_{LL}$ and $4 \in W_{UR}$, so the basis is infeasible as $1 + 1 - 3 < 0$. For $n \geq 6$, we implemented an algorithm that runs through all combinations $(s, \pi, U, L)$ and checks the feasibility conditions, using Theorems 4.1, 4.3 and 4.3 to limit the search space. The results for $n \in \{6, \ldots, 10\}$ are presented in Tables 1 to 5. Here the columns (except the
first, and for even \( n \) the last) correspond to the values of \( s \), and the rows correspond to cycle types, where a tuple \((l_1, \ldots, l_t)\) indicates that \( \pi \) has \( t \) cycles with lengths \( l_1, \ldots, l_t \), and \( l_1 \) is the length of the cycle containing node 1. In particular, the first row corresponds to the quasi-Hamiltonian feasible bases. The shaded cells indicate the number of quasi-Hamiltonian feasible bases and the total number of feasible bases.

**Table 1. Numbers of feasible bases for \( n = 6 \).**

| \( s \) | 2 | 3 | 4 | 5 | 6 | total |
|--------|---|---|---|---|---|-------|
| (6)    | 17 | 6 | 5 | 6 | 3 | 36    |
| (4,2)  | 6  | 5 | 5 | 6 | 3 | 25    |
| (3,3)  | 3  | 2 | 1 | 1 | 0 | 7     |
| (2,4)  | 0  | 2 | 1 | 1 | 0 | 4     |
| (2,2,2)| 0  | 3 | 1 | 2 | 1 | 7     |
| total  | 26 | 17| 14| 15| 7 | 79    |

**Table 2. Numbers of feasible bases for \( n = 7 \).**

| \( s \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | total |
|--------|---|---|---|---|---|---|---|-------|
| (7)    | 35|
| (5,2)  | 18|
| (4,3)  | 11|
| (3,4)  | 7 |
| (3,2,2)| 3 |
| total  | 74|

**Table 3. Numbers of feasible bases for \( n = 8 \).**

| \( s \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | total |
|--------|---|---|---|---|---|---|---|---|-------|
| (8)    | 1026| 458| 460| 422| 418| 338| 219| 3341|
| (6,2)  | 516 | 333| 339| 321| 313| 300| 263| 169 |2723 |
| (5,3)  | 337 | 145| 134| 121| 115| 106| 106 |1038 |
| (4,4)  | 247 | 120| 110| 113| 108| 107| 107 |1181 |
| (3,5)  | 178 | 116| 88 | 92 | 82 | 81 | 54 | 638 |
| (2,6)  | 0   | 119| 75 | 72 | 71 | 70 | 28 | 435 |
| (4,2,2)| 128 | 112| 110| 110| 117| 81 | 55 | 713 |
| (2,4,2)| 0   | 115| 83 | 83 | 84 | 74 | 47 | 473 |
| (3,3,2)| 152 | 128| 104| 108| 99 | 74 | 47 | 712 |
| (2,3,3)| 0   | 38 | 19 | 17 | 16 | 22 | 11 | 123 |
| (2,2,2)| 0   | 27 | 22| 22 | 26 | 17 | 10 | 124 |
| total  | 2584| 1711|1544|1483|1466|1175|733 |10696|

**Table 4. Numbers of feasible bases for \( n = 9 \).**

| \( s \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | total |
|--------|---|---|---|---|---|---|---|---|-------|
| (9)    | 3891|
| (7,2)  | 1932|
| (6,3)  | 1294|
| (5,4)  | 954 |
| (4,5)  | 788 |
| (3,6)  | 490 |
| (5,2,2)| 468 |
| (4,3,2)| 651 |
| (3,4,2)| 357 |
| (3,3,3)| 159 |
| (3,2,2)| 56  |
| total  | 11040|

In these tables, we observe that the number of feasible bases decreases as the number of cycles increases, and that for a fixed number of cycles the number of feasible bases is larger if node 1 is on a long cycle. In view of Conjecture 1, we are interested in the ratio between the number of quasi-Hamiltonian bases and the total number of feasible bases, and we would like this ratio to be bounded below by 1 divided by a polynomial function of \( n \). Defining \( a_n \) and \( b_n \) to be the numbers of quasi-Hamiltonian feasible bases and the total number of feasible bases in the considered class, respectively, we can summarize our counting results as shown in Table 6.

5. Summary and conclusion

In this paper we continue to study the polytope \( \text{WH}_\beta(G) \) which was introduced by Eshragh et al. [12] in connection with a random walk based approach to the Hamilton Cycle Problem, and investigated further in [10, 11]. Two ingredients are needed in order to make this approach work: (1) if the input graph is Hamiltonian then there need to be sufficiently many feasible bases corresponding to Hamiltonian cycles, and (2) the graph of the polytope needs to have good mixing properties. In order to make progress towards establishing these two properties, a good understanding of the combinatorial structure of \( \text{WH}_\beta(G) \) is needed, and in this paper we present significant results in this direction:
Table 5. Numbers of feasible bases for $n = 10$.

|    | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | total |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-------|
| (10)| 163701 | 83664 | 79720 | 74812 | 72468 | 69116 | 58400 | 50696 | 36127 | 688704 |
| (8,2)| 81890 | 56040 | 53980 | 51041 | 49613 | 47595 | 40438 | 34783 | 24939 | 440319 |
| (7,3)| 55099 | 27526 | 25563 | 23849 | 23073 | 22028 | 18711 | 16537 | 12276 | 224662 |
| (6,4)| 40832 | 21170 | 20070 | 18836 | 18200 | 17624 | 14926 | 12958 | 9268  | 173884 |
| (5,5)| 32416 | 16844 | 15986 | 15197 | 14749 | 13744 | 11212 | 9870  | 7110  | 137128 |
| (4,6)| 27019 | 14178 | 13004 | 12550 | 12077 | 11491 | 9689  | 8390  | 5851  | 114249 |
| (3,7)| 20834 | 14224 | 11289 | 10585 | 9816  | 7704  | 6305  | 4088  | 95815 |
| (2,8)| 0    | 14343 | 10520 | 10046 | 9722  | 8834  | 7518  | 7011  | 3856  | 71850  |
| (6,2,2)| 20428 | 17493 | 16895 | 16071 | 15672 | 15245 | 13041 | 11121 | 7987  | 133953 |
| (2,6,2)| 0    | 11948 | 9180  | 8849  | 8592  | 7871  | 6843  | 6247  | 3621  | 63151  |
| (5,3,2)| 27320 | 18625 | 17596 | 16720 | 16241 | 15256 | 12735 | 11215 | 8349  | 144057 |
| (3,5,2)| 14514 | 12835 | 10735 | 10524 | 10258 | 9461  | 7399  | 6148  | 4185  | 86059  |
| (2,5,3)| 0    | 7619  | 5464  | 5217  | 5082  | 4527  | 3732  | 3592  | 2083  | 37307  |
| (4,4,2)| 20235 | 14289 | 13344 | 12873 | 12409 | 12060 | 10245 | 8775  | 6223  | 110453 |
| (2,4,4)| 0    | 3637  | 2662  | 2527  | 2441  | 2277  | 1979  | 1810  | 1010  | 18343  |
| (4,3,3)| 9194  | 4600  | 3994  | 3814  | 3674  | 3497  | 2972  | 2674  | 2025  | 36453  |
| (3,4,3)| 12247 | 8298  | 6394  | 6180  | 5943  | 5584  | 4415  | 3683  | 2529  | 55273  |
| (4,2,2,2)| 3354  | 3504  | 3324  | 3219  | 3124  | 3118  | 2675  | 2254  | 1607  | 26179  |
| (2,4,2,2)| 0    | 5365  | 4233  | 4096  | 3987  | 3751  | 3350  | 2959  | 1777  | 29158  |
| (3,3,2,2)| 6061  | 6490  | 5518  | 5430  | 5315  | 4896  | 3879  | 3274  | 2346  | 43209  |
| (2,3,3,2)| 0    | 3963  | 2958  | 2842  | 2789  | 2455  | 2045  | 1967  | 1232  | 20251  |
| (2,2,2,2)| 0    | 582   | 472   | 463   | 448   | 446   | 406   | 343   | 213   | 3373   |

Table 6. The proportion of quasi-Hamiltonian feasible bases in the class of feasible bases considered in this section.

| $n$ | $a_n$ | $b_n$ | $na_n/b_n$ |
|-----|-------|-------|-------------|
| 5   | 1     | 1     | 5.0000      |
| 6   | 36    | 79    | 2.7342      |
| 7   | 35    | 74    | 3.3108      |
| 8   | 3341  | 10696 | 2.4989      |
| 9   | 3891  | 11040 | 3.1720      |
| 10  | 688704| 2754190| 2.5006     |
| 11  | 801114| 2884325| 3.0552     |
| 12  | 234123800| 1113400022| 2.5233   |
| 13  | 269326587| 1172169769| 2.9870    |

(1) The set of feasible bases does not depend on the value of $\beta$ as long as it is sufficiently close to 1.
(2) $WH_\beta(G)$ can be interpreted as a generalized network flow polytope, and this interpretation leads to a nice graph theoretical interpretation of the structure of feasible bases.
(3) For a special class of bases, we prove a complete characterization of the feasible bases.
(4) We illustrate the characterization of feasible bases for small values of $n$ and present computational results supporting Conjecture 1.

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